



## รายงานวิจัยฉบับสมบูรณ์

การวิเคราะห์สมบัติของแบริอันในการกระจายร่วมแบบ  
ไฮรัลและส่วนกลับเลขค่อนต้มสี

โดย ดร. ดริศ สามารถ

เมษายน 2561

สัญญาเลขที่ MRG5980255

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สาขาวิชาพิสิกส์ประยุกต์ คณะวิทยาศาสตร์และศิลปศาสตร์  
มหาวิทยาลัยเทคโนโลยีราชมงคลลีสาน

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย  
สกอ.และต้นสังกัดไม่จำเป็นต้องเห็นด้วยเสมอไป)

## กิตติกรรมประกาศ

หัวหน้าโครงการวิจัยขอขอบพระคุณสำนักงานกองทุนสนับสนุนการวิจัย (สกว.) ที่ค่อยสนับสนุนและให้โอกาสหัวหน้าโครงการวิจัยได้การจัดทำโครงการวิจัยนี้มาตลอดสองปีที่ได้รับทุน หัวหน้าโครงการวิจัยขอขอบพระคุณนักวิจัยที่ปรึกษา ศ. ดร. ยุ่ง แยน ที่ค่อยช่วยเหลือสนับสนุนให้คำชี้แนะปรึกษาและกำลังใจตลอดโครงการวิจัย อีกทั้งคณะผู้บริหารและเพื่อนร่วมงานที่มหาวิทยาลัยเทคโนโลยีราชมงคลอีสานที่ช่วยจัดสรรเวลาและโอกาสในการทำงาน นอกจากนี้หัวหน้าโครงการขอขอบพระคุณเจ้าหน้าที่ฝ่ายวิชาการของสกว. ที่ค่อยช่วยเหลือและแก้ปัญหาหลายๆอย่างที่เกิดขึ้นตั้งแต่เริ่มต้นจนเสร็จสิ้นในการดำเนินโครงการวิจัยนี้ ที่สำคัญที่สุดคือคุณแม่ของหัวหน้าโครงการวิจัยที่ค่อยเป็นแรงผลักดันให้มีกำลังกายและใจในการทำวิจัยโดยตลอด รวมทั้งบุคคลอื่นๆที่ไม่ได้เอียนมาม ณ ที่นี้หัวหน้าโครงการวิจัยขอขอบคุณในทุกๆสิ่งทุกอย่าง ที่ช่วยให้โครงการวิจัยนี้สำเร็จลุล่วงทุกประการ

### Abstract

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Project Code : MRG5980255

Project Title : Analysis of baryon properties in combined chiral and  $1/N_c$  expansions

Investigator : Dr. Daris Samart

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Project Period : 2 years

In this project, we studied the  $N^*(1875)(3/2^-)$  resonance with a coupled channel unitary scheme, considering the  $\Delta\pi$  and  $\Sigma^*K$ , with their interaction extracted from SU(3) chiral Lagrangians, and then added two more channels, the  $N^*(1535)\pi$  and  $N\sigma$ , which proceed via triangle diagrams involving the  $\Sigma^*K$  and  $\Delta\pi$  respectively in the intermediate states. The triangle diagram in the  $N^*(1535)\pi$  case develops a singularity at the same energy as the resonance mass of the  $N^*(1875)(3/2^-)$  state.

On the other hand,  $1/N_c$  expansion is very useful tool to study octet-octet baryon potential in SU(3) flavor symmetry. By constructing octet-octet baryon potential in framework of the  $1/N_c$  expansion, we obtain the  $N_c$  scale of coupling constants from chiral Lagrangian for contact term interactions up to next-to-next-to leading order (NNLO). The application of large- $N_c$  sum rules to the hyperon-nucleon (YN) potential at leading order (LO) of the chiral expansion reduces the model parameters to 3 from 5 at the LO of  $1/N_c$  expansion.

Keywords : Chiral SU(3) lagrangian, triangle singularity, coupled channel unitary scheme,  $1/N_c$  expansion, SU(3) flavor symmetry, baryon-baryon interactions

## บทคัดย่อ

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รหัสโครงการ : MRG5980255

ชื่อโครงการ : การวิเคราะห์สมบัติของแบริออนในการกระจายร่วมแบบไครัลและส่วนกลับเลขค่าอนตัมสี

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ระยะเวลาโครงการ : 2 ปี

ในโครงการวิจัยนี้ เราได้ศึกษาการสั่นพ้องของอนุภาค  $N^*(1875)(3/2^-)$  ด้วยวิธีการยูนิทารีปฏิกิริยาคู่คุบโดย พิจารณาคุณตรกิริยาของ  $\Delta\pi$  และ  $\Sigma^*K$  ที่หาได้จากไครัล  $SU(3)$  ลากรางเจียน และจากนั้นเราได้เพิ่มการ พิจารณาการคู่คุบของอันตรกิริยาคู่คุบของ  $N^*(1535)\pi$  และ  $N\sigma$  เข้าไปด้วยซึ่งจะเกี่ยวพันผ่านแผนภาพแบบสามเหลี่ยมที่มี  $\Sigma^*K$  และ  $\Delta\pi$  เป็นสถานะระหว่างกลาง แผนภาพแบบสามเหลี่ยมในกรณีของ  $N^*(1535)\pi$  สามารถทำให้เกิดภาวะเอกฐานได้ที่ตำแหน่งพลังงานเดียวกันมวลสั่นพ้องของสถานะ  $N^*(1875)(3/2^-)$

ในขณะเดียวกัน การกระจายแบบ  $1/Nc$  เป็นวิธีการที่เป็นประโยชน์ในการศึกษาคักย์ระหว่างอคเตท-อคเตท แบริออนในสมมاثรเฟลเวอร์  $SU(3)$  จากการสร้างคักย์อคเตท-อคเตทแบริออนด้วยวิธีการกระจาย  $1/Nc$  เราสามารถทราบค่ามาตรา  $Nc$  ของค่าคงที่คุ่คุบของอันตรกิริยาตอนแทคจากไครัล  $SU(3)$  ลากรางเจียนจนถึง อันดับถัดจาก-ถัดจากแรกเริ่ม การประยุกต์ของขั้นรุ่ลของ  $Nc$  ขนาดใหญ่กับศักย์ไฮเปอรอน-นิวเคลียนที่อันดับ แรกเริ่มของการกระจายแบบไครัลสามารถลดจำนวนพารามิเตอร์ของแบบจำลองได้เหลือ 3 จาก 5 ตัว

Keywords : ไครัล  $SU(3)$  ลากรางเจียน, ภาวะเอกฐานสามเหลี่ยม, วิธีการยูนิทารีปฏิกิริยาคู่คุบ, การกระจายส่วนกลับเลขค่าอนตัมสี, สมมاثรเฟลเวอร์  $SU(3)$ , อันตรกิริยาของแบริออน-แบริออน

## วัตถุประสงค์

- เพื่อศึกษาการเกิดการสั่นพ้องของอนุภาค  $N^*(1875)$  ด้วยวิธีการยูนิฟารีคู่ควบจากแอมพลิจูดของแผนภาพสามเหลี่ยม
- เพื่อใช้วิธีการการกระจายส่วนกลับเลขค่าอนตัมสีในทฤษฎีคิวชีดีในศักย์ของแบริอันอคเตทที่ได้จากโครงสร้าง SU(3) ลักษณะเจียน

## บทนำ

ทฤษฎีคุณตัมของรังคพลศาสตร์ (Quantum Chromo-Dynamics) หรือเรียกโดยย่อว่าคิวชีดี (QCD) เป็นทฤษฎีพื้นฐานยุคใหม่ของแรงนิวเคลียร์อย่างเข้ม โดยในทฤษฎีดังกล่าวมีอนุภาคมูลฐานที่เรียกว่า ควาร์ก และมีพาราของแรงที่เรียกว่า กลูอน ทั้งนี้เมื่อรวมคิวชีดีกับทฤษฎีสามคุณตัมของแรงนิวเคลียร์อย่างอ่อนและแรงแม่เหล็กไฟฟ้าได้ทำให้เกิดแบบจำลองมาตรฐานของอนุภาคขึ้นเพื่อใช้อธิบายลักษณะและอันตรกิริยาของอนุภาคมูลฐานทั้งหมดในเอกสาร ทฤษฎีคิวชีดีประสบความสำเร็จอย่างยิ่งเมื่อใช้อธิบายรูปแบบของแรงนิวเคลียร์อย่างเข้มที่ระดับพลังงานสูงๆ ซึ่งค่าความแรงของอันตรกิริยานั้นในทฤษฎีคิวชีดีมีค่าน้อยๆ จึงทำให้สามารถประยุกต์ทฤษฎีการก่อกรณีในสามคุณตัมในการคำนวณและทำนายผลการทดลองต่างๆ ได้อย่างแม่นยำ จนทำให้รางวัลโนเบลสาขาฟิสิกส์ปี 2006 ได้มอบรางวัลให้แก่นักฟิสิกส์ผู้ทำนายผลของทฤษฎีคิวชีดีที่ระดับพลังงานสูงได้ แต่อย่างไรก็ตามเนื่องค่าความแรงของอันตรกิริยานั้นในทฤษฎีคิวชีดีมีค่ามาก ซึ่งทำให้เกิดปัญหาขึ้นนั่นคือฟิสิกส์ไม่สามารถใช้รูปแบบการคำนวณแบบเดียวกันกับทฤษฎีคิวชีดีที่พลังงานสูงๆ ได้ นอกจากนั้นแล้วปรากฏการณ์ต่างๆ ของแรงนิวเคลียร์อย่างเข้มยังไม่สามารถใช้ทฤษฎีคิวชีดีอธิบายได้โดยตรง ด้วยเหตุนี้จึงเกิดคำถามที่ตามมาดังต่อไปนี้ เราจะสามารถใช้คิวชีดีที่พลังงานต่ำอธิบายปรากฏการณ์ของแรงนิวเคลียร์อย่างเข้มที่มีอยู่อย่างมากมายได้อย่างไร เราสามารถทำความเข้าใจการกำหนดและการกักกันในคิวชีดีโดยใช้แฮดรอนแทนที่ควาร์กและกลูอนได้อย่างเป็นระบบหรือไม่ และตัวทฤษฎีคิวชีดีสามารถทำนายโครงสร้างการเกิดแฮดรอนที่แปลงประخلافได้หรือไม่ เป็นต้น

