รายงานวิจัยฉบับสมบูรณ์

โครงการ

การระบายสีจุดยอดแบบเท่าเทียมในบางคลาสของกราฟ

Equitable vertex colorings in some classes of graphs

(ทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่)

โดย

ผศ.ดร. เกียรติสุดา นาคประสิทธิ์ และคณะ

กรกฎาคม 2557

สัญญาเลขที่ MRG5580003

รายงานวิจัยฉบับสมบูรณ์

โครงการ : การระบายสีจุดยอดแบบเท่าเทียมในบางคลาสของกราฟ
Equitable vertex colorings in some classes of graphs

(ทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่)

ผศ.ดร. เกียรติสุดา นาคประสิทธิ์ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์
(หัวหน้าโครงการวิจัยผู้รับทุน) มหาวิทยาลัยขอนแก่น

ศาตราจารย์ ดร. ณรงค์ ปั้นนิ่ม ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์
(นักวิจัยที่ปรึกษา) มหาวิทยาลัยศรีนครินทรวิโรฒ

สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา
และ สำนักงานกองทุนสนับสนุนการวิจัย และ มหาวิทยาลัยขอนแก่น
(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ สกว.ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณสำนักงานคณะกรรมการการอุดมศึกษา สำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยขอนแก่น ที่ได้ให้โอกาสผู้วิจัยได้รับทุนเพื่อเป็นการพัฒนาศักยภาพในการทำงานวิจัยอาจารย์รุ่นใหม่ในการทำงานวิจัยครั้งนี้

ศาสตราจารย์ ดร.ณรงค์ ปั้นนิ่ม นักวิจัยที่ปรึกษาของโครงการนี้ ผู้ซึ่งให้คำแนะนำ สั่งสอน และถ่ายทอดความรู้ทางด้านทฤษฎีกราฟ รวมถึงคำแนะนำเกี่ยวกับการส่งบทความตีพิมพ์ อันส่งผลให้ผู้วิจัยสามารถทำงานวิจัยได้สำเร็จตามเป้าหมาย ผู้ช่วยศาสตราจารย์ ดร. กิตติกร นาคประสิทธิ์ และ นางสาวสุกานดา คำเมือง ผู้ร่วมวิจัย ที่ได้ร่วมงานวิจัยในโครงการวิจัยนี้

คณะผู้ประเมินของวารสารวิชาการระดับนานาชาติที่ได้ให้คำแนะนำ ข้อเสนอแนะ ตลอดทั้งปรับปรุงต้นฉบับของบทความที่ส่งไปตีพิมพ์ในวารสารเหล่านั้น

คณาจารย์ นักศึกษาและเจ้าหน้าที่ฝ่ายสนับสนุน ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยขอนแก่น ที่ได้สนับสนุนการศึกษาวิจัยในโครงการวิจัยนี้เป็นอย่างดี

ผศ.ดร. เกียรติสุดา นาคประสิทธิ์
หัวหน้าโครงการวิจัย

บทคัดย่อ

รหัสโครงการ : MRG5580003

ชื่อโครงการ : การระบายสีจุดยอดแบบเท่าเทียมในบางคลาสของกราฟ

ชื่อนักวิจัย : ผศ.ดร. เกียรติสุดา นาคประสิทธิ์
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยขอนแก่น

นักวิจัยที่ปรึกษา: ศาตราจารย์ ดร. ณรงค์ ปั้นนิ่ม
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยศรีนครินทรวิโรฒ

อีเมล์: kmaneeruk@hotmail.com (เกียรติสุดา นาคประสิทธิ์)
punnim@gmail.com (ณรงค์ ปั้นนิ่ม)

ระยะเวลาโครงการ : 2 กรกฎาคม 2555 – 1 กรกฎาคม 2557

กราฟ G มี*การระบายสี k-ดีเฟกทีฟแบบเท่าเทียมด้วยสี m สี* เมื่อจุดยอดสามารถให้สีด้วยสี m สี ที่ระดับขั้นสูงสุดของกราฟย่อยอินดิวซ์โดยเซตของจุดยอดซึ่งได้รับสีเดียวกันมีค่าไม่เกิน k และสองเซตใด ๆ ของจุดยอดที่ได้รับสีเดียวกันมีจำนวนสมาชิกแตกต่างกันไม่เกินหนึ่ง *รงคเลข k-ดีเฟกทีฟแบบเท่าเทียม*ของกราฟ G เขียนแทนด้วย $\chi_{ED,k}(G)$ คือจำนวนเต็มบวก m ที่น้อยที่สุด ซึ่งกราฟ G มีการระบายสี k-ดีเฟกทีฟแบบเท่าเทียมด้วยสี m สี

*การระบายสีเส้นเชื่อมแบบเข้ม*คือการระบายสีเส้นเชื่อมโดยแท้ซึ่งเส้นเชื่อมสองเส้นใด ๆ ที่ได้รับสีเดียวกันจะไม่อยู่บนวิถีความยาวสาม *รงคดัชนีแบบเข้ม*ของกราฟ G เขียนแทนด้วย $s'(G)$ คือจำนวนสีที่น้อยที่สุดในการระบายสีเส้นเชื่อมแบบเข้ม กำหนดให้ $d(v)$ แทนระดับขั้นของจุดยอด v และให้ *ระดับขั้นแบบออ*ของกราฟ G คือค่าที่มากที่สุดของ $d(u) + d(v)$ เมื่อ u และ v เป็นจุดยอดที่ประชิดกันในกราฟ G

ในงานวิจัยนี้ เรานำเสนอค่ารงคเลข k-ดีเฟกทีฟแบบเท่าเทียมของกราฟสองส่วนแบบบริบูรณ์ สำหรับ k=1 และ k=2 และแสดงว่าสำหรับแต่ละกราฟ G ซึ่งมีระดับขั้นแบบออเท่ากับ 6 จะได้ว่า $s'(G) \leq 10$ และเมื่อเพิ่มเงื่อนไข G เป็นกราฟสองส่วน จะได้ว่า $s'(G) \leq 9$

คำหลัก : การระบายสีดีเฟกทีฟแบบเท่าเทียม, การระบายสีเส้นเชื่อมแบบเข้ม, ระดับขั้นแบบออ

Abstract

Project Code: MRG5580003

Project Title: Equitable vertex coloring in some classes of graphs

Investigator: Asst. Prof. Dr. Keaitsuda Nakprasit
Department of Mathematics, Faculty of Science,
Khon Kaen University

Mentor: Professor Dr. Narong Punnim
Department of Mathematics, Faculty of Science,
Srinakharinwirot University

E-mail Address: kmaneeruk@hotmail.com (Keaitsuda Nakprasit)
punnim@gmail.com (Narong Punnim)

Project Period: July 2, 2012 - July 1, 2014

A graph G has an *equitable k-defective coloring in m colors* if its vertices can be colored with m colors such that the maximum degree of any subgraph induced by vertices assigned to the same color is at most k and the numbers of vertices in any two sets composed of the vertices that are assigned to the same color differ by at most one. The *equitable k-defective chromatic number* of a graph G, denoted by $\chi_{ED,k}(G)$, is the smallest positive integer m for which G has an equitable k-defective coloring in m colors.

A *strong edge-coloring* is a proper edge-coloring such that two edges with the same color are not allowed to lie on a path of length three. The *strong chromatic index* of a graph G denoted by $s'(G)$ is the minimum number of colors in a strong edge-coloring. We denote the degree of a vertex v by $d(v)$. Let the *Ore-degree* of a graph G be the maximum values of $d(u) + d(v)$ where u and v are adjacent vertices in G.

In this research, we present the equitable k-defective chromatic numbers of complete bipartite graphs for k=1 and k=2 and show that each graph G with Ore-degree 6 has $s'(G) \leq 10$. With the further condition that G is bipartite, we have $s'(G) \leq 9$.

Keywords: equitable defective coloring, strong edge-coloring, Ore-degree

Contents

Chapter 1

Introduction

A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$. In this research, graphs are considered to be finite, undirected, and simple. Let $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer that is not less than x and the largest integer that is not greater than x, respectively. We refer the reader to [3] for terminology in graph theory.

An *equitable m-coloring* of a graph G is a proper m-coloring of G such that the numbers of vertices in any two sets composed of the vertices that are assigned to the same color differ by at most one. A graph G is said to be *equitably m-colorable* if G has an equitable m-coloring. The *equitable chromatic number* of a graph G, denoted by $\chi_=(G)$, is the smallest positive integer m for which G is equitably m-colorable. The equitable coloring introduced first by Meyer [1] in 1973.

A graph G has an *equitable k-defective coloring in m colors* if its vertices can be colored with m colors such that the maximum degree of any subgraph induced by vertices assigned to the same color is at most k and the numbers of vertices in any two sets composed of the vertices that are assigned to the same color differ by at most one. The *equitable k-defective chromatic number* of a graph G, denoted by $\chi_{ED,k}$, is the smallest positive integer m for which G has an equitable k-defective coloring in m colors.

A *strong edge-coloring* is a proper edge-coloring such that two edges with the same color are not allowed to lie on a path of length three. The *strong chromatic index* of a graph G denoted by $s'(G)$ is the minimum number of colors in a strong edge-coloring. We denote the degree of a vertex v by $d(v)$. Let the *Ore-degree* of a graph G be the maximum values of $d(u) + d(v)$ where u and v are adjacent vertices in G.

An *equitable k-coloring game* on a graph G is a coloring game that Alice and Bob play with a color set $C = \{1, 2, \ldots, k\}$. They take turns coloring the vertices of G one at a time with colors from C, with Alice has the first move, under the rule that adjacent vertices have different colors. Each color cannot be used more than b times where $b = \lceil |V(G)|/k \rceil$, and the coloring is allowed to have at most $|V(G)| - k(b-1)$ colors to be used b times. When there are no legal colorings left, Alice wins if all the vertices are colored and Bob wins otherwise. By this rule, we always has an equitable coloring when all vertices are colored.

In this research, we investigate three main results. First, we present the equitable k-defective chromatic numbers of complete bipartite graphs for $k = 1$ and $k = 2$. Second, we show that each graph G with Ore-degree 6 has $s'(G) \leq 10$, and with the further condition that G is bipartite, we have $s'(G) \leq 9$. Finally, we characterize complete bipartite graphs that Alice has a winning strategy in the equitable coloring game.

Chapter 2
Methodology

Methodology

(1) Investigate equitable vertex coloring techniques from literatures.

(2) Characterize properties of equitable vertex coloring in class of graphs.

(3) Solve some conjectures concerning the equitable vertex coloring in class of graphs.

(4) Submit research papers to international mathematics journals.

(5) Summarize results of the research and doing report.

(6) Discuss with mentor during project duration.

Chapter 3

Main Results

In this chapter, we present our main results. We separate this chapter into 3 sections. First, the equitable k-defective coloring. Second, the strong chromatic index of graphs with restricted Ore-degrees. Third, equitable Coloring Games on Complete Bipartite Graphs.

3.1 The equitable k-defective coloring

In this section, we present the equitable k-defective chromatic number of complete bipartite graphs $K_{m,n}$ with $m \leq n$ for $k = 1$ and $k = 2$.

Theorem 3.1.1 *For a complete bipartite graph $K_{m,n}$ with $m \geq 2$,*

$$\chi_{ED,1}(K_{m,n}) = \lceil m/(M+1) \rceil + \lceil n/(M+1) \rceil$$

where M is the largest integer such that $m (mod\, M) < \lceil m/M \rceil$ and $n (mod\, M) < \lceil n/M \rceil$.

Nakprasit and Saigrasun [2] characterized the complete bipartite graph $K_{m,n}$ with $m \leq n$ such that $\chi_{=}(K_{m,n}) = 1 + \lceil n/(m+1) \rceil$ and found the smallest integer C such that for every integer $n \geq C$ implies $\chi_{=}(K_{m,n}) = 1 + \lceil n/(m+1) \rceil$.

Theorem 3.1.2 *For a complete bipartite graph $K_{m,n}$ with $m \leq n$,*
$\chi_{ED,2}(K_{m,n}) = \chi_{=}(K_{m,n})$ except

(1) $\chi_{ED,2}(K_{1,n}) = 1 + \lceil (n-2)/4 \rceil$;

(2) $\chi_{ED,2}(K_{2,n}) = 1 + \lceil (n-2)/5 \rceil$;

(3) $\chi_{ED,2}(K_{m,n}) = 1 + \lceil (m-2)/5 \rceil + \lceil (n-2)/5 \rceil$,
where $(m, n) = (7, 10), (7, 11), (7, 17), (11, 14),$ *or* $(11, 19)$;

(4) $\chi_{ED,2}(K_{5,7}) = 4$;

(5) $\chi_{ED,2}(K_{6,9}) = 4$.

3.2 The strong chromatic index of graphs with restricted Ore-degrees

A *strong edge-coloring* is a proper edge-coloring such that two edges with the same color are not allowed to lie on a path of length three. The *strong chromatic index* of a graph G denoted by $s'(G)$ is the minimum number of colors in a strong edge-coloring.

We denote the degree of a vertex v by $d(v)$. Let the *Ore-degree* of a graph G be the maximum values of $d(u) + d(v)$ where u and v are adjacent vertices in G. Let F_3 denote the graph obtained from a 5-cycle by adding a new vertex and joining it to a pair of nonadjacent vertices of the 5-cycle. In 2008, Wu and Lin [4] studied the strong chromatic index with respect to the Ore-degree. Their main result states that if a connected graph G is not F_3 and its Ore-degree is 5, then $s'(G) \leq 6$. Inspired by the result of Wu and Lin, we investigate the strong edge-coloring of graphs with Ore-degree 6. We show that each graph G with Ore-degree 6 has $s'(G) \leq 10$. With the further condition that G is bipartite, we have $s'(G) \leq 9$. Our results give general forms of previous results about strong chromatic indices of graphs with maximum degree 3.

Graphs in this section are finite, undirected, and loopless, but multiple edges are allowed. We always assume that graphs are connected unless the context implies otherwise. Note that some results that we refer to may not consider multiple edges, but these results can be extended easily to graphs with multiple edges. Throughout this section, the term coloring means strong edge-coloring, unless the coloring is specified to be other type of coloring.

For graphs with small Ore-degrees, we have the followings.

