รายงานวิจัยฉบับสมบูรณ์

โครงการ : การลู่เข้าแบบเข้มสำหรับการหาค่าที่เหมาะที่สุดของปัญหาระดับขั้นในปริภูมิฮิลแบรต์

ดร. ธัญญรัตน์ จิตรพีระ

มีนาคม 2558

สัญญาเลขที่ MRG5680157

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ดร. ธัญญรัตน์ จิตรพีระ
มหาวิทยาลัยเทคโนโลยีราชมงคลล้านนา

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว.ไม่จำเป็นต้องเห็นด้วยเสมอไป)

บทคัดย่อ

รหัสโครงการ : MRG5680157

ชื่อโครงการ : การลู่เข้าแบบเข้มสำหรับการหาค่าที่เหมาะที่สุดของปัญหาระดับขั้นในปริภูมิฮิลแบรต์

ชื่อนักวิจัย : ดร. ธัญญรัตน์ จิตรพีระ มหาวิทยาลัยเทคโนโลยีราชมงคลล้านนา

E-mail Address : t.jitpeera@hotmail.com

ระยะเวลาโครงการ : 3 มิถุนายน 2556 ถึง 2 มิถุนายน 2558

โครงการนี้มุ่งเน้นด้วยเทคนิค viscosity, steepest descent, project สำหรับการหาคำตอบของปัญหาระดับขั้น ปัญหาจุดตรึง ปัญหาอสมการเชิงดุลยภาพ เราจะแสดงทฤษฎีที่ลู่เข้าแบบเข้มโดยใช้การส่งแบบไม่ขยาย การส่งแบบเข้มทางเดียว การส่งแบบต่อเนื่องลิปชิกส์และการส่งเชิงเส้นมีขอบในปริภูมิฮิลแบรต์ สำหรับบทประยุกต์เรานำผลลัพธ์เพื่อใช้ศึกษาปัญหาของอสมการเชิงแปรผันทางเดียวและค่าต่ำสุดของปัญหาระดับขั้นบนเซตจุดตรึง

คำหลัก : จุดตรึง ปัญหาระดับขั้น ปริภูมิฮิลแบรต์

Abstract

Project Code : MRG5680157

Project Title : Strong convergence for hierarchical constrained optimization problems in Hilbert spaces

Investigator : Dr. Thanyarat Jitpeera, Rajamangala University Technology of Lanna

E-mail Address : t.jitpeera@hotmail.com

Project Period : 3 June 2013 – 2 June 2015

This project focus on the techniques of viscosity method, steepest descent method, projection method for solving the hierarchical problem, fixed point and variational inequalities. We show the strong convergent theorem by using nonexpansive mapping, strongly monotone, Lipschitz continuous and linear bounded operator in Hilbert space. As applications, we use the results to study problems of the monotone variational inequality and the hierarchical minimization over fixed point sets.

Keywords : Fixed Point, Hierarchical Problem, Hilbert Spaces

วัตถุประสงค์ของโครงการ

- สร้างทฤษฎีบทใหม่สำหรับการหาและแสดงการพิสูจน์การลู่เข้าแบบเข้มของกระบวนการทำซ้ำของปัญหาระดับขั้นของจุดตรึง และปัญหาระดับขั้นของดุลยภาพที่เป็นการส่งแบบไม่ขยาย
- นำผลเฉลยที่ได้ไปประยุกต์ใช้เพื่อที่จะประมาณค่าปัญหาระดับขั้นของจุดตรึงและปัญหาระดับขั้นของดุลยภาพ

วิธีการทดลอง

1. วิจัยและศึกษาพื้นฐานทฤษฎีบทของจุดตรึง ปัญหาดุลยภาพทั่วไปผสม ปัญหาอสมการเชิงแปรผันและปัญหาจุดตรึงระดับขั้นจากหนังสือและวารสารงานวิจัย
2. ค้นหาเงื่อนไขของการส่งแบบเข้มทางเดียวและการส่งแบบลิปชิกส์เพื่อใช้ในการพิสูจน์การลู่เข้าแบบเข้ม
3. แนะนำกระบวนการทำซ้ำแบบใหม่และพิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มเพื่อหาเซตคำตอบของปัญหาจุดตรึงในปริภูมิฮิลแบรต์
4. ส่งผลงานวิจัยเพื่อให้ได้รับการตีพิมพ์ในวารสารระดับนานาชาติ

ผลการทดลอง

สามารถสร้างกระบวนการทำซ้ำแบบใหม่ ให้ C เป็นเซตปิด นูน โดยเป็นเซตย่อยของปริภูมิฮิลแบรต์ ให้ $F:C\to C$ เป็นการส่งลิปชิกส์และการส่งแบบเข้มทางเดียว $\phi:C\to C$ เป็นการส่งแบบหดตัว ให้ $T:C\to C$ เป็นการส่งแบบไม่ขยายโดยที่ $F(T)\neq\phi$ และ $S:H\to H$ เป็นการส่งแบบไม่ขยาย สมมติให้ $\{x_n\}$ เป็นลำดับโดยที่ $x_0\in C$ เป็นค่าคงที่

$$y_n = P_C[\beta_n Sx_n + (1-\beta_n)x_n],$$
$$x_{n+1} = \gamma\lambda_n\phi(x_n) + (I-\lambda_n\mu F)Ty_n, \forall n\geq 0.$$

โดยที่ $\{\beta_n\},\{\lambda_n\}\subset(0,1)$ สอดคล้องกับเงื่อนไข

$(C1):\beta_n\leq k\lambda_n;$

$(C2):\lim_{n\to\infty}\lambda_n=0, \lim_{n\to\infty}((\lambda_n-\lambda_{n-1})/\lambda_n)=0, \sum_{n=0}^{\infty}\lambda_n=\infty;$

$(C3):\lim_{n\to\infty}((\beta_n-\beta_{n-1})/\beta_n)=0.$

แล้ว $\{x_n\}$ ลู่เข้าแบบเข้มไปยัง $x^*\in\Omega,$ ที่เป็นเพียงคำตอบเดียวของปัญหาอสมการเชิงแปรผัน $\langle(\mu F-\gamma\phi)x^*, x-x^*\rangle\geq 0, \forall x\in\Omega, \quad \Omega=VI(F(T),S)\neq\phi$

สามารถสร้างกระบวนการทำซ้ำแบบใหม่ ให้ C เป็นเซตปิด นูน โดยเป็นเซตย่อยของปริภูมิฮิลแบรต์ ให้ $F: C \times C \to R$ เป็น bifunction สอดคล้อง (A1)-(A5) ให้ $\varphi: C \to R \cup \{+\infty\}$ เป็น proper lower semicontinuous and convex function ให้ $T_i : C \to C$ เป็นการส่งแบบไม่ขยายโดยที่

$$\Theta = \bigcap_{i=1}^{\infty} F(T_i) \cap SQVI(B_1, M_1, B_2, M_2) \cap MEP(F, \varphi) \neq \phi$$

และ $f: C \to C$ เป็นการส่งแบบหดตัว ให้ Q, E_1, E_2 เป็นการส่งแบบเข้มทางเดียว ให้ A เป็น strongly positive linear bounded linear self-adjoint ให้ $M_1, M_2 : H \to 2^H$ เป็น maximal monotone สมมติให้ $\{x_n\}$ เป็นลำดับโดยที่ $x_0 \in C$ เป็นค่าคงที่

$$\begin{aligned}
&F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \forall y \in C \\
&z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\
&y_n = J_{M_1,\lambda}(z_n - \mu E_1 z_n), \\
&x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((I - \beta_n) I - \alpha_n A) W_n y_n, \forall n \geq 0.
\end{aligned}$$

โดยที่ $\{\beta_n\}, \{\lambda_n\} \subset (0,1), \lambda \in (0, 2\eta_1), \mu \in (0, 2\eta_2), r \in (0, 2\delta)$ สอดคล้องกับเงื่อนไข

$$(C1): \sum_{n=0}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0;$$

$$(C2): 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1;$$

$$(C3): \lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0, \forall i = 1, 2, \ldots N.$$

แล้ว $\{x_n\}$ ลู่เข้าแบบเข้มไปยัง $x^* \in \Theta, \quad x^* = P_{\Theta}(\gamma f + I - A)(x^*)$

สรุปผลการทดลอง

ผลการดำเนินงานของโครงการ การลู่เข้าแบบเข้มสำหรับการค่าที่เหมาะที่สุดของปัญหาระดับขั้นในปริภูมิฮิลแบร์ต เราสร้างทฤษฎีบทใหม่ที่พิสูจน์การลู่เข้าแบบเข้ม โดยหาผลเฉลยของปัญหาระดับขั้นจุดตรึง ปัญหาระดับขั้นดุลยภาพ เริ่มจากศึกษาความรู้พื้นฐานทางทฤษฎีจุดตรึงเพื่อใช้ในการแก้ปัญหาที่ต่างกัน และพิจารณาการส่งชนิดต่างๆ ที่สามารถพิสูจน์การลู่เข้าแบบเข้มได้ เพื่อจะสร้างกระบวนการทำซ้ำแบบใหม่นัยทั่วไป เพื่อใช้ในการประมาณค่าที่เหมาะที่สุดของปัญหาภายใต้เงื่อนไขที่เหมาะสม

ข้อเสนอแนะ

พัฒนาโครงการอย่างต่อเนื่อง เพื่อแสดงการลู่เข้าแบบเข้มสำหรับประยุกต์ใช้กับปัญหาอื่นๆ เป็นการปรับปรุง ต่อยอดให้งานวิจัยสำหรับการส่งชนิดอื่น สำหรับกระบวนการทำซ้ำทั่วไป

ภาคผนวก

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Research Article

Strong Convergence of an Iterative Algorithm for Hierarchical Problems

Poom Kumam[1] and Thanyarat Jitpeera[2]

[1] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Bang Mod, Thung Khru, Bangkok 10140, Thailand*
[2] *Department of Mathematics, Faculty of Science and Agriculture, Rajamangala University of Technology Lanna, Phan, Chiangrai 57120, Thailand*

Correspondence should be addressed to Thanyarat Jitpeera; t.jitpeera@hotmail.com

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We introduce the triple hierarchical problem over the solution set of the variational inequality problem and the fixed point set of a nonexpansive mapping. The strong convergence of the algorithm is proved under some mild conditions. Our results extend those of Yao et al., Iiduka, Ceng et al., and other authors.

1. Introduction

Let C be a closed convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\rightarrow$, respectively. Let A be a nonlinear mapping. The *Hartman-Stampacchia variational inequality* [1] is to find $x \in C$ such that $\langle Ax, y - x \rangle \geq 0, \forall y \in C$. The set of solutions is denoted by $\mathrm{VI}(C, A)$. $f : C \rightarrow C$ is said to be a *ρ-contraction* if there exists a constant $\rho \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|, \forall x, y \in C$. A mapping $A : H \rightarrow H$ is said to be *monotone* if $\langle Ax - Ay, x - y \rangle \geq 0, \forall x, y \in H$. A mapping $A : H \rightarrow H$ is said to be *α- strongly monotone* if there exists a positive real number α such that $\langle Ax - Ay, x - y \rangle \geq \alpha\|x - y\|^2, \forall x, y \in H$. A mapping $A : H \rightarrow H$ is said to be *β-inverse-strongly monotone* if there exists a positive real number β such that $\langle Ax - Ay, x - y \rangle \geq \beta\|Ax - Ay\|^2, \forall x, y \in H$. A mapping $A : H \rightarrow H$ is said to be *L-Lipschitz continuous* if there exists a positive real number L such that $\|Ax - Ay\| \leq L\|x - y\|, \forall x, y \in H$. A linear bounded operator A is said to be *strongly positive* on H if there exists a constant $\overline{\gamma} > 0$ with the property $\langle Ax, x \rangle \geq \overline{\gamma}\|x\|^2, \forall x \in H$. A mapping $T : C \rightarrow C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$.

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. If C is bounded closed convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty (see [2]).

We discuss the following variational inequality problem over the fixed point set of a nonexpansive mapping (see [3–16]), which is said to be the *hierarchical problem*. Let a monotone, continuous mapping $A : H \rightarrow H$ and a nonexpansive mapping $T : H \rightarrow H$. Find $x \in \mathrm{VI}(F(T), A) = \{x \in F(T) : \langle Ax, y - x \rangle \geq 0, \forall y \in F(T)\}$, where $F(T) \neq \emptyset$. This solution set is denoted by Ξ.

We introduce the following variational inequality problem over the solution set of variational inequality problem and the fixed point set of a nonexpansive mapping (see [17, 18]), which is said to be the *triple hierarchical problem*. Let an inverse-strongly monotone $A : H \rightarrow H$, a strongly monotone and Lipschitz continuous $B : H \rightarrow H$, and a nonexpansive mapping $T : H \rightarrow H$. Find $x \in \mathrm{VI}(\Xi, B) = \{x \in \Xi : \langle Bx, y - x \rangle \geq 0, \forall y \in \Xi\}$, where $\Xi := \mathrm{VI}(F(T), A) \neq \emptyset$.

In 2009, Yao et al. [19] considered the following two-step iterative algorithm with the initial guess $x_0 \in C$ which is chosen arbitrarily:

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) T y_n, \\ y_n &= \beta_n S x_n + (1 - \beta_n) x_n, \quad \forall n \geq 0, \end{aligned} \tag{1}$$

where $\alpha_n, \beta_n \in (0,1)$ satisfies certain assumptions. Let S, T be two nonexpansive mappings and let $f : C \to C$ be a contraction mapping. Then, they proved that the above iterative sequence $\{x_n\}$ converges strongly to fixed point.

Next, Iiduka [17] introduced a monotone variational inequality with variational inequality constraint over the fixed point set of a nonexpansive mapping; the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_1 \in H$, is chosen arbitrarily:

$$y_n = T(x_n - \lambda_n A_1 x_n),$$
$$x_{n+1} = y_n - \mu\alpha_n A_2 y_n, \quad \forall n \geq 0, \tag{2}$$

where $\alpha_n \in (0,1]$ and $\lambda_n \in (0, 2\alpha]$ satisfy certain conditions, $A_1 : H \to H$ is an inverse-strongly monotone, $A_2 : H \to H$ is a strongly monotone and Lipschitz continuous, and $T : H \to H$ is a nonexpansive mapping; then the strongly convergence analysis of the sequence generated by (2) is proved under some appropriate conditions.

In 2011, Yao et al. [20] studied the hierarchical problem over the fixed point set. Let the sequences $\{x_n\}$ be generated by these two following algorithms:

implicit algorithm $x_t = TP_C[I - t(A - \gamma f)]x_t, \forall t \in (0,1)$

explicit algorithm $x_{n+1} = \beta_n x_n + (1-\beta_n)TP_C[I - \alpha_n(A - \gamma f)]x_n, \forall n \geq 0$.

They illustrated that these two algorithms converge strongly to the unique solution of the variational inequality which is to find $x^* \in F(T)$ such that

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T), \tag{3}$$

where $A : C \to H$ is a strongly positive linear bounded operator, $f : C \to H$ is a ρ-contraction, and $T : C \to C$ is a nonexpansive mapping satisfying some conditions.

Very recently, Ceng et al. [21] studied the following new algorithms. For $x_0 \in C$ is chosen arbitrarily, they defined a sequence $\{x_n\}$ by

$$x_{n+1} = P_C[\lambda_n\gamma(\alpha_n f(x_n) + (1-\alpha_n)Sx_n) + (I - \lambda_n\mu F)Tx_n], \quad \forall n \geq 0, \tag{4}$$

where the mappings S, T are nonexpansive mappings with $F(T) \neq \emptyset$. Let $F : C \to H$ be a Lipschitzian and strongly monotone operator and let $f : C \to H$ be a contraction mapping satisfying some appropriate conditions. They proved that the proposed algorithms strongly converge to the minimum norm fixed point of T.

