รายงานวิจัยฉบับสมบูรณ์

โครงการ กระบวนการมาร์คอฟและการวิเคราะห์เคอร์-เนลความร้อน - สองทฤษฎีหนึ่งมุมมอง

โดย นายสันติ ทาเสนา

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สัญญาเลขที่ MRG5580135

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นายสันติ ทาเสนา มหาวิทยาลัยเชียงใหม่

สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา
สำนักงานกองทุนสนับสนุนการวิจัย และ
มหาวิทยาลัยเชียงใหม่

บทคัดย่อ

รหัสโครงการ MRG5580135

ชื่อโครงการ กระบวนการมาร์คอฟและการวิเคราะห์เคอร์เนลความร้อน - สองทฤษฎี หนึ่งมุมมอง

ชื่อนักวิจัย สันติ ทาเสนา มหาวิทยาลัยเชียงใหม่

อีเมลล์ santi.tasena@cmu.ac.th

ระยะเวลาโครงการ 2 ปี

บทคัดย่อ ในงานวิจัยชิ้นนี้ เราได้ทำการศึกษาเพอร์เทอร์เบชันของสมการความร้อนภายใต้การเปลี่ยนแปลงของฟังก์ชันระยะทาง โดยให้ความสนใจกับอสมการฮาร์แนคเป็นพิเศษ เราได้หาเงื่อนไขที่จำเป็นและเพียงพอที่จะทำให้ปริภูมิดิริชเลท์ควอซิคอนฟอมอลสอดคล้องกับอสมการฮาร์แนคเมื่อไหร่ก็ตามที่ปริภูมิดิริชเรท์เริ่มต้นมีสมบัตินี้

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Abstract

Project Code: MRG5580135

Project Title: Markov Process vs. Heat Kernel Analysis - Two theories, one approach

Investigator: Santi Tasena, Chiang Mai University

E-mail Address: santi.tasena@cmu.ac.th

Project Period: 2 years

Abstract: In this work, we study the perturbation of the heat equation under the change of metric / distance function. We particularly interested in the stability of the Harnack inequality. We are able to give a sufficient and necessary condition for a large class of functions in which the Harnack inequality of the quasi-conformal Dirichlet space follows from that of the original Dirichlet space.

Keywords: Doubling Property, Harnack Inequality, Poincare Inequality

1. ความสำคัญและที่มาของปัญหา

Introduction to heat kernels

Heat equation is an equation describing the dynamical system of the thermal conductivity. It represents the change or the transfer of heat across space and time. A solution of heat equation can be interpreted as the temperature of the material at future time. Heat equation can be solved under the condition that the temperature of the material at present time is known i.e. under the present of Initial condition. Since solutions of heat equation can be written as an integral of the product of the initial temperature of the material and the heat kernel, the behavior of the heat kernel directly affects the behavior of the solution of heat equation.

For composite materials, the heat equation is quite complicated due to the fact that transfer rates might be different at different points and different directions. Even in non-composite materials, the shape and the thickness of materials also affect the transfer rate. Therefore, it is inevitable to study many different heat operators so that one would able to understand more about the thermal conductivity.

Introduction to Markov processes

Markov process is a type of stochastic processes with the assumption that the future behavior of the process will only depend on the past through the present value of the process. If the present value is x, then the probability that at the future time t, the value will change to y is denoted by $p(t, x, y)$, the so called transition density of Markov process. If one were able to compute the transition function, then one would be able to do all computations necessary to predict the future behaviors of the process. The famous example of this is the Brownian motion which has been extensively studied and applied in physics, engineering, biology, economics, finance, insurance, etc.

However, it is not possible to compute transition functions for all Markov processes. When this happens, one is left with two choices, use numerical methods to predict local behaviors and use analytical methods to predict global behaviors and rough behaviors of the processes.

Markov processes vs. heat kernels

The transition function of Markov process is nothing but the fundamental solution of heat equation. It is proved that on a large class of Markov processes called Hunt processes, there is a one-one correspondence between the distributions of Hunt processes and Heat operators. Actually, there is a one-one correspondence between the following objects:

1. The distributions of Hunt processes,
2. Tile transition functions,
3. The Markov semigroups,
4. The resolvent semigroups,
5. The heat operators,
6. The Dirichlet forms.

This result leads to the possibility of study Markov processes using analytic approach from the study of heat kernel.

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2. วัตถุประสงค์งานวิจัย

The goal of this work is to study the perturbation of heat equations under the change of metric / distance function. We particularly interested in the stability of the Harnack inequality. We will use the analytical approach, especially the knowledge and techniques developed by Saloff-Coste and Grigor'yan. Using Sturm's results, this is equivalent to proving the doubling property and the Poincare inequality on the quasi-conformal Dirichlet spaces.

3. ระเบียบวิธีวิจัย

1. Investigate the necessary background knowledge as much as possible.
2. State the conditions and try to prove the doubling property and Poincare inequality under conformal change of metric / distance function.
3. Send the work to mentors for advice and revising.
4. Write articles and submit to international journals.

4. ผลงานวิจัย

Denote $(E, D(E))$ a strongly local regular Dirichlet form on $L^2(X, \nu)$ where X is a locally compact but non-compact, second countable, Hausdorff space and ν is a Borel measure on X. Any such $(E, D(E))$ is associated with an energy measure, viz., a measure-valued function Γ with domain $D(E) \times D(E)$ such that

$$E(u, v) = \int d\Gamma(u, v)$$

for all $u, v \in D(E)$. The intrinsic distance ρ associated to $(E, D(E))$ is defined by setting

$$\rho(x, y) = \sup\{|u(x) - u(y)| : d\Gamma(u, u) \leq d\nu\}$$

for all $x, y \in X$. A basic assumption which we will be assuming throughout this work is that the intrinsic distance ρ is complete and it actually generates the topology of X.

A closed subset Z of X is said to be (λ, Λ)-accessible covering radius $R: Z \to [0, \infty]$, where $0 < \lambda < \Lambda \leq 1$, if it satisfies the following two conditions.

a. For any $z \in Z$ and $0 < r < R(z)$, the set $S(z,r) = S_\Lambda(z,r) = \{x \in X : \Lambda r \leq \rho(x,Z) \leq \rho(x,z) \leq r\}$ is nonempty.

b. There is a constant $c_\lambda > 0$ such that any point in $C(z,r) = C_\Lambda(z,r) = \bigcup_{s \leq r} S_\Lambda(z,s)$ can be connected to z via a path lying entirely in $C_\lambda(z, c_\lambda r)$ whenever $z \in Z$ and $0 < r < R(z)$.

With these terminologies, we are able to prove the following result (see Appendix 6.1).

Theorem 4.1. *Let* $(E, D(E))$ *a strongly local regular Dirichlet form on* $L^2(X, \nu)$ *and* Z *be an accessible subset of* X *covering radius* $R: Z \to [0, \infty]$. *Assume that* ν *is doubling and that the Poincare inequality holds for* Z*-remote balls and there are constants* $C_P > 0$ *and* $\delta \in (0,1)$ *such that*

$$\frac{1}{\nu(B(x,\delta r))\nu(B(y,\delta r))} \int_{B(x,\delta r)} \int_{B(y,\delta r)} |u(x) - u(y)|^2 d\nu(x) d\nu(y) \leq C_P r^2 \int_{B(z,r)} d\Gamma(u,u)$$

for all $z \in Z$, $0 < r < R(z)$, $x, y \in S(z,r)$, *and* $u \in D(E)$.

Then there are constants $\eta_1 > \eta_2 > 0$ *such that the (weak) Poincare inequality holds for the family of balls*

$$\{B(x,r) \colon z \in Z, d(x,Z) = d(x,z) < \eta_1 R(z) and\ r < \eta_2 R(z)\}.$$

Particulary, the (weak) Poincare inequality holds for Z*-anchored balls with radius smaller than* $\eta_2 R$.

If $R = \infty$, *then the (weak) Poincare inequality holds for all balls. Moreover,* $(E, D(E))$ *also satisfies the (parabolic) Harnack inequality and the Gaussian estimates in this case.*

Next, we consider operator $(E_h, D(E_h))$ based on $(E, D(E))$ defined by

$$E_h(u,v) = \int h^{-2} d\Gamma(u,v)$$

for any $u, v \in D(E)$ such that the above integration is defined. We give a mild sufficient condition for which the above $(E_h, D(E_h))$ can be extended to a Dirichlet form called a quasi-conformal Dirichlet form since it changes the intrinsic distance on X. The intrinsic distance ρ_h of $(E_h, D(E_h))$ can be defined in term of ρ by

$$\rho_h(x,y) = \inf \int h(\gamma(t)) |\gamma|'(t) dt$$

where the infimum is taken over all path γ connecting x and y. Here, $|\gamma|'(t) = \limsup_{\epsilon \to 0} \frac{\rho(\gamma(t), \gamma(t+\epsilon))}{\epsilon}$.

We are interested to know when the Harnack inequality remains stable under the quasi-conformal change of metric.

Based on Theorem 4.1, we are able to provide a sufficient and necessary condition for a large class of quasi-conformal density h (see Appendix 6.2).

A quasi-conformal Dirichlet form $(E_h, D(E_h))$ satisfies (Rh) condition if there is a constant $C_h > 0$ such that $\rho_h(x,z) \le C_h h(x)\rho(x,z)$ for all $x \in C(z)$ and $z \in Z$.

A quasi-conformal Dirichlet form $(E_h, D(E_h))$ satisfies (WP) condition if there is a constant $C_P > 0$ such that

$$\frac{1}{\nu\big(B(x,\delta r)\big)\nu\big(B(y,\delta r)\big)} \int_{B(x,\delta r)} \int_{B(y,\delta r)} |u(x)-u(y)|^2 d\nu(x) d\nu(y) \le C_P r^2\, h(x)h(y) \int_{B(z,r)} h^{-2} d\Gamma(u,u)$$

for all $z \in Z$, $r > 0$, $x, y \in S(z,r)$, *and* $u \in D(E)$.

Theorem 4.2. *Let* $(E, D(E))$ *a strongly local regular Dirichlet form on* $L^2(X,\nu)$ *and* Z *be a discrete accessible subset of* X *covering infinite radius.*
If h^{-2} *is locally integrable with singularity set* Z, *then* $(E_h, D(E_h))$ *is a quasi-conformal Dirichlet form satisfying the basic assumption.*

Assume further that h *have polynomial growth in the* Z*-normal direction and* $(E, D(E))$ *satisfies the Harnack inequality. Then,* $(E_h, D(E_h))$ *satisfies the Harnack inequality if and only if both (Rh) and (WP) hold.*

5. สรุปผลและอภิปรายผล

Even though we are able to prove the stability of Harnack inequality on quasi-conforrmal Dirichlt spaces for a large class of quasi-conformal density, there are still many improvements we can considered. One example to combine this result with many know results for weighted Dirichlet space and prove the stability of Harnack inequality on conformal Dirichlet form. Another example is to weaken the accessibility assumption. We believe that the accessibility assumption can be weaken to at least the case of a finite union of accessible sets. It would be interesting to see whether this can be weaken further.

All these require further investigation, however.

6. ภาคผนวก

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Poincaré Inequality: From remote balls to all balls[☆]

Santi Tasena[a,*], Laurent Saloff-Coste[b], Sompong Dhompongsa[a]

[a] *Department of Mathematics, Faculty of Sciences, Chiang Mai University, Chiang Mai, 50200 Thailand*
[b] *Department of Mathematics, Malott Hall, Cornell University, Ithaca, New York, 14853-4201 USA*

Abstract

It is shown that the Poincaré inequality holds uniformly for all balls whenever it holds for **Z**-remote balls providing the set **Z** satisfies some additional conditions including the condition that **Z** does not separate the space. The aims of this paper is to prove similar results without using this assumption.

Keywords:
Anchored Ball, Doubling Measure, Poincaré Inequality, Remote Ball

1. Introduction

In 2005 Grigor'yan and Saloff-Coste [3] proved that if the Poincaré inequality holds uniformly for **Z**-remote balls and for **Z**-anchored balls in a geodesic space $(\mathbf{X}, d)$, then it holds uniformly for all balls in $(\mathbf{X}, d)$. Furthermore, the assumption on **Z**-anchored balls may be dropped when the subset **Z** is a singleton and the space **X** satisfies the RCA condition with respect to the subset **Z**. Three years later, Gyrya and Saloff-Coste [4] (see also [5]) extended the result to the case that **Z** is the boundary of an inner uniform domain. However, all of these results rely on the assumption that any pair of points outside **Z** can be connected by a path which is relatively far away from **Z**, that is, a path γ for which $d(\gamma(t), \mathbf{Z}) \geq \delta \min(d(\gamma(0), \mathbf{Z}), d(\gamma(1), \mathbf{Z}))$

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*Corresponding author
Email addresses: `santi.tasena@cmu.ac.th` (Santi Tasena), `lsc@math.cornell.edu` (Laurent Saloff-Coste), `sompongd@chiangmai.ac.th` (Sompong Dhompongsa)

where $\delta > 0$ is a fixed constant. This assumption does not hold for the real line. It could also fail for graphs and one dimensional simplexes. It is therefore natural to ask whether this assumption can be dropped or be replaced by other assumptions that would allow the set $\mathbf{Z}$ to separate the space. We show in this work that this is indeed possible.

2. Preliminaries

In this section we shall review concepts used throughout this work. Since the subject has been progressed in many directions, it would not be possible to give a complete review here and we apologize for any missing work that is not mentioned. For more information on the subject, see, e.g., [9, 1, 10, 11, 2, 6].

Henceforth, a topological (metric) space refers to a *locally compact but non-compact, second countable, and Hausdorff (metric) space*, a measure means a *Borel measure with full support*, and a Dirichlet form means a *regular, strongly local* Dirichlet form *admitting the carré du champ operator*. Details of these terminologies are given below.

Let $(\mathbf{X}, d)$ be a metric space. Denote $\mathbf{B}(x, r) \overset{\text{df}}{=} \{y \in \mathbf{X} \ : \ d(x, y) < r\}$. A non-zero measure ν on a metric space is said to be *doubling* if there is a constant $C_D \geq 1$ such that

$$\nu(\mathbf{B}(x, 2r)) \leq C_D \nu(\mathbf{B}(x, r))$$

for all $x \in \mathbf{X}$ and $r > 0$. The constant C_D is called a doubling constant associated to ν. Examples of doubling measures include the Lebesgue measures on Euclidean spaces and, more generally, volume measures on Riemannian manifolds with nonnegative Ricci curvature.

Definition 2.1. A *Dirichlet form* is a pair $(E, \mathrm{D}(E))$ where E is a closed, symmetric, densely defined bilinear form on the space of square integrable functions $\mathrm{L}^2(\mathbf{X}, \nu)$ with domain $\mathrm{D}(E)$ such that for all $u \in \mathrm{D}(E)$, the function $v \overset{\text{df}}{=} (u \vee 0) \wedge 1$ belongs to $\mathrm{D}(E)$ and $E(v, v) \leq E(u, u)$.

The domain $\mathrm{D}(E)$ equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathrm{L}^2(\mathbf{X}, \nu)} + E(\cdot, \cdot)$ is a Hilbert space, called a *Dirichlet space* and its norm is called the *Dirichlet norm* on $\mathrm{D}(E)$. A Dirichlet form $(E, \mathrm{D}(E))$ is *regular* if the intersection of its domain $\mathrm{D}(E)$ with $\mathrm{C_c}(\mathbf{X})$ is dense in $\mathrm{C_c}(\mathbf{X})$ under the uniform metric. Here, $\mathrm{C_c}(\mathbf{X})$ is the space of continuous functions with compact supports in $\mathbf{X}$.

The *energy measure* Γ associated to $(E, \mathrm{D}(E))$ is a measure-valued function with domain $\mathrm{D}(E) \times \mathrm{D}(E)$ such that $\int \mathrm{d}\Gamma(u,v) = E(u,v)$ for all $u, v \in \mathrm{D}(E)$. The *intrinsic distance* d on $\mathbf{X}$ is defined using the energy measure by setting

$$d(x,y) \stackrel{\mathrm{df}}{=} \sup\{|u(x) - u(y)| \ : \ u \in \mathrm{D}(E) \cap \mathrm{C_c}(\mathbf{X}) \text{ and } \mathrm{d}\Gamma(u,u) \leq \mathrm{d}\nu \ a.e.\}$$

for all $x, y \in \mathbf{X}$. Here $\mathrm{d}\Gamma(u,u) \leq \mathrm{d}\nu$ a.e. means $\frac{\mathrm{d}\Gamma(u,u)}{\mathrm{d}\nu} \leq 1 \, a.e.$ If d is a complete metric generating the topology of $\mathbf{X}$, then $(\mathbf{X}, d)$ is a geodesic space [13, 7], i.e., any two points in $\mathbf{X}$ can be joined by a path with length exactly equal to the distance between those two points. In fact, we will assume this is true throughout this work.

Assumption 2.2. The intrinsic distance associated to a Dirichlet form on $\mathrm{L}^2(\mathbf{X}, \nu)$ is a complete metric and it generates the topology of $\mathbf{X}$.

