รายงานวิจัยฉบับสมบูรณ์

โครงการ

ระเบียบวิธีการกระทำซ้ำเพื่อแก้ปัญหาดุลยภาพแบบแยกและปัญหาที่เกี่ยวข้องกับการประยุกต์

Iterative methods for solving split equilibrium problems and other relevant problems with applications

โดย ดร.เอื้อมพร วิทยารัฐ

พฤษภาคม 2561

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หัวหน้าโครงการวิจัย
ดร.เอื้อมพร วิทยารัฐ มหาวิทยาลัยพะเยา
นักวิจัยที่ปรึกษา
ศ.ดร.ภูมิ คำเอม มหาวิทยาลัยเทคโนโลยีพระจอมเกล้าธนบุรี

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ชื่อนักวิจัย: ดร.เอื้อมพร วิทยารัฐ
สาขาคณิตศาสตร์ คณะวิทยาศาสตร์
มหาวิทยาลัยพะเยา
uamporn.wi@up.ac.th, u.witthayarat@hotmail.com

นักวิจัยที่ปรึกษา: ศาสตราจารย์ ดร.ภูมิ คำเอม
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเทคโนโลยีพระจอมเกล้าธนบุรี
Poom.kum@kmutt.ac.th

ระยะเวลาโครงการ: 2 พฤษภาคม 2559 – 1 พฤษภาคม 2561

ในงานวิจัยนี้ได้ทำการปรับปรุงระเบียบวิธีการกระทำซ้ำที่ปรากฏในงานวิจัยที่มีมาการหน้า เพื่อทำการประมาณค่าคำตอบร่วมของปัญหาดุลยภาพแบบแยกและปัญหาจุดตรึงในปริภูมิฮิลเบิร์ต ซึ่งลำดับที่สร้างขึ้นโดยระเบียบวิธีการกระทำซ้ำดังกล่าวนี้จะลู่เข้าอย่างเข้มสู่คำตอบร่วมของทั้งสองปัญหาโดยปราศจากสมมติฐานเกี่ยวกับนอร์มของตัวดำเนินการ นอกจากนี้ยังได้นำระเบียบวิธีที่ได้ไปประยุกต์ใช้ในการแก้ปัญหาความเป็นไปได้แบบแยก และ ปัญหาการค่าต่ำสุดเชิงนูนแบบแยกพร้อมทั้งนำเสนอตัวอย่างเพื่อเป็นการสนับสนุนทฤษฎีบทหลัก งานวิจัยนี้เป็นการขยายและปรับปรุงระเบียบวิธีการกระทำซ้ำของ Kazmi และ Rizvi

คำหลัก: ปัญหาดุลยภาพแบบแยก, ปัญหาจุดตรึง, ระเบียบวิธีการกระทำซ้ำ, การลู่เข้าอย่างเข้ม, ปริภูมิฮิลเบิร์ต

Abstract

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Investigator:	Dr. Uamporn Witthayarat Department of Mathematics School of Science University of Phayao uamporn.wi@up.ac.th, u.witthayarat@hotmail.com
Mentor:	Professor Dr.Poom Kumam Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi Poom.kum@kmutt.ac.th
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In this research, we modified the previous iterative methods in the literature for approximating a common solution of split equilibrium problem together with a fixed point problem in the framework of Hilbert spaces. Without the assumption on the norm of the operator, we prove that the sequence generated by our algorithms strongly converge to a solution of the problems. Furthermore, we give some applications to solve split feasibility problem and split convex minimization problem and give some numerical examples which support our main theorem. Our result mainly extends and improves the results obtained by Kazmi and Rizvi.

Keywords: Split Equilibrium Problem, Fixed Point Problem, Iterative Method, Strong Convergence, Hilbert Spaces

Contents

CHAPTER 1
Executive Summary

1.1 Introduction

Since Equilibrium problem was introduced by Ky Fan [1] and Boom and Oetli [2], it becomes the most attractive topic for many mathematicians. We found that many problems such as variational inequality problem, saddle point problem, minimization problem including the problem in physics, optimization theory and economics can be reformulated to the equilibrium problem. As its first generalization, many authors mentioned how to approximate the common solutions of some equilibrium problems but they still mostly observed in the same subset in the same space. However, in real world problem, we normally found that some equilibrium problems are not necessary to be considered in the same subset of the same space. Therefore, the split equilibrium problem (SEP) which contains two equilibrium problems was introduced and mentioned in this case. The relation between solutions of these two equilibrium problems in SEP is that the image of solution of one equilibrium problem under the bounded linear operator is the solution of another one with no need to consider in the same space. That makes SEP more general than classical equilibrium problems. Furthermore, we found that split variational inequality problem [introduced by Censors et al.] is the special case of SEP and we can also exactly link these problems to the split fixed point problems. Due to its applications, these all enable us to be more widely solve the real world problems in the future.

According to its most significance, many methods have been proposed to approximate its solutions, for example iterative methods generated by Mann, Halpern, Ishikawa including the CQ method, viscosity approximation method, hybrid projection methods and many others. Actually, these methods are commonly used for solving the equilibrium but when we apply these methods to split equilibrium problems there are some conditions that we have to mention dues to the different spaces. Some methods that are proved in different ways may get the better sufficient conditions than the previous ones and exactly we can deduce our method to solve the previous classical ones. Furthermore, there are several improvements and generalizations of the methods for solving the split equilibrium problems and related problems that have been suggested in many different ways.

These are the main objectives in this research which are to construct the new iteration methods for solving various kinds of split equilibrium problem and study convergence theorems which admit the better sufficient conditions. Our main results can extend and improve the corresponding previous results in this area and can apply to solve several problems in applied sciences and other related branches.

Based on the objective we mention above, we propose the new iteration scheme that improve the previous one in the literatures to solve the split equilibrium problem together with fixed point problem under the better conditions that avoid the norm of operators. The convergence theorem are proved and the applications are also presented, application to split feasibility problem and application to split convex minimization problem, respectively. Finally, numerical examples are given to support our main theorem.

1.2 Literature Review

Throughout this work, let H_1 and H_2 be two real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be two nonempty closed and convex subsets of H_1 and H_2, respectively. Let $\{x_n\}$ be a sequence in H_1, we also denote "$x_n \to x$" as strong convergence and "$x_n \rightharpoonup x$" as weak convergence of the sequence $\{x_n\}$ to a point $x \in H_1$.

A mapping $S : C \to C$ is called nonexpansive, if

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in C.$$

The fixed point problem for the mapping $S : C \to C$ is to find $x \in C$ such that $Sx = x$. We denote the set of fixed point of S as $Fix(S)$.

The equilibrium problem was first introduced and studied by Blum and Oettli [1] which is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C,$$

where F is a bifunction from $C \times C$ to $\mathbb{R}$. Its solution set is denoted by $EP(F)$.

There are many authors take the most interest to solve the equilibrium with their algorithm. In 2005, Combettes and Hirstoaga [2] introduced their algorithm to approximate the solution of equilibrium problem by using iterative method and proved the strong convergence theorem. Later, in 2007, Takahashi and Takahashi [3] also proposed the new iterative method called "viscosity approximation method" for finding a common solution of equilibrium problem together with fixed point problem. Moreover, based on the idea of Takahashi and Takahashi [3], PlubTieng and PunPaeng [4] improved and introduced the new scheme for solving the equilibrium problem. Recently, Liu et al. [5] extended the viscosity approximation method to find a common solution of the infinite family of fixed point problems together with equilibrium problem and other relevant problem.

The split feasibility problem is to find $x \in C$ such that $Ax \in Q$. We denote by $C \cap A^{-1}Q$ its solution set. The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [12] for modeling inverse problems which arise from phase retrievals, medical image reconstruction and recently in modeling of intensity modulated radiation therapy. The SFP attracts the attention of many authors due to its application in signal processing. In order to solve the split feasibility problem (SFP), Byrne [11] proposed the following iterative algorithm in the framework of Hilbert spaces: $x_1 \in C$ and

$$x_{n+1} = P_C(x_n - \lambda A^*(I - P_Q)Ax_n), \; n \geq 1, \tag{1.1}$$

which is often called the CQ algorithm, where $\lambda > 0$, P_C and P_Q are the metric projections on C and Q, respectively. It was shown that the sequence $\{x_n\}$ converges weakly to a solution of SFP provided $0 < r < 2/\|A\|^2$. Since then several iterations have been invented for solving the SFP (see, for example, [9, 16, 17, 22]).

Recently, Censor and Segal [6] proposed the iterative scheme to approximate a solution of split common fixed point problem which is a generalized of split feasibility problem and convex feasibility problem:

Let A be a real $m \times n$ matrix and let $U : \mathbb{R}^n \to \mathbb{R}^n$ and $T : \mathbb{R}^m \to \mathbb{R}^m$ be operators with nonempty $Fix(U) = C$ and $Fix(T) = Q$. The problem is to find $x^* \in C$ such that $Ax^* \in Q$.

Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, the split equilibrium problem (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \;\; \forall x \in C,$$

and such that

$$y^* = Ax^* \in Q \;\text{ solves }\; F_2(y^*, y) \geq 0 \;\; \forall y \in Q.$$

We can see that the first part of SEP seems like the classical equilibrium problem EP where we can denote its solution set as $EP(F_1)$. The SEP looks like a pair of equilibrium problems which have to be solved so that the image $y^* = Ax^*$ under the given bounded operator A. We denote the solution set of the second EP in SEP as $EP(F_2)$. The solution set of SEP is denoted by $\Omega = \{p \in EP(F_1) : Ap \in EP(F_2)\}$.

In 2013, Kazmi and Rizvi [7] focus on how to approximate a common solution of split equilibrium problem, variational inequality problem and fixed point problem by stating the strong convergence theorem of their iterative algorithm as shown in the following:

$$\begin{aligned} u_n &= J_{r_n}^{F_1}(x_n + \gamma A^*(J_{r_n}^{F_2} - I)Ax_n); \\ y_n &= P_C(u_n - \lambda_n D u_n); \\ x_{n+1} &= \alpha_n \nu + \beta_n x_n + \gamma_n S y_n. \end{aligned} \tag{1.2}$$

They proved that $\{x_n\}$ generated by (1.2) converges strongly to the common solution of SEP, FPP and VI under some appropriate conditions of the sequences $\{\alpha_n\}, \{\gamma_n\}, \{\beta_n\}$ and $\{\lambda_n\}$.

However, it was observed that the step size γ depends on the computation of the operator norm A^*A which is not an easy task in practice. To overcome this difficulty, Lopez et al. [8] suggested a new way of stepsize τ_n as follows:

$$\tau_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \ \rho_n \in (0, 4).$$

CHAPTER 2
Preliminaries and lemmas

In this section we recall some definitions and lemmas which will be needed in the next section. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we know from [18] that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle; \tag{2.3}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.4}$$

Furthermore, for $x, y, u, v \in H$,

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.5}$$

The nearest point projection of a nonempty, closed and convex set C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x-y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x-y\rangle \tag{2.6}$$

for all $x, y \in H$. Moreover $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [18].

2.1 Lemmas

Lemma 2.1.1. *[10] Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, that is, if the sequence x_n converges weakly to $x \in C$ and $\|x_n - Tx_n\| \to 0$, then $x = Tx$.*

Assumption 1. *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:*

(i) $F(x, x) = 0, \ \forall x \in C$;

(ii) F *is monotone, i.e.,* $F(x, y) + F(y, x) \leq 0, \ \forall x \in C$;

(iii) For each $x, y, z \in C, \limsup_{t \to 0} F(tz + (1-t)x, y) \leq F(x, y)$;

(iv) For each $x \in C, y \to F(x, y)$ *is convex and lower semicontinuous.*

(v) Fixed $r > 0$ *and* $z \in C$*, there exists a nonempty compact convex subset* K *of* H_1 *and* $x \in C \cap K$ *such that*

$$F(y, x) + \frac{1}{r}\langle y - x, x - z < 0\rangle, \ \forall y \in C \setminus K.$$

Lemma 2.1.2. *Let $T : H \to H$ be an operator. The following statements are equivalent.*

(i) T *is firmly nonexpansive.*

(ii) $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \ x, y \in H$.