Observation 3.2.1 *(Characterization of graphs with small Ore-degrees)*

(1) *The only graph with Ore-degree 0 is K_1.*

(2) *No graph has Ore-degree 1.*

(3) *The only graph with Ore-degree 2 is a path with one edge.*

(4) *The only graph with Ore-degree 3 is a path with two edges.*

(5) *A graph G has Ore-degree 4 if and only if G is a path of length at least 3, $K_{1,3}$, a cycle, or a graph with two vertices and two multiple edges.*

Next, we proceed to investigate strong chromatic indices in terms of Ore-degree of graphs in general.

Lemma 3.2.2 *Let G be a graph with Ore-degree at most R. If M is the set of vertices of G with degree $R-2$, then $s'(G) \leq \max\{s'(G-M), 3R-8\}$.*

A path ww_1w_2 is a *special 2-path* if $d(w_1) = d(w_2) = 2$ and w is an $(R-2)$-vertex.

Lemma 3.2.3 *Let G be a graph with Ore-degree at most R with a special 2-path ww_1w_2. Then $s'(G) \leq \max\{s'(G-w_1), 2R-3\}$.*

Theorem 3.2.4 *If a graph G has Ore-degree at most 6, then $s'(G) \leq 10$. With the futher condition that G is bipartite, we have $s'(G) \leq 9$.*

3.3 Equitable coloring games on complete bipartite graphs

In a *coloring game*, Alice wins if and only if all vertices are colored. To enforce a coloring of all vertices to be an equitable k-coloring, we propose the additional rules for an *equitable k-coloring game* as follows. Let $b = \lceil |V(G)|/k \rceil$ and $d = \lfloor |V(G)|/k \rfloor$. A color that has been used b times is a *major color*. Each color cannot be used more than b times and the coloring is allowed to have at most $|V(G)| - k(b-1)$ major colors.

From now on, a game means an equitable k-coloring game on a complete bipartite $K_{m,n}$. Let X and Y be partite sets of $K_{m,n}$ of size m and n, respectively, with $m \leq n$. A partite set is called *even* (respectively, *odd*) if its size is even (respectively, odd).

3.3.1 The games with $b \leq 2$

Lemma 3.3.1 *If $d \leq 1$ then Alice has a winning strategy.*

Lemma 3.3.2 *If $b = d = 2$ then Bob has a winning strategy.*

Lemma 3.3.3 *Let $m = d = b - 1 = 2$. Alice has a winning strategy if and only if (i) $n = 3(k-1)$ and n is even, or (ii) $n < 3(k-1)$ and n is odd.*

3.3.2 The games with $d \geq 3$

The *X-maximizing tactic* is a strategy of Bob defined as follows. Bob plays a new color in X if there exists an unused color, and fewer than d uncolored vertices are in X or Alice played a new color in the turn immediately before. Otherwise, Bob plays a legal color with the largest class size in X. If all vertices in X are colored, then Bob plays arbitrarily. The *Y-maximizing tactic* is defined similarly.

Remark. If Bob always has new colors to use for the X-maximizing tactic before all vertices in X are colored, then there is a color class of size less than d. Thus if we assume that a coloring is completed despite the X-maximizing tactic used by Bob, then all colors are used before all vertices in X are colored.

Lemma 3.3.4 *If $d \leq 1$ then Alice has a winning strategy.*

Lemma 3.3.5 *If $b = d = 2$ then Bob has a winning strategy.*

Lemma 3.3.6 *Let $m = d = b - 1 = 2$. Alice has a winning strategy if and only if (i) $n = 3(k-1)$ and n is even, or (ii) $n < 3(k-1)$ and n is odd.*

3.3.3 The games with $d \geq 3$

The *X-maximizing tactic* is a strategy of Bob defined as follows. Bob plays a new color in X if there exists an unused color, and fewer than d uncolored vertices are in X or Alice played a new color in the turn immediately before. Otherwise, Bob plays a legal color with the largest class size in X. If all vertices in X are colored, then Bob plays arbitrarily. The *Y-maximizing tactic*is defined similarly.

Remark. If Bob always has new colors to use for the X-maximizing tactic before all vertices in X are colored, then there is a color class of size less than d. Thus if we assume that a coloring is completed despite the X-maximizing tactic used by Bob, then all colors are used before all vertices in X are colored.

Lemma 3.3.7 *Let $d \geq 3$. If Alice colors a vertex in X in the first turn, then Bob has a winning strategy.*

Remark. In view of Lemma 3.3.7, we assume that Alice colors a vertex in Y in the first turn for a game with $d \geq 3$.

Before we investigate the equitable game coloring further, we define two conditions of m and n which are refered in later parts.

Definition 3.3.8 *Let $k = 2t + 1$ where t is an integer. We say m and n satisfy* condition *(A) if $m = rb + (t - r)d$ and $n = rb + (t + 1 - r)d$ where r is an integer satisfying $1 \leq r \leq t$.*

Definition 3.3.9 *Let $k = 2t + 1$ where t is an integer. We say m and n satisfy* condition *(B) if one of the following holds:*

(1) $m = tb$ *and* $n = (t - 1)b + 2d$,

(2) $m = (t - 1)b + d$ *and* $n = (t + 1)b$,

(3) $m = (t - 1)b + d$ *and* $n = tb + d$,

(4) $m = (t - 2)b + 2d$ *and* $n = (t + 1)b$,

(5) $m = (t - 2)b + 2d$ *and* $n = tb + d$, *or*

(6) $m = (t - 2)b + 2d$ *and* $n = (t - 1)b + 2d$.

Lemma 3.3.10 *Let $d \geq 3$. If k is even, then Bob has a winning strategy.*

Lemma 3.3.11 *Let $d \geq 3$. If k is odd and Alice has a winning strategy, then m and n satisfy condition (A), (B), or $n - m \geq d + 1$.*

Lemma 3.3.12 *Let $d \geq 3$. If k is odd and Alice has a winning strategy, then m and n satisfy condition (A), (B), or $n - m \leq d - 1$.*

Combining Lemmas 3.3.11 and 3.3.12, we immediately have the following result.

Corollary 3.3.13 *Let $d \geq 3$. If k is odd and Alice has a winning strategy, then m and n satisfy condition (A) or (B).*

Corollary 3.3.13 gives necessary condition for complete bipartite graphs that Alice has a winning strategy. Next we investigate which necessary condtions are also sufficient.

Lemma 3.3.14 *Let odd $k = 2t + 1$, $d \geq 3$, while m and n satisfy condition (A) or (B). If b is even, then Bob has a winning strategy. (Note that b and d maybe equal in this Lemma.)*

Lemma 3.3.15 *Let odd $k = 2t + 1$, $d \geq 2$, $m = (t - 1)b + d$, and $n = tb + d$, or $(t + 1)b$. If b is odd, then Alice has a winning strategy. (Note that b and d maybe equal in this Lemma.)*

Lemma 3.3.16 *Let odd $k = 2t + 1$ and odd $b > d \geq 2$. Bob has a winning strategy if m and n satisfy one of the following:*
(i) $m = tb$, and $n = (t - 1)b + 2d$,
(ii) $m = (t - 2)b + 2d$, and $n = (t - 1)b + 2d$,
(iii) $m = (t - 2)b + 2d$, and $n = tb + d$ or $(t + 1)b$.

Lemma 3.3.17 *Let odd $k = 2t + 1$, odd $b > d \geq 3$, $m = rb + (t - r)d \geq 3$, and $n = rb + (t - r + 1)d$ where $1 \leq r \leq t$. Alice has a winning strategy if and only if $r = t$.*

3.3.4 The games with $b = d + 1 = 3$

The main idea of this section is similar to one in Subection 3.3.3. However we need tactics other than the maximizing tactic which is not effective anymore; now

Bob may not be able to play a new color to make a color class of size less than d even when some colors are unused. For example, Bob uses the X-maximizing tactic with $|X| = 6$, and Alice can counter the tactic by playing the same color after each play of Bob. This play results in three color classes of size 2 in X even if some colors are unused. To emphasize the difference and to prevent confusion, we separate the games with $b = d + 1 = 3$ to be considered in this section.

To define the next tactic, we needs two new deifinitions. If a color c appears twice and it is played by Bob first and Alice later while there is an unused color to play, then we call c a *bad* color unless stated otherwise. A *good* color is a color that is not bad. (Note that an unused color is also a good color.) Let $f'(X)$ be the number of good colors in X, and $g'(X)$ be the number of good colors of size 3 in X. Bob has four types of colors for playing in the the *X-optimizing tactic*: (1) a new color in X, (2) a legal good color in X with the largest size, (3) a legal bad color in X, or (4) a legal color in Y. Table 3.1 lists these four types of colors from the most preference to the least preference according to situations. Bob always plays the most prefered legal color. The *Y-optimizing tactic*, $f'(Y)$, and $g'(Y)$ can be defined similarly to previous definitions.

Conditions	Bob's preference
Alice plays a new color	(1), (2), (3), (4)
Alice plays a used color in X	(1), (2), (3), (4)
If Bob plays (2) then there are at most one color class of size 1 in X	(1), (2), (3), (4)
Alice plays a used color in Y, and if Bob responds by playing (2) then there are at least two color classes of size 1 in X	(2), (1), (3), (4)

Table 3.1: The X-optimizing tactic

Remark. Assume that in a certain stage of the game, two (or more) color classes of size 1 appear in X. If Bob always has a new color to use for the X-optimizing tactic before all vertices in X are colored, then there is a color class of size 1 in the endgame. Thus if we suppose that a coloring is completed despite the X-

optimizing tactic used by Bob, then all colors are used before all vertices in X are colored.

Lemma 3.3.18 *Let $b = d + 1 = 3$. If Alice colors a vertex in X in the first turn, then Bob has a winning strategy.*

Remark. In view of Lemma 3.3.18, we assume that Alice colors a vertex in Y in the first turn for a game in this section.

Lemma 3.3.19 *Let $b = d + 1 = 3$. If k is even, then Bob has a winning strategy.*

Lemma 3.3.20 *Let $b = d + 1 = 3$. If k is odd and Alice has a winning strategy, then m and n satisfy condition (A), (B), or $n - m \geq d + 1$.*

Lemma 3.3.21 *Let $b = d + 1 = 3$. If k is odd and Alice has a winning strategy, then m and n satisfy condition (A), (B), or $n - m \leq d - 1$.*

Combining Lemmas 3.3.20 and 3.3.21, we immediately have the following result.

Corollary 3.3.22 *Let $b = d + 1 = 3$. If k is odd and Alice has a winning strategy, then m and n satisfy condition (A) or (B).*

Corollary 3.3.22 gives necessary condition for complete bipartite graphs that Alice has a winning strategy. Next we investigate which necessary condtions are also sufficient.

Lemma 3.3.23 *Let odd $k = 2t + 1, b = d + 1 = 3$, $m = rb + (t - r)d \geq 3$, and $n = rb + (t - r + 1)d$ where $1 \leq r \leq t$. Alice has a winning strategy if and only if $r = t$.*

3.3.5 Conclusion

To characterize complete bipartite graphs that Alice has a winning strategy, we refer to

(i) Lemmas 3.3.4, 3.3.5, and 3.3.6 for $b \leq 2$,

(ii) Corollary 3.3.13 and Lemmas 3.3.10, 3.3.14, 3.3.15, 3.3.16, and 3.3.17 for $d \geq 3$,

(iii) Corollary 3.3.22 and Lemmas 3.3.15, 3.3.16, 3.3.19, and 3.3.23 for $b = d + 1 = 3$.

Theorem 3.3.24 *Alice has a winning strategy for an equitable k-coloring game on $K_{m,n}$ if and only if one of the following holds:*

(i) $d \leq 1$,

(ii) $m = d = b - 1 = 2, n = 3(k-1), n$ *is even,*

(iii) $m = d = b - 1 = 2, n < 3(k-1), n$ *is odd, or*

(iv) k *is odd* ($k = 2t+1$), b *is odd,* $m = (t-i)b + id \geq 3$, *and* $n = (t+j)b + (1-j)d$ *where* $i, j = 0$ *or* 1.

Bibliography

[1] W. Meyer, Equitable coloring, *American Mathematical Monthly*, **80** (1973), 920-922.

[2] K. Nakprasit, W. Saigrasun, On equitable coloring of complete r-partite graphs, *International Journal of Pure and Applied Mathematics*, **71**, No. 2 (2011), 229-239.

[3] D. B. West, *Introduction to graph theory*, 2-nd ed., Prentice-Hall, Upper Saddle River, NJ, USA (2001).

[4] J. Wu and W. Lin, The strong chromatic index of a class of graphs, *Discrete Mathematics*, **308** (2008), 6254–6261.

Output

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ

1.1 Nakprasit, K., Cummuang, S., *Equitable defective colorings of complete bipartite graphs*, International Journal of Pure and Applied Mathematics, **2014**; 92: 73-86.

1.2 Nakprasit, K., Nakprasit, K., *The strong chromatic index of graphs with restricted Ore-degrees*, Ars Combinatoria, **accepted**.

1.3 Nakprasit, K., Nakprasit, K., *Equitable coloring games on complete bipartite graphs*, submitted to Discrete Applied Mathematics.

2. การนำผลงานวิจัยไปใช้ประโยชน์

ผลงานวิจัยที่ได้มามีการนำไปใช้ประโยชน์ทั้งเชิงวิชาการ และเชิงสาธารณะโดยทำให้มีการพัฒนาการเรียนการสอนและมีเครือข่ายความร่วมมือสร้างกระแสความสนใจในวงกว้าง

3. อื่น ๆ: การเสนอผลงานในที่ประชุมวิชาการ

3.1 วันที่ 20 - 22 มีนาคม 2557

หัวข้อ: Equitable Coloring Games on Complete Bipartite Graphs

ชื่อการประชุม : งานประชุมวิชาการคณิตศาสตร์ประจำปี 2557 ครั้งที่ 19

(The 19th Annual Meeting in Mathematics: AMM2014)

ระหว่างวันที่ 20 – 22 มีนาคม 2557 ณ โรงแรม เอ-วัน เดอะ รอยัล ครูส พัทยา

(A-One, The Royal Cruise Hotel Pattaya)

Appendix

A1. **Keaitsuda Nakprasit** and Sukanda Cummuang, Equitable defective colorings of complete bipartite graphs, International Journal of Pure and Applied Mathematics, 2014; 92: 73-86.

A2. **Keaitsuda Nakprasit** and Kittikorn Nakprasit, The strong chromatic index of graphs with restricted Ore-degrees, Ars Combinatoria, accepted.

