รายงานวิจัยฉบับสมบูรณ์

วิธีการปกติแบบปรับปรุงสำหรับแก้ปัญหาความเป็นไปได้แบบแยกส่วนเชิงจุด

The modified regularization method for solving proximal split feasibility problems

โดย รองศาสตราจารย์ ดร.ประสิทธิ์ ช่อลำเจียก

เมษายน 2561

สัญญาเลขที่ MRG5980248

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและสำนักงานคณะกรรมการการอุดมศึกษา

กิตติกรรมประกาศ

งานวิจัยเรื่อง วิธีการปกติแบบปรับปรุงสำหรับแก้ปัญหาความเป็นไปได้แบบแยกส่วนเชิงจุด (MRG5980248) นี้ ประสบความสำเร็จลุล่วงได้ด้วยดีจากการได้รับทุนอุดหนุนการวิจัยจาก สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) สำนักงานคณะกรรมการการอุดมศึกษา (สกอ.) และมหาวิทยาลัยพะเยา ประจำปี 2559 - 2561 ผู้วิจัยขอขอบพระคุณ ศาสตราจารย์ ดร.สุเทพ สวนใต้ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ นักวิจัยที่ปรึกษา สำหรับการให้คำแนะนำและข้อเสนอแนะในการทำวิจัยด้วยดีตลอดมา

บทคัดย่อ

รหัสโครงการ: MRG5980248

ชื่อโครงการ: วิธีการปกติแบบปรับปรุงสำหรับแก้ปัญหาความเป็นไปได้แบบแยกส่วนเชิงจุด

ชื่อนักวิจัย และสถาบัน: รองศาสตราจารย์ ดร.ประสิทธิ์ ช่อลำเจียก มหาวิทยาลัยพะเยา

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บทคัดย่อ: จุดประสงค์ของงานวิจัยนี้ คือ การสร้างวิธีการทำซ้ำแบบใหม่ เพื่อใช้แก้ปัญหาความเป็นไปได้แบบแยกส่วนเชิงจุดและปัญหาที่เกี่ยวข้อง การศึกษาทฤษฎีบทการลู่เข้าภายใต้เงื่อนไขที่เหมาะสม และการให้ผลลัพธ์เชิงตัวเลขเพื่อสนับสนุนงานวิจัย

คำสำคัญ: วิธีการแบบปกติ / วิธีการแบบปกติ / ปัญหาความเป็นไปได้แบบแยกส่วนเชิงจุด / ปัญหาค่าต่ำสุด / ทฤษฎีบทการลู่เข้า/ ระเบียบวิธีทำซ้ำ

Abstract

Project Code: MRG5980248

Project Title: The modified regularization method for solving proximal split feasibility problems

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Project Period: June 1, 2016 – May 31, 2018

Abstract: The purposes of this research are to introduce new kinds of iterative methods for solving the split proximal feasibility problems as well as the related problems, to investigate the convergence theorems under suitable conditions and to provide some numerical examples to support our main results.

Keywords: Regularization Method/ Proximal Splitting Feasibility Problem/ Minimization Problem/ Convergence Theorem/ Iterative Method

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CHAPTER I

INTRODUCTION

In optimization theory, a major problem is the proximal split feasibility problem (PSFP). Numerous problems in applied science, economics, engineering and other related fields can be reformulated as this problem. To be more precise, the proximal split feasibility problem includes, as special cases, the convex minimization problem, the min-max problem, the complementarity problem, the linear inverse problem, the fixed point problem of some nonlinear operators, the illposed problem and the variational inequality problem. The regularization technique is a powerful tool in handling for solving such problem in some certain spaces. In the literature, Censor-Elfving introduced a notion of the split feasibility problem (SFP), which is to find an element of a closed convex subset of the Euclidean space whose image under a linear operator is an element of 9 another closed convex subset of a Euclidean space. Byrne subsequently proposed the CQ-method for solving this problem and established the weak convergence of sequences generated by this method to a solution of SFP. However, it is noted that this method requires a computation on the operator norm which is in general not an easy task in practice. Subsequently, Moudafi-Thakur presented the notion of the proximal split feasibility problem (PSFP), which is quite more general and flexible than the split feasibility problem, in Hilbert spaces. The PSFP is to find a minimizer of the objective convex function whose image under a linear operator is also a minimizer of another convex function. However, only the weak convergence was obtained in some suitable conditions. Since then, due to its applications in various areas, there have been several modifications and generalizations of these method suggested and invented independently for solving the problem in many different contexts. It is therefore the main objective in this research to develop and modify new regularization methods and study convergence theorems which admit less stringent and/or more constructive requirements on solving the proximal split feasibility problem in a certain space. The main results established in this research can improve and generalize the corresponding results in this area and, of course, can be applied to solve major problems existed in science, engineering, economics and other related branches.

CHAPTER II

LITERATURE REVIEW

Let H_1 and H_2 be real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semi-continuous and convex functions. Let $A : H_1 \to H_2$ be a bounded linear operator. The proximal split feasibility problem is to find a minimizer x^* of f such that Ax^* minimizes g, that is, find $x^* \in \arg\min f$ such that

$$Ax^* \in \arg\min g, \tag{2.1}$$

where $\arg\min f = \{x \in H_1 : f(x) \leq f(y), \ \forall y \in H_1\}$ and $\arg\min g = \{x \in H_2 : g(x) \leq g(y), \ \forall y \in H_2\}$.

In what follows, $\Omega = \arg\min f \cap A^{-1}(\arg\min g)$ will denote the solution set of the problem (2.1).

The split feasibility problem in finite dimensional Hilbert spaces was first introduced by Censor-Elfving [9] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction, especially, intensity-modulated therapy [8]. Due to its applications, there have been many works rapidly established in the recent years (see, for instance, [6, 12, 30, 40, 47]).

Let C be a nonempty closed and convex subset of a real Hilbert space H with the norm $\|\cdot\|$ and the inner product $\langle \cdot, \cdot \rangle$. For each $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|. \tag{2.2}$$

Then P_C is called the *metric projection* of H onto C. For $x \in H$, we know that

$$\langle x - P_C x, y - P_C x \rangle \leq 0 \tag{2.3}$$

for all $y \in C$. If f and g are the indicator functions of two nonempty closed and convex sets $C \subset H_1$ and $Q \subset H_2$, that is,

$$f(x) = \delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$g(x) = \delta_Q(x) = \begin{cases} 0, & \text{if } x \in Q, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the problem (2.1) becomes the following convex minimization problem:

Find $x^* \in C$ such that

$$Ax^* \in Q. \tag{2.4}$$

This problem is called *the split feasibility problem*. A classical way to solve the problem (2.4) is to use the CQ-algorithm which was introduced by Byrne [4], which is defined in the following manner: $x_1 \in H_1$ and

$$x_{n+1} = P_C(x_n - \mu_n A^*(I - P_Q)Ax_n) \tag{2.5}$$

for each $n \geq 1$, where the step-size $\mu_n \in (0, \frac{2}{\|A\|^2})$ and P_C, P_Q are the metric projections on C and Q, respectively.

It is noted that the operator norm $\|A\|$ or the largest eigenvalue of A^*A may not be calculated easily in general. To overcome this difficulty, Lopez et al. [23] suggested the following algorithm: let $x_1 \in H_1$ and assume that $\{x_n\} \subset C$ has been constructed and $\nabla h(x_n) \neq 0$. Then compute x_{n+1} via the rule

$$x_{n+1} = P_C\big(x_n - \mu_n A^*(I - P_Q)Ax_n\big) \tag{2.6}$$

for each $n \geq 1$, where $\mu_n = \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}$ with $0 < \rho_n < 4$ and

$$h(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2.$$

It was proved that, if $\inf_n \rho_n(4 - \rho_n) > 0$, then the sequence $\{x_n\}$ defined by (2.6) converges weakly to a solution of (2.4).

Recall that the *subdifferential* of $F : H \to \mathbb{R} \cup \{+\infty\}$ at x is defined by

$$\partial F(x) = \{y \in H \ : \ F(x) + \langle y, z - x\rangle \leq F(z), \ \forall z \in H\}. \tag{2.7}$$

The *proximity operator* of F is defined by

$$\mathrm{prox}_{\lambda F}(x) = \arg\min_{y \in H} \left\{ F(y) + \frac{1}{2\lambda}\|x - y\|^2 \right\} \tag{2.8}$$

for any $\lambda > 0$. It is seen that

$$0 \in \partial F(x^*) \iff x^* = \mathrm{prox}_{\lambda F}(x^*). \tag{2.9}$$

Hence the minimizers of any functions are the fixed point of its proximity operator. Moreover, the proximity operator of F is firmly nonexpansive, namely,

$$\langle \mathrm{prox}_{\lambda F}(x) - \mathrm{prox}_{\lambda F}(y), x - y\rangle \geq \|\mathrm{prox}_{\lambda F}(x) - \mathrm{prox}_{\lambda F}(y)\|^2 \tag{2.10}$$

for all $x, y \in H$, which is equivalent to

$$\begin{aligned} & \|\mathrm{prox}_{\lambda F}(x) - \mathrm{prox}_{\lambda F}(y)\|^2 \\ \leq \ & \|x - y\|^2 - \|(I - \mathrm{prox}_{\lambda F})(x) - (I - \mathrm{prox}_{\lambda F})(y)\|^2 \end{aligned} \tag{2.11}$$

for all $x, y \in H$. Also, the complement $I - \mathrm{prox}_{\lambda F}$ is firmly nonexpansive. This suggests us to employ the technique in fixed point theory for solving the convex minimization feasibility problem. See [18].

Recently, Moudafi-Thakur [33] proposed the following split proximal algorithm: $x_1 \in H_1$ and

$$x_{n+1} = \mathrm{prox}_{\lambda \mu_n f}\big(x_n - \mu_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n\big), \tag{2.12}$$

where the step-size

$$\mu_n = \rho_n \frac{h(x_n) + l(x_n)}{\theta^2(x_n)} \tag{2.13}$$

with

$$0 < \rho_n < 4, \quad h(x_n) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda g})Ax_n\|^2, \tag{2.14}$$

$$l(x_n) = \frac{1}{2}\|(I - \mathrm{prox}_{\mu_n \lambda f})x_n\|^2 \tag{2.15}$$

and

$$\theta(x_n) = \sqrt{\|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2}. \tag{2.16}$$

They proved that, if $\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n)+l(x_n)} - \epsilon$ for some $\epsilon > 0$ small enough, then the sequence $\{x_n\}$ generated by (2.12) converges weakly to a solution of (2.1). However, we observe that the step-size sequence $\{\mu_n\}$ appeared in (2.13) seems to be implicit because of the terms $l(x_n)$ and $\theta(x_n)$.

In order to solve the proximal split feasibility problem, we introduce a Halpern-type algorithm and prove its strong convergence under the condition on the step size suggested by Lopez et al. [23]. Finally, we provide some numerical experiments to support our main result.

We next consider another type of the proximal split feasibility problem.

Let H_1 and H_2 be real Hilbert spaces. Let $t \geq 1$ and $r \geq 1$ be given integers and let $\{C_i\}_{i=1}^t$ and $\{Q_j\}_{j=1}^r$ be nonempty, closed and convex subsets of H_1 and H_2, respectively. The Multiple-sets Split Feasibility Problem (MSFP) which is the problem of finding a point x^* such that

$$x^* \in C := \bigcap_{i=1}^{t} C_i, \quad Ax^* \in Q := \bigcap_{j=1}^{r} Q_j, \tag{2.17}$$

where A is a given bounded linear operator (denote A^* by the adjoint operator of A). This problem was first introduced, in finite-dimensional Hilbert spaces, by Censor et al. in [10] for modeling inverse problems which arise in modeling of intensity modulated radiation therapy [8], and signal processing and image reconstruction [5, 25]. Due to its applications, there have been many algorithms invented to solve MSFP (see, for instance, [39, 41, 46, 47, 48]). In particular, when $t = r = 1$, the MSFP (2.17) becomes the split feasibility problem (SFP) which was introduced in [9].

Throughout this work, we always assume that the MSFP (2.17) is consistent and also denote the solution set by S. It is known that the MSFP is equivalent to the following minimization problem:

$$\min \frac{1}{2}\|x - P_C(x)\|^2 + \frac{1}{2}\|Ax - P_Q(Ax)\|^2, \tag{2.18}$$

where P_C and P_Q are the metric projections onto C and Q, respectively. It should be noted that the computation of a projection onto a general closed convex subset is difficult because of its closed form. To overcome this difficulty, Fukushima [16] suggested a so-called relaxed projection method to calculate the projection onto a level set of a convex function by computing a sequence of projections onto half-spaces containing the original level set. In the setting of finite-dimensional Hilbert spaces, Yang [44] introduced the relaxed CQ algorithms for solving SFP where the closed convex subsets C and Q are level sets of convex functions given as follows:

$$C = \{x \in H_1 : c(x) \leq 0\} \textit{ and } Q = \{y \in H_2 : q(y) \leq 0\}, \tag{2.19}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex functions. We assume that both c and q are subdifferentiable on H_1 and H_2, respectively, and that ∂c and ∂q are bounded operators (*i.e.*, bounded on bounded sets). It is known that every convex function defined on a finite-dimensional Hilbert space is subdifferentiable and its subdifferential operator is a bounded operator (see [3]). In what follows, we define two sets at point x_n by

$$C_n = \{x \in H_1 : c(x_n) \leq \langle \xi_n, x_n - x\rangle\}, \tag{2.20}$$

where $\xi_n \in \partial c(x_n)$, and

$$Q_n = \{y \in H_2 : q(Ax_n) \leq \langle \varepsilon_n, Ax_n - y\rangle\}, \tag{2.21}$$

where $\varepsilon_n \in \partial q(Ax_n)$. It is clear that C_n and Q_n are half-spaces and $C_n \supset C$ and $Q_n \supset Q$ for every $n \geq 1$. The specific form of the metric projections onto C_n and Q_n can be found in [3]. In fact, Yang [44] constructed a relaxed CQ algorithm for solving the SFP by using the half-spaces C_n and Q_n instead of the sets C and Q in the CQ algorithm, respectively and proved its convergence under some suitable choices of the step-sizes.

In order to achieve the convergence, in such algorithms mentioned above, the selection of the step-sizes requires prior information on the norm of the bounded linear operator (matrix in the finite-dimensional framework), which is not always possible in practice. To avoid this computation, there have been worthwhile works that the convergence is guaranteed without any prior information of the matrix norm (see, for examples [38, 41, 42, 45]). Among these works, López et al. [25] introduced a new way to select the step-size by replacing the parameter μ_n appeared in (2.30) by

$$\mu_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \quad n \geq 1, \tag{2.22}$$

where $\rho_n \in (0, 4)$, $f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ and $\nabla f(x_n) = A^*(I - P_Q)Ax_n$ for all $n \geq 1$. They also practised this way of selecting step-sizes for variants of the CQ algorithm, including a relaxed CQ algorithm, and a Halpern-type algorithm and proved both weak and strong convergence. Subsequently, in 2013, He and Zhao [21] introduced the following Halpern-relaxed CQ algorithm in Hilbert spaces: take $x_1 \in H_1$ and generate $\{x_n\}$ by

$$x_{n+1} = P_{C_n}[\alpha_n u + (1 - \alpha_n)(x_n - \tau_n \nabla f_n(x_n))], \tag{2.23}$$

where C_n and Q_n are, respectively, given as in (2.31) and (2.32), $\{\alpha_n\} \subset (0, 1)$, $\{\rho_n\} \subset (0, 4)$ and the sequence $\{\tau_n\}$ is given by

$$\tau_n = \frac{\rho_n f_n(x_n)}{\|\nabla f_n(x_n)\|^2} \tag{2.24}$$

and

$$f_n(x_n) = \frac{1}{2}\|(I - P_{Q_n})Ax_n\|^2, \quad n \geq 1. \tag{2.25}$$

In this case, we have

$$\nabla f_n(x_n) = A^*(I - P_{Q_n})Ax_n. \tag{2.26}$$

They obtained the strong convergence provided $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and the step-size is chosen such that $\inf_{n\in\mathbb{N}} \rho_n(4 - \rho_n) > 0$.

For solving the MSFP, following [10], we define the level sets of convex functions by

$$C_i = \{x \in H_1 : c_i(x) \leq 0\} \ \ and \ \ Q_j = \{y \in H_2 : q_j(y) \leq 0\}, \tag{2.27}$$

where $c_i : H_1 \to \mathbb{R}$ $(i = 1, ..., t)$ and $q_j : H_2 \to \mathbb{R}$ $(j = 1, ..., r)$ are weakly lower semi-continuous and convex functions. We assume that c_i $(i = 1, ..., t)$ and q_j $(j = 1, ..., r)$ are subdifferentiable on H_1 and H_2, respectively, and that ∂c_i $(i = 1, ..., t)$ and ∂q_j $(j = 1, ..., r)$ are bounded on bounded sets. Censor et al. [10] also defined the following proximity function:

$$f(x) := \frac{1}{2}\sum_{i=1}^{t} l_i \|x - P_{C_i}(x)\|^2 + \frac{1}{2}\sum_{j=1}^{r} \lambda_j \|Ax - P_{Q_j}(Ax)\|^2, \tag{2.28}$$

where l_i $(i = 1, ..., t)$ and λ_j $(i = 1, ..., r)$ are all positive constants such that $\sum_{i=1}^{t} l_i + \sum_{j=1}^{r} \lambda_j = 1$. In this case, we also have

$$\nabla f(x) = \sum_{i=1}^{t} l_i(x - P_{C_i}(x)) + \sum_{j=1}^{r} \lambda_j A^*(I - P_{Q_j})Ax. \tag{2.29}$$

They introduced the following projection algorithm:

$$x_{n+1} = P_\Omega(x_n - \rho \nabla f(x_n)), \tag{2.30}$$

where $\rho > 0$ and $\Omega \subseteq \mathbb{R}^N$ is an auxiliary simple nonempty closed convex set such that $\Omega \cap S \neq \emptyset$. It was proved that if $\rho \in (0, 2/L)$ with L being the Lipschitz constant of ∇f, then the sequence $\{x_n\}$ generated by (2.30) converges to a solution in MSFP.

As observed in the results of Byrne [4], we see that the selection of the step-sizes ρ in (2.30) depends on the largest eigenvalue (spectral radius) of the matrix A^*A which is not always possible in practice. To avoid this computation, there have been worthwhile works that the convergence is guaranteed without any prior information of the matrix norm (see, for examples [38, 41, 42, 45]). Among these works, López et al. [25] introduced a new way to select the step-size and also practised this way of selecting step-sizes for variants of the CQ algorithm, including a relaxed CQ algorithm, and a Halpern-type algorithm and proved both weak and strong convergence. Combining the relaxed CQ algorithm with that of López et al. [25], in 2013, He and Zhao [21] introduced a new relaxed CQ algorithm such that the strong convergence is guaranteed in infinite-dimensional Hilbert spaces. With this choice of the step-sizes, the estimation of the norm of operators is avoided and the metric projections are easily to be calculated.

In what follows, we define two sets at point x_n by

$$C_i^n = \{x \in H_1 : c_i(x_n) \leq \langle \xi_i^n, x_n - x \rangle\}, \tag{2.31}$$

where $\xi_i^n \in \partial c_i(x_n)$ for $i = 1, ..., t$, and

$$Q_j^n = \{y \in H_2 : q_j(Ax_n) \leq \langle \zeta_j^n, Ax_n - y \rangle\}, \tag{2.32}$$

where $\zeta_j^n \in \partial q_j(Ax_n)$ for $j = 1, ..., r$. We see that C_i^n $(i = 1, ..., t)$ and Q_j^n $(j = 1, ..., r)$ are half-spaces and $C_i^n \supset C_i$ $(i = 1, ..., t)$ and $Q_j^n \supset Q_j$ $(j = 1, ..., r)$ for all $n \geq 1$. We define

$$f_n(x) := \frac{1}{2} \sum_{i=1}^{t} l_i \|x - P_{C_i^n}(x)\|^2 + \frac{1}{2} \sum_{j=1}^{r} \lambda_j \|Ax - P_{Q_j^n}(Ax)\|^2, \tag{2.33}$$

where $C_i^n(i = 1, ..., t)$ and $Q_j^n(j = 1, ..., r)$ are given as in (2.31) and (2.32), respectively.

We then have

$$\nabla f_n(x) := \sum_{i=1}^{t} l_i(x - P_{C_i^n}(x)) + \sum_{j=1}^{r} \lambda_j A^*(I - P_{Q_j^n})Ax, \tag{2.34}$$

where A^* is the adjoint operator of A.

For obtaining the strong convergence, recently, inspired by the algorithms proposed by Zhao et al. [48] and López et al. [25], He et al. [22] introduced a new relaxed self-adaptive CQ algorithm for solving the MSFP such that the strong convergence is guaranteed by using Halpern's iteration process. Let $u \in H_1$ be fixed, and choose an initial guess $x_1 \in H_1$ arbitrarily. Let $\{x_n\}$ be the sequence generated by the following manner:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(x_n - \tau_n \nabla f_n(x_n)), \quad n \geq 1, \tag{2.35}$$

where f_n is given as in (2.33), $\{\alpha_n\} \subset (0,1)$ and $\tau_n = \rho_n \frac{f_n(x_n)}{\|\nabla f_n(x_n)\|^2}$ with $0 < \rho_n < 4$ for all $n \in \mathbb{N}$. It was proved that if $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\inf_{n\in\mathbb{N}} \rho_n(4-\rho_n) > 0$, then $\{x_n\}$ generated by (2.35) converges strongly to a solution in MSFP.

In this research, motivated by the previous works, we propose the following inertial relaxed CQ algorithm which combines the inertial technique with the relaxed CQ method:

Algorithm 3.1 Let $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\{\rho_n\} \subset (0,4)$. Let $u \in H_1$ be fixed and take $x_0, x_1 \in H_1$ arbitrarily. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by the following manner:

$$\begin{aligned} x_{n+1} &= \alpha_n u + (1-\alpha_n)(y_n - \tau_n \nabla f_n(y_n)), \\ y_n &= x_n + \beta_n(x_n - x_{n-1}), \quad n \geq 1, \end{aligned} \tag{2.36}$$

where f_n is given as in (2.33) and $\tau_n = \rho_n \frac{f_n(y_n)}{\|\nabla f_n(y_n)\|^2}$ for all $n \in \mathbb{N}$. If $\nabla f_n(y_n) = 0$, then y_n is a solution of MSFP. Here β_n is an extrapolation factor and the inertia is represented by the term $\beta_n(x_n - x_{n-1})$. It is remarkable that the inertial terminology greatly improves the performance of the algorithm and has a nice convergence properties [1, 14, 15, 24, 35, 36] and also [27, 28]. Using the inertial technique and Halpern's idea, we prove its strong convergence of the sequence generated by our proposed scheme. Our algorithm is easily to be implemented since it involves the metric projections onto half-spaces which have exact forms and has no need to know a priori information of the norm of bounded linear operators. Numerical experiments are included to show the effectiveness of the our algorithm. The obtained results mainly extend and improve that of He et al. [22] and also complement the corresponding results of [4, 25, 48].

CHAPTER III

PRELIMINARIES

3.1 Preliminaries and lemmas

In this section, we provide some basic concepts, definitions and lemmas which will be used in the sequel.

Definition 3.1.1. (Fixed point)

Let X be a nonempty set and $T : X \to X$ a self-mapping. We say that $x \in X$ is a fixed point of T if

$$T(x) = x \tag{3.37}$$

and denote by $Fix(T)$ the set of all fixed points of T.

Example 3.1.2. *1. If $X = \mathbb{R}$ and $T(x) = x^2 + 5x + 4$, then $Fix(T) = \{-2\}$;*

2. If $X = \mathbb{R}$ and $T(x) = x^2 - x$, then $Fix(T) = \{0, 2\}$;

3. If $X = \mathbb{R}$ and $T(x) = x + 5$, then $Fix(T) = \emptyset$;

4. If $X = \mathbb{R}$ and $T(x) = x$, then $Fix(T) = \mathbb{R}$;.

Definition 3.1.3. (Metric space)

Let X be a nonempty set and $d : X \times X \to [0, \infty)$ a function. Then d is called a *metric* on X if the following properties hold:

1. $d(x, y) \geq 0$ for all $x, y \in X$;
2. $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
3. $d(x, y) = d(y, x)$ for all $x, y \in X$;
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The value of metric d at (x, y) is called distance between x and y, and the ordered pair (X, d) is called a *metric space*.

Example 3.1.4. *The real line $\mathbb{R}$ and define*

$$d(x, y) = |x - y| \; for \; all \; x, y \in \mathbb{R}. \tag{3.38}$$

Then $(\mathbb{R}, d)$ *is a metric space.*

Example 3.1.5. *The Euclidean plane* $\mathbb{R}^2$ *and define*

$$d(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2} \tag{3.39}$$

where $x = (\xi_1, \xi_2), y = (\eta_1, \eta_2) \in \mathbb{R}^2$*. Then* $(\mathbb{R}^2, d)$ *is a metric space.*

Example 3.1.6. *The Euclidean space* $\mathbb{R}^n$ *and define*

$$d(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2 + ... + (\xi_n - \eta_n)^2} \tag{3.40}$$

where $x = (\xi_1, \xi_2, \xi_3, ..., \xi_n), y = (\eta_1, \eta_2, \eta_3, ..., \eta_n) \in \mathbb{R}^n$*. Then* $(\mathbb{R}^n, d)$ *is a metric space.*

Example 3.1.7. *Let* X *be the set of all bounded sequences of complex numbers; that is every element of* X *is a complex sequence*

$$x = (\xi_1, \xi_2, ...)$$

such that $|\xi_j| \leq c_x$ *for all* $j = 1, 2, ...$ *and* c_x *is a real number which may depend on* x*, but does not depend on* j *and define*

$$d(x,y) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j| \tag{3.41}$$

where $y = (\eta_j) \in X$ *and* $\mathbb{N} = 1, 2, ...$ *Then* (X, d) *is a metric space.*

Definition 3.1.8. (Closed set)

Let (X, d) be a metric space. A subset $U \subseteq X$ is open if for every $x \in X$ there exists $r > 0$ such that $B(x, r) \subseteq U$. A set U is closed if its complement $X \setminus U$ is open.

Theorem 3.1.9. *Let* M *be a nonempty subset of a metric space* X*. Then* M *is closed if and only if there exists a sequence* $\{x_n\} \subseteq M$ *and* $x_n \to x$ *implies that* $x \in M$*.*

Definition 3.1.10. (Convergent sequence)

A sequence $\{x_n\}$ in a metric space X is said to be convergent to $x \in \mathbb{R}$ if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ if $n > N$ then $d(x, y) < \epsilon$. In this case, we write $x_n \to x$

Definition 3.1.11. (Cauchy sequence)

A sequence $\{x_n\}$ in a metric space X is said to be Cauchy if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ if $m, n > N$ then $d(x_m, x_n) < \epsilon$.

Definition 3.1.12. (Bounded sequence)

A sequence $\{x_n\}$ in X is bounded if there exists $M > 0$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Definition 3.1.13. (Lipschitzian mapping)

Let (X, d) be a metric space. Then a map $T : X \to X$ is called a lipschitzian mapping on X if there exists $L > 0$ such that

$$d(T(x), T(y)) \leq Ld(x, y) \; for \; all \; x, y \in X.$$

Definition 3.1.14. (Nonexpansive mapping)

Let (X, d) be a metric space. Then a map $T : X \to X$ is called a nonexpansive mapping on X if

$$d(T(x), T(y)) \leq d(x, y) \; for \; all \; x, y \in X.$$

Definition 3.1.15. (Contraction mapping)

Let (X, d) be a metric space. Then a map $T : X \to X$ is called a contraction mapping on X if there exists $q \in [0, 1)$ such that

$$d(T(x), T(y)) \leq qd(x, y) \; for \; all \; x, y \in X.$$

Theorem 3.1.16. ***(The Banach contraction principle)***

Let X be a complete metric space and let T be a contraction of X into itself. Then T has a unique fixed point.

Definition 3.1.17. (Vector space)

A vector space or linear space X over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) is a set X together with an internal binary operation "+" called addition and a scalar multiplication carrying (α, x) in $\mathbb{K} \times X$ to αx in X satisfying the following for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{K}$:

1. $x + y = y + x$;
2. $(x + y) + z = x + (y + z)$;
3. there exists an element $0 \in X$ call the *zero vector* of X such that $x+0 = x$ for all $x \in X$;
4. for every element $x \in X$, there exists an element $-x \in X$ called *the additive inverse* or *the negative* of x such that $x + (-x) = 0$;
5. $\alpha(x + y) = \alpha x + \alpha y$;
6. $(\alpha + \beta)x = \alpha x + \beta y$;
7. $(\alpha\beta)x = \alpha(\beta x)$;
8. $1 \cdot x = x$.

The elements of a vector space X are called vectors, and the elements of $\mathbb{K}$ are called scalars.

Example 3.1.18. *The Euclidean space $\mathbb{R}^n$ and define*

$$\begin{aligned} x + y &= (\xi_1 + \eta_1, \xi_2 + \eta_2, \xi_3 + \eta_3, ..., \xi_n + \eta_n) \\ \alpha x &= (\alpha\xi_1, \alpha\xi_2, \alpha\xi_3, ..., \alpha\xi_n) \end{aligned}$$

where $x = (\xi_1, \xi_2, \xi_3, ..., \xi_n), y = (\eta_1, \eta_2, \eta_3, ..., \eta_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then, space $\mathbb{R}^n$ is a real vector space.

Definition 3.1.19. (Convex set)

Let C be a subset of a linear space X. Then C is said to be convex if $(1 - \lambda)x + \lambda y \in C$ for all x, y and all scalar $\lambda \in [0, 1]$.

Example 3.1.20. *1. Every subspace of vector space is convex set.*

2. $\overline{B}(x; r) = \{x : \|x\| \leq r\}$ *is convex set.*

3. $[0, 1]^N = [1, 0] \times [1, 0] \times ... \times [1, 0]$ *is convex set in* $\mathbb{R}^N$.

Proposition 3.1.21. *Let C be a subset of a linear space X. Then C is convex if and only if $\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n \in C$ for any finite set $\{x_1, x_2, ..., x_n\} \subseteq C$ and scalars $\lambda_i \geq 0$ with $\lambda_1 + \lambda_2 + ... + \lambda_n = 1$.*

Definition 3.1.22. (Convex function)

Let X be a linear space and $f : X \to (-\infty, \infty]$ a function. Then f is said to be convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in X$ and $\lambda \in [0,1]$.

Example 3.1.23. *1. $F(x) = |x|^p$ where $p \geq 1$ is convex function in $\mathbb{R}$.*

2. $F(x) = x^3 - x^2$ is convex function in $[\frac{1}{3}, \infty)$.

3. $F(x) = x \log x$ where $p \geq 1$ is convex function in $\mathbb{R}^+$.

Definition 3.1.24. (Normed space)

let X be a norm linear space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\|\cdot\| : X \to \mathbb{R}^+$ a function. Then $\|\cdot\|$ is said to be a norm if the following properties hold:

1. $\|x\| \geq 0$, and $\|x\| = 0 \Leftrightarrow x = 0$;
2. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$;
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The ordered pair $(X, \|\cdot\|)$ is called a normed space.

Example 3.1.25. *$\mathbb{R}^n$ is a normed space with the following norms:*

$$\begin{aligned}
\|x\|_1 &= \sum_{i=1}^{n} |x_i| \ for\ all\ \ x = (x_1, x_2, .., x_n) \in \mathbb{R}^n; \\
\|x\|_p &= \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p} \ for\ all\ \ x = (x_1, x_2, .., x_n) \in \mathbb{R}^n \ and\ p \in (1, \infty); \\
\|x\|_\infty &= \max_{1\leq i\leq n} |x_i| \ for\ all\ \ x = (x_1, x_2, .., x_n) \in \mathbb{R}^n.
\end{aligned}$$

Remark 3.1.26. 1. $\mathbb{R}^n$ equipped with the norm defined by $\|x\|_p = \big(\sum_{i=1}^n |x_i|^p\big)^{1/p}$ is denoted by l_q^n for all $1 \leq p < \infty$.

2. $\mathbb{R}^n$ equipped with the norm defined by $\|x\|_\infty = \max_{1\leq i\leq n} |x_i|$ is denoted by l_∞^n.

Example 3.1.27. *Let $X = l_1$, the linear space whose elements consist of all absolutely convergent sequences $(x_1, x_2, ..., x_i, ...)$ of scalars ($\mathbb{R}$ or $\mathbb{C}$),*

$$l_1 = \{x : x = (x_1, x_2, ..., x_i, ...) \ and \ \sum_{i=1}^{\infty} |x_i| < \infty\}.$$

Then l_1 is a normed space with the norm defined by $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$.

Example 3.1.28. *let* $X = l_p$ $(1 < p < \infty)$, *the linear space whose elements consist of all p-summable sequences* $(x_1, x_2, ..., x_i, ...)$ *of scalars* ($\mathbb{R}$ *or* $\mathbb{C}$),

$$l_p = \{x : x = (x_1, x_2, ..., x_i, ...) \text{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty\}.$$

Then l_p *is a normed space with the norm defined by* $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$.

Example 3.1.29. *let* $X = l_\infty$, *the linear space whose elements consist of all bounded sequences* $(x_1, x_2, ..., x_i, ...)$ *of scalars* ($\mathbb{R}$ *or* $\mathbb{C}$),

$$l_\infty = \{x : x = (x_1, x_2, ..., x_i, ...) \text{ and } \{x_i\}_{i=1}^{\infty} \text{ is bounded}\}.$$

Then l_∞ *is a normed space with the norm defined by* $\|x\|_\infty = \sup_{i\in\mathbb{N}} |x_i|$.

Definition 3.1.30. (Completeness)

The space X is said to be complete if every Cauchy sequence in X converges.

Example 3.1.31. *The Euclidean space* $\mathbb{R}^n$ *is complete with*

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2 + ... + (\xi_n - \eta_n)^2} \tag{3.42}$$

where $x = (\xi_1, \xi_2, \xi_3, ..., \xi_n), y = (\eta_1, \eta_2, \eta_3, ..., \eta_n) \in \mathbb{R}^n$.

Example 3.1.32. *The sequence space* l_∞ *is complete.*

Example 3.1.33. *The sequence space* l_p *is complete.*

Definition 3.1.34. (Banach space)

A normed space which is complete with respect to the metric induced by the norm is called a Banach space.

Example 3.1.35. *The Euclidean space* $\mathbb{R}^n$ *is a Banach space with the norm defined by*

$$\|x\| = \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2},$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

Example 3.1.36. *The space l_p, $1 \leq p < \infty$ is a Banach space with the norm defined by*

$$\|x\|_p = \Big(\sum_{i=1}^{\infty} |x_i|^p\Big)^{1/p},$$

where $x = (x_1, x_2, ..., x_n, ...)$ and $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

Example 3.1.37. *The space l_∞ of all bounded sequence $x = (x_1, x_2, ..., x_n, ...)$ is a Banach space with the norm defined by*

$$\|x\| = \sup_i |x_i|.$$

Definition 3.1.38. (Inner product space)

An inner product space is a vector space X with an inner product defined on X. Here, an inner product on X is a mapping of $X \times X$ into the scalar field $\mathbb{K}$ of X; that is, with every pair of vectors x and y there is associated a scalar which is written

$$\langle x, y \rangle \tag{3.43}$$

and is called the inner product of x and y, such that for all vectors x, y, z and scalars a we have

(IP1) $\langle x, x \rangle \geq 0$;

(IP2) $\langle x, x \rangle = 0 \Leftrightarrow x = 0$;

(IP3) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$;

(IP4) $\langle x, y \rangle = \overline{\langle y, x \rangle}$;

(IP5) $\langle x + y, z \rangle = \langle x, z \rangle + \langle z, y \rangle$.

Example 3.1.39. *The function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \; for \; all \; x = (x_1, x_2, ..., x_n), \; y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n \tag{3.44}$$

is an inner product on $\mathbb{R}^n$. In this case $\mathbb{R}^n$ with this inner product is called real Euclidean n-space.

Example 3.1.40. *Let $\mathbb{C}^n$ be the set of n-tuples of complex numbers. Then the function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i} \; for \; all \; x = (x_1, x_2, ..., x_n), \; y = (y_1, y_2, ..., y_n) \in \mathbb{C}^n \tag{3.45}$$

is an inner product on $\mathbb{C}^n$*. In this case* $\mathbb{C}^n$ *with this inner product is called complex Euclidean n-space.*

Example 3.1.41. *Let* l_2 *be the set of all sequences of complex numbers* $(a_1, a_2, \ldots, a_i, \ldots)$ *with* $\sum_{i=1}^{\infty} |a_i|^2 < \infty$*. Then the function* $\langle \cdot, \cdot \rangle : l_2 \times l_2 \to \mathbb{C}$ *defined by*

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i} \text{ for all } x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in l_2 \tag{3.46}$$

is an inner product on l_2*.*

Proposition 3.1.42. ***(The Cauchy-Schwarz inequality)***

Let X *be an inner product space. Then the following holds:*

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle \text{ for all } x, y \in X, \tag{3.47}$$

i.e.,

$$|\langle x, y \rangle| \leq \|x\| \|y\| \text{ for all } x, y \in X. \tag{3.48}$$

Definition 3.1.43. (Hilbert space)

An inner product space which is complete with respect to the induced norm is called a Hilbert space.

Example 3.1.44. *The Euclidean space* $\mathbb{R}^n$ *is a Hilbert space with inner product defined by*

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

where $x = (x_1, x_2, \ldots, x_n)$*,* $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$

Example 3.1.45. *The space* l_2 *is a Hilbert space with inner product defined by*

$$\langle x, y \rangle = \sum_{j=1}^{\infty} x_j \overline{y_j},$$

where $x, y \in l_2$*.*

Let H be a Hilbert space. Recall that a mapping $T : H \to H$ is said to be nonexpansive if, for all $x, y \in H$,

$$\|Tx - Ty\| \leq \|x - y\| \tag{3.49}$$

$T : H \to H$ is said to be firmly nonexpansive if, for all $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \tag{3.50}$$

or equivalently

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2 \tag{3.51}$$

for all $x, y \in H$. It is known that T is firmly nonexpansive if and only if $I - T$ is firmly nonexpansive. We know that the metric projection P_C from H onto a nonempty, closed and convex subset $C \subset H$ is a typical example of a firmly nonexpansive mapping, which is defined by

$$P_C x := \arg\min_{y \in C} \|x - y\|^2, \quad x \in H. \tag{3.52}$$

It is well known that P_C is characterized by the inequality, for $x \in H$

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall y \in C. \tag{3.53}$$

In a real Hilbert space H, we have the following equality:

$$\langle x, y\rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2 \tag{3.54}$$

and the subdifferential inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \tag{3.55}$$

for all $x, y \in H$.

Definition 3.1.46. (Proper function)

Let function $f : X \to (-\infty, \infty]$. Then f is said to be proper if there exists $x \in X$ with $f(x) < \infty$.

Definition 3.1.47. (Lower semicontinuous function)

Let X be a linear space and $f : X \to (-\infty, \infty]$ a proper function. Then f is said to be lower semicontinuous (l.s.c.) at $x_0 \in X$ if

$$f(x_0) \leq \liminf_{x \to x_0} f(x_0) = \sup_{V \in U_{x_0}} \inf_{x \in V} f(x), \tag{3.56}$$

where U_{x_0} is a base of neighborhoods of the point $x_0 \in X$. f is said to be lower semicontinuous on X if it is lower semicontinuous on each point of X, *i.e.*, for each $x \in X$,

$$x \to x_0 \Rightarrow f(x) \leq \liminf_{n\to\infty} f(x_n). \tag{3.57}$$

Example 3.1.48. *Let $(X, \|\cdot\|)$ be normed space. If $F(x) = \|x\|$ for all $x \in X$ then F is lower semicontinuous function.*

Definition 3.1.49. (Bounded linear operator)

Let X and Y be normed spaces and $T : X \to Y$ a linear operator. The operator T is said to be bounded if there is a real number c such that for all $x \in X$,

$$\|Tx\| \leq c\|x\|. \tag{3.58}$$

The *subdifferential* of a proper convex function $f : X \to (-\infty, +\infty]$ is the set-valued operator $\partial f : X \to 2^X$ defined as

$$\partial f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle + f(x) \leq f(y)\}.$$

If f is proper convex and lower semicontinuous, then the subdifferential $\partial f(x) \neq \emptyset$ for any $x \in int\mathcal{D}(f)$, the interior of the domain of f.

Lemma 3.1.50. [10] *Let $\{C_i\}_{i=1}^t$ and $\{Q_j\}_{j=1}^r$ be closed convex subsets of H_1 and H_2 respectively and $A : H_1 \to H_2$ a bounded linear operator. Let $f(x)$ be the function defined as in (2.28). Then $\nabla f(x)$ is Lipschitz continuous with $L = \sum_{i=1}^t l_i + \|A\|^2 \sum_{j=1}^r \lambda_j$ as the Lipschitz constant.*

Lemma 3.1.51. *Let $f : H \to \mathbb{R}$ be given by $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$. Then*

(i) *f is convex and differential.*

(ii) *$\nabla f(x) = A^*(I - P_Q)Ax$, $x \in H$.*

(iii) *f is weakly lower semi-continuous on H.*

(iv) *$\|\nabla f(x) - \nabla f(y)\| \leq \|A\|^2\|x - y\|$ for all $x, y \in H$.*

Lemma 3.1.52. [29, 43] *Let* $\{a_n\}$ *and* $\{c_n\}$ *are sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\delta_n)a_n + b_n + c_n, \quad n \geq 1, \tag{3.59}$$

where $\{\delta_n\}$ *is a sequence in* $(0,1)$ *and* $\{b_n\}$ *is a real sequence. Assume* $\sum_{n=1}^{\infty} c_n < \infty$. *Then the following results hold:*

(i) *If* $b_n \leq \delta_n M$ *for some* $M \geq 0$, *then* $\{a_n\}$ *is a bounded sequence.*

(ii) *If* $\sum_{n=1}^{\infty} \delta_n = \infty$ *and* $\limsup_{n\to\infty} b_n/\delta_n \leq 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 3.1.53. [30] *Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence* $\{\Gamma_{n_i}\}$ *of* $\{\Gamma_n\}$ *which satisfies* $\Gamma_{n_i} < \Gamma_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Define the sequence* $\{\tau(n)\}_{n\geq n_0}$ *of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}, \tag{3.60}$$

where $n_0 \in \mathbb{N}$ *such that* $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. *Then, the following hold:*

(i) $\tau(n_0) \leq \tau(n_0+1) \leq \ldots$ *and* $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

Lemma 3.1.54. [19] *Assume that* $\{s_n\}$ *is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\delta_n \tag{3.61}$$

and

$$s_{n+1} \leq s_n - \eta_n + t_n \tag{3.62}$$

for each $n \geq 0$, *where* $\{\gamma_n\}$ *is a sequence in* $(0,1)$, $\{\eta_n\}$ *is a sequence of nonnegative real numbers,* $\{\delta_n\}$ *and* $\{t_n\}$ *are real sequences such that*

(a) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(b) $\lim_{n\to\infty} t_n = 0$,

(c) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for any subsequence* $\{n_k\}$ *of* $\{n\}$.

Then $\lim_{n\to\infty} s_n = 0$.

CHAPTER IV

MAIN RESULTS

4.1 On solving proximal split feasibility problems and applications

4.1.1 Algorithms and Convergence Theorem

Let H_1 and H_2 be real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R}\cup\{+\infty\}$ be proper, lower semi-continuous and convex functions and $A : H_1 \to H_2$ be a bounded linear operator. Denote by Ω the solution set of the split proximal feasibility problem. We introduce the following results:

Algorithm I.

Step 1. Choose an initial point $x_0 \in H_1$;

Step 2. Assume that $\{x_n\}$ has been constructed. Set

$$h(x_n) = \frac{1}{2}\|(I - \text{prox}_{\lambda g})Ax_n\|^2 \tag{4.63}$$

with $\|\nabla h(x_n)\| \neq 0$ for each $n \geq 1$.

We compute x_{n+1} via the following manner:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\text{prox}_{\lambda f}\left(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2} A^*(I - \text{prox}_{\lambda g})Ax_n\right), \tag{4.64}$$

for each $n \geq 1$, where $u \in H_1$ is fixed, $\lambda > 0$, $\{\alpha_n\} \subset (0, 1)$ and $\{\rho_n\} \subset (0, 4)$.

Theorem 4.1.1. *Suppose that $\Omega \neq \emptyset$ and assume that $\{\alpha_n\}$ and $\{\rho_n\}$ satisfy the conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\liminf_{n\to\infty} \rho_n(4 - \rho_n) > 0$.

Then the sequence $\{x_n\}$ converges strongly to $z = P_\Omega u$.

Proof. Let $z = P_\Omega u$. Then $z = \text{prox}_{\lambda f} z$ and $Az = \text{prox}_{\lambda g} Az$. Note that

$$\nabla h(x_n) = A^*(I - \text{prox}_{\lambda g})Ax_n. \tag{4.65}$$

So, since $I - \text{prox}_{\lambda g}$ is firmly nonexpansive, using (2.10), we have

$$\begin{aligned} \langle \nabla h(x_n), x_n - z\rangle &= \big\langle A^*(I - \text{prox}_{\lambda g})Ax_n, x_n - z\big\rangle \\ &= \big\langle (I - \text{prox}_{\lambda g})Ax_n, Ax_n - Az\big\rangle \\ &= \big\langle (I - \text{prox}_{\lambda g})Ax_n - (I - \text{prox}_{\lambda g})Az, Ax_n - Az\big\rangle \\ &\geq \|(I - \text{prox}_{\lambda g})Ax_n\|^2 = 2h(x_n). \end{aligned} \tag{4.66}$$

Using (4.66), we obtain

$$\begin{aligned} &\left\| x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) - z\right\|^2 \\ =\ & \|x_n - z\|^2 + \rho_n^2 \frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} - 2\rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\big\langle \nabla h(x_n), x_n - z\big\rangle \\ \leq\ & \|x_n - z\|^2 + \rho_n^2 \frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} - 4\rho_n \frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} \\ =\ & \|x_n - z\|^2 - \rho_n(4 - \rho_n)\frac{h^2(x_n)}{\|\nabla h(x_n)\|^2}. \end{aligned} \tag{4.67}$$

Since $\{\rho_n\} \subset (0, 4)$, it then follows that

$$\left\| x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) - z\right\| \leq \|x_n - z\|. \tag{4.68}$$

Next, we show that $\{x_n\}$ is bounded. Consider

$$\begin{aligned} &\|x_{n+1} - z\| \\ =\ & \left\| \alpha_n(u - z) + (1 - \alpha_n)\big(\text{prox}_{\lambda f}(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2} A^*(I - \text{prox}_{\lambda g})Ax_n) - z\big)\right\| \\ \leq\ & \alpha_n\|u - z\| + (1 - \alpha_n)\left\| x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) - z\right\| \\ \leq\ & \alpha_n\|u - z\| + (1 - \alpha_n)\|x_n - z\|. \end{aligned} \tag{4.69}$$

It follows, by induction, that

$$\|x_n - z\| \leq \max\{\|u - z\|, \|x_0 - z\|\} \tag{4.70}$$

and hence $\{x_n\}$ is bounded. Using (2.11) and (4.67), we see that

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
\leq\ & (1-\alpha_n)\Big\|\mathrm{prox}_{\lambda f}\big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\big) - z\Big\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\
\leq\ & (1-\alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) - z\Big\|^2 \\
& - (1-\alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) - \mathrm{prox}_{\lambda f}\big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\big)\Big\|^2 \\
& + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\
\leq\ & (1-\alpha_n)\|x_n - z\|^2 - (1-\alpha_n)\rho_n(4-\rho_n)\frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} \qquad (4.71) \\
& - (1-\alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) - \mathrm{prox}_{\lambda f}\big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\big)\Big\|^2 \\
& + 2\alpha_n\langle u - z, x_{n+1} - z\rangle.
\end{aligned}$$

Set

$$s_n = \|x_n - z\|^2, \quad \gamma_n = \alpha_n, \qquad (4.72)$$

$$\delta_n = 2\langle u - z, x_{n+1} - z\rangle, \quad t_n = 2\alpha_n\langle u - z, x_{n+1} - z\rangle \qquad (4.73)$$

and

$$\begin{aligned}
\eta_n \ =\ & (1-\alpha_n)\rho_n(4-\rho_n)\frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} + (1-\alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) \\
& - \mathrm{prox}_{\lambda f}\big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\big)\Big\|^2. \qquad (4.74)
\end{aligned}$$

From (4.71), it follows that

$$\|x_{n+1} - z\|^2 \ \leq\ (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \qquad (4.75)$$

and

$$\begin{aligned}
\|x_{n+1} - z\|^2 \ \leq\ & \|x_n - z\|^2 - (1-\alpha_n)\rho_n(4-\rho_n)\frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} \\
& - (1-\alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) \\
& - \mathrm{prox}_{\lambda f}\big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\big)\Big\|^2 \\
& + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \qquad (4.76)
\end{aligned}$$

It is easy to cheek that $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$ by using (C1) and (C2), respectively. In order to apply Lemma 3.1.54, we need to show that $\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ for any subsequence $\{n_k\}$ of $\{n\}$.

Suppose that $\lim_{k\to\infty} \eta_{n_k} = 0$ for any subsequence $\{n_k\}$ of $\{n\}$. By (C1) and (C3), it follows that

$$\lim_{k\to\infty} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|} = 0 \tag{4.77}$$

and

$$\lim_{k\to\infty} \left\| x_{n_k} - \rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2} \nabla h(x_{n_k}) - \text{prox}_{\lambda f}\big(x_{n_k} - \rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2} \nabla h(x_{n_k})\big) \right\| = 0. \tag{4.78}$$

We note that $\{\nabla h(x_{n_k})\}$ is bounded. Indeed, by the Lipschitzian continuity of ∇h and the boundedness of $\{x_{n_k}\}$, we obtain

$$\begin{aligned} \|\nabla h(x_{n_k})\| &\leq \|\nabla h(x_{n_k}) - \nabla h(z)\| + \|\nabla h(z)\| \\ &\leq \|A\|^2 \|x_{n_k} - z\| + \|\nabla h(z)\|. \end{aligned} \tag{4.79}$$

So, by (4.77), we obtain

$$\lim_{k\to\infty} h(x_{n_k}) = 0 \tag{4.80}$$

for any subsequence $\{n_k\}$ of $\{n\}$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \rightharpoonup x^*$ and

$$\limsup_{n\to\infty} \langle u - z, x_n - z\rangle = \lim_{i\to\infty} \langle u - z, x_{n_i} - z\rangle. \tag{4.81}$$

By the lower semi-continuity of h, we have

$$0 \leq h(x^*) \leq \liminf_{i\to\infty} h(x_{n_i}) = \lim_{i\to\infty} h(x_{n_i}) = 0. \tag{4.82}$$

Hence we have

$$h(x^*) = \frac{1}{2}\|(I - \text{prox}_{\lambda g})Ax^*\| = 0. \tag{4.83}$$

Thus Ax^* is a fixed point of the proximity operator g, that is, $0 \in \partial g(Ax^*)$ or Ax^* is a minimizer of g.

Next, we show that x^* is also a minimizer of f. Observe that

$$\begin{aligned}
&\|x_{n_k} - \text{prox}_{\lambda f} x_{n_k}\| \\
\leq\ & \Big\| x_{n_k} - (x_{n_k} - \rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2} \nabla h(x_{n_k})) \Big\| \\
&+ \Big\| x_{n_k} - \rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2} \nabla h(x_{n_k}) - \text{prox}_{\lambda f}\big(x_{n_k} - \rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2} \nabla h(x_{n_k})\big) \Big\| \\
&+ \Big\| \text{prox}_{\lambda f}(x_{n_k} - \rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2} \nabla h(x_{n_k})) - \text{prox}_{\lambda f} x_{n_k} \Big\| \\
\leq\ & 2\rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|} \qquad (4.84) \\
&+ \Big\| x_{n_k} - \rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2} \nabla h(x_{n_k}) - \text{prox}_{\lambda f}\big(x_{n_k} - \rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2} \nabla h(x_{n_k})\big) \Big\|.
\end{aligned}$$

This implies, by (4.77) and (4.78), that

$$\lim_{k\to\infty} \|x_{n_k} - \text{prox}_{\lambda f} x_{n_k}\| = 0 \qquad (4.85)$$

for any subsequence $\{n_k\}$ of $\{n\}$. Note that $\text{prox}_{\lambda f}$ is nonexpansive and $x_{n_i} \rightharpoonup x^*$. So, by the demiclosedness principle [17], we conclude that x^* is a fixed point of the proximity operator of f. This shows that x^* is also a minimizer of f. Therefore, $x^* \in \Omega$. On the other hand, we observe that

$$\begin{aligned}
&\|x_{n_k+1} - x_{n_k}\| \\
\leq\ & \alpha_{n_k}\|u - x_{n_k}\| + (1-\alpha_{n_k}) \Big\| \text{prox}_{\lambda f}\big(x_{n_k} - \rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2} \nabla h(x_{n_k})\big) - x_{n_k} \Big\| \\
\leq\ & \alpha_{n_k}\|u - x_{n_k}\| \\
&+ (1-\alpha_{n_k}) \Big\| \text{prox}_{\lambda f}\big(x_{n_k} - \rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2} \nabla h(x_{n_k})\big) \\
&- \big(x_{n_k} - \rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2} \nabla h(x_{n_k})\big) \Big\| \\
&+ (1-\alpha_{n_k}) \Big\| \big(x_{n_k} - \rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2} \nabla h(x_{n_k})\big) - x_{n_k} \Big\| \\
=\ & \alpha_{n_k}\|u - x_{n_k}\| \\
&+ (1-\alpha_{n_k}) \Big\| \text{prox}_{\lambda f}\big(x_{n_k} - \rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2} \nabla h(x_{n_k})\big) \\
&- \big(x_{n_k} - \rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2} \nabla h(x_{n_k})\big) \Big\| \\
&+ (1-\alpha_{n_k})\rho_{n_k} \frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|} \to 0 \qquad (4.86)
\end{aligned}$$

as $k \to \infty$. Hence, by (2.3), we obtain

$$\begin{aligned}
\limsup_{k\to\infty}\langle u-z, x_{n_k+1}-z\rangle &= \limsup_{k\to\infty}\langle u-z, x_{n_k}-z\rangle \\
&\leq \limsup_{n\to\infty}\langle u-z, x_n-z\rangle \\
&= \lim_{i\to\infty}\langle u-z, x_{n_i}-z\rangle \\
&= \langle u-z, x^*-z\rangle \\
&\leq 0. \qquad (4.87)
\end{aligned}$$

This implies that

$$\limsup_{k\to\infty}\delta_{n_k}\leq 0 \qquad (4.88)$$

for any subsequence $\{n_k\}$. Therefore, by using Lemma 3.1.54, we conclude that $s_n = \|x_n - z\|^2 \to 0$. Hence $x_n \to z = P_\Omega u$. This completes the proof. □

When $f = \delta_C$, $g = \delta_Q$ the indicators functions of nonempty closed and convex sets C, Q of H_1 and H_2, respectively, we obtain the following results:

Algorithm II.

Step 1. Choose an initial point $x_0 \in H_1$;

Step 2. Assume that $\{x_n\} \subseteq C$ has been constructed. Set $h(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ with $\|\nabla h(x_n)\| \neq 0$. We compute x_{n+1} via the following manner:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)P_C\Big(x_n - \rho_n\frac{h(x_n)}{\|\nabla h(x_n)\|^2}A^*(I-P_Q)Ax_n\Big) \qquad (4.89)$$

for each $n \geq 1$, where $u \in C$ is fixed, $\{\alpha_n\} \subset (0,1)$ and $\{\rho_n\} \subset (0,4)$.

Corollary 4.1.2. *Suppose that $\Theta = C \cap A^{-1}(Q) \neq \emptyset$ and assume that $\{\alpha_n\}$ and $\{\rho_n\}$ satisfy the conditions (C1)–(C3). Then the sequence $\{x_n\}$ converges strongly to $z = P_\Theta u$.*

Remark 4.1.3. In the case of $\|\nabla h(x_n)\| = 0$, we see that Algorithm I reduces to the following: $x_0 \in H_1$ and

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\mathrm{prox}_{\lambda f} x_n \qquad (4.90)$$

for each $n \geq 1$, where $u \in H_1$ is fixed, $\{\alpha_n\} \subset (0,1)$ and $\lambda > 0$. If the sequences $\{\alpha_n\}$ satisfies (C1) and (C2), then the sequence $\{x_n\}$ converges strongly to $z = P_{\arg\min f} u$. Since ∇h is continuous, it follows that $\nabla h(x_n) \to \nabla h(z)$. So, we obtain $\nabla h(z) = 0$ because $\|\nabla h(x_n)\| = 0$. This shows that Az is a minimizer of g. Hence $\{x_n\}$ converges strongly to a solution of (2.1).

Remark 4.1.4. We highlight our work in the following inclusions:

(1) The strong convergence theorems for solving the proximal split feasibility problem of two convex functions established in this paper mainly improve and generalize the results obtained by Byrne [4], Lopez et al. [23] and Moudafi-Thakur [33].

(2) We obtain strong convergence theorem by using a simpler and more explicitly than that of Moudafi-Thakur [33] which may be required an implicit computation.

4.1.2 Numerical examples

In this section, we give numerical examples to support our main theorem.

Example 4.1.5. Let $f : \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$f(x) = \|x\|^2 + (2, 4, -5)x + 10 \tag{4.91}$$

and

$$g(x) = \|x\|^2 - (8, 10, -8)x - 5, \tag{4.92}$$

respectively. Let $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Solve the following proximal split feasibility problem:

Find $x^* \in \mathbb{R}^3$ such that x^* minimizes f and Ax^* also minimizes g.

We can check that $x^* = (-1, -2, 2.5)$ is a minimizer of f and $Ax^* = (4, 5, -4)$ minimizes g. We next show the convergence behavior of the sequence in Algorithm I by using our conditions. Let $u = (1, 1, 1)$ and $x_0 = (-2, 4, -3)$. Choose $\lambda = 1$, $\alpha_n = \frac{10^{-3}}{n+1}$ and $\rho_n = 2$ for all $n \in \mathbb{N}$. Computing Algorithm I, iteratively, we obtain the following numerical results.

n	x_n	Ax_n	$f(x_n)$	$g(A(x_n))$
1	(-2.00000,4.00000,-3.00000)	(-8.00000,2.00000,0.00000)	66.000000	107.000000
5	(-1.00362,-1.95651,2.47069)	(3.93775,5.01684,-3.96376)	-1.247236	-61.994527
10	(-0.99977,-1.99961,2.49974)	(3.99971,4.99992,-3.99915)	-1.250000	-61.999999
15	(-0.99987,-1.99973,2.49987)	(3.99988,5.00016,-3.99947)	-1.250000	-62.000000
20	(-0.99989,-1.99981,2.49988)	(3.99987,4.99997,-3.99960)	-1.250000	-62.000000
25	(-0.99992,-1.99984,2.49992)	(3.99992,5.00009,-3.99969)	-1.250000	-62.000000
30	(-0.99993,-1.99988,2.49992)	(3.99991,4.99998,-3.99973)	-1.250000	-62.000000
35	(-0.99994,-1.99989,2.49995)	(3.99995,5.00006,-3.99978)	-1.250000	-62.000000
40	(-0.99995,-1.99991,2.49994)	(3.99994,4.99999,-3.99980)	-1.250000	-62.000000
45	(-0.99996,-1.99991,2.49996)	(3.99996,5.00005,-3.99983)	-1.250000	-62.000000
50	(-0.99996,-1.99993,2.49995)	(3.99995,4.99999,-3.99984)	-1.250000	-62.000000

Table 1 Numerical results for Algorithm I

From Table 1, the minimum values of f and g are -1.25 and -62, respectively. The errors of $\|x_{n+1} - x_n\|_2$ are plotted in the following figure.

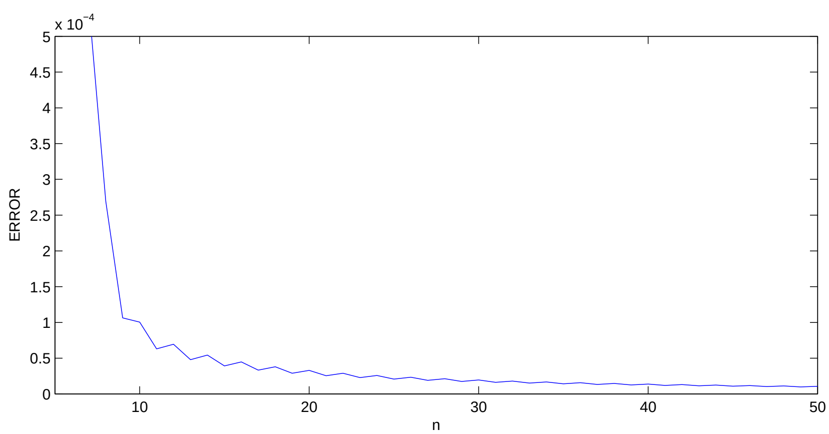


Figure 1 Errors plotting of Table 1

Example 4.1.6. Solve the following unconstrained linear equation system: find x^* in $\mathbb{R}^5$ such that $Ax^* = b$, where

$$A = \begin{pmatrix} 2 & 3 & -1 & 1 & 6 \\ -2 & -4 & 1 & -2 & 5 \\ -1 & -2 & -2 & -5 & 2 \\ 5 & 1 & -3 & 3 & -3 \\ 4 & 2 & 4 & 2 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} -20 \\ 21 \\ 6 \\ -15 \\ 18 \end{pmatrix}.$$

Let $u = (1, 1, 1, 1, 1)^T$ and $x_0 = (-3, 1, 4, -2, 0)^T$. Choose $\lambda = 1$, $\alpha_n = \frac{10^{-5}}{\sqrt{n+1}}$ and $\rho_n = 2$ for all $n \in \mathbb{N}$. Computing Algorithm II iteratively, we obtain the following numerical results.

n	x_n^T	$\|x_{n+1} - x_n\|_2$
1	(-3.00000,1.00000,4.00000,-2.00000,0.00000)	1.906981E+01
50	(0.70266,-1.46469,2.52074,-0.85842,2.42700)	1.678374E-02
100	(0.82396,-1.68393,2.71636,-0.91545,2.25244)	5.873192E-03
150	(0.89597,-1.81311,2.83231,-0.94998,2.14917)	2.054463E-03
200	(0.93856,-1.88954,2.90090,-0.97041,2.08809)	7.183362E-04
250	(0.96375,-1.93475,2.94148,-0.98250,2.05196)	2.510188E-04
300	(0.97865,-1.96149,2.96548,-0.98965,2.03060)	8.764489E-05
350	(0.98746,-1.97732,2.97968,-0.99389,2.01797)	3.056299E-05
400	(0.99268,-1.98669,2.98809,-0.99640,2.01049)	1.063548E-05
450	(0.99577,-1.99226,2.99308,-0.99789,2.00606)	3.687309E-06
500	(0.99761,-1.99558,2.99606,-0.99878,2.00341)	1.269242E-06
550	(0.99874,-1.99762,2.99789,-0.99934,2.00180)	4.299900E-07

Table 2 Numerical results for Algorithm II

From Table 2, the solution of the linear equation system is $(1, -2, 3, -1, 2)^T$.

