**Final Report**

# Some behaviors of solutions of Rational Difference Equations and Piecewise Linear Systems of Difference Equations

By Wirot Tikjha

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# Final Report

## Some behaviors of solutions of Rational Difference Equations and Piecewise Linear Systems of Difference Equations

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**บทคัดย่อ**

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**ชื่อโครงการ:** พฤติกรรมของผลเฉลยบางประการของสมการเชิงผลต่างตรรกยะและสมการเชิงผลต่างเชิงเส้นเป็นช่วง

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**บทคัดย่อ:** เราทำการศึกษาการมีขอบเขตของกรณีเฉพาะของผลเฉลยของสมการเชิงผลต่างตรรกยะ $z_{n+1}=(\alpha+\beta z_{n-1}+\gamma z_{n-1}z_n+\delta z_{n-1}z_{n-2})/(A+Bz_{n-1}+Cz_{n-1}z_n+Dz_{n-1}z_{n-2})$ $n$ = 0, 1, 2, ... ที่มีพารามิเตอร์และเงื่อนไขเริ่มต้นไม่เป็นลบและตัวส่วนไม่เป็นศูนย์ เราใช้วิธีการทำซ้ำในการหาขอบเขตของสมการเชิงผลต่างตรรกยะดังกล่าว จากนั้นเรายังศึกษาความเสถียรวงกว้าง ลักษณะการเป็นคาบ ผลเฉลยที่เป็นจุดสมดุล รวมถึงขอบเขตของระบบสมการเชิงผลต่างเชิงเส้นเป็นช่วง $x_{n+1}=|x_n|+ay_n+b$ และ $y_{n+1}=x_n+c|y_n|+d$ , n = 0, 1, 2, ... เมื่อเงื่อนไขเริ่มต้น $x_0$ และ $y_0$ เป็นจำนวนจริงใด ๆ และพารามิเตอร์ $a, b, c$ และ $d$ เป็นจำนวนเต็ม -1, 0 หรือ 1 เราพบลักษณะการเป็นคาบของผลเฉลยของระบบสมการเชิงผลต่าง $x_{n+1}=|x_n|-y_n-1$ และ $y_{n+1}=x_n+|y_n|-1$, n = 0, 1, 2, ... . นอกจากนั้นเรายังสามารถพิสูจน์ได้ว่าผลเฉลยของระบบสมการเชิงผลต่าง $x_{n+1}=|x_n|-y_n+\zeta$ and $y_{n+1}=x_n+|y_n|-\varphi$ เป็นจุดสมดุลในที่สุดเมื่อ $\zeta$ และ $\varphi$ เป็นจำนวนจริงบวกใด ๆ

**คำหลัก :** สมการเชิงผลต่าง สมการเชิงผลต่างตรรกยะ ผลเฉลยที่เป็นคาบ จุดสมดุล

## Abstract

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**Project Code:** MRG5980053

**Project Title:** Some behaviors of solutions of Rational Difference Equations and Piecewise Linear Systems of Difference Equations

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**Abstract:** We investigate the boundedness of special cases of solutions to rational difference equation $z_{n+1} = (\alpha + \beta z_{n-1} + \gamma z_{n-1} z_n + \delta z_{n-1} z_{n-2}) / (A + B z_{n-1} + C z_{n-1} z_n + D z_{n-1} z_{n-2})$ n = 0, 1, 2, ... with nonnegative parameters and with nonnegative initial conditions and positive denominators. We used the iteration method to prove the boundedness of the equations. We also investigate the global stability, the periodic nature, equilibrium solutions and the boundedness of solutions to system $x_{n+1} = |x_n| + a y_n + b$ and $y_{n+1} = x_n + c|y_n| + d$, n = 0, 1, 2, ... where the initial conditions $x_0$ and $y_0$ are arbitrary real numbers and the parameters $a, b, c$ and $d$ are integers -1, 0 or 1. We found some periodic characters of solutions to system $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n + |y_n| - 1$, n = 0, 1, 2, ... . Moreover we can prove that every solution of system $x_{n+1} = |x_n| - y_n + \zeta$ and $y_{n+1} = x_n + |y_n| - \varphi$ is eventually equilibrium point where the parameters $\zeta$ and $\varphi$ are any positive real numbers.

**Keywords:** Difference equation, Rational difference equation, Periodic solution, Equilibrium point

## Executive summary

**Introduction to the research problem and its significance:**

In recent history there has been a surge of interest in systems of both rational and piecewise linear difference equations due to their practical applications in evolutional biology, neural networks [1], economics [2], and population modeling [3]. Dr. Awerbuch's team at The Harvard School of Public Health used systems of difference equations to model the growth rate of mosquitoes while conducting research to determine the most effective method of mosquito abatement [4]. There are many models in biology and ecology [5, 6, 7] that use difference equations. The following models are examples: a discrete analogue of a model of Haematopoiesis, a discrete Baleen whale model, neural networks, a host-parasitoid phenolmenological model, a model of flour beetle populations growth, a discrete delay logistic model, a simple genotype selection model, and a model of the spread of an epidemic.

The boundedness character of the solution to a system of difference equations is necessary for understanding the global behavior of the system, including its global stability. It is also essential in the study of most applications. There are many open problems, see Camouzis and Ladas [8], in the area of pure difference equations. There are also many open problems and conjectures [6] in the applied aspect of difference equations. Biologists, economists, ecologists and other researchers are waiting for mathematicians around the globe to solve their models, or to help them create more accurate models for their applications.

**Literature review:**

In 2008, Amleh et al. [9, 10] considered the rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, 2, \ldots \tag{1}$$

with nonnegative parameters and with arbitrary nonnegative initial conditions such that the denominator is always positive. They investigated the global stability character, the periodic nature, and the boundedness of solutions to Eq.(1). Some patterns of boundedness are included in Camouzis et al. [5]. The methods and techniques that they developed to better understand the dynamics of this system are also useful in the analysis of mathematical models which involve difference equations. They believe that the special cases of System in [5, 9, 10] are genuine examples which provide prototypes for the development of the basic theory of nonlinear difference equations.

For the study of piecewise linear difference equations, Devaney [12, 13] investigated the equation, known as the gingerbreadman map,

$$x_{n+1} = |x_n| - x_{n-1} + 1, \quad n = 0, 1, 2, \ldots \tag{2}$$

which has been shown to be chaotic in certain regions and stable in others. The name of this equation is due to the fact that solutions in the plane look like a “gingerbreadman" [14] when graphed. Eq.(2) is equivalent to the piecewise linear system,

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ y_{n+1} = x_n \end{cases}, \quad n = 0, 1, 2, \dots .$$

Gerasimos Ladas and his team made significant contributions to the generalized system of gingerbreadman map and gave the system to be an open problem to investigate systems in the form of 81 piecewise linear systems:

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b \\ y_{n+1} = x_n + c|y_n| + d \end{cases}, \quad n = 0, 1, 2, \dots \tag{3}$$

where the initial conditions $x_0$ and $y_0$ are arbitrary real numbers. They numbered the above system by using their parameters a, b, c, and d. The system's number N is given by

$$N = 27(a + 1) + 9(b + 1) + 3(c + 1) + (d + 1) + 1$$

the parameters a, b, c, and d are integers between -1 and 1, inclusively. There are several researchers study about system (3) such as Grove et al.[15] studied the system number 8, a = b = -1 c = 1 and d = 0,

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ y_{n+1} = x_n + |y_n| \end{cases}, \quad n = 0, 1, 2, \dots . \tag{4}$$

They found that the system (4) has the unique equilibrium point (-2/5, -1/5) and there are two prime period 3 cycles. They has been shown that every solution to system (4) is either equilibrium point or periodic with prime period-3. Grove and Ladas [14] studied system number 4, a = b = d = -1 and c = 0,

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ y_{n+1} = x_n - 1 \end{cases}, \quad n = 0, 1, 2, \dots . \tag{5}$$

They found that the system (5) has the unique equilibrium point (0, -1) and system (5) can be reduced to second order difference equation $x_{n+1} = |x_n| - x_{n-1}$ which is periodic with prime period-9. Lapierre[17] studied system number 10, a = c = d = -1 and b = 0,

$$\begin{cases} x_{n+1} = |x_n| - y_n \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \quad n = 0, 1, 2, \dots . \tag{6}$$

She showed that every solution to system (6) is eventually the equilibrium point (1, 0) for every initial condition $(x_0, y_0) \in R^2$. Tikjha et al. [18] studied the system number 1, a = b = c = d = -1,

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \ n = 0, 1, 2, \dots . \tag{7}$$

They found that the system (7) has the unique equilibrium point (1, -1) and there are two prime period 6 cycles. They has been shown that every solution of system (7) is either equilibrium point or periodic with prime period-6.

There are still some interesting systems that are special cases of system (3) such that we can investigate the boundedness character of solutions, the global stability, and periodic nature of the solutions. Robert M. May [5] said that "...The response to large amplitude disturbances requires a nonlinear or global analysis, for which no general techniques are available ...".

**Objectives:**

1) To investigate the boundedness of a generalization of Eq.(1), which is the rational difference equation

$$z_{n+1} = \frac{\alpha + \beta z_{n-1} + \gamma z_{n-1} z_n + \delta z_{n-1} z_{n-2}}{A + B z_{n-1} + C z_{n-1} z_n + D z_{n-1} z_{n-2}}, \ n = 0, 1, 2, \dots \tag{8}$$

with nonnegative parameters

2) To discover the global stability character, the periodic nature, and the boundedness of special cases of system (3).

**Methodology:**

1) Write computer programs to simulate the behavior of rational difference equations and piecewise linear systems of difference equations.
2) Simulate the behavior of solutions to both rational difference equations and piecewise linear systems of difference equations by changing initial conditions and parameters.
3) Analyze the behavior of solutions by recognizing the pattern in each equations or systems.
4) Make conjectures by using analyzing results.
5) Prove the conjectures. The common idea of proofs is to separate initial condition into few regions and find some characters of solution to the system of each region and then establishing lemma in each region and finally summarizing the behaviors of each system to be a theorem.

## Results

The results of this research are separated into two parts which consists of boundedness of special cases of rational difference equations in (8) and global character of special cases of piecewise linear system (3).

**Boundedness of rational difference equations:**

In this section, we will show the boundedness of rational difference equations as follows:

$$z_{n+1}=\frac{\alpha+z_nz_{n-1}}{z_{n-1}z_{n-2}} \tag{9}$$

$$z_{n+1}=\frac{\alpha+z_nz_{n-1}+\delta z_{n-1}z_{n-2}}{A+z_{n-1}z_{n-2}} \tag{10}$$

$$z_{n+1}=\frac{\alpha+\beta z_{n-1}+\gamma z_nz_{n-1}+z_{n-1}z_{n-2}}{Bz_{n-1}+z_{n-1}z_{n-2}} \tag{11}$$

**Theorem 1** Let $\{z_n\}$ be a solution of System(9) with nonnegative parameters and with nonnegative initial conditions and the denominator is always positive. Then System(9) has unbounded solutions.

**Proof.** Consider

$$z_{n+1}=\frac{\alpha+z_nz_{n-1}}{z_{n-1}z_{n-2}}=\frac{\alpha}{z_{n-1}z_{n-2}}+\frac{z_nz_{n-1}}{z_{n-1}z_{n-2}}>\frac{\alpha}{z_{n-1}z_{n-2}}.$$

We see that $y_{n+1}=\frac{y_{n-1}y_{n-2}}{\alpha}\to 0$ by choosing initial condition $y_0,y_{-1},y_{-2}\in(0,1)$. Then

$$z_{n+1}=\frac{\alpha+z_nz_{n-1}}{z_{n-1}z_{n-2}}=\frac{\alpha}{z_{n-1}z_{n-2}}+\frac{z_nz_{n-1}}{z_{n-1}z_{n-2}}>\frac{\alpha}{z_{n-1}z_{n-2}}\uparrow\infty.$$

Hence $\{z_n\}$ has unbounded solutions. □

**Theorem2** Let $\{z_n\}$ be a solution of System(10) with nonnegative parameters and with nonnegative initial conditions and the denominator is always positive. Then System(10) is bounded and permanent.

**Proof.** Consider

$$z_{n+1}=\frac{\alpha+z_nz_{n-1}+\delta z_{n-1}z_{n-2}}{A+z_{n-1}z_{n-2}}\geq\frac{\max\{\alpha,\delta\}}{\min\{A,1\}}:=m.$$

Then for $n\geq 4$, we have

$$z_{n+1}<\frac{\alpha}{A+z_{n-1}z_{n-2}}+\frac{z_n}{z_{n-2}}+\delta$$

$$< \frac{\alpha}{A+z_{n-1}z_{n-2}} + \frac{1}{z_{n-2}}\left(\frac{\alpha}{A+z_{n-2}z_{n-3}} + \frac{z_{n-1}}{z_{n-3}} + \delta\right) + \delta$$

$$< \frac{\alpha}{A+z_{n-1}z_{n-2}} + \frac{1}{z_{n-2}}\left(\frac{\alpha}{A+z_{n-2}z_{n-3}} + \frac{1}{z_{n-3}}\left(\frac{\alpha}{A+z_{n-3}z_{n-4}} + \frac{z_{n-2}}{z_{n-4}} + \delta\right) + \delta\right) + \delta$$

$$< \frac{\alpha}{A+z_{n-1}z_{n-2}} + \frac{\alpha}{z_{n-2}\left(A+z_{n-2}z_{n-3}\right)} + \frac{\alpha}{z_{n-2}z_{n-3}\left(A+z_{n-3}z_{n-4}\right)} + \frac{z_{n-2}}{z_{n-2}z_{n-3}z_{n-4}} + \frac{\delta}{z_{n-2}z_{n-3}} + \frac{\delta}{z_{n-2}} + \delta$$

$$< \frac{\alpha}{A+m^2} + \frac{\alpha}{m\left(A+m^2\right)} + \frac{\alpha}{m\left(A+m^2\right)} + \frac{1}{m^2} + \frac{\delta}{m^2} + \frac{\delta}{m} + \delta .$$

Hence $\{z_n\}$ is bounded and permanent. □

**Theorem3** Let $\{z_n\}$ be a solution of System(11) with nonnegative parameters and with nonnegative initial conditions and the denominator is always positive. Then System(10) is bounded and permanent.