A regular Dirichlet form $(E, \mathrm{D}(E))$ is *strongly local* if $E(u,v) = 0$ whenever u is constant on a neighborhood of the support of v. It is shown that strong locality, *Leibniz rule*, and *chain rule* are equivalent [12]. Examples of strongly local Dirichlet forms are those constructed from differential operators.

A Dirichlet form $(E, \mathrm{D}(E))$ on $\mathrm{L}^2(\mathbf{X}, \nu)$ with associated carré du champ operator Γ is said to satisfy the *(weak uniform) Poincaré inequality* on a family $\mathcal{F}$ of balls if there are constants $k \geq 1$ and $C_P > 0$ such that

$$\inf_{\xi \in \mathbb{R}} \int_{\mathbf{B}(x,r)} (u - \xi)^2 \mathrm{d}\nu \leq C_P r^2 \int_{\mathbf{B}(x,kr)} \Gamma(u,u) \mathrm{d}\nu$$

for all $\mathbf{B}(x,r) \in \mathcal{F}$ and $u \in \mathrm{D}(E)$. The constant C_P is called a Poincaré constant and k is called a scaling constant. If $k = 1$, then $(E, \mathrm{D}(E))$ is said to satisfy the *(uniform) Poincaré inequality* on a family $\mathcal{F}$. If $\mathcal{F}$ is not mentioned, then $\mathcal{F}$ is assumed to be the family of all balls.

Under the doubling property and Assumption 2.2, the weak Poincaré inequality and the Poincaré inequality are equivalent [9].

Two more families of balls that will be discussed in this work are the families of remote balls and of anchored balls. Fix a closed subset $\mathbf{Z}$ of $\mathbf{X}$. A family of $\mathbf{Z}$-*remote* balls refers to any family $\mathcal{F}_\varepsilon \stackrel{\mathrm{df}}{=} \{\mathbf{B}(x,r) \ : \ r \leq \varepsilon d(x, \mathbf{Z})\}$ where $\varepsilon \in (0,1)$ and the family of $\mathbf{Z}$-*anchored* balls with covering radius $R : \mathbf{Z} \to (0, \infty]$ refers to the family of balls with center $z \in \mathbf{Z}$ and radius $r < R(z)$.

In 2005 Grigorýan and Saloff-Coste [3] proved the following statement.

Theorem 2.3 ([3, Proposition 4.2]). *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strongly local, regular Dirichlet form on $L^2(\mathbf{X}, \nu)$ and $\mathbf{Z} \subsetneq \mathbf{X}$ be closed. Assume that the (weak) Poincaré inequality holds for $\mathbf{Z}$-remote balls $\mathcal{F}_\varepsilon$ and $\mathbf{Z}$-anchored balls with covering radius $R : \mathbf{Z} \to (0, \infty]$, that is, there are constants $\kappa_0, C_P \geq 1$ such that*

$$\inf_{\xi \in \mathbb{R}} \int_{\mathbf{B}(x,r)} |u - \xi|^2 d\mu \leq C_P r^2 \int_{\mathbf{B}(x,\kappa_0 r)} d\Gamma(u,u)$$

for all $u \in \mathcal{D}(\mathcal{E})$ and all ball $\mathbf{B}(x,r)$ with either $x \in \mathbf{Z}$ and $r < R(x)$ or $r < \varepsilon d(x, \mathbf{Z})$. Then the (weak) Poincaré inequality holds for the family of balls

$$\{\mathbf{B}(x,r) \ : \ \exists z \in \mathbf{Z}, d(x,\mathbf{Z}) = d(x,z) < \frac{1}{4} R(z) \text{ and } r < \frac{\varepsilon}{4\kappa_0} R(z)\}.$$

Actually, there are two differences between the above theorem and Proposition 4.2 in [3]. First, Proposition 4.2 in [3] stated the result only in Riemannian manifold setting but its proof, however, does not use any specific manifold assumption. So the proof can be trivially ported to the Dirichlet form setting. Second, the proof of Proposition 4.2 in [3] is only provided when $R = \infty$. However, the same proof can be applied to general $R : \mathbf{Z} \to (0, \infty]$. Therefore, we will not repeat the proof here.

3. Main Result

The goal of this work is to provide additional conditions for the following assertion to be true. If the Poincaré inequality holds for remote balls, then it also holds for all balls. The standard argument is to use chains of balls to estimate the Poincaré inequality. The difficulty, however, is whether we can do that without blowing up the estimation. This is guaranteed provided that the following two conditions are met. First the balls we used in the estimation must not intersect each other too many times. Second we must be able to estimate the Poincaré inequality for any anchored ball using that of remote ball. To fulfill the first condition, we must choose chains of balls according to paths with specific properties. This can be done by considering the condition (AC2) given below. To conform with the second condition, any anchored ball should contain at least one remote ball. This is an implication of the condition (AC1) given in Definition 3.1. Note that these conditions are generalizations of the fully accessible condition provided in [3].

Definition 3.1. Fix $0 < \lambda < \Lambda \leq 1$. A closed subset $\mathbf{Z}$ of a geodesic space $(\mathbf{X}, d)$ is said to be (λ, Λ)-*accessible covering distance* $R : \mathbf{Z} \to (0, \infty]$ if

(AC1) the Λ-strip $\mathbf{S}(z, r) = \mathbf{S}_\Lambda(z, r, \mathbf{Z}) \stackrel{\text{df}}{=} \{x \in \mathbf{X} : \Lambda r \leq d(x, \mathbf{Z}) \leq d(x, z) \leq r\} \neq \emptyset$ for all $z \in \mathbf{Z}$ and $r < R(z)$, and

(AC2) there is a constant $c_\lambda \geq 1$ such that any point in the λ-cone $\mathbf{C}(z, r) = \mathbf{C}_\lambda(z, r, \mathbf{Z})$ defined by $\mathbf{C}(z, r) \stackrel{\text{df}}{=} \cup_{0 \leq q \leq r} \mathbf{S}_\lambda(z, q, \mathbf{Z})$ can be connected to z via a path in $\mathbf{C}(z, c_\lambda r)$.

If $\mathbf{Z} = \{z\}$ is a singleton and for some $r_0 > 0$ the set $\mathbf{S}(z, r)$ is connected for all $r > r_0$, then $(\mathbf{X}, d)$ is said to have *relatively connected annuli (RCA) with respect to* z.

For example, any singleton is (λ, Λ)-accessible covering distance $R = \infty$ for all $0 < \lambda < \Lambda \leq 1$. In Euclidean space, any proper linear subspace is (λ, Λ)-accessible covering distance $R = \infty$ for all $0 < \lambda < \Lambda \leq 1$. Any discrete subset $\mathbf{Z}$ is (λ, Λ)-accessible covering distance $R_\epsilon(z) \stackrel{\text{df}}{=} \epsilon d(z, \mathbf{Z} - \{z\})$ for all $0 < \lambda < \Lambda \leq 1$ and $\epsilon \in (0, 1)$. It is possible, however, for a discrete subset $\mathbf{Z}$ to be (λ, Λ)-accessible covering distance $R(z) > d(z, \mathbf{Z} - \{z\})$. A discrete set $\mathbf{Z} \stackrel{\text{df}}{=} \mathbb{Z}^k \times \{0\}^n \subseteq \mathbb{R}^k \times \mathbb{R}^n$, for example, is (λ, Λ)-accessible covering distance $R = \infty$ even though $d(z, \mathbf{Z} - \{z\}) = 1$ for all $z \in \mathbf{Z}$. Any boundary of an (inner) uniform domain is also (λ, Λ)-accessible inside that domain.

Next is the main theorem of this work.

Theorem 3.2 (Main Theorem)**.** *Let $(E, \mathrm{D}(E))$ be a strongly local, regular, Dirichlet form on $\mathrm{L}^2(\mathbf{X}, \nu)$ where the measure ν is doubling. Denote Γ its associated energy measure. Let also $\mathbf{Z} \subsetneq \mathbf{X}$ be (λ, Λ)-accessible covering distance $R : \mathbf{Z} \to (0, \infty]$. Assume that the Poincaré inequality holds for $\mathbf{Z}$-remote balls and there are constants $C_P > 0$ and $\delta \in (0, 1)$ such that*

$$\fint_{\mathbf{B}(x,\delta r)} \fint_{\mathbf{B}(y,\delta r)} |f(u) - f(v)|^2 \mathrm{d}\nu(u) \mathrm{d}\nu(v) \leq C_P r^2 \fint_{\mathbf{B}(z,r)} \mathrm{d}\Gamma(f, f) \tag{3.1}$$

for all $z \in \mathbf{Z}$, $0 < r < R(z)$, $x, y \in \mathbf{S}_\lambda(z, r)$, and $f \in \mathrm{D}(E)$. Then there is a constant $\eta > 0$ such that the (weak) Poincaré inequality holds for $\mathbf{Z}$-anchored balls with covering radius ηR.

The inequality (3.1) is necessary for linking information related to the Poincaré inequality between different components of the space. It should be intuitively clear why this condition would lead to the uniform Poincaré inequality for all balls. Nevertheless, showing that this is true is not an easy task. It should also be noted here that the Poincaré inequality for all balls also implies the inequality (3.1). Therefore, this condition is actually neccessary and we do not believe that it can be replaced by any weaker condition. An interesting question we chould ask instead is whether the accessibility condition can be replaced by a weaker condition so that the above result remains true.

Combining the main thoerem with Theorem 2.3, we immediately have the following corollary.

Corollary 3.3. *Let $(E, D(E))$ be a strongly local, regular, Dirichlet form in $L^2(\mathbf{X}, \nu)$ where the measure ν is doubling. Denote Γ its associated energy measure. Let also $\mathbf{Z} \subset \mathbf{X}$ be (λ, Λ)-accessible covering distance $R : \mathbf{Z} \to (0, \infty]$. Assume that the Poincaré inequality holds for $\mathbf{Z}$-remote balls and there are constants $C_P > 0$ and $\delta \in (0, 1)$ such that the inequality (3.1) holds for all $z \in \mathbf{Z}$, $0 < r < R(z)$, $x, y \in \mathbf{S}_\lambda(z, r)$, and $f \in D(E)$.*

Then there are constants $\eta_1 > \eta_2 > 0$ such that the (weak) Poincaré inequality holds for the family of balls $\{\mathbf{B}(x, r) : \exists z \in \mathbf{Z}, d(x, \mathbf{Z}) = d(x, z) < \eta_1 R(z)$ and $r < \eta_2 R(z)\}$.

When applying to a discrete set $\mathbf{Z}$, we have the following corollary.

Corollary 3.4. *Let $(E, D(E))$ be a strongly local, regular, Dirichlet form in $L^2(\mathbf{X}, \nu)$ where the measure ν is doubling. Denote Γ its associated energy measure. Let also $\mathbf{Z} \subset \mathbf{X}$ be such that $d_{\mathbf{Z}} = \inf_{z \in \mathbf{Z}} d(z, \mathbf{Z} - \{z\}) > 0$. Assume that the Poincaré inequality holds for $\mathbf{Z}$-remote balls and there are constants $C_P > 0$ and $\delta \in (0, 1)$ such that the inequality (3.1) holds for any $z \in \mathbf{Z}$, $0 < r < d_{\mathbf{Z}}$, $x, y \in \mathbf{S}_\lambda(z, r)$, and $f \in D(E)$.*

Then there is a constant $\eta > 0$ such that the (weak) Poincaré inequality holds for the family of balls with radius at most $\eta d_{\mathbf{Z}}$.

Next we give an example showing that these results are helpful for proving that the Poicaré inequality holds uniformly for all balls. Here, we present a case for which it is easy to prove the Poincaré inequality for remote balls and the inequality (3.1) but not so easy to prove the Poincaré inequality for all balls. The idea is to consider a weighted Dirichlet space in which the weight function h is unbounded and is roughly constant on balls remoted to a closed set $\mathbf{Z}$, that is, there is a constant $C > 0$ such that $\sup_B h \leq C \inf_B h$ for any

Z-remote ball B. Since the weight function is roughly constant on **Z**-remote balls, the Poincaré inequality for **Z**-remote balls and the inequality (3.1) can be proved easily. The Poincaré inequality for all balls, on the other hand, is hard to prove because the function h is unbounded.

A Simple Example

In the following example, let $\mathbf{X} = \mathbb{R}^k \times \mathbb{R}^n$ and $\mathbf{Z} = \mathbb{Z}^k \times \{0\}^n$. It is easy to see that $\mathbf{Z}$ is $(\lambda, 1)$-accessible covering infinite distance for any $\lambda \in (0, 1)$. Let the Borel measure $\nu_{\alpha,\beta}$ be defined by

$$\mathrm{d}\nu_{\alpha,\beta}(x) \overset{\mathrm{df}}{=} (\beta + d(x, \mathbf{Z})^2)^{\frac{\alpha}{2}} \mathrm{d}x.$$

By computation, we can show that

(a) there is a constant $C_1 = C_1(\alpha, \beta) > 0$ such that

$$\frac{1}{C_1}(\beta + d(x, \mathbf{Z})^2)^{\frac{\alpha}{2}} r^{n+k} \leq \nu_{\alpha,\beta}(\mathbf{B}(x, r)) \leq C_1(\beta + d(x, \mathbf{Z})^2)^{\frac{\alpha}{2}} r^{n+k}$$

for all $r \leq d(x, \mathbf{Z})/2$ and $x \in \mathbf{X}$,

(b) there is a constant $C_2 = C_2(\alpha, \beta) > 0$ such that

$$\frac{1}{C_2}(\beta + r^2)^{\frac{\alpha}{2}} r^{n+k} \leq \nu_{\alpha,\beta}(\mathbf{B}(z, r)) \leq C_2(\beta + r^2)^{\frac{\alpha}{2}} r^{n+k}$$

for all $z \in \mathbf{Z}$ and either $r > 0$ (in case $\alpha \geq 0$) or $0 < r \leq 1$ (in case $\alpha < 0$),

(c) if $\alpha < 0$, there is a constant $C_3 = C_3(\alpha, \beta) > 0$ such that

$$\frac{1}{C_3}(r^k + (\beta + r^2)^{\frac{\alpha}{2}} r^{n+k}) \leq \nu_{\alpha,\beta}(\mathbf{B}(z, r)) \leq C_3(r^k + (\beta + r^2)^{\frac{\alpha}{2}} r^{n+k})$$

for all $r \geq 1$ and $z \in \mathbf{Z}$.

This implies $\nu_{\alpha,\beta}$ satisfies the doubling condition for both **Z**-remote and **Z**-anchored balls as well as the volume comparison condition. By Proposition 4.7 in [3], $\nu_{\alpha,\beta}$ is doubling for all $\alpha > -n$. Moreover, there is a constant $C_4 > 0$ such that

$$\frac{1}{C_4}(\beta + r + d(x, \mathbf{Z}))^{\alpha} r^{n+k} \leq \nu_{\alpha,\beta}(\mathbf{B}(x, r)) \leq C_4(\beta + r + d(x, \mathbf{Z}))^{\alpha} r^{n+k}$$

for all $x \in \mathbf{X}$ and $r > 0$.

Next, consider a heat operator

$$L_{\alpha,\beta} \stackrel{\text{df}}{=} (\beta + d(\cdot, \mathbf{Z})^2)^{-\frac{\alpha}{2}} \sum_{i=1}^{n+k} \frac{\partial}{\partial x_i} (\beta + d(\cdot, \mathbf{Z})^2)^{\frac{\alpha}{2}} \frac{\partial}{\partial x_i}$$

defined on the space $L^2(\mathbf{X}, \nu_{\alpha,\beta})$. It's associated Dirichlet form is defined by the formula

$$E_{\alpha,\beta}(f,g) \stackrel{\text{df}}{=} \sum_{i=1}^{n+k} \int (\beta + d(x, \mathbf{Z})^2)^{\frac{\alpha}{2}} \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_i}(x) \mathrm{d}x$$

where its domain is the completion of the set of all smooth functions with compact supports under the Dirichlet norm and its intrinsic distance is the Euclidean distance.

The (uniform) Poincaré inequality for the Laplacian immediately implies that $(E_{\alpha,\beta}, D(E_{\alpha,\beta}))$ satisfies all assumptions in Corollary 3.3. Thus, $(E_{\alpha,\beta}, D(E_{\alpha,\beta}))$ satisfies the Poincaré inequality. Moreover, its heat kernel $p : (0, \infty) \times \mathbf{X} \times \mathbf{X} \to [0, \infty)$ has Gaussian behavior [13], specifically, there are constants $c_1, c_2, c_3, c_4 > 0$ such that for all $t > 0$ and $x, y \in \mathbf{X}$,

$$\begin{aligned} p(t,x,y) &\leq \frac{c_1 e^{-c_2|x-y|^2/t}}{t^{\frac{n+k}{2}} (\beta + \sqrt{t} + d(x, \mathbf{Z}))^\alpha (\beta + \sqrt{t} + d(y, \mathbf{Z}))^\alpha} \\ p(t,x,y) &\geq \frac{c_3 e^{-c_4|x-y|^2/t}}{t^{\frac{n+k}{2}} (\beta + \sqrt{t} + d(x, \mathbf{Z}))^\alpha (\beta + \sqrt{t} + d(y, \mathbf{Z}))^\alpha}. \end{aligned}$$

Note that an analogous analysis can also be done for $\mathbf{Z} = \mathbb{R}^k \times \{0\}^n$. Also, both cases are generalizations of examples from [3, 8].

