รายงานวิจัยฉบับสมบูรณ์

โครงการ จำนวนเกมโดมิเนชันของฟอเรสของวิถีและจำนวนเกมโดมิเนชันของกราฟที่มีดีกรีขนาดใหญ่

โดย นายเฉลิมพงศ์ วรวรรโณทัย

กรกฎาคม 2561

สัญญาเลขที่ MRG5980091

รายงานวิจัยฉบับสมบูรณ์

โครงการ จำนวนเกมโดมิเนชันของฟอเรสของวิถีและจำนวนเกมโดมิเนชันของกราฟที่มีดีกรีขนาดใหญ่

นายเฉลิมพงศ์ วรวรรโณทัย

ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยศิลปากร

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

และมหาวิทยาลัยศิลปากร

กิตติกรรมประกาศ

กระผมขอขอบคุณสำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยศิลปากร ที่ให้ทุนวิจัยแก่กระผมตามสัญญาเลขที่ MRG5980091 ขอบคุณ รศ.ดร.นวรัตน์ อนันต์ชื่น นักวิจัยที่ปรึกษาที่คอยให้คำแนะนำและแง่คิดการทำวิจัยในโครงการนี้

เฉลิมพงศ์ วรวรรโณทัย

บทคัดย่อ

รหัสโครงการ: MRG5980091

ชื่อโครงการ: จำนวนเกมโดมิเนชันของฟอเรสของวิถีและจำนวนเกมโดมิเนชันของกราฟที่มีดีกรีขนาดใหญ่

ชื่อนักวิจัย: นายเฉลิมพงศ์ วรวรรโณทัย

ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยศิลปากร

E-mail Address: worawannotai_c@silpakorn.edu

ระยะเวลาโครงการ: 2 ปี (2 พฤษภาคม 2559 – 1 พฤษภาคม 2561)

เกมโดมิเนชันเป็นเกมที่เล่นบนกราฟ G ประกอบด้วยผู้เล่นสองฝ่าย ฝ่ายหนึ่งเรียกว่า Dominator และอีกฝ่ายเรียกว่า Staller โดยผู้เล่นแต่ละฝ่ายจะผลัดกันเลือกจุด ๆ หนึ่งบนกราฟ G เมื่อเลือกเสร็จ เราจะกล่าวว่าจุดนั้นและจุดเพื่อนบ้านของมันจะถูกครอบคลุมโดยจุดที่ถูกเลือกไป ผู้เล่นจะเลือกได้เฉพาะจุดที่ครอบคลุมจุดเพิ่มอย่างน้อยหนึ่งจุด เกมจะจบลงเมื่อจุดทั้งหมดในกราฟถูกครอบคลุม Dominator มีจุดหมายที่จะให้เกมจบลงเร็วที่สุด แต่ Staller มีจุดมุ่งหมายที่จะให้เกมจบช้าที่สุด จำนวนเกมโดมิเนชัน $\gamma_g(G)$ ($\gamma'_g(G)$) ชนิดที่ 1 (ชนิดที่ 2) คือจำนวนครั้งที่ใช้ในการเล่นเกมโดมิเนชันบนกราฟ G จนจบโดยที่ทั้งคู่เล่นดีที่สุดและ Dominaotr (Staller) เป็นฝ่ายเริ่มก่อน

ในโครงการนี้ เรามีจุดมุ่งหมายที่จะหาจำนวนเกมโดมิเนชันทั้งสองชนิดของกราฟ G เมื่อ G เป็นฟอเรสของวิถี และเมื่อ G เป็นกราฟที่มีดีกรีสูงสุดขนาดใหญ่

คำหลัก: เกมโดมิเนชัน, ฟอเรสของวิถี, ดีกรีสูงสุด

Abstract

Project Code: MRG5980091

Project Title: Game domination numbers of forests of paths and game domination numbers of graphs with large maximum degrees

Investigator: Chelermpong Worawannotai

Department of Mathematics, Faculty of Science, Silpakorn University

E-mail Address: worawannotai_c@silpakorn.edu

Project Period: 2 years (May 2, 2016 – May 1, 2018)

The domination game played on a graph G consists of two players, Dominator and Staller, who alternately choose a vertex of G. The chosen vertex and its neighbors are said to be dominated by the chosen vertex. A player can only choose a vertex that dominates at least one new vertex. The game ends when all vertices are dominated. Dominator aims to finish the game in as few turns as possible while Staller aims to finish the game in as many turns as possible. The game domination number $\gamma_g(G)$ (respectively $\gamma'_g(G)$) of type 1 (respectively type 2) is the total number of turns both players use in a game which Dominator (respectively Staller) starts and both players use optimal strategies.

In this project we aim to determine the game domination numbers $\gamma_g(G)$ and $\gamma'_g(G)$ when G is a forest of paths and when G has large maximum degree with respect to the number of vertices.

Keywords: Game domination, forest of paths, maximum degree

1 Executive Summary

A set S in a graph G is a *dominating set* if any vertex of G not in S is adjacent to some vertex in S. Domination has applications in many resource allocation problems such as transceivers installation. For example, in installing wifi access points on a building, the common requirements are that the wifi signal should cover every place on the building and the cost of installation should be minimal. Domination is a widely studied topic with over one thousand research papers dedicated to this topic. For more information on domination, we refer the reader to [7, 8].

There are many variations of domination. In this project, we study the game version called domination game which was introduced by Brešar, Klavžar and Rall [2] in 2010. The *domination game* is played on a graph by two players, *Dominator* and *Staller*, who alternately chooses a vertex of the graph. After a player chooses a vertex, that vertex and its neighbors are said to be *dominated*. A vertex is *valid* to choose if its closed neighborhood contains at least one undominated vertex. The game ends when all vertices are dominated. Dominator aims to finish the game in as few moves as possible while Staller aims to finish the game in as many moves as possible. The game domination number $\gamma_g(G)$, (respectively $\gamma_g'(G)$) is the total number of moves both players use in a game played on a graph G which Dominator (respectively Staller) starts and both players use optimal strategies.

The first bound of the game domination number was obtained in term of the domination number: $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$ for any graph G [2]. From [2, 10], the two game domination numbers of a graph can differ by at most one. In fact for any pair of positive integers (k, l) that differ by at most one except for $(2, 1)$ there is a graph G such that $\gamma_g(G) = k$ and $\gamma_g'(G) = l$ [2, 10, 11].

A *partially-dominated graph* is a graph whose some vertices are declared dominated from the beginning. The notion of game domination numbers extends naturally to partially-dominated graphs by considering the numbers of moves to dominate the remaining undominated vertices.

A partially-dominated graph H is the *residual graph* of a partially-dominated graph G if H is obtained from G by deleting all vertices which are invalid moves and deleting all edges joining dominated vertices. Let H be the residual graph of a partially-dominated graph G. Since removing vertices that are invalid moves does not affect the game, $\gamma_g(G) = \gamma_g(H)$, $\gamma_g'(G) = \gamma_g'(H)$ and we can replace the game played on G by the game played on H.

A fundamental tool for analyzing domination game is proved in [10]:

Theorem 1. *[10] (Continuation Principle) Let G be a (partially-dominated) graph and let A and B be subsets of $V(G)$. Let G_A and G_B be the partially-dominated graphs in which the sets A and B have already been dominated, respectively. If $B \subseteq A$, then $\gamma_g(G_A) \leq \gamma_g(G_B)$ and $\gamma_g'(G_A) \leq \gamma_g'(G_B)$.*

Let $G = (V, E)$ and $H = (V', E')$ be partially-dominated graphs where A and B are the sets of dominated vertices of G and H, respectively. The *union of G and H*, denoted by $G \cup H$, is the partially-dominated graph with the vertex set $V \cup V'$, the edge set $E \cup E'$ and the set of dominated vertices $A \cup B$. If V and V' are disjoint, then the union is disjoint, denoted by $G + H$. P. Dorbec, G. Košmrlj and G. Renault [6] found bounds for the game domination number of a disjoint union of two graphs in terms of the game domination number of each graph.

Determining game domination numbers of graphs is not an easy task even for the simplest connected graphs such as paths and cycles [12]. In this project, we divide our results in two parts. In the first part, we determine the game domination numbers of a disjoint union of paths and cycles together with optimal strategies for both players. Our proofs rely on the following observation.

When the domination game is played on a disjoint union of paths and cycles, at any stage of the game, the residual graph is a disjoint union of cycles and partially-dominated paths with some endpoints dominated. In other words, the type of the graph does not change during the game. Therefore, if we can find an optimal first move, we have an optimal strategy for the whole game.

In the second part, we give a recursive formula for computing the game domination numbers of a galaxy (a forest of stars). Our proofs make use of the observation that we can assume that the centers of the stars are dominated without affecting the game domination numbers.

2 Game domination numbers of a disjoint union of paths and cycles

In this section, we give the formula for the game domination numbers of a disjoint union of paths and cycles. For the details of the proofs we refer the reader to our manuscript given in the appendix.

Definition 2. Let P_n denote a path with n vertices. Let P'_n denote a partially-dominated path P_{n+1} with the left end vertex dominated. Let P''_n denote a partially-dominated path P_{n+2} with both end vertices dominated.

Observe that each of P_n, P'_n and P''_n has n undominated vertices.

Definition 3. A partially-dominated graph is PC if each of its component is either P_n, P'_n, P''_n or C_n for some positive integer n.

Definition 4. Let G be a PC graph. A component of G is called a *path-component* if it is a P_n, P'_n or P''_n for some positive integer n. A component of G is called a *cycle-component* if it is a cycle.

Definition 5. For $i \in \{0, 1, 2, 3\}$, a path P_n is said to be in class $[i]$ if $n \equiv i \pmod 4$, a partially-dominated P'_n (or P''_n) is said to be in class $[i]^*$ if $n \equiv i \pmod 4$ and a cycle C_n is said to be in class $\textcircled{i}$ if $n \equiv i \pmod 4$. The classes $[i]_>$ and $[i]^*_>$ are defined similarly but they only consist (partially-dominated) paths with at least 5 undominated vertices.

The following paramenters will be useful for describing the game domination numbers of a PC graph.

Definition 6. For a partially-dominated graph G, let $a(G), b(G), c(G), d(G)$, and $e(G)$ be the numbers of components of G that are in $[2]^*$, $[3]^*$, $[3]$, $\textcircled{1} \cup \textcircled{2}$, and $\textcircled{3}$, respectively.

Definition 7. Let a, b and c be integers. Define

$$f(a, b, c) = \left\lceil \frac{a - c - 1}{2} + \frac{b}{4} \right\rceil .$$

Definition 8. Let G be a PC graph. Define

$$\delta(G) = \begin{cases} 1 & \text{if } d(G) \neq 0 \text{ and } e(G) \equiv \theta + f(a,b,c) - \lceil \frac{b}{2} \rceil \pmod 2 \\ 0 & \text{else} \end{cases},$$

and

$$\delta'(G) = \begin{cases} 1 & \text{if } d(G) \neq 0 \text{ and } e(G) \not\equiv \theta + f(a,b+1,c) - \lceil \frac{b-1}{2} \rceil \pmod 2 \\ 0 & \text{else} \end{cases}.$$

Definition 9. Let $G = P_{n_1} + \cdots + P_{n_k} + P'_{m_1} + \cdots + P'_{m_r} + P''_{s_1} + \cdots + P''_{s_l} + C_{t_1} + \cdots + C_{t_q}$. Define $\theta(G) = \lceil \frac{n_1}{2} \rceil + \cdots + \lceil \frac{n_k}{2} \rceil + \lceil \frac{m_1}{2} \rceil + \cdots + \lceil \frac{m_r}{2} \rceil + \lceil \frac{s_1}{2} \rceil + \cdots + \lceil \frac{s_l}{2} \rceil + \lceil \frac{t_1}{2} \rceil + \cdots + \lceil \frac{t_q}{2} \rceil$.

For a PC graph G the number $\theta(G)$ is the sum of the ceiling of half the number of undominated vertices of each component of G.

The Continuation Principle allows us to make the following assumption.

Assumption 10. Throughout this section assume that at any stage of the game Dominator plays in such a way that the set of additional vertices dominated by his move is not properly contained in that of other choice's and Staller plays in such a way that the set of additional vertices dominated by his move does not properly contain that of other choice's.

At any stage of the domination game played on a disjoint union of paths and cycles, the residual graph is always a PC graph. This reduces our analysis to just determining an optimal first move for each player in any PC graph. Recall that by our convention the left most vertex of P'_n is dominated. The following theorem gives the game domination numbers of a PC graph and optimal strategies.

Theorem 11. *Let G be a PC graph. Let $\theta = \theta(G), a = a(G), b = b(G), c = c(G), d = d(G), e = e(G), \delta = \delta(G)$ and $\delta' = \delta'(G)$. Then*

$$\gamma_g(G) = \theta + f(a,b,c) - \left\lceil \frac{b}{2} \right\rceil - d - e + \delta$$

and

$$\gamma'_g(G) = \theta + f(a,b+1,c) - \left\lceil \frac{b-1}{2} \right\rceil - d - e + \delta'.$$

Moreover, an optimal strategy for each player is as follows.
A Dominator's optimal strategy: Each turn Dominator plays on a component of the residual graph of G chosen from the class in the following order.

D1. $[2]^*_>$ *or* $[3]^*$

D2. $[1]_>, [1]^*_>$, $[3]$ *or* $\{P'_2, P''_2\}$

D3. $[2]_>$

D4. $[0], [0]^*$ *or* $\{P_1, P'_1, P''_1, P_2\}$

D5. 0 *or* 3

D6. ①

D7. ②

When Dominator plays on a path-component, he plays to dominate the left most undominated vertices.

A Staller's optimal strategy: Each turn Staller plays on a component of the residual graph of G chosen from the class in the following order.