(iii) I-T *is firmly nonexpansive.*

Lemma 2.1.3. *Assume that $F_1 : C \times C \to \mathbb{R}$ satisfying Assumption 1. For $r > 0$ and for all $x \in H_1$, define a mapping $J_r^{F_1} : H \to C$ as follows:*

$$J_r^{F_1} x = \{z \in C : F_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}.$$

Then the following hold:

(i) $J_r^{F_1}$ *is nonempty and single-valued;*

(ii) $J_r^{F_1}$ *is firmly nonexpansive, i.e.,*

$$\|J_r^{F_1} x - J_r^{F_1} y\| \leq \langle J_r^{F_1} x - J_r^{F_1} y, x - y\rangle, \ \forall x, y \in H_1;$$

(iii) $Fix(J_r^{F_1}) = EP(F_1)$;

(iv) $EP(F_1)$ *is closed and convex.*

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ satisfying Assumption 1. For $s > 0$ and for all $w \in H_2$, define a mapping $J_s^{F_2} : H_2 \to Q$ as follows:

$$J_s^{F_2}(w) = \{d \in Q : F_2(d,e) + \frac{1}{s}\langle e - d, d - w\rangle \geq 0, \ \forall e \in Q\}.$$

Then we easily observe that $J_s^{F_2}$ is nonempty, single-valued and firmly nonexpansive, $EP(F_2, Q)$ is closed and convex and $Fix(J_s^{F_2}) = EP(F_2, Q)$, where $EP(F_2, Q)$ is the solution set of the following equilibrium problem:

$$\text{Find } y^* \in Q \text{ such that } F_2(y^*, y) \geq 0, \quad \forall y \in Q.$$

Lemma 2.1.4. *Let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying Assumption 1 hold and let* $J_r^{F_1}$ *be defined as in Lemma 2.1.3 for* $r > 0$. *Let* $x, y \in H_1$ *and* $r_1, r_2 > 0$. *Then:*

$$\|J_{r_2}^{F_1} y - J_{r_1}^{F_1} x\| \leq \|y - x\| + \frac{r_2 - r_1}{r_2}\|J_{r_2}^{F_1} y - y\|.$$

Lemma 2.1.5. *Let* $\{s_n\}$ *be a real sequence that does not decrease at infinity, in the sense that there exists a subsequence* $\{s_{n_k}\}$ *so that*

$$s_{n_k} \leq s_{n_k+1}, \ \forall k \geq 0.$$

For every $n \geq n_0$ *define an integer sequence* $\{\Gamma(n)\}$ *as*

$$\Gamma(n) = \max\{n_0 \leq k \leq n : s_k < s_{k+1}\}.$$

Then $\Gamma(n) \to \infty$ *as* $n \to \infty$ *and for all* $n > n_0$

$$\max\{s_{\Gamma(n),s_n}\} \leq s_{\Gamma(n)+1}.$$

CHAPTER 3
Main Results

3.1 Strong convergence theorem

In this Chapter, we divide into three sections including strong convergence theorem, applicationt to split feasibility problem and application to split convex minimization problem, respectively. We state the convergence theorem which shows that the sequence generated by this iteration method strongly converges to a common solution of the problems we mentioned. Moreover, we give two applications with numerical examples for supporting our main theorem.

Let H_1 and H_2 be two real Hilbert spaces with the nonempty closed and convex subsets C and Q, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption 1 and F_2 is upper semicontinuous in the first argument. Let $S : C \to C$ be a nonexpansive mapping such that $\Theta := Fix(S) \cap \Omega \neq \emptyset$. We further define

$$f(x_n) = \frac{1}{2}\|(I - J_{r_n}^{F_2})Ax_n\|^2, \ n \geq 0,$$

and

$$\nabla f(x_n) = A^*(I - J_{r_n}^{F_2})Ax_n.$$

Algorithm 1 Choose an arbitrary initial guess $x_0 \in C$, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{aligned} u_n &= J_{r_n}^{F_1}(x_n - \tau_n \nabla f(x_n)); \\ x_{n+1} &= \alpha_n g(x_n) + \beta_n x_n + \gamma_n S u_n; \end{aligned}$$

where g is a contraction on C, $\tau_n = \dfrac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}$, $\rho_n \in (0, 4)$.

Theorem 3.1.1. *Assume that* $r_n \in (0, \infty)$ *and* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are the sequences in* $(0, 1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $\liminf_{n\to\infty} \beta_n > 0$*;*

(C3) $\liminf_{n\to\infty} \gamma_n > 0$*;*

(C4) $\liminf_{n\to\infty} \rho_n(4 - \rho_n) > 0$*;*

(C5) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ *generated by Algorithm 1 converges strongly to* $z = P_\Theta g(z)$.

3.2 Application to split feasibility problem

For obtaining the result for the split feasibility problem, let the solution set $\Theta := Fix(S) \cap \Gamma \neq \emptyset$, and define

$$f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2, \; n \geq 0,$$

and

$$\nabla f(x_n) = A^*(I - P_Q)Ax_n.$$

Algorithm 2 Choose an arbitrary initial guess $x_0 \in C$, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{aligned} u_n &= P_C(x_n - \tau_n \nabla f(x_n)); \\ x_{n+1} &= \alpha_n g(x_n) + \beta_n x_n + \gamma_n S u_n; \end{aligned}$$

where g is a contraction on C, $\tau_n = \dfrac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}$, $\rho_n \in (0, 4)$.

Theorem 3.2.1. *Assume that* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are the sequences in* $(0, 1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1$ *satisfying the same conditions (C1)-(C4) in Theorem 3.1.1. Then the sequence* $\{x_n\}$ *generated by Algorithm 2 converges strongly to* $z = P_\Theta g(z)$.

Example 3.2.2. *Let* $H_1 = H_2 = \mathbb{R}^3$*. Define* $C = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq 1\}$ *and*

$$Q = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 + x_2 + 4x_3 \geq 1\}.$$

Let

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & -2 & 0 \end{pmatrix}.$$

Let $S : C \to C$ *be defined by* $Sx = (-x_1, x_2, -x_3)$ *and* $g : C \to C$ *by* $g(x) = \frac{x}{2}$ *where* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Choose $\alpha_n = \frac{1}{n+1}$, $\beta_n = 0.5$, $\gamma_n = 0.5 - \frac{1}{n+1}$ and $E_n = \|x_{n+1} - x_n\|_2 < 10^{-4}$ for all $n \in \mathbb{N}$.

We now study the effect (in terms of convergence, stability, number of iterations required and the cpu time) of the sequence $\{\rho_n\} \subset (0, 4)$ on the iterative scheme by choosing different ρ_n such that $\inf_n \rho_n(4 - \rho_n) > 0$. We choose different choices of x_1 as

Choice 1: $x_1 = (0, 0, 1)$;

Choice 2: $x_1 = (0.5, 0.5, 0.5)$;

Choice 3: $x_1 = (0.2, 0.6, 0.1)$;

Choice 4: $x_1 = (0.8, 0.6, 0)$.

The numerical experiments, using our Algorithm 2 in Theorem 3.2.1, for each choice are reported in the following Table 1.

The convergence behavior of the error E_n for each choice of x_1 is shown in Figure 1-4, respectively.

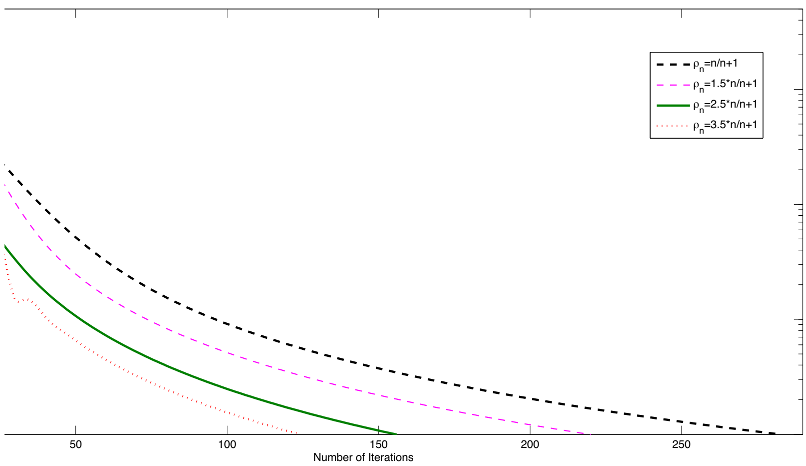

Table 3.1: Algorithm 3.7 with different cases of ρ_n and different choices of x_1

		$\rho_n = \frac{n}{n+1}$	$\rho_n = \frac{1.5n}{n+1}$	$\rho_n = \frac{2.5n}{n+1}$	$\rho_n = \frac{3.5n}{n+1}$
Choice 1	No. of Iter.	97	74	52	29
	cpu (Time)	0.018883	0.015916	0.011313	0.005729
Choice 2	No. of Iter.	97	74	52	41
	cpu (Time)	0.026899	0.019866	0.016848	0.010174
Choice 3	No. of Iter.	97	74	52	41
	cpu (Time)	0.026644	0.015758	0.011374	0.017603
Choice 4	No. of Iter.	97	74	52	30
	cpu (Time)	0.023669	0.016431	0.010965	0.007143

Figure 1: Error plotting E_n for Choice 1 in Example 3.3.2

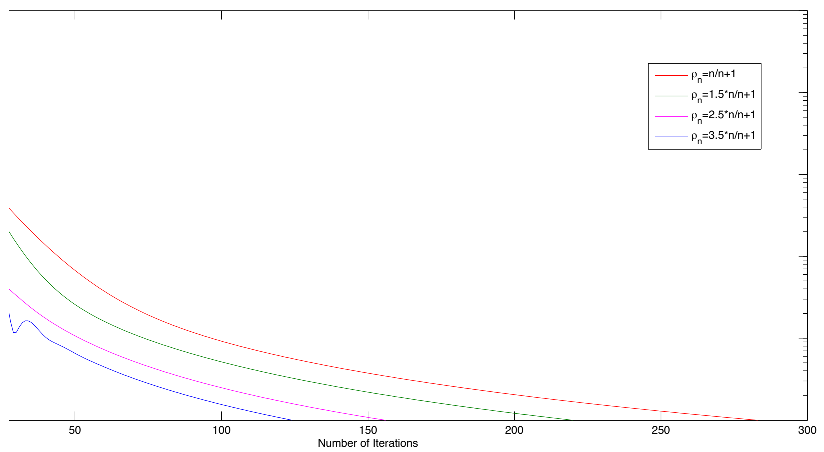

Figure 2: Error plotting E_n for Choice 2 in Example 3.3.2

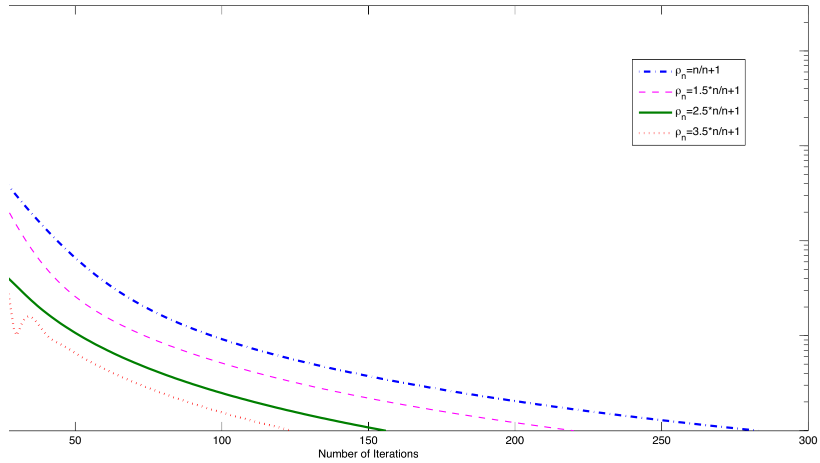

Figure 3: Error plotting E_n for Choice 3 in Example 3.3.2

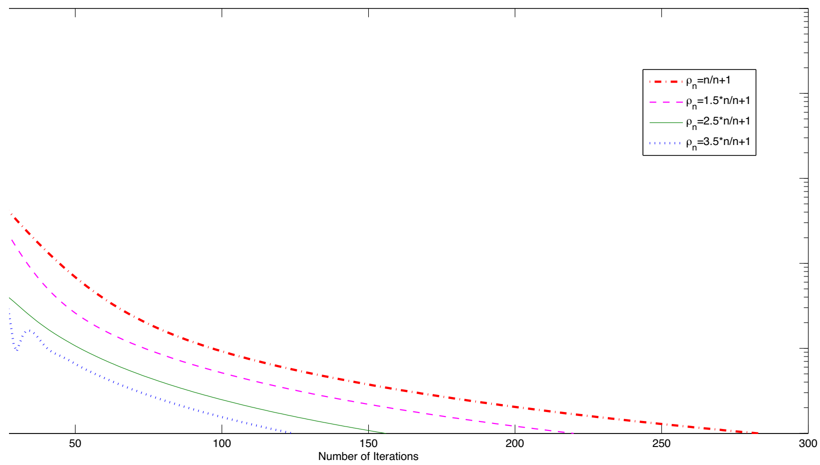

Figure 4: Error plotting E_n for Choice 4 in Example 3.3.2

3.3 Application to split convex minimization problem

In this section, we consider the following split convex minimization problem as follows:

The proximity operator of F is defined by

$$prox_{\lambda F}(x) = \arg\min_{y \in H}\{F(y) + \frac{1}{2\lambda}\|x-y\|^2\} \tag{3.7}$$

for any $\lambda > 0$. It is seen that

$$0 \in \partial F(x^*) \Leftrightarrow x^* = prox_{\lambda F}(x^*). \tag{3.8}$$

Let $f_1, f_2 : C \to \mathbb{R} \cup \{\infty\}$ be convex and lower semicontinuous. The split convex minimization problem is to find a minimizer x^* of f_1 that Ax^* is a minimizer of f_2, where A is a bounded linear operator.

To this end, we define

$$f(x_n) = \frac{1}{2}\|(I - prox_{\lambda f_2})Ax_n\|^2, \; n \geq 0,$$

and

$$\nabla f(x_n) = A^*(I - prox_{\lambda f_2})Ax_n.$$

Algorithm 3 Choose an arbitrary initial guess $x_0 \in C$, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{aligned} u_n &= prox_{\lambda f_1}(x_n - \tau_n \nabla f(x_n)); \\ x_{n+1} &= \alpha_n g(x_n) + \beta_n x_n + \gamma_n S u_n; \end{aligned}$$

where g is a contraction on C, $\tau_n = \dfrac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}$, $\rho_n \in (0, 4)$.

Theorem 3.3.1. *Assume that $\lambda > 0$ and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ satisfying the same conditions (C1)-(C4) in Theorem 3.1.1. Then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $z = P_\Theta g(z)$.*

Example 3.3.2. *Let $H_1 = H_2 = \mathbb{R}^3$. Let $f_1 : \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ be defined by*

$$f_1(x) = \|x\|_2^2 + (2, 4, -5)x + 10$$

and let $f_2 : \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$f_2(x) = \|x\|_2^2 - (8, 10, -8)x - 5.$$

Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \\ 2 & 1 & 0 \end{pmatrix}.$$

Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ *be defined by* $Sx = (-2 - x_1, -4 - x_2, 0.5x_3 + 1.25)$ *and* $g : \mathbb{R}^3 \to \mathbb{R}^3$ *by* $g(x) = \frac{x}{2}$ *where* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. *Find* $x \in \mathbb{R}^3$ *such that* x *minimizes* f_1 *and* Ax *minimize* f_2 *and* x *is also a fixed point of* S. *Choose* $\alpha_n = \frac{1}{n+1}$, $\beta_n = 0.1$, $\gamma_n = 0.9 - \frac{1}{n+1}$, $\lambda = 1$ *and* $E_n = \|x_{n+1} - x_n\|_2 < 10^{-4}$ *for all* $n \in \mathbb{N}$.

The numerical experiments, using our Algorithm 3 in Theorem 3.3.1, for each choice are reported in the following Table 2. We choose different choices of x_1 as

Choice 1: $x_1 = (0, 0, 1)$; Choice 2: $x_1 = (0.5, 0.5, 0.5)$;
Choice 3: $x_1 = (0.2, 0.6, 0.1)$; Choice 4: $x_1 = (0.8, 0.6, 0)$.

The numerical experiments, using our Algorithm 3 in Theorem 3.3.1, for each choice are reported in the following Table 2.

Table 3.2: Algorithm 3.7 with different cases of ρ_n and different choices of x_1

		$\rho_n = \frac{n}{n+1}$	$\rho_n = \frac{1.5n}{n+1}$	$\rho_n = \frac{2.5n}{n+1}$	$\rho_n = \frac{3.5n}{n+1}$
Choice 1	No. of Iter.	97	74	52	29
	cpu (Time)	0.018883	0.015916	0.011313	0.005729
Choice 2	No. of Iter.	97	74	52	41
	cpu (Time)	0.026899	0.019866	0.016848	0.010174
Choice 3	No. of Iter.	97	74	52	41
	cpu (Time)	0.026644	0.015758	0.011374	0.017603
Choice 4	No. of Iter.	97	74	52	30
	cpu (Time)	0.023669	0.016431	0.010965	0.007143

The convergence behavior of the error E_n for each choice of x_1 is shown in Figure 1-4, respectively.

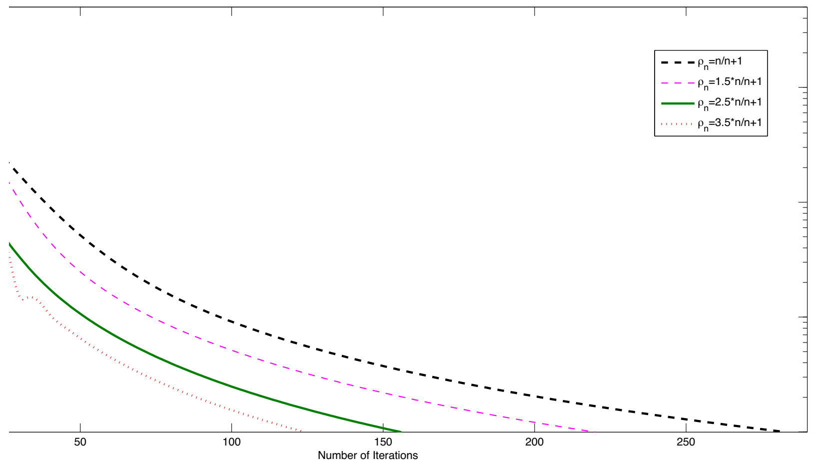

Figure 1: Error plotting E_n for Choice 1 in Example 3.3.2

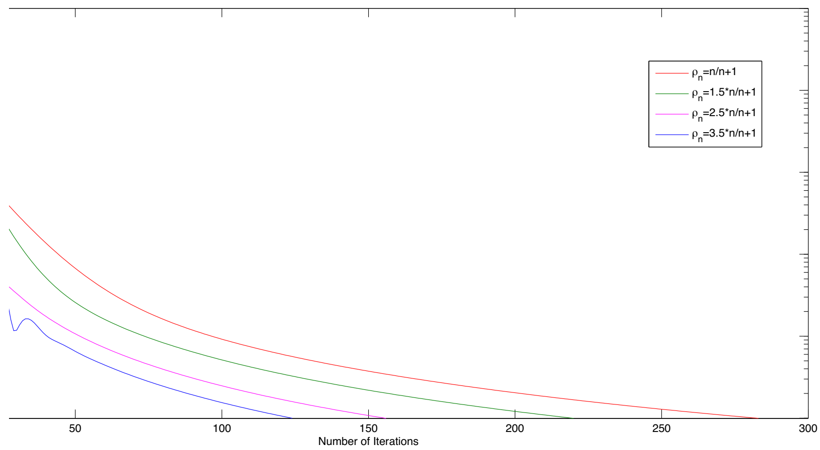

Figure 2: Error plotting E_n for Choice 2 in Example 3.3.2

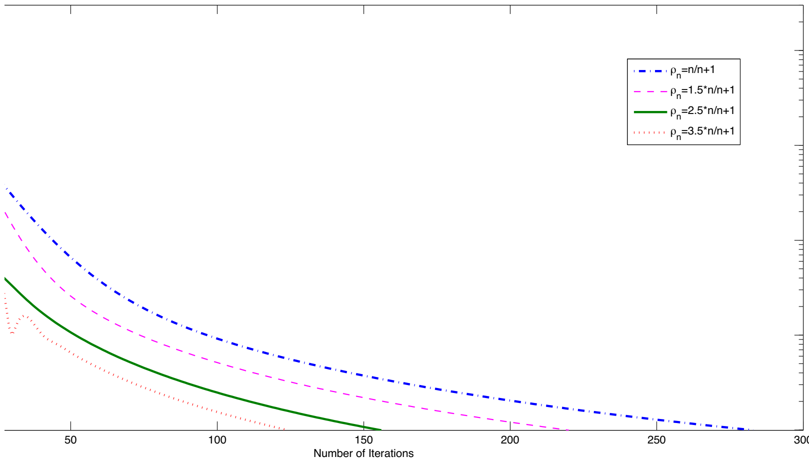

Figure 3: Error plotting E_n for Choice 3 in Example 3.3.2

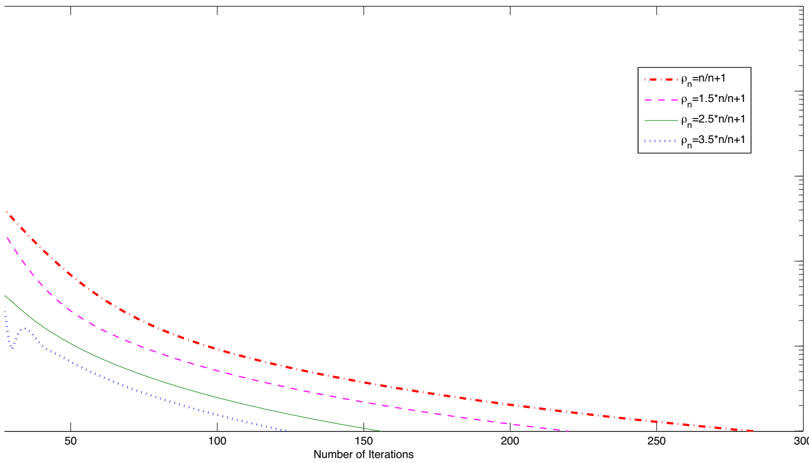

Figure 4: Error plotting E_n for Choice 4 in Example 3.3.2

Remark 3.3.3. *From our numerical experiments, it is observed that the different choices of x_1 have no effect in terms of CPU runtime for the convergence of our algorithm. However, if the stepsizes $\{\rho_n\}$ is taken close to 4, then the number of iterations and the CPU runtime have small reduction.*

Appendix

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ (ระบุชื่อผู้แต่ง ชื่อเรื่อง ชื่อวารสาร ปี เล่มที่ เลขที่ และหน้า) หรือผลงานตามที่คาดไว้ในสัญญาโครงการ

   Uamporn Witthayarat, Poom Kumam and Prasit Cholamjiak: A modified iterative method for approximating a common solution of split equilibrium problem and fixed point problem in Hilbert spaces

   หมายเหตุ * อยู่ระหว่างส่งตีพิมพ์

2. อื่นๆ (เช่น ผลงานตีพิมพ์ในวารสารวิชาการในประเทศ การเสนอผลงานในที่ประชุมวิชาการ หนังสือ การจดสิทธิบัตร)