**Proof.** Consider

$$z_{n+1} = \frac{\alpha+\beta z_{n-1}+\gamma z_n z_{n-1}+z_{n-1}z_{n-2}}{Bz_{n-1}+z_{n-1}z_{n-2}} \geq \frac{\max\{\beta,1\}}{\min\{A,1\}} := m .$$

Then for $n \geq 4$, we have

$$z_{n+1} < \frac{\alpha}{Bz_{n-1}+z_{n-1}z_{n-2}} + \frac{\beta}{B+z_{n-2}} + \frac{\gamma z_n}{B+z_{n-2}} + 1$$

$$< \frac{\alpha}{Bz_{n-1}+z_{n-1}z_{n-2}} + \frac{\beta}{B+z_{n-2}} + \frac{\gamma}{B+z_{n-2}}\left(\frac{\alpha}{Bz_{n-2}+z_{n-2}z_{n-3}} + \frac{\beta}{B+z_{n-3}} + \frac{\gamma z_{n-1}}{B+z_{n-3}} + 1\right) + 1$$

$$< \frac{\alpha}{Bz_{n-1}+z_{n-1}z_{n-2}} + \frac{\beta}{B+z_{n-2}} + \frac{\gamma}{B+z_{n-2}}\left(\frac{\alpha}{Bz_{n-2}+z_{n-2}z_{n-3}} + \frac{\beta}{B+z_{n-3}} + \frac{\gamma}{B+z_{n-3}}\left(\frac{\alpha}{Bz_{n-3}+z_{n-3}z_{n-4}}\right.\right.$$

$$\left.\left. + \frac{\beta}{B+z_{n-4}} + \frac{\gamma z_{n-2}}{B+z_{n-4}} + 1\right) + 1\right) + 1$$

$$= \frac{\alpha}{Bz_{n-1}+z_{n-1}z_{n-2}} + \frac{\beta}{B+z_{n-2}} + \frac{\alpha\gamma}{\left(B+z_{n-2}\right)\left(Bz_{n-2}+z_{n-2}z_{n-3}\right)} + \frac{\beta\gamma}{\left(B+z_{n-2}\right)\left(B+z_{n-3}\right)}$$

$$+ \frac{\alpha\gamma^2}{\left(B+z_{n-2}\right)\left(B+z_{n-3}\right)\left(Bz_{n-3}+z_{n-3}z_{n-4}\right)} + \frac{\beta\gamma^2}{\left(B+z_{n-2}\right)\left(B+z_{n-3}\right)\left(B+z_{n-4}\right)} + \frac{\gamma^3 z_{n-2}}{\left(B+z_{n-2}\right)\left(B+z_{n-3}\right)\left(B+z_{n-4}\right)}$$

$$+ \frac{\gamma^2}{\left(B+z_{n-2}\right)\left(B+z_{n-3}\right)} + \frac{\gamma}{B+z_{n-2}} + 1$$

$$< \frac{\alpha}{Bm+m^2} + \frac{\beta}{B+m} + \frac{\alpha\beta}{(B+m)(Bm+m^2)} + \frac{\beta\gamma}{(B+m)(B+m)} + \frac{\alpha\gamma^2}{(B+m)^2(Bm+m^2)} + \frac{\beta\gamma^2}{(B+m)^3}$$

$$+ \frac{\gamma^3}{(B+m)^2} + \frac{\gamma^2}{(B+m)^2} + \frac{\gamma}{(B+m)} + 1.$$

Hence $\{z_n\}$ is bounded and permanent. □

**Global character of piecewise linear systems:**

In this section, we will show the global characters of two piecewise linear systems as follows:

$$x_{n+1} = |x_n| - y_n - 1 \text{ and } y_{n+1} = x_n + |y_n| - 1. \qquad (11)$$

It is easy to verify that the following cycles being periodic solutions of System(11):

$$P_{3.1} = \begin{pmatrix} -\frac{1}{3}, & -1 \\ \frac{1}{3}, & -\frac{1}{3} \\ -\frac{1}{3}, & -\frac{1}{3} \end{pmatrix}, P_{3.2} = \begin{pmatrix} \frac{3}{5}, & \frac{1}{5} \\ -\frac{3}{5}, & -\frac{1}{5} \\ -\frac{1}{5}, & -\frac{7}{5} \end{pmatrix}, P_{4.1} = \begin{pmatrix} -1, & -1 \\ 1, & -1 \\ 1, & 1 \\ -1, & 1 \end{pmatrix}, P_{4.2} = \begin{pmatrix} 1, & -3 \\ 3, & 3 \\ -1, & 5 \\ -5, & 3 \end{pmatrix}.$$

We will separate initial condition of system (11) into 8 regions as follows:

$L_1$ = {(x, y): x ≥ 0 and y = 0}

$L_2$ = {(x, y): x = 0 and y ≥ 0}

$L_3$ = {(x, y): x ≤ 0 and y = 0}

$L_4$ = {(x, y): x = 0 and y ≤ 0}

$Q_1$ = {(x, y): x > 0 and y > 0}

$Q_2$ = {(x, y): x < 0 and y > 0}

$Q_3$ = {(x, y): x < 0 and y < 0}

$Q_4$ = {(x, y): x > 0 and y < 0}.

In this report, we will choose initial condition in $Q_1$ and $Q_3$. The following lemmas will be tools for investigating the behaviors of solutions to the system.

**Lemma 1** [19] Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(11). Suppose the initial condition $(x_N, y_N) \in \{(1, y) \mid y \in R\}$ for some positive integer $N$. Then $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is eventually the prime period-4 solution.

**Lemma 2** [19] Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(11). Suppose the initial condition $(x_N, y_N) \in \{(-1, y) \mid y \in R\}$ for some positive integer $N$. Then $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is eventually the prime period-4 solution.

**Theorem 4** [19] Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(11). Suppose the initial condition $(x_0, y_0) \in Q_2 \cup Q_4 \cup L_1 \cup L_2 \cup L_3 \cup L_4$. Then $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is eventually the prime period-3 solution or prime period-3 solution.

Now we ready to investigate behaviors of solutions to System(11). Let $(x_0, y_0) \in Q_1$. Then $x_0 > 0$ and $y_0 > 0$. Thus

$$x_1 = |x_0| - y_0 - 1 = x_0 - y_0 - 1$$
$$y_1 = x_0 + |y_0| - 1 = x_0 + y_0 - 1.$$

We separate the possible solutions into 3 cases.

$Case\ \mathrm{I}: x_0 - y_0 \geq 1$ and so $x_0 + y_0 > 1$. Then $x_1 \geq 0$ and $y_1 > 0$. We have

$$x_2 = |x_1| - y_1 - 1 = x_0 - y_0 - 1 - (x_0 + y_0 - 1) - 1 = -2y_0 - 1 < 0$$
$$y_2 = x_1 + |y_1| - 1 = x_0 - y_0 - 1 + x_0 + y_0 - 1 - 1 = 2x_0 - 3.$$

$Case\ 1.1: 2x_0 - 3 \geq 0 \left( x_0 \geq \dfrac{3}{2} \right)$. We have $y_2 = 2x_0 - 3 \geq 0$ and so

$$x_3 = |x_2| - y_2 - 1 = 2y_0 + 1 - 2x_0 + 3 - 1 = -2x_0 + 2y_0 + 3$$
$$y_3 = x_2 + |y_2| - 1 = -2y_0 - 1 + 2x_0 - 3 - 1 = 2x_0 - 2y_0 - 5.$$

$Case\ 1.1.1: -2x_0 + 2y_0 + 3 \leq 0$ and $2x_0 - 2y_0 - 5 \geq 0$

$$x_4 = |x_3| - y_3 - 1 = 2x_0 - 2y_0 - 3 - 2x_0 + 2y_0 + 5 - 1 = 1$$
$$y_4 = x_3 + |y_3| - 1 = -2x_0 + 2y_0 + 3 + 2x_0 - 2y_0 - 5 - 1 = -3.$$

$Case\ 1.1.2: -2x_0 + 2y_0 + 3 \leq 0$ and $2x_0 - 2y_0 - 5 < 0$

$$x_4 = |x_3| - y_3 - 1 = 2x_0 - 2y_0 - 3 - 2x_0 + 2y_0 + 5 - 1 = 1$$
$$y_4 = x_3 + |y_3| - 1 = -2x_0 + 2y_0 + 3 - 2x_0 + 2y_0 + 5 - 1 = -4x_0 + 4y_0 + 7$$
$$x_5 = |x_4| - y_4 - 1 = 1 + 4x_0 - 4y_0 - 7 - 1 = 4x_0 - 4y_0 - 7 > 0$$
$$y_5 = x_4 + |y_4| - 1 = 1 + 4x_0 - 4y_0 - 7 - 1 = 4x_0 - 4y_0 - 7 > 0$$
$$x_6 = |x_5| - y_5 - 1 = -1$$
$$y_6 = x_5 + |y_5| - 1 = 8x_0 - 8y_0 - 15.$$

We apply Lemma 1 or Lemma 2 to conclude that the solution is eventually prime period 4.

$Case\ 1.2: 2x_0 - 3 < 0 \left( x_0 < \dfrac{3}{2} \right)$ จะได้ว่า $y_2 < 0$

$$x_3 = |x_2| - y_2 - 1 = 2y_0 + 1 - 2x_0 + 3 - 1 = -2x_0 + 2y_0 + 3 > 0$$
$$y_3 = x_2 + |y_2| - 1 = -2y_0 - 1 - 2x_0 + 3 - 1 = -2x_0 - 2y_0 + 1 < 0$$
$$x_4 = |x_3| - y_3 - 1 = -2x_0 + 2y_0 + 3 + 2x_0 + 2y_0 - 1 - 1 = 4y_0 + 1 > 0$$
$$y_4 = x_3 + |y_3| - 1 = -2x_0 + 2y_0 + 3 + 2x_0 + 2y_0 - 1 - 1 = 4y_0 + 1 > 0$$

$$x_5 = |x_4| - y_4 - 1 = 4y_0 + 1 - 4y_0 - 1 - 1 = -1$$
$$y_5 = x_4 + |y_4| - 1 = 4y_0 + 1 + 4y_0 + 1 - 1 = 8y_0 + 1 > 0$$
$$x_6 = |x_5| - y_5 - 1 = 1 - 8y_0 - 1 - 1 = -8y_0 - 1 < 0$$
$$y_6 = x_5 + |y_5| - 1 = -1 + 8y_0 + 1 - 1 = 8y_0 - 1.$$

*Case* 1.2.1.1: $y_6 = 8y_0 - 1 \geq 0 \quad \left( y_0 \geq \frac{1}{8} \right)$

$$x_7 = |x_6| - y_6 - 1 = 8y_0 + 1 - 8y_0 + 1 - 1 = 1$$
$$y_7 = x_6 + |y_6| - 1 = -8y_0 - 1 + 8y_0 - 1 - 1 = -3.$$

*Case* 1.2.1.2 : $y_6 = 8y_0 - 1 < 0 \quad \left( y_0 < \frac{1}{8} \right)$

$$x_7 = |x_6| - y_6 - 1 = 8y_0 + 1 - 8y_0 + 1 - 1 = 1$$
$$y_7 = x_6 + |y_6| - 1 = -8y_0 - 1 - 8y_0 + 1 - 1 = -16y_0 - 1 < 0$$
$$x_8 = |x_7| - y_7 - 1 = 1 + 16y_0 + 1 - 1 = 16y_0 + 1 > 0$$
$$y_8 = x_7 + |y_7| - 1 = 1 + 16y_0 + 1 - 1 = 16y_0 + 1 > 0$$
$$x_9 = |x_8| - y_8 - 1 = 16y_0 + 1 - 16y_0 - 1 - 1 = -1$$
$$y_9 = x_8 + |y_8| - 1 = 16y_0 + 1 + 16y_0 + 1 - 1 = 32y_0 + 1 > 0$$
$$x_{10} = |x_9| - y_9 - 1 = 1 - 32y_0 - 1 - 1 = -32y_0 - 1 < 0$$
$$y_{10} = x_9 + |y_9| - 1 = -1 + 32y_0 + 1 - 1 = 32y_0 - 1.$$

*Case* 1.2.1.2.1: $y_{10} = 32y_0 - 1 \geq 0 \quad \left( \frac{1}{8} > y_0 \geq \frac{1}{32} \right)$

$$x_{11} = |x_{10}| - y_{10} - 1 = 32y_0 + 1 - 32y_0 + 1 - 1 = 1$$
$$y_{11} = x_{10} + |y_{10}| - 1 = -32y_0 - 1 + 32y_0 - 1 - 1 = -3.$$

*Case* 1.2.1.2.2 : $y_{10} = 32y_0 - 1 < 0 \quad \left( y < \frac{1}{32} \right)$

$$x_{11} = |x_{10}| - y_{10} - 1 = 32y_0 + 1 - 32y_0 + 1 - 1 = 1$$
$$y_{11} = x_{10} + |y_{10}| - 1 = -32y_0 - 1 - 32y_0 + 1 - 1 = -64y_0 - 1 < 0$$
$$x_{12} = |x_{11}| - y_{11} - 1 = 1 + 64y_0 + 1 - 1 = 64y_0 + 1 > 0$$
$$y_{12} = x_{11} + |y_{11}| - 1 = 1 + 64y_0 + 1 - 1 = 64y_0 + 1 > 0$$
$$x_{13} = |x_{12}| - y_{12} - 1 = 64y_0 + 1 - 64y_0 - 1 - 1 = -1$$
$$y_{13} = x_{12} + |y_{12}| - 1 = 64y_0 + 1 + 64y_0 + 1 - 1 = 128y_0 + 1 > 0$$
$$x_{14} = |x_{13}| - y_{13} - 1 = 1 - 128y_0 - 1 - 1 = -128y_0 - 1 < 0$$
$$y_{14} = x_{13} + |y_{13}| - 1 = -1 + 128y_0 + 1 - 1 = 128y_0 - 1.$$

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So we will have a pattern of solutions to system(11).