In this paper, we consider a new iterative algorithm for solving the triple hierarchical problem over the solution set of the variational inequality problem and the fixed point set of a nonexpansive mapping which contain algorithms (1) and (4) as follows:

$$y_n = P_C[\beta_n Sx_n + (1-\beta_n)x_n],$$
$$x_{n+1} = \gamma\lambda_n\phi(x_n) + (I - \lambda_n\mu F)Ty_n, \quad \forall n \geq 0, \tag{5}$$

where the mappings S, T are nonexpansive mappings with $F(T) \neq \emptyset$. Let $F : C \to H$ be a Lipschitzian and strongly monotone operator, and let $\phi : H \to H$ be a contraction mapping satisfying some mild conditions. Find a point $x^* \in F(T)$ such that

$$\langle (I - S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{6}$$

This solution set of (6) is denoted by $\Omega := VI(F(T), S)$. The strong convergence for the proposed algorithms to the solution is solved under some appropriate assumptions. Our results improve the results of Ceng et al. [21], Iiduka [17], Yao et al. [19], Yao et al. [20], and some authors.

2. Preliminaries

Let C be a nonempty closed convex subset of H. There holds the following inequality in an inner product space $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \forall x, y \in H$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{7}$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \tag{8}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{9}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{10}$$

for all $x \in H, y \in C$. Let B be a monotone mapping of C into H. In the context of the variational inequality problem the characterization of projection (9) implies the following:

$$u \in VI(C, B) \Longleftrightarrow u = P_C(u - \lambda Bu), \quad \lambda > 0. \tag{11}$$

It is also known that H satisfies the Opial's condition [22]; that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$ holds for every $y \in H$ with $x \neq y$.

Lemma 1 (see [23]). *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero; that is, $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ imply $x = Tx$.*

Lemma 2 (see [24]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 3 (see [10]). *Let $B : H \to H$ be β-strongly monotone and L-Lipschitz continuous and let $\mu \in (0, 2\beta/L^2)$. For $\lambda \in [0,1]$, define $T_\lambda : H \to H$ by $T_\lambda(x) := x - \lambda\mu B(x)$ for all $x \in H$. Then, for all $x, y \in H$, $\|T_\lambda(x) - T_\lambda(y)\| \leq (1 - \lambda\tau)\|x - y\|$ hold, where $\tau := 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0,1]$.*

Lemma 4 (see [25]). *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n) a_n + \delta_n, \quad \forall n \geq 0, \tag{12}$$

where $\{\gamma_n\} \subset (0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathscr{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty}(\delta_n/\gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Strong Convergence Theorem

In this section, we introduce an iterative algorithm of triple hierarchical for solving monotone variational inequality problems for κ-Lipschitzian and η-strongly monotone operators over the solution set of variational inequality problems and the fixed point set of a nonexpansive mapping.

Theorem 5. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F : C \to C$ be κ-Lipschitzian and η-strongly monotone operators with constant κ and $\eta > 0$, respectively, and let $\phi : C \to C$ be a ρ-contraction with coefficient $\rho \in [0,1)$. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and let $S : H \to H$ be a nonexpansive mapping. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma < \tau$, where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)}$. Suppose that $\{x_n\}$ is a sequence generated by the following algorithm where $x_0 \in C$ is chosen arbitrarily:*

$$\begin{aligned} y_n &= P_C [\beta_n S x_n + (1-\beta_n) x_n], \\ x_{n+1} &= \gamma\lambda_n \phi(x_n) + (I - \lambda_n \mu F) T y_n, \quad \forall n \geq 0, \end{aligned} \tag{13}$$

where $\{\beta_n\}, \{\lambda_n\}, \subset (0,1)$ *satisfy the following conditions:*

(C1): $\beta_n \leq k\lambda_n$;

(C2): $\lim_{n\to\infty}\lambda_n = 0$, $\lim_{n\to\infty}((\lambda_n - \lambda_{n-1})/\lambda_n) = 0$, $\sum_{n=0}^{\infty}\lambda_n = \infty$;

(C3): $\lim_{n\to\infty}((\beta_n - \beta_{n-1})/\beta_n) = 0$.

Then $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *which is the unique solution of another variational inequality:*

$$\langle (\mu F - \gamma\phi) x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega, \tag{14}$$

where $\Omega := VI(F(T), S) \neq \emptyset$.

Proof. We will divide the proof into four steps.

Step 1. We will show that $\{x_n\}$ is bounded. Indeed, for any $x^* \in F(T)$, we have

$$\begin{aligned} &\|y_n - x^*\| \\ &= \|P_C [\beta_n S x_n + (1-\beta_n) x_n] - P_C x^*\| \\ &\leq \|\beta_n S x_n + (1-\beta_n) x_n - x^*\| \\ &= \|\beta_n (Sx_n - Sx^*) + (1-\beta_n)(x_n - x^*) + \beta_n (Sx^* - x^*)\| \\ &\leq \beta_n \|x_n - x^*\| + (1-\beta_n)\|x_n - x^*\| + \beta_n \|Sx^* - x^*\| \\ &\leq \|x_n - x^*\| + \beta_n \|Sx^* - x^*\|. \end{aligned} \tag{15}$$

From (13), we deduce that

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &= \|\gamma\lambda_n\phi(x_n) + (I - \lambda_n\mu F) T y_n - x^*\| \\ &= \|\gamma\lambda_n (\phi(x_n) - \phi(x^*)) + (I - \lambda_n \mu F)(T y_n - x^*) \\ &\qquad + \lambda_n (\gamma\phi(x^*) - \mu F x^*)\| \\ &\leq \gamma\lambda_n \|\phi(x_n) - \phi(x^*)\| + (I - \lambda_n\mu F)\|T y_n - x^*\| \\ &\quad + \lambda_n \|\gamma\phi(x^*) - \mu F x^*\| \\ &\leq \gamma\rho\lambda_n \|x_n - x^*\| + (1-\lambda_n\tau)\|y_n - x^*\| \\ &\quad + \lambda_n \|\gamma\phi(x^*) - \mu F x^*\|. \end{aligned} \tag{16}$$

Substituting (15) into (16), we obtain

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &\leq \gamma\rho\lambda_n \|x_n - x^*\| \\ &\quad + (1-\lambda_n\tau)\{\|x_n - x^*\| + \beta_n \|Sx^* - x^*\|\} \\ &\quad + \lambda_n \|\gamma\phi(x^*) - \mu F x^*\| \\ &\leq \gamma\rho\lambda_n \|x_n - x^*\| + (1-\lambda_n\tau)\|x_n - x^*\| \\ &\quad + \beta_n \|Sx^* - x^*\| + \lambda_n \|\gamma\phi(x^*) - \mu F x^*\| \\ &\leq [1 - \lambda_n(\tau - \gamma\rho)]\|x_n - x^*\| + k\lambda_n \|Sx^* - x^*\| \\ &\quad + \lambda_n \|\gamma\phi(x^*) - \mu F x^*\| \\ &\leq [1 - \lambda_n(\tau - \gamma\rho)]\|x_n - x^*\| \\ &\quad + \lambda_n (k\|Sx^* - x^*\| + \|\gamma\phi(x^*) - \mu F x^*\|) \\ &\leq \max\left\{\|x_n - x^*\| + \frac{1}{\tau - \gamma\rho}\right. \\ &\qquad \left. \times (k\|Sx^* - x^*\| + \|\gamma\phi(x^*) - \mu F x^*\|)\right\}. \end{aligned} \tag{17}$$

By induction, it follows that

$$\begin{aligned} &\|x_n - x^*\| \\ &\quad \le \max\left\{\|x_0 - x^*\| + \frac{1}{\tau - \gamma\rho} \right. \\ &\qquad \left. \times \left(k\|Sx^* - x^*\| + \|\gamma\phi(x^*) - \mu F x^*\|\right)\right\}, \\ &\qquad n \ge 0. \end{aligned} \tag{18}$$

Therefore, $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{Ty_n\}$, $\{Sx_n\}$, $\{\phi(x_n)\}$, and $\{FT(y_n)\}$.

Step 2. We will show that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Setting $v_n := \beta_n S x_n + (1-\beta_n)x_n$, we obtain

$$\begin{aligned} &\|v_n - v_{n-1}\| \\ &\quad = \|\beta_n S x_n + (1-\beta_n)x_n - \beta_{n-1}Sx_{n-1} - (1-\beta_{n-1})x_{n-1}\| \\ &\quad = \|\beta_n(Sx_n - Sx_{n-1}) + (\beta_n - \beta_{n-1})Sx_{n-1} \\ &\qquad + (1-\beta_n)(x_n - x_{n-1}) + (\beta_{n-1} - \beta_n)x_{n-1}\| \\ &\quad \le \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|x_{n-1}\|) \\ &\qquad + (1-\beta_n)\|x_n - x_{n-1}\| \\ &\quad \le \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|x_{n-1}\|), \end{aligned} \tag{19}$$

which implies that

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|P_C v_n - P_C v_{n-1}\| \\ &\le \|v_n - v_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|x_{n-1}\|). \end{aligned} \tag{20}$$

It follows from (13) that

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &\quad = \|\gamma\lambda_n\phi(x_n) + (I - \lambda_n\mu F)Ty_n - \gamma\lambda_{n-1}\phi(x_{n-1}) \\ &\qquad - (I - \lambda_{n-1}\mu F)Ty_{n-1}\| \\ &\quad = \|\gamma\lambda_n(\phi(x_n) - \phi(x_{n-1})) + (\lambda_n - \lambda_{n-1})\gamma\phi(x_{n-1}) \\ &\qquad + (I - \lambda_n\mu F)Ty_n - (I - \lambda_{n-1}\mu F)Ty_{n-1}\| \\ &\quad \le \gamma\rho\lambda_n\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\gamma\|\phi(x_{n-1})\| \\ &\qquad + \|(I - \lambda_n\mu F)Ty_n - (I - \lambda_n\mu F)Ty_{n-1} \\ &\qquad + (I - \lambda_n\mu F)Ty_{n-1} - (I - \lambda_{n-1}\mu F)Ty_{n-1}\| \\ &\quad \le \gamma\rho\lambda_n\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\gamma\|\phi(x_{n-1})\| \\ &\qquad + (1 - \lambda_n\tau)\|y_n - y_{n-1}\| + |\lambda_n - \lambda_{n-1}|\mu\|FTy_{n-1}\| \\ &\quad \le \gamma\rho\lambda_n\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}| \\ &\qquad \times (\gamma\|\phi(x_{n-1})\| + \mu\|FTy_{n-1}\|) \\ &\qquad + (1 - \lambda_n\tau)\{\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \\ &\qquad \times (\|Sx_{n-1}\| + \|x_{n-1}\|)\} \\ &\quad \le [1 - \lambda_n(\tau - \gamma\rho)]\|x_n - x_{n-1}\| \\ &\qquad + |\lambda_n - \lambda_{n-1}|(\gamma\|\phi(x_{n-1})\| + \mu\|FTy_{n-1}\|) \\ &\qquad + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|x_{n-1}\|) \\ &\quad = [1 - \lambda_n(\tau - \gamma\rho)]\|x_n - x_{n-1}\| \\ &\qquad + \left(\frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} + \frac{|\beta_n - \beta_{n-1}|}{\lambda_n}\right)\lambda_n M_1 \\ &\quad \le [1 - \lambda_n(\tau - \gamma\rho)]\|x_n - x_{n-1}\| \\ &\qquad + \left(\frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} + \frac{k|\beta_n - \beta_{n-1}|}{\beta_n}\right)\lambda_n M_1, \end{aligned} \tag{21}$$

where M_1 is a constant such that

$$\sup_{n\ge 0}\{(\gamma\|\phi(x_n)\| + \mu\|FTy_n\|), (\|Sx_n\| + \|x_n\|)\} \le M_1. \tag{22}$$

Hence, conditions (C2) and (C3) allow us to apply Lemma 4; then we get

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{23}$$

By (21), we get

$$\begin{aligned} &\frac{\|x_{n+1} - x_n\|}{\lambda_n} \\ &\quad \le [1 - \lambda_n(\tau - \gamma\rho)]\frac{\|x_n - x_{n-1}\|}{\lambda_n} \\ &\qquad + \frac{|\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}|}{\lambda_n}M_1 \\ &\quad = [1 - \lambda_n(\tau - \gamma\rho)]\frac{\|x_n - x_{n-1}\|}{\lambda_{n-1}} \\ &\qquad + [1 - \lambda_n(\tau - \gamma\rho)]\left(\frac{\|x_n - x_{n-1}\|}{\lambda_n} - \frac{\|x_n - x_{n-1}\|}{\lambda_{n-1}}\right) \\ &\qquad + \frac{|\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}|}{\lambda_n}M_1 \\ &\quad \le [1 - \lambda_n(\tau - \gamma\rho)]\frac{\|x_n - x_{n-1}\|}{\lambda_{n-1}} \\ &\qquad + \lambda_n\|x_n - x_{n-1}\|\frac{1}{\lambda_n}\left|\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}}\right| \\ &\qquad + M_1\lambda_n\frac{|\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}|}{\lambda_n^2}. \end{aligned} \tag{24}$$

Using the conditions (C2) and (C3), we can apply Lemma 4 to conclude that

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\lambda_n}=0. \tag{25}$$

By (13), we compute

$$\begin{aligned}\|x_{n+1}-Ty_n\| &= \|\gamma\lambda_n\phi(x_n)+(I-\lambda_n\mu F)Ty_n-Ty_n\| \\ &= \|\gamma\lambda_n\phi(x_n)+Ty_n-\lambda_n\mu FTy_n-Ty_n\| \\ &\le \lambda_n\|\gamma\phi(x_n)-\mu FTy_n\|.\end{aligned} \tag{26}$$

From the condition (C2), we note that $\lim_{n\to\infty}\|x_{n+1}-Ty_n\|=0$. At the same time, from (13), we also have

$$\begin{aligned}\|y_n-x_n\| &= \|P_C[\beta_nSx_n+(1-\beta_n)x_n]-P_Cx_n\| \\ &\le \|\beta_nSx_n+(1-\beta_n)x_n-x_n\| \\ &\le \beta_n\|Sx_n-x_n\|.\end{aligned} \tag{27}$$

By the conditions (C1) and (C2), we note that $\lim_{n\to\infty}\|y_n-x_n\|=0$. Consider

$$\begin{aligned}\|y_n-Ty_n\| &\le \|y_n-x_n\|+\|x_n-x_{n+1}\| \\ &\quad+\|x_{n+1}-Ty_n\|\longrightarrow 0.\end{aligned} \tag{28}$$

From (23), (26), and (27), we obtain

$$\lim_{n\to\infty}\|y_n-Ty_n\|=0. \tag{29}$$

We set $v_n=\beta_nSx_n+(1-\beta_n)x_n$; then we get

$$\begin{aligned}\|y_n-v_n\| &= \|P_Cv_n-v_n\| \\ &\le \|v_n-v_n\|\longrightarrow 0, \quad \text{as } n\longrightarrow\infty.\end{aligned} \tag{30}$$

From (13), we have

$$\begin{aligned}\|Ty_n-Tx_n\| &= \|TP_C[\beta_nSx_n+(1-\beta_n)x_n]-TP_Cx_n\| \\ &\le \|\beta_nSx_n+(1-\beta_n)x_n-x_n\| \\ &\le \beta_n\|Sx_n-x_n\|.\end{aligned} \tag{31}$$

By the conditions (C1) and (C2) again, we note that $\lim_{n\to\infty}\|Ty_n-Tx_n\|=0$. Consider

$$\|x_n-Tx_n\|\le\|x_n-y_n\|+\|y_n-Ty_n\|+\|Ty_n-Tx_n\|\longrightarrow 0. \tag{32}$$

From (29), $\lim_{n\to\infty}\|x_n-y_n\|=0$, and $\lim_{n\to\infty}\|Ty_n-Tx_n\|=0$, we obtain

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{33}$$

Step 3. We will show that $\limsup_{n\to\infty}\langle\mu Fx^*-\gamma\phi(x^*),x_n-x^*\rangle\le 0$. Rewrite (13) as

$$\begin{aligned}x_{n+1} &= \gamma\lambda_n\phi(x_n)+(I-\mu\lambda_nF)Ty_n \\ &\quad-v_n+\beta_nSx_n+(1-\beta_n)x_n.\end{aligned} \tag{34}$$