S1. $[0]^*$ *or* $[3]^*$

S2. $[0]$, $[2]^*$, $[2]$ *or* $[3]$

S3. $[1]^*$ *or* $[1]$

S4. ⓪ *or* ③

S5. ①

S6. ②

When Staller plays on a component from $[1]$ *or* $[2]$*, he plays in such a way that the resulting residual graph of this component contains a component from* $[1]^*$*. When Staller plays on a component from* $[0]$ *or* $[3]$*, he plays to dominate the two left most vertices. When Staller plays on any other path-component, he plays to dominate one new vertex.*

3 Game domination numbers of a galaxy

In this section, we give a recursive formula for the game domination numbers of a galaxy. For the details of the proofs we refer the reader to our manuscript given in the appendix.

A *star* is a tree which has a vertex that is adjacent to all other vertices, called a *center*. We denote a star with k leaves by S_k. For convenience, when considering S_1, we let one vertex be its center and another is not. Therefore every star has a unique center. A *forest* is a graph whose components are trees. A *galaxy* is a graph whose components are stars. In a galaxy, it does not matter whether the centers are dominated initially or not.

Lemma 12. *Let F be a galaxy and let C be a set of some centers of F that are not isolated vertices. Then $\gamma_g(F) = \gamma_g(F|C)$ and $\gamma'_g(F) = \gamma'_g(F|C)$.*

By Lemma 12, we can make the following assumptions without affecting the game domination numbers.

Assumption 13. Throughout this section, when considering a galaxy, we assume that all isolated centers are already dominated.

In general, for a given graph G and a subgraph H of G, it is not necessary true that $\gamma_g(G) \geq \gamma_g(H)$ or $\gamma'_g(G) \geq \gamma'_g(H)$. For example, let $G = S_3$ and H be the subgraph of G consisting of the three leaves. Then $\gamma_g(G) = 1 < 3 = \gamma_g(H)$ and $\gamma'_g(G) = 2 < 3 = \gamma'_g(H)$. However, for a forest of stars, if the subgraph has fewer or equal number of components, then the inequalities hold.

Lemma 14. *Let G be a forest of stars and F be a subgraph of G. If the number of components of F is no more than the number of components of G, then $\gamma_g(G) \geq \gamma_g(F)$ and $\gamma'_g(G) \geq \gamma'_g(F)$.*

Now we compare two galaxies that differ in only one component.

Lemma 15. *Let F be a forest of stars. For positive integers a and b, let $F_a = F + S_a$ and $F_b = F + S_b$. If $a \geq b$, then $\gamma_g(F_a) \geq \gamma_g(F_b)$ and $\gamma'_g(F_a) \geq \gamma'_g(F_b)$.*

By the Continuation Principle, we can assume that Dominator always plays on a center of some star and Staller always plays on a leaf of some star. Next, we show how Dominator chooses a vertex optimally.

Theorem 16. *Let F be a partially-dominated forest of stars. Then a Dominator's optimal strategy is to play on a center with the most number of undominated neighbors.*

In the next lemma, we compare two galaxies with equal number of components that satisfy a certain ordering. As a consequence, we obtain a Staller's optimal strategy.

Lemma 17. *Let $F = S_{n_1} + \cdots + S_{n_m}$ and $G = S_{t_1} + \cdots + S_{t_m}$ where $n_1 \leq \cdots \leq n_m$ and $t_1 \leq \cdots \leq t_m$. If $\sum_{i=1}^{j} n_i \geq \sum_{i=1}^{j} t_i$ for all $1 \leq j \leq m$ and $\sum_{i=1}^{m} n_i = \sum_{i=1}^{m} t_i$, then $\gamma_g(F) \geq \gamma_g(G)$ and $\gamma'_g(F) \geq \gamma'_g(G)$.*

Theorem 18. *Let F be a forest of stars. Then a Staller's optimal strategy is to play on a leaf vertex adjacent to a center with the most number of undominated neighbors.*

The following theorem give a recursive formula for computing the game domination numbers of a galaxy.

Theorem 19. *Let $F = r_1 S_{n_1} + \cdots + r_m S_{n_m}$ and $G = r_1 S_{n_1} + \cdots + r_{m-1} S_{n_{m-1}}$ where $1 \leq n_1 < \cdots < n_m$. Then*

$$\gamma_g(F) = \begin{cases} 1 & \text{if } m = 1 \text{ and } r_m = 1 \\ r_1 & \text{if } n_m = 1 \\ r_m + \gamma_g(G + \frac{r_m}{2} S_{n_m - 1}) & \text{if } n_m > 1 \text{ and } r_m \text{ is even} \\ r_m + \gamma'_g(G + \frac{r_m - 1}{2} S_{n_m - 1}) & \text{if } n_m > 1 \text{ and } r_m \text{ is odd} \end{cases}$$

and

$$\gamma'_g(F) = \begin{cases} 2 & \text{if } m = 1, r_m = 1 \text{ and } n_m \geq 2 \\ r_1 & \text{if } n_m = 1 \\ r_m + \gamma'_g(G + \frac{r_m}{2} S_{n_m - 1}) & \text{if } n_m > 1 \text{ and } r_m \text{ is even} \\ r_m + \gamma_g(G + \frac{r_m + 1}{2} S_{n_m - 1}) & \text{if } n_m > 1 \text{ and } r_m \text{ is odd} \end{cases}.$$

Now we find some minimal galaxies that has largest possible game domination numbers with respect to the number of components. First let's consider when all stars in a galaxy that have different numbers of leaves.

Theorem 20. *Let $F_m = S_{n_1} + \cdots + S_{n_m}$ where $1 \leq n_1 < \cdots < n_m$. Then*

$$\gamma_g(F_m) = \begin{cases} 1 & \text{if } m = 1 \\ 2m-2 & \text{if } m > 1 \text{ and } n_1 = 1 \\ 2m-1 & \text{if } m > 1 \text{ and } n_1 > 1 \end{cases}$$

and

$$\gamma_g'(F_m) = \begin{cases} 2m-1 & \text{if } n_1 = 1 \\ 2m & \text{if } n_1 > 1. \end{cases}$$

Corollary 21. *Let $F_m = S_{n_1} + \cdots + S_{n_m}$ where $m > 1$ and $1 \leq n_1 < \cdots < n_m$. Then*

$$\gamma_g(F_m) = \begin{cases} 2m-2 & \text{if } n_1 = 1 \\ 2m-1 & \text{if } n_1 > 1 \end{cases}$$

and

$$\gamma_g'(F_m) = \begin{cases} 2m-1 & \text{if } n_1 = 1 \\ 2m & \text{if } n_1 > 1. \end{cases}$$

In particular, $\gamma_g(F_m) < \gamma_g'(F_m)$.

Lemma 22. *Let $F_m = S_2+S_3+\cdots+S_m+S_m$ where $m > 1$. Then $\gamma_g(F_m) = \gamma_g'(F_m) = 2m-1$.*

Corollary 23. *For a positive integer m, the forest $F_m = S_2 + S_3 + \cdots + S_{m+1}$ is a minimal galaxy that realizes the pair $(2m-1, 2m)$.*

Lemma 24. *For an integer m and a positive integer n, we have*

$$\left\lceil \frac{m}{2^{n-1}} \right\rceil = \left\lfloor \frac{m-1}{2^{n-1}} \right\rfloor + 1.$$

Finally we find minimal graphs among galaxies consisting isomorphic stars that satisfy the upperbounds of [2].

Theorem 25. *Let m and n be positive integers. Then*

$$\gamma_g(mS_n) = 2m - \left\lceil \frac{m}{2^{n-1}} \right\rceil$$

and

$$\gamma_g'(mS_n) = 2m - \left\lfloor \frac{m}{2^{n-1}} \right\rfloor.$$

Corollary 26. *Let m and n be positive integers. Then $\gamma_g(mS_n) = \gamma_g'(mS_n)$ if and only if $2^{n-1}|m$.*

Proof. By Theorem 25 and since $\left\lceil \frac{m}{2^{n-1}} \right\rceil = \left\lfloor \frac{m}{2^{n-1}} \right\rfloor$ if and only if $2^{n-1}|m$. □

Corollary 27. *Among the forests of m isomorphic stars, the forest mS_n is a minimal forest that realizes the pair $(2m-1, 2m)$ where n is the smallest integer greater than $1 + \log_2 m$.*

References

[1] B. Brešar, P. Dorbec, S. Klavžar and G. Komrlj, Domination game: effect of edge- and vertex-removal, Discrete Math. 330 (2014) 1–10.

[2] B. Brešar, S. Klavžar, D. F. Rall, Domination game and an imagination strategy, SIAM J. Discrete Math. 24 (2010) 979–991.

[3] B. Brešar, S. Klavžar, D. F. Rall, Domination game played on trees and spanning subgraphs, Discrete Math. 313(2013) 915–923.

[4] B. Brešar, S. Klavžar, G. Košmrlj and D.F. Rall, Domination game: extremal families of graphs for the 3/5-conjectures, Discrete Appl. Math. 161 (2013) 1308–1316

[5] Cs. Bujtás, Domination game of forests, Discrete Math., 338 (2015), 2220–2228.

[6] P. Dorbec, G. Košmrlj and G. Renault, The domination game played on unions of graphs, Discrete Math. 338 (2015) 71–79.

[7] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Domination in Graph : Advanced Topics*, Marcel Dekker, New York, 1998.

[8] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[9] M. A. Henning and W. B. Kinnersley, Domination game: A proof of the 3/5 -Conjecture for graphs with minimum degree at least two, SIAM J. Discrete Math., 30 (2016), 20–35, doi:10.1137/140976935.

[10] W.B. Kinnersley, D.B. West and R. Zamani, Extremal problems for game domination number, SIAM J. Discrete Math. 27 (2013) 2090–2107.

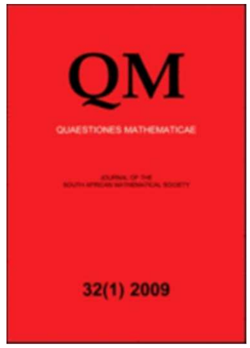
[11] G. Komrlj, Realizations of the game domination number, J. Comb. Optim. 28 (2014) 447–461.

[12] R. Zamani, Hamiltonian cycles through specified edges in bipartite graphs, domination game, and the game of revolutionaries and spies, Ph.D. Thesis, University of Illinois at Urbana-Champaign, 2011. Pro-Quest/UMI, Ann Arbor (Publication No. AAT 3496787)

Output

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ
    a. W. Ruksasakchai, K. Onphaeng, C. Worawannotai, Game domination numbers of a disjoint union of paths and cycles, Quaestiones Mathematicae (submitted).
    b. K. Laopreeda, C. Worawannotai, Game domination numbers of galaxies, Bulletin of the Korean Mathematical Society (submitted).
2. การนำผลงานวิจัยไปใช้ประโยชน์
    a. ใช้ในการพัฒนาการเรียนการสอน โดยนำเนื้อหาบางส่วนให้นักศึกษาศึกษาในวิชาสัมมนา
    b. ใช้แนะแนวทางการทำวิจัยและให้นักศึกษานำไปต่อยอดเป็นวิทยานิพนธ์ได้

ภาคผนวก



Game domination numbers of a disjoint union of paths and cycles

Journal:	*Quaestiones Mathematicae*
Manuscript ID	TQMA-2018-0099
Manuscript Type:	Original Article
Date Submitted by the Author:	13-Mar-2018
Complete List of Authors:	Ruksasakchai, Watcharintorn; Kasetsart University Kamphaeng Saen Campus Faculty of Liberal Arts and Science, Department of Mathematics, Statistic and Computer Science Onphaeng, Kritkhajohn; Silpakorn University Faculty of Science, Department of Mathematics Worawannotai, Chalermpong; Silpakorn University Faculty of Science, Department of Mathematics
Keywords:	domination game, game domination number, disjoint union of paths and cycles
Mathematic Subject Classification Code:	05 COMBINATORICS {For finite fields, see 11Txx}, 91 GAME THEORY, ECONOMICS, SOCIAL AND BEHAVIORAL SCIENCES

Game domination numbers of a disjoint union of paths and cycles

Watcharintorn Ruksasakchai
Department of Mathematics, Statistic and Computer Science, Faculty of Liberal Arts and Science
Kasetsart University, Kamphaeng Saen Campus, Nakhon Pathom, Thailand
faaswtr@ku.ac.th
Kritkhajohn Onphaeng Chalermpong Worawannotai*
Department of Mathematics, Faculty of Science, Silpakorn University
Nakhon Pathom, Thailand
dome3579@gmail.com, worawannotai_c@silpakorn.edu

March 14, 2018

Abstract

The domination game is played on a graph G by two players, Dominator and Staller, who alternately chooses a vertex of G in such a way that at least one new vertex is dominated. The game ends when all vertices are dominated. Dominator aims to finish the game in as few moves as possible while Staller aims to finish the game in as many moves as possible. The game domination number $\gamma_g(G)$ (respectively $\gamma'_g(G)$) is the total number of moves both players use in a game which Dominator (respectively Staller) starts and both players use optimal strategies.

In this paper we determine the game domination numbers of a disjoint union of paths and cycles.

Keywords: domination game, game domination number, disjoint union of paths and cycles
AMS 2010 Subject Classification: 05C57, 91A43, 05C69

1 Introduction

The *domination game* is played on a graph by two players, *Dominator* and *Staller*, who alternately chooses a vertex of the graph. After a player chooses a vertex, that vertex and its neighbors are said to be *dominated*. A vertex is *valid* to choose if its closed neighborhood contains at least one undominated vertex. The game ends when all vertices are dominated. Dominator aims to finish the game in as few moves as possible while Staller aims to finish the game in as many moves as possible. The game domination number $\gamma_g(G)$, (respectively $\gamma'_g(G)$) is the total number of moves both players use in a game played on a graph G which Dominator (respectively Staller) starts and both players use optimal strategies.

*corresponding author, supported by The Thailand Research Fund (research grant no. MRG5980091).

The *domination game* was introduced by Brešar, Klavžar and Rall [1] in 2010 where the first bound of the game domination number was obtained in term of the domination number: $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$ for any graph G. From [1, 3], the two game domination numbers of a graph can differ by at most one. In fact for any pair of positive integers (k, l) that differ by at most one except for $(2, 1)$ there is a graph G such that $\gamma_g(G) = k$ and $\gamma'_g(G) = l$ [1, 3, 4].