$Case\,\mathrm{II}: x_0 - y_0 < 1, x_0 + y_0 \geq 1.$ Then $x_1 < 0$ and $y_1 \geq 0$. Hence $(x_1, y_1) \in Q_2 \cup L_3$.

$$x_2 = |x_1| - y_1 - 1 = -x_0 + y_0 + 1 - x_0 - y_0 + 1 - 1 = -2x_0 + 1$$

$$y_2 = x_1 + |y_1| - 1 = x_0 - y_0 - 1 + x_0 + y_0 - 1 - 1 = 2x_0 - 3.$$

$Case\,1.1: x_2 = -2x_0 + 1 \geq 0$ and $y_2 = 2x_0 - 3 < 0$. We have

$$x_3 = |x_2| - y_2 - 1 = -2x_0 + 1 - 2x_0 + 3 - 1 = -4x_0 + 3 > 0$$

$$y_3 = x_2 + |y_2| - 1 = -2x_0 + 1 - 2x_0 + 3 - 1 = -4x_0 + 3 > 0$$

$$x_4 = |x_3| - y_3 - 1 = -4x_0 + 3 + 4x_0 - 3 - 1 = -1$$

$$y_4 = x_3 + |y_3| - 1 = -4x_0 + 3 - 4x_0 + 3 - 1 = -8x_0 + 5 > 0$$

$$x_5 = |x_4| - y_4 - 1 = 1 + 8x_0 - 5 - 1 = 8x_0 - 5 < 0$$

$$y_5 = x_4 + |y_4| - 1 = -1 - 8x_0 + 5 - 1 = -8y_0 + 3.$$

$Case\,1.1.1: y_5 = -8y_0 + 3 \geq 0 \quad \left(x_0 \leq \frac{3}{8}\right)$

$$x_6 = |x_5| - y_5 - 1 = -8x_0 + 5 + 8x_0 - 3 - 1 = 1$$

$$y_6 = x_5 + |y_5| - 1 = 8x_0 - 5 - 8x_0 + 3 - 1 = -3.$$

$Case\,1.1.2: y_5 = -8y_0 + 3 < 0 \quad \left(\frac{3}{8} < x_0 \leq \frac{1}{2}\right)$

$$x_6 = |x_5| - y_5 - 1 = -8x_0 + 5 + 8x_0 - 3 - 1 = 1$$

$$y_6 = x_5 + |y_5| - 1 = 8x_0 - 5 + 8x_0 - 3 - 1 = 16x_0 - 9 < 0$$ by testing in interval $\left(\frac{3}{8}, \frac{1}{2}\right]$

$$x_7 = |x_6| - y_6 - 1 = 1 - 16x_0 + 9 - 1 = -16x_0 + 9 > 0$$

$$y_7 = x_6 + |y_6| - 1 = 1 + 16x_0 - 9 - 1 = -16x_0 + 9 > 0$$

$$x_8 = |x_7| - y_7 - 1 = -16x_0 + 9 + 16x_0 - 9 - 1 = -1$$

$$y_8 = x_7 + |y_7| - 1 = -16x_0 + 9 - 16x_0 + 9 - 1 = -32x_0 + 17 > 0$$ by testing in interval $\left(\frac{3}{8}, \frac{1}{2}\right]$

$$x_9 = |x_8| - y_8 - 1 = 1 + 32x_0 - 17 - 1 = 32x_0 - 17 < 0$$

$$y_9 = x_8 + |y_8| - 1 = -1 - 32x_0 + 17 - 1 = -32x_0 + 15$$

$$x_{10} = |x_9| - y_9 - 1 = -32x_0 + 17 + 32x_0 - 15 - 1 = 1.$$

We apply Lemma 1 to conclude that the solution is eventually prime period 4.

$Case\,\mathrm{III}: x_0 - y_0 < 1, x_0 + y_0 < 1.$ Then $x_1 < 0$ and $y_1 < 0$.

$$x_2 = -2x_0 + 1$$

$$y_2 = -2y_0 - 1 < 0.$$

If $x_2 \geq 0$ then $(x_2, y_2) \in Q_4 \cup L_4$ .

Suppose that $x_2 = -2x_0 + 1 < 0 \quad \left(x_0 > \frac{1}{2}\right)$

$$x_3 = 2x_0 + 2y_0 - 1 > 0$$
$$y_3 = 2x_0 + 2y_0 + 1.$$

If $y_3 \le 0$ then $(x_3, y_3) \in Q_4 \cup L_1$.

Suppose that $y_3 = 2x_0 + 2y_0 + 1 > 0 \quad \left(x_0 - y_0 < \frac{1}{2}\right)$

$$x_4 = 4x_0 - 3$$
$$y_4 = 4y_0 - 1.$$

If $x_4 \ge 0, y_4 \ge 0$ then it will contradict with condition $x_0 + y_0 < 1$.

Suppose that $x_4 = 4x_0 - 3 < 0, y_4 = 4y_0 - 1 < 0 \quad \left(\frac{1}{2} < x_0 < \frac{3}{4}, y_0 < \frac{1}{4}\right)$. Then

$$x_5 = -4x_0 - 4y_0 + 3$$
$$y_5 = 4x_0 - 4y_0 - 3 < 0.$$

If $x_5 \ge 0$ then $(x_5, y_5) \in Q_4 \cup L_4$.

Suppose that $x_5 = -4x_0 - 4y_0 + 3 < 0 \quad \left(\frac{3}{4} < x_0 + y_0 < 1\right)$. Then

$$x_6 = 8y_0 - 1$$
$$y_6 = -8x_0 + 5.$$

If $x_6 < 0, y_6 < 0$ then it will contradict with condition $x_0 - y_0 < \frac{1}{2}$.

Suppose that $x_6 = 8y_0 - 1 > 0$, $y_6 = -8x_0 + 5 > 0 \quad \left(\frac{1}{2} < x_0 < \frac{5}{8}, \frac{1}{8} < y_0 < \frac{1}{4}\right)$. Then

$$x_7 = 8x_0 + 8y_0 - 7 < 0$$
$$y_7 = -8x_0 + 8y_0 + 3.$$

If $y_7 \ge 0$ then $(x_7, y_7) \in Q_2 \cup L_3$.

Suppose that $y_7 = -8x_0 + 8y_0 + 3 < 0 \quad \left(\frac{3}{8} < x_0 - y_0 < \frac{1}{2}\right)$. Then

$$x_8 = -16y_0 + 3$$
$$y_8 = 16x_0 - 11 < 0.$$

If $x_8 \ge 0$ then $(x_8, y_8) \in Q_4 \cup L_4$.

Suppose that $x_8 = -16y_0 + 3 < 0 \quad \left(\frac{3}{16} < y_0 < \frac{1}{4}\right)$. Then

$$x_9 = -16x_0 + 16y_0 + 7 > 0$$
$$y_9 = -16x_0 - 16y_0 + 13.$$

If $y_9 \le 0$ then $(x_9, y_9) \in Q_4 \cup L_1$.

Suppose that $y_9 = -16x_0 - 16y_0 + 13 > 0$ $\left(\frac{3}{4} < x_0 + y_0 < \frac{13}{16}\right)$. Then

$$x_{10} = 32y_0 - 7$$
$$y_{10} = -32x_0 + 19.$$

If $x_{10} \ge 0, y_{10} \ge 0$ then it will contradict with condition $x_0 - y_0 < \frac{3}{8}$.

Suppose that $x_{10} = 32y_0 - 7 < 0$, $y_{10} = -32x_0 + 19 < 0$ $\left(\frac{19}{32} < x_0 \le \frac{5}{8}, \frac{1}{8} \le y_0 < \frac{7}{32}\right)$. Then

$$x_{11} = 32x_0 - 32y_0 - 13$$
$$y_{11} = 32x_0 + 32y_0 - 27 < 0.$$

If $x_{11} \ge 0$ then $(x_{11}, y_{11}) \in Q_4 \cup L_4$.

Suppose that $x_{11} = 32x_0 - 32y_0 - 13 < 0$ $\left(\frac{3}{8} < x_0 - y_0 < \frac{13}{32}\right)$. Then

$$x_{12} = -64x_0 + 39$$
$$y_{12} = -64y_0 + 13.$$

If $x_{12} < 0, y_{12} < 0$ then it will contradict with condition $x_0 + y_0 < \frac{13}{16}$.

Suppose that $x_{12} = -64x_0 + 39 \ge 0$, $y_{12} = -64y_0 + 13 \ge 0$ $\left(\frac{19}{32} < x_0 \le \frac{39}{64}, \frac{1}{8} \le y_0 < \frac{13}{64}\right)$. Then

$$x_{13} = -64x_0 + 64y_0 + 25 < 0$$
$$y_{13} = -64x_0 - 64y_0 + 51.$$

If $y_{13} \ge 0$ then $(x_{13}, y_{13}) \in Q_2 \cup L_3$.

Suppose that $y_{13} = -64x_0 - 64y_0 + 51 < 0$ $\left(\frac{51}{64} < x_0 + y_0 < \frac{13}{16}\right)$.

$$x_{14} = 128x_0 - 77$$
$$y_{14} = 128y_0 - 27 < 0.$$

If $x_{14} \ge 0$ then $(x_{14}, y_{14}) \in Q_4 \cup L_4$.

Suppose that $x_{14} = 128x_0 - 77 < 0$ $\left(\frac{19}{32} < x_0 < \frac{77}{128}\right)$. Then

$$x_{15} = -128x_0 - 128y_0 + 103$$
$$y_{15} = 128x_0 - 128y_0 - 51.$$

If $x_{15} < 0, y_{15} < 0$ then it will contradict with condition $y_0 < \frac{26}{128}$. Suppose that

$x_{15} = -128x_0 - 128y_0 + 103 \geq 0$ , $y_{15} = 128x_0 - 128y_0 - 51 \geq 0 \left( \frac{51}{64} < x_0 + y_0 \leq \frac{103}{128}, \frac{51}{128} \leq x_0 - y_0 < \frac{13}{32} \right)$.

Then

$$x_{16} = -256x_0 + 153$$
$$y_{16} = -256y_0 + 51.$$

If $x_{16} \geq 0, y_{16} \geq 0$ then it will contradict with condition $x_0 + y_0 < \frac{51}{64}$.

Suppose that $x_{16} = -256x_0 + 153 < 0$ , $y_{16} = -256y_0 + 51 < 0 \left( \frac{153}{256} < x_0 \leq \frac{77}{128}, \frac{51}{256} \leq x_0 - y_0 < \frac{27}{128} \right)$.

Then

$$x_{17} = 256x_0 + 256y_0 - 205$$
$$y_{17} = -256x_0 + 256y_0 + 101 < 0.$$

If $x_{17} \geq 0$ then $(x_{17}, y_{17}) \in Q_4 \cup L_4$.

Suppose that $x_{17} = 256x_0 + 256y_0 - 205 < 0 \left( \frac{51}{64} < x_0 + y_0 < \frac{205}{256} \right)$.

$$x_{18} = -512y_0 + 103$$
$$y_{18} = 512x_0 - 307.$$

If $x_{18} < 0, y_{18} < 0$ then it will contradict with condition $x_0 - y_0 > \frac{51}{128}$. Suppose that

$x_{18} = -512y_0 + 103 > 0$ , $y_{18} = 512x_0 - 307 > 0$ $\left( \frac{307}{512} < x_0 < \frac{77}{128}, \frac{51}{256} < y_0 < \frac{103}{512} \right)$. Then

$$x_{19} = -512x_0 - 512y_0 + 409 < 0$$
$$y_{19} = 512x_0 - 512y_0 - 205.$$

If $y_{19} \geq 0$ then $(x_{19}, y_{19}) \in Q_2 \cup L_3$.

Suppose that $y_{19} = 512x_0 - 512y_0 - 205 < 0 \left( \frac{51}{128} < x_0 - y_0 < \frac{205}{512} \right)$. Then

$$x_{20} = 1024y_0 - 205$$
$$y_{20} = -1024x_0 + 613 < 0.$$

If $x_{20} \geq 0$ then $(x_{20}, y_{20}) \in Q_4 \cup L_4$.

Suppose that $x_{20} = 1024y_0 - 205 < 0$ $\left( \frac{51}{256} < y_0 < \frac{205}{1024} \right)$. Then

$$x_{21} = 1024x_0 - 1024y_0 - 409 > 0$$
$$y_{21} = 1024x_0 + 1024y_0 - 819.$$

If $y_{21} \geq 0$ then $(x_{21}, y_{21}) \in Q_2$.

Suppose that $y_{21} = 1024x_0 + 1024y_0 - 819 > 0 \left( \frac{819}{1024} < x_0 + y_0 < \frac{205}{256} \right)$. Then

$x_{22} = -2048y_0 + 409$

$y_{22} = 2048x_0 - 1229$.

If $x_{22} \ge 0, y_{22} \ge 0$ then it will contradict with condition $x_0 - y_0 < \frac{205}{512}$. Suppose that $x_{22} = -2048y_0 + 409 < 0,\ y_{22} = 2048x_0 - 1229 < 0$ $\left(\frac{307}{512} < x_0 < \frac{1229}{2048}, \frac{409}{2048} < y_0 < \frac{205}{1024}\right)$. Then

$x_{23} = -2048x_0 + 2048y_0 + 819$

$y_{23} = -2048x_0 - 2048y + 1637 < 0$.