We observe that

$$\begin{aligned}&x_n-x_{n+1}\\ &= x_n-\gamma\lambda_n\phi(x_n)\\ &\quad-(I-\mu\lambda_nF)Ty_n+v_n-\beta_nSx_n-x_n+\beta_nx_n\\ &= \lambda_n(\mu F-\gamma\phi)x_n\\ &\quad-\lambda_n\mu Fx_n-(I-\mu\lambda_nF)Ty_n+(I-\mu\lambda_nF)y_n\\ &\quad-(I-\mu\lambda_nF)y_n+v_n+\beta_n(I-S)x_n\\ &= \lambda_n(\mu F-\gamma\phi)x_n+\lambda_n\mu(Fy_n-Fx_n)+(y_n-Ty_n)\\ &\quad-\mu\lambda_nF(y_n-Ty_n)+(v_n-y_n)+\beta_n(I-S)x_n\\ &= \lambda_n(\mu F-\gamma\phi)x_n+\lambda_n\mu(Fy_n-Fx_n)+(y_n-Ty_n)\\ &\quad-\mu\lambda_nF(y_n-Ty_n)+\lambda_n(y_n-Ty_n)\\ &\quad-\lambda_n(y_n-Ty_n)+(v_n-y_n)+\beta_n(I-S)x_n\\ &= \lambda_n(\mu F-\gamma\phi)x_n+\lambda_n\mu(Fy_n-Fx_n)\\ &\quad+\lambda_n(I-\mu F)(y_n-Ty_n)+(1-\lambda_n)(y_n-Ty_n)\\ &\quad+(v_n-y_n)+\beta_n(I-S)x_n.\end{aligned} \tag{35}$$

Set

$$z_n=\frac{x_n-x_{n+1}}{\lambda_n}, \quad \forall n\ge 0. \tag{36}$$

We note from (35) that

$$\begin{aligned}z_n = &\ (\mu F-\gamma\phi)x_n+\mu(Fy_n-Fx_n)+(I-\mu F)(y_n-Ty_n)\\ &+\frac{1-\lambda_n}{\lambda_n}(y_n-Ty_n)\\ &+\frac{1}{\lambda_n}(v_n-y_n)+\frac{\beta_n}{\lambda_n}(I-S)x_n.\end{aligned} \tag{37}$$

This yields that, for each $x^*\in F(T)$,

$$\begin{aligned}&\langle z_n,x_n-x^*\rangle\\ &= \langle(\mu F-\gamma\phi)x_n,x_n-x^*\rangle+\mu\langle(Fy_n-Fx_n),x_n-x^*\rangle\\ &\quad+\langle(I-\mu F)y_n-(I-\mu F)Ty_n,x_n-x^*\rangle\\ &\quad+\frac{1-\lambda_n}{\lambda_n}\langle y_n-Ty_n,x_n-x^*\rangle\\ &\quad+\frac{1}{\lambda_n}\langle v_n-y_n,x_n-x^*\rangle+\frac{\beta_n}{\lambda_n}\langle(I-S)x_n,x_n-x^*\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle (\mu F - \gamma\phi) x^*, x_n - x^* \rangle \\
&\quad + \langle (\mu F - \gamma\phi) x_n - (\mu F - \gamma\phi) x^*, x_n - x^* \rangle \\
&\quad + \mu \langle (F y_n - F x_n), x_n - x^* \rangle \\
&\quad + \langle (I - \mu F) y_n - (I - \mu F) T y_n, x_n - x^* \rangle \\
&\quad + \frac{1-\lambda_n}{\lambda_n} \langle y_n - T y_n, x_n - x^* \rangle + \frac{1}{\lambda_n} \langle v_n - y_n, x_n - x^* \rangle \\
&\quad + \frac{\beta_n}{\lambda_n} \langle (I - S) x_n, x_n - x^* \rangle.
\end{aligned} \tag{38}
$$

In view of (38), $\langle (\mu F - \gamma\phi)x_n - (\mu F - \gamma\phi)x^*, x_n - x^* \rangle$ is nonnegative due to the monotonicity of $\mu F - \gamma\phi$. From (38), we derive that

$$
\begin{aligned}
\langle z_n, x_n - x^* \rangle \geq{}& \langle (\mu F - \gamma\phi) x^*, x_n - x^* \rangle \\
&+ \mu \langle (F y_n - F x_n), x_n - x^* \rangle \\
&+ \langle (I - \mu F) y_n - (I - \mu F) T y_n, x_n - x^* \rangle \\
&+ \frac{1-\lambda_n}{\lambda_n} \langle y_n - T y_n, x_n - x^* \rangle \\
&+ \frac{1}{\lambda_n} \langle v_n - y_n, x_n - x^* \rangle \\
&+ \frac{\beta_n}{\lambda_n} \langle (I - S) x_n, x_n - x^* \rangle.
\end{aligned} \tag{39}
$$

Since (29) implies $\|(I-\mu F)y_n - (I-\mu F)Ty_n\| \to 0$, as $n \to \infty$, from (25), then we get $z_n \to 0$. Using (C1) and (30), $\|y_n - x_n\| \to 0$, as $n \to \infty$ and $\{x_n\}$ is bounded. We obtain from (39) that

$$
\limsup_{n\to\infty} \langle (\mu F - \gamma\phi) x^*, x_n - x^* \rangle \leq 0, \quad \forall x^* \in F(T). \tag{40}
$$

Since the sequence $\{x_n\}$ is bounded, we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$
\begin{aligned}
&\limsup_{n\to\infty} \langle (\mu F - \gamma\phi) x^*, x_n - x^* \rangle \\
&\quad = \limsup_{j\to\infty} \langle (\mu F - \gamma\phi) x^*, x_{n_j} - x^* \rangle
\end{aligned} \tag{41}
$$

and $x_{n_j} \rightharpoonup \tilde{x}$. From (33), by the demiclosed principle of the nonexpansive mapping, it follows that $\tilde{x} \in F(T)$. Then

$$
\begin{aligned}
&\limsup_{j\to\infty} \langle (\mu F - \gamma\phi) x^*, x_{n_j} - x^* \rangle \\
&\quad = \langle (\mu F - \gamma\phi) x^*, \tilde{x} - x^* \rangle \leq 0.
\end{aligned} \tag{42}
$$

Step 4. Finally, we will prove $x_{n+1} \to x^*$. From (13), we note that

$$
\begin{aligned}
\|y_n - x^*\|^2 &= \|P_C[\beta_n S x_n + (1-\beta_n)x_n] - P_C x^*\|^2 \\
&\leq \|[\beta_n S x_n + (1-\beta_n)x_n] - x^*\|^2 \\
&\leq \|\beta_n (S x_n - S x^*) + (1-\beta_n)(x_n - x^*) \\
&\qquad + \beta_n (S x^* - x^*)\|^2 \\
&\leq \|\beta_n (S x_n - S x^*) + (1-\beta_n)(x_n - x^*)\|^2 \\
&\quad + 2\beta_n \langle S x^* - x^*, y_n - x^* \rangle \\
&\leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 \\
&\quad + 2\beta_n \langle S x^* - x^*, y_n - x^* \rangle \\
&\leq \|x_n - x^*\|^2 + 2\beta_n \|S x^* - x^*\| \|y_n - x^*\|.
\end{aligned} \tag{43}
$$

Using (43), we compute

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\gamma\lambda_n \phi(x_n) + (I - \lambda_n \mu F) T y_n - x^*\|^2 \\
&= \|\gamma\lambda_n (\phi(x_n) - \phi(x^*)) \\
&\qquad + (I - \lambda_n \mu F) T y_n - (I - \lambda_n \mu F) x^* \\
&\qquad + (I - \lambda_n \mu F) x^* - x^* + \gamma\lambda_n \phi(x^*)\|^2 \\
&= \|\gamma\lambda_n (\phi(x_n) - \phi(x^*)) + (I - \lambda_n \mu F)(T y_n - x^*) \\
&\qquad + \lambda_n (\gamma\phi(x^*) - \mu F x^*)\|^2 \\
&\leq \|\gamma\lambda_n (\phi(x_n) - \phi(x^*)) + (I - \lambda_n \mu F)(T y_n - x^*)\|^2 \\
&\quad + 2\lambda_n \langle \gamma\phi(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \\
&\leq \gamma^2 \lambda_n^2 \|\phi(x_n) - \phi(x^*)\|^2 + (1 - \lambda_n \tau)^2 \|T y_n - x^*\|^2 \\
&\quad + 2\lambda_n \langle \gamma\phi(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \\
&\quad + 2 \langle \gamma\lambda_n (\phi(x_n) - \phi(x^*)), (I - \mu\lambda_n F)(T y_n - x^*) \rangle \\
&\leq \gamma^2 \rho^2 \lambda_n^2 \|x_n - x^*\|^2 + (1 - 2\lambda_n \tau + \lambda_n^2 \tau^2) \|y_n - x^*\|^2 \\
&\quad + 2\lambda_n \langle \gamma\phi(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \\
&\quad + 2\gamma\lambda_n \langle \phi(x_n) - \phi(x^*), (I - \mu\lambda_n F) T y_n - (I - \mu\lambda_n F) x^* \rangle \\
&= \gamma^2 \rho^2 \lambda_n^2 \|x_n - x^*\|^2 + (1 - 2\lambda_n \tau + \lambda_n^2 \tau^2) \|y_n - x^*\|^2 \\
&\quad + 2\lambda_n \langle \gamma\phi(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \\
&\quad + 2\gamma\lambda_n \langle \phi(x_n) - \phi(x^*), (T y_n - x^*) - \mu\lambda_n F (T y_n - x^*) \rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \gamma^2\rho^2\lambda_n^2\|x_n - x^*\|^2 + \left(1 - 2\lambda_n\tau + \lambda_n^2\tau^2\right)\|y_n - x^*\|^2 \\
&\quad + 2\lambda_n\langle\gamma\phi(x^*) - \mu Fx^*, x_{n+1} - x^*\rangle \\
&\quad + 2\gamma\lambda_n\langle\phi(x_n) - \phi(x^*), Ty_n - x^*\rangle \\
&\quad - 2\gamma\lambda_n\langle\phi(x_n) - \phi(x^*), \mu\lambda_n F(Ty_n - x^*)\rangle \\
&\le \gamma^2\rho^2\lambda_n^2\|x_n - x^*\|^2 + \left(1 - 2\lambda_n\tau + \lambda_n^2\tau^2\right) \\
&\quad \times \left\{\|x_n - x^*\|^2 + 2\beta_n\|Sx^* - x^*\|\|y_n - x^*\|\right\} \\
&\quad + 2\lambda_n\langle\gamma\phi(x^*) - \mu Fx^*, x_{n+1} - x^*\rangle \\
&\quad + 2\gamma\rho\lambda_n\|x_n - x^*\|\|Ty_n - x^*\| \\
&\quad - 2\gamma\rho\mu\lambda_n^2\|x_n - x^*\|\|F(Ty_n - x^*)\| \\
&\le \left[1 - \lambda_n\left(2\tau - \lambda_n\tau^2 - \lambda_n\gamma^2\rho^2\right)\right]\|x_n - x^*\|^2 \\
&\quad + 2\varepsilon_n\lambda_n\|Sx^* - x^*\|\|y_n - x^*\| \\
&\quad + 2\lambda_n\langle\gamma\phi(x^*) - \mu Fx^*, x_{n+1} - x^*\rangle \\
&\quad + 2\gamma\rho\lambda_n\|x_n - x^*\|\|Ty_n - x^*\| \\
&\quad - 2\gamma\rho\mu\lambda_n^2\|x_n - x^*\|\|F(Ty_n - x^*)\|.
\end{aligned}
\tag{44}
$$

Since $\{x_n\}$, $\{Ty_n\}$, and $\{FTy_n\}$ are all bounded, we can choose a constant $M_2 > 0$ such that

$$
\begin{aligned}
&\sup_{n\ge 0}\frac{1}{2\tau - \lambda_n\tau^2 - \lambda_n\gamma^2\rho^2} \\
&\quad \times \left\{2\gamma\rho\mu\|x_n - x^*\|\|F(Ty_n - x^*)\|\right\} \le M_2.
\end{aligned}
\tag{45}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \le{}& \left[1 - \lambda_n\left(2\tau - \lambda_n\tau^2 - \lambda_n\gamma^2\rho^2\right)\right]\|x_n - x^*\|^2 \\
&+ \lambda_n\left(2\tau - \lambda_n\tau^2 - \lambda_n\gamma^2\rho^2\right)\delta_n,
\end{aligned}
\tag{46}
$$

where

$$
\begin{aligned}
\delta_n ={}& \frac{2\varepsilon_n}{2\tau - \lambda_n\tau^2 - \lambda_n\gamma^2\rho^2}\|Sx^* - x^*\|\|y_n - x^*\| \\
&+ \frac{2}{2\tau - \lambda_n\tau^2 - \lambda_n\gamma^2\rho^2}\langle\gamma\phi(x^*) - \mu Fx^*, x_{n+1} - x^*\rangle \\
&+ \frac{2}{2\tau - \lambda_n\tau^2 - \lambda_n\gamma^2\rho^2}\gamma\rho\|x_n - x^*\|\|Ty_n - x^*\| \\
&- \lambda_n M_2.
\end{aligned}
\tag{47}
$$

Now, applying Lemma 4 and (35), we conclude that $x_n \to x^*$. This completes the proof. □

Corollary 6. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F : C \to C$ be κ-Lipschitzian and η-strongly monotone operators with constant κ and $\eta > 0$, respectively. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and let $S : H \to H$ be a nonexpansive mapping. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily:*

$$
x_{n+1} = (I - \lambda_n\mu F)TP_C\left[\beta_n Sx_n + (1 - \beta_n)x_n\right], \quad \forall n \ge 0,
\tag{48}
$$

where $\{\beta_n\}, \{\lambda_n\} \subset (0,1)$ satisfy the following conditions (C1)–(C3). Then $\{x_n\}$ converges strongly to $x^* \in \Omega$, which is the unique solution of variational inequality:

$$
\langle(I - \mu F)x^*, x - x^*\rangle \ge 0, \quad \forall x \in \Omega,
\tag{49}
$$

where $\Omega := VI(F(T), S) \neq \emptyset$.

Proof. Putting $\phi \equiv 0$ in Theorem 5, we can obtain the desired conclusion immediately. □

Corollary 7. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $\phi : H \to H$ be a ρ-contraction with coefficient $\rho \in [0,1)$, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $S : H \to H$ a nonexpansive mapping. Suppose $\{x_n\}$ is a sequence generated by the following algorithm, $x_0 \in C$, arbitrarily:*

$$
\begin{aligned}
y_n &= P_C\left[\beta_n Sx_n + (1 - \beta_n)x_n\right], \\
x_{n+1} &= \lambda_n\phi(x_n) + (1 - \lambda_n)Ty_n, \quad \forall n \ge 0,
\end{aligned}
\tag{50}
$$

where $\{\beta_n\}, \{\lambda_n\} \subset (0,1)$ satisfy the following conditions (C1)–(C3). Then $\{x_n\}$ converges strongly to $x^* \in \Omega$, which is the unique solution of variational inequality:

$$
\langle(I - \phi)x^*, x - x^*\rangle \ge 0, \quad \forall x \in \Omega,
\tag{51}
$$

where $\Omega := VI(F(T), S) \neq \emptyset$.

