รายงานวิจัยฉบับสมบูรณ์

โครงการ

วิธีการทำซ้ำสำหรับปัญหาอสมการแปรผันทั่วไปบนเซตไม่คอนเวกซ์

Iterative method for general nonconvex variational Inequalities problems

โดย ผู้ช่วยศาสตราจารย์ ดร.อิสระ อินจันทร์

กรกฎาคม 2557

สัญญาเลขที่ MRG5580080

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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา และสำนักงานกองทุนสนับสนุนการวิจัย

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งานวิจัยเรื่อง วิธีการทำซ้ำสำหรับปัญหาอสมการแปรผันทั่วไปบนเซตไม่คอนเวกซ์(MRG5580080) นี้สำเร็จลุล่วงด้วยดี จากการได้รับทุนอุดหนุนการวิจัยจากสำนักงานกองทุนสนับสนุนการวิจัย (สกว.) และสำนักงานคณะกรรมการอุดมศึกษา (สกอ.) และมหาวิทยาลัยราชภัฏอุตรดิตถ์ประจำปี 2555-2557 ข้าพเจ้าขอขอบคุณ ศาสตราจารย์ ดร.สมยศ พลับเที่ยง ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยนเรศวร นักวิจัยที่ปรึกษา ที่ได้ให้คำแนะนำและข้อเสนอแนะในการทำวิจัยด้วยดีตลอดมา

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Abstract

The aim of this project is to consider and study a new iterative scheme for solving the general nonconvex variational inequalities. We plan to prove the convergence criteria for the suggested iterative methods under suitable conditions.

In the first year, we will study and discuss some important basic results and consider some new theorems about the general iterative scheme for general nonconvex variational inequalities in the normed spaces.

In the second year, we will focus our study to the heart of our project, that is, we will suggest and analyze the iterative scheme for finding the approximation solvability of the system of general nonconvex variational inequalities in the normed spaces.

In conclusion, we point out that the results of this project are the extension and improvements of the earlier and recent results in this field. Much work is needed to develop this interesting subject.

Keywords: the system of general nonconvex variational inequalities / fixed point problems/ Nonexpansive mapping / Optimization problem

บทคัดย่อ

จุดประสงค์ของงานวิจัยนี้ คือ การศึกษากระบวนการทำซ้ำเพื่อหาผลเฉลยของระบบอสมการแปรผันทั่วไปแบบไม่คอนเวกซ์และปัญหาจุดตรึง โดยใช้เทคนิดการลู่เข้าของกระบวนการทำซ้ำ และจะได้เสนอการประยุกต์ใช้ปัญหาดังกล่าว

โดยในปีแรกจะได้ศึกษาพื้นฐานและสร้างทฤษฎีบทใหม่ของอสมการแปรผันทั่วไปแบบไม่คอนเวกซ์บนปริภูมินอร์มและการประยุกต์

ในปีที่สองจะได้ศึกษาพื้นฐานและสร้างทฤษฎีบทใหม่ระบบอสมการแปรผันทั่วไปแบบไม่คอนเวกซ์ในปริภูมินอร์มและการประยุกต์

ผลที่ได้รับจากการศึกษานี้คือการขยายและสร้างทฤษฎีบทใหม่เกี่ยวกับระบบอสมการแปรผันทั่วไปแบบไม่คอนเวกซ์ของนักวิจัยหลายๆ ท่านเพื่อพัฒนาองค์ความรู้ให้ดีขึ้น

คำสำคัญ : ระบบอสมการแปรผันทั่วไปแบบไม่คอนเวกซ์/ ปัญหาจุดตรึง/ การส่งแบบไม่ขยาย / ปัญหาค่าเหมาะสมที่สุด

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บทที่ 1

บทนำ(Introduction)

ปัญหาอสมการแปรผันเริ่มศึกษาโดย Stampacchia [1] ซึ่งต่อมาได้มีผลกระทบและมีอิทธิพลในการพัฒนาเกือบทุกสาขาทั้งสาขาวิทยาศาสตร์บริสุทธิ์และวิทยาศาสตร์ประยุกต์และและได้มีการศึกษาเรื่อยมา ซึ่งทั้งสาขาต่างๆ ไม่ว่าจะเป็นคณิตศาสตร์และวิทยาศาสตร์ได้ศึกษาปัญหาที่คล้ายคลึงกัน ซึ่งผลมาจากการทำงานร่วมกันระหว่างสาขาต่างๆ ของคณิตศาสตร์และวิทยาศาสตร์วิศวกรรมตอนนี้เรามีความหลากหลายของเทคนิคที่จะแนะนำและวิเคราะห์ขั้นตอนวิธีการต่างๆ ในการแก้ปัญหาอสมการแปรผันทั่วไป และการเพิ่มประสิทธิภาพที่เกี่ยวข้อง

ต่อมาเราได้ศึกษาว่าปัญหาอสมการแปรผันและปัญหาจุดตรึงร่วมกัน อย่างไรก็ตามงานวิจัยหลายๆ ชิ้นในทางนี้ได้ศึกษาภายใต้เงื่อนไขของเซตที่เป็นเซตคอนเวกซ์ (Convex set) ซึ่งผลลัพธ์ที่ได้นั้นอาจไม่จริงหรือไม่สามารถนำไปประยุกต์ใช้บนเซตไม่คอนเวกซ์ (nonconvex set) โดย Noor [4] ได้เริ่มต้นศึกษาคลาสของปัญหาอสมการแปรผัน ซึ่งเราเรียกว่า อสมการแปรผันไม่คอนเวกซ์ทั่วไป (general nonconvex variational inequality) บนเซต uniformly prox-regular ซึ่งเซตนี้เป็นเซตไม่คอนเวกซ์ และมีเซตคอนเวกซ์เป็นกรณีหนึ่งของมัน ศึกษาเพิ่มเติมได้ใน [5,6,7] โดยอาศัย projection operator, Noor [8] ได้ศึกษาการสมนัยระหว่างอสมการแปรผันไม่คอนเวกซ์ทั่วไปกับปัญหาจุดตรึง

สำหรับงานวิจัยนี้จะได้อธิบายลักษณะของ projection operator สำหรับ เซต prox-regular และจะอาศัยลักษณะดังกล่าวสร้างกระบวนการทำซ้ำแล้วแสดงการหาผลเฉลยของอสมการแปรผันไม่คอนเวกซ์ทั่วไปและปัญหาจุดตรึงของฟังก์ชัน Lipschitz continuous

ในปี 2011, I. Inchan และ N. Petrot [31], ให้ $T_1, T_2, T_3, g_1, g_2, g_3 : H \to H$ be การส่งไม่เชิงเส้น C เป็นคอนเวกซ์สับเซตของ H และ r_1, r_2, r_3 เป็นจำนวนจริงบวก ***the system of general variational inequalities involving three different nonlinear operators*** สำหรับ r_1, r_2, r_3 กำหนดโดย สามารถหา $(x^*, y^*, z^*) \in H \times H \times H$ ซึ่ง

$$\begin{aligned}
&\langle r_1 T_1 y^* + g_1(x^*) - g_1(y^*), g_1(x) - g_1(x^*) \rangle \geq 0, \forall g_1(x) \in C, \\
&\langle r_2 T_2 z^* + g_2(y^*) - g_2(z^*), g_2(x) - g_2(y^*) \rangle \geq 0, \forall g_2(x) \in C, \qquad (4) \\
&\langle r_3 T_3 x^* + g_3(z^*) - g_3(x^*), g_3(x) - g_3(z^*) \rangle \geq 0, \forall g_3(x) \in C
\end{aligned}$$

จาก (1) จะเห็นว่าปัญหา (4) สมนัยกับ:

$$\begin{aligned}
&g_1(x^*) = P_C[g_1(y^*) - r_1 T_1 y^*], \\
&g_2(y^*) = P_C[g_2(z^*) - r_2 T_2 z^*], \qquad (5) \\
&g_3(z^*) = P_C[g_3(x^*) - r_3 T_3 x^*],
\end{aligned}$$

ซึ่ง $C \subset g_i(H), i = 1,2,3$ แล้วลำดับ $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ generated by (5) ลู่เข้าอย่างเข้มไปยัง x^*, y^*, z^* ตามลำดับ ซึ่ง (x^*, y^*, z^*) เป็นผลเฉลยของ system of general variational inequalities involving three different nonlinear operators (4)

ต่อมา Noor [32] ใช้เทคนิคของกระบวนการท้ำสามขั้นตอนเพื่อหาผลเฉลยของ general nonconvex variational inequalities

ให้ C เป็นเซตปิดที่ไม่เป็นเซตว่างและเป็นสับเซตของ H **proximal normal cone** ของ C ที่ $u \in H$ กำหนดโดย

$$N_C^P(u) := \{\xi \in H : u = P_C[u + \alpha\xi]\},$$

เมื่อ $\alpha > 0$ เป็นค่าคงที่และ

$$P_C[u^* \in K : d_C(u) = \|u - u^*\|\}$$

เมื่อ $d_C(\cdot)$ เป็นระยะทางปกติของ C นั่นคือ

$$d_C(u) = \inf_{v \in C} \|v - u\|$$

สำหรับ $r \in (0, \infty]$ เซต $C_r = \{u \in H : d_C(u) < r\}$ กำหนดให้การส่งไม่เชิงเส้น $T, g : H \to H$ สำหรับการหา $u \in H$: $g(u) \in C_r$ ซึ่งทำให้

$$\langle \rho Tu, g(v) - g(u) \rangle + \gamma \|g(v) - g(u)\|^2 \geq 0, \forall v \in H : g(v) \in C_r \quad (6)$$

ซึ่งจะเรียกว่า ***general nonconvex variational inequality*** ได้นำเสนอและศึกษาโดย Noor [23] เมื่อ $\gamma > 0$ และ $\rho > 0$ เป็นค่าคงที่ ซึ่งถ้า $C_r \equiv C$ เป็นคอนเวกซ์สับเซตของ *H* แล้วปัญหา (6) สมนัยกับ (3) สำหรับ $u \in H : g(u) \in C_r$ เป็นผลเฉลยของ general nonconvex variational inequality (6) ก็ต่อเมื่อ $u \in C_r$ ซึ่ง

$$g(u) = P_{C_r}[g(u) - \rho Tu], \quad (7)$$

เมื่อ P_{C_r} เป็น projection ของ H ไปทั่วถึง uniformly Prox-regular set C_r แล้วได้แสดงการลู่เข้าของกระบวนการทำซ้ำสามขั้นตอนเพื่อหาผลเฉลยของ general nonconvex variational inequalities สำหรับการส่งแบบ strongly monotone and Lipschitz continuous สำหรับ $u_0 \in H$ สามารถสร้าง u_{n+1} โดย

$$\begin{aligned}
&\langle \rho Tu_n + g(w_n) - g(u_n), g(v) - g(w_n) \rangle + \gamma \|g(v) - g(w_n)\|^2 \geq 0, \forall g(v) \in C_r, \\
&\langle \rho Tw_n + g(y_n) - g(w_n), g(v) - g(y_n) \rangle + \gamma \|g(v) - g(y_n)\|^2 \geq 0, \forall g(v) \in C_r, \quad (8) \\
&\langle \rho Ty_n + g(u_{n+1}) - g(y_n), g(v) - g(u_{n+1}) \rangle + \gamma \|g(v) - g(u_{n+1})\|^2 \geq 0, \forall g(v) \in C_r,
\end{aligned}$$

เมื่อ $\rho > 0$ และ $\gamma > 0$ เป็นค่าคงที่ แล้ว $\{u_n\}$ ลู่เข้าไปยัง $u \in H$

จากการศึกษา (4) และ (8) จะได้ว่างานวิจัยนี้จะได้ศึกษา $T, g_1, g_2, g_3 : H \to H$ เป็นการส่งไม่เชิงเส้น และกำหนด ***the system of general nonconvex variational inequalities involving three different nonlinear operators*** กำหนดโดย สำหรับการหา $(x^*, y^*, z^*) \in H \times H \times H$ ซึ่งทำให้

$$\begin{aligned}
&\langle \rho Ty^* + g_1(x^*) - g_1(y^*), g_1(x) - g_1(x^*) \rangle + \gamma \|g_1(x) - g_1(x^*)\|^2 \geq 0, \forall g_1(x) \in C_r, \\
&\langle \rho Tz^* + g_2(y^*) - g_2(z^*), g_2(x) - g_2(y^*) \rangle + \gamma \|g_2(x) - g_2(y^*)\|^2 \geq 0, \forall g_2(x) \in C_r, \quad (9) \\
&\langle \rho Tx^* + g_3(z^*) - g_3(x^*), g_3(x) - g_3(z^*) \rangle + \gamma \|g_3(x) - g_3(z^*)\|^2 \geq 0, \forall g_3(x) \in C_r
\end{aligned}$$

เมื่อ $\rho > 0$ และ $\gamma > 0$ โดยใช้ (7) จะได้ว่า (9) สมนัยกับ

$$\begin{aligned}
g_1(x^*) &= P_{C_r}[g_1(y^*) - \rho Ty^*], \\
g_2(y^*) &= P_{C_r}[g_2(z^*) - \rho Tz^*], \quad (10) \\
g_3(z^*) &= P_{C_r}[g_3(x^*) - \rho Tx^*].
\end{aligned}$$

สำหรับ $u_0 \in H$ สามารถสร้าง u_{n+1} กำหนดโดย

$$\begin{aligned}
&\langle \rho Tu_n + g_1(w_n) - g_1(u_n), g_1(v) - g_1(w_n) \rangle + \gamma \|g_1(v) - g_1(w_n)\|^2 \geq 0, \forall g(v) \in C_r, \\
&\langle \rho Tw_n + g_2(y_n) - g_2(w_n), g_2(v) - g_2(y_n) \rangle + \gamma \|g_2(v) - g_2(y_n)\|^2 \geq 0, \forall g(v) \in C_r, \quad (11) \\
&\langle \rho Ty_n + g_3(u_{n+1}) - g_3(y_n), g_3(v) - g_3(u_{n+1}) \rangle + \gamma \|g_3(v) - g_3(u_{n+1})\|^2 \geq 0, \forall g(v) \in C_r,
\end{aligned}$$

เมื่อ $\rho > 0$ และ $\gamma > 0$ เป็นค่าคงที่ และจาก (7) เราจะได้ว่า

$$
\begin{aligned}
g_1(w_n) &= P_{C_r}[g_1(u_n) - \rho T u_n], \\
g_2(y_n) &= P_{C_r}[g_2(w_n) - \rho T w_n], \\
g_3(u_{n+1}) &= P_{C_r}[g_3(y_n) - \rho T y_n],
\end{aligned}
\tag{12}
$$

แล้วจะได้แสดงว่า $\{w_n\}$, $\{y_n\}$ และ $\{u_n\}$ กำหนดโดย (10) ลู่เข้าอย่างเข้มไปยัง x^*, y^*, z^* ตามลำดับซึ่ง (x^*, y^*, z^*) เป็นผลเฉลยของ system of general nonconvex variational inequalities involving three different nonlinear operators (9).

บทที่ 2

Preliminaries

In this chapter, we give some defnitions, notations, and some useful results that will be used in the later chapters.

2.1 Useful lemmas.

Lemma 2.1.1. [21] Let $T : C \to C$ be a k-strictly pseudo-contraction. Defined $S_\lambda : C \to C$ by $S_\lambda x = \lambda x + (1-\lambda)Tx$ for each $x \in C$. Then, as $\lambda \in [k, 1]$, S_λ is nonexpansive mapping and $F(T) = F(S_\lambda)$.

Lemma 2.1.2. In a real Hilbert space H, there holds the inequality

1. $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle \quad x, y \in H$ and $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$,
2. $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \forall t \in [0,1]$.

Definition 2.1.3. [1] Let C be nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mapping of C into itself. For each $j = 1, 2, \ldots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in I \equiv [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. We define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I \\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I \\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I \\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I \\
&\vdots \\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I \\
S &= U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called $S - mapping$ generated by $T_1, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Lemma 2.1.4. [10] Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a self-mapping of C. If S is a k-strict pseudo-contraction mapping, then S satisfies the Lipschitz condition

$$\|S_x - S_y\| \leq \tfrac{1+k}{1-k}\|x-y\|, \quad \forall x, y \in C.$$

Lemma 2.1.5. [20] Let $\{s_n\}$ be a sequence of nonnegative real number satisfying

$$s_{n+1} = (1-\alpha_n)s_n + \delta_n + \eta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

1. $\sum_{n=1}^{\infty} \alpha_n = \infty$,
2. $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$,
3. $\sum_{n=1}^{\infty} |\eta_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} \alpha_n = 0$.

Lemma 2.1.6. Let H be a real Hilbert space. There hold the following identities

1. $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$ and $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$
2. $\| \sum_{i=0}^{m} \alpha_i x_i\|^2 = \sum_{i=0}^{m} \alpha_i \|x_i\|^2 - \sum_{i=0}^{m} \alpha_i\alpha_j \|x_i - x_j\|^2$ for $\sum_{i=0}^{m} \alpha_i = 1, \alpha_i \in [0,1]$, $\forall i \in \{0, 1, 2, \ldots, m\}$.

Lemma 2.1.7. [1] Let C be a nonempty closed convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of k_i-strictly pseudo-contractive mapping of C into C with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $k = \max\{k_i : i = 1, 2, \ldots, N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, j = 1, 2, 3, \ldots, N,$ where $I = [0,1], \alpha_1^j + \alpha_2^j + \alpha_3^j = 1, \alpha_1^j, \alpha_3^j \in (k, 1)$ for all $j = 1, 2, \ldots, N-1$ and $\alpha_1^N \in (k, 1], \alpha_3^N \in (k, 1], \alpha_2^j \in (k, 1]$ for all $j = 1, 2, \ldots, N$. Let S be the mapping generated by $T_1, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^{N} F(T_i)$ and S is a nonexpansive mapping.

Lemma 2.1.8. [19] A real Hilbert space H satisfies Opial's condition, i.e, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for each $y \in H$ with $x \neq y$.

Lemma 2.1.9. [18] Let C be a nonempty closed convex subset of a real Hilbert and $T : C \longrightarrow C$ be a nonexpansive mapping. Then T is demi-closed on C,i.e.,if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \longrightarrow 0,\ then\ x = Tx$.

2.2 Nonconvex Variational

Let C be a closed subset of a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ respectively. Let us recall the following well-known definitions and some auxiliary results of nonlinear convex analysis and nonsmooth analysis.

Definition 2.2.1. Let $u \in H$ be a point not lying in C. A point $v \in C$ is called a closest point or a projection of u onto C if $d_C(u) = \|u - v\|$ when d_C is a usual distance. The set of all such closest points is denoted by $P_C(u)$; that is,

$$P_C(u) = \{v \in C : d_C(u) = \|u - v\|\}. \tag{2.2.1}$$

Definition 2.2.2. Let C be a subset of H. The proximal normal cone to C at x is given by

$$N_C^P(x) = \{z \in H : \exists\rho > 0; x \in P_C(x + \rho z)\}. \tag{2.2.2}$$

The following characterization of $N_C^P(x)$ can be found in [43].

Lemma 2.2.3. Let C be a closed subset of a Hilbert space H. Then

$$z \in N_C^P(x) \text{ if and only if } \exists \sigma > 0, \langle z, y - x \rangle \leq \sigma \|y - x\|^2, \quad \forall y \in C. \tag{2.2.3}$$

Clark et al. [44] and Poliquin et al. [38] have introduced and atudied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class or uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems, and differential inclusions.

Definition 2.2.4. For a given $r \in (0, +\infty]$, a subset C of H is said to be uniformly prox-regular with respect to r if, for all $\overline{x} \in C$ and for all $0 \neq z \in N_C^P(x)$, one has

$$\langle \frac{z}{\|z\|}, x - \overline{x} \rangle \leq \frac{1}{2r} \|x - \overline{x}\|^2, \quad \forall x \in C. \tag{2.2.4}$$

It is well known that a closed subset of a Hilbert space is convex if and only if it is proximally smooth of radius $r > 0$. Thus, in Definition 2.2.4, in the case of $r = \infty$, the uniform r-prox-regularity C is equivalent to convexity of C. Then, it is clear that the class of uniformly prox-regular sets is sufficiently large to include the class p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets, and many other nonconvex sets; see [44, 38].

Lemma 2.2.5. [39] Let C be a nonempty closed subset of H, $r \in (0, +\infty]$ and set $C_r; = \{x \in H : d(x, C) < r\}$. If C is uniform r-uniformly prox-regular, then the following hold:

(1) for all $x \in C_r, P_C(x) \neq \emptyset$,

(2) for all $s \in (0, r), P_C$ is Lipschitz continuous with constant $t_s = \frac{r}{r-s}$ on C_s,

(3) the proximal normal cone is closed as a set-valued mapping.