4.2 The Modified Inertial Relaxed CQ Algorithm for Solving the Split Feasibility Problems

4.2.1 Algorithms and Convergence Theorem

We propose the modified inertial relaxed CQ algorithm as follows:

Algorithm 3.1 Let $f : H_1 \to H_1$ be a contraction (*i.e.* there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$ for all $x, y \in H_1$) and let $\{\alpha_n\} \subset (0,1)$, $\{\theta_n\} \subset [0,1)$ and $\{\rho_n\} \subset (0,4)$. Take $x_0, x_1 \in H_1$ arbitrarily and generate the sequences $\{x_n\}$ and $\{y_n\}$ by the following manner:

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= P_{C_n}[\alpha_n f(y_n) + (1-\alpha_n)(y_n - \tau_n \nabla f_n(y_n))], \quad n \geq 1. \end{aligned} \tag{4.93}$$

Here we set

$$\tau_n = \rho_n \frac{f_n(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2}. \tag{4.94}$$

for all $n \in \mathbb{N}$. We remark that if $\nabla f_n(y_n) = \nabla g_n(y_n) = 0$, then y_n is a solution of SFP.

We next prove the strong convergence of the sequence generated by the proposed algorithm.

Theorem 4.2.1. *Assume that* $\{\alpha_n\} \subset (0,1)$, $\{\rho_n\} \subset (0,4)$ *and* $\{\theta_n\} \subset [0,\theta)$, *where* $\theta \in [0,1)$ *satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $\inf_{n\in\mathbb{N}} \rho_n(4-\rho_n) > 0$*;*

(C3) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$

Then the sequence $\{x_n\}$ *generated by Algorithm 3.1 converges strongly to a solution in SFP.*

Proof. Let $z = P_S f(z)$. Then $z \in C \subset C_n$ and $Az \in Q \subset Q_n$ for all $n \in \mathbb{N}$. It means $z = P_{C_n}z$ and $Az = P_{Q_n}Az$ for all $n \in \mathbb{N}$. Set $v_n = y_n - \tau_n \nabla f_n(y_n)$ and $w_n = \alpha_n f(y_n) + (1-\alpha_n)v_n$ for all $n \in \mathbb{N}$. Then we obtain

$$\begin{aligned}\|y_n - z\| &= \|x_n - z + \theta_n(x_n - x_{n-1})\| \\ &\leq \|x_n - z\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{4.95}$$

Since $(I - P_{Q_n})$ is firmly nonexpansive,

$$\begin{aligned}\langle \nabla f_n(y_n), y_n - z\rangle &= \langle (I - P_{Q_n})Ay_n, Ay_n - Az\rangle \\ &\geq \|(I - P_{Q_n})Ay_n\|^2 \\ &= 2f_n(y_n). \end{aligned} \tag{4.96}$$

Using (4.94) and (4.96), it follows that

$$\begin{aligned}\|v_n - z\|^2 &= \|y_n - \tau_n\nabla f_n(y_n) - z\|^2 \\ &= \|y_n - z\|^2 + \tau_n^2\|\nabla f_n(y_n)\|^2 - 2\tau_n\langle \nabla f_n(y_n), y_n - z\rangle \\ &\leq \|y_n - z\|^2 + \tau_n^2\|\nabla f_n(y_n)\|^2 - 4\tau_n f_n(y_n) \\ &= \|y_n - z\|^2 + \rho_n^2 \frac{f_n^2(y_n)}{(\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2)^2}\|\nabla f_n(y_n)\|^2 \\ &\quad - 4\rho_n \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} \\ &\leq \|y_n - z\|^2 + \rho_n^2 \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} \\ &\quad - 4\rho_n \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} \\ &= \|y_n - z\|^2 - \rho_n(4 - \rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2}. \end{aligned} \tag{4.97}$$

So, since $\rho_n \in (0, 4)$, we have for all $n \in \mathbb{N}$,

$$\|v_n - z\| \leq \|y_n - z\|. \tag{4.98}$$

Thus, using (4.98) and the nonexpansiveness of P_{C_n}, we obtain

$$\begin{aligned}\|x_{n+1} - z\| &= \|P_{C_n}w_n - P_{C_n}z\| \\ &\leq \|w_n - z\| \\ &= \|\alpha_n(f(y_n) - f(z)) + \alpha_n(f(z) - z) + (1-\alpha_n)(v_n - z)\| \end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\alpha\|y_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|v_n - z\| \\
&\leq \alpha_n\alpha\|y_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|y_n - z\| \\
&= (1-\alpha_n(1-\alpha))\|y_n - z\| + \alpha_n\|f(z) - z\|. \qquad (4.99)
\end{aligned}
$$

Combining (4.95) and (4.99), we immediately obtain

$$
\|x_{n+1} - z\| \leq (1-\alpha_n(1-\alpha))\|x_n - z\| + (1-\alpha_n(1-\alpha))\theta_n\|x_n - x_{n-1}\| + \alpha_n\|f(z) - z\|. \qquad (4.100)
$$

By conditions (C1) and (C3), we see that

$$
\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \left(\frac{1-\alpha_n(1-\alpha)}{1-\alpha}\right)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0, \qquad (4.101)
$$

which implies that the sequence $\{\sigma_n\}$ is bounded. Putting

$$
M = \max\left\{\frac{\|f(z) - z\|}{1-\alpha}, \sup_{n\in\mathbb{N}}\sigma_n\right\}
$$

and using Lemma 3.1.52 (i), we conclude that the sequence $\{\|x_n - z\|\}$ is bounded. This shows that the sequence $\{x_n\}$ is bounded and so is $\{y_n\}$. On the other hand, we see that

$$
\begin{aligned}
\|y_n - z\|^2 &= \|x_n - z + \theta_n(x_n - x_{n-1})\|^2 \\
&= \|x_n - z\|^2 + 2\theta_n\langle x_n - z, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2 \qquad (4.102)
\end{aligned}
$$

and, from (3.54)

$$
\langle x_n - z, x_n - x_{n-1}\rangle = -\frac{1}{2}\|x_{n-1} - z\|^2 + \frac{1}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_n - x_{n-1}\|^2. \qquad (4.103)
$$

Combining (4.102) and (4.103), we obtain, since $\theta_n \in [0,1)$,

$$
\begin{aligned}
\|y_n - z\|^2 &= \|x_n - z\|^2 + \theta_n(-\|x_{n-1} - z\|^2 + \|x_n - z\|^2 + \|x_n - x_{n-1}\|^2) \\
&\quad +\theta_n^2\|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\
&\quad +2\theta_n\|x_n - x_{n-1}\|^2. \qquad (4.104)
\end{aligned}
$$

Using (3.55), (4.97) and the firm nonexpansiveness of P_{C_n}, we also have

$$
\|x_{n+1} - z\|^2 = \|P_{C_n}w_n - P_{C_n}z\|^2
$$

$$
\begin{aligned}
&\leq \|w_n - z\|^2 - \|P_{C_n} w_n - w_n\|^2 \\
&= \|\alpha_n(f(y_n) - z) + (1-\alpha_n)(v_n - z)\|^2 - \|P_{C_n} w_n - w_n\|^2 \\
&\leq (1-\alpha_n)\|v_n - z\|^2 + 2\alpha_n\langle f(y_n) - z, w_n - z\rangle - \|P_{C_n} w_n - w_n\|^2 \\
&\leq (1-\alpha_n)\|y_n - z\|^2 - (1-\alpha_n)\rho_n(4-\rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} \\
&\quad + 2\alpha_n\langle f(y_n) - z, w_n - z\rangle - \|P_{C_n} w_n - w_n\|^2. \qquad (4.105)
\end{aligned}
$$

Combining (4.104) and (4.105), we thus have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 \leq\ & (1-\alpha_n)\|x_n - z\|^2 + (1-\alpha_n)\theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\
& + 2(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2 \\
& -(1-\alpha_n)\rho_n(4-\rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} \\
& + 2\alpha_n\langle f(y_n) - z, w_n - z\rangle - \|P_{C_n} w_n - w_n\|^2. \qquad (4.106)
\end{aligned}
$$

Set $\Gamma_n = \|x_n - z\|^2$ for all $n \in \mathbb{N}$. We next consider the following two cases.

Case 1: Suppose that there exists a natural number N such that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq N$. In this case, $\{\Gamma_n\}$ is convergent. From (C1) and (C2), we can find a constant σ such that $(1-\alpha_n)\rho_n(4-\rho_n) \geq \sigma > 0$ for all $n \in \mathbb{N}$. So (4.106) reduces to

$$
\begin{aligned}
\Gamma_{n+1} \leq\ & (1-\alpha_n)\Gamma_n + (1-\alpha_n)\theta_n(\Gamma_n - \Gamma_{n-1}) + 2(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2 \\
& - \sigma\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} + 2\alpha_n\langle f(y_n) - z, w_n - z\rangle \\
& - \|P_{C_n} w_n - w_n\|^2, \qquad (4.107)
\end{aligned}
$$

which gives

$$
\begin{aligned}
\sigma\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} \leq\ & (\Gamma_n - \Gamma_{n+1}) + (1-\alpha_n)\theta_n(\Gamma_n - \Gamma_{n-1}) \\
& + 2(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2 \\
& + 2\alpha_n\langle f(y_n) - z, w_n - z\rangle. \qquad (4.108)
\end{aligned}
$$

It is easy to see that (C3) implies $\theta_n\|x_n - x_{n-1}\| \to 0$ since $\{\alpha_n\}$ is bounded. Since $\{\Gamma_n\}$ converges and $\alpha_n \to 0$,

$$
\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} \to 0 \qquad (4.109)
$$

as $n \to \infty$. It is easily checked that $\{\nabla g_n(y_n)\}$ is bounded. Also, we have $\{\nabla f_n(y_n)\}$ is bounded since $\{y_n\}$ is bounded. Indeed, by Lemma 3.1.51 (iv), we have

$$\|\nabla f_n(y_n)\| = \|\nabla f_n(y_n) - \nabla f_n(z)\| \leq \|A\|^2 \|y_n - z\|. \tag{4.110}$$

So from (4.109), we conclude that $f_n(y_n) \to 0$ as $n \to \infty$, *i.e.*,

$$\|(I - P_{Q_n})Ay_n\| \to 0, \tag{4.111}$$

as $n \to \infty$. Since ∂q is bounded on bounded sets, there exists a constant $\mu > 0$ such that $\|\varepsilon_n\| \leq \mu$ for all $n \in \mathbb{N}$. From (4.111) and $P_{Q_n}(Ay_n) \in Q_n$, we have

$$\begin{aligned} q(Ay_n) &\leq \langle \varepsilon_n, Ay_n - P_{Q_n}(Ay_n)\rangle \\ &\leq \mu\|(I - P_{Q_n})Ay_n\| \\ &\to 0, \end{aligned} \tag{4.112}$$

as $n \to \infty$. Since $\{y_n\}$ is bounded, there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \rightharpoonup x^* \in H_1$. It also follows that $Ay_{n_k} \rightharpoonup Ax^* \in H_2$. By the lower-semicontinuity of q, we have

$$q(Ax^*) \leq \liminf_{k\to\infty} q(Ay_{n_k}) \leq 0. \tag{4.113}$$

This shows that $Ax^* \in Q$. We next prove that $x^* \in C$. Again, using (4.107), we have

$$\begin{aligned} (1-\alpha_n)\|P_{C_n}w_n - w_n\|^2 &\leq (\Gamma_n - \Gamma_{n+1}) + (1-\alpha_n)\theta_n(\Gamma_n - \Gamma_{n-1}) \\ &\quad + 2(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2 \\ &\quad + 2\alpha_n\langle f(y_n) - z, w_n - z\rangle, \end{aligned} \tag{4.114}$$

consequently, as $n \to \infty$,

$$\|P_{C_n}w_n - w_n\| \to 0. \tag{4.115}$$

By the definition of C_n, we obtain

$$c(w_n) \leq \langle \xi_n, w_n - P_{C_n}w_n\rangle \leq \kappa\|w_n - P_{C_n}w_n\| \to 0, \tag{4.116}$$

as $n \to \infty$, where κ is a constant such that $\|\xi_n\| \leq \kappa$ for all $n \in \mathbb{N}$. We next consider the following estimation:

$$\begin{aligned}
\|v_n - y_n\| &= \|y_n - \tau_n \nabla f_n(y_n) - y_n\| \\
&= \tau_n \|\nabla f_n(y_n)\| \\
&= \rho_n \frac{f_n(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} \|\nabla f_n(y_n)\| \\
&\to 0, \qquad (4.117)
\end{aligned}$$

as $n \to \infty$. We also have

$$\|w_n - y_n\| \leq \alpha_n \|f(y_n) - y_n\| + (1 - \alpha_n)\|v_n - y_n\| \to 0, \qquad (4.118)$$

as $n \to \infty$. Hence, since $y_{n_k} \rightharpoonup x^*$, there is a corresponding subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \rightharpoonup x^*$. From (4.116), it follows that

$$c(x^*) \leq \liminf_{k \to \infty} c(w_{n_k}) = 0. \qquad (4.119)$$

So we obtain $x^* \in C$ and hence $x^* \in S$. From (3.53) we obtain

$$\begin{aligned}
\limsup_{n \to \infty} \langle f(z) - z, w_n - z\rangle &= \lim_{k \to \infty} \langle f(z) - z, w_{n_k} - z\rangle \\
&= \langle f(z) - z, x^* - z\rangle \\
&\leq 0. \qquad (4.120)
\end{aligned}$$

On the other hand, we see that

$$\begin{aligned}
\|w_n - z\|^2 &= \langle w_n - z, w_n - z\rangle \\
&= \alpha_n \langle f(y_n) - f(z), w_n - z\rangle + \alpha_n \langle f(z) - z, w_n - z\rangle \\
&\quad + (1 - \alpha_n)\langle v_n - z, w_n - z\rangle \\
&\leq \alpha_n \alpha \|y_n - z\|\|w_n - z\| + \alpha_n \langle f(z) - z, w_n - z\rangle \\
&\quad + (1 - \alpha_n)\|v_n - z\|\|w_n - z\| \\
&\leq (1 - \alpha_n(1 - \alpha))\|y_n - z\|\|w_n - z\| + \alpha_n \langle f(z) - z, w_n - z\rangle \\
&\leq (1 - \alpha_n(1 - \alpha))\Big(\frac{\|y_n - z\|^2}{2} + \frac{\|w_n - z\|^2}{2}\Big) \\
&\quad + \alpha_n \langle f(z) - z, w_n - z\rangle, \qquad (4.121)
\end{aligned}$$

which gives

$$\begin{aligned}
\|w_n - z\|^2 &\leq \frac{1-\alpha_n(1-\alpha)}{1+\alpha_n(1-\alpha)}\|y_n - z\|^2 \\
&\quad + \frac{2\alpha_n}{1+\alpha_n(1-\alpha)}\langle f(z) - z, w_n - z\rangle \\
&\leq \frac{1-\alpha_n(1-\alpha)}{1+\alpha_n(1-\alpha)}(\|x_n - z\| + \theta_n\|x_n - x_{n-1}\|)^2 \\
&\quad + \frac{2\alpha_n}{1+\alpha_n(1-\alpha)}\langle f(z) - z, w_n - z\rangle \\
&= \left(1 - \frac{2\alpha_n(1-\alpha)}{1+\alpha_n(1-\alpha)}\right)(\|x_n - z\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - z\| \\
&\quad + \theta_n^2\|x_n - x_{n-1}\|^2) + \frac{2\alpha_n}{1+\alpha_n(1-\alpha)}\langle f(z) - z, w_n - z\rangle. \quad (4.122)
\end{aligned}$$

Then, by (4.122), we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|P_{C_n} w_n - z\|^2 \\
&\leq \|w_n - z\|^2 \\
&\leq \left(1 - \frac{2\alpha_n(1-\alpha)}{1+\alpha_n(1-\alpha)}\right)(\|x_n - z\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - z\| \\
&\quad + \theta_n^2\|x_n - x_{n-1}\|^2) + \frac{2\alpha_n}{1+\alpha_n(1-\alpha)}\langle f(z) - z, w_n - z\rangle. \quad (4.123)
\end{aligned}$$

Put $M_1 = \sup_{n\in\mathbb{N}}\|x_n - z\|$ and $\gamma_n = \frac{2\alpha_n(1-\alpha)}{1+\alpha_n(1-\alpha)}$ for all $n \in \mathbb{N}$. It is easily checked that $\gamma_n \in (0,1)$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}\gamma_n = \infty$. From (4.123), it follows that

$$\begin{aligned}
\Gamma_{n+1} &\leq (1-\gamma_n)\Gamma_n + 2\theta_n\|x_n - x_{n-1}\|M_1 + \theta_n\|x_n - x_{n-1}\|^2 \\
&\quad + \frac{2\alpha_n}{1+\alpha_n(1-\alpha)}\langle f(z) - z, w_n - z\rangle. \quad (4.124)
\end{aligned}$$

Applying Lemma 3.1.52 (ii) and using (4.120) and the conditions (C1) and (C3), we conclude that $\Gamma_n = \|x_n - z\|^2 \to 0$ and thus $x_n \to z$ as $n \to \infty$.

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\psi : \mathbb{N} \to \mathbb{N}$ as in (3.60). Then, by Lemma 3.1.54, we have $\Gamma_{\psi(n)} \leq \Gamma_{\psi(n)+1}$. From (4.106), it follows that

$$\begin{aligned}
\Gamma_{\psi(n)+1} &\leq (1-\alpha_{\psi(n)})\Gamma_{\psi(n)} + (1-\alpha_{\psi(n)})\theta_{\psi(n)}(\Gamma_{\psi(n)} - \Gamma_{\psi(n)-1}) \\
&\quad + 2(1-\alpha_{\psi(n)})\theta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|^2
\end{aligned}$$

$$-\sigma\frac{f^2{}_{\psi(n)}(y_{\psi(n)})}{\|\nabla f_{\psi(n)}(y_{\psi(n)})\|^2+\|\nabla g_{\psi(n)}(y_{\psi(n)})\|^2}$$
$$-(1-\alpha_{\psi(n)})\|P_{C_{\psi(n)}}w_{\psi(n)}-w_{\psi(n)}\|^2$$
$$+2\alpha_{\psi(n)}\langle f(y_{\psi(n)})-z,w_{\psi(n)}-z\rangle, \quad (4.125)$$

which gives

$$\begin{aligned}\sigma\frac{f^2_{\psi(n)}(y_{\psi(n)})}{\|\nabla f_{\psi(n)}(y_{\psi(n)})\|^2+\|\nabla g_{\psi(n)}(y_{\psi(n)})\|^2} &\le (1-\alpha_{\psi(n)})\theta_{\psi(n)}(\Gamma_{\psi(n)}-\Gamma_{\psi(n)-1})\\ &\quad+2(1-\alpha_{\psi(n)})\theta_{\psi(n)}\|x_{\psi(n)}-x_{\psi(n)-1}\|^2\\ &\quad+2\alpha_{\psi(n)}\langle f(y_{\psi(n)})-z,w_{\psi(n)}-z\rangle\\ &\le (1-\alpha_{\psi(n)})\theta_{\psi(n)}\|x_{\psi(n)}-x_{\psi(n)-1}\|\\ &\quad(\sqrt{\Gamma_{\psi(n)}}+\sqrt{\Gamma_{\psi(n)-1}})\\ &\quad+2(1-\alpha_{\psi(n)})\theta_{\psi(n)}\|x_{\psi(n)}-x_{\psi(n)-1}\|^2\\ &\quad+2\alpha_{\psi(n)}\langle f(y_{\psi(n)})-z,w_{\psi(n)}-z\rangle\\ &\to 0, \quad (4.126)\end{aligned}$$

as $n\to\infty$. It follows that $f_{\psi(n)}(y_{\psi(n)})=\|(I-P_{Q_{\psi(n)}})Ay_{\psi(n)}\|\to 0$. Similarly, by (4.125), we can show that

$$\lim_{n\to\infty}\|P_{C_{\psi(n)}}w_{\psi(n)}-w_{\psi(n)}\|=0 \quad (4.127)$$

and by (4.118)

$$\lim_{n\to\infty}\|w_{\psi(n)}-y_{\psi(n)}\|=0. \quad (4.128)$$

Now repeating the argument of the proof in Case 1, we obtain

$$\limsup_{n\to\infty}\langle f(z)-z,w_{\psi(n)}-z\rangle\le 0. \quad (4.129)$$

On the other hand, observe that

$$\|y_{\psi(n)}-x_{\psi(n)}\|=\theta_{\psi(n)}\|x_{\psi(n)}-x_{\psi(n)-1}\|\to 0, \quad (4.130)$$

as $n\to\infty$. It follows that $\|x_{\psi(n)+1}-x_{\psi(n)}\|\to 0$ as $n\to\infty$. Indeed, by (4.127), (4.128) and (4.130), we have

$$\|x_{\psi(n)+1}-x_{\psi(n)}\| = \|P_{C_{\psi(n)}}w_{\psi(n)}-x_{\psi(n)}\|$$

$$
\begin{aligned}
&\leq \|P_{C_{\psi(n)}} w_{\psi(n)} - w_{\psi(n)}\| + \|w_{\psi(n)} - y_{\psi(n)}\| + \|y_{\psi(n)} - x_{\psi(n)}\| \\
&\to 0, \qquad (4.131)
\end{aligned}
$$

as $n \to \infty$. Using (4.124), we have

$$
\begin{aligned}
\Gamma_{\psi(n)+1} &\leq (1-\gamma_{\psi(n)})\Gamma_{\psi(n)} + 2\theta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|M_1 + \theta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|^2 \\
&\quad + \frac{2\alpha_{\psi(n)}}{1+\alpha_{\psi(n)}(1-\alpha)}\langle f(z) - z, w_{\psi(n)} - z\rangle, \qquad (4.132)
\end{aligned}
$$

which implies

$$
\begin{aligned}
\gamma_{\psi(n)}\Gamma_{\psi(n)} &\leq 2\theta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|M_1 + \theta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|^2 \\
&\quad + \frac{2\alpha_{\psi(n)}}{1+\alpha_{\psi(n)}(1-\alpha)}\langle f(z) - z, w_{\psi(n)} - z\rangle. \qquad (4.133)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Gamma_{\psi(n)} &\leq \frac{2\theta_{\psi(n)}}{\gamma_{\psi(n)}}\|x_{\psi(n)} - x_{\psi(n)-1}\|M_1 + \frac{\theta_{\psi(n)}}{\gamma_{\psi(n)}}\|x_{\psi(n)} - x_{\psi(n)-1}\|^2 \\
&\quad + \frac{1}{1-\alpha}\langle f(z) - z, w_{\psi(n)} - z\rangle. \qquad (4.134)
\end{aligned}
$$

Hence from (C3), (4.129) and (4.131), we obtain

$$
\limsup_{n\to\infty} \Gamma_{\psi(n)} \leq 0. \qquad (4.135)
$$

This means $\lim_{n\to\infty} \Gamma_{\psi(n)} = \lim_{n\to\infty} \|x_{\psi(n)} - z\|^2 = 0$. So we have $x_{\psi(n)} \to z$ as $n \to \infty$. On the other hand, we see that

$$
\begin{aligned}
\|x_{\psi(n)+1} - z\| &\leq \|x_{\psi(n)+1} - x_{\psi(n)}\| + \|x_{\psi(n)} - z\| \\
&\to 0, \qquad (4.136)
\end{aligned}
$$

as $n \to \infty$. By Lemma 3.1.54, we have $\Gamma_n \leq \Gamma_{\psi(n)+1}$ and thus

$$
\Gamma_n = \|x_n - z\|^2 \leq \|x_{\psi(n)+1} - z\|^2 \to 0. \qquad (4.137)
$$

This concludes that $x_n \to z$ as $n \to \infty$. We thus complete the proof. □

Remark 4.2.2. We remark here that the condition (C3) is easily implemented in numerical computation since the valued of $\|x_n - x_{n-1}\|$ is known before choosing

θ_n. Indeed, the parameter θ_n can be chosen such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{\omega_n}{\|x_n - x_{n-1}\|}, \theta\right\} & if \ x_n \neq x_{n-1}, \\ \theta & otherwise, \end{cases}$$

where $\{\omega_n\}$ is a positive sequence such that $\omega_n = o(\alpha_n)$.

We next consider the case when the norm of operators can be easily calculated.

Algorithm 3.2 Take $x_0, x_1 \in H_1$ and generate the sequence $\{x_n\}$ by the following manner:

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= P_{C_n}[\alpha_n f(y_n) + (1-\alpha_n)(y_n - \lambda_n \nabla f_n(y_n))], \end{aligned} \tag{4.138}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\theta_n\} \subset [0,1)$ and $\{\lambda_n\} \subset (0,\infty)$.

Theorem 4.2.3. *Assume that $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,\infty)$ and $\{\theta_n\} \subset [0,\theta)$, where $\theta \in [0,1)$ satisfy the following conditions:*

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\inf\limits_{n\in\mathbb{N}} \lambda_n(2 - \lambda_n\|A\|^2) > 0$;

(C3) $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to the solution of SFP.

Proof. Since the proof line is closed to that of Theorem 4.3.1, we just give a sketch proof. Let $z = P_S f(z)$. Set $v_n = y_n - \lambda_n \nabla f_n(y_n)$ and $w_n = \alpha_n f(y_n) + (1-\alpha_n)v_n$ for all $n \in \mathbb{N}$. We first show that the sequence $\{x_n\}$ is bounded. To this end, it suffices to show that $\|v_n - z\| \leq \|y_n - z\|$ for all $n \in \mathbb{N}$. By using the argument as in Theorem 4.3.1, we can show that $\langle \nabla f_n(y_n), y_n - z\rangle \geq 2 f_n(y_n)$. It follows that

$$\begin{aligned} \|v_n - z\|^2 &= \|y_n - z\|^2 + \lambda_n^2\|\nabla f_n(y_n)\|^2 - 2\lambda_n\langle \nabla f_n(y_n), y_n - z\rangle \\ &\leq \|y_n - z\|^2 + \lambda_n^2\|\nabla f_n(y_n)\|^2 - 4\lambda_n f_n(y_n) \end{aligned}$$

$$
\begin{aligned}
&\le \|y_n - z\|^2 + \lambda_n^2\|A\|^2\|(I-P_{Q_n})Ay_n\|^2 - 4\lambda_n f_n(y_n)\\
&= \|y_n - z\|^2 + 2\lambda_n^2\|A\|^2 f_n(y_n) - 4\lambda_n f_n(y_n)\\
&\le \|y_n - z\|^2 - 2\lambda_n(2-\lambda_n\|A\|^2) f_n(y_n). \qquad (4.139)
\end{aligned}
$$

From (C2), we have $\|v_n - z\| \le \|y_n - z\|$ for all $n \in \mathbb{N}$. By (4.105) and (4.139), we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 \le\ & (1-\alpha_n)\|x_n - z\|^2 + (1-\alpha_n)\theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2)\\
& + 2(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2 - (1-\alpha_n)\lambda_n(2-\lambda_n\|A\|^2) f_n(y_n)\\
& + 2\alpha_n\langle f(y_n) - z, w_n - z\rangle - \|P_{C_n}w_n - w_n\|^2. \qquad (4.140)
\end{aligned}
$$

Set $\Gamma_n = \|x_n - z\|^2$ for all $n \in \mathbb{N}$. We next consider the following two cases.

Case 1: Suppose that there exists a natural number N such that $\Gamma_{n+1} \le \Gamma_n$ for all $n \ge N$. In this case, $\{\Gamma_n\}$ is convergent. From (C1) and (C2), we can find a constant σ such that $(1-\alpha_n)\lambda_n(2-\lambda_n\|A\|^2) \ge \sigma > 0$ for all $n \in \mathbb{N}$. So we obtain

$$
\begin{aligned}
\Gamma_{n+1} \le\ & (1-\alpha_n)\Gamma_n + (1-\alpha_n)\theta_n(\Gamma_n - \Gamma_{n-1}) + 2(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2\\
& - \sigma f_n(y_n) + 2\alpha_n\langle f(y_n) - z, w_n - z\rangle - \|P_{C_n}w_n - w_n\|^2, \qquad (4.141)
\end{aligned}
$$

which implies

$$
\begin{aligned}
\sigma f_n(y_n) \le\ & (\Gamma_n - \Gamma_{n+1}) + (1-\alpha_n)\theta_n(\Gamma_n - \Gamma_{n-1}) + 2(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2\\
& + 2\alpha_n\langle f(y_n) - z, w_n - z\rangle. \qquad (4.142)
\end{aligned}
$$

This shows, by (C1) and (C3), that $f_n(y_n) = \|(I-P_{Q_n})Ay_n\| \to 0$ as $n \to \infty$. Similarly, we can show that $\|P_{C_n}w_n - w_n\| \to 0$ $n \to \infty$. Following the proof line as in Theorem 4.3.1, we can prove that $\{x_n\}$ converges strongly to z.

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. This case can be done by a similar argument as in Case 1. So we omit the rest of proof. We thus complete the proof. $\square$

4.2.2 Numerical examples

In this section, we provide some numerical examples and illustrate its performance by using the modified inertial relaxed CQ method (Algorithm 3.1).

Example 4.2.4. *Let* $H_1 = H_2 = \mathbb{R}^3$, $C = \{x = (a,b,c)^T \in \mathbb{R}^3 : a^2 + b^2 - 5 \leq 0\}$ *and* $Q = \{y = (p,q,r)^T \in \mathbb{R}^3 : p + r^2 - 2 \leq 0\}$. *Let* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *be defined by* $f(x) = \frac{x}{2}$. *Find* $x^* \in C$ *such that* $Ax^* \in Q$, *where* $A = \begin{pmatrix} 1 & 2 & 7 \\ 1 & 3 & 0 \\ 4 & 1 & 2 \end{pmatrix}$.

Choose $\alpha_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$ and $\theta = 0.5$. For each $n \in \mathbb{N}$, let $\omega_n = \frac{1}{(n+1)^3}$ and define θ_n as in Remark 4.3.2. We now study the effect (in terms of convergence, stability, number of iterations required and the cpu time) of the sequence $\{\rho_n\} \subset (0,4)$ on the iterative scheme by choosing different ρ_n such that $\inf_n \rho_n(4 - \rho_n) > 0$ in the following cases.

Case 1: $\rho_n = \frac{n}{2n+1}$;

Case 2: $\rho_n = \frac{n}{n+1}$;

Case 3: $\rho_n = \frac{2n}{n+1}$;

Case 4: $\rho_n = \frac{3n}{n+1}$.

The stopping criterion is defined by

$$E_n = \frac{1}{2}\|x_n - P_{C_n}x_n\|^2 + \frac{1}{2}\|Ax_n - P_{Q_n}Ax_n\|^2 < 10^{-4}.$$

We consider different choices of x_0 and x_1 as

Choice 1: $x_0 = (-7,-2,-6)^T$ and $x_1 = (-2,2,-6)^T$;

Choice 2: $x_0 = (1,2,-5)^T$ and $x_1 = (0,1,-7)^T$;

Choice 3: $x_0 = (1,5,-1)^T$ and $x_1 = (-3,4,-7)^T$;

Choice 4: $x_0 = (1,5,2)^T$ and $x_1 = (3,2,7)^T$

The numerical experiments for each case of ρ_n are shown in Figure 1-4, respectively.

Table 1: Algorithm 3.1 with different cases of ρ_n and different choices of x_0 and x_1

		Case 1	Case 2	Case 3	Case 4
Choice 1	No. of Iter.	12	8	5	4
	cpu (Time)	0.003553	0.002377	0.002195	0.002075
Choice 2	No. of Iter.	7	6	4	4
	cpu (Time)	0.002799	0.002769	0.002357	0.002184
Choice 3	No. of Iter.	12	9	6	4
	cpu (Time)	0.003828	0.002602	0.002401	0.002142
Choice 4	No. of Iter.	27	17	11	9
	cpu (Time)	0.007181	0.00343	0.002612	0.002431

Figure 1: Comparison of the itertions for Choice 1 in Example 4.1

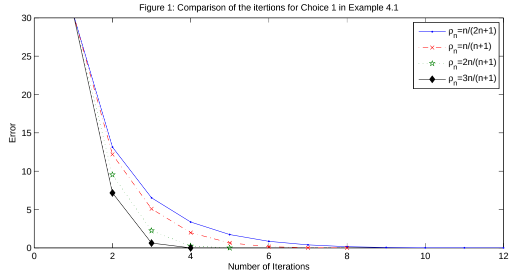


Figure 2: Comparison of the itertions for Choice 2 in Example 4.1

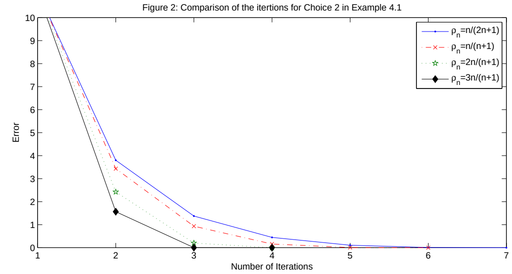


Figure 3: Comparison of the itertions for Choice 3 in Example 4.1

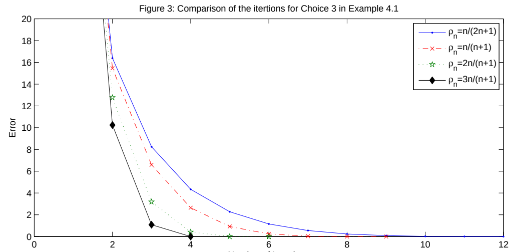
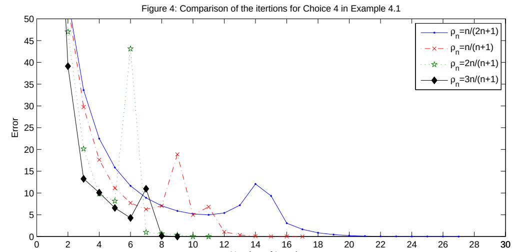
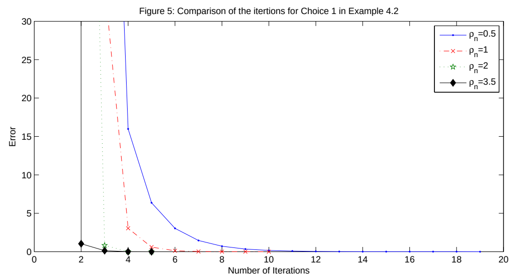
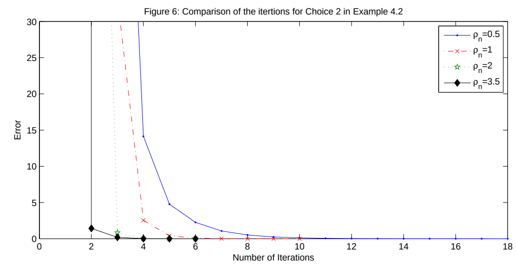


Figure 4: Comparison of the itertions for Choice 4 in Example 4.1







Example 4.2.5. *Let* $H_1 = H_2 = \mathbb{R}^5$, $C = \{x = (a,b,c,d,e)^T \in \mathbb{R}^5 : a^2 + b^2 + c^2 + d^2 + e^2 - 0.4 \leq 0\}$ *and* $Q = \{y = (p,q,r,s,t)^T \in \mathbb{R}^5 : p + q + r + s - 0.75 \leq 0\}$. *Let* $f : \mathbb{R}^5 \to \mathbb{R}^5$ *be defined by* $f(x) = \frac{x}{2}$. *Find* $x^* \in C$ *such that* $Ax^* \in Q$, *where*

$$A = \begin{pmatrix} 3 & -2 & 5 & -2 & 3 \\ 2 & -2 & 5 & -2 & 9 \\ 2 & -3 & 5 & -1 & -3 \\ -2 & -2 & 8 & -7 & -2 \end{pmatrix}.$$

Let α_n, θ_n and E_n be as in Example 4.2.4. We choose different cases of ρ_n as follows:

Case 1: $\rho_n = 0.5$;

Case 2: $\rho_n = 1$;

Case 3: $\rho_n = 2$;

Case 4: $\rho_n = 3.5$.

The different choices of x_0 and x_1 are given as follows:

Choice 1: $x_0 = (-3.2, -1, -2.5, 5, -3.7)^T$ and $x_1 = (-2.3, -1.5, 5.2, -7.5, 7.3)^T$;

Choice 2: $x_0 = (-2, -5, -3, 2, -3)^T$ and $x_1 = (-5, -4, 5, -7, 7)^T$;

Choice 3: $x_0 = (3, 8, 5, -2, 8)^T$ and $x_1 = (-2, -5, 5, -9, 9)^T$;

Choice 4: $x_0 = (4.5, 0, -2.5, 1, 3)^T$ and $x_1 = (-3.6, -4.2, 1, 1.5, 8)^T$.

The numerical experiments are shown in Figure 5-8, respectively.

Remark 4.2.6.

We finally make the following conclusions from the numerical experiments in Examples 4.2.4 and 4.2.5.

Table 2: Algorithm 3.1 with different cases of ρ_n and different choices of x_0 and x_1

		Case 1	Case 2	Case 3	Case 4
Choice 1	No. of Iter.	19	10	5	5
	cpu (Time)	0.005632	0.003408	0.003223	0.002791
Choice 2	No. of Iter.	18	10	6	6
	cpu (Time)	0.00391	0.002683	0.002447	0.002381
Choice 3	No. of Iter.	19	10	6	6
	cpu (Time)	0.004233	0.003016	0.002601	0.002575
Choice 4	No. of Iter.	13	7	6	6
	cpu (Time)	0.004812	0.003559	0.002922	0.002412

Figure 7: Comparison of the itertions for Choice 3 in Example 4.2

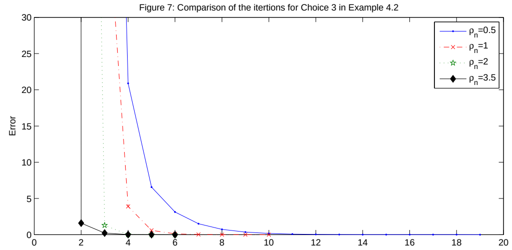
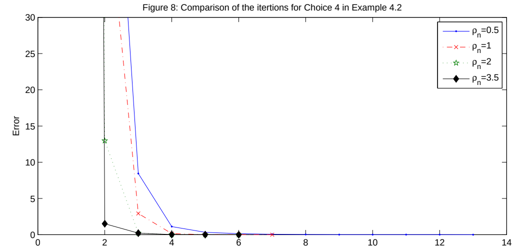


Figure 8: Comparison of the itertions for Choice 4 in Example 4.2



1. For each different Cases and different Choices, it is shown that Algorithm 3.1 has a good convergence speed. Indeed, we see that it is fast, stable and required small number of iterations for seeking solutions.

2. It is observed that the number of iterations and the cpu run time are significantly decreasing starting from Case 1 to Case 4. However, there is no significant difference in both cpu run time and number of iterations for each choice of x_0 and x_1. So, initial guess does not have any significant effect on the convergence of the algorithm.

3. The conditions in Theorem 4.3.1 are easily implemented in numerical computations and need no estimation on the spectral radius of $A^T A$.

4. The restriction of metric projections onto C and Q is relaxed by using those of C_n and Q_n which have specific forms.

We finally end this section by providing a comparison of convergence of Algorithm 3.1 with Halpern-relaxed CQ algorithm (2.23) defined by He and Zhao [21] through Examples 4.2.4 and 4.2.5. For the convenience, let us denote Algorithm 3.1 and Algorithm (2.23) by MIner-R-Iter and H-R-Iter, respectively. Let the contraction f be defined by $f(x) = 0.5x$. Set $\alpha_n = \frac{1}{n+1}$, $\rho_n = \frac{3n}{n+1}$ and $\omega_n = \frac{1}{(n+1)^3}$ for all $n \in \mathbb{N}$. Set $\beta = 0.5$ and $\beta_n = \bar{\beta}_n$ as in Remark 4.3.2. The stopping criterion E_n is defined as in Example 4.2.4. For points u, x_0 and x_1 picked randomly, we obtain the following numerical results.

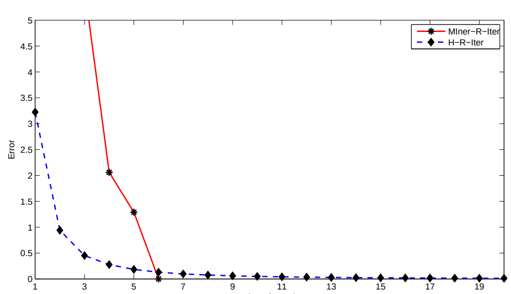
The error plotting of E_n of MIner-R-Iter and H-R-Iter for each choice in Table 3 is shown in the following figures, respectively.

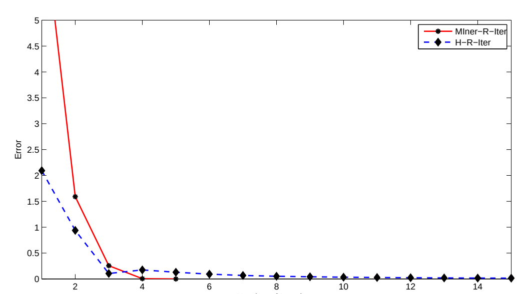
Error plotting of E_n for Table 3

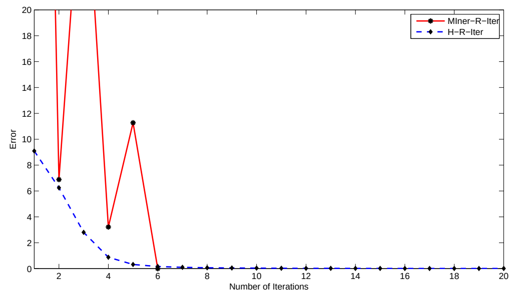
Remark 4.2.7. In numerical experiment, it is revealed that the sequence generated by MIner-R-Iter involving the viscosity term and the inertial technique converges more quickly than by H-R-Iter of He and Zhao [21] does.

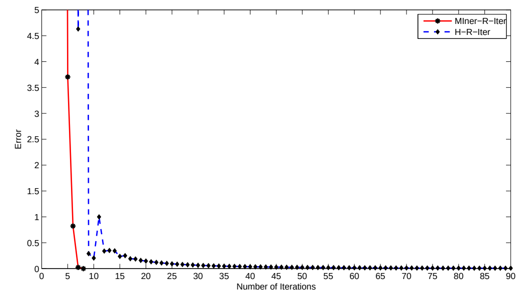
Table 3: Comparison of MIner-R-Iter and H-R-Iter in Example 4.2.4

			MIner-R-Iter	H-R-Iter
Choice 1	$u = (0, -1, -5)^T$	No. of Iter.	6	223
	$x_0 = (2, 6, -3)^T$	cpu (Time)	0.001384	0.064889
	$x_1 = (-2, -1, 8)^T$			
Choice 2	$u = (1, -2, 1)^T$	No. of Iter.	5	181
	$x_0 = (-3, -4, -1)^T$	cpu (Time)	0.000836	0.037471
	$x_1 = (-5, 2, -1)^T$			
Choice 3	$u = (5, -3, -1)^T$	No. of Iter.	6	140
	$x_0 = (2, 1, -1)^T$	cpu (Time)	0.000963	0.026824
	$x_1 = (-5, 3, 5)^T$			
Choice 4	$u = (-2, -1, 4)^T$	No. of Iter.	8	763
	$x_0 = (7.35, 1.75, -3.24)^T$	cpu (Time)	0.001311	0.687214
	$x_1 = (-6.34, 0.42, 7.36)^T$			









4.3 Relaxed CQ Algorithms Involving the Inertial Technique for Multiple-sets Split Feasibility Problems

4.3.1 Algorithms and Convergence Theorem

We study the inertial relaxed self-adaptive CQ algorithm in Hilbert spaces for solving MSFP (2.17). Denote by S the solution set of the SFP.

Theorem 4.3.1. *Let H_1 and H_2 be real Hilbert spaces and let $\{C_i\}_{i=1}^t$ and $\{Q_j\}_{j=1}^r$ be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that $\{\alpha_n\}, \{\beta_n\}$ and $\{\rho_n\}$ satisfy the following assumptions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $\inf_{n\in\mathbb{N}} \rho_n(4-\rho_n) > 0$*;*

(C3) $\{\beta_n\} \subset [0,\beta]$*, where* $\beta \in [0,1)$ *and* $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$

Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $P_S u$.

Proof. Set $z = P_S u$. We note that $I - P_{C_i^n}$, $(i = 1, ..., t)$ and $I - P_{Q_j^n}$, $(j = 1, ..., r)$ are firmly nonexpansive and $\nabla f_n(z) = 0$ for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}\langle \nabla f_n(y_n), y_n - z\rangle &= \Big\langle \sum_{i=1}^{t} l_i(y_n - P_{C_i^n}(y_n)) + \sum_{j=1}^{r} \lambda_j A^*(I - P_{Q_j^n})Ay_n, y_n - z\Big\rangle \\ &= \sum_{i=1}^{t} l_i\langle (I - P_{C_i^n})y_n, y_n - z\rangle + \sum_{j=1}^{r} \lambda_j\langle (I - P_{Q_j^n})Ay_n, Ay_n - Az\rangle\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{i=1}^{t} l_i\|(I-P_{C_i^n})y_n\|^2 + \sum_{j=1}^{r}\lambda_j\|(I-P_{Q_j^n})Ay_n\|^2 \\
&= 2f_n(y_n). \qquad (4.143)
\end{aligned}$$

So we have

$$\begin{aligned}
\|y_n - \tau_n\nabla f_n(y_n) - z\|^2 &= \|y_n - z\|^2 + \tau_n^2\|\nabla f_n(y_n)\|^2 - 2\tau_n\langle\nabla f_n(y_n), y_n - z\rangle \\
&\leq \|y_n - z\|^2 + \tau_n^2\|\nabla f_n(y_n)\|^2 - 4\tau_n f_n(y_n) \\
&= \|y_n - z\|^2 - \rho_n^2\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2} - 4\rho_n\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2} \\
&= \|y_n - z\|^2 - \rho_n(4-\rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2}. \qquad (4.144)
\end{aligned}$$

Hence we obtain, for each $n \in \mathbb{N}$, since $\rho_n \in (0,4)$

$$\|y_n - \tau_n\nabla f_n(y_n) - z\| \leq \|y_n - z\|. \qquad (4.145)$$

On the other hand, we also have

$$\begin{aligned}
\|y_n - z\| &= \|x_n - z + \beta_n(x_n - x_{n-1})\| \\
&\leq \|x_n - z\| + \beta_n\|x_n - x_{n-1}\|. \qquad (4.146)
\end{aligned}$$

Combining (4.145) and (4.146), we obtain

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n(u - z) + (1-\alpha_n)(y_n - \tau_n\nabla f_n(y_n) - z)\| \\
&\leq \alpha_n\|u - z\| + (1-\alpha_n)\|y_n - z\| \qquad (4.147) \\
&\leq \alpha_n\|u - z\| + (1-\alpha_n)\|x_n - z\| + (1-\alpha_n)\beta_n\|x_n - x_{n-1}\|.
\end{aligned}$$

By (C3), we see that $\delta_n = \frac{(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|}{\alpha_n} \to 0$. Hence it is bounded. Put

$$M = \max\left\{\|u - z\|, \sup_{n\geq 1}\delta_n\right\}.$$

So (4.147) becomes

$$\|x_{n+1} - z\| \leq (1-\alpha_n)\|x_n - z\| + \alpha_n M. \qquad (4.148)$$

Applying Lemma 3.1.52 (i), we can conclude that $\{x_n\}$ is bounded and also $\{y_n\}$ is bounded. By Lemma 3.1.50, we see that

$$\|\nabla f_n(y_n)\| = \|\nabla f_n(y_n) - \nabla f_n(z)\| \leq L\|y_n - z\|, \qquad (4.149)$$

where $L = \sum_{i=1}^{t} l_i + \|A\|^2 \sum_{j=1}^{r} \lambda_j$. This shows that $\{\nabla f_n(y_n)\}$ is bounded.

We next compute the following estimation:

$$\begin{aligned}\|y_n - z\|^2 &= \|x_n - z + \beta_n(x_n - x_{n-1})\|^2 \\ &= \|x_n - z\|^2 + 2\beta_n\langle x_n - x_{n-1}, x_n - z\rangle + \beta_n^2\|x_n - x_{n-1}\|^2 \end{aligned} \tag{4.150}$$

Using (3.54), we have

$$\langle x_n - x_{n-1}, x_n - z\rangle = -\frac{1}{2}\|x_{n-1} - z\|^2 + \frac{1}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_n - x_{n-1}\|^2. \tag{4.151}$$

Combining (4.150) and (4.151), we obtain

$$\begin{aligned}\|y_n - z\|^2 &= \|x_n - z\|^2 + \beta_n(-\|x_{n-1} - z\|^2 + \|x_n - z\|^2 + \|x_n - x_{n-1}\|^2) \\ &\quad +\beta_n^2\|x_n - x_{n-1}\|^2 \qquad (4.152) \\ &\leq \|x_n - z\|^2 + \beta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + 2\beta_n\|x_n - x_{n-1}\|^2. \end{aligned}$$

Using (3.55) and (4.144), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n(u - z) + (1 - \alpha_n)(y_n - \tau_n\nabla f_n(y_n) - z)\|^2 \\ &\leq (1 - \alpha_n)\|y_n - \tau_n\nabla f_n(y_n) - z\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\leq (1 - \alpha_n)\|y_n - z\|^2 - (1 - \alpha_n)\rho_n(4 - \rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2} \\ &\quad +2\alpha_n\langle u - z, x_{n+1} - z\rangle. \end{aligned} \tag{4.153}$$

Combining (4.152) and (4.153), we derive

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1 - \alpha_n)\|x_n - z\|^2 + (1 - \alpha_n)\beta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\ &\quad + 2(1 - \alpha_n)\beta_n\|x_n - x_{n-1}\|^2 - (1 - \alpha_n)\rho_n(4 - \rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2} \\ &\quad + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \end{aligned} \tag{4.154}$$

Set $\Gamma_n = \|x_n - z\|^2$ for all $n \in \mathbb{N}$. We note, by (C1) and (C2), that there is a constant σ such that $(1 - \alpha_n)\rho_n(4 - \rho_n) \geq \sigma > 0$ for all $n \in \mathbb{N}$. So from (4.154) we get

$$\Gamma_{n+1} \leq (1 - \alpha_n)\Gamma_n + (1 - \alpha_n)\beta_n(\Gamma_n - \Gamma_{n-1}) \tag{4.155}$$

$$+ 2(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|^2 - \sigma\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2} + 2\alpha_n\langle u - z, x_{n+1} - z\rangle.$$

We next consider the following two cases:

Case 1: Suppose that there exists a natural number N such that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq N$. In this case, $\lim_{n\to\infty}\Gamma_n$ exists. From (4.155), we have

$$\begin{aligned} \sigma\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2} \leq\ & (\Gamma_n - \Gamma_{n+1}) + (1-\alpha_n)\beta_n(\Gamma_n - \Gamma_{n-1}) \\ & + 2(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \end{aligned} \tag{4.156}$$

It is easy to check that (C3) implies $\beta_n\|x_n - x_{n-1}\| \to 0$ since $\{\alpha_n\}$ is bounded. So, by (C1) and the boundedness of $\{x_n\}$, we have from (4.156)

$$\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2} \to 0 \quad as \quad n \to \infty.$$

Since $\{\|\nabla f_n(y_n)\|\}$ is bounded, it follows that $f_n(y_n) \to 0$ as $n \to \infty$. This shows that

$$\lim_{n\to\infty}\|(I - P_{C_i^n})y_n\| = 0 \quad (i = 1, 2, ..., t) \tag{4.157}$$

and

$$\lim_{n\to\infty}\|(I - P_{Q_j^n})Ay_n\| = 0 \quad (j = 1, 2, ..., r). \tag{4.158}$$

Since ∂q_j $(j = 1, ..., r)$ are bounded on bounded sets, there exists a constant $\mu > 0$ such that $\|\zeta_j^n\| \leq \mu$ $(j = 1, ..., r)$ for all $n \in \mathbb{N}$. From (4.158) and $P_{Q_j^n}(Ay_n) \in Q_j^n$ $(j = 1, ..., r)$, we obtain

$$q_j(Ay_n) \leq \langle\zeta_j^n, Ay_n - P_{Q_j^n}(Ay_n)\rangle \leq \mu\|(I - P_{Q_j^n})Ay_n\| \to 0, \tag{4.159}$$

as $n \to \infty$. Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \rightharpoonup x^*$. Then $Ay_{n_k} \rightharpoonup Ax^*$. Since q_j is weakly lower semi-continuous,

$$q_j(Ax^*) \leq \liminf_{k\to\infty} q_j(Ay_{n_k}) \leq 0. \tag{4.160}$$

Therefore $Ax^* \in Q_j$ $(j = 1, ..., r)$.

We next show that $x^* \in C_i$ $(i = 1, ..., t)$. By the definition of C_i^n $(i = 1, ..., t)$ and (4.157), we see that

$$c_i(y_n) \leq \langle \xi_i^n, y_n - P_{C_i^n}(y_n)\rangle \leq \delta\|y_n - P_{C_i^n}y_n\| \to 0, \tag{4.161}$$

as $n \to \infty$, where δ is a constant such that $\|\xi_i^n\| \leq \delta$ $(i = 1, ..., t)$ for all $n \in \mathbb{N}$. By the weak lower semi-continuity of c_i $(i = 1, ..., t)$ and $y_{n_k} \rightharpoonup x^*$, we have

$$c_i(x^*) \leq \liminf_{k\to\infty} c_i(y_{n_k}) \leq 0. \tag{4.162}$$

Hence $x^* \in C_i$ $(i = 1, ..., t)$ and consequently, $x^* \in S$. From (3.53), it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - z, y_n - z\rangle &= \lim_{k\to\infty}\langle u - z, y_{n_k} - z\rangle \\ &= \langle u - z, x^* - z\rangle \leq 0. \end{aligned}\tag{4.163}$$

On the other hand, we see that

$$\|y_n - x_n\| = \beta_n\|x_n - x_{n-1}\| \to 0. \tag{4.164}$$

Hence, by (4.163) and (4.164), we obtain

$$\limsup_{n\to\infty}\langle u - z, x_n - z\rangle \leq 0. \tag{4.165}$$

Again from (4.155) we have

$$\begin{aligned}\Gamma_{n+1} &\leq (1-\alpha_n)\Gamma_n + (1-\alpha_n)\beta_n(\Gamma_n - \Gamma_{n-1}) + 2(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|^2 \\ &\quad + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)\Gamma_n + (1-\alpha_n)\beta_n\|x_n - x_{n-1}\|(\sqrt{\Gamma_n} + \sqrt{\Gamma_{n-1}}) \\ &\quad +2(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|^2 + \ 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \end{aligned}\tag{4.166}$$

From (4.165) and conditions (C1) and (C3), using Lemma 3.1.52 (ii), we conclude that $\Gamma_n = \|x_n - z\|^2 \to 0$ and thus $x_n \to z$ as $n \to \infty$.

Case 2 : Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ as in (3.60). Then, by Lemma 3.1.54, we have $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$. From (4.155), it follows that

$$\Gamma_{\tau(n)+1} \leq (1-\alpha_\tau(n))\Gamma_{\tau(n)} + (1-\alpha_\tau(n))\beta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|(\sqrt{\Gamma_{\tau(n)}} + \sqrt{\Gamma_{\tau(n)-1}})$$

$$+ 2(1-\alpha_\tau(n))\beta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|^2 - \sigma\frac{f^2_{\tau(n)}(y_\tau(n))}{\|\nabla f_{\tau(n)}(y_\tau(n))\|^2}$$
$$+ 2\alpha_{\tau(n)}\langle u-z, x_{\tau(n)+1} - z\rangle, \tag{4.167}$$

which gives

$$\begin{aligned}\sigma\frac{f^2_{\tau(n)}(y_\tau(n))}{\|\nabla f_{\tau(n)}(y_\tau(n))\|^2} \leq\ & (1-\alpha_\tau(n))\beta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|(\sqrt{\Gamma_{\tau(n)}} + \sqrt{\Gamma_{\tau(n)-1}}) \\ & + 2(1-\alpha_\tau(n))\beta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|^2 \\ & + 2\alpha_{\tau(n)}\langle u-z, x_{\tau(n)+1} - z\rangle. \end{aligned}\tag{4.168}$$

Using a similar argument as in the proof of Case 1, we can show that

$$\lim_{n\to\infty}\|(I - P_{C_i^{\tau(n)}})y_{\tau(n)}\| = 0,$$

$$\lim_{n\to\infty}\|(I - P_{Q_j^{\tau(n)}})y_{\tau(n)}\| = 0,$$

and

$$\limsup_{n\to\infty}\langle u-z, x_{\tau(n)} - z\rangle \leq 0. \tag{4.169}$$

On the other hand, we see that

$$\begin{aligned}\|x_{\tau(n)+1} - x_{\tau(n)}\| \leq\ & \alpha_{\tau(n)}\|u - x_{\tau(n)}\| + (1-\alpha_n)\|y_{\tau(n)} - x_{\tau(n)}\| \\ & + (1-\alpha_{\tau(n)})\tau_{\tau(n)}\|\nabla f_{\tau(n)}(y_{\tau(n)})\| \\ =\ & \alpha_{\tau(n)}\|u - x_{\tau(n)}\| + (1-\alpha_{\tau(n)})\beta_{\tau_n}\|x_{\tau(n)} - x_{\tau(n)-1}\| \\ & + (1-\alpha_{\tau(n)})\rho_{\tau(n)}\frac{f_{\tau(n)}(y_{\tau(n)})}{\|\nabla f_{\tau(n)}(y_{\tau(n)})\|} \\ \to\ & 0. \end{aligned}\tag{4.170}$$

as $n \to \infty$. Using (4.169) and (4.170), we obtain

$$\limsup_{n\to\infty}\langle u-z, x_{\tau(n)+1} - z\rangle \leq 0. \tag{4.171}$$

Again from (4.167) we see that

$$\begin{aligned}\alpha_{\tau(n)}\Gamma_{\tau(n)} \leq\ & (1-\alpha_\tau(n))\beta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|(\sqrt{\Gamma_{\tau(n)}} + \sqrt{\Gamma_{\tau(n)-1}}) \\ & + 2(1-\alpha_\tau(n))\beta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|^2\end{aligned}$$

$$+ 2\alpha_{\tau(n)}\langle u - z, x_{\tau(n)+1} - z\rangle, \tag{4.172}$$

which gives

$$\begin{aligned}\Gamma_{\tau(n)} \;\leq\; & (1-\alpha_\tau(n))\frac{\beta_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\|(\sqrt{\Gamma_{\tau(n)}} + \sqrt{\Gamma_{\tau(n)-1}}) \\ & + 2(1-\alpha_\tau(n))\frac{\beta_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\|^2 \\ & + 2\langle u - z, x_{\tau(n)+1} - z\rangle. \end{aligned} \tag{4.173}$$

This shows that, by (4.171) and (C3)

$$\limsup_{n\to\infty} \Gamma_{\tau(n)} \leq 0. \tag{4.174}$$

Thus $\|x_{\tau(n)} - z\| \to 0$. We see that

$$\sqrt{\Gamma_{\tau(n)+1}} = \|x_{\tau(n)+1} - z\| \leq \|x_{\tau(n)+1} - x_{\tau(n)}\| + \|x_{\tau(n)} - z\| \to 0, \tag{4.175}$$

as $n \to \infty$. By Lemma 3.1.54, we also have

$$\Gamma_n \leq \Gamma_{\tau(n)+1} \to 0. \tag{4.176}$$

So we can conclude that $x_n \to z$ as $n \to \infty$. We thus complete the proof. □

Remark 4.3.2. We remark here that the conditions (C3) is easily implemented in numerical computation since the valued of $\|x_n - x_{n-1}\|$ is known before choosing β_n. Indeed, the parameter β_n can be chosen such that $0 \leq \beta_n \leq \bar{\beta}_n$, where

$$\bar{\beta}_n = \begin{cases} \min\left\{\frac{\omega_n}{\|x_n - x_{n-1}\|}, \beta\right\} & if\ x_n \neq x_{n-1}, \\ \beta & otherwise, \end{cases}$$

where $\{\omega_n\}$ is a positive sequence such that $\omega_n = o(\alpha_n)$.

4.3.2 Numerical examples

We provide some numerical examples and illustrate its performance by using Algorithm 3.1. Firstly, numerical results are shown in different choices of the step-size ρ_n with different values u, x_1 and x_2. Secondly, the comparison of convergence rate is made by Example 4.3.3 to show that our algorithm has a better convergence than that of He et al. [22] defined in (2.35). For this convenience, we denote algorithm (2.35) by Algorithm 3.2.

Example 4.3.3. [22] *Let* $H_1 = H_2 = \mathbb{R}^3$, $r = t = 2$ *and* $l_1 = l_2 = \lambda_1 = \lambda_2 = \frac{1}{4}$. *Define*

$$C_1 = \{x = (a, b, c)^T \in \mathbb{R}^3 : a + b^2 + 2c \leq 0\},$$

$$C_2 = \{x = (a, b, c)^T \in \mathbb{R}^3 : \tfrac{a^2}{16} + \tfrac{b^2}{9} + \tfrac{c^2}{4} - 1 \leq 0\},$$

$$Q_1 = \{x = (a, b, c)^T \in \mathbb{R}^3 : a^2 + b - c \leq 0\},$$

$$Q_2 = \{x = (a, b, c)^T \in \mathbb{R}^3 : \tfrac{a^2}{4} + \tfrac{b^2}{4} + \tfrac{c^2}{9} - 1 \leq 0\}.$$

and $A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 2 & 0 & 2 \end{pmatrix}$. *Find* $x^* \in C_1 \cap C_2$ *such that* $Ax^* \in Q_1 \cap Q_2$.

Choose $\alpha_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$ and $\beta = 0.5$. For each $n \in \mathbb{N}$, let $\omega_n = \frac{1}{(n+1)^{1.2}}$ and define $\beta_n = \bar{\beta}_n$ as in Remark 4.3.2. We now study the effect (in terms of convergence, stability, number of iterations required and the cpu time) of the sequence $\{\rho_n\} \subset (0, 4)$ on the iterative scheme by choosing different ρ_n such that $\inf_n \rho_n(4 - \rho_n) > 0$ in the following cases.

Case 1: $\rho_n = 1$; Case 2: $\rho_n = 2$; Case 3: $\rho_n = 3$; Case 4: $\rho_n = 3.95$.

The stopping criterion is defined by

$$E_n = \frac{1}{2}\sum_{i=1}^{2} \|x_n - P_{C_i^n} x_n\|^2 + \frac{1}{2}\sum_{j=1}^{2} \|Ax_n - P_{Q_j^n} Ax_n\|^2 < 10^{-4}.$$

We choose different choices of u, x_0 and x_1 as

Choice 1: $u = (2, 2, -2)^T$, $x_0 = (1, 1, 5)^T$ and $x_1 = (5, -3, 2)^T$;

Choice 2: $u = (1, 3, -2)^T$, $x_0 = (-4, 3, -2)^T$ and $x_1 = (-5, 2, 1)^T$;

Choice 3: $u = (4, -3, -6)^T$, $x_0 = (7, 5, 1)^T$ and $x_1 = (7, -3, -1)^T$;

Choice 4: $u = (7, -4, -3)^T$, $x_0 = (5.32, 2.33, 7.75)^T$ and $x_1 = (3.23, 3.75, -3.86)^T$.

The numerical experiments, using our Algorithm 3.1, for each case and choice are reported in the following Table 1. Table 1: Algorithm 3.1 with different cases of ρ_n and different choices of u, x_0 and x_1

		Case 1	Case 2	Case 3	Case 4
Choice 1	No. of Iter.	244	122	81	25
	cpu (Time)	0.05129	0.027395	0.015663	0.00472
Choice 2	No. of Iter.	392	196	131	13
	cpu (Time)	0.090982	0.04594	0.02693	0.002119
Choice 3	No. of Iter.	351	175	105	22
	cpu (Time)	0.099001	0.034915	0.02138	0.00473
Choice 4	No. of Iter.	444	178	88	27
	cpu (Time)	0.108428	0.036239	0.016809	0.005466

The convergence behavior of the error E_n for each choice of u, x_0 and x_1 is shown in Figure 1-4, respectively.