4. Proof of the main result

The proof of the main theorem is based on the following facts.

Theorem 4.1 ([3, Lemma 2.9])**.** *Fixed a doubling measure ν on a geodesic space $(\mathbf{X}, d)$ with doubling constant C_D. Then for any $x, y \in \mathbf{X}$ and $0 < s \leq r$,*

$$\frac{\nu(\mathbf{B}(x,r))}{\nu(\mathbf{B}(y,s))} \leq C_D \Big(\frac{r + d(x,y)}{s} \Big)^{\log_2 C_D}.$$

Theorem 4.2 ([3, Lemma 2.10]). *Let ν be a doubling measure on a geodesic space $(\mathbf{X}, d)$ with doubling constant C_D. Then for any $0 < s \le r$ and $x \in \mathbf{X}$ with $\mathbf{B}(x, r) \neq \mathbf{X}$,*

$$\frac{\nu(\mathbf{B}(x,r))}{\nu(\mathbf{B}(x,s))} \ge \left(1 + C_D^{-1}\right)^{-1} \left(\frac{r}{s}\right)^{\log_3(1+C_D^{-1})}.$$

Theorem 4.3 ([9, Lemma 5.3.12]). *Assume that ν is a doubling measure in a geodesic space. Then for any $s > 0$, there exists a constant $c_s > 0$ such that for any sequence of balls $\{B_i\}_{i=1}^{\infty}$ and any sequence of nonnegative numbers $\{a_i\}_{i=1}^{\infty}$,*

$$\int \left(\sum_{i=1}^{\infty} a_i \mathbf{1}_{sB_i}\right)^2 \le c_s \int \left(\sum_{i=1}^{\infty} a_i \mathbf{1}_{B_i}\right)^2.$$

Here $s\mathbf{B}(x,r) \stackrel{df}{=} \mathbf{B}(x, sr)$ for any ball $\mathbf{B}(x,r)$ and any $s \ge 0$.

Recall that a *(strict) ϵ-Whitney covering $\mathcal{W}$* of an open set G is a countable family of disjoint balls $\mathbf{B}(x,r)$ such that $r = \epsilon d(x, \mathbf{X}-G)$ and $\cup_{B\in\mathcal{W}} 3B = G$. For any $z \in \mathbf{X}-G$, denote

$$\mathcal{W}_z \stackrel{\text{df}}{=} \{B \in \mathcal{W} : 3B \cap \mathbf{C}_\lambda(z, R(z)) \neq \emptyset\}.$$

Theorem 4.4 ([5, Lemma 3.18]). *Let $(\mathbf{X}, d)$ be a doubling geodesic space, i.e., $(\mathbf{X}, d)$ is a geodesic space admitting doubling measures. For any $\epsilon < 1/4$ and any open subset G of $\mathbf{X}$, there is a strict ϵ-Whitney covering $\mathcal{W}$ of G for which*

$$c_b = c_b(\epsilon) \stackrel{df}{=} \sup_{s \le \frac{1}{10\epsilon}} \sup_{x\in\mathbf{X}} \sum_{B\in\mathcal{W}} \mathbf{1}_{sB}(x) < \infty.$$

Next, we gather some lemmas that will be useful in the proof of the main theorem.

Lemma 4.5. *Assume ν is a doubling measure with doubling constant C_D on a geodesic space $(\mathbf{X}, d)$. Given any $\varepsilon \in (0,1]$, define $n_b = n_b(C_D, \epsilon) \stackrel{df}{=} \left\lfloor C_D \left(\frac{5}{\varepsilon}\right)^{\log_2 C_D} \right\rfloor$, then any family of disjoint balls $\mathcal{B}$ with radius between εr and r, and intersecting a fixed ball of radius r has at most n_b balls.*

Proof. Assume that $B \cap \mathbf{B}(z,r) \neq \emptyset$ for all $B \in \mathcal{B}$. Then $B \subseteq \mathbf{B}(z,3r)$. By Theorem 4.1,

$$\begin{aligned} \frac{\nu(\mathbf{B}(z,3r))}{\nu(\mathbf{B}(x,s))} &\leq C_D \left(\frac{3r + d(x,z)}{s} \right)^{\log_2 C_D} \\ &\leq C_D \left(\frac{5}{\varepsilon} \right)^{\log_2 C_D} \end{aligned}$$

for any $\mathbf{B}(x,s) \in \mathcal{B}$. Let n be the cardinality of $\mathcal{B}$. Using the fact that all balls in $\mathcal{B}$ are pairwise disjoint, we have

$$\begin{aligned} n\nu(\mathbf{B}(z,3r)) &\leq C_D \left(\frac{5}{\varepsilon} \right)^{\log_2 C_D} \sum_{B \in \mathcal{B}} \nu(B) \\ &\leq C_D \left(\frac{5}{\varepsilon} \right)^{\log_2 C_D} \nu(\mathbf{B}(z,3r)). \end{aligned}$$

Thus, $n \leq C_D \left(\frac{5}{\varepsilon}\right)^{\log_2 C_D}$. □

An immediate application of the above lemma is the following result.

Lemma 4.6. *Let $\mathbf{Z}$ be (λ, Λ)-accessible covering distance $R : \mathbf{Z} \to (0,\infty]$ in a geodesic space $(\mathbf{X}, d)$ admitting a doubling measure ν with doubling constant C_D. Given any $\varepsilon \in (0,1]$, there is a constant $n_s > 1$ depends only on $C_D, \epsilon, \varepsilon, \lambda$ such that the number of $\mathbf{B}(x,r) \in \mathcal{W}_z$ where $z \in \mathbf{Z}$ for which $\varepsilon r_0 \leq r \leq r_0$ is at most n_s.*

Proof. Let $\mathcal{B}$ denote the collection of all such $\mathbf{B}(x,r)$. For any $B = \mathbf{B}(x,r) \in \mathcal{B}$, $\mathbf{C}_\lambda(z, R(z)) \cap 3B$ is nonempty. Pick $y \in \mathbf{C}_\lambda(z, R(z)) \cap 3B$. It follows that

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \\ &\leq 3r + \lambda^{-1}(d(y,x) + d(x,\mathbf{Z})) \\ &\leq (3 + 3\lambda^{-1} + \lambda^{-1}\epsilon^{-1}) r_0. \end{aligned}$$

This means $\mathbf{B}(x,r) \subseteq \mathbf{B}(z, (4 + 3\lambda^{-1} + \lambda^{-1}\epsilon^{-1}) r_0)$. By Lemma 4.5, there is a constant n_s depends only on $C_D, \epsilon, \varepsilon$, and λ such that the cardinality of such $\mathcal{B}$ is always at most n_s. □

Lemma 4.7. *Let* $(\mathbf{X}, d)$ *be a doubling geodesic space and* $\mathbf{Z} \subseteq \mathbf{X}$ *be* (λ, Λ)*-accessible covering distance* $R : \mathbf{Z} \to (0, \infty]$. *Then there are constants* $n_c, c_1, c_2 \geq 1 \geq c_3 > 0$, *depend only on* $\epsilon, \lambda, c_\lambda$, *and a doubling constant* C_D, *such that for any* $x \in \mathbf{C}_\lambda(z, R(z)) - \{z\}, z \in \mathbf{Z}$, *there is a sequence of balls* $\{B_i = \mathbf{B}(x_i, r_i)\}_{i=1}^{\infty} \subseteq \mathcal{W}$ *with the following properties*

(CB1) $x \in 3B_1$,

(CB2) $3B_i \cap 3B_j \neq \emptyset$ *if and only if* $|i - j| = 1$,

(CB3) there exists a partition $\mathcal{P}_k$ *of* $\{B_i\}_{i=1}^{\infty}$, *each with at most* n_c *elements such that*

$$\frac{1}{c_1}\epsilon^k d(x, z) \leq r_i \leq c_1 \epsilon^k d(x, z)$$

for all $B_i = \mathbf{B}(x_i, r_i) \in \mathcal{P}_k$,

(CB4) $3B_i \cap \mathbf{C}_\lambda(z, c_\lambda d(x, z)) \neq \emptyset$ *for all* i,

(CB5) $r_i \geq c_3 d(x_i, z)$ *for all* i,

(CB6) $B_j \subseteq c_2 B_i$ *for any* $j > i$.

Proof. Denote $d(x, z) = r > 0$. By assumptions, there is a path $\gamma : [0, 1] \to \mathbf{C}_\lambda(z, c_\lambda r)$ from x to z. Without loss of generality, we can assume that there is a unique $t_i > t_{i-1}$ with $d(\gamma(t_i), z) = \epsilon^i r$ and $\gamma([t_i, 1]) \subset \mathbf{S}_\lambda(z, c_\lambda \epsilon^i r)$ for all i. This can be done by recursively define $t_{i+1} \overset{\text{df}}{=} \inf\{t > t_i \ : \ \gamma(t) \notin \mathbf{S}_\lambda(z, \epsilon^i r)\}$ while replacing the subpath $\gamma_{|[t_{i+1}, 1]}$ with a path connecting $\gamma(t_{i+1})$ and z lying entirely in $\mathbf{C}_\lambda(z, c_\lambda \epsilon^{i+1} r)$.

For each $B = \mathbf{B}(y, s) \in \mathcal{W}$ such that $3B \cap \gamma([t_i, t_{i+1}]) \neq \emptyset$, $\frac{c_\lambda \epsilon^{i+1}}{1+\epsilon} r \leq s \leq \frac{c_\lambda \epsilon^i}{1-\epsilon} r$. To see this, let $w \in B \cap \gamma([t_i, t_{i+1}])$. Then

$$\begin{aligned} s &= \epsilon d(y, z) \\ &\leq \epsilon\left(d(y, w) + d(w, z)\right) \\ &\leq \epsilon s + c_\lambda \epsilon^i r. \end{aligned}$$

Therefore, $s \leq \frac{c_\lambda \epsilon^i}{1-\epsilon} r$. The other inequality can be proved similarly. By Lemma 4.6, there exists $n_c \geq 1$, independent of i, such that at most n_c such B cover $\gamma([t_i, t_{i+1}])$.

For each i, choose $A_{1,i} \in \mathcal{W}$ so that $\gamma(t_i) \in 3A_{1,i}$ then choose $A_{j+1,i}$ recursively as follows. Let $s_{j+1,i}$ be the last s so that $d(\gamma(s), 3A_{j,i}) = 0$. If

$s_{j+1,i} \geq t_i$, we are done. Otherwise, choose $A_{j+1,i} \in \mathcal{W}$ such that $3A_{j+1,i}$ contains $\gamma(s_{j+1,i})$. This process will be done in at most n_c steps since there are at most n_c such balls. Next, if $3A_{l,i} \cap 3A_{k,i} \neq \emptyset$ where $k < l$, replace the whole finite chain $\{A_{j,i}\}$ with $\{A_{j,i}\} - \{A_{k+1,i}, \ldots, A_{l-1,i}\}$. Repeat the process until $3A_{l,i} \cap 3A_{k,i} \neq \emptyset$ if and only if $|l-k| = 1$. If $3A_{l,m} \cap 3A_{k,n} \neq \emptyset$ where $m < n$, replace the whole finite chain $\{A_{j,m}\}$ with $\{A_{1,m}, \ldots, A_{l,m}\}$, $\{A_{j,n}\}$ with $\{A_{j,n}\} - \{A_{1,n}, \ldots, A_{k-1,n}\}$, and $\{A_{j,i}\}$ with empty set for all $i = m+1, \ldots, n-1$. Repeat the process until each $A_{l,m}$ has unique $A_{k,n}, n > m$ such that $3A_{l,m} \cap 3A_{k,n} \neq \emptyset$. Define $B_{n_i+j} = A_{j,i}$ where n_i is the cardinality of $\cup_{k<i}\{A_{j,k}\}$. Set $\mathcal{P}_k = \{B_i \ : \ B_i \cap \mathbf{S}_\lambda(z, \epsilon^i r) \neq \emptyset, B_i \notin \cup_{i<k}\mathcal{P}_i\}$. Argue as before, each $\mathcal{P}_k$ has at most n_c elements and it satisfies *(CB3)* with constant $c_1 = \max(\frac{1+\epsilon}{c_\lambda \epsilon}, \frac{c_\lambda}{1-\epsilon})$.

Clearly, $3B_i \cap \mathbf{C}_\lambda(z, c_\lambda r) \neq \emptyset$. Next, pick $y \in 3B_i \cap \mathbf{C}_\lambda(z, c_\lambda r)$, then

$$\begin{aligned} r_i &= \frac{1}{\epsilon} d(x_i, \mathbf{Z}) \\ &\geq \frac{1}{\epsilon}\left(d(y, \mathbf{Z}) - d(y, x_i)\right) \\ &\geq \frac{1}{\epsilon}\left(\frac{1}{\lambda} d(y, z) - 3r_i\right) \end{aligned}$$

which implies

$$\begin{aligned} (3+\epsilon) r_i &\geq \frac{1}{\lambda}\left(d(x_i, z) - d(x_i, y)\right) \\ &\geq \frac{1}{\lambda}\left(d(x_i, z) - 3r_i\right). \end{aligned}$$

Solving this inequality gives $r_i \geq \frac{1}{3\lambda+\epsilon\lambda+3} d(x_i, z)$.

For (CB6), the distance from x_i to any $w \in B_j = \mathbf{B}(x_j, r_j)$, $i < j$, is at most

$$\begin{aligned}\sum_{k=i}^{j} 3r_k + r_j &\leq 4\sum_{k=i}^{j} r_k \\ &\leq 4n_c c_1^2 \sum_{k=i}^{j} \epsilon^{k-i} r_i \\ &\leq 4n_c c_1^2 r_i \sum_{k=0}^{\infty} \epsilon^k \\ &\leq \frac{4n_c c_1^2 r_i}{(1-\epsilon)}.\end{aligned}$$

Let $c_2 = \frac{4n_c c_1^2}{(1-\epsilon)}$ and we are done. □

Lemma 4.8. *For any balls* $\mathbf{B}(x, r_x), \mathbf{B}(y, r_y)$ *such that* $\mathbf{B}(x, 3r_x) \cap \mathbf{B}(y, 3r_y)$ *is nonempty,*

$$(1-3\epsilon)\max(r_x, r_y) \leq (1+3\epsilon)\min(r_x, r_y)$$

where $r_x = \epsilon d(x, \mathbf{Z})$ *and* $r_y = \epsilon d(y, \mathbf{Z})$*.*

Proof. Let $w \in \mathbf{B}(x, 3r_x) \cap \mathbf{B}(y, 3r_y)$ and assume, without loss of generality, that $r_x \geq r_y$. Then

$$\begin{aligned}d(x, \mathbf{Z}) &\leq d(x, w) + d(w, \mathbf{Z}) \\ &\leq d(x, w) + d(w, y) + d(y, \mathbf{Z}).\end{aligned}$$

Thus,

$$\begin{aligned}r_x - r_y &\leq \epsilon(d(x, w) + d(w, y)) \\ &\leq \epsilon(3r_x + 3r_y),\end{aligned}$$

i.e., $(1-3\epsilon)r_x \leq (1+3\epsilon)r_y$. □

Combination of Lemma 4.8 and Theorem 4.1 implies the following lemma.

Lemma 4.9. *Let* ν *be a doubling measure with doubling constant* C_D*. For any* $\mathbf{B}(x, r_x), \mathbf{B}(y, r_y) \in \mathcal{W}$ *such that* $\mathbf{B}(x, 3r_x) \cap \mathbf{B}(y, 3r_y)$ *is nonempty,*

$$\max(\nu(\mathbf{B}(x, 4r_x)), \nu(\mathbf{B}(y, 4r_y))) \leq C_D \left(\frac{7+21\epsilon}{1-3\epsilon}\right)^{\log_2 C_D} \nu(\mathbf{B}(x, 4r_x) \cap \mathbf{B}(y, 4r_y)).$$

Proof. Let $z \in \mathbf{B}(x, 3r_x) \cap \mathbf{B}(y, 3r_y)$ and $r = \min(r_x, r_y)$. Then

$$\mathbf{B}(z, r) \subseteq \mathbf{B}(x, 4r_x) \cap \mathbf{B}(y, 4r_y) \subseteq \mathbf{B}(z, \frac{1+3\epsilon}{1-3\epsilon}(7r)).$$

Therefore,

$$\begin{aligned} \max(\nu(\mathbf{B}(x, 4r_x)), \nu(\mathbf{B}(y, 4r_y))) &\leq \nu(\mathbf{B}(z, \frac{1+3\epsilon}{1-3\epsilon}(7r))) \\ &\leq C_D \left(\frac{7+21\epsilon}{1-3\epsilon}\right)^{\log_2 C_D} \nu(\mathbf{B}(z, r)) \\ &\leq C_D \left(\frac{7+21\epsilon}{1-3\epsilon}\right)^{\log_2 C_D} \nu(\mathbf{B}(x, r_x) \cap \mathbf{B}(y, r_y)). \end{aligned}$$

□

Now we are in a position to prove the main theorem. Note that, the statement of Theorem 4.10 given below is different from that of Theorem 3.2 given earlier but they are the same statements. Here, Theorem 4.10 contains details and labels that will be useful in the proof.

