



รายงานวิจัยฉบับสมบูรณ์

โครงการ

การออกแบบตัวควบคุมต้นทุนที่เหมาะสมสำหรับการทำให้เสถียรภาพและการทำให้เกิดความพร้อมกันของโครงข่ายพลวัตซับซ้อนที่มีตัวหน่วงโดยใช้ตัวเชื่อมต่อแบบผสมและการประยุกต์

Optimal Cost Controller Design for Stabilization and

Synchronization of Delayed Complex Dynamical

Networks with Hybrid Connection and Application

(ทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่)

โดย

อ.ดร.ธงชัย บทมาตย์

มิถุนายน 2557

รายงานวิจัยฉบับสมบูรณ์

การออกแบบตัวควบคุมต้นทุนที่เหมาะสมสำหรับการทำให้เสถียรภาพและ
การทำให้เกิดความพร้อมกันของโครงข่ายพลวัตซับซ้อนที่มีตัวหน่วง
โดยใช้ตัวเชื่อมต่อแบบผสมและการประยุกต์

**Optimal Cost Controller Design for Stabilization and Synchronization of
Delayed Complex Dynamical Networks with Hybrid Connection
and Application**

(ทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่)

ดร.ธงชัย บทมาตย์

ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์

(หัวหน้าโครงการวิจัยผู้รับทุน)

มหาวิทยาลัยศรีนครินทรวิโรฒ

รศ.ดร.ปิยะพงศ์ เนียมทรัพย์

ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์

(นักวิจัยที่ปรึกษา)

มหาวิทยาลัยเชียงใหม่

สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา สำนักงานกองทุน

สนับสนุนการวิจัยและมหาวิทยาลัยศรีนครินทรวิโรฒ

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

บทคัดย่อ

รหัสโครงการ : MRG5580081

ชื่อโครงการ : การออกแบบตัวควบคุมต้นทุนที่เหมาะสมสำหรับการทำให้เสถียรภาพ และการทำให้เกิดความพร้อมกันของโครงข่ายพลวัตซับซ้อนที่มีตัว หน่วงโดยใช้ตัวเชื่อมต่อแบบผสมและการประยุกต์

ชื่อนักวิจัย : ดร. ธงชัย บทมาตย์

นักวิจัยที่ปรึกษา: ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยศรีนครินทรวิโรฒ
รศ.ดร. ปิยะพงศ์ เนียมทรัพย์

อีเมล: thongchaib@swu.ac.th (ธงชัย บทมาตย์)
piyapong.n@cmu.ac.th (ปิยะพงศ์ เนียมทรัพย์)

ระยะเวลาโครงการ : 2 กรกฎาคม 2555 – 1 กรกฎาคม 2557

ในงานวิจัยนี้ งานวิจัยนี้ได้ศึกษาเพื่อออกแบบและสร้างตัวควบคุมใหม่ที่ทำให้ การทำงานพร้อมกันของโครงข่ายพลศาสตร์ที่ซับซ้อนที่มีตัวหน่วงที่ขึ้นกับเวลาแบบผสม และ การเชื่อมต่อแบบผสมโดยที่ตัวเชื่อมต่อประกอบด้วย ตัวเชื่อมต่อสถานะ ตัวเชื่อมต่อที่มีตัว หน่วงที่ขึ้นกับเวลาแบบช่วง ตัวเชื่อมต่อที่มีตัวหน่วงที่ขึ้นกับเวลาแบบกระจาย มีความ เสถียรภาพแบบเลขชี้กำลัง อีกทั้งยังได้ออกแบบตัวควบคุมแบบย้อนกลับและตัวควบคุมแบบ ย้อนกลับที่เป็นคาบ เราได้ใช้เมตริกซ์หนึ่งหน่วยในการแปลงระบบของโครงข่ายพลศาสตร์ที่ ซับซ้อนให้เป็นระบบเชิงเส้นที่มีตัวหน่วงที่ขึ้นกับเวลา และเราได้สร้างฟังก์ชันไลปูนอฟ-คราซอฟ สกีใหม่และรูปแบบของไลบ์นิช-นิวตันรวมทั้งการใช้เทคนิคของการอินทิกรัล เพื่อให้ได้มาซึ่ง หลักเกณฑ์เสถียรภาพแบบเลขชี้กำลังรูปแบบใหม่ที่อยู่ในรูปอสมการเมตริกซ์เชิงเส้น และได้ เขียนโปรแกรมเชิงตัวเลขของแบบจำลองทางคณิตศาสตร์ของการทำงานพร้อมกันของโครงข่าย พลศาสตร์ที่ซับซ้อนที่มีการเชื่อมต่อแบบผสมและตัวหน่วงที่ขึ้นกับเวลาเพื่อยืนยันประสิทธิภาพ ของตัวควบคุมที่สร้างขึ้น สุดท้ายได้ประยุกต์ตัวควบคุมดังกล่าวกับปัญหาการทำงานพร้อมกัน ของโครงข่ายประสาทเทียมที่มีตัวหน่วงที่ขึ้นกับเวลาแบบผสม เพื่อมีความเสถียรภาพแบบเลขชี้ กำลัง

คำหลัก : การทำให้เกิดความพร้อมกันแบบเลขชี้กำลัง, โครงข่ายพลวัตซับซ้อน, ตัวหน่วงที่ แปรผันตามเวลาแบบผสม, ตัวเชื่อมต่อแบบผสม, ตัวควบคุมที่เป็นคาบ

Abstract

Project Code: MRG5580081

Project Title: Optimal Cost Controller Design for Stabilization and Synchronization of Delayed Complex Dynamical Networks with Hybrid Connection and Application

Investigator: Dr. Thongchai Botmart
Department of Mathematics, Faculty of Science, Srinakharinwirot University

Mentor: Assoc. Prof. Dr. Piyapong Niamsup
Department of Mathematics, Faculty of Science, Chiang Mai University

E-mail Address: thongchaib@swu.ac.th (Thongchai Botmart)
piyapong.n@cmu.ac.th (Piyapong Niamsup)

Project Period: July 2, 2012 - July 1, 2014

This research work investigate the problem of exponential synchronization for complex dynamical network with mixed time-varying and hybrid coupling delays, which is composed of state coupling, interval time-varying delay coupling and distributed time-varying delay coupling. The designed controller ensures that the synchronization of delayed complex dynamical network are proposed via either feedback control or intermittent feedback control. We use common unitary matrices, and the problem of synchronization is transformed into the stability analysis of some linear time-varying delay systems. This is based on the construction of an improved Lyapunov-Krasovskii functional combined with the Leibniz-Newton formula and the technique of dealing with some integral terms. New synchronization criteria are derived in terms of LMIs which can be solved efficiently by standard convex optimization algorithms. Numerical examples are included to show the effectiveness of the proposed feedback control and intermittent feedback control scheme. Moreover, we apply the controller for problem of exponential synchronization for master-slave neural networks with mixed time-varying delays via hybrid intermittent feedback control.

Keywords: Exponential synchronization; complex dynamical network; mixed time-varying delays; hybrid coupling; intermittent control

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ

สำนักงานคณะกรรมการการอุดมศึกษา สำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยศรีนครินทรวิโรฒ ที่ได้ให้โอกาสผู้วิจัยได้รับทุนเพื่อเป็นการพัฒนาศักยภาพในการทำงานวิจัยอาจารย์รุ่นใหม่ในการทำงานวิจัยครั้งนี้

รองศาสตราจารย์ ดร.ปิยะพงศ์ เนียมทรัพย์ นักวิจัยที่ปรึกษาให้กับโครงการนี้ผู้ซึ่งอบรม สอนและถ่ายทอดความรู้ด้านต่างๆ จนผู้วิจัยสามารถทำงานวิจัยได้สำเร็จตามเป้าหมาย

คณะผู้ประเมินของวารสารวิชาการต่างๆ ที่ได้ให้คำแนะนำ ตลอดทั้งปรับปรุงต้นฉบับของบทความที่ส่งไปตีพิมพ์ในวารสารนั้นๆ

คณาจารย์ นักศึกษาและเจ้าหน้าที่ฝ่ายสนับสนุน ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยศรีนครินทรวิโรฒ ได้ร่วมศึกษาวิจัยและช่วยเหลือโครงการวิจัยในครั้งนี้

ดร.ธงชัย บทมาตย์
หัวหน้าโครงการวิจัย

Contents

บทคัดย่อ	iii
Abstract	iv
กิตติกรรมประกาศ	v
Chapter 1 Executive Summary	1
1.1 Synchronization of delayed complex dynamical network	1
1.2 Synchronization of master-slave neural networks	5
Chapter 2 Main Results	9
2.1 Synchronization of delayed complex dynamical network via delayed feedback control and intermittent control	9
2.2 Synchronization of master-slave neural networks with mixed time-varying delays via hybrid intermittent feedback control	15
Output	18
Appendix	19

Chapter 1

Executive Summary

1.1 Synchronization of delayed complex dynamical network

Consider a complex dynamical network consisting of N identical coupled nodes, with each node being an n -dimensional dynamical system

$$\begin{aligned}
 \dot{x}_i(t) &= f(x_i(t), x_i(t-h(t)), \int_{t-k_1(t)}^t x_i(s) ds) + c_1 \sum_{j=1}^N a_{ij} G_1 x_j(t) \\
 &\quad + c_2 \sum_{j=1}^N b_{ij} G_2 x_j(t-h(t)) + c_3 \sum_{j=1}^N c_{ij} G_3 \int_{t-k_1(t)}^t x_j(s) ds + \mathcal{U}_i(t), \\
 t &\geq 0, \quad i = 1, 2, \dots, N, \\
 x_i(t) &= \phi_i(t), \quad t \in [-\tau_{\max}, 0], \quad \tau_{\max} = \max\{h_2, d, k_1, k_2\},
 \end{aligned} \tag{1.1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in \mathbb{R}^n$ is the state vector of i th node; $\mathcal{U}_i(t) \in \mathbb{R}^m$ are the control input of the node i ; the constant $c_1, c_2, c_3 > 0$ are the coupling strength; $G_1 = (g_{1ij})_{n \times n}$, $G_2 = (g_{2ij})_{n \times n}$, $G_3 = (g_{3ij})_{n \times n} \in \mathbb{R}^{n \times n}$ are a constant inner-coupling matrix, if some pairs (i, j) , $1 \leq i, j \leq n$, with $g_{1ij} \neq 0$, $g_{2ij} \neq 0$ and $g_{3ij} \neq 0$ which means two coupled nodes are linked through their i th and j th state variables, otherwise $g_{1ij} = 0$, $g_{2ij} = 0$, $g_{3ij} = 0$; $A = (a_{ij})_{N \times N}$, $B = (b_{ij})_{N \times N}$, $C = (c_{ij})_{N \times N} \in \mathbb{R}^{N \times N}$ are the outer-coupling matrix of the network, in which a_{ij} , b_{ij} are defined as follows: if there are a connection between node i and node j ($j \neq i$), then $a_{ij} = a_{ji} = 1$, $b_{ij} = b_{ji} = 1$, $c_{ij} = c_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0$, $b_{ij} = b_{ji} = 0$, $c_{ij} = c_{ji} = 0$ ($j \neq i$), and the diagonal elements of matrix A , B and C are defined

by

$$\begin{aligned} a_{ii} &= - \sum_{j=1, i \neq j}^N a_{ij} = - \sum_{j=1, i \neq j}^N a_{ji}, & b_{ii} &= - \sum_{j=1, i \neq j}^N b_{ij} = - \sum_{j=1, i \neq j}^N b_{ji}, \\ c_{ii} &= - \sum_{j=1, i \neq j}^N c_{ij} = - \sum_{j=1, i \neq j}^N c_{ji}, & i &= 1, 2, \dots, N. \end{aligned} \quad (1.2)$$

It is assumed that network (1.1) is connected in the sense that there are no isolated clusters, that is, A , B , C are irreducible matrix.

Definition 1.1.1 *The delayed dynamical network (1.1) is said to achieve asymptotical synchronization if*

$$x_1(t) = x_2(t) = \dots = s(t) \quad \text{as } t \longrightarrow \infty, \quad (1.3)$$

where $s(t)$ is a solution of an isolated node, satisfying

$$\dot{s}(t) = f(s(t), s(t - h(t)), \int_{t-k_1(t)}^t s(\theta) d\theta).$$

In order to stabilize the origin of dynamical network (1.1) by means of the state feedback controller $\mathcal{U}_i(t)$ satisfying either (H1) or (H2) for $i = 1, 2, \dots, n$:

$$\begin{aligned} (H1) &: \mathcal{U}_i(t) = D_{1i}u_i(t) + D_{2i}u_i(t - d(t)) + D_{3i} \int_{t-k_2(t)}^t u_i(s) ds, \quad \forall t \geq t_0, \\ (H2) &: \mathcal{U}_i(t) = \begin{cases} D_{4i}u_i(t) + D_{5i}u_i(t - d(t)) \\ + D_{6i} \int_{t-k_2(t)}^t u_i(s) ds, & n\omega \leq t \leq n\omega + \delta \\ 0, & n\omega + \delta < t \leq (n+1)\omega \end{cases} \end{aligned}$$

where D_{ji} , $j = 1, 2, \dots, 6$ are given matrices of appropriate dimensions, $u_i(t) = K_i(x_i(t) - s(t))$ and K_i is a constant matrix control gain, $\omega > 0$ is the control period and $\delta > 0$ is called the control width (control duration) and n is a non-negative integer. Then, substituting it into dynamical network (1.1), it is easy to get the following:

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t), x_i(t - h(t)), \int_{t-k_1(t)}^t x_i(s) ds) + c_1 \sum_{j=1}^N a_{ij} G_1 x_j(t) \\ &+ c_2 \sum_{j=1}^N b_{ij} G_2 x_j(t - h(t)) + c_3 \sum_{j=1}^N c_{ij} G_3 \int_{t-k_1(t)}^t x_j(s) ds \\ &+ D_{1i} K_i (x_i(t) - s(t)) + D_{2i} u_i(t - d(t)) + D_{3i} \int_{t-k_2(t)}^t u_i(s) ds. \end{aligned} \quad (1.4)$$

Namely, the dynamical network (1.1) is governed by the following system:

$$\begin{aligned}
\dot{x}_i(t) = & f(x_i(t), x_i(t-h(t)), \int_{t-k_1(t)}^t x_i(s) ds) + c_1 \sum_{j=1}^N a_{ij} G_1 x_j(t) \\
& + c_2 \sum_{j=1}^N b_{ij} G_2 x_j(t-h(t)) + c_3 \sum_{j=1}^N c_{ij} G_3 \int_{t-k_1(t)}^t x_j(s) ds \\
& + D_{4i} K_i(x_i(t) - s(t)) + D_{5i} u_i(t-d(t)) + D_{6i} \int_{t-k_2(t)}^t u_i(s) ds, \\
n\omega \leq t \leq n\omega + \delta, & \tag{1.5} \\
\dot{x}_i(t) = & f(x_i(t), x_i(t-h(t)), \int_{t-k_1(t)}^t x_i(s) ds) + c_1 \sum_{j=1}^N a_{ij} G_1 x_j(t) \\
& + c_2 \sum_{j=1}^N b_{ij} G_2 x_j(t-h(t)) + c_3 \sum_{j=1}^N c_{ij} G_3 \int_{t-k_1(t)}^t x_j(s) ds, \\
n\omega + \delta < t \leq (n+1)\omega, \quad i = 1, 2, \dots, N.
\end{aligned}$$

It is clear that, if the zero solution of the dynamical network (1.4) and (1.5) are globally exponentially stable, then exponential synchronization of the controlled dynamical network (1.1) is achieved. The time-varying delay functions $h(t)$, $d(t)$, $k_1(t)$ and $k_2(t)$ satisfy the condition

$$0 \leq h_1 \leq h(t) \leq h_2, \quad 0 \leq d(t) \leq d, \quad 0 \leq k_1(t) \leq k_1, \quad 0 \leq k_2(t) \leq k_2. \tag{1.6}$$

The initial condition function $\phi_i(t)$ denotes a continuous vector-valued initial function of $t \in [-\tau_{\max}, 0]$.

In this paper, we assume that $s(t)$ is an orbitally stable solution of the above system. Clearly, the stability of the synchronized states (1.3) of network (1.1) is determined by the dynamics of the isolate node, the coupling strength c_1 , c_2 and c_3 , the inner-coupling matrix G_1 , G_2 and G_3 , the outer-coupling matrix A , B and C .

The following definition and lemmas are used in the proof of the main result.

Definition 1.1.2 *A functional $V : \mathbb{R}^+ \times C \rightarrow \mathbb{R}^+$ is called a Lyapunov-Krasovskii functional for the system (1.1) if it has the following properties. There exist $\lambda_1, \lambda_2, \lambda_3 > 0$ such that*

$$(i) \quad \lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|^2,$$

$$(ii) \quad \dot{V}(t, x_t) \leq -\lambda_3 \|x(t)\|^2.$$

Lemma 1.1.3 *Consider the non autonomous time-delay system (1.1). If there exist a Lyapunov function $V(t, x_t)$ and $\lambda_1, \lambda_2 > 0$ such that for every solution $x(t)$ of the system, the following conditions hold,*

$$(i) \quad \lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|^2,$$

$$(ii) \quad \dot{V}(t, x_t) \leq 0,$$

then the solution of the system is bounded, i.e., there exists $N > 0$ such that $\|x(t, \phi)\| \leq N \|\phi\|, \forall t \geq 0$.

Lemma 1.1.4 *Consider the autonomous time-delay system (1.1). If there exist a Lyapunov-Krasovskii function $V(x_t)$ and $\lambda_1, \lambda_2, \lambda_3 > 0$ such that for every solution $x(t)$ of the system, the following conditions hold,*

$$(i) \quad \lambda_1 \|x(t)\|^2 \leq V(x_t) \leq \lambda_2 \|x_t\|^2,$$

$$(ii) \quad \dot{V}(x_t) \leq -\lambda_3 \|x(t)\|^2,$$

then the solution of the system (1.1) is exponentially stable.

Proposition 1.1.5 *(Cauchy inequality) For any symmetric positive definite matrix $N \in M^{n \times n}$ and $x, y \in \mathbb{R}^n$ we have*

$$\pm 2x^T y \leq x^T N x + y^T N^{-1} y.$$

Lemma 1.1.6 *(Schur complement lemma). Given constant symmetric matrices X, Y, Z with appropriate dimensions satisfying $X = X^T, Y = Y^T > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0.$$

Lemma 1.1.7 *For any constant symmetric matrix $M \in R^{n \times n}$, $M = M^T > 0$, $0 \leq h_m \leq h(t) \leq h_M$, $t \geq 0$, and any differentiable vector function $x(t) \in R^n$, we have*

$$\begin{aligned}
 (a) \quad & \left[\int_{t-h_m}^t \dot{x}(s) ds \right]^T M \left[\int_{t-h_m}^t \dot{x}(s) ds \right] \leq h_m \int_{t-h_m}^t \dot{x}^T(s) M \dot{x}(s) ds, \\
 (b) \quad & \left[\int_{t-h(t)}^{t-h_m} \dot{x}(s) ds \right]^T M \left[\int_{t-h(t)}^{t-h_m} \dot{x}(s) ds \right] \leq (h(t) - h_m) \int_{t-h(t)}^{t-h_m} \dot{x}^T(s) M \dot{x}(s) ds \\
 & \leq (h_M - h_m) \int_{t-h(t)}^{t-h_m} \dot{x}^T(s) M \dot{x}(s) ds.
 \end{aligned}$$

Definition 1.1.8 *If the matrix $A \in M_{n \times n}$ is similar to a diagonal matrix, then A is said to be diagonalizable.*

Lemma 1.1.9 *Let G be a family of diagonalizable matrices. Then G is a commuting family (under multiplication) if and only if it is a simultaneously diagonalizable family.*

Lemma 1.1.10 *Let A, B be a family of diagonalizable matrices. Then A, B is a commuting family (under multiplication) if and only if it is a simultaneously diagonalizable family.*

1.2 Synchronization of master-slave neural networks

We consider the master-slave cellular neural networks (MSCNNs) with mixed time-varying delays are described as follows:

$$\begin{aligned}
 \dot{x}(t) &= -Ax(t) + C\tilde{f}(x(t)) + D\tilde{g}(x(t - h_1(t))) \\
 &\quad + E \int_{t-k_1(t)}^t \tilde{h}(x(s)) ds + I(t), \\
 x(t) &= \phi_1(t), \quad t \in [-d, 0],
 \end{aligned} \tag{1.7}$$

$$\begin{aligned}
 \dot{y}(t) &= -Ay(t) + C\tilde{f}(y(t)) + D\tilde{g}(y(t - h_1(t))) \\
 &\quad + E \int_{t-k_1(t)}^t \tilde{h}(y(s)) ds + I(t) + \mathcal{U}(t), \\
 y(t) &= \phi_2(t), \quad t \in [-d, 0],
 \end{aligned} \tag{1.8}$$

where $x(t) = [x_1(t), x_2(t), \dots, x_n(t)] \in R^n$, $y(t) = [y_1(t), y_2(t), \dots, y_n(t)] \in R^n$ are the master systems state vector and the slave systems state vector of the neural

networks, respectively. n is the number of neural, and

$$\begin{aligned}\tilde{f}(x(t)) &= [\tilde{f}_1(x_1(t)), \tilde{f}_2(x_2(t)), \dots, \tilde{f}_n(x_n(t))]^T, \\ \tilde{g}(x(t)) &= [\tilde{g}_1(x_1(t)), \tilde{g}_2(x_2(t)), \dots, \tilde{g}_n(x_n(t))]^T, \\ \tilde{h}(x(t)) &= [\tilde{h}_1(x_1(t)), \tilde{h}_2(x_2(t)), \dots, \tilde{h}_n(x_n(t))]^T,\end{aligned}$$

are the activation functions, $A = \text{diag}(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n)$, $\bar{a}_i > 0$ represents the self-feedback term and C, D, E denote the connection weights, the discretely delayed connection weights and the distributively delayed connection weight, respectively.

The synchronization error $e(t)$ is the form $e(t) = y(t) - x(t)$. Therefore, the cellular neural networks with mixed time-varying delays of synchronization error between the master-slave systems given in (1.7) and (1.8) can be described by

$$\begin{aligned}\dot{e}(t) &= -Ae(t) + Cf(e(t)) + Dg(e(t - h_1(t))) \\ &\quad + E \int_{t-k_1(t)}^t h(e(s))ds + \mathcal{U}(t), \\ e(t) &= \phi_2(t) - \phi_1(t) = \phi(t), \quad t \in [-d, 0],\end{aligned}\tag{1.9}$$

where $f(e(t)) = \tilde{f}(e(t) + x(t)) - \tilde{f}(x(t))$, $g(e(t - h_1(t))) = \tilde{g}(e(t - h_1(t)) + x(t - h_1(t))) - \tilde{g}(x(t - h_1(t)))$, $\int_{t-k_1(t)}^t h(e(s))ds = \int_{t-k_1(t)}^t h(e(s) + x(s)) - h(x(s))ds$. The state hybrid feedback controller $\mathcal{U}(t)$ satisfying :

$$\mathcal{U}(t) = \begin{cases} B_1 u(t) + B_2 u(t - h_2(t)) \\ + B_3 \int_{t-k_2(t)}^t u(s)ds, & n\omega \leq t \leq n\omega + \delta, \\ 0, & n\omega + \delta < t \leq (n+1)\omega. \end{cases}\tag{1.10}$$

where $u(t) = Ke(t)$ and K is a constant matrix control gain, $\omega > 0$ is the control period and $\delta > 0$ is called the control width (control duration) and n is a non-negative integer. In this paper, our goal is to design suitable K such that system (1.8) synchronizes with system (1.7). Then, substituting it into (1.9), it is easy to

get the following:

$$\begin{aligned}
\dot{e}(t) &= -Ae(t) + Cf(e(t)) + Dg(e(t - h_1(t))) + E \int_{t-k_1(t)}^t h(e(s))ds \\
&\quad + B_1Ke(t) + B_2Ke(t - h_2(t)) + B_3K \int_{t-k_2(t)}^t e(s)ds, \\
n\omega &\leq t \leq n\omega + \delta, \\
\dot{e}(t) &= -Ae(t) + Cf(e(t)) + Dg(e(t - h_1(t))) + E \int_{t-k_1(t)}^t h(e(s))ds \\
n\omega + \delta &< t \leq (n+1)\omega, \\
e(t) &= \phi_2(t) - \phi_1(t) = \phi(t), \quad t \in [-d, 0].
\end{aligned} \tag{1.11}$$

Throughout this paper, we consider various activation functions and the activation functions $\tilde{f}(\cdot)$, $\tilde{g}(\cdot)$ and $\tilde{h}(\cdot)$ satisfy the following assumption:

(A1) The activation functions $\tilde{f}(\cdot)$, $\tilde{g}(\cdot)$ and $\tilde{h}(\cdot)$ satisfy Lipschitzian with the Lipschitz constants $\hat{f}_i$, $\hat{g}_i > 0$ and $\hat{h}_i > 0$:

$$\begin{aligned}
|\tilde{f}_i(\xi_1) - \tilde{f}_i(\xi_2)| &\leq \hat{f}_i|\xi_1 - \xi_2|, & i = 1, 2, \dots, n, \forall \xi_1, \xi_2 \in R, \\
|\tilde{g}_i(\xi_1) - \tilde{g}_i(\xi_2)| &\leq \hat{g}_i|\xi_1 - \xi_2|, & i = 1, 2, \dots, n, \forall \xi_1, \xi_2 \in R, \\
|\tilde{h}_i(\xi_1) - \tilde{h}_i(\xi_2)| &\leq \hat{h}_i|\xi_1 - \xi_2|, & i = 1, 2, \dots, n, \forall \xi_1, \xi_2 \in R,
\end{aligned} \tag{1.12}$$

and we denote

$$\begin{aligned}
F &= \text{diag}\{\hat{f}_i, \quad i = 1, 2, \dots, n\}, \\
G &= \text{diag}\{\hat{g}_i, \quad i = 1, 2, \dots, n\}, \\
H &= \text{diag}\{\hat{h}_i, \quad i = 1, 2, \dots, n\}.
\end{aligned}$$

The time-varying delay functions $h_i(t)$, $k_i(t)$, $i = 1, 2$ satisfy the condition

$$\begin{aligned}
0 &\leq h_{1m} \leq h_1(t) \leq h_{1M}, \quad 0 \leq h_2(t) \leq h_2, \\
0 &\leq k_1(t) \leq k_1, \quad 0 \leq k_2(t) \leq k_2.
\end{aligned} \tag{1.13}$$

It is worth noting that the time delay is assumed to be a continuous function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, but the delay function is bounded but not restricted

to being zero. The initial functions $\phi(t) \in C^1([-d, 0], R^n)$, $d = \max\{h_{1M}, h_2, k_1, k_2\}$ with the norm

$$\|\phi\| = \sup_{t \in [-d, 0]} \sqrt{\|\phi(t)\|^2 + \|\dot{\phi}(t)\|^2}.$$

Definition 1.2.1 *Given $\alpha > 0$. The zero solution of system (1.11) with $u(t) = Ke(t)$ is α -stable if there exist a positive number $N > 0$ such that every solution $e(t, \phi)$ satisfies the following condition:*

$$\|e(t, \phi)\| \leq Ne^{-\alpha t} \|\phi\|, \quad \forall t \geq 0.$$

This research work investigate the problem of exponential synchronization for complex dynamical network with mixed time-varying and hybrid coupling delays, which is composed of state coupling, interval time-varying delay coupling and distributed time-varying delay coupling. The designed controller ensures that the synchronization of delayed complex dynamical network are proposed via either feedback control or intermittent feedback control. We use common unitary matrices, and the problem of synchronization is transformed into the stability analysis of some linear time-varying delay systems. This is based on the construction of an improved Lyapunov-Krasovskii functional combined with the Leibniz-Newton formula and the technique of dealing with some integral terms. New synchronization criteria are derived in terms of LMIs which can be solved efficiently by standard convex optimization algorithms. Numerical examples are included to show the effectiveness of the proposed feedback control and intermittent feedback control scheme. Moreover, we apply the controller for problem of exponential synchronization for master-slave neural networks with mixed time-varying delays via hybrid intermittent feedback control.

Chapter 2

Main Results

2.1 Synchronization of delayed complex dynamical network via delayed feedback control and intermittent control

In this section, we shall obtain some delay-dependent exponential synchronization criteria for general complex dynamical network with discrete and distributed time-varying delays and hybrid coupling delays (1.1) by strict LMI approaches. Let us set

$$\tilde{A}_i = J(t) + c_1 \lambda_{1i} G_1, \quad \tilde{B}_i = J_h(t) + c_2 \lambda_{2i} G_2, \quad \tilde{C}_i = J_{k_1}(t) + c_3 \lambda_{3i} G_3,$$

and

- (1) $J(t) = f'(s(t), s(t-h(t)), \int_{t-k_1(t)}^t s(\xi) d\xi) \in R^{n \times n}$ is the Jacobian of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^t x(s) ds)$ at $s(t)$ with the derivative of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^t x(s) ds)$ respect to $x(t)$,
- (2) $J_h(t) = f'(s(t), s(t-h(t)), \int_{t-k_1(t)}^t s(\xi) d\xi) \in R^{n \times n}$ is the Jacobian of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^t x(s) ds)$ at $s(t-h(t))$ with the derivative of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^t x(s) ds)$ respect to $x(t-h(t))$,
- (3) $J_{k_1}(t) = f'(s(t), s(t-h(t)), \int_{t-k_1(t)}^t s(\xi) d\xi) \in R^{n \times n}$ is the Jacobian of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^t x(s) ds)$ at $\int_{t-k_1(t)}^t s(\xi) d\xi$ with the derivative of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^t x(s) ds)$ respect to $\int_{t-k_1(t)}^t x(s) ds$.