Figure 1: Comparison of the itertions for Choice 1 in Example 4.1

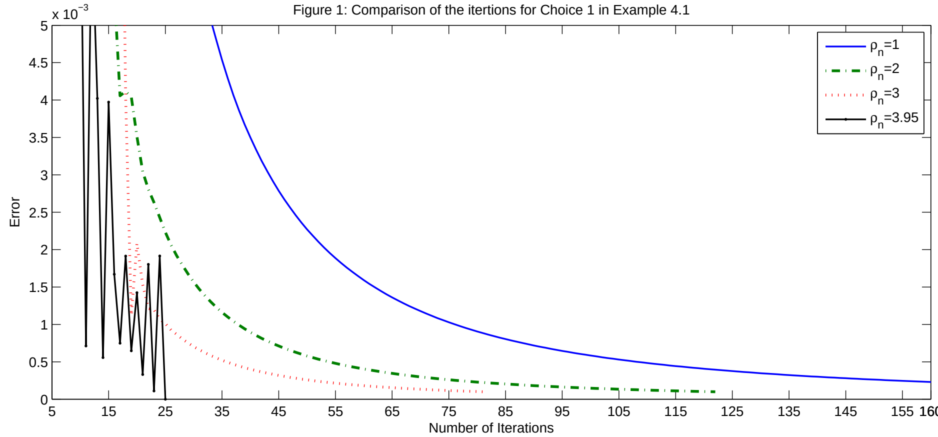


Figure 2: Comparison of the itertions for Choice 2 in Example 4.1

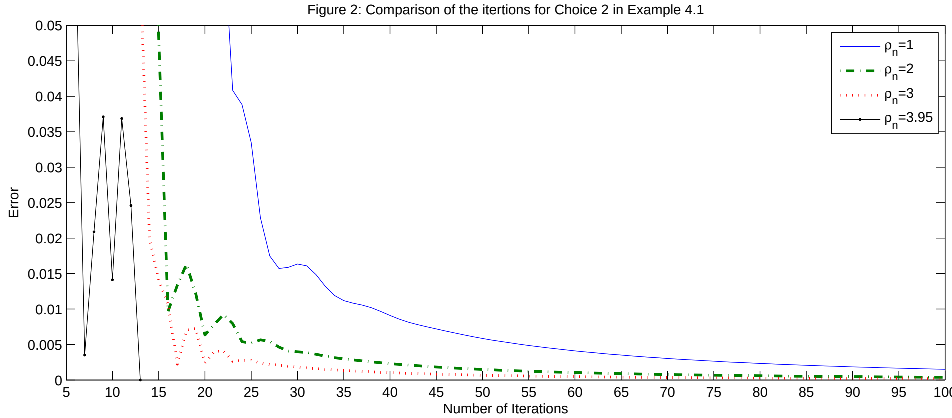
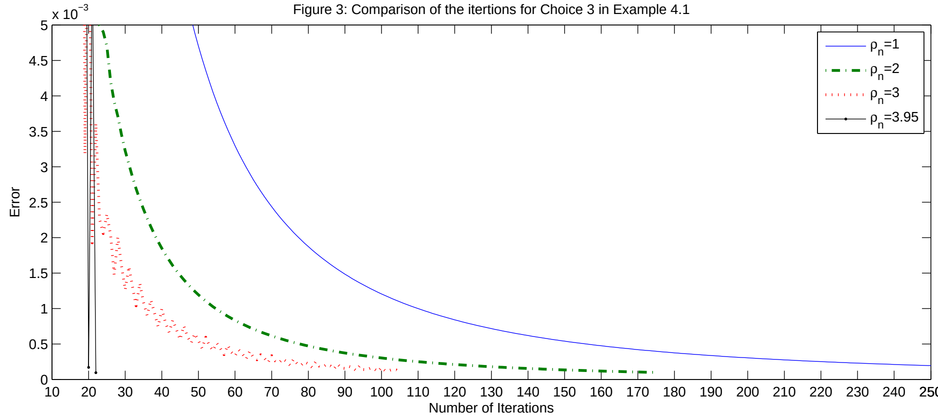
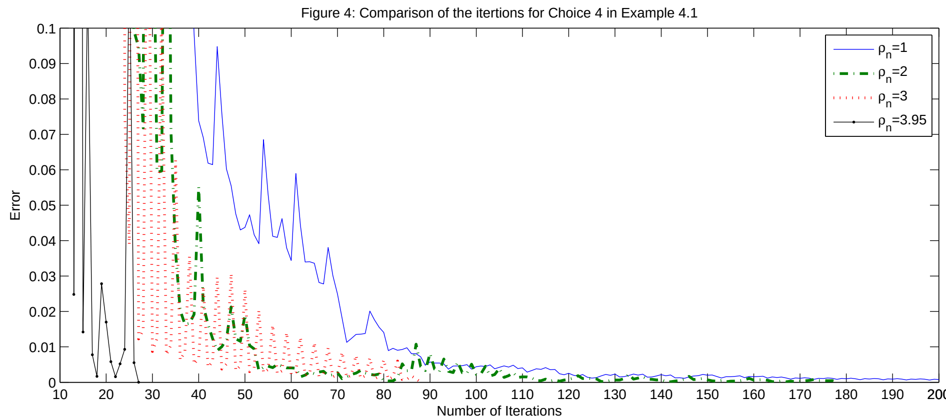


Figure 3: Comparison of the itertions for Choice 3 in Example 4.1





Remark 4.3.4. We make the following observations from our numerical experiments in Example 4.3.3.

1. For each different cases and different choices, we see that our algorithm is effective. It appears that Algorithm 3.1 has a good convergence speed and requires small number of iterations in the experiment.

2. It is observed that the number of iterations and the cpu run time are significantly decreasing starting from Case 1 to Case 4. However, there is no significant difference in both cpu run time and number of iterations for each choice of x_0 and x_1. So, initial guess does not have any significant effect on the convergence of the algorithm. However, we note that the sequence $\{x_n\}$ converges to a solution in MSFP which is of the form $P_S u$. Since the solution set S is not singleton, so the choice of u effects on the convergence behavior of the algorithm.

3. Our conditions appeared in Theorem 4.3.1 are easily implemented in numerical computations. This is because it needs no estimation on the spectral radius or the largest eigenvalue of $A^T A$ and the restriction of metric projections onto C and Q is relaxed by using those of C_n and Q_n which have specific forms in computation.

We finally end this section by providing a comparison of convergence of Algorithm 3.1 and Algorithm 3.2. Let $\alpha_n = \frac{1}{n+1}$, $\rho_n = 3.95$ and $\omega_n = \frac{1}{(n+1)^{1.2}}$ for all $n \in \mathbb{N}$. Set $\beta = 0.5$ and $\beta_n = \bar{\beta}_n$ as in Remark 4.3.2. For points u, x_0 and x_1 randomly, we obtain the following numerical results.

Table 2: Comparison of Algorithm 3.1 and Algorithm 3.2 in Example 4.1

			Algor 3.1	Algor 3.2
Choice 1	$u = (0, 1, 2)^T$	No. of Iter.	21	31
	$x_0 = (-4, -2, 3)^T$	cpu (Time)	0.004364	0.006537
	$x_1 = (-1, 2, 0)^T$			
Choice 2	$u = (-1, 3, 1)^T$	No. of Iter.	22	69
	$x_0 = (-1, 2, 3)^T$	cpu (Time)	0.004626	0.013906
	$x_1 = (-7, -4, -5)^T$			
Choice 3	$u = (3, 1, 3)^T$	No. of Iter.	97	287
	$x_0 = (-5, 1, -4)^T$	cpu (Time)	0.021787	0.074538
	$x_1 = (-5, -2, -3)^T$			
Choice 4	$u = (-1, 3, -3)^T$	No. of Iter.	18	161
	$x_0 = (3.2645, -2.3458, -5.3245)^T$	cpu (Time)	0.003854	0.034188
	$x_1 = (-2.5891, -3.2654, -3.2564)^T$			

The error plotting of E_n of Algorithm 3.1 and Algorithm 3.2 for each choice is shown in Figure 5-8, respectively.

Figure 5: Comparison of Algorithm 3.1 and 3.2 for Choice 1 in Example 4.1

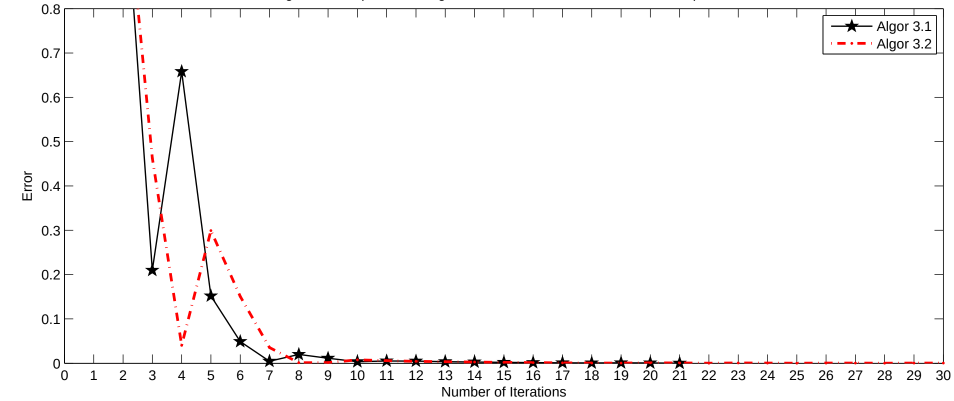


Figure 6: Comparison of Algorithm 3.1 and 3.2 for Choice 2 in Example 4.1

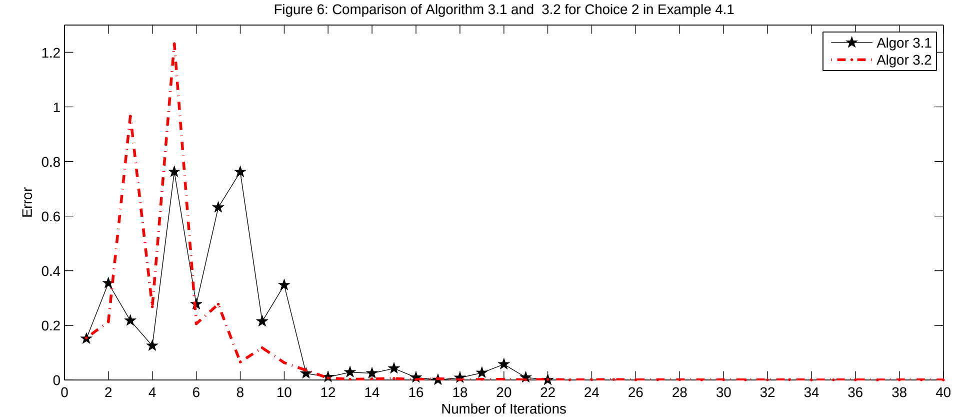
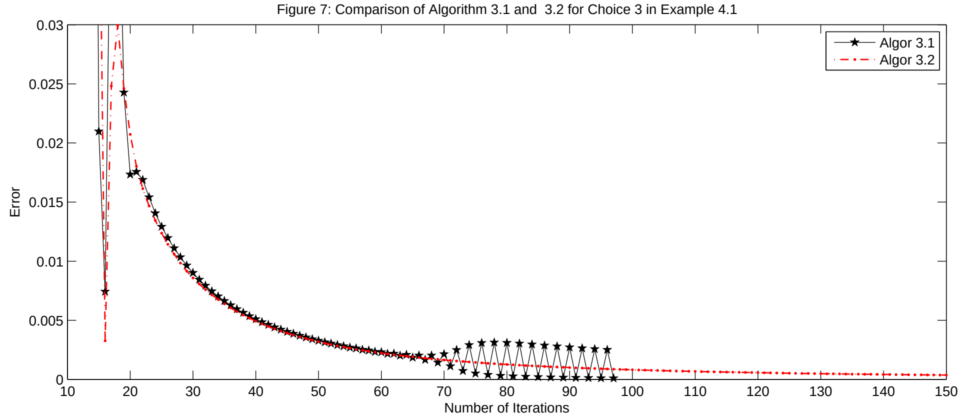
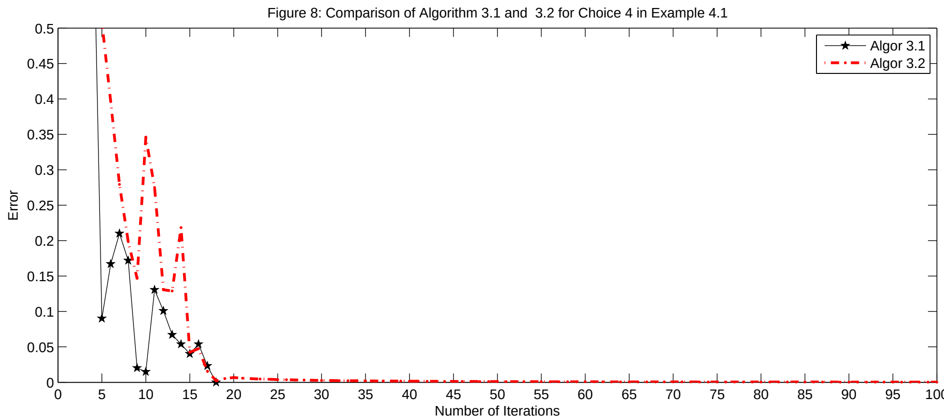


Figure 7: Comparison of Algorithm 3.1 and 3.2 for Choice 3 in Example 4.1





Remark 4.3.5. In numerical experiment, it is revealed that the sequence generated by our proposed Algorithm 3.1 involving the inertial technique converges more quickly than by Algorithm 3.2 of He et al. [22] does. This concludes that the inertial term constructed in Algorithm 3.1 improves the speed of convergence for solving the MSFP.

CHAPTER V

CONCLUSION

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APPENDIX

Ann. Funct. Anal.◇
http://dx.doi.org/10.1215/20088752-2017-0028
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

ON SOLVING PROXIMAL SPLIT FEASIBILITY PROBLEMS AND APPLICATIONS

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Communicated by B. Ricceri

Abstract. We study the problem of proximal split feasibility of two objective convex functions in Hilbert spaces. We prove that, under suitable conditions, certain strong convergence theorems of the Halpern-type algorithm present solutions to the proximal split feasibility problem. Finally, we provide some related applications as well as numerical experiments.

1. Introduction and preliminaries

Let H_1 and H_2 be real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous and convex functions. Let $A : H_1 \to H_2$ be a bounded linear operator.

Now we consider the *proximal split feasibility problem*. Find a minimizer x^* of f such that Ax^* minimizes g; that is, find $x^* \in \arg\min f$ such that

$$Ax^* \in \arg\min g, \tag{1.1}$$

where $\arg\min f = \{x \in H_1 : f(x) \le f(y), \forall y \in H_1\}$, and where $\arg\min g = \{x \in H_2 : g(x) \le g(y), \forall y \in H_2\}$. In what follows, $\Omega = \arg\min f \cap A^{-1}(\arg\min g)$ will denote the solution set of the problem (1.1).

The split feasibility problem in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [4]. It concerns modeling inverse problems which

Received Nov. 14, 2016; Accepted Feb. 28, 2017.
◇Advance publication—final volume, issue, and page numbers to be assigned.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47H09; Secondary 47H05, 47J25.
Keywords. split feasibility problem, strong convergence, Halpern-type algorithm, proximity operator.

arise from phase retrievals and in medical image reconstruction, especially intensity-modulated therapy (see [3]). Due to its applications, this problem has been discussed in many works published in recent years (see, for example, [2], [5], [10], [12], [13]).

Let C be a nonempty, closed, and convex subset of a real Hilbert space H with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. For each $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|. \tag{1.2}$$

Then P_C is called the *metric projection* of H onto C. For any $x \in H$, we know that

$$\langle x - P_C x, y - P_C x\rangle \leq 0 \tag{1.3}$$

for all $y \in C$.

If f and g are the indicator functions of two nonempty, closed, and convex sets $C \subset H_1$ and $Q \subset H_2$, respectively, then

$$f(x) = \delta_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$g(x) = \delta_Q(x) = \begin{cases} 0 & \text{if } x \in Q, \\ +\infty & \text{otherwise.} \end{cases}$$

Then the problem (1.1) becomes the following convex minimization problem. Find $x^* \in C$ such that

$$Ax^* \in Q. \tag{1.4}$$

This problem is called *the split feasibility problem*. A classical way to solve the problem (1.4) is to use the CQ algorithm introduced by Byrne [1, p. 442], which is defined in the following manner: $x_1 \in H_1$, and

$$x_{n+1} = P_C\big(x_n - \mu_n A^*(I - P_Q)Ax_n\big) \tag{1.5}$$

for each $n \geq 1$, where the stepsize $\mu_n \in (0, \frac{2}{\|A\|^2})$ and P_C, P_Q are the metric projections on C and Q, respectively.

It is noted that, in general, the operator norm $\|A\|$ or the largest eigenvalue of A^*A may not be calculated easily. To overcome this difficulty, López et al. [9, Algorithm 3.1] suggested the following algorithm: let $x_1 \in H_1$, and assume that $\{x_n\} \subset C$ has been constructed and that $\nabla h(x_n) \neq 0$. Then compute x_{n+1} via the rule

$$x_{n+1} = P_C\big(x_n - \mu_n A^*(I - P_Q)Ax_n\big) \tag{1.6}$$

for each $n \geq 1$, where $\mu_n = \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}$ with $0 < \rho_n < 4$ and $h(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$. It was proved that, if $\inf_n \rho_n(4 - \rho_n) > 0$, then the sequence $\{x_n\}$ defined by (1.6) converges weakly to a solution of (1.4).

Recall that the *subdifferential* of $F : H \to \mathbb{R} \cup \{+\infty\}$ at x is defined by

$$\partial F(x) = \big\{y \in H : F(x) + \langle y, z - x\rangle \leq F(z), \forall z \in H\big\}. \tag{1.7}$$

The *proximity operator* of F is defined by

$$\operatorname{prox}_{\lambda F}(x) = \arg\min_{y\in H}\Big\{F(y) + \frac{1}{2\lambda}\|x-y\|^2\Big\} \tag{1.8}$$

for any $\lambda > 0$. It is seen that

$$0 \in \partial F(x^*) \Longleftrightarrow x^* = \operatorname{prox}_{\lambda F}(x^*). \tag{1.9}$$

Hence the minimizers of any functions are the fixed point of its proximity operator. Moreover, the proximity operator of F is firmly nonexpansive, namely,

$$\big\langle \operatorname{prox}_{\lambda F}(x) - \operatorname{prox}_{\lambda F}(y), x - y\big\rangle \geq \big\|\operatorname{prox}_{\lambda F}(x) - \operatorname{prox}_{\lambda F}(y)\big\|^2 \tag{1.10}$$

for all $x, y \in H$, which is equivalent to

$$\begin{aligned}&\big\|\operatorname{prox}_{\lambda F}(x) - \operatorname{prox}_{\lambda F}(y)\big\|^2\\ &\quad\leq \|x-y\|^2 - \big\|(I-\operatorname{prox}_{\lambda F})(x) - (I-\operatorname{prox}_{\lambda F})(y)\big\|^2\end{aligned} \tag{1.11}$$

for all $x, y \in H$. Also, the complement $I - \operatorname{prox}_{\lambda F}$ is firmly nonexpansive. This suggests that we should employ the technique in fixed point theory for solving the convex minimization feasibility problem (see [6]).

Recently, Moudafi and Thakur [11, p. 2102] proposed the following split proximal algorithm: $x_1 \in H_1$ and

$$x_{n+1} = \operatorname{prox}_{\lambda\mu_n f}\big(x_n - \mu_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n\big), \tag{1.12}$$

where the stepsize

$$\mu_n = \rho_n \frac{h(x_n) + l(x_n)}{\theta^2(x_n)} \tag{1.13}$$

with

$$0 < \rho_n < 4, \qquad h(x_n) = \frac{1}{2}\big\|(I - \operatorname{prox}_{\lambda g})Ax_n\big\|^2, \tag{1.14}$$

$$l(x_n) = \frac{1}{2}\big\|(I - \operatorname{prox}_{\mu_n\lambda f})x_n\big\|^2 \tag{1.15}$$

and

$$\theta(x_n) = \sqrt{\big\|\nabla h(x_n)\big\|^2 + \big\|\nabla l(x_n)\big\|^2}. \tag{1.16}$$

They proved that, if $\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n)+l(x_n)} - \epsilon$ for some $\epsilon > 0$ small enough, then the sequence $\{x_n\}$ generated by (1.12) converges weakly to a solution of (1.1). We observe, however, that the stepsize sequence $\{\mu_n\}$, which appeared in (1.13), seems to be implicit because of the terms $l(x_n)$ and $\theta(x_n)$.

In order to solve the proximal split feasibility problem, we introduce a Halpern-type algorithm and prove its strong convergence under the condition on the stepsize suggested by López et al. [9, Theorem 3.5]. Then we provide some numerical experiments to support our main result. In order to complete the proof, we need the following lemma proved by He and Yang [8].

Lemma 1.1 ([8, Lemma 7])**.** *Assume that* $\{s_n\}$ *is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\delta_n, \tag{1.17}$$

and

$$s_{n+1} \leq s_n - \eta_n + t_n \tag{1.18}$$

for each $n \geq 0$*, where* $\{\gamma_n\}$ *is a sequence in* $(0,1)$*, where* $\{\eta_n\}$ *is a sequence of nonnegative real numbers, and where* $\{\delta_n\}$ *and* $\{t_n\}$ *are real sequences such that*

(a) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(b) $\lim_{n\to\infty} t_n = 0$, *and*
(c) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for any subsequence* $\{n_k\}$ *of* $\{n\}$*.*

Then $\lim_{n\to\infty} s_n = 0$.

2. Main results

Let H_1 and H_2 be real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous, and convex functions, and let $A : H_1 \to H_2$ be a bounded linear operator. We introduce the following results.

Algorithm I.
Step 1. Choose an initial point $x_0 \in H_1$.
Step 2. Assume that $\{x_n\}$ has been constructed. Set

$$h(x_n) = \frac{1}{2}\left\|(I - \operatorname{prox}_{\lambda g})Ax_n\right\|^2 \tag{2.1}$$

with $\|\nabla h(x_n)\| \neq 0$ for each $n \geq 1$.

We compute x_{n+1} in the following manner:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\operatorname{prox}_{\lambda f}\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2} A^*(I - \operatorname{prox}_{\lambda g})Ax_n\Big), \tag{2.2}$$

for each $n \geq 1$, where $u \in H_1$ is fixed, $\lambda > 0$, $\{\alpha_n\} \subset (0,1)$, $\{\rho_n\} \subset (0,4)$.

Theorem 2.1. *Suppose that* $\Omega \neq \emptyset$*, and assume that* $\{\alpha_n\}$ *and* $\{\rho_n\}$ *satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$,
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$, *and*
(C3) $\liminf_{n\to\infty} \rho_n(4-\rho_n) > 0$.

Then the sequence $\{x_n\}$ *converges strongly to* $z = P_\Omega u$.

Proof. Let $z = P_\Omega u$. Then $z = \operatorname{prox}_{\lambda f} z$, and $Az = \operatorname{prox}_{\lambda g} Az$. Note that

$$\nabla h(x_n) = A^*(I - \operatorname{prox}_{\lambda g})Ax_n. \tag{2.3}$$

Thus, since $I - \mathrm{prox}_{\lambda g}$ is firmly nonexpansive, by using (1.10), we have

$$\begin{aligned}\big\langle \nabla h(x_n), x_n - z\big\rangle &= \big\langle A^*(I - \mathrm{prox}_{\lambda g})Ax_n, x_n - z\big\rangle \\ &= \big\langle (I - \mathrm{prox}_{\lambda g})Ax_n, Ax_n - Az\big\rangle \\ &= \big\langle (I - \mathrm{prox}_{\lambda g})Ax_n - (I - \mathrm{prox}_{\lambda g})Az, Ax_n - Az\big\rangle \\ &\geq \big\|(I - \mathrm{prox}_{\lambda g})Ax_n\big\|^2 = 2h(x_n). \end{aligned} \tag{2.4}$$

Then by using (2.4), we obtain

$$\begin{aligned}&\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) - z\Big\|^2 \\ &\quad= \|x_n - z\|^2 + \rho_n^2 \frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} - 2\rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\big\langle \nabla h(x_n), x_n - z\big\rangle \\ &\quad\leq \|x_n - z\|^2 + \rho_n^2 \frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} - 4\rho_n \frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} \\ &\quad= \|x_n - z\|^2 - \rho_n(4 - \rho_n)\frac{h^2(x_n)}{\|\nabla h(x_n)\|^2}. \end{aligned} \tag{2.5}$$

Since $\{\rho_n\} \subset (0,4)$, it then follows that

$$\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) - z\Big\| \leq \|x_n - z\|. \tag{2.6}$$

Next we show that $\{x_n\}$ is bounded. Consider

$$\begin{aligned}&\|x_{n+1} - z\| \\ &\quad= \Big\|\alpha_n(u - z) \\ &\qquad + (1 - \alpha_n)\Big(\mathrm{prox}_{\lambda f}\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}A^*(I - \mathrm{prox}_{\lambda g})Ax_n\Big) - z\Big)\Big\| \\ &\quad\leq \alpha_n\|u - z\| + (1 - \alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) - z\Big\| \\ &\quad\leq \alpha_n\|u - z\| + (1 - \alpha_n)\|x_n - z\|. \end{aligned} \tag{2.7}$$

It follows, by induction, that

$$\|x_n - z\| \leq \max\big\{\|u - z\|, \|x_0 - z\|\big\}; \tag{2.8}$$

hence $\{x_n\}$ is bounded. Using (1.11) and (2.5), we see that

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ &\quad\leq (1 - \alpha_n)\Big\|\mathrm{prox}_{\lambda f}\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\Big) - z\Big\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\quad\leq (1 - \alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) - z\Big\|^2 \\ &\qquad - (1 - \alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) \end{aligned}$$

$$
\begin{aligned}
&\quad - \operatorname{prox}_{\lambda f}\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\Big)\Big\|^2 \\
&\quad + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\
&\le (1-\alpha_n)\|x_n - z\|^2 - (1-\alpha_n)\rho_n(4-\rho_n)\frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} \\
&\quad - (1-\alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) \\
&\quad - \operatorname{prox}_{\lambda f}\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\Big)\Big\|^2 \\
&\quad + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \qquad (2.9)
\end{aligned}
$$

Next we set

$$
s_n = \|x_n - z\|^2, \qquad \gamma_n = \alpha_n, \qquad (2.10)
$$

$$
\delta_n = 2\langle u - z, x_{n+1} - z\rangle, \qquad t_n = 2\alpha_n\langle u - z, x_{n+1} - z\rangle \qquad (2.11)
$$

and

$$
\begin{aligned}
\eta_n = (1-\alpha_n)\rho_n(4-\rho_n)\frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} + (1-\alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) \\
- \operatorname{prox}_{\lambda f}\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\Big)\Big\|^2. \qquad (2.12)
\end{aligned}
$$

From (2.9), it follows that

$$
\|x_{n+1} - z\|^2 \le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \qquad (2.13)
$$

and that

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\quad \le \|x_n - z\|^2 - (1-\alpha_n)\rho_n(4-\rho_n)\frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} \\
&\qquad - (1-\alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) \\
&\qquad - \operatorname{prox}_{\lambda f}\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\Big)\Big\|^2 \\
&\qquad + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \qquad (2.14)
\end{aligned}
$$

It is easy to check that $\lim_{n\to\infty} t_n = 0$ and that $\sum_{n=0}^{\infty}\gamma_n = \infty$ by using (C1) and (C2), respectively. In order to apply Lemma 1.1, we need to show that $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty}\delta_{n_k} \le 0$ for any subsequence $\{n_k\}$ of $\{n\}$.

Suppose that $\lim_{k\to\infty}\eta_{n_k} = 0$ for any subsequence $\{n_k\}$ of $\{n\}$. By (C1) and (C3), it follows that

$$
\lim_{k\to\infty}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|} = 0 \qquad (2.15)
$$

and that

$$\lim_{k\to\infty}\Big\|x_{n_k}-\rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k}) - \operatorname{prox}_{\lambda f}\Big(x_{n_k}-\rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big)\Big\| = 0. \tag{2.16}$$

We note that $\{\nabla h(x_{n_k})\}$ is bounded. Indeed, by the Lipschitzian continuity of ∇h and by the boundedness of $\{x_{n_k}\}$, we obtain

$$\begin{aligned}\big\|\nabla h(x_{n_k})\big\| &\le \big\|\nabla h(x_{n_k})-\nabla h(z)\big\| + \big\|\nabla h(z)\big\| \\ &\le \|A\|^2\|x_{n_k}-z\| + \big\|\nabla h(z)\big\|.\end{aligned} \tag{2.17}$$

Hence, by (2.15), we obtain

$$\lim_{k\to\infty} h(x_{n_k}) = 0 \tag{2.18}$$

for any subsequence $\{n_k\}$ of $\{n\}$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \rightharpoonup x^*$, and

$$\limsup_{n\to\infty}\langle u-z, x_n - z\rangle = \lim_{i\to\infty}\langle u-z, x_{n_i}-z\rangle. \tag{2.19}$$

By the lower semicontinuity of h, we have

$$0 \le h(x^*) \le \liminf_{i\to\infty} h(x_{n_i}) = \lim_{i\to\infty} h(x_{n_i}) = 0. \tag{2.20}$$

Hence we have

$$h(x^*) = \frac{1}{2}\big\|(I-\operatorname{prox}_{\lambda g})Ax^*\big\| = 0. \tag{2.21}$$

Thus Ax^* is a fixed point of the proximity operator g; that is, $0 \in \partial g(Ax^*)$, or Ax^* is a minimizer of g.

Next we show that x^* is also a minimizer of f. Observe that

$$\begin{aligned}&\|x_{n_k} - \operatorname{prox}_{\lambda f} x_{n_k}\| \\ &\le \Big\|x_{n_k} - \Big(x_{n_k}-\rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big)\Big\| \\ &\quad + \Big\|x_{n_k}-\rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k}) \\ &\quad - \operatorname{prox}_{\lambda f}\Big(x_{n_k}-\rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big)\Big\| \\ &\quad + \Big\|\operatorname{prox}_{\lambda f}\Big(x_{n_k}-\rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big) - \operatorname{prox}_{\lambda f} x_{n_k}\Big\| \\ &\le 2\rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|} + \Big\|x_{n_k}-\rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k}) \\ &\quad - \operatorname{prox}_{\lambda f}\Big(x_{n_k}-\rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big)\Big\|.\end{aligned} \tag{2.22}$$

This implies, by (2.15) and (2.16), that

$$\lim_{k\to\infty}\|x_{n_k} - \operatorname{prox}_{\lambda f} x_{n_k}\| = 0 \tag{2.23}$$

for any subsequence $\{n_k\}$ of $\{n\}$. Note that $\text{prox}_{\lambda f}$ is nonexpansive and that $x_{n_i} \rightharpoonup x^*$. Thus, by the demiclosedness principle (see [7]), we conclude that x^* is a fixed point of the proximity operator of f. This shows that x^* is also a minimizer of f. Hence $x^* \in \Omega$. On the other hand, we observe that

$$\begin{aligned}
&\|x_{n_k+1} - x_{n_k}\| \\
&\quad \le \alpha_{n_k}\|u - x_{n_k}\| \\
&\qquad + (1-\alpha_{n_k})\Big\|\text{prox}_{\lambda f}\Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big) - x_{n_k}\Big\| \\
&\quad \le \alpha_{n_k}\|u - x_{n_k}\| + (1-\alpha_{n_k})\Big\|\text{prox}_{\lambda f}\Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big) \\
&\qquad - \Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big)\Big\| \\
&\qquad + (1-\alpha_{n_k})\Big\|\Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big) - x_{n_k}\Big\| \\
&\quad = \alpha_{n_k}\|u - x_{n_k}\| + (1-\alpha_{n_k})\Big\|\text{prox}_{\lambda f}\Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big) \\
&\qquad - \Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big)\Big\| \\
&\qquad + (1-\alpha_{n_k})\rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|} \\
&\to 0 \qquad (2.24)
\end{aligned}$$

as $k \to \infty$. Thus, by (1.3), we obtain

$$\begin{aligned}
\limsup_{k\to\infty}\langle u - z, x_{n_k+1} - z\rangle &= \limsup_{k\to\infty}\langle u - z, x_{n_k} - z\rangle \\
&\le \limsup_{n\to\infty}\langle u - z, x_n - z\rangle \\
&= \lim_{i\to\infty}\langle u - z, x_{n_i} - z\rangle \\
&= \langle u - z, x^* - z\rangle \\
&\le 0. \qquad (2.25)
\end{aligned}$$

This implies that

$$\limsup_{k\to\infty}\delta_{n_k} \le 0 \qquad (2.26)$$

for any subsequence $\{n_k\}$. Therefore, by using Lemma 1.1, we conclude that $s_n = \|x_n - z\|^2 \to 0$. Hence $x_n \to z = P_\Omega u$. This completes the proof. □

When $f = \delta_C$ and $g = \delta_Q$ are indicator functions of nonempty, closed, and convex sets C and Q of H_1 and H_2, respectively, we obtain the following results.

Algorithm II.

Step 1. Choose an initial point $x_0 \in H_1$.

Step 2. Assume that $\{x_n\} \subseteq C$ has been constructed. Set $h(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ with $\|\nabla h(x_n)\| \neq 0$. Compute x_{n+1} in the following manner:

$$x_{n+1} = \alpha_n u + (1-\alpha_n) P_C\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2} A^*(I - P_Q)Ax_n\Big) \tag{2.27}$$

for each $n \geq 1$, where $u \in C$ is fixed, where $\{\alpha_n\} \subset (0,1)$ and where $\{\rho_n\} \subset (0,4)$.

Corollary 2.2. *Suppose that $\Theta = C \cap A^{-1}(Q) \neq \emptyset$, and assume that $\{\alpha_n\}$ and $\{\rho_n\}$ satisfy the conditions (C1)–(C3). Then the sequence $\{x_n\}$ converges strongly to $z = P_\Theta u$.*

Remark 2.3. In the case of $\|\nabla h(x_n)\| = 0$, we see that Algorithm I reduces to the following: $x_0 \in H_1$, and

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\operatorname{prox}_{\lambda f} x_n \tag{2.28}$$

for each $n \geq 1$, where $u \in H_1$ is fixed, where $\{\alpha_n\} \subset (0,1)$, and where $\lambda > 0$. If the sequence $\{\alpha_n\}$ satisfies (C1) and (C2), then the sequence $\{x_n\}$ converges strongly to $z = P_{\arg\min f} u$. Since ∇h is continuous, it follows that $\nabla h(x_n) \to \nabla h(z)$. Thus we obtain $\nabla h(z) = 0$ because $\|\nabla h(x_n)\| = 0$. This shows that Az is a minimizer of g. Hence $\{x_n\}$ converges strongly to a solution of (1.1).

Remark 2.4. We highlight our work with the following conclusions.

(1) In this paper, we have established strong convergence theorems for solving the proximal split feasibility problem of two convex functions. These theorems mainly improve and generalize the results obtained by Byrne [1], López et al. [9], and Moudafi and Thakur [11].

(2) We obtain strong convergence theorems by using a simpler and more explicit method than that of Moudafi and Thakur [11] whose approach may require an implicit computation.

3. Numerical examples

In this section, we give numerical examples to support our main theorem.

Example 3.1. Let $f : \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$, and let $g : \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$f(x) = \|x\|^2 + (2, 4, -5)x + 10 \tag{3.1}$$

and

$$g(x) = \|x\|^2 - (8, 10, -8)x - 5, \tag{3.2}$$

respectively. Let $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Solve the following proximal split feasibility problem:

Find $x^* \in \mathbb{R}^3$ such that x^* minimizes f and Ax^* minimizes g.

We can check that $x^* = (-1, -2, 2.5)$ is a minimizer of f such that $Ax^* = (4, 5, -4)$ minimizes g. We next show the convergence behavior of the sequence in Algorithm I by using our conditions. Let $u = (1, 1, 1)$, and let $x_0 = (-2, 4, -3)$.

TABLE 1. Numerical results for Algorithm I

n	x_n	Ax_n	$f(x_n)$	$g(A(x_n))$
1	$(-2.00000, 4.00000, -3.00000)$	$(-8.00000, 2.00000, 0.00000)$	66.000000	107.000000
5	$(-1.00362, -1.95651, 2.47069)$	$(3.93775, 5.01684, -3.96376)$	-1.247236	-61.994527
10	$(-0.99977, -1.99961, 2.49974)$	$(3.99971, 4.99992, -3.99915)$	-1.250000	-61.999999
15	$(-0.99987, -1.99973, 2.49987)$	$(3.99988, 5.00016, -3.99947)$	-1.250000	-62.000000
20	$(-0.99989, -1.99981, 2.49988)$	$(3.99987, 4.99997, -3.99960)$	-1.250000	-62.000000
25	$(-0.99992, -1.99984, 2.49992)$	$(3.99992, 5.00009, -3.99969)$	-1.250000	-62.000000
30	$(-0.99993, -1.99988, 2.49992)$	$(3.99991, 4.99998, -3.99973)$	-1.250000	-62.000000
35	$(-0.99994, -1.99989, 2.49995)$	$(3.99995, 5.00006, -3.99978)$	-1.250000	-62.000000
40	$(-0.99995, -1.99991, 2.49994)$	$(3.99994, 4.99999, -3.99980)$	-1.250000	-62.000000
45	$(-0.99996, -1.99991, 2.49996)$	$(3.99996, 5.00005, -3.99983)$	-1.250000	-62.000000
50	$(-0.99996, -1.99993, 2.49995)$	$(3.99995, 4.99999, -3.99984)$	-1.250000	-62.000000

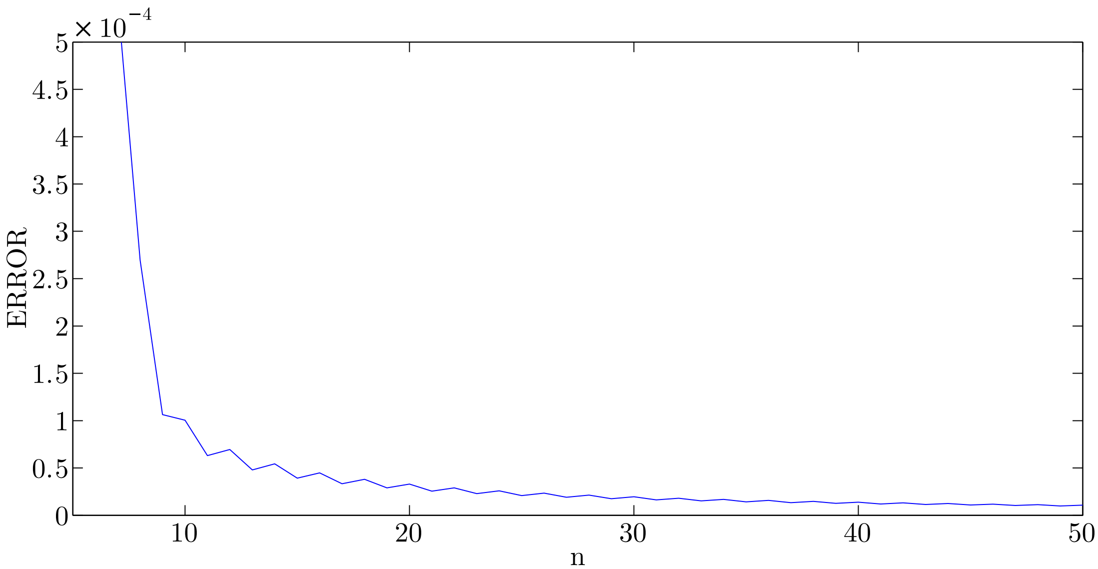

FIGURE 1. Error plotting of Table 1.

Choose $\lambda = 1$, $\alpha_n = \frac{10^{-3}}{n+1}$, and $\rho_n = 2$ for all $n \in \mathbb{N}$. Computing Algorithm I iteratively, we obtain the following numerical results.

From Table 1, the minimum values of f and g are -1.25 and -62, respectively. The errors of $\|x_{n+1} - x_n\|_2$ are plotted in Figure 1.

Example 3.2. Solve the following unconstrained linear equation system: find x^* in $\mathbb{R}^5$ such that $Ax^* = b$, where

$$A = \begin{pmatrix} 2 & 3 & -1 & 1 & 6 \\ -2 & -4 & 1 & -2 & 5 \\ -1 & -2 & -2 & -5 & 2 \\ 5 & 1 & -3 & 3 & -3 \\ 4 & 2 & 4 & 2 & 4 \end{pmatrix}, \qquad b = \begin{pmatrix} -20 \\ 21 \\ 6 \\ -15 \\ 18 \end{pmatrix}.$$

TABLE 2. Numerical results for Algorithm II

n	x_n^T	$\|x_{n+1} - x_n\|_2$
1	$(-3.00000, 1.00000, 4.00000, -2.00000, 0.00000)$	1.906981E+01
50	$(0.70266, -1.46469, 2.52074, -0.85842, 2.42700)$	1.678374E−02
100	$(0.82396, -1.68393, 2.71636, -0.91545, 2.25244)$	5.873192E−03
150	$(0.89597, -1.81311, 2.83231, -0.94998, 2.14917)$	2.054463E−03
200	$(0.93856, -1.88954, 2.90090, -0.97041, 2.08809)$	7.183362E−04
250	$(0.96375, -1.93475, 2.94148, -0.98250, 2.05196)$	2.510188E−04
300	$(0.97865, -1.96149, 2.96548, -0.98965, 2.03060)$	8.764489E−05
350	$(0.98746, -1.97732, 2.97968, -0.99389, 2.01797)$	3.056299E−05
400	$(0.99268, -1.98669, 2.98809, -0.99640, 2.01049)$	1.063548E−05
450	$(0.99577, -1.99226, 2.99308, -0.99789, 2.00606)$	3.687309E−06
500	$(0.99761, -1.99558, 2.99606, -0.99878, 2.00341)$	1.269242E−06
550	$(0.99874, -1.99762, 2.99789, -0.99934, 2.00180)$	4.299900E−07

Let $u = (1, 1, 1, 1, 1)^T$, and let $x_0 = (-3, 1, 4, -2, 0)^T$. Choose $\lambda = 1$, $\alpha_n = \frac{10^{-5}}{\sqrt{n+1}}$ and $\rho_n = 2$ for all $n \in \mathbb{N}$. Computing Algorithm II iteratively, we obtain the following numerical results.

From Table 2, the solution of the linear equation system is $(1, -2, 3, -1, 2)^T$.

Acknowledgments. Cholamjiak's research was supported by the Thailand Research Fund and the Commission on Higher Education under Grant MRG5980248. Furthermore, Cho's work was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF), which is funded by the Ministry of Science, ICT, and Future Planning (2014R1A2A2A01002100).

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Bull. Malays. Math. Sci. Soc.
https://doi.org/10.1007/s40840-018-0614-0

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A New CQ Algorithm for Solving Split Feasibility Problems in Hilbert Spaces

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Received: 14 November 2017 / Revised: 14 February 2018

Abstract In this paper, we propose a CQ-type algorithm for solving the split feasibility problem (SFP) in real Hilbert spaces. The algorithm is designed such that the stepsizes are directly computed at each iteration. We will show that the sequence generated by the proposed algorithm converges in norm to the minimum-norm solution of the SFP under appropriate conditions. In addition, we give some numerical examples to verify the implementation of our method. Our result improves and complements many known related results in the literature.

Keywords Split feasibility problem · Variational inequality · Gradient projection method · Weak convergence · Strong convergence · minimum-norm solution

Mathematics Subject Classification (2000) 47H10 · 90C25

Communicated by Mohammad Sal Moslehian.

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Published online: 03 March 2018

1 Introduction

Let H be a real Hilbert space with the inner product $\langle ., . \rangle$ and the induced norm $||.||$. Let I denote the identity operator on H. Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) was first introduced by Censor and Elfving [6], and it can be formulated as follows:

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q, \tag{1.1}$$

if such points exist, where $A : H_1 \to H_2$ is a bounded linear operator.

We will use Ω to denote the solution set of (1.1), i.e.,

$$\Omega := \{x^* \in C : Ax^* \in Q\}.$$

The problem (1.1) arises in signal processing and image reconstruction with particular progress in intensity modulated therapy, and many iterative algorithms have been established for it (see, e.g., [3,4,6–8,11,15,17,20]).

From an optimization point of view, $x^* \in \Omega$ if and only if x^* is a solution of the following minimization problem with zero optimal value:

$$\min_{x \in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2. \tag{1.2}$$

Note that the function f is differentiable convex and has a Lipschitz gradient given by $\nabla f(x) = A^*(I - P_Q)Ax$. Hence, x^* solves the SFP if and only if x^* solves the variational inequality problem of finding $x \in C$ such that

$$\langle \nabla f(x), y - x \rangle \geq 0 \quad \forall y \in C. \tag{1.3}$$

A popular algorithm was known under the name of CQ algorithm introduced by Byrne [3,4] as follows:

$$x^{k+1} = P_C \left(I - \gamma A^*(I - P_Q)A\right) x^k, \quad k \in \mathbb{N}, \tag{1.4}$$

where $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$.

In fact, the CQ algorithm is the gradient projection method for the variational inequality problem (1.3). For more details on the SFP and the CQ algorithm, the interested reader is referred to see [1,3–5,10,13,19,22,23] and the references therein. Xu [22] proved the weak convergence of (1.4) in the setting of Hilbert spaces. In order to obtain strong convergence, Wang and Xu [18] proposed the following algorithm:

$$x^{k+1} = P_C \left[(1 - \alpha_k)(x^k - \gamma \nabla f(x^k))\right], \quad k \geq 0. \tag{1.5}$$

Wang and Xu [18] proved that the above iterative sequence converges strongly to the minimum-norm solution of the SFP (1.1) provided that the sequence $\{\alpha_k\}$ and parameter γ satisfy the following conditions:

(1) $\alpha_k \to 0$ and $0 < \gamma < \frac{2}{\|A\|^2}$;
(2) $\sum_{k=0}^{\infty} \alpha_k = \infty$;
(3) either $\sum_{k=0}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty$ or $\lim_{k\to\infty} |\alpha_{k+1} - \alpha_k|/\alpha_k = 0$.

In 2012, Yu et al. [20] proved the strong convergence of (1.5) without the condition (3). It is worth mentioning that the determination of the step-size in (1.5) depends on the Lipschitz constant $L = \|A\|^2$ of gradient ∇f, which is in general not easy to compute in practice. This leads us to the following question.

Question *Can we design a self-adaptive scheme for the algorithm* (1.5) *above?*
In this paper, we give a positive answer to this question. Motivated and inspired by the works of Lopéz et al. [13], Tian and Zhang [16], Wang and Xu [18], Xu [22], Yao et al. [24] and Zhou et al. [25], we will introduce a self-adaptive CQ-type algorithm for finding a solution of the SFP in the setting of infinite-dimensional real Hilbert spaces. The advantage of our algorithm lies in the fact that step-sizes are dynamically chosen and do not depend on the operator norm. Moreover, we will prove that the proposed algorithm converges strongly to the minimum-norm solution of the SFP.

The rest of the paper is organized as follows. Some useful definitions and results are collected in Sect. 2 for the convergence analysis of the proposed algorithm. In Sect. 3, we introduce a new self-adaptive CQ-type algorithm for finding an element of the set Ω and prove strong convergence of the method. Our result improves the corresponding results of Chuang [9], Wang and Xu [18], Xu [22] and Yao et al. [24]. We also consider the relaxation version for the proposed method in Sect. 4. Finally in Sect. 5, we provide some numerical experiments to illustrate the performance of the proposed algorithms.

2 Preliminaries

Let C be a closed convex subset of a real Hilbert space H. It is easy to see that

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2, \tag{2.1}$$

for all $x, y \in H$ and for all $t \in [0, 1]$.

In what follows, the strong (weak) convergence of a sequence $\{x^k\}$ to x will be denoted by $x^k \to x$ ($x^k \rightharpoonup x$), respectively. For a given sequence $\{x^k\} \subset H$, $\omega_w(x^k)$ denotes the weak ω-limit set of $\{x^k\}$, that is,

$$\omega_w(x^k) := \{x \in H : x^{k_j} \rightharpoonup x \text{ for some subsequence } \{k_j\} \text{ of } \{k\}\}.$$

For every element $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$||x - P_C x|| = \inf\{||x - y|| : y \in C\}.$$

P_C is called the metric projection of H onto C.

Lemma 2.1 *The metric projection P_C has the following basic properties:*

(1) $\langle x - P_C x, y - P_C x\rangle \le 0$ *for all* $x \in H$ *and* $y \in C$*;*
(2) $\|P_C x - P_C y\| \le \|x - y\|$ *for all* $x, y \in H$*;*
(3) $\|P_C x - P_C y\|^2 \le \langle x - y, P_C x - P_C y\rangle$ *for every* $x, y \in H$*;*

Let C and Q be nonempty closed convex subsets of the infinite-dimensional real Hilbert spaces H_1 and H_2, respectively, $A \in B(H_1, H_2)$, where $B(H_1, H_2)$ denotes the family of all bounded linear operators from H_1 to H_2.

Lemma 2.2 (see [2]) *Let* $f : H_1 \to \mathbb{R}$ *be a function defined by* $f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2$*. Then*

(1) f *is convex and differentiable;*
(2) f *is w-lsc on* H_1*;*
(3) $\nabla f(x) = A^*(I - P_Q)Ax$, $x \in H_1$*;*
(4) ∇f *is* $\frac{1}{\|A\|^2}$*-inverse strongly monotone, i.e.,*

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \ge \frac{1}{\|A\|^2}\|\nabla f(x) - \nabla f(y)\|^2 \quad \forall x, y \in H_1.$$

Remark 2.1 From (4) of Lemma 2.2, it is easy to see that ∇f is $\|A\|^2$-Lipschitz, that is,

$$\|\nabla f(x) - \nabla f(y)\| \le \|A\|^2\|x - y\| \quad \forall x, y \in H_1.$$

In convergence analysis of the proposed algorithms, we will use the well-known lemmas.

Lemma 2.3 (Maingé [14]) *Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence* $\{\Gamma_{n_j}\}$ *of* $\{\Gamma_n\}$ *such that* $\Gamma_{n_j} < \Gamma_{n_j+1}$ *for all* $j \ge 0$*. Also consider the sequence of integers* $\{\tau(n)\}_{n\ge n_0}$ *defined by*

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n\ge n_0}$ *is a nondecreasing sequence verifying* $\lim_{n\to\infty} \tau(n) = \infty$ *and, for all* $n \ge n_0$*,*

$$\max\{\Gamma_{\tau(n)}, \Gamma_n\} \le \Gamma_{\tau(n)+1}.$$

Lemma 2.4 (Xu [21]) *Assume that* $\{a_k\}$ *is a sequence of nonnegative real numbers such that*

$$a_{k+1} \le (1 - \alpha_k)a_k + \alpha_k\gamma_k + b_k, \quad k \in \mathbb{N},$$

where $\{\alpha_k\}$ *is a sequence in* $(0, 1)$*,* $\{b_k\}$ *is a sequence of nonnegative real numbers and* $\{\gamma_k\}$ *is a sequence of real numbers such that*

(1) $\sum_{k=0}^{\infty} \alpha_k = \infty$,

(2) $\sum_{k=0}^{\infty} b_k < \infty$,
(3) $\limsup_{k\to\infty} \gamma_k \leq 0$.

Then $\lim_{k\to\infty} a_k = 0$.

We end this section by recalling a new fundamental tool which will be helpful for proving strong convergence of our relaxation CQ algorithm.

Lemma 2.5 (He and Yang 2013 [12]) *Assume that* $\{s_k\}$ *is a sequence of nonnegative real numbers such that for all* $k \in \mathbb{N}$

$$s_{k+1} \leq (1-\alpha_k)s_k + \alpha_k \delta_k,$$
$$s_{k+1} \leq s_k - \eta_k + \gamma_k,$$

where $\{\alpha_k\}$ *is a sequence in* $(0, 1)$, $\{\eta_k\}$ *is a sequence of nonnegative real numbers, and* $\{\delta_k\}$ *and* $\{\gamma_k\}$ *are two sequences in* $\mathbb{R}$ *such that*

(1) $\sum_{k=0}^{\infty} \alpha_k = \infty$,
(2) $\lim_{k\to\infty} \gamma_k = 0$,
(3) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies that* $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for any subsequence* $\{n_k\}$ *of* $\{n\}$.

Then $\lim_{s\to\infty} s_k = 0$.

3 A New Modification of CQ Algorithm and Its Convergence

In this section, we introduce a CQ-type algorithm with self-adaptive step-sizes for solving the SFP (1.1) and establish its strong convergence under some mild conditions. The algorithm is designed as follows.

Algorithm 3.1 [CQ-type algorithm for the SFP (1.1)]
Initialization Take two positive sequences $\{\beta_k\}$ and $\{\rho_k\}$ satisfying the following conditions:

$$\{\beta_k\} \subset (0, 1), \quad \lim_{k\to\infty} \beta_k = 0, \quad \sum_{k=0}^{\infty} \beta_k = \infty, \tag{3.1}$$

$$\rho_k(4-\rho_k) > 0. \tag{3.2}$$

Select initial $x^0 \in H_1$ and set $k := 0$.
Iterative Step Given x^k, if $\nabla f(x^k) = 0$ then stop [x^k is a solution to the SFP (1.1)]. Otherwise, compute

$$\lambda_k = \frac{\rho_k f(x^k)}{\|\nabla f(x^k)\|^2}$$

and

$$x^{k+1} = P_C\left[(1-\beta_k)(x^k - \lambda_k \nabla f(x^k))\right]. \tag{3.3}$$

Let $k := k + 1$ and return to *Iterative Step*.

For the convergence analysis of Algorithm 3.1, we need the following results.

Lemma 3.1 *Let $\{x^k\}$ be the sequence generated by Algorithm 3.1. Then, for each $z \in \Omega$, the following inequality holds:*

$$\|x^{k+1} - z\|^2 \leq \beta_k \|z\|^2 + (1 - \beta_k)\|x^k - z\|^2 - \rho_k(4 - \rho_k)(1 - \beta_k)\frac{f^2(x^k)}{\|\nabla f(x^k)\|^2}.$$

Proof By Lemma 2.1 (2) and (3.3), we have

$$\begin{aligned}
\|x^{k+1} - z\|^2 &= \|P_C\left[(1 - \beta_k)\left(x^k - \lambda_k \nabla f(x^k)\right)\right] - P_C z\|^2 \\
&\leq \|(1 - \beta_k)\left(x^k - \lambda_k \nabla f(x^k)\right) - z\|^2 \\
&= \|\beta_k(-z) + (1 - \beta_k)\left(x^k - \lambda_k \nabla f(x^k) - z\right)\|^2 \qquad (3.4) \\
&\leq \beta_k \|z\|^2 + (1 - \beta_k)\|x^k - \lambda_k \nabla f(x^k) - z\|^2. \qquad (3.5)
\end{aligned}$$

Note that

$$\begin{aligned}
\langle \nabla f(x^k), x^k - z\rangle &= \langle (I - P_Q)Ax^k, Ax^k - Az\rangle \\
&= \langle (I - P_Q)Ax^k - (I - P_Q)Az, Ax^k - Az\rangle \\
&\geq \|(I - P_Q)Ax^k\|^2 = 2f(x^k). \qquad (3.6)
\end{aligned}$$

We now estimate the second term on the right-hand side of (3.5) as follows:

$$\begin{aligned}
&\left\|x^k - \lambda_k \nabla f(x^k) - z\right\|^2 \\
&= \|x^k - z\|^2 + \lambda_k^2 \|\nabla f(x^k)\|^2 - 2\lambda_k \langle \nabla f(x^k), x^k - z\rangle \\
&\leq \|x^k - z\|^2 + \lambda_k^2 \|\nabla f(x^k)\|^2 - 4\lambda_k f(x^k) \\
&\leq \|x^k - z\|^2 + \frac{\rho_k^2 f^2(x^k)}{\|\nabla f(x^k)\|^2} - \frac{4\rho_k f^2(x^k)}{\|\nabla f(x^k)\|^2}. \qquad (3.7)
\end{aligned}$$

From (3.5) and (3.7), we arrive at

$$\begin{aligned}
\|x^{k+1} - z\|^2 &\leq \beta_k \|z\|^2 + (1 - \beta_k)\|x^k - z\|^2 \\
&\quad + (1 - \beta_k)\left[\frac{\rho_k^2 f^2(x^k)}{\|\nabla f(x^k)\|^2} - \frac{4\rho_k f^2(x^k)}{\|\nabla f(x^k)\|^2}\right] \\
&= \beta_k \|z\|^2 + (1 - \beta_k)\|x^k - z\|^2 - \rho_k(4 - \rho_k)(1 - \beta_k)\frac{f^2(x^k)}{\|\nabla f(x^k)\|^2}.
\end{aligned}$$

This completes the proof. □

Lemma 3.2 *The sequence* $\{x^k\}$ *generated by Algorithm 3.1 is bounded.*

Proof By Lemmas 3.1 and (3.2), we have

$$\begin{aligned}\|x^{k+1}-z\|^2 &\le \beta_k\|z\|^2+(1-\beta_k)\|x^k-z\|^2-\rho_k(4-\rho_k)(1-\beta_k)\frac{f^2(x^k)}{\|\nabla f(x^k)\|^2}\\ &\le \beta_k\|z\|^2+(1-\beta_k)\|x^k-z\|^2.\end{aligned}$$

So, we get

$$\|x^{k+1}-z\|^2\le \max\{\|z\|^2, \|x^k-z\|^2\}.$$

By induction,

$$\|x^{k+1}-z\|^2\le \max\{\|z\|^2, \|x^0-z\|^2\},$$

this implies that sequence $\{x^k\}$ is bounded. □

Lemma 3.3 *Let* $\{x^k\}$ *be the sequence generated by Algorithm 3.1. Then the following inequality holds for all* $z\in\Omega$ *and* $k\in\mathbb{N}$,

$$\begin{aligned}\|x^{k+1}-z\|^2 \le (1-\beta_k)\|x^k-z\|^2+\beta_k\Big[\beta_k\|z\|^2+2(1-\beta_k)\langle x^k-z,-z\rangle\\ +2\lambda_k(1-\beta_k)\langle\nabla f(x^k),z\rangle\Big].\end{aligned}$$

Proof By (3.2) and (3.7), we have

$$\begin{aligned}\|x^k-\lambda_k\nabla f(x^k)-z\|^2 &\le \|x^k-z\|^2-\rho_k(4-\rho_k)\frac{f^2(x^k)}{\|\nabla f(x^k)\|^2}\\ &\le \|x^k-z\|^2.\end{aligned}$$

Combining with (3.4) of Lemma 3.1, we obtain

$$\begin{aligned}\|x^{k+1}-z\|^2 &\le \left\|\beta_k(-z)+(1-\beta_k)\left(x^k-\lambda_k\nabla f(x^k)-z\right)\right\|^2\\ &\le\beta_k^2\|z\|^2+(1-\beta_k)^2\left\|x^k-\lambda_k\nabla f(x^k)-z\right\|^2\\ &\quad+2\beta_k(1-\beta_k)\left\langle x^k-\lambda_k\nabla f(x^k)-z,-z\right\rangle\\ &\le \beta_k^2\|z\|^2+(1-\beta_k)^2\|x^k-z\|^2+2\beta_k(1-\beta_k)\langle x^k-z,-z\rangle\\ &\quad+2\beta_k\lambda_k(1-\beta_k)\langle\nabla f(x^k),z\rangle\\ &\le (1-\beta_k)\|x^k-z\|^2+\beta_k\Big[\beta_k\|z\|^2+2(1-\beta_k)\langle x^k-z,-z\rangle\\ &\quad+2\lambda_k(1-\beta_k)\langle\nabla f(x^k),z\rangle\Big].\end{aligned}$$

The proof is complete. □

We are now in a position to establish the strong convergence of the sequence generated by Algorithm 3.1.

Theorem 3.1 *Assume that* $\inf_k \rho_k(4-\rho_k) > 0$. *Then the sequence* $\{x^k\}$ *generated by Algorithm 3.1 converges strongly to the minimum-norm element of* Ω.

Proof Let $z := P_\Omega 0$. From Lemma 3.1, we have

$$\|x^{k+1}-z\|^2 \le \beta_k\|z\|^2 + (1-\beta_k)\|x^k-z\|^2 - \rho_k(4-\rho_k)(1-\beta_k)\frac{f^2(x^k)}{\|\nabla f(x^k)\|^2}. \tag{3.8}$$

From (3.2) and the assumption $\inf_k \rho_k(4-\rho_k) > 0$, we can find a constant σ such that $(1-\beta_k)\rho_k(4-\rho_k) \ge \sigma > 0$ for all $k \in \mathbb{N}$. Hence

$$\|x^{k+1}-z\|^2 \le \beta_k\|z\|^2 + (1-\beta_k)\|x^k-z\|^2 - \sigma\frac{f^2(x^k)}{\|\nabla f(x^k)\|^2} \tag{3.9}$$

or

$$\sigma\frac{f^2(x^k)}{\|\nabla f(x^k)\|^2} \le \beta_k\|z\|^2 + (1-\beta_k)\|x^k-z\|^2 - \|x^{k+1}-z\|^2.$$

So, we obtain

$$\sigma\frac{f^2(x^k)}{\|\nabla f(x^k)\|^2} \le \|x^k-z\|^2 - \|x^{k+1}-z\|^2 + \beta_k\|z\|^2. \tag{3.10}$$

Now, we consider two possible cases
Case 1 Put $\Gamma_k := \|x^k-z\|^2$ for all $k \in \mathbb{N}$. Assume that there is a $k_0 \ge 0$ such that for each $k \ge n_0$, $\Gamma_{k+1} \le \Gamma_k$. In this case, $\lim_{k\to\infty}\Gamma_k$ exists and $\lim_{k\to\infty}(\Gamma_k - \Gamma_{k+1}) = 0$.
Since $\lim_{k\to\infty}\beta_k = 0$, it follows from (3.10) that

$$\lim_{k\to\infty}\sigma\frac{f^2(x^k)}{\|\nabla f(x^k)\|^2} = 0. \tag{3.11}$$

It follows from (3.11) that

$$\lim_{k\to\infty}\lambda_k\left\|\nabla f(x^k)\right\| = \lim_{k\to\infty}\frac{f(x^k)}{\|\nabla f(x^k)\|} = 0.$$

Since ∇f is Lipschitz, we have

$$\|\nabla f(x^k)\| = \|\nabla f(x^k) - \nabla f(z)\| \le \|A\|^2\|x^k-z\| \quad \forall z \in \Omega.$$

Hence, $\{\nabla f(x^k)\}$ is bounded. This together with (3.11) implies that $f(x^k) \to 0$ as $k \to \infty$. We now show that $\omega_w(x^k) \subset \Omega$. Let $\bar{x} \in \omega_w(x^k)$ be an arbitrary element.

Since $\{x^k\}$ is bounded (by Lemma 3.2), there exists a subsequence $\{x^{k_j}\}$ of $\{x^k\}$ such that $x^{k_j} \rightharpoonup \bar{x}$. With regard to the weak lower semicontinuity of f, we obtain

$$0 \le f(\bar{x}) \le \liminf_{j\to\infty} f(x^{k_j}) = \lim_{k\to\infty} f(x^k) = 0.$$

We immediately deduce that $f(\bar{x}) = 0$, i.e., $A\bar{x} \in Q$. The choice of $\bar{x}$ in $\omega_w(x^k)$ was arbitrary, and so we conclude that $\omega_w(x^k) \subset \Omega$.