การเสนอผลงานในที่ประชุมวิชาการนานาชาติ

2.1 ชื่อการจัดการประชุม	The 10^{th} International Conference on Nonlinear Analysis and Convex Analysis
สถานที่จัดประชุม	Chitose City Cultural Center, Hokkaido JAPAN, July 4-9, 2017.
ชื่อเรื่องที่นำเสนอ	A modified self-adaptive method for the split feasibility problem and the fixed point problem in Banach spaces

การเสนอผลงานในที่ประชุมวิชาการนานาชาติ

2.2 ชื่อการจัดการประชุม	The 6^{th} Asian Conference on Nonlinear Analysis and Optimiztion
สถานที่จัดประชุม	OIST&ANA Intercontinental Menza Beach Resort, November 5-9 2018
ชื่อเรื่องที่นำเสนอ	On solving split equilibrium problem in real Hilbert spaces with its applications

3.การนำผลงานวิจัยไปใช้ประโยชน์

จากผลงานวิจัยที่ได้ศึกษาในโครงการ “ระเบียบวิธีการกระทำซ้ำสำหรับแก้ปัญหาดุลยภาพแบบแยกและปัญหาที่เกี่ยวข้องกับการประยุกต์” ในเบื้องต้นนั้นผู้วิจัยได้นำผลการศึกษามาใช้ต่อยอดในการทำงานวิจัยร่วมกับกลุ่มวิจัย หน่วยความเป็นเลิศทางด้านคณิตศาสตร์ สาขาวิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยพะเยา ซึ่งทำให้เกิดการพูดคุยและแบ่งปันองค์ความรู้ในงานวิจัยร่วมกัน อันนำไปสู่การสร้างแนวทางและพัฒนางานวิจัยให้สามารถนำไปใช้ประโยชน์ได้มากขึ้น ซึ่งในกิจกรรมของกลุ่มนั้นได้มีการเชิญนักวิจัยจากทั้งภายในและภายนอกมหาวิทยาลัย รวมถึงนักวิจัยที่มีความเชี่ยวชาญ

จากต่างประเทศมาบรรยายและแบ่งปันประสบการณ์รวมถึงแนวทางในทำงานวิจัย อันก่อให้เกิดเครือข่ายความร่วมมือการวิจัยในอนาคต นอกจากนี้ผู้วิจัยยังได้นำผลการวิจัยมาประยุกต์ใช้ในด้านการเรียนการสอน ทั้งในรายวิชาการวิเคราะห์เชิงคณิตศาสตร์ รายวิชาหัวข้อปัจจุบันทางคณิตศาสตร์ รวมไปถึงการถ่ายทอดองค์ความรู้งานวิจัยด้านนี้ให้กับนิสิตในระดับปริญญาตรีผ่านรายวิชาสัมมนาและการศึกษาอิสระ

ผลงานวิจัยที่ตีพิมพ์

ผู้แต่ง: Uamporn Witthayarat, Poom Kumam and Prasit Cholamjiak

ชื่อเรื่องงานวิจัยที่ได้: A modified iterative method for approximating a common solution of split equilibrium problem and fixed point problem in Hilbert spaces

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ฐานข้อมูล: ISI

A modified iterative method for approximating a common solution of split equilibrium problem and fixed point problem in Hilbert spaces

Uamporn Witthayarat[1], Poom Kumam[2], Prasit Cholamjiak[1, *]

[1] *School of Science, University of Phayao, Phayao 56000, Thailand*

[2] *Department of Mathematics, King Mongkut's University of Technology Thonburi, Bangkok 10140, Thailand*

Abstract

In this work, we modify the iterative method for approximating a common solution of a split equilibrium problem together with a fixed point problem in the framework of Hilbert spaces. Without the assumption on the norm of the operator, we prove that the sequence generated by our algorithms strongly converge to a solution of the problems. Furthermore, we also give numerical examples which support our main theorem. Our result mainly extends and improves the results obtained by Kazmi and Rizvi.

Keywords: split equilibrium problem; strong convergence; iterative method; fixed point problem; Hilbert space.

AMS Subject Classification: 47H04, 47H10, 54H25.

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1 Introduction

Throughout this work, let H_1 and H_2 be two real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be two nonempty closed and convex subsets of H_1 and H_2, respectively. Let $\{x_n\}$ be a sequence in H_1, we also denote "$x_n \to x$" as strong convergence and "$x_n \rightharpoonup x$" as weak convergence of the sequence $\{x_n\}$ to a point $x \in H_1$.

A mapping $S : C \to C$ is called *nonexpansive*, if

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in C.$$

The *fixed point problem* for the mapping $S : C \to C$ is to find $x \in C$ such that $Sx = x$. We denote the set of fixed point of S as $Fix(S)$.

*Corresponding author: u.witthayarat@hotmail.com.com (U. Witthayarat)

The *equilibrium problem* was first introduced and studied by Blum and Oettli[1] which is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C,$$

where F is a bifunction from $C \times C$ to $\mathbb{R}$. Its solution set is denoted by $EP(F)$.

There are many authors take the most interest to solve the equilibrium with their algorithm. In 2005, Combettes and Hirstoaga [2] introduced their algorithm to approximate the solution of equilibrium problem by using iterative method and proved the strong convergence theorem. Later, in 2007, Takahashi and Takahashi [3] also proposed the new iterative method called "*viscosity approximation method*" for finding a common solution of equilibrium problem together with fixed point problem. Moreover, based on the idea of Takahashi and Takahashi [3], PlubTieng and PunPaeng[4] improved and introduced the new scheme for solving the equilibrium problem. Recently, Liu et al.[5] extended the viscosity approximation method to find a common solution of the infinite family of fixed point problems together with equilibrium problem and other relevant problem.

The split feasibility problem is to find $x \in C$ such that $Ax \in Q$. We denote by $C \cap A^{-1}Q$ its solution set. The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [12] for modeling inverse problems which arise from phase retrievals, medical image reconstruction and recently in modeling of intensity modulated radiation therapy. The SFP attracts the attention of many authors due to its application in signal processing. In order to solve the split feasibility problem (SFP), Byrne [11] proposed the following iterative algorithm in the framework of Hilbert spaces: $x_1 \in C$ and

$$x_{n+1} = P_C(x_n - \lambda A^*(I - P_Q)Ax_n), \ n \geq 1, \tag{1.1}$$

which is often called the CQ algorithm, where $\lambda > 0$, P_C and P_Q are the metric projections on C and Q, respectively. It was shown that the sequence $\{x_n\}$ converges weakly to a solution of SFP provided $0 < r < 2/\|A\|^2$. Since then several iterations have been invented for solving the SFP (see, for example, [9, 16, 17, 22]).

Recently, Censor and Segal[6] proposed the iterative scheme to approximate a solution of *split common fixed point problem* which is a generalized of split feasibility problem and convex feasibility problem:

Let A be a real $m \times n$ matrix and let $U : \mathbb{R}^n \to \mathbb{R}^n$ and $T : \mathbb{R}^m \to \mathbb{R}^m$ be operators with nonempty $Fix(U) = C$ and $Fix(T) = Q$. The problem is to find $x^* \in C$ such that $Ax^* \in Q$.

Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, the *split equilibrium problem (SEP)* is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \ \forall x \in C,$$

and such that

$$y^* = Ax^* \in Q \ \text{ solves } \ F_2(y^*, y) \geq 0 \ \forall y \in Q.$$

We can see that the first part of SEP seems like the classical equilibrium problem EP where we can denote its solution set as $EP(F_1)$. The SEP looks like a pair of equilibrium problems which have to be solved so that the image $y^* = Ax^*$ under the given bounded operator A. We denote the solution set of the second EP in SEP as $EP(F_2)$. The solution set of SEP is denoted by $\Omega = \{p \in EP(F_1) : Ap \in EP(F_2)\}$.

In 2013, Kazmi and Rizvi[7] focus on how to approximate a common solution of split equilibrium problem, variational inequality problem and fixed point problem by stating the strong convergence theorem of their iterative algorithm as shown in the following:

$$\begin{aligned} u_n &= J_{r_n}^{F_1}(x_n + \gamma A^*(J_{r_n}^{F_2} - I)Ax_n); \\ y_n &= P_C(u_n - \lambda_n D u_n); \\ x_{n+1} &= \alpha_n \nu + \beta_n x_n + \gamma_n S y_n. \end{aligned} \tag{1.2}$$

They proved that $\{x_n\}$ generated by (1.2) converges strongly to the common solution of SEP, FPP and VI under some appropriate conditions of the sequences $\{\alpha_n\}, \{\gamma_n\}, \{\beta_n\}$ and $\{\lambda_n\}$.

However, it was observed that the step size γ depends on the computation of the operator norm A^*A which is not an easy task in practice. To overcome this difficulty, Lopez et al. [8] suggested a new way of stepsize τ_n as follows:

$$\tau_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \ \rho_n \in (0, 4).$$

In this work, motivated by the previous works, we introduce the modified iterative methods for solving the split equilibrium problem and the fixed point problem in Hilbert spaces and then prove its strong convergence of the sequence generated by our schemes without prior knowledge of the operator norm. Our main results complements the results of Kazmi and Rizvi[7] and other relevant work in the literature. Finally, we give some experiments to show the efficiency and the implementation of our purpose method.

2 Preliminaries and lemmas

In this section we recall some definitions and lemmas which will be needed in the next section. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we know from [18] that

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle; \tag{2.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore, for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

The nearest point projection of a nonempty, closed and convex set C is denoted by P_C, that is, $\|x - P_C x\| \le \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know the metric projection P_C is firmly nonexpansive, *i.e.*,

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle \tag{2.4}$$

for all $x, y \in H$. Moreover $\langle x - P_C x, y - P_C x\rangle \le 0$ holds for all $x \in H$ and $y \in C$; see [18].

Lemma 2.1. *[10] Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, that is, if the sequence x_n converges weakly to $x \in C$ and $\|x_n - Tx_n\| \to 0$, then $x = Tx$.*

Assumption 2.2. *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:*

(i) $F(x, x) = 0, \ \forall x \in C$;

(ii) *F is monotone, i.e.,* $F(x, y) + F(y, x) \leq 0, \ \forall x \in C$;

(iii) *For each* $x, y, z \in C, \limsup_{t\to 0} F(tz + (1-t)x, y) \leq F(x, y)$;

(iv) *For each $x \in C, y \to F(x, y)$ is convex and lower semicontinuous.*

(v) *Fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H_1 and $x \in C \cap K$ such that*
$$F(y, x) + \frac{1}{r}\langle y - x, x - z < 0\rangle, \ \forall y \in C \setminus K.$$

Lemma 2.3. *Let $T : H \to H$ be an operator. The following statements are equivalent.*

(i) *T is firmly nonexpansive.*

(ii) $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \ x, y \in H$.