If $x_{23} \ge 0$ and $(x_{23}, y_{23}) \in Q_4 \cup L_4$.

Suppose that $x_{23} = -2048x_0 + 2048y_0 + 819 < 0$ $\left(\frac{819}{2048} < x_0 - y_0 < \frac{205}{512}\right)$. Then

$x_{24} = 4096x_0 - 2457$

$y_{24} = 4096y_0 - 819$.

If $x_{24} < 0, y_{24} < 0$ then it will contradict with condition $x_0 + y_0 > \frac{819}{1024}$. Suppose that $x_{24} = 4096x_0 - 2457 \ge 0,\ y_{24} = 4096y_0 - 819 \ge 0 \left(\frac{2457}{4096} \le x_0 < \frac{1229}{2048}, \frac{819}{4096} < y_0 < \frac{205}{1024}\right)$. Then

$x_{25} = 4096x_0 - 4096y_0 - 1639 < 0$

$y_{25} = 4096x_0 + 4096y - 3277$.

If $y_{25} \ge 0$ then $(x_{25}, y_{25}) \in Q_2 \cup L_3$.

Suppose that $y_{25} = 4096x_0 + 4096y - 3277 < 0 \left(\frac{1637}{2048} < x_0 + y_0 < \frac{3277}{4096}\right)$. Then

$x_{26} = -8192x_0 + 4915$

$y_{26} = -8192y_0 + 1637 < 0$.

If $x_{26} \ge 0$ then $(x_{26}, y_{26}) \in Q_4 \cup L_4$.

Suppose that $x_{26} = -8192x_0 + 4915 < 0$ $\left(\frac{4915}{8192} < x_0 < \frac{1229}{2048}\right)$. Then

$x_{27} = 8192x_0 + 8192y_0 - 6553$

$y_{27} = -8192x_0 + 8192y_0 + 3277$.

If $x_{27} < 0, y_{27} < 0$ then it will contradict with condition $y_0 > \frac{819}{4096}$.

Suppose that

$x_{27} = 8192x_0 + 8192y_0 - 6553 \ge 0,\ y_{27} = -8192x_0 + 8192y_0 + 3277 \ge 0$

$\left(\frac{6553}{8192} \le x_0 + y_0 < \frac{3277}{4096}, \frac{819}{2048} < x_0 - y_0 \le \frac{3277}{8192}\right)$.

$x_{28} = 16384x_0 - 9831$

$$y_{28} = 16384 y_0 - 3277.$$

If $x_{28} \geq 0, y_{28} \geq 0$ then it will contradict with condition $x_0 + y_0 < \dfrac{3277}{4096}$. Suppose that

$$x_{28} = 16384 x_0 - 9831 < 0,\ y_{28} = 16384 y_0 - 3277 < 0 \left( \frac{4915}{8192} < x_0 < \frac{9831}{16384}, \frac{1637}{8192} < y_0 < \frac{205}{1024} \right).$$ Then

$$x_{29} = -16384 x_0 - 16384 y_0 + 13107$$

$$y_{29} = 16384 x_0 - 16384 y_0 - 6555.$$

If $x_{29} \geq 0, y_{29} \geq 0$ then it will contradict with condition $x_0 < \dfrac{9831}{16384}$. Suppose that

$$x_{29} = -16384 x_0 - 16384 y_0 + 13107 < 0,\ y_{29} = 16384 x_0 - 16384 y_0 - 6555 < 0$$

$$\left( \frac{13107}{16384} < x_0 + y_0 < \frac{3277}{4096}, \frac{819}{2048} < x_0 - y_0 < \frac{6555}{16384} \right)$$

$$x_{30} = 32768 y_0 - 6553$$

$$y_{30} = -32768 x_0 + 19661.$$

If $x_{30} < 0, y_{30} < 0$ then it will contradict with condition $x_0 - y_0 < \dfrac{3277}{8192}$. Suppose that

$$x_{30} = 32768 y_0 - 6553 \geq 0,\ y_{30} = -32768 x_0 + 19661 \geq 0 \left( \frac{4915}{8192} < x_0 \leq \frac{19661}{32768}, \frac{6553}{32768} \leq y_0 < \frac{205}{1024} \right).$$

Then

$$x_{31} = 32768 x_0 + 32768 y_0 - 26215$$

$$y_{31} = -32768 x_0 + 32768 y_0 + 13107.$$

If $x_{31} \geq 0, y_{31} \geq 0$ then it will contradict with condition $x_0 < \dfrac{19661}{32768}$. Suppose that

$$x_{31} = 32768 x_0 + 32768 y_0 - 26215 < 0,\ y_{31} = -32768 x_0 + 32768 y_0 + 13107 < 0$$

$$\left( \frac{13107}{16384} < x_0 + y_0 < \frac{26215}{32768}, \frac{13107}{32768} \leq x_0 - y_0 < \frac{6555}{16384} \right).$$

Then

$$x_{32} = -65536 y_0 + 13107$$

$$y_{32} = 65536 x_0 - 39323 < 0.$$

If $x_{32} \geq 0$ then $(x_{32}, y_{32}) \in Q_4 \cup L_4$.

Suppose that $x_{32} = -65536 y_0 + 13107 < 0 \quad \left( \dfrac{13107}{65536} < y_0 < \dfrac{205}{1024} \right)$. Then

$$x_{33} = -65536 x_0 + 65536 y_0 + 26215 > 0$$

$$y_{33} = -65536 x_0 - 65536 y_0 + 52429.$$

If $y_{33} \leq 0$ then $(x_{33}, y_{33}) \in Q_4 \cup L_1$.

Suppose that $y_{33} = -65536x_0 - 65536y_0 + 52429 > 0$ $\left(\frac{13107}{16384} < x_0 + y_0 < \frac{52429}{65536}\right)$. Then

$$x_{34} = 131072y_0 - 26215$$
$$y_{34} = -131072x_0 + 78643.$$

If $x_{34} \geq 0, y_{34} \geq 0$ then it will contradict with condition $x_0 - y_0 > \frac{13107}{32768}$. Suppose that

$x_{34} = 131072y_0 - 26215 < 0,\ y_{34} = -131072x_0 + 78643 < 0\left(\frac{78643}{131072} < x_0 \leq \frac{19661}{32768}, \frac{13107}{65536} < y_0 < \frac{26215}{131072}\right)$.

Then

$$x_{35} = 131072x_0 - 131072y_0 - 52429$$
$$y_{35} = 131072x_0 + 131072y_0 - 104859 < 0.$$

If $x_{35} \geq 0$ then $(x_{35}, y_{35}) \in Q_4 \cup L_4$.

Suppose that $x_{35} = 131072x_0 - 131072y_0 - 52429 < 0$ $\left(\frac{13107}{32768} < x_0 + y_0 < \frac{52429}{131072}\right)$. Then

$$x_{36} = -262144x_0 + 157287$$
$$y_{36} = -262144y_0 + 52429.$$

If $x_{36} < 0, y_{36} < 0$ then it will contradict with condition $x_0 + y_0 < \frac{52429}{65536}$. Suppose that

$x_{36} = -262144x_0 + 157287 > 0,\ y_{36} = -262144y_0 + 52429 > 0$

$\left(\frac{78643}{131072} < x_0 < \frac{157287}{262144}, \frac{13107}{65536} < y_0 < \frac{52429}{262144}\right)$.

Then $x_{37} = -262144x_0 + 262144y_0 + 104857 < 0$ and $y_{37} = -262144x_0 - 262144y_0 + 209715$.

If $y_{37} \geq 0$ then $(x_{37}, y_{37}) \in Q_2 \cup L_3$.

Suppose that $y_{37} = -262144x_0 - 262144y_0 + 209715 < 0$ $\left(\frac{209715}{262144} < x_0 + y_0 < \frac{104859}{131072}\right)$. Then

$$x_{38} = 524288x_0 - 314573$$
$$y_{38} = 524288y_0 - 104859 < 0.$$

If $x_{38} \geq 0$ then $(x_{38}, y_{38}) \in Q_4 \cup L_4$.

Suppose that $x_{38} = 524288x_0 - 314573 < 0$ $\left(\frac{78643}{131072} < x_0 < \frac{314573}{524288}\right) x_0 = 0.599995, y_0 = 0.2$

$$x_{39} = -524288x_0 - 524288y_0 + 419431 > 0$$
$$y_{39} = 524288x_0 - 524288y_0 - 209715$$

If $y_{39} \leq 0$ then $(x_{39}, y_{39}) \in Q_4 \cup L_1$.

Suppose that $y_{39} = 524288x_0 - 524288y_0 - 209715 > 0$ $\left(\frac{209715}{524288} < x_0 - y_0 < \frac{6555}{16384}\right)$. Then

$$x_{40} = -1048576x_0 + 629145$$

$y_{40} = -1048576y_0 + 209715 < 0$.

If $x_{40} \geq 0$ then $(x_{40}, y_{40}) \in Q_4 \cup L_4$. Suppose that $x_{40} = -1048576x_0 + 629145 < 0$. Then $\left(\frac{629145}{1048576} < x_0 < \frac{314573}{524288}\right)\left(\frac{209715}{1048576} < y_0 < \frac{52429}{262144}\right)$

$x_{41} = 1048576x_0 + 1048576y_0 - 838861$

$y_{41} = -1048576x_0 + 1048576y_0 + 419429 < 0$

If $x_{40} \geq 0$ then $(x_{41}, y_{41}) \in Q_4 \cup L_4$.

Suppose that $x_{41} = 1048576x_0 + 1048576y_0 - 838861 < 0$. Then

$x_{42} = -2097152y_0 + 499431$

$y_{42} = 2097152x_0 - 1258291$.

If $x_{42} < 0, y_{42} < 0$ then it will contradict with condition $x_0 - y_0 > \frac{209715}{524288}$. Suppose that $x_{42} = -2097152y_0 + 499431 > 0$ $y_{42} = 2097152x_0 - 1258291 > 0$.Then $\left(\frac{1258291}{2097152} < x_0 < \frac{314573}{524288}\right)\left(\frac{209715}{1048576} < y_0 < \frac{419431}{2097152}\right)$

$x_{43} = -2097152x_0 - 2097152y_0 + 1677721 \leq 0$

$y_{43} = 2097152x_0 - 2097152y_0 - 838861$.

If $y_{43} \geq 0$ then $(x_{43}, y_{43}) \in Q_2 \cup L_3$.

Suppose that $y_{43} = 2097152x_0 - 2097152y_0 - 838861 < 0$ $\left(\frac{209751}{524288} < x_0 - y_0 < \frac{838861}{2097152}\right)$. Then

$x_{44} = 2^{22}y_0 - 838861$

$y_{44} = 2^{22}x_0 + 2516581 < 0$.

If $x_{44} \geq 0$ then $(x_{44}, y_{44}) \in Q_2 \cup L_3$. $\left(\frac{209715}{2^{20}} < y_0 < \frac{838861}{2^{22}}\right)$. Then

$x_{45} = 2^{22}x_0 - 2^{22}y_0 - 1677721 > 0$

$y_{45} = 2^{22}x_0 - 2^{22}y_0 - 3355443$.

If $y_{45} \leq 0$ then $(x_{41}, y_{41}) \in Q_4 \cup L_1$. Then

$x_{46} = -2^{23}y_0 + 1677721$

$y_{46} = 2^{23}x_0 - 5033165$.

If $x_{46} \geq 0, y_{46} \geq 0$ then it will contradict with condition $x_0 - y_0 > \frac{838861}{2097152}$. Suppose that $x_{46} = -2^{23}y_0 + 1677721 < 0$ and $y_{46} = 2^{23}x_0 - 5033165 < 0$ $\left(\frac{1258291}{2^{21}} < x_0 < \frac{5033165}{2^{23}}, \frac{1677721}{2^{23}} < y_0 < \frac{838861}{2^{22}}\right)$. Then

$x_{47} = -2^{23}x_0 + 2^{23}y_0 + 3355443$

$$y_{47} = -2^{23}x_0 - 2^{23}y_0 + 6710885 < 0.$$

If $x_{47} \geq 0$ then $(x_{47}, y_{47}) \in Q_4 \cup L_4$. Then

$$x_{48} = 2^{24}x_0 - 10066329$$
$$y_{48} = 2^{24}y_0 - 3355443.$$

If $x_{48} \leq 0, y_{48} \leq 0$ then it will contradict with condition $x_0 + y_0 > \frac{3355443}{2^{22}}$. Suppose that $x_{48} = 2^{24}x_0 - 10066329 > 0$ and $y_{48} = 2^{24}y_0 - 3355443 > 0$ $\left(\frac{10066329}{2^{24}} < x_0 < \frac{5033165}{2^{23}}, \frac{3355443}{2^{24}} < y_0 < \frac{838861}{2^{22}}\right)$. Then

$$x_{49} = 2^{24}x_0 - 2^{24}y_0 - 6710887 < 0$$
$$y_{49} = 2^{24}x_0 + 2^{24}y_0 - 13421773.$$

If $y_{49} \geq 0$ then $(x_{47}, y_{47}) \in Q_4 \cup L_3$. Then

$$x_{50} = -2^{25}x_0 + 20132659$$
$$y_{50} = -2^{25}y_0 + 6710885 < 0.$$

If $x_{50} \geq 0$ then $(x_{50}, y_{50}) \in Q_4 \cup L_4$ $\left(\frac{20132659}{2^{25}} < x_0 < \frac{5033165}{2^{23}}\right)$. Then

$$x_{51} = 2^{25}x_0 + 2^{25}y_0 - 26843545$$
$$y_{51} = -2^{25}x_0 + 2^{25}y_0 + 13421773.$$