Theorem 4.10. *Let $(E, \mathrm{D}(E))$ be a strongly local, regular, Dirichlet form in $\mathrm{L}^2(\mathbf{X}, \nu)$ where the measure ν is doubling with doubling constant C_D. Denote Γ its associated energy measure. Let also $\mathbf{Z} \subset \mathbf{X}$ be (λ, Λ)-accessible covering distance $R : \mathbf{Z} \to (0, \infty]$. Assume that,*

(PI1) there are constants $k \geq 1$, $P_1 > 0$ and $\epsilon_0 \in (0, 1]$ such that for all $f \in \mathrm{D}(E)$, $x \in \mathbf{X} - \mathbf{Z}$, and $r \leq \epsilon_0 d(x, \mathbf{Z})$,

$$\inf_{\xi \in \mathbb{R}} \int_{\mathbf{B}(x,r)} |f - \xi|^2 d\nu \leq P_1 r^2 \int_{\mathbf{B}(x,kr)} d\Gamma(f, f),$$

(PI2) there is a constant $P_2 > 0$ such that for any $z \in \mathbf{Z}$, $0 < r < R(z)$, $x, y \in \mathbf{S}_\lambda(z, r)$, and $f \in \mathrm{D}(E)$

$$\fint_{\mathbf{B}(x,\delta r)} \fint_{\mathbf{B}(y,\delta r)} |f(u) - f(v)|^2 d\nu(u) d\nu(v) \leq P_2 r^2 \fint_{\mathbf{B}(z,r)} d\Gamma(f, f).$$

Then there are constants $P, \kappa \geq 1 \geq \eta > 0$ such that for all $f \in \mathrm{D}(E)$, $z \in \mathbf{Z}$, and $r < \eta R(z)$,

$$\inf_{\xi \in \mathbb{R}} \int_{\mathbf{B}(z,r)} |f - \xi|^2 d\nu \leq P r^2 \int_{\mathbf{B}(z,\kappa r)} d\Gamma(f, f).$$

Proof. To simplify the writing, the notation $\alpha \lesssim \beta$ will stand for $\alpha \le c\beta$ for some constant c depends only on $C_D, P_1, P_2, \lambda, \Lambda, \epsilon, \epsilon_0, k$.

First choose $0 < \epsilon < \min(\frac{1}{200}, \frac{1}{8k}, \frac{\epsilon_0}{8})$ small enough so that $\frac{8\epsilon}{(1-3\epsilon)^2} \le \frac{1}{\lambda} - \frac{1}{\Lambda}$ and denote $\mathcal{W}$ the strict ϵ-Whitney covering of $\mathbf{X}-\mathbf{Z}$. Denote also $\eta = \frac{2\epsilon\Lambda}{1-3\epsilon}$. Fixed $z \in \mathbf{Z}$ and $r \in (0, \eta R(z))$. Denote $\mathcal{A} = \{B \in \mathcal{W} : 3B \cap \mathbf{B}(z,r) \neq \emptyset\}$. For each $A = \mathbf{B}(\xi_A, r_A) \in \mathcal{A}$, choose $\zeta_A \in \mathbf{Z}$ so that $d(\xi_A, \zeta_A) = d(\xi_A, \mathbf{Z})$. Notice that

$$\begin{aligned} d(\xi_A, \mathbf{Z}) &\le d(\xi_A, z) \\ &\le 3r_A + r \\ &= 3\epsilon d(\xi_A, \mathbf{Z}) + r, \end{aligned}$$

so $d(\xi_A, \mathbf{Z}) \le \frac{r}{1-3\epsilon} \le 2r$ and $r_A \le \frac{\epsilon r}{1-3\epsilon} \le r$. This implies

$$\begin{aligned} d(z, \zeta_A) &\le d(z, \xi_A) + d(\xi_A, \zeta_A) \\ &\le 3r_A + r + d(\xi_A, \mathbf{Z}) \\ &\le \frac{2}{1-3\epsilon} r, \end{aligned}$$

i.e., $\zeta_A \in \mathbf{Z} \cap \mathbf{B}(z, \frac{2}{1-3\epsilon} r)$.

By assumption, there is $x_z \in \mathbf{S}_\Lambda(z, \frac{(1-3\epsilon)r}{2\epsilon\Lambda}) \subseteq \mathbf{C}_\Lambda(z, R(z))$. For each ζ_A,

$$\begin{aligned} \frac{(1-3\epsilon)r}{2\epsilon\lambda} &\ge \frac{(1-3\epsilon)r}{2\epsilon}\left(\frac{1}{\Lambda} + \frac{4\epsilon}{(1-3\epsilon)^2}\right) \\ &\ge d(x_z, z) + d(z, \zeta_A) \\ &\ge d(x_z, \mathbf{Z}) \\ &\ge \frac{(1-3\epsilon)r}{2\epsilon} \end{aligned}$$

which means $x_z \in \mathbf{S}_\lambda(\zeta_A, \frac{(1-3\epsilon)r}{2\epsilon\lambda}) \subseteq \mathbf{C}_\lambda(\zeta_A, R(\zeta_A))$. Therefore, we can choose, for each ζ_A, a sequence of remote balls $\mathcal{B}^A = \{B_i^A = \mathbf{B}(x_i^A, r_i^A)\}_{i=1}^\infty \subseteq \mathcal{W}$ connecting x_z and ζ_A according to Lemma 4.7.

By construction, $3B_i^A \cap \mathbf{C}_\lambda(\zeta_A, \frac{c_\lambda(1-3\epsilon)r}{2\epsilon\lambda}) \neq \emptyset$ and $r_i^A \le c_1 \frac{(1-3\epsilon)r}{2\epsilon\lambda}$ for all i. Thus, $16B_i^A \subseteq \mathbf{B}(\zeta_A, (19c_1+1)\frac{c_\lambda(1-3\epsilon)r}{2\epsilon\lambda})$ which implies

$$16B_i^A \subseteq \mathbf{B}(z, \left((19c_1+1)\frac{c_\lambda(1-3\epsilon)}{2\epsilon\lambda} + \frac{2}{1-3\epsilon}\right) r)$$

for all i and A. Denote $\kappa = \left(\max\left(\frac{c_\lambda(19c_1+1)(1-3\epsilon)}{2\epsilon\lambda}, \frac{1}{\epsilon} + 4k + 3\right) + \frac{2}{1-3\epsilon}\right)$.

Since $\frac{8\epsilon}{(1-3\epsilon)^2} \leq \frac{1}{\lambda}$, $\frac{(1-3\epsilon)r}{2\epsilon\lambda} \geq \frac{4r}{1-3\epsilon} \geq \frac{r_A}{\epsilon}$. Hence, there must be an i_A for which $3B^A_{i_A}$ intersects $\mathbf{S}_\lambda(\zeta_A, \frac{r_A}{\epsilon})$. For any $x^A \in 3B^A_{i_A} \cap \mathbf{S}_\lambda(\zeta_A, \frac{r_A}{\epsilon})$, $\mathbf{B}(x^A, 3\epsilon d(x^A, \mathbf{Z})) \cap 3B^A_{i_A} \neq \emptyset$. By Lemma 4.8, $\max(r^A_{i_A}, \epsilon d(x^A, \mathbf{Z})) \leq \left(\frac{1-3\epsilon}{1+3\epsilon}\right) \min(r^A_{i_A}, \epsilon d(x^A, \mathbf{Z}))$. But $\lambda r_A \leq \epsilon d(x^A, \mathbf{Z}) \leq r_A$. Thus, $\left(\frac{\lambda+3\lambda\epsilon}{1-3\epsilon}\right) r_A \leq r^A_{i_A} \leq \left(\frac{1-3\epsilon}{1+3\epsilon}\right) r_A$. Particularly, $\lambda r_A \leq r^A_{i_A} \leq r_A$.

Claim that there is a number n_0 depends only on C_D and ϵ, λ such that for each $\mathbf{B}(x, s) \in \mathcal{W}$, there is at most n_0 balls $A \in \mathcal{A}$ such that $B^A_{i_A} = \mathbf{B}(x, s)$. To see this, notice that $s \leq r_A \leq \frac{s}{\lambda}$ and

$$\begin{aligned} d(x, \xi_A) &\leq d(x, \zeta_A) + d(\zeta_A, \xi_A) \\ &\leq \frac{1}{c_3} r^A_{i_A} + d(\mathbf{Z}, \xi_A) \\ &\leq \frac{1}{c_3} s + \epsilon r_A \\ &\leq \left(\frac{1}{c_3} + \frac{\epsilon}{\lambda}\right) s \end{aligned}$$

which implies $A \subseteq \mathbf{B}(x, (\frac{1}{c_3} + \frac{\epsilon}{\lambda} + \frac{1}{\lambda})s)$. Since all such A are disjoint and each has radius at least s, we can apply Lemma 4.5 and obtain n_0. In fact, $n_0 = \lfloor C_D(5(\frac{1}{c_3} + \frac{\epsilon}{\lambda} + \frac{1}{\lambda})^{\log_2 C_D}) \rfloor$.

Denote $f_B = \fint_{4B} f d\nu$ for any ball B and pick any ball $W = \mathbf{B}(\xi_W, r_W) \in \mathcal{W}$ for which $x_z \in 3W$. For any $A \in \mathcal{A}$,

$$|f - f_W|^2 \leq 4\left[|f - f_A|^2 + |f_A - f_{B^A_{i_A}}|^2 + |f_{B^A_{i_A}} - f_{B^A_1}|^2 + |f_{B^A_1} - f_W|^2\right].$$

Therefore,

$$\int_{\mathbf{B}(z,r)} |f - f_W|^2 d\nu \leq 4 \sum_{A \in \mathcal{A}} \int_{4A} \left[|f - f_A|^2 + |f_A - f_{B^A_{i_A}}|^2 + |f_{B^A_{i_A}} - f_{B^A_1}|^2 + |f_{B^A_1} - f_W|^2\right] d\nu.$$

If we can show that each term is bounded by $r^2 \int_{\mathbf{B}(z,\kappa r)} d\Gamma(f, f)$ upto some constant, then the result follows.

(I) The first term is bounded easily. For any $A \in \mathcal{A}$, $4kA \subseteq \mathbf{B}(z, \frac{1+4k\epsilon}{1-3\epsilon} r)$. This follows from the fact that $3A \cap \mathbf{B}(z, r) \neq \emptyset$ and $r_A \leq \frac{\epsilon r}{1-3\epsilon}$ so

$$\begin{aligned} d(w, z) &< 4kr_A + 3r_A + r \\ &\leq \frac{1 + 4k\epsilon}{1 - 3\epsilon} r \end{aligned}$$

for all $w \in 4A$. The assumption *(PI1)* then implies

$$\begin{aligned}\sum_{A\in\mathcal{A}}\int_{4A}|f-f_A|^2 d\nu &\le P_1 r^2 \sum_{A\in\mathcal{A}}\int_{4kA} d\Gamma(f,f)\\ &\le P_1 r^2 \int_{\mathbf{B}(z,\frac{1+4k\epsilon}{1-3\epsilon}r)}\left(\sum_{A\in\mathcal{A}}\mathbf{1}_{4kA}\right) d\Gamma(f,f)\\ &\le c_b P_1 r^2 \int_{\mathbf{B}(z,\frac{1+4k\epsilon}{1-3\epsilon}r)} d\Gamma(f,f)\\ &\le c_b P_1 r^2 \int_{\mathbf{B}(z,\kappa r)} d\Gamma(f,f).\end{aligned}$$

For the other terms, we need the following fact. For any $U = \mathbf{B}(x,s), V = \mathbf{B}(y,t)$ with $s = \epsilon d(x,\mathbf{Z})$, $t = \epsilon d(y,\mathbf{Z})$, and $3U \cap 3V \neq \emptyset$,

$$|f_U - f_V|^2 \lesssim \frac{\min(s,t)^2}{\max(\nu(U),\nu(V))}\int_{4kU\cup 4kV} d\Gamma(f,f). \tag{4.1}$$

Note that the present of maximum and minimum is, actually, not neccessary since those numbers are roughly the same by Lemma 4.8 and Theorem 4.1. To prove the above inequality, observe that

$$\begin{aligned}\nu(4U\cap 4V)|f_U - f_V|^2 &\le \int_{4U\cap 4V}|f-f_U|^2 + \int_{4U\cap 4V}|f-f_V|^2\\ &\le \int_{4U}|f-f_U|^2 + \int_{4V}|f-f_V|^2\\ &\le P_1 s^2\int_{4kU} d\Gamma(f,f) + P_1 t^2 \int_{4kV} d\Gamma(f,f)\end{aligned}$$

which gives the inequality (4.1) after applying Lemma 4.8 and Lemma 4.9.

(II) Next, we show that $\sum_{A\in\mathcal{A}}\int_{4A}|f_A - f_{B^A_{i_A}}|^2 \lesssim r^2\int_{\mathbf{B}(z,\kappa r)} d\Gamma(f,f)$. Recall that we can pick $x^A \in 3B^A_{i_A}\cap \mathbf{S}_\lambda(\zeta_A, \frac{r_A}{\epsilon})$. Denote $s^A = \epsilon d(x^A,\mathbf{Z})$. Note that $s^A, r^A_{i_A} \in [\lambda r_A, r_A]$. Also, $4kA, 4k\mathbf{B}(x^A,s^A) \subseteq \mathbf{B}(\zeta_A,(4k+\frac{1}{\epsilon})r_A)$ and $4kB^A_{i_A} \subseteq \mathbf{B}(\zeta_A, \left(\frac{1}{\epsilon}+4k+3\right) r_A)$.

The inequality (4.1) implies that

$$\begin{aligned}
\int_{4A} |f_{\mathbf{B}(x^A,s^A)} - f_{B^A_{i_A}}|^2 d\nu &\lesssim \nu(4A)\frac{r_A^2}{\nu(\mathbf{B}(x^A,s^A))}\int_{4k\mathbf{B}(x^A,s^A)\cup 4kB^A_{i_A}} d\Gamma(f,f) \\
&\leq C_D\left(\frac{4r_A+d(x^A,\xi_A)}{s^A}\right)^{\log_2 C_D} r_A^2 \int_{\mathbf{B}(\zeta_A,\left(\frac{1}{\epsilon}+4k+3\right)r_A)} d\Gamma(f,f) \\
&\leq C_D\left(\frac{4r_A+2r_A/\epsilon}{\lambda r_A}\right)^{\log_2 C_D} r_A^2 \int_{\mathbf{B}(\zeta_A,\left(\frac{1}{\epsilon}+4k+3\right)r_A)} d\Gamma(f,f) \\
&\leq C_D\left(\frac{4\epsilon+2}{\epsilon\lambda}\right)^{\log_2 C_D} r_A^2 \int_{\mathbf{B}(\zeta_A,\left(\frac{1}{\epsilon}+4k+3\right)r_A)} d\Gamma(f,f).
\end{aligned}$$

By assumption *(PI2)*,

$$\begin{aligned}
\int_{4A} |f_A - f_{\mathbf{B}(x^A,s^A)}|^2 d\nu &= \nu(4A)\left|\fint_{4A} f d\nu - \fint_{4\mathbf{B}(x^A,s^A)} f d\nu\right|^2 \\
&\leq \nu(4A)\fint_{4A}\fint_{4\mathbf{B}(x^A,s^A)} |f(u)-f(v)|^2 d\nu(u)d\nu(v) \\
&\leq \nu(4A)C_D^2\left[\left(\frac{1}{4k\epsilon}\right)\left(\frac{1}{4k\epsilon\lambda}\right)\right]^{\log_2 C_D}\fint_{\mathbf{B}(\xi_A,\frac{r_A}{k\epsilon})}\fint_{\mathbf{B}(x^A,\frac{r_A}{k\epsilon})} |f(u)-f(v)|^2 d\nu(\\
&\leq P_2 r_A^2 \int_{\mathbf{B}(\zeta_A,\frac{r_A}{\epsilon})} d\Gamma(f,f).
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{4A} |f_A - f_{B^A_{i_A}}|^2 &\leq 2\int_{4A} |f_{\mathbf{B}(x^A,s^A)} - f_{B^A_{i_A}}|^2 d\nu + 2\int_{4A} |f_A - f_{\mathbf{B}(x^A,s^A)}|^2 d\nu \\
&\lesssim r_A^2 \int_{\mathbf{B}(\zeta_A,\left(\frac{1}{\epsilon}+4k+3\right)r_A)} d\Gamma(f,f) + r_A^2\int_{\mathbf{B}(\zeta_A,\frac{r_A}{\epsilon})} d\Gamma(f,f) \\
&\lesssim r_A^2 \int_{\mathbf{B}(\zeta_A,\left(\frac{1}{\epsilon}+4k+3\right)r_A)} d\Gamma(f,f).
\end{aligned}$$

Denote $\mathcal{A}_k$ the set of all those balls in $\mathcal{A}$ with radius between $r/2^k$ and $r/2^{k+1}$. Claim that $\sum_{A\in\mathcal{A}_k} \mathbf{1}_{\mathbf{B}(\zeta_A,\left(\frac{1}{\epsilon}+4k+3\right)r_A)}$ can be uniformly bounded.

