**Final Report**

**Project Title Best Proximity Points and Global Optimal Approximate Solutions**

**By Poom Kumam**

**May 2014**

Contract No. MRG5580213

# Final Report

**Project Title Best Proximity Points and Global Optimal Approximate Solutions**


| Researcher | Institute |
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**This project granted by the Thailand Research Fund**

## Abstract

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**Project Code : MRG5580213**

**Project Title : Best Proximity Points and Global Optimal Approximate Solutions**

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**Project Period : 1 June 2012 – 30 May 2014**

**Abstract:**

The aim of this project is to establish the best proximity point theorems for generalized contractions mappings; there by producing optimal approximate solutions of certain fixed point equations. In addition to exploring the existence of the best proximity point for generalized contractions, an iterative algorithm is also presented to determine such an optimal approximate solution. Moreover, we also extend and generalize the notion of proximal contractions of self-contractions and establish the best proximity point theorems for these non-self mappings and also give examples to validate our main results.

Firstly, in this project focuses on the best proximity point theorems for proximal contractions of generalized contraction mappings which serve as non-self mapping analogues of generalized contraction self-mappings. Also, necessary and sufficient conditions are established for non-self contraction mappings to have the best proximity point, a common best proximity point and a couple best proximity point for pairs of contractive non-self mappings and for pairs of contraction non-self mappings, yielding common optimal approximate solutions of certain fixed point equations. Besides establishing the existence of common best proximity points, iterative algorithms are also furnished to determine such optimal approximate solutions. Secondly, in this project we also establish the best proximity point theorems for generalized contractive non-self for multi-valued mappings, yielding global optimal approximate solutions of certain fixed point equations. As a consequence, it ascertains the existence of an optimal approximate solution to some equations for which it is plausible that there is no solution. An algorithm is exhibited to determine such an optimal approximate solution designated as the best proximity point. It is interesting to observe that the preceding best proximity point theorem includes the famous Banach contraction principle.

**Keywords :** best proximity points, fixed points, proximal contractions, contraction mappings, common best proximity points

# Content

## 2. Executive Summary

**1. Introduction to the research problem and its significance**

Best proximity point theorems provide sufficient conditions that ensure the existence of approximate solutions, which are optimal as well. In fact, if there is no solution to the fixed point equation Tx = x for a non-self-mapping $T: A \longrightarrow B$, then it is desirable to determine an approximate solution x such that the error d(x,Tx) is minimum. In light of the fact that d(x,Tx) = d(A,B), an absolute optimal approximate solution is an element x for which the error d(x,Tx) assumes the least possible value d(A,B). As a result, the best proximity pair theorem furnishes sufficient conditions for the existence of an optimal approximate solution x, known as the best proximity point of the mapping T, satisfying the condition that d(x,Tx) = d(A,B). Interestingly, the best proximity theorems also serve as a natural generalization of fixed point theorems. Indeed, the best proximity point becomes a fixed point if the mapping under consideration is a self-mapping. On the other hand, though the best approximation theorems ensure the existence of approximate solutions, such results need not yield optimal solutions. But, best proximity point theorems furnish sufficient conditions that assure the existence of approximate solutions which are optimal as well.

The classical and well-known Banach's contraction principle states that every contraction on a complete metric space has a unique fixed point that is realizable as the limit of Picard iterates. Numerous interesting extensions and variants of the aforesaid result exist in the literature. However, the mappings involved in all these results are self-mappings. So, it is contemplated in this project to derive some best proximity point theorems which furnish non-self-mapping analogues of the aforesaid Banach's contraction principle. A point $x$ in $A$ is said to be the *best proximity point* for $T$ provided that the distance of $x$ to $T x$ equals the distance of $A$ to $B$. Indeed, theorems on the best proximity point are connected to optimal approximate solutions of some equations. Consequently, the results established in this project guarantee the existence of optimal approximate solutions for certain fixed point equations when there is no solution. Despite the fact that the best approximation theorems render approximate solutions to the equation $T x = x$ when there is no solution, such results do not necessarily produce approximate solutions that are optimal in the sense that the incurred error $d(x, T x)$ is minimum. On the other hand, in the case of non-existence of a solution to the equation $T x = x$, the best proximity point theorems yield optimal approximate solutions. In other words, given a non-self-mapping $T : A \longrightarrow B$, the best proximity point theorem presents sufficient conditions that ascertain the existence of an element $x$ in $A$ that minimizes the real valued function $x \longrightarrow d(x, T x)$. On account of the fact that, for any element $x$ in $A$, $d(x, T x)$ is at least $d(A, B)$, the best proximity point theorem assures the global minimum, and therefore an absolute minimum, of the error $d(x, T x)$ by dint of restricting an approximate solution of $T x = x$ to meet the requirement that $d(x, T x) = d(A, B)$. Essentially, the best proximity point theorem seeks an optimal approximate solution $x$, called the best proximity point of the mapping $T$, satisfying the condition that $d(x, T x) = d(A, B)$.

Fixed point theory is an indispensable tool for solving the equation $Tx = x$ for a mapping T defined on a subset of a metric space, a normed linear space or a topological vector space. As a non-self-mapping $T : A \longrightarrow B$ does not necessarily have a fixed point, one often tries to determine an element x which is in some sense closest to $Tx$. The best approximation theorems and the best proximity point theorems are pertinent in this perspective. An interesting best approximation theorem, due to Fan, asserts that if A is a non-empty compact convex subset of a Hausdorff locally convex topological vector space X with a semi-norm p and $T : A \longrightarrow X$ is a continuous mapping, then there is an element x in A satisfying the condition that dp(x,Tx) = dp(Tx,A). On the other hand, despite the fact that best approximation theorems assure the existence of approximate solutions, such results need not produce optimal solutions.

First, in this project we focus on the best proximity point theorems for proximal contractions of generalized contraction mappings which serve as non-self-mapping analogues of generalized contraction self-mappings. Also, necessary and sufficient conditions are established for a non-self-contraction mapping to have a best proximity point, a common best proximity point and a couple best proximity point for pairs of contractive non-self-mappings and for pairs of contraction non-self-mappings, yielding common optimal approximate solutions of certain fixed point equations. Besides establishing the existence of common best proximity points, iterative algorithms are also furnished to determine such optimal approximate solutions.

Second, in this project we establish the best proximity point theorems for generalized contractive non-self for multi-valued mappings, yielding global optimal approximate solutions of certain fixed point equations. Besides establishing the existence of best proximity points, iterative algorithms are also furnished to determine such optimal approximate solutions. Moreover, an algorithm is exhibited to determine such an optimal approximate solution designated as the best proximity point. It is interesting to observe that the preceding best proximity point theorem includes the famous Banach contraction principle. The main objective of this article is to resolve an optimization problem in the setting of a metric space that is endowed with a partial order. Under appropriate conditions, we also study the existence of solutions for the minimization problem $\min_{x\in A} \|x - Tx\|$. In addition to exploring the existence of a best proximity point for generalized contractions, an iterative algorithm is also presented to determine such an optimal approximate solution.

**2. Objectives of Project**

2.1) To investigate the significance of some geometric properties. For examples, the existence of the best proximity points for extending and generalizing the class of proximal contraction under some conditions related to some geometric properties.

2.2) To establish new algorithm for approximating global solution to the best proximity points of mappings which related to the convex minimization problems and optimization problems.

2.3) To obtain couple (common) best proximity point theorems without some conditions in some metric spaces and for set-valued non-self maps.

2.4) To establish the existence of common best proximity points; iterative algorithms are also furnished to determine such optimal approximate solutions.

2.5) To resolve an optimization problem in the setting of a metric space that is endowed with a partial order.

2.6) Study the existence of solutions for the minimization problem $\min_{x\in A} \|x - Tx\|$

**3. Methodology**

3.1) Studying and investigating on the best proximity point theorems for proximal contraction mappings.

3.2) Studying and investigating best proximity point, a common best proximity point and a couple best proximity points.

3.3) We find necessary and sufficient conditions which are established for a non-self-contraction mapping to have the best proximity point, a common best proximity point and a couple best proximity points for pairs of contractive non-self-mappings and for pairs of contraction non-self-mappings, yielding common optimal approximate solutions of certain fixed point equations.

3.4) Establishing the existence of common best proximity points; iterative algorithms are also furnished to determine such optimal approximate solutions. In addition to exploring the existence of the best proximity point for generalized contractions, an iterative algorithm is also presented to determine such an optimal approximate solution.

3.5) Writing and submitting papers.

**4. Schedule for the entire project and expected outputs**

**First year (2012)**

| **Schedule for the entire project** | **2012 (6 months)** | | | | | |
|---|---|---|---|---|---|---|
| | **1** | **2** | **3** | **4** | **5** | **6** |
| 1. Find data and papers. | | | | | | |
| 2. Study and prove theory of best proximity. | | | | | | |
| 4. Writing and submitting papers. | | | | | | |
| **Procedures** | **2012 (12 months)** | | | | | |
| | **7** | **8** | **9** | **10** | **11** | **12** |
| 1. Find data and more relevant papers. | | | | | | |
| 2. Innovation and research to find new knowledge continuously (and also from mentor). | | | | | | |
| 3. 4. Writing and submitting papers. | | | | | | |
| 4. Progress report 1 year. | | | | | | |

**Expected outputs (2 papers)**:

1. Coupled best proximity point theorem in metric spaces, Fixed Point Theory and Applications (2011 Impact Factor =1.63)

2. Best proximity point Theorems for generalized cyclic contractions in ordered metric Spaces, Journal of Optimization Theory and Applications (2011 Impact Factor= 1.062)

**Second year (2013)**

| **Schedule for the entire project** | **2513 ( 6 months)** | | | | | |
|---|---|---|---|---|---|---|
| | **1** | **2** | **3** | **4** | **5** | **6** |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1. Find data, more relevant papers and to meet mentor. | | | | | | |
| 2.Joint and presentation conference to find new knowledge. | | | | | | |
| 3. Writing papers and its applications. | | | | | | |
| 4. Progress report 1 year. | | | | | | |

| Procedures | 2013 ( 12 months) | | | | | |
|---|---|---|---|---|---|---|
| | 7 | 8 | 9 | 10 | 11 | 12 |
| 1. Writing papers. | | | | | | |
| 2.Submitting papers. | | | | | | |
| 3. Full report. | | | | | | |

**Expected outputs (2 papers)**:
1. Some common best proximity points for proximity commuting mappings, Optimization Letter, (2011 Impact Factor= 0.952)
2. Best proximity points and optimal approximate solutions, Journal Global Optimization (2011 Impact Factor= 1.196)

**6. Budget details**

**Total budget can not exceed 480,000 baht for 2 years.**

| | Month | year 1 | year 2 | Total |
|---|---|---|---|---|
| **Honorarium** (principal investigator) | | | | |
| Dr. Poom Kumam | 10,000 | 120,000 | 120,000 | 240,000 |
| **Materials** | | | | |
| 1 Copy all papers and Journals | | 20,000 | 20,000 | 40,000 |
| 2 Books | | 5,000 | 5,000 | 10,000 |
| 3 Page charge (reprints) | | 20,000 | 20,000 | 40,000 |
| 4 Office material, A4 | | 10,000 | 10,000 | 20,000 |
| 5 Toner print, Color tonner print, computers material data, Flash drive, ect. | | 10,000 | 10,000 | 20,000 |
| **Other expenses** | | | | |
| 1 Analysis Data | | 15,000 | 15,000 | 30,000 |
| 2 Joint Conference | | 20,000 | 20,000 | 40,000 |
| 3 To find a Mentor (travels, hotel, ect.) | | 20,000 | 20,000 | 40,000 |
| **Total** | | **240,000** | **240,000** | **480,000** |

## 3. Objective

3.1) To investigate the significance of some geometric properties. For examples, the existence of the best proximity points for extending and generalizing the class of proximal contraction under some conditions related to some geometric properties.

3.2) To establish new algorithm for approximating global solution to the best proximity points of mappings which related to the convex minimization problems and optimization problems.

3.3) To obtain couple (common or triple) best proximity point theorems without some conditions in some metric spaces and for set-valued non-self maps.

3.4) To establish the existence of common best proximity points; iterative algorithms are also furnished to determine such optimal approximate solutions.

3.5) To resolve an optimization problem in the setting of a metric space that is endowed with a partial order.

3.6) Study the existence of solutions for the minimization problem $\min\{||x - Tx||;\ x \in A\}$.

## 4. Research methodology

4.1) Studying and investigating on the best proximity point theorems for proximal contraction mappings.

4.2) Studying and investigating best proximity point, a common best proximity point and a couple best proximity points.

4.3) We find necessary and sufficient conditions which are established for a non-self-contraction mapping to have the best proximity point, a common best proximity point and a couple best proximity points for pairs of contractive non-self-mappings and for pairs of contraction non-self-mappings, yielding common optimal approximate solutions of certain fixed point equations.

4.4) Establishing the existence of common best proximity points; iterative algorithms are also furnished to determine such optimal approximate solutions. In addition to exploring the existence of the best proximity point for generalized contractions, an iterative algorithm is also presented to determine such an optimal approximate solution.

4.5) Writing and submitting papers.

# 5.1 BEST PROXIMITY POINT

## 5.1.1 Best proximity points for asymptotic proximal pointwise weaker Meir-Keeler-type $\psi$-contraction mappings

**Definition 5.1.1.1.** Let (A,B) be a nonempty pair in Banach space $X$, and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a weaker Meir-Keeler-type function. A mapping $T : A \cup B \to A \cup B$ is said to be an asymptotic proximal pointwise weaker Meir-Keeler-type $\psi$-contraction, if for each $n \in \mathbb{N}$ and $(x, y) \in A \times B$,

$$\|T^{2n}x - T^{2n}y\| \leq \max\{\psi^n(\|x\|)\|x - y\|, dist(A, B)\} \text{ for all } y \in B,$$
$$\|T^{2n}x - T^{2n}y\| \leq \max\{\psi^n(\|y\|)\|x - y\|, dist(A, B)\} \text{ for all } x \in A.$$

**Theorem 5.1.1.2.** *Let (A,B) be a nonempty bounded closed convex pair in a uniformly convex Banach space $X$ and $T : A \cup B \to A \cup B$ be an asymptotic proximal pointwise weaker Meir-Keeler-type $\psi$-contraction. If $T$ is a relatively nonexpansive mapping, then there exists a unique pair $(u_0, v_0) \in A \times B$ such that*

$$\|u_0 - Tu_0\| = \|v_0 - Tv_0\| = dist(A, B).$$

*Moreover, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges in norm to $u$ and $\{x_{2n+1}\}$ converges in norm to $v$.*

**Proof.** Fix an $x_0 \in A$ and define a function $f : B \to [0, \infty)$ by

$$f(u) = \limsup_{n\to\infty}\|T^{2n}(x_0) - u\| \text{ for } u \in B.$$

Since $X$ is uniformly convex and $B$ is bounded closed and convex, it follow that $f$ has unique minimizer over $B$; that is, we have a unique point $u_0 \in B$ satisfying

$$f(u_0) = \inf_{u\in B} f(u).$$

Indeed, for all $m \geq 1$ and $u \in B$, we have

$$\begin{aligned}
f(T^{2m}(u)) &= \limsup_{n\to\infty}\|T^{2n}(x_0) - T^{2m}u\| \\
&= \limsup_{n\to\infty}\|T^{2n+2m}(x_0) - T^{2m}u\| \\
&= \limsup_{n\to\infty}\|T^{2m}(T^{2n}(x_0)) - T^{2m}u\| \qquad (5.1.1)\\
&\leq \limsup_{n\to\infty}\max\{\psi^m(\|u\|)\|T^{2n}(x_0) - u\|, \operatorname{dist}(A,B)\} \\
&= \max\{\psi^m(\|u\|)f(u), \operatorname{dist}(A,B)\}.
\end{aligned}$$

Since $u_0 \in B$ is the minimum of $f$, for all $m \geq 1$, we have

$$f(u_0) \leq f(T^{2m}u_0) \leq \max\{\psi^m(\|u_0\|)f(u_0), \operatorname{dist}(A,B)\}. \qquad (5.1.2)$$

We now claim that $f(u_0) = \operatorname{dist}(A,B)$. Since for each $u \in B$, $\{\psi^m(\|u\|)\}$ is non-increasing, it must converges to some $\eta \geq 0$. Suppose that $\eta > 0$, by definition of weaker Meir-Keeler-type function, there exists $\delta > \eta$ such that for $u \in B$ with $\eta \leq \|u\| < \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(\|u\|) < \eta$. Since $\lim_{m\to\infty}\psi^m(\|u\|) = \eta$ there exists $m_0 \in \mathbb{N}$ such that $\eta \leq \psi^m(\|u\|) < \delta$, for all $m \geq m_0$. Thus we conclude that $\psi^{m_0+n_0}(\|u\|) < \eta$, thus we get the contradiction. So

$$\lim_{m\to\infty}\psi^m(\|u\|) = 0. \qquad (5.1.3)$$

Taking $m \to \infty$ in the inequality (5.1.2),we get

$$f(u_0) = \operatorname{dist}(A,B).$$

On the other hand, by the relatively nonexpansive of $T$, we have

$$f(T^2u_0) = \limsup_{n\to\infty}\|(T^{2n}(x_0)) - T^2u_0\| \leq \limsup_{n\to\infty}\|(T^{2n-2}(x_0)) - u_0\| = f(u_0),$$

which implies that $T^2u_0 = u_0$, by the uniqueness of minimum of $f$, then $u_0$ is a fixed point of $T^2$ in $B$. Hence,

$$\lim_{m\to\infty}\sup_{n\geq m}\|(T^{2m}(x_0)) - T^{2n}u_0\| = \lim_{m\to\infty}\|(T^{2m}(x_0)) - u_0\| = f(u_0) = \operatorname{dist}(A,B).$$

By the property $UC$ of (A,B), it follows from Lemma 4.0.0.4 that $\{T^{2n}(x_0)\}$ is a Cauchy sequence, so there exists $x' \in A$ such that $T^{2n}x_0 \to x'$ as $n \to \infty$. By the similar argument as above, if $y_0 \in B$ and $g : A \to [0,\infty)$ is given by $g(v) = \limsup_{n\to\infty}\|T^{2n}(y_0) - v\|$ for $v \in B$, we get $v_0$ is a fixed point of $T^2$, where $v_0$ is a minimum in exactly one point in $A$, and also $T^{2n}y_0 \to y' \in B$. Hence, we obtain

$$u_0 = T^{2n}u_0 \to x' \text{ and } v_0 = T^{2n}v_0 \to y'.$$

This show that $(u_0, v_0) = (x', y')$, and $T^{2n}x_0 \to u_0$, $T^{2n}y_0 \to v_0$. Moreover,

$$\begin{aligned} \|u_0 - v_0\| &= \|T^{2n}(u_0) - T^{2n}v_0\| \\ &\leq \max\{\psi^n(\|u_0\|)\|u_0 - v_0\|, \operatorname{dist}(A, B)\}. \end{aligned} \tag{5.1.4}$$

Taking $n \to \infty$ in the inequality (5.1.4), by (5.1.3) and definition of $\operatorname{dist}(A, B)$, we get

$$\|u_0 - v_0\| = \operatorname{dist}(A, B).$$

Since $T$ is relatively nonexpansive mapping, we have

$$\operatorname{dist}(A, B) \leq \|Tu_0 - Tv_0\| \leq \|u_0 - v_0\| = \operatorname{dist}(A, B),$$

Therefore $Tu_0 = v_0$ and $Tv_0 = u_0$. This implies that

$$\|Tu_0 - u_0\| = \|v_0 - Tv_0\| = \operatorname{dist}(A, B).$$

$\square$

**Example 1.** Consider $X = \mathbb{R}^2$ with Euclidean metric and let

$$A = \{(1, a) : a \geq 0\} \text{ and } B = \{(-1, b) : b \geq 0\},$$

then $A$ and $B$ be a nonempty closed and convex subset of $X$ and $d(A, B) = 2$. Define $T : A \cup B \to A \cup B$, by

$$T(1, a) = (-1, \tfrac{a}{2}) \text{ and } T(-1, b) = (1, \tfrac{b}{2}) \text{ for all } a, b \geq 0.$$

Then $T$ is a cyclic mapping and for each $(1, a) \in A$ and $(-1, b) \in B$, we have

$$T^{2n}(1, a) = (1, \tfrac{a}{2^{2n}}) \text{ and } T^{2n}(-1, b) = (-1, \tfrac{b}{2^{2n}}).$$

Next, we will show that $T$ is an asymptotic proximal pointwise weaker Meir-Keeler-type $\psi$-contraction with weaker Meir-Keeler-type function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\psi(t) = \begin{cases} 0, & when\ \ 0 \leq t \leq 1 \\ 2t, & when\ \ 1 < t < 2 \\ 1, & when\ \ t \geq 2. \end{cases}$$

Since,

$$\begin{aligned} d(T^{2n}(1,a),T^{2n}(-1,b)) &= d((1,\frac{a}{2^{2n}}),(-1,\frac{b}{2^{2n}})) \\ &= \sqrt{4+\left(\frac{a-b}{2^{2n}}\right)^2} \\ &\leq \sqrt{4+(a-b)^2} \\ &\leq \max\{\psi^n(d((0,0),(1,a))d((1,a),(-1,b)),dist(A,B)\}. \end{aligned}$$

Similarly, we can conclude that

$d(T^{2n}(1,a),T^{2n}(-1,b)) \leq \max\{\psi^n(d((0,0),(-1,b))d((1,a),(-1,b)),dist(A,B)\}$,

and hence $T$ is an asymptotic proximal pointwise weaker Meir-Keeler-type

$\psi$-contraction. Moreover

$((1,0),(-1,0)) \in A \times B$ is a pair of best proximity point of $T$, because

$$d((1,0),T(1,0)) = d((-1,0),T(-1,0)) = 2 = d(A,B).$$

### 5.1.2 Generalized proximal $\psi-$contraction mappings for Best proximity points

**Definition 5.1.2.1.** A mapping $S : A \to B$ is said to be a *generalized proximal $\psi-$contraction of the first kind*, if for all $u, v, x, y \in A$ satisfies

$$d(u,Sx) = d(v,Sy) = d(A,B) \Longrightarrow d(u,v) \leq \psi(d(x,y)),$$

where $\psi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous function from the right such that $\psi(t) < t$ for all $t > 0$.

**Definition 5.1.2.2.** A mapping $S : A \to B$ is said to be a *generalized proximal $\psi-$contraction of the second kind* if for all $u, v, x, y \in A$ satisfies

$$d(u,Sx) = d(v,Sy) = d(A,B) \Longrightarrow d(Su,Sv) \leq \psi(d(Sx,Sy)),$$

where $\psi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous from the right such that $\psi(t) < t$ for all $t > 0$.

It is easy to see that, if we take $\psi(t) = \alpha t$, where $\alpha \in [0,1)$, then a generalized proximal $\psi-$contraction of the first kind and generalized proximal $\psi-$contraction of the second kind reduces to a proximal contraction of the first kind and a proximal contraction of the second kind, respectively. Moreover, it is easy to see that a self-mapping generalized proximal $\psi-$contraction of the first kind and the second kind reduces to the condition of Boy and Wong' s fixed point theorem[15].

**Theorem 5.1.2.3.** *Let $(X, d)$ be a complete metric space and $A$ and $B$ be non-empty, closed subsets of $X$ such that $A_0$ and $B_0$ are non-empty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions:*

*(a) $S$ and $T$ are generalized proximal $\psi-$contraction of the first kind;*

*(b) $g$ is an isometry;*

*(c) The pair $(S, T)$ is a proximal cyclic contraction;*

*(d) $S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*

*(e) $A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$.*

*Then, there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B),$$

*converges to the element $x$. For any fixed $y_0 \in B_0$, the sequence $\{y_n\}$, defined by*

$$d(gy_{n+1}, Ty_n) = d(A, B),$$

*converges to the element $y$.*

*On the other hand, a sequence $\{u_n\}$ in $A$ converges to $x$ if there is a sequence of positive numbers $\{\epsilon_n\}$ such that*

$$\lim_{n\to\infty} \epsilon_n = 0 \text{ and } d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

*where $z_{n+1} \in A$ satisfies the condition that $d(z_{n+1}, Su_n) = d(A, B)$.*

*Proof.* Let $x_0$ a fixed element in $A_0$. In view of the fact that $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, it is ascertained that there exists an element $x_1 \in A_0$ such that

$$d(gx_1, Sx_0) = d(A, B). \tag{5.1.5}$$

Again, since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element $x_2 \in A_0$ such that

$$d(gx_2, Sx_1) = d(A, B). \tag{5.1.6}$$

By similar fashion, we can find $x_n$ in $A_0$. Having chosen $x_n$, one can determine an element $x_{n+1} \in A_0$ such that

$$d(gx_{n+1}, Sx_n) = d(A, B). \tag{5.1.7}$$

Because of the facts that $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, by a generalized proximal $\psi-$contraction of the first kind of $S$, $g$ is an isometry and property of $\psi$, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(gx_{n+1}, gx_n) \\ &\leq \psi(d(x_n, x_{n-1})) \\ &\leq d(x_n, x_{n-1}), \end{aligned} \tag{5.1.8}$$

this mean that the sequence $\{d(x_{n+1}, x_n)\}$ is non-increasing and bounded below. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r. \tag{5.1.9}$$

If $r > 0$, then

$$\begin{aligned} r &= \lim_{n\to\infty} d(x_{n+1}, x_n) \\ &\leq \lim_{n\to\infty} \psi(d(x_n, x_{n-1})) \\ &= \psi(r) \\ &< r, \end{aligned}$$

which is a contradiction unless $r = 0$. Therefore,

$$\alpha_n := \lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{5.1.10}$$

We claim that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequence $\{x_{m_k}\}, \{x_{n_k}\}$ of $\{x_n\}$ such that $n_k > m_k \geq k$ with

$$r_k := d(x_{m_k}, x_{n_k}) \geq \varepsilon \quad \text{and} \quad d(x_{m_k}, x_{n_k-1}) < \varepsilon \tag{5.1.11}$$

for $k \in \{1, 2, 3, \ldots\}$. Thus

$$\begin{aligned} \varepsilon \leq r_k &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \varepsilon + \alpha_{n_k-1}, \end{aligned}$$

$$\tag{5.1.12}$$

it follows from (5.1.10), implies that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{5.1.13}$$

Notice also that

$$\begin{aligned} \varepsilon \le r_k &\le d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k}) + d(x_{m_k+1}, x_{n_k+1}) \\ &= \alpha_{m_k} + \alpha_{n_k} + d(x_{m_k+1}, x_{n_k+1}) \\ &\le \alpha_{m_k} + \alpha_{n_k} + \psi(d(x_{m_k}, x_{n_k})). \end{aligned} \tag{5.1.14}$$

Taking $k \to \infty$ in above inequality, by (5.1.10), (5.1.13) and property of $\psi$, we get $\varepsilon \le \psi(\varepsilon)$. Therefore, $\varepsilon = 0$, which is a contradiction. So we obtain the claim and hence converge to some element $x \in A$. Similarly, in view of the fact that $T(B_0) \subseteq A_0$ and $A_0 \subseteq g(A_0)$, we can conclude that there is a sequence $\{y_n\}$ such that converge to some element $y \in B$. Since the pair $(S,T)$ is a proximal cyclic contraction and $g$ is an isometry, we have

$$d(x_{n+1}, y_{n+1}) = d(gx_{n+1}, gy_{n+1}) \le \alpha d(x_n, y_n) + (1-\alpha)d(A,B). \tag{5.1.15}$$

We take limit in (5.1.15) as $n \to \infty$, it follows that

$$d(x,y) = d(A,B), \tag{5.1.16}$$

so, we concluded that $x \in A_0$ and $y \in B_0$. Since $S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$ that there is $u \in A$ and $v \in B$ such that

$$d(u, Sx) = d(A,B) \tag{5.1.17}$$

and

$$d(v, Ty) = d(A,B). \tag{5.1.18}$$

From (5.1.7), (5.1.17) and the notion of generalized proximal $\psi-$contraction of first kind of $S$, we get

$$d(u, gx_{n+1}) \le \psi(d(x, x_n)) \tag{5.1.19}$$

Letting $n \to \infty$, we get $d(u, gx) \le \psi(0) = 0$ and thus $u = gx$. Therefore

$$d(gx, Sx) = d(A,B). \tag{5.1.20}$$

Similarly, we can show that $v = gy$ and then

$$d(gy, Ty) = d(A, B). \tag{5.1.21}$$

From (5.1.16), (5.1.20) and (5.1.21), we get

$$d(x, y) = d(gx, Sx) = d(gy, Ty) = d(A, B).$$

Next, to prove the uniqueness, let us suppose that there exist $x^* \in A$ and $y^* \in B$ with $x \neq x^*, y \neq y^*$ such that

$$d(gx^*, Sx^*) = d(A, B)$$

and

$$d(gy^*, Ty^*) = d(A, B).$$

Since $g$ is an isometry, $S$ and $T$ are generalized proximal $\psi-$contractions of the first kind and the property of $\psi$, it follows that

$$d(x, x^*) = d(gx, gx^*) \leq \psi(d(x, x^*)) < d(x, x^*)$$

and

$$d(y, y^*) = d(gy, gy^*) \leq \psi(d(y, y^*)) < d(y, y^*),$$

which is a contradiction, so we have $x = x^*$ and $y = y^*$. On the other hand, let $\{u_n\}$ be a sequence in $A$ and $\{\epsilon_n\}$ a sequence of positive real numbers such that

$$\lim_{n\to\infty} \epsilon_n = 0 \text{ and } d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

where $z_{n+1} \in A$ satisfies the condition that $d(z_{n+1}, Su_n) = d(A, B)$. Since $S$ is a generalized proximal $\psi-$contraction of first kind, we have

$$d(x_{n+1}, z_{n+1}) \leq \psi(d(x_n, u_n)). \tag{5.1.22}$$

Given $\epsilon > 0$, we choose a positive integer $N$ such that $\epsilon_n \leq \epsilon$ for all $n \geq N$, we obtain that

$$\begin{aligned} d(x_{n+1}, u_{n+1}) &\leq d(x_{n+1}, z_{n+1}) + d(z_{n+1}, u_{n+1}) \\ &\leq \psi(d(x_n, u_n)) + \epsilon_n. \end{aligned}$$

Therefore, we get

$$\begin{aligned} d(u_{n+1}, x) &\leq d(u_{n+1}, x_{n+1}) + d(x_{n+1}, x) \\ &\leq \psi(d(x_n, u_n)) + \epsilon_n + d(x_{n+1}, x). \end{aligned} \tag{5.1.23}$$

We claim that $d(u_n, x) \to 0$ as $n \to \infty$, suppose the contrary, by a inequality (5.1.23) and property of $\psi$, we get

$$\begin{aligned}
\lim_{n\to\infty} d(u_{n+1}, x) &\leq \lim_{n\to\infty} (d(u_{n+1}, x_{n+1}) + d(x_{n+1}, x)) \\
&\leq \lim_{n\to\infty} (\psi(d(x_n, u_n)) + \epsilon_n + d(x_{n+1}, x)) \\
&= \psi((\lim_{n\to\infty} d(x_n, u_n)) \\
&\leq \psi((\lim_{n\to\infty} (d(x_n, x) + d(x, u_n))) \\
&< \lim_{n\to\infty} d(x, u_n)
\end{aligned}$$

which is a contradiction, so we have $\{u_n\}$ is convergent and it converges to $x$. This completes the proof of the theorem. □

**Corollary 5.1.2.4.** *Let $(X, d)$ be a complete metric space and $A$ and $B$ be non-empty, closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are non-empty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions:*

*(a) $S$ and $T$ are generalized proximal $\psi-$contraction of the first kind;*

*(b) $S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$ ;*

*(c) The pair $(S, T)$ is a proximal cyclic contraction.*

*Then, there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

If take $\psi(t) = \alpha t$, where $0 \leq \alpha < 1$, we obtain following corollary;

**Corollary 5.1.2.5.** *Let $(X, d)$ be a complete metric space and $A$ and $B$ be non-empty, closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are non-empty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions:*

*(a) $S$ and $T$ are proximal contractions of first kind;*

*(b) $S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*

*(c) The pair $(S, T)$ is a proximal cyclic contraction.*

*Then, there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

For a self-mapping, includes the Boy and Wong' s fixed point theorem [15] as followings:

**Corollary 5.1.2.6.** *Let $(X,d)$ be a complete metric space and $T : X \to X$ a mapping that satisfies $d(Tx,Ty) \le \psi(d(x,y))$ for all $x,y \in X$, where $\psi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous function from the right such that $\psi(t) < t$ for all $t > 0$. Then $T$ has a unique fixed point $v \in X$. More over, for each $x \in X$, $\{T^n x\}$ converges to $v$.*

**Example 2.** Consider the complete metric space $\mathbb{R}^2$ with metric defined by

$$d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|,$$

for all $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$. Let

$$A = \{(0,y) : -1 \le y \le 1\}, \quad B = \{(1,y) : -1 \le y \le 1\}.$$

Then $d(A,B) = 1$. Define the mappings $S : A \to B$ as follows:

$$S((0,y)) = \begin{cases} (1,0) & ; \quad -1 \le y < 0, \\ \left(1, y - \dfrac{y^2}{2}\right), & ; \quad 0 \le y \le 1. \end{cases}$$

First, we show that $S$ is generalized proximal $\psi-$contraction of the first kind with the function $\psi : [0,\infty) \to [0,\infty)$ defined by

$$\psi(t) = \begin{cases} t - \dfrac{t^2}{2} & ; \quad 0 \le t \le 1, \\ t - 1, & ; \quad t > 1. \end{cases}$$

Let $(0,x_1),(0,x_2),(0,a_1)$ and $(0,a_2)$ be elements in $A$ satisfying

$$d((0,x_1),S(0,a_1)) = d(A,B) = 1, \quad d((0,x_2),S(0,a_2)) = d(A,B) = 1.$$

It follow that

$$x_i = 0 \quad \text{when } -1 \le a_i < 0, \ \text{ for } \ i = 1,2$$

and

$$x_i = a_i - \frac{a_i^2}{2} \quad \text{when } 0 \le a_i \le 1, \ \text{ for } \ i = 1,2.$$

Clearly, in case $x_1 = x_2 = 0$ and $x_1 = 0$, $x_2 = a_2 - \dfrac{a_2^2}{2}$(or $x_1 = a_1 - \dfrac{a_1^2}{2}$, $x_2 = 0$ ).

Assume that $x_i = a_i - \dfrac{a_i^2}{2}$ for $i = 1, 2$ with $a_1 - a_2 > 0$, then we have

$$\begin{aligned} d((0,x_1),(0,x_2)) &= d\big((0, a_1 - \frac{a_1^2}{2}), (0, a_2 - \frac{a_2^2}{2})\big) \\ &= |(a_1 - \frac{a_1^2}{2}) - (a_2 - \frac{a_2^2}{2})| \\ &= (a_1 - a_2) - (\frac{a_1^2}{2} - \frac{a_2^2}{2}) \\ &\leq (a_1 - a_2) - \frac{1}{2}(a_1 - a_2)^2 \\ &= \psi(d((0,a_1),(0,a_2)) \end{aligned}$$

Thus $S$ is a generalized proximal $\psi-$contraction of the first kind.

Next, we prove that $S$ is not a proximal contraction. Suppose $S$ is proximal contraction, then for each $(0,x),(0,y),(0,a),(0,b) \in A$ satisfying

$$d((0,x), S(0,a)) = d(A,B) = 1 \text{ and } d((0,y), S(0,b)) = d(A,B) = 1, \qquad (5.1.24)$$

there exists $k \in [0,1)$ such that

$$d((0,x),(0,y)) \leq kd((0,a),(0,b)).$$

From (5.1.80), we get

$$x = 0 \quad \text{when } -1 \leq a < 0, \quad y = 0 \quad \text{when } -1 \leq b < 0$$

and

$$x = a - \frac{a^2}{2} \quad \text{when } 0 \leq a \leq 1, \quad y = b - \frac{b^2}{2} \quad \text{when } 0 \leq b \leq 1.$$

Suppose that $x = a - \dfrac{a^2}{2}$ and $y = b - \dfrac{b^2}{2}$, then we have

$$\begin{aligned} \big|\big(a - \frac{a^2}{2}\big) - \big(b - \frac{b^2}{2}\big)\big| &= d((0,x),(0,y)) \\ &\leq kd((0,a),(0,b)) \\ &= k|a - b|. \end{aligned}$$

Letting $b = 0$ with $a \neq 0$, we get

$$1 = \lim_{a \to 0^+} (1 - \frac{a}{2}) \leq k < 1,$$

which is a contradiction. Therefore $S$ is not a proximal contraction.

**Example 3.** Consider the complete metric space $\mathbb{R}^2$ with metric defined by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - x_2| + |y_1 - y_2|,$$

for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. Let $A = \{(0, y) : -1 \leq y \leq 1\}$ and $B = \{(1, x) : -1 \leq x \leq 1\}$. Define two mappings $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ as follows: $g((x, y)) = (x, -y)$,

$$S((0, y)) = \begin{cases} (1, 0) & ; \quad -1 \leq y < 0, \\ \left(1, \frac{y}{2}(1 - \frac{y}{2})\right), & ; \quad 0 \leq y \leq 1 \end{cases}$$

and

$$T((1, x)) = \begin{cases} (0, 0) & ; \quad -1 \leq x < 0, \\ \left(0, \frac{x}{2}(1 - \frac{x}{2})\right), & ; \quad 0 \leq x \leq 1. \end{cases}$$

Then it is easy to see that $d(A, B) = 1$, $A_0 = A$, $B_0 = B$ and the mapping $g$ is an isometry.

Next, we claim that $S$ and $T$ are generalized proximal $\psi-$contractions of the first kind with the function $\psi : [0, \infty) \to [0, \infty)$ defined by

$$\psi(t) = \begin{cases} \frac{t}{2} - \frac{t^2}{4} & ; \quad 0 \leq t \leq 1, \\ t - 1, & ; \quad t > 1. \end{cases}$$

If $(0, y_1), (0, y_2) \in A$ such that

$$d((0, a), S(0, y_1)) = d(A, B) = 1 \text{ and } d((0, b), S(0, y_2)) = d(A, B) = 1$$

for all $(0, a), (0, b) \in A$, then we have

$$a = \begin{cases} 0 & ; \quad -1 \leq y_1 < 0, \\ \frac{y_1}{2}(1 - \frac{y_1}{2}), & ; \quad 0 \leq y_1 \leq 1, \end{cases}$$

and

$$b = \begin{cases} 0 & ; \quad -1 \leq y_2 < 0, \\ \dfrac{y_2}{2}(1 - \dfrac{y_2}{2}), & ; \quad 0 \leq y_2 \leq 1, \end{cases}$$

Clearly, in case $a = b = 0$ and $a = 0, b = \frac{y_2}{2}(1 - \frac{y_2}{2})$( or $a = \frac{y_1}{2}(1 - \frac{y_1}{2}), b = 0$ ). Suppose $a = \frac{y_1}{2}(1 - \frac{y_1}{2})$ and $b = \frac{y_2}{2}(1 - \frac{y_2}{2})$ with $y_1 - y_2 = t > 0$, we have

$$\begin{aligned} d((0,a),(0,b)) &= \left|(\frac{y_1}{2} - \frac{y_1^2}{4}) - (\frac{y_2}{2} - \frac{y_2^2}{4})\right| \\ &= \left|\frac{(y_1 - y_2)}{2}(1 - \frac{1}{2}(y_1 + y_2))\right| \\ &\leq \left|\frac{y_1 - y_2}{2}(1 - \frac{1}{2}(y_1 - y_2))\right| \\ &= \psi(d((0,y_1),(0,y_2))) \end{aligned}$$

Hence $S$ is a generalized proximal $\psi-$contraction of the first kind. In the same way, we can prove that $T$ is a generalized proximal $\psi-$contraction of the first kind. Moreover, we can see that the pair $(S,T)$ forms a proximal cyclic contraction. Further, it is see that the unique element $(0,0) \in A$ and $(1,0) \in B$ such that

$$d(g(0,0), S(0,0)) = d(g(1,0), T(1,0)) = d((0,0),(1,0)) = d(A,B).$$

**Theorem 5.1.2.7.** *Let $(X,d)$ be a complete metric space and $A$ and $B$ be non-empty, closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are non-empty. Let $S : A \to B$ and $g : A \to A$ satisfy the following conditions:*

*(a) $S$ is a generalized proximal $\psi-$contractions of first and second kinds;*

*(b) $g$ is an isometry;*

*(c) $S$ preserves isometric distance with respect to $g$;*

*(d) $S(A_0) \subseteq B_0$;*

*(e) $A_0 \subseteq g(A_0)$.*

*Then, there exists a unique point $x \in A$ such that*

$$d(gx, Sx) = d(A,B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(gx_{n+1}, Sx_n) = d(A,B),$$

*converges to the element* $x$.

*On the other hand, a sequence* $\{u_n\}$ *in* $A$ *converges to* $x$ *if there is a sequence of positive numbers* $\{\epsilon_n\}$ *such that*

$$\lim_{n\to\infty} \epsilon_n = 0 \text{ and } d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

*where* $z_{n+1} \in A$ *satisfies the condition that* $d(z_{n+1}, Su_n) = d(A, B)$.

*Proof.* Since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, similarly in the proof, we can construct the sequence $\{x_n\}$ of element in $A_0$ such that

$$d(gx_{n+1}, Sx_n) = d(A, B) \tag{5.1.25}$$

for all non-negative number $n$. It follows from $g$ is an isometry and the virtue of a generalized proximal $\psi-$contraction of the first kind of $S$, we see that

$$d(x_n, x_{n+1}) = d(gx_n, gx_{n+1}) \leq \psi(d(x_n, x_{n-1}))$$

for all $n \in \mathbb{N}$. Similarly to the proof, we can conclude that the sequence $\{x_n\}$ is a Cauchy sequence and converges to some $x \in A$. Since $S$ is a generalized proximal $\psi-$contraction of the second kind and preserves isometric distance with respect to $g$ that

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &= d(Sgx_n, Sgx_{n+1}) \\ &\leq \psi(d(Sx_{n-1}, Sx_n)) \\ &\leq d(Sx_{n-1}, Sx_n) \end{aligned}$$

this mean that the sequence $\{d(Sx_{n+1}, Sx_n)\}$ is non-increasing and bounded below. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = r. \tag{5.1.26}$$

If $r > 0$, then

$$\begin{aligned} r &= \lim_{n\to\infty} d(Sx_{n+1}, Sx_n) \\ &\leq \lim_{n\to\infty} \psi(d(Sx_{n-1}, Sx_n)) \\ &= \psi(r) \\ &< r, \end{aligned}$$

which is a contradiction, unless $r = 0$. Therefore

$$\beta_n := \lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = 0. \tag{5.1.27}$$

We claim that $\{Sx_n\}$ is a Cauchy sequence. Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequence $\{Sx_{m_k}\}, \{Sx_{n_k}\}$ of $\{Sx_n\}$ such that $n_k > m_k \geq k$ with

$$r_k := d(Sx_{m_k}, Sx_{n_k}) \geq \varepsilon \text{ and } d(Sx_{m_k}, Sx_{n_k-1}) < \varepsilon \tag{5.1.28}$$

for $k \in \{1, 2, 3, \ldots\}$. Thus

$$\begin{aligned} \varepsilon \leq r_k &\leq d(Sx_{m_k}, Sx_{n_k-1}) + d(Sx_{n_k-1}, Sx_{n_k}) \\ &< \varepsilon + \beta_{n_k-1}, \end{aligned} \tag{5.1.29}$$

it follows from (5.1.27), implies that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{5.1.30}$$

Notice also that

$$\begin{aligned} \varepsilon \leq r_k &\leq d(Sx_{m_k}, Sx_{m_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) + d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &= \beta_{m_k} + \beta_{n_k} + d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &\leq \beta_{m_k} + \beta_{n_k} + \psi(d(Sx_{m_k}, Sx_{n_k})). \end{aligned} \tag{5.1.31}$$

Taking $k \to \infty$ in above inequality, by (5.1.27), (5.1.30) and property of $\psi$, we get $\varepsilon \leq \psi(\varepsilon)$. Hence, $\varepsilon = 0$, which is a contradiction. So we obtain the claim and then it converges to some $y \in B$. Therefore, we can conclude that

$$d(gx, y) = \lim_{n\to\infty} d(gx_{n+1}, Sx_n) = d(A, B) \tag{5.1.32}$$

that is $gx \in A_0$. Since $A_0 \subseteq g(A_0)$, we have $gx = gz$ for some $z \in A_0$ and then $d(gx, gz) = 0$. By the fact that $g$ is an isometry, we have $d(x, z) = d(gx, gz) = 0$. Hence $x = z$ and so $x$ becomes to a point in $A_0$. As $S(A_0) \subseteq B_0$ that

$$d(u, Sx) = d(A, B) \tag{5.1.33}$$

for some $u \in A$. It follows from (5.1.25), (5.1.33) and $S$ is a generalized proximal $\psi-$contraction of the first kind that

$$d(u, gx_{n+1}) \leq \psi(d(x, x_n)) \tag{5.1.34}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$, we get the sequence $\{gx_n\}$ converge to a point $u$. By the fact that $g$ is continuous, we have

$$gx_n \to gx \text{ as } n \to \infty.$$

By the uniqueness of limit of the sequence, we conclude that $u = gx$. Therefore, it results that $d(gx, Sx) = d(u, Sx) = d(A, B)$. The uniqueness and the remaining part of the proof follows as in Theorem. This completes the proof of the theorem. $\square$

If $g$ is assumed to be the identity mapping, then the Theorem 5.3.1.3, we obtain the following corollary:

**Corollary 5.1.2.8.** *Let $(X, d)$ be a complete metric space and $A$ and $B$ be non-empty, closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are non-empty. Let $S : A \to B$ satisfy the following conditions:*

*(a) $S$ is a generalized proximal $\psi-$contractions of first and second kinds;*

*(b) $S(A_0) \subseteq B_0$.*

*Then, there exists a unique point $x \in A$ such that*

$$d(x, Sx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Sx_n) = d(A, B),$$

*converges to the best proximity point $x$ of $S$.*

**Corollary 5.1.2.9.** *Let $(X, d)$ be a complete metric space and $A$ and $B$ be non-empty, closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are non-empty. Let $S : A \to B$ satisfy the following conditions:*

*(a) $S$ is a proximal contractions of first and second kinds;*

*(b) $S(A_0) \subseteq B_0$ .*

*Then, there exists a unique point $x \in A$ such that*

$$d(x, Sx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Sx_n) = d(A, B),$$

*converges to the best proximity point $x$ of $S$.*

### 5.1.3 Best proximity points for generalized proximal $C$-contraction mappings in metric spaces with partial orders

**Definition 5.1.3.1.** A mapping $T : A \to B$ is said to be a *generalized proximal $C$-contraction* if, for all $u, v, x, y \in A$, satisfies

$$\left.\begin{aligned} & x \preceq y \\ & d(u, Tx) = d(A, B) \\ & d(v, Ty) = d(A, B) \end{aligned}\right\} \implies d(u, v) \leq \frac{1}{2}(d(x, v) + d(y, u)) - \psi(d(x, v), d(y, u)). \tag{5.1.35}$$

where $\psi : [0, \infty)^2 \to [0, \infty)$ is continuous and nondecreasing function such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

**Theorem 5.1.3.2.** *Let $X$ be a nonempty set such that $(X, \preceq)$ is a partially ordered set and $(X, d)$ be a complete metric space. Let $A$ and $B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Let $T : A \to B$ satisfy the following conditions:*

(a) *$T$ is a continuous, proximally order-preserving and generalized proximal $C$-contraction such that $T(A_0) \subseteq B_0$;*

(b) *there exist element $x_0$ and $x_1$ in $A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B).$$

*Then there exists a point $x \in A$ and such that*

$$d(x, Tx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by*

$$d(x_{n+1}, Tx_n) = d(A, B)$$

*converges to the point $x$.*

*Proof.* By the hypothesis $(b)$, there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and

$$d(x_1, Tx_0) = d(A, B).$$

Since $T(A_0) \subseteq B_0$, there exists a point $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

By the proximally order-preserving of $T$, we get $x_1 \preceq x_2$. Continuing this process, we can find a sequence $\{x_n\}$ in $A_0$ such that $x_{n-1} \preceq x_n$ and

$$d(x_n, Tx_{n-1}) = d(A, B).$$

Having found the point $x_n$, one can choose a point $x_{n+1} \in A_0$ such that $x_n \preceq x_{n+1}$ and

$$d(x_{n+1}, Tx_n) = d(A, B). \tag{5.1.36}$$

Since $T$ is a generalized proximal $C$-contraction, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{1}{2}(d(x_{n-1}, x_{n+1}) + d(x_n, x_n)) - \psi(d(x_{n-1}, x_{n+1}), d(x_n, x_n)) \\ &= \frac{1}{2}d(x_{n-1}, x_{n+1}) - \psi(d(x_{n-1}, x_{n+1}), 0) \\ &\leq \frac{1}{2}d(x_{n-1}, x_{n+1}) \\ &\leq \frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \end{aligned} \tag{5.1.37}$$

and so it follow that $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$, that is, the sequence $\{d(x_{n+1}, x_n)\}$ is non-increasing and bounded bolow. Then there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r. \tag{5.1.38}$$

Taking $n \to \infty$ in (5.1.37), we have

$$r \leq \lim_{n\to\infty} \frac{1}{2}d(x_{n-1}, x_{n+1}) \leq \frac{1}{2}(r + r) = r$$

and so

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2r. \tag{5.1.39}$$

Again, taking $n \to \infty$ in (5.1.37) and using (5.1.38), (5.1.39) and the continuity of $\psi$, we get

$$r \leq \frac{1}{2}(2r) = r - \psi(2r, 0) \leq r$$

and hence $\psi(2r, 0) = 0$. So, by the property of $\psi$, we have $r = 0$, which implies that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{5.1.40}$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequence $\{x_{m_k}\}, \{x_{n_k}\}$ of $\{x_n\}$ such that $n_k > m_k \geq k$ with

$$r_k := d(x_{m_k}, x_{n_k}) \geq \varepsilon, \quad d(x_{m_k}, x_{n_k-1}) < \varepsilon \tag{5.1.41}$$

for each $k \in \{1, 2, 3, \cdots\}$. For each $n \geq 1$, let $\alpha_n := d(x_{n+1}, x_n)$. So, we have

$$\begin{aligned}\varepsilon \leq r_k &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})\\ &< \varepsilon + \alpha_{n_k-1},\end{aligned}$$

it follows from (5.1.40) that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{5.1.42}$$

Notice also that

$$\begin{aligned}r_k = d(x_{n_k}, x_{m_k}) &\leq d(x_{n_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k})\\ &= d(x_{n_k}, x_{m_k+1}) + \alpha_{m_k}\\ &\leq d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k+1}) + \alpha_{m_k}\\ &= r_k + \alpha_{m_k} + \alpha_{m_k}.\end{aligned} \tag{5.1.43}$$

Taking $k \to \infty$ in (5.1.43), by (5.1.40) and (5.1.42), we conclude that

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \varepsilon. \tag{5.1.44}$$

Similarly, we can show that

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \varepsilon. \tag{5.1.45}$$

On the other hand, by the construction of $\{x_n\}$, we may assume that $x_{m_k} \preceq x_{n_k}$ such that

$$d(x_{n_k+1}, Tx_{n_k}) = d(A, B) \tag{5.1.46}$$

and

$$d(x_{m_k+1}, Tx_{m_k}) = d(A, B). \tag{5.1.47}$$

By the triangle inequality, (5.1.46), (5.1.47) and the generalized proximal $C$-contraction of $T$, we have

$$\begin{aligned}\varepsilon \leq r_k &\leq d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k}) + d(x_{m_k+1}, x_{n_k+1})\\ &= \alpha_{m_k} + \alpha_{n_k} + d(x_{m_k+1}, x_{n_k+1})\\ &\leq \alpha_{m_k} + \alpha_{n_k} + \frac{1}{2}[d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})]\\ &\quad -\psi(d(x_{n_k}, x_{m_k+1}), d(x_{m_k}, x_{n_k+1}))\end{aligned}$$

Taking $k \to \infty$ in the above inequality, by (5.1.40), (5.1.44), (5.1.45) and the continuity of $\psi$, we get

$$\varepsilon \leq \frac{1}{2}(\varepsilon + \varepsilon) - \psi(\varepsilon, \varepsilon) \leq \varepsilon.$$

Therefore, $\psi(\varepsilon, \varepsilon) = 0$. By the property of $\psi$, we have that $\varepsilon = 0$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since $A$ is a closed subset of the complete metric space $X$, there exist $x \in A$ such that

$$\lim_{n\to\infty} x_n = x. \tag{5.1.48}$$

Letting $n \to \infty$ in (5.1.36) , by (5.1.48) and the continuity of $T$, it follows that

$$d(x, Tx) = d(A, B).$$

□

**Corollary 5.1.3.3.** *Let $X$ be a nonempty set such that $(X, \preceq)$ is a partially ordered set and $(X, d)$ be a complete metric space. Let $A$ and $B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Let $T : A \to B$ satisfy the following conditions:*

(a) *$T$ is a continuous, increasing such that $T(A_0) \subseteq B_0$ and*

$$\left.\begin{array}{l} x \preceq y \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \implies d(u, v) \leq \alpha(d(x, v) + d(y, u)), \tag{5.1.49}$$

*where $\alpha \in (0, \frac{1}{2})$;*

(b) *there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B).$$

*Then there exists a point $x \in A$ and such that*

$$d(x, Tx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by*

$$d(x_{n+1}, Tx_n) = d(A, B)$$

*converges to the point $x$.*

*Proof.* Let $\alpha \in (0, \frac{1}{2})$ and the function $\psi$ in Theorem 5.1.3.2 be defined by

$$\psi(a,b) = (\frac{1}{2} - \alpha)(a+b).$$

Obviously, it follows that $\psi(a,b) = 0$ if and only if $a = b = 0$ and (5.1.35) become to (5.1.49). Hence we obtain the Corollary 5.1.3.3. □

Now, we give an example to illustrate Theorem 5.1.3.2

**Example 4.** Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Let

$$A = \{(x,0) : x \in \mathbb{R}\}, \quad B = \{(0,y) : y \in \mathbb{R}\ , y \geq 1\}.$$

Then $d(A,B) = 1$, $A_0 = \{(0,0)\}$ and $B_0 = \{(0,1)\}$. Define a mapping $T : A \to B$ as follows:

$$T((x,0)) = (0, 1 + |x|)$$

for all $(x,0) \in A$. Clearly, $T$ is continuous and $T(A_0) \subseteq B_0$. If $x_1 \preceq x_2$ and

$$d(u_1, Tx_1) = d(A,B) = 1, \quad d(u_2, Tx_2) = d(A,B) = 1$$

for some $u_1, u_2, x_1, x_2 \in A$, then we have

$$u_1 = u_1 = (0,0), \quad x_1 = x_2 = (0,0).$$

Therefore, $T$ is a generalized proximal $C$-contraction with $\psi : [0,\infty)^2 \to [0,\infty)$ defined by

$$\psi(a,b) = \frac{1}{4}(a+b).$$

Further, observe that $(0,0) \in A$ such that

$$d((0,0), T(0,0)) = d(A,B) = 1.$$

**Theorem 5.1.3.4.** *Let $X$ be a nonempty set such that $(X, \preceq)$ is a partially ordered set and $(X,d)$ be a complete metric space. Let $A$ and $B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Let $T : A \to B$ satisfy the following conditions:*

(a) *$T$ is an proximally order-preserving and generalized proximal $C$-contraction such that $T(A_0) \subseteq B_0$;*

(b) *there exist element $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B);$$

(c) *if* $\{x_n\}$ *is an increasing sequence in* $A$ *converges to* $x$*, then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$.

*Then there exists a point* $x \in A$ *and such that*

$$d(x, Tx) = d(A, B).$$

*Proof.* As in the proof of Theorem 5.1.3.2, we have

$$d(x_{n+1}, Tx_n) = d(A, B) \tag{5.1.50}$$

for all $n \geq 0$ . Moreover, $\{x_n\}$ is a Cauchy sequence and converges to some point $x \in A$. Observe that, for each $n \in \mathbb{N}$,

$$\begin{aligned} d(A, B) = d(x_{n+1}, Tx_n) &\leq d(x_{n+1}, x) + d(x, Tx_n) \\ &\leq d(x, x_{n+1}) + d(x, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &\leq d(x, x_{n+1}) + d(x, x_{n+1}) + d(A, B). \end{aligned}$$

Taking $n \to \infty$ in the above inequality, we obtain $\lim_{n\to\infty} d(x, Tx_n) = d(A, B)$ and hence $x \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $v \in A$ such that

$$d(v, Tx) = d(A, B). \tag{5.1.51}$$

Next, we prove that $x = v$. By the condition $(c)$, we have $x_n \preceq x$ for all $n \in \mathbb{N}$. Using (5.1.50), (5.1.49) and the generalized proximal $C$-contraction of $T$, we have

$$d(x_{n+1}, v) \leq \frac{1}{2}[d(x_n, v) + d(x, x_{n+1})] - \psi(d(x_n, v), d(x, x_{n+1})). \tag{5.1.52}$$

Letting $n \to \infty$ in (5.1.52), we get

$$d(x, v) \leq \frac{1}{2}d(x, v) - \psi(d(x, v), 0),$$

which implies that $d(x, v) = 0$, that is, $x = v$. If we replace $v$ by $x$ in (5.1.49), we have

$$d(x, Tx) = d(A, B).$$

$\square$

**Corollary 5.1.3.5.** *Let $X$ be a nonempty set such that $(X, \preceq)$ is a partially ordered set and let $(X, d)$ be a complete metric space. Let $A$ and $B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Let $T : A \to B$ satisfy the following conditions:*

(a) *$T$ is an increasing mapping such that $T(A_0) \subseteq B_0$ and*

$$\left.\begin{array}{l} x \preceq y \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \implies d(u, v) \leq \alpha(d(x, v) + d(y, u)), \tag{5.1.53}$$

*where $\alpha \in (0, \frac{1}{2})$;*

(b) *there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B);$$

(c) *if $\{x_n\}$ is an increasing sequence in $A$ converges to a point $x \in X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

*Then there exists a point $x \in A$ and such that*

$$d(x, Tx) = d(A, B).$$

$$\text{For } x, y \in X \ \text{ there exists } \ z \in X \ \text{ which is comparable to } \ x \ \text{ and } \ y. \tag{5.1.54}$$

**Theorem 5.1.3.6.** *Let $X$ be a nonempty set such that $(X, \preceq)$ is a partially ordered set and $(X, d)$ be a complete metric space. Let $A$ and $B$ be nonempty closed subsets of $X$ and $A_0$ and $B_0$ are nonempty such that $A_0$ satisfying condition (5.1.54). Let $T : A \to B$ satisfy the following conditions:*

(a) *$T$ is a continuous, proximally order-preserving and generalized proximal $C$-contraction such that $T(A_0) \subseteq B_0$;*

(b) *there exist element $x_0$ and $x_1$ in $A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B).$$

*Then there exists a unique point $x \in A$ and such that*

$$d(x, Tx) = d(A, B).$$

*Proof.* We shall only proof the part of uniqueness part. Suppose that there exist $x$ and $x^*$ in $A$ which are best proximity point, that is

$$d(x, Tx) = d(A, B) \quad \text{and} \quad d(x^*, Tx^*) = d(A, B).$$

**Case I** $x$ is comparable to $x^*$, that is $x \preceq x^*$ (or $x^* \preceq x$ ), by the generalized proximal $C$-contraction of $T$, we have

$$d(x, x^*) \;\; \leq \frac{1}{2}[d(x, x^*) + d(x^*, x)] - \psi(d(x, x^*), d(x^*, x)) \;\; \leq \;\; d(x^*, x),$$

which implies that $\psi(d(x, x^*), d(x^*, x)) = 0$. Using the property of $\psi$, we get $d(x^*, x) = 0$ and hence $x = x^*$.

**Case II** $x$ is not comparable to $x^*$. Since $A_0$ satisfying condition (5.1.54), then there exist $z \in A_0$ such that $z$ comparable to $x$ and $x^*$, that is $x \preceq z$ (or $z \preceq x$ ) and $x^* \preceq z$ (or $z \preceq x^*$ ). Suppose that $x \preceq z$ and $x^* \preceq z$. Since $T(A_0) \subseteq B_0$, there exists a point $v_0 \in A_0$ such that

$$d(v_0, Tz) = d(A, B).$$

By proximally order-preserving, we get $x \preceq v_0$ and $x^* \preceq v_0$. Since $T(A_0) \subseteq B_0$, there exists a point $v_1 \in A_0$ such that

$$d(v_1, Tv_0) = d(A, B).$$

Again, by proximally order-preserving, we get $x \preceq v_1$ and $x^* \preceq v_1$. One can proceed further in a similar fashion to find $v_n$ in $A_0$ with $v_{n+1} \in A_0$ such that

$$d(v_{n+1}, Tv_n) = d(A, B).$$

Hence $x \preceq v_n$ and $x^* \preceq v_n$ for all $n \in \mathbb{N}$. By the generalized proximal $C$-contraction of $T$, we have

$$d(v_{n+1}, x) \;\; \leq \;\; \frac{1}{2}[d(v_n, x) + d(x, v_{n+1})] - \psi(d(v_n, x), d(x, v_{n+1})), \tag{5.1.55}$$

$$d(v_{n+1}, x^*) \;\; \leq \;\; \frac{1}{2}[d(v_n, x^*) + d(x^*, v_{n+1})] - \psi(d(v_n, x^*), d(x^*, v_{n+1})), \tag{5.1.56}$$

it follow from (5.1.55), (5.1.56) and property of $\psi$, we get

$$v_n \to x \quad \text{and} \quad v_n \to x^* \quad \text{as } n \to \infty.$$

By the uniqueness of limit, we conclude that $x = x^*$. Other cases can we proved similarly and this completes the proof. $\square$

**Theorem 5.1.3.7.** *Let $X$ be a nonempty set such that $(X, \preceq)$ is a partially ordered set and $(X, d)$ be a complete metric space. Let $A$ and $B$ be nonempty closed subsets of $X$ and $A_0$ and $B_0$ are nonempty such that $A_0$ satisfying condition (5.1.54). Let $T : A \to B$ satisfy the following conditions:*

(a) *$T$ is an proximally order-preserving and generalized proximal $C$-contraction such that $T(A_0) \subseteq B_0$;*

(b) *there exist element $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B);$$

(c) *if $\{x_n\}$ is an increasing sequence in $A$ converges to $x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

*Then there exists a unique point $x \in A$ and such that*

$$d(x, Tx) = d(A, B).$$

*Proof.* For the proof, combine the proofs of Theorem 5.1.3.4 and Theorem 5.1.3.7. $\square$

### 5.1.4 Best proximity point theorems for rational proximal contractions

**Definition 5.1.4.1.** Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{A}$ and $\mathcal{B}$ be two non-empty subsets of $\mathcal{X}$. Then, $\mathcal{T} : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be a rational proximal contraction of the first kind if there exist non-negative real numbers $\alpha, \beta, \gamma, \delta$ with $\alpha+\beta+2\gamma+2\delta < 1$, such that the conditions

$$d(u_1, \mathcal{T}x_1) = d(\mathcal{A}, \mathcal{B}) \text{ and } d(u_2, \mathcal{T}x_2) = d(\mathcal{A}, \mathcal{B})$$

imply that

$$\begin{aligned} d(u_1, u_2) \leq\ & \alpha d(x_1, x_2) + \frac{\beta[1 + d(x_1, u_1)]d(x_2, u_2)}{1 + d(x_1, x_2)} + \gamma[d(x_1, u_1) + d(x_2, u_2)] \\ & + \delta[d(x_1, u_2) + d(x_2, u_1)] \end{aligned} \tag{5.1.57}$$

for all $u_1, u_2, x_1, x_2 \in \mathcal{A}$.

**Definition 5.1.4.2.** Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{A}$ and $\mathcal{B}$ be two non-empty subsets of $\mathcal{X}$. Then, $\mathcal{T} : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be a rational proximal contraction of the second kind if there exist non-negative real numbers $\alpha, \beta, \gamma, \delta$ with $\alpha+\beta+2\gamma+2\delta < 1$ such that the conditions

$$d(u_1, \mathcal{T}x_1) = d(\mathcal{A}, \mathcal{B}) \text{ and } d(u_2, \mathcal{T}x_2) = d(\mathcal{A}, \mathcal{B})$$

imply that

$$\begin{aligned} d(\mathcal{T}u_1, \mathcal{T}u_2) \quad &\leq \quad \alpha d(\mathcal{T}x_1, \mathcal{T}x_2) + \frac{\beta[1 + d(\mathcal{T}x_1, Tu_1)]d(\mathcal{T}x_2, Tu_2)}{1 + d(\mathcal{T}x_1, \mathcal{T}x_2)} \\ &\quad +\gamma[d(\mathcal{T}x_1, \mathcal{T}u_1) + d(\mathcal{T}x_2, \mathcal{T}u_2)] \\ &\quad +\delta[d(\mathcal{T}x_1, \mathcal{T}u_2) + d(\mathcal{T}x_2, \mathcal{T}u_1)] \end{aligned} \tag{5.1.58}$$

for all $u_1, u_2, x_1, x_2 \in \mathcal{A}$.

**Theorem 5.1.4.3.** *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{A}$ and $\mathcal{B}$ be two non-empty, closed subsets of $\mathcal{X}$ such that $\mathcal{B}$ is approximatively compact with respect to $\mathcal{A}$. Assume that $\mathcal{A}_0$ and $\mathcal{B}_0$ are non-empty and $\mathcal{T} : \mathcal{A} \longrightarrow \mathcal{B}$ is a non-self-mapping such that:*

*(a) $\mathcal{T}$ is a rational proximal contraction of the first kind;*

*(b) $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$.*

*Then, there exists $x \in \mathcal{A}$ such that $B_{est}(\mathcal{T}) = \{x\}$. Further, for any fixed $x_0 \in \mathcal{A}_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B})$, converges to $x$.*

*Proof.* Let $x_0 \in \mathcal{A}_0$ (such a point there exists since $\mathcal{A}_0 \neq \emptyset$). Since $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$, then by the definition of $\mathcal{B}_0$, there exists $x_1 \in \mathcal{A}_0$ such that $d(x_1, \mathcal{T}x_0) = d(\mathcal{A}, \mathcal{B})$. Again, since $\mathcal{T}x_1 \in \mathcal{B}_0$, it follows that there is $x_2 \in \mathcal{A}_0$ such that $d(x_2, \mathcal{T}x_1) = d(\mathcal{A}, \mathcal{B})$. Continuing this process, we can construct a sequence $\{x_n\}$ in $\mathcal{A}_0$, such that

$$d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B}),$$

for every non-negative integer $n$. Using the fact that $\mathcal{T}$ is a rational proximal contraction of the first kind, we have

$$\begin{aligned} d(x_n, x_{n+1}) \quad &\leq \quad \alpha d(x_{n-1}, x_n) + \frac{\beta[1 + d(x_{n-1}, x_n)]d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)} \\ &\quad +\gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \delta d(x_{n-1}, x_{n+1}) \\ &\leq \quad \alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}) + \gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\quad +\delta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \end{aligned}$$

It follows that

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)$$

where $k = \frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta} < 1$. Therefore, $\{x_n\}$ is a Cauchy sequence and, since $(\mathcal{X}, d)$ is complete and $\mathcal{A}$ is closed, the sequence $\{x_n\}$ converges to some $x \in \mathcal{A}$.
Moreover, we have

$$\begin{aligned} d(x, \mathcal{B}) &\leq d(x, \mathcal{T}x_n) \\ &\leq d(x, x_{n+1}) + d(x_{n+1}, \mathcal{T}x_n) \\ &= d(x, x_{n+1}) + d(\mathcal{A}, \mathcal{B}) \\ &\leq d(x, x_{n+1}) + d(x, \mathcal{B}). \end{aligned}$$

Taking the limit as $n \longrightarrow +\infty$, we get $d(x, \mathcal{T}x_n) \longrightarrow d(x, \mathcal{B})$. Since $\mathcal{B}$ is approximatively compact with respect to $\mathcal{A}$, then the sequence $\{\mathcal{T}x_n\}$ has a subsequence $\{\mathcal{T}x_{n_k}\}$ that converges to some $y \in \mathcal{B}$. Therefore

$$d(x, y) = \lim_{k \longrightarrow +\infty} d(x_{n_k+1}, \mathcal{T}x_{n_k}) = d(\mathcal{A}, \mathcal{B}),$$

and hence $x$ must be in $\mathcal{A}_0$. Since $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$, then $d(u, \mathcal{T}x) = d(\mathcal{A}, \mathcal{B})$ for some $u \in \mathcal{A}$. Again, using the fact that $\mathcal{T}$ is a rational proximal contraction of the first kind, we get

$$\begin{aligned} d(u, x_{n+1}) \quad \leq \quad & \alpha d(x, x_n) + \frac{\beta[1 + d(x, u)]d(x_n, x_{n+1})}{1 + d(x, x_n)} \\ & + \gamma[d(x, u) + d(x_n, x_{n+1})] + \delta[d(x, x_{n+1}) + d(x_n, u)]. \end{aligned}$$

Taking the limit as $n \longrightarrow +\infty$, we have

$$d(u, x) \leq (\gamma + \delta)d(u, x),$$

which implies $x = u$, since $\gamma + \delta < 1$. Thus, it follows that

$$d(x, \mathcal{T}x) = d(u, \mathcal{T}x) = d(\mathcal{A}, \mathcal{B}),$$

that is, $x \in B_{est}(\mathcal{T})$. Now, to prove the uniqueness of the best proximity point (i.e., $B_{est}(\mathcal{T})$ is singleton), assume that $z$ is another best proximity point of $\mathcal{T}$ so that

$$d(z, \mathcal{T}z) = d(\mathcal{A}, \mathcal{B}).$$

Since $\mathcal{T}$ is a rational proximal contraction of the first kind, we have

$$d(x,z) \leq \alpha d(x,z) + \frac{\beta[1+d(x,x)]d(z,z)}{1+d(x,z)} + \gamma[d(x,x)+d(z,z)] + \delta[d(x,z)+d(z,x)]$$

which implies

$$d(x,z) \leq (\alpha + 2\delta) d(x,z).$$

It follows immediately that $x = z$, since $\alpha + 2\delta < 1$. Hence, $\mathcal{T}$ has a unique best proximity point. $\square$

As consequences of the Theorem 5.1.4.3, we state the following corollaries.

**Corollary 5.1.4.4.** *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{A}$ and $\mathcal{B}$ be two non-empty, closed subsets of $\mathcal{X}$ such that $\mathcal{B}$ is approximatively compact with respect to $\mathcal{A}$. Assume that $\mathcal{A}_0$ and $\mathcal{B}_0$ are non-empty and $\mathcal{T} : \mathcal{A} \longrightarrow \mathcal{B}$ is a non-self-mapping such that:*

*(a) $\mathcal{T}$ is a generalized proximal contraction of the first kind, with $\alpha + 2\gamma + 2\delta < 1$;*

*(b) $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$.*

*Then, there exists $x \in \mathcal{A}$ such that $B_{est}(\mathcal{T}) = \{x\}$. Further, for any fixed $x_0 \in \mathcal{A}_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B})$, converges to the best proximity point $x$.*

**Corollary 5.1.4.5.** *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{A}$ and $\mathcal{B}$ be two non-empty, closed subsets of $\mathcal{X}$ such that $\mathcal{B}$ is approximatively compact with respect to $\mathcal{A}$. Assume that $\mathcal{A}_0$ and $\mathcal{B}_0$ are non-empty and $\mathcal{T} : \mathcal{A} \longrightarrow \mathcal{B}$ is a non-self-mapping such that:*

*(a) There exists a non-negative real number $\alpha < 1$ such that, for all $u_1, u_2, x_1, x_2$ in $\mathcal{A}$, the conditions $d(u_1, \mathcal{T}x_1) = d(\mathcal{A}, \mathcal{B})$ and $d(u_2, \mathcal{T}x_2) = d(\mathcal{A}, \mathcal{B})$ imply that $d(u_1, u_2) \leq \alpha d(x_1, x_2)$;*

*(b) $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$.*

*Then, there exists $x \in \mathcal{A}$ such that $B_{est}(\mathcal{T}) = \{x\}$. Further, for any fixed $x_0 \in \mathcal{A}_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B})$, converges to the best proximity point $x$.*

The following fixed point result can be considered as a special case of the Theorem 5.1.4.3, when $\mathcal{T}$ is a self-mapping.

**Corollary 5.1.4.6.** *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T}$ be a self-mapping on $\mathcal{X}$. Assume that there exist non-negative real numbers $\alpha, \beta, \gamma, \delta$ with $\alpha+\beta+2\gamma+2\delta < 1$ such that*

$$\begin{aligned} d(\mathcal{T}x_1, \mathcal{T}x_2) \;\; &\leq \;\; \alpha d(x_1, x_2) + \frac{\beta[1 + d(x_1, \mathcal{T}x_2)]d(x_2, \mathcal{T}x_2)}{1 + d(x_1, x_2)} \\ &\quad +\gamma[d(x_1, \mathcal{T}x_1) + d(x_2, \mathcal{T}x_2)] \quad + \delta[d(x_1, \mathcal{T}x_2) + d(x_2, \mathcal{T}x_1)]. \end{aligned}$$

*for all $x_1, x_2 \in \mathcal{X}$. Then the mapping $\mathcal{T}$ has a unique fixed point.*

**Remark 5.1.4.7.** Note that the Corollary 5.1.4.6 is a proper extension of the contraction mapping principle [34] because the continuity of the mapping $\mathcal{T}$ is not required. It is well known that a contraction mapping must be continuous.

**Theorem 5.1.4.8.** *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{A}$ and $\mathcal{B}$ be two non-empty, closed subsets of $\mathcal{X}$ such that $\mathcal{A}$ is approximatively compact with respect to $\mathcal{B}$. Assume that $\mathcal{A}_0$ and $\mathcal{B}_0$ are non-empty and $\mathcal{T} : \mathcal{A} \longrightarrow \mathcal{B}$ is a non-self-mapping such that:*

*(a) $\mathcal{T}$ is a continuous rational proximal contraction of the second kind;*

*(b) $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$.*

*Then, there exists $x \in B_{est}(\mathcal{T})$ and for any fixed $x_0 \in \mathcal{A}_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B})$, converges to $x$, and $\mathcal{T}x = \mathcal{T}z$ for all $x, z \in B_{est}(\mathcal{T})$.*

*Proof.* Following the same lines of the proof of the Theorem 5.1.4.3, it is possible to construct a sequence $\{x_n\}$ in $\mathcal{A}_0$ such that

$$d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B}),$$

for every non-negative integer $n$. Using the fact that $\mathcal{T}$ is a rational proximal contraction of the second kind, we have

$$\begin{aligned} d(\mathcal{T}x_n, \mathcal{T}x_{n+1}) \;\; &\leq \;\; \alpha d(\mathcal{T}x_{n-1}, \mathcal{T}x_n) + \frac{\beta[1 + d(\mathcal{T}x_{n-1}, \mathcal{T}x_n)]d(\mathcal{T}x_n, \mathcal{T}x_{n+1})}{1 + d(\mathcal{T}x_{n-1}, \mathcal{T}x_n)} \\ &\quad +\gamma[d(\mathcal{T}x_{n-1}, \mathcal{T}x_n) + d(\mathcal{T}x_n, \mathcal{T}x_{n+1})] + \delta d(\mathcal{T}x_{n-1}, \mathcal{T}x_{n+1}) \\ &\leq \;\; \alpha d(\mathcal{T}x_{n-1}, \mathcal{T}x_n) + \beta d(\mathcal{T}x_n, \mathcal{T}x_{n+1}) + \gamma[d(\mathcal{T}x_{n-1}, \mathcal{T}x_n) + \\ &\quad\;\; d(\mathcal{T}x_n, \mathcal{T}x_{n+1})] + \delta[d(\mathcal{T}x_{n-1}, \mathcal{T}x_n) + d(\mathcal{T}x_n, \mathcal{T}x_{n+1})]. \end{aligned}$$

It follows that

$$d(\mathcal{T}x_n, \mathcal{T}x_{n+1}) \leq kd(\mathcal{T}x_{n-1}, \mathcal{T}x_n)$$

where $k = \frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta} < 1$. Therefore, $\{\mathcal{T}x_n\}$ is a Cauchy sequence and, since $(\mathcal{X}, d)$ is complete, then the sequence $\{\mathcal{T}x_n\}$ converges to some $y \in \mathcal{B}$.
Moreover, we have

$$\begin{aligned} d(y, \mathcal{A}) &\leq d(y, x_{n+1}) \\ &\leq d(y, \mathcal{T}x_n) + d(\mathcal{T}x_n, x_{n+1}) \\ &= d(y, \mathcal{T}x_n) + d(\mathcal{A}, \mathcal{B}) \\ &\leq d(y, \mathcal{T}x_n) + d(y, \mathcal{A}). \end{aligned}$$

Taking the limit as $n \longrightarrow +\infty$, we get $d(y, x_n) \longrightarrow d(y, \mathcal{A})$. Since $\mathcal{A}$ is approximatively compact with respect to $\mathcal{B}$, then the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some $x \in \mathcal{A}$. Now, using the continuity of $\mathcal{T}$, we obtain that

$$d(x, \mathcal{T}x) = \lim_{k \longrightarrow +\infty} d(x_{n_k+1}, \mathcal{T}x_{n_k}) = d(\mathcal{A}, \mathcal{B}),$$

that is, $x \in B_{est}(\mathcal{T})$. Finally, to prove the last assertion of the present theorem, assume that $z$ is another best proximity point of $\mathcal{T}$ so that

$$d(z, \mathcal{T}z) = d(\mathcal{A}, \mathcal{B}).$$

Since $\mathcal{T}$ is a rational proximal contraction of the second kind, we have

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}z) \leq\ & \alpha d(\mathcal{T}x, \mathcal{T}z) + \frac{\beta[1 + d(\mathcal{T}x, \mathcal{T}x)]d(\mathcal{T}z, \mathcal{T}z)}{1 + d(\mathcal{T}x, \mathcal{T}z)} + \gamma[d(\mathcal{T}x, \mathcal{T}x) + \\ & d(\mathcal{T}z, \mathcal{T}z)] + \delta[d(\mathcal{T}x, \mathcal{T}z) + d(\mathcal{T}z, \mathcal{T}x)] \end{aligned}$$

which implies

$$d(\mathcal{T}x, \mathcal{T}z) \leq (\alpha + 2\delta)d(\mathcal{T}x, \mathcal{T}z).$$

It follows immediately that $\mathcal{T}x = \mathcal{T}z$, since $\alpha + 2\delta < 1$. $\square$

As consequences of the Theorem 5.1.4.8, we state the following corollaries.

**Corollary 5.1.4.9.** *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{A}$ and $\mathcal{B}$ be two non-empty, closed subsets of $\mathcal{X}$ such that $\mathcal{A}$ is approximatively compact with respect to $\mathcal{B}$. Assume that $\mathcal{A}_0$ and $\mathcal{B}_0$ are non-empty and $\mathcal{T} : \mathcal{A} \longrightarrow \mathcal{B}$ is a non-self-mapping such that:*

*(a)* $\mathcal{T}$ *is a continuous generalized proximal contraction of the second kind, with* $\alpha + 2\gamma + 2\delta < 1$*;*

*(b)* $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$.

*Then, there exists* $x \in B_{est}(\mathcal{T})$ *and for any fixed* $x_0 \in \mathcal{A}_0$*, the sequence* $\{x_n\}$*, defined by* $d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B})$*, converges to* $x$*. Further,* $\mathcal{T}x = \mathcal{T}z$ *for all* $x, z \in B_{est}(\mathcal{T})$*.*

**Corollary 5.1.4.10.** *Let* $(\mathcal{X}, d)$ *be a complete metric space and* $\mathcal{A}$ *and* $\mathcal{B}$ *be two non-empty, closed subsets of* $\mathcal{X}$ *such that* $\mathcal{A}$ *is approximatively compact with respect to* $\mathcal{B}$*. Assume that* $\mathcal{A}_0$ *and* $\mathcal{B}_0$ *are non-empty and* $\mathcal{T} : \mathcal{A} \longrightarrow \mathcal{B}$ *is a non-self-mapping such that:*

*(a) There exists a non-negative real number* $\alpha < 1$ *such that, for all* $u_1, u_2, x_1, x_2$ *in* $\mathcal{A}$*, the conditions* $d(u_1, \mathcal{T}x_1) = d(\mathcal{A}, \mathcal{B})$ *and* $d(u_2, \mathcal{T}x_2) = d(\mathcal{A}, \mathcal{B})$ *imply that* $d(\mathcal{T}u_1, \mathcal{T}u_2) \leq \alpha d(\mathcal{T}x_1, \mathcal{T}x_2)$*;*

*(b)* $\mathcal{T}$ *is continuous;*

*(c)* $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$.

*Then, there exists* $x \in B_{est}(\mathcal{T})$ *and for any fixed* $x_0 \in \mathcal{A}_0$*, the sequence* $\{x_n\}$*, defined by* $d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B})$*, converges to* $x$*. Further,* $\mathcal{T}x = \mathcal{T}z$ *for all* $x, z \in B_{est}(\mathcal{T})$*.*

**Remark 5.1.4.11.** Note that in the Theorem 5.1.4.3 is not required the continuity of the mapping $\mathcal{T}$. On the contrary, the continuity of $\mathcal{T}$ is an hypothesis of the Theorem 5.1.4.8.

Our next theorem concerns a non-self-mapping that is a rational proximal contraction of the first kind as well as a rational proximal contraction of the second kind. In this theorem we consider only a completeness hypothesis without assuming the continuity of the non-self-mapping.

**Theorem 5.1.4.12.** *Let* $(\mathcal{X}, d)$ *be a complete metric space and* $\mathcal{A}$ *and* $\mathcal{B}$ *be two non-empty, closed subsets of* $\mathcal{X}$*. Assume that* $\mathcal{A}_0$ *and* $\mathcal{B}_0$ *are non-empty and* $\mathcal{T} : \mathcal{A} \longrightarrow \mathcal{B}$ *is a non-self-mapping such that:*

*(a)* $\mathcal{T}$ *is a rational proximal contraction of the first and second kinds;*

*(b)* $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$.

*Then, there exists a unique* $x \in B_{est}(\mathcal{T})$. *Further, for any fixed* $x_0 \in \mathcal{A}_0$, *the sequence* $\{x_n\}$, *defined by* $d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B})$, *converges to* $x$.

*Proof.* Following the same lines of the proof of the Theorem 5.1.4.3, it is possible to construct a sequence $\{x_n\}$ in $\mathcal{A}_0$ such that

$$d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B}),$$

for every non-negative integer $n$. Also using the same arguments in the proof of the Theorem 5.1.4.3, we deduce that the sequence $\{x_n\}$ is a Cauchy sequence and hence converges to some $x \in \mathcal{A}$. Moreover, on the lines of the proof of the Theorem 5.1.4.8, we obtain that the sequence $\{\mathcal{T}x_n\}$ is a Cauchy sequence and hence converges to some $y \in \mathcal{B}$. Therefore, we have

$$d(x, y) = \lim_{n \longrightarrow +\infty} d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B}),$$

and hence $x$ must be in $\mathcal{A}_0$. Since $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$, then $d(u, \mathcal{T}x) = d(\mathcal{A}, \mathcal{B})$ for some $u \in \mathcal{A}$. Using the fact that $\mathcal{T}$ is a rational proximal contraction of the first kind, we get

$$\begin{aligned} d(u, x_{n+1}) \quad &\leq \quad \alpha d(x, x_n) + \frac{\beta[1 + d(x, u)]d(x_n, x_{n+1})}{1 + d(x, x_n)} \\ &\quad + \gamma\gamma[d(x, u) + d(x_n, x_{n+1})] + \delta[d(x, x_{n+1}) + d(x_n, u)]. \end{aligned}$$

Taking the limit as $n \longrightarrow +\infty$, we have

$$d(u, x) \leq (\gamma + \delta)d(u, x),$$

which implies that $x = u$, since $\gamma + \delta < 1$. Thus, it follows that

$$d(x, \mathcal{T}x) = d(u, \mathcal{T}x) = d(\mathcal{A}, \mathcal{B}),$$

that is, $x \in B_{est}(\mathcal{T})$. Again, following the same lines of the proof of the Theorem 5.1.4.3, we prove the uniqueness of the best proximity point of the mapping $\mathcal{T}$. To avoid repetitions we omit the details. $\square$

**Example 5.** Let $\mathcal{X} = \mathbb{R}$ endowed with the usual metric $d(x,y) = |x-y|$, for all $x, y \in \mathcal{X}$. Define $\mathcal{A} = [-1,1]$ and $\mathcal{B} = [-3,-2] \cup [2,3]$. Then, $d(\mathcal{A},\mathcal{B}) = 1$, $\mathcal{A}_0 = \{-1,1\}$ and $\mathcal{B}_0 = \{-2,2\}$. Also define $\mathcal{T} : \mathcal{A} \longrightarrow \mathcal{B}$ by

$$\mathcal{T}x = \begin{cases} 2 & \text{if } x \text{ is rational} \\ 3 & \text{otherwise.} \end{cases}$$

It is easy to show that $\mathcal{T}$ is a rational proximal contraction of the first and second kinds and $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$. Then, all the hypotheses of the Theorem 5.1.4.12 are satisfied and $d(1, \mathcal{T}(1)) = d(\mathcal{A},\mathcal{B})$. Clearly, the Theorem 5.1.4.8 is not applicable in this case.

### 5.1.5 Best proximity points for Geraghty's proximal contraction mappings

**Definition 5.1.5.1.** A mapping $T : A \to B$ is called a *Geraghty's proximal contraction of the first kind* if, for all $u, v, x, y \in A$, there exist $\beta \in \mathcal{S}$ such that

$$\left.\begin{array}{l} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \implies d(u,v) \le \beta(d(x,y))d(x,y).$$

**Definition 5.1.5.2.** A mapping $T : A \to B$ is called a *Geraghty's proximal contraction of the second kind* if, for all $u, v, x, y \in A$, there exist $\beta \in \mathcal{S}$ such that

$$\left.\begin{array}{l} d(u,Tx) = d(A,B)) \\ d(v,Ty) = d(A,B) \end{array}\right\} \implies d(Tu,Tv) \le \beta(d(Tx,Ty))d(Tx,Ty).$$

It is easy to see that, if we take $\beta(t) = k$, where $k \in [0,1)$, then a Geraghty's proximal contraction of the first kind and Geraghty's proximal contraction of the second kind reduce to a proximal contraction of the first kind and a proximal contraction of the second kind, respectively.

Next, we extend the result of Sadiq Basha [14] and Banach's fixed point theorem to the case of nonself-mappings satisfying Geraghty's proximal contraction condition.

**Theorem 5.1.5.3.** *Let $(X,d)$ be a complete metric space and $A$, $B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions:*

(a) *$S$ and $T$ are Geraghty's proximal contraction of the first kind;*

(b) *$g$ is an isometry;*

(c) *the pair $(S,T)$ is a proximal cyclic contraction;*

(d) *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*

(e) *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$.*

*Then there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

*converges to the element $x$. For any fixed $y_0 \in B_0$, the sequence $\{y_n\}$ defined by*

$$d(gy_{n+1}, Ty_n) = d(A, B)$$

*converges to the element $y$.*

*On the other hand, a sequence $\{u_n\}$ in $A$ converges to $x$ if there exists a sequence of positive numbers $\{\epsilon_n\}$ such that*

$$\lim_{n\to\infty} \epsilon_n = 0, \quad d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

*where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A, B)$.*

*Proof.* Let $x_0$ be a fixed element in $A_0$. In view of the fact that $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, it follows that there exists an element $x_1 \in A_0$ such that

$$d(gx_1, Sx_0) = d(A, B).$$

Again, since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element $x_2 \in A_0$ such that

$$d(gx_2, Sx_1) = d(A, B).$$

By the same method, we can find $x_n$ in $A_0$ such that

$$d(gx_n, Sx_{n-1}) = d(A, B).$$

So, inductively, one can determine an element $x_{n+1} \in A_0$ such that

$$d(gx_{n+1}, Sx_n) = d(A, B). \tag{5.1.59}$$

Since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, $S$ is a Geraghty's proximal contraction of the first kind, $g$ is an isometry and the property of $\beta$, it follows that, for each $n \geq 1$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(gx_{n+1}, gx_n) \\ &\leq \beta(d(x_n, x_{n-1}))d(x_n, x_{n-1}) \\ &\leq d(x_n, x_{n-1}), \end{aligned}$$

which implies that the sequence $\{d(x_{n+1}, x_n)\}$ is non-increasing and bounded below. Hence there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$. Suppose that $r > 0$. Observe that

$$\frac{d(x_{n+1}, x_n)}{d(x_n, x_{n-1})} \leq \beta(d(x_n, x_{n-1})),$$

which implies that $\lim_{n\to\infty} \beta(d(x_n, x_{n-1})) = 1$, Since $\beta \in \mathcal{S}$, we have $r = 0$ and so

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0. \tag{5.1.60}$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequences $\{x_{m_k}\}, \{x_{n_k}\}$ of $\{x_n\}$ such that, for any $n_k > m_k \geq k$,

$$r_k := d(x_{m_k}, x_{n_k}) \geq \varepsilon, \quad d(x_{m_k}, x_{n_k-1}) < \varepsilon$$

for any $k \in \{1, 2, 3, \cdots\}$. For each $n \geq 1$, let $\alpha_n := d(x_{n-1}, x_n)$. Then we have

$$\begin{aligned} \varepsilon \leq r_k &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \varepsilon + \alpha_{n_k-1} \end{aligned} \tag{5.1.61}$$

and so it follows from (5.2.2) and (5.2.3) that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{5.1.62}$$

Notice also that

$$\begin{aligned} &\varepsilon \leq r_k \\ &\quad \leq d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k}) + d(x_{m_k+1}, x_{n_k+1}) \\ &\quad = \alpha_{m_k} + \alpha_{n_k} + d(x_{m_k+1}, x_{n_k+1}) \\ &\quad \leq \alpha_{m_k} + \alpha_{n_k} + \beta(d(x_{m_k}, x_{n_k}))d(x_{m_k}, x_{n_k}) \end{aligned}$$

and so

$$\frac{r_k - \alpha_{m_k} - \alpha_{n_k}}{d(x_{m_k}, x_{n_k})} \leq \beta(d(x_{m_k}, x_{n_k})).$$

Taking $k \to \infty$ in the above inequality, by (5.2.2), (5.2.4) and $\beta \in \mathcal{S}$, we get $\varepsilon = 0$, which is a contradiction. So we know that the sequence $\{x_n\}$ is a Cauchy sequence. Hence $\{x_n\}$ converges to some element $x \in A$.

Similarly, in view of the fact that $T(B_0) \subseteq A_0$ and $A_0 \subseteq g(A_0)$, we can conclude that there exists a sequence $\{y_n\}$ such that converge to some element $y \in B$. Since the pair $(S, T)$ is a proximal cyclic contraction and $g$ is an isometry, we have

$$d(x_{n+1}, y_{n+1}) = d(gx_{n+1}, gy_{n+1}) \leq kd(x_n, y_n) + (1-k)d(A, B). \tag{5.1.63}$$

Taking $n \to \infty$ in (5.2.5), it follows that

$$d(x, y) = d(A, B) \tag{5.1.64}$$

and so $x \in A_0$ and $y \in B_0$. Since $S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$, there exist $u \in A$ and $v \in B$ such that

$$d(u, Sx) = d(A, B), \quad d(v, Ty) = d(A, B). \tag{5.1.65}$$

From (5.2.1) and (5.2.7), since $S$ is a Geraghty's proximal contraction of first kind of $S$, we get

$$d(u, gx_{n+1}) \leq \beta(d(x, x_n))d(x, x_n). \tag{5.1.66}$$

Letting $n \to \infty$ in the above inequality, we get $d(u, gx) \leq 0$ and so $u = gx$. Therefore, we have

$$d(gx, Sx) = d(A, B). \tag{5.1.67}$$

Similarly, we can show that $v = gy$ and so

$$d(gy, Ty) = d(A, B). \tag{5.1.68}$$

From (5.2.6), (5.2.9) and (5.2.10), we get

$$d(x, y) = d(gx, Sx) = d(gy, Ty) = d(A, B).$$

Next, to prove the uniqueness, suppose that there exist $x^* \in A$ and $y^* \in B$ with $x \neq x^*$ and $y \neq y^*$ such that

$$d(gx^*, Sx^*) = d(A, B), \quad d(gy^*, Ty^*) = d(A, B).$$

Since $g$ is an isometry and $S$ is a Geraghty's proximal contraction of the first kind, it follows that

$$d(x,x^*) = d(gx,gx^*) \leq \beta(d(x,x^*))d(x,x^*)$$

and hence

$$1 = \frac{d(x,x^*)}{d(x,x^*)} \leq \beta(d(x,x^*)) < 1,$$

which is a contradiction. Thus we have $x = x^*$. Similarly, we can prove that $y = y^*$.

On the other hand, let $\{u_n\}$ be a sequence in $A$ and $\{\epsilon_n\}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \epsilon_n = 0, \quad d(u_{n+1}, z_{n+1}) \leq \epsilon_n, \tag{5.1.69}$$

where $z_{n+1} \in A$ satisfies the condition that

$$d(gz_{n+1}, Su_n) = d(A,B). \tag{5.1.70}$$

By (5.2.1) and (5.2.11), since $S$ is a Geraghty's proximal contraction of first kind and $g$ is an isometry, we have

$$d(x_{n+1}, z_{n+1}) = d(gx_{n+1}, gz_{n+1}) \leq \beta(d(x_n,u_n))d(x_n,u_n).$$

For any $\epsilon > 0$, choose a positive integer $N$ such that $\epsilon_n \leq \epsilon$ for all $n \geq N$. Observe that

$$\begin{aligned} d(x_{n+1}, u_{n+1}) &\leq d(x_{n+1}, z_{n+1}) + d(z_{n+1}, u_{n+1}) \\ &\leq \beta(d(x_n,u_n))d(x_n,u_n) + \epsilon_n, \\ &\leq d(x_n,u_n) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it can conclude that, for all $n \geq N$, the sequence $\{d(x_n,u_n)\}$ is non-increasing and bounded below and hence converges to some nonnegative real number $r'$. Since the sequence $\{x_n\}$ converges to $x$, we get

$$\lim_{n\to\infty} d(u_n, x) = \lim_{n\to\infty} d(u_n, x_n) = r'. \tag{5.1.71}$$

Suppose that $r' > 0$. Since

$$\begin{aligned} d(u_{n+1}, x) &\leq d(u_{n+1}, x_{n+1}) + d(x_{n+1}, x) \\ &\leq \beta(d(x_n,u_n))d(x_n,u_n) + \epsilon_n + d(x_{n+1}, x) \\ &\leq \beta(d(x_n,u_n))d(x_n,u_n) + \epsilon_n + d(x_{n+1}, x), \end{aligned} \tag{5.1.72}$$

it follow from the inequality (5.1.69), (5.1.71) and (5.2.12) that

$$\frac{d(u_{n+1}, x) - \epsilon_n - d(x_{n+1}, x)}{d(x_n, u_n)} \leq \beta(d(x_n, u_n)) < 1, \tag{5.1.73}$$

which implies that $\beta(d(x_n, u_n)) \to 1$ and so $d(u_n, x_n) \to 0$, that is,

$$\lim_{n\to\infty} d(u_n, x) = \lim_{n\to\infty} d(u_n, x_n) = 0,$$

which is a contradiction. Thus $r' = 0$ and hence $\{u_n\}$ is convergent to the point $x$. This completes the proof. □

If $g$ is the identity mapping in Theorem 5.1.5.3, then we obtain the following:

**Corollary 5.1.5.4.** *Let $(X, d)$ be a complete metric space and $A$, $B$ be nonempty closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ be the mappings satisfying the following conditions:*

(a) *$S$ and $T$ are Geraghty's proximal contractions of the first kind;*

(b) *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*

(c) *the pair $(S, T)$ is a proximal cyclic contraction.*

*Then there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

If take $\beta(t) = k$, where $0 \leq k < 1$, we obtain following:

**Corollary 5.1.5.5.** *Let $(X, d)$ be a complete metric space and $A$, $B$ be nonempty closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ be the mappings satisfying the following conditions:*

(a) *$S$ and $T$ are proximal contractions of first kind;*

(b) *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*

(c) *the pair $(S, T)$ is a proximal cyclic contraction.*

*Then there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

**Theorem 5.1.5.6.** *Let $(X, d)$ be a complete metric space and $A$, $B$ be nonempty closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$ and $g : A \to A$ be the mappings satisfying the following conditions:*

(a) *$S$ is a Geraghty's proximal contraction of first and second kinds;*

(b) *$g$ is an isometry;*

(c) *$S$ preserves isometric distance with respect to $g$;*

(d) *$S(A_0) \subseteq B_0$;*

(e) *$A_0 \subseteq g(A_0)$.*

*Then there exists a unique point $x \in A$ such that*

$$d(gx, Sx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

*converges to the element $x$.*

*On the other hand, a sequence $\{u_n\}$ in $A$ converges to $x$ if there exists a sequence $\{\epsilon_n\}$ of positive numbers such that*

$$\lim_{n\to\infty} \epsilon_n = 0, \quad d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

*where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A, B)$.*

*Proof.* Since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, as in the proof of Theorem 5.3.1.5, we can construct the sequence $\{x_n\}$ in $A_0$ such that

$$d(gx_{n+1}, Sx_n) = d(A, B) \tag{5.1.74}$$

for each $n \geq 1$. Since $g$ is an isometry and $S$ is a Geraghty's proximal contraction of the first kind, we see that

$$d(x_n, x_{n+1}) = d(gx_n, gx_{n+1}) \leq \beta(d(x_n, x_{n-1}))d(x_n, x_{n-1})$$

for all $n \geq 1$. Again, similarly, we can show that the sequence $\{x_n\}$ is a Cauchy sequence and so it converges to some $x \in A$. Since $S$ is a Geraghty's proximal contraction of the second kind and preserves the isometric distance with respect to $g$, we have

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &= d(Sgx_n, Sgx_{n+1}) \\ &\leq \beta(d(Sx_{n-1}, Sx_n))d(Sx_{n-1}, Sx_n) \\ &\leq d(Sx_{n-1}, Sx_n), \end{aligned}$$

which means that the sequence $\{d(Sx_{n+1}, Sx_n)\}$ is non-increasing and bounded below. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = r.$$

Suppose that $r > 0$. Observe that

$$\frac{d(Sx_n, Sx_{n+1})}{d(Sx_{n-1}, Sx_n)} \leq \beta(d(Sx_{n-1}, Sx_n)).$$

Taking $k \to \infty$ in the above inequality, we get $\beta(d(Sx_{n-1}, Sx_n)) \to 1$ and so, since $\beta \in \mathcal{S}$, we have $r = 0$. Thus we have

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = 0. \tag{5.1.75}$$

Now, we claim that $\{Sx_n\}$ is a Cauchy sequence. Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequences $\{Sx_{m_k}\}, \{Sx_{n_k}\}$ of $\{Sx_n\}$ such that, for any $n_k > m_k \geq k$,

$$r_k := d(Sx_{m_k}, Sx_{n_k}) \geq \varepsilon, \quad d(Sx_{m_k}, Sx_{n_k-1}) < \varepsilon$$

for any $k \in \{1, 2, 3, \cdots\}$. For each $n \geq 1$, let $\gamma_n := d(Sx_{n-1}, Sx_n)$. Then we have

$$\begin{aligned} \varepsilon \leq r_k &\leq d(Sx_{m_k}, Sx_{n_k-1}) + d(Sx_{n_k-1}, Sx_{n_k}) \\ &< \varepsilon + \gamma_{n_k-1} \end{aligned} \tag{5.1.76}$$

and so it follows from (5.1.75) and (5.2.18) that

$$\lim_{k\to\infty} r_k = \varepsilon.$$

Notice also that

$$\begin{aligned} &\varepsilon \leq r_k \\ &\quad \leq d(Sx_{m_k}, Sx_{m_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) + d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &\quad = \gamma_{m_k} + \gamma_{n_k} + d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &\quad \leq \gamma_{m_k} + \gamma_{n_k} + \beta(d(Sx_{m_k}, Sx_{n_k}))d(Sx_{m_k}, Sx_{n_k}). \end{aligned}$$

So, it follow that

$$1 = \lim_{k\to\infty} \frac{r_k - \gamma_{m_k} - \gamma_{n_k}}{d(Sx_{m_k}, Sx_{n_k})} \leq \lim_{k\to\infty} \beta(d(Sx_{m_k}, Sx_{n_k})) < 1$$

and so $\lim_{k\to\infty} \beta(d(Sx_{m_k}, Sx_{n_k})) = 1$. Since $\beta \in \mathcal{S}$, we have $\lim_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) = 0$, that is, $\varepsilon = 0$, which is a contradiction. So, we obtain the claim and then it converges to some $y \in B$. Therefore, we can conclude that

$$d(gx, y) = \lim_{n\to\infty} d(gx_{n+1}, Sx_n) = d(A, B),$$

which implies that $gx \in A_0$. Since $A_0 \subseteq g(A_0)$, we have $gx = gz$ for some $z \in A_0$ and then $d(gx, gz) = 0$. By the fact that $g$ is an isometry, we have $d(x, z) = d(gx, gz) = 0$. Hence $x = z$ and so $x \in A_0$. Since $S(A_0) \subseteq B_0$, there exists $u \in A$ that

$$d(u, Sx) = d(A, B). \tag{5.1.77}$$

Since $S$ is a Geraghty's proximal contraction of the first kind, it follows from (5.2.16) and (5.2.19) that

$$d(u, gx_{n+1}) \leq \beta(d(x, x_n))d(x, x_n) \tag{5.1.78}$$

for all $n \geq 1$. Taking $n \to \infty$ in (5.2.20), it follows that the sequence $\{gx_n\}$ converge to a point $u$. Since $g$ is continuous and $\lim_{n\longrightarrow\infty} x_n = x$, we have $gx_n \to gx$ as $n \to \infty$. By the uniqueness of the limit, we conclude that $u = gx$. Therefore, it follows that $d(gx, Sx) = d(u, Sx) = d(A, B)$.

The uniqueness and the remaining part of the proof follow from the proof of Theorem 5.1.5.3. This completes the proof. □

If $g$ is the identity mapping in Theorem 5.1.5.6, then we obtain the following:

**Corollary 5.1.5.7.** *Let $(X, d)$ be a complete metric space and $A$, $B$ be nonempty closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$ be the mappings satisfying the following conditions:*

(a) *$S$ is a Geraghty's proximal contraction of first and second kinds;*

(b) *$S(A_0) \subseteq B_0$.*

*Then there exists a unique point $x \in A$ such that*

$$d(x, Sx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by*

$$d(x_{n+1}, Sx_n) = d(A, B)$$

*converges to the best proximity point $x$ of $S$.*

**Corollary 5.1.5.8.** *Let $(X, d)$ be a complete metric space and $A$, $B$ be nonempty closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$ be a mapping satisfying the following conditions:*

(a) *$S$ is a proximal contractions of first and second kinds;*

(b) $S(A_0) \subseteq B_0$ .

*Then there exists a unique point $x \in A$ such that*

$$d(x, Sx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by*

$$d(x_{n+1}, Sx_n) = d(A, B)$$

*converges to the best proximity point $x$ of $S$.*

**Proposition 5.1.5.9.** *Let $f : [0, \infty) \to [0, \infty)$ be a function defined by $f(t) = \ln(1 + t)$. Then we have the following inequality:*

$$f(a) - f(b) \leq f(|a - b|) \tag{5.1.79}$$

*for all $a, b \in [0, \infty)$.*

*Proof.* If $x = y$, we have done. Suppose that $x > y$. Then since we have

$$\frac{1 + x}{1 + y} = \frac{1 + x + y - y}{1 + y} = 1 + \frac{x - y}{1 + y} < 1 + |x - y|,$$

it follow that $\ln(1 + x) - \ln(1 + y) < \ln(1 + |x - y|)$. In case $x < y$, by the similar argument, we can prove that the inequality (5.2.14) holds. □

**Proposition 5.1.5.10.** *For each $x, y \in \mathbb{R}$, we have the following inequality:*

$$\frac{1}{(1 + |x|)(1 + |y|)} \leq \frac{1}{1 + |x - y|}$$

*holds.*

*Proof.* Since

$$\begin{aligned}1+|x-y| &\le 1+|x|+|y|\\ &\le 1+|x|+|y|+|x||y|\\ &= (1+|x|)(1+|y|),\end{aligned}$$

so that

$$\frac{1}{(1+|x|)(1+|y|)} \le \frac{1}{1+|x-y|}.$$

□

**Example 6.** Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Let

$$A = \{(0,x) : x \in \mathbb{R}\}, \quad B = \{(2,y) : y \in \mathbb{R}\}.$$

Then $d(A,B) = 2$. Define the mappings $S : A \to B$ as follows:

$$S((0,x)) = \big(2, \ln(1+|x|)\big).$$

First, we show that $S$ is Geraghty's proximal contractions the first kind with $\beta \in \mathcal{S}$ defined by

$$\beta(t) = \begin{cases} 1, & t = 0,\\ \dfrac{\ln(1+t)}{t}, & t > 0.\end{cases}$$

Let $(0,x_1), (0,x_2), (0,a_1)$ and $(0,a_1)$ be elements in $A$ satisfying

$$d((0,a_1), S(0,x_1)) = d(A,B) = 2, \quad d((0,a_2), S(0,x_2)) = d(A,B) = 2.$$

Then we have $a_i = \ln(1+|x_i|)$ for $i = 1,2$. If $x_1 = x_2$, we have done. Assume that $x_1 \neq x_2$. Then, by Proposition 5.1.5.9 and the fact that the function $f(x) = \ln(1+t)$ is increasing, we have

$$\begin{aligned} d((0,a_1),(0,a_2)) &= d((0,\ln(1+|x_1|)),(0,\ln(1+|x_2|)))\\ &= |\ln(1+|x_1|) - \ln(1+|x_2|)|\\ &\le |\ln(1+||x_1|-|x_2||)|\\ &\le |\ln(1+|x_1-x_2|)|\\ &= \frac{|\ln(1+|x_1-x_2|)|}{|x_1-x_2|}|x_1-x_2|\\ &= \beta(d((0,x_1),(0,x_2))d((0,x_1),(0,x_2)).\end{aligned}$$

Thus $S$ is a Geraghty's proximal contraction of the first kind.

Next, we prove that $S$ is not a proximal contraction. Suppose $S$ is proximal contraction, then for each $(0,x^*),(0,y^*),(0,a^*),(0,b^*) \in A$ satisfying

$$d((0,x^*),S(0,a^*)) = d(A,B) = 2 \text{ and } d((0,y^*),S(0,b^*)) = d(A,B) = 2, \quad (5.1.80)$$

there exists $k \in [0,1)$ such that

$$d((0,x^*),(0,y^*)) \leq kd((0,a^*),(0,b^*)).$$

From (5.1.80), we get $x^* = \ln(1+|a^*|)$ and $y^* = \ln(1+|b^*|)$ and so

$$\begin{aligned}\big|\ln(1+|a^*|) - \ln(1+|b^*|)\big| &= d((0,x^*),(0,y^*)) \\ &\leq kd((0,a^*),(0,b^*)) \\ &= k|a^* - b^*|.\end{aligned}$$

Letting $b^* = 0$, we get

$$1 = \lim_{|a^*|\to 0^+} \frac{\big|\ln(1+|a^*|)\big|}{|a^*|} \leq k < 1,$$

which is a contradiction. Thus $S$ is not a proximal contraction.