If $x_{51} \leq 0, y_{51} \leq 0$ then it will contradict with condition $y_0 > \frac{3355443}{2^{24}}$. Suppose that $x_{51} = 2^{25}x_0 + 2^{25}y_0 - 26843545 > 0$ and $y_{51} = -2^{25}x_0 + 2^{25}y_0 + 13421773 > 0$. Then

$$x_{52} = 2^{26}x_0 - 40265319$$
$$y_{52} = 2^{26}y_0 - 13421773.$$

If $x_{52} \geq 0, y_{52} \geq 0$ then it will contradict with condition $x_0 + y_0 < \frac{13421773}{2^{24}}$. Suppose that $x_{52} = 2^{26}x_0 - 40265319 < 0$ and $y_{52} = 2^{26}y_0 - 13421773 < 0$ $\left(\frac{20132659}{2^{25}} < x_0 < \frac{40265319}{2^{26}}, \frac{3355443}{2^{24}} < y_0 < \frac{13421773}{2^{26}}\right)$. Then

$$x_{53} = -2^{26}x_0 - 2^{26}y_0 + 53687091$$
$$y_{53} = 2^{26}x_0 - 2^{26}y_0 - 26843547.$$

If $x_{53} \geq 0, y_{53} \geq 0$ then it will contradict with condition $x_0 < \frac{40265319}{2^{26}}$. Suppose that $x_{53} = -2^{26}x_0 - 2^{26}y_0 + 53687091 < 0$ and $y_{53} = 2^{26}x_0 - 2^{26}y_0 - 26843547 < 0$. Then

$$x_{54} = 2^{27}y_0 - 26843545$$
$$y_{54} = -2^{27}x_0 + 80530637.$$

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We can formulate that formula of above solutions $\{(x_n, y_n)\}_{n=7}^{\infty}$ as follows: for $n \geq 1$, we have

$x_{24n-17} = 2^{10(n-1)+3} x_0 + 2^{10(n-1)+3} y_0 - (2\delta_n + 1)$ and
$y_{24n-17} = -2^{10(n-1)+3} x_0 + 2^{10(n-1)+3} y_0 + \delta_n$,
$x_{24n-16} = -2^{10(n-1)+4} y_0 + \delta_n < 0$ and
$y_{24n-16} = 2^{10(n-1)+4} y_0 - 3\delta_n - 2 < 0$,

$x_{24n-15} = -2^{10(n-1)+4} x_0 + -2^{10(n-1)+4} y_0 + 2\delta_n + 1$ and
$y_{24n-15} = -2^{10(n-1)+4} x_0 - 2^{10(n-1)+4} y_0 + 4\delta_n + 1$,

$x_{24n-14} = 2^{10(n-1)+5} y_0 - (2\delta_n + 1)$ and
$y_{24n-14} = -2^{10(n-1)+5} x_0 + 6\delta_n + 1$,

$x_{24n-13} = 2^{10(n-1)+5} x_0 - 2^{10(n-1)+5} y_0 - (4\delta_n + 1)$ and
$y_{24n-13} = 2^{10(n-1)+5} x_0 + 2^{10(n-1)+5} y_0 - (8\delta_n + 3)$,

$x_{24n-12} = 2^{10(n-1)+6} x_0 + 12\delta_n + 3$ and
$y_{24n-12} = 2^{10(n-1)+6} + (4\delta_n + 3)$,

$x_{24n-11} = -2^{10(n-1)+6} x_0 + 2^{10(n-1)+6} y_0 + 8\delta_n + 1 < 0$ and
$y_{24n-11} = -2^{10(n-1)+6} x_0 - 2^{10(n-1)+6} y_0 + 16\delta_n + 3 < 0$,

$x_{24n-10} = 2^{10(n-1)+7} x_0 - (24\delta_n + 5)$ and
$y_{24n-10} = 2^{10(n-1)+7} y_0 - (8\delta_n + 3)$,

$x_{24n-9} = -2^{10(n-1)+7} x_0 - 2^{10(n-1)+7} y_0 - (32\delta_n + 7)$ and
$y_{24n-9} = 2^{10(n-1)+7} x_0 - 2^{10(n-1)+7} y_0 - (16\delta_n + 3)$,

$x_{24n-8} = -2^{10(n-1)+8} x_0 + 48\delta_n + 9$ and
$y_{24n-8} = 2^{10(n-1)+8} y_0 + (16\delta_n + 3)$,

$x_{24n-7} = 2^{10(n-1)+8} x_0 + 2^{10(n-1)+8} y_0 - (64\delta_n + 13)$ and
$y_{24n-7} = -2^{10(n-1)+8} x_0 + 2^{10(n-1)+8} y_0 + (32\delta_n + 5) < 0$,

$x_{24n-6} = -2^{10(n-1)+9} y_0 + 32\delta_n + 7$ and

$y_{24n-6} = 2^{10(n-1)+9} y_0 - (96\delta_n + 19)$,

$x_{24n-5} = -2^{10(n-1)+9} x_0 - 2^{10(n-1)+9} y_0 + 128\delta_n + 25$ and
$y_{24n-5} = 2^{10(n-1)+9} x_0 - 2^{10(n-1)+9} y_0 - (64\delta_n + 13)$,

$x_{24n-4} = -2^{10(n-1)+10} y_0 - (64\delta_n + 13)$ and
$y_{24n-4} = -2^{10(n-1)+10} x_0 + 613 < 0$,

$x_{24n-3} = 2^{10(n-1)+10} x_0 - 2^{10(n-1)+10} y_0 + (128\delta_n + 25)$ and
$y_{24n-3} = 2^{10(n-1)+10} x_0 + 2^{10(n-1)+10} y_0 - (256\delta_n + 51)$,

$x_{24n-2} = -2^{10(n-1)+11} y_0 + (128\delta_n + 25)$ and
$y_{24n-2} = 2^{10(n-1)+11} x_0 - (384\delta_n + 77)$,

$x_{24n-1} = -2^{10(n-1)+11} x_0 + 2^{10(n-1)+11} y_0 + (256\delta_n + 51)$ and
$y_{24n-1} = -2^{10(n-1)+11} x_0 - 2^{10(n-1)+11} y_0 + (512\delta_n + 101)$,

$x_{24n} = -2^{10(n-1)+12} x_0 - (768\delta_n + 153)$ and
$y_{24n} = -2^{10(n-1)+12} x_0 - (256\delta_n + 51)$,

$x_{24n+1} = 2^{10(n-1)+12} x_0 - 2^{10(n-1)+12} x_0 - (512\delta_n + 103)$ and
$y_{24n+1} = 2^{10(n-1)+12} x_0 + 2^{10(n-1)+12} y_0 - (1024\delta_n + 205)$,

$x_{24n+2} = -2^{10(n-1)+13} x_0 + (1536\delta_n + 307)$ and
$y_{24n+2} = -2^{10(n-1)+13} y_0 + (512\delta_n + 101)$,

$x_{24n+3} = 2^{10(n-1)+13} x_0 - (2048\delta_n + 409)$ and
$y_{24n+3} = 2^{10(n-1)+13} x_0 + (1024\delta_n + 205)$,

$x_{24n+4} = 2^{10(n-1)+14} x_0 - (3072\delta_n + 615)$ and
$y_{24n+4} = 2^{10(n-1)+14} x_0 - (1024\delta_n + 205)$,

$x_{24n+5} = -2^{10(n-1)+14} x_0 - 2^{10(n-1)+14} y_0 + (4096\delta_n + 819)$ and
$y_{24n+5} = 2^{10(n-1)+14} x_0 - 2^{10(n-1)+14} y_0 - (2048\delta_n + 411)$,

$x_{24n+6} = 2^{10(n-1)+15} y_0 - (2048\delta_n + 409)$ and

$$y_{24n+6} = -2^{10(n-1)+15} x_0 + (6144\delta_n + 1229),$$

for $\delta_n = 3, 13107, \ldots$ . It is easy to see that the limit of regions trend to the point (0.6, 0.2) which is a member of period 3 cycle $P_{3.2}$. So we can conclude that solution of the system is either eventually go out of $Q_1 \cup Q_3$, solution is eventually prime period 3 or prime period 4, or stay in $Q_1 \cup Q_3$ which solution satisfies the above pattern.

Next, we will investigate solutions to the system when initial condition in third quadrant and we will focus on the pattern of solutions which only lie in $Q_3$. Let $(x_0, y_0) \in Q_3$. Then $x_0 < 0$ and $y_0 < 0$. Thus

$$x_1 = |x_0| - y_0 - 1 = -x_0 - y_0 - 1$$
$$y_1 = x_0 + |y_0| - 1 = x_0 - y_0 - 1.$$

If $(x_1, y_1) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_1, y_1) \in Q_3$, then

$$x_2 = |x_1| - y_1 - 1 = x_0 + y_0 + 1 - x_0 + y_0 + 1 - 1 = 2y_0 + 1$$
$$y_2 = x_1 + |y_1| - 1 = -x_0 - y_0 - 1 - x_0 + y_0 + 1 - 1 = -2x_0 - 1.$$

If $(x_2, y_2) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_2, y_2) \in Q_3$, then

$$x_3 = |x_2| - y_2 - 1 = 2x_0 - 2y_0 - 1$$
$$y_3 = x_2 + |y_2| - 1 = 2x_0 + 2y_0 + 1.$$

If $(x_3, y_3) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_3, y_3) \in Q_3$, then

$$x_4 = |x_3| - y_3 - 1 = -4x_0 - 1$$
$$y_4 = x_3 + |y_3| - 1 = -4y_0 - 3.$$

If $(x_4, y_4) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_4, y_4) \in Q_3$, then

$$x_5 = |x_4| - y_4 - 1 = 4x_0 + 4y_0 + 3$$
$$y_5 = x_4 + |y_4| - 1 = -4x_0 + 4y_0 + 1.$$

If $(x_5, y_5) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_5, y_5) \in Q_3$, then

$$x_6 = |x_5| - y_5 - 1 = -8y_0 - 5$$
$$y_6 = x_5 + |y_5| - 1 = 8x_0 + 1.$$

If $(x_6, y_6) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_6, y_6) \in Q_3$, then

$$x_7 = |x_6| - y_6 - 1 = -8x_0 + 8y_0 + 3$$
$$y_7 = x_6 + |y_6| - 1 = -8x_0 - 8y_0 - 7.$$

If $(x_7, y_7) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_7, y_7) \in Q_3$, then

$$x_8 = |x_7| - y_7 - 1 = 16x_0 + 3$$
$$y_8 = x_7 + |y_7| - 1 = 16y_0 + 9.$$

If $(x_8, y_8) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_8, y_8) \in Q_3$, then

$$x_9 = |x_8| - y_8 - 1 = -16x_0 - 16y_0 - 13$$
$$y_9 = x_8 + |y_8| - 1 = 16x_0 - 16y_0 - 7.$$

If $(x_9, y_9) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_9, y_9) \in Q_3$, then

$$x_{10} = |x_9| - y_9 - 1 = 32y_0 + 19$$
$$y_{10} = x_9 + |y_9| - 1 = -32x_0 - 7.$$

If $(x_{10}, y_{10}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{10}, y_{10}) \in Q_3$, then

$$x_{11} = |x_{10}| - y_{10} - 1 = 32x_0 - 32y_0 - 13$$
$$y_{11} = x_{10} + |y_{10}| - 1 = 32x_0 + 32y_0 + 25.$$

If $(x_{11}, y_{11}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{11}, y_{11}) \in Q_3$, then

$$x_{12} = |x_{11}| - y_{11} - 1 = -64x_0 - 13$$
$$y_{12} = x_{11} + |y_{11}| - 1 = -64y_0 - 39.$$

If $(x_{12}, y_{12}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{12}, y_{12}) \in Q_3$, then

$$x_{13} = |x_{12}| - y_{12} - 1 = 64x_0 + 64y_0 + 51$$
$$y_{13} = x_{12} + |y_{12}| - 1 = -64x_0 + 64y_0 + 25.$$

If $(x_{13}, y_{13}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{13}, y_{13}) \in Q_3$, then

$$x_{14} = |x_{13}| - y_{13} - 1 = -128y_0 - 77$$
$$y_{14} = x_{13} + |y_{13}| - 1 = 128x_0 + 25.$$

If $(x_{14}, y_{14}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{14}, y_{14}) \in Q_3$, then

$$x_{15} = |x_{14}| - y_{14} - 1 = -128x_0 + 128y_0 + 51$$
$$y_{15} = x_{14} + |y_{14}| - 1 = -128x_0 - 128y_0 - 103.$$

If $(x_{15}, y_{15}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{15}, y_{15}) \in Q_3$, then

$$x_{16} = |x_{15}| - y_{15} - 1 = 256x_0 + 51$$
$$y_{16} = x_{15} + |y_{15}| - 1 = 256y_0 + 153.$$

If $(x_{16}, y_{16}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{16}, y_{16}) \in Q_3$, then

$$x_{17} = |x_{16}| - y_{16} - 1 = -256x_0 - 256y_0 - 205$$
$$y_{17} = x_{16} + |y_{16}| - 1 = 256x_0 - 256y_0 - 103.$$

If $(x_{17}, y_{17}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{17}, y_{17}) \in Q_3$, then

$$x_{18} = |x_{17}| - y_{17} - 1 = 512y_0 + 307$$
$$y_{18} = x_{17} + |y_{17}| - 1 = -512x_0 - 103.$$

If $(x_{18}, y_{18}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{18}, y_{18}) \in Q_3$, then

$$x_{19} = |x_{18}| - y_{18} - 1 = 512x_0 - 512y_0 - 205$$
$$y_{19} = x_{18} + |y_{18}| - 1 = 512x_0 + 512y_0 + 409.$$