A *partially-dominated graph* is a graph whose some vertices are declared dominated from the beginning. The notion of game domination numbers extends naturally to partially-dominated graphs by considering the numbers of moves to dominate the remaining undominated vertices.

A partially-dominated graph H is the *residual graph* of a partially-dominated graph G if H is obtained from G by deleting all vertices which are invalid moves and deleting all edges joining dominated vertices. Let H be the residual graph of a partially-dominated graph G. Since removing vertices that are invalid moves does not affect the game, $\gamma_g(G) = \gamma_g(H)$, $\gamma'_g(G) = \gamma'_g(H)$ and we can replace the game played on G by the game played on H.

A fundamental tool for analyzing domination game is proved in [3]:

Theorem 1. *[3] (Continuation Principle) Let G be a (partially-dominated) graph and let A and B be subsets of $V(G)$. Let G_A and G_B be the partially-dominated graphs in which the sets A and B have already been dominated, respectively. If $B \subseteq A$, then $\gamma_g(G_A) \leq \gamma_g(G_B)$ and $\gamma'_g(G_A) \leq \gamma'_g(G_B)$.*

Let $G = (V, E)$ and $H = (V', E')$ be partially-dominated graphs where A and B are the sets of dominated vertices of G and H, respectively. The *union of G and H*, denoted by $G \cup H$, is the partially-dominated graph with the vertex set $V \cup V'$, the edge set $E \cup E'$ and the set of dominated vertices $A \cup B$. If V and V' are disjoint, then the union is disjoint, denoted by $G + H$. P. Dorbec, G. Košmrlj and G. Renault [2] found bounds for the game domination number of a disjoint union of two graphs in terms of the game domination number of each graph.

Determining game domination numbers of graphs is not an easy task even for the simplest connected graphs such as paths and cycles [5]. In this paper, we determine the game domination numbers of a disjoint union of paths and cycles together with optimal strategies for both players. Our proofs rely on the following observation.

When the domination game is played on a disjoint union of paths and cycles, at any stage of the game, the residual graph is a disjoint union of cycles and partially-dominated paths with some endpoints dominated. In other words, the type of the graph does not change during the game. Therefore, if we can find an optimal first move, we have an optimal strategy for the whole game.

In Section 2, we define related parameters and give lemmas that will be used for comparing choices of moves. In Section, 3 we determine the game domination numbers of a disjoint union of paths and cycles together with optimal strategies.

2 Parameters for a disjoint union of paths and cycles

In this section, we introduce notation and parameters for describing the game domination numbers of a disjoint union of paths and cycles. Moreover, lemmas that are useful for comparing different moves are given.

Definition 2. Let P_n denote a path with n vertices. Let P'_n denote a partially-dominated path P_{n+1} with the left end vertex dominated. Let P''_n denote a partially-dominated path P_{n+2} with both end vertices dominated.

Observe that each of P_n, P'_n and P''_n has n undominated vertices.

Definition 3. A partially-dominated graph is PC if each of its component is either P_n, P'_n, P''_n or C_n for some positive integer n.

Definition 4. Let G be a PC graph. A component of G is called a *path-component* if it is a P_n, P'_n or P''_n for some positive integer n. A component of G is called a *cycle-component* if it is a cycle.

Definition 5. For $i \in \{0, 1, 2, 3\}$, a path P_n is said to be in class $[i]$ if $n \equiv i \pmod 4$, a partially-dominated P'_n (or P''_n) is said to be in class $[i]^*$ if $n \equiv i \pmod 4$ and a cycle C_n is said to be in class (i) if $n \equiv i \pmod 4$. The classes $[i]_>$ and $[i]^*_>$ are defined similarly but they only consist (partially-dominated) paths with at least 5 undominated vertices.

The following paramenters will be useful for describing the game domination numbers of a PC graph.

Definition 6. For a partially-dominated graph G, let $a(G), b(G), c(G), d(G)$, and $e(G)$ be the numbers of components of G that are in $[2]^*, [3]^*, [3], (1) \cup (2)$, and (3), respectively.

Definition 7. Let a, b and c be integers. Define

$$f(a,b,c) = \left\lceil \frac{a-c-1}{2} + \frac{b}{4} \right\rceil.$$

Definition 8. Let G be a PC graph. Define

$$\delta(G) = \begin{cases} 1 & \text{if } d(G) \neq 0 \text{ and } e(G) \equiv \theta + f(a,b,c) - \lceil \frac{b}{2} \rceil \pmod 2 \\ 0 & \text{else} \end{cases},$$

and

$$\delta'(G) = \begin{cases} 1 & \text{if } d(G) \neq 0 \text{ and } e(G) \not\equiv \theta + f(a,b+1,c) - \lceil \frac{b-1}{2} \rceil \pmod 2 \\ 0 & \text{else} \end{cases}.$$

Definition 9. Let $G = P_{n_1} + \cdots + P_{n_k} + P'_{m_1} + \cdots + P'_{m_r} + P''_{s_1} + \cdots + P''_{s_l} + C_{t_1} + \cdots + C_{t_q}$. Define $\theta(G) = \lceil \frac{n_1}{2} \rceil + \cdots + \lceil \frac{n_k}{2} \rceil + \lceil \frac{m_1}{2} \rceil + \cdots + \lceil \frac{m_r}{2} \rceil + \lceil \frac{s_1}{2} \rceil + \cdots + \lceil \frac{s_l}{2} \rceil + \lceil \frac{t_1}{2} \rceil + \cdots + \lceil \frac{t_q}{2} \rceil$.

For a PC graph G the number $\theta(G)$ is the sum of the ceiling of half the number of undominated vertices of each component of G.

In the remaining of this section, we present lemmas that will be useful for comparing a player's moves.

Lemma 10. *Let a, b, c and i be integers. Then the following statements hold.*

(i) $f(a+i, b, c) = f(a, b+2i, c) = f(a, b, c-i)$.

(ii) $f(a, b, c) + i = f(a+2i, b, c) = f(a, b+4i, c) = f(a, b, c-2i)$.

(iii) *If* $i \geq 0$, *then* $f(a+i,b,c) \geq f(a,b,c)$.

(iv) *If* $i \geq 0$, *then* $f(a,b+i,c) \geq f(a,b,c)$.

Proof. The results follow from direct computation. □

Lemma 11. *For an integer* n*, the following statements hold.*

(i) $\lceil \frac{n}{2} + \frac{1}{2} \rceil = \lceil \frac{n}{2} + \frac{1}{4} \rceil$.

(ii) $\lceil \frac{n}{2} + \frac{1}{4} \rceil - \lceil \frac{n}{2} \rceil = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$.

(iii) $\lceil \frac{n}{2} \rceil - \lceil \frac{n-1}{2} \rceil = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$.

Proof. The results follow from direct computation. □

Lemma 12. *Let* a *and* c *be integers. Then* $f(a,0,c)+1 = f(a+1,1,c)$.

Proof. Since $f(a,0,c)+1 = \lceil \frac{a-c-1}{2} \rceil + 1 = \lceil \frac{a-c}{2} + \frac{1}{2} \rceil$ and $f(a+1,1,c) = \lceil \frac{a-c}{2} + \frac{1}{4} \rceil$, the result follows from Lemma 11(i). □

Lemma 13. *Let* a, b *and* c *be integers. Then the following values are either* 0 *or* 1.

(i) $\left(f(a,b+1,c) - \lceil \frac{b-1}{2} \rceil\right) - \left(f(a,b+2,c) - \lceil \frac{b}{2} \rceil\right)$.

(ii) $\left(f(a+1,b+2,c) - \lceil \frac{b-1}{2} \rceil\right) - \left(f(a+2,b,c) - \lceil \frac{b}{2} \rceil\right)$.

(iii) $\left(f(a+1,b+1,c) - \lceil \frac{b-1}{2} \rceil\right) - \left(f(a,b,c) - \lceil \frac{b-2}{2} \rceil\right)$.

Proof.

(i) Since $(f(a,b+1,c) - \lceil \frac{b-1}{2} \rceil) - (f(a,b+2,c) - \lceil \frac{b}{2} \rceil) = \lceil \frac{a-c-1}{2} + \frac{b+1}{4} \rceil - \lceil \frac{a-c-1}{2} + \frac{b+2}{4} \rceil - \lceil \frac{b-1}{2} \rceil + \lceil \frac{b}{2} \rceil$, the result follows from Lemma 11.

(ii) By Lemma 10(i), $f(a+2,b,c) = f(a+1,b+2,c)$. Then the result follows from Lemma 11(iii).

(iii) In (i), replace a with $a+1$ and then use Lemma 10(ii) to obtain the result. □

3 Main results

In this section we find the game domination numbers of a disjoint union of paths and cycles together with optimal strategies for both players. We start by invoking the Continuation Principle to simplify our analysis.

When comparing two choices of moves, if the set of additional vertices dominated by making the first choice is contained in the set of additional vertices dominated by making the second choice, then the first choice is not worse than the second choice for Dominator (and the second choice is not worse than the first choice for Staller). Therefore the Continuation Principle allows us to make the following assumption.

Assumption 14. Throughout this paper assume that at any stage of the game Dominator plays in such a way that the set of additional vertices dominated by his move is not properly contained in that of other choice's and Staller plays in such a way that the set of additional vertices dominated by his move does not properly contain that of other choice's.

At any stage of the domination game played on a disjoint union of paths and cycles, the residual graph is always a PC graph. This reduces our analysis to just determining an optimal first move for each player in any PC graph. Recall that by our convention the left most vertex of P'_n is dominated. The following theorem gives the game domination numbers of a PC graph and optimal strategies.

Theorem 15. *Let G be a PC graph. Let $\theta = \theta(G), a = a(G), b = b(G), c = c(G), d = d(G), e = e(G), \delta = \delta(G)$ and $\delta' = \delta'(G)$. Then*

$$\gamma_g(G) = \theta + f(a, b, c) - \left\lceil \frac{b}{2} \right\rceil - d - e + \delta$$

and

$$\gamma'_g(G) = \theta + f(a, b+1, c) - \left\lceil \frac{b-1}{2} \right\rceil - d - e + \delta'.$$

Moreover, an optimal strategy for each player is as follows.
A Dominator's optimal strategy: Each turn Dominator plays on a component of the residual graph of G chosen from the class in the following order.

D1. $[2]^*_>$ *or* $[3]^*$

D2. $[1]_>, [1]^*_>$, $[3]$ *or* $\{P'_2, P''_2\}$

D3. $[2]_>$

D4. $[0], [0]^*$ *or* $\{P_1, P'_1, P''_1, P_2\}$

D5. 0 *or* 3

D6. 1

D7. 2

When Dominator plays on a path-component, he plays to dominate the left most undominated vertices.

A Staller's optimal strategy: Each turn Staller plays on a component of the residual graph of G chosen from the class in the following order.

S1. $[0]^*$ *or* $[3]^*$

S2. $[0]$, $[2]^*$, $[2]$ *or* $[3]$

S3. $[1]^*$ *or* $[1]$

S4. 0 *or* 3

S5. 1

S6. 2

When Staller plays on a component from [1] or [2], he plays in such a way that the resulting residual graph of this component contains a component from $[1]^*$. When Staller plays on a component from [0] or [3], he plays to dominate the two left most vertices. When Staller plays on any other path-component, he plays to dominate one new vertex.

Proof. Let A and B be the desired values of $\gamma_g(G)$ and $\gamma_g'(G)$, respectively. We induct on the number of undominated vertices of graphs. One can check that the theorem holds for any graph with fewer than 4 undominated vertices. Assume that G has at least 4 undominated vertices. First, we show that $\gamma_g(G) = A$. To prove this, we find Dominator's optimal first move by considering all his valid first moves on a Dominator-start game.

Let H be the residual graph of G after Dominator plays his first move on G. Then $\gamma_g(G) \leq 1 + \gamma_g'(H)$ with equality if Dominator plays his first move optimally. We divide our arguments based on the choice of Dominator's first move. In each case, we count the number of moves of the game with specified Dominator's first move and the remaining moves are played optimally by both players. After Dominator makes his first move, the component in G on which he plays will either be

1. reduced to nothing in H if Dominator plays his first move on a component of G with at most three undominated vertices,
2. reduced to one component in H if Dominator plays his first move on a cycle-component of G, or his first move dominates the first three undominated vertices or the last three undominated vertices of a path-component of G with at least four undominated vertices, or
3. reduced to two components in H if his first move does not dominate the first undominated vertex nor the last undominated vertex of a path-component of G with at least five undominated vertices.

Table 1 and Table 2 show the values of $1+\gamma_g'(H)$ for all residual graphs H obtained from Dominator making first moves on G where Table 1 deals with the case that Dominator makes his first move on a path-component of G and Table 2 deals with the case that Dominator makes his first move on a cycle-component of G. The first column of each table shows the classes of components on which Dominator plays his first move. The second column shows the classes of residual graphs of the components that were played on. The third to eighth columns show the changes in values of parameters $t \in \{\theta, a, b, c, d, e\}$ where $\Delta t = t(H) - t(G)$. In Table 1 the columns corresponding to Δd and Δe are not shown because Δd and Δe are always 0 there. The last column shows the values of $1 + \gamma_g'(H)$ and how they compare.

Now we show how to obtain the entries in Table 1 and Table 2. First let's consider when the component on which Dominator plays his first move is in $[0] \cup [0]^*$.

Case 1 The component in G on which Dominator played is reduced to one component in H. Then that component of H is in $[1]^*$. Therefore $\theta(H) = \theta - 1, a(H) = a$, $b(H) = b$, $c(H) = c$, $d(H) = d$ and $e(H) = e$. By the induction hypothesis we have $\gamma_g'(H) = \theta - 1 + f(a, b+1, c) - \lceil \frac{b-1}{2} \rceil - d - e + \delta'(H)$. Hence the number of moves of the game in this case is equal to

$$\theta + f(a, b+1, c) - \left\lceil \frac{b-1}{2} \right\rceil - d - e + \delta'(H). \tag{1}$$

For convenience, let H_1 denote the graph H in Case 1.