Fix $A \in \mathcal{A}_k$. For any $A' \in \mathcal{A}_k$ such that $\mathbf{B}(\zeta_A, \left(\frac{1}{\epsilon}+4k+3\right) r_A)\cap\mathbf{B}(\zeta_{A'}, \left(\frac{1}{\epsilon}+4k+3\right) r_{A'}) \neq \emptyset$, $A' \subseteq \mathbf{B}(\zeta_A, \left(\frac{1}{\epsilon}+4k+3\right)\frac{3r}{2^k})$. Since the radius of such A' is at least $\frac{r}{2^{k+1}}$, Lemma 4.5 implies that there can only be at most n_1 such A' where n_1 depends only on C_D, δ and ϵ. Therefore, $\sum_{A\in\mathcal{A}_k} \mathbf{1}_{\mathbf{B}(\zeta_A,\left(\frac{1}{\epsilon}+4k+3\right)r_A)} \leq n_1$.

Now,

$$\begin{aligned}
\sum_{A\in\mathcal{A}} \int_{4A} |f_A - f_{B^A_{i_A}}|^2 d\nu &= \sum_k \sum_{A\in\mathcal{A}_k} \int_{4A} |f_A - f_{B^A_{i_A}}|^2 d\nu \\
&\lesssim \sum_k \sum_{A\in\mathcal{A}_k} \left(\frac{r}{2^k}\right)^2 \int_{\mathbf{B}(\zeta_A,\left(\frac{1}{\epsilon}+4k+3\right)r_A)} \Gamma(f,f) d\nu \\
&\lesssim \sum_k \left(\frac{r}{2^k}\right)^2 \int_{\mathbf{B}(z,\kappa r)} d\Gamma(f,f) \\
&\lesssim r^2 \int_{\mathbf{B}(z,\kappa r)} d\Gamma(f,f).
\end{aligned}$$

(III) Next we prove the bound for the third term. By Lemma 4.5, there must be an $n_3 \geq 1$, independent of z and r, such that $\{B^A_1 \ : \ A \in \mathcal{A}\} = \{D_1, \ldots, D_n\}$, $\exists n \leq n_3$. This follows from the fact that the radius of B^A_1 is at least $\frac{1-3\epsilon}{2c_1} r$ while lying inside $\mathbf{B}(z,\kappa r)$. Denote $\mathcal{G}_i \stackrel{\text{df}}{=} \cup_{B^A_1 = D_i} \mathcal{B}^A$ and $\mathcal{D}_i \stackrel{\text{df}}{=} \{B^A_{i_A} \ : \ A \in \mathcal{A}, B^A_1 = D_i\}$. For each $B^A_{i_A} \in \mathcal{D}_i$, $B^A_{i_A} \subset c_2 B^A_j$ for all $j < i_A$, and $4B^A_{j-1} \subseteq 16 B^A_j$ by Lemma 4.8. Hence,

$$\begin{aligned}
|f_{B^A_{i_A}} - f_{B^A_1}| \mathbf{1}_{B^A_{i_A}} &\leq \sum_{j=2}^{i_A} |f_{B^A_j} - f_{B^A_{j-1}}| \mathbf{1}_{B^A_{i_A}} \\
&\lesssim \sum_{j=2}^{i_A} r^A_j \Big(\frac{1}{\nu(B^A_j)} \int_{16 B^A_j} d\Gamma(f,f) \Big)^{1/2} \mathbf{1}_{B^A_{i_A}} \mathbf{1}_{c_2 B^A_j} \\
&\leq \sum_{G\in\mathcal{G}_i} r \Big(\frac{1}{\nu(G)} \int_{16G} d\Gamma(f,f) \Big)^{1/2} \mathbf{1}_{B^A_{i_A}} \mathbf{1}_{c_2 G}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{D\in\mathcal{D}_i}\int_{4D}|f_D-f_{D_i}|^2d\nu &\lesssim \sum_{D\in\mathcal{D}_i}\int_D|f_D-f_{D_i}|^2d\nu\\
&\lesssim \int\sum_{D\in\mathcal{D}_i}\left(\sum_{G\in\mathcal{G}_i}r\left(\frac{1}{\nu(G)}\int_{16G}d\Gamma(f,f)\right)^{1/2}1_D1_{c_2G}\right)^2 1_Dd\nu\\
&\lesssim r^2\int\left(\sum_{D\in\mathcal{W}}1_D\right)\left(\sum_{G\in\mathcal{G}_i}\left(\frac{1}{\nu(G)}\int_{16G}d\Gamma(f,f)\right)^{1/2}1_{c_2G}\right)^2d\nu\\
&\lesssim r^2\int\left(\sum_{G\in\mathcal{G}_i}\left(\frac{1}{\nu(G)}\int_{16G}d\Gamma(f,f)\right)^{1/2}1_{c_2G}\right)^2d\nu\\
&\lesssim r^2\int\left(\sum_{G\in\mathcal{G}_i}\left(\frac{1}{\nu(G)}\int_{16G}d\Gamma(f,f)\right)^{1/2}1_G\right)^2d\nu
\end{aligned}$$

where the last inequality follows from Lemma 4.3.

Since balls in $\mathcal{G}_i$ are disjoint,

$$\begin{aligned}
\sum_{D\in\mathcal{D}_i}\int_{4D}|f_D-f_{D_i}|^2d\nu &\lesssim r^2\int\sum_{G\in\mathcal{G}_i}\left(\frac{1}{\nu(G)}\int_{16G}d\Gamma(f,f)\right)1_Gd\nu\\
&\lesssim r^2\sum_{G\in\mathcal{G}_i}\left(\frac{1}{\nu(G)}\int_{16G}d\Gamma(f,f)\right)\int 1_Gd\nu\\
&\lesssim r^2\int\left(\sum_{G\in\mathcal{G}_i}1_{16G}\right)d\Gamma(f,f)\\
&\lesssim r^2\int_{\mathbf{B}(z,\kappa r)}d\Gamma(f,f).
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{A\in\mathcal{A}}\int_{4A}|f_{B^A_{i_A}}-f_{B^A_1}|^2 &= \sum_{i=1}^n\sum_{D\in\mathcal{D}_i}\int_{4D}|f_D-f_{D_i}|^2d\nu\\
&\le n_3r^2\int_{\mathbf{B}(z,\kappa r)}d\Gamma(f,f)\\
&\lesssim r^2\int_{\mathbf{B}(z,\kappa r)}d\Gamma(f,f).
\end{aligned}$$

(IV) For the last term, we have

$$\begin{aligned}
\sum_{A\in\mathcal{A}}\int_{4A}|f_{B_1^A}-f_W|^2 d\nu &= \sum_i |f_{D_i}-f_W|^2\left(\sum_{B_1^A=D_i}\nu(4A)\right)\\
&\lesssim \nu(\mathbf{B}(z,\frac{1+4\epsilon}{1-3\epsilon}r))\sum_i |f_{D_i}-f_W|^2\\
&\lesssim \frac{\nu(\mathbf{B}(z,\frac{1+4\epsilon}{1-3\epsilon}r))}{\nu(W)} n_3 r^2\int_{\mathbf{B}(z,\frac{1+4\epsilon}{2\epsilon\Lambda}r)} d\Gamma(f,f)\\
&\lesssim \left(\frac{\frac{1+4\epsilon}{1-3\epsilon}r+\frac{(1-3\epsilon)}{2\epsilon\Lambda}r+r_W}{r_W}\right)^{\log_2 C_D} r^2\int_{\mathbf{B}(z,\frac{1+4\epsilon}{2\epsilon\Lambda}r)} d\Gamma(f,f).
\end{aligned}$$

Since $x_z\in 3W$,

$$\begin{aligned}
r_W &= \epsilon d(\xi_W,\mathbf{Z})\\
&\geq \epsilon\left(d(x_z,\mathbf{Z})-d(\xi_W,x_z)\right)\\
&\geq \epsilon\left(d(x_z,\mathbf{Z})-3r_W\right)
\end{aligned}$$

and hence

$$\begin{aligned}
r_W &\geq \frac{\epsilon}{1+3\epsilon}d(x_z,\mathbf{Z})\\
&\geq \left(\frac{\epsilon}{1+3\epsilon}\right)\left(\frac{1-3\epsilon}{2\epsilon}r\right)\\
&\geq \frac{r}{4}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{A\in\mathcal{A}}\int_{4A}|f_{B_1^A}-f_W|^2 d\nu &\lesssim \left(\frac{\frac{1+4\epsilon}{1-3\epsilon}r+\frac{(1-3\epsilon)}{2\epsilon\Lambda}r}{\frac{r}{4}}+1\right)^{\log_2 C_D} r^2\int_{\mathbf{B}(z,\frac{1+4\epsilon}{2\epsilon\Lambda}r)} d\Gamma(f,f)\\
&\lesssim r^2\int_{\mathbf{B}(z,\kappa r)} d\Gamma(f,f).
\end{aligned}$$

This completes the proof. □

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6.2 ผลงานอื่นๆ ได้แก่ ผลงานวิจัยชื่อ

Harnack Inequality under the Change of Metric

ซึ่งในขณะนี้อยู่ระหว่างการแก้ไขขัดเกลาสำนวน ก่อนที่จะส่งตีพิมพ์ต่อไป

Harnack Inequality under the Change of Metric*

Santi Tasena†, Sompong Dhompongsa‡, and Laurent Saloff-Coste§

Abstract

In this work, we give a sufficient and necessary condition in which the Harnack inequality of the original space would imply that of the quasi-conformal space. The result can be applied to a large class of function including those that their singularities separate the space into dijoint connected components.

1 Introduction

One question generally asked regarding a heat equation is whether its solutions exhibit the same behavior as that of the classical heat equation. For example, whether the Harnack inequality holds or whether the Harnack inequality of a heat equation implies that of its variants. Grigory'an and Saloff-Coste[1], for instance, give a sufficient condition for which the Harnack inequality of the heat equation on a manifold remains holds on a weighted manifolds (see also [2, 3, 7]).

In this work, we ask a similar question: whether the Harnack inequality is stable under the conformal metrics. Unlike the weighted manifolds where the change lies only on the volume measure, the conformal metric changes both the volume measure and the intrinsic distance.

Given a complete Riemannian manifold (M, g) with dimension n and a conformal metric $\tilde{g} = e^{2\omega} g$ of g, the volume measure $\mathrm{v}_{\tilde{g}}$ of $(M, \tilde{g})$ is related to the volume measure v_g of (M, g) via the formula $\mathrm{dv}_{\tilde{g}} = e^{n\omega}\mathrm{dv}_g$ and the length $\mathrm{Length}_{\tilde{g}}(\gamma)$ of a curve $\gamma : [0, 1] \to M$ under $(M, \tilde{g})$ is given by $\mathrm{Length}_{\tilde{g}}(\gamma) = \int_0^1 e^{\omega(\gamma(t))}|\gamma'(t)|\mathrm{d}t$ while the length of γ under (M, g) is given by $\mathrm{Length}_g(\gamma) = \int_0^1 |\gamma'(t)|\mathrm{d}t$. The Laplace-Beltrami operator of (M, g) and $(M, \tilde{g})$ can be written in terms of local coordinates as

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†Corresponding author. Address: Department of Mathematics, Faculty of Sciences, Chiang Mai University, Chiang Mai, 50200 Thailand

‡Department of Mathematics, Faculty of Sciences, Chiang Mai University, Chiang Mai, 50200 Thailand

§Department of Mathematics, Malott Hall, Cornell University, Ithaca, New York, 14853-4201 USA

$$\triangle_g = -\frac{1}{\sqrt{\det(g)}} \sum_{i,j} \frac{\partial}{\partial x_j} \left(g^{ij} \det(g) \frac{\partial}{\partial x_i} \right),$$

$$\triangle_{\tilde{g}} = -\frac{e^{-n\omega}}{\sqrt{\det(g)}} \sum_{i,j} \frac{\partial}{\partial x_j} \left(e^{(n-2)\omega} g^{ij} \det(g) \frac{\partial}{\partial x_i} \right),$$

respectively. As a result, the associated Dirichlet form $E_{\tilde{g}}$ of $\triangle_{\tilde{g}}$ can be written as $E_{\tilde{g}}(u,v) = \int_M e^{(n-2)\omega} g(\nabla_g u, \nabla_g v) \mathrm{dv}_g$ whereas $E_g(u,v) = \int_M g(\nabla_g u, \nabla_g v) \mathrm{dv}_g$.

This situation extends naturally to the setting on Dirichlet spaces. Given a strongly local regular Dirichlet form $(E, \mathbb{D}(E))$ on $L^2(X, \nu)$ with the associative carre du champ operator Γ and a function $h : X \to [0, \infty]$ with mild conditions, it is possible to construct another strongly local regular Dirichlet form $\left(E^h, \mathbb{D}(E_h)\right)$ on $L^2(X, \nu_h)$ where $\mathrm{d}\nu_h = h^n \mathrm{d}\nu$ such that

$$E_h(u,v) = \int h^{(n-2)} \Gamma(u,v) \mathrm{d}\nu$$

whenever defined. We can then ask whether $(E_h, \mathbb{D}(E_h))$ satisfies the Harnack inequality given that $(E, \mathbb{D}(E))$ does. Note that, on Dirichlet spaces, the function h does not have to be smooth. It does not even have to be continuous. For convenience, we will assume h is semi-continuous, however.[1]

In this work, the authors prove this result for a class of functions with polynomial growth in asymtotically Z-normal direction where Z is its singularity set. Since this class of functions is dense in the space of weighted doubling measures[10, Proposition 3.3], this result should be applicable in many situations. To guarantee that the conformal distance is complete, however, we have to assume that the singularity set is discrete.

2 Preliminaries

In this work, a (topological) space is assume to be locally compact but non-compact, second countable and Hausdorff, a measure means a Borel measure with full support, and a Dirichlet form means a regular, strongly local Dirichlet form admitting the carré du champ operator.

Let Γ be the carré du champ operator associated to a Dirichlet form $(E, \mathbb{D}(E))$ on $\mathbb{L}^2(X, \nu)$. Denote $\mathbb{D}_{loc}(E)$, the set of all $u \in \mathbb{L}^2(X, \nu)$ such that for all relatively compact open set V, we can find $u_V \in \mathbb{D}(E)$ such that $u = u_V$ a.e on V. By strong locality, we may extend the domain of Γ to $\mathbb{D}_{loc}(E)$ by setting $\Gamma(u,u) = \Gamma(u_V, u_V)$ on V. Denote also $\mathbb{C}(X)$ the space of all continuous functions on X and $\mathbb{C}_c(X)$ the space of continuous functions with compact support on X. The *intrinsic distance* ρ on X is defined via

$$\rho(x,y) \stackrel{\mathrm{df}}{=} \sup \left\{ |u(x) - u(y)| \ : \ u \in \mathbb{D}(E) \cap \mathbb{C}_c(X) \text{ and } \Gamma(u,u) \leq 1 \ a.e. \right\}$$

[1] It has been proved that weight functions of weighted doubling measures are equivalent to (lower or upper) semi-continuous functions[10, Theorem 3.6].

for all $x, y \in X$. *A common assumption, which we will assume in this work, is that ρ is a complete metric and metrises the topology of X.* Under this assumption, (X, ρ) is a geodesic space[8, 4], i.e., any two points in X can be joined by a curve with length exactly equal to the distance between those two points. Also,

$$\rho(x, y) = \sup \{|u(x) - u(y)| \ : \ u \in \mathbb{D}_{loc}(E) \cap \mathbb{C}(X) \text{ and } \Gamma(u, u) \leq 1 \ a.e.\}$$

for all $x, y \in X$.

A non-zero σ-finite measure μ on the metric space (X, ρ) is said to be *doubling* if there is a constant $C_D \geq 1$ such that

$$\mu(B(x, 2r)) \leq C_D \mu(B(x, r))$$

for all $x \in X$ and $r > 0$. Examples of doubling measures include Lebesgue measure on Euclidean spaces, Haar measures on Lie groups and volume measures on Riemannian manifolds with nonnegative Ricci curvature. It can be proved that any doubling measure has full support.

Strong locality and regularity of $(E, \mathbb{D}(E))$ implies that the set of Lipchitz function with compact support, $\mathbb{L}\text{ip}_c(X, \rho)$, is a subset of $\mathbb{D}(E)$ and the set of Lipchitz function, $\mathbb{L}\text{ip}(X, \rho)$, is a subset of $\mathbb{D}_{loc}(E)$. Moreover, $\Gamma(u, u) \leq l^2$ a.e. for all $u \in \mathbb{L}\text{ip}(X, \nu)$ with Lipchitz constant l. Since $\mathbb{L}\text{ip}_c(X, \rho)$ is dense in $\mathbb{C}_c(X)$, proving $\mathbb{L}\text{ip}_c(X, \rho)$ is a subset of a Dirichlet form's domain also guarantee its regularity.