If $(x_{19}, y_{19}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{19}, y_{19}) \in Q_3$, then

$$x_{20} = |x_{19}| - y_{19} - 1 = -1024x_0 - 205$$
$$y_{20} = x_{19} + |y_{19}| - 1 = -1024y_0 - 615.$$

If $(x_{10}, y_{10}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{10}, y_{10}) \in Q_3$, then

$$x_{21} = |x_{20}| - y_{20} - 1 = 1024x_0 + 1024y_0 + 819$$
$$y_{21} = x_{20} + |y_{20}| - 1 = -1024x_0 + 1024y_0 + 409.$$

If $(x_{11}, y_{11}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{11}, y_{11}) \in Q_3$, then

$$x_{22} = |x_{21}| - y_{21} - 1 = -2048y_0 - 1229$$

$$y_{22} = x_{21} + |y_{21}| - 1 = 2048x_0 + 409.$$

If $(x_{12}, y_{12}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{12}, y_{12}) \in Q_3$, then

$$x_{33} = |x_{22}| - y_{22} - 1 = -2048x_0 + 2048y_0 + 819$$
$$y_{23} = x_{22} + |y_{22}| - 1 = -2048x_0 - 2048y_0 - 1639.$$

If $(x_{23}, y_{23}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{23}, y_{23}) \in Q_3$, then

$$x_{24} = |x_{23}| - y_{23} - 1 = 4096x_0 + 819$$
$$y_{24} = x_{23} + |y_{23}| - 1 = 4096y_0 + 2457.$$

If $(x_{24}, y_{24}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{24}, y_{24}) \in Q_3$, then

$$x_{25} = |x_{24}| - y_{24} - 1 = -4096x_0 - 4096y_0 - 3277$$
$$y_{25} = x_{24} + |y_{24}| - 1 = 4096x_0 - 4096y_0 - 1639.$$

If $(x_{25}, y_{25}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{25}, y_{25}) \in Q_3$, then

$$x_{26} = |x_{25}| - y_{25} - 1 = 8192y_0 + 4915$$
$$y_{26} = x_{25} + |y_{25}| - 1 = -8192x_0 - 1639.$$

If $(x_{26}, y_{26}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{26}, y_{26}) \in Q_3$, then

$$x_{27} = |x_{26}| - y_{26} - 1 = 8192x_0 - 8192y_0 - 3277$$
$$y_{27} = x_{26} + |y_{26}| - 1 = 8192x_0 + 8192y_0 + 6553.$$

If $(x_{27}, y_{27}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{27}, y_{27}) \in Q_3$, then

$$x_{28} = |x_{27}| - y_{27} - 1 = -16384x_0 - 3277$$
$$y_{28} = x_{27} + |y_{27}| - 1 = -16384y_0 - 9831.$$

If $(x_{18}, y_{18}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{18}, y_{18}) \in Q_3$, then

$$x_{29} = |x_{28}| - y_{28} - 1 = 16384x_0 + 16384y_0 + 13107$$
$$y_{29} = x_{28} + |y_{28}| - 1 = -16384x_0 + 16384y_0 + 6553.$$

If $(x_{19}, y_{19}) \notin Q_3$ then we can conclude that solution of the system is eventually prime period 3 or prime period 4. Suppose that $(x_{19}, y_{19}) \in Q_3$.

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We can see the pattern of solution $\{(x_n, y_n)\}_{n=6}^{\infty}$ as follows: for $n \ge 1$ and $x_0 \in (a_n, b_n)$ and $y_0 \in (c_n, d_n)$

$$x_{8n-2} = -2^{4n-1} y_0 - (3\delta_n + 2) \text{ and}$$
$$y_{8n-2} = 2^{4n-1} x_0 + \delta_n ,$$

$$x_{8n-1} = -2^{4n-1} x_0 + 2^{4n-1} y_0 + 2\delta_n + 1 \text{ and}$$
$$y_{8n-1} = -2^{4n-1} x_0 - 2^{4n-1} y_0 - (4\delta_n + 3),$$

$$x_{8n} = 2^{4n} x_0 + 2\delta_n + 1 \text{ and}$$
$$y_{8n} = 2^{4n} y_0 + 6\delta_n + 3,$$

$$x_{8n+1} = -2^{4n} x_0 - 2^{4n} y_0 - (8\delta_n + 5) \text{ and}$$
$$y_{8n+1} = 2^{4n} x_0 - 2^{4n} y_0 - (4\delta_n + 3),$$

$$x_{8n+2} = 2^{4n+1} y_0 + 12\delta_n + 7 \text{ and}$$
$$y_{8n+2} = -2^{4n+1} x_0 - (4\delta_n + 3),$$

$$x_{8n+3} = 2^{4n+1} x_0 - 2^{4n+1} y_0 - (8\delta_n + 5) \text{ and}$$
$$y_{8n+3} = 2^{4n+1} x_0 + 2^{4n+1} y_0 + 16\delta_n + 9,$$

$$x_{8n+4} = -2^{4n+2} x_0 - (8\delta_n + 5) \text{ and}$$
$$y_{8n+4} = -2^{4n+2} y_0 - (24\delta_n + 15),$$

$$x_{8n+5} = 2^{4n+2} x_0 + 2^{4n+2} y_0 + 32\delta_n + 19 \text{ and}$$
$$y_{8n+5} = -2^{4n+2} x_0 + 2^{4n+2} y_0 + 16\delta_n + 9,$$

where $a_n = -\frac{(2^{4n-2}+1)}{5\times 2^{4n-2}}$, $b_n = -\frac{(2^{4n-1}-3)}{5\times 2^{4n-1}}$, $c_n = -\frac{(3\times 2^{4n-1}+1)}{5\times 2^{4n-1}}$, $d_n = -\frac{(3\times 2^{4n-3}-1)}{5\times 2^{4n-3}}$.

Note that If $x_2 = 2y_0 + 1 > 0$ and $y_2 = -2x_0 - 1 > 0$ which mean that $(x_2, y_2) \in Q_1$ then we also have interesting solutions, which initial condition $\left(x_0 + y_0 \ge \frac{-3}{4}\right)$ as follows:

$$x_3 = |x_2| - y_2 - 1 = 2y_0 + 1 + 2x_0 + 1 - 1 = 2x_0 + 2y_0 + 1 < 0$$
$$y_3 = x_2 + |y_2| - 1 = 2y_0 + 1 - 2x_0 - 1 - 1 = -2x_0 + 2y_0 - 1$$

$$x_4 = |x_3| - y_3 - 1 = -2x_0 - 2y_0 - 1 + 2x_0 - 2y_0 + 1 - 1 = -4y_0 - 1$$

$$y_4 = x_3 + |y_3| - 1 = 2x_0 + 2y_0 + 1 + 2x_0 - 2y + 1 - 1 = 4x_0 + 1 < 0$$

$$Case\,1: x_4 = -4y_0 - 1 \geq 0 \quad \left(\frac{-1}{2} \leq y_0 \leq \frac{-1}{4}\right)$$

$$x_5 = |x_4| - y_4 - 1 = -4y_0 - 1 - 4x_0 - 1 - 1 = -4x_0 - 4y_0 - 3 \geq 0$$

$$y_5 = x_4 + |y_4| - 1 = -4y_0 - 1 - 4x_0 - 1 - 1 = -4x_0 - 4y_0 - 3 \geq 0$$

$$x_6 = |x_5| - y_5 - 1 = -4x_0 - 4y_0 - 3 + 4x_0 + 4y_0 + 3 - 1 = -1$$

We apply Lemma 2 to conclude that the solution is eventually prime period 4.

$$Case\,2: x_4 = -4y_0 - 1 < 0 \quad \left(y_0 > \frac{-1}{4}\right)$$

$$x_5 = |x_4| - y_4 - 1 = 4y_0 + 1 - 4x_0 - 1 - 1 = -4x_0 + 4y_0 - 1 \geq 0$$

$$y_5 = x_4 + |y_4| - 1 = -4y_0 - 1 - 4x_0 - 1 - 1 = -4x_0 - 4y_0 - 3$$

$$Case\,2.1: y_5 = -4x_0 - 4y_0 - 3 \leq 0 \quad \left(x_0 + y_0 \geq \frac{-3}{4}\right)$$

$$x_6 = |x_5| - y_5 - 1 = -4x_0 + 4y_0 - 1 + 4x_0 + 4y_0 + 3 - 1 = 8y_0 + 1$$

$$y_6 = x_5 + |y_5| - 1 = -4x_0 + 4y_0 - 1 + 4x_0 + 4y_0 + 3 - 1 = 8y_0 + 1$$

$$Case\,2.1.1: x_6 = 8y_0 + 1 \geq 0 \quad \left(y_0 \geq \frac{-1}{8}\right)$$

$$x_7 = |x_6| - y_6 - 1 = 8y_0 + 1 - 8y_0 - 1 - 1 = -1$$

We apply Lemma 2 to conclude that the solution is eventually prime period 4.

$$Case\,2.1.2: x_6 = 8y_0 + 1 < 0 \quad \left(\frac{-1}{4} < y_0 < \frac{-1}{8}\right)$$

$$x_7 = |x_6| - y_6 - 1 = -8y_0 - 1 - 8y_0 - 1 - 1 = -16y_0 - 3$$

$$y_7 = x_6 + |y_6| - 1 = 8y_0 + 1 - 8y_0 - 1 - 1 = -1$$

$$Case\,2.1.2.1: x_7 = -16y_0 - 3 \geq 0 \quad \left(\frac{-1}{4} < y_0 \leq \frac{-3}{16}\right)$$

$$x_8 = |x_7| - y_7 - 1 = -16y_0 - 3 + 1 - 1 = -16y_0 - 3 \geq 0$$

$$y_8 = x_7 + |y_7| - 1 = -16y_0 - 3 + 1 - 1 = -16y_0 - 3 \geq 0$$

$$x_9 = |x_8| - y_8 - 1 = -16y_0 - 3 + 16y_0 + 3 - 1 = -1$$

We apply Lemma 2 to conclude that the solution is eventually prime period 4.

$$Case\,2.1.2.2: x_7 = -16y_0 - 3 < 0 \quad \left(\frac{-3}{16} < y_0 \leq \frac{-1}{8}\right)$$

$$x_8 = |x_7| - y_7 - 1 = 16y_0 + 3 + 1 - 1 = 16y_0 + 3 > 0$$

$$y_8 = x_7 + |y_7| - 1 = -16y_0 - 3 + 1 - 1 = -16y_0 - 3 < 0$$

$x_9 = |x_8| - y_8 - 1 = 16y_0 + 3 + 16y_0 + 3 - 1 = 32y_0 + 5$

$y_9 = x_8 + |y_8| - 1 = 16y_0 + 3 + 16y_0 + 3 - 1 = 32y_0 + 5$

*Case* 2.1.2.2.1: $x_9 = 32y_0 + 5 \geq 0 \quad \left(\frac{-5}{32} \leq y_0 \leq \frac{-1}{8}\right)$

$x_{10} = |x_9| - y_9 - 1 = -1$

We apply Lemma 2 to conclude that the solution is eventually prime period 4.

*Case* 2.1.2.2.2: $x_9 = 32y_0 + 5 < 0 \quad \left(\frac{-3}{16} < y_0 < \frac{-5}{32}\right)$

$x_{10} = |x_9| - y_9 - 1 = -64y_0 - 11$

$y_{10} = x_9 + |y_9| - 1 = -1$

*Case* 2.1.2.2.2.1: $x_{10} = -64y_0 - 11 \geq 0 \quad \left(\frac{-3}{16} < y_0 < \frac{-11}{64}\right)$

$x_{11} = |x_{10}| - y_{10} - 1 = -64y_0 - 11 \geq 0$

$y_{11} = x_{10} + |y_{10}| - 1 = -64y_0 - 11 \geq 0$

$x_{12} = |x_{11}| - y_{11} - 1 = -1$

We apply Lemma 2 to conclude that the solution is eventually prime period 4.

*Case* 2.1.2.2.2.2: $x_{10} = -64y_0 - 11 < 0 \quad \left(\frac{-11}{64} < y_0 < \frac{-5}{32}\right)$

$x_{11} = |x_{10}| - y_{10} - 1 = 64y_0 + 11 > 0$

$y_{11} = x_{10} + |y_{10}| - 1 = -64y_0 - 11 < 0$

$x_{12} = |x_{11}| - y_{11} - 1 = 128y_0 + 21$

$y_{12} = x_{11} + |y_{11}| - 1 = 128y_0 + 21$

*Case* 2.1.2.2.2.2.1: $x_{12} = 128y_0 + 21 \geq 0 \quad \left(\frac{-21}{128} < y_0 < \frac{-5}{32}\right)$

$x_{13} = |x_{12}| - y_{12} - 1 = -1$

We apply Lemma 2 to conclude that the solution is eventually prime period 4.

*Case* 2.1.2.2.2.2.2: $x_{12} = 128y_0 + 21 < 0 \quad \left(\frac{-11}{64} < y_0 < \frac{-21}{128}\right)$

$x_{13} = |x_{12}| - y_{12} - 1 = -256y_0 - 43$

$y_{13} = x_{12} + |y_{12}| - 1 = -1$

*Case* 2.1.2.2.2.2.2.1: $x_{12} = -256y_0 - 43 \geq 0 \quad \left(\frac{-11}{64} < y_0 \leq \frac{-43}{256}\right)$

$x_{14} = |x_{13}| - y_{13} - 1 = -256y_0 - 43 \geq 0$

$y_{14} = x_{13} + |y_{13}| - 1 = -256y_0 - 43 \geq 0$

$x_{12} = |x_{11}| - y_{11} - 1 = -1$

We apply Lemma 2 to conclude that the solution is eventually prime period 4.

$$Case\ 2.1.2.2.2.2.2.2.2 : x_{12} = -256y_0 - 43 < 0 \quad \left( \frac{-43}{256} < y_0 < \frac{-21}{128} \right)$$

$\vdots$

So we will have a pattern of solutions to system(11).