Proof. Putting $\gamma = 1$, $\mu = 2$, and $F \equiv I/2$ in Theorem 5, we can obtain the desired conclusion immediately. □

Corollary 8. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $S : H \to H$ be a nonexpansive mapping. Suppose $\{x_n\}$ is a sequence generated by the following algorithm, $x_0 \in C$, arbitrarily:*

$$
x_{n+1} = (1 - \lambda_n)TP_C\left[\beta_n Sx_n + (1 - \beta_n)x_n\right], \quad \forall n \ge 0,
\tag{52}
$$

where $\{\beta_n\}, \{\lambda_n\} \subset (0,1)$ satisfy the following conditions (C1)–(C3). Then $\{x_n\}$ converges strongly to $x^* \in F(T)$, which is the unique solution of variational inequality:

$$
\langle(I - S)x^*, x - x^*\rangle \ge 0, \quad \forall x \in F(T).
\tag{53}
$$

Proof. Putting $\phi \equiv 0$ in Corollary 7, we can obtain the desired conclusion immediately. □

Corollary 9. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $\phi : H \to H$ be a ρ-contraction with coefficient $\rho \in [0,1)$, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $S : C \to C$ a nonexpansive mapping. Suppose $\{x_n\}$ is a sequence generated by the following algorithm, $x_0 \in C$, arbitrarily:*

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n) T [\beta_n S x_n + (1-\beta_n) x_n], \quad \forall n \geq 0, \tag{54}$$

where $\{\beta_n\}, \{\lambda_n\} \subset (0,1)$ satisfy the following conditions (C1)–(C3). Then $\{x_n\}$ converges strongly to $x^* \in F(T)$, which is the unique solution of variational inequality:

$$\langle (I - S) x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{55}$$

Proof. Putting $P_C \equiv I$ in Corollary 7, we can obtain the desired conclusion immediately. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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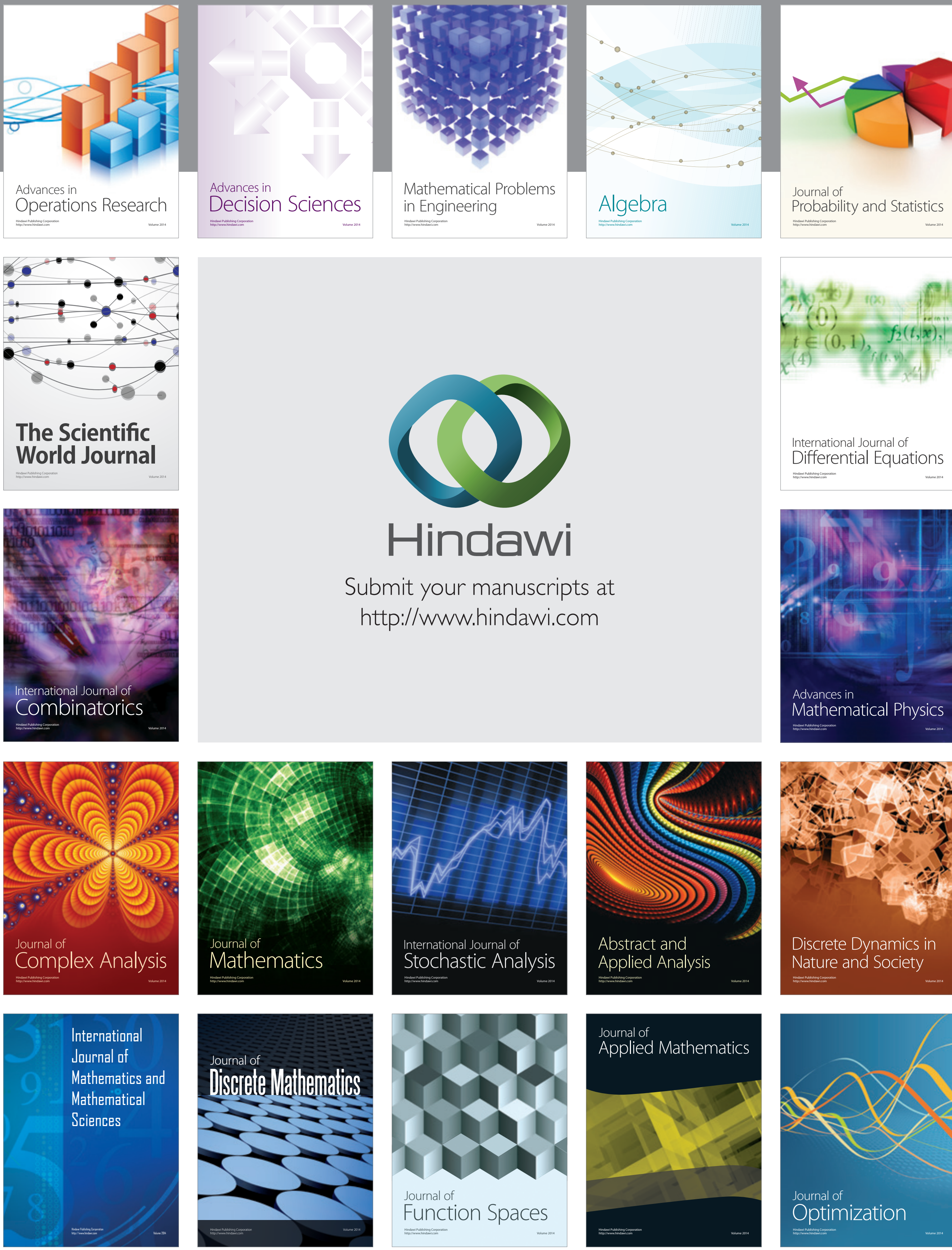
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Research Article

Iterative Algorithms for Mixed Equilibrium Problems, System of Quasi-Variational Inclusion, and Fixed Point Problem in Hilbert Spaces

Poom Kumam[1,2] and Thanyarat Jitpeera[3]

[1] *Computational Science and Engineering Research Cluster (CSEC), King Mongkut's University of Technology Thonburi (KMUTT), Bang Mod, Thrung Khru, Bangkok 10140, Thailand*
[2] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bang Mod, Thrung Khru, Bangkok 10140, Thailand*
[3] *Department of Mathematics, Faculty of Science and Agriculture, Rajamangala University of Technology Lanna, Phan, Chiangrai 57120, Thailand*

Correspondence should be addressed to Thanyarat Jitpeera; t.jitpeera@hotmail.com

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We introduce a new iterative algorithm for approximating a common element of the set of solutions for mixed equilibrium problems, the set of solutions of a system of quasi-variational inclusion, and the set of fixed points of an infinite family of nonexpansive mappings in a real Hilbert space. Strong convergence of the proposed iterative algorithm is obtained. Our results generalize, extend, and improve the results of Peng and Yao, 2009, Qin et al. 2010 and many authors.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product and norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$, $\forall x, y \in C$. They use $F(T)$ to denote the set of *fixed points* of T; that is, $F(T) = \{x \in C : Tx = x\}$. It is assumed throughout the paper that T is a nonexpansive mapping such that $F(T) \neq \emptyset$. Recall that a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in [0,1)$, and $x, y \in C$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$.

Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function and let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Ceng and Yao [1] considered the following *mixed equilibrium problem* for finding $x \in C$ such that

$$F(x,y) + \varphi(y) \ge \varphi(x), \quad \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $\mathrm{MEP}(F,\varphi)$. We see that x is a solution of problem (1) which implies that $x \in \operatorname{dom}\varphi = \{x \in C \mid \varphi(x) < +\infty\}$. If $\varphi \equiv 0$, then the mixed equilibrium problem (1) becomes the following *equilibrium problem* for finding $x \in C$ such that

$$F(x,y) \ge 0, \quad \forall y \in C. \tag{2}$$

The set of solutions of (2) is denoted by $\mathrm{EP}(F)$. The mixed equilibrium problems include fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems, and the equilibrium problem as special cases. Numerous problems in physics, optimization, and economics reduce to find a solution of (2). Some methods have been proposed to solve the equilibrium problem (see [2–14]).

Let $B : C \to H$ be a mapping. The *variational inequality problem,* denoted by $\mathrm{VI}(C,B)$, is for finding $x \in C$ such that

$$\langle Bx, y - x\rangle \ge 0, \tag{3}$$

for all $y \in C$. The variational inequality problem has been extensively studied in the literature. See, for example, [15, 16] and the references therein. A mapping B of C into H is called *monotone* if

$$\langle Bx - By, x - y \rangle \geq 0, \tag{4}$$

for all $x, y \in C$. B is called *β-inverse-strongly monotone* if there exists a positive real number $\beta > 0$ such that for all $x, y \in C$

$$\langle Bx - By, x - y \rangle \geq \beta \|Bx - By\|^2. \tag{5}$$

We consider a *system of quasi-variational inclusion* for finding $(x^*, y^*) \in H \times H$ such that

$$\begin{gathered} \theta \in x^* - y^* + \rho_1 (B_1 y^* + M_1 x^*), \\ \theta \in y^* - x^* + \rho_2 (B_2 x^* + M_2 y^*), \end{gathered} \tag{6}$$

where $B_i : H \to H$ and $M_i : H \to 2^H$ are nonlinear mappings for each $i = 1, 2$. The set of solutions of problem (6) is denoted by $\mathrm{SQVI}(B_1, M_1, B_2, M_2)$. As special cases of problem (6), we have the following.

(1) If $B_1 = B_2 = B$ and $M_1 = M_2 = M$, then problem (6) is reduced to (7) for finding $(x^*, y^*) \in H \times H$ such that

$$\begin{gathered} \theta \in x^* - y^* + \rho_1 (By^* + Mx^*), \\ \theta \in y^* - x^* + \rho_2 (Bx^* + My^*). \end{gathered} \tag{7}$$

(2) Further, if $x^* = y^*$, then problem (7) is reduced to (8) for finding $x^* \in H$ such that

$$\theta \in Bx^* + Mx^*, \tag{8}$$

where θ is the zero vector in H. The set of solutions of problem (8) is denoted by $I(B, M)$. A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H, f \in M(x)$ and $g \in M(y)$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping M is *maximal* if its graph $G(M) := \{(f, x) \in H \times H : f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for all $(y, g) \in G(M)$ imply $f \in M(x)$. Let B be a monotone mapping of C into H and let $N_C \overline{y}$ be the *normal cone* to C at $\overline{y} \in C$; that is, $N_C \overline{y} = \{w \in H : \langle u - \overline{y}, w \rangle \leq 0, \forall u \in C\}$, and define

$$M\overline{y} = \begin{cases} B\overline{y} + N_C \overline{y}, & \overline{y} \in C; \\ \emptyset, & \overline{y} \notin C. \end{cases} \tag{9}$$

Then, M is the *maximal monotone* and $\theta \in M\overline{y}$ if and only if $\overline{y} \in \mathrm{VI}(C, B)$; see [17].

Let $M : H \to 2^H$ be a set-valued maximal monotone mapping; then, the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda} x^* = (I + \lambda M)^{-1} x^*, \quad x^* \in H \tag{10}$$

is called the *resolvent operator* associated with M, where λ is any positive number and I is the identity mapping. The following characterizes the resolvent operator.

(R1) The resolvent operator $J_{M,\lambda}$ is single-valued and nonexpansive for all $\lambda > 0$; that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\| \leq \|x - y\|, \quad \forall x, y \in H, \ \forall \lambda > 0. \tag{11}$$

(R2) The resolvent operator $J_{M,\lambda}$ is 1-inverse-strongly monotone; see [18]; that is,

$$\begin{aligned} &\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \\ &\quad \leq \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y) \rangle, \quad \forall x, y \in H. \end{aligned} \tag{12}$$

(R3) The solution of problem (8) is a fixed point of the operator $J_{M,\lambda}(I - \lambda B)$ for all $\lambda > 0$; see also [19]; that is,

$$I(B, M) = F(J_{M,\lambda}(I - \lambda B)), \quad \forall \lambda > 0. \tag{13}$$

(R4) If $0 < \lambda \leq 2\beta$, then the mapping $J_{M,\lambda}(I - \lambda B) : H \to H$ is nonexpansive.

(R5) $I(B, M)$ is closed and convex.

Let A be a strongly positive linear bounded operator on H; that is, there exists a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{14}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{15}$$

where A is a strongly positive linear bounded operator and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In 2007, Plubtieng and Punpaeng [20] proposed the following iterative algorithm:

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A) T u_n. \end{gathered} \tag{16}$$

They proved that if the sequences $\{\epsilon_n\}$ and $\{r_n\}$ of parameters satisfy appropriate conditions, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution z of the variational inequality

$$\langle (A - \gamma f) z, x - z \rangle \geq 0, \quad \forall x \in F(T) \cap \mathrm{EP}(F), \tag{17}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(T)\cap \mathrm{EP}(F)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{18}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In 2009, Peng and Yao [21] introduced an iterative algorithm based on extragradient method which solves the problem for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a family of finitely nonexpansive mappings, and the set of the variational inequality for a monotone, Lipschitz continuous mapping in a real Hilbert space. The sequences generated by $v \in C$ are

$$\begin{gathered} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \gamma_n B u_n), \\ x_{n+1} = \alpha_n v + \beta_n x_n + (1-\alpha_n-\beta_n) W_n P_C(u_n - \lambda_n B y_n), \end{gathered} \tag{19}$$

for all $n \geq 1$, where W_n is W-mapping. They proved the strong convergence theorems under some mild conditions.

In 2010, Qin et al. [22] introduced an iterative method for finding solutions of a generalized equilibrium problem, the set of fixed points of a family of nonexpansive mappings, and the common variational inclusions. The sequences generated by $x_1 \in C$ and $\{x_n\}$ are a sequence generated by

$$\begin{gathered} F(u_n, y) + \langle A_3 x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n A_2 u_n), \\ y_n = P_C(z_n - \eta_n A_1 z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n, \quad \forall n \geq 1, \end{gathered} \tag{20}$$

where f is a contraction and A_i is inverse-strongly monotone mappings for $i = 1, 2, 3$ and W_n is called a W-mapping generated by $S_n, S_{n_1}, \ldots, S_1$ and $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$. They proved the strong convergence theorems under some mild conditions. Liou [23] introduced an algorithm for finding a common element of the set of solutions of a mixed equilibrium problem and the set of variational inclusion in a real Hilbert space. The sequences generated by $x_0 \in C$ are

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \quad \forall y \in C, \tag{21}$$

$$x_{n+1} = P_C[(1-\alpha_n A) J_{M,\lambda}(u_n - \lambda B u_n)],$$

for all $n \geq 1$, where A is a strongly positive bounded linear operator and B, Q are inverse-strongly monotone. They proved the strong convergence theorems under some suitable conditions.

Next, Petrot et al. [24] introduced the new following iterative process for finding the set of solutions of quasi-variational inclusion problem and the set of fixed point of a nonexpansive mapping. The sequence is generated by

$$\begin{gathered} x_0 \in H, \quad \text{chosen arbitrary}, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S z_n, \\ z_n = J_{M,\lambda}(y_n - \lambda A y_n), \\ y_n = J_{M,\rho}(x_n - \rho A x_n), \end{gathered} \tag{22}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ and $\lambda \in (0, 2\alpha]$. They proved that $\{x_n\}$ generated by (22) converges strongly to z_0 which is the unique solution in $F(S) \cap I(A, M)$.

In 2011, Jitpeera and Kumam [25] introduced a shrinking projection method for finding the common element of the common fixed points of nonexpansive semigroups, the set of common fixed point for an infinite family, the set of solutions of a system of mixed equilibrium problems, and the set of solution of the variational inclusion problem. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0 \in C, C_1 = C$, $x_1 = P_{C_1} x_0$, $u_n \in C$, and

$$\begin{gathered} x_0 = x \in C \quad \text{chosen arbitrary}, \\ u_n = K_{r_{N,n}}^{F_N} K_{r_{N-1,n}}^{F_{N-1}} K_{r_{N-2,n}}^{F_{N-2}} \cdots K_{r_{2,n}}^{F_2} K_{r_{1,n}}^{F_1} x_n, \\ y_n = J_{M_2,\delta_n}(u_n - \delta_n B u_n), \\ v_n = J_{M_1,\lambda_n}(y_n - \lambda_n A y_n), \\ z_n = \alpha_n v_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} S(s) W_n v_n \, ds, \\ C_{n+1} = \left\{ z \in C_n : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n(1-\alpha_n) \times \left\| v_n - \frac{1}{t_n}\int_0^{t_n} S(s) W_n v_n \, ds \right\|^2 \right\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{gathered} \tag{23}$$

where $K_{r_k}^{F_k} : C \to C, k = 1, 2, \ldots, N$. We proved the strong convergence theorem under certain appropriate conditions.

In this paper, motivated by the above results, we introduce a new iterative method for finding a common element of the set of solutions for mixed equilibrium problems, the set of solutions of a system of quasi-variational inclusions, and the set of fixed points of an infinite family of nonexpansive mappings in a real Hilbert space. Then, we prove strong convergence theorems which are connected with [5, 26–29]. Our results extend and improve the corresponding results of

Jitpeera and Kumam [25], Liou [23], Plubtieng and Punpaeng [20], Petrot et al. [24], Peng and Yao [21], Qin et al. [22], and some authors.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H. Then,

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle,$$
$$\|\lambda x + (1-\lambda) y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda) \times \|x-y\|^2, \quad \forall x, y \in H, \ \lambda \in [0,1]. \tag{24}$$

For every point $x \in H$, there exists a unique *nearest point* in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{25}$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{26}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \le 0, \tag{27}$$

$$\|x-y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, \ y \in C. \tag{28}$$

Let B be a monotone mapping of C into H. In the context of the variational inequality problem, the characterization of projection (27) implies the following:

$$u \in \mathrm{VI}(C,B) \Longleftrightarrow u = P_C(u - \lambda B u), \quad \lambda > 0. \tag{29}$$

It is also known that H satisfies the Opial condition [30]; that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{30}$$

holds for every $y \in H$ with $x \ne y$.