(iii) *I-T is firmly nonexpansive.*

Lemma 2.4. *Assume that $F_1 : C \times C \to \mathbb{R}$ satisfying Assumption 2.2. For $r > 0$ and for all $x \in H_1$, define a mapping $J_r^{F_1} : H \to C$ as follows:*
$$J_r^{F_1} x = \{z \in C : F_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \ \forall y \in C\}.$$
Then the following hold:

(i) *$J_r^{F_1}$ is nonempty and single-valued;*

(ii) *$J_r^{F_1}$ is firmly nonexpansive, i.e.,*
$$\|J_r^{F_1}x - J_r^{F_1}y\| \leq \langle J_r^{F_1}x - J_r^{F_1}y, x - y\rangle, \ \forall x, y \in H_1;$$

(iii) $Fix(J_r^{F_1}) = EP(F_1)$;

(iv) *$EP(F_1)$ is closed and convex.*

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ satisfying Assumption 2.2. For $s > 0$ and for all $w \in H_2$, define a mapping $J_s^{F_2} : H_2 \to Q$ as follows:
$$J_s^{F_2}(w) = \{d \in Q : F_2(d, e) + \frac{1}{s}\langle e - d, d - w\rangle \geq 0, \ \forall e \in Q\}.$$
Then we easily observe that $J_s^{F_2}$ is nonempty, single-valued and firmly nonexpansive, $EP(F_2, Q)$ is closed and convex and $Fix(J_s^{F_2}) = EP(F_2, Q)$, where $EP(F_2, Q)$ is the solution set of the following equilibrium problem:
$$\text{Find } y^* \in Q \text{ such that } F_2(y^*, y) \geq 0, \ \ \forall y \in Q.$$

Lemma 2.5. *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.2 hold and let $J_r^{F_1}$ be defined as in Lemma 2.4 for $r > 0$. Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then:*

$$\|J_{r_2}^{F_1}y - J_{r_1}^{F_1}x\| \leq \|y - x\| + \frac{r_2 - r_1}{r_2}\|J_{r_2}^{F_1}y - y\|.$$

Lemma 2.6. *Let $\{s_n\}$ be a real sequence that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{n_k}\}$ so that*

$$s_{n_k} \leq s_{n_k+1}, \ \ \forall k \geq 0.$$

For every $n \geq n_0$ define an integer sequence $\{\Gamma(n)\}$ as

$$\Gamma(n) = \max\{n_0 \leq k \leq n : s_k < s_{k+1}\}.$$

Then $\Gamma(n) \to \infty$ as $n \to \infty$ and for all $n > n_0$

$$\max\{s_{\Gamma(n),s_n}\} \leq s_{\Gamma(n)+1}.$$

3 Strong convergence theorem

In this section, we show the strong convergence theorem of the our generated iterative scheme in the framework of the real Hilbert spaces.

Let H_1 and H_2 be two real Hilbert spaces with the nonempty closed and convex subsets C and Q, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption 1 and F_2 is upper semicontinuous in the first argument. Let $S : C \to C$ be a nonexpansive mapping such that $\Theta := Fix(S) \cap \Omega \neq \emptyset$. We further define

$$f(x_n) = \frac{1}{2}\|(I - J_{r_n}^{F_2})Ax_n\|^2, \ n \geq 0,$$

and

$$\nabla f(x_n) = A^*(I - J_{r_n}^{F_2})Ax_n.$$

Algorithm 1 Choose an arbitrary initial guess $x_0 \in C$, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{aligned} u_n &= J_{r_n}^{F_1}(x_n - \tau_n \nabla f(x_n)); \\ x_{n+1} &= \alpha_n g(x_n) + \beta_n x_n + \gamma_n S u_n; \end{aligned}$$

where g is a contraction on C, $\tau_n = \dfrac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}$, $\rho_n \in (0, 4)$.

Theorem 3.1. *Assume that $r_n \in (0, \infty)$ and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $\liminf_{n\to\infty} \beta_n > 0$*;*

(C3) $\liminf_{n\to\infty} \gamma_n > 0$*;*

(C4) $\liminf_{n\to\infty} \rho_n(4-\rho_n) > 0$;

(C5) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to $z = P_\Theta g(z)$.

Proof. Firstly, we would claim that the generated sequence $\{x_n\}$ is bounded. Put $w_n = x_n - \tau_n \nabla f(x_n)$. We note that $I - J_{r_n}^{F_1}$ is firmly nonexpansive and $\nabla f(z) = 0$.So, by Lemma 2.3 we have the following,

$$\begin{aligned}\langle \nabla f(x_n), x_n - z\rangle &= \langle (I - J_{r_n}^{F_1})Ax_n, Ax_n - Az\rangle\\ &\geq \|(I - J_{r_n}^{F_1})Ax_n\|^2\\ &= 2f(x_n).\end{aligned}$$

This implies that

$$\begin{aligned}\|x_n - \tau_n\nabla f(x_n) - z\|^2 &= \|x_n - z\|^2 + \|\tau_n \nabla f(x_n)\|^2 - 2\tau_n\langle \nabla f(x_n), x_n - z\rangle\\ &\leq \|x_n - z\|^2 + \tau_n^2\|\nabla f(x_n)\|^2 - 4\tau_n f(x_n)\\ &= \|x_n - z\|^2 - \rho_n(4-\rho_n)\frac{f^2(x_n)}{\|\nabla f(x_n)\|^2}.\end{aligned}$$

We thus obtain, since $J_{r_n}^{F_1}$ is firmly nonexpansive,

$$\begin{aligned}\|u_n - z\|^2 &= \|J_{r_n}^{F_1}w_n - z\|^2\\ &\leq \|w_n - z\|^2 - \|J_{r_n}^{F_1}w_n - w_n\|^2\\ &= \|x_n - \tau_n\nabla f(x_n) - z\|^2 - \|J_{r_n}^{F_1}w_n - w_n\|^2\\ &= \|x_n - z\|^2 - \rho_n(4-\rho_n)\frac{f^2(x_n)}{\|\nabla f(x_n)\|^2} - \|J_{r_n}^{F_1}w_n - w_n\|^2.\end{aligned}$$

We see that

$$\begin{aligned}\|x_{n+1} - z\| &\leq \alpha_n\|g(x_n) - z\| + \beta_n\|x_n - z\| + \gamma_n\|Su_n - z\|\\ &\leq \alpha_n\|g(x_n) - g(z)\| + \alpha_n\|g(z) - z\| + \beta_n\|x_n - z\| + \gamma_n\|u_n - z\|\\ &\leq \alpha_n\alpha\|x_n - z\| + \alpha_n\|g(z) - z\| + \beta_n\|x_n - z\| + \gamma_n\|x_n - z\|\\ &= (1 - \alpha_n(1-\alpha))\|x_n - z\| + \alpha_n\|g(z) - z\|\\ &\leq \max\{\|x_n - z\|, \|\frac{g(z) - z}{1-\alpha}\|\}.\end{aligned}$$

Hence, the sequence $\{x_n\}$ is bounded by induction.

Put $M = \sup_{n\in\mathbb{N}} \|g(x_n) - z\|\|x_{n+1} - z\|$. We next investigate the following

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \|\beta_n x_n + \gamma_n S u_n - z\|^2 + 2\alpha_n \langle g(x_n) - z, x_{n+1} - z\rangle \\
&= \beta_n(\beta_n + \gamma_n)\|x_n - z\|^2 + \gamma_n(\beta_n + \gamma_n)\|Su_n - z\|^2 \\
&\quad -\beta_n\gamma_n\|x_n - Su_n\|^2 + \alpha_n M \\
&\leq \beta_n(1 - \alpha_n)\|x_n - z\|^2 + \gamma_n(1 - \alpha_n)\|u_n - z\|^2 \\
&\quad -\beta_n\gamma_n\|x_n - Su_n\|^2 + \alpha_n M \\
&\leq (1 - \alpha_n)^2\|x_n - z\|^2 - \gamma_n(1 - \alpha_n)(\rho_n(4 - \rho_n)\frac{f^2(x_n)}{\|\nabla f(x_n)\|^2} + \|J_{r_n}^{F_1} w_n - w_n\|^2) \\
&\quad -\beta_n\gamma_n\|x_n - Su_n\|^2 + \alpha_n M \\
&\leq (1 - \alpha_n)\|x_n - z\|^2 - \gamma_n(1 - \alpha_n)(\rho_n(4 - \rho_n)\frac{f^2(x_n)}{\|\nabla f(x_n)\|^2} + \|J_{r_n}^{F_1} w_n - w_n\|^2) \\
&\quad -\beta_n\gamma_n\|x_n - Su_n\|^2 + \alpha_n M. \qquad (3.1)
\end{aligned}$$

For convenience, let $s_n = \|x_n - z\|^2$ and separate the behavior of $\{s_n\}$ into two different cases as follow.

Case I: $\{s_n\}$ is a decreasing sequence, that is, the limit of the sequence $\{s_n\}$ exists. Hence, it follows that

$$\begin{aligned}
s_{n+1} &\leq (1 - \alpha_n)s_n - \gamma_n(1 - \alpha_n)(\rho_n(4 - \rho_n)\frac{f^2(x_n)}{\|\nabla f(x_n)\|^2} + \|J_{r_n}^{F_1} w_n - w_n\|^2) \\
&\quad -\beta_n\gamma_n\|x_n - Su_n\|^2 + \alpha_n M. \qquad (3.2)
\end{aligned}$$

It follows that

$$\gamma_n(1 - \alpha_n)\rho_n(4 - \rho_n)\frac{f^2(x_n)}{\|\nabla f(x_n)\|^2} \leq (s_n - s_{n+1}) + \alpha_n M.$$

Hence $\dfrac{f(x_n)}{\|\nabla f(x_n)\|} \to 0$ by conditions (C1), (C3) and (C4). It follows that $f(x_n) = \frac{1}{2}\|(I - J_{r_n}^{F_2})Ax_n\|^2 \to 0$, since $\{x_n\}$ is bounded. Similarly, we can show that $\|J_{r_n}^{F_1} w_n - w_n\| \to 0$ and $\|x_n - Su_n\| \to 0$.

Then,

$$\begin{aligned}
\|w_n - x_n\| &= \|x_n - \tau_n\nabla f(x_n) - x_n\| \\
&= \rho_n\frac{f(x_n)}{\|\nabla f(x_n)\|} \\
&\to 0.
\end{aligned}$$

So,

$$\begin{aligned}
\|u_n - x_n\| &\leq \|u_n - w_n\| + \|w_n - x_n\| \\
&= \|J_{r_n}^{F_1} w_n - w_n\| + \|w_n - x_n\| \\
&\to 0.
\end{aligned}$$

It follows that $\|u_n - w_n\| \to 0$ as $n \to \infty$. Also, we see that

$$\begin{aligned}\|x_n - Sx_n\| &\leq \|x_n - Su_n\| + \|Su_n - Sx_n\| \\ &\leq \|x_n - Su_n\| + \|u_n - x_n\| \\ &\to 0.\end{aligned}$$

Next, we show that $\limsup_{n\to\infty}\langle g(z)-z, x_{n+1}-z\rangle \leq 0$, where $z = P_\Theta g(z)$. To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle g(z) - z, x_n - z\rangle = \lim_{i\to\infty}\langle g(z) - z, x_{n_i} - z\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some $w \in C$. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$.

Now, we prove that $w \in Fix(S) \cap \Omega$. Let us show that $w \in Fix(S)$. Assume that $w \notin Fix(S)$. Since $x_{n_i} \rightharpoonup w$ and $Sw \neq w$. From Opial's condition, we have

$$\begin{aligned}\liminf_{i\to\infty}\|x_{n_i} - w\| &< \liminf_{i\to\infty}\|x_{n_i} - Sw\| \\ &\leq \liminf_{i\to\infty}(\|x_{n_i} - Sx_{n_i}\| + \|Sx_{n_i} - Sw\|) \\ &\leq \liminf_{i\to\infty}\|x_{n_i} - w\|,\end{aligned}$$

which is a contradiction. Thus, we obtain $w \in Fix(S)$.

Next, we show that $w \in EP(F_1)$. Since $u_n = J_{r_n}^{F_1} w_n$, we have

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from the monotonicity of F_1 that

$$\frac{1}{r_{n_i}}\langle y - u_n, u_n - w_n\rangle \geq F_1(y, u_n)$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - w_{n_i}}{r_{n_i}}\rangle \geq F_1(y, u_{n_i}).$$

Since $\|u_n - w_n\| \to 0$ and $\liminf r_n > 0$, we get $\dfrac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$. It follows by Assumption 2.1(iv) that $0 \geq F_1(y, w)$, $\forall w \in C$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C, w \in C$, we get $y_t \in C$ and hence $F_1(y_t, w) \leq 0$. So from Assumption 2.1(i) and (iv) we have

$$0 = F_1(y_t, y_t) \leq tF_1(y_t, y) + (1-t)F_1(y_t, w) \leq tF_1(y_t, y).$$

Therefore $0 \leq F_1(y_t, y)$. From Assumption 2.1(iii), we have $0 \leq F_1(w, y)$. This implies that $w \in EP(F_1)$.