A3. **Keaitsuda Nakprasit** and Kittikorn Nakprasit, Equitable coloring games on complete bipartite graphs, Manuscript.

A1. **Nakprasit, K.**, Cummuang, S., *Equitable defective colorings of complete bipartite graphs*, International Journal of Pure and Applied Mathematics, **2014**; 92: 73-86.

International Journal of Pure and Applied Mathematics
Volume 92 No. 1 2014, 73-86
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v92i1.6

ijpam.eu

EQUITABLE DEFECTIVE COLORINGS OF COMPLETE BIPARTITE GRAPHS

Keaitsuda Nakprasit[1 §], Sukanda Cummuang[2]

[1]Department of Mathematics
Faculty of Science
Khon Kaen University
40002, THAILAND

[2]Centre of Excellence in Mathematics
PERDO, Bangkok, 10400, THAILAND

Abstract: A graph G has an equitable k-defective coloring in m colors if its vertices can be colored with m colors such that the maximum degree of any subgraph induced by vertices assigned to the same color is at most k and the numbers of vertices in any two sets composed of the vertices that are assigned to the same color differ by at most one. The equitable k-defective chromatic number of a graph G, denoted by $\chi_{ED,k}(G)$, is the smallest positive integer m for which G has an equitable k-defective coloring in m colors. In this paper, we present the equitable k-defective chromatic numbers of complete bipartite graphs for $k = 1$ and $k = 2$.

AMS Subject Classification: 05C15, 05C35
Key Words: equitable defective coloring, complete bipartite graph

1. Introduction

A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$. In this paper,

Received: December 11, 2013

[§]Correspondence author

graphs are considered to be finite, undirected, and simple. Let $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer that is not less than x and the largest integer that is not greater than x, respectively. We refer the reader to [15] for terminology in graph theory.

An *equitable* m*-coloring* of a graph G is a proper m-coloring of G such that the numbers of vertices in any two sets composed of the vertices that are assigned to the same color differ by at most one. A graph G is said to be *equitably* m*-colorable* if G has an equitable m-coloring. The *equitable chromatic number* of a graph G, denoted by $\chi_=(G)$, is the smallest positive integer m for which G is equitably m-colorable. The equitable coloring introduced first by Meyer [13] in 1973.

Lih and Wu [9] investigated the equitable chromatic number of a connected bipartite graph. The authors proved that every complete bipartite graph $K_{n,n}$ can be equitably colored using k colors if and only if $\lceil n/\lfloor k/2 \rfloor \rceil - \lfloor n/\lceil k/2 \rceil \rfloor \leq 1$. Moreover, if G is a connected bipartite graph with partite sets X, Y and ϵ edges such that $\epsilon < \lfloor n/(m+1) \rfloor (n-m) + 2n$, then $\chi_=(G) \leq 1 + \lceil n/(m+1) \rceil$, where $|X| = m \leq n = |Y|$.

Lam et. al. [8] determined the equitable chromatic number of a complete r-partite graph $K_{m_1,\ldots,m_r}$. They showed that $\chi_=(K_{m_1,\ldots,m_r}) = \sum_{i=1}^{r} \lceil m_i/(M+1) \rceil$ where M is the largest integer such that $m_i (\text{mod } M) \leq \lceil m_i/M \rceil$ $(i = 1, \ldots, r)$.

Nakprasit and Saigrasun [14] characterized the complete bipartite graph $K_{m,n}$ with $m \leq n$ such that $\chi_=(K_{m,n}) = 1 + \lceil n/(m+1) \rceil$ and found the smallest integer C such that for every integer $n \geq C$ implies $\chi_=(K_{m,n}) = 1 + \lceil n/(m+1) \rceil$.

For more on the equitable coloring of graphs see [6] and [10].

A subset U of $V(G)$ is said to be k*-independent* if the maximum degree of an induced subgraph $G[U]$ is at most k. A k*-defective coloring in* m *colors* of a graph G is an m-coloring of G such that the set of vertices that are assigned to the same color is k-independent. A graph G is (m, k)*-colorable* if G has an k-defective coloring in m colors. The k*-defective chromatic number* of a graph G, denoted by $\chi_k(G)$, is the smallest positive integer m for which G is (m, k)-colorable. Note that $\chi_0(G)$ is the usual chromatic number of G. It is clear that $\chi_k(G) \leq \lceil n/(k+1) \rceil$, where n is the order of G.

The concept of (m, k)-coloring has been studied by several authors. Hopkins and Staton [7] refered to a k-independent set as a k-small set. Maddox ([11], [12]) and Andrews and Jacobson [2] refered to this set as a k-dependent set. The k-defective chromatic number has been investigated as the k-partition number by Frick [4], Frick and Henning [5], Maddox ([11], [12]), Hopkins and Staton [7] and under the name k-chromatic number by Andrews and Jacobson [2].

Achuthan et al. [1] determined the smallest order of a triangle-free graph such that $\chi_k(G) = m$, denoted by $f(m,k)$. They showed that $f(3,2) = 13$. Moreover, they presented a lower bound for $f(m,k)$ for $m \geq 3$ and also an upper bound for $f(3,k)$.

A graph G has an *equitable k-defective coloring* in m colors if G has a k-defective coloring in m colors and the numbers of vertices in any two sets composed of the vertices that are assigned to the same color differ by at most one. The *equitable k-defective chromatic number* of a graph G, denoted by $\chi_{ED,k}(G)$, is the smallest positive integer m for which G has an equitably k-defective coloring in m colors.

Cummuang and Nakprasit [3] presented the equitable k-defective chromatic numbers of paths, cycles, complete graphs, hypercubes, stars, and wheels for any positive integer k.

Williams, Vandenbussche, and Yu [16] studied the equitable defective coloring of sparse planar graph by using the discharing method. The authors proved that every planar graph with minimum degree at least 2 and girth at least 10 has an equitable 1-defective coloring in m colors for $m \geq 3$.

In this paper, we show that $\chi_{ED,1}(K_{m,n}) = \chi_=(K_{m,n})$ for $2 \leq m \leq n$ and $\chi_{ED,2}(K_{m,n}) = \chi_=(K_{m,n})$ for all but a finite number of (m,n) pairs.

2. Preliminary Results

Let $K_{m,n}$ be a complete bipartite graph with partite sets X and Y, where $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, and $m \leq n$. Lemmas 1 – 4 can be found in [14].

Lemma 1. *Let $K_{m,n}$ be a complete bipartite graph. If $V_1, V_2, \ldots, V_{1+\lceil n/(m+1)\rceil}$ are equitable color classes of $K_{m,n}$, then $X = V_i$ for some $i \in \{1, 2, \ldots, 1 + \lceil n/(m+1)\rceil\}$.*

For Lemmas 2 to 4, we let $n = a(m+1) + b, 0 \leq b \leq m$ where a and b are integers.

Lemma 2. *The complete bipartite graph $K_{m,n}$ has $1 + \lceil n/(m+1)\rceil$ equitable color classes of size m or $m+1$ if and only if $b = 0$ or $m - a \leq b \leq m$.*

Lemma 3. *The equitable chromatic number $\chi_=(K_{m,n}) = 1 + \lceil n/(m+1)\rceil$ if and only if $b = 0$ or $m - 2a - 1 \leq b \leq m$.*

Lemma 4. *Given a positive integer m, let C be the smallest positive integer such that $\chi_=(Km,n) = 1 + \lceil n/(m+1)\rceil$ for every integer $n \geq C$. Then $C = (m-1)(\lceil m/2\rceil - 1)$.*

Lemma 5. *Let $k, n \in \mathbb{N}$ and $n \geq k$ such that $n = a(k+1) + b$, $0 \leq b \leq k$. There exist nonnegative integers s and t such that $n = s(k+1) + tk$ if and only if $b = 0$ or $k - a \leq b \leq k$.*

Proof. To prove the theorem, we first assume that there exist nonnegative integers s and t such that $n = s(k+1) + tk$. By choosing s and t such that $s + t$ is minimum, we can show that $(a, b) = (s, 0)$ for $t = 0$ and $(a, b) = (s + t - 1, k - t + 1)$ for $t > 0$.

Suppose that $b = 0$ or $k - a \leq b \leq k$. We can show that $n = s(k+1) + tk$ where (s, t) are $(a, 0)$, $(0, a+1)$, and $(a + b - k, 1 + k - b)$ for $b = 0$, $b = k - a$, and $k - a + 1 \leq b \leq k$, respectively. This completes the proof. □

Lemma 6. *Let $k, n \in \mathbb{N}$ with $n \geq k(k-1)$. Then there exist nonnegative integers s and t such that $n = s(k+1) + tk$ and $s + t = \lceil n/(k+1) \rceil$.*

Proof. By the division algorithm, $n = a(k+1) + b$ where $0 \leq b \leq k$. We consider two cases.

Case 1 : $a = k - 2$. Then $b \geq 2$. Consequently, $k - a = 2 \leq b \leq k$.

Case 2 : $a \geq k - 1$. For $b = 0$, we have $s = a$ and $t = 0$.

For $1 \leq b \leq k$, we have $k - a + 1 \leq b \leq k$. Lemma 5 implies there exist nonnegative integers s and t such that $n = s(k+1) + tk$. We choose s and t with minimum $s + t$ to obtain $t \leq k$ which implies $s + t = \lceil n/(k+1) \rceil$. □

3. The Equitable 1-Defective Coloring of $K_{m,n}$

In this section, we investigate the equitable 1-defective chromatic numbers of complete bipartite graphs $K_{m,n}$ with $m \leq n$.

Theorem 7. *For a complete bipartite graph $K_{1,n}$, $\chi_{ED,1}(K_{1,n}) = 1 + \lceil (n-1)/3 \rceil$.*

Proof. The proof follows from Corollary 2 in [3]. □

Theorem 8. *For a complete bipartite graph $K_{m,n}$ with $m \geq 2$, $\chi_{ED,1}(K_{m,n}) = \lceil m/(M+1) \rceil + \lceil n/(M+1) \rceil$ where M is the largest integer such that $m (\mathrm{mod}\ M) < \lceil m/M \rceil$ and $n (\mathrm{mod}\ M) < \lceil n/M \rceil$.*

Proof. We first define $f(t) = t+\lceil (m-t)/3\rceil+\lceil (n-t)/3\rceil$ for $t = 0, 1, 2, \ldots, m$.

Observe that the equitable 1-defective coloring with t non-independent color classes, where $t \geq 1$, has at least $f(t)$ color classes.

One can verify that $\min\{f(t) : t = 0, 1, 2, \ldots, m\} = f(1)$ for $m \equiv n \equiv 1(\text{mod } 3)$, and otherwise $\min\{f(t) : t = 0, 1, 2, \ldots, m\} = f(0)$. We consider two cases.

Case 1 : $m \equiv n \equiv 1(\text{mod } 3)$.

In this case, the minimum of $f(t)$ is attained at $t = 1$ and $f(1) = 1 + \lceil (m-1)/3\rceil + \lceil (n-1)/3\rceil \geq (m + n + 1)/3$.

Since $m \equiv 1(\text{mod } 3) < \lceil m/3\rceil$ and $n \equiv 1(\text{mod } 3) < \lceil n/3\rceil$, we obtain $M \geq 3$. This implies that $\lceil m/(M + 1)\rceil + \lceil n/(M + 1)\rceil \leq \lceil m/4\rceil + \lceil n/4\rceil \leq [(m + 3)/4] + [(n + 3)/4]$.

Since $[(m + n + 1)/3] - [((m + 3)/4) + ((n + 3)/4)] = [(m + n - 14)/12] \geq 0$ for $m + n \geq 14$, the theorem holds. The remaining (m, n) are $(4, 4)$ and $(4, 7)$ for which $\chi_{ED,1}(K_{m,n}) = \lceil m/(M + 1)\rceil + \lceil n/(M + 1)\rceil$.

Case 2 : $m \not\equiv 1(\text{mod } 3)$ or $n \not\equiv 1(\text{mod } 3)$.

In this case, the minimum of $f(t)$ is attained at $t = 0$. Since $\lceil m/(M+1)\rceil + \lceil n/(M+1)\rceil \leq \lceil m/3\rceil + \lceil n/3\rceil = f(0)$, therefore $\chi_{ED,1}(K_{m,n}) = \lceil m/(M+1)\rceil + \lceil n/(M + 1)\rceil$. □

4. The Equitable 2-Defective Coloring of $K_{m,n}$

In this section, we investigate the equitable 2-defective chromatic numbers of complete bipartite graphs $K_{m,n}$ with $m \leq n$. First, we introduce some definitions that will be used in later arguments.

Definition 9. Let P be a color class in an equitable 2-defective coloring of $K_{m,n}$. We say that

1. P is a **color class of type *A*** if it comprises two vertices of X and two vertices of Y;
2. P is a **color class of type *B*** if it comprises two vertices of X and one vertex of Y;
3. P is a **color class of type *C*** if it comprises one vertex of X and two vertices of Y.

Definition 10. Let c be an equitable 2-defective coloring of $K_{m,n}$. We say that

1. c is a **coloring of type *A5*** if its maximum color class size is 5 and it contains a color class of type ***A*** as its only non-independent color class;
2. c is a **coloring of type *A4*** if its maximum color class size is 4 and it contains a color class of type ***A*** as its only non-independent color class;
3. c is a **coloring of type *B4*** if its maximum color class size is 4 and it contains a color class of type ***B*** as its only non-independent color class;
4. c is a **coloring of type *B3*** if its maximum color class size is 3 and it contains a color class of type ***B*** as its only non-independent color class;
5. c is a **coloring of type *C4*** if its maximum color class size is 4 and it contains a color class of type ***C*** as its only non-independent color class;
6. c is a **coloring of type *C3*** if its maximum color class size is 3 and it contains a color class of type ***C*** as its only non-independent color class.

Lemma 11. *The following statements hold for the equitable 2-defective coloring of $K_{m,n}$.*

(i) Every equitable 2-defective coloring of $K_{m,n}$ with $\chi_{ED,2}(K_{m,n})$ color classes has a color class which induces K_2 if and only if $(m,n) = (1,1)$ or $(1,3)$.

(ii) If $K_{m,n}$ has an equitable 2-defective coloring in c colors, then it has an equitable 2-defective coloring in c colors that has at most one non-independent color classes.

Proof. We shall prove only the sufficiency of (i) because the necessity can be easily verified.
Consider an equitable 2-defective coloring with $\chi_{ED,2}(K_{m,n})$ color classes with a color class that induces K_2.