For the infinite family of nonexpansive mappings of $T_1, T_2, \ldots$, and sequence $\{\lambda_i\}_{i=1}^{\infty}$ in $[0,1)$, see [31]; we define the mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n,0} &= I, \\ U_{n,1} &= \lambda_1 T_1 U_{n,0} + (1-\lambda_1) U_{n,0}, \\ U_{n,2} &= \lambda_2 T_2 U_{n,1} + (1-\lambda_2) U_{n,1}, \\ &\vdots \\ U_{n,N-1} &= \lambda_{N-1} T_{N-1} U_{n,N-2} + (1-\lambda_{N-1}) U_{n,N-2}, \\ W_n = U_{n,N} &= \lambda_N T_N U_{n,N-1} + (1-\lambda_N) U_{n,N-1}. \end{aligned} \tag{31}$$

Lemma 1 (Shimoji and Takahashi [32]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\mathcal{T} = \{T_i\}_{i=1}^{N}$ be a family of infinitely nonexpanxive mappings with $F(\mathcal{T}) = \bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$ and let $\{\lambda_i\}$ be a real sequence such that $0 < \lambda_i \le b < 1$ for every $i \ge 1$. Then*

(1) *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$ for each $n \ge 1$;*

(2) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k} x$ exists;*

(3) *the mapping $W : C \to C$ defined by $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x$ is a nonexpansive mapping satisfying $F(W) = F(\mathcal{T})$ and it is called the W-mapping generated by $T_1, T_2, \ldots$, and $\lambda_1, \lambda_2, \ldots$;*

(4) *if K is any bounded subset of C, then $\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0$.*

For solving the mixed equilibrium problem, let us give the following assumptions for a bifunction $F : C \times C \to \mathbb{R}$ and a proper extended real-valued function $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone; that is, $F(x,y) + F(y,x) \le 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \le F(x,y)$;

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous;

(A5) for each $y \in C$, $x \mapsto F(x,y)$ is weakly upper semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z); \tag{32}$$

(B2) C is a bounded set.

We need the following lemmas for proving our main results.

Lemma 2 (Peng and Yao [21]). *Let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction that satisfies (A1)–(A5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \Big\{ z \in C : F(z,y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \ge \varphi(z), \forall y \in C \Big\}, \tag{33}$$

for all $x \in H$. Then, the following hold:

(1) *for each $x \in H$, $T_r(x) \ne \emptyset$;*

(2) *T_r is single-valued;*

(3) T_r *is firmly nonexpansive; that is, for any* $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;

(4) $F(T_r) = MEP(F, \varphi)$;

(5) $MEP(F, \varphi)$ *is closed and convex.*

Lemma 3 (Xu [33]). *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \delta_n, \quad n \geq 0, \tag{34}$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty}(\delta_n/\alpha_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 4 (Suzuki [34]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. *Suppose* $x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 5 (Marino and Xu [35]). *Assume* A *is a strongly positive linear bounded operator on a Hilbert space* H *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \rho \leq \|A\|^{-1}$. *Then,* $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.

Lemma 6. *For given* $x^*, y^* \in C \times C$, (x^*, y^*) *is a solution of problem* (6) *if and only if* x^* *is a fixed point of the mapping* $G : C \to C$ *defined by*

$$G(x) = J_{M_1,\lambda}\left[J_{M_2,\mu}(x - \mu E_2 x) - \lambda E_1 J_{M_2,\mu}(x - \mu E_2 x)\right], \quad \forall x \in C, \tag{35}$$

where $y^* = J_{M_2,\mu}(x - \mu E_2 x)$, λ, μ *are positive constants, and* $E_1, E_2 : C \to H$ *are two mappings.*

Proof.

$$\begin{gathered} \theta \in x^* - y^* + \lambda(E_1 y^* + M_1 x^*), \\ \theta \in y^* - x^* + \mu(E_2 x^* + M_2 y^*) \end{gathered} \tag{36}$$

$\Leftrightarrow$

$$\begin{gathered} x^* = J_{M_1,\lambda}(y^* - \lambda E_1 y^*), \\ y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*) \end{gathered} \tag{37}$$

$\Leftrightarrow$

$$\begin{aligned} G(x^*) = J_{M_1,\lambda}\big[&J_{M_2,\mu}(x^* - \mu E_2 x^*) \\ &-\lambda E_1 J_{M_2,\mu}(x^* - \mu E_2 x^*)\big] = x^*. \end{aligned} \tag{38}$$

This completes the proof. □

Now, we prove the following lemmas which will be applied in the main theorem.

Lemma 7. *Let* $G : C \to C$ *be defined as in Lemma 6. If* $E_1, E_2 : C \to H$ *is* η_1, η_2*-inverse-strongly monotone and* $\lambda \in (0, 2\eta_1)$, *and* $\mu \in (0, 2\eta_2)$, *respectively, then* G *is nonexpansive.*

Proof. For any $x, y \in C$ and $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, we have

$$\begin{aligned}
&\|G(x) - G(y)\|^2 \\
&= \big\|J_{M_1,\lambda}\left[J_{M_2,\mu}(x - \mu E_2 x) - \lambda E_1 J_{M_2,\mu}(x - \mu E_2 x)\right] \\
&\qquad - J_{M_1,\lambda}\left[J_{M_2,\mu}(y - \mu E_2 y) - \lambda E_1 J_{M_2,\mu}(y - \mu E_2 y)\right]\big\|^2 \\
&\leq \big\|\left[J_{M_2,\mu}(x - \mu E_2 x) - \lambda E_1 J_{M_2,\mu}(x - \mu E_2 x)\right] \\
&\qquad - \left[J_{M_2,\mu}(y - \mu E_2 y) - \lambda E_1 J_{M_2,\mu}(y - \mu E_2 y)\right]\big\|^2 \\
&= \big\|\left[J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y)\right] \\
&\qquad - \lambda\left[E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y)\right]\big\|^2 \\
&= \left\|J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&\quad - 2\lambda\big\langle J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y), \\
&\qquad\qquad E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y)\big\rangle \\
&\quad + \lambda^2\left\|E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&\leq \left\|J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&\quad - 2\lambda\eta_1\left\|E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&\quad + \lambda^2\left\|E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&= \left\|J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&\quad + \lambda(\lambda - 2\eta_1)\left\|E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&\leq \left\|J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&\leq \|(x - \mu E_2 x) - (y - \mu E_2 y)\|^2 \\
&= \|(x - y) - \mu(E_2 x - E_2 y)\|^2 \\
&= \|x - y\|^2 - 2\mu\langle x - y, E_2 x - E_2 y\rangle + \mu^2\|E_2 x - E_2 y\|^2 \\
&\leq \|x - y\|^2 - 2\eta_2\mu\|E_2 x - E_2 y\|^2 + \mu^2\|E_2 x - E_2 y\|^2 \\
&= \|x - y\|^2 + \mu(\mu - 2\eta_2)\|E_2 x - E_2 y\|^2 \\
&\leq \|x - y\|^2.
\end{aligned} \tag{39}$$

This shows that G is nonexpansive on C. □

3. Main Results

In this section, we show a strong convergence theorem for finding a common element of the set of solutions for mixed equilibrium problems, the set of solutions of a system of quasi-variational inclusion, and the set of fixed points of a infinite family of nonexpansive mappings in a real Hilbert space.

Theorem 8. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuos and convex function. Let $T_i : C \to C$ be nonexpansive mappings for all $i = 1, 2, 3, \ldots$, such that $\Theta := \cap_{i=1}^{\infty} F(T_i) \cap SQVI(B_1, M_1, B_2, M_2) \cap MEP(F, \varphi) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$ and let Q, E_1, E_2 be δ, η_1, η_2-inverse-strongly monotone mapping of C into H. Let A be a strongly positive bounded linear self-adjoint on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$, let $M_1, M_2 : H \to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2 holds and let W_n be the W-mapping defined by* (31). *Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$, and*

$$\begin{aligned} & F(u_n, y) + \varphi(y) - \varphi(u_n) \\ & \quad + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \quad \forall y \in C, \\ & \qquad z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ & \qquad y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ & x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A) W_n y_n, \\ & \qquad \forall n \geq 0, \end{aligned} \tag{40}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, and $r \in (0, 2\delta)$ satisfy the following conditions:

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(C3) $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ converges strongly to $x^* \in \Theta$, where $x^* = P_\Theta(\gamma f + I - A)(x^*)$, P_Θ is the metric projection of H onto Θ and (x^*, y^*), where $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ is solution to the problem (6).

Proof. Let $x^* \in \Theta$; that is $T_r(x^* - rQx^*) = J_{M_1,\lambda}[J_{M_2,\mu}(x^* - \mu B_2 x^*) - \lambda B_1 J_{M_2,\mu}(x^* - \mu B_2 x^*)] = T_i(x^*) = x^*, i \geq 1$. Putting $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$, one can see that $x^* = J_{M_1,\lambda}(y^* - \lambda B_1 y^*)$.

We divide our proofs into the following steps:

(1) sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ are bounded;

(2) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;

(3) $\lim_{n\to\infty} \|Qx_n - Qx^*\| = 0$, $\lim_{n\to\infty} \|E_1 z_n - E_1 x^*\| = 0$ and $\lim_{n\to\infty} \|E_2 u_n - E_2 x^*\| = 0$;

(4) $\lim_{n\to\infty} \|x_n - Wx_n\| = 0$;

(5) $\limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle \leq 0$, where $x^* = P_\Theta(\gamma f + I - A)x^*$;

(6) $\lim_{n\to\infty} \|x_n - x^*\| = 0$.

Step 1. From conditions (C1) and (C2), we may assume that $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$. By the same argument as that in [9], we can deduce that $(1-\beta_n)I - \alpha_n A$ is positive and $\|(1-\beta_n)I - \alpha_n A\| \leq 1 - \beta_n - \alpha_n \overline{\gamma}$. For all $x, y \in C$ and $r \in (0, 2\delta)$. since Q is a δ-inverse-strongly monotone and B_1, B_2 are η_1, η_2-inverse-strongly monotone, we have

$$\begin{aligned} & \|(I - rQ)x - (I - rQ)y\|^2 \\ & \quad = \|(x - y) - r(Qx - Qy)\|^2 \\ & \quad = \|x - y\|^2 - 2r\langle x - y, Qx - Qy\rangle + r^2\|Qx - Qy\|^2 \\ & \quad \leq \|x - y\|^2 - 2r\delta\|Qx - Qy\|^2 + r^2\|Qx - Qy\|^2 \\ & \quad = \|x - y\|^2 + r(r - 2\delta)\|Qx - Qy\|^2 \\ & \quad \leq \|x - y\|^2. \end{aligned} \tag{41}$$

It follows that $\|(I - rQ)x - (I - rQ)y\| \leq \|x - y\|$; hence $I - rQ$ is nonexpansive.

In the same way, we conclude that $\|(I - \lambda E_1)x - (I - \lambda E_1)y\| \leq \|x - y\|$ and $\|(I - \mu E_2)x - (I - \mu E_2)y\| \leq \|x - y\|$; hence $I - \lambda E_1, I - \mu E_2$ are nonexpansive. Let $y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), n \geq 0$. It follows that

$$\begin{aligned} \|y_n - x^*\| & = \|J_{M_1,\lambda}(z_n - \lambda E_1 z_n) - J_{M_1,\lambda}(y^* - \lambda E_1 y^*)\| \\ & \leq \|(z_n - \lambda E_1 z_n) - (y^* - \lambda E_1 y^*)\| \\ & \leq \|z_n - y^*\|, \\ \|z_n - y^*\| & = \|J_{M_2,\mu}(u_n - \mu E_2 u_n) - J_{M_2,\mu}(x^* - \mu E_2 x^*)\| \\ & \leq \|(u_n - \mu E_2 u_n) - (x^* - \mu E_2 x^*)\| \\ & \leq \|u_n - x^*\|. \end{aligned} \tag{42}$$

By Lemma 2, we have $u_n = T_r(x_n - rQx_n)$ for all $n \geq 0, \forall x, y \in C$. Then, for $r \in (0, 2\delta)$, we obtain

$$\begin{aligned} \|u_n - x^*\|^2 & = \|T_r(x_n - rQx_n) - T_r(x^* - rQx^*)\|^2 \\ & \leq \|(x_n - rQx_n) - (x^* - rQx^*)\|^2 \\ & \leq \|x_n - x^*\|^2 + r(r - 2\delta)\|Qx_n - Qx^*\|^2 \\ & \leq \|x_n - x^*\|^2. \end{aligned} \tag{43}$$

Hence, we have

$$\|y_n - x^*\| \leq \|x_n - x^*\|. \tag{44}$$

From (40) and (44), we deduce that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_n(\gamma f(x_n)-Ax^*)+\beta_n(x_n-x^*)\\&\quad+((1-\beta_n)I-\alpha_nA)(W_ny_n-x^*)\|\\&\le \alpha_n\|\gamma f(x_n)-Ax^*\|+\beta_n\|x_n-x^*\|\\&\quad+(1-\beta_n-\alpha_n\overline{\gamma})\|y_n-x^*\|\\&\le \alpha_n\|\gamma f(x_n)-Ax^*\|+\beta_n\|x_n-x^*\|\\&\quad+(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-x^*\|\\&\le \alpha_n\gamma\|f(x_n)-f(x^*)\|+\alpha_n\|\gamma f(x^*)-Ax^*\|\\&\quad+(1-\alpha_n\overline{\gamma})\|x_n-x^*\|\\&\le \alpha_n\gamma\alpha\|x_n-x^*\|+\alpha_n\|\gamma f(x^*)-Ax^*\|\\&\quad+(1-\alpha_n\overline{\gamma})\|x_n-x^*\|\\&=(1-\alpha_n(\overline{\gamma}-\gamma\alpha))\|x_n-x^*\|\\&\quad+\alpha_n(\overline{\gamma}-\gamma\alpha)\frac{\|\gamma f(x^*)-Ax^*\|}{(\overline{\gamma}-\gamma\alpha)}\\&\le \max\left\{\|x_n-x^*\|,\frac{\|\gamma f(x^*)-Ax^*\|}{(\overline{\gamma}-\gamma\alpha)}\right\}.\end{aligned}\tag{45}$$

It follows by mathematical induction that

$$\|x_{n+1}-x^*\|\le\max\left\{\|x_0-x^*\|,\frac{\|\gamma f(x^*)-Ax^*\|}{(\overline{\gamma}-\gamma\alpha)}\right\},\quad n\ge 0.\tag{46}$$

Hence, $\{x_n\}$ is bounded and also $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, $\{W_ny_n\}$, $\{AW_ny_n\}$, and $\{fx_n\}$ are all bounded.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Putting $t_n=(x_{n+1}-\beta_nx_n)/(1-\beta_n)=(\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)W_ny_n)/(1-\beta_n)$, we get $x_{n+1}=(1-\beta_n)t_n+\beta_nx_n$, $n\ge 1$. We note that

$$\begin{aligned}t_{n+1}-t_n&=\frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}A)W_{n+1}y_{n+1}}{1-\beta_{n+1}}\\&\quad-\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)W_ny_n}{1-\beta_n}\\&=\frac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1})-\frac{\alpha_n}{1-\beta_n}\gamma f(x_n)\\&\quad+W_{n+1}y_{n+1}-W_ny_n\\&\quad-\frac{\alpha_{n+1}}{1-\beta_{n+1}}AW_{n+1}y_{n+1}+\frac{\alpha_n}{1-\beta_n}AW_ny_n\\&=\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1})-AW_{n+1}y_{n+1})\\&\quad+\frac{\alpha_n}{1-\beta_n}(AW_ny_n-\gamma f(x_n))\\&\quad+W_{n+1}y_{n+1}-W_{n+1}y_n+W_{n+1}y_n-W_ny_n.\end{aligned}\tag{47}$$