Let C be a closed subset of a real Hilbert space H. A mapping $T : C \to H$ is called $\gamma-strongly$ $monotone$ if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \gamma \|x - y\|^2, \tag{2.2.5}$$

for all $x, y \in C$. A mapping T is called $\mu - Lipschitz$ if there exists a constant $\mu > 0$ such that

$$\|Tx - Ty\| \leq \mu \|x - y\|, \tag{2.2.6}$$

for all $x, y \in C$.

Lemma 2.2.6. In a real Hilbert space H, there holds the inequality

1. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \quad x, y \in H$ and $\|x - y\|^2 = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2$,
2. $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \forall t \in [0, 1]$.

บทที่ 3

Main Results

3.1 A general hierarchical problem

In this section, we introduced the iterative scheme for finite family of k-strictly pseudo-contractive mappings. Then we prove strong convergence of algorithm (1.6) and solving a common solution of a general hierarchical problem and fixed point problems of finite family of k-strictly pseudo-contractive mappings.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. The hierarchical problem is of finding $\tilde{x} \in Fix(T)$ such that

$$\langle S\tilde{x} - \tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \forall x \in Fix(T), \tag{3.1.1}$$

where S, T are two nonexpansive mappings and $Fix(T)$ is the set of fixed points of T. Recently, this problem has been studied by many authors (see,[2]-[17]). Now, we briefly recall some historic results which relate to the problem (3.1.1). For solving the problem (3.1.1), in 2006, Moudafi and Mainge [4] first introduced an implicit iterative algorithm:

$$x_{t,s} = sQ(x_{t,s}) + (1-s)[tS(x_{t,s}) + (1-t)T(x_{t,s})] \tag{3.1.2}$$

and proved that the net $\{x_{t,s}\}$ defined by (1.2) strongly converges to x_t as $s \to 0$, where x_t satisfies $x_t{=}proj_{Fix(P_t)}Q(x_t)$, where $P_t{:}C \to C$ is a mapping defined by

$$P_t(x) = tS(x) + (1-t)T(x), \forall x \in C, t \in (0,1),$$

or, equivalently, x_t is the unique solution of the quasivariational inequality:

$$0 \in (I-Q)x_t + N_{Fix(P_t)}(x_t),$$

where the normal cone to $Fix(P_t), N_{Fix(P_t)}$ is defined as follows:

$$N_{Fix(P_t)} : x \to \begin{cases} \{u \in H : \langle y - x, u\rangle \leq 0\}, & \text{if } x \in Fix(P_t), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Moreover, as $t \to 0$, the net $\{x_t\}$ in turn weakly converges to the unique solution x_∞ of the fixed point equation $x_\infty{=}proj_\Omega Q(x_\infty)$ or, equivalently, x_∞ is the unique solution of the variational inequality:

$$0 \in (I-Q)x_\infty + N_\Omega(x_\infty).$$

Recall that a mapping $f : C \longrightarrow C$ is said to be contractive if there exists a constant $\gamma \in (0,1)$ such that

$$\|fx - fy\| \leq \gamma\|x - y\|, \quad \forall x, y \in C.$$

A mapping $T : C \longrightarrow C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A mapping T is said to be k-strict pseudo-contractive if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \forall x, y \in D(T). \tag{3.1.3}$$

Note that the class of k-strict pseudo-contraction strictly includes the class of nonexpansive mappings. Forward, we use $Fix(T)$ to denote the fixed points set of T, that is $Fix(T) = \{x \in C : Tx = x\}$. we see that, if $S_k : C \to C$ defined by $S_k x = kx + (1 - k)Tx$ for all $x \in C$ where T is k-strict pseudo-contractive then S_k is nonexpansive mapping [21].

In this paper, motivate by Kangtunkarn and Suantai [1], we introduce a mapping for finding a common fixed point of T is a λ-strict pseudo-contractive mapping and $\{T_i\}_{i=1}^N$ a finite family of k_i-strict pseudo-contractive mappings of C into itself. For each $n \in \mathbb{N}$, and $j = 1, 2, ..., N$, let $\alpha_j^n = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in [0, 1] \times [0, 1] \times [0, 1]$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$ with $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. We define the mapping $S_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,0} &= I; \\
U_{n,1} &= \alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I; \\
U_{n,2} &= \alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I; \\
U_{n,3} &= \alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I; \\
&\vdots \quad ; \\
U_{n,N-1} &= \alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I; \\
S_n &= U_{n,N} = \alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I.
\end{aligned} \tag{3.1.4}$$

Motivated and inspired by the results in the literature, in this paper, we consider a general hierarchical problem of finding $x^* \in F(T)$ such that, for any $n \geq 1$,

$$\langle S_n x^* - x^*, x - x^* \rangle \leq 0, \forall x \in F(S_\lambda), \tag{3.1.5}$$

where S_n is the S-mapping defined by (3.1.4) and S_λ is a nonexpansive mapping defined in Lemma 2.2.6.

Algorithm 3.1.1. Let C be a nonempty closed convex subset of a real Hilbert space H and let T is a λ-strict pseudo-contractive mapping with $S_\lambda x = \lambda x + (1 - \lambda)Tx$ and $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mapping of C into itself. Let $f : C \longrightarrow C$ be a contraction with coefficient $\gamma \in (0, 1)$. For any $x_0 \in C$, let $\{x_n\}$ be the sequence generated by

$$x_{n+1} = \alpha_n S_n x_n + (1 - \alpha_n) S_\lambda(\beta_n f(x_n) + (1 - \beta_n)x_n), \quad \forall n \geq 0, \tag{3.1.6}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real numbers in $(0, 1)$ and S_n is the $S - mapping$ defined by (3.1.4).

We show that an explicit iterative algorithm which converges strongly to a solution x^* of the general hierarchical problem (3.1.5).

Lemma 3.1.2. Let H be a Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contraction of H into itself for some $k_i \in [0,1)$ and $k = \max\{k_i : i = 1,2,\dots,N\}$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let S_n be the S-mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $k < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all , $k < c \leq \alpha_1^{n,N} \leq 1$, $k \leq \alpha_3^{n,N} \leq d < 1$, $k \leq \alpha_2^{n,j} \leq e < 1$ for all $j = 1,2,\dots,N$ and $\sum_{n=1}^\infty |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty$, $\sum_{n=1}^\infty |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$ for all $j = \{1,2,3,\dots,N\}$. Then for all $x \in H$, $\sum_{n=1}^\infty \|S_{n+1}x - S_n x\| < \infty$.

พิสูจน์. For each $x \in C$ and $n \in \mathbb{N}$, we have .

$$\begin{aligned}
\|U_{n+1,1}x - U_{n,1}x\| &= \|\alpha_1^{n+1,1}T_1x + (1 - \alpha_1^{n+1,1}x) - \alpha_1^{n,1}T_1x + (1 - \alpha_1^{n,1})x\| \\
&= \|\alpha_1^{n+1,1}T_1x - \alpha_1^{n+1,1}x - \alpha_1^{n,1}T_1x + \alpha_1^{n,1}x\| \\
&= \|(\alpha_1^{n+1,1} - \alpha_1^{n,1})T_1x - (\alpha_1^{n+1,1} - \alpha_1^{n,1})x\| \\
&= |\alpha_1^{n+1,1} - \alpha_1^{n,1}|\|T_1x - x\| \qquad (3.1.7)
\end{aligned}$$

and for $n \in \mathbb{N}$, and for $k \in \{2,3,\dots,N\}$, we have

$$\begin{aligned}
\|U_{n+1,k}x - U_{n,k}x\| &= \|\alpha_1^{n+1,k}T_kU_{n+1,k-1}x + \alpha_2^{n+1,k}U_{n+1,k-1}x + \alpha_3^{n+1,k}x \\
&\quad -\alpha_1^{n,k}T_kU_{n,k-1}x + \alpha_2^{n,k}U_{n,k-1}x + \alpha_3^{n,k}x\| \\
&= \|\alpha_1^{n+1,k}T_kU_{n+1,k-1}x + \alpha_3^{n+1,k}x - \alpha_1^{n,k}T_kU_{n,k-1}x - \alpha_3^{n,k}x \\
&\quad +\alpha_2^{n+1,k}U_{n+1,k-1}x - \alpha_2^{n,k}U_{n,k-1}x\| \\
&= \|\alpha_1^{n+1,k}T_kU_{n+1,k-1}x - \alpha_1^{n+1,k}T_kU_{n,k-1}x + \alpha_1^{n+1,k}T_kU_{n,k-1}x \\
&\quad -\alpha_1^{n,k}T_kU_{n,k-1}x + (\alpha_3^{n+1,k} - \alpha_3^{n,k})x + \alpha_2^{n+1,k}U_{n+1,k-1}x - \alpha_2^{n,k}U_{k-1}x\| \\
&= \|\alpha_1^{n+1,k}(T_kU_{n+1,k-1}x - T_kU_{n,k-1}x) + (\alpha_1^{n+1,k} - \alpha_1^{n,k})T_kU_{n,k-1}x \\
&\quad +(\alpha_3^{n+1,k} - \alpha_3^{n,k})x + \alpha_2^{n+1,k}U_{n+1,k-1}x - \alpha_2^{n,k}U_{n,k-1}x\| \\
&= \|\alpha_1^{n+1,k}(T_kU_{n+1,k-1}x - T_kU_{n,k-1}x) + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) \\
&\quad \times T_kU_{n,k-1}x + (\alpha_3^{n+1,k} - \alpha_3^{n,k})x + \alpha_2^{n+1,k}U_{n+1,k-1}x \\
&\quad -\alpha_2^{n+1,k}U_{n,k-1}x + \alpha_2^{n+1,k}U_{n,k-1}x - \alpha_2^{n,k}U_{n,k-1}x\| \\
&= \|\alpha_1^{n+1,k}(T_kU_{n+1,k-1}x - T_kU_{n,k-1}x) + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) \\
&\quad \times T_kU_{n,k-1}x + (\alpha_3^{n+1,k} - \alpha_3^{n,k})x + \alpha_2^{n+1,k}(U_{n+1,k-1}x \\
&\quad -U_{n,k-1}x) + (\alpha_2^{n+1,k} - \alpha_2^{n,k})U_{n,k-1}x\| \\
&\leq \alpha_1^{n+1,k}\|T_kU_{n+1,k-1}x - T_kU_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}x\| \\
&\quad +|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x\| \\
&\quad +\alpha_2^{n+1,k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}x\|
\end{aligned}$$

(3.1.8)

$$
\begin{aligned}
&= \alpha_1^{n+1,k}\|T_kU_{n+1,k-1}x - T_kU_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}x\| \\
&\quad +\alpha_2^{n+1,k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |1 - \alpha_1^{n+1,k} \\
&\quad -\alpha_3^{n+1,k} - 1 + \alpha_1^{n,k} + \alpha_3^{n,k}|\|U_{n,k-1}x\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x\| \\
&\leq \alpha_1^{n+1,k}\frac{1+k}{1-k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}x\| \\
&\quad +\alpha_2^{n+1,k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + (|\alpha_1^{n,k} \\
&\quad -\alpha_1^{n+1,k}| + |\alpha_3^{n,k} - \alpha_3^{n+1,k}|)\|U_{n,k-1}x\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x\| \\
&\leq \frac{1+k}{1-k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}x\| \\
&\quad +\frac{1+k}{1-k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + (|\alpha_1^{n,k} - \alpha_1^{n+1,k}| \\
&\quad +|\alpha_3^{n,k} - \alpha_3^{n+1,k}|)\|U_{n,k-1}x\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x\| \\
&= \frac{2}{1-k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|(\|T_kU_{n,k-1}x \\
&\quad +\|U_{n,k-1}x\|) + |\alpha_3^{n,k} - \alpha_3^{n+1,k}|(\|U_{n,k-1}x\| + \|x\|).
\end{aligned}
$$

By (3.1.7) and (3.1.8), we have

$$
\begin{aligned}
\|S_{n+1}x - S_nx\| &= \|U_{n+1,N}x - U_{n,N}x\| \\
&\leq \frac{2}{1-k}\|U_{n+1,N-1}x - U_{n,N-1}x\| + |\alpha_1^{n+1,N} - \alpha_1^{n,N}|(\|T_NU_{n,N-1}x\| \\
&\quad +\|U_{n,N-1}x\|) + |\alpha_3^{n+1,N} - \alpha_3^{n,N}|(\|U_{n,N-1}x\| + \|x\|) \\
&\leq \frac{2}{1-k}\Bigg(\frac{2}{1-k}\|U_{n+1,N-2}x - U_{n,N-2}x\| \\
&\quad +|\alpha_1^{n+1,N-1} - \alpha_1^{n,N-1}|(\|T_{N-1}U_{n,N-2}x\| + \|U_{n,N-2}x\|) \\
&\quad +|\alpha_3^{n+1,N-1} - \alpha_3^{n,N-1}|[\|U_{n,N-2}x\| + \|x\|]\Bigg) \\
&\quad +|\alpha_1^{n+1,N} - \alpha_1^{n,N}|(\|T_NU_{n,N-1}x\| + \|U_{n,N-1}x\|) \\
&\quad +|\alpha_3^{n+1,N} - \alpha_3^{n,N}|(\|U_{n,N-1}x\| + \|x\|) \\
&= \left(\frac{2}{1-k}\right)^2\|U_{n+1,N-2}x - U_{n,N-2}x\| + \sum_{j=N-1}^{N}\left(\frac{2}{1-k}\right)^{N-j}|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}x\| \\
&\quad +\|U_{n,j-1}x\|) + \sum_{j=N-1}^{N}\left(\frac{2}{1-k}\right)^{N-j}|\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|U_{n,j-1}x\| + \|x\|) \\
&\quad \vdots \\
&\leq \left(\frac{2}{1-k}\right)^{N-1}\|U_{n+1,1}x - U_{n,1}x\| + \sum_{j=2}^{N}\left(\frac{2}{1-k}\right)^{N-j}|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}x\| \\
&\quad +\|U_{n,j-1}x\|) + \sum_{j=2}^{N}\left(\frac{2}{1-k}\right)^{N-j}|\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|U_{n,j-1}x\| + \|x\|)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{2}{1-k}\right)^{N-1}|\alpha_1^{n+1,1}-\alpha_1^{n,1}|\|T_1x-x\|+\sum_{j=2}^{N}\left(\frac{2}{1-k}\right)^{N-j}\\
&\quad +|\alpha_1^{n+1,j}-\alpha_1^{n,j}|(\|T_jU_{n,j-1}x\|+\|U_{n,j-1}x\|+\|x\|)+\sum_{j=2}^{N}\left(\frac{2}{1-k}\right)^{N-j}\\
&\quad +|\alpha_3^{n+1,j}-\alpha_3^{n,j}|(\|U_{n,j-1}x\|+\|x\|).
\end{aligned}
$$

This implies by assumption we have that

$$\sum_{n=1}^{\infty}\|S_{n+1}x-S_nx\|<\infty.$$

This complete the proof. □

Lemma 3.1.3. Let H be a Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contraction of H into itself for some $k_i\in[0,1)$ and $k=\max\{k_i : i=1,2,\dots,N\}$ with $\bigcap_{i=1}^N F(T_i)\neq\emptyset$. Let S_n be the S-mapping generated by $T_1,T_2,\dots,T_N$ and $\alpha_1^{(n)},\alpha_2^{(n)},\dots,\alpha_N^{(n)}$, where $\alpha_j^{(n)}=(\alpha_1^{n,j},\alpha_2^{n,j},\alpha_3^{n,j})\in I\times I\times I, I=[0,1], \alpha_1^{n,j}+\alpha_2^{n,j}+\alpha_3^{n,j}=1$ and satisfy condition:

(1) $k<a\leq\alpha_1^{n,j},\alpha_3^{n,j}\leq b<1$ for all $, k<c\leq\alpha_1^{n,N}\leq 1, k\leq\alpha_3^{n,N}\leq d<1, k\leq\alpha_2^{n,j}\leq e<1$ for all $j=1,2,\dots,N$

(2) $\sum_{n=1}^{\infty}|\alpha_1^{n,j}-\alpha_1^j|<\infty, \sum_{n=1}^{\infty}|\alpha_2^{n,j}-\alpha_2^j|<\infty, \sum_{n=1}^{\infty}|\alpha_3^{n,j}-\alpha_3^j|<\infty$ for all $j=\{1,2,3,\dots,N\}$.

Then for all $x\in H, \lim_{n\to\infty}\|S_nx-Sx\|=0$

พิสูจน์. Let $x\in C$ and for each $n\in\mathbb{N}$, from the definition of S mapping and Lemma 2.1.4, we have

$$
\begin{aligned}
\|U_{n,1}x-U_1x\| &= \|\alpha_1^{n,1}T_1U_{n,0}x+\alpha_2^{n,1}U_{n,0}x+\alpha_3^{n,1}x-(\alpha_1^1T_1U_0x+\alpha_2^1U_0x+\alpha_3^1x)\|\\
&\leq |\alpha_1^{n,1}-\alpha_1^1|\|T_1x\|+|\alpha_2^{n,1}-\alpha_2^1|\|x\|+|\alpha_3^{n,1}-\alpha_3^1|\|x\|.
\end{aligned}
$$

From boundedness and condition (2) we have

$$\lim_{n\to\infty}\|U_{n,1}x-U_1x\|=0. \tag{3.1.9}$$

Next, consider

$$
\begin{aligned}
\|U_{n,2}x-U_2x\| &= \|\alpha_1^{n,2}T_2U_{n,1}x+\alpha_2^{n,2}U_{n,1}x+\alpha_3^{n,2}x-(\alpha_1^2T_2U_1x+\alpha_2^2U_1x+\alpha_3^2x)\|\\
&\leq \|\alpha_1^{n,2}T_2U_{n,1}x-\alpha_1^{n,2}T_2U_1x+\alpha_1^{n,2}T_2U_1x+\alpha_2^{n,2}U_{n,1}x+\alpha_3^{n,2}x\\
&\quad -(\alpha_1^2T_2U_1x+\alpha_2^2U_1x+\alpha_3^2x)\|\\
&\leq \|\alpha_1^{n,2}(T_2U_{n,1}x-T_2U_1x)\|+\|(\alpha_3^{n,2}-\alpha_3^2)(x)\|+\|(\alpha_1^{n,2}-\alpha_1^2)(T_2U_1x)\|\\
&\quad +\|\alpha_2^{n,2}U_{n,1}x-\alpha_2^2U_1x\|\\
&\leq \alpha_1^{n,2}\|T_2U_{n,1}x-T_2U_1x\|+|\alpha_3^{n,2}-\alpha_3^2|\|x\|+|\alpha_1^{n,2}-\alpha_1^2|\|T_2U_1x\|\\
&\quad +\alpha_2^{n,2}\|U_{n,1}x-U_1x\|+|\alpha_2^{n,2}-\alpha_2^2|\|U_1x\|\\
&\leq \alpha_1^{n,2}\frac{1+k}{1-k}\|U_{n,1}x-U_1x\|+|\alpha_3^{n,2}-\alpha_3^2|\|x\|+|\alpha_1^{n,2}-\alpha_1^2|\|T_2U_1x\|\\
&\quad +\alpha_2^{n,2}\|U_{n,1}x-U_1x\|+|\alpha_2^{n,2}-\alpha_2^2|\|U_1x\|.
\end{aligned}
$$

From boundedness, condition (2) and equation (3.1.9), we have

$$\lim_{n\to\infty} \|U_{n,2}x - U_2x\| = 0. \tag{3.1.10}$$

Similarly of the proof, we have

$$\lim_{n\to\infty} \|U_{n,N}x - U_Nx\| = 0. \tag{3.1.11}$$

Since $\|S_nx - Sx\| = \|U_{n,N}x - U_Nx\|$, we have

$$\lim_{n\to\infty} \|S_nx - Sx\| = 0. \tag{3.1.12}$$

This complete the proof. □

Theorem 3.1.4. Let C be a nonempty closed convex subset of a real Hilbert space H, let T be a λ-strictly pseudo-contractive mapping and $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mappings of C into itself for some $k_i \in [0,1)$ and $k = \max\{k_i : i = 1,2,...,N\}$ which $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let S_n be the S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1^n, \alpha_2^n, ..., \alpha_N^n$ where $\alpha_j^n = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $k < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1,2,...,N-1$, $k < c \leq \alpha_1^{n,N} \leq 1, k \leq \alpha_3^{n,N} \leq d < 1, k \leq \alpha_2^{n,j} \leq e < 1$ for all $j = 1,2,...N$. Assume that set Ω of solution of general hierarchical problem (3.1.5) is nonempty. For a mapping $f : C \to C$ is a contraction with $\gamma \in (0,1)$, sequence $\{\alpha_n\}, \{\beta_n\}$ are two real number in $(0,1)$ and assume that the following condition hold:

(1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\lim_{n\longrightarrow\infty} \frac{\beta_n}{\alpha_n} = 0$,

(2) $\sum_{n=1}^{\infty} \beta_n = \infty$,

(3) $\lim_{n\longrightarrow\infty} \frac{1}{\beta_n}|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}| = 0$, and $\lim_{n\longrightarrow\infty} \frac{1}{\alpha_n}|1 - \frac{\beta_{n-1}}{\beta_n}| = 0$

(4) $\sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty, \sum_{n=1}^{\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$ for all $j = \{1,2,3,\ldots,N\}$,

(5) $\sum_{n=1}^{\infty} |\alpha_1^{n,j} - \alpha_1^j| < \infty, \sum_{n=1}^{\infty} |\alpha_2^{n,j} - \alpha_2^j| < \infty, \sum_{n=1}^{\infty} |\alpha_3^{n,j} - \alpha_3^j| < \infty$ for all $j = \{1,2,3,\ldots,N\}$.