ก่อนปี ค.ศ. 2012 ฐานข้อมูล Particle Data Group (PDG) ซึ่งเป็นแหล่งรวมข้อมูลทั้งหมดของอนุภาคมูลฐานได้รายงานการค้นพบอนุภาคประหลาดที่มีลักษณะคล้ายกับว่ามีอนุภาคควาร์กห้าก้อนที่เรียกว่าเพนตควาร์ก (Penta-quark) ที่เรียกว่า  $N^*(1875)$  ที่มีแพร็ตี้เป็นคี่และมีเลขคุณตัมสปิน  $3/2$  และมีมวลอยู่ที่ประมาณ  $1800$  MeV เมื่อเร็วๆ นี้ PDG ได้มีการปรับปรุงชุดข้อมูลที่เป็นปัจจุบันของอนุภาคดังกล่าวและพบว่า  $N^*(1875)$  มีมวลอยู่ที่ช่วง  $1875 - 1920$  MeV และมีช่วงอายุของการสลายตัวที่  $250 \pm 70$  MeV ซึ่งทางของการสลายตัวของอนุภาคนี้ได้แก่  $N\omega$  (15-25%),  $\Delta\pi$  (10-35%) และ  $N\sigma$  (30-60%) มีสิ่งที่น่าสนใจเกี่ยวกับสำหรับการค้นพบอนุภาค  $N^*(1875)$  จากการสังเกตการเกิดกำหนดซึ่งจะมีจุดยอดอยู่ที่  $1875$  MeV จากการศึกษาอันตรกิริยาของซูโดสเกลาร์เมซอน-แบริอันพบว่าในกรณีที่เลขคุณตัม  $S = 0$  และ  $I = 1/2$  โดยใช้การคำนวณแบบคู่คุบอันตรกิริยา  $\Delta\pi$  และ  $\Sigma^*K$  สามารถทำให้เกิดการกำหนดของอนุภาค  $N^*(1875)$  อย่างไรก็ได้สมบัติโครงสร้างและองค์ประกอบของอนุภาค  $N^*(1875)$  ยังไม่สามารถยืนยันได้อย่างชัดเจนว่าเป็นสถานะควาร์กห้าก้อน, เป็นสถานะไม่เลกูลของเมซอน (แฮดรอนที่ประกอบด้วยควาร์กสองก้อน) กับแบริอัน (แฮดรอนที่ประกอบด้วยควาร์กสามก้อน) หรือ เกิดจากการผลทางจลนศาสตร์ของการกระจายเงินกันแน่ ด้วยเหตุนี้การศึกษารูปแบบการสลายตัวของ  $N^*(1875)$  ทั้งไปเป็นเมซอนและแบริอันชนิดอื่นๆ จะช่วยให้เราทราบข้อมูลเพิ่มเติมเกี่ยวกับธรรมชาติของอนุภาคดังกล่าวได้ นอกจากนี้ยังเป็นช่วยหาค่าตอบที่หรือค่าอธิบายส่วนเติมเต็มของแบบจำลองมาตรฐานฟิสิกส์อนุภาคในส่วน

## ของแรงนิวเคลียร์อย่างเข้มได้อีกทางหนึ่ง

อันตรกิริยาระหว่างไฮเปอรอนกับนิวเคลียอนหรือไฮเปอรอนกับไฮเปอรอนยังไม่ได้รับการศึกษาอย่างกว้างขวางและลึกซึ้งเหมือนกับอันตรกิริยาระหว่างนิวเคลียอนกับนิวเคลียอนทั้งเชิงทฤษฎีและการทดลองทั้งที่อันตรกิริยาหรือแรงของไฮเปอรอนกับนิวเคลียอนมีความสำคัญในการทำความเข้าใจเกี่ยวกับสารที่เรียกว่าควาร์กกลุ่อนพลาสมาชีที่เป็นสถานะของสารที่อยู่ในเอกภพระยะแรกเริ่มหรือแกนกลางของดาวนิวตรอน นอกจากนี้ยังมีส่วนในการทำความเข้าใจเกี่ยวกับแผนภาพสภาวะสารคิวชีดีอีกทางหนึ่งอีกด้วย ทฤษฎีสนามยังผลไครัลเป็นวิธีการที่ได้รับความนิยมและประสบความสำเร็จในการศึกษาอันตรกิริยาระหว่างนิวเคลียอนนิวเคลียอนเป็นอย่างยิ่งและได้ถูกนำมาใช้ในการศึกษาอันตรกิริยาระหว่างไฮเปอรอนกับนิวเคลียอนหรือไฮเปอรอนกับไฮเปอรอนอีกด้วย อย่างไรก็ได้เนื่องจากข้อมูลทางการทดลองที่มีอยู่น้อยมาก ไม่สามารถทำให้การศึกษาอันตรกิริยาดังกล่าวมีความก้าวหน้ามากเนื่องจากทฤษฎีสนามยังผลไครัลนั่นมีพารามิเตอร์อิสระที่อยู่เป็นจำนวนมากจึงเป็นการยากในการศึกษา ดังนั้นการใช้การกระจายส่วนกลับเลขคุณต้มสีในคิวชีดีสามารถช่วยให้เราสามารถลดจำนวนพารามิเตอร์อิสระลงได้ซึ่งจะเป็นประโยชน์ต่อการใช้ทฤษฎีสนามยังผลไครัลในการศึกษาอันตรกิริยาระหว่างไฮเปอรอนกับนิวเคลียอนหรือไฮเปอรอนกับไฮเปอรอนได้ดียิ่งขึ้น

### ระเบียบวิธีวิจัย

ในโครงการวิจัยนี้ได้แบ่งการดำเนินงานออกเป็น 2 ส่วนหลัก ดังนี้คือ

#### 1. การคำนวณด้วยวิธีการยูนิทารีปฏิกริยาคู่ควบจากแผนภาพไฟยน์แมนสามเหลี่ยม

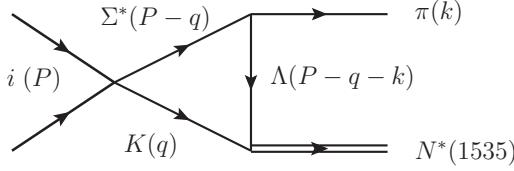
##### 1.1. วิธีการยูนิทารีปฏิกริยาคู่ควบ

วิธีการยูนิทารีปฏิกริยาคู่ควบเป็นรูปแบบหนึ่งในการคำนวณปฏิกริยาของกราฟิกที่เกี่ยวกับการกระเจิงกันของอนุภาคเพื่อใช้อธิบายการเกิดการสั่นพ้องของอนุภาคเมzonหรือแบริโอนที่เกิดขึ้นในห้องปฏิบัติการการทดลองของการชั่นกันของอนุภาค แนวคิดหลักของวิธีการยูนิทารีปฏิกริยาคู่ควบนี้มีอยู่ว่าการสั่นพ้องของเมzonหรือแบริโอนเกิดจากผลของการคู่ควบคู่กันระหว่างอันตรกิริยาของกราฟิกที่กระเจิงกันของอนุภาคหลายๆคู่ปฏิกริยาที่มีเลขคุณต้มที่เท่ากันในแต่ละกรณีที่เราพิจารณาการเกิดการสั่นพ้องของอนุภาคนั้นโดยรูปแบบและข้อมูลอันตรกิริยาของกราฟิกระหว่างอนุภาคสามารถหาได้จากไครัลลาราเงินในทฤษฎีสนามยังผลไครัล [1] และผนวกรวมกับการพิจารณาการควบคุมของหลักการยูนิทารีจากกลศาสตร์คุณต้มในแอมเพลจุตการกระเจิงจึงทำให้เกิดรูปแบบการคำนวณที่เรียกว่าวิธีการยูนิทารีปฏิกริยาคู่ควบนั้นเอง [2] ซึ่งวิธีการดังกล่าวกล่าวประสบความสำเร็จอย่างมากในการอธิบายรูปแบบการเกิดการสั่นพ้องได้เกือบทั้งหมดของ การสั่นพ้องของอนุภาคเมazon(เมazonและแบริโอน)ที่ตรวจวัดได้ในห้องปฏิบัติการ (ดูบทความบหวนวรรณกรรมของที่มาความสำคัญ หลักการ รูปแบบการคำนวณและการนำไปใช้ได้ที่อ้างอิง [3]) โดยข้อมูลต่างๆของการสั่นพ้องของอนุภาคเมazonได้ถูกบรรจุไว้ในฐานข้อมูล Particle Data Group หรือ PDG [4]

ในหัวข้อนี้เราจะใช้วิธีการคำนวณด้วยวิธีการยูนิทารีปฏิกริยาคู่ควบตาม [5] และในส่วนของเลขคุณต้ม  $S = 0, I = \frac{1}{2}$  ก็จะมีคู่ปฏิกริยาอนุภาค  $\Delta\pi, \Sigma^*K$  สำหรับอันตรกิริยาของ  $s$ -wave ที่ทำให้เกิดสถานะที่มีเลขคุณต้มเป็น  $J^P = 3/2^-$  และอันตรกิริยาสามารถเขียนได้ดังนี้

$$V_{ij} = -C_{ij} \frac{1}{4f^2} (k^0 + k'^0), \quad (1)$$

	$\Delta\pi$	$\Sigma^*K$
$\Delta\pi$	5	2
$\Sigma^*K$	2	2

Table 1: The  $C_{ij}$  coefficients of the Eq. (1).Figure 1: แผนภาพสามเหลี่ยมสำหรับสถานะส่งผ่านของ  $i(\Delta\pi, \Sigma^*K) \rightarrow N^*(1535)\pi$ 

เมื่อ  $k^0, k'^0$  คือพลังงานของเม่อนเริ่มต้นและเม่อนสุดท้ายตามลำดับและ  $f$  คือค่าคงที่การสลายตัวของเม่อนส่วนสัมประสิทธิ์  $C_{ij}$  ได้ถูกแสดงในตารางที่ 1 เมทริกซ์การกระเจิงสามารถหาได้จากสมการ Bethe-Salpeter เมทริกซ์การกระเจิงที่อยู่ในรูปเมทริกซ์ดังนี้

$$T = [1 - VG]^{-1}V, \quad (2)$$

โดยที่  $G$  คือฟังก์ชันลูปของคู่เม่อนเบริอันซึ่งสามารถนิยามได้ดังนี้

$$G = \int \frac{d^4q}{(2\pi)^4} \frac{M}{E(\vec{q})} \frac{1}{k^0 + p^0 - q^0 - E(\vec{q}) + i\epsilon} \frac{1}{q^2 - m^2 + i\epsilon} \quad (3)$$

เมื่อ  $M$  และ  $m$  คือมวลของเบริอันและเม่อนตามลำดับ

### 1.2. ภาวะเอกฐานสามเหลี่ยม (Triangle singularity)

ภาวะเอกฐานสามเหลี่ยมเป็นปรากฏการณ์ที่เกิดขึ้นการที่ตัวหารที่อยู่ในรูปของผลิตผลของการกระเจิงของอนุภาคสามตัวที่มีค่าเป็นศูนย์ซึ่งจะทำให้เราสามารถได้รับผลการสั่นพอง [6, 7] เนื่องไปในการเกิดภาวะเอกฐานสามเหลี่ยมและผลิตผลของแผนภาพสามเหลี่ยมสามารถเขียนได้ดังนี้ [8]

$$q_{on+} - q_{a-} = 0, \quad (4)$$

เมื่อ  $q_{on+}$  และ  $q_{a-}$  คือโมเมนต์ที่ถูกนิยามในอ้างอิง [8] ซึ่งเป็นการแสดงในรูปของพลังงานในกรอบอ้างอิงศูนย์กลาง  $\sqrt{s}$  ที่ทำให้ภาวะเอกฐานสามเหลี่ยมปรากฏขึ้นในรูปของผลิตผล เมื่อเราตรวจสอบสมการ 4 สำหรับมวลของ  $N^*(1535)$  ที่มากกว่าผลรวมของ  $m_\Lambda + m_K$  ที่ระดับมวล 1615 MeV ซึ่งอยู่ในช่วงของค่ามวล  $N^*(1535)$  เมื่อพิจารณาค่าคงที่ความกว้างของการสลายตัว 150 MeV และสมการ 4 ให้ผลเฉลยค่ามวลที่ 1878 MeV ด้วยเหตุนี้เราจึงพิจารณาแผนภาพใหม่แทนดังรูปที่ 1. รูปแสดงผลิตผลของแผนภาพสามเหลี่ยมสามารถเขียนได้ในรูป

$$\begin{aligned} -it_{\Delta\pi, \pi N^*} &= V_{\Delta\pi, \Sigma^* K}^{(I=1/2)} g_{\Sigma^*, \pi\Lambda} g_{N^*, K\Lambda} (\vec{S} \cdot \vec{k}) 2M_\Lambda 2M_{\Sigma^*} \int \frac{d^4q}{(2\pi)^4} \frac{i}{(P - q)^2 - M_{\Sigma^*}^2 + i\epsilon} \\ &\quad \times \frac{i}{(P - q - k)^2 - M_\Lambda^2 + i\epsilon} \frac{i}{q^2 - m_K^2 + i\epsilon} \\ &\equiv V_{\Delta\pi, \Sigma^* K}^{(I=1/2)} g_{\Sigma^*, \pi\Lambda} g_{N^*, K\Lambda} (\vec{S} \cdot \vec{k}) 2M_\Lambda 2M_{\Sigma^*} t_T, \end{aligned} \quad (5)$$