**Example 7.** Consider the complete metric space $\mathbb{R}^2$ with metric defined by

$$d((x_1,x_2),(y_1,y_2)) = |x_1 - x_2| + |y_1 - y_2|,$$

for all $(x_1,x_2),(y_1,y_2) \in \mathbb{R}^2$. Let

$$A = \{(0,x) : x \in \mathbb{R}\}, \quad B = \{(2,y) : y \in \mathbb{R}\}.$$

Define two mappings $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ as follows:

$$S((0,x)) = \big(2, \frac{1}{2+|x|}\big), \quad T((2,y)) = \big(0, \frac{1}{2+|y|}\big), \quad g((x,y)) = (x,-y).$$

Then $d(A,B) = 2$, $A_0 = A$, $B_0 = B$ and the mapping $g$ is an isometry.

Next, we show that $S$ and $T$ are Geraghty's proximal contractions the first kind with $\beta \in \mathcal{S}$ defined by

$$\beta(t) = \frac{1}{1+t} \quad \text{for all } t \geq 0.$$

Let $(0,x_1),(0,x_2),(0,a_1)$ and $(0,a_1)$ be elements in $A$ satisfying

$$d((0,a_1),S(0,x_1)) = d(A,B) = 2, \quad d((0,a_2),S(0,x_2)) = d(A,B) = 2.$$

Then we have

$$a_i = \frac{1}{2+|x_i|} \text{ for } i = 1, 2.$$

If$x_1 = x_2$, we have done. Assume that $x_1 \neq x_2$, Then, by Proposition 5.1.5.10, we have

$$\begin{aligned}
d((0,a_1),(0,a_2)) &= d\big((0,\frac{1}{2+|x_1|}),(0,\frac{1}{2+|x_2|})\big) \\
&= \left|\frac{1}{2+|x_1|} - \frac{1}{2+|x_2|}\right| \\
&= \left|\frac{|x_1|-|x_2|}{(2+|x_1|)(2+|x_2|)}\right| \\
&\leq \left|\frac{x_1-x_2}{(1+|x_1|)(1+|x_2|)}\right| \\
&\leq \frac{1}{1+|x_1-x_2|}\big|x_1-x_2\big| \\
&= \beta(d((0,x_1),(0,x_2))d((0,x_1),(0,x_2)).
\end{aligned}$$

Thus $S$ is a Geraghty's proximal contraction of the first kind. Similarly, we can see that $T$ is a Geraghty's proximal contraction of the first kind. Next, we show that the pair $(S,T)$ is a proximal cyclic contraction. Let $(0,u),(0,x) \in A$ and $(2,v),(2,y) \in B$ be such that

$$d((0,u),S(0,x)) = d(A,B) = 2, \quad d((2,v),T(2,y)) = d(A,B) = 2.$$

Then we get

$$u = \frac{1}{2+|x|}, \quad v = \frac{1}{2+|y|}.$$

In case $x = y$, clear. Suppose that $x \neq y$, then we have

$$\begin{aligned}
d((0,u),(2,v)) &= |u - v| + 2 \\
&= \Big|\frac{1}{2+|x|} - \frac{1}{2+|y|}\Big| + 2 \\
&= \left|\frac{|x| - |y|}{(2+|x|)(2+|y|)}\right| + 2 \\
&\leq \frac{|x-y|}{(2+|x|)(2+|y|)} + 2 \\
&\leq \frac{1}{4}\big|x - y\big| + 2 \\
&\leq k\big(\big|x-y\big| + 2\big) + (1-k)2 \\
&= kd((0,x),(2,y)) + (1-k)d(A,B),
\end{aligned}$$

where $k = [\frac{1}{4}, 1)$. Hence the pair $(S,T)$ is a proximal cyclic contraction. Therefore, all the hypothesis of Theorem 5.1.5.3 are satisfied. Further, it is easy to see that $(0,0) \in A$ and $(2,0) \in B$ are the unique element such that

$$d(g(0,0), S(0,0)) = d(g(2,0), T(2,0)) = d((0,0),(2,0)) = d(A,B).$$

## 5.2 COMMON BEST PROXIMITY POINT

### 5.2.1 Common best proximity point

**Theorem 5.2.1.1.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A$ is approximatively compact with respect to $B$. Also, assume that $A_0$ and $B_0$ are nonempty. Let the non-self mapping $S : A \to B$, $T : A \to B$ satisfy the following conditions:*

*(a) For each $x$ and $y$ are elements in $A$,*

$$d(Sx, Sy) \leq d(Tx, Ty) - \varphi(d(Tx, Ty)),$$

*where, $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.*

*(b) $T$ is continuous.*

*(c) $S$ and $T$ commute proximally.*

*(d) $S$ and $T$ can be swapped proximally.*

*(e) $S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$ .*

*Then, there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B) \text{ and } d(x, Sx) = d(A, B).$$

*Moreover, if $x^*$ is another common best proximity point of the mappings $S$ and $T$, then it is necessary that*

$$d(x, x^*) \leq 2d(A, B).$$

*Proof.* Let $x_0$ a fixed element in $A_0$. In view of the fact that $S(A_0) \subseteq T(A_0)$ it is ascertained that there exists an element $x_1 \in A_0$ such that $Sx_0 = Tx_1$. Again, since $S(A_0) \subseteq T(A_0)$, there exists an element $x_2 \in A_0$ such that $Sx_1 = Tx_2$. By similar fashion we can find $x_n$ in $A_0$ such that

$$Sx_{n-1} = Tx_n, \tag{5.2.1}$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &\leq d(Tx_n, Tx_{n+1}) - \varphi(d(Tx_n, Tx_{n+1})) \\ &\leq d(Sx_{n-1}, Sx_n) - \varphi(d(Sx_{n-1}, Sx_n)) \\ &\leq d(Sx_{n-1}, Sx_n) \end{aligned} \tag{5.2.2}$$

this mean that the sequence $\{d(Sx_{n-1}, Sx_n)\}$ is non-increasing and bounded below. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(Sx_{n-1}, Sx_n) = r. \tag{5.2.3}$$

If $r > 0$, then

$$d(Sx_n, Sx_{n+1}) \leq d(Sx_{n-1}, Sx_n) - \varphi(d(Sx_{n-1}, Sx_n)). \tag{5.2.4}$$

Taking $n \to \infty$, in inequality (5.2.4), by the continuities of $\varphi$, we get $r \leq r - \varphi(r) < r$, which is a contradiction unless $r = 0$. Therefore

$$\lim_{n\to\infty} d(Sx_{n-1}, Sx_n) = 0. \tag{5.2.5}$$

Next, we will prove that $\{Sx_n\}$ is a Cauchy sequence. We distinguish two cases.

**Case I** Suppose there exits $n \in \mathbb{N}$ such that $Sx_n = Sx_{n+1}$, we observe that

$$\begin{aligned} d(Sx_{n+1}, Sx_{n+2}) &\leq d(Tx_{n+1}, Tx_{n+2}) - \varphi(d(Tx_{n+1}, Tx_{n+2})) \\ &\leq d(Sx_n, Sx_{n+1}) - \varphi(d(Sx_n, Sx_{n+1})) \\ &= 0, \end{aligned}$$

which implies that $Sx_{n+1} = Sx_{n+2}$. So, for every $m > n$, we conclude that $Sx_m = Sx_n$. Hence $\{Sx_n\}$ is a Cauchy sequence in $B$.

**Case II** The successive terms of $\{Sx_n\}$ are different. Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequence $\{Sx_{m_k}\}, \{Sx_{n_k}\}$ of $\{Sx_n\}$ with $n_k > m_k \geq k$ such that

$$d(Sx_{m_k}, Sx_{n_k}) \geq \varepsilon \quad \text{and} \quad d(Sx_{m_k}, Sx_{n_k-1}) < \varepsilon. \tag{5.2.6}$$

By using (5.2.6) and triangular inequality, we get

$$\begin{aligned} \varepsilon &\leq d(Sx_{m_k}, Sx_{n_k}) \\ &\leq d(Sx_{m_k}, Sx_{n_k-1}) + d(Sx_{n_k-1}, Sx_{n_k}) \\ &\leq \varepsilon + d(Sx_{n_k-1}, Sx_{n_k}). \end{aligned} \tag{5.2.7}$$

Using (5.2.7) and (5.2.5), we have

$$d(Sx_{m_k}, Sx_{n_k}) \to \varepsilon \quad \text{as} \quad k \to \infty. \tag{5.2.8}$$

Again, by the triangular inequality, we get

$$d(Sx_{m_k}, Sx_{n_k}) \leq d(Sx_{m_k}, Sx_{m_k+1}) + d(Sx_{m_k+1}, Sx_{n_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) \tag{5.2.9}$$

and

$$d(Sx_{m_k+1}, Sx_{n_k+1}) \leq d(Sx_{m_k+1}, Sx_{m_k}) + d(Sx_{m_k}, Sx_{n_k}) + d(Sx_{n_k}, Sx_{n_k+1}). \tag{5.2.10}$$

From, (5.2.5), (5.2.8), (5.2.9) and (5.2.10), we obtain that

$$d(Sx_{m_k+1}, Sx_{n_k+1}) \to \varepsilon \quad \text{as} \quad k \to \infty. \tag{5.2.11}$$

In view of the fact that

$$\begin{aligned} d(Sx_{m_k+1}, Sx_{n_k+1}) &\leq d(Tx_{m_k+1}, Tx_{n_k+1}) - \varphi(d(Tx_{m_k+1}, Tx_{n_k+1})) \\ &\leq d(Sx_{m_k}, Sx_{n_k}) - \varphi(d(Sx_{m_k}, Sx_{n_k})). \end{aligned} \tag{5.2.12}$$

Letting, $k \to \infty$, in inequality (5.2.12), we obtain

$$\varepsilon \leq \varepsilon - \varphi(\varepsilon),$$

which is a contradiction, by a property of $\varphi$. Then, we deduce that $\{Sx_n\}$ is a Cauchy sequence in $B$. Since $B$ is closed subset a complete metric space $X$, then there exists $y \in B$ such that $Sx_n \to y$ as $n \to \infty$. Consequently, we have that the sequence $\{Tx_n\}$ also converges to $y$. From $S(A_0) \subseteq B_0$, there exists an element $u_n \in A$ such that

$$d(Sx_n, u_n) = d(A, B), \tag{5.2.13}$$

for all $n \in \mathbb{N}$. So, it follows from (5.2.1) and (5.2.13) that

$$d(Tx_n, u_{n-1}) = d(Sx_{n-1}, u_{n-1}) = d(A, B), \tag{5.2.14}$$

for all $n \in \mathbb{N}$. By (5.2.13), (5.2.14) and the fact that the mappings $S$ and $T$ are commuting proximally, we obtain

$$Tu_n = Su_{n-1} \tag{5.2.15}$$

for all $n \in \mathbb{N}$. Moreover, we have

$$\begin{aligned} d(y,A) &\leq d(y,u_n) \\ &\leq d(y,Sx_n) + d(Sx_n,u_n) \\ &= d(y,Sx_n) + d(A,B) \\ &\leq d(y,Sx_n) + d(y,A). \end{aligned} \tag{5.2.16}$$

Therefore $d(y,u_n) \to d(y,A)$ as $n \to \infty$. Since $A$ is approximatively compact with respect to $B$, then there exists subsequence $\{u_{n_k}\}$ of sequence $\{u_n\}$ such that converging to some element $u \in A$. Further, since $d(y,u_{n_k-1}) \to d(y,A)$ and $A$ is approximatively compact with respect to $B$, then there exists subsequence $\{u_{n_{k_j}-1}\}$ of sequence $\{u_{n_k-1}\}$ such that converging to some element $v \in A$. By the continuity of the mappings $S$ and $T$, we have

$$Tu = \lim_{j\to\infty} Tu_{n_{k_j}} = \lim_{k\to\infty} Su_{n_{k_j}-1} = Sv \tag{5.2.17}$$

and

$$\begin{aligned} d(y,u) &= \lim_{k\to\infty} d(Sx_{n_k},u_{n_k}) = d(A,B), \\ d(y,v) &= \lim_{j\to\infty} d(Tx_{n_{k_j}},u_{n_{k_j}-1}) = d(A,B). \end{aligned} \tag{5.2.18}$$

Because $S$ and $T$ can be swapped proximally, we get

$$Tv = Su. \tag{5.2.19}$$

Next, we will prove that $Su = Sv$, suppose the contrary, by (5.2.17), (5.2.18) (5.2.19) and property of $\varphi$ , we have

$$\begin{aligned} d(Su,Sv) &\leq d(Tu,Tv) - \varphi(d(Tu,Tv)) \\ &\leq d(Sv,Su) - \varphi(d(Sv,Su)) \\ &< d(Sv,Su) \end{aligned}$$

which is a contradiction. Thus $Su = Sv$ and also $Tu = Su$. Since $S(A_0)$ is contained in $B_0$, there exists an element $x$ in A such that

$$d(x,Tu) = d(A,B) \text{ and } d(x,Su) = d(A,B).$$

By the commuting proximally of $S$ and $T$, $Sx = Tx$. Consequently, we have

$$\begin{aligned} d(Su,Sx) &\leq d(Tu,Tx) - \varphi(d(Tu,Tx)) \\ &\leq d(Su,Sx) - \varphi(d(Su,Sx)). \end{aligned} \tag{5.2.20}$$

In inequality (5.2.20), if $Su \neq Sx$, then $d(Su, Sx) \leq d(Su, Sx) - \varphi(d(Su, Sx)) < d(Su, Sx)$, it is impossible, So, we have $Su = Sx$ and hence $Tu = Tx$. It follows that

$$d(x, Tx) \quad = d(x, Tu) \quad = d(A, B)$$

and

$$d(x, Sx) \quad = d(x, Su) \quad = d(A, B).$$

Therefore, $x$ is a common best proximity point of $S$ and $T$. Suppose that $x^*$ is another common best proximity point of the mappings $S$ and $T$, so that

$$d(x^*, Tx^*) \quad = \quad d(A, B)$$

and

$$d(x^*, Sx^*) \quad = \quad d(A, B).$$

By the commuting proximally of $S$ and $T$, we get $Sx = Tx$ and $Sx^* = Tx^*$. Consequently, we have

$$\begin{aligned} d(Sx^*, Sx) &\leq d(Tx^*, Tx) - \varphi(d(Tx^*, Tx)) \\ &\leq d(Sx^*, Sx) - \varphi(d(Sx^*, Sx)). \end{aligned} \tag{5.2.21}$$

In inequality (5.2.21) , if $Sx^* \neq Sx$, then

$$d(Sx^*, Sx) \leq d(Sx^*, Sx) - \varphi(d(Sx^*, Sx)) < d(Sx^*, Sx),$$

it is impossible. So, we have $Sx = Sx^*$. Moreover, it can be concluded that

$$\begin{aligned} d(x, x^*) &\leq d(x, Sx) + d(Sx, Sx^*) + d(Sx^*, x^*) \\ &= d(A, B) + d(A, B) \\ &= 2d(A, B) \end{aligned}$$

and the proof is completes. □

If we take $\varphi(t) = (1 - \alpha)t$, where $0 \leq \alpha < 1$ in Theorem 5.2.1.1, we obtain following corollary:

**Corollary 5.2.1.2.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A$ is approximatively compact with respect to $B$. Also, assume that $A_0$ and $B_0$ are nonempty. Let the non-self mapping $S : A \to B$, $T : A \to B$ satisfy the following conditions.*

*(a) There is a non-negative real number $\alpha < 1$ such that*

$$d(Sx_1, Sx_2) \leq \alpha d(Tx_1, Tx_2)$$

*for all $x_1, x_2 \in A$.*

*(b) $T$ is continuous.*

*(c) $S$ and $T$ commute proximally.*

*(d) $S$ and $T$ can be swapped proximally.*

*(e) $S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$ . Then, there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B) \text{ and } d(x, Sx) = d(A, B).$$

*Further, if $x^*$ is another common best proximity point of the mappings $S$ and $T$, then it is necessary that*

$$d(x, x^*) \leq 2d(A, B).$$

**Example 8.** Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Let

$$A = \{(x, 1) \, : \, 0 \leq x \leq 1\}$$

and

$$B = \{(x, -1) : 0 \leq x \leq 1\}.$$

Define two mappings $S : A \to B$, $T : A \to B$ as follows:

$$S(x, 1) = \big(x - \frac{x^2}{2}, -1\big)$$

and

$$T((x, 1)) = \big(x, -1\big).$$

It is easy to see that $d(A, B) = 2$, $A_0 = A$ and $B_0 = B$. Further, $S$ and $T$ are continuous and $A$ is approximatively compact with respect to $B$.

First, we will show that $S$ and $T$ are satisfy the condition $(a)$ of of Theorem 5.2.1.1 with $\varphi : [0, \infty) \to [0, \infty)$ defined by $\varphi(t) = \dfrac{t^2}{2}$, for all $t \in [0, \infty)$.

Let $(x,1),(y,1) \in A$, without loss generality we can take that $x > y$, then, we have

$$\begin{aligned} d(S(x,1),S(y,1)) &= \left|(x-\frac{x^2}{2})-(y-\frac{y^2}{2})\right| \\ &= \left(x-y\right)-\frac{1}{2}\left(x^2-y^2\right) \\ &= \left(x-y\right)-\frac{1}{2}\left((x-y)(x+y)\right) \\ &\leq \left(x-y\right)-\frac{1}{2}\left(x-y\right)^2 \\ &= d(T(x,1),T(y,1))-\varphi(d(T(x,1),T(y,1))). \end{aligned}$$

Next, we will show that $S$ and $T$ commute proximally. Let $(u,1),(v,1),(x,1) \in A$ are satisfying

$$d((u,1),S(x,1)) = d(A,B) = 2 \text{ and } d((v,1),T(x,1)) = d(A,B) = 2.$$

It follows that

$$u = x - \frac{x^2}{2} \text{ and } v = x,$$

and hence

$$S(v,1) = \left(v-\frac{v^2}{2},-1\right) = \left(x-\frac{x^2}{2},-1\right) = (u,-1) = T(u,1).$$

Finally, we will show that $S$ and $T$ swapped proximally. If it is true that

$$d((u,1),(y,-1)) = d((v,1),(y,-1)) = d(A,B) = 2 \text{ and } S(u,1) = T(v,1),$$

for some $(u,1),(v,1) \in A$ and $(y,-1) \in B$, then we get $u = v = 0$ and thus

$$S(v,1) = T(u,1).$$

Therefore, all hypothesis of Theorem 5.2.1.1 are satisfied.
Furthermore, $(0,1) \in A$ is a common best proximity point of $S$ and $T$, because

$$d((0,1),S(0,1)) = d((0,1),(0,-1)) = d((0,1),T((0,1)) = d(A,B).$$

On the other hand, suppose that there exists $k \in [0,1)$ such that

$$d(S(x,1),S(y,1)) \leq kd(T(x,1),T(y,1)),$$

that is

$$\left|(x - \frac{x^2}{2}) - (y - \frac{y^2}{2})\right| \leq k\left|x - y\right|.$$

Putting $y = 0$, it follow that

$$1 = \lim_{x \to 0^+} \left|(1 - \frac{x}{2})\right| \leq k < 1,$$

which is a contradiction. Therefore, the results of Sadiq Basha in [14] can not be applied to this example and our main result Theorem 5.2.1.1.

### 5.2.2 Tripled best proximity point

**Theorem 5.2.2.1.** *Let $A, B$ be nonempty closed subsets of a metric space $X$ and $F : A^3 \to B, G : B^3 \to A$ be two mappings such that the ordered pair $(F, G)$ is a cyclic contraction. For any $(x_0, y_0, z_0) \in A^3$, we define the sequence $\{x_n\}, \{y_n\}, \{z_n\}$ in $X$ by*

$$x_{2n+1} = F(x_{2n}, y_{2n}, z_{2n}), \quad x_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1}),$$

$$y_{2n+1} = F(y_{2n}, x_{2n}, y_{2n}), \quad y_{2n+2} = G(y_{2n+1}, x_{2n+1}, y_{2n+1}),$$

$$z_{2n+1} = F(z_{2n}, y_{2n}, x_{2n}), \quad z_{2n+2} = G(z_{2n+1}, y_{2n+1}, x_{2n+1})$$

*for all $n \in \mathbb{N} \cup \{0\}$. If $d(A, B) = 0$, then $F$ has a tripled fixed point $(p, q, r) \in A^3$ and $G$ has a tripled fixed point $(p', q', r') \in B^3$. Moreover, we have*

$$x_{2n} \to p, \quad y_{2n} \to q, \quad z_{2n} \to r, \quad x_{2n+1} \to p', \quad y_{2n+1} \to q', \quad z_{2n+1} \to r'.$$

*Furthermore, if $q = r$ and $q' = r'$, then $F$ and $G$ have a common tripled fixed point in $(A \cap B)^3$.*

*Proof.* Since $d(A, B) = 0$, it follows that the pairs $(A, B)$ and $(B, A)$ satisfy the property UC*. Therefore, we claim that $F$ has a tripled best proximity point $(p, q, r) \in A^3$, that is,

$$d(p, F(p, q, r)) = d(q, F(q, p, q)) = d(r, F(r, q, p)) = d(A, B) \tag{5.2.22}$$

and $G$ has a tripled best proximity point $(p', q', r') \in B^3$, that is,

$$d(p', G(p', q', r')) = d(q', G(q', p', q')) = d(r', G(r', q', p')) = d(A, B). \tag{5.2.23}$$

From (5.2.22) and $d(A,B)=0$ , we conclude that

$$p=F(p,q,r),\quad q=F(q,p,q),\quad r=F(r,q,p),$$

that is, $(p,q,r)$ is a tripled fixed point of $F$. It follows from (5.2.23) and $d(A,B)=0$ that

$$p'=G(p',q',r'),\quad q'=G(q',p',q'),\quad r'=G(r',q',p'),$$

that is, $(p',q',r')$ is a tripled fixed point of $G$.

Next, we assume that $q=r$ and $q'=r'$ and then we show that $F$ and $G$ have a unique common tripled fixed point in $(A\cap B)^3$. From Theorem, we get

$$d(p,p')+d(q,q')+d(r,r')=3d(A,B). \tag{5.2.24}$$

Since $d(A,B)=0$, we get

$$d(p,p')+d(q,q')+d(r,r')=0$$

which implies that $p=p'$, $q=q'$ and $r=r'$. Therefore, we conclude that $(p,q,r)\in(A\cap B)^3$ is common tripled fixed point of $F$ and $G$. This completes the proof. $\square$

**Example 9.** Consider a space $X=\mathbb{R}$ with the usual metric and let $A=[-2,0]$ and $B=[0,2]$. Define two mappings $F:A^3\to B$ and $G:B^3\to A$ by

$$F(x,y,z)=-\frac{x+y+z}{6},\quad G(x,y,z)=-\frac{u+v+w}{6}$$

for all $(x,y,z)\in A^3$ and $(u,v,w)\in B^3$, respectively. Then $d(A,B)=0$ and the ordered pair $(F,G)$ is a cyclic contraction with $\alpha=\frac{1}{2}$. Indeed, for any $(x,y,z)\in A^3$ and $(u,v,w)\in B^3$, we have

$$\begin{aligned} d(F(x,y,z),G(u,v,w)) &= \left|-\frac{x+y+z}{6}+\frac{u+v+w}{6}\right| \\ &\leq \frac{1}{6}(\mid x-u\mid+\mid y-v\mid+\mid z-w\mid) \\ &\leq \frac{\alpha}{3}[d(x,u)+d(y,v)+d(z,w)]+(1-\alpha)d(A,B). \end{aligned}$$

Therefore, all the hypothesis of Theorem 5.2.2.1 hold. Therefore, $F$ and $G$ have a common tripled fixed point and this point is $(0,0,0)\in(A\cap B)^3$.

**Corollary 5.2.2.2.** *Let $A$ be a nonempty closed subset of a complete metric space $X$ and $F:A^3\to A, G:A^3\to A$ be two mappings such that the ordered pair $(F,G)$*

*be a cyclic contraction. For any* $(x_0, y_0, z_0) \in A^3$, *we define the sequences* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *in* $X$ *by*

$$x_{2n+1} = F(x_{2n}, y_{2n}, z_{2n}), \quad x_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1}),$$

$$y_{2n+1} = F(y_{2n}, x_{2n}, y_{2n}), \quad y_{2n+2} = G(y_{2n+1}, x_{2n+1}, y_{2n+1}),$$

$$z_{2n+1} = F(z_{2n}, y_{2n}, x_{2n}), \quad z_{2n+2} = G(z_{2n+1}, y_{2n+1}, x_{2n+1})$$

*for all* $n \in \mathbb{N} \cup \{0\}$. *Then* $F$ *has a tripled fixed point* $(p, q, r) \in A^3$ *and* $G$ *has a tripled fixed point* $(p', q', r') \in A^3$. *Moreover, we have*

$$x_{2n} \to p, \quad y_{2n} \to q, \quad z_{2n} \to r, \quad x_{2n+1} \to p', \quad y_{2n+1} \to q', \quad z_{2n+1} \to r'.$$

*Furthermore, if* $q = r$ *and* $q' = r'$, *then* $F$ *and* $G$ *have a common tripled fixed point in* $A^3$.

### 5.2.3 Coupled best proximity point

**Definition 5.2.3.1.** Let $A$ and $B$ be nonempty subsets of a metric space $X$ and $F : A \times A \to B$. A point $(x, x') \in A \times A$ is called a *coupled best proximity point* of $F$ if

$$d(x, F(x, x')) = d(x', F(x', x)) = d(A, B).$$

It is easy to see that if $A = B$ in definition 5.2.3.1, then a coupled best proximity point reduces to a coupled fixed point.

Next, we introduce the notion of a cyclic contraction for a pair of two binary mappings.

**Definition 5.2.3.2.** Let $A$ and $B$ be nonempty subsets of a metric space $X$, $F : A \times A \to B$ and $G : B \times B \to A$. The ordered pair $(F, G)$ is said to be a *cyclic contraction* if there exists a non-negative number $\alpha < 1$ such that

$$d(F(x, x'), G(y, y')) \leq \frac{\alpha}{2}[d(x, y) + d(x', y')] + (1 - \alpha)d(A, B)$$

for all $(x, x') \in A \times A$ and $(y, y') \in B \times B$.

Note that if $(F, G)$ is a cyclic contraction, then $(G, F)$ is also a cyclic contraction. The following lemma plays an important role in our main results.

**Lemma 5.2.3.3.** *Let $A$ and $B$ be nonempty subsets of a metric space $X$, $F : A \times A \to B$, $G : B \times B \to A$ and $(F, G)$ be a cyclic contraction. If $(x_0, x_0') \in A \times A$ and we define*

$$x_{2n+1} = F(x_{2n}, x_{2n}'), x_{2n+1}' = F(x_{2n}', x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x_{2n+1}'), x_{2n+2}' = G(x_{2n+1}', x_{2n+1})$$

*for all $n \in \mathbb{N} \cup \{0\}$, then $d(x_{2n}, x_{2n+1}) \to d(A, B)$, $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$, $d(x_{2n}', x_{2n+1}') \to d(A, B)$ and $d(x_{2n+1}', x_{2n+2}') \to d(A, B)$.*

*Proof.* For each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
d(x_{2n}, x_{2n+1}) &= d(x_{2n}, F(x_{2n}, x_{2n}')) \\
&= d(G(x_{2n-1}, x_{2n-1}'), F(G(x_{2n-1}, x_{2n-1}'), G(x_{2n-1}', x_{2n-1}))) \\
&\le \frac{\alpha}{2}[d(x_{2n-1}, G(x_{2n-1}, x_{2n-1}')) + d(x_{2n-1}', G(x_{2n-1}', x_{2n-1}))] \\
&\quad +(1-\alpha)d(A, B) \\
&= \frac{\alpha}{2}\Big[d(F(x_{2n-2}, x_{2n-2}'), G(F(x_{2n-2}, x_{2n-2}'), F(x_{2n-2}', x_{2n-2}))) \\
&\quad +d(F(x_{2n-2}', x_{2n-2}), G(F(x_{2n-2}', x_{2n-2}), F(x_{2n-2}, x_{2n-2}')))\Big] \\
&\quad +(1-\alpha)d(A, B) \\
&\le \frac{\alpha}{2}\Big[\frac{\alpha}{2}[d(x_{2n-2}, F(x_{2n-2}, x_{2n-2}')) + d(x_{2n-2}', F(x_{2n-2}', x_{2n-2})) \\
&\quad +(1-\alpha)d(A, B)] \\
&\quad +\frac{\alpha}{2}[d(x_{2n-2}', F(x_{2n-2}', x_{2n-2})) + d(x_{2n-2}, F(x_{2n-2}, x_{2n-2}')) + \\
&\quad (1-\alpha)d(A, B)\Big] + (1-\alpha)d(A, B) \\
&= \frac{\alpha^2}{2}[d(x_{2n-2}, F(x_{2n-2}, x_{2n-2}')) + d(x_{2n-2}', F(x_{2n-2}', x_{2n-2}))] \\
&\quad +(1-\alpha^2)d(A, B).
\end{aligned}$$

By induction, we see that

$$d(x_{2n}, x_{2n+1}) \le \frac{\alpha^{2n}}{2}[d(x_0, F(x_0, x_0')) + d(x_0', F(x_0', x_0))] + (1-\alpha^{2n})d(A, B).$$

Taking $n \to \infty$, we obtain

$$d(x_{2n}, x_{2n+1}) \to d(A, B). \tag{5.2.25}$$

For each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(x_{2n+1}, G(x_{2n+1}, x'_{2n+1})) \\
&= d(F(x_{2n}, x'_{2n}), G(F(x_{2n}, x'_{2n}), F(x'_{2n}, x_{2n}))) \\
&\leq \frac{\alpha}{2}[d(x_{2n}, F(x_{2n}, x'_{2n})) + d(x'_{2n}, F(x'_{2n}, x_{2n}))] + (1-\alpha)d(A, B) \\
&= \frac{\alpha}{2}\Big[d(G(x_{2n-1}, x'_{2n-1}), F(G(x_{2n-1}, x'_{2n-1}), G(x'_{2n-1}, x_{2n-1}))) \\
&\quad + d(G(x'_{2n-1}, x_{2n-1}), F(G(x'_{2n-1}, x_{2n-1}), G(x_{2n-1}, x'_{2n-1})))\Big] \\
&\quad + (1-\alpha)d(A, B) \\
&\leq \frac{\alpha}{2}\Big[\frac{\alpha}{2}[d(x_{2n-1}, G(x_{2n-1}, x'_{2n-1})) + d(x'_{2n-1}, G(x'_{2n-1}, x_{2n-1})) \\
&\quad + (1-\alpha)d(A, B)] \\
&\quad + \frac{\alpha}{2}[d(x'_{2n-1}, G(x'_{2n-1}, x_{2n-1})) + d(x_{2n-1}, G(x_{2n-1}, x'_{2n-1})) \\
&\quad + (1-\alpha)d(A, B)\Big] + (1-\alpha)d(A, B) \\
&= \frac{\alpha^2}{2}[d(x_{2n-1}, G(x_{2n-1}, x'_{2n-2})) + d(x'_{2n-1}, G(x'_{2n-1}, x_{2n-1}))] \\
&\quad + (1-\alpha^2)d(A, B).
\end{aligned}$$

By induction, we see that

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{\alpha^{2n}}{2}[d(x_1, G(x_1, x'_1)) + d(x'_1, G(x'_1, x_1))] + (1-\alpha^{2n})d(A, B).$$

Setting $n \to \infty$, we obtain

$$d(x_{2n+1}, x_{2n+2}) \to d(A, B). \tag{5.2.26}$$

By similar argument, we also have $d(x'_{2n}, x'_{2n+1}) \to d(A, B)$ and $d(x'_{2n+1}, x'_{2n+2}) \to d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$. □

**Lemma 5.2.3.4.** *Let $A$ and $B$ be nonempty subsets of a metric space $X$ such that $(A, B)$ and $(B, A)$ have a property UC, $F : A \times A \to B$, $G : B \times B \to A$ and let the ordered pair $(F, G)$ is a cyclic contraction. If $(x_0, x'_0) \in A \times A$ and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all $n \in \mathbb{N} \cup \{0\}$, then for $\epsilon > 0$, there exists a positive integer $N_0$ such that for all $m > n \geq N_0$,*

$$\frac{1}{2}[d(x'_{2m}, x'_{2n+1}) + d(x_{2m}, x_{2n+1})] < d(A, B) + \epsilon. \tag{5.2.27}$$

*Proof.* By Lemma 5.2.3.3, we have $d(x_{2n}, x_{2n+1}) \to d(A, B)$ and $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$. Since $(A, B)$ has a property UC, we get $d(x_{2n}, x_{2n+2}) \to 0$. A similar argument shows that $d(x'_{2n}, x'_{2n+2}) \to 0$. As $(B, A)$ has a property UC, we also have $d(x_{2n+1}, x_{2n+3}) \to 0$ and $d(x'_{2n+1}, x'_{2n+3}) \to 0$. Suppose that (5.2.27) does not hold. Then there exists $\epsilon' > 0$ such that for all $k \in \mathbb{N}$, there is $m_k > n_k \geq k$ satisfying

$$\frac{1}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1})] \geq d(A, B) + \epsilon'$$

and

$$\frac{1}{2}[d(x'_{2m_k-2}, x'_{2n_k+1}) + d(x_{2m_k-2}, x_{2n_k+1})] < d(A, B) + \epsilon'.$$

Therefore, we get

$$\begin{aligned} d(A, B) + \epsilon' &\leq \frac{1}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1})] \\ &\leq \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k-2}) + d(x'_{2m_k-2}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k-2}) \\ &\quad + d(x_{2m_k-2}, x_{2n_k+1})] \\ &< \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k-2}) + d(x_{2m_k}, x_{2m_k-2})] + d(A, B) + \epsilon'. \end{aligned}$$

Letting $k \to \infty$, we obtain to see that

$$\frac{1}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1})] \to d(A, B) + \epsilon'. \tag{5.2.28}$$

By using the triangle inequality we get

$$
\begin{aligned}
&\tfrac{1}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1})] \\
&\leq \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2m_k+2}, x'_{2n_k+3}) + d(x'_{2n_k+3}, x'_{2n_k+1}) \\
&\quad + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&= \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(G(x'_{2m_k+1}, x_{2m_k+1}), F(x'_{2n_k+2}, x_{2n_k+2})) + d(x'_{2n_k+3}, x'_{2n_k+1}) \\
&\quad + d(x_{2m_k}, x_{2m_k+2}) + d(G(x_{2m_k+1}, x'_{2m_k+1}), F(x_{2n_k+2}, x'_{2n_k+2})) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\leq \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + \frac{\alpha}{2}[d(x'_{2m_k+1}, x'_{2n_k+2}) + d(x_{2m_k+1}, x_{2n_k+2})] \\
&\quad + (1-\alpha)d(A,B) + d(x'_{2n_k+3}, x'_{2n_k+1}) \\
&\quad + d(x_{2m_k}, x_{2m_k+2}) + \frac{\alpha}{2}[d(x_{2m_k+1}, x_{2n_k+2}) + d(x'_{2m_k+1}, x'_{2n_k+2})] \\
&\quad + (1-\alpha)d(A,B) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&= \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2n_k+3}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\quad + \frac{\alpha}{2}[d(x_{2m_k+1}, x_{2n_k+2}) + d(x'_{2m_k+1}, x'_{2n_k+2})] + (1-\alpha)d(A,B) \\
&= \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2n_k+3}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\quad + \frac{\alpha}{2}[d(F(x_{2m_k}, x'_{2m_k}), G(x_{2n_k+1}, x'_{2n_k+1})) + d(F(x'_{2m_k}, x_{2m_k}), G(x'_{2n_k+1}, x_{2n_k+1}))] \\
&\quad + (1-\alpha)d(A,B) \\
&\leq \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2n_k+3}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\quad + \frac{\alpha}{2}\Bigg[\frac{\alpha}{2}[d(x_{2m_k}, x_{2n_k+1}) + d(x'_{2m_k}, x'_{2n_k+1}) + (1-\alpha)d(A,B)] \\
&\quad + \frac{\alpha}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1}) + (1-\alpha)d(A,B)]\Bigg] \\
&\quad + (1-\alpha)d(A,B) \\
&= \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2n_k+3}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\quad + \frac{\alpha^2}{2}[d(x_{2m_k}, x_{2n_k+1}) + d(x'_{2m_k}, x'_{2n_k+1})] + (1-\alpha^2)d(A,B).
\end{aligned}
$$

Taking $k \to \infty$, we get

$$d(A,B) + \epsilon' \leq \alpha^2[d(A,B) + \epsilon'] + (1-\alpha^2)d(A,B) = d(A,B) + \alpha^2\epsilon'$$

which contradicts. Therefore, we can conclude that (5.2.27) holds. □

**Lemma 5.2.3.5.** *Let $A$ and $B$ be nonempty subsets of a metric space $X$, $(A,B)$ and $(B,A)$ satisfy the property UC*. Let $F : A \times A \to B$, $G : B \times B \to A$ and $(F,G)$ be a cyclic contraction. If $(x_0, x'_0) \in A \times A$ and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all* $n \in \mathbb{N} \cup \{0\}$*, then* $\{x_{2n}\}$*,* $\{x'_{2n}\}$*,* $\{x_{2n+1}\}$ *and* $\{x'_{2n+1}\}$ *are Cauchy sequences.*

*Proof.* By Lemma 5.2.3.3, we have $d(x_{2n}, x_{2n+1}) \to d(A, B)$ and $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$. Since $(A, B)$ has a property UC$^*$, we get $d(x_{2n}, x_{2n+2}) \to 0$. As $(B, A)$ has a property UC$^*$, we also have $d(x_{2n+1}, x_{2n+3}) \to 0$.

We now show that for every $\epsilon > 0$ there exists $N$ such that

$$d(x_{2m}, x_{2n+1}) \leq d(A, B) + \epsilon \tag{5.2.29}$$

for all $m > n \geq N$.

Suppose (5.2.29) not, then there exists $\epsilon > 0$ such that for all $k \in \mathbb{N}$ there exists $m_k > n_k \geq k$ such that

$$d(x_{2m_k}, x_{2n_k+1}) > d(A, B) + \epsilon. \tag{5.2.30}$$

Now we have

$$\begin{aligned} d(A, B) + \epsilon &< d(x_{2m_k}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k+1}) \\ &\leq d(A, B) + \epsilon + d(x_{2n_k-1}, x_{2n_k+1}) \end{aligned}$$

Taking $k \to \infty$, we have $d(x_{2m_k}, x_{2n_k+1}) \to d(A, B) + \epsilon$.

By Lemma 5.2.3.4, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1})] < d(A, B) + \epsilon. \tag{5.2.31}$$

for all $m > n \geq N$. By using the triangle inequality we get

$d(x_{2m_k}, x_{2n_k+1})$

$$
\begin{aligned}
&\leq d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1}) \\
&= d(x_{2m_k}, x_{2m_k+2}) + d(G(x_{2m_k+1}, x'_{2m_k+1}), F(x_{2n_k+2}, x'_{2n_k+2})) + d(x_{2n_k+3}, x_{2n_k+1}) \\
&\leq d(x_{2m_k}, x_{2m_k+2}) + \frac{\alpha}{2}[d(x_{2m_k+1}, x_{2n_k+2}) + d(x'_{2m_k+1}, x'_{2n_k+2})] + (1-\alpha)d(A,B) \\
&\quad + d(x_{2n_k+3}, x_{2n_k+1}) \\
&= \frac{\alpha}{2}[d(F(x_{2m_k}, x'_{2m_k}), G(x_{2n_k+1}, x'_{2n_k+1})) + d(F(x'_{2m_k}, x_{2m_k}), G(x'_{2n_k+1}, x_{2n_k+1}))] \\
&\quad + (1-\alpha)d(A,B) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\
&\leq \frac{\alpha}{2}\Bigg[\frac{\alpha}{2}[d(x_{2m_k}, x_{2n_k+1}) + d(x'_{2m_k}, x'_{2n_k+1}) + (1-\alpha)d(A,B)] \\
&\quad + \frac{\alpha}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1}) + (1-\alpha)d(A,B)]\Bigg] \\
&\quad + (1-\alpha)d(A,B) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\
&= \alpha^2\frac{1}{2}[d(x_{2m_k}, x_{2n_k+1}) + d(x'_{2m_k}, x'_{2n_k+1})] + (1-\alpha^2)d(A,B) \\
&\quad + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\
&< \alpha^2(d(A,B)+\epsilon) + (1-\alpha^2)d(A,B) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\
&= d(A,B) + \alpha^2\epsilon + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})
\end{aligned}
$$

Taking $k \to \infty$, we get

$$d(A,B) + \epsilon \leq d(A,B) + \alpha^2\epsilon$$

which contradicts. Therefore, condition (5.2.29) holds. Since (5.2.29) holds and $d(x_{2n}, x_{2n+1}) \to d(A,B)$, by using property UC* of $(A,B)$, we have $d(x_{2n}, x_{2m}) \to 0$. Therefore $\{x_{2n}\}$ is a Cauchy sequence. In similar way, we can prove that $\{x'_{2n}\}$, $\{x_{2n+1}\}$ and $\{x'_{2n+1}\}$ are Cauchy sequences. □

Here we state the main results of this paper on the existence and convergence of coupled best proximity points for cyclic contraction pairs on nonempty subsets of metric spaces satisfying the property UC*.

**Theorem 5.2.3.6.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $(A,B)$ and $(B,A)$ satisfy the property UC*. Let $F : A \times A \to B$, $G : B \times B \to A$ and $(F,G)$ be a cyclic contraction. Let $(x_0, x'_0) \in A \times A$ and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all* $n \in \mathbb{N} \cup \{0\}$*. Then* $F$ *has a coupled best proximity point* $(p, q) \in A \times A$ *and* $G$ *has a coupled best proximity point* $(p', q') \in B \times B$ *such that*

$$d(p, p') + d(q, q') = 2d(A, B).$$

*Moreover, we have* $x_{2n} \to p$*,* $x'_{2n} \to q$*,* $x_{2n+1} \to p'$ *and* $x'_{2n+1} \to q'$*.*

*Proof.* By Lemma 5.2.3.3, we get $d(x_{2n}, x_{2n+1}) \to d(A, B)$. Using Lemma 5.2.3.5, we have $\{x_{2n}\}$ and $\{x'_{2n}\}$ are Cauchy sequences. Thus, there exists $p, q \in A$ such that $x_{2n} \to p$ and $x'_{2n} \to q$. We obtain that

$$d(A, B) \leq d(p, x_{2n-1}) \leq d(p, x_{2n}) + d(x_{2n}, x_{2n-1}). \tag{5.2.32}$$

Letting $n \to \infty$ in (5.2.32), we have $d(p, x_{2n-1}) \to d(A, B)$. By a similar argument we also have $d(q, x'_{2n-1}) \to d(A, B)$. It follows that

$$\begin{aligned} d(x_{2n}, F(p, q)) &= d(G(x_{2n-1}, x'_{2n-1}), F(p, q)) \\ &\leq \frac{\alpha}{2}[d(x_{2n-1}, p) + d(x'_{2n-1}, q)] + (1 - \alpha)d(A, B). \end{aligned}$$

Taking $n \to \infty$, we get $d(p, F(p, q)) = d(A, B)$. Similarly, we can prove that $d(q, F(q, p)) = d(A, B)$. Therefore, we have $(p, q)$ is a coupled best proximity point of $F$.

In similar way, we can prove that there exists $p', q' \in B$ such that $x_{2n+1} \to p'$ and $x'_{2n+1} \to q'$. Moreover, we also have $d(p', G(p', q')) = d(A, B)$ and $d(q', G(q', p')) = d(A, B)$ and so $(p', q')$ is a coupled best proximity point of $G$.

Finally, we show that $d(p, p') + d(q, q') = 2d(A, B)$. For $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(G(x_{2n-1}, x'_{2n-1}), F(x_{2n}, x'_{2n})) \\ &\leq \frac{\alpha}{2}[d(x_{2n-1}, x_{2n}) + d(x'_{2n-1}, x'_{2n})] + (1 - \alpha)d(A, B). \end{aligned}$$

Letting $n \to \infty$, we have

$$d(p, p') \leq \frac{\alpha}{2}[d(p, p') + d(q, q')] + (1 - \alpha)d(A, B). \tag{5.2.33}$$

For $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} d(x'_{2n}, x'_{2n+1}) &= d(G(x'_{2n-1}, x_{2n-1}), F(x'_{2n}, x_{2n})) \\ &\leq \frac{\alpha}{2}[d(x'_{2n-1}, x'_{2n}) + d(x_{2n-1}, x_{2n})] + (1 - \alpha)d(A, B). \end{aligned}$$

Letting $n \to \infty$, we have

$$d(q, q') \leq \frac{\alpha}{2}[d(q, q') + d(p, p')] + (1 - \alpha)d(A, B). \tag{5.2.34}$$

It follows from (5.2.33) and (5.2.34) that

$$d(p, p') + d(q, q') \leq \alpha[d(p, p') + d(q, q')] + 2(1 - \alpha)d(A, B),$$

which implies that

$$d(p, p') + d(q, q') \leq 2d(A, B). \tag{5.2.35}$$

Since $d(A, B) \leq d(p, p')$ and $d(A, B) \leq d(q, q')$, we have

$$2d(A, B) \leq d(p, p') + d(q, q'). \tag{5.2.36}$$

From (5.2.35) and (5.2.36), we get

$$d(p, p') + d(q, q') = 2d(A, B).$$

This complete the proof. $\square$

Note that every pair of nonempty closed subsets $A, B$ of a uniformly convex Banach space $X$ such that $A$ is convex satisfies the property UC*. Therefore, we obtain the following corollary.

**Corollary 5.2.3.7.** *Let $A$ and $B$ be nonempty closed convex subsets of a uniformly convex Banach space $X$, $F : A \times A \to B$, $G : B \times B \to A$ and $(F, G)$ be a cyclic contraction. Let $(x_0, x'_0) \in A \times A$ and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all $n \in \mathbb{N} \cup \{0\}$. Then $F$ has a coupled best proximity point $(p, q) \in A \times A$ and $G$ has a coupled best proximity point $(p', q') \in B \times B$ such that*

$$d(p, p') + d(q, q') = 2d(A, B).$$

*Moreover, we have $x_{2n} \to p$, $x'_{2n} \to q$, $x_{2n+1} \to p'$ and $x'_{2n+1} \to q'$.*

**Example 10.** Consider uniformly convex Banach space $X = \mathbb{R}$ with the usual norm. Let $A = [1, 2]$ and $B = [-2, -1]$. Thus $d(A, B) = 2$. Define $F : A \times A \to B$ and $G : B \times B \to A$ by

$$F(x, x') = \frac{-x - x' - 2}{4}$$

and

$$G(x, x') = \frac{-x - x' + 2}{4}.$$

For arbitrary $(x, x') \in A \times A$ and $(y, y') \in B \times B$ and fixed $\alpha = \frac{1}{2}$, we get

$$\begin{aligned} d(F(x, x'), G(y, y')) &= \left| \frac{-x - x' - 2}{4} - \frac{-y - y' + 2}{4} \right| \\ &\leq \frac{|x - y| + |x' - y|}{4} + 1 \\ &= \frac{\alpha}{2}[d(x, y) + d(x', y')] + (1 - \alpha)d(A, B). \end{aligned}$$

This implies that $(F, G)$ is a cyclic contraction with $\alpha = \frac{1}{2}$. Since $A$ and $B$ are convex, we have $(A, B)$ and $(B, A)$ satisfy the property UC$^*$. Therefore, all hypothesis of Corollary 5.2.3.7 hold. So $F$ has a coupled best proximity point and $G$ has a coupled best proximity point. We note that a point $(1, 1) \in A \times A$ is a unique coupled best proximity point of $F$ and a point $(-1, -1) \in B \times B$ is a coupled best proximity point of $G$. Furthermore, we get

$$d(1, -1) + d(1, -1) = 4 = 2d(A, B).$$

**Theorem 5.2.3.8.** *Let $A$ and $B$ be nonempty compact subsets of a metric space $X$, $F : A \times A \to B$, $G : B \times B \to A$ and $(F, G)$ be a cyclic contraction pair. If $(x_0, x'_0) \in A \times A$ and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all $n \in \mathbb{N} \cup \{0\}$, then $F$ has a coupled best proximity point $(p, q) \in A \times A$ and $G$ has a coupled best proximity point $(p', q') \in B \times B$ such that*

$$d(p, p') + d(q, q') = 2d(A, B).$$

*Proof.* Since $x_0, x'_0 \in A$ and

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

and

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$, we have $x_{2n}, x'_{2n} \in A$ and $x_{2n+1}, x'_{2n+1} \in B$ for all $n \in \mathbb{N} \cup \{0\}$. As $A$ is compact, the sequence $\{x_{2n}\}$ and $\{x'_{2n}\}$ have convergent subsequences $\{x_{2n_k}\}$ and $\{x'_{2n_k}\}$, respectively, such that

$$x_{2n_k} \to p \in A \text{ and } x'_{2n_k} \to q \in A.$$

Now, we have

$$d(A, B) \leq d(p, x_{2n_k-1}) \leq d(p, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}). \tag{5.2.37}$$

By Lemma 5.2.3.3, we have $d(x_{2n_k}, x_{2n_k-1}) \to d(A, B)$. Taking $k \to \infty$ in (5.3.1), we get $d(p, x_{2n_k-1}) \to d(A, B)$. By a similar argument we observe that $d(q, x_{2n_k-1}) \to d(A, B)$. Note that

$$\begin{aligned} d(A, B) &\leq d(x_{2n_k}, F(p, q)) \\ &= d(G(x_{2n_k-1}, x'_{2n_k-1}), F(p, q)) \\ &\leq \frac{\alpha}{2}[d(x_{2n_k-1}, p) + d(x'_{2n_k-1}, q)] + (1-\alpha)d(A, B). \end{aligned}$$

Taking $k \to \infty$, we get $d(p, F(p, q)) = d(A, B)$. Similarly, we can prove that $d(q, F(q, p)) = d(A, B)$. Thus $F$ has a coupled best proximity $(p, q) \in A \times A$. In similar way, since $B$ is compact, we can also prove that $G$ has a coupled best proximity point in $(p', q') \in B \times B$. For $d(p, p') + d(q, q') = 2d(A, B)$ similar to the final step of the proof of Theorem 5.2.3.6. This complete the proof. □

**Theorem 5.2.3.9.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$, $F : A \times A \to B$, $G : B \times B \to A$ and $(F, G)$ be a cyclic contraction. Let $(x_0, x'_0) \in A \times A$ and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all* $n \in \mathbb{N} \cup \{0\}$*. If* $d(A, B) = 0$*, then* $F$ *and* $G$ *have a unique common coupled fixed point* $(p, q) \in A \cap B \times A \cap B$*. Moreover, we have* $x_{2n} \to p$*,* $x'_{2n} \to q$*,* $x_{2n+1} \to p$ *and* $x'_{2n+1} \to q$*.*

*Proof.* Since $d(A, B) = 0$, we get $(A, B)$ and $(B, A)$ have the property UC*. Therefore, by Theorem 5.2.3.6 claim that $F$ has a coupled best proximity point $(p, q) \in A \times A$ that is

$$d(p, F(p, q)) = d(A, B) \text{ and } d(q, F(q, p)) = d(A, B) \tag{5.2.38}$$

and $G$ has a coupled best proximity point $(p', q') \in B \times B$ that is

$$d(p', G(p', q')) = d(A, B) \text{ and } d(q', G(q', p')) = d(A, B). \tag{5.2.39}$$

Moreover, we have

$$d(p, p') + d(q, q') = 2d(A, B). \tag{5.2.40}$$

From (5.2.38) and $d(A, B) = 0$, we conclude that

$$p = F(p, q) \text{ and } q = F(q, p)$$

that is $(p, q)$ is a coupled fixed point of $F$. It follows from (5.2.39) and $d(A, B) = 0$, we get

$$p' = G(p', q') \text{ and } q' = G(q', p')$$

that is $(p', q')$ is a coupled fixed point of $G$. Using (5.2.40) and the fact that $d(A, B) = 0$, we have

$$d(p, p') + d(q, q') = 0$$

which implies that $p = p'$ and $q = q'$. Therefore, we conclude that $(p, q) \in A \cap B \times A \cap B$ is a common coupled fixed point of $F$ and $G$.

Finally, we show the uniqueness of common coupled fixed point of $F$ and $G$. Let $(\hat{p}, \hat{q})$ be another common coupled fixed point of $F$ and $G$. So $\hat{p} = G(\hat{p}, \hat{q})$ and $\hat{q} = G(\hat{q}, \hat{p})$. Now, we obtain that

$$d(p, \hat{p}) = d(F(p, q), G(\hat{p}, \hat{q})) \leq \frac{\alpha}{2}[d(p, \hat{p}) + d(q, \hat{q})] \tag{5.2.41}$$

and also

$$d(q, \hat{q}) = d(F(q, p), G(\hat{q}, \hat{p})) \leq \frac{\alpha}{2}[d(q, \hat{q}) + d(p, \hat{p})]. \tag{5.2.42}$$

It follows from (5.3.6) and (5.3.7) that

$$d(p, \hat{p}) + d(q, \hat{q}) \leq \alpha[d(p, \hat{p}) + d(q, \hat{q})],$$

which implies that $d(p, \hat{p}) + d(q, \hat{q}) = 0$ and so $d(p, \hat{p}) = 0$ and $d(q, \hat{q}) = 0$. Therefore, $(p, q)$ is the unique common coupled fixed point in $A \cap B \times A \cap B$. $\square$

**Example 11.** Consider $X = \mathbb{R}$ with the usual metric, $A = [-1, 0]$ and $B = [0, 1]$. Define $F : A \times A \to B$ by $F(x, y) = -\dfrac{x + y}{4}$ and $G(x, y) = -\dfrac{x + y}{8}$. Then $d(A, B) = 0$ and $(F, G)$ is a cyclic contraction with $\alpha = \frac{1}{2}$. Indeed,

$$\begin{aligned} d(F(x, x'), G(y, y')) &= \left| -\frac{x + x'}{4} + \frac{y + y'}{8} \right| \\ &\leq \left| -\frac{x + x'}{4} + \frac{2y + 2y'}{8} \right| \\ &= \frac{1}{4}(|x - y| + |x' - y'|) \\ &= \frac{\alpha}{2}[d(x, y) + d(x', y')] + (1 - \alpha)d(A, B) \end{aligned}$$

for all $(x, x') \in A \times A$ and $(y, y') \in B \times B$. Therefore, all hypothesis of Theorem 5.2.3.9 hold. So $F$ and $G$ have a unique common coupled fixed point and this point is $(0, 0) \in A \cap B \times A \cap B$.

If we take $A = B$ in Theorem 5.2.3.9, then we get the following results.

**Corollary 5.2.3.10.** *Let $A$ be nonempty closed subsets of a complete metric space $X$, $F : A \times A \to A$ and $G : A \times A \to A$ and let the order pair $(F, G)$ is a cyclic contraction. Let $(x_0, x'_0) \in A \times A$ and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all $n \in \mathbb{N} \cup \{0\}$. Then $F$ and $G$ have a unique common coupled fixed point $(p, q) \in A \times A$. Moreover, we have $x_{2n} \to p$, $x'_{2n} \to q$, $x_{2n+1} \to p$ and $x'_{2n+1} \to q$*

We take $F = G$ in Corollary 5.2.3.10, then we get the following results.

**Corollary 5.2.3.11.** *Let $A$ be nonempty closed subsets of a complete metric space $X$, $F : A \times A \to A$ and*

$$d(F(x,x'),F(y,y')) \leq \frac{\alpha}{2}[d(x,y)+d(x',y')] \tag{5.2.43}$$

*for all $(x,x'),(y,y') \in A \times A$. Then $F$ has a unique coupled fixed point $(p,q) \in A \times A$.*

**Example 12.** Consider $X = \mathbb{R}$ with the usual metric and $A = [0, \frac{1}{2}]$. Define $F : A \times A \to A$ by

$$F(x,y) = \begin{cases} \dfrac{x^2-y^2}{4} & ; x \geq y \\ 0 & ; x < y. \end{cases}$$

We show that $F$ satisfies (5.2.43) with $\alpha = \frac{1}{2}$. Let $(x,x'),(y,y') \in A \times A$.

Case 1: If $x < x'$ and $y < y'$, then

$$d(F(x,x'),F(y,y')) = 0 \leq \frac{1}{4}[|x-y|+|x'-y'|] = \frac{\alpha}{2}[d(x,y)+d(x',y')].$$

Case 2: If $x < x'$ and $y \geq y'$, then

$$\begin{aligned} d(F(x,x'),F(y,y')) &= \left|0 - \frac{y^2-y'^2}{4}\right| \\ &\leq \frac{1}{4}[|y-y'||y+y'|] \\ &\leq \frac{1}{4}|y-y'| \\ &\leq \frac{1}{4}[|x-x'|+|y-y'|] \\ &= \frac{\alpha}{2}[d(x,y)+d(x',y')]. \end{aligned}$$

Case 3: If $x \geq x'$ and $y < y'$. In this case we can prove by a similar argument as in case 2.

Case 4: If $x \geq x'$ and $y \geq y'$, then

$$\begin{aligned} d(F(x,x'),F(y,y')) &= \left|\frac{x^2-x'^2}{4} - \frac{y^2-y'^2}{4}\right| \\ &\leq \frac{1}{4}[|x-x'||x+x'|+|y-y'||y+y'|] \\ &\leq \frac{1}{4}[|x-x'|+|y-y'|] \\ &= \frac{\alpha}{2}[d(x,y)+d(x',y')]. \end{aligned}$$

Thus condition (5.2.43) holds with $\alpha = \frac{1}{2}$. Therefore, by Corollary 5.2.3.11 $F$ has the unique coupled fixed point in $A$ that is a point $(0,0)$.

## 5.3 OPTIMAL APPROXIMATE SOLUTION

### 5.3.1 The existence theorems of an optimal approximate solution for generalized proximal contraction mappings

**Definition 5.3.1.1.** Let $A, B$ be nonempty subset of metric space $(X, d)$, $T : A \to B$ and $\mathcal{K} : A \to [0, 1)$. A mapping $T$ is said to be a generalized proximal contraction of the first kind with respect to $\mathcal{K}$ if

$$\begin{cases} d(a, Tx) = d(A, B), \\ d(b, Ty) = d(A, B) \end{cases} \implies d(a, b) \le \mathcal{K}(x)d(x, y)$$

for all $a, b, x, y \in A$.

**Example 13.** Consider the metric space $\mathbb{R}^2$ with Euclidean metric. Let $A = \{(0, y) : -1 < y < 1\}$ and $B = \{(1, y) : -1 < y < 1\}$. Define a mapping $T : A \to B$ as follows:

$$T((0, y)) = \left(1, \frac{y^2}{2}\right)$$

for all $(0, y) \in A$.

It easy to check that there is no $\alpha \in [0, 1)$ satisfies

$$d(a, Tx) = d(b, Ty) = d(A, B) \implies d(a, b) \le \alpha d(x, y)$$

for all $a, b, x, y \in A$. Therefore, $T$ is not a proximal contractions of the first kind.

Consider a function $\mathcal{K} : A \to [0, 1)$ defined by

$$\mathcal{K}((0, y)) = \frac{|y| + 1}{2}.$$

Next, we claim that $T$ is a generalized proximal contractions of the first kind with respect to $\mathcal{K}$.

If $(0, y_1), (0, y_2) \in A$ such that

$$d(a, T(0, y_1)) = d(A, B) = 1 \text{ and } d(b, T(0, y_2)) = d(A, B) = 1$$

for all $a, b \in A$, then we have

$$a = \left(0, \frac{y_1^2}{2}\right), \quad b = \left(0, \frac{y_2^2}{2}\right).$$

Therefore, it follows that

$$\begin{aligned}
d(a,b) &= d\left(\left(0, \frac{y_1^2}{2}\right), \left(0, \frac{y_2^2}{2}\right)\right) \\
&= \left|\frac{y_1^2}{2} - \frac{y_2^2}{2}\right| \\
&= \left(\frac{|y_1 + y_2|}{2}\right)|y_1 - y_2| \\
&\leq \left(\frac{|y_1| + |y_2|}{2}\right)|y_1 - y_2| \\
&\leq \left(\frac{|y_1| + 1}{2}\right)|y_1 - y_2| \\
&= \mathcal{K}((0, y_1))d((0, y_1), (0, y_2)).
\end{aligned}$$

This implies that $T$ is a generalized proximal contraction of the first kind with respect to $\mathcal{K}$.

**Definition 5.3.1.2.** Let $A, B$ be nonempty subset of metric space $(X, d)$, $T : A \to B$ and $\mathcal{K} : A \to [0, 1)$. A mapping $T$ is said to be a generalized proximal contraction of the second kind with respect to $\mathcal{K}$ if

$$\begin{cases} d(a, Tx) = d(A, B), \\ d(b, Ty) = d(A, B) \end{cases} \implies d(a, b) \leq \mathcal{K}(x)d(Tx, Ty)$$

for all $a, b, x, y \in A$.

**Theorem 5.3.1.3.** *Let $(X, d)$ be a complete metric space and $A$, $B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Suppose that $T : A \to B$, $g : A \to A$, $\mathcal{K} : A \to [0, 1)$ are mappings satisfying the following conditions:*

*(a) $T$ is a continuous generalized proximal contraction of first kind with respect to $\mathcal{K}$;*

*(b) $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$;*

*(c) $g$ is an isometry;*

*(d) $\mathcal{K}(x) \leq \mathcal{K}(y)$, whenever $d(gx, Ty) = d(A, B)$.*

*Then there exists a unique point $x \in A$ such that $d(gx, Tx) = d(A, B)$.*

*Proof.* Let $x_0$ be a fixed element in $A_0$. From $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, it follows that there exists a point $x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B). \tag{5.3.1}$$

Again, since $Tx_1 \in T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists a point $x_2 \in A_0$ such that

$$d(gx_2, Tx_1) = d(A, B). \tag{5.3.2}$$

Continuing this process, we can construct the sequence $\{x_n\}$ in $A_0$ such that

$$d(gx_n, Tx_{n-1}) = d(A, B) \tag{5.3.3}$$

for all $n \in \mathbb{N}$. Since $T$ is a generalized proximal contraction of the first kind with respect to $\mathcal{K}$, it follows that

$$d(gx_{n+1}, gx_n) \leq \mathcal{K}(x_n)d(x_n, x_{n-1}) \tag{5.3.4}$$

for all $n \in \mathbb{N}$. Also, since $g$ is an isometry, we have

$$d(x_{n+1}, x_n) \leq \mathcal{K}(x_n)d(x_n, x_{n-1}) \tag{5.3.5}$$

for all $n \in \mathbb{N}$. By using (5.3.3) and (d), we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \mathcal{K}(x_n)d(x_n, x_{n-1}) \\ &\leq \mathcal{K}(x_{n-1})d(x_n, x_{n-1}) \\ &\leq \mathcal{K}(x_{n-2})d(x_n, x_{n-1}) \\ &\vdots \\ &\leq \mathcal{K}(x_0)d(x_n, x_{n-1}) \end{aligned} \tag{5.3.6}$$

for all $n \in \mathbb{N}$. By repeating (5.3.6), we get

$$d(x_{n+1}, x_n) \leq (\mathcal{K}(x_0))^n d(x_1, x_0) \tag{5.3.7}$$

for all $n \in \mathbb{N}$. Now, we let $k := \mathcal{K}(x_0) \in [0, 1)$. For positive integers $m$ and $n$ with $n > m$, it follows from (5.3.7) that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq k^{n-1}d(x_1, x_0) + k^{n-2}d(x_1, x_0) + \cdots + k^m d(x_1, x_0) \\ &\leq \left(\frac{k^m}{1-k}\right)d(x_1, x_0). \end{aligned} \tag{5.3.8}$$

Since $k \in [0,1)$, we have $\left(\frac{k^m}{1-k}\right)d(x_1, x_0) \to 0$ as $m \to \infty$, which implies that $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, it follows that the sequence $\{x_n\}$ converges to point $x \in X$. Since $T$ and $g$ are continuous, we get

$$d(gx, Tx) = \lim_{n\to\infty} d(gx_{n+1}, Tx_n) = d(A, B). \tag{5.3.9}$$

Next, we suppose that $x^*$ is another point in $X$ such that

$$d(gx^*, Tx^*) = d(A, B). \tag{5.3.10}$$

Since $T$ is a generalized proximal contraction of the first kind with respect to $\mathcal{K}$, by using (5.3.9) and (5.3.10), we get

$$d(gx, gx^*) \le \mathcal{K}(x)d(x, x^*). \tag{5.3.11}$$

Since $g$ is an isometry, it follows that

$$d(x, x^*) \le \mathcal{K}(x)d(x, x^*), \tag{5.3.12}$$

which implies that $x = x^*$. This completes the proof. $\square$

**Example 14.** Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Let $A = \{(0, y) : -1 \le y \le 1\}$ and $B = \{(1, y) : -1 \le y \le 1\}$. Define two mappings $T : A \to B$ and $g : A \to A$ as follows:

$$T((0, y)) = \left(1, \frac{y^2}{4}\right), \quad g((0, y)) = (0, -y)$$

for all $(0, y) \in A$. Then it is easy to see that $d(A, B) = 1$, $A_0 = A$, $B_0 = B$ and the mapping $g$ is an isometry.

Consider a function $\mathcal{K} : A \to [0, 1)$ defined by

$$\mathcal{K}((0, y)) = \frac{|y| + 1}{4}.$$

Next, we claim that $T$ is a generalized proximal contractions of the first kind with respect to $\mathcal{K}$. If $(0, y_1), (0, y_2) \in A$ such that

$$d(a, T(0, y_1)) = d(A, B) = 1 \text{ and } d(b, T(0, y_2)) = d(A, B) = 1$$

for all $a, b \in A$, then we have

$$a = \left(0, \frac{y_1^2}{4}\right), \quad b = \left(0, \frac{y_2^2}{4}\right).$$

Therefore, it follows that

$$\begin{aligned}
d(a,b) &= d\left(\left(0, \frac{y_1^2}{4}\right), \left(0, \frac{y_2^2}{4}\right)\right) \\
&= \left|\frac{y_1^2}{4} - \frac{y_2^2}{4}\right| \\
&= \left(\frac{|y_1 + y_2|}{4}\right)|y_1 - y_2| \\
&\leq \left(\frac{|y_1| + |y_2|}{4}\right)|y_1 - y_2| \\
&\leq \left(\frac{|y_1| + 1}{4}\right)|y_1 - y_2| \\
&= \mathcal{K}((0, y_1))d((0, y_1), (0, y_2)).
\end{aligned}$$

This implies that the non-self-mapping $T$ is a generalized proximal contraction of the first kind with respect to $\mathcal{K}$. It easy to see that $\mathcal{K}(x) \leq \mathcal{K}(y)$ whenever $d(gx, Ty) = d(A, B)$. Moreover, since $T$ is continuous and $g$ is an isometry, all the conditions of Theorem 5.3.1.3 are satisfied and so $T$ has an unique element $(0, 0) \in A$ such that

$$d(g(0,0), T(0,0)) = d(A, B).$$

**Corollary 5.3.1.4.** *Let $(X, d)$ be a complete metric space and $A$, $B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Suppose that $T : A \to B$ and $\mathcal{K} : A \to [0, 1)$ are mappings satisfying the following conditions:*

*(a) $T$ is a continuous generalized proximal contraction of first kind with respect to $\mathcal{K}$;*

*(b) $T(A_0) \subseteq B_0$;*

*(c) $\mathcal{K}(x) \leq \mathcal{K}(y)$, whenever $d(x, Ty) = d(A, B)$.*

*Then $T$ has a unique best proximity point in $A$.*

**Theorem 5.3.1.5.** *Let $(X, d)$ be a complete metric space and $A$, $B$ be nonempty closed subsets of $X$ such that $A$ is approximatively compact with respect to $B$. Suppose that $A_0$ and $B_0$ are nonempty and $T : A \to B$, $g : A \to A$, $\mathcal{K} : A \to [0, 1)$ are mappings satisfying the following conditions:*

(*a*) *$T$ is a continuous generalized proximal contraction of the second kind with respect to $\mathcal{K}$;*

(*b*) *$T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$;*

(*c*) *$g$ is an isometry;*

(*d*) *$T$ preserves isometric distance with respect to $g$;*

(*e*) *$\mathcal{K}(x) \leq \mathcal{K}(y)$, whenever $d(gx, Ty) = d(A, B)$.*

*Then there exists a point $x \in A$ such that $d(gx, Tx) = d(A, B)$. Moreover, if $x^*$ is another point in $A$ for which $d(gx^*, Tx^*) = d(A, B)$, then $Tx = Tx^*$.*

*Proof.* As in the proof of Theorem 5.3.1.3, for fixed $x_0 \in A_0$, we can define a sequence $\{x_n\}$ in $A_0$ such that

$$d(gx_n, Tx_{n-1}) = d(A, B) \tag{5.3.13}$$

for all $n \in \mathbb{N}$. Since $T$ is a generalized proximal contraction of the second kind with respect to $\mathcal{K}$, it follows that

$$d(Tgx_{n+1}, Tgx_n) \leq \mathcal{K}(x_n) d(Tx_n, Tx_{n-1}). \tag{5.3.14}$$

Since $T$ preserves isometric distance with respect to $g$, we have

$$d(Tx_{n+1}, Tx_n) \leq \mathcal{K}(x_n) d(Tx_n, Tx_{n-1}) \tag{5.3.15}$$

for all $n \in \mathbb{N}$. By using (5.3.13) and (e), we have

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &\leq \mathcal{K}(x_n) d(Tx_n, Tx_{n-1}) \\ &\leq \mathcal{K}(x_{n-1}) d(Tx_n, Tx_{n-1}) \\ &\leq \mathcal{K}(x_{n-2}) d(Tx_n, Tx_{n-1}) \\ &\vdots \\ &\leq \mathcal{K}(x_0) d(Tx_n, Tx_{n-1}) \end{aligned} \tag{5.3.16}$$

for all $n \in \mathbb{N}$. By repeating (5.3.16), we get

$$d(Tx_{n+1}, Tx_n) \leq (\mathcal{K}(x_0))^n d(Tx_1, Tx_0) \tag{5.3.17}$$

for all $n \in \mathbb{N}$. Now, we let $k := \mathcal{K}(x_0) \in [0, 1)$. For positive integers $m$ and $n$ with

$n > m$, it follows from (5.3.17) that

$$\begin{aligned} d(Tx_n, Tx_m) &\leq d(Tx_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_{n-2}) + \cdots + d(Tx_{m+1}, Tx_m) \\ &\leq k^{n-1} d(Tx_1, Tx_0) + k^{n-2} d(Tx_1, Tx_0) + \cdots + k^m d(Tx_1, Tx_0) \\ &\leq \left(\frac{k^m}{1-k}\right) d(Tx_1, Tx_0). \end{aligned} \tag{5.3.18}$$

Since $k \in [0,1)$, we have $\left(\frac{k^m}{1-k}\right) d(Tx_1, Tx_0) \to 0$ as $m \to \infty$, which is implies that $\{Tx_n\}$ is a Cauchy sequence in $B$. By completeness of $B \subseteq X$, there exists a point $y \in B$ such that $Tx_n \to y$ as $n \to \infty$. By (5.3.13) and the triangle inequality, we have

$$\begin{aligned} d(y, A) &\leq d(y, gx_n) \\ &\leq d(y, Tx_{n-1}) + d(Tx_{n-1}, gx_n) \\ &= d(y, Tx_{n-1}) + d(A, B) \\ &\leq d(y, Tx_{n-1}) + d(y, A). \end{aligned} \tag{5.3.19}$$

Letting $n \to \infty$ in (3.19), we get $d(y, gx_n) \to d(y, A)$. Since $A$ is approximatively compact with respect to $B$, it follows that $\{gx_n\}$ has a convergence subsequence $\{gx_{n_k}\}$, say $gx_{n_k} \to z \in A$ as $k \to \infty$. Thus we have

$$d(z, y) = \lim_{k \to \infty} d(gx_{n_k}, Tx_{n_k - 1}) = d(A, B), \tag{5.3.20}$$

which implies that $z \in A_0$. Since $A_0 \subseteq g(A_0)$, we have $z = gx$ for some $x \in A_0$. Therefore, $gx_{n_k} \to gx$ as $k \to \infty$. Since $g$ is an isometry, we get $x_{n_k} \to x$ as $k \to \infty$. By the continuity of $T$, we have $Tx_{n_k} \to Tx$ as $k \to \infty$ and then $y = Tx$. From (5.3.20), we can conclude that

$$d(gx, Tx) = d(A, B). \tag{5.3.21}$$

Next, we suppose that $x^*$ is another point in $X$ such that

$$d(gx^*, Tx^*) = d(A, B). \tag{5.3.22}$$

Since $T$ is a generalized proximal contraction of the second kind with respect to $\mathcal{K}$, by the virtue of (5.3.21) and (5.3.22), we get

$$d(Tgx, Tgx^*) \leq \mathcal{K}(x) d(Tx, Tx^*). \tag{5.3.23}$$

Since $T$ preserve isometric distance with respect to $g$, it follows that

$$d(Tx, Tx^*) \leq \mathcal{K}(x)d(Tx, Tx^*), \tag{5.3.24}$$

which implies that $Tx = Tx^*$. This completes the proof. $\square$

**Corollary 5.3.1.6.** *Let $(X, d)$ be a complete metric space and $A$, $B$ be nonempty closed subsets of $X$ such that $A$ is approximatively compact with respect to $B$. Suppose that $A_0$ and $B_0$ are nonempty and $T : A \to B$ and $\mathcal{K} : A \to [0, 1)$ are mappings satisfying the following conditions:*

*(a) $T$ is a continuous generalized proximal contraction of the second kind with respect to $\mathcal{K}$;*

*(b) $T(A_0) \subseteq B_0$;*

*(c) $\mathcal{K}(x) \leq \mathcal{K}(y)$, whenever $d(x, Ty) = d(A, B)$.*

*Then $T$ has a best proximity point. Moreover, if $x^*$ is another best proximity point of $T$, then $Tx = Tx^*$.*

**Theorem 5.3.1.7.** *Let $(X, d)$ be a complete metric space, $A$ and $B$ be nonempty closed subsets of $X$ and $\mathcal{K} : A \cup B \to [0, 1)$. Suppose that $S : A \to B$ is a mapping satisfying*

$$d(Sx, Sy) \leq \mathcal{K}(x)d(x, y) \tag{5.3.25}$$

*for all $x, y \in A$. Then the following holds:*

**(A)** *There exists a nonexpansive mapping $T : B \to A$ such that $(S, T)$ satisfies the min-max condition whenever $S$ has a best proximity point.*

**(B)** *If there exists a nonexpansive mapping $T : B \to A$ such that $(S, T)$ satisfies the min-max condition and $\mathcal{K}(Sx) \leq \mathcal{K}(x)$ and $\mathcal{K}(Tx) \leq \mathcal{K}(x)$ for all $x \in A$, then $S$ has a best proximity point.*

**(C)** *For two any best proximity points $z$ and $z^*$ of $S$, we have*

$$d(z, z^*) \leq \Big(\frac{2}{1 - \mathcal{K}(z)}\Big)d(A, B).$$

*Proof.* **(A)**: Let $S$ has a best proximity point $a \in A$. We define a mapping $T : B \to A$ by $Ty = a$ for all $y \in B$. Clearly, $T$ is a nonexpansive mapping. It follows from the definition of $T$ that

$$d(Ty, STy) = d(a, Sa) = d(A, B) \tag{5.3.26}$$

for all $y \in B$. Thus we can conclude that $\min(Sx, Ty) = d(A, B)$ for all $x \in A$ and $y \in B$.

Next, we show that $(S, T)$ satisfies the min-max condition. Suppose that $x \in A$ and $y \in B$ such that $d(A, B) < d(x, y)$. Then we have

$$\min(Sx, Ty) = d(A, B) < d(x, y) \leq \max(Sx, Ty), \tag{5.3.27}$$

which implies that the pair $(S, T)$ satisfies the min-max condition. Therefore, we can find a nonexpansive mapping $T : B \to A$ such that $(S, T)$ satisfies the min-max condition.

**(B)**: Fix $x_0 \in A$ and define a sequence $\{x_n\}$ in $A \cup B$ by

$$x_{2n-1} = Sx_{2n-2}, \quad x_{2n} = Tx_{2n-1}$$

for all $n \in \mathbb{N}$. Since $T$ is nonexpansive, it follows from (5.3.25) that

$$\begin{aligned} d(x_{2n-2}, x_{2n}) &= d(Tx_{2n-3}, Tx_{2n-1}) \\ &\leq d(x_{2n-3}, x_{2n-1}) \\ &= d(Sx_{2n-4}, Sx_{2n-2}) \\ &\leq \mathcal{K}(x_{2n-4}) d(x_{2n-4}, x_{2n-2}) \\ &= \mathcal{K}(Tx_{2n-5}) d(x_{2n-4}, x_{2n-2}) \\ &\leq \mathcal{K}(x_{2n-5}) d(x_{2n-4}, x_{2n-2}) \\ &= \mathcal{K}(Sx_{2n-6}) d(x_{2n-4}, x_{2n-2}) \\ &\leq \mathcal{K}(x_{2n-6}) d(x_{2n-4}, x_{2n-2}) \\ &\vdots \\ &\leq \mathcal{K}(x_0) d(x_{2n-4}, x_{2n-2}) \end{aligned} \tag{5.3.28}$$

for all $n \in \mathbb{N}$. By repeating the above argument, we have

$$d(x_{2n-2}, x_{2n}) \leq (\mathcal{K}(x_0))^{n-1} d(x_0, x_2) \tag{5.3.29}$$

for all $n \in \mathbb{N}$, which implies that the sequence $\{x_{2n}\}$ is a Cauchy sequence in $X$. A similar argument asserts that the sequence $\{x_{2n-1}\}$ is a Cauchy sequence in $X$. By the completeness of $X$, we conclude that $\{x_{2n}\}$ converges to a point $a \in A$ and $\{x_{2n-1}\}$ converges to a point $b \in B$. Since $S$ is continuous, $\{Sx_{2n}\}$ converges to $Sa$, which implies that $\{x_{2n-1}\}$ converges to $Sa$. Thus $Sa = b$.

Similarly, it easy to check that $Tb = a$. Therefore, we have

$$TSa = Tb = a, \quad STb = Sa = b. \tag{5.3.30}$$

Now, we can conclude that

$$\min(Sa, Tb) = d(a, b) = \max(Sa, Tb). \tag{5.3.31}$$

By the virtue of the min-max condition of $(S, T)$, we get $d(a, b) \leq d(A, B)$. Since $d(A, B) \leq d(a, b)$, we have $d(a, b) = d(A, B)$. Therefore, we have

$$d(a, Sa) = d(a, b) = d(A, B), \tag{5.3.32}$$

which implies that $S$ has a best proximity point in $A$.

**(C)**: let $z$ and $z^*$ are best proximity points of $S$. Then $d(z, Sz) = d(A, B)$ and $d(z^*, Sz^*) = d(A, B)$. Using the triangle inequality and (5.3.25), we have

$$\begin{aligned} d(z, z^*) &\leq d(z, Sz) + d(Sz, Sz^*) + d(Sz^*, z^*) \\ &\leq \mathcal{K}(z)d(z, z^*) + 2d(A, B). \end{aligned} \tag{5.3.33}$$

This implies that $d(z, z^*) \leq \big(\frac{2}{1-\mathcal{K}(z)}\big)d(A, B)$. This completes the proof. □

### 5.3.2 Some fixed point results for weakly isotone mappings in ordered Banach spaces

Let $E$ be a nonempty set and $h : E \to E$ be a given mapping. For every $x \in E$, we denote by $h^{-1}(x)$ the subset of $E$ defined by:

$$h^{-1}(x) := \{u \in E \,|\, hu = x\}.$$

**Definition 5.3.2.1.** Let $E$ be an ordered Banach space and $f, g, h : E \to E$ be given mappings such that $f(E) \subseteq h(E)$ and $g(E) \subseteq h(E)$. We say that $f$ and $g$ are *weakly isotone increasing with respect to $h$* if and only if for all $x \in E$, we have:

$$f(x) \preceq g(y), \ \forall\, y \in h^{-1}(f(x)) \tag{5.3.34}$$

and

$$g(x) \preceq f(y), \ \forall\, y \in h^{-1}(g(x)). \tag{5.3.35}$$

Similarly $f$ and $g$ are said to be *weakly isotone decreasing with respect to $h$* if

$$f(x) \succeq g(y), \ \forall\, y \in h^{-1}(f(x)) \tag{5.3.36}$$

and

$$g(x) \succeq f(y), \ \forall\, y \in h^{-1}(g(x)). \tag{5.3.37}$$

for all $x \in E$. If $f$ and $g$ are either weakly isotone increasing with respect to $h$ or weakly isotone decreasing with respect to $h$, then it is said that $f$ and $g$ are *weakly isotone with respect to $h$*.

**Remark 5.3.2.2.** If $h : E \to E$ is the identity mapping ($h(x) = x$ for all $x \in E$, shortly $h = I_E$), then the fact that $f$ and $g$ are weakly isotone increasing with respect to $h$ implies that $f$ and $g$ are weakly isotone increasing mappings.

**Example 15.** Consider $E = \mathbb{R}_+$ endowed with the usual norm and

$$P = \{z \in \mathbb{R} \,|\, z \geq 0\}.$$

Let $f, g, h : E \to E$ by

$$f(x) = 5 \text{ for all } x \in E, \quad g(x) = \begin{cases} x & \text{if } x \in [0,5] \\ 5 & \text{if } x > 5 \end{cases}$$

and

$$h(x) = x \text{ for all } x \in E.$$

Then, we obtain that $h^{-1}(f(x)) = \{5\}$ and $h^{-1}(g(x)) = \{g(x) \,|\, x \in E\}$. Since $f(E), g(E) \subseteq h(E)$ and conditions (5.3.34) and (5.3.35) hold, we have $f$ and $g$ are weakly isotone increasing with respect to $h$.

**Example 16.** Consider $E = [0, 2]$ endowed with the usual norm and

$$P = \{z \in \mathbb{R} \,|\, z \geq 0\}.$$

Let $f, g, h : E \to E$ by

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \in [0,1) \\ 0 & \text{if } x \in [1,2], \end{cases}$$

$$g(x) = \begin{cases} x & \text{if } x \in [0,1) \\ 0 & \text{if } x \in [1,2] \end{cases}$$

and

$$h(x) = \begin{cases} x^2 & \text{if } x \in [0,1) \\ 0 & \text{if } x \in [1,2]. \end{cases}$$

Now we show that the condition (5.3.34) holds. We discuss this in two cases.

Case 1: ($x = 0$ or $x \geq 1$) In this case, we get $f(x) = 0$. It is easy to see that (5.3.34) holds.

Case 2: ($x \in (0,1)$) In this case, we get $f(x) = \sqrt{x}$. Let $y \in h^{-1}(f(x))$ and then $h(y) = f(x) = \sqrt{x}$. By definition of mapping $h$, we have $y = \sqrt[4]{x}$. Thus

$$f(x) = \sqrt{x} \preceq \sqrt[4]{x} = g(\sqrt[4]{x}) = g(y).$$

Next, we will show that the condition (5.3.35) holds. We discuss this in two cases.

Case 1: ($x = 0$ or $x \geq 1$) In this case, we get $g(x) = 0$. It obvious that (5.3.35) holds.

Case 2: ($x \in (0,1)$) In this case, we get $g(x) = x$. Let $y \in h^{-1}(g(x))$ and then $h(y) = g(x) = x$. By definition of mapping $h$, we have $y = \sqrt{x}$. Thus

$$g(x) = x \preceq \sqrt[4]{x} = f(\sqrt{x}) = f(y).$$

Moreover, we obtain that $f(E) = g(E) \subseteq h(E)$. Therefore, $f$ and $g$ are weakly isotone increasing with respect to $h$.

**Definition 5.3.2.3.** Let $X$ be subset of an ordered Banach space $E$ and $f, g, h : X \to X$. Two mappings $f, g : X \to X$ are said to satisfy the ***weak-condition*** $D_X$ ***with respect to*** $h$ if for any monotone sequence $\{h(x_n)\}$ and for any fixed $a \in X$ the condition

$$\{h(x_1), h(x_2), h(x_3), ...\} \subseteq \{a\} \cup f(\{x_1, x_2, x_3, ...\}) \cup g(\{x_1, x_2, x_3, ...\})$$

implies $\{x_n\}$ is convergent.

**Remark 5.3.2.4.** If $h : E \to E$ is the identity mapping, then the $f$ and $g$ are weak-condition $D_X$ with respect to $h$ implies that $f$ and $g$ are weak-condition $D_X$.

Let $X$ be a closed subset of an ordered Banach space $E$ and $f, g, h : X \to X$ be three mappings such that $f$ and $g$ are weakly isotone with respect to $h$. Given $x_0 \in X$ we define a sequence $\{h(x_n)\}$ in $X$ as follows:

$$h(x_{2n-1}) = f(x_{2n-2}), \quad h(x_{2n}) = g(x_{2n-1})$$

for all $n \in \mathbb{N}$. We say that $\{h(x_n)\}$ is an $(f, g, h)-$sequence with initial point $x_0$. If $h$ is an identity mapping on $X$, then we write $(f, g)-$sequence with initial point $x_0$. Later on, we denote by $C(f, g, h)$ the set of coincidence points of $f, g$ and $h$, that is, $C(f, g, h) = \{x \in E : h(x) = f(x) = g(x)\}$.

Next, we give the coincidence point theorems for weakly isotone mappings under the certain conditions.

**Theorem 5.3.2.5.** *Let $X$ be a closed subset of an ordered Banach space $E$ and $f, g, h : X \to X$ be three continuous mappings. If $f$ and $g$ are weakly isotone with respect to $h$ and if $f$ and $g$ satisfy weak-condition $D_X$ with respect to $h$, then $f, g$ and $h$ have a coincidence point. Moreover, for every $x_0 \in X$, we have $\lim_{n\to\infty} h(x_n) \in C(f, g, h)$ for every $(f, g, h)-$sequence $\{h(x_n)\}$ with initial point $x_0$.*

**Proof.** Assume that $f$ and $g$ are weakly isotone increasing with respect to $h$. By Definition 5.3.2.1, it follows that $f(X) \subseteq h(X)$ and $g(X) \subseteq h(X)$. Let $x_0$ be an arbitrary point in $X$. Since $f(X) \subseteq h(X)$, there exists $x_1 \in X$ such that $h(x_1) = f(x_0)$. Since $g(X) \subseteq h(X)$, there exists $x_2 \in X$ such that $h(x_2) = g(x_1)$.

Continuing this process, we can construct a sequence $\{h(x_n)\}$ in $X$ defined by

$$h(x_{2n-1}) = f(x_{2n-2}), \quad h(x_{2n}) = g(x_{2n-1}), \quad \forall\, n \in \mathbb{N}. \tag{5.3.38}$$

By the construction, we have $x_1 \in h^{-1}(f(x_0))$ and $x_2 \in h^{-1}(g(x_1))$, then using the fact that $f$ and $g$ are weakly isotone increasing with respect to $h$, we get that

$$h(x_1) = f(x_0) \preceq g(x_1) = h(x_2) \preceq f(x_2) = h(x_3).$$

We continue this process to get

$$h(x_1) \preceq h(x_2) \preceq \cdots \preceq h(x_{2n-1}) \preceq h(x_{2n}) \preceq \cdots \tag{5.3.39}$$

for all $n \in \mathbb{N}$ Now, we have

$$\begin{aligned} \{h(x_1), h(x_2), h(x_3), ...\} &= \{h(x_1)\} \cup \{h(x_3), h(x_5), ...\} \cup \{h(x_2), h(x_4), ...\} \\ &\subseteq \{h(x_1)\} \cup f(\{x_1, x_2, x_3, ...\}) \cup g(\{x_1, x_2, x_3, ...\}). \end{aligned}$$

From the hypothesis that is $f$ and $g$ are weak-condition $D_X$ with respect to $h$, we have the sequence $\{x_n\}$ converge to some $x \in X$. Since $f, g, h$ are continuous, we get

$$h(x) = \lim_{n\to\infty} h(x_{2n-1}) = \lim_{n\to\infty} f(x_{2n-2}) = f(x)$$

and

$$h(x) = \lim_{n\to\infty} h(x_{2n}) = \lim_{n\to\infty} g(x_{2n-1}) = g(x)$$

It follows that $f(x) = g(x) = h(x)$ and so $x$ is a coincidence point of $f, g$ and $h$. For the case when $f$ and $g$ are weakly isotone decreasing with respect to $h$ is similar.

Next theorem, we propose the weaken assumption of continuous of $f$ and $g$ in Theorem 5.3.2.5 by monotone-continuous condition.

**Theorem 5.3.2.6.** *Let $X$ be a closed subset of an ordered Banach space $E$ and let $f, g : X \to X$ be two monotone-continuous mappings. If $f$ and $g$ are weakly isotone and if $f$ and $g$ satisfy weak-condition $D_X$, then $f$ and $g$ have a common fixed point. Moreover, for every $x_0 \in X$, we have $\lim_{n\to\infty} x_n \in F(f, g)$ for every $(f, g)-$sequence $\{x_n\}$ with initial point $x_0$, where $F(f, g)$ is set of common fixed point of $f$ and $g$.*

**Proof.** With the same argument of Theorem 5.3.2.5 we can prove this theorem if a mapping $h$ is an identity mapping on $X$.

Since the condition $D_X$ implies the weak-condition $D_X$, we get the following result.

**Theorem 5.3.2.7.** *Let $X$ be a closed subset of an ordered Banach space $E$ and let $f, g : X \to X$ be two monotone-continuous mappings. If $f$ and $g$ are weakly isotone and if $f$ and $g$ satisfy condition $D_X$, then $f$ and $g$ have a common fixed point. Moreover, for every $x_0 \in X$, we have $\lim_{n\to\infty} x_n \in F(f, g)$ for every $(f, g)-$sequence $\{x_n\}$ with initial point $x_0$, where $F(f, g)$ is set of common fixed point of $f$ and $g$.*

Since the continuous mapping implies the monotone-continuous mapping, we get the Dhage et al.'s results in [30].

**Corollary 5.3.2.8.** *[30, Theorem 2.1] Let $X$ be a closed subset of an ordered Banach space $E$ and let $f, g : X \to X$ be two continuous mappings. If $f$ and $g$ are weakly isotone and if $f$ and $g$ satisfy condition $D_X$, then $f$ and $g$ have a common fixed point.*

**Corollary 5.3.2.9.** *[30, Corollary 2.1] Let $X$ be a closed subset of an ordered Banach space $E$ and let $f, g : X \to X$ be two continuous mappings. If $f$ and $g$ are weakly isotone and if $f$ and $g$ are countably condensing, then $f$ and $g$ have a common fixed point.*

In this section, we give the coincidence and common fixed point results for single valued and multivalued mapping. Let $E$ be an ordered Banach space, $P$ be a cone in $E$ and $\preceq$ is a partial ordering with respect to $P$. For $X, Y \in 2^E$, we will write $X \precapprox Y$ mean that $x \preceq y$ for all $x \in X$ and $y \in Y$. Moreover, $X \succapprox Y$ mean that $Y \precapprox X$.

**Example 17.** Consider $E = \mathbb{R}^2_+$ endowed with the usual norm and

$$P = \{(z, z) \in \mathbb{R}^2_+ \,|\, z \geq 0\}.$$

Let

$$X := \{(x, x) \in E : \|(x, x)\| \leq 1\}$$

and

$$Y := \{(y, y) \in E : 9 \leq \|(y, y)\| \leq 16\}.$$

Then, we have $X \precapprox Y$. Let $E$ be a nonempty set and $h : E \to E$ be a given mapping. For every $A \subseteq E$, we denote by $h^{-1}(A)$ the subset of $E$ defined by:

$$h^{-1}(A) := \{x \in E \,|\, h(x) \in A\}.$$

**Definition 5.3.2.10.** Let $E$ be an ordered Banach space, $F, G : E \to 2^E$ and $h : E \to E$ be given mappings such that $F(E) \subseteq h(E)$ and $G(E) \subseteq h(E)$. We say that $F$ and $G$ are *weakly isotone increasing with respect to* $h$ if and only if for all $x \in E$, we have:

$$F(x) \precapprox G(y), \ \forall\, y \in h^{-1}(F(x)) \tag{5.3.40}$$

and

$$G(x) \precapprox F(y), \ \forall\, y \in h^{-1}(G(x)). \tag{5.3.41}$$

Similarly $F$ and $G$ are said to be *weakly isotone decreasing with respect to* $h$ if

$$F(x) \succapprox G(y), \ \forall\, y \in h^{-1}(F(x)) \tag{5.3.42}$$

and

$$G(x) \gtrapprox F(y), \ \forall\, y \in h^{-1}(G(x)) \tag{5.3.43}$$

for all $x \in E$. We say that $F$ and $G$ are *weakly isotone with respect to* $h$ if $F$ and $G$ are weakly isotone increasing with respect to $h$ or weakly isotone decreasing with respect to $h$.

**Remark 5.3.2.11.** For $h = I_E$, we obtain that $F$ and $G$ are weakly isotone with respect to $h$ implies that $F$ and $G$ are weakly isotone.

**Definition 5.3.2.12.** Let $E$ be an ordered Banach space. A mapping $F : E \to 2^E$ is said to be *closed* if for each sequence $\{x_n\}$ in $E$ with $\lim\limits_{n\to\infty} x_n = x_0$ for some $x_0 \in E$, and for each sequence $\{y_n\}$ in $E$ with $y_n \in F(x_n)$ and $\lim\limits_{n\to\infty} y_n = y_0$ for some $y_0 \in E$, we have $y_0 \in F(x_0)$.

**Definition 5.3.2.13.** Let $E$ be an ordered Banach space. A mapping $F : E \to 2^E$ is said to be *monotone-closed* if for each increasing or decreasing sequence $\{x_n\}$ in $E$ with $\lim\limits_{n\to\infty} x_n = x_0$ for some $x_0 \in E$, and for each sequence $\{y_n\}$ in $E$ with $y_n \in F(x_n)$ and $\lim\limits_{n\to\infty} y_n = y_0$ for some $y_0 \in E$, we have $y_0 \in F(x_0)$.

**Definition 5.3.2.14.** Let $E$ be an ordered Banach space and $X \subseteq E$. Two multivalued mappings $F, G : X \to 2^X$ are said to satisfy the ***condition*** $D_X$ if for any countable set $A$ of $X$ and for any fixed $a \in X$ the condition

$$A \subseteq \{a\} \cup F(A) \cup G(A)$$

implies $\overline{A}$ is compact.

**Definition 5.3.2.15.** Let $E$ be an ordered Banach space, $X \subseteq E$. Two multivalued mappings $F, G : X \to 2^X$ are said to satisfy the ***weak-condition*** $D_X$ if for any monotone sequence $\{x_n\}$ and for any fixed $a \in X$ the condition

$$\{x_1, x_2, x_3, ...\} \subseteq \{a\} \cup F(\{x_1, x_2, x_3, ...\}) \cup G(\{x_1, x_2, x_3, ...\})$$

implies $\{x_n\}$ is convergent.

**Remark 5.3.2.16.** We obtain that condition $D_X$ implies weak-condition $D_X$ in case of multivalued mappings.

**Definition 5.3.2.17.** Let $E$ be an ordered Banach space, $X \subseteq E$ and $h : X \to X$. Two multivalued mappings $F, G : X \to 2^X$ are said to satisfy the ***weak-condition*** $D_X$ ***with respect to*** $h$ if for any monotone sequence $\{h(x_n)\}$ and for any fixed $a \in X$ the condition

$$\{h(x_1), h(x_2), h(x_3), ...\} \subseteq \{a\} \cup F(\{x_1, x_2, x_3, ...\}) \cup G(\{x_1, x_2, x_3, ...\})$$

implies $\{x_n\}$ is convergent.

**Remark 5.3.2.18.** If we take $h = I_E$, then $F$ and $G$ are weak-condition $D_X$ with respect to $h$ implies that $F$ and $G$ are weak-condition $D_X$.

Let $X$ be a closed subset of an ordered Banach space $E$ and $h : X \to X$. Given $F, G : X \to 2^X$ be two multivalued mappings such that $F$ and $G$ are weakly isotone with respect to $h$ and given $x_0 \in X$ we define a sequence $\{h(x_n)\}$ in $X$ as follows:

$$h(x_{2n-1}) \in F(x_{2n-2}), \quad h(x_{2n}) \in G(x_{2n-1})$$

for all $n \in \mathbb{N}$. We say that $\{h(x_n)\}$ is a $(F, G, h)-$sequence with initial point $x_0$. If $h$ is an identity mapping on $X$, then we write $(F, G)-$sequence with initial point $x_0$. Later on, we denote

$$\mathcal{CO}(F, G, h) := \{x \in E \,|\, h(x) \in F(x) \text{ and } h(x) \in G(x)\}$$

and denote $\mathcal{F}(F, G)$ is set of all common fixed point of $F$ and $G$, that is,

$$\mathcal{F}(F, G) := \{x \in E \,|\, x \in F(x) \text{ and } x \in G(x)\}.$$

Now, we establish the coincidence point theorems for weakly isotone increasing for single valued and multivalued mappings under certain conditions.

**Theorem 5.3.2.19.** *Let $X$ be a closed subset of an ordered Banach space $E$ and let $F, G : X \to 2^X$ be two closed mappings and $h : X \to X$ be a continuous mapping. If $F$ and $G$ are weakly isotone with respect to $h$ and satisfy weak-condition $D_X$ with respect to $h$, then there is a point $z \in X$ such that $z \in \mathcal{CO}(F, G, h)$. Moreover, for every $x_0 \in X$, we have $\lim\limits_{n\to\infty} h(x_n) \in \mathcal{CO}(F, G, h)$ for every $(F, G, h)-$sequence $\{h(x_n)\}$ with initial point $x_0$.*

**Proof.** Assume that $F$ and $G$ are weakly isotone increasing with respect to $h$. By Definition 5.3.2.10, we have $F(X) \subseteq h(X)$ and $G(X) \subseteq h(X)$. Let $x_0$ be an arbitrary point in $X$. Since $F(X) \subseteq h(X)$, we get $F(x_0) \subseteq h(X)$ and so there exists $x_1 \in X$ such that $h(x_1) \in F(x_0)$. It follows from $G(X) \subseteq h(X)$ that $G(x_1) \subseteq h(X)$ and then there exists $x_2 \in X$ such that $h(x_2) \in G(x_1)$. Continuing this process, we can construct a sequence $\{h(x_n)\}$ in $X$ defined by

$$h(x_{2n-1}) \in F(x_{2n-2}), \quad h(x_{2n}) \in G(x_{2n-1}), \quad \forall\, n \in \mathbb{N}. \tag{5.3.44}$$

By construction, we have $x_1 \in h^{-1}(F(x_0))$ and $x_2 \in h^{-1}(G(x_1))$, then using the hypothesis that $F$ and $G$ are weakly isotone increasing with respect to $h$, we have $F(x_0) \precapprox G(x_1)$ and $G(x_1) \precapprox F(x_2)$. We continue this process to get

$$F(x_0) \precapprox G(x_1) \precapprox \cdots \precapprox F(x_{2n-2}) \precapprox G(x_{2n-1}) \precapprox \cdots \tag{5.3.45}$$

for all $n \in \mathbb{N}$. From (5.3.44) and the definition of $\precapprox$, we get

$$h(x_1) \preceq h(x_2) \preceq \cdots \preceq h(x_{2n-1}) \preceq h(x_{2n}) \preceq \cdots \tag{5.3.46}$$

for all $n \in \mathbb{N}$. Now, we have

$$\begin{aligned} \{h(x_1), h(x_2), h(x_3), ...\} &= \{h(x_1)\} \cup \{h(x_3), h(x_5), ...\} \cup \{h(x_2), h(x_4), ...\} \\ &\subseteq \{h(x_1)\} \cup F(\{x_1, x_2, x_3, ...\}) \cup G(\{x_1, x_2, x_3, ...\}). \end{aligned}$$

From the hypothesis that is $F$ and $G$ are weak-condition $D_X$ with respect to $h$, we get the sequence $\{x_n\}$ converge to some $x \in X$. Since $F, G$ are closed mappings and $h$ is continuous mapping, we get

$$h(x) = \lim_{n\to\infty} h(x_{2n-1}) \in \lim_{n\to\infty} F(x_{2n-2}) = F(x)$$

and

$$h(x) = \lim_{n\to\infty} h(x_{2n}) \in \lim_{n\to\infty} G(x_{2n-1}) = G(x).$$

Therefore, $x \in \mathcal{CO}(F, G, h)$. For the case when $F$ and $G$ are weakly isotone decreasing with respect to $h$ is similar. If we take the mapping $h$ in Theorem 5.3.2.19 as the identity mapping on $X$, we get the following result for two monotone-closed mappings.

**Theorem 5.3.2.20.** *Let $X$ be a closed subset of an ordered Banach space $E$ and let $F, G : X \to 2^X$ be two monotone-closed mappings. If $F$ and $G$ are weakly isotone and satisfy weak-condition $D_X$, then $F$ and $G$ have a common fixed point. Moreover, for every $x_0 \in X$, we have $\lim_{n\to\infty} x_n \in \mathcal{F}(F, G)$ for every $(F, G)-$sequence with initial point $x_0$.*

Since the condition $D_X$ implies the weak-condition $D_X$, we can omit the proof of the following result.

**Theorem 5.3.2.21.** *Let $X$ be a closed subset of an ordered Banach space $E$ and let $F, G : X \to 2^X$ be two monotone-closed mappings. If $F$ and $G$ are weakly isotone and satisfy condition $D_X$, then $F$ and $G$ have a common fixed point. Moreover, for every $x_0 \in X$, we have $\lim_{n\to\infty} x_n \in \mathcal{F}(F, G)$ for every $(F, G)-$sequence with initial point $x_0$.*

From the fact that every closed mapping is a monotone-closed mapping, we get the following results of Dhage et al. [30].

**Corollary 5.3.2.22.** *[30, Theorem 3.1] Let $X$ be a closed subset of an ordered Banach space $E$ and $F, G : X \to C(X)$. If $F$ and $G$ are closed and weakly isotone mappings satisfy condition $D_X$, then $F$ and $G$ have a common fixed point.*

**Corollary 5.3.2.23.** *[30, Corollary 3.1] Let $X$ be a closed subset of an ordered Banach space $E$ and $F, G : X \to C(X)$. If $F$ and $G$ are closed, weakly isotone and countably condensing mappings, then $F$ and $G$ have a common fixed point.*

### 5.3.3 Coupled coincidence point and common coupled fixed point theorems with lacking the mixed monotone property

**Theorem 5.3.3.1.** *Let $(X, d, \preceq)$ be an ordered cone metric space over a solid cone $P$ and let $g : X \longrightarrow X$ and $F : X \times X \longrightarrow X$. Suppose that the following hold:*

*(i) $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of $X$;*

*(ii) $g$ and $F$ satisfy property;*

*(iii) there exist $x_0, y_0 \in X$ such that $gx_0 \asymp F(x_0, y_0)$ and $gy_0 \asymp F(y_0, x_0)$;*

*(iv) there exists* $a_i \geq 0$*, for* $i = 1, 2, \ldots, 6$ *and* $\sum_{i=1}^{6} a_i < 1$ *such that for all* $x, y, u, v \in X$ *satisfying* $gx \asymp gu$ *and* $gy \asymp gv$*,*

$$\begin{aligned} d(F(x,y), F(u,v)) \quad \preceq_P \quad & a_1 d(gx, gu) + a_2 d(F(x,y), gx) + a_3 d(gy, gv) \\ & + a_4 d(F(u,v), gu) + a_5 d(F(x,y), gu) + a_6 d(F(u,v), gx), \end{aligned} \tag{5.3.47}$$

*holds;*

*(v) if* $x_n \longrightarrow x$ *when* $n \longrightarrow \infty$ *in* $X$*, then* $x_n \asymp x$ *for* $n$ *sufficiently large.*

*Then there exist* $x, y \in X$ *such that*

$$F(x,y) = gx \quad \text{and} \quad F(y,x) = gy,$$

*that is,* $F$ *and* $g$ *have a coupled coincidence point* $(x, y) \in X \times X$*.*

*Proof.* Starting from $x_0$, $y_0$ (condition (iii)) and using the fact that $F(X \times X) \subseteq g(X)$ (condition (i)), we can construct sequences $\{gx_n\}$ and $\{gy_n\}$ in $X$ such that

$$gx_n = F(x_{n-1}, y_{n-1}) \quad \text{and} \quad gy_n = F(y_{n-1}, x_{n-1}) \tag{5.3.48}$$

for all $n \in \mathbb{N}$. By (iii), we get $gx_0 \asymp F(x_0, y_0) = gx_1$ and by condition (ii) implies that

$$gx_1 = F(x_0, y_0) \asymp F(x_1, y_1) = gx_2.$$

Proceeding by induction we get that $gx_{n-1} \asymp gx_n$ and, similarly, $gy_{n-1} \asymp gy_n$ for all $n \in \mathbb{N}$. Therefore, we can apply condition (5.3.47) to obtain

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\preceq_P a_1 d(gx_{n-1}, gx_n) + a_2 d(F(x_{n-1}, y_{n-1}), gx_{n-1}) \\ &\quad + a_3 d(gy_{n-1}, gy_n) + a_4 d(F(x_n, y_n), gx_n) \\ &\quad + a_5 d(F(x_{n-1}, y_{n-1}), gx_n) + a_6 d(F(x_n, y_n), gx_{n-1}) \\ &= a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) \\ &\quad + a_4 d(gx_{n+1}, gx_n) + a_5 d(gx_n, gx_n) + a_6 d(gx_{n+1}, gx_{n-1}) \\ &\preceq_P a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) \\ &\quad + a_4 d(gx_{n+1}, gx_n) + a_6 [d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})] \\ &\preceq_P (a_1 + a_2 + a_6) d(gx_{n-1}, gx_n) + a_3 d(gy_{n-1}, gy_n) \\ &\quad + (a_4 + a_6) d(gx_n, gx_{n+1}) \end{aligned}$$

which implies that

$$(1-a_4-a_6)d(gx_n, gx_{n+1}) \preceq_P (a_1+a_2+a_6)d(gx_{n-1}, gx_n)+a_3d(gy_{n-1}, gy_n). \quad (5.3.49)$$

Similarly, starting with $d(gy_n, gy_{n+1}) = d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))$ and using that $gx_{n-1} \asymp gx_n$ and $gy_{n-1} \asymp gy_n$ for all $n \in \mathbb{N}$, we get

$$(1-a_4-a_6)d(gy_n, gy_{n+1}) \preceq_P (a_1+a_2+a_6)d(gy_{n-1}, gy_n)+a_3d(gx_{n-1}, gx_n). \quad (5.3.50)$$

Combine (5.3.49) and (5.3.50) we obtain that

$$\begin{aligned}(1 - a_4 - a_6)&[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] \\ &\preceq_P (a_1 + a_2 + a_3 + a_6)[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)].\end{aligned} \quad (5.3.51)$$

Now, starting from $d(gx_{n+1}, gx_n) = d(F(x_n, y_n), F(x_{n-1}, y_{n-1})$ and using that $gx_{n-1} \asymp gx_n$ and $gy_{n-1} \asymp gy_n$ for all $n \in \mathbb{N}$, we get that

$$(1 - a_2 - a_5)d(gx_n, gx_{n+1}) \preceq_P (a_1 + a_4 + a_5)d(gx_{n-1}, gx_n) + a_3d(gy_{n-1}, gy_n).$$

Similarly, starting from $d(gy_{n+1}, gy_n) = d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))$ and using that $gx_{n-1} \asymp gx_n$ and $gy_{n-1} \asymp gy_n$ for all $n \in \mathbb{N}$, we get that

$$(1 - a_2 - a_5)d(gy_n, gy_{n+1}) \preceq_P (a_1 + a_4 + a_5)d(gy_{n-1}, gy_n) + a_3d(gx_{n-1}, gx_n).$$

Again adding up, we obtain that

$$\begin{aligned}(1 - a_2 - a_5)&[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] \\ &\preceq_P (a_1 + a_3 + a_4 + a_5)[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)].\end{aligned} \quad (5.3.52)$$

Finally, adding up (5.3.51) and (5.3.52), it follows that

$$d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) \preceq_P \lambda[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)], \quad (5.3.53)$$

with

$$0 \leq \lambda = \frac{2a_1 + a_2 + 2a_3 + a_4 + a_5 + a_6}{2 - a_2 - a_4 - a_5 - a_6} < 1, \quad (5.3.54)$$

since $\sum_{i=1}^{n} a_i < 1$.