Lemma 2.1.1 *Consider the hybrid coupling delays dynamical network in (1.1). Let $0 = \lambda_{j1} > \lambda_{j2} \geq \lambda_{j3} \geq \dots \geq \lambda_{jN}$, $j = \{1, 2, 3\}$ be the eigenvalues of the outer-coupling matrix A , B and C , respectively. If the $N - 1$ following n -dimensional*

linear time-varying delays differential equations are delay-dependent exponentially stable about their zero solutions:

$$\begin{aligned}
\dot{z}_i(t) &= (\tilde{A}_i + D_{4i}K_i)z_i(t) + \tilde{B}_iz_i(t - h(t)) + \tilde{C}_i \int_{t-k_1(t)}^t z_i(s) ds \\
&\quad + D_{5i}K_iz_i(t - d(t)) + D_{6i}K_i \int_{t-k_2(t)}^t z_i(s) ds, \quad n\omega \leq t \leq n\omega + \delta, \\
i &= 2, \dots, N, \\
\dot{z}_i(t) &= \tilde{A}_iz_i(t) + \tilde{B}_iz_i(t - h(t)) + \tilde{C}_i \int_{t-k_1(t)}^t z_i(s) ds, \quad n\omega + \delta < t \leq (n+1)\omega, \\
i &= 2, \dots, N,
\end{aligned} \tag{2.14}$$

then the dynamical networks (1.5) is exponentially stable, then exponential synchronization of the controlled dynamical networks (1.1) is achieved.

2.1.1 Linear delayed feedback control

Let us denote

$$\begin{aligned}
\|\phi_i\| &= \|z_i(0)\|, \quad \|\varphi_i\| = \sup_{-\tau_{\max} \leq s \leq 0} \|z_i(s)\|, \quad K_i = -L_iP_i^{-1}, \\
\gamma_i &= \lambda_{\min}(P_i^{-1}), \\
\ell_i &= \lambda_{\max}(P_i^{-1}) + \left[2h_2\lambda_{\max}(P_i^{-1}R_iP_i^{-1}) + h_2\lambda_{\max}(P_i^{-1}U_iP_i^{-1}) \right] \frac{1 - e^{-2\alpha h_2}}{2\alpha} \\
&\quad + d\lambda_{\max}(P_i^{-1}L_i^T T_i^{-1} L_i P_i^{-1}) \frac{1 - e^{-2\alpha d}}{2\alpha}, \\
\xi_i &= \left[2\lambda_{\max}(P_i^{-1}Q_iP_i^{-1}) + h_2\lambda_{\max}(P_i^{-1}R_iP_i^{-1}) + h_2\lambda_{\max}(P_i^{-1}U_iP_i^{-1}) \right] \\
&\quad \times \frac{1 - e^{-2\alpha h_2}}{2\alpha} + k_1\lambda_{\max}(P_i^{-1}S_iP_i^{-1}) \frac{1 - e^{-2\alpha k_1}}{2\alpha} \\
&\quad + d\lambda_{\max}(P_i^{-1}L_i^T T_i^{-1} L_i P_i^{-1}) \frac{1 - e^{-2\alpha h_2}}{2\alpha} \\
&\quad + k_2\lambda_{\max}(P_i^{-1}L_i^T W_i^{-1} L_i P_i^{-1}) \frac{1 - e^{-2\alpha d}}{2\alpha}, \\
\mathcal{N}_i &= \ell_i \|\phi_i\|^2 + \xi_i \|\varphi_i\|^2, \\
\gamma &= \min\{\gamma_i, \quad i = 2, 3, \dots, N\}, \quad \mathcal{N} = \max\{\mathcal{N}_i, \quad i = 2, 3, \dots, N\}.
\end{aligned}$$

Theorem 2.1.2 For some given scalars $0 < \alpha$, the dynamical networks (2.14) with time-varying delay satisfying (1.6) are exponentially stable if there exist symmetric

positive definite matrices $P_i > 0$, $Q_i > 0$, $R_i > 0$, $S_i > 0$, $U_i > 0$, $T_i > 0$, $W_i > 0$ and a matrix L_i with appropriately dimensioned such that the following symmetric linear matrix inequality holds:

$$\Sigma_{i1} = \Sigma_i - \begin{bmatrix} 0 & 0 & I & -I & 0 \end{bmatrix}^T e^{-2\alpha h_2} U_i \begin{bmatrix} 0 & 0 & I & -I & 0 \end{bmatrix} < 0, \quad (2.15)$$

$$\Sigma_{i2} = \Sigma_i - \begin{bmatrix} 0 & 0 & 0 & I & -I \end{bmatrix}^T e^{-2\alpha h_2} U_i \begin{bmatrix} 0 & 0 & 0 & I & -I \end{bmatrix} < 0, \quad (2.16)$$

$$\Sigma_{i3} = \begin{bmatrix} -0.5(e^{-2\alpha h_1} + e^{-2\alpha h_2})R_i & 2k_1\tilde{C}_iP_i & k_2L_i^T & 2L_i^T \\ * & -2k_1e^{-2\alpha k_1}S_i & 0 & 0 \\ * & * & -k_2W_i & 0 \\ * & * & * & -2e^{-2\alpha d}T_i \end{bmatrix} < 0, \quad (2.17)$$

$$\Sigma_{i4} = \begin{bmatrix} -0.5P_i & 2k_1\tilde{C}_iP_i & d^2L_i^T & 3D_{2i}^T & 2k_2D_{3i}^T \\ * & -2k_1e^{-2\alpha k_1}S_i & 0 & 0 & 0 \\ * & * & -d^2T_i & 0 & 0 \\ * & * & * & -3e^{-2\alpha d}T_i & 0 \\ * & * & * & * & -2k_2e^{-2\alpha k_2}W_i \end{bmatrix} < 0 \quad (2.18)$$

$i = 2, \dots, N$, where

$$\Sigma_i = \begin{bmatrix} \Sigma_{i11} & \Sigma_{i12} & \Sigma_{i13} & \Sigma_{i14} & \Sigma_{i15} \\ * & \Sigma_{i22} & 0 & \Sigma_{i24} & 0 \\ * & * & \Sigma_{i33} & \Sigma_{i34} & 0 \\ * & * & * & \Sigma_{i44} & \Sigma_{i45} \\ * & * & * & * & \Sigma_{i55} \end{bmatrix},$$

$$\begin{aligned}
\Sigma_{i11} &= P_i^T (\tilde{A}_i + \alpha I) + (\tilde{A}_i + \alpha I)^T P_i - D_{1i} L_i - L_i^T D_{1i}^T + 3e^{2\alpha d} D_{2i}^T T_i D_{2i} \\
&\quad + 2k_2 e^{2\alpha k_2} D_{3i}^T W_i D_{3i} + 2Q_i + k_1 S_i - 0.5e^{-2\alpha h_1} R_i - 0.5e^{-2\alpha h_2} R_i, \\
\Sigma_{i12} &= P_i \tilde{A}_i^T, \\
\Sigma_{i13} &= e^{-2\alpha h_1} R_i, \\
\Sigma_{i14} &= \tilde{B}_i P_i, \\
\Sigma_{i15} &= e^{-2\alpha h_2} R_i, \\
\Sigma_{i22} &= h_1^2 R_i + h_2^2 R_i + \eta^2 U_i - 1.5P_i, \\
\Sigma_{i24} &= \tilde{B}_i P_i, \\
\Sigma_{i33} &= -e^{-2\alpha h_1} Q_i - e^{-2\alpha h_1} R_i - e^{-2\alpha h_2} U_i, \\
\Sigma_{i34} &= e^{-2\alpha h_2} U_i, \\
\Sigma_{i44} &= -2e^{-2\alpha h_2} U_i, \\
\Sigma_{i45} &= e^{-2\alpha h_2} U_i, \\
\Sigma_{i55} &= -2e^{-2\alpha h_2} U_i - 2e^{-2\alpha h_2} Q_i - 2e^{-2\alpha h_2} R_i,
\end{aligned}$$

then, the dynamical networks (2.14) are exponential synchronization. Moreover, the feedback control is

$$u_i(t) = -L_i P_i^{-1} z_i(t). \quad (2.19)$$

2.1.2 Intermittent delayed feedback control

Theorem 2.1.3 For some given scalars $0 < \alpha < \varepsilon$, the dynamical networks (2.14) with time-varying delay satisfying (1.6) are exponentially stable if there exist symmetric positive definite matrices $P_i > 0$, $Q_i > 0$, $R_i > 0$, $S_i > 0$, $U_i > 0$, $T_i > 0$, $W_i > 0$ and a matrix L_i with appropriately dimensioned such that the following symmetric linear matrix inequality holds:

$$\Pi_{i1} = \Pi_i - \begin{bmatrix} 0 & 0 & I & -I & 0 \end{bmatrix}^T e^{-2\alpha h_2} U_i \begin{bmatrix} 0 & 0 & I & -I & 0 \end{bmatrix} < 0, \quad (2.20)$$

$$\Pi_{i2} = \Pi_i - \begin{bmatrix} 0 & 0 & 0 & I & -I \end{bmatrix}^T e^{-2\alpha h_2} U_i \begin{bmatrix} 0 & 0 & 0 & I & -I \end{bmatrix} < 0, \quad (2.21)$$

$$\Pi_{i3} = \tilde{\Pi}_i - \begin{bmatrix} 0 & 0 & I & -I & 0 \end{bmatrix}^T e^{-2\alpha h_2} U_i \begin{bmatrix} 0 & 0 & I & -I & 0 \end{bmatrix} < 0, \quad (2.22)$$

$$\Pi_{i4} = \tilde{\Pi}_i - \begin{bmatrix} 0 & 0 & 0 & I & -I \end{bmatrix}^T e^{-2\alpha h_2} U_i \begin{bmatrix} 0 & 0 & 0 & I & -I \end{bmatrix} < 0, \quad (2.23)$$

$$\Pi_{i5} = \begin{bmatrix} -0.5(e^{-2\alpha h_1} + e^{-2\alpha h_2})R_i & 2k_1\tilde{C}_iP_i & k_2L_i^T & 2L_i^T \\ * & -2k_1e^{-2\alpha k_1}S_i & 0 & 0 \\ * & * & -k_2W_i & 0 \\ * & * & * & -2e^{-2\alpha d}T_i \end{bmatrix} < 0, \quad (2.24)$$

$$\Pi_{i6} = \begin{bmatrix} -0.5P_i & 2k_1\tilde{C}_iP_i & d^2L_i^T & 3D_{5i}^T & 2k_2D_{6i}^T \\ * & -2k_1e^{-2\alpha k_1}S_i & 0 & 0 & 0 \\ * & * & -d^2T_i & 0 & 0 \\ * & * & * & -3e^{-2\alpha d}T_i & 0 \\ * & * & * & * & -2k_2e^{-2\alpha k_2}W_i \end{bmatrix} < 0 \quad (2.25)$$

$$\Pi_{i7} = \begin{bmatrix} -0.5(e^{-2\alpha h_1} + e^{-2\alpha h_2})R_i - 2\varepsilon P_i & 2k_1\tilde{C}_iP_i \\ * & -2k_1e^{-2\alpha k_1}S_i \end{bmatrix} < 0, \quad (2.26)$$

$$\Pi_{i8} = \begin{bmatrix} -0.5P_i & 2k_1\tilde{C}_iP_i \\ * & -2k_1e^{-2\alpha k_1}S_i \end{bmatrix} < 0, \quad (2.27)$$

and

$$-\alpha\delta + (\varepsilon - \alpha)(\omega - \delta) < 0, \quad (2.28)$$

$i = 2, \dots, N$, where

$$\Pi_i = \begin{bmatrix} \Pi_{i11} & \Pi_{i12} & \Pi_{i13} & \Pi_{i14} & \Pi_{i15} \\ * & \Pi_{i22} & 0 & \Pi_{i24} & 0 \\ * & * & \Pi_{i33} & \Pi_{i34} & 0 \\ * & * & * & \Pi_{i44} & \Pi_{i45} \\ * & * & * & * & \Pi_{i55} \end{bmatrix},$$

$$\begin{aligned}
\tilde{\Pi}_i &= \begin{bmatrix} \tilde{\Pi}_{i11} & \Pi_{i12} & \Pi_{i13} & \Pi_{i14} & \Pi_{i15} \\ * & \Pi_{i22} & 0 & \Pi_{i24} & 0 \\ * & * & \Pi_{i33} & \Pi_{i34} & 0 \\ * & * & * & \Pi_{i44} & \Pi_{i45} \\ * & * & * & * & \Pi_{i55} \end{bmatrix}, \\
\Pi_{i11} &= P_i^T (\tilde{A}_i + \alpha I) + (\tilde{A}_i + \alpha I)^T P_i - D_{4i} L_i - L_i^T D_{4i}^T + 3e^{2\alpha d} D_{5i}^T T_i D_{5i} \\
&\quad + 2k_2 e^{2\alpha k_2} D_{6i}^T W_i D_{6i} + 2Q_i + k_1 S_i - 0.5e^{-2\alpha h_1} R_i - 0.5e^{-2\alpha h_2} R_i, \\
\tilde{\Pi}_{i11} &= P_i^T (\tilde{A}_i + \alpha I) + (\tilde{A}_i + \alpha I)^T P_i + 2Q_i + k_1 S_i - 0.5e^{-2\alpha h_1} R_i - 0.5e^{-2\alpha h_2} R_i, \\
\Pi_{i12} &= P_i \tilde{A}_i^T, \\
\Pi_{i13} &= e^{-2\alpha h_1} R_i, \\
\Pi_{i14} &= \tilde{B}_i P_i, \\
\Pi_{i15} &= e^{-2\alpha h_2} R_i, \\
\Pi_{i22} &= h_1^2 R_i + h_2^2 R_i + \eta^2 U_i - 1.5P_i, \\
\Pi_{i24} &= \tilde{B}_i P_i, \\
\Pi_{i33} &= -e^{-2\alpha h_1} Q_i - e^{-2\alpha h_1} R_i - e^{-2\alpha h_2} U_i, \\
\Pi_{i34} &= e^{-2\alpha h_2} U_i, \\
\Pi_{i44} &= -2e^{-2\alpha h_2} U_i, \\
\Pi_{i45} &= e^{-2\alpha h_2} U_i, \\
\Pi_{i55} &= -2e^{-2\alpha h_2} U_i - 2e^{-2\alpha h_2} Q_i - 2e^{-2\alpha h_2} R_i,
\end{aligned}$$

then, the dynamical networks (2.14) are exponential synchronization. Moreover, the feedback control is

$$u_i(t) = \begin{cases} -L_i P_i^{-1} z_i(t), & n\omega \leq t \leq n\omega + \delta, \\ 0, & n\omega + \delta < t \leq (n+1)\omega. \end{cases} \quad (2.29)$$

2.2 Synchronization of master-slave neural networks with mixed time-varying delays via hybrid intermittent feedback control

Let us set

$$\begin{aligned}
\Upsilon &= e^{-2\alpha h_{1M}}, \quad \lambda_1 = \lambda_{\min}(P^{-1}), \\
\lambda_2 &= \lambda_{\max}(P^{-1}) + (h_{1m} + h_{1M})\lambda_{\max}(P^{-1}QP^{-1}) \\
&\quad + (h_{1m}^3 + h_{1M}^3)\lambda_{\max}(P^{-1}RP^{-1}) + \delta^3\lambda_{\max}(P^{-1}UP^{-1}) \\
&\quad + h_2^3\lambda_{\max}(P^{-1}Y^TS_1^{-1}YP^{-1}) + k_1^2\lambda_{\max}(HU_3^{-1}H) \\
&\quad + k_2^2\lambda_{\max}(P^{-1}Y^TS_2^{-1}YP^{-1}).
\end{aligned}$$

Theorem 2.2.1 *For some given scalars $0 < \alpha < \varepsilon$, the error system (1.11) with time-varying delay satisfying (1.13) are exponentially stable if there exist symmetric positive definite matrices P, Q, R, U, S_1, S_2 , diagonal matrices U_i , $i = 1, 2, 3$ and a matrix Y with appropriately dimensioned such that the following LMI holds:*

$$\Gamma_1 = \Gamma_1 - \begin{bmatrix} 0 & 0 & -I & I & 0 \end{bmatrix}^T \Upsilon U \begin{bmatrix} 0 & 0 & -I & I & 0 \end{bmatrix} < 0, \quad (2.30)$$

$$\Gamma_2 = \Gamma_1 - \begin{bmatrix} 0 & 0 & 0 & I & -I \end{bmatrix}^T \Upsilon U \begin{bmatrix} 0 & 0 & 0 & I & -I \end{bmatrix} < 0, \quad (2.31)$$

$$\Gamma_3 = \begin{bmatrix} \Pi_{11} & 2PF^T & PH^T & 2Y \\ * & -2U_1 & 0 & 0 \\ * & * & -U_3 & 0 \\ * & * & * & -2e^{-2\alpha h_2}S_1 \end{bmatrix} < 0, \quad (2.32)$$

$$\Gamma_4 = \begin{bmatrix} -0.1P & h_2^2Y^T \\ * & -h_2^2S_1 \end{bmatrix} < 0, \quad (2.33)$$

$$\Gamma_5 = \begin{bmatrix} -0.1e^{-2\alpha h_{1M}}U & 2PG^T \\ * & -2U_2 \end{bmatrix} < 0, \quad (2.34)$$

$$\Gamma_6 = \Gamma_2 - \begin{bmatrix} 0 & 0 & -I & I & 0 \end{bmatrix}^T \Upsilon U \begin{bmatrix} 0 & 0 & -I & I & 0 \end{bmatrix} < 0, \quad (2.35)$$

$$\Gamma_7 = \Gamma_2 - \begin{bmatrix} 0 & 0 & 0 & I & -I \end{bmatrix}^T \Upsilon U \begin{bmatrix} 0 & 0 & 0 & I & -I \end{bmatrix} < 0, \quad (2.36)$$

$$\Gamma_8 = \begin{bmatrix} \Pi_{11} & 2PF^T & k_1PH^T \\ * & -2U_1 & 0 \\ * & * & -k_1U_3 \end{bmatrix} < 0, \quad (2.37)$$

$$-\alpha\delta + (\varepsilon - \alpha)(\omega - \delta) < 0, \quad (2.38)$$

$$\Gamma_1 = \begin{bmatrix} \Gamma_{111} & \Gamma_{112} & \Gamma_{13} & 0 & \Gamma_{15} \\ * & \Gamma_{122} & 0 & 0 & 0 \\ * & * & \Gamma_{33} & \Gamma_{34} & 0 \\ * & * & * & \Gamma_{44} & \Gamma_{45} \\ * & * & * & * & \Gamma_{55} \end{bmatrix},$$

$$\Gamma_2 = \begin{bmatrix} \Gamma_{211} & \Gamma_{212} & \Gamma_{13} & 0 & \Gamma_{15} \\ * & \Gamma_{222} & 0 & 0 & 0 \\ * & * & \Gamma_{33} & \Gamma_{34} & 0 \\ * & * & * & \Gamma_{44} & \Gamma_{45} \\ * & * & * & * & \Gamma_{55} \end{bmatrix},$$

where

$$\begin{aligned} \Pi_{11} &= -0.1(e^{-2\alpha h_{1m}} + e^{-2\alpha h_{1M}})R \\ \Gamma_{111} &= [-A + \alpha I]P + P[-A + \alpha I]^T - B_1Y - Y^TB_1^T + 2Q \\ &\quad + C^TU_1C + D^TU_2D + 2k_1e^{2\alpha k_1}E^TU_3E + 3e^{2\alpha h_2}B_2^TS_1B_2 \\ &\quad + 2k_2e^{2\alpha k_2}B_3^TS_2B_3 - 0.9e^{-2\alpha h_{1m}}R - 0.9e^{-2\alpha h_{1M}}R, \\ \Gamma_{211} &= [-A + \alpha I]P + P[-A + \alpha I]^T + 2Q - 2\varepsilon P + C^TU_1C \\ &\quad + D^TU_2D + 2k_1e^{2\alpha k_1}E^TU_3E - 0.9e^{-2\alpha h_{1m}}R - 0.9e^{-2\alpha h_{1M}}R, \\ \Gamma_{112} &= -PA^T - Y^TB^T, \quad \Gamma_{212} = -PA^T, \quad \Gamma_{13} = e^{-2\alpha h_{1m}}R, \\ \Gamma_{15} &= e^{-2\alpha h_{1M}}R, \\ \Gamma_{122} &= h_{1m}^2R + h_{1M}^2R + \delta^2U - 1.9P + C^TU_1C + D^TU_2D \\ &\quad + 2k_1e^{2\alpha k_1}E^TU_3E + 3e^{2\alpha h_2}B_2^TS_1B_2 \\ &\quad + 2k_2e^{2\alpha k_2}B_3^TS_2B_3, \end{aligned}$$

$$\begin{aligned}
\Gamma_{222} &= h_{1m}^2 R + h_{1M}^2 R + \delta^2 U - 2P + C^T U_1 C + D^T U_2 D \\
&\quad + 2k_1 e^{2\alpha k_1} E^T U_3 E, \\
\Gamma_{33} &= -e^{-2\alpha h_{1m}} Q - e^{-2\alpha h_{1m}} R - e^{-2\alpha h_{1M}} U, \\
\Gamma_{34} &= e^{-2\alpha h_{1M}} U, \quad \Gamma_{44} = -1.9e^{-2\alpha h_{1M}} U, \quad \Gamma_{45} = e^{-2\alpha h_{1M}} U, \\
\Gamma_{55} &= -e^{-2\alpha h_{1M}} Q - e^{-2\alpha h_{1M}} R - e^{-2\alpha h_{1M}} U,
\end{aligned}$$

then the error system (1.11) have exponential synchronization. Moreover, the feedback control is

$$\mathcal{U}(t) = \begin{cases} -B_1 Y P^{-1} e(t) - B_2 Y P^{-1} e(t - h_2(t)) \\ -B_3 Y P^{-1} \int_{t-k_2(t)}^t e(s) ds, & n\omega \leq t \leq n\omega + \delta, \\ 0, & n\omega + \delta < t \leq (n+1)\omega. \end{cases} \quad (2.39)$$

Output

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ

1.1 **Thongchai Botmart** and Piyapong Niamsup, Exponential synchronization of complex dynamical network with mixed time-varying and hybrid coupling delays via intermittent control, *Advances in Difference Equations*, 2014, 2014:116, 1–33.

1.2 **Thongchai Botmart**, Exponential synchronization of master–slave neural networks with mixed time-varying delays via hybrid intermittent feedback control, *International Journal of Pure and Applied Mathematics*, 2014, Submitted.

2. การนำผลงานวิจัยไปใช้ประโยชน์

ผลงานวิจัยที่ได้มา มีการนำไปใช้ประโยชน์ทั้งเชิงวิชาการ และเชิงสาธารณะโดยทำให้มีการพัฒนาการเรียนการสอนและมีเครือข่ายความร่วมมือสร้างกระแสความสนใจในวงกว้าง

3. อื่นๆ: การเสนอผลงานในที่ประชุมวิชาการ

(1) ผลงานอื่นๆ คือ การนำเสนอผลงานงานวิจัย

ชื่อการจัดการประชุม : The Asian Mathematical Conference 2013 (AMC 2013)

สถานที่จัดประชุม : June 30, 2013 to July 4, 2013, BEXCO, Busan, Korea

ชื่อเรื่องที่น่าสนใจ : Exponential synchronization of complex dynamical network with mixed time-varying and hybrid coupling delays via feedback control

Appendix

- A1 **Thongchai Botmart** and Piyapong Niamsup, Exponential synchronization of complex dynamical network with mixed time-varying and hybrid coupling delays via intermittent control, *Advances in Difference Equations*, 2014, 2014:116, 1-33.
- A2 **Thongchai Botmart**, Exponential synchronization of master-slave neural networks with mixed time-varying delays via hybrid intermittent feedback control, *International Journal of Pure and Applied Mathematics*, 2014, Submitted.

- A1. **Thongchai Botmart** and Piyapong Niamsup, Exponential synchronization of complex dynamical network with mixed time-varying and hybrid coupling delays via intermittent control, *Advances in Difference Equations*, 2014, 2014:116 , 1-33.

RESEARCH

Open Access

Exponential synchronization of complex dynamical network with mixed time-varying and hybrid coupling delays via intermittent control

Thongchai Botmart^{1,2*} and Piyapong Niamsup^{2,3}

*Correspondence:

thongchaib@swu.ac.th

¹Department of Mathematics,
Srinakharinwirot University,
Sukhumvit 23 Road, Bangkok,
10110, Thailand

²Centre of Excellence in
Mathematics, CHE, Si Ayutthaya
Road, Bangkok, 10400, Thailand
Full list of author information is
available at the end of the article

Abstract

In this paper, we shall investigate the problem of exponential synchronization for complex dynamical network with mixed time-varying and hybrid coupling delays, which is composed of state coupling, interval time-varying delay coupling and distributed time-varying delay coupling. The designed controller ensures that the synchronization of delayed complex dynamical network are proposed via either feedback control or intermittent feedback control. The constraint on the derivative of the time-varying delay is not required which allows the time-delay to be a fast time-varying function. We use common unitary matrices, and the problem of synchronization is transformed into the stability analysis of some linear time-varying delay systems. This is based on the construction of an improved Lyapunov-Krasovskii functional combined with the Leibniz-Newton formula and the technique of dealing with some integral terms. New synchronization criteria are derived in terms of LMIs which can be solved efficiently by standard convex optimization algorithms. Two numerical examples are included to show the effectiveness of the proposed feedback control and intermittent feedback control scheme.

Keywords: exponential synchronization; complex dynamical network; mixed time-varying delays; hybrid coupling; intermittent control

1 Introduction

Complex dynamical network, as an interesting subject, has been thoroughly investigated for decades. These networks show very complicated behavior and can be used to model and explain many complex systems in nature such as computer networks [1], the world wide web [2], food webs [3], cellular and metabolic networks [4], social networks [5], electrical power grids [6] *etc.* In general, a complex network is a large set of interconnected nodes, in which a node is a fundamental unit with specific contents. As an implicit assumption, these networks are described by the mathematical term *graph*. In such graphs, each vertex represents an individual element in the system, while edges represent the relations between them. Two nodes are joined by an edge if and only if they interact.

In the last decade, the synchronization of complex dynamic networks has attracted much attention of researchers in this field [7–18]. Because the synchronization of complex dynamical networks can well explain many natural phenomena observed and is one of the important dynamical mechanisms for creating order in complex dynamical networks, the

synchronization of coupled dynamical networks has come to be a focal point in the study of nonlinear science. Wang and Chen introduced a uniform dynamical network model and also investigated its synchronization [11–13]. They have shown that the synchronizability of a scale-free dynamical network is robust against random removal of nodes, and yet it is fragile to specific removal of the most highly connected nodes [12]. The authors in [14, 15] investigated synchronization of general complex dynamical network models with coupling delays. Li and Chen [8] considered the synchronization stability of complex dynamical network models with coupling delays for both continuous- and discrete-time, and they derived some synchronization conditions for both delay-independent and delay-dependent asymptotical stabilities. By utilizing Lyapunov functional method, Wang *et al.* [16] introduced several synchronization criteria for both delay-independent and delay-dependent asymptotical stability. Li and Yi [17] investigated synchronization of complex networks with time-varying couplings, the stability criteria were obtained by using Lyapunov-Krasovskii function method and subspace projection method. Yue and Li [18] studied the synchronization stability of continuous and discrete complex dynamical networks with interval time-varying delays in the dynamical nodes and the coupling term simultaneously, delay-dependent synchronization stability are derived in the form of linear matrix inequalities.

It is well known that the existence of time-delay in a system may cause instability and an example of oscillations can be found in systems such as chemical engineering systems, biological modeling, electrical networks, physical networks, and many others [19–25]. The stability criteria for a system with time-delays can be classified into two categories: delay-independent and delay-dependent. Delay-independent criteria do not employ any information on the size of the delay; while delay-dependent criteria make use of such information at different levels. Delay-dependent stability conditions are generally less conservative than delay-independent ones especially when the delay is small [25]. Recently, the delay-dependent stability for interval time-varying delay was investigated in [6, 18, 20–22]. Interval time-varying delay is a time-delay that varies in an interval in which the lower bound is not restricted to be 0. Jiang and Han [22] considered the problem of robust H_∞ control for uncertain linear systems with interval time-varying delay based on Lyapunov functional approach in which restriction on the differentiability of the interval time-varying delay was removed. Shao [24] presented a new delay-dependent stability criterion for linear systems with interval time-varying delay, and stability criteria are derived in terms of linear matrix inequalities without introducing any free-weighting matrices. In order to reduce further the conservatism introduced by the descriptor model transformation and bounding techniques, a free-weighting matrix method is proposed in [20, 26–29]. In [18], the synchronization problem has been investigated for continuous/discrete complex dynamical networks with interval time-varying delays. Based on a piecewise analysis method and the Lyapunov functional method, some new delay-dependent synchronization criteria are derived in the form of LMIs by introducing free-weighting matrices. It will be pointed out later that some existing results require more free-weighting matrix variables than our result.

Intermittent control is one of discontinuous control and has a nonzero control width. It is an engineering approach that has been widely used in engineering fields, such as manufacturing, air-quality control, transportation, and communication in practice. However, results using intermittent control to study exponential synchronization are few. In recent

years, several synchronization criteria for complex dynamical networks with or without time-delays via feedback control or intermittent control have been presented; see [30–41] and the references therein. Synchronization of a complex dynamical network with delayed nodes by pinning periodically intermittent control was also reported in [31]. A periodically intermittent control was applied to the complex dynamical networks with both time-varying delays dynamical nodes and time-varying delays coupling in [32, 33]. In [34], the authors investigated exponential synchronization of a complex network with nonidentical time-delayed dynamical nodes by applying open-loop control to all nodes and adding some intermittent controllers to partial nodes. The authors in [31] investigated synchronization of a general model of complex delayed dynamical networks. The periodically intermittent control scheme is introduced to drive the network to achieve synchronization. Based on the Lyapunov stability theory and pinning control method, some novel synchronization criteria for such dynamical network are derived. To the best of the authors' knowledge, the problem of exponential synchronization for a complex dynamical network with mixed time-varying delays in the network hybrid coupling and time-varying delays in the dynamical nodes has not been fully investigated yet and remains open.