It follows that

$$\begin{aligned}&\|t_{n+1}-t_n\|-\|x_{n+1}-x_n\|\\&\quad\le\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\|+\|AW_{n+1}y_{n+1}\|)\\&\qquad+\frac{\alpha_n}{1-\beta_n}(\|AW_ny_n\|+\|\gamma f(x_n)\|)\\&\qquad+\|W_{n+1}y_{n+1}-W_{n+1}y_n\|\\&\qquad+\|W_{n+1}y_n-W_ny_n\|-\|x_{n+1}-x_n\|\\&\quad\le\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\|+\|AW_{n+1}y_{n+1}\|)\\&\qquad+\frac{\alpha_n}{1-\beta_n}(\|AW_ny_n\|+\|\gamma f(x_n)\|)\\&\qquad+\|y_{n+1}-y_n\|+\|W_{n+1}y_n-W_ny_n\|\\&\qquad-\|x_{n+1}-x_n\|.\end{aligned}\tag{48}$$

By the definition of W_n,

$$\begin{aligned}&\|W_{n+1}y_n-W_ny_n\|\\&\quad=\|\lambda_{n+1,N}T_NU_{n+1,N-1}y_n+(1-\lambda_{n+1,N})y_n\\&\qquad-\lambda_{n,N}T_NU_{n,N-1}y_n-(1-\lambda_{n,N})y_n\|\\&\quad\le|\lambda_{n+1,N}-\lambda_{n,N}|\|y_n\|\\&\qquad+\|\lambda_{n+1,N}T_NU_{n+1,N-1}y_n-\lambda_{n,N}T_NU_{n,N-1}y_n\|\\&\quad\le|\lambda_{n+1,N}-\lambda_{n,N}|\|y_n\|\\&\qquad+\|\lambda_{n+1,N}(T_NU_{n+1,N-1}y_n-T_NU_{n,N-1}y_n)\|\\&\qquad+|\lambda_{n+1,N}-\lambda_{n,N}|\|T_NU_{n,N-1}y_n\|\\&\quad\le 2M|\lambda_{n+1,N}-\lambda_{n,N}|\\&\qquad+\lambda_{n+1,N}\|U_{n+1,N-1}y_n-U_{n,N-1}y_n\|,\end{aligned}\tag{49}$$

where M is an approximate constant such that $M\ge\max\{\sup_{n\ge1}\{\|y_n\|\},\sup_{n\ge1}\{\|T_mU_{n,m-1}y_n\|\}\mid m=1,2,\ldots,N\}$. Since $0<\lambda_{n_i}\le 1$ for all $n\ge 1$ and $i=1,2,\ldots,N$, we compute

$$\begin{aligned}&\|U_{n+1,N-1}y_n-U_{n,N-1}y_n\|\\&\quad=\|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}y_n+(1-\lambda_{n+1,N-1})y_n\\&\qquad-\lambda_{n,N-1}T_{N-1}U_{n,N-2}y_n-(1-\lambda_{n,N-1})y_n\|\end{aligned}$$

$$\begin{aligned}
&\leq \left|\lambda_{n+1,N-1}-\lambda_{n,N-1}\right|\left\|y_n\right\| \\
&\quad+\left\|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}y_n-\lambda_{n,N-1}T_{N-1}U_{n,N-2}y_n\right\| \\
&\leq \left|\lambda_{n+1,N-1}-\lambda_{n,N-1}\right|\left\|y_n\right\| \\
&\quad+\left\|\lambda_{n+1,N-1}\left(T_{N-1}U_{n+1,N-2}y_n-T_{N-1}U_{n,N-2}y_n\right)\right\| \\
&\quad+\left|\lambda_{n+1,N-1}-\lambda_{n,N-1}\right|\left\|T_{N-1}U_{n,N-2}y_n\right\| \\
&\leq 2M\left|\lambda_{n+1,N-1}-\lambda_{n,N-1}\right|+\left\|U_{n+1,N-2}y_n-U_{n,N-2}y_n\right\|.
\end{aligned} \tag{50}$$

It follows that

$$\begin{aligned}
&\left\|U_{n+1,N-1}y_n-U_{n,N-1}y_n\right\| \\
&\quad\leq 2M\left|\lambda_{n+1,N-1}-\lambda_{n,N-1}\right|+2M\left|\lambda_{n+1,N-2}-\lambda_{n,N-2}\right| \\
&\qquad+\left\|U_{n+1,N-3}y_n-U_{n,N-3}y_n\right\| \\
&\quad\leq 2M\sum_{i=2}^{N-1}\left|\lambda_{n+1,i}-\lambda_{n,i}\right|+\left\|U_{n+1,1}y_n-U_{n,1}y_n\right\| \\
&\quad= 2M\sum_{i=2}^{N-1}\left|\lambda_{n+1,i}-\lambda_{n,i}\right| \\
&\qquad+\left\|\lambda_{n+1,1}T_1y_n+\left(1-\lambda_{n+1,1}\right)y_n\right. \\
&\qquad\left.-\lambda_{n,1}T_1y_n-\left(1-\lambda_{n,1}\right)y_n\right\| \\
&\quad\leq 2M\sum_{i=1}^{N-1}\left|\lambda_{n+1,i}-\lambda_{n,i}\right|.
\end{aligned} \tag{51}$$

Substituting (51) into (49),

$$\begin{aligned}
&\left\|W_{n+1}y_n-W_ny_n\right\| \\
&\quad\leq 2M\left|\lambda_{n+1,N}-\lambda_{n,N}\right|+2\lambda_{n+1,N}M\sum_{i=1}^{N-1}\left|\lambda_{n+1,i}-\lambda_{n,i}\right| \\
&\quad\leq 2M\sum_{i=1}^{N}\left|\lambda_{n+1,i}-\lambda_{n,i}\right|.
\end{aligned} \tag{52}$$

We note that

$$\begin{aligned}
&\left\|y_{n+1}-y_n\right\| \\
&\quad=\left\|J_{M_1,\lambda}\left(z_{n+1}-\lambda E_1z_{n+1}\right)-J_{M_1,\lambda}\left(z_n-\lambda E_1z_n\right)\right\| \\
&\quad\leq\left\|\left(z_{n+1}-\lambda E_1z_{n+1}\right)-\left(z_n-\lambda E_1z_n\right)\right\| \\
&\quad\leq\left\|z_{n+1}-z_n\right\| \\
&\quad=\left\|J_{M_2,\mu}\left(u_{n+1}-\mu E_2u_{n+1}\right)-J_{M_2,\mu}\left(u_n-\mu E_2u_n\right)\right\| \\
&\quad\leq\left\|\left(u_{n+1}-\mu E_2u_{n+1}\right)-\left(u_n-\mu E_2u_n\right)\right\| \\
&\quad\leq\left\|u_{n+1}-u_n\right\| \\
&\quad=\left\|T_r\left(x_{n+1}-rDx_{n+1}\right)-T_r\left(x_n-rDx_n\right)\right\| \\
&\quad\leq\left\|\left(x_{n+1}-rDx_{n+1}\right)-\left(x_n-rDx_n\right)\right\| \\
&\quad\leq\left\|x_{n+1}-x_n\right\|.
\end{aligned} \tag{53}$$

Applying (52) and (53) in (48), we get

$$\begin{aligned}
&\left\|t_{n+1}-t_n\right\|-\left\|x_{n+1}-x_n\right\| \\
&\quad\leq\frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f\left(x_{n+1}\right)\right\|+\left\|AW_{n+1}y_{n+1}\right\|\right) \\
&\qquad+\frac{\alpha_n}{1-\beta_n}\left(\left\|AW_ny_n\right\|+\left\|\gamma f\left(x_n\right)\right\|\right)+\left\|x_{n+1}-x_n\right\| \\
&\qquad+2M\sum_{i=1}^{N}\left|\lambda_{n+1,i}-\lambda_{n,i}\right|-\left\|x_{n+1}-x_n\right\|.
\end{aligned} \tag{54}$$

By conditions (C1)–(C3), imply that

$$\limsup_{n\to\infty}\left(\left\|t_{n+1}-t_n\right\|-\left\|x_{n+1}-x_n\right\|\right)\leq 0. \tag{55}$$

Hence, by Lemma 4, we obtain

$$\lim_{n\to\infty}\left\|t_n-x_n\right\|=0. \tag{56}$$

It follows that

$$\lim_{n\to\infty}\left\|x_{n+1}-x_n\right\|=\lim_{n\to\infty}\left(1-\beta_n\right)\left\|t_n-x_n\right\|=0. \tag{57}$$

We obtain that

$$\lim_{n\to\infty}\left\|x_{n+1}-x_n\right\|=0. \tag{58}$$

Step 3. We can rewrite (40) as $x_{n+1}=\alpha_n(\gamma f(x_n)-AW_ny_n)+\beta_n(x_n-W_ny_n)+W_ny_n$. We observe that

$$\begin{aligned}
\left\|x_n-W_ny_n\right\|&\leq\left\|x_n-x_{n+1}\right\|+\left\|x_{n+1}-W_ny_n\right\| \\
&\leq\left\|x_n-x_{n+1}\right\|+\alpha_n\left\|\gamma f\left(x_n\right)-AW_ny_n\right\| \\
&\quad+\beta_n\left\|x_n-W_ny_n\right\|;
\end{aligned} \tag{59}$$

it follows that

$$\begin{aligned}
&\left\|x_n-W_ny_n\right\| \\
&\quad\leq\frac{1}{1-\beta_n}\left\|x_n-x_{n+1}\right\|+\frac{\alpha_n}{1-\beta_n}\left\|\gamma f\left(x_n\right)-AW_ny_n\right\|.
\end{aligned} \tag{60}$$

By conditions (C1), (C2), and (58), imply that

$$\lim_{n\to\infty}\left\|x_n-W_ny_n\right\|=0. \tag{61}$$

From (42) and (43), we get

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&= \|J_{M_1,\lambda}(z_n - \lambda E_1 z_n) - J_{M_1,\lambda}(x^* - \lambda E_1 x^*)\|^2 \\
&\le \|(z_n - \lambda E_1 z_n) - (x^* - \lambda E_1 x^*)\|^2 \\
&\le \|z_n - x^*\|^2 + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&\le \|J_{M_2,\mu}(u_n - \mu E_2 u_n) - J_{M_2,\mu}(x^* - \mu E_2 x^*)\|^2 \\
&\quad + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&\le \|(u_n - \mu E_2 u_n) - (x^* - \mu E_2 x^*)\|^2 \\
&\quad + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&\le \|u_n - x^*\|^2 + \mu(\mu - 2\eta_2)\|E_2 u_n - E_2 x^*\|^2 \\
&\quad + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&\le \|x_n - x^*\|^2 + r(r - 2\delta)\|Qx_n - Qx^*\|^2 \\
&\quad + \mu(\mu - 2\eta_2)\|E_2 u_n - E_2 x^*\|^2 \\
&\quad + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2.
\end{aligned} \tag{62}$$

By (40), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - W_n y_n) \\
&\quad + (I - \alpha_n A)(W_n y_n - x^*)\|^2 \\
&\le \|(I - \alpha_n A)(W_n y_n - x^*) + \beta_n(x_n - W_n y_n)\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - Ax^*, x_{n+1} - x^*\rangle \\
&\le \|(I - \alpha_n A)(y_n - x^*) + \beta_n(x_n - W_n y_n)\|^2 \\
&\quad + 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|x_{n+1} - x^*\| \\
&= (1 - \alpha_n\overline{\gamma})^2\|y_n - x^*\|^2 + \beta_n^2\|x_n - W_n y_n\|^2 \\
&\quad + 2(1 - \alpha_n\overline{\gamma})\beta_n\|y_n - x^*\|\|x_n - W_n y_n\| \\
&\quad + 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|x_{n+1} - x^*\|.
\end{aligned} \tag{63}$$

Substituting (62) into (63), imply that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 + r(r - 2\delta)\|Qx_n - Qx^*\|^2 \\
&\quad + \mu(\mu - 2\eta_2)\|E_2 u_n - E_2 x^*\|^2 \\
&\quad + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&\quad + \beta_n^2\|x_n - W_n y_n\|^2 \\
&\quad + 2(1 - \alpha_n\overline{\gamma})\beta_n\|y_n - x^*\|\|x_n - W_n y_n\| \\
&\quad + 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|x_{n+1} - x^*\|.
\end{aligned} \tag{64}$$

Thus,

$$\begin{aligned}
&r(2\delta - r)\|Qx_n - Qx^*\|^2 + \mu(2\eta_2 - \mu)\|E_2 u_n - E_2 x^*\|^2 \\
&\quad + \lambda(2\eta_1 - \lambda)\|E_1 z_n - E_1 x^*\|^2 \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \beta_n^2\|x_n - W_n y_n\|^2 \\
&\quad + 2(1 - \alpha_n\overline{\gamma})\beta_n\|y_n - x^*\|\|x_n - W_n y_n\| \\
&\quad + 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|x_{n+1} - x^*\| \\
&\le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| \\
&\quad + \beta_n^2\|x_n - W_n y_n\|^2 \\
&\quad + 2(1 - \alpha_n\overline{\gamma})\beta_n\|y_n - x^*\|\|x_n - W_n y_n\| \\
&\quad + 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|x_{n+1} - x^*\|.
\end{aligned} \tag{65}$$

By conditions (C1), (C2), (58), and (61), we deduce immediately that

$$\begin{aligned}
\lim_{n\to\infty}\|Qx_n - Qx^*\| &= \lim_{n\to\infty}\|E_1 z_n - E_1 x^*\| \\
&= \lim_{n\to\infty}\|E_2 u_n - E_2 x^*\| = 0.
\end{aligned} \tag{66}$$

Step 4. We show that $\lim_{n\to\infty}\|x_n - Wx_n\| = 0$. Since T_r is firmly nonexpansive, we have

$$\begin{aligned}
&\|u_n - x^*\|^2 \\
&= \|T_r(x_n - rQx_n) - T_r(x^* - rQx^*)\|^2 \\
&\le \langle (x_n - rQx_n) - (x^* - rQx^*), u_n - x^*\rangle \\
&= \frac{1}{2}\{\|(x_n - rQx_n) - (x^* - rQx^*)\|^2 + \|u_n - x^*\|^2\} \\
&\quad - \frac{1}{2}\{\|(x_n - rQx_n) - (x^* - rQx^*) - (u_n - x^*)\|^2\} \\
&= \frac{1}{2}\{\|x_n - x^*\|^2 + \|u_n - x^*\|^2 \\
&\qquad - \|(x_n - u_n) - r(Qx_n - Qx^*)\|^2\}
\end{aligned}$$

$$= \frac{1}{2}\left\{\left\|x_n - x^*\right\|^2 + \left\|u_n - x^*\right\|^2 - \left(\left\|x_n - u_n\right\|^2 + r^2\left\|Qx_n - Qx^*\right\|^2 - 2r\left\langle x_n - u_n, Qx_n - Qx^*\right\rangle\right)\right\}$$
$$\leq \frac{1}{2}\left\{\left\|x_n - x^*\right\|^2 + \left\|u_n - x^*\right\|^2 - \left\|x_n - u_n\right\|^2 - r^2\left\|Qx_n - Qx^*\right\|^2 + 2r\left\|x_n - u_n\right\|\left\|Qx_n - Qx^*\right\|\right\}, \tag{67}$$

which implies that

$$\left\|u_n - x^*\right\|^2 \leq \left\|x_n - x^*\right\|^2 - \left\|x_n - u_n\right\|^2 + 2r\left\|x_n - u_n\right\|\left\|Qx_n - Qx^*\right\|. \tag{68}$$