From the relation (5.3.53), we have

$$\begin{aligned}d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) &\preceq_P \lambda[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)] \\ &\preceq_P \lambda^2[d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})] \\ &\vdots \\ &\preceq_P \lambda^n[d(gx_0, gx_1) + d(gy_0, gy_1)].\end{aligned}$$

If $d(gx_0, gx_1) + d(gy_0, gy_1) = 0_E$ then $(x_0, y_0)$ is a coupled coincidence point of $F$ and $g$. So let $0_E <_P d(gx_0, gx_1) + d(gy_0, gy_1)$.

For any $m > n \geq 1$, repeated use of triangle inequality gives

$$\begin{aligned} d(gx_n, gx_m) &+ d(gy_n, gy_m) \\ &\leq_P d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{m-1}, gx_m) \\ &\quad + d(gy_n, gy_{n+1}) + d(gy_{n+1}, gy_{n+2}) + \cdots + d(gy_{m-1}, gy_m) \\ &\leq_P [\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}][d(gx_0, gx_1) + d(gy_0, gy_1)] \\ &\leq_P \frac{\lambda^n}{1-\lambda}[d(gx_0, gx_1) + d(gy_0, gy_1)]. \end{aligned}$$

Since $\frac{\lambda^n}{1-\lambda} \to 0$ as $n \to \infty$, we get $\frac{\lambda^n}{1-\lambda}[d(gx_0, gx_1) + d(gy_0, gy_1)] \to 0_E$ as $n \to \infty$.

From $(p_4)$, we have for $0_E \ll c$ and large $n$:

$$\frac{\lambda^n}{1-\lambda}[d(gx_0, gx_1) + d(gy_0, gy_1)] \ll c.$$

By $(p_3)$, we get

$$d(gx_n, gx_m) + d(gy_n, gy_m) \ll c.$$

Since

$$d(gx_n, gx_m) \leq_P d(gx_n, gx_m) + d(gy_n, gy_m)$$

and

$$d(gy_n, gy_m) \leq_P d(gx_n, gx_m) + d(gy_n, gy_m)$$

then by $(p_3)$, we get $d(gx_n, gx_m) \ll c$ and $d(gy_n, gy_m) \ll c$ for $n$ large enough. Therefore, we get $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. By completeness of $g(X)$, there exists $gx, gy \in g(X)$ such that $gx_n \longrightarrow gx$ and $gy_n \longrightarrow gy$ as $n \longrightarrow \infty$.

By (v), we have $gx_n \asymp gx$ and $gy \asymp gy_n$ for all $n \geq 0$. Now we prove that $F(x, y) = gx$ and $F(y, x) = gy$.

If $gx_n = gx$ and $gy_n = gy$ for some $n \geq 0$. From (5.3.47), we have

$$\begin{aligned}
d(F(x,y),gx) &\leq_P d(F(x,y),gx_{n+1}) + d(gx_{n+1},gx) \\
&= d(F(x,y),F(x_n,y_n)) + d(gx_{n+1},gx) \\
&\leq_P a_1 d(gx,gx_n) + a_2 d(F(x,y),gx) + a_3 d(gy,gy_n) \\
&\quad + a_4 d(F(x_n,y_n),gx_n) + a_5 d(F(x,y),gx_n) \\
&\quad + a_6 d(F(x_n,y_n),gx) + d(gx_{n+1},gx) \\
&\leq_P a_1 d(gx,gx_n) + a_2 d(F(x,y),gx) + a_3 d(gy,gy_n) \\
&\quad + a_4 d(gx_{n+1},gx) + a_4 d(gx,gx_n) + a_5 d(F(x,y),gx) + a_5 d(gx,gx_n) \\
&\quad + a_6 d(gx_{n+1},gx) + d(gx_{n+1},gx) \\
&= a_2 d(F(x,y),gx) + a_4 d(gx_{n+1},gx) + a_5 d(F(x,y),gx) \\
&\quad + a_6 d(gx_{n+1},gx) + d(gx_{n+1},gx),
\end{aligned}$$

which further implies that

$$d(F(x,y),gx) \leq_P \frac{1+a_4+a_6}{1-a_2-a_5} d(gx_{n+1},gx).$$

Since $gx_n \longrightarrow gx$ then for $0_E \ll c$ there exists $N \in \mathbb{N}$ such that

$$d(gx_{n+1},gx) \ll \frac{(1-a_2-a_5)c}{1+a_4+a_6},$$

for all $n \geq N$. Therefore, ,

$$d(F(x,y),gx) \ll c.$$

Now, according to $(p_2)$ it follows that $d(F(x,y),gx) = 0_E$, and $F(x,y) = gx$. Similarly, we can prove that $F(y,x) = gy$. Hence $(x,y)$ is coupled coincidence point of mappings $F$ and $g$.

So we suppose that $(gx_n, gy_n) \neq (gx, gy)$ for all $n \geq 0$. Using (5.3.47), we get

$$\begin{aligned}
d(F(x,y),gx) &\leq_P d(F(x,y),gx_{n+1}) + d(gx_{n+1},gx) \\
&= d(F(x,y),F(x_n,y_n)) + d(gx_{n+1},gx) \\
&\leq_P a_1 d(gx,gx_n) + a_2 d(F(x,y),gx) + a_3 d(gy,gy_n) \\
&\quad + a_4 d(F(x_n,y_n),gx_n) + a_5 d(F(x,y),gx_n) \\
&\quad + a_6 d(F(x_n,y_n),gx) + d(gx_{n+1},gx) \\
&\leq_P a_1 d(gx,gx_n) + a_2 d(F(x,y),gx) + a_3 d(gy,gy_n) \\
&\quad + a_4 d(gx_{n+1},gx) + a_4 d(gx,gx_n) + a_5 d(F(x,y),gx) + a_5 d(gx,gx_n) \\
&\quad + a_6 d(gx_{n+1},gx) + d(gx_{n+1},gx),
\end{aligned}$$

which further implies that

$$\begin{aligned}
&d(F(x,y),gx) \\
&\leq_P \frac{a_1+a_4+a_5}{1-a_2-a_5} d(gx,gx_n) + \frac{1+a_4+a_6}{1-a_2-a_5} d(gx_{n+1},gx) + \frac{a_3}{1-a_2-a_5} d(gy,gy_n).
\end{aligned}$$

Since $gx_n \longrightarrow gx$ and $gy_n \longrightarrow gy$ then for $0_E \ll c$ there exists $N \in \mathbb{N}$ such that $d(gx_n, gx) \ll \frac{(1-a_2-a_5)c}{3(a_1+a_4+a_5)}$, $d(gx_{n+1}, gx) \ll \frac{(1-a_2-a_5)c}{3(1+a_4+a_6)}$, and $d(gy_n, gy) \ll \frac{(1-a_2-a_5)c}{3a_3}$, for all $n \geq N$. Thus,

$$d(F(x,y),gx) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c.$$

Now, according to $(p_2)$ it follows that $d(F(x,y), gx) = 0_E$, and $F(x,y) = gx$. Similarly, $F(y,x) = gy$. Hence $(x,y)$ is coupled coincidence point of mappings $F$ and $g$. □

**Remark 5.3.3.2.** In Theorem 5.3.3.1, condition (ii) is a substitution for the mixed $g$-monotone property that was used in most of the coupled coincidence point theorems so far. Therefore, Theorem improve results of Nashine *et al.* [22]. Moreover, it is an ordered version extension of the results of Abbas et al. [21].

**Corollary 5.3.3.3.** *Let $(X, d, \preceq)$ be an ordered cone metric space over a solid cone $P$ and let $g : X \longrightarrow X$ and $F : X \times X \longrightarrow X$. Suppose that the following hold:*

*(i) $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of $X$;*

*(ii) $g$ and $F$ satisfy property;*

*(iii) there exist* $x_0, y_0 \in X$ *such that* $gx_0 \asymp F(x_0, y_0)$ *and* $gy_0 \asymp F(y_0, x_0)$*;*

*(iv) there exists* $\alpha, \beta, \gamma \geq 0$ *and* $\alpha + \beta + \gamma < 1$ *such that for all* $x, y, u, v \in X$ *satisfying* $gx \asymp gu$ *and* $gy \asymp gv$*,*

$$d(F(x,y), F(u,v)) \leq_P \alpha d(gx, gu) + \beta d(gy, gv) + \gamma d(F(x,y), gu), \tag{5.3.55}$$

*holds;*

*(v) if* $x_n \longrightarrow x$ *when* $n \longrightarrow \infty$ *in* $X$*, then* $x_n \asymp x$ *for* $n$ *sufficiently large.*

*Then there exist* $x, y \in X$ *such that*

$$F(x,y) = gx \quad \text{and} \quad F(y,x) = gy,$$

*that is,* $F$ *and* $g$ *have a coupled coincidence point* $(x, y) \in X \times X$*.*

Putting $g = I_X$, where $I_X$ is the identity mapping from $X$ into $X$ in Theorem 5.3.3.1 we get the following corollary.

**Corollary 5.3.3.4.** *Let* $(X, d, \preceq)$ *be an ordered cone metric space over a solid cone* $P$ *and let* $F : X \times X \longrightarrow X$*. Suppose that the following hold:*

*(i)* $X$ *is a complete;*

*(ii)* $g$ *and* $F$ *satisfy property;*

*(iii) there exist* $x_0, y_0 \in X$ *such that* $x_0 \asymp F(x_0, y_0)$ *and* $y_0 \asymp F(y_0, x_0)$*;*

*(iv) there exists* $a_i \geq 0$*, for* $i = 1, 2, \ldots, 6$ *and* $\sum_{i=1}^{6} a_i < 1$ *such that for all* $x, y, u, v \in X$ *satisfying* $x \asymp u$ *and* $y \asymp v$*,*

$$\begin{aligned} d(F(x,y), F(u,v)) \ \leq_P \ & a_1 d(x,u) + a_2 d(F(x,y), x) + a_3 d(y,v) \\ & + a_4 d(F(u,v), u) + a_5 d(F(x,y), u) + a_6 d(F(u,v), x), \end{aligned} \tag{5.3.56}$$

*holds;*

*(v) if* $x_n \longrightarrow x$ *when* $n \longrightarrow \infty$ *in* $X$*, then* $x_n \asymp x$ *for* $n$ *sufficiently large.*

*Then there exist* $x, y \in X$ *such that*

$$F(x,y) = x \quad \textit{and} \quad F(y,x) = y,$$

*that is,* $F$ *has a coupled fixed point* $(x,y) \in X \times X$.

Our second main result is the following

**Theorem 5.3.3.5.** *Let* $(X, d, \preceq)$ *be an ordered cone metric space over a solid cone* $P$. *Let* $F : X \times X \longrightarrow X$ *and* $g : X \longrightarrow X$ *be mappings. Suppose that the following hold:*

*(i)* $F(X \times X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* $X$*;*

*(ii)* $g$ *and* $F$ *satisfy property;*

*(iii) there exist* $x_0, y_0 \in X$ *such that* $gx_0 \asymp F(x_0, y_0)$ *and* $gy_0 \asymp F(y_0, x_0)$*;*

*(iv) there is some* $h \in [0, 1/2)$ *such that for all* $x, y, u, v \in X$ *satisfying* $gx \asymp gu$ *and* $gy \asymp gv$*, there exists*

$$\Theta_{x,y,u,v} \in \{d(gx, gu), d(gy, gv), d(F(x,y), gu)\}$$

*such that*

$$d(F(x,y), F(u,v)) \leq_P h\Theta_{x,y,u,v}.$$

*(v) if* $x_n \longrightarrow x$ *when* $n \longrightarrow \infty$ *in* $X$*, then* $x_n \asymp x$ *for* $n$ *sufficiently large.*

*Then there exist* $x, y \in X$ *such that*

$$F(x,y) = gx \quad \textit{and} \quad F(y,x) = gy,$$

*that is,* $F$ *and* $g$ *have a coupled coincidence point* $(x,y) \in X \times X$.

*Proof.* Since $F(X \times X) \subseteq g(X)$ (condition (i)), we can starting from $x_0$, $y_0$ (condition (iii)) and construct sequences $\{gx_n\}$ and $\{gy_n\}$ in $X$ such that

$$gx_n = F(x_{n-1}, y_{n-1}) \quad \text{and} \quad gy_n = F(y_{n-1}, x_{n-1}) \tag{5.3.57}$$

for all $n \in \mathbb{N}$. From (iii), we get $gx_0 \asymp F(x_0, y_0) = gx_1$ and by condition (ii) implies that

$$gx_1 = F(x_0, y_0) \asymp F(x_1, y_1) = gx_2.$$

By repeating this process, we have $gx_{n-1} \asymp gx_n$. Similarly, we can prove that $gy_{n-1} \asymp gy_n$ for all $n \in \mathbb{N}$.

Since $gx_{n-1} \asymp gx_n$ and $gy_{n-1} \asymp gy_n$ for all $n \in \mathbb{N}$, from (iv), we have that there exists $h \in [0, 1/2)$ and

$$\begin{aligned}\Theta_1 &\in \{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(F(x_{n-1}, y_{n-1}), gx_n)\}\\ &= \{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), 0_E\}\end{aligned}$$

such that

$$d(gx_n, gx_{n+1}) = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \preceq_P h\Theta_1.$$

Similarly, one can show that there exists

$$\Theta_2 \in \{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), 0_E)\}$$

such that

$$d(gy_n, gy_{n+1}) = d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \preceq_P h\Theta_2.$$

Now, denote $\delta_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})$. Since the cases $\Theta_1 = 0_E$ and $\Theta_2 = 0_E$ are trivial, we have to consider the following four possibilities.

**Case 1.** $d(gx_n, gx_{n+1}) \preceq_P hd(gx_{n-1}, gx_n)$ and $d(gy_n, gy_{n+1}) \preceq_P hd(gy_{n-1}, gy_n)$. Adding up, we get that

$$\delta_n \preceq_P h\delta_{n-1} \preceq_P 2h\delta_{n-1}.$$

**Case 2.** $d(gx_n, gx_{n+1}) \preceq_P hd(gx_{n-1}, gx_n)$ and $d(gy_n, gy_{n+1}) \preceq_P hd(gx_{n-1}, gx_n)$. Then

$$\delta_n \preceq_P 2hd(gx_{n-1}, gx_n) \preceq_P 2hd(gx_{n-1}, gx_n) + 2hd(gy_{n-1}, gy_n) = 2h\delta_{n-1}.$$

**Case 3.** $d(gx_n, gx_{n+1}) \preceq_P hd(gy_{n-1}, gy_n)$ and $d(gy_n, gy_{n+1}) \preceq_P hd(gx_{n-1}, gx_n)$. This case is treated analogously to Case 1.

**Case 4.** $d(gx_n, gx_{n+1}) \preceq_P hd(gy_{n-1}, gy_n)$ and $d(gy_n, gy_{n+1}) \preceq_P hd(gy_{n-1}, gy_n)$. This case is treated analogously to Case 2.

Thus, in all case, we get $\delta_n \preceq_P 2h\delta_{n-1}$ for all $n \in \mathbb{N}$, where $0 \leq 2h < 1$. Therefore,

$$\delta_n \preceq_P 2h\delta_{n-1} \preceq_P (2h)^2\delta_{n-2} \preceq_P \cdots \preceq_P (2h)^n\delta_0,$$

and by the same argument as in Theorem it is proved that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. By the completeness of $g(X)$, there exists $gx, gy \in g(X)$ such that $gx_n \longrightarrow gx$ and $gy_n \longrightarrow gy$.

From (v), we get $gx_n \asymp gx$ and $gy \asymp gy_n$ for all $n \geq 0$. Now we prove that $F(x,y) = gx$ and $F(y,x) = gy$.

If $gx_n = gx$ and $gy_n = gy$ for some $n \geq 0$. From (iv), we have

$$\begin{aligned} d(F(x,y), gx) &\leq_P d(F(x,y), gx_{n+1}) + d(gx_{n+1}, gx) \\ &= d(F(x,y), F(x_n, y_n)) + d(gx_{n+1}, gx) \\ &\leq_P h\Theta_{x,y,x_n,y_n} + d(gx_{n+1}, gx), \end{aligned}$$

where $\Theta_{x,y,x_n,y_n} \in \{d(gx, gx_n), d(gy, gy_n), d(F(x,y), gx_n)\}$. Let $c \in \operatorname{int}(P)$ be fixed. If $\Theta_{x,y,x_n,y_n} = d(gx, gx_n) = 0_E$ or $\Theta_{x,y,x_n,y_n} = d(gy, gy_n) = 0_E$, then for $n$ sufficiently large we have that

$$d(F(x,y), gx) \ll c.$$

By property $(p_2)$, it follows that $F(x,y) = gx$. If $\Theta_{x,y,x_n,y_n} = d(F(x,y), gx_n)$, then we get that

$$\begin{aligned} d(F(x,y), gx) &\leq_P hd(F(x,y), gx_n) + d(gx_{n+1}, gx) \\ &\leq_P hd(F(x,y), gx) + hd(gx, gx_n) + d(gx_{n+1}, gx) \\ &= hd(F(x,y), gx) + d(gx_{n+1}, gx). \end{aligned}$$

Now it follows that for $n$ sufficiently large,

$$\begin{aligned} d(F(x,y), gx) &\leq_P \frac{1}{1-h} d(gx_{n+1}, gx) \\ &\leq_P \frac{1}{1-h}(1-h)c \\ &= c. \end{aligned}$$

Therefore, again by property $(p_2)$, we get that $F(x,y) = gx$. Similarly, we can prove that $F(y,x) = gy$. Hence $(x,y)$ is a coupled point of coincidence of $F$ and $g$.

Then, we suppose that $(gx_n, gy_n) \neq (gx, gy)$ for all $n \geq 0$. For this, consider

$$\begin{aligned} d(F(x,y), gx) &\leq_P d(F(x,y), gx_{n+1}) + d(gx_{n+1}, gx) \\ &= d(F(x,y), F(x_n, y_n)) + d(gx_{n+1}, gx) \\ &\leq_P h\Theta_{x,y,x_n,y_n} + d(gx_{n+1}, gx), \end{aligned}$$

where $\Theta_{x,y,x_n,y_n} \in \{d(gx, gx_n), d(gy, gy_n), d(F(x, y), gx_n)\}$. Let $c \in \operatorname{int}(P)$ be fixed. If $\Theta_{x,y,x_n,y_n} = d(gx, gx_n)$ or $\Theta_{x,y,x_n,y_n} = d(gy, gy_n)$, then for $n$ sufficiently large we have that

$$d(F(x, y), gx) \ll h \cdot \frac{c}{2h} + \frac{c}{2} = c.$$

By property $(p_2)$, it follows that $F(x, y) = gx$. If $\Theta_{x,y,x_n,y_n} = d(F(x, y), gx_n)$, then we get that

$$\begin{aligned} d(F(x, y), gx) &\leq_P hd(F(x, y), gx_n) + d(gx_{n+1}, gx) \\ &\leq_P hd(F(x, y), gx) + hd(gx, gx_n) + d(gx_{n+1}, gx). \end{aligned}$$

Now it follows that for $n$ sufficiently large,

$$\begin{aligned} d(F(x, y), gx) &\leq_P \frac{h}{1-h} d(gx, gx_n) + \frac{1}{1-h} d(gx_{n+1}, gx) \\ &\ll \frac{h}{1-h} \cdot \frac{1-h}{h} \cdot \frac{c}{2} + \frac{1}{1-h}(1-h)\frac{c}{2} = c. \end{aligned}$$

Thus, again by property $(p_2)$, we get that $F(x, y) = gx$.

Similarly, $F(y, x) = gy$ is obtained. Hence $(x, y)$ is a coupled point of coincidence of the mappings $F$ and $g$. □

**Remark 5.3.3.6.** It would be interesting to relate our Theorem 5.3.3.5 with Theorem 2.1 of Long *et al.* [23].

Putting $g = I_X$, where $I_X$ is the identity mapping from $X$ into $X$ in Theorem 5.3.3.5 we get the following corollary.

**Corollary 5.3.3.7.** *Let $(X, d, \preceq)$ be an ordered cone metric space over a solid cone $P$. Let $F : X \times X \longrightarrow X$ be mappings. Suppose that the following hold:*

*(i) $X$ is complete;*

*(ii) $F$ satisfy property;*

*(iii) there exist $x_0, y_0 \in X$ such that $x_0 \asymp F(x_0, y_0)$ and $y_0 \asymp F(y_0, x_0)$;*

*(iv) there is some $h \in [0, 1/2)$ such that for all $x, y, u, v \in X$ satisfying $x \asymp u$ and $y \asymp v$, there exists*

$$\Theta_{x,y,u,v} \in \{d(x, u), d(y, v), d(F(x, y), u)\}$$

*such that*

$$d(F(x,y),F(u,v)) \preceq_P h\Theta_{x,y,u,v}.$$

*(v) if $x_n \longrightarrow x$ when $n \longrightarrow \infty$ in $X$, then $x_n \asymp x$ for $n$ sufficiently large.*

*Then there exist $x, y \in X$ such that*

$$F(x,y) = x \quad \textit{and} \quad F(y,x) = y,$$

*that is, $F$ has a coupled fixed point $(x,y) \in X \times X$.*

For the given partial order $\preceq$ on the set $X$, we shall denote also by $\preceq$ the order on $X \times X$ given by

$$(x_1,y_1) \preceq (x_2,y_2) \iff x_1 \preceq x_2 \text{ and } y_1 \succeq y_2. \tag{5.3.58}$$

**Theorem 5.3.3.8.** *In addition to the hypotheses of Theorem 5.3.3.1, suppose that for every $(x,y),(x^*,y^*) \in X \times X$ there exists $(u,v) \in X \times X$ such that*

$$(F(u,v),F(v,u)) \asymp (F(x,y),F(y,x))$$

*and*

$$(F(u,v),F(v,u)) \asymp (F(x^*,y^*),F(y^*,x^*)).$$

*If $F$ and $g$ are $w^*$-compatible, then $F$ and $g$ have a unique common coupled fixed point, that is, there exists a unique $(\widehat{u},\widehat{v}) \in X \times X$ such that*

$$\widehat{u} = g\widehat{u} = F(\widehat{u},\widehat{v}) \quad \textit{and} \quad \widehat{v} = g\widehat{v} = F(\widehat{v},\widehat{u}).$$

*Proof.* From Theorem 5.3.3.1, the set of coupled coincidence points of $F$ and $g$ is nonempty. Suppose $(x,y)$ and $(x^*,y^*)$ are coupled coincidence points of $F$, that is, $gx = F(x,y)$, $gy = F(y,x)$, $gx^* = F(x^*,y^*)$ and $gy^* = F(y^*,x^*)$. We will prove that

$$gx = gx^* \quad \text{and} \quad gy = gy^*. \tag{5.3.59}$$

By assumption, there exists $(u,v) \in X \times X$ such that

$$(F(u,v),F(v,u)) \asymp (F(x,y),F(y,x))$$

and

$$(F(u,v),F(v,u)) \asymp (F(x^*,y^*),F(y^*,x^*))$$

Put $u_0 = u$, $v_0 = v$, and choose $u_1, v_1 \in X$ so that $gu_1 = F(u_0, v_0)$ and $gv_1 = F(v_0, u_0)$. Then, similarly as in the proof of Theorem, we can inductively define sequences $\{gu_n\}$, $\{gv_n\}$ with

$$gu_{n+1} = F(u_n, v_n) \quad \text{and} \quad gv_{n+1} = F(v_n, u_n)$$

for all $n$. Further, set $x_0 = x$, $y_0 = y$, $x_0^* = x^*$, $y_0^* = y^*$ and, in a similar way, define the sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gx_n^*\}$, $\{gy_n^*\}$. Then it is easy to show that

$$gx_n \to F(x, y), \quad gy_n \to F(y, x)$$

and

$$gx_n^* \to F(x^*, y^*), \quad gy_n^* \to F(y^*, x^*)$$

as $n \longrightarrow \infty$.

Since

$$(gx, gy) = (gx_1, gy_1) = (F(x, y), F(y, x)) \asymp (F(u, v), F(v, u)) = (gu_1, gv_1),$$

we have $gx \asymp gu_1$ and $gy \asymp gv_1$. It is easy to show that, similarly,

$$(gx, gy) \asymp (gu_n, gv_n),$$

for all $n \geq 1$, that is, $gx \asymp gu_n$ and $gy \asymp gv_n$ for all $n \geq 1$. Thus from (5.3.47), we have

$$\begin{aligned}
d(gu_{n+1}, gx) &= d(F(u_n, v_n), F(x, y)) \\
&\leq_P a_1 d(gu_n, gx) + a_2 d(F(u_n, v_n), gu_n) + a_3 d(gv_n, gy) \\
&\quad + a_4 d(F(x, y), gx) + a_5 d(F(u_n, v_n), gx) + a_6 d(F(x, y), gu_n) \\
&= a_1 d(gu_n, gx) + a_2 d(gu_{n+1}, gu_n) + a_3 d(gv_n, gy) \\
&\quad + a_4 d(gx, gx) + a_5 d(gu_{n+1}, gx) + a_6 d(gx, gu_n) \\
&\leq_P a_1 d(gu_n, gx) + a_2 [d(gu_{n+1}, gx) + d(gx, gu_n)] + a_3 d(gv_n, gy) \\
&\quad + a_5 d(gu_{n+1}, gx) + a_6 d(gx, gu_n),
\end{aligned}$$

that is,

$$(1 - a_2 - a_5) d(gu_{n+1}, gx) \leq_P (a_1 + a_2 + a_6) d(gu_n, gx) + a_3 d(gv_n, gy).$$

In the same way, starting from $d(gv_{n+1}, gy)$, we can show that

$$(1 - a_2 - a_5)d(gv_{n+1}, gy) \leq_P (a_1 + a_2 + a_6)d(gv_n, gy) + a_3 d(gu_n, gx).$$

Thus

$$(1-a_2-a_5)[d(gu_{n+1}, gx)+d(gv_{n+1}, gy)] \leq_P (a_1+a_2+a_3+a_6)[d(gu_n, gx)+d(gv_n, gy)] \tag{5.3.60}$$

In a similar way, starting from $d(gx, gu_{n+1})$, resp. $d(gy, gv_{n+1})$, and adding up the obtained inequalities, one gets that

$$(1-a_4-a_6)[d(gx, gu_{n+1})+d(gy, gv_{n+1})] \leq_P (a_1+a_3+a_4+a_5)[d(gx, gu_n)+d(gy, gv_n)]. \tag{5.3.61}$$

Finally, adding up (5.3.60) and (5.3.61), we obtain that

$$d(gu_{n+1}, gx) + d(gv_{n+1}, gy) \leq_P \lambda[d(gu_n, gx) + d(gv_n, gy)], \tag{5.3.62}$$

where $\lambda$ is determined as in (5.3.54), and hence $0 \leq \lambda < 1$.

By inequality (5.3.62) $n$ time, we have

$$\begin{aligned} d(gu_n, gx) + d(gv_n, gy) &\leq_P \lambda[d(gu_{n-1}, gx) + d(gv_{n-1}, gy)] \\ &\leq_P \lambda^2[d(gu_{n-2}, gx) + d(gv_{n-2}, gy)] \\ &\;\;\vdots \\ &\leq_P \lambda^n[d(gu_0, gx) + d(gv_0, gy)] \end{aligned}$$

It follows from $\lambda^n[d(gu_0, gx) + d(gv_0, gy)] \to 0_E$ as $n \to \infty$ that

$$d(gu_n, gx) + d(gv_n, gy) \ll c$$

for all $c \in \operatorname{int}(P)$ and large $n$. Since

$$0_E \leq_P d(gu_n, gx) \leq_P d(gu_n, gx) + d(gv_n, gy),$$

it follows by $(p_3)$ that $d(gu_n, gx) \ll c$ for large $n$ and so $gu_n \longrightarrow gx$ when $n \longrightarrow \infty$. Similarly, $gv_n \longrightarrow gy$ when $n \longrightarrow \infty$. By the same procedure one can show that $gu_n \longrightarrow gx^*$ and $gv_n \longrightarrow gy^*$ as $n \longrightarrow \infty$. By the uniqueness of the limit, we get $gx = gx^*$ and $gy = gy^*$, i.e., (5.3.59) is proved. Therefore, $(gx, gy)$ is the unique coupled point of coincidence of $F$ and $g$.

Note that if $(gx, gy)$ is a coupled point of coincidence of $F$ and $g$, then $(gy, gx)$ is also a coupled points of coincidence of $F$ and $g$. Then $gx = gy$ and therefore $(gx, gx)$ is the unique coupled point of coincidence of $F$ and $g$.

Next, we show that $F$ and $g$ have a common coupled fixed point. Let $\widehat{u} := gx$. Then we have $\widehat{u} = gx = F(x, x)$. Sine $F$ and $g$ are $w^*$-compatible, we have

$$g\widehat{u} = ggx = gF(x, x) = F(gx, gx) = F(\widehat{u}, \widehat{u}).$$

Thus $(g\widehat{u}, g\widehat{u})$ is a coupled point of coincidence of $F$ and $g$. By the uniqueness of a coupled point of coincidence of $F$ and $g$, we get $g\widehat{u} = gx$. Therefore $\widehat{u} = g\widehat{u} = F(\widehat{u}, \widehat{u})$ that is $(\widehat{u}, \widehat{u})$ is a common coupled fixed point of $F$ and $g$.

Finally, we show that the uniqueness of a common coupled fixed point of $F$ and $g$. Let $(\widetilde{u}, \widetilde{u}) \in X \times X$ is a another common coupled fixed point of $F$ and $g$. So

$$\widetilde{u} = g\widetilde{u} = F(\widetilde{u}, \widetilde{u}).$$

Then $(g\widehat{u}, g\widehat{u})$ and $(g\widetilde{u}, g\widetilde{u})$ are two common coupled points of coincidence of $F$ and $g$ and, as was previously proved, it must be $g\widehat{u} = g\widetilde{u}$, and so $\widehat{u} = g\widehat{u} = g\widetilde{u} = \widetilde{u}$. This complete the prove. $\square$

**Example 18.** Let $X = \mathbb{R}$ be ordered by the following relation

$$x \preceq y \iff x \geq y$$

Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|f\| = \|f\|_\infty + \|f'\|_\infty$ for all $f \in E$ and

$$P = \{f \in E : f(t) \geq 0 \text{ for } t \in [0, 1]\}.$$

It is well known that the cone $P$ is not normal. Let

$$d(x, y) = |x - y|\varphi$$

for all $x, y \in X$, for a fixed $\varphi \in P$ (e.g., $\varphi(t) = e^t$ for $t \in [0, 1]$). Then $(X, d)$ is a complete ordered cone metric space over a nonnormal solid cone.

Let $g : X \longrightarrow X$ and $F : X \times X \longrightarrow X$ be defined by

$$gx = \frac{x^2}{2} \qquad \text{and} \qquad F(x, y) = \frac{x^2 + y^2}{8}.$$

Consider $y_1 = 2$ and $y_2 = 1$, we have for $x = 3$, we get $y_1 = 2 \preceq 1 = y_2$ but

$$F(x, y_1) = \frac{13}{8} \preceq \frac{10}{8} = F(x, y_2).$$

So the mapping $F$ does not satisfy the mixed $g$-monotone property. Therefore, Theorem of Nashine et al.cannot be used to reach this conclusion.

Now, we show that Theorem can be used for this case.

Take $a_1 = a_3 = \frac{1}{4}$ and $a_2 = a_4 = a_5 = a_6 = 0$. We will check that condition (5.3.47) in Theorem is holds.

For $x, y, u, v \in X$ satisfying $gu \asymp gx$ and $gv \asymp gy$, we have

$$\begin{aligned} d(F(x,y), F(u,v)) &= \left| \frac{x^2+y^2}{8} - \frac{u^2+v^2}{8} \right| \varphi \\ &\preceq_P \frac{1}{4}\left| \frac{x^2}{2} - \frac{u^2}{2} \right| \varphi + \frac{1}{4}\left| \frac{y^2}{2} - \frac{v^2}{2} \right| \varphi \\ &= \frac{1}{4}d(gx, gu) + \frac{1}{4}d(gy, gv) \\ &= a_1 d(gx, gu) + a_3 d(gy, gv). \end{aligned}$$

Next, we show that $F$ and $g$ are $w^*$-compatible. We note that if $gx = F(x,x)$ then we get only one case is $x = 0$ and hence

$$gF(x,x) = gF(0,0) = g0 = 0 = F(0,0) = F(g0, g0) = F(gx, gx).$$

Therefore, $F$ and $g$ are $w^*$-compatible.

Moreover, others condition in Theorem 5.3.3.8 are also satisfied. Now, we can apply Theorems 5.3.3.8 to conclude that the existence of unique common coupled fixed point of $F$ and $g$ that is a point $(0,0)$.

**Theorem 5.3.3.9.** *In addition to the hypotheses of Theorem 5.3.3.8, suppose that for every $(x,y), (x^*, y^*) \in X \times X$ there exists $(u,v) \in X \times X$ such that*

$$(F(u,v), F(v,u)) \asymp (F(x,y), F(y,x))$$

*and*

$$(F(u,v), F(v,u)) \asymp (F(x^*,y^*), F(y^*,x^*)).$$

*If $F$ and $g$ are $w^*$-compatible, then $F$ and $g$ have a unique coupled common fixed point, that is, there exists a unique $(\widehat{u}, \widehat{v}) \in X \times X$ such that*

$$\widehat{u} = g\widehat{u} = F(\widehat{u}, \widehat{v}) \quad \textit{and} \quad \widehat{v} = g\widehat{v} = F(\widehat{v}, \widehat{u}).$$

### 5.3.4 On $P$-contractions in ordered metric space

**Theorem 5.3.4.1.** *Let $(X, \sqsubseteq, d)$ be a complete partially ordered metric space and $f : X \to X$ be a continuous and nondecreasing $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of $f$ in $X$.*

**Proof** For the existence of the fixed point, we choose $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$. If $fx_0 = x_0$, then the proof is finished. Suppose that $fx_0 \neq x_0$. We define a sequence $\{x_n\}_{n=1}^{+\infty}$ such that $x_n = f^n x_0$. Since $x_0 \sqsubseteq fx_0$ and $f$ is nondecreasing w.r.t. $\sqsubseteq$, we obtain

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots .$$

If there exists $n_0 \in \mathbb{N}$ such that $\varrho(x_{n_0}, x_{n_0+1}) = d(x_{n_0}, x_{n_0+1})$, then by the notion of $\mathcal{P}$-contractivity, the proof is finished. Therefore, we assume that $\varrho(x_n, x_{n+1}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Also, assume that $\varrho(x_n, x_{n+1}) \neq 0$ for all $n \in \mathbb{N}$, otherwise we can find $n_0 \in \mathbb{N}$ with $x_{n_0} = x_{n_0+1}$, that is $x_{n_0} = fx_{n_0}$ and the proof is finished. Hence, we consider only in the case of which $0 < \varrho(x_n, x_{n+1}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

Since $x_n \sqsubseteq x_{n+1}$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq d(x_{n-1}, x_n) - \varrho(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, we have $\{d(x_n, x_{n+1})\}_{n=1}^{+\infty}$ nonincreasing. Since $\{d(x_n, x_{n+1})\}_{n=1}^{+\infty}$ is bounded, there exists $l \geq 0$ such that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = l$. Thus, there exists $q \geq 0$ such that $\lim_{n\to+\infty} \varrho(x_n, x_{n+1}) = q$.

Assume that $l > 0$. Then, by the $\mathcal{P}$-contractivity of $f$, we have

$$l \leq l - q.$$

Hence $q = 0$, which implies that $l = 0$, a contradiction. Therefore, we have

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{5.3.63}$$

Now we show that $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence in $X$. Assume the contrary. Then, there exists $\epsilon_0 > 0$ for which we can define subsequences $\{x_{m_k}\}_{k=1}^{+\infty}$ and

$\{x_{n_k}\}_{k=1}^{+\infty}$ of $\{x_n\}_{n=1}^{+\infty}$ such that $n_k$ is minimal in the sense that $n_k > m_k > k$ and $d(x_{m_k}, x_{n_k}) \geq \epsilon_0$. Therefore, $d(x_{m_k}, x_{n_k-1}) < \epsilon_0$. Observe that

$$\begin{aligned} \epsilon_0 &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \epsilon_0 + d(x_{n_k-1}, x_{n_k}). \end{aligned}$$

Letting $k \to +\infty$, we obtain $\epsilon_0 \leq \lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) \leq \epsilon_0$ and so that

$$\lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) = \epsilon_0. \tag{5.3.64}$$

By the two following inequalities:

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})$$

and

$$d(x_{m_k-1}, x_{n_k-1}) \leq d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}),$$

we can apply (5.3.63) and (5.3.64) to obtain

$$\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon_0. \tag{5.3.65}$$

Furthermore, we deduce that the limit $\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1})$ also exists. Now, by the $\mathcal{P}$-contractivity, we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k-1}, x_{n_k-1}) - \varrho(x_{m_k-1}, x_{n_k-1}).$$

From (5.3.64) and (5.3.65), we may find that

$$0 \leq -\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1}),$$

which further implies that $\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1}) = 0$. Notice that $x_{m_k-1} \sqsubseteq x_{n_k-1}$ at each $k \in \mathbb{N}$. Consequently, we obtain that $\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = 0$, which is a contradiction. So, $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence. Since $X$ is complete, there exists $x^*$ such that $x_n = f^n x_0 \to x^*$ as $n \to +\infty$. Finally the continuity of $f$ and $ff^n x_0 = f^{n+1}x_0 \to x^*$ imply that $fx^* = x^*$. Therefore, $x^*$ is a fixed point of $f$. $\square$

Next, we drop the continuity of $f$ in the Theorem 5.3.4.1, and find out that we can still guarantee a fixed point if we strengthen the condition of a partially ordered set to a sequentially ordered set.

**Theorem 5.3.4.2.** *Let* $(X, \sqsubseteq, d)$ *be a complete sequentially ordered metric space and* $f : X \to X$ *be a nondecreasing* $\mathcal{P}$*-contraction of type (A) w.r.t.* $\sqsubseteq$*. If there exists* $x_0 \in X$ *with* $x_0 \sqsubseteq fx_0$*, then* $\{f^n x_0\}_{n=1}^{+\infty}$ *converges to a fixed point of* $f$ *in* $X$*.*

**Proof** If we take $x_n = f^n x_0$ in the proof of Theorem 5.3.4.1, then we conclude that $\{x_n\}_{n=1}^{+\infty}$ converges to a point $x^*$ in $X$.

Next, we prove that $x^*$ is a fixed point of $f$ in $X$. Indeed, suppose that $x^*$ is not a fixed point of $f$, i.e., $d(x^*, fx^*) \neq 0$. Since $x^*$ is comparable with $x_n$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x^*, fx^*) &\leq d(x^*, fx_n) + d(fx^*, fx_n) \\ &\leq d(x^*, fx_n) + d(x^*, x_n) - \varrho(x^*, x_n) \\ &\leq d(x^*, fx_n) + d(x^*, x_n) \\ &= d(x^*, x_{n+1}) + d(x^*, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. By the definition of a convergent sequence, we have for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x^*) < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$ with $n \geq N$. Therefore, we have

$$\begin{aligned} d(x^*, fx^*) &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

As easily seen, $d(x^*, fx^*)$ is less than any nonnegative real number, so $d(x^*, fx^*) = 0$, which is a contradiction. Hence, $x^*$ is a fixed point of $f$. □

**Corollary 5.3.4.3.** *Let* $(X, \sqsubseteq, d)$ *be a complete totally ordered metric space and* $f : X \to X$ *be a nondecreasing* $\mathcal{P}$*-contraction of type (A) w.r.t.* $\sqsubseteq$*. If there exists* $x_0 \in X$ *with* $x_0 \sqsubseteq fx_0$*, then* $\{f^n x_0\}_{n=1}^{+\infty}$ *converges to a unique fixed point of* $f$ *in* $X$*.*

**Proof** Take $x_n = f^n x_0$ as in the proof of Theorem 5.3.4.1. Since the total ordering implies the partial ordering, we conclude that $\{x_n\}_{n=1}^{+\infty}$ converges to a fixed point.

Next, we show that the fixed point of $f$ is unique. Assume that $u$ and $v$ are two distinct fixed points of $f$, i.e., $d(u, v) \neq 0$. Since $X$ is totally ordered, $u$ and $v$ are comparable. Thus, we have

$$\begin{aligned} d(u, v) &= d(fu, fv) \\ &\leq d(u, v) - \varrho(u, v), \end{aligned} \tag{5.3.66}$$

which is a contradiction. Therefore, $u = v$ and the fixed point of $f$ is unique. $\square$

We can still guarantee the uniqueness of a fixed point by weaken the total ordering condition as stated and proved in the next theorem.

**Theorem 5.3.4.4.** *Let $(X, \sqsubseteq, d)$ be a complete partially ordered metric space and $f : X \to X$ be a continuous and nondecreasing $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$. Suppose that for each $x, y \in X$, there exists $w \in X$ which is comparable to both $x$ and $y$. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of $f$ in $X$.*

**Proof** If we take $x_n = f^n x_0$ in the proof of Theorem 5.3.4.1, then we conclude that $x_n$ converges to a fixed point of $f$ in $X$.

Next, we show that the fixed point of $f$ is unique. Assume that $u$ and $v$ be two distinct fixed points of $f$, i.e., $d(u, v) \neq 0$. We will prove this part by showing that the sequence $\{w_n\}_{n=1}^{+\infty}$ given by $w_n = f^n w$ converges to both $u$ and $v$. Since $u, v \in X$, there exists $w \in X$ such that $w$ is comparable to both $u$ and $v$. Therefore, we have

$$\begin{aligned} d(u, f^n w) &\leq d(u, f^{n-1} w) - \varrho(u, f^{n-1} w) \\ &\leq d(u, f^{n-1} w). \end{aligned} \tag{5.3.67}$$

If we define a sequence $y_n = d(u, f^n w)$ and $z_n = \varrho(u, f^n w)$, we may obtain from (5.3.67) that $\{y_n\}_{n=1}^{+\infty}$ is nonincreasing and there exists $l, q \geq 0$ such that $\lim_{n\to+\infty} y_n = l$ and $\lim_{n\to+\infty} z_n = q$.

Assume that $l > 0$. Then by the $\mathcal{P}$-contractivity of $f$, we have

$$l \leq l - q,$$

which is a contradiction. Hence, $\lim_{n\to+\infty} y_n = 0$. In the same way, we can also show that $\lim_{n\to+\infty} d(v, f^n w) = 0$. That is, $\{w_n\}_{n=1}^{+\infty}$ converges to both $u$ and $v$. Since the limit of a convergent sequence in a metric space is unique, we conclude that $u = v$. Hence, this yields the uniqueness of the fixed point. $\square$

**Theorem 5.3.4.5.** *Let $(X, \sqsubseteq, d)$ be a complete sequentially ordered metric space and $f : X \to X$ be a nondecreasing $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$. Suppose that for each $x, y \in X$, there exists $w \in X$ which is comparable to both $x$ and $y$. If* *there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of $f$ in $X$.*

**Proof** If we take $x_n = f^n x_0$ in the proof of Theorem 5.3.4.1, then we conclude that $x_n$ converges to a fixed point of $f$ in $X$. The rest of the proof is similar to the proof of Theorem 5.3.4.4. $\square$

In this section, we drop the monotonicity conditions of $f$ and finds out that we still can apply our results to confirm the existence and uniqueness of fixed point of $f$.

**Theorem 5.3.4.6.** *Let $(X, \sqsubseteq, d)$ be a complete partially ordered metric space and $f : X \to X$ be a continuous $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. If there exists $x_0 \in X$ such that $x_0$ and $fx_0$ are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of $f$ in $X$.*

**Proof** For the existence of the fixed point, we choose $x_0 \in X$ such that $x_0$ and $fx_0$ are comparable. If $fx_0 = x_0$, then the proof is finished. Suppose that $fx_0 \neq x_0$. We define a sequence $\{x_n\}_{n=1}^{+\infty}$ such that $x_n = f^n x_0$. Since $x_0$ and $fx_0$ are comparable, we have $x_n$ and $x_{n+1}$ comparable for all $n \in \mathbb{N}$.

If there exists $n_0 \in \mathbb{N}$ such that $\varrho(x_{n_0}, x_{n_0+1}) = d(x_{n_0}, x_{n_0+1})$, then by the notion of $\mathcal{P}$-contractivity, the proof is finished. Therefore, we assume that $\varrho(x_n, x_{n+1}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Also, assume that $\varrho(x_n, x_{n+1}) \neq 0$ for all $n \in \mathbb{N}$, otherwise we can find $n_0 \in \mathbb{N}$ with $x_{n_0} = x_{n_0+1}$, that is $x_{n_0} = fx_{n_0}$ and the proof is finished. Hence, we consider only in the case of which $0 < \varrho(x_n, x_{n+1}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

Since $x_n$ and $x_{n+1}$ are comparable for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq d(x_{n-1}, x_n) - \varrho(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, we have $\{d(x_n, x_{n+1})\}_{n=1}^{+\infty}$ nonincreasing. Since $\{d(x_n, x_{n+1})\}_{n=1}^{+\infty}$ is bounded, there exists $l \geq 0$ such that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = l$. Thus, there exists $q \geq 0$ such that $\lim_{n\to+\infty} \varrho(x_n, x_{n+1}) = q$.

Assume that $l > 0$. Then, by the $\mathcal{P}$-contractivity of $f$, we have

$$l \leq l - q.$$

Hence $q = 0$, which implies that $l = 0$, a contradiction. Hence, $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$.

Now we show that $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence in $X$. Assume the contrary. Then, there exists $\epsilon_0 > 0$ for which we can define subsequences $\{x_{m_k}\}_{k=1}^{+\infty}$ and $\{x_{n_k}\}_{k=1}^{+\infty}$ of $\{x_n\}_{n=1}^{+\infty}$ such that $n_k$ is minimal in the sense that $n_k > m_k > k$ and $d(x_{m_k}, x_{n_k}) \geq \epsilon_0$. Therefore, $d(x_{m_k}, x_{n_k-1}) < \epsilon_0$. Observe that

$$\begin{aligned} \epsilon_0 &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \epsilon_0 + d(x_{n_k-1}, x_{n_k}). \end{aligned}$$

Letting $k \to +\infty$, we obtain $\epsilon_0 \leq \lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) \leq \epsilon_0$ and so that

$$\lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) = \epsilon_0. \tag{5.3.68}$$

By the two following inequalities:

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})$$

and

$$d(x_{m_k-1}, x_{n_k-1}) \leq d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}),$$

we can apply the fact that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$ and (5.3.68) to obtain

$$\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon_0. \tag{5.3.69}$$

Furthermore, we deduce that the limit $\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1})$ also exists. Now, by the $\mathcal{P}$-contractivity, we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k-1}, x_{n_k-1}) - \varrho(x_{m_k-1}, x_{n_k-1}).$$

From (5.3.68) and (5.3.69), we may find that

$$0 \leq -\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1}),$$

which further implies that $\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1}) = 0$. Notice that $x_{m_k-1} \sqsubseteq x_{n_k-1}$ at each $k \in \mathbb{N}$. Consequently, we obtain that $\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = 0$, which is a contradiction. So, $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence. Since $X$ is complete, there exists $x^*$ such that $x_n = f^n x_0 \to x^*$ as $n \to +\infty$. Finally the continuity of $f$ and $ff^n x_0 = f^{n+1}x_0 \to x^*$ imply that $fx^* = x^*$. Therefore, $x^*$ is a fixed point of $f$. $\square$

Further results can be proved using the same plots of the earlier theorems in this paper, so we are omitting them.

**Theorem 5.3.4.7.** *Let $(X, \sqsubseteq, d)$ be a complete sequentially ordered metric space and $f : X \to X$ be a $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. If there exists $x_0 \in X$ such that $x_0$ and $fx_0$ are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of $f$ in $X$.*

**Corollary 5.3.4.8.** *Let $(X, \sqsubseteq, d)$ be a complete totally ordered metric space with and $f : X \to X$ be a $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. If there exists $x_0 \in X$ such that $x_0$ and $fx_0$ are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of $f$ in $X$.*

**Theorem 5.3.4.9.** *Let $(X, \sqsubseteq, d)$ be a complete partially ordered metric space and $f : X \to X$ be a continuous $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. Suppose that each $x, y \in X$, there exists $w \in X$ which is comparable to both $x$ and $y$. If there exists $x_0 \in X$ such that $x_0$ and $fx_0$ are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of $f$ in $X$.*

**Theorem 5.3.4.10.** *Let $(X, \sqsubseteq, d)$ be a complete sequentially ordered metric space and $f : X \to X$ be a $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. Suppose that each $x, y \in X$, there exists $w \in X$ which is comparable to both $x$ and $y$. If there exists $x_0 \in X$ such that $x_0$ and $fx_0$ are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of $f$ in $X$.*

We give an example to ensure the applicability of our theorems.

**Example 19.** Let $X = [0,1] \times [0,1]$ and suppose that we write $x = (x_1, x_2)$ and $y = (y_1, y_2)$ for $x, y \in X$.

Define $d, \varrho : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 2\max\{x_1 + y_1, x_2 + y_2\} & \text{otherwise} \end{cases}$$

and

$$\varrho(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{x_1, x_2 + y_2\} & \text{otherwise} \end{cases}.$$

Let $\sqsubseteq$ be an ordering in $X$ such that for $x, y \in X$, $x \sqsubseteq y$ if and only if $x_1 = y_1$ and $x_2 \leq y_2$. Then, $(X, \sqsubseteq, d)$ is a partially ordered metric space with $\varrho$ as a $\mathcal{P}$-function of type (A) w.r.t. $\sqsubseteq$ in $X$.

Now, let $f$ be a self mapping on $X$ defined by $fx = f((x_1, x_2)) = (0, \frac{x_2^2}{2})$ for all $x \in X$. It is obvious that $f$ is continuous and nondecreasing w.r.t. $\sqsubseteq$.

Let $x, y \in X$ be comparable w.r.t $\sqsubseteq$. If $x = y$, then they clearly satisfy the inequality. On another hand, if $x \neq y$, we have

$$\begin{aligned}
d(fx, fy) &= d\big(f((x_1, x_2)), f((y_1, y_2))\big) \\
&= d\big((0, \frac{x_2^2}{2}), (0, \frac{y_2^2}{2})\big) \\
&= 2\max\left\{0, \frac{x_2^2}{2} + \frac{y_2^2}{2}\right\} \\
&= x_2^2 + y_2^2 \\
&\leq x_2 + y_2 \\
&\leq \max\{2x_1, x_2 + y_2\} \\
&= 2\max\{2x_1, x_2 + y_2\} - \max\{2x_1, x_2 + y_2\} \\
&\leq 2\max\{2x_1, x_2 + y_2\} - \max\{x_1, x_2 + y_2\} \\
&= d(x,y) - \varrho(x,y).
\end{aligned}$$

Therefore, the inequality is satisfied for every comparable $x, y \in X$. So, $f$ is a continuous and nondecreasing $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$. Let $x_0 = (0,0)$, so we have $x_0 \sqsubseteq fx_0$. Now, applying Theorem 5.3.4.1, we conclude that $f$ has a fixed point in $X$ which is the point $(0,0)$.

### 5.3.5 Existence and uniqueness of best proximity points for generalized almost contractions

In this section, we introduce the new class of the generalized Banach contraction for non-self mappings so called generalized almost $(\varphi,\theta)_\alpha$ contraction and we also sutdy the best proximity theorems for this classes. First, we recall the notion of $(\varphi,\theta)$ contraction defined by Samet [33] as follow:

**Definition 5.3.5.1.** ([31]) Let $A$ and $B$ be nonempty subsets of metric space $X$. A mapping $T : A \to B$ is said to be a *almost* $(\varphi,\theta)$ *contraction* if and only if there exist $\varphi \in \Psi$ and $\theta \in \Theta$ such that, for all $x, y \in A$,

$$\begin{aligned} d(Tx,Ty) \;\leq\; & \varphi\big(d(x,y)) + \theta(d(y,Tx) - d(A,B), d(x,Ty) - d(A,B), \\ & d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)\big), \end{aligned} \tag{5.3.70}$$

The existence

**Definition 5.3.5.2.** Let $A$ and $B$ be nonempty subsets of metric space $X$. A mapping $T : A \to B$ is said to be a *generalized almost* $(\varphi,\theta)_\alpha$ *contraction* if and only if

$$\begin{aligned} \alpha(x,y)d(Tx,Ty) \;\leq\; & \varphi\big(M(x,y)) + \theta(d(y,Tx) - d(A,B), d(x,Ty) - d(A,B), \\ & d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)\big), \end{aligned} \tag{5.3.71}$$

for all $x, y \in A$, where $\alpha : A \times A \to [0,\infty)$, $\varphi \in \Psi$ $\theta \in \Theta$ and

$$\begin{aligned} M(x,y) \;=\; & \max\{d(x,y), d(x,Tx) - d(A,B), d(y,Ty) - d(A,B), \\ & \tfrac{1}{2}[d(x,Ty) + d(y,Tx)] - d(A,B)\}. \end{aligned}$$

Clearly, if we take $\alpha(x,y) = 1$ for all $x, y \in A$ and $M(x,y) = d(x,y)$, the generalized almost $(\varphi,\theta)_\alpha$ contraction reduce to almost $(\varphi,\theta)$ contraction.

**Theorem 5.3.5.3.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A,B)$ has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:*

*(a) $T$ are $\alpha$-proximal admissible and generalized almost $(\varphi,\theta)_\alpha-$contraction;*

*(b) $T$ is continuous;*

*(c) there exist element $x_0$ and $x_1$ in $A_0$ such that $d(x_1,Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq 1$;*

*(d) $T(A_0) \subseteq B_0$.*

*Then there exists an element* $x \in A$ *such that*

$$d(x, Tx) = d(A, B).$$

*Moreover, for any fixed* $x_0 \in A_0$*, the sequence* $\{x_n\}$*, defined by*

$$d(x_{n+1}, Tx_n) = d(A, B),$$

*converges to the element* $x$*.*

*Proof.* By the hypothesis $(c)$, there exist $x_0$ and $x_1$ in $A_0$ such that

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1. \tag{5.3.72}$$

From the fact that $T(A_0) \subseteq B_0$, there exists an element $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B). \tag{5.3.73}$$

By (5.3.72), (5.3.73) and the $\alpha$-proximal admissible, we get

$$\alpha(x_1, x_2) \geq 1.$$

Since $T(A_0) \subseteq B_0$, we can find an element $x_3 \in A_0$ such that

$$d(x_3, Tx_2) = d(A, B). \tag{5.3.74}$$

Again, by (5.3.73), (5.3.74) and the $\alpha$-proximal admissible, we have

$$\alpha(x_2, x_3) \geq 1$$

By similar fashion, we can find $x_n$ in $A_0$. Having chosen $x_n$, one can determine an element $x_{n+1} \in A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A, B) \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geq 1. \tag{5.3.75}$$

In view the facts that, the pair $(A, B)$ has $P-$ property and generalized almost

$(\varphi,\theta)_\alpha-$contraction of $T$, we have

$$\begin{aligned} d(x_1,x_2) &= d(Tx_0,Tx_1) \\ &\leq \alpha(x_0,x_1)d(Tx_0,Tx_1) \\ &\leq \varphi(M(x_0,x_1)) \\ &\quad +\theta(d(x_1,Tx_0)-d(A,B),d(x_0,Tx_1)-d(A,B), \\ &\quad d(x_0,Tx_0)-d(A,B),d(x_1,Tx_1)-d(A,B)) \\ &= \varphi(M(x_0,x_1)) \\ &\quad +\theta(0,d(x_0,Tx_1)-d(A,B),d(x_0,Tx_0)-d(A,B),d(x_1,Tx_1)-d(A,B)) \\ &= \varphi(M(x_0,x_1)). \end{aligned} \tag{5.3.76}$$

Since

$$\begin{aligned} M(x_0,x_1) &= \max\{d(x_0,x_1),d(x_0,Tx_0)-d(A,B),d(x_1,Tx_1)-d(A,B), \\ &\quad \tfrac{1}{2}[d(x_0,Tx_1)+d(x_1,Tx_0)]-d(A,B)\} \\ &\leq \max\{d(x_0,x_1),d(x_0,x_1)+d(x_1,Tx_0)-d(A,B),d(x_1,x_2) \\ &\quad +d(x_2,Tx_1)-d(A,B),\ \tfrac{1}{2}[d(x_0,x_1)+d(x_1,x_2)+d(x_2,Tx_1) \\ &\quad +d(A,B)]-d(A,B)\} \\ &= \max\{d(x_0,x_1),d(x_1,x_2),\tfrac{1}{2}[d(x_0,x_1)+d(x_1,x_2) \\ &\quad +d(A,B)+d(A,B)]-d(A,B)\} \\ &= \max\{d(x_0,x_1),d(x_1,x_2),\tfrac{1}{2}[d(x_0,x_1)+d(x_1,x_2)]\} \\ &= \max\{d(x_0,x_1),d(x_1,x_2)\}. \end{aligned} \tag{5.3.77}$$

By (5.3.76) and (5.3.77), we get

$$d(x_1,x_2) \leq \varphi(\max\{d(x_0,x_1),d(x_1,x_2)\}). \tag{5.3.78}$$

If there exist $n_0 \in \mathbb{N}\cup\{0\}$ such that $x_{n_0+1}=x_{n_0}$, by (5.3.75) we obtain the best proximity point. Suppose that $x_{n+1}\neq x_n$ for all $n\in\mathbb{N}\cup\{0\}$, then $d(x_n,x_{n+1})>0$ for all $n\in\mathbb{N}\cup\{0\}$. If $\max\{d(x_0,x_1),d(x_1,x_2)\}=d(x_1,x_2)$, by the property $\varphi(t)<t$ for all $t>0$ , we get

$$d(x_1,x_2) \leq \varphi(\max\{d(x_0,x_1),d(x_1,x_2)\}) < d(x_1,x_2)$$

which is a contradiction and hence $\max\{d(x_0,x_1),d(x_1,x_2)\}=d(x_0,x_1)$. That is

$$d(x_1,x_2) \leq \varphi(d(x_0,x_1)). \tag{5.3.79}$$

Again, since the pair $(A,B)$ has $P-$ property, $\alpha$-proximal admissible and generalized almost $(\varphi,\theta)_\alpha-$contraction of $T$, we have

$$\begin{aligned} d(x_2,x_3) &= d(Tx_1,Tx_2) \\ &\leq \alpha(x_1,x_2)d(Tx_1,Tx_2) \\ &\leq \varphi(M(x_1,x_2)) \\ &\quad +\theta(d(x_2,Tx_1)-d(A,B),d(x_1,Tx_2)-d(A,B), \\ &\quad d(x_1,Tx_1)-d(A,B),d(x_2,Tx_2)-d(A,B)) \qquad (5.3.80) \\ &= \varphi(M(x_1,x_2)) \\ &\quad +\theta(0,d(x_1,Tx_2)-d(A,B),d(x_1,Tx_1)-d(A,B), \\ &\quad d(x_2,Tx_2)-d(A,B)) \\ &= \varphi(M(x_1,x_2)) \end{aligned}$$

and since

$$\begin{aligned} M(x_1,x_2) &= \max\{d(x_1,x_2),d(x_1,Tx_1)-d(A,B),d(x_2,Tx_2)-d(A,B), \\ &\quad \tfrac{1}{2}[d(x_1,Tx_2)+d(x_2,Tx_1)]-d(A,B)\} \\ &\leq \max\{d(x_1,x_2),d(x_1,x_2)+d(x_2,Tx_1)-d(A,B), \\ &\quad d(x_2,x_3)+d(x_3,Tx_2)-d(A,B),\tfrac{1}{2}[d(x_1,x_2)+d(x_2,x_3)+d(x_3,Tx_2) \\ &\quad +d(A,B)]-d(A,B)\} \\ &= \max\{d(x_1,x_2),d(x_2,x_3),\tfrac{1}{2}[d(x_1,x_2)+d(x_2,x_3)+d(A,B)+d(A,B)] \\ &\quad -d(A,B)\} \\ &= \max\{d(x_1,x_2),d(x_2,x_3),\tfrac{1}{2}[d(x_1,x_2)+d(x_2,x_3)]\} \\ &= \max\{d(x_1,x_2),d(x_2,x_3)\}. \end{aligned} \qquad (5.3.81)$$

By (5.3.76) and (5.3.77), we get

$$d(x_2,x_3) \leq \varphi(\max\{d(x_1,x_2),d(x_2,x_3)\}). \qquad (5.3.82)$$

By similar argument as above, we can conclude that, $\max\{d(x_1,x_2),d(x_2,x_3)\} = d(x_1,x_2)$ and thus

$$d(x_2,x_3) \leq \varphi(d(x_1,x_2)). \qquad (5.3.83)$$

Using (5.3.79) and (5.3.83) and the nondecreasing of $\varphi$, we get

$$d(x_2,x_3) \leq \varphi^2(d(x_0,x_1)).$$

Continuing this process, by induction we have that

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)) \tag{5.3.84}$$

for all $n \in \mathbb{N} \cup \{0\}$. Fix $\varepsilon > 0$ and let $h = h(\varepsilon)$ be a positive integer such that

$$\sum_{n \geq h} \varphi^n(d(x_0, x_1)) < \varepsilon. \tag{5.3.85}$$

Let $m > n > h$, using the triangular inequality, (5.3.84) and (5.3.85), we obtain

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \varphi^k(d(x_0, x_1)) \leq \sum_{n \geq h} \varphi^n(d(x_0, x_1)) < \varepsilon.$$

This show that $\{x_n\}$ is a Cauchy sequence. Since A is a closed subset of complete metric spaces $X$, then there exists $x \in A$ such that

$$\lim_{n \to \infty} d(x_n, x) = 0. \tag{5.3.86}$$

By (5.3.75), (5.3.86) and the continuity of $T$, we get

$$d(x, Tx) = \lim_{n \to \infty} d(x_{n+1}, Tx_n) = d(A, B)$$

and the proof is completes.

Next, we remove condition $T$ is continuous in Theorem 5.3.5.3, by assuming the following condition which was defined by Jleli et al. [32] for proving the new best proximity point theorem.

$(H)$ : If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x$ for some $x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k$

**Theorem 5.3.5.4.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A, B)$ has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:*

*(a) $T$ are $\alpha$-proximal admissible and generalized almost $(\varphi, \theta)_\alpha-$contraction*

*(b) $A$ satisfies condition $(H)$*

*(c) there exist element $x_0$ and $x_1$ in $A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha((x_0, x_1)) \geq 1$;*

*(d)* $T(A_0) \subseteq B_0$.

*Then there exists an element* $x \in A$ *such that*

$$d(x, Tx) = d(A, B).$$

*Moreover, for any fixed* $x_0 \in A_0$*, the sequence* $\{x_n\}$*, defined by*

$$d(x_{n+1}, Tx_n) = d(A, B),$$

*converges to the element* $x$.

*Proof.* As in the proof of Theorem 5.3.5.3, we have

$$d(x_{n+1}, Tx_n) = d(A, B)$$

for all $n \geq 0$ . Moreover, $\{x_n\}$ is a Cauchy sequence and converges to some point $x \in A$. By the $P-$ property and (5.3.84), we have

$$d(Tx_{n-1}, Tx_n) = d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)) \tag{5.3.87}$$

for all $n \in \mathbb{N} \cup \{0\}$. That is $\lim_{n\to\infty} d(Tx_{n-1}, Tx_n) = 0$ and by the same argument as proof of Theorem 5.3.5.3, we obtain that $\{Tx_n\}$ is a Cauchy sequence. Since $B$ is a closed subset of the complete metric space $(X, d)$, there exists $x_\star \in B$ such that $Tx_n$ converges to $x_\star$. Therefore

$$d(x, x_\star) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(A, B) \tag{5.3.88}$$

On the other hand, from the condition $(H)$ of $T$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k$. the pair $(A, B)$ has $P-$ property and property of mapping $T$, we get

$$\begin{aligned} d(x_{n_k+1}, x) &= d(Tx_{n_k}, Tx) \\ &\leq \alpha(x_{n_k}, x) d(Tx_{n_k}, Tx) \\ &\leq \varphi(M(x_{n_k}, x)) \\ &\quad +\theta(d(x_{n_k}, Tx) - d(A, B), d(x, Tx_{n_k}) - d(A, B), \\ &\quad d(x, Tx) - d(A, B), d(x_{n_k}, Tx_{n_k}) - d(A, B)) \end{aligned} \tag{5.3.89}$$

Indeed,

$$\begin{aligned}
M(x_{n_k}, x) &= \max\{d(x_{n_k}, x), d(x_{n_k}, Tx_{n_k}) - d(A,B), d(x,Tx) - d(A,B), \\
&\quad \tfrac{1}{2}[d(x_{n_k}, Tx) + d(x, Tx_{n_k})] - d(A,B)\} \\
&\leq \max\{d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k}) - d(A,B), d(x,Tx) \\
&\quad -d(A,B), \tfrac{1}{2}[d(x_{n_k}, x) + d(x,Tx) + d(x, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k})] \\
&\quad -d(A,B)\} \\
&\leq \max\{d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}), d(x,Tx) - d(A,B), \\
&\quad \tfrac{1}{2}[d(x_{n_k}, x) + d(x,Tx) + d(x, x_{n_k+1}) + d(A,B)] - d(A,B)\} \\
&:= \mathcal{M}(x_{n_k}, x).
\end{aligned} \tag{5.3.90}$$

From the definition of $\mathcal{M}(x_{n_k}, x)$, we get

$$\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x) = d(x,Tx) - d(A,B). \tag{5.3.91}$$

Since

$$\begin{aligned}
d(x,Tx) &\leq d(x, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k}) + d(Tx_{n_k}, Tx) \\
&\leq d(x, x_{n_k+1}) + d(A,B) + d(Tx_{n_k}, Tx)
\end{aligned}$$

it follows that

$$\begin{aligned}
d(x,Tx) - d(x, x_{n_k+1}) - d(A,B) &\leq d(Tx_{n_k}, Tx) \\
&\leq \alpha(x_{n_k}, x) d(Tx_{n_k}, Tx) \\
&\leq \varphi(M(x_{n_k}, x)) \\
&\quad +\theta(d(x_{n_k}, Tx) - d(A,B), d(x, Tx_{n_k}) - d(A,B), \\
&\quad d(x,Tx) - d(A,B), d(x_{n_k}, Tx_{n_k}) - d(A,B)) \\
&\leq \varphi(\mathcal{M}(x_{n_k}, x)) \\
&\quad +\theta(d(x_{n_k}, Tx) - d(A,B), d(x, Tx_{n_k}) - d(A,B), \\
&\quad d(x,Tx) - d(A,B), d(x_{n_k}, Tx_{n_k}) - d(A,B))
\end{aligned} \tag{5.3.92}$$

Suppose that,

$$d(x,Tx) - d(A,B) > 0,$$

then for $k$ large enough, we have $\mathcal{M}(x_{n_k}, x) > 0$. Using the property $\varphi(t) < t$ for all

$t > 0$ , we get

$$\begin{aligned} d(x,Tx) - d(x,x_{n_k+1}) - d(A,B) &< \mathcal{M}(x_{n_k},x) \\ &\quad +\theta(d(x_{n_k},Tx) - d(A,B), d(x,Tx_{n_k}) - d(A,B), \\ &\quad d(x,Tx) - d(A,B), d(x_{n_k},Tx_{n_k}) - d(A,B)) \end{aligned} \tag{5.3.93}$$

Combining (5.3.88), (5.3.91) with (5.3.93) and the proprety of $\theta$, we obtain that

$$\begin{aligned} d(x,Tx) - d(A,B) &= \lim_{k\to\infty} d(x,Tx) - d(x,x_{n_k+1}) - d(A,B) \\ &< \lim_{k\to\infty} \mathcal{M}(x_{n_k},x) \\ &\quad + \lim_{k\to\infty} \theta(d(x_{n_k},Tx) - d(A,B), d(x,Tx_{n_k}) - d(A,B), \\ &\quad d(x,Tx) - d(A,B), d(x_{n_k},Tx_{n_k}) - d(A,B)) \\ &= \lim_{k\to\infty} \mathcal{M}(x_{n_k},x) \\ &= d(x,Tx) - d(A,B) \end{aligned}$$

which is a contradiction and thus $d(x,Tx)-d(A,B) = 0$. Hence, $d(x,Tx) = d(A,B)$ and the proof is complete. □

The uniqueness Next, we present an example where it can be appreciated that hypotheses in Theorems 5.3.5.3 and 5.3.5.4 do not guarantee uniqueness of the best proximity point.

**Example 20.** Let $X = R^2$ with the Euclidean metric. Consider $A := \{(2,0),(0,2)\}$ and $B := \{(-2,0),(0,-2)\}$. Obviously, (A,B) satisfies the $P$-property and $d(A,B) = 2\sqrt{2}$, furthermore $A_0 = A$ and $B_0 = B$. Define $T : A \to B$ by $T(x,y) = (\frac{-y}{2}, \frac{-x}{2})$ for all $x, y \in A$, clearly $T$ is continuous. Let $\alpha : A \times A \longrightarrow [0,\infty)$ define by

$$\alpha(x,y) = \begin{cases} 2 & ; x = y, \\ \frac{1}{2} & ; x \neq y \end{cases} \tag{5.3.94}$$

We can show that $T$ are $\alpha$-proximal admissible and generalized almost $(\varphi,\theta)_\alpha-$contraction with $\varphi(t) = t/2$ for all $t \geq 0$ and for all $\theta \in \Theta$. Furthermore,

$$d((2,0),T(2,0)) = d((2,0),(0,-2)) = d((0,2),(-2,0)) = d((0,2),T(0,2)) = d(A,B) = 2\sqrt{2}.$$

Therefore, $(2,0)$ and $(0,2)$ are a best proximity point of mapping $T$,

Now, we need a sufficient condition for give uniqueness of the best proximity point as follows :

**Definition 5.3.5.5.** ([32]) Let $T : A \to B$ be a non-self mapping and $\alpha : A \times A \to [0, \infty)$. We say that T is $(\alpha, d)-$regular if for all $(x, y) \in \alpha^{-1}([0, 1))$, there exists $z \in A_0$ such that

$$\alpha((x, z)) \geq 1 \quad \text{and} \quad \alpha(y, z) \geq 1.$$

**Theorem 5.3.5.6.** *Adding condition $(\alpha, d)-$regular of $T$ to the hypotheses of Theorem 5.3.5.3, then we obtain the uniqueness of the best proximity point of $T$.*

*Proof.* We shall only proof the part of uniqueness. Suppose that there exist $x$ and $x^*$ in $A$ which are distinct best proximity points, that is

$$d(x, Tx) = d(A, B) \quad \text{and} \quad d(x^*, Tx^*) = d(A, B).$$

Using the pair $(A, B)$ has $P-$ property, we have

$$d(x, x^*) = d(Tx, Tx^*). \tag{5.3.95}$$

**Case I** If $\alpha(x, x^*) \geq 1$. By (5.3.95) and generalized almost $(\varphi, \theta)_\alpha-$contraction of $T$, we have

$$\begin{aligned} d(x, x^*) &= d(Tx, Tx^*) \\ &\leq \alpha(x, x^*)d(Tx, Tx^*) \\ &\leq \varphi(M(x, x^*)) \\ &\quad +\theta(d(x^*, Tx) - d(A, B), d(x, Tx^*) - d(A, B), \\ &\quad d(x, Tx) - d(A, B), d(x^*, Tx^*) - d(A, B)) \\ &= \varphi(M(x, x^*)) \\ &\quad +\theta(d(x^*, Tx) - d(A, B), d(x, Tx^*) - d(A, B), 0, 0) \\ &= \varphi(M(x, x^*)) \end{aligned} \tag{5.3.96}$$

and since

$$\begin{aligned} M(x, x^*) &= \max\{d(x, x^*), d(x, Tx) - d(A, B), d(x^*, Tx^*) - d(A, B), \\ &\quad \tfrac{1}{2}[d(x, Tx^*) + d(x^*, Tx)] - d(A, B)\} \\ &= \max\{d(x, x^*), 0, 0, \tfrac{1}{2}[d(x, Tx^*) + d(x^*, Tx)] - d(A, B)\} \\ &\leq \max\{d(x, x^*), \tfrac{1}{2}[d(x, x^*) + d(x^*, Tx^*) + d(x^*, x) + d(x, Tx)] - d(A, B)\} \\ &= \max\{d(x, x^*), \tfrac{1}{2}[d(x, x^*) + d(x^*, x)]\} \\ &= d(x, x^*). \end{aligned} \tag{5.3.97}$$

Combining (5.3.96) with (5.3.97) and using the property $\varphi(t) < t$ for all $t > 0$ , we get

$$d(x, x^*) \leq \varphi(M(x, x^*)) = \varphi(d(x, x^*)) < d(x, x^*)$$

which is a contradiction and hence $x = x^*$.

**Case II** If $\alpha(x, x^*) < 1$. By the $(\alpha, d)-$regular of $T$, there exists $z \in A_0$ such that

$$\alpha((x, z)) \geq 1 \quad \text{and} \quad \alpha(x^*, z) \geq 1.$$

Since $T(A_0) \subseteq B_0$, there exists a point $v_0 \in A_0$ such that

$$d(v_0, Tz) = d(A, B).$$

From $\alpha((x, z)) \geq 1$, $d(x, Tx) = d(A, B)$ and $d(v_0, Tz) = d(A, B)$ and by the $\alpha$-proximal admissible, we have

$$\alpha(x, v_0) \geq 1.$$

Since $T(A_0) \subseteq B_0$, there exists a point $v_1 \in A_0$ such that

$$d(v_1, Tv_0) = d(A, B).$$

By similar argument as above, we can conclude that $\alpha(x, v_1) \geq 1$. One can proceed further in a similar fashion to find $v_n$ in $A_0$ with $v_{n+1} \in A_0$ such that

$$d(v_{n+1}, Tv_n) = d(A, B) \quad \text{and} \quad \alpha(x, v_n) \geq 1. \tag{5.3.98}$$

for all $n \in \mathbb{N}$. By (5.3.98), the pair $(A, B)$ has $P-$ property and property of mapping $T$, we get

$$d(x, v_{n+1}) = d(Tx, Tv_n). \tag{5.3.99}$$

Using the property of mapping $T$, we get

$$\begin{aligned}
d(x, v_{n+1}) &= d(Tx, Tv_n) \\
&\leq \alpha(x, v_n) d(Tx, Tv_n) \\
&\leq \varphi(M(x, v_n)) \\
&\quad +\theta(d(v_n, Tx) - d(A, B), d(x, Tv_n) - d(A, B), \\
&\quad d(x, Tx) - d(A, B), d(v_n, Tv_n) - d(A, B)) \\
&= \varphi(M(x, v_n)) \\
&\quad +\theta(d(v_n, Tx) - d(A, B), d(x, Tv_n) - d(A, B), \\
&\quad 0, d(v_n, Tv_n) - d(A, B)) \\
&= \varphi(M(x, v_n))
\end{aligned}$$

and since

$$\begin{aligned} M(x, v_n) &= \max\{d(x, v_n), d(x, Tx) - d(A, B), d(v_n, Tv_n) - d(A, B), \\ &\quad \tfrac{1}{2}[d(x, Tv_n) + d(v_n, Tx)] - d(A, B)\} \\ &= \max\{d(x, v_n), 0, 0, \tfrac{1}{2}[d(x, Tv_n) + d(v_n, Tx)] - d(A, B)\} \\ &\leq \max\{d(x, v_n), \tfrac{1}{2}[d(x, v_{n+1}) + d(v_{n+1}, Tv_n) + d(v_n, x) + d(x, Tx)] - d(A, B)\} \\ &= \max\{d(x, v_n), \tfrac{1}{2}[d(x, v_{n+1}) + d(v_n, x)]\} \\ &= \max\{d(x, v_n), d(x, v_{n+1})\}. \end{aligned}$$

Thus

$$d(x, v_{n+1}) \leq \varphi(M(x, v_n)) \leq \varphi(\max\{d(x, v_n), d(x, v_{n+1})\}).$$

If $v_N = x$, for some $N \in \mathbb{N}$. By (5.3.99), we get

$$d(x, v_{N+1}) = d(Tx, Tv_N) = 0$$

which implies that $v_{N+1} = x$. Moreover, we obtain $v_n = x$ for all $n \geq N$ and thus $v_n \to x$ as $n \to \infty$. Suppose that $v_n \neq x$ for all $n \in \mathbb{N}$, then $d(v_n, x) > 0$ for all $n$. If $\max\{d(x, v_n), d(x, v_{n+1})\} = d(x, v_{n+1})$, by the property $\varphi(t) < t$ for all $t > 0$ , we get

$$d(x, v_{n+1}) \leq \varphi(M(x, v_n)) = \varphi(d(x, v_{n+1})) < d(x, v_{n+1})$$

which is a contradiction and hence $\max\{d(x, v_n), d(x, v_{n+1})\} = d(x, v_n)$. That is

$$d(x, v_{n+1}) \leq \varphi(M(x, v_n)) = \varphi(d(x, v_n)) \tag{5.3.100}$$

for all $n \geq N$. By induction of (5.3.100), we have

$$d(x, v_{n+1}) \leq \varphi^n(d(x, v_1)).$$

Taking $n \to \infty$, we obtain that $v_n \to x$ as $n \to \infty$. So, in all cases, we have $v_n \to x$ as $n \to \infty$. Similarly, we can prove that $v_n \to x^*$ as $n \to \infty$. By the uniqueness of limit, we conclude that $x = x^*$ and this completes the proof. □

beginthm Adding condition $(\alpha, d)-$regular of $T$ to the hypotheses of Theorem 5.3.5.4, then we obtain the uniqueness of the best proximity point of $T$.

*Proof.* Combine the proofs of Theorem 5.3.5.6 and Theorem 5.3.5.4. □

Best proximity points Theorems If we take $\varphi(t) = kt$, where $0 \leq k < 1$ and $\theta(t_1, t_2, t_3, t_4) = L\min\{t_1, t_2, t_3, t_4\}$, then Theorem 5.3.72 and Theorem 5.3.75, we get the following.

**Theorem 5.3.5.7.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A, B)$ has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:*

*(a) $T$ is $\alpha$-proximal admissible and*

$$\begin{aligned}\alpha(x,y)d(Tx,Ty) \;\; &\leq \;\; kM(x,y) + L\min\{d(x,Ty) - d(A,B), d(y,Tx) - d(A,B)\\ &\qquad d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)\}\end{aligned}$$

*for all $x, y \in A$.*

*(b) $T$ is continuous (or $A$ satisfies condition $(H)$ );*

*(c) there exist element $x_0$ and $x_1$ in $A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha((x_0, x_1)) \geq 1$;*

*(d) $T(A_0) \subseteq B_0$.*

*Then there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Tx_n) = d(A, B),$$

*converges to the element $x$.*

If we add the condition that $T$ is $(\alpha, d)-$regular in Theorem 5.3.5.7, therefore we can obtain the uniqueness of the best proximity point.

If we take $\alpha(x, y) = 1$, for all $x, y \in A$ in Theorem 5.3.5.3 and Theorem 5.3.5.4, we get the following Theorems.

**Theorem 5.3.5.8.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A, B)$has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:*

*(a)*

$$\begin{aligned}d(Tx,Ty) \;\; &\leq \;\; \varphi(M(x,y)) + \theta(d(x,Ty) - d(A,B), d(y,Tx) - d(A,B)\\ &\qquad d(x,Tx) - d(A,B), d(y,Ty) - d(A,B))\end{aligned}$$

*for all $x, y \in A$.*

*(b) $T$ is continuous (or $A$ satisfies condition $(H)$ );*

*(c) $T(A_0) \subseteq B_0$.*

*Then there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Tx_n) = d(A, B),$$

*converges to the element $x$.*

If $M(x, y) = d(x, y)$, then Theorem 5.3.5.8, include the following.

**Theorem 5.3.5.9.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A, B)$has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:*

*(a)*

$$\begin{aligned} d(Tx, Ty) \quad \leq \quad & \varphi(d(x, y)) + \theta(d(x, Ty) - d(A, B), d(y, Tx) - d(A, B) \\ & d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)) \end{aligned}$$

*for all $x, y \in A$.*

*(b) $T$ is continuous (or $A$ satisfies condition $(H)$ );*

*(c) $T(A_0) \subseteq B_0$.*

*Then there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Tx_n) = d(A, B),$$

*converges to the element $x$.*

If we take $\varphi(t) = kt$ and $\theta(t_1, t_2, t_3, t_4) = L \min\{t_1, t_2, t_3, t_4\}$, for all $x, y \in A$ in Theorem 5.3.5.9, we obtain the following theorem.

**Theorem 5.3.5.10.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A, B)$has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:*

*(a)*

$$\begin{aligned} d(Tx, Ty) \leq\ & kM(x, y) + L \min\{d(x, Ty) - d(A, B), d(y, Tx) - d(A, B) \\ & d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)\} \end{aligned}$$

*for all $x, y \in A$.*

*(b) $T$ is continuous (or $A$ satisfies condition $(H)$ );*

*(c) $T(A_0) \subseteq B_0$.*

*Then there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Tx_n) = d(A, B),$$

*converges to the element $x$.*

If $M(x, y) = d(x, y)$ and putting $L = 0$ in Theorem 5.3.5.10, we obtain the following.

**Theorem 5.3.5.11.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A, B)$has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:*

*(a)*

$$d(Tx, Ty) \leq kd(x, y)$$

*for all $x, y \in A$.*

*(b) $T$ is continuous (or $A$ satisfies condition $(H)$ );*

*(c) there exist element $x_0$ and $x_1$ in $A_0$ such that $d(x_1, Tx_0) = d(A, B)$;*

*(d) $T(A_0) \subseteq B_0$.*

*Then there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B).$$

*Moreover, for any fixed* $x_0 \in A_0$*, the sequence* $\{x_n\}$*, defined by*

$$d(x_{n+1}, Tx_n) = d(A, B),$$

*converges to the element* $x$*.*

If $M(x,y) = \frac{k}{2}[d(x,Ty) + d(y,Tx)] - d(A,B)$ and putting $L = 0$ in Theorem 5.3.5.10, we obtain the following theorem:

**Theorem 5.3.5.12.** *Let* $A$ *and* $B$ *be nonempty closed subsets of a complete metric space* $X$ *such that* $A_0$ *is nonempty and the pair* $(A, B)$*has the* $P-$ *property. Let* $T : A \to B$ *satisfy the following conditions:*

*(a)*

$$d(Tx, Ty) \leq \frac{k}{2}[d(x,Ty) + d(y,Tx)] - d(A,B)$$

*for all* $x, y \in A$*.*

*(b)* $T$ *is continuous(or* $A$ *satisfies condition* $(H)$ *);*

*(c) there exist element* $x_0$ *and* $x_1$ *in* $A_0$ *such that* $d(x_1, Tx_0) = d(A, B)$*;*

*(d)* $T(A_0) \subseteq B_0$*.*

*Then there exists an element* $x \in A$ *such that*

$$d(x, Tx) = d(A, B).$$

*Moreover, for any fixed* $x_0 \in A_0$*, the sequence* $\{x_n\}$*, defined by*

$$d(x_{n+1}, Tx_n) = d(A, B),$$

*converges to the element* $x$*.*

Fixed points Theorem It is easy to observe that for self-mappings, our results includes the following:

**Theorem 5.3.5.13.** *Let* $A$ *be nonempty closed subsets of a complete metric space* $X$ *and* $T : A \to A$ *such that*

$$d(Tx, Ty) \leq \varphi(M(x,y)) + \theta(\{d(x,Ty), d(y,Tx), d(x,Tx), d(y,Ty)\}),$$

*for all* $x, y \in A$*, where* $\varphi \in \Psi$ $\theta \in \Theta$*. Then* $T$ *has a unique fixed point* $x \in A$*. Moreover, for any fixed* $x_0 \in A$*, the sequence* $\{x_n\}$*, defined by* $x_{n+1} = Tx_n$ *converges to the element* $x$*.*

**Theorem 5.3.5.14.** *Let $A$ be nonempty closed subsets of a complete metric space $X$ and $T : A \to A$ such that*

$$d(Tx, Ty) \leq kM(x, y) + L \min\{d(x, Ty), d(y, Tx), d(x, Tx), d(y, Ty)\}.$$

*Then $T$ has a unique fixed point $x \in A$. Moreover, for any fixed $x_0 \in A$, the sequence $\{x_n\}$, defined by $x_{n+1} = Tx_n$ converges to the element $x$.*

**Theorem 5.3.5.15.** *Let $A$ be nonempty closed subsets of a complete metric space $X$ and $T : A \to A$ such that*

$$d(Tx, Ty) \leq kd(x, y) + L \min\{d(x, Ty), d(y, Tx), d(x, Tx), d(y, Ty)\}$$

*for all $x, y \in A$. Then $T$ has a unique fixed point $x \in A$. Moreover, for any fixed $x_0 \in A$, the sequence $\{x_n\}$, defined by $x_{n+1} = Tx_n$ converges to the element $x$.*

We recall some preliminaries from (see, [33] also) as follows:

Let $(X, d)$ be a metric space and $\mathcal{R}$ be a binary relation over $X$. Denote

$$\mathcal{S} = \mathcal{R} \cup \mathcal{R}^{-1}$$

this is the symmetric relation attached to R. Clearly,

$$x, y \in X, x\mathcal{S}y \Longleftrightarrow x\mathcal{R}y \text{ or } y\mathcal{R}x.$$

**Definition 5.3.5.16.** (*[32]*) A mapping $T : A \to B$ is said to be *proximal comparative* if and only if

$$\left.\begin{array}{l} x_1\mathcal{S}x_2 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array}\right\} \quad \Longrightarrow \quad u_1\mathcal{S}u_2.$$

**Corollary 5.3.5.17.** *Let $(X, d)$ be a complete metric space, $\mathcal{R}$ be a binary relation over $X$, and $A$ and $B$ be two non-empty, closed subsets of $X$ such that $A_0$ are non-empty and the pair $(A, B)$ has the $P-$ property. Let $T : A \longrightarrow B$ such that the following conditions holds:*

*(a) $T$ is a continuous proximal comparative mapping;*

*(b) there exist element $x_0$ and $x_1$ in $A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \mathcal{S} x_1$;*

*(c) there exist $\varphi \in \Psi$ and $\theta \in \Theta$ such that $x, y \in A, x\mathcal{S}y$ implies that*

$$\begin{aligned} d(Tx, Ty) &\leq \varphi\big(M(x,y)) + \theta(d(y,Tx) - d(A,B), d(x,Ty) \\ &\quad -d(A,B), d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)\big) \end{aligned}$$

*(d) $T(A_0) \subseteq B_0$.*

*Then there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B).$$

*Proof.* Define the mapping $\alpha : A \times A \longrightarrow [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & ; x\mathcal{S}y, \\ 0 & ; \text{ otherwise.} \end{cases} \tag{5.3.101}$$

Since $T$ is proximal comparative, we have

$$\left.\begin{array}{l} x\mathcal{S}y \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \Longrightarrow u\mathcal{S}v,$$

for all $u, v, x, y \in A$. Using the definition of $\alpha$, we get

$$\left.\begin{array}{l} \alpha(x,y) \geq 1, \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{array}\right\} \Longrightarrow \alpha(u,v) \geq 1,$$

for all $u, v, x, y \in A$ and hence $T$ is $\alpha$-proximal admissible. By the condition $(b)$ implies that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$. By the condition $(c)$, we get

$$\begin{aligned} \alpha(x,y)d(Tx,Ty) &\leq \varphi\big(M(x,y)) + \theta(d(y,Tx) - d(A,B), d(x,Ty) - d(A,B) \\ &\quad , d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)\big), \end{aligned}$$

that is, $T$ is, generalized almost $(\varphi, \theta)_\alpha-$contraction. Therefore, all hypothesisses of Theorem 5.3.72 are satisfied, and the desired result follows immediately. □

Next, below we give an example to illustrate the main result Theorem 5.3.72.

**Example 21.** Consider $X = R^4$ with the metric defined by

$$d((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |x_4 - y_4|$$

for all $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in R^4$. Let $A, B \subset X$ defined by

$$A := \left\{\left(0, 0, \frac{1}{n}, \frac{-1}{n}\right)\right\} \cup \{(0, 0, 0, 0)\},$$

$$B := \left\{\left(1, -1, \frac{1}{n}, \frac{-1}{n}\right)\right\} \cup \{(1, -1, 0, 0)\}.$$

Then $A$ and $B$ are nonempty closed subsets of $X$ and $d(A, B) = 2$. Moreover $A_0 = A$ and $B_0 = B$. Suppose

$$d((0, 0, x_1, x_2), (1, -1, y_1, y_2)) = d(A, B) = 2$$

and

$$d((0, 0, x_1', x_2'), (1, -1, y_1', y_2')) = d(A, B) = 2,$$

then we get $x_1 = y_1, x_2 = y_2$ and $x_1' = y_1', x_2' = y_2'$. Hence, the pair $(A, B)$ has the $P$-property. Let $T : A \to B$ be a mapping defined as

$$T\big(0, 0, x, y\big) = \big(0, 0, \frac{x}{2}, \frac{y}{2}\big)$$

for all $\big(0, 0, x, y\big) \in A$. We define the mapping $\alpha : A \times A \to [0, \infty)$ by

$$\alpha(x, y) = 1 \quad \text{for all}\, x, y \in A.$$

We can see that $T$ is generalized almost $(\varphi, \theta)_\alpha-$contraction with $\varphi \in \Psi$ is given by $\varphi(t) = t/2$ for all $t \geq 0$ and for all $\theta \in \Theta$. Furthermore, $(0, 0, 0, 0) \in A$ is a best proximity point of mapping $T$.

## 6. Conclusions and Discussions

We find necessary and sufficient conditions which are established for a non-self-contraction mapping to have the best proximity point, a common best proximity point and a couple best proximity points for pairs of contractive non-self-mappings and for pairs of contraction non-self-mappings, yielding common optimal approximate solutions of certain fixed point equations. We introduced a new type of a contractive condition defined on an ordered space, namely a P -contraction, which generalizes the weak contraction. We also proved some fixed-point theorems for such a condition in ordered metric spaces. We establishing the existence of common best proximity points; iterative algorithms are also furnished to determine such optimal approximate solutions. In addition to exploring the existence of the best proximity point for generalized contractions, an iterative algorithm is also presented to determine such an optimal approximate solution. We study the new class of an asymptotic proximal pointwise weaker Meir–Keeler-type w-contraction and prove the existence of solutions for the minimization problem in a uniformly convex Banach space. We generalized the notion of proximal contractions of the first and the second kinds and established the best proximity point theorems for these classes. We extend the notion of weakly C-contraction mappings to the case of non-self mappings and establish the best proximity point theorems for this class. We provide sufficient conditions, which warrant the existence and uniqueness of thebest proximity point for two new types of contractions in the setting of metric spaces. The presented results extend, generalize and improve some known results from bestproximity point theory and fixed-point theory. We also give some examples to illustrate and validate our definitions and results. We show that the mixed g-monotone property in common coupled fixed point theorems in ordered cone metric spaces can be replaced by another property. We also extend the notion of weakly isotone mappings in an orderedBanach space in the case of single valued and

multivalued mappings and obtain the coincidence and common fixed-point theorems in an ordered Banach space. Moreover, we also obtain the existence theorem for a common solution of two integral equations.

7. Outputs (Acknowledge the Thailand Research Fund: **MRG5580213**)

7.1 International Journal Publications (ISI 15 papers and Scopus 1 paper)

1) Wutiphol Sintunavarat and **Poom Kumam**, Coupled best proximity point theorem in metric spaces, Fixed Point Theory and Applications, 2012, 2012:93. **(2012 Impact Factor 1.78**

2) Chirasak Mongkolkeha and **Poom Kumam**, "Some common best proximity points for proximity commuting mappings", Optimization Letters (2013) 7:1825–1836. **(2012 Impact Factor 1.654)**

3) Winate Sanhan, Chirasak Mongkolkeha and **Poom Kumam**", Generalized proximal \psi-contraction mappings and Best proximity points," Abstract and Applied Analysis, Volume 2012, Article ID 896912, 19 pages. **(2012 Impact Factor 1.102)**

4) Parin Chaipunya, Wutiphol Sintunavarat and **Poom Kumam**, On P-contractions in ordered metric spaces, Fixed Point Theory and Applications, 2012, **2012**:219. **(2012 Impact Factor 1.87)**

5) Chirasak Mongkolkeha and **Poom Kumam**, Best proximity points for asymptotic proximal pointwise weaker Meir-Keeler-type $\psi$-contraction mappings, Journal of the Egyptian Mathematical Society (2013) 21, 13–16. (Scopus)

6) Chirasak Mongkolkeha, Yeol Je Cho and Poom Kumam, Best Proximity Points for Generalized Proxinal C-Contraction Mappings in Metric Spaces with Partial Orders, Journal of Inequalities and Applications, 2013, **2013**:94. **(2012 Impact Factor 0.82)**

7) Hemant Kumar Nashine, Calogero Vetro and **Poom Kumam**, Best proximity point theorems for rational proximal contractions, Fixed Point Theory and Applications, 2013, **2013**:95. **(2012 Impact Factor 1.87)**

8) Ravi P Agarwal, Wutiphol Sintunavarat and **Poom Kumam**, Coupled coincidence point and common coupled fixed point theorems with lacking the mixed monotone property, Fixed Point Theory and Applications 2013, 2013:22. **(2012 Impact Factor 1.87)**

9) Wutiphol Sintunavarat and **Poom Kumam**, Some fixed point results for weakly isotone mappings in ordered Banach spaces, Applied Mathematics and Computation 224 (2013) 826–834. **(2012 Impact Factor 1.349)**

10) Chirasak Mongkolkeha, Yeol Je Cho and **Poom Kumam**, Best proximity points for Geraghty's proximal contraction mappings, *Fixed Point Theory and Applications* 2013, **2013**:180 **(2012 Impact Factor 1.87)**

11) Wutiphol Sintunavarat and **Poom Kumam**, "The existence theorems of an optimal approximate solution for generalized proximal contraction mappings", Abstract and Applied Analysis, Volume 2013 (2013), Article ID 375604, 8 pages. **(2012 Impact Factor 1.102)**

12) Y. J. Cho, A. Gupta, E. Karapinar, **P. Kumam** and W. Sintunavarat, Tripled Best Proximity Point Theorem in Metric Space, Mathematical Inequalities & Applications, Volume 16, Number 4 (2013), 1197–1216. **(2012 Impact Factor 0.558)**

13) Chirasak Mongkolkeha, Chayut Kongban and **Poom Kumam**", The existence and uniqueness of best proximity point theorems for generalized almost contraction," Abstract and Applied Analysis, Volume 2014, Article ID 813614, 11 pages **(2012 Impact Factor 1.102)**

14) Poom Kumam, Hassen Aydi, Erdal Karapinar and Wutiphol Sintunavarat, Best proximity points and extension of Mizoguchi-Takahashi's fixed point theorems, Fixed Point Theory and Applications, 2013, **2013**:242 **(2012 Impact Factor 1.87)**

15) P. Kumam, P. Salimi and C, Vetro, Best proximity point results for modified $\alpha$-proximal C-contraction mappings. Fixed Point Theory and Applications 2014, 2014:99. **(2012 Impact Factor 1.87)**

7.2 Others e.g. national journal publication, proceeding, international conference, book chapter, patent

Invited Speaker in International Conferences

I. Invited Speaker topic: *Generalized Vector Equilibrium for Multifunctions in Topological Vector Spaces,* **Spring School Workshop: Analysis and approximation in**

**optimization under uncertainty,** Vietnam Institute for Advanced Study in Mathematics, 18-22 March, 2013, Hanoi, **Viatnam**.

II. Main Speaker topic: Common Fixed Point Theorems for Generalized Nonlinear Contraction Mappings with Applications, **International Conference Anatolian Communications in Nonlinear Analysis**, July 03-06, 2013, “Abant Izzet Baysal University, Bulo, **Turkey**.

III. Invited Speaker topic: *“Some Fixed Point Results for Generalized Nonlinear Contraction Mappings and its Applications”,* **The 7th Asian Conference on Fixed Point Theory and Optimizations** (ACFPTO 2013), 18-20 July 2013 @ Kasetsart University, KamphaengSaen Campus, **Thailand**.

Oral Presentation International Conferences

IV. **Poom Kumam*,** *The best proximity point theorems for generalized proximal contraction mappings,* **2013 ANZIAM Conference, 3rd-7th**, **February, 2013,** Newcastle, **Australia**.

V. **Poom Kumam*,** *Global Optimal Solutions of Best Proximity Points for Generalized Contraction Proximally Maps*, **South Pacific Optimization Meeting: SPOM 2013, 9 – 12 February 2013,** Newcastle, **Australia**.

VI. Chirasak Mongkolkeha and **Poom Kumam*,** *Global Optimal Solutions For Generalized Contractions With Commute Proximally Mappings*, IAENG International Conference on Operations Research (ICOR'13), 13-15 March, 2013, **Hong Kong**.

### 7.3 Workshop and Seminar (and spacial talk)

- Prof. Dr. Anthony To-Ming Lau, Spacial talk Workshop on “Fixed Point Theory and Amenable Semigroup”, 17-18 January 2013 @ Fixed Point Laboratory SCL-802, KMUTT.

- Prof. Brailey Sims and Prof. Phan Quoc Khanh, Special Talk on “Recent Developments in Optimization and Fixed Point Theory”, 16-17 July 2013 @ Fixed Point Lab, SCL-802, KMUTT.

- Workshop on “Nonlinear Analysis and Variational Problems”, 6 – 8 December 2556 @ KMUTT by Professor Yeol Je Cho from Korea.

- Workshop on “Advance in Metric Fixed Point Theory” 19-21 FEB 2014@KMUTT by Professor Andrzej Wisnicki from Poland.

# Some common best proximity points for proximity commuting mappings


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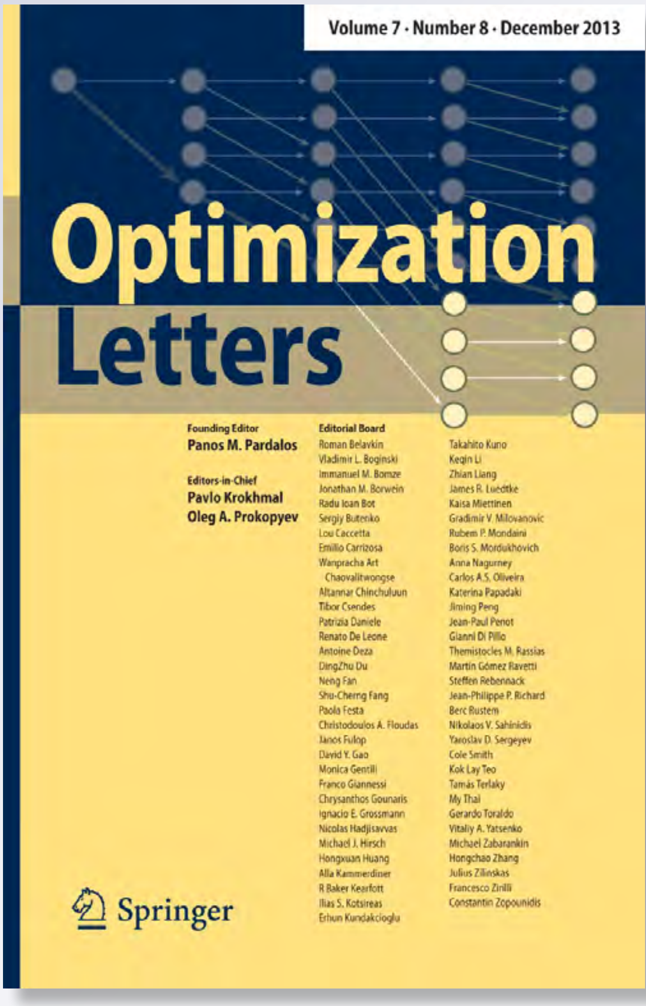
**Optimization Letters**

ISSN 1862-4472
Volume 7
Number 8

Optim Lett (2013) 7:1825-1836
DOI 10.1007/s11590-012-0525-1



Springer

Optim Lett (2013) 7:1825–1836
DOI 10.1007/s11590-012-0525-1

ORIGINAL PAPER

# Some common best proximity points for proximity commuting mappings

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Received: 20 January 2012 / Accepted: 28 June 2012 / Published online: 19 July 2012


**Abstract** In this paper, we prove new common best proximity point theorems for a proximity commuting mapping in a complete metric space. Our results generalized a recent result of Sadiq Basha [Common best proximity points: global minimization of multi-objective functions, J. Glob. Optim., (2011)] and some results in the literature.

**Keywords** Best proximity point · Common best proximity point · Common fixed point · Proximally commuting mappings

**Mathematics Subject Classification (2000)** 47H10 · 47H09

## 1 Introduction

Best proximity point theorems provide sufficient conditions that ensure the existence of approximate solutions which are optimal as well. In fact, if there is no solution to the fixed point equation $Tx = x$ for a non-self mapping $T : A \to B$, then it is desirable to determine an approximate solution $x$ such that the error $d(x, Tx)$ is minimum. A classical best approximation theorem was introduced by Fan [8], that is, if $A$ is a non-empty compact convex subset of a Hausdorff locally convex topological vector space $B$ and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterward, several authors, including

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Prolla [14], Reich [15], Sehgal and Singh [23,24], have derived extensions of Fan's theorem in many directions. Other works of of the existence of a best proximity point for contractions can be seen in [1,4,7,10]. In 2005, Anthony Eldred et al. [5] have obtained best proximity point theorems for relatively non-expansive mappings. Best proximity point theorems for several types of contractions have been established in [2,3,11–13,17–19,25–29].

Recently, Sadiq Basha in [20] gave necessary and sufficient to claimed that the existence of best proximity point for proximal contraction of first kind and the second kind which are non-self mapping analogues of contraction self-mappings and also established some best proximity and convergence theorems and very recently, he gave common best proximity point theorems for proximity commuting mapping of multi-objective function as follows.

**Theorem 1.1** [21, Theorem 3.1] *Let $A$ and $B$ be non-empty closed subsets of a complete metric space $X$ such that $A$ is approximatively compact with respect to $B$. Also, assume that $A_0$ and $B_0$ are non-empty. Let the non-self mapping $S : A \to B$, $T : A \to B$ satisfy the following conditions.*

(a) *There is a non-negative real number $\alpha < 1$ such that*

$$d(Sx_1, Sx_2) \leq \alpha d(Tx_1, Tx_2)$$

*for all $x_1, x_2 \in A$.*

(b) *$T$ is continuous.*
(c) *$S$ and $T$ commute proximally.*
(d) *$S$ and $T$ can be swapped proximally.*
(e) *$S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$.*

*Then, there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B) \quad \text{and } d(x, Sx) = d(A, B).$$

*Further, if $x^*$ is another common best proximity point of the mappings $S$ and $T$, then it is necessary that*

$$d(x, x^*) \leq 2d(A, B).$$

The aim of this paper is to generalizes the condition of Theorem 1.1 above for obtain the existence of a best proximity theorem and we also give an illustrative example for support our main result. The result of this paper are extension and generalization of the result of Sadiq Basha in [21] and some results in the literature.

## 2 Preliminaries

Let $A$ and $B$ be non-empty subsets of a metric space $(X, d)$, we recall the following notations and notions that will be used in what follows.

$$d(A,B) := \inf\{d(x,y) : x \in A \text{ and } y \in B\},$$
$$A_0 := \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\},$$
$$B_0 := \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\}.$$

If $A \cap B \neq \emptyset$, then $A_0$ and $B_0$ are non-empty. Further, it is interesting to notice that $A_0$ and $B_0$ are contained in the boundaries of $A$ and $B$ respectively, provided $A$ and $B$ are closed subsets of a normed linear space such that $d(A,B) > 0$ (see [18]).

**Definition 2.1** A mapping $T : X \to X$ is said to be *contraction* , if for each $x, y \in X$, there exists a constant $k \in [0,1)$ such that

$$d(Tx, Ty) \leq k(dx, y). \tag{1}$$

**Definition 2.2** A mapping $T : X \to X$ is said to be *weak contraction*, if for each $x, y \in X$,

$$d(Tx, Ty) \leq d(x,y) - \varphi(d(x,y)), \tag{2}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

In fact, if we take $\varphi(t) = (1-k)t$ for all $t \geq 0$, where $0 \leq k < 1$ then we have (2) become (1).

**Definition 2.3** A point $x \in A$ is said to be a best proximity point of the mapping $S : A \to B$ if it satisfies the following condition

$$d(x, Sx) = d(A,B).$$

It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

**Definition 2.4** Let $S : A \to B$ and $T : A \to B$. An element $x^* \in A$ is said to be a common best proximity point if it satisfies the following condition:

$$d(x^*, Sx^*) = d(x^*, Tx^*) = d(A,B).$$

Observed that a common best proximity point is an element at which the multi-objective functions $x \to d(x, Sx)$ and $x \to d(x, Tx)$ attain common global minimum, since $d(x, Sx) \geq d(A,B)$ and $d(x, Tx) \geq d(A,B)$ for all $x$.

**Definition 2.5** [21, Definition 2.3] A mapping $S : A \to B$ and $T : A \to B$ is said to be a *commute proximally* if they satisfy the following condition:

$$[d(u, Sx) = d(v, Tx) = d(A,B)] \Longrightarrow Sv = Tu$$

for all $u, v, x, \in A$.

It is easy to see that proximal commutativity of self-mappings become commutativity of the mappings.

**Definition 2.6** [21, Definition 2.4] A mapping $S : A \to B$ and $T : A \to B$ is said to be a *swapped proximally* if they satisfy the following condition

$$[d(y,u) = d(y,v) = d(A,B) \quad \text{and } Su = Tv] \Longrightarrow Sv = Tu$$

for all $u, v, \in A$ and $y \in B$.

**Definition 2.7** $A$ is said to be *approximatively compact with respect to* $B$ if every sequence $\{x_n\}$ in $A$ satisfies the condition that $d(y, x_n) \to d(y, A)$ for some $y \in B$ has a convergent subsequence.

We observe that every set is approximatively compact with respect to itself. Also, every compact set is approximatively compact with respect to any set. Moreover, $A_0$ and $B_0$ are non-empty set if $A$ is compact and $B$ is approximatively compact with respect to $A$.