Then the sequence $\{x_n\}$ in (3.1.6) solve the following variational inequality:

$$\begin{cases} \tilde{x} \in \Omega \\ \langle (I - f)\tilde{x}, x - \tilde{x} \rangle \geq 0, \ \ \forall x \in \Omega. \end{cases} \tag{3.1.13}$$

พิสูจน์. From (3.1.6), let $y_n = \beta_n f(x_n) + (1 - \beta_n)x_n$ and $x^* \in \Omega$ we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n S_n x_n + (1 - \alpha_n) S_k y_n - x^*\| \\ &\leq \alpha_n \|S_n x_n - x^*\| + (1 - \alpha_n)\|S_k y_n - x^*\| \\ &\leq \alpha_n \|x_n - x^*\| + (1 - \alpha_n)\|y_n - x^*\|. \end{aligned} \tag{3.1.14}$$

Consider,

$$\begin{aligned} \|y_n - x^*\| &= \|\beta_n f(x_n) + (1-\beta_n)x_n - x^*\| \\ &\leq \|\beta_n \gamma\|x_n - x^*\| + \|f(x^*) - x^*\| + (1-\beta_n)\|x_n - x^*\| \\ &= (1-(1-\gamma)\beta_n)\|x_n - x^*\| + \|f(x^*) - x^*\|. \end{aligned} \tag{3.1.15}$$

From (3.1.14) and (3.1.15), we have

$$\begin{aligned} \therefore\ \|x_{n+1} - x^*\| &\leq \alpha_n\|x_n - x^*\| + (1-\alpha_n)\big[(1-(1-\gamma)\beta_n)\|x_n - x^*\| + \|f(x^*) - x^*\|\big] \\ &\leq \alpha_n\|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\| + (1-\alpha_n)\|f(x^*) - x^*\| \\ &= \|x_n - x^*\| + (1-\alpha_n)\|f(x^*) - x^*\| \\ &\leq \max\{\|x_0 - x^*\|, \|f(x^*) - x^*\|\}. \end{aligned}$$

Then $\{x_n\}$ and $\{y_n\}$ are bounded and hence $\{f(x_n)\}, \{S_n x_n\}, \{S_\lambda y_n\}$ are also.

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|\beta_n f(x_n) - \beta_n f(x_{n-1}) + \beta_n f(x_{n-1}) - \beta_{n-1} f(x_{n-1}) + (1-\beta_n)x_n \\ &\quad -(1-\beta_n)x_{n-1} + (1-\beta_n)x_{n-1} - (1-\beta_{n-1})x_{n-1}\| \\ &\leq \beta_n\gamma\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|f(x_{n-1})\| + (1-\beta_n)\|x_n - x_{n-1}\| \\ &\quad +|\beta_n - \beta_{n-1}|\|x_{n-1}\| \\ &= (1-(1-\gamma)\beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|f(x_{n-1})\| + \|x_{n-1}\|). \end{aligned}$$

From definition of $\{x_n\}$ and nonexpansiveness of S_n, we have

$$\begin{aligned} \|x_n - x_{n-1}\| &= \|\alpha_n S_n x_n + (1-\alpha_n)S_\lambda y_n - \alpha_{n-1}S_{n-1}x_{n-1} + (1-\alpha_{n-1})S_\lambda y_{n-1}\| \\ &= \|\alpha_n S_n x_n - \alpha_n S_n x_{n-1} + \alpha_n S_n x_{n-1} - \alpha_{n-1}S_n x_{n-1} + \alpha_{n-1}S_n x_{n-1} \\ &\quad -\alpha_{n-1}S_{n-1}x_{n-1} + (1-\alpha_n)S_\lambda y_{n-1} + (1-\alpha_n)S_\lambda y_{n-1} - (1-\alpha_{n-1})S_\lambda y_{n-1}\| \\ &\leq \alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|S_n x_{n-1}\| + \alpha_{n-1}\|S_n x_{n-1} - S_{n-1}x_{n-1}\| \\ &\quad +(1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|S_\lambda y_{n-1}\| \\ &\leq \alpha_n\|x_n - x_{n-1}\| + (1-\alpha_n)\big[(1-(1-\gamma)\beta_n)\|x_n - x_{n-1}\| \\ &\quad +|\beta_n - \beta_{n-1}|(\|f(x_{n-1})\| + \|x_{n-1}\|)\big] + |\alpha_n - \alpha_{n-1}|(\|S_n x_{n-1}\| + \|S_\lambda y_{n-1}\|) \\ &\quad +\alpha_{n-1}\|S_n x_{n-1} - S_{n-1}x_{n-1}\| \\ &\leq \big[\alpha_n + (1-\alpha_n)(1-(1-\gamma)\beta_n)\big]\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|f(x_{n-1})\| + \|x_{n-1}\|) \\ &\quad +|\alpha_n - \alpha_{n-1}|(\|S_n x_{n-1}\| + \|S_\lambda y_{n-1}\|) + \alpha_{n-1}\|S_n x_{n-1} - S_{n-1}x_{n-1}\| \\ &= \big[1-(1-\gamma)\beta_n(1-\alpha_n)\big]\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|f(x_{n-1})\| + \|x_{n-1}\|) \\ &\quad +|\alpha_n - \alpha_{n-1}|(\|S_n x_{n-1}\| + \|S_\lambda y_{n-1}\|) + \alpha_{n-1}\|S_n x_{n-1} - S_{n-1}x_{n-1}\|. \end{aligned}$$

Put $M = \sup\Big\{\|f(x_{n-1})\|, \|S_n x_{n-1}\|, \|S_\lambda y_{n-1}\|\Big\}, \quad n \geq 1$, it follows that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq [1-(1-\gamma)\beta_n((1-\alpha_n)]\|x_n - x_{n-1}\| + (|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|)M \\ &\quad +\alpha_{n-1}\|S_n x_{n-1} - S_{n-1}x_{n-1}\|. \end{aligned}$$

Put $\delta_n = \|S_n x_{n-1} - S_{n-1}x_{n-1}\|$, from Lemma 3.1.2, we have $\Sigma_{n=1}^{\infty}\delta_n < \infty$, it follows that

$$\begin{aligned}
\frac{\|x_{n+1}-x_n\|}{\alpha_n} &= [1-(1-\gamma)\beta_n(1-\alpha_n)]\frac{\|x_n - x_{n-1}\|}{\alpha_n} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n}M + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} + \alpha_{n-1}\frac{\delta_n}{\alpha_n} \\
&= [1-(1-\gamma)\beta_n(1-\alpha_n)]\frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \\
&\quad +[1-(1-\gamma)\beta_n(1-\alpha_n)]\left(\frac{\|x_n - x_{n-1}\|}{\alpha_n} - \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}}\right) \\
&\quad +\frac{|\beta_n - \beta_{n-1}|}{\alpha_n}M + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n}M + \alpha_{n-1}\frac{\delta_n}{\alpha_n} \\
&\leq [1-(1-\gamma)\beta_n(1-\alpha_n)]\frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \\
&\quad +\left(\left|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}\right| + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} + \frac{\delta_n}{\alpha_n}\right)M \\
&= [1-(1-\gamma)\beta_n(1-\alpha_n)]\frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \\
&\quad +(1-\gamma)\beta_n(1-\alpha_n)\left\{\frac{M}{(1-\gamma)(1-\alpha_n)}\left(\frac{1}{\beta_n}\left|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}\right| + \frac{1}{\beta_n}\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n}\right.\right. \\
&\quad \left.\left. +\frac{1}{\beta_n}\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \frac{1}{\beta_n}\frac{|\beta_n - \beta_{n-1}|}{\alpha_n} + \frac{\delta_n}{\alpha_n}\right)\right\}.
\end{aligned}$$

From Lemma 2.2.5, we obtain that

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\alpha_n} = 0. \tag{3.1.16}$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.1.17}$$

From (3.1.6) and (3.3.19), we have that

$$\lim_{n\to\infty}\|x_n - S_\lambda y_n\| = 0. \tag{3.1.18}$$

It follows that

$$y_n - x_n = \beta_n(f(x_n) - x_n) \to 0. \tag{3.1.19}$$

It implies that

$$\|y_n - S_\lambda y_n\| \leq \|y_n - x_n\| + \|x_n - S_\lambda y_n\| \to 0. \tag{3.1.20}$$

Sine the sequence $\{x_n\}$ and $\{y_n\}$ are also bounded. Thus there exists a subsequence of $\{y_n\}$, which is still denoted by $\{y_{n_i}\}$ which converges weakly to a point $\tilde{x} \in H$. Therefore, $\tilde{x} \in Fix(T)$ by (3.1.6), we observe that

$$x_{n+1} - x_n = \alpha_n(S_n x_n - x_n) + (1-\alpha_n)(S_\lambda y_n - y_n) + (1-\alpha_n)\beta_n(fx_n - x_n),$$

that is,

$$\frac{x_n - x_{n+1}}{\alpha_n} = (I - S_n)x_n + \frac{1-\alpha_n}{\alpha_n}(I - S_\lambda)y_n + \frac{\beta_n(1-\alpha_n)}{\alpha_n}(I-f)x_n.$$

Set $z_n = \frac{(x_n - x_{n+1})}{\alpha_n}$ for each $n \geq 1$, that is,

$$z_n = (I - S_n)x_n + \frac{1-\alpha_n}{\alpha_n}(I - S_\lambda)y_n + \frac{\beta_n(1-\alpha_n)}{\alpha_n}(I-f)x_n.$$

Using monotonicity of $I - S_\lambda$ and $I - S_n$, we derive that, for all $u \in Fix(T)$,

$$\begin{aligned}\langle z_n, x_n - u\rangle &= \langle (I-S_n)x_n, x_n - u\rangle + \frac{1-\alpha_n}{\alpha_n}\langle (I-S_\lambda)y_n - (I-S_\lambda)u, y_n - u\rangle \\ &\quad + \frac{1-\alpha_n}{\alpha_n}\langle (I-S_\lambda)y_n, x_n - y_n\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-f)x_n, x_n - u\rangle \\ &\geq \langle (I-S_n)u, x_n - u\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-f)x_n, x_n - u\rangle \\ &\quad + \frac{(1-\alpha_n)\beta_n}{\alpha_n}\langle (I-S_\lambda y_n, x_n - fx_n)\rangle \\ &= \langle (I-S)u, x_n - u\rangle + \langle (S-S_n)u, x_n - u\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-f)x_n, x_n - u\rangle \\ &\quad + \frac{(1-\alpha_n)\beta_n}{\alpha_n}\langle (I-S_\lambda)y_n, x_n - fx_n\rangle.\end{aligned}$$

But, since $z_n \longrightarrow 0$, $\frac{\beta_n}{\alpha_n} \longrightarrow 0$ and $\lim_{n\longrightarrow\infty}\|S_n u - Su\| = 0$, it follows from the above inequality that

$$\limsup_{n\longrightarrow\infty}\langle (I-S)u, x_n - u\rangle \leq 0, \quad \forall u \in Fix(T).$$

It suffices to guarantee that $\omega_w(x_n) \subset \Omega$. As a matter of fact, if we take any $x^* \in \omega_w(x_n)$, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^*$. Therefore, we have

$$\langle (I-S)u, x^* - u\rangle = \lim_{j\longrightarrow\infty}\langle (I-S)u, x_{n_j} - u\rangle \leq 0, \quad \forall u \in Fix(T).$$

Note that $x^* \in Fix(T)$. Hence x^* solves the following problem:

$$\begin{cases} x^* \in Fix(T) \\ \langle (I-S)u, x^* - u\rangle \geq 0, \ \ \forall u \in Fix(T). \end{cases}$$

It is obvious that this equivalent to the problem (3.1.5) by Lemma 3.1.3, we have $S_n \longrightarrow S$ uniformly in any bounded set. Thus $x^* \in \Omega$. Let $\tilde{x}$ be the solution of the variational inequality (3.3.12), by Lemma 2.1.8 we have $\tilde{x}$ is unique. Now, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle (I-f)\tilde{x}, x_n - \tilde{x}\rangle = \lim_{i\longrightarrow\infty}\langle (I-f)\tilde{x}, x_{n_i} - \tilde{x}\rangle.$$

Without loss of generality, we can assume that $x_{n_i} \rightharpoonup x^*$. Then $x^* \in \Omega$. Therefore, we have

$$\limsup_{n\longrightarrow\infty}\langle (I-f)\tilde{x}, x_n - \tilde{x}\rangle = \langle (I-f)\tilde{x}, x^* - \tilde{x}\rangle \geq 0.$$

This completes the proof. □

Theorem 3.1.5. Let C be a nonempty closed convex subset of a real Hilbert space H, let T be a λ-strictly pseudo-contractive mapping and $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mappings of C into itself for some $k_i \in [0,1)$ and $k = \max\{k_i : i = 1,2,...,N\}$ which $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let S_n be the S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1^n, \alpha_2^n, ..., \alpha_N^n$ where $\alpha_j^n = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I\times I\times I$, $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $k < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1,2,...,N-1$, $k < c \leq \alpha_1^{n,N} \leq 1$, $k \leq \alpha_3^{n,N} \leq d < 1$, $k \leq \alpha_2^{n,j} \leq e < 1$ for all $j = 1,2,...N$. Assume that set Ω of solution of generalized hierarchical problem (3.1.5) is nonempty. For a mapping $f : C \to C$ is a contraction with $\gamma \in (0,1)$, sequence $\{\alpha_n\}, \{\beta_n\}$ are two real number in $(0,1)$ and assume that the following condition hold:

(1) $\lim_{n\longrightarrow\infty}\alpha_n = 0$ *and* $\lim_{n\longrightarrow\infty}\frac{\beta_n}{\alpha_n} = 0$,

(2) $\sum_{n=1}^{\infty}\beta_n = \infty$,

(3) $\lim_{n\longrightarrow\infty}\frac{1}{\beta_n}|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}| = 0$ and $\lim_{n\longrightarrow\infty}\frac{1}{\alpha_n}|1 - \frac{\beta_{n-1}}{\beta_n}| = 0$,

(4) $\sum_{n=1}^{\infty}|\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty, \sum_{n=1}^{\infty}|\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$ for all $j = \{1, 2, 3, \ldots, N\}$,

(5) $\sum_{n=1}^{\infty}|\alpha_1^{n,j} - \alpha_1^{j}| < \infty, \sum_{n=1}^{\infty}|\alpha_2^{n,j} - \alpha_2^{j}| < \infty, \sum_{n=1}^{\infty}|\alpha_3^{n,j} - \alpha_3^{j}| < \infty$ for all $j = \{1, 2, 3, \ldots, N\}$,

(6) there exists a constant $d > 0$ such that $\|x - S_\lambda x\| \geq \rho Dist(x, F(S_\lambda))$, where

$$Dist(x, F(S_\lambda)) = \inf_{y\in F(S_\lambda)}\|x - y\|.$$

Then the sequence $\{x_n\}$ difined by (3.1.6) converges strongly to a point $\widetilde{x} \in Fix(T)$, which solve the variational inequality problem (3.3.12).

พิสูจน์. From (3.1.6), we have

$$x_{n+1} - \tilde{x} = \alpha_n(S_n x_n - S_n\tilde{x}) + \alpha_n(S_n\tilde{x} - \tilde{x}) + (1 - \alpha_n)(S_\lambda y_n - \tilde{x}).$$

Thus we have

$$\begin{aligned} \|x_{n+1} - \tilde{x}\|^2 &\leq \|\alpha_n(S_n x_n - S_n\tilde{x}) + (1 - \alpha_n)(S_\lambda y_n - \tilde{x})\|^2 + 2\alpha_n\langle S_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x}\rangle \\ &\leq (1 - \alpha_n)\|S_\lambda y_n - \tilde{x}\|^2 + \alpha_n\|S_n x_n - S_n\tilde{x}\|^2 + 2\alpha_n\langle S_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x}\rangle \quad (3.1.21) \\ &\leq (1 - \alpha_n)\|y_n - \tilde{x}\|^2 + \alpha_n\|x_n - \tilde{x}\|^2 + 2\alpha_n\langle S_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x}\rangle. \end{aligned}$$

Now we consider

$$\begin{aligned} \|y_n - \tilde{x}\|^2 &= \|(1 - \beta_n)(x_n - \tilde{x}) + \beta_n(fx_n - f\tilde{x}) + \beta_n(f\tilde{x} - \tilde{x})\|^2 \\ &\leq \|(1 - \beta_n)(x_n - \tilde{x}) + \beta_n(fx_n - f\tilde{x})\|^2 + 2\beta_n\langle f\tilde{x} - \tilde{x}, y_n - \tilde{x}\rangle \\ &\leq (1 - \beta_n)\|x_n - \tilde{x}\|^2 + \beta_n\|(fx_n - f\tilde{x})\|^2 + 2\beta_n\langle f\tilde{x} - \tilde{x}, y_n - \tilde{x}\rangle \quad (3.1.22) \\ &\leq (1 - \beta_n)\|x_n - \tilde{x}\|^2 + \beta_n\gamma^2\|x_n - \tilde{x}\|^2 + 2\beta_n\langle f\tilde{x} - \tilde{x}, y_n - \tilde{x}\rangle \\ &= \big[1 - \big(1 - \gamma^2\big)\beta_n\big]\|x_n - \tilde{x}\|^2 + 2\beta_n\langle f\tilde{x} - \tilde{x}, y_n - \tilde{x}\rangle. \end{aligned}$$

Substituting (3.1.22) into (3.1.21), we get

$$\begin{aligned} \|x_{n+1} - \tilde{x}\|^2 &= \alpha_n\|x_n - \tilde{x}\|^2 + (1 - \alpha_n)\big[1 - \big(1 - \gamma^2\big)\beta_n\big]\|x_n - \tilde{x}\|^2 \\ &\quad +2\beta_n(1 - \alpha_n)\langle f\tilde{x} - \tilde{x}, y_n - \tilde{x}\rangle + 2\alpha_n\langle S_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x}\rangle \\ &= \big[1 - \big(1 - \gamma^2\big)\beta_n(1 - \alpha_n)\big]\|x_n - \tilde{x}\|^2 + 2\beta_n(1 - \alpha_n)\langle f\tilde{x} - \tilde{x}, y_n - \tilde{x}\rangle \\ &\quad +2\alpha_n\langle S_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x}\rangle \\ &= \big[1 - \big(1 - \gamma^2\big)\beta_n(1 - \alpha_n)\big]\|x_n - \tilde{x}\|^2 + \big(1 - \gamma^2\big)\beta_n(1 - \alpha_n) \\ &\quad \times\Big\{\frac{1}{1 - \gamma^2}\langle f\tilde{x} - \tilde{x}, y_n - \tilde{x}\rangle + \frac{2}{(1 - \gamma^2)(1 - \alpha_n)} \times \frac{\alpha_n}{\beta_n}\langle S_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x}\rangle\Big\} \quad (3.1.23) \end{aligned}$$

By Theorem 3.1.4, we note that every weak cluster point of the sequence $\{x_n\}$ is in Ω. Since $y_n - x_n \to 0$, then every weak cluster point of $\{y_n\}$ is also in Ω. Consequently, since $\tilde{x} = proj_{\Omega}(f\tilde{x})$, we easily have

$$\limsup_{n\to\infty} \langle f\tilde{x} - \tilde{x}, y_n - \tilde{x}\rangle \leq 0. \tag{3.1.24}$$

On the other hand, we observe that

$$\langle S_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x}\rangle = \left\langle S_n\tilde{x} - \tilde{x}, proj_{Fix(S_\lambda)}x_{n+1} - \tilde{x}\right\rangle + \left\langle S_n\tilde{x} - \tilde{x}, x_{n+1} - proj_{Fix(S_\lambda)}x_{n+1}\right\rangle$$

Since $\tilde{x}$ is a solution of the problem(3.1.5) and $proj_{Fix(S_\lambda)}x_{n+1} \in Fix(S_\lambda)$, we have

$$\left\langle S_n\tilde{x} - \tilde{x}, proj_{Fix(S_\lambda)}x_{n+1} - \tilde{x}\right\rangle \leq 0.$$