เมื่อเราทำการอินทิเกรตทั่วทั้งตัวแปร  $q^0$  เราจะได้ว่า [8]

$$\begin{aligned} t_T &= \int \frac{d^3q}{(2\pi)^3} \frac{1}{8\omega_K E_{\Sigma^*} E_\Lambda} \frac{1}{k^0 - E_\Lambda - E_{\Sigma^*}} \\ &\quad \times \frac{1}{P^0 + \omega_K + E_\Lambda - k^0} \frac{1}{P^0 - \omega_K - E_\Lambda - k^0 + i\epsilon} \frac{1}{P^0 - E_{\Sigma^*} - \omega_K + i\epsilon} \\ &\quad \times \{2P^0\omega_K + 2k^0E_\Lambda - 2(\omega_K + E_\Lambda)(\omega_K + E_\Lambda + E_{\Sigma^*})\}, \end{aligned} \quad (6)$$

where

$$\begin{aligned} \omega_K &= \sqrt{m_K^2 + \vec{q}^2}, & E_{\Sigma^*} &= \sqrt{M_{\Sigma^*}^2 + \vec{q}^2} + \frac{i\Gamma_{\Sigma^*}}{2}, & E_\Lambda &= \sqrt{M_\Lambda^2 + (\vec{q} + \vec{k})^2}, \\ k^0 &= \frac{s + m_\pi^2 - M_{N^*}^2}{2\sqrt{s}}, & |\vec{k}| &= \frac{\lambda^{\frac{1}{2}}(s, m_\pi^2, M_{N^*}^2)}{2\sqrt{s}}, \\ \lambda(x, y, z) &= x^2 + y^2 + z^2 - 2xy - 2yz - 2xz \end{aligned} \quad (7)$$

นอกจากนี้เรายังได้เพิ่มการพิจารณาผลจากภาวะเอกฐานสามเหลี่ยมของคู่ปฏิกิริยาอนุภาค  $N\sigma$  ซึ่งสามารถแสดงได้ดังรูปที่ 2

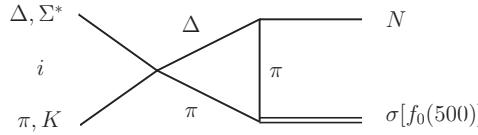


Figure 2: แผนภาพสามเหลี่ยมสำหรับสถานะส่งผ่านของ  $i(\Delta\pi, \Sigma^*K) \rightarrow N\sigma$ .

## 2. การวิเคราะห์คักย์อุดเตหเบริอนสองก้อนโดยใช้วิธีการการกระจายส่วนกลับเลขค่อนต้มสีในทฤษฎีคิวชีดี

การกระจายส่วนกลับพารามิเตอร์จำนวนของสีในทฤษฎีคิวชีดี คือ วิธีการหนึ่งที่ใช้ในการศึกษาผลของคิวชีดีในแอดรอนที่ระดับพลังงานงานต่ำๆ โดยการพิจารณาให้ค่าคงที่คู่คุบของอัตราการรั่วของควาร์กและกลุ่มอน ( $g_s$ ) ให้อยู่ในมาตรฐานประจุสี ( $N_c$ ) กล่าวคือ  $g_s \rightarrow g_s/\sqrt{N_c}$  [9] ซึ่งจากการทดลองที่ดีที่สุดที่บ่งชี้ว่า  $N_c = 3$  ให้กลไกเป็นจำนวนขนาดใหญ่ได้ ดังนั้นโครงสร้างภายในของนิวเคลียส หรือเบริอนชนิดอื่นๆ สามารถทำความเข้าใจได้ง่ายขึ้น เมื่อมีจำนวนของสีเป็น  $N_c$  ซึ่งมีค่ามากๆ [10] เนื่องจากค่า  $g_s/\sqrt{N_c}$  มีค่าน้อยจึงทำให้เราสามารถประยุกต์ทฤษฎีคิวชีดีก่อการในส่วนควบคุมต้มในการคำนวณได้แม้จะอยู่ในระดับพลังงานน้อยๆ ตาม นอกจากนี้การกระจายจำนวนสีแบบ  $1/N_c$  [11] สามารถพิสูจน์ให้เห็นได้ว่ามันมีสมมาตรเทียบเท่ากันกับแบบจำลองควาร์กที่ไม่เป็นสัมพัทธภาพซึ่งเป็นแบบจำลองโครงสร้างของแอดรอนเชิงประภูมิการณ์ที่ไม่ได้มีความเชื่อมโยงกับคิวชีดีโดยตรง [12]

ผลจากการวิเคราะห์การประมาณจำนวนของสีขนาดใหญ่ทำให้เราสามารถทำการกระจายส่วนประกอบแม่ทริกซ์ของเบริอนให้อยู่ในรูปการกระจาย  $1/N_c$  ได้ ซึ่งความสามารถเขียนรูปแบบทั่วไปของวิธีการกระจายส่วนกลับพารามิเตอร์จำนวนของสีในทฤษฎีคิวชีดีได้ดังนี้ [13, 14, 15, 16]

$$\langle \bar{p}, \bar{\chi} | C_{QCD} | p, \chi \rangle = \sum_r c_r(\bar{p}, p) (\bar{\chi} | \mathcal{O}_{\text{eff}}^{(r)} | \chi). \quad (8)$$

โดยที่  $|p, \chi\rangle$  คือสถานะจริงของเบริอนในคิวชีดี และ  $C_{QCD}$  คือฟังก์ชันสหสมพันธ์ของตัวดำเนินการคิวชีดีซึ่งสามารถคำนวณได้โดยตรงจากไครลัล  $SU(3)$  ลาการางเจียน ข้อมูลที่ซับซ้อนจากคิวชีดี

สามารถเขียนให้อยู่ในฟังก์ชันไม่ทราบค่า  $c_r(\bar{p}, p)$  และ  $|\chi\rangle$  คือสถานะยังผลที่แสดงสมบัติของสpin และ เฟลเวอร์เท่านั้น ตัวดำเนินการยังผล  $\mathcal{O}_{\text{eff}}^{(r)}$  สามารถเขียนให้อยู่ในรูปผลคูณดังนี้  $(J)^l (T)^m (G)^n$  เมื่อ  $l + m + n = r$  โดยที่ตัวดำเนินการ  $J$ ,  $T$  และ  $G$  คือ ตัวดำเนินการ สpin เฟลเวอร์ และ สpin-เฟลเวอร์ ตามลำดับ [16] มาตร  $N_c$  ของตัวดำเนินการดังกล่าวมีประโยชน์เป็นอย่างยิ่งในการกระจายส่วนประกอบ เมทริกซ์ของแบริอันในรูปกำลังของ  $1/N_c$  ซึ่งสามารถสรุปได้ดังนี้ [15]:

$$J^i \sim \frac{1}{N_c}, \quad T^a \sim N_c^0, \quad G^{ia} \sim N_c^0. \quad (9)$$

ในหัวข้อนี้เราจะศึกษาศักย์ของแบริอันแบริอันโดยที่ส่วนประกอบเมทริกซ์ของศักย์ในระบบแบริอันสองก้อนสามารถนิยามได้ดังนี้ [17, 18]

$$V(\mathbf{p}_-, \mathbf{p}_+) = \langle (\mathbf{p}'_1, C), (\mathbf{p}'_2, D) | H | (\mathbf{p}_1, A), (\mathbf{p}_2, B) \rangle. \quad (10)$$

ในที่นี้  $A, \dots, D$  แสดงแทนองค์ประกอบของสpin และ เฟลเวอร์ของแบริอันแต่ละตัว ส่วนตัวแบริอัน ( $\mathbf{p}'_i$ ) ไม่ เมนตัมของแบริอันตัวที่  $i = 1, 2$  ก่อน(หลัง)การชน เราได้นิยามตัวแปรโมเมนตัมที่เหมาะสมใน การศึกษาการชนของแบริอันสองก้อนได้ดังนี้

$$\mathbf{p}_\pm = \mathbf{p}' \pm \mathbf{p}, \quad (11)$$

โดยที่

$$\mathbf{p} = \frac{1}{2} (\mathbf{p}_1 - \mathbf{p}_2), \quad \mathbf{p}' = \frac{1}{2} (\mathbf{p}'_1 - \mathbf{p}'_2). \quad (12)$$

และมีเงื่อนไข  $\mathbf{p}_+ \cdot \mathbf{p}_- = 0$  ขนาดโมเมนตัมในการศึกษานี้จะอยู่ในระดับ  $p \sim \Lambda_{QCD}$

แม้มิล็อตเนียนสำหรับแรงนิวเคลียร์ในรูปแบบการกระจายของฮาร์ทีที่ขึ้นกัดของจำนวนของสีมีขนาดใหญ่สามารถเขียนให้อยู่ในรูป [16, 18]

$$H = N_c \sum_{s,t,u} v_{stu} \left( \frac{J}{N_c} \right)^s \left( \frac{T}{N_c} \right)^t \left( \frac{G}{N_c} \right)^u, \quad (13)$$

โดยที่ ฟังก์ชันสัมประสิทธิ์  $v_{stu}$  เป็นฟังก์ชันของโมเมนตัม  $\mathbf{p}_\pm$  ส่วนตัวดำเนินการยังผลในวิธีการกระจายวนกลับพารามิเตอร์จำนวนของสีในทฤษฎีคิวชีดี  $J$  (สpin)  $T$  (เฟลเวอร์) และ  $G$  (สpin-เฟลเวอร์) ได้ถูกนิยามได้ดังนี้

$$J^i = q^\dagger \frac{\sigma^i}{2} q, \quad T^a = q^\dagger \frac{\lambda^a}{2} q, \quad G^{ia} = q^\dagger \frac{\sigma^i \lambda^a}{4} q, \quad (14)$$