Case 2 The component in G on which Dominator played is reduced to two components in H.

Case 2.1 One of the two components of H is in $[0]^*$ and the other is in $[1]^*$. Then $\theta(H) = \theta - 1, a(H) = a$, $b(H) = b$, $c(H) = c$, $d(H) = d$ and $e(H) = e$. By the induction hypothesis we have $\gamma_g'(H) = \theta - 1 + f(a, b+1, c) - \lceil \frac{b-1}{2} \rceil - d - e + \delta'(H)$. Moreover $\delta'(H) = \delta'(H_1)$. Hence the number of moves of the game in this case is equal to (1)

Case 2.2 One of the two components of H is in $[2]^*$ and the other is in $[3]^*$. Then $\theta(H) = \theta - 1, a(H) = a+1$, $b(H) = b+1$, $c(H) = c$, $d(H) = d$ and $e(H) = e$. By the induction hypothesis we have $\gamma_g'(H) = \theta - 1 + f(a+1, b+2, c) - \lceil \frac{b}{2} \rceil - d - e + \delta'(H)$. Hence the number of moves of the game in this case is equal to

$$\theta + f(a+1, b+2, c) - \left\lceil \frac{b}{2} \right\rceil - d - e + \delta'(H). \tag{*}$$

Now we compare (*) and (1). By Lemma 13(iii) the difference

$$\left(\theta + f(a+1, b+2, c) - \left\lceil \frac{b}{2} \right\rceil\right) - \left(\theta + f(a, b+1, c) - \left\lceil \frac{b-1}{2} \right\rceil\right)$$

is either 0 or 1. If the difference is 1, then $(*) - (1) \geq 0$ since $\delta'(H), \delta'(H_1) \in \{0, 1\}$. If the difference is 0, then $\delta'(H) = \delta'(H_1)$ and $(*) - (1) = 0$. So $(*) \geq (1)$. Therefore, Dominator's optimal first move on $[0] \cup [0]^*$ is to follow Case 1 or Case 2.1 which results in the total of (1) moves.

The remaining entries on the tables and the comparisons can be obtained in a similar manner. Therefore, when Dominator plays on a path-component, his optimal move on that component is to dominate the left most undominated vertices. Now we compare the optimal moves on components from different classes.

Claim 1. (i) $(3) = (5)$

(ii) $(2) = (6)$

Proof of claim. First we show $(3) = (5)$. Let H_3 and H_5 be the H's in (3) and (5), respectively. By Lemma 10(i) the difference

$$\left(\theta + f(a-1, b+2, c) - \left\lceil \frac{b}{2} \right\rceil - d - e\right) - \left(\theta - 1 + f(a, b, c) - \left\lceil \frac{b-2}{2} \right\rceil - d - e\right)$$

is equal to 0. It follows that $\delta'(H_3) = \delta'(H_5)$. Therefore $(3) = (5)$. Similarly, we have $(2) = (6)$.

■

Claim 2. $(5) \leq (2) \leq (4) \leq (1) \leq (7) \leq (8) \leq (9)$.

Proof of claim. First we show $(5) \leq (2)$. Let H_2 and H_5 be the H's in (2) and (5), respectively. By Lemma 13(iii) the difference

$$\left(\theta - 1 + f(a+1, b+1, c) - \left\lceil \frac{b-1}{2} \right\rceil - d - e\right) - \left(\theta - 1 + f(a, b, c) - \left\lceil \frac{b-2}{2} \right\rceil - d - e\right)$$

is either 0 or 1. If the difference is 1, then $(2) - (5) \geq 0$ since $\delta'(H_2), \delta'(H_5) \in \{0, 1\}$. If the difference is 0, then $\delta'(H_2) = \delta'(H_5)$ and $(2) - (5) = 0$. Therefore $(2) \geq (5)$. Similarly, the other inequalities can be shown. ■

1st move	Residual	$\Delta\theta$	Δa	Δb	Δc	$1+\gamma_g'(H)$	Remark
$[0]\cup[0]^*$	$[1]^*$ $[0]^*,[1]^*$	-1	0	0	0	$\theta+f(a,b+1,c)-\lceil\frac{b-1}{2}\rceil-d-e+\delta'(H)$	$=: (1)$
	$[2]^*,[3]^*$	-1	1	1	0	$\theta+f(a+1,b+2,c)-\lceil\frac{b}{2}\rceil-d-e+\delta'(H)$	$\geq (1)$
$[1]_>\cup[1]^*_>$	$[2]^*$ $[0]^*,[2]^*$	-2	1	0	0	$\theta-1+f(a+1,b+1,c)-\lceil\frac{b-1}{2}\rceil-d-e+\delta'(H)$	$=: (2)$
	$[1]^*,[1]^*$	-1	0	0	0	$\theta+f(a,b+1,c)-\lceil\frac{b-1}{2}\rceil-d-e+\delta'(H)$	$\geq (2)$
	$[3]^*,[3]^*$	-1	0	2	0	$\theta+f(a,b+3,c)-\lceil\frac{b+1}{2}\rceil-d-e+\delta'(H)$	$= (2)$
$[2]^*_>$	$[3]^*$ $[0]^*,[3]^*$	-1	-1	1	0	$\theta+f(a-1,b+2,c)-\lceil\frac{b}{2}\rceil-d-e+\delta'(H)$	$=: (3)$
	$[1]^*,[2]^*$	-1	0	0	0	$\theta+f(a,b+1,c)-\lceil\frac{b-1}{2}\rceil-d-e+\delta'(H)$	$\geq (3)$
$[2]_>$	$[3]^*$ $[0]^*,[3]^*$	-1	0	1	0	$\theta+f(a,b+2,c)-\lceil\frac{b}{2}\rceil-d-e+\delta'(H)$	$=: (4)$
	$[1]^*,[2]^*$	-1	1	0	0	$\theta+f(a+1,b+1,c)-\lceil\frac{b-1}{2}\rceil-d-e+\delta'(H)$	$\geq (4)$
$[3]^*$	$[0]^*$ $[0]^*,[0]^*$	-2	0	-1	0	$\theta-1+f(a,b,c)-\lceil\frac{b-2}{2}\rceil-d-e+\delta'(H)$	$=: (5)$
	$[1]^*,[3]^*$	-1	0	0	0	$\theta+f(a,b+1,c)-\lceil\frac{b-1}{2}\rceil-d-e+\delta'(H)$	$\geq (5)$
	$[2]^*,[2]^*$	-2	2	-1	0	$\theta-1+f(a+2,b,c)-\lceil\frac{b-2}{2}\rceil-d-e+\delta'(H)$	$\geq (5)$
$[3]$	$[0]^*$ $[0]^*,[0]^*$	-2	0	0	-1	$\theta-1+f(a,b+1,c-1)-\lceil\frac{b-1}{2}\rceil-d-e+\delta'(H)$	$=: (6)$
	$[1]^*,[3]^*$	-1	0	1	-1	$\theta+f(a,b+2,c-1)-\lceil\frac{b}{2}\rceil-d-e+\delta'(H)$	$\geq (6)$
	$[2]^*,[2]^*$	-2	2	0	-1	$\theta-1+f(a+2,b+1,c-1)-\lceil\frac{b-1}{2}\rceil-d-e+\delta'(H)$	$\geq (6)$
P_1,P_1',P_1'',P_2	–	-1	0	0	0	$\theta+f(a,b+1,c)-\lceil\frac{b-1}{2}\rceil-d-e+\delta'(H)$	$= (1)$
P_2',P_2''	–	-1	-1	0	0	$\theta+f(a-1,b+1,c)-\lceil\frac{b-1}{2}\rceil-d-e+\delta'(H)$	$= (2)$

Table 1: Effect of Dominator's first moves on a path-component

1st move	Residual	$\Delta\theta$	Δa	Δb	Δc	Δd	Δe	$1+\gamma_g'(H)$	Remark
0	$[1]^*$	-1	0	0	0	0	0	$\theta+f(a,b+1,c)-\lceil\frac{b-1}{2}\rceil-d-e+\delta'(H)$	$=: (7)$
1	$[2]^*$	-2	1	0	0	-1	0	$\theta-1+f(a+1,b+1,c)-\lceil\frac{b-1}{2}\rceil-(d-1)-e+\delta'(H)$	$=: (8)$
2	$[3]^*$	-1	0	1	0	-1	0	$\theta+f(a,b+2,c)-\lceil\frac{b}{2}\rceil-(d-1)-e+\delta'(H)$	$=: (9)$
3	$[0]^*$	-2	0	0	0	0	-1	$\theta-1+f(a,b+1,c)-\lceil\frac{b-1}{2}\rceil-d-(e-1)+\delta'(H)$	$= (7)$

Table 2: Effect of Dominator's first moves on a cycle-component

Claim 3. (i) If G has a component in $[2]^*_>$ or $[3]^*$, then $A = (3) = (5)$.

(ii) If G has no components in $[2]^*_>$ or $[3]^*$ but G has a component in $[1]_>, [1]^*_>$, $[3]$ or $\{P_2', P_2''\}$ then $A = (2) = (6)$.

(iii) If G has no components in $[2]^*_>$, $[3]^*$, $[1]_>, [1]^*_>$, $[3]$ or $\{P_2', P_2''\}$ but G has a component in $[2]_>$, then $A = (4)$.

(iv) If G has no components in $[2]^*_>$, $[3]^*$, $[1]_>, [1]^*_>$, $[3]$, $\{P_2', P_2''\}$ or $[2]_>$ but G has a component in $[0], [0]^*$ or $\{P_1, P_1', P_1'', P_2\}$ then $A = (1)$.

(v) If G has no path-components but G has a component in 0 or 3, then $A = (7)$.

(vi) If G has no path-components and no components in 0 or 3, but G has a component in 1, then $A = (8)$.

(vii) If G only has components in 2, then $A = (9)$.

Proof of claim. (i) Suppose G has a component in $[3]^*$. By Claim 1 and Claim 2 we can assume that Dominator's optimal first move is to play on this component. Let H_5 be the H in (5). By Table 1 we have $\theta(H_5) = \theta - 2$, $a(H_5) = a$, $b(H_5) = b - 1$, $c(H_5) = c$, $d(H_5) = d$ and $e(H_5) = e$. Consider $\delta(G)$ and $\delta'(H_5)$. Notice that
$e(G) \equiv \theta + f(a, b, c) - \lceil \frac{b}{2} \rceil \pmod 2$
$\Leftrightarrow e(H_5) \equiv \theta - 2 + f(a, b, c) - \lceil \frac{b-2}{2} \rceil + 1 \pmod 2$
$\Leftrightarrow e(H_5) \equiv \theta(H_5) + f(a(H_5), b(H_5) + 1, c(H_5)) - \left\lceil \frac{b(H_5)-1}{2} \right\rceil + 1 \pmod 2$
$\Leftrightarrow e(H_5) \not\equiv \theta(H_5) + f(a(H_5), b(H_5) + 1, c(H_5)) - \left\lceil \frac{b(H_5)-1}{2} \right\rceil \pmod 2$.
Therefore $\delta(G) = \delta'(H_5)$. Hence $\theta + f(a, b, c) - \lceil \frac{b}{2} \rceil - d - e + \delta(G) = \theta - 1 + f(a, b, c) - \lceil \frac{b-2}{2} \rceil - d - e + \delta'(H_5)$. That is $A = (5)$. Similarly, if G has a component in $[2]^*_>$, one can show that $A = (3)$.

(ii) Since G has no components in $[2]^*_>$ or $[3]^*$, we have $b = 0$. By Claim 1 and Claim 2 we can assume that Dominator's optimal first move is to play on this component. Suppose G has a component in $[1]_> \cup [1]^*_>$. Let H_2 be the H in (2). By Table 1, we have $\theta(H_2) = \theta - 2$, $a(H_2) = a + 1$, $b(H_2) = b = 0$, $c(H_2) = c$, $d(H_2) = d$ and $e(H_2) = e$.

Consider $\delta(G)$ and $\delta'(H_2)$. By Lemma 12, we have
$e(G) \equiv \theta + f(a, 0, c) \pmod 2$
$\Leftrightarrow e(H_2) \equiv \theta - 1 + f(a + 1, 1, c) \pmod 2$
$\Leftrightarrow e(H_2) \not\equiv \theta - 2 + f(a + 1, 1, c) \pmod 2$
$\Leftrightarrow e(H_2) \not\equiv \theta(H_2) + f(a(H_2), 1, c(H_2)) \pmod 2$.
Therefore $\delta(G) = \delta'(H_2)$. Hence $\theta + f(a, 0, c) - d - e + \delta(G) = \theta - 1 + f(a+1, 1, c) - \lceil \frac{-1}{2} \rceil - d - e + \delta'(H_2)$. That is $A = (2)$. Similarly, if G has a component in $[3]$ or $\{P_2', P_2''\}$, one can show that $A = (6)$.
(iii)-(vii) Apply the same process as the proof of (i) and (ii).

■

In the above argument we have considered Dominator's all possible first moves. By Claim 1 and Claim 2, we have $(3) = (5) \leq (2) = (6) \leq (4) \leq (1) \leq (7) \leq (8) \leq (9)$. By this and Claim 3 we have $\gamma_g(G) = A$ and the stated Dominator's strategy is optimal.