$Case 2.2 : y_5 = -4x_0 - 4y_0 - 3 > 0 \quad \left( x_0 + y_0 < \frac{-3}{4} \right)$ which is out of the scope of initial condition.

Next, we will consider the solution of system

$$x_{n+1} = |x_n| - y_n - \varsigma \text{ and } y_{n+1} = x_n + |y_n| - \varphi \qquad (12)$$

where any initial condition $(x_0, y_0) \in R^2$ and parameter $\zeta, \varphi \in (0,\infty)$. The following solutions are some examples of System(12).

| Iteration | xn | yn |
|---|---|---|
| 1 | -0.1 | -0.4 |
| 2 | 0.6 | -0.7 |
| 3 | 1.4 | -0.3 |
| 4 | 1.8 | 0.9 |
| 5 | 1 | 0.7 |
| 6 | 0.4 | 0.1 |
| 7 | 0.4 | 0.1 |
| 8 | 0.4 | 0.1 |
| 9 | 0.4 | 0.1 |
| 10 | 0.4 | 0.1 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 0.1 | 0.2 | 0.1 | 0.3 |

| Iteration | xn | yn |
|---|---|---|
| 1 | -28.9 | -31.2 |
| 2 | 60.2 | -60.3 |
| 3 | 120.6 | -0.3 |
| 4 | 121 | 120.1 |
| 5 | 1 | 0.7 |
| 6 | 0.4 | 0.1 |
| 7 | 0.4 | 0.1 |
| 8 | 0.4 | 0.1 |
| 9 | 0.4 | 0.1 |
| 10 | 0.4 | 0.1 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 0.1 | 0.2 | -1 | 30 |

| Iteration | xn | yn |
|---|---|---|
| 1 | 1.4 | -1.5 |
| 2 | 3 | -0.3 |
| 3 | 3.4 | 2.5 |
| 4 | 1 | 0.7 |
| 5 | 0.4 | 0.1 |
| 6 | 0.4 | 0.1 |
| 7 | 0.4 | 0.1 |
| 8 | 0.4 | 0.1 |
| 9 | 0.4 | 0.1 |
| 10 | 0.4 | 0.1 |

| $\zeta$ | $\varphi$ | x0 | y0 |
|---|---|---|---|
| 0.1 | 0.2 | -1 | -0.3 |

| Iteration | xn | yn |
|---|---|---|
| 1 | -29.8 | -30.3 |
| 2 | 60.2 | -60.3 |
| 3 | 120.6 | -0.3 |
| 4 | 121 | 120.1 |
| 5 | 1 | 0.7 |
| 6 | 0.4 | 0.1 |
| 7 | 0.4 | 0.1 |
| 8 | 0.4 | 0.1 |
| 9 | 0.4 | 0.1 |
| 10 | 0.4 | 0.1 |

| $\zeta$ | $\varphi$ | x0 | y0 |
|---|---|---|---|
| 0.1 | 0.2 | -0.1 | 30 |

| teration | xn | yn |
|---|---|---|
| 1 | -19.9 | -50.1 |
| 2 | 80 | -90 |
| 3 | 180 | -30 |
| 4 | 220 | 130 |
| 5 | 100 | 70 |
| 6 | 40 | 10 |
| 7 | 40 | 10 |
| 8 | 40 | 10 |
| 9 | 40 | 10 |
| 10 | 40 | 10 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 10 | 20 | -0.1 | 30 |

| Iteration | xn | yn |
|---|---|---|
| 1 | 11.3 | -21.3 |
| 2 | 42.6 | -30 |
| 3 | 82.6 | -7.4 |
| 4 | 100 | 55.2 |
| 5 | 54.8 | 24.8 |
| 6 | 40 | 10 |
| 7 | 40 | 10 |
| 8 | 40 | 10 |
| 9 | 40 | 10 |
| 10 | 40 | 10 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 10 | 20 | -1 | -0.3 |

| Iteration | xn | yn |
|---|---|---|
| 1 | -19 | -51 |
| 2 | 80 | -90 |
| 3 | 180 | -30 |
| 4 | 220 | 130 |
| 5 | 100 | 70 |
| 6 | 40 | 10 |
| 7 | 40 | 10 |
| 8 | 40 | 10 |
| 9 | 40 | 10 |
| 10 | 40 | 10 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 10 | 20 | -1 | 30 |

| Iteration | xn | yn |
|---|---|---|
| 1 | 9.8 | -20.2 |
| 2 | 40 | -30.4 |
| 3 | 80.4 | -10.4 |
| 4 | 100.8 | 50 |
| 5 | 60.8 | 30.8 |
| 6 | 40 | 10 |
| 7 | 40 | 10 |
| 8 | 40 | 10 |
| 9 | 40 | 10 |
| 10 | 40 | 10 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 10 | 20 | 0.1 | 0.3 |

| Iteration | xn | yn |
|---|---|---|
| 1 | -0.1 | -20.2 |
| 2 | 20.4 | -40.3 |
| 3 | 60.8 | -39.9 |
| 4 | 100.8 | 0.9 |
| 5 | 100 | 79.9 |
| 6 | 20.2 | 0.1 |
| 7 | 20.2 | 0.1 |
| 8 | 20.2 | 0.1 |
| 9 | 20.2 | 0.1 |
| 10 | 20.2 | 0.1 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 0.1 | 20 | 0.1 | 0.3 |

| Iteration | xn | yn |
|---|---|---|
| 1 | -28.9 | -51 |
| 2 | 80 | -99.9 |
| 3 | 180 | -39.9 |
| 4 | 220 | 120.1 |
| 5 | 100 | 79.9 |
| 6 | 20.2 | 0.1 |
| 7 | 20.2 | 0.1 |
| 8 | 20.2 | 0.1 |
| 9 | 20.2 | 0.1 |
| 10 | 20.2 | 0.1 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 0.1 | 20 | -1 | 30 |

| Iteration | xn | yn |
|---|---|---|
| 1 | 1.4 | -21.3 |
| 2 | 22.8 | -39.9 |
| 3 | 62.8 | -37.1 |
| 4 | 100 | 5.7 |
| 5 | 94.4 | 74.3 |
| 6 | 20.2 | 0.1 |
| 7 | 20.2 | 0.1 |
| 8 | 20.2 | 0.1 |
| 9 | 20.2 | 0.1 |
| 10 | 20.2 | 0.1 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 0.1 | 20 | -1 | -0.3 |

| Iteration | xn | yn |
|---|---|---|
| 1 | -29.8 | -50.1 |
| 2 | 80 | -99.9 |
| 3 | 180 | -39.9 |
| 4 | 220 | 120.1 |
| 5 | 100 | 79.9 |
| 6 | 20.2 | 0.1 |
| 7 | 20.2 | 0.1 |
| 8 | 20.2 | 0.1 |
| 9 | 20.2 | 0.1 |
| 10 | 20.2 | 0.1 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 0.1 | 20 | -0.1 | 30 |

| Iteration | xn | yn |
|---|---|---|
| 1 | 1.4 | -21.3 |
| 2 | 22.8 | -39.9 |
| 3 | 62.8 | -37.1 |
| 4 | 100 | 5.7 |
| 5 | 94.4 | 74.3 |
| 6 | 20.2 | 0.1 |
| 7 | 20.2 | 0.1 |
| 8 | 20.2 | 0.1 |
| 9 | 20.2 | 0.1 |
| 10 | 20.2 | 0.1 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 0.1 | 20 | -1 | -0.3 |

| Iteration | xn | yn |
|---|---|---|
| 1 | -28.9 | -51 |
| 2 | 80 | -99.9 |
| 3 | 180 | -39.9 |
| 4 | 220 | 120.1 |
| 5 | 100 | 79.9 |
| 6 | 20.2 | 0.1 |
| 7 | 20.2 | 0.1 |
| 8 | 20.2 | 0.1 |
| 9 | 20.2 | 0.1 |
| 10 | 20.2 | 0.1 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 0.1 | 20 | -1 | 30 |

| Iteration | xn | yn |
|---|---|---|
| 1 | -0.1 | -20.2 |
| 2 | 20.4 | -40.3 |
| 3 | 60.8 | -39.9 |
| 4 | 100.8 | 0.9 |
| 5 | 100 | 79.9 |
| 6 | 20.2 | 0.1 |
| 7 | 20.2 | 0.1 |
| 8 | 20.2 | 0.1 |
| 9 | 20.2 | 0.1 |
| 10 | 20.2 | 0.1 |

| $\zeta$ | $\varphi$ | x0 | y0 |
|---|---|---|---|
| 0.1 | 20 | 0.1 | 0.3 |

| Iteration | xn | yn |
|---|---|---|
| 1 | 9.8 | -0.4 |
| 2 | 20.2 | 9.2 |
| 3 | 21 | 10.8 |
| 4 | 20.2 | 10 |
| 5 | 20.2 | 10 |
| 6 | 20.2 | 10 |
| 7 | 20.2 | 10 |
| 8 | 20.2 | 10 |
| 9 | 20.2 | 10 |
| 10 | 20.2 | 10 |

| $\zeta$ | $\varphi$ | x0 | y0 |
|---|---|---|---|
| 10 | 0.2 | 0.1 | 0.3 |

| Iteration | xn | yn |
|---|---|---|
| 1 | -19 | -31.2 |
| 2 | 60.2 | -50.4 |
| 3 | 120.6 | 9.6 |
| 4 | 121 | 110.8 |
| 5 | 20.2 | 10 |
| 6 | 20.2 | 10 |
| 7 | 20.2 | 10 |
| 8 | 20.2 | 10 |
| 9 | 20.2 | 10 |
| 10 | 20.2 | 10 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 10 | 0.2 | -1 | 30 |

| Iteration | xn | yn |
|---|---|---|
| 1 | 11.3 | -1.5 |
| 2 | 22.8 | 9.6 |
| 3 | 23.2 | 13 |
| 4 | 20.2 | 10 |
| 5 | 20.2 | 10 |
| 6 | 20.2 | 10 |
| 7 | 20.2 | 10 |
| 8 | 20.2 | 10 |
| 9 | 20.2 | 10 |
| 10 | 20.2 | 10 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 10 | 0.2 | -1 | -0.3 |

| Iteration | xn | yn |
|---|---|---|
| 1 | -19.9 | -30.3 |
| 2 | 60.2 | -50.4 |
| 3 | 120.6 | 9.6 |
| 4 | 121 | 110.8 |
| 5 | 20.2 | 10 |
| 6 | 20.2 | 10 |
| 7 | 20.2 | 10 |
| 8 | 20.2 | 10 |
| 9 | 20.2 | 10 |
| 10 | 20.2 | 10 |

| ζ | φ | x0 | y0 |
|---|---|---|---|
| 10 | 0.2 | -0.1 | 30 |

From the above observations, we see that the equilibrium point of system is depend on the parameters $\zeta$ and $\varphi$. It is easy to verify that the equilibrium point of System(12) is $(2\zeta+\varphi,\zeta)$. Moreover, the solutions of the system will reach the equilibrium within six iterations. First of all we will give lemmas for proving the main result.

For convenient to investigate solutions, we will use a notation refer to a set of points in complex plan define as $C^1=\{(x,y):|x|-x+|y|-y+2\zeta-\varphi\geq|x-|y|-\varphi|-||x|-y+\zeta|\}$.

**Lemma 3** Let $\{(x_n,y_n)\}_{n=0}^{\infty}$ and be a solution of System(12). Suppose the initial condition $(x_0,y_0)\in Q_1\cup L_1\cup L_2$. Then $(x_1,y_1)\in C^1$.

**Proof.** Suppose $(x_0,y_0)\in Q_1\cup L_1\cup L_3$ then $x_0\geq 0$ and $y_0\geq 0$. Thus

**Case 1:** Suppose further $x_0 \geq y_0+\varphi$. We have $x_1 = x_0 - y_0 + \zeta > 0$ and $y_1 = x_0 - y_0 - \varphi \geq 0$. Note that

$$|x_1| - x_1 + |y_1| - y_1 + 2\zeta - \varphi = 2\zeta - \varphi$$

and

$$|x_1 - |y_1| - \varphi| - ||x_1| - y_1 + \zeta| = -\zeta - \varphi.$$

Hence $(x_1, y_1)$ is an element of $C^1$ and Case 1 is complete.

**Case 2:** Suppose $x_0 < y_0+\varphi$ and $x_0 + \zeta \geq y_0$. We have $x_1 = x_0 - y_0 + \zeta \geq 0$ and $y_1 = x_0 - y_0 - \varphi < 0$.

Note that

$$|x_1| - x_1 + |y_1| - y_1 + 2\zeta - \varphi = -2x_0 + 2y_0 + 2\zeta + \varphi$$

and

$$|x_1 - |y_1| - \varphi| - |\,|x_1| - y_1 + \zeta| = |2x_0 - 2y_0 + \zeta - 2\varphi| - 2\zeta - \varphi.$$

**Case 2A:** Suppose further $2x_0 - 2y_0 + \zeta - 2\varphi \geq 0$. Then

$$|x_1 - |y_1| - \varphi| - |\,|x_1| - y_1 + \zeta| = 2x_0 - 2y_0 - 2\varphi - \zeta - \varphi.$$

Since $x_0 - y_0 - \varphi < 0$, we have $2x_0 - 2y_0 - 2\varphi - \zeta < 0$. Also note that $|y_1| - y_1 + 2\zeta > 0$, so

$$\begin{aligned} |x_1| - x_1 + |y_1| - y_1 + 2\zeta - \varphi &= |y_1| - y_1 + 2\zeta - \varphi \\ &> 2x_0 - 2y_0 - 2\varphi - \zeta - \varphi \\ &= |x_1 - |y_1| - \varphi| - |\,|x_1| - y_1 + \zeta|. \end{aligned}$$

**Case 2B:** Suppose further $2x_0 - 2y_0 + \zeta - 2\varphi < 0$. Then

$$\begin{aligned} |x_1 - |y_1| - \varphi| - |\,|x_1| - y_1 + \zeta| &= -2x_0 + 2y_0 - 3\zeta + \varphi \\ &< -2x_0 + 2y_0 + 2\zeta + \varphi \\ &= |x_1| - x_1 + |y_1| - y_1 + 2\zeta - \varphi. \end{aligned}$$

Hence $(x_1, y_1)$ is an element of $C^1$ and Case 2 is complete.