Thus it follows that

$$\begin{aligned}
\langle S_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x}\rangle &\leq \langle S_n\tilde{x} - \tilde{x}, x_{n+1} - proj_{Fix(S_\lambda)}x_{n+1}\rangle \\
&\leq \|S_n\tilde{x} - \tilde{x}\|\left\|x_{n+1} - proj_{Fix(S_\lambda)}x_{n+1}\right\| \\
&= \|S_n\tilde{x} - \tilde{x}\| \times Dist(x_{n+1}, Fix(S_\lambda)) \\
&\leq \frac{1}{\rho}\|S_n\tilde{x} - \tilde{x}\|\|x_{n+1} - S_\lambda x_{n+1}\|.
\end{aligned}$$

We note that

$$\begin{aligned}
\|x_{n+1} - S_\lambda x_{n+1}\| &\leq \|x_{n+1} - S_\lambda x_n\| + \|S_\lambda x_n - S_\lambda x_{n+1}\| \\
&\leq \alpha_n\|S_n x_n - S_\lambda x_n\| + (1-\alpha_n)\|S_\lambda y_n - S_\lambda x_n\| + \|x_{n+1} - x_n\| \\
&\leq \alpha_n\|S_n x_n - S_\lambda x_n\| + \|y_n - x_n\| + \|x_{n+1} - x_n\| \\
&\leq \alpha_n\|S_n x_n - S_\lambda x_n\| + \beta_n\|f x_n - x_n\| + \|x_{n+1} - x_n\|.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\frac{\alpha_n}{\beta_n}\langle S_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x}\rangle \leq\ & \frac{\alpha_n^2}{\beta_n}\left(\frac{1}{\rho}\|S_n\tilde{x} - \tilde{x}\|\|S_n x_n - S_\lambda x_n\|\right) \\
&+\alpha_n\left(\frac{1}{\rho}\|S_n\tilde{x} - \tilde{x}\|\|f x_n - x_n\|\right) \\
&+\frac{\alpha_n^2}{\beta_n}\frac{\|x_{n+1} - x_n\|}{\alpha_n}\left(\frac{1}{\rho}\|S_n\tilde{x} - \tilde{x}\|\right).
\end{aligned}$$

From Theorem 3.1.4 we have $\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\alpha_n} = 0$. And then, we note that $\{\frac{1}{\rho}\|S_n\tilde{x} - \tilde{x}\|\|S_n x_n - S_\lambda x_n\|\}$, $\{\frac{1}{\rho}\|S_n\tilde{x} - \tilde{x}\|\|f x_n - x_n\|\}$, and $\{\frac{1}{\rho}\|S_n\tilde{x} - \tilde{x}\|\}$ are all bounded. Hence it follows from (i) and the above inequality that

$$\limsup_{n\to\infty}\frac{\alpha_n}{\beta_n}\langle S_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x}\rangle \leq 0.$$

Finally, by (3.1.23) and Lemma 2.2.5, we conclude that the sequence $\{x_n\}$ converges strongly to a point $\tilde{x} \in Fix(S_\lambda) = Fix(T)$. This completes the proof. □

3.2 Existence Theorems for Nonconvex variational Inequalities Problems

In this section, we prove the existence theorem for a mapping defined by $T = T_1 + T_2$ when T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping, we have a mapping T is Lipschitz continuous but not strongly monotone mapping. This work is extend and improve the result of N. Petrot [39].

Let H be a real Hilbert space, and let C be a nonempty closed subset of H. In this section, will consider the following problem: find $x^* \in C$ such that

$$-Tx^* \in N_C^P(x^*). \tag{3.2.1}$$

The problem of type (3.2.1) was studied by Noor [29] but in a finite dimension Hilbert space setting. In 2010 [39] Petrot intend to consider the problem (3.2.1) in an infinite dimension Hilbert space for a mapping T satisfied μ-Lipschitz continuous and γ-strongly monotone. In this section we extended the result of [39] Petrot to a mapping $T = T_1 + T_2$ with T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. We see that T is Lipschitz continuous but not strongly monotone mapping. To do this, the following remark is useful.

Remark 3.2.1. Let T_1 be a μ_1-Lipschitz continuous and γ-strongly monotone mapping, and let T_2 be a μ_2-Lipschitz continuous mapping. Then the function $f : (1, M) \to (0, \infty)$ which defined by

$$f(t) = \frac{\sqrt{(t\gamma - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t^2 - 1)}}{t(\mu_1^2 - \mu_2^2)}, \forall\ t \in M,$$

where $M = \frac{\gamma\mu_2 + \sqrt{(\mu_1^2 - \gamma^2)(\mu_1^2 - \mu_2^2)}}{\gamma^2 - (\mu_1^2 - \mu_2^2)}$.

In this work, we have to assume that $\mu_2 < \mu_1$. Thus, from now on, without loss of generality we will always assume that $\mu_2 < \mu_1$.

Theorem 3.2.2. Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T_1, T_2 : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. If $T = T_1 + T_2$ and $0 < \delta_{T(C)} \leq \gamma r$, then the problem (3.2.1) has a solution.

Proof. We first, defined a function $h : [1, M) \to (0, \infty)$ which is defined by

$$h(x) = \frac{r(t-1)}{t\delta_{T(C)}} + f(t), \forall t \in [1, M). \tag{3.2.2}$$

We see that the net $\{t_s\}_{s\in(0,r)}$ which is defined by $t_s = \frac{r}{r-s}$ converges to 1 as $s \downarrow 0$. It follows that $h(t) \downarrow \frac{\gamma-\mu}{\mu_1^2-\mu_2^2}$ as $t_s \downarrow 1$, we can find $s^* \in (0, r)$ such that $\frac{\gamma-\mu_2}{\mu_1^2-\mu_2^2} < h(t)$, Then we have

$$\frac{t_{s^*}\gamma -_2}{t_{s^*}(\mu_1^2 - \mu_2^2)} - f(t_{s^*}) < h(t_{s^*}) - f(t_{s^*}) = \frac{r(t_{s^*} - 1)}{t_{s^*}\delta_{T(C)}} = \frac{s^*}{\delta_{T(C)}}.$$

Now, we choose a fixed positive real number ρ such that

$$\frac{t_s\gamma - \mu_2}{\mu_1^2 - \mu_2^2} - f(t_{s^*}) < \rho < \min\{\frac{t_s\gamma - \mu_2}{\mu_1^2 - \mu_2^2} + f(t_{s^*}), \frac{s^*}{\delta_{T(C)}}\}. \tag{3.2.3}$$

Next, for an element $x_0 \in C$ and use an induction process to obtain a sequence $\{x_n\} \subset C$ satisfying

$$x_{n+1} = proj_C(x_n - \rho T x_n), \quad \forall n = 0, 1, 2, ... \tag{3.2.4}$$

Consequently, from (3.3.9) and Lemma 2.2.5, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|proj_C(x_n - \rho T x_n) - proj_C(x_{n-1} - \rho T x_{n-1})\| \\ &= t_s\|(x_n - \rho T x_n) - (x_{n-1} - \rho T x_{n-1})\| \\ &= t_s\|(x_n - x_{n-1}) - \rho(T x_n - T x_{n-1})\| \\ &\leq t_s[\|x_n - x_{n-1} - \rho(T_1 x_n - T_1 x_{n-1})\| + \rho\|T_2 x_n - T_2 x_{n-1}\|] \\ &\leq t_s[\|x_n - x_{n-1} - \rho(T_1 x_n - T_1 x_{n-1})\| + \rho\mu_2\|x_n - x_{n-1}\|]. \end{aligned} \tag{3.2.5}$$

Since the mapping T_1 is γ-strongly monotone and μ_1-Lipschitz continuous, we obtain

$$\begin{aligned} \|x_n - x_{n-1} - \rho(T_1 x_n - T_1 x_{n-1})\|^2 &= \|x_n - x_{n-1}\|^2 - 2\rho\langle x_n - x_{n-1}, T_1 x_n - T_1 x_{n-1}\rangle \\ &\quad + \rho^2\|T_1 x_n - T_1 x_{n-1}\|^2 \\ &\leq \|x_n - x_{n-1}\|^2 - 2\rho\gamma\|x_n - x_{n-1}\|^2 + \rho^2\mu_1^2\|x_n - x_{n-1}\|^2 \\ &= (1 - 2\rho\gamma + \rho\mu_1^2)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.2.6}$$

It follows that

$$\|x_n - x_{n-1} - \rho(T_1 x_n - T_1 x_{n-1})\| \leq \sqrt{1 - 2\rho\gamma + \rho\mu_1^2}\|x_n - x_{n-1}\|^2. \tag{3.2.7}$$

From (3.3.10) and (3.3.13), we get

$$\|x_{n+1} - x_n\| \leq t_s(\rho\mu_2 + \sqrt{1 - 2\rho\gamma + \rho\mu_1^2})\|x_n - x_{n-1}\|. \tag{3.2.8}$$

Now, we see that for the choice of ρ, we know that $\{x_n\}$ is a Cauchy sequence in C. Then $\{x_n\}$ is a convergence sequence, it follows that, if $x_n \to x^*$ as $n \to \infty$ we have $x^* \in proj_C(x^* + \rho(-Tx^*))$ for some $\rho > 0$. From definition 2.2.2, we have $-Tx^* \in N_C^P(x^*)$. This completes the proof. □

Corollary 3.2.3. [39] Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T : C \to H$ be a γ-strongly monotone mapping and μ-Lipschitz continuous mapping. If $0 < \delta_{T(C)} \leq \gamma r$, then the problem (3.2.1) has a solution.

Proof. From Theorem 3.3.4, if $T_2 \equiv 0$ we have a result. □

3.3 Iterative Algorithm for Nonconvex Variational Inequalities

In this section, we suggest and analyze an iterative scheme for solving the system of nonconvex variational inequalities by using projection technique. We prove strong convergence of iterative scheme to the solution of the system of nonconvex variational inequalities requires to the modified mapping T which is Lipschitz continuous but not strongly monotone mapping. Our result can be viewed and improvement the result of N. Petrot [39].

Let C_r be a uniformly r-prox-regular(nonconvex) set. For given nonlinear mappings $T : C_r \to H$, we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\langle \rho T y^* + x^* - y^*, x - x^* \rangle \geq 0, \forall x \in C_r, \rho > 0$$
$$\langle \eta T x^* + y^* - x^*, x - y^* \rangle \geq 0, \forall x \in C_r, \eta > 0, \tag{3.3.1}$$

which is called the *system of nonconvex variational inequalities*.

It is worth mentioning that if $T_1 = T_2 = T, x^* = y^* = u$ and $\rho = \eta$, then problem (3.3.1) is equivalent to finding $u \in C_r$ such that

$$\langle Tu, v - u \rangle \geq 0, \forall v \in C_r, \tag{3.3.2}$$

which is known as *nonconvex variational inequalities* introduced and studied by Bounkhel et. al. [22] and Noor [30, 31].

It is known that problem (3.3.2) is equivalent to finding $u \in C_r$ such that

$$0 \in Tu + N^P_{C_r}(u), \tag{3.3.3}$$

which $N^P_{C_r}(u)$ denote the normal cone of C_r at u. The problem (3.3.3) is called the *variational inclusion associated with nonconvex variational inequalities* (3.3.2).

Lemma 3.3.1. For given $x^*, y^* \in C_r$ is a solution of system of nonconvex variational inequalities (3.3.1), if and only if

$$x^* = P_C[y^* - \rho T y^*],$$
$$y^* = P_C[x^* - \eta T x^*], \tag{3.3.4}$$

where P_C is the projection of H onto the uniformly prox-regular set C_r.

พิสูจน์. Let $x^*, y^* \in C_r$ be a solution of (3.3.1), from (3.3.3), for a constant $\rho > 0$, we have

$$0 \in \rho T_1 y^* + x^* - y^* + \rho N^P_{C_r}(x^*) = (I + \rho N^P_{C_r})(x^*) - [y^* - \rho T_1 y^*]$$

if and only if

$$x^* = (I + \rho N^P_{C_r})^{-1}[y^* - \rho T_1 y^*] = P_C[y^* - \rho T_1 y^*],$$

where we have used the well-known fact that $P_C = (I + \rho N^P_{C_r})^{-1}$.

Similarly, we obtain

$$y^* = P_C[x^* - \eta T_2 x^*].$$

This prove our assertions. □

Algorithm 3.3.2. For arbitrarily chosen initial points $x_0, y_0 \in C_r$, the sequence $\{x_n\}$ and $\{y_n\}$ in the following way:

$$\begin{aligned} y_n &= P_C[x_n - \eta T x_n], \eta > 0 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n P_C[y_n - \rho T y_n], \rho > 0, \end{aligned} \tag{3.3.5}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$.

Remark 3.3.3. [39] Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T_1, T_2 : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. Let

$$\xi = r[\mu_1^2 - \gamma \frac{\mu_2 - \sqrt{(\mu_1^2 - \gamma\mu_2)^2 - \mu_1^2(\gamma - \mu_2)^2}}{\mu_1^2}] \tag{3.3.6}$$

then for each $s \in (0, \xi)$, we have

$$\gamma t_s - \mu_2 > \sqrt{(\mu_1^2 - \mu_2^2)(t_s^2 - 1)}, \tag{3.3.7}$$

where $t_s = \frac{r}{r-s}$.

In this paper, we may assume that $M^{\rho,\eta}\delta_{T(C)} < \xi$, we see that for any $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$ it satisfy the inequality 3.3.7 too. where $M^{\rho,\eta} = \min\{\rho, \eta\}, \delta_{T(C)} = \sup\{\|u - v\| : u, v \in T(C)\}$.

Now, we suggest and analyze the following explicit projection method (3.3.2) for solving the system of nonconvex variational inequalities (3.3.1). Thus, from now on, without loss of generality we will always assume that $\mu_2 < \mu_1$.

Theorem 3.3.4. Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T_1, T_2 : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. If $T = T_1 + T_2$ and there exists constant $\rho, \eta > 0$ and $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$, such that

$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \triangle_{t_s} < \rho, \eta < \min\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \triangle_{t_s}, \frac{1}{t_s \mu_2}\}, \tag{3.3.8}$$

where $\triangle_{t_s} = \frac{\sqrt{(\gamma t_s - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t_s^2 - 1)}}{t_s(\mu_1^2 - \mu_2^2)}$. If the sequence of positive real number $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, then the sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 3.3.2 converge to a solution of the system of nonconvex variational inequalities (3.3.1).

พิสูจน์. Let $x^*, y^* \in C_r$ be a solution of (3.3.1) and from Lemma 3.3.1, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n P_C[y_n - \rho T y_n] - x^*\| \\
&= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(P_C[y_n - \rho T y_n] - P_C[y^* - \rho T y^*])\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|P_C[y_n - \rho T y_n] - P_C[y^* - \rho T y^*]\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n t_s \|(y_n - \rho T y_n) - (y^* - \rho T y^*)\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n t_s [\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| + \rho\|(T_2 y_n - T_2 y^*)\|] \qquad (3.3.9)
\end{aligned}$$

From T_1 are both μ_1-Lipschitz continuous and γ-strongly monotone mapping and from Lemma 2.2.6, we obtain

$$\begin{aligned}
\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\|^2 &= \|y_n - y^*\|^2 - 2\rho\langle y_n - y^*, T_1 y_n - T_1 y^*\rangle + \rho^2\|T_1 y_n - T_1 y^*\|^2 \\
&\leq \|y_n - y^*\|^2 - 2\rho\gamma\|y_n - y^*\|^2 + \rho^2\mu_1^2\|y_n - y^*\|^2 \\
&= (1 - 2\rho\gamma + \rho^2\mu_1^2)\|y_n - y^*\|^2.
\end{aligned}$$

It follows that

$$\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| \leq \sqrt{1 - 2\rho\gamma + \rho^2\mu_1^2}\|y_n - y^*\|. \qquad (3.3.10)$$

On the other hand, from T_2 is μ_2-Lipschitz continuous, we have

$$\|T_2 y_n - T_2 y^*\| \leq \mu_2\|y_n - y^*\|. \qquad (3.3.11)$$

Thus, by (3.3.9), (3.3.10) and (3.3.11), we have

$$\|x_{n+1} - x^*\| \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n t_s(\rho\mu_2 + \sqrt{1 - 2\rho\gamma + \rho^2\mu_1^2})\|y_n - y^*\|. \qquad (3.3.12)$$

Similarly, we have

$$\begin{aligned}
\|y_n - y^*\| &= \|P_C[x_n - \eta T x_n] - y^*\| \\
&= \|P_C[x_n - \eta T x_n] - P_C[x^* - \eta T x^*]\| \\
&\leq t_s\|(x_n - \eta T x_n) - (x^* - \eta x^*)\| \\
&\leq t_s[\|(x_n - x^*) - \eta(T_1 x_n - T_1 x^*)\| + \eta\|T_2 x_n - T_2 x^*\|]. \qquad (3.3.13)
\end{aligned}$$

Similarly, from T_1 are both μ_1-Lipschitz continuous and γ-strongly monotone mapping, we obtain

$$\begin{aligned}
\|(x_n - x^*) - \eta(T_1 x_n - T_1 x^*)\|^2 &= \|x_n - x^*\|^2 - 2\eta\langle x_n - x^*, T_1 x_n - T_1 x^*\rangle + \eta^2\|T_1 x_n - T_1 x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\eta\gamma\|x_n - x^*\|^2 + \eta^2\mu_1^2\|x_n - x^*\|^2 \\
&= (1 - 2\eta\gamma + \eta^2\mu_1^2)\|x_n - x^*\|^2.
\end{aligned}$$

It follows that

$$\|(x_n - x^*) - \eta(T_1 x_n - T_1 x^*)\| \leq \sqrt{1 - 2\eta\gamma + \eta^2\mu_1^2}\|x_n - x^*\|. \qquad (3.3.14)$$

On the other hand, from T_2 is μ_2-Lipschitz continuous, we have

$$\|T_2 x_n - T_2 x^*\| \leq \mu_2\|x_n - x^*\|. \qquad (3.3.15)$$

Thus, by (3.3.13), (3.3.14) and (3.3.15), we have

$$\|y_n - y^*\| \leq t_s(\eta\mu_2 + \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_1^2})\|x_n - x^*\|. \tag{3.3.16}$$

Moreover, from (3.3.12) and (3.3.16) we put $\theta_1 = t_s(\rho\mu_2 + \sqrt{1 - 2\rho\gamma + \rho^2\mu_1^2}), \theta_2 = t_s(\eta\mu_2 + \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_1^2})$, it follows that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\theta_2\|x_n - x^*\| \\ &= (1 - (1-\theta_1\theta_2)\alpha_n)\|x_n - x^*\| \\ &\leq \prod_{i=0}^{n}(1 - (1-\theta_1\theta_2)\alpha_i)\|x_0 - x^*\|. \end{aligned} \tag{3.3.17}$$

Since $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ and conditions (3.3.8), we obtain

$$\lim_{n\to\infty}\prod_{i=0}^{n}(1 - (1-\theta_1\theta_2)\alpha_i) = 0. \tag{3.3.18}$$

It follows from (3.3.18) and (3.3.17), we have

$$\lim_{n\to\infty}\|x_n - x^*\| = 0. \tag{3.3.19}$$

From (3.3.16) and (3.3.19), we have

$$\lim_{n\to\infty}\|y_n - y^*\| = 0. \tag{3.3.20}$$

Which is $x^*, y^* \in C_r$ satisfying the system of nonconvex variational inequalities (3.3.1). This completes the proof. □

Corollary 3.3.5. Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T : C \to H$ be such that T is a μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ and $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$, such that

$$\frac{\gamma}{\mu^2} - \triangle_{t_s} < \rho, \eta < \frac{\gamma}{\mu^2} + \triangle_{t_s}, \tag{3.3.21}$$

where $\triangle_{t_s} = \frac{\sqrt{(\gamma t_s)^2 - (\mu_1^2)(t_s^2 - 1)}}{t_s(\mu_1^2)}$. If the sequence of positive real number $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, and $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, then the sequence $\{x_n\}$ and $\{y_n\}$ is generated by for $x_0, y_0 \in C_r$,

$$\begin{aligned} y_n &= P_C[x_n - \eta T x_n], \eta > 0 \\ x_{n+1} &= P_C[y_n - \rho T y_n], \rho > 0, \end{aligned} \tag{3.3.22}$$

strongly converge to a solution of the system of nonconvex variational inequalities (3.3.1).