It is well-known that there is a one-one correspondence between a Dirichlet form $(E, \mathbb{D}(E))$ and a *heat operator*, i.e., a nonpositive, densely defined, self-adjoint operator $(L, \mathbb{D}(L))$ on $\mathbb{L}^2(X, \nu)$ so that

(DHC1) the domain $\mathbb{D}(L)$ of L dense in $\mathbb{D}(E)$ under the Dirichlet norm, and

(DHC2) $E(u, v) = -\langle u, Lv \rangle_{\mathbb{L}^2(X,\nu)}$ for all $u, v \in \mathbb{D}(L)$.

With this correspondence, a (local) weak solution u of the heat equation $\partial_t u = -Lu$ corresponding to L can be defined. For the details, please see [3].

A Dirichlet form $(E, \mathbb{D}(E))$ associated to a heat operator $(L, \mathbb{D}(L))$ on $\mathbb{L}^2(X, \nu)$ is said to satisfies *(uniform) parabolic Harnack inequality*, or *PHI* for short, if there exists a constant $c \geq 1$ such that for $x \in X, r > 0$ and any nonnegative weak solution u of the heat equation $\partial_t u = Lu$ on $(0, r^2) \times B(x, r)$,

$$\sup_{Q^-} u \leq c \inf_{Q^+} u$$

where $Q^- = (r^2/4, r^2/2) \times B(x, r/2)$, $Q^+ = (3r^2/4, r^2) \times B(x, r/2)$, and both supremum and infimum are computed up to measure zero.

One consequence of PHI is that all solutions of the heat equation have continuous representations. Hence, we may as well assume that they are continuous. For further information on Harnack inequality, see [5].

The heat kernel associated to a heat operator L is a function $p : (0, \infty) \times X \times X$ such that

$$u(t,x) = \int u(0,y)p(t,x,y)\mathrm{d}\nu(y) \quad \forall t > 0, X \in X$$

for all solutions u of the heat equation associated to L.

A heat kernel p is said to have *Gaussian upper bound* if there exist constants $c_1, c_2 > 0$ such that

$$p(t,x,y) \leq \frac{c_1}{\sqrt{\nu(B(x,\sqrt{t}))\nu(B(y,\sqrt{t}))}} e^{-\frac{\rho^2(x,y)}{c_2 t}}$$

It is is said to have *Gaussian lower bound* if there exist constants $c_1, c_2 > 0$ such that instead

$$p(t,x,y) \geq \frac{c_1}{\sqrt{\nu(B(x,\sqrt{t}))\nu(B(y,\sqrt{t}))}} e^{-\frac{\rho^2(x,y)}{c_2 t}}$$

It is said to have *Gaussian behavior* or *Gaussian estimates* if it satisfies both Gaussian upper bound and Gaussian lower bound. A heat operator or a Dirichlet form is said to have *Gaussian behavior* if its associated heat kernel exists and has Gaussian behavior.

It turn out that PHI and Gaussian behavior are equivalent. Moreover, there are equivalent to doubling property and Poincaré inequality defined below.

A Dirichlet form $(E, \mathbb{D}(E))$ on $\mathbb{L}^2(X, \nu)$ with associated carré du champ operator Γ is said to satisfies *(weak) Poincaré inequality* if for some constant $k \geq 1$, there is a constant $C_P > 0$ such that for all $x \in X$, $r > 0$, and $u \in \mathbb{D}(E)$,

$$\inf_{\xi \in \mathbb{R}} \int_{B(x,r)} (u - \xi)^2 \mathrm{d}\nu \leq C_P r^2 \int_{B(x,kr)} \Gamma(u,u)\mathrm{d}\nu$$

If $k = 1$, then $(E, \mathbb{D}(E))$ is said to satisfies *Poincaré inequality.* Under doubling property, the weak Poincaré inequality and Poincaré inequality are equivalent.

Theorem 2.1 (Sturm 1995)**.** *Let $(E, \mathbb{D}(E))$ be a strongly local, regular Dirichlet form on $\mathbb{L}^2(X, \nu)$. Then the following are equivalent.*

(a) $(E, \mathbb{D}(E))$ satisfies parabolic Harnack inequality.

(b) $(E, \mathbb{D}(E))$ has Gaussian behavior.

(c) $(E, \mathbb{D}(E))$ satisfies the Poincaré inequality and ν is doubing.

The strategies used in this work is then to show that the weighted measure is doubling under the conformal metric and the conformal Dirichlet space satisfies the Poincaré inequality.

The proof of doubling property shall be based on the volume comparison condition introduce by Li and Tam[6] using the result of Grigor'yan and Saloff-Coste[1].

Let $0 < \delta \leq 1$ and A be a closed subset of (X, ρ). A Borel measure ν on X is said to satisfies *δ-volume comparison condition on A for R-small balls*[2] if there are constants $C_\nu > 0$ and $\epsilon \in (0, 1]$ such that for any $x \in X$ and $a \in A$,

$$\nu(B(a, \rho(a, x))) \leq C_\nu \nu(B(x, \epsilon\delta\rho(a, x)))$$

provided that $R > \rho(x, A) \geq \delta\rho(x, a)$. The Borel measure ν is said to be *doubling for A-remote R-small balls* if there is a constant $C_\nu > 0$ such that for any $x \in X$,

$$\nu(B(x, r) \leq C_\nu \nu(B(x, \frac{1}{2}r)$$

provided that $r \leq \delta\rho(x, A) < \delta R$. Lastly, the Borel measure ν is said to be *doubling for A-anchored R-small balls* if there is a constant $C_\nu > 0$ such that for any $a \in A$ and $r > 0$,

$$\nu(B(a, r) \leq C_\nu \nu(B(a, \frac{1}{2}r)$$

provided that $r < R$. If $A = X$, then we simply say ν is *doubling for R-small balls*

It turn out that the specific values of δ and ϵ are unimportant in the definition of volume comparison condition and doubling property for remote balls – it can be proved that these results hold for specific $\delta, \epsilon \in (0, 1]$ if and only if the same result holds for all $\delta, \epsilon \in (0, 1]$. Moreover, these concepts are related as follow.

Theorem 2.2. *[1, Lemma 4.4] Fix a closed subset A of (X, d) and $R > 0$. If a Borel measure on (X, d) is doubling for A-remote $4R$-small balls and A-anchored $4R$-small balls and it also satisfies volume comparison condition on A for $4R$-small balls, then ν is doubling for $\hat{A}$-anchored R-small balls where $\hat{A} = \{x \in X \ : \ d(x, A) \leq R\}$.*

Note that Grigor'yan and Saloff-Coste[1] only prove the result for $R = \infty$ but general result can be proved using exactly the same argument.

The proof of the Poincaré inequality will be based on a simple application of Corollary 3.2 in [9] to the case $R = \infty$. Henceforth, denote

$$S(z, r) = S_\lambda(z, r; Z) \stackrel{\text{df}}{=} \{x \in Z : \lambda r \leq \rho(x, Z) \leq \rho(x, z) \leq r\},$$

$C(z, r) = C_\lambda(z, r; Z) \stackrel{\text{df}}{=} \cup_{r \leq s \geq 0} S(z, s)$, and $C(z) \stackrel{\text{df}}{=} C(z, \infty)$ for all $z \in Z$. A subset $Z \subseteq X$ is said to satisfy *λ-skew condition* where $\lambda \in (0, 1]$ if $S_{\lambda, Z}(z, r) \neq \emptyset$ for all $z \in Z$ and $r > 0$. A subset $Z \subseteq X$ is said to be *(λ, Λ)-accessible* where $0 < \lambda < \Lambda \leq 1$ if Z satisfies Λ-skew condition and there is a constant $c_\Lambda \geq 1$ such that all points in $C_\lambda(z, r)$ can be connected to z via a path in $C_\lambda(z, c_\lambda r)$.

A Dirichlet form $(E, \mathbb{D}(E))$ on $\mathbb{L}^2(X, \nu)$ with associated carré du champ operator Γ is said to satisfies *(weak) Poincaré inequality for Z-remote balls* if

[2] The statement given here is stronger than that defined in [1]. Therefore, it will not effect the validity of Theorem 2.2.

for some constants $k \geq 1 \geq \epsilon > 0$, there is a constant $C_P > 0$ such that

$$\inf_{\xi \in \mathbb{R}} \int_{B(x,r)} (u - \xi)^2 \mathrm{d}\nu \leq C_P r^2 \int_{B(x,kr)} \Gamma(u,u) \mathrm{d}\nu$$

for all $x \in X$, $r \leq \epsilon\rho(x, Z)$, and $u \in \mathbb{D}(E)$.

Theorem 2.3. *[9, Corollary 3.2] Let Γ be the carré du champ operator associated to a Dirichlet form $(E, \mathbb{D}(E))$ on $\mathbb{L}^2(X, \nu)$ in which the measure ν is doubling. Let also Z be an acessible subset of X. Assume that the Poincaré inequality holds for Z-remote balls and there are constants $C_P > 0$ and $\delta \in (0, 1)$ such that*

$$\fint_{B(x,\delta r)} \fint_{B(y,\delta r)} (u(v) - u(w))^2 \mathrm{d}\nu(v)\mathrm{d}\nu(w) \leq C_P r^2 \fint_{B(z,r)} \Gamma(u,u)\mathrm{d}\nu \qquad (1)$$

for all $x, y \in S(z, r)$ and $u \in \mathbb{D}(E)$. Then $(E, \mathbb{D}(E))$ satisfies the Poincaré inequality for all balls.

We end this section by the discussion of polynomial growth functions in the asymptotically Z-normal direction.

Definition 2.4. Fix $\delta \in (0, 1)$ and $\alpha \geq \beta$. A function $h : X \to [0, \infty]$ on a geodesic space (X, ρ) is said to have (α, β)-polynomial growth in the (δ-asymptotically) Z-normal direction if there is a constant $c_h > 0$ such that

$$c_h^{-1} \left(\frac{\rho(x, Z)}{\rho(y, Z)} \right)^{\beta} \leq \frac{h(x)}{h(y)} \leq c_h \left(\frac{\rho(x, Z)}{\rho(y, Z)} \right)^{\alpha} \qquad (2)$$

for all $x, y \in C(z)$ in which $\rho(y, z) \leq \rho(x, z)$ where $z \in Z$ and that $h(x)$ is finite and nonzero for some $x \in X$.

Note that the exact value of δ is the above definition is not important. It can be proved that a function has polynomial growth in an δ-asymptotically Z-normal direction for some $\delta \in (0, 1)$ if and only if that function has the same property for all $\delta \in (0, 1)$. Moreover, α and β remains invariant under this fact – the only constant changed after δ is c_h.

This class of functions might seem to be limited. It is, however, closed under multiplication, division, finite maximum and minimum, as well as addition. Moreover, all weighted doubling measure is a limit of weighted doubling measures with this property[10, Proposition 3.3]. Since doubling property is a neccessary condition for the Harnack inequality, assuming that a function satisfies equation 2 should be considered mildly.

3 Main Results

The statement of main reults will be based on the following conditions.

(Rh) There is a constant $C_h > 0$ such that $\rho^h(x,z) \le C_h h(x)\rho(x,z)$ whenever $x \in C(z), z \in Z$.

(WP) There is a constant $P > 0$ such that

$$\fint_{B(x,\delta r)} \fint_{B(y,\delta r)} (u(v) - u(w))^2 \mathrm{d}\nu(v)\mathrm{d}\nu(w) \le Pr^2 h(x)h(y) \fint_{B(z,r)} h^{-2}\Gamma(u,u)\mathrm{d}\nu \tag{3}$$

for all $x, y \in S(z,r)$ and $u \in \mathbb{D}(E)$.

Further details on these conditions will be given in the next section. It turn out that both (Rh) and (WP) are neccessary conditions for the Harnack inequality on $(E_h, \mathbb{D}(E_h))$.

Theorem 3.1 (Main Theorem)**.** *Let $(E, \mathbb{D}(E))$ be a strongly local regular Dirichlet form on $\mathbb{L}^2(X,\nu)$ with the associated carre du champ Γ, and that h^{-2} is locally integrable with discrete singulairity. Denote*

$$E_h(u,v) \stackrel{df}{=} \int h^{-2}\Gamma(u,v)\mathrm{d}\nu \tag{4}$$

whenever defined. Then the above formula defined a strongly local regular Dirichlet form on $\mathbb{L}^2(X,\nu)$.

Assume further that h have polynomial growth in the Z-normal direction where Z is a discrete accessible subset of X, and $(E, \mathbb{D}(E))$ satisfies Harnack inequality. Then, $(E_h, \mathbb{D}(E_h))$ satisfies Harnack inequality if and only if both (Rh) and (WP) hold.

4 Quasi-Conformal Dirichlet Forms

In this section, fixed a Dirichlet form $(E, \mathbb{D}(E))$ on $\mathbb{L}^2(X,\nu)$ and denote Γ its carré du champ operator. For each measurable function $h : X \to [0,\infty]$, denote

$$E_h(u,v) \stackrel{\text{df}}{=} \int h^{-2}\Gamma(u,v)\mathrm{d}\nu$$

for all $u, v \in \mathbb{D}_{loc}(E)$ whenever defined and denote $\mathbb{D}(E_h)$ the set of all $u \in \mathbb{L}^2(X,\nu) \cap \mathbb{D}_{loc}(E)$ such that $E_h(u,u) < \infty$. The function h will be called the *(quasi-conformal) density* of $(E_h, \mathbb{D}(E_h))$.

Definition 4.1. If $(E_h, \mathbb{D}(E_h))$ is a Dirichlet form on $\mathbb{L}^2(X,\nu)$, then it is called a *quasi-conformal Dirichlet form* of $(E, \mathbb{D}(E))$. Its associated carré du champ operator $h^{-2}\Gamma$ will be denoted by Γ_h.

We will show in this section that the following condition is sufficient to show that a function h is a quasi-conformal density.

(LI) the function h^{-2} is locally integrable.

The condition (LI) is a necessity. For $(E_h, \mathbb{D}(E_h))$ to be densely defined, h^{-2} must be locally integrable.

Lemma 4.2. *For any function $h : X \to [0, \infty]$ satisfying (LI), $\mathbb{L}\mathrm{ip}_c(X, \rho) \subseteq \mathbb{D}(E_h)$ and hence the set $\mathbb{D}(E_h)$ is dense in $\mathbb{L}^2(X, \nu)$.*

Proof. Let $u \in \mathbb{L}\mathrm{ip}_c(X, \rho)$ with Lipchitz constant l and the support of u is a subset of some relatively compact open set V. Then

$$\begin{aligned} E_h(u, u) &= \int h^{-2}\Gamma(u, u) \\ &\leq l^2 \int_V h^{-2} \mathrm{d}\nu \\ &< \infty \end{aligned}$$

Therefore, $u \in \mathbb{D}(E_h)$. □

As a consequence of the above lemma, $(E_h, \mathbb{D}(E_h))$ is a closed symmetric bilinear form on $\mathbb{L}^2(X, \nu)$. Strong locality of $(E, \mathbb{D}(E)$ implies that

$$\Gamma((u \vee 0) \wedge 1, (u \vee 0) \wedge 1) \leq \Gamma(u, u) \text{ a.e.}$$

for all $u \in \mathbb{D}_{loc}(E)$. Thus, $(E_h, \mathbb{D}(E_h))$ is a Dirichlet form on $\mathbb{L}^2(X, \nu)$. Moreover, strong locality of $(E_h, \mathbb{D}(E_h))$ immediately follows. If we can show that its intrinsic metric ρ_h is also complete and metrises the original topology of X, then $\mathbb{L}\mathrm{ip}_c(X, \rho_h) \subseteq \mathbb{D}(E_h)$ and hence $(E_h, \mathbb{D}(E_h))$ must also be regular.

From now on, to differentiate the ball under ρ_h from the ball under ρ, we will use the notation $B_h(x, r)$ for balls under ρ_h while reserve $B(x, r)$ for balls under ρ.

Lemma 4.3. *For any upper semi-continuous function $h : X \to [0, \infty]$ satisfying (LI) and (DS), ρ_h is a metric metrises the original topology on $X - Z$. Moreover,*

$$\rho_h(x, y) = \inf \left\{ \int_0^1 h(\gamma(t))|\gamma|'(t)\mathrm{d}t : \gamma \textit{ is a path connecting } x = \gamma(0), y = \gamma(1) \right\}$$

for all $x, y \in X$.

Proof. Using the fact that any upper semi-continuous function is a monotone limit of a sequence of continuous functions, we may assume that h is continuous. Denote

$$d(x, y) = \inf \left\{ \int_0^1 h(\gamma(t))|\gamma|'(t)\mathrm{d}t : \gamma \text{ is a path connecting } x = \gamma(0), y = \gamma(1) \right\}$$

for all $x, y \in X - Z$. Since h is finite and nonzero on $X - Z$, d is a metric on $X - Z$.