Next, we show that $Aw \in EP(F_2)$. Since $x_{n_i} \rightharpoonup w$ and A is a bounded linear operator, we obtain $Ax_{n_i} \rightharpoonup Aw$.

Now set $\nu_{n_i} = Ax_{n_i} - J^{F_2}_{r_{n_i}} Ax_{n_i}$. It follows that $\lim_{i\to\infty} \nu_{n_i} = 0$ and $Ax_{n_i} - \nu_{n_i} = J^{F_2}_{r_{n_i}} Ax_{n_i}$. Therefore from Lemma 2.4, we have

$$F_2(Ax_{n_i} - \nu_{n_i}, z) + \frac{1}{r_{n_i}}\langle z - (Ax_{n_i} - \nu_{n_i}), (Ax_{n_i} - \nu_{n_i}) - Ax_{n_i}\rangle \geq 0, \ \forall z \in Q.$$

Since F_2 is upper semicontinuous in the first argument, taking $\limsup$ to above inequality as $i \to \infty$ and using condition (iv), we obtain

$$F_2(Aw, z) \geq 0, \ \forall z \in Q,$$

which means that $Aw \in EP(F_2)$ and hence $w \in \Omega$.

Now from (2.2), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle g(z) - z, x_n - z\rangle &= \limsup_{i\to\infty}\langle g(z) - z, x_{n_i} - z\rangle \\
&= \langle g(z) - z, w - z\rangle \\
&\leq 0.
\end{aligned}$$

Thus,

$$\limsup_{n\to\infty}\langle g(z) - z, x_{n+1} - z\rangle \leq 0. \tag{3.3}$$

We see that,

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \alpha_n\langle g(x_n - z, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle \\
&\quad +\gamma_n\langle Su_n - z, x_{n+1} - z\rangle \\
&= \alpha_n\langle g(x_n) - g(z), x_{n+1} - z\rangle + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\
&\quad +\beta_n\langle x_n - z, x_{n+1} - z\rangle + \gamma_n\langle Su_n - z, x_{n+1} - z\rangle \\
&\leq \alpha_n\alpha\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\
&\quad +\beta_n\|x_n - z\|\|x_{n+1} - z\| + \gamma_n\|Su_n - z\|\|x_{n+1} - z\| \\
&\leq \alpha_n\alpha\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\
&\quad +\beta_n\|x_n - z\|\|x_{n+1} - z\| + \gamma_n\|x_n - z\|\|x_{n+1} - z\| \\
&= (1 - \alpha_n(1-\alpha))(\|x_n - z\|\|x_{n+1} - z\|) + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\
&\leq (1 - \alpha_n(1-\alpha))(\frac{\|x_n - z\|^2 + \|x_{n+1} - z\|^2}{2}) + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle.
\end{aligned}$$

So,

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \frac{(1 - \alpha_n(1-\alpha))}{1 + \alpha_n(1-\alpha)}\|x_n - z\|^2 + 2\alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\
&= \Big(1 - \frac{2\alpha_n(1-\alpha)}{1 + \alpha_n(1-\alpha)}\Big)\|x_n - z\|^2 + 2\alpha_n\langle g(z) - z, x_{n+1} - z\rangle.
\end{aligned} \tag{3.4}$$

By using condition (C1), and (3.3) we can conclude that $x_n \to z$.

Case II: $\{s_n\}$ is not a decreasing sequence. Hence we can find a subsequence $\{s_{n_k}\}$ so that

$s_{n_k} \leq s_{n_{k+1}}$. In this case, we define an integer sequence $\{\Gamma(n)\}$ as in Lemma 2.6. Since $s_{\Gamma(n)} \leq s_{\Gamma(n)+1}$, $\forall n \geq n_0$. It follows that by (3.2),

$$\begin{aligned} s_{\Gamma(n)+1} \leq\ & (1-\alpha_{\Gamma(n)})s_{\Gamma(n)} - \gamma_{\Gamma(n)}(1-\alpha_{\Gamma(n)})(\rho_{\Gamma(n)}(4-\rho_{\Gamma(n)})\frac{f^2(x_{\Gamma(n)})}{\|\nabla f(x_{\Gamma(n)})\|^2} - \|J^{F_1}_{r_{\Gamma(n)}}w_{\Gamma(n)} - w_{\Gamma(n)}\|^2) \\ & -\beta_{\Gamma(n)}\gamma_{\Gamma(n)}\|x_{\Gamma(n)} - Su_{\Gamma(n)}\|^2 + \alpha_{\Gamma(n)}M. \end{aligned}$$

Hence, $\dfrac{f(x_{\Gamma(n)})}{\|\nabla f(x_{\Gamma(n)})\|} \to 0$, $\|J^{F_1}_{r_{\Gamma(n)}}w_{\Gamma(n)} - w_{\Gamma(n)}\| \to 0$ and $\|x_{\Gamma(n)} - Su_{\Gamma(n)}\| \to 0$. Similar to Case I, we can show that $\limsup_{n\to\infty}\langle g(z)-z, x_{\Gamma(n)} - z\rangle \leq 0$. It is easy to see that $\|x_{\Gamma(n)} - x_{\Gamma(n)+1}\| \to 0$. Hence, $\limsup_{n\to\infty}\langle g(z)-z, x_{\Gamma(n)+1} - z\rangle \leq 0$. From (3.4), we have

$$s_{\Gamma(n)+1} \leq (1-\delta_{\Gamma(n)})s_{\Gamma(n)} + 2\alpha_{\Gamma(n)}\langle g(z)-z, x_{\Gamma(n)+1}-z\rangle$$

where $\delta_{\Gamma(n)} = \dfrac{2\alpha_{\Gamma(n)}(1-\alpha)}{1+\alpha_{\Gamma(n)}(1-\alpha)}$. So, $\delta_{\Gamma(n)}s_{\Gamma(n)} \leq 2\alpha_{\Gamma(n)}\langle g(z)-z, x_{\Gamma(n)+1}-z\rangle$, yields

$$s_{\Gamma(n)} \leq \frac{1+\alpha_{\Gamma(n)}(1-\alpha)}{1-\alpha}\langle g(z)-z, x_{\Gamma(n)+1}-z\rangle.$$

Hence, $\limsup_{n\to\infty} s_{\Gamma(n)} \leq 0$. By Lemma (2.6), we have $s_n \leq s_{\Gamma(n)}$. Then $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\|x_n - z\|^2 = 0$. So, $x_n \to z$. This completes the proof. □

4 Applications

We next give some applications to split feasibility problem and the split convex minimization problem.

4.1 Application to split feasibility problem

For obtaining the result for the split feasibility problem, let the solution set $\Theta := Fix(S) \cap \Gamma \neq \emptyset$, and define

$$f(x_n) = \frac{1}{2}\|(I-P_Q)Ax_n\|^2,\ n \geq 0,$$

and

$$\nabla f(x_n) = A^*(I-P_Q)Ax_n.$$

Algorithm 2 Choose an arbitrary initial guess $x_0 \in C$, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{aligned} u_n &= P_C(x_n - \tau_n\nabla f(x_n)); \\ x_{n+1} &= \alpha_n g(x_n) + \beta_n x_n + \gamma_n Su_n; \end{aligned}$$

where g is a contraction on C, $\tau_n = \dfrac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}$, $\rho_n \in (0,4)$.

Theorem 4.1. *Assume that* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are the sequences in* $(0,1)$ *with* $\alpha_n+\beta_n+\gamma_n=1$ *satisfying the same conditions (C1)-(C4) in Theorem 3.1. Then the sequence* $\{x_n\}$ *generated by Algorithm 2 converges strongly to* $z = P_\Theta g(z)$.

Example 4.2. *Let* $H_1 = H_2 = \mathbb{R}^3$. *Define* $C = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq 1\}$ *and*

$$Q = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 + x_2 + 4x_3 \geq 1\}.$$

Let

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & -2 & 0 \end{pmatrix}.$$

Let $S : C \to C$ *be defined by* $Sx = (-x_1, x_2, -x_3)$ *and* $g : C \to C$ *by* $g(x) = \frac{x}{2}$ *where* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Choose $\alpha_n = \frac{1}{n+1}$, $\beta_n = 0.5$, $\gamma_n = 0.5 - \frac{1}{n+1}$ and $E_n = \|x_{n+1} - x_n\|_2 < 10^{-4}$ for all $n \in \mathbb{N}$.

We now study the effect (in terms of convergence, stability, number of iterations required and the cpu time) of the sequence $\{\rho_n\} \subset (0,4)$ on the iterative scheme by choosing different ρ_n such that $\inf_n \rho_n(4-\rho_n) > 0$. We choose different choices of x_1 as
Choice 1: $x_1 = (0,0,1)$;
Choice 2: $x_1 = (0.5, 0.5, 0.5)$;
Choice 3: $x_1 = (0.2, 0.6, 0.1)$;
Choice 4: $x_1 = (0.8, 0.6, 0)$.

The numerical experiments, using our Algorithm 2 in Theorem 4.1, for each choice are reported in the following Table 1.

Table 1: Algorithm 3.1 with different cases of ρ_n and different choices of x_1

		$\rho_n = \frac{n}{n+1}$	$\rho_n = \frac{1.5n}{n+1}$	$\rho_n = \frac{2.5n}{n+1}$	$\rho_n = \frac{3.5n}{n+1}$
Choice 1	No. of Iter.	97	74	52	29
	cpu (Time)	0.018883	0.015916	0.011313	0.005729
Choice 2	No. of Iter.	97	74	52	41
	cpu (Time)	0.026899	0.019866	0.016848	0.010174
Choice 3	No. of Iter.	97	74	52	41
	cpu (Time)	0.026644	0.015758	0.011374	0.017603
Choice 4	No. of Iter.	97	74	52	30
	cpu (Time)	0.023669	0.016431	0.010965	0.007143

The convergence behavior of the error E_n for each choice of x_1 is shown in Figure 1-4, respectively.