พิสูจน์. From Theorem 3.3.4, if $T_2 \equiv 0$ and $\alpha_n = 1$ for any $n \geq 0$, we have a result. □

We can applied Theorem 3.3.4 to the system of general of nonconvex variational inequalities, for given nonlinear mappings $T, g : C_r \to H$, we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T g(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T g(x^*) + g(y^*) - g(x^*), x - g(y^*)\rangle \geq 0, \forall x \in C_r, \eta > 0, \end{aligned} \tag{3.3.23}$$

which is called the *system of general nonconvex variational inequalities*. Similar of the proof of Lemma 3.3.1, we can proof that

Lemma 3.3.6. For given $x^*, y^* \in C_r$ is a solution of system of nonconvex variational inequalities (3.3.23), if and only if

$$g(x^*) = P_C[g(y^*) - \rho Tg(y^*)],$$
$$g(y^*) = P_C[g(x^*) - \eta Tg(x^*)], \tag{3.3.24}$$

where P_C is the projection of H onto the uniformly prox-regular set C_r.

Theorem 3.3.7. Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, let $g : C \to H$ is injective mapping and let $T_1, T_2 : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. If $T = T_1 + T_2$ and there exists constant $\rho, \eta > 0$ and $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$, such that

$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \triangle_{t_s} < \rho, \eta < \min\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \triangle_{t_s}, \frac{1}{t_s \mu_2}\}, \tag{3.3.25}$$

where $\triangle_{t_s} = \frac{\sqrt{(\gamma t_s - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t_s^2 - 1)}}{t_s(\mu_1^2 - \mu_2^2)}$. If the sequence of positive real number $\alpha_n \in [0, 1]$ with $\Sigma_{n=0}^{\infty} \alpha_n = 0$, then the sequence $\{x_n\}$ and $\{y_n\}$ is generated by for $x_0, y_0 \in C_r$,

$$\begin{aligned} g(y_n) &= P_C[g(x_n) - \eta Tg(x_n)], \eta > 0 \\ g(x_{n+1}) &= (1 - \alpha_n)g(x_n) + \alpha_n P_C[g(y_n) - \rho Tg(y_n)], \rho > 0, \end{aligned} \tag{3.3.26}$$

strongly converge to a solution of the system of nonconvex variational inequalities (3.3.23).

พิสูจน์. Similar the proof in Theorem 3.3.4, let $x^*, y^* \in C_r$ be a solution of (3.3.23) and from Lemma 3.3.6, we can compute that

$$\|g(x_{n+1}) - g(x^*)\| \leq \prod_{i=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_i)\|g(x_0) - g(x^*)\|. \tag{3.3.27}$$

where $\theta_1 = t_s(\rho\mu_2 + \sqrt{1 - 2\rho\gamma + \rho^2\mu_1^2})$ From $\Sigma_{n=0}^{\infty} \alpha_n = \infty$ and conditions (3.3.25), we obtain

$$\lim_{n\to\infty} \prod_{i=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_i) = 0. \tag{3.3.28}$$

It follows from (3.3.27) and (3.3.28), we have

$$\lim_{n\to\infty} \|g(x_n) - g(x^*)\| = 0. \tag{3.3.29}$$

And we can compute that

$$\|g(y_n) - g(y^*)\| \leq \theta_2\|g(x_n) - g(x^*)\|, \tag{3.3.30}$$

where $\theta_2 = t_s(\eta\mu_2 + \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_1^2})$, it follows that

$$\lim_{n\to\infty} \|g(y_n) - g(y^*)\| = 0. \tag{3.3.31}$$

From g is injective mapping, we have $\lim_{n\to\infty} \|x_n - x^*\| = 0$ and $\lim_{n\to\infty} \|y_n - y^*\| = 0$ satisfying the system of general nonconvex variational inequalities (3.3.23). This complete the proof. □

Corollary 3.3.8. Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, let $g : C \to H$ is injective mapping and let $T : C \to H$ be such that T is a μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ and $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$, such that

$$\frac{\gamma}{\mu^2} - \triangle_{t_s} < \rho, \eta < \frac{\gamma}{\mu^2} + \triangle_{t_s}, \tag{3.3.32}$$

where $\triangle_{t_s} = \frac{\sqrt{(\gamma t_s)^2 - (\mu_1^2)(t_s^2 - 1)}}{t_s(\mu_1^2)}$. If the sequence of positive real number $\alpha_n \in [0, 1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, and $\alpha_n \in [0, 1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, then the sequence $\{x_n\}$ and $\{y_n\}$ is generated by for $x_0, y_0 \in C_r$,

$$\begin{aligned} g(y_n) &= P_C[g(x_n) - \eta T g(x_n)], \eta > 0 \\ g(x_{n+1}) &= P_C[g(y_n) - \rho T g(y_n)], \rho > 0, \end{aligned} \tag{3.3.33}$$

strongly converge to a solution of the system of nonconvex variational inequalities (3.3.23).

พิสูจน์. From Theorem 3.3.4, if $T_2 \equiv 0$ and $\alpha_n = 1$ for any $n \geq 0$, we have a result. □

บทที่ 4

CONCLUSIONS

4.1 Outputs 3 papers (Supported by TRF: MRG5580080)

1. Iterative Algorithm for Finite Family of k_i-Strictly Pseudo-Contractive Mappings for a General Hierarchical Problem in Hilbert Spaces. Thai Journal of Mathematics, Articles in Press.
2. Existence Theorems for Nonconvex Variational Inequalities Problems. Applied Mathematical Sciences, Vol. 7, 2013, no. 31, 1515 - 1522.
3. Strong Convergence Theorems of Iterative Algorithm for Nonconvex Variational Inequalities. Submitted to Thai Journal of Mathematics.

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Iterative Algorithm for Finite Family of k_i-Strictly Pseudo-Contractive Mappings for a General Hierarchical Problem in Hilbert Spaces[1]

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Abstract : In this work, we introduced the iterative scheme for finite family of k-strictly pseudo-contractive mappings. Then we prove strong convergence of algorithm (1.6) and solving a common solution of a general hierarchical problem and fixed point problems of finite family of k-strictly pseudo-contractive mappings.

Keywords : k-strictly pseudo-contractive mappings; hierarchical problem; variational inequality; fixed point; Hilbert spaces.
2010 Mathematics Subject Classification : 47H09; 47H10 (2000 MSC)

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. The hierarchical problem is of finding $\tilde{x} \in Fix(T)$ such that

$$\langle S\tilde{x} - \tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in Fix(T), \tag{1.1}$$

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where S, T are two nonexpansive mappings and $Fix(T)$ to denote the fixed points set of T, that is $Fix(T) = \{x \in C : Tx = x\}$. Recently, this problem has been studied by many authors (see,[2]-[17]).

Now, we briefly recall some historic results which relate to the problem (1.1).

For solving the problem (1.1), in 2006, Moudafi and Mainge [4] first introduced an implicit iterative algorithm:

$$x_{t,s} = sQ(x_{t,s}) + (1-s)[tS(x_{t,s}) + (1-t)T(x_{t,s})] \tag{1.2}$$

and proved that the net $\{x_{t,s}\}$ defined by (1.2) strongly converges to x_t as $s \to 0$, where x_t satisfies $x_t{=}proj_{Fix(P_t)}Q(x_t)$, where $P_t{:}C \to C$ is a mapping defined by

$$P_t(x) = tS(x) + (1-t)T(x), \forall x \in C, t \in (0,1),$$

or, equivalently, x_t is the unique solution of the quasivariational inequality:

$$0 \in (I-Q)x_t + N_{Fix(P_t)}(x_t),$$

where the normal cone to $Fix(P_t), N_{Fix(P_t)}$ is defined as follows:

$$N_{Fix(P_t)} : x \to \begin{cases} \{u \in H : \langle y-x, u\rangle \leq 0\}, & \text{if } x \in Fix(P_t), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Moreover, as $t \to 0$, the net $\{x_t\}$ in turn weakly converges to the unique solution x_∞ of the fixed point equation $x_\infty{=}proj_\Omega Q(x_\infty)$ or, equivalently, x_∞ is the unique solution of the variational inequality:

$$0 \in (I-Q)x_\infty + N_\Omega(x_\infty).$$

Recall that a mapping $f : C \longrightarrow C$ is said to be contractive if there exists a constant $\gamma \in (0,1)$ such that

$$\|fx - fy\| \leq \gamma\|x-y\|, \quad \forall x, y \in C.$$

A mapping $T : C \longrightarrow C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x-y\|, \quad \forall x, y \in C.$$

A mapping T is said to be k-strict pseudo-contractive if there exists $k \in [0,1)$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2, \forall x, y \in D(T). \tag{1.3}$$

Note that the class of k-strict pseudo-contraction strictly includes the class of nonexpansive mappings. We see that, if $S_k : C \to C$ defined by $S_kx = kx + (1-k)Tx$ for all $x \in C$ where T is k-strict pseudo-contractive then S_k is nonexpansive mapping [21].

In this paper, motivate by Kangtunkarn and Suantai [1], we introduce a mapping for finding a common fixed point of T is a λ-strict pseudo-contractive mapping and $\{T_i\}_{i=1}^N$ a finite family of k_i-strict pseudo-contractive mappings of C into itself. For each $n \in \mathbb{N}$, and $j = 1, 2, ..., N$, let $\alpha_j^n = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in [0,1] \times [0,1] \times [0,1]$ with $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. We define the mapping $S_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,0} &= I; \\
U_{n,1} &= \alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I; \\
U_{n,2} &= \alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I; \\
U_{n,3} &= \alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I; \\
&\vdots \quad ; \\
U_{n,N-1} &= \alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I; \\
S_n &= U_{n,N} = \alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I. \qquad (1.4)
\end{aligned}$$

Motivated and inspired by the results in the literature, in this paper, we consider a general hierarchical problem of finding $x^* \in F(T)$ such that, for any $n \geq 1$,

$$\langle S_n x^* - x^*, x - x^* \rangle \leq 0, \forall x \in F(S_\lambda), \qquad (1.5)$$

where S_n is the S-mapping defined by (1.4) and S_λ is a nonexpansive mapping defined in Lemma 2.1.

Algorithm 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let T is a λ-strict pseudo-contractive mapping with $S_\lambda x = \lambda x + (1-\lambda)Tx$ and $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mapping of C into itself. Let $f : C \to C$ be a contraction with coefficient $\gamma \in (0,1)$. For any $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \alpha_n S_n x_n + (1-\alpha_n) S_\lambda(\beta_n f(x_n) + (1-\beta_n)x_n), \quad \forall n \geq 0, \qquad (1.6)$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real numbers in $(0,1)$ and S_n is the $S-$mapping defined by (1.4).

We show that an explicit iterative algorithm which converges strongly to a solution x^* of the general hierarchical problem (1.5).

2 Preliminaries

In this section, we collect and give some definition and useful lemmas that will be used for our main results in the next section.

Lemma 2.1. *[21] Let $T : C \to C$ be a k-strictly pseudo-contraction. Defined $S_\lambda : C \to C$ by $S_\lambda x = \lambda x + (1-\lambda)Tx$ for each $x \in C$. Then, as $\lambda \in [k, 1]$, S_λ is nonexpansive mapping and $F(T) = F(S_\lambda)$.*

Lemma 2.2. *In a real Hilbert space H, there holds the inequality*

1. $\|x+y\|^2 \leq \|x\|^2+2\langle y,x+y\rangle$ *and* $\|x-y\|^2 = \|x\|^2-2\langle x,y\rangle+\|y\|^2, \forall x,y \in H.$
2. $\|tx+(1-t)y\|^2 = t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2, \forall t \in [0,1], \forall x,y \in H.$
3. $\|\sum_{i=0}^m \alpha_i x_i\|^2 = \sum_{i=0}^m \alpha_i\|x_i\|^2 - \sum_{i=0}^m \alpha_i\alpha_j\|x_i-x_j\|^2$ *for* $\sum_{i=0}^m \alpha_i = 1, \alpha_i \in [0,1], \forall i \in \{0,1,2,\ldots,m\}.$

Definition 2.3. *[1] Let C be nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mapping of C into itself. For each $j = 1,2,\ldots,N$, let $\alpha_j = (\alpha_1^j,\alpha_2^j,\alpha_3^j) \in I \times I \times I$ where $\alpha_1^j,\alpha_2^j,\alpha_3^j \in I \equiv [0,1]$ and $\alpha_1^j+\alpha_2^j+\alpha_3^j = 1$. We define the mapping $S : C \to C$ as follows:*

$$\begin{aligned}
U_0 &= I\\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I\\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I\\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I\\
&\vdots\\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I\\
S &= U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called $S-$mapping generated by $T_1,\ldots,T_N$ and $\alpha_1,\alpha_2,\ldots,\alpha_N$.

Lemma 2.4. *[10] Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a self-mapping of C. If S is a k-strict pseudo-contraction mapping, then S satisfies the Lipschitz condition*

$$\|S_x - S_y\| \leq \frac{1+k}{1-k}\|x-y\|, \quad \forall x,y \in C.$$

Lemma 2.5. *[20] Let $\{s_n\}$ be a sequence of nonnegative real number satisfying*

$$s_{n+1} = (1-\alpha_n)s_n + \delta_n + \eta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

1. $\sum_{n=1}^\infty \alpha_n = \infty,$
2. $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^\infty |\delta_n| < \infty,$
3. $\sum_{n=1}^\infty |\eta_n| < \infty.$

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.6. *[1] Let C be a nonempty closed convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mapping of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $k = \max\{k_i : i = 1,2,\ldots,N\}$ and let $\alpha_j = (\alpha_1^j,\alpha_2^j,\alpha_3^j) \in I\times I\times I, j = 1,2,3,\ldots,N$, where $I = [0,1], \alpha_1^j+\alpha_2^j+\alpha_3^j = 1, \alpha_1^j,\alpha_3^j \in (k,1)$ for all $j = 1,2,\ldots,N-1$ and $\alpha_1^N \in (k,1], \alpha_3^N \in (k,1], \alpha_2^j \in (k,1]$ for all $j = 1,2,\ldots,N$. Let S be the mapping generated by $T_1,\ldots,T_N$ and $\alpha_1,\alpha_2,\ldots,\alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a nonexpansive mapping.*

Lemma 2.7. *[19] A real Hilbert space H satisfies Opial's condition, i.e, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for each $y \in H$ with $x \neq y$.

Lemma 2.8. *[18] Let C be a nonempty closed convex subset of a real Hilbert and $T : C \to C$ be a nonexpansive mapping. Then T is demi-closed on C,i.e.,if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

3 Main Results

In this section, we prove strong convergence of algorithm (1.6) and solving a common solution of a general hierarchical problems and fixed point problems of finite family of strict pseudo-contractive mappings. First, we can prove the lemmas that will be used in the main theorem.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contraction of C into itself for some $k_i \in [0,1)$ and $k = \max\{k_i : i = 1,2,\dots,N\}$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let S_n be the S-mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I, I = [0,1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $k < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $k < c \leq \alpha_1^{n,N} \leq 1, k \leq \alpha_3^{n,N} \leq d < 1, k \leq \alpha_2^{n,j} \leq e < 1$ for all $j = 1,2,\dots,N$ and $\sum_{n=1}^\infty |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty, \sum_{n=1}^\infty |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$ for all $j = \{1,2,3,\dots,N\}$. Then for all $x \in H, \sum_{n=1}^\infty \|S_{n+1}x - S_nx\| < \infty$.*

Proof. For each $x \in C$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|U_{n+1,1}x - U_{n,1}x\| &= \|\alpha_1^{n+1,1}T_1x + (1-\alpha_1^{n+1,1}x) - \alpha_1^{n,1}T_1x + (1-\alpha_1^{n,1})x\| \\
&= \|\alpha_1^{n+1,1}T_1x - \alpha_1^{n+1,1}x - \alpha_1^{n,1}T_1x + \alpha_1^{n,1}x\| \\
&= \|(\alpha_1^{n+1,1} - \alpha_1^{n,1})T_1x - (\alpha_1^{n+1,1} - \alpha_1^{n,1})x\| \\
&= |\alpha_1^{n+1,1} - \alpha_1^{n,1}|\|T_1x - x\| \qquad (3.1)
\end{aligned}$$

and for $n \in \mathbb{N}$, and for $k \in \{2, 3, \ldots, N\}$, we have

$$
\begin{aligned}
\|U_{n+1,k}x - U_{n,k}x\| &= \|\alpha_1^{n+1,k} T_k U_{n+1,k-1}x + \alpha_2^{n+1,k} U_{n+1,k-1}x + \alpha_3^{n+1,k} x \\
&\quad -\alpha_1^{n,k} T_k U_{n,k-1}x + \alpha_2^{n,k} U_{n,k-1}x + \alpha_3^{n,k} x\| \\
&= \|\alpha_1^{n+1,k} T_k U_{n+1,k-1}x + \alpha_3^{n+1,k} x - \alpha_1^{n,k} T_k U_{n,k-1}x - \alpha_3^{n,k} x \\
&\quad +\alpha_2^{n+1,k} U_{n+1,k-1}x - \alpha_2^{n,k} U_{n,k-1}x\| \\
&= \|\alpha_1^{n+1,k} T_k U_{n+1,k-1}x - \alpha_1^{n+1,k} T_k U_{n,k-1}x + \alpha_1^{n+1,k} T_k U_{n,k-1}x \\
&\quad -\alpha_1^{n,k} T_k U_{n,k-1}x + (\alpha_3^{n+1,k} - \alpha_3^{n,k})x + \alpha_2^{n+1,k} U_{n+1,k-1}x - \alpha_2^{n,k} U_{k-1}x\| \\
&= \|\alpha_1^{n+1,k}(T_k U_{n+1,k-1}x - T_k U_{n,k-1}x) + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) T_k U_{n,k-1}x \\
&\quad +(\alpha_3^{n+1,k} - \alpha_3^{n,k})x + \alpha_2^{n+1,k} U_{n+1,k-1}x - \alpha_2^{n,k} U_{n,k-1}x\| \\
&= \|\alpha_1^{n+1,k}(T_k U_{n+1,k-1}x - T_k U_{n,k-1}x) + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) \\
&\quad \times T_k U_{n,k-1}x + (\alpha_3^{n+1,k} - \alpha_3^{n,k})x + \alpha_2^{n+1,k} U_{n+1,k-1}x \\
&\quad -\alpha_2^{n+1,k} U_{n,k-1}x + \alpha_2^{n+1,k} U_{n,k-1}x - \alpha_2^{n,k} U_{n,k-1}x\| \\
&= \|\alpha_1^{n+1,k}(T_k U_{n+1,k-1}x - T_k U_{n,k-1}x) + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) \\
&\quad \times T_k U_{n,k-1}x + (\alpha_3^{n+1,k} - \alpha_3^{n,k})x + \alpha_2^{n+1,k}(U_{n+1,k-1}x \\
&\quad -U_{n,k-1}x) + (\alpha_2^{n+1,k} - \alpha_2^{n,k}) U_{n,k-1}x\| \\
&\leq \alpha_1^{n+1,k}\|T_k U_{n+1,k-1}x - T_k U_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}x\| \\
&\quad +|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x\| \\
&\quad +\alpha_2^{n+1,k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}x\| \\
&= \alpha_1^{n+1,k}\|T_k U_{n+1,k-1}x - T_k U_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}x\| \\
&\quad +\alpha_2^{n+1,k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |1 - \alpha_1^{n+1,k} \\
&\quad -\alpha_3^{n+1,k} - 1 + \alpha_1^{n,k} + \alpha_3^{n,k}|\|U_{n,k-1}x\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x\| \\
&\leq \alpha_1^{n+1,k}\frac{1+k}{1-k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}x\| \\
&\quad +\alpha_2^{n+1,k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + (|\alpha_1^{n,k} \\
&\quad -\alpha_1^{n+1,k}| + |\alpha_3^{n,k} - \alpha_3^{n+1,k}|)\|U_{n,k-1}x\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x\| \\
&\leq \frac{1+k}{1-k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}x\| \\
&\quad +\frac{1+k}{1-k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + (|\alpha_1^{n,k} - \alpha_1^{n+1,k}| \\
&\quad +|\alpha_3^{n,k} - \alpha_3^{n+1,k}|)\|U_{n,k-1}x\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x\| \\
&= \frac{2}{1-k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|(\|T_k U_{n,k-1}x \\
&\quad +\|U_{n,k-1}x\|) + |\alpha_3^{n,k} - \alpha_3^{n+1,k}|(\|U_{n,k-1}x\| + \|x\|). \qquad (3.2)
\end{aligned}
$$