In this paper, inspired by the above discussions, we shall investigate the problem of exponential synchronization for a complex dynamical network with mixed time-varying and hybrid coupling delays, which is composed of constant coupling, interval time-varying delay coupling, and distributed time-varying delay coupling. The designed controller ensures that the synchronization of a delayed complex dynamical network is proposed via either feedback control or intermittent feedback control. The constraint on the derivative of the time-varying delay is not required, which allows the time-delay to be a fast time-varying function. We use common unitary matrices, and the problem of synchronization is transformed into the stability analysis of some linear time-varying delay systems. Based on the construction of an improved Lyapunov-Krasovskii functional is combined with the Leibniz-Newton formula and the technique of dealing with some integral terms. New synchronization criteria are derived in terms of LMIs which can be solved efficiently by standard convex optimization algorithms. Two numerical examples are included to show the effectiveness of the proposed feedback control and intermittent feedback control scheme.

The organization of the remaining part is as follows. In Section 2, a class of general complex dynamical network model with mixed time-varying and hybrid coupling delays and some useful lemmas are given. In Section 3, synchronization stability in complex dynamical network with mixed time-varying and hybrid coupling delays via feedback control and intermittent feedback control are investigated. Numerical examples illustrated the obtained results are given in Section 4. The paper ends with conclusions in Section 5.

2 Network model and mathematic preliminaries

Consider a complex dynamical network consisting of N identical coupled nodes, with each node being an n -dimensional dynamical system

$$\begin{aligned} \dot{x}_i(t) = & f\left(x_i(t), x_i(t-h(t)), \int_{t-k_1(t)}^t x_i(s) ds\right) + c_1 \sum_{j=1}^N a_{ij} G_1 x_j(t) + c_2 \sum_{j=1}^N b_{ij} G_2 x_j(t-h(t)) \\ & + c_3 \sum_{j=1}^N c_{ij} G_3 \int_{t-k_1(t)}^t x_j(s) ds + \mathcal{U}_i(t), \quad t \geq 0, i = 1, 2, \dots, N, \\ x_i(t) = & \phi_i(t), \quad t \in [-\tau_{\max}, 0], \tau_{\max} = \max\{h_2, d, k_1, k_2\}, \end{aligned} \quad (1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in \mathbb{R}^n$ is the state vector of i th node; $\mathcal{U}_i(t) \in \mathbb{R}^m$ are the control input of the node i ; the constants $c_1, c_2, c_3 > 0$ are the coupling strength; $G_1 = (g_{1ij})_{n \times n}$, $G_2 = (g_{2ij})_{n \times n}$, $G_3 = (g_{3ij})_{n \times n} \in \mathbb{R}^{n \times n}$ are constant inner-coupling matrices, if some pairs (i, j) , $1 \leq i, j \leq n$, with $g_{1ij} \neq 0$, $g_{2ij} \neq 0$, and $g_{3ij} \neq 0$, which means two coupled nodes are linked through their i th and j th state variables, otherwise $g_{1ij} = 0$, $g_{2ij} = 0$, $g_{3ij} = 0$; $A = (a_{ij})_{N \times N}$, $B = (b_{ij})_{N \times N}$, and $C = (c_{ij})_{N \times N} \in \mathbb{R}^{N \times N}$ are the outer-coupling matrices of the network, in which a_{ij} , b_{ij} are defined as follows: if there are a connection between node i and node j ($j \neq i$), then $a_{ij} = a_{ji} = 1$, $b_{ij} = b_{ji} = 1$, $c_{ij} = c_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0$, $b_{ij} = b_{ji} = 0$, $c_{ij} = c_{ji} = 0$ ($j \neq i$), and the diagonal elements of matrices A , B , and C are defined by

$$\begin{aligned} a_{ii} &= - \sum_{j=1, i \neq j}^N a_{ij} = - \sum_{j=1, i \neq j}^N a_{ji}, \\ b_{ii} &= - \sum_{j=1, i \neq j}^N b_{ij} = - \sum_{j=1, i \neq j}^N b_{ji}, \\ c_{ii} &= - \sum_{j=1, i \neq j}^N c_{ij} = - \sum_{j=1, i \neq j}^N c_{ji}, \quad i = 1, 2, \dots, N. \end{aligned} \quad (2)$$

It is assumed that network (1) is connected in the sense that there are no isolated clusters, that is, A , B , C are irreducible matrices.

Definition 2.1 [18] The delayed dynamical network (1) is said to achieve asymptotical synchronization if

$$x_1(t) = x_2(t) = \dots = s(t) \quad \text{as } t \rightarrow \infty, \quad (3)$$

where $s(t)$ is a solution of an isolated node, satisfying

$$\dot{s}(t) = f\left(s(t), s(t-h(t)), \int_{t-k_1(t)}^t s(\theta) d\theta\right).$$

In order to stabilize the origin of dynamical network (1) by means of the state feedback controller $\mathcal{U}_i(t)$ satisfying either (H1) or (H2), for $i = 1, 2, \dots, n$,

$$\begin{aligned} \text{(H1): } \mathcal{U}_i(t) &= D_{1i}u_i(t) + D_{2i}u_i(t-d(t)) \\ &\quad + D_{3i} \int_{t-k_2(t)}^t u_i(s) ds, \quad \forall t \geq t_0, \\ \text{(H2): } \mathcal{U}_i(t) &= \begin{cases} D_{4i}u_i(t) + D_{5i}u_i(t-d(t)) \\ \quad + D_{6i} \int_{t-k_2(t)}^t u_i(s) ds, & n\omega \leq t \leq n\omega + \delta, \\ 0, & n\omega + \delta < t \leq (n+1)\omega, \end{cases} \end{aligned}$$

where D_{ji} , $j = 1, 2, \dots, 6$ are given matrices of appropriate dimensions, $u_i(t) = K_i(x_i(t) - s(t))$ and K_i is a constant matrix control gain, $\omega > 0$ is the control period and $\delta > 0$ is called the control width (control duration) and n is a non-negative integer. Then substituting it into

dynamical network (1), it is easy to get the following:

$$\begin{aligned}\dot{x}_i(t) = & f\left(x_i(t), x_i(t-h(t)), \int_{t-k_1(t)}^t x_i(s) ds\right) + c_1 \sum_{j=1}^N a_{ij} G_1 x_j(t) \\ & + c_2 \sum_{j=1}^N b_{ij} G_2 x_j(t-h(t)) + c_3 \sum_{j=1}^N c_{ij} G_3 \int_{t-k_1(t)}^t x_j(s) ds \\ & + D_{1i} K_i(x_i(t) - s(t)) + D_{2i} u_i(t-d(t)) + D_{3i} \int_{t-k_2(t)}^t u_i(s) ds.\end{aligned}\quad (4)$$

Namely, the dynamical network (1) is governed by the following system:

$$\begin{aligned}\dot{x}_i(t) = & f\left(x_i(t), x_i(t-h(t)), \int_{t-k_1(t)}^t x_i(s) ds\right) + c_1 \sum_{j=1}^N a_{ij} G_1 x_j(t) \\ & + c_2 \sum_{j=1}^N b_{ij} G_2 x_j(t-h(t)) + c_3 \sum_{j=1}^N c_{ij} G_3 \int_{t-k_1(t)}^t x_j(s) ds \\ & + D_{4i} K_i(x_i(t) - s(t)) + D_{5i} u_i(t-d(t)) + D_{6i} \int_{t-k_2(t)}^t u_i(s) ds, \\ n\omega \leq & t \leq n\omega + \delta,\end{aligned}\quad (5)$$

$$\begin{aligned}\dot{x}_i(t) = & f\left(x_i(t), x_i(t-h(t)), \int_{t-k_1(t)}^t x_i(s) ds\right) + c_1 \sum_{j=1}^N a_{ij} G_1 x_j(t) \\ & + c_2 \sum_{j=1}^N b_{ij} G_2 x_j(t-h(t)) + c_3 \sum_{j=1}^N c_{ij} G_3 \int_{t-k_1(t)}^t x_j(s) ds, \\ n\omega + \delta < & t \leq (n+1)\omega, i = 1, 2, \dots, N.\end{aligned}$$

It is clear that, if the zero solutions of the dynamical network (4) and (5) are globally exponentially stable, then exponential synchronization of the controlled dynamical network (1) is achieved. The time-varying delay functions $h(t)$, $d(t)$, $k_1(t)$, and $k_2(t)$ satisfy the conditions

$$0 \leq h_1 \leq h(t) \leq h_2, \quad 0 \leq d(t) \leq d, \quad 0 \leq k_1(t) \leq k_1, \quad 0 \leq k_2(t) \leq k_2. \quad (6)$$

The initial condition function $\phi_i(t)$ denotes a continuous vector-valued initial function of $t \in [-\tau_{\max}, 0]$.

In this paper, we assume that $s(t)$ is an orbitally stable solution of the above system. Clearly, the stability of the synchronized states (3) of network (1) is determined by the dynamics of the isolate node, the coupling strength c_1 , c_2 , and c_3 , the inner-coupling matrices G_1 , G_2 , and G_3 , and the outer-coupling matrices A , B , and C .

The following lemmas are used in the proof of the main result.

Lemma 2.2 [42] *Let A, B be a family of diagonalizable matrices. Then A, B is a commuting family (under multiplication) if and only if it is a simultaneously diagonalizable family.*

Lemma 2.3 [19] *For any constant symmetric matrix $M \in \mathbb{R}^{n \times n}$, $M = M^T > 0$, $0 \leq h_1 \leq h(t) \leq h_2$, $t \geq 0$, and any differentiable vector function $x(t) \in \mathbb{R}^n$, we have*

$$\begin{aligned} \text{(a)} \quad & \left[\int_{t-h_1}^t \dot{x}(s) ds \right]^T M \left[\int_{t-h_1}^t \dot{x}(s) ds \right] \leq h_1 \int_{t-h_1}^t \dot{x}^T(s) M \dot{x}(s) ds, \\ \text{(b)} \quad & \left[\int_{t-h(t)}^{t-h_1} \dot{x}(s) ds \right]^T M \left[\int_{t-h(t)}^{t-h_1} \dot{x}(s) ds \right] \leq (h(t) - h_1) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s) M \dot{x}(s) ds \\ & \leq (h_2 - h_1) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s) M \dot{x}(s) ds. \end{aligned}$$

Lemma 2.4 (Cauchy inequality [19]) *For any symmetric positive definite matrix $N \in \mathbb{R}^{n \times n}$ and $x, y \in \mathbb{R}^n$ we have*

$$\pm 2x^T y \leq x^T N x + y^T N^{-1} y.$$

3 Synchronization of delayed complex dynamical network via delayed feedback control and intermittent control

In this section, we shall obtain some delay-dependent exponential synchronization criteria for general complex dynamical network with discrete and distributed time-varying delays and hybrid coupling delays (1) by strict LMI approaches. Let us set

$$\tilde{A}_i = J(t) + c_1 \lambda_{1i} G_1, \quad \tilde{B}_i = J_h(t) + c_2 \lambda_{2i} G_2, \quad \tilde{C}_i = J_{k_1}(t) + c_3 \lambda_{3i} G_3$$

and

1. $J(t) = f'(s(t), s(t-h(t)), \int_{t-k_1(t)}^t s(\xi) d\xi) \in \mathbb{R}^{n \times n}$ is the Jacobian of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^t x(s) ds)$ at $s(t)$ with the derivative of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^t x(s) ds)$ respect to $x(t)$,
2. $J_h(t) = f'(s(t), s(t-h(t)), \int_{t-k_1(t)}^t s(\xi) d\xi) \in \mathbb{R}^{n \times n}$ is the Jacobian of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^t x(s) ds)$ at $s(t-h(t))$ with the derivative of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^t x(s) ds)$ respect to $x(t-h(t))$,
3. $J_{k_1}(t) = f'(s(t), s(t-h(t)), \int_{t-k_1(t)}^t s(\xi) d\xi) \in \mathbb{R}^{n \times n}$ is the Jacobian of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^t x(s) ds)$ at $\int_{t-k_1(t)}^t s(\xi) d\xi$ with the derivative of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^t x(s) ds)$ respect to $\int_{t-k_1(t)}^t x(s) ds$.

Lemma 3.1 *Consider the hybrid coupling delays dynamical network in (1). Let $0 = \lambda_{j1} > \lambda_{j2} \geq \lambda_{j3} \geq \dots \geq \lambda_{jN}$, $j = \{1, 2, 3\}$, be the eigenvalues of the outer-coupling matrices A , B , and C , respectively. If the $N-1$ following n -dimensional linear time-varying delays differential equations are delay-dependent exponentially stable about their zero solutions:*

$$\begin{aligned} \dot{z}_i(t) = & (\tilde{A}_i + D_{4i} K_i) z_i(t) + \tilde{B}_i z_i(t-h(t)) + \tilde{C}_i \int_{t-k_1(t)}^t z_i(s) ds \\ & + D_{5i} K_i z_i(t-d(t)) + D_{6i} K_i \int_{t-k_2(t)}^t z_i(s) ds, \quad n\omega \leq t \leq n\omega + \delta, i = 2, \dots, N, \quad (7) \\ \dot{z}_i(t) = & \tilde{A}_i z_i(t) + \tilde{B}_i z_i(t-h(t)) + \tilde{C}_i \int_{t-k_1(t)}^t z_i(s) ds, \quad n\omega + \delta < t \leq (n+1)\omega, i = 2, \dots, N, \end{aligned}$$

then the dynamical networks (5) is exponentially stable, and then exponential synchronization of the controlled dynamical networks (1) is achieved.

Proof To investigate the stability of the synchronized states (3), set

$$e_i(t) = x_i(t) - s(t), \quad i = 1, 2, \dots, N. \quad (8)$$

Substituting (8) into (5), for $1 \leq i \leq N$, we have

$$\begin{aligned} \dot{e}_i(t) = & f\left(x_i(t), x_i(t-h(t)), \int_{t-k_1(t)}^t x_i(s) ds\right) - f\left(s(t), s(t-h(t)), \int_{t-k_1(t)}^t s_i(\xi) d\xi\right) \\ & + c_1 \sum_{j=1}^N a_{ij} G_1 e_j(t) + c_2 \sum_{j=1}^N b_{ij} G_2 e_j(t-h(t)) \\ & + c_3 \sum_{j=1}^N c_{ij} G_3 \int_{t-k_1(t)}^t e_j(s) ds + D_{4i} K_i(e_i(t)) + D_{5i} K_i(e_i(t-d(t))) \\ & + D_{6i} K_i \int_{t-k_2(t)}^t e_j(s) ds, \quad n\omega \leq t \leq n\omega + \delta, \\ \dot{e}_i(t) = & f\left(x_i(t), x_i(t-h(t)), \int_{t-k_1(t)}^t x_i(s) ds\right) - f\left(s(t), s(t-h(t)), \int_{t-k_1(t)}^t s_i(\xi) d\xi\right) \\ & + c_1 \sum_{j=1}^N a_{ij} G_1 e_j(t) + c_2 \sum_{j=1}^N b_{ij} G_2 e_j(t-h(t)) \\ & + c_3 \sum_{j=1}^N c_{ij} G_3 \int_{t-k_1(t)}^t e_j(s) ds, \quad n\omega + \delta < t \leq (n+1)\omega, i = 1, 2, \dots, N. \end{aligned} \quad (9)$$

Since $f(\cdot)$ is continuous differentiable, it is easy to know that the origin of the nonlinear system (9) is an asymptotically stable equilibrium point if it is an asymptotically stable equilibrium point of the following linear time-varying delays systems:

$$\begin{aligned} \dot{e}_i(t) = & J(t)e_i(t) + J_h(t)e_i(t-h(t)) + J_{k_1}(t) \int_{t-k_1(t)}^t e_i(s) ds \\ & + c_1 G_1(e_1(t), e_2(t), \dots, e_N(t))(a_{i1}, \dots, a_{iN})^T \\ & + c_2 G_2(e_1(t-h(t)), \dots, e_N(t-h(t)))(b_{i1}, \dots, b_{iN})^T \\ & + c_3 G_3 \int_{t-k_1(t)}^t (e_1(s), e_2(s), \dots, e_N(s))(c_{i1}, \dots, c_{iN})^T ds \\ & + D_{4i} K_i e_i(t) + D_{5i} K_i e_i(t-d(t)) + D_{6i} K_i \int_{t-k_2(t)}^t e_j(s) ds, \\ & n\omega \leq t \leq n\omega + \delta, \\ \dot{e}_i(t) = & J(t)e_i(t) + J_h(t)e_i(t-h(t)) + J_{k_1}(t) \int_{t-k_1(t)}^t e_i(s) ds \\ & + c_1 G_1(e_1(t), e_2(t), \dots, e_N(t))(a_{i1}, \dots, a_{iN})^T \\ & + c_2 G_2(e_1(t-h(t)), \dots, e_N(t-h(t)))(b_{i1}, \dots, b_{iN})^T \end{aligned}$$

$$+ c_3 G_3 \int_{t-k_1(t)}^t (e_1(s), e_2(s), \dots, e_N(s))(c_{i1}, \dots, c_{iN})^T ds,$$

$$n\omega + \delta < t \leq (n+1)\omega.$$

Letting $e(t) = (e_1(t), \dots, e_N(t)) \in R^{n \times N}$, $e(t-h(t)) = (e_1(t-h(t)), \dots, e_N(t-h(t))) \in R^{n \times N}$, $\int_{t-k_1(t)}^t e(s) ds = \int_{t-k_1(t)}^t (e_1(s), e_2(s), \dots, e_N(s)) ds \in R^{n \times N}$, $K = \text{diag}\{K_1, K_2, \dots, K_N\}$, and $D_j = \text{diag}\{D_{j1}, D_{j2}, \dots, D_{jN}\}$, $j = \{4, 5, 6\}$, we have

$$\begin{aligned} \dot{e}(t) &= (J(t) + DK)e(t) + J_h(t)e(t-h(t)) + J_{k_1}(t) \int_{t-k_1(t)}^t e(s) ds + c_1 G_1 e(t) A^T \\ &\quad + c_2 G_2 e(t-h(t)) B^T + c_3 G_3 \int_{t-k_1(t)}^t e(s) C^T ds + D_5 K e(t-d(t)) \\ &\quad + D_6 K \int_{t-k_2(t)}^t e(s) ds, \quad n\omega \leq t \leq n\omega + \delta, \\ \dot{e}(t) &= J(t)e(t) + J_h(t)e(t-h(t)) + J_{k_1}(t) \int_{t-k_1(t)}^t e(s) ds + c_1 G_1 e(t) A^T \\ &\quad + c_2 G_2 e(t-h(t)) B^T + c_3 G_3 \int_{t-k_1(t)}^t e(s) C^T ds, \quad n\omega + \delta < t \leq (n+1)\omega. \end{aligned} \quad (10)$$

Obviously, A, B, C are diagonalizable. If A, B , and C commute pairwise, i.e., $AB = BA$, then based on Lemma 2.2, one can get a common unitary matrix $\hat{U} \in R^{N \times N}$ with $\hat{u}_i \in \mathbb{R}^n$ such that

$$\hat{U}^T A \hat{U} = \Gamma_1, \quad \hat{U}^T B \hat{U} = \Gamma_2, \quad \hat{U}^T C \hat{U} = \Gamma_3,$$

where $\hat{U}^T \hat{U} = I$, $\Gamma_j = \text{diag}\{\lambda_{1j}, \dots, \lambda_{Nj}\}$, $j = \{1, 2, 3\}$. In addition, with (2) and the irreducible feature of A, B , and C we can select with $\hat{u}_1 = \frac{1}{\sqrt{N}}(1, 1, \dots, 1)^T$ such that $\lambda_{1j} = 0$, $j = \{1, 2, 3\}$.

Using the nonsingular transform $e(t)\hat{U} = z(t) = (z_1(t), \dots, z_N(t)) \in R^{N \times N}$, from (10), we have the following matrix equation:

$$\begin{aligned} \dot{z}(t) &= (J(t) + DK)z(t) + J_h(t)z(t-h(t)) + J_{k_1}(t) \int_{t-k_1(t)}^t z(s) ds + c_1 G_1 z(t) \Gamma_1 \\ &\quad + c_2 G_2 z(t-h(t)) \Gamma_2 + c_3 G_3 \int_{t-k(t)}^t z(s) \Gamma_3 ds + D_5 K z(t-d(t)) \\ &\quad + D_6 K \int_{t-k_2(t)}^t z(s) ds, \quad n\omega \leq t \leq n\omega + \delta, \\ \dot{z}(t) &= J(t)z(t) + J_h(t)z(t-h(t)) + J_{k_1}(t) \int_{t-k_1(t)}^t z(s) ds + c_1 G_1 z(t) \Gamma_1 \\ &\quad + c_2 G_2 z(t-h(t)) \Gamma_2 + c_3 G_3 \int_{t-k_1(t)}^t z(s) \Gamma_3 ds, \quad n\omega + \delta < t \leq (n+1)\omega, \end{aligned}$$

that is,

$$\begin{aligned} \dot{z}_i(t) &= (\tilde{A}_i + D_{4i} K_i) z_i(t) + \tilde{B}_i z_i(t-h(t)) + \tilde{C}_i \int_{t-k_1(t)}^t z_i(s) ds \\ &\quad + D_{5i} K_i z_i(t-d(t)) + D_{6i} K_i \int_{t-k_2(t)}^t z_i(s) ds, \quad n\omega \leq t \leq n\omega + \delta, \end{aligned}$$

$$\dot{z}_i(t) = \tilde{A}_i z_i(t) + \tilde{B}_i z_i(t - h(t)) + \tilde{C}_i \int_{t-k_1(t)}^t z_i(s) ds,$$

$$n\omega + \delta < t \leq (n+1)\omega, i = 1, \dots, N.$$

Thus, we have transformed the stability problem of the dynamical networks (5) to the stability problem of the N pieces of n -dimensional linear time-varying delays differential equations. Note that $\lambda_{1k} = 0$ corresponding to the synchronization of the dynamical networks (5), where the state $s(t)$ is an orbitally stable solution of the isolate node as assumed above in (3). If the following $N - 1$ pieces of n -dimensional linear switched time-varying delays systems:

$$\begin{aligned} \dot{z}_i(t) &= (\tilde{A}_i + D_{4i}K_i)z_i(t) + \tilde{B}_i z_i(t - h(t)) + \tilde{C}_i \int_{t-k_1(t)}^t z_i(s) ds \\ &\quad + D_{5i}K_i z_i(t - d(t)) + D_{6i}K_i \int_{t-k_2(t)}^t z_i(s) ds, \quad n\omega \leq t \leq n\omega + \delta, \\ \dot{z}_i(t) &= \tilde{A}_i z_i(t) + \tilde{B}_i z_i(t - h(t)) + \tilde{C}_i \int_{t-k_1(t)}^t z_i(s) ds, \\ n\omega + \delta &< t \leq (n+1)\omega, i = 2, \dots, N, \end{aligned}$$

are exponentially stable, then $e(t)$ will tend to the origin exponentially, which is equivalent to the synchronization of the dynamical networks (5) being exponentially stable. This completes the proof. $\square$

Lemma 3.2 Consider the hybrid coupling delays dynamical network in (1). Let $0 = \lambda_{j1} > \lambda_{j2} \geq \lambda_{j3} \geq \dots \geq \lambda_{jN}$, $j = \{1, 2, 3\}$, be the eigenvalues of the outer-coupling matrices A , B , and C , respectively. If the $N - 1$ following n -dimensional linear time-varying delays differential equations are delay-dependent exponentially stable about their zero solutions:

$$\begin{aligned} \dot{z}_i(t) &= (\tilde{A}_i + D_{1i}K_i)z_i(t) + \tilde{B}_i z_i(t - h(t)) + \tilde{C}_i \int_{t-k_1(t)}^t z_i(s) ds \\ &\quad + D_{2i}K_i z_i(t - d(t)) + D_{3i}K_i \int_{t-k_2(t)}^t z_i(s) ds, \quad i = 2, \dots, N, \end{aligned} \quad (11)$$

then the dynamical networks (4) is exponentially stable, then exponential synchronization of the controlled dynamical networks (1) is achieved.

3.1 Linear delayed feedback control

Let us denote

$$\begin{aligned} \|\phi_i\| &= \|z_i(0)\|, \quad \|\varphi_i\| = \sup_{-\tau_{\max} \leq s \leq 0} \|z_i(s)\|, \quad K_i = -L_i P_i^{-1}, \\ \gamma_i &= \lambda_{\min}(P_i^{-1}), \\ \ell_i &= \lambda_{\max}(P_i^{-1}) + [2h_2 \lambda_{\max}(P_i^{-1} R_i P_i^{-1}) + h_2 \lambda_{\max}(P_i^{-1} U_i P_i^{-1})] \frac{1 - e^{-2\alpha h_2}}{2\alpha} \\ &\quad + d \lambda_{\max}(P_i^{-1} L_i^T T_i^{-1} L_i P_i^{-1}) \frac{1 - e^{-2\alpha d}}{2\alpha}, \end{aligned}$$

$$\begin{aligned}\xi_i &= [2\lambda_{\max}(P_i^{-1}Q_iP_i^{-1}) + h_2\lambda_{\max}(P_i^{-1}R_iP_i^{-1}) + h_2\lambda_{\max}(P_i^{-1}U_iP_i^{-1})] \\ &\quad \times \frac{1 - e^{-2\alpha h_2}}{2\alpha} + k_1\lambda_{\max}(P_i^{-1}S_iP_i^{-1})\frac{1 - e^{-2\alpha k_1}}{2\alpha} \\ &\quad + d\lambda_{\max}(P_i^{-1}L_i^T T_i^{-1} L_i P_i^{-1})\frac{1 - e^{-2\alpha h_2}}{2\alpha} \\ &\quad + k_2\lambda_{\max}(P_i^{-1}L_i^T W_i^{-1} L_i P_i^{-1})\frac{1 - e^{-2\alpha d}}{2\alpha},\end{aligned}$$

$$\mathcal{N}_i = \ell_i \|\phi_i\|^2 + \xi_i \|\varphi_i\|^2,$$

$$\gamma = \min\{\gamma_i, i = 2, 3, \dots, N\}, \quad \mathcal{N} = \max\{\mathcal{N}_i, i = 2, 3, \dots, N\}.$$

Theorem 3.3 For some given scalars $0 < \alpha$, the dynamical networks (11) with time-varying delay satisfying (6) are exponentially stable if there exist symmetric positive definite matrices $P_i > 0$, $Q_i > 0$, $R_i > 0$, $S_i > 0$, $U_i > 0$, $T_i > 0$, $W_i > 0$, and a matrix L_i appropriately dimensioned such that the following symmetric linear matrix inequality holds:

$$\Sigma_{i1} = \Sigma_i - [0 \quad 0 \quad I \quad -I \quad 0]^T e^{-2\alpha h_2} U_i [0 \quad 0 \quad I \quad -I \quad 0] < 0, \quad (12)$$

$$\Sigma_{i2} = \Sigma_i - [0 \quad 0 \quad 0 \quad I \quad -I]^T e^{-2\alpha h_2} U_i [0 \quad 0 \quad 0 \quad I \quad -I] < 0, \quad (13)$$

$$\Sigma_{i3} = \begin{bmatrix} -0.5(e^{-2\alpha h_1} + e^{-2\alpha h_2})R_i & 2k_1\tilde{C}_iP_i & k_2L_i^T & 2L_i^T \\ * & -2k_1e^{-2\alpha k_1}S_i & 0 & 0 \\ * & * & -k_2W_i & 0 \\ * & * & * & -2e^{-2\alpha d}T_i \end{bmatrix} < 0, \quad (14)$$

$$\Sigma_{i4} = \begin{bmatrix} -0.5P_i & 2k_1\tilde{C}_iP_i & d^2L_i^T & 3D_{2i}^T & 2k_2D_{3i}^T \\ * & -2k_1e^{-2\alpha k_1}S_i & 0 & 0 & 0 \\ * & * & -d^2T_i & 0 & 0 \\ * & * & * & -3e^{-2\alpha d}T_i & 0 \\ * & * & * & * & -2k_2e^{-2\alpha k_2}W_i \end{bmatrix} < 0, \quad (15)$$