1st move	Residual	$\Delta\theta$	Δa	Δb	Δc	$1+\gamma_g(H')$	Remark
$[0]^*$	$[3]^*$	0	0	1	0	$\theta+1+f(a,b+1,c)-\lceil\frac{b+1}{2}\rceil-d-e+\delta(H')$	$=: (10)$
	$[2]^*$	-1	1	0	0	$\theta+f(a+1,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$\leq (10)$
	$[0]^*, [1]^*$	-1	0	0	0	$\theta+f(a,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$\leq (10)$
	$[2]^*, [3]^*$	-1	1	1	0	$\theta+f(a+1,b+1,c)-\lceil\frac{b+1}{2}\rceil-d-e+\delta(H')$	$\leq (10)$
$[1]^*$	$[0]^*$	-1	0	0	0	$\theta+f(a,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$=: (11)$
	$[3]^*$	-1	0	1	0	$\theta+f(a,b+1,c)-\lceil\frac{b+1}{2}\rceil-d-e+\delta(H')$	$\leq (11)$
	$[0]^*, [2]^*$	-2	1	0	0	$\theta-1+f(a+1,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$\leq (11)$
	$[1]^*, [1]^*$	-1	0	0	0	$\theta+f(a,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$\leq (11)$
	$[3]^*, [3]^*$	-1	0	2	0	$\theta+f(a,b+2,c)-\lceil\frac{b+2}{2}\rceil-d-e+\delta(H')$	$\leq (11)$
$[2]^*$	$[1]^*$	0	-1	0	0	$\theta+1+f(a-1,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$=: (12)$
	$[0]^*$	-1	-1	0	0	$\theta+f(a-1,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$\leq (12)$
	$[0]^*, [3]^*$	-1	-1	1	0	$\theta+f(a-1,b+1,c)-\lceil\frac{b+1}{2}\rceil-d-e+\delta(H')$	$\leq (12)$
	$[1]^*, [2]^*$	-1	0	0	0	$\theta+f(a,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$\leq (12)$
$[3]^*$	$[2]^*$	-1	1	-1	0	$\theta+f(a+1,b-1,c)-\lceil\frac{b-1}{2}\rceil-d-e+\delta(H')$	$=: (13)$
	$[1]^*$	-1	0	-1	0	$\theta+f(a,b-1,c)-\lceil\frac{b-1}{2}\rceil-d-e+\delta(H')$	$\leq (13)$
	$[0]^*, [0]^*$	-2	0	-1	0	$\theta-1+f(a,b-1,c)-\lceil\frac{b-1}{2}\rceil-d-e+\delta(H')$	$\leq (13)$
	$[1]^*, [3]^*$	-1	0	0	0	$\theta+f(a,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$\leq (13)$
	$[2]^*, [2]^*$	-2	2	-1	0	$\theta-1+f(a+2,b-1,c)-\lceil\frac{b-1}{2}\rceil-d-e+\delta(H')$	$\leq (13)$
$[0]$	$[2]^*$	-1	1	0	0	$\theta+f(a+1,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$= (12)$
	$[0]^*, [1]^*$	-1	0	0	0	$\theta+f(a,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$\leq (12)$
	$[2]^*, [3]^*$	-1	1	1	0	$\theta+f(a+1,b+1,c)-\lceil\frac{b+1}{2}\rceil-d-e+\delta(H')$	$\leq (12)$
$[1]$	$[3]^*$	-1	0	1	0	$\theta+f(a,b+1,c)-\lceil\frac{b+1}{2}\rceil-d-e+\delta(H')$	$\leq (11)$
	$[0]^*, [2]^*$	-2	1	0	0	$\theta-1+f(a+1,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$\leq (11)$
	$[1]^*, [1]^*$	-1	0	0	0	$\theta+f(a,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$= (11)$
	$[3]^*, [3]^*$	-1	0	2	0	$\theta+f(a,b+2,c)-\lceil\frac{b+2}{2}\rceil-d-e+\delta(H')$	$\leq (11)$
$[2]$	$[0]^*$	-1	0	0	0	$\theta+f(a,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$\leq (12)$
	$[0]^*, [3]^*$	-1	0	1	0	$\theta+f(a,b+1,c)-\lceil\frac{b+1}{2}\rceil-d-e+\delta(H')$	$\leq (12)$
	$[1]^*, [2]^*$	-1	1	0	0	$\theta+f(a+1,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$= (12)$
$[3]$	$[1]^*$	-1	0	0	-1	$\theta+f(a,b,c-1)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$= (12)$
	$[0]^*, [0]^*$	-2	0	0	-1	$\theta-1+f(a,b,c-1)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$\leq (12)$
	$[1]^*, [3]^*$	-1	0	1	-1	$\theta+f(a,b+1,c-1)-\lceil\frac{b+1}{2}\rceil-d-e+\delta(H')$	$\leq (12)$
	$[2]^*, [2]^*$	-2	2	0	-1	$\theta-1+f(a+2,b,c-1)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$\leq (12)$
P_1	$-$	-1	0	0	0	$\theta+f(a,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$= (11)$

Table 3: Effect of Staller's first moves on a path-component

1st move	Residual	$\Delta\theta$	Δa	Δb	Δc	Δd	Δe	$1+\gamma_g(H')$	Remark
0	$[1]^*$	-1	0	0	0	0	0	$\theta+f(a,b,c)-\lceil\frac{b}{2}\rceil-d-e+\delta(H')$	$=: (14)$
1	$[2]^*$	-2	1	0	0	-1	0	$\theta-1+f(a+1,b,c)-\lceil\frac{b}{2}\rceil-(d-1)-e+\delta(H')$	$=: (15)$
2	$[3]^*$	-1	0	1	0	-1	0	$\theta+f(a,b+1,c)-\lceil\frac{b+1}{2}\rceil-(d-1)-e+\delta(H')$	$=: (16)$
3	$[0]^*$	-2	0	0	0	0	-1	$\theta-1+f(a,b,c)-\lceil\frac{b}{2}\rceil-d-(e-1)+\delta(H')$	$=: (17)$

Table 4: Effect of Staller's first moves on a cycle-component

Next, we show that $\gamma_g'(G) = B$. To prove this, we find Staller's optimal first move by considering all his valid first moves on a Staller-start game.

Let H' be the residual graph of G after Staller plays his first move on G. Then $\gamma_g'(G) \geq 1 + \gamma_g(H')$ with equality if Staller plays his first move optimally. We divide our arguments based on the choice of Staller's first move. In each case, we count the number of moves with specified Staller's first move and the remaining moves are played optimally by both players. After Staller makes his first move, the component in G on which he plays will either be

1. reduced to nothing in H' if Staller plays his first move on a component of G that is P_1, P_1', P_1'', P_2 or C_3,

2. reduced to one component in H' if Staller plays his first move on a cycle-component of G with at least four vertices, or his first move dominates the first undominated vertex or the last undominated vertex of a path-component of G with at least two undominated vertices (excluding P_2), or

3. reduced to two components in H' if his first move does not dominate the first undominated vertex and the last undominated vertex of a path-component of G with at least five undominated vertices.

Table 3 and Table 4 show the values of $1 + \gamma_g(H')$ for all residual graphs H' obtained from Staller making first moves on G where Table 3 deals with the case that Staller makes his first move on a path-component of G and Table 4 deals with the case that Staller makes his first move on a cycle-component of G. The first column of each table shows the classes of components on which Staller plays his first move. The second column shows the classes of residual graphs of the component that were played on. The third to eighth columns show the changes in values of parameter $t \in \{\theta, a, b, c, d, e\}$ where $\Delta t = t(H') - t(G)$. (In Table 3 the columns corresponding to Δd and Δe are not shown because Δd and Δe are always 0.) The last column shows the values of $1 + \gamma_g(H')$ and how they compare.

Now we show how to obtain the entries in the tables. Let's consider when the component on which Staller plays his first move is in $[0]^*$.

Case 1 The component in G on which Staller played is reduced to one component in H'. Then that component of H' is in $[3]^*$ or $[2]^*$.

Case 1.1 The component of H' is in $[3]^*$. Therefore $\theta(H') = \theta$, $a(H') = a$, $b(H') = b + 1$, $c(H') = c$, $d(H') = d$ and $e(H') = e$. By the induction hypothesis we have $\gamma_g(H') = \theta + f(a, b+1, c) - \lceil \frac{b+1}{2} \rceil - d - e + \delta(H')$. Hence the number of moves of the game in this case is equal to

$$\theta + 1 + f(a, b+1, c) - \left\lceil \frac{b+1}{2} \right\rceil - d - e + \delta(H'). \tag{10}$$

For convenience, let H_1' denote the graph H' in Case 1.1.

Case 1.2 The component of H' is in $[2]^*$. Therefore $\theta(H') = \theta - 1$, $a(H') = a + 1$, $b(H') = b$, $c(H') = c$, $d(H') = d$ and $e(H') = e$. By the induction hypothesis we have $\gamma_g(H') = \theta - 1 + f(a+1, b, c) - \lceil \frac{b}{2} \rceil - d - e + \delta(H')$. Hence the number of moves of the game in this case is equal to

$$\theta + f(a+1, b, c) - \left\lceil \frac{b}{2} \right\rceil - d - e + \delta(H'). \tag{**}$$

Now we compare (10) and (**). Consider the difference

$$\left(\theta + 1 + f(a, b+1, c) - \left\lceil \frac{b+1}{2} \right\rceil\right) - \left(\theta + f(a+1, b, c) - \left\lceil \frac{b}{2} \right\rceil\right)$$

which is 0 or 1 by Lemma 13(i). If the difference is 1, then $(10)-(**) \geq 0$ since $\delta(H'), \delta(H'_1) \in \{0, 1\}$. If the difference is 0, then $\delta(H'_1) = \delta(H')$ and $(10) - (**) = 0$. So $(10) \geq (**)$.

Case 2 The component in G on which Staller played is reduced to two components in H'.

Case 2.1 One component of H' is in $[0]^*$ and the other is in $[1]^*$. Then $\theta(H') = \theta - 1$, $a(H') = a$, $b(H') = b$, $c(H') = c$, $d(H') = d$ and $e(H') = e$. By the induction hypothesis we have $\gamma_g(H') = \theta - 1 + f(a, b, c) - \left\lceil \frac{b}{2} \right\rceil - d - e + \delta(H')$. Hence the number of moves of the game in this case is equal to

$$\theta + f(a, b, c) - \left\lceil \frac{b}{2} \right\rceil - d - e + \delta(H'). \tag{†}$$

Now we compare (10) and (†). Consider the difference

$$\left(\theta + 1 + f(a, b+1, c) - \left\lceil \frac{b+1}{2} \right\rceil\right) - \left(\theta + f(a, b, c) - \left\lceil \frac{b}{2} \right\rceil\right)$$

which is greater than or equal to 0 by Lemma 13(i). If the difference is greater than or equal to 1, then $(10)-(†) \geq 0$ since $\delta(H'), \delta(H'_1) \in \{0, 1\}$. If the difference is 0, then $\delta(H'_1) = \delta(H')$ and $(10) - (†) = 0$. So $(10) \geq (†)$.

Case 2.2 One component of H' is in $[2]^*$ and the other is in $[3]^*$. Then $\theta(H') = \theta - 1$, $a(H') = a + 1$, $b(H') = b + 1$, $c(H') = c$, $d(H') = d$ and $e(H') = e$. By the induction hypothesis we have $\gamma_g(H') = \theta - 1 + f(a+1, b+1, c) - \left\lceil \frac{b+1}{2} \right\rceil - d - e + \delta(H')$. Hence the number of moves of the game in this case is equal to

$$\theta + f(a+1, b+1, c) - \left\lceil \frac{b+1}{2} \right\rceil - d - e + \delta(H'). \tag{††}$$

Now we compare (10) and (††). Consider the difference

$$\left(\theta + 1 + f(a, b+1, c) - \left\lceil \frac{b+1}{2} \right\rceil\right) - \left(\theta + f(a+1, b+1, c) - \left\lceil \frac{b+1}{2} \right\rceil\right)$$

which is greater than or equal to 0 by Lemma 10. If the difference is greater than 0, then $(10) - (††) \geq 0$ since $\delta(H'), \delta(H'_1) \in \{0, 1\}$. If the difference is 0, then $\delta(H'_1) = \delta(H')$ and $(10) - (††) = 0$. So $(10) \geq (††)$.

Therefore, Staller's optimal first move on $[0]^*$ is to follow Case 1.1 which results in the total of (10) moves.

The remaining entries on the tables and the comparisons can be obtained in a similar manner. Therefore, we have the following Staller's optimal first move on each class of path-components. When Staller plays on a component from [1] or [2], he plays in such a way that the resulting residual graph of this component contains a component from $[1]^*$. When Staller plays on a component from [0] or [3], he plays to dominate the two left most vertices. When Staller plays on any other path-component, he plays to dominate one new vertex.

Now we compare the optimal move from each class.

Claim 4. (i) (10) = (13)

(ii) (14) = (17)

Proof of claim. Let H'_{10} and H'_{13} be the H''s in (10) and (13), respectively. Consider the difference

$$\left(\theta + 1 + f(a, b+1, c) - \left\lceil \frac{b+1}{2} \right\rceil - d - e\right) - \left(\theta + f(a+1, b-1, c) - \left\lceil \frac{b-1}{2} \right\rceil - d - e\right)$$

which is equal to 0 by Lemma 10(i). It follows that $\delta(H'_{10}) = \delta(H'_{13})$. Therefore (10) = (13). Similarly, we have (14) = (17). ■

Claim 5. (10) $\geq$ (12) $\geq$ (11) $\geq$ (14) $\geq$ (15) $\geq$ (16).

Proof of claim. Let H'_{10} and H'_{12} be the H''s in (10) and (12), respectively. Consider the difference

$$\left(\theta + 1 + f(a, b+1, c) - \left\lceil \frac{b+1}{2} \right\rceil - d - e\right) - \left(\theta + 1 + f(a-1, b, c) - \left\lceil \frac{b}{2} \right\rceil - d - e\right)$$

which is 0 or 1 by Lemma 13(iii). If the difference is 1, then (10) $-$ (12) ≥ 0 since $\delta(H'_{10}), \delta(H'_{12}) \in \{0, 1\}$. If the difference is 0, then $\delta(H'_{10}) = \delta(H'_{12})$ and (10) = (12). Therefore (10) $\geq$ (12). Similarly, the other inequalities can be shown. ■

Claim 6. (i) If G has a component in $[0]^*$ or $[3]^*$, then B = (10) = (13).

(ii) If G has no components in $[0]^*$ or $[3]^*$ but G has a component in $[2]^*$, $[0]$, $[2]$ or $[3]$, then B = (12).

(iii) If G has no components in $[0]^*$, $[3]^*$, $[2]^*$, $[0]$, $[2]$ or $[3]$ but G has a component in $[1]^*$ or $[1]$, then B = (11).