Let $P = \{x_1, y_1\}$ and Q be non-independent color classes that result from this coloring. We repartition $P \cup Q$ into two equitable independent sets and leave all other color classes unchanged. By continuing this process, we obtain an equitable 2-defective coloring that has at most one non-independent set that is a color class that induces K_2.

Let $\{x_1, y_1\}, A_1, A_2, \ldots, A_r, B_1, B_2, \ldots, B_s$ be color classes that result from the above coloring where $A_i \subseteq X, B_j \subseteq Y$ and $|A_1| \leq |A_i|, |B_1| \leq |B_j|$ for $1 \leq i \leq r, 1 \leq j \leq s$.

Note that $1 + r + s = \chi_{ED,2}(K_{m,n})$. Since $m \geq 2$, we have $r \geq 1$ and $s \geq 1$. Then $A_1 \cup \{x_1\}, A_2, \ldots, A_r, B_1 \cup \{y_1\}, B_2, \ldots, B_s$ are equitable color classes of $K_{m,n}$. This contradicts $\chi_{ED,2}(K_{m,n}) = 1 + r + s$. Thus $m = 1$.

For $n = 2$ or $n \geq 4$, it is straightforward to verify that a coloring of type ***C3*** or ***C4*** has $\chi_{ED,2}(K_{1,n})$ color classes. Thus, the only possible (m, n) are $(1, 1)$ and $(1, 3)$.

Next, we prove (ii) by considering an equitable 2-defective coloring with (at least) two non-independent color classes, say P and Q. We can repartition $P \cup Q$ into two color classes with fewer non-independent color classes and leave all other color classes unchanged. By continuing this process, we obtain an equitable 2-defective coloring with at most one non-independent color class. □

Lemma 12. *For a complete bipartite graph* $K_{1,n}$, $\chi_{ED,2}(K_{1,n}) = 1+\lceil (n-2)/4 \rceil$.

Proof. The proof follows from Corollary 2 in [3]. □

Lemma 13. *For a complete bipartite graph* $K_{2,n}$, $\chi_{ED,2}(K_{2,n}) = 1+\lceil (n-2)/5 \rceil$.

Proof. For $1 \le n \le 6$, it is easy to see that $\chi_{ED,2}(K_{2,n}) = 1 + \lceil (n-2)/5 \rceil$. We consider the case when $n \ge 7$. By the division algorithm, $n - 2 = 5q + r, 0 \le r \le 4$. Therefore, $r = 0$ or $3 - 2q \le r \le 4$. Lemma 3 implies that $\chi_=(K_{4,n-2}) = 1 + \lceil (n-2)/5 \rceil$.

Consider a coloring of type ***C4*** or ***C3***. Note that the minimum number of color classes that result from this coloring is $\chi_=(K_{4,n-2}) = 1+\lceil (n-2)/5 \rceil$, but $\chi_=(K_{2,n}) = 1 + \lceil n/3 \rceil$. Thus, $\chi_{ED,2}(K_{2,n}) = 1 + \lceil (n-2)/5 \rceil$ by Lemma 11. □

Lemma 14. *[coloring of type **A5**] Suppose that* $m, n \neq 2$. *The number of color classes in a coloring of type* **A5** *is less than* $\chi_=(K_{m,n})$ *if and only if* $(m, n) = (7, 10), (7, 11), (7, 17), (11, 14)$, *or* $(11, 19)$.

Proof. We begin by assuming that the number of color classes in a coloring of type ***A5*** is less than $\chi_=(K_{m,n})$. Then there exist nonnegative integers q, r, s, and t such that $m = 2+4q+5r$, $n = 2+4s+5t$, and $\chi_=(K_{m,n}) > q+r+s+t+1$.

Let $A = \{x \in \mathbb{N} : x \ge 20\} \cup \{5, 6, 10, 11, 12, 15, 16, 17, 18\}$.

Observe that if the smallest color class size of an equitable coloring is at least 5, then the number of color classes in the equitable coloring is less than the number of color classes in a coloring of type ***A5***. Combining this with Lemma 6, we have a contradiction for $m, n \in A$. Thus, we consider only the cases when $m \in \{7, 14, 19\}$ or $n \in \{7, 14, 19\}$.

Case 1 : $m \in \{7, 14, 19\}$. We consider two subcases.

Subcase 1.1 : $m = 7$. If $n \ge 18$, then Lemmas 1 and 4 indicte that $K_{7,n}$ has an equitable coloring with color classes of size 7, which is a contradiction. The remaining cases are $n = 7, 10, 11, 12, 14, 15, 16,$ or 17. We can verify directly that the only possible (m,n) are $(7,10),(7,11)$, and $(7,17)$.

Subcase 1.2 : $m = 14$ or 19. It is easy to show that $\chi_=(K_{m,n}) \leq q+r+s+t+1$ which is a contradiction.

Case 2 : $n \in \{7,14,19\}$. We consider two subcases.

Subcase 2.1 : $n = 7$. We consider only $(m,n) = (6,7)$ and $(7,7)$. The number of color classes that result from a coloring of type ***A5*** on $K_{m,7}$ is 3. Because this result is greater than $\chi_=(K_{m,7}) = 2$, we have a contradiction.

Subcase 2.2 : $m = 14$ or 19. If $q \geq 2$ or $r \geq 2$, then $\chi_=(K_{m,n}) \leq q+r+s+t+1$ which is a contradiction. It is sufficient to consider $q \leq 1$ and $r \leq 1$. That is $m = 6, 7$, or 11. With the exception of the cases $(m,n) = (11,14)$ and $(11,19)$, the numbers of color classes resulting from a coloring of type ***A5*** is greater than $\chi_=(K_{m,n})$.

Conversely, we can easily verify that the numbers of color classes resulting from a coloring of type ***A5*** are $4, 4, 5, 6$ and 7 for $(m,n) = (7,10)$, $(7,11)$, $(7,17)$, $(11,14)$, and $(11,19)$, respectively. Each of the numbers of color classes is less than the equitable chromatic number of the corresponding graph. □

Lemma 15. *[coloring of type **A4**] Suppose that $m,n \neq 2$. The number of color classes in a coloring of type* **A4** *is less than $\chi_=(K_{m,n})$ if and only if $(m,n) = (6,9)$.*

Proof. We begin by assuming that the number of color classes in a coloring of type ***A4*** is less than $\chi_=(K_{m,n})$. Then there exist nonnegative integers q, r, s, and t such that $m = 2+3q+4r$, $n = 2+3s+4t$, and $\chi_=(K_{m,n}) > q+r+s+t+1$.

Let $A = \{x \in \mathbb{N} : x \geq 12\} \cup \{4,5,8,9,10\}$.

Observe that if the smallest color class size of an equitable coloring is at least 4, then the number of color classes in the equitable coloring is less than the number of color classes in a coloring of type ***A4***. Combining this with Lemma 6, we have a contradiction for $m,n \in A$. Therefore, we consider only the cases when $m \in \{6,11\}$ or $n \in \{6,11\}$.

Case 1 : $m \in \{6,11\}$. We consider two subcases.

Subcase 1.1 : $m = 6$. For $t \geq 2$, Lemmas 1 and 4 imply that $\chi_=(K_{6,n}) < q+r+s+t+1$ which is a contradiction. Because $\chi_=(K_{6,n}) \leq q+r+s+t+1$ when$s \geq 2$, we consider only $(m,n) = (6,6)$ and $(6,9)$. The number of color classes that result from a coloring of type ***A4*** on $K_{6,6}$ is 3, which is greater than $\chi_=(K_{6,6}) = 2$. Thus, the remaining (m,n) is $(6,9)$.

Subcase 1.2 : $m = 11$. Since $\chi_=(K_{11,n}) \leq q+r+s+t+1$ for all n, we have a contradiction.

Case 2 : $n \in \{6,11\}$. We can show that $\chi_=(K_{m,n}) \leq q+r+s+t+1$ except the cases $0 \leq q \leq 1$ and $r = 0$. That is, $m = 2$ or 5. However, each of these cases is eliminated.

Hence, the only possible (m, n) in this case is $(6, 9)$.

Conversely, we can verify that the number of color classes that result from a coloring of type ***A4*** on $K_{6,9}$ is 4, which is less than $\chi_=(K_{6,9}) = 5$. □

Lemma 16. *[coloring of type **B4**] Suppose that $m, n \neq 2$. The number of color classes in a coloring of type* **B4** *is less than $\chi_=(K_{m,n})$ if and only if $(m, n) = (5, 7)$ or $(6, 9)$.*

Proof. We begin by assuming that the number of color classes in a coloring of type ***B4*** is less than $\chi_=(K_{m,n})$. Then there exist nonnegative integers q, r, s, and t such that $m = 2+3q+4r$, $n = 1+3s+4t$, and $\chi_=(K_{m,n}) > q+r+s+t+1$.

Let $A = \{x \in \mathbb{N} : x \geq 12\} \cup \{4, 5, 8, 9, 10\}$.

Observe that if the smallest color class size of an equitable coloring is at least 4, then the number of color classes in the equitable coloring is less than the number of color classes in a coloring of type ***B4***. Combining this with Lemma 6, we have a contradiction for $m, n \in A$. Therefore, we consider only the cases when $m \in \{6, 11\}$ or $n \in \{1, 7, 11\}$.

Case 1 : $m \in \{6, 11\}$. We consider two subcases.

Subcase 1.1 : $m = 6$. By Lemmas 1 and 4, $\chi_=(K_{6,n}) < q+r+s+t+1$ where $n \geq 10$. Therefore, $n = 7, 8$, or 9. However, $\chi_=(K_{6,7}) = 2$ and $\chi_=(K_{6,8}) = 4$ are not less than the number of color classes that result from a coloring of type ***B4***. Thus, the only possibility is $(m, n) = (6, 9)$.

Subcase 1.2 : $m = 11$. Since $\chi_=(K_{11,n}) \leq q + r + s + t + 1$ for all n, we have a contradiction.

Case 2 : $n \in \{1, 7, 11\}$. Because $K_{1,1}$ cannot be assigned by a coloring of type ***B4***, we consider only $n = 7$ or 11. We can show that $\chi_=(K_{m,n}) \leq q+r+s+t+1$ if $m \neq 5$.

The number of color classes that result from a coloring of type ***B4*** on $K_{5,11}$ is 5, which is greater than $\chi_=(K_{5,11}) = 3$. Thus, the only possible (m, n) in this case is $(5, 7)$.

Conversely, for both $(m, n) = (5, 7)$ and $(6, 9)$, the number of color classes that result from a coloring of type ***B4*** on $K_{m,n}$ is 4, which is less than $\chi_=(K_{m,n}) = 5$. □

Lemma 17. *[coloring of type **B3**] Suppose $m, n \neq 2$. The number of color classes in a coloring of type* **B3** *is less than $\chi_=(K_{m,n})$ if and only if $(m, n) = (5, 7)$.*

Proof. We first assume that the number of color classes in a coloring of type ***B3*** is less than $\chi_=(K_{m,n})$. Then there exist nonnegative integers q, r, s, and t

such that $m = 2 + 2q + 3r$, $n = 1 + 2s + 3t$, and $\chi_=(K_{m,n}) > q + r + s + t + 1$.

Let $A = \{x \in \mathbb{N} : x \geq 6\} \cup \{3, 4\}$.

Observe that if the smallest color class size of an equitable coloring is at least 3, then the number of color classes in the equitable coloring is less than the number of color classes in a coloring of type ***B3***. Combining this with Lemma 6, we have a contradiction for $m, n \in A$. We consider only the cases when $m = 5$ or $n = 5$.

Case 1 : $m = 5$. Lemmas 1 and 4 imply that $\chi_=(K_{5,n}) < q + r + s + t + 1$ for $n \geq 8$. If $n = 5$ or 6, then Lemma 2 indicates that $\chi_=(K_{5,n}) < q + r + s + t + 1$. Thus, the only possibility is $(m, n) = (5, 7)$.

Case 2 : $n = 5$. For both $(m, n) = (4, 5)$ and $(5, 5)$, the number of color classes that result from a coloring of type ***B3*** is 4, which is greater than $\chi_=(K_{m,n}) = 2$. This is a contradiction.

Conversely, the number of color classes the result from a coloring of type ***B3*** on $K_{5,7}$ is 4, which is less than $\chi_=(K_{5,7}) = 5$. □

Lemma 18. *[coloring of type* ***C4****] Suppose that* $m, n \notin \{1, 2\}$*. The number of color classes in a coloring of type* **C4** *is not less than* $\chi_=(K_{m,n})$*.*

Proof. Suppose that the number of color classes in a coloring of type ***C4*** is less than $\chi_=(K_{m,n})$. Then there exist nonnegative integers q, r, s, and t such that $m = 1 + 3q + 4r$, $n = 2 + 3s + 4t$, and $\chi_=(K_{m,n}) > q + r + s + t + 1$.

Let $A = \{x \in \mathbb{N} : x \geq 12\} \cup \{4, 5, 8, 9, 10\}$.

Observe that if the smallest color class size of an equitable coloring is at least 4, then the number of color classes in the equitable coloring is less than the number of color classes in a coloring of type ***C4***. Combining this with Lemma 6, we have a contradiction for $m, n \in A$. We consider only the cases when $m \in \{7, 11\}$ or $n \in \{6, 11\}$.

Case 1 : $m \in \{7, 11\}$. We can show that $\chi_=(K_{m,n}) \leq q + r + s + t + 1$ if $n \neq 2, 5$. However, $n = 2, 5$ violates the condition $m \leq n$.

Case 2 : $n \in \{6, 11\}$. Because $\chi_=(K_{4,6}) = 3$ and $\chi_=(K_{5,6}) = 2$ are not greater than the number of color classes that result from a coloring of type ***C4***, we consider only $n = 11$.

Because $m = 7$ and 11 have been considered, only the cases when $m = 4, 5, 8, 9$, or 10 remain to be considered. We can show that the numbers of color classes that result from a coloring of type ***C4*** are $5, 5, 6, 6$, and 7 for $(m, n) = (4, 11)$, $(5, 11)$, $(8, 11)$, $(9, 11)$, and $(10, 11)$, respectively. Each of the numbers of color classes is not less than the equitable chromatic number of the corresponding graph.