เมื่อ  $q$  และ  $q^\dagger$  คือสนามควาร์กที่อยู่ขึ้นกัดของจำนวนพารามิเตอร์ในขนาดใหญ่ นอกจานี้มาตร  $N_c$  ของ การกระจายของตัวดำเนินการดังกล่าวบนสถานะของแบริอันแต่ละตัวมีค่ามาตรดังนี้ [18]

$$\langle N' | J^i | N \rangle \sim 1, \quad \langle N' | T^a | N \rangle \sim N_c, \quad \langle N' | G^{ia} | N \rangle \sim N_c. \quad (15)$$

โดยที่มาตร  $N_c$  ของตัวดำเนินการเหล่านี้มีความสำคัญในการสร้างศักย์แบริอันในแต่ละอันดับ  $N_c$

นอกจากนี้แล้วในการวิเคราะห์ด้วยวิธีการ  $N_c$  ขนาดใหญ่ ตัวแปรโมเมนตัม  $\mathbf{p}_\pm$  และมวลแบริอัน  $m_B$  นีมาตร  $N_c$  ได้แก่

$$\mathbf{p}_+ \sim 1/N_c, \quad \mathbf{p}_- \sim 1, \quad m_B \sim N_c \quad (16)$$

ในขั้นต่อไปเราจะสร้างแม้มิล็อตเนียน  $H$  ของศักย์ระบบแบริอันสองตัวจากตัวดำเนินการในสมการ (14) และโมเมนตัม  $\mathbf{p}_\pm$  ที่ไม่เปลี่ยนภายในการหมุนและการสลับกันของอนุภาค ( $P_{12}$ ) ในแต่ละอันดับของ

$\mathbf{p}_+$	$\mathbf{p}_-$	$\vec{\sigma}_1 \cdot \vec{\sigma}_2$	$(\vec{\sigma}_1 + \vec{\sigma}_2)$	$(\vec{\sigma}_1 - \vec{\sigma}_2)$	$\vec{\sigma}_1 \times \vec{\sigma}_2$	$[\sigma_1 \sigma_2]_2^{ij}$
T	-	+	+	-	-	+
P	-	-	+	+	+	+
$P_{12}$	-	-	+	+	-	+

Table 2: สมบัติการแปลงของโมเมนตัมและตัวดำเนินการสปินภายใต้ การผันกลับของเวลา (T) แพริtie (P) และ การสลับกันของอนุภาค ( $P_{12}$ )

	$\lambda_1 \cdot \lambda_2$	$(\lambda_1 + \lambda_2)^a$	$(\lambda_1 - \lambda_2)^a$	$(\lambda_1 \times \lambda_2)^a$	$[\lambda_1 \lambda_2]_2^{ab}$
$T$	+	+	+	-	+
$P$	+	+	+	+	+
$P_{12}$	+	+	-	-	+

Table 3: สมบัติการแปลงของโมเมนตัมและตัวดำเนินการเฟลเวอร์ภายใต้ การผันกลับของเวลา (T) แฟริที (P) และ การสลับกันของอนุภาค ( $P_{12}$ )

$N_c$  ตั้งแต่ อันดับ  $N_c$  จนถึง  $1/N_c$  สมบัติการแปลงของໂປຣເຕອຣ์ต່າງໆແລະ ໂມມັນທັນກາຍໃຫ້ ການຝັ້ນ ກລັບຂອງເວລາ (T) ແພຣີຕີ (P) ແລະ ການສັບກັນຂອງອນຸການ ( $P_{12}$ ) ໄດ້ຖືກແສດງໄວ້ໃນຕາງໆທີ່ 2 ແລະ 3 ສັງເກດ ວ່າສັງຄຣ  $[AB]_2^{ij}$  ແທນເທນເຫວົ້ວອັນດັບສອງສົມມາຕຣແລະ ເທຣະເປັນຄູນຍໍທີ່ເກີດຈາກເວກເຕອຣ໌ສອງຕ້ວ່າ  $A^i$ ,  $B^j$  ນັ້ນຄືວ່າ

$$[AB]_2^{ij} \equiv A^i B^j + A^j B^i - \frac{2}{3} \delta^{ij} \vec{A} \cdot \vec{B} \quad (17)$$

## ผลการทดสอบ

ในโครงการวิจัยนี้ได้แบ่งผลการวิจัยออกเป็น 2 ส่วนหลัก ดังนี้คือ

## 1. การคำนวณด้วยวิธีการยนต์หารีปฏิกริยาค่าควบคุมจากแผนภาพไฟยน์แม่นสามเหลี่ยม

เมื่อพิจารณาแ้อมพลิจูดจากแผนภาพสามเหลี่ยมในรูปที่ 1 และ 2 ในกรณีของภาวะเอกฐานสามเหลี่ยมของปฏิกิริยาการกระเจิงในรูปทั้งสองจะเกิดขึ้นเมื่ออนุภาคนายในแผนภาพเมื่อโมเมนตัมสี่มิติของอนุภาคนั้นสามมีค่าเท่ากับมวลนิ่งของพวกรั้นเองจะเป็นได้ว่าส่วนจินตภาพของแ้อมพลิจูดสามารถทำให้เกิดการสั่นพ้องที่ระดับพลังงานเดียวกันกับมวลการสั่นพ้องของอนุภาคนั้น ( $N^*(1875)$ ) ดังรูปที่ 3 ด้วยเหตุนี้เราจึงพิจารณาแ้อมพลิจูดของแผนภาพสามเหลี่ยมดังกล่าวในส่วนที่เป็นค่าจินตภาพตามทฤษฎีบีทอฟทิลอลในทฤษฎีสามควบตัมเพื่อนำไปใช้ในการสร้างอันตรกิริยาเครอร์เนล ( $V$ ) สำหรับการคำนวณในสมการ Bethe-Salpeter ของปฏิกิริยาคู่ควบ จากแ้อมพลิจูดในสมการ 6 เราจะได้ว่า

$$\begin{aligned} \text{Im} \tilde{t}_{i,j} &= V_{j,\Sigma^* K} V_{i,\Sigma^* K} (g_{\Sigma^*, \pi\Lambda} g_{N^*, K\Lambda} 2M_\Lambda 2M_{\Sigma^*})^2 \\ &\quad \times \frac{1}{N_{N^*}} \int d\tilde{m} \frac{1}{3} \vec{k}^2 S_{N^*}(\tilde{m}) |\text{Im} G_{\pi N^*}(s, \tilde{m})| |t_T(s, \tilde{m})|^2, \end{aligned} \quad (18)$$

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$$\text{Im}G_{\pi N^*} = -\frac{1}{4\pi}\frac{M_{N^*}}{\sqrt{s}}|\vec{k}|, \quad |\vec{k}| = \frac{\lambda^{\frac{1}{2}}(s, m_\pi^2, M_{N^*}^2)}{2\sqrt{s}}, \quad (19)$$

$$S_{N^*}(\tilde{m}) = -\frac{1}{\pi} \text{Im} \frac{1}{\tilde{m} - M_{N^*} + i\frac{\Gamma_{N^*}}{2}}, \quad (20)$$

และ

$$N_{N^*} = \int S_{N^*}(\tilde{m}) d\tilde{m}. \quad (21)$$

โดยเราจะทำการอินทิเกรตทั่วทั้งการกระจายตัวของมวล  $\tilde{m}$  ตั้งแต่  $1535 \pm 150$  MeV ในที่สุดอันตรกิริยา เคอร์เนลสำหรับปฏิกิริยาคู่คิวบ (V) สามารถหาได้จาก

$$\tilde{V}_{i,\pi N^*} = V_{i,\Sigma^* K}^{(I=1/2)} g_{\Sigma^*,\pi\Lambda} g_{N^*,K\Lambda} 2M_\Lambda 2M_{\Sigma^*} \tilde{V} \quad (22)$$

โดยที่

$$\tilde{V} = \sqrt{\frac{\text{Im}\tilde{t}}{\text{Im}G_{\pi N^*}}} \quad (23)$$

จากนั้นเราจะทำการคำนวณแอมเพลจูดด้วยวิธีการรูนิทารีบปฏิกิริยาคู่คิวบโดยพิจารณาคู่ปฏิกิริยาทั้งหมด

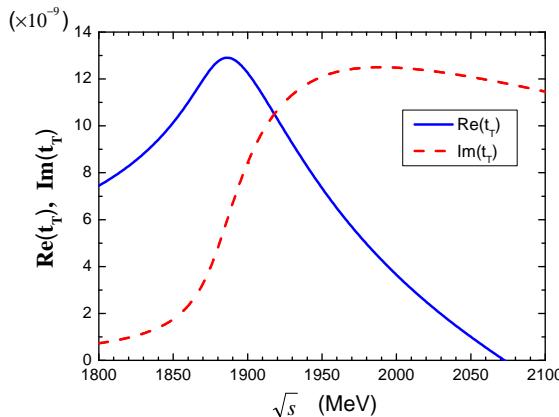


Figure 3: ส่วนจริงและจินตภาพของแอมเพลจูด  $t_T$  ของแผนภาพสามเหลี่ยมตามสมการที่ 6 ที่มีคู่อนุภาค  $\Sigma^* K$  เป็นสถานะส่งผ่าน

สี่คู่นั้นคือ  $\Delta \pi, \Sigma^* K, N^*(1535) \pi$  และ  $N \sigma$  ซึ่งสามารถเขียนให้อยู่ในรูปเมทริกซ์ได้ดังนี้

$$\begin{pmatrix} \Sigma^* K \rightarrow \Sigma^* K & \Delta \pi \rightarrow \Sigma^* K & N^*(1535) \pi \rightarrow \Sigma^* K & N \sigma \rightarrow \Sigma^* K \\ \Sigma^* K \rightarrow \Delta \pi & \Delta \pi \rightarrow \Delta \pi & N^*(1535) \pi \rightarrow \Delta \pi & N \sigma \rightarrow \Delta \pi \\ \Sigma^* K \rightarrow N^*(1535) \pi & \Delta \pi \rightarrow N^*(1535) \pi & 0 & 0 \\ \Sigma^* K \rightarrow N \sigma & \Delta \pi \rightarrow N \sigma & 0 & 0 \end{pmatrix} \quad (24)$$

เมื่อเราคำนวณแอมเพลจูดที่ใส่ค่าอันตรกิริยาเคอร์เนล V จากสมการ 22 ลงในสมการ 2 ผลการคำนวณเชิงตัวเลขสามารถแสดงให้เห็นในรูปที่ 4, 5, 6 และ 7 เราจะสังเกตได้ว่าที่ซองคู่ปฏิกิริยา  $\Sigma^* K \rightarrow \Sigma^* K$  เป็นซองที่ทำให้เกิดการสั่นพ้องในรูปแบบของอนุภาค  $N^*(1875)$  ได้

จากนั้นเราจะทำการคำนวณค่าคงที่คู่คิวบของแต่ละซองปฏิกิริยาการกระจายเจิงเพื่อนำไปคำนวณหาสมบัติต่างๆที่เกี่ยวข้องกับการกระจายเจิงหรือการสลายตัวของอนุภาค  $N^*(1875)$  ได้ โดยค่าคงที่คู่คิวบจาก