$i = 2, \dots, N$, where

$$\Sigma_i = \begin{bmatrix} \Sigma_{i11} & \Sigma_{i12} & \Sigma_{i13} & \Sigma_{i14} & \Sigma_{i15} \\ * & \Sigma_{i22} & 0 & \Sigma_{i24} & 0 \\ * & * & \Sigma_{i33} & \Sigma_{i34} & 0 \\ * & * & * & \Sigma_{i44} & \Sigma_{i45} \\ * & * & * & * & \Sigma_{i55} \end{bmatrix},$$

$$\begin{aligned}\Sigma_{i11} &= P_i^T(\tilde{A}_i + \alpha I) + (\tilde{A}_i + \alpha I)^T P_i - D_{1i}L_i - L_i^T D_{1i}^T + 3e^{2\alpha d}D_{2i}^T T_i D_{2i} \\ &\quad + 2k_2e^{2\alpha k_2}D_{3i}^T W_i D_{3i} + 2Q_i + k_1S_i - 0.5e^{-2\alpha h_1}R_i - 0.5e^{-2\alpha h_2}R_i,\end{aligned}$$

$$\Sigma_{i12} = P_i\tilde{A}_i^T,$$

$$\Sigma_{i13} = e^{-2\alpha h_1}R_i,$$

$$\Sigma_{i14} = \tilde{B}_iP_i,$$

$$\Sigma_{i15} = e^{-2\alpha h_2}R_i,$$

$$\Sigma_{i22} = h_1^2R_i + h_2^2R_i + \eta^2U_i - 1.5P_i,$$

$$\begin{aligned}\Sigma_{i24} &= \tilde{B}_i P_i, \\ \Sigma_{i33} &= -e^{-2\alpha h_1} Q_i - e^{-2\alpha h_1} R_i - e^{-2\alpha h_2} U_i, \\ \Sigma_{i34} &= e^{-2\alpha h_2} U_i, \\ \Sigma_{i44} &= -2e^{-2\alpha h_2} U_i, \\ \Sigma_{i45} &= e^{-2\alpha h_2} U_i, \\ \Sigma_{i55} &= -2e^{-2\alpha h_2} U_i - 2e^{-2\alpha h_2} Q_i - 2e^{-2\alpha h_2} R_i,\end{aligned}$$

then the dynamical networks (11) have exponential synchronization. Moreover, the feedback control is

$$u_i(t) = -L_i P_i^{-1} z_i(t). \quad (16)$$

Proof Let $Y_i = P_i^{-1}$, $y_i(t) = Y_i z_i(t)$. Using the feedback control (16) we consider the following Lyapunov-Krasovskii functional:

$$\begin{aligned}V_i(z_i(t)) &= V_{i1}(t) + V_{i2}(t) + V_{i3}(t) + V_{i4}(t) + V_{i5}(t) + V_{i6}(t) + V_{i7}(t) \\ &\quad + V_{i8}(t) + V_{i9}(t),\end{aligned} \quad (17)$$

where

$$\begin{aligned}V_{i1}(t) &= z_i^T(t) Y_i z_i(t), \\ V_{i2}(t) &= \int_{t-h_1}^t e^{2\alpha(s-t)} z_i^T(s) Y_i Q_i Y_i z_i(s) ds, \\ V_{i3}(t) &= \int_{t-h_2}^t e^{2\alpha(s-t)} z_i^T(s) Y_i Q_i Y_i z_i(s) ds, \\ V_{i4}(t) &= h_1 \int_{-h_1}^0 \int_{t+s}^t e^{2\alpha(\tau-t)} \dot{z}_i^T(\tau) Y_i R_i Y_i \dot{z}_i(\tau) d\tau ds, \\ V_{i5}(t) &= h_2 \int_{-h_2}^0 \int_{t+s}^t e^{2\alpha(\tau-t)} \dot{z}_i^T(\tau) Y_i R_i Y_i \dot{z}_i(\tau) d\tau ds, \\ V_{i6}(t) &= (h_2 - h_1) \int_{t-h_2}^{t-h_1} \int_{t+s}^t e^{2\alpha(\tau-t)} \dot{z}_i^T(\tau) Y_i U_i Y_i \dot{z}_i(\tau) d\tau ds, \\ V_{i7}(t) &= \int_{-k_1}^0 \int_{t+s}^t e^{2\alpha(\tau-t)} \dot{z}_i^T(\tau) Y_i S_i Y_i \dot{z}_i(\tau) d\tau ds, \\ V_{i8}(t) &= d \int_{-d}^0 \int_{t+s}^t e^{2\alpha(\tau-t)} \dot{z}_i^T(\tau) K_i^T T_i^{-1} K_i \dot{z}_i(\tau) d\tau ds, \\ V_{i9}(t) &= \int_{-k_2}^0 \int_{t+s}^t e^{2\alpha(\tau-t)} \dot{z}_i^T(\tau) K_i^T W_i^{-1} K_i \dot{z}_i(\tau) d\tau ds.\end{aligned}$$

It easy to check that

$$\gamma \|z_i(t)\|^2 \leq V_i(z_i(t)), \quad \forall t \geq 0. \quad (18)$$

By taking the derivative of $V_{i1}(t)$ along the trajectories of system (11), we have the following:

$$\begin{aligned}\dot{V}_{i1}(t) &= 2z_i^T(t)Y_i\dot{z}_i(t) \\ &= 2y_i^T(t)\left[(\tilde{A}_i + D_{1i}K_i)z_i(t) + \tilde{B}_iz_i(t-h(t)) + \tilde{C}_i\int_{t-k_1(t)}^t z_i(s)ds\right. \\ &\quad \left.+ D_{2i}K_iz_i(t-d(t)) + D_{3i}K_i\int_{t-k_2(t)}^t z_i(s)ds\right] \\ &= y_i^T(t)[P_i\tilde{A}_i + \tilde{A}_i^TP_i]y_i(t) + 2y_i^T(t)\tilde{B}_iP_iy_i(t-h(t)) \\ &\quad + 2y_i^T(t)\tilde{C}_iP_i\int_{t-k_1(t)}^t y_i(s)ds - 2y_i^T(t)D_{1i}L_i^Ty_i(t) + 2y_i^T(t)D_{2i}u_i(t-d(t)) \\ &\quad + 2y_i^T(t)D_{3i}\int_{t-k_2(t)}^t u_i(s)ds + 2y_i^T(t)\alpha P_iy_i(t) - 2y_i^T(t)\alpha P_iy_i(t).\end{aligned}$$

Applying Lemma 2.4 and Lemma 2.3 gives

$$\begin{aligned}2y_i^T(t)\tilde{C}_iP_i\int_{t-k_1(t)}^t y_i(s)ds &\leq 2k_1e^{2\alpha k_1}y_i^T(t)\tilde{C}_iP_iS_i^{-1}P_i\tilde{C}_i^Ty_i(t) \\ &\quad + \frac{e^{-2\alpha k_1}}{2k_1}\left(\int_{t-k_1(t)}^t y_i(s)ds\right)^TS_i\left(\int_{t-k_1(t)}^t y_i(s)ds\right) \\ &\leq 2k_1e^{2\alpha k_1}y_i^T(t)\tilde{C}_iP_iS_i^{-1}P_i\tilde{C}_i^Ty_i(t) \\ &\quad + \frac{1}{2}e^{-2\alpha k_1}\int_{t-k_1(t)}^t y_i^T(s)S_iy_i(s)ds, \\ 2y_i^T(t)D_{2i}u_i(t-d(t)) &\leq 3e^{2\alpha d}y_i^T(t)D_{2i}T_iD_{2i}^Ty_i(t) \\ &\quad + \frac{e^{-2\alpha d}}{3}u_i^T(t-d(t))T_i^{-1}u_i(t-d(t)), \\ 2y_i^T(t)D_{3i}\int_{t-k_2(t)}^t u_i(s)ds &\leq 2k_2e^{2\alpha k_2}y_i^T(t)D_{3i}W_iD_{3i}^Ty_i(t) \\ &\quad + \frac{e^{2\alpha k_2}}{2k_2}\left(\int_{t-k_2(t)}^t u_i(s)ds\right)^TW_i^{-1}\left(\int_{t-k_2(t)}^t u_i(s)ds\right) \\ &\leq 2k_2e^{2\alpha k_2}y_i^T(t)D_{3i}W_iD_{3i}^Ty_i(t) \\ &\quad + \frac{e^{2\alpha k_2}}{2}\int_{t-k_2(t)}^t u_i^T(s)W_i^{-1}u_i(s)ds.\end{aligned}$$

Therefore

$$\begin{aligned}\dot{V}_{i1}(t) + 2\alpha V_{i1}(t) &\leq y_i^T(t)[P_i\tilde{A}_i + \tilde{A}_i^TP_i]y_i(t) + 2y_i^T(t)\alpha P_iy_i(t) \\ &\quad + 2y_i^T(t)\tilde{B}_iP_iy_i(t-h(t)) - 2y_i^T(t)D_{1i}L_i^Ty_i(t) \\ &\quad + 2k_1e^{2\alpha k_1}y_i^T(t)\tilde{C}_iP_iS_i^{-1}P_i\tilde{C}_i^Ty_i(t) \\ &\quad + \frac{1}{2}e^{-2\alpha k_1}\int_{t-k_1(t)}^t y_i^T(s)S_iy_i(s)ds \\ &\quad + 3e^{2\alpha d}y_i^T(t)D_{2i}T_iD_{2i}^Ty_i(t) \\ &\quad + \frac{e^{-2\alpha d}}{3}u_i^T(t-d(t))T_i^{-1}u_i(t-d(t))\end{aligned}$$

$$+ 2k_2 e^{2\alpha k_2} y_i^T(t) D_{3i} W_i D_{3i}^T y_i(t) + \frac{e^{2\alpha k_2}}{2} \int_{t-k_2(t)}^t u_i^T(s) W_i^{-1} u_i(s) ds. \quad (19)$$

Next, by taking the derivative of $V_{ij}(t)$, $j = 2, 3, \dots, 9$ along the trajectories of system (11), we have the following:

$$\begin{aligned} \dot{V}_{i2}(t) &\leq y_i^T(t) Q_i y_i(t) - e^{-2\alpha h_1} y_i^T(t-h_1) Q_i y_i(t-h_1) - 2\alpha V_{i2}(t), \\ \dot{V}_{i3}(t) &\leq y_i^T(t) Q_i y_i(t) - e^{-2\alpha h_2} y_i^T(t-h_2) Q_i y_i(t-h_2) - 2\alpha V_{i3}(t), \\ \dot{V}_{i4}(t) &\leq h_1^2 \dot{y}_i^T(t) R_i \dot{y}_i(t) - h_1 e^{-2\alpha h_1} \int_{t-h_1}^t \dot{y}_i^T(s) R_i \dot{y}_i(s) ds - 2\alpha V_{i4}(t), \\ \dot{V}_{i5}(t) &\leq h_2^2 \dot{y}_i^T(t) R_i \dot{y}_i(t) - h_2 e^{-2\alpha h_2} \int_{t-h_2}^t \dot{y}_i^T(s) R_i \dot{y}_i(s) ds - 2\alpha V_{i5}(t), \\ \dot{V}_{i6}(t) &\leq \eta^2 \dot{y}_i^T(t) U_i \dot{y}_i(t) - \eta e^{-2\alpha h_2} \int_{t-h_2}^{t-h_1} \dot{y}_i^T(s) U_i \dot{y}_i(s) ds - 2\alpha V_{i6}(t), \\ \dot{V}_{i7}(t) &\leq k_1 y_i^T(t) S_i y_i(t) - e^{-2\alpha k_1} \int_{t-k_1(t)}^t y_i^T(s) S_i y_i(s) ds - 2\alpha V_{i7}(t), \\ \dot{V}_{i8}(t) &\leq d^2 \dot{z}_i^T(t) K_i^T T_i^{-1} K_i \dot{z}_i(t) - d e^{-2\alpha d} \int_{t-d}^t \dot{z}_i^T(s) K_i^T T_i^{-1} K_i \dot{z}_i(s) ds - 2\alpha V_{i8}(t) \\ &\leq d^2 \dot{y}_i^T(t) P_i K_i^T T_i^{-1} K_i P_i \dot{y}_i(t) - d(t) e^{-2\alpha d} \int_{t-d(t)}^t \dot{u}_i^T(s) T_i^{-1} \dot{u}_i(s) ds - 2\alpha V_{i8}(t) \\ &= d^2 \dot{y}_i^T(t) L_i^T T_i^{-1} L_i \dot{y}_i(t) - d(t) e^{-2\alpha d} \int_{t-d(t)}^t \dot{u}_i^T(s) T_i^{-1} \dot{u}_i(s) ds - 2\alpha V_{i8}(t), \\ \dot{V}_{i9}(t) &\leq k_2 z_i^T(t) K_i^T W_i^{-1} K_i z_i(t) - e^{-2\alpha k_2} \int_{t-k_2}^t z_i^T(s) K_i^T W_i^{-1} K_i z_i(s) ds - 2\alpha V_{i9}(t) \\ &\leq k_2 y_i^T(t) P_i K_i^T W_i^{-1} K_i P_i y_i(t) - e^{-2\alpha k_2} \int_{t-k_2(t)}^t u_i^T(s) W_i^{-1} u_i(s) ds - 2\alpha V_{i9}(t) \\ &\leq k_2 y_i^T(t) L_i^T W_i^{-1} L_i y_i(t) - e^{-2\alpha k_2} \int_{t-k_2(t)}^t u_i^T(s) W_i^{-1} u_i(s) ds - 2\alpha V_{i9}(t). \end{aligned} \quad (20)$$

Applying Lemma 2.3 and the Leibniz-Newton formula, we have

$$\begin{aligned} -h_1 \int_{t-h_1}^t \dot{y}_i^T(s) R_i \dot{y}_i(s) ds &\leq -\left[\int_{t-h_1}^t \dot{y}_i(s) ds \right]^T R_i \left[\int_{t-h_1}^t \dot{y}_i(s) ds \right] \\ &\leq -[y_i(t) - y_i(t-h_1)]^T R_i [y_i(t) - y_i(t-h_1)] \\ &= -y_i^T(t) R_i y_i(t) + 2y_i^T(t) R_i y_i(t-h_1) \\ &\quad - y_i^T(t-h_1) R_i y_i(t-h_1) \end{aligned} \quad (21)$$

and

$$\begin{aligned} -h_2 \int_{t-h_2}^t \dot{y}_i^T(s) R_i \dot{y}_i(s) ds &\leq -\left[\int_{t-h_2}^t \dot{y}_i(s) ds \right]^T R_i \left[\int_{t-h_2}^t \dot{y}_i(s) ds \right] \\ &\leq -[y_i(t) - y_i(t-h_2)]^T R_i [y_i(t) - y_i(t-h_2)] \end{aligned}$$

$$\begin{aligned}
 &= -y_i^T(t)R_i y_i(t) + 2y_i^T(t)R_i y_i(t-h_2) \\
 &\quad - y_i^T(t-h_2)R_i y_i(t-h_2).
 \end{aligned} \tag{22}$$

On the other hand,

$$\begin{aligned}
 -(h_2 - h_1) \int_{t-h_2}^{t-h_1} \dot{y}_i^T(s) U_i \dot{y}_i(s) ds &= -(h_2 - h_1) \int_{t-h_2}^{t-h(t)} \dot{y}_i^T(s) U_i \dot{y}_i(s) ds \\
 &\quad - (h_2 - h_1) \int_{t-h(t)}^{t-h_1} \dot{y}_i^T(s) U_i \dot{y}_i(s) ds \\
 &= -(h_2 - h(t)) \int_{t-h_2}^{t-h(t)} \dot{y}_i^T(s) U_i \dot{y}_i(s) ds \\
 &\quad - (h(t) - h_1) \int_{t-h_2}^{t-h(t)} \dot{y}_i^T(s) U_i \dot{y}_i(s) ds \\
 &\quad - (h(t) - h_1) \int_{t-h(t)}^{t-h_1} \dot{y}_i^T(s) U_i \dot{y}_i(s) ds \\
 &\quad - (h_2 - h(t)) \int_{t-h(t)}^{t-h_1} \dot{y}_i^T(s) U_i \dot{y}_i(s) ds.
 \end{aligned}$$

Using Lemma 2.3 gives

$$\begin{aligned}
 -(h_2 - h(t)) \int_{t-h_2}^{t-h(t)} \dot{y}_i^T(s) U_i \dot{y}_i(s) ds &\leq - \left[\int_{t-h_2}^{t-h(t)} \dot{y}_i(s) ds \right]^T U_i \left[\int_{t-h_2}^{t-h(t)} \dot{y}_i(s) ds \right] \\
 &\leq - [y_i(t-h(t)) - y_i(t-h_2)]^T U_i \\
 &\quad \times [y_i(t-h(t)) - y_i(t-h_2)] \\
 &= -y_i^T(t-h(t)) U_i y_i(t-h(t)) \\
 &\quad + 2y_i^T(t-h(t)) U_i y_i(t-h_2) \\
 &\quad - y_i^T(t-h_2) U_i y_i(t-h_2)
 \end{aligned} \tag{23}$$

and

$$\begin{aligned}
 -(h(t) - h_1) \int_{t-h(t)}^{t-h_1} \dot{y}_i^T(s) U_i \dot{y}_i(s) ds &\leq - \left[\int_{t-h(t)}^{t-h_1} \dot{y}_i(s) ds \right]^T U_i \left[\int_{t-h(t)}^{t-h_1} \dot{y}_i(s) ds \right] \\
 &\leq - [y_i(t-h_1) - y_i(t-h(t))]^T U_i \\
 &\quad \times [y_i(t-h_1) - y_i(t-h(t))] \\
 &= -y_i^T(t-h_1) U_i y_i(t-h_1) + 2y_i^T(t-h_1) U_i y_i(t-h(t)) \\
 &\quad - y_i^T(t-h(t)) U_i y_i(t-h(t)).
 \end{aligned} \tag{24}$$

Let $\beta = \frac{h_2-h(t)}{h_2-h_1} \leq 1$. Then

$$\begin{aligned}
 -(h_2 - h(t)) \int_{t-h(t)}^{t-h_1} \dot{y}_i^T(s) U_i \dot{y}_i(s) ds &= -\beta \int_{t-h(t)}^{t-h_1} (h_2 - h_1) \dot{y}_i^T(s) U_i \dot{y}_i(s) ds \\
 &\leq -\beta \int_{t-h(t)}^{t-h_1} (h(t) - h_1) \dot{y}_i^T(s) U_i \dot{y}_i(s) ds
 \end{aligned}$$

$$\begin{aligned} &\leq -\beta[y_i(t-h_1) - y_i(t-h(t))]^T U_i \\ &\quad \times [y_i(t-h_1) - y_i(t-h(t))] \end{aligned} \quad (25)$$

and

$$\begin{aligned} -(h(t) - h_1) \int_{t-h_2}^{t-h(t)} \dot{y}_i^T(s) U_i \dot{y}_i(s) ds &= -(1-\beta) \int_{t-h_2}^{t-h(t)} (h_2 - h_1) \dot{y}_i^T(s) U_i \dot{y}_i(s) ds \\ &\leq -(1-\beta) \int_{t-h_2}^{t-h(t)} (h_2 - h(t)) \dot{y}_i^T(s) U_i \dot{y}_i(s) ds \\ &\leq -(1-\beta)[y_i(t-h(t)) - y_i(t-h_2)]^T U_i \\ &\quad \times [y_i(t-h(t)) - y_i(t-h_2)]. \end{aligned} \quad (26)$$

Therefore from (23)-(26), we obtain

$$\begin{aligned} -(h_2 - h_1) \int_{t-h_2}^{t-h_1} \dot{y}_i^T(s) U_i \dot{y}_i(s) ds &\leq -[y_i(t-h(t)) - y_i(t-h_2)]^T U_i \\ &\quad \times [y_i(t-h(t)) - y_i(t-h_2)] \\ &\quad - [y_i(t-h_1) - y_i(t-h(t))]^T U_i \\ &\quad \times [y_i(t-h_1) - y_i(t-h(t))] \\ &\quad - \beta[y_i(t-h_1) - y_i(t-h(t))]^T U_i \\ &\quad \times [y_i(t-h_1) - y_i(t-h(t))] \\ &\quad - (1-\beta)[y_i(t-h(t)) - y_i(t-h_2)]^T U_i \\ &\quad \times [y_i(t-h(t)) - y_i(t-h_2)]. \end{aligned} \quad (27)$$

From $\dot{V}_{18}(t)$, applying Lemma 2.3 and the Leibniz-Newton formula gives

$$\begin{aligned} -d(t)e^{-2\alpha d} \int_{t-d(t)}^t \dot{u}_i^T(s) T_i^{-1} \dot{u}_i(s) ds &\leq -e^{-2\alpha d} \left(\int_{t-d(t)}^t \dot{u}_i(s) ds \right)^T T_i^{-1} \left(\int_{t-d(t)}^t \dot{u}_i(s) ds \right) \\ &\leq -e^{-2\alpha d} u_i^T(t) T_i^{-1} u_i(t) \\ &\quad + 2e^{-2\alpha d} u_i^T(t) T_i^{-1} u_i(t-d(t)) \\ &\quad - e^{-2\alpha d} u_i^T(t-d(t)) T_i^{-1} u_i(t-d(t)) \\ &\leq -e^{-2\alpha d} u_i^T(t) T_i^{-1} u_i(t) + 3e^{-2\alpha d} u_i^T(t) T_i^{-1} u_i(t) \\ &\quad + \frac{e^{-2\alpha d}}{3} u_i^T(t-d(t)) T_i^{-1} T_i T_i^{-1} u_i(t-d(t)) \\ &\quad - e^{-2\alpha d} u_i^T(t-d(t)) T_i^{-1} u_i(t-d(t)) \\ &= 2e^{-2\alpha d} z_i^T(t) K_i^T T_i^{-1} K_i z_i(t) \\ &\quad + \frac{e^{-2\alpha d}}{3} u_i^T(t-d(t)) T_i^{-1} T_i T_i^{-1} u_i(t-d(t)) \\ &\quad - e^{-2\alpha d} u_i^T(t-d(t)) T_i^{-1} u_i(t-d(t)) \end{aligned}$$

$$\begin{aligned}
&= 2e^{-2\alpha d} y_i^T(t) L_i^T T_i^{-1} L_i y_i(t) \\
&\quad + \frac{e^{-2\alpha d}}{3} u_i^T(t-d(t)) T_i^{-1} u_i(t-d(t)) \\
&\quad - e^{-2\alpha d} u_i^T(t-d(t)) T_i^{-1} u_i(t-d(t)). \tag{28}
\end{aligned}$$

By using the following identity relation:

$$\begin{aligned}
&-\dot{z}_i(t) + (\tilde{A}_i + D_{1i}K_i)z_i(t) + \tilde{B}_i z_i(t-h(t)) + \tilde{C}_i \int_{t-k_1(t)}^t z_i(s) ds \\
&\quad + D_{2i}K_i z_i(t-d(t)) + D_{3i}K_i \int_{t-k_2(t)}^t z_i(s) ds = 0,
\end{aligned}$$

we have

$$\begin{aligned}
&-2\dot{y}_i^T(t) P_i \dot{y}_i(t) + 2\dot{y}_i^T(t) \tilde{A}_i P_i y_i(t) - 2\dot{y}_i^T(t) D_{1i} L_i y_i(t) + 2\dot{y}_i^T(t) \tilde{B}_i P_i y_i(t-h(t)) \\
&\quad + 2\dot{y}_i^T(t) \tilde{C}_i P_i \int_{t-k_1(t)}^t y_i(s) ds + 2\dot{y}_i^T(t) D_{2i} u_i(t-d(t)) \\
&\quad + 2\dot{y}_i^T(t) D_{3i} \int_{t-k_2(t)}^t u_i(s) ds = 0. \tag{29}
\end{aligned}$$

Applying Lemma 2.4 and Lemma 2.3 gives

$$\begin{aligned}
2\dot{y}_i^T(t) \tilde{C}_i P_i \int_{t-k_1(t)}^t y_i(s) ds &\leq 2k_1 e^{2\alpha k_1} \dot{y}_i^T(t) \tilde{C}_i P_i S_i^{-1} P_i \tilde{C}_i^T \dot{y}_i(t) \\
&\quad + \frac{1}{2k_1} e^{-2\alpha k_1} \left(\int_{t-k_1(t)}^t y_i(s) ds \right)^T S_i \\
&\quad \times \left(\int_{t-k_1(t)}^t y_i(s) ds \right) \\
&\leq 2k_1 e^{2\alpha k_1} \dot{y}_i^T(t) \tilde{C}_i P_i S_i^{-1} P_i \tilde{C}_i^T \dot{y}_i(t) \\
&\quad + \frac{1}{2} e^{-2\alpha k_1} \int_{t-k_1(t)}^t y_i^T(s) S_i y_i(s) ds, \tag{30}
\end{aligned}$$

$$\begin{aligned}
2\dot{y}_i^T(t) D_{2i} u_i(t-d(t)) &\leq 3e^{2\alpha d} \dot{y}_i^T(t) D_{2i}^T T_i^{-1} D_{2i} \dot{y}_i(t) \\
&\quad + \frac{e^{-2\alpha d}}{3} u_i^T(t-d(t)) T_i^{-1} u_i(t-d(t)), \tag{31}
\end{aligned}$$

$$\begin{aligned}
3\dot{y}_i^T(t) D_{3i} \int_{t-k_2(t)}^t u_i(s) ds &\leq 2k_2 e^{2\alpha k_2} \dot{y}_i^T(t) D_{3i}^T W_i^{-1} D_{3i} \dot{y}_i(t) \\
&\quad + \frac{1}{2k_2} e^{-2\alpha k_2} \left(\int_{t-k_2(t)}^t u_i(s) ds \right)^T W_i \\
&\quad \times \left(\int_{t-k_2(t)}^t u_i(s) ds \right) \\
&\leq 2k_2 e^{2\alpha k_2} \dot{y}_i^T(t) D_{3i}^T W_i^{-1} D_{3i} \dot{y}_i(t) \\
&\quad + \frac{e^{2\alpha k_2}}{2} \int_{t-k_2(t)}^t u_i^T(s) W_i^{-1} u_i(s) ds. \tag{32}
\end{aligned}$$

Hence, according to (19)-(28), (30)-(32), and adding the zero items of (29) we have

$$\begin{aligned} \dot{V}_i(z_i(t)) + 2\alpha V_i(z_i(t)) &\leq \xi_i^T(t) [(1-\beta)\Sigma_{1i} + \beta\Sigma_{2i}] \xi_i(t) + y_i^T(t) \mathcal{M}_{3i} y_i(t) \\ &\quad + \dot{y}_i^T(t) \mathcal{M}_{4i} \dot{y}_i(t), \end{aligned} \quad (33)$$

where Σ_{1i} and Σ_{2i} are defined as in (12) and (13), respectively, and

$$\begin{aligned} \xi_i^T(t) &= [y_i^T(t) \quad \dot{y}_i^T(t) \quad y_i^T(t-h_1) \quad y_i^T(t-h(t)) \quad y_i^T(t-h_2)], \\ \mathcal{M}_{3i} &= -0.5(e^{-2\alpha h_1} + e^{-2\alpha h_2})R_i + 2k_1 e^{2\alpha k_1} \tilde{C}_i P_i S_i^{-1} P_i \tilde{C}_i^T + k_2 L_i^T W_i^{-1} L_i \\ &\quad + 2e^{-2\alpha d} L_i^T T_i^{-1} L_i, \\ \mathcal{M}_{4i} &= -0.5P_i + 2k_1 e^{2\alpha k_1} \tilde{C}_i P_i S_i^{-1} P_i \tilde{C}_i^T + d^2 L_i^T T_i^{-1} L_i + 3e^{2\alpha d} D_{2i}^T T_i^{-1} D_{2i} \\ &\quad + 2k_2 e^{2\alpha k_2} D_{3i}^T W_i^{-1} D_{3i}. \end{aligned}$$

By $(1-\beta)\Sigma_{1i} + \beta\Sigma_{2i} < 0$ holds if and only if $\Sigma_{1i} < 0$ and $\Sigma_{2i} < 0$. Applying the Schur complement lemma, the inequalities $\mathcal{M}_{3i} < 0$ and $\mathcal{M}_{4i} < 0$ are equivalent to $\Sigma_{3i} < 0$ and $\Sigma_{4i} < 0$, respectively. Therefore, it follows from (12)-(15), and (33), we obtain

$$\dot{V}_i(z_i(t)) + 2\alpha V_i(z_i(t)) \leq 0, \quad \forall t \geq 0. \quad (34)$$

Integrating both sides of (34) from 0 to t , we have

$$V_i(z_i(t)) \leq V_i(z_i(0))e^{-2\alpha t}, \quad \forall t \geq 0.$$

On the other hand, using the condition (18), we have

$$\|z_i(t)\| \leq \sqrt{\frac{V_i(z_i(0))}{\gamma}} e^{-\alpha t}, \quad \forall t \geq 0.$$

Estimating $V_i(z_i(0))$ gives

$$\begin{aligned} V_{i1}(z_i(0)) &= z_i^T(0) P_i^{-1} z_i(0) \leq \lambda_{\max}(P_i^{-1}) \|\phi_i\|^2, \\ V_{i2}(z_i(0)) &= \int_{-h_1}^0 e^{2\alpha s} z_i^T(s) Y_i Q_i Y_i z_i(s) ds \leq \lambda_{\max}(P_i^{-1} Q_i P_i^{-1}) \int_{-h_1}^0 e^{2\alpha s} ds \|\phi_i\|^2 \\ &= \lambda_{\max}(P_i^{-1} Q_i P_i^{-1}) \frac{1 - e^{-2\alpha h_1}}{2\alpha} \|\phi_i\|^2 \leq \lambda_{\max}(P_i^{-1} Q_i P_i^{-1}) \frac{1 - e^{-2\alpha h_2}}{2\alpha} \|\phi_i\|^2, \\ V_{i3}(z_i(0)) &\leq \lambda_{\max}(P_i^{-1} Q_i P_i^{-1}) \frac{1 - e^{-2\alpha h_2}}{2\alpha} \|\phi_i\|^2, \\ V_{i4}(z_i(0)) &= h_1 \int_{-h_1}^0 \int_s^0 e^{2\alpha \tau} \dot{z}_i^T(\tau) Y_i R_i Y_i \dot{z}_i(\tau) d\tau ds \\ &= h_1 \int_{-h_1}^0 e^{2\alpha s} [z_i^T(0) Y_i R_i Y_i z_i(0) - z_i^T(s) Y_i R_i Y_i z_i(s)] ds \\ &\leq h_2 \lambda_{\max}(Y_i R_i Y_i) \int_{-h_1}^0 e^{2\alpha s} ds \|\phi_i\|^2 - h_2 \lambda_{\max}(Y_i R_i Y_i) \int_{-h_1}^0 e^{2\alpha s} ds \|\phi_i\|^2 \end{aligned}$$

$$\begin{aligned}
&= h_2 \lambda_{\max}(Y_i R_i Y_i) \frac{1 - e^{-2\alpha h_1}}{2\alpha} \|\phi_i\|^2 - h_2 \lambda_{\max}(Y_i R_i Y_i) \frac{1 - e^{-2\alpha h_1}}{2\alpha} \|\varphi_i\|^2 \\
&\leq h_2 \lambda_{\max}(P_i^{-1} R_i P_i^{-1}) \frac{1 - e^{-2\alpha h_2}}{2\alpha} \|\phi_i\|^2 + h_2 \lambda_{\max}(P_i^{-1} R_i P_i^{-1}) \\
&\quad \times \frac{1 - e^{-2\alpha h_2}}{2\alpha} \|\varphi_i\|^2, \\
V_{i5}(z_i(0)) &\leq h_2 \lambda_{\max}(P_i^{-1} R_i P_i^{-1}) \frac{1 - e^{-2\alpha h_2}}{2\alpha} \|\phi_i\|^2 + h_2 \lambda_{\max}(P_i^{-1} R_i P_i^{-1}) \\
&\quad \times \frac{1 - e^{-2\alpha h_2}}{2\alpha} \|\varphi_i\|^2, \\
V_{i6}(z_i(0)) &\leq h_2 \lambda_{\max}(P_i^{-1} U_i P_i^{-1}) \frac{1 - e^{-2\alpha h_2}}{2\alpha} \|\phi_i\|^2 + h_2 \lambda_{\max}(P_i^{-1} U_i P_i^{-1}) \\
&\quad \times \frac{1 - e^{-2\alpha h_2}}{2\alpha} \|\varphi_i\|^2, \\
V_{i7}(z_i(0)) &= \int_{-k_1}^0 \int_s^0 e^{2\alpha\tau} z_i^T(\tau) Y_i S_i Y_i z_i(\tau) d\tau ds \\
&\leq \int_{-k_1}^0 \int_{-k_1}^0 e^{2\alpha s} z_i^T(\tau) Y_i S_i Y_i z_i(\tau) d\tau ds \\
&\leq \lambda_{\max}(Y_i S_i Y_i) \int_{-k_1}^0 \int_{-k_1}^0 e^{2\alpha\tau} d\tau ds \|\varphi_i\|^2 \\
&= k_1 \lambda_{\max}(P_i^{-1} S_i P_i^{-1}) \frac{1 - e^{-2\alpha k_1}}{2\alpha} \|\varphi_i\|^2, \\
V_{i8}(z_i(0)) &\leq d \lambda_{\max}(P_i^{-1} L_i^T T_i^{-1} L_i P_i^{-1}) \frac{1 - e^{-2\alpha d}}{2\alpha} \|\phi_i\|^2 \\
&\quad + d \lambda_{\max}(P_i^{-1} L_i^T T_i^{-1} L_i P_i^{-1}) \frac{1 - e^{-2\alpha d}}{2\alpha} \|\varphi_i\|^2, \\
V_{i9}(z_i(0)) &\leq k_2 \lambda_{\max}(P_i^{-1} L_i^T W_i^{-1} L_i P_i^{-1}) \frac{1 - e^{-2\alpha k_2}}{2\alpha} \|\varphi_i\|^2,
\end{aligned}$$

we have

$$\|z_i(t)\| \leq \sqrt{\frac{\mathcal{N}}{\gamma}} e^{-\alpha t}, \quad \forall t \geq 0,$$

which implies the dynamical networks (11) is globally exponentially stable under the controller H1, then exponential synchronization of the controlled dynamical networks (4) is achieved. The proof is thus completed. $\square$