Hence, the numbers of color classes that result from a coloring of type ***C4*** are not less than $\chi_=(K_{m,n})$. □

Lemma 19. *[coloring of type **C3**] Suppose $m, n \notin \{1, 2\}$. The number of color classes in a coloring of type* **C3** *is not less than $\chi_=(K_{m,n})$.*

Proof. Suppose that the number of color classes in a coloring of type ***C3*** is less than $\chi_=(K_{m,n})$. Then there exist nonnegative integers q, r, s, and t such that $m = 1 + 2q + 3r$, $n = 2 + 2s + 3t$, and $\chi_=(K_{m,n}) > q + r + s + t + 1$.

Let $A = \{x \in \mathbb{N} : x \geq 6\} \cup \{3, 4\}$.

Observe that if the smallest color class size of an equitable coloring is at least 3, then the number of color classes in the equitable coloring is less than the number of color classes in a coloring of type ***C3***. Combining this with Lemma 6, we have a contradiction for $m, n \in A$. We consider only the cases when $m = 5$ or $n = 5$.

Because $\chi_=(K_{5,n}) \leq q+r+s+t+1$ for all n, we consider only the case when $n = 5$. The numbers of color classes that result from a coloring of type ***C3*** on $K_{3,5}$, $K_{4,5}$, and $K_{5,5}$ are $3, 3$, and 4, respectively. Each of the numbers of color classes is not less than the equitable chromatic number of the corresponding graph.

Hence, the number of color classes in a coloring of type ***C3*** is not less than $\chi_=(K_{m,n})$. □

Now, we are ready to prove our main Theorem.

Theorem 20. *For a complete bipartite graph $K_{m,n}$ with $m \leq n$, $\chi_{ED,2}(K_{m,n}) = \chi_=(K_{m,n})$ except*

1. $\chi_{ED,2}(K_{1,n}) = 1 + \lceil (n-2)/4 \rceil$;

2. $\chi_{ED,2}(K_{2,n}) = 1 + \lceil (n-2)/5 \rceil$;

3. $\chi_{ED,2}(K_{m,n}) = 1 + \lceil (m-2)/5 \rceil + \lceil (n-2)/5 \rceil$,
   where $(m, n) = (7, 10), (7, 11), (7, 17), (11, 14),$ *or* $(11, 19)$;

4. $\chi_{ED,2}(K_{5,7}) = 4$;

5. $\chi_{ED,2}(K_{6,9}) = 4$.

Proof. We have by Lemma 12, $\chi_{ED,2}(K_{1,n}) = 1 + \lceil (n-2)/4 \rceil$. By Lemma 13, $\chi_{ED,2}(K_{2,n}) = 1 + \lceil (n-2)/5 \rceil$. In addition, $\chi_{=}(K_{1,n}) = 1 + \lceil n/2 \rceil$ and $\chi_{=}(K_{2,n}) = 1 + \lceil n/3 \rceil$.

Consider $K_{m,n}$ where $(m,n) = (7,10), (7,11), (7,17), (11,14)$, or $(11,19)$. By Lemma 14, the number of color classes that result from a coloring of type ***A5*** on $K_{m,n}$ is less than $\chi_{=}(K_{m,n})$. By Lemmas 15 – 19, colorings of other types on $K_{m,n}$ do not exist or the resulting numbers of color classes are less than $\chi_{=}(K_{m,n})$. From the proof of Lemma 14, the numbers of color classes that result from colorings of type ***A5*** on $K_{m,n}$ are $4, 4, 5, 6$, and 7 for $(m,n) = (7,10), (7,11), (7,17), (11,14)$, and $(11,19)$, respectively. Hence, $\chi_{ED,2}(K_{m,n}) = 1 + \lceil (m-2)/5 \rceil + \lceil (n-2)/5 \rceil$ for (m,n) in this case.

For $K_{5,7}$, Lemma 17 implies that the number of color classes that result from a coloring of type ***B3*** on $K_{5,7}$ is less than $\chi_{=}(K_{5,7})$. By Lemmas 14 – 19, colorings of other types on $K_{5,7}$ do not exist or the resulting numbers of color classes are less than $\chi_{=}(K_{5,7})$. From the proof of Lemma 17, the number of color classes that result from a coloring of type ***B3*** on $K_{5,7}$ is 4. Hence, $\chi_{ED,2}(K_{5,7}) = 4$.

For $K_{6,9}$, Lemmas 15 and 16 imply that the numbers of color classes that result from colorings of types ***A4*** and ***B4*** on $K_{6,9}$ are less than $\chi_{=}(K_{6,9})$. By Lemmas 14 – 19, colorings of other types on $K_{6,9}$ do not exist or the resulting numbers of color classes are less than $\chi_{=}(K_{6,9})$. The proofs of Lemmas 15 and 16 show that the number of color classes resulting from a coloring of types ***A4*** and ***B4*** are equal to 4. Hence, $\chi_{ED,2}(K_{6,9}) = 4$.

If $K_{m,n}$ is not addressed by the previous cases, then Lemmas 14 – 19 indicate that $\chi_{=}(K_{m,n})$ is not greater than the number of color classes that result from a coloring of type ***A5***, ***A4***, ***B4***, ***B3***, ***C4***, or ***C3***. Hence, $\chi_{ED,2}(K_{m,n}) = \chi_{=}(K_{m,n})$. □

Acknowledgments

The first author was supported by the Thailand Research Fund under grant MRG5580003. The second author was supported by the Project for the Promotion of Science and Mathematics Talented Teachers and the Centre of Excellence in Mathematics, Thailand.

References

[1] N. Achuthan, N. R. Achuthan, M. Simanihuruk, On minimal triangle-free graphs with prescribed k-defective chromatic number, *Discrete Mathematics*, **311** (2011), 1119-1127.

[2] J. A. Andrews, M. S. Jacobson, On a generalization of chromatic number, *Congressus Numerantium*, **47** (1985), 33-48.

[3] S. Cummuang, K. Nakprasit, Equitable defective colorings of some classes of graphs, *KKU Science Journal*, **39**, No. 1 (2011), 687-694.

[4] M. Frick, A survey of (m, k)-colourings, *Annals of Discrete Mathematics*, **55** (1993), 45-58.

[5] M. Frick, M. A. Henning, Extremal results on defective colourings of graphs, *Discrete Mathematics*, **126** (1994), 151-158.

[6] H. Furmańczyk, Equitable coloring of graph, in *Graph colorings*, Vol. 352 of *Contemporary Mathematics*, 35-50, American Mathematical Society, Providence, RI, USA (2004).

[7] G. Hopkins, W. Staton, Vertex partitions and k-small subsets of graphs, *Ars Combinatoria*, **22** (1986), 19-24.

[8] P. C. B. Lam, W. C. Shiu, C. S. Tong, Z. F. Zhang, On the equitable chromatic number of complete r-partite graphs, *Discrete Applied Mathematics*, **113** (2001), 307-310.

[9] K. W. Lih, P. L. Wu, On equitable coloring of bipartite graphs, *Discrete Mathematics*, **151**, No. 1-3 (1996), 155-160, **doi:** 10.1016/0012-365X(94)00092-W.

[10] K. W. Lih, The equitable coloring of graphs, in *Handbook of Combinatorial Optimization*, Kluwer Academic Publishers, **3** (1998), 543-566.

[11] R. B. Maddox, Vertex partitions and transition parameters, *Ph.D. Thesis, The University of Mississippi*, Oxford, MS, USA (1988).

[12] R. B. Maddox, On k-dependent subsets and partitions of k-degenerate graphs, *Congressus Numerantium*, **66** (1988), 11-14.

[13] W. Meyer, Equitable coloring, *American Mathematical Monthly*, **80** (1973), 920-922, **doi:** 10.2307/2319405.

[14] K. Nakprasit, W. Saigrasun, On equitable coloring of complete r-partite graphs, *International Journal of Pure and Applied Mathematics*, **71**, No. 2 (2011), 229-239.

[15] D. B. West, *Introduction to graph theory*, 2-nd ed., Prentice-Hall, Upper Saddle River, NJ, USA (2001).

[16] L. Williams, J. Vandenbussche, G. Yu, Equitable defective coloring of sparse planar graph, *Discrete Mathematics*, **312** (2012), 957-962, **doi:** 10.1016/j.disc.2011.10.024.

A2. **Nakprasit, K.**, Nakprasit, K., *The strong chromatic index of graphs with restricted Ore-degrees*, Ars Combinatoria, **accepted**.

The strong chromatic index of graphs with restricted Ore-degrees

Keaitsuda Nakprasit[†] and Kittikorn Nakprasit[‡],[1]

[†]Department of Mathematics, Faculty of Science,
Khon Kaen University, Khon Kaen 40002,
Thailand
e-mail : kmaneeruk@hotmail.com
[‡]Department of Mathematics, Faculty of Science,
Khon Kaen University, Khon Kaen 40002,
Thailand
e-mail : kitnak@hotmail.com

Abstract

A strong edge-coloring is a proper edge-coloring such that two edges with the same color are not allowed to lie on a path of length three. The strong chromatic index of a graph G denoted by $s'(G)$ is the minimum number of colors in a strong edge-coloring.

We denote the degree of a vertex v by $d(v)$. Let the Ore-degree of a graph G be the maximum values of $d(u)+d(v)$ where u and v are adjacent vertices in G. Let F_3 denote the graph obtained from a 5-cycle by adding a new vertex and joining it to a pair of nonadjacent vertices of the 5-cycle. In 2008, Wu and Lin [J. Wu and W. Lin, The strong chromatic index of a class of graphs, Discrete Math., 308 (2008), 6254–6261] studied the strong chromatic index with respect to the Ore-degree. Their main result states that if a connected graph G is not F_3 and its Ore-degree is 5, then $s'(G) \leq 6$. Inspired by the result of Wu and Lin, we investigate the strong edge-coloring of graphs with Ore-degree 6. We show that each graph G with Ore-degree 6 has $s'(G) \leq 10$. With the further condition that G is bipartite, we have $s'(G) \leq 9$. Our results give general forms of previous results about strong chromatic indices of graphs with maximum degree 3.

[1]Corresponding author email:kitnak@hotmail.com

1 Introduction

Graphs in this paper are finite, undirected, and loopless, but multiple edges are allowed. We always assume that graphs are connected unless the context implies otherwise. Note that some results that we refer to may not consider multiple edges, but these results can be extended easily to graphs with multiple edges.

Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G respectively. We use $d(x)$ to denote the degree of a vertex x and $\Delta(G)$ to denote the maximum degree of a graph G. Let the *Ore-degree* of a graph G be the maximum values of $d(u) + d(v)$ where u and v are adjacent vertices in G. A k-vertex is a vertex of degree k.

The *distance* between edges e_1 and e_2 in a graph G is the distance between the corresponding two vertices in the line graph of G. A *strong edge-coloring* of a graph G is an edge-coloring in which two distinct edges with distance at most 2 have different colors. A *strong k-edge-coloring* is a strong edge-coloring using at most k colors. The *strong chromatic index* $s'(G)$ is the minimum k such that G has a strong k-edge-coloring. Throughout this paper, the term coloring means strong edge-coloring, unless the coloring is specified to be other type of coloring.

Erdős and Nešetřil [4] conjectured that $s'(G) \leq 5D^2/4 - D/2 + 1/4$, if D is odd and $s'(G) \leq 5D^2/4$, if D is even, where $D = \Delta(G)$. Andersen [1] and Horák, Qing, and Trotter [7] settled the case $D = 3$ of the conjecture by showing the following.

Theorem 1.1 ([1, 7]) *If a graph G has a maximum degree three, then $s'(G) \leq 10$.*

Horák [6] showed that there is a strong 23-edge-coloring for graphs with maximum degree four. Cranston [3] improved the bound to 22. The conjecture for $D = 4$ which has $s'(G) \leq 20$ remains unsolved.

Faudree et al. [5] formulated a bipartite version of this problem. They conjectured that $s'(G) \leq D^2$, if G is a bipartite graph. Steger and Yu [14] settled the conjecture for $\Delta(G) = 3$ which is the first non-trivial case of the second conjecture by showing the following.

Theorem 1.2 ([14]) *If a bipartite graph G has a maximum degree three, then $s'(G) \leq 9$.*

A stronger version of the second conjecture, due to Brualdi and Massey [2], states that $s'(G)$ is bounded by D_1D_2, where D_1 and D_2 are the maximum degrees among vertices in the two partite sets, respectively. Quinn and Benjamin [12] proved this for a special class of bipartite graphs whose partite sets are the k-sets and l-sets in $[m]$, adjacent when the two sets share exactly j elements. Quinn and Sundberg [13] proved it for the incidence bigraph of the k-sets in $[m]$. Nakprasit [10] gave the affirmative answer to the conjecture for $D_1 = 2$.

Note that there are researches focusing to colorings related to Ore-degrees of graphs instead of maximum degrees. For example, Kierstead and Kostochka [8, 9] studied the relation of ordinary coloring, equitable coloring, and nearly-equitable coloring to Ore-degrees of graphs. In [5], Faudree et al. conjectured that if G is a bipartite graph with Ore-degree at most 5, then $s'(G) \leq 6$. Let F_D denote the graph obtained from a 5-cycle by adding $D - 2$ new vertices and joining them to a pair of nonadjacent vertices of the 5-cycle. Wu and Lin [15] obtained the main result in their paper which verified the conjecture in a stronger form as follows.

Theorem 1.3 ([15]) *If a graph G is not F_3 and its Ore-degree is at most 5, then $s'(G) \leq 6$.*

Let H_1 denote the graph obtained from a 8-cycle $C = v_1v_2 \dots v_8$ by adding two vertices v_1' and v_5' and joining v_1' to v_2, v_8, and v_5' to v_4, v_6. Wu and Lin [15] noted that they did not know any graphs with Ore-degree 5 to have $s'(G) \geq 6$ except F_3, H_1, and $K_{2,3}$. The result of Wu and Lin is generalized by Nakprasit and Nakprasit [11] as follows.

Theorem 1.4 ([11]) *If each edge xy of a graph G has $d(x)+d(y) \leq D+2$ and $\min\{d(x), d(y)\} \leq 2$, then $s'(G) \leq 2D + 1$. With the further conditon that G is not F_D, we have $s'(G) \leq 2D$.*

However, the stronger form of Theorem 1.3 in terms of Ore-degrees is not known. For graphs with small Ore-degrees, we have the followings.

Observation 1.5 *(Characterization of graphs with small Ore-degrees)*

(i) The only graph with Ore-degree 0 is K_1.

(ii) No graph has Ore-degree 1.

(iii) The only graph with Ore-degree 2 is a path with one edge.

(iv) The only graph with Ore-degree 3 is a path with two edges.