Using Lemma 3.3, we have

$$\begin{aligned}\|x^{k+1} - z\|^2 &\le (1-\beta_k)\|x^k - z\|^2 + \beta_k \Big[\beta_k\|z\|^2 + 2(1-\beta_k)\langle x^k - z, -z\rangle \\ &\quad + 2\lambda_k(1-\beta_k)\langle \nabla f(x^k), z\rangle\Big] \\ &\le (1-\beta_k)\|x^k - z\|^2 + \beta_k \Big[\beta_k\|z\|^2 + 2(1-\beta_k)\langle x^k - z, -z\rangle \\ &\quad + 2(1-\beta_k)\lambda_k\|\nabla f(x^k)\|\|z\|\Big]. \end{aligned} \tag{3.12}$$

To apply Lemma 2.4, it remains to show that $\limsup_{k\to\infty}\langle x^k - z, -z\rangle \le 0$. Indeed, since $z = P_\Omega 0$, by using the property of the projection [Lemma 2.1 (1)], we arrive at

$$\limsup_{k\to\infty}\langle x^k - z, -z\rangle = \max_{\hat{z}\in\omega_w(x^k)} \langle \hat{z} - z, -z\rangle \le 0.$$

By applying Lemma 2.4 to (3.12) with the data:

$$\begin{aligned} &a_k := \|x^k - z\|^2, \quad \alpha_k := \beta_k, \quad b_k := 0, \\ &\gamma_k := \beta_k\|z\|^2 + 2(1-\beta_k)\langle x^k - z, -z\rangle + 2\lambda_k\|\nabla f(x^k)\|\|z\|, \end{aligned}$$

we immediately deduce that the sequence $\{x^k\}$ converges strongly to $z = P_\Omega 0$. Furthermore, it follows again from Lemma 2.1 (1) that

$$\langle p - z, -z\rangle \le 0 \quad \forall p \in \Omega.$$

Hence

$$\|z\|^2 \le \langle p, z\rangle \le \|z\|\|p\| \quad \forall p \in \Omega,$$

from which we infer that z is the minimum-norm solution of the SFP (1.1).

Case 2 Assume that there exists a subsequence $\{\Gamma_{k_m}\} \subset \{\Gamma_k\}$ such that $\Gamma_{k_m} \le \Gamma_{k_m+1}$ for all $m \in \mathbb{N}$. In this case, we can define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(k) = \max\{n \le k : \Gamma_n < \Gamma_{n+1}\}.$$

Then we have from Lemma 2.3 that $\tau(k) \to \infty$ as $k \to \infty$ and $\Gamma_{\tau(k)} < \Gamma_{\tau(k)+1}$. So, we have from (3.10) that

$$\sigma \frac{f^2(x^{\tau(k)})}{\|\nabla f(x^{\tau(k)})\|^2} \leq \|x^{\tau(k)} - z\|^2 - \|x^{\tau(k)+1} - z\|^2 + \beta_{\tau(k)}\|z\|^2 \\ \leq \beta_{\tau(k)}\|z\|^2.$$

Following the same way as the proof of Case 1, we have that

$$\lim_{k\to\infty} \frac{f^2(x^{\tau(k)})}{\|\nabla f(x^{\tau(k)})\|^2} = 0, \\ \limsup_{k\to\infty}\langle x^{\tau(k)} - z, -z\rangle = \max_{\widetilde{z}\in\omega_w(x\{^{\tau(k)}\})} \langle \widetilde{z} - z, -z\rangle \leq 0 \tag{3.13}$$

and

$$\|x^{\tau(k)+1} - z\|^2 \leq (1 - \beta_{\tau(k)})\|x^{\tau(k)} - z\|^2 \\ + \beta_{\tau(k)} \Big[\beta_{\tau(k)}\|z\|^2 + 2(1 - \beta_{\tau(k)})\langle x^{\tau(k)} - z, -z\rangle \\ + 2(1 - \beta_{\tau(k)})\lambda_{\tau(k)} \left\|\nabla f(x^{\tau(k)})\right\| \|z\|\Big], \tag{3.14}$$

where $\beta_{\tau(k)} \to 0$.

Since $\Gamma_{\tau(k)} < \Gamma_{\tau(k)+1}$, we have from (3.14) that

$$\|x^{\tau(k)} - z\|^2 \leq \beta_{\tau(k)}\|z\|^2 + 2(1 - \beta_{\tau(k)})\langle x^{\tau(k)} - z, -z\rangle \\ + 2(1 - \beta_{\tau(k)})\lambda_{\tau(k)}\|\nabla f(x^{\tau(k)})\|\|z\|, \tag{3.15}$$

Combining (3.13) and (3.15) yields

$$\limsup_{k\to\infty} \|x^{\tau(k)} - z\|^2 \leq 0,$$

and hence

$$\lim_{k\to\infty} \|x^{\tau(k)} - z\|^2 = 0.$$

From (3.14), we have

$$\limsup_{k\to\infty} \|x^{\tau(k)+1} - z\|^2 \leq \limsup_{k\to\infty} \|x^{\tau(k)} - z\|^2.$$

Thus

$$\lim_{k\to\infty} \|x^{\tau(k)+1} - z\|^2 = 0.$$

Therefore, by Lemma 2.3, we obtain

$$0 \leq \|x^k - z\| \leq \max\{\|x^{\tau(k)} - z\|, \|x^k - z\|\} \leq \|x^{\tau(k)+1} - z\| \to 0.$$

Consequently, $\{x^k\}$ converges strongly to $z = P_\Omega 0$. The proof is complete. □

Remark 3.1 One main advantage of our algorithm compared to others is that step-sizes are directly computed in each iteration and do not depend on the norm of A. Therefore, Theorem 3.1 improves Theorem 5.5 of Chuang [9], Theorem 4.3 of Wang and Xu [18], Theorem 5.5 of Xu [22], and Theorem 3.1 of Yao et al. [24].

4 A Relaxation Algorithm

When the sets C and Q are complicated, the computation of P_C and P_Q is expensive. This may affect the applicability of Algorithm 3.1. To overcome this drawback, we will use relaxation method of Yang [23] as follows: Consider the split feasibility problem (1.1) in which the involved sets C and Q are given as sub-level sets of convex functions, i.e.,

$$C = \{x \in H_1 : c(x) \leq 0\} \quad \text{and} \quad Q = \{y \in H_2 : q(y) \leq 0\},$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are lower semicontinuous convex functions. We assume that ∂c and ∂q are bounded operators (i.e., bounded on bounded sets). Set

$$C_k = \{x \in H_1 : c(x^k) \leq \langle \xi^k, x^k - x \rangle\}, \tag{4.1}$$

where $\xi^k \in \partial c(x^k)$, and

$$Q_k = \{y \in H_2 : q(Ax^k) \leq \langle \zeta^k, Ax^k - y \rangle\}, \tag{4.2}$$

where $\zeta^k \in \partial q(Ax^k)$. Obviously, C_k and Q_k are half-spaces, and it is easy to check that $C_k \supset C$ and $Q_k \supset Q$ hold for every $k \geq 0$ from the subdifferentiable inequality. We now define

$$f_k(x) = \frac{1}{2}\|(I - P_{Q_k})Ax\|^2, \quad k \geq 0, \tag{4.3}$$

where Q_k is given as in (4.2). We have

$$\nabla f_k(x) = A^*(I - P_{Q_k})Ax.$$

Now we introduce the following relaxation version of Algorithm 3.1.

Algorithm 4.1 (A relaxation CQ algorithm for SFP (1.1))

Initialization Take two positive sequences $\{\beta_k\}$ and $\{\rho_k\}$ satisfying the following conditions:

$$\{\beta_k\} \subset (0, 1), \quad \lim_{k\to\infty} \beta_k = 0, \quad \sum_{k=0}^{\infty} \beta_k = \infty, \tag{4.4}$$

$$\rho_k(4 - \rho_k) > 0. \tag{4.5}$$

Select initial $x^0 \in H_1$ and set $k := 0$.
Iterative Step Given x^k, if $\nabla f_k(x^k) = 0$ then stop [x^k is a solution to the SFP (1.1)]. Otherwise, compute

$$\lambda_k = \frac{\rho_k f_k(x^k)}{\|\nabla f_k(x^k)\|^2}$$

and

$$x^{k+1} = P_{C_k}\left[(1 - \beta_k)(x^k - \lambda_k \nabla f_k(x^k))\right]. \tag{4.6}$$

Let $k := k + 1$ and return to *Iterative Step*.

The following lemma is quite helpful to analyze the convergence of Algorithm 4.1.

Lemma 4.1 *If* $\nabla f_k(x^k) = 0$, *then* $x^k \in \Omega$.

Proof If $\nabla f_k(x^k) = 0$ for some $x^k \in C_k$, then

$$A^*(I - P_{Q_k})Ax^k = 0.$$

It is easy to see that $Ax^k \in Q_k$. By (4.1) and (4.2) we have $c(x^k) \leq 0$ and $q(Ax^k) \leq 0$. So $x^k \in C$ and $Ax^k \in Q$ and the proof is complete. □

The strong convergence of Algorithm 4.1 is proved below.

Theorem 4.1 *Assume that* $\inf_k \rho_k(4 - \rho_k) > 0$. *Then the sequence* $\{x^k\}$ *generated by Algorithm 4.1 converges strongly to the minimum-norm element of* Ω.

Proof Let $z := P_\Omega 0$. Since $\inf_k \rho_k(4 - \rho_k) > 0$, we may assume without loss of generality that there exists $\epsilon > 0$ such that $\rho_k(4 - \rho_k)(1 - \beta_k) \geq \epsilon$. Arguing as the proof in the proof of Theorem 3.1 and replacing f, C and Q with f_k, C_k and Q_k, respectively, we have

$$\|x^{k+1} - z\|^2 \leq \beta_k\|z\|^2 + (1 - \beta_k)\|x^k - z\|^2 - \frac{\epsilon f_k^2(x^k)}{\|\nabla f_k(x^k)\|^2}. \tag{4.7}$$

From (4.7) and (3.12), we obtain the following two inequalities:

$$\|x^{k+1} - z\|^2 \leq (1 - \beta_k)\|x^k - z\|^2 + \beta_k\delta_k,$$
$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \eta_k + \beta_k\|z\|^2,$$

where

$$\delta_k := \beta_k \|z\|^2 + 2(1-\beta_k)\langle x^k - z, -z\rangle + 2(1-\beta_k)\lambda_k \left\|\nabla f_k(x^k)\right\| \|z\|,$$
$$\eta_k = \frac{\epsilon f_k^2(x^k)}{\|\nabla f_k(x^k)\|^2}, \quad \{\beta_k\} \subset (0,1), \quad \lim_{k\to\infty} \beta_k = 0, \quad \sum_{k=0}^{\infty} \beta_k = \infty.$$

In order to use Lemma 2.5 with the data $s_k := \|x^k - z\|^2$, it remains to show that for any subsequence $\{k_l\}$ of $\{k\}$,

$$\eta_{k_l} \to 0 \Longrightarrow \limsup_{l\to\infty} \delta_{k_l} \le 0.$$

A similar argument as in the proof of Theorem 3.1 shows that

$$\lim_{l\to\infty} f_{k_l}(x^{k_l}) = 0. \tag{4.8}$$

or equivalently,

$$\lim_{l\to\infty} \left\|(I - P_{Q_{k_l}})Ax^{k_l}\right\|^2 = 0. \tag{4.9}$$

Since $\{x^{k_l}\}$ is bounded, there exists a subsequence $\{x^{k_{l_m}}\}$ of $\{x^{k_l}\}$ which converges weakly to $\bar{x}$. Without loss of generality, we can assume that $x^{k_l} \rightharpoonup \bar{x}$. Since $P_{Q_{k_l}} Ax^{k_l} \in Q_{k_l}$, we have

$$q(Ax^{k_l}) \le \left\langle \zeta^{k_l}, Ax^{k_l} - P_{Q_{k_l}} Ax^{k_l} \right\rangle, \tag{4.10}$$

where $\zeta^{k_l} \in \partial q(Ax^{k_l})$. From the boundedness assumption of ζ^{k_l} and (4.9), we have

$$q(Ax^{k_l}) \le \|\zeta^{k_l}\| \left\|Ax^{k_l} - P_{Q_{k_l}} Ax^{k_l}\right\| \to 0. \tag{4.11}$$

From the weak lower semicontinuity of the convex function $q(x)$ and since $x^{k_l} \rightharpoonup \bar{x}$, it follows from (4.13) that

$$q(A\bar{x}) \le \liminf_{l\to\infty} q(Ax^{k_l}) \le 0,$$

which means that $A\bar{x} \in Q$.

We will prove that

$$\lim_{l\to\infty} \|x^{k_l} - x^{k_l+1}\| = 0. \tag{4.12}$$

Indeed, from (4.6) we obtain

$$\begin{aligned}\|x^{k_l+1} - x^{k_l}\| &= \left\| P_{C_{k_l}}\left[(1-\beta_{k_l})\left(x^{k_l} - \lambda_{k_l}\nabla f_{k_l}(x^{k_l})\right)\right] - x^{k_l}\right\| \\ &\leq \left\|(1-\beta_{k_l})\left(x^{k_l} - \lambda_{k_l}\nabla f_{k_l}(x^{k_l})\right) - x^{k_l}\right\| \\ &\leq \beta_{k_l}\left\|x^{k_l} - \lambda_{k_l}\nabla f_{k_l}(x^{k_l})\right\| + \lambda_{k_l}\left\|\nabla f_{k_l}(x^{k_l})\right\| \to 0,\end{aligned}$$

as $l \to \infty$.

Further, using the fact that $x^{k_l+1} \in C_{k_l}$ and by the definition of C_{k_l}, we get

$$c(x^{k_l}) \leq \langle \xi^{k_l}, x^{k_l} - x^{k_l+1}\rangle,$$

where $\xi^{k_l} \in \partial c(x^{k_l})$. Due to the boundedness of ξ^{k_l} and (4.12), we have

$$c(x^{k_l}) \leq \|\xi^{k_l}\|\left\|x^{k_l} - x^{k_l+1}\right\| \to 0 \tag{4.13}$$

as $l \to \infty$. Similarly, we obtain that $c(\bar{x}) \leq 0$, i.e., $\bar{x} \in C$.

We now deduce that

$$\begin{aligned}\limsup_{l\to\infty}\delta_{k_l} &= \limsup_{l\to\infty}\Big[\beta_{k_l}\|z\|^2 + 2(1-\beta_{k_l})\langle x^{k_l} - z, -z\rangle \\ &\quad + 2(1-\beta_{k_l})\lambda_{k_l}\left\|\nabla f(x^{k_l})\right\|\|z\|\Big] \\ &= 2\limsup_{l\to\infty}\langle x^{k_l} - z, -z\rangle \\ &= 2\max_{\bar{z}\in\omega_w(x\{^{k_l}\})}\langle \bar{z} - z, -z\rangle \leq 0.\end{aligned}$$

Finally, using Lemma 2.5, we have $\|x^k - z\| \to 0$. We thus complete the proof. □

5 Numerical Experiments

In this section, we provide the numerical examples and illustrate its performance by using Algorithm 3.1.

Example 5.1 Let $H_1 = H_2 = L_2[0, 1]$ with the inner product given by

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\mathrm{d}t.$$

Let $C = \{x \in L_2[0, 1] : \|x\|_{L_2} \leq 1\}$ and $Q = \{x \in L_2[0, 1] : \langle x, \frac{t}{2}\rangle = 0\}$. Find $x \in C$ such that $Ax \in Q$, where $(Ax)(t) = \frac{x(t)}{2}$.

Table 1 Algorithm 3.1 with different cases of ρ_k

		$\rho_k = \frac{0.5k}{k+1}$	$\rho_k = \frac{k}{k+1}$	$\rho_k = \frac{2k}{k+1}$	$\rho_k = \frac{3.5k}{k+1}$
$x^1 = \sin(t) + t^2$	No. of Iter.	7	5	3	2
	cpu (time)	0.0285646	0.0211888	0.0119378	0.0081063
$x^1 = e^t + 2t$	No. of Iter.	10	6	4	2
	cpu (time)	0.0405886	0.0236906	0.0155129	0.0088102

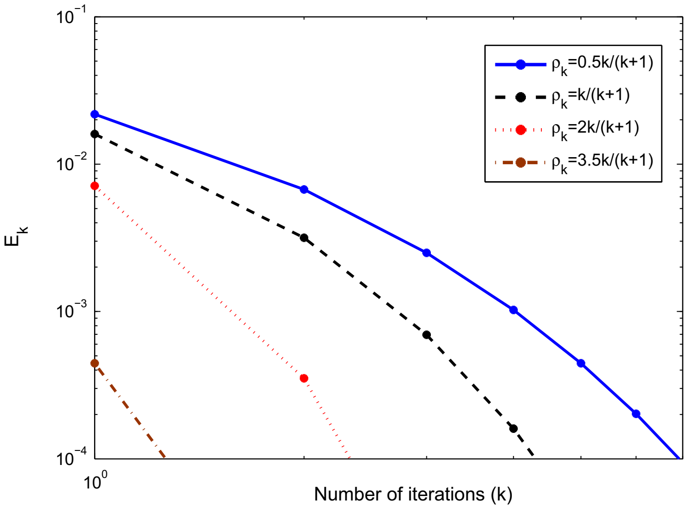

Fig. 1 Error plotting with $x^1 = \sin(t) + t^2$

Choose $\beta_k = \frac{1}{k+1}$ for all $k \in \mathbb{N}$. The stopping criterion is defined by

$$E_k = \frac{1}{2} \left\| Ax^k - P_Q Ax^k \right\|_{L_2}^2 < 10^{-4}.$$

We now study its convergence in terms of the number of iterations and the cpu time with different step-sizes of $\{\rho_k\}$ as reported in Table 1.

The error plotting of E_k for each choice of x^1 are shown in Figs. 1 and 2, respectively.

We next provide some numerical examples and illustrate its performance by using the modified relaxed CQ method (Algorithm 4.1).

Example 5.2 Let $H_1 = H_2 = \mathbb{R}^3$, $C = \{x = (a, b, c)^T \in \mathbb{R}^3 : a^2 + b^2 - 4 \leq 0\}$ and $Q = \{x = (a, b, c)^T \in \mathbb{R}^3 : a + c^2 - 1 \leq 0\}$. Find $x \in C$ such that $Ax \in Q$, where $A = \begin{pmatrix} -1 & 3 & 5 \\ 5 & 3 & 2 \\ 2 & 1 & 0 \end{pmatrix}$.

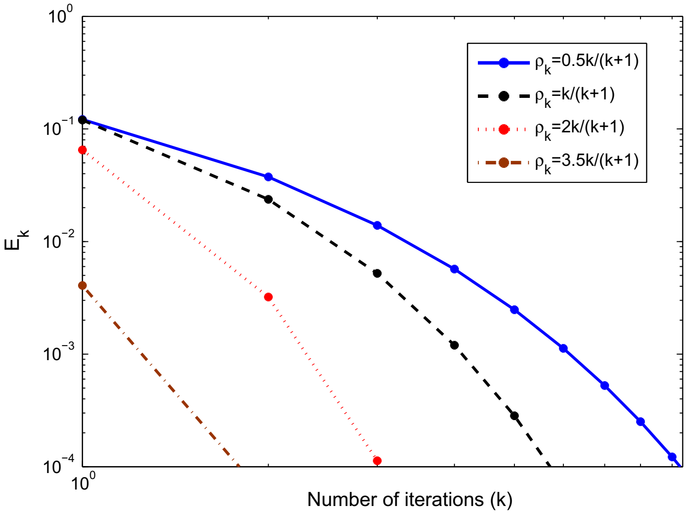

Fig. 2 Error plotting with $x^1 = e^t + 2t$

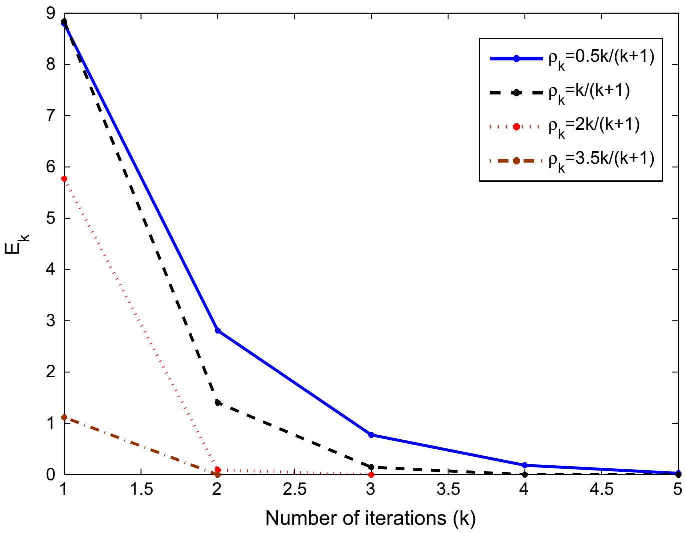

Fig. 3 Error plotting with $x^1 = [0, 1, 2]^T$

Choose $\beta_k = \frac{1}{k+1}$ for all $k \in \mathbb{N}$. The stopping criterion is defined by

$$E_k = \frac{1}{2}\left\|Ax^k - P_{Q_k}Ax^k\right\|_2^2 < 10^{-4}.$$

The numerical experiments for each case of ρ_k are shown in Figs. 3 and 4, respectively (Table 2).

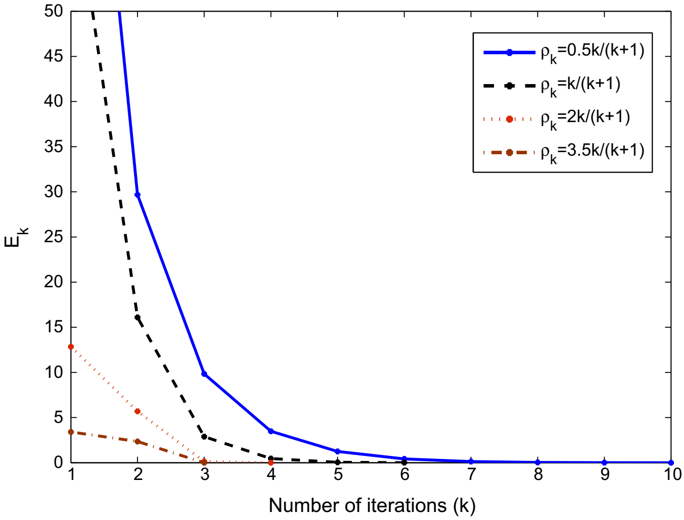

Fig. 4 Error plotting with $x^1 = [-2, 5, 4]^T$

Table 2 Algorithm 4.1 with different cases of ρ_k

		$\rho_k = \frac{0.5k}{k+1}$	$\rho_k = \frac{k}{k+1}$	$\rho_k = \frac{2k}{k+1}$	$\rho_k = \frac{3.5k}{k+1}$
$x^1 = [0, 1, 2]^T$	No. of Iter.	7	5	3	2
	cpu (time)	0.003993	0.003588	0.002996	0.002916
$x^1 = [-2, 5, 4]^T$	No. of Iter.	10	6	4	3
	cpu (time)	0.005002	0.004193	0.003783	0.003639

Remark 5.1 From our numerical experiments, it is observed that the different choices of x^1 have no effect in terms of cpu run time for the convergence of our algorithm. However, if the step-sizes $\{\rho_k\}$ is taken close to 4, then the number of iterations and the cpu time have small reduction.

Acknowledgements P. Cholamjiak was supported by the Thailand Research Fund and the Commission on Higher Education under Grant MRG5980248. S. Suantai was partially supported by Chiang Mai University.

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JOURNAL OF INDUSTRIAL AND
MANAGEMENT OPTIMIZATION

doi:10.3934/jimo.2018023

THE MODIFIED INERTIAL RELAXED CQ ALGORITHM FOR SOLVING THE SPLIT FEASIBILITY PROBLEMS

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(Communicated by Christian Clason)

Abstract. In this work, we propose a new version of inertial relaxed CQ algorithms for solving the split feasibility problems in the frameworks of Hilbert spaces. We then prove its strong convergence by using a viscosity approximation method under some weakened assumptions. To be more precisely, the computation on the norm of operators and the metric projections is relaxed. Finally, we provide numerical experiments to illustrate the convergence behavior and to show the effectiveness of the sequences constructed by the inertial technique.

1. **Introduction.** Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. In this research, we study the Split Feasibility Problem (SFP) which is the problem of finding a point $x \in C$ such that

$$Ax \in Q \tag{1}$$

where $A : H_1 \to H_2$ is a given bounded linear operator (here we denote A^* by its adjoint operator). This problem was first proposed, in finite-dimensional Hilbert spaces, by Censer and Elfving in [7] for modeling inverse problems which arise from phase retrievals, medical image reconstruction and recently in modeling of intensity modulated radiation therapy. The SFP attracts the attention of many authors due to its application in signal processing and image recovery [13]. Various algorithms have been invented to solve it (see, for examples, [4, 6, 25, 26]).

We assume the SFP (1) is consistent, and let S be the solution set, *i.e.*,

$$S = \{x \in C : Ax \in Q\}.$$

It is easily seen that S is closed and convex. In Hilbert spaces, a classical way to solve the SFP is to employ the CQ algorithm which was introduced by Byrne

2010 *Mathematics Subject Classification.* 65K05, 65K10, 49J52.

Key words and phrases. Inertial method, viscosity approximation method, relaxed CQ algorithm, split feasibility problem, hilbert space.

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[4] and is defined in the following manner: take an initial point x_1 arbitrarily and generate the sequence $\{x_n\}$ by

$$x_{n+1} = P_C(x_n - \mu_n A^*(I - P_Q)Ax_n), \quad n \geq 1, \tag{2}$$

where the step-size $\mu_n \in (0, \frac{2}{\|A\|^2})$ and P_C, P_Q are the metric projections on C and Q, respectively. We note that this algorithm is found to be a gradient-projection method in convex minimization as a spacial case. It was proved that $\{x_n\}$ generated by (2) converges weakly to a solution of SFP.

However, in general, the computation of a projection onto a general closed convex subset is difficult because of its closed form. To overcome this difficulty, Fukushima [11] suggested a so-called relaxed projection method to calculate the projection onto a level set of a convex function by computing a sequence of projections onto half-spaces containing the original level set. In the setting of finite-dimensional Hilbert spaces, Yang [28] introduced the relaxed CQ algorithms for solving SFP where the closed convex subsets C and Q are level sets of convex functions given as follows:

$$C = \{x \in H_1 : c(x) \leq 0\} \textit{ and } Q = \{y \in H_2 : q(y) \leq 0\}, \tag{3}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex functions. We assume that both c and q are subdifferentiable on H_1 and H_2, respectively, and that ∂c and ∂q are bounded operators (*i.e.*, bounded on bounded sets). It is known that every convex function defined on a finite-dimensional Hilbert space is subdifferentiable and its subdifferential operator is a bounded operator (see [3]). In what follows, we define two sets at point x_n by

$$C_n = \{x \in H_1 : c(x_n) \leq \langle \xi_n, x_n - x\rangle\}, \tag{4}$$

where $\xi_n \in \partial c(x_n)$, and

$$Q_n = \{y \in H_2 : q(Ax_n) \leq \langle \varepsilon_n, Ax_n - y\rangle\}, \tag{5}$$

where $\varepsilon_n \in \partial q(Ax_n)$. It is clear that C_n and Q_n are half-spaces and $C_n \supset C$ and $Q_n \supset Q$ for every $n \geq 1$. The specific form of the metric projections onto C_n and Q_n can be found in [3]. In fact, Yang [28] constructed a relaxed CQ algorithm for solving the SFP by using the half-spaces C_n and Q_n instead of the sets C and Q in the CQ algorithm, respectively and proved its convergence under some suitable choices of the step-sizes.

In order to achieve the convergence, in such algorithms mentioned above, the selection of the step-sizes requires prior information on the norm of the bounded linear operator (matrix in the finite-dimensional framework), which is not always possible in practice. To avoid this computation, there have been worthwhile works that the convergence is guaranteed without any prior information of the matrix norm (see, for examples [24, 25, 26, 29]). Among these works, López et al. [13] introduced a new way to select the step-size by replacing the parameter μ_n appeared in (2) by

$$\mu_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \quad n \geq 1, \tag{6}$$

where $\rho_n \in (0, 4)$, $f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ and $\nabla f(x_n) = A^*(I - P_Q)Ax_n$ for all $n \geq 1$. They also practised this way of selecting step-sizes for variants of the CQ algorithm, including a relaxed CQ algorithm, and a Halpern-type algorithm and proved both weak and strong convergence. Subsequently, in 2013, He and Zhao

[12] introduced the following Halpern-relaxed CQ algorithm in Hilbert spaces: take $x_1 \in H_1$ and generate $\{x_n\}$ by

$$x_{n+1} = P_{C_n}[\alpha_n u + (1-\alpha_n)(x_n - \tau_n \nabla f_n(x_n))], \tag{7}$$

where C_n and Q_n are, respectively, given as in (4) and (5), $\{\alpha_n\} \subset (0,1)$, $\{\rho_n\} \subset (0,4)$ and the sequence $\{\tau_n\}$ is given by

$$\tau_n = \frac{\rho_n f_n(x_n)}{\|\nabla f_n(x_n)\|^2} \tag{8}$$

and

$$f_n(x_n) = \frac{1}{2}\|(I - P_{Q_n})Ax_n\|^2, \quad n \geq 1. \tag{9}$$

In this case, we have

$$\nabla f_n(x_n) = A^*(I - P_{Q_n})Ax_n. \tag{10}$$

They obtained the strong convergence provided $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and the step-size is chosen such that $\inf_{n\in\mathbb{N}} \rho_n(4-\rho_n) > 0$.

In optimization theory, to speed up the convergence rate, Polyak [22] firstly proposed the heavy ball method of the two-order time dynamical system which is a two-step iterative method for minimizing a smooth convex function f. In order to improve the convergence rate, Nesterov [21] introduced a modified heavy ball method as follows:

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= y_n - \lambda_n \nabla f(y_n), \quad n \geq 1, \end{aligned} \tag{11}$$

where $\theta_n \in [0,1)$ is an extrapolation factor and λ_n is a positive sequence. Here, the inertia is represented by the term $\theta_n(x_n - x_{n-1})$. It is remarkable that the inertial terminology greatly improves the performance of the algorithm and has a nice convergence properties (see [9, 10, 14]). In [1], Alvarez and Attouch employed the idea of the heavy ball method to the setting of a general maximal monotone operator using the framework of the proximal point algorithm [23]. This method is called the inertial proximal point algorithm and it is of the following form:

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= (I + \lambda_n T)^{-1}(y_n), \quad n \geq 1, \end{aligned} \tag{12}$$

where T is a maximal monotone operator. It was proved that if λ_n is non-decreasing and $\theta_n \in [0,1)$ is chosen such that

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty,$$

then $\{x_n\}$ generated by (12) converges to a zero point of T. See also [20].

In subsequent work, Maingé [15] (see also [16]) introduced the inertial Mann algorithm for solving the fixed point problem of nonexpansive mappings in Hilbert spaces as follows: take $x_0, x_1 \in H_1$ and generate the sequence $\{x_n\}$ by

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= y_n + \alpha_n(Ty_n - y_n), \quad n \geq 1, \end{aligned} \tag{13}$$

where T is a nonexpansive mapping on H_1, $\theta_n \in [0,1)$ and $\alpha_n \in (0,1)$. It was shown that the sequence $\{x_n\}$ converges weakly to a fixed point of T under the following conditions:

(A) $\theta_n \in [0, \theta)$ where $\theta \in [0, 1)$;
(B) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$;
(C) $0 < \inf_{n\geq 1} \alpha_n \leq \sup_{n\geq 1} \alpha_n < 1$.

Very recently, Dang et al. [9] proposed two kinds of the inertial relaxed CQ algorithms for solving SFP in Hilbert spaces as follows: take $x_0, x_1 \in H_1$ and generate the sequence $\{x_n\}$ iteratively by

$$\begin{aligned} y_n &= x_n + \phi_n(x_n - x_{n-1}), \\ x_{n+1} &= P_{C_n}(y_n - \gamma A^T(I - P_{Q_n})Ay_n), \quad n \geq 1, \end{aligned} \tag{14}$$

and

$$\begin{aligned} y_n &= x_n + \phi_n(x_n - x_{n-1}), \\ x_{n+1} &= (1-\beta_n)y_n + \beta_n P_{C_n}(y_n - \gamma A^T(I - P_{Q_n})Ay_n), \quad n \geq 1. \end{aligned} \tag{15}$$

It was proved that if $\gamma \in (0, 2/L)$ where L denotes the spectral radius of $A^T A$ and $\phi_n \in [0, \bar{\phi}_n)$ where $\bar{\phi}_n = \min\{\phi, 1/\max\{n^2\|x_n - x_{n-1}\|, n^2\|x_n - x_{n-1}\|^2\}\}$, $\phi \in [0, 1)$, then $\{x_n\}$ defined by (14) converges weakly to a solution in SFP. Moreover, in addition, if $0 < \inf_{n\geq 0} \beta_n < R < 1$, then $\{x_n\}$ defined by the modified inertial relaxed CQ algorithm (15) converges weakly to a solution in SFP. See also [8].

In this work, we suggest the modified inertial relaxed CQ algorithm with a new adaptive way of determining the step-size sequence for solving the SFP. Using the viscosity approximation method introduced by [19], we then prove its strong convergence of the sequence generated by the proposed scheme. Our algorithm can be implemented easily since it involves the metric projections onto half-spaces which have exact forms and has no need to know a priori information of the norm of bounded linear operators. Numerical experiments are included to illustrate the effectiveness of our algorithm. The main results complement the results in [4, 9, 12, 13] and others. To this end, for $x \in H_1$, we now define

$$g_n(x) = \frac{1}{2}\|(I - P_{C_n})x\|^2, \quad n \geq 1, \tag{16}$$

where C_n is given as in (4). We then have, for $x \in H_1$

$$\nabla g_n(x) = (I - P_{C_n})x, \quad n \geq 1. \tag{17}$$

The rest of this paper is organized as follows: Some basic concepts and lemmas are provided in Section 2. The modified inertial relaxed CQ algorithm is presented and the strong convergence result of this paper is proved in Section 3. Finally, in Section 4, numerical experiments are shown to support our proposed algorithm.

2. **Preliminaries.** In this section, we give some preliminaries which will be used in the sequel. Let H be a Hilbert space. Recall that a mapping $T : H \to H$ is said to be nonexpansive if, for all $x, y \in H$,

$$\|Tx - Ty\| \leq \|x - y\| \tag{18}$$

$T : H \to H$ is said to be firmly nonexpansive if, for all $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \tag{19}$$

or equivalently

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2 \tag{20}$$

for all $x, y \in H$. It is known that T is firmly nonexpansive if and only if $I - T$ is firmly nonexpansive. We know that the metric projection P_C from H onto a nonempty, closed and convex subset $C \subset H$ is a typical example of a firmly nonexpansive mapping, which is defined by

$$P_C x := \arg\min_{y \in C} \|x - y\|^2, \quad x \in H. \tag{21}$$

It is well known that P_C is characterized by the inequality, for $x \in H$

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \quad \forall y \in C. \tag{22}$$

In a real Hilbert space H, we have the following equality:

$$\langle x, y \rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2 \tag{23}$$

and the subdifferential inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \tag{24}$$

for all $x, y \in H$.

Definition 2.1. Let $f : H \to \mathbb{R}$ be a convex function. The subdifferential of f at x is defined as

$$\partial f(x) = \{\xi \in H : f(y) \geq f(x) + \langle \xi, y - x \rangle, \quad \forall y \in H\}. \tag{25}$$

A function $f : H \to \mathbb{R}$ is said to be weakly lower semi-continuous at x if x_n converges weakly to x implies

$$f(x) \leq \liminf_{n \to \infty} f(x_n). \tag{26}$$

We know the following results (see [2, 5]).

Lemma 2.2. *Let $f : H \to \mathbb{R}$ be given by $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$. Then*
(i) *f is convex and differential.*
(ii) *$\nabla f(x) = A^*(I - P_Q)Ax$, $x \in H$.*
(iii) *f is weakly lower semi-continuous on H.*
(iv) *$\|\nabla f(x) - \nabla f(y)\| \leq \|A\|^2 \|x - y\|$ for all $x, y \in H$.*

Lemma 2.3. [17, 27] *Let $\{a_n\}$ and $\{c_n\}$ are sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \quad n \geq 1, \tag{27}$$

where $\{\delta_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a real sequence. Assume $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:
(i) *If $b_n \leq \delta_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*
(ii) *If $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} b_n/\delta_n \leq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.4. [18] *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\psi(n)\}_{n \geq n_0}$ of integers as follows:*

$$\psi(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}, \tag{28}$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:
(i) *$\Gamma(n_0) \leq \Gamma(n_0 + 1) \leq \ldots$ and $\Gamma(n) \to \infty$;*
(ii) *$\Gamma_{\psi(n)} \leq \Gamma_{\psi(n)+1}$ and $\Gamma_n \leq \Gamma_{\psi(n)+1}$, $\forall n \geq n_0$.*

3. **Strong convergence theorem.** In this section, we propose the modified inertial relaxed CQ algorithm as follows:

Algorithm 3.1 Let $f : H_1 \to H_1$ be a contraction (*i.e.* there exists a constant $\alpha \in (0,1)$ such that $\|f(x)-f(y)\| \leq \alpha\|x-y\|$ for all $x, y \in H_1$) and let $\{\alpha_n\} \subset (0,1)$, $\{\theta_n\} \subset [0,1)$ and $\{\rho_n\} \subset (0,4)$. Take $x_0, x_1 \in H_1$ arbitrarily and generate the sequences $\{x_n\}$ and $\{y_n\}$ by the following manner:

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= P_{C_n}[\alpha_n f(y_n) + (1-\alpha_n)(y_n - \tau_n \nabla f_n(y_n))], \quad n \geq 1. \end{aligned} \tag{29}$$

Here we set

$$\tau_n = \rho_n \frac{f_n(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2}. \tag{30}$$

for all $n \in \mathbb{N}$. We remark that if $\nabla f_n(y_n) = \nabla g_n(y_n) = 0$, then y_n is a solution of SFP.

We next prove the strong convergence of the sequence generated by the proposed algorithm.

Theorem 3.1. *Assume that* $\{\alpha_n\} \subset (0,1)$, $\{\rho_n\} \subset (0,4)$ *and* $\{\theta_n\} \subset [0,\theta)$, *where* $\theta \in [0,1)$ *satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\inf_{n\in\mathbb{N}} \rho_n(4-\rho_n) > 0$;

(C3) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then the sequence $\{x_n\}$ *generated by Algorithm 3.1 converges strongly to a solution in SFP.*

Proof. Let $z = P_S f(z)$. Then $z \in C \subset C_n$ and $Az \in Q \subset Q_n$ for all $n \in \mathbb{N}$. It means $z = P_{C_n} z$ and $Az = P_{Q_n} Az$ for all $n \in \mathbb{N}$. Set $v_n = y_n - \tau_n \nabla f_n(y_n)$ and $w_n = \alpha_n f(y_n) + (1-\alpha_n)v_n$ for all $n \in \mathbb{N}$. Then we obtain

$$\begin{aligned} \|y_n - z\| &= \|x_n - z + \theta_n(x_n - x_{n-1})\| \\ &\leq \|x_n - z\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{31}$$

Since $(I - P_{Q_n})$ is firmly nonexpansive,

$$\begin{aligned} \langle \nabla f_n(y_n), y_n - z\rangle &= \langle (I-P_{Q_n})Ay_n, Ay_n - Az\rangle \\ &\geq \|(I-P_{Q_n})Ay_n\|^2 \\ &= 2f_n(y_n). \end{aligned} \tag{32}$$

Using (30) and (32), it follows that

$$\begin{aligned} \|v_n - z\|^2 &= \|y_n - \tau_n\nabla f_n(y_n) - z\|^2 \\ &= \|y_n - z\|^2 + \tau_n^2\|\nabla f_n(y_n)\|^2 - 2\tau_n\langle \nabla f_n(y_n), y_n - z\rangle \\ &\leq \|y_n - z\|^2 + \tau_n^2\|\nabla f_n(y_n)\|^2 - 4\tau_n f_n(y_n) \\ &= \|y_n - z\|^2 + \rho_n^2 \frac{f_n^2(y_n)}{(\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2)^2}\|\nabla f_n(y_n)\|^2 \\ &\quad - 4\rho_n \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} \end{aligned}$$

$$\begin{aligned}
&\leq \|y_n - z\|^2 + \rho_n^2 \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} \\
&\quad - 4\rho_n \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} \\
&= \|y_n - z\|^2 - \rho_n(4-\rho_n) \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2}. \qquad (33)
\end{aligned}$$

So, since $\rho_n \in (0,4)$, we have for all $n \in \mathbb{N}$,

$$\|v_n - z\| \leq \|y_n - z\|. \qquad (34)$$

Thus, using (34) and the nonexpansiveness of P_{C_n}, we obtain

$$\begin{aligned}
\|x_{n+1} - z\| &= \|P_{C_n} w_n - P_{C_n} z\| \\
&\leq \|w_n - z\| \\
&= \|\alpha_n(f(y_n) - f(z)) + \alpha_n(f(z) - z) + (1-\alpha_n)(v_n - z)\| \\
&\leq \alpha_n \alpha \|y_n - z\| + \alpha_n \|f(z) - z\| + (1-\alpha_n)\|v_n - z\| \\
&\leq \alpha_n \alpha \|y_n - z\| + \alpha_n \|f(z) - z\| + (1-\alpha_n)\|y_n - z\| \\
&= (1 - \alpha_n(1-\alpha))\|y_n - z\| + \alpha_n \|f(z) - z\|. \qquad (35)
\end{aligned}$$

Combining (31) and (35), we immediately obtain

$$\|x_{n+1} - z\| \leq (1-\alpha_n(1-\alpha))\|x_n - z\| + (1-\alpha_n(1-\alpha))\theta_n \|x_n - x_{n-1}\| + \alpha_n \|f(z) - z\|. \qquad (36)$$

By conditions (C1) and (C3), we see that

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \left(\frac{1-\alpha_n(1-\alpha)}{1-\alpha} \right) \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0, \qquad (37)$$

which implies that the sequence $\{\sigma_n\}$ is bounded. Putting

$$M = \max \left\{ \frac{\|f(z)-z\|}{1-\alpha}, \sup_{n\in\mathbb{N}} \sigma_n \right\}$$

and using Lemma 2.3 (i), we conclude that the sequence $\{\|x_n - z\|\}$ is bounded. This shows that the sequence $\{x_n\}$ is bounded and so is $\{y_n\}$. On the other hand, we see that

$$\begin{aligned}
\|y_n - z\|^2 &= \|x_n - z + \theta_n(x_n - x_{n-1})\|^2 \\
&= \|x_n - z\|^2 + 2\theta_n \langle x_n - z, x_n - x_{n-1} \rangle + \theta_n^2 \|x_n - x_{n-1}\|^2 \qquad (38)
\end{aligned}$$

and, from (23)

$$\langle x_n - z, x_n - x_{n-1} \rangle = -\frac{1}{2}\|x_{n-1} - z\|^2 + \frac{1}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_n - x_{n-1}\|^2. \qquad (39)$$

Combining (38) and (39), we obtain, since $\theta_n \in [0,1)$,

$$\begin{aligned}
\|y_n - z\|^2 &= \|x_n - z\|^2 + \theta_n(-\|x_{n-1} - z\|^2 + \|x_n - z\|^2 + \|x_n - x_{n-1}\|^2) \\
&\quad + \theta_n^2 \|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\
&\quad + 2\theta_n \|x_n - x_{n-1}\|^2. \qquad (40)
\end{aligned}$$

Using (24), (33) and the firm nonexpansiveness of P_{C_n}, we also have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|P_{C_n}w_n - P_{C_n}z\|^2 \\
&\le \|w_n - z\|^2 - \|P_{C_n}w_n - w_n\|^2 \\
&= \|\alpha_n(f(y_n)-z) + (1-\alpha_n)(v_n - z)\|^2 - \|P_{C_n}w_n - w_n\|^2 \\
&\le (1-\alpha_n)\|v_n - z\|^2 + 2\alpha_n\langle f(y_n)-z, w_n - z\rangle - \|P_{C_n}w_n - w_n\|^2 \\
&\le (1-\alpha_n)\|y_n - z\|^2 \\
&\quad -(1-\alpha_n)\rho_n(4-\rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} \\
&\quad + 2\alpha_n\langle f(y_n)-z, w_n - z\rangle - \|P_{C_n}w_n - w_n\|^2. \qquad (41)
\end{aligned}$$

Combining (40) and (41), we thus have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le (1-\alpha_n)\|x_n - z\|^2 + (1-\alpha_n)\theta_n(\|x_n - z\|^2 - \|x_{n-1}-z\|^2) \\
&\quad + 2(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2 \\
&\quad -(1-\alpha_n)\rho_n(4-\rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} \\
&\quad + 2\alpha_n\langle f(y_n)-z, w_n - z\rangle - \|P_{C_n}w_n - w_n\|^2. \qquad (42)
\end{aligned}$$

Set $\Gamma_n = \|x_n - z\|^2$ for all $n \in \mathbb{N}$. We next consider the following two cases.

Case 1. Suppose that there exists a natural number N such that $\Gamma_{n+1} \le \Gamma_n$ for all $n \ge N$. In this case, $\{\Gamma_n\}$ is convergent. From (C1) and (C2), we can find a constant σ such that $(1-\alpha_n)\rho_n(4-\rho_n) \ge \sigma > 0$ for all $n \in \mathbb{N}$. So (42) reduces to

$$\begin{aligned}
\Gamma_{n+1} &\le (1-\alpha_n)\Gamma_n + (1-\alpha_n)\theta_n(\Gamma_n - \Gamma_{n-1}) + 2(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2 \\
&\quad - \sigma\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} + 2\alpha_n\langle f(y_n)-z, w_n - z\rangle \\
&\quad -\|P_{C_n}w_n - w_n\|^2, \qquad (43)
\end{aligned}$$

which gives

$$\begin{aligned}
\sigma\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} &\le (\Gamma_n - \Gamma_{n+1}) + (1-\alpha_n)\theta_n(\Gamma_n - \Gamma_{n-1}) \\
&\quad +2(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2 \\
&\quad +2\alpha_n\langle f(y_n)-z, w_n - z\rangle. \qquad (44)
\end{aligned}$$

It is easy to see that (C3) implies $\theta_n\|x_n - x_{n-1}\| \to 0$ since $\{\alpha_n\}$ is bounded. Since $\{\Gamma_n\}$ converges and $\alpha_n \to 0$,

$$\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2} \to 0 \qquad (45)$$

as $n \to \infty$. It is easily checked that $\{\nabla g_n(y_n)\}$ is bounded. Also, we have $\{\nabla f_n(y_n)\}$ is bounded since $\{y_n\}$ is bounded. Indeed, by Lemma 2.2 (iv), we have

$$\|\nabla f_n(y_n)\| = \|\nabla f_n(y_n) - \nabla f_n(z)\| \le \|A\|^2\|y_n - z\|. \qquad (46)$$

So from (45), we conclude that $f_n(y_n) \to 0$ as $n \to \infty$, *i.e.*,

$$\|(I - P_{Q_n})Ay_n\| \to 0, \qquad (47)$$

as $n \to \infty$. Since ∂q is bounded on bounded sets, there exists a constant $\mu > 0$ such that $\|\varepsilon_n\| \le \mu$ for all $n \in \mathbb{N}$. From (47) and $P_{Q_n}(Ay_n) \in Q_n$, we have

$$\begin{aligned} q(Ay_n) &\le \langle \varepsilon_n, Ay_n - P_{Q_n}(Ay_n)\rangle \\ &\le \mu\|(I - P_{Q_n})Ay_n\| \\ &\to 0, \end{aligned} \tag{48}$$

as $n \to \infty$. Since $\{y_n\}$ is bounded, there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \rightharpoonup x^* \in H_1$. It also follows that $Ay_{n_k} \rightharpoonup Ax^* \in H_2$. By the lower-semicontinuity of q, we have

$$q(Ax^*) \le \liminf_{k\to\infty} q(Ay_{n_k}) \le 0. \tag{49}$$

This shows that $Ax^* \in Q$. We next prove that $x^* \in C$. Again, using (43), we have

$$\begin{aligned} (1-\alpha_n)\|P_{C_n}w_n - w_n\|^2 &\le (\Gamma_n - \Gamma_{n+1}) + (1-\alpha_n)\theta_n(\Gamma_n - \Gamma_{n-1}) \\ &\quad + 2(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2 \\ &\quad + 2\alpha_n\langle f(y_n) - z, w_n - z\rangle, \end{aligned} \tag{50}$$

consequently, as $n \to \infty$,

$$\|P_{C_n}w_n - w_n\| \to 0. \tag{51}$$

By the definition of C_n, we obtain

$$c(w_n) \le \langle \xi_n, w_n - P_{C_n}w_n\rangle \le \kappa\|w_n - P_{C_n}w_n\| \to 0, \tag{52}$$

as $n \to \infty$, where κ is a constant such that $\|\xi_n\| \le \kappa$ for all $n \in \mathbb{N}$. We next consider the following estimation:

$$\begin{aligned} \|v_n - y_n\| &= \|y_n - \tau_n\nabla f_n(y_n) - y_n\| \\ &= \tau_n\|\nabla f_n(y_n)\| \\ &= \rho_n\frac{f_n(y_n)}{\|\nabla f_n(y_n)\|^2 + \|\nabla g_n(y_n)\|^2}\|\nabla f_n(y_n)\| \\ &\to 0, \end{aligned} \tag{53}$$

as $n \to \infty$. We also have

$$\|w_n - y_n\| \le \alpha_n\|f(y_n) - y_n\| + (1-\alpha_n)\|v_n - y_n\| \to 0, \tag{54}$$

as $n \to \infty$. Hence, since $y_{n_k} \rightharpoonup x^*$, there is a corresponding subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \rightharpoonup x^*$. From (52), it follows that

$$c(x^*) \le \liminf_{k\to\infty} c(w_{n_k}) = 0. \tag{55}$$

So we obtain $x^* \in C$ and hence $x^* \in S$. From (22) we obtain

$$\begin{aligned} \limsup_{n\to\infty}\langle f(z) - z, w_n - z\rangle &= \lim_{k\to\infty}\langle f(z) - z, w_{n_k} - z\rangle \\ &= \langle f(z) - z, x^* - z\rangle \\ &\le 0. \end{aligned} \tag{56}$$

On the other hand, we see that

$$\begin{aligned}
\|w_n - z\|^2 &= \langle w_n - z, w_n - z\rangle \\
&= \alpha_n\langle f(y_n) - f(z), w_n - z\rangle + \alpha_n\langle f(z) - z, w_n - z\rangle \\
&\quad +(1-\alpha_n)\langle v_n - z, w_n - z\rangle \\
&\leq \alpha_n\alpha\|y_n - z\|\|w_n - z\| + \alpha_n\langle f(z) - z, w_n - z\rangle \\
&\quad +(1-\alpha_n)\|v_n - z\|\|w_n - z\| \\
&\leq (1-\alpha_n(1-\alpha))\|y_n - z\|\|w_n - z\| + \alpha_n\langle f(z) - z, w_n - z\rangle \\
&\leq (1-\alpha_n(1-\alpha))\Big(\frac{\|y_n - z\|^2}{2} + \frac{\|w_n - z\|^2}{2}\Big) \\
&\quad +\alpha_n\langle f(z) - z, w_n - z\rangle, \qquad (57)
\end{aligned}$$

which gives

$$\begin{aligned}
\|w_n - z\|^2 &\leq \frac{1-\alpha_n(1-\alpha)}{1+\alpha_n(1-\alpha)}\|y_n - z\|^2 + \frac{2\alpha_n}{1+\alpha_n(1-\alpha)}\langle f(z) - z, w_n - z\rangle \\
&\leq \frac{1-\alpha_n(1-\alpha)}{1+\alpha_n(1-\alpha)}(\|x_n - z\| + \theta_n\|x_n - x_{n-1}\|)^2 \\
&\quad +\frac{2\alpha_n}{1+\alpha_n(1-\alpha)}\langle f(z) - z, w_n - z\rangle \\
&= \left(1 - \frac{2\alpha_n(1-\alpha)}{1+\alpha_n(1-\alpha)}\right)(\|x_n - z\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - z\| \\
&\quad + \theta_n^2\|x_n - x_{n-1}\|^2) \\
&\quad + \frac{2\alpha_n}{1+\alpha_n(1-\alpha)}\langle f(z) - z, w_n - z\rangle. \qquad (58)
\end{aligned}$$

Then, by (58), we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|P_{C_n} w_n - z\|^2 \\
&\leq \|w_n - z\|^2 \\
&\leq \left(1 - \frac{2\alpha_n(1-\alpha)}{1+\alpha_n(1-\alpha)}\right)(\|x_n - z\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - z\| \\
&\quad + \theta_n^2\|x_n - x_{n-1}\|^2) \\
&\quad + \frac{2\alpha_n}{1+\alpha_n(1-\alpha)}\langle f(z) - z, w_n - z\rangle. \qquad (59)
\end{aligned}$$

Put $M_1 = \sup_{n\in\mathbb{N}}\|x_n - z\|$ and $\gamma_n = \frac{2\alpha_n(1-\alpha)}{1+\alpha_n(1-\alpha)}$ for all $n \in \mathbb{N}$. It is easily checked that $\gamma_n \in (0, 1)$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}\gamma_n = \infty$. From (59), it follows that

$$\begin{aligned}
\Gamma_{n+1} &\leq (1-\gamma_n)\Gamma_n + 2\theta_n\|x_n - x_{n-1}\|M_1 + \theta_n\|x_n - x_{n-1}\|^2 \\
&\quad + \frac{2\alpha_n}{1+\alpha_n(1-\alpha)}\langle f(z) - z, w_n - z\rangle. \qquad (60)
\end{aligned}$$

Applying Lemma 2.3 (ii) and using (56) and the conditions (C1) and (C3), we conclude that $\Gamma_n = \|x_n - z\|^2 \to 0$ and thus $x_n \to z$ as $n \to \infty$.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\psi : \mathbb{N} \to \mathbb{N}$ as in (28). Then, by Lemma 2.4, we have $\Gamma_{\psi(n)} \leq \Gamma_{\psi(n)+1}$. From (42), it follows that

$$\begin{aligned}\Gamma_{\psi(n)+1} \leq\ & (1-\alpha_{\psi(n)})\Gamma_{\psi(n)} + (1-\alpha_{\psi(n)})\theta_{\psi(n)}(\Gamma_{\psi(n)} - \Gamma_{\psi(n)-1}) \\ & + 2(1-\alpha_{\psi(n)})\theta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|^2 \\ & -\sigma\frac{f^2{}_{\psi(n)}(y_{\psi(n)})}{\|\nabla f_{\psi(n)}(y_{\psi(n)})\|^2 + \|\nabla g_{\psi(n)}(y_{\psi(n)})\|^2} \\ & - (1-\alpha_{\psi(n)})\|P_{C_{\psi(n)}}w_{\psi(n)} - w_{\psi(n)}\|^2 \\ & +2\alpha_{\psi(n)}\langle f(y_{\psi(n)}) - z, w_{\psi(n)} - z\rangle, \end{aligned} \tag{61}$$

which gives

$$\begin{aligned} & \sigma\frac{f^2_{\psi(n)}(y_{\psi(n)})}{\|\nabla f_{\psi(n)}(y_{\psi(n)})\|^2 + \|\nabla g_{\psi(n)}(y_{\psi(n)})\|^2} \\ \leq\ & (1-\alpha_{\psi(n)})\theta_{\psi(n)}(\Gamma_{\psi(n)} - \Gamma_{\psi(n)-1}) \\ & + 2(1-\alpha_{\psi(n)})\theta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|^2 \\ & + 2\alpha_{\psi(n)}\langle f(y_{\psi(n)}) - z, w_{\psi(n)} - z\rangle \\ \leq\ & (1-\alpha_{\psi(n)})\theta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|(\sqrt{\Gamma_{\psi(n)}} + \sqrt{\Gamma_{\psi(n)-1}}) \\ & + 2(1-\alpha_{\psi(n)})\theta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|^2 \\ & + 2\alpha_{\psi(n)}\langle f(y_{\psi(n)}) - z, w_{\psi(n)} - z\rangle \\ \to\ & 0, \end{aligned} \tag{62}$$

as $n \to \infty$. It follows that $f_{\psi(n)}(y_{\psi(n)}) = \|(I - P_{Q_{\psi(n)}})Ay_{\psi(n)}\| \to 0$. Similarly, by (61), we can show that

$$\lim_{n\to\infty}\|P_{C_{\psi(n)}}w_{\psi(n)} - w_{\psi(n)}\| = 0 \tag{63}$$

and by (54)

$$\lim_{n\to\infty}\|w_{\psi(n)} - y_{\psi(n)}\| = 0. \tag{64}$$

Now repeating the argument of the proof in Case 1, we obtain

$$\limsup_{n\to\infty}\langle f(z) - z, w_{\psi(n)} - z\rangle \leq 0. \tag{65}$$

On the other hand, observe that

$$\|y_{\psi(n)} - x_{\psi(n)}\| = \theta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\| \to 0, \tag{66}$$

as $n \to \infty$. It follows that $\|x_{\psi(n)+1} - x_{\psi(n)}\| \to 0$ as $n \to \infty$. Indeed, by (63), (64) and (66), we have

$$\begin{aligned}\|x_{\psi(n)+1} - x_{\psi(n)}\| =\ & \|P_{C_{\psi(n)}}w_{\psi(n)} - x_{\psi(n)}\| \\ \leq\ & \|P_{C_{\psi(n)}}w_{\psi(n)} - w_{\psi(n)}\| + \|w_{\psi(n)} - y_{\psi(n)}\| \\ & +\|y_{\psi(n)} - x_{\psi(n)}\| \\ \to\ & 0, \end{aligned} \tag{67}$$

as $n \to \infty$. Using (60), we have

$$\begin{aligned}\Gamma_{\psi(n)+1} \leq\ & (1-\gamma_{\psi(n)})\Gamma_{\psi(n)} + 2\theta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|M_1 \\ & +\theta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|^2 \\ & + \frac{2\alpha_{\psi(n)}}{1+\alpha_{\psi(n)}(1-\alpha)}\langle f(z) - z, w_{\psi(n)} - z\rangle, \end{aligned} \tag{68}$$

which implies

$$\begin{aligned}\gamma_{\psi(n)}\Gamma_{\psi(n)} &\leq 2\theta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|M_1 + \theta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|^2 \\ &\quad + \frac{2\alpha_{\psi(n)}}{1+\alpha_{\psi(n)}(1-\alpha)}\langle f(z) - z, w_{\psi(n)} - z\rangle. \end{aligned} \tag{69}$$

Hence

$$\begin{aligned}\Gamma_{\psi(n)} &\leq \frac{2\theta_{\psi(n)}}{\gamma_{\psi(n)}}\|x_{\psi(n)} - x_{\psi(n)-1}\|M_1 + \frac{\theta_{\psi(n)}}{\gamma_{\psi(n)}}\|x_{\psi(n)} - x_{\psi(n)-1}\|^2 \\ &\quad + \frac{1}{1-\alpha}\langle f(z) - z, w_{\psi(n)} - z\rangle. \end{aligned} \tag{70}$$

Hence from (C3), (65) and (67), we obtain

$$\limsup_{n\to\infty}\Gamma_{\psi(n)} \leq 0. \tag{71}$$

This means $\lim_{n\to\infty}\Gamma_{\psi(n)} = \lim_{n\to\infty}\|x_{\psi(n)} - z\|^2 = 0$. So we have $x_{\psi(n)} \to z$ as $n \to \infty$. On the other hand, we see that

$$\begin{aligned}\|x_{\psi(n)+1} - z\| &\leq \|x_{\psi(n)+1} - x_{\psi(n)}\| + \|x_{\psi(n)} - z\| \\ &\to 0, \end{aligned} \tag{72}$$

as $n \to \infty$. By Lemma 2.4, we have $\Gamma_n \leq \Gamma_{\psi(n)+1}$ and thus

$$\Gamma_n = \|x_n - z\|^2 \leq \|x_{\psi(n)+1} - z\|^2 \to 0. \tag{73}$$

This concludes that $x_n \to z$ as $n \to \infty$. We thus complete the proof. □

Lemma 3.2. *We remark here that the condition (C3) is easily implemented in numerical computation since the valued of $\|x_n - x_{n-1}\|$ is known before choosing θ_n. Indeed, the parameter θ_n can be chosen such that $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{\omega_n}{\|x_n - x_{n-1}\|}, \theta\right\} & if\ x_n \neq x_{n-1}, \\ \theta & otherwise, \end{cases}$$

where $\{\omega_n\}$ is a positive sequence such that $\omega_n = o(\alpha_n)$.

We next consider the case when the norm of operators can be easily calculated.

Algorithm 3.2 Take $x_0, x_1 \in H_1$ and generate the sequence $\{x_n\}$ by the following manner:

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= P_{C_n}[\alpha_n f(y_n) + (1-\alpha_n)(y_n - \lambda_n \nabla f_n(y_n))], \end{aligned} \tag{74}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\theta_n\} \subset [0,1)$ and $\{\lambda_n\} \subset (0,\infty)$.

Theorem 3.3. *Assume that $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,\infty)$ and $\{\theta_n\} \subset [0,\theta)$, where $\theta \in [0,1)$ satisfy the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $\inf_{n\in\mathbb{N}}\lambda_n(2 - \lambda_n\|A\|^2) > 0$;

(C3) $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to the solution of SFP.

Proof. Since the proof line is closed to that of Theorem 3.1, we just give a sketch proof. Let $z = P_S f(z)$. Set $v_n = y_n - \lambda_n \nabla f_n(y_n)$ and $w_n = \alpha_n f(y_n) + (1-\alpha_n)v_n$ for all $n \in \mathbb{N}$. We first show that the sequence $\{x_n\}$ is bounded. To this end, it suffices to show that $\|v_n - z\| \leq \|y_n - z\|$ for all $n \in \mathbb{N}$. By using the argument as in Theorem 3.1, we can show that $\langle \nabla f_n(y_n), y_n - z\rangle \geq 2f_n(y_n)$. It follows that

$$\begin{aligned}
\|v_n - z\|^2 &= \|y_n - z\|^2 + \lambda_n^2\|\nabla f_n(y_n)\|^2 - 2\lambda_n\langle \nabla f_n(y_n), y_n - z\rangle \\
&\leq \|y_n - z\|^2 + \lambda_n^2\|\nabla f_n(y_n)\|^2 - 4\lambda_n f_n(y_n) \\
&\leq \|y_n - z\|^2 + \lambda_n^2\|A\|^2\|(I - P_{Q_n})Ay_n\|^2 - 4\lambda_n f_n(y_n) \\
&= \|y_n - z\|^2 + 2\lambda_n^2\|A\|^2 f_n(y_n) - 4\lambda_n f_n(y_n) \\
&\leq \|y_n - z\|^2 - 2\lambda_n(2 - \lambda_n\|A\|^2)f_n(y_n). \qquad (75)
\end{aligned}$$

From (C2), we have $\|v_n - z\| \leq \|y_n - z\|$ for all $n \in \mathbb{N}$. By (41) and (75), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 \leq\; & (1-\alpha_n)\|x_n - z\|^2 + (1-\alpha_n)\theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\
& + 2(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2 - (1-\alpha_n)\lambda_n(2 - \lambda_n\|A\|^2)f_n(y_n) \\
& + 2\alpha_n\langle f(y_n) - z, w_n - z\rangle - \|P_{C_n}w_n - w_n\|^2. \qquad (76)
\end{aligned}$$

Set $\Gamma_n = \|x_n - z\|^2$ for all $n \in \mathbb{N}$. We next consider the following two cases.