3.2 Intermittent delayed feedback control

Theorem 3.4 *For some given scalars $0 < \alpha < \varepsilon$, the dynamical networks (7) with time-varying delay satisfying (6) are exponentially stable if there exist symmetric positive definite matrices $P_i > 0$, $Q_i > 0$, $R_i > 0$, $S_i > 0$, $U_i > 0$, $T_i > 0$, $W_i > 0$, and a matrix L_i with appropriately dimensioned such that the following symmetric linear matrix inequality holds:*

$$\Pi_{i1} = \Pi_i - [0 \quad 0 \quad I \quad -I \quad 0]^T e^{-2\alpha h_2} U_i [0 \quad 0 \quad I \quad -I \quad 0] < 0, \quad (35)$$

$$\Pi_{i2} = \Pi_i - [0 \quad 0 \quad 0 \quad I \quad -I]^T e^{-2\alpha h_2} U_i [0 \quad 0 \quad 0 \quad I \quad -I] < 0, \quad (36)$$

$$\Pi_{i3} = \tilde{\Pi}_i - [0 \ 0 \ I \ -I \ 0]^T e^{-2\alpha h_2} U_i [0 \ 0 \ I \ -I \ 0] < 0, \quad (37)$$

$$\Pi_{i4} = \tilde{\Pi}_i - [0 \ 0 \ 0 \ I \ -I]^T e^{-2\alpha h_2} U_i [0 \ 0 \ 0 \ I \ -I] < 0, \quad (38)$$

$$\Pi_{i5} = \begin{bmatrix} -0.5(e^{-2\alpha h_1} + e^{-2\alpha h_2})R_i & 2k_1 \tilde{C}_i P_i & k_2 L_i^T & 2L_i^T \\ * & -2k_1 e^{-2\alpha k_1} S_i & 0 & 0 \\ * & * & -k_2 W_i & 0 \\ * & * & * & -2e^{-2\alpha d} T_i \end{bmatrix} < 0, \quad (39)$$

$$\Pi_{i6} = \begin{bmatrix} -0.5P_i & 2k_1 \tilde{C}_i P_i & d^2 L_i^T & 3D_{5i}^T & 2k_2 D_{6i}^T \\ * & -2k_1 e^{-2\alpha k_1} S_i & 0 & 0 & 0 \\ * & * & -d^2 T_i & 0 & 0 \\ * & * & * & -3e^{-2\alpha d} T_i & 0 \\ * & * & * & * & -2k_2 e^{-2\alpha k_2} W_i \end{bmatrix} < 0, \quad (40)$$

$$\Pi_{i7} = \begin{bmatrix} -0.5(e^{-2\alpha h_1} + e^{-2\alpha h_2})R_i - 2\varepsilon P_i & 2k_1 \tilde{C}_i P_i \\ * & -2k_1 e^{-2\alpha k_1} S_i \end{bmatrix} < 0, \quad (41)$$

$$\Pi_{i8} = \begin{bmatrix} -0.5P_i & 2k_1 \tilde{C}_i P_i \\ * & -2k_1 e^{-2\alpha k_1} S_i \end{bmatrix} < 0 \quad (42)$$

and

$$-\alpha\delta + (\varepsilon - \alpha)(\omega - \delta) < 0, \quad (43)$$

$i = 2, \dots, N$, where

$$\Pi_i = \begin{bmatrix} \Pi_{i11} & \Pi_{i12} & \Pi_{i13} & \Pi_{i14} & \Pi_{i15} \\ * & \Pi_{i22} & 0 & \Pi_{i24} & 0 \\ * & * & \Pi_{i33} & \Pi_{i34} & 0 \\ * & * & * & \Pi_{i44} & \Pi_{i45} \\ * & * & * & * & \Pi_{i55} \end{bmatrix},$$

$$\tilde{\Pi}_i = \begin{bmatrix} \tilde{\Pi}_{i11} & \Pi_{i12} & \Pi_{i13} & \Pi_{i14} & \Pi_{i15} \\ * & \Pi_{i22} & 0 & \Pi_{i24} & 0 \\ * & * & \Pi_{i33} & \Pi_{i34} & 0 \\ * & * & * & \Pi_{i44} & \Pi_{i45} \\ * & * & * & * & \Pi_{i55} \end{bmatrix},$$

$$\begin{aligned} \Pi_{i11} &= P_i^T (\tilde{A}_i + \alpha I) + (\tilde{A}_i + \alpha I)^T P_i - D_{4i} L_i - L_i^T D_{4i}^T + 3e^{2\alpha d} D_{5i}^T T_i D_{5i} \\ &\quad + 2k_2 e^{2\alpha k_2} D_{6i}^T W_i D_{6i} + 2Q_i + k_1 S_i - 0.5e^{-2\alpha h_1} R_i - 0.5e^{-2\alpha h_2} R_i, \end{aligned}$$

$$\tilde{\Pi}_{i11} = P_i^T (\tilde{A}_i + \alpha I) + (\tilde{A}_i + \alpha I)^T P_i + 2Q_i + k_1 S_i - 0.5e^{-2\alpha h_1} R_i - 0.5e^{-2\alpha h_2} R_i,$$

$$\Pi_{i12} = P_i \tilde{A}_i^T,$$

$$\Pi_{i13} = e^{-2\alpha h_1} R_i,$$

$$\Pi_{i14} = \tilde{B}_i P_i,$$

$$\Pi_{i15} = e^{-2\alpha h_2} R_i,$$

$$\Pi_{i22} = h_1^2 R_i + h_2^2 R_i + \eta^2 U_i - 1.5P_i,$$

$$\begin{aligned}\Pi_{i24} &= \tilde{B}_i P_i, \\ \Pi_{i33} &= -e^{-2\alpha h_1} Q_i - e^{-2\alpha h_1} R_i - e^{-2\alpha h_2} U_i, \\ \Pi_{i34} &= e^{-2\alpha h_2} U_i, \\ \Pi_{i44} &= -2e^{-2\alpha h_2} U_i, \\ \Pi_{i45} &= e^{-2\alpha h_2} U_i, \\ \Pi_{i55} &= -2e^{-2\alpha h_2} U_i - 2e^{-2\alpha h_2} Q_i - 2e^{-2\alpha h_2} R_i,\end{aligned}$$

then the dynamical networks (7) have exponential synchronization. Moreover, the feedback control is

$$u_i(t) = \begin{cases} -L_i P_i^{-1} z_i(t), & n\omega \leq t \leq n\omega + \delta, \\ 0, & n\omega + \delta < t \leq (n+1)\omega. \end{cases} \quad (44)$$

Proof Case I: for $n\omega \leq t \leq n\omega + \delta$, we choose the Lyapunov-Krasovskii functional as in (17) and using the feedback control (44), we may proof this case by using a similar argument as in the proof of Theorem 3.3. By replacing D_{1i} , D_{2i} and D_{3i} in (12)-(15) with D_{4i} , D_{5i} , and D_{6i} , respectively. We have

$$\begin{aligned}\dot{V}_i(z_i(t)) + 2\alpha V_i(z_i(t)) &\leq \xi_i^T(t) [(1-\beta)\Pi_{1i} + \beta\Pi_{2i}] \xi_i(t) + y_i^T(t) \mathcal{N}_{3i} y_i(t) \\ &\quad + \dot{y}_i^T(t) \mathcal{N}_{4i} \dot{y}_i(t),\end{aligned} \quad (45)$$

where Π_{1i} and Π_{2i} are defined as in (35) and (36), respectively, and

$$\begin{aligned}\xi_i^T(t) &= [y_i^T(t) \quad \dot{y}_i^T(t) \quad y_i^T(t-h_1) \quad y_i^T(t-h(t)) \quad y_i^T(t-h_2)], \\ \mathcal{N}_{3i} &= -0.5(e^{-2\alpha h_1} + e^{-2\alpha h_2}) R_i + 2k_1 e^{2\alpha k_1} \tilde{C}_i P_i S_i^{-1} P_i \tilde{C}_i^T \\ &\quad + k_2 L_i^T W_i^{-1} L_i + 2e^{-2\alpha d} L_i^T T_i^{-1} L_i, \\ \mathcal{N}_{4i} &= -0.5P_i + 2k_1 e^{2\alpha k_1} \tilde{C}_i P_i S_i^{-1} P_i \tilde{C}_i^T + d^2 L_i^T T_i^{-1} L_i + 3e^{2\alpha d} D_{5i}^T T_i^{-1} D_{5i} \\ &\quad + 2k_2 e^{2\alpha k_2} D_{6i}^T W_i^{-1} D_{6i}.\end{aligned}$$

By $(1-\beta)\Pi_{1i} + \beta\Pi_{2i} < 0$ holds if and only if $\Pi_{1i} < 0$ and $\Pi_{2i} < 0$. Applying the Schur complement lemma, the inequalities $\mathcal{N}_{5i} < 0$ and $\mathcal{N}_{6i} < 0$ are equivalent to $\Pi_{5i} < 0$ and $\Pi_{6i} < 0$, respectively. Therefore, it follows from (35)-(36), (39)-(40), and (45), we obtain

$$\dot{V}_i(z_i(t)) + 2\alpha V_i(z_i(t)) \leq 0 \quad \text{for } n\omega \leq t \leq n\omega + \delta. \quad (46)$$

Thus, by the above differential inequality (46), we have

$$V_i(z_i(t)) \leq V_i(z_i(n\omega)) e^{-2\alpha(t-n\omega)} \quad \text{for } n\omega \leq t \leq n\omega + \delta. \quad (47)$$

Case II: for $n\omega + \delta \leq t \leq (n+1)\omega$, we choose the Lyapunov-Krasovskii functional having the following form:

$$V_i(z_i(t)) = V_{i1}(t) + V_{i2}(t) + V_{i3}(t) + V_{i4}(t) + V_{i5}(t) + V_{i6}(t) + V_{i7}(t),$$

where $V_{ij}(t), j = 1, 2, \dots, 7$ are defined similar in (17). We are able to do a similar estimation as we did for Theorem 3.3, and we have the following:

$$\begin{aligned} \dot{V}_i(z_i(t)) + 2\alpha V_i(z_i(t)) &\leq \xi_i^T(t) [(1-\beta)\Pi_{3i} + \beta\Pi_{4i}] \xi_i(t) + y_i^T(t) \mathcal{N}_{7i} y_i(t) + \dot{y}_i^T(t) \mathcal{N}_{8i} \dot{y}_i(t) \\ &\leq \xi_i^T(t) [(1-\beta)\Pi_{3i} + \beta\Pi_{4i}] \xi_i(t) + y_i^T(t) \mathcal{N}_{7i} y_i(t) \\ &\quad + \dot{y}_i^T(t) \mathcal{N}_{8i} \dot{y}_i(t) + 2\varepsilon V_i(z_i(t)) - 2\varepsilon V_{1i}(t) \\ &= \xi_i^T(t) [(1-\beta)\Pi_{3i} + \beta\Pi_{4i}] \xi_i(t) + y_i^T(t) \mathcal{N}_{7i} y_i(t) \\ &\quad + \dot{y}_i^T(t) \mathcal{N}_{8i} \dot{y}_i(t) + 2\varepsilon V_i(z_i(t)) - 2\varepsilon y_i^T(t) P_i y_i(t), \\ \dot{V}_i(z_i(t)) - 2(\varepsilon - \alpha) V_i(z_i(t)) &\leq \xi_i^T(t) [(1-\beta)\Pi_{3i} + \beta\Pi_{4i}] \xi_i(t) \\ &\quad + y_i^T(t) (\mathcal{N}_{7i} - 2\varepsilon P_i) y_i(t) + \dot{y}_i^T(t) \mathcal{N}_{8i} \dot{y}_i(t), \end{aligned} \quad (48)$$

where Π_{1i} and Π_{2i} are defined as in (37) and (38), respectively, and

$$\begin{aligned} \xi_i^T(t) &= [y_i^T(t) \quad \dot{y}_i^T(t) \quad y_i^T(t-h_1) \quad y_i^T(t-h(t)) \quad y_i^T(t-h_2)], \\ \mathcal{N}_{7i} &= -0.1(e^{-2\alpha h_1} + e^{-2\alpha h_2}) R_i + 2k_1 e^{2\alpha k_1} \tilde{C}_i P_i S_i^{-1} P_i \tilde{C}_i^T, \\ \mathcal{N}_{8i} &= -0.5 P_i + 2k_1 e^{2\alpha k_1} \tilde{C}_i P_i S_i^{-1} P_i \tilde{C}_i^T. \end{aligned}$$

Now $(1-\beta)\Pi_{3i} + \beta\Pi_{4i} < 0$ holds if and only if $\Pi_{3i} < 0$ and $\Pi_{4i} < 0$. Applying the Schur complement lemma, the inequalities $(\mathcal{N}_{7i} - 2\varepsilon P_i) < 0$ and $\mathcal{N}_{8i} < 0$ are equivalent to $\Pi_{7i} < 0$ and $\Pi_{8i} < 0$, respectively. Therefore, it follows from (37)-(38), (41)-(42), and (48), that we obtain

$$\dot{V}_i(z_i(t)) - 2(\varepsilon - \alpha) V_i(z_i(t)) \leq 0 \quad \text{for } n\omega + \delta < t \leq (n+1)\omega. \quad (49)$$

From the above differential inequality (49), we have

$$V_i(z_i(t)) \leq V_i(z_i(n\omega + \delta)) e^{2(\varepsilon - \alpha)(t - n\omega - \delta)} \quad \text{for } n\omega + \delta < t \leq (n+1)\omega. \quad (50)$$

By (47) and (50), we have

$$\begin{aligned} V_i(z_i((n+1)\omega)) &\leq V_i(z_i(n\omega + \delta)) e^{2(\varepsilon - \alpha)(\omega - \delta)} \\ &\leq V_i(z_i(n\omega)) e^{-2\alpha\delta} e^{2(\varepsilon - \alpha)(\omega - \delta)} \\ &= V_i(z_i(n\omega)) e^{-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta)} \\ &\leq V_i(z_i((n-1)\omega + \delta)) e^{2\rho(\omega - \delta)} e^{-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta)} \\ &\leq V_i(z_i((n-1)\omega)) e^{-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta)} e^{-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta)} \\ &= V_i(z_i((n-1)\omega)) e^{2(-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta))} \\ &\quad \vdots \\ &\leq V_i(z_i(0)) e^{(-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta))(n+1)}. \end{aligned}$$

For any $t > 0$, there is a $n_0 \geq 0$, such that $n_0\omega \leq t \leq (n_0 + 1)\omega$.

Case 1. For $n_0\omega + \delta \leq t \leq (n_0 + 1)\omega$, using condition (43), we have

$$\begin{aligned} V_i(z_i(t)) &\leq V_i(z_i(n_0\omega + \delta))e^{2(\varepsilon-\alpha)(t-(n_0\omega+\delta))} \\ &\leq V_i(z_i(n_0\omega))e^{-2\alpha\delta}e^{2(\varepsilon-\alpha)(t-(n_0\omega+\delta))} \\ &\leq V_i(z_i(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))n_0}e^{-2\alpha\delta}e^{2(\varepsilon-\alpha)(t-(n_0\omega+\delta))} \\ &\leq V_i(z_i(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))n_0}e^{-2\alpha\delta}e^{2(\varepsilon-\alpha)((n_0+1)\omega-(n_0\omega+\delta))} \\ &= V_i(z_i(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n_0+1)} \\ &= V_i(z_i(0))e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n_0+1)\omega}{\omega}} \\ &\leq V_i(z_i(0))e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))t}{\omega}}. \end{aligned} \quad (51)$$

Case 2. For $n_0\omega \leq t \leq n_0\omega + \delta$, using condition (43), we have

$$\begin{aligned} V_i(z_i(t)) &\leq V_i(z_i(n_0\omega))e^{-2\alpha(t-n_0\omega)} \\ &\leq V_i(z_i(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))n_0}e^{-2\alpha(t-n_0\omega)} \\ &\leq V_i(z_i(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))n_0} \\ &= V_i(z_i(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n_0+1)} \\ &= V_i(z_i(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n_0+1)\omega}{\omega}} \\ &\leq V_i(z_i(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))t}{\omega}}. \end{aligned} \quad (52)$$

Let $\xi = e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}$. By (51) and (52), we have

$$V_i(z_i(t)) \leq \xi V_i(z_i(0))e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))t}{\omega}}, \quad \forall t \geq 0.$$

On the other hand, using the condition (18), we have obtained the following:

$$\|z_i(t)\| \leq \sqrt{\frac{\mathcal{N}\xi}{\gamma}} e^{\frac{(-\alpha\delta+(\varepsilon-\alpha)(\omega-\delta))t}{\omega}}, \quad \forall t \geq 0.$$

which implies the dynamical networks (7) is exponentially stable under the controller H2, then exponential synchronization of the controlled dynamical networks (5) is achieved. The proof is thus completed. $\square$

Remark 3.5 It is clear that as $\delta \rightarrow \omega$ the intermittent feedback control will reduce to a continuous feedback. In this case, presented in Theorem 3.3.

Remark 3.6 In [14–16], the authors investigated synchronization of complex dynamical network with coupling time-delay, but the time-delay considered in these three works are assumed to be constants delay. In [9], Li *et al.* presented synchronization in complex dynamical networks with time-varying delays in the network couplings and time-varying delays in the dynamical nodes, but the time-varying delays are required to be differentiable, which is a very strict condition. Obviously, we do not need these limit condition in this paper.

Remark 3.7 If $k_1(t) = 0$, $c_1 = 0$, $c_3 = 0$, and $\mathcal{U}_i(t) = 0$, then system (1) reduces to the following system presented in [9, 18]:

$$\dot{x}_i(t) = f(x_i(t)) + c_2 \sum_{j=1}^N b_{ij} G_2 x_j(t - h(t)), \quad t > 0, i = 1, 2, \dots, N. \quad (53)$$

According to Theorem 3.3, we obtain the following corollary for the synchronization of network (53).

Corollary 3.8 For some given scalars $0 < \alpha$, the dynamical networks (53) with time-varying delay $h(t)$ satisfying (6) are exponentially synchronization if there exist symmetric positive definite matrices $P_i > 0$, $Q_i > 0$, $R_i > 0$, $U_i > 0$, such that the following symmetric linear matrix inequality holds:

$$\Gamma_{i1} = \Gamma_i - [0 \quad 0 \quad I \quad -I \quad 0]^T e^{-2\alpha h_2} U_i [0 \quad 0 \quad I \quad -I \quad 0] < 0, \quad (54)$$

$$\Gamma_{i2} = \Gamma_i - [0 \quad 0 \quad 0 \quad I \quad -I]^T e^{-2\alpha h_2} U_i [0 \quad 0 \quad 0 \quad I \quad -I] < 0, \quad (55)$$

where

$$\Gamma_{i11} = P_i^T (J(t) + \alpha I) + (J(t) + \alpha I)^T P_i + 2Q_i - e^{-2\alpha h_1} R_i - e^{-2\alpha h_2} R_i,$$

$$\Gamma_{i12} = P_i J^T(t),$$

$$\Gamma_{i13} = e^{-2\alpha h_1} R_i,$$

$$\Gamma_{i14} = c_2 \lambda_{2i} G_2 P_i,$$

$$\Gamma_{i15} = e^{-2\alpha h_2} R_i,$$

$$\Gamma_{i22} = h_1^2 R_i + h_2^2 R_i + \eta^2 U_i - 2P_i,$$

$$\Gamma_{i24} = c_2 \lambda_{2i} G_2 P_i,$$

$$\Gamma_{i33} = -e^{-2\alpha h_1} Q_i - e^{-2\alpha h_1} R_i - e^{-2\alpha h_2} U_i,$$

$$\Gamma_{i34} = e^{-2\alpha h_2} U_i,$$

$$\Gamma_{i44} = -2e^{-2\alpha h_2} U_i,$$

$$\Gamma_{i45} = e^{-2\alpha h_2} U_i,$$

$$\Gamma_{i55} = -2e^{-2\alpha h_2} U_i - 2e^{-2\alpha h_2} Q_i - 2e^{-2\alpha h_2} R_i.$$

Proof Similar to proof of Theorem 3.3. Indeed, by setting $S_i = 0$, $T_i = 0$, and $W_i = 0$ in (17), one may easily derive the result and hence the proof is omitted. $\square$

Remark 3.9 In [31–34], the authors investigated synchronization of complex dynamical network with coupling time-delay based on intermittent control, but the controller is presented in terms of nominal state-delayed systems. On the other hands, we have considered more complicated problem, namely, synchronization of complex dynamical network with hybrid coupling delay and mixed time-varying delay (interval time-varying delay and distributed time-varying delay), which time-varying delay using both state-delayed feedback control as well as intermittent state-delayed feedback control. It should be pointed out that the synchronization problem for complex dynamical networks with both interval

and distributed time-varying delays has not received much attention in the literature, not to mention the case when the coupling and controller are also involved.

4 Numerical examples

In this section, we now provide an example to show the effectiveness of the result in Theorem 3.3 and Theorem 3.4.

Example 4.1 We first consider the perturbed Chua circuit system with mixed time-varying delays is used as uncoupled node in the network (1) to show the effectiveness of the proposed control scheme. The perturbed Chua circuit system with mixed time-varying delays is given by [43]

$$\begin{aligned}\dot{x}_1(t) &= p \left(x_2(t-h(t)) - \frac{1}{7} (2x_1^3(t) - x_1(t)) \right), \\ \dot{x}_2(t) &= x_1(t) - sx_2(t) + x_3(t-h(t)), \\ \dot{x}_3(t) &= qx_2(t) + r \int_{t-k_1(t)}^t x_1^2(s) ds,\end{aligned}\tag{56}$$

where p, q, r , and s are real positive constants. It is well known that the system (56) exhibits chaotic behavior with the parameters p, q, r , and s are chosen as $p = 7$, $q = -\frac{100}{7}$, $r = 0.07$, and $s = 1.5$, the initial condition function $\phi(t) = [0.65 \cos t, \sin t, \sin t]^T$, the time-varying delay functions $h(t) = 0.1 + 0.1|\sin t|$ and $k_1(t) = 0.1|\cos t|$ is shown in Figure 1. The solution of the system (56) is denoted by $s(t) = (s_1(t), s_2(t), s_3(t))^T$, which is shown in Figure 2. It is stable at the equilibrium point $s(t) = 0$, $s(t-h(t)) = 0$, $\int_{t-k_1(t)}^t s(\theta) d\theta = 0$, and the Jacobian matrices are

$$J(t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1.5 & 0 \\ 0 & -\frac{100}{7} & 0 \end{bmatrix}, \quad J_h(t) = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_{k_1}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

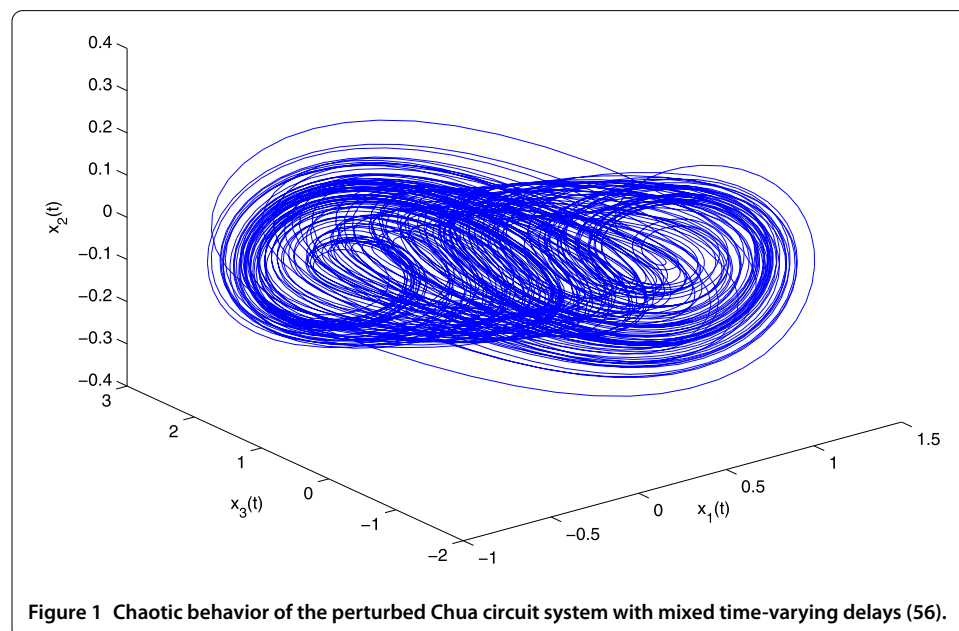


Figure 1 Chaotic behavior of the perturbed Chua circuit system with mixed time-varying delays (56).

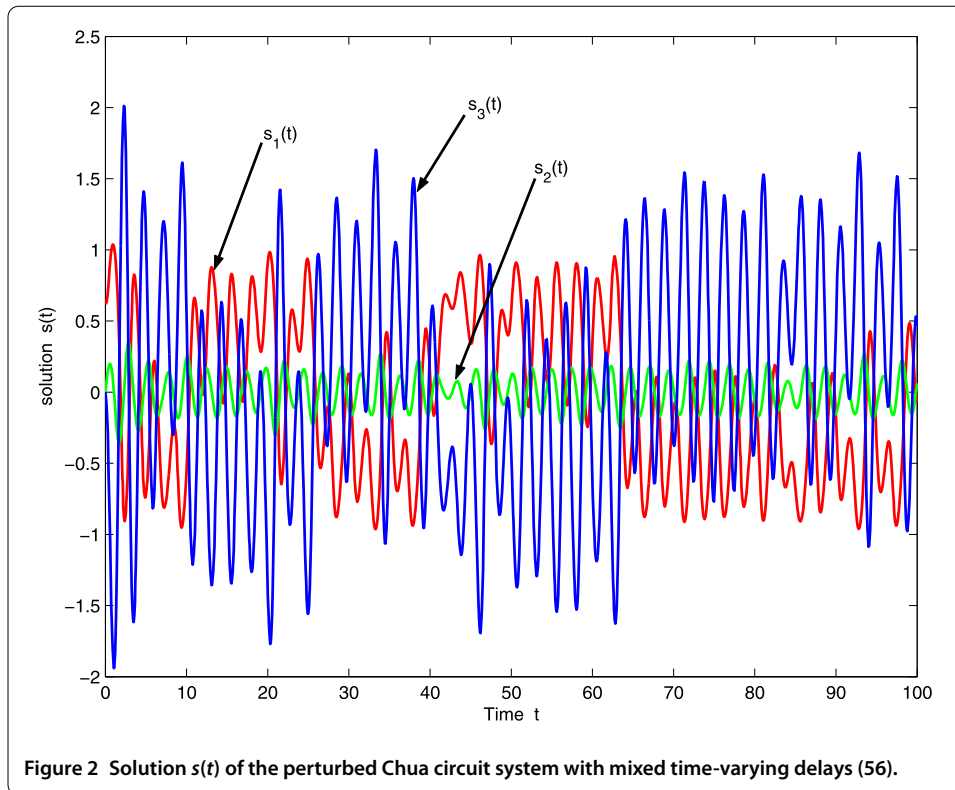


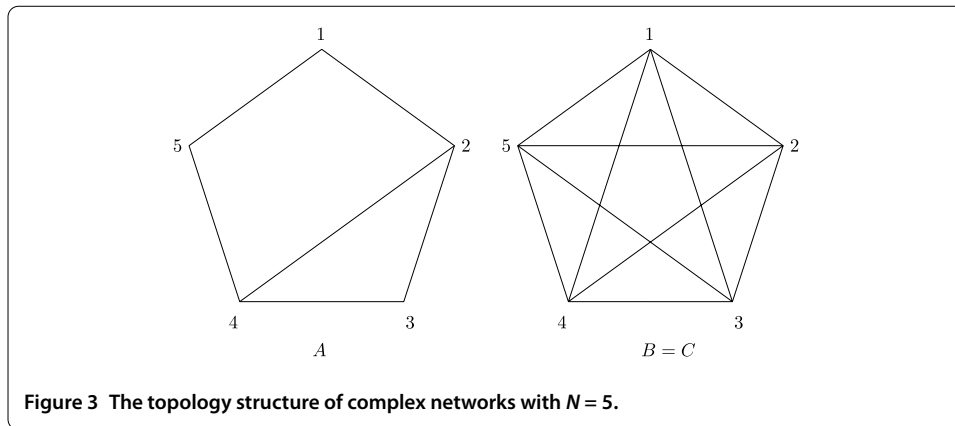
Figure 2 Solution $s(t)$ of the perturbed Chua circuit system with mixed time-varying delays (56).