## 3 Main result

**Theorem 3.1** *Let $A$ and $B$ be non-empty closed subsets of a complete metric space $X$ such that $A$ is approximatively compact with respect to $B$. Also, assume that $A_0$ and $B_0$ are non-empty. Let the non-self mapping $S : A \to B$, $T : A \to B$ satisfy the following conditions:*

(a) *For each $x$ and $y$ are elements in $A$,*

$$d(Sx, Sy) \leq d(Tx, Ty) - \varphi(d(Tx, Ty)),$$

*where, $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.*

(b) *$T$ is continuous.*

(c) *$S$ and $T$ commute proximally.*

(d) *$S$ and $T$ can be swapped proximally.*

(e) *$S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$.*

*Then, there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B) \quad \text{and } d(x, Sx) = d(A, B).$$

*Moreover, if $x^*$ is another common best proximity point of the mappings $S$ and $T$, then it is necessary that*

$$d(x, x^*) \leq 2d(A, B).$$

*Proof* Let $x_0$ a fixed element in $A_0$. In view of the fact that $S(A_0) \subseteq T(A_0)$ it is ascertained that there exists an element $x_1 \in A_0$ such that $Sx_0 = Tx_1$. Again, since $S(A_0) \subseteq T(A_0)$, there exists an element $x_2 \in A_0$ such that $Sx_1 = Tx_2$. By similar fashion we can find $x_n$ in $A_0$ such that

$$Sx_{n-1} = Tx_n, \tag{3}$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &\leq d(Tx_n, Tx_{n+1}) - \varphi(d(Tx_n, Tx_{n+1})) \\ &\leq d(Sx_{n-1}, Sx_n) - \varphi(d(Sx_{n-1}, Sx_n)) \\ &\leq d(Sx_{n-1}, Sx_n) \end{aligned} \tag{4}$$

this mean that the sequence $\{d(Sx_{n-1}, Sx_n)\}$ is non-increasing and bounded below. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(Sx_{n-1}, Sx_n) = r. \tag{5}$$

If $r > 0$, then

$$d(Sx_n, Sx_{n+1}) \leq d(Sx_{n-1}, Sx_n) - \varphi(d(Sx_{n-1}, Sx_n)). \tag{6}$$

Taking $n \to \infty$, in inequality (6), by the continuities of $\varphi$, we get $r \leq r - \varphi(r) < r$, which is a contradiction unless $r = 0$. Therefore

$$\lim_{n\to\infty} d(Sx_{n-1}, Sx_n) = 0. \tag{7}$$

Next, we will prove that $\{Sx_n\}$ is a Cauchy sequence. We distinguish two cases.

**Case I** Suppose there exits $n \in \mathbb{N}$ such that $Sx_n = Sx_{n+1}$, we observe that

$$\begin{aligned} d(Sx_{n+1}, Sx_{n+2}) &\leq d(Tx_{n+1}, Tx_{n+2}) - \varphi(d(Tx_{n+1}, Tx_{n+2})) \\ &\leq d(Sx_n, Sx_{n+1}) - \varphi(d(Sx_n, Sx_{n+1})) \\ &= 0, \end{aligned}$$

which implies that $Sx_{n+1} = Sx_{n+2}$. So, for every $m > n$, we conclude that $Sx_m = Sx_n$. Hence $\{Sx_n\}$ is a Cauchy sequence in $B$.

**Case II** The successive terms of $\{Sx_n\}$ are different. Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequence $\{Sx_{m_k}\}$, $\{Sx_{n_k}\}$ of $\{Sx_n\}$ with $n_k > m_k \geq k$ such that

$$d(Sx_{m_k}, Sx_{n_k}) \geq \varepsilon \quad \text{and} \quad d(Sx_{m_k}, Sx_{n_k-1}) < \varepsilon. \tag{8}$$

By using (8) and triangular inequality, we get

$$\begin{aligned}\varepsilon &\leq d(Sx_{m_k}, Sx_{n_k})\\ &\leq d(Sx_{m_k}, Sx_{n_k-1}) + d(Sx_{n_k-1}, Sx_{n_k})\\ &\leq \varepsilon + d(Sx_{n_k-1}, Sx_{n_k}).\end{aligned} \tag{9}$$

Using (9) and (7), we have

$$d(Sx_{m_k}, Sx_{n_k}) \to \varepsilon \quad \text{as} \quad k \to \infty. \tag{10}$$

Again, by the triangular inequality, we get

$$d(Sx_{m_k}, Sx_{n_k}) \leq d(Sx_{m_k}, Sx_{m_k+1}) + d(Sx_{m_k+1}, Sx_{n_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) \tag{11}$$

and

$$d(Sx_{m_k+1}, Sx_{n_k+1}) \leq d(Sx_{m_k+1}, Sx_{m_k}) + d(Sx_{m_k}, Sx_{n_k}) + d(Sx_{n_k}, Sx_{n_k+1}). \tag{12}$$

From, (7), (10), (11) and (12), we obtain that

$$d(Sx_{m_k+1}, Sx_{n_k+1}) \to \varepsilon \quad \text{as} \quad k \to \infty. \tag{13}$$

In view of the fact that

$$\begin{aligned}d(Sx_{m_k+1}, Sx_{n_k+1}) &\leq d(Tx_{m_k+1}, Tx_{n_k+1}) - \varphi(d(Tx_{m_k+1}, Tx_{n_k+1}))\\ &\leq d(Sx_{m_k}, Sx_{n_k}) - \varphi(d(Sx_{m_k}, Sx_{n_k})).\end{aligned} \tag{14}$$

Letting, $k \to \infty$, in inequality (14), we obtain

$$\varepsilon \leq \varepsilon - \varphi(\varepsilon),$$

which is a contradiction, by a property of $\varphi$. Then, we deduce that $\{Sx_n\}$ is a Cauchy sequence in $B$. Since $B$ is closed subset a complete metric space $X$, then there exists $y \in B$ such that $Sx_n \to y$ as $n \to \infty$. Consequently, we have that the sequence $\{Tx_n\}$ also converges to $y$. From $S(A_0) \subseteq B_0$, there exists an element $u_n \in A$ such that

$$d(Sx_n, u_n) = d(A, B), \tag{15}$$

for all $n \in \mathbb{N}$. So, it follows from (3) and (15) that

$$d(Tx_n, u_{n-1}) = d(Sx_{n-1}, u_{n-1}) = d(A, B), \tag{16}$$

for all $n \in \mathbb{N}$. By (15), (16) and the fact that the mappings $S$ and $T$ are commuting proximally, we obtain

$$Tu_n = Su_{n-1} \tag{17}$$

for all $n \in \mathbb{N}$. Moreover, we have

$$\begin{aligned} d(y, A) &\leq d(y, u_n) \\ &\leq d(y, Sx_n) + d(Sx_n, u_n) \\ &= d(y, Sx_n) + d(A, B) \\ &\leq d(y, Sx_n) + d(y, A). \end{aligned} \tag{18}$$

Therefore $d(y, u_n) \to d(y, A)$ as $n \to \infty$. Since $A$ is approximatively compact with respect to $B$, then there exists subsequence $\{u_{n_k}\}$ of sequence $\{u_n\}$ such that converging to some element $u \in A$. Further, since $d(y, u_{n_k - 1}) \to d(y, A)$ and $A$ is approximatively compact with respect to $B$, then there exists subsequence $\{u_{n_{k_j}-1}\}$ of sequence $\{u_{n_k-1}\}$ such that converging to some element $v \in A$. By the continuity of the mappings $S$ and $T$, we have

$$Tu = \lim_{j\to\infty} Tu_{n_{k_j}} = \lim_{k\to\infty} Su_{n_{k_j}-1} = Sv \tag{19}$$

and

$$\begin{aligned} d(y, u) &= \lim_{k\to\infty} d(Sx_{n_k}, u_{n_k}) = d(A, B), \\ d(y, v) &= \lim_{j\to\infty} d(Tx_{n_{k_j}}, u_{n_{k_j}-1}) = d(A, B). \end{aligned} \tag{20}$$

Because $S$ and $T$ can be swapped proximally, we get

$$Tv = Su. \tag{21}$$

Next, we will prove that $Su = Sv$, suppose the contrary, by (19), (20), (21) and property of $\varphi$, we have

$$\begin{aligned} d(Su, Sv) &\leq d(Tu, Tv) - \varphi(d(Tu, Tv)) \\ &\leq d(Sv, Su) - \varphi(d(Sv, Su)) \\ &< d(Sv, Su) \end{aligned}$$

which is a contradiction. Thus $Su = Sv$ and also $Tu = Su$. Since $S(A_0)$ is contained in $B_0$, there exists an element $x$ in A such that

$$d(x, Tu) = d(A, B) \quad \text{and} \quad d(x, Su) = d(A, B).$$

By the commuting proximally of $S$ and $T$, $Sx = Tx$. Consequently, we have

$$\begin{aligned} d(Su, Sx) &\leq d(Tu, Tx) - \varphi(d(Tu, Tx)) \\ &\leq d(Su, Sx) - \varphi(d(Su, Sx)). \end{aligned} \tag{22}$$

In inequality (22), if $Su \neq Sx$, then $d(Su, Sx) \leq d(Su, Sx) - \varphi(d(Su, Sx)) < d(Su, Sx)$, it is impossible, So, we have $Su = Sx$ and hence $Tu = Tx$. It follows that

$$d(x, Tx) = d(x, Tu) = d(A, B)$$

and

$$d(x, Sx) = d(x, Su) = d(A, B).$$

Therefore, $x$ is a common best proximity point of $S$ and $T$. Suppose that $x^*$ is another common best proximity point of the mappings $S$ and $T$, so that

$$d(x^*, Tx^*) = d(A, B)$$

and

$$d(x^*, Sx^*) = d(A, B).$$

By the commuting proximally of $S$ and $T$, we get $Sx = Tx$ and $Sx^* = Tx^*$. Consequently, we have

$$\begin{aligned} d(Sx^*, Sx) &\leq d(Tx^*, Tx) - \varphi(d(Tx^*, Tx)) \\ &\leq d(Sx^*, Sx) - \varphi(d(Sx^*, Sx)). \end{aligned} \tag{23}$$

In inequality (23) , if $Sx^* \neq Sx$, then

$$d(Sx^*, Sx) \leq d(Sx^*, Sx) - \varphi(d(Sx^*, Sx)) < d(Sx^*, Sx),$$

it is impossible. So, we have $Sx = Sx^*$. Moreover, it can be concluded that

$$\begin{aligned} d(x, x^*) &\leq d(x, Sx) + d(Sx, Sx^*) + d(Sx^*, x^*) \\ &= d(A, B) + d(A, B) \\ &= 2d(A, B) \end{aligned}$$

and the proof is completes. □

If we take $\varphi(t) = (1 - \alpha)t$, where $0 \leq \alpha < 1$ in Theorem 3.1, we obtain following corollary.

**Corollary 3.1** [21, Theorem 3.1] *Let $A$ and $B$ be non-empty closed subsets of a complete metric space $X$ such that $A$ is approximatively compact with respect to $B$. Also, assume that $A_0$ and $B_0$ are non-empty. Let the non-self mapping $S : A \to B$, $T : A \to B$ satisfy the following conditions.*

(a) *There is a non-negative real number $\alpha < 1$ such that*

$$d(Sx_1, Sx_2) \leq \alpha d(Tx_1, Tx_2)$$

*for all $x_1, x_2 \in A$.*

(b) *$T$ is continuous.*

(c) *$S$ and $T$ commute proximally.*
(d) *$S$ and $T$ can be swapped proximally.*
(e) *$S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$. Then, there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B) \quad and \quad d(x, Sx) = d(A, B).$$

*Further, if $x^*$ is another common best proximity point of the mappings $S$ and $T$, then it is necessary that*

$$d(x, x^*) \leq 2d(A, B).$$

## 4 An example

Now, below we give an example to illustrate Theorem 3.1.

*Example 4.1* Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Let

$$A = \{(x, 1) \ : \ 0 \leq x \leq 1\}$$

and

$$B = \{(x, -1) : 0 \leq x \leq 1\}.$$

Define two mappings $S : A \to B, \ T : A \to B$ as follows:

$$S(x, 1) = \left(x - \frac{x^2}{2}, -1\right)$$

and

$$T((x, 1)) = (x, -1).$$

It is easy to see that $d(A, B) = 2, \ A_0 = A$ and $B_0 = B$. Further, $S$ and $T$ are continuous and $A$ is approximatively compact with respect to $B$.
First, we will show that $S$ and $T$ are satisfy the condition $(a)$ of of Theorem 3.1 with $\varphi : [0, \infty) \to [0, \infty)$ defined by $\varphi(t) = \dfrac{t^2}{2}$, for all $t \in [0, \infty)$.
Let $(x, 1), (y, 1) \in A$, without loss generality we can take that $x > y$, then, we have

$$\begin{aligned} d(S(x, 1), S(y, 1)) &= \left|\left(x - \frac{x^2}{2}\right) - \left(y - \frac{y^2}{2}\right)\right| \\ &= (x - y) - \frac{1}{2}(x^2 - y^2) \\ &= (x - y) - \frac{1}{2}((x - y)(x + y)) \end{aligned}$$

$$
\begin{aligned}
&\leq (x-y) - \frac{1}{2}(x-y)^2 \\
&= d(T(x,1), T(y,1)) - \varphi(d(T(x,1), T(y,1))).
\end{aligned}
$$

Next, we will show that $S$ and $T$ commute proximally. Let $(u, 1), (v, 1), (x, 1) \in A$ are satisfying

$$
d((u,1), S(x,1)) = d(A,B) = 2 \quad \text{and} \quad d((v,1), T(x,1)) = d(A,B) = 2.
$$

It follows that

$$
u = x - \frac{x^2}{2} \quad \text{and} \quad v = x,
$$

and hence

$$
S(v,1) = \left(v - \frac{v^2}{2}, -1\right) = \left(x - \frac{x^2}{2}, -1\right) = (u, -1) = T(u,1).
$$

Finally, we will show that $S$ and $T$ swapped proximally. If it is true that

$$
d((u,1),(y,-1)) = d((v,1),(y,-1)) = d(A,B) = 2 \quad \text{and } S(u,1) = T(v,1),
$$

for some $(u, 1), (v, 1) \in A$ and $(y, -1) \in B$, then we get $u = v = 0$ and thus

$$
S(v,1) = T(u,1).
$$

Therefore, all hypothesis of Theorem 3.1 are satisfied.
Furthermore, $(0, 1) \in A$ is a common best proximity point of $S$ and $T$, because

$$
d((0,1), S(0,1)) = d((0,1),(0,-1)) = d((0,1), T((0,1)) = d(A,B).
$$

On the other hand, suppose that there exists $k \in [0, 1)$ such that

$$
d(S(x,1), S(y,1)) \leq kd(T(x,1), T(y,1)),
$$

that is

$$
\left| \left(x - \frac{x^2}{2}\right) - \left(y - \frac{y^2}{2}\right) \right| \leq k|x-y|.
$$

Putting $y = 0$ and $x \geq 0$, it follows that

$$
1 = \lim_{x \to 0^+} \left| \left(1 - \frac{x}{2}\right) \right| \leq k < 1,
$$

which is a contradiction. Therefore, the results of Sadiq Basha in [21] cannot be applied to this example and our main result Theorem 3.1.

## 5 Conclusions and remarks

Our main result, Theorem 3.1, extended and improved the condition (a) of Theorem 1.1 by using the concept of weak contraction and k-contraction maps which are more weaker than contractive maps with the function $\varphi(t) = (1-k)t$. In fact, the class of weakly contractive maps lies between the classes of contraction mappings and contractive mappings. ($d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, when $T$ is self-mapping on $X$). Mostly, the generalization and the extension of Banach's contraction principle focus on weakening the contractive condition of the operator or weakening the completeness of the metric space; see for examples [6,9,16,22,30] and references therein.

**Acknowledgments** The authors are grateful for the reviewers for the careful reading of the paper and for the suggestions which improved the quality of this work. Mr. Chirasak Mongkolkeha was supported by the Thailand Research Fund through the the Royal Golden Jubilee Ph.D Program (Grant no. PHD/0029/2553). Poom Kumam was supported by the Commission on Higher Education, the Thailand Research Fund and the King Mongkut's University of Technology Thonburi (Grant no. MRG5580213).

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Sintunavarat and Kumam *Fixed Point Theory and Applications* 2012, **2012**:93
http://www.fixedpointtheoryandapplications.com/content/2012/1/93

Fixed Point Theory and Applications
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# Coupled best proximity point theorem in metric Spaces

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**Abstract**

In this article the concept of coupled best proximity point and cyclic contraction pair are introduced and then we study the existence and convergence of these points in metric spaces. We also establish new results on the existence and convergence in a uniformly convex Banach spaces. Furthermore, we give new results of coupled fixed points in metric spaces and give some illustrative examples. An open problems are also given at the end for further investigation.

## 1 Introduction

The Banach contraction principle [1] states that if $(X, d)$ is a complete metric space and $T : X \to X$ is a contraction mapping (i.e., $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$, where $\alpha$ is a non-negative number such that $\alpha < 1$), then $T$ has a unique fixed point. This principle has been generalized in many ways over the years [2-15].

One of the most interesting is the study of the extension of Banach contraction principle to the case of non-self mappings. In fact, given nonempty closed subsets $A$ and $B$ of a complete metric space $(X, d)$, a contraction non-self-mapping $T : A \to B$ does not necessarily has a fixed point.

Eventually, it is quite natural to find an element $x$ such that $d(x, Tx)$ is minimum for a given problem which implies that $x$ and $Tx$ are in close proximity to each other.

A point $x$ in $A$ for which $d(x, Tx) = d(A, B)$ is call a best proximity point of $T$. Whenever a non-self-mapping $T$ has no fixed point, a best proximity point represent an optimal approximate solution to the equation $Tx = x$. Since a best proximity point reduces to a fixed point if the underlying mapping is assumed to be self-mappings, the best proximity point theorems are natural generalizations of the Banach contraction principle.

In 1969, Fan [16] introduced and established a classical best approximation theorem, that is, if $A$ is a nonempty compact convex subset of a Hausdorff locally convex topological vector space $B$ and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterward, many authors have derived extensions of Fan's Theorem and the best approximation theorem in many directions such as Prolla [17], Reich [18], Sehgal and Singh [19,20], Wlodarczyk and Plebaniak [21-24], Vetrivel et al. [25], Eldred and Veeramani [26], Mongkolkeha and Kumam [27] and Sadiq Basha and Veeramani [28-31].

On the other hand, Bhaskar and Lakshmikantham [32] introduced the notions of a mixed monotone mapping and proved some coupled fixed point theorems for mappings satisfying the mixed monotone property. They have observation that their theorem can be used to investigate a large class of problems and discussed the existence and uniqueness of a solution for a periodic boundary value problem. For several improvements and generalizations see in [33-36] and reference therein.

The purpose of this article is to first introduce the notion of coupled best proximity point and cyclic contraction pair. We also establish the existence and convergence theorem of coupled best proximity points in metric spaces. Moreover, we apply this results in uniformly convex Banach space. We also study some results on the existence and convergence of coupled fixed point in metric spaces and give illustrative examples of our theorems. An open problem are also given at the end for further investigations.

## 2 Preliminaries

In this section, we give some basic definitions and concepts related to the main results of this article. Throughout this article we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. For nonempty subsets $A$ and $B$ of a metric space $(X, d)$, we let

$$d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\}$$

stands for the distance between $A$ and $B$.

A Banach space $X$ is said to be

(1) *strictly convex* if the following implication holds for all $x, y \in X$:

$$\|x\| = \|y\| = 1 \text{ and } x \neq y \Rightarrow \left\|\frac{x+y}{2}\right\| < 1.$$

(2) *uniformly convex* if for each $\varepsilon$ with $0 < \varepsilon \leq 2$, there exists $\delta > 0$ such that the following implication holds for all $x, y \in X$:

$$||x|| \leq 1, ||y|| \leq 1 \text{ and } ||x-y|| \geq \varepsilon \Rightarrow \left\|\frac{x+y}{2}\right\| < 1-\delta.$$

It easily to see that a uniformly convex Banach space $X$ is strictly convex but the converse is not true.

**Definition 2.1**. [37] Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. The ordered pair $(A, B)$ satisfies the *property UC* if the following holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in $A$ and $\{y_n\}$ is a sequence in $B$ such that $d(x_n, y_n) \to d(A, B)$ and $d(z_n, y_n) \to d(A, B)$, then $d(x_n, z_n) \to 0$.

**Example 2.2**. [37]*The following are examples of a pair of nonempty subsets $(A, B)$ satisfying the property UC.*

*(1) Every pair of nonempty subsets $A$, $B$ of a metric space $(X, d)$ such that $d(A, B) = 0$.*

*(2) Every pair of nonempty subsets $A$, $B$ of a uniformly convex Banach space $X$ such that $A$ is convex.*

*(3) Every pair of nonempty subsets $A$, $B$ of a strictly convex Banach space which $A$ is convex and relatively compact and the closure of $B$ is weakly compact.*

**Definition 2.3**. Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $T : A \to B$ be a mapping. A point $x \in A$ is said to be a *best proximity point* of $T$ if it satisfies the condition that

$$d(x,\ Tx) = d(A,\ B).$$

It can be observed that a best proximity point reduces to a fixed point if the underlying mapping is a self-mapping.

**Definition 2.4**. [32] Let $A$ be a nonempty subset of a metric space $X$ and $F : A \text{ X } A \to A$. A point $(x, x') \in A \times A$ is called a *coupled fixed point* of $F$ if

$$x = F(x, x') \text{ and } x' = F(x', x).$$

## 3 Coupled best proximity point theorem

In this section, we study the existence and convergence of coupled best proximity points for cyclic contraction pairs. We begin by introducing the notion of property UC* and a coupled best proximity point.

**Definition 3.1**. Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. The ordered pair $(A, B)$ satisfies the *property UC** if $(A, B)$ has property UC and the following condition holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in $A$ and $\{y_n\}$ is a sequence in $B$ satisfying:

(1) $d(z_n, y_n) \to d(A, B)$.

(2) For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_m,\ y_n) \leq d(A,\ B) + \varepsilon$$

for all $m > n \geq N$,

then, for every $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that

$$d(x_m,\ z_n) \leq d(A,\ B) + \varepsilon$$

for all $m > n \geq N_1$.

**Example 3.2**. *The following are examples of a pair of nonempty subsets* $(A, B)$ *satisfying the property UC**.

*(1) Every pair of nonempty subsets* $A$, $B$ *of a metric space* $(X, d)$ *such that* $d(A, B) = 0$.

*(2) Every pair of nonempty closed subsets* $A$, $B$ *of a uniformly convex Banach space* $X$ *such that* $A$ *is convex* [[38], Lemma 3.7].

**Definition 3.3**. Let $A$ and $B$ be nonempty subsets of a metric space $X$ and $F : A \times A \to B$. A point $(x, x') \in A \times A$ is called a *coupled best proximity point* of $F$ if

$$d(x,\ F(x,\ x')) = d(x',\ F(x',\ x)) = d(A,\ B).$$

It is easy to see that if $A = B$ in Definition 3.3, then a coupled best proximity point reduces to a coupled fixed point.

Next, we introduce the notion of a cyclic contraction for a pair of two binary mappings.

**Definition 3.4**. Let $A$ and $B$ be nonempty subsets of a metric space $X$, $F : A \times A \to B$ and $G : B \times B \to A$. The ordered pair $(F, G)$ is said to be a *cyclic contraction* if there exists a non-negative number $\alpha < 1$ such that

$$d(F(x,\ x'),\ G(y,\ y')) \leq \frac{\alpha}{2}[d(x,\ y) + d(x',\ y')] + (1-\alpha)d(A,\ B)$$

for all $(x, x') \in A \times A$ and $(y, y') \in B \times B$.

Note that if $(F, G)$ is a cyclic contraction, then $(G, F)$ is also a cyclic contraction.

**Example 3.5**. *Let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$ and let $A = [2,4]$ and $B = [-4, -2]$. It easy to see that $d(A, B) = 4$. Define $F : A \times A \to B$ and $G : B \times B \to A$ by*

$$F(x,\ x') = \frac{-x - x' - 4}{4}$$

*and*

$$G(x,\ x') = \frac{-x - x' + 4}{4}.$$

*For arbitrary $(x, x') \in A \times A$ and $(y, y') \in B \times B$ and fixed $\alpha = \frac{1}{2}$, we get*

$$\begin{aligned} d(F(x,\ x'),\ G(y,\ y')) &= \left| \frac{-x - x' - 4}{4} - \frac{-y - y' + 4}{4} \right| \\ &\leq \frac{|x - y| + |x' - y'|}{4} + 2 \\ &= \frac{\alpha}{2}[d(x,\ y) + d(x',\ y')] + (1-\alpha)d(A,\ B). \end{aligned}$$

*This implies that $(F, G)$ is a cyclic contraction with $\alpha = \frac{1}{2}$.*

**Example 3.6**. *Let $X = \mathbb{R}^2$ with the metric $d((x, y), (x', y')) = \max\{|x - x'|, |y - y'|\}$ and let $A = \{(x, 0) : 0 \leq x \leq 1\}$ and $B = \{(x, 1) : 0 \leq x \leq 1\}$. It easy to prove that $d(A, B) = 1$. Define $F : A \times A \to B$ and $G : B \times B \to A$ by*

$$F((x,\ 0),\ (x',\ 0)) = \left(\frac{x + x'}{2},\ 1\right)$$

*and*

$$G((x,\ 1),\ (x',\ 1)) = (\frac{x + x'}{2},\ 0).$$

*We obtain that*

$$d(F((x,\ 0),\ (x',\ 0)),\ G((y,\ 1),\ (y',\ 1))) = d((\frac{x + x'}{2},\ 1),\ (\frac{y + y'}{2},\ 0)) = 1$$

*Also for all $\alpha > 0$, we get*

$$\begin{aligned} &\frac{\alpha}{2}[d((x,\ 0), (y,\ 1)) + d((x',\ 0),\ (y',\ 1))] + (1-\alpha)d(A,\ B) \\ &\qquad = \frac{\alpha}{2}[\max\{|x - y|,\ 1\} + \max\{|x' - y'|,\ 1\}] + (1-\alpha)d(A,\ B) \\ &\qquad = \frac{\alpha}{2} \times 2 + (1-\alpha) \\ &\qquad = 1. \end{aligned}$$

*This implies that* $(F, G)$ *is cyclic contraction.*

The following lemma plays an important role in our main results.

**Lemma 3.7**. *Let* $A$ *and* $B$ *be nonempty subsets of a metric space* $X$, $F : A \times A \to B$, $G : B \times B \to A$ *and* $(F, G)$ *be a cyclic contraction. If* $(x_0, x'_0) \in A \times A$ *and we define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), \; x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), \; x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all* $n \lfloor \mathbb{N} \cup \{0\}$, *then* $d(x_{2n}, x_{2n+1}) \to d(A, B)$, $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$, $d(x'_{2n}, x'_{2n+1}) \to d(A, B)$ *and* $d(x'_{2n+1}, x'_{2n+2}) \to d(A, B)$.

*Proof*. For each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
d(x_{2n}, x_{2n+1}) &= d(x_{2n}, F(x_{2n}, x'_{2n})) \\
&= d(G(x_{2n-1}, x'_{2n-1}), F(G(x_{2n-1}, x'_{2n-1}), G(x'_{2n-1}, x_{2n-1}))) \\
&\leq \frac{\alpha}{2}[d(x_{2n-1}, G(x_{2n-1}, x'_{2n-1})) + d(x'_{2n-1}, G(x'_{2n-1}, x_{2n-1}))] + (1-\alpha)d(A, B) \\
&= \frac{\alpha}{2}\big[d(F(x_{2n-2}, x'_{2n-2}), G(F(x_{2n-2}, x'_{2n-2}), F(x'_{2n-2}, x_{2n-2}))) \\
&\quad + d(F(x'_{2n-2}, x_{2n-2}), G(F(x'_{2n-2}, x_{2n-2}), F(x_{2n-2}, x'_{2n-2})))\big] + (1-\alpha)d(A, B) \\
&\leq \frac{\alpha}{2}\Big[\frac{\alpha}{2}[d(x_{2n-2}, F(x_{2n-2}, x'_{2n-2})) + d(x'_{2n-2}, F(x'_{2n-2}, x_{2n-2})) + (1-\alpha)d(A, B)] \\
&\quad + \frac{\alpha}{2}[d(x'_{2n-2}, F(x'_{2n-2}, x_{2n-2})) + d(x_{2n-2}, F(x_{2n-2}, x'_{2n-2})) + (1-\alpha)d(A, B)\Big] \\
&\quad + (1-\alpha)d(A, B) \\
&= \frac{\alpha^2}{2}[d(x_{2n-2}, F(x_{2n-2}, x'_{2n-2})) + d(x'_{2n-2}, F(x'_{2n-2}, x_{2n-2}))] + (1-\alpha^2)d(A, B).
\end{aligned}$$

By induction, we see that

$$d(x_{2n}, x_{2n+1}) \leq \frac{\alpha^{2n}}{2}[d(x_0, F(x_0, x'_0)) + d(x'_0, F(x'_0, x_0))] + (1-\alpha^{2n})d(A, B).$$

Taking $n \to \infty$, we obtain

$$d(x_{2n}, x_{2n+1}) \to d(A, B). \tag{3.1}$$

For each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(x_{2n+1}, G(x_{2n+1}, x'_{2n+1})) \\
&= d(F(x_{2n}, x'_{2n}), G(F(x_{2n}, x'_{2n}), F(x'_{2n}, x_{2n}))) \\
&\leq \frac{\alpha}{2}[d(x_{2n}, F(x_{2n}, x'_{2n})) + d(x'_{2n}, F(x'_{2n}, x_{2n}))] + (1-\alpha)d(A, B) \\
&= \frac{\alpha}{2}\big[d(G(x_{2n-1}, x'_{2n-1}), F(G(x_{2n-1}, x'_{2n-1}), G(x'_{2n-1}, x_{2n-1}))) \\
&\quad + d(G(x'_{2n-1}, x_{2n-1}), F(G(x'_{2n-1}, x_{2n-1}), G(x_{2n-1}, x'_{2n-1})))\big] + (1-\alpha)d(A, B) \\
&\leq \frac{\alpha}{2}\Big[\frac{\alpha}{2}[d(x_{2n-1}, G(x_{2n-1}, x'_{2n-1})) + d(x'_{2n-1}, G(x'_{2n-1}, x_{2n-1})) + (1-\alpha)d(A, B)] \\
&\quad + \frac{\alpha}{2}[d(x'_{2n-1}, G(x'_{2n-1}, x_{2n-1})) + d(x_{2n-1}, G(x_{2n-1}, x'_{2n-1})) + (1-\alpha)d(A, B)\Big] \\
&\quad + (1-\alpha)d(A, B) \\
&= \frac{\alpha^2}{2}[d(x_{2n-1}, G(x_{2n-1}, x'_{2n-2})) + d(x'_{2n-1}, G(x'_{2n-1}, x_{2n-1}))] + (1-\alpha^2)d(A, B).
\end{aligned}$$

By induction, we see that

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{\alpha^{2n}}{2}[d(x_1, G(x_1, x'_1)) + d(x'_1, G(x'_1, x_1))] + (1 - \alpha^{2n})d(A, B).$$

Setting $n \to \infty$, we obtain

$$d(x_{2n+1}, x_{2n+2}) \to d(A, B). \tag{3.2}$$

By similar argument, we also have $d(x'_{2n}, x'_{2n+1}) \to d(A, B)$ and $d(x'_{2n+1}, x'_{2n+2}) \to d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$. □

**Lemma 3.8**. *Let $A$ and $B$ be nonempty subsets of a metric space $X$ such that $(A, B)$ and $(B, A)$ have a property UC, $F : A \times A \to B$, $G : B \times B \to A$ and let the ordered pair $(F, G)$ is a cyclic contraction. If $(x_0, x'_0) \in A \times A$ and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all $n \in \mathbb{N} \cup \{0\}$, then for $\varepsilon > 0$, there exists a positive integer $N_0$ such that for all $m > n \geq N_0$,*

$$\frac{1}{2}[d(x'_{2m}, x'_{2n+1}) + d(x_{2m}, x_{2n+1})] < d(A, B) + \varepsilon. \tag{3.3}$$

*Proof*. By Lemma 3.7, we have $d(x_{2n}, x_{2n+1}) \to d(A, B)$ and $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$. Since $(A, B)$ has a property UC, we get $d(x_{2n}, x_{2n+2}) \to 0$. A similar argument shows that $d(x'_{2n}, x'_{2n+2}) \to 0$. As $(B, A)$ has a property UC, we also have $d(x_{2n+1}, x_{2n+3}) \to 0$ and $d(x'_{2n+1}, x'_{2n+3}) \to 0$. Suppose that (3.3) does not hold. Then there exists $\varepsilon' > 0$ such that for all $k \in \mathbb{N}$, there is $m_k > n_k \geq k$ satisfying

$$\frac{1}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1})] \geq d(A, B) + \varepsilon'.$$

and

$$\frac{1}{2}[d(x'_{2m_k-2}, x'_{2n_k+1}) + d(x_{2m_k-2}, x_{2n_k+1})] < d(A, B) + \varepsilon'.$$

Therefore, we get

$$\begin{aligned} d(A, B) + \varepsilon' &\leq \frac{1}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1})] \\ &\leq \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k-2}) + d(x'_{2m_k-2}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k-2}) + d(x_{2m_k-2}, x_{2n_k+1})] \\ &< \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k-2}) + d(x_{2m_k}, x_{2m_k-2})] + d(A, B) + \varepsilon'. \end{aligned}$$

Letting $k \to \infty$, we obtain to see that

$$\frac{1}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1})] \to d(A, B) + \varepsilon'. \tag{3.4}$$

By using the triangle inequality we get

$$
\begin{aligned}
&\tfrac{1}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k},\ x_{2n_k+1})] \\
&\quad \le \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2m_k+2}, x'_{2n_k+3}) + d(x'_{2n_k+3}, x'_{2n_k+1}) \\
&\quad\quad + d(x_{2m_k-2}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1})\big] \\
&\quad = \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(G(x'_{2m_k+1}, x_{2m_k+1}), F(x'_{2n_k+2}, x_{2n_k+2})) + d(x'_{2n_k+3}, x'_{2n_k+1}) \\
&\quad + d(x_{2m_k}, x_{2m_k+2}) + d(G(x_{2m_k+1}, x_{2m_k+1}), F(x_{2n_k+2}, x'_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\quad \le \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + \frac{\alpha}{2}[d(x'_{2m_k+1}, x'_{2n_k+2}) + d(x_{2m_k+1}, x_{2n_k+2})] \\
&\quad\quad + (1-\alpha)d(A, B) + d(x'_{2n_k+3}, x'_{2n_k+1}) \\
&\quad\quad + d(x_{2m_k-2}, x_{2m_k+2}) + \frac{\alpha}{2}[d(x_{2m_k+1}, x_{2n_k+2}) + d(x'_{2m_k+1}, x'_{2n_k+2})] \\
&\quad\quad + (1-\alpha)d(A, B) + d(x_{2n_k+3}, x_{2n_k+1})\big] \\
&\quad = \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2n_k+3}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\quad\quad + \frac{\alpha}{2}[d(x_{2m_k+1}, x_{2n_k+2}) + d(x'_{2m_k+1}, x'_{2n_k+2})] + (1-\alpha)d(A, B) \\
&\quad = \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2n_k+3}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\quad\quad + \frac{\alpha}{2}[d(F(x_{2m_k}, x'_{2m_k}), G(x_{2n_k+1}, x'_{2n_k+1})) + d(F(x'_{2m_k}, x_{2m_k}), G(x'_{2n_k+1}, x_{2n_k+1}))] \\
&\quad\quad + (1-\alpha)d(A, B) \\
&\quad \le \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2n_k+3}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\quad\quad + \frac{\alpha}{2}\left[\frac{\alpha}{2}\ [d(x_{2m_k}, x_{2n_k+1}) + d(x'_{2m_k+1}, x'_{2n_k+1}) + (1-\alpha)d(A, B)]\right. \\
&\quad + \left.\frac{\alpha}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k+1}, x_{2n_k+1}) + (1-\alpha)d(A, B)]\right] \\
&\quad\quad + (1-\alpha)d(A, B) \\
&\quad = \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2n_k+3}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\quad\quad + \frac{\alpha^2}{2}[d(x_{2m_k}, x_{2n_k+1}) + d(x'_{2m_k}, x'_{2n_k+1})] + (1-\alpha^2)d(A, B).
\end{aligned}
$$

Taking $k \to \infty$, we get

$$d(A, B) + \varepsilon' \le \alpha^2[d(A, B) + \varepsilon'] + (1-\alpha^2)d(A, B) = d(A, B) + \alpha^2\varepsilon'$$

which contradicts. Therefore, we can conclude that (3.3) holds. □

**Lemma 3.9**. *Let $A$ and $B$ be nonempty subsets of a metric space $X$, $(A, B)$ and $(B, A)$ satisfy the property UC*. Let $F : A \times A \to B$, $G : B \times B \to A$ and $(F, G)$ be a cyclic contraction. If $(x_0, x'_0) \in A \times A$ and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all $n \in \mathbb{N} \cup \{0\}$, then $\{x_{2n}\}$, $\{x'_{2n}\}$, $\{x_{2n+1}\}$ and $\{x'_{2n+1}\}$ are Cauchy sequences.*

*Proof*. By Lemma 3.7, we have $d(x_{2n}, x_{2n+1}) \to d(A, B)$ and $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$. Since $(A, B)$ has a property UC*, we get $d(x_{2n}, x_{2n+2}) \to 0$. As $(B, A)$ has a property UC*, we also have $d(x_{2n+1}, x_{2n+3}) \to 0$.

We now show that for every $\varepsilon > 0$ there exists $N$ such that

$$d(x_{2m}, x_{2n+1}) \leq d(A,B) + \varepsilon \tag{3.5}$$

for all $m > n \geq N$.

Suppose (3.5) not, then there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ there exists $m_k > n_k \geq k$ such that

$$d(x_{2m_k}, x_{2n_k+1}) > d(A,B) + \varepsilon. \tag{3.6}$$

Now we have

$$\begin{aligned} d(A,B) + \varepsilon &< d(x_{2m_k}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k+1}) \\ &\leq d(A,B) + \varepsilon + d(x_{2n_k-1}, x_{2n_k+1}) \end{aligned}$$

Taking $k \to \infty$, we have $d(x_{2m_k}, x_{2n_k+1}) \to d(A,B) + \varepsilon$.

By Lemma 3.8, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1})] < d(A,B) + \varepsilon \tag{3.7}$$

for all $m > n \geq N$. By using the triangle inequality we get

$$\begin{aligned} & d(x_{2m_k}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &= d(x_{2m_k}, x_{2m_k+2}) + d(G(x_{2m_k+1}, x'_{2m_k}+1), F(x_{2n_k+2}, x'_{2n_k}+2)) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2m_k+2}) + \frac{\alpha}{2}[d(x_{2m_k+1}, x_{2n_k+2}) + d(x'_{2m_k}+1, x'_{2n_k}+2)] + (1-\alpha)d(A,B) \\ &\quad + d(x_{2n_k+3}, x_{2n_k+1}) \\ &= \frac{\alpha}{2}[d(F(x_{2m_k}, x'_{2m_k}), G(x_{2n_k+1}, x'_{2n_k}+1)) + d(F(x'_{2m_k}, x_{2m_k}), G(x'_{2n_k}+1, x_{2n_k+1}))] \\ &\quad + (1-\alpha)d(A,B) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &\leq \frac{\alpha}{2}\Big[\frac{\alpha}{2}\,[d(x_{2m_k}, x_{2n_k+1}) + d(x'_{2m_k}, x'_{2n_k}+1) + (1-\alpha)d(A,B)] \\ &\quad + \frac{\alpha}{2}[d(x'_{2m_k}, x'_{2n_k}+1) + d(x_{2m_k}, x_{2n_k+1}) + (1-\alpha)d(A,B)]\Big] \\ &\quad + (1-\alpha)d(A,B) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &= \alpha^2\frac{1}{2}[d(x_{2m_k}, x_{2n_k+1}) + d(x'_{2m_k}, x'_{2n_k}+1)] + (1-\alpha^2)d(A,B) \\ &\quad + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &< \alpha^2(d(A,B)+\varepsilon) + (1-\alpha^2)d(A,B) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &= d(A,B) + \alpha^2\varepsilon + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \end{aligned}$$

Taking $k \to \infty$, we get

$$d(A,B) + \varepsilon \leq d(A,B) + \alpha^2\varepsilon$$

which contradicts. Therefore, condition (3.5) holds. Since (3.5) holds and $d(x_{2n}, x_{2n+1}) \to d(A, B)$, by using property UC* of $(A, B)$, we have $\{x_{2n}\}$ is a Cauchy sequence. In similar way, we can prove that $\{x'_{2n}\}$, $\{x_{2n+1}\}$ and $\{x'_{2n+1}\}$ are Cauchy sequences. □

Here we state the main results of this article on the existence and convergence of coupled best proximity points for cyclic contraction pairs on nonempty subsets of metric spaces satisfying the property UC*.

**Theorem 3.10**. *Let A and B be nonempty closed subsets of a complete metric space X such that (A, B) and (B, A) satisfy the property UC*. Let* $F : A \times A \to B$, $G : B \times B \to A$ *and (F, G) be a cyclic contraction. Let* $(x_0, x'_0) \in A \times A$ *and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all* $n \in \mathbb{N} \cup \{0\}$. *Then F has a coupled best proximity point* $(p, q) \in A \times A$ *and G has a coupled best proximity point* $(p', q') \in B \times B$ *such that*

$$d(p, p') + d(q, q') = 2d(A, B).$$

*Moreover, we have* $x_{2n} \to p$, $x'_{2n} \to q$, $x_{2n+1} \to p'$ *and* $x'_{2n+1} \to q'$.

*Proof*. By Lemma 3.7, we get $d(x_{2n}, x_{2n+1}) \to d(A, B)$. Using Lemma 3.9, we have $\{x_{2n}\}$ and $\{x'_{2n}\}$ are Cauchy sequences. Thus, there exists $p, q \in A$ such that $x_{2n} \to p$ and $x'_{2n} \to q$. We obtain that

$$d(A, B) \le d(p, x_{2n-1}) \le d(p, x_{2n}) + d(x_{2n}, x_{2n-1}). \tag{3.8}$$

Letting $n \to \infty$ in (3.8), we have $d(p, x_{2n-1}) \to d(A, B)$. By a similar argument we also have $d(q, x'_{2n-1}) \to d(A, B)$. It follows that

$$\begin{aligned} d(x_{2n}, F(p, q)) &= d(G(x_{2n-1}, x'_{2n-1}), F(p, q)) \\ &\le \frac{\alpha}{2}[d(x_{2n-1}, p) + d(x'_{2n-1}, q)] + (1-\alpha)d(A, B). \end{aligned}$$

Taking $n \to \infty$, we get $d(p, F(p, q)) = d(A, B)$. Similarly, we can prove that $d(q, F(q, p)) = d(A, B)$. Therefore, we have $(p, q)$ is a coupled best proximity point of $F$.

In similar way, we can prove that there exists $p', q' \in B$ such that $x_{2n+1} \to p'$ and $x'_{2n+1} \to q'$. Moreover, we also have $d(p', G(p', q')) = d(A, B)$ and $d(q', G(q', p')) = d(A, B)$ and so $(p', q')$ is a coupled best proximity point of $G$.

Finally, we show that $d(p, p') + d(q, q') = 2d(A, B)$. For $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(G(x_{2n-1}, x'_{2n-1}), F(x_{2n}, x'_{2n})) \\ &\le \frac{\alpha}{2}[d(x_{2n-1}, x_{2n}) + d(x'_{2n-1}, x'_{2n})] + (1-\alpha)d(A, B). \end{aligned}$$

Letting $n \to \infty$, we have

$$d(p, p') \le \frac{\alpha}{2}[d(p, p') + d(q, q')] + (1-\alpha)d(A, B). \tag{3.9}$$

For $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} d(x'_{2n}, x'_{2n+1}) &= d(G(x'_{2n-1}, x_{2n-1}), F(x'_{2n}, x_{2n})) \\ &\le \frac{\alpha}{2}[d(x'_{2n-1}, x'_{2n}) + d(x_{2n-1}, x_{2n})] + (1-\alpha)d(A, B). \end{aligned}$$

Letting $n \to \infty$, we have

$$d(q, q') \le \frac{\alpha}{2}[d(q, q') + d(p, p')] + (1-\alpha)d(A, B). \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$d(p,p') + d(q,q') \le \alpha[d(p,p') + d(q,q')] + 2(1-\alpha)d(A,B).$$

which implies that

$$d(p,p') + d(q,q') \le 2d(A,B). \tag{3.11}$$

Since $d(A, B) \le d(p, p')$ and $d(A, B) \le d(q, q')$, we have

$$2d(A,B) \le d(p,p') + d(q,q'). \tag{3.12}$$

From (3.11) and (3.12), we get

$$d(p,p') + d(q,q') = 2d(A,B).$$

This complete the proof. □

Note that every pair of nonempty closed subsets $A$, $B$ of a uniformly convex Banach space $X$ such that $A$ is convex satisfies the property UC*. Therefore, we obtain the following corollary.

**Corollary 3.11**. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X,* $F : A \times A \to B$, $G : B \times B \to A$ *and* $(F, G)$ *be a cyclic contraction. Let* $(x_0, x_0') \in A \times A$ *and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all* $n \in \mathbb{N} \cup \{0\}$. *Then F has a coupled best proximity point* $(p, q) \in A \times A$ *and G has a coupled best proximity point* $(p', q') \in B \times B$ *such that*

$$d(p,p') + d(q,q') = 2d(A,B).$$

*Moreover, we have* $x_{2n} \to p$, $x'_{2n} \to q$, $x_{2n+1} \to p'$ *and* $x'_{2n+1} \to q'$.

Next, we give some illustrative example of Corollary 3.11.

**Example 3.12**. *Consider uniformly convex Banach space* $X = \mathbb{R}$ *with the usual norm. Let* $A = [1,2]$ *and* $B = [-2, -1]$. *Thus* $d(A, B) = 2$. *Define* $F : A \times A \to B$ *and* $G : B \times B \to A$ *by*

$$F(x,x') = \frac{-x - x' - 2}{4}$$

*and*

$$G(x,x') = \frac{-x - x' + 2}{4}.$$

*For arbitrary* $(x, x') \in A \times A$ *and* $(y, y') \in B \times B$ *and fixed* $\alpha = \frac{1}{2}$, *we get*

$$\begin{aligned} d(F(x,x'), G(y,y')) &= \left| \frac{-x-x'-2}{4} - \frac{-y-y'+2}{4} \right| \\ &\le \frac{|x-y| + |x'-y'|}{4} + 1 \\ &= \frac{\alpha}{2}[d(x,y) + d(x',y')] + 1(1-\alpha)d(A,B). \end{aligned}$$

*This implies that* $(F, G)$ *is a cyclic contraction with* $\alpha = \frac{1}{2}$. *Since A and B are convex, we have* $(A, B)$ *and* $(B, A)$ *satisfy the property UC*. Therefore, all hypothesis of Corollary 3.11 hold. So F has a coupled best proximity point and G has a coupled best proximity point. We note that a point* $(1, 1) \in A \times A$ *is a unique coupled best proximity point of F and a point* $(-1, -1) \in B \times B$ *is a unique coupled best proximity point of G. Furthermore, we get*

$$d(1,-1) + d(1,-1) = 4 = 2d(A,B).$$

**Theorem 3.13**. *Let A and B be nonempty compact subsets of a metric space X,* $F : A\times A \to B$, $G : B \times B \to A$ *and* $(F, G)$ *be a cyclic contraction pair. If* $(x_0, x_0') \in A \times A$ *and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all* $n \in \mathbb{N} \cup \{0\}$, *then F has a coupled best proximity point* $(p, q) \in A \times A$ *and G has a coupled best proximity point* $(p', q') \in B \times B$ *such that*

$$d(p,p') + d(q,q') = 2d(A,B).$$

*Proof.* Since $x_0, x_0' \in A$ and

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

and

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$, we have $x_{2n}, x'_{2n} \in A$ and $x_{2n+1}, x'_{2n+1} \in B$ for all $n \in \mathbb{N} \cup \{0\}$. As $A$ is compact, the sequence $\{x_{2n}\}$ and $\{x'_{2n}\}$ have convergent subsequences $\{x_{2n_k}\}$ and $\{x'_{2n_k}\}$, respectively, such that

$$x_{2n_k} \to p \in A \text{ and } x'_{2n_k} \to q \in A.$$

Now, we have

$$d(A,B) \le d(p, x_{2n_k-1}) \le d(p, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}). \tag{3.13}$$

By Lemma 3.7, we have $d(x_{2n_k}, x_{2n_k-1}) \to d(A,B)$. Taking $k \to \infty$ in (3.13), we get $d(p, x_{2n_k-1}) \to d(A,B)$. By a similar argument we observe that $d(q, x_{2n_k-1}) \to d(A,B)$. Note that

$$\begin{aligned} d(A,B) &\le d((x_{2n_k}, F(p,q)) \\ &= d(G(x_{2n_k-1}, x'_{2n_k-1}), F(p,q)) \\ &\le \frac{\alpha}{2}[d(x_{2n_k-1}, p) + d(x'_{2n_k-1}, q)] + (1-\alpha)d(A,B). \end{aligned}$$

Taking $k \to \infty$, we get $d(p, F(p, q)) = d(A, B)$. Similarly, we can prove that $d(q, F(q, p)) = d(A, B)$. Thus $F$ has a coupled best proximity $(p, q) \in A \times A$. In similar way, since $B$ is compact, we can also prove that $G$ has a coupled best proximity point in $(p', q') \in B \times B$.

For $d(p, p') + d(q, q') = 2d(A, B)$ similar to the final step of the proof of Theorem 3.10. This complete the proof. □

## 4 Coupled fixed point theorem

In this section, we give the new coupled fixed point theorem for a cyclic contraction pair.

**Theorem 4.1**. *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$, $F : A \times A \to B$, $G : B \times B \to A$ and $(F, G)$ be a cyclic contraction. Let $(x_0, x'_0) \in A \times A$ and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all $n \in \mathbb{N} \cup \{0\}$. If $d(A, B) = 0$, then $F$ and $G$ have a unique common coupled fixed point $(p, q) \in A \cap B \times A \cap B$. Moreover, we have $x_{2n} \to p$, $x'_{2n} \to q$, $x_{2n+1} \to p$ and $x'_{2n+1} \to q$.*

*Proof*. Since $d(A, B) = 0$, we get $(A, B)$ and $(B, A)$ have the property UC*. Therefore, by Theorem 3.10 claim that $F$ has a coupled best proximity point $(p, q) \in A \times A$ that is

$$d(p, F(p, q)) = d(q, F(q, p)) = d(A, B) \tag{4.1}$$

and $G$ has a coupled best proximity point $(p', q') \in B \times B$ that is

$$d(p', G(p', q')) = d(q', G(q', p')) = d(A, B). \tag{4.2}$$

Moreover, we have

$$d(p, p') + d(q, q') = 2d(A, B). \tag{4.3}$$

From (4.1) and $d(A, B) = 0$, we conclude that

$$p = F(p, q) \text{ and } q = F(q, p)$$

that is $(p, q)$ is a coupled fixed point of $F$ . It follows from (4.2) and $d(A, B) = 0$, we get

$$p' = G(p', q') \text{ and } q' = G(q', p')$$

that is $(p', q')$ is a coupled fixed point of $G$. Using (4.3) and the fact that $d(A, B) = 0$, we have

$$d(p, p') + d(q, q') = 0$$

which implies that $p = p'$ and $q = q'$. Therefore, we conclude that $(p, q) \in A \cap B \times A \cap B$ *i*s a common coupled fixed point of $F$ and $G$.

Finally, we show the uniqueness of common coupled fixed point of $F$ and $G$. Let $(\hat{p}, \hat{q})$ be another common coupled fixed point of $F$ and $G$. So $\hat{p} = G(\hat{p}, \hat{q})$ and $\hat{q} = G(\hat{q}, \hat{p})$. Now, we obtain that

$$d(p, \hat{p}) = d(F(p, q), G(\hat{p}, \hat{q})) \leq \frac{\alpha}{2}[d(p, \hat{p}) + d(q, \hat{q})] \tag{4.4}$$

and also

$$d(q,\hat{q}) = d(F(q,p), G(\hat{q},\hat{p})) \leq \frac{\alpha}{2}[d(q,\hat{q}) + d(p,\hat{p})]. \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$d(p,\hat{p}) + d(q,\hat{q}) \leq \alpha[d(p,\hat{p}) + d(q,\hat{q})],$$

which implies that $d(\hat{p},\hat{q}) + d(q,\hat{q}) = 0$ and so $d(p,\hat{p}) = 0$ and $d(q,\hat{q}) = 0$. Therefore, $(p, q)$ is a unique common coupled fixed point in $A \cap B \times A \cap B$. □

**Example 4.2**. *Consider* $X = \mathbb{R}$ *with the usual metric,* $A = [-1, 0]$ *and* $B = [0,1]$ *. Define* $F : A \times A \to B$ *by* $F(x,y) = -\frac{x+y}{4}$ *and* $G(x,y) = -\frac{x+y}{8}$*. Then* $d(A, B) = 0$ *and* $(F, G)$ *is a cyclic contraction with* $\alpha = \frac{1}{2}$*. Indeed, for arbitrary* $(x, x') \in A \times A$ *and* $(y, y') \in B \times B$*, we have*

$$\begin{aligned} d(F(x,x'), G(y,y')) &= \left| -\frac{x+x'}{4} + \frac{y+y'}{4} \right| \\ &\leq \left| -\frac{x+x'}{4} + \frac{2y+2y'}{8} \right| \\ &= \frac{1}{4}(|x-y| + |x'-y'|) \\ &= \frac{\alpha}{2}[d(x,y) + d(x',y')] + (1-\alpha)d(A,B). \end{aligned}$$

*Therefore, all hypothesis of Theorem* 4.1 *hold. So F and G have a unique common coupled fixed point and this point is* $(0, 0) \in A \cap B \times A \cap B$.

If we take $A = B$ in Theorem 4.1, then we get the following results.

**Corollary 4.3**. *Let A be nonempty closed subsets of a complete metric space X,* $F : A \times A \to A$ *and* $G : A \times A \to A$ *and let the order pair* $(F, G)$ *is a cyclic contraction. Let* $(x_0, x'_0) \in A \times A$ *and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

*and*

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

*for all* $n \in \mathbb{N} \cup \{0\}$*. Then F and G have a unique common coupled fixed point* $(p, q) \in A \times A$*. Moreover, we have* $x_{2n} \to p$, $x'_{2n} \to q$ , $x_{2n+1} \to p$ *and* $x'_{2n+1} \to q$

We take $F = G$ in Corollary 4.3, then we get the following results.

**Corollary 4.4**. *Let A be nonempty closed subsets of a complete metric space X,* $F : A \times A \to A$ *and*

$$d(F(x,x'), F(y,y')) \leq \frac{\alpha}{2}[d(x,y) + d(x',y')] \tag{4.6}$$

*for all* $(x, x'), (y, y') \in A \times A$*. Then F has a unique coupled fixed point* $(p, q) \in A \times A$.

**Example 4.5**. *Consider* $X = \mathbb{R}$ *with the usual metric and* $A = [0, \frac{1}{2}]$*. Define* $F : A \times A \to A$ *by*

$$F(x, y) = \begin{cases} \dfrac{x^2 - y^2}{4}; & x \geq y \\ 0; & x < y. \end{cases}$$

*We show that F satisfies* (4.6) *with* $\alpha = \frac{1}{2}$. *Let* $(x, x'), (y, y') \in A \times A$.

**Case 1:** *If* $x < x'$ *and* $y < y'$, *then*

$$d(F(x, x'), F(y, y')) = 0 \leq \frac{1}{4}[|x - y| + |x' - y'|] = \frac{\alpha}{2}[d(x, y) + d(x', y')].$$

**Case 2:** *If* $x < x'$ *and* $y \geq y'$, *then*

$$\begin{aligned} d(F(x, x'), F(y, y')) &= \left|0 - \frac{y^2 - y'^2}{4}\right| \\ &\leq \frac{1}{4}[|y - y'|\ |y + y'|] \\ &\leq \frac{1}{4}|y - y'| \\ &= \frac{1}{4}(y - y') \\ &< \frac{1}{4}[(y - y') + (x' - x)] \\ &\leq \frac{1}{4}[|x - y| + |x' - y'|] \\ &= \frac{\alpha}{2}[d(x, y) + d(x', y')]. \end{aligned}$$

**Case 3:** *If* $x \geq x'$ *and* $y < y'$. *In this case we can prove by a similar argument as in case* 2.

**Case 4:** *If* $x \geq x'$ *and* $y \geq y'$, *then*

$$\begin{aligned} d(F(x, x'), F(y, y')) &= \left|\frac{x^2 - x'^2}{4} - \frac{y^2 - y'^2}{4}\right| \\ &\leq \frac{1}{4}[|x - x'|\ |x + x'| + |y - y'|\ |y + y'|] \\ &\leq \frac{1}{4}[|x - x'| + |y - y'|] \\ &= \frac{\alpha}{2}[d(x, y) + d(x', y')]. \end{aligned}$$

*Thus condition* (4.6) *holds with* $\alpha = \frac{1}{2}$. *Therefore, by Corollary* 4.4 *F has the unique coupled fixed point in A that is a point* (0, 0).

**Open problems:**

- In Theorem 3.10, can be replaced the property UC* by a more general condition ?
- In Theorem 3.10, can be drop the property UC* ?
- Can be extend the result in this article to another spaces ?

**Acknowledgements**

The first author would like to thank the Research Professional Development Project Under the Science Achievement Scholarship of Thailand (SAST) for some financial support. Furthermore, the second author was supported by the

Commission on Higher Education, the Thailand Research Fund and the King Mongkut's University of Technology Thonburi for financial support during the preparation of this manuscript.

### Authors' contributions
All authors read and approved the final manuscript.

### Competing interests
The authors declare that they have no competing interests.

Received: 4 November 2011 Accepted: 7 June 2012 Published: 7 June 2012

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doi:10.1186/1687-1812-2012-93
**Cite this article as:** Sintunavarat and Kumam: **Coupled best proximity point theorem in metric Spaces.** *Fixed Point Theory and Applications* 2012 **2012**:93.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 896912, 19 pages
doi:10.1155/2012/896912

*Research Article*

# Generalized Proximal $\psi$-Contraction Mappings and Best Proximity Points

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Received 19 July 2012; Accepted 24 September 2012

Academic Editor: Haiyun Zhou



We generalized the notion of proximal contractions of the first and the second kinds and established the best proximity point theorems for these classes. Our results improve and extend recent result of Sadiq Basha (2011) and some authors.

## 1. Introduction

The significance of fixed point theory stems from the fact that it furnishes a unified treatment and is a vital tool for solving equations of form $Tx = x$ where $T$ is a self-mapping defined on a subset of a metric space, a normed linear space, topological vector space or some suitable space. Some applications of fixed point theory can be found in [1–12]. However, almost all such results dilate upon the existence of a fixed point for self-mappings. Nevertheless, if $T$ is a non-self-mapping, then it is probable that the equation $Tx = x$ has no solution, in which case best approximation theorems explore the existence of an approximate solution whereas best proximity point theorems analyze the existence of an approximate solution that is optimal. A classical best approximation theorem was introduced by Fan [13]; that is, if $A$ is a nonempty compact convex subset of a Hausdorff locally convex topological vector space $B$ and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterward, several authors, including Prolla [14], Reich [15], Sehgal, and Singh [16, 17], have derived extensions of Fan's theorem in many directions. Other works of the existence of a best proximity point for contractions can be seen in [18–21]. In 2005, Eldred et al. [22] have obtained best proximity point theorems for relatively nonexpansive mappings. Best proximity point theorems for several types of contractions have been established in [23–36].

Recently, Sadiq Basha in [37] gave necessary and sufficient to claimed that the existence of best proximity point for proximal contraction of first kind and the second kind which are non-self mapping analogues of contraction self-mappings and also established some best proximity and convergence theorem as follow.

**Theorem 1.1** (see [37, Theorem 3.1]). *Let $(X, d)$ be a complete metric space and let $A$ and $B$ be nonempty, closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions.*

(a) *$S$ and $T$ are proximal contractions of first kind.*

(b) *$g$ is an isometry.*

(c) *The pair $(S, T)$ is a proximal cyclic contraction.*

(d) *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$.*

(e) *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$.*

*Then, there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B). \tag{1.1}$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B), \tag{1.2}$$

*converges to the element $x$. For any fixed $y_0 \in B_0$, the sequence $\{y_n\}$, defined by*

$$d(gy_{n+1}, Ty_n) = d(A, B), \tag{1.3}$$

*converges to the element $y$.*

*On the other hand, a sequence $\{u_n\}$ in $A$ converges to $x$ if there is a sequence of positive numbers $\{\epsilon_n\}$ such that*

$$\lim_{n \to \infty} \epsilon_n = 0, \qquad d(u_{n+1}, z_{n+1}) \leq \epsilon_n, \tag{1.4}$$

*where $z_{n+1} \in A$ satisfies the condition that $d(z_{n+1}, Su_n) = d(A, B)$.*

**Theorem 1.2** (see [37, Theorem 3.4]). *Let $(X, d)$ be a complete metric space and let $A$ and $B$ be nonempty, closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$ and $g : A \to A$ satisfy the following conditions.*

(a) *$S$ is proximal contractions of first and second kinds.*

(b) *$g$ is an isometry.*

(c) *$S$ preserves isometric distance with respect to $g$.*

(d) $S(A_0) \subseteq B_0$.

(e) $A_0 \subseteq g(A_0)$.

*Then, there exists a unique point $x \in A$ such that*

$$d(gx, Sx) = d(A, B). \tag{1.5}$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B), \tag{1.6}$$

*converges to the element $x$.*

*On the other hand, a sequence $\{u_n\}$ in $A$ converges to $x$ if there is a sequence of positive numbers $\{\epsilon_n\}$ such that*

$$\lim_{n \longrightarrow \infty} \epsilon_n = 0, \qquad d(u_{n+1}, z_{n+1}) \leq \epsilon_n, \tag{1.7}$$

*where $z_{n+1} \in A$ satisfies the condition that $d(z_{n+1}, Su_n) = d(A, B)$.*

The aim of this paper is to introduce the new classes of proximal contractions which are more general than class of proximal contraction of first and second kinds, by giving the necessary condition to have best proximity points and we also give some illustrative examples of our main results. The results of this paper are extension and generalizations of main result of Sadiq Basha in [37] and some results in the literature.

## 2. Preliminaries

Given nonvoid subsets $A$ and $B$ of a metric space $(X, d)$, we recall the following notations and notions that will be used in what follows:

$$\begin{gathered} d(A, B) := \inf\{d(x, y) : x \in A, \ \ y \in B\}, \\ A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}, \\ B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{gathered} \tag{2.1}$$

If $A \cap B \neq \emptyset$, then $A_0$ and $B_0$ are nonempty. Further, it is interesting to notice that $A_0$ and $B_0$ are contained in the boundaries of $A$ and $B$, respectively, provided $A$ and $B$ are closed subsets of a normed linear space such that $d(A, B) > 0$ (see [31]).

*Definition 2.1* ([37, Definition 2.2]). A mapping $S : A \to B$ is said to be a *proximal contraction of the first kind* if there exists $\alpha \in [0, 1)$ such that

$$d(u, Sx) = d(v, Sy) = d(A, B) \Longrightarrow d(u, v) \leq \alpha d(x, y) \tag{2.2}$$

for all $u, v, x, y \in A$.

It is easy to see that a self-mapping that is a proximal contraction of the first kind is precisely a contraction. However, a non-self-proximal contraction is not necessarily a contraction.

*Definition 2.2* (see [37, Definition 2.3]). A mapping $S : A \to B$ is said to be a *proximal contraction of the second kind* if there exists $\alpha \in [0, 1)$ such that

$$d(u, Sx) = d(v, Sy) = d(A, B) \Longrightarrow d(Su, Sv) \leq \alpha d(Sx, Sy) \tag{2.3}$$

for all $u, v, x, y \in A$.

*Definition 2.3.* Let $S : A \to B$ and $T : B \to A$. The pair $(S, T)$ is said to be a *proximal cyclic contraction pair* if there exists a nonnegative number $\alpha < 1$ such that

$$d(a, Sx) = d(b, Ty) = d(A, B) \Longrightarrow d(a, b) \leq \alpha d(x, y) + (1 - \alpha)d(A, B) \tag{2.4}$$

for all $a, x \in A$ and $b, y \in B$.

*Definition 2.4.* Leting $S : A \to B$ and an isometry $g : A \to A$, the mapping $S$ is said to preserve isometric distance with respect to $g$ if

$$d(Sgx, Sgy) = d(Sx, Sy) \tag{2.5}$$

for all $x, y \in A$.

*Definition 2.5.* A point $x \in A$ is said to be a *best proximity point* of the mapping $S : A \to B$ if it satisfies the condition that

$$d(x, Sx) = d(A, B). \tag{2.6}$$

It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

*Definition 2.6.* $A$ is said to be *approximatively compact with respect to B* if every sequence $\{x_n\}$ in $A$ satisfies the condition that $d(y, x_n) \to d(y, A)$ for some $y \in B$ has a convergent subsequence.

We observe that every set is approximatively compact with respect to itself and that every compact set is approximatively compact. Moreover, $A_0$ and $B_0$ are nonempty set if $A$ is compact and $B$ is approximatively compact with respect to $A$.

## 3. Main Results

*Definition 3.1.* A mapping $S : A \to B$ is said to be *a generalized proximal $\psi$-contraction of the first kind*, if for all $u, v, x, y \in A$ satisfies

$$d(u, Sx) = d(v, Sy) = d(A, B) \Longrightarrow d(u, v) \leq \psi(d(x, y)), \tag{3.1}$$

where $\psi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function from the right such that $\psi(t) < t$ for all $t > 0$.

*Definition 3.2.* A mapping $S : A \to B$ is said to be *a generalized proximal $\psi$-contraction of the second kind*, if for all $u, v, x, y \in A$ satisfies

$$d(u, Sx) = d(v, Sy) = d(A, B) \Longrightarrow d(Su, Sv) \leq \psi(d(Sx, Sy)), \tag{3.2}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a upper semicontinuous from the right such that $\psi(t) < t$ for all $t > 0$.

It is easy to see that if we take $\psi(t) = \alpha t$, where $\alpha \in [0, 1)$, then a generalized proximal $\psi$-contraction of the first kind and generalized proximal $\psi$-contraction of the second kind reduce to a proximal contraction of the first kind Definition 2.1 and a proximal contraction of the second kind Definition 2.2, respectively. Moreover, it is easy to see that a self-mapping generalized proximal $\psi$-contraction of the first kind and the second kind reduces to the condition of Boy and Wong' s fixed point theorem [3].

Next, we extend the result of Sadiq Basha [37] and the Banach's contraction principle to the case of non-self-mappings which satisfy generalized proximal $\psi$-contraction condition.

**Theorem 3.3.** *Let $(X, d)$ be a complete metric space and let $A$ and $B$ be nonempty, closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$, $T : B \to A$, and $g : A \cup B \to A \cup B$ satisfy the following conditions:*

(a) *$S$ and $T$ are generalized proximal $\psi$-contraction of the first kind;*

(b) *$g$ is an isometry;*

(c) *The pair $(S, T)$ is a proximal cyclic contraction;*

(d) *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*

(e) *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$.*

*Then, there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B). \tag{3.3}$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B), \tag{3.4}$$

*converges to the element $x$. For any fixed $y_0 \in B_0$, the sequence $\{y_n\}$, defined by*

$$d(gy_{n+1}, Ty_n) = d(A, B), \tag{3.5}$$

*converges to the element $y$.*

*On the other hand, a sequence $\{u_n\}$ in $A$ converges to $x$ if there is a sequence of positive numbers $\{\epsilon_n\}$ such that*

$$\lim_{n \to \infty} \epsilon_n = 0, \qquad d(u_{n+1}, z_{n+1}) \leq \epsilon_n, \tag{3.6}$$

*where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A, B)$.*

*Proof.* Let $x_0$ be a fixed element in $A_0$. In view of the fact that $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, it is ascertained that there exists an element $x_1 \in A_0$ such that

$$d(gx_1, Sx_0) = d(A, B). \tag{3.7}$$

Again, since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element $x_2 \in A_0$ such that

$$d(gx_2, Sx_1) = d(A, B). \tag{3.8}$$

By similar fashion, we can find $x_n$ in $A_0$. Having chosen $x_n$, one can determine an element $x_{n+1} \in A_0$ such that

$$d(gx_{n+1}, Sx_n) = d(A, B). \tag{3.9}$$

Because of the facts that $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, by a generalized proximal $\psi$-contraction of the first kind of $S$, $g$ is an isometry and property of $\psi$, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(gx_{n+1}, gx_n) \\ &\leq \psi(d(x_n, x_{n-1})) \\ &\leq d(x_n, x_{n-1}). \end{aligned} \tag{3.10}$$

This means that the sequence $\{d(x_{n+1}, x_n)\}$ is nonincreasing and bounded. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r. \tag{3.11}$$

If $r > 0$, then

$$\begin{aligned} r &= \lim_{n\to\infty} d(x_{n+1}, x_n) \\ &\leq \lim_{n\to\infty} \psi(d(x_n, x_{n-1})) \\ &= \psi(r) \\ &< r, \end{aligned} \tag{3.12}$$

which is a contradiction unless $r = 0$. Therefore,

$$\alpha_n := \lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.13}$$

We claim that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequence $\{x_{m_k}\}, \{x_{n_k}\}$ of $\{x_n\}$ such that $n_k > m_k \geq k$ with

$$r_k := d(x_{m_k}, x_{n_k}) \geq \varepsilon, \qquad d(x_{m_k}, x_{n_k-1}) < \varepsilon \tag{3.14}$$

for $k \in \{1,2,3,\dots\}$. Thus

$$\begin{aligned} \varepsilon \le r_k &\le d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \varepsilon + \alpha_{n_k-1}. \end{aligned} \tag{3.15}$$

It follows from (3.13) that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{3.16}$$

On the other hand, by constructing the sequence $\{x_n\}$, we have

$$d(gx_{m_k+1}, Sx_{m_k}) = d(A,B), \qquad d(gx_{n_k+1}, Sx_{n_k}) = d(A,B). \tag{3.17}$$

Sine $S$ is a generalized proximal $\psi$-contraction of the first kind and $g$ is an isometry, we have

$$d(x_{m_k+1}, x_{n_k+1}) = d(gx_{m_k+1}, gx_{n_k+1}) \le \psi(d(x_{m_k}, x_{n_k})). \tag{3.18}$$

Notice also that

$$\begin{aligned} \varepsilon \le r_k &\le d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k}) + d(x_{m_k+1}, x_{n_k+1}) \\ &= \alpha_{m_k} + \alpha_{n_k} + d(x_{m_k+1}, x_{n_k+1}) \\ &\le \alpha_{m_k} + \alpha_{n_k} + \psi(d(x_{m_k}, x_{n_k})). \end{aligned} \tag{3.19}$$

Taking $k \to \infty$ in above inequality, by (3.13), (3.16), and property of $\psi$, we get $\varepsilon \le \psi(\varepsilon)$. Therefore, $\varepsilon = 0$, which is a contradiction. So we obtain the claim and hence converge to some element $x \in A$. Similarly, in view of the fact that $T(B_0) \subseteq A_0$ and $A_0 \subseteq g(A_0)$, we can conclude that there is a sequence $\{y_n\}$ such that $d(gy_{n+1}, Sy_n) = d(A,B)$ and converge to some element $y \in B$. Since the pair $(S,T)$ is a proximal cyclic contraction and $g$ is an isometry, we have

$$d(x_{n+1}, y_{n+1}) = d(gx_{n+1}, gy_{n+1}) \le \alpha d(x_n, y_n) + (1-\alpha) d(A,B). \tag{3.20}$$

We take limit in (3.20) as $n \to \infty$; it follows that

$$d(x,y) = d(A,B), \tag{3.21}$$

so, we concluded that $x \in A_0$ and $y \in B_0$. Since $S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$, there is $u \in A$ and $v \in B$ such that

$$d(u, Sx) = d(A,B) \tag{3.22}$$

$$d(v, Ty) = d(A,B). \tag{3.23}$$

From (3.9), (3.22), and the notion of generalized proximal $\psi$-contraction of first kind of $S$, we get

$$d(u, gx_{n+1}) \leq \psi(d(x, x_n)). \tag{3.24}$$

Letting $n \to \infty$, we get $d(u, gx) \leq \psi(0) = 0$ and thus $u = gx$. Therefore

$$d(gx, Sx) = d(A, B). \tag{3.25}$$

Similarly, we can show that $v = gy$ and then

$$d(gy, Ty) = d(\mathrm{A}, B). \tag{3.26}$$

From (3.21), (3.25), and (3.26), we get

$$d(x, y) = d(gx, Sx) = d(gy, Ty) = d(A, B). \tag{3.27}$$

Next, to prove the uniqueness, let us suppose that there exist $x^* \in A$ and $y^* \in B$ with $x \neq x^*, y \neq y^*$ such that

$$\begin{aligned} d(gx^*, Sx^*) &= d(A, B), \\ d(gy^*, Ty^*) &= d(A, B). \end{aligned} \tag{3.28}$$

Since $g$ is an isometry, $S$ and $T$ are generalized proximal $\psi$-contractions of the first kind and the property of $\psi$; it follows that

$$\begin{aligned} d(x, x^*) &= d(gx, gx^*) \leq \psi(d(x, x^*)) < d(x, x^*), \\ d(y, y^*) &= d(gy, gy^*) \leq \psi(d(y, y^*)) < d(y, y^*), \end{aligned} \tag{3.29}$$

which is a contradiction, so we have $x = x^*$ and $y = y^*$. On the other hand, let $\{u_n\}$ be a sequence in $A$ and let $\{\epsilon_n\}$ be a sequence of positive real numbers such that

$$\lim_{n \to \infty} \epsilon_n = 0, \quad d(u_{n+1}, z_{n+1}) \leq \epsilon_n, \tag{3.30}$$

where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A, B)$. Since $S$ is a generalized proximal $\psi$-contraction of first kind and $g$ is an isometry, we have

$$d(x_{n+1}, z_{n+1}) \leq \psi(d(x_n, u_n)). \tag{3.31}$$

Given $\epsilon > 0$, we choose a positive integer $N$ such that $\epsilon_n \leq \epsilon$ for all $n \geq N$; we obtain that

$$\begin{aligned} d(x_{n+1}, u_{n+1}) &\leq d(x_{n+1}, z_{n+1}) + d(z_{n+1}, u_{n+1}) \\ &\leq \psi(d(x_n, u_n)) + \epsilon_n. \end{aligned} \tag{3.32}$$

Therefore, we get

$$\begin{aligned} d(u_{n+1}, x) &\le d(u_{n+1}, x_{n+1}) + d(x_{n+1}, x) \\ &\le \psi(d(x_n, u_n)) + \epsilon_n + d(x_{n+1}, x). \end{aligned} \tag{3.33}$$

We claim that $d(u_n, x) \to 0$ as $n \to \infty$; supposing the contrary, by inequality (3.33) and property of $\psi$, we get

$$\begin{aligned} \lim_{n\to\infty} d(u_{n+1}, x) &\le \lim_{n\to\infty} (d(u_{n+1}, x_{n+1}) + d(x_{n+1}, x)) \\ &\le \lim_{n\to\infty} \big(\psi(d(x_n, u_n)) + \epsilon_n + d(x_{n+1}, x)\big) \\ &= \psi\Big(\lim_{n\to\infty} d(x_n, u_n)\Big) \\ &< \lim_{n\to\infty} d(x_n, u_n) \\ &\le \lim_{n\to\infty} (d(x_n, x) + d(x, u_n)) \\ &= \lim_{n\to\infty} d(x, u_n), \end{aligned} \tag{3.34}$$

which is a contradiction, so we have $\{u_n\}$ is convergent and it converges to $x$. This completes the proof of the theorem. □

If $g$ is assumed to be the identity mapping, then by Theorem 3.3, we obtain the following corollary.

**Corollary 3.4.** *Let $(X, d)$ be a complete metric space and let $A$ and $B$ be nonempty, closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S : A \to B, T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions:*

(a) *$S$ and $T$ are generalized proximal $\psi$-contraction of the first kind;*

(b) *$S(A_0) \subseteq B_0,\ T(B_0) \subseteq A_0$;*

(c) *the pair $(S, T)$ is a proximal cyclic contraction.*

*Then, there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B). \tag{3.35}$$

If we take $\psi(t) = \alpha t$, where $0 \le \alpha < 1$, we obtain following corollary.

**Corollary 3.5** (see [37, Theorem 3.1]). *Let $(X, d)$ be a complete metric space and $A$ and $B$ be non-empty, closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are non-empty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions:*

(a) *$S$ and $T$ are proximal contractions of first kind;*

(b) *$g$ is an isometry;*

(c) *the pair* $(S,T)$ *is a proximal cyclic contraction;*

(d) $S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$*;*

(e) $A_0 \subseteq g(A_0)$ *and* $B_0 \subseteq g(B_0)$*.*

*Then, there exists a unique point* $x \in A$ *and there exists a unique point* $y \in B$ *such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B). \tag{3.36}$$

*Moreover, for any fixed* $x_0 \in A_0$*, the sequence* $\{x_n\}$*, defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B), \tag{3.37}$$

*converges to the element* $x$*. For any fixed* $y_0 \in B_0$*, the sequence* $\{y_n\}$*, defined by*

$$d(gy_{n+1}, Ty_n) = d(A, B), \tag{3.38}$$

*converges to the element* $y$*.*

If $g$ is assumed to be the identity mapping in Corollary 3.5, we obtain the following corollary.

**Corollary 3.6.** *Let* $(X, d)$ *be a complete metric space and let* $A$ *and* $B$ *be nonempty, closed subsets of* $X$*. Further, suppose that* $A_0$ *and* $B_0$ *are nonempty. Let* $S : A \to B$*,* $T : B \to A$*, and* $g : A \cup B \to A \cup B$ *satisfy the following conditions:*

(a) $S$ *and* $T$ *are proximal contractions of first kind;*

(b) $S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$*;*

(c) *the pair* $(S,T)$ *is a proximal cyclic contraction.*

*Then, there exists a unique point* $x \in A$ *and there exists a unique point* $y \in B$ *such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B). \tag{3.39}$$

For a self-mapping, Theorem 3.3 includes the Boy and Wong' s fixed point theorem [3] as follows.

**Corollary 3.7.** *Let* $(X, d)$ *be a complete metric space and let* $T : X \to X$ *be a mapping that satisfies* $d(Tx, Ty) \le \varphi(d(x, y))$ *for all* $x, y \in X$*, where* $\varphi : [0, \infty) \to [0, \infty)$ *is an upper semicontinuous function from the right such that* $\varphi(t) < t$ *for all* $t > 0$*. Then* $T$ *has a unique fixed point* $v \in X$*. Moreover, for each* $x \in X$*,* $\{T^n x\}$ *converges to* $v$*.*

Next, we give an example to show that Definition 3.1 is different form Definition 2.1; moreover we give an example which supports Theorem 3.3.

*Example 3.8.* Consider the complete metric space $\mathbb{R}^2$ with metric defined by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|, \tag{3.40}$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Let

$$A = \{(0, y) : 0 \le y \le 1\}, \quad B = \{(1, y) : 0 \le y \le 1\}. \tag{3.41}$$

Then $d(A, B) = 1$. Define the mappings $S : A \to B$ as follows:

$$S((0, y)) = \left(1, y - \frac{y^2}{2}\right). \tag{3.42}$$

First, we show that $S$ is generalized proximal $\psi$-contraction of the first kind with the function $\psi : [0, \infty) \to [0, \infty)$ defined by

$$\psi(t) = \begin{cases} t - \dfrac{t^2}{2}, & 0 \le t \le 1, \\ t - 1, & t > 1. \end{cases} \tag{3.43}$$

Let $(0, x_1), (0, x_2), (0, a_1)$ and $(0, a_2)$ be elements in $A$ satisfying

$$d((0, x_1), S(0, a_1)) = d(A, B) = 1, \quad d((0, x_2), S(0, a_2)) = d(A, B) = 1. \tag{3.44}$$

It follows that

$$x_i = a_i - \frac{a_i^2}{2} \quad \text{for } i = 1, 2. \tag{3.45}$$

Without loss of generality, we may assume that $a_1 - a_2 > 0$, so we have

$$\begin{aligned} d((0, x_1), (0, x_2)) &= d\left(\left(0, a_1 - \frac{a_1^2}{2}\right), \left(0, a_2 - \frac{a_2^2}{2}\right)\right) \\ &= \left|\left(a_1 - \frac{a_1^2}{2}\right) - \left(a_2 - \frac{a_2^2}{2}\right)\right| \\ &= (a_1 - a_2) - \left(\frac{a_1^2}{2} - \frac{a_2^2}{2}\right) \\ &\le (a_1 - a_2) - \frac{1}{2}(a_1 - a_2)^2 \\ &= \psi(d((0, a_1), (0, a_2))). \end{aligned} \tag{3.46}$$

Thus $S$ is a generalized proximal $\psi$-contraction of the first kind.

Next, we prove that $S$ is not a proximal contraction. Suppose $S$ is proximal contraction then for each $(0, x), (0, y), (0, a), (0, b) \in A$ satisfying

$$d((0, x), S(0, a)) = d(A, B) = 1, \qquad d((0, y), S(0, b)) = d(A, B) = 1, \tag{3.47}$$

there exists $k \in [0,1)$ such that

$$d\big((0,x),(0,y)\big) \le kd((0,a),(0,b)). \tag{3.48}$$

From (3.47), we get

$$x = a - \frac{a^2}{2}, \qquad y = b - \frac{b^2}{2}, \tag{3.49}$$

and thus

$$\begin{aligned}\left|\left(a-\frac{a^2}{2}\right)-\left(b-\frac{b^2}{2}\right)\right| &= d\big((0,x),(0,y)\big)\\ &\le kd((0,a),(0,b))\\ &= k|a-b|.\end{aligned} \tag{3.50}$$

Letting $b = 0$ with $a \neq 0$, we get

$$1 = \lim_{a\to 0^+}\left(1-\frac{a}{2}\right) \le k < 1, \tag{3.51}$$

which is a contradiction. Therefore $S$ is not a proximal contraction and Definition 3.1 is different form Definition 2.1.

*Example 3.9.* Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Let

$$\begin{aligned} A &= \{(0,y) : y \in \mathbb{R}\},\\ B &= \{(1,y) : y \in \mathbb{R}\}.\end{aligned} \tag{3.52}$$

Define two mappings $S : A \to B, T : B \to A$ and $g : A\cup B \to A\cup B$ as follows:

$$S\big((0,y)\big) = \left(1,\frac{y}{4}\right), \quad T\big((1,y)\big) = \left(0,\frac{y}{4}\right), \quad g\big((x,y)\big) = (x,-y). \tag{3.53}$$

Then it is easy to see that $d(A,B) = 1$, $A_0 = A$, $B_0 = B$ and the mapping $g$ is an isometry.

Next, we claim that $S$ and $T$ are generalized proximal $\psi$-contractions of the first kind. Consider a function $\psi : [0,\infty) \to [0,\infty)$ defined by $\psi(t) = t/2$ for all $t \ge 0$. If $(0,y_1),(0,y_2) \in A$ such that

$$d\big(a,S(0,y_1)\big) = d(A,B) = 1, \qquad d\big(b,S(0,y_2)\big) = d(A,B) = 1 \tag{3.54}$$

for all $a,b \in A$, then we have

$$a = \left(0,\frac{y_1}{4}\right), \qquad b = \left(0,\frac{y_2}{4}\right). \tag{3.55}$$

Because,

$$\begin{aligned} d(a,b) &= d\left(\left(0,\frac{y_1}{4}\right),\left(0,\frac{y_2}{4}\right)\right) \\ &= \left|\frac{y_1}{4}-\frac{y_2}{4}\right| \\ &= \frac{1}{4}|y_1-y_2| \\ &\le \frac{1}{2}|y_1-y_2| \\ &= \frac{1}{2}d((0,y_1),(0,y_2)) \\ &= \psi(d((0,y_1),(0,y_2))). \end{aligned} \tag{3.56}$$

Hence $S$ is a generalized proximal $\psi$-contraction of the first kind. If $(1,y_1),(1,y_2) \in B$ such that

$$d(a,T(1,y_1)) = d(A,B) = 1, \qquad d(b,T(1,y_2)) = d(A,B) = 1 \tag{3.57}$$

for all $a,b \in B$, then we get

$$a = \left(1,\frac{y_1}{4}\right), \qquad b = \left(1,\frac{y_2}{4}\right). \tag{3.58}$$

In the same way, we can see that $T$ is a generalized proximal $\psi$-contraction of the first kind. Moreover, the pair $(S,T)$ forms a proximal cyclic contraction and other hypotheses of Theorem 3.3 are also satisfied. Further, it is easy to see that the unique element $(0,0) \in A$ and $(1,0) \in B$ such that

$$d(g(0,0),S(0,0)) = d(g(1,0),T(1,0)) = d((0,0),(1,0)) = d(A,B). \tag{3.59}$$

Next, we establish a best proximity point theorem for non-self-mappings which are generalized proximal $\psi$-contractions of the first kind and the second kind.

**Theorem 3.10.** *Let $(X,d)$ be a complete metric space and let $A$ and $B$ be non-empty, closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are non-empty. Let $S : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(a) *$S$ is a generalized proximal $\psi$-contraction of first and second kinds;*

(b) *$g$ is an isometry;*

(c) *$S$ preserves isometric distance with respect to $g$;*

(d) *$S(A_0) \subseteq B_0$;*

(e) *$A_0 \subseteq g(A_0)$.*

*Then, there exists a unique point $x \in A$ such that*

$$d(gx, Sx) = d(A, B). \tag{3.60}$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B), \tag{3.61}$$

*converges to the element $x$.*

*On the other hand, a sequence $\{u_n\}$ in $A$ converges to $x$ if there is a sequence of positive numbers $\{\epsilon_n\}$ such that*

$$\lim_{n\to\infty} \epsilon_n = 0, \qquad d(u_{n+1}, z_{n+1}) \leq \epsilon_n, \tag{3.62}$$

*where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A, B)$.*

*Proof.* Since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, similarly in the proof of Theorem 3.3, we can construct the sequence $\{x_n\}$ of element in $A_0$ such that

$$d(gx_{n+1}, Sx_n) = d(A, B) \tag{3.63}$$

for nonnegative number $n$. It follows from $g$ that is an isometry and the virtue of a generalized proximal $\psi$-contraction of the first kind of $S$; we see that

$$d(x_n, x_{n+1}) = d(gx_n, gx_{n+1}) \leq \psi(d(x_n, x_{n-1})) \tag{3.64}$$

for all $n \in \mathbb{N}$. Similarly to the proof of Theorem 3.3, we can conclude that the sequence $\{x_n\}$ is a Cauchy sequence and converges to some $x \in A$. Since $S$ is a generalized proximal $\psi$-contraction of the second kind and preserves isometric distance with respect to $g$ that

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &= d(Sgx_n, Sgx_{n+1}) \\ &\leq \psi(d(Sx_{n-1}, Sx_n)) \\ &\leq d(Sx_{n-1}, Sx_n), \end{aligned} \tag{3.65}$$

this means that the sequence $\{d(Sx_{n+1}, Sx_n)\}$ is nonincreasing and bounded below. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = r. \tag{3.66}$$

If $r > 0$, then

$$\begin{aligned} r &= \lim_{n\to\infty} d(Sx_{n+1}, Sx_n) \\ &\le \lim_{n\to\infty} \psi(d(Sx_{n-1}, Sx_n)) \\ &= \psi(r) \\ &< r, \end{aligned} \tag{3.67}$$

which is a contradiction, unless $r = 0$. Therefore

$$\beta_n := \lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = 0. \tag{3.68}$$

We claim that $\{Sx_n\}$ is a Cauchy sequence. Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequence $\{Sx_{m_k}\}, \{Sx_{n_k}\}$ of $\{Sx_n\}$ such that $n_k > m_k \ge k$ with

$$r_k := d(Sx_{m_k}, Sx_{n_k}) \ge \varepsilon, \qquad d(Sx_{m_k}, Sx_{n_k-1}) < \varepsilon \tag{3.69}$$

for $k \in \{1, 2, 3, \ldots\}$. Thus

$$\begin{aligned} \varepsilon \le r_k &\le d(Sx_{m_k}, Sx_{n_k-1}) + d(Sx_{n_k-1}, Sx_{n_k}) \\ &< \varepsilon + \beta_{n_k-1}, \end{aligned} \tag{3.70}$$

it follows from (3.68) that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{3.71}$$

Notice also that

$$\begin{aligned} \varepsilon \le r_k &\le d(Sx_{m_k}, Sx_{m_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) + d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &= \beta_{m_k} + \beta_{n_k} + d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &\le \beta_{m_k} + \beta_{n_k} + \psi(d(Sx_{m_k}, Sx_{n_k})). \end{aligned} \tag{3.72}$$

Taking $k \to \infty$ in previous inequality, by (3.68), (3.71), and property of $\psi$, we get $\varepsilon \le \psi(\varepsilon)$. Hence, $\varepsilon = 0$, which is a contradiction. So we obtain the claim and then it converges to some $y \in B$. Therefore, we can conclude that

$$d(gx, y) = \lim_{n\to\infty} d(gx_{n+1}, Sx_n) = d(A, B). \tag{3.73}$$

That is $gx \in A_0$. Since $A_0 \subseteq g(A_0)$, we have $gx = gz$ for some $z \in A_0$ and then $d(gx, gz) = 0$. By the fact that $g$ is an isometry, we have $d(x, z) = d(gx, gz) = 0$. Hence $x = z$ and so $x$ becomes to a point in $A_0$. As $S(A_0) \subseteq B_0$ that

$$d(u, Sx) = d(A, B) \tag{3.74}$$

for some $u \in A$. It follows from (3.63) and (3.74) that $S$ is a generalized proximal $\psi$-contraction of the first kind that

$$d(u, gx_{n+1}) \leq \psi(d(x, x_n)) \tag{3.75}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$, we get the sequence $\{gx_n\}$ converging to a point $u$. By the fact that $g$ is continuous, we have

$$gx_n \longrightarrow gx \quad \text{as } n \longrightarrow \infty. \tag{3.76}$$

By the uniqueness of limit of the sequence, we conclude that $u = gx$. Therefore, it results that $d(gx, Sx) = d(u, Sx) = d(A, B)$. The uniqueness and the remaining part of the proof follow as in Theorem 3.3. This completes the proof of the theorem. □

If $g$ is assumed to be the identity mapping, then by Theorem 3.10, we obtain the following corollary.

**Corollary 3.11.** *Let $(X, d)$ be a complete metric space and let $A$ and $B$ be nonempty, closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$ satisfy the following conditions:*

(a) *$S$ is a generalized proximal $\psi$-contraction of first and second kinds;*

(b) *$S(A_0) \subseteq B_0$.*

*Then, there exists a unique point $x \in A$ such that*

$$d(x, Sx) = d(A, B). \tag{3.77}$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Sx_n) = d(A, B), \tag{3.78}$$

*converges to the best proximity point $x$ of $S$.*

If we take $\psi(t) = \alpha t$, where $0 \leq \alpha < 1$ in Theorem 3.10, we obtain following corollary.

**Corollary 3.12** (see [37, Theorem 3.4]). *Let $(X, d)$ be a complete metric space and let $A$ and $B$ be non-empty, closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are non-empty. Let $S : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(a) *$S$ is a proximal contraction of first and second kinds;*

(b) *$g$ is an isometry;*

(c) *$S$ preserves isometric distance with respect to $g$;*

(d) $S(A_0) \subseteq B_0$;

(e) $A_0 \subseteq g(A_0)$.

*Then, there exists a unique point $x \in A$ such that*

$$d(gx, Sx) = d(A, B). \tag{3.79}$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B), \tag{3.80}$$

*converges to the element $x$.*

If $g$ is assumed to be the identity mapping in Corollary 3.12, we obtain the following corollary.

**Corollary 3.13.** *Let $(X, d)$ be a complete metric space and let $A$ and $B$ be non-empty, closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are non-empty. Let $S : A \to B$ satisfy the following conditions:*

(a) *$S$ is a proximal contraction of first and second kinds;*

(b) $S(A_0) \subseteq B_0$.

*Then, there exists a unique point $x \in A$ such that*

$$d(x, Sx) = d(A, B). \tag{3.81}$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Sx_n) = d(A, B), \tag{3.82}$$

*converges to the best proximity point $x$ of $S$.*

## Acknowledgments

W. Sanhan would like to thank Faculty of Liberal Arts and Science, Kasetsart University, Kamphaeng-Saen Campus and Centre of Excellence in Mathematics, CHE, Sriayudthaya Rd., Bangkok, Thailand. C. Mongkolkeha was supported from the Thailand Research Fund through the the Royal Golden Jubilee Ph.D. Program (Grant no. PHD/0029/2553). P. Kumam was supported by the Commission on Higher Education, the Thailand Research Fund, and the King Mongkuts University of Technology Thonburi (Grant no. MRG5580213).

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# On $\mathcal{P}$-contractions in ordered metric spaces

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**Abstract**

In this paper, we introduced a new type of a contractive condition defined on an ordered space, namely a $\mathcal{P}$-contraction, which generalizes the weak contraction. We also proved some fixed point theorems for such a condition in ordered metric spaces. A supporting example of our results is provided in the last part of our paper as well.

**MSC:** 06A05; 06A06; 47H09; 47H10; 54H25

**Keywords:** $\mathcal{P}$-function; $\mathcal{P}$-contraction; metric space; partially ordered set

## 1 Introduction and preliminaries

It is well known that the Banach contraction principle has been improved in different directions in different spaces by mathematicians over the years. Even in the contemporary research, it remains a heavily investigated branch. Thus, several authors have generalized the principle in various ways (see, for example, [1–14]).

In 1997, Alber and Guerre-Delabriere [15] have introduced the concept of weak contraction in Hilbert spaces. Later, Rhoades [16] showed, in 2001, that these results are also valid in complete metric spaces. We state the result of Rhoades in the following.

A mapping $f : X \to X$, where $(X, d)$ is a metric space, is said to be weakly contractive if

$$d(fx, fy) \le d(x, y) - \varphi\big(d(x, y)\big) \tag{1.1}$$

for all $x, y \in X$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a function satisfying:

(i) $\varphi$ is continuous and nondecreasing;
(ii) $\varphi(t) = 0$ if and only if $t = 0$;
(iii) $\lim_{t\to+\infty} \varphi(t) = +\infty$.

Note that (1.1) reduces to an ordinary contraction when $\varphi(t) := kt$, where $0 \le k < 1$.

**Theorem 1.1** ([16]) *Let $(X, d)$ be a complete metric space and $f$ be a weakly contractive mapping. Then $f$ has a unique fixed point $x^*$ in $X$.*

An interesting way to generalize this theorem is to consider it in case a partial ordering is defined on the space. Recall that a relation $\sqsubseteq$ is a partial ordering on a set $X$ if it is reflexive, antisymmetric and transitive. By this meaning, we write $b \sqsupseteq a$ instead of $a \sqsubseteq b$ to emphasize some particular cases. Any $a, b \in X$ are said to be *comparable* if $a \sqsupseteq b$ or $a \sqsubseteq b$. If a set $X$ has a partial ordering $\sqsubseteq$ defined on it, we say that it is a *partially ordered set* (*w.r.t.* $\sqsubseteq$) and denote it by $(X, \sqsubseteq)$. $(X, \sqsubseteq)$ is said to be a *totally ordered set* if any two elements in $X$ are comparable. Moreover, it is said to be a *sequentially ordered set* if each element of a convergent sequence in $X$ is comparable with its limit. Yet, if $(X, d)$ is a metric space and $(X, \sqsubseteq)$ is a partially ordered (totally ordered, sequentially ordered) set, we say that $X$ is a *partially ordered (totally ordered, sequentially ordered, respectively) metric space*, and it will be denoted by $(X, \sqsubseteq, d)$.

In 2009, Harjani and Sadarangani [17] carried the work of Rhoades [16] into partially ordered metric spaces. We now state the result proved in [17] as follows.

**Theorem 1.2** ([17]) *Let $(X, \sqsubseteq, d)$ be a complete partially ordered metric space and let $f : X \to X$ be a continuous and nondecreasing mapping such that*

$$d(fx, fy) \le d(x, y) - \varphi\big(d(x, y)\big)$$

*for $x \sqsubseteq y$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a function satisfying*:

(i) *$\varphi$ is continuous and nondecreasing*;
(ii) *$\varphi(t) = 0$ if and only if $t = 0$*;
(iii) $\lim_{t\to+\infty} \varphi(t) = +\infty$.

*If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then $f$ has a fixed point.*

Harjini and Sadarangani [17] also proved fixed point theorems for noncontinuous mappings, nonincreasing mappings and even for non-monotonic mappings.

The aim of this paper is to introduce a weak condition which resulted in the concept called a $\mathcal{P}$*-contraction*.

## 2 $\mathcal{P}$-functions

In this section, we introduce our concept of a $\mathcal{P}$-function and some of its fundamental properties. Not to be ambiguous, we assume that $\mathbb{R}$ represents the set of all real numbers while $\mathbb{N}$ represents the set of all positive integers.

**Definition 2.1** Let $(X, \sqsubseteq, d)$ be a partially ordered metric space. A function $\varrho : X \times X \to \mathbb{R}$ is called a $\mathcal{P}$*-function w.r.t.* $\sqsubseteq$ in $X$ if it satisfies the following conditions:

(i) $\varrho(x, y) \ge 0$ for every comparable $x, y \in X$;
(ii) for any sequences $\{x_n\}_{n=1}^{+\infty}$, $\{y_n\}_{n=1}^{+\infty}$ in $X$ such that $x_n$ and $y_n$ are comparable at each $n \in \mathbb{N}$, if $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} y_n = y$, then $\lim_{n\to+\infty} \varrho(x_n, y_n) = \varrho(x, y)$;
(iii) for any sequences $\{x_n\}_{n=1}^{+\infty}$, $\{y_n\}_{n=1}^{+\infty}$ in $X$ such that $x_n$ and $y_n$ are comparable at each $n \in \mathbb{N}$, if $\lim_{n\to+\infty} \varrho(x_n, y_n) = 0$, then $\lim_{n\to+\infty} d(x_n, y_n) = 0$.

If, in addition, the following condition is also satisfied:

(A) for any sequences $\{x_n\}_{n=1}^{+\infty}$, $\{y_n\}_{n=1}^{+\infty}$ in $X$ such that $x_n$ and $y_n$ are comparable at each $n \in \mathbb{N}$, if the limit $\lim_{n\to+\infty} d(x_n, y_n)$ exists, then the limit $\lim_{n\to+\infty} \varrho(x_n, y_n)$ also exists,

then $\varrho$ is said to be a $\mathcal{P}$*-function of type (A) w.r.t.* $\sqsubseteq$ in $X$.

**Example 2.2** Let $(X, \sqsubseteq, d)$ be a partially ordered metric space. Suppose that the function $\varphi : [0, +\infty) \to [0, +\infty)$ is defined as in Theorem 1.2. Then $\varphi \circ d$ is a $\mathcal{P}$-function of type (A) w.r.t. $\sqsubseteq$ in $X$.

**Proposition 2.3** *Let $(X, \sqsubseteq, d)$ be a partially ordered metric space and $\varrho : X \times X \to \mathbb{R}$ be a $\mathcal{P}$-function w.r.t. $\sqsubseteq$ in $X$. If $x, y \in X$ are comparable and $\varrho(x, y) = 0$, then $x = y$.*

*Proof* Let $x, y \in X$ be comparable and $\varrho(x, y) = 0$. Define $\{x_n\}_{n=1}^{+\infty}$ and $\{y_n\}_{n=1}^{+\infty}$ to be two constant sequences in $X$ such that $x_n = x$ and $y_n = y$ for all $n \in \mathbb{N}$. It follows from the definition of a $\mathcal{P}$-function, since $x$ and $y$ are comparable, that $d(x, y) = 0$. That is, $x = y$. □

**Corollary 2.4** *Let $(X, \sqsubseteq, d)$ be a totally ordered metric space and $\varrho : X \times X \to \mathbb{R}$ be a $\mathcal{P}$-function w.r.t. $\sqsubseteq$ in $X$. If $x, y \in X$ and $\varrho(x, y) = 0$, then $x = y$.*

*Proof* Since $X$ is totally ordered, any $x, y \in X$ are comparable. The rest of the proof is straightforward. □

**Example 2.5** Let $X = \mathbb{R}$. Define $d, \varrho : X \times X \to \mathbb{R}$ with $d(x, y) = |x - y|$ and $\varrho(x, y) = 1 + |x - y|$. If $X$ is endowed with a usual ordering $\leq$, then $(X, \leq, d)$ is a totally ordered metric space with $\varrho$ as a $\mathcal{P}$-function of type (A) w.r.t. $\leq$ in $X$. Note that $\varrho(x, y) \neq 0$ for all $x, y \in X$, even when $x = y$.

This example shows that the converse of Proposition 2.3 and that of Corollary 2.4 are not generally true.

**Definition 2.6** Let $(X, \sqsubseteq, d)$ be a partially ordered metric space, a mapping $f : X \to X$ is called a *$\mathcal{P}$-contraction w.r.t.* $\sqsubseteq$ if there exists a $\mathcal{P}$-function $\varrho : X \times X \to \mathbb{R}$ w.r.t. $\sqsubseteq$ in $X$ such that

$$d(fx, fy) \leq d(x, y) - \varrho(x, y) \tag{2.1}$$

for any comparable $x, y \in X$. Naturally, if there exists a $\mathcal{P}$-function of type (A) w.r.t. $\sqsubseteq$ in $X$ such that the inequality (2.1) holds for any comparable $x, y \in X$, then $f$ is said to be a *$\mathcal{P}$-contraction of type (A) w.r.t.* $\sqsubseteq$.

**Remark 2.7** From Example 2.2, it follows that in partially ordered metric spaces, a weak contraction is also a $\mathcal{P}$-contraction.

## 3 Fixed point results

### 3.1 Fixed point theorems for monotonic mappings

**Theorem 3.1** *Let $(X, \sqsubseteq, d)$ be a complete partially ordered metric space and $f : X \to X$ be a continuous and nondecreasing $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of $f$ in $X$.*

*Proof* For the existence of the fixed point, we choose $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$. If $fx_0 = x_0$, then the proof is finished. Suppose that $fx_0 \neq x_0$. We define a sequence $\{x_n\}_{n=1}^{+\infty}$ such that $x_n = f^n x_0$. Since $x_0 \sqsubseteq fx_0$ and $f$ is nondecreasing w.r.t. $\sqsubseteq$, we obtain

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots .$$

If there exists $n_0 \in \mathbb{N}$ such that $\varrho(x_{n_0}, x_{n_0+1}) = d(x_{n_0}, x_{n_0+1})$, then by the notion of $\mathcal{P}$-contractivity, the proof is finished. Therefore, we assume that $\varrho(x_n, x_{n+1}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Also, assume that $\varrho(x_n, x_{n+1}) \neq 0$ for all $n \in \mathbb{N}$. Otherwise, we can find $n_0 \in \mathbb{N}$ with $x_{n_0} = x_{n_0+1}$, that is, $x_{n_0} = fx_{n_0}$, and the proof is finished. Hence, we consider only the case where $0 < \varrho(x_n, x_{n+1}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

Since $x_n \sqsubseteq x_{n+1}$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq d(x_{n-1}, x_n) - \varrho(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, we have $\{d(x_n, x_{n+1})\}_{n=1}^{+\infty}$ nonincreasing. Since $\{d(x_n, x_{n+1})\}_{n=1}^{+\infty}$ is bounded, there exists $l \geq 0$ such that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = l$. Thus, there exists $q \geq 0$ such that $\lim_{n\to+\infty} \varrho(x_n, x_{n+1}) = q$.

Assume that $l > 0$. Then, by the $\mathcal{P}$-contractivity of $f$, we have

$$l \leq l - q.$$

Hence, $q = 0$, which implies that $l = 0$, a contradiction. Therefore, we have

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{3.1}$$

Now we show that $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence in $X$. Assume the contrary. Then there exists $\epsilon_0 > 0$ for which we can define subsequences $\{x_{m_k}\}_{k=1}^{+\infty}$ and $\{x_{n_k}\}_{k=1}^{+\infty}$ of $\{x_n\}_{n=1}^{+\infty}$ such that $n_k$ is minimal in the sense that $n_k > m_k > k$ and $d(x_{m_k}, x_{n_k}) \geq \epsilon_0$. Therefore, $d(x_{m_k}, x_{n_k-1}) < \epsilon_0$. Observe that

$$\begin{aligned} \epsilon_0 &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \epsilon_0 + d(x_{n_k-1}, x_{n_k}). \end{aligned}$$

Letting $k \to +\infty$, we obtain $\epsilon_0 \leq \lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) \leq \epsilon_0$ and so

$$\lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) = \epsilon_0. \tag{3.2}$$

By the two following inequalities:

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})$$

and

$$d(x_{m_k-1}, x_{n_k-1}) \leq d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}),$$

we can apply (3.1) and (3.2) to obtain

$$\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon_0. \tag{3.3}$$

Furthermore, we deduce that the limit $\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1})$ also exists. Now, by the $\mathcal{P}$-contractivity, we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k-1}, x_{n_k-1}) - \varrho(x_{m_k-1}, x_{n_k-1}).$$

From (3.2) and (3.3), we may find that

$$0 \leq -\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1}),$$

which further implies that $\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1}) = 0$. Notice that $x_{m_k-1} \sqsubseteq x_{n_k-1}$ at each $k \in \mathbb{N}$. Consequently, we obtain that $\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = 0$, which is a contradiction. So, $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence. Since $X$ is complete, there exists $x^*$ such that $x_n = f^n x_0 \to x^*$ as $n \to +\infty$. Finally, the continuity of $f$ and $ff^n x_0 = f^{n+1}x_0 \to x^*$ imply that $fx^* = x^*$. Therefore, $x^*$ is a fixed point of $f$. □

**Remark 3.2** In the setting of Remark 2.7, Theorem 3.1 reduces to Theorem 1.2 of [17].

Next, we drop the continuity of $f$ in the Theorem 3.1, and find out that we can still guarantee a fixed point if we strengthen the condition of a partially ordered set to a sequentially ordered set.

**Theorem 3.3** *Let $(X, \sqsubseteq, d)$ be a complete sequentially ordered metric space and $f : X \to X$ be a nondecreasing $\mathcal{P}$-contraction of type $(A)$ w.r.t. $\sqsubseteq$. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of $f$ in $X$.*

*Proof* If we take $x_n = f^n x_0$ in the proof of Theorem 3.1, then we conclude that $\{x_n\}_{n=1}^{+\infty}$ converges to a point $x^*$ in $X$.

Next, we prove that $x^*$ is a fixed point of $f$ in $X$. Indeed, suppose that $x^*$ is not a fixed point of $f$, *i.e.*, $d(x^*, fx^*) \neq 0$. Since $x^*$ is comparable with $x_n$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x^*, fx^*) &\leq d(x^*, fx_n) + d(fx^*, fx_n) \\ &\leq d(x^*, fx_n) + d(x^*, x_n) - \varrho(x^*, x_n) \\ &\leq d(x^*, fx_n) + d(x^*, x_n) \\ &= d(x^*, x_{n+1}) + d(x^*, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. By the definition of a convergent sequence, we have, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x^*) < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$ with $n \geq N$. Therefore, we have

$$\begin{aligned} d(x^*, fx^*) &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

As easily seen, $d(x^*, fx^*)$ is less than any nonnegative real number, so $d(x^*, fx^*) = 0$, which is a contradiction. Hence, $x^*$ is a fixed point of $f$. □

**Corollary 3.4** *Let $(X, \sqsubseteq, d)$ be a complete totally ordered metric space and $f : X \to X$ be a nondecreasing $\mathcal{P}$-contraction of type $(A)$ w.r.t. $\sqsubseteq$. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of $f$ in $X$.*

*Proof* Take $x_n = f^n x_0$ as in the proof of Theorem 3.1. Since the total ordering implies the partial ordering, we conclude that $\{x_n\}_{n=1}^{+\infty}$ converges to a fixed point.

Next, we show that the fixed point of $f$ is unique. Assume that $u$ and $v$ are two distinct fixed points of $f$, *i.e.*, $d(u,v) \neq 0$. Since $X$ is totally ordered, $u$ and $v$ are comparable. Thus, we have

$$\begin{aligned} d(u,v) &= d(fu,fv) \\ &\leq d(u,v) - \varrho(u,v), \end{aligned} \tag{3.4}$$

which is a contradiction. Therefore, $u = v$ and the fixed point of $f$ is unique. □

We can still guarantee the uniqueness of a fixed point by weakening the total ordering condition as stated and proved in the next theorem.