Case 1. Suppose that there exists a natural number N such that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq N$. In this case, $\{\Gamma_n\}$ is convergent. From (C1) and (C2), we can find a constant σ such that $(1-\alpha_n)\lambda_n(2 - \lambda_n\|A\|^2) \geq \sigma > 0$ for all $n \in \mathbb{N}$. So we obtain

$$\begin{aligned}
\Gamma_{n+1} \leq\; & (1-\alpha_n)\Gamma_n + (1-\alpha_n)\theta_n(\Gamma_n - \Gamma_{n-1}) + 2(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2 \\
& - \sigma f_n(y_n) + 2\alpha_n\langle f(y_n) - z, w_n - z\rangle - \|P_{C_n}w_n - w_n\|^2, \qquad (77)
\end{aligned}$$

which implies

$$\begin{aligned}
\sigma f_n(y_n) \leq\; & (\Gamma_n - \Gamma_{n+1}) + (1-\alpha_n)\theta_n(\Gamma_n - \Gamma_{n-1}) + 2(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2 \\
& + 2\alpha_n\langle f(y_n) - z, w_n - z\rangle. \qquad (78)
\end{aligned}$$

This shows, by (C1) and (C3), that $f_n(y_n) = \|(I - P_{Q_n})Ay_n\| \to 0$ as $n \to \infty$. Similarly, we can show that $\|P_{C_n}w_n - w_n\| \to 0$ $n \to \infty$. Following the proof line as in Theorem 3.1, we can prove that $\{x_n\}$ converges strongly to z.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. This case can be done by a similar argument as in Case 1. So we omit the rest of proof. We thus complete the proof. □

4. **Numerical experiments.** In this section, we provide some numerical examples and illustrate its performance by using the modified inertial relaxed CQ method (Algorithm 3.1).

Examples 1. Let $H_1 = H_2 = \mathbb{R}^3$, $C = \{x = (a,b,c)^T \in \mathbb{R}^3 : a^2 + b^2 - 5 \leq 0\}$ and $Q = \{y = (p,q,r)^T \in \mathbb{R}^3 : p + r^2 - 2 \leq 0\}$. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f(x) = \frac{x}{2}$. Find $x^* \in C$ such that $Ax^* \in Q$, where $A = \begin{pmatrix} 1 & 2 & 7 \\ 1 & 3 & 0 \\ 4 & 1 & 2 \end{pmatrix}$.

Choose $\alpha_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$ and $\theta = 0.5$. For each $n \in \mathbb{N}$, let $\omega_n = \frac{1}{(n+1)^3}$ and define θ_n as in Remark 3.2. We now study the effect (in terms of convergence, stability, number of iterations required and the cpu time) of the sequence $\{\rho_n\} \subset$

$(0,4)$ on the iterative scheme by choosing different ρ_n such that $\inf_n \rho_n(4-\rho_n) > 0$ in the following cases.

Case 1. $\rho_n = \frac{n}{2n+1}$;

Case 2. $\rho_n = \frac{n}{n+1}$;

Case 3. $\rho_n = \frac{2n}{n+1}$;

Case 4. $\rho_n = \frac{3n}{n+1}$.

The stopping criterion is defined by

$$E_n = \frac{1}{2}\|x_n - P_{C_n}x_n\|^2 + \frac{1}{2}\|Ax_n - P_{Q_n}Ax_n\|^2 < 10^{-4}.$$

We consider different choices of x_0 and x_1 as
Choice 1: $x_0 = (-7,-2,-6)^T$ and $x_1 = (-2,2,-6)^T$;
Choice 2: $x_0 = (1,2,-5)^T$ and $x_1 = (0,1,-7)^T$;
Choice 3: $x_0 = (1,5,-1)^T$ and $x_1 = (-3,4,-7)^T$;
Choice 4: $x_0 = (1,5,2)^T$ and $x_1 = (3,2,7)^T$

TABLE 1. Algorithm 3.1 with different cases of ρ_n and different choices of x_0 and x_1

		Case 1	Case 2	Case 3	Case 4
Choice 1	No. of Iter.	11	8	5	4
	cpu (Time)	0.003553	0.002377	0.002195	0.002075
Choice 2	No. of Iter.	7	6	4	4
	cpu (Time)	0.002799	0.002769	0.002357	0.002184
Choice 3	No. of Iter.	12	9	6	4
	cpu (Time)	0.003828	0.002602	0.002401	0.002142
Choice 4	No. of Iter.	27	17	11	9
	cpu (Time)	0.007181	0.00343	0.002612	0.002431

The numerical experiments for each case of ρ_n are shown in Figure 1-4, respectively.

Examples 2. Let $H_1 = H_2 = \mathbb{R}^5$, $C = \{x = (a,b,c,d,e)^T \in \mathbb{R}^5 : a^2+b^2+c^2+d^2+e^2-0.4 \le 0\}$ and $Q = \{y = (p,q,r,s,t)^T \in \mathbb{R}^5 : p+q+r+s-0.75 \le 0\}$. Let $f : \mathbb{R}^5 \to \mathbb{R}^5$ be defined by $f(x) = \frac{x}{2}$. Find $x^* \in C$ such that $Ax^* \in Q$, where

$$A = \begin{pmatrix} 3 & -2 & 5 & -2 & 3 \\ 2 & -2 & 5 & -2 & 9 \\ 2 & -3 & 5 & -1 & -3 \\ -2 & -2 & 8 & -7 & -2 \end{pmatrix}.$$

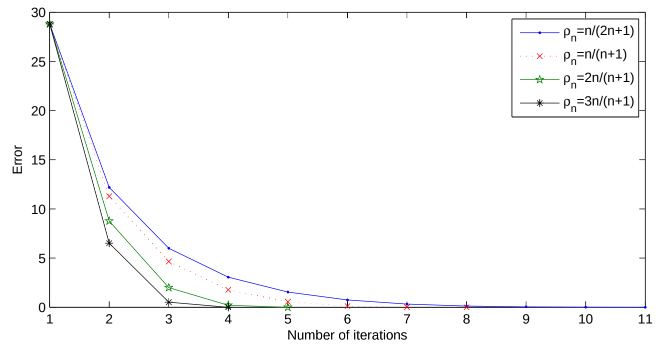

FIGURE 1. Comparison of the iterations of Choice 1 in Example 1

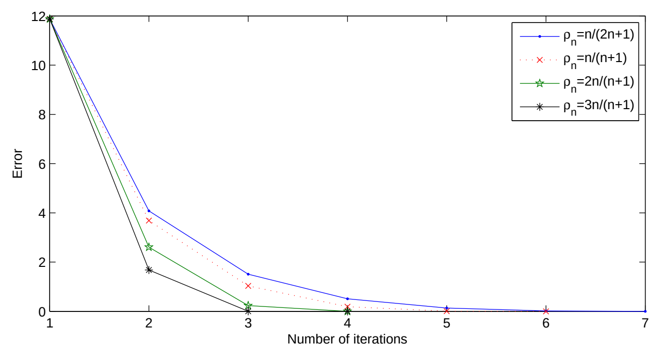

FIGURE 2. Comparison of the iterations of Choice 2 in Example 1

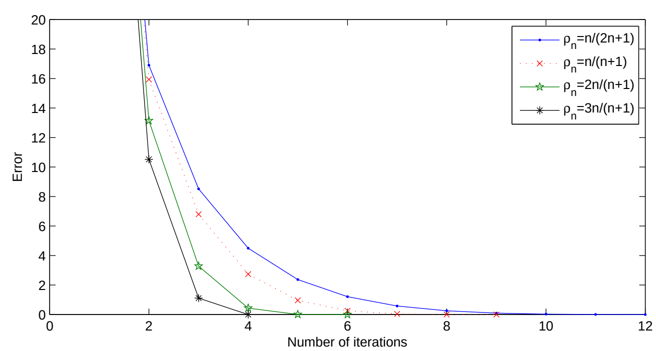

FIGURE 3. Comparison of the iterations of Choice 3 in Example 1

Let α_n, θ_n and E_n be as in Example 1. We choose different cases of ρ_n as follows:

Case 1. $\rho_n = 0.5$;

Case 2. $\rho_n = 1$;

Case 3. $\rho_n = 2$;

Case 4. $\rho_n = 3.5$.

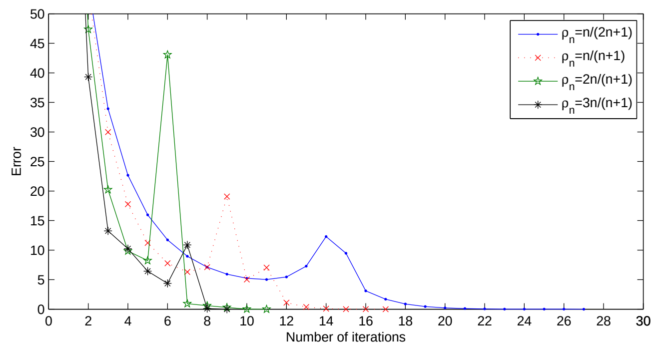

FIGURE 4. Comparison of the iterations of Choice 4 in Example 1

The different choices of x_0 and x_1 are given as follows:
Choice 1: $x_0 = (-3.2, -1, -2.5, 5, -3.7)^T$ and $x_1 = (-2.3, -1.5, 5.2, -7.5, 7.3)^T$;
Choice 2: $x_0 = (-2, -5, -3, 2, -3)^T$ and $x_1 = (-5, -4, 5, -7, 7)^T$;
Choice 3: $x_0 = (3, 8, 5, -2, 8)^T$ and $x_1 = (-2, -5, 5, -9, 9)^T$;
Choice 4: $x_0 = (4.5, 0, -2.5, 1, 3)^T$ and $x_1 = (-3.6, -4.2, 1, 1.5, 8)^T$.

TABLE 2. Algorithm 3.1 with different cases of ρ_n and different choices of x_0 and x_1

		Case 1	Case 2	Case 3	Case 4
Choice 1	No. of Iter.	19	10	5	5
	cpu (Time)	0.005632	0.003408	0.003223	0.002791
Choice 2	No. of Iter.	18	10	6	6
	cpu (Time)	0.00391	0.002683	0.002447	0.002381
Choice 3	No. of Iter.	19	10	6	6
	cpu (Time)	0.004233	0.003016	0.002601	0.002575
Choice 4	No. of Iter.	13	7	6	6
	cpu (Time)	0.004812	0.003559	0.002922	0.002412

The numerical experiments are shown in Figure 5-8, respectively.

Remark 1. We finally make the following conclusions from the numerical experiments in Examples 1 and 2.

1. For each different Cases and different Choices, it is shown that Algorithm 3.1 has a good convergence speed. Indeed, we see that it is fast, stable and required small number of iterations for seeking solutions.
2. It is observed that the number of iterations and the cpu run time are significantly decreasing starting from Case 1 to Case 4. However, there is no significant difference in both cpu run time and number of iterations for each

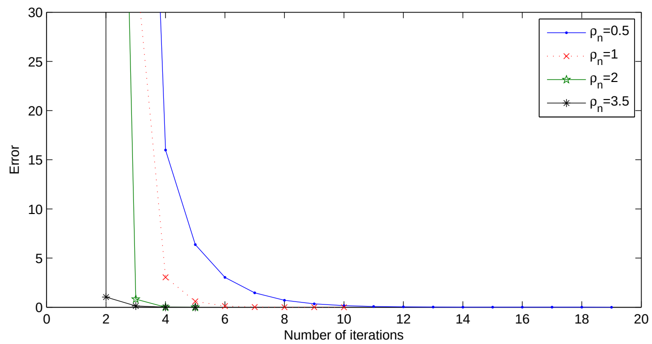

FIGURE 5. Comparison of the iterations of Choice 1 in Example 2

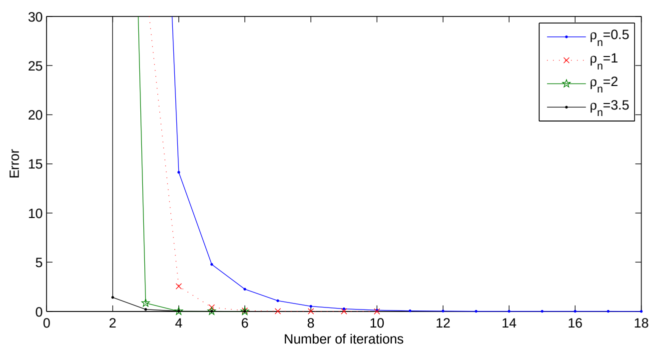

FIGURE 6. Comparison of the iterations of Choice 2 in Example 2

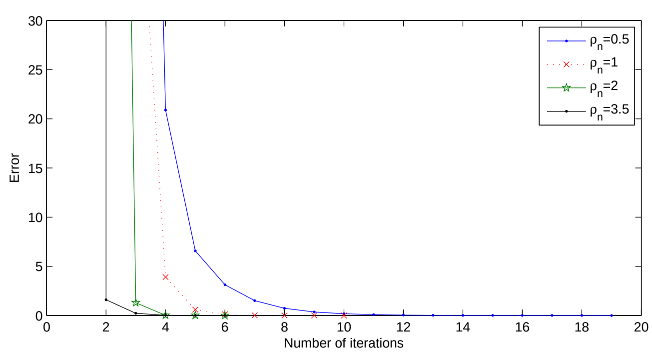

FIGURE 7. Comparison of the iterations of Choice 3 in Example 2

choice of x_0 and x_1. So, initial guess does not have any significant effect on the convergence of the algorithm.

3. The conditions in Theorem 3.1 are easily implemented in numerical computations and need no estimation on the spectral radius of $A^T A$.

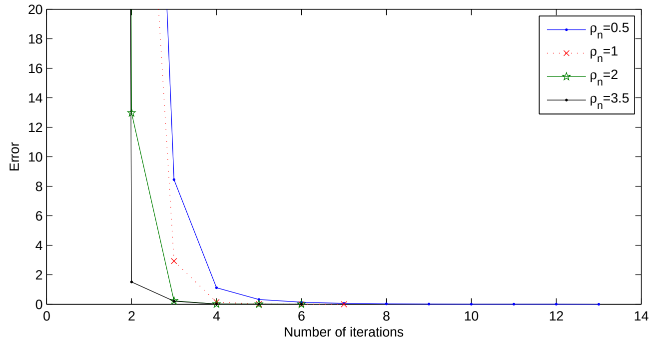

FIGURE 8. Comparison of the iterations of Choice 4 in Example 2

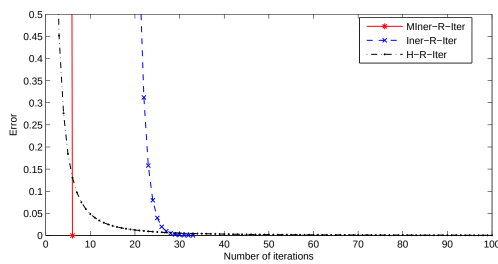

FIGURE 9. Error ploting of Choice 1 in Example 1

4. The restriction of metric projections onto C and Q is relaxed by using those of C_n and Q_n which have specific forms.

We finally end this section by providing a comparison of convergence of Algorithm 3.1 with the modified relaxed CQ algorithms defined by He and Zhao [12] and Dang et al. [9] through the example. For the convenience, let us denote Algorithm 3.1, Algorithm (7) and Algorithm (14) by MIner-R-Iter, Iner-R-Iter and H-R-Iter, respectively.

Examples 3. Let H_1, H_2, C, Q, A and f be as in Example 1.

Choose $\alpha_n = \frac{1}{n+1}$, $\rho_n = \frac{3n}{n+1}$ and $\omega_n = \frac{1}{(n+1)^2}$ for all $n \in \mathbb{N}$. Set $\theta = \phi = 0.8$ and $\theta_n = \bar{\theta}_n$ as in Remark 3.2. Let $\gamma = \frac{1}{\|A^T A\|}$ and $\phi_n = 0.4$ if $\phi \leq \frac{1}{\max\{n^2\|x_n - x_{n-1}\|^2, n^2\|x_n - x_{n-1}\|\}}$ and $\phi_n = \frac{1}{\max\{(n+1)^2\|x_n - x_{n-1}\|^2, (n+1)^2\|x_n - x_{n-1}\|\}}$; otherwise. The stopping criterion E_n is defined as in Example 1. For points u, x_0 and x_1 picked randomly, we obtain the following numerical results.

Remark 2. In numerical experiment, it is revealed that the sequence generated by MIner-R-Iter involving the viscosity term and the inertial technique converges more quickly than by H-R-Iter of He and Zhao [12] and Iner-R-Iter of Dang et al. [9] do.

Acknowledgments. This research was supported by the Thailand Research Fund and the Commission on Higher Education under Grant MRG5980248. S. Suantai wishes to thank Chiang Mai University. N. Pholasa was supported by School of Science, University of Phayao.

TABLE 3. Comparison of MIner-R-Iter, Iner-R-Iter and H-R-Iter in Example 1

			MIner-R-Iter	Iner-R-Iter	H-R-Iter
Choice 1	$u = (0, -1, -5)^T$	No. of Iter.	6	33	223
	$x_0 = (2, 6, -3)^T$	cpu (Time)	0.000737	0.007677	0.064889
	$x_1 = (-2, -1, 8)^T$				
Choice 2	$u = (2, 1, 0)^T$	No. of Iter.	4	26	378
	$x_0 = (3, 4, -1)^T$	cpu (Time)	0.000522	0.004861	0.137471
	$x_1 = (-5, -2, 1)^T$				
Choice 3	$u = (5, -3, -1)^T$	No. of Iter.	9	29	140
	$x_0 = (2, 1, -1)^T$	cpu (Time)	0.001458	0.005175	0.026824
	$x_1 = (-5, 3, 5)^T$				
Choice 4	$u = (-2, -1, 4)^T$	No. of Iter.	9	34	763
	$x_0 = (7.35, 1.75, -3.24)^T$	cpu (Time)	0.001481	0.008058	0.687214
	$x_1 = (-6.34, 0.42, 7.36)^T$				

The error plotting of E_n of MIner-R-Iter, Iner-R-Iter and H-R-Iter for each choice in Table 3 is shown in the following figures, respectively.

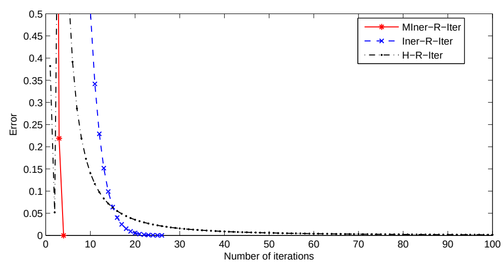

FIGURE 10. Error ploting of Choice 2 in Example 1

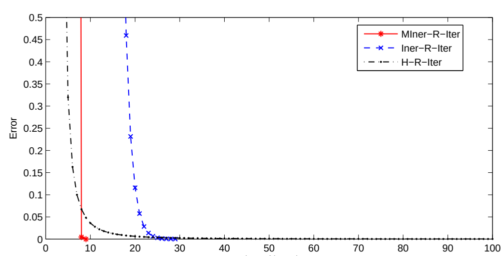

FIGURE 11. Error ploting of Choice 3 in Example 1

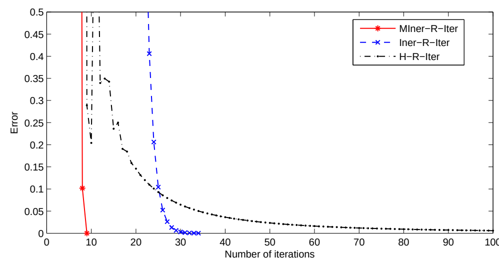

FIGURE 12. Error ploting of Choice 4 in Example 1

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Received April 2017; 1st revision June 2017; 2nd revision August 2017.

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J. Fixed Point Theory Appl. 19 (2017) 3097–3118
DOI 10.1007/s11784-017-0472-7
Published online August 28, 2017

Journal of Fixed Point Theory and Applications

A strong convergence result involving an inertial forward–backward algorithm for monotone inclusions

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Abstract. Our interest in this paper is to prove a strong convergence result for finding a zero of the sum of two monotone operators, with one of the two operators being co-coercive using an iterative method which is a combination of Nesterov's acceleration scheme and Haugazeau's algorithm in real Hilbert spaces. Our numerical results show that the proposed algorithm converges faster than the un-accelerated Haugazeau's algorithm.

Mathematics Subject Classification. 47H06, 47H09, 47J05, 47J25.

Keywords. Splitting algorithm, Nesterov method, Haugazeau algorithm, Projection, Strong Convergence.

1. Introduction

Let H be a real Hilbert space. We study the following inclusion problem: find $\hat{x} \in H$ such that

$$0 \in A\hat{x} + B\hat{x} \tag{1.1}$$

where $A{:}H \to H$ is an operator and $B{:}H \to 2^H$ is a set-valued operator. This problem includes, as special cases, convex programming, variational inequalities, split feasibility problem and minimization problem. To be more precise, some concrete problems in machine learning, image processing and linear inverse problem can be modeled mathematically as this form.

A classical method for solving problem (1.1) is the forward–backward splitting method [6,27,34,36,48,49,53,56,61] which is defined by the following manner: $x_1 \in H$ and

$$x_{n+1} = (I + rB)^{-1}(x_n - rAx_n),\ n \geq 1, \tag{1.2}$$

where $r > 0$. We see that each step of iterates involves only A as the forward step and B as the backward step, but not the sum of A and B. This method includes, in particular, the proximal point algorithm [17,47,54,59,62,64] and the gradient method [58,63,67–69]. In [35], Lions and Mercier introduced the following splitting iterative methods in a real Hilbert space:

$$x_{n+1} = (2J_r^A - I)(2J_r^B - I)x_n, \ n \geq 1 \tag{1.3}$$

and

$$x_{n+1} = J_r^A(2J_r^B - I)x_n + (I - J_r^B)x_n, \ n \geq 1, \tag{1.4}$$

where $J_r^T = (I + rT)^{-1}$. The first one is often called Peaceman–Rachford algorithm [49] and the second one is called Douglas–Rachford algorithm [28]. We note that both algorithms are weakly convergent in general [5,35]. There have been many works concerning the problem of finding zero points of the sum of two monotone operators (in Hilbert spaces) and accretive operators (in Banach spaces); see [25,26,34,36,48,53,65,66].

Let H be a real Hilbert space and f and g two proper, convex and lower semi continuous functions from H to $\mathbb{R} \cup \{+\infty\}$ such that f is differentiable with L-Lipschitz continuous gradient, and the proximal map of g is "simple", meaning that its "proximal map"

$$x \mapsto \arg\min_{y \in H} g(y) + \frac{||x - y||^2}{2\gamma}$$

can be easily computed.

In particular, if $A := \nabla f$ and $B := \partial g$, where ∇f is the gradient of f and ∂g is the subdifferential of g which is defined by $\partial g(x) := \big\{ s \in H : g(y) \geq g(x) + \langle s, y - x \rangle, \ \ \forall y \in H \big\}$ then problem (1.1) becomes the following minimization problem:

$$\min_{x \in H} f(x) + g(x) \tag{1.5}$$

and (1.2) also becomes

$$x_{n+1} = \text{prox}_{rg}(x_n - r\nabla f(x_n)), n \geq 1,$$

where $r > 0$. Among the many algorithms which exist to tackle such problems, the proximal splitting algorithms, which perform alternating descents in f and in g, are frequently used, because of their simplicity and relatively small per-iteration complexity. One can mention the forward–backward (FB) splitting, the Douglas–Rachford splitting, the ADMM (alternating direction method of multipliers), which all have been proved to be efficient in many imaging problems such as denoising, inpainting, deconvolution, colour transfer and many others.

Let us recall that the inertial term is based upon a discrete version of a second order dissipative dynamical system [1,2] and can be regarded as a procedure of speeding up the convergence properties (see, e.g., [4,7,37,38,52]). Recently, there have been increasing interests in studying inertial type algorithms, see, for example, inertial forward–backward splitting methods [36,46], inertial Douglas–Rachford splitting method [13], inertial ADMM [14,22], and inertial forward–backward–forward method [15]. Some inertial algorithms for solving nonsmooth and nonconvex optimization problems have been recently studied in [11,12]. For example, it is known that acceleration scheme developed by Nesterov improves the theoretical rate of convergence of forward–backward method from the standard $O(k^{-1})$ down to $O(k^{-2})$ and the inertial extrapolation scheme of Nesterov's accelerated forward–backward method is actually $o(k^{-2})$ rather than $O(k^{-2})$ (see [3]). These results and other related ones analyzed the convergence properties of inertial type algorithms and demonstrated their performance numerically on some imaging and data analysis problems.

In [4], Alvarez and Attouch translated the idea of the heavy ball method in [51,52] to the setting of a general maximal monotone operator using the framework of the proximal point algorithm. The resulting algorithm is called the inertial proximal point algorithm and it is written as:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) \\ x_{n+1} = (I + r_n B)^{-1} y_n, \ n \geq 1. \end{cases} \tag{1.6}$$

Alvarez and Attouch [4], proved that under the condition

$$\sum \alpha_n \|x_n - x_{n-1}\|^2 < \infty, \tag{1.7}$$

the algorithm (1.6) converges weakly to a zero of B.

In [41], Moudafi and Oliny introduced an additional single-valued, co-coercive and Lipschitz continuous operator A into the inertial proximal point algorithm:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) \\ x_{n+1} = (I + r_n B)^{-1}(y_n - r_n A x_n), \ n \geq 1. \end{cases} \tag{1.8}$$

Moudafi and Oliny [41] obtained a weak convergence result using algorithm (1.8) under the same condition (1.7) imposed above in [4]. As remarked in [36], the algorithm (1.8) does not take the form of a forward–backward splitting algorithm, since operator A is still evaluated at the point x_n for $\alpha_n > 0$.

We note that there are many problems that arise in infinite dimensional spaces. In such problems norm convergence is often much more desirable than weak convergence (see [5] and references therein). For this reason algorithms that provide strong convergence result is better than forward–backward splitting (and its inertial extrapolation type) method that provides weak convergence in infinite dimensional real Hilbert spaces. Another reason to study their strong convergence result is an academic interest.

In order to obtain the strong convergence, in his unpublished 1968 dissertation, Haugazeau [32] (see also p. 42 in [29]) proposed independently a strongly convergent variant of a periodic projection algorithm for finding a common point of m intersecting closed convex sets $\{S_i\}_{i=1}^m$ in H, requiring essentially the same kind of computations. To describe his method, let us define, for a given ordered triplet $(x, y, z) \in H^3$,

$$R(x, y) = \{u \in H : \langle u - y, x - y\rangle \leq 0\},$$

and let us denote by $Q(x, y, z)$ the projection of x onto $R(x, y) \cap R(y, z)$. Thus, $R(x, x) = H$ and, if $x \neq y$, $R(x, y)$ is a closed affine half space onto which y is the projection of x. Haugazeau [32] showed that, given an arbitrary starting point $x_0 \in H$, the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = Q(x_0, x_n, P_{n(mod m)+1} x_n), \ \forall n \geq 1$$

converges strongly to the projection of x_0 onto the set of common points of m intersecting closed convex sets $\{S_i\}_{i=1}^m$. Many modifications of Haugazeau's method have been studied and considered by many authors for solving fixed point problems and optimization problems in the literature. (see, for example, [5,33,39,40,42,43,57] and the references contained therein for more details.) In this work, we study and prove strong convergence results, under some mild conditions, using a combination of Haugazeau's algorithm and Nesterov's acceleration scheme for solving the inclusion problem (1.1) in the framework of real Hilbert spaces. Our work is motivated by the accelerated variant of the forward–backward algorithm proposed by Lorenz and Pock [36], which in turn generalizes the works of Beck and Teboulle [7], Nesterov [44,45] and Güler [30]. Our results are new, interesting and complement (in terms of mode of convergence) many recent results previously obtained in this direction in the literature.

2. Preliminaries

Let C be a nonempty, closed and convex subset of real Hilbert space H with inner product $\langle .,. \rangle$ and norm $||.||$.

Definition 2.1. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C.$$

Construction of fixed points of nonexpansive mappings is an important subject in nonlinear mapping theory and its applications; in particular, in image recovery and signal processing (see, for example, [18,50,70]). For the past 50 years or so, the approximation of fixed points of nonexpansive mappings and fixed points of some of their generalizations and approximation of zeros of monotone-type operators in Hilbert spaces have been a flourishing area of research for many mathematicians. For example, the reader can consult the recent monographs of Bauschke and Combettes [6], Berinde [9] and Chidume [24].

For any point $u \in H$, there exists a unique point $P_C u \in C$ such that

$$||u - P_C u|| \leq ||u - y||, \ \forall y \in C.$$

P_C is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq ||P_C x - P_C y||^2, \tag{2.1}$$

for all $x, y \in H$. Furthermore, $P_C x$ is characterized by the properties $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y \rangle \geq 0, \tag{2.2}$$

for all $y \in C$. We also know that all Hilbert space has the Kadec–Klee property, that is, $\{x_n\}$ converges weakly to x and $\|x_n\| \to \|x\|$ imply $\{x_n\}$ converges strongly to x.

Definition 2.2. A mapping $T{:}H \to H$ is said to be *firmly nonexpansive* if and only if $2T - I$ is nonexpansive, or equivalently

$$\langle x - y, Tx - Ty\rangle \geq ||Tx - Ty||^2, \ \forall x, y \in H.$$

Alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive. Projections are firmly nonexpansive.

Definition 2.3. A nonlinear operator T whose domain $D(T) \subset H$ and range $R(T) \subset H$ is said to be:
(a) *monotone* if

$$\langle x - y, Tx - Ty\rangle \geq 0, \forall x, y \in D(T),$$

(b) *β-strongly monotone* if there exists $\beta > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \beta||x - y||^2, \forall x, y \in D(T),$$

(c) *ν-inverse strongly monotone* (for short, ν-ism) if there exists $\nu > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \nu||Tx - Ty||^2, \forall x, y \in D(T).$$

It can be easily seen that (i) if T is nonexpansive, then $I - T$ is monotone; (ii) the projection mapping P_C is a 1-ism. The inverse strongly monotone (also referred to as co-coercive) operators have been widely used to solve practical problems in various fields, for instance, in traffic assignment problems; see, for example, [10,31] and the references therein.

The following lemmas will be needed in the sequel.

Lemma 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H and $P_C{:}H \to C$ be the metric projection from H onto C. Then the following inequality holds:*

$$||y - P_C x||^2 + ||x - P_C x||^2 \leq ||x - y||^2, \quad \forall x \in H, \forall y \in C.$$

Lemma 2.5. (Sahu et al. [55]) *Let C be a closed and convex subset of a real Hilbert space H. For any $x, y, z \in H$, and a real number $a \in \mathbb{R}$, the set*

$$\{v \in C : ||y - v||^2 \leq ||x - v||^2 + \langle z, v\rangle + a\}$$

is closed and convex.

Lemma 2.6. (Lopez et al. [34]) *Let E be a real Banach space. Let $A : H \to 2^H$ be a maximal monotone operator and $B : H \to H$ be an α-inverse strongly monotone mapping on H. Define $T_r := (I + rB)^{-1}(x - rAx), \ r > 0$. Then we have,*

(i) *for $r > 0, \ F(T_r) = (A + B)^{-1}(0)$.*
(ii) *for $0 < s \leq r$ and $x \in E, ||x - T_s x|| \leq 2||x - T_r x||$.*

We shall adopt the following notation in this paper:

- $x_n \to x$ means that $x_n \to x$ strongly.
- $x_n \rightharpoonup x$ means that $x_n \to x$ weakly;

- $w_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ is the weak w-limit set of the sequence $\{x_n\}_{n=1}^{\infty}$.

Lemma 2.7. (Browder [16]) *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I-T)x_n \to y$, then $(I-T)x = y$. In particular, if $y = 0$, then $x \in F(T)$.*

3. Main results

Let H be a real Hilbert space. Let $A : H \to H$ be an α-ism and $B : H \to 2^H$ a maximal monotone operator such that $\Omega := (A+B)^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\} \subset \mathbb{R}$ and let a sequence $\{x_n\}_{n=0}^{\infty}$ in H be generated by $x_0, x_1 \in H$ and for all $n \geq 1$,

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = (I + r_n B)^{-1}(y_n - r_n A y_n), \\ C_n = \{u \in H : ||z_n - u||^2 \leq ||x_n - u||^2 - 2\alpha_n \langle x_n - u, x_{n-1} - x_n\rangle \\ \qquad + \alpha_n^2 ||x_{n-1} - x_n||^2\}, \\ Q_n = \{u \in H : \langle u - x_n, x_0 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{3.1}$$

Remark 3.1. We make the following remarks about our iterative method (3.1).

(1) We observe that for any $u \in H$,

$$||z_n - u||^2 \leq ||x_n - u||^2 - 2\alpha_n \langle x_n - u, x_{n-1} - x_n\rangle + \alpha_n^2 ||x_{n-1} - x_n||^2$$
$$\Leftrightarrow$$
$$\|z_n\|^2 - 2\langle u, z_n\rangle \leq \|x_n\|^2 - 2\langle u, x_n\rangle + 2\alpha_n \langle u, x_{n-1} - x_n\rangle$$
$$-2\alpha_n \langle x_n, x_{n-1} - x_n\rangle + \alpha_n^2 ||x_{n-1} - x_n||^2$$
$$\Leftrightarrow$$
$$2\langle u, x_n - z_n - \alpha_n(x_{n-1} - x_n)\rangle \leq \|x_n\|^2 - \|z_n\|^2$$
$$-2\alpha_n \langle x_n, x_{n-1} - x_n\rangle + \alpha_n^2 ||x_{n-1} - x_n||^2$$
$$\Leftrightarrow$$
$$2\langle u, y_n - z_n\rangle \leq \|x_n\|^2 - \|z_n\|^2 - 2\alpha_n \langle x_n, x_{n-1} - x_n\rangle + \alpha_n^2 ||x_{n-1} - x_n||^2$$
$$\Leftrightarrow$$
$$\langle u, x_n - z_n\rangle \leq \frac{1}{2}\Big[\|x_n\|^2 - \|z_n\|^2 - 2\alpha_n \langle x_n, x_{n-1} - x_n\rangle + \alpha_n^2 ||x_{n-1} - x_n||^2\Big].$$

Therefore, the set C_n defined in our iterative method (3.1) is a half space. Hence, the metric projection P_{C_n} has a closed-form expression and can be easily computed (see [19]).

(2) For any $u \in H$,

$$\langle u - x_n, x_0 - x_n\rangle \leq 0$$
$$\Leftrightarrow$$
$$\langle u, x_0 - x_n\rangle \leq \langle x_n, x_0 - x_n\rangle.$$

Therefore. the set Q_n defined in iterative method (3.1) is a half space and the metric projection P_{Q_n} has a closed-form expression and can also be easily computed.

(3) Since both C_n and Q_n are half-spaces, then the closed-form expressions for the projections onto the intersection of two half-spaces C_n and Q_n are given in Propositions 28.18 and 28.19 of [6]. Therefore, the iterate x_{n+1} in our iterative method (3.1) can be easily computed.

We now give our main result of this paper.

Theorem 3.2. *Let H be a real Hilbert space. Let $A : H \to H$ be an α-ism and $B : H \to 2^H$ a maximal monotone operator such that $\Omega := (A+B)^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\}$ be a bounded real sequence. Let a sequence $\{x_n\}_{n=0}^{\infty}$ in H be generated by (3.1). If $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\alpha$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\overline{x} = P_\Omega x_0$.*

Proof. We divide our proof into these steps.
Step 1 Show that $\{x_n\}_{n=0}^{\infty}$ is well defined and $\Omega \subset C_n \cap Q_n,\ \forall n \geq 0$.
By Lemma 2.5, it is obvious that C_n is closed and convex for all $n \geq 0$. Furthermore, Q_n is closed and convex for all $n \geq 0$. So, $C_n \cap Q_n$ is closed and convex for all $n \geq 0$.
Let $u \in \Omega$. Then by a direct computation, we obtain,

$$\begin{aligned}||y_n - u||^2 &= ||(x_n - u) - \alpha_n(x_{n-1} - x_n)||^2\\ &= ||x_n - u||^2 - 2\alpha_n\langle x_n - u, x_{n-1} - x_n\rangle + \alpha_n^2||x_{n-1} - x_n||^2.\end{aligned}$$

Furthermore, we have,

$$\begin{aligned}||z_n - u||^2 &= ||(I + r_nB)^{-1}(y_n - r_nAy_n) - (I + r_nB)^{-1}(u - r_nAu)||^2\\ &\leq ||y_n - u - r_n(Ay_n - Au)||^2\\ &= ||y_n - u||^2 - 2r_n\langle Ay_n - Au, y_n - u\rangle + r_n^2||Ay_n - Au||^2\\ &\leq ||y_n - u||^2 - 2r_n\alpha||Ay_n - Au||^2 + r_n^2||Ay_n - Au||^2\\ &= ||y_n - u||^2 - (2\alpha - r_n)r_n||Ay_n - Au||^2\\ &\leq ||y_n - u||^2\\ &= ||x_n - u||^2 - 2\alpha_n\langle x_n - u, x_{n-1} - x_n\rangle + \alpha_n^2||x_{n-1} - x_n||^2.\end{aligned}$$

Therefore, $u \in C_n,\ \forall n \geq 1$. Clearly, $u \in C_0$. So, $u \in C_n,\ \forall n \geq 0$. Thus, $\Omega \subset C_n,\ \forall n \geq 0$. For $n = 0$, we have that $x_0 \in H$ and $Q_0 = H$ and hence $\Omega \subset H = C_0 \cap Q_0$. Suppose that x_k is given and $\Omega \subset C_k \cap Q_k$ for some $k \in \{0, 1, 2, \ldots\}$. Since $C_k \cap Q_k$ is nonempty, closed and convex, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k\cap Q_k}(x_0)$. It follows that,

$$\langle z - x_{k+1}, x_0 - x_{k+1}\rangle \leq 0,\ \forall z \in C_k \cap Q_k.$$

Since $\Omega \subset C_k \cap Q_k$, we have in particular that,

$$\langle z - x_{k+1}, x_0 - x_{k+1}\rangle \leq 0,\ \forall z \in \Omega.$$

This implies that $\Omega \subset C_{k+1}$. Hence $\Omega \subset C_{k+1} \cap Q_{k+1}$. By induction, $\Omega \subset C_n \cap Q_n,\ \forall n \geq 0$ and $\{x_n\}_{n=0}^{\infty}$ is well defined.

Step 2 Show that $\{x_n\}_{n=0}^{\infty}$ is bounded.
From our iterative scheme (3.1), we observe that,

$$\langle y - x_n, x_0 - x_n \rangle \leq 0,\ \forall y \in Q_n (n \geq 1).$$

This implies that $x_n = P_{Q_n}(x_0)$ and hence,

$$||x_n - x_0|| \leq ||x_0 - y||,\ \forall y \in Q_n.$$

Since $\Omega \subset Q_n$, we have

$$||x_n - x_0|| \leq ||x_0 - y||,\ \forall y \in \Omega. \tag{3.2}$$

In particular, we have (since $x_{n+1} \in Q_n$)

$$||x_n - x_0|| \leq ||x_{n+1} - x_0||. \tag{3.3}$$

By (3.2) and (3.3), we obtain $\lim_{n\to\infty} ||x_n - x_0||$ exists. This implies that $\{x_n\}$ is bounded.
Step 3 Show that $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$.
By Lemma 2.4 and the fact that $x_n = P_{Q_n}(x_0)$, we see that,

$$||x_{n+1} - x_n||^2 \leq ||x_{n+1} - x_0||^2 - ||x_n - x_0||^2.$$

Since $\lim_{n\to\infty} ||x_n - x_0||$ exists, it follows that $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$.
Step 4 Show that $\lim_{n\to\infty} x_n = \overline{x}$, where $\overline{x} = P_\Omega(x_0)$.
We obtain from (3.1) and Step 3 that,

$$\|y_n - x_n\| = |\alpha_n| \|x_n - x_{n-1}\| \to 0,\ n \to \infty.$$

Hence,

$$\|x_{n+1} - y_n\| \leq ||x_{n+1} - x_n|| + \|y_n - x_n\| \to 0,\ n \to \infty.$$

Since $x_{n+1} \in C_n$, we have that,

$$\begin{aligned} ||z_n - x_{n+1}||^2 &\leq ||x_n - x_{n+1}||^2 - 2\alpha_n \langle x_n - x_{n+1}, x_{n-1} - x_n \rangle \\ &\quad + \alpha_n^2 ||x_{n-1} - x_n||^2 \\ &\leq ||x_n - x_{n+1}||^2 + 2|\alpha_n| \|x_n - x_{n+1}\| x_{n-1} - x_n \| \\ &\quad + \alpha_n^2 ||x_{n-1} - x_n||^2 \to 0,\ n \to \infty. \end{aligned}$$

Furthermore, we have,

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_n - x_{n+1}\| \to 0,\ n \to \infty,$$

and

$$\|z_n - y_n\| \leq \|z_n - x_n\| + \|x_n - y_n\| \to 0,\ n \to \infty.$$

Take $z_n := T_n y_n$, where $T_n := (I + r_n B)^{-1}(I - r_n A)$. Therefore,

$$\|T_n y_n - y_n\| = \|z_n - y_n\| \to 0,\ n \to \infty.$$

Since $\liminf_{n\to\infty} r_n > 0$, there exists $\epsilon > 0$ such that $r_n \geq \epsilon$, $\forall n \geq 1$. Then, by Lemma 2.6, we have,

$$\lim_{n\to\infty} ||T_\epsilon y_n - y_n|| \leq 2 \lim_{n\to\infty} ||T_n y_n - y_n|| = 0.$$

By Lemmas 3.3 and 3.1 of [34], T_ϵ is nonexpansive and $F(T_\epsilon) = (A+B)^{-1}(0)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$x_{n_i} \rightharpoonup w \in H$. Using the fact that $\|y_n - x_n\| \to 0, n \to \infty$ and $x_{n_i} \rightharpoonup w \in H$, we have $y_{n_i} \rightharpoonup w \in H$. We can therefore make use of Lemma 2.7 to assure that $w \in \Omega$.

If $\overline{x} = P_\Omega(x_0)$, it follows from (3.2), the fact that $w \in \Omega$ and the lower semicontinuity of the norm that,

$$\begin{aligned}||x_0 - \overline{x}|| \leq ||x_0 - w|| &\leq \liminf_{i\to\infty} ||x_0 - x_{n_i}|| \\ &\leq \limsup_{i\to\infty} ||x_0 - x_{n_i}|| \leq ||x_0 - \overline{x}||.\end{aligned}$$

Thus, we have that $\lim_{i\to\infty} ||x_{n_i} - x_0|| = ||x_0 - w|| = ||x_0 - \overline{x}||$. This implies that $x_{n_i} \to w = \overline{x},\ i \to \infty$. It follows that $\{x_n\}$ converges weakly to $\overline{x}$. So we have,

$$\begin{aligned}||x_0 - \overline{x}|| &\leq \liminf_{n\to\infty} ||x_0 - x_n|| \\ &\leq \limsup_{n\to\infty} ||x_0 - x_n|| \leq ||x_0 - \overline{x}||.\end{aligned}$$

This shows that $\lim_{n\to\infty} ||x_n - x_0|| = ||x_0 - \overline{x}||$. From $x_n \rightharpoonup \overline{x}$, we also have $x_n - x_0 \rightharpoonup \overline{x} - x_0$. Since H satisfies the Kadec–Klee property, it follows that $x_n - x_0 \to \overline{x} - x_0$. Therefore $x_n \to \overline{x}$ as $n \to \infty$. We thus complete the proof. □

If we take $A := 0$ in Theorem 3.2, then we obtain the following corollary which is new in its own right.

Corollary 3.3. *Let H be a real Hilbert space. Let $B : H \to 2^H$ be a maximal monotone operator such that $B^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\}$ be a bounded real sequence. Let a sequence $\{x_n\}_{n=0}^\infty$ in H be generated by $x_0, x_1 \in H$ and for all $n \geq 1$,*

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = (I + r_n B)^{-1} y_n, \\ C_n = \{u \in H : ||z_n - u||^2 \leq ||x_n - u||^2 - 2\alpha_n\langle x_n - u, x_{n-1} - x_n\rangle \\ \qquad + \alpha_n^2 ||x_{n-1} - x_n||^2\}, \\ Q_n = \{u \in H : \langle u - x_n, x_0 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases}$$

If $\liminf_{n\to\infty} r_n > 0$, then $\{x_n\}_{n=0}^\infty$ converges strongly to $\overline{x} = P_\Omega x_0$.

Remark 3.4. We remark here that our Theorem 3.2 and Corollary 3.3 complement many weak convergence results for monotone inclusion problems using inertial-type algorithms obtained in [4,13–15,21,36,37,41,46] in the sense that we obtain strong convergence results using the modified inertial extrapolation method in real Hilbert spaces.

4. Applications

(1) **Application to convex minimization problems**

Let f and g be two proper, convex and lower semicontinuous functions from H to $\mathbb{R}\cup\{+\infty\}$ such that f is differentiable with L-Lipschitz continuous gradient, and g is such that its proximal map can be easily computed. Assume that Ω is the set of solutions of problem (1.5) and $\Omega \neq \emptyset$. In Theorem 3.2, take $A := \nabla f$ and $B := \partial g$. Therefore, we obtain the following strong convergence result with inertial for solving problem (1.5).

Theorem 4.1. *Let H be a real Hilbert space. Let f and g be two proper, convex and lower semicontinuous functions from H to $\mathbb{R} \cup \{+\infty\}$ such that f is differentiable with L-Lipschitz continuous gradient, and g is such that its proximal map can be easily computed. Assume that Ω is the set of solutions of problem (1.5) and $\Omega \neq \emptyset$. Let $\{\alpha_n\}$ be a bounded real sequence and $\gamma \in (0, \frac{2}{L})$. Let a sequence $\{x_n\}_{n=0}^{\infty}$ in H be generated by $x_0, x_1 \in H$ and for all $n \geq 1$,*

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = \text{prox}_{\gamma g}(y_n - \gamma \nabla f(y_n)), \\ C_n = \{u \in H : ||z_n - u||^2 \leq ||x_n - u||^2 - 2\alpha_n \langle x_n - u, x_{n-1} - x_n \rangle \\ \qquad + \alpha_n^2 ||x_{n-1} - x_n||^2\}, \\ Q_n = \{u \in H : \langle u - x_n, x_0 - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases}$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\overline{x} = P_\Omega x_0$.

(2) **Application to split feasibility problems**

Let H_1 and H_2 be real Hilbert spaces and $T : H_1 \to H_2$ a bounded linear operator. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. The split feasibility problem (SFP) is the problem of finding a point $x \in C$ such that

$$Tx \in Q.$$

We denote the solution sets by $\Omega := C \cap T^{-1}(Q) = \{y \in C : Ty \in Q\}$. This problem was first introduced by Censor and Elfving [20], in a finite dimensional Hilbert space, for solving the inverse problems in the context of phase retrievals, medical image reconstruction and also in modeling of intensity modulated radiation therapy.

Recall that the indicator function on C is the function i_C, defined as

$$i_C(x) := \begin{cases} 0, & x \in C \\ \infty, & \text{otherwise}. \end{cases} \tag{4.1}$$

It is well known that the proximal mapping of i_C is the metric projection on C; i.e,

$$\begin{aligned} \text{prox}_{i_C}(x) &= \arg\min_{u \in C} ||u - x|| \\ &= P_C(x). \end{aligned}$$

Let H_1 and H_2 be Hilbert spaces. Let $T : H_1 \to H_2$ be a bounded linear operator and T^* the adjoint of T. Let P_Q be the projection of H_2 onto nonempty, closed and convex subset Q. Take $f(x) = \frac{1}{2}||Tx - P_Q Tx||^2$ and $g(x) = i_C(x)$. Therefore, from Theorem 4.1, we obtain the following theorem for solving split feasibility problems:

Corollary 4.2. *Let H_1 and H_2 be real Hilbert spaces. Let $T : H_1 \to H_2$ be a bounded linear operator and T^* the adjoint of T. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $\Omega = C \cap T^{-1}(Q) \neq \emptyset$. Let $\{\alpha_n\}$ be a bounded real sequence and $r \in (0, \frac{2}{\|T\|^2})$. Let a sequence $\{x_n\}_{n=0}^{\infty}$ in H be generated by $x_0, x_1 \in H$ and for all $n \geq 1$,*

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = P_C(y_n - rT^*(I - P_Q)Ty_n), \\ C_n = \{u \in H : ||z_n - u||^2 \leq ||x_n - u||^2 - 2\alpha_n\langle x_n - u, x_{n-1} - x_n\rangle \\ \qquad + \alpha_n^2||x_{n-1} - x_n||^2\}, \\ Q_n = \{u \in H : \langle u - x_n, x_0 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{4.2}$$

for all $n \geq 1$. Then $\{x_n\}$ converges strongly to $q := P_\Omega x_0$, where P_Ω is the metric projection from H_1 onto Ω.

(3) **Application to LASSO problem**

The l_1-norm regularized least squares model is

$$min_{x \in \mathbb{R}^n} \frac{1}{2}||Ax - b||_2^2 + \lambda||x||_1, \tag{4.3}$$

where $A \in \mathbb{R}^{m \times n}$ is a given matrix, b is a given vector and λ a positive scalar. Let Ω be the solution set of (4.3).

The concept of l_1 regularization has been studied for many years. The least square problem with l_1 penalty was presented and popularized independently under names, Least Absolute Selection and Shrinkage Operator (LASSO)[60], and Basis Pursuit Denoising [23].

The interest in compressed sensing, is in recovering a solution x to an undetermined system of linear equations $Ax = b$ in the case where $n \gg m$. It is known from linear algebra that this linear system either does not exist or is not unique when the number of unknowns is greater than the number of equations. The system is usually solved by finding the minimum l_2-norm solution, also known as linear least squares. If x is sparse, as is usually the case in applications, then x can be recovered by computing the above l_1-norm regularized least squares model (4.3). This (4.3) model is most often referred to as LASSO. The LASSO problem can be cast as a second order cone programming and solve by standard general algorithms like an interior point method [8], but the computational complexity of such traditional methods is too high to handle large-scale data encountered in many real applications.

Two notable algorithms that take advantage of special structure of LASSO problems are iterative shrinkage thresholding algorithm (ISTA) and its accelerated version fast iterative shrinkage thresholding algorithm (FISTA). The computation of ISTA, which is also known as the proximal gradient method, only involves matrix and vector multiplication, and has great advantage over standard convex algorithms by avoiding a matrix factorization [47]. Beck and

Teboulle [7] put forward an accelerated ISTA named as FISTA, in which a relaxation parameter is chosen. Meanwhile, Nesterov [44,45] had earlier developed a similar algorithm to FISTA. These two algorithms are designed for solving problems containing convex differentiable objectives combined with an l_1 regularization terms as the following problem:

$$\min\{f(x)+g(x) : x \in \mathbb{R}^n\}, \tag{4.4}$$

where f is a smooth convex function and g is continuous function but possibly nonsmooth. Clearly, LASSO problem is a special case of (4.4), formulation with $f(x)=\frac{1}{2}||Ax-b||^2,\ g(x)=\lambda||x||_1$. Its gradient $\nabla f = A^*Ax - A^*b$ is Lipschitz continuous with Lipschitz constant $L(f)=\|A^*A\|$. The proximal map with $g(x)=\lambda||x||_1$ is given as $\text{prox}_g(x) = arg\min_u \lambda||x||_1 + \dfrac{1}{2}||u-x||_2^2$, which is separable in indices. Thus, for $x \in \mathbb{R}^n$,

$$\begin{aligned}\text{prox}_g(x) = \text{prox}_{\lambda||.||_1}(x) &= \Big(\text{prox}_{\lambda|.|_1}(x_1), \dots, \text{prox}_{\lambda|.|_1}(x_n)\Big)\\ &= (\alpha_1, \dots, \alpha_n),\end{aligned}$$

where $\alpha_k = \text{sgn}(x_k)\max\{|x_k|-\lambda, 0\}$ for $k=1,2,\dots,n$. Thus we get from Theorem 4.1 the following theorem for solving the Lasso problem in infinite dimensional Hilbert spaces.

Corollary 4.3. *Let H be a real Hilbert space and f and g from H to $\mathbb{R}$ such that $f(x)=\frac{1}{2}\|Ax-b\|^2,\ g(x)=\lambda||x||_1$. Suppose $\Omega \neq \emptyset$. Let $\{\alpha_n\}$ be a bounded real sequence and $r \in (0, \frac{2}{\|A^*A\|})$. Let a sequence $\{x_n\}_{n=0}^{\infty}$ in H be generated by $x_0, x_1 \in H$ and for all $n \geq 1$,*

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}),\\ z_n = \text{prox}_{\gamma g}(y_n - rA^*(Ay_n - b)),\\ C_n = \{u \in H : ||z_n - u||^2 \leq ||x_n - u||^2 - 2\alpha_n\langle x_n - u, x_{n-1} - x_n\rangle \\ \qquad + \alpha_n^2||x_{n-1} - x_n||^2\},\\ Q_n = \{u \in H : \langle u - x_n, x_0 - x_n\rangle \leq 0\},\\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{4.5}$$

for all $n \geq 1$. Then $\{x_n\}$ converges strongly to $q := P_\Omega x_0$, where P_Ω is the metric projection from H onto Ω.

5. Numerical example

In this section, we present some numerical examples to illustrate the performance of our algorithm. We consider the following simple numerical example to demonstrate the effectiveness of the algorithm (4.2). We apply the algorithm (4.2) to solve the split feasibility problem and compare the numerical results with the standard form (i.e., $\alpha_n = 0$) defined as follows: $x_1 \in H$ and

$$\begin{cases} z_n = P_C(x_n - rA^*(I - P_Q)Ax_n),\\ C_n = \{u \in H : ||z_n - u|| \leq ||x_n - u||\},\\ Q_n = \{u \in H : \langle u - x_n, x_0 - x_n\rangle \leq 0\},\\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{5.1}$$

In the numerical results listed in the following tables, 'Iter.' and 'Sec.' denote the number of iterations and the cpu time in seconds, respectively.

Example 5.1. Let $H_1 = L_2([\alpha, \beta]) = H_2$ and we give a numerical example in $(L_2([\alpha, \beta]), ||.||_{L_2})$ of the problem considered in Corollary 4.2 in this section. Now take

$$C := \{x \in L_2([\alpha, \beta]) : \langle a, x\rangle \leq b\},$$

where $0 \neq a \in L_2([\alpha, \beta])$ and $b \in \mathbb{R}$, then (see [19])

$$P_C(x) = \begin{cases} \frac{b-\langle a,x\rangle}{||a||_{L_2}^2} a + x, & \langle a, x\rangle > b \\ x, & \langle a, x\rangle \leq b. \end{cases}$$

Let

$$Q = \{x \in L_2([\alpha, \beta]) : ||x - d||_{L_2} \leq r\}$$

be a closed ball centered at $d \in L_2([\alpha, \beta])$ with radius $r > 0$, then

$$P_Q(x) = \begin{cases} d + r\frac{x-d}{||x-d||}, & x \notin Q \\ x, & x \in Q. \end{cases}$$

Now, suppose

$$C := \left\{x \in L_2([0, 2\pi]) : \int_0^{2\pi} x(t)dt \leq 1\right\}$$

and

$$Q = \left\{x \in L_2([0, 2\pi]) : \int_0^{2\pi} |x(t) - \sin(t)|^2 dt \leq 16\right\}$$

and $A : L^2([0, 2\pi]) \to L^2([0, 2\pi]),\ (Ax)(s) = x(s),\ \forall x \in L^2([0, 2\pi])$. Then $(A^*x)(s) = x(s)$ and $\|A\| = 1$. Let us consider the following problem:

$$\text{find} \quad x^* \in C \quad \text{such that} \quad Ax^* \in Q. \tag{5.2}$$

Observe that the set of solutions of problem (5.2) is nonempty (since $x(t) = 0,\ a.e.$ is in the set of solutions).

In Corollary 4.2, $x_0(t) = x_1(t),\ t \in [0, 2\pi]$. Take $r = 1.90$, $\alpha_n := \frac{n-1}{n+\alpha-1}$, $\forall n \geq 1$ with $\alpha = 3$ and $\alpha_0 = 0.84$. We take $E(x_n) = \frac{1}{2}\|P_C(x_n) - x_n\|^2 + \frac{1}{2}\|P_Q(Ax_n) - Ax_n\|^2 \leq \varepsilon = 10^{-3}$ as the stopping criterion. We test several initial values and compare iterative method (4.2) with the un-accelerated one defined by (5.1). The results are listed in Table 1.

We take $E(x_n) = \frac{1}{2}\|P_C(x_n) - x_n\|^2 + \frac{1}{2}\|P_Q(Ax_n) - Ax_n\|^2 \leq \varepsilon = 10^{-i}(i = 0, 1, 2, 3)$ as the stopping criterion.We choose $x_0^1 = \frac{t^2}{10}$, $x_0^2 = \frac{2^t}{16}$,

Table 1. Computational results for Example 5.1

x_0	Un-accelerated algorithm 4.2		Algorithm 4.2	
	Sec.	Iter.	Sec.	Iter.
$\frac{t^2}{10}$	27.5781	18	12.6094	12
$\frac{2^t}{16}$	35.1094	12	17.5469	8
$\frac{1}{2}e^{\frac{t}{3}}$	1.3281	9	0.6406	3
$\frac{1}{2}e^{\frac{t}{4}} + \frac{t^2}{24}$	3.7969	13	2.7813	6
$\frac{1}{2}\log_2(t) + \frac{t^2}{24}$	23.9844	13	13.2813	6
$2\sin^4 5t - 3\cos 2t$	49.7656	15	21.3906	11

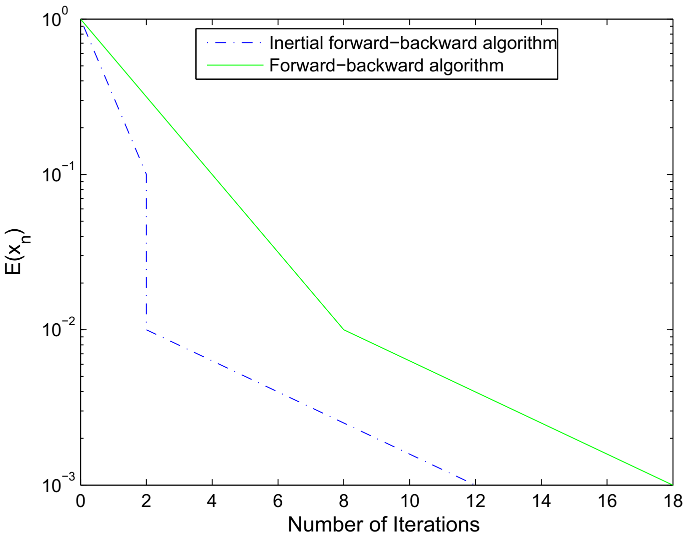

Figure 1. Comparison of the number of iterations, for example 5.1 with x_0^1

$x_0^3 = \frac{1}{2}e^{\frac{t}{3}}$, $x_0^4 = \frac{1}{2}e^{\frac{t}{4}} + \frac{t^2}{24}$, $x_0^5 = \frac{1}{2}\log_2(t) + \frac{t^2}{24}$ and $x_0^6 = 2\sin^4 5t - 3\cos 2t$ as initial values, and the results are presented in Fig. 1, 2, 3, 4, 5 and 6, respectively.

6. Conclusion and final remarks

In this paper, we consider an iterative method which is a combination of the inertial forward–backward algorithm and Haugazeau's algorithm for solving monotone inclusions given by the sum of two monotone operators with an easy-to-compute resolvent operator and another monotone operator which is co-coercive and prove the strong convergence of the sequence of iterates generated by our proposed algorithm to a solution of monotone inclusions in

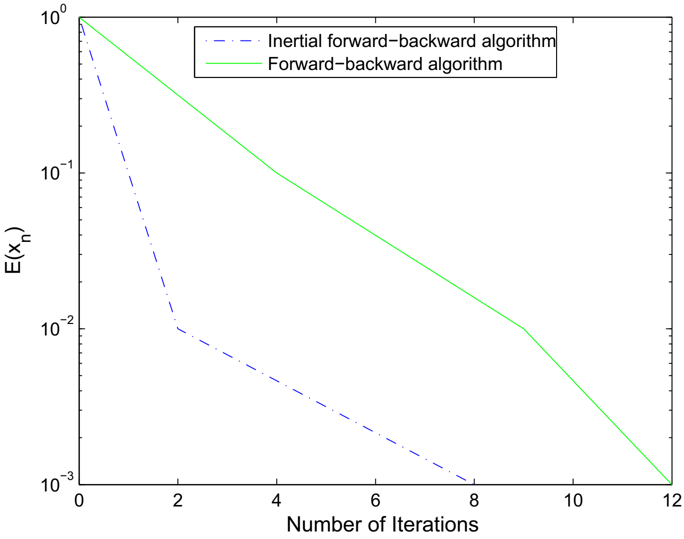

FIGURE 2. Comparison of the number of iterations, for example 5.1 with x_0^2

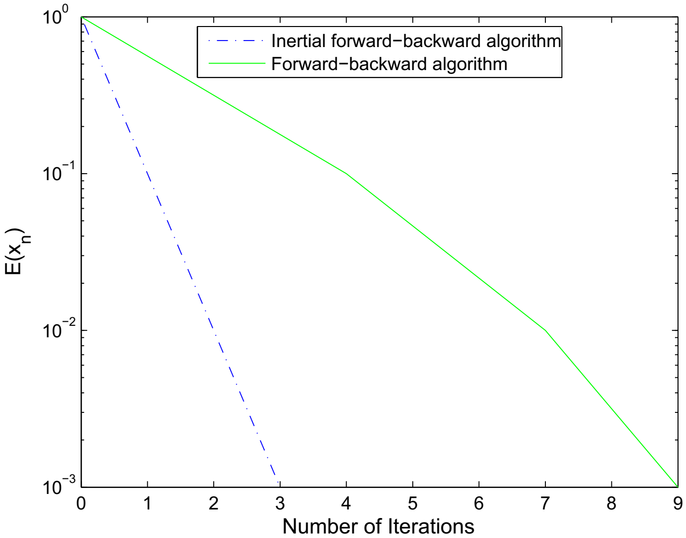

FIGURE 3. Comparison of the number of iterations for Example 5.1 with x_0^3

real Hilbert spaces. Our results in this paper complement, in terms of the mode of convergence in infinite dimensional real Hilbert spaces, the results of Beck and Teboulle [7], the primal-dual algorithm of Chambolle and Pock [21] and Lorenz and Pock [36]. We can also obtain a strong convergence result using a combination of inertial primal-dual algorithm and Haugazeau's

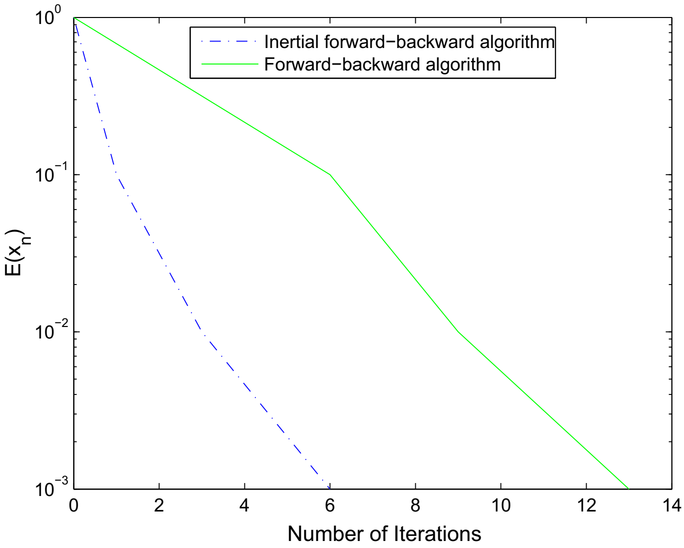

Figure 4. Comparison of the number of iterations for Example 5.1 with x_0^4

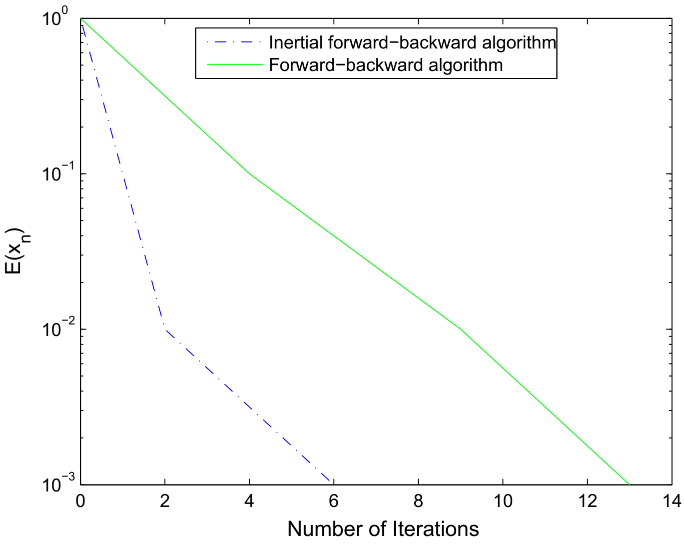

Figure 5. Comparison of the number of iterations for Example 5.1 with x_0^5

algorithm for convex concave programming by adapting appropriately our iterative method in this paper. From our numerical experiment, we see that the inertial term leads to faster convergence.