Consider a network consisting of five identical perturbed Chua circuit system with mixed time-varying and hybrid coupling delays. The corresponding controlled dynamical network (4) can be described as

$$\begin{aligned} \dot{x}_i(t) = & f\left(x_i(t), x_i(t-h(t)), \int_{t-k_1(t)}^t x_i(s) ds\right) + c_1 \sum_{j=1}^N a_{ij} G_1 x_j(t) \\ & + c_2 \sum_{j=1}^N b_{ij} G_2 x_j(t-h(t)) + c_3 \sum_{j=1}^N c_{ij} G_3 \int_{t-k_1(t)}^t x_j(s) ds \\ & + D_{1i} K_i (x_i(t) - s(t)) + D_{2i} K_i (x_i(t-d(t)) - s(t-d(t))) \\ & + D_{3i} K_i \left(\int_{t-k_2(t)}^t x_i(s) ds - \int_{t-k_2(t)}^t s(\theta) d\theta \right), \quad i = 1, 2, \dots, 5. \end{aligned}$$

Assume that $D_{1i} = \text{diag}\{3, 3, 3\}$, $D_{2i} = \text{diag}\{0.1, 0.1, 0.1\}$, $D_{3i} = \text{diag}\{0.1, 0.1, 0.1\}$, $i = 1, 2, \dots, 5$, the coupling strength $c_1 = 0.3$, $c_2 = 0.2$, $c_3 = 0.4$, the inner-coupling matrices are

$$\begin{aligned} G_1 &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, & G_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ G_3 &= \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \end{aligned}$$



and the outer-coupling matrices are given by the following irreducible symmetric matrices satisfying condition (2):

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}, \quad B = C = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix},$$

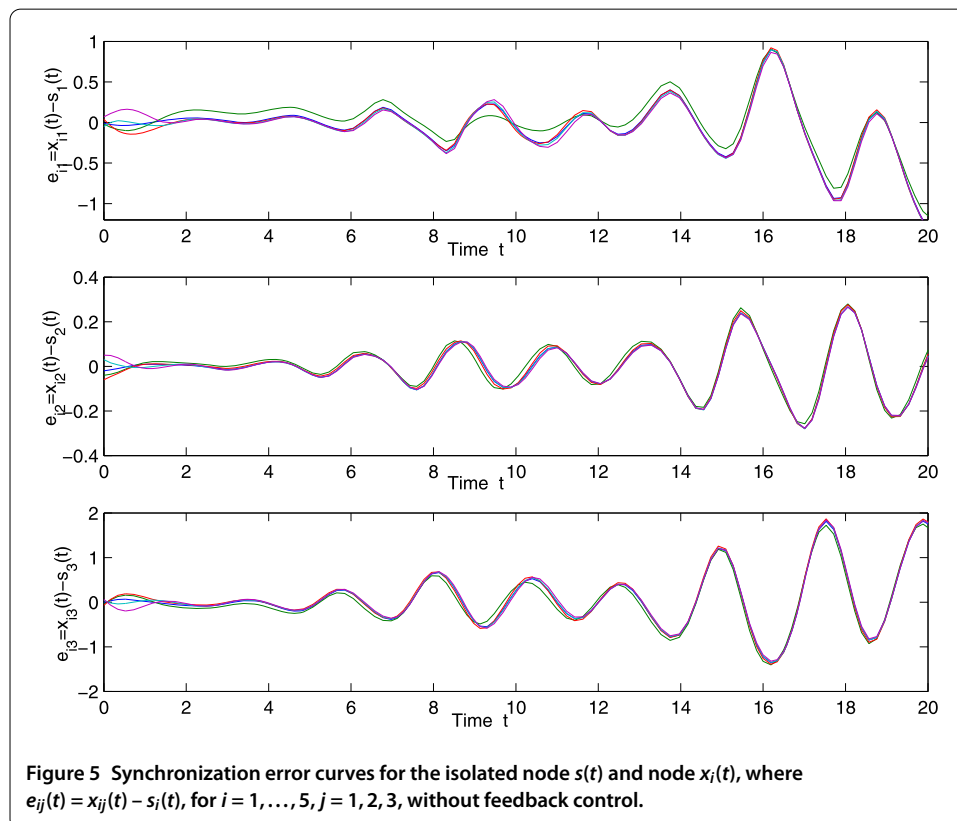
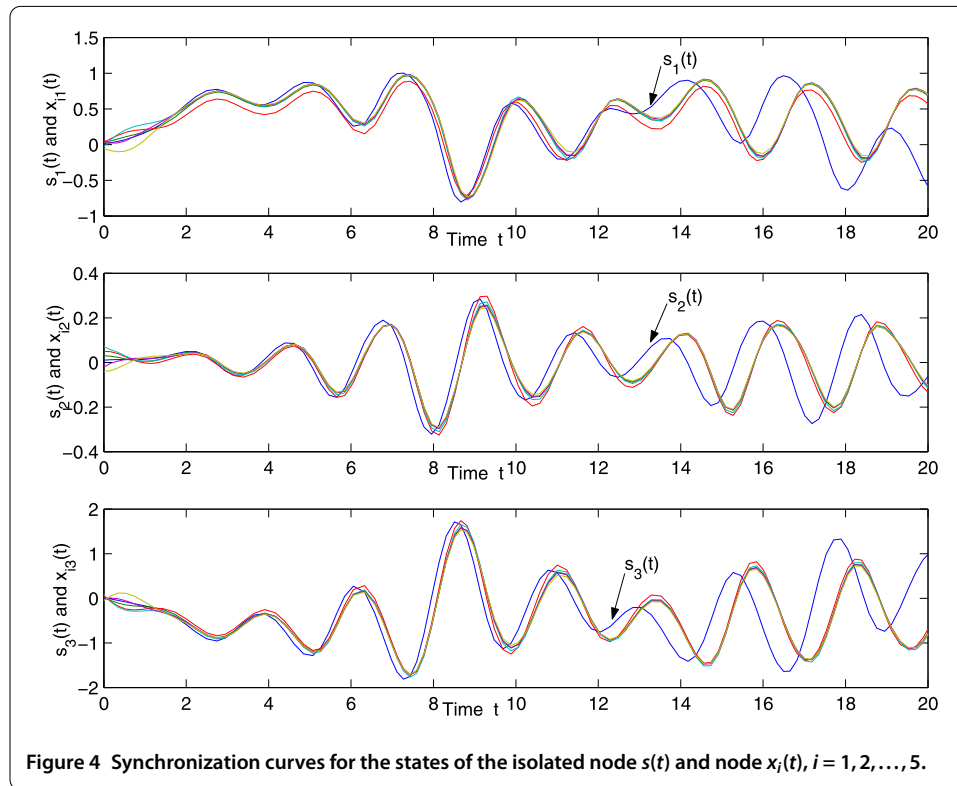
and the topology structure of complex networks is shown in Figure 3.

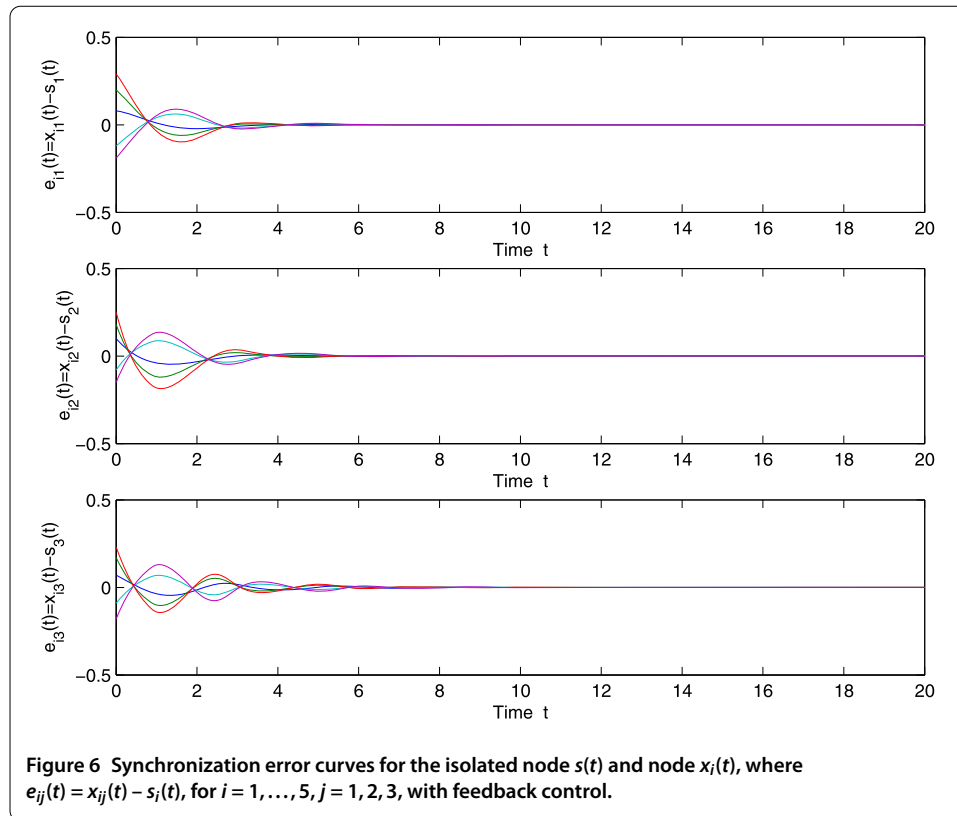
The eigenvalues of A , B , and C are $\lambda_1 = \{0, -1.382, -2.382, -3.618, -4.618\}$, $\lambda_2 = \{0, -5, -5, -5, -5\}$, and $\lambda_3 = \{0, -5, -5, -5, -5\}$, respectively.

Solution: From the conditions (12)-(15) of Theorem 3.3, we let $\alpha = 0.02$, $h_1 = 0.1$, $h_2 = 0.2$, $k_1 = 0.1$, $k_2 = 0.1$, $d = 0.3$, the gain matrices of the desired controllers can be obtained as follows:

$$\begin{aligned} K_1 &= \begin{bmatrix} -4.6038 & -0.2266 & -0.8478 \\ -0.1812 & -15.2461 & 2.5773 \\ -1.0007 & 3.6578 & -4.2867 \end{bmatrix}, & K_2 &= \begin{bmatrix} -3.8398 & -0.1242 & -1.0323 \\ -0.0015 & -15.0343 & 1.9854 \\ -1.0438 & 2.7597 & -4.1461 \end{bmatrix}, \\ K_3 &= \begin{bmatrix} -3.7311 & 0.0828 & -1.2030 \\ 0.2016 & -14.8969 & 1.8453 \\ -1.1970 & 2.4629 & -4.2466 \end{bmatrix}, & K_4 &= \begin{bmatrix} -3.9213 & 0.3286 & -1.4385 \\ 0.5011 & -15.4924 & 1.7291 \\ -1.4346 & 2.1419 & -4.6878 \end{bmatrix}, \\ K_5 &= \begin{bmatrix} -4.3268 & 0.5100 & -1.6409 \\ 0.7973 & -16.7178 & 1.6821 \\ -1.6546 & 1.9279 & -5.3012 \end{bmatrix}. \end{aligned}$$

The numerical simulations are carried out using the explicit Runge-Kutta-like method (dde45), interpolation and extrapolation by spline of the third order. Figure 4 shows the synchronization between the states of isolate node $s(t)$ and node $x_i(t)$, $i = 1, 2, \dots, 5$. Figure 5 shows the synchronization errors between the states of isolate node $s(t)$ and node $x_i(t)$, where $e_{ij}(t) = x_{ij}(t) - s_i(t)$, for $i = 1, \dots, 5$, $j = 1, 2, 3$, without feedback control. Figure 6 shows the synchronization errors between the states of isolated node $s(t)$ and node $x_i(t)$,





where $e_{ij}(t) = x_{ij}(t) - s_i(t)$, for $i = 1, \dots, 5, j = 1, 2, 3$, with feedback control. We see that the synchronization errors converge to zero under the above conditions.

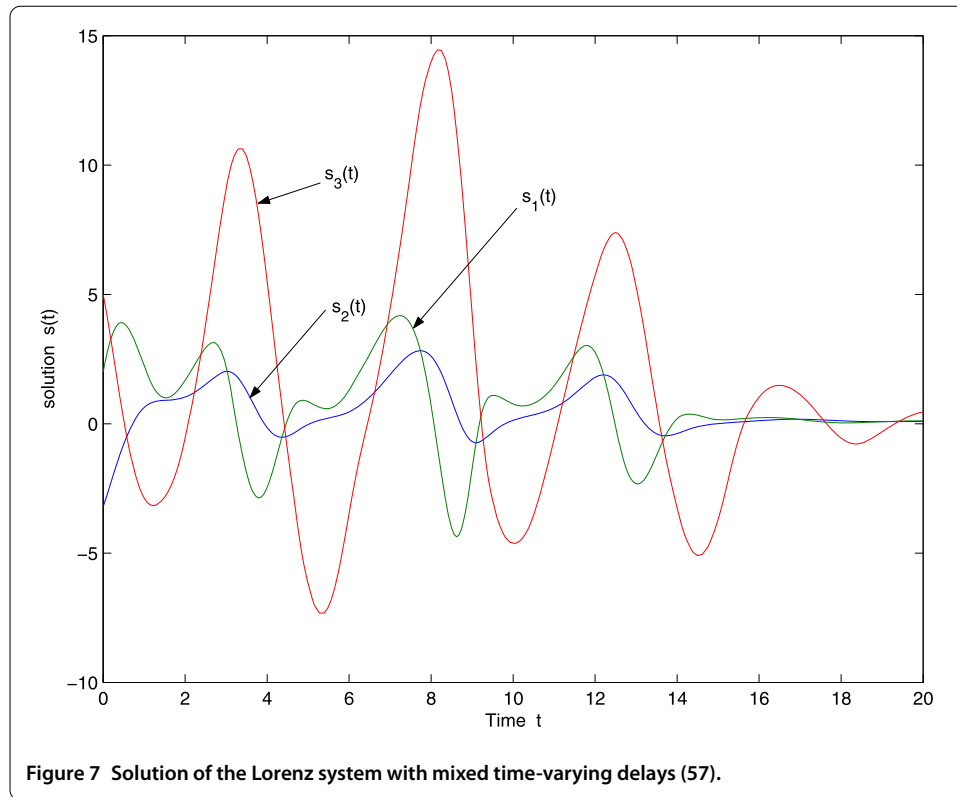
Example 4.2 We consider the nonlinear network model with five nodes, in which each node is a Lorenz system with mixed time-varying delay described by [7]

$$\begin{aligned}\dot{x}_{i1}(t) &= a(x_{i2}(t) - x_{i1}(t)), \\ \dot{x}_{i2}(t) &= cx_{i1}(t - h(t)) - x_{i2}(t) - x_{i1}(t)x_{i3}(t - h(t)), \\ \dot{x}_{i3}(t) &= x_{i1}(t) \int_{t-k_1(t)}^t x_{i2}(s) ds - bx_{i3}(t - h(t)),\end{aligned}\tag{57}$$

where $a = 0.9$, $b = 1.3$, and $c = -1$. For the initial function $\phi(t) = [-3.2 \cos t, 2 \cos t, 5 \cos t]^T$ the solution of system (57) is denoted by $s(t) = (s_1(t), s_2(t), s_3(t))^T$, which is shown in Figure 7. It is asymptotically stable at the equilibrium point $s(t) = 0$, $s(t - h(t)) = 0$, $\int_{t-k_1(t)}^t s(\theta) d\theta = 0$ and its Jacobian matrices are

$$J(t) = \begin{bmatrix} -0.9 & 0.9 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_{h(t)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1.3 \end{bmatrix}, \quad J_{k_1}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Assume that $D_{4i} = \text{diag}\{2, 2, 2\}$, $D_{5i} = \text{diag}\{0.1, 0.1, 0.1\}$, $D_{6i} = \text{diag}\{0.1, 0.1, 0.1\}$, $i = 1, 2, \dots, 5$, the coupling strength $c_1 = 0.1$, $c_2 = 0.2$, $c_3 = 0.3$, the inner-coupling matrices



are

$$G_1 = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix},$$

and the outer-coupling matrices are given by the following irreducible symmetric matrices satisfying condition (2):

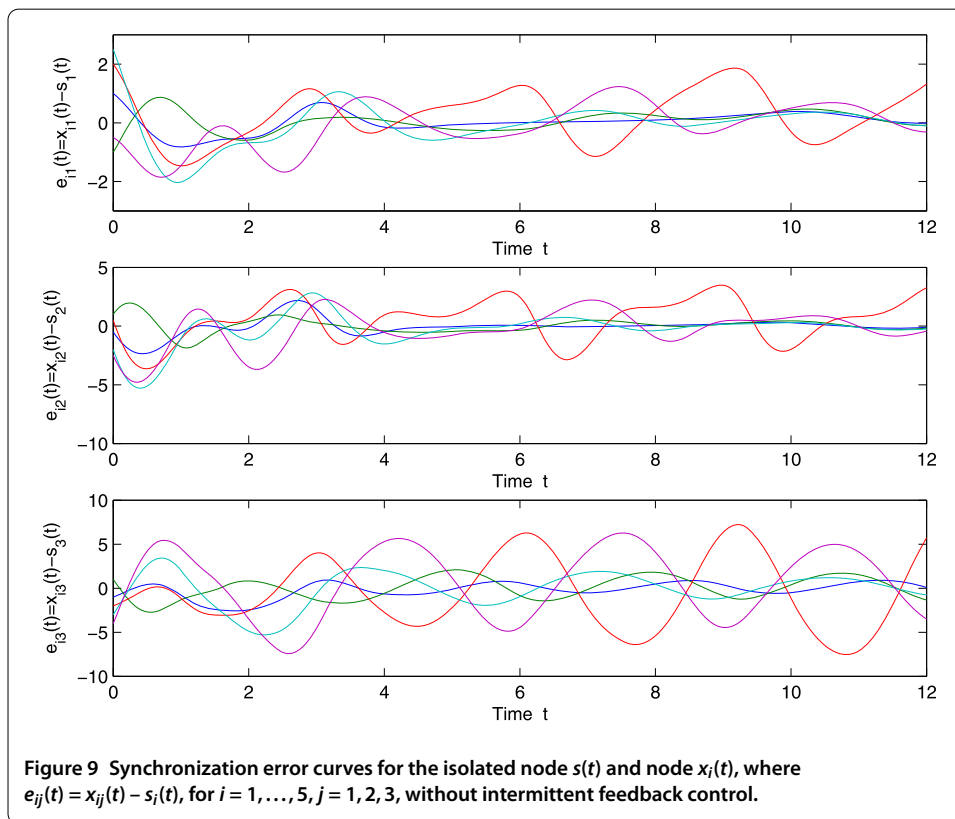
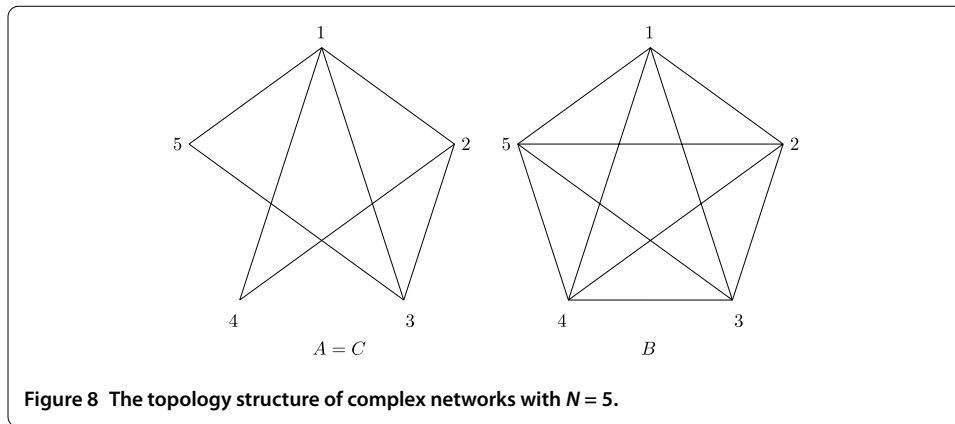
$$A = C = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 1 & 1 & -3 & 0 & 1 \\ 1 & 1 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix},$$

and the topology structure of complex networks is shown in Figure 8.

The eigenvalues of A , B , and C are $\lambda_1 = \{0, -1.5858, 3, -4.4142, -5\}$, $\lambda_2 = \{0, -5, -5, -5, -5\}$, and $\lambda_3 = \{0, -1.5858, 3, -4.4142, -5\}$, respectively.

Solution: From the conditions (35)-(43) of Theorem 3.4, we let $\varepsilon = 0.09$, $\alpha = 0.07$, $\omega = 4$, $\delta = 2.5$, $h_1 = 0.1$, $h_2 = 0.2$, $k_1 = 0.1$, $k_2 = 0.12$, $d = 0.3$; the gain matrices of the desired controllers can be obtained as follows:

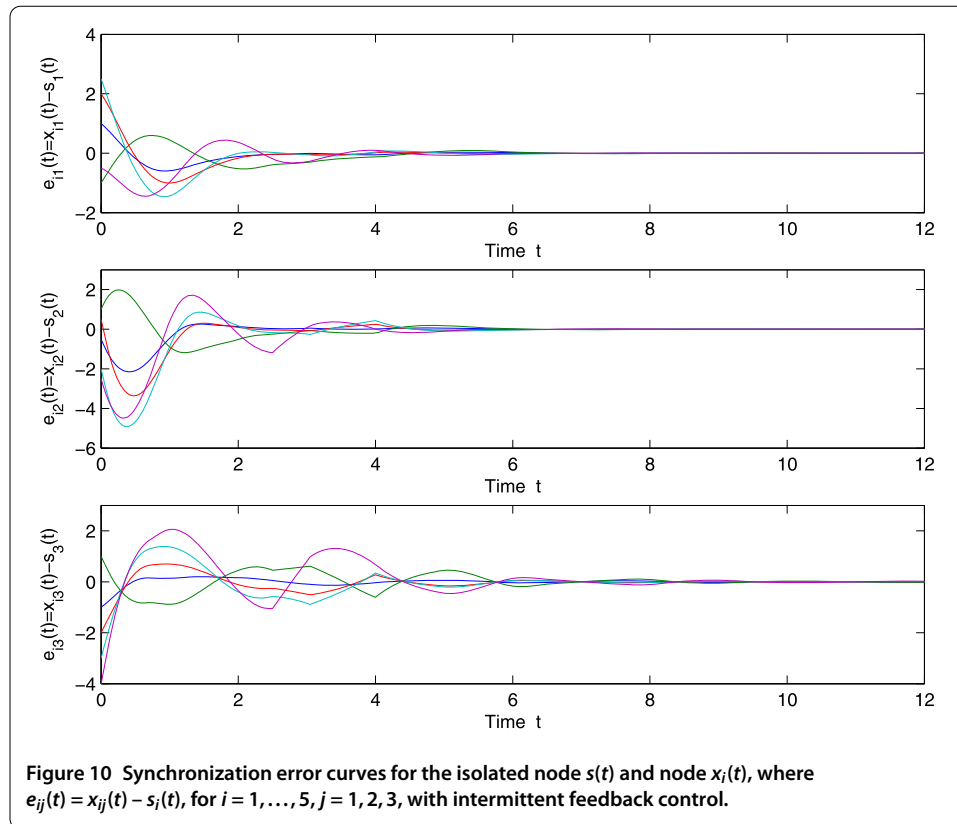
$$K_1 = \begin{bmatrix} -0.1592 & -0.0094 & 0 \\ -0.0155 & -0.1306 & 0 \\ 0 & 0 & -0.3712 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.2479 & -0.0265 & 0 \\ -0.0336 & -0.2273 & 0 \\ 0 & 0 & -0.7249 \end{bmatrix},$$



$$K_3 = \begin{bmatrix} -0.2325 & -0.0247 & 0 \\ -0.0306 & -0.2134 & 0 \\ 0 & 0 & -0.7079 \end{bmatrix}, \quad K_4 = \begin{bmatrix} -0.2188 & -0.0234 & 0 \\ -0.0286 & -0.2006 & 0 \\ 0 & 0 & -0.6986 \end{bmatrix},$$

$$K_5 = \begin{bmatrix} -0.2132 & -0.0230 & 0 \\ -0.0279 & -0.1953 & 0 \\ 0 & 0 & -0.6962 \end{bmatrix}.$$

Figure 9 shows the synchronization errors between the states of the isolated node $s(t)$ and node $x_i(t)$, where $e_{ij}(t) = x_{ij}(t) - s_i(t)$, for $i = 1, \dots, 5$, $j = 1, 2, 3$, without intermittent feedback control. Figure 10 shows the synchronization errors between the states of the



isolated node $s(t)$ and node $x_i(t)$, where $e_{ij}(t) = x_{ij}(t) - s_i(t)$, for $i = 1, \dots, 5, j = 1, 2, 3$, with intermittent feedback control. We see that the synchronization errors converge to zero under the above conditions.

Remark 4.1 In Example 4.1 and Example 4.2, each of them to consider general complex networks in which every dynamical node has mixed time-varying delays (interval time-varying delay and distributed time-varying delay), and the complex networks have state coupling, interval time-varying delay coupling and distributed time-varying delay coupling.

Example 4.3 Consider a network model with five nodes, where each node is a three-dimensional stable linear system described by [9, 18]

$$\begin{aligned}\dot{x}_{i1}(t) &= -x_{i1}(t), \\ \dot{x}_{i2}(t) &= -2x_{i2}(t), \\ \dot{x}_{i3}(t) &= -3x_{i3}(t),\end{aligned}\tag{58}$$

which is asymptotically stable at the equilibrium point $s(t) = 0$, and its Jacobian matrix is $J(t) = \text{diag}\{-1, -2, -3\}$. Assume that the network coupling is the same as that in Example 4.1. The upper bounds on the time-delay obtained from Corollary 3.8 are listed in Table 1. We see that Corollary 3.8 provides a less conservative result than those obtained via the methods of [9, 18]. When $h_m \neq 0$ especially, the result in [9] is not discussed while Corollary 3.8 in this paper also considers the case $h_m \neq 0$.

Table 1 Comparison of the maximum value h_M ($h_m = 0$) for difference c_2

c_2	0.3	0.4	0.5	0.6
Li <i>et al.</i> [9]	0.960	0.710	0.562	0.464
Yue and Li [18]	1.345	0.950	0.731	0.587
Corollary 3.8	1.9707	1.2848	0.8712	0.5941

Remark 4.2 In [9] presented the synchronization problem of general complex dynamical networks with time-varying delays in the network couplings and time-varying delays in the dynamical nodes, respectively. But the time-varying delays are required to be differentiable, however, in most cases, these conditions are difficult to satisfy. Therefore, in this paper we will employ some new techniques so that the above conditions can be removed.

5 Conclusions

This paper has investigated synchronization for complex dynamical network with mixed time-varying and hybrid coupling delays, which is composed of state coupling, interval time-varying delay coupling, and distributed time-varying delay coupling. The time-varying delay function is not necessary to be differentiable which allows the time-delay function to be a fast time-varying function. We transformed the synchronization problem of the complex network into the stability analysis of linear systems. A new class of Lyapunov-Krasovskii functionals is constructed; new delay-dependent sufficient conditions for the exponential synchronization of complex dynamical network have been derived by a set of LMIs without introducing any free-weighting matrices. The delay feedback controllers H1 and H2 designed can guarantee exponential synchronization of the complex dynamical network. Simulation results have been given to illustrate the effectiveness of the proposed method.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details

¹Department of Mathematics, Srinakharinwirot University, Sukhumvit 23 Road, Bangkok, 10110, Thailand. ²Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok, 10400, Thailand. ³Department of Mathematics, Chiang Mai University, Huay Kaew Road, Chiang Mai, 50200, Thailand.