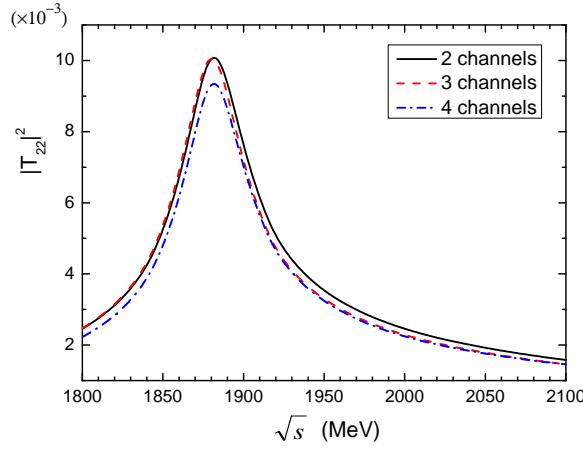


Figure 4: ค่า  $|T_{22}|^2$  ที่เป็นฟังก์ชันของพลังงานในกรอบอ้างอิงศูนย์กลางมวล ( $\sqrt{s}$ ) สำหรับสองคู่ปฏิกิริยา ( $\Delta\pi, \Sigma^*K$ ) สามคู่ปฏิกิริยา ( $\Delta\pi, \Sigma^*K, N^*\pi$ ) และสี่คู่ปฏิกิริยา ( $\Delta\pi, \Sigma^*K, N^*\pi, N\sigma$ )

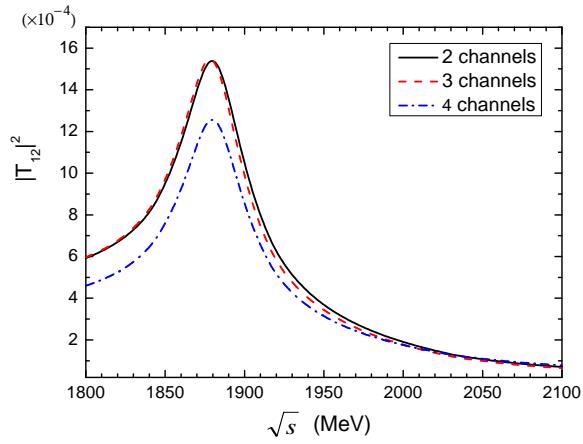


Figure 5: ค่า  $|T_{12}|^2$  ที่เป็นฟังก์ชันของพลังงานในกรอบอ้างอิงศูนย์กลางมวล ( $\sqrt{s}$ ) สำหรับปฏิกิริยาส่งผ่าน  $\Delta\pi \rightarrow \Sigma^*K$

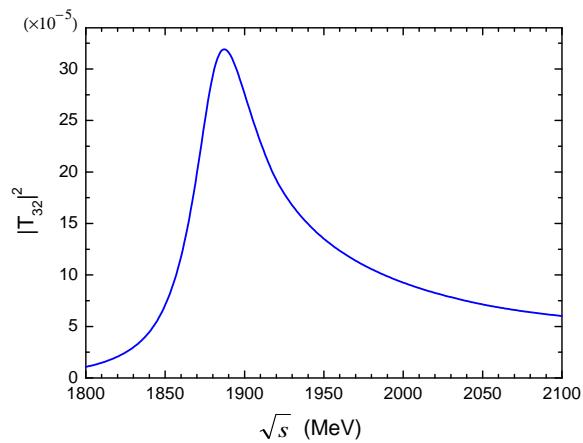


Figure 6: ค่า  $|T_{32}|^2$  ที่เป็นฟังก์ชันของพลังงานในกรอบอ้างอิงศูนย์กลางมวล ( $\sqrt{s}$ ) สำหรับปฏิกิริยาส่งผ่าน  $\Sigma^*K \rightarrow N^*\pi$

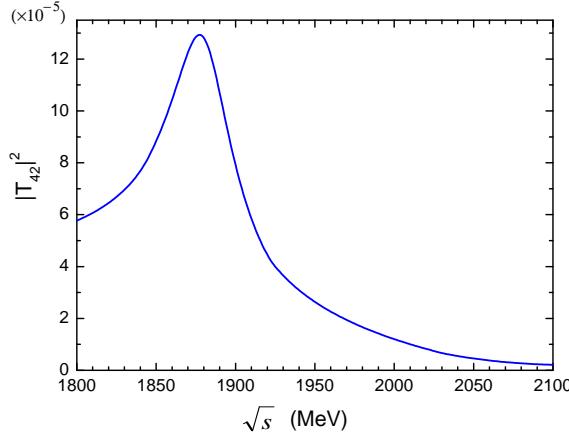


Figure 7: ค่า  $|T_{42}|^2$  ที่เป็นฟังก์ชันของพลังงานในกรอบอ้างอิงศูนย์กลางมวล ( $\sqrt{s}$ ) สำหรับปฏิกิริยาสั่งผ่าน  $\Sigma^* K \rightarrow N\sigma$

การกระเจิงแบบบูนิทารีปฏิกิริยาคู่คิวบสามารถคำนวณได้จาก  $T_{ij}$  ในสมการ 2 โดยที่  $i, j = \Delta\pi, \Sigma^* K, N^* \pi, N\sigma$  และเราจะหาค่ามวลและความกว้างของการสั่นพ้องจากตำแหน่งของจุดสูงสุดของการสั่นพ้องและค่าความกว้างในแต่ละช่องคู่ปฏิกิริยา เราจะได้ว่า

$$T_{ij} = \frac{g_i g_j}{\sqrt{s} - M_R + i \frac{\Gamma_R}{2}}. \quad (25)$$

เมื่อ  $M_R$  และ  $\Gamma_R$  คือค่าตำแหน่งมวลและความกว้างของการสลายตัวของการสั่นพ้องตามลำดับ และเราจะให้ช่องปฏิกิริยา  $\Sigma^* K$  หรือ  $i, j = 2$  เป็นช่องปฏิกิริยาอ้างอิงดังต่อไปนี้

$$T_{22}|_{\text{peak}} = \frac{g_2^2}{i \frac{\Gamma_R}{2}}, \quad g_2^2 = i \frac{\Gamma_R}{2} T_{22}|_{\text{peak}} \quad (26)$$

ในการนี้เราจะให้  $g_2$  มีค่าบวกหรือลบก็ได้และเราสามารถคำนวณค่าคงที่คู่คิวบ  $g_i$  ของช่องปฏิกิริยาอื่นๆ ได้จากสมการข้างล่างนี้

$$\frac{g_i}{g_2} = \frac{T_{i2}}{T_{22}}|_{\text{peak}}. \quad (27)$$

เมื่อเราได้ค่าคงที่คู่คิวบของช่องคู่ปฏิกิริยาทั้งหมด เราจึงสามารถหาค่าความกว้างของการสลายตัวในแต่ละช่องปฏิกิริยาได้จาก

$$\Gamma_i = \frac{1}{2\pi} \frac{M_B}{M_R} |g_i|^2 p_i, \quad (28)$$

โดยที่  $M_B$  คือมวลของแบริออน

$$p_i = \frac{\lambda^{1/2}(M_R^2, M_m^2, M_B^2)}{2M_R} \quad (29)$$

เมื่อ  $M_m$  มวลของเมชอนในแต่ละช่องปฏิกิริยา  $\Delta\pi, \Sigma^* K, N^* \pi, N\sigma$  ค่าคงที่คู่คิวบของช่องปฏิกิริยาทั้งเมื่อใช้สมการ 26 และ 27 มีค่าดังต่อไปนี้

$$\begin{aligned} g_{R, \Sigma^* K} &= 1.72 - 0.70i, & g_{R, \Delta\pi} &= 0.34 + 0.59i, \\ g_{R, N^* \pi} &= -0.29 + 0.17i, & g_{R, N\sigma} &= 0.22 - 0.01i. \end{aligned} \quad (30)$$

จากค่าที่ได้ดังกล่าวมานี้ค่าคงที่การสลายตัวก็สามารถหาได้จาก  $\Gamma_i = \frac{1}{2\pi} \frac{M_i}{M_R} |g_i|^2 p_i$ , เมื่อ  $M_i$  คือมวลของ แบริโอนในช่อง  $i$  และ  $p_i$  คือโมเมนตัมของมัน เราสามารถหาค่าคงที่การสลายตัวย่อยได้ดังนี้

$$\begin{aligned} \Gamma_{\Delta\pi} &= 25.2 \text{ MeV}, & \Gamma_{\Sigma^*K} &= 13.1 \text{ MeV}, \\ \Gamma_{N^*\pi} &= 4.2 \text{ MeV}, & \Gamma_{N\sigma} &= 2.3 \text{ MeV}. \end{aligned} \quad (31)$$

เราสังเกตได้ว่าผลรวมของค่าคงที่การสลายตัวย่อยมีค่าเท่ากับ  $44.8 \text{ MeV}$  ซึ่งน้อยกว่าค่า  $\Gamma_R = 71.2 \text{ MeV}$  ที่เราหาได้จากในรูปที่ 4 อย่างไรก็ตามจุดยอดของ  $N^*(1875)$  มีการกระจายตัวเนื่องจากมวลของ  $\Sigma^*$  ที่ มีค่าความกว้าง  $\Gamma_{\Sigma^*} = 36 \text{ MeV}$  [4] ดังนั้นเราจึงต้องทำการหาค่าค่อนโวลุชั่นของช่องการสลายตัวย่อยที่  $\Gamma_{\Sigma^*K}$

$$\begin{aligned} \tilde{\Gamma}_{\Sigma^*K} &= \frac{1}{N_R} \int_{M_R-2\Gamma_R}^{M_R+2\Gamma_R} d\tilde{M}_R \int_{M_{\Sigma^*}-2\Gamma_{\Sigma^*}}^{M_{\Sigma^*}+2\Gamma_{\Sigma^*}} d\tilde{M}_{\Sigma^*} \\ &\times S_R(\tilde{M}_R) S_{\Sigma^*}(\tilde{M}_{\Sigma^*}) \Gamma_{\Sigma^*K}(\tilde{M}_R, \tilde{M}_{\Sigma^*}, m_K), \end{aligned} \quad (32)$$

เมื่อ  $S_R(\tilde{M}_R)$  (หรือ  $S_{\Sigma^*}(\tilde{M}_{\Sigma^*})$ ) คือ พังก์ชั่นสเปกทรัลของอนุภาค  $N^*(1875)$  (หรือ  $\Sigma^*$ ) และ

$$N_R = \int_{M_R-2\Gamma_R}^{M_R+2\Gamma_R} d\tilde{M}_R \int_{M_{\Sigma^*}-2\Gamma_{\Sigma^*}}^{M_{\Sigma^*}+2\Gamma_{\Sigma^*}} d\tilde{M}_{\Sigma^*} S_R(\tilde{M}_R) S_{\Sigma^*}(\tilde{M}_{\Sigma^*}),$$

ค่าคงที่การสลายตัวย่อยของช่องปฏิกิริยา  $\Sigma^*K$  ที่ผ่านการหาค่าค่อนโวลุชั่นของช่องการสลายตัวย่อยคือ

$$\Gamma_{\Sigma^*K}(\tilde{M}_R, \tilde{M}_{\Sigma^*}, m_K) = \frac{1}{2\pi} \frac{\tilde{M}_{\Sigma^*}}{\tilde{M}_R} g_{R,\Sigma^*K}^2 \tilde{p}, \quad (33)$$