Let $\epsilon > 0$. For each $x \in X - Z$,

$$\sup_{B(x, r_x)} h \leq (1 + \epsilon) \inf_{B(x, r_x)} h$$

for some $r_x = r_x(\epsilon) > 0$. This follows from the continuity of h. Denote $M_x = \sup_{B(x,r_x)} h^{-1}$ and $m_x = \inf_{B(x,r_x)} h^{-1}$. For any $y, z \in B(x, r_x)$,

$$\begin{aligned}
&\{u \in \mathbb{D}(E) \cap \mathbb{C}_c(X) \ : \ \Gamma(M_x u, M_x u) \leq 1 \ a.e.\} \\
&= \left\{u \in \mathbb{D}(E) \cap \mathbb{C}_c(X) \ : \ M_x^2 \Gamma(u,u) \leq 1 \ a.e.\right\} \\
&\subseteq \left\{u \in \mathbb{D}(E) \cap \mathbb{C}_c(X) \ : \ h^{-2} \Gamma(u,u) \leq 1 \ a.e.\right\} \\
&\subseteq \left\{u \in \mathbb{D}(E) \cap \mathbb{C}_c(X) \ : \ m_x^2 \Gamma(u,u) \leq 1 \ a.e.\right\} \\
&= \{u \in \mathbb{D}(E) \cap \mathbb{C}_c(X) \ : \ \Gamma(m_x u, m_x u) \leq 1 \ a.e.\}
\end{aligned}$$

Thus, $\frac{1}{M_x}\rho(y,z) \leq \rho_h(y,z) \leq \frac{1}{m_x}\rho(y,z) \leq \frac{1+\epsilon}{M_x}\rho(y,z)$ for all $y, z \in B(x, r_x)$. However, $\frac{1}{M_x}\rho(y,z) \leq d(y,z) \leq \frac{1+\epsilon}{M_x}\rho(y,z)$ for all $y, z \in B(x, r_x)$ too. Therefore,

$$(1+\epsilon)^{-1} d(y,z) \leq \rho_h(y,z) \leq (1+\epsilon) d(y,z)$$

for all $y, z \in B(x, r_x)$. This particularly implies that both ρ_h and d are metric metrise the original topology on $X - Z$.

Next, we show that $\rho_h = d$. Let $\gamma : [0,1] \to X$ be a geodesic such that $\gamma([0,1)) \cap Z = \emptyset$. Then there is a partition $0 = t_0 < t_1 < \ldots$ of $[0,1)$ such that the image of γ on $[t_i, t_{i+1}]$ is a subset of some $B(x, r_x)$. It follows that

$$\begin{aligned}
\rho_h(\gamma(0), \gamma(1)) &\leq \sum_i \rho_h(\gamma(t_i), \gamma(t_{i+1})) \\
&\leq (1+\epsilon) \sum_i d(\gamma(t_i), \gamma(t_{i+1})) \\
&= (1+\epsilon) d(\gamma(0), \gamma(1))
\end{aligned}$$

Since Z is discrete, this inequality can be extended to all geodesic γ. Letting $\epsilon \to 0$ yields $\rho_h \leq d$.

For the converse inequality, denote $d_x = d(x, \cdot)$ where $x \in X$. For each $y \in X - Z$ and $z \in B(y, r_y)$

$$\begin{aligned}
\limsup_{w \to z} \frac{|d_x(z) - d_x(w)|}{\rho(z,w)} &\leq \limsup_{w \to z} \frac{|d(z,w)|}{\rho(z,w)} \\
&\leq \limsup_{w \to z} \frac{(1+\epsilon)\rho(z,w)}{M_y \rho(z,w)} \\
&\leq \frac{1+\epsilon}{M_y}
\end{aligned}$$

This implies the Lipchitz constant of d_x is at most $\frac{1+\epsilon}{M_y}$ on $B(y, r_y)$. By strong locality, $\Gamma(d_x, d_x)(z) \leq \frac{(1+\epsilon)^2}{M_y^2} \leq (1+\epsilon)^2 h^2(z)$ a.e. $z \in B(y, r_y)$. Therefore,

$$\Gamma_h((1+\epsilon)^{-1} d_x, (1+\epsilon)^{-1} d_x) \leq 1 \ a.e.$$

By definition of ρ_h, $\rho_h(x,y) \geq (1+\epsilon)^{-1} d_x(y)$. Let $\epsilon \to 0$ yields $\rho_h \geq d$.

$\square$

It is worth mention that the above representation of ρ_h only depends on h and ρ with out explicitly involves $(E, \mathbb{D}(E))$. Thus, it is possible to directly defined $(\rho_h)^{1/h}$ via this formula.

Corollary 4.4. *For any upper semi-continuous function $h : X \to [0, \infty]$ satisfying (LI) and having discrete singularity,, $\rho = (\rho_h)^{1/h}$ on $X - Z$, i.e.,*

$$\rho(x, y) = \inf \left\{ \int_0^1 \frac{1}{h(\gamma(t))} |\gamma|'_h(t) \mathrm{d}t : \gamma \textit{ is a path connecting } x = \gamma(0), y = \gamma(1) \right\}$$

for all $x, y \in X - Z$.

Lemma 4.5. *For any upper semi-continuous function $h : X \to [0, \infty]$ satisfying (LI) and having discrete singularity, the set Z is discrete with respected to ρ_h. Moreover,*

$$b_Z \overset{df}{=} \min_{z \in Z} \rho_h(z, Z - \{z\}) > 0.$$

Proof. Let $\gamma : [0, 1] \to X$ be a path connecting two different points in Z. Since $\epsilon_h < \frac{1}{2} a_h$, there is $t \in (0, 1)$ such that $\rho(\gamma(t), Z) \geq \epsilon_h$. Then

$$\int_0^1 h(\gamma(t)|\gamma|'(t))\mathrm{d}t \geq \int_{\{t \,:\, \rho(\gamma(t), S) \geq \epsilon_h\}} h(\gamma(t)|\gamma|'(t))\mathrm{d}t \geq \underline{h}(a_h - 2\epsilon_h).$$

Thus, $b_Z \geq \underline{h}(a_h - 2\epsilon_h) > 0$. □

Lemma 4.6. *For any upper semi-continuous function $h : X \to [0, \infty]$ satisfying (LI), and having discrete singularity, its intrinsic metric ρ_h is a complete metric on X. Particularly, ρ_h metrises the original topology on X.*

Proof. Let (x_k) be a Cauchy sequence in (X, ρ_h). Denote $d = \liminf_{k \to \infty} \rho_h(x_k, Z)$.

Case I: $d > 0$.

Since (x_k) is Cauchy, there is a l such that $\rho_h(x_l, Z) > \frac{1}{2} d$ and $\rho_h(x_m, x_n) \leq \frac{1}{4} d$ for all $m, n \geq l$. It follows that $\rho_h(x_m, Z) > \frac{1}{4} d$ for all $m \geq l$. Since $B_h(x_l, \frac{1}{2} d) \cap Z = \emptyset$, $m \overset{\text{df}}{=} \inf_{x \in B_h(x_l, \frac{1}{4} d)} h(x) > 0$. Now, $\rho(x_n, x_m) \leq \frac{1}{m} \rho_h(x_m, x_n)$ for all $m, n \geq l$ implies (x_l) is Cauchy under ρ. Thus, (x_k) must converge to a point $x \in X - Z$ under ρ. Since ρ_h and ρ metrises the same topology on $X - Z$, (x_k) converge to x under ρ_h also.

Case II: $d = 0$.

This means we can construct, recursively, a subsequence (y_k) of (x_k) such that $\rho_h(y_1, Z) \leq \frac{1}{8} b_Z$ and $\rho_h(y_{k+1}, Z) \leq \frac{1}{2} \rho_h(y_k, Z)$ for all k. Since (y_k) is Cauchy, there is a k such that $\rho_h(y_m, y_n) \leq \frac{1}{8} b_Z$ for all $m, n \geq k$. Let $z \in Z$ be such that $\rho_h(z, y_k) = \rho_h(y_k, Z)$. Then $\rho_h(y_m, z) \leq \frac{1}{4} b_Z$ which implies $\rho_h(z, y_m) = \rho_h(y_m, Z)$ for all $m \geq k$ too. Clearly, (y_k) converge to z which implies (x_k) converges as well. □

Combining all the above results and we have the following.

Proposition 4.7. *For any $h : X \to [0, \infty]$ satisfies (LI), and having discrete singularity, $(E_h, \mathbb{D}(E_h))$ defined a strongly local regular Dirichlet form on $\mathbb{L}^2(X, \nu)$. Moreover, its intrinsic metric ρ_h is complete and metrises the original topology of X.*

5 Characterization of functions with polynomial growth on some normal directions

Henceforth, we write $f(x) \lesssim g(x)$, or $f \lesssim g$ for short, if there is a constant c independent of x in the common domain of f and g such that $f(x) \leq cg(x)$, and write $f(x) \sim g(x)$, or $f \sim g$ for short, if $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$.

To simplify the arguments in the rest of this work, we will consider the implication of assuming functions to have polynomial growth in some normal direction.

Proposition 5.1. *Assume that $h : X \to [0, \infty]$ has (α, β)-polynomial growth function in the Z-normal direction. Then there is $c = c(\beta, c_h) > 0$ such that for any $\epsilon \in (0, 1/2]$,*

$$\sup_{B(x,\epsilon\rho(x,Z))} h \leq c \inf_{B(x,\epsilon\rho(x,Z))} h$$

for all $x \in X$.

Proof. Let $y \in B(x, \epsilon\rho(x, Z))$. Then $\rho(y, Z) \leq \rho(y, x) + \rho(x, Z) \leq (1 + \epsilon)\rho(x, Z)$ and $\rho(y, Z) \geq \rho(x, Z) - \rho(x, y) \geq (1 - \epsilon)\rho(x, Z)$. Thus,

$$c_h^{-1} \left(\frac{1-\epsilon}{1+\epsilon}\right)^\beta \leq \frac{h(y_1)}{h(y_2)}$$

for all $y_1, y_2 \in B(x, \epsilon\rho(x, Z))$. Therefore, we may choose $c = c_h \max_{0<\epsilon\leq 1/2} \left(\frac{1+\epsilon}{1-\epsilon}\right)^\beta = c_h 3^\beta$. □

Proposition 5.2. *Assume that $h : X \to [0, \infty]$ has (α, β)-polynomial growth function in the Z-normal direction. Then there is a function $a_h : [0, \infty) \to [0, \infty]$ having (α, β)-polynomial growth function in the $\{0\}$-normal direction and $a_h(\rho(x, Z)) \sim h(x)$.*

Proof. Define $a(r) = a_h(r) = \sup_{r\leq\rho(x,Z)\leq 2r} h(x)$. Obviously, $a(\rho(\cdot, Z)) \geq h$. Then use the fact that $\sup_{r\leq\rho(x,Z)\leq 2r} h(x) \leq 2^\alpha c_h \inf_{r\leq\rho(x,Z)\leq 2r} h(x)$ to conclude that $a(\rho(x, Z)) \leq 2^\alpha c_h h(x)$ for all $x \in X$.

For any $s, t > 0$,

$$\begin{aligned}\frac{a(s)}{a(t)} &= \frac{\sup_{s\leq\rho(x,Z)\leq 2s} h(x)}{\sup_{t\leq\rho(x,Z)\leq 2t} h(x)} \\ &\leq c_h \left(\frac{2s}{t}\right)^\alpha \\ &\leq (2^\alpha c_h) \left(\frac{s}{t}\right)^\alpha.\end{aligned}$$

Similarly, $\frac{a(s)}{a(t)} \geq (2^\beta c_h)^{-1} \left(\frac{s}{t}\right)^\beta \geq (2^\alpha c_h)^{-1} \left(\frac{s}{t}\right)^\beta$.

Last, a is not the zero function, otherwise, h would be too. Similarly, a is not infinite everywhere. □

The combination of the above two propositions particularly implies that

$$\sup_{r\le s\le 3r} a_h(s) \le c_a \inf_{r\le s\le 3r} a_h(s), \quad \forall r>0$$

for some fixed constant $c_a > 0$.

The function a_h in the above theorem allow us to simplify many arguments and conditions. For example, the condition (Rh) is equivalent to the following condition.

(Ra) There is a constant $C_a > 0$ such that $\int_0^r a_h(t)\mathrm{d}t \le C_a a_h(r) r$ for all $r \ge 0$.

The proof of this fact will be relying on the following fact.

Lemma 5.3. *Assume that $h : X \to [0,\infty]$ has (α,β)-polynomial growth function in the Z-normal direction. Then $\rho_h(\cdot, Z) \sim \int_0^{\rho(\cdot,Z)} a_h(t)\mathrm{d}t$.*

Proof. For each pair of $x \in X$ and $z \in Z$ such that $\rho(x,z) = \rho(x,Z)$, let γ be a normalized geodesic connecting x and z. Then

$$\begin{aligned}
\rho_h(x,Z) &\le \rho_h(x,z) \\
&\le \int_0^1 h(\gamma(t))|\gamma|'(t)\mathrm{d}t \\
&= \rho(x,z)\int_0^1 h(\gamma(t))\mathrm{d}t \\
&\lesssim \rho(x,Z)\int_0^1 a_h(\rho(\gamma(t),Z))\mathrm{d}t \\
&= \rho(x,Z)\int_0^1 a_h(t\rho(x,Z))\mathrm{d}t \\
&= \int_0^{\rho(x,Z)} a_h(t)\mathrm{d}t
\end{aligned}$$

Next, let γ be instead a ρ_h-geodesic connecting x and z for which $\rho_h(x,z) = \rho_h(x,Z)$. It follows that $\rho_h(\gamma(t),z) = \rho_h(\gamma(t),Z)$ for all t as well. Set $t_0 = 0$ and choose recursively t_{k+1} to be the first $t > t_k$ such that $\rho(\gamma(t),Z) \ge \rho(\gamma(t_k),Z)/2$.

Now

$$\begin{aligned}
\rho_h(x,Z) &= \sum_{k=0}^{\infty}\int_{t_{k+1}}^{t_k} h(\gamma(t))|\gamma|'(t)\mathrm{d}t \\
&\gtrsim \sum_{k=0}^{\infty} h(\gamma(t_k))\int_{t_{k+1}}^{t_k} |\gamma|'(t)\mathrm{d}t \\
&\geq \sum_{k=0}^{\infty} h(\gamma(t_k))\rho(\gamma(t_k),\gamma(t_{k+1})) \\
&\gtrsim \sum_{k=0}^{\infty} a_h(\rho(\gamma(t_k),Z))|\rho(\gamma(t_k),Z)-\rho(\gamma(t_{k+1},Z))| \\
&\sim \sum_{k=0}^{\infty} a_h(\rho(\gamma(t_k),Z))\int_{\rho(\gamma(t_{k+1}),Z)}^{\rho(\gamma(t_k),Z)} \mathrm{d}t \\
&\sim \sum_{k=0}^{\infty}\int_{\rho(\gamma(t_{k+1}),Z)}^{\rho(\gamma(t_k),Z)} a_h(t)\mathrm{d}t \\
&= \int_0^{\rho(x,Z)} a_h(t)\mathrm{d}t
\end{aligned}$$

Therefore, $\rho_h(\cdot,Z) \sim \int_0^{\rho(\cdot,Z)} a_h(t)\mathrm{d}t$. □

Proposition 5.4. *The condition (Rh) is equivalent to the condition (Ra).*

Proof. This follows from the fact that $\rho_h(\cdot,Z) \sim \int_0^{\rho(\cdot,Z)} a_h(t)\mathrm{d}t$ and $h \sim a_h(\rho(\cdot,Z))$. □

Lemma 5.5. *Assume that $h : X \to [0,\infty]$ has (α,β)-polynomial growth function in the Z-normal direction. Then there is $c_1 = c_1(c_h,\beta,\delta_Z,C_\rho) > 0$ and $c_2 = c_2(c_h,\beta,\delta_Z,C_\rho) > 0$ such that*

$$\rho_h(x,Z) \geq \max\left(c_1 h(x)\rho(x,Z), c_2 a_h(\rho(x,Z))\rho(x,Z)\right)$$

for all $x \in X$.

Proof. Let γ be a ρ_h-geodesic connecting x and $z \in Z$. Then

$$\begin{aligned}
\rho_h(x,z) &\geq \int_{\{\gamma\in B(x,\rho(x,Z)/2)\}} h(\gamma(t))|\gamma|'(t)\mathrm{d}t \\
&\geq c_h^{-1}3^{-\beta}h(x)\int_{\{\gamma\in B(x,\rho(x,Z)/2)\}} |\gamma|'(t)\mathrm{d}t \\
&\geq \frac{1}{2}c_h^{-1}3^{-\beta}h(x)\rho(x,Z)
\end{aligned}$$

The rest follows from Proposition 5.2. □

Lemma 5.6. *Assume that $h : X \to [0,\infty]$ has (α,β)-polynomial growth function in the Z-normal direction and (Rh) holds. There is $c = c(\delta_Z, C_h, c_h, \beta) \geq 1$ such that for any $\epsilon \in (0, 1/2]$,*

$$B_h(x, \epsilon c^{-1}\rho_h(x,Z)) \subseteq B(x, \epsilon\rho(x,Z)) \subseteq B_h(x, \epsilon c\rho_h(x,Z))$$

for all $x \in X$.