Acknowledgements

We would like to thank referees for their valuable comments and suggestions. This work is supported by the Thailand Research Fund (TRF), the Office of the Higher Education Commission (OHEC), Srinakharinwirot University (grant number MRG5580081), and the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

Received: 8 December 2013 Accepted: 22 April 2014 Published: 06 May 2014

References

1. Faloutsos, M, Faloutsos, P, Faloutsos, C: On power-law relationships of the Internet topology. *Comput. Commun. Rev.* **29**, 251-263 (1999)
2. Albert, R, Jeong, H, Barabási, AL: Diameter of the world wide web. *Nature* **401**, 130-131 (1999)
3. Williams, RJ, Martinez, ND: Simple rules yield complex food webs. *Nature* **404**, 180-183 (2000)
4. Jeong, H, Tombor, B, Albert, R, Oltvai, Z, Barabási, AL: The large-scale organization of metabolic network. *Nature* **407**, 651-653 (2000)
5. Wasserman, S, Faust, K: *Social Network Analysis*. Cambridge University Press, Cambridge (1994)
6. Strogatz, SH: Exploring complex networks. *Nature* **410**, 268-276 (2001)
7. Zhang, Q, Lu, J, Lu, J, Tse, CK: Adaptive feedback synchronization of a general complex dynamical network with delayed nodes. *IEEE Trans. Circuits Syst. II* **55**, 183-187 (2008)
8. Li, C, Chen, G: Synchronization in general complex dynamical networks with coupling delays. *Physica A* **343**, 263-278 (2004)

9. Li, K, Guan, S, Gong, X, Lai, CH: Synchronization stability of general complex dynamical networks with time-varying delays. *Phys. Lett. A* **372**, 7133-7139 (2008)
10. Liu, T, Zhao, J: Synchronization of complex switched delay dynamical networks with simultaneously diagonalizable coupling matrices. *J. Control Theory Appl.* **6**, 351-356 (2008)
11. Wang, XF, Chen, G: Pinning control of scale-free dynamical networks. *Physica A* **310**, 521-531 (2002)
12. Wang, XF, Chen, G: Synchronization scale-free dynamical networks: robustness and fragility. *IEEE Trans. Circuits Syst. I* **49**, 54-62 (2002)
13. Wang, XF, Chen, G: Synchronization in small-world dynamical networks. *Int. J. Bifurc. Chaos* **12**, 187-192 (2002)
14. Gao, H, Lam, J, Chen, G: New criteria for synchronization stability of general complex dynamical networks with coupling delays. *Phys. Lett. A* **306**, 263-273 (2006)
15. Zhou, J, Xiang, L, Liu, Z: Global synchronization in general complex delayed dynamical networks and its applications. *Phys. Lett. A* **385**, 729-742 (2007)
16. Wang, L, Dai, HP, Sun, YX: Synchronization criteria for a generalized complex delayed dynamical network model. *Phys. Lett. A* **383**, 703-713 (2007)
17. Li, P, Yi, Z: Synchronization analysis of delayed complex networks with time-varying couplings. *Phys. Lett. A* **387**, 3729-3737 (2008)
18. Yue, D, Li, H: Synchronization stability of continuous/discrete complex dynamical networks with interval time-varying delays. *Neurocomputing* **73**, 809-819 (2010)
19. Gu, K, Kharitonov, VL, Chen, J: *Stability of Time-Delay System*. Birkhäuser, Boston (2003)
20. Han, QL: Robust stability for a class of linear systems with time varying delay and nonlinear perturbation. *Comput. Math. Appl.* **47**, 1201-1209 (2004)
21. Han, QL, Gu, K: Stability of linear systems with time-varying delay: a generalized discretized Lyapunov functional approach. *Asian J. Control* **3**, 170-180 (2001)
22. Jiang, X, Han, QL: On H_∞ control for linear systems with interval time-varying delay. *Automatica* **41**, 2099-2106 (2005)
23. Park, P: A delay-invariant stability criterion for systems with uncertain time-invariant delays. *IEEE Trans. Autom. Control* **44**, 876-877 (1999)
24. Shao, HY: New delay-dependent stability criteria for systems with interval delay. *Automatica* **45**, 744-749 (2009)
25. Xu, S, Lam, J, Zou, Y: Further results on delay-dependent robust stability conditions of uncertain neutral systems. *Int. J. Robust Nonlinear Control* **15**, 233-246 (2005)
26. Gu, K, Niculescu, SI: Additional dynamics in transformed time-delay systems. *IEEE Trans. Autom. Control* **45**, 572-575 (2000)
27. Han, QL: A descriptor system approach to robust stability of uncertain neutral systems with discrete and distributed delays. *Automatica* **40**, 1791-1796 (2004)
28. Peng, C, Tian, YC: Delay-dependent robust stability criteria for uncertain systems with interval time-varying delay. *J. Comput. Appl. Math.* **214**, 480-494 (2008)
29. Tian, J, Xiong, L, Liu, J, Xie, X: Novel delay-dependent robust stability criteria for uncertain neutral systems with time-varying delay. *Chaos Solitons Fractals* **40**, 1858-1866 (2009)
30. Huang, T, Li, C: Chaotic synchronization by the intermittent feedback method. *J. Comput. Appl. Math.* **234**, 1097-1104 (2010)
31. Xia, W, Cao, J: Pinning synchronization of delayed dynamical networks via periodically intermittent control. *Chaos* **19**, 013120 (2009)
32. Cai, S, Liu, Z, Xu, F, Shen, J: Periodically intermittent controlling complex dynamical networks with time-varying delays to a desired orbit. *Phys. Lett. A* **373**, 3846-3854 (2009)
33. Cai, S, Hao, J, He, Q, Liu, Z: Exponential synchronization of complex delayed dynamical networks via pinning periodically intermittent control. *Phys. Lett. A* **375**, 1965-1971 (2011)
34. Cai, S, He, Q, Hao, J, Liu, Z: Exponential synchronization of complex networks with nonidentical time-delayed dynamical nodes. *Phys. Lett. A* **374**, 2539-2550 (2010)
35. Yang, X, Cao, J: Stochastic synchronization of coupled neural networks with intermittent control. *Phys. Lett. A* **373**, 3259-3272 (2009)
36. Wang, Y, Hao, J, Zuo, Z: A new method for exponential synchronization of chaotic delayed systems via intermittent control. *Phys. Lett. A* **374**, 2024-2029 (2010)
37. Zhang, W, Huang, J, Wei, P: Weak synchronization of chaotic neural networks with parameter mismatch via periodically intermittent control. *Appl. Math. Model.* **35**, 612-620 (2011)
38. Yu, J, Hu, C, Jiang, H, Teng, Z: Exponential synchronization of Cohen-Grossberg neural networks via periodically intermittent control. *Neurocomputing* **74**, 1776-1782 (2011)
39. Zhu, H, Cui, B: Stabilization and synchronization of chaotic systems via intermittent control. *Commun. Nonlinear Sci. Numer. Simul.* **15**, 3577-3586 (2010)
40. Zhang, G, Lin, X, Zhang, X: Exponential stabilization of neutral-type neural networks with mixed interval time-varying delays by intermittent control. *Circuits Syst. Signal Process.* **33**, 371-391 (2014)
41. Huang, J, Li, C, Han, G: Stabilization of delayed chaotic neural networks by periodically intermittent control. *Circuits Syst. Signal Process.* **28**, 567-579 (2009)
42. Horn, RA, Johnson, CR: *Matrix Analysis*. Cambridge University Press, Cambridge (1985)
43. Botmart, T, Niamsup, P: Adaptive control and synchronization of the perturbed Chua system. *Math. Comput. Simul.* **75**, 37-55 (2007)

10.1186/1687-1847-2014-116

Cite this article as: Botmart and Niamsup: Exponential synchronization of complex dynamical network with mixed time-varying and hybrid coupling delays via intermittent control. *Advances in Difference Equations* 2014, **2014**:116

- A2. **Thongchai Botmart**, Exponential synchronization of master-slave neural networks with mixed time-varying delays via hybrid intermittent feedback control, International Journal of Pure and Applied Mathematics, 2014, Submitted.

Exponential synchronization of master-slave neural networks with mixed time-varying delays via hybrid intermittent feedback control

T. Botmart¹

¹Department of Mathematics Srinakharinwirot
University, Bangkok 10110, Thailand.
Corresponding author: thongchaib@swu.ac.th

June 16, 2014

Abstract

In this paper, we investigate the problem of exponential synchronization for master-slave neural networks with mixed time-varying delays via hybrid intermittent feedback control. The constraint on the derivative of the time-varying delay is not required which allows the time-delay to be a fast time-varying function. Based on the construction of improved Lyapunov-Krasovskii functionals is combined with Leibniz-Newtons formula and the technique of dealing with some integral terms. New delay-dependent sufficient conditions for the exponential synchronization of the error systems with memoryless hybrid feedback control are first established in terms of LMIs without introducing any free-weighting matrices. The designed controller ensures that the synchronization of the error systems are proposed via hybrid intermittent feedback control. Numerical simulations are presented to illustrate the effectiveness of these synchronization criteria.

AMS Subject Classification: 34D06; 92B20; 93B52

Key Words and Phrases: master-slave neural networks; exponential synchronization; mixed time-varying delays, intermittent feedback control.

1 Introduction

In the past decade, synchronization in neural networks (NNs), such as cellular NNs, hopfield NNs and bi-directional associative memory networks, has received a great deal of interest among scientists in a variety of areas, such as signal processing, pattern recognition, static image processing, associative memory, content-addressable memory and combinatorial optimization [1, 2, 3, 4]. In performing a periodicity or stability analysis of a neural network, the conditions to be imposed on the neural network are determined by the characteristics of various activation functions and network parameters. When neural networks are created for problem solving, it is desirable for their activation functions are not too restrictive. As a result, there has been considerable research work on the stability of neural networks with various activation functions and more general conditions [5, 6]. The first concept of chaos synchronization is making two chaotic systems oscillate in a synchronized manner was introduced by [2] and many different methods have been applied theoretically and experimentally to synchronize chaotic systems, for example active control [7], adaptive control [7], time-delay feedback control [?, 8] and intermittent control [9], etc. Moreover, neural networks with distributed delays have been extensively discussed [8, 10, 11, 12]. In [12], a neural circuit has been designed with distributed delays, which solves the general problem of recognized patterns in a time-dependent signal. The master-slave synchronization problem has been investigated for neural networks with discrete and distributed time-varying delays in [10], based on the drive-response concept, LMI approach and the Lyapunov stability theorem, several delay-dependent feedback controllers were derived to achieve the exponential synchronization of the chaotic neural networks.

Intermittent control is one of discontinuous control and has a nonzero control width. It is an engineering approach that has been widely used in engineering fields, such as manufacturing, air-quality control, transportation and communication in practice. However,

results using intermittent control to study exponential synchronization are few. In fact, to use synchronization of NNs in some intervals only may prove to be more cost effective than using synchronization of NNs at all times. Another reason could be that the intermittent control scheme might be useful in communications, where synchronization of NNs between transmitter and receiver can be used as means of transmitting information [13]. In recent years, several synchronization criteria for neural networks with or without time delays via intermittent control have been presented, see [14, 15, 16, 17, 18]. In [16], the problem of synchronization for a class of CohenGrossberg neural networks with time delays under periodical intermittent control has been investigated. The quasi-synchronization problem has been investigated for chaotic systems with parameter mismatch by means of periodically intermittent control and design a general periodically-intermittent controller for chaotic systems in [15]. To the best of the author's knowledge, the problem of exponential synchronization for master-slave neural networks with mixed time-varying delays in state and control have not been fully investigated yet and remains open.

This paper, inspired by the above discussions, we shall investigate the problem of exponential synchronization for master-slave neural networks with mixed time-varying delays, which is composed of discrete interval time-varying delay and distributed time-varying delay. The designed controller ensures that the synchronization of delayed master-slave neural networks are proposed via hybrid intermittent feedback control. There are various activation functions which are considered in the system and the restriction on differentiability of interval time-varying delays is removed. Based on the construction of improved Lyapunov-Krasovskii functional is combined with Leibniz-Newton formula and the technique of dealing with some integral terms. New synchronization criteria are derived in terms of LMIs which can be solved efficiently by standard convex optimization algorithms. A numerical example is also given to illustrate the effectiveness of the proposed method.

The rest of this paper is organized as follows. In Section 2, we give notations, definition, propositions and lemma for using in the proof of the main results. In Section 3, synchronization stability in master-slave neural networks with mixed time-varying delays via hybrid intermittent feedback control are investigated. Numerical

examples illustrated the obtained results are given in Section 4. The paper ends with conclusions in Section 5.

2 Preliminaries

The following notation will be used in this paper: R^+ denotes the set of all real non-negative numbers; R^n denotes the n -dimensional space and the vector norm $\| \cdot \|$; $M^{n \times r}$ denotes the space of all matrices of $(n \times r)$ -dimensions.

A^T denotes the transpose of matrix A ; A is symmetric if $A = A^T$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A ; $\lambda_{\max}(A) = \max\{\operatorname{Re}\lambda; \lambda \in \lambda(A)\}$.

Matrix A is called semi-positive definite ($A \geq 0$) if $\langle Ax, x \rangle \geq 0$, for all $x \in R^n$; A is positive definite ($A > 0$) if $\langle Ax, x \rangle > 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$. The symmetric term in a matrix is denoted by $*$.

In this paper, the master-slave cellular neural networks (MSC-NNs) with mixed time-varying delays are described as follows:

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + C\tilde{f}(x(t)) + D\tilde{g}(x(t - h_1(t))) \\ &\quad + E \int_{t-k_1(t)}^t \tilde{h}(x(s))ds + I(t), \\ x(t) &= \phi_1(t), \quad t \in [-d, 0], \end{aligned} \tag{1}$$

$$\begin{aligned} \dot{y}(t) &= -Ay(t) + C\tilde{f}(y(t)) + D\tilde{g}(y(t - h_1(t))) \\ &\quad + E \int_{t-k_1(t)}^t \tilde{h}(y(s))ds + I(t) + \mathcal{U}(t), \\ y(t) &= \phi_2(t), \quad t \in [-d, 0], \end{aligned} \tag{2}$$

where $x(t) = [x_1(t), x_2(t), \dots, x_n(t)] \in R^n$, $y(t) = [y_1(t), y_2(t), \dots, y_n(t)] \in R^n$ are the master systems state vector and the slave systems state vector of the neural networks, respectively. n is the number of neural, and

$$\begin{aligned} \tilde{f}(x(t)) &= [\tilde{f}_1(x_1(t)), \tilde{f}_2(x_2(t)), \dots, \tilde{f}_n(x_n(t))]^T, \\ \tilde{g}(x(t)) &= [\tilde{g}_1(x_1(t)), \tilde{g}_2(x_2(t)), \dots, \tilde{g}_n(x_n(t))]^T, \\ \tilde{h}(x(t)) &= [\tilde{h}_1(x_1(t)), \tilde{h}_2(x_2(t)), \dots, \tilde{h}_n(x_n(t))]^T, \end{aligned}$$

are the activation functions, $A = \operatorname{diag}(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n)$, $\bar{a}_i > 0$ represents the self-feedback term and C, D, E denote the connection

weights, the discretely delayed connection weights and the distributively delayed connection weight, respectively.

The synchronization error $e(t)$ is the form $e(t) = y(t) - x(t)$. Therefore, the cellular neural networks with mixed time-varying delays of synchronization error between the master-slave systems given in (1) and (2) can be described by

$$\begin{aligned}\dot{e}(t) &= -Ae(t) + Cf(e(t)) + Dg(e(t - h_1(t))) \\ &\quad + E \int_{t-k_1(t)}^t h(e(s))ds + \mathcal{U}(t), \\ e(t) &= \phi_2(t) - \phi_1(t) = \phi(t), \quad t \in [-d, 0],\end{aligned}\quad (3)$$

where $f(e(t)) = \tilde{f}(e(t) + x(t)) - \tilde{f}(x(t))$, $g(e(t - h_1(t))) = \tilde{g}(e(t - h_1(t)) + x(t - h_1(t))) - \tilde{g}(x(t - h_1(t)))$, $\int_{t-k_1(t)}^t h(e(s))ds = \int_{t-k_1(t)}^t h(e(s) + x(s)) - h(x(s))ds$. The state hybrid feedback controller $\mathcal{U}(t)$ satisfying :

$$\mathcal{U}(t) = \begin{cases} B_1 u(t) + B_2 u(t - h_2(t)) \\ \quad + B_3 \int_{t-k_2(t)}^t u(s)ds, & n\omega \leq t \leq n\omega + \delta, \\ 0, & n\omega + \delta < t \leq (n+1)\omega. \end{cases} \quad (4)$$

where $u(t) = Ke(t)$ and K is a constant matrix control gain, $\omega > 0$ is the control period and $\delta > 0$ is called the control width (control duration) and n is a non-negative integer. In this paper, our goal is to design suitable K such that system (2) synchronizes with system (1). Then, substituting it into (3), it is easy to get the following:

$$\begin{aligned}\dot{e}(t) &= -Ae(t) + Cf(e(t)) + Dg(e(t - h_1(t))) + E \int_{t-k_1(t)}^t h(e(s))ds \\ &\quad + B_1 Ke(t) + B_2 Ke(t - h_2(t)) + B_3 K \int_{t-k_2(t)}^t e(s)ds, \\ n\omega \leq t \leq n\omega + \delta, \\ \dot{e}(t) &= -Ae(t) + Cf(e(t)) + Dg(e(t - h_1(t))) + E \int_{t-k_1(t)}^t h(e(s))ds \\ n\omega + \delta < t \leq (n+1)\omega, \\ e(t) &= \phi_2(t) - \phi_1(t) = \phi(t), \quad t \in [-d, 0].\end{aligned}\quad (5)$$

Throughout this paper, we consider various activation functions and the activation functions $\tilde{f}(\cdot)$, $\tilde{g}(\cdot)$ and $\tilde{h}(\cdot)$ satisfy the following assumption:

(A1) The activation functions $\tilde{f}(\cdot)$, $\tilde{g}(\cdot)$ and $\tilde{h}(\cdot)$ satisfy Lipschitzian with the Lipschitz constants $\hat{f}_i, \hat{g}_i > 0$ and $\hat{h}_i > 0$:

$$\begin{aligned} |\tilde{f}_i(\xi_1) - \tilde{f}_i(\xi_2)| &\leq \hat{f}_i |\xi_1 - \xi_2|, & i = 1, 2, \dots, n, \forall \xi_1, \xi_2 \in R, \\ |\tilde{g}_i(\xi_1) - \tilde{g}_i(\xi_2)| &\leq \hat{g}_i |\xi_1 - \xi_2|, & i = 1, 2, \dots, n, \forall \xi_1, \xi_2 \in R, \\ |\tilde{h}_i(\xi_1) - \tilde{h}_i(\xi_2)| &\leq \hat{h}_i |\xi_1 - \xi_2|, & i = 1, 2, \dots, n, \forall \xi_1, \xi_2 \in R, \end{aligned} \quad (6)$$

and we denote

$$\begin{aligned} F &= \text{diag}\{\hat{f}_i, \quad i = 1, 2, \dots, n\}, \\ G &= \text{diag}\{\hat{g}_i, \quad i = 1, 2, \dots, n\}, \\ H &= \text{diag}\{\hat{h}_i, \quad i = 1, 2, \dots, n\}. \end{aligned}$$

The time-varying delay functions $h_i(t), k_i(t)$, $i = 1, 2$ satisfy the condition

$$\begin{aligned} 0 &\leq h_{1m} \leq h_1(t) \leq h_{1M}, \quad 0 \leq h_2(t) \leq h_2, \\ 0 &\leq k_1(t) \leq k_1, \quad 0 \leq k_2(t) \leq k_2. \end{aligned} \quad (7)$$

It is worth noting that the time delay is assumed to be a continuous function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, but the delay function is bounded but not restricted to being zero. The initial functions $\phi(t) \in C^1([-d, 0], R^n)$, $d = \max\{h_{1M}, h_2, k_1, k_2\}$ with the norm

$$\|\phi\| = \sup_{t \in [-d, 0]} \sqrt{\|\phi(t)\|^2 + \|\dot{\phi}(t)\|^2}.$$

Definition 1. Given $\alpha > 0$. The zero solution of system (5) with $u(t) = Ke(t)$ is α -stable if there exist a positive number $N > 0$ such that every solution $e(t, \phi)$ satisfies the following condition:

$$\|e(t, \phi)\| \leq Ne^{-\alpha t} \|\phi\|, \quad \forall t \geq 0.$$

We introduce the following technical well-known lemma, which will be used in the proof of our results.

Lemma 1. [19] (Cauchy inequality) For any symmetric positive definite matrix $N \in M^{n \times n}$ and $x, y \in R^n$ we have

$$\pm 2x^T y \leq x^T N x + y^T N^{-1} y.$$

Lemma 2. [19] For any symmetric positive definite matrix $M > 0$, scalar $\gamma > 0$ and vector function $\omega : [0, \gamma] \rightarrow R^n$ such that the integrations concerned are well defined, the following inequality holds

$$\left(\int_0^\gamma \omega(s) ds \right)^T M \left(\int_0^\gamma \omega(s) ds \right) \leq \gamma \left(\int_0^\gamma \omega^T(s) M \omega(s) ds \right).$$

Lemma 3. [19] (Schur complement lemma). Given constant symmetric matrices X, Y, Z with appropriate dimensions satisfying $X = X^T, Y = Y^T > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0.$$

3 Main Results

Let us set

$$\begin{aligned} \Upsilon &= e^{-2\alpha h_{1M}}, \quad \lambda_1 = \lambda_{\min}(P^{-1}), \\ \lambda_2 &= \lambda_{\max}(P^{-1}) + (h_{1m} + h_{1M})\lambda_{\max}(P^{-1}QP^{-1}) \\ &\quad + (h_{1m}^3 + h_{1M}^3)\lambda_{\max}(P^{-1}RP^{-1}) + \delta^3\lambda_{\max}(P^{-1}UP^{-1}) \\ &\quad + h_2^3\lambda_{\max}(P^{-1}Y^T S_1^{-1}YP^{-1}) + k_1^2\lambda_{\max}(HU_3^{-1}H) \\ &\quad + k_2^2\lambda_{\max}(P^{-1}Y^T S_2^{-1}YP^{-1}). \end{aligned}$$

Theorem 2. For some given scalars $0 < \alpha < \varepsilon$, the error system (5) with time-varying delay satisfying (7) are exponentially stable if there exist symmetric positive definite matrices P, Q, R, U, S_1, S_2 , diagonal matrices $U_i, i = 1, 2, 3$ and a matrix Y with appropriately dimensioned such that the following LMI holds:

$$\Gamma_1 = \Gamma_1 - \begin{bmatrix} 0 & 0 & -I & I & 0 \end{bmatrix}^T \Upsilon U \begin{bmatrix} 0 & 0 & -I & I & 0 \end{bmatrix} < 0, \quad (8)$$

$$\Gamma_2 = \Gamma_1 - \begin{bmatrix} 0 & 0 & 0 & I & -I \end{bmatrix}^T \Upsilon U \begin{bmatrix} 0 & 0 & 0 & I & -I \end{bmatrix} < 0, \quad (9)$$

$$\Gamma_3 = \begin{bmatrix} \Pi_{11} & 2PF^T & PH^T & 2Y \\ * & -2U_1 & 0 & 0 \\ * & * & -U_3 & 0 \\ * & * & * & -2e^{-2\alpha h_2}S_1 \end{bmatrix} < 0, \quad (10)$$

$$\Gamma_4 = \begin{bmatrix} -0.1P & h_2^2 Y^T \\ * & -h_2^2 S_1 \end{bmatrix} < 0, \quad (11)$$

$$\Gamma_5 = \begin{bmatrix} -0.1e^{-2\alpha h_{1M}}U & 2PG^T \\ * & -2U_2 \end{bmatrix} < 0, \quad (12)$$

$$\Gamma_6 = \Gamma_2 - \begin{bmatrix} 0 & 0 & -I & I & 0 \end{bmatrix}^T \Upsilon U \begin{bmatrix} 0 & 0 & -I & I & 0 \end{bmatrix} < 0, \quad (13)$$

$$\Gamma_7 = \Gamma_2 - \begin{bmatrix} 0 & 0 & 0 & I & -I \end{bmatrix}^T \Upsilon U \begin{bmatrix} 0 & 0 & 0 & I & -I \end{bmatrix} < 0, \quad (14)$$

$$\Gamma_8 = \begin{bmatrix} \Pi_{11} & 2PF^T & k_1PH^T \\ * & -2U_1 & 0 \\ * & * & -k_1U_3 \end{bmatrix} < 0, \quad (15)$$

$$-\alpha\delta + (\varepsilon - \alpha)(\omega - \delta) < 0, \quad (16)$$

$$\Gamma_1 = \begin{bmatrix} \Gamma_{111} & \Gamma_{112} & \Gamma_{13} & 0 & \Gamma_{15} \\ * & \Gamma_{122} & 0 & 0 & 0 \\ * & * & \Gamma_{33} & \Gamma_{34} & 0 \\ * & * & * & \Gamma_{44} & \Gamma_{45} \\ * & * & * & * & \Gamma_{55} \end{bmatrix},$$

$$\Gamma_2 = \begin{bmatrix} \Gamma_{211} & \Gamma_{212} & \Gamma_{13} & 0 & \Gamma_{15} \\ * & \Gamma_{222} & 0 & 0 & 0 \\ * & * & \Gamma_{33} & \Gamma_{34} & 0 \\ * & * & * & \Gamma_{44} & \Gamma_{45} \\ * & * & * & * & \Gamma_{55} \end{bmatrix},$$

where

$$\begin{aligned} \Pi_{11} &= -0.1(e^{-2\alpha h_{1m}} + e^{-2\alpha h_{1M}})R \\ \Gamma_{111} &= [-A + \alpha I]P + P[-A + \alpha I]^T - B_1Y - Y^TB_1^T + 2Q \\ &\quad + C^TU_1C + D^TU_2D + 2k_1e^{2\alpha k_1}E^TU_3E + 3e^{2\alpha h_2}B_2^TS_1B_2 \\ &\quad + 2k_2e^{2\alpha k_2}B_3^TS_2B_3 - 0.9e^{-2\alpha h_{1m}}R - 0.9e^{-2\alpha h_{1M}}R, \\ \Gamma_{211} &= [-A + \alpha I]P + P[-A + \alpha I]^T + 2Q - 2\varepsilon P + C^TU_1C \\ &\quad + D^TU_2D + 2k_1e^{2\alpha k_1}E^TU_3E - 0.9e^{-2\alpha h_{1m}}R - 0.9e^{-2\alpha h_{1M}}R, \\ \Gamma_{112} &= -PA^T - Y^TB^T, \quad \Gamma_{212} = -PA^T, \quad \Gamma_{13} = e^{-2\alpha h_{1m}}R, \\ \Gamma_{15} &= e^{-2\alpha h_{1M}}R, \\ \Gamma_{122} &= h_{1m}^2R + h_{1M}^2R + \delta^2U - 1.9P + C^TU_1C + D^TU_2D \\ &\quad + 2k_1e^{2\alpha k_1}E^TU_3E + 3e^{2\alpha h_2}B_2^TS_1B_2 \\ &\quad + 2k_2e^{2\alpha k_2}B_3^TS_2B_3, \end{aligned}$$

$$\begin{aligned}
\Gamma_{222} &= h_{1m}^2 R + h_{1M}^2 R + \delta^2 U - 2P + C^T U_1 C + D^T U_2 D \\
&\quad + 2k_1 e^{2\alpha k_1} E^T U_3 E, \\
\Gamma_{33} &= -e^{-2\alpha h_{1m}} Q - e^{-2\alpha h_{1m}} R - e^{-2\alpha h_{1M}} U, \\
\Gamma_{34} &= e^{-2\alpha h_{1M}} U, \quad \Gamma_{44} = -1.9e^{-2\alpha h_{1M}} U, \quad \Gamma_{45} = e^{-2\alpha h_{1M}} U, \\
\Gamma_{55} &= -e^{-2\alpha h_{1M}} Q - e^{-2\alpha h_{1M}} R - e^{-2\alpha h_{1M}} U,
\end{aligned}$$

then the error system (5) have exponential synchronization. Moreover, the feedback control is

$$\mathcal{U}(t) = \begin{cases} -B_1 Y P^{-1} e(t) - B_2 Y P^{-1} e(t - h_2(t)) \\ -B_3 Y P^{-1} \int_{t-k_2(t)}^t e(s) ds, & n\omega \leq t \leq n\omega + \delta, \\ 0, & n\omega + \delta < t \leq (n+1)\omega. \end{cases} \quad (17)$$

Proof. Case I: for $n\omega \leq t \leq n\omega + \delta$, let $W = P^{-1}$, $z(t) = We(t)$. Using the feedback control (17) we consider the following Lyapunov-Krasovskii functional

$$V(e(t)) = \sum_{i=1}^9 V_i, \quad (18)$$

where

$$\begin{aligned}
V_1 &= e^T(t) W e(t), \\
V_2 &= \int_{t-h_{1m}}^t e^{2\alpha(s-t)} e^T(s) W Q W e(s) ds, \\
V_3 &= \int_{t-h_{1M}}^t e^{2\alpha(s-t)} e^T(s) W Q W e(s) ds, \\
V_4 &= h_{1m} \int_{-h_{1m}}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{e}^T(\theta) W R W \dot{e}(\theta) d\theta ds, \\
V_5 &= h_{1M} \int_{-h_{1M}}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{e}^T(\theta) W R W \dot{e}(\theta) d\theta ds, \\
V_6 &= \delta \int_{-h_{1M}}^{-h_{1m}} \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{e}^T(\theta) W U W \dot{e}(\theta) d\theta ds, \\
V_7 &= \int_{-k_1}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} h^T(e(\theta)) U_3^{-1} h(e(\theta)) d\theta ds, \\
V_8 &= h_2 \int_{h_2}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{u}^T(\theta) S_1^{-1} \dot{u}(\theta) d\theta ds,
\end{aligned}$$

$$V_9 = \int_{-k_2}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} u^T(\theta) S_2^{-1} u(\theta) d\theta ds.$$

It easy to check that

$$\lambda_1 \|e(t)\|^2 \leq V(t, e(t)) \leq \lambda_2 \|e(t)\|^2, \quad \forall t \geq 0. \quad (19)$$