และ

$$\tilde{p} = \frac{\lambda^{1/2}(\tilde{M}_R^2, \tilde{M}_{\Sigma^*}^2, m_K^2)}{2\tilde{M}_R} \theta(\tilde{M}_R - \tilde{M}_{\Sigma^*} - m_K).$$

และการคำนวณโดยการแทนค่าตัวเลขของแต่ละตัวแปรจาก [4] จะได้ว่า

$$\tilde{\Gamma}_{\Sigma^*K} = 33.2 \text{ MeV}. \quad (34)$$

เมื่อนำค่าที่ได้ดังกล่าวไปหาผลรวมของค่าคงที่การสลายตัวย่อยของช่องปฏิกิริยาทั้งหมดจะได้เท่ากับ  $64.9 \text{ MeV}$  ซึ่งมีค่าใกล้เคียงค่าที่เราได้จาก  $\Gamma_R = 71.2 \text{ MeV}$

เพื่อเป็นการทำนายผลการคำนวณของเราให้สอดคล้องกับผลการทดลองในอนาคตเราจะเลือกในการศึกษาการสลายตัวของ  $N^*(1875)$  ที่สลายตัวไปเป็น  $\pi^+ \pi^- p$  ในสถานะสุดท้าย ซึ่งรูปแบบการสลายตัวที่เป็นไปได้ทั้งหมดได้ถูกแสดงไว้ในรูปที่ 8 เมื่อพิจารณาผลการรวมของเลขค้อนตั้มสปินและไอโซสปินแต่ละ

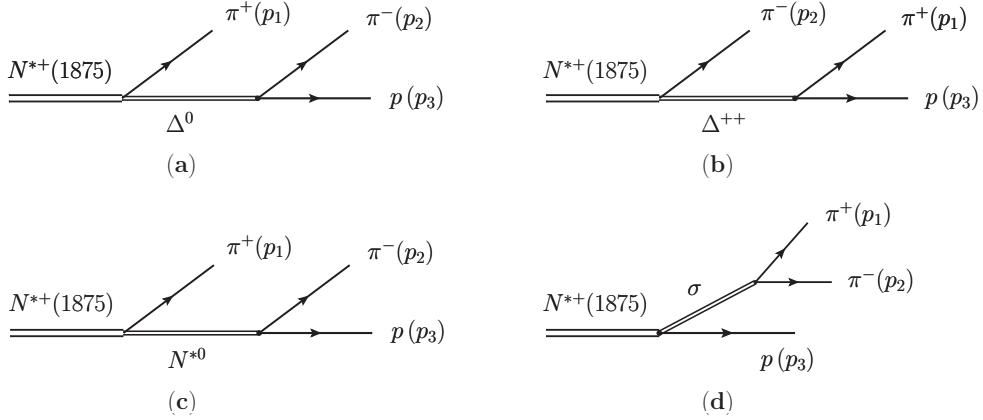


Figure 8: แผนภาพ Feynman แสดงสำหรับการสลายตัวของการสัมพ้อง  $N^*(1875)$  ไปสู่  $\pi\Delta$  [(a) และ (b)],  $\pi N^*$  [(c)] และ  $\sigma p$  [(d)].

กระบวนการสลายตัวแล้วเราจะได้ว่า

$$\begin{aligned}
 -it_{R^+ \rightarrow \pi^+ \Delta^0 \rightarrow \pi^+ \pi^- p} &= g_{R,\pi\Delta} \left( -\sqrt{\frac{1}{6}} \right) \sqrt{\frac{1}{3}} \\
 &\times \frac{1}{M_{23} - M_\Delta + i\frac{\Gamma_\Delta}{2}} \vec{S} \cdot \vec{p}_2 \frac{f_{\pi N \Delta}}{m_\pi}, \\
 -it_{R^+ \rightarrow \pi^- \Delta^{++} \rightarrow \pi^- \pi^+ p} &= g_{R,\pi\Delta} \sqrt{\frac{1}{2}} (-1) \vec{S} \cdot \vec{p}_1 \\
 &\times \frac{1}{M_{13} - M_\Delta + i\frac{\Gamma_\Delta}{2}} \frac{f_{\pi N \Delta}}{m_\pi}, \\
 -it_{R^+ \rightarrow \pi^+ N^{*0} \rightarrow \pi^+ \pi^- p} &= g'_{R,\pi\Delta} \left( \sqrt{\frac{2}{3}} \right) \left( -\sqrt{\frac{2}{3}} \right) \vec{S} \cdot \vec{p}_1 \\
 &\times \frac{1}{M_{23} - M_{N^*} + i\frac{\Gamma_{N^*}}{2}} g_{N^*,\pi N}, \\
 -it_{R^+ \rightarrow \sigma p \rightarrow \pi^+ \pi^- p} &= g'_{R,\sigma N} \left( -\sqrt{\frac{2}{3}} \right) \vec{S} \cdot (\vec{p}_1 + \vec{p}_2) (-1) \\
 &\times \frac{1}{M_{12}^2 - m_\sigma^2 + im_\sigma \Gamma_{N^*}} g_{\sigma,\pi\pi},
 \end{aligned} \tag{35}$$

เมื่อ  $M_{ij}^2 = (p_i + p_j)^2$ ;  $i, j = 1, 2, 3$  สำหรับ  $\pi^+$ ,  $\pi^-$ ,  $p$

ในการคำนวณนี้เราจะใช้ค่า  $g_{N^*,\pi N} = 0.70$  จาก [19] และ

$$g'_{R,\pi N^*} = \frac{\sqrt{3}}{p_1} g_{R,\pi N^*}, \quad g'_{R,\sigma p} = \frac{\sqrt{3}}{|\vec{p}_1 + \vec{p}_2|} g_{R,\sigma p}. \tag{36}$$

ผลตอนนี้เราพร้อมที่จะเขียนและพิจารณาตั้งหนึ่งหนึ่งสำหรับการสลายตัวในกระบวนการ  $R^+ \rightarrow \pi^+ \pi^- p$  จากแผนภาพทั้งหมดในรูปที่ 8 ดังนี้

$$\begin{aligned}
 -it_{\text{tot}} &= (B + C + D) \vec{S} \cdot \vec{p}_1 + (A + D) \vec{S} \cdot \vec{p}_2 \\
 &= A' \vec{S} \cdot \vec{p}_1 + B' \vec{S} \cdot \vec{p}_2,
 \end{aligned} \tag{37}$$

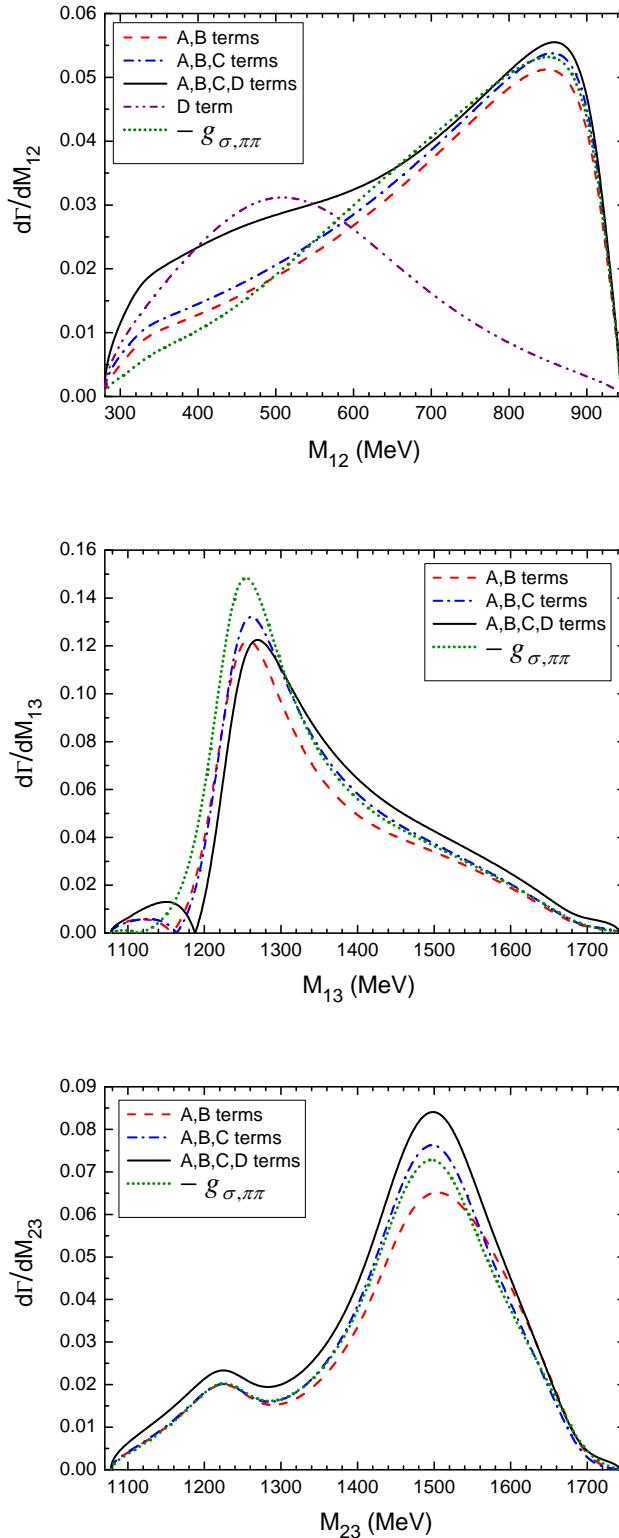


Figure 9: ผลการคำนวณการกระจายตัวเชิงมวลของแต่ละช่องปฏิกิริยา  $\frac{d\Gamma}{dM_{12}}$ ,  $\frac{d\Gamma}{dM_{13}}$  และ  $\frac{d\Gamma}{dM_{23}}$  สำหรับการสลายตัว  $N^{*+}(1875) \rightarrow \pi^+ \pi^- p$  เมื่อ 1,2,3 แทนด้วยอนุภาค  $\pi^+$ ,  $\pi^-$  และ  $p$  และเส้นประสีแดงแทนด้วยผลจากการผลิตคู่อนุภาค  $\pi\Delta$  (นั่นคือพจน์  $A$  และ  $B$  ในสมการ 37), และเส้นประสีน้ำเงินแทนด้วยผลจากการผลิตคู่อนุภาค  $\pi\Delta$  และ  $\pi N^*$  (นั่นคือพจน์  $A$ ,  $B$  และ  $C$  ในสมการ 37) ส่วนเส้นประสีดำแทนด้วยผลจากการผลิตคู่อนุภาค  $\pi\Delta$ ,  $\pi N^*$  and  $\sigma N$  (นั่นคือพจน์  $A$ ,  $B$ ,  $C$  และ  $D$  ในสมการ 37), เส้นประสีเขียวคือกรณีที่พิจารณาค่าสัมประสิทธิ์จากบวกเป็นลบของค่าคงที่  $g_{\sigma, \pi\pi}$