By (3.1) and (3.2), we have

$$
\begin{aligned}
\|S_{n+1}x - S_nx\| &= \|U_{n+1,N}x - U_{n,N}x\| \\
&\leq \frac{2}{1-k}\|U_{n+1,N-1}x - U_{n,N-1}x\| + |\alpha_1^{n+1,N} - \alpha_1^{n,N}|(\|T_NU_{n,N-1}x\| \\
&\quad +\|U_{n,N-1}x\|) + |\alpha_3^{n+1,N} - \alpha_3^{n,N}|(\|U_{n,N-1}x\| + \|x\|) \\
&\leq \frac{2}{1-k}\Bigg(\frac{2}{1-k}\|U_{n+1,N-2}x - U_{n,N-2}x\| \\
&\quad +|\alpha_1^{n+1,N-1} - \alpha_1^{n,N-1}|(\|T_{N-1}U_{n,N-2}x\| + \|U_{n,N-2}x\|) \\
&\quad +|\alpha_3^{n+1,N-1} - \alpha_3^{n,N-1}|[\|U_{n,N-2}x\| + \|x\|]\Bigg) \\
&\quad +|\alpha_1^{n+1,N} - \alpha_1^{n,N}|(\|T_NU_{n,N-1}x\| + \|U_{n,N-1}x\|) \\
&\quad +|\alpha_3^{n+1,N} - \alpha_3^{n,N}|(\|U_{n,N-1}x\| + \|x\|) \\
&= \left(\frac{2}{1-k}\right)^2\|U_{n+1,N-2}x - U_{n,N-2}x\| + \sum_{j=N-1}^{N}\left(\frac{2}{1-k}\right)^{N-j}|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}x\| \\
&\quad +\|U_{n,j-1}x\|) + \sum_{j=N-1}^{N}\left(\frac{2}{1-k}\right)^{N-j}|\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|U_{n,j-1}x\| + \|x\|) \\
&\quad \vdots \\
&\leq \left(\frac{2}{1-k}\right)^{N-1}\|U_{n+1,1}x - U_{n,1}x\| + \sum_{j=2}^{N}\left(\frac{2}{1-k}\right)^{N-j}|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}x\| \\
&\quad +\|U_{n,j-1}x\|) + \sum_{j=2}^{N}\left(\frac{2}{1-k}\right)^{N-j}|\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|U_{n,j-1}x\| + \|x\|) \\
&= \left(\frac{2}{1-k}\right)^{N-1}|\alpha_1^{n+1,1} - \alpha_1^{n,1}|\|T_1x - x\| + \sum_{j=2}^{N}\left(\frac{2}{1-k}\right)^{N-j} \\
&\quad +|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}x\| + \|U_{n,j-1}x\| + \|x\|) + \sum_{j=2}^{N}\left(\frac{2}{1-k}\right)^{N-j} \\
&\quad +|\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|U_{n,j-1}x\| + \|x\|).
\end{aligned}
$$

This implies by assumption we have that

$$\sum_{n=1}^{\infty}\|S_{n+1}x - S_nx\| < \infty.$$

This complete the proof. □

Lemma 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contraction of C into itself* *for some* $k_i \in [0,1)$ *and* $k = \max\{k_i : i = 1, 2, \ldots, N\}$ *with* $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* S_n *be the* S*-mapping generated by* $T_1, T_2, \ldots, T_N$ *and* $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, *where* $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I, I = [0,1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ *and satisfy conditions:*

(1) $k < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ *for all* $k < c \leq \alpha_1^{n,N} \leq 1, k \leq \alpha_3^{n,N} \leq d < 1, k \leq \alpha_2^{n,j} \leq e < 1$ *for all* $j = 1, 2, \ldots, N$

(2) $\sum_{n=1}^{\infty} |\alpha_1^{n,j} - \alpha_1^j| < \infty, \sum_{n=1}^{\infty} |\alpha_2^{n,j} - \alpha_2^j| < \infty, \sum_{n=1}^{\infty} |\alpha_3^{n,j} - \alpha_3^j| < \infty$ *for all* $j = \{1, 2, 3, \ldots, N\}$.

Then for all $x \in H, \lim_{n\to\infty} \|S_n x - Sx\| = 0$.

Proof. Let $x \in C$ and for each $n \in \mathbb{N}$, from the definition of S mapping and Lemma 2.4, we have

$$\begin{aligned}
\|U_{n,1}x - U_1x\| &= \|\alpha_1^{n,1}T_1U_{n,0}x + \alpha_2^{n,1}U_{n,0}x + \alpha_3^{n,1}x - (\alpha_1^1T_1U_0x + \alpha_2^1U_0x + \alpha_3^1x)\| \\
&\leq |\alpha_1^{n,1} - \alpha_1^1|\|T_1x\| + |\alpha_2^{n,1} - \alpha_2^1|\|x\| + |\alpha_3^{n,1} - \alpha_3^1|\|x\|.
\end{aligned}$$

From boundedness and condition (2) we have

$$\lim_{n\to\infty} \|U_{n,1}x - U_1x\| = 0. \tag{3.3}$$

Next, consider

$$\begin{aligned}
\|U_{n,2}x - U_2x\| &= \|\alpha_1^{n,2}T_2U_{n,1}x + \alpha_2^{n,2}U_{n,1}x + \alpha_3^{n,2}x - (\alpha_1^2T_2U_1x + \alpha_2^2U_1x + \alpha_3^2x)\| \\
&\leq \|\alpha_1^{n,2}T_2U_{n,1}x - \alpha_1^{n,2}T_2U_1x + \alpha_1^{n,2}T_2U_1x + \alpha_2^{n,2}U_{n,1}x + \alpha_3^{n,2}x \\
&\quad -(\alpha_1^2T_2U_1x + \alpha_2^2U_1x + \alpha_3^2x)\| \\
&\leq \|\alpha_1^{n,2}(T_2U_{n,1}x - T_2U_1x)\| + \|(\alpha_3^{n,2} - \alpha_3^2)(x)\| + \|(\alpha_1^{n,2} - \alpha_1^2)(T_2U_1x)\| \\
&\quad +\|\alpha_2^{n,2}U_{n,1}x - \alpha_2^2U_1x\| \\
&\leq \alpha_1^{n,2}\|T_2U_{n,1}x - T_2U_1x\| + |\alpha_3^{n,2} - \alpha_3^2|\|x\| + |\alpha_1^{n,2} - \alpha_1^2|\|T_2U_1x\| \\
&\quad +\alpha_2^{n,2}\|U_{n,1}x - U_1x\| + |\alpha_2^{n,2} - \alpha_2^2|\|U_1x\| \\
&\leq \alpha_1^{n,2}\frac{1+k}{1-k}\|U_{n,1}x - U_1x\| + |\alpha_3^{n,2} - \alpha_3^2|\|x\| + |\alpha_1^{n,2} - \alpha_1^2|\|T_2U_1x\| \\
&\quad +\alpha_2^{n,2}\|U_{n,1}x - U_1x\| + |\alpha_2^{n,2} - \alpha_2^2|\|U_1x\|.
\end{aligned}$$

From boundedness, condition (2) and equation (3.3), we have

$$\lim_{n\to\infty} \|U_{n,2}x - U_2x\| = 0. \tag{3.4}$$

Similarly of the proof, we have

$$\lim_{n\to\infty} \|U_{n,N}x - U_Nx\| = 0. \tag{3.5}$$

Since $\|S_nx - Sx\| = \|U_{n,N}x - U_Nx\|$, we have

$$\lim_{n\to\infty} \|S_nx - Sx\| = 0. \tag{3.6}$$

This complete the proof. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, let T be a λ-strictly pseudo-contractive mapping and $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mappings of C into itself for some $k_i \in [0,1)$ and $k = \max\{k_i : i = 1,2,...,N\}$ which $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let S_n be the S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1^n, \alpha_2^n, ..., \alpha_N^n$ where $\alpha_j^n = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $I = [0,1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $k < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1,2,...,N-1, k < c \leq \alpha_1^{n,N} \leq 1, k \leq \alpha_3^{n,N} \leq d < 1, k \leq \alpha_2^{n,j} \leq e < 1$ for all $j = 1,2,...N$. Assume that set Ω of solution of general hierarchical problem (1.5) is nonempty. For a mapping $f : C \to C$ is a contraction with $\gamma \in (0,1)$, sequence $\{\alpha_n\}, \{\beta_n\}$ are two real number in $(0,1)$ and assume that the following condition hold:*

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$,

(2) $\sum_{n=1}^{\infty} \beta_n = \infty$,

(3) $\lim_{n\to\infty} \frac{1}{\beta_n}|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}| = 0$, *and* $\lim_{n\to\infty} \frac{1}{\alpha_n}|1 - \frac{\beta_{n-1}}{\beta_n}| = 0$

(4) $\sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty, \sum_{n=1}^{\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$ *for all* $j = \{1,2,3,\dots,N\}$,

(5) $\sum_{n=1}^{\infty} |\alpha_1^{n,j} - \alpha_1^j| < \infty, \sum_{n=1}^{\infty} |\alpha_2^{n,j} - \alpha_2^j| < \infty, \sum_{n=1}^{\infty} |\alpha_3^{n,j} - \alpha_3^j| < \infty$ *for all* $j = \{1,2,3,\dots,N\}$.

Then the sequence $\{x_n\}$ in (1.6) solve the following variational inequality:

$$\begin{cases} \tilde{x} \in \Omega \\ \langle (I-f)\tilde{x}, x - \tilde{x} \rangle \geq 0, \ \ \forall x \in \Omega. \end{cases} \tag{3.7}$$

Proof. From (1.6), let $y_n = \beta_n f(x_n) + (1-\beta_n)x_n$ and $x^* \in \Omega$ we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n S_n x_n + (1-\alpha_n) S_k y_n - x^* \\ &\leq \alpha_n \|S_n x_n - x^*\| + (1-\alpha_n)\|S_k y_n - x^*\| \\ &\leq \alpha_n \|x_n - x^*\| + (1-\alpha_n)\|y_n - x^*\|. \end{aligned} \tag{3.8}$$

Consider,

$$\begin{aligned} \|y_n - x^*\| &= \|\beta_n f(x_n) + (1-\beta_n)x_n - x^*\| \\ &\leq \|\beta_n \gamma \|x_n - x^*\| + \|f(x^*) - x^*\| + (1-\beta_n)\|x_n - x^*\| \\ &= (1 - (1-\gamma)\beta_n)\|x_n - x^*\| + \|f(x^*) - x^*\|. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \alpha_n \|x_n - x^*\| + (1-\alpha_n)\big[(1-(1-\gamma)\beta_n)\|x_n - x^*\| + \|f(x^*) - x^*\|\big] \\ &\leq \alpha_n \|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\| + (1-\alpha_n)\|f(x^*) - x^*\| \\ &= \|x_n - x^*\| + (1-\alpha_n)\|f(x^*) - x^*\| \\ &\leq \max\{\|x_0 - x^*\|, \|f(x^*) - x^*\|\}. \end{aligned}$$

Then $\{x_n\}$ and $\{y_n\}$ are bounded and hence $\{f(x_n)\}, \{S_n x_n\}, \{S_\lambda y_n\}$ are also. Now we consider

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|\beta_n f(x_n) - \beta_n f(x_{n-1}) + \beta_n f(x_{n-1}) - \beta_{n-1} f(x_{n-1}) + (1-\beta_n)x_n \\
&\quad -(1-\beta_n)x_{n-1} + (1-\beta_n)x_{n-1} - (1-\beta_{n-1})x_{n-1}\| \\
&\leq \beta_n \gamma \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|f(x_{n-1})\| + (1-\beta_n)\|x_n - x_{n-1}\| \\
&\quad +|\beta_n - \beta_{n-1}|\|x_{n-1}\| \\
&= (1-(1-\gamma)\beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|f(x_{n-1})\| + \|x_{n-1}\|).
\end{aligned}$$

From definition of $\{x_n\}$ and nonexpansiveness of S_n, we have

$$\begin{aligned}
\|x_n - x_{n-1}\| &= \|\alpha_n S_n x_n + (1-\alpha_n)S_\lambda y_n - \alpha_{n-1} S_{n-1} x_{n-1} + (1-\alpha_{n-1})S_\lambda y_{n-1}\| \\
&= \|\alpha_n S_n x_n - \alpha_n S_n x_{n-1} + \alpha_n S_n x_{n-1} - \alpha_{n-1} S_n x_{n-1} + \alpha_{n-1} S_n x_{n-1} \\
&\quad -\alpha_{n-1} S_{n-1} x_{n-1} + (1-\alpha_n) S_\lambda y_{n-1} + (1-\alpha_n) S_\lambda y_{n-1} - (1-\alpha_{n-1}) S_\lambda y_{n-1}\| \\
&\leq \alpha_n \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|S_n x_{n-1}\| + \alpha_{n-1}\|S_n x_{n-1} - S_{n-1} x_{n-1}\| \\
&\quad +(1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|S_\lambda y_{n-1}\| \\
&\leq \alpha_n \|x_n - x_{n-1}\| + (1-\alpha_n)\big[(1-(1-\gamma)\beta_n)\|x_n - x_{n-1}\| \\
&\quad +|\beta_n - \beta_{n-1}|(\|f(x_{n-1})\| + \|x_{n-1}\|)\big] + |\alpha_n - \alpha_{n-1}|(\|S_n x_{n-1}\| + \|S_\lambda y_{n-1}\|) \\
&\quad +\alpha_{n-1}\|S_n x_{n-1} - S_{n-1} x_{n-1}\| \\
&\leq \big[\alpha_n + (1-\alpha_n)(1-(1-\gamma)\beta_n)\big]\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|f(x_{n-1})\| + \|x_{n-1}\|) \\
&\quad +|\alpha_n - \alpha_{n-1}|(\|S_n x_{n-1}\| + \|S_\lambda y_{n-1}\|) + \alpha_{n-1}\|S_n x_{n-1} - S_{n-1} x_{n-1}\| \\
&= \big[1-(1-\gamma)\beta_n(1-\alpha_n)\big]\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|f(x_{n-1})\| + \|x_{n-1}\|) \\
&\quad +|\alpha_n - \alpha_{n-1}|(\|S_n x_{n-1}\| + \|S_\lambda y_{n-1}\|) + \alpha_{n-1}\|S_n x_{n-1} - S_{n-1} x_{n-1}\|.
\end{aligned}$$

Put $M = \sup\left\{\|f(x_{n-1})\|, \|S_n x_{n-1}\|, \|S_\lambda y_{n-1}\|\right\}, \quad n \geq 1$, it follows that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq [1-(1-\gamma)\beta_n((1-\alpha_n)]\|x_n - x_{n-1}\| + (|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|)M \\
&\quad +\alpha_{n-1}\|S_n x_{n-1} - S_{n-1} x_{n-1}\|.
\end{aligned}$$

Put $\delta_n = \|S_n x_{n-1} - S_{n-1} x_{n-1}\|$, from Lemma 3.1, we have $\Sigma_{n=1}^{\infty} \delta_n < \infty$, it follows that

$$\begin{aligned}
\frac{\|x_{n+1}-x_n\|}{\alpha_n} &= [1-(1-\gamma)\beta_n(1-\alpha_n)]\frac{\|x_n-x_{n-1}\|}{\alpha_n}+\frac{|\beta_n-\beta_{n-1}|}{\alpha_n}M+\frac{|\beta_n-\beta_{n-1}|}{\alpha_n}+\alpha_{n-1}\frac{\delta_n}{\alpha_n} \\
&= [1-(1-\gamma)\beta_n(1-\alpha_n)]\frac{\|x_n-x_{n-1}\|}{\alpha_{n-1}} \\
&\quad +[1-(1-\gamma)\beta_n(1-\alpha_n)]\left(\frac{\|x_n-x_{n-1}\|}{\alpha_n}-\frac{\|x_n-x_{n-1}\|}{\alpha_{n-1}}\right) \\
&\quad +\frac{|\beta_n-\beta_{n-1}|}{\alpha_n}M+\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}M+\alpha_{n-1}\frac{\delta_n}{\alpha_n} \\
&\leq [1-(1-\gamma)\beta_n(1-\alpha_n)]\frac{\|x_n-x_{n-1}\|}{\alpha_{n-1}} \\
&\quad +\left(\left|\frac{1}{\alpha_n}-\frac{1}{\alpha_{n-1}}\right|+\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}+\frac{|\beta_n-\beta_{n-1}|}{\alpha_n}+\frac{\delta_n}{\alpha_n}\right)M \\
&= [1-(1-\gamma)\beta_n(1-\alpha_n)]\frac{\|x_n-x_{n-1}\|}{\alpha_{n-1}} \\
&\quad +(1-\gamma)\beta_n(1-\alpha_n)\left\{\frac{M}{(1-\gamma)(1-\alpha_n)}\left(\frac{1}{\beta_n}\left|\frac{1}{\alpha_n}-\frac{1}{\alpha_{n-1}}\right|+\frac{1}{\beta_n}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}\right.\right. \\
&\quad \left.\left.+\frac{1}{\beta_n}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}+\frac{1}{\beta_n}\frac{|\beta_n-\beta_{n-1}|}{\alpha_n}+\frac{\delta_n}{\alpha_n}\right)\right\}.
\end{aligned}$$

From Lemma 2.5, we obtain that

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\alpha_n}=0. \tag{3.10}$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.11}$$

From (1.6) and (3.11), we have that

$$\lim_{n\to\infty}\|x_n-S_\lambda y_n\|=0. \tag{3.12}$$

It follows that

$$y_n-x_n=\beta_n(f(x_n)-x_n)\to 0. \tag{3.13}$$

It implies that

$$\|y_n-S_\lambda y_n\|\leq\|y_n-x_n\|+\|x_n-S_\lambda y_n\|\to 0. \tag{3.14}$$

Since the sequence $\{x_n\}$ and $\{y_n\}$ are also bounded. Thus there exists a subsequence of $\{y_n\}$, which is still denoted by $\{y_{n_i}\}$ which converges weakly to a point $\tilde{x}\in H$. Therefore, $\tilde{x}\in Fix(T)$ by (1.6), we observe that

$$x_{n+1}-x_n=\alpha_n(S_nx_n-x_n)+(1-\alpha_n)(S_\lambda y_n-y_n)+(1-\alpha_n)\beta_n(fx_n-x_n),$$

that is,

$$\frac{x_n - x_{n+1}}{\alpha_n} = (I - S_n)x_n + \frac{1-\alpha_n}{\alpha_n}(I - S_\lambda)y_n + \frac{\beta_n(1-\alpha_n)}{\alpha_n}(I - f)x_n.$$

Set $z_n = \frac{(x_n - x_{n+1})}{\alpha_n}$ for each $n \geq 1$, that is

$$z_n = (I - S_n)x_n + \frac{1-\alpha_n}{\alpha_n}(I - S_\lambda)y_n + \frac{\beta_n(1-\alpha_n)}{\alpha_n}(I - f)x_n.$$

Using monotonicity of $I - S_\lambda$ and $I - S_n$, we derive that, for all $u \in Fix(T)$,

$$\begin{aligned}
\langle z_n, x_n - u\rangle &= \langle (I - S_n)x_n, x_n - u\rangle + \frac{1-\alpha_n}{\alpha_n}\langle (I - S_\lambda)y_n - (I - S_\lambda)u, y_n - u\rangle \\
&\quad + \frac{1-\alpha_n}{\alpha_n}\langle (I - S_\lambda)y_n, x_n - y_n\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I - f)x_n, x_n - u\rangle \\
&\geq \langle (I - S_n)u, x_n - u\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I - f)x_n, x_n - u\rangle \\
&\quad + \frac{(1-\alpha_n)\beta_n}{\alpha_n}\langle (I - S_\lambda y_n, x_n - fx_n)\rangle \\
&= \langle (I - S)u, x_n - u\rangle + \langle (S - S_n)u, x_n - u\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I - f)x_n, x_n - u\rangle \\
&\quad + \frac{(1-\alpha_n)\beta_n}{\alpha_n}\langle (I - S_\lambda)y_n, x_n - fx_n\rangle.
\end{aligned}$$

But, since $z_n \to 0$, $\frac{\beta_n}{\alpha_n} \to 0$ and $\lim_{n\to\infty} \|S_n u - Su\| = 0$, it follows from the above inequality that

$$\limsup_{n\to\infty}\langle (I - S)u, x_n - u\rangle \leq 0, \quad \forall u \in Fix(T).$$

It suffices to guarantee that $\omega_w(x_n) \subset \Omega$. As a matter of fact, if we take any $x^* \in \omega_w(x_n)$, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^*$. Therefore, we have

$$\langle (I - S)u, x^* - u\rangle = \lim_{j\to\infty}\langle (I - S)u, x_{n_j} - u\rangle \leq 0, \quad \forall u \in Fix(T).$$

Note that $x^* \in Fix(T)$. Hence x^* solves the following problem:

$$\begin{cases} x^* \in Fix(T) \\ \langle (I - S)u, x^* - u\rangle \geq 0, \ \forall u \in Fix(T). \end{cases}$$

It is obvious that this equivalent to the problem (1.5) by Lemma 3.2, we have $S_n \to S$ uniformly in any bounded set. Thus $x^* \in \Omega$. Let $\tilde{x}$ be the solution of the variational inequality (3.7), by Lemma 2.7 we have $\tilde{x}$ is unique. Now, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (I - f)\tilde{x}, x_n - \tilde{x}\rangle = \lim_{i\to\infty}\langle (I - f)\tilde{x}, x_{n_i} - \tilde{x}\rangle.$$

Without loss of generality, we can assume that $x_{n_i} \rightharpoonup x^*$. Then $x^* \in \Omega$. Therefore, we have

$$\limsup_{n\to\infty}\langle (I-f)\tilde{x}, x_n-\tilde{x}\rangle \;=\; \langle (I-f)\tilde{x}, x^*-\tilde{x}\rangle \;\geq 0.$$

This completes the proof. □

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H, let T be a λ-strictly pseudo-contractive mapping and $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mappings of C into itself for some $k_i \in [0,1)$ and $k = \max\{k_i : i = 1,2,...,N\}$ which $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let S_n be the S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1^n, \alpha_2^n, ..., \alpha_N^n$ where $\alpha_j^n = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $I = [0,1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $k < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1,2,...,N-1, k < c \leq \alpha_1^{n,N} \leq 1, k \leq \alpha_3^{n,N} \leq d < 1, k \leq \alpha_2^{n,j} \leq e < 1$ for all $j = 1,2,...N$. Assume that set Ω of solution of generalized hierarchical problem (1.5) is nonempty. For a mapping $f : C \to C$ is a contraction with $\gamma \in (0,1)$, sequence $\{\alpha_n\}, \{\beta_n\}$ are two real number in $(0,1)$ and assume that the following condition hold:*

1. *(1)* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\lim_{n\to\infty}\frac{\beta_n}{\alpha_n} = 0$,
2. *(2)* $\sum_{n=1}^{\infty}\beta_n = \infty$,
3. *(3)* $\lim_{n\to\infty}\frac{1}{\beta_n}|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}| = 0$ *and* $\lim_{n\to\infty}\frac{1}{\alpha_n}|1 - \frac{\beta_{n-1}}{\beta_n}| = 0$,
4. *(4)* $\sum_{n=1}^{\infty}|\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty, \sum_{n=1}^{\infty}|\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$ *for all* $j = \{1,2,3,\ldots,N\}$,
5. *(5)* $\sum_{n=1}^{\infty}|\alpha_1^{n,j} - \alpha_1^j| < \infty, \sum_{n=1}^{\infty}|\alpha_2^{n,j} - \alpha_2^j| < \infty, \sum_{n=1}^{\infty}|\alpha_3^{n,j} - \alpha_3^j| < \infty$ *for all* $j = \{1,2,3,\ldots,N\}$,
6. *(6) there exists a constant $d > 0$ such that $\|x - S_\lambda x\| \geq \rho Dist(x, F(S_\lambda))$, where*

$$Dist(x, F(S_\lambda)) = \inf_{y\in F(S_\lambda)}\|x-y\|.$$

Then the sequence $\{x_n\}$ difined by (1.6) converges strongly to a point $\widetilde{x} \in Fix(T)$, which solve the variational inequality problem (3.7).