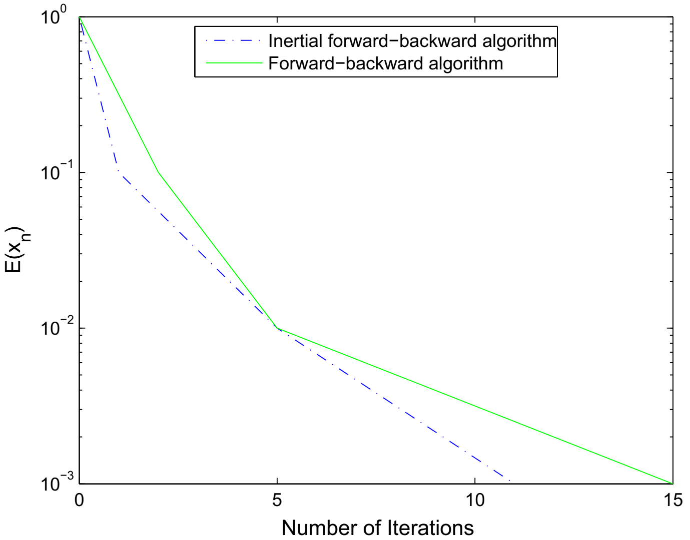

FIGURE 6. Comparison of the number of iterations for Example 5.1 with x_0^6

In our future research, we shall develop iterative method with inertial extrapolation term which does not involve the construction of sets C_n and Q_n as given in (3.1) and the sequence of iterates generated by the method converges strongly to a solution $x^* \in (A+B)^{-1}(0) \neq \emptyset$. When this is achieved, we would compare numerically the new proposed accelerated method with the un-accelerated method (see, e.g., [64]) of solving monotone inclusions in real Hilbert spaces. For the time being, our result in this paper obtains strong convergence result using Haugazeau's algorithm involving inertial extrapolation term and show numerically that our proposed scheme converges faster than the un-accelerated Haugazeau's algorithm for solving monotone inclusions in real Hilbert spaces.

Acknowledgements

The research was carried out when the Fourth Author was an Alexander von Humboldt Postdoctoral Fellow at the Institute of Mathematics, University of Wurzburg, Germany. He is grateful to the Alexander von Humboldt Foundation, Bonn, for the fellowship and the Institute of Mathematics, Julius Maximilian University of Wurzburg, Germany for the hospitality and facilities. P. Cholamjiak was supported by the Thailand Research Fund and the Commission on Higher Education under Grant MRG5980248.

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DOI 10.1007/s11075-017-0411-0

ORIGINAL PAPER

Iterative methods for solving quasi-variational inclusion and fixed point problem in q-uniformly smooth Banach spaces

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Received: 17 October 2016 / Accepted: 31 August 2017

Abstract In this work, we introduce implicit and explicit iterations for solving the variational inclusion problem for the sum of two operators and the fixed point problem of nonexpansive mappings. We then prove its strong convergence theorems in the framework of Banach spaces. We finally provide some applications of the main results.

Keywords Variational inequality · Banach space · Strong convergence · Iterative method · m-accretive operator

Mathematics Subject Classification (2010) 47H09 · 47H10 · 47H17 · 47J25 · 49J40

1 Introduction

Let X be real Banach space, we consider the following so-called *variational inclusion problem*: Find $x \in X$ such that

$$0 \in Ax + Bx, \tag{1.1}$$

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Published online: 08 September 2017

where $A : C \longrightarrow X$ is a single-valued mapping, $B : X \longrightarrow 2^X$ is a set-valued mapping and 0 is a zero vector in X. The set of solutions of (1.1) is denoted by $(A + B)^{-1}0$. It is well known that the problem (1.1) has wide applications in the fields of economics, structural analysis, mechanics, optimization problems, signal processing, image recovery, and applied sciences (see, e.g., [11, 17, 21, 23, 24], and the references therein).

A classical method for solving this problem is the forward-backward splitting method [18, 25, 29, 37] which is defined by the following manner: $x_1 \in X$ and

$$x_{n+1} = (I + rB)^{-1}(x_n - rAx_n), \ n \geq 1, \tag{1.2}$$

where $r > 0$. We see that each step of iterates involves only with A as the forward step and B as the backward step, but not the sum of A and B. This method includes, in particular, the proximal point algorithm [12, 13, 22, 27, 32] and the gradient method [9, 19]. Lions-Mercier [25] introduced the following splitting iterative methods in a real Hilbert space:

$$x_{n+1} = (2J_r^A - I)(2J_r^B - I)x_n, \ n \geq 1 \tag{1.3}$$

and

$$x_{n+1} = J_r^A(2J_r^B - I)x_n + (I - J_r^B)x_n, \ n \geq 1, \tag{1.4}$$

where $J_r^T = (I + rT)^{-1}$. The first one is often called Peaceman-Rachford algorithm [30] and the second one is called Douglas-Rachford algorithm [20]. We note that both algorithms can be weakly convergent in general [29].

Recently, López et al. [26] introduced the following Halpern-type forward-backward method: $x_1 \in X$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J_{r_n}^B(x_n - r_n(Ax_n + a_n)) + b_n), \tag{1.5}$$

where J_r^B is the resolvent of B, $\{r_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1]$ and $\{a_n\}$, $\{b_n\}$ are error sequences in X. It was proved that the sequence $\{x_n\}$ generated by (1.5) strongly converges to a zero point of the sum of A and B under some appropriate conditions.

Very recently, Abdou et al. [1] introduced the following two algorithms for solving the fixed point problem of a nonexpansive mapping and the variational inclusion problem in Hilbert spaces:

$$x_t = (1 - \kappa)Sx_t + \kappa J_\lambda^B(t\gamma f(x_t) + (1 - t)x_t - \lambda Ax_t), \tag{1.6}$$

for all $t \in (0, 1)$ and

$$x_{n+1} = (1 - \kappa)Sx_n + \kappa J_{\lambda_n}^B(\alpha_n\gamma f(x_n) + (1 - \alpha_n)x_n - \lambda_n Ax_n), \tag{1.7}$$

for all $n \geq 1$. It was proved that the sequences generated by (1.6) and (1.7) converge strongly to a common solution.

There have been many works concerning the problem of finding zero points of the sum of two monotone operators (in Hilbert spaces) and accretive operators (in Banach spaces) (see [14, 16, 36–38, 42]).

In this work, motivated by the previous work, we study implicit and explicit iteration methods for solving the inclusion problem for the sum of accretive and m-accretive operators in the framework of Banach spaces. We then prove its strong convergence under some mild conditions. Finally, we provide some applications including its experiments to support the main results. Our results extend and improve many results in the literature.

2 Preliminaries

Throughout this paper, we denote by X and X^* a real Banach space and the dual space of X, respectively. Let $q > 1$ be a real number. The *generalized duality mapping* $J_q : X \longrightarrow 2^{X^*}$ is defined by

$$J_q(x) = \{j_q(x) \in X^* : \langle x, j_q(x)\rangle = \|x\|^q, \|j_q(x)\| = \|x\|^{q-1}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. In particular, $J_q = J_2$ is called the *normalized duality mapping* and $J_q(x) = \|x\|^{q-2}J_2(x)$ for $x \neq 0$. If $X := H$ is a real Hilbert space, then $J = I$, where I is the identity mapping. It is well known that if X is smooth, then J_q is single-valued, which is denoted by j_q (see [35]).

The *modulus of convexity* of X is the function $\delta : (0, 2] \longrightarrow [0, 1]$ defined by

$$\delta(\epsilon) = \inf\left\{1 - \tfrac{\|x+y\|}{2} : x, y \in X, \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\right\}.$$

A Banach space X is said to be *uniformly convex* if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

The *modulus of smoothness* of X is the function $\rho : \mathbb{R}^+ := [0, \infty) \longrightarrow \mathbb{R}^+$ defined by

$$\rho(t) = \sup\left\{\tfrac{\|x+ty\|+\|x-ty\|}{2} - 1 : x, y \in X, \|x\| = \|y\| = 1\right\}.$$

A Banach space X is said to be *uniformly smooth* if $\frac{\rho_X(t)}{t} \longrightarrow 0$ as $t \longrightarrow 0$. Suppose that $1 < q \leq 2$, then X is said to be *q-uniformly smooth* if there exists $c > 0$ such that $\rho(t) \leq ct^q$ for all $t > 0$. If X is q-uniformly smooth, then X is uniformly smooth. It is well known that each uniformly convex Banach space (uniformly smooth Banach space) is reflexive and strictly convex (see [15, 35]).

Let $A : X \longrightarrow 2^X$ be a set-valued mapping. We denote the domain and range of an operator $A : X \longrightarrow 2^X$ by $D(A) = \{x \in X : Ax \neq \emptyset\}$ and $R(A) = \bigcup\{Az : z \in D(A)\}$, respectively. Let $q > 1$. A set-valued mapping $A : D(A) \subset X \longrightarrow 2^X$ is said to be *accretive* of order q if for each $x, y \in D(A)$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq 0, u \in Ax \text{ and } v \in Ay.$$

An accretive operator A is said to be *m-accretive* if $R(I + rA) = X$ for all $r > 0$. Let $A : D(A) \subset X \longrightarrow 2^X$ be an m-accretive operator. The *resolvent operator* of A, denoted by $J_\lambda^A : X \longrightarrow D(A)$ is defined by

$$J_\lambda^A = (I + \lambda A)^{-1},$$

where λ is any positive number and also denote $A^{-1}0$ by the set of zeros of A, that is, $A^{-1}0 = \{x \in D(A) : 0 \in Ax\}$. Let C be a nonempty subset of a real Banach space

X. A mapping $S : C \longrightarrow C$ is said to be *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\|, \forall x, y \in C.$$

If $0 < L < 1$, then S is a contraction and if $L = 1$, then S is a nonexpansive mapping. We denote the fixed points set of the mapping S by $Fix(S) = \{x \in C : Sx = x\}$.

Let $\alpha > 0$ and $q > 1$. A mapping $A : C \longrightarrow X$ is said to be *α-inverse strongly accreive* (α-isa) of order q if for each $x, y \in D(A)$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq \alpha\|Ax - Ay\|^q, u \in Ax \text{ and } v \in Ay.$$

Lemma 2.1 ([26]) *Let X be a real q-uniformly smooth Banach space and $A : X \longrightarrow X$ be an α-isa of order q. Then, the following inequality holds:*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^q \leq \|x - y\|^q - \lambda(\alpha q - \kappa_q \lambda^{q-1})\|Ax - Ay\|^q$$

for all $x, y \in X$. In particular, if $0 < \lambda \leq \left(\frac{\alpha q}{\kappa_q}\right)^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Lemma 2.2 [39] *Let $1 < q \leq 2$ and X be a Banach space. Then, the following are equivalent.*

(i) *X is q-uniformly smooth.*
(ii) *There is a constant $\kappa_q > 0$ such that for all $x, y \in X$*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + \kappa_q\|y\|^q. \tag{2.1}$$

Remark 2.3 The constant κ_q satisfying (2.1) is called the *q-uniform smoothness coefficient of X*.

Lemma 2.4 ([39]) *Let $p > 1$ and $r > 0$ be two fixed real numbers and X be a Banach space. Then, the following are equivalent.*

(i) *X is uniformly convex.*
(ii) *There is a strictly increasing, continuous, and convex function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \|x\|^p - p\langle x, j_p(y)\rangle + (p - 1)\|y\|^p, \forall x, y \in B_r.$$

We use the notation $x_n \rightharpoonup x$ stands for weak convergence of $\{x_n\}$ to x and $x_n \longrightarrow x$ stands for the strong convergence of $\{x_n\}$ to x.

Lemma 2.5 ([10]) Let C be a nonempty, closed, and convex subset of a uniformly convex Banach space X and $S : C \longrightarrow C$ be a nonexpansive mapping. Then, $I - S$ is demiclosed at zero, *i.e.*, $x_n \rightharpoonup x$ and $x_n - Sx_n \longrightarrow 0$ implies $x = Sx$.

Following the proof line as in Lemma 2.7 of [41], we obtain the following results.

Lemma 2.6 *Let C be a nonempty, closed, and convex subset of a real smooth Banach space X and let $j_q : X \longrightarrow X^*$ be a generalized duality mapping. Assume that the mapping $F : C \longrightarrow X$ is accretive and weakly continuous along segments, that is, $F(x+ty) \rightharpoonup F(x)$ as $t \longrightarrow 0$. Then, the variational inequality*

$$x^* \in C, \langle Fx^*, j_q(x-x^*)\rangle \geq 0, x \in C$$

is equivalent to the dual variational inequality

$$x^* \in C, \langle Fx, j_q(x-x^*)\rangle \geq 0, x \in C.$$

Lemma 2.7 ([34]) *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\longrightarrow\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\longrightarrow\infty} \|l_n - x_n\| = 0$.*

Lemma 2.8 ([40]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\longrightarrow\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\delta_n| < \infty$.

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.9 ([28]) *Let $q > 1$. Then, the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \left(\frac{q-1}{q}\right)b^{\frac{q}{q-1}},$$

for arbitrary positive real numbers a, b.

Proposition 2.10 ([28]) *Let $q > 1$. Then, the following inequality holds:*

$$a^q - b^q \leq qa^{q-1}(a-b),$$

for arbitrary positive real numbers a, b.

Lemma 2.11 (*The resolvent identity* [7]) *For $\lambda > 0$, $\mu > 0$ and $x \in X$, then*

$$J_\lambda^B x = J_\mu^B\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda^B x\right).$$

From Lemma above, we have the following fact.

Lemma 2.12 *For each $r, s > 0$ then*

$$\|J_r^B x - J_s^B x\| \le \left|1 - \frac{s}{r}\right| \|J_r^B x - x\| \text{for all} x \in X.$$

Proposition 2.13 *Let X be a real q-uniformly smooth Banach space. Let B be an m-accretive operator on X and let J_λ^B be the resolvent operator associated with B and λ. Then, we have*

$$\|J_\lambda^B x - J_\lambda^B y\|^q \le \left\langle x - y, j_q\left(J_\lambda^B x - J_\lambda^B y\right)\right\rangle, \forall x, y \in X.$$

Proof. For any $x, y \in X$ and $\lambda > 0$, we set $u = J_\lambda^B x$ and $v = J_\lambda^B y$. By definition of the accretive operator, we have $x - u \in \lambda Bu$ and $y - v \in \lambda Bv$. Since B is m-accretive,

$$\begin{aligned} 0 &\le \langle x - u - (y - v), j_q(u - v)\rangle \\ &= \langle x - y, j_q(u - v)\rangle - \langle u - v, j_q(u - v)\rangle \\ &= \langle x - y, j_q(u - v)\rangle - \|u - v\|^q. \end{aligned}$$

It follows that

$$\|u - v\|^q \le \langle x - y, j_q(u - v)\rangle,$$

i.e.,

$$\|J_\lambda^B x - J_\lambda^B y\|^q \le \left\langle x - y, j_q\left(J_\lambda^B x - J_\lambda^B y\right)\right\rangle, \forall x, y \in X.$$

This completes the proof. □

3 Main results

In this section, we prove the convergence theorem by using an implicit iteration.

3.1 Convergence theorem for implicit iteration scheme

Let X be a uniformly convex and q-uniformly smooth Banach space which admits a weakly sequentially continuous generalized duality mapping $j_q : X \longrightarrow X^*$. Let $f : X \longrightarrow X$ be a ρ-contraction, $A : X \longrightarrow X$ be an α-isa of order q and $B : X \longrightarrow 2^X$ be an m-accretive operator. Let $J_\lambda^B = (I + \lambda B)^{-1}$ be a resolvent of B for $\lambda > 0$ and $S : X \longrightarrow X$ be a nonexpansive mapping such that $Fix(S) \cap (A + B)^{-1}0 \neq \emptyset$. Let $0 < \gamma < 1$. For $t \in (0, 1)$, consider the following mapping S_t on X defined by

$$S_t x := (1 - \gamma)Sx + \gamma J_{\lambda_t}^B(tf(x) + (1 - t)x - \lambda_t Ax), \forall x \in X,$$

where $0 < a \leq \lambda_t < \frac{\lambda_t}{1-t} \leq b < \left(\frac{\alpha q}{\kappa_q}\right)^{\frac{1}{q-1}}$. It is observed that S_t is a contraction. Indeed, by the nonexpansiveness of $J_{\lambda_t}^B$ and Lemma 2.1, for all $x, y \in X$, we have

$$\begin{aligned}
&\|S_t x - S_t y\| \\
&= \left\|\big((1-\gamma)Sx + \gamma J_{\lambda_t}^B(tf(x) + (1-t)x - \lambda_t Ax)\big) - \big((1-\gamma)Sy + \gamma J_{\lambda_t}^B(tf(y) + (1-t)y - \lambda_t Ay)\big)\right\| \\
&= \left\|(1-\gamma)(Sx - Sy) + \gamma\left[J_{\lambda_t}^B\left(tf(x) + (1-t)\left(I - \frac{\lambda_t}{1-t}A\right)x\right)\right.\right. \\
&\quad \left.\left. - J_{\lambda_t}^B\left(tf(y) + (1-t)\left(I - \frac{\lambda_t}{1-t}A\right)y\right)\right]\right\| \\
&\leq (1-\gamma)\|Sx - Sy\| + \gamma\left\|t(f(x) - f(y)) + (1-t)\left[\left(I - \frac{\lambda_t}{1-t}A\right)x - \left(I - \frac{\lambda_t}{1-t}A\right)y\right]\right\| \\
&\leq (1-\gamma)\|x - y\| + \gamma t\|f(x) - f(y)\| + (1-t)\gamma\left\|\left(I - \frac{\lambda_t}{1-t}A\right)x - \left(I - \frac{\lambda_t}{1-t}A\right)y\right\| \\
&\leq (1-\gamma)\|x - y\| + \gamma t\rho\|x - y\| + (1-t)\gamma\|x - y\| \\
&= \big(1 - (1-\rho)\gamma t\big)\|x - y\|,
\end{aligned}$$

which implies that the mapping S_t is a contraction. Hence, S_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation

$$x_t = (1-\gamma)Sx_t + \gamma J_{\lambda_t}^B(tf(x_t) + (1-t)x_t - \lambda_t Ax_t). \tag{3.1}$$

Our first main result is to show that the net $\{x_t\}$ defined by (3.1) converges strongly, as $t \longrightarrow 0$, to a point in $Fix(S) \cap (A+B)^{-1}0$ which is also a solution of the variational inequality.

Theorem 3.1 *Suppose that $Fix(S) \cap (A+B)^{-1}0 \neq \emptyset$. Then, the net $\{x_t\}$ defined by (3.1) converges strongly, as $t \longrightarrow 0$, to a point $x^* \in Fix(S) \cap (A+B)^{-1}0$, which is the unique solution of the variational inequality*

$$\langle f(x^*) - x^*, j_q(z - x^*)\rangle \leq 0, \forall z \in Fix(S) \cap (A+B)^{-1}0. \tag{3.2}$$

Proof. First, we show the uniqueness of a solution of the variational inequality (3.2). If $x^* \in Fix(S) \cap (A+B)^{-1}0$ and $\hat{x} \in Fix(S) \cap (A+B)^{-1}0$ both are solutions to (3.2), then we obtain

$$\langle f(\hat{x}) - \hat{x}, j_q(x^* - \hat{x})\rangle \leq 0$$

and

$$\langle f(x^*) - x^*, j_q(\hat{x} - x^*)\rangle \leq 0.$$

Adding up above two inequalities, we have

$$\langle x^* - \hat{x} - \big(f(x^*) - f(\hat{x})\big), j_q(x^* - \hat{x})\rangle \leq 0,$$

and hence,

$$\|x^* - \hat{x}\|^q \leq \rho\|x^* - \hat{x}\|^q.$$

This implies that $\hat{x} = x^*$ and the uniqueness is proved.

Next, we show that $\{x_t\}$ is bounded. Set $y_t = J^B_{\lambda_t}(tf(x_t) + (1-t)x_t - \lambda Ax_t)$. Taking $p \in Fix(S) \cap (A+B)^{-1}0$, we see that

$$p = Sp = J^B_{\lambda_t}(p - \lambda_t Ap) = J^B_{\lambda_t}\Big(tp + (1-t)\Big(I - \frac{\lambda_t}{1-t}A\Big)p\Big).$$

Since $J^B_{\lambda_t}$ and $I - \frac{\lambda_t}{1-t}A$ are nonexpansive,

$$\begin{aligned}
&\|y_t - p\| \\
&= \Big\|J^B_{\lambda_t}\Big(tf(x_t) + (1-t)\Big(I - \frac{\lambda_t}{1-t}A\Big)x_t\Big) - J^B_{\lambda_t}\Big(tp + (1-t)\Big(I - \frac{\lambda_t}{1-t}A\Big)p\Big)\Big\| \\
&\le \Big\|t(f(x_t) - p) + (1-t)\Big[\Big(I - \frac{\lambda_t}{1-t}A\Big)x_t - \Big(I - \frac{\lambda_t}{1-t}A\Big)p\Big]\Big\| \\
&\le t\|f(x_t) - f(p)\| + t\|f(p) - p\| + (1-t)\Big\|\Big(I - \frac{\lambda_t}{1-t}A\Big)x_t - \Big(I - \frac{\lambda_t}{1-t}A\Big)p\Big\| \\
&\le t\rho\|x_t - p\| + t\|f(p) - p\| + (1-t)\|x_t - p\| \\
&= \big(1 - (1-\rho)t\big)\|x_t - p\| + t\|f(p) - p\|. \qquad (3.3)
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
\|x_t - p\| &= \|(1-\gamma)(Sx_t - p) + \gamma(y_t - p)\| \\
&\le (1-\gamma)\|Sx_t - p\| + \gamma\|y_t - p\| \\
&\le (1-\gamma)\|x_t - p\| + \gamma\|y_t - p\| \\
&\le (1-\gamma)\|x_t - p\| + \gamma\big[\big(1 - (1-\rho)t\big)\|x_t - p\| + t\|f(p) - p\|\big] \\
&= \big(1 - (1-\rho)\gamma t\big)\|x_t - p\| + \gamma t\|f(p) - p\|, \qquad (3.4)
\end{aligned}$$

which implies that

$$\|x_t - p\| \le \frac{1}{1-\rho}\|f(p) - p\|.$$

Hence, $\{x_t\}$ is bounded and so are $\{f(x_t)\}$, $\{Ax_t\}$, and $\{Sx_t\}$.

Next, we show that $\lim_{t\longrightarrow 0}\|x_t - Sx_t\| = 0$. From (3.4), we know that $\|x_t - p\| \le \|y_t - p\|$. Then, by the convexity of $\|\cdot\|^q$ for all $q > 1$ and Lemma 2.2, we have

$$\begin{aligned}
\|x_t - p\|^q &\le \|y_t - p\|^q \\
&\le \Big\|(1-t)\Big[\Big(x_t - \frac{\lambda_t}{1-t}Ax_t\Big) - \Big(p - \frac{\lambda_t}{1-t}Ap\Big)\Big] + t(f(x_t) - p)\Big\|^q \\
&\le (1-t)\Big\|\Big(x_t - \frac{\lambda_t}{1-t}Ax_t\Big) - \Big(p - \frac{\lambda_t}{1-t}Ap\Big)\Big\|^q + t\|f(x_t) - p\|^q \\
&= (1-t)\Big\|(x_t - p) - \frac{\lambda_t}{1-t}(Ax_t - Ap)\Big\|^q + t\|f(x_t) - p\|^q \\
&\le (1-t)\Big[\|x_t - p\|^q - \frac{q\lambda_t}{1-t}\langle Ax_t - Ap, j_q(x_t - p)\rangle + \frac{\kappa_q\lambda_t^q}{(1-t)^q}\|Ax_t - Ap\|^q\Big] + t\|f(x_t) - p\|^q \\
&\le (1-t)\Big[\|x_t - p\|^q - \frac{\alpha q\lambda_t}{1-t}\|Ax_t - Ap\|^q + \frac{\kappa_q\lambda_t^q}{(1-t)^q}\|Ax_t - Ap\|^q\Big] + t\|f(x_t) - p\|^q \\
&= (1-t)\Big[\|x_t - p\|^q - \frac{\lambda_t}{1-t}\Big(\alpha q - \frac{\kappa_q\lambda_t^{q-1}}{(1-t)^{q-1}}\Big)\|Ax_t - Ap\|^q\Big] + t\|f(x_t) - p\|^q \\
&\le \|x_t - p\|^q - \lambda_t\Big(\alpha q - \frac{\kappa_q\lambda_t^{q-1}}{(1-t)^{q-1}}\Big)\|Ax_t - Ap\|^q + t\|f(x_t) - p\|^q,
\end{aligned}$$

which implies that

$$\lambda_t\left(\alpha q-\frac{\kappa_q\lambda_t^{q-1}}{(1-t)^{q-1}}\right)\|Ax_t-Ap\|^q\leq t\|f(x_t)-p\|^q.$$

By our assumption, we obtain

$$\lim_{t\longrightarrow 0}\|Ax_t-Ap\|=0. \tag{3.5}$$

On the other hand, from Proposition 2.13 and Lemma 2.4, we have

$$\begin{aligned}
&\|y_t-p\|^q\\
=\ &\|J_{\lambda_t}^B(tf(x_t)+(1-t)x_t-\lambda_tAx_t)-J_{\lambda_t}^B(p-\lambda_tAp)\|^q\\
\leq\ &\langle tf(x_t)+(1-t)x_t-\lambda_tAx_t-(p-\lambda_tAp),\, j_q(y_t-p)\rangle\\
\leq\ &\frac{1}{q}\Big[\|tf(x_t)+(1-t)x_t-\lambda_tAx_t-(p-\lambda_tAp)\|^q+(q-1)\|y_t-p\|^q\\
&-g(\|tf(x_t)+(1-t)x_t-\lambda_t(Ax_t-Ap)-y_t\|)\Big],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|y_t-p\|^q\\
\leq\ &\|tf(x_t)+(1-t)x_t-\lambda_tAx_t-(p-\lambda_tAp)\|^q-g(\|tf(x_t)+(1-t)x_t-\lambda_t(Ax_t-Ap)-y_t\|)\\
=\ &\left\|(1-t)\left[\left(I-\frac{\lambda_t}{1-t}A\right)x_t-\left(I-\frac{\lambda_t}{1-t}A\right)p\right]+t(f(x_t)-p)\right\|^q\\
&-g(\|tf(x_t)+(1-t)x_t-\lambda_t(Ax_t-Ap)-y_t\|)\\
\leq\ &(1-t)\left\|\left(I-\frac{\lambda_t}{1-t}A\right)x_t-\left(I-\frac{\lambda_t}{1-t}A\right)p\right\|^q+t\|f(x_t)-p\|^q\\
&-g(\|tf(x_t)+(1-t)x_t-\lambda_t(Ax_t-Ap)-y_t\|)\\
\leq\ &(1-t)\|x_t-p\|^q+t\|f(x_t)-p\|^q-g(\|tf(x_t)+(1-t)x_t-\lambda_t(Ax_t-Ap)-y_t\|)\\
\leq\ &\|x_t-p\|^q+t\|f(x_t)-p\|^q-g(\|tf(x_t)+(1-t)x_t-\lambda_t(Ax_t-Ap)-y_t\|)\\
\leq\ &\|y_t-p\|^q+t\|f(x_t)-p\|^q-g(\|tf(x_t)+(1-t)x_t-\lambda_t(Ax_t-Ap)-y_t\|).
\end{aligned}$$

This gives

$$g(\|tf(x_t)+(1-t)x_t-\lambda_t(Ax_t-Ap)-y_t\|)\leq t\|f(x_t)-p\|^q.$$

Hence,

$$\lim_{t\longrightarrow 0}g(\|tf(x_t)+(1-t)x_t-\lambda_t(Ax_t-Ap)-y_t\|)=0.$$

Since g is a continuous function, by (3.5), we obtain that

$$\lim_{t\longrightarrow 0}\|x_t-y_t\|=0. \tag{3.6}$$

From (3.1), we note that

$$x_t=(1-\gamma)Sx_t+\gamma y_t$$

and hence,

$$\lim_{t \longrightarrow 0} \|y_t - Sx_t\| = 0.$$

It follows that, as $t \to 0$,

$$\|Sx_t - x_t\| \leq \|Sx_t - Sy_t\| + \|Sy_t - x_t\| \leq \|x_t - y_t\| + \|Sy_t - x_t\| \to 0. \tag{3.7}$$

For $z \in Fix(S) \cap (A+B)^{-1}0$, we see that

$$\begin{aligned}
&\|y_t - z\|^q \\
&= \left\| J_{\lambda_t}^B\left(tf(x_t) + (1-t)\left(I - \frac{\lambda_t}{1-t}A\right)x_t\right) - J_{\lambda_t}^B\left(tz + (1-t)\left(I - \frac{\lambda_t}{1-t}A\right)z\right)\right\|^q \\
&\leq \left\langle tf(x_t) + (1-t)\left(I - \frac{\lambda_t}{1-t}A\right)x_t - tz - (1-t)\left(I - \frac{\lambda_t}{1-t}A\right)z, j_q(y_t - z)\right\rangle \\
&= (1-t)\left\langle\left(I - \frac{\lambda_t}{1-t}A\right)x_t - \left(I - \frac{\lambda_t}{1-t}A\right)z, j_q(y_t - z)\right\rangle + t\langle f(x_t) - f(z), j_q(y_t - z)\rangle \\
&\quad + t\langle f(z) - z, j_q(y_t - z)\rangle \\
&\leq (1-t)\|x_t - z\|\|y_t - z\|^{q-1} + t\rho\|x_t - z\|\|y_t - z\|^{q-1} + t\langle f(z) - z, j_q(y_t - z)\rangle \\
&\leq (1-t)\|y_t - z\|^q + t\rho\|y_t - z\|^q + t\langle f(z) - z, j_q(y_t - z)\rangle \\
&= (1-(1-\rho)t)\|y_t - z\|^q + t\langle f(z) - z, j_q(y_t - z)\rangle,
\end{aligned}$$

which implies that

$$\|x_t - z\|^q \leq \|y_t - z\|^q \leq \frac{1}{1-\rho}\langle f(z) - z, j_q(y_t - z)\rangle. \tag{3.8}$$

Next, we show that $\{x_t\}$ is relatively norm-compact. Assume that $t_n \in (0, 1)$ is a sequence such that $t_n \longrightarrow 0$ as $n \longrightarrow \infty$. Put $x_n := x_{t_n}$, $y_n := y_{t_n}$ and $\lambda_n := \lambda_{t_n}$. From (3.8), we have

$$\|x_n - z\|^q \leq \frac{1}{1-\rho}\langle f(z) - z, j_q(y_n - z)\rangle. \tag{3.9}$$

By the reflexivity of a Banach space X and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^* \in X$ as $i \longrightarrow \infty$. So there exists a corresponding subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup x^* \in X$ as $i \longrightarrow \infty$. From (3.7), we have $\lim_{n \longrightarrow \infty} \|x_n - Sx_n\| = 0$. It follows from Lemma 2.5 that $x^* \in Fix(S)$. Further, we show that $x^* \in (A+B)^{-1}0$. Let $v \in Bu$. Note that

$$y_n = J_{\lambda_n}^B(t_n f(x_n) + (1-t_n)x_n - \lambda_n Ax_n).$$

Then, we have

$$t_n f(x_n) + (1-t_n)x_n - \lambda_n Ax_n \in (I + \lambda_n B)y_n \Longleftrightarrow \frac{1}{\lambda_n}\big(t_n f(x_n) + (1-t_n)x_n - \lambda_n Ax_n - y_n\big) \in By_n.$$

Note that, by the boundedness of $\{x_n\}$, we can find a positive constant M_1 such that

$$M_1 = \max\left\{\sup_{n\geq 1}\|y_n - u\|^{q-1}, \sup_{n\geq 1}\|f(x_n) - x_n\|\|y_n - u\|^{q-1}\right\} < \infty.$$

Since B is m-accretive, we have for all $(u, v) \in B$,

$$\left\langle \frac{1}{\lambda_n}\big(t_n f(x_n) + (1 - t_n)x_n - \lambda_n Ax_n - y_n\big) - v, j_q(y_n - u)\right\rangle \geq 0$$
$$\Longleftrightarrow \langle t_n f(x_n) + (1 - t_n)x_n - \lambda_n Ax_n - y_n - \lambda_n v, j_q(y_n - u)\rangle \geq 0,$$

which implies that

$$\begin{aligned}\langle Ax_n + v, j_q(y_n - u)\rangle &\leq \frac{1}{\lambda_n}\langle x_n - y_n, j_q(y_n - u)\rangle + \frac{t_n}{\lambda_n}\langle f(x_n) - x_n, j_q(y_n - u)\rangle \\ &\leq \frac{1}{\lambda_n}\|x_n - y_n\|\|y_n - u\|^{q-1} + \frac{t_n}{\lambda_n}\|f(x_n) - x_n\|\|y_n - u\|^{q-1} \\ &\leq \frac{1}{\lambda_n}\big(\|x_n - y_n\| + t_n\big)M_1. \end{aligned} \tag{3.10}$$

Since $\langle Ax_{n_i} - Ax^*, j_q(x_{n_i} - x^*)\rangle \geq \alpha\|Ax_{n_i} - Ax^*\|^q$, and $x_{n_i} \rightharpoonup x^*$, we have $Ax_{n_i} \to Ax^*$ since j_q is weakly sequentially continuous. Then, by (3.6), it follows that $\langle Ax^*+v, j_q(x^*-u)\rangle \leq 0$. Hence, $\langle -Ax^*-v, j_q(x^*-u)\rangle \geq 0$ and consequently, $-Ax^* \in Bx^*$. So we have $x^* \in (A + B)^{-1}0$ and hence $x^* \in Fix(S) \cap (A + B)^{-1}0$.

From (3.9), in particular, replacing n with n_i and z with x^*, we have

$$\|x_{n_i} - x^*\|^q \leq \frac{1}{1-\rho}\langle f(x^*) - x^*, j_q(y_{n_i} - x^*)\rangle. \tag{3.11}$$

Since $y_{n_i} \rightharpoonup x^*$ and j_q is weakly sequentially continuous, we get $x_{n_i} \longrightarrow x^*$. Let $\{s_k\} \subset (0, 1)$ be another sequence such that $s_k \to 0$ as $k \to \infty$. Put $x_k := x_{s_k}$, $y_k := y_{s_k}$, and $\lambda_k := \lambda_{s_k}$. Let $\{x_{k_j}\}$ be a subsequence of $\{x_k\}$ such that $x_{k_j} \rightharpoonup \hat{x}$. In a similar way, we can show that $\hat{x} \in Fix(S) \cap (A + B)^{-1}0$ and also $x^* = \hat{x}$.

Now, returning to (3.9) with $n = n_i$ and taking the limit as $i \longrightarrow \infty$, we have

$$\|x^* - z\|^q \leq \frac{1}{1-\rho}\langle f(z) - z, j_q(x^* - z)\rangle.$$

In particular, x^* solves the variational inequality

$$\langle f(z) - z, j_q(z - x^*)\rangle \leq 0, \forall z \in Fix(S) \cap (A + B)^{-1}0,$$

which is equivalent to the following dual variational inequality (see Lemma 2.6)

$$\langle f(x^*) - x^*, j_q(z - x^*)\rangle \leq 0, \forall z \in Fix(S) \cap (A + B)^{-1}0. \tag{3.12}$$

This shows that the net $\{x_t\}$, as $t \longrightarrow 0$, converges strongly to $x^* \in Fix(S) \cap (A + B)^{-1}0$ which is also a solution of (3.2). This completes the proof. □

3.2 Convergence theorem for explicit iteration scheme

In this section, we establish the strong convergence theorem of an explicit iteration in Banach spaces.

Theorem 3.2 *Let X be a uniformly convex and q-uniformly smooth Banach space which admits a weakly sequentially continuous generalized duality mapping $j_q : X \longrightarrow X^*$. Let $f : X \longrightarrow X$ be a ρ-contraction, $A : X \longrightarrow X$ be an α-isa of order q and $B : X \longrightarrow 2^X$ be an m-accretive operator. Let $J_\lambda^B = (I + \lambda B)^{-1}$ be a resolvent of B for $\lambda > 0$ and $S : X \longrightarrow X$ be a nonexpansive mapping such that $Fix(S) \cap (A + B)^{-1}0 \neq \emptyset$. For given $x_1 \in X$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = (1-\gamma)Sx_n + \gamma J_{\lambda_n}^B(\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n Ax_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \forall n \geq 1, \end{cases} \tag{3.13}$$

where $\gamma \in (0,1)$, $\{\lambda_n\} \subset \left(0, \left(\alpha q/\kappa_q\right)^{1/(q-1)}\right)$, $\{\alpha_n\} \subset (0,1)$, and $\{\beta_n\} \subset (0,1)$ satisfying the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a \leq \beta_n \leq b < 1$;
(C3) $0 < a' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq b' < \left(\frac{\alpha q}{\kappa_q}\right)^{\frac{1}{q-1}}$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then, the sequence $\{x_n\}$ defined by (3.13) converges strongly to a point $x^* \in Fix(S) \cap (A+B)^{-1}0$, where x^* is the unique solution of the variational inequality (3.2).

Proof. First, we show that $\{x_n\}$ is bounded. Set $z_n = J_{\lambda_n}^B(\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n Ax_n)$ for all $n \in \mathbb{N}$. Taking $p \in Fix(S) \cap (A+B)^{-1}0$, we obtain

$$p = Sp = J_{\lambda_n}^B(p - \lambda_n Ap) = J_{\lambda_n}^B\left(\alpha_n p + (1-\alpha_n)\left(p - \frac{\lambda_n}{1-\alpha_n}Ap\right)\right).$$

Since $J_{\lambda_n}^B$ and $I - \frac{\lambda_n}{1-\alpha_n}A$ are nonexpansive, it follows that

$$\begin{aligned}
&\|z_n - p\| \\
= &\left\| J_{\lambda_n}^B\left(\alpha_n f(x_n) + (1-\alpha_n)\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n\right) - J_{\lambda_n}^B\left(\alpha_n p + (1-\alpha_n)\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)p\right)\right\| \\
\leq &\left\|\alpha_n(f(x_n) - p) + (1-\alpha_n)\left[\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)p\right]\right\| \\
\leq &\ \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\left\|\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)p\right\| \\
\leq &\ \alpha_n\rho\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\
= &\ \big(1 - (1-\rho)\alpha_n\big)\|x_n - p\| + \alpha_n\|f(p) - p\|.
\end{aligned} \tag{3.14}$$

Hence, we have

$$\begin{aligned}\|y_n - p\| &= \|(1-\gamma)(Sx_n - p) + \gamma(z_n - p)\| \\ &\le (1-\gamma)\|Sx_n - p\| + \gamma\|z_n - p\| \\ &\le (1-\gamma)\|x_n - p\| + \gamma\big[\big(1-(1-\rho)\alpha_n\big)\|x_n - p\| + \alpha_n\|f(p) - p\|\big] \\ &= \big(1-(1-\rho)\alpha_n\gamma\big)\|x_n - p\| + \alpha_n\gamma\|f(p) - p\|.\end{aligned}$$

Then, it follows that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\beta_n(x_n - p) + (1-\beta_n)(y_n - p)\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\big[\big(1-(1-\rho)\alpha_n\gamma\big)\|x_n - p\| + \alpha_n\gamma\|f(p) - p\|\big] \\ &= \big[1-(1-\rho)(1-\beta_n)\alpha_n\gamma\big]\|x_n - p\| + (1-\beta_n)\alpha_n\gamma\|f(p) - p\| \\ &\le \max\left\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-\rho}\right\}.\end{aligned}$$

By induction, we have

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1-\rho}\right\}, \forall n \ge 1.$$

Hence, $\{x_n\}$ is bounded. So are $\{f(x_n)\}$, $\{Ax_n\}$ and $\{Sx_n\}$.

Next, we show that $\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = 0$. Set $z_n = J^B_{\lambda_n}u_n$, where $u_n = \alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n Ax_n$. We observe that

$$\begin{aligned}&\|z_{n+1} - z_n\| \\ &= \|J^B_{\lambda_{n+1}}u_{n+1} - J^B_{\lambda_n}u_n\| \\ &\le \|J^B_{\lambda_{n+1}}u_{n+1} - J^B_{\lambda_{n+1}}u_n\| + \|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\| \\ &\le \|u_{n+1} - u_n\| + \|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\| \\ &= \|\alpha_n f(x_{n+1}) + (1-\alpha_{n+1})x_{n+1} - \lambda_{n+1}Ax_{n+1} - (\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n Ax_n)\| \\ &\quad + \|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\|\end{aligned}$$

$$\begin{aligned}&= \Bigg\|\alpha_{n+1}(f(x_{n+1}) - f(x_n)) + (\alpha_{n+1} - \alpha_n)(f(x_n) - x_n) \\ &\quad + (1-\alpha_{n+1})\left[\left(I - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\right)x_{n+1} - \left(I - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\right)x_n\right] + (\lambda_n - \lambda_{n+1})Ax_n\Bigg\| \\ &\quad + \|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\| \\ &\le \alpha_{n+1}\|f(x_{n+1}) - f(x_n)\| + |\alpha_{n+1} - \alpha_n|\big(\|f(x_n)\| + \|x_n\|\big) \\ &\quad + (1-\alpha_{n+1})\left\|\left(I - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\right)x_{n+1} - \left(I - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\right)x_n\right\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\ &\quad + \|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\| \\ &\le \big(1-(1-\rho)\alpha_{n+1}\big)\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\big(\|f(x_n)\| + \|x_n\|\big) + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\ &\quad + \|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\|.\end{aligned}$$

By Lemma 2.12, we have

$$\|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\| \le \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|J^B_{\lambda_{n+1}}u_n - u_n\|.$$

Then, it follows that

$$\begin{aligned}\|z_{n+1}-z_n\| \le\ & \big(1-(1-\rho)\alpha_{n+1}\big)\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\big(\|f(x_n)\|+\|x_n\|\big) + |\lambda_{n+1}-\lambda_n|\|Ax_n\| \\ & + \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|J^B_{\lambda_{n+1}}u_n - u_n\|. \end{aligned} \tag{3.15}$$

Since $x_{n+1} = \beta_n x_n + (1-\beta_n)y_n$, where $y_n = (1-\gamma)Sx_n + \gamma z_n$, it follows that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|(1-\gamma)(Sx_{n+1}-Sx_n) + \gamma(z_{n+1}-z_n)\| \\ &\le (1-\gamma)\|Sx_{n+1}-Sx_n\| + \gamma\|z_{n+1}-z_n\| \\ &\le (1-\gamma)\|x_{n+1}-x_n\| + \gamma\|z_{n+1}-z_n\|. \end{aligned} \tag{3.16}$$

Note that, by the boundedness of $\{x_n\}$, we can find a positive constant M_2 such that

$$M_2 = \max\left\{\sup_{n\ge1}(\|f(x_n)\|+\|x_n\|), \sup_{n\ge1}\|Ax_n\|, \sup_{n\ge1}\|J^B_{\lambda_{n+1}}u_n - u_n\|\right\} < \infty.$$

Substituting (3.15) into (3.16), we have

$$\begin{aligned}&\|y_{n+1}-y_n\| \\ \le\ & (1-\gamma)\|x_{n+1}-x_n\| + \gamma\Big[\big(1-(1-\rho)\alpha_{n+1}\big)\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\big(\|f(x_n)\|+\|x_n\|\big) \\ & +|\lambda_{n+1}-\lambda_n|\|Ax_n\| + \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|J^B_{\lambda_{n+1}}u_n - u_n\|\Big] \\ \le\ & \big(1-(1-\rho)\alpha_{n+1}\gamma\big)\|x_{n+1}-x_n\| + \left(|\alpha_{n+1}-\alpha_n| + |\lambda_{n+1}-\lambda_n| + \frac{|\lambda_{n+1}-\lambda_n|}{a'}\right)M_2.\end{aligned}$$

From $(C1)-(C3)$, we have

$$\limsup_{n\longrightarrow\infty}\big(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|\big) \le 0.$$

Hence, from Lemma 2.7, we obtain that

$$\lim_{n\longrightarrow\infty}\|x_n - y_n\| = 0. \tag{3.17}$$

Consequently,

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\| = \lim_{n\longrightarrow\infty}(1-\beta_n)\|x_n-y_n\| = 0. \tag{3.18}$$

Next, we show that $\lim_{n\longrightarrow\infty}\|x_n - Sx_n\| = 0$. By the convexity of $\|\cdot\|^q$ for all $q > 1$ and Lemma 2.2, we have

$$\begin{aligned}
\|u_n - p\|^q &= \left\|(1-\alpha_n)\left[\left(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n\right) - \left(p - \frac{\lambda_n}{1-\alpha_n}Ap\right)\right] + \alpha_n(f(x_n)-p)\right\|^q \\
&\le (1-\alpha_n)\left\|\left(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n\right) - \left(p - \frac{\lambda_n}{1-\alpha_n}Ap\right)\right\|^q + \alpha_n\|f(x_n)-p\|^q \\
&= (1-\alpha_n)\left\|(x_n - p) - \frac{\lambda_n}{1-\alpha_n}(Ax_n - Ap)\right\|^q + \alpha_n\|f(x_n)-p\|^q \\
&\le (1-\alpha_n)\left[\|x_n-p\|^q - \frac{q\lambda_n}{1-\alpha_n}\langle Ax_n - Ap, j_q(x_n-p)\rangle + \frac{\kappa_q\lambda_n^q}{(1-\alpha_n)^q}\|Ax_n - Ap\|^q\right] \\
&\quad +\alpha_n\|f(x_n)-p\|^q \\
&\le (1-\alpha_n)\left[\|x_n-p\|^q - \frac{\alpha q\lambda_n}{1-\alpha_n}\|Ax_n-Ap\|^q + \frac{\kappa_q\lambda_n^q}{(1-\alpha_n)^q}\|Ax_n - Ap\|^q\right] + \alpha_n\|f(x_n)-p\|^q \\
&= (1-\alpha_n)\left[\|x_n-p\|^q - \frac{\lambda_n}{1-\alpha_n}\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q\right] + \alpha_n\|f(x_n)-p\|^q \\
&\le \|x_n-p\|^q - \lambda_n\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q + \alpha_n\|f(x_n)-p\|^q. \qquad (3.19)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\|y_n - p\|^q \\
&= \|(1-\gamma)(Sx_n - p) + \gamma(J_{\lambda_n}^B u_n - p)\|^q \\
&\le (1-\gamma)\|Sx_n - p\|^q + \gamma\|J_{\lambda_n}^B u_n - p\|^q \\
&\le (1-\gamma)\|x_n - p\|^q + \gamma\|u_n - p\|^q \\
&\le (1-\gamma)\|x_n - p\|^q + \gamma\left[\|x_n-p\|^q - \lambda_n\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q + \alpha_n\|f(x_n)-p\|^q\right] \\
&= \|x_n-p\|^q - \lambda_n\gamma\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q + \alpha_n\gamma\|f(x_n)-p\|^q.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|x_{n+1} - p\|^q \\
&\le \beta_n\|x_n - p\|^q + (1-\beta_n)\|y_n - p\|^q \\
&\le \beta_n\|x_n - p\|^q + (1-\beta_n)\left[\|x_n-p\|^q - \lambda_n\gamma\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q + \alpha_n\gamma\|f(x_n)-p\|^q\right] \\
&= \|x_n-p\|^q - \lambda_n(1-\beta_n)\gamma\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q + \alpha_n(1-\beta_n)\gamma\|f(x_n)-p\|^q,
\end{aligned}$$

which implies from $(C2)$, $(C3)$ and Proposition 2.10 that

$$\begin{aligned}
a'(1-b)\gamma\big(\alpha q - \kappa_q(b')^{q-1}\big)\|Ax_n - Ap\|^q &\le \|x_n-p\|^q - \|x_{n+1}-p\|^q + (1-\beta_n)\alpha_n\gamma\|f(x_n)-p\|^q \\
&\le q\|x_n - p\|^{q-1}\big(\|x_n-p\| - \|x_{n+1}-p\|\big) + (1-\beta_n)\alpha_n\gamma\|f(x_n)-p\|^q \\
&\le q\|x_n - p\|^{q-1}\|x_{n+1}-x_n\| + (1-\beta_n)\alpha_n\gamma\|f(x_n)-p\|^q.
\end{aligned}$$

Then, by $(C1)$ and (3.18), we obtain that

$$\lim_{n\longrightarrow\infty} \|Ax_n - Ap\| = 0. \tag{3.20}$$

On the other hand, from Proposition 2.13 and Lemma 2.4, we have

$$\begin{aligned}
&\|z_n - p\|^q \\
&= \|J^B_{\lambda_n}(\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n Ax_n) - J^B_{\lambda_n}(p - \lambda_n Ap)\|^q \\
&\le \langle \alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n Ax_n - (p - \lambda_n Ap), j_q(z_n - p)\rangle \\
&\le \frac{1}{q}\Bigg[\|\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n Ax_n - (p-\lambda_n Ap)\|^q + (q-1)\|z_n - p\|^q \\
&\quad - g(\|\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - z_n\|)\Bigg],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|z_n - p\|^q \\
&\le \|\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n Ax_n - (p-\lambda_n Ap)\|^q - g(\|\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - z_n\|) \\
&\le \alpha_n\|f(x_n) - p\|^q + \|x_n - p\|^q - g(\|\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - z_n\|).
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
\|y_n - p\|^q &\le (1-\gamma)\|Sx_n - p\|^q + \gamma\|z_n - p\|^q \\
&\le (1-\gamma)\|x_n - p\|^q + \gamma\|z_n - p\|^q \\
&\le (1-\gamma)\|x_n - p\|^q \\
&\quad + \gamma\Big[\alpha_n\|f(x_n) - p\|^q + \|x_n - p\|^q - g(\|\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - z_n\|)\Big] \\
&= \|x_n - p\|^q + \alpha_n\gamma\|f(x_n) - p\|^q - \gamma g(\|\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - z_n\|).
\end{aligned} \tag{3.21}$$

Consequently,

$$\begin{aligned}
&\|x_{n+1} - p\|^q \\
&\le \beta_n\|x_n - p\|^q + (1-\beta_n)\|y_n - p\|^q \\
&\le \beta_n\|x_n - p\|^q \\
&\quad + (1-\beta_n)\Big[\|x_n - p\|^q + \alpha_n\gamma\|f(x_n) - p\|^q - \gamma g(\|\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - z_n\|)\Big] \\
&= \|x_n - p\|^q + (1-\beta_n)\alpha_n\gamma\|f(x_n) - p\|^q - (1-\beta_n)\gamma g(\|\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - z_n\|),
\end{aligned}$$

which implies from $(C2)$ that

$$\begin{aligned}
&(1-b)\gamma g(\|\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - z_n\|) \\
&\le \|x_n - p\|^q - \|x_{n+1} - p\|^q + \alpha_n(1-\beta_n)\gamma\|f(x_n) - p\|^q \\
&\le q\|x_n - p\|^{q-1}\big(\|x_n - p\| - \|x_{n+1} - p\|\big) + (1-\beta_n)\alpha_n\gamma\|f(x_n) - p\|^q \\
&\le q\|x_n - p\|^{q-1}\|x_n - x_{n+1}\| + (1-\beta_n)\alpha_n\gamma\|f(x_n) - p\|^q.
\end{aligned}$$

Then, by $(C1)$ and (3.18), we have

$$\lim_{n\longrightarrow\infty} g(\|\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - z_n\|) = 0.$$

Since g is a continuous function, by (3.20), we obtain that

$$\lim_{n\longrightarrow\infty} \|x_n - z_n\| = 0. \tag{3.22}$$

Since $y_n = (1-\gamma)Sx_n + \gamma z_n$, it follows that $(1-\gamma)(x_n - Sx_n) = x_n - y_n + \gamma(z_n - x_n)$. Hence,

$$(1-\gamma)\|x_n - Sx_n\| \leq \|x_n - y_n\| + \gamma\|x_n - z_n\|.$$

From (3.17) and (3.22), we obtain that

$$\lim_{n\longrightarrow\infty} \|x_n - Sx_n\| = 0. \tag{3.23}$$

Next, we show that

$$\limsup_{n\longrightarrow\infty} \langle f(x^*) - x^*, j_q(z_n - x^*)\rangle \leq 0,$$

where x^* is the same as in Theorem 3.1. Since $\{z_n\}$ is bounded and X is reflexive, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\longrightarrow\infty} \langle f(x^*) - x^*, j_q(z_n - x^*)\rangle = \lim_{i\longrightarrow\infty} \langle f(x^*) - x^*, j_q(z_{n_i} - x^*)\rangle$$

and $z_{n_i} \rightharpoonup z \in X$ as $i \longrightarrow \infty$. Also, we have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in X$. From (3.23) and Lemma 2.5, we have $z \in Fix(S)$. Further, by the same agrument as in the proof of Theorem 3.1, we can show that $z \in (A+B)^{-1}0$. Hence, we obtain that $z \in Fix(S) \cap (A+B)^{-1}0$. Since j_q is weakly sequentially continuous, it follows that

$$\begin{aligned}\limsup_{n\longrightarrow\infty} \langle f(x^*) - x^*, j_q(z_n - x^*)\rangle &= \lim_{i\longrightarrow\infty} \langle f(x^*) - x^*, j_q(z_{n_i} - x^*)\rangle \\ &= \langle f(x^*) - x^*, j_q(z - x^*)\rangle \leq 0. \end{aligned} \tag{3.24}$$

Finally, we show that $x_n \longrightarrow x^*$. We see that

$$\begin{aligned}\|z_n - x^*\|^q &= \|J^B_{\lambda_n}(\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n Ax_n) - J^B_{\lambda_n}(x^* - \lambda_n Ax^*)\|^q \\ &\leq \langle \alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n Ax_n - x^* + \lambda_n Ax^*, j_q(z_n - x^*)\rangle \\ &= \left\langle \alpha_n f(x_n) + (1-\alpha_n)\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n - \alpha_n x^* - (1-\alpha_n)\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x^*, j_q(z_n - x^*)\right\rangle \\ &= (1-\alpha_n)\left\langle\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x^*, j_q(z_n - x^*)\right\rangle \\ &\quad + \alpha_n\langle f(x_n) - f(x^*), j_q(z_n - x^*)\rangle + \alpha_n\langle f(x^*) - x^*, j_q(z_n - x^*)\rangle \\ &\leq (1-\alpha_n)\|x_n - x^*\|\|z_n - x^*\|^{q-1} + \alpha_n\rho\|x_n - x^*\|\|z_n - x^*\|^{q-1} + \alpha_n\langle f(x^*) - x^*, j_q(z_n - x^*)\rangle \\ &= (1-\alpha_n(1-\rho))\|x_n - x^*\|\|z_n - x^*\|^{q-1} + \alpha_n\langle f(x^*) - x^*, j_q(z_n - x^*)\rangle \\ &\leq (1-\alpha_n(1-\rho))\left(\frac{1}{q}\|x_n - x^*\|^q + \frac{q-1}{q}\|z_n - x^*\|^q\right) + \alpha_n\langle f(x^*) - x^*, j_q(z_n - x^*)\rangle, \end{aligned}$$

which yields that

$$\|z_n - x^*\|^q \le (1-\alpha_n(1-\rho))\|x_n - x^*\|^q + q\alpha_n\langle f(x^*) - x^*, j_q(z_n - x^*)\rangle. \quad (3.25)$$

It follows, by (3.25), that

$$\begin{aligned}\|y_n - x^*\|^q &\le (1-\gamma)\|Sx_n - x^*\|^q + \gamma\|z_n - x^*\|^q \\ &\le (1-\gamma)\|x_n - x^*\|^q + \gamma\Big((1-\alpha_n(1-\rho))\|x_n - x^*\|^q + q\alpha_n\langle f(x^*) - x^*, j_q(z_n - x^*)\rangle\Big) \\ &= (1-\gamma\alpha_n(1-\rho))\|x_n - x^*\|^q + \gamma q\alpha_n\langle f(x^*) - x^*, j_q(z_n - x^*)\rangle.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\|x_{n+1} - x^*\|^q \\ \le\ & \beta_n\|x_n - x^*\|^q + (1-\beta_n)\|y_n - x^*\|^q \\ \le\ & \beta_n\|x_n - x^*\|^q + (1-\beta_n)\Big((1-\gamma\alpha_n(1-\rho))\|x_n - x^*\|^q + \gamma q\alpha_n\langle f(x^*) - x^*, j_q(z_n - x^*)\rangle\Big) \\ =\ & \Big(1-(1-\beta_n)\gamma\alpha_n(1-\rho)\Big)\|x_n - x^*\|^q + (1-\beta_n)\gamma q\alpha_n\langle f(x^*) - x^*, j_q(z_n - x^*)\rangle.\end{aligned}$$

Set $\gamma_n = (1-\beta_n)\gamma\alpha_n(1-\rho)$ and $\delta_n = \frac{q}{1-\rho}\langle f(x^*) - x^*, j_q(z_n - x^*)\rangle$. From $(C1)$ and (3.24), it is easily seen that $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\longrightarrow\infty}\delta_n \le 0$. Therefore, by Lemma 2.8, we conclude that $x_n \longrightarrow x^*$. This completes the proof. □

We next provide the example and its numerical experiments to support our main theorem.

Example 3.3 *Let $X = \mathbb{R}^3$ and let $x = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$. Define $S : \mathbb{R}^3 \to \mathbb{R}^3$ by $S(x) = \begin{pmatrix} -1 \\ 1-y_1 \\ y_2-1 \end{pmatrix}$. Let $F(x) = \frac{1}{2}\|Cx - d\|^2$ where $C = \begin{pmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -2 & 1 & -2 \end{pmatrix}$ and $d = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$. Let $G(x) = 7y_1 + 3y_3 + 5$. Find $x^* \in \mathbb{R}^3$ such that $x^* \in Fix(S)\bigcap(\nabla F + \partial G)^{-1}(0)$, that is, find $x^* \in Fix(S)$ which is also a minimizer of the following minimization problem:*

$$\min_{x\in\mathbb{R}^3}\frac{1}{2}\|Cx - d\|^2 + 7y_1 + 3y_3 + 5. \quad (3.26)$$

It is known that $\nabla F(x) = C^T(Cx - d)$ and ∇F is $1/K$-isa of order 2, where K is the largest eigenvalue of C^TC (see [11]). Moreover, by [2], ∂G is maximal monotone since G is convex and lower semicontinuous and hence, it is m-accretive. Putting $A = \nabla F$ and $B = \partial G$, by Theorem 3.2, our algorithm becomes

$$\begin{cases} y_n = (1-\gamma)Sx_n + \gamma J_{\lambda_n}^{\partial G}(\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n C^T(Cx_n - d)), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \forall n \ge 1, \end{cases} \quad (3.27)$$

where $\gamma \in (0,1)$, $\{\lambda_n\} \subset \big(0, 2/K\big)$, $\{\alpha_n\} \subset (0,1)$ *and* $\{\beta_n\} \subset (0,1)$ *which satisfy the following conditions:*

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $0 < a \leq \beta_n \leq b < 1$*;*
(C3) $0 < a' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq b' < 2/K$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Choose $\alpha_n = \frac{1}{50n+1}$, $\beta_n = \frac{2n-1}{4n+2}$, $\gamma = 0.5$ *and* $\lambda_n = \lambda \in (0, 0.05) \subset (0, 2/K)$ *for all* $n \in \mathbb{N}$. *Let* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *be defined by* $f(x) = 0.5x$. *So our algorithm* (3.27) *has the following form:*

$$\begin{cases} \begin{pmatrix} z_1^n \\ z_2^n \\ z_3^n \end{pmatrix} = 0.5 \begin{pmatrix} -1 \\ 1 - y_1^n \\ y_2^n - 1 \end{pmatrix} + 0.5 J_\lambda^{\partial G} \Bigg[\left(\frac{1}{50n+1}\right) \begin{pmatrix} 0.5y_1^n \\ 0.5y_2^n \\ 0.5y_3^n \end{pmatrix} + \left(\frac{50n}{50n+1}\right) \begin{pmatrix} y_1^n \\ y_2^n \\ y_3^n \end{pmatrix} \\ \qquad -\lambda \begin{pmatrix} 3 & 2 & -2 \\ 2 & -2 & 1 \\ -1 & 4 & -2 \end{pmatrix} \begin{pmatrix} 3y_1^n + 2y_2^n - y_3^n - 1 \\ 2y_1^n - 2y_2^n + 4y_3^n + 2 \\ -2y_1^n + y_2^n - 2y_3^n \end{pmatrix} \Bigg], \\ \begin{pmatrix} y_1^{n+1} \\ y_2^{n+1} \\ y_3^{n+1} \end{pmatrix} = \left(\frac{2n-1}{4n+2}\right) \begin{pmatrix} y_1^n \\ y_2^n \\ y_3^n \end{pmatrix} + \left(\frac{2n+3}{4n+2}\right) \begin{pmatrix} z_1^n \\ z_2^n \\ z_3^n \end{pmatrix}, \forall n \geq 1, \end{cases} \tag{3.28}$$

where $J_\lambda^{\partial G}(x) = \begin{pmatrix} y_1 - 7\lambda \\ y_2 \\ y_3 - 3\lambda \end{pmatrix}$. *Let* $x_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ *be the initial point. Then, we obtain the following numerical results.*

From Tables 1, 2 and 3, *we see that* $x^* = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ *is an approximation solution of* $Fix(S) \bigcap (\nabla F + \partial G)^{-1}(0)$.