โดยที่

$$\begin{aligned}
 A &= -\frac{1}{3\sqrt{2}} \frac{g_{R,\pi\Delta} \cdot f_{\pi N\Delta}}{m_\pi} \frac{1}{M_{23} - M_\Delta + i\frac{\Gamma_\Delta}{2}}, \\
 B &= -\frac{1}{\sqrt{2}} \frac{g_{R,\pi\Delta} \cdot f_{\pi N\Delta}}{m_\pi} \frac{1}{M_{13} - M_\Delta + i\frac{\Gamma_\Delta}{2}}, \\
 C &= -\frac{2}{\sqrt{3}} \frac{g_{R,\pi N^*} \cdot g_{N^*,\pi N}}{p_1} \frac{1}{M_{23} - M_{N^*} + i\frac{\Gamma_{N^*}}{2}}, \\
 D &= \sqrt{2} \frac{g_{R,\sigma N} \cdot g_{\sigma,\pi\pi}}{|\vec{p}_1 + \vec{p}_2|} \frac{1}{M_{12}^2 - m_\sigma^2 + im_\sigma\Gamma_\sigma}, \\
 A' &= B + C + D, \quad B' = A + D.
 \end{aligned} \tag{38}$$

การกระจายตัวเชิงอนุพันธ์ของมวลสามารถหาได้จาก [4]

$$\frac{d^2\Gamma}{dM_{12}dM_{23}} = \frac{1}{(2\pi)^3} \frac{4M_p M_R}{32M_R^3} \overline{\sum} \sum \left| t_{\text{tot}} \right|^2 4M_{12}M_{23}, \tag{39}$$

และเราจะพบว่า

$$\overline{\sum} \sum \left| t_{\text{tot}} \right|^2 = \frac{1}{3} \left[ |A'|^2 \vec{p}_1^2 + |B'|^2 \vec{p}_2^2 + 2\text{Re}(A'B'^*) \vec{p}_1 \cdot \vec{p}_2 \right], \tag{40}$$

เมื่อ  $\vec{p}_1 \cdot \vec{p}_2$  สามารถเขียนให้อยู่ในรูปของ  $M_{12}$  ดังนี้

$$2\vec{p}_1 \cdot \vec{p}_2 = m_1^2 + m_2^2 + 2E_1 E_2 - M_{12}^2 \tag{41}$$

และ  $E_1, E_2$  คือ

$$E_1 = \frac{M_R^2 + m_1^2 - M_{23}^2}{2M_R}, \quad E_2 = \frac{M_R^2 + m_2^2 - M_{13}^2}{2M_R} \tag{42}$$

ในการหาค่า  $\frac{d\Gamma}{dM_{12}}$  เราอินทิเกรตสมการ (39) ทั่วทั้ง  $M_{23}$  และขอบเขตของมันได้ถูกกำหนดไว้ใน PDG [4]. ใน  $t_{\text{tot}}$  เราต้องการ  $M_{12}, M_{13}, M_{23}$  ให้เป็นตัวแปร ตัวแปร  $M_{13}$  สามารถเขียนให้อยู่ในรูป

$$M_{13}^2 = M_R^2 + m_1^2 + m_2^2 + m_3^2 - M_{12}^2 - M_{23}^2 \tag{43}$$

จากผลการคำนวณเชิงตัวเลขของการกระจายเชิงมวลในรูปที่ 9 จะพบว่าที่การกระจายของการสลายตัว  $\pi\pi$  ที่  $M_{12}$  นั้นจะเกิดการสั่นพ้องขึ้นที่ตรงกับระดับพลังงานของมวลอนุภาค  $\sigma$  นั่นหมายความว่าอันตรกิริยาของปฏิกิริยาคู่คุบที่ซ่อง  $N\sigma$  นั้นมีความสำคัญมากที่ช่วยทำให้ค่าความกว้างของการสลายตัวมีค่าตรงกันกับผลการทดลองสำหรับ  $N^*(1875)$  และมีส่วนสำคัญในการผลิตการสั่นพ้องที่เกิดขึ้นในกรอบวิธีการยุนิทรีปฏิกิริยาคู่คุบอีกด้วย

## 2. การวิเคราะห์ศักย์อคเตทแบริออนสองก้อนโดยใช้วิธีการการกระจายส่วนกลับเลขค่วนต้มสีในทฤษฎีวีดี

ในหัวข้อนี้เราจะเริ่มต้นด้วยโครงสร้าง SU(3) ลักษณะเจียนสำหรับอันตรกิริยาของอคเตทแบริออนสองก้อนที่ติดกันที่มีอันดับการกระจายโครงสร้างโครงสร้างที่อันดับแรกเริ่ม (LO, Leading Order) ซึ่งไม่มีอนุพันธ์ประกูลในลักษณะเจียน [20]

$$\begin{aligned}
 \mathcal{L}^{(1)} &= C_i^{(1)} \langle \bar{B}_1 \bar{B}_2 (\Gamma_i B)_2 (\Gamma_i B)_1 \rangle, \\
 \mathcal{L}^{(2)} &= C_i^{(2)} \langle \bar{B}_1 (\Gamma_i B)_1 \bar{B}_2 (\Gamma_i B)_2 \rangle, \\
 \mathcal{L}^{(3)} &= C_i^{(3)} \langle \bar{B}_1 (\Gamma_i B)_1 \rangle \langle \bar{B}_2 (\Gamma_i B)_2 \rangle,
 \end{aligned} \tag{44}$$

โดยที่ 1 และ 2 แทนอนุภาคในระบบการการกระเจิง และ  $B$  คืออคเตทแบริอันที่อยู่ในรูปเมทริกซ์ของกลุ่ม  $SU(3)$  ซึ่งจะอยู่ในรูป

$$B = \frac{1}{\sqrt{2}} \sum_{a=1}^8 \lambda^a B^a = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & \frac{-\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ -\Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix}, \quad (45)$$

และสัญลักษณ์วงเล็บ  $\langle \dots \rangle$  คือการแทรซในเมทริกซ์  $3 \times 3$  และนอร์มอลไลซ์ของเมทริกซ์ของแกลมัมน์คือ  $\langle \lambda^a \lambda^b \rangle = 2 \delta^{ab}$  นอกจากนี้สัญกรณ์  $\Gamma_i$  คือสมาชิกของพีชคณิต Clifford สำหรับสpinเนอร์ ดังนี้

$$\Gamma_1 = 1, \quad \Gamma_2 = \gamma^\mu, \quad \Gamma_3 = \sigma^{\mu\nu}, \quad \Gamma_4 = \gamma^\mu \gamma_5, \quad \Gamma_5 = i \gamma_5. \quad (46)$$

เราสามารถหาศักย์การกระเจิงของอคเตทแบริอันได้โดยทำการคำนวณตามวิธีการใน [21] ด้วยการกำหนดสมบัติ โควาเรียนที่เชิงสัมพัทธภาพอีกทั้งเอกลักษณ์แบบ Fierz และ Claley-Hamilton ได้ถูกใช้เพื่อกำจัดตัวเนิน การที่ซ้ำซ้อนอีกด้วย โดยกำหนดให้  $\mathcal{H} = -\mathcal{L}$  ศักย์การกระเจิงของอคเตทแบริอันที่อยู่ในอันดับการกระจายของโมเมนตัมอันดับสองสามารถเขียนได้ให้อยู่ในรูป

$$\begin{aligned} V^{(1)} &= \langle \bar{\chi}_2, d; \bar{\chi}_1, c | \mathcal{H}^{(1)} | a, \chi_1; b, \chi_2 \rangle \\ &= \left\{ \frac{1}{3} \delta^{cd} \delta^{ba} + \frac{1}{2} (d^{cde} + i f^{cde}) (d^{eba} + i f^{eba}) \right\} \\ &\times \left\{ c_S^{(1)} \tilde{O}_S + c_T^{(1)} \tilde{O}_T + \left( c_1^{(1)} p_-^2 + c_2^{(1)} p_+^2 \right) \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} + \left( c_3^{(1)} p_-^2 + c_4^{(1)} p_+^2 \right) \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right. \\ &+ \left. c_5^{(1)} \frac{i}{2} (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{p}_+ \times \vec{p}_-) + c_6^{(1)} (\vec{p}_- \cdot \vec{\sigma}_1) (\vec{p}_- \cdot \vec{\sigma}_2) + c_7^{(1)} (\vec{p}_+ \cdot \vec{\sigma}_1) (\vec{p}_+ \cdot \vec{\sigma}_2) \right\}, \end{aligned} \quad (47)$$

โดยที่

$$\begin{aligned} \tilde{O}_S &= \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} + \frac{i}{2M^2} (\vec{p}_+ \times \vec{p}_-) \cdot (\vec{\sigma}_1 - \vec{\sigma}_2), \\ \tilde{O}_T &= \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{i}{2M^2} (\vec{p}_+ \times \vec{p}_-) \cdot (\vec{\sigma}_1 - \vec{\sigma}_2), \end{aligned} \quad (48)$$

และ  $\vec{\sigma}_i \equiv \vec{\sigma}_{\bar{\chi}_i \chi_i}$  เมื่อ  $i = 1, 2$  ด้วย  $a(c), b(d), \chi_1(\bar{\chi}_1)$  และ  $\chi_2(\bar{\chi}_2)$  คือ เฟลเวอร์และสpinของแบริอันที่พุ่งเข้า(พุ่งออก)หมายเลขที่ 1 และ 2 ตามลำดับ ส่วน  $M$  คือมวลของแบริอันอคเตทในจีดจำกัด สมมาตรแบบ  $SU(3)$  สำหรับค่าคงที่คู่คู่  $c_i^{(1)}$  คือผลบวกเชิงเส้นของค่าคงที่  $C_i^{(1)}$  ดังต่อไปนี้

$$\begin{aligned} c_S^{(1)} &= C_1^{(1)} + C_2^{(1)}, \quad c_T^{(1)} = C_3^{(1)} - C_4^{(1)}, \\ c_1^{(1)} &= -\frac{1}{4M^2} (C_2^{(1)} + C_3^{(1)}), \quad c_2^{(1)} = -\frac{1}{2M^2} (C_1^{(1)} - C_2^{(1)}), \\ c_3^{(1)} &= -\frac{1}{4M^2} (C_2^{(1)} + C_3^{(1)}), \quad c_4^{(1)} = \frac{1}{4M^2} (C_3^{(1)} - C_4^{(1)}), \\ c_5^{(1)} &= -\frac{1}{2M^2} (C_1^{(1)} - 3C_2^{(1)} - 3C_3^{(1)} - C_4^{(1)}), \\ c_6^{(1)} &= \frac{1}{4M^2} (C_2^{(1)} + C_3^{(1)} + C_4^{(1)} + C_5^{(1)}), \quad c_7^{(1)} = -\frac{1}{4M^2} (C_3^{(1)} + C_4^{(1)}). \end{aligned} \quad (49)$$

สัญลักษณ์ของโมเมนตัมในหัวข้อนี้สามารถนิยามได้ดังนี้

$$\vec{p}_+ = \frac{1}{2} (\vec{p}' + \vec{p}), \quad p_+^2 = \vec{p}_+ \cdot \vec{p}_+,$$

$$\begin{aligned}\vec{p}_- &= \vec{p}' - \vec{p}, & p_-^2 &= \vec{p}_- \cdot \vec{p}_-, \\ \vec{n} &= \vec{p} \times \vec{p}' = \vec{p}_+ \times \vec{p}_-, \end{aligned}\quad (50)$$