**Case 3:** Suppose $x_0 < y_0 + \varphi$ and $x_0 + \zeta < y_0$. We have $x_1 = x_0 - y_0 + \zeta < 0$ and $y_1 = x_0 - y_0 - \varphi < 0$. Note that

$$|x_1| - x_1 + |y_1| - y_1 + 2\zeta - \varphi = -4x_0 + 4y_0 + \varphi$$

and

$$|x_1 - |y_1| - \varphi| - |\,|x_1| - y_1 + \zeta| = \varphi - \zeta.$$

Since $x_0 + \zeta < y_0$, we have $y_0 > x_0$. Thus $-4x_0 + 4y_0 > 0$. Then

$$\begin{aligned} |x_1| - x_1 + |y_1| - y_1 + 2\zeta - \varphi &= -4x_0 + 4y_0 + \varphi \\ &> \varphi - \zeta \\ &= |x_1 - |y_1| - \varphi| - |\,|x_1| - y_1 + \zeta|. \end{aligned}$$

Hence $(x_1, y_1)$ is an element of $C^1$ and Case 3 is complete.

□

**Lemma 4** Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(12). Suppose the initial condition $(x_0, y_0) \in Q_2 \cup L_3$. Then $(x_1, y_1) \in C^1$.

**Proof.** Suppose $(x_0, y_0) \in Q_2 \cup L_3$ then $x_0 < 0$ and $y_0 \geq 0$. Thus

**Case 1:** Suppose further $-x_0 + \zeta < y_0$. We have $x_1 = -x_0 - y_0 + \zeta < 0$ and $y_1 = x_0 - y_0 - \varphi < 0$. Note that

$$|x_1| - x_1 + |y_1| - y_1 + 2\zeta - d = 4y_0 + \varphi$$

and

$$|x_1 - |y_1| - \varphi| - |\,|x_1| - y_1 + \zeta| = \varphi - \zeta.$$

Hence $(x_1, y_1)$ is an element of $C^1$ and Case 1 is complete.

**Case 2:** Suppose $-x_0 + \zeta \geq y_0$. We have $x_1 = -x_0 - y_0 + \zeta \geq 0$ and $y_1 = x_0 - y_0 - \varphi < 0$. Note that

$$|x_1| - x_1 + |y_1| - y_1 + 2\zeta - \varphi = -2x_0 + 2y_0 + 2\zeta + \varphi$$

and

$$|x_1 - |y_1| - \varphi| - ||x_1| - y_1 + \zeta| = |-2y_0 + \zeta - 2\varphi| + 2x_0 - 2\zeta - \varphi.$$

**Case 2A:** Suppose further $-2y_0 + \zeta - 2\varphi \geq 0$. Then

$$|x_1 - |y_1| - \varphi| - ||x_1| - y_1 + \zeta| = 2x_0 - 2y_0 - \zeta - 3\varphi.$$

Since $y_1 = x_0 - y_0 - \varphi < 0$, we have $2x_0 - 2y_0 - \zeta - 3\varphi < 0$. Hence $(x_1, y_1)$ is an element of $C^1$ and Case 2A is complete.

**Case 2B:** Suppose further $-2y_0 + \zeta - 2\varphi < 0$. Then

$$|x_1 - |y_1| - \varphi| - ||x_1| - y_1 + \zeta| = 2x_0 + 2y_0 - 3\zeta + \varphi$$

Since $x_1 = -x_0 - y_0 + \zeta \geq 0$, we have $2x_0 + 2y_0 - 3\zeta < 0$. Hence $(x_1, y_1)$ is an element of $C^1$ and Case 2 is complete. □

**Lemma 5** Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(12). Suppose the initial condition $(x_0, y_0) \in Q_3 \cup L_4$. Then $(x_1, y_1) \in C^1$.

**Proof.** Suppose $(x_0, y_0) \in Q_3 \cup L_4$ then $x_0 \leq 0$ and $y_0 < 0$. Thus

$$|x_1| - x_1 + |y_1| - y_1 + 2\zeta - \varphi = -2x_0 + 2y_0 + 2\zeta + \varphi > 0$$

and

$$|x_1 - |y_1| - \varphi| - ||x_1| - y_1 + \zeta| = |\zeta - 2\varphi| - (-2x_0 - 2y_0 + 2\zeta + \varphi).$$

**Case 1:** Suppose that $\zeta - 2\varphi \geq 0$. Then

$$|x_1 - |y_1| - \varphi| - ||x_1| - y_1 + \zeta| = 2x_0 + 2y_0 - \zeta - 3\varphi.$$

Hence $(x_1, y_1)$ is an element of $C^1$ and Case 1 is complete.

**Case 2:** Suppose that $\zeta - 2\varphi < 0$. Then

$$|x_1 - |y_1| - \varphi| - ||x_1| - y_1 + \zeta| = 2x_0 + 2y_0 - 3\zeta + \varphi.$$

Since $-2x_0 - 2y_0 + 2\zeta > 0$, we have $2x_0 + 2y_0 - 3\zeta < 0$. Hence $(x_1, y_1)$ is an element of $C^1$ and Case 2 is complete. □

**Lemma 6** Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(12). Suppose the initial condition $(x_0, y_0) \in Q_4$. Then $(x_1, y_1) \in C^1$.

**Proof.** Suppose $(x_0, y_0) \in Q_4$ then $x_0 > 0$ and $y_0 < 0$. Thus

**Case 1:** Suppose further $x_0 + y_0 \geq \varphi$. We have $x_1 = x_0 - y_0 + \zeta > 0$ and $y_1 = x_0 + y_0 - \varphi \geq 0$.

Note that

$$|x_1| - x_1 + |y_1| - y_1 + 2\zeta - \varphi = 2\zeta - \varphi$$

and

$$|x_1 - |y_1| - \varphi| - ||x_1| - y_1 + \zeta| = -\zeta - \varphi.$$

Hence $(x_1, y_1)$ is an element of $C^1$ and Case 1 is complete.

**Case 2:** Suppose $x_0 + y_0 < \varphi$. We have $x_1 = x_0 - y_0 + \zeta > 0$ and $y_1 = x_0 + y_0 - \varphi < 0$. Note that

$$|x_1| - x_1 + |y_1| - y_1 + 2\zeta - \varphi = -2x_0 - 2y_0 + 2\zeta + \varphi$$

and

$$|x_1 - |y_1| - \varphi| - ||x_1| - y_1 + \zeta| = |2x_0 + \zeta - 2\varphi| + 2y_0 - 2\zeta - \varphi.$$

**Case 2A:** Suppose further $-2x_0 + \zeta - 2\varphi \geq 0$. Then

$$|x_1 - |y_1| - \varphi| - ||x_1| - y_1 + \zeta| = 2x_0 + 2y_0 - \zeta - 3\varphi.$$

Since $2x_0 + \zeta - 2\varphi \geq 0$ and $\zeta > -2x_0$. Thus $-2x_0 - 2y_0 + 2\zeta + \varphi > 0$. Since $y_1 = x_0 + y_0 - \varphi <$ 0, we have $2x_0 + 2y_0 - \zeta - 3\varphi < 0$. Hence $(x_1, y_1)$ is an element of $C^1$ and Case 2A is complete.

**Case 2B:** Suppose $-2x_0 + \zeta - 2\varphi < 0$. Then

$$|x_1 - |y_1| - \varphi| - ||x_1| - y_1 + \zeta| = -2x_0 + 2y_0 - 3\zeta + \varphi.$$

Since $y_0 < 0$ and $\zeta > 0$, we have $-2x_0 + 2y_0 - 3\zeta < 0$. Hence $(x_1, y_1)$ is an element of $C^1$ and Case 2 is complete.

□

**Theorem 5** Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(12). Suppose the initial condition $(x_0, y_0) \in R^2$ and $\zeta, \varphi \in (0, \infty)$. Then $\{(x_n, y_n)\}_{n=6}^{\infty}$ is the equilibrium $(2\zeta + \varphi, \zeta)$.

**Proof.** Suppose $(x_0, y_0) \in R^2$. To show that condition:

$$x_2 + |x_2| \geq y_2 + |y_2| - \zeta + 2\varphi \tag{13}$$

is true. By Lemmas 3 through 6 we know that $(x_1, y_1)$ is an element of $C^1$, so we have

$$|x_1| - x_1 + |y_1| - y_1 + 2\zeta - \varphi \geq |x_1 - |y_1| - \varphi| - ||x_1| - y_1 + \zeta|.$$

Then

$$|x_1| - y_1 + \zeta + ||x_1| - y_1 + \zeta| \geq x_1 - |y_1| - \varphi + |x_1 - |y_1| - \varphi| - \zeta + 2\varphi.$$

Hence condition (13) is true. Next to show that condition:

$$x_3 \geq |y_3| + \varphi$$

(14)

is true. Since condition (13) is true, we have

$$x_2 - |y_2| - \varphi \geq - |x_2| + y_2 - \zeta + \varphi$$

and we always have

$$x_2 - |y_2| - \varphi \leq |x_2| - y_2 + \zeta - \varphi.$$

Then

$$|x_2 - |y_2| - \varphi| \leq |x_2| - y_2 + \zeta - \varphi.$$

Hence condition (14) is true. Next to show that condition:

$$x_4 \geq 0, y_4 \geq 0 \text{ and } x_4 \geq |y_4| + \varphi \tag{15}$$

is true. Since condition (14) is true, we have

$$| x_3| + x_3 \geq y_3 + |y_3| - \zeta + 2\varphi$$

and so

$$x_3 - |y_3| - \varphi \geq - |x_3| + y_3 - \zeta + \varphi$$

and we always have

$$x_3 - |y_3| - \varphi \leq |x_3| - y_3 + \zeta - \varphi.$$

Then

$$|x_3 - |y_3| - \varphi| \leq |x_3| - y_3 + \zeta - \varphi.$$

It is easy to verify that $x_4 \geq 0$, $y_4 \geq 0$. Hence condition (15) is true. Next condition:

$$x_5 \geq 0, y_5 \geq 0 \text{ and } x_5 = |y_5| + \zeta + \varphi$$

(16)

is true. Finally, it is easy to show by direct computations that $(x_6, y_6) = (2\zeta + \varphi, \zeta)$.This completes the proof of the theorem. □

## Conclusion and Discussion

We use iteration method and specific choosing initial condition to prove the boundedness of rational difference equation (9) – (11). By choosing initial condition $y_0, y_{-1}, y_{-2} \in (0,1)$, we can show that equation (9) has unbounded solutions. Then we use the iteration method to show that equation (10) and (11) are bounded and permanent. This two methods could be apply to another family of special cases to main equation (8).

After that we investigate piecewise linear systems of difference equations. As we know, investigating stability of system of difference equations requires theorems that involve Jacobian matrix. So the functions of the system must are differentiable. Unfortunately, piecewise linear systems of difference equations are the system with absolute value. So we can't apply the stability theorem to the piecewise linear systems. The common idea of proofs of the above systems of piecewise linear systems is to separate initial condition into few regions and find some characters of solution to the system of each region and then establishing lemmas and finally summarizing the behaviors of each system to be a theorem. System(11) has periodic with period 3 and periodic with period 4 solutions and we can show that every solution, initial condition in $Q_1$ and $Q_4$, is eventually the prime period-3 solution or prime period-4 solution by applying Lemma 1 Lemma 2 and Theorem 4**.** The solutions are separate into many cases and by looking at the pattern of solutions we also have many patterns of solutions in each region. These patterns could be proved by using mathematical induction the same as [15, 18, 19]. In System(12), we investigate the solution by changing parameters $\zeta, \varphi \in (0,\infty)$ and initial condition $(x_0, y_0)$ in each quadrant. The equilibrium point of System(12) is $(2\zeta + \varphi, \zeta)$. We see that every solution is eventually equilibrium point within six iterations. We proved by finding the common conditions of the solutions to System(12) in $(x_1, y_1)$ to $(x_6, y_6)$ and proving that each condition is true. The solution of System(12) can be reach equilibrium point before six iterations when initial conditions satisfy condition (12) – (16). If initial conditions satisfy condition (14) then the solution will be equilibrium point within 3 iterations and if initial conditions satisfy condition (16) then the solution will be equilibrium point by only 1 iteration.

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# Appendix

**International Journal Publication:**

1) Tikjha, W., Lapierre, E., Sitthiwirattham, T., The stable equilibrium of a system of piecewise linear difference equations, Adv. Difference Equ. (2017) 2017:67 doi: 10.1186/s13662-017-1117-2

**Proceeding:**

1) Tikjha W. and Lapierre E. (2017), On the periodic behavior of a system of piecewise linear difference equations, Advances in Difference Equations and Discrete Dynamical Systems. Springer Proceedings in Mathematics & Statistics.

**International conference:**

1) Tikjha, W., Lapierre, E, and Grove E.A. (2016). Prime period 4 behavior of certain piecewise linear system of difference equations where initial condition are some points in positive x-axis. International Conference on Difference Equations and Applications – ICDEA, Japan, 2016.

2) Tikjha, W., Lapierre, E, and Grove E.A. (2017). On a Family of First Order Piecewise Linear Systems. International Conference on Difference Equations and Applications – ICDEA, Romania, 2017.