(v) A graph G has Ore-degree 4 if and only if G is a path of length at least 3, $K_{1,3}$, a cycle, or a graph with two vertices and two multiple edges.

Since graphs with the above Ore-degrees can be classified explicitly, we can find their strong chromatic indices easily. Thus Theorem 1.3 by Wu and Lin is the first non-trivial result about the strong chromatic index in terms of Ore-degrees.

Inspired by the result of Wu and Lin, we show that each graph G with Ore-degree at most 6 has $s'(G) \leq 10$. With the further condition that G is bipartite, we have $s'(G) \leq 9$. Our results give general forms of Theorems 1.1 and 1.2.

2 The strong edge-colorings of graphs with restricted Ore-degrees

Note again that we assume that each graph is connected unless the context implies otherwise.

Next, we proceed to investigate strong chromatic indices in terms of Ore-degree of graphs in general.

Lemma 2.1 *Let G be a graph with Ore-degree at most R. If M is the set of vertices of G with degree $R-2$, then $s'(G) \leq \max\{s'(G-M), 3R-8\}$.*

Proof. If $\Delta(G) = R-1$, then $G = K_{1,R-1}$ which has $s'(G) = R-1$. If $\Delta(G) \leq R-3$, then $G = G-M$. Thus $\Delta(G) = R-2$ which implies $M \neq \emptyset$. Let G_1 be the graph induced by the edges incident to M.

First, note that each edge of G_1 has at most $2(R-3)+(R-2) = 3R-8$ edges within distance two. If each edge of one component of G_1 has $3R-8$ edges within distance two, then G satisfies the condition of Theorem 1.4 which implies $s'(G) \leq 2(R-2)+1 = 2R-3$. Now we may assume that every component of G_1 has

Apply strong edge-coloring with $s'(G-M)$ colors to $E(G-M)$. It can be see that we can use this as a partial strong edge-coloring in G. Now, we want to extend the coloring to edges in the component of $E(G_1)$ with an edge e having at most $3R-9$ edges within distance two. To greedily color them one by one, we give an ordering of the edges of this component in the following way.

If the distance from e_1 to e is greater than the distance from e_2 to e, then we color e_1 before e_2. Since every edge has at most $3R-9$ colored edges within distance two at each step, we can color every edge of such component of G_1. Using similar method to all components to complete the coloring. □

A path ww_1w_2 is a *special 2-path* if $d(w_1) = d(w_2) = 2$ and w is an $(R-2)$-vertex.

Lemma 2.2 *Let G be a graph with Ore-degree at most R with a special 2-path ww_1w_2. Then $s'(G) \leq \max\{s'(G-w_1), 2R-3\}$.*

Proof. Apply strong edge-coloring with $s'(G-w_1)$ colors to $G-w_1$. Now ww_1 has at most $2(R-3)+1 = 2R-5$ colored edges within distance two and w_1w_2 has at most $(R-3)+(R-2) = 2R-5$ colored edges within distance two. Since we have at least $2R-3$ available colors, we can extend the coloring to ww_1 and w_1w_2. As a result, we have a required coloring. □

Theorem 2.3 *If a graph G has Ore-degree at most 6, then $s'(G) \leq 10$. With the futher condition that G is bipartite, we have $s'(G) \leq 9$.*

Proof. Let G be a graph with Ore-degree 6. If $\Delta(G) = 5$, then $G = K_{1,5}$ which has $s'(G) = 5$. So we can assume that $\Delta(G) \leq 4$. Let M be the set of vertices with degree 4. Lemma 2.1 yields that $s'(G) \leq \max\{s'(G-M), 10\}$. Since $\Delta(G-M) \leq 3$, we have $s'(G-M) \leq 10$ by Theorem 1.1. Thus we have $s'(G) \leq 10$.

Now, it remains to show that $s'(G) \leq 9$ when G is bipartite. Suppose that G is a minimal counterexample to the theorem. Consider the case that G contains two distinct edges e_1, e_2 with a pair of common endpoints. Since $s'(G-e_1) \leq 9$ by minimality and e_1 has at most seven edges within distance two, we have $s'(G) \leq 9$. Thus we may assume G has no multiple edges. If $\Delta(G) = 5$, then $G = K_{1,5}$ which has $s'(G) = 5$. If $\Delta(G) \leq 3$, then Theorem 1.2 yields $s'(G) \leq 9$. Consequently, we assume that $\Delta(G) = 4$. Since G is not $K_{1,4}$ which has $s'(G) = 4$, the graph G contains a 4-vertex adjacent to a 2-vertex. If G has no 3-vertices, then Theorem 1.4 yields $s'(G) \leq 9$. Thus G contains a 3-vertex and a 4-vertex. Since G is connected and has Ore-degree 6, the graph G has a path of length at least two with every internal vertex is 2-vertex whereas one endpoint is 3-vertex and the other is 4-vertex. If G contains a special 2-path, then $s'(G) \leq 9$ by minimality of G and Lemma 2.2. Thus G contains a path uvw with u is a 4-vertex, v is a 2-vertex, and w is a 3-vertex.

Consider such u with its four neighbors v_1, v_2, v_3, and v_4. Since G is bipartite, the set $\{v_1, v_2, v_3, v_4\}$ is independent. Suppose some v_i is a 1-vertex. Since $s'(G - uv_i) \leq 9$ by minimality and uv_i has at most six edges within distance two, we have $s'(G) \leq 9$. Let w_i different from u be the other neighbor of the 2-vertex v_i $(1 \leq i \leq 4)$. Note that w_1, w_2, w_3, w_4 are not necessarily pairwise distinct. We have $d(w_i) \neq 1$ as before. Moreover, $d(w_i) \neq 2$ because G has no special 2-edges. Combinining with the fact that G has Ore-degree at most 6, we have each $d(w_i) = 3$ or 4. From the choice of u, some w_i is a 3-vertex. Let $W = \{w_1, w_2, w_3, w_4\}$.

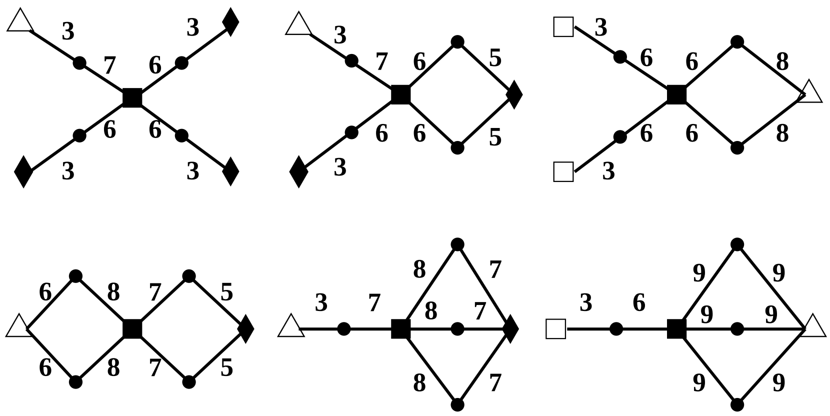

Figure 1: Configurations in a minimum counterexample.

We claim that the counterexample G must contain one of the configurations in Fig. 1, where a black square is the vertex u, a dot indicates a vertex of degree 2 (that is some v_i), a hollow triangle indicates a vertex of degree 3, a hollow square indicates a vertex of degree 4, the degree of a black diamond is at least the number of edges incident to the black diamond in the figure, and all vertices are distinct. If $|W| = 4$, that is all w_1, w_2, w_3, w_4 are distinct, then G contains the first configuration. Consider the case $|W| = 3$ where $w_3 = w_4$. If $d(w_1)$ or $d(w_2)$ is 3, then G

contains the second configuration, otherwise G contains the third configuration. The case $|W| = 2$ where $w_1 = w_2$ and $w_3 = w_4$ implies G contains the fourth configuration. Consider the case $|W| = 2$ where $w_2 = w_3 = w_4$. If $d(w_1) = 3$, then G contains the fifth configuration, otherwise G contains the sixth configuration. The case that $|W| = 1$ contradicts the fact that $d(w_i) = 3$.

After some partial strong k-edge coloring on G, we use $A(e)$ denote the number of legal colors from k colors that can be assigned to e. Consider a coloring of all edges in G except edges in a configurtion. Each edge e is the figure is shown with a lower bound for $A(e)$ that is calculated from 9 minus the number of edges with distance within two from the edge e,

Since the number of legal colors for each edge not incident to u is large enough, the sets of legal colors of those edges cannot be all pairwise disjoint. Thus we can assign some color to two of those edges simultaneously. Next, we color other two edges that are not incident to the vertex u. Note that the lower bound for $A(e)$ in each uncolored edge e is now decreased by at most three. Finally, we color four edges incident to the vertex u sequentially from an edge with the least number of legal colors to the most one. Since the number of legal colors for each edge is large enough, the strong edge-coloring using at most nine colors can be completed. □

Acknowledgements

Both authors would like to thank the anonymous referee for helpful comments and suggestions which much improve our presentation of the paper, especially the proofs.

The first author was supported by The Thailand Research Fund under grant MRG5580003. The second author was supported by National Research Council of Thailand and Khon Kaen University, Thailand (Grant number: kku fmis (570018)).

References

[1] L.D. Andersen, The strong chromatic index of a cubic graph is at most 10, Discrete Math., 108 (1992), 231–252.

[2] R.A. Brualdi and J.Q. Massey, Incidence and strong edge colorings of graphs, Discrete Math., 122 (1993), 51–58.

[3] D.W. Cranston, Strong edge-coloring of graphs with maximum degree 4 using 22 colors. Discrete Math., 306 (2006), 2772–2778.

[4] P.Erdős and J. Nešetřil, Problem, in: G. Halász and V. T. Sós, eds., Irregularities of partitions, (Springer, New York, 1989), 83–87.

[5] R.J. Faudree, A. Gyárfàs, R.H. Schelp, and Z. Tuza, Induced matchings in bipartite graphs, Discrete Math., 78 (1989), 83–87.

[6] P. Horák, The strong chromatic index of graphs with maximum degree four, Contemporary methods in graph theory, 399–403, Bibliographisches Inst., Mannheim, 1990.

[7] P. Horák, H. Qing, and W.T. Trotter, Induced matchings in cubic graphs, J. Graph Theory, 17 (1993), 151–160.

[8] H.A. Kierstead and A.V. Kostochka, An Ore-type theorem on equitable coloring, J. Combin. Theory Ser. B, 98 (2008), 226–234.

[9] H.A. Kierstead and A.V. Kostochka, Ore-type versions of Brooks theorem, J. Combin. Theory Ser. B, 99 (2009) 298–305.

[10] K. Nakprasit, A note on the strong chromatic index of bipartite graphs, Discrete Math., 308 (2008), 3726–3728.

[11] K. Nakprasit and K. Nakprasit, The strong chromatic index of graphs and subdivisions, Discrete Math., 317 (2014), 75–78.

[12] J.J. Quinn and A.T. Benjamin, Strong chromatic index of subset graphs, J. Graph Theory, 24 (1997), 267–273.

[13] J.J. Quinn and E.L. Sundberg, Strong chromatic index in subset graphs, Ars Combin., 49 (1998), 155–159.

[14] A. Steger and M.L. Yu, On induced matchings, Discrete Math., 120 (1993), 291–295.

[15] J. Wu and W. Lin, The strong chromatic index of a class of graphs, Discrete Math. 308 (2008), 6254–6261.

Date: Fri, 13 Jun 2014 13:40:05 -0500

To: kitnak@hotmail.com

From: MathJournals@attglobal.net

Subject: Ars Comb paper

Professor Nakprasit,

I have now received the referees third report on your paper with Keaitsuda Nakprasit entitled, "The strong chromatic index of graphs with restricted Ore-degrees".
The report is attached below. The referee now recommends publication. Therefore, I am happy to accept it for publication once the corrections have been made.

Would you please send me a new copy: single spaced, in 10 point Times or Computer Modern font, 4.5 x 7.125 inches (11.4 x 18.1 cm) in text size. I use your copy as camera-ready for publication. A .pdf file by email will suffice.
Yours sincerely,
J.L. Allston
Managing Editor
Ars Combinatoria

A3. **Nakprasit, K.**, Nakprasit, K., *Equitable coloring games on complete bipartite graphs*, manuscript.

Equitable Coloring Games on Complete Bipartite Graphs

Keaitsuda Nakprasit

Department of Mathematics, Faculty of Science, Khon Kaen University, 40002, Thailand

E-mail address: kmaneeruk@hotmail.com

Kittikorn Nakprasit *

Department of Mathematics, Faculty of Science, Khon Kaen University, 40002, Thailand

E-mail address: kitnak@hotmail.com

Abstract

An *equitable k-coloring game* on a graph G is a game that Alice and Bob play with a color set $C = \{1, 2, \ldots, k\}$. They take turns coloring the vertices of G one at a time with colors from C, with Alice has the first move, under the rule that adjacent vertices have different colors. Each color cannot be used more than b times where $b = \lceil |V(G)|/k \rceil$, and the coloring is allowed to have at most $|V(G)| - k(b-1)$ colors to be used b times. When there are no legal colorings left, Alice wins if all the vertices are colored and Bob wins otherwise. By this rule, we always has an equitable coloring when all vertices are colored.

In this paper, we have a characterization for complete bipartite graphs that Alice has a winning strategy in the equitable coloring game.

1 Introduction

A k-coloring game on a graph G is a game that Alice and Bob play with a color set $C = \{1, 2, \ldots, k\}$. They take turns coloring vertices in G one at a time with colors from C, with Alice has the first move, under the rule that adjacent vertices have different colors. When there are no legal colorings left, Alice wins if all the vertices are colored and Bob wins otherwise.

In 1981, Steven Brams in [7] considered the coloring game on maps which is equivalent to the coloring game on planar graphs. In 1991, Bodlaender [1] introduced the formal concept of the coloring game in general graphs. From then, the research in the coloring game has flourished with many interesting results. The *game chromatic number* of a graph G is the least k that Alice has a winning strategy for the k-coloring game. One of the most studied problems in this topic is to find the game chromatic number of planar graphs. Upper bounds

*Corresponding Author

for the game chromatic number have been determined for outerplanar graphs [8] and planar graphs [5, 13, 14, 19, 20]. Guan and Zhu [8] proved that the game chromatic number of an outerplanar graph is at most 7. Zhu [20] proved that the game chromatic number of a planar graph is at most 17. Currently, both results are the best-known bounds for respective classes of graphs. Upper bounds for the game chromatic numbers have been determined for other graphs including forests [6], line graphs of k-degenerate graphs [2], and graphs embeddable on orientable surfaces with bounded genus [13].