Proof. From (1.6), we have

$$x_{n+1} - \tilde{x} = \alpha_n(S_n x_n - S_n\tilde{x}) + \alpha_n(S_n\tilde{x} - \tilde{x}) + (1-\alpha_n)(S_\lambda y_n - \tilde{x}).$$

Thus we have

$$\begin{aligned}\|x_{n+1}-\tilde{x}\|^2 &\leq \|\alpha_n(S_n x_n - S_n\tilde{x}) + (1-\alpha_n)(S_\lambda y_n - \tilde{x})\|^2 + 2\alpha_n\langle S_n\tilde{x}-\tilde{x}, x_{n+1}-\tilde{x}\rangle \\ &\leq (1-\alpha_n)\|S_\lambda y_n - \tilde{x}\|^2 + \alpha_n\|S_n x_n - S_n\tilde{x}\|^2 + 2\alpha_n\langle S_n\tilde{x}-\tilde{x}, x_{n+1}-\tilde{x}\rangle \quad (3.15) \\ &\leq (1-\alpha_n)\|y_n-\tilde{x}\|^2 + \alpha_n\|x_n-\tilde{x}\|^2 + 2\alpha_n\langle S_n\tilde{x}-\tilde{x}, x_{n+1}-\tilde{x}\rangle.\end{aligned}$$

Now we consider

$$\begin{aligned} \|y_n-\tilde{x}\|^2 &= \|(1-\beta_n)(x_n-\tilde{x})+\beta_n(fx_n-f\tilde{x})+\beta_n(f\tilde{x}-\tilde{x})\|^2 \\ &\le \|(1-\beta_n)(x_n-\tilde{x})+\beta_n(fx_n-f\tilde{x})\|^2+2\beta_n\langle f\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle \\ &\le (1-\beta_n)\|x_n-\tilde{x}\|^2+\beta_n\|(fx_n-f\tilde{x})\|^2+2\beta_n\langle f\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle \\ &\le (1-\beta_n)\|x_n-\tilde{x}\|^2+\beta_n\gamma^2\|x_n-\tilde{x}\|^2+2\beta_n\langle f\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle \\ &= \big[1-\big(1-\gamma^2\big)\beta_n\big]\|x_n-\tilde{x}\|^2+2\beta_n\langle f\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle. \end{aligned} \tag{3.16}$$

Substituting (3.16) into (3.15), we get

$$\begin{aligned} \|x_{n+1}-\tilde{x}\|^2 &= \alpha_n\|x_n-\tilde{x}\|^2+(1-\alpha_n)\big[1-\big(1-\gamma^2\big)\beta_n\big]\|x_n-\tilde{x}\|^2 \\ &\quad +2\beta_n(1-\alpha_n)\langle f\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle+2\alpha_n\langle S_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle \\ &= \big[1-\big(1-\gamma^2\big)\beta_n(1-\alpha_n)\big]\|x_n-\tilde{x}\|^2+2\beta_n(1-\alpha_n)\langle f\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle \\ &\quad +2\alpha_n\langle S_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle \\ &= \big[1-\big(1-\gamma^2\big)\beta_n(1-\alpha_n)\big]\|x_n-\tilde{x}\|^2+\big(1-\gamma^2\big)\beta_n(1-\alpha_n) \\ &\quad \times\Big\{\frac{1}{1-\gamma^2}\langle f\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle+\frac{2}{(1-\gamma^2)(1-\alpha_n)}\times\frac{\alpha_n}{\beta_n}\langle S_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle\Big\} \end{aligned} \tag{3.17}$$

By Theorem 3.3, we note that every weak cluster point of the sequence $\{x_n\}$ is in Ω. Since $y_n-x_n\to 0$, then every weak cluster point of $\{y_n\}$ is also in Ω. Consequently, since $\tilde{x}=proj_{\Omega}(f\tilde{x})$, we easily have

$$\limsup_{n\to\infty}\langle f\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle\le 0. \tag{3.18}$$

On the other hand, we observe that

$$\langle S_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle=\Big\langle S_n\tilde{x}-\tilde{x},proj_{Fix(S_\lambda)}x_{n+1}-\tilde{x}\Big\rangle+\Big\langle S_n\tilde{x}-\tilde{x},x_{n+1}-proj_{Fix(S_\lambda)}x_{n+1}\Big\rangle$$

Since $\tilde{x}$ is a solution of the problem(1.5) and $proj_{Fix(S_\lambda)}x_{n+1}\in Fix(S_\lambda)$, we have

$$\big\langle S_n\tilde{x}-\tilde{x},proj_{Fix(S_\lambda)}x_{n+1}-\tilde{x}\big\rangle\le 0.$$

Thus it follows that

$$\begin{aligned} \langle S_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle &\le \langle S_n\tilde{x}-\tilde{x},x_{n+1}-proj_{Fix(S_\lambda)}x_{n+1}\rangle \\ &\le \|S_n\tilde{x}-\tilde{x}\|\big\|x_{n+1}-proj_{Fix(S_\lambda)}x_{n+1}\big\| \\ &= \|S_n\tilde{x}-\tilde{x}\|\times Dist(x_{n+1},Fix(S_\lambda)) \\ &\le \frac{1}{\rho}\|S_n\tilde{x}-\tilde{x}\|\|x_{n+1}-S_\lambda x_{n+1}\|. \end{aligned}$$

We note that

$$\begin{aligned} \|x_{n+1}-S_\lambda x_{n+1}\| &\le \|x_{n+1}-S_\lambda x_n\|+\|S_\lambda x_n-S_\lambda x_{n+1}\| \\ &\le \alpha_n\|S_nx_n-S_\lambda x_n\|+(1-\alpha_n)\|S_\lambda y_n-S_\lambda x_n\|+\|x_{n+1}-x_n\| \\ &\le \alpha_n\|S_nx_n-S_\lambda x_n\|+\|y_n-x_n\|+\|x_{n+1}-x_n\| \\ &\le \alpha_n\|S_nx_n-S_\lambda x_n\|+\beta_n\|fx_n-x_n\|+\|x_{n+1}-x_n\|. \end{aligned}$$

Hence we have

$$\begin{aligned}\frac{\alpha_n}{\beta_n}\langle S_n\tilde{x}-\tilde{x}, x_{n+1}-\tilde{x}\rangle \quad \leq \quad & \frac{\alpha_n^2}{\beta_n}\left(\frac{1}{\rho}\|S_n\tilde{x}-\tilde{x}\|\|S_nx_n-S_\lambda x_n\|\right)\\ & +\alpha_n\left(\frac{1}{\rho}\|S_n\tilde{x}-\tilde{x}\|\|fx_n-x_n\|\right)\\ & +\frac{\alpha_n^2}{\beta_n}\frac{\|x_{n+1}-x_n\|}{\alpha_n}\left(\frac{1}{\rho}\|S_n\tilde{x}-\tilde{x}\|\right).\end{aligned}$$

From Theorem 3.3 we have $\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\alpha_n} = 0$. And then, we note that $\{\frac{1}{\rho}\|S_n\tilde{x}-\tilde{x}\|\|S_nx_n-S_\lambda x_n\|\}$, $\{\frac{1}{\rho}\|S_n\tilde{x}-\tilde{x}\|\|fx_n-x_n\|\}$, and $\{\frac{1}{\rho}\|S_n\tilde{x}-\tilde{x}\|\}$ are all bounded. Hence it follows from (1) and the above inequality that

$$\limsup_{n\to\infty}\frac{\alpha_n}{\beta_n}\langle S_n\tilde{x}-\tilde{x}, x_{n+1}-\tilde{x}\rangle \leq 0.$$

Finally, by (3.17) and Lemma 2.5, we conclude that the sequence $\{x_n\}$ converges strongly to a point $\tilde{x}\in Fix(S_\lambda)=Fix(T)$. This completes the proof. □

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Existence Theorems for Nonconvex Variational Inequalities Problems

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Abstract

In this paper, we prove the existence theorem for a mapping defined by $T = T_1 + T_2$ when T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping, we have a mapping T is Lipschitz continuous but not strongly monotone mapping. This work is extend and improve the result of N. Petrot [17].

Mathematics Subject Classification: 46C05, 47D03, 47H09, 47H10, 47H20

Keywords: asymptotically k-strict pseudo-contractive mapping in the intermediate sense; Mann's iteration method

1 Introduction

The theory of variational inequalities is a branch of the mathematical sciences dealing with general equilibrium problems. It has a wide range of applications in economics, operations research, industry, physical, and engineering sciences. Many research papers have been written lately, both on the theory and applications of this field. Important connection with main areas of pure and applied science have been made, see for example [2, 5, 6] and the references cited therein.

Variational inequalities theory, which was introduce by Stampacchia [18], provides us with a simple, natural general and unified framework to study a wide class of problems arising in pure and applied science. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solutions to important classes of problems. On the other hand, it also enables us to develop highly efficient and powerful new numerical methods for solving, for example, obstacle, unilateral, free, moving, and complex equilibrium problems.

In this work we consider the condition for existence solution of variational inequalities problems in nonconvex sets. We will proved that a mapping $T = T_1+T_2$ when T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping has a solution on nonconvex satisfying uniformly r-prox regular subset of Hilbert space. The result extended and improved result of N. Petrot [17].

Let H be a real Hilbert space and let C be a nonempty closed subset of H. A mapping T of C into H is called $\gamma -$ *strongly monotone* if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \gamma\|x - y\|^2, \tag{1}$$

for all $x, y \in C$. T is called $\mu -$ *Lipschitz* if there exists a constant $\mu > 0$ such that

$$\|Tx - Ty\| \leq \mu\|x - y\|, \tag{2}$$

for all $x, y \in C$.

2 Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ respectively. Let C be a closed convex subset of H, let P_C be the matric projection of H onto C i.e. for $x \in H, P_Cx$ satisfies the property

$$\|x - P_Cx\| = \min_{y\in C}\|x - y\|.$$

It is know that P_C is nonexpansive. Further, for $x \in H$ and $z \in C$

$$z = P_C x \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \forall y \in C.$$

Definition 2.1. *Let $u \in H$ be a point not lying in C. A point $v \in C$ is called a closest point or a projection of u onto C if $d_C(u) = \|u - v\|$ when d_C is a usual distance. The set of all such closest points is denoted by $proj_C(u)$; that is,*

$$proj_C(u) = \{v \in C : d_C(u) = \|u - v\|\}. \quad (3)$$

Definition 2.2. *Let C be a subset of H. The proximal normal cone to C at x is given by*

$$N_C^P(x) = \{z \in H : \exists \rho > 0; x \in proj_C(x + \rho z)\} \quad (4)$$

The following characterization of $N_C^P(x)$ can be found in [3].

Lemma 2.3. *Let C be a closed subset of a Hilbert space H. Then*

$$z \in N_C^P(x) \text{ if and only if } \exists \sigma > 0, \langle z, y - x \rangle \leq \sigma \|y - x\|^2, \quad \forall y \in C. \quad (5)$$

Clark et al. [4] and Poliquin et al. [16] have introduced and atudied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class or uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems, and differential inclusions.

Definition 2.4. *For a given $r \in (0, +\infty]$, a subset C of H is said to be uniformly prox-regular with respect to r if, for all $\overline{x} \in C$ and for all $0 \neq z \in N_C^P(x)$, one has*

$$\langle \frac{z}{\|z\|}, x - \overline{x} \rangle \leq \frac{1}{2r} \|x - \overline{x}\|^2, \quad \forall x \in C. \quad (6)$$

It is well known that a closed subset of a Hilbert space is convex if and only if it is proximally smooth of radius $r > 0$. Thus, in definition 2.4, in the case of $r = \infty$, the uniform r-prox-regularity C is equivalent to convexity of C. Then, it is clear that the class of uniformly prox-regular sets is sufficiently large to include the class p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets, and many other nonconvex sets; see [4, 16].

Lemma 2.5. *[17] Let C be a nonempty closed subset of H, $r \in (0, +\infty]$ and set $C_r; = \{x \in H : d(x, C) < r\}$. If C is uniform r-uniformly prox-regular, then the following hold:*

(1) for all $x \in C_r, proj_C(x) \neq \emptyset$,

(2) for all $s \in (0, r), proj_C$ *is Lipschitz continuous with constant* $\frac{r}{r-s}$ *on* C_s,

(3) the proximal normal cone is closed as a set-valued mapping.

For a given nonlinear operator T, we consider the problem of finding $u \in C_r$ such that

$$\langle Tu, v - u \rangle \geq 0 \quad \forall\, v \in C_r \tag{7}$$

which is called the nonconvex variational inequality. For the existence of a solution and other aspects of the nonconvex variational inequalities and their generalization, see [9, 15].

Similarly, if C_r is a nonconvex (uniformly prox-regular) set, then problem (7) is equivalent to finding $u \in C_r$ such that

$$0 \in Tu + N^P_{C_r}(u) \tag{8}$$

where $N^P_{C_r}(u)$ denotes the normal cone of C_r at u in the sense of nonconvex analysis. Problem (8) is called the nonconvex variational inclusion problem associated with nonconvex variational inequality (7).

3 Main Result

Let H be a real Hilbert space, and let C be a nonempty closed subset of H. In this section, will consider the following problem: find $x^* \in C$ such that

$$-Tx^* \in N^P_C(x^*). \tag{9}$$

The problem of type (9) was studied by Noor [7] but in a finite dimension Hilbert space setting. In 2010 [17] Petrot intend to consider the problem (9) in an infinite dimension Hilbert space for a mapping T satisfied μ-Lipschitz continuous and γ-strongly monotone. In this section we extended the result of [17] Petrot to a mapping $T = T_1 + T_2$ with T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. We see that T is Lipschitz continuous but not strongly monotone mapping. To do this, the following remark is useful.

Remark 3.1. *Let* T_1 *be a* μ_1*-Lipschitz continuous and* γ*-strongly monotone mapping, and let* T_2 *be a* μ_2*-Lipschitz continuous mapping. Then the function* $f : (1, M) \to (0, \infty)$ *which defined by*

$$f(t) = \frac{\sqrt{(t\gamma - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t^2 - 1)}}{t(\mu_1^2 - \mu_2^2)}, \forall\ t \in M,$$

where $M = \frac{\gamma\mu_2 + \sqrt{(\mu_1^2 - \gamma^2)(\mu_1^2 - \mu_2^2)}}{\gamma^2 - (\mu_1^2 - \mu_2^2)}$.

In this work, we have to assume that $\mu_2 < \mu_1$. Thus, from now on, without loss of generality we will always assume that $\mu_2 < \mu_1$.

Theorem 3.2. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T_1, T_2 : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. If $T = T_1 + T_2$ and $0 < \delta_{T(C)} \leq \gamma r$, then the problem (9) has a solution.*

Proof. We first, defined a function $h : [1, M) \to (0, \infty)$ which is defined by

$$h(x) = \frac{r(t-1)}{t\delta_{T(C)}} + f(t), \forall t \in [1, M). \tag{10}$$

We see that the net $\{t_s\}_{s\in(0,r)}$ which is defined by $t_s = \frac{r}{r-s}$ converges to 1 as $s \downarrow 0$. It follows that $h(t) \downarrow \frac{\gamma - \mu}{\mu_1^2 - \mu_2^2}$ as $t_s \downarrow 1$, we can find $s^* \in (0, r)$ such that $\frac{\gamma - \mu_2}{\mu_1^2 - \mu_2^2} < h(t)$, Then we have

$$\frac{t_{s^*}\gamma - {}_2}{t_{s^*}(\mu_1^2 - \mu_2^2)} - f(t_{s^*}) < h(t_{s^*}) - f(t_{s^*}) = \frac{r(t_{s^*} - 1)}{t_{s^*}\delta_{T(C)}} = \frac{s^*}{\delta_{T(C)}}.$$

Now, we choose a fixed positive real number ρ such that

$$\frac{t_s\gamma - \mu_2}{\mu_1^2 - \mu_2^2} - f(t_{s^*}) < \rho < \min\{\frac{t_s\gamma - \mu_2}{\mu_1^2 - \mu_2^2} + f(t_{s^*}), \frac{s^*}{\delta_{T(C)}}\}. \tag{11}$$

Next, for an element $x_0 \in C$ and use an induction process to obtain a sequence $\{x_n\} \subset C$ satisfying

$$x_{n+1} = proj_C(x_n - \rho T x_n), \quad \forall n = 0, 1, 2, ... \tag{12}$$

Consequently, from (12) and Lemma 2.5, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|proj_C(x_n - \rho T x_n) - proj_C(x_{n-1} - \rho T x_{n-1})\| \\ &= t_s\|(x_n - \rho T x_n) - (x_{n-1} - \rho T x_{n-1})\| \\ &= t_s\|(x_n - x_{n-1}) - \rho(T x_n - T x_{n-1})\| \\ &\leq t_s[\|x_n - x_{n-1} - \rho(T_1 x_n - T_1 x_{n-1})\| + \rho\|T_2 x_n - T_2 x_{n-1}\|] \\ &\leq t_s[\|x_n - x_{n-1} - \rho(T_1 x_n - T_1 x_{n-1})\| + \rho\mu_2\|x_n - x_{n-1}\|]. \end{aligned} \tag{13}$$

Since the mapping T_1 is γ-strongly monotone and μ_1-Lipschitz continuous, we obtain

$$\begin{aligned}\|x_n - x_{n-1} - \rho(T_1x_n - T_1x_{n-1})\|^2 &= \|x_n - x_{n-1}\|^2 - 2\rho\langle x_n - x_{n-1}, T_1x_n - T_1x_{n-1}\rangle \\ &\quad +\rho^2\|T_1x_n - T_1x_{n-1}\|^2 \\ &\leq \|x_n - x_{n-1}\|^2 - 2\rho\gamma\|x_n - x_{n-1}\|^2 + \rho^2\mu_1^2\|x_n - x_{n-1}\|^2 \\ &= (1 - 2\rho\gamma + \rho\mu_1^2)\|x_n - x_{n-1}\|^2.\end{aligned} \tag{14}$$

It follows that

$$\|x_n - x_{n-1} - \rho(T_1x_n - T_1x_{n-1})\| \leq \sqrt{1 - 2\rho\gamma + \rho\mu_1^2}\|x_n - x_{n-1}\|^2. \tag{15}$$

From (14) and (15), we get

$$\|x_{n+1} - x_n\| \leq t_s(\rho\mu_2 + \sqrt{1 - 2\rho\gamma + \rho\mu_1^2})\|x_n - x_{n-1}\|. \tag{16}$$

Now, we see that for the choice of ρ, we know that $\{x_n\}$ is a Cauchy sequence in C. Then $\{x_n\}$ is a convergence sequence, it follows that, if $x_n \to x^*$ as $n \to \infty$ we have $x^* \in proj_C(x^* + \rho(-Tx^*))$ for some $\rho > 0$. From definition 2.2, we have $-Tx^* \in N_C^P(x^*)$. This completes the proof. □

Corollary 3.3. *[17] Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T : C \to H$ be a γ-strongly monotone mapping and μ-Lipschitz continuous mapping. If $0 < \delta_{T(C)} \leq \gamma r$, then the problem (9) has a solution.*

Proof. From Theorem 3.2, if $T_2 \equiv 0$ we have a result. □

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Strong Convergence Theorems of Iterative Algorithm for Nonconvex Variational Inequalities*

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Abstract

In this work, we suggest and analyze an iterative scheme for solving the system of nonconvex variational inequalities by using projection technique. We prove strong convergence of iterative scheme to the solution of the system of nonconvex variational inequalities requires to the modified mapping T which is Lipschitz continuous but not strongly monotone mapping. Our result can be viewed and improvement the result of N. Petrot [18].