Taking the derivative of $V(t, e(t))$ along the solution of system (5) we have

$$\begin{aligned} \dot{V}_1 &= z^T(t)[-AP - PA^T - 2B_1Y]z(t) + 2z^T(t)Cf(e(t)) \\ &\quad + 2z^T(t)Dg(e(t - h_1(t))) + 2z^T(t)E \int_{t-k_1(t)}^t h(e(s))ds \\ &\quad + 2z^T(t)B_2u(t - h_2(t)) + 2z^T(t)B_3 \int_{t-k_2(t)}^t u(s)ds, \\ \dot{V}_2 &= z^T(t)Qz(t) - e^{-2\alpha h_{1m}} z^T(t - h_{1m})Qz(t - h_{1m}) - 2\alpha V_2, \\ \dot{V}_3 &= z^T(t)Qz(t) - e^{-2\alpha h_{1M}} z^T(t - h_{1M})Qz(t - h_{1M}) - 2\alpha V_3, \\ \dot{V}_4 &\leq h_{1m}^2 \dot{z}^T(t)R\dot{z}(t) - h_{1m}e^{-2\alpha h_{1m}} \int_{t-h_{1m}}^t \dot{z}^T(s)R\dot{z}(s)ds - 2\alpha V_4, \\ \dot{V}_5 &\leq h_{1M}^2 \dot{z}^T(t)R\dot{z}(t) - h_{1M}e^{-2\alpha h_{1M}} \int_{t-h_{1M}}^t \dot{z}^T(s)R\dot{z}(s)ds - 2\alpha V_5, \\ \dot{V}_6 &\leq \delta^2 \dot{z}^T(t)U\dot{z}(t) - \delta e^{-2\alpha h_{1M}} \int_{t-h_{1M}}^{t-h_{1m}} \dot{z}^T(s)R\dot{z}(s)ds - 2\alpha V_6, \\ \dot{V}_7 &\leq k_1 h^T(e(t))U_3^{-1}h(e(t)) - e^{-2\alpha k_1} \int_{t-k_1}^t h^T(e(s))U_3^{-1}h(e(s))ds \\ &\quad - 2\alpha V_7, \\ \dot{V}_8 &\leq h_2^2 \dot{u}^T(t)S_1^{-1}\dot{u}(t) - h_2e^{-2\alpha h_2} \int_{t-h_2}^t \dot{u}^T(s)S_1^{-1}\dot{u}(s)ds - 2\alpha V_8, \\ \dot{V}_9 &\leq k_2^2 u^T(t)S_2^{-1}u(t) - k_2e^{-2\alpha k_2} \int_{t-k_2}^t u^T(s)S_2^{-1}u(s)ds - 2\alpha V_9. \end{aligned} \quad (20)$$

For assumption A1, we can obtain the following three inequalities:

$$\begin{aligned} |f_i(e_i(t))| &\leq \hat{f}_i|e_i(t) + x_i(t) - x_i(t)| = \hat{f}_i|e_i(t)|, \\ |g_i(e_i(t))| &\leq \hat{g}_i|e_i(t) + x_i(t) - x_i(t)| = \hat{g}_i|e_i(t)|, \\ |h_i(e_i(t))| &\leq \hat{h}_i|e_i(t) + x_i(t) - x_i(t)| = \hat{h}_i|e_i(t)|. \end{aligned} \quad (21)$$

Applying Lemma (1) and Lemma (2) and since the matrices $U_i, i = 1, 2, 3$ are diagonal, we have

$$\begin{aligned}
2z^T(t)Cf(e(t)) &\leq z^T(t)C^T U_1 C z(t) + z^T(t)P F^T U_1^{-1} F P z(t) \\
2z^T(t)Dg(e(t-h_1(t))) &\leq z^T(t)D^T U_2 D z(t) \\
&\quad + z^T(t-h_1(t))P G^T U_2^{-1} G P z(t-h_1(t)), \\
k_1 h^T(e(t))U_3^{-1}h(e(t)) &\leq k_1 z^T(t)P H^T U_3^{-1} H P z(t), \\
2z^T(t)E \int_{t-k_1(t)}^t h(e(s))ds &\leq 2k_1 e^{2\alpha k_1} z^T(t)E^T U_3 E z(t) \\
&\quad + \frac{e^{-2\alpha k_1}}{2} \int_{t-k_1(t)}^t h^T(e(s))U_3^{-1}h(e(s))ds, \\
2z^T(t)B_2 u(t-h_2(t)) &\leq 3e^{2\alpha h_2} z^T(t)B_2^T S_1 B_2 z(t) \\
&\quad + \frac{e^{-2\alpha h_2}}{3} u^T(t-h_2(t))S_1^{-1}u(t-h_2(t)), \\
2z^T(t)B_3 \int_{t-k_2(t)}^t u(s)ds &\leq 2k_2 e^{2\alpha k_2} z^T(t)B_3^T S_2 B_3 z(t) \\
&\quad + \frac{e^{-2\alpha k_2}}{2} \int_{t-k_2(t)}^t u^T(s)S_2^{-1}u(s)ds, \\
h_2^2 \dot{u}^T(t)S_1^{-1}\dot{u}(t) &= h_2^2 \dot{z}^T(t)Y^T S_1^{-1}Y \dot{z}(t).
\end{aligned}$$

and the Leibniz-Newton formula gives

$$\begin{aligned}
&-h_2 e^{-2\alpha h_2} \int_{t-h_2}^t \dot{u}^T(s)S_1^{-1}\dot{u}(s)ds \\
&\leq 2e^{-2\alpha h_2} z^T(t)Y^T S_1^{-1}Y z(t) + \frac{e^{-2\alpha h_2}}{3} u^T(t-h_2(t))S_1^{-1}u(t-h_2(t)) \\
&\quad - e^{-2\alpha h_2} u^T(t-h_2(t))S_1^{-1}u(t-h_2(t)). \tag{22}
\end{aligned}$$

Applying Lemma 2 and the Leibniz-Newton formula, we have

$$\begin{aligned}
-h_{1m} e^{-2\alpha h_{1m}} \int_{t-h_{1m}}^t \dot{z}^T(s)R \dot{z}(s)ds &\leq -e^{-2\alpha h_{1m}} \left[z^T(t)R z(t) \right. \\
&\quad - 2z^T(t)R z(t-h_{1m}) \tag{23} \\
&\quad \left. + z^T(t-h_{1m})R z(t-h_{1m}) \right],
\end{aligned}$$

$$\begin{aligned}
-h_{1M}e^{-2\alpha h_{1M}} \int_{t-h_{1M}}^t \dot{z}^T(s)R\dot{z}(s)ds &\leq -e^{-2\alpha h_{1M}} \left[z^T(t)Rz(t) \right. \\
&\quad -2z^T(t)Rz(t-h_{1M}) \quad (24) \\
&\quad \left. +z^T(t-h_{1M})Rz(t-h_{1M}) \right].
\end{aligned}$$

Note that

$$\begin{aligned}
-\delta \int_{t-h_{1M}}^{t-h_{1m}} \dot{z}^T(s)U\dot{z}(s)ds &= -(h_{1M}-h(t)) \int_{t-h_{1M}}^{t-h(t)} \dot{z}^T(s)U\dot{z}(s)ds \\
&\quad -(h(t)-h_{1m}) \int_{t-h_{1M}}^{t-h(t)} \dot{z}^T(s)U\dot{z}(s)ds \\
&\quad -(h(t)-h_{1m}) \int_{t-h(t)}^{t-h_{1m}} \dot{z}^T(s)U\dot{z}(s)ds \\
&\quad -(h_{1M}-h(t)) \int_{t-h(t)}^{t-h_{1m}} \dot{z}^T(s)U\dot{z}(s)ds.
\end{aligned}$$

Using Lemma 2 and let $\beta = \frac{h_{1M}-h(t)}{h_{1M}-h_{1m}} \leq 1$. Then

$$\begin{aligned}
&-\delta \int_{t-h_{1M}}^{t-h_{1m}} \dot{z}^T(s)U\dot{z}(s)ds \\
&\leq -[z(t-h(t)) - z(t-h_{1M})]^T U [z(t-h(t)) - z(t-h_{1M})] \\
&\quad -[z(t-h_{1m}) - z(t-h(t))]^T U [z(t-h_{1m}) - z(t-h(t))] \quad (25) \\
&\quad -\beta[z(t-h_{1m}) - z(t-h(t))]^T U [z(t-h_{1m}) - z(t-h(t))] \\
&\quad -(1-\beta)[z(t-h(t)) - z(t-h_{1M})]^T U [z(t-h(t)) - z(t-h_{1M})].
\end{aligned}$$

By using the following identity relation

$$\begin{aligned}
0 &= -2\dot{z}^T(t)P\dot{z}(t) - 2\dot{z}^T(t)APz(t) + 2\dot{z}^T(t)Cf(e(t)) \\
&\quad +2\dot{z}^T(t)Dg(e(t-h_1(t))) + 2\dot{z}^T(t)E \int_{t-k_1(t)}^t h(e(s))ds \\
&\quad -2\dot{z}^T(t)B_1Yz(t) + 2\dot{z}^T(t)B_2(t)u(t-h_2(t)) \\
&\quad +2\dot{z}^T(t)B_3 \int_{t-k_2(t)}^t u(s)ds. \quad (26)
\end{aligned}$$

By using Lemma 1 and 2, we have

$$\begin{aligned}
2\dot{z}^T(t)Cf(e(t)) &\leq \dot{z}^T(t)C^TU_1C\dot{z}(t) \\
&\quad + z^T(t)PF^TU_1^{-1}FPz(t), \\
2\dot{z}^T(t)Dg(e(t-h_1(t))) &\leq \dot{z}^T(t)D^TU_2D\dot{z}(t) \\
&\quad + z^T(t-h_1(t))PG^TU_2^{-1}GPz(t-h_1(t)), \\
2\dot{z}^T(t)E \int_{t-k_1(t)}^t h(e(s))ds &\leq 2k_1e^{2\alpha k_1}\dot{z}^T(t)E^TU_3E\dot{z}(t) \quad (27) \\
&\quad + \frac{e^{-2\alpha k_1}}{2} \int_{t-k_1(t)}^t h^T(e(s))U_3^{-1}h(e(s))ds, \\
2\dot{z}^T(t)B_2(t)u(t-h_2(t)) &\leq 3e^{2\alpha h_2}\dot{z}^T(t)B_2^TS_1B_2\dot{z}(t) \\
&\quad + \frac{e^{-2\alpha h_2}}{3}u^T(t-h_2(t))S_1^{-1}u(t-h_2(t)), \\
2\dot{z}^T(t)B_3 \int_{t-k_2(t)}^t u(s)ds &\leq 2k_2e^{2\alpha k_2}\dot{z}^T(t)B_3^TS_2B_3\dot{z}(t) \\
&\quad + \frac{e^{-2\alpha k_2}}{2} \int_{t-k_2(t)}^t u^T(s)S_2^{-1}u(s)ds,
\end{aligned}$$

From (20) - (27), we obtain

$$\begin{aligned}
\dot{V}(e(t)) + 2\alpha V(e(t)) &\leq \xi^T(t)((1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2)\xi(t) \\
&\quad + z^T(t)\mathcal{M}_3z(t) + \dot{z}^T(t)\mathcal{M}_4\dot{z}(t) \\
&\quad + z^T(t-h_1(t))\mathcal{M}_5z(t-h_1(t)), \quad (28)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}_3 &= -0.1R(e^{-2\alpha h_{1m}} + e^{-2\alpha h_{1M}}) \\
&\quad + 2PF^TU_1^{-1}FP + k_1PH^TU_3^{-1}HP + 2e^{-2\alpha h_2}Y^TS_1^{-1}Y \\
\mathcal{M}_4 &= -0.1P + h_2^2Y^TS_1^{-1}Y, \\
\mathcal{M}_5 &= -0.1e^{-2\alpha h_{1M}}U + 2PG^TU_2^{-1}GP, \\
\mathcal{M}_6 &= Q_1 + P^{-1}Y^TQ_2YP^{-1}, \\
\xi(t) &= [z(t), \dot{z}(t), z(t-h_{1m}), z(t-h(t)), z(t-h_{1M})].
\end{aligned}$$

Since $0 \leq \beta \leq 1$, $(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2$ is a convex combination of $\mathcal{M}_1$ and $\mathcal{M}_2$. Therefore, $(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2 < 0$ is equivalent to $\Gamma_1 < 0$ and $\Gamma_2 < 0$. Applying Schur complement lemma, the inequalities $\mathcal{M}_3 < 0$, $\mathcal{M}_4 < 0$ and $\mathcal{M}_5 < 0$ are equivalent to

$\Gamma_3 < 0$, $\Gamma_4 < 0$ and $\Gamma_5 < 0$, respectively. Thus, it follows from (8) - (12) and (28), we obtain

$$\dot{V}(e(t)) + 2\alpha V(e(t)) \leq 0, \quad \text{for } n\omega \leq t \leq n\omega + \delta. \quad (29)$$

Thus, by the above differential inequality (29), we have

$$\dot{V}(e(t)) \leq V(e(n\omega))e^{-2\alpha t - n\omega}, \quad \text{for } n\omega \leq t \leq n\omega + \delta. \quad (30)$$

Case II: for $n\omega + \delta \leq t \leq (n+1)\omega$, we choose Lyapunov-Krasovskii functional having the following form :

$$V(e(t)) = \sum_{i=1}^7 V_i, \quad (31)$$

where $V_i(t)$, $i = 1, 2, \dots, 7$ are defined similar in (18). We are able to do similar estimation as we did for Case I, we have the following

$$\begin{aligned} \dot{V}(e(t)) + 2(\alpha - \varepsilon)V(e(t)) &\leq \xi^T(t)((1 - \beta)\mathcal{N}_1 + \beta\mathcal{N}_2)\xi(t) \\ &\quad + z^T(t)\mathcal{N}_3 z(t) \\ &\quad + z^T(t - h_1(t))\mathcal{N}_4 z(t - h_1(t)), \end{aligned} \quad (32)$$

where

$$\begin{aligned} \mathcal{N}_3 &= -0.1R(e^{-2\alpha h_{1m}} + e^{-2\alpha h_{1M}}) \\ &\quad + 2PF^T U_1^{-1}FP + k_1PH^T U_3^{-1}HP, \\ \mathcal{N}_4 &= \mathcal{M}_4. \end{aligned}$$

Since $0 \leq \beta \leq 1$, $(1 - \beta)\mathcal{N}_1 + \beta\mathcal{N}_2$ is a convex combination of $\mathcal{N}_1$ and $\mathcal{N}_2$. Therefore, $(1 - \beta)\mathcal{N}_1 + \beta\mathcal{N}_2 < 0$ is equivalent to $\Gamma_6 < 0$ and $\Gamma_7 < 0$. Applying Schur complement lemma, the inequalities $\mathcal{N}_3 < 0$, $\mathcal{N}_4 < 0$ are equivalent to $\Gamma_8 < 0$, $\Gamma_5 < 0$, respectively. Thus, it follows from (12) - (15) and (32), we obtain

$$\dot{V}(e(t)) - 2(\varepsilon - \alpha)V(e(t)) \leq 0, \quad \text{for } n\omega \leq t \leq n\omega + \delta. \quad (33)$$

Thus, by the above differential inequality (29), we have

$$\dot{V}(e(t)) \leq V(e(n\omega + \delta))e^{2(\varepsilon - \alpha)(t - n\omega - \delta)}, \quad \text{for } n\omega \leq t \leq n\omega + \delta. \quad (34)$$

By (30) and (34), we have

$$\begin{aligned}
V(e((n+1)\omega)) &\leq V(e(n\omega + \delta))e^{2(\varepsilon-\alpha)(\omega-\delta)} \\
&\leq V(e(n\omega))e^{-2\alpha\delta}e^{2(\varepsilon-\alpha)(\omega-\delta)} \\
&= V(e(n\omega))e^{-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta)} \\
&\leq V(e((n-1)\omega + \delta))e^{2\rho(\omega-\delta)}e^{-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta)} \\
&\leq V(e((n-1)\omega))e^{-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta)}e^{-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta)} \\
&= V(e((n-1)\omega))e^{2(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))} \\
&\vdots \\
&\leq V(e(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n+1)}.
\end{aligned}$$

For any $t > 0$, there is a $n_0 \geq 0$, such that $n_0\omega \leq t \leq (n_0 + 1)\omega$.

Case 1. For $n_0\omega + \delta \leq t \leq (n_0 + 1)\omega$, using condition (16), we have

$$\begin{aligned}
V(e(t)) &\leq V(e(n_0\omega + \delta))e^{2(\varepsilon-\alpha)(t-(n_0\omega+\delta))} \\
&\leq V(e(n_0\omega))e^{-2\alpha\delta}e^{2(\varepsilon-\alpha)(t-(n_0\omega+\delta))} \\
&\leq V(e(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))n_0}e^{-2\alpha\delta}e^{2(\varepsilon-\alpha)(t-(n_0\omega+\delta))} \\
&\leq V(e(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))n_0}e^{-2\alpha\delta}e^{2(\varepsilon-\alpha)((n_0+1)\omega-(n_0\omega+\delta))} \\
&= V(e(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n_0+1)} \\
&= V(e(0))e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n_0+1)\omega}{\omega}} \\
&\leq V(e(0))e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))t}{\omega}}. \tag{35}
\end{aligned}$$

Case 2. For $n_0\omega \leq t \leq n_0\omega + \delta$, using condition (16), we have

$$\begin{aligned}
V(e(t)) &\leq V(e(n_0\omega))e^{-2\alpha(t-n_0\omega)} \\
&\leq V(e(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))n_0}e^{-2\alpha(t-n_0\omega)} \\
&\leq V(e(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))n_0} \\
&= V(e(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n_0+1)} \\
&= V(e(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n_0+1)\omega}{\omega}} \\
&\leq V(e(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))t}{\omega}}. \tag{36}
\end{aligned}$$

Let $\xi = e^{-(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}$. By (35) and (36), we have

$$V(e(t)) \leq \xi V(e(0))e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))t}{\omega}}, \quad \forall t \geq 0.$$

On the other hand, using the condition (19), we have obtained the following:

$$\|e(t)\| \leq \sqrt{\frac{V(e(0))\xi}{\gamma}} e^{\frac{(-\alpha\delta + (\varepsilon - \alpha)(\omega - \delta))t}{\omega}}, \quad \forall t \geq 0.$$

which implies the error system (5) is exponentially stable under the controller $H1$, then the controlled slave system (1) is synchronized with the master system (2). The proof is thus completed. $\square$

4 Numerical examples

In this section, we now provide an example to show the effectiveness of the result in Theorem 2.

Example 4.1 Consider the cellular neural networks with various activation functions and mixed time-varying delays using hybrid intermittent feedback control with the following parameters :

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + C\tilde{f}(x(t)) + D\tilde{g}(x(t - h_1(t))) \\ &\quad + E \int_{t-k_1(t)}^t \tilde{h}(x(s))ds + I(t), \end{aligned} \quad (37)$$

$$\begin{aligned} x(t) &= \phi_1(t), \quad t \in [-d, 0], \\ \dot{y}(t) &= -Ay(t) + C\tilde{f}(y(t)) + D\tilde{g}(y(t - h_1(t))) \\ &\quad + E \int_{t-k_1(t)}^t \tilde{h}(y(s))ds + I(t) + \mathcal{U}(t) \\ y(t) &= \phi_2(t), \quad t \in [-d, 0], \end{aligned} \quad (38)$$

where

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3 & -0.2 \\ 0.1 & -0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 0.8 & 0.4 \\ -0.3 & 0.5 \end{bmatrix}, \\ E &= \begin{bmatrix} 0.5 & 0.2 \\ -0.3 & 0.5 \end{bmatrix}, \quad F = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad G = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix}, \\ H &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \\ B_3 &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \\ \phi_1(t) &= [-0.4 \cos t, 0.5 \cos t], \quad \phi_2(t) = [\sin t, \sin t]. \end{aligned}$$

Solution: From the conditions (8)-(15) of Theorem 2, we let $\alpha = 0.07$, $\varepsilon = 0.09$, $\omega = 4$, $\delta = 2.5$, $h_{1m} = 0.1$, $h_{1M} = 0.2$, $h_2 = 0.3$, $k_1 = 0.15$, $k_2 = 0.2$. By using the LMI Toolbox in MATLAB, we obtain

$$\begin{aligned}
P &= \begin{bmatrix} 1.5072 & 0.0147 \\ 0.0147 & 0.8106 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.3204 & 0.0027 \\ 0.0027 & 0.1821 \end{bmatrix}, \\
R &= \begin{bmatrix} 1.5954 & 0.0073 \\ 0.0073 & 0.9829 \end{bmatrix}, \quad U = \begin{bmatrix} 6.5228 & 0.1067 \\ 0.1067 & 3.2384 \end{bmatrix}, \\
S_1 &= 10^{-3} \begin{bmatrix} 0.0446 & -0.4554 \\ -0.4554 & 1.2455 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.0003 & -0.0017 \\ -0.0017 & 0.0311 \end{bmatrix}, \\
U_1 &= \begin{bmatrix} 2.6046 & 0 \\ 0 & 0.5215 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0.6454 & 0 \\ 0 & 0.1738 \end{bmatrix}, \\
U_3 &= \begin{bmatrix} 0.4273 & 0 \\ 0 & 0.1726 \end{bmatrix}, \quad Y = 10^{-3} \begin{bmatrix} 0.0362 & 0.0994 \\ 0.3908 & 2.7357 \end{bmatrix}, \\
K &= 10^{-3} \begin{bmatrix} -0.0252 & 0.1231 \\ 0.2923 & -3.3803 \end{bmatrix}.
\end{aligned}$$

We let $h_1(t) = 0.1 + 0.1|\sin t|$, $h_2(t) = 0.3e^{|\sin t|}$, $k_1(t) = 0.15|\cos t|$, $k_2(t) = 0.2e^{|\cos t|}$, $\phi_1(t) = [-0.4 \cos t, 0.5 \cos t]$, $\phi_2(t) = [\sin t, \sin t]$, $\forall t \in [-0.3, 0]$ and the activation function as follows:

$$\begin{aligned}
f_1(x_1(t)) &= 0.2(|x_1(t) + 1| - |x_1(t) - 1|), \\
f_2(x_2(t)) &= 0.1(|x_1(t) + 1| - |x_1(t) - 1|), \\
g_1(x_1(t)) &= 0.15(|x_1(t) + 1| - |x_1(t) - 1|), \\
g_2(x_2(t)) &= 0.1(|x_2(t) + 1| - |x_2(t) - 1|), \\
h_1(x_1(s)) &= 0.1 \tanh(-4x_1(s)), \\
h_2(x_2(s)) &= 0.15 \tanh(5x_2(s)).
\end{aligned}$$

Figure 1. shows the trajectories of solutions $e_1(t)$ and $e_2(t)$ of the cellular neural networks with various activation functions and mixed time-varying delays without hybrid intermittent feedback control ($\mathcal{U}(t) = 0$). Figure 2. shows the trajectories of solutions $e_1(t)$ and $e_2(t)$ of the the cellular neural networks with various activation functions and mixed time-varying delays with hybrid inter-

mittent feedback control

$$\mathcal{U}(t) = \begin{cases} -10^{-3} \begin{bmatrix} -0.1010 & 0.4924 \\ 0.2923 & -3.3803 \end{bmatrix} e(t) \\ -10^{-3} \begin{bmatrix} -0.0757 & 0.3693 \\ 0.2923 & -3.3803 \end{bmatrix} e(t - h_2(t)) \\ -10^{-3} \begin{bmatrix} -0.0505 & 0.2462 \\ 0.2923 & -3.3803 \end{bmatrix} \int_{t-k_2(t)}^t e(s) ds, \\ n\omega \leq t \leq n\omega + \delta, \\ 0, \quad n\omega + \delta < t \leq (n+1)\omega. \end{cases} \quad (39)$$

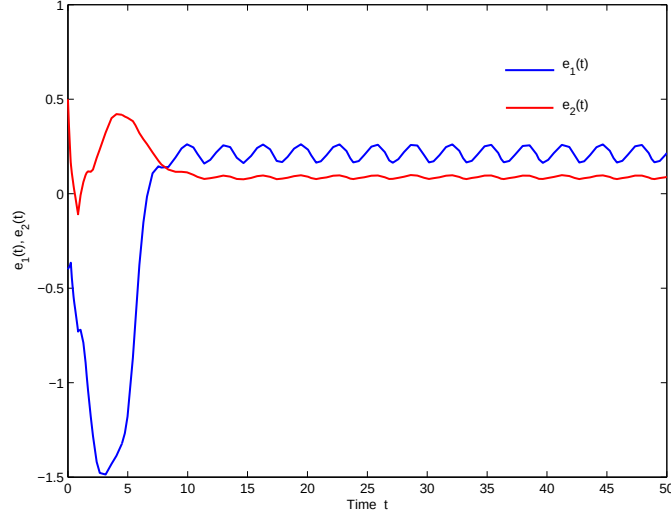


Figure 1: shows the trajectories of solutions $e_1(t)$ and $e_2(t)$ of the cellular neural networks without hybrid intermittent feedback control ($\mathcal{U}(t) = 0$).

5 Conclusions

In this paper, we have investigated the exponential synchronization of cellular neural networks with various activation functions and mixed time-varying delays via hybrid intermittent feedback control. The interval time-varying delay function is not necessary to

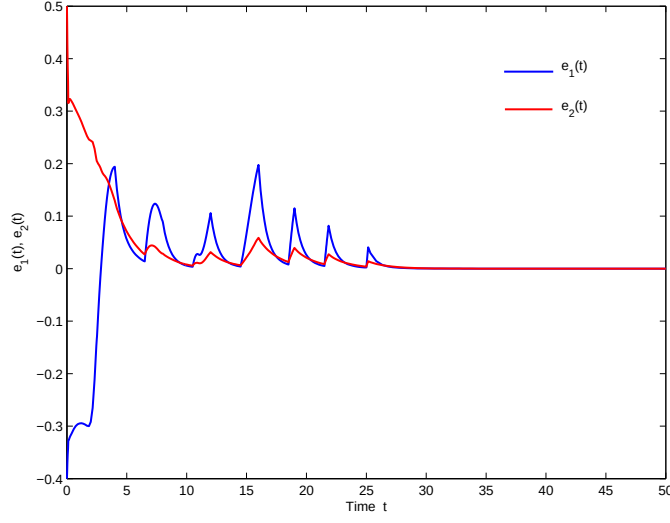


Figure 2: shows the trajectories of solutions $e_1(t)$ and $e_2(t)$ of the cellular neural networks with hybrid intermittent feedback control (39).

be differentiable which allows time-delay function to be a fast time-varying function. A new class of Lyapunov-Krasovskii functional is constructed to new delay-dependent sufficient conditions for the exponential synchronization of the error systems have been derived by a set of LMIs without introducing any free-weighting matrices. The hybrid intermittent feedback controller designed can guarantee exponential stability of the error system. Simulation results have been given to illustrate the effectiveness of the proposed method.

6 Acknowledgements.

We would like to thank referees for their valuable comments and suggestions. This work is supported by the Thailand Research Fund(TRF), the Office of the Higher Education Commission(OHEC), Srinakharinwirot University (grant number MRG5580081). We would like to thank Assoc. Prof. Dr. Piyapong Niamsup for valuable comments and suggestions

References

- [1] M. M. Gupta, L. Jin and N. Homma, *Static and Dynamic Neural Networks: From Fundamentals to Advanced Theory*, New York: Wiley, 2003.
- [2] L. M. Pecora and T. L. Carroll, Synchronization in chaotic systems, *Phys. Rev. Lett.*, **64** (1990), 821-824.
- [3] J. Liang and J. Cao, Global asymptotic stability of bi-directional associative memory networks with distributed delays, *Appl. Math. Comput.*, **152** (2004), 415-424.
- [4] H. Zhao, Global asymptotic stability of Hopfield neural network involving distributed delays, *Neural Netw.*, **17** (2004), 47-53.
- [5] M. V. Thuan, Guaranteed cost control of neural networks with various activation functions and mixed time-varying delays in state and control, *Differential equations and control processes*, **3** (2011), 18-29.
- [6] V. N. Phat and H. Trinh, Exponential stabilization of neural networks with various activation functions and mixed time-varying delays, *IEEE Trans. Neural Networks*, **21** (2010), 1180-1184.
- [7] T. Botmart and P. Niamsup, Adaptive control and synchronization perturbed Chua's system, *Math. Comput. Simulation*, **75** (2007), 37-55.
- [8] T. Botmart and W. Weera, Guaranteed Cost Control for Exponential Synchronization of Cellular Neural Networks with Mixed Time-Varying Delays via Hybrid Feedback Control, *Abstr. Appl. Anal.*, **2013** (2013), 1-12.
- [9] T. Botmart, P. Niamsup and X. Liu, Synchronization of non-autonomous chaotic systems with time-varying delay via delayed feedback control, *Commun. Nonlinear Sci. Numer. Simulation*, **17** (2012), 189-1907.

- [10] T. Li, S.-M. Fei and K.-J. Zhang, Synchronization control of recurrent neural networks with distributed delays, *Physica A*, **387** (2008), 982-996.
- [11] Z.-G. Wu, P. Shi, H. Su and J. Chu, Exponential synchronization of neural networks with discrete and distributed delays under time-varying sampling, *IEEE Trans. Neural Netw*, **23** (2012), 1368-1376.
- [12] J. H. Park, On global stability criterion of neural networks with continuously distributed delays, *Chaos Solitons Fractals*, **37** (2008), 444-449.
- [13] Zochowski M, Intermittent dynamical control, *Phys D.*, **145** (2000), 181-90.
- [14] X. Yang, J. Cao, Stochastic synchronization of coupled neural networks with intermittent control, *Phys. Lett. A.*, **373** (2009), 3359-3272.
- [15] W. Zhang, J. Huang, P. Wei, Weak synchronization of chaotic neural networks with parameter mismatch via periodically intermittent control, *Appl. Math. Model.*, **35** (2011), 612-620.
- [16] J. Yu, C. Hu, H. Jiang, Z. Teng, Exponential synchronization of Cohen-Grossberg neural networks via periodically intermittent control, *Neurocomputing*, **74** (2011), 1776-1782.
- [17] H. Zhu, B. Cui, Stabilization and synchronization of chaotic systems via intermittent control, *Commun Nonlinear Sci Numer Simulat.*, **15** (2010), 3577-3586.
- [18] G. Zhang, X. Lin, X. Zhang, Exponential Stabilization of Neutral-Type Neural Networks with Mixed Interval Time-Varying Delays by Intermittent Control, *Circuits Systems Signal Process.*, **33** (2014), 371-391.
- [19] K. Gu, V.L. Kharitonov and J.Chen, *Stability of time-delay system*, Boston: Birkhauser; 2003.