An *equitable coloring* of a graph is a proper vertex coloring such that the sizes of every two color classes differ by at most 1. A graph G is *equitably k-colorable* if there is an equitable coloring of k colors for G. The smallest integer k for which G is equitably k-colorable is called the *equitable chromatic number* of G. Meyer [16] introduced this notion of equitable colorability in 1973. However, a work in 1970 of Hajnal and Szemerédi [9] actually settled a conjecture of Erdős by showing that a graph G with maximum degree $\Delta(G)$ is equitably k-colorable if $k \geq \Delta(G)+1$. The simpler proof of Hajnal-Szemerédi theorem was given later by Kierstead and Kostochka [11].

The bound of the Hajnal-Szemerédi theorem is sharp. However, Chen, Lih, and Wu [4] conjectured that every connected graph G with maximum degree $\Delta \geq 3$ has an equitable coloring with Δ colors, except when G is $K_{\Delta+1}$ or $K_{\Delta,\Delta}$. They proved the conjecture for graphs with maximum degree at most three. Later, Kierstead and Kostochka [12] proved the conjecture for graphs with maximum degree at most four. Yap and Zhang [18] proved that the conjecture holds for outerplanar graphs. Later, Kostochka [10] determined the sharp upper bound of equitable chromatic number for outerplanar graph which is $1+\Delta/2$. Lam et al. [15] determined which k that each n-partie graph can have an equitable k-coloring. This result verifies the conjecture for complete n-partite graphs. The conjecture is also verified for various graphs including trees [3] and planar graphs with maximum degree at least nine [17].

In this paper, we introduce an equitable version of a coloring game. In a coloring game, Alice wins if and only if all vertices are colored. To enforce a coloring of all vertices to be an equitable k-coloring, we propose the additional rules for an equitable k-coloring game as follows. Let $b = \lceil |V(G)|/k \rceil$ and $d = \lfloor |V(G)|/k \rfloor$. A color that has been used b times is a *major color*. Each color cannot be used more than b times and the coloring is allowed to have at most $|V(G)| - k(b-1)$ major colors.

From now on, a game means an equitable k-coloring game on a complete bipartite $K_{m,n}$. Let X and Y be partite sets of $K_{m,n}$ of size m and n, respectively, with $m \leq n$. A partite set is called *even* (respectively, *odd*) if its size is even (respectively, odd). Let $f(X)$ be the number of colors that appear in X and $g(X)$ be the number of major colors that appear in X. Functions $f(Y)$ and $g(Y)$ are defined similarly.

We study the games with $b \leq 2$ in Section 2, the games with $d \geq 3$ in Section 3, and the games with $b = d+1 = 3$ in Section 4. In Section 5, we conclude our finding by characterizing complete bipartite graphs that Alice has a winning strategy in the equitable coloring game.

2 The games with $b \leq 2$

Lemma 1. If $d \leq 1$ then Alice has a winning strategy.

Proof. Since Alice wins trivially when $b = 1$, we consider only the case $b = 2$ and $d = 1$.

Consider Alice's strategy as follows. If it is possible, then Alice assigns a used color to a vertex preferably in an even partite set. Otherwise she assigns a new color to a vertex preferably in an odd partite set.

Note that when X and Y have uncolored vertices the number of major colors in every two turns always increases by Alice's strategy. If $m+n-k$ major colors occur during gameplay, then the size of each remaining color class can be at most 1. Thus remaining vertices can be arbitrarily colored. Now suppose that fewer than $m+n-k$ major colors occur during a game but all vertices in one partite set, say X, are colored. We analyze this situation into two cases.

Case 1: m is odd. In this case, there exists exactly one color class of current size 1 in X by Alice's strategy. Then the remaining vertices can be colored arbitrarily to complete an equitable coloring.

Case 2: m is even. Note that there are at most two color classes of current size 1 by Alice's strategy. If there is no color class of current size 1 in X, then the remaining vertices can be colored arbitrarily to complete an equitable coloring. Suppose there are two color classes of current size 1 in X. Deducing from Alice's strategy, we have that n is also even. So there must be at least two color classes of size 1 in an equitable k-coloring of $K_{m,n}$. Thus an equitable coloring will be completed by arbitrary coloring of the remaining vertices. □

Lemma 2. If $b = d = 2$ then Bob has a winning strategy.

Proof. Obviously Bob wins when m or n is odd. In the case that m and n are even, Bob imitates Alice's play every turn until one partite set has exactly one vertex uncolored, then he plays a new color in such partite set. This move makes Bob win the game. □

Lemma 3. Let $m = d = b-1 = 2$. Alice has a winning strategy if and only if (i) $n = 3(k-1)$ and n is even, or (ii) $n < 3(k-1)$ and n is odd.

Proof. Let $m = d = b-1 = 2$.

Neccesity: Assume (i) and (ii) do not hold. Consider Bob's strategy as follows. Bob colors a vertex in Y preferably by a used color until Alice plays a vertex in X or plays the $(k-1)$st color. If Alice plays a vertex in X, then Bob assigns a new color to another vertex in X. If Alice plays the $(k-1)$st color, then Alice assigns the last color to a vertex in the same partite set. Bob wins by this strategy.

Sufficiency: Assume (i) or (ii) holds. Consider Alice's strategy as follows. Alice colors a vertex in Y preferably by a used color until Bob plays a vertex in X or plays the $(k-1)$st color. If Bob plays a vertex in X, then Alice assigns the same color to another vertex in X. If Bob plays the $(k-1)$st color, then Alice assigns the last color to a vertex in the other partite set. Alice wins by this strategy. □

3 The games with $d \geq 3$

The *X-maximizing tactic* is a strategy of Bob defined as follows. Bob plays a new color in X if there exists an unused color, and fewer than d uncolored vertices are in X or Alice played a new color in the turn immediately before. Otherwise, Bob plays a legal color with

the largest class size in X. If all vertices in X are colored, then Bob plays arbitrarily. The *Y-maximizing tactic*is defined similarly.
Remark. If Bob always has new colors to use for the X-maximizing tactic before all vertices in X are colored, then there is a color class of size less than d. Thus if we assume that a coloring is completed despite the X-maximizing tactic used by Bob, then all colors are used before all vertices in X are colored.

Lemma 4. Let $d \geq 3$. If Alice colors a vertex in X in the first turn, then Bob has a winning strategy.

Proof. Assume that Alice colors a vertex in X in the first turn. We claim that Bob can win by the X-maximizing tactic. Suppose all vertices are colored. Note that all colors are used before all vertices in X are colored by the previous remark. Bob's strategy yields $f(X) \geq f(Y) + 1$ and $g(X) \geq g(Y) - 1$. Thus $m = df(X) + (b - d)g(X) > df(Y) + (b - d)g(Y) = n$ which is a contradiction. Hence Bob has a winning strategy. □

Remark. In view of Lemma 4, we assume that Alice colors a vertex in Y in the first turn for a game with $d \geq 3$.

Before we investigate the equitable game coloring further, we define two conditions of m and n which are refered in later parts.

Definition 1. Let $k = 2t + 1$ where t is an integer. We say m and n satisfy *condition* (A) if $m = rb + (t - r)d$ and $n = rb + (t + 1 - r)d$ where r is an integer satisfying $1 \leq r \leq t$.

Definition 2. Let $k = 2t + 1$ where t is an integer. We say m and n satisfy *condition* (B) if one of the following holds:

(i) $m = tb$ and $n = (t - 1)b + 2d$,

(ii) $m = (t - 1)b + d$ and $n = (t + 1)b$,

(iii) $m = (t - 1)b + d$ and $n = tb + d$,

(iv) $m = (t - 2)b + 2d$ and $n = (t + 1)b$,

(v) $m = (t - 2)b + 2d$ and $n = tb + d$, or

(vi) $m = (t - 2)b + 2d$ and $n = (t - 1)b + 2d$.

Lemma 5. Let $d \geq 3$. If k is even, then Bob has a winning strategy.

Proof. We consider two cases.

Case 1: $n - m \leq 2$. We claim that Bob can win by the Y-maximizing tactic. Suppose all vertices are colored. Bob's strategy yields $f(Y) \geq f(X) + 2$ and $g(Y) \geq g(X) - 1$. Thus $n - m \geq 3d - b \geq 3$ which is a contradiction.

Case 2: $n - m \geq 3$. We claim that Bob can win by the X-maximizing tactic. Suppose all vertices are colored. Bob's strategy yields $f(X) \geq f(Y)$ and $g(X) \geq g(Y) - 2$. (Note that the equality of the second bound is attained only if $g(X) + 2 = f(X) = f(Y) = g(Y)$.) Thus $n - m \leq 2$ which is a contradiction.

Hence Bob has a winning strategy. □

Lemma 6. Let $d \geq 3$. If k is odd and Alice has a winning strategy, then m and n satisfy condition (A), (B), or $n - m \geq d + 1$.

Proof. Suppose Bob uses the Y-maximizing tactic. Since Alice has a winning strategy, all vertices can be colored in the end. Bob's strategy yields $f(Y) \geq f(X) + 3$ and $g(Y) \geq g(X) - 1$, or $f(Y) = f(X) + 1$ and $g(Y) \geq g(X)$. If $f(Y) \geq f(X) + 3$ and $g(Y) \geq g(X) - 1$, then $n - m \geq 3d - b \geq d + 1$. Now suppose $f(Y) = f(X) + 1$ and $g(Y) \geq g(X)$. Then $n - m \geq d + 1$, or m and n satisfy condition (A) or (B). □

Lemma 7. Let $d \geq 3$. If k is odd and Alice has a winning strategy, then m and n satisfy condition (A), (B), or $n - m \leq d - 1$.

Proof. Suppose Bob uses the X-maximizing tactic. Since Alice has a winning strategy, all vertices can be colored in the end. Bob's strategy yields that m and n satisfy condtion (B), or $f(X) \geq f(Y) + 1$ and $g(X) \geq g(Y) - 1$, or $f(X) = f(Y) - 1$ and $g(X) \geq g(Y)$. If $f(X) \geq f(Y) + 1$ and $g(X) \geq g(Y) - 1$, then $m > n$ which is a contradiction. Now we consider the case $f(X) = f(Y) - 1$ and $g(X) \geq g(Y)$. We have $n - m \leq d$ and the equality is attained only if m and n satisfy condition (A) or (B). □

Combining Lemmas 6 and 7, we immediately have the following result.

Corollary 1. Let $d \geq 3$. If k is odd and Alice has a winning strategy, then m and n satisfy condition (A) or (B).

Corollary 1 gives necessary condition for complete bipartite graphs that Alice has a winning strategy. Next we investigate which necessary condtions are also sufficient.

Lemma 8. Let odd $k = 2t + 1$, $d \geq 3$, while m and n satisfy condition (A) or (B). If b is even, then Bob has a winning strategy. (Note that b and d maybe equal in this lemma.)

Proof. Since b is even, Bob can play the first color until Alice plays the second color in a partite set. Then Bob uses the maximizing tactic on such partite set. Suppose all vertices are colored.

Case 1: The second color appears in X. Bob's strategy yields $f(X) \geq f(Y) + 1$ and $g(X) \geq g(Y) - 1$. Thus $m > n$ which is a contradiction.

Case 2: The second color appears in Y. Bob's strategy yields $f(Y) \geq f(X) + 3$ and $g(Y) \geq g(X) - 1$. Thus $m - n \geq 4d - b$ which is a contradiction.

Hence Bob has a winning strategy. □

Lemma 9. Let odd $k = 2t + 1$, $d \geq 2$, $m = (t - 1)b + d$, and $n = tb + d$, or $(t + 1)b$. If b is odd, then Alice has a winning strategy. (Note that b and d maybe equal in this lemma.)

Proof. Alice begins the game by playing a vertex in Y then she responds to Bob's moves. If Bob plays the first color, then Alice also plays the first color in the next turn. If Bob plays the $(2r)$th color (respectively the $(2r+1)$st color), then Alice plays the $(2r+1)$st color (respectively the $(2r)$th color) to a vertex in the other partite set. Alice can carry on this strategy until all vertices are colored. Thus Alice has a winning strategy. □

Lemma 10. Let odd $k = 2t + 1$ and odd $b > d \geq 2$. Bob has a winning strategy if m and n satisfy one of the following:
(i) $m = tb$, and $n = (t-1)b + 2d$,
(ii) $m = (t-2)b + 2d$, and $n = (t-1)b + 2d$,
(iii) $m = (t-2)b + 2d$, and $n = tb + d$ or $(t+1)b$.

Proof. Bob sets the trap by playing a new color in Y in each of his first $t-1$ turns. If Alice fails to respond by playing $t-1$ new colors on X, then Bob can continue playing new colors to make at least $t+2$ colors in Y. Thus vertices in X can be colored by at most $t-1$ colors which are not enough.

If Alice avoids the trap, then we have the situation that Y has t colored vertices and X has $t-1$ colored vertices in which all used colors are different.

Case 1: m and n satisfy (i). Bob continues playing a legal used color in Y with the largest current class size until Alice plays a new color (that is the $(2t)$th color). This leads to the situation that t used colors in Y become major colors or Alice is the first person who play the $(2t)$th color. In the former situation, each remaining color class in Y has class size less than d which results in Bob's victory. In the latter situation, Bob plays the last color in the same partite set as of the $(2t)$th color. This make the other partite set has too few colors to be assigned. Thus Bob wins.

Case 2: m and n satisfy (ii) or (iii). Bob continues playing a legal used color in X with the largest current class size until Alice plays a new color (that is the $(2t)$th color). If Alice plays the $(2t)$th color, then Bob plays the last color in the same partite set. We have that Bob wins as in Case 1. □

Lemma 11. Let odd $k = 2t+1$, odd $b > d \geq 3$, $m = rb+(t-r)d \geq 3$, and $n = rb+(t-r+1)d$ where $1 \leq r \leq t$. Alice has a winning strategy if and only if $r = t$.

Proof. One can prove the sufficiency part by the strategy as in the proof of Lemma 9.

Neccesity: Assume $1 \leq r \leq t-1$. Bob uses the strategy in a series of steps.