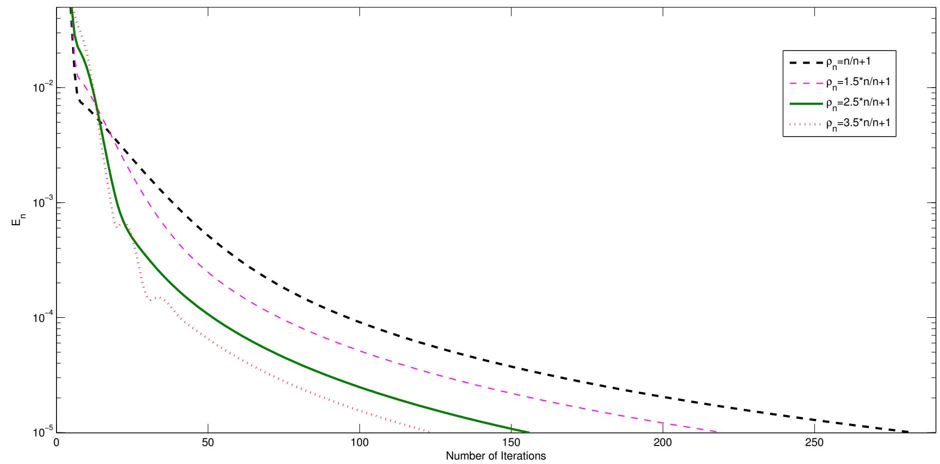

Figure 1: Error plotting E_n for Choice 1 in Example 4.4

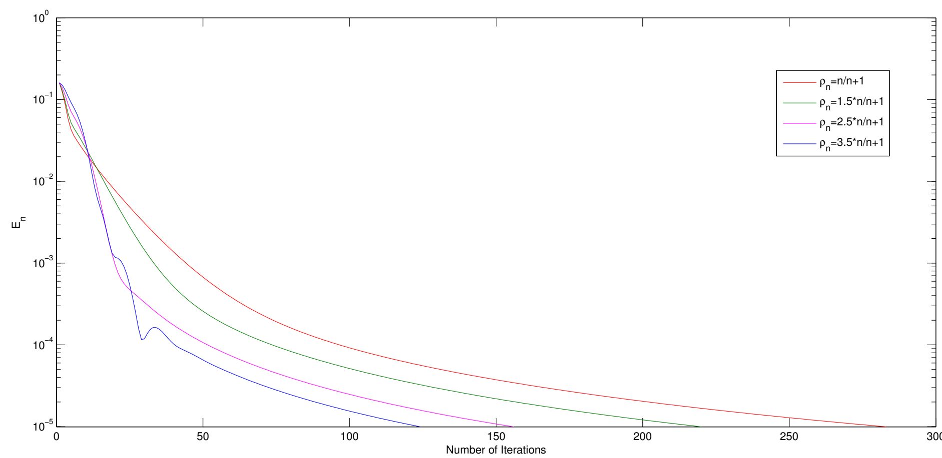

Figure 2: Error plotting E_n for Choice 2 in Example 4.4

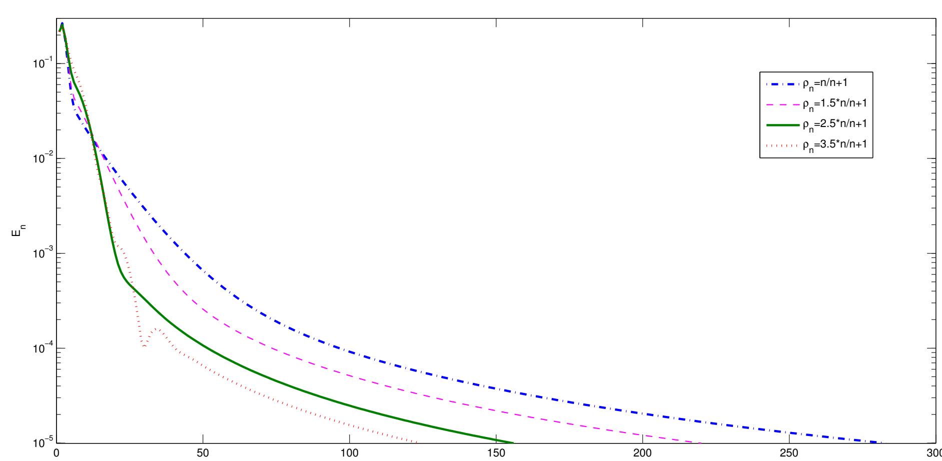

Figure 3: Error plotting E_n for Choice 3 in Example 4.4

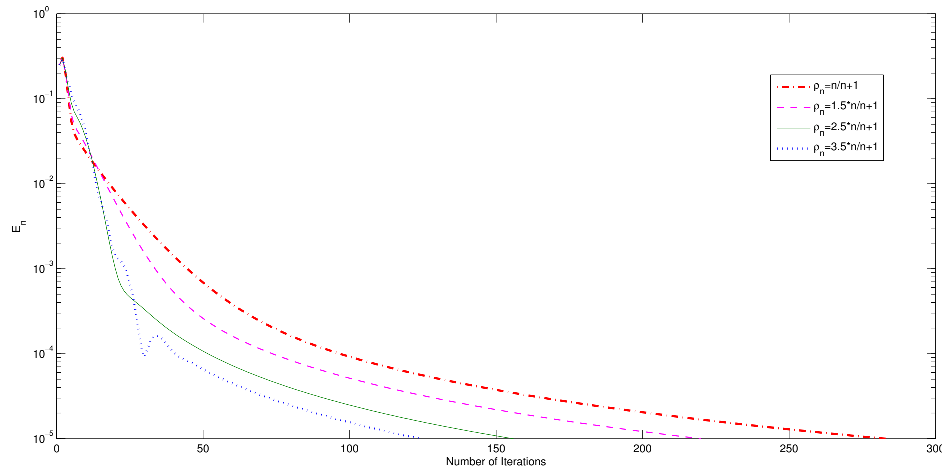

Figure 4: Error plotting E_n for Choice 4 in Example 4.4

4.2 Application to split convex minimization problem

In this section, we consider the following split convex minimization problem as follows:

The proximity operator of F is defined by

$$prox_{\lambda F}(x) = \arg\min_{y\in H}\{F(y) + \frac{1}{2\lambda}\|x-y\|^2\} \tag{4.1}$$

for any $\lambda > 0$. It is seen that

$$0 \in \partial F(x^*) \Leftrightarrow x^* = prox_{\lambda F}(x^*). \tag{4.2}$$

Let $f_1, f_2 : C \to \mathbb{R} \cup \{\infty\}$ be convex and lower semicontinuous. The split convex minimization problem is to find a minimizer x^* of f_1 that Ax^* is a minimizer of f_2, where A is a bounded linear operator.

To this end, we define

$$f(x_n) = \frac{1}{2}\|(I - prox_{\lambda f_2})Ax_n\|^2,\ n \geq 0,$$

and

$$\nabla f(x_n) = A^*(I - prox_{\lambda f_2})Ax_n.$$

Algorithm 3 Choose an arbitrary initial guess $x_0 \in C$, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{aligned} u_n &= prox_{\lambda f_1}(x_n - \tau_n \nabla f(x_n)); \\ x_{n+1} &= \alpha_n g(x_n) + \beta_n x_n + \gamma_n S u_n; \end{aligned}$$

where g is a contraction on C, $\tau_n = \dfrac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}$, $\rho_n \in (0, 4)$.

Theorem 4.3. *Assume that $\lambda > 0$ and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ satisfying the same conditions (C1)-(C4) in Theorem 3.1. Then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $z = P_\Theta g(z)$.*

Example 4.4. *Let $H_1 = H_2 = \mathbb{R}^3$. Let $f_1 : \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ be defined by*

$$f_1(x) = \|x\|_2^2 + (2, 4, -5)x + 10$$

and let $f_2 : \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$f_2(x) = \|x\|_2^2 - (8, 10, -8)x - 5.$$

Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \\ 2 & 1 & 0 \end{pmatrix}.$$

Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $Sx = (-2 - x_1, -4 - x_2, 0.5x_3 + 1.25)$ and $g : \mathbb{R}^3 \to \mathbb{R}^3$ by $g(x) = \frac{x}{2}$ where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Find $x \in \mathbb{R}^3$ such that x minimizes f_1 and Ax minimize f_2 and x is also a fixed point of S. Choose $\alpha_n = \frac{1}{n+1}$, $\beta_n = 0.1$, $\gamma_n = 0.9 - \frac{1}{n+1}$, $\lambda = 1$ and $E_n = \|x_{n+1} - x_n\|_2 < 10^{-4}$ for all $n \in \mathbb{N}$.

The numerical experiments, using our Algorithm 3 in Theorem 4.3, for each choice are reported in the following Table 2. We choose different choices of x_1 as
Choice 1: $x_1 = (0, 0, 1)$; Choice 2: $x_1 = (0.5, 0.5, 0.5)$;
Choice 3: $x_1 = (0.2, 0.6, 0.1)$; Choice 4: $x_1 = (0.8, 0.6, 0)$.

The numerical experiments, using our Algorithm 3 in Theorem 4.3, for each choice are reported in the following Table 2.

Table 2: Algorithm 3.1 with different cases of ρ_n and different choices of x_1

		$\rho_n = \frac{n}{n+1}$	$\rho_n = \frac{1.5n}{n+1}$	$\rho_n = \frac{2.5n}{n+1}$	$\rho_n = \frac{3.5n}{n+1}$
Choice 1	No. of Iter.	97	74	52	29
	cpu (Time)	0.018883	0.015916	0.011313	0.005729
Choice 2	No. of Iter.	97	74	52	41
	cpu (Time)	0.026899	0.019866	0.016848	0.010174
Choice 3	No. of Iter.	97	74	52	41
	cpu (Time)	0.026644	0.015758	0.011374	0.017603
Choice 4	No. of Iter.	97	74	52	30
	cpu (Time)	0.023669	0.016431	0.010965	0.007143

The convergence behavior of the error E_n for each choice of x_1 is shown in Figure 1-4, respectively.