(iv) If G has no path-components but G has a component in 0 or 3, then B = (14) = (17).

(v) If G has no path-components and no components in 0 or 3 but G has a component in 1, then B = (15).

(vi) If G only has components in 2, then B = (16).

Proof of claim. (i) Suppose G has a component in $[0]^*$. By Claim 4 and Claim 5 we can assume that Staller's optimal first move is to play on this component. Let H'_{10} be the H' in (10). By Table 3 we have $\theta(H'_{10}) = \theta$, $a(H'_{10}) = a$, $b(H'_{10}) = b + 1$, $c(H'_{10}) = c$, $d(H'_{10}) = d$ and $e(H'_{10}) = e$. Consider $\delta'(G)$ and $\delta(H'_{10})$. Notice that

$e(G) \not\equiv \theta + f(a, b+1, c) - \left\lceil \frac{b-1}{2} \right\rceil \pmod 2$

$\Leftrightarrow e(G) \equiv \theta + f(a, b+1, c) - \left\lceil \frac{b-1}{2} \right\rceil - 1 \pmod 2$

$\Leftrightarrow e(H'_{10}) \equiv \theta(H'_{10}) + f(a(H'_{10}), b(H'_{10}), c(H'_{10})) - \left\lceil \frac{b(H'_{10})-2}{2} \right\rceil - 1 \pmod 2$

$\Leftrightarrow e(H'_{10}) \equiv \theta(H'_{10}) + f(a(H'_{10}), b(H'_{10}), c(H'_{10})) - \left\lceil \frac{b(H'_{10})}{2} \right\rceil \pmod 2$.

Therefore $\delta'(G) = \delta(H'_{10})$. Hence $\theta + f(a, b+1, c) - \left\lceil \frac{b-1}{2} \right\rceil - d - e + \delta'(G) = \theta + 1 + f(a, b+1, c) - \left\lceil \frac{b+1}{2} \right\rceil - d - e + \delta(H'_{10})$. That is B = (10). Similarly, if G has a component in $[3]^*$, one can show that $B = (13)$.

(ii) Since G has no components in $[0]^*$ or $[3]^*$, we have $b = 0$. Suppose G has a component in $[2]^*$. By Claim 4 and Claim 5 we can assume that Staller's optimal first move is to play on this component. Let H'_{12} be the H' in (12). By Table 3, we have $\theta(H'_{12}) = \theta$, $a(H'_{12}) = a-1$, $b(H'_{12}) = b = 0$, $c(H'_{12}) = c$, $d(H'_{12}) = d$ and $e(H'_{12}) = e$.

Consider $\delta'(G)$ and $\delta(H'_{12})$. By Lemma 12
$e(G) \not\equiv \theta + f(a,1,c) \pmod 2$
$\Leftrightarrow e(G) \equiv \theta + f(a,1,c) - 1 \pmod 2$
$\Leftrightarrow e(H'_{12}) \equiv \theta(H'_{12}) + f(a(H'_{12})+1, 1, c(H'_{12})) - 1 \pmod 2$
$\Leftrightarrow e(H'_{12}) \equiv \theta(H'_{12}) + f(a(H'_{12}), 0, c(H'_{12})) \pmod 2$.
Therefore $\delta'(G) = \delta(H'_{12})$. Hence $\theta + f(a, b+1, c) - \left\lceil \frac{b-1}{2} \right\rceil - d - e + \delta'(G) = \theta + f(a+1, b, c) - \left\lceil \frac{b}{2} \right\rceil - d - e + \delta(H'_{12})$. That is $B = (12)$. Similarly, if G has a component in [0], [2] or [3], one can show that $B = (12)$.

(ii)-(vii) Apply the same process.

■

In the above argument we have considered Staller's all possible first moves. By Claim 4 and Claim 5, we have $(10) = (13) \geq (12) \geq (11) \geq (14) = (17) \geq (15) \geq (16)$. By this and Claim 6 we have $\gamma'_g(G) = B$ and the stated Staller's strategy is optimal.

□

Since a disjoint union of paths and cycles is a PC graph with no partially-dominated paths, we have the following theorem.

Theorem 16. *Let G be a disjoint union of paths and cycles. Let $\theta = \theta(G), c = c(G), d = d(G), e = e(G), \delta = \delta(G)$ and $\delta' = \delta'(G)$. Then*

$$\gamma_g(G) = \theta + \left\lceil \frac{-c-1}{2} \right\rceil - d - e + \delta$$

and

$$\gamma'_g(G) = \theta + \left\lceil \frac{-c}{2} - \frac{1}{4} \right\rceil - d - e + \delta'.$$

Proof. Note that $a(G) = b(G) = 0$. By Theorem 15, we have $\gamma_g(G) = \theta + f(0,0,c) - d - e + \delta = \theta + \left\lceil \frac{-c-1}{2} \right\rceil - d - e + \delta$ and $\gamma'_g(G) = \theta + f(0,1,c) - \left\lceil \frac{-1}{2} \right\rceil - d - e + \delta' = \theta + \left\lceil \frac{-c-1}{2} + \frac{1}{4} \right\rceil - d - e + \delta' = \theta + \left\lceil \frac{-c}{2} - \frac{1}{4} \right\rceil - d - e + \delta'$. □

Corollary 17. *Let G be a forest of paths. Let $\theta = \theta(G)$ and $c = c(G)$. Then*

$$\gamma_g(G) = \theta + \left\lceil \frac{-c-1}{2} \right\rceil$$

and

$$\gamma'_g(G) = \theta + \left\lceil \frac{-c}{2} - \frac{1}{4} \right\rceil.$$

Proof. By Theorem 16. □

Corollary 18. *Let G be a disjoint union of cycles. Let $\theta = \theta(G), d = d(G), e = e(G), \delta = \delta(G)$ and $\delta' = \delta'(G)$. Then*

$$\gamma_g(G) = \theta - d - e + \delta$$

and

$$\gamma'_g(G) = \theta - d - e + \delta'.$$

Proof. By Theorem 16. □

References

[1] B. Brešar, S. Klavžar, D. F. Rall, Domination game and an imagination strategy, SIAM J. Discrete Math. 24 (2010) 979–991.

[2] P. Dorbec, G. Košmrlj and G. Renault, The domination game played on unions of graphs, Discrete Math. 338 (2015) 71–79.

[3] W.B. Kinnersley, D.B. West and R. Zamani, Extremal problems for game domination number, SIAM J. Discrete Math. 27 (2013) 2090–2107.

[4] G. Komrlj, Realizations of the game domination number, J. Comb. Optim. 28 (2014) 447–461.

[5] R. Zamani, Hamiltonian cycles through specified edges in bipartite graphs, domination game, and the game of revolutionaries and spies, Ph.D. Thesis, University of Illinois at Urbana-Champaign, 2011. Pro-Quest/UMI, Ann Arbor (Publication No. AAT 3496787)

Bulletin of the
Korean Mathematical Society

Contact Information

Corresponding Author	**Name: Chalermpong Worawannotai** **Affiliation:** Department of Mathematics, Faculty of Science, Silpakorn University **Address:** [73000] 6 Ratchamanka Nai Road, Pra Pathom Jedi, Muang, Nakhon Pathom **E-mail**: worawannotai_c@silpakorn.edu **Country**: Thailand

Manuscript Information

Manuscript ID	**B18-0694**	**Date Submitted**	**2018-07-24** 06:50	**Date Received**	**None (or N/A)**
Document type	Original Article	**Date Revised**	-	**Date Accepted**	**None (or N/A)**
Category	Algebra and Number Theory				
Title	**Game domination numbers of a galaxy**				

All Author(s)

Name	Affiliation	E-mail
Kraiwit **Laopreeda**	Department of Mathematics, Faculty of Science, Silpakorn University	kraiwit12415@gmail.com
Chalermpong **Worawannotai**	Department of Mathematics, Faculty of Science, Silpakorn University	worawannotai_c@silpakorn.edu

Abstract

Domination game is a game played on a graph by two players, Dominator and Staller.
They alternately choose a vertex on the graph;
the chosen vertex and all of its neighbors will be dominated.
A vertex is valid to choose if at least one vertex in its closed neighborhood is undominated.
The game ends when all the vertices on the graph are dominated.
Dominator's goal is to minimize the total number of chosen vertices so that the game ends as soon as possible.
On the other hand, Staller's goal is to maximize the total number of chosen vertices so that the game is prolonged as much as possible.
A game domination number is the total number of vertices chosen to finish a domination game when Dominator and Staller play optimally.

In this paper, we determine the game domination numbers of a galaxy (a forest of stars).

MSC Number	05C57, 91A43, 05C69
Keyword	domination game, game domination number, galaxy
Cover Letter	**None (or N/A)**

Handling Editor(s) recommended by the author

Name	Affiliation
Seog-Jin Kim	Department of Mathematics Education, Konkuk University, Seoul 05029, Korea

Reviewer(s) recommended by the author

Name	Affiliation	Reason for Recommendation (Briefly)	E-mail
Csilla Bujtás	Department of Computer Science and Systems Technology, University of Pannonia, Veszprém, Hungary	An expert in domination game	bujtas@dcs.uni-pannon.hu
Sandi Klavžar	Faculty of Mathematics and Physics, University of Ljubljana, Slovenia	An expert in domination game	sandi.klavzar@fmf.uni-lj.si
Boštjan Brešar	Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia	An expert in domination game	bostjan.bresar@uni-mb.si

Unwanted Reviewer(s) by the Author

Name	Affiliation	Reason for Recommendation (Briefly)	E-mail
None (or N/A)			

Files

Order	File Name	Total Page	Author Response	Date
Original Manuscript	galaxy-2018-07-24-kms.pdf (262kb)	12		**2018-07-24** 06:46

OK

Bull. Korean Math. Soc.

GAME DOMINATION NUMBERS OF A GALAXY

Kraiwit Laopreeda and Chalermpong Worawannotai

Abstract. Domination game is a game played on a graph by two players, Dominator and Staller. They alternately choose a vertex on the graph; the chosen vertex and all of its neighbors will be dominated. A vertex is valid to choose if at least one vertex in its closed neighborhood is undominated. The game ends when all the vertices on the graph are dominated. Dominator's goal is to minimize the total number of chosen vertices so that the game ends as soon as possible. On the other hand, Staller's goal is to maximize the total number of chosen vertices so that the game is prolonged as much as possible. A game domination number is the total number of vertices chosen to finish a domination game when Dominator and Staller play optimally.

In this paper, we determine the game domination numbers of a galaxy (a forest of stars).

1. Introduction

A *dominating set* of a graph $G = (V, E)$ is a subset S of V such that all vertices of G are either in S or adjacent to some member of S. The *domination number* of a graph G, denoted by $\gamma(G)$, is the cardinality of a minimum dominating set of G. Domination can be applied to allocate resource efficiently and thoroughly. Because of such application, domination is a widely studied topic in graph theory. For more information about domination, see [7, 8].

In 2010, Brešar [3] has introduced a variation of domination as a game called *domination game*. The game is played on a graph by two players, *Dominator* and *Staller*. They alternately choose a vertex on the graph; the chosen vertex and all of its neighbors will be dominated. A vertex is *valid* to choose if at least one vertex in its closed neighborhood is undominated. The game *ends* when all the vertices on the graph are dominated; in other words, the set of chosen vertices becomes a dominating set. Dominator's goal is to minimize the total number of chosen vertices so that the game ends as soon as possible. On the other hand, Staller's goal is to maximize the total number of chosen vertices so that the game is prolonged as much as possible. For a graph G, we let $DS(G)$ denote the domination game on G which Dominator starts the game, and let $SD(G)$ denote the domination game on G which Staller starts

2010 *Mathematics Subject Classification.* 05C57, 91A43, 05C69.

Key words and phrases. domination game, game domination number, galaxy.

This work was financially supported by The Thailand Research Fund (research grant no. MRG5980091).

the game. A *game domination number* is the total number of vertices chosen to finish a domination game when Dominator and Staller play optimally. Game domination numbers of $DS(G)$ and $SD(G)$ are denoted by $\gamma_g(G)$ and $\gamma'_g(G)$ respectively.

Domination games played on trees and forests have been studied by many authors [2, 4, 5, 9]. Effect of graph operations such as vertex-removal, edge-removal and union are also studied [1, 6]. In this paper, we focus on finding the game domination numbers of a galaxy (a forest of stars). Section 2 recalls related definitions and well-known results. Section 3 presents Dominator's and Staller's optimal strategies for playing domination games on a galaxy. We will use these strategies to find recursive formulas of the game domination numbers. Finally, in Section 4, we find some minimal galaxies that have largest possible game domination numbers with respect to the number of components.

2. Preliminaries

In this section, we recall some definitions and useful results.

A *star* is a tree which has a vertex that is adjacent to all other vertices, called a *center*. We denote a star with k leaves by S_k. For convenience, when considering S_1, we let one vertex be its center and another is not. Therefore every star has a unique center. A *forest* is a graph whose components are trees. A *galaxy* is a graph whose components are stars.

Brešar, Klavžar and Rall gave the bounds of the game domination number in terms of the domination number.

Theorem 2.1. [3] *For any graph G, we have $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$.*

The two types of game domination numbers of a graph differ by at most one.

Theorem 2.2. [3, 9, 11] *For any graph G, we have $|\gamma_g(G) - \gamma'_g(G)| \leq 1$.*

For a pair of positive integers k and l, we say that the pair (k, l) is *realizable* if there exists a graph G such that $\gamma_g(G) = k$ and $\gamma'_g(G) = l$. By Theorem 2.2, we know that $k - 1 \leq l \leq k + 1$ holds for any realizable pair (k, l). In fact, any such pair except $(2, 1)$ is realizable [3, 9, 10].

For a graph G and a subset A of $V(G)$, let $G|A$ be the *partially-dominated* graph arising from G with A dominated. To find the game domination numbers of $G|A$, we consider only the number of vertices chosen after A is dominated.

The proofs in our paper require comparing choices of a move. One crucial tool for analyzing such choices is the Continuation Principle which was introduced by Brešar, Klavžar and Rall [3] and was formally proved later by Kinnersley [9].