**Theorem 3.5** *Let $(X,\sqsubseteq,d)$ be a complete partially ordered metric space and $f : X \to X$ be a continuous and nondecreasing $\mathcal{P}$-contraction of type $(A)$ w.r.t. $\sqsubseteq$. Suppose that for each $x,y \in X$, there exists $w \in X$ which is comparable to both $x$ and $y$. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of $f$ in $X$.*

*Proof* If we take $x_n = f^n x_0$ in the proof of Theorem 3.1, then we conclude that $x_n$ converges to a fixed point of $f$ in $X$.

Next, we show that the fixed point of $f$ is unique. Assume that $u$ and $v$ are two distinct fixed points of $f$, *i.e.*, $d(u,v) \neq 0$. Since $u,v \in X$, there exists $w \in X$ such that $w$ is comparable to both $u$ and $v$. We will prove this part by showing that the sequence $\{w_n\}_{n=1}^{+\infty}$ given by $w_n = f^n w$ converges to both $u$ and $v$. Therefore, we have

$$\begin{aligned} d\big(u,f^n w\big) &\leq d\big(u,f^{n-1}w\big) - \varrho\big(u,f^{n-1}w\big) \\ &\leq d\big(u,f^{n-1}w\big). \end{aligned} \tag{3.5}$$

If we define a sequence $y_n = d(u,f^n w)$ and $z_n = \varrho(u,f^n w)$, we may obtain from (3.5) that $\{y_n\}_{n=1}^{+\infty}$ is nonincreasing and there exist $l,q \geq 0$ such that $\lim_{n\to+\infty} y_n = l$ and $\lim_{n\to+\infty} z_n = q$.

Assume that $l > 0$. Then, by the $\mathcal{P}$-contractivity of $f$, we have

$$l \leq l - q,$$

which is a contradiction. Hence, $\lim_{n\to+\infty} y_n = 0$. In the same way, we can also show that $\lim_{n\to+\infty} d(v,f^n w) = 0$. That is, $\{w_n\}_{n=1}^{+\infty}$ converges to both $u$ and $v$. Since the limit of a convergent sequence in a metric space is unique, we conclude that $u = v$. Hence, this yields the uniqueness of the fixed point. □

**Theorem 3.6** *Let $(X,\sqsubseteq,d)$ be a complete sequentially ordered metric space and $f : X \to X$ be a nondecreasing $\mathcal{P}$-contraction of type $(A)$ w.r.t. $\sqsubseteq$. Suppose that for each $x,y \in X$, there exists $w \in X$ which is comparable to both $x$ and $y$. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of $f$ in $X$.*

*Proof* If we take $x_n = f^n x_0$ in the proof of Theorem 3.1, then we conclude that $x_n$ converges to a fixed point of $f$ in $X$. The rest of the proof is similar to the proof of Theorem 3.5. □

**Remark 3.7** In parallel with the study of Theorems 3.1, 3.3, 3.5 and 3.6, we can also prove in the same way that if the mapping $f$ is nonincreasing, the above theorems still hold. However, we will omit the result for nonincreasing mappings.

### 3.2 Fixed point theorems for mappings with the lack of monotonicity

In this section, we drop the monotonicity conditions of $f$ and find out that we can still apply our results to confirm the existence and uniqueness of a fixed point of $f$.

**Theorem 3.8** *Let $(X, \sqsubseteq, d)$ be a complete partially ordered metric space and $f : X \to X$ be a continuous $\mathcal{P}$-contraction of type $(A)$ w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. If there exists $x_0 \in X$ such that $x_0$ and $fx_0$ are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of $f$ in $X$.*

*Proof* For the existence of the fixed point, we choose $x_0 \in X$ such that $x_0$ and $fx_0$ are comparable. If $fx_0 = x_0$, then the proof is finished. Suppose that $fx_0 \neq x_0$. We define a sequence $\{x_n\}_{n=1}^{+\infty}$ such that $x_n = f^n x_0$. Since $x_0$ and $fx_0$ are comparable, we have $x_n$ and $x_{n+1}$ comparable for all $n \in \mathbb{N}$.

If there exists $n_0 \in \mathbb{N}$ such that $\varrho(x_{n_0}, x_{n_0+1}) = d(x_{n_0}, x_{n_0+1})$, then by the notion of $\mathcal{P}$-contractivity, the proof is finished. Therefore, we assume that $\varrho(x_n, x_{n+1}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Also, assume that $\varrho(x_n, x_{n+1}) \neq 0$ for all $n \in \mathbb{N}$. Otherwise, we can find $n_0 \in \mathbb{N}$ with $x_{n_0} = x_{n_0+1}$, that is, $x_{n_0} = fx_{n_0}$, and the proof is finished. Hence, we consider only the case where $0 < \varrho(x_n, x_{n+1}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

Since $x_n$ and $x_{n+1}$ are comparable for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq d(x_{n-1}, x_n) - \varrho(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, we have $\{d(x_n, x_{n+1})\}_{n=1}^{+\infty}$ is nonincreasing. Since $\{d(x_n, x_{n+1})\}_{n=1}^{+\infty}$ is bounded, there exists $l \geq 0$ such that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = l$. Thus, there exists $q \geq 0$ such that $\lim_{n\to+\infty} \varrho(x_n, x_{n+1}) = q$.

Assume that $l > 0$. Then, by the $\mathcal{P}$-contractivity of $f$, we have

$$l \leq l - q.$$

Hence, $q = 0$, which implies that $l = 0$, a contradiction. Hence, $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$.

Now we show that $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence in $X$. Assume the contrary. Then there exists $\epsilon_0 > 0$ for which we can define subsequences $\{x_{m_k}\}_{k=1}^{+\infty}$ and $\{x_{n_k}\}_{k=1}^{+\infty}$ of $\{x_n\}_{n=1}^{+\infty}$ such that $n_k$ is minimal in the sense that $n_k > m_k > k$ and $d(x_{m_k}, x_{n_k}) \geq \epsilon_0$. Therefore, $d(x_{m_k}, x_{n_k-1}) < \epsilon_0$. Observe that

$$\begin{aligned} \epsilon_0 &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \epsilon_0 + d(x_{n_k-1}, x_{n_k}). \end{aligned}$$

Letting $k \to +\infty$, we obtain $\epsilon_0 \leq \lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) \leq \epsilon_0$ and so

$$\lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) = \epsilon_0. \tag{3.6}$$

By the two following inequalities:

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})$$

and

$$d(x_{m_k-1}, x_{n_k-1}) \leq d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}),$$

we can apply the fact that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$ and (3.6) to obtain

$$\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon_0. \tag{3.7}$$

Furthermore, we deduce that the limit $\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1})$ also exists. Now, by the $\mathcal{P}$-contractivity, we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k-1}, x_{n_k-1}) - \varrho(x_{m_k-1}, x_{n_k-1}).$$

From (3.6) and (3.7), we may find that

$$0 \leq -\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1}),$$

which further implies that $\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1}) = 0$. Notice that $x_{m_k-1} \sqsubseteq x_{n_k-1}$ at each $k \in \mathbb{N}$. Consequently, we obtain that $\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = 0$, which is a contradiction. So, $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence. Since $X$ is complete, there exists $x^*$ such that $x_n = f^n x_0 \to x^*$ as $n \to +\infty$. Finally, the continuity of $f$ and $ff^n x_0 = f^{n+1} x_0 \to x^*$ imply that $fx^* = x^*$. Therefore, $x^*$ is a fixed point of $f$. □

Further results can be proved using the same plots as those of the earlier theorems in this paper, so we omit them.

**Theorem 3.9** *Let $(X, \sqsubseteq, d)$ be a complete sequentially ordered metric space and $f : X \to X$ be a $\mathcal{P}$-contraction of type $(A)$ w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. If there exists $x_0 \in X$ such that $x_0$ and $fx_0$ are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of $f$ in $X$.*

**Corollary 3.10** *Let $(X, \sqsubseteq, d)$ be a complete totally ordered metric space with and $f : X \to X$ be a $\mathcal{P}$-contraction of type $(A)$ w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. If there exists $x_0 \in X$ such that $x_0$ and $fx_0$ are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of $f$ in $X$.*

**Theorem 3.11** *Let $(X, \sqsubseteq, d)$ be a complete partially ordered metric space and $f : X \to X$ be a continuous $\mathcal{P}$-contraction of type $(A)$ w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. Suppose that for each $x, y \in X$, there exists $w \in$*

*$X$ which is comparable to both $x$ and $y$. If there exists $x_0 \in X$ such that $x_0$ and $fx_0$ are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of $f$ in $X$.*

**Theorem 3.12** *Let $(X, \sqsubseteq, d)$ be a complete sequentially ordered metric space and $f : X \to X$ be a $\mathcal{P}$-contraction of type $(A)$ w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. Suppose that for each $x, y \in X$, there exists $w \in X$ which is comparable to both $x$ and $y$. If there exists $x_0 \in X$ such that $x_0$ and $fx_0$ are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of $f$ in $X$.*

## 4 Example

We give an example to ensure the applicability of our theorems.

**Example 4.1** Let $X = [0,1] \times [0,1]$ and suppose that we write $x = (x_1, x_2)$ and $y = (y_1, y_2)$ for $x, y \in X$.

Define $d, \varrho : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 2\max\{x_1 + y_1, x_2 + y_2\} & \text{otherwise} \end{cases}$$

and

$$\varrho(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{x_1, x_2 + y_2\} & \text{otherwise.} \end{cases}$$

Let $\sqsubseteq$ be an ordering in $X$ such that for $x, y \in X$, $x \sqsubseteq y$ if and only if $x_1 = y_1$ and $x_2 \leq y_2$. Then $(X, \sqsubseteq, d)$ is a partially ordered metric space with $\varrho$ as a $\mathcal{P}$-function of type (A) w.r.t. $\sqsubseteq$ in $X$.

Now, let $f$ be a self mapping on $X$ defined by $fx = f((x_1, x_2)) = (0, \frac{x_2^2}{2})$ for all $x \in X$. It is obvious that $f$ is continuous and nondecreasing w.r.t. $\sqsubseteq$.

Let $x, y \in X$ be comparable w.r.t. $\sqsubseteq$. If $x = y$, then they clearly satisfy the inequality (2.1). On the other hand, if $x \neq y$, we have

$$\begin{aligned} d(fx, fy) &= d\big(f\big((x_1, x_2)\big), f\big((y_1, y_2)\big)\big) \\ &= d\left(\left(0, \frac{x_2^2}{2}\right), \left(0, \frac{y_2^2}{2}\right)\right) \\ &= 2\max\left\{0, \frac{x_2^2}{2} + \frac{y_2^2}{2}\right\} \\ &= x_2^2 + y_2^2 \\ &\leq x_2 + y_2 \\ &\leq \max\{2x_1, x_2 + y_2\} \\ &= 2\max\{2x_1, x_2 + y_2\} - \max\{2x_1, x_2 + y_2\} \\ &\leq 2\max\{2x_1, x_2 + y_2\} - \max\{x_1, x_2 + y_2\} \\ &= d(x,y) - \varrho(x,y). \end{aligned}$$

Therefore, the inequality (2.1) is satisfied for every comparable $x, y \in X$. So, $f$ is a continuous and nondecreasing $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$. Let $x_0 = (0, 0)$, so we have $x_0 \sqsubseteq fx_0$. Now, applying Theorem 3.1, we conclude that $f$ has a fixed point in $X$ which is the point $(0, 0)$.

## 5 Conclusion

It is undeniable that Rhoades's weak contraction is one of the earliest and the most important extensions of the contraction principle. The results in this paper give a new direction to expanding the framework of contractive type mappings in metric spaces. Still, there is a question to be raised from this paper onwards.

**Question** Are our results still true for any $\mathcal{P}$-contractions (not necessarily of type (A))?

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed significantly in writing this paper. They have also read and approved the final manuscript.

**Acknowledgements**
This work was partially supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission (NRU-CSEC No. 55000613). The second author would like to thank the Research Professional Development Project under the Science Achievement Scholarship of Thailand (SAST). Also, the third author was supported by the Commission on Higher Education, the Thailand Research Fund and the King Mongkut's University of Technology Thonburi (Grant No. MRG5580213).

Received: 19 March 2012 Accepted: 16 November 2012 Published: 5 December 2012

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doi:10.1186/1687-1812-2012-219
**Cite this article as:** Chaipunya et al.: **On $\mathcal{P}$-contractions in ordered metric spaces.** *Fixed Point Theory and Applications* 2012 **2012**:219.


ISSN: 1110-256X
JOURNAL
OF THE
EGYPTIAN
MATHEMATICAL
SOCIETY
Vol. 21 (2)
2013
July 2013
www.etms-eg.org
EGYPTIAN MATHEMATICAL SOCIETY
CAIRO
Available online at www.sciencedirect.com
SciVerse ScienceDirect

Journal of the Egyptian Mathematical Society (2013) **21**, 13–16

Egyptian Mathematical Society

**Journal of the Egyptian Mathematical Society**

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

# Best proximity points for asymptotic proximal pointwise weaker Meir–Keeler-type $\psi$-contraction mappings

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Received 1 December 2012; accepted 23 December 2012
Available online 20 February 2013

**KEYWORDS**

Best proximity point
Property *UC*
Asymptotic proximal pointwise weaker Meir–Keeler-type $\psi$-contraction

**Abstract** In this paper, we study the new class of an asymptotic proximal pointwise weaker Meir–Keeler-type $\psi$-contraction and prove the existence of solutions for the minimization problem in a uniformly convex Banach space. Also, we give some an example for support our main result.

**2000 MATHEMATICS SUBJECT CLASSIFICATION:** 47H09, 47H10



## 1. Introduction and preliminaries

The best proximity theorem furnishes sufficient conditions for the existence of an optimal approximate solution $x$, known as the best proximity point of the non-self mapping $T$, satisfying the condition that $d(x, Tx) = dist(A, B)$. Interestingly, the best proximity theorems also serve as a natural generalization of fixed point theorems. Indeed, the best proximity point becomes a fixed point if the mapping under consideration is a self-mapping. On the other hand, though the best proximity theorems ensure the existence of approximate solutions, such results need not yield optimal solutions. But, best proximity point theorems furnish sufficient conditions that assure the existence of approximate solutions which are optimal as well.

The classical and well-known Banach's contraction principle states that if a self-mapping $T$ of a complete metric space $X$ is a contraction mapping (i.e., $d(Tx, Ty) \leqslant \alpha d(x, y)$ for all $x, y \in X$, where $\alpha \in [0, 1)$), then $T$ has a unique fixed point. This principle has been extended in several ways such as [1–6]. In 2003, Kirk, Srinivasan, and Veeramani [7] extended the Banach's contraction principle to case of cyclic mappings. Let $(X, d)$ be a metric space and let $A$, $B$, be a non-empty subset of $X$. A mapping $T$: $A \cup B \to A \cup B$ is called a *cyclic mapping* if $T(A) \subset B$ and $T(B) \subset A$. A point $x \in A$ is called a *best proximity point* of $T$ in $A$ if $d(x, Tx) = \text{dist}(A, B)$, where $\text{dist}(A, B) = \inf\{d(x, y): \ x \in A, y \in B\}$. A cyclic mapping $T$: $A \cup B \to A \cup B$ is said to be a *relatively non-expansive* if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x \in A$ and $y \in B$ (notice that a relatively non-expansive mapping need not be a continuous in general). In 2005, Eldred, Kirk and Veeramani [8] proved the existence of a best proximity point for relatively non-expansive

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2012.12.002

mappings by using the notion of proximal normal structure. In 2006, Eldred and Veeramani [9] introduced the notion called *cyclic contraction* and gave sufficient condition for the existence of a best proximity point for a cyclic contraction mapping $T$ on a uniformly convex Banach space. In 2009, Suzuki et al. [10] introduced the notion of the property $UC$ as follow :

**Definition 1.1** 10. Let A and B be non-empty subsets of a metric space $(X,d)$. Then $(A,B)$ is said to be satisfy the property $UC$, if the following holds: If $\{x_n\}$ and $\{\acute{x}_n\}$ are sequences in $A$ and $\{y_n\}$ is a sequence in $B$ such that

$$\lim_{n\to\infty} d(x_n, y_n) = dist(A,B) \text{ and } \lim_{n\to\infty} d(\acute{x}_n, y_n) = dist(A,B),$$

then $\lim_{n\to\infty} d(x_n, \acute{x}_n) = 0$.

Also, they extended the result in [9] to metric spaces with the property $UC$. The following lemma plays an important role in next sections;

**Lemma 1.2** 10. *Let A and B be subsets of a metric space $(X,d)$. Assume that $(A,B)$ has the property UC. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and B, respectively, such that either of the following holds:*

$$\lim_{m\to\infty} \sup_{n\geqslant m} d(x_m, y_n) = dist(A,B) \text{ or } \lim_{n\to\infty} \sup_{m\geqslant n} d(x_m, y_n) = dist(A,B).$$

*Then $\{x_n\}$ is Cauchy.*

On the other hand, in 2003, Kirk [11], introduced the notion of an asymptotic contraction mapping as follows:

**Definition 1.3** 11. Let $(X,d)$ be a metric space. A mapping $T$: $X\to X$ is said to be an asymptotic contraction if

$$d(T^n(x), T^n(y)) \leqslant \phi_n(d(x,y)) \quad \text{for all } x, y \in X,$$

where $\phi_n$: $[0,\infty)\to[0,\infty)$ and $\phi_n \to \phi$ uniformly on the range of $d$ in which $\phi$: $[0,\infty)\to[0,\infty)$ is continuous and $\phi(s) < s$ for all $s > 0$.

In 2007, Kirk [12], introduced the notion of an asymptotic pointwise contraction mapping as follows:

**Definition 1.4** 12. Let $(X,d)$ be a metric space. A mapping $T$: $X\to X$is said to be an asymptotic pointwise contraction if there exists a sequence of functions $\alpha_n : X \to \mathbb{R}^+$ such that $\alpha_n \to \alpha$ pointwise on $X$ and for each integer $n \geqslant 1$,

$$d(T^n(x), T^n(y)) \leqslant \alpha_n(x)(d(x,y)) \quad \text{for all } x, y \in X.$$

In 2008, Kirk and Xu [13], introduced the notion of a pointwise asymptotically non-expansive mapping as follows:

**Definition 1.5** 13. Let $K$ be a non-empty subset of Banach space $X$. A mapping $T$: $K\to K$ is said to be a pointwise asymptotically non-expansive, if for each integer $n \geqslant 1$,

$$\|T^n(x) - T^n(y)\| \leqslant \alpha_n(x)\|x-y\| \quad \text{for all } x, y \in K,$$

where $\alpha_n \to 1$ pointwise on $K$.

In 2009, Anuradha and Veeramani in [14] introduced a new class of mappings; they called each mapping of this class a proximal pointwise contraction:

**Definition 1.6** 14. Let A and B be non-empty subsets of a metric space $(X,d)$. Let $T$: $A\cup B\to A\cup B$ be a cyclic mapping. The mapping $T$ is said to be a proximal pointwise contraction if for each $(x,y)\in A\times B$ there exist $0\leqslant \alpha(x) < 1$, $0 \leqslant \alpha(y) < 1$ such that

$$d(T(x), T(y)) \leqslant \max\{\alpha(x)d(x,y), dist(A,B)\} \quad \text{for all } y \in B,$$
$$d(T(x), T(y)) \leqslant \max\{\alpha(y)d(x,y), dist(A,B)\} \quad \text{for all } x \in A.$$

Recently, Abkar and Gabeleh [15] introduced a new notion of an asymptotic proximal pointwise contraction mapping as follows:

**Definition 1.7** 15. Let $(A,B)$ be a non-empty pair in a Banach space $X$. A mapping $T$: $A\cup B\to A\cup B$ is said to be an asymptotic proximal pointwise contraction if $T$ is cyclic and there exists a function $\alpha$: $A\cup B\to[0,1)$ such that for any integer $n\geqslant 1$ and $(x,y)\in A\times B$,

$$\|T^{2n}x - T^{2n}y\| \leqslant \max\{\alpha_n(x)\|x-y\|, dist(A,B)\} \quad \text{for all } y \in B,$$
$$\|T^{2n}x - T^{2n}y\| \leqslant \max\{\alpha_n(y)\|x-y\|, dist(A,B)\} \quad \text{for all } x \in A,$$

where $\alpha_n \to \alpha$ pointwise on $A\cup B$.

Just recently, Chen [16] defined the following new notion of the weaker Meir–Keeler-type function and an asymptotic pointwise weaker Meir–Keeler-type contraction, $\mathbb{R}_+$ denoted the set of all non-negative numbers.

**Definition 1.8** 16. The function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a weaker Meir–Keeler-type function, if for each $\eta > 0$, there exists $\delta > \eta$ such that for $t\in\mathbb{R}_+$ with $\eta \leqslant t < \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(t) < \eta$.

**Definition 1.9** 16. Let $X$ be a Banach space, and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a weaker Meir–Keeler-type function. A mapping $T$: $X\to X$ is said to be an asymptotic pointwise weaker Meir–Keeler-type $\psi$-contraction, if for each $n\in\mathbb{N}$,

$$\|T^n x - T^n y\| \leqslant \psi^n(\|x\|)\|x-y\| \quad \text{for all } x, y \in X.$$

For example of a weaker Meir–Keeler-type mapping and a weaker Meir–Keeler-type mapping which is not a Meir–Keeler-type mapping, we can see in [17]. Best proximity point theorems for several types of contractions, for examples see in [18–23].

In this paper, we give the notion of new class of an asymptotic proximal pointwise weaker Meir–Keeler-type $\psi$-contraction and prove the existence of a best proximity point theorem for this mapping. Also, we give some an example for support our main Theorem.

## 2. Asymptotic proximal pointwise weaker Meir–Keeler-type $\psi$-contraction

In this section, we prove the existence of a best proximity point for an asymptotic proximal pointwise weaker Meir–Keeler-type $\psi$-contraction in a uniformly convex Banach space. First, we introduce below notion of an asymptotic proximal pointwise weaker Meir–Keeler-type $\psi$-contraction mapping.

**Definition 2.1.** Let $(A, B)$ be a non-empty pair in Banach space $X$, and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a weaker Meir–Keeler-type function. A mapping $T$: $A \cup B \to A \cup B$ is said to be an asymptotic proximal pointwise weaker Meir–Keeler-type $\psi$-contraction, if for each $n \in \mathbb{N}$ and $(x, y) \in A \times B$,

$$\|T^{2n}x - T^{2n}y\| \leqslant \max\{\psi^n(\|x\|)\|x - y\|, dist(A, B)\}$$

for all $y \in B$,

$$\|T^{2n}x - T^{2n}y\| \leqslant \max\{\psi^n(\|y\|)\|x - y\|, dist(A, B)\}$$

for all $x \in A$.

Before stating the main result, we recall definition and fact of asymptotic centers. Let $X$ be a Banach space, $C$ subset of $X$ and $\{x_n\}$ is a bounded sequence in $X$. The asymptotic centers of $\{x_n\}$ relative to $C$ denoted by $A_C(x_n)$ is the set of minimizers in $A$ (if any) of the function $f$ given by

$$f(x) = \limsup_{n\to\infty}\|x_n - x\|.$$

That is,

$$A_C(x_n) = \{x \in C : f(x) = inf_{u \in C} f(u)\},$$

and we can see that, if $X$ is uniformly convex and $C$ is closed and convex, then $A_C(x_n)$ consists of exactly one point.

**Theorem 2.2.** *Let $(A, B)$ be a non-empty bounded closed convex pair in a uniformly convex Banach space $X$ and $T$: $A \cup B \to A \cup B$ be an asymptotic proximal pointwise weaker Meir–Keeler-type $\psi$-contraction. If $T$ is a relatively non-expansive mapping, then there exists a unique pair $(v_0, u_0) \in A \times B$ such that*

$$\|u_0 - Tu_0\| = \|v_0 - Tv_0\| = dist(A, B).$$

*Moreover, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges in norm to $v_0$ and $\{x_{2n+1}\}$ converges in norm to $u_0$.*

**Proof.** Fix an $x_0 \in A$ and define a function $f$: $B \to [0, \infty)$ by

$$f(u) = \limsup_{n\to\infty}\|T^{2n}(x_0) - u\| \text{ for } u \in B.$$

Since $X$ is uniformly convex and $B$ is bounded closed and convex, it follow that $f$ has unique minimizer over $B$; that is, we have a unique point $u_0 \in B$ satisfying

$$f(u_0) = \inf_{u\in B} f(u).$$

Indeed, for all $m \geqslant 1$ and $u \in B$, we have

$$\begin{aligned} f(T^{2m}(u)) &= \limsup_{n\to\infty}\|T^{2n}(x_0) - T^{2m}u\| \\ &= \limsup_{n\to\infty}\|T^{2n+2m}(x_0) - T^{2m}u\| \\ &= \limsup_{n\to\infty}\|T^{2m}(T^{2n}(x_0)) - T^{2m}u\| \\ &\leqslant \limsup_{n\to\infty}\max\{\psi^m(\|u\|)\|T^{2n}(x_0) \\ &\quad - u\|, \mathrm{dist}(A, B)\} \\ &= \max\{\psi^m(\|u\|)f(u), \mathrm{dist}(A, B)\}. \end{aligned} \tag{2.1}$$

Since $u_0 \in B$ is the minimum of $f$, for all $m \geqslant 1$, we have

$$f(u_0) \leqslant f(T^{2m}u_0) \leqslant \max\{\psi^m(\|u_0\|)f(u_0), \mathrm{dist}(A, B)\}. \tag{2.2}$$

We now claim that $f(u_0) = \mathrm{dist}(A, B)$. Since for each $u \in B$, $\{\psi^m(\|u\|)\}$ is non-increasing, it must converges to some $\eta \geqslant 0$. Suppose that $\eta > 0$, by definition of weaker Meir–Keeler-type function, there exists $\delta > \eta$ such that for $u \in B$ with $\eta \leqslant \|u\| < \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(\|u\|) < \eta$. Since $\lim_{m\to\infty}\psi^m(\|u\|) = \eta$ there exists $m_0 \in \mathbb{N}$ such that $\eta \leqslant \psi^m(\|u\|) < \delta$, for all $m \geqslant m_0$. Thus we conclude that $\psi^{m_0+n_0}(\|u\|) < \eta$, thus we get the contradiction. So

$$\lim_{m\to\infty}\psi^m(\|u\|) = 0. \tag{2.3}$$

Taking $m \to \infty$ in the inequality (2.2),we get

$$f(u_0) = \mathrm{dist}(A, B).$$

On the other hand, by the relatively non-expansive of $T$, we have

$$\begin{aligned} f(T^2u_0) &= \limsup_{n\to\infty}\|(T^{2n}(x_0)) - T^2u_0\| \\ &\leqslant \limsup_{n\to\infty}\|(T^{2n-2}(x_0)) - u_0\| = f(u_0), \end{aligned}$$

which implies that $T^2u_0 = u_0$, by the uniqueness of minimum of $f$, then $u_0$ is a fixed point of $T^2$ in $B$. Hence,

$$\begin{aligned} \lim_{m\to\infty}\sup_{n\geqslant m}\|(T^{2m}(x_0)) - T^{2n}u_0\| &= \lim_{m\to\infty}\|(T^{2m}(x_0)) - u_0\| \\ &= f(u_0) = \mathrm{dist}(A, B). \end{aligned}$$

By the property $UC$ of $(A, B)$, it follows from Lemma 1.2 that $\{T^{2n}(x_0)\}$ is a Cauchy sequence, so there exists $x' \in A$ such that $T^{2n}x_0 \to x'$ as $n \to \infty$. By the similar argument as above, if $y_0 \in B$ and $g$: $A \to [0, \infty)$ is given by $g(v) = \limsup_{n\to\infty}\|T^{2n}(y_0) - v\|$ for $v \in A$, we get $v_0$ is a fixed point of $T^2$, where $v_0$ is a minimum in exactly one point in $A$, and also $T^{2n}y_0 \to y' \in B$. Hence, we obtain

$$u_0 = T^{2n}u_0 \to y' \text{ and } v_0 = T^{2n}v_0 \to x'.$$

This show that $(v_0, u_0) = (x', y')$, and $T^{2n}x_0 \to v_0$, $T^{2n}y_0 \to u_0$. Moreover,

$$\begin{aligned} \|u_0 - v_0\| &= \|T^{2n}(u_0) - T^{2n}v_0\| \\ &\leqslant \max\{\psi^n(\|u_0\|)\|u_0 - v_0\|, \mathrm{dist}(A, B)\}. \end{aligned} \tag{2.4}$$

Taking $n \to \infty$ in the inequality (2.4), by (2.3) and definition of $\mathrm{dist}(A, B)$, we get

$$\|u_0 - v_0\| = \mathrm{dist}(A, B).$$

Since $T$ is relatively non-expansive mapping, we have

$$\mathrm{dist}(A, B) \leqslant \|Tu_0 - Tv_0\| \leqslant \|u_0 - v_0\| = \mathrm{dist}(A, B).$$

Therefore $Tu_0 = v_0$ and $Tv_0 = u_0$. This implies that

$$\|Tu_0 - u_0\| = \|v_0 - Tv_0\| = \mathrm{dist}(A, B). \quad \square$$

Now, we shall give a validate example of Theorem 2.2.

**Example 2.3.** Consider $X = \mathbb{R}^2$ with the metric $d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}_+$. Let

$$A = \{(1, a) : a \geqslant 0\} \text{ and } B = \{(-1, b) : b \geqslant 0\},$$

then $A$ and $B$ be a non-empty closed and convex subset of $X$ and $dist(A, B) = 2$. Define $T$:$A \cup B \to A \cup B$, by

$$T(1, a) = \left(-1, \frac{a}{2}\right) \text{ and } T(-1, b) = \left(1, \frac{b}{2}\right) \quad \text{for all } a, b \geqslant 0.$$

Then $T$ is a cyclic mapping, relatively non-expansive and for each $(1, a) \in A$ and $(-1, b) \in B$, we have

$T^{2n}(1,a) = \left(1, \frac{a}{2^{2n}}\right)$ and $T^{2n}(-1,b) = \left(-1, \frac{b}{2^{2n}}\right)$. Next, we will show that $T$ is an asymptotic proximal pointwise weaker Meir–Keeler-type $\psi$-contraction with weaker Meir–Keeler-type function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$defined by

$$\psi(t) = \frac{t}{2} \text{ for all } t \geq 0.$$

Since,

$$\begin{aligned} d(T^{2n}(1,a), T^{2n}(-1,b)) &= d((1, \frac{a}{2^{2n}}), (-1, \frac{b}{2^{2n}})) \\ &= \max\{2, |\frac{a-b}{2^{2n}}|\} \\ &\leq \max\{2, |\frac{a-b}{2^{n}}|\} \\ &\leq \max\{2, \psi^n(d((0,0),(1,a))|a-b|\} \\ &\leq \max\{\psi^n(d((0,0),(1,a))d((1,a),(-1,b)),\ dist(A,B)\}. \end{aligned}$$

Similarly, we can conclude that

$$d(T^{2n}(1,a), T^{2n}(-1,b)) \leqslant \max\{\psi^n(d((0,0), (-1,b))d((1,a),(-1,b))), dist(A,B)\},$$

and hence $T$ is an asymptotic proximal pointwise weaker Meir–Keeler-type $\psi$-contraction. Moreover $((1,0),(-1,0)) \in A \times B$ is a pair of best proximity point of $T$, because

$$d((1,0), T(1,0)) = d((-1,0), T(-1,0)) = 2 = dist(A,B).$$


## Acknowledgements

Mr. Chirasak Mongkolkeha was supported from the Thailand Research Fund through the Royal Golden Jubilee Ph.D. Program (Grant No. PHD/0029/2553). The second author was supported by the Commission on Higher Education, the Thailand Research Fund and the King Mongkuts University of Technology Thonburi (Grant No. MRG550085).


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# Best proximity points for generalized proximal $C$-contraction mappings in metric spaces with partial orders

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**Abstract**
In this paper we extend the notion of weakly $C$-contraction mappings to the case of non-self mappings and establish the best proximity point theorems for this class. Our results generalize the result due to Harjani *et al.* (Comput. Math. Appl. 61:790-796, 2011) and some other authors.

## 1 Introduction and preliminaries

In 1922, Banach proved that every contractive mapping in a complete metric space has a unique fixed point, which is called Banach's fixed point theorem or Banach's contraction principle. Since Banach's fixed point theorem, many authors have extended, improved and generalized this theorem in several ways and, further, some applications of Banach's fixed point theorem can be found in [1–6] and many others.

In 1972, Chatterjea [7] introduced the following definition.

**Definition 1.1** Let $(X,d)$ be a metric space. A mapping $T: X \to X$ is called a *$C$-contraction* if there exists $\alpha \in (0, \frac{1}{2})$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq \alpha \bigl(d(x, Ty) + d(y, Tx)\bigr).$$

In 2009, Choudhury [8] introduced a generalization of $C$-contraction given by the following definition.

**Definition 1.2** Let $(X,d)$ be a metric space. A mapping $T: X \to X$ is called a *weakly $C$-contraction* if, for all $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}\bigl[d(x, Ty) + d(y, Tx)\bigr] - \psi\bigl(d(x, Ty), d(y, Tx)\bigr), \tag{1.1}$$

where $\psi: [0,\infty)^2 \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(x,y) = 0$ if and only if $x = y = 0$.

In 2011, Harjani *et al.* [9] presented some fixed point results for weakly $C$-contraction mappings in a complete metric space endowed with a partial order as follows.

**Theorem 1.3** *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric $d$ in $X$ such that $(X,d)$ is a complete metric space. Let $T: X \to X$ be a continuous and nondecreasing mapping such that*

$$d(Tx,Ty) \le \frac{1}{2}\big[d(x,Ty)+d(y,Tx)\big] - \psi\big(d(x,Ty),d(y,Tx)\big)$$

*for $x \preceq y$, where $\psi : [0,\infty)^2 \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(x,y) = 0$ if and only if $x = y = 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then $T$ has a fixed point.*

On the other hand, most of the results on Banach's fixed point theorem dilate upon the existence of a fixed point for self-mappings. Nevertheless, if $T$ is a non-self mapping, then it is probable that the equation $Tx = x$ has no solution, in which case best approximation theorems explore the existence of an approximate solution, whereas best proximity point theorems analyze the existence of an approximate solution that is optimal.

A classical best approximation theorem was introduced by Fan [10], that is, if $A$ is a nonempty compact convex subset of a Hausdorff locally convex topological vector space $B$ and $T: A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x,Tx) = d(Tx,A)$. Afterward, several authors including Prolla [11], Reich [12], Sehgal and Singh [13, 14] have derived the extensions of Fan's theorem in many directions. Other works on the existence of a best proximity point for some contractions can be seen in [15–19]. In 2005, Eldred, Kirk and Veeramani [20] obtained best proximity point theorems for relatively nonexpansive mappings, and some authors have proved best proximity point theorems for several types of contractions (see, for example, [21–26]).

Let $X$ be a nonempty set such that $(X,\preceq)$ is a partially ordered set and let $(X,d)$ be a complete metric space. Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. Now, we recall the following notions:

$$\begin{aligned}
&d(A,B) := \inf\big\{d(x,y) : x \in A \text{ and } y \in B\big\},\\
&A_0 := \big\{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\big\},\\
&B_0 := \big\{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\big\}.
\end{aligned}$$

If $A \cap B \neq \emptyset$, then $A_0$ and $B_0$ are nonempty. Further, it is interesting to notice that $A_0$ and $B_0$ are contained in the boundaries of $A$ and $B$, respectively, provided $A$ and $B$ are closed subsets of a normed linear space such that $d(A,B) > 0$ (see [27]).

**Definition 1.4** A mapping $T: A \to B$ is said to be *increasing* if

$$x \preceq y \quad\Longrightarrow\quad Sx \preceq Sy$$

for all $x, y \in A$.

**Definition 1.5** [28] A mapping $T : A \to B$ is said to be *proximally order-preserving* if and only if it satisfies the condition that

$$\left.\begin{array}{l} x \preceq y \\ d(u, Tx) = d(A,B) \\ d(v, Ty) = d(A,B) \end{array}\right\} \quad \Longrightarrow \quad u \preceq v \tag{1.2}$$

for all $u, v, x, y \in A$.

It is easy to observe that for a self-mapping, the notion of a proximally order-preserving mapping reduces to that of an increasing mapping.

**Definition 1.6** A point $x \in A$ is called a *best proximity point* of the mapping $T : A \to B$ if

$$d(x, Tx) = d(A,B).$$

In view of the fact that $d(x, Tx) \geq d(A,B)$ for all $x$ in $A$, it can be observed that the global minimum of the mapping $x \mapsto d(x, Tx)$ is attained from a best proximity point. Moreover, it is easy to see that the best proximity point reduces to a fixed point if the underlying mapping $T$ is a self-mapping.

In this paper, we introduce a new class of proximal contractions, which extends the class of weakly $C$-contractive mappings to the class of non-self mappings, and also give some examples to illustrate our main results. Our results extend and generalize the corresponding results given by Harjani *et al.* [9] and some authors in the literature.

## 2 Main results

In this section, we first introduce the notion of a generalized proximal $C$-contraction mapping and establish the best proximity point theorems.

**Definition 2.1** A mapping $T : A \to B$ is said to be a *generalized proximal $C$-contraction* if, for all $u, v, x, y \in A$, it satisfies

$$\left.\begin{array}{l} x \preceq y \\ d(u, Tx) = d(A,B) \\ d(v, Ty) = d(A,B) \end{array}\right\} \quad \Longrightarrow \quad d(u,v) \leq \frac{1}{2}\bigl(d(x,v) + d(y,u)\bigr) - \psi\bigl(d(x,v), d(y,u)\bigr), \tag{2.1}$$

where $\psi : [0,\infty)^2 \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(x,y) = 0$ if and only if $x = y = 0$.

For a self-mapping, it is easy to see that equation (2.1) reduces to (1.1).

**Theorem 2.2** *Let $X$ be a nonempty set such that $(X, \preceq)$ is a partially ordered set and let $(X,d)$ be a complete metric space. Let $A$ and $B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Let $T : A \to B$ satisfy the following conditions*:

(a) *$T$ is a continuous, proximally order-preserving and generalized proximal $C$-contraction such that $T(A_0) \subseteq B_0$;*

(b) *there exist elements* $x_0$ *and* $x_1$ *in* $A_0$ *such that* $x_0 \preceq x_1$ *and*

$$d(x_1, Tx_0) = d(A, B).$$

*Then there exists a point* $x \in A$ *such that*

$$d(x, Tx) = d(A, B).$$

*Moreover, for any fixed* $x_0 \in A_0$, *the sequence* $\{x_n\}$ *defined by*

$$d(x_{n+1}, Tx_n) = d(A, B)$$

*converges to the point* $x$.

*Proof* By the hypothesis (b), there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and

$$d(x_1, Tx_0) = d(A, B).$$

Since $T(A_0) \subseteq B_0$, there exists a point $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

By the proximally order-preserving property of $T$, we get $x_1 \preceq x_2$. Continuing this process, we can find a sequence $\{x_n\}$ in $A_0$ such that $x_{n-1} \preceq x_n$ and

$$d(x_n, Tx_{n-1}) = d(A, B).$$

Having found the point $x_n$, one can choose a point $x_{n+1} \in A_0$ such that $x_n \preceq x_{n+1}$ and

$$d(x_{n+1}, Tx_n) = d(A, B). \tag{2.2}$$

Since $T$ is a generalized proximal $C$-contraction, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{1}{2}\big(d(x_{n-1}, x_{n+1}) + d(x_n, x_n)\big) - \psi\big(d(x_{n-1}, x_{n+1}), d(x_n, x_n)\big) \\ &= \frac{1}{2} d(x_{n-1}, x_{n+1}) - \psi\big(d(x_{n-1}, x_{n+1}), 0\big) \\ &\leq \frac{1}{2} d(x_{n-1}, x_{n+1}) \\ &\leq \frac{1}{2}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) \end{aligned} \tag{2.3}$$

and so it follows that $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$, that is, the sequence $\{d(x_{n+1}, x_n)\}$ is non-increasing and bounded below. Then there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r. \tag{2.4}$$

Taking $n \to \infty$ in (2.3), we have

$$r \leq \lim_{n\to\infty} \frac{1}{2} d(x_{n-1}, x_{n+1}) \leq \frac{1}{2}(r + r) = r$$

and so

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2r. \tag{2.5}$$

Again, taking $n \to \infty$ in (2.3) and using (2.4), (2.5) and the continuity of $\psi$, we get

$$r \leq \frac{1}{2}(2r) = r - \psi(2r, 0) \leq r$$

and hence $\psi(2r, 0) = 0$. So, by the property of $\psi$, we have $r = 0$, which implies that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.6}$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and subsequences $\{x_{m_k}\}$, $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_k > m_k \geq k$ with

$$r_k := d(x_{m_k}, x_{n_k}) \geq \varepsilon, \qquad d(x_{m_k}, x_{n_k-1}) < \varepsilon \tag{2.7}$$

for each $k \in \{1, 2, 3, \ldots\}$. For each $n \geq 1$, let $\alpha_n := d(x_{n+1}, x_n)$. So, we have

$$\begin{aligned} \varepsilon \leq r_k &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \varepsilon + \alpha_{n_k-1}. \end{aligned}$$

It follows from (2.6) that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{2.8}$$

Notice also that

$$\begin{aligned} r_k &= d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k}) \\ &= d(x_{n_k}, x_{m_k+1}) + \alpha_{m_k} \\ &\leq d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k+1}) + \alpha_{m_k} \\ &= r_k + \alpha_{m_k} + \alpha_{m_k}. \end{aligned} \tag{2.9}$$

Taking $k \to \infty$ in (2.9), by (2.6) and (2.8), we conclude that

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \varepsilon. \tag{2.10}$$

Similarly, we can show that

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \varepsilon. \tag{2.11}$$

On the other hand, by the construction of $\{x_n\}$, we may assume that $x_{m_k} \preceq x_{n_k}$ such that

$$d(x_{n_k+1}, Tx_{n_k}) = d(A,B) \tag{2.12}$$

and

$$d(x_{m_k+1}, Tx_{m_k}) = d(A,B). \tag{2.13}$$

By the triangle inequality, (2.12), (2.13) and the generalized proximal $C$-contraction of $T$, we have

$$\begin{aligned}
\varepsilon \le r_k &\le d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k}) + d(x_{m_k+1}, x_{n_k+1}) \\
&= \alpha_{m_k} + \alpha_{n_k} + d(x_{m_k+1}, x_{n_k+1}) \\
&\le \alpha_{m_k} + \alpha_{n_k} + \frac{1}{2}\big[d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})\big] \\
&\quad - \psi\big(d(x_{n_k}, x_{m_k+1}), d(x_{m_k}, x_{n_k+1})\big).
\end{aligned}$$

Taking $k \to \infty$ in the above inequality, by (2.6), (2.10), (2.11) and the continuity of $\psi$, we get

$$\varepsilon \le \frac{1}{2}(\varepsilon + \varepsilon) - \psi(\varepsilon, \varepsilon) \le \varepsilon.$$

Therefore, $\psi(\varepsilon, \varepsilon) = 0$. By the property of $\psi$, we have that $\varepsilon = 0$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since $A$ is a closed subset of the complete metric space $X$, there exists $x \in A$ such that

$$\lim_{n\to\infty} x_n = x. \tag{2.14}$$

Letting $n \to \infty$ in (2.2), by (2.14) and the continuity of $T$, it follows that

$$d(x, Tx) = d(A,B). \qquad \square$$

**Corollary 2.3** *Let $X$ be a nonempty set such that $(X, \preceq)$ is a partially ordered set and let $(X,d)$ be a complete metric space. Let $A$ and $B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Let $T : A \to B$ satisfy the following conditions*:

(a) *$T$ is continuous, increasing such that $T(A_0) \subseteq B_0$ and*

$$\left.\begin{aligned} &x \preceq y \\ &d(u, Tx) = d(A,B) \\ &d(v, Ty) = d(A,B) \end{aligned}\right\} \implies d(u,v) \le \alpha\big(d(x,v) + d(y,u)\big), \tag{2.15}$$

*where $\alpha \in (0, \frac{1}{2})$*;

(b) *there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A,B).$$

*Then there exists a point $x \in A$ such that*

$$d(x, Tx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by*

$$d(x_{n+1}, Tx_n) = d(A, B)$$

*converges to the point $x$.*

*Proof* Let $\alpha \in (0, \frac{1}{2})$ and the function $\psi$ in Theorem 2.2 be defined by

$$\psi(a, b) = \left(\frac{1}{2} - \alpha\right)(a + b).$$

Obviously, it follows that $\psi(a, b) = 0$ if and only if $a = b = 0$ and (2.1) become (2.15). Hence we obtain Corollary 2.3. □

For a self-mapping, the condition (b) implies that $x_0 \preceq Tx_0$ and so Theorem 2.2 includes the results of Harjani *et al.* [9] as follows.

**Corollary 2.4** [9] *Let $X$ be a nonempty set such that $(X, \preceq)$ is a partially ordered set and let $(X, d)$ be a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that, for all $x, y \in X$,*

$$d(Tx, Ty) \leq \frac{1}{2}\big[d(x, Ty) + d(y, Tx)\big] - \psi\big(d(x, Ty), d(y, Tx)\big)$$

*for $x \preceq y$, where $\psi : [0, \infty)^2 \to [0, \infty)$ is a continuous and nondecreasing function such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then $T$ has a fixed point.*

Now, we give an example to illustrate Theorem 2.2.

**Example 2.5** Consider the complete metric space $\mathbb{R}^2$ with an Euclidean metric. Let

$$A = \big\{(x, 0) : x \in \mathbb{R}\big\}, \qquad B = \big\{(0, y) : y \in \mathbb{R}, y \geq 1\big\}.$$

Then $d(A, B) = 1$, $A_0 = \{(0, 0)\}$ and $B_0 = \{(0, 1)\}$. Define a mapping $T : A \to B$ as follows:

$$T\big((x, 0)\big) = \big(0, 1 + |x|\big)$$

for all $(x, 0) \in A$. Clearly, $T$ is continuous and $T(A_0) \subseteq B_0$. If $x_1 \preceq x_2$ and

$$d(u_1, Tx_1) = d(A, B) = 1, \qquad d(u_2, Tx_2) = d(A, B) = 1$$

for some $u_1, u_2, x_1, x_2 \in A$, then we have

$$u_1 = u_1 = (0, 0), \qquad x_1 = x_2 = (0, 0).$$

Therefore, $T$ is a generalized proximal $C$-contraction with $\psi : [0,\infty)^2 \to [0,\infty)$ defined by

$$\psi(a,b) = \frac{1}{4}(a+b).$$

Further, observe that $(0,0) \in A$ such that

$$d\big((0,0), T(0,0)\big) = d(A,B) = 1.$$

In Theorem 2.6, we do not need the condition that $T$ is continuous. Now, we improve the condition in Theorem 2.2 to prove the new best proximity point theorem as follows.

**Theorem 2.6** *Let $X$ be a nonempty set such that $(X,\preceq)$ is a partially ordered set and let $(X,d)$ be a complete metric space. Let $A$ and $B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Let $T : A \to B$ satisfy the following conditions*:

(a) *$T$ is a proximally order-preserving and generalized proximal $C$-contraction such that $T(A_0) \subseteq B_0$;*

(b) *there exist elements $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A,B);$$

(c) *if $\{x_n\}$ is an increasing sequence in $A$ converging to $x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

*Then there exists a point $x \in A$ such that*

$$d(x, Tx) = d(A,B).$$

*Proof* As in the proof of Theorem 2.2, we have

$$d(x_{n+1}, Tx_n) = d(A,B) \tag{2.16}$$

for all $n \geq 0$. Moreover, $\{x_n\}$ is a Cauchy sequence and converges to some point $x \in A$. Observe that for each $n \in \mathbb{N}$,

$$\begin{aligned} d(A,B) &= d(x_{n+1}, Tx_n) \leq d(x_{n+1}, x) + d(x, Tx_n) \\ &\leq d(x, x_{n+1}) + d(x, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &\leq d(x, x_{n+1}) + d(x, x_{n+1}) + d(A,B). \end{aligned}$$

Taking $n \to \infty$ in the above inequality, we obtain $\lim_{n\to\infty} d(x, Tx_n) = d(A,B)$ and hence $x \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $v \in A$ such that

$$d(v, Tx) = d(A,B). \tag{2.17}$$

Next, we prove that $x = v$. By the condition (c), we have $x_n \preceq x$ for all $n \in \mathbb{N}$. Using (2.16), (2.17) and the generalized proximal $C$-contraction of $T$, we have

$$d(x_{n+1}, v) \leq \frac{1}{2}\big[d(x_n, v) + d(x, x_{n+1})\big] - \psi\big(d(x_n, v), d(x, x_{n+1})\big). \tag{2.18}$$

Letting $n \to \infty$ in (2.18), we get

$$d(x,v) \le \frac{1}{2}d(x,v) - \psi\bigl(d(x,v),0\bigr),$$

which implies that $d(x,v) = 0$, that is, $x = v$. If we replace $v$ by $x$ in (2.17), we have

$$d(x,Tx) = d(A,B). \qquad \square$$

**Corollary 2.7** *Let $X$ be a nonempty set such that $(X,\preceq)$ is a partially ordered set and let $(X,d)$ be a complete metric space. Let $A$ and $B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Let $T : A \to B$ satisfy the following conditions*:

(a) *$T$ is an increasing mapping such that $T(A_0) \subseteq B_0$ and*

$$\left.\begin{aligned} &x \preceq y\\ &d(u,Tx) = d(A,B)\\ &d(v,Ty) = d(A,B)\end{aligned}\right\} \implies d(u,v) \le \alpha\bigl(d(x,v) + d(y,u)\bigr), \tag{2.19}$$

*where $\alpha \in (0,\frac{1}{2})$;*

(b) *there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1,Tx_0) = d(A,B);$$

(c) *if $\{x_n\}$ is an increasing sequence in $A$ converging to a point $x \in X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

*Then there exists a point $x \in A$ such that*

$$d(x,Tx) = d(A,B).$$

**Corollary 2.8** [9] *Let $X$ be a nonempty set such that $(X,\preceq)$ is a partially ordered set and let $(X,d)$ be a complete metric space. Assume that if $\{x_n\} \subseteq X$ is a nondecreasing sequence such that $x_n \to x$ in $X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$. Let $T : X \to X$ be a nondecreasing mapping such that*

$$d(Tx,Ty) \le \frac{1}{2}\bigl[d(x,Ty) + d(y,Tx)\bigr] - \psi\bigl(d(x,Ty),d(y,Tx)\bigr)$$

*for $x \preceq y$, where $\psi : [0,\infty)^2 \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(x,y) = 0$ if and only if $x = y = 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then $T$ has a fixed point.*

Now, we recall the condition defined by Nieto and Rodríguez-López [3] for the uniqueness of the best proximity point in Theorems 2.2 and 2.6.

$$\text{For } x,y \in X, \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{2.20}$$

**Theorem 2.9** *Let $X$ be a nonempty set such that $(X,\preceq)$ is a partially ordered set and let $(X,d)$ be a complete metric space. Let $A$ and $B$ be nonempty closed subsets of $X$ and let $A_0$*

*and $B_0$ be nonempty such that $A_0$ satisfies the condition* (2.20). *Let $T : A \to B$ satisfy the following conditions*:

(a) *$T$ is a continuous, proximally order-preserving and generalized proximal $C$-contraction such that $T(A_0) \subseteq B_0$;*

(b) *there exist elements $x_0$ and $x_1$ in $A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B).$$

*Then there exists a unique point $x \in A$ such that*

$$d(x, Tx) = d(A, B).$$

*Proof* We will only prove the uniqueness of the point $x \in A$ such that $d(x, Tx) = d(A, B)$. Suppose that there exist $x$ and $x^*$ in $A$ which are best proximity points, that is,

$$d(x, Tx) = d(A, B) \quad \text{and} \quad d\bigl(x^*, Tx^*\bigr) = d(A, B).$$

Case I: $x$ is comparable to $x^*$, that is, $x \preceq x^*$ (or $x^* \preceq x$). By the generalized proximal $C$-contraction of $T$, we have

$$d\bigl(x, x^*\bigr) \le \frac{1}{2}\bigl[d\bigl(x, x^*\bigr) + d\bigl(x^*, x\bigr)\bigr] - \psi\bigl(d\bigl(x, x^*\bigr), d\bigl(x^*, x\bigr)\bigr) \le d\bigl(x^*, x\bigr),$$

which implies that $\psi(d(x, x^*), d(x^*, x)) = 0$. Using the property of $\psi$, we get $d(x^*, x) = 0$ and hence $x = x^*$.

Case II: $x$ is not comparable to $x^*$. Since $A_0$ satisfies the condition (2.20), then there exists $z \in A_0$ such that $z$ is comparable to $x$ and $x^*$, that is, $x \preceq z$ (or $z \preceq x$) and $x^* \preceq z$ (or $z \preceq x^*$). Suppose that $x \preceq z$ and $x^* \preceq z$. Since $T(A_0) \subseteq B_0$, there exists a point $v_0 \in A_0$ such that

$$d(v_0, Tz) = d(A, B).$$

By the proximally order-preserving property of $T$, we get $x \preceq v_0$ and $x^* \preceq v_0$. Since $T(A_0) \subseteq B_0$, there exists a point $v_1 \in A_0$ such that

$$d(v_1, Tv_0) = d(A, B).$$

Again, by the proximally order-preserving property of $T$, we get $x \preceq v_1$ and $x^* \preceq v_1$. One can proceed further in a similar fashion to find $v_n$ in $A_0$ with $v_{n+1} \in A_0$ such that

$$d(v_{n+1}, Tv_n) = d(A, B).$$

Hence $x \preceq v_n$ and $x^* \preceq v_n$ for all $n \in \mathbb{N}$. By the generalized proximal $C$-contraction of $T$, we have

$$d(v_{n+1}, x) \le \frac{1}{2}\bigl[d(v_n, x) + d(x, v_{n+1})\bigr] - \psi\bigl(d(v_n, x), d(x, v_{n+1})\bigr), \tag{2.21}$$

$$d\bigl(v_{n+1}, x^*\bigr) \le \frac{1}{2}\bigl[d\bigl(v_n, x^*\bigr) + d\bigl(x^*, v_{n+1}\bigr)\bigr] - \psi\bigl(d\bigl(v_n, x^*\bigr), d\bigl(x^*, v_{n+1}\bigr)\bigr). \tag{2.22}$$

It follows from (2.21), (2.22) and the property of $\psi$ that

$$v_n \to x \quad \text{and} \quad v_n \to x^* \quad \text{as } n \to \infty.$$

By the uniqueness of limit, we conclude that $x = x^*$. Other cases can we proved similarly and this completes the proof. □

**Theorem 2.10** *Let $X$ be a nonempty set such that $(X, \preceq)$ is a partially ordered set and let $(X, d)$ be a complete metric space. Let $A$ and $B$ be nonempty closed subsets of $X$ and let $A_0$ and $B_0$ be nonempty such that $A_0$ satisfies the condition* (2.20). *Let $T : A \to B$ satisfy the following conditions*:

(a) *$T$ is a proximally order-preserving and generalized proximal $C$-contraction such that $T(A_0) \subseteq B_0$;*

(b) *there exist elements $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B);$$

(c) *if $\{x_n\}$ is an increasing sequence in $A$ converging to $x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

*Then there exists a unique point $x \in A$ such that*

$$d(x, Tx) = d(A, B).$$

*Proof* For the proof, combine the proofs of Theorems 2.6 and 2.9. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

**Acknowledgements**
This research was partially finished at the Department of Mathematics Education, Gyeongsang National University, Republic of Korea. Mr. Chirasak Mongkolkeha was supported by the Thailand Research Fund through the Royal Golden Jubilee Program under Grant PHD/0029/2553 for the Ph.D. program at KMUTT, Thailand. Also, the second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science, and Technology (NRF-2012-0008170). The third author was supported by the Commission on Higher Education, the Thailand Research Fund, and the King Mongkut's University of Technology Thonburi (KMUTT) (Grant No. MRG5580213).

Received: 28 October 2012 Accepted: 18 February 2013 Published: 6 March 2013

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doi:10.1186/1029-242X-2013-94
**Cite this article as:** Mongkolkeha et al.: Best proximity points for generalized proximal *C*-contraction mappings in metric spaces with partial orders. *Journal of Inequalities and Applications* 2013 **2013**:94.

Fixed Point Theory and Applications
a SpringerOpen Journal

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# Best proximity point theorems for rational proximal contractions

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**Abstract**
We provide sufficient conditions which warrant the existence and uniqueness of the best proximity point for two new types of contractions in the setting of metric spaces. The presented results extend, generalize and improve some known results from best proximity point theory and fixed-point theory. We also give some examples to illustrate and validate our definitions and results.

**MSC:** 41A65; 46B20; 47H10

**Keywords:** best proximity point; contraction; fixed point; generalized proximal contraction; optimal approximate solution

## 1 Introduction

Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{T}$ be a self-mapping defined on a subset of $\mathcal{X}$. In this setting, the fixed-point theory is an important tool for solving equations of the kind $\mathcal{T}x = x$, whose solutions are the fixed points of the mapping $\mathcal{T}$. On the other hand, if $\mathcal{T}$ is not a self-mapping, say $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are nonempty subsets of $\mathcal{X}$, then $\mathcal{T}$ does not necessarily have a fixed point. Consequently, the equation $\mathcal{T}x = x$ could have no solutions, and in this case, it is of a certain interest to determine an element $x$ that is in some sense closest to $\mathcal{T}x$. Thus, we can say that the aim of the best proximity point theorems is to provide sufficient conditions to solve a minimization problem. In view of the fact that $d(x, \mathcal{T}x)$ is at least $d(\mathcal{A}, \mathcal{B}) := \inf\{d(x, y) : x \in \mathcal{A} \text{ and } y \in \mathcal{B}\}$, a best proximity point theorem concerns the global minimum of the real valued function $x \to d(x, \mathcal{T}x)$, that is, an indicator of the error involved for an approximate solution of the equation $\mathcal{T}x = x$, by complying the condition $d(x, \mathcal{T}x) = d(\mathcal{A}, \mathcal{B})$. The notation of best proximity point is introduced in [1] but one of the most interesting results in this direction is due to Fan [2] and can be stated as follows.

**Theorem 1.1** *Let $\mathcal{K}$ be a nonempty, compact and convex subset of a normed space $\mathcal{E}$. Then for any continuous mapping $\mathcal{T} : \mathcal{K} \to \mathcal{E}$, there exists $x \in \mathcal{K}$ with $\|x - \mathcal{T}x\| = \inf_{y\in\mathcal{K}} \|\mathcal{T}x - y\|$.*

Some generalizations and extensions of this theorem appeared in the literature by Prolla [3], Reich [4], Sehgal and Singh [5, 6], Vetrivel *et al.* [7] and others. It turns out that many of the contractive conditions which are investigated for fixed points ensure the existence of best proximity points. Some results of this kind are obtained in [1, 5–40]. Note that the authors often, in proving these results, assume restrictive compactness hypotheses on the domain and codomain of the involved nonself-mapping. Inspired by [29], we consider these hypotheses too restrictive in dealing with proximal contractions and so we prove that the compactness hypotheses can be successfully replaced by standard completeness hypotheses. Following this idea, we propose a new type of condition to study the existence and uniqueness of the best proximity point of a nonself-mapping by assuming both compactness hypotheses and standard completeness hypotheses. Precisely, we introduce the notions of rational proximal contractions of the first and second kinds, then we establish some corresponding best proximity point theorems for such contractions. Our definitions include some earlier definitions as special cases. In particular, the presented theorems contain the results given in [29].

## 2 Preliminaries

In this section, we give some basic notations and definitions that will be used in the sequel.

Let $(\mathcal{X}, d)$ be a metric space, $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of $\mathcal{X}$ and $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ be a nonself-mapping. We denote by $B_{\text{est}}(\mathcal{T})$ the set of all best proximity points of $\mathcal{T}$, that is,

$$B_{\text{est}}(\mathcal{T}) := \{x \in \mathcal{A} \text{ such that } d(x, \mathcal{T}x) = d(\mathcal{A}, \mathcal{B})\}.$$

Also, let

$$\mathcal{A}_0 := \{x \in \mathcal{A} : d(x, y) = d(\mathcal{A}, \mathcal{B}) \text{ for some } y \in \mathcal{B}\}$$

and

$$\mathcal{B}_0 := \{y \in \mathcal{B} : d(x, y) = d(\mathcal{A}, \mathcal{B}) \text{ for some } x \in \mathcal{A}\}.$$

Sufficient conditions to ensure that $\mathcal{A}_0$ and $\mathcal{B}_0$ are nonempty are given in [41]. Also, observe that if $\mathcal{A}$ and $\mathcal{B}$ are closed subsets of a normed linear space such that $d(\mathcal{A}, \mathcal{B}) > 0$, then $\mathcal{A}_0$ and $\mathcal{B}_0$ are contained in the boundaries of $\mathcal{A}$ and $\mathcal{B}$, respectively; see [27].

Now, we give sequentially two definitions that are essential to state and prove our main results.

**Definition 2.1** Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of $\mathcal{X}$. Then $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ is said to be a rational proximal contraction of the first kind if there exist nonnegative real numbers $\alpha$, $\beta$, $\gamma$, $\delta$ with $\alpha + \beta + 2\gamma + 2\delta < 1$, such that the conditions

$$d(u_1, \mathcal{T}x_1) = d(\mathcal{A}, \mathcal{B}) \quad \text{and} \quad d(u_2, \mathcal{T}x_2) = d(\mathcal{A}, \mathcal{B})$$

imply that

$$\begin{aligned} d(u_1, u_2) \le{}& \alpha d(x_1, x_2) + \frac{\beta[1 + d(x_1, u_1)]d(x_2, u_2)}{1 + d(x_1, x_2)} + \gamma\bigl[d(x_1, u_1) + d(x_2, u_2)\bigr] \\ &+ \delta\bigl[d(x_1, u_2) + d(x_2, u_1)\bigr] \end{aligned} \tag{1}$$

for all $u_1, u_2, x_1, x_2 \in \mathcal{A}$.

Note that, if $\beta = 0$, then from (1) we get the definition of the generalized proximal contraction of the first kind with $\alpha + 2\gamma + 2\delta < 1$; see [29].

Moreover, if $\mathcal{T}$ is a self-mapping on $\mathcal{A}$, then the requirement in Definition 2.1 reduces to the following generalized contractive condition of rational type useful in establishing a fixed-point theorem:

$$d(\mathcal{T}x_1,\mathcal{T}x_2) \le \alpha d(x_1,x_2) + \frac{\beta[1+d(x_1,\mathcal{T}x_1)]d(x_2,\mathcal{T}x_2)}{1+d(x_1,x_2)} + \gamma\bigl[d(x_1,\mathcal{T}x_1)+d(x_2,\mathcal{T}x_2)\bigr] + \delta\bigl[d(x_1,\mathcal{T}x_2)+d(x_2,\mathcal{T}x_1)\bigr].$$

**Definition 2.2** Let $(\mathcal{X},d)$ be a metric space and $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of $\mathcal{X}$. Then $\mathcal{T}:\mathcal{A}\to\mathcal{B}$ is said to be a rational proximal contraction of the second kind if there exist nonnegative real numbers $\alpha$, $\beta$, $\gamma$, $\delta$ with $\alpha+\beta+2\gamma+2\delta<1$ such that the conditions

$$d(u_1,\mathcal{T}x_1)=d(\mathcal{A},\mathcal{B}) \quad \text{and} \quad d(u_2,\mathcal{T}x_2)=d(\mathcal{A},\mathcal{B})$$

imply that

$$\begin{aligned} &d(\mathcal{T}u_1,\mathcal{T}u_2) \\ &\quad\le \alpha d(\mathcal{T}x_1,\mathcal{T}x_2) + \frac{\beta[1+d(\mathcal{T}x_1,\mathcal{T}u_1)]d(\mathcal{T}x_2,\mathcal{T}u_2)}{1+d(\mathcal{T}x_1,\mathcal{T}x_2)} \\ &\qquad + \gamma\bigl[d(\mathcal{T}x_1,\mathcal{T}u_1)+d(\mathcal{T}x_2,\mathcal{T}u_2)\bigr] + \delta\bigl[d(\mathcal{T}x_1,\mathcal{T}u_2)+d(\mathcal{T}x_2,\mathcal{T}u_1)\bigr] \end{aligned} \tag{2}$$

for all $u_1,u_2,x_1,x_2\in\mathcal{A}$.

Note that, if $\beta=0$, then from (2) we get the definition of the generalized proximal contraction of the second kind with $\alpha+2\gamma+2\delta<1$, see [29].

The following example illustrates that a rational proximal contraction of the second kind is not necessarily a rational proximal contraction of the first kind. Therefore, both Definitions 2.1 and 2.2 are consistent.

**Example 2.1** Let $\mathcal{X}=\mathbb{R}\times\mathbb{R}$ endowed with the usual metric

$$d\bigl((x_1,x_2),(y_1,y_2)\bigr)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2},$$

for all $(x_1,x_2),(y_1,y_2)\in\mathbb{R}\times\mathbb{R}$. Define $\mathcal{A}:=\{(x,1):x\in\mathbb{R}\}$ and $\mathcal{B}:=\{(x,-1):x\in\mathbb{R}\}$. Also define $\mathcal{T}:\mathcal{A}\to\mathcal{B}$ by

$$\mathcal{T}\bigl((x,1)\bigr)=\begin{cases}(-1,-1) & \text{if } x \text{ is rational},\\ (1,-1) & \text{otherwise.}\end{cases}$$

Then $d(\mathcal{A},\mathcal{B})=2$ and $\mathcal{T}$ is a rational proximal contraction of the second kind but not a rational proximal contraction of the first kind. Indeed, using Definition 2.2 and after routine calculations, one can show that the left-hand side of inequality (2) is equal to 0. On the other hand, using Definition 2.1 and after routine calculations, one can show that the left-hand side of inequality (1) is equal to 2 and so inequality (1) is not satisfied for all nonnegative real numbers $\alpha$, $\beta$, $\gamma$, $\delta$ with $\alpha+\beta+2\gamma+2\delta<1$.

It is well known that the notion of approximative compactness plays an important role in the theory of approximation [12]. In particular, the notion of an approximatively compact set was introduced by Efimov and Stechkin [16] and the properties of approximatively compact sets have been largely studied. The boundendly compact sets that are the sets whose intersection with any closed ball is compact are useful examples of approximatively compact sets. It is shown in [14] that in every infinite-dimensional separable Banach space there exists a bounded approximatively compact set, which is not compact.

**Remark 2.1** Since $(\mathcal{X}, d)$ is a metric space, the bounded compactness of a set is equivalent to its closure and the possibility of selecting from any bounded sequence contained in it a converging subsequence.

Here, for our further use, we give the following definition.

**Definition 2.3** Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of $\mathcal{X}$. Then $\mathcal{B}$ is said to be approximatively compact with respect to $\mathcal{A}$ if every sequence $\{y_n\}$ of $\mathcal{B}$, satisfying the condition $d(x, y_n) \to d(x, \mathcal{B})$ for some $x$ in $\mathcal{A}$, has a convergent subsequence.

Obviously, any set is approximatively compact with respect to itself.

## 3 Rational proximal contractions

Our first main result is the following best proximity point theorem for a rational proximal contraction of the first kind.

**Theorem 3.1** *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{A}$ and $\mathcal{B}$ be two nonempty, closed subsets of $\mathcal{X}$ such that $\mathcal{B}$ is approximatively compact with respect to $\mathcal{A}$. Assume that $\mathcal{A}_0$ and $\mathcal{B}_0$ are nonempty and $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ is a nonself-mapping such that*:

(a) *$\mathcal{T}$ is a rational proximal contraction of the first kind*;

(b) $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$.

*Then there exists $x \in \mathcal{A}$ such that $B_{\mathrm{est}}(\mathcal{T}) = \{x\}$. Further, for any fixed $x_0 \in \mathcal{A}_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B})$, converges to $x$.*

*Proof* Let $x_0 \in \mathcal{A}_0$ (such a point there exists since $\mathcal{A}_0 \neq \emptyset$). Since $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$, then by the definition of $\mathcal{B}_0$, there exists $x_1 \in \mathcal{A}_0$ such that $d(x_1, \mathcal{T}x_0) = d(\mathcal{A}, \mathcal{B})$. Again, since $\mathcal{T}x_1 \in \mathcal{B}_0$, it follows that there is $x_2 \in \mathcal{A}_0$ such that $d(x_2, \mathcal{T}x_1) = d(\mathcal{A}, \mathcal{B})$. Continuing this process, we can construct a sequence $\{x_n\}$ in $\mathcal{A}_0$, such that

$$d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B}),$$

for every nonnegative integer $n$. Using the fact that $\mathcal{T}$ is a rational proximal contraction of the first kind, we have

$$\begin{aligned} d(x_n, x_{n+1}) \leq{}& \alpha d(x_{n-1}, x_n) + \frac{\beta[1 + d(x_{n-1}, x_n)]d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)} + \gamma\bigl[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\bigr] \\ &+ \delta d(x_{n-1}, x_{n+1}) \\ \leq{}& \alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}) + \gamma\bigl[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\bigr] \\ &+ \delta\bigl[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\bigr]. \end{aligned}$$

It follows that

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n),$$

where $k = \frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta} < 1$. Therefore, $\{x_n\}$ is a Cauchy sequence and, since $(\mathcal{X}, d)$ is complete and $\mathcal{A}$ is closed, the sequence $\{x_n\}$ converges to some $x \in \mathcal{A}$.

Moreover, we have

$$\begin{aligned} d(x, \mathcal{B}) &\leq d(x, \mathcal{T}x_n) \\ &\leq d(x, x_{n+1}) + d(x_{n+1}, \mathcal{T}x_n) \\ &= d(x, x_{n+1}) + d(\mathcal{A}, \mathcal{B}) \\ &\leq d(x, x_{n+1}) + d(x, \mathcal{B}). \end{aligned}$$

Taking the limit as $n \to +\infty$, we get $d(x, \mathcal{T}x_n) \to d(x, \mathcal{B})$. Since $\mathcal{B}$ is approximatively compact with respect to $\mathcal{A}$, then the sequence $\{\mathcal{T}x_n\}$ has a subsequence $\{\mathcal{T}x_{n_k}\}$ that converges to some $y \in \mathcal{B}$. Therefore,

$$d(x, y) = \lim_{k \to +\infty} d(x_{n_k+1}, \mathcal{T}x_{n_k}) = d(\mathcal{A}, \mathcal{B}),$$

and hence $x$ must be in $\mathcal{A}_0$. Since $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$, then $d(u, \mathcal{T}x) = d(\mathcal{A}, \mathcal{B})$ for some $u \in \mathcal{A}$. Again, using the fact that $\mathcal{T}$ is a rational proximal contraction of the first kind, we get

$$\begin{aligned} d(u, x_{n+1}) \leq{}& \alpha d(x, x_n) + \frac{\beta[1 + d(x, u)]d(x_n, x_{n+1})}{1 + d(x, x_n)} + \gamma\bigl[d(x, u) + d(x_n, x_{n+1})\bigr] \\ &+ \delta\bigl[d(x, x_{n+1}) + d(x_n, u)\bigr]. \end{aligned}$$

Taking the limit as $n \to +\infty$, we have

$$d(u, x) \leq (\gamma + \delta)d(u, x),$$

which implies $x = u$, since $\gamma + \delta < 1$. Thus, it follows that

$$d(x, \mathcal{T}x) = d(u, \mathcal{T}x) = d(\mathcal{A}, \mathcal{B}),$$

that is, $x \in B_{\text{est}}(\mathcal{T})$. Now, to prove the uniqueness of the best proximity point (*i.e.*, $B_{\text{est}}(\mathcal{T})$ is singleton), assume that $z$ is another best proximity point of $\mathcal{T}$ so that

$$d(z, \mathcal{T}z) = d(\mathcal{A}, \mathcal{B}).$$

Since $\mathcal{T}$ is a rational proximal contraction of the first kind, we have

$$d(x, z) \leq \alpha d(x, z) + \frac{\beta[1 + d(x, x)]d(z, z)}{1 + d(x, z)} + \gamma\bigl[d(x, x) + d(z, z)\bigr] + \delta\bigl[d(x, z) + d(z, x)\bigr]$$

which implies

$$d(x, z) \leq (\alpha + 2\delta)d(x, z).$$

It follows immediately that $x = z$, since $\alpha + 2\delta < 1$. Hence, $\mathcal{T}$ has a unique best proximity point. □

As consequences of the Theorem 3.1, we state the following corollaries.

**Corollary 3.1** *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{A}$ and $\mathcal{B}$ be two nonempty, closed subsets of $\mathcal{X}$ such that $\mathcal{B}$ is approximatively compact with respect to $\mathcal{A}$. Assume that $\mathcal{A}_0$ and $\mathcal{B}_0$ are nonempty and $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ is a nonself-mapping such that:*

(a) *$\mathcal{T}$ is a generalized proximal contraction of the first kind, with $\alpha + 2\gamma + 2\delta < 1$;*

(b) $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$.

*Then, there exists $x \in \mathcal{A}$ such that $B_{\mathrm{est}}(\mathcal{T}) = \{x\}$. Further, for any fixed $x_0 \in \mathcal{A}_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B})$, converges to the best proximity point $x$.*

**Corollary 3.2** *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{A}$ and $\mathcal{B}$ be two nonempty, closed subsets of $\mathcal{X}$ such that $\mathcal{B}$ is approximatively compact with respect to $\mathcal{A}$. Assume that $\mathcal{A}_0$ and $\mathcal{B}_0$ are nonempty and $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ is a nonself-mapping such that:*

(a) *There exists a nonnegative real number $\alpha < 1$ such that, for all $u_1$, $u_2$, $x_1$, $x_2$ in $\mathcal{A}$, the conditions $d(u_1, \mathcal{T}x_1) = d(\mathcal{A}, \mathcal{B})$ and $d(u_2, \mathcal{T}x_2) = d(\mathcal{A}, \mathcal{B})$ imply that $d(u_1, u_2) \le \alpha d(x_1, x_2)$;*

(b) $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$.

*Then there exists $x \in \mathcal{A}$ such that $B_{\mathrm{est}}(\mathcal{T}) = \{x\}$. Further, for any fixed $x_0 \in \mathcal{A}_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B})$, converges to the best proximity point $x$.*

The following fixed-point result can be considered as a special case of the Theorem 3.1, when $\mathcal{T}$ is a self-mapping.

**Corollary 3.3** *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T}$ be a self-mapping on $\mathcal{X}$. Assume that there exist nonnegative real numbers $\alpha$, $\beta$, $\gamma$, $\delta$ with $\alpha + \beta + 2\gamma + 2\delta < 1$ such that*

$$\begin{aligned} d(\mathcal{T}x_1, \mathcal{T}x_2) \le{}& \alpha d(x_1, x_2) + \frac{\beta[1 + d(x_1, \mathcal{T}x_2)]d(x_2, \mathcal{T}x_2)}{1 + d(x_1, x_2)} + \gamma\big[d(x_1, \mathcal{T}x_1) + d(x_2, \mathcal{T}x_2)\big] \\ &+ \delta\big[d(x_1, \mathcal{T}x_2) + d(x_2, \mathcal{T}x_1)\big] \end{aligned}$$

*for all $x_1, x_2 \in \mathcal{X}$. Then the mapping $\mathcal{T}$ has a unique fixed point.*

**Remark 3.1** Note that the Corollary 3.3 is a proper extension of the contraction mapping principle [13] because the continuity of the mapping $\mathcal{T}$ is not required. It is well known that a contraction mapping must be continuous.

Now, we state and prove a best proximity point theorem for a rational proximal contraction of the second kind.

**Theorem 3.2** *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{A}$ and $\mathcal{B}$ be two nonempty, closed subsets of $\mathcal{X}$ such that $\mathcal{A}$ is approximatively compact with respect to $\mathcal{B}$. Assume that $\mathcal{A}_0$ and $\mathcal{B}_0$ are nonempty and $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ is a nonself-mapping such that:*

(a) *$\mathcal{T}$ is a continuous rational proximal contraction of the second kind;*

(b) $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$.

*Then there exists $x \in B_{\mathrm{est}}(\mathcal{T})$ and for any fixed $x_0 \in \mathcal{A}_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B})$, converges to $x$, and $\mathcal{T}x = \mathcal{T}z$ for all $x, z \in B_{\mathrm{est}}(\mathcal{T})$.*

*Proof* Following the same lines of the proof of the Theorem 3.1, it is possible to construct a sequence $\{x_n\}$ in $\mathcal{A}_0$ such that

$$d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A}, \mathcal{B}),$$

for every nonnegative integer $n$. Using the fact that $\mathcal{T}$ is a rational proximal contraction of the second kind, we have

$$\begin{aligned}
&d(\mathcal{T}x_n, \mathcal{T}x_{n+1}) \\
&\quad \leq \alpha d(\mathcal{T}x_{n-1}, \mathcal{T}x_n) + \frac{\beta[1 + d(\mathcal{T}x_{n-1}, \mathcal{T}x_n)]d(\mathcal{T}x_n, \mathcal{T}x_{n+1})}{1 + d(\mathcal{T}x_{n-1}, \mathcal{T}x_n)} \\
&\qquad + \gamma\big[d(\mathcal{T}x_{n-1}, \mathcal{T}x_n) + d(\mathcal{T}x_n, \mathcal{T}x_{n+1})\big] + \delta d(\mathcal{T}x_{n-1}, \mathcal{T}x_{n+1}) \\
&\quad \leq \alpha d(\mathcal{T}x_{n-1}, \mathcal{T}x_n) + \beta d(\mathcal{T}x_n, \mathcal{T}x_{n+1}) + \gamma\big[d(\mathcal{T}x_{n-1}, \mathcal{T}x_n) + d(\mathcal{T}x_n, \mathcal{T}x_{n+1})\big] \\
&\qquad + \delta\big[d(\mathcal{T}x_{n-1}, \mathcal{T}x_n) + d(\mathcal{T}x_n, \mathcal{T}x_{n+1})\big].
\end{aligned}$$

It follows that

$$d(\mathcal{T}x_n, \mathcal{T}x_{n+1}) \leq k d(\mathcal{T}x_{n-1}, \mathcal{T}x_n),$$

where $k = \frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta} < 1$. Therefore, $\{\mathcal{T}x_n\}$ is a Cauchy sequence and, since $(\mathcal{X}, d)$ is complete, then the sequence $\{\mathcal{T}x_n\}$ converges to some $y \in \mathcal{B}$.

Moreover, we have

$$\begin{aligned}
d(y, \mathcal{A}) &\leq d(y, x_{n+1}) \leq d(y, \mathcal{T}x_n) + d(\mathcal{T}x_n, x_{n+1}) \\
&= d(y, \mathcal{T}x_n) + d(\mathcal{A}, \mathcal{B}) \leq d(y, \mathcal{T}x_n) + d(y, \mathcal{A}).
\end{aligned}$$

Taking the limit as $n \to +\infty$, we get $d(y, x_n) \to d(y, \mathcal{A})$. Since $\mathcal{A}$ is approximatively compact with respect to $\mathcal{B}$, then the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some $x \in \mathcal{A}$. Now, using the continuity of $\mathcal{T}$, we obtain that

$$d(x, \mathcal{T}x) = \lim_{k \to +\infty} d(x_{n_k+1}, \mathcal{T}x_{n_k}) = d(\mathcal{A}, \mathcal{B}),$$

that is, $x \in B_{\mathrm{est}}(\mathcal{T})$. Finally, to prove the last assertion of the present theorem, assume that $z$ is another best proximity point of $\mathcal{T}$ so that

$$d(z, \mathcal{T}z) = d(\mathcal{A}, \mathcal{B}).$$

Since $\mathcal{T}$ is a rational proximal contraction of the second kind, we have

$$\begin{aligned}
d(\mathcal{T}x, \mathcal{T}z) \leq{}& \alpha d(\mathcal{T}x, \mathcal{T}z) + \frac{\beta[1 + d(\mathcal{T}x, \mathcal{T}x)]d(\mathcal{T}z, \mathcal{T}z)}{1 + d(\mathcal{T}x, \mathcal{T}z)} + \gamma\big[d(\mathcal{T}x, \mathcal{T}x) + d(\mathcal{T}z, \mathcal{T}z)\big] \\
&+ \delta\big[d(\mathcal{T}x, \mathcal{T}z) + d(\mathcal{T}z, \mathcal{T}x)\big]
\end{aligned}$$

which implies

$$d(\mathcal{T}x,\mathcal{T}z) \le (\alpha + 2\delta)d(\mathcal{T}x,\mathcal{T}z).$$

It follows immediately that $\mathcal{T}x = \mathcal{T}z$, since $\alpha + 2\delta < 1$. □

As consequences of the Theorem 3.2, we state the following corollaries.

**Corollary 3.4** *Let $(\mathcal{X},d)$ be a complete metric space and $\mathcal{A}$ and $\mathcal{B}$ be two nonempty, closed subsets of $\mathcal{X}$ such that $\mathcal{A}$ is approximatively compact with respect to $\mathcal{B}$. Assume that $\mathcal{A}_0$ and $\mathcal{B}_0$ are nonempty and $\mathcal{T}:\mathcal{A}\to\mathcal{B}$ is a nonself-mapping such that*:

(a) *$\mathcal{T}$ is a continuous generalized proximal contraction of the second kind, with $\alpha + 2\gamma + 2\delta < 1$*;

(b) *$\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$.*

*Then, there exists $x \in B_{\text{est}}(\mathcal{T})$ and for any fixed $x_0 \in \mathcal{A}_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1},\mathcal{T}x_n) = d(\mathcal{A},\mathcal{B})$, converges to $x$. Further, $\mathcal{T}x = \mathcal{T}z$ for all $x,z \in B_{\text{est}}(\mathcal{T})$.*

**Corollary 3.5** *Let $(\mathcal{X},d)$ be a complete metric space and $\mathcal{A}$ and $\mathcal{B}$ be two nonempty, closed subsets of $\mathcal{X}$ such that $\mathcal{A}$ is approximatively compact with respect to $\mathcal{B}$. Assume that $\mathcal{A}_0$ and $\mathcal{B}_0$ are nonempty and $\mathcal{T}:\mathcal{A}\to\mathcal{B}$ is a nonself-mapping such that*:

(a) *There exists a nonnegative real number $\alpha < 1$ such that, for all $u_1, u_2, x_1, x_2$ in $\mathcal{A}$, the conditions $d(u_1,\mathcal{T}x_1) = d(\mathcal{A},\mathcal{B})$ and $d(u_2,\mathcal{T}x_2) = d(\mathcal{A},\mathcal{B})$ imply that $d(\mathcal{T}u_1,\mathcal{T}u_2) \le \alpha d(\mathcal{T}x_1,\mathcal{T}x_2)$*;

(b) *$\mathcal{T}$ is continuous*;

(c) *$\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$.*

*Then there exists $x \in B_{\text{est}}(\mathcal{T})$ and for any fixed $x_0 \in \mathcal{A}_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1},\mathcal{T}x_n) = d(\mathcal{A},\mathcal{B})$, converges to $x$. Further, $\mathcal{T}x = \mathcal{T}z$ for all $x,z \in B_{\text{est}}(\mathcal{T})$.*

**Remark 3.2** Note that in the Theorem 3.1 is not required the continuity of the mapping $\mathcal{T}$. On the contrary, the continuity of $\mathcal{T}$ is an hypothesis of the Theorem 3.2.

Our next theorem concerns a nonself-mapping that is a rational proximal contraction of the first kind as well as a rational proximal contraction of the second kind. In this theorem, we consider only a completeness hypothesis without assuming the continuity of the nonself-mapping.

**Theorem 3.3** *Let $(\mathcal{X},d)$ be a complete metric space and $\mathcal{A}$ and $\mathcal{B}$ be two nonempty, closed subsets of $\mathcal{X}$. Assume that $\mathcal{A}_0$ and $\mathcal{B}_0$ are nonempty and $\mathcal{T}:\mathcal{A}\to\mathcal{B}$ is a nonself-mapping such that*:

(a) *$\mathcal{T}$ is a rational proximal contraction of the first and second kinds*;

(b) *$\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$.*

*Then there exists a unique $x \in B_{\text{est}}(\mathcal{T})$. Further, for any fixed $x_0 \in \mathcal{A}_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1},\mathcal{T}x_n) = d(\mathcal{A},\mathcal{B})$, converges to $x$.*

*Proof* Following the same lines of the proof of the Theorem 3.1, it is possible to construct a sequence $\{x_n\}$ in $\mathcal{A}_0$ such that

$$d(x_{n+1},\mathcal{T}x_n) = d(\mathcal{A},\mathcal{B}),$$

for every nonnegative integer $n$. Also, using the same arguments in the proof of the Theorem 3.1, we deduce that the sequence $\{x_n\}$ is a Cauchy sequence, and hence converges to some $x \in \mathcal{A}$. Moreover, on the lines of the proof of the Theorem 3.2, we obtain that the sequence $\{\mathcal{T}x_n\}$ is a Cauchy sequence and hence converges to some $y \in \mathcal{B}$. Therefore, we have

$$d(x,y) = \lim_{n\to+\infty} d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{A},\mathcal{B}),$$

and hence $x$ must be in $\mathcal{A}_0$. Since $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$, then $d(u,\mathcal{T}x) = d(\mathcal{A},\mathcal{B})$ for some $u \in \mathcal{A}$. Using the fact that $\mathcal{T}$ is a rational proximal contraction of the first kind, we get

$$\begin{aligned} d(u,x_{n+1}) \leq{}& \alpha d(x,x_n) + \frac{\beta[1+d(x,u)]d(x_n,x_{n+1})}{1+d(x,x_n)} \\ &+ \gamma\big[d(x,u)+d(x_n,x_{n+1})\big] + \delta\big[d(x,x_{n+1})+d(x_n,u)\big]. \end{aligned}$$

Taking the limit as $n \to +\infty$, we have

$$d(u,x) \leq (\gamma+\delta)d(u,x),$$

which implies that $x = u$, since $\gamma + \delta < 1$. Thus, it follows that

$$d(x,\mathcal{T}x) = d(u,\mathcal{T}x) = d(\mathcal{A},\mathcal{B}),$$

that is, $x \in B_{\text{est}}(\mathcal{T})$. Again, following the same lines of the proof of the Theorem 3.1, we prove the uniqueness of the best proximity point of the mapping $\mathcal{T}$. To avoid repetitions, we omit the details. □

**Example 3.1** Let $\mathcal{X} = \mathbb{R}$ endowed with the usual metric $d(x,y) = |x-y|$, for all $x,y \in \mathcal{X}$. Define $\mathcal{A} = [-1,1]$ and $\mathcal{B} = [-3,-2] \cup [2,3]$. Then, $d(\mathcal{A},\mathcal{B}) = 1$, $\mathcal{A}_0 = \{-1,1\}$ and $\mathcal{B}_0 = \{-2,2\}$. Also define $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ by

$$\mathcal{T}x = \begin{cases} 2 & \text{if } x \text{ is rational}, \\ 3 & \text{otherwise}. \end{cases}$$

It is easy to show that $\mathcal{T}$ is a rational proximal contraction of the first and second kinds and $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$. Then all the hypotheses of the Theorem 3.3 are satisfied and $d(1,\mathcal{T}(1)) = d(\mathcal{A},\mathcal{B})$. Clearly, the Theorem 3.2 is not applicable in this case.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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**Acknowledgements**
The third author is supported by Università degli Studi di Palermo, Local University Project R. S. ex 60%. The second author was supported by the Commission on Higher Education, the Thailand Research Fund, and the King Mongkuts University of Technology Thonburi (Grant No. MRG5580213).

Received: 12 January 2013 Accepted: 26 March 2013 Published: 12 April 2013

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doi:10.1186/1687-1812-2013-95
**Cite this article as:** Nashine et al.: Best proximity point theorems for rational proximal contractions. *Fixed Point Theory and Applications* 2013 **2013**:95.

Fixed Point Theory and Applications
a SpringerOpen Journal

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# Coupled coincidence point and common coupled fixed point theorems lacking the mixed monotone property

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**Abstract**
In this paper, we prove the coupled coincidence point theorems for a $w^*$-compatible mapping in partially ordered cone metric spaces over a solid cone without the mixed $g$-monotone property. In the case of a totally ordered space, these results are automatically obvious under the assumption given. Therefore, these results can be applied in a much wider class of problems. We also prove the uniqueness of a common coupled fixed point in this setup and give some example which is not applied to the existence of a common coupled fixed point by using the mixed $g$-monotone property but can be applied to our results.
**MSC:** 47H10; 54H25

**Keywords:** cone metric spaces; common coupled fixed point; coupled coincidence point; $w^*$-compatible mappings; mixed $g$-monotone property

## 1 Introduction

The famous Banach contraction principle states that if $(X, d)$ is a complete metric space and $T : X \to X$ is a contraction mapping (*i.e.*, $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$, where $\alpha$ is a non-negative number such that $\alpha < 1$), then $T$ has a unique fixed point. This principle is one of the cornerstones in the development of nonlinear analysis. Fixed point theorems have applications not only in the various branches of mathematics, but also in economics, chemistry, biology, computer science, engineering, and others. Due to the importance, generalizations of Banach's contraction principle have been investigated heavily by several authors.

Following this trend, the problem of existence and uniqueness of fixed points in partially ordered sets has been studied thoroughly because of its interesting nature. In 1986, Turinici [1] presented the first result in this direction. Afterward, Ran and Reurings [2] gave some applications of Turinici's theorem to matrix equations. The results of Ran and Reurings were further extended to ordered cone metric spaces in [3–5]. In 2005, Nieto and Rodríguez-López [6] extended Ran and Reurings's theorems for nondecreasing mappings and obtained a unique solution for a first-order ordinary differential equation with periodic boundary conditions.

The notion of coupled fixed points was introduced by Guo and Lakshmikantham [7]. Since then, the concept has been of interest to many researchers in metrical fixed point theory. In 2006, Bhaskar and Lakshmikantham [8] introduced the concept of a mixed monotone property (see further Definition 2.4). They proved classical coupled fixed point theorems for mappings satisfying the mixed monotone property and also discussed an application of their result by investigating the existence and uniqueness of a solution of the periodic boundary value problem. Following this result, Harjani *et al.* [9] (see also [10, 11]) studied the existence and uniqueness of solutions of a nonlinear integral equation as an application of coupled fixed points. Very recently, motivated by the work of Caballero *et al.* [12], Jleli and Samet [13] discussed the existence and uniqueness of a positive solution for the singular nonlinear fractional differential equation boundary value problem

$$\begin{aligned} &D^{\alpha}_{0^+}u(t) = f\bigl(t, u(t), u(t)\bigr), \quad 0 < t < 1, \\ &u(a) = u'(b) = 0, \quad a, b \in \{0, 1\}, \end{aligned} \tag{1.1}$$

where $\alpha \in \mathbb{R}$ such that $3 < \alpha \leq 4$, $D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative and $f : (0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous, $\lim_{t\to 0^+} f(t,\cdot,\cdot) = +\infty$ ($f$ is singular at $t = 0$) for all $t \in (0,1]$, $f(t,\cdot,\cdot)$ is nondecreasing with respect to the first component and decreasing with respect to its second and third components.

Since their important role in the study of the existence and uniqueness of a solution of the periodic boundary value problem, a nonlinear integral equation, and the existence and uniqueness of a positive solution for the singular nonlinear fractional differential equation boundary value problem, a wide discussion on coupled fixed point theorems aimed the interest of many scientists.

In 2009, Lakshmikantham and Ćirić [14] extended the concept of a mixed monotone property to a mixed $g$-monotone mapping and proved coupled coincidence point and common coupled fixed point theorems which are more general than the result of Bhaskar and Lakshmikantham in [8]. A number of articles on coupled fixed point, coupled coincidence point, and common coupled fixed point theorems have been dedicated to the improvement; see [15–30] and the references therein.

On the other hand, in 2007, Huang and Zhang [31] have re-introduced the concept of a cone metric space which is replacing the set of real numbers by an ordered Banach space $E$. They went further and defined the convergence via interior points of the cone by which the order in $E$ is defined. This approach allows the investigation of cone spaces in the case when the cone is not necessarily normal. They also continued with results concerned with the normal cones only. One of the main results from [31] is the Banach contraction principle in the setting of normal cone spaces. Afterward, many authors generalized their fixed point theorems in cone spaces with normal cones. In other words, the fixed point problem in the setting of cone metric spaces is appropriate only in the case when the underlying cone is non-normal but just has interior that is nonempty. In this case only, proper generalizations of results from the ordinary metric spaces can be obtained. In 2011, Janković *et al.* [32] gave some examples showing that theorems from ordinary metric spaces cannot be applied in the setting of cone metric spaces, when the cone is non-normal.

Recently, Nashine *et al.* [33] established common coupled fixed point theorems for mixed $g$-monotone and $w^*$-compatible mappings satisfying more general contractive conditions in ordered cone metric spaces over a cone that is only solid (*i.e.*, has a nonempty interior) which improve works of Karapınar [34] and Shatanawi [35]. This result is an ordered version extension of the results of Abbas *et al.* [36].

In this work, we show that the mixed $g$-monotone property in common coupled fixed point theorems in ordered cone metric spaces can be replaced by another property due to Đorić *et al.* [37]. This property is automatically satisfied in the case of a totally ordered space. Therefore, these results can be applied in a much wider class of problems. Our results generalize and extend many well-known comparable results in the literature. An illustrative example is presented in this work when our results can be used in proving the existence of a common coupled fixed point, while the results of Nashine *et al.* [33] cannot.

## 2 Preliminaries

In this section, we give some notations and a property that are useful for our main results. Let $E$ be a real Banach space with respect to a given norm $\|\cdot\|_E$ and let $0_E$ be a zero vector of $E$. A nonempty subset $P$ of $E$ is called a *cone* if the following conditions hold:

1. $P$ is closed and $P \neq \{0_E\}$;
2. $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Longrightarrow ax + by \in P$;
3. $x \in P$, $-x \in P \Longrightarrow x = 0_E$.

Given a cone $P \subset E$, a partial ordering $\preceq_P$ with respect to $P$ is naturally defined by $x \preceq_P y$ if and only if $y - x \in P$ for $x, y \in E$. We will write $x \prec_P y$ to indicate that $x \preceq_P y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int}(P)$, where $\operatorname{int}(P)$ denotes the interior of $P$.

The cone $P$ is said to be *normal* if there exists a real number $K > 0$ such that for all $x, y \in E$,

$$0_E \preceq_P x \preceq_P y \quad \Longrightarrow \quad \|x\|_E \leq K\|y\|_E.$$

The least positive number $K$ satisfying the above statement is called a *normal constant* of $P$. In 2008, Rezapour and Hamlbarani [38] showed that there are no normal cones with a normal constant $K < 1$.

In what follows, we always suppose that $E$ is a real Banach space with cone $P$ satisfying $\operatorname{int}(P) \neq \emptyset$ (such cones are called *solid*).

**Definition 2.1** ([31]) Let $X$ be a nonempty set and $d : X \times X \to E$ satisfy

1. $0_E \preceq_P d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_E$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \preceq_P d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space.

**Definition 2.2** ([31]) Let $(X, d)$ be a cone metric space, $\{x_n\}$ be a sequence in $X$, and $x \in X$.

1. If for every $c \in E$ with $0_E \ll_P c$, there is $N \in \mathbb{N}$ such that $d(x_n, x) \ll_P c$ for all $n \geq N$, then $\{x_n\}$ is said to *converge* to $x$. This limit is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
2. If for every $c \in E$ with $0_E \ll_P c$, there is $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll_P c$ for all $n, m > N$, then $\{x_n\}$ is called a *Cauchy sequence* in $X$.
3. If every Cauchy sequence in $X$ is convergent in $X$, then $(X, d)$ is called a *complete* cone metric space.

Let $(X, d)$ be a cone metric space. Then the following properties are often used (particularly when dealing with cone metric spaces in which the cone need not be normal):

($p_1$) if $a \preceq_P ka$, where $a \in P$ and $k \in [0,1)$, then $a = 0_E$;
($p_2$) if $0_E \preceq_P u \ll c$ for each $0_E \ll c$, then $u = 0_E$;
($p_3$) if $u, v, w \in E$, $u \preceq_P v$ and $v \ll w$, then $u \ll w$;
($p_4$) if $c \in \operatorname{int}(P)$, $0_E \preceq_P a_n \in E$ and $a_n \to 0_E$, then there exists $k \in \mathbb{N}$ such that for all $n > k$, we have $a_n \ll c$.

**Definition 2.3** Let $X$ be a nonempty set. Then $(X, d, \preceq)$ is called an *ordered cone metric space* if

(i) $(X, d)$ is a cone metric space,
(ii) $(X, \preceq)$ is a partially ordered set.

Let $(X, \preceq)$ be a partially ordered set. By $x \succeq y$, we mean $y \preceq x$ for $x, y \in X$. Elements $x, y \in X$ are called *comparable* if $x \preceq y$ or $y \preceq x$ holds. A mapping $f$ is said to be *g-nondecreasing* (resp., *g-nonincreasing*) if, for all $x, y \in X$, $gx \preceq gy$ implies $f(x) \preceq f(y)$ (resp., $f(y) \preceq f(x)$). If $g$ is the identity mapping, then $f$ is said to be *nondecreasing* (resp., *nonincreasing*).

**Definition 2.4** ([8, 14]) Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \to X$ and $g : X \to X$. The mapping $F$ is said to have a *mixed g-monotone property* if $F$ is monotone $g$-nondecreasing in its first argument and monotone $g$-nonincreasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad gx_1 \preceq gx_2 \quad \Longrightarrow \quad F(x_1, y) \preceq F(x_2, y) \tag{2.1}$$

and

$$y_1, y_2 \in X, \quad gy_1 \preceq gy_2 \quad \Longrightarrow \quad F(x, y_1) \succeq F(x, y_2) \tag{2.2}$$

hold. If in the previous relations $g$ is the identity mapping, then it is said that $F$ has a *mixed monotone property*.

**Definition 2.5** ([8, 14]) Let $X$ be a nonempty set and $F : X \times X \to X$, $g : X \to X$. An element $(x, y) \in X \times X$ is called

($C_1$) a *coupled fixed point* of $F$ if $x = F(x, y)$ and $y = F(y, x)$;
($C_2$) a *coupled coincidence point* of mappings $g$ and $F$ if

$$gx = F(x, y) \quad \text{and} \quad gy = F(y, x),$$

and in this case $(gx, gy)$ is called a *coupled point of coincidence*;
($C_3$) a *common coupled fixed point* of mappings $g$ and $F$ if

$$x = gx = F(x, y) \quad \text{and} \quad y = gy = F(y, x).$$

**Definition 2.6** ([36]) Let $X$ be a nonempty set. Mappings $F : X \times X \to X$ and $g : X \to X$ are called

($W_1$) *w-compatible* if $gF(x, y) = F(gx, gy)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$;
($W_2$) *$w^*$-compatible* if $gF(x, x) = F(gx, gx)$ whenever $gx = F(x, x)$.

It is easy to see that $w$-compatible implies $w^*$-compatible. The following example shows that the converse of the above argument is not true.

**Example 2.7** Let $X = [0, \infty)$ and $F : X \times X \to X$ and $g : X \to X$ be defined by

$$F(x,y) = \begin{cases} \pi, & (x,y) = (0,1), \\ 2, & (x,y) = (1,0), \\ 8, & \text{otherwise}, \end{cases} \qquad gx = \begin{cases} \pi, & x = 0, \\ 2, & x = 1, \\ 8, & x \in \{4,6,8,\dots\}, \\ 5, & \text{otherwise}. \end{cases}$$

It is easy to see that $g0 = \pi = F(0,1)$ and $g1 = 2 = F(1,0)$, but $gF(0,1) = 5 \neq 8 = F(g0,g1)$. Hence, $F$ and $g$ are not $w$-compatible.

However, $F(x,x) = gx$ is possible only if $x \in \{4,6,8,\dots\}$ and for all points in this case, we get $gF(x,x) = 8 = F(gx,gx)$. Therefore, $F$ and $g$ are $w^*$-compatible.

For elements $x$, $y$ of a partially ordered set $(X, \preceq)$, we will write $x \asymp y$ whenever $x$ and $y$ are comparable (*i.e.*, $x \preceq y$ or $y \preceq x$ holds).

Next, we give a new property due to Đorić *et al.* [37].

Let $X$ be a nonempty set and let $g : X \to X$ and $F : X \times X \to X$. We will consider the following condition:

$$\text{if } x,y,u,v \in X \text{ are such that } gx \asymp F(x,y) = gu, \text{ then } F(x,y) \asymp F(u,v). \tag{2.3}$$

In particular, when $g = I_X$, it reduces to

$$\text{for all } x,y,v, \text{ if } x \asymp F(x,y), \text{ then } F(x,y) \asymp F\big(F(x,y),v\big). \tag{2.4}$$

**Remark 2.8** We obtain that the conditions (2.3) and (2.4) are trivially satisfied if $(X, \preceq)$ is the totally ordered.

The following examples show that the condition (2.3) ((2.4), resp.) may be satisfied when $F$ does not have the mixed $g$-monotone property (monotone property, resp.).

**Example 2.9** Let $X = \{a,b,c,d\}$, $\preceq = \{(a,a),(b,b),(c,c),(d,d),(a,b),(c,d)\}$,

$$g : \begin{pmatrix} a & b & c & d \\ c & d & c & d \end{pmatrix}, \qquad F : \begin{pmatrix} (a,y) & (b,y) & (c,y) & (d,y) \\ b & a & c & d \end{pmatrix}$$

for all $y \in X$. Since $ga = c \preceq d = gb$ but $F(a,y) \succeq F(b,y)$ for all $y \in X$, the mapping $F$ does not have the mixed $g$-monotone property. But it has property (2.3) since

(1) For each $y \in X$, we get $gc \asymp F(c,y) = gc$ and $F(c,y) \asymp F(c,v)$ for all $v \in X$.
(2) For each $y \in X$, we get $gd \asymp F(d,y) = gd$ and $F(d,y) \asymp F(d,v)$ for all $v \in X$.

**Example 2.10** Let $X = \{a,b,c,d\}$, $\preceq = \{(a,a),(b,b),(c,c),(d,d),(a,b),(c,d)\}$,

$$F : \begin{pmatrix} (a,y) & (b,y) & (c,y) & (d,y) \\ b & a & c & d \end{pmatrix}$$

for all $y \in X$. Since $a \preceq b$ but $F(a,y) = b \succeq a = F(b,y)$ for all $y \in X$, the mapping $F$ does not have the mixed monotone property. But it has property (2.4) since

(1) For each $y \in X$, we get $a \asymp F(a,y)$ and $F(a,y) = b \asymp a = F(F(a,y),v)$ for all $v \in X$.
(2) For each $y \in X$, we get $b \asymp F(b,y)$ and $F(b,y) = a \asymp b = F(F(b,y),v)$ for all $v \in X$.
(3) The other two cases are trivial.

## 3 Coupled coincidence point theorems lacking the mixed $g$-monotone property

In this section, we give the existence of coupled coincidence point theorems in ordered cone metric spaces lacking the mixed $g$-monotone property. Our first main result is the following theorem.

**Theorem 3.1** *Let $(X,d,\preceq)$ be an ordered cone metric space over a solid cone $P$ and let $g : X \to X$ and $F : X \times X \to X$. Suppose that the following hold*:

(i) *$F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of $X$*;
(ii) *$g$ and $F$ satisfy property* (2.3);
(iii) *there exist $x_0, y_0 \in X$ such that $gx_0 \asymp F(x_0,y_0)$ and $gy_0 \asymp F(y_0,x_0)$*;
(iv) *there exists $a_i \geq 0$ for $i = 1,2,\ldots,6$ and $\sum_{i=1}^{6} a_i < 1$ such that for all $x,y,u,v \in X$ satisfying $gx \asymp gu$ and $gy \asymp gv$,*

$$\begin{aligned} &d\bigl(F(x,y),F(u,v)\bigr) \\ &\quad \leq_P a_1 d(gx,gu) + a_2 d\bigl(F(x,y),gx\bigr) + a_3 d(gy,gv) \\ &\qquad + a_4 d\bigl(F(u,v),gu\bigr) + a_5 d\bigl(F(x,y),gu\bigr) + a_6 d\bigl(F(u,v),gx\bigr) \end{aligned} \tag{3.1}$$

*holds*;

(v) *if $x_n \to x$ when $n \to \infty$ in $X$, then $x_n \asymp x$ for $n$ sufficiently large.*

*Then there exist $x,y \in X$ such that*

$$F(x,y) = gx \quad \text{and} \quad F(y,x) = gy,$$

*that is, $F$ and $g$ have a coupled coincidence point $(x,y) \in X \times X$.*

*Proof* Starting from $x_0$, $y_0$ (condition (iii)) and using the fact that $F(X \times X) \subseteq g(X)$ (condition (i)), we can construct sequences $\{gx_n\}$ and $\{gy_n\}$ in $X$ such that

$$gx_n = F(x_{n-1},y_{n-1}) \quad \text{and} \quad gy_n = F(y_{n-1},x_{n-1}) \tag{3.2}$$

for all $n \in \mathbb{N}$. By (iii), we get $gx_0 \asymp F(x_0,y_0) = gx_1$, and the condition (ii) implies that

$$gx_1 = F(x_0,y_0) \asymp F(x_1,y_1) = gx_2.$$

Proceeding by induction, we get that $gx_{n-1} \asymp gx_n$ and, similarly, $gy_{n-1} \asymp gy_n$ for all $n \in \mathbb{N}$. Therefore, we can apply the condition (3.1) to obtain

$$\begin{aligned} d(gx_n,gx_{n+1}) &= d\bigl(F(x_{n-1},y_{n-1}),F(x_n,y_n)\bigr) \\ &\leq_P a_1 d(gx_{n-1},gx_n) + a_2 d\bigl(F(x_{n-1},y_{n-1}),gx_{n-1}\bigr) \end{aligned}$$

$$\begin{aligned}
&+ a_3 d(gy_{n-1}, gy_n) + a_4 d\big(F(x_n, y_n), gx_n\big) \\
&+ a_5 d\big(F(x_{n-1}, y_{n-1}), gx_n\big) + a_6 d\big(F(x_n, y_n), gx_{n-1}\big) \\
= {}& a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) \\
&+ a_4 d(gx_{n+1}, gx_n) + a_5 d(gx_n, gx_n) + a_6 d(gx_{n+1}, gx_{n-1}) \\
\preceq_P {}& a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) \\
&+ a_4 d(gx_{n+1}, gx_n) + a_6 \big[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})\big] \\
\preceq_P {}& (a_1 + a_2 + a_6) d(gx_{n-1}, gx_n) + a_3 d(gy_{n-1}, gy_n) \\
&+ (a_4 + a_6) d(gx_n, gx_{n+1}),
\end{aligned}$$

which implies that

$$(1 - a_4 - a_6) d(gx_n, gx_{n+1}) \preceq_P (a_1 + a_2 + a_6) d(gx_{n-1}, gx_n) + a_3 d(gy_{n-1}, gy_n). \tag{3.3}$$

Similarly, starting with $d(gy_n, gy_{n+1}) = d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))$ and using $gx_{n-1} \asymp gx_n$ and $gy_{n-1} \asymp gy_n$ for all $n \in \mathbb{N}$, we get

$$(1 - a_4 - a_6) d(gy_n, gy_{n+1}) \preceq_P (a_1 + a_2 + a_6) d(gy_{n-1}, gy_n) + a_3 d(gx_{n-1}, gx_n). \tag{3.4}$$

Combining (3.3) and (3.4), we obtain that

$$\begin{aligned}
&(1 - a_4 - a_6)\big[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\big] \\
&\quad \preceq_P (a_1 + a_2 + a_3 + a_6)\big[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)\big].
\end{aligned} \tag{3.5}$$

Now, starting from $d(gx_{n+1}, gx_n) = d(F(x_n, y_n), F(x_{n-1}, y_{n-1}))$ and using $gx_{n-1} \asymp gx_n$ and $gy_{n-1} \asymp gy_n$ for all $n \in \mathbb{N}$, we get that

$$(1 - a_2 - a_5) d(gx_n, gx_{n+1}) \preceq_P (a_1 + a_4 + a_5) d(gx_{n-1}, gx_n) + a_3 d(gy_{n-1}, gy_n).$$

Similarly, starting from $d(gy_{n+1}, gy_n) = d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))$ and using $gx_{n-1} \asymp gx_n$ and $gy_{n-1} \asymp gy_n$ for all $n \in \mathbb{N}$, we get that

$$(1 - a_2 - a_5) d(gy_n, gy_{n+1}) \preceq_P (a_1 + a_4 + a_5) d(gy_{n-1}, gy_n) + a_3 d(gx_{n-1}, gx_n).$$

Again adding up, we obtain that

$$\begin{aligned}
&(1 - a_2 - a_5)\big[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\big] \\
&\quad \preceq_P (a_1 + a_3 + a_4 + a_5)\big[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)\big].
\end{aligned} \tag{3.6}$$

Finally, adding up (3.5) and (3.6), it follows that

$$d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) \preceq_P \lambda\big[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)\big] \tag{3.7}$$

with

$$0 \leq \lambda = \frac{2a_1 + a_2 + 2a_3 + a_4 + a_5 + a_6}{2 - a_2 - a_4 - a_5 - a_6} < 1, \tag{3.8}$$

since $\sum_{i=1}^{6} a_i < 1$.