Step 1: Bob sets up a trap by playing the first color until Alice plays the second color. Suppose Alice is the first person who plays the second color. Then Bob uses the maximizing tactic on the partite set with the second color. Suppose furthermore that all vertices are colored in the end. If the second color appears in X, then Bob's strategy yields $f(X) \geq f(Y) + 1$ and $g(X) \geq g(Y) - 2$. Thus $m \geq n$ which is a contradiction. If the second color appears in Y, then Bob's strategy yields $f(Y) \geq f(X) + 3$ and $g(Y) \geq g(X) - 1$. Thus $n - m \geq d + 1$ which is also a contradiction. Thus Bob wins if Alice is the first person who plays the second color. Now we suppose that Alice can avoid this trap. This leads to the situation that the first color class has size b.

Step i for $2 \leq i \leq r+1$: Assume that Alice avoids all traps in previous steps. Bob sets up the first trap in Step i by assigning the $(2i-2)$nd color to a vertex in X. If Alice fails to respond by assigning the $(2i-1)$st color to a vertex in Y, then Bob uses the X-maximizing tactic. One can deduce that Bob wins as in Step 1. Suppose Alice assigns the $(2i-1)$st color to a vertex in Y. Bob sets up the second trap in this step by playing used colors preferably the $(2i-2)$nd color, until Alice plays the $(2i)$th color. Note that if Alice can avoid all traps from each Step j where $1 \leq j \leq i$, then i color classes of size b are in Y and $i-1$ color classes of size b are in X. Since there are exactly $2r$ major colors in an equitable coloring of this graph, Alice must be the first person who plays the $(2i)$th color in Step i for some i. When this happens, Bob uses the maximizing tactic on the partite set with the $(2i)$th color. This tactic results in Bob's victory as in Step 1.

Hence Bob can win by using his trap in some step. □

4 The games with $b = d + 1 = 3$

The main idea of this section is similar to one in Section 3. However we need tactics other than the maximizing tactic which is not effective anymore; now Bob may not be able to play a new color to make a color class of size less than d even when some colors are unused. For example, Bob uses the X-maximizing tactic with $|X| = 6$, and Alice can counter the tactic by playing the same color after each play of Bob. This play results in three color classes of size 2 in X even if some colors are unused. To emphasize the difference and to prevent confusion, we separate the games with $b = d + 1 = 3$ to be considered in this section.

To define the next tactic, we needs two new deifinitions. If a color c appears twice and it is played by Bob first and Alice later while there is an unused color to play, then we call c a *bad* color unless stated otherwise. A *good* color is a color that is not bad. (Note that an unused color is also a good color.) Let $f'(X)$ be the number of good colors in X, and $g'(X)$ be the number of good colors of size 3 in X. Bob has four types of colors for playing in the the *X-optimizing tactic*: (1) a new color in X, (2) a legal good color in X with the largest size, (3) a legal bad color in X, or (4) a legal color in Y. Table 1 lists these four types of colors from the most preference to the least preference according to situations. Bob always plays the most prefered legal color. The *Y-optimizing tactic*, $f'(Y)$, and $g'(Y)$ can be defined similarly to previous definitions.

Conditions	Bob's preference
Alice plays a new color	(1), (2), (3), (4)
Alice plays a used color in X	(1), (2), (3), (4)
If Bob plays (2) then there are at most one color class of size 1 in X	(1), (2), (3), (4)
Alice plays a used color in Y, and if Bob responds by playing (2) then there are at least two color classes of size 1 in X	(2), (1), (3), (4)

Table 1: The X-optimizing tactic

Remark. Assume that in a certain stage of the game, two (or more) color classes of size 1 appear in X. If Bob always has a new color to use for the X-optimizing tactic before all vertices in X are colored, then there is a color class of size 1 in the endgame. Thus if we suppose that a coloring is completed despite the X-optimizing tactic used by Bob, then all colors are used before all vertices in X are colored.

Lemma 12. Let $b = d + 1 = 3$. If Alice colors a vertex in X in the first turn, then Bob has a winning strategy.

Proof. Assume that Alice colors a vertex in X in the first turn. We claim that Bob can win by the X-optimizing tactic. Suppose all vertices are colored. Note that all colors are used before all vertices in X are colored by the previous remark. Bob's strategy yields $f'(X) \geq f(Y)+1$ and $g'(X) \geq g(Y)-1$. Thus

$$m = df(X) + (b-d)g(X) \geq df'(X) + (b-d)g'(X) > df(Y) + (b-d)g(Y) = n$$

which is a contradiction. Hence Bob has a winning strategy. □

Remark. In view of Lemma 12, we assume that Alice colors a vertex in Y in the first turn for a game in this section.

Lemma 13. Let $b = d+1 = 3$. If k is even, then Bob has a winning strategy.

Proof. We consider two cases. **Case 1:** $n - m \leq 2$. We claim that Bob can win by the Y-optimizing tactic. Suppose all vertices are colored. Bob's strategy yields $f'(Y) \geq f(X)+2$ and $g'(Y) \geq g(X)-1$. Thus $n - m \geq 3d - b \geq 3$ which is a contradiction.

Case 2: $n - m \geq 3$. Bob begins his turn by playing a new color in X. If Alice does not use the same color in X in her second turn, then Bob uses the X-optimizing tactic. In the situation that Alice uses the same color in X in her second turn, Bob regards this particular color as a good color and he plays his second turn according to the parity of m as follows.

- If m is odd, then he plays a new color in X.
- If m is even, then he plays a used color in X.

After that he uses the X-optimizing tactic. Suppose all vertices are colored. Note that in each scenario of this case, if Bob always has a new color to use for the X-optimizing tactic before all vertices in X are colored, then there is a color class of size 1 in the endgame; this is a contradiction. Thus we assume that all colors are used before all vertices in X are colored. In each scenario, Bob's strategy yields $f'(X) \geq f(Y)$ and $g'(X) \geq g(Y)-2$. Thus $n-m \leq 2$ which is a contradiction. Hence Bob has a winning strategy. □

Lemma 14. Let $b = d+1 = 3$. If k is odd and Alice has a winning strategy, then m and n satisfy condition (A), (B), or $n - m \geq d+1$.

Proof. Suppose Bob uses the Y-optimizing tactic. Since Alice has a winning strategy, all vertices can be colored in the end. Bob's strategy yields $f(Y) \geq f(X)+2$ and $g'(Y) \geq g(X)-1$, or $f(Y) = f(X)+1$ and $g'(Y) \geq g(X)$. If $f(Y) \geq f(X)+2$ and $g'(Y) \geq g(X)-1$, then $n - m \geq 3d - b \geq d+1$. Now suppose $f(Y) = f(X)+1$ and $g'(Y) \geq g(X)$. We have $n - m \geq d+1$, or m and n satisfy condition (A) or (B). □

Lemma 15. Let $b = d+1 = 3$. If k is odd and Alice has a winning strategy, then m and n satisfy condition (A), (B), or $n - m \leq d-1$.

Proof. Suppose Bob uses the strategy as in Case 2 in the proof of Lemma 13. Since Alice has a winning strategy, all vertices can be colored in the end. If Bob always has a new color to use for the X-optimizing tactic before all vertices in X are colored, then there is a color class of size 1 in the endgame; this is a contradiction. Thus we assume that all colors are used before all vertices in X are colored. Bob's strategy yields m and n satisfy condtion (B), or $f(X) \geq f(Y)+1$ and $g'(X) \geq g(Y)-1$, or $f(X) = f(Y)-1$ and $g'(X) = g(X) \geq g(Y)$. If $f(X) \geq f(Y)+1$ and $g'(X) \geq g(Y)-1$, then $m > n$ which is a contradiction. Now consider the case $f(X) = f(Y)-1$ and $g'(X) \geq g(Y)$. We have $n-m \leq d$ and the equality is attained only if m and n satisfy condition (A) or (B). □

Combining Lemmas 14 and 15, we immediately have the following result.

Corollary 2. Let $b = d+1 = 3$. If k is odd and Alice has a winning strategy, then m and n satisfy condition (A) or (B).

Corollary 2 gives necessary condition for complete bipartite graphs that Alice has a winning strategy. Next we investigate which necessary condtions are also sufficient.

Lemma 16. Let odd $k = 2t+1, b = d+1 = 3$, $m = rb+(t-r)d \geq 3$, and $n = rb+(t-r+1)d$ where $1 \leq r \leq t$. Alice has a winning strategy if and only if $r = t$.

Proof. One can prove the sufficiency part by the strategy as in the proof of Lemma 9.

Neccesity: Assume $1 \leq r \leq t-1$. Bob uses the strategy in a series of steps.

Step 1. Bob sets up a trap by playing the first color until Alice plays the second color. Suppose Alice is the first person who plays the second color and all vertices are colored in the end. If the second color appears in Y. Bob's Y-optimizing strategy yields $f(Y) \geq f(X)+3$ and $g'(Y) \geq g(X)-1$. Thus $n-m \geq d+1$ which is a contradiction. If the second color appears in X. Bob uses the strategy as in Case 2 in the proof of Lemma 13. This yields $f(X) \geq f(Y)+1$ and $g'(X) \geq g(Y)-2$. Thus $m \geq n$ which is also a contradiction.

Thus Bob wins if Alice is the first person who plays the second color. Now we suppose that Alice can avoid this trap. This leads to the situation that the first color class has size 3.

Step i for $2 \leq i \leq r+1$. Assume that Alice avoid all traps in previous steps. Bob sets up the first trap in Step i by assigning the $(2i-2)$nd color to a vertex in X. If Alice fails to respond by assigning the $(2i-1)$st color to a vertex in Y, then Bob uses the strategy similar to one in Case 2 of the proof of Lemma 13 except that he considers the parity of the number of uncolored vertices in X instead of m. One can deduce that Bob wins as in Step 1. Suppose Alice assigns the $(2i-1)$st color to a vertex in Y. Bob sets up the second trap in this step by playing used colors preferably the $(2i-2)$nd color until Alice plays the $(2i)$th color. Note that if Alice can avoid all traps from each Step j where $1 \leq j \leq i$, then i color classes of size b are in Y and $i-1$ color classes of size b are in X. Since there are exactly $2r$ major colors in an equitable coloring of this graph, Alice must be the first person who plays the $(2i)$th color in Step i for some i. When this happens, Bob uses the optimizing tactic on the partite set with the $(2i)$th color. This tactic results in Bob's victory as in Step 1.

Hence Bob can win by using his trap in some step. □

5 Conclusion

To characterize complete bipartite graphs that Alice has a winning strategy, we refer to
(i) Lemmas 1, 2, and 3 for $b \leq 2$,
(ii) Corollary 1 and Lemmas 5, 8, 9, 10, and 11 for $d \geq 3$,
(iii) Corollary 2 and Lemmas 9, 10, 13, and 16 for $b = d+1 = 3$.

Theorem 1. Alice has a winning strategy for an equitable k-coloring game on $K_{m,n}$ if and only if one of the following holds:
(i) $d \leq 1$,
(ii) $m = d = b - 1 = 2, n = 3(k-1), n$ is even,
(iii) $m = d = b - 1 = 2, n < 3(k-1), n$ is odd, or
(iv) k is odd ($k = 2t+1$), b is odd, $m = (t-i)b + id \geq 3$, and $n = (t+j)b + (1-j)d$ where $i, j = 0$ or 1.

acknowledgements

The first author was supported by the Thailand Research Fund under Grant MRG5580003.

References

[1] H.L. Bodlaender, On the complexity of some coloring games, *Int. J. Found. Comput. Sci.*, **2** (1991), 133–147.

[2] L. Cai and X. Zhu, Game chromatic index of k-degenerate graphs, *J. Graph Theory*, **36** (2001), 144–155.

[3] B.-L. Chen and K.-W. Lih, Equitable coloring of trees, *J. Combin. Theory Ser. B*, **61** (1994), 83–87

[4] B.-L. Chen, K.-W. Lih, and P.-L. Wu, Equitable coloring and the maximum degree, *Europ. J. Combinatorics*, **15** (1994), 443–447.

[5] T. Dinski and X. Zhu, A bound for the game chromatic number of graphs, *Discrete Math.*, **196** (1999), 109–115.

[6] U. Faigle, W. Kern, H. Kierstead, W.T. Trotter, On the game chromatic number of some classes of graphs, *Ars Combin.*, **35** (1993), 143–150.

[7] M. Gardner, Mathematical games, *Scientific American*, (April, 1981) **23**.

[8] D.J. Guan and X. Zhu, Game chromatic number of outerplanar graphs, *J. Graph Theory*, **30** (1999), 67–70.

[9] A. Hajnal and E. Szemerédi, Proof of conjecture of Erdős, in: *Combinatorial Theory and its Applications, Vol. II* (P. Erdős, A. Rényi and V. T. Sós Editors), (North-Holland, 1970), 601–603.

[10] A. V. Kostochka, Equitable colorings of outerplanar graphs, *Discrete Math.*, **258** (2002), 373–377.

[11] H. A. Kierstead and A. V. Kostochka, A short proof of the Hajnal-Szemerdi theorem on equitable colouring, *Combinatorics, Probability and Computing*, **17** (2008), 265–270

[12] H. A. Kierstead and A. V. Kostochka, Every 4-colorable graph with maximum degree 4 has an equitable 4-coloring. *J. Graph Theory*, **71** (2012), 31–48.

[13] H.A. Kierstead, A simple competitive graph coloring algorithm, *J. Combin. Theory, Ser. B*, **78** (2000), 57–68.

[14] H.A. Kierstead and W.T. Trotter, Planar graph coloring with an uncooperative partner, *J. Graph Theory*, **18** (1994), 569–584.

[15] P. C. B. Lam , W. C. Shiua, C. S. Tonga, and Z. F. Zhangb, On the equitable chromatic number of complete n-partite graphs, *Discrete Applied Mathematics*, **113** (2001), 307-310.

[16] W. Meyer, Equitable Coloring, *American Math. Monthly*, **80** (1973), 143–149.

[17] K. Nakprasit, Equitable colorings of planar graphs with maximum degree at least nine, *Discrete Mathematics*, **312** (2012), 1019–1024.

[18] H.-P. Yap and Y. Zhang, The equitable Δ-colouring conjecture holds for outerplanar graphs, *Bull. Inst. Math. Acad. Sin.*, **25** (1997), 143–149.

[19] X. Zhu, The game coloring number of planar graphs, *J. Combin. Theory, Ser. B*, **75** (1999), 245–258.

[20] X. Zhu, Refined activation strategy for the marking game, *J. Combin. Theory, Ser. B*, **98** (2008), 1–18.