Proof. Assume that $y \in B(x, \epsilon\rho(x,Z))$ and γ is a ρ-geodesic connecting x and y. Then

$$\begin{aligned} \rho_h(x,y) &\leq \int_0^1 h(\gamma(t))|\gamma|'(t)\mathrm{d}t \\ &\leq c_h 3^\beta h(x) \int_0^1 |\gamma|'(t)\mathrm{d}t \\ &\leq c_h 3^\beta h(x)\rho(x,y) \\ &\leq \epsilon c_h 3^\beta h(x)\rho(x,Z) \\ &\leq \epsilon C_h c_h 3^\beta \rho_h(x,Z). \end{aligned}$$

Next, assume that $y \notin B(x, \epsilon\rho(x,Z))$ and γ is a ρ_h-geodesic connecting x and y. Then

$$\begin{aligned} \rho_h(x,y) &= \int_0^1 h(\gamma(t))|\gamma|'(t)\mathrm{d}t \\ &\geq c_h^{-1} 3^{-\beta} h(x) \int_{\{\gamma \in B(x,\epsilon\rho(x,Z))\}} |\gamma|'(t)\mathrm{d}t \\ &\geq \epsilon c_h^{-1} 3^{-\beta} h(x)\rho(x,Z) \\ &\geq \epsilon C_h^{-1} c_h^{-1} 3^{-\beta} \rho_h(x,Z). \end{aligned}$$

This completes the proof. □

6 Proof of the Main Theorem

First, we verify the neccessity of (Rh) and (Ra).

Lemma 6.1. *Let ν be a doulbing measure on a geodesic space (X,ρ). If ν is also doubling under ρ_h where h has (α,β)-polynomial growth in the Z-normal direction, then (Ra) holds.*

Proof. Since $0 < a_h(t) < \infty$ for all $t \neq 0$, $r \mapsto \int_0^r a_h(t)\mathrm{d}t$ is strictly increasing and hence there is an bijection $A : [0, \int_0^\infty a_h(t)\mathrm{d}t) \to [0,\infty)$ for which $\int_0^{A(r)} a_h(t)\mathrm{d}t = r$ for all $r \geq 0$. Clearly, A is also increasing.

Let $c \geq 1$ be a constant for which $c^{-1} \int_0^{\rho(\cdot,Z)} a_h(t)\mathrm{d}t \leq \rho_h(\cdot, Z) \leq c \int_0^{\rho(\cdot,Z)} a_h(t)\mathrm{d}t$. Then $A(c^{-1}\rho_h(\cdot, Z)) \leq \rho(\cdot, Z) \leq A(c\rho_h(\cdot, Z))$.

Let $z \in Z, r > 0$, and $x \in S(z, A(r))$. Then $\rho_h(x, z) \leq c'r$.*(This has to be proved)* Since ν is doubling under ρ_h, $\nu(B_h(z, r)) \sim \nu(B_h(x, r))$. Using the fact that ν is doubling under both ρ_h and ρ together with Lemma 5.6 yields $\nu(B_h(z, r)) \sim \nu(B(x, A(r))) \sim \nu(B(z, A(r)))$. The doubling property of ν under ρ_h then implies $A(2r) \leq c_1 A(r)$ for some fixed constant $c_1 > 0$ independent of r and z. Thus,

$$\begin{aligned} 2r &= \int_0^{A(2r)} a_h(t)\mathrm{d}t \\ 2\int_0^{A(r)} a_h(t)\mathrm{d}t &\leq \int_0^{c_1 A(r)} a_h(t)\mathrm{d}t \\ \int_0^{A(r)} a_h(t)\mathrm{d}t &\leq \int_{A(r)}^{c_1 A(r)} a_h(t)\mathrm{d}t \\ &\lesssim a_h(A(r))A(r) \end{aligned}$$

Since A is surjective, (Ra) follows. □

Lemma 6.2. *Assume that Z is accessible. The there exist constants $c_1, c_2 > 0$ such that*

$$\rho_h(x, z) \leq c_1 \int_0^{c_2 r} a_h(t)\mathrm{d}t$$

for all $x \in S(z, r)$ where $z \in Z$ and $r > 0$.

Proof. Fix $\epsilon < \frac{1}{10}$. By Lemma 4.7 in [9], there is a sequence of balls $B_i = B(x_i, r_i)$ such that $r_i = \epsilon\rho(x_i, Z)$, $x \in B(x_0, 3r_0)$, $B(x_i, 3r_i) \cap B(x_{i+1}, 3r_{i+1}) \neq \emptyset$, and $\sup_k \#\left\{i : \frac{1}{c_1}\epsilon^k \rho(x, z) \leq r_i \leq c_1\epsilon^k\rho(x, z)\right\} \leq N$ for some fixed constants $c_1, N \geq 1 > \epsilon > 0$ independent of x, z, and r. Using these balls, it is possible to construct a path γ connecting x and z such that there are $t_0 < t_1 < t_2 < \ldots$ for which

(i) $\gamma(t_i) \in B(x_i, 3r_i) \cap B(x_{i+1}, 3r_{i+1})$, and

(ii) $\gamma_{|[t_i, t_{i+1}]}$ is a geodesic connecting $\gamma(t_i)$ and $\gamma(t_{i+1})$

for all $i = 0, 1, 2, \ldots$. Set $t_{-1} = 0$ and $N_k = \left\{i : \frac{1}{c_1}\epsilon^k\rho(x, z) \leq r_i \leq c_1\epsilon^k\rho(x, z)\right\}$. For $t \in [t_i, t_{i+1}]$ where $i + 1 \in N_k$, $\gamma(t) \in B(x_{i+1}, r_{i+1})$ and hence

$$\begin{aligned} \rho(\gamma(t), Z) &\leq 3r_{i+1} + \rho(x_{i+1}, Z) \\ &= (3 + \epsilon^{-1})r_{i+1} \\ &\leq (3\epsilon + 1)c_1\epsilon^{k-1}\rho(x, z), \end{aligned}$$

and similarly, $\rho(\gamma(t), Z) \geq (1-3\epsilon)c_1\epsilon^{k-1}\rho(x,z)$. Therefore, $\rho(\gamma(t), Z) \sim \epsilon^{k-1}\rho(x,z)$ whenever $t \in [t_i, t_{i+1}]$ and $i+1 \in N_k$. It follows that

$$
\begin{aligned}
\rho_h(x,z) &\leq \int h(\gamma(t))|\gamma|'(t)\mathrm{d}t \\
&= \sum_{k=0}^{\infty}\sum_{i\in N_k}\int_{t_{i-1}}^{t_i} h(\gamma(t))|\gamma|'(t)\mathrm{d}t \\
&\sim \sum_{k=0}^{\infty}\sum_{i\in N_k} a_h(\epsilon^{k-1}\rho(x,z))\int_{t_{i-1}}^{t_i}|\gamma|'(t)\mathrm{d}t \\
&\sim \sum_{k=0}^{\infty}\sum_{i\in N_k} a_h(\epsilon^{k-1}\rho(x,z))\rho(\gamma(t_{i-1}),\gamma(t_i)) \\
&\leq \sum_{k=0}^{\infty}\sum_{i\in N_k} 3c_1 a_h(\epsilon^{k-1}\rho(x,z))\epsilon^k\rho(x,z) \\
&\leq \frac{3c_1N\epsilon}{1-\epsilon}\sum_{k=0}^{\infty} a_h(\epsilon^{k-1}\rho(x,z))\epsilon^{k-1}(1-\epsilon)\rho(x,z) \\
&\sim \sum_{k=0}^{\infty}\int_{\epsilon^k\rho(x,z)}^{\epsilon^{k-1}\rho(x,z)} a_h(t)\mathrm{d}t \\
&= \int_0^{\epsilon^{-1}\rho(x,z)} a_h(t)\mathrm{d}t \\
&\leq \int_0^{\epsilon^{-1}r} a_h(t)\mathrm{d}t
\end{aligned}
$$

□

Theorem 6.3. *Let ν be a doulbing measure on a geodesic space (X,ρ) and Z be an accessible subset of X. If ν is also doubling under ρ_h where h has (α,β)-polynomial growth in the Z-normal direction, then (Rh) holds.*

Proof. Let $x \in S(z,r)$ where $z \in Z$. By Lemma 6.2, there is a constantsc $c > 0$ such that $\rho_h(x,z) \leq \int_0^{cr} a_h(t)\mathrm{d}t$. By Lemma 6.1, $\int_0^{cr} a_h(t)\mathrm{d}t \lesssim a_h(cr)r \sim a_h(r)r \sim a_h(\rho(x,Z))\rho(x,Z) \sim h(x)\rho(x,Z)$. Therefore, $\rho_h(x,z) \lesssim h(x)\rho(x,Z)$. □

Lemma 6.4. *Let Z be an accessible subset of a geodesic space (X,ρ). Assume that the function h has (α,β)-polynomial growth in the Z-normal direction and that (Rh) holds. If Z is accessible under ρ, then it is also accessible under ρ_h.*

Proof. First, we show that Z satisfies the skew condition under ρ_h. Fix $z \in Z$ and $r > 0$. By assumption, there is $x \in X$ such that $\Lambda\rho(x,z) \leq \rho(x,Z)$ and $\Lambda r \leq \rho(x,Z) \leq r$. Since (Rh) holds, $\rho_h(x,Z) \sim a_h(r)r \sim \int_0^r a_h(t)$. By Lemma 6.2, $\rho_h(x,z) \lesssim \int_0^r a_h(t)\mathrm{d}t$. Therefore, $\int_0^r a_h(t)\mathrm{d}t \sim \rho_h(x,z) \sim \rho_h(x,Z)$ which implies the Λ'-skew condition for some constant $\Lambda' \in (0,1)$.

Next, let γ be a path connecting $\gamma(1) \in Z$ and $\gamma(0) \in C_\Lambda(z,r)$ lying in $C_\lambda(z, c_\lambda r)$. This means $\lambda\rho(\gamma(t),z) \le \rho(\gamma(t),Z)$ for all t. Using the same reasoning as in the previous paragraph, there must exists a constant $0 < \lambda' < \Lambda'$ such that $\lambda'\rho_h(\gamma(t),z) \le \rho_h(\gamma(t),Z)$ for all t. Therefore, Z satisfies the (λ',Λ')-accessible condition. □

Lemma 6.5. *Let Z be an (λ,Λ)-accessible subset of a geodesic space (X,ρ). Then there exists a constant $\kappa \ge 1$ such that*

$$C_\lambda(z_1,\kappa r) \cap C_\lambda(z_2,\kappa r) \neq \emptyset$$

whenever $\rho(z_1,z_2) \le r$.

Proof. Consider $x \in S_\Lambda(z_1,\kappa r)$. Then

$$\begin{aligned} \rho(x,z_2) &\ge \rho(x,z_1) - \rho(z_1,z_2) \\ &\ge \Lambda\kappa r - r \\ &= (\Lambda\kappa - 1)r. \end{aligned}$$

If we choose $\kappa = (\Lambda-\lambda)^{-1}$, then $\rho(x,z_2) \ge \lambda\kappa r$ which directly implies $x \in C_\lambda(z_2,r)$. Particularly, $C_\lambda(z_1,\kappa r) \cap C_\lambda(z_2,\kappa r) \neq \emptyset$. □

Lemma 6.6. *Let Z be an accessible subset of a geodesic space (X,ρ). Assume that the function h has (α,β)-polynomial growth in the Z-normal direction and that (Rh) holds. Then there is a constant $c_\rho > 0$ such that*

$$B(z, c_\rho^{-1} r) \subseteq B_h(z, \int_0^r a_h(t)\mathrm{d}t) \subseteq B(z, c_\rho r)$$

for all $z \in Z$ and $r > 0$.

Proof. Let $x \in B_h(z,s)$ where $s = \int_0^r a_h(t)\mathrm{d}t$ and choose $z' \in Z$ such that $\rho_h(x,z') = \rho_h(x,Z)$. Then $\rho_h(z,z') \le 2s$. By the previous lemma, there is $y \in X$ such that $\lambda(\Lambda-\lambda)^{-1}s \le \rho_h(y,Z) \le \rho_h(y,z) \le (\Lambda-\lambda)^{-1}s$ and $\lambda(\Lambda-\lambda)^{-1}s \le \rho_h(y,Z) \le \rho_h(y,z') \le (\Lambda-\lambda)^{-1}s$. Therefore, $\rho(y,z) \sim r \sim \rho(y,z')$. Similarly, $\rho(x,z') \lesssim r$. Therefore, $\rho(x,z) \lesssim r$.

The proof for the first subset relation can be done similarly. □

Theorem 6.7. *Let ν be a doubling measure on a geodesic space (X,ρ) and Z be an accessible subset of X. Assume that the function h has (α,β)-polynomial growth in the Z-normal direction. If (Rh) holds, then ν is also doubling under ρ_h.*

Proof. First, we show that ν is doubling on Z-remote balls with respect to the distance ρ_h. Let $c = c(\delta_Z, C_h, c_h, \beta)$ be the constant in Lemma 5.6. Fix $x \in X - Z$ and $r < \frac{1}{4}c^{-1}\rho_h(x,Z)$. Choose $\epsilon \in (0,\frac{1}{2})$ such that $2r = \epsilon c^{-1}\rho_h(x,Z)$.

Then

$$\begin{aligned}
\nu(B_h(x,2r)) &= \nu(B_h(x,\epsilon c^{-1}\rho_h(x,Z))) \\
&\leq \nu(B(x,\epsilon\rho(x,Z))) \\
&\leq C_D^{2\log_2 c+2}\nu(B(x,\frac{\epsilon c^{-2}}{2}\rho(x,Z))) \\
&\leq C_D^{2\log_2 c+2}\nu(B_h(x,\frac{\epsilon c^{-1}}{2}\rho(x,Z))) \\
&\leq C_D^{2\log_2 c+2}\nu(B_h(x,r))
\end{aligned}$$

This shows that ν is doubling on Z-remote balls with constant $C_D^{2\log_2 c+2}$.

Next, we show that ν satisfies the volume comparison condition with respect to the distance ρ_h. Let $z \in Z$ and $x \in X$ be such that $\rho_h(x,z) \geq \Lambda\rho_h(x,Z)$. Set $s = \int_0^r a_h(t)\mathrm{d}t = \rho_h(x,Z)$. Then $B_h(z,s) \subseteq B(z,c_\rho r)$ and $B(x,\frac{\rho(x,Z)}{64c}) \subseteq B_h(x,s/64)$. Since $r \sim \rho(x,Z)$, we have

$$\begin{aligned}
\nu(B_h(z,s)) &\leq \nu(B(z,c_\rho r)) \\
&\lesssim \nu(B(z,\frac{r}{64c})) \\
&\sim \nu(B(z,\frac{\rho(x,Z)}{64c})) \\
&\leq \nu(B_h(x,\frac{s}{64}))
\end{aligned}$$

Therefore, ν satisfies the volume comparison condition with respect to the distance ρ_h.

Last, combining what we have proved with the fact that Z satisfies the skew condition to conclude that ν is doubling for all balls.

□

Proof of Theorem 3.1

First, we show that $(E^h,\mathbb{D}(E^h))$ satisfies the Poincaré inequality for Z-remote balls under ρ_h. Let $x \in X$ and $r = \epsilon c^{-1}\rho_h(x,Z)$ where $\epsilon \in (0,1/4)$. Then

$$\begin{aligned}
\inf_{\xi\in\mathbb{R}}\int_{B_h(x,r)}(u-\xi)^2\mathrm{d}\nu &\leq \inf_{\xi\in\mathbb{R}}\int_{B(x,\epsilon\rho(x,Z))}(u-\xi)^2\mathrm{d}\nu \\
&\lesssim (\epsilon\rho(x,Z))^2\int_{B(x,\epsilon\rho(x,Z))}\Gamma(u,u)\mathrm{d}\nu \\
&\lesssim (\epsilon\rho(x,Z))^2h^2(x)\int_{B(x,\epsilon\rho(x,Z))}h^{-2}\Gamma(u,u)\mathrm{d}\nu \\
&\lesssim (\epsilon\rho_h(x,Z))^2\int_{B_h(x,\epsilon c\rho_h(x,Z))}h^{-2}\Gamma(u,u)\mathrm{d}\nu \\
&\lesssim r^2\int_{B_h(x,\epsilon c\rho_h(x,Z))}h^{-2}\Gamma(u,u)\mathrm{d}\nu
\end{aligned}$$

Last, we proof the inequality (1). Let $x, y \in S(z,r)$ and $u \in \mathbb{D}(E^h)$. Let s be such that $\int_0^s a_h(t)\mathrm{d}t = r$.

$$\begin{aligned}
\fint_{B_h(x,\delta r)} \fint_{B_h(y,\delta r)} (u(v)-u(w))^2 \mathrm{d}\nu(v)\mathrm{d}\nu(w) &\lesssim \fint_{B(x,\delta s)} \fint_{B(y,\delta s)} (u(v)-u(w))^2 \mathrm{d}\nu(v)\mathrm{d}\nu(w) \\
&\lesssim s^2 \fint_{B(z,s)} \Gamma(u,u)\mathrm{d}\nu \\
&\lesssim s^2 a_h^2(s) \fint_{B(z,s)} h^{-2}\Gamma(u,u)\mathrm{d}\nu \\
&\lesssim r^2 \fint_{B_h(z,r)} h^{-2}\Gamma(u,u)\mathrm{d}\nu
\end{aligned}$$

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