From the relation (3.7), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) &\preceq_P \lambda\big[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)\big] \\ &\preceq_P \lambda^2\big[d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})\big] \\ &\vdots \\ &\preceq_P \lambda^n\big[d(gx_0, gx_1) + d(gy_0, gy_1)\big]. \end{aligned}$$

If $d(gx_0, gx_1) + d(gy_0, gy_1) = 0_E$, then $(x_0, y_0)$ is a coupled coincidence point of $F$ and $g$. So, let $0_E <_P d(gx_0, gx_1) + d(gy_0, gy_1)$.

For any $m > n \geq 1$, repeated use of the triangle inequality gives

$$\begin{aligned} &d(gx_n, gx_m) + d(gy_n, gy_m) \\ &\quad\preceq_P d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{m-1}, gx_m) \\ &\qquad + d(gy_n, gy_{n+1}) + d(gy_{n+1}, gy_{n+2}) + \cdots + d(gy_{m-1}, gy_m) \\ &\quad\preceq_P \big[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\big]\big[d(gx_0, gx_1) + d(gy_0, gy_1)\big] \\ &\quad\preceq_P \frac{\lambda^n}{1-\lambda}\big[d(gx_0, gx_1) + d(gy_0, gy_1)\big]. \end{aligned}$$

Since $\frac{\lambda^n}{1-\lambda} \to 0$ as $n \to \infty$, we get $\frac{\lambda^n}{1-\lambda}[d(gx_0, gx_1) + d(gy_0, gy_1)] \to 0_E$ as $n \to \infty$.

From $(p_4)$, we have for $0_E \ll c$ and large $n$,

$$\frac{\lambda^n}{1-\lambda}\big[d(gx_0, gx_1) + d(gy_0, gy_1)\big] \ll c.$$

By $(p_3)$, we get

$$d(gx_n, gx_m) + d(gy_n, gy_m) \ll c.$$

Since

$$d(gx_n, gx_m) \preceq_P d(gx_n, gx_m) + d(gy_n, gy_m)$$

and

$$d(gy_n, gy_m) \preceq_P d(gx_n, gx_m) + d(gy_n, gy_m),$$

then by $(p_3)$, we get $d(gx_n, gx_m) \ll c$ and $d(gy_n, gy_m) \ll c$ for $n$ large enough. Therefore, we get $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. By completeness of $g(X)$, there exist $gx, gy \in g(X)$ such that $gx_n \to gx$ and $gy_n \to gy$ as $n \to \infty$.

By (v), we have $gx_n \asymp gx$ and $gy \asymp gy_n$ for all $n \geq 0$. Now, we prove that $F(x,y) = gx$ and $F(y,x) = gy$.

If $gx_n = gx$ and $gy_n = gy$ for some $n \geq 0$, from (3.1) we have

$$\begin{aligned} d\big(F(x,y),gx\big) &\preceq_P d\big(F(x,y),gx_{n+1}\big) + d(gx_{n+1},gx) \\ &= d\big(F(x,y),F(x_n,y_n)\big) + d(gx_{n+1},gx) \\ &\preceq_P a_1 d(gx,gx_n) + a_2 d\big(F(x,y),gx\big) + a_3 d(gy,gy_n) \\ &\quad + a_4 d\big(F(x_n,y_n),gx_n\big) + a_5 d\big(F(x,y),gx_n\big) \\ &\quad + a_6 d\big(F(x_n,y_n),gx\big) + d(gx_{n+1},gx) \\ &\preceq_P a_1 d(gx,gx_n) + a_2 d\big(F(x,y),gx\big) + a_3 d(gy,gy_n) \\ &\quad + a_4 d(gx_{n+1},gx) + a_4 d(gx,gx_n) + a_5 d\big(F(x,y),gx\big) + a_5 d(gx,gx_n) \\ &\quad + a_6 d(gx_{n+1},gx) + d(gx_{n+1},gx) \\ &= a_2 d\big(F(x,y),gx\big) + a_4 d(gx_{n+1},gx) + a_5 d\big(F(x,y),gx\big) \\ &\quad + a_6 d(gx_{n+1},gx) + d(gx_{n+1},gx), \end{aligned}$$

which further implies that

$$d\big(F(x,y),gx\big) \preceq_P \frac{1 + a_4 + a_6}{1 - a_2 - a_5} d(gx_{n+1},gx).$$

Since $gx_n \to gx$, then for $0_E \ll c$, there exists $N \in \mathbb{N}$ such that

$$d(gx_{n+1},gx) \ll \frac{(1 - a_2 - a_5)c}{1 + a_4 + a_6}$$

for all $n \geq N$. Therefore,

$$d\big(F(x,y),gx\big) \ll c.$$

Now, according to $(p_2)$, it follows that $d(F(x,y),gx) = 0_E$ and $F(x,y) = gx$. Similarly, we can prove that $F(y,x) = gy$. Hence, $(x,y)$ is a coupled coincidence point of the mappings $F$ and $g$.

So, we suppose that $(gx_n,gy_n) \neq (gx,gy)$ for all $n \geq 0$. Using (3.1), we get

$$\begin{aligned} d\big(F(x,y),gx\big) &\preceq_P d\big(F(x,y),gx_{n+1}\big) + d(gx_{n+1},gx) \\ &= d\big(F(x,y),F(x_n,y_n)\big) + d(gx_{n+1},gx) \\ &\preceq_P a_1 d(gx,gx_n) + a_2 d\big(F(x,y),gx\big) + a_3 d(gy,gy_n) \\ &\quad + a_4 d\big(F(x_n,y_n),gx_n\big) + a_5 d\big(F(x,y),gx_n\big) \\ &\quad + a_6 d\big(F(x_n,y_n),gx\big) + d(gx_{n+1},gx) \\ &\preceq_P a_1 d(gx,gx_n) + a_2 d\big(F(x,y),gx\big) + a_3 d(gy,gy_n) \\ &\quad + a_4 d(gx_{n+1},gx) + a_4 d(gx,gx_n) + a_5 d\big(F(x,y),gx\big) + a_5 d(gx,gx_n) \\ &\quad + a_6 d(gx_{n+1},gx) + d(gx_{n+1},gx), \end{aligned}$$

which further implies that

$$d\big(F(x,y),gx\big) \\ \preceq_P \frac{a_1+a_4+a_5}{1-a_2-a_5}d(gx,gx_n)+\frac{1+a_4+a_6}{1-a_2-a_5}d(gx_{n+1},gx)+\frac{a_3}{1-a_2-a_5}d(gy,gy_n).$$

Since $gx_n \to gx$ and $gy_n \to gy$, then for $0_E \ll c$, there exists $N \in \mathbb{N}$ such that $d(gx_n,gx) \ll \frac{(1-a_2-a_5)c}{3(a_1+a_4+a_5)}$, $d(gx_{n+1},gx) \ll \frac{(1-a_2-a_5)c}{3(1+a_4+a_6)}$, and $d(gy_n,gy) \ll \frac{(1-a_2-a_5)c}{3a_3}$ for all $n \geq N$. Thus,

$$d\big(F(x,y),gx\big) \ll \frac{c}{3}+\frac{c}{3}+\frac{c}{3}=c.$$

Now, according to $(p_2)$, it follows that $d(F(x,y),gx)=0_E$ and $F(x,y)=gx$. Similarly, $F(y,x)=gy$. Hence, $(x,y)$ is a coupled coincidence point of the mappings $F$ and $g$. □

**Remark 3.2** In Theorem 3.1, the condition (ii) is a substitution for the mixed $g$-monotone property that has been used in most of the coupled coincidence point theorems so far. Therefore, Theorem 3.1 improves the results of Nashine *et al.* [33]. Moreover, it is an ordered version extension of the results of Abbas *et al.* [36].

**Corollary 3.3** *Let $(X,d,\preceq)$ be an ordered cone metric space over a solid cone $P$ and let $g:X\to X$ and $F:X\times X\to X$. Suppose that the following hold*:

(i) *$F(X\times X)\subseteq g(X)$ and $g(X)$ is a complete subspace of $X$*;
(ii) *$g$ and $F$ satisfy property* (2.3);
(iii) *there exist $x_0,y_0\in X$ such that $gx_0 \asymp F(x_0,y_0)$ and $gy_0 \asymp F(y_0,x_0)$*;
(iv) *there exist $\alpha,\beta,\gamma\geq 0$ and $\alpha+\beta+\gamma<1$ such that for all $x,y,u,v\in X$ satisfying $gx\asymp gu$ and $gy\asymp gv$,*

$$d\big(F(x,y),F(u,v)\big) \preceq_P \alpha d(gx,gu)+\beta d(gy,gv)+\gamma d\big(F(x,y),gu\big) \tag{3.9}$$

*holds*;
(v) *if $x_n\to x$ when $n\to\infty$ in $X$, then $x_n\asymp x$ for $n$ sufficiently large.*

*Then there exist $x,y\in X$ such that*

$$F(x,y)=gx \quad \textit{and} \quad F(y,x)=gy,$$

*that is, $F$ and $g$ have a coupled coincidence point $(x,y)\in X\times X$.*

Putting $g=I_X$, where $I_X$ is the identity mapping from $X$ into $X$ in Theorem 3.1, we get the following corollary.

**Corollary 3.4** *Let $(X,d,\preceq)$ be an ordered cone metric space over a solid cone $P$ and let $F:X\times X\to X$. Suppose that the following hold*:

(i) *$X$ is complete*;
(ii) *$g$ and $F$ satisfy property* (2.4);
(iii) *there exist $x_0,y_0\in X$ such that $x_0\asymp F(x_0,y_0)$ and $y_0\asymp F(y_0,x_0)$*;

(iv) *there exists* $a_i \geq 0$ *for* $i = 1, 2, \dots, 6$ *and* $\sum_{i=1}^{6} a_i < 1$ *such that for all* $x, y, u, v \in X$ *satisfying* $x \asymp u$ *and* $y \asymp v$,

$$\begin{aligned} &d\bigl(F(x,y), F(u,v)\bigr) \\ &\quad \preceq_P a_1 d(x,u) + a_2 d\bigl(F(x,y), x\bigr) + a_3 d(y,v) \\ &\qquad + a_4 d\bigl(F(u,v), u\bigr) + a_5 d\bigl(F(x,y), u\bigr) + a_6 d\bigl(F(u,v), x\bigr) \end{aligned} \tag{3.10}$$

*holds*;

(v) *if* $x_n \to x$ *when* $n \to \infty$ *in* $X$, *then* $x_n \asymp x$ *for* $n$ *sufficiently large*.

*Then there exist* $x, y \in X$ *such that*

$$F(x,y) = x \quad \text{and} \quad F(y,x) = y,$$

*that is*, $F$ *has a coupled fixed point* $(x,y) \in X \times X$.

Our second main result is the following.

**Theorem 3.5** *Let* $(X, d, \preceq)$ *be an ordered cone metric space over a solid cone* $P$. *Let* $F : X \times X \to X$ *and* $g : X \to X$ *be mappings. Suppose that the following hold*:

(i) $F(X \times X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* $X$;

(ii) $g$ *and* $F$ *satisfy property* (2.3);

(iii) *there exist* $x_0, y_0 \in X$ *such that* $gx_0 \asymp F(x_0, y_0)$ *and* $gy_0 \asymp F(y_0, x_0)$;

(iv) *there is some* $h \in [0, 1/2)$ *such that for all* $x, y, u, v \in X$ *satisfying* $gx \asymp gu$ *and* $gy \asymp gv$, *there exists*

$$\Theta_{x,y,u,v} \in \bigl\{d(gx, gu), d(gy, gv), d\bigl(F(x,y), gu\bigr)\bigr\}$$

*such that*

$$d\bigl(F(x,y), F(u,v)\bigr) \preceq_P h\Theta_{x,y,u,v};$$

(v) *if* $x_n \to x$ *when* $n \to \infty$ *in* $X$, *then* $x_n \asymp x$ *for* $n$ *sufficiently large*.

*Then there exist* $x, y \in X$ *such that*

$$F(x,y) = gx \quad \text{and} \quad F(y,x) = gy,$$

*that is*, $F$ *and* $g$ *have a coupled coincidence point* $(x,y) \in X \times X$.

*Proof* Since $F(X \times X) \subseteq g(X)$ (condition (i)), we can start from $x_0$, $y_0$ (condition (iii)) and construct sequences $\{gx_n\}$ and $\{gy_n\}$ in $X$ such that

$$gx_n = F(x_{n-1}, y_{n-1}) \quad \text{and} \quad gy_n = F(y_{n-1}, x_{n-1}) \tag{3.11}$$

for all $n \in \mathbb{N}$. From (iii), we get $gx_0 \asymp F(x_0, y_0) = gx_1$ and the condition (ii) implies that

$$gx_1 = F(x_0, y_0) \asymp F(x_1, y_1) = gx_2.$$

By repeating this process, we have $gx_{n-1} \asymp gx_n$. Similarly, we can prove that $gy_{n-1} \asymp gy_n$ for all $n \in \mathbb{N}$.

Since $gx_{n-1} \asymp gx_n$ and $gy_{n-1} \asymp gy_n$ for all $n \in \mathbb{N}$, from (iv), we have that there exist $h \in [0, 1/2)$ and

$$\begin{aligned}\Theta_1 &\in \{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(F(x_{n-1}, y_{n-1}), gx_n)\} \\ &= \{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), 0_E\}\end{aligned}$$

such that

$$d(gx_n, gx_{n+1}) = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \preceq_P h\Theta_1.$$

Similarly, one can show that there exists

$$\Theta_2 \in \{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), 0_E)\}$$

such that

$$d(gy_n, gy_{n+1}) = d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \preceq_P h\Theta_2.$$

Now, denote $\delta_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})$. Since the cases $\Theta_1 = 0_E$ and $\Theta_2 = 0_E$ are trivial, we have to consider the following four possibilities.

Case 1. $d(gx_n, gx_{n+1}) \preceq_P hd(gx_{n-1}, gx_n)$ and $d(gy_n, gy_{n+1}) \preceq_P hd(gy_{n-1}, gy_n)$. Adding up, we get that

$$\delta_n \preceq_P h\delta_{n-1} \preceq_P 2h\delta_{n-1}.$$

Case 2. $d(gx_n, gx_{n+1}) \preceq_P hd(gx_{n-1}, gx_n)$ and $d(gy_n, gy_{n+1}) \preceq_P hd(gx_{n-1}, gx_n)$. Then

$$\delta_n \preceq_P 2hd(gx_{n-1}, gx_n) \preceq_P 2hd(gx_{n-1}, gx_n) + 2hd(gy_{n-1}, gy_n) = 2h\delta_{n-1}.$$

Case 3. $d(gx_n, gx_{n+1}) \preceq_P hd(gy_{n-1}, gy_n)$ and $d(gy_n, gy_{n+1}) \preceq_P hd(gx_{n-1}, gx_n)$. This case is treated analogously to Case 1.

Case 4. $d(gx_n, gx_{n+1}) \preceq_P hd(gy_{n-1}, gy_n)$ and $d(gy_n, gy_{n+1}) \preceq_P hd(gy_{n-1}, gy_n)$. This case is treated analogously to Case 2.

Thus, in all cases, we get $\delta_n \preceq_P 2h\delta_{n-1}$ for all $n \in \mathbb{N}$, where $0 \leq 2h < 1$. Therefore,

$$\delta_n \preceq_P 2h\delta_{n-1} \preceq_P (2h)^2\delta_{n-2} \preceq_P \cdots \preceq_P (2h)^n\delta_0,$$

and by the same argument as in Theorem 3.1, it is proved that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. By the completeness of $g(X)$, there exist $gx, gy \in g(X)$ such that $gx_n \to gx$ and $gy_n \to gy$.

From (v), we get $gx_n \asymp gx$ and $gy \asymp gy_n$ for all $n \geq 0$. Now, we prove that $F(x, y) = gx$ and $F(y, x) = gy$.

If $gx_n = gx$ and $gy_n = gy$ for some $n \geq 0$, from (iv) we have

$$\begin{aligned} d\big(F(x,y),gx\big) &\preceq_P d\big(F(x,y),gx_{n+1}\big) + d(gx_{n+1},gx) \\ &= d\big(F(x,y),F(x_n,y_n)\big) + d(gx_{n+1},gx) \\ &\preceq_P h\Theta_{x,y,x_n,y_n} + d(gx_{n+1},gx), \end{aligned}$$

where $\Theta_{x,y,x_n,y_n} \in \{d(gx,gx_n), d(gy,gy_n), d(F(x,y),gx_n)\}$. Let $c \in \operatorname{int}(P)$ be fixed. If $\Theta_{x,y,x_n,y_n} = d(gx,gx_n) = 0_E$ or $\Theta_{x,y,x_n,y_n} = d(gy,gy_n) = 0_E$, then for $n$ sufficiently large, we have that

$$d\big(F(x,y),gx\big) \ll c.$$

By property $(p_2)$, it follows that $F(x,y) = gx$. If $\Theta_{x,y,x_n,y_n} = d(F(x,y),gx_n)$, then we get that

$$\begin{aligned} d\big(F(x,y),gx\big) &\preceq_P hd\big(F(x,y),gx_n\big) + d(gx_{n+1},gx) \\ &\preceq_P hd\big(F(x,y),gx\big) + hd(gx,gx_n) + d(gx_{n+1},gx) \\ &= hd\big(F(x,y),gx\big) + d(gx_{n+1},gx). \end{aligned}$$

Now, it follows that for $n$ sufficiently large,

$$\begin{aligned} d\big(F(x,y),gx\big) &\preceq_P \frac{1}{1-h} d(gx_{n+1},gx) \\ &\preceq_P \frac{1}{1-h}(1-h)c \\ &= c. \end{aligned}$$

Therefore, again by property $(p_2)$, we get that $F(x,y) = gx$. Similarly, we can prove that $F(y,x) = gy$. Hence, $(x,y)$ is a coupled point of coincidence of $F$ and $g$.

Then, we suppose that $(gx_n, gy_n) \neq (gx, gy)$ for all $n \geq 0$. For this, consider

$$\begin{aligned} d\big(F(x,y),gx\big) &\preceq_P d\big(F(x,y),gx_{n+1}\big) + d(gx_{n+1},gx) \\ &= d\big(F(x,y),F(x_n,y_n)\big) + d(gx_{n+1},gx) \\ &\preceq_P h\Theta_{x,y,x_n,y_n} + d(gx_{n+1},gx), \end{aligned}$$

where $\Theta_{x,y,x_n,y_n} \in \{d(gx,gx_n), d(gy,gy_n), d(F(x,y),gx_n)\}$. Let $c \in \operatorname{int}(P)$ be fixed. If $\Theta_{x,y,x_n,y_n} = d(gx,gx_n)$ or $\Theta_{x,y,x_n,y_n} = d(gy,gy_n)$, then for $n$ sufficiently large, we have that

$$d\big(F(x,y),gx\big) \ll h \cdot \frac{c}{2h} + \frac{c}{2} = c.$$

By property $(p_2)$, it follows that $F(x,y) = gx$. If $\Theta_{x,y,x_n,y_n} = d(F(x,y),gx_n)$, then we get that

$$\begin{aligned} d\big(F(x,y),gx\big) &\preceq_P hd\big(F(x,y),gx_n\big) + d(gx_{n+1},gx) \\ &\preceq_P hd\big(F(x,y),gx\big) + hd(gx,gx_n) + d(gx_{n+1},gx). \end{aligned}$$

Now, it follows that for $n$ sufficiently large,

$$\begin{aligned} d\big(F(x,y),gx\big) &\preceq_P \frac{h}{1-h}d(gx,gx_n)+\frac{1}{1-h}d(gx_{n+1},gx) \\ &\ll \frac{h}{1-h}\cdot\frac{1-h}{h}\cdot\frac{c}{2}+\frac{1}{1-h}(1-h)\frac{c}{2}=c. \end{aligned}$$

Thus, again by property $(p_2)$, we get that $F(x,y)=gx$.

Similarly, $F(y,x)=gy$ is obtained. Hence, $(x,y)$ is a coupled point of coincidence of the mappings $F$ and $g$. □

**Remark 3.6** It would be interesting to relate our Theorem 3.5 with Theorem 2.1 of Long *et al.* [39].

Putting $g=I_X$, where $I_X$ is the identity mapping from $X$ into $X$ in Theorem 3.5, we get the following corollary.

**Corollary 3.7** *Let $(X,d,\preceq)$ be an ordered cone metric space over a solid cone $P$. Let $F: X\times X\to X$ be mappings. Suppose that the following hold:*

(i) *$X$ is complete;*
(ii) *$F$ satisfies property* (2.4);
(iii) *there exist $x_0,y_0\in X$ such that $x_0\asymp F(x_0,y_0)$ and $y_0\asymp F(y_0,x_0)$;*
(iv) *there is some $h\in[0,1/2)$ such that for all $x,y,u,v\in X$ satisfying $x\asymp u$ and $y\asymp v$, there exists*

$$\Theta_{x,y,u,v}\in\big\{d(x,u),d(y,v),d\big(F(x,y),u\big)\big\}$$

*such that*

$$d\big(F(x,y),F(u,v)\big)\preceq_P h\Theta_{x,y,u,v}.$$

(v) *if $x_n\to x$ when $n\to\infty$ in $X$, then $x_n\asymp x$ for $n$ sufficiently large.*

*Then there exist $x,y\in X$ such that*

$$F(x,y)=x \quad \textit{and} \quad F(y,x)=y,$$

*that is, $F$ has a coupled fixed point $(x,y)\in X\times X$.*

## 4 Common coupled fixed point theorems lacking the mixed monotone property

Some questions arise naturally from Theorems 3.1 and 3.5. For example, one may ask if there are necessary conditions for the existence and uniqueness of a common coupled fixed point of $F$ and $g$?

The next theorem provides a positive answer to this question with additional hypotheses to Theorems 3.1 and 3.5.

For the given partial order $\preceq$ on the set $X$, we will denote also by $\preceq$ the order on $X\times X$ given by

$$(x_1,y_1)\preceq(x_2,y_2) \quad\Longleftrightarrow\quad x_1\preceq x_2 \quad\text{and}\quad y_1\succeq y_2. \tag{4.1}$$

**Theorem 4.1** *In addition to the hypotheses of Theorem* 3.1, *suppose that for every* $(x,y)$, $(x^*,y^*) \in X \times X$, *there exists* $(u,v) \in X \times X$ *such that*

$$\big(F(u,v),F(v,u)\big) \asymp \big(F(x,y),F(y,x)\big)$$

*and*

$$\big(F(u,v),F(v,u)\big) \asymp \big(F\big(x^*,y^*\big),F\big(y^*,x^*\big)\big).$$

*If $F$ and $g$ are $w^*$-compatible, then $F$ and $g$ have a unique common coupled fixed point, that is, there exists a unique* $(\hat{u},\hat{v}) \in X \times X$ *such that*

$$\hat{u} = g\hat{u} = F(\hat{u},\hat{v}) \quad \text{and} \quad \hat{v} = g\hat{v} = F(\hat{v},\hat{u}).$$

*Proof* From Theorem 3.1, the set of coupled coincidence points of $F$ and $g$ is nonempty. Suppose $(x,y)$ and $(x^*,y^*)$ are coupled coincidence points of $F$, that is, $gx = F(x,y)$, $gy = F(y,x)$, $gx^* = F(x^*,y^*)$ and $gy^* = F(y^*,x^*)$. We will prove that

$$gx = gx^* \quad \text{and} \quad gy = gy^*. \tag{4.2}$$

By assumption, there exists $(u,v) \in X \times X$ such that

$$\big(F(u,v),F(v,u)\big) \asymp \big(F(x,y),F(y,x)\big)$$

and

$$\big(F(u,v),F(v,u)\big) \asymp \big(F\big(x^*,y^*\big),F\big(y^*,x^*\big)\big).$$

Put $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ so that $gu_1 = F(u_0,v_0)$ and $gv_1 = F(v_0,u_0)$. Then, similarly as in the proof of Theorem 3.1, we can inductively define sequences $\{gu_n\}$, $\{gv_n\}$ with

$$gu_{n+1} = F(u_n,v_n) \quad \text{and} \quad gv_{n+1} = F(v_n,u_n)$$

for all $n$. Further, set $x_0 = x$, $y_0 = y$, $x_0^* = x^*$, $y_0^* = y^*$ and, in a similar way, define the sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gx_n^*\}$, $\{gy_n^*\}$. Then it is easy to show that

$$gx_n \to F(x,y), \qquad gy_n \to F(y,x)$$

and

$$gx_n^* \to F\big(x^*,y^*\big), \qquad gy_n^* \to F\big(y^*,x^*\big)$$

as $n \to \infty$.

Since

$$(gx,gy) = (gx_1,gy_1) = \big(F(x,y),F(y,x)\big) \asymp \big(F(u,v),F(v,u)\big) = (gu_1,gv_1),$$

we have $gx \asymp gu_1$ and $gy \asymp gv_1$. It is easy to show that, similarly,

$$(gx, gy) \asymp (gu_n, gv_n)$$

for all $n \geq 1$, that is, $gx \asymp gu_n$ and $gy \asymp gv_n$ for all $n \geq 1$. Thus, from (3.1), we have

$$\begin{aligned} d(gu_{n+1}, gx) &= d\big(F(u_n, v_n), F(x, y)\big) \\ &\preceq_P a_1 d(gu_n, gx) + a_2 d\big(F(u_n, v_n), gu_n\big) + a_3 d(gv_n, gy) \\ &\quad + a_4 d\big(F(x, y), gx\big) + a_5 d\big(F(u_n, v_n), gx\big) + a_6 d\big(F(x, y), gu_n\big) \\ &= a_1 d(gu_n, gx) + a_2 d(gu_{n+1}, gu_n) + a_3 d(gv_n, gy) \\ &\quad + a_4 d(gx, gx) + a_5 d(gu_{n+1}, gx) + a_6 d(gx, gu_n) \\ &\preceq_P a_1 d(gu_n, gx) + a_2 \big[d(gu_{n+1}, gx) + d(gx, gu_n)\big] + a_3 d(gv_n, gy) \\ &\quad + a_5 d(gu_{n+1}, gx) + a_6 d(gx, gu_n), \end{aligned}$$

that is,

$$(1 - a_2 - a_5) d(gu_{n+1}, gx) \preceq_P (a_1 + a_2 + a_6) d(gu_n, gx) + a_3 d(gv_n, gy).$$

In the same way, starting from $d(gv_{n+1}, gy)$, we can show that

$$(1 - a_2 - a_5) d(gv_{n+1}, gy) \preceq_P (a_1 + a_2 + a_6) d(gv_n, gy) + a_3 d(gu_n, gx).$$

Thus,

$$\begin{aligned} &(1 - a_2 - a_5)\big[d(gu_{n+1}, gx) + d(gv_{n+1}, gy)\big] \\ &\quad \preceq_P (a_1 + a_2 + a_3 + a_6)\big[d(gu_n, gx) + d(gv_n, gy)\big]. \end{aligned} \tag{4.3}$$

In a similar way, starting from $d(gx, gu_{n+1})$, resp. $d(gy, gv_{n+1})$, and adding up the obtained inequalities, one gets that

$$\begin{aligned} &(1 - a_4 - a_6)\big[d(gx, gu_{n+1}) + d(gy, gv_{n+1})\big] \\ &\quad \preceq_P (a_1 + a_3 + a_4 + a_5)\big[d(gx, gu_n) + d(gy, gv_n)\big]. \end{aligned} \tag{4.4}$$

Finally, adding up (4.3) and (4.4), we obtain that

$$d(gu_{n+1}, gx) + d(gv_{n+1}, gy) \preceq_P \lambda\big[d(gu_n, gx) + d(gv_n, gy)\big], \tag{4.5}$$

where $\lambda$ is determined as in (3.8), and hence $0 \leq \lambda < 1$.

By inequality (4.5) $n$ time, we have

$$\begin{aligned} &d(gu_n, gx) + d(gv_n, gy) \\ &\quad \preceq_P \lambda\big[d(gu_{n-1}, gx) + d(gv_{n-1}, gy)\big] \end{aligned}$$

$$\begin{aligned}
&\preceq_P \lambda^2\bigl[d(gu_{n-2},gx)+d(gv_{n-2},gy)\bigr]\\
&\vdots\\
&\preceq_P \lambda^n\bigl[d(gu_0,gx)+d(gv_0,gy)\bigr].
\end{aligned}$$

It follows from $\lambda^n[d(gu_0,gx)+d(gv_0,gy)] \to 0_E$ as $n\to\infty$ that

$$d(gu_n,gx)+d(gv_n,gy) \ll c$$

for all $c \in \operatorname{int}(P)$ and large $n$. Since

$$0_E \preceq_P d(gu_n,gx) \preceq_P d(gu_n,gx)+d(gv_n,gy),$$

it follows by $(p_3)$ that $d(gu_n,gx) \ll c$ for large $n$, and so $gu_n \to gx$ when $n\to\infty$. Similarly, $gv_n \to gy$ when $n \to \infty$. By the same procedure, one can show that $gu_n \to gx^*$ and $gv_n \to gy^*$ as $n\to\infty$. By the uniqueness of the limit, we get $gx = gx^*$ and $gy = gy^*$, *i.e.*, (4.2) is proved. Therefore, $(gx,gy)$ is the unique coupled point of coincidence of $F$ and $g$.

Note that if $(gx,gy)$ is a coupled point of coincidence of $F$ and $g$, then $(gy,gx)$ is also a coupled point of coincidence of $F$ and $g$. Then $gx = gy$ and therefore $(gx,gx)$ is the unique coupled point of coincidence of $F$ and $g$.

Next, we show that $F$ and $g$ have a common coupled fixed point. Let $\hat{u} := gx$. Then we have $\hat{u} = gx = F(x,x)$. Since $F$ and $g$ are $w^*$-compatible, we have

$$g\hat{u} = ggx = gF(x,x) = F(gx,gx) = F(\hat{u},\hat{u}).$$

Thus, $(g\hat{u},g\hat{u})$ is a coupled point of coincidence of $F$ and $g$. By the uniqueness of a coupled point of coincidence of $F$ and $g$, we get $g\hat{u} = gx$. Therefore, $\hat{u} = g\hat{u} = F(\hat{u},\hat{u})$, that is, $(\hat{u},\hat{u})$ is a common coupled fixed point of $F$ and $g$.

Finally, we show the uniqueness of a common coupled fixed point of $F$ and $g$. Let $(\tilde{u},\tilde{u}) \in X \times X$ be another common coupled fixed point of $F$ and $g$. So,

$$\tilde{u} = g\tilde{u} = F(\tilde{u},\tilde{u}).$$

Then $(g\hat{u},g\hat{u})$ and $(g\tilde{u},g\tilde{u})$ are two common coupled points of coincidence of $F$ and $g$ and, as was previously proved, it must be $g\hat{u} = g\tilde{u}$, and so $\hat{u} = g\hat{u} = g\tilde{u} = \tilde{u}$. This completes the proof. □

Next, we give some illustrative example which supports Theorem 4.1, while the results of Nashine *et al.* [33] do not.

**Example 4.2** Let $X = \mathbb{R}$ be ordered by the following relation:

$$x \preceq y \quad \Longleftrightarrow \quad x \geq y.$$

Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|f\| = \|f\|_\infty + \|f'\|_\infty$ for all $f \in E$ and

$$P = \bigl\{f \in E : f(t) \geq 0 \text{ for } t \in [0,1]\bigr\}.$$

It is well known (see, *e.g.*, [40]) that the cone $P$ is not normal. Let

$$d(x,y) = |x-y|\varphi$$

for all $x,y \in X$, for a fixed $\varphi \in P$ (*e.g.*, $\varphi(t) = e^t$ for $t \in [0,1]$). Then $(X,d)$ is a complete ordered cone metric space over a non-normal solid cone.

Let $g : X \to X$ and $F : X \times X \to X$ be defined by

$$gx = \frac{x^2}{2} \quad \text{and} \quad F(x,y) = \frac{x^2+y^2}{8}.$$

Consider $y_1 = 2$ and $y_2 = 1$, we have for $x = 3$, we get $y_1 = 2 \preceq 1 = y_2$, but

$$F(x,y_1) = \frac{13}{8} \preceq \frac{10}{8} = F(x,y_2).$$

So, the mapping $F$ does not satisfy the mixed $g$-monotone property. Therefore, Theorems 3.1 and 3.2 of Nashine *et al.* [33] cannot be used to reach this conclusion.

Now, we show that Theorem 4.1 can be used for this case.

Take $a_1 = a_3 = \frac{1}{4}$ and $a_2 = a_4 = a_5 = a_6 = 0$. We will check that the condition (3.1) in Theorem 3.1 holds.

For $x,y,u,v \in X$ satisfying $gu \asymp gx$ and $gv \asymp gy$, we have

$$\begin{aligned} d\bigl(F(x,y),F(u,v)\bigr) &= \left|\frac{x^2+y^2}{8} - \frac{u^2+v^2}{8}\right|\varphi \\ &\leq_P \frac{1}{4}\left|\frac{x^2}{2} - \frac{u^2}{2}\right|\varphi + \frac{1}{4}\left|\frac{y^2}{2} - \frac{v^2}{2}\right|\varphi \\ &= \frac{1}{4}d(gx,gu) + \frac{1}{4}d(gy,gv) \\ &= a_1 d(gx,gu) + a_3 d(gy,gv). \end{aligned}$$

Next, we show that $F$ and $g$ are $w^*$-compatible. We note that if $gx = F(x,x)$, then we get only one case, that is, $x = 0$, and hence

$$gF(x,x) = gF(0,0) = g0 = 0 = F(0,0) = F(g0,g0) = F(gx,gx).$$

Therefore, $F$ and $g$ are $w^*$-compatible.

Moreover, other conditions in Theorem 4.1 are also satisfied. Now, we can apply Theorem 4.1 to conclude the existence of a unique common coupled fixed point of $F$ and $g$ that is a point $(0,0)$.

The following uniqueness result corresponding to Theorem 3.5 can be proved in the same way as Theorem 4.1.

**Theorem 4.3** *In addition to the hypotheses of Theorem* 3.5, *suppose that for every* $(x,y)$, $(x^*,y^*) \in X \times X$, *there exists* $(u,v) \in X \times X$ *such that*

$$\bigl(F(u,v),F(v,u)\bigr) \asymp \bigl(F(x,y),F(y,x)\bigr)$$

*and*

$$\big(F(u,v),F(v,u)\big) \asymp \big(F\big(x^*,y^*\big),F\big(y^*,x^*\big)\big).$$

*If $F$ and $g$ are $w^*$-compatible, then $F$ and $g$ have a unique coupled common fixed point, that is, there exists a unique $(\hat{u},\hat{v}) \in X \times X$ such that*

$$\hat{u} = g\hat{u} = F(\hat{u},\hat{v}) \quad \textit{and} \quad \hat{v} = g\hat{v} = F(\hat{v},\hat{u}).$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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**Acknowledgements**
The authors would like to thank Professor Hichem Ben-El-Mechaiekh and the referee for valuable comments. The second author would like to thank the Research Professional Development Project under the Science Achievement Scholarship of Thailand (SAST) and the third author would like to thank the Commission on Higher Education, the Thailand Research Fund and KMUTT under Grant No. MRG5580213 for financial support during the preparation of this manuscript.

Received: 7 July 2012 Accepted: 15 January 2013 Published: 31 January 2013

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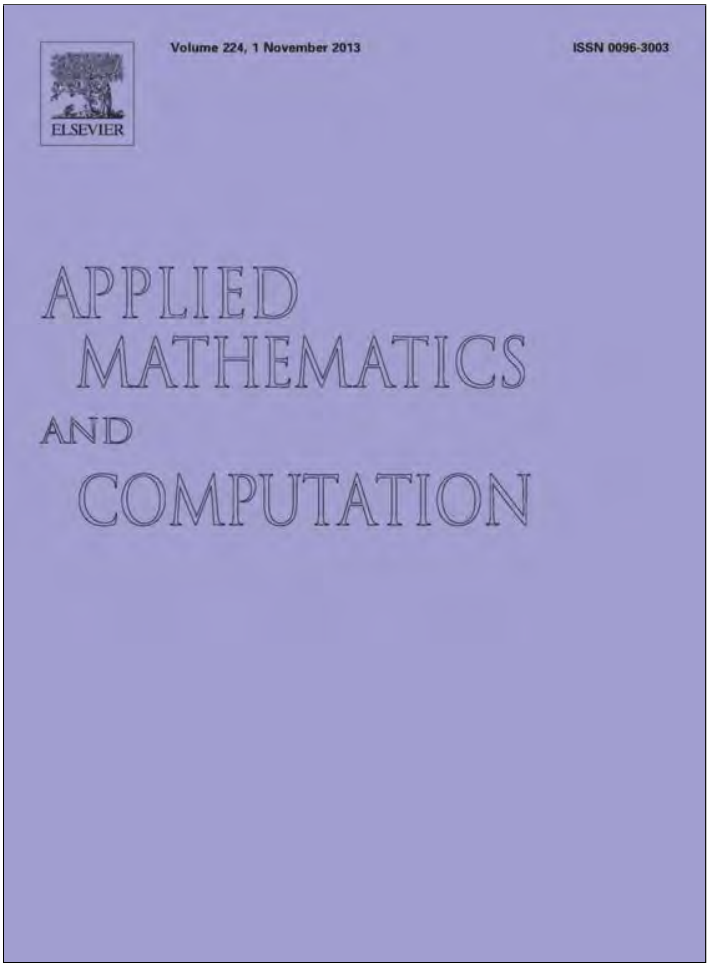
doi:10.1186/1687-1812-2013-22
**Cite this article as:** Agarwal et al.: **Coupled coincidence point and common coupled fixed point theorems lacking the mixed monotone property.** *Fixed Point Theory and Applications* 2013 **2013**:22.


Volume 224, 1 November 2013
ISSN 0096-3003
ELSEVIER
APPLIED MATHEMATICS AND COMPUTATION

Applied Mathematics and Computation 224 (2013) 826–834

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# Some fixed point results for weakly isotone mappings in ordered Banach spaces

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ARTICLE INFO

*Keywords:*
An ordered Banach space
Weakly isotone mapping
Condensing weakly isotone mapping
Condition $D_X$
Common fixed points

ABSTRACT

In this paper, we introduce the new notion which is more general than the previous works on the weakly isotone mappings in the case of single valued mappings and multivalued mappings in an ordered Banach space. We also extend some common fixed point theorems of Dhage et al. [4] and give some applications of the main results. Furthermore, at the end of this paper, we give an open problem for further investigation.



## 1. Introduction

Let $X$ be an arbitrary nonempty set and $f : X \to X$. A fixed point for a mapping $f$ is a point $x \in X$ such that $fx = x$. Fixed point theorem plays the most important role in many fields, so the discussions and studies on its concept provide wide applications in various areas not only in mathematics but also in other branches. For example, in mathematics, fixed point theorems are vital for the existence of a solution to boundary valued problems, integral equations (see e.g.[12,15]). In economics, fixed point results are incredibly useful when it comes to prove the existence of a solution for various types of Nash equilibria (see e.g. [2]). Moreover, there are some applications in chemistry, biology, computer science, and engineering (see e.g. [3,5,8]). The classical contraction mapping principle of Banach is one of the most powerful theorems in fixed point theory. A number of articles in the fixed point theory have been dedicated to the improvement and generalization see in [1,6,7,10,11] and references therein.

In 2003, Dhage et al. [4] introduced the class of weak isotone mappings and the class of countably condensing mappings in an ordered Banach space and they proved some common fixed point theorems for weakly isotone mappings in the case of single valued and multivalued.

**Theorem 1** [4, Theorem 2.1]. *Let $X$ be a closed subset of an ordered Banach space $E$ and let $f, g : X \to X$ be two continuous mappings. If $f$ and $g$ are weakly isotone mappings that satisfy condition $D_X$, then $f$ and $g$ have a common fixed point.*

**Theorem 2** [4, Corollary 2.1]. *Let $X$ be a closed subset of an ordered Banach space $E$ and let $f, g : X \to X$ be two continuous mappings. If $f$ and $g$ are weakly isotone and countably condensing mappings, then $f$ and $g$ have a common fixed point.*

**Theorem 3** [4, Theorem 3.1]. *Let $X$ be a closed subset of an ordered Banach space $E$ and $F, G : X \to C(X)$, where $C(X)$ is class of a nonempty closed subset of $X$. If $F$ and $G$ are closed and weakly isotone mappings that satisfy condition $D_X$, then $F$ and $G$ have a common fixed point.*

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0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2013.09.004

**Theorem 4** [4, Corollary 3.1]. *Let $X$ be a closed subset of an ordered Banach space $E$ and $F, G : X \to C(X)$, where $C(X)$ is class of nonempty closed subset of $X$. If $F$ and $G$ are closed, weakly isotone and countably condensing mappings, then $F$ and $G$ have a common fixed point.*

Inspired by results of Dhage et al. [4], some authors extend these theorems and apply their results to the common solutions for a pair of integral inclusions (see in [13,14]).

Motivated by the interesting works of Dhage et al. [4], we extend the notion of weakly isotone mappings in an ordered Banach space in the case of single valued and multivalued mappings and obtain the coincidence and common fixed point theorems in an ordered Banach space. Moreover, we also obtain the existence theorem for a common solution of two integral equations by using our result. At the end of this paper, we give an open problem for further investigation.

## 2. Preliminaries

In this section, we shall recall some definitions and mathematical preliminaries. In the sequel, $\mathbb{R}, \mathbb{R}_+$ and $\mathbb{N}$ denote the set of real numbers, the set of nonnegative real numbers and the set of positive integers respectively.

**Definition 5.** A binary relation $\preceq$ on a nonempty set $X$ is said to be an *order relation* (and $X$ equipped with $\preceq$ is called a partially ordered set) if it satisfies the following three properties:

(1) reflexivity: $x \preceq x$ for all $x \in X$,
(2) antisymmetry: $x \preceq y$ and $y \preceq x$ imply $x = y$,
(3) transitivity: $x \preceq y$ and $y \preceq z$ imply $x \preceq z$.

Let $(X, \preceq)$ be a partially ordered set and $x, y \in X$. By $x \succeq y$ holds, we mean that $y \preceq x$ holds and by $x \prec y$ holds, we mean that $x \preceq y$ holds but $x \neq y$.

Let $(E, \|\cdot\|_E)$ be a Banach space, $0_E$ be a zero element in $E$ and $P$ be a subset of $E$. Then the subset $P$ is called a *cone* if the following conditions are satisfied:

$(C_1)$ $P$ is nonempty closed and $P \neq \{0_E\}$;
$(C_2)$ if $a, b$ are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$;
$(C_3)$ $P \cap (-P) = \{0_E\}$.

For a cone $P$, define the *partial ordering* $\preceq$ with respect to $P$ by $x \preceq y$ if and only if $y - x \in P$. We write $x \ll y$ to indicate that $y - x \in \operatorname{int}(P)$, where $\operatorname{int}(P)$ stands for the interior of $P$.

The cone $P$ is called *normal* if there is a number $K > 0$ such that for all $x, y \in E$,

$$0_E \preceq x \preceq y \Rightarrow \|x\|_E \leqslant K\|y\|_E.$$

The least positive number satisfying the above is called the *normal constant* of $P$. In 2008, Rezapour and Hamlbarani [9] showed that there are no normal cones with normal constant $K < 1$.

The cone $P$ is said to be *regular* if every increasing sequence which is bounded from the above is convergent, that is, if $\{x_n\}$ is a sequence in $E$ such that

$$x_1 \preceq x_2 \preceq \cdots \preceq y$$

for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty}\|x_n - x\|_E = 0$. Equivalently, the cone $P$ is said to be regular if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone (see also [9]).

In what follows we always suppose that $E$ is a real Banach space with cone $P$ satisfying $\operatorname{int}(P) \neq \emptyset$ (such cones are called *solid*) and $\preceq$ is a partial ordering with respect to $P$.

**Definition 6.** Let $E$ be an ordered Banach space and let $f, g : E \to E$ be two mappings. It is said that $f$ and $g$ are *weakly isotone increasing* if $f(x) \preceq g(f(x))$ and $g(x) \preceq f(g(x))$ for all $x \in E$. Similarly $f$ and $g$ are said to be *weakly isotone decreasing* if $f(x) \succeq g(f(x))$ and $g(x) \succeq f(g(x))$ for all $x \in E$. If $f$ and $g$ are either weakly isotone increasing or weakly isotone decreasing, then it is said that $f$ and $g$ are *weakly isotone*.

**Definition 7.** Let $E$ be an ordered Banach space and let $x \in E$. A mapping $f : E \to E$ is said to be *continuous in $x$* if $f(x_n) \to f(x)$ as $n \to \infty$ for each sequence $\{x_n\}$ that converges to $x$.

**Definition 8.** Let $E$ be an ordered Banach space and let $x \in E$. A mapping $f : E \to E$ is said to be *monotone-continuous in $x$* if $f(x_n) \to f(x)$ as $n \to \infty$ for each increasing or decreasing sequence $\{x_n\}$ that converges to $x$.

**Definition 9.** Let $E$ be an ordered Banach space and $X \subseteq E$. Two mappings $f, g : X \to X$ are said to satisfy the *condition $D_X$* if for any countable set $A$ of $X$ and for any fixed $a \in X$ the condition

$$A \subseteq \{a\} \cup f(A) \cup g(A),$$

implies $\overline{A}$ is compact, where $\overline{A}$ denotes the closure of $A$.

**Definition 10.** Let $E$ be an ordered Banach space and $X \subseteq E$. Two mappings $f, g : X \to X$ are said to satisfy the *weak-condition* $D_X$ if for any monotone sequence $\{x_n\}$ and for any fixed $a \in X$ the condition

$$\{x_1, x_2, x_3, \ldots\} \subseteq \{a\} \cup f(\{x_1, x_2, x_3, \ldots\}) \cup g(\{x_1, x_2, x_3, \ldots\}),$$

implies $\{x_n\}$ is convergent.

We note that if $f$ and $g$ satisfy the condition $D_X$, then they satisfy the weak-condition $D_X$. In the next example, we show that the converse of the previous sentence is not true.

**Example 1.** Let $E = \mathbb{R}^2$ with the usual norm and $P = \{(z, z) \in \mathbb{R}^2 \,|\, z \geqslant 0\}$. Let

$$X = \{x \in E \,|\, \|x\| \leqslant 2\} \cup \{(u, 0) \,|\, u \in \mathbb{R},\ u > 2\}$$

and $f, g : X \to X$ be defined by

$$f(x) = \begin{cases} (1,1) & \text{if } \|x\| \leqslant 2, \\ (0,0) & \text{if } \|x\| > 2 \end{cases}$$

and $g(x) = x$ for all $x \in X$. Then mappings $f$ and $g$ satisfy the weak-condition $D_X$, but do not verify the condition $D_X$.

For any subset $A$ of $E$, $diam(A)$ denotes the diameter of $A$, that is

$$diam(A) := sup\{d(x, y) \,|\, x, y \in A\}$$

and $\circ$ denotes the composition of mappings. The Kuratowski measure of noncompactness for a bounded subset $A$ of $E$ is defined by

$$\alpha(A) := \inf\{r > 0 \,|\, A \subseteq \cup_{i=1}^{n} A_i \text{ and } diam(A) \leqslant r \quad \text{for } i = 1, 2, 3, \ldots, n\}.$$

**Definition 11.** Let $E$ be an ordered Banach space and let $X \subseteq E$. A mapping $f : X \to E$ is said to be *countably condensing* if $f(X)$ is bounded and for every countably bounded set $A \subseteq X$ with $\alpha(A) > 0$ we have $\alpha(f(A)) < \alpha(A)$.

**Definition 12.** Let $E$ be an ordered Banach space and let $X \subseteq E$. Two mappings $f, g : X \to E$ are said to be a *monotone-condensing* if $f(X)$ and $(g \circ f)(X)$ are bounded and for every bounded monotone sequence $\{x_n\}$ with

$\alpha(\{x_1, x_2, x_3, \ldots\}) > 0$ and $\alpha(f(\{x_1, x_2, x_3, \ldots\})) > 0$,

we have

$\alpha((g \circ f)(\{x_1, x_2, x_3, \ldots\})) < \alpha(\{x_1, x_2, x_3, \ldots\})$.

**Remark 13.** Let $E$ be an ordered Banach space and let $X \subseteq E$ and $f, g : X \to E$. If $f$ and $g$ are countably condensing, then they are a monotone-condensing.

For a nonempty set $A$, we let $2^A$ stands for the family of all nonempty subsets of $A$ and $C(A)$ stands for the class of closed subset of $A$. For a multivalued mapping $F : A \to 2^A$, we use the notation $F(A) := \cup_{x \in A} F(x)$.

**Definition 14.** Let $E$ be an ordered Banach space and let $X \subseteq E$. A mapping $F : X \to 2^E$ is said to be *countably condensing* if $F(X)$ is bounded and for every countably bounded set $A \subseteq X$ with $\alpha(A) > 0$ we have $\alpha(F(A)) < \alpha(A)$.

**Definition 15.** Let $X$ be a closed subset of an ordered Banach space $E$. Two mappings $F, G : X \to 2^E$ are said to be a *monotone-condensing* if $F(X)$ and $(G \circ F)(X)$ are bounded and for every bounded monotone sequence $\{x_n\}$ with

$\alpha(\{x_1, x_2, x_3, \ldots\}) > 0$ and $\alpha(F(\{x_1, x_2, x_3, \ldots\})) > 0$,

we have

$\alpha((G \circ F)(\{x_1, x_2, x_3, \ldots\})) < \alpha(\{x_1, x_2, x_3, \ldots\})$.

## 3. Coincidence and common fixed point theorems for single valued mappings

We begin this section by introducing the notion of weakly isotone and weak-condition $D_X$ for tree single mappings. In the following, we always assume that $E$ is an ordered Banach space, $P$ is a cone in $E$ and $\preceq$ is a partial ordering with respect to $P$.

Let $E$ be a nonempty set and $h : E \to E$ be a given mapping. For every $x \in E$, we denote by $h^{-1}(x)$ the subset of $E$ defined by:

$$h^{-1}(x) := \{u \in E \,|\, hu = x\}.$$

**Definition 16.** Let $E$ be an ordered Banach space and $f, g, h : E \to E$ be given mappings such that $f(E) \subseteq h(E)$ and $g(E) \subseteq h(E)$. We say that $f$ and $g$ are *weakly isotone increasing with respect to h* if and only if for all $x \in E$, we have:

$$f(x) \preceq g(y), \quad \forall\, y \in h^{-1}(f(x)) \tag{1}$$

and

$$g(x) \preceq f(y), \quad \forall\, y \in h^{-1}(g(x)). \tag{2}$$

Similarly $f$ and $g$ are said to be *weakly isotone decreasing with respect to h* if

$$f(x) \succeq g(y), \quad \forall\, y \in h^{-1}(f(x)) \tag{3}$$

and

$$g(x) \succeq f(y), \quad \forall\, y \in h^{-1}(g(x)) \tag{4}$$

for all $x \in E$. If $f$ and $g$ are either weakly isotone increasing with respect to $h$ or weakly isotone decreasing with respect to $h$, then it is said that $f$ and $g$ are *weakly isotone with respect to h*.

**Remark 17.** If $h : E \to E$ is the identity mapping ($h(x) = x$ for all $x \in E$, shortly $h = I_E$), then the fact that $f$ and $g$ are weakly isotone increasing with respect to $h$ implies that $f$ and $g$ are weakly isotone increasing mappings.

**Example 2.** Consider $E = \mathbb{R}_+$ endowed with the usual norm and

$$P = \{z \in \mathbb{R} \,|\, z \geqslant 0\}.$$

Let $f, g, h : E \to E$ by

$$f(x) = 5 \quad \text{for all } x \in E, \quad g(x) = \begin{cases} x & \text{if } x \in [0, 5] \\ 5 & \text{if } x > 5 \end{cases}$$

and

$$h(x) = x \quad \text{for all } x \in E.$$

Then, we obtain that $h^{-1}(f(x)) = \{5\}$ and $h^{-1}(g(x)) = \{g(x) \,|\, x \in E\}$. Since $f(E), g(E) \subseteq h(E)$ and conditions (1) and (2) hold, we have $f$ and $g$ are weakly isotone increasing with respect to $h$.

**Example 3.** Consider $E = [0, 2]$ endowed with the usual norm and

$$P = \{z \in \mathbb{R} \,|\, z \geqslant 0\}.$$

Let $f, g, h : E \to E$ by

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \in [0, 1), \\ 0 & \text{if } x \in [1, 2], \end{cases}$$

$$g(x) = \begin{cases} x & \text{if } x \in [0, 1), \\ 0 & \text{if } x \in [1, 2] \end{cases}$$

and

$$h(x) = \begin{cases} x^2 & \text{if } x \in [0, 1), \\ 0 & \text{if } x \in [1, 2]. \end{cases}$$

Now we show that the condition (1) holds. We discuss this in two cases.

Case 1: ($x = 0$ or $x \geqslant 1$) In this case, we get $f(x) = 0$. It is easy to see that (1) holds.
Case 2: ($x \in (0, 1)$) In this case, we get $f(x) = \sqrt{x}$. Let $y \in h^{-1}(f(x))$ and then $h(y) = f(x) = \sqrt{x}$. By definition of mapping $h$, we have $y = \sqrt[4]{x}$. Thus

$$f(x) = \sqrt{x} \preceq \sqrt[4]{x} = g(\sqrt[4]{x}) = g(y).$$

Next, we will show that the condition (2) holds. We discuss this in two cases.

Case 1: ($x = 0$ or $x \geqslant 1$) In this case, we get $g(x) = 0$. It obvious that (2) holds.
Case 2: ($x \in (0, 1)$) In this case, we get $g(x) = x$. Let $y \in h^{-1}(g(x))$ and then $h(y) = g(x) = x$. By definition of mapping $h$, we have $y = \sqrt{x}$. Thus

$$g(x) = x \preceq \sqrt[4]{x} = f(\sqrt{x}) = f(y).$$

Moreover, we obtain that $f(E) = g(E) \subseteq h(E)$. Therefore, $f$ and $g$ are weakly isotone increasing with respect to $h$.

**Definition 18.** Let $X$ be subset of an ordered Banach space $E$ and $f, g, h : X \to X$. Two mappings $f, g : X \to X$ are said to satisfy the *weak-condition $D_X$ with respect to h* if for any monotone sequence $\{h(x_n)\}$ and for any fixed $a \in X$ the condition

$$\{h(x_1), h(x_2), h(x_3), \ldots\} \subseteq \{a\} \cup f(\{x_1, x_2, x_3, \ldots\}) \cup g(\{x_1, x_2, x_3, \ldots\})$$

implies $\{x_n\}$ is convergent.

**Remark 19.** If $h : E \to E$ is the identity mapping, then the $f$ and $g$ are weak-condition $D_X$ with respect to $h$ implies that $f$ and $g$ are weak-condition $D_X$.

Let $X$ be a closed subset of an ordered Banach space $E$ and $f, g, h : X \to X$ be three mappings such that $f$ and $g$ are weakly isotone with respect to $h$. Given $x_0 \in X$ we define a sequence $\{h(x_n)\}$ in $X$ as follows:

$$h(x_{2n-1}) = f(x_{2n-2}), \quad h(x_{2n}) = g(x_{2n-1})$$

for all $n \in \mathbb{N}$. We say that $\{h(x_n)\}$ is an $(f, g, h)-$sequence with initial point $x_0$. If $h$ is an identity mapping on $X$, then we write $(f, g)$-sequence with initial point $x_0$. Later on, we denote by $C(f, g, h)$ the set of coincidence points of $f, g$ and $h$, that is, $C(f, g, h) = \{x \in E : h(x) = f(x) = g(x)\}$.

Next, we give the coincidence point theorems for weakly isotone mappings under certain conditions.

**Theorem 20.** *Let $X$ be a closed subset of an ordered Banach space $E$ and $f, g, h : X \to X$ be three continuous mappings. If $f$ and $g$ are weakly isotone with respect to $h$ and if $f$ and $g$ satisfy weak-condition $D_X$ with respect to $h$, then $f, g$ and $h$ have a coincidence point. Moreover, for every $x_0 \in X$, we have $\lim_{n\to\infty} h(x_n) \in C(f, g, h)$ for every $(f, g, h)$-sequence $\{h(x_n)\}$ with initial point $x_0$.*

**Proof.** Assume that $f$ and $g$ are weakly isotone increasing with respect to $h$. By Definition 16, it follows that $f(X) \subseteq h(X)$ and $g(X) \subseteq h(X)$. Let $x_0$ be an arbitrary point in $X$. Since $f(X) \subseteq h(X)$, there exists $x_1 \in X$ such that $h(x_1) = f(x_0)$. Since $g(X) \subseteq h(X)$, there exists $x_2 \in X$ such that $h(x_2) = g(x_1)$.

Continuing this process, we can construct a sequence $\{h(x_n)\}$ in $X$ defined by

$$h(x_{2n-1}) = f(x_{2n-2}), \quad h(x_{2n}) = g(x_{2n-1}), \quad \forall\, n \in \mathbb{N}. \tag{5}$$

By the construction, we have $x_1 \in h^{-1}(f(x_0))$ and $x_2 \in h^{-1}(g(x_1))$, then using the fact that $f$ and $g$ are weakly isotone increasing with respect to $h$, we get that

$$h(x_1) = f(x_0) \preceq g(x_1) = h(x_2) \preceq f(x_2) = h(x_3).$$

We continue this process to get

$$h(x_1) \preceq h(x_2) \preceq \cdots \preceq h(x_{2n-1}) \preceq h(x_{2n}) \preceq \cdots \tag{6}$$

for all $n \in \mathbb{N}$ Now, we have

$$\{h(x_1), h(x_2), h(x_3), \ldots\} = \{h(x_1)\} \cup \{h(x_3), h(x_5), \ldots\} \cup \{h(x_2), h(x_4), \ldots\} \subseteq \{h(x_1)\} \cup f(\{x_1, x_2, x_3, \ldots\}) \cup g(\{x_1, x_2, x_3, \ldots\}).$$

From the hypothesis that is $f$ and $g$ are weak-conditions $D_X$ with respect to $h$, we have the sequence $\{x_n\}$ converge to some $x \in X$. Since $f, g, h$ are continuous, we get

$$h(x) = \lim_{n\to\infty} h(x_{2n-1}) = \lim_{n\to\infty} f(x_{2n-2}) = f(x)$$

and

$$h(x) = \lim_{n\to\infty} h(x_{2n}) = \lim_{n\to\infty} g(x_{2n-1}) = g(x)$$

It follows that $f(x) = g(x) = h(x)$ and so $x$ is a coincidence point of $f, g$ and $h$. For the case when $f$ and $g$ are weakly isotone decreasing with respect to $h$ is similar. □

Next theorem, we propose the weakening assumption of continuous of $f$ and $g$ in Theorem 20 by monotone-continuous condition.

**Theorem 21.** *Let $X$ be a closed subset of an ordered Banach space $E$ and let $f, g : X \to X$ be two monotone-continuous mappings. If $f$ and $g$ are weakly isotone and if $f$ and $g$ satisfy weak-condition $D_X$, then $f$ and $g$ have a common fixed point. Moreover, for every $x_0 \in X$, we have $\lim_{n\to\infty} x_n \in F(f, g)$ for every $(f, g)$-sequence $\{x_n\}$ with initial point $x_0$, where $F(f, g)$ is set of common fixed points of $f$ and $g$.*

**Proof.** With the same argument of Theorem 20 we can prove this theorem if a mapping $h$ is an identity mapping on $X$. □

Since the condition $D_X$ implies the weak-condition $D_X$, we get the following results.

**Theorem 22.** *Let $X$ be a closed subset of an ordered Banach space $E$ and let $f, g : X \to X$ be two monotone-continuous mappings. If $f$ and $g$ are weakly isotone and if $f$ and $g$ satisfy condition $D_X$, then $f$ and $g$ have a common fixed point. Moreover, for every $x_0 \in X$, we have $\lim_{n\to\infty} x_n \in F(f,g)$ for every $(f,g)$-sequence $\{x_n\}$ with initial point $x_0$, where $F(f,g)$ is set of common fixed point of $f$ and $g$.*

Since the continuous mapping implies the monotone-continuous mapping, we get Dhage et al.'s results in [4].

**Corollary 23** [4, Theorem 2.1]. *Let $X$ be a closed subset of an ordered Banach space $E$ and let $f, g : X \to X$ be two continuous mappings. If $f$ and $g$ are weakly isotone and if $f$ and $g$ satisfy condition $D_X$, then $f$ and $g$ have a common fixed point.*

**Corollary 24** [4, Corollary 2.1]. *Let $X$ be a closed subset of an ordered Banach space $E$ and let $f, g : X \to X$ be two continuous mappings. If $f$ and $g$ are weakly isotone and if $f$ and $g$ are countably condensing, then $f$ and $g$ have a common fixed point.*

## 4. Coincidence and common fixed point theorems for single valued and multivalued mappings

In this section, we give the coincidence and common fixed point results for single valued and multivalued mapping. Let $E$ be an ordered Banach space, $P$ be a cone in $E$ and $\preceq$ is a partial ordering with respect to $P$. For $X, Y \in 2^E$, we will write $X \precapprox Y$ mean that $x \preceq y$ for all $x \in X$ and $y \in Y$. Moreover, $X \succapprox Y$ mean that $Y \precapprox X$.

**Example 4.** Consider $E = \mathbb{R}^2_+$ endowed with the usual norm and

$$P = \{(z,z) \in \mathbb{R}^2_+ \,|\, z \geqslant 0\}.$$

Let

$$X := \{(x,x) \in E : \|(x,x)\| \leqslant 1\}$$

and

$$Y := \{(y,y) \in E : 9 \leqslant \|(y,y)\| \leqslant 16\}.$$

Then, we have $X \precapprox Y$.

Let $E$ be a nonempty set and $h : E \to E$ be a given mapping. For every $A \subseteq E$, we denote by $h^{-1}(A)$ the subset of $E$ defined by:

$$h^{-1}(A) := \{x \in E \,|\, h(x) \in A\}.$$

**Definition 25.** Let $E$ be an ordered Banach space, $F, G : E \to 2^E$ and $h : E \to E$ be given mappings such that $F(E) \subseteq h(E)$ and $G(E) \subseteq h(E)$. We say that $F$ and $G$ are *weakly isotone increasing with respect to h* if and only if for all $x \in E$, we have:

$$F(x) \precapprox G(y), \quad \forall\, y \in h^{-1}(F(x)) \tag{7}$$

and

$$G(x) \precapprox F(y), \quad \forall\, y \in h^{-1}(G(x)). \tag{8}$$

Similarly $F$ and $G$ are said to be *weakly isotone decreasing with respect to h* if

$$F(x) \succapprox G(y), \quad \forall\, y \in h^{-1}(F(x)) \tag{9}$$

and

$$G(x) \succapprox F(y), \quad \forall\, y \in h^{-1}(G(x)) \tag{10}$$

for all $x \in E$. We say that $F$ and $G$ are *weakly isotone with respect to h* if $F$ and $G$ are weakly isotone increasing with respect to $h$ or weakly isotone decreasing with respect to $h$.

**Remark 26.** For $h = I_E$, we obtain that $F$ and $G$ are weakly isotone with respect to $h$ implies that $F$ and $G$ are weakly isotone.

**Definition 27.** Let $E$ be an ordered Banach space. A mapping $F : E \to 2^E$ is said to be *closed* if for each sequence $\{x_n\}$ in $E$ with $\lim_{n\to\infty} x_n = x_0$ for some $x_0 \in E$, and for each sequence $\{y_n\}$ in $E$ with $y_n \in F(x_n)$ and $\lim_{n\to\infty} y_n = y_0$ for some $y_0 \in E$, we have $y_0 \in F(x_0)$.

**Definition 28.** Let $E$ be an ordered Banach space. A mapping $F : E \to 2^E$ is said to be *monotone-closed* if for each increasing or decreasing sequence $\{x_n\}$ in $E$ with $\lim_{n\to\infty} x_n = x_0$ for some $x_0 \in E$, and for each sequence $\{y_n\}$ in $E$ with $y_n \in F(x_n)$ and $\lim_{n\to\infty} y_n = y_0$ for some $y_0 \in E$, we have $y_0 \in F(x_0)$.

**Definition 29.** Let $E$ be an ordered Banach space and $X \subseteq E$. Two multivalued mappings $F, G : X \to 2^X$ are said to satisfy the *condition* $D_X$ if for any countable set $A$ of $X$ and for any fixed $a \in X$ the condition

$$A \subseteq \{a\} \cup F(A) \cup G(A)$$

implies $\overline{A}$ is compact.

**Definition 30.** Let $E$ be an ordered Banach space, $X \subseteq E$. Two multivalued mappings $F, G : X \to 2^X$ are said to satisfy the *weak-condition* $D_X$ if for any monotone sequence $\{x_n\}$ and for any fixed $a \in X$ the condition

$$\{x_1, x_2, x_3, \ldots\} \subseteq \{a\} \cup F(\{x_1, x_2, x_3, \ldots\}) \cup G(\{x_1, x_2, x_3, \ldots\})$$

implies $\{x_n\}$ is convergent.

**Remark 31.** We obtain that condition $D_X$ implies a weak-condition $D_X$ in the case of multivalued mappings.

**Definition 32.** Let $E$ be an ordered Banach space, $X \subseteq E$ and $h : X \to X$. Two multivalued mappings $F, G : X \to 2^X$ are said to satisfy the *weak-condition* $D_X$ *with respect to* $h$ if for any monotone sequence $\{h(x_n)\}$ and for any fixed $a \in X$ the condition

$$\{h(x_1), h(x_2), h(x_3), \ldots\} \subseteq \{a\} \cup F(\{x_1, x_2, x_3, \ldots\}) \cup G(\{x_1, x_2, x_3, \ldots\})$$

implies $\{x_n\}$ is convergent.

**Remark 33.** If we take $h = I_E$, then $F$ and $G$ are weak-condition $D_X$ with respect to $h$ implies that $F$ and $G$ are weak-conditions $D_X$.

Let $X$ be a closed subset of an ordered Banach space $E$ and $h : X \to X$. Given $F, G : X \to 2^X$ be two multivalued mappings such that $F$ and $G$ are weakly isotone with respect to $h$ and given $x_0 \in X$ we define a sequence $\{h(x_n)\}$ in $X$ as follows:

$$h(x_{2n-1}) \in F(x_{2n-2}), \quad h(x_{2n}) \in G(x_{2n-1})$$

for all $n \in \mathbb{N}$. We say that $\{h(x_n)\}$ is a $(F, G, h)-$sequence with initial point $x_0$. If $h$ is an identity mapping on $X$, then we write $(F, G)-$sequence with initial point $x_0$. Later on, we denote

$$\mathcal{CO}(F, G, h) := \{x \in E \,|\, h(x) \in F(x) \text{ and } h(x) \in G(x)\}$$

and denote $\mathcal{F}(F, G)$ is the set of all common fixed points of $F$ and $G$, that is,

$$\mathcal{F}(F, G) := \{x \in E \,|\, x \in F(x) \text{ and } x \in G(x)\}.$$

Now, we establish the coincidence point theorems for weakly isotone increasing for single valued and multivalued mappings under certain conditions.

**Theorem 34.** *Let $X$ be a closed subset of an ordered Banach space $E$ and let $F, G : X \to 2^X$ be two closed mappings and $h : X \to X$ be a continuous mapping. If $F$ and $G$ are weakly isotone with respect to $h$ and satisfy weak-condition $D_X$ with respect to $h$, then there is a point $z \in X$ such that $z \in \mathcal{CO}(F, G, h)$. Moreover, for every $x_0 \in X$, we have $\lim_{n\to\infty} h(x_n) \in \mathcal{CO}(F, G, h)$ for every $(F, G, h)-$sequence $\{h(x_n)\}$ with initial point $x_0$.*

**Proof.** Assume that $F$ and $G$ are weakly isotone increasing with respect to $h$. By Definition 25, we have $F(X) \subseteq h(X)$ and $G(X) \subseteq h(X)$. Let $x_0$ be an arbitrary point in $X$. Since $F(X) \subseteq h(X)$, we get $F(x_0) \subseteq h(X)$ and so there exists $x_1 \in X$ such that $h(x_1) \in F(x_0)$. It follows from $G(X) \subseteq h(X)$ that $G(x_1) \subseteq h(X)$ and then there exists $x_2 \in X$ such that $h(x_2) \in G(x_1)$. Continuing this process, we can construct a sequence $\{h(x_n)\}$ in $X$ defined by

$$h(x_{2n-1}) \in F(x_{2n-2}), \quad h(x_{2n}) \in G(x_{2n-1}), \quad \forall\, n \in \mathbb{N}. \tag{11}$$

By construction, we have $x_1 \in h^{-1}(F(x_0))$ and $x_2 \in h^{-1}(G(x_1))$, then using the hypothesis that $F$ and $G$ are weakly isotone increasing with respect to $h$, we have $F(x_0) \precsim G(x_1)$ and $G(x_1) \precsim F(x_2)$. We continue this process to get

$$F(x_0) \precsim G(x_1) \precsim \cdots \precsim F(x_{2n-2}) \precsim G(x_{2n-1}) \precsim \cdots \tag{12}$$

for all $n \in \mathbb{N}$. From (11) and the definition of $\precsim$, we get

$$h(x_1) \preceq h(x_2) \preceq \cdots \preceq h(x_{2n-1}) \preceq h(x_{2n}) \preceq \cdots \tag{13}$$

for all $n \in \mathbb{N}$. Now, we have

$$\{h(x_1), h(x_2), h(x_3), \ldots\} = \{h(x_1)\} \cup \{h(x_3), h(x_5), \ldots\} \cup \{h(x_2), h(x_4), \ldots\} \subseteq \{h(x_1)\} \cup F(\{x_1, x_2, x_3, \ldots\}) \cup G(\{x_1, x_2, x_3, \ldots\}).$$

From the hypothesis that is $F$ and $G$ are weak-conditions $D_X$ with respect to $h$, we get the sequence $\{x_n\}$ converge to some $x \in X$. Since $F, G$ are closed mappings and $h$ is a continuous mapping, we get

$$h(x) = \lim_{n\to\infty} h(x_{2n-1}) \in \lim_{n\to\infty} F(x_{2n-2}) = F(x)$$

and

$$h(x) = \lim_{n\to\infty} h(x_{2n}) \in \lim_{n\to\infty} G(x_{2n-1}) = G(x).$$

Therefore, $x \in \mathcal{CO}(F, G, h)$. For the case when $F$ and $G$ are weakly isotone decreasing with respect to $h$ is similar.

If we take the mapping $h$ in Theorem 34 as the identity mapping on $X$, we get the following results for two monotone-closed mappings. $\square$

**Theorem 35.** *Let $X$ be a closed subset of an ordered Banach space $E$ and let $F, G : X \to 2^X$ be two monotone-closed mappings. If $F$ and $G$ are weakly isotone and satisfy weak-condition $D_X$, then $F$ and $G$ have a common fixed point. Moreover, for every $x_0 \in X$, we have $\lim_{n\to\infty} x_n \in \mathcal{F}(F, G)$ for every $(F, G)$-sequence with initial point $x_0$.*

Since the condition $D_X$ implies the weak-condition $D_X$, we can omit the proof of the following result.

**Theorem 36.** *Let $X$ be a closed subset of an ordered Banach space $E$ and let $F, G : X \to 2^X$ be two monotone-closed mappings. If $F$ and $G$ are weakly isotone and satisfy the condition $D_X$, then $F$ and $G$ have a common fixed point. Moreover, for every $x_0 \in X$, we have $\lim_{n\to\infty} x_n \in \mathcal{F}(F, G)$ for every $(F, G)-$sequence with initial point $x_0$.*

From the fact that every closed mapping is a monotone-closed mapping, we get the following results of Dhage et al. [4].

**Corollary 37** [4,Theorem 3.1]. *Let $X$ be a closed subset of an ordered Banach space $E$ and $F, G : X \to C(X)$. If $F$ and $G$ are closed and weakly isotone mappings that satisfy condition $D_X$, then $F$ and $G$ have a common fixed point.*

**Corollary 38** [4,Corollary 3.1]. *Let $X$ be a closed subset of an ordered Banach space $E$ and $F, G : X \to C(X)$. If $F$ and $G$ are closed, weakly isotone and countably condensing mappings, then $F$ and $G$ have a common fixed point.*

## 5. Some applications

Let $\mathbb{R}$ be the real line, $E$ be a Banach space with norm $\|\cdot\|_E$ and let $C(E)$ denote the class of all nonempty closed subsets of $E$. Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$.

In this section, we consider the integral inclusions

$$x(t) \in q(t) + \int_0^{\sigma(t)} k(t, s) F(s, x(s))\, ds \tag{14}$$

$$x(t) \in q(t) + \int_0^{\sigma(t)} k(t, s) G(s, x(s))\, ds \tag{15}$$

for $t \in J$, where $\sigma : J \to J, q : J \to E, k : J \times J \to \mathbb{R}$ are continuous and $F, G : J \times E \to C(E)$.

By a common solution for the integral inclusions (14) and (15), we mean a continuous function $x : J \to E$ such that

$$x(t) = q(t) + \int_0^{\sigma(t)} k(t, s) v_1(s)\, ds$$

and

$$x(t) = q(t) + \int_0^{\sigma(t)} k(t, s) v_2(s)\, ds$$

for some $v_1, v_2 \in B(J, E)$ satisfying $v_1(t) \in F(t, x(t))$ and $v_2(t) \in G(t, x(t))$, for all $t \in J$, where $B(J, E)$ is the space of all $E$-valued Bochner integrable functions on $J$.

In 2007 Turkoglu and Altun [13] proved an existence theorem of common solutions for the integral inclusions (14) and (15) via a common fixed point theorem of Dhage et al. [4]. We can obtain a similar result as a consequence of Theorem 4.2.

## 6. Open problems:

- Can the notion of monotone-condensing for two single valued mappings (see Definition 12) be extended to the case of three single valued mappings?
- Can the notion of monotone-condensing for two multivalued mappings (see Definition 15) be extended to cases of single valued and multivalued mappings?
- Can the idea in this paper be applied to the condition $R$ for single valued and multivalued mappings (see in the work of Dhage et al. [4])?
- Can main theorems in this paper be applied to other integral inclusions (14) and (15)?

## Acknowledgements

The second author would like to thank the Commission on Higher Education, the Thailand Research Fund and the King Mongkut's University of Technology Thonburi (KMUTT) under Grant No. MRG5580213 for financial support during the preparation of this manuscript. Moreover, the authors thank the editors and referees for their insightful comments.

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Fixed Point Theory and Applications
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# Best proximity points for Geraghty's proximal contraction mappings

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**Abstract**
In this paper, we generalized the notion of proximal contractions of the first and second kinds by using Geraghty's theorem and establish best proximity point theorems for proximal contractions. Our results improve and extend the recent results of Sadiq Basha and some others.
**MSC:** 47H09; 47H10

**Keywords:** fixed point; best proximity point; Geraghty's proximal contraction mapping; proximal contraction mapping

## 1 Introduction

Several problems can be modeled as equations of the form $Tx = x$, where $T$ is a given self-mapping defined on a subset of a metric space, a normed linear space, a topological vector space or some suitable space. However, if $T$ is a nonself-mapping from $A$ to $B$, then the aforementioned equation does not necessarily admit a solution. In this case, it is contemplated to find an approximate solution $x$ in $A$ such that the error $d(x, Tx)$ is minimum, where $d$ is the distance function. In view of the fact that $d(x, Tx)$ is at least $d(A, B)$, a best proximity point theorem guarantees the global minimization of $d(x, Tx)$ by the requirement that an approximate solution $x$ satisfies the condition $d(x, Tx) = d(A, B)$. Such optimal approximate solutions are called best proximity points of the mapping $T$. Interestingly, best proximity theorems also serve as a natural generalization of fixed point theorems, for a best proximity point becomes a fixed point if the mapping under consideration is a self-mapping.

A classical best approximation theorem was introduced by Fan [1], that is, if $A$ is a nonempty compact convex subset of a Hausdorff locally convex topological vector space $B$ and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterward, several authors, including Prolla [2], Reich [3], Sehgal and Singh [4, 5], derived the extensions of Fan's theorem in many directions. Other works on the existence of a best proximity point for contractions can be seen in [6–14].

In 1922, Banach proved that every contractive mapping in a complete metric spaces has a unique fixed point, which is called Banach's fixed point theorem or Banach's contraction principle. Since Banach's fixed point theorem, many authors have extended, improved and generalized this theorem in several ways. Some applications of Banach's fixed point theorem can be found in [15–18]. One of such generalizations is due to Geraghty [19] as follows.

**Theorem 1.1** [19] *Let $(X,d)$ be a complete metric space and let $f$ be a self-mapping on $X$ such that for each $x,y \in X$ satisfying*

$$d(fx,fy) \leq \alpha\big(d(x,y)\big)d(x,y), \tag{1.1}$$

*where $\alpha \in \mathcal{S}$, $\mathcal{S}$ is the family of functions from $[0,\infty)$ into $[0,1)$ which satisfies the condition*

$$\alpha(t_n) \to 1 \quad \Rightarrow \quad t_n \to 0.$$

*Then the sequence $\{f_n\}$ converges to the unique fixed point of $f$ in $X$.*

In 2005, Eldred *et al.* [20] obtained best proximity point theorems for relatively nonexpansive mappings. Best proximity point theorems for several types of contractions were established in [21–25].

Recently, Sadiq Basha in [26] gave necessary and sufficient conditions to claim the existence of a best proximity point for proximal contractions of the first kind and the second kind, which are non-self mapping analogues of contraction self-mappings, and also established some best proximity and convergence theorems.

The aim of this paper is to introduce the new classes of proximal contractions, which are more general than a class of proximal contractions of the first and second kinds, by giving the necessary condition to have best proximity points, and we also give some illustrative example of our main results. The results of this paper are extension and generalizations of the main result of Sadiq Basha in [26] and some results in the literature.

## 2 Preliminaries

Given nonempty subsets $A$ and $B$ of a metric space $(X,d)$, we recall the following notations and notions that will be used in what follows.

$$\begin{aligned}
&d(A,B) := \inf\big\{d(x,y) : x \in A \text{ and } y \in B\big\},\\
&A_0 := \big\{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\big\},\\
&B_0 := \big\{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\big\}.
\end{aligned}$$

If $A \cap B \neq \emptyset$, then $A_0$ and $B_0$ are nonempty. Further, it is interesting to notice that $A_0$ and $B_0$ are contained in the boundaries of $A$ and $B$, respectively, provided $A$ and $B$ are closed subsets of a normed linear space such that $d(A,B) > 0$ (see [27]).

**Definition 2.1** [26] A mapping $T : A \to B$ is called a *proximal contraction of the first kind* if there exists $k \in [0,1)$ such that

$$\left.\begin{aligned} d(u,Tx) = d(A,B)\\ d(v,Ty) = d(A,B) \end{aligned}\right\} \quad \Longrightarrow \quad d(u,v) \leq kd(x,y)$$

for all $u,v,x,y \in A$.

It is easy to see that a self-mapping that is a proximal contraction of the first kind is precisely a contraction. However, a nonself-proximal contraction is not necessarily a contraction.

**Definition 2.2** [26] A mapping $T : A \to B$ is called a *proximal contraction of the second kind* if there exists $k \in [0,1)$ such that

$$\left.\begin{aligned} d(u, Tx) = d(A,B) \\ d(v, Ty) = d(A,B) \end{aligned}\right\} \quad \Longrightarrow \quad d(Tu, Tv) \le kd(Tx, Ty)$$

for all $a, b, x, y \in A$.

**Definition 2.3** Let $S : A \to B$ and $T : B \to A$ be mappings. The pair $(S, T)$ is called a *proximal cyclic contraction pair* if there exists $k \in [0,1)$ such that

$$\left.\begin{aligned} d(a, Sx) = d(A,B) \\ d(b, Ty) = d(A,B) \end{aligned}\right\} \quad \Longrightarrow \quad d(a,b) \le kd(x,y) + (1-k)d(A,B)$$

for all $a, x \in A$ and $b, y \in B$.

**Definition 2.4** Let $S : A \to B$ and $g : A \to A$ be an isometry. The mapping $S$ is said to preserve the *isometric distance* with respect to $g$ if

$$d(Sgx, Sgy) = d(Sx, Sy)$$

for all $x, y \in A$.

**Definition 2.5** A point $x \in A$ is called a *best proximity point* of the mapping $S : A \to B$ if it satisfies the condition that

$$d(x, Sx) = d(A,B).$$

It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

## 3 Main results

In this section, we introduce a new class of proximal contractions, the so-called Geraghty's proximal contraction mappings, and prove best proximity theorems for this class.

**Definition 3.1** A mapping $T : A \to B$ is called *Geraghty's proximal contraction of the first kind* if, there exists $\beta \in \mathcal{S}$ such that

$$\left.\begin{aligned} d(u, Tx) = d(A,B) \\ d(v, Ty) = d(A,B) \end{aligned}\right\} \quad \Longrightarrow \quad d(u,v) \le \beta\big(d(x,y)\big)d(x,y)$$

for all $u, v, x, y \in A$.

**Definition 3.2** A mapping $T : A \to B$ is called *Geraghty's proximal contraction of the second kind* if, there exists $\beta \in \mathcal{S}$ such that

$$\left.\begin{aligned} d(u, Tx) = d(A,B) \\ d(v, Ty) = d(A,B) \end{aligned}\right\} \quad \Longrightarrow \quad d(Tu, Tv) \le \beta\big(d(Tx, Ty)\big)d(Tx, Ty)$$

for all $u, v, x, y \in A$.

It is easy to see that if we take $\beta(t) = k$, where $k \in [0,1)$, then Geraghty's proximal contraction of the first kind and Geraghty's proximal contraction of the second kind reduce to a proximal contraction of the first kind (Definition 2.1) and a proximal contraction of the second kind (Definition 2.2), respectively.

Next, we extend the result of Sadiq Basha [26] and Banach's fixed point theorem to the case of nonself-mappings satisfying Geraghty's proximal contraction condition.

**Theorem 3.3** *Let $(X,d)$ be a complete metric space and let $A, B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Let $S: A \to B$, $T: B \to A$ and $g: A \cup B \to A \cup B$ satisfy the following conditions*:

(a) *$S$ and $T$ are Geraghty's proximal contractions of the first kind*;
(b) *$g$ is an isometry*;
(c) *the pair $(S,T)$ is a proximal cyclic contraction*;
(d) *$S(A_0) \subseteq B_0$, $T(B_0) \subseteq A_0$*;
(e) *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$*.

*Then there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that*

$$d(gx, Sx) = d(gy, Ty) = d(x,y) = d(A,B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by*

$$d(gx_{n+1}, Sx_n) = d(A,B)$$

*converges to the element $x$. For any fixed $y_0 \in B_0$, the sequence $\{y_n\}$ defined by*

$$d(gy_{n+1}, Ty_n) = d(A,B)$$

*converges to the element $y$.*

*On the other hand, a sequence $\{u_n\}$ in $A$ converges to $x$ if there exists a sequence of positive numbers $\{\epsilon_n\}$ such that*

$$\lim_{n\to\infty} \epsilon_n = 0, \qquad d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

*where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A,B)$.*

*Proof* Let $x_0$ be a fixed element in $A_0$. In view of the fact that $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, it follows that there exists an element $x_1 \in A_0$ such that

$$d(gx_1, Sx_0) = d(A,B).$$

Again, since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element $x_2 \in A_0$ such that

$$d(gx_2, Sx_1) = d(A,B).$$

By the same method, we can find $x_n$ in $A_0$ such that

$$d(gx_n, Sx_{n-1}) = d(A,B).$$

So, inductively, one can determine an element $x_{n+1} \in A_0$ such that

$$d(gx_{n+1}, Sx_n) = d(A, B). \tag{3.1}$$

Since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, $S$ is Geraghty's proximal contraction of the first kind, $g$ is an isometry and the property of $\beta$, it follows that for each $n \geq 1$

$$\begin{aligned} d(x_{n+1}, x_n) &= d(gx_{n+1}, gx_n) \\ &\leq \beta\big(d(x_n, x_{n-1})\big)d(x_n, x_{n-1}) \\ &\leq d(x_n, x_{n-1}), \end{aligned}$$

which implies that the sequence $\{d(x_{n+1}, x_n)\}$ is non-increasing and bounded below. Hence there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$. Suppose that $r > 0$. Observe that

$$\frac{d(x_{n+1}, x_n)}{d(x_n, x_{n-1})} \leq \beta\big(d(x_n, x_{n-1})\big),$$

which implies that $\lim_{n\to\infty} \beta(d(x_n, x_{n-1})) = 1$. Since $\beta \in \mathcal{S}$, we have $r = 0$ which is a contradiction and hence

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0. \tag{3.2}$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequences $\{x_{m_k}\}$, $\{x_{n_k}\}$ of $\{x_n\}$ such that for any $n_k > m_k \geq k$

$$r_k := d(x_{m_k}, x_{n_k}) \geq \varepsilon, \qquad d(x_{m_k}, x_{n_k-1}) < \varepsilon$$

for any $k \in \{1, 2, 3, \dots\}$. For each $n \geq 1$, let $\alpha_n := d(x_{n+1}, x_n)$. Then we have

$$\begin{aligned} \varepsilon \leq r_k &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \varepsilon + \alpha_{n_k-1} \end{aligned} \tag{3.3}$$

and so it follows from (3.2) and (3.3) that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{3.4}$$

Notice also that

$$\begin{aligned} \varepsilon &\leq r_k \\ &\leq d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k}) + d(x_{m_k+1}, x_{n_k+1}) \\ &= \alpha_{m_k} + \alpha_{n_k} + d(x_{m_k+1}, x_{n_k+1}) \\ &\leq \alpha_{m_k} + \alpha_{n_k} + \beta\big(d(x_{m_k}, x_{n_k})\big)d(x_{m_k}, x_{n_k}) \end{aligned}$$

and so

$$\frac{r_k - \alpha_{m_k} - \alpha_{n_k}}{d(x_{m_k}, x_{n_k})} \le \beta\bigl(d(x_{m_k}, x_{n_k})\bigr).$$

Taking $k \to \infty$ in the above inequality, by (3.2), (3.4) and $\beta \in \mathcal{S}$, we get $\varepsilon = 0$, which is a contradiction. So we know that the sequence $\{x_n\}$ is a Cauchy sequence. Hence $\{x_n\}$ converges to some element $x \in A$.

Similarly, in view of the fact that $T(B_0) \subseteq A_0$ and $A_0 \subseteq g(A_0)$, we can conclude that there exists a sequence $\{y_n\}$ such that it converges to some element $y \in B$. Since the pair $(S, T)$ is a proximal cyclic contraction and $g$ is an isometry, we have

$$d(x_{n+1}, y_{n+1}) = d(gx_{n+1}, gy_{n+1}) \le kd(x_n, y_n) + (1-k)d(A, B). \tag{3.5}$$

Taking $n \to \infty$ in (3.5), it follows that

$$d(x, y) = d(A, B) \tag{3.6}$$

and so $x \in A_0$ and $y \in B_0$. Since $S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$, there exist $u \in A$ and $v \in B$ such that

$$d(u, Sx) = d(A, B), \qquad d(v, Ty) = d(A, B). \tag{3.7}$$

From (3.1) and (3.7), since $S$ is Geraghty's proximal contraction of the first kind of $S$, we get

$$d(u, gx_{n+1}) \le \beta\bigl(d(x, x_n)\bigr)d(x, x_n). \tag{3.8}$$

Letting $n \to \infty$ in the above inequality, we get $d(u, gx) \le 0$ and so $u = gx$. Therefore, we have

$$d(gx, Sx) = d(A, B). \tag{3.9}$$

Similarly, we can show that $v = gy$ and so

$$d(gy, Ty) = d(A, B). \tag{3.10}$$

From (3.6), (3.9) and (3.10), we get

$$d(x, y) = d(gx, Sx) = d(gy, Ty) = d(A, B).$$

Next, to prove the uniqueness, suppose that there exist $x^* \in A$ and $y^* \in B$ with $x \neq x^*$ and $y \neq y^*$ such that

$$d\bigl(gx^*, Sx^*\bigr) = d(A, B), \qquad d\bigl(gy^*, Ty^*\bigr) = d(A, B).$$

Since $g$ is an isometry and $S$ is Geraghty's proximal contraction of the first kind, it follows that

$$d(x,x^*) = d(gx,gx^*) \leq \beta\bigl(d(x,x^*)\bigr)d(x,x^*)$$

and hence

$$1 = \frac{d(x,x^*)}{d(x,x^*)} \leq \beta\bigl(d(x,x^*)\bigr) < 1,$$

which is a contradiction. Thus we have $x = x^*$. Similarly, we can prove that $y = y^*$.

On the other hand, let $\{u_n\}$ be a sequence in $A$ and $\{\epsilon_n\}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \epsilon_n = 0, \qquad d(u_{n+1}, z_{n+1}) \leq \epsilon_n, \tag{3.11}$$

where $z_{n+1} \in A$ satisfies the condition that

$$d(gz_{n+1}, Su_n) = d(A,B). \tag{3.12}$$

By (3.1) and (3.12), since $S$ is Geraghty's proximal contraction of the first kind and $g$ is an isometry, we have

$$d(x_{n+1}, z_{n+1}) = d(gx_{n+1}, gz_{n+1}) \leq \beta\bigl(d(x_n,u_n)\bigr)d(x_n,u_n).$$

For any $\epsilon > 0$, choose a positive integer $N$ such that $\epsilon_n \leq \epsilon$ for all $n \geq N$. Observe that

$$\begin{aligned} d(x_{n+1}, u_{n+1}) &\leq d(x_{n+1}, z_{n+1}) + d(z_{n+1}, u_{n+1}) \\ &\leq \beta\bigl(d(x_n,u_n)\bigr)d(x_n,u_n) + \epsilon_n \\ &\leq d(x_n,u_n) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we can conclude that for all $n \geq N$ the sequence $\{d(x_n,u_n)\}$ is nonincreasing and bounded below and hence converges to some nonnegative real number $r'$. Since the sequence $\{x_n\}$ converges to $x$, we get

$$\lim_{n\to\infty} d(u_n,x) = \lim_{n\to\infty} d(u_n,x_n) = r'. \tag{3.13}$$

Suppose that $r' > 0$. Since

$$\begin{aligned} d(u_{n+1},x) &\leq d(u_{n+1},x_{n+1}) + d(x_{n+1},x) \\ &\leq \beta\bigl(d(x_n,u_n)\bigr)d(x_n,u_n) + \epsilon_n + d(x_{n+1},x), \end{aligned} \tag{3.14}$$

it follows from inequalities (3.11), (3.13) and (3.14) that

$$\frac{d(u_{n+1},x) - \epsilon_n - d(x_{n+1},x)}{d(x_n,u_n)} \leq \beta\bigl(d(x_n,u_n)\bigr) < 1, \tag{3.15}$$

which implies that $\beta(d(x_n, u_n)) \to 1$ and so $d(u_n, x_n) \to 0$, that is,

$$\lim_{n\to\infty} d(u_n, x) = \lim_{n\to\infty} d(u_n, x_n) = 0,$$

which is a contradiction. Thus $r' = 0$ and hence $\{u_n\}$ is convergent to the point $x$. This completes the proof. □

If $g$ is the identity mapping in Theorem 3.3, then we obtain the following.

**Corollary 3.4** *Let $(X, d)$ be a complete metric space and let $A$, $B$ be nonempty closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ be the mappings satisfying the following conditions*:

(a) *$S$ and $T$ are Geraghty's proximal contractions of the first kind*;
(b) *$S(A_0) \subseteq B_0$, $T(B_0) \subseteq A_0$*;
(c) *the pair $(S, T)$ is a proximal cyclic contraction.*

*Then there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that*

$$d(x, Sx) = d(y, Ty) = d(x, y) = d(A, B).$$

If we take $\beta(t) = k$, where $0 \leq k < 1$, we obtain the following corollary.

**Corollary 3.5** [26] *Let $(X, d)$ be a complete metric space and let $A$, $B$ be nonempty closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ be the mappings satisfying the following conditions*:

(a) *$S$ and $T$ are proximal contractions of the first kind*;
(b) *$g$ is an isometry*;
(c) *the pair $(S, T)$ is a proximal cyclic contraction*;
(d) *$S(A_0) \subseteq B_0$, $T(B_0) \subseteq A_0$*;
(e) *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$.*

*Then there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

*converges to the element $x$. For any fixed $y_0 \in B_0$, the sequence $\{y_n\}$ defined by*

$$d(gy_{n+1}, Ty_n) = d(A, B)$$

*converges to the element $y$.*

If $g$ is the identity mapping in Corollary 3.5, we obtain the following corollary.

**Corollary 3.6** *Let $(X, d)$ be a complete metric space and let $A$, $B$ be nonempty closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ be the mappings satisfying the following conditions*:

(a) *$S$ and $T$ are proximal contractions of the first kind*;
(b) $S(A_0) \subseteq B_0$, $T(B_0) \subseteq A_0$;
(c) *the pair $(S, T)$ is a proximal cyclic contraction.*

*Then there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that*

$$d(x, Sx) = d(y, Ty) = d(x, y) = d(A, B).$$

Next, we establish a best proximity point theorem for nonself-mappings which are Geraghty's proximal contractions of the first kind and the second kind.

**Theorem 3.7** *Let $(X, d)$ be a complete metric space and let $A, B$ be nonempty closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$ and $g : A \to A$ be the mappings satisfying the following conditions*:
(a) *$S$ is Geraghty's proximal contraction of the first and second kinds*;
(b) *$g$ is an isometry*;
(c) *$S$ preserves isometric distance with respect to $g$*;
(d) $S(A_0) \subseteq B_0$;
(e) $A_0 \subseteq g(A_0)$.

*Then there exists a unique point $x \in A$ such that*

$$d(gx, Sx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

*converges to the element $x$.*

*On the other hand, a sequence $\{u_n\}$ in $A$ converges to $x$ if there exists a sequence $\{\epsilon_n\}$ of positive numbers such that*

$$\lim_{n\to\infty} \epsilon_n = 0, \qquad d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

*where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A, B)$.*

*Proof* Since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, as in the proof of Theorem 3.3, we can construct the sequence $\{x_n\}$ in $A_0$ such that

$$d(gx_{n+1}, Sx_n) = d(A, B) \tag{3.16}$$

for each $n \geq 1$. Since $g$ is an isometry and $S$ is Geraghty's proximal contraction of the first kind, we see that

$$d(x_n, x_{n+1}) = d(gx_n, gx_{n+1}) \leq \beta\big(d(x_n, x_{n-1})\big)d(x_n, x_{n-1})$$

for all $n \geq 1$. Again, similarly, we can show that the sequence $\{x_n\}$ is a Cauchy sequence and so it converges to some $x \in A$. Since $S$ is Geraghty's proximal contraction of the second kind and preserves the isometric distance with respect to $g$, we have

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &= d(Sgx_n, Sgx_{n+1}) \\ &\le \beta\big(d(Sx_{n-1}, Sx_n)\big)d(Sx_{n-1}, Sx_n) \\ &\le d(Sx_{n-1}, Sx_n), \end{aligned}$$

which means that the sequence $\{d(Sx_{n+1}, Sx_n)\}$ is non-increasing and bounded below. Hence there exists $r \ge 0$ such that

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = r.$$

Suppose that $r > 0$. Observe that

$$\frac{d(Sx_n, Sx_{n+1})}{d(Sx_{n-1}, Sx_n)} \le \beta\big(d(Sx_{n-1}, Sx_n)\big).$$

Taking $k \to \infty$ in the above inequality, we get $\beta(d(Sx_{n-1}, Sx_n)) \to 1$. Since $\beta \in \mathcal{S}$, we have $r = 0$ which is a contradiction and thus

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = 0. \tag{3.17}$$

Now, we claim that $\{Sx_n\}$ is a Cauchy sequence. Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequences $\{Sx_{m_k}\}$, $\{Sx_{n_k}\}$ of $\{Sx_n\}$ such that, for any $n_k > m_k \ge k$,

$$r_k := d(Sx_{m_k}, Sx_{n_k}) \ge \varepsilon, \qquad d(Sx_{m_k}, Sx_{n_k-1}) < \varepsilon$$

for any $k \in \{1, 2, 3, \ldots\}$. For each $n \ge 1$, let $\gamma_n := d(Sx_{n+1}, Sx_n)$. Then we have

$$\begin{aligned} \varepsilon \le r_k &\le d(Sx_{m_k}, Sx_{n_k-1}) + d(Sx_{n_k-1}, Sx_{n_k}) \\ &< \varepsilon + \gamma_{n_k-1} \end{aligned} \tag{3.18}$$

and so it follows from (3.17) and (3.18) that

$$\lim_{k\to\infty} r_k = \varepsilon.$$

Notice also that

$$\begin{aligned} \varepsilon &\le r_k \\ &\le d(Sx_{m_k}, Sx_{m_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) + d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &= \gamma_{m_k} + \gamma_{n_k} + d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &\le \gamma_{m_k} + \gamma_{n_k} + \beta\big(d(Sx_{m_k}, Sx_{n_k})\big)d(Sx_{m_k}, Sx_{n_k}). \end{aligned}$$

So, it follows that

$$1 = \lim_{k\to\infty} \frac{r_k - \gamma_{m_k} - \gamma_{n_k}}{d(Sx_{m_k}, Sx_{n_k})} \le \lim_{k\to\infty} \beta\big(d(Sx_{m_k}, Sx_{n_k})\big) < 1$$

and so $\lim_{k\to\infty}\beta(d(Sx_{m_k},Sx_{n_k}))=1$. Since $\beta\in\mathcal{S}$, we have $\lim_{k\to\infty}d(Sx_{m_k},Sx_{n_k})=0$, that is, $\varepsilon=0$, which is a contradiction. So, we obtain the claim and then it converges to some $y\in B$. Therefore, we can conclude that

$$d(gx,y)=\lim_{n\to\infty}d(gx_{n+1},Sx_n)=d(A,B),$$

which implies that $gx\in A_0$. Since $A_0\subseteq g(A_0)$, we have $gx=gz$ for some $z\in A_0$ and then $d(gx,gz)=0$. By the fact that $g$ is an isometry, we have $d(x,z)=d(gx,gz)=0$. Hence $x=z$ and so $x\in A_0$. Since $S(A_0)\subseteq B_0$, there exists $u\in A$ such that

$$d(u,Sx)=d(A,B). \tag{3.19}$$

Since $S$ is Geraghty's proximal contraction of the first kind, it follows from (3.16) and (3.19) that

$$d(u,gx_{n+1})\le\beta\bigl(d(x,x_n)\bigr)d(x,x_n) \tag{3.20}$$

for all $n\ge1$. Taking $n\to\infty$ in (3.20), it follows that the sequence $\{gx_n\}$ converges to a point $u$. Since $g$ is continuous and $\lim_{n\to\infty}x_n=x$, we have $gx_n\to gx$ as $n\to\infty$. By the uniqueness of the limit, we conclude that $u=gx$. Therefore, it follows that $d(gx,Sx)=d(u,Sx)=d(A,B)$.

The uniqueness and the remaining part of the proof follow from the proof of Theorem 3.3. This completes the proof. □

If $g$ is the identity mapping in Theorem 3.7, then we obtain the following.

**Corollary 3.8** *Let $(X,d)$ be a complete metric space and let $A$, $B$ be nonempty closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S:A\to B$ be the mappings satisfying the following conditions*:

(a) *$S$ is Geraghty's proximal contraction of the first and second kinds*;
(b) *$S(A_0)\subseteq B_0$.*

*Then there exists a unique point $x\in A$ such that*

$$d(x,Sx)=d(A,B).$$

*Moreover, for any fixed $x_0\in A_0$, the sequence $\{x_n\}$ defined by*

$$d(x_{n+1},Sx_n)=d(A,B)$$

*converges to the best proximity point $x$ of $S$.*

If we take $\beta(t)=k$ in Theorem 3.7, where $0\le k<1$, we obtain the following.

**Corollary 3.9** [26] *Let $(X,d)$ be a complete metric space and let $A$, $B$ be nonempty closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S:A\to B$ and $g:A\to A$ be the mappings satisfying the following conditions*:

(a) *$S$ is a proximal contraction of the first and second kinds*;

(b) *$g$ is an isometry*;
(c) *$S$ preserves isometric distance with respect to $g$*;
(d) $S(A_0) \subseteq B_0$;
(e) $A_0 \subseteq g(A_0)$.

*Then there exists a unique point $x \in A$ such that*

$$d(gx, Sx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

*converges to the element $x$.*

If $g$ is the identity mapping in Corollary 3.9, then we obtain the following.

**Corollary 3.10** *Let $(X, d)$ be a complete metric space and let $A$, $B$ be nonempty closed subsets of $X$. Further, suppose that $A_0$ and $B_0$ are nonempty. Let $S : A \to B$ be a mapping satisfying the following conditions*:

(a) *$S$ is a proximal contraction of the first and second kinds*;
(b) $S(A_0) \subseteq B_0$.

*Then there exists a unique point $x \in A$ such that*

$$d(x, Sx) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by*

$$d(x_{n+1}, Sx_n) = d(A, B)$$

*converges to the best proximity point $x$ of $S$.*

## 4 Examples

Next, we give an example to show that Definition 3.1 is different from Definition 2.1; moreover, we give an example which supports Theorem 3.3. First, we give some proposition for our example as follows.

**Proposition 4.1** *Let $f : [0, \infty) \to [0, \infty)$ be a function defined by $f(t) = \ln(1 + t)$. Then we have the following inequality*:

$$f(a) - f(b) \le f\bigl(|a - b|\bigr) \tag{4.1}$$

*for all $a, b \in [0, \infty)$.*

*Proof* If $x = y$, we have done. Suppose that $x > y$. Then since we have

$$\frac{1+x}{1+y} = \frac{1+x+y-y}{1+y} = 1 + \frac{x-y}{1+y} < 1 + |x - y|,$$

it follows that $\ln(1+x) - \ln(1+y) < \ln(1+|x-y|)$. In the case $x < y$, by a similar argument, we can prove that inequality (4.1) holds. □

**Proposition 4.2** *For each $x, y \in \mathbb{R}$, we have that the following inequality holds*:

$$\frac{1}{(1+|x|)(1+|y|)} \leq \frac{1}{1+|x-y|}.$$

*Proof* Since

$$\begin{aligned} 1+|x-y| &\leq 1+|x|+|y| \\ &\leq 1+|x|+|y|+|x||y| \\ &= \bigl(1+|x|\bigr)\bigl(1+|y|\bigr), \end{aligned}$$

so that

$$\frac{1}{(1+|x|)(1+|y|)} \leq \frac{1}{1+|x-y|}.$$ □

**Example 4.3** Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Let

$$A = \bigl\{(0,x) : x \in \mathbb{R}\bigr\}, \qquad B = \bigl\{(2,y) : y \in \mathbb{R}\bigr\}.$$

Then $d(A,B) = 2$. Define the mappings $S : A \to B$ as follows:

$$S\bigl((0,x)\bigr) = \bigl(2, \ln\bigl(1+|x|\bigr)\bigr).$$

First, we show that $S$ is Geraghty's proximal contractions of the first kind with $\beta \in \mathcal{S}$ defined by

$$\beta(t) = \begin{cases} 1, & t = 0, \\ \frac{\ln(1+t)}{t}, & t > 0. \end{cases}$$

Let $(0,x_1)$, $(0,x_2)$, $(0,a_1)$ and $(0,a_2)$ be elements in $A$ satisfying

$$d\bigl((0,a_1), S(0,x_1)\bigr) = d(A,B) = 2, \qquad d\bigl((0,a_2), S(0,x_2)\bigr) = d(A,B) = 2.$$

Then we have $a_i = \ln(1+|x_i|)$ for $i = 1,2$. If $x_1 = x_2$, we have done. Assume that $x_1 \neq x_2$. Then, by Proposition 4.1 and the fact that the function $f(x) = \ln(1+t)$ is increasing, we have

$$\begin{aligned} d\bigl((0,a_1),(0,a_2)\bigr) &= d\bigl(\bigl(0,\ln\bigl(1+|x_1|\bigr)\bigr),\bigl(0,\ln\bigl(1+|x_2|\bigr)\bigr)\bigr) \\ &= \bigl|\ln\bigl(1+|x_1|\bigr) - \ln\bigl(1+|x_2|\bigr)\bigr| \\ &\leq \bigl|\ln\bigl(1+\bigl||x_1|-|x_2|\bigr|\bigr)\bigr| \\ &\leq \bigl|\ln\bigl(1+|x_1-x_2|\bigr)\bigr| \end{aligned}$$

$$\begin{aligned} &= \frac{|\ln(1+|x_1-x_2|)|}{|x_1-x_2|}|x_1-x_2| \\ &= \beta\bigl(d\bigl((0,x_1),(0,x_2)\bigr)\bigr)d\bigl((0,x_1),(0,x_2)\bigr). \end{aligned}$$

Thus $S$ is Geraghty's proximal contraction of the first kind.

Next, we prove that $S$ is not a proximal contraction of the first kind. Suppose $S$ is a proximal contraction of the first kind, then for each $(0,x^*),(0,y^*),(0,a^*),(0,b^*)\in A$ satisfying

$$d\bigl((0,x^*),S(0,a^*)\bigr)=d(A,B)=2 \quad\text{and}\quad d\bigl((0,y^*),S(0,b^*)\bigr)=d(A,B)=2, \tag{4.2}$$

there exists $k\in[0,1)$ such that

$$d\bigl((0,x^*),(0,y^*)\bigr)\le kd\bigl((0,a^*),(0,b^*)\bigr).$$

From (4.2), we get $x^*=\ln(1+|a^*|)$ and $y^*=\ln(1+|b^*|)$ and so

$$\begin{aligned} \bigl|\ln\bigl(1+\bigl|a^*\bigr|\bigr)-\ln\bigl(1+\bigl|b^*\bigr|\bigr)\bigr| &= d\bigl((0,x^*),(0,y^*)\bigr) \\ &\le kd\bigl((0,a^*),(0,b^*)\bigr) \\ &= k\bigl|a^*-b^*\bigr|. \end{aligned}$$

Letting $b^*=0$, we get

$$1=\lim_{|a^*|\to 0^+}\frac{|\ln(1+|a^*|)|}{|a^*|}\le k<1,$$

which is a contradiction. Thus $S$ is not a proximal contraction of the first kind.