Since $J_{M_1,\lambda}$ is 1-inverse-strongly monotone, we have

$$\left\|y_n - x^*\right\|^2$$
$$= \left\|J_{M_1,\lambda}\left(z_n - \lambda E_1 z_n\right) - J_{M_1,\lambda}\left(x^* - \lambda E_1 x^*\right)\right\|^2$$
$$\leq \left\langle \left(z_n - \lambda E_1 z_n\right) - \left(x^* - \lambda E_1 x^*\right), y_n - x^*\right\rangle$$
$$= \frac{1}{2}\left\{\left\|\left(z_n - \lambda E_1 z_n\right) - \left(x^* - \lambda E_1 x^*\right)\right\|^2 + \left\|y_n - x^*\right\|^2\right\} - \frac{1}{2}\left\{\left\|\left(z_n - \lambda E_1 z_n\right) - \left(x^* - \lambda E_1 x^*\right) - \left(y_n - x^*\right)\right\|^2\right\}$$
$$= \frac{1}{2}\left\{\left\|z_n - x^*\right\|^2 + \left\|y_n - x^*\right\|^2 - \left\|\left(z_n - y_n\right) - \lambda\left(E_1 z_n - E_1 x^*\right)\right\|^2\right\}$$
$$= \frac{1}{2}\left\{\left\|z_n - x^*\right\|^2 + \left\|y_n - x^*\right\|^2 - \left(\left\|z_n - y_n\right\|^2 + \lambda^2\left\|E_1 z_n - E_1 x^*\right\|^2 - 2\lambda\left\langle z_n - y_n, E_1 z_n - E_1 x^*\right\rangle\right)\right\}$$
$$\leq \frac{1}{2}\left\{\left\|z_n - x^*\right\|^2 + \left\|y_n - x^*\right\|^2 - \left\|z_n - y_n\right\|^2 - \lambda^2\left\|E_1 z_n - E_1 x^*\right\|^2 + 2\lambda\left\|z_n - y_n\right\|\left\|E_1 z_n - E_1 x^*\right\|\right\}, \tag{69}$$

which implies that

$$\left\|y_n - x^*\right\|^2 \leq \left\|z_n - x^*\right\|^2 - \left\|z_n - y_n\right\|^2 + 2\lambda\left\|z_n - y_n\right\|\left\|E_1 z_n - E_1 x^*\right\|. \tag{70}$$

In the same way with (70), we can get

$$\left\|z_n - x^*\right\|^2 \leq \left\|u_n - x^*\right\|^2 - \left\|u_n - z_n\right\|^2 + 2\mu\left\|u_n - z_n\right\|\left\|E_2 u_n - E_2 x^*\right\|. \tag{71}$$

Substituting (71) into (70), imply that

$$\left\|y_n - x^*\right\|^2 \leq \left\|u_n - x^*\right\|^2 - \left\|u_n - z_n\right\|^2 + 2\mu\left\|u_n - z_n\right\|\left\|E_2 u_n - E_2 x^*\right\| - \left\|z_n - y_n\right\|^2 + 2\lambda\left\|z_n - y_n\right\|\left\|E_1 z_n - E_1 x^*\right\|. \tag{72}$$

Again, substituting (68) into (72), we get

$$\left\|y_n - x^*\right\|^2$$
$$\leq \left\{\left\|x_n - x^*\right\|^2 - \left\|x_n - u_n\right\|^2 + 2r\left\|x_n - u_n\right\|\left\|Qx_n - Qx^*\right\|\right\} - \left\|u_n - z_n\right\|^2 + 2\mu\left\|u_n - z_n\right\|\left\|E_2 u_n - E_2 x^*\right\| - \left\|z_n - y_n\right\|^2 + 2\lambda\left\|z_n - y_n\right\|\left\|E_1 z_n - E_1 x^*\right\|. \tag{73}$$

Substituting (73) into (63), imply that

$$\left\|x_{n+1} - x^*\right\|^2$$
$$\leq \left(1 - \alpha_n\overline{\gamma}\right)^2\left\{\left\|x_n - x^*\right\|^2 - \left\|x_n - u_n\right\|^2 + 2r\left\|x_n - u_n\right\|\left\|Qx_n - Qx^*\right\| - \left\|u_n - z_n\right\|^2 + 2\mu\left\|u_n - z_n\right\|\left\|E_2 u_n - E_2 x^*\right\| - \left\|z_n - y_n\right\|^2 + 2\lambda\left\|z_n - y_n\right\|\left\|E_1 z_n - E_1 x^*\right\|\right\}$$
$$+ \beta_n^2\left\|x_n - W_n y_n\right\|^2 + 2\left(1 - \alpha_n\overline{\gamma}\right)\beta_n\left\|y_n - x^*\right\|\left\|x_n - W_n y_n\right\| + 2\alpha_n\left\|\gamma f\left(x_n\right) - Ax^*\right\|\left\|x_{n+1} - x^*\right\|. \tag{74}$$

Then, we derive

$$\left(1 - \alpha_n\overline{\gamma}\right)^2\left(\left\|x_n - u_n\right\|^2 + \left\|u_n - z_n\right\|^2 + \left\|z_n - y_n\right\|^2\right)$$
$$\leq \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 + 2r\left\|x_n - u_n\right\|\left\|Qx_n - Qx^*\right\| + 2\mu\left\|u_n - z_n\right\|\left\|E_2 u_n - E_2 x^*\right\|$$
$$+ 2\lambda\left\|z_n - y_n\right\|\left\|E_1 z_n - E_1 x^*\right\| + \beta_n^2\left\|x_n - W_n y_n\right\|^2 + 2\left(1 - \alpha_n\overline{\gamma}\right)\beta_n\left\|y_n - x^*\right\|\left\|x_n - W_n y_n\right\|$$
$$+ 2\alpha_n\left\|\gamma f\left(x_n\right) - Ax^*\right\|\left\|x_{n+1} - x^*\right\|$$

$$\begin{aligned}&\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\| \\ &\quad + 2r \|x_n - u_n\| \|Qx_n - Qx^*\| \\ &\quad + 2\mu \|u_n - z_n\| \|E_2u_n - E_2x^*\| \\ &\quad + 2\lambda \|z_n - y_n\| \|E_1z_n - E_1x^*\| + \beta_n^2\|x_n - W_ny_n\|^2 \\ &\quad + 2(1 - \alpha_n\overline{\gamma}) \beta_n \|y_n - x^*\| \|x_n - W_ny_n\| \\ &\quad + 2\alpha_n \|\gamma f(x_n) - Ax^*\| \|x_{n+1} - x^*\|.\end{aligned} \tag{75}$$

By conditions (C1), (C2), (58), (61), and (66), we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|u_n - z_n\| = \lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{76}$$

Observe that

$$\begin{aligned}\|W_ny_n - y_n\| \leq \|W_ny_n - x_n\| + \|x_n - u_n\| \\ + \|u_n - z_n\| + \|z_n - y_n\|.\end{aligned} \tag{77}$$

By (61) and (76), we have

$$\lim_{n\to\infty} \|W_ny_n - y_n\| = 0. \tag{78}$$

Note that

$$\|Wy_n - y_n\| \leq \|Wy_n - W_ny_n\| + \|W_ny_n - y_n\|. \tag{79}$$

From Lemma 1, we get

$$\lim_{n\to\infty} \|Wy_n - W_ny_n\| = 0. \tag{80}$$

By (78) and (80), we have $\lim_{n\to\infty}\|Wy_n - y_n\| = 0$. It follows that $\lim_{n\to\infty}\|Wx_n - x_n\| = 0$.

Step 5. We show that $\limsup_{n\to\infty}\langle(\gamma f - A)z, x_n - z\rangle \leq 0$, where $z = P_\Theta(\gamma f + I - A)z$. It is easy to see that $P_\Theta(\gamma f + (I - A))$ is a contraction of H into itself. Indeed, since $0 < \gamma < \overline{\gamma}/\alpha$, we have

$$\begin{aligned}&\|P_\Theta(\gamma f + (I - A))x - P_\Theta(\gamma f + (I - A))y\| \\ &\quad \leq \|(\gamma f + (I - A))x - (\gamma f + (I - A))y\| \\ &\quad \leq \gamma \|f(x) - f(y)\| + |I - A| \|x - y\| \\ &\quad \leq \gamma\alpha \|x - y\| + (1 - \overline{\gamma}) \|x - y\| \\ &\quad = (1 - \overline{\gamma} + \gamma\alpha) \|x - y\|.\end{aligned} \tag{81}$$

Since H is complete, there exists a unique fixed point $z \in H$ such that $z = P_\Theta(\gamma f + I - A)(z)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, such that

$$\lim_{i\to\infty} \langle (A - \gamma f)z, z - x_{n_i}\rangle = \limsup_{n\to\infty} \langle (A - \gamma f)z, z - x_n\rangle. \tag{82}$$

Also, since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $w \in C$. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. From $\|Wx_n - x_n\| \to 0$, we obtain $Wx_{n_i} \rightharpoonup w$. Then, by the demiclosed principle of nonexpansive mappings, we obtain $w \in \cap_{i=1}^{\infty}F(T_i)$.

Next, we show that $w \in \mathrm{MEP}(F, \varphi)$. Since $u_n = T_r(x_n - rQx_n)$, we obtain

$$\begin{aligned}&F(u_n, y) + \varphi(y) - \varphi(u_n) \\ &\quad + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \quad \forall y \in C.\end{aligned} \tag{83}$$

From (A2), we also have

$$\varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq F(y, u_n), \quad \forall y \in C, \tag{84}$$

and hence,

$$\begin{aligned}&\varphi(y) - \varphi(u_{n_i}) + \left\langle y - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rQx_{n_i})}{r}\right\rangle \\ &\quad \geq F(y, u_{n_i}), \quad \forall y \in C.\end{aligned} \tag{85}$$

For t with $0 < t \leq 1$ and $y \in H$, let $y_t = ty + (1 - t)w$. From (85) we have

$$\begin{aligned}\langle y_t - u_{n_i}, Qy_t\rangle &\geq \langle y_t - u_{n_i}, Qy_t\rangle - \varphi(y_t) + \varphi(u_{n_i}) \\ &\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rQx_{n_i})}{r}\right\rangle \\ &\quad + F(y_t, u_{n_i}) \\ &= \langle y_t - u_{n_i}, Qy_t - Qu_{n_i}\rangle \\ &\quad + \langle y_t - u_{n_i}, Qu_{n_i} - Qx_{n_i}\rangle \\ &\quad - \varphi(y_t) + \varphi(u_{n_i}) \\ &\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r}\right\rangle + F(y_t, u_{n_i}).\end{aligned} \tag{86}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Qu_{n_i} - Qx_{n_i}\| \to 0$. Further, from an inverse-strongly monotonicity of Q, we have $\langle y_t - u_{n_i}, Qy_t - Qu_{n_i}\rangle \geq 0$. So, from (A4), (A5), and the weakly lower semicontinuity of φ, $\langle u_{n_i} - x_{n_i}\rangle/r \to 0$ and $u_{n_i} \to w$ weakly, we have

$$\langle y_t - w, Qy_t\rangle \geq -\varphi(y_t) + \varphi(w) + F(y_t, w). \tag{87}$$

From (A1), (A4), and (87), we also have

$$\begin{aligned}0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t)\\ &\le tF(y_t, y) + (1-t)F(y_t, w) + t\varphi(y)\\ &\quad + (1-t)\varphi(w) - \varphi(y_t)\\ &= t(F(y_t, y) + \varphi(y) - \varphi(y_t))\\ &\quad + (1-t)(F(y_t, w) + \varphi(w) - \varphi(y_t))\\ &\le t(F(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)\langle y_t - w, Qy_t\rangle\\ &= t(F(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)t\langle y - w, Qy_t\rangle,\end{aligned} \tag{88}$$

and hence,

$$0 \le F(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - w, Qy_t\rangle. \tag{89}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F(w, y) + \varphi(y) - \varphi(w) + \langle y - w, Qw\rangle \ge 0. \tag{90}$$

This implies that $w \in \mathrm{MEP}(F, \varphi)$.

Lastly, we show that $w \in \mathrm{SQVI}(B_1, M_1, B_2, M_2)$. Since $\|u_n - z_n\| \to 0$ and $\|z_n - y_n\| \to 0$ as $n \to \infty$, we get

$$\|u_n - y_n\| \le \|u_n - z_n\| + \|z_n - y_n\|, \tag{91}$$

we conclude that $\|u_n - y_n\| \to 0$ as $n \to \infty$. Moreover, by the nonexpansivity of G in Lemma 6, we have

$$\begin{aligned}&\|y_n - G(y_n)\|\\ &\quad = \left\|J_{M_1,\lambda}\left[J_{M_2,\mu}(u_n - \mu E_2 u_n) - \lambda E_1 J_{M_2,\mu}(u_n - \mu E_2 u_n)\right]\right.\\ &\qquad \left. - G(y_n)\right\|\\ &\quad = \|G(u_n) - G(y_n)\|\\ &\quad \le \|u_n - y_n\|.\end{aligned} \tag{92}$$

Thus, $\lim_{n\to\infty}\|y_n - G(y_n)\| = 0$. According to Lemma 7, we obtain that $w \in \mathrm{SQVI}(B_1, M_1, B_2, M_2)$. Hence, $w \in \Theta$. Since $z = P_\Theta(I - A + \gamma f)(z)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma f - A)z, x_n - z\rangle &= \limsup_{i\to\infty}\langle(\gamma f - A)z, x_{n_i} - z\rangle\\ &= \langle(\gamma f - A)z, w - z\rangle\\ &\le 0.\end{aligned} \tag{93}$$

Step 6. We show that $\{x_n\}$ converges strongly to z; we compute that

$$\begin{aligned}&\|x_{n+1} - z\|^2\\ &\quad = \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n y_n - z\|^2\\ &\quad = \|\alpha_n(\gamma f(x_n) - Az) + \beta_n(x_n - z)\\ &\qquad + ((1-\beta_n)I - \alpha_n A)(W_n y_n - z)\|^2\\ &\quad = \alpha_n^2\|\gamma f(x_n) - Az\|^2\\ &\qquad + \|\beta_n(x_n - z) + ((1-\beta_n)I - \alpha_n A)(W_n y_n - z)\|^2\\ &\qquad + 2\langle\beta_n(x_n - z) + ((1-\beta_n)I - \alpha_n A)\\ &\qquad\qquad \times(W_n y_n - z), \alpha_n(\gamma f(x_n) - Az)\rangle\\ &\quad \le \alpha_n^2\|\gamma f(x_n) - Az\|^2\\ &\qquad + \{\beta_n\|x_n - z\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|y_n - z\|\}^2\\ &\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(x_n) - Az\rangle\\ &\qquad + 2\alpha_n(1 - \beta_n - \alpha_n\overline{\gamma})\langle W_n y_n - z, \gamma f(x_n) - Az\rangle\\ &\quad \le \alpha_n^2\|\gamma f(x_n) - Az\|^2\\ &\qquad + \{\beta_n\|x_n - z\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - z\|\}^2\\ &\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(x_n) - \gamma f(z)\rangle\\ &\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(z) - Az\rangle\\ &\qquad + 2\alpha_n(1 - \beta_n - \alpha_n\overline{\gamma})\langle W_n y_n - z, \gamma f(x_n) - \gamma f(z)\rangle\\ &\qquad + 2\alpha_n(1 - \beta_n - \alpha_n\overline{\gamma})\langle W_n y_n - z, \gamma f(z) - Az\rangle\\ &\quad \le \alpha_n^2\|\gamma f(x_n) - Az\|^2 + (1 - \alpha_n\overline{\gamma})^2\|x_n - z\|^2\\ &\qquad + 2\alpha_n\beta_n\gamma\|x_n - z\|\|f(x_n) - f(z)\|\\ &\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(z) - Az\rangle\\ &\qquad + 2\alpha_n(1 - \beta_n - \alpha_n\overline{\gamma})\gamma\|W_n y_n - z\|\|f(x_n) - f(z)\|\\ &\qquad + 2\alpha_n(1 - \beta_n - \alpha_n\overline{\gamma})\langle W_n y_n - z, \gamma f(z) - Az\rangle\\ &\quad \le \alpha_n^2\|\gamma f(x_n) - Az\|^2 + (1 - \alpha_n\overline{\gamma})^2\|x_n - z\|^2\\ &\qquad + 2\alpha_n\beta_n\gamma\alpha\|x_n - z\|^2\\ &\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(z) - Az\rangle\\ &\qquad + 2\alpha_n(1 - \beta_n - \alpha_n\overline{\gamma})\gamma\alpha\|x_n - z\|^2\\ &\qquad + 2\alpha_n(1 - \beta_n - \alpha_n\overline{\gamma})\langle W_n y_n - z, \gamma f(z) - Az\rangle\\ &\quad = \alpha_n^2\|\gamma f(x_n) - Az\|^2\end{aligned}$$

$$
\begin{aligned}
&+\left(1-2\alpha_n\overline{\gamma}+\alpha_n^2\overline{\gamma}^2+2\alpha_n\gamma\alpha-2\alpha_n^2\overline{\gamma}\gamma\alpha\right) \\
&\times\left\|x_n-z\right\|^2+2\alpha_n\beta_n\left\langle x_n-z,\gamma f(z)-Az\right\rangle \\
&+2\alpha_n\left(1-\beta_n-\alpha_n\overline{\gamma}\right)\left\langle W_ny_n-z,\gamma f(z)-Az\right\rangle \\
\leq&\left\{1-\alpha_n\left(2\overline{\gamma}-\alpha_n\overline{\gamma}^2-2\gamma\alpha+2\alpha_n\overline{\gamma}\gamma\alpha\right)\right\}\left\|x_n-z\right\|^2 \\
&+\alpha_n^2\left\|\gamma f(x_n)-Az\right\|^2 \\
&+2\alpha_n\beta_n\left\langle x_n-z,\gamma f(z)-Az\right\rangle \\
&+2\alpha_n\left(1-\beta_n-\alpha_n\overline{\gamma}\right)\left\langle W_ny_n-z,\gamma f(z)-Az\right\rangle \\
\leq&\left\{1-\alpha_n\left(2\overline{\gamma}-\alpha_n\overline{\gamma}^2-2\gamma\alpha+2\alpha_n\overline{\gamma}\gamma\alpha\right)\right\}\left\|x_n-z\right\|^2 \\
&+\alpha_n\sigma_n,
\end{aligned}
\tag{94}
$$

where $\sigma_n = \alpha_n\|\gamma f(x_n) - Az\|^2 + 2\beta_n\langle x_n - z, \gamma f(z) - Az\rangle + 2(1-\beta_n-\alpha_n\overline{\gamma})\langle W_ny_n - z, \gamma f(z) - Az\rangle$. It is easy to see that $\limsup_{n\to\infty}\sigma_n \leq 0$. Applying Lemma 3 to (94), we conclude that $x_n \to z$. This completes the proof. □

Next, the following example shows that all conditions of Theorem 8 are satisfied.