1 Introduction

The theory of variational inequalities is a branch of the mathematical sciences dealing with general equilibrium problems. It has a wide range of applications in economics, operations research, industry, physical, and engineering sciences. Many research papers have been written lately, both on the theory and applications of this field. Important connection with main areas of pure and applied science have been made, see for example [3, 6, 7] and the references cited therein.

Variational inequalities theory, which was introduce by Stampacchia [19], provides us with a simple, natural general and unified framework to study a wide class of problems arising in pure and applied science. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solutions to important classes of problems. On the other hand, it also enables us to develop highly efficient and powerful new numerical methods for solving, for example, obstacle, unilateral, free, moving, and complex equilibrium problems.

In 2010, N. Petrot [18], introduced some existence theorems and provide the conditions for existence solutions of the variational inequalities problems in nonconvex setting and prove the strongly monotonic assumption of the mapping may not need for the existence of solutions.

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In this work we consider the iterative scheme for modified mapping is Lipschitz continuous but not strongly monotone mapping and we can prove strong convergence of iterative to the solution of the system of nonconvex variational inequalities.

2 Preliminaries

Let C be a closed subset of a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ respectively. Let us recall the following well-known definitions and some auxiliary results of nonlinear convex analysis and nonsmooth analysis.

Definition 2.1. Let $u \in H$ be a point not lying in C. A point $v \in C$ is called a closest point or a projection of u onto C if $d_C(u) = \|u - v\|$ when d_C is a usual distance. The set of all such closest points is denoted by $P_C(u)$; that is,

$$P_C(u) = \{v \in C : d_C(u) = \|u - v\|\}. \tag{2.1}$$

Definition 2.2. Let C be a subset of H. The proximal normal cone to C at x is given by

$$N_C^P(x) = \{z \in H : \exists \rho > 0; x \in P_C(x + \rho z)\}. \tag{2.2}$$

The following characterization of $N_C^P(x)$ can be found in [4].

Lemma 2.3. *Let C be a closed subset of a Hilbert space H. Then*

$$z \in N_C^P(x) \text{ if and only if } \exists \sigma > 0, \langle z, y - x\rangle \leq \sigma\|y - x\|^2, \quad \forall y \in C. \tag{2.3}$$

Clark et al. [5] and Poliquin et al. [17] have introduced and atudied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class or uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems, and differential inclusions.

Definition 2.4. For a given $r \in (0, +\infty]$, a subset C of H is said to be uniformly prox-regular with respect to r if, for all $\overline{x} \in C$ and for all $0 \neq z \in N_C^P(x)$, one has

$$\langle \frac{z}{\|z\|}, x - \overline{x}\rangle \leq \frac{1}{2r}\|x - \overline{x}\|^2, \quad \forall x \in C. \tag{2.4}$$

It is well known that a closed subset of a Hilbert space is convex if and only if it is proximally smooth of radius $r > 0$. Thus, in Definition 2.4, in the case of $r = \infty$, the uniform r-prox-regularity C is equivalent to convexity of C. Then, it is clear that the class of uniformly prox-regular sets is sufficiently large to include the class p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets, and many other nonconvex sets; see [5, 17].

Let C_r be a uniformly r-prox-regular(nonconvex) set. For given nonlinear mappings $T : C_r \to H$, we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} \langle \rho T y^* + x^* - y^*, x - x^*\rangle \geq 0, \forall x \in C_r, \rho > 0 \\ \langle \eta T x^* + y^* - x^*, x - y^*\rangle \geq 0, \forall x \in C_r, \eta > 0, \end{aligned} \tag{2.5}$$

which is called the *system of nonconvex variational inequalities*.

It is worth mentioning that if $T_1 = T_2 = T, x^* = y^* = u$ and $\rho = \eta$, then problem (2.5) is equivalent to finding $u \in C_r$ such that

$$\langle Tu, v - u\rangle \geq 0, \forall v \in C_r, \tag{2.6}$$

which is known as *nonconvex variational inequalities* introduced and studied by Bounkhel et. al. [1] and Noor [9, 10].

It is known that problem (2.6) is equivalent to finding $u \in C_r$ such that

$$0 \in Tu + N^P_{C_r}(u), \tag{2.7}$$

which $N^P_{C_r}(u)$ denote the normal cone of C_r at u. The problem (2.7) is called the *variational inclusion associated with nonconvex variational inequalities* (2.6).

Lemma 2.5. *[18] Let C be a nonempty closed subset of H, $r \in (0, +\infty]$ and set $C_r; = \{x \in H : d(x, C) < r\}$. If C is uniform r-uniformly prox-regular, then the following hold:*

(1) for all $x \in C_r, P_C(x) \neq \emptyset$,

(2) for all $s \in (0, r), P_C$ is Lipschitz continuous with constant $t_s = \frac{r}{r-s}$ on C_s,

(3) the proximal normal cone is closed as a set-valued mapping.

Let C be a closed subset of a real Hilbert space H. A mapping $T : C \to H$ is called $\gamma -$ *strongly monotone* if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \gamma \|x - y\|^2, \tag{2.8}$$

for all $x, y \in C$. A mapping T is called $\mu -$ *Lipschitz* if there exists a constant $\mu > 0$ such that

$$\|Tx - Ty\| \leq \mu\|x - y\|, \tag{2.9}$$

for all $x, y \in C$.

Lemma 2.6. *In a real Hilbert space H, there holds the inequality*

1. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \quad x, y \in H$ *and* $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$,
2. $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \forall t \in [0, 1]$.

3 Main Results

In this section we first establish the equivalent between the system of nonconvex variational inequalities (2.5) anf the fixed point problem with the projection technique.

Lemma 3.1. *For given $x^*, y^* \in C_r$ is a solution of system of nonconvex variational inequalities (2.5), if and only if*

$$\begin{aligned} x^* &= P_C[y^* - \rho T y^*], \\ y^* &= P_C[x^* - \eta T x^*], \end{aligned} \tag{3.1}$$

where P_C is the projection of H onto the uniformly prox-regular set C_r.

Proof. Let $x^*, y^* \in C_r$ be a solution of (2.5), from (2.7), for a constant $\rho > 0$, we have

$$0 \in \rho T_1 y^* + x^* - y^* + \rho N^P_{C_r}(x^*) = (I + \rho N^P_{C_r})(x^*) - [y^* - \rho T_1 y^*]$$

if and only if

$$x^* = (I + \rho N^P_{C_r})^{-1}[y^* - \rho T_1 y^*] = P_C[y^* - \rho T_1 y^*],$$

where we have used the well-known fact that $P_C = (I + \rho N^P_{C_r})^{-1}$.

Similarly, we obtain

$$y^* = P_C[x^* - \eta T_2 x^*].$$

This prove our assertions. □

Algorithm 3.2. *For arbitrarily chosen initial points $x_0, y_0 \in C_r$, the sequence $\{x_n\}$ and $\{y_n\}$ in the following way:*

$$\begin{aligned} y_n &= P_C[x_n - \eta T x_n], \eta > 0 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n P_C[y_n - \rho T y_n], \rho > 0, \end{aligned} \tag{3.2}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$.

Remark 3.3. [18] Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T_1, T_2 : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. Let

$$\xi = r[\mu_1^2 - \gamma \frac{\mu_2 - \sqrt{(\mu_1^2 - \gamma\mu_2)^2 - \mu_1^2(\gamma - \mu_2)^2}}{\mu_1^2}] \tag{3.3}$$

then for each $s \in (0, \xi)$, we have

$$\gamma t_s - \mu_2 > \sqrt{(\mu_1^2 - \mu_2^2)(t_s^2 - 1)}, \tag{3.4}$$

where $t_s = \frac{r}{r-s}$.

In this paper, we may assume that $M^{\rho,\eta}\delta_{T(C)} < \xi$, we see that for any $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$ it satisfy the inequality 3.4 too. where $M^{\rho,\eta} = \min\{\rho, \eta\}, \delta_{T(C)} = \sup\{\|u - v\| : u, v \in T(C)\}$.

Now, we suggest and analyze the following explicit projection method (3.2) for solving the system of nonconvex variational inequalities (2.5). Thus, from now on, without loss of generality we will always assume that $\mu_2 < \mu_1$.

Theorem 3.4. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T_1, T_2 : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. If $T = T_1 + T_2$ and there exists constant $\rho, \eta > 0$ and $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$, such that*

$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \triangle_{t_s} < \rho, \eta < \min\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \triangle_{t_s}, \frac{1}{t_s \mu_2}\}, \tag{3.5}$$

where $\triangle_{t_s} = \frac{\sqrt{(\gamma t_s - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t_s^2 - 1)}}{t_s(\mu_1^2 - \mu_2^2)}$. If the sequence of positive real number $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, then the sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 3.2 converge to a solution of the system of nonconvex variational inequalities (2.5).

Proof. Let $x^*, y^* \in C_r$ be a solution of (2.5) and from Lemma 3.1, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n P_C[y_n - \rho T y_n] - x^*\| \\ &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(P_C[y_n - \rho T y_n] - P_C[y^* - \rho T y^*])\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|P_C[y_n - \rho T y_n] - P_C[y^* - \rho T y^*]\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n t_s\|(y_n - \rho T y_n) - (y^* - \rho T y^*)\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n t_s[\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| + \rho\|(T_2 y_n - T_2 y^*)\|.] \end{aligned} \tag{3.6}$$

From T_1 are both μ_1-Lipschitz continuous and γ-strongly monotone mapping and from Lemma 2.6, we obtain

$$\begin{aligned} \|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\|^2 &= \|y_n - y^*\|^2 - 2\rho\langle y_n - y^*, T_1 y_n - T_1 y^*\rangle + \rho^2\|T_1 y_n - T_1 y^*\|^2 \\ &\le \|y_n - y^*\|^2 - 2\rho\gamma\|y_n - y^*\|^2 + \rho^2\mu_1^2\|y_n - y^*\|^2 \\ &= (1 - 2\rho\gamma + \rho^2\mu_1^2)\|y_n - y^*\|^2. \end{aligned}$$

It follows that

$$\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| \leq \sqrt{1 - 2\rho\gamma + \rho^2\mu_1^2}\|y_n - y^*\|. \tag{3.7}$$

On the other hand, from T_2 is μ_2-Lipschitz continuous, we have

$$\|T_2 y_n - T_2 y^*\| \leq \mu_2\|y_n - y^*\|. \tag{3.8}$$

Thus, by (3.6), (3.7) and (3.8), we have

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n t_s(\rho\mu_2 + \sqrt{1 - 2\rho\gamma + \rho^2\mu_1^2})\|y_n - y^*\|. \tag{3.9}$$

Similarly, we have

$$\begin{aligned} \|y_n - y^*\| &= \|P_C[x_n - \eta T x_n] - y^*\| \\ &= \|P_C[x_n - \eta T x_n] - P_C[x^* - \eta T x^*]\| \\ &\leq t_s\|(x_n - \eta T x_n) - (x^* - \eta x^*)\| \\ &\leq t_s[\|(x_n - x^*) - \eta(T_1 x_n - T_1 x^*)\| + \eta\|T_2 x_n - T_2 x^*\|]. \end{aligned} \tag{3.10}$$

Similarly, from T_1 are both μ_1-Lipschitz continuous and γ-strongly monotone mapping, we obtain

$$\begin{aligned} \|(x_n - x^*) - \eta(T_1 x_n - T_1 x^*)\|^2 &= \|x_n - x^*\|^2 - 2\eta\langle x_n - x^*, T_1 x_n - T_1 x^*\rangle + \eta^2\|T_1 x_n - T_1 x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - 2\eta\gamma\|x_n - x^*\|^2 + \eta^2\mu_1^2\|x_n - x^*\|^2 \\ &= (1 - 2\eta\gamma + \eta^2\mu_1^2)\|x_n - x^*\|^2. \end{aligned}$$

It follows that

$$\|(x_n - x^*) - \eta(T_1 x_n - T_1 x^*)\| \leq \sqrt{1 - 2\eta\gamma + \eta^2\mu_1^2}\|x_n - x^*\|. \tag{3.11}$$

On the other hand, from T_2 is μ_2-Lipschitz continuous, we have

$$\|T_2 x_n - T_2 x^*\| \leq \mu_2\|x_n - x^*\|. \tag{3.12}$$

Thus, by (3.10), (3.11) and (3.12), we have

$$\|y_n - y^*\| \leq t_s(\eta\mu_2 + \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_1^2})\|x_n - x^*\|. \tag{3.13}$$

Moreover, from (3.9) and (3.13) we put $\theta_1 = t_s(\rho\mu_2 + \sqrt{1 - 2\rho\gamma + \rho^2\mu_1^2}), \theta_2 = t_s(\eta\mu_2 + \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_1^2})$, it follows that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\theta_2\|x_n - x^*\| \\ &= (1 - (1 - \theta_1\theta_2)\alpha_n)\|x_n - x^*\| \\ &\leq \prod_{i=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_i)\|x_0 - x^*\|. \end{aligned} \tag{3.14}$$

Since $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ and conditions (3.5), we obtain

$$\lim_{n\to\infty}\prod_{i=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_i) = 0. \tag{3.15}$$

It follows from (3.15) and (3.14), we have

$$\lim_{n\to\infty}\|x_n - x^*\| = 0. \tag{3.16}$$

From (3.13) and (3.16), we have

$$\lim_{n\to\infty}\|y_n - y^*\| = 0. \tag{3.17}$$

Which is $x^*, y^* \in C_r$ satisfying the system of nonconvex variational inequalities (2.5). This completes the proof. □

Corollary 3.5. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T : C \to H$ be such that T is a μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ and $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$, such that*

$$\frac{\gamma}{\mu^2} - \triangle_{t_s} < \rho, \eta < \frac{\gamma}{\mu^2} + \triangle_{t_s}, \tag{3.18}$$

where $\triangle_{t_s} = \frac{\sqrt{(\gamma t_s)^2 - (\mu_1^2)(t_s^2 - 1)}}{t_s(\mu_1^2)}$. If the sequence of positive real number $\alpha_n \in [0, 1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, and $\alpha_n \in [0, 1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, then the sequence $\{x_n\}$ and $\{y_n\}$ is generated by for $x_0, y_0 \in C_r$,

$$\begin{aligned} y_n &= P_C[x_n - \eta T x_n], \eta > 0 \\ x_{n+1} &= P_C[y_n - \rho T y_n], \rho > 0, \end{aligned} \tag{3.19}$$

strongly converge to a solution of the system of nonconvex variational inequalities (2.5).

Proof. From Theorem 3.4, if $T_2 \equiv 0$ and $\alpha_n = 1$ for any $n \geq 0$, we have a result. □

4 Applications

In this section, we can applied Theorem 3.4 to the system of general of nonconvex variational inequalities, for given nonlinear mappings $T, g : C_r \to H$, we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T g(y^*) + g(x^*) - g(y^*), x - g(x^*) \rangle \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T g(x^*) + g(y^*) - g(x^*), x - g(y^*) \rangle \geq 0, \forall x \in C_r, \eta > 0, \end{aligned} \tag{4.1}$$

which is called the *system of general nonconvex variational inequalities*. Similar of the proof of Lemma 3.1, we can proof that

Lemma 4.1. *For given $x^*, y^* \in C_r$ is a solution of system of nonconvex variational inequalities (4.1), if and only if*

$$\begin{aligned} g(x^*) &= P_C[g(y^*) - \rho T g(y^*)], \\ g(y^*) &= P_C[g(x^*) - \eta T g(x^*)], \end{aligned} \tag{4.2}$$

where P_C is the projection of H onto the uniformly prox-regular set C_r.

Theorem 4.2. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, let $g : C \to H$ is injective mapping and let $T_1, T_2 : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. If $T = T_1 + T_2$ and there exists constant $\rho, \eta > 0$ and $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$, such that*

$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \triangle_{t_s} < \rho, \eta < \min\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \triangle_{t_s}, \frac{1}{t_s \mu_2}\}, \tag{4.3}$$

where $\triangle_{t_s} = \frac{\sqrt{(\gamma t_s - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t_s^2 - 1)}}{t_s(\mu_1^2 - \mu_2^2)}$. If the sequence of positive real number $\alpha_n \in [0, 1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, then the sequence $\{x_n\}$ and $\{y_n\}$ is generated by for $x_0, y_0 \in C_r$,

$$\begin{aligned} g(y_n) &= P_C[g(x_n) - \eta T g(x_n)], \eta > 0 \\ g(x_{n+1}) &= (1 - \alpha_n) g(x_n) + \alpha_n P_C[g(y_n) - \rho T g(y_n)], \rho > 0, \end{aligned} \tag{4.4}$$

strongly converge to a solution of the system of nonconvex variational inequalities (4.1).

Proof. Similar the proof in Theorem 3.4, let $x^*, y^* \in C_r$ be a solution of (4.1) and from Lemma 4.1, we can compute that

$$\|g(x_{n+1}) - g(x^*)\| \leq \prod_{i=0}^{n}(1-(1-\theta_1\theta_2)\alpha_i)\|g(x_0) - g(x^*)\|. \tag{4.5}$$

where $\theta_1 = t_s(\rho\mu_2 + \sqrt{1-2\rho\gamma+\rho^2\mu_1^2})$ From $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ and conditions (4.3), we obtain

$$\lim_{n\to\infty}\prod_{i=0}^{n}(1-(1-\theta_1\theta_2)\alpha_i) = 0. \tag{4.6}$$

It follows from (4.5) and (4.6), we have

$$\lim_{n\to\infty}\|g(x_n) - g(x^*)\| = 0. \tag{4.7}$$

And we can compute that

$$\|g(y_n) - g(y^*)\| \leq \theta_2\|g(x_n) - g(x^*)\|, \tag{4.8}$$

where $\theta_2 = t_s(\eta\mu_2 + \sqrt{1-2\eta\gamma_2+\eta^2\mu_1^2})$, it follows that

$$\lim_{n\to\infty}\|g(y_n) - g(y^*)\| = 0. \tag{4.9}$$

From g is injective mapping, we have $\lim_{n\to\infty}\|x_n - x^*\| = 0$ and $\lim_{n\to\infty}\|y_n - y^*\| = 0$ satisfying the system of general nonconvex variational inequalities (4.1). This complete the proof. □

Corollary 4.3. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, let $g : C \to H$ is injective mapping and let $T : C \to H$ be such that T is a μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ and $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$, such that*

$$\frac{\gamma}{\mu^2} - \triangle_{t_s} < \rho, \eta < \frac{\gamma}{\mu^2} + \triangle_{t_s}, \tag{4.10}$$

where $\triangle_{t_s} = \frac{\sqrt{(\gamma t_s)^2 - (\mu_1^2)(t_s^2-1)}}{t_s(\mu_1^2)}$. If the sequence of positive real number $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, and $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, then the sequence $\{x_n\}$ and $\{y_n\}$ is generated by for $x_0, y_0 \in C_r$,

$$\begin{aligned} g(y_n) &= P_C[g(x_n) - \eta T g(x_n)], \eta > 0 \\ g(x_{n+1}) &= P_C[g(y_n) - \rho T g(y_n)], \rho > 0, \end{aligned} \tag{4.11}$$

strongly converge to a solution of the system of nonconvex variational inequalities (4.1).

Proof. From Theorem 3.4, if $T_2 \equiv 0$ and $\alpha_n = 1$ for any $n \geq 0$, we have a result. □

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