**Example 4.4** Consider the complete metric space $\mathbb{R}^2$ with metric defined by

$$d\bigl((x_1,x_2),(y_1,y_2)\bigr)=|x_1-y_1|+|x_2-y_2|$$

for all $(x_1,x_2),(y_1,y_2)\in\mathbb{R}^2$. Let

$$A=\bigl\{(0,x):x\in\mathbb{R}\bigr\},\qquad B=\bigl\{(2,y):y\in\mathbb{R}\bigr\}.$$

Define two mappings $S:A\to B$, $T:B\to A$ and $g:A\cup B\to A\cup B$ as follows:

$$S\bigl((0,x)\bigr)=\left(2,\frac{|x|}{2(1+|x|)}\right),\qquad T\bigl((2,y)\bigr)=\left(0,\frac{|y|}{2(1+|y|)}\right),\qquad g\bigl((x,y)\bigr)=(x,-y).$$

Then $d(A,B)=2$, $A_0=A$, $B_0=B$ and the mapping $g$ is an isometry.

Next, we show that $S$ and $T$ are Geraghty's proximal contractions of the first kind with $\beta\in\mathcal{S}$ defined by

$$\beta(t)=\frac{1}{1+t}\quad\text{for all } t\ge 0.$$

Let $(0,x_1)$, $(0,x_2)$, $(0,a_1)$ and $(0,a_2)$ be elements in $A$ satisfying

$$d\big((0,a_1),S(0,x_1)\big) = d(A,B) = 2, \qquad d\big((0,a_2),S(0,x_2)\big) = d(A,B) = 2.$$

Then we have

$$a_i = \frac{|x_i|}{2(1+|x_i|)} \quad \text{for } i = 1,2.$$

If $x_1 = x_2$, we have done. Assume that $x_1 \neq x_2$, then, by Proposition 4.2, we have

$$\begin{aligned}
d\big((0,a_1),(0,a_2)\big) &= d\left(\left(0,\frac{|x_1|}{2(1+|x_1|)}\right),\left(0,\frac{|x_2|}{2(1+|x_2|)}\right)\right) \\
&= \left|\frac{|x_1|}{2(1+|x_1|)} - \frac{|x_2|}{2(1+|x_2|)}\right| \\
&= \left|\frac{|x_1|-|x_2|}{2(1+|x_1|)(1+|x_2|)}\right| \\
&\leq \left|\frac{x_1-x_2}{(1+|x_1|)(1+|x_2|)}\right| \\
&\leq \frac{1}{1+|x_1-x_2|}|x_1-x_2| \\
&= \beta\big(d\big((0,x_1),(0,x_2)\big)\big)d\big((0,x_1),(0,x_2)\big).
\end{aligned}$$

Thus $S$ is Geraghty's proximal contraction of the first kind. Similarly, we can see that $T$ is Geraghty's proximal contraction of the first kind. Next, we show that the pair $(S,T)$ is a proximal cyclic contraction. Let $(0,u),(0,x) \in A$ and $(2,v),(2,y) \in B$ be such that

$$d\big((0,u),S(0,x)\big) = d(A,B) = 2, \qquad d\big((2,v),T(2,y)\big) = d(A,B) = 2.$$

Then we get

$$u = \frac{|x|}{2(1+|x|)}, \qquad v = \frac{|y|}{2(1+|y|)}.$$

In the case $x = y$, clear. Suppose that $x \neq y$, then we have

$$\begin{aligned}
d\big((0,u),(2,v)\big) &= |u-v| + 2 \\
&= \left|\frac{|x|}{2(1+|x|)} - \frac{|y|}{2(1+|y|)}\right| + 2 \\
&= \left|\frac{|x|-|y|}{2(1+|x|)(1+|y|)}\right| + 2 \\
&\leq \frac{|x-y|}{2(1+|x|)(1+|y|)} + 2 \\
&\leq \frac{1}{2}|x-y| + 2 \\
&\leq k\big(|x-y|+2\big) + (1-k)2 \\
&= kd\big((0,x),(2,y)\big) + (1-k)d(A,B),
\end{aligned}$$

where $k = [\frac{1}{2}, 1)$. Hence the pair $(S, T)$ is a proximal cyclic contraction. Therefore, all the hypotheses of Theorem 3.3 are satisfied. Further, it is easy to see that $(0,0) \in A$ and $(2,0) \in B$ are the unique elements such that

$$d\bigl(g(0,0), S(0,0)\bigr) = d\bigl(g(2,0), T(2,0)\bigr) = d\bigl((0,0),(2,0)\bigr) = d(A,B).$$

## 5 Conclusions

This article has investigated the existence of an optimal approximate solution, the so-called best proximity point, for the generalized notion of proximal contractions of the first and second kinds, which were defined by Sadiq Basha in [26]. Furthermore, an algorithm for computing such an optimal approximate solution and example which supports our main results have been presented.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in this research. All authors read and approved the final manuscript.

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**Acknowledgements**
Mr. Chirasak Mongkolkeha was supported from the Thailand Research Fund through the Royal Golden Jubilee Program under Grant PHD/0029/2553 for the Ph.D. Program at KMUTT, Thailand. This research was partially finished at Department of Mathematics Education, Gyeongsang National University, Republic of Korea. Also, the second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science, and Technology (Grant No. 2012-0008170). The third author was supported by the Commission on Higher Education, the Thailand Research Fund and the King Mongkut's University of Technology Thonburi (Grant No. MRG5580213).

Received: 20 February 2013 Accepted: 15 June 2013 Published: 8 July 2013

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doi:10.1186/1687-1812-2013-180
**Cite this article as:** Mongkolkeha et al.: Best proximity points for Geraghty's proximal contraction mappings. *Fixed Point Theory and Applications* 2013 **2013**:180.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 375604, 8 pages
http://dx.doi.org/10.1155/2013/375604

*Research Article*

# The Existence Theorems of an Optimal Approximate Solution for Generalized Proximal Contraction Mappings

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Received 3 June 2013; Accepted 31 July 2013

Academic Editor: Calogero Vetro



Recently, Basha (2011) established the best proximity point theorems for proximal contractions of the first and second kinds which are extension of Banach's contraction principle in the case of non-self-mappings. The aim of this paper is to extend and generalize the notions of proximal contractions of the first and second kinds which are more general than the notion of self-contractions, establish the existence of an optimal approximate solution theorems for these non-self-mappings, and also give examples to validate our main results.

## 1. Introduction

Since Banach's contraction principle [1] first appeared, several authors have generalized this principle in different directions. However, they have shown the existence of a fixed point for self-mappings. One of the most interesting results on Banach's contraction principle is the case of non-self-mappings. In fact, for any nonempty closed subsets $A$ and $B$ of a complete metric space $(X, d)$, a contractive non-self-mapping $T : A \to B$ does not necessarily have a fixed point $Tx = x$. In this case, a best proximity point, that is, a point $x \in A$ for which $d(x, Tx) = d(A, B) := \inf\{d(x, y) : x \in A \quad y \in B\}$ represents an optimal approximate solution to the equation $Tx = x$. It is well known that a best proximity point reduces to a fixed point if the underlying mapping is assumed to be a self-mapping. Consequently, best proximity point theorems are improvement of Banach's contraction principle in case of non-self-mappings.

A classical best approximation theorem was introduced by Fan [2]. Afterward, several authors including Prolla [3], Reich [4], and Sehgal and Singh [5, 6] have derived extensions of Fan's Theorem in many directions. Other works of the existence of a best proximity point for contractive mappings can be found in [7–13]. On the other hand, many best proximity point theorems for set-valued mappings have been established in [14–19]. In particular, Eldred et al. [20] have obtained best proximity point theorems for relatively nonexpansive mappings.

Recently, Basha [21] gave necessary and sufficient conditions to claim the existence of best proximity point for proximal contraction of first and second kinds which are non-self-mapping analogues of contraction self-mappings, and they also established some best proximity theorems. Afterward, several mathematicians extended and improved these results in many ways (see in [22–25]).

The purpose of this paper is to extend and generalize the class of proximal contraction of first and second kinds which are different from another type in the literature. For such mappings, we seek the necessary condition for these classes to have best proximity points and also give some examples to illustrate our main results. The results of this paper are generalizations of results of Basha in [21] and some results of the fundamental metrical fixed point and best proximity point theorems in the literature.

## 2. Preliminaries

Throughout this paper, suppose that $A$ and $B$ are nonempty subsets of a metric space $(X, d)$. We use the following notations:

$$d(A,B) := \inf\{d(x,y) : x \in A \text{ and } y \in B\},$$
$$A_0 := \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\}, \quad (1)$$
$$B_0 := \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\}.$$

*Remark 1.* It is easy to see that $A_0$ and $B_0$ are nonempty whenever $A \cap B \neq \emptyset$. Further, if $A$ and $B$ are closed subsets of a normed linear space such that $d(A,B) > 0$, then $A_0 \subseteq Bdr(A)$ and $B_0 \subseteq Bdr(B)$, where $Bdr(A)$ is a boundary of $A$.

*Definition 2* (see [21]). A mapping $T : A \to B$ is called a proximal contraction of the first kind if there exists $\alpha \in [0,1)$ such that, for all $a, b, x, y \in A$,

$$\begin{matrix} d(a,Tx) = d(A,B), \\ d(b,Ty) = d(A,B) \end{matrix} \Longrightarrow d(a,b) \leq \alpha d(x,y). \quad (2)$$

*Remark 3.* If $T$ is self-mapping, then $T$ is a proximal contraction of the first kind deduced to $T$ which is a contraction mapping. But a non-self-proximal contraction is not necessarily a contraction.

*Definition 4* (see [21]). A mapping $T : A \to B$ is said to be a proximal contraction of the second kind if there exists $\alpha \in [0,1)$ such that, for all $a, b, x, y \in A$,

$$\begin{matrix} d(a,Tx) = d(A,B), \\ d(b,Ty) = d(A,B) \end{matrix} \Longrightarrow d(Ta,Tb) \leq \alpha d(Tx,Ty). \quad (3)$$

The necessary condition for a self-mapping $T$ to be a proximal contraction of the second kind is that

$$d(TTx, TTy) \leq \alpha d(Tx, Ty) \quad (4)$$

for all $x$, $y$ in the domain of $T$. Therefore, every contraction self-mapping is a proximal contraction of the second kind, but the converse is not true (see Example 5).

*Example 5.* Consider $\mathbb{R}$ endowed with the Euclidean metric. Let the self-mapping $T : [0,1] \to [0,1]$ be defined as follows:

$$T(x) = \begin{cases} 0 & \text{if } x \text{ is rational}, \\ 1 & \text{otherwise}. \end{cases} \quad (5)$$

It is easy to prove that $T$ is a proximal contraction of the second kind. However, $T$ is not a contraction mapping.

The above example also exhibits that a self-mapping, that is, a proximal contraction of the second kind, is not necessarily continuous.

*Definition 6.* Let $S : A \to B$ and $\mathrm{T} : B \to A$ be mappings. The pair $(S, T)$ is said to be

(1) a cyclic contractive pair if $d(A,B) < d(x,y) \Rightarrow d(Sx,Ty) < d(x,y)$ for all $x \in A$ and $y \in B$;

(2) a cyclic expansive pair if $d(A,B) < d(x,y) \Rightarrow d(Sx,Ty) > d(x,y)$ for all $x \in A$ and $y \in B$;

(3) a cyclic inequality pair if $d(A,B) < d(x,y) \Rightarrow d(Sx,Ty) \neq d(x,y)$ for all $x \in A$ and $y \in B$.

*Definition 7.* Let $S : A \to B$ and $T : B \to A$ be mappings. The pair $(S, T)$ is said to satisfy min-max condition if, for all $x \in A$ and $y \in B$,

$$d(A,B) < d(x,y) \Longrightarrow \min(Sx, \mathrm{T}y) \neq \max(Sx, Ty), \quad (6)$$

where $\min(Sx, Ty)$ and $\max(Sx, Ty)$ are defined by

$$\begin{aligned} \min(Sx,Ty) = \min\{&d(x,y), d(x,Sx), d(y,Ty), \\ &d(Sx,Ty), d(x,STy), \\ &d(y,TSx), d(Sx,TSx), \\ &d(Ty,STy), d(TSx,STy)\}, \\ \max(Sx,Ty) = \max\{&d(x,y), d(x,Sx), d(y,Ty), \\ &d(x,Ty), d(y,Sx), d(Sx,Ty), \\ &d(x,TSx), d(y,STy), d(x,STy), \\ &d(y,TSx), d(Sx,TSx), \\ &d(Ty,STy), d(TSx,STy)\}. \end{aligned} \quad (7)$$

We observe that the cyclic contractive pairs, cyclic expansive pairs, and cyclic inequality pairs satisfy the min-max condition.

*Definition 8.* Let $T : A \to B$ a mapping and $g : A \to A$ be an isometry. The mapping $T$ is said to preserve isometric distance with respect to $g$ if

$$d(Tgx, Tgy) = d(Tx, Ty) \quad (8)$$

for all $x, y \in A$.

*Definition 9.* A point $x \in A$ is said to be a best proximity point of a mapping $T : A \to B$ if it satisfies the condition that

$$d(x, Tx) = d(A,B). \quad (9)$$

Observe that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

*Definition 10.* $A$ is said to be approximatively compact with respect to $B$ if every sequence $\{x_n\}$ in $A$ satisfies the condition that $d(y, x_n) \to d(y, A)$ for some $y \in B$ has a convergent subsequence.

*Remark 11.* Any nonempty subset of metric space $(X, d)$ is approximatively compact with respect to itself.

## 3. Main Results

In this section, we introduce the notions of generalized proximal contraction mappings of the first and second kinds which are different from another type in the literature. We also give the existence theorems of an optimal approximate solution for these mappings.

*Definition 12.* Let $A$, $B$ be nonempty subset of metric space $(X, d)$, $T : A \to B$ and $\mathscr{K} : A \to [0, 1)$. A mapping $T$ is said to be a generalized proximal contraction of the first kind with respect to $\mathscr{K}$ if

$$\begin{array}{l} d(a, Tx) = d(A, B), \\ d(b, Ty) = d(A, B) \end{array} \Longrightarrow d(a, b) \leq \mathscr{K}(x) d(x, y) \tag{10}$$

for all $a, b, x, y \in A$.

*Remark 13.* If we take $\mathscr{K}(x) = \alpha$ for all $x \in A$, where $\alpha \in [0, 1)$, then a generalized proximal contraction of the first kind with respect to $\mathscr{K}$ reduces to a proximal contraction of the first kind (Definition 2). In case of a self-mapping, it is apparent that the class of contraction mapping is contained in the class of generalized proximal contraction of the first kind with respect to $\mathscr{K}$ mapping.

Now, we give an example to claim that the class of proximal contraction mapping of the first kind is a proper subclass of the class of generalized proximal contractions of the first kind with respect to $\mathscr{K}$ mapping.

*Example 14.* Consider the metric space $\mathbb{R}^2$ with Euclidean metric. Let $A = \{(0, y) : -1 < y < 1\}$ and $B = \{(1, y) : -1 < y < 1\}$. Define a mapping $T : A \to B$ as follows:

$$T((0, y)) = \left(1, \frac{y^2}{2}\right) \tag{11}$$

for all $(0, y) \in A$.

It is easy to check that there is no $\alpha \in [0, 1)$ satisfing

$$d(a, Tx) = d(b, Ty) = d(A, B) \Longrightarrow d(a, b) \leq \alpha d(x, y) \tag{12}$$

for all $a, b, x, y \in A$. Therefore, $T$ is not a proximal contraction of the first kind.

Consider a function $\mathscr{K} : A \to [0, 1)$ defined by

$$\mathscr{K}((0, y)) = \frac{|y| + 1}{2}. \tag{13}$$

Next, we claim that $T$ is a generalized proximal contraction of the first kind with respect to $\mathscr{K}$.

If $(0, y_1), (0, y_2) \in A$ such that

$$\begin{aligned} d(a, T((0, y_1))) &= d(A, B) = 1, \\ d(b, T((0, y_2))) &= d(A, B) = 1, \end{aligned} \tag{14}$$

for all $a, b \in A$, then we have

$$a = \left(0, \frac{y_1^2}{2}\right), \qquad b = \left(0, \frac{y_2^2}{2}\right). \tag{15}$$

Therefore, it follows that

$$\begin{aligned} d(a, b) &= d\left(\left(0, \frac{y_1^2}{2}\right), \left(0, \frac{y_2^2}{2}\right)\right) \\ &= \left|\frac{y_1^2}{2} - \frac{y_2^2}{2}\right| \\ &= \left(\frac{|y_1 + y_2|}{2}\right)|y_1 - y_2| \\ &\leq \left(\frac{|y_1| + |y_2|}{2}\right)|y_1 - y_2| \\ &\leq \left(\frac{|y_1| + 1}{2}\right)|y_1 - y_2| \\ &= \mathscr{K}((0, y_1)) d((0, y_1), (0, y_2)). \end{aligned} \tag{16}$$

This implies that $T$ is a generalized proximal contraction of the first kind with respect to $\mathscr{K}$.

*Definition 15.* Let $A$, $B$ be nonempty subset of metric space $(X, d)$, $T : A \to B$ and $\mathscr{K} : A \to [0, 1)$. A mapping $T$ is said to be a generalized proximal contraction of the second kind with respect to $\mathscr{K}$ if

$$\begin{array}{l} d(a, Tx) = d(A, B), \\ d(b, Ty) = d(A, B) \end{array} \Longrightarrow d(a, b) \leq \mathscr{K}(x) d(Tx, Ty) \tag{17}$$

for all $a, b, x, y \in A$.

Clearly, a proximal contraction of the second kind (Definition 4) is a generalized proximal contraction of the second kind.

Next, we extend the results of Basha [21] and many results in the literature.

**Theorem 16.** *Let $(X, d)$ a complete metric space and $A$, $B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Suppose that $T : A \to B$, $g : A \to A$, and $\mathscr{K} : A \to [0, 1)$ are mappings satisfying the following conditions:*

- (a) *$T$ is a continuous generalized proximal contraction of first kind with respect to $\mathscr{K}$;*
- (b) *$T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$;*
- (c) *$g$ is an isometry;*
- (d) *$\mathscr{K}(x) \leq \mathscr{K}(y)$, whenever $d(gx, Ty) = d(A, B)$.*

*Then there exists a unique point $x \in A$ such that $d(gx, Tx) = d(A, B)$.*

*Proof.* Let $x_0$ be a fixed element in $A_0$. From $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, it follows that there exists a point $x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B). \tag{18}$$

Again, since $Tx_1 \in T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists a point $x_2 \in A_0$ such that

$$d(gx_2, Tx_1) = d(A, B). \tag{19}$$

Continuing this process, we can construct the sequence $\{x_n\}$ in $A_0$ such that

$$d(gx_n, Tx_{n-1}) = d(A, B) \tag{20}$$

for all $n \in \mathbb{N}$. Since $T$ is a generalized proximal contraction of the first kind with respect to $\mathcal{K}$, it follows that

$$d(gx_{n+1}, gx_n) \leq \mathcal{K}(x_n)\, d(x_n, x_{n-1}) \tag{21}$$

for all $n \in \mathbb{N}$. Also, since $g$ is an isometry, we have

$$d(x_{n+1}, x_n) \leq \mathcal{K}(x_n)\, d(x_n, x_{n-1}) \tag{22}$$

for all $n \in \mathbb{N}$. By using (20) and (d), we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \mathcal{K}(x_n)\, d(x_n, x_{n-1}) \\ &\leq \mathcal{K}(x_{n-1})\, d(x_n, x_{n-1}) \\ &\leq \mathcal{K}(x_{n-2})\, d(x_n, x_{n-1}) \\ &\ \vdots \\ &\leq \mathcal{K}(x_0)\, d(x_n, x_{n-1}) \end{aligned} \tag{23}$$

for all $n \in \mathbb{N}$. By repeating (23), we get

$$d(x_{n+1}, x_n) \leq (\mathcal{K}(x_0))^n d(x_1, x_0) \tag{24}$$

for all $n \in \mathbb{N}$. Now, we let $k := \mathcal{K}(x_0) \in [0, 1)$. For positive integers $m$ and $n$ with $n > m$, it follows from (24) that

$$\begin{aligned} & d(x_n, x_m) \\ &\quad \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\quad \leq k^{n-1} d(x_1, x_0) + k^{n-2} d(x_1, x_0) + \cdots + k^m d(x_1, x_0) \\ &\quad \leq \left(\frac{k^m}{1-k}\right) d(x_1, x_0). \end{aligned} \tag{25}$$

Since $k \in [0, 1)$, we have $(k^m/(1-k))d(x_1, x_0) \to 0$ as $m \to \infty$, which implies that $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, it follows that the sequence $\{x_n\}$ converges to point $x \in X$. Since $T$ and $g$ are continuous, we get

$$d(gx, Tx) = \lim_{n\to\infty} d(gx_{n+1}, Tx_n) = d(A, B). \tag{26}$$

Next, we suppose that $x^*$ is another point in $X$ such that

$$d(gx^*, Tx^*) = d(A, B). \tag{27}$$

Since $T$ is a generalized proximal contraction of the first kind with respect to $\mathcal{K}$, by using (26) and (27), we get

$$d(gx, gx^*) \leq \mathcal{K}(x)\, d(x, x^*). \tag{28}$$

Since $g$ is an isometry, it follows that

$$d(x, x^*) \leq \mathcal{K}(x)\, d(x, x^*), \tag{29}$$

which implies that $x = x^*$. This completes the proof. □

Now, we give an example to illustrate Theorem 16.

*Example 17.* Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Let $A = \{(0, y) : -1 \leq y \leq 1\}$ and $B = \{(1, y) : -1 \leq y \leq 1\}$. Define two mappings $T : A \to B$ and $g : A \to A$ as follows:

$$T((0, y)) = \left(1, \frac{y^2}{4}\right), \qquad g((0, y)) = (0, -y) \tag{30}$$

for all $(0, y) \in A$. Then it is easy to see that $d(A, B) = 1$, $A_0 = A$, $B_0 = B$, and the mapping $g$ is an isometry.

Consider a function $\mathcal{K} : A \to [0, 1)$ defined by

$$\mathcal{K}((0, y)) = \frac{|y| + 1}{4}. \tag{31}$$

Next, we claim that $T$ is a generalized proximal contraction of the first kind with respect to $\mathcal{K}$. If $(0, y_1), (0, y_2) \in A$ such that

$$\begin{aligned} d(a, T((0, y_1))) &= d(A, B) = 1, \\ d(b, T((0, y_2))) &= d(A, B) = 1, \end{aligned} \tag{32}$$

for all $a, b \in A$, then we have

$$a = \left(0, \frac{y_1^2}{4}\right), \qquad b = \left(0, \frac{y_2^2}{4}\right). \tag{33}$$

Therefore, it follows that

$$\begin{aligned} d(a, b) &= d\left(\left(0, \frac{y_1^2}{4}\right), \left(0, \frac{y_2^2}{4}\right)\right) \\ &= \left|\frac{y_1^2}{4} - \frac{y_2^2}{4}\right| \\ &= \left(\frac{|y_1 + y_2|}{4}\right)|y_1 - y_2| \\ &\leq \left(\frac{|y_1| + |y_2|}{4}\right)|y_1 - y_2| \\ &\leq \left(\frac{|y_1| + 1}{4}\right)|y_1 - y_2| \\ &= \mathcal{K}((0, y_1))\, d((0, y_1), (0, y_2)). \end{aligned} \tag{34}$$

This implies that the non-self-mapping $T$ is a generalized proximal contraction of the first kind with respect to $\mathcal{K}$. It is easy to see that $\mathcal{K}(x) \leq \mathcal{K}(y)$ whenever $d(gx, Ty) = d(A, B)$. Moreover, since $T$ is continuous and $g$ is an isometry, all the conditions of Theorem 16 are satisfied, and so $T$ has a unique element $(0, 0) \in A$ such that

$$d(g((0, 0)), T((0, 0))) = d(A, B). \tag{35}$$

**Corollary 18** (see [21, Theorem 3.3]). *Let $(X, d)$ be a complete metric space and $A$, $B$ nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty. Suppose that $T : A \to B$ and $g : A \to A$ are mappings satisfying the following conditions:*

(a) *$T$ is a continuous proximal contraction of the first kind;*

(b) $T(A_0) \subseteq B_0$ *and* $A_0 \subseteq g(A_0)$;

(c) $g$ *is an isometry.*

*Then there exists a unique element* $x \in A$ *such that* $d(gx, Tx) = d(A, B)$.

*Proof.* Since a proximal contraction of the first kind is a special case of a generalized proximal contraction of the first kind, we can prove this result by applying Theorem 16. □

In Theorem 16, if $g$ is the identity mapping, then it yields the following best proximity point theorem.

**Corollary 19.** *Let* $(X, d)$ *a complete metric space and* $A$, $B$ *be nonempty closed subsets of* $X$ *such that* $A_0$ *and* $B_0$ *are nonempty. Suppose that* $T : A \to B$ *and* $\mathscr{K} : A \to [0, 1)$ *are mappings satisfying the following conditions:*

(a) $T$ *is a continuous generalized proximal contraction of first kind with respect to* $\mathscr{K}$;

(b) $T(A_0) \subseteq B_0$;

(c) $\mathscr{K}(x) \leq \mathscr{K}(y)$, *whenever* $d(x, Ty) = d(A, B)$.

*Then* $T$ *has a unique best proximity point in* $A$.

**Corollary 20** (see [21, Corollary 3.4]). *Let* $(X, d)$ *be a complete metric space and* $A$, $B$ *nonempty closed subsets of* $X$ *such that* $A_0$ *and* $B_0$ *are nonempty. Let* $T : A \to B$ *be a mapping satisfying the following conditions:*

(a) $T$ *is a continuous proximal contraction of the first kind;*

(b) $T(A_0) \subseteq B_0$.

*Then* $T$ *has a unique best proximity point in* $A$.

*Proof.* Since a proximal contraction of the first kind is a special case of a generalized proximal contraction of the first kind with respect to $\mathscr{K}$, we can prove this result by applying Corollary 19. □

Next, we prove the second main result for generalized proximal contraction of the second kind with respect to $\mathscr{K}$ mapping.

**Theorem 21.** *Let* $(X, d)$ *a complete metric space and* $A$, $B$ *be nonempty closed subsets of* $X$ *such that* $A$ *is approximatively compact with respect to* $B$. *Suppose that* $A_0$ *and* $B_0$ *are nonempty and* $T : A \to B$, $g : A \to A$, *and* $\mathscr{K} : A \to [0, 1)$ *are mappings satisfying the following conditions:*

(a) $T$ *is a continuous generalized proximal contraction of the second kind with respect to* $\mathscr{K}$;

(b) $T(A_0) \subseteq B_0$ *and* $A_0 \subseteq g(A_0)$;

(c) $g$ *is an isometry;*

(d) $T$ *preserves isometric distance with respect to* $g$;

(e) $\mathscr{K}(x) \leq \mathscr{K}(y)$, *whenever* $d(gx, Ty) = d(A, B)$.

*Then there exists a point* $x \in A$ *such that* $d(gx, Tx) = d(A, B)$. *Moreover, if* $x^*$ *is another point in* $A$ *for which* $d(gx^*, Tx^*) = d(A, B)$, *then* $Tx = Tx^*$.

*Proof.* As in the proof of Theorem 16, for fixed $x_0 \in A_0$, we can define a sequence $\{x_n\}$ in $A_0$ such that

$$d(gx_n, Tx_{n-1}) = d(A, B) \tag{36}$$

for all $n \in \mathbb{N}$. Since $T$ is a generalized proximal contraction of the second kind with respect to $\mathscr{K}$, it follows that

$$d(Tgx_{n+1}, Tgx_n) \leq \mathscr{K}(x_n)\, d(Tx_n, Tx_{n-1}). \tag{37}$$

Since $T$ preserves isometric distance with respect to $g$, we have

$$d(Tx_{n+1}, Tx_n) \leq \mathscr{K}(x_n)\, d(Tx_n, Tx_{n-1}) \tag{38}$$

for all $n \in \mathbb{N}$. By using (36) and (e), we have

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &\leq \mathscr{K}(x_n)\, d(Tx_n, Tx_{n-1}) \\ &\leq \mathscr{K}(x_{n-1})\, d(Tx_n, Tx_{n-1}) \\ &\leq \mathscr{K}(x_{n-2})\, d(Tx_n, Tx_{n-1}) \\ &\vdots \\ &\leq \mathscr{K}(x_0)\, d(Tx_n, Tx_{n-1}) \end{aligned} \tag{39}$$

for all $n \in \mathbb{N}$. By repeating (39), we get

$$d(Tx_{n+1}, Tx_n) \leq (\mathscr{K}(x_0))^n d(Tx_1, Tx_0) \tag{40}$$

for all $n \in \mathbb{N}$. Now, we let $k := \mathscr{K}(x_0) \in [0, 1)$. For positive integers $m$ and $n$ with $n > m$, it follows from (40) that

$$\begin{aligned} d(Tx_n, Tx_m) &\leq d(Tx_n, Tx_{n-1}) \\ &\quad + d(Tx_{n-1}, Tx_{n-2}) + \cdots + d(Tx_{m+1}, Tx_m) \\ &\leq k^{n-1} d(Tx_1, Tx_0) + k^{n-2} d(Tx_1, Tx_0) \\ &\quad + \cdots + k^m d(Tx_1, Tx_0) \\ &\leq \left(\frac{k^m}{1-k}\right) d(Tx_1, Tx_0). \end{aligned} \tag{41}$$

Since $k \in [0, 1)$, we have $(k^m/(1-k)) d(Tx_1, Tx_0) \to 0$ as $m \to \infty$, which implies that $\{Tx_n\}$ is a Cauchy sequence in $B$. By completeness of $B \subseteq X$, there exists a point $y \in B$ such that $Tx_n \to y$ as $n \to \infty$. By (36) and the triangle inequality, we have

$$\begin{aligned} d(y, A) &\leq d(y, gx_n) \\ &\leq d(y, Tx_{n-1}) + d(Tx_{n-1}, gx_n) \\ &= d(y, Tx_{n-1}) + d(A, B) \\ &\leq d(y, Tx_{n-1}) + d(y, A). \end{aligned} \tag{42}$$

Letting $n \to \infty$ in (42), we get $d(y, gx_n) \to d(y, A)$. Since $A$ is approximatively compact with respect to $B$, it follows that

$\{gx_n\}$ has a convergence subsequence $\{gx_{n_k}\}$; say $gx_{n_k} \to z \in A$ as $k \to \infty$. Thus we have

$$d(z, y) = \lim_{k\to\infty} d\left(gx_{n_k}, Tx_{n_k - 1}\right) = d(A, B), \quad (43)$$

which implies that $z \in A_0$. Since $A_0 \subseteq g(A_0)$, we have $z = gx$ for some $x \in A_0$. Therefore, $gx_{n_k} \to gx$ as $k \to \infty$. Since $g$ is an isometry, we get $x_{n_k} \to x$ as $k \to \infty$. By the continuity of $T$, we have $Tx_{n_k} \to Tx$ as $k \to \infty$ and then $y = Tx$. From (43), we can conclude that

$$d(gx, Tx) = d(A, B). \quad (44)$$

Next, we suppose that $x^*$ is another point in $X$ such that

$$d(gx^*, Tx^*) = d(A, B). \quad (45)$$

Since $T$ is a generalized proximal contraction of the second kind with respect to $\mathscr{K}$, by the virtue of (44) and (45), we get

$$d(Tgx, Tgx^*) \le \mathscr{K}(x)\, d(Tx, Tx^*). \quad (46)$$

Since $T$ preserves isometric distance with respect to $g$, it follows that

$$d(Tx, Tx^*) \le \mathscr{K}(x)\, d(Tx, Tx^*), \quad (47)$$

which implies that $Tx = Tx^*$. This completes the proof. □

**Corollary 22** (see [21, Theorem 3.1]). *Let $(X, d)$ be a complete metric space and $A$, $B$ nonempty closed subsets of $X$ such that $A$ is approximatively compact with respect to $B$. Suppose that $A_0$ and $B_0$ are nonempty and $T : A \to B$ and $g : A \to A$ are mappings satisfying the following conditions:*

(a) *$T$ is a continuous proximal contraction of the second kind;*

(b) *$T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$;*

(c) *$g$ is an isometry;*

(d) *$T$ preserves isometric distance with respect to $g$.*

*Then there exists a point $x \in A$ such that $d(gx, Tx) = d(A, B)$. Moreover, if $x^*$ is another point in $A$ for which $d(gx^*, Tx^*) = d(A, B)$, then $Tx = Tx^*$.*

*Proof.* Since a proximal contraction of the second kind is a special case of a generalized proximal contraction of the second kind with respect to $\mathscr{K}$, we can prove this result by applying Theorem 21. □

**Corollary 23.** *Let $(X, d)$ be a complete metric space and $A$, $B$ nonempty closed subsets of $X$ such that $A$ is approximatively compact with respect to $B$. Suppose that $A_0$ and $B_0$ are nonempty and $T : A \to B$ and $\mathscr{K} : A \to [0, 1)$ are mappings satisfying the following conditions:*

(a) *$T$ is a continuous generalized proximal contraction of the second kind with respect to $\mathscr{K}$;*

(b) *$T(A_0) \subseteq B_0$;*

(c) *$\mathscr{K}(x) \le \mathscr{K}(y)$, whenever $d(x, Ty) = d(A, B)$.*

*Then $T$ has a best proximity point. Moreover, if $x^*$ is another best proximity point of $T$, then $Tx = Tx^*$.*

*Proof.* We can prove this result by applying Theorem 21 with $g = I_A$, where $I_A$ is an identity mapping on $A$. □

**Corollary 24** (see [21, Corollary 3.2]). *Let $(X, d)$ be a complete metric space and $A$, $B$ nonempty closed subsets of $X$ such that $A$ is approximatively compact with respect to $B$. Suppose that $A_0$ and $B_0$ are nonempty and $T : A \to B$ is mapping satisfying the following conditions:*

(a) *$T$ is a continuous generalized proximal contraction of the second kind;*

(b) *$T(A_0) \subseteq B_0$.*

*Then $T$ has a best proximity point. Moreover, if $x^*$ is another best proximity point of $T$, then $Tx = Tx^*$.*

*Proof.* Since a proximal contraction of the second kind is a special case of a generalized proximal contraction of the second kind, we can prove this result by applying Corollary 23. □

Here, we give the last result in this work.

**Theorem 25.** *Let $(X, d)$ be a complete metric space, $A$ and $B$ nonempty closed subsets of $X$, and $\mathscr{K} : A \cup B \to [0, 1)$. Suppose that $S : A \to B$ is a mapping satisfying*

$$d(Sx, Sy) \le \mathscr{K}(x)\, d(x, y) \quad (48)$$

*for all $x, y \in A$. Then the following holds.*

**(A)** *There exists a nonexpansive mapping $T : B \to A$ such that $(S, T)$ satisfies the min-max condition whenever $S$ has a best proximity point.*

**(B)** *If there exists a nonexpansive mapping $T : B \to A$ such that $(S, T)$ satisfies the min-max condition and $\mathscr{K}(Sx) \le \mathscr{K}(x)$ and $\mathscr{K}(Tx) \le \mathscr{K}(x)$ for all $x \in A$, then $S$ has a best proximity point.*

**(C)** *For two any best proximity points $z$ and $z^*$ of $S$, we have*

$$d(z, z^*) \le \left(\frac{2}{1 - \mathscr{K}(z)}\right) d(A, B). \quad (49)$$

*Proof.* **(A)** Let $S$ has a best proximity point $a \in A$. We define a mapping $T : B \to A$ by $Ty = a$ for all $y \in B$. Clearly, $T$ is a nonexpansive mapping. It follows from the definition of $T$ that

$$d(Ty, STy) = d(a, Sa) = d(A, B) \quad (50)$$

for all $y \in B$. Thus we can conclude that $\min(Sx, Ty) = d(A, B)$ for all $x \in A$ and $y \in B$.

Next, we show that $(S, T)$ satisfies the min-max condition. Suppose that $x \in A$ and $y \in B$ such that $d(A, B) < d(x, y)$. Then we have

$$\min(Sx, Ty) = d(A, B) < d(x, y) \le \max(Sx, Ty), \quad (51)$$

which implies that the pair $(S,T)$ satisfies the min-max condition. Therefore, we can find a nonexpansive mapping $T : B \to A$ such that $(S,T)$ satisfies the min-max condition.

**(B)** Fix $x_0 \in A$ and define a sequence $\{x_n\}$ in $A \cup B$ by

$$x_{2n-1} = Sx_{2n-2}, \qquad x_{2n} = Tx_{2n-1} \tag{52}$$

for all $n \in \mathbb{N}$. Since $T$ is nonexpansive, it follows from (48) that

$$\begin{aligned} d(x_{2n-2}, x_{2n}) &= d(Tx_{2n-3}, Tx_{2n-1}) \\ &\le d(x_{2n-3}, x_{2n-1}) \\ &= d(Sx_{2n-4}, Sx_{2n-2}) \\ &\le \mathscr{K}(x_{2n-4})\, d(x_{2n-4}, x_{2n-2}) \\ &= \mathscr{K}(Tx_{2n-5})\, d(x_{2n-4}, x_{2n-2}) \\ &\le \mathscr{K}(x_{2n-5})\, d(x_{2n-4}, x_{2n-2}) \\ &= \mathscr{K}(Sx_{2n-6})\, d(x_{2n-4}, x_{2n-2}) \\ &\le \mathscr{K}(x_{2n-6})\, d(x_{2n-4}, x_{2n-2}) \\ &\;\;\vdots \\ &\le \mathscr{K}(x_0)\, d(x_{2n-4}, x_{2n-2}) \end{aligned} \tag{53}$$

for all $n \in \mathbb{N}$. By repeating the above argument, we have

$$d(x_{2n-2}, x_{2n}) \le (\mathscr{K}(x_0))^{n-1} d(x_0, x_2) \tag{54}$$

for all $n \in \mathbb{N}$, which implies that the sequence $\{x_{2n}\}$ is a Cauchy sequence in $X$. A similar argument asserts that the sequence $\{x_{2n-1}\}$ is a Cauchy sequence in $X$. By the completeness of $X$, we conclude that $\{x_{2n}\}$ converges to a point $a \in A$ and $\{x_{2n-1}\}$ converges to a point $b \in B$. Since $S$ is continuous, $\{Sx_{2n}\}$ converges to $Sa$, which implies that $\{x_{2n-1}\}$ converges to $Sa$. Thus $Sa = b$.

Similarly, it is easy to check that $Tb = a$. Therefore, we have

$$TSa = Tb = a, \qquad STb = Sa = b. \tag{55}$$

Now, we can conclude that

$$\min(Sa, Tb) = d(a,b) = \max(Sa, Tb). \tag{56}$$

By the virtue of the min-max condition of $(S,T)$, we get $d(a,b) \le d(A,B)$. Since $d(A,B) \le d(a,b)$, we have $d(a,b) = d(A,B)$. Therefore, we have

$$d(a, Sa) = d(a,b) = d(A,B), \tag{57}$$

which implies that $S$ has a best proximity point in $A$.

**(C)** Let $z$ and $z^*$ be best proximity points of $S$. Then $d(z,Sz) = d(A,B)$ and $d(z^*, Sz^*) = d(A,B)$. Using the triangle inequality and (48), we have

$$\begin{aligned} d(z,z^*) &\le d(z,Sz) + d(Sz,Sz^*) + d(Sz^*, z^*) \\ &\le \mathscr{K}(z)\, d(z,z^*) + 2d(A,B). \end{aligned} \tag{58}$$

This implies that $d(z,z^*) \le (2/(1-\mathscr{K}(z)))d(A,B)$. This completes the proof. □

**Corollary 26** (see [21, Theorem 3.6]). *Let $(X,d)$ be a complete metric space and $A$ and $B$ nonempty closed subsets of $X$. Suppose that $S : A \to B$ is a contraction mapping. Then $S$ has a best proximity point if and only if there exists a nonexpansive mapping $T : B \to A$ such that $(S,T)$ satisfies the min-max condition.*

*Moreover, $d(z,z^*) \le (2/(1-\alpha))d(A,B)$ for some $\alpha \in [0,1)$ and any two best proximity points $z$ and $z^*$ of $S$.*

*Proof.* Since $S$ is a contraction mapping, we have $d(Sx,Sy) \le \alpha d(x,y)$ for some $\alpha \in [0,1)$ and all $x, y \in A$. Now, we can prove this result by applying Theorem 25 with a function $\mathscr{K} : A \cup B \to [0,1)$ defined by $\mathscr{K}(x) = \alpha$ for all $x \in A \cup B$. □

## Acknowledgments

The second author would like to thank the Commission on Higher Education, the Thailand Research Fund, and the King Mongkut's University of Technology Thonburi (Grant no. MRG5580213) for financial support during the preparation of this paper.

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The Scientific World Journal
Hindawi Publishing Corporation
http://www.hindawi.com
Volume 2013

Mathematical
Inequalities
& Applications
Volume 16, Number 4 (2013), 1197–1216 doi:10.7153/mia-16-93

# TRIPLED BEST PROXIMITY POINT THEOREM IN METRIC SPACES

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(*Communicated by Josip Pečarić*)

*Abstract.* The purpose of this article is to first introduce the notion of tripled best proximity point and cyclic contraction pair. We also establish the existence and convergence theorems of tripled best proximity points in metric spaces. Moreover, we apply our results to setting of uniformly convex Banach space. Finally, we obtain some results on the existence and convergence of tripled fixed point in metric spaces and give illustrative examples of our theorems.

## 1. Introduction and preliminaries

In the two last decades, the theory of fixed points has appeared as a crucial technique in the study of nonlinear functional analysis. In particular, the techniques and tools in fixed point theory have application in many branches of applied mathematics and also in many research fields such as physics, chemistry, biology, economics, computer sciences, and many branches of engineering. The most significant result in fixed point theory, known as the Banach Contraction Mapping Principle (BCMP) is given by Banach in [4]. BCMP states that every contraction (self-mapping) $T : X \to X$ on a complete metric space $(X,d)$ has a unique fixed point, that is, $Tx = x$. Due to its wide application potential, this celebrated principle has been generalized in many ways over the years [2, 10, 11, 23, 33].

On the other hand, the study of the existence of fixed point for non-self mapping on various abstract spaces is also very interesting. More precisely, for a given nonempty closed subsets $A$ and $B$ of a complete metric space $(X,d)$, a contraction nonself mapping $T : A \to B$ does not necessarily yields a fixed point, that is, $d(Tx,x) \neq 0$. In this case, it is quite natural to investigate an element $x \in X$ such that $d(x,Tx)$ is minimum, that is, the points $x$ and $Tx$ are close proximity to each other.

Let $A$ and $B$ be closed subsets of a metric space $(X,d)$ and $T : A \to B$ be a nonself mapping. A point $x$ in $A$ for which $d(x,Tx) = d(A,B)$ is called a best proximity point of $T$. If $A \cap B \neq \emptyset$ then the best proximity point becomes a fixed point of $T$. In other words, since a best proximity point reduces to a fixed point if the underlying mapping is assumed to be self mappings, the best proximity point theorems are natural generalizations of the BCMP. In this direction, the first result was given by Fan [13] in 1969. In these pioneering work, the author introduced and established a classical best approximation theorem:

*Mathematics subject classification* (2010): 47H10, 54H25, 46J10, 46J15.
*Keywords and phrases*: Tripled fixed point, tripled Common Fixed Point, tripled best proximity point.


Paper MIA-16-93

THEOREM 1. ([13]) *If $A$ is a nonempty compact convex subset of a Hausdorff locally convex topological vector space $B$ and $T : A \to B$ is continuous mapping, then there exists an element $x \in A$ such that $d(x,Tx) = d(Tx,A)$.*

Following this initial paper, a number of authors have derived extensions of Fan's Theorem and best approximation theorem in many directions such as Prolla [27], Sehgal and Singh [28, 29], Wlodarczyk and Plebaniak [36, 37, 38, 39], Vetrivel et al. [35], Eldred and Veramani [12], Mongkolkeha and Kumam [24, 25, 26] and Basha and Veeramani [5, 6, 7, 8] (see also [3, 16, 17, 18, 19, 20, 21, 22] and reference therein).

One interesting and crucial notion is the one of coupled fixed point, introduced by Guo and Lakshmikantham [15] in 1987. Bhaskar and Laksmikantham [14] introduced the notion of mixed monotone mapping and proved some coupled fixed point theorems for mappings satisfying the mixed monotone property. In [14], the authors observed that their theorems can be used to investigate a large class of problems and discuss the existence and uniqueness of solution for a periodic boundary value problem. Several improvements and generalizations of [14] have recently appeared in the literature (see [1, 30, 31] and references therein).

Very recently, Berinde and Borcut [9] introduced the notions of tripled fixed point. They proved existence and uniqueness results of tripled fixed point in a partially ordered complete metric space. On the other hand, the concept of coupled best proximity point and property UC$^*$ are first introduced by Sintunavarat and Kumam [32]. They also give existence and convergence theorems of coupled best proximity point for cyclic contraction pairs.

Motivated by the interesting works [9] and [32], we first introduce the notions of tripled best proximity point and later establish the existence and convergence theorems of tripled best proximity point in metric spaces. Moreover, we apply these results in uniformly convex Banach space. We also study some results on the existence and convergence of tripled fixed point in metric spaces and give illustrative examples of our theorems.

We recall some basic definitions and examples that are related to the main results of this article. Throughout this article we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. For nonempty subsets $A$ and $B$ of a metric space $(X,d)$, we set

$$d(A,B) = \inf\{d(x,y) : x \in A,\ \ y \in B\} \tag{1}$$

stands for the distance between $A$ and $B$.

A Banach spaces $X$ is said to be:

(1) *strictly convex* if the following implication holds: for all $x,y \in X$,

$$\|x\| = \|y\| = 1 \text{ and } x \neq y \Longrightarrow \left\|\frac{x+y}{2}\right\| < 1.$$

(2) *uniformly convex* if, for any $\varepsilon$ with $0 < \varepsilon \leqslant 2$, there exists $\delta > 0$ such that the following implication holds: for all $x,y \in X$,

$$\|x\| \leqslant 1,\ \|y\| \leqslant 1 \text{ and } \|x-y\| \geqslant \varepsilon \Longrightarrow \left\|\frac{x+y}{2}\right\| < 1-\delta.$$

It is easily to see that a uniformly convex Banach space $X$ is strictly convex but the converge is not true.

DEFINITION 1. ([34]) Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. We say that the ordered pair $(A,B)$ satisfies the *property UC* if the following holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in $A$ and $\{y_n\}$ is a sequence in $B$ such that $d(x_n,y_n) \to d(A,B)$ and $d(z_n,y_n) \to d(A,B)$, then $d(x_n,z_n) \to 0$.

EXAMPLE 1. Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. The following statements are examples of pairs of nonempty subsets $(A,B)$ satisfying the property UC.

(1) A pair $(A,B)$ of nonempty subsets $A,B$ of a metric space $(X,d)$ such that $d(A,B) = 0$.

(2) A pair $(A,B)$ of nonempty subsets $A,B$ of a uniformly convex Banach space $X$ such that $A$ is convex.

(3) A pair $(A,B)$ of nonempty subsets $A,B$ of a strictly convex Banach space such that $A$ is convex and relatively compact and the closure of $B$ is weakly compact.

DEFINITION 2. ([32]) Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. We say that the ordered pair $(A,B)$ satisfies the *property UC*$^*$ if $(A,B)$ has property UC and the following condition holds:

If $\{x_n\}$, $\{z_n\}$ are two sequences in $A$ and $\{y_n\}$ is a sequence in $B$ satisfying the following conditions:

(1) $d(z_n,y_n) \to d(A,B)$,

(2) for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_m,y_n) \leqslant d(A,B) + \varepsilon$ for all $m > n \geqslant N$,

then exists $N_1 \in \mathbb{N}$ such that $d(x_m,z_n) \leqslant d(A,B) + \varepsilon$ for all $m > n \geqslant N_1$.

EXAMPLE 2. ([32]) Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. The following statements are examples of a pair of nonempty subsets $(A,B)$ satisfying the property UC$^*$.

(1) A pair $(A,B)$ of nonempty subsets $A,B$ of a metric space $(X,d)$ such that $d(A,B) = 0$.

(2) A pair $(A,B)$ of nonempty closed subsets $A,B$ of a uniformly convex Banach space $X$ such that $A$ is convex.

DEFINITION 3. Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$ and $T : A \to B$ be a mapping. A point $x \in A$ is called a *best proximity point* of $T$ if the following condition holds:

$$d(x,Tx) = d(A,B).$$

It can be observed that a best proximity point reduces to a fixed point if the underlying mapping is a self-mapping.

DEFINITION 4. ([9]) Let $A$ be a nonempty subset of a metric space $(X,d)$ and $F : A^3 \to A$ be a mapping. A point $(x,y,z) \in A^3$ is called a *tripled fixed point* of $F$ if the following conditions hold:

$$x = F(x,y,z), \quad y = F(y,x,y), \quad z = F(z,y,x).$$

## 2. Tripled best proximity point theorems

In this section we study the existence and convergence of tripled best proximity points for cyclic contraction pairs in metric spaces.

DEFINITION 5. Let $A$, $B$ be nonempty subsets of a metric space $(X,d)$ and $F : A^3 \to B$ be mapping. An ordered tripled $(x,y,z) \in A^3$ is called a *tripled best proximity point* of $F$ if,

$$d(x,F(x,y,z)) = d(y,F(y,x,y)) = d(z,F(z,y,x)) = d(A,B).$$

It is easy to see that, if $A = B$ in Definition 5, then a tripled best proximity point reduces to a tripled fixed point.

Next, we introduce the notions of a cyclic contractions for a pair of mappings.

DEFINITION 6. Let $A$, $B$ be nonempty subsets of a metric space $(X,d$ and $F : A^3 \to B$, $G : B^3 \to A$ be two mappings. The ordered pair $(F,G)$ is called a *cyclic contraction* if there exists a non-negative number $\alpha < 1$ such that

$$d(F(x,y,z),G(u,v,w)) \leqslant \frac{\alpha}{3}[d(x,u)+d(y,v)+d(z,w)]+(1-\alpha)d(A,B)$$

for all $(x,y,z) \in A^3$ and $(u,v,w) \in B^3$.

Note that, if $(F,G)$ is a cyclic contraction, then the pair $(G,F)$ is also a cyclic contraction.

EXAMPLE 3. Let $X = \mathbb{R}$ with the usual metric $d(x,y) = |x-y|$ and let $A = [2,6]$ and $B = [-6,-2]$. It easy to see that $d(A,B) = 4$. Define two mappings $F : A^3 \to B$ and $G : B^3 \to A$ by

$$F(x,y,z) = \frac{-x-y-z-6}{6}, \quad G(u,v,w) = \frac{-u-v-w+6}{6}$$

for all $(x,y,z) \in A^3$ and $(u,v,w) \in B^3$, respectively. For any $(x,y,z) \in A^3$, $(u,v,w) \in B^3$ and fixed $\alpha = \frac{1}{2}$, we get

$$\begin{aligned} d(F(x,y,z),F(u,v,w)) &= \left| \frac{-x-y-z-6}{6} - \frac{-u-v-w+6}{6} \right| \\ &\leqslant \frac{|x-u|+|y-v|+|z-w|}{6} + 2 \\ &= \frac{\alpha}{3}[d(x,u)+d(y,v)+d(z,w)]+(1-\alpha)d(A,B). \end{aligned}$$

This implies that the pair $(F,G)$ is a cyclic contraction with $\alpha = \frac{1}{2}$.

EXAMPLE 4. Let $X=\mathbb{R}^3$ with the metric

$$d((x,y,z),(u,v,w))=\max\{|x-u|,|y-v|,|z-w|\}$$

for $(x,y,z),(u,v,w)\in X$ and let

$$A=\{(x,0,0)\in X:0\leqslant x\leqslant 1\},\quad B=\{(x,1,1)\in X:0\leqslant x\leqslant 1\}.$$

It easy to prove that $d(A,B)=1$. Define two mappings $F:A^3\to B$ and $G:B^3\to A$ by

$$\begin{aligned}F((x,0,0),(y,0,0),(z,0,0))&=\left(\frac{x+y+z}{3},1,1\right),\\ G((u,1,1),(v,1,1),(w,1,1))&=\left(\frac{u+v+w}{3},0,0\right),\end{aligned}$$

respectively. Then we obtain

$$\begin{aligned}&d(F((x,0,0),(y,0,0),(z,0,0)),G((u,1,1),(v,1,1),(w,1,1)))\\ &=d\left(\left(\frac{x+y+z}{3},1,1\right),\left(\frac{u+v+w}{3},0,0\right)\right)\\ &=1.\end{aligned}$$

Also, if $(x,0,0),(y,0,0),(z,0,0)\in A$ and $(u,1,1),(v,1,1),(w,1,1)\in B$, then we have

$$\begin{aligned}&\frac{\alpha}{3}[d((x,0,0),(u,1,1))+d((y,0,0),(v,1,1))+d((z,0,0),(w,1,1))]+(1-\alpha)d(A,B)\\ &=\frac{\alpha}{3}[\max\{|x-u|,1,1\}+\max\{|y-v|,1,1\}+\max\{|z-w|,1,1\}]+(1-\alpha)d(A,B)\\ &=\frac{\alpha}{3}\times 3+(1-\alpha)\\ &=1\end{aligned}$$

for any non-negative real number $\alpha<1$. Therefore, letting

$$\begin{aligned}(\mathbf{X},\mathbf{Y},\mathbf{Z})&=((x,0,0),(y,0,0),(z,0,0))\in A^3,\\ (\mathbf{U},\mathbf{V},\mathbf{W})&=((u,1,1),(v,1,1),(w,1,1))\in B^3,\end{aligned}$$

we get

$$d(F(\mathbf{X},\mathbf{Y},\mathbf{Z}),G(\mathbf{U},\mathbf{V},\mathbf{W}))\leqslant\frac{\alpha}{3}[d(\mathbf{X},\mathbf{U})+d(\mathbf{Y},\mathbf{V})+d(\mathbf{Z},\mathbf{W})]+(1-\alpha)d(A,B)$$

for any non-negative real number $\alpha<1$. This implies that the pair $(F,G)$ is a cyclic contraction.

The following lemmas play an important role in our main results.

LEMMA 1. *Let $A$, $B$ be nonempty subsets of a metric space $(X,d)$ and $F : A^3 \to B$, $G : B^3 \to A$ be two mappings such that the ordered pair $(F,G)$ is a cyclic contraction. If $(x_0,y_0,z_0) \in A^3$ and we define the sequence $\{x_n\},\{y_n\},\{z_n\}$ in $X$ by*

$$\begin{aligned}
x_{2n+1} &= F(x_{2n},y_{2n},z_{2n}), & x_{2n+2} &= G(x_{2n+1},y_{2n+1},z_{2n+1})\\
y_{2n+1} &= F(y_{2n},x_{2n},y_{2n}), & y_{2n+2} &= G(y_{2n+1},x_{2n+1},y_{2n+1})\\
z_{2n+1} &= F(z_{2n},y_{2n},x_{2n}), & z_{2n+2} &= G(z_{2n+1},y_{2n+1},x_{2n+1})
\end{aligned}$$

*for all $n \in \mathbb{N} \cup \{0\}$, then we have*

$$\begin{aligned}
d(x_{2n},x_{2n+1}) &\to d(A,B), & d(x_{2n+1},x_{2n+2}) &\to d(A,B),\\
d(y_{2n},y_{2n+1}) &\to d(A,B), & d(y_{2n+1},y_{2n+2}) &\to d(A,B),\\
d(z_{2n},z_{2n+1}) &\to d(A,B), & d(z_{2n+1},z_{2n+2}) &\to d(A,B).
\end{aligned}$$

*Proof.* For all $n \in \mathbb{N}$, we have

$$\begin{aligned}
& d(x_{2n},x_{2n+1})\\
&= d(x_{2n},F(x_{2n},y_{2n},z_{2n}))\\
&= d(G(x_{2n-1},y_{2n-1},z_{2n-1}),\\
&\qquad F(G(x_{2n-1},y_{2n-1},z_{2n-1}),G(y_{2n-1},x_{2n-1},y_{2n-1}),G(z_{2n-1},y_{2n-1},x_{2n-1})))\\
&\leqslant \frac{\alpha}{3}[d(x_{2n-1},G(x_{2n-1},y_{2n-1},z_{2n-1}))+d(y_{2n-1},G(y_{2n-1},x_{2n-1},y_{2n-1}))\\
&\quad +d(z_{2n-1},G(z_{2n-1},y_{2n-1},x_{2n-1}))]+(1-\alpha)d(A,B)\\
&= \frac{\alpha}{3}\Big[d(F(x_{2n-2},y_{2n-2},z_{2n-2}),\\
&\qquad G(F(x_{2n-2},y_{2n-2},z_{2n-2}),F(y_{2n-2},x_{2n-2},y_{2n-2}),F(z_{2n-2},y_{2n-2},x_{2n-2})))\\
&\quad +d(F(y_{2n-2},x_{2n-2},y_{2n-2}),\\
&\qquad G(F(y_{2n-2},x_{2n-2},y_{2n-2}),F(x_{2n-2},y_{2n-2},z_{2n-2}),F(y_{2n-2},x_{2n-2},y_{2n-2})))\\
&\quad +d(F(z_{2n-2},y_{2n-2},x_{2n-2}),\\
&\qquad G(F(z_{2n-2},y_{2n-2},x_{2n-2}),F(y_{2n-2},x_{2n-2},y_{2n-2}),F(x_{2n-2},y_{2n-2},z_{2n-2})))\Big]\\
&\quad +(1-\alpha)d(A,B)\\
&\leqslant \frac{\alpha}{3}\Big[\frac{\alpha}{3}[d(x_{2n-2},F(x_{2n-2},y_{2n-2},z_{2n-2}))+d(y_{2n-2},F(y_{2n-2},x_{2n-2},y_{2n-2}))\\
&\quad +d(z_{2n-2},F(z_{2n-2},y_{2n-2},x_{2n-2}))]+(1-\alpha)d(A,B)\\
&\quad +\frac{\alpha}{3}[d(y_{2n-2},F(y_{2n-2},x_{2n-2},y_{2n-2}))+d(z_{2n-2},F(z_{2n-2},y_{2n-2},x_{2n-2}))\\
&\quad +d(x_{2n-2},F(x_{2n-2},y_{2n-2},z_{2n-2}))]+(1-\alpha)d(A,B)\\
&\quad +\frac{\alpha}{3}[d(z_{2n-2},F(z_{2n-2},y_{2n-2},x_{2n-2}))+d(x_{2n-2},F(x_{2n-2},y_{2n-2},z_{2n-2}))\\
&\quad +d(y_{2n-2},F(y_{2n-2},x_{2n-2},y_{2n-2}))]+(1-\alpha)d(A,B)\Big]\\
&\quad +(1-\alpha)d(A,B)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\alpha^2}{3}[d(z_{2n-2},F(z_{2n-2},y_{2n-2},x_{2n-2}))+d(x_{2n-2},F(x_{2n-2},y_{2n-2},z_{2n-2}))\\
&\quad +d(y_{2n-2},F(y_{2n-2},x_{2n-2},y_{2n-2}))]+(1-\alpha^2)d(A,B).
\end{aligned}$$

By induction, we see that

$$\begin{aligned}
d(x_{2n},x_{2n+1}) \leqslant \frac{\alpha^{2n}}{3}&[d(x_0,F(x_0,y_0,z_0))+d(y_0,F(y_0,x_0,y_0))+d(z_0,F(z_0,y_0,x_0))]\\
&+(1-\alpha^{2n})d(A,B)
\end{aligned}$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$, we obtain

$$d(x_{2n},x_{2n+1}) \to d(A,B). \tag{2}$$

For all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&d(x_{2n+1},x_{2n+2})\\
&= d(x_{2n+1},G(x_{2n+1},y_{2n+1},z_{2n+1}))\\
&= d(F(x_{2n},y_{2n},z_{2n}),G(F(x_{2n},y_{2n},z_{2n}),F(y_{2n},x_{2n},y_{2n}),F(z_{2n},y_{2n},x_{2n})))\\
&\leqslant \frac{\alpha}{3}\big[d(x_{2n},F(x_{2n},y_{2n},z_{2n}))+d(y_{2n},F(y_{2n},x_{2n},y_{2n}))\\
&\quad +d(z_{2n},F(z_{2n},y_{2n},x_{2n})\big]+(1-\alpha)d(A,B)\\
&= \frac{\alpha}{3}\big[d(G(x_{2n-1},y_{2n-1},z_{2n-1}),\\
&\qquad F(G(x_{2n-1},y_{2n-1},z_{2n-1}),G(y_{2n-1},x_{2n-1},y_{2n-1}),G(z_{2n-1},y_{2n-1},x_{2n-1})))\\
&\quad +d(G(y_{2n-1},x_{2n-1},y_{2n-1}),\\
&\qquad F(G(y_{2n-1},x_{2n-1},y_{2n-1}),G(x_{2n-1},y_{2n-1},z_{2n-1}),G(y_{2n-1},x_{2n-1},y_{2n-1})))\\
&\quad +d(G(z_{2n-1},y_{2n-1},x_{2n-1}),\\
&\qquad F(G(z_{2n-1},y_{2n-1},x_{2n-1}),G(y_{2n-1},x_{2n-1},y_{2n-1}),G(x_{2n-1},y_{2n-1},z_{2n-1})))\big]\\
&\quad +(1-\alpha)d(A,B)\\
&\leqslant \frac{\alpha}{3}\Big[\frac{\alpha}{3}[d(x_{2n-1},G(x_{2n-1},y_{2n-1},z_{2n-1}))+d(y_{2n-1},G(y_{2n-1},x_{2n-1},y_{2n-1}))\\
&\quad +d(z_{2n-1},G(z_{2n-1},y_{2n-1},x_{2n-1}))]+(1-\alpha)d(A,B)\\
&\quad +\frac{\alpha}{3}[d(y_{2n-1},G(y_{2n-1},x_{2n-1},y_{2n-1}))+d(z_{2n-1},G(z_{2n-1},y_{2n-1},x_{2n-1}))\\
&\quad +d(x_{2n-1},G(x_{2n-1},y_{2n-1},z_{2n-1}))]+(1-\alpha)d(A,B)\\
&\quad +\frac{\alpha}{3}[d(z_{2n-1},G(z_{2n-1},y_{2n-1},x_{2n-1}))+d(x_{2n-1},G(x_{2n-1},y_{2n-1},z_{2n-1}))\\
&\quad +d(y_{2n-1},G(y_{2n-1},x_{2n-1},y_{2n-1}))]+(1-\alpha)d(A,B)\Big]\\
&\quad +(1-\alpha)d(A,B)\\
&= \frac{\alpha^2}{3}[d(z_{2n-1},G(z_{2n-1},y_{2n-1},x_{2n-1}))+d(x_{2n-1},G(x_{2n-1},y_{2n-1},z_{2n-1}))\\
&\quad +d(y_{2n-1},G(y_{2n-1},x_{2n-1},y_{2n-1}))]+(1-\alpha^2)d(A,B).
\end{aligned}$$

By induction, we see that

$$d(x_{2n+1},x_{2n+2}) \leqslant \frac{\alpha^{2n}}{3}[d(x_1,G(x_1,y_1,z_1))+d(y_1,G(y_1,x_1,y_1))+d(z_1,G(z_1,y_1,x_1))]$$
$$+(1-\alpha^{2n})d(A,B)$$

for all $n \in \mathbb{N}$. Therefore, letting $n \to \infty$, we obtain

$$d(x_{2n+1},x_{2n+2}) \to d(A,B). \tag{3}$$

By the similar argument, we also have

$$d(y_{2n},y_{2n+1}) \to d(A,B), \quad d(y_{2n+1},y_{2n+2}) \to d(A,B),$$
$$d(z_{2n+1},z_{2n+2}) \to d(A,B), \quad d(z_{2n},z_{2n+1}) \to d(A,B).$$

This completes the proof. □

LEMMA 2. *Let $A$, $B$ be nonempty subsets of a metric space $(X,d)$ such that the pairs $(A,B)$ and $(B,A)$ have the property UC and $F : A^3 \to B$, $G : B^3 \to A$ be two mappings such that the ordered pair $(F,G)$ is a cyclic contraction. For any $(x_0,y_0,z_0) \in A^3$, we define the sequence $\{x_n\},\{y_n\},\{z_n\}$ in $X$ by*

$$\begin{array}{ll} x_{2n+1} = F(x_{2n},y_{2n},z_{2n}), & x_{2n+2} = G(x_{2n+1},y_{2n+1},z_{2n+1}) \\ y_{2n+1} = F(y_{2n},x_{2n},y_{2n}), & y_{2n+2} = G(y_{2n+1},x_{2n+1},y_{2n+1}) \\ z_{2n+1} = F(z_{2n},y_{2n},x_{2n}), & z_{2n+2} = G(z_{2n+1},y_{2n+1},x_{2n+1}) \end{array}$$

*for all $n \in \mathbb{N} \cup \{0\}$. Then, for any $\varepsilon > 0$, there exists a positive integer $N_0$ such that, for all $m > n \geqslant N_0$,*

$$\frac{1}{3}[d(x_{2m},x_{2n+1})+d(y_{2m},y_{2n+1})+d(z_{2m},z_{2n+1})] < d(A,B)+\varepsilon. \tag{4}$$

*Proof.* By Lemma 1, we have

$$\begin{array}{ll} d(x_{2n},x_{2n+1}) \to d(A,B), & d(x_{2n+1},x_{2n+2}) \to d(A,B), \\ d(y_{2n},y_{2n+1}) \to d(A,B), & d(y_{2n+1},y_{2n+2}) \to d(A,B), \\ d(z_{2n},z_{2n+1}) \to d(A,B), & d(z_{2n+1},z_{2n+2}) \to d(A,B). \end{array}$$

Since $(A,B)$ has the property UC, we get

$$d(x_{2n},x_{2n+2}) \to 0.$$

The similar argument shows that

$$d(y_{2n},y_{2n+2}) \to 0 \text{ and } d(z_{2n},z_{2n+2}) \to 0.$$

Since $(B,A)$ has the property UC, we also have

$$d(x_{2n+1},x_{2n+3}) \to 0, \quad d(y_{2n+1},y_{2n+3}) \to 0, \quad d(z_{2n+1},z_{2n+3}) \to 0.$$

Suppose that (4) does not hold. Then there exists $\varepsilon' > 0$ such that, for all $k \in \mathbb{N}$, there exists $m_k > n_k \geqslant k$ satisfying

$$\frac{1}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] \geqslant d(A,B) + \varepsilon'.$$

Further, corresponding to $n_k$, we can choose $m_k$ in such a way that it is the smallest integer with $m_k > n_k$ and satisfying above relation. Then

$$\frac{1}{3}[d(x_{2m_k-2}, x_{2n_k+1}) + d(y_{2m_k-2}, y_{2n_k+1}) + d(z_{2m_k-2}, z_{2n_k+1})] < d(A,B) + \varepsilon'.$$

Therefore, we get

$$\begin{aligned}
& d(A,B) + \varepsilon' \\
\leqslant\ & \frac{1}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] \\
\leqslant\ & \frac{1}{3}[d(x_{2m_k}, x_{2m_k-2}) + d(x_{2m_k-2}, x_{2n_k+1}) \\
& + d(y_{2m_k}, y_{2m_k-2}) + d(y_{2m_k-2}, y_{2n_k+1}) + d(z_{2m_k}, z_{2m_k-2}) + d(z_{2m_k-2}, z_{2n_k+1})] \\
<\ & \frac{1}{3}[d(x_{2m_k}, x_{2m_k-2}) + d(y_{2m_k}, y_{2m_k-2}) + d(z_{2m_k}, z_{2m_k-2})] + d(A,B) + \varepsilon'.
\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\frac{1}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] \to d(A,B) + \varepsilon'.$$

By using the triangle inequality, we get

$$\begin{aligned}
& \frac{1}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] \\
\leqslant\ & \frac{1}{3}[d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1}) \\
& + d(y_{2m_k}, y_{2m_k+2}) + d(y_{2m_k+2}, y_{2n_k+3}) + d(y_{2n_k+3}, y_{2n_k+1}) \\
& + d(z_{2m_k}, z_{2m_k+2}) + d(z_{2m_k+2}, z_{2n_k+3}) + d(z_{2n_k+3}, z_{2n_k+1})] \\
=\ & \frac{1}{3}[d(x_{2m_k}, x_{2m_k+2}) \\
& + d(G(x_{2m_k+1}, y_{2m_k+1}, z_{2m_k+1}), F(x_{2n_k+2}, y_{2n_k+2}, z_{2n_k+2})) + d(x_{2n_k+3}, x_{2n_k+1}) \\
& + d(y_{2m_k}, y_{2m_k+2}) \\
& + d(G(y_{2m_k+1}, x_{2m_k+1}, y_{2m_k+1}), F(y_{2n_k+2}, x_{2n_k+2}, y_{2n_k+2})) + d(y_{2n_k+3}, y_{2n_k+1}) \\
& + d(z_{2m_k}, z_{2m_k+2}) \\
& + d(G(z_{2m_k+1}, y_{2m_k+1}, x_{2m_k+1}), F(z_{2n_k+2}, y_{2n_k+2}, x_{2n_k+2})) + d(z_{2n_k+3}, z_{2n_k+1})] \\
\leqslant\ & \frac{1}{3}\Big[d(x_{2m_k}, x_{2m_k+2}) + \frac{\alpha}{3}(d(x_{2m_k+1}, x_{2n_k+2}) + d(y_{2m_k+1}, y_{2n_k+2}) \\
& + d(z_{2m_k+1}, z_{2n_k+2})) + (1-\alpha)d(A,B) + d(x_{2n_k+3}, x_{2n_k+1})
\end{aligned}$$

$$
\begin{aligned}
&+d(y_{2m_k},y_{2m_k+2})+\frac{\alpha}{3}(d(y_{2m_k+1},y_{2n_k+2})+d(x_{2m_k+1},x_{2n_k+2})\\
&+d(y_{2m_k+1},y_{2n_k+2}))+(1-\alpha)d(A,B)+d(y_{2n_k+3},y_{2n_k+1})\\
&+d(z_{2m_k},z_{2m_k+2})+\frac{\alpha}{3}(d(z_{2m_k+1},z_{2n_k+2})+d(y_{2m_k+1},y_{2n_k+2})\\
&+d(x_{2m_k+1},x_{2n_k+2}))+(1-\alpha)d(A,B)+d(z_{2n_k+3},z_{2n_k+1})\Big]\\
=&\ \frac{1}{3}[d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})+d(y_{2m_k},y_{2m_k+2})\\
&+d(y_{2n_k+3},y_{2n_k+1})+d(z_{2m_k},z_{2m_k+2})+d(z_{2n_k+3},z_{2n_k+1})]\\
&+\frac{\alpha}{3}(d(x_{2m_k+1},x_{2n_k+2})+d(y_{2m_k+1},y_{2n_k+2})+d(z_{2m_k+1},z_{2n_k+2}))\\
&+(1-\alpha)d(A,B)\\
=&\ \frac{1}{3}[d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})+d(y_{2m_k},y_{2m_k+2})\\
&+d(y_{2n_k+3},y_{2n_k+1})+d(z_{2m_k},z_{2m_k+2})+d(z_{2n_k+3},z_{2n_k+1})]\\
&+\frac{\alpha}{3}[d(F(x_{2m_k},y_{2m_k},z_{2m_k}),G(x_{2n_k+1},y_{2n_k+1},z_{2n_k+1}))\\
&+d(F(y_{2m_k},x_{2m_k},y_{2m_k}),G(y_{2n_k+1},x_{2n_k+1},y_{2n_k+1}))\\
&+d(F(z_{2m_k},y_{2m_k},x_{2m_k}),G(z_{2n_k+1},y_{2n_k+1},x_{2n_k+1}))]\\
&+(1-\alpha)d(A,B)\\
\leqslant&\ \frac{1}{3}[d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})+d(y_{2m_k},y_{2m_k+2})\\
&+d(y_{2n_k+3},y_{2n_k+1})+d(z_{2m_k},z_{2m_k+2})+d(z_{2n_k+3},z_{2n_k+1})]\\
&+\frac{\alpha^2}{3}[d(x_{2m_k},x_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})+d(z_{2m_k},z_{2n_k+1})]\\
&+(1-\alpha^2)d(A,B).
\end{aligned}
$$

Taking $k\to\infty$, we get

$$d(A,B)+\varepsilon'\leqslant\alpha^2[d(A,B)+\varepsilon']+(1-\alpha^2)d(A,B)=d(A,B)+\alpha^2\varepsilon',$$

which is a contradiction. Therefore, we can conclude that (4) holds. This completes the proof. □

LEMMA 3. *Let $A$, $B$ be nonempty subsets of a metric space $(X,d)$ such that the pairs $(A,B)$, $(B,A)$ satisfy the property $UC^*$. Let $F:A^3\to B$, $G:B^3\to A$ be two mappings such that the ordered pair $(F,G)$ is a cyclic contraction. For $(x_0,y_0,z_0)\in A^3$, we define the sequence $\{x_n\},\{y_n\},\{z_n\}$ in $X$ by*

$$
\begin{aligned}
x_{2n+1}&=F(x_{2n},y_{2n},z_{2n}), & x_{2n+2}&=G(x_{2n+1},y_{2n+1},z_{2n+1}),\\
y_{2n+1}&=F(y_{2n},x_{2n},y_{2n}), & y_{2n+2}&=G(y_{2n+1},x_{2n+1},y_{2n+1}),\\
z_{2n+1}&=F(z_{2n},y_{2n},x_{2n}), & z_{2n+2}&=G(z_{2n+1},y_{2n+1},x_{2n+1})
\end{aligned}
$$

*for all $n\in\mathbb{N}\cup\{0\}$. Then the sequences $\{x_{2n}\}$, $\{y_{2n}\}$, $\{z_{2n}\}$, $\{x_{2n+1}\}$, $\{y_{2n+1}\}$ and $\{z_{2n+1}\}$ are Cauchy sequences.*

*Proof.* By Lemma 1, we have

$$d(x_{2n},x_{2n+1}) \to d(A,B), \quad d(x_{2n+1},x_{2n+2}) \to d(A,B).$$

Since the pair $(A,B)$ satisfies the property UC, we get $d(x_{2n},x_{2n+2}) \to 0$. Similarly, we also have $d(x_{2n+1},x_{2n+3}) \to 0$ since the pair $(B,A)$ satisfies the property UC.

Now, we show that, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_{2m},x_{2n+1}) \leqslant d(A,B)+\varepsilon \tag{5}$$

for all $m > n \geqslant N$. Suppose that (5) does not hold. Then there exists $\varepsilon > 0$ such that, for all $k \in \mathbb{N}$, there exists $m_k > n_k \geqslant k$ such that

$$d(x_{2m_k},x_{2n_k+1}) > d(A,B)+\varepsilon. \tag{6}$$

Further, corresponding to $n_k$, we can choose $m_k$ in such a way that it is the smallest integer with $m_k > n_k$ and satisfying above relation. Now, we have

$$\begin{aligned} d(A,B)+\varepsilon &< d(x_{2m_k},x_{2n_k+1}) \\ &\leqslant d(x_{2m_k},x_{2m_k-2})+d(x_{2m_k-2},x_{2n_k+1}) \\ &\leqslant d(x_{2m_k},x_{2m_k-2})+d(A,B)+\varepsilon. \end{aligned}$$

Taking $k \to \infty$, we have $d(x_{2m_k},x_{2n_k+1}) \to d(A,B)+\varepsilon$. By Lemma 2, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{3}[d(x_{2m_k},x_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})+d(z_{2m_k},z_{2n_k+1})] < d(A,B)+\varepsilon \tag{7}$$

for all $m > n \geqslant \mathbb{N}$. By using the triangle inequality, we get

$$\begin{aligned} & d(A,B)+\varepsilon \\ &< d(x_{2m_k},x_{2n_k+1}) \\ &\leqslant d(x_{2m_k},x_{2m_k+2})+d(x_{2m_k+2},x_{2n_k+3})+d(x_{2n_k+3},x_{2n_k+1}) \\ &= d(x_{2m_k},x_{2m_k+2}) \\ &\quad +d(G(x_{2m_k+1},y_{2m_k+1},z_{2m_k+1}),F(x_{2n_k+2},y_{2n_k+2},z_{2n_k+2}))+d(x_{2n_k+3},x_{2n_k+1}) \\ &\leqslant d(x_{2m_k},x_{2m_k+2})+\frac{\alpha}{3}[d(x_{2m_k+1},x_{2n_k+2})+d(y_{2m_k+1},y_{2n_k+2})+d(z_{2m_k+1},z_{2n_k+2})] \\ &\quad +(1-\alpha)d(A,B)+d(x_{2n_k+3},x_{2n_k+1}) \\ &= \frac{\alpha}{3}[d(F(x_{2m_k},y_{2m_k},z_{2m_k}),G(x_{2n_k+1},y_{2n_k+1},z_{2n_k+1})) \\ &\quad +d(F(y_{2m_k},x_{2m_k},y_{2m_k}),G(y_{2n_k+1},x_{2n_k+1},y_{2n_k+1})) \\ &\quad +d(F(z_{2m_k},y_{2m_k},x_{2m_k}),G(z_{2n_k+1},y_{2n_k+1},x_{2n_k+1}))] \\ &\quad +(1-\alpha)d(A,B)+d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1}) \\ &\leqslant \frac{\alpha}{3}\Big[\frac{\alpha}{3}[d(x_{2m_k},x_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})+d(z_{2m_k},z_{2n_k+1})]+(1-\alpha)d(A,B) \end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha}{3}[d(y_{2m_k},y_{2n_k+1})+d(x_{2m_k},x_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})]+(1-\alpha)d(A,B)\\
&+\frac{\alpha}{3}[d(z_{2m_k},z_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})+d(x_{2m_k},x_{2n_k+1})]+(1-\alpha)d(A,B)\Big]\\
&+(1-\alpha)d(A,B)+d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})\\
=\ &\frac{\alpha^2}{3}[d(x_{2m_k},x_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})+d(z_{2m_k},z_{2n_k+1})]\\
&+(1-\alpha^2)d(A,B)+d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})\\
<\ &\alpha^2(d(A,B)+\varepsilon)+(1-\alpha^2)d(A,B)+d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})\\
=\ &\alpha^2\varepsilon+d(A,B)+d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1}).
\end{aligned}
$$

Taking $k\to\infty$, we get

$$d(A,B)+\varepsilon\leqslant d(A,B)+\alpha^2\varepsilon,$$

which is a contradiction. Therefore, the condition (5) holds. Since (5) holds and $d(x_{2n},x_{2n+1})\to d(A,B)$, by using the property UC$^*$ of $(A,B)$, we deduce that $\{x_{2n}\}$ is a Cauchy sequence. In a similar way, we can prove that $\{y_{2n}\}$, $\{z_{2n}\}$, $\{x_{2n+1}\}$, $\{y_{2n+1}\}$ and $\{z_{2n+1}\}$ are Cauchy sequences. This completes the proof. □

Here, we state the main result of this article on the existence and convergence of tripled best proximity points for cyclic contraction pairs on nonempty subsets of metric spaces satisfying the property UC$^*$.

THEOREM 2. *Let $A$, $B$ be nonempty closed subsets of a metric space $(X,d)$ such that the pairs $(A,B)$ and $(B,A)$ have the property UC$^*$ and $F:A^3\to B$, $G:B^3\to A$ be two mappings such that the ordered pair $(F,G)$ is a cyclic contraction. For any $(x_0,y_0,z_0)\in A^3$, we define the sequence $\{x_n\},\{y_n\},\{z_n\}$ in $X$ by*

$$
\begin{aligned}
x_{2n+1}&=F(x_{2n},y_{2n},z_{2n}), & x_{2n+2}&=G(x_{2n+1},y_{2n+1},z_{2n+1}),\\
y_{2n+1}&=F(y_{2n},x_{2n},y_{2n}), & y_{2n+2}&=G(y_{2n+1},x_{2n+1},y_{2n+1}),\\
z_{2n+1}&=F(z_{2n},y_{2n},x_{2n}), & z_{2n+2}&=G(z_{2n+1},y_{2n+1},x_{2n+1})
\end{aligned}
$$

*for all $n\in\mathbb{N}\cup\{0\}$. Then $F$ has a tripled best proximity point $(p,q,r)\in A^3$ and $G$ has a tripled best proximity point $(p',q',r')\in B^3$. Moreover, we have*

$$x_{2n}\to p,\ \ y_{2n}\to q,\ \ z_{2n}\to r,\ \ x_{2n+1}\to p',\ \ y_{2n+1}\to q',\ \ z_{2n+1}\to r'.$$

*Furthermore, if $q=r$ and $q'=r'$, then*

$$d(p,p')+d(q,q')+d(r,r')=3d(A,B).$$

*Proof.* By Lemma 1, we get $d(x_{2n},x_{2n+1})\to d(A,B)$. Using Lemma 3, we see that $\{x_{2n}\}$, $\{y_{2n}\}$ and $\{z_{2n}\}$ are Cauchy sequences. Thus there exist $p,q,r\in A$ such that $x_{2n}\to p$, $y_{2n}\to q$ and $z_{2n}\to r$. Now, we obtain

$$d(A,B)\leqslant d(p,x_{2n-1})\leqslant d(p,x_{2n})+d(x_{2n},x_{2n-1}). \tag{8}$$

Letting $n \to \infty$ in (8), we have $d(p, x_{2n-1}) \to d(A,B)$. By the similar argument, we also have $d(q, y_{2n-1}) \to d(A,B)$ and $d(r, z_{2n-1}) \to d(A,B)$. It follows that

$$\begin{aligned} d(x_{2n}, F(p,q,r)) &= d(G(x_{2n-1}, y_{2n-1}, z_{2n-1}), F(p,q,r)) \\ &\leqslant \frac{\alpha}{3}[d(x_{2n-1}, p) + d(y_{2n-1}, q) + d(z_{2n-1}, r)] + (1-\alpha)d(A,B). \end{aligned}$$

Taking $n \to \infty$, we get $d(p, F(p,q,r)) = d(A,B)$. Similarly, we can prove that

$$d(q, F(q,p,q)) = d(A,B), \quad d(r, F(r,q,p)) = d(A,B).$$

Therefore, $(p,q,r)$ is a tripled best proximity point of $F$.

By the similar way, we can prove that there exist $p', q', r' \in B$ such that $x_{2n+1} \to p'$, $y_{2n+1} \to q'$ and $z_{2n+1} \to r'$. Moreover, we have

$$d(p', G(p', q', r')) = d(A,B), \quad d(q', F(q', p', q')) = d(A,B)$$

and

$$d(r', F(r', q', p')) = d(A,B)$$

and so $(p', q', r')$ is a tripled best proximity point of $G$.

Finally, we assume that $q = r$ and $q' = r'$ and then we show that

$$d(p,p') + d(q,q') + d(r,r') = 3d(A,B).$$

For all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(G(x_{2n-1}, y_{2n-1}, z_{2n-1}), F(x_{2n}, y_{2n}, z_{2n})) \\ &\leqslant \frac{\alpha}{3}[d(x_{2n-1}, x_{2n}) + d(y_{2n-1}, y_{2n}) + d(z_{2n-1}, z_{2n})] + (1-\alpha)d(A,B). \end{aligned}$$

Letting $n \to \infty$, we have

$$d(p,p') \leqslant \frac{\alpha}{3}[d(p,p') + d(q,q') + d(r,r')] + (1-\alpha)d(A,B). \tag{9}$$

For all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(G(y_{2n-1}, x_{2n-1}, y_{2n-1}), F(y_{2n}, x_{2n}, y_{2n})) \\ &\leqslant \frac{\alpha}{3}[d(y_{2n-1}, y_{2n}) + d(x_{2n-1}, x_{2n}) + d(y_{2n-1}, y_{2n})] + (1-\alpha)d(A,B). \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(q,q') &\leqslant \frac{\alpha}{3}[d(q,q') + d(p,p') + d(q,q')] + (1-\alpha)d(A,B) \\ &= \frac{\alpha}{3}[d(q,q') + d(p,p') + d(r,r')] + (1-\alpha)d(A,B). \end{aligned} \tag{10}$$

Similarly, we have,

$$d(r,r') \leqslant \frac{\alpha}{3}[d(p,p') + d(q,q') + d(r,r')] + (1-\alpha)d(A,B). \tag{11}$$

It follows from (9), (10) and (11) that

$$d(p,p') + d(q,q') + d(r,r') \leqslant \alpha[d(p,p') + d(q,q') + d(r,r')] + 3(1-\alpha)d(A,B)$$

which implies that

$$d(p,p') + d(q,q') + d(r,r') \leqslant 3d(A,B). \tag{12}$$

Since $d(A,B) \leqslant d(p,p')$, $d(A,B) \leqslant d(q,q')$ and $d(A,B) \leqslant d(r,r')$, we have

$$d(p,p') + d(q,q') + d(r,r') \geqslant 3d(A,B). \tag{13}$$

From (12) and (13), we get

$$d(p,p') + d(q,q') + d(r,r') = 3d(A,B). \tag{14}$$

This completes the proof. □

Note that every pair of nonempty closed subsets $A,B$ of a uniformly convex Banach space $X$ such that $A$ is convex satisfies the property UC$^*$. Therefore, we obtain the following corollary.

COROLLARY 1. *Let $A$ and $B$ be nonempty closed convex subsets of a uniformly convex Banach space $X$ and $F : A^3 \to B, G : B^3 \to A$ be two mappings such that the ordered pair $(F,G)$ is a cyclic contraction. For any $(x_0,y_0,z_0) \in A^3$, we define the sequence $\{x_n\},\{y_n\},\{z_n\}$ in $X$ by*

$$\begin{aligned}
x_{2n+1} &= F(x_{2n},y_{2n},z_{2n}), \quad & x_{2n+2} &= G(x_{2n+1},y_{2n+1},z_{2n+1}),\\
y_{2n+1} &= F(y_{2n},x_{2n},y_{2n}), \quad & y_{2n+2} &= G(y_{2n+1},x_{2n+1},y_{2n+1}),\\
z_{2n+1} &= F(z_{2n},y_{2n},x_{2n}), \quad & z_{2n+2} &= G(z_{2n+1},y_{2n+1},x_{2n+1})
\end{aligned}$$

*for all $n \in \mathbb{N} \cup \{0\}$. Then $F$ has a tripled best proximity point $(p,q,r) \in A^3$ and $G$ has a tripled best proximity point $(p',q',r') \in B^3$. Moreover, we have*

$$x_{2n} \to p,\ \ y_{2n} \to q,\ \ z_{2n} \to r,\ \ x_{2n+1} \to p',\ \ y_{2n+1} \to q',\ \ z_{2n+1} \to r'.$$

*Furthermore, if $q = r$ and $q' = r'$, then*

$$d(p,p') + d(q,q') + d(r,r') = 3d(A,B).$$

Next, we give an example to illustrate Corollary 1.

EXAMPLE 5. Consider a uniformly convex Banach space $X = \mathbb{R}$ with the usual norm and let $A = [1,3]$ and $B = [-3,-1]$. Thus $d(A,B) = 2$. Define two mappings $F : A^3 \to B$ and $G : B^3 \to A$ by

$$F(x,y,z) = \frac{-x-y-z-3}{6}, \quad G(u,v,w) = \frac{-u-v-w+3}{6}$$

for all $(x,y,z) \in A^3$ and $(u,v,w) \in B^3$, respectively. For any $(x,y,z) \in A^3$ and $(u,v,w) \in B^3$ and fixed $\alpha = \frac{1}{2}$, we get

$$\begin{aligned} d(F(x,y,z),G(u,v,w)) &= \left|\frac{-x-y-z-3}{6} - \frac{-u-v-w+3}{6}\right| \\ &\leqslant \frac{|x-u|+|y-v|+|z-w|}{6} + 1 \\ &= \frac{\alpha}{3}[d(x,u)+d(y,v)+d(z,w)] + (1-\alpha)d(A,B) \end{aligned}$$

This implies that $(F,G)$ is a cyclic contraction with $\alpha = \frac{1}{2}$. Since $A$ and $B$ are closed convex, the pairs $(A,B)$ and $(B,A)$ satisfy the property $\mathrm{UC}^*$. Therefore, all the hypothesis of Corollary 1 hold. Therefore, $F$ has a tripled best proximity point and $G$ has a tripled best proximity point. We note that a point $(1,1,1) \in A^3$ is a unique tripled best proximity point of $F$ and a point $(-1,-1,-1) \in B^3$ is a unique tripled best proximity point of G. Furthermore, we get

$$d(1,-1)+d(1,-1)+d(1,-1) = 6 = 3d(A,B).$$

Next, we give the tripled best proximity point result in compact subsets of metric spaces.

THEOREM 3. *Let $A$, $B$ be nonempty compact subsets of a metric space $(X,d)$ and $F: A^3 \to B$, $G: B^3 \to A$ be two mappings such that the ordered pair $(F,G)$ is a cyclic contraction. For any $(x_0,y_0,z_0) \in A^3$ we define the sequence $\{x_n\},\{y_n\},\{z_n\}$ in $X$ by*

$$\begin{aligned} x_{2n+1} &= F(x_{2n},y_{2n},z_{2n}), \quad & x_{2n+2} &= G(x_{2n+1},y_{2n+1},z_{2n+1}), \\ y_{2n+1} &= F(y_{2n},x_{2n},y_{2n}), \quad & y_{2n+2} &= G(y_{2n+1},x_{2n+1},y_{2n+1}), \\ z_{2n+1} &= F(z_{2n},y_{2n},x_{2n}), \quad & z_{2n+2} &= G(z_{2n+1},y_{2n+1},x_{2n+1}) \end{aligned}$$

*for all $n \in \mathbb{N}\cup\{0\}$. Then $F$ has a tripled best proximity point $(p,q,r) \in A^3$ and $G$ has a tripled best proximity point $(p',q',r') \in B^3$. Moreover, we have*

$$x_{2n} \to p,\ y_{2n} \to q,\ z_{2n} \to r,\ x_{2n+1} \to p',\ y_{2n+1} \to q',\ z_{2n+1} \to r'.$$

*Furthermore, if $q = r$ and $q' = r'$, then*

$$d(p,p') + d(q,q') + d(r,r') = 3d(A,B).$$

*Proof.* Since $x_0,y_0,z_0 \in A$ and

$$\begin{aligned} x_{2n+1} &= F(x_{2n},y_{2n},z_{2n}), \quad & x_{2n+2} &= G(x_{2n+1},y_{2n+1},z_{2n+1}) \\ y_{2n+1} &= F(y_{2n},x_{2n},y_{2n}), \quad & y_{2n+2} &= G(y_{2n+1},x_{2n+1},y_{2n+1}) \\ z_{2n+1} &= F(z_{2n},y_{2n},x_{2n}), \quad & z_{2n+2} &= G(z_{2n+1},y_{2n+1},x_{2n+1}) \end{aligned}$$

for all $n \in \mathbb{N}\cup\{0\}$, we have $x_{2n},y_{2n},z_{2n} \in A$ and $x_{2n+1},y_{2n+1},z_{2n+1} \in A$ for all $n \in \mathbb{N}\cup\{0\}$. Since $A$ is compact, the sequences $\{x_{2n}\}$, $\{y_{2n}\}$ and $\{z_{2n}\}$ have the convergent subsequences $\{x_{2n_k}\}$, $\{y_{2n_k}\}$ and $\{z_{2n_k}\}$, respectively, such that

$$x_{2n_k} \to p \in A, \ \ y_{2n_k} \to q \in A, \ \ z_{2n_k} \to r \in A.$$

Now, we have

$$d(A,B) \leqslant d(p, x_{2n_k-1}) \leqslant d(p, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}). \tag{15}$$

By Lemma 1, we have $d(x_{2n_k}, x_{2n_k-1}) \to d(A,B)$. Taking $k \to \infty$ in (15), we get

$$d(p, x_{2n_k-1}) \to d(A,B).$$

By the similar argument, we observe that

$$d(q, x_{2n_k-1}) \to d(A,B), \ \ d(r, x_{2n_k-1}) \to d(A,B).$$

Note that

$$\begin{aligned} d(A,B) &\leqslant d(x_{2n_k}, F(p,q,r)) = d(G(x_{2n_k-1}, y_{2n_k-1}, z_{2n_k-1}), F(p,q,r)) \\ &\leqslant \frac{\alpha}{3}[d(x_{2n_k-1}, p) + d(y_{2n_k-1}, q) + d(z_{2n_k-1}, r)] + (1-\alpha)d(A,B). \end{aligned}$$

Taking $k \to \infty$, we get $d(p, F(p,q,r)) = d(A,B)$. Similarly, we can prove that

$$d(q, F(q,p,q)) = d(A,B), \quad d(r, F(r,q,p)) = d(A,B).$$

Thus $F$ has a tripled best proximity $(p,q,r) \in A^3$. In a similar way, since $B$ is compact, we can also prove that $G$ has a tripled best proximity point $(p',q',r') \in B^3$. To prove

$$d(p,p') + d(q,q') + d(r,r') = 3d(A,B),$$

we follows the step of the proof of Theorem 2. This completes the proof. □

## 3. Tripled fixed point theorems

In this section, we give a new tripled fixed point theorem for a cyclic contraction pair and give one example to illustrate the result.

THEOREM 4. *Let $A$, $B$ be nonempty closed subsets of a metric space $(X,d)$ and $F: A^3 \to B$, $G: B^3 \to A$ be two mappings such that the ordered pair $(F,G)$ is a cyclic contraction. For any $(x_0,y_0,z_0) \in A^3$, we define the sequence $\{x_n\}, \{y_n\}, \{z_n\}$ in $X$ by*

$$\begin{aligned} x_{2n+1} &= F(x_{2n}, y_{2n}, z_{2n}), \quad x_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1}), \\ y_{2n+1} &= F(y_{2n}, x_{2n}, y_{2n}), \quad y_{2n+2} = G(y_{2n+1}, x_{2n+1}, y_{2n+1}), \\ z_{2n+1} &= F(z_{2n}, y_{2n}, x_{2n}), \quad z_{2n+2} = G(z_{2n+1}, y_{2n+1}, x_{2n+1}) \end{aligned}$$

*for all $n \in \mathbb{N} \cup \{0\}$. If $d(A,B) = 0$, then $F$ has a tripled fixed point $(p,q,r) \in A^3$ and $G$ has a tripled fixed point $(p',q',r') \in B^3$. Moreover, we have*

$$x_{2n} \to p, \ \ y_{2n} \to q, \ \ z_{2n} \to r, \ \ x_{2n+1} \to p', \ \ y_{2n+1} \to q', \ \ z_{2n+1} \to r'.$$

*Furthermore, if $q = r$ and $q' = r'$, then $F$ and $G$ have a common tripled fixed point in $(A \cap B)^3$.*

*Proof.* Since $d(A,B)=0$, it follows that the pairs $(A,B)$ and $(B,A)$ satisfy the property UC$^*$. Therefore, by Theorem 2, we claim that $F$ has a tripled best proximity point $(p,q,r)\in A^3$, that is,

$$d(p,F(p,q,r)) = d(q,F(q,p,q)) = d(r,F(r,q,p)) = d(A,B) \tag{16}$$

and $G$ has a tripled best proximity point $(p',q',r')\in B^3$, that is,

$$d(p',G(p',q',r')) = d(q',G(q',p',q')) = d(r',G(r',q',p')) = d(A,B). \tag{17}$$

From (16) and $d(A,B)=0$, we conclude that

$$p=F(p,q,r), \quad q=F(q,p,q), \quad r=F(r,q,p),$$

that is, $(p,q,r)$ is a tripled fixed point of $F$. It follows from (17) and $d(A,B)=0$ that

$$p'=G(p',q',r'), \quad q'=G(q',p',q'), \quad r'=G(r',q',p'),$$

that is, $(p',q',r')$ is a tripled fixed point of $G$.

Next, we assume that $q=r$ and $q'=r'$ and then we show that $F$ and $G$ have a unique common tripled fixed point in $(A\cap B)^3$. From Theorem 2, we get

$$d(p,p')+d(q,q')+d(r,r') = 3d(A,B). \tag{18}$$

Since $d(A,B)=0$, we get

$$d(p,p')+d(q,q')+d(r,r') = 0$$

which implies that $p=p'$, $q=q'$ and $r=r'$. Therefore, we conclude that $(p,q,r)\in(A\cap B)^3$ is common tripled fixed point of $F$ and $G$. This completes the proof. □

Next, we give one example to illustrate Theorem 4.

EXAMPLE 6. Consider a space $X=\mathbb{R}$ with the usual metric and let $A=[-2,0]$ and $B=[0,2]$. Define two mappings $F:A^3\to B$ and $G:B^3\to A$ by

$$F(x,y,z) = -\frac{x+y+z}{6}, \quad G(x,y,z) = -\frac{u+v+w}{6}$$

for all $(x,y,z)\in A^3$ and $(u,v,w)\in B^3$, respectively. Then $d(A,B)=0$ and the ordered pair $(F,G)$ is a cyclic contraction with $\alpha=\frac{1}{2}$. Indeed, for any $(x,y,z)\in A^3$ and $(u,v,w)\in B^3$, we have

$$\begin{aligned} d(F(x,y,z),G(u,v,w)) &= \left|-\frac{x+y+z}{6}+\frac{u+v+w}{6}\right| \\ &\leqslant \frac{1}{6}(|x-u|+|y-v|+|z-w|) \\ &\leqslant \frac{\alpha}{3}[d(x,u)+d(y,v)+d(z,w)]+(1-\alpha)d(A,B). \end{aligned}$$

Therefore, all the hypothesis of Theorem 4 hold. Therefore, $F$ and $G$ have a common tripled fixed point and this point is $(0,0,0)\in(A\cap B)^3$.

If we take $A = B$ in Theorem 4, then we get the following results.

COROLLARY 2. *Let $A$ be a nonempty closed subset of a complete metric space $(X,d)$ and $F : A^3 \to A$, $G : A^3 \to A$ be two mappings such that the ordered pair $(F,G)$ be a cyclic contraction. For any $(x_0,y_0,z_0) \in A^3$, we define the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ in $X$ by*

$$\begin{aligned} x_{2n+1} &= F(x_{2n},y_{2n},z_{2n}), \quad & x_{2n+2} &= G(x_{2n+1},y_{2n+1},z_{2n+1}),\\ y_{2n+1} &= F(y_{2n},x_{2n},y_{2n}), \quad & y_{2n+2} &= G(y_{2n+1},x_{2n+1},y_{2n+1}),\\ z_{2n+1} &= F(z_{2n},y_{2n},x_{2n}), \quad & z_{2n+2} &= G(z_{2n+1},y_{2n+1},x_{2n+1}) \end{aligned}$$

*for all $n \in \mathbb{N} \cup \{0\}$. Then $F$ has a tripled fixed point $(p,q,r) \in A^3$ and $G$ has a tripled fixed point $(p',q',r') \in A^3$. Moreover, we have*

$$x_{2n} \to p,\ y_{2n} \to q,\ z_{2n} \to r,\ x_{2n+1} \to p',\ y_{2n+1} \to q',\ z_{2n+1} \to r'.$$

*Furthermore, if $q = r$ and $q' = r'$, then $F$ and $G$ have a common tripled fixed point in $A^3$.*

If we take $F = G$ in Corollary 2, then we get the following results.

COROLLARY 3. *Let $A$ be nonempty closed subsets of a complete metric space $(X,d)$ and $F : A^3 \to A$ be a mapping satisfying*

$$d(F(x,y,z),F(u,v,w)) \leqslant \frac{\alpha}{3}[d(x,u)+d(y,v)+d(z,w)]$$

*for all $(x,y,z),(u,v,w) \in A^3$. Then $F$ has a tripled fixed point $(p,q,r) \in A^3$.*

*Acknowledgements.*

The first author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant Number: NRF-2012-0008170), the fourth author was supported by the Commission on Higher Education, the Thailand Research Fund and the King Mongkut's University of Technology Thonburi (Grant No.MRG5580213).

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(Received November 25, 2012)

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 813614, 11 pages
http://dx.doi.org/10.1155/2014/813614

*Research Article*

# Existence and Uniqueness of Best Proximity Points for Generalized Almost Contractions

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Received 5 August 2013; Revised 19 November 2013; Accepted 26 December 2013; Published 16 March 2014

Academic Editor: Adrian Petrusel



The purpose of this paper is to elicit some interesting extensions of generalized almost contraction mappings to the case of non-self-mappings with $\alpha$-proximal admissible and prove best proximity point theorems for this classes. Moreover, we also give some examples and applications to support our main results.

## 1. Introduction

Many problems can be formulated as equations of the form $Tx = x$, where $T$ is a self-mapping with some suitable domains. From the fact that fixed point theory plays an important role in furnishing a uniform treatment to solve various equations of the form $Tx = x$ However, in the case that $T$ is non-self-mapping, the aforementioned equation does not necessarily have a fixed point. In such case, it is worthy to determine an approximate solution $x$ such that the error $d(x, Tx)$ is minimum. This is the idea behind best approximation theory. $A$ classical best approximation theorem was introduced by Fan [1]; that is, if $A$ is a nonempty compact convex subset of a Hausdorff locally convex topological vector space $B$ and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterward, several authors, including Prolla [2], Reich [3], and Sehgal and Singh [4, 5], have derived extensions of Fan's Theorem in many directions. Moreover, for a detailed account of global optimization and the existence of a best proximity point, one can refer to [5–15]. In 2013, Samet [16] studied the existence and uniqueness of best proximity points for almost $(\varphi, \theta)$-contractive mappings in complete metric spaces. Recently, Jleli et al. [17] introduced a new class of non-self-contractive mappings with generalization of $\alpha$-proximal admissible defined by Samet et al. [18] which is called $\alpha - \psi$-proximal contractive type mappings and proved existence and uniqueness of best proximity points.

Motivated from the above results, we will study the best proximity point theorem for new classes as generalized almost contraction in metric spaces by using the $\alpha$-proximal admissible of Jleli et al. [17]. Also, we give some illustrative examples and applications to support our main results.

## 2. Preliminaries

Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$; we recall the following notations and notions that will be used in what follows:

$$\begin{gathered} d(A, B) := \inf \{d(x, y) : x \in A \text{ and } y \in B\}, \\ A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}, \quad (1) \\ B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{gathered}$$

If $A \cap B \neq \emptyset$, then $A_0$ and $B_0$ are nonempty. Further, it is interesting to notice that $A_0$ and $B_0$ are contained in the boundaries of $A$ and $B$, respectively, provided $A$ and $B$ are closed subsets of a normed linear space such that $d(A,B)>0$ (see [19]).

*Definition 1.* A point $x \in A$ is said to be a *best proximity point* of the mapping $S: A \to B$ if it satisfies the following condition:

$$d(x,Sx)=d(A,B). \tag{2}$$

It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

*Definition 2* (see [13]). Let $(A,B)$ be a pair of nonempty subsets of $X$ with $A_0 \neq \emptyset$. Then the pair $(A,B)$ is said to have the *P-property*if and only if

$$\begin{cases} d(x_1,y_1)=d(A,B) \\ d(x_2,y_2)=d(A,B) \end{cases} \Longrightarrow d(x_1,x_2)=(y_1,y_2), \tag{3}$$

where $x_1,x_2 \in A_0$ and $y_1,y_2 \in B_0$.

It is easy to see that, for any nonempty subset $A$ of $X$, the pair $(A,A)$ has the *P-property*.

*Example 3* (see [13]). Let $A,B$ be two nonempty closed convex subsets of a Hilbert space $X$. Then $(A,B)$ satisfies the *P-property*.

*Example 4* (see [20]). Let $A$, $B$ be two nonempty, bounded, closed, and convex subsets of a uniformly convex Banach space $X$. Then $(A,B)$ has the *P-property*.

*Example 5* (see [20]). Let $X=R^2$ with the metric defined by

$$d((x_1,y_1),(x_2,y_2))=\max\{|x_1-y_1|,|x_2-y_2|\}. \tag{4}$$

Let $A:=\{(x,0):-1\le x\le 1\}$ and $B:=\{(0,y):-1\le y\le 1\}$. Then $(A,B)$ satisfies the *P-property*.

*Definition 6* (see [18]). A self-mapping $T: X \to X$ is said to be $\alpha$-admissible, where $\alpha: X\times X \to [0,\infty)$, if

$$x,y\in X,\quad \alpha(x,y)\ge 1 \Longrightarrow \alpha(Tx,Ty)\ge 1. \tag{5}$$

*Definition 7* (see [17]). Let $T: A \to B$ and $\alpha: A\times A \to [0,\infty)$. One says that $T$ is $\alpha$-proximal admissible, if

$$\begin{cases} \alpha(x_1,x_2)\ge 1 \\ d(u_1,Tx_1)=d(A,B) \\ d(u_2,Tx_2)=d(A,B) \end{cases} \Longrightarrow \alpha(u_1,u_2)\ge 1 \tag{6}$$

for all $x_1,x_2,u_1,u_2\in A$.

Clearly, for self-mapping, $T$ being $\alpha$-proximal admissible implies that $T$ is $\alpha$-admissible.

*Definition 8.* One says the function $\varphi: [0,\infty) \to [0,\infty)$ is a (c)-comparison function if and only if the following conditions hold:

($\Phi_1$) $\varphi$ is a nondecreasing function,

($\Phi_2$) for any $t>0$, $\sum_{n=1}^{\infty}\varphi^n(t)$ is a convergent series.

One denotes the set of (c)-comparison function by $\Psi$.

It is easily proved that if $\varphi$ is a (c)-comparison function, then $\varphi(t)<t$ for all $t>0$.

*Definition 9* (see [16]). Let $\theta: [0,\infty)^4 \to [0,\infty)$ satisfy the following conditions:

(1) $\theta$ is continuous,

(2) $\theta(a,b,c,d)=0$ if and only if the product $abcd=0$.

One denotes the class of function $\theta$ by $\Theta$.

*Example 10* (see [16]). The following functions belong to $\Theta$:

(1) $\theta(t_1,t_2,t_3,t_4)=\tau\min\{t_1,t_2,t_3,t_4\}$, $\tau>0$;

(2) $\theta(t_1,t_2,t_3,t_4)=\tau\ln(1+t_1t_2t_3t_4)$, $\tau>0$;

(3) $\theta(t_1,t_2,t_3,t_4)=\tau t_1t_2t_3t_4$, $\tau>0$.

## 3. The Existence and Uniqueness of Best Proximity Points

In this section, we introduce the new class of the generalized Banach contraction for non-self-mappings so-called generalized almost $(\varphi,\theta)_\alpha$ contraction and we also study the best proximity theorems for these classes. First, we recall the notion of $(\varphi,\theta)$ contraction defined by Samet [16] as follows.

*Definition 11* (see [16]). Let $A$ and $B$ be nonempty subsets of metric space $X$. A mapping $T: A \to B$ is said to be an *almost* $(\varphi,\theta)$ *contraction* if and only if there exist $\varphi\in\Psi$ and $\theta\in\Theta$ such that, for all $x,y\in A$,

$$\begin{aligned} d(Tx,Ty) &\le \varphi(d(x,y)) \\ &\quad + \theta\big(d(y,Tx) \\ &\qquad - d(A,B), d(x,Ty)-d(A,B), d(x,Tx) \\ &\qquad - d(A,B), d(y,Ty)-d(A,B)\big). \end{aligned} \tag{7}$$

### 3.1. The Existence

*Definition 12.* Let $A$ and $B$ be nonempty subsets of metric space $X$. A mapping $T: A \to B$ is said to be a *generalized almost* $(\varphi,\theta)_\alpha$ *contraction* if and only if

$$\begin{aligned} &\alpha(x,y)\,d(Tx,Ty) \\ &\quad \le \varphi(M(x,y)) \\ &\qquad + \theta\big(d(y,Tx)-d(A,B), d(x,Ty) \\ &\qquad\quad - d(A,B), d(x,Tx)-d(A,B), d(y,Ty) \\ &\qquad\quad - d(A,B)\big), \end{aligned} \tag{8}$$

for all $x, y \in A$, where $\alpha : A \times A \to [0, \infty)$, $\varphi \in \Psi$ $\theta \in \Theta$, and

$$\begin{aligned} M(x,y) &= \max\Big\{d(x,y), d(x,Tx) - d(A,B), d(y,Ty) \\ &\quad -d(A,B), \frac{1}{2}[d(x,Ty) + d(y,Tx)] - d(A,B)\Big\}. \end{aligned} \tag{9}$$

Clearly, if we take $\alpha(x, y) = 1$ for all $x, y \in A$ and $M(x, y) = d(x, y)$, the generalized almost $(\varphi, \theta)_\alpha$ contraction reduces to almost $(\varphi, \theta)$ contraction.