Lemma 2.3. [9, Lemma 2.1] *(Continuation Principle). Let G be a graph, and fix A, $B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma'_g(G|A) \leq \gamma'_g(G|B)$.*

Recall that at any moment in the domination game, a vertex is valid if itself and its neighbors are not completely dominated. The *residual graph* of a partially-dominated graph G is a partially-dominated graph obtained by removing all invalid vertices and all edges joining dominated vertices in G. Removing invalid vertices and all edges joining dominate vertices do not affect the game as stated by the following lemma.

Lemma 2.4. [9, p7] *Let H be the residual graph of G. Then $\gamma_g(G) = \gamma_g(H)$ and $\gamma'_g(G) = \gamma'_g(H)$.*

3. Game domination numbers of a galaxy

In this section we find the optimal strategies for Dominator and Staller to play the domination games on a galaxy. We will use those strategies to find recursive formulas for computing the game domination numbers.

First, we show that on a galaxy, it does not matter whether the centers are dominated initially or not.

Lemma 3.1. *Let F be a galaxy and let C be a set of some centers of F that are not isolated vertices. Then $\gamma_g(F) = \gamma_g(F|C)$ and $\gamma'_g(F) = \gamma'_g(F|C)$.*

Proof. Let's consider a star with at least one leaf. If a player plays on the center, all of the vertices of the star will be dominated. On the other hand, if a player plays on a leaf, that leaf and the center will be dominated. The center of the star and at least one additional vertex will be dominated by a first move on the star. We can conclude that dominated centers do not affect the game. Therefore, $\gamma_g(F) = \gamma_g(F|C)$ and $\gamma'_g(F) = \gamma'_g(F|C)$. □

By Lemma 3.1, we can make the following assumptions without affecting the game domination numbers.

Assumption 3.2. *From now on when we consider galaxies, we assume that all non-isolated centers are already dominated.*

In general, for a given graph G and a subgraph H of G, it is not necessary true that $\gamma_g(G) \geq \gamma_g(H)$ or $\gamma'_g(G) \geq \gamma'_g(H)$. For example, let $G = S_3$ and H be the subgraph of G consisting of the three leaves. Then $\gamma_g(G) = 1 < 3 = \gamma_g(H)$ and $\gamma'_g(G) = 2 < 3 = \gamma'_g(H)$. However, for a galaxy, if the subgraph has fewer or equal number of components, then the inequalities hold.

Lemma 3.3. *Let G be a galaxy and F be a subgraph of G. If the number of components of F is at most the number of components of G, then $\gamma_g(G) \geq \gamma_g(F)$ and $\gamma'_g(G) \geq \gamma'_g(F)$.*

Proof. For a positive integer m, let $G = S_{n_1} + S_{n_2} + \cdots + S_{n_m}$ be a galaxy of m stars where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_m$ and $F = S_{t_1} + S_{t_2} + \cdots + S_{t_k}$ be a subgraph of G where $k \leq m$ and $1 \leq t_1 \leq t_2 \cdots \leq t_k$. For $i = \{1, 2, \dots, k\}$, observe that $t_i \leq n_{i+m-k}$ so we can view S_{t_i} in F as a subgraph of $S_{n_{i+m-k}}$ in

G. Note that F is the residual graph of $G|(V(G) \setminus V(F))$. By Lemma 2.3 and Lemma 2.4, we have

$$\gamma_g(G) \geq \gamma_g(G|V(G) \setminus V(F)) = \gamma_g(F)$$

and

$$\gamma'_g(G) \geq \gamma'_g(G|V(G) \setminus V(F)) = \gamma'_g(F).$$

□

Now we compare two galaxies that differ in only one component.

Lemma 3.4. *Let F be a galaxy. For positive integers $a \geq b$, let $F_a = F + S_a$ and $F_b = F + S_b$. Then $\gamma_g(F_a) \geq \gamma_g(F_b)$ and $\gamma'_g(F_a) \geq \gamma'_g(F_b)$.*

Proof. By Lemma 3.3. □

By Lemma 2.3, we can assume that Dominator always plays on a center of some star and Staller always plays on a leaf of some star. Next, we show how Dominator chooses a vertex optimally.

Theorem 3.5. *Let F be a partially-dominated galaxy. Then a Dominator's optimal strategy is to play on a center with the most number of undominated neighbors.*

Proof. Without loss of generality we consider only the DS game. Let S_a and S_b be components in F where S_a has the most number of undominated leaves. We will compare the effect of playing the center of S_a and the center of S_b. Let F_a and F_b be the residual graphs after Dominator chose the center on S_a and S_b respectively. We can view F_a as a subgraph of F_b so $\gamma'_g(F_a) \leq \gamma'_g(F_b)$ by Lemma 3.4. Therefore, a Dominator's optimal strategy is to play on a center with the most number of undominated neighbors. □

In the next lemma, we compare two galaxies with the same number of components and the same number of leaves that satisfy a certain ordering. As a consequence, we obtain a Staller's optimal strategy.

Lemma 3.6. *Let $F = S_{n_1} + \cdots + S_{n_m}$ and $G = S_{t_1} + \cdots + S_{t_m}$ where $n_1 \leq \cdots \leq n_m$ and $t_1 \leq \cdots \leq t_m$. If $\sum_{i=1}^{j} n_i \geq \sum_{i=1}^{j} t_i$ for all $1 \leq j \leq m$ and $\sum_{i=1}^{m} n_i = \sum_{i=1}^{m} t_i$, then $\gamma_g(F) \geq \gamma_g(G)$ and $\gamma'_g(F) \geq \gamma'_g(G)$.*

Proof. We will prove by induction on the total numbers of leaves in the graphs. For $F = G = S_1$, we get $\gamma_g(F) \geq \gamma_g(G)$ and $\gamma'_g(F) \geq \gamma'_g(G)$. Now let $F = S_{n_1} + \cdots + S_{n_m}$, and $G = S_{t_1} + \cdots + S_{t_m}$ where $\sum_{i=1}^{j} n_i \geq \sum_{i=1}^{j} t_i$ for all $1 \leq j \leq m$ and $\sum_{i=1}^{m} n_i = \sum_{i=1}^{m} t_i = v$. Let's consider the SD games. In the first move, Staller must choose a leaf on some component that satisfies

$$\gamma'_g(F) = 1 + \max_{i \in \{1,2,\ldots,m\}} \gamma_g(S_{n_1} + \cdots + S_{n_{i-1}} + S_{n_i - 1} + S_{n_{i+1}} + \cdots + S_{n_m}).$$

Without loss of generality, let Staller choose a leaf on S_{n_j} where $j = 1$ or $n_{j-1} < n_j$. For $i \in \{1, 2, \ldots, m\}$, let r_i be the number of undominated leaves on S_{n_i} after Staller chose a leaf on S_{n_j}. That is

$$r_i = \begin{cases} n_i, & i \neq j \\ n_i - 1, & i = j. \end{cases}$$

That gives $r_1 \leq \cdots \leq r_m$. For $i \in \{1, 2, \ldots, m\}$, let u_i be the number of undominated leaves on S_{n_i} after Staller chose a leaf on S_{n_m}. We get $\sum_{i=1}^{k} r_i \leq \sum_{i=1}^{k} u_i$ for $k \in \{1, 2, \ldots, m\}$. Since $\sum_{i=1}^{m} r_i = \sum_{i=1}^{m} u_i = v - 1$, by the induction hypothesis we have

$$\gamma_g'(F) = 1 + \gamma_g(S_{n_1} + \cdots + S_{n_{m-1}} + S_{n_m - 1}). \tag{1}$$

In the same way, $\gamma_g'(G) = 1 + \gamma_g(S_{t_1} + \cdots + S_{t_{m-1}} + S_{t_m - 1})$. Since $\sum_{i=1}^{k} n_i \geq \sum_{i=1}^{k} t_i$ for $k \in \{1, 2, \ldots, m\}$ and $(\sum_{i=1}^{m} n_i) - 1 = (\sum_{i=1}^{m} t_i) - 1 = v - 1$, by the induction hypothesis, we get

$$\gamma_g(S_{n_1} + \cdots + S_{n_{m-1}} + S_{n_m - 1}) \geq \gamma_g(S_{t_1} + \cdots + S_{t_{m-1}} + S_{t_m - 1}).$$

Therefore,

$$\gamma_g'(F) \geq \gamma_g'(G).$$

Now let's consider the DS games. Since $\sum_{i=1}^{m} n_i = \sum_{i=1}^{m} t_i$ and $\sum_{i=1}^{m-1} n_i \geq \sum_{i=1}^{m-1} t_i$, we have $n_m \leq t_m$. By Theorem 3.5,

$$\gamma_g(F) = 1 + \gamma_g'(S_{n_1} + \cdots + S_{n_{m-1}})$$

and

$$\gamma_g(G) = 1 + \gamma_g'(S_{t_1} + \cdots + S_{t_{m-1}}).$$

Let $H = S_{t_1} + \cdots + S_{t_{m-2}} + S_{t_{m-1} + (t_m - n_m)}$. Observe that the numbers of leaves in $S_{n_1} + \cdots + S_{n_{m-1}}$ and H are equal. By the induction hypothesis, we get

$$\gamma_g'(S_{n_1} + \cdots + S_{n_{m-1}}) \geq \gamma_g'(H).$$

By Lemma 3.4, we get

$$\gamma_g'(H) \geq \gamma_g'(S_{t_1} + \cdots + S_{t_{m-1}}).$$

Therefore, $\gamma_g(F) \geq \gamma_g(G)$. The proof is completed by the mathematical induction. □

Theorem 3.7. *Let F be a galaxy. Then a Staller's optimal strategy is to play on a leaf vertex adjacent to a center with the most number of undominated neighbors.*

Proof. By the Equation (1) in the proof of Lemma 3.6. □

The following theorem give a recursive formula for computing the game domination numbers of a galaxy. Since $\gamma_g(S_0) = \gamma_g'(S_0) = \gamma_g(S_1) = \gamma_g'(S_1)$, we may only consider galaxies without isolated vertices.

Theorem 3.8. *Let* $F = r_1S_{n_1} + \cdots + r_mS_{n_m}$ *and* $G = r_1S_{n_1} + \cdots + r_{m-1}S_{n_{m-1}}$ *where* $1 \leq n_1 < \cdots < n_m$*. Then*

$$\gamma_g(F) = \begin{cases} 1 & \textit{if } m = 1 \textit{ and } r_m = 1 \\ r_1 & \textit{if } n_m = 1 \\ r_m + \gamma_g(G + \frac{r_m}{2}S_{n_m-1}) & \textit{if } n_m > 1 \textit{ and } r_m \textit{ is even} \\ r_m + \gamma_g'(G + \frac{r_m-1}{2}S_{n_m-1}) & \textit{if } n_m > 1 \textit{ and } r_m \textit{ is odd} \end{cases}$$

and

$$\gamma_g'(F) = \begin{cases} 2 & \textit{if } m = 1, r_m = 1 \textit{ and } n_m \geq 2 \\ r_1 & \textit{if } n_m = 1 \\ r_m + \gamma_g'(G + \frac{r_m}{2}S_{n_m-1}) & \textit{if } n_m > 1 \textit{ and } r_m \textit{ is even} \\ r_m + \gamma_g(G + \frac{r_m+1}{2}S_{n_m-1}) & \textit{if } n_m > 1 \textit{ and } r_m \textit{ is odd} \end{cases}.$$

Proof. Clearly the statement holds for a star and for a galaxy consisting of only S_1's. Now assume $n_m > 1$. Consider how both players play during the first r_m turns. By Theorem 3.5 and Theorem 3.7, both players will play on a component S_{n_m} which has not been played. When Dominator plays, all of the vertices of an S_{n_m} will be dominated. When Staller plays, an S_{n_m} will be reduced to an S_{n_m-1}. Let's consider the DS game first. If r_m is even, Dominator and Staller will play for $\frac{r_m}{2}$ turns each. After these first r_m moves, F is reduced to $r_1S_{n_1} + \cdots + r_{m-1}S_{n_{m-1}} + \frac{r_m}{2}S_{n_m-1}$ or $G + \frac{r_m}{2}S_{n_m-1}$. Since Dominator plays first, the $(r_m + 1)$th turn is Dominator's turn. If r_m is odd, Dominator will play for $\frac{r_m+1}{2}$ turns and Staller will play for $\frac{r_m-1}{2}$ turns. After these first r_m moves, F is reduced to $r_1S_{n_1} + \cdots + r_{m-1}S_{n_{m-1}} + \frac{r_m-1}{2}S_{n_m-1}$ or $G + \frac{r_m-1}{2}S_{n_m-1}$. Since Dominator plays first, the $(r_m + 1)$th turn is Staller's turn. Therefore

$$\gamma_g(F) = \begin{cases} r_m + \gamma_g(G + \frac{r_m}{2}S_{n_m-1}) & \text{if } r_m \text{ is even} \\ r_m + \gamma_g'(G + \frac{r_m-1}{2}S_{n_m-1}) & \text{if } r_m \text{ is odd} \end{cases}.$$

By applying the similar argument to the SD game, we get

$$\gamma_g'(F) = \begin{cases} r_m + \gamma_g'(G + \frac{r_m}{2}S_{n_m-1}) & \text{if } r_m \text{ is even} \\ r_m + \gamma_g(G + \frac{r_m+1}{2}S_{n_m-1}) & \text{if } r_m \text{ is odd} \end{cases}.$$

□

4. Extremal galaxies

Note that the domination number of a galaxy equals its number of components. By Theorem 2.1 and Theorem 2.2, for a galaxy F with m components,

$$\gamma_g(F) \leq 2m - 1 \tag{2}$$

and

$$\gamma'_g(F) \leq 2m. \tag{3}$$

In this section, we find some minimal galaxies that achieve these bounds. First let's consider galaxies whose stars have different numbers of leaves.