Example 9. For instance, let $\alpha_n = 1/2(n+1)$, let $\beta_n = (2n+2)/2(2n)$, let $\lambda_n = n/(n+1)$. Then, we will show that the sequences $\{\alpha_n\}$ satisfy condition (C1). Indeed, we take $\alpha_n = 1/2(n+1)$; then, we have

$$
\begin{aligned}
\sum_{n=1}^{\infty}\alpha_n &= \sum_{n=1}^{\infty}\frac{1}{2(n+1)} = \infty, \\
\lim_{n\to\infty}\alpha_n &= \lim_{n\to\infty}\frac{1}{2(n+1)} = 0.
\end{aligned}
\tag{95}
$$

We will show that the sequences $\{\beta_n\}$ satisfy condition (C2). Indeed, we set $\beta_n = (2n+2)/2(2n) = (1/2) + (1/2n)$. It is easy to see that $0 < \liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n < 1$.

Next, we will show the condition (C3) is satisfied. We take $\lambda_n = n/(n+1)$; then we compute

$$
\begin{aligned}
\lim_{n\to\infty}\left|\lambda_n-\lambda_{n-1}\right| &= \lim_{n\to\infty}\left|\frac{n}{n+1}-\frac{n-1}{(n-1)+1}\right| \\
&= \lim_{n\to\infty}\left|\frac{n(n)-(n-1)(n+1)}{(n+1)n}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^2-n^2+1}{(n+1)n}\right| \\
&= \lim_{n\to\infty}\left|\frac{1}{n(n+1)}\right|.
\end{aligned}
\tag{96}
$$

Then, we have $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$. The sequence $\{\lambda_n\}$ satisfies condition (C3).

Using Theorem 8, we obtain the following corollaries.

Corollary 10. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C\times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuos and convex function. Let $T_i : C \to C$ be nonexpansive mappings for all $i = 1, 2, 3, \ldots$, such that $\Theta := \cap_{i=1}^{\infty}F(T_i) \cap SQVI(B_1, M_1, B_2, M_2) \cap MEP(F, \varphi) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$ and let Q, E_1, E_2 be δ, η_1, η_2-inverse-strongly monotone mapping of C into H. Let $M_1, M_2 : H \to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2 holds and let W_n be the W-mapping defined by* (31). *Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$, and*

$$
\begin{aligned}
&F(u_n, y)+\varphi(y)-\varphi(u_n) \\
&\quad+\frac{1}{r}\left\langle y-u_n, u_n-(x_n-rQx_n)\right\rangle \geq 0, \quad \forall y\in C, \\
&z_n = J_{M_2,\mu}\left(u_n-\mu E_2u_n\right), \\
&y_n = J_{M_1,\lambda}\left(z_n-\lambda E_1z_n\right), \\
&x_{n+1} = \alpha_nf(x_n)+\beta_nx_n+\left(1-\beta_n-\alpha_n\right)W_ny_n, \quad \forall n\geq 0,
\end{aligned}
\tag{97}
$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, and $r \in (0, 2\delta)$ satisfy the following conditions:

(C1) $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ converges strongly to $x^* \in \Theta$, where $x^* = P_\Theta(f + I)(x^*)$, P_Θ is the metric projection of H onto Θ and (x^*, y^*), where $y^* = J_{M_2,\mu}(x^* - \mu E_2x^*)$ is solution to the problem (6).

Proof. Taking $\gamma \equiv 1$ and $A \equiv I$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 11. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C\times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuos and convex function. Let $T_i : C \to C$ be a nonexpansive mappings for all $i = 1, 2, 3, \ldots$, such that $\Theta := \cap_{i=1}^{\infty}F(T_i)\cap SQVI(B_1, M_1, B_2, M_2)\cap MEP(F, \varphi) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$ and let E_1, E_2 be η_1, η_2-inverse-strongly monotone mapping of C into H. Let A be strongly positive bounded linear self-adjoint on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$, let $M_1, M_2 : H \to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2 holds and let W_n be the W-mapping defined* by (31). Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$, and

$$\begin{gathered} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ \forall y \in C, \\ z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A) W_n y_n, \\ \forall n \geq 0, \end{gathered} \tag{98}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, and $r \in (0,\infty)$ satisfy the following conditions:

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ converges strongly to $x^* \in \Theta$, where $x^* = P_\Theta(\gamma f + I - A)(x^*)$, P_Θ is the metric projection of H onto Θ and (x^*, y^*), where $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ is solution to the problem (6).

Proof. Taking $Q \equiv 0$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 12. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuos and convex function such that $\Theta := SQVI(B_1, M_1, B_2, M_2) \cap MEP(F, \varphi) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$ and let Q, E_1, E_2 be δ, η_1, η_2-inverse-strongly monotone mapping of C into H. Let A be a strongly positive bounded linear self-adjoint on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$, let $M_1, M_2 : H \to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2 holds, let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$, and*

$$\begin{gathered} F(u_n, y) + \varphi(y) - \varphi(u_n) \\ + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \quad \forall y \in C, \\ z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A) y_n, \quad \forall n \geq 0, \end{gathered} \tag{99}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, and $r \in (0, 2\delta)$ satisfy the following conditions:

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ converges strongly to $x^* \in \Theta$, where $x^* = P_\Theta(\gamma f + I - A)(x^*)$, P_Θ is the metric projection of H onto Θ and (x^*, y^*), where $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ is solution to the problem (7), which is the unique solution of the variational inequality

$$\langle (\gamma f - A) x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Theta, \tag{100}$$

Proof. Taking $W_n \equiv I$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 13. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5). Let $T_i : C \to C$ be nonexpansive mappings for all $i = 1, 2, 3, \ldots$, such that $\Theta := \cap_{i=1}^{\infty} F(T_i) \cap SQVI(B_1, M_1, B_2, M_2) \cap EP(F) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$ and let Q, E_1, E_2 be δ, η_1, η_2-inverse-strongly monotone mapping of C into H. Let A be a strongly positive bounded linear self-adjoint on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$, let $M_1, M_2 : H \to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2 holds and let W_n be the W-mapping defined by (31). Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$, and*

$$\begin{gathered} F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \\ \forall y \in C, \\ z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A) W_n y_n, \\ \forall n \geq 0, \end{gathered} \tag{101}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, and $r \in (0, 2\delta)$ satisfy the following conditions:

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ converges strongly to $x^* \in \Theta$, where $x^* = P_\Theta(\gamma f + I - A)(x^*)$, P_Θ is the metric projection of H onto Θ and (x^*, y^*), where $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ is solution to the problem (6).

Proof. Taking $\varphi \equiv 0$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 14. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) such that $\Theta := SQVI(B_1, M_1, B_2, M_2) \cap EP(F) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$ and let Q, E_1, E_2 be δ, η_1, η_2-inverse-strongly monotone mapping of C into H. Let A be a strongly positive bounded linear self-adjoint on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$, let $M_1, M_2 : H \to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2*

holds, let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $u_n \in C$, *and*

$$\begin{aligned} F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle &\geq 0, \\ &\forall y \in C, \\ z_n &= J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ y_n &= J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\, y_n&, \\ &\forall n \geq 0, \end{aligned} \tag{102}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, *and* $r \in (0, 2\delta)$ *satisfy the following conditions:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ *converges strongly to* $x^* \in \Theta$, *where* $x^* = P_\Theta(\gamma f + I - A)(x^*)$, P_Θ *is the metric projection of* H *onto* Θ *and* (x^*, y^*), *where* $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ *is solution to the problem* (6).

Proof. Taking $\varphi \equiv 0$ and $W_n \equiv I$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 15. *Let* C *be a nonempty closed convex subset of a real Hilbert Space* H. *Let* F *be a bifunction of* $C \times C$ *into real numbers* $\mathbb{R}$ *satisfying (A1)–(A5) such that* $\Theta := SQVI(B_1, M_1, B_2, M_2) \cap EP(F) \neq \emptyset$. *Let* f *be a contraction of* C *into itself with coefficient* $\alpha \in (0,1)$ *and let* Q, E_1, E_2 *be* δ, η_1, η_2*-inverse-strongly monotone mapping of* C *into* H. *Let* $M_1, M_2 : H \to 2^H$ *be a maximal monotone mapping. Let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $u_n \in C$, *and*

$$\begin{aligned} F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle &\geq 0, \\ &\forall y \in C, \\ z_n &= J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ y_n &= J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\beta_n-\alpha_n)\, y_n&, \\ &\forall n \geq 0, \end{aligned} \tag{103}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, *and* $r \in (0, 2\delta)$ *satisfy the following conditions:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ *converges strongly to* $x^* \in \Theta$, *where* $x^* = P_\Theta(f + I)(x^*)$, P_Θ *is the metric projection of* H *onto* Θ *and* (x^*, y^*), *where* $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ *is solution to the problem* (6).

Proof. Taking $\gamma \equiv 1$, $A \equiv I$, $\varphi \equiv 0$, and $W_n \equiv I$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 16. *Let* C *be a nonempty closed convex subset of a real Hilbert Space* H. *Let* F *be a bifunction of* $C \times C$ *into real numbers* $\mathbb{R}$ *satisfying (A1)–(A5) such that* $\Theta := SQVI(B_1, M_1, B_2, M_2) \cap EP(F) \neq \emptyset$. *Let* f *be a contraction of* C *into itself with coefficient* $\alpha \in (0,1)$ *and let* Q, E *be* δ, η*-inverse-strongly monotone mapping of* C *into* H. *Let* $M_1, M_2 : H \to 2^H$ *be a maximal monotone mapping. Let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $u_n \in C$, *and*

$$\begin{aligned} F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle &\geq 0, \\ &\forall y \in C, \\ z_n &= J_{M_2,\mu}(u_n - \mu E u_n), \\ y_n &= J_{M_1,\lambda}(z_n - \lambda E z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\beta_n-\alpha_n)\, y_n&, \\ &\forall n \geq 0, \end{aligned} \tag{104}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta)$, $\mu \in (0, 2\eta)$, *and* $r \in (0, 2\delta)$ *satisfy the following conditions:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ *converges strongly to* $x^* \in \Theta$, *where* $x^* = P_\Theta(f + I)(x^*)$, P_Θ *is the metric projection of* H *onto* Θ *and* (x^*, y^*), *where* $y^* = J_{M_2,\mu}(x^* - \mu E x^*)$ *is solution to the problem* (6).

Proof. Taking $E_1 = E_2 = E$ in Corollary 15, we can conclude the desired conclusion easily. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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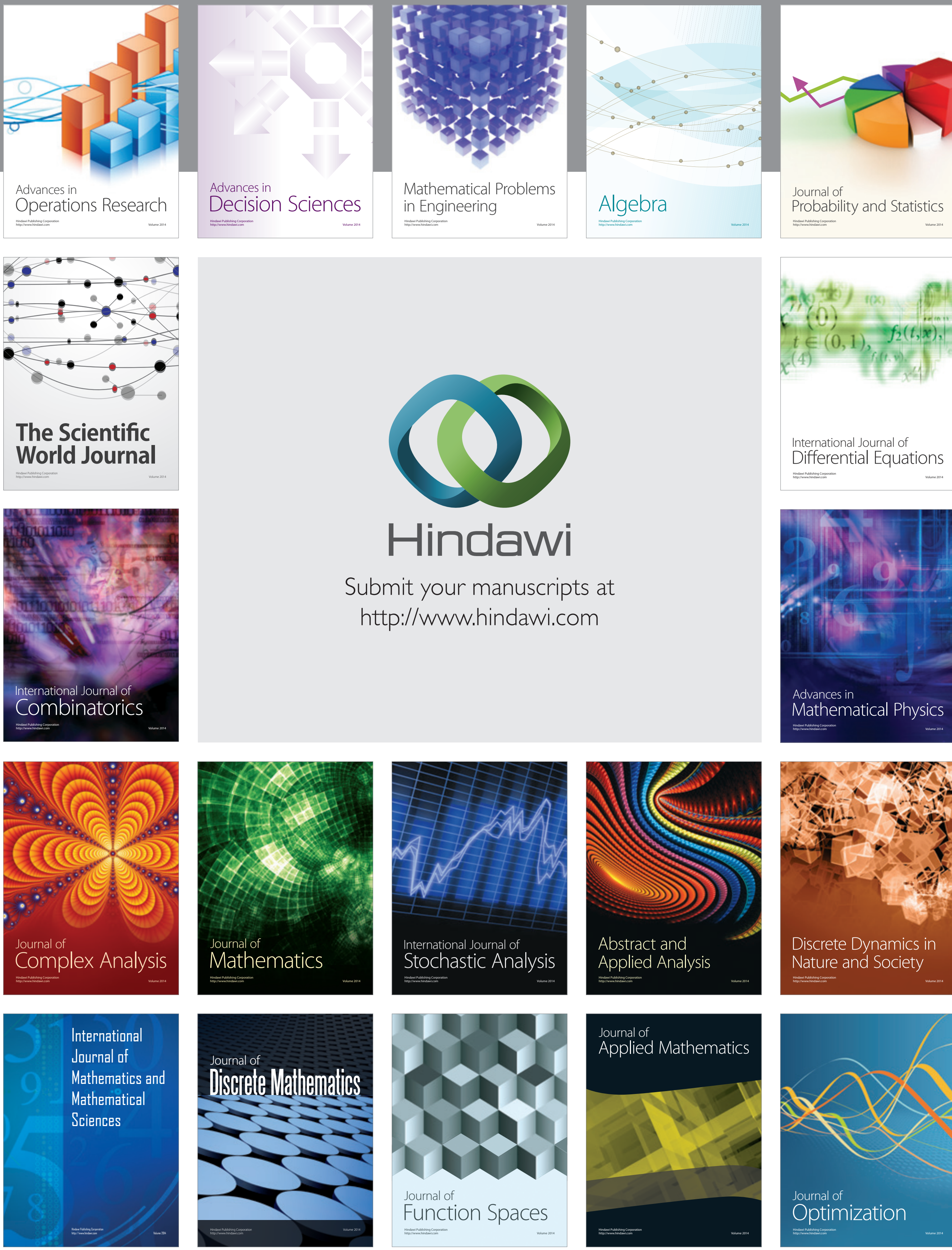
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