**Theorem 13.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A, B)$ has the $P$-property. Let $T : A \to B$ satisfy the following conditions:*

(a) *$T$ are $\alpha$-proximal admissible and generalized almost $(\varphi, \theta)_\alpha$-contraction;*

(b) *$T$ is continuous;*

(c) *there exist elements $x_0$ and $x_1$ in $A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$;*

(d) *$T(A_0) \subseteq B_0$.*

*Then there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B). \tag{10}$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Tx_n) = d(A, B), \tag{11}$$

*converges to the element $x$.*

*Proof.* By the hypothesis $(c)$, there exist $x_0$ and $x_1$ in $A_0$ such that

$$d(x_1, Tx_0) = d(A, B), \quad \alpha(x_0, x_1) \geq 1. \tag{12}$$

From the fact that $T(A_0) \subseteq B_0$, there exists an element $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B). \tag{13}$$

By (12), (13), and the $\alpha$-proximal admissible, we get

$$\alpha(x_1, x_2) \geq 1. \tag{14}$$

Since $T(A_0) \subseteq B_0$, we can find an element $x_3 \in A_0$ such that

$$d(x_3, Tx_2) = d(A, B). \tag{15}$$

Again, by (13), (15), and the $\alpha$-proximal admissible, we have

$$\alpha(x_2, x_3) \geq 1. \tag{16}$$

By similar fashion, we can find $x_n$ in $A_0$. Having chosen $x_n$, one can determine an element $x_{n+1} \in A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A, B), \quad \alpha(x_n, x_{n+1}) \geq 1. \tag{17}$$

In view of the fact that the pair $(A, B)$ has $P$-*property* and generalized almost $(\varphi, \theta)_\alpha$-contraction of $T$, we have

$$\begin{aligned} d(x_1, x_2) &= d(Tx_0, Tx_1) \\ &\leq \alpha(x_0, x_1) d(Tx_0, Tx_1) \\ &\leq \varphi(M(x_0, x_1)) \\ &\quad + \theta(d(x_1, Tx_0) - d(A,B), d(x_0, Tx_1) - d(A,B), \\ &\qquad d(x_0, Tx_0) - d(A,B), d(x_1, Tx_1) - d(A,B)) \\ &= \varphi(M(x_0, x_1)) \\ &\quad + \theta(0, d(x_0, Tx_1) - d(A,B), d(x_0, Tx_0) \\ &\qquad -d(A,B), d(x_1, Tx_1) - d(A,B)) \\ &= \varphi(M(x_0, x_1)). \end{aligned} \tag{18}$$

Since

$$\begin{aligned} M(x_0, x_1) &= \max\Big\{d(x_0, x_1), d(x_0, Tx_0) - d(A,B), d(x_1, Tx_1) \\ &\qquad - d(A,B), \frac{1}{2}[d(x_0, Tx_1) + d(x_1, Tx_0)] \\ &\qquad -d(A,B)\Big\} \\ &\leq \max\Big\{d(x_0, x_1), d(x_0, x_1) + d(x_1, Tx_0) \\ &\qquad - d(A,B), d(x_1, x_2) + d(x_2, Tx_1) \\ &\qquad - d(A,B), \frac{1}{2}[d(x_0, x_1) + d(x_1, x_2) \\ &\qquad\qquad +d(x_2, Tx_1) + d(A,B)] \\ &\qquad -d(A,B)\Big\} \\ &= \max\Big\{d(x_0, x_1), d(x_1, x_2), \\ &\qquad \frac{1}{2}[d(x_0, x_1) + d(x_1, x_2) + d(A,B) + d(A,B)] \\ &\qquad -d(A,B)\Big\} \\ &= \max\Big\{d(x_0, x_1), d(x_1, x_2), \frac{1}{2}[d(x_0, x_1) + d(x_1, x_2)]\Big\} \\ &= \max\{d(x_0, x_1), d(x_1, x_2)\}. \end{aligned} \tag{19}$$

By (18) and (19), we get

$$d(x_1, x_2) \leq \varphi(\max\{d(x_0, x_1), d(x_1, x_2)\}). \tag{20}$$

If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, by (17), we obtain the best proximity point. Suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$; then $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. If $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_1, x_2)$, by the property $\varphi(t) < t$ for all $t > 0$, we get

$$d(x_1, x_2) \leq \varphi(\max\{d(x_0, x_1), d(x_1, x_2)\}) < d(x_1, x_2), \tag{21}$$

which is a contradiction and hence $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_0, x_1)$. That is,

$$d(x_1, x_2) \leq \varphi(d(x_0, x_1)). \tag{22}$$

Again, since the pair $(A, B)$ has *P-property*, is $\alpha$-proximal admissible, and generalized almost $(\varphi, \theta)_\alpha$-contraction of $T$, we have

$$\begin{aligned} d(x_2, x_3) &= d(Tx_1, Tx_2) \\ &\leq \alpha(x_1, x_2) d(Tx_1, Tx_2) \\ &\leq \varphi(M(x_1, x_2)) \\ &\quad + \theta(d(x_2, Tx_1) - d(A, B), d(x_1, Tx_2) - d(A, B), \\ &\qquad d(x_1, Tx_1) - d(A, B), d(x_2, Tx_2) - d(A, B)) \\ &= \varphi(M(x_1, x_2)) \\ &\quad + \theta(0, d(x_1, Tx_2) - d(A, B), d(x_1, Tx_1) - d(A, B), \\ &\qquad d(x_2, Tx_2) - d(A, B)) \\ &= \varphi(M(x_1, x_2)) \end{aligned} \tag{23}$$

and since

$$\begin{aligned} M(x_1, x_2) &= \max\Big\{d(x_1, x_2), d(x_1, Tx_1) - d(A, B), d(x_2, Tx_2) \\ &\qquad - d(A, B), \frac{1}{2}[d(x_1, Tx_2) + d(x_2, Tx_1)] \\ &\qquad - d(A, B)\Big\} \\ &\leq \max\Big\{d(x_1, x_2), d(x_1, x_2) + d(x_2, Tx_1) - d(A, B), \\ &\qquad d(x_2, x_3) + d(x_3, Tx_2) - d(A, B), \\ &\qquad \frac{1}{2}[d(x_1, x_2) + d(x_2, x_3) \\ &\qquad + d(x_3, Tx_2) + d(A, B)] \\ &\qquad - d(A, B)\Big\} \\ &= \max\Big\{d(x_1, x_2), d(x_2, x_3), \\ &\qquad \frac{1}{2}[d(x_1, x_2) + d(x_2, x_3) + d(A, B) + d(A, B)] \\ &\qquad - d(A, B)\Big\} \\ &= \max\Big\{d(x_1, x_2), d(x_2, x_3), \\ &\qquad \frac{1}{2}[d(x_1, x_2) + d(x_2, x_3)]\Big\} \\ &= \max\{d(x_1, x_2), d(x_2, x_3)\}. \end{aligned} \tag{24}$$

By (23) and (24), we get

$$d(x_2, x_3) \leq \varphi(\max\{d(x_1, x_2), d(x_2, x_3)\}). \tag{25}$$

By similar argument as above, we can conclude that $\max\{d(x_1, x_2), d(x_2, x_3)\} = d(x_1, x_2)$ and thus

$$d(x_2, x_3) \leq \varphi(d(x_1, x_2)). \tag{26}$$

Using (22) and (26) and the nondecreasing of $\varphi$, we get

$$d(x_2, x_3) \leq \varphi^2(d(x_0, x_1)). \tag{27}$$

Continuing this process, by induction we have that

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)) \tag{28}$$

for all $n \in \mathbb{N} \cup \{0\}$. Fix $\varepsilon > 0$ and let $h = h(\varepsilon)$ be a positive integer such that

$$\sum_{n \geq h} \varphi^n(d(x_0, x_1)) < \varepsilon. \tag{29}$$

Let $m > n > h$; using the triangular inequality, (28) and (29), we obtain

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \varphi^k(d(x_0, x_1)) \leq \sum_{n \geq h} \varphi^n(d(x_0, x_1)) < \varepsilon. \end{aligned} \tag{30}$$

This shows that $\{x_n\}$ is a Cauchy sequence. Since $A$ is a closed subset of complete metric spaces $X$, then there exists $x \in A$ such that

$$\lim_{n \to \infty} d(x_n, x) = 0. \tag{31}$$

By (17), (31), and the continuity of $T$, we get

$$d(x, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(A, B) \tag{32}$$

and the proof is complete. □

Next, we remove condition $T$ is continuous in Theorem 13, by assuming the following condition which was defined by Jleli et al. [17] for proving the new best proximity point theorem.

($H$) If $\{x_n\}$ is a sequence in $A$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x$ for some $x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k$.

**Theorem 14.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A, B)$ has the $P$-property. Let $T : A \to B$ satisfy the following conditions:*

(a) *$T$ are $\alpha$-proximal admissible and generalized almost $(\varphi, \theta)_\alpha$-contraction;*

(b) *$A$ satisfies condition ($H$);*

(c) *there exist elements $x_0$ and $x_1$ in $A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha((x_0, x_1)) \geq 1$;*

*Then there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B). \tag{33}$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Tx_n) = d(A, B), \tag{34}$$

*converges to the element $x$.*

*Proof.* As in the proof of Theorem 13, we have

$$d(x_{n+1}, Tx_n) = d(A, B) \tag{35}$$

for all $n \geq 0$. Moreover, $\{x_n\}$ is a Cauchy sequence and converges to some point $x \in A$. By the $P$-*property* and (28), we have

$$d(Tx_{n-1}, Tx_n) = d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)) \tag{36}$$

for all $n \in \mathbb{N} \cup \{0\}$. That is, $\lim_{n\to\infty} d(Tx_{n-1}, Tx_n) = 0$ and, by the same argument as proof of Theorem 13, we obtain that $\{Tx_n\}$ is a Cauchy sequence. Since $B$ is a closed subset of the complete metric space $(X, d)$, there exists $x_* \in B$ such that $Tx_n$ converges to $x_*$. Therefore

$$d(x, x_*) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(A, B). \tag{37}$$

On the other hand, from the condition ($H$) of $T$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k$. The pair $(A, B)$ has $P$-*property* and property of mapping $T$; we get

$$\begin{aligned} d(x_{n_k+1}, x) &= d(Tx_{n_k}, Tx) \\ &\leq \alpha(x_{n_k}, x)\, d(Tx_{n_k}, Tx) \\ &\leq \varphi(M(x_{n_k}, x)) \\ &\quad + \theta\big(d(x_{n_k}, Tx) - d(A, B), d(x, Tx_{n_k}) - d(A, B), \\ &\qquad d(x, Tx) - d(A, B), d(x_{n_k}, Tx_{n_k}) - d(A, B)\big). \end{aligned} \tag{38}$$

Indeed,

$$\begin{aligned} &M(x_{n_k}, x) \\ &= \max\Big\{d(x_{n_k}, x), d(x_{n_k}, Tx_{n_k}) - d(A, B), d(x, Tx) \\ &\qquad - d(A, B), \frac{1}{2}\left[d(x_{n_k}, Tx) + d(x, Tx_{n_k})\right] \\ &\qquad -d(A, B)\Big\} \\ &\leq \max\Big\{d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k}) \\ &\qquad - d(A, B), d(x, Tx) - d(A, B), \\ &\qquad \frac{1}{2}\big[d(x_{n_k}, x) + d(x, Tx) + d(x, x_{n_k+1}) \\ &\qquad + d(x_{n_k+1}, Tx_{n_k})\big] - d(A, B)\Big\} \\ &\leq \max\Big\{d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}), d(x, Tx) - d(A, B), \\ &\qquad \frac{1}{2}\left[d(x_{n_k}, x) + d(x, Tx) + d(x, x_{n_k+1}) + d(A, B)\right] \\ &\qquad - d(A, B)\Big\} := \mathscr{M}(x_{n_k}, x). \end{aligned} \tag{39}$$

From the definition of $\mathscr{M}(x_{n_k}, x)$, we get

$$\lim_{k\to\infty} \mathscr{M}(x_{n_k}, x) = d(x, Tx) - d(A, B). \tag{40}$$

Since

$$\begin{aligned} d(x, Tx) &\leq d(x, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k}) + d(Tx_{n_k}, Tx) \\ &\leq d(x, x_{n_k+1}) + d(A, B) + d(Tx_{n_k}, Tx) \end{aligned} \tag{41}$$

it follows that

$$\begin{aligned}
&d(x,Tx)-d\left(x,x_{n_k+1}\right)-d(A,B)\\
&\quad\le d\left(Tx_{n_k},Tx\right)\\
&\quad\le \alpha\left(x_{n_k},x\right)d\left(Tx_{n_k},Tx\right)\\
&\quad\le \varphi\left(M\left(x_{n_k},x\right)\right)\\
&\qquad+\theta\Big(d\left(x_{n_k},Tx\right)-d(A,B),d\left(x,Tx_{n_k}\right)-d(A,B),\\
&\qquad\qquad d(x,Tx)-d(A,B),d\left(x_{n_k},Tx_{n_k}\right)-d(A,B)\Big)\\
&\quad\le \varphi\left(\mathcal{M}\left(x_{n_k},x\right)\right)\\
&\qquad+\theta\Big(d\left(x_{n_k},Tx\right)-d(A,B),d\left(x,Tx_{n_k}\right)-d(A,B),\\
&\qquad\qquad d(x,Tx)-d(A,B),d\left(x_{n_k},Tx_{n_k}\right)-d(A,B)\Big).
\end{aligned}\tag{42}$$

Suppose that

$$d(x,Tx)-d(A,B)>0.\tag{43}$$

Then for $k$ large enough, we have $\mathcal{M}(x_{n_k},x)>0$. Using the property $\varphi(t)<t$ for all $t>0$, we get

$$\begin{aligned}
&d(x,Tx)-d\left(x,x_{n_k+1}\right)-d(A,B)\\
&\quad<\mathcal{M}\left(x_{n_k},x\right)\\
&\qquad+\theta\Big(d\left(x_{n_k},Tx\right)-d(A,B),d\left(x,Tx_{n_k}\right)-d(A,B),\\
&\qquad\qquad d(x,Tx)-d(A,B),d\left(x_{n_k},Tx_{n_k}\right)-d(A,B)\Big).
\end{aligned}\tag{44}$$

Combining (37) and (40) with (44) and the property of $\theta$, we obtain that

$$\begin{aligned}
&d(x,Tx)-d(A,B)\\
&\quad=\lim_{k\to\infty}d(x,Tx)-d\left(x,x_{n_k+1}\right)-d(A,B)\\
&\quad<\lim_{k\to\infty}\mathcal{M}\left(x_{n_k},x\right)\\
&\qquad+\lim_{k\to\infty}\theta\Big(d\left(x_{n_k},Tx\right)-d(A,B),d\left(x,Tx_{n_k}\right)-d(A,B),\\
&\qquad\qquad d(x,Tx)-d(A,B),d\left(x_{n_k},Tx_{n_k}\right)-d(A,B)\Big)\\
&\quad=\lim_{k\to\infty}\mathcal{M}\left(x_{n_k},x\right)\\
&\quad=d(x,Tx)-d(A,B)
\end{aligned}\tag{45}$$

which is a contradiction and thus $d(x,Tx)-d(A,B)=0$. Hence, $d(x,Tx)=d(A,B)$ and the proof is complete. □

*3.2. The Uniqueness.* Next, we present an example where it can be appreciated that hypotheses in Theorems 13 and 14 do not guarantee uniqueness of the best proximity point.

*Example 15.* Let $X=R^2$ with the Euclidean metric. Consider $A:=\{(2,0),(0,2)\}$ and $B:=\{(-2,0),(0,-2)\}$. Obviously, $(A,B)$ satisfies the $P$-*property* and $d(A,B)=2\sqrt{2}$; furthermore $A_0=A$ and $B_0=B$. Define $T:A\to B$ by $T(x,y)=(-y/2,-x/2)$ for all $x,y\in A$; clearly $T$ is continuous. Let $\alpha:A\times A\to[0,\infty)$ be defined by

$$\alpha(x,y)=\begin{cases}2; & x=y,\\ \dfrac{1}{2}; & x\neq y.\end{cases}\tag{46}$$

We can show that $T$ are $\alpha$-proximal admissible and generalized almost $(\varphi,\theta)_\alpha$-contraction with $\varphi(t)=t/2$ for all $t\ge 0$ and for all $\theta\in\Theta$. Furthermore,

$$\begin{aligned}
d((2,0),T(2,0))&=d((2,0),(0,-2))=d((0,2),(-2,0))\\
&=d((0,2),T(0,2))=d(A,B)=2\sqrt{2}.
\end{aligned}\tag{47}$$

Therefore, $(2,0)$ and $(0,2)$ are a best proximity point of mapping $T$.

Now, we need a sufficient condition to give uniqueness of the best proximity point as follows.

*Definition 16* (see [17]). Let $T:A\to B$ be a non-self-mapping and $\alpha:A\times A\to[0,\infty)$. One says that $T$ is $(\alpha,d)$-regular if, for all $(x,y)\in\alpha^{-1}([0,1))$, there exists $z\in A_0$ such that

$$\alpha(x,z)\ge 1,\qquad \alpha(y,z)\ge 1.\tag{48}$$

**Theorem 17.** *Adding condition $(\alpha,d)$-regular of $T$ to the hypotheses of Theorem 13, then one obtains the uniqueness of the best proximity point of $T$.*

*Proof.* We will only prove the part of uniqueness. Suppose that there exist $x$ and $x^*$ in $A$ which are distinct best proximity points; that is,

$$d(x,Tx)=d(A,B),\qquad d\left(x^*,Tx^*\right)=d(A,B).\tag{49}$$

Using the pair $(A,B)$ that has $P$-*property*, we have

$$d\left(x,x^*\right)=d\left(Tx,Tx^*\right).\tag{50}$$

*Case 1* (if $\alpha(x,x^*) \geq 1$). By (50) and generalized almost $(\varphi,\theta)_\alpha$-contraction of $T$, we have

$$\begin{aligned}
d(x,x^*) &= d(Tx,Tx^*) \\
&\leq \alpha(x,x^*)\, d(Tx,Tx^*) \\
&\leq \varphi(M(x,x^*)) \\
&\quad + \theta(d(x^*,Tx) - d(A,B), d(x,Tx^*) - d(A,B), \\
&\qquad d(x,Tx) - d(A,B), d(x^*,Tx^*) - d(A,B)) \\
&= \varphi(M(x,x^*)) \\
&\quad + \theta(d(x^*,Tx) - d(A,B), d(x,Tx^*) - d(A,B), 0, 0) \\
&= \varphi(M(x,x^*))
\end{aligned} \tag{51}$$

since

$$\begin{aligned}
M(x,x^*) &= \max\Big\{d(x,x^*), d(x,Tx) - d(A,B), d(x^*,Tx^*) \\
&\qquad -d(A,B), \frac{1}{2}[d(x,Tx^*) + d(x^*,Tx)] - d(A,B)\Big\} \\
&= \max\Big\{d(x,x^*), 0, 0, \frac{1}{2}[d(x,Tx^*) + d(x^*,Tx)] \\
&\qquad -d(A,B)\Big\} \\
&\leq \max\Big\{d(x,x^*), \frac{1}{2}\Big[d(x,x^*) + d(x^*,Tx^*) + d(x^*,x) \\
&\qquad +d(x,Tx)\Big] - d(A,B)\Big\} \\
&= \max\Big\{d(x,x^*), \frac{1}{2}[d(x,x^*) + d(x^*,x)]\Big\} \\
&= d(x,x^*).
\end{aligned} \tag{52}$$

Combining (51) with (52) and using the property $\varphi(t) < t$ for all $t > 0$, we get

$$d(x,x^*) \leq \varphi(M(x,x^*)) = \varphi(d(x,x^*)) < d(x,x^*) \tag{53}$$

which is a contradiction and hence $x = x^*$.

*Case 2* (if $\alpha(x,x^*) < 1$). By the $(\alpha,d)$-regular of $T$, there exists $z \in A_0$ such that

$$\alpha((x,z)) \geq 1, \qquad \alpha(x^*,z) \geq 1. \tag{54}$$

Since $T(A_0) \subseteq B_0$, there exists a point $v_0 \in A_0$ such that

$$d(v_0,Tz) = d(A,B). \tag{55}$$

From $\alpha(x,z) \geq 1$, $d(x,Tx) = d(A,B)$, and $d(v_0,Tz) = d(A,B)$ and by the $\alpha$-proximal admissible, we have

$$\alpha(x,v_0) \geq 1. \tag{56}$$

Since $T(A_0) \subseteq B_0$, there exists a point $v_1 \in A_0$ such that

$$d(v_1,Tv_0) = d(A,B). \tag{57}$$

By similar argument as above, we can conclude that $\alpha(x,v_1) \geq 1$. One can proceed further in a similar fashion to find $v_n$ in $A_0$ with $v_{n+1} \in A_0$ such that

$$d(v_{n+1},Tv_n) = d(A,B), \qquad \alpha(x,v_n) \geq 1, \tag{58}$$

for all $n \in \mathbb{N}$. By (58), the pair $(A,B)$ has *P-property* and property of mapping $T$; we get

$$d(x,v_{n+1}) = d(Tx,Tv_n). \tag{59}$$

Using the property of mapping $T$, we get

$$\begin{aligned}
d(x,v_{n+1}) &= d(Tx,Tv_n) \\
&\leq \alpha(x,v_n)\, d(Tx,Tv_n) \\
&\leq \varphi(M(x,v_n)) \\
&\quad + \theta(d(v_n,Tx) - d(A,B), d(x,Tv_n) - d(A,B), \\
&\qquad d(x,Tx) - d(A,B), d(v_n,Tv_n) - d(A,B)) \\
&= \varphi(M(x,v_n)) \\
&\quad + \theta(d(v_n,Tx) - d(A,B), d(x,Tv_n) - d(A,B), \\
&\qquad 0, d(v_n,Tv_n) - d(A,B)) \\
&= \varphi(M(x,v_n))
\end{aligned} \tag{60}$$

since

$$\begin{aligned}
M(x,v_n) &= \max\Big\{d(x,v_n), d(x,Tx) - d(A,B), d(v_n,Tv_n) \\
&\qquad - d(A,B), \frac{1}{2}[d(x,Tv_n) + d(v_n,Tx)] - d(A,B)\Big\} \\
&= \max\Big\{d(x,v_n), 0, 0, \frac{1}{2}[d(x,Tv_n) + d(v_n,Tx)] \\
&\qquad -d(A,B)\Big\} \\
&\leq \max\Big\{d(x,v_n), \frac{1}{2}[d(x,v_{n+1}) + d(v_{n+1},Tv_n) \\
&\qquad +d(v_n,x) + d(x,Tx)] - d(A,B)\Big\} \\
&= \max\Big\{d(x,v_n), \frac{1}{2}[d(x,v_{n+1}) + d(v_n,x)]\Big\} \\
&\leq \max\{d(x,v_n), d(x,v_{n+1})\}.
\end{aligned} \tag{61}$$

Thus

$$d(x, v_{n+1}) \le \varphi(M(x, v_n)) \le \varphi(\max\{d(x, v_n), d(x, v_{n+1})\}). \quad (62)$$

If $v_N = x$, for some $N \in \mathbb{N}$. By (59), we get

$$d(x, v_{N+1}) = d(Tx, Tv_N) = 0 \quad (63)$$

which implies that $v_{N+1} = x$. Moreover, we obtain $v_n = x$ for all $n \ge N$ and thus $v_n \to x$ as $n \to \infty$. Suppose that $v_n \ne x$ for all $n \in \mathbb{N}$; then $d(v_n, x) > 0$ for all $n$. If $\max\{d(x, v_n), d(x, v_{n+1})\} = d(x, v_{n+1})$, by the property $\varphi(t) < t$ for all $t > 0$, we get

$$d(x, v_{n+1}) \le \varphi(M(x, v_n)) = \varphi(d(x, v_{n+1})) < d(x, v_{n+1}) \quad (64)$$

which is a contradiction and hence $\max\{d(x, v_n), d(x, v_{n+1})\} = d(x, v_n)$. That is,

$$d(x, v_{n+1}) \le \varphi(M(x, v_n)) = \varphi(d(x, v_n)) \quad (65)$$

for all $n \ge N$. By induction of (65), we have

$$d(x, v_{n+1}) \le \varphi^n(d(x, v_1)). \quad (66)$$

Taking $n \to \infty$, we obtain that $v_n \to x$ as $n \to \infty$. So, in all cases, we have $v_n \to x$ as $n \to \infty$. Similarly, we can prove that $v_n \to x^*$ as $n \to \infty$. By the uniqueness of limit, we conclude that $x = x^*$ and this completes the proof. □

**Theorem 18.** *Adding condition $(\alpha, d)$–regular of $T$ to the hypotheses of Theorem 14, then we obtain the uniqueness of the best proximity point of $T$.*

*Proof.* Combine the proofs of Theorems 17 and 14. □

## 4. Consequences

*4.1. Best Proximity Points Theorems.* If we take $\varphi(t) = kt$, where $0 \le k < 1$ and $\theta(t_1, t_2, t_3, t_4) = L\min\{t_1, t_2, t_3, t_4\}$, then Theorem 13 and Theorem 14, we get the following.

**Theorem 19.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A, B)$ has the $P$-property. Let $T : A \to B$ satisfy the following conditions:*

(a) *$T$ is $\alpha$-proximal admissible and*

$$\alpha(x, y)\, d(Tx, Ty) \le kM(x, y) + L\min\{d(x, Ty) - d(A, B), d(y, Tx) - d(A, B), d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)\} \quad (67)$$

*for all $x, y \in A$;*

(b) *$T$ is continuous (or $A$ satisfies condition $(H)$);*

(c) *there exist elements $x_0$ and $x_1$ in $A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha((x_0, x_1)) \ge 1$;*

(d) *$T(A_0) \subseteq B_0$.*

*Then there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B). \quad (68)$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Tx_n) = d(A, B), \quad (69)$$

*converges to the element $x$.*

If we add the condition that $T$ is $(\alpha, d)$-regular in Theorem 19, therefore we can obtain the uniqueness of the best proximity point.

If we take $\alpha(x, y) = 1$, for all $x, y \in A$ in Theorems 13 and 14, we get the following Theorems.

**Theorem 20.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A, B)$ has the $P$-property. Let $T : A \to B$ satisfy the following conditions:*

(a)

$$d(Tx, Ty) \le \varphi(M(x, y)) + \theta(d(x, Ty) - d(A, B), d(y, Tx) - d(A, B)\, d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)) \quad (70)$$

*for all $x, y \in A$;*

(b) *$T$ is continuous (or $A$ satisfies condition $(H)$);*

(c) *$T(A_0) \subseteq B_0$.*

*Then there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B). \quad (71)$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Tx_n) = d(A, B), \quad (72)$$

*converges to the element $x$.*

If $M(x, y) = d(x, y)$, then Theorem 20 includes the following.

**Theorem 21.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A, B)$ has the $P$-property. Let $T : A \to B$ satisfy the following conditions:*

(a)

$$d(Tx,Ty) \leq \varphi(d(x,y)) + \theta(d(x,Ty) - d(A,B), d(y,Tx) - d(A,B) d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)) \quad (73)$$

*for all $x, y \in A$;*

(b) *$T$ is continuous (or $A$ satisfies condition $(H)$);*

(c) *$T(A_0) \subseteq B_0$.*

*Then there exists an element $x \in A$ such that*

$$d(x,Tx) = d(A,B). \quad (74)$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Tx_n) = d(A,B), \quad (75)$$

*converges to the element $x$.*

If we take $\varphi(t) = kt$ and $\theta(t_1,t_2,t_3,t_4) = L\min\{t_1,t_2,t_3,t_4\}$, for all $x, y \in A$ in Theorem 21, we obtain the following theorem.

**Theorem 22.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A,B)$ has the $P$-property. Let $T : A \to B$ satisfy the following conditions:*

(a)

$$d(Tx,Ty) \leq kM(x,y) + L\min\{d(x,Ty) - d(A,B), d(y,Tx) - d(A,B), d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)\} \quad (76)$$

*for all $x, y \in A$;*

(b) *$T$ is continuous (or $A$ satisfies condition $(H)$);*

(c) *$T(A_0) \subseteq B_0$.*

*Then there exists an element $x \in A$ such that*

$$d(x,Tx) = d(A,B). \quad (77)$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Tx_n) = d(A,B), \quad (78)$$

*converges to the element $x$.*

If $M(x,y) = d(x,y)$ and putting $L = 0$ in Theorem 22, we obtain the following.

**Theorem 23.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A,B)$ has the $P$-property. Let $T : A \to B$ satisfy the following conditions:*

(a)

$$d(Tx,Ty) \leq kd(x,y) \quad (79)$$

*for all $x, y \in A$;*

(b) *$T$ is continuous (or $A$ satisfies condition $(H)$);*

(c) *there exist elements $x_0$ and $x_1$ in $A_0$ such that $d(x_1,Tx_0) = d(A,B)$;*

(d) *$T(A_0) \subseteq B_0$.*

*Then there exists an element $x \in A$ such that*

$$d(x,Tx) = d(A,B). \quad (80)$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Tx_n) = d(A,B), \quad (81)$$

*converges to the element $x$.*

If $M(x,y) = (k/2)[d(x,Ty) + d(y,Tx)] - d(A,B)$ and putting $L = 0$ in Theorem 22, we obtain the following theorem.

**Theorem 24.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty and the pair $(A,B)$ has the $P$-property. Let $T : A \to B$ satisfy the following conditions:*

(a)

$$d(Tx,Ty) \leq \frac{k}{2}[d(x,Ty) + d(y,Tx)] - d(A,B) \quad (82)$$

*for all $x, y \in A$;*

(b) *$T$ is continuous (or $A$ satisfies condition $(H)$);*

(c) *there exist elements $x_0$ and $x_1$ in $A_0$ such that $d(x_1,Tx_0) = d(A,B)$;*

(d) *$T(A_0) \subseteq B_0$.*

*Then there exists an element $x \in A$ such that*

$$d(x,Tx) = d(A,B). \quad (83)$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Tx_n) = d(A,B), \quad (84)$$

*converges to the element $x$.*

*4.2. Fixed Points Theorem.* It is easy to observe that, for self-mappings, our results include the following.

**Theorem 25.** *Let $A$ be nonempty closed subsets of a complete metric space $X$ and $T : A \to A$ such that*

$$\begin{aligned} d(Tx, Ty) &\le \varphi(M(x,y)) \\ &\quad + \theta(\{d(x,Ty), d(y,Tx), d(x,Tx), d(y,Ty)\}), \end{aligned} \tag{85}$$

*for all $x, y \in A$, where $\varphi \in \Psi$ $\theta \in \Theta$. Then $T$ has a unique fixed point $x \in A$. Moreover, for any fixed $x_0 \in A$, the sequence $\{x_n\}$, defined by $x_{n+1} = Tx_n$, converges to the element $x$.*

**Theorem 26.** *Let $A$ be nonempty closed subsets of a complete metric space $X$ and $T : A \to A$ such that*

$$\begin{aligned} d(Tx, Ty) &\le kM(x,y) \\ &\quad + L\min\{d(x,Ty), d(y,Tx), d(x,Tx), d(y,Ty)\}. \end{aligned} \tag{86}$$

*Then $T$ has a unique fixed point $x \in A$. Moreover, for any fixed $x_0 \in A$, the sequence $\{x_n\}$, defined by $x_{n+1} = Tx_n$, converges to the element $x$.*

**Theorem 27.** *Let $A$ be nonempty closed subsets of a complete metric space $X$ and $T : A \to A$ such that*

$$\begin{aligned} d(Tx, Ty) &\le kd(x,y) \\ &\quad + L\min\{d(x,Ty), d(y,Tx), d(x,Tx), d(y,Ty)\} \end{aligned} \tag{87}$$

*for all $x, y \in A$. Then $T$ has a unique fixed point $x \in A$. Moreover, for any fixed $x_0 \in A$, the sequence $\{x_n\}$, defined by $x_{n+1} = Tx_n$, converges to the element $x$.*

## 5. Some Applications and an Example

We recall some preliminaries from (see, [6, 17] also) as follows.

Let $(X, d)$ be a metric space and $\mathscr{R}$ a binary relation over $X$. Denote

$$\mathcal{S} = \mathscr{R} \cup \mathscr{R}^{-1}; \tag{88}$$

this is the symmetric relation attached to $\mathscr{R}$. Clearly,

$$x, y \in X, \quad x\mathcal{S}y \Longleftrightarrow x\mathscr{R}y \quad \text{or} \quad y\mathscr{R}x. \tag{89}$$

*Definition 28* (see [17]). A mapping $T : A \to B$ is said to be *proximal comparative* if and only if

$$\left\{\begin{array}{l} x_1\mathcal{S}x_2 \\ d(u_1, Tx_1) = d(A,B) \\ d(u_2, Tx_2) = d(A,B) \end{array}\right. \Longrightarrow u_1\mathcal{S}u_2. \tag{90}$$

**Corollary 29.** *Let $(X, d)$ be a complete metric space, $\mathscr{R}$ a binary relation over $X$, and $A$ and $B$ two nonempty, closed subsets of $X$ such that $A_0$ are nonempty and the pair $(A, B)$ has the $P$-property. Let $T : A \to B$ such that the following conditions hold:*

(a) *$T$ is a continuous proximal comparative mapping;*

(b) *there exist elements $x_0$ and $x_1$ in $A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0\mathcal{S}x_1$;*

(c) *there exist $\varphi \in \Psi$ and $\theta \in \Theta$ such that $x, y \in A$, $x\mathcal{S}y$ implies that*

$$\begin{aligned} d(Tx, Ty) &\le \varphi(M(x,y)) \\ &\quad + \theta(d(y,Tx) - d(A,B), d(x,Ty) - d(A,B), \\ &\qquad d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)) \end{aligned} \tag{91}$$

(d) $T(A_0) \subseteq B_0$.

*Then there exists an element $x \in A$ such that*

$$d(x, Tx) = d(A, B). \tag{92}$$

*Proof.* Define the mapping $\alpha : A \times A \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1; & x\mathcal{S}y, \\ 0; & \text{otherwise.} \end{cases} \tag{93}$$

Since $T$ is proximal comparative, we have

$$\left\{\begin{array}{l} x_1\mathcal{S}x_2 \\ d(u_1, Tx_1) = d(A,B) \\ d(u_2, Tx_2) = d(A,B) \end{array}\right. \Longrightarrow u_1\mathcal{S}u_2. \tag{94}$$

for all $u, v, x, y \in A$. Using the definition of $\alpha$, we get

$$\left\{\begin{array}{l} \alpha(x,y) \ge 1, \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B), \end{array}\right. , \Longrightarrow \alpha(u,v) \ge 1, \tag{95}$$

for all $u, v, x, y \in A$ and hence $T$ is $\alpha$-proximal admissible. Condition $(b)$ implies that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \ge 1$. By condition (c), we get

$$\begin{aligned} \alpha(x,y)\, d(Tx, Ty) &\le \varphi(M(x,y)) \\ &\quad + \theta(d(y,Tx) - d(A,B), d(x,Ty) - d(A,B), \\ &\qquad d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)); \end{aligned} \tag{96}$$

that is, $T$ is generalized almost $(\varphi, \theta)_\alpha$-contraction. Therefore, all hypotheses of Theorem 13 are satisfied, and the desired result follows immediately. □

Next, below we give an example to illustrate the main result of Theorem 13.

*Example 30.* Consider $X = R^4$ with the metric defined by

$$d((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |x_4 - y_4| \quad (97)$$

for all $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in R^4$. Let $A, B \subset X$ defined by

$$A := \left\{\left(0, 0, \frac{1}{n}, \frac{-1}{n}\right)\right\} \cup \{(0, 0, 0, 0)\},$$
$$B := \left\{\left(1, -1, \frac{1}{n}, \frac{-1}{n}\right)\right\} \cup \{(1, -1, 0, 0)\}. \quad (98)$$

Then $A$ and $B$ are nonempty closed subsets of $X$ and $d(A, B) = 2$. Moreover $A_0 = A$ and $B_0 = B$. Suppose

$$d((0, 0, x_1, x_2), (1, -1, y_1, y_2)) = d(A, B) = 2,$$
$$d\left(\left(0, 0, x_1', x_2'\right), \left(1, -1, y_1', y_2'\right)\right) = d(A, B) = 2; \quad (99)$$

then we get $x_1 = y_1, x_2 = y_2$ and $x_1' = y_1', x_2' = y_2'$. Hence, the pair $(A, B)$ has the $P$-$property$. Let $T : A \to B$ be a mapping defined as

$$T(0, 0, x, y) = \left(0, 0, \frac{x}{2}, \frac{y}{2}\right) \quad (100)$$

for all $(0, 0, x, y) \in A$. We define the mapping $\alpha : A \times A \to [0, \infty)$ by

$$\alpha(x, y) = 1 \quad \forall x, y \in A. \quad (101)$$

We can see that $T$ is generalized almost $(\varphi, \theta)_\alpha$-contraction with $\varphi \in \Psi$ given by $\varphi(t) = t/2$ for all $t \geq 0$ and for all $\theta \in \Theta$. Furthermore, $(0, 0, 0, 0) \in A$ is a best proximity point of mapping $T$.

## 6. Conclusions

We introduce the new class of generalized almost $(\varphi, \theta)_\alpha$ contraction and presented sufficient conditions for proving the existence and uniqueness of the best proximity point. Moreover, we also gave some applications and examples to support our results.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The authors thank the referee for comments and suggestions on this paper. The third author was supported by the Commission on Higher Education, the Thailand Research Fund, and the King Mongkuts University of Technology Thonburi (Grant no. MRG550085).

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Fixed Point Theory and Applications
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# Best proximity points and extension of Mizoguchi-Takahashi's fixed point theorems

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Dedicated to Prof. W Takahashi on the occasion of his 70th birthday

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**Abstract**
In this paper, we introduce a multi-valued cyclic generalized contraction by extending the Mizoguchi and Takahashi's contraction for non-self mappings. We also establish a best proximity point for such type contraction mappings in the context of metric spaces. Later, we characterize this result to investigate the existence of best proximity point theorems in uniformly convex Banach spaces. We state some illustrative examples to support our main theorems. Our results extend, improve and enrich some celebrated results in the literature, such as Nadler's fixed point theorem, Mizoguchi and Takahashi's fixed point theorem.
**MSC:** 41A65; 46B20; 47H09; 47H10

**Keywords:** best proximity points; multi-valued contraction; cyclic contraction; $\mathcal{MT}$-function (or $\mathcal{R}$-function)

## 1 Introduction

It is evident that the fixed point theory is one of the fundamental tools in nonlinear functional analysis. The celebrated Banach contraction mapping principle [1] is the most known and crucial result in fixed point theory. It says that each contraction in a complete metric space has a unique fixed point. This theorem not only guarantees the existence and uniqueness of the fixed point but also shows how to evaluate this point. By virtue of this fact, the Banach contraction mapping principle has been generalized in many ways over the years (see *e.g.*, [2–5]).

Investigation of the existence and uniqueness of a fixed point of non-self mappings is one of the interesting subjects in fixed point theory. In fact, given nonempty closed subsets $A$ and $B$ of a complete metric space $(X,d)$, a contraction non-self-mapping $T : A \to B$ does not necessarily yield a fixed point $Tx = x$. In this case, it is very natural to investigate whether there is an element $x$ such that $d(x, Tx)$ is minimum. A notion of best proximity point appears at this point. A point $x$ is called best proximity point of $T : A \to B$ if

$$d(x, Tx) = d(A,B) = \inf\{d(x,y) : x \in A \text{ and } y \in B\},$$

where $(X,d)$ is a metric space, and $A$, $B$ are subsets of $X$. A best proximity point represents an optimal approximate solution to the equation $Tx = x$ whenever a non-self-mapping $T$ has no fixed point. It is clear that a fixed point coincides with a best proximity point if $d(A,B) = 0$. Since a best proximity point reduces to a fixed point if the underlying mapping is assumed to be self-mappings, the best proximity point theorems are natural generalizations of the Banach's contraction principle.

In 1969, Fan [6] introduced the notion of a best proximity and established a classical best approximation theorem. More precisely, if $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$, where $A$ is a nonempty compact convex subset of a Hausdorff locally convex topological vector space $B$. Subsequently, many researchers have studied the best proximity point results in many ways (see in [7–14] and the references therein).

In the same year, Nadler [15] gave a useful lemma about Hausdorff metric. In paper [15], the author also characterized the celebrated Banach fixed point theorem in the context of multi-valued mappings.

**Lemma 1.1** (Nadler [15]) *If $A, B \in CB(X)$ and $a \in A$, then for each $\epsilon > 0$, there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$.*

**Theorem 1.2** (Nadler [15]) *Let $(X, d)$ be a complete metric space and $T : X \to CB(X)$. If there exists $r \in [0, 1)$ such that*

$$H(Tx, Ty) \leq rd(x, y), \tag{1.1}$$

*for all $x, y \in X$, then $T$ has at least one fixed point, that is, there exists $z \in X$ such that $z \in Tz$.*

The theory of multi-valued mappings has applications in many areas such as in optimization problem, control theory, differential equations, economics and many branches in analysis. Due to this fact, a number of authors have focused on the topic and have published some interesting fixed point theorems in this frame (see [16–19] and references therein). Following this trend, in 1989, Mizoguchi and Takahashi [17] proved a generalization (Theorem 1.3 below) of Theorem 1.2; see Theorem 2 in Alesina *et al.* [20]. Theorem 2 is a partial answer of Problem 9 in Reich [21]. See also [22–24].

**Theorem 1.3** (Mizoguchi and Takahashi [17]) *Let $(X, d)$ be a complete metric space and $T : X \to CB(X)$. Assume that*

$$H(Tx, Ty) \leq \alpha\big(d(x, y)\big)d(x, y), \tag{1.2}$$

*for all $x, y \in X$, where $\alpha : [0, \infty) \to [0, 1)$ is $\mathcal{MT}$-function (or $\mathcal{R}$-function), i.e.,*

$$\limsup_{x \to t^+} \alpha(x) < 1$$

*for all $t \in [0, \infty)$. Then $T$ has at least one fixed point, that is, there exists $z \in X$ such that $z \in Tz$.*

**Remark 1.4** In original statement of Mizoguchi and Takahashi [17], the domain $\alpha$ is $(0, \infty)$. However both are equivalent, because $d(x, y) = 0$ implies that $H(Tx, Ty) = 0$.

**Remark 1.5** We obtain that if $\alpha : [0,\infty) \to [0,1)$ is a nondecreasing function or a nonincreasing function, then $\alpha$ is a $\mathcal{MT}$-function. Therefore, the class of $\mathcal{MT}$-functions is a rich class, and so this class has been investigated heavily by many authors.

In 2007, Eldred *et al.* [25] claimed that Theorem 1.3 is equivalent to Theorem 1.2 in the following sense:

If a mapping $T : X \to CB(X)$ satisfies (1.2), then there exists a nonempty complete subset $M$ of $X$ satisfying the following:

(i) $M$ is $T$-invariant, that is, $Tx \subseteq M$ for all $x \in M$,

(ii) $T$ satisfies (1.1) for all $x, y \in M$.

Very recently, Suzuki [26] gave an example which says that Mizoguchi-Takahashi's fixed point theorem for multi-valued mappings is a real generalization of Nadler's result. In his remarkable paper, Suzuki also gave a very simple proof of Mizoguchi-Takahashi's theorem.

On the other hand, Kirk-Srinavasan-Veeramani [27] introduced the concept of a cyclic contraction.

Let $A$ and $B$ be two nonempty subsets of a metric space $(X,d)$, and let $T : A \cup B \to A \cup B$ be a mapping. Then $T$ is called a *cyclic map* if $T(A) \subseteq B$ and $T(B) \subseteq A$. In addition, if $T$ is a contraction, then $T$ is called *cyclic contraction*.

The authors [27] give a characterization of Banach contraction mapping principle in complete metric spaces. After this initial paper, a number of papers has appeared on the topic in literature (see, *e.g.*, [27–39]).

In this paper, we introduce the notion of a generalized multi-valued cyclic contraction pair, which is an extension of Mizoguchi-Takahashi's contraction mappings for non-self version and establish a best proximity point of such mappings in metric spaces via property UC* due to Sintunavarat and Kumam [40]. Further, by applying the main results, we investigate best proximity point theorems in a uniformly convex Banach space. We also give some illustrative examples, which support our main results. Our results generalize, improve and enrich some well-known results in literature.

## 2 Preliminaries

In this section, we recall some basic definitions and elementary results in literature. Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers, by $\mathbb{R}$ the set of all real numbers and by $\mathbb{R}_+$ the set of all nonnegative real numbers. We denote by $CB(X)$ the class of all nonempty closed bounded subsets of a metric space $(X,d)$. The Hausdorff metric induced by $d$ on $CB(X)$ is given by

$$H(A,B) = \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Big\},$$

for every $A, B \in CB(X)$, where $d(a,B) = \inf\{d(a,b) : b \in B\}$ is the distance from $a$ to $B \subseteq X$.

**Remark 2.1** The following properties of the Hausdorff metric induced by $d$ are well known:

(i) $H$ is a metric on $CB(X)$.

(ii) If $A, B \in CB(X)$ and $q > 1$ is given, then for every $a \in A$, there exists $b \in B$ such that $d(a,b) \leq qH(A,B)$.

**Definition 2.2** Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$ and let $T : A \to 2^B$ be a multi-valued mapping. A point $x \in A$ is said to be a *best proximity point* of a multi-valued mapping $T$ if it satisfies the condition that

$$d(x, Tx) = d(A,B).$$

We notice that a best proximity point reduces to a fixed point for a multi-valued mapping if the underlying mapping is a self-mapping.

A Banach space $X$ is said to be

(i) *strictly convex* if the following implication holds for all $x,y \in X$:

$$\|x\| = \|y\| = 1 \quad \text{and} \quad x \neq y \quad \Longrightarrow \quad \left\| \frac{x+y}{2} \right\| < 1;$$

(ii) *uniformly convex* if for each $\epsilon$ with $0 < \epsilon \leq 2$, there exists $\delta > 0$ such that the following implication holds for all $x,y \in X$:

$$\|x\| \leq 1, \qquad \|y\| \leq 1 \quad \text{and} \quad \|x-y\| \geq \epsilon \quad \Longrightarrow \quad \left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

It is easy to see that a uniformly convex Banach space $X$ is strictly convex, but the converse is not true.

**Definition 2.3** [41] Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. The ordered pair $(A,B)$ is said to satisfy the *property UC* if the following holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in $A$, and $\{y_n\}$ is a sequence in $B$ such that $d(x_n,y_n) \to d(A,B)$ and $d(z_n,y_n) \to d(A,B)$, then $d(x_n,z_n) \to 0$.

**Example 2.4** [41] The following are examples of a pair of nonempty subsets $(A,B)$ satisfying the property UC.

(i) Every pair of nonempty subsets $A$, $B$ of a metric space $(X,d)$ such that $d(A,B) = 0$.

(ii) Every pair of nonempty subsets $A$, $B$ of a uniformly convex Banach space $X$ such that $A$ is convex.

(iii) Every pair of nonempty subsets $A$, $B$ of a strictly convex Banach space, where $A$ is convex and relatively compact and the closure of $B$ is weakly compact.

**Definition 2.5** [40] Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. The ordered pair $(A,B)$ satisfies the *property UC** if $(A,B)$ has property UC, and the following condition holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in $A$, and $\{y_n\}$ is a sequence in $B$ satisfying

(i) $d(z_n,y_n) \to d(A,B)$.

(ii) For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_m,y_n) \leq d(A,B) + \epsilon$$

for all $m > n \geq N$,

then $d(x_n,z_n) \to 0$.

**Example 2.6** The following are examples of a pair of nonempty subsets $(A, B)$ satisfying the property UC*.

(i) Every pair of nonempty subsets $A$, $B$ of a metric space $(X, d)$ such that $d(A, B) = 0$.

(ii) Every pair of nonempty closed subsets $A$, $B$ of uniformly convex Banach space $X$ such that $A$ is convex (see Lemma 3.7 in [42]).

## 3 Best proximity point for multi-valued mapping theorems

In this section, we investigate the existence and convergence of best proximity points for generalized multi-valued cyclic contraction pairs and obtain some new results on fixed point theorems for such mappings. We begin by introducing the notion of multi-valued cyclic contraction.

**Definition 3.1** Let $A$ and $B$ be nonempty subsets of a metric space $X$, $T : A \to 2^B$ and $S : B \to 2^A$. The ordered pair $(T, S)$ is said to be a *generalized multi-valued cyclic contraction* if there exists a function $\alpha : [d(A, B), \infty) \to [0, 1)$ with

$$\limsup_{x \to t^+} \alpha(x) < 1$$

for each $t \in [d(A, B), \infty)$ such that

$$H(Tx, Sy) \leq \alpha\bigl(d(x, y)\bigr)d(x, y) + \bigl(1 - \alpha\bigl(d(x, y)\bigr)\bigr)d(A, B) \tag{3.1}$$

for all $x \in A$ and $y \in B$.

Note that if $(T, S)$ is a generalized multi-valued cyclic contraction, then $(S, T)$ is also a generalized multi-valued cyclic contraction. Here, we state the main results of this paper on the existence of best proximity points for a generalized multi-valued cyclic contraction pair, which satisfies the property UC* in metric spaces.

**Theorem 3.2** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $(A, B)$ and $(B, A)$ satisfy the property UC*. Let $T : A \to CB(B)$ and $S : B \to CB(A)$. If $(T, S)$ is a generalized multi-valued cyclic contraction pair, then $T$ has a best proximity point in $A$, or $S$ has a best proximity point in $B$.*

*Proof* We consider two cases separately.

*Case* 1. Suppose that $d(A, B) = 0$. Define the function $\beta : [d(A, B), \infty) \to [0, 1)$ by

$$\beta(t) = \frac{\alpha(t) + 1}{2}$$

for $t \in [d(A, B), \infty) = [0, \infty)$. Then we obtain that

$$\limsup_{s \to t^+} \beta(s) < 1$$

for all $t \in [0, \infty)$.

Now, we will construct the sequence $\{x_n\}$ in $X$. Let $x_0 \in A$ be an arbitrary point. Since $Tx_0 \in CB(B)$, we can choose $x_1 \in Tx_0$. If $x_1 = x_0$, we have $x_0 \in Tx_0$, and then $x_0$ is a best proximity point of $T$. Also, it follows from (3.1) with $x = x_0$ and $y = x_1$ that $Tx_0 = Sx_1$. This implies that $x_1 \in Sx_1$. Therefore, $x_1$ is a best proximity point of $S$, and we finish the proof. Otherwise, if $x_0 \neq x_1$, by Lemma 1.1, there exists $x_2 \in Sx_1$ such that

$$\begin{aligned} d(x_1,x_2) &\leq H(Tx_0,Sx_1) + \left[\frac{1-\alpha(d(x_0,x_1))}{2}\right] d(x_0,x_1) \\ &\leq \alpha\bigl(d(x_0,x_1)\bigr)d(x_0,x_1) + \bigl(1-\alpha\bigl(d(x_0,x_1)\bigr)\bigr)d(A,B) \\ &\quad + \left[\frac{1-\alpha(d(x_0,x_1))}{2}\right] d(x_0,x_1) \\ &= \left[\frac{1+\alpha(d(x_0,x_1))}{2}\right] d(x_0,x_1) \\ &= \beta\bigl(d(x_0,x_1)\bigr)d(x_0,x_1). \end{aligned}$$

If $x_2 = x_1$, we have $x_1 \in Sx_1$, and then $x_1$ is a best proximity point of $S$. Also, it follows from (3.1) with $x = x_2$ and $y = x_1$ that $Tx_2 = Sx_1$. This implies that $x_2 \in Tx_2$. Therefore, $x_2$ is a best proximity point of $T$, and we finish the proof. Otherwise, if $x_2 \neq x_1$, by Lemma 1.1, there exists $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2,x_3) &\leq H(Sx_1,Tx_2) + \left[\frac{1-\alpha(d(x_1,x_2))}{2}\right] d(x_1,x_2) \\ &= H(Tx_2,Sx_1) + \left[\frac{1-\alpha(d(x_2,x_1))}{2}\right] d(x_2,x_1) \\ &\leq \alpha\bigl(d(x_2,x_1)\bigr)d(x_2,x_1) + \bigl(1-\alpha\bigl(d(x_2,x_1)\bigr)\bigr)d(A,B) \\ &\quad + \left[\frac{1-\alpha(d(x_2,x_1))}{2}\right] d(x_2,x_1) \\ &= \left[\frac{1+\alpha(d(x_2,x_1))}{2}\right] d(x_2,x_1) \\ &= \beta\bigl(d(x_2,x_1)\bigr)d(x_2,x_1) \\ &= \beta\bigl(d(x_1,x_2)\bigr)d(x_1,x_2). \end{aligned}$$

By repeating this process, we can find $x_n$ such that

$$d(x_{n+1},x_{n+2}) \leq \beta\bigl(d(x_n,x_{n+1})\bigr)d(x_n,x_{n+1}) < d(x_n,x_{n+1})$$

for all $n \in \mathbb{N}$.

Thus, for fixed $x_0 \in A$, we can define a sequence $\{x_n\}$ in $X$ satisfying

$$x_{2n} \in Sx_{2n-1} \subseteq A \quad \text{and} \quad x_{2n-1} \in Tx_{2n-2} \subseteq B$$

such that

$$d(x_{n+1},x_{n+2}) \leq \beta\bigl(d(x_n,x_{n+1})\bigr)d(x_n,x_{n+1}) < d(x_n,x_{n+1})$$

for $n \in \mathbb{N}$. Therefore, $\{d(x_n,x_{n+1})\}$ is a strictly decreasing sequence in $\mathbb{R}_+$. So $\{d(x_n,x_{n+1})\}$ converges to some nonnegative real number $\rho$. Since $\limsup_{s\to\rho^+}\beta(s) < 1$ and $\beta(\rho) < 1$, there exist $r \in [0,1)$ and $\eta > 0$ such that $\beta(s) \leq r$ for all $s \in [\rho, \rho + \eta]$. We can take $\nu \in \mathbb{N}$ such that

$$\rho \leq d(x_n, x_{n+1}) \leq \rho + \eta$$

for all $n \in \mathbb{N}$ with $n \geq \nu$. Then since

$$d(x_{n+1}, x_{n+2}) \leq \beta\big(d(x_n, x_{n+1})\big) d(x_n, x_{n+1}) \leq r d(x_n, x_{n+1})$$

for $n \in \mathbb{N}$ with $n \geq \nu$, we have

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=1}^{\nu} d(x_n, x_{n+1}) + \sum_{n=\nu}^{\infty} d(x_n, x_{n+1}) < \infty,$$

that is, $\{x_n\}$ is a Cauchy sequence. Since $X$ is complete, $\{x_n\}$ converges to some point $z \in X$. Clearly, the subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ converge to the same point $z$. Since $A$ and $B$ are closed, we derive that $z \in A \cap B$. We consider that

$$\begin{aligned} d(Tz, z) &= \lim_{n\to\infty} d(Tz, x_{2n}) \\ &\leq \lim_{n\to\infty} H(Tz, Sx_{2n-1}) \\ &\leq \lim_{n\to\infty} \beta\big(d(z, x_{2n-1})\big) d(z, x_{2n-1}) \\ &\leq \lim_{n\to\infty} d(z, x_{2n-1}) \\ &= 0 \\ &= d(A, B). \end{aligned}$$

Hence we get $d(z, Tz) = d(A, B)$. Analogously, we also obtain $d(z, Sz) = d(A, B)$.

*Case* 2. We will show that $T$ or $S$ have best proximity points in $A$ and $B$, respectively, under the assumption of $d(A, B) > 0$. Suppose, to the contrary, that for all $a \in A$, $d(a, Ta) > d(A, B)$ and for all $b' \in B$, $d(Sb', b') > d(A, B)$.

Next, we define a function $\beta : [d(A, B), \infty) \to [0, 1)$ by

$$\beta(t) = \frac{\alpha(t) + 1}{2}$$

for all $t \in [d(A, B), \infty)$. So we derive $\limsup_{x\to t^+} \beta(x) < 1$ and $\alpha(t) < \beta(t)$ for all $t \in [d(A, B), \infty)$.

For each $a \in A$ and $b \in Ta$, we have

$$d(A, B) < d(a, Ta) \leq d(a, b).$$

Therefore,

$$\big[\beta\big(d(a, b)\big) - \alpha\big(d(a, b)\big)\big] d(A, B) < \big[\beta\big(d(a, b)\big) - \alpha\big(d(a, b)\big)\big] d(a, b),$$

and then we get

$$\begin{aligned}&\alpha\big(d(a,b)\big)d(a,b)+\big(1-\alpha\big(d(a,b)\big)\big)d(A,B)\\&\quad<\beta\big(d(a,b)\big)d(a,b)+\big(1-\beta\big(d(a,b)\big)\big)d(A,B).\end{aligned}\tag{3.2}$$

Since $(T,S)$ is a generalized multi-valued cyclic contraction pair, by (3.2), we conclude

$$\begin{aligned}H(Ta,Sb)&\le\alpha\big(d(a,b)\big)d(a,b)+\big(1-\alpha\big(d(a,b)\big)\big)d(A,B)\\&<\beta\big(d(a,b)\big)d(a,b)+\big(1-\beta\big(d(a,b)\big)\big)d(A,B)\end{aligned}\tag{3.3}$$

for all $a\in A$ and $b\in Ta$.

Similarly, we obtain that for each $b'\in B$ and $a'\in Sb'$, we have

$$H\big(Ta',Sb'\big)<\beta\big(d\big(a',b'\big)\big)d\big(a',b'\big)+\big(1-\beta\big(d\big(a',b'\big)\big)\big)d(A,B).\tag{3.4}$$

Next, we will construct the sequence $\{x_n\}$ in $A\cup B$. Let $x_0$ be an arbitrary point in $A$ and $x_1\in Tx_0\subseteq B$. From (3.3), there exists $x_2\in Sx_1\subseteq A$ such that

$$d(x_1,x_2)<\beta\big(d(x_0,x_1)\big)d(x_0,x_1)+\big(1-\beta\big(d(x_0,x_1)\big)\big)d(A,B).\tag{3.5}$$

Since $x_1\in B$ and $x_2\in Sx_1$, from (3.4), we can find $x_3\in Tx_2$ such that

$$d(x_2,x_3)<\beta\big(d(x_1,x_2)\big)d(x_1,x_2)+\big(1-\beta\big(d(x_1,x_2)\big)\big)d(A,B).\tag{3.6}$$

Analogously, we can define the sequence $\{x_n\}$ in $A\cup B$ such that

$$x_{2n-1}\in Tx_{2n-2},\qquad x_{2n}\in Sx_{2n-1}$$

and

$$d(x_n,x_{n+1})<\beta\big(d(x_{n-1},x_n)\big)d(x_{n-1},x_n)+\big(1-\beta\big(d(x_{n-1},x_n)\big)\big)d(A,B)\tag{3.7}$$

for all $n\in\mathbb{N}$. Since $\beta(d(x_{n-1},x_n))<1$ and $d(A,B)<d(x_{n-1},x_n)$ for all $n\in\mathbb{N}$, we get

$$\begin{aligned}d(x_n,x_{n+1})&<\beta\big(d(x_{n-1},x_n)\big)d(x_{n-1},x_n)+\big(1-\beta\big(d(x_{n-1},x_n)\big)\big)d(x_{n-1},x_n)\\&=d(x_{n-1},x_n)\end{aligned}\tag{3.8}$$

for all $n\in\mathbb{N}$. Therefore, $\{d(x_{n-1},x_n)\}$ is a strictly decreasing sequence in $\mathbb{R}_+$ and bounded below. So the sequence $\{d(x_{n-1},x_n)\}$ converges to some nonnegative real number $d$. Since $\limsup_{x\to d^+}\beta(x)<1$ and $\beta(d)<1$, there exist $d_0\in[0,1)$ and $\epsilon>0$ such that $\beta(s)\le d_0$ for all $s\in[d,d+\epsilon]$. Now, we can take $N_0\in\mathbb{N}$ such that

$$d\le d(x_{n-1},x_n)\le d+\epsilon$$

for all $n \geq N_0$. From (3.7), we have

$$d(x_n, x_{n+1}) < d_0 d(x_{n-1}, x_n) + (1 - d_0) d(A, B) \tag{3.9}$$

for all $n \geq N_0$. By the same consideration, we obtain

$$d(A, B) < d(x_n, x_{n+1}) < d_0^{n-N_0} d(x_{N_0}, x_{N_0+1}) + \left(1 - d_0^{n-N_0}\right) d(A, B) \tag{3.10}$$

for all $n \geq N_0$. Since $d_0 \in [0, 1)$, we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B). \tag{3.11}$$

From (3.11), we conclude that

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = d(A, B), \tag{3.12}$$

and

$$\lim_{n\to\infty} d(x_{2n+2}, x_{2n+1}) = d(A, B). \tag{3.13}$$

Since $\{x_{2n}\}$ and $\{x_{2n+2}\}$ are two sequences in $A$, and $\{x_{2n+1}\}$ is sequence in $B$ with $(A, B)$ satisfies the property UC*, we derive that

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = 0. \tag{3.14}$$

Since $(B, A)$ satisfies the property UC*, and by (3.11), we find that

$$\lim_{n\to\infty} d(x_{2n-1}, x_{2n+1}) = 0. \tag{3.15}$$

Next, we show that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m > n \geq N$, we have

$$d(x_{2m}, x_{2n+1}) \leq d(A, B) + \epsilon. \tag{3.16}$$

Suppose, to the contrary, that there exists $\epsilon_0 > 0$ such that for each $k \geq 1$, there is $m_k > n_k \geq k$ such that

$$d(x_{2m_k}, x_{2n_k+1}) > d(A, B) + \epsilon_0. \tag{3.17}$$

Further, corresponding to $n_k$, we can choose $m_k$ in such a way that it is the smallest integer with $m_k > n_k \geq k$ satisfying (3.17). Then we have

$$d(x_{2m_k}, x_{2n_k+1}) > d(A, B) + \epsilon_0 \tag{3.18}$$

and

$$d(x_{2(m_k-1)}, x_{2n_k+1}) \leq d(A, B) + \epsilon_0. \tag{3.19}$$

From (3.18), (3.19) and the triangle inequality, we have

$$\begin{aligned} d(A,B) + \epsilon_0 &< d(x_{2m_k}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2(m_k-1)}) + d(x_{2(m_k-1)}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2(m_k-1)}) + d(A,B) + \epsilon_0. \end{aligned} \tag{3.20}$$

Using the fact that $\lim_{k\to\infty} d(x_{2m_k}, x_{2(m_k-1)}) = 0$. Letting $k \to \infty$ in (3.20), we have

$$\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = d(A,B) + \epsilon_0. \tag{3.21}$$

From (3.8), (3.9) and $(T,S)$ is a generalized multi-valued cyclic contraction pair, we get

$$\begin{aligned} d(x_{2m_k}, x_{2n_k+1}) &\leq d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &< d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+1}, x_{2n_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) + \alpha\big(d(x_{2m_k}, x_{2n_k+1})\big) d(x_{2m_k}, x_{2n_k+1}) \\ &\quad + \big(1 - \alpha\big(d(x_{2m_k}, x_{2n_k+1})\big)\big) d(A,B) \\ &< d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+2}, x_{2n_k+1}) + \beta\big(d(x_{2m_k}, x_{2n_k+1})\big) d(x_{2m_k}, x_{2n_k+1}) \\ &\quad + \big(1 - \beta\big(d(x_{2m_k}, x_{2n_k+1})\big)\big) d(A,B) \\ &\leq d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+2}, x_{2n_k+1}) + d_0 d(x_{2m_k}, x_{2n_k+1}) \\ &\quad + (1 - d_0) d(A,B). \end{aligned} \tag{3.22}$$

Letting $k \to \infty$ in (3.22) and using (3.14), (3.15) and (3.21), we have

$$d(A,B) + \epsilon_0 \leq d_0\big(d(A,B) + \epsilon_0\big) + (1 - d_0) d(A,B) = d(A,B) + d_0\epsilon_0,$$

which is a contradiction. Therefore, (3.16) holds.

Since (3.12) and (3.16) hold, by using property UC* of $(A,B)$, we have $d(x_{2n}, x_{2m}) \to 0$. Therefore, $\{x_{2n}\}$ is a Cauchy sequence. By the completeness of $X$ and since $A$ is closed, we get

$$\lim_{n\to\infty} x_{2n} = p \tag{3.23}$$

for some $p \in \overline{A} = A$. But

$$\begin{aligned} d(A,B) &\leq d(p, x_{2n-1}) \\ &\leq d(p, x_{2n}) + d(x_{2n}, x_{2n-1}) \end{aligned}$$

for all $n \in \mathbb{N}$. From (3.11) and (3.23),

$$\lim_{n\to\infty} d(p, x_{2n-1}) = d(A,B). \tag{3.24}$$

Since

$$\begin{aligned} d(A,B) &< d(x_{2n}, Tp) \\ &\le H(Sx_{2n-1}, Tp) \\ &= H(Tp, Sx_{2n-1}) \\ &\le \alpha\big(d(p,x_{2n-1})\big)d(p,x_{2n-1}) + \big(1 - d(p,x_{2n-1})\big)d(A,B) \\ &\le d(p,x_{2n-1}) \end{aligned} \tag{3.25}$$

for all $n \in \mathbb{N}$. By (3.23) and (3.24), we get

$$d(p, Tp) = d(A,B). \tag{3.26}$$

In a similar mode, we can conclude that the sequence $\{x_{2n-1}\}$ is a Cauchy sequence in $B$. Since $X$ is complete, and since $B$ is closed, we have

$$\lim_{n\to\infty} x_{2n-1} = q \tag{3.27}$$

for some $q \in \overline{B} = B$. Since

$$\begin{aligned} d(A,B) &\le d(x_{2n}, q) \\ &\le d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, q) \end{aligned}$$

for all $n \in \mathbb{N}$. It follows from (3.11) and (3.27) that

$$\lim_{n\to\infty} d(x_{2n}, q) = d(A,B). \tag{3.28}$$

Since

$$\begin{aligned} d(A,B) &< d(Sq, x_{2n+1}) \\ &\le H(Sq, Tx_{2n}) \\ &= H(Tx_{2n}, Sq) \\ &\le \alpha\big(d(x_{2n},q)\big)d(x_{2n},q) + \big(1 - d(x_{2n},q)\big)d(A,B) \\ &\le d(x_{2n},q) \end{aligned} \tag{3.29}$$

for all $n \in \mathbb{N}$, then by (3.27) and (3.28), we have

$$d(q, Sq) = d(A,B). \tag{3.30}$$

From (3.26) and (3.30), we have a contradiction. Therefore, $T$ has a best proximity point in $A$ or $S$ has a best proximity point in $B$. This completes the proof. □

**Remark 3.3** If $d(A,B) = 0$, then Theorem 3.2 yields existence of a fixed point in $A \cap B$ of two multi-valued non-self mappings $S$ and $T$. Moreover, if $A = B = X$ and $T = S$, then Theorem 3.2 reduces to Mizoguchi-Takahashi's fixed point theorem [17].

Note that every pair of nonempty closed subsets $A$, $B$ of a uniformly convex Banach space such that $A$ is convex satisfies the property UC*. Therefore, we obtain the following corollary.

**Corollary 3.4** *Let $A$ and $B$ be nonempty closed convex subsets of a uniformly convex Banach space $X$, $T : A \to CB(B)$ and $S : B \to CB(A)$. If $(T,S)$ is a generalized multi-valued cyclic contraction pair, then $T$ has a best proximity point in $A$ or $S$ has a best proximity point in $B$.*

Next, we give some illustrative examples of Corollary 3.4.

**Example 3.5** Consider the uniformly convex Banach space $X = \mathbb{R}$ with Euclidean norm. Let $A = [1,2]$ and $B = [-2,-1]$. Then $A$ and $B$ are nonempty closed and convex subsets of $X$ and $d(A,B) = 2$. Since $A$ and $B$ are convex, we have $(A,B)$ and $(B,A)$ satisfy the property UC*.

Let $T : A \to CB(B)$ and $S : B \to CB(A)$ be defined as

$$Tx = \left[\frac{-x-1}{2}, -1\right]$$

for all $x \in A$ and

$$Sy = \left[1, \frac{-y+1}{2}\right]$$

for all $y \in B$.

Let $\alpha : [d(A,B),\infty) \to [0,1)$ be defined by $\alpha(t) = \frac{1}{2}$ for all $t \in [d(A,B),\infty) = [2,\infty)$. Next, we show that $(T,S)$ is a generalized multi-valued cyclic contraction pair with $\alpha(t) = \frac{1}{2}$ for all $t \in [2,\infty)$.

For each $x \in A$ and $y \in B$, we have

$$\begin{aligned} H(Tx,Sy) &= H\left(\left[\frac{-x-1}{2}, -1\right], \left[1, \frac{-y+1}{2}\right]\right) \\ &\leq \left|\left(\frac{-x-1}{2}\right) - \left(\frac{-y+1}{2}\right)\right| \\ &= \left|\frac{-x+y-2}{2}\right| \\ &\leq \frac{1}{2}|x-y| + 1 \\ &= \frac{1}{2}d(x,y) + \frac{1}{2}d(A,B) \\ &= \alpha\big(d(x,y)\big)d(x,y) + \big(1-\alpha\big(d(x,y)\big)\big)d(A,B). \end{aligned}$$

Therefore, all assumptions of Corollary 3.4 are satisfied, and then $T$ has a best proximity point in $A$, that is, a point $x = 1$. Moreover, $S$ also has a best proximity point in $B$, that is, a point $y = -1$.

**Example 3.6** Consider the uniformly convex Banach space $X = \mathbb{R}^2$ with Euclidean norm. Let

$$A := \{(0,x) : x \geq 0\}$$

and

$$B = \{(2,y) : y \geq 0\}.$$

Then $A$ and $B$ are nonempty closed and convex subsets of $X$ and $d(A,B) = 2$. Since $A$ and $B$ are convex, we have $(A,B)$ and $(B,A)$ satisfy the property UC*.

Let $T : A \to CB(B)$ and $S : B \to CB(A)$ be defined as

$$T(0,x) = \{2\} \times \left[0, \frac{x}{2}\right]$$

and

$$S(2,y) = \{0\} \times \left[0, \frac{y}{2}\right]$$

for all $x, y \geq 0$.

Let $\alpha : [d(A,B),\infty) \to [0,1)$ define by $\alpha(t) = \frac{1}{2}$ for all $t \in [d(A,B),\infty) = [2,\infty)$. Next, we show that $(T,S)$ is a generalized multi-valued cyclic contraction pair with mapping $\alpha(t) = \frac{1}{2}$ for all $t \in [2,\infty)$.

For each $(0,x) \in A$ and $(2,y) \in B$, we have

$$\begin{aligned}
H\big(T(0,x), S(2,y)\big) &= H\left(\{2\} \times \left[0, \frac{x}{2}\right], \{0\} \times \left[0, \frac{y}{2}\right]\right) \\
&= \sqrt{4 + \left(\frac{|x-y|}{2}\right)^2} \\
&\leq \frac{1}{2}\left(\sqrt{4 + |x-y|^2}\right) + 1 \\
&= \frac{1}{2} d\big((0,x),(2,y)\big) + \frac{1}{2} d(A,B) \\
&= \alpha\big(d\big((0,x),(2,y)\big)\big) d\big((0,x),(2,y)\big) + \big(1 - \alpha\big(d\big((0,x),(2,y)\big)\big)\big) d(A,B).
\end{aligned}$$

Therefore, all assumptions of Corollary 3.4 are satisfied, and then $T$ has a best proximity point in $A$ that is a point $(0,0)$. Furthermore, $S$ also has a best proximity point in $B$ that is a point $(2,0)$.

**Open problems**

- In Theorem 3.2, can we replace the property UC* by a more general property?
- In Theorem 3.2, can we drop the property UC*?
- Can we extend the result in this paper to another spaces?

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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**Acknowledgements**
Poom Kumam was supported by the Commission on Higher Education, the Thailand Research Fund and the King Mongkut's University of Technology Thonburi (Grant No. MRG5580213).

Received: 30 May 2013 Accepted: 26 September 2013 Published: 07 Nov 2013

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10.1186/1687-1812-2013-242
**Cite this article as:** Kumam et al.: **Best proximity points and extension of Mizoguchi-Takahashi's fixed point theorems.** *Fixed Point Theory and Applications* 2013, 2013:242

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# Best proximity point results for modified $\alpha$-proximal $C$-contraction mappings

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**Abstract**

First we introduce new concepts of contraction mappings, then we establish certain best proximity point theorems for such kind of mappings in metric spaces. Finally, as consequences of these results, we deduce best proximity point theorems in metric spaces endowed with a graph and in partially ordered metric spaces. Moreover, we present an example and some fixed point results to illustrate the usability of the obtained theorems.

**MSC:** 46N40; 46T99; 47H10; 54H25

**Keywords:** best proximity point; fixed point; metric space

## 1 Introduction

A wide variety of problems arising in different areas of pure and applied mathematics, such as difference and differential equations, discrete and continuous dynamic systems, and variational analysis, can be modeled as fixed point equations of the form $x = Tx$. Therefore, fixed point theory plays a crucial role for solving equations of above kind, whose solutions are the fixed points of the mapping $T : X \to X$, where $X$ is a nonempty set. Areas of potential applications of this theory include physics, economics, and engineering in dealing with the study of equilibrium points (which are fixed points of certain mappings). On the other hand, if $T$ is a nonself-mapping, the above fixed point equation could have no solutions and, in this case, it is of a certain interest to determine an approximate solution $x$ that is optimal in the sense that the distance between $x$ and $Tx$ is minimum. In this context, best proximity point theory is an useful tool in studying such kind of element. We recall the following concept.

**Definition 1.1** Let $A$, $B$ be two nonempty subsets of a metric space $(X, d)$ and $T : A \to B$ be a nonself-mapping. An element $x \in A$ such that $d(x, Tx) = d(A, B)$ is a best proximity point of the nonself-mapping $T$.

Clearly, if $T$ is a self-mapping, a best proximity point is a fixed point, that is, $x = Tx$.

From the beginning, best proximity point theory of nonself-mappings has been studied by many authors; see the pioneering papers of Fan [1] and Kirk *et al.* [2]. The investigation of several variants of conditions for the existence of a best proximity point can be found in [3–12]. In particular, some significant best proximity point results for multivalued mappings are presented in [13]; see also the references therein.

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Inspired and motivated by the above facts, in this paper, we introduce new concepts of contraction mappings. Then we establish certain best proximity point theorems for such kind of mappings in metric spaces. As consequences of these results, we deduce best proximity point theorems in metric spaces endowed with a graph and in partially ordered metric spaces. Moreover, we present an example and some fixed point results to illustrate the usability of the obtained theorems.

## 2 Preliminaries

In this section, we collect some useful definitions and results from fixed point theory.

Samet *et al.* [14] defined the notion of $\alpha$-admissible mapping as follows.

**Definition 2.1** ([14]) Let $\alpha : X \times X \to [0, +\infty)$ be a function. We say that a self-mapping $T : X \to X$ is $\alpha$-admissible if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

By using this concept, they proved some fixed point results.

**Theorem 2.1** ([14]) *Let $(X, d)$ be a complete metric space and $T : X \to X$ be an $\alpha$-admissible mapping. Assume that the following conditions hold*:

(i) *for all $x, y \in X$ we have*

$$\alpha(x, y)d(Tx, Ty) \leq \psi\bigl(d(x, y)\bigr), \tag{1}$$

*where $\psi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(iii) *either $T$ is continuous or for any sequence $\{x_n\}$ in $X$ with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

*Then $T$ has a fixed point.*

Later on, working on these ideas a wide variety of papers appeared in the literature; see for instance [15–17]. Finally, we recall that Karapinar *et al.* [18] introduced the notion of triangular $\alpha$-admissible mapping as follows.

**Definition 2.2** ([18]) Let $\alpha : X \times X \to (-\infty, +\infty)$ be a function. We say that a self-mapping $T : X \to X$ is triangular $\alpha$-admissible if

$$\text{(i)} \quad x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1,$$

$$\text{(ii)} \quad x, y, z \in X, \quad \begin{cases} \alpha(x, z) \geq 1, \\ \alpha(z, y) \geq 1 \end{cases} \quad \Longrightarrow \quad \alpha(x, y) \geq 1.$$

For more details and applications of this line of research, we refer the reader to some related papers of the authors and others [19–25].

## 3 Main results in metric spaces

Let $A$, $B$ be two nonempty subsets of a metric space $(X,d)$. Following the usual notation, we put

$$A_0 := \{x \in A : d(x,y) = d(A,B), \text{ for some } y \in B\},$$
$$B_0 := \{y \in B : d(x,y) = d(A,B), \text{ for some } x \in A\}.$$

If $A \cap B \neq \emptyset$, then $A_0$ and $B_0$ are nonempty. Further, it is interesting to notice that $A_0$ and $B_0$ are contained in the boundaries of $A$ and $B$, respectively, provided $A$ and $B$ are closed subsets of a normed linear space such that $d(A,B) > 0$ (see [26]). Also, we will use the following definition; see [27] for more details.

**Definition 3.1** Let $A$, $B$ be two nonempty subsets of a metric space $(X,d)$. The pair $(A,B)$ is said to have the $V$-property if, for every sequence $\{y_n\}$ of $B$ that satisfies the condition $d(x,y_n) \to d(x,B)$ for some $x \in A$, there is $y \in B$ such that $d(x,y) = d(x,B)$.

From now on, denote with $\Psi$ the family of all continuous and nondecreasing functions $\psi : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ such that $\psi(x,y) = 0$ if and only if $x = y = 0$.

**Definition 3.2** Let $A$, $B$ be two nonempty subsets of a metric space $(X,d)$ and $\alpha : A \times A \to [0,+\infty)$ be a function. We say that a nonself-mapping $T : A \to B$ is triangular $\alpha$-proximal admissible if, for all $x, y, z, x_1, x_2, u_1, u_2 \in A$,

$$\text{(T1)} \quad \begin{cases} \alpha(x_1,x_2) \geq 1, \\ d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B) \end{cases} \implies \alpha(u_1,u_2) \geq 1,$$

$$\text{(T2)} \quad \begin{cases} \alpha(x,z) \geq 1, \\ \alpha(z,y) \geq 1 \end{cases} \implies \alpha(x,y) \geq 1.$$

**Definition 3.3** Let $A$, $B$ be two nonempty subsets of a metric space $(X,d)$ and $\alpha : A \times A \to [0,+\infty)$ be a function. We say that a nonself-mapping $T : A \to B$ is

(i) a modified $\alpha$-proximal $C$-contraction if, for all $u, v, x, y \in A$,

$$\begin{cases} \alpha(x,y) \geq 1, \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B) \end{cases}$$
$$\implies d(u,v) \leq \frac{1}{2}\big(d(x,v) + d(y,u)\big) - \psi\big(d(x,v), d(y,u)\big), \tag{2}$$

(ii) an $\alpha$-proximal $C$-contraction of type (I) if, for all $u, v, x, y \in A$,

$$\begin{cases} d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B) \end{cases}$$
$$\implies \alpha(x,y)d(u,v) \leq \frac{1}{2}\big(d(x,v) + d(y,u)\big) - \psi\big(d(x,v), d(y,u)\big),$$

where $0 \leq \alpha(x,y) \leq 1$ for all $x, y \in A$,

(iii) an $\alpha$-proximal $C$-contraction of type (II) if, for all $u, v, x, y \in A$,

$$\begin{cases} d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B) \end{cases}$$
$$\Longrightarrow \quad \bigl(\alpha(x, y) + \ell\bigr)^{d(u,v)} \leq (\ell + 1)^{\frac{1}{2}(d(x,v)+d(y,u))-\psi(d(x,v),d(y,u))},$$

where $\ell > 0$.

**Remark 3.1** Every $\alpha$-proximal $C$-contraction of type (I) and $\alpha$-proximal $C$-contraction of type (II) mappings are modified $\alpha$-proximal $C$-contraction mappings.

Now we give our main result.

**Theorem 3.1** *Let $A$, $B$ be two nonempty subsets of a metric space $(X, d)$ such that $A$ is complete and $A_0$ is nonempty. Assume that $T : A \to B$ is a continuous modified $\alpha$-proximal $C$-contraction such that the following conditions hold*:

(i) *$T$ is a triangular $\alpha$-proximal admissible mapping and $T(A_0) \subseteq B_0$,*
(ii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1.$$

*Then $T$ has a best proximity point. Further, the best proximity point is unique if, for every $x, y \in A$ such that $d(x, Tx) = d(A, B) = d(y, Ty)$, we have $\alpha(x, y) \geq 1$.*

*Proof* By (ii) there exist $x_0, x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1.$$

On the other hand, $T(A_0) \subseteq B_0$, then there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

Now, since $T$ is triangular $\alpha$-proximal admissible, we have $\alpha(x_1, x_2) \geq 1$. Thus

$$d(x_2, Tx_1) = d(A, B) \quad \text{and} \quad \alpha(x_1, x_2) \geq 1.$$

Since $T(A_0) \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3, Tx_2) = d(A, B).$$

Then we have

$$d(x_2, Tx_1) = d(A, B), \qquad d(x_3, Tx_2) = d(A, B), \qquad \alpha(x_1, x_2) \geq 1.$$

Again, since $T$ is triangular $\alpha$-proximal admissible, we obtain $\alpha(x_2, x_3) \geq 1$ and hence

$$d(x_3, Tx_2) = d(A, B), \qquad \alpha(x_2, x_3) \geq 1.$$

By continuing this process, we construct a sequence $\{x_n\}$ such that

$$\begin{cases} \alpha(x_{n-1}, x_n) \geq 1, \\ d(x_n, Tx_{n-1}) = d(A, B), \\ d(x_{n+1}, Tx_n) = d(A, B), \end{cases} \tag{3}$$

for all $n \in \mathbb{N}$. Now, from (2) with $u = x_n$, $v = x_{n+1}$, $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{1}{2}\big(d(x_{n-1}, x_{n+1}) + d(x_n, x_n)\big) - \psi\big(d(x_{n-1}, x_{n+1}), d(x_n, x_n)\big) \\ &= \frac{1}{2}d(x_{n-1}, x_{n+1}) - \psi\big(d(x_{n-1}, x_{n+1}), 0\big) \\ &\leq \frac{1}{2}d(x_{n-1}, x_{n+1}) \\ &\leq \frac{1}{2}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big), \end{aligned} \tag{4}$$

which implies $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$. It follows that the sequence $\{d_n\}$, where $d_n := d(x_n, x_{n+1})$, is decreasing and so there exists $d \geq 0$ such that $d_n \to d$ as $n \to +\infty$. Then, taking the limit as $n \to +\infty$ in (4), we obtain

$$d \leq \frac{1}{2} \lim_{n \to +\infty} d(x_{n-1}, x_{n+1}) \leq \frac{1}{2}(d + d) = d,$$

that is,

$$\lim_{n \to +\infty} d(x_{n-1}, x_{n+1}) = 2d. \tag{5}$$

Again taking the limit as $n \to +\infty$ in (4), by (5) and the continuity of $\psi$, we get

$$d \leq d - \psi(2d, 0),$$

and so $\psi(2d, 0) = 0$. Therefore, by the property of $\psi$, we get $d = 0$, that is,

$$\lim_{n \to +\infty} d(x_{n+1}, x_n) = 0. \tag{6}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there are $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers $k$

$$n(k) > m(k) > k, \quad d(x_{n(k)}, x_{m(k)}) \geq \varepsilon, \qquad d(x_{n(k)-1}, x_{m(k)}) < \varepsilon.$$

This implies that, for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon &\leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \\ &< d(x_{n(k)}, x_{n(k)-1}) + \varepsilon. \end{aligned}$$

Taking the limit as $k \to +\infty$ in the above inequality and using (6), we get

$$\lim_{k\to+\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{7}$$

Again, from

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})$$

and

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)+1}, x_{n(k)}),$$

taking the limit as $k \to +\infty$, by (6) and (7) we deduce

$$\lim_{k\to+\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon. \tag{8}$$

Similarly, we deduce

$$\lim_{k\to+\infty} d(x_{n(k)}, x_{m(k)+1}) = \varepsilon \tag{9}$$

and

$$\lim_{k\to+\infty} d(x_{m(k)}, x_{n(k)+1}) = \varepsilon. \tag{10}$$

We shall show that

$$\alpha(x_{m(k)}, x_{n(k)}) \geq 1, \quad \text{where } n(k) > m(k) > k. \tag{11}$$

Since $T$ is a triangular $\alpha$-proximal admissible mapping and

$$\begin{cases} \alpha(x_{m(k)}, x_{m(k)+1}) \geq 1, \\ \alpha(x_{m(k)+1}, x_{m(k)+2}) \geq 1, \end{cases}$$

by (T2) of Definition 3.2, we have

$$\alpha(x_{m(k)}, x_{m(k)+2}) \geq 1.$$

Again, since $T$ is a triangular $\alpha$-proximal admissible mapping and

$$\begin{cases} \alpha(x_{m(k)}, x_{m(k)+2}) \geq 1, \\ \alpha(x_{m(k)+2}, x_{m(k)+3}) \geq 1, \end{cases}$$

by (T2) of Definition 3.2 we have

$$\alpha(x_{m(k)}, x_{m(k)+3}) \geq 1.$$

Thus, by continuing this process, we get (11).

On the other hand, we know that

$$\begin{cases} d(x_{m(k)+1}, Tx_{m(k)}) = d(A,B), \\ d(x_{n(k)+1}, Tx_{n(k)}) = d(A,B). \end{cases}$$

Therefore, from (2) we have

$$\begin{aligned} d(x_{m(k)+1}, x_{n(k)+1}) \le{}& \frac{1}{2}\big(d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1})\big) \\ &- \psi\big(d(x_{m(k)}, x_{n(k)+1}), d(x_{n(k)}, x_{m(k)+1})\big). \end{aligned}$$

Taking the limit as $k \to +\infty$ in the above inequality and using (8), (9), (10) and the continuity of $\psi$, we get

$$\varepsilon \le \frac{1}{2}(\varepsilon + \varepsilon) - \psi(\varepsilon, \varepsilon)$$

and hence $\psi(\varepsilon, \varepsilon) = 0$, which leads to the contradiction $\varepsilon = 0$. Thus, $\{x_n\}$ is a Cauchy sequence. Since $A$ is complete, then there is $z \in A$ such that $x_n \to z$. Now, from

$$d(x_{n+1}, Tx_n) = d(A,B), \quad \text{for all } n \in \mathbb{N} \cup \{0\},$$

taking the limit as $n \to +\infty$, we deduce $d(z, Tz) = d(A,B)$, because of the continuity of $T$.

Finally we prove the uniqueness of the point $x \in A$ such that $d(x, Tx) = d(A,B)$. Indeed, suppose that there exist $x, y \in A$ which are best proximity points, that is, $d(x, Tx) = d(A,B) = d(y, Ty)$. Since $\alpha(x,y) \ge 1$, we have

$$\begin{aligned} d(x,y) &\le \frac{1}{2}\big(d(x,y) + d(y,x)\big) - \psi\big(d(x,y), d(y,x)\big) \\ &= d(x,y) - \psi\big(d(x,y), d(x,y)\big), \end{aligned}$$

which implies $d(x,y) = 0$, that is, $x = y$. □

**Corollary 3.1** *Let $A$, $B$ be two nonempty subsets of a metric space $(X,d)$ such that $A$ is complete and $A_0$ is nonempty. Assume that $T : A \to B$ is a continuous $\alpha$-proximal $C$-contraction mapping of type $(I)$ or a continuous $\alpha$-proximal $C$-contraction mapping of type $(II)$ such that the following conditions hold*:

(i) *$T$ is a triangular $\alpha$-proximal admissible mapping and $T(A_0) \subseteq B_0$*,

(ii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B) \quad \textit{and} \quad \alpha(x_0, x_1) \ge 1.$$

*Then $T$ has a best proximity point. Further, the best proximity point is unique if, for every $x, y \in A$ such that $d(x, Tx) = d(A,B) = d(y, Ty)$, we have $\alpha(x,y) \ge 1$.*

In analogy to the main result but omitting the continuity hypothesis of $T$, we can state the following theorem.

**Theorem 3.2** *Let $A$, $B$ be two nonempty subsets of a metric space $(X,d)$ such that $A$ is complete, the pair $(A,B)$ has the $V$-property and $A_0$ is nonempty. Assume that $T : A \to B$ is a modified $\alpha$-proximal $C$-contraction such that the following conditions hold*:

(i) *$T$ is a triangular $\alpha$-proximal admissible mapping and $T(A_0) \subseteq B_0$,*

(ii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1,$$

(iii) *if $\{x_n\}$ is a sequence in $A$ such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x \in A$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

*Then $T$ has a best proximity point. Further, the best proximity point is unique if, for every $x, y \in A$ such that $d(x, Tx) = d(A,B) = d(y, Ty)$, we have $\alpha(x,y) \geq 1$.*

*Proof* Following the proof of Theorem 3.1, there exist a Cauchy sequence $\{x_n\} \subseteq A$ and $z \in A$ such that (3) holds and $x_n \to z$ as $n \to +\infty$. On the other hand, for all $n \in \mathbb{N}$, we can write

$$\begin{aligned} d(z,B) &\leq d(z, Tx_n) \\ &\leq d(z, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &= d(z, x_{n+1}) + d(A,B). \end{aligned}$$

Taking the limit as $n \to +\infty$ in the above inequality, we get

$$\lim_{n \to +\infty} d(z, Tx_n) = d(z,B) = d(A,B). \tag{12}$$

Since the pair $(A,B)$ has the $V$-property, then there exists $w \in B$ such that $d(z,w) = d(A,B)$ and hence $z \in A_0$. Moreover, since $T(A_0) \subseteq B_0$, then there exists $v \in A$ such that

$$d(v, Tz) = d(A,B).$$

Now, by (iii) and (3), we have $\alpha(x_n, z) \geq 1$ and $d(x_{n+1}, Tx_n) = d(A,B)$ for all $n \in \mathbb{N} \cup \{0\}$. Also, since $T$ is a modified $\alpha$-proximal $C$-contraction, we get

$$d(x_{n+1}, v) \leq \frac{1}{2}\big(d(x_n, v) + d(z, x_{n+1})\big) - \psi\big(d(x_n, v), d(z, x_{n+1})\big).$$

Taking the limit as $n \to +\infty$ in the above inequality, we have

$$d(z,v) \leq \frac{1}{2} d(z,v) - \psi\big(d(z,v), 0\big)$$

which implies, $d(z,v) = 0$, that is, $v = z$. Hence $z$ is a best proximity point of $T$. The uniqueness of the best proximity point follows easily proceeding as in Theorem 3.1. □

Next, we use an example to illustrate the efficiency of the new theorem.

**Example 3.1** Let $X = \mathbb{R}$ be endowed with the usual metric $d(x,y) = |x - y|$, for all $x, y \in X$. Consider $A = (-\infty, -1]$, $B = [1, +\infty)$ and define $T : A \to B$ by

$$Tx = \begin{cases} -x + 1, & \text{if } x \in (-\infty, -14), \\ x^2 + 1, & \text{if } x \in [-14, -12), \\ 4x^4 + 5, & \text{if } x \in [-12, -10), \\ -x^3 + 2, & \text{if } x \in [-10, -8), \\ 10, & \text{if } x \in [-8, -6), \\ \ln(|x| + 1), & \text{if } x \in [-6, -4), \\ -x + |x + 3||x + 4|e^{-x}, & \text{if } x \in [-4, -2), \\ 1, & \text{if } x \in [-2, -1]. \end{cases}$$

Also, define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 4, & \text{if } x, y \in [-2, -1], \\ \frac{1}{2}, & \text{otherwise,} \end{cases}$$

and $\psi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ by

$$\psi(s,t) = \frac{1}{2}(s + t), \quad \text{for all } s, t \in X.$$

Clearly, the pair $(A, B)$ has the $V$-property and $d(A, B) = 2$. Now, we have

$$A_0 = \bigl\{x \in A : d(x,y) = d(A,B) = 2, \text{ for some } y \in B\bigr\} = \{-1\},$$
$$B_0 = \bigl\{y \in B : d(x,y) = d(A,B) = 2, \text{ for some } x \in A\bigr\} = \{1\}.$$

It is immediate to see that $T(A_0) \subseteq B_0$, $d(-1, T(-1)) = d(A, B) = 2$ and $\alpha(-1, -1) \geq 1$.

Now, let $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$. Therefore, $x, y, z \in [-2, -1]$, that is, $\alpha(x, z) \geq 1$. Also suppose

$$\begin{cases} \alpha(x,y) \geq 1, \\ d(u, Tx) = d(A,B) = 2, \\ d(v, Ty) = d(A,B) = 2, \end{cases}$$

then

$$\begin{cases} x, y \in [-2, -1], \\ d(u, Tx) = 2, \\ d(v, Ty) = 2. \end{cases}$$

Hence, $u = v = -1$, that is, $\alpha(u, v) \geq 1$. Further,

$$d(u,v) = 0 \leq \frac{1}{2}\bigl(d(x,v) + d(y,u)\bigr) - \psi\bigl(d(x,v), d(y,u)\bigr),$$

that is, $T$ is a triangular $\alpha$-proximal admissible and modified $\alpha$-proximal $C$-contraction mapping. Moreover, if $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\{x_n\} \subseteq [-2, -1]$ and hence $x \in [-2, -1]$. Consequently, $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore all the conditions of Theorem 3.2 hold for this example and $T$ has a best proximity point. Here $z = -1$ is the best proximity point of $T$.

We conclude this section with another corollary.

**Corollary 3.2** *Let $A$, $B$ be two nonempty subsets of a metric space $(X, d)$ such that $A$ is complete, the pair $(A, B)$ has the $V$-property and $A_0$ is nonempty. Assume that $T : A \to B$ is a continuous $\alpha$-proximal $C$-contraction mapping of type (I) or a continuous $\alpha$-proximal $C$-contraction mapping of type (II) such that the following conditions hold*:

(i) *$T$ is a triangular $\alpha$-proximal admissible mapping and $T(A_0) \subseteq B_0$,*

(ii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1,$$

(iii) *if $\{x_n\}$ is a sequence in $A$ such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x \in A$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

*Then $T$ has a best proximity point. Further, the best proximity point is unique if, for every $x, y \in A$ such that $d(x, Tx) = d(A, B) = d(y, Ty)$, we have $\alpha(x, y) \geq 1$.*

## 4 Some results in metric spaces endowed with a graph

Consistent with Jachymski [28], let $(X, d)$ be a metric space and $\Delta$ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph $G$ such that the set $V(G)$ of its vertices coincides with $X$, and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. We assume that $G$ has no parallel edges, so we can identify $G$ with the pair $(V(G), E(G))$. Moreover, we may treat $G$ as a weighted graph (see [29], p.309) by assigning to each edge the distance between its vertices. If $x$ and $y$ are vertices in a graph $G$, then a path in $G$ from $x$ to $y$ of length $N$ ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N + 1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, N$. A graph $G$ is connected if there is a path between any two vertices. $G$ is weakly connected if $\tilde{G}$ is connected (see for details [28, 30]).

Recently, some results have appeared providing sufficient conditions for a mapping to be a Picard operator if $(X, d)$ is endowed with a graph. The first result in this direction was given by Jachymski [28].

**Definition 4.1** ([28]) Let $(X, d)$ be a metric space endowed with a graph $G$. We say that a self-mapping $T : X \to X$ is a Banach $G$-contraction or simply a $G$-contraction if $T$ preserves the edges of $G$, that is,

$$\text{for all } x, y \in X, \quad (x, y) \in E(G) \quad \Longrightarrow \quad (Tx, Ty) \in E(G)$$

and $T$ decreases weights of the edges of $G$ in the following way:

$$\exists \alpha \in (0, 1), \quad \text{for all } x, y \in X, (x, y) \in E(G) \quad \Longrightarrow \quad d(Tx, Ty) \leq \alpha d(x, y).$$

**Definition 4.2** Let $A, B$ be two nonempty closed subsets of a metric space $(X, d)$ endowed with a graph $G$. We say that a nonself-mapping $T : A \to B$ is a $G$-proximal $C$-contraction if, for all $u, v, x, y \in A$,

$$\begin{cases} (x,y) \in E(G), \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B) \end{cases}$$
$$\implies \quad d(u,v) \leq \frac{1}{2}\bigl(d(x,v) + d(y,u)\bigr) - \psi\bigl(d(x,v), d(y,u)\bigr)$$

and

$$\begin{cases} (x,y) \in E(G), \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B) \end{cases} \quad \implies \quad (u,v) \in E(G).$$

**Theorem 4.1** *Let $A$, $B$ be two nonempty closed subsets of a metric space $(X, d)$ endowed with a graph $G$. Assume that $A$ is complete, $A_0$ is nonempty and $T : A \to B$ is a continuous $G$-proximal $C$-contraction mapping such that the following conditions hold*:

(i) $T(A_0) \subseteq B_0$,

(ii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B) \quad \textit{and} \quad (x_0, x_1) \in E(G),$$

(iii) *for all $(x,y) \in E(G)$ and $(y,z) \in E(G)$, we have $(x,z) \in E(G)$.*

*Then $T$ has a best proximity point. Further, the best proximity point is unique if, for every $x, y \in A$ such that $d(x, Tx) = d(A,B) = d(y, Ty)$, we have $(x,y) \in E(G)$.*

*Proof* Define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Firstly we prove that $T$ is a triangular $\alpha$-proximal admissible mapping. To this aim, assume

$$\begin{cases} \alpha(x,y) \geq 1, \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B). \end{cases}$$

Therefore, we have

$$\begin{cases} (x,y) \in E(G), \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B). \end{cases}$$

Since $T$ is a $G$-proximal $C$-contraction mapping, we get $(u,v) \in E(G)$, that is, $\alpha(u,v) \geq 1$ and

$$d(u,v) \leq \frac{1}{2}\big(d(x,v) + d(y,u)\big) - \psi\big(d(x,v), d(y,u)\big).$$

Also, let $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$, then $(x,z) \in E(G)$ and $(z,y) \in E(G)$. Consequently, from (iii), we deduce that $(x,y) \in E(G)$, that is, $\alpha(x,y) \geq 1$. Thus $T$ is a triangular $\alpha$-proximal admissible mapping with $T(A_0) \subseteq B_0$. Moreover, $T$ is a continuous modified $\alpha$-proximal $C$-contraction. From (ii) there exist $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A,B)$ and $(x_0,x_1) \in E(G)$, that is, $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq 1$. Hence, all the conditions of Theorem 3.1 are satisfied and $T$ has a unique fixed point. □

Similarly, by using Theorem 3.2, we can prove the following theorem.

**Theorem 4.2** *Let $A$, $B$ be two nonempty closed subsets of a metric space $(X,d)$ endowed with a graph $G$. Assume that $A$ is complete, the pair $(A,B)$ has the $V$-property and $A_0$ is nonempty. Also suppose that $T : A \to B$ is a $G$-proximal $C$-contraction mapping such that the following conditions hold*:

(i) $T(A_0) \subseteq B_0$,

(ii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B) \quad \textit{and} \quad (x_0,x_1) \in E(G),$$

(iii) *for all $(x,y) \in E(G)$ and $(y,z) \in E(G)$, we have $(x,z) \in E(G)$*,

(iv) *if $\{x_n\}$ is a sequence in $X$ such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$.*

*Then $T$ has a best proximity point. Further, the best proximity point is unique if, for every $x, y \in A$ such that $d(x, Tx) = d(A,B) = d(y, Ty)$, we have $(x,y) \in E(G)$.*

## 5 Some results in partially ordered metric spaces

In recent years, Ran and Reurings [31] initiated the study of weaker contraction conditions by considering self-mappings in partially ordered metric space. Further these results were generalized by many authors; see for instance [32, 33]. Here we consider some recent results of Mongkolkeha *et al.* [34] and Sadiq Basha *et al.* [35].

**Definition 5.1** ([35]) Let $(X,d,\preceq)$ be a partially ordered metric space. We say that a nonself-mapping $T : A \to B$ is proximally ordered-preserving if and only if, for all $x_1, x_2, u_1, u_2 \in A$,

$$\begin{cases} x_1 \preceq x_2, \\ d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B) \end{cases} \implies \quad u_1 \preceq u_2.$$

**Theorem 5.1** (Theorem 2.2 of [34]) *Let $A$, $B$ be two nonempty closed subsets of a partially ordered complete metric space $(X,d,\preceq)$ such that $A_0$ is nonempty. Assume that $T : A \to B$ satisfies the following conditions*:

(i) *$T$ is continuous and proximally ordered-preserving such that $T(A_0) \subseteq B_0$*,

(ii) *there exist elements* $x_0, x_1 \in A_0$ *such that*

$$d(x_1, Tx_0) = d(A,B) \quad \textit{and} \quad x_0 \preceq x_1,$$

(iii) *for all* $x, y, u, v \in A$,

$$\begin{cases} x \preceq y, \\ d(u,Tx) = d(A,B), \\ d(y,Ty) = d(A,B) \end{cases} \implies d(u,v) \leq \frac{1}{2}\big(d(x,v) + d(y,u)\big) - \psi\big(d(x,v), d(y,u)\big).$$

*Then* $T$ *has a best proximity point.*

*Proof* Define $\alpha : A \times A \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

Firstly we prove that $T$ is a triangular $\alpha$-proximal admissible mapping. To this aim, assume

$$\begin{cases} \alpha(x,y) \geq 1, \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B). \end{cases}$$

Therefore, we have

$$\begin{cases} x \preceq y, \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B). \end{cases}$$

Now, since $T$ is proximally ordered-preserving, then $u \preceq v$, that is, $\alpha(u,v) \geq 1$. Consequently, condition (T1) of Definition 3.2 holds. Also, assume

$$\begin{cases} \alpha(x,z) \geq 1, \\ \alpha(z,y) \geq 1, \end{cases}$$

so that $\left\{\begin{smallmatrix} x \preceq z, \\ z \preceq y, \end{smallmatrix}\right.$ and consequently $x \preceq y$, that is, $\alpha(x,y) \geq 1$. Hence, condition (T2) of Definition 3.2 holds. Further, by (ii) we have

$$d(x_1, Tx_0) = d(A,B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1.$$

Moreover, from (iii) we get

$$\begin{cases} \alpha(x,y) \geq 1, \\ d(u,Tx) = d(A,B), \\ d(y,Ty) = d(A,B) \end{cases} \implies d(u,v) \leq \frac{1}{2}\big(d(x,v) + d(y,u)\big) - \psi\big(d(x,v), d(y,u)\big).$$

Thus all the conditions of Theorem 3.1 hold and $T$ has a best proximity point. □

Similarly, omitting the continuity hypothesis of $T$, we can give the following result.

**Theorem 5.2** (see Theorem 2.6 of [34]) *Let $A$, $B$ be two nonempty closed subsets of a partially ordered complete metric space $(X,d,\preceq)$ such that $A_0$ is nonempty and the pair $(A,B)$ has the $V$-property. Assume that $T:A\to B$ satisfies the following conditions*:

(i) *$T$ is proximally ordered-preserving such that $T(A_0)\subseteq B_0$,*

(ii) *there exist elements $x_0,x_1\in A_0$ such that*

$$d(x_1,Tx_0)=d(A,B) \quad \textit{and} \quad x_0\preceq x_1,$$

(iii) *for all $x,y,u,v\in A$,*

$$\begin{cases} x\preceq y, \\ d(u,Tx)=d(A,B), \\ d(y,Ty)=d(A,B) \end{cases} \implies d(u,v)\le\frac{1}{2}\bigl(d(x,v)+d(y,u)\bigr)-\psi\bigl(d(x,v),d(y,u)\bigr),$$

(iv) *if $\{x_n\}$ is an increasing sequence in $A$ converging to $x\in A$, then $x_n\preceq x$ for all $n\in\mathbb{N}$.*

*Then $T$ has a best proximity point.*

## 6 Application to fixed point theorems

In this section we briefly collect some fixed point results which are consequences of the results presented in the main section. Stated precisely, from Theorem 3.1, we obtain the following theorems.

**Theorem 6.1** *Let $(X,d)$ be a complete metric space. Assume that $T:X\to X$ is a continuous self-mapping satisfying the following conditions*:

(i) *$T$ is triangular $\alpha$-admissible,*

(ii) *there exists $x_0$ in $X$ such that $\alpha(x_0,Tx_0)\ge 1$,*

(iii) *for all $x,y\in X$,*

$$\alpha(x,y)d(Tx,Ty)\le\frac{1}{2}\bigl(d(x,Ty)+d(y,Tx)\bigr)-\psi\bigl(d(x,Ty),d(y,Tx)\bigr).$$

*Then $T$ has a fixed point.*

**Theorem 6.2** *Let $(X,d)$ be a complete metric space. Assume that $T:X\to X$ is a continuous self-mapping satisfying the following conditions*:

(i) *$T$ is triangular $\alpha$-admissible,*

(ii) *there exists $x_0$ in $X$ such that $\alpha(x_0,Tx_0)\ge 1$,*

(iii) *for all $x,y\in X$,*

$$\bigl(\alpha(x,y)+\ell\bigr)^{d(Tx,Ty)}\le(\ell+1)^{\frac{1}{2}(d(x,Ty)+d(y,Tx))-\psi(d(x,Ty),d(y,Tx))},$$

*where $\ell>0$.*

*Then $T$ has a fixed point.*

Analogously, from Theorem 3.2, we obtain the following theorems, which do not require the continuity of $T$.

**Theorem 6.3** *Let* $(X,d)$ *be a complete metric space. Assume that* $T : X \to X$ *is a self-mapping satisfying the following conditions*:

(i) $T$ *is triangular* $\alpha$*-admissible*,

(ii) *there exists* $x_0$ *in* $X$ *such that* $\alpha(x_0, Tx_0) \geq 1$,

(iii) *for all* $x, y \in X$,

$$\alpha(x,y)d(Tx,Ty) \leq \frac{1}{2}\bigl(d(x,Ty) + d(y,Tx)\bigr) - \psi\bigl(d(x,Ty), d(y,Tx)\bigr),$$

(iv) *if* $\{x_n\}$ *is a sequence in* $X$ *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *and* $x_n \to x$ *as* $n \to +\infty$, *then* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$.

*Then* $T$ *has a fixed point*.

**Theorem 6.4** *Let* $(X,d)$ *be a complete metric space. Assume that* $T : X \to X$ *is a self-mapping satisfying the following conditions*:

(i) $T$ *is triangular* $\alpha$*-admissible*,

(ii) *there exists* $x_0$ *in* $X$ *such that* $\alpha(x_0, Tx_0) \geq 1$,

(iii) *for all* $x, y \in X$,

$$\bigl(\alpha(x,y) + 1\bigr)^{d(Tx,Ty)} \leq 2^{[\frac{1}{2}(d(x,Ty)+d(y,Tx)) - \psi(d(x,Ty),d(y,Tx))]},$$

(iv) *if* $\{x_n\}$ *is a sequence in* $A$ *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *and* $x_n \to x \in A$ *as* $n \to +\infty$, *then* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$.

*Then* $T$ *has a fixed point*.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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**Acknowledgements**
First author was supported by the Commission on Higher Education, the Thailand Research Fund and the King Mongkut's University of Technology Thonburi (Grant no. MRG5580213). Third author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Received: 14 October 2013 Accepted: 20 March 2014 Published: 16 Apr 2014

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10.1186/1687-1812-2014-99
**Cite this article as:** Kumam et al.: Best proximity point results for modified $\alpha$-proximal $C$-contraction mappings. *Fixed Point Theory and Applications* 2014, 2014:99