From Fig. 1, *we observe that the convergence rate of the algorithm depends significantly on the step size* λ_n. *In fact, from the view of our numerical experience, when* λ_n *is taken close to zero, we have small reduction in the number of iterations.*

Table 1 Numerical results of Example 3.3 for iteration process (3.28) with $\lambda = 0.01$

n	$x_n = (y_1^n, y_2^n, y_3^n)^T$	$\|x_{n+1} - x_n\|$
1	$(1.000000000, 0.000000000, 2.00000000)^T$	1.830712169
10	$(-0.895390857, 1.51603082, 0.27313376)^T$	0.108303666
50	$(-0.990134038, 1.866451538, 0.875031661)^T$	0.00017108
100	$(-0.990401512, 1.867307766, 0.876376227)^T$	$5.97E-06$
150	$(-0.990443477, 1.867427173, 0.876522724)^T$	$2.48E-06$
200	$(-0.990463745, 1.867484472, 0.87659212)^T$	$1.36E-06$
250	$(-0.990475702, 1.867518171, 0.876632687)^T$	$8.54E-07$
300	$(-0.990483593, 1.867540367, 0.876659307)^T$	$5.87E-07$

Table 2 Numerical results of Example 3.3 for iteration process (3.28) with $\lambda = 0.001$

n	$x_n = (y_1^n, y_2^n, y_3^n)^T$	$\|x_{n+1} - x_n\|$
1	$(1.000000000, 0.000000000, 2.00000000)^T$	1.830712169
10	$(-0.928728283, 1.640399583, 0.249059878)^T$	0.133590425
50	$(-0.999664389, 1.987139714, 0.986861236)^T$	$5.57E-05$
100	$(-0.999787071, 1.98753753, 0.987530573)^T$	$5.89E-06$
150	$(-0.999823615, 1.987648759, 0.98768394)^T$	$2.49E-06$
200	$(-0.999841469, 1.987702684, 0.987757125)^T$	$1.36E-06$
250	$(-0.999852055, 1.987734531, 0.987800007)^T$	$8.61E-07$
300	$(-0.999859061, 1.987755558, 0.987828184)^T$	$5.92E-07$

4 Convergence theorem for a family of nonexpansive mappings

In this section, we provide some applications to a countable family of nonexpansive mappings.

Definition 4.1 Let C be a nonempty subset of a real Banach space X. Let $\{S_n\}_{n=1}^{\infty} : C \longrightarrow C$ be a sequence of mappings with $\bigcap_{n=1}^{\infty} Fix(S_n) \neq \emptyset$. Suppose that for any bounded subset B of C. We say that

(i) $\{S_n\}_{n=1}^{\infty}$ satisfies the *AKTT-condition* (see [5]), if

$$\sum_{n=1}^{\infty} \sup_{x \in B} \|S_{n+1}x - S_n x\| < \infty; \tag{4.1}$$

(ii) $\{S_n\}_{n=1}^{\infty}$ satisfies the *PU-condition* (see [31]), if there exists a continuous and increasing function $h_B : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and for all $k, l \in \mathbb{N}$ such that

$$h_B(0) = 0 \text{and} \lim_{k,l \longrightarrow \infty} \sup_{x \in B} h_B(\|S_k x - S_l x\|) = 0. \tag{4.2}$$

Table 3 Numerical results of Example 3.3 for iteration process (3.28) with $\lambda = 0.0001$

n	$x_n = (y_1^n, y_2^n, y_3^n)^T$	$\|x_{n+1} - x_n\|$
1	$(1.000000000, 0.000000000, 2.00000000)^T$	1.830712169
10	$(-0.931946516, 1.652071984, 0.24421647)^T$	0.136327693
50	$(-0.999779317, 1.998110712, 0.997754605)^T$	$4.43E-05$
100	$(-0.999894464, 1.998480861, 0.998363889)^T$	$5.87E-06$
150	$(-0.999930274, 1.998590869, 0.998517819)^T$	$2.48E-06$
200	$(-0.999947795, 1.998644262, 0.99859132)^T$	$1.36E-06$
250	$(-0.99995819, 1.998675811, 0.998634398)^T$	$8.60E-07$
300	$(-0.999965071, 1.998696646, 0.998662709)^T$	$5.92E-07$

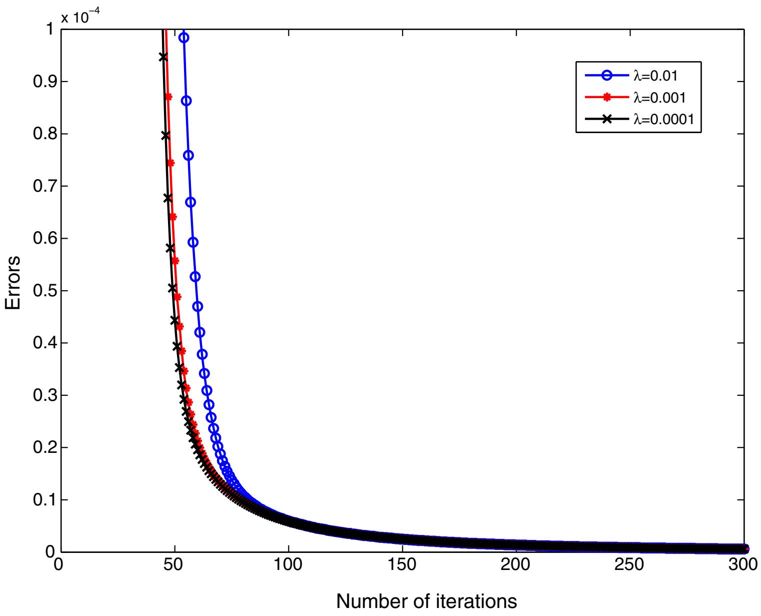

Fig. 1 The convergence behavior of the iteration process with different λ

Remark 4.2 If $\{S_n\}_{n=1}^{\infty}$ satisfies the $AKTT$-condition, then $\{S_n\}_{n=1}^{\infty}$ satisfies the PU-condition (see [31], Remark 3.2).

Lemma 4.3 ([31]) *Let $\{S_n\}_{n=1}^{\infty} : C \longrightarrow C$ be a sequence of mappings. Suppose that for any bounded subset B of C, there exists a continuous and increasing function $h_B : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $h_B(0) = 0$ satisfying (4.2). If the mapping $S : C \longrightarrow C$ be defined by $Sx = \lim_{n\longrightarrow\infty} S_n x$ for all $x \in C$. Then,*

$$\lim_{n\longrightarrow\infty} \sup_{x\in B}\{h_B(\|Sx - S_n x\|)\} = 0.$$

Theorem 4.4 *Let X be a uniformly convex and q-uniformly smooth Banach space which admits a weakly sequentially continuous generalized duality mapping $j_q : X \longrightarrow X^*$. Let $f : X \longrightarrow X$ be a ρ-contraction, $A : X \longrightarrow X$ be an α-isa of order q and $B : X \longrightarrow 2^X$ be an m-accretive operator. Let $J_\lambda^B = (I + \lambda B)^{-1}$ be a resolvent of B for $\lambda > 0$ and let $\{S_n\}_{n=1}^{\infty} : X \longrightarrow X$ be a family of nonexpansive mappings such that $\Omega := \bigcap_{n=1}^{\infty} Fix(S_n) \cap (A + B)^{-1}0 \neq \emptyset$. For an initial guess $x_1 \in X$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = (1-\gamma)S_n x_n + \gamma J_{\lambda_n}^B(\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \forall n \geq 1, \end{cases} \tag{4.3}$$

where $\gamma \in (0, 1)$, $\{\lambda_n\} \subset \left(0, \left(\alpha q/\kappa_q\right)^{1/(q-1)}\right)$, $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ which satisfy the conditions $(C1)$-$(C3)$. Suppose, in addition, that $\{S_n\}_{n=1}^{\infty}$ satisfies the PU-condition and $S : X \longrightarrow X$ be a mapping defined by $Sx = \lim_{n\longrightarrow\infty} S_n x$

for all $x \in X$ such that $Fix(S) = \bigcap_{n=1}^{\infty} F(S_n)$. Then, the sequence $\{x_n\}$ defined by (4.3) converges strongly to a point $x^* \in \Omega$, where x^* is the unique solution of the variational inequality (3.2).

Proof By using the same arguments and techniques as those of Theorem 3.2, we know that

$$\lim_{n \longrightarrow \infty} \|x_n - S_n x_n\| = 0.$$

Now, it suffices to show that $\lim_{n \longrightarrow \infty} \|x_n - Sx_n\| = 0$. We see that

$$\|x_n - Sx_n\| \leq \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\|. \tag{4.4}$$

Since $\{S_n\}_{n=1}^{\infty}$ satisfies the PU-condition, by Lemma 4.3, it follows that

$$\lim_{n \longrightarrow \infty} h_B(\|S_n x_n - Sx_n\|) = 0,$$

which implies by the property of h_B that $\lim_{n \longrightarrow \infty} \|S_n x_n - Sx_n\| = 0$. Then, from (4.4), we get that

$$\lim_{n \longrightarrow \infty} \|x_n - Sx_n\| = 0.$$

This completes the proof. □

5 Convergence theorem for a nonexpansive semigroup

Definition 5.1 Let C be a nonempty, closed, and convex subset of a real Banach space X. A one-parameter family $\mathcal{S} = \{S(t) : t \geq 0\} : C \longrightarrow C$ is said to be a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $S(0)x = x$for all$x \in C$;
(ii) $S(s+t)x = S(s)S(t)x$ for all $x \in C$ and $s, t \geq 0$;
(iii) for each $x \in C$ the mapping $t \mapsto S(t)x$ is continuous;
(iv) $\|S(t)x - S(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t > 0$.

Remark 5.2 We denote by $Fix(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$,that is $Fix(\mathcal{S}) := \bigcap_{t>0} Fix(S(t)) = \{x \in C : x = S(t)x\}$.

Definition 5.3 ([3, 4, 8]) Let C be a nonempty, closed, and convex subset of a real Banach space X, $\mathcal{S} = \{S(t) : t > 0\}$ be a continuous operator semigroup on C. Then, $\mathcal{S}$ is said to be *uniformly asymptotically regular* (in short, u.a.r.) on C if for all $h \geq 0$ and any bounded subset B of C such that

$$\lim_{t \longrightarrow \infty} \sup_{x \in B} \|S(h)S(t)x - S(t)x\| = 0.$$

The nonexpansive semigroup $\{\sigma_t : t > 0\}$ defined by the following lemma is an example of u.a.r. operator semigroup.

Lemma 5.4 ([33]) Let C be a nonempty, closed, and convex subset of a uniformly convex Banach space X and let B be a bounded, closed, and convex subset of C.

If we denote $\mathcal{S} = \{S(t) : t > 0\}$ is a nonexpansive semigroup on C such that $Fix(\mathcal{S}) = \bigcap_{t>0} Fix(S(t)) \neq \emptyset$. For all $h > 0$, the set $\sigma_t(x) = \frac{1}{t}\int_0^t S(s)xds$,then

$$\lim_{t\longrightarrow\infty} \sup_{x\in B} \|\sigma_t(x) - S(h)\sigma_t(x)\| = 0.$$

Example 5.5 *The set* $\{\sigma_t : t > 0\}$ *defined by Lemma 5.4 is a u.a.r. nonexpansive semigroup. In fact, it is obvious that* $\{\sigma_t : t > 0\}$ *is a nonexpansive semigroup. For each* $h > 0$*, we have*

$$\begin{aligned}\|\sigma_t(x) - \sigma_h\sigma_t(x)\| &= \left\|\sigma_t(x) - \tfrac{1}{h}\int_0^h S(s)\sigma_t(x)ds\right\| \\ &= \left\|\tfrac{1}{h}\int_0^h (\sigma_t(x) - S(s)\sigma_t(x))ds\right\| \\ &\leq \tfrac{1}{h}\int_0^h \|\sigma_t(x) - S(s)\sigma_t(x)\|ds.\end{aligned}$$

It follows from Lemma 5.4 that

$$\lim_{t\longrightarrow\infty} \sup_{x\in B} \|\sigma_t(x) - S(s)\sigma_t(x)\|ds = 0.$$

Theorem 5.6 *Let* X *be a uniformly convex and* q*-uniformly smooth Banach space which admits a weakly sequentially continuous generalized duality mapping* $j_q : X \longrightarrow X^*$*. Let* $f : X \longrightarrow X$ *be a* ρ*-contraction,* $A : X \longrightarrow X$ *be an* α*-isa of order* q *and* $B : X \longrightarrow 2^X$ *be an m-accretive operator. Let* $J_\lambda^B = (I + \lambda B)^{-1}$ *be a resolvent of* B *for* $\lambda > 0$ *and let* $\mathcal{S} = \{S(t) : t > 0\}$ *be a u.a.r nonexpansive semigroup such that* $\Omega := \bigcap_{t>0} Fix\big(S(t)\big) \cap (A+B)^{-1}0 \neq \emptyset$*. For an initial guess* $x_1 \in X$*, define the sequence* $\{x_n\}$ *by*

$$\begin{cases} y_n = (1-\gamma)S(t_n)x_n + \gamma J_{\lambda_n}^B(\alpha_n f(x_n) + (1-\alpha_n)x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \forall n \geq 1, \end{cases} \tag{5.1}$$

where $\gamma \in (0, 1)$*,* $\{\lambda_n\} \subset \left(0, \left(\alpha q/\kappa_q\right)^{1/(q-1)}\right)$*,* $\{\alpha_n\} \subset (0, 1)$*, and* $\{\beta_n\} \subset (0, 1)$ *which satisfy the conditions* $(C1)$*-*$(C3)$ *and* $\{t_n\}$ *is a positive real divergent sequence. Then, the sequence* $\{x_n\}$ *defined by* (5.1) *converges strongly as* $n \longrightarrow \infty$ *to a point* $x^* \in \Omega$*, where* x^* *is the unique solution of the variational inequality* (refeq:3.2).

Proof By using the same arguments and techniques as those of Theorem 3.2, we know that $\lim_{n\longrightarrow\infty} \|x_n - S(t_n)x_n\| = 0$. Now, we only show that $\lim_{n\longrightarrow\infty} \|x_n - S(h)x_n\| = 0$ for all $h \geq 0$. Then, we have

$$\begin{aligned}\|x_n - S(h)x_n\| &\leq \|x_n - S(t_n)x_n\| + \|S(t_n)x_n - S(h)S(t_n)x_n\| + \|S(h)S(t_n)x_n - S(h)x_n\| \\ &\leq 2\|x_n - S(t_n)x_n\| + \sup_{x\in x_{\{n\}}} \|S(t_n)x - S(h)S(t_n)x\|. \end{aligned} \tag{5.2}$$

Since $\{S(t) : t \geq 0\}$ is a u.a.r. nonexpansive semigroup and $t_n \longrightarrow \infty$ then for all $h \geq 0$ and for any bounded subset C of X containing $\{x_n\}$, we have

$$\lim_{n\longrightarrow\infty} \|S(t_n)x_n - S(h)S(t_n)x_n\| \leq \lim_{n\longrightarrow\infty} \sup_{x\in C} \|S(t_n)x - S(h)S(t_n)x\| = 0. \tag{5.3}$$

Then, from (5.2) and (5.3), we get that

$$\lim_{n\longrightarrow\infty} \|x_n - S(h)x_n\| = 0,$$

for all $h \geq 0$. This completes the proof. □

6 Some applications

In this section, we will utilize Theorems 3.1 and 3.2 to study some convergence theorem in L_p and l_p spaces with $1 < p < \infty$. It well known that spaces of Hilbert H, L_p, and l_p with $1 < p < \infty$ are q-uniformly smooth, i.e.,

$$H, L_p, \text{ and } l_p \text{ are } \begin{cases} 2\text{–uniformly smooth, if } 2 \leq p < \infty, \\ p\text{–uniformly smooth, if } 1 < p \leq 2. \end{cases}$$

Furthermore, the following facts are well known (see [6, 39]).

(1) For $2 \leq p < \infty$, the spaces of L_p and l_p are 2-uniformly smooth with $\kappa_q = \kappa_2 = p - 1$.

(2) For $1 < p \leq 2$, the spaces of L_p and l_p are p-uniformly smooth with $\kappa_q = \kappa_p = (1 + t_p^{p-1})(1 + t_p)^{1-p}$, where t_p is the unique solution of the equation

$$(p - 2)t^{p-1} + (p - 1)t^{p-2} - 1 = 0, 0 < t < 1.$$

(3) Every Hilbert space is 2-uniformly smooth with $\kappa_q = \kappa_2 = 1$.

(4) For $1 < p < \infty$, the spaces of L_p and l_p are q-uniformly smooth and uniformly convex.

(5) For $1 < p < \infty$, the space of l_p has weakly sequentially continuous generalized duality mappings, but L_p space ($1 < p < \infty, p \neq 2$) does not have weakly sequentially continuous generalized duality mappings.

Remark 6.1 Theorems 3.1 and 3.2 hold for l_p space with $1 < p < \infty$ and also hold for L_p space with $1 < p < \infty, p \neq 2$ if L_p has a weakly sequentially continuous generalized duality mapping.

Acknowledgments The authors would like to thank editor and reviewers for valuable comments. P. Sunthrayuth was supported by RMUTT research foundation scholarship of the Rajamangala University of Technology Thanyaburi (Grant No. NRF04066005). P. Cholamjiak was supported by the Thailand Research Fund and the Commission on Higher Education under Grant MRG5980248.

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RACSAM
DOI 10.1007/s13398-017-0465-9

ORIGINAL PAPER

Iterative methods with perturbations for the sum of two accretive operators in q-uniformly smooth Banach spaces

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Received: 6 February 2017 / Accepted: 30 October 2017

Abstract In this work, we introduce implicit and explicit iteration processes with perturbations for solving the fixed point problem of nonexpansive mappings and the quasi-variational inclusion problem. We then prove its strong convergence under some suitable conditions. In the last section of the paper, some applications are given also. The results obtained in this paper extend and improve some known others presented in the literature.

Keywords Variational inclusion · Banach space · Strong convergence · Iterative method · m-Accretive operator

Mathematics Subject Classification 47H09 · 47H10 · 47H17 · 47J25 · 49J40

1 Introduction

Let C be a nonempty, closed and convex of a real Banach space X. Let $S : C \longrightarrow C$ be a mapping. We use $F(S)$ to denote the set of all fixed points of S, i.e., $F(S) = \{x \in C : x = Sx\}$. Recall that a mapping $S : C \longrightarrow C$ is said to be L-Lipschitzian if there exists $L > 0$ such

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Published online: 04 November 2017

that

$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

A mapping $S : C \longrightarrow C$ is said to be nonexpansive, if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A popular way to solve the fixed point problem for nonexpansive mappings is to employ iterative methods which now have received vast investigations. This is because of its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery, and signal processing.

Let C be a nonempty closed convex subset of a real Banach space X. Let $A : C \to X$ be a single-valued nonlinear mapping and let $B : X \to 2^X$ be a multi-valued mapping. The so called quasi-variational inclusion problem is to find a point $x \in X$ such that

$$0 \in (A + B)x. \tag{1.1}$$

We denote the solution set of (1.1) by $(A+B)^{-1}0$. A number of problems arising in structural analysis, mechanics, and economics can be studied in the framework of this kind of variational inclusions; see, for instance [1–3]. The problem (1.1) includes many optimization problems as special cases.

Takahashi et al. [4] proved the following theorem for maximal monotone operators with nonlinear operator in Hilbert spaces:

Theorem T Let C be a closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping of C into H and let B be a maximal monotone operator on H such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$ and let S be a nonexpansive mapping of C into itself such that $F(S) \cap (A+B)^{-1}0 \neq \emptyset$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) S(\alpha_n x + (1 - \alpha_n) J_{\lambda_n}(x_n - \lambda_n A x_n)), \quad \forall n \geq 1, \tag{1.2}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

(i) $0 < a \leq \lambda_n \leq b < 2\alpha$;
(ii) $0 < c \leq \beta_n \leq d < 1$;
(iii) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$;
(iv) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to a point of $F(S) \cap (A + B)^{-1}0$.

Manaka–Takahashi [5] introduced the following iteration process in Hilbert spaces $H : x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S J_{\lambda_n}(x_n - \lambda_n A x_n), \ \ \forall n \geq 1, \tag{1.3}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\}$ is a positive sequence, S is a nonexpansive mapping on C, $A : C \to H$ is an inverse-strongly monotone mapping, $B : D(B) \subset C \to 2^H$ is a maximal monotone operator, and S is a nonexpansive mapping on C. They showed that the sequence $\{x_n\}$ generated by (1.3) converges weakly to a point in $F(S) \cap (A + B)^{-1}0$ under some mild conditions.

Recently, Lopez et al. [6] considered the following iteration process in the framework of Banach spaces: $u, x_1 \in X$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J_{\lambda_n}(x_n - \lambda_n(A x_n + a_n)) + b_n), \quad \forall n \geq 1, \tag{1.4}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in X. They proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a solution of $(A+B)^{-1}0$.

We note that, in applications, there are perturbations always occurring in the iterative processes because the manipulations are inaccurate. It is no doubt that researching the convergent problems of iterative methods with perturbation members is a significant job. This leads us, in this paper, to introduce implicit and explicit iterative schemes with perturbations for solving the fixed point problem for nonexpansive mappings and the quasi-variational inclusion problem. We then prove its strong convergence under some suitable conditions. Finally, we provide some applications to the main result. The obtained results improve and extend some known results appeared in the literature.

2 Preliminaries

In this section, we collect some definitions and lemmas which will be used in the sequel. In what follows, we shall use the following notations: $x_n \to x$ mean that $\{x_n\}$ converges strongly to x; $x_n \rightharpoonup x$ mean that $\{x_n\}$ converges weakly to x.

A Banach space X is said to be strictly convex, if whenever x and y are not collinear, then: $\|x+y\| < \|x\| + \|y\|$. Let $S(X) = \{x \in X : \|x\| = 1\}$ denote the unit sphere of X. The *modulus of convexity* of X is the function $\delta : (0, 2] \longrightarrow [0, 1]$ defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S(X), \|x-y\| \geq \epsilon\right\}.$$

A Banach space X is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

The *modulus of smoothness* of X is the function $\rho : \mathbb{R}^+ := [0, \infty) \longrightarrow \mathbb{R}^+$ defined by

$$\rho(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : x, y \in S(X)\right\}.$$

A Banach space X is said to be *uniformly smooth* if $\frac{\rho(t)}{t} \longrightarrow 0$ as $t \longrightarrow 0$. Suppose that $q > 1$, a Banach space X is said to be *q-uniformly smooth* if there exists a fixed constant $c > 0$ such that $\rho(t) \leq ct^q$ for all $t > 0$. If X is q-uniformly smooth, then $q \leq 2$ and X is uniformly smooth.

Let X^* be a dual space of a Banach space X. Let $q > 1$ be a real number. The *generalized duality mapping* $J_q : X \longrightarrow 2^{X^*}$ is defined by

$$J_q(x) = \{j_q(x) \in X^* : \langle x, j_q(x)\rangle = \|x\|^q, \|j_q(x)\| = \|x\|^{q-1}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. In particular, $J_q = J_2$ is called the *normalized duality mapping* and $J_q(x) = \|x\|^{q-2} J_2(x)$ for $x \neq 0$. If X is a real Hilbert space, then $J_q = I$, where I is the identity mapping. It is well known that if X is smooth, then J_q is single-valued, which is denoted by j_q. The generalized duality mapping j_q is said to be *weakly sequentially continuous generalized duality mapping* if for each $\{x_n\}$ in X with $x_n \rightharpoonup x$, we have $j_q(x_n) \rightharpoonup^* j_q(x)$.

The following facts are well known (see [7,8]):

(1) Each uniformly convex Banach space (uniformly smooth Banach space) is reflexive and strictly convex.
(2) If a Banach space X admits a weakly sequentially continuous generalized duality mapping, then X satisfies Opials condition, and X is smooth smooth.

(3) All Hilbert spaces, L_p (or l_p) spaces and the Sobolev spaces W_m^p with $p \geq 2$ are 2-uniformly smooth, while L_p (or l_p) spaces and the Sobolev spaces W_m^p with $1 < p \leq 2$ are p-uniformly smooth.
(4) Typical examples of both uniformly convex and uniformly smooth Banach spaces are L_p, where $p > 1$. More precisely, L_p is $\min\{p, 2\}$ -uniformly smooth for each $p > 1$.

Let $A : X \longrightarrow 2^X$ be a set-valued mapping. We denote the domain and range of an operator $A : X \longrightarrow 2^X$ by $D(A) = \{x \in X : Ax \neq \emptyset\}$ and $R(A) = \bigcup\{Az : z \in D(A)\}$, respectively. Let $q > 1$. A set-valued mapping $A : D(A) \subset X \longrightarrow 2^X$ is said to be *accretive* of order q if for each $x, y \in D(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq 0, \quad u \in Ax \text{ and } v \in Ay.$$

An accretive operator A is said to be *m-accretive* if $R(I + \lambda A) = X$ for all $\lambda > 0$. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone (see [8]).

Let A be an m-accretive operator on X, we use $A^{-1}0$ to denote the set of all zeros of A, i.e., $A^{-1}0 = \{x \in D(A) : 0 \in Ax\}$. For an accretive operator A, we can define a single valued operator $J_\lambda^A : R(I + \lambda A) \longrightarrow D(A)$ by $J_\lambda^A = (I + \lambda A)^{-1}$ for each $\lambda > 0$, which is called the resolvent of A for λ. It is well known that J_λ^A is a nonexpansive mapping with $F(J_\lambda^A) = A^{-1}0$.

Let $\alpha > 0$ and $q > 1$. A mapping $A : C \longrightarrow X$ is said to be *α-inverse strongly accretive* (α-isa) of order q if for each $x, y \in X$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Ax - Ay, j_q(x - y)\rangle \geq \alpha\|Ax - Ay\|^q.$$

It is obvious that A is also $1/\alpha$-Lipschitz continuous. If $X := H$ is a real Hilbert space, then $A : C \longrightarrow H$ is called α-inverse strongly monotone (α-ism).

Lemma 2.1 [6] *Let C be a subset of a real q-uniformly smooth Banach space X and $A : C \longrightarrow X$ be an α-isa of order q. Then the following inequality holds:*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^q \leq \|x - y\|^q - \lambda(\alpha q - \kappa_q \lambda^{q-1})\|Ax - Ay\|^q.$$

for all $x, y \in X$. In particular, if $0 < \lambda \leq \left(\frac{\alpha a}{\kappa_q}\right)^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Using the concept of sub-differentials, we have the following inequality:

Lemma 2.2 [9] *Let $q > 1$ and X be a real normed space with the generalized duality mapping J_q. Then, for any $x, y \in X$, we have*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y)\rangle, \tag{2.1}$$

where $j_q(x + y) \in J_q(x + y)$.

Lemma 2.3 [10] *Let $1 < q \leq 2$ and X be a Banach space. Then the following are equivalent.*

(i) *X is q-uniformly smooth.*
(ii) *There is a constant $\kappa_q > 0$ which is called the q*-uniform smoothness coefficient of X *such that for all $x, y \in X$*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + \kappa_q\|y\|^q.$$

In particular, if X is a real 2-uniformly smooth Banach space, then there exists a constant $K > 0$ such that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + 2\|Ky\|^2.$$

Lemma 2.4 [10] *Let $p > 1$ and $r > 0$ be two fixed real numbers and X be a Banach space. Then the following are equivalent.*

(i) *X is uniformly convex.*
(ii) *There is a strictly increasing, continuous and convex function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $g(0) = 0$ and*

$$g(\|x-y\|) \leq \|x\|^p - p\langle x, j_p(y)\rangle + (p-1)\|y\|^p, \quad \forall x, y \in B_r.$$

Lemma 2.5 [11] *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X and $S : C \longrightarrow C$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero, i.e., $x_n \rightharpoonup x$ and $x_n - Sx_n \longrightarrow 0$ implies $x = Sx$.*

Following the proof line as in Lemma 2.7 of [12], we obtain the following result.

Lemma 2.6 *Let C be a nonempty, closed and convex subset of a real smooth Banach space X and let $j_q : X \longrightarrow X^*$ be a generalized duality mapping. Assume that the mapping $F : C \longrightarrow X$ is accretive and weakly continuous along segments, that is, $F(x+ty) \rightharpoonup F(x)$ as $t \longrightarrow 0$. Then the variational inequality*

$$x^* \in C, \quad \langle Fx^*, j_q(x-x^*)\rangle \geq 0, \quad x \in C$$

is equivalent to the dual variational inequality

$$x^* \in C, \quad \langle Fx, j_q(x-x^*)\rangle \geq 0, \quad x \in C.$$

Proposition 2.7 [13] *Let $q > 1$. Then the following inequality holds:*

$$a^q - b^q \leq qa^{q-1}(a-b),$$

for arbitrary positive real numbers a, b.

Lemma 2.8 [14] *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\longrightarrow\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\longrightarrow\infty} \|l_n - x_n\| = 0$.*

Lemma 2.9 [15] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\longrightarrow\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\delta_n| < \infty$.

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.10 (The Resolvent Identity [16]) *Let X be a real Banach space. Let A be an m-accretive operator. For $\lambda, \mu > 0$ and $x \in X$, then*

$$J_\lambda^A x = J_\mu^A\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda^A x\right),$$

where $J_\lambda^A = (I + \lambda A)^{-1}$ and $J_\mu^A = (I + \mu A)^{-1}$.

From the Resolvent Identity, we also have the following result.

Lemma 2.11 *For each $r, s > 0$ then*

$$\|J_r^A x - J_s^A x\| \le \left|1 - \frac{s}{r}\right| \|J_r^A x - x\| \text{ for all } x \in X.$$

Proposition 2.12 *Let X be a real q-uniformly smooth Banach space. Let A be an m-accretive operator on X and let J_λ^A be the resolvent operator associated with A and λ. Then J_λ^A is firmly nonexpansive,* i.e.,

$$\|J_\lambda^A x - J_\lambda^A y\|^q \le \langle x - y, j_q(J_\lambda^A x - J_\lambda^A y)\rangle, \quad \forall x, y \in X.$$

Proof For each $x, y \in X$ and $\lambda > 0$, we set $u = J_\lambda^A x$ and $v = J_\lambda^A y$. By definition of the accretive operator, we have $x - u \in \lambda A u$ and $y - v \in \lambda A v$. Since A is m-accretive, we also have

$$\begin{aligned} 0 &\le \langle x - u - (y - v), j_q(u - v)\rangle \\ &= \langle x - y, j_q(u - v)\rangle - \langle u - v, j_q(u - v)\rangle \\ &= \langle x - y, j_q(u - v)\rangle - \|u - v\|^q, \end{aligned}$$

which implies that

$$\|u - v\|^q \le \langle x - y, j_q(u - v)\rangle,$$

i.e.,

$$\|J_\lambda^A x - J_\lambda^A y\|^q \le \langle x - y, j_q(J_\lambda^A x - J_\lambda^A y)\rangle, \quad \forall x, y \in X.$$

This completes the proof. □

3 Main results

In this section, we prove a strong convergence theorem which is generated by an implicit iteration process.

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real uniformly convex and q-uniformly smooth Banach space X which admits a weakly sequentially continuous generalized duality mapping j_q. Let $A : C \longrightarrow X$ be an α-isa of order q and let $B : D(B) \longrightarrow 2^X$ be an m-accretive operator such that $D(B) \subset C$. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $\Omega := F(S) \cap (A + B)^{-1}0 \neq \emptyset$. Let λ be a real positive constant such that $0 < \lambda < \left(\frac{\alpha q}{\kappa_q}\right)^{\frac{1}{q-1}}$ and let $\{u_t\} \subset X$ be a perturbation with $\lim_{t \longrightarrow 0^+} u_t = u' \in X$. For each $0 < t < 1 - \lambda\left(\frac{\kappa_q}{\alpha q}\right)^{\frac{1}{q-1}}$, let $\{x_t\}$ be a net defined by*

$$x_t = S J_\lambda^B (t u_t + (1 - t) x_t - \lambda A x_t), \tag{3.1}$$

where $J_\lambda^B = (I + \lambda B)^{-1}$. Then the net $\{x_t\}$ converges strongly as $t \longrightarrow 0^+$ to a point $x^* \in \Omega$, which solves uniquely the following variational inequality:

$$\langle u' - x^*, j_q(z - x^*)\rangle \le 0, \quad \forall z \in \Omega. \tag{3.2}$$

Proof We first show that the net $\{x_t\}$ is well defined. For each $t \in (0, 1 - \lambda\left(\frac{\kappa_q}{\alpha q}\right)^{\frac{1}{q-1}})$, we define a mapping $S_t : C \longrightarrow C$ by

$$S_t x := S J_\lambda^B (t u_t + (1 - t) x - \lambda A x), \quad \forall x \in C.$$

Since S, J_λ^B and $I - \frac{\lambda}{1-t}A$ (see Lemma 2.1) are nonexpansive. For each $x, y \in C$, we have

$$\begin{aligned}\|S_t x - S_t y\| &= \|SJ_\lambda^B(tu_t + (1-t)x - \lambda Ax) - SJ_\lambda^B(tu_t + (1-t)y - \lambda Ay)\| \\ &\le \|(tu_t + (1-t)x - \lambda Ax) - (tu_t + (1-t)y - \lambda Ay)\| \\ &= (1-t)\left\|\left(I - \frac{\lambda}{1-t}A\right)x - \left(I - \frac{\lambda}{1-t}A\right)y\right\| \\ &\le (1-t)\|x-y\|,\end{aligned}$$

which implies that S_t is a contraction. Hence, S_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point Eq. (3.1). Therefore, $\{x_t\}$ is well defined.

Take any $p \in \Omega$. It is observed that

$$\begin{aligned}p &= Sp = SJ_\lambda^B(p - \lambda Ap) \\ &= SJ_\lambda^B\left(tp + (1-t)\left(p - \frac{\lambda}{1-t}Ap\right)\right), \quad \forall t \in \left(0, 1 - \lambda\left(\frac{\kappa_q}{\alpha q}\right)^{\frac{1}{q-1}}\right).\end{aligned}$$

Set $x_t = Sy_t$, where $y_t = J_\lambda^B(tu_t + (1-t)x_t - \lambda Ax_t)$. Since S, J_λ^B and $I - \frac{\lambda}{1-t}A$ (see Lemma 2.1) are nonexpansive, we have

$$\begin{aligned}\|y_t - p\| &= \left\|J_\lambda^B\left(tu_t + (1-t)\left(I - \frac{\lambda}{1-t}A\right)x_t\right) - J_\lambda^B\left(tp + (1-t)\left(I - \frac{\lambda}{1-t}A\right)p\right)\right\| \\ &\le \left\|t(u_t - p) + (1-t)\left[\left(I - \frac{\lambda}{1-t}A\right)x_t - \left(I - \frac{\lambda}{1-t}A\right)p\right]\right\| \\ &\le t\|u_t - p\| + (1-t)\left\|\left(I - \frac{\lambda}{1-t}A\right)x_t - \left(I - \frac{\lambda}{1-t}A\right)p\right\| \\ &\le t\|u_t - p\| + (1-t)\|x_t - p\|. \qquad (3.3)\end{aligned}$$

It follows that

$$\begin{aligned}\|x_t - p\| &= \|Sy_t - Sp\| \\ &\le \|y_t - p\| \\ &\le t\|u_t - p\| + (1-t)\|x_t - p\|,\end{aligned}$$

which implies that

$$\|x_t - p\| \le \|u_t - p\|.$$

Since $\lim_{t \longrightarrow 0^+} u_t = u'$, then there exists a constant $K_1 > 0$ such that $K_1 = \sup_{t>0}\{\|u_t\|\}$. Hence, $\{x_t\}$ is bounded, so are $\{y_t\}$, $\{Sx_t\}$ and $\{Ax_t\}$.

Next, we show that $\lim_{t\longrightarrow 0^+}\|x_t - Sx_t\| = 0$. Since $\|x_t - p\| \le \|y_t - p\|$. By using the convexity of $\|\cdot\|^q$ for all $q > 1$ and Lemma 2.3, we derive

$$\begin{aligned}
\|x_t - p\|^q &\le \|y_t - p\|^q \\
&\le \left\|(1-t)\left[\left(x_t - \frac{\lambda}{1-t}Ax_t\right) - \left(p - \frac{\lambda}{1-t}Ap\right)\right] + t(u_t - p)\right\|^q \\
&\le (1-t)\left\|\left(x_t - \frac{\lambda}{1-t}Ax_t\right) - \left(p - \frac{\lambda}{1-t}Ap\right)\right\|^q + t\|u_t - p\|^q \\
&= (1-t)\left\|(x_t - p) - \frac{\lambda}{1-t}(Ax_t - Ap)\right\|^q + t\|u_t - p\|^q \\
&\le (1-t)\left[\|x_t - p\|^q - \frac{q\lambda}{1-t}\langle Ax_t - Ap, j_q(x_t - p)\rangle\right. \\
&\quad \left. + \frac{\kappa_q\lambda^q}{(1-t)^q}\|Ax_t - Ap\|^q\right] + t\|u_t - p\|^q \\
&\le (1-t)\left[\|x_t - p\|^q - \frac{\alpha q\lambda}{1-t}\|Ax_t - Ap\|^q\right. \\
&\quad \left. + \frac{\kappa_q\lambda^q}{(1-t)^q}\|Ax_t - Ap\|^q\right] + t\|u_t - p\|^q \\
&= (1-t)\left[\|x_t - p\|^q - \frac{\lambda}{1-t}\left(\alpha q - \frac{\kappa_q\lambda^{q-1}}{(1-t)^{q-1}}\right)\|Ax_t - Ap\|^q + t\|u_t - p\|^q\right. \\
&\le \|x_t - p\|^q - \lambda\left(\alpha q - \frac{\kappa_q\lambda^{q-1}}{(1-t)^{q-1}}\right)\|Ax_t - Ap\|^q + t\|u_t - p\|^q,
\end{aligned}$$

which implies that

$$\lambda\left(\alpha q - \frac{\kappa_q\lambda^{q-1}}{(1-t)^{q-1}}\right)\|Ax_t - Ap\|^q \le t\|u_t - p\|^q. \tag{3.4}$$

Since $t \in \left(0, 1 - \lambda\left(\frac{\kappa_q}{\alpha q}\right)^{\frac{1}{q-1}}\right)$, we have $\alpha q - \frac{\kappa_q\lambda^{q-1}}{(1-t)^{q-1}} > 0$. Also, it follows from (3.4) that

$$\lim_{t\longrightarrow 0^+}\|Ax_t - Ap\| = 0.$$

By Proposition 2.12 and Lemma 2.4, we have

$$\begin{aligned}
\|y_t - p\|^q &= \|J_\lambda^B(tu_t + (1-t)x_t - \lambda Ax_t) - J_\lambda^B(p - \lambda Ap)\|^q \\
&\le \langle tu_t + (1-t)x_t - \lambda Ax_t - (p - \lambda Ap), j_q(y_t - p)\rangle \\
&\le \frac{1}{q}[\|tu_t + (1-t)x_t - \lambda Ax_t - (p - \lambda Ap)\|^q \\
&\quad + (q-1)\|y_t - p\|^q - g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|)],
\end{aligned}$$

which implies that

$$\begin{aligned}\|y_t - p\|^q &\le \|tu_t + (1-t)x_t - \lambda Ax_t - (p - \lambda Ap)\|^q - g(\|tu_t \\ &\quad + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|) \\ &= \left\|(1-t)\left[\left(I - \frac{\lambda}{1-t}A\right)x_t - \left(I - \frac{\lambda}{1-t}A\right)p\right]\right. \\ &\quad \left. + t(u_t - p)\right\|^q - g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|) \\ &\le (1-t)\left\|\left(I - \frac{\lambda}{1-t}A\right)x_t - \left(I - \frac{\lambda}{1-t}A\right)p\right\|^q \\ &\quad + t\|u_t - p\|^q - g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|) \\ &\le (1-t)\|x_t - p\|^q + t\|u_t - p\|^q - g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|) \\ &\le \|x_t - p\|^q + t\|u_t - p\|^q - g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|) \\ &\le \|y_t - p\|^q + t\|u_t - p\|^q - g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|).\end{aligned}$$

Hence, we have

$$g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|) \le t\|u_t - p\|^q,$$

and so

$$\lim_{t \longrightarrow 0^+} g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|) = 0.$$

By the property of g, we have

$$\lim_{t \longrightarrow 0^+} \|x_t - y_t\| = 0. \tag{3.5}$$

Also, we obtain

$$\lim_{t \longrightarrow 0^+} \|y_t - Sy_t\| = \lim_{t \longrightarrow 0^+} \|y_t - x_t\| = 0.$$

Moreover, we observe that

$$\begin{aligned}\|x_t - Sx_t\| &\le \|x_t - y_t\| + \|y_t - Sy_t\| + \|Sy_t - Sx_t\| \\ &\le 2\|x_t - y_t\| + \|y_t - Sy_t\| \longrightarrow 0 \text{ as } t \longrightarrow 0^+.\end{aligned} \tag{3.6}$$

For any $z \in \Omega$, we note that

$$\begin{aligned}\|x_t - z\|^q &\le \left\|(1-t)\left[\left(x_t - \frac{\lambda}{1-t}Ax_t\right) - \left(z - \frac{\lambda}{1-t}Az\right)\right] + t(u_t - z)\right\|^q \\ &\le (1-t)^q\left\|\left(x_t - \frac{\lambda}{1-t}Ax_t\right) - \left(z - \frac{\lambda}{1-t}Az\right)\right\|^q + qt\langle u_t - z, j_q(x_t - z)\rangle \\ &\le (1-t)\|x_t - z\|^q + qt\langle u' - z, j_q(x_t - z)\rangle + qt\langle u_t - u', j_q(x_t - z)\rangle,\end{aligned}$$

which implies that

$$\|x_t - z\|^q \le q\langle u' - z, j_q(x_t - z)\rangle + q\langle u_t - u', j_q(x_t - z)\rangle. \tag{3.7}$$

Next, we show that the net $\{x_t\}$ is relatively norm-compact. Assume that $\{t_n\} \subset (0, 1)$ is a sequence such that $t_n \longrightarrow 0^+$ as $n \longrightarrow \infty$. Put $x_n := x_{t_n}$, $y_n := y_{t_n}$, $\lambda_n := \lambda_{t_n}$ and $u_n := u_{t_n}$. From (3.7), we have

$$\|x_n - z\|^q \le q\langle u' - z, j_q(x_n - z)\rangle + q\langle u_n - u', j_q(x_n - z)\rangle. \tag{3.8}$$

By the reflexivity of X and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^* \in C$. In addition, by (3.6), we also have $\lim_{n\longrightarrow\infty} \|x_n - Sx_n\| = 0$. It follows from Lemma 2.5 that $x^* \in F(S)$. Furthermore, we show that $x^* \in (A+B)^{-1}0$. Let $v \in Bu$. Since

$$y_n = J^B_{\lambda_n}(t_n u_n + (1-t_n)x_n - \lambda_n Ax_n).$$

It is observed that

$$\begin{aligned} &t_n u_n + (1-t_n)x_n - \lambda_n Ax_n \in (I + \lambda_n B)y_n \\ \Longleftrightarrow\ &\frac{1}{\lambda_n}\big(t_n u_n + (1-t_n)x_n - \lambda_n Ax_n - y_n\big) \in By_n. \end{aligned}$$

Since B is accretive, we have for all $(u, v) \in B$,

$$\begin{aligned} &\left\langle \frac{1}{\lambda_n}\big(t_n u_n + (1-t_n)x_n - \lambda_n Ax_n - y_n\big) - v,\ j_q(y_n - u)\right\rangle \geq 0 \\ \Longleftrightarrow\ &\langle t_n u_n + (1-t_n)x_n - \lambda_n Ax_n - y_n - \lambda_n v,\ j_q(y_n - u)\rangle \geq 0, \end{aligned}$$

which implies that

$$\begin{aligned} \langle Ax_n + v, j_q(y_n - u)\rangle &\leq \frac{1}{\lambda_n}\langle x_n - y_n, j_q(y_n - u)\rangle + \frac{t_n}{\lambda_n}\langle u_n - x_n, j_q(y_n - u)\rangle \\ &\leq \frac{1}{\lambda_n}\|x_n - y_n\|\|y_n - u\|^{q-1} + \frac{t_n}{\lambda_n}\|u_n - x_n\|\|y_n - u\|^{q-1} \\ &\leq (\|x_n - y_n\| + t_n)K_2, \end{aligned} \tag{3.9}$$

where $K_2 > 0$ is a constant such that $K_2 = \sup_{n\geq 1}\left\{\frac{1}{\lambda_n}\big(\|y_n - u\|^{q-1}, \|u_n - x_n\|\|y_n - u\|^{q-1}\big)\right\}$.

Since a Banach space X has a weakly sequentially continuous generalized duality mapping and from (3.5), we get $\langle Ax^* + v, j_q(x^* - u)\rangle \leq 0$, or $\langle -Ax^* - v, j_q(x^* - u)\rangle \geq 0$. Since B is m-accretive, we have $-Ax^* \in Bx^*$. This shows that $x^* \in (A+B)^{-1}0$. Thus $x^* \in \Omega := F(S) \cap (A+B)^{-1}0$.

Now, replacing z in (3.8) with x^*, we have

$$\|x_n - x^*\|^q \leq \langle u' - x^*, j_q(x_n - x^*)\rangle + \langle u_n - u', j_q(x_n - x^*)\rangle. \tag{3.10}$$

Since $x_n \rightharpoonup x^*$, we get $x_n \longrightarrow x^*$. This proves the relatively norm compactness of the net $\{x_t\}$ as $t \longrightarrow 0^+$.

Now, returning to (3.8) and taking the limit as $n \longrightarrow \infty$, we have

$$\|x^* - z\|^q \leq \langle u' - z, \quad j_q(x^* - z)\rangle.$$

In particular, x^* solves the variational inequality

$$\langle u' - z, j_q(z - x^*)\rangle \leq 0, \quad \forall z \in \Omega, \tag{3.11}$$

which is equivalent to the dual variational inequality (see Lemma 2.6):

$$\langle u' - x^*, j_q(z - x^*)\rangle \leq 0, \quad \forall z \in \Omega. \tag{3.12}$$

Hence, $x^* \in \Omega$ is a solution of variational inequality (3.2). Furthermore, we show that the solution of (3.2) is singleton. Assume that $\hat{x}, x^* \in \Omega$ are solutions of (3.2). Then, we have

$$\langle u' - \hat{x}, j_q(x^* - \hat{x})\rangle \leq 0$$

and

$$\langle u' - x^*, j_q(\hat{x} - x^*)\rangle \le 0.$$

Adding up above two inequalities, we have

$$\|x^* - \hat{x}\|^q \le 0,$$

which implies that $\hat{x} = x^*$ and the uniqueness is proved.

In summary, we have shown that each cluster point of $\{x_t\}$ equal to x^* as $t \longrightarrow 0^+$. Therefore, we can conclude that the net $\{x_t\}$ converges strongly to x^*. This completes the proof. □

Next, we prove a strong convergence theorem which is generated by an explicit iteration process.

Theorem 3.2 *Let C be a nonempty, closed and convex subset of a real uniformly convex and q-uniformly smooth Banach space X which admits a weakly sequentially continuous generalized duality mapping j_q. Let $A : C \longrightarrow X$ be an α-isa of order q and let $B : D(B) \longrightarrow 2^X$ be an m-accretive operator such that $D(B) \subset C$. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $\Omega := F(S) \cap (A + B)^{-1}0 \neq \emptyset$. Let $\{\lambda_n\}$ be a real positive sequence and let $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. For an initial guess $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J^B_{\lambda_n}(\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n Ax_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)Sy_n, \quad \forall n \ge 1, \end{cases} \tag{3.13}$$

where $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}$ and $\{u_n\} \subset X$ is a perturbation for the n-step iteration with $\lim_{n\longrightarrow\infty} u_n = u' \in X$. Suppose that the following conditions are satisfied:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a' \le \beta_n \le b' < 1$;
(C3) $0 < c' \le \lambda_n < \frac{\lambda_n}{1-\alpha_n} \le d' < \left(\frac{\alpha q}{\kappa_q}\right)^{\frac{1}{q-1}}$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ defined by (3.13) converges strongly to a point $x^* \in \Omega$, which solves uniquely the variational inequality (3.2).

Proof We first show that $\{x_n\}$ is bounded. Since $\lim_{n\longrightarrow\infty} u_n = u' \in X$, which implies that $\{u_n\}$ is bounded. Take any $p \in \Omega$, then there exists a constant $M_1 > 0$ such that $M_1 = \sup_{n\ge1}\{\|u_n - p\|\}$. It is observed that

$$p = Sp = J^B_{\lambda_n}(p - \lambda_n Ap) = J^B_{\lambda_n}\left(\alpha_n p + (1 - \alpha_n)\left(p - \frac{\lambda_n}{1 - \alpha_n} Ap\right)\right).$$

Since S, $J^B_{\lambda_n}$ and $I - \frac{\lambda_n}{1-\alpha_n}A$ are nonexpansive (see Lemma 2.1), we have

$$\begin{aligned}\|y_n - p\| &= \left\| J^B_{\lambda_n}\left(\alpha_n u_n + (1-\alpha_n)\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n\right)\right.\\ &\quad \left. - J^B_{\lambda_n}\left(\alpha_n p + (1-\alpha_n)\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)p\right)\right\|\\ &\le \left\|\alpha_n(u_n - p) + (1-\alpha_n)\left[\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)p\right]\right\|\\ &\le \alpha_n\|u_n - p\| + (1-\alpha_n)\left\|\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)p\right\|\\ &\le \alpha_n\|u_n - p\| + (1-\alpha_n)\|x_n - p\|. \end{aligned} \tag{3.14}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\beta_n(x_n - p) + (1-\beta_n)(Sy_n - p)\|\\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|Sy_n - p\|\\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\|\\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\Big[\alpha_n\|u_n - p\| + (1-\alpha_n)\|x_n - p\|\Big]\\ &= \big(1 - (1-\beta_n)\alpha_n\big)\|x_n - p\| + (1-\beta_n)\alpha_n\|u_n - p\|\\ &\le \max\{\|x_n - p\|, M_1\}.\end{aligned}$$

By the mathematical induction, we have

$$\|x_n - p\| \le \max\{\|x_1 - p\|, M_1\}, \quad \forall n \ge 1.$$

Thus, $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{Ax_n\}$ and $\{Sx_n\}$.

Next, we show that $\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = 0$. Set $y_n = J^B_{\lambda_n}z_n$, where $z_n = \alpha_n u_n + (1-\alpha_n)x_n - \lambda_n Ax_n$. Then, we have

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|J^B_{\lambda_{n+1}}z_{n+1} - J^B_{\lambda_n}z_n\| \le \|J^B_{\lambda_{n+1}}z_{n+1} - J^B_{\lambda_{n+1}}z_n\| + \|J^B_{\lambda_{n+1}}z_n - J^B_{\lambda_n}z_n\|\\ &\le \|z_{n+1} - z_n\| + \|J^B_{\lambda_{n+1}}z_n - J^B_{\lambda_n}z_n\|\\ &= \|\alpha_{n+1}u_{n+1} + (1-\alpha_{n+1})x_{n+1}\\ &\quad - \lambda_{n+1}Ax_{n+1} - (\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n Ax_n)\| + \|J^B_{\lambda_{n+1}}z_n - J^B_{\lambda_n}z_n\|\\ &= \left\|\alpha_{n+1}(u_{n+1} - u_n) + (\alpha_{n+1} - \alpha_n)(u_n - x_n)\right.\\ &\quad + (1-\alpha_{n+1})\left[\left(I - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\right)x_{n+1} - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n\right]\\ &\quad \left. + (\lambda_n - \lambda_{n+1})Ax_n\right\| + \|J^B_{\lambda_{n+1}}z_n - J^B_{\lambda_n}z_n\| \le \alpha_{n+1}\big(\|u_{n+1}\| + \|u_n\|\big)\\ &\quad + |\alpha_{n+1} - \alpha_n|\big(\|u_n\| + \|x_n\|\big) + (1-\alpha_{n+1})\left\|\left(I - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\right)x_{n+1}\right.\\ &\quad \left. - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n\right\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \|J^B_{\lambda_{n+1}}z_n - J^B_{\lambda_n}z_n\|\\ &\le (1-\alpha_{n+1})\|x_{n+1} - x_n\| + \alpha_{n+1}\big(\|u_{n+1}\| + \|u_n\|\big) + |\alpha_{n+1}\\ &\quad - \alpha_n|\big(\|u_n\| + \|x_n\|\big) + |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \|J^B_{\lambda_{n+1}}z_n - J^B_{\lambda_n}z_n\|.\end{aligned}$$

By Lemma 2.11, we have

$$\|J_{\lambda_{n+1}}^B z_n - J_{\lambda_n}^B z_n\| \leq \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \|J_{\lambda_{n+1}}^B z_n - z_n\|.$$

It follows that

$$\begin{aligned}\|y_{n+1} - y_n\| &\leq (1-\alpha_{n+1})\|x_{n+1} - x_n\| + \alpha_{n+1}\big(\|u_{n+1}\| + \|u_n\|\big) + |\alpha_{n+1} \\ &\quad - \alpha_n|\big(\|u_n\| + \|x_n\|\big) + |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \|J_{\lambda_{n+1}}^B z_n - z_n\| \\ &\leq (1-\alpha_{n+1})\|x_{n+1} - x_n\| + \Bigg(\alpha_{n+1} + |\alpha_{n+1} - \alpha_n| + |\lambda_{n+1} - \lambda_n| \\ &\quad + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}}\Bigg) M_2,\end{aligned}$$

where $M_2 = \sup_{n\geq 1}\big\{\|u_{n+1}\| + \|u_n\|, \|u_n\| + \|x_n\|, \|Ax_n\|, \|J_{\lambda_{n+1}}^B z_n - z_n\|\big\}$. Then, we have

$$\begin{aligned}\|Sy_{n+1} - Sy_n\| \leq \|y_{n+1} - y_n\| &\leq (1-\alpha_{n+1})\|x_{n+1} - x_n\| \\ &\quad + \Bigg(\alpha_{n+1} + |\alpha_{n+1} - \alpha_n| + |\lambda_{n+1} - \lambda_n| + \frac{|\lambda_{n+1} - \lambda_n|}{a'}\Bigg) M_2.\end{aligned}$$

From (C1) and (C3), we have

$$\limsup_{n\longrightarrow\infty} \big(\|Sy_{n+1} - Sy_n\| - \|x_{n+1} - x_n\|\big) \leq 0.$$

By Lemma 2.8, we get

$$\lim_{n\longrightarrow\infty} \|Sy_n - x_n\| = 0. \tag{3.15}$$

Consequently,

$$\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = \lim_{n\longrightarrow\infty} (1-\beta_n)\|Sy_n - x_n\| = 0. \tag{3.16}$$

Next, we show that $\lim_{n\longrightarrow\infty} \|x_n - Sx_n\| = 0$. By the convexity of $\|\cdot\|^q$ for all $q > 1$ and Lemma 2.3, we have

$$\begin{aligned}\|y_n - p\|^q &= \left\|(1-\alpha_n)\left[\left(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n\right) - \left(p - \frac{\lambda_n}{1-\alpha_n}Ap\right)\right] + \alpha_n(u_n - p)\right\|^q \\ &\leq (1-\alpha_n)\left\|\left(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n\right) - \left(p - \frac{\lambda_n}{1-\alpha_n}Ap\right)\right\|^q + \alpha_n\|u_n - p\|^q \\ &= (1-\alpha_n)\left\|(x_n - p) - \frac{\lambda_n}{1-\alpha_n}(Ax_n - Ap)\right\|^q + \alpha_n\|u_n - p\|^q \\ &\leq (1-\alpha_n)\Bigg[\|x_n - p\|^q - \frac{q\lambda_n}{1-\alpha_n}\langle Ax_n - Ap, j_q(x_n - p)\rangle \\ &\quad + \frac{\kappa_q\lambda_n^q}{(1-\alpha_n)^q}\|Ax_n - Ap\|^q\Bigg] + \alpha_n\|u_n - p\|^q \leq (1-\alpha_n)\Bigg[\|x_n - p\|^q \\ &\quad - \frac{\alpha q\lambda_n}{1-\alpha_n}\|Ax_n - Ap\|^q + \frac{\kappa_q\lambda_n^q}{(1-\alpha_n)^q}\|Ax_n - Ap\|^q\Bigg] \\ &\quad + \alpha_n\|u_n - p\|^q = (1-\alpha_n)\Bigg[\|x_n - p\|^q\end{aligned}$$

$$-\frac{\lambda_n}{1-\alpha_n}\left(\alpha q-\frac{\kappa_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n-Ap\|^q\Bigg]+\alpha_n\|u_n-p\|^q$$
$$\leq \|x_n-p\|^q-\lambda_n\left(\alpha q-\frac{\kappa_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n-Ap\|^q+\alpha_n\|u_n-p\|^q. \tag{3.17}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|^q &\leq \beta_n\|x_n-p\|^q+(1-\beta_n)\|Sy_n-p\|^q\\ &\leq \beta_n\|x_n-p\|^q+(1-\beta_n)\|y_n-p\|^q\leq \beta_n\|x_n-p\|^q+(1-\beta_n)\\ &\quad\times\left[\|x_n-p\|^q-\lambda_n\left(\alpha q-\frac{\kappa_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n-Ap\|^q+\alpha_n\|u_n-p\|^q\right]\\ &= \|x_n-p\|^q-\lambda_n(1-\beta_n)\left(\alpha q-\frac{\kappa_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n-Ap\|^q\\ &\quad+\alpha_n(1-\beta_n)\|u_n-p\|^q,\end{aligned}$$

which implies from (C2), (C3) and Proposition 2.7 that

$$\begin{aligned}&c'(1-b')\big(\alpha q-\kappa_q(d')^{q-1}\big)\|Ax_n-Ap\|^q\\ &\quad\leq \|x_n-p\|^q-\|x_{n+1}-p\|^q+\alpha_n(1-\beta_n)\|u_n-p\|^q\\ &\quad\leq q\|x_n-p\|^{q-1}\big(\|x_n-p\|-\|x_{n+1}-p\|\big)+\alpha_n(1-\beta_n)\|u_n-p\|^q\\ &\quad\leq q\|x_n-p\|^{q-1}\|x_{n+1}-x_n\|+\alpha_n(1-\beta_n)\|u_n-p\|^q.\end{aligned}$$

Moreover, from (C1), (C3) and (3.16), we have

$$\lim_{n\longrightarrow\infty}\|Ax_n-Ap\|=0. \tag{3.18}$$

By Proposition 2.12 and Lemma 2.4, we have

$$\begin{aligned}\|y_n-p\|^q &= \|J_{\lambda_n}^B(\alpha_nu_n+(1-\alpha_n)x_n-\lambda_nAx_n)-J_{\lambda_n}^B(p-\lambda_nAp)\|^q\\ &\leq \langle\alpha_nu_n+(1-\alpha_n)x_n-\lambda_nAx_n-(p-\lambda_nAp),\,j_q(y_n-p)\rangle\\ &\leq \frac{1}{q}\Big[\|\alpha_nu_n+(1-\alpha_n)x_n-\lambda_nAx_n-(p-\lambda_nAp)\|^q+(q-1)\|y_n-p\|^q\\ &\quad-g(\|\alpha_nu_n+(1-\alpha_n)x_n-\lambda_n(Ax_n-Ap)-y_n\|)\Big],\end{aligned}$$

which implies that

$$\begin{aligned}\|y_n-p\|^q &\leq \|\alpha_nu_n+(1-\alpha_n)x_n-\lambda_nAx_n-(p-\lambda_nAp)\|^q-g(\|\alpha_nu_n\\ &\quad+(1-\alpha_n)x_n-\lambda_n(Ax_n-Ap)-y_n\|)\\ &\leq \alpha_n\|u_n-p\|^q+\|x_n-p\|^q\\ &\quad-g(\|\alpha_nu_n+(1-\alpha_n)x_n-\lambda_n(Ax_n-Ap)-y_n\|).\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|^q &\leq \beta_n\|x_n-p\|^q+(1-\beta_n)\|y_n-p\|^q\leq \beta_n\|x_n-p\|^q\\ &\quad+(1-\beta_n)\Big[\alpha_n\|u_n-p\|^q+\|x_n-p\|^q-g(\|\alpha_nu_n\end{aligned}$$

$$
\begin{aligned}
&\quad + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|)\Big] \\
&= \|x_n - p\|^q + \alpha_n(1-\beta_n)\|u_n - p\|^q - (1-\beta_n)g(\|\alpha_n u_n \\
&\quad + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|),
\end{aligned}
$$

which implies by (C2) and Proposition 2.7 that

$$
\begin{aligned}
&(1-b')g(\|\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|) \\
&\quad \le \|x_n - p\|^q - \|x_{n+1} - p\|^q + \alpha_n(1-\beta_n)\|u_n - p\|^q \\
&\quad \le q\|x_n - p\|^{q-1}\big(\|x_n - p\| - \|x_{n+1} - p\|\big) + \alpha_n(1-\beta_n)\|u_n - p\|^q \\
&\quad \le q\|x_n - p\|^{q-1}\|x_{n+1} - x_n\| + \alpha_n(1-\beta_n)\|u_n - p\|^q.
\end{aligned}
$$

Then, from (C1), (C2) and (3.16), we have

$$
\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0. \tag{3.19}
$$

Consequently,

$$
\begin{aligned}
\|x_n - Sx_n\| &\le \|x_n - Sy_n\| + \|Sy_n - Sx_n\| \\
&\le \|x_n - Sy_n\| + \|y_n - x_n\| \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty.
\end{aligned} \tag{3.20}
$$

Next, we show that

$$
\limsup_{n\longrightarrow\infty} \langle u' - x^*, j_q(y_n - x^*)\rangle \le 0,
$$

where x^* is the same as in Theorem 3.1. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$
\limsup_{n\longrightarrow\infty} \langle u' - x^*, j_q(x_n - x^*)\rangle = \lim_{i\longrightarrow\infty} \langle u' - x^*, j_q(x_{n_i} - x^*)\rangle.
$$

By the reflexivity of X and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in C$. From (3.19) and (3.20), we also have $y_n - Sy_n \longrightarrow 0$. Then from Lemma 2.5, we have $z \in F(S)$. Furthermore, by the similar method in the proof of Theorem 3.1, we can show that $z \in \Omega$. Since a Banach space X has a weakly sequentially continuous generalized duality mapping. Then, we have

$$
\begin{aligned}
\limsup_{n\longrightarrow\infty} \langle u' - x^*, j_q(y_n - x^*)\rangle &= \limsup_{n\longrightarrow\infty} \langle u' - x^*, j_q(x_n - x^*)\rangle \\
&= \langle u' - x^*, j_q(z - x^*)\rangle \le 0.
\end{aligned} \tag{3.21}
$$

Finally, we show that $x_n \longrightarrow x^*$. From (3.14) and Lemma 2.2, we have

$$
\begin{aligned}
\|y_n - x^*\|^q &= \left\|(1-\alpha_n)\left[\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x^*\right] + \alpha_n(u_n - x^*)\right\|^q \\
&\le (1-\alpha_n)^q\left\|\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x^*\right\|^q \\
&\quad + q\alpha_n\langle u_n - x^*, j_q(y_n - x^*)\rangle \le (1-\alpha_n)^q\|x_n - x^*\|^q \\
&\quad + q\alpha_n\langle u_n - x^*, j_q(y_n - x^*)\rangle.
\end{aligned}
$$

Then, it follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^q &\le \beta_n\|x_n-x^*\|^q+(1-\beta_n)\|Sy_n-x^*\|^q\\ &\le \beta_n\|x_n-x^*\|^q+(1-\beta_n)\|y_n-x^*\|^q\\ &\le \beta_n\|x_n-x^*\|^q+(1-\beta_n)\big[(1-\alpha_n)^q\|x_n-x^*\|^q\\ &\quad +q\alpha_n\langle u_n-x^*, j_q(y_n-x^*)\rangle\big]\\ &\le \big(1-\alpha_n(1-\beta_n)\big)\|x_n-x^*\|^q+q\alpha_n(1-\beta_n)\langle u_n-u', j_q(y_n-x^*)\rangle\\ &\quad +q\alpha_n(1-\beta_n)\langle u_n-x^*, j_q(y_n-x^*)\rangle\\ &\le \big(1-\alpha_n(1-\beta_n)\big)\|x_n-x^*\|^q\\ &\quad +q\alpha_n(1-\beta_n)\|u_n-u'\|\|y_n-x^*\|^{q-1}\\ &\quad +q\alpha_n(1-\beta_n)\langle u_n-x^*, j_q(y_n-x^*)\rangle. \end{aligned} \tag{3.22}$$

Then (3.22) reduces to

$$\|x_{n+1}-x^*\|^q \le (1-\gamma_n)\|x_n-x^*\|^q+\gamma_n\delta_n,$$

where $\gamma_n := \alpha_n(1-\beta_n)$ and $\delta_n := q\|u_n-u'\|\|y_n-x^*\|^{q-1}+q\langle u'-x^*, j_q(y_n-x^*)\rangle$. It is easily seen that $\sum_{n=1}^{\infty}\gamma_n=\infty$ and $\limsup_{n\longrightarrow\infty}\delta_n\le 0$. We can therefore apply Lemma 2.9 to conclude that $x_n \longrightarrow x^*$. This completes the proof. □

Corollary 3.3 *Let C be a nonempty, closed and convex subset of a real uniformly convex and 2-uniformly smooth Banach space X which admits a weakly sequentially continuous duality mapping j. Let $A : C \longrightarrow X$ be an α-isa of order 2 and let $B : D(B) \longrightarrow 2^X$ be an m-accretive operator such that $D(B)\subset C$. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $\Omega := F(S)\cap(A+B)^{-1}0 \ne \emptyset$. Let $\{\lambda_n\}$ be a real positive sequence and let $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. For an initial guess $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J^B_{\lambda_n}(\alpha_n u_n+(1-\alpha_n)x_n-\lambda_n Ax_n),\\ x_{n+1}=\beta_n x_n+(1-\beta_n)Sy_n, \quad \forall n\ge 1, \end{cases} \tag{3.23}$$

where $J^B_{\lambda_n}=(I+\lambda_n B)^{-1}$ and $\{u_n\}\subset X$ is a perturbation for the n-step iteration with $\lim_{n\longrightarrow\infty}u_n=u'\in X$. Suppose that the following conditions are satisfied:

(C1) $\lim_{n\longrightarrow\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$*;*
(C2) $0<a'\le\beta_n\le b'<1$*;*
(C3) $0<c'\le\lambda_n<\frac{\lambda_n}{1-\alpha_n}\le d'<\frac{\alpha}{K^2}$ *and* $\lim_{n\longrightarrow\infty}|\lambda_{n+1}-\lambda_n|=0$.