Theorem 4.1. *Let $F_m = S_{n_1} + \cdots + S_{n_m}$ where $1 \leq n_1 < \cdots < n_m$. Then*

$$\gamma_g(F_m) = \begin{cases} 1 & \textit{if } m = 1 \\ 2m-2 & \textit{if } m > 1 \textit{ and } n_1 = 1 \\ 2m-1 & \textit{if } m > 1 \textit{ and } n_1 > 1 \end{cases}$$

and

$$\gamma'_g(F_m) = \begin{cases} 2m-1 & \textit{if } n_1 = 1 \\ 2m & \textit{if } n_1 > 1 \end{cases}.$$

Proof. We induct on m. Clearly, $\gamma_g(S_{n_1}) = 1$ and

$$\gamma'_g(S_{n_1}) = \begin{cases} 1 & \text{if } n_1 = 1 \\ 2 & \text{if } n_1 > 1. \end{cases}$$

By applying Theorem 3.5 and Theorem 3.7 we get

$$\begin{aligned} \gamma_g(S_{n_1} + S_{n_2}) &= 1 + \gamma'_g(S_{n_1}) \\ &= \begin{cases} 2 & \text{if } n_1 = 1 \\ 3 & \text{if } n_1 > 1 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma'_g(S_{n_1} + S_{n_2}) &= 1 + \gamma_g(S_{n_1} + S_{n_2 - 1}) \\ &= 2 + \gamma'_g(S_{n_1}) \\ &= \begin{cases} 3 & \text{if } n_1 = 1 \\ 4 & \text{if } n_1 > 1. \end{cases} \end{aligned}$$

For an integer $k \geq 2$, suppose

$$\gamma_g(F_k) = \begin{cases} 2k-2 & \text{if } n_1 = 1 \\ 2k-1 & \text{if } n_1 > 1 \end{cases}$$

and

$$\gamma'_g(F_k) = \begin{cases} 2k-1 & \text{if } n_1 = 1 \\ 2k & \text{if } n_1 > 1 \end{cases}.$$

By applying Theorem 3.5 and Theorem 3.7 we get

$$\begin{aligned} \gamma_g(F_{k+1}) &= 1 + \gamma'_g(F_k) \\ &= \begin{cases} 2(k+1)-2 & \text{if } n_1 = 1 \\ 2(k+1)-1 & \text{if } n_1 > 1 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma'_g(F_{k+1}) &= 2 + \gamma'_g(F_k) \\ &= \begin{cases} 2(k+1)-1 & \text{if } n_1 = 1 \\ 2(k+1) & \text{if } n_1 > 1 \end{cases}. \end{aligned}$$

The proof is completed by the mathematical induction. □

When $m > 1$ and $n_1 > 1$, the graph F_m in Theorem 4.1 satisfies the upper-bounds (2) and (3). We now find a minimal galaxy that satisfies the bounds.

Lemma 4.2. *Let* $F_m = (S_2 + S_3 + \cdots + S_m) + S_m$ *where* $m > 1$. *Then* $\gamma_g(F_m) = \gamma'_g(F_m) = 2m - 1$.

Proof. We induct on m. For $m = 2$, we get

$$\gamma_g(S_2 + S_2) = 3 = 2(2) - 1,$$

and

$$\gamma'_g(S_2 + S_2) = 3 = 2(2) - 1.$$

For an integer $k \geq 2$, suppose $\gamma_g((S_2 + S_3 + \cdots + S_k) + S_k) = \gamma'_g((S_2 + S_3 + \cdots + S_k) + S_k) = 2k - 1$. By Theorem 3.8 and Theorem 4.1, we get

$$\begin{aligned} \gamma_g((S_2 + S_3 + \cdots + S_{k+1}) + S_{k+1}) &= 2 + \gamma_g((S_2 + S_3 + \cdots + S_k) + S_k) \\ &= 2 + (2k-1) \\ &= 2k+1 \\ &= 2(k+1) - 1 \end{aligned}$$

and

$$\begin{aligned} \gamma'_g((S_2 + S_3 + \cdots + S_{k+1}) + S_{k+1}) &= 2 + \gamma'_g((S_2 + S_3 + \cdots + S_k) + S_k) \\ &= 2 + (2k-1) \\ &= 2k+1 \\ &= 2(k+1) - 1. \end{aligned}$$

The proof is completed by the mathematical induction. □

Theorem 4.3. *For a positive integer m, the galaxy $F_m = S_2 + S_3 + \cdots + S_{m+1}$ is a minimal galaxy that realizes the pair $(2m-1, 2m)$.*

Proof. By Theorem 4.1, F_m realizes the pair $(2m-1, 2m)$. We will show that any proper subgraph G of F_m has $\gamma'_g(G) < 2m$. By Lemma 3.3, it suffices to consider subgraphs with m components. Let $G = S_{n_1} + S_{n_2} + \cdots + S_{n_m}$ be a proper subgraph of F_m, where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_m$. Observe that $n_i \leq i+1$ for $1 \leq i \leq m$. Let $j = \max\{i | n_i \neq i+1\}$. Let F'_m be the graph obtained from F_m by removing a leaf in S_{j+1}, that is $F'_m = S_2 + S_3 + \cdots + S_j + S_j + S_{j+2} + \cdots + S_{m+1}$. Observe that G is a subgraph of F'_m. By Lemma 3.3, we get $\gamma'_g(G) \leq \gamma'_g(F'_m)$. Let F''_m be the graph obtained from F_m by removing a leaf in S_{m+1}, that is $F''_m = S_2 + S_3 + \cdots + S_m + S_m$. By Lemma 4.2, we have $\gamma'_g(F''_m) = 2m-1$. By Lemma 3.6, we have $\gamma'_g(F'_m) \leq \gamma'_g(F''_m)$. Thus $\gamma'_g(G) \leq \gamma'_g(F'_m) \leq \gamma'_g(F''_m) = 2m-1$ so G cannot realize the pair $(2m-1, 2m)$. Therefore, F_m is a minimal galaxy that realizes the pair $(2m-1, 2m)$. □

Next we consider galaxies that contain only one type of stars and find minimal graphs among these galaxies that satisfy the upperbounds (2) and (3).

Lemma 4.4. *For an integer m and a positive integer n, we have*

$$\left\lceil \frac{m}{2^{n-1}} \right\rceil = \left\lfloor \frac{m-1}{2^{n-1}} \right\rfloor + 1.$$

Proof. Write $m = b2^{n-1} + r$ where b is an integer and $0 \leq r < 2^{n-1}$. We have

$$\left\lceil \frac{m}{2^{n-1}} \right\rceil = \left\lceil \frac{b2^{n-1}+r}{2^{n-1}} \right\rceil = b + \left\lceil \frac{r}{2^{n-1}} \right\rceil$$

and

$$\left\lfloor \frac{m-1}{2^{n-1}} \right\rfloor + 1 = \left\lfloor \frac{b2^{n-1}+r-1}{2^{n-1}} \right\rfloor + 1 = b + \left\lfloor \frac{r-1}{2^{n-1}} \right\rfloor + 1.$$

Therefore, $\left\lceil \frac{m}{2^{n-1}} \right\rceil = \left\lfloor \frac{m-1}{2^{n-1}} \right\rfloor + 1$. □

Theorem 4.5. *Let m and n be positive integers. Then*

$$\gamma_g(mS_n) = 2m - \left\lceil \frac{m}{2^{n-1}} \right\rceil$$

and

$$\gamma'_g(mS_n) = 2m - \left\lfloor \frac{m}{2^{n-1}} \right\rfloor.$$

Proof. We induct on n. Let $n = 1$. We get

$$\gamma_g(mS_1) = m = 2m - \left\lceil \frac{m}{2^{1-1}} \right\rceil$$

and

$$\gamma'_g(mS_1) = m = 2m - \left\lfloor \frac{m}{2^{1-1}} \right\rfloor.$$

Now, for a positive integer k, suppose $\gamma_g(mS_k) = 2m - \left\lceil \frac{m}{2^{k-1}} \right\rceil$ and $\gamma'_g(mS_k) = 2m - \left\lfloor \frac{m}{2^{k-1}} \right\rfloor$. We use Theorem 3.8, Lemma 4.4 and the induction hypothesis

to find $\gamma_g(mS_{k+1})$ and $\gamma'_g(mS_{k+1})$ as follows.
When m is even,

$$\begin{aligned}\gamma_g(mS_{k+1}) &= m + \gamma_g\left(\frac{m}{2}S_k\right) \\ &= m + 2\left(\frac{m}{2}\right) - \left\lceil \frac{\frac{m}{2}}{2^{k-1}} \right\rceil \\ &= 2m - \left\lceil \frac{m}{2^k} \right\rceil\end{aligned}$$

and

$$\begin{aligned}\gamma'_g(mS_{k+1}) &= m + \gamma'_g\left(\frac{m}{2}S_k\right) \\ &= m + 2\left(\frac{m}{2}\right) - \left\lfloor \frac{\frac{m}{2}}{2^{k-1}} \right\rfloor \\ &= 2m - \left\lfloor \frac{m}{2^k} \right\rfloor.\end{aligned}$$

When m is odd,

$$\begin{aligned}\gamma_g(mS_{k+1}) &= m + \gamma'_g\left(\frac{m-1}{2}S_k\right) \\ &= m + 2\left(\frac{m-1}{2}\right) - \left\lfloor \frac{\frac{m-1}{2}}{2^{k-1}} \right\rfloor \\ &= 2m - 1 - \left\lfloor \frac{m-1}{2^k} \right\rfloor \\ &= 2m - 1 - \left\lceil \frac{m}{2^k} \right\rceil + 1 \\ &= 2m - \left\lceil \frac{m}{2^k} \right\rceil\end{aligned}$$

and

$$\begin{aligned}\gamma'_g(mS_{k+1}) &= m + \gamma_g\left(\frac{m+1}{2}S_k\right) \\ &= m + 2\left(\frac{m+1}{2}\right) - \left\lceil \frac{\frac{m+1}{2}}{2^{k-1}} \right\rceil \\ &= 2m + 1 - \left\lceil \frac{m+1}{2^k} \right\rceil \\ &= 2m + 1 - \left\lfloor \frac{m}{2^k} \right\rfloor - 1 \\ &= 2m - \left\lfloor \frac{m}{2^k} \right\rfloor.\end{aligned}$$

The proof is completed by the mathematical induction. □

Corollary 4.6. *Let m and n be positive integers. Then $\gamma_g(mS_n) = \gamma'_g(mS_n)$ if and only if $2^{n-1}|m$.*

Proof. By Theorem 4.5 and since $\left\lceil \frac{m}{2^{n-1}} \right\rceil = \left\lfloor \frac{m}{2^{n-1}} \right\rfloor$ if and only if $2^{n-1}|m$. □

Corollary 4.7. *Among the galaxies of m isomorphic stars, the galaxy mS_n is the minimal galaxy that realizes the pair $(2m-1, 2m)$ where n is the smallest integer greater than $1 + \log_2 m$.*

Proof. Observe that $0 < m < 2^{n-1}$. Therefore $\left\lceil \frac{m}{2^{n-1}} \right\rceil = 1$ and $\left\lfloor \frac{m}{2^{n-1}} \right\rfloor = 0$. By Theorem 4.5, we have

$$\gamma_g(mS_n) = 2m - \left\lceil \frac{m}{2^{n-1}} \right\rceil = 2m - 1$$

and

$$\gamma'_g(mS_n) = 2m - \left\lfloor \frac{m}{2^{n-1}} \right\rfloor = 2m.$$

Next, we show that mS_n is the minimal galaxy of m isomorphic stars that realizes the pair $(2m-1, 2m)$. Let $G = mS_{n-l}$ be a graph where $l \in \{1, 2, \cdots, n-1\}$. Since $n - l \leq 1 + \log_2 m$, we have $\frac{m}{2^{n-l-1}} \geq 1$. Thus

$$\left\lfloor \frac{m}{2^{n-l-1}} \right\rfloor \geq 1.$$

Therefore

$$\gamma'_g(G) = 2m - \left\lfloor \frac{m}{2^{n-l-1}} \right\rfloor < 2m.$$

Hence G cannot realize the pair $(2m-1, 2m)$. Therefore mS_n is the minimal galaxy of m isomorphic stars which realizes the pair $(2m-1, 2m)$. □

References

[1] B. Brešar, P. Dorbec, S. Klavžar, and G. Košmrlj, "Domination game: effect of edge- and vertex-removal," *Discreate Math.* **330** (2014) 1–10.

[2] B. Brešar, S. Klavžar, G. Košmrlj and D.F. Rall, "Domination game: External families of graph for 3/5-conjectures," *Discrete Appl. Math.* **161** (2013) 1308–1316.

[3] B. Brešar, S. Klavžar, and D.F. Rall, "Domination game and imagination strategy," *SIAM J. Discrete Math.* **24** (2010) 979–991.

[4] B. Brešar, S. Klavžar, and D.F. Rall, "Domination game played on trees and spanning," *Discrete Math.* **313** (2013) 915–923.

[5] C. Bujtás, "Domination game on forests," *Discrete Math.* **338** (2015) 2220–2228.

[6] P. Dorbec, G. Košmrlj, and G. Renault "The domination game played on unions of graphs," *Discrete Math.* **338** (2015) 71–79.

[7] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Domination in Graph: Advanced Topics*, Marcel Dekker, New York, 1998.

[8] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[9] W.B. Kinnersley, D.B. West, and R. Zamani, "Extremal problems for game domination Number", *SIAM J. Discrete Math.* **27** (2013) 2090–2107.

[10] G. Košmrlj, "Realizations of the game domination number," *J. Comb. Optim.* **28** (2014) 447–461.

[11] R. Zamani, "Hamiltonian cycles through specified edges in bipartite graphs, domination game, and the game of revolutionaries and spies", Ph.D. dissertion, University of Illinois at Urbana-Champaign, 2011.

Kraiwit Laopreeda
Department of Mathematics, Faculty of Science, Silpakorn University
Nakhon Pathom, Thailand
Email address: kraiwit12415@gmail.com

Chalermpong Worawannotai
Department of Mathematics, Faculty of Science, Silpakorn University
Nakhon Pathom, Thailand
Email address: worawannotai_c@silpakorn.edu