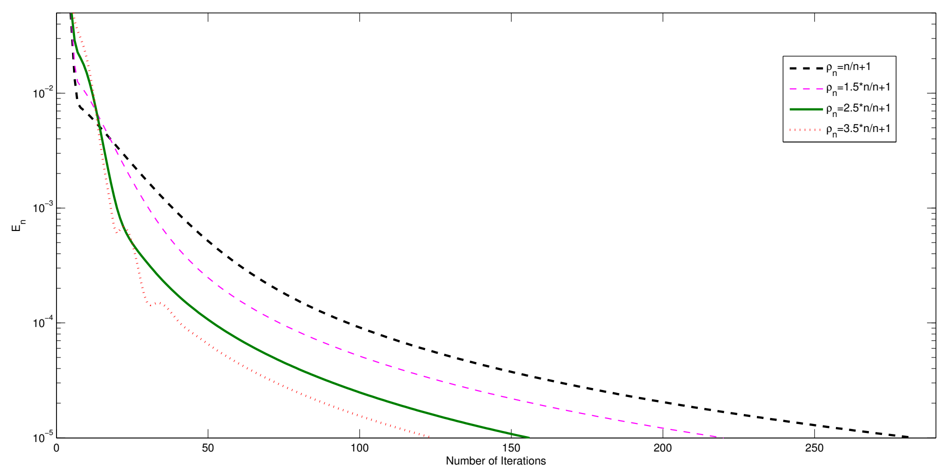

Figure 1: Error plotting E_n for Choice 1 in Example 4.4

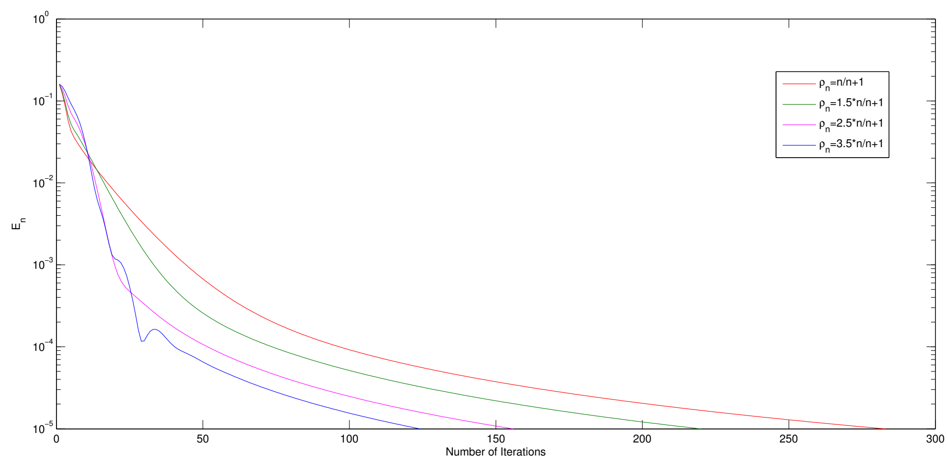

Figure 2: Error plotting E_n for Choice 2 in Example 4.4

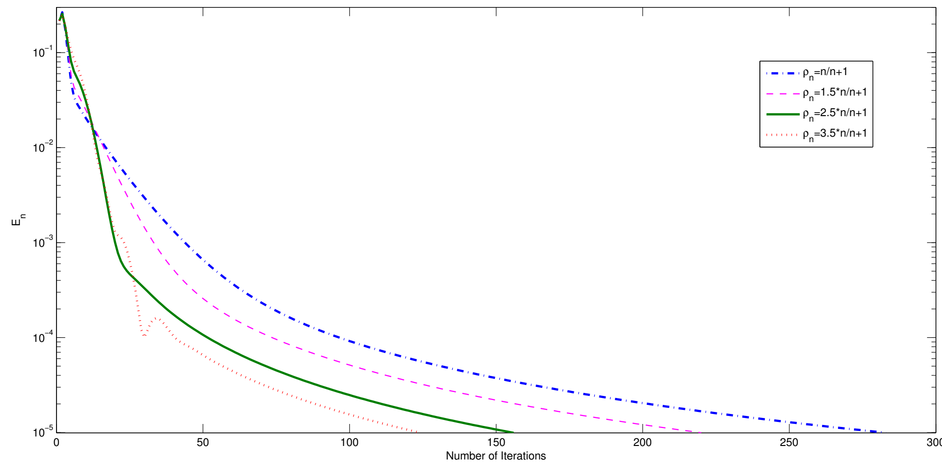

Figure 3: Error plotting E_n for Choice 3 in Example 4.4

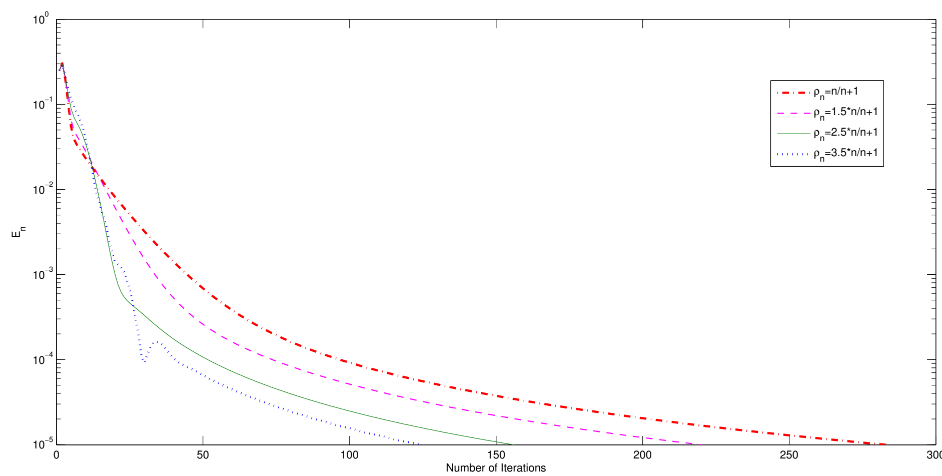

Figure 4: Error plotting E_n for Choice 4 in Example 4.4

Remark 4.5. *From our numerical experiments, it is observed that the different choices of x_1 have no effect in terms of CPU runtime for the convergence of our algorithm. However, if the stepsizes $\{\rho_n\}$ is taken close to 4, then the number of iterations and the CPU runtime have small reduction.*

5 Acknowledgment

The authors would like to express their special thanks to the Thailand's research fund under the scholarship no. MRG590140.

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การเสนอผลงานในที่ประชุมวิชาการนานาชาติ

2.1 ชื่อการจัดการประชุม The 10th International Conference on Nonlinear Analysis and Convex Analysis

สถานที่จัดประชุม Chitose City Cultural Center, Hokkaido JAPAN, July 4-9, 2017.

ชื่อเรื่องที่นำเสนอ A modified self-adaptive method for the split feasibility problem and the fixed point problem in Banach spaces

The International Research Working Group in
Nonlinear Analysis ans Convex Analysis

NACA2017 Organizers
W. Takahashi, S. Akashi, T. Tanaka

May 16, 2017

Dr. Uamporn Witthayarat
University of Phayao
Thailand
E-mail: `u.witthayarat@hotmail.com`

Official invitation to the 10^{th} International Conference on Nonlinear Analysis and Convex Analysis (NACA2017) held at Chitose City Cultural Center in Hokkaido, Japan and in the period July 4–9, 2017.

Organizers: Prof. Wataru Takahashi, Prof. Shigeo Akashi, Prof. Tamaki Tanaka

---

Dear Dr. Uamporn Witthayarat

On behalf of the conference organizing members of NACA2017, we would like to invite you to "**the 10^{th} International Conference on Nonlinear Analysis and Convex Analysis** (NACA2017)" for the purpose to get together world-wide experts on nonlinear analysis and convex analysis. We would like to invite you to Japan for the period of **from July 4 to 9**. Your paper

Registration No.:	**NACA2017reg202**
Title:	**A modified self-adaptive method for the split feasibility problem and the fixed point problem in Banach spaces**
Talked by:	**Suthep Suantai, Uamporn Uamporn Witthayarat Witthayarat, Yekini Shehu and Prasit Cholamjiak**

has been accepted for an oral presentation in the conference.

For more information,

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We wish you will enjoy several topics in this conference and also get new experience in Hokkaido, Japan. We are looking forward to meeting/seeing you soon.

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การเสนอผลงานในที่ประชุมวิชาการนานาชาติ

2.1 ชื่อการจัดการประชุม	The 6th Asian Conference on Nonlinear Analysis and Optimiztion
สถานที่จัดประชุม	OIST&ANA Intercontinental Menza Beach Resort, November 5-9 2018
ชื่อเรื่องที่นำเสนอ	On solving split equilibrium problem in real Hilbert spaces with its applications

The 6th Asian Conference on

Nao-Asia 2018

Okinawa Japan

Abstracts

The 6th Asian Conference on

Nonlinear Analysis and Optimization

5-9 November 2018 at OIST & ANA InterContinental Manza Beach Resort

November 7 (Wednesday)

(OIST)

TIME	Auditorium AU			
9:00 10:05	Plenary (WEO1AU) Cenference Photo (2)			
TIME	Meeting Room 1	Meeting Room 2	Meeting Room 3	Meeting Room 4
10:05	Special Session (WEO2M1)	Regular (WEO2M2)	Regular (WEO2M3)	Special Session† (WEO2M4)
	Seminar Room C209	Seminar Room C210	Seminar Room B250	Auditorium
11:20	Regular (WEO2S1)	Regular (WEO2S2)	Regular (WEO2S3)	
11:20 11:45	Coffee Break			
TIME	Seminar Room C209	Seminar Room C210	Seminar Room B250	Auditorium
11:45 12:55	Keynote (WEO3S1)	Keynote (WEO3S2)	Keynote (WEO3S3)	
12:55 14:30	Lunch (Grano@OIST)			
TIME	Seminar Room C209	Seminar Room C210	Seminar Room B250	Auditorium
14:30 16:15	Keynote (WEO4S1)	Keynote (WEO4S2)	Keynote (WEO4S3)	
16:15 16:35	Coffee Break			
TIME	Seminar Room C209	Seminar Room C210	Seminar Room B250	Auditorium
16:35	Regular (WEO5S1)	Regular (WEO5S2)	Regular (WEO5S3)	
	Meeting Room 1	Meeting Room 2	Meeting Room 3	Meeting Room 4
17:50	Regular (WEO5M1)	Regular (WEO5M2)	Regular (WEO5M3)	

November 7 (Wednesday), Morning

[WEO1AU] Plenary (chair: R. T. Rockafellar).

(1) **Wataru Takahashi** (p.112)
New classes of nonlinear operators and weak and strong convergence theorems in Hilbert spaces and Banach spaces

[WEO2M1] Special Session (chair: Nobusumi Sagara).

Special Session on Mathematical Economics

(1) **Yuhki Hosoya** (p.27)
Shephard's lemma and nonlinear partial differential equations

(2) **Naoki Yoshihara* and Se Ho Kwak** (p.131)
Sraffian indeterminacy in general equilibrium

(3) **Mitsunori Noguchi** (p.81)
Essential stability of purifiable alpha-core strategies of games with incomplete information

[WEO2M2] Regular Talks (chair: Aoi Honda).

(1) **Kazuhiro Hishinuma* and Hideaki Iiduka** (p.23)
Convergence property, computational performance, and usability of fixed point quasiconvex subgradient method

(2) **Aliyu Muhammed Awwal* and Poom Kumam** (p.4)
A projection Hestenes-Stiefel-like method for monotone nonlinear equations with convex constraints

(3) **Julalak Prabseang* and Kamsing Nonlaopon** (p.88)
Quantum Hermite-Hadamard inequalities for double integral and q-differentiable convex functions

[WEO2M3] Regular Talks (chair: Masahiro Inuiguchi).

(1) **Kosuke Togashi and Hiroaki Mohri*◇** (p.76)
Analysis for mechanism of commitment problems of separatist conflicts by 2 level game theory and coalition cooperation degree

(2) **Hiroaki Mohri and Jun-ichi Takeshita*** (p.113)
Erlang distribution damage analysis on failures immediately after shocks by two factors

(3) **Yi Chou Chen** (p.10)
A monopolist's optimal production rate

[WEO2M4] Special Session (2) (chair: Shuyu Sun).

Diffuse Interface Methods for Modeling Multi-Phase Mixture

(1) **Shuyu Sun*, Jisheng Kou and Zhonghua Qiao** (p.109)
Multi-scale simulation of two-phase flow with partial miscibility
(2) **Yuanqing Wu*, Jisheng Kou and Maoqing Ye** (p.126)
A Darcy-Brinkman-Forchheimer framework meeting Newton's second law in matrix acidization simulation
(3) **Tao Zhang* and Shuyu Sun** (p.135)
Lattice Boltzmann method for phase field model with Peng- Robinson equation of state

[WEO2S1] Regular Talks (chair: Narin Petrot).

(1) **Yirmeyahu Jeremy Kaminski** (p.37)
Equilibrium locus of the flow on circular networks of cells
(2) **Seiichi Iwamoto, Yutaka Kimura*$^{\diamondsuit}$ and Toshiharu Fujita** (p.51)
Two dualities – complementary versus shift –
(3) **Mitsuhiro Hoshino$^{\diamondsuit}$** (p.26)
Local behavior of monotonization and an index of ordering in learning processes of basic self-organizing maps

[WEO2S2] Regular Talks (chair: Daishi Kuroiwa).

(1) **C. Castaing, C. Godet-Thobie*, P. Dinh Phung and L. Xuan Truong** (p.21)
On fractional differential inclusions with nonlocal boundary conditions
(2) **Panatda Boonman* and Rabian Wangkeeree** (p.6)
Levitin-Polyak well-posedness for parametric quasivariational inclusion and disclusion problems
(3) **Mohammed Harunor Rashid** (p.91)
Convergence analysis of a restricted inexact Newton-type method for generalized equations

[WEO2S3] Regular Talks (chair: Yasunori Kimura).

(1) **Prasit Cholamjiak** (p.13)
The modified forward-backward method with linesearches
(2) **Mayumi Hojo* and Wataru Takahashi** (p.24)
Fixed point and weak convergence theorems for noncommutative two extended generalized hybrid mappings in Banach spaces
(3) **Uamporn Witthayarat* Poom Kumam and Prasit Cholamjiak** (p.124)
On solving split equilibrium problem in real Hilbert spaces with its applications

[WEO3S1] Keynote (chair: Tomonari Suzuki).

(1) **Ryszard Płuciennik** (p.87)
On some modifications of n-th von Neumann-Jordan constant in Banach spaces
(2) **Kichi-Suke Saito*, Naoto Komuro and Ryotaro Tanaka** (p.96)
On the symmetry of Banach soaces

ON SOLVING SPLIT EQUILIBRIUM PROBLEM IN REAL HILBERT SPACES WITH ITS APPLICATIONS

UAMPORN WITTHAYARAT*, POOM KUMAM, AND PRASIT CHOLAMJIAK

(Uamporn Witthayarat)
DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE, PHAYAO 56000, THAILAND
Email: u.witthayarat@hotmail.com

(Poom Kumam)
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, BANGKOK 10140, THAILAND
Email: poom.kum@kmutt.ac.th

(Prasit Cholamjiak)
DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE, PHAYAO 56000, THAILAND
Email: prasitch2008@yahoo.com

In this talk, we propose a new iterative scheme for finding a common solution of split equilibrium problem and fixed point problem in real Hilbert spaces. We prove the strong convergence theorem under the suitable conditions and especially with our the assumption of the norm of operator. Moreover, we discuss about its applications together with some numerical examples with support the main theorem in addition. Our results improve and extend the result optained by Kazmi and Rizvi and many previous ones in the literatures.

2010 *Mathematics Subject Classification.* Primary 47H05; Secondary 41A65.

Key words and phrases. Fixed point problem, Hilbert space, iterative method, split equilibrium problem, strong convergence.

*Presenting author.

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