Then the sequence $\{x_n\}$ defined by (3.23) converges strongly to a point $x^*\in\Omega$, which solves uniquely the following variational inequality:

$$\langle u'-x^*, j(z-x^*)\rangle\le 0, \quad \forall z\in\Omega.$$

Corollary 3.4 *Let C be a nonempty, closed and convex subset of a real Hilbert H. Let $A : C \longrightarrow H$ be an α-ism and let $B : D(B) \longrightarrow 2^H$ be a maximal monotone operator such that $D(B)\subset C$. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $\Omega := F(S)\cap(A+B)^{-1}0\ne\emptyset$. Let $\{\lambda_n\}$ be a real positive sequence and let $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. For an initial guess $x_1\in C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J^B_{\lambda_n}(\alpha_n u_n+(1-\alpha_n)x_n-\lambda_n Ax_n),\\ x_{n+1}=\beta_n x_n+(1-\beta_n)Sy_n, \quad \forall n\ge 1, \end{cases} \tag{3.24}$$

where $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}$ *and* $\{u_n\} \subset H$ *is a perturbation for the* n*-step iteration with* $\lim_{n\longrightarrow\infty} u_n = u' \in H$. *Suppose that the following conditions are satisfied:*

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a' \leq \beta_n \leq b' < 1$;
(C3) $0 < c' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq d' < 2\alpha$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ *defined by* (3.24) *converges strongly to a point* $x^* \in \Omega$, *which solves uniquely the following variational inequality:*

$$\langle u' - x^*, z - x^*\rangle \leq 0, \quad \forall z \in \Omega.$$

4 Applications

In this section, we give some applications of Theorem 3.2 in the framework of Hilbert spaces. Throughout this section, let C be a nonempty, closed and convex subset of a real Hilbert space H.

4.1 Application to variational inequality problems

Let $A : C \longrightarrow H$ be a nonlinear monotone operator. The *variational inequality problem* is to find $z \in C$ such that

$$\langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{4.1}$$

The set of solutions of problem (4.1) is denoted by $VI(C, A)$. In the context of the variational inequality problem, it well known that

$$z \in VI(C, A) \Longleftrightarrow z = P_C(z - \lambda Az), \quad \forall \lambda > 0,$$

where P_C is the metric projection from H onto C.

Let $g : H \longrightarrow (-\infty, \infty]$ be a proper convex lower semi-continuous function. Then the *subdifferential* ∂g of g is defined as follows:

$$\partial g(x) = \{y \in H : g(z) \geq g(x) + \langle z - x, y\rangle, \quad \forall z \in H\}, \quad \forall x \in H.$$

It is known that ∂g is maximal monotone (see [17]). Let i_C be the indicator function of C defined by

$$i_C(x) = \begin{cases} 0, & x \in C; \\ \infty, & x \notin C. \end{cases} \tag{4.2}$$

Since i_C is a proper lower semi-continuous convex function on H, then subdifferential ∂i_C of i_C is a maximal monotone operator. So, we can define the resolvent $J^{\partial i_C}_\lambda$ of ∂i_C for $\lambda > 0$ by

$$J^{\partial i_C}_\lambda x = (I + \lambda \partial i_C)^{-1} x$$

for all $x \in H$.

Lemma 4.1 [18] *Let* ∂i_C *be the subdifferential of* i_C, *where* i_C *defined as in* (4.2) *and let* $J^{\partial i_C}_\lambda$ *be the resolvent of* ∂i_C *for* $\lambda > 0$. *Then, we have*

$$y = J^{\partial i_C}_\lambda x \Longleftrightarrow y = P_C x, \quad \forall x \in H, y \in C.$$

Further, we have $(A + \partial i_C)^{-1} 0 = VI(C, A)$.

Theorem 4.2 *Let $A : C \longrightarrow H$ be an α-ism. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{\lambda_n\}$ be a real positive sequence and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$. For an initial guess $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = P_C(\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n Ax_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Sy_n, \quad \forall n \geq 1, \end{cases} \tag{4.3}$$

where $\{u_n\} \subset H$ is a perturbation for the n-step iteration with $\lim_{n\longrightarrow\infty} u_n = u' \in H$. Suppose that the following conditions are satisfied:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $0 < a' \leq \beta_n \leq b' < 1$*;*
(C3) $0 < c' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq d' < 2\alpha$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ defined by (4.3) converges strongly to a point $x^* \in F(S) \cap VI(C, A)$.

4.2 Application to equilibrium problems

Let $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of all real numbers. The *equilibrium problem* is to find $z \in C$ such that

$$G(z, y) \geq 0, \tag{4.4}$$

for all $y \in C$. The set of solutions of problem (4.6) is denoted by $EP(G)$. For solving the equilibrium problem, let us assume that a bifunction $G : C \times C \longrightarrow \mathbb{R}$ satisfies the following conditions:

(A1) $G(x, x) = 0$ for all $x \in C$;
(A2) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0$ for all $x \in C$;
(A3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} G(tz + (1-t)y, y) \leq G(x, y)$;
(A4) for all $x \in C$, $G(x, \cdot)$ is convex and lower semi-continuous.

Lemma 4.3 [19] *Let $G : C \times C \longrightarrow \mathbb{R}$ satisfying the conditions $(A1)-(A4)$. Let $\lambda > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$G(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

Lemma 4.4 [20] *Assume that $G : C \times C \longrightarrow \mathbb{R}$ satisfies the conditions $(A1)-(A4)$. For $\lambda > 0$ and $x \in H$, define a mapping $T_\lambda : H \longrightarrow C$ as follows:*

$$T_\lambda(x) = \left\{z \in C : G(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\right\}, \ \forall x \in H.$$

Then, the following hold:

(1) *T_λ is single-valued;*
(2) *T_λ is firmly nonexpansive, i.e., for each $x, y \in H$,*

$$\|T_\lambda x - T_\lambda y\|^2 \leq \langle T_\lambda x - T_\lambda y, x - y\rangle;$$

(3) *$F(T_\lambda) = EP(G)$;*
(4) *$EP(G)$ is closed and convex.*

We call such T_λ the resolvent of G for $\lambda > 0$.

Lemma 4.5 [18] *Let $G : C \times C \longrightarrow \mathbb{R}$ satisfies the conditions $(A1)-(A4)$. Let A_G be a multivalued mapping of H into itself defined by*

$$A_G x = \begin{cases} \{z \in H : G(x, y) \geq \langle y - x, z \rangle, \ \forall y \in C\}, & x \in C; \\ \emptyset, & x \notin C. \end{cases}$$

Then, $EP(G) = A_G^{-1}0$ and A_G is a maximal monotone operator with $D(A_G) \subset C$. Further, for any $x \in H$ and $\lambda > 0$, the resolvent T_λ of G coincides with the resolvent of A_G, that is,

$$T_\lambda x = (I + \lambda A_G)^{-1} x.$$

Theorem 4.6 *Let $A : C \longrightarrow H$ be an α-ism. Let $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction which satisfies the conditions $(A1) - -(A4)$. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $F(S) \cap EP(G) \neq \emptyset$. Let $\{\lambda_n\}$ be a real positive sequence and let $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. For an initial guess $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = T_{\lambda_n}(\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S y_n, \ \ \forall n \geq 1, \end{cases} \tag{4.5}$$

where $\{u_n\} \subset H$ is a perturbation for the n-step iteration with $\lim_{n\longrightarrow\infty} u_n = u' \in H$. Suppose that the following conditions are satisfied:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a' \leq \beta_n \leq b' < 1$;
(C3) $0 < c' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq d' < 2\alpha$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ defined by (4.5) converges strongly to a point $x^* \in F(S) \cap EP(G)$.

4.3 Application to convex minimization problems

Let $f : H \longrightarrow \mathbb{R}$ be a convex smooth function and $g : H \longrightarrow \mathbb{R}$ be a convex, lower-semicontinuous and nonsmooth function. The *convex minimization problem* is to find $z \in C$ such that

$$f(z) + g(z) \leq f(x) + g(x), \tag{4.6}$$

for all $x \in C$. The set of solutions of problem (4.6) is denoted by $CMP(f, g)$. By Fermat's rule, it is known that the problem (4.6) is equivalent to the problem of finding $z \in C$ such that

$$0 \in \nabla f(z) + \partial g(z),$$

where ∇f is a gradient of f and ∂g is a subdifferential of g. In fact, we can set $A = \nabla f$ and $B = \partial g$ in Theorem 3.2. It is also known ∇f is $(1/L)$-Lipschitz continuous, then it is also L-ism (see [21]). Further, ∂g is maximal monotone (see [17]). So we obtain the following result.

Theorem 4.7 *Let $f : H \longrightarrow \mathbb{R}$ be a convex and differentiable function with $(1/L)$-Lipschitz continuous gradient ∇f and $G : H \longrightarrow \mathbb{R}$ be a convex and lower semi-continuous function such that $D(\partial G) \subset C$. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $F(S) \cap CMP(f, g) \neq \emptyset$. Let $\{\lambda_n\}$ be a real positive sequence and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$. For an initial guess $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J_{\lambda_n}(\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n \nabla f(x_n)), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S y_n, \ \ \forall n \geq 1, \end{cases} \tag{4.7}$$

where $\{u_n\} \subset H$ *is a perturbation for the n-step iteration with* $\lim_{n\longrightarrow\infty} u_n = u' \in H$. *Suppose that the following conditions are satisfied:*

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a' \leq \beta_n \leq b' < 1$;
(C3) $0 < c' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq d' < 2L$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ *defined by* (4.7) *converges strongly to a point* $x^* \in F(S) \cap CMP(f, g)$.

4.4 Application to linear inverse problems

Let T be a bounded linear operator on H and $b \in H$. The unconstrained linear problem is to find $x \in H$ such that

$$Tx = b. \tag{4.8}$$

The set of solutions of problem (4.8) is denoted by $\Gamma = \{x \in H : x = T^{-1}b\}$. For each $x \in H$, we define $f : H \longrightarrow \mathbb{R}$ by

$$f(x) = \frac{1}{2}\|Tx - b\|^2.$$

It is well known that $\nabla f = T^t(Tx - b)$ and ∇f is K-Lipschitz continuous with K the largest eigenvalue of T^tT [22]. So we obtain immediately the following result.

Theorem 4.8 *Let* $T : H \longrightarrow H$ *be a bonded linear operator and* $b \in H$ *with* K *the largest eigenvalue of* T^tT. *Let* $S : H \longrightarrow H$ *be a nonexpansive mapping such that* $F(S) \cap \Gamma \neq \emptyset$. *Let* $\{\lambda_n\}$ *be a real positive sequence and let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $(0, 1)$. *For an initial guess* $x_1 \in H$, *define the sequence* $\{x_n\}$ *by*

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n T^t(Tx_n - b), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)Sy_n, \quad \forall n \geq 1, \end{cases} \tag{4.9}$$

where $\{u_n\} \subset H$ *is a perturbation for the n-step iteration with* $\lim_{n\longrightarrow\infty} u_n = u' \in H$. *Suppose that the following conditions are satisfied:*

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a' \leq \beta_n \leq b' < 1$;
(C3) $0 < c' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq d' < \frac{2}{K}$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ *defined by* (4.9) *converges strongly to a point* $x^* \in F(S) \cap \Gamma$.

Acknowledgements S. Suantai would like to thank Chiang Mai University for financial supports, P. Cholamjiak was supported by the Thailand Research Fund and the Commission on Higher Education under Grant MRG5980248 and P. Sunthrayuth was supported by RMUTT research foundation scholarship of Rajamangala University of Technology Thanyaburi under Grant NRF04066005.

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Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas

Relaxed CQ Algorithms Involving the Inertial Technique for Multiple-sets Split Feasibility Problems

--Manuscript Draft--

Manuscript Number:	
Full Title:	Relaxed CQ Algorithms Involving the Inertial Technique for Multiple-sets Split Feasibility Problems
Article Type:	Original Paper
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Order of Authors Secondary Information:	
Funding Information:	
Abstract:	In this work, we introduce the inertial relaxed CQ algorithms for solving the multiple-sets split feasibility problems (MSFP) in the frameworks of Hilbert spaces. By mixing the inertial technique with the self-adaptive method, not only the computation on the matrix norm and the orthogonal projection is relaxed but also the convergence speed is improved. We then establish the strong convergence theorem by combining the relaxed CQ algorithm with Halpern's iteration process. Finally, we provide numerical experiments to illustrate the convergence behavior and the effectiveness of our proposed algorithm. The main result extends and improves the corresponding results.

Relaxed CQ Algorithms Involving the Inertial Technique for Multiple-sets Split Feasibility Problems

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Abstract

In this work, we introduce the inertial relaxed CQ algorithms for solving the multiple-sets split feasibility problems (MSFP) in the frameworks of Hilbert spaces. By mixing the inertial technique with the self-adaptive method, not only the computation on the matrix norm and the orthogonal projection is relaxed but also the convergence speed is improved. We then establish the strong convergence theorem by combining the relaxed CQ algorithm with Halpern's iteration process. Finally, we provide numerical experiments to illustrate the convergence behavior and the effectiveness of our proposed algorithm. The main result extends and improves the corresponding results.

Keywords: Inertial relaxed CQ algorithm; Halpern's iteration process; Multiple-sets split feasibility problem; Self-adaptive method.

AMS Subject Classification: 65K05, 65K10, 49J52.

1 Introduction

Let H_1 and H_2 be real Hilbert spaces. Let $t \geq 1$ and $r \geq 1$ be given integers and let $\{C_i\}_{i=1}^t$ and $\{Q_j\}_{j=1}^r$ be nonempty, closed and convex subsets of H_1 and H_2, respectively.

In this research, we study the Multiple-sets Split Feasibility Problem (MSFP) which is the problem of finding a point x^* such that

$$x^* \in C := \bigcap_{i=1}^{t} C_i, \quad Ax^* \in Q := \bigcap_{j=1}^{r} Q_j, \tag{1.1}$$

where A is a given bounded linear operator (denote A^* by the adjoint operator of A). This problem was first introduced, in finite-dimensional Hilbert spaces, by Censor et al. in [6] for modeling inverse

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problems which arise in modeling of intensity modulated radiation therapy [7], and signal processing and image reconstruction [4, 14]. Due to its applications, there have been many algorithms invented to solve MSFP (see, for instance, [21, 23, 28, 29, 30]). In particular, when $t = r = 1$, the MSFP (1.1) becomes the split feasibility problem (SFP) which was introduced in [5].

Throughout this work, we always assume that the MSFP (1.1) is consistent and also denote the solution set by S. It is known that the MSFP is equivalent to the following minimization problem:

$$\min \frac{1}{2}\|x - P_C(x)\|^2 + \frac{1}{2}\|Ax - P_Q(Ax)\|^2, \tag{1.2}$$

where P_C and P_Q are the metric projections onto C and Q, respectively. It should be noted that the computation of a projection onto a general closed convex subset is difficult because of its closed form. To overcome this difficulty, Fukushima [10] suggested a so-called relaxed projection method to calculate the projection onto a level set of a convex function by computing a sequence of projections onto half-spaces containing the original level set. In the setting of finite-dimensional Hilbert spaces, Yang [26] introduced the relaxed CQ algorithms for solving SFP where the closed convex subsets C and Q are level sets of convex functions given as follows:

$$C = \{x \in H_1 : c(x) \leq 0\} \textit{ and } Q = \{y \in H_2 : q(y) \leq 0\}, \tag{1.3}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are weakly lower semi-continuous and convex functions. It is assumed that both c and q are subdifferentiable on H_1 and H_2, respectively, and that ∂c and ∂q are bounded operators (*i.e.*, bounded on bounded sets). It is known that every convex function defined on a finite-dimensional Hilbert space is subdifferentiable and its subdifferential operator is a bounded operator (see [2]). Define two sets at point x_n by

$$C_n = \{x \in H_1 : c(x_n) \leq \langle \xi_n, x_n - x\rangle\}, \tag{1.4}$$

where $\xi_n \in \partial c(x_n)$, and

$$Q_n = \{y \in H_2 : q(Ax_n) \leq \langle \zeta_n, Ax_n - y\rangle\}, \tag{1.5}$$

where $\zeta_n \in \partial q(Ax_n)$. It is clear that C_n and Q_n are half-spaces and $C_n \supset C$ and $Q_n \supset Q$ for every $n \geq 1$. In this case, the metric projections onto the sets C_n and Q_n can be easily calculated since it has the specific form which can be found in [2]. Employing this tool, Yang [26] constructed a relaxed CQ algorithm for solving the SFP by using the half-spaces C_n and Q_n instead of the sets C and Q, respectively and then proved its convergence under some suitable choices of the step-sizes.

For solving the MSFP, following [6], we define the level sets of convex functions by

$$C_i = \{x \in H_1 : c_i(x) \leq 0\} \quad \textit{and} \quad Q_j = \{y \in H_2 : q_j(y) \leq 0\}, \tag{1.6}$$

where $c_i : H_1 \to \mathbb{R}$ $(i = 1, ..., t)$ and $q_j : H_2 \to \mathbb{R}$ $(j = 1, ..., r)$ are weakly lower semi-continuous and convex functions. We assume that c_i $(i = 1, ..., t)$ and q_j $(j = 1, ..., r)$ are subdifferentiable on H_1 and H_2, respectively, and that ∂c_i $(i = 1, ..., t)$ and ∂q_j $(j = 1, ..., r)$ are bounded on bounded sets. Censor et al. [6] also defined the following proximity function:

$$f(x) := \frac{1}{2}\sum_{i=1}^{t} l_i\|x - P_{C_i}(x)\|^2 + \frac{1}{2}\sum_{j=1}^{r} \lambda_j\|Ax - P_{Q_j}(Ax)\|^2, \tag{1.7}$$

where l_i $(i = 1, ..., t)$ and λ_j $(i = 1, ..., r)$ are all positive constants such that $\sum_{i=1}^{t} l_i + \sum_{j=1}^{r} \lambda_j = 1$. In this case, we also have

$$\nabla f(x) = \sum_{i=1}^{t} l_i(x - P_{C_i}(x)) + \sum_{j=1}^{r} \lambda_j A^*(I - P_{Q_j})Ax. \tag{1.8}$$

They introduced the following projection algorithm:

$$x_{n+1} = P_\Omega(x_n - \rho\nabla f(x_n)), \tag{1.9}$$

where $\rho > 0$ and $\Omega \subseteq \mathbb{R}^N$ is an auxiliary simple nonempty closed convex set such that $\Omega \cap S \neq \emptyset$. It was proved that if $\rho \in (0, 2/L)$ with L being the Lipschitz constant of ∇f, then the sequence $\{x_n\}$ generated by (1.9) converges to a solution in MSFP.

As observed in the results of Byrne [3], we see that the selection of the step-sizes ρ in (1.9) depends on the largest eigenvalue (spectral radius) of the matrix A^*A which is not always possible in practice. To avoid this computation, there have been worthwhile works that the convergence is guaranteed without any prior information of the matrix norm (see, for examples [22, 23, 24, 27]). Among these works, López et al. [14] introduced a new way to select the step-size and also practised this way of selecting step-sizes for variants of the CQ algorithm, including a relaxed CQ algorithm, and a Halpern-type algorithm and proved both weak and strong convergence. Combining the relaxed CQ algorithm with that of López et al. [14], in 2013, He and Zhao [11] introduced a new relaxed CQ algorithm such that the strong convergence is guaranteed in infinite-dimensional Hilbert spaces. With this choice of the step-sizes, the estimation of the norm of operators is avoided and the metric projections are easily to be calculated.

In what follows, we define two sets at point x_n by

$$C_i^n = \{x \in H_1 : c_i(x_n) \leq \langle \xi_i^n, x_n - x\rangle\}, \tag{1.10}$$

where $\xi_i^n \in \partial c_i(x_n)$ for $i = 1, ..., t$, and

$$Q_j^n = \{y \in H_2 : q_j(Ax_n) \leq \langle \zeta_j^n, Ax_n - y\rangle\}, \tag{1.11}$$

where $\zeta_j^n \in \partial q_j(Ax_n)$ for $j = 1, ..., r$. We see that C_i^n $(i = 1, ..., t)$ and Q_j^n $(j = 1, ..., r)$ are half-spaces and $C_i^n \supset C_i$ $(i = 1, ..., t)$ and $Q_j^n \supset Q_j$ $(j = 1, ..., r)$ for all $n \geq 1$. We define

$$f_n(x) := \frac{1}{2}\sum_{i=1}^{t} l_i\|x - P_{C_i^n}(x)\|^2 + \frac{1}{2}\sum_{j=1}^{r} \lambda_j\|Ax - P_{Q_j^n}(Ax)\|^2, \tag{1.12}$$

where $C_i^n(i = 1, ..., t)$ and $Q_j^n(j = 1, ..., r)$ are given as in (1.10) and (1.11), respectively.
We then have

$$\nabla f_n(x) := \sum_{i=1}^{t} l_i(x - P_{C_i^n}(x)) + \sum_{j=1}^{r} \lambda_j A^*(I - P_{Q_j^n})Ax, \tag{1.13}$$

where A^* is the adjoint operator of A.

For obtaining the strong convergence, recently, inspired by the algorithms proposed by Zhao et al. [30] and López et al. [14], He et al. [12] introduced a new relaxed self-adaptive CQ algorithm for solving the MSFP such that the strong convergence is guaranteed by using Halpern's iteration process. Let $u \in H_1$ be fixed, and choose an initial guess $x_1 \in H_1$ arbitrarily. Let $\{x_n\}$ be the sequence generated by the following manner:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(x_n - \tau_n \nabla f_n(x_n)), \quad n \geq 1, \tag{1.14}$$

where f_n is given as in (1.12), $\{\alpha_n\} \subset (0,1)$ and $\tau_n = \rho_n \frac{f_n(x_n)}{\|\nabla f_n(x_n)\|^2}$ with $0 < \rho_n < 4$ for all $n \in \mathbb{N}$. It was proved that if $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\inf_{n\in\mathbb{N}} \rho_n(4-\rho_n) > 0$, then $\{x_n\}$ generated by (1.14) converges strongly to a solution in MSFP.

In this paper, motivated by the previous works, we propose the following inertial relaxed CQ algorithm which combines the inertial technique with the relaxed CQ method:

Algorithm 3.1 Let $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\{\rho_n\} \subset (0,4)$. Let $u \in H_1$ be fixed and take $x_0, x_1 \in H_1$ arbitrarily. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by the following manner:

$$\begin{aligned} x_{n+1} &= \alpha_n u + (1-\alpha_n)(y_n - \tau_n \nabla f_n(y_n)), \\ y_n &= x_n + \beta_n(x_n - x_{n-1}), \quad n \geq 1, \end{aligned} \tag{1.15}$$

where f_n is given as in (1.12) and $\tau_n = \rho_n \frac{f_n(y_n)}{\|\nabla f_n(y_n)\|^2}$ for all $n \in \mathbb{N}$. If $\nabla f_n(y_n) = 0$, then y_n is a solution of MSFP. Here β_n is an extrapolation factor and the inertia is represented by the term $\beta_n(x_n - x_{n-1})$. It is remarkable that the inertial terminology greatly improves the performance of the algorithm and has a nice convergence properties [1, 8, 9, 13, 19, 20] and also [15, 16]. Using the inertial technique and Halpern's idea, we prove its strong convergence of the sequence generated by our proposed scheme. Our algorithm is easily to be implemented since it involves the metric projections onto half-spaces which have exact forms and has no need to know a priori information of the norm of bounded linear operators. Numerical experiments are included to show the effectiveness of the our algorithm. The obtained results mainly extend and improve that of He et al. [12] and also complement the corresponding results of [3, 14, 30].

The rest of this paper is organized as follows: Some basic concepts and lemmas are provided in Section 2. The strong convergence result of this paper is proved in Section 3. Finally, in Section 4, numerical experiments are demonstrated for supporting the main theorem.

2 Preliminaries and lemmas

In this section, we give some preliminaries which will be used in the sequel. Let H be a Hilbert space. Recall that a mapping $T : H \to H$ is said to be nonexpansive if, for all $x, y \in H$,

$$\|Tx - Ty\| \leq \|x - y\| \tag{2.1}$$

$T : H \to H$ is said to be firmly nonexpansive if, for all $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I-T)x - (I-T)y\|^2, \tag{2.2}$$

or equivalently

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2 \tag{2.3}$$

for all $x, y \in H$. It is known that T is firmly nonexpansive if and only if $I - T$ is firmly nonexpansive. We know that the metric projection P_C from H onto a nonempty closed convex subset $C \subset H$ is a typical example of a firmly nonexpansive mapping, which is defined by

$$P_C x := \arg\min_{y \in C} \|x - y\|^2, \quad x \in H. \tag{2.4}$$

It is well known that P_C is characterized by the inequality, for $x \in H$

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \quad \forall y \in C. \tag{2.5}$$

In a real Hilbert space H, we know the following results:

$$\langle x, y \rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2 \tag{2.6}$$

and the subdifferential inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \tag{2.7}$$

for all $x, y \in H$.

Definition 2.1. *Let $f : H \to \mathbb{R}$ be a convex function. The subdifferential of f at x is defined as*

$$\partial f(x) = \{\xi \in H : f(y) \geq f(x) + \langle \xi, y - x \rangle, \quad \forall y \in H\}. \tag{2.8}$$

A function $f : H \to \mathbb{R}$ is said to be weakly lower semi-continuous at x if x_n converges weakly to x implies

$$f(x) \leq \liminf_{n \to \infty} f(x_n). \tag{2.9}$$

Lemma 2.2. [6] *Let $\{C_i\}_{i=1}^t$ and $\{Q_j\}_{j=1}^r$ be closed convex subsets of H_1 and H_2 respectively and $A : H_1 \to H_2$ a bounded linear operator. Let $f(x)$ be the function defined as in (1.7). Then $\nabla f(x)$ is Lipschitz continuous with $L = \sum_{i=1}^t l_i + \|A\|^2 \sum_{j=1}^r \lambda_j$ as the Lipschitz constant.*

Lemma 2.3. [17, 25] *Let $\{a_n\}$ and $\{c_n\}$ are sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \quad n \geq 1, \tag{2.10}$$

where $\{\delta_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a real sequence. Assume $\sum_{n=1}^\infty c_n < \infty$. Then the following results hold:

(i) *If $b_n \leq \delta_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*

(ii) *If $\sum_{n=1}^\infty \delta_n = \infty$ and $\limsup_{n\to\infty} b_n/\delta_n \leq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.4. [18] *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}, \tag{2.11}$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) *$\Gamma(n_0) \leq \Gamma(n_0 + 1) \leq \ldots$ and $\Gamma(n) \to \infty$;*

(ii) *$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.*

3 Strong Convergence Theorem

In this section, we are in position to study the inertial relaxed self-adaptive CQ algorithm in Hilbert spaces for solving MSFP (1.1).

Theorem 3.1. *Let H_1 and H_2 be real Hilbert spaces and let $\{C_i\}_{i=1}^t$ and $\{Q_j\}_{j=1}^r$ be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that $\{\alpha_n\}, \{\beta_n\}$ and $\{\rho_n\}$ satisfy the following assumptions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $\inf_{n\in\mathbb{N}} \rho_n(4-\rho_n) > 0$*;*

(C3) $\{\beta_n\} \subset [0, \beta]$*, where* $\beta \in [0, 1)$ *and* $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0.$

Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $P_S u$.

Proof. Set $z = P_S u$. We note that $I - P_{C_i^n}$, $(i = 1, ..., t)$ and $I - P_{Q_j^n}$, $(j = 1, ..., r)$ are firmly nonexpansive and $\nabla f_n(z) = 0$ for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}
\langle \nabla f_n(y_n), y_n - z\rangle &= \Big\langle \sum_{i=1}^{t} l_i(y_n - P_{C_i^n}(y_n)) + \sum_{j=1}^{r} \lambda_j A^*(I - P_{Q_j^n})Ay_n, y_n - z\Big\rangle \\
&= \sum_{i=1}^{t} l_i\langle (I - P_{C_i^n})y_n, y_n - z\rangle + \sum_{j=1}^{r} \lambda_j \langle (I - P_{Q_j^n})Ay_n, Ay_n - Az\rangle \\
&\geq \sum_{i=1}^{t} l_i \|(I - P_{C_i^n})y_n\|^2 + \sum_{j=1}^{r} \lambda_j \|(I - P_{Q_j^n})Ay_n\|^2 \\
&= 2f_n(y_n). \qquad (3.1)
\end{aligned}$$

So we have

$$\begin{aligned}
\|y_n - \tau_n \nabla f_n(y_n) - z\|^2 &= \|y_n - z\|^2 + \tau_n^2 \|\nabla f_n(y_n)\|^2 - 2\tau_n \langle \nabla f_n(y_n), y_n - z\rangle \\
&\leq \|y_n - z\|^2 + \tau_n^2 \|\nabla f_n(y_n)\|^2 - 4\tau_n f_n(y_n) \\
&= \|y_n - z\|^2 - \rho_n^2 \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2} - 4\rho_n \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2} \\
&= \|y_n - z\|^2 - \rho_n(4 - \rho_n) \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2}. \qquad (3.2)
\end{aligned}$$

Hence we obtain, for each $n \in \mathbb{N}$, since $\rho_n \in (0, 4)$

$$\|y_n - \tau_n \nabla f_n(y_n) - z\| \leq \|y_n - z\|. \qquad (3.3)$$

On the other hand, we also have

$$\begin{aligned}
\|y_n - z\| &= \|x_n - z + \beta_n(x_n - x_{n-1})\| \\
&\leq \|x_n - z\| + \beta_n \|x_n - x_{n-1}\|. \qquad (3.4)
\end{aligned}$$

Combining (3.3) and (3.4), we obtain

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n(u - z) + (1 - \alpha_n)(y_n - \tau_n \nabla f_n(y_n) - z)\| \\
&\leq \alpha_n \|u - z\| + (1 - \alpha_n)\|y_n - z\| \\
&\leq \alpha_n \|u - z\| + (1 - \alpha_n)\|x_n - z\| + (1 - \alpha_n)\beta_n \|x_n - x_{n-1}\|. \qquad (3.5)
\end{aligned}$$

By (C3), we see that $\delta_n = \frac{(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|}{\alpha_n} \to 0$. Hence it is bounded. Put

$$M = \max\big\{\|u - z\|, \sup_{n\geq 1} \delta_n\big\}.$$

So (3.5) becomes

$$\|x_{n+1} - z\| \leq (1-\alpha_n)\|x_n - z\| + \alpha_n M. \tag{3.6}$$

Applying Lemma 2.3 (i), we can conclude that $\{x_n\}$ is bounded and also $\{y_n\}$ is bounded. By Lemma 2.2, we see that

$$\|\nabla f_n(y_n)\| = \|\nabla f_n(y_n) - \nabla f_n(z)\| \leq L\|y_n - z\|, \tag{3.7}$$

where $L = \sum_{i=1}^{t} l_i + \|A\|^2 \sum_{j=1}^{r} \lambda_j$. This shows that $\{\nabla f_n(y_n)\}$ is bounded.

We next compute the following estimation:

$$\begin{aligned} \|y_n - z\|^2 &= \|x_n - z + \beta_n(x_n - x_{n-1})\|^2 \\ &= \|x_n - z\|^2 + 2\beta_n\langle x_n - x_{n-1}, x_n - z\rangle + \beta_n^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.8}$$

Using (2.6), we have

$$\langle x_n - x_{n-1}, x_n - z\rangle = -\frac{1}{2}\|x_{n-1} - z\|^2 + \frac{1}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_n - x_{n-1}\|^2. \tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$\begin{aligned} \|y_n - z\|^2 &= \|x_n - z\|^2 + \beta_n(-\|x_{n-1} - z\|^2 + \|x_n - z\|^2 + \|x_n - x_{n-1}\|^2) + \beta_n^2\|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - z\|^2 + \beta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + 2\beta_n\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.10}$$

Using (2.7) and (3.2), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n(u - z) + (1-\alpha_n)(y_n - \tau_n\nabla f_n(y_n) - z)\|^2 \\ &\leq (1-\alpha_n)\|y_n - \tau_n\nabla f_n(y_n) - z\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)\|y_n - z\|^2 - (1-\alpha_n)\rho_n(4-\rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2} + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we derive

$$\begin{aligned} \|x_{n+1} - z\|^2 \leq\ & (1-\alpha_n)\|x_n - z\|^2 + (1-\alpha_n)\beta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\ & + 2(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|^2 - (1-\alpha_n)\rho_n(4-\rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2} \\ & + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \end{aligned} \tag{3.12}$$

Set $\Gamma_n = \|x_n - z\|^2$ for all $n \in \mathbb{N}$. We note, by (C1) and (C2), that there is a constant σ such that $(1-\alpha_n)\rho_n(4-\rho_n) \geq \sigma > 0$ for all $n \in \mathbb{N}$. So from (3.12) we get

$$\begin{aligned} \Gamma_{n+1} \leq\ & (1-\alpha_n)\Gamma_n + (1-\alpha_n)\beta_n(\Gamma_n - \Gamma_{n-1}) \\ & + 2(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|^2 - \sigma\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2} + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \end{aligned} \tag{3.13}$$

We next consider the following two cases:

Case 1: Suppose that there exists a natural number N such that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq N$. In this case, $\lim_{n\to\infty} \Gamma_n$ exists. From (3.13), we have

$$\sigma \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2} \leq (\Gamma_n - \Gamma_{n+1}) + (1-\alpha_n)\beta_n(\Gamma_n - \Gamma_{n-1}) + 2(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \tag{3.14}$$

It is easy to check that (C3) implies $\beta_n\|x_n - x_{n-1}\| \to 0$ since $\{\alpha_n\}$ is bounded. So, by (C1) and the boundedness of $\{x_n\}$, we have from (3.14)

$$\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2} \to 0 \quad as \quad n \to \infty.$$

Since $\{\|\nabla f_n(y_n)\|\}$ is bounded, it follows that $f_n(y_n) \to 0$ as $n \to \infty$. This shows that

$$\lim_{n\to\infty} \|(I - P_{C_i^n})y_n\| = 0 \quad (i = 1, 2, ..., t) \tag{3.15}$$

and

$$\lim_{n\to\infty} \|(I - P_{Q_j^n})Ay_n\| = 0 \quad (j = 1, 2, ..., r). \tag{3.16}$$

Since ∂q_j $(j = 1, ..., r)$ are bounded on bounded sets, there exists a constant $\mu > 0$ such that $\|\zeta_j^n\| \leq \mu$ $(j = 1, ..., r)$ for all $n \in \mathbb{N}$. From (3.16) and $P_{Q_j^n}(Ay_n) \in Q_j^n$ $(j = 1, ..., r)$, we obtain

$$q_j(Ay_n) \leq \langle \zeta_j^n, Ay_n - P_{Q_j^n}(Ay_n)\rangle \leq \mu\|(I - P_{Q_j^n})Ay_n\| \to 0, \tag{3.17}$$

as $n \to \infty$. Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \rightharpoonup x^*$. Then $Ay_{n_k} \rightharpoonup Ax^*$. Since q_j is weakly lower semi-continuous,

$$q_j(Ax^*) \leq \liminf_{k\to\infty} q_j(Ay_{n_k}) \leq 0. \tag{3.18}$$

Therefore $Ax^* \in Q_j$ $(j = 1, ..., r)$.

We next show that $x^* \in C_i$ $(i = 1, ..., t)$. By the definition of C_i^n $(i = 1, ..., t)$ and (3.15), we see that

$$c_i(y_n) \leq \langle \xi_i^n, y_n - P_{C_i^n}(y_n)\rangle \leq \delta\|y_n - P_{C_i^n}y_n\| \to 0, \tag{3.19}$$

as $n \to \infty$, where δ is a constant such that $\|\xi_i^n\| \leq \delta$ $(i = 1, ..., t)$ for all $n \in \mathbb{N}$. By the weak lower semi-continuity of c_i $(i = 1, ..., t)$ and $y_{n_k} \rightharpoonup x^*$, we have

$$c_i(x^*) \leq \liminf_{k\to\infty} c_i(y_{n_k}) \leq 0. \tag{3.20}$$

Hence $x^* \in C_i$ $(i = 1, ..., t)$ and consequently, $x^* \in S$. From (2.5), it follows that

$$\limsup_{n\to\infty}\langle u - z, y_n - z\rangle = \lim_{k\to\infty}\langle u - z, y_{n_k} - z\rangle = \langle u - z, x^* - z\rangle \leq 0. \tag{3.21}$$

On the other hand, we see that

$$\|y_n - x_n\| = \beta_n \|x_n - x_{n-1}\| \to 0. \tag{3.22}$$

Hence, by (3.21) and (3.22), we obtain

$$\limsup_{n\to\infty} \langle u - z, x_n - z\rangle \leq 0. \tag{3.23}$$

Again from (3.13) we have

$$\begin{aligned}
\Gamma_{n+1} &\leq (1-\alpha_n)\Gamma_n + (1-\alpha_n)\beta_n(\Gamma_n - \Gamma_{n-1}) + 2(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|^2 \\
&\quad + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\
&\leq (1-\alpha_n)\Gamma_n + (1-\alpha_n)\beta_n\|x_n - x_{n-1}\|(\sqrt{\Gamma_n} + \sqrt{\Gamma_{n-1}}) + 2(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|^2 \\
&\quad + 2\alpha_n \langle u - z, x_{n+1} - z\rangle. \qquad (3.24)
\end{aligned}$$

From (3.23) and conditions (C1) and (C3), using Lemma 2.3 (ii), we conclude that $\Gamma_n = \|x_n - z\|^2 \to 0$ and thus $x_n \to z$ as $n \to \infty$.

Case 2 : Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ as in (2.11). Then, by Lemma 2.4, we have $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$. From (3.13), it follows that

$$\begin{aligned}
\Gamma_{\tau(n)+1} &\leq (1-\alpha_\tau(n))\Gamma_{\tau(n)} + (1-\alpha_\tau(n))\beta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|(\sqrt{\Gamma_{\tau(n)}} + \sqrt{\Gamma_{\tau(n)-1}}) \\
&\quad + 2(1-\alpha_\tau(n))\beta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|^2 - \sigma \frac{f^2_{\tau(n)}(y_\tau(n))}{\|\nabla f_{\tau(n)}(y_\tau(n))\|^2} \\
&\quad + 2\alpha_{\tau(n)}\langle u - z, x_{\tau(n)+1} - z\rangle, \qquad (3.25)
\end{aligned}$$

which gives

$$\begin{aligned}
\sigma \frac{f^2_{\tau(n)}(y_\tau(n))}{\|\nabla f_{\tau(n)}(y_\tau(n))\|^2} &\leq (1-\alpha_\tau(n))\beta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|(\sqrt{\Gamma_{\tau(n)}} + \sqrt{\Gamma_{\tau(n)-1}}) \\
&\quad + 2(1-\alpha_\tau(n))\beta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|^2 \\
&\quad + 2\alpha_{\tau(n)}\langle u - z, x_{\tau(n)+1} - z\rangle. \qquad (3.26)
\end{aligned}$$

Using a similar argument as in the proof of Case 1, we can show that

$$\lim_{n\to\infty} \|(I - P_{C_i^{\tau(n)}})y_{\tau(n)}\| = 0,$$

$$\lim_{n\to\infty} \|(I - P_{Q_j^{\tau(n)}})y_{\tau(n)}\| = 0,$$

and

$$\limsup_{n\to\infty} \langle u - z, x_{\tau(n)} - z\rangle \leq 0. \tag{3.27}$$

On the other hand, we see that

$$\begin{aligned}\|x_{\tau(n)+1}-x_{\tau(n)}\| &= \|\alpha_{\tau(n)}(u-x_{\tau(n)})+(1-\alpha_{\tau(n)})(y_{\tau(n)}-\tau_{\tau(n)}\nabla f_{\tau(n)}(y_{\tau(n)})-x_{\tau(n)})\| \\ &\leq \alpha_{\tau(n)}\|u-x_{\tau(n)}\|+(1-\alpha_n)\|y_{\tau(n)}-x_{\tau(n)}\| \\ &\quad + (1-\alpha_{\tau(n)})\tau_{\tau(n)}\|\nabla f_{\tau(n)}(y_{\tau(n)})\| \\ &= \alpha_{\tau(n)}\|u-x_{\tau(n)}\|+(1-\alpha_{\tau(n)})\beta_{\tau_n}\|x_{\tau(n)}-x_{\tau(n)-1}\| \\ &\quad + (1-\alpha_{\tau(n)})\rho_{\tau(n)}\frac{f_{\tau(n)}(y_{\tau(n)})}{\|\nabla f_{\tau(n)}(y_{\tau(n)})\|} \\ &\to 0. \qquad (3.28)\end{aligned}$$

as $n\to\infty$. Using (3.27) and (3.28), we obtain

$$\limsup_{n\to\infty}\langle u-z, x_{\tau(n)+1}-z\rangle \leq 0. \qquad (3.29)$$

Again from (3.25) we see that

$$\begin{aligned}\alpha_{\tau(n)}\Gamma_{\tau(n)} &\leq (1-\alpha_\tau(n))\beta_{\tau(n)}\|x_{\tau(n)}-x_{\tau(n)-1}\|(\sqrt{\Gamma_{\tau(n)}}+\sqrt{\Gamma_{\tau(n)-1}}) \\ &\quad + 2(1-\alpha_\tau(n))\beta_{\tau(n)}\|x_{\tau(n)}-x_{\tau(n)-1}\|^2 \\ &\quad + 2\alpha_{\tau(n)}\langle u-z, x_{\tau(n)+1}-z\rangle, \qquad (3.30)\end{aligned}$$

which gives

$$\begin{aligned}\Gamma_{\tau(n)} &\leq (1-\alpha_\tau(n))\frac{\beta_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)}-x_{\tau(n)-1}\|(\sqrt{\Gamma_{\tau(n)}}+\sqrt{\Gamma_{\tau(n)-1}}) \\ &\quad + 2(1-\alpha_\tau(n))\frac{\beta_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)}-x_{\tau(n)-1}\|^2 \\ &\quad + 2\langle u-z, x_{\tau(n)+1}-z\rangle. \qquad (3.31)\end{aligned}$$

This shows that, by (3.29) and (C3)

$$\limsup_{n\to\infty}\Gamma_{\tau(n)} \leq 0. \qquad (3.32)$$

Thus $\|x_{\tau(n)}-z\|\to 0$. We see that

$$\sqrt{\Gamma_{\tau(n)+1}} = \|x_{\tau(n)+1}-z\| \leq \|x_{\tau(n)+1}-x_{\tau(n)}\|+\|x_{\tau(n)}-z\|\to 0, \qquad (3.33)$$

as $n\to\infty$. By Lemma 2.4, we also have

$$\Gamma_n \leq \Gamma_{\tau(n)+1}\to 0. \qquad (3.34)$$

So we can conclude that $x_n\to z$ as $n\to\infty$. We thus complete the proof. □

Remark 3.2. *We remark here that the conditions (C3) is easily implemented in numerical computation since the valued of $\|x_n-x_{n-1}\|$ is known before choosing β_n. Indeed, the parameter β_n can be chosen such that $0\leq\beta_n\leq\bar{\beta}_n$, where*

$$\bar{\beta}_n = \begin{cases}\min\left\{\frac{\omega_n}{\|x_n-x_{n-1}\|},\beta\right\} & if\ x_n\neq x_{n-1}, \\ \beta & otherwise,\end{cases}$$

where $\{\omega_n\}$ is a positive sequence such that $\omega_n=o(\alpha_n)$.

4 Numerical Experiments

In this section, we provide some numerical examples and illustrate its performance by using Algorithm 3.1. Firstly, numerical results are shown in different choices of the step-size ρ_n with different values u, x_1 and x_2. Secondly, the comparison of convergence rate is made by Example 4.1 to show that our algorithm has a better convergence than that of He et al. [12] defined in (1.14). For this convenience, we denote algorithm (1.14) by Algorithm 3.2.

Example 4.1. [12] *Let $H_1 = H_2 = \mathbb{R}^3$, $r = t = 2$ and $l_1 = l_2 = \lambda_1 = \lambda_2 = \frac{1}{4}$. Define*

$C_1 = \{x = (a, b, c)^T \in \mathbb{R}^3 : a + b^2 + 2c \le 0\}$,

$C_2 = \{x = (a, b, c)^T \in \mathbb{R}^3 : \frac{a^2}{16} + \frac{b^2}{9} + \frac{c^2}{4} - 1 \le 0\}$,

$Q_1 = \{x = (a, b, c)^T \in \mathbb{R}^3 : a^2 + b - c \le 0\}$,

$Q_2 = \{x = (a, b, c)^T \in \mathbb{R}^3 : \frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{9} - 1 \le 0\}$.

and $A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 2 & 0 & 2 \end{pmatrix}$. *Find $x^* \in C_1 \cap C_2$ such that $Ax^* \in Q_1 \cap Q_2$.*

Choose $\alpha_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$ and $\beta = 0.5$. For each $n \in \mathbb{N}$, let $\omega_n = \frac{1}{(n+1)^{1.2}}$ and define $\beta_n = \bar{\beta}_n$ as in Remark 3.2. We now study the effect (in terms of convergence, stability, number of iterations required and the cpu time) of the sequence $\{\rho_n\} \subset (0, 4)$ on the iterative scheme by choosing different ρ_n such that $\inf_n \rho_n(4 - \rho_n) > 0$ in the following cases.

Case 1: $\rho_n = 1$; Case 2: $\rho_n = 2$; Case 3: $\rho_n = 3$; Case 4: $\rho_n = 3.95$.

The stopping criterion is defined by

$$E_n = \frac{1}{2}\sum_{i=1}^{2} \|x_n - P_{C_i^n} x_n\|^2 + \frac{1}{2}\sum_{j=1}^{2} \|Ax_n - P_{Q_j^n} Ax_n\|^2 < 10^{-4}.$$

We choose different choices of u, x_0 and x_1 as

Choice 1: $u = (2, 2, -2)^T$, $x_0 = (1, 1, 5)^T$ and $x_1 = (5, -3, 2)^T$;

Choice 2: $u = (1, 3, -2)^T$, $x_0 = (-4, 3, -2)^T$ and $x_1 = (-5, 2, 1)^T$;

Choice 3: $u = (4, -3, -6)^T$, $x_0 = (7, 5, 1)^T$ and $x_1 = (7, -3, -1)^T$;

Choice 4: $u = (7, -4, -3)^T$, $x_0 = (5.32, 2.33, 7.75)^T$ and $x_1 = (3.23, 3.75, -3.86)^T$.

The numerical experiments, using our Algorithm 3.1, for each case and choice are reported in the following Table 1.

Table 1: Algorithm 3.1 with different cases of ρ_n and different choices of u, x_0 and x_1

		Case 1	Case 2	Case 3	Case 4
Choice 1	No. of Iter.	244	122	81	25
	cpu (Time)	0.05129	0.027395	0.015663	0.00472
Choice 2	No. of Iter.	392	196	131	13
	cpu (Time)	0.090982	0.04594	0.02693	0.002119
Choice 3	No. of Iter.	351	175	105	22
	cpu (Time)	0.099001	0.034915	0.02138	0.00473
Choice 4	No. of Iter.	444	178	88	27
	cpu (Time)	0.108428	0.036239	0.016809	0.005466

The convergence behavior of the error E_n for each choice of u, x_0 and x_1 is shown in Figure 1-4, respectively.

Figure 1: Comparison of the itertions for Choice 1 in Example 4.1

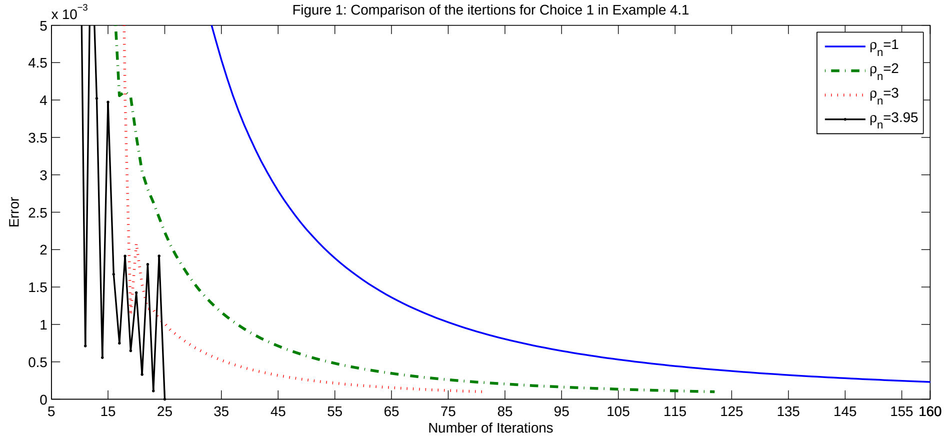


Figure 2: Comparison of the itertions for Choice 2 in Example 4.1

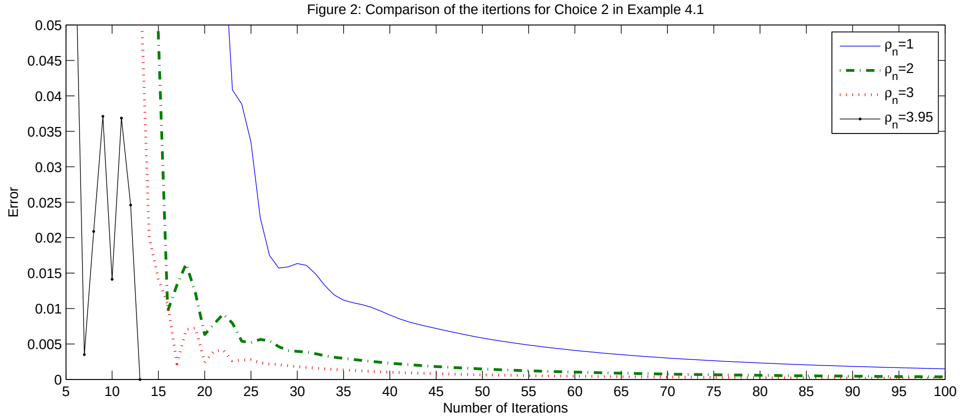


Figure 3: Comparison of the itertions for Choice 3 in Example 4.1

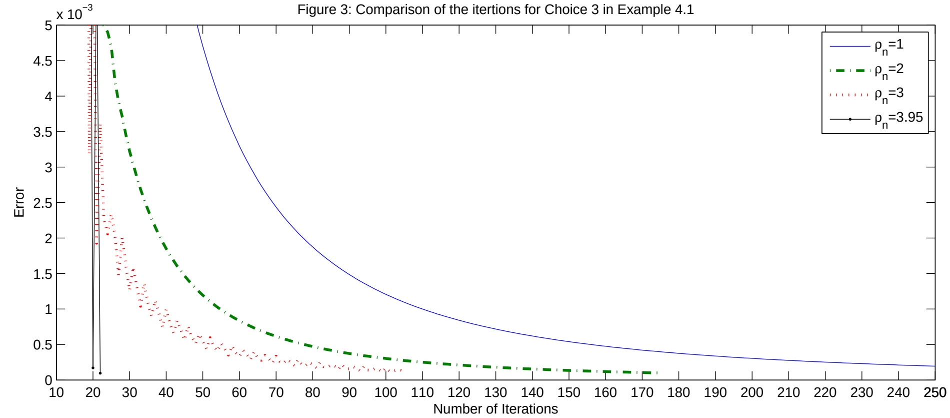
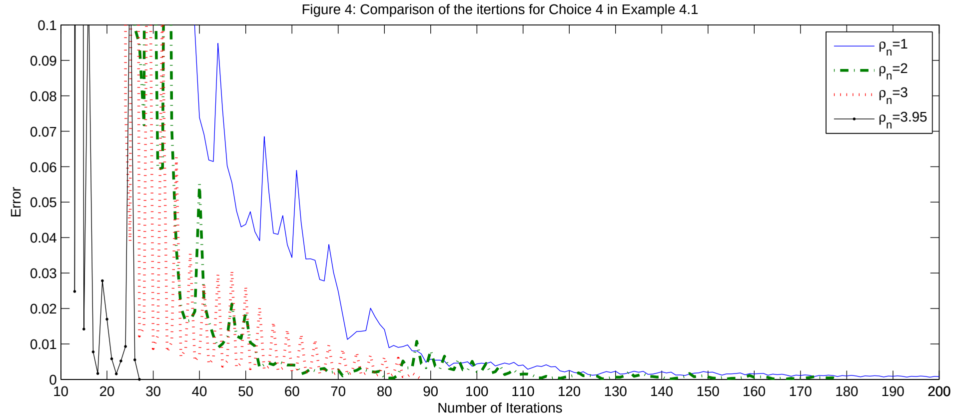


Figure 4: Comparison of the itertions for Choice 4 in Example 4.1



Remark 4.2. *We make the following observations from our numerical experiments in Example 4.1.*

1. *For each different cases and different choices, we see that our algorithm is effective. It appears that Algorithm 3.1 has a good convergence speed and requires small number of iterations in the experiment.*

2. *It is observed that the number of iterations and the cpu run time are significantly decreasing starting from Case 1 to Case 4. However, there is no significant difference in both cpu run time and number of iterations for each choice of x_0 and x_1. So, initial guess does not have any significant effect on the convergence of the algorithm. However, we note that the sequence $\{x_n\}$ converges to a solution in MSFP which is of the form $P_S u$. Since the solution set S is not singleton, so the choice of u effects on the convergence behavior of the algorithm.*

3. *Our conditions appeared in Theorem 3.1 are easily implemented in numerical computations. This is because it needs no estimation on the spectral radius or the largest eigenvalue of $A^T A$ and the restriction of metric projections onto C and Q is relaxed by using those of C_n and Q_n which have specific forms in computation.*

We finally end this section by providing a comparison of convergence of Algorithm 3.1 and Algorithm 3.2. Let $\alpha_n = \frac{1}{n+1}$, $\rho_n = 3.95$ and $\omega_n = \frac{1}{(n+1)^{1.2}}$ for all $n \in \mathbb{N}$. Set $\beta = 0.5$ and $\beta_n = \bar{\beta}_n$ as in Remark 3.2. For points u, x_0 and x_1 randomly, we obtain the following numerical results.

Table 2: Comparison of Algorithm 3.1 and Algorithm 3.2 in Example 4.1

			Algor 3.1	Algor 3.2
Choice 1	$u = (0, 1, 2)^T$	No. of Iter.	21	31
	$x_0 = (-4, -2, 3)^T$	cpu (Time)	0.004364	0.006537
	$x_1 = (-1, 2, 0)^T$			
Choice 2	$u = (-1, 3, 1)^T$	No. of Iter.	22	69
	$x_0 = (-1, 2, 3)^T$	cpu (Time)	0.004626	0.013906
	$x_1 = (-7, -4, -5)^T$			
Choice 3	$u = (3, 1, 3)^T$	No. of Iter.	97	287
	$x_0 = (-5, 1, -4)^T$	cpu (Time)	0.021787	0.074538
	$x_1 = (-5, -2, -3)^T$			
Choice 4	$u = (-1, 3, -3)^T$	No. of Iter.	18	161
	$x_0 = (3.2645, -2.3458, -5.3245)^T$	cpu (Time)	0.003854	0.034188
	$x_1 = (-2.5891, -3.2654, -3.2564)^T$			

The error plotting of E_n of Algorithm 3.1 and Algorithm 3.2 for each choice is shown in Figure 5-8, respectively.

Figure 5: Comparison of Algorithm 3.1 and 3.2 for Choice 1 in Example 4.1

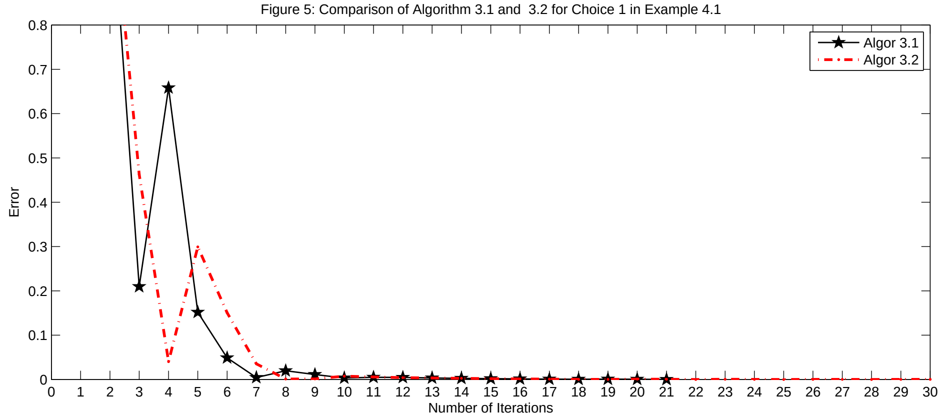


Figure 6: Comparison of Algorithm 3.1 and 3.2 for Choice 2 in Example 4.1

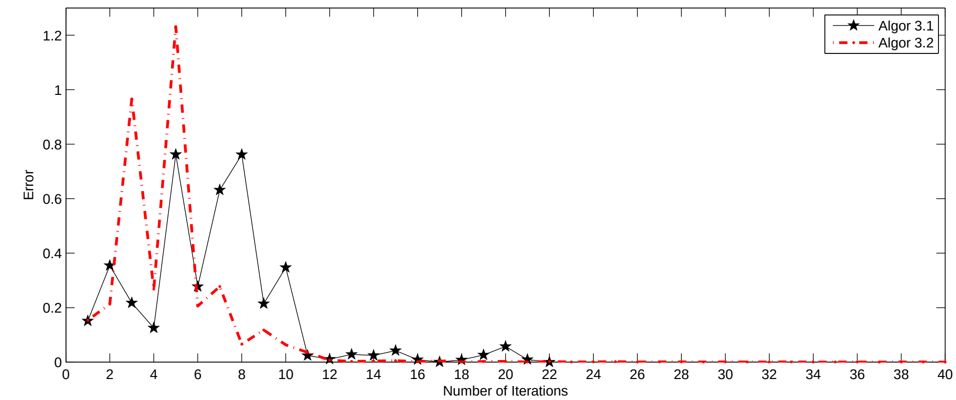


Figure 7: Comparison of Algorithm 3.1 and 3.2 for Choice 3 in Example 4.1

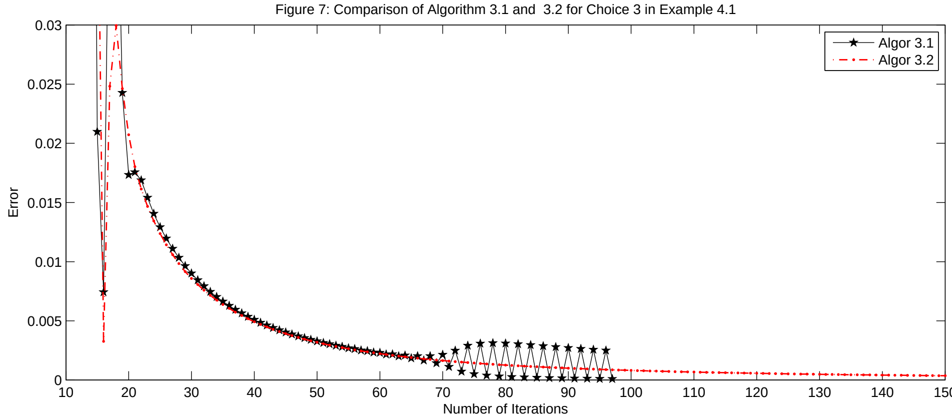
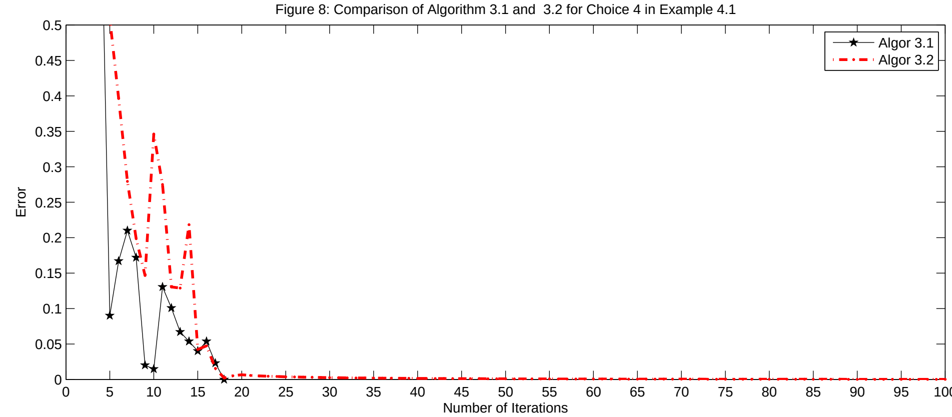


Figure 8: Comparison of Algorithm 3.1 and 3.2 for Choice 4 in Example 4.1



Remark 4.3. *In numerical experiment, it is revealed that the sequence generated by our proposed Algorithm 3.1 involving the inertial technique converges more quickly than by Algorithm 3.2 of He et al. [12] does. This concludes that the inertial term constructed in Algorithm 3.1 improves the speed of convergence for solving the MSFP.*

ACKNOWLEDGMENTS: This research was supported by the Thailand Research Fund and the Commission on Higher Education under Grant MRG5980248.

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