โดยที่  $\vec{p}(\vec{p}')$  คือโมเมนตัมสามมิติที่พุ่งเข้า (พุ่งออก) ในการอบศูนย์กลางมวล และเงื่อนไขตั้งจากของโมเมนตัมทั้งสองสามารถนิยามได้ดังนี้

$$\vec{p}_+ \cdot \vec{p}_- = 0. \quad (51)$$

ด้วยวิธีการเดียวกันเรารสามารถหาศักย์แปรร้อนของค่าคงที่คู่คบ  $C_i^{(2)}$  และ  $C_i^{(3)}$  ได้ดังนี้

$$\begin{aligned}V^{(2)} &= \langle \bar{\chi}_2, d; \bar{\chi}_1, c | \mathcal{H}^{(2)} | a, \chi_1; b, \chi_2 \rangle \\ &= \left\{ \frac{1}{3} \delta^{ca} \delta^{bd} + \frac{1}{2} (d^{cae} + i f^{cae}) (d^{edb} + i f^{edb}) \right\} \\ &\times \left\{ c_S^{(2)} \tilde{O}_S + c_T^{(2)} \tilde{O}_T + \left( c_1^{(2)} p_-^2 + c_2^{(2)} p_+^2 \right) \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} + \left( c_3^{(2)} p_-^2 + c_4^{(2)} p_+^2 \right) \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right. \\ &+ \left. c_5^{(2)} \frac{i}{2} (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{p}_+ \times \vec{p}_-) + c_6^{(2)} (\vec{p}_- \cdot \vec{\sigma}_1) (\vec{p}_- \cdot \vec{\sigma}_2) + c_7^{(2)} (\vec{p}_+ \cdot \vec{\sigma}_1) (\vec{p}_+ \cdot \vec{\sigma}_2) \right\}, \end{aligned}\quad (52)$$

และ

$$\begin{aligned}V^{(3)} &= \langle \bar{\chi}_2, d; \bar{\chi}_1, c | \mathcal{H}^{(3)} | a, \chi_1; b, \chi_2 \rangle \\ &= \delta^{ca} \delta^{bd} \\ &\times \left\{ c_S^{(3)} \tilde{O}_S + c_T^{(3)} \tilde{O}_T + \left( c_1^{(3)} p_-^2 + c_2^{(3)} p_+^2 \right) \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} + \left( c_3^{(3)} p_-^2 + c_4^{(3)} p_+^2 \right) \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right. \\ &+ \left. c_5^{(3)} \frac{i}{2} (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{p}_+ \times \vec{p}_-) + c_6^{(3)} (\vec{p}_- \cdot \vec{\sigma}_1) (\vec{p}_- \cdot \vec{\sigma}_2) + c_7^{(3)} (\vec{p}_+ \cdot \vec{\sigma}_1) (\vec{p}_+ \cdot \vec{\sigma}_2) \right\}, \end{aligned}\quad (53)$$

โดยที่ค่าคงที่คู่คบ  $c_i^{(2,3)}$  ในสมการ (52) และ (53) คือผลรวมเชิงเส้นของค่าคงที่  $C_i^{(2,3)}$  เข่นเดียวกันกับกรณีของ  $c_i^{(1)}$

ขั้นต่อไปเราจะทำการสร้างศักย์แปรร้อนอโศกเทด้วยตัวดำเนินการในวิธีการกระจายส่วนกลับของเลขค่วนตัมสี ( $1/N_c$ ) และมิลโลเนียนของศักย์แปรร้อนจากสมการ 13 ด้วยตัวดำเนินการในสมการ 14 ที่อันดับแรกเริ่มของการกระจาย  $1/N_c$  ซึ่งสามารถเขียนให้อยู่ในรูป

$$\hat{H}_{\text{LO}} = U_1^{\text{LO}}(p_-^2) \mathbb{1}_1 \cdot \mathbb{1}_2 + U_2^{\text{LO}}(p_-^2) T_1 \cdot T_2 + U_3^{\text{LO}}(p_-^2) G_1 \cdot G_2 + U_4^{\text{LO}}(p_-^2) (p_-^i p_-^j)_{(2)} \cdot (G_1^{i,a} G_2^{j,a}) \quad (54)$$

โดยที่  $T_1 \cdot T_2 = T_1^a T_2^a$  และ  $G_1 \cdot G_2 = G_1^{i,a} G_2^{i,a}$  ส่วน  $U_i^{\text{LO}}(p_-^2)$  คือฟังก์ชันใดๆที่ขึ้นอยู่กับตัวแปร  $p_-^2$  และมีมा�ตรเป็น  $N_c^0$

ณ อันดับถัดจากถัดจากแรกเริ่ม (Next-to-Next-to-leading order, NNLO) การกระจาย  $1/N_c$  และมิลโลเนียนจะได้ว่า

$$\begin{aligned}\hat{H}_{\text{NNLO}} &= U_1^{\text{NNLO}}(p_-^2) p_+^2 \mathbb{1}_1 \cdot \mathbb{1}_2 + U_2^{\text{NNLO}}(p_-^2) \vec{J}_1 \cdot \vec{J}_2 + U_3^{\text{NNLO}}(p_-^2) \vec{J}_1 \cdot \vec{J}_2 T_1 \cdot T_2 \\ &+ U_4^{\text{NNLO}}(p_-^2) p_+^2 T_1 \cdot T_2 + U_5^{\text{NNLO}}(p_-^2) p_+^2 G_1 \cdot G_2 \\ &+ U_6^{\text{NNLO}}(p_-^2) i (\vec{p}_+ \times \vec{p}_-) \cdot (\vec{J}_1 + \vec{J}_2) + U_7^{\text{NNLO}}(p_-^2) i (\vec{p}_+ \times \vec{p}_-) \cdot (T_1^a \vec{G}_2^a + \vec{G}_1^a T_2^a) \\ &+ U_8^{\text{NNLO}}(p_-^2) i (\vec{p}_+ \times \vec{p}_-) \cdot (\vec{J}_1 + \vec{J}_2) T_1 \cdot T_2 + U_9^{\text{NNLO}}(p_-^2) (p_-^i p_-^j)_{(2)} \cdot (J_1^i J_2^j)_{(2)} \\ &+ U_{10}^{\text{NNLO}}(p_-^2) (p_-^i p_-^j)_{(2)} \cdot (J_1^i J_2^j)_{(2)} T_1 \cdot T_2 + U_{11}^{\text{NNLO}}(p_-^2) (p_+^i p_+^j)_{(2)} \cdot (G_1^{i,a} G_2^{j,a})_{(2)} \end{aligned}\quad (55)$$

เราจะพบร่วมมือและมีผลต่อการเปลี่ยนของอุคเตทแบบเบริลล์ในอันดับถัดจากแรกเริ่ม (Next-to-leading order, NLO) ใน การกระจาย  $1/N_c$  นี้ การดำเนินการของตัวดำเนินการบนสถานะแบบเบริลล์สามารถหาได้ดังนี้ [13]

$$\begin{aligned} \mathbb{1} |a, \chi) &= 3 |a, \bar{\chi}), \\ J_i |a, \chi) &= \frac{1}{2} \sigma_{\bar{\chi}\chi}^{(i)} |a, \bar{\chi}), \\ T^a |b, \chi) &= i f^{bca} |c, \chi), \\ G_i^a |b, \chi) &= \sigma_{\bar{\chi}\chi}^{(i)} \left( \frac{1}{2} d^{bca} + \frac{i}{3} f^{bca} \right) |c, \bar{\chi}), \end{aligned} \quad (56)$$

และศักย์เบริลล์สามารถหาได้โดย

$$V = (\bar{\chi}_2, d; \bar{\chi}_1, c | \hat{H} | a, \chi_1; b, \chi_2), \quad (57)$$

ก่อนทำการวิเคราะห์ศักย์เบริลล์ในอันดับตัวดำเนินการกระจาย  $1/N_c$  เราจะทำการกำหนดรูปแบบของฟังก์ชันใดๆ  $U_i^{LO}$  and  $U_i^{NNLO}$  ให้อยู่ในรูป

$$U_i^{LO}(p_-^2) = g_i, \quad U_i^{NNLO}(p_-^2) = h_i. \quad (58)$$

จากการใช้สมการที่ 56 ในสมการที่ 54 และ 55 ศักย์เบริลล์การกระจาย  $1/N_c$  ที่อันดับ LO คือ

$$\begin{aligned} V_{LO} &= 9 g_1 \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} \delta^{cd} \delta^{bd} + g_2 i^2 f^{ace} f^{bde} \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} \\ &+ g_3 \vec{\sigma}_1 \cdot \vec{\sigma}_2 \left( \frac{1}{2} d^{ace} + \frac{i}{3} f^{ace} \right) \left( \frac{1}{2} d^{bde} + \frac{i}{3} f^{bde} \right) \\ &+ g_4 (p_-^i p_-^j)_{(2)} \cdot (\sigma_1^i \sigma_2^j)_{(2)} \left( \frac{1}{2} d^{ace} + \frac{i}{3} f^{ace} \right) \left( \frac{1}{2} d^{bde} + \frac{i}{3} f^{bde} \right), \end{aligned} \quad (59)$$

และที่อันดับ NNLO จะอยู่ในรูป

$$\begin{aligned} V_{NNLO} &= 9 h_1 p_+^2 \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} \delta^{cd} \delta^{bd} + \frac{1}{4} h_2 \vec{\sigma}_1 \cdot \vec{\sigma}_2 \delta^{cd} \delta^{bd} + \frac{1}{4} h_3 \vec{\sigma}_1 \cdot \vec{\sigma}_2 i^2 f^{ace} f^{bde} \\ &+ h_4 p_+^2 i^2 f^{ace} f^{bde} \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} + h_5 p_+^2 \vec{\sigma}_1 \cdot \vec{\sigma}_2 \left( \frac{1}{2} d^{ace} + \frac{i}{3} f^{ace} \right) \left( \frac{1}{2} d^{bde} + \frac{i}{3} f^{bde} \right) \\ &+ \frac{3}{2} i h_6 (\vec{p}_+ \times \vec{p}_-) \cdot (\vec{\sigma}_1 + \vec{\sigma}_2) \delta^{cd} \delta^{bd} \\ &+ i h_7 (\vec{p}_+ \times \vec{p}_-) \cdot \left[ \vec{\sigma}_1 \left( \frac{1}{2} d^{ace} + \frac{i}{3} f^{ace} \right) i f^{bde} + \vec{\sigma}_2 i f^{ace} \left( \frac{1}{2} d^{bde} + \frac{i}{3} f^{bde} \right) \right] \\ &+ \frac{3}{2} i h_8 (\vec{p}_+ \times \vec{p}_-) \cdot (\vec{\sigma}_1 + \vec{\sigma}_2) i^2 f^{ace} f^{bde} + \frac{1}{4} h_9 (p_-^i p_-^j)_{(2)} \cdot (\sigma_1^i \sigma_2^j)_{(2)} \delta^{cd} \delta^{bd} \\ &+ \frac{1}{4} h_{10} (p_-^i p_-^j)_{(2)} \cdot (\sigma_1^i \sigma_2^j)_{(2)} i^2 f^{ace} f^{bde} \\ &+ h_{11} (p_+^i p_+^j)_{(2)} \cdot (\sigma_1^i \sigma_2^j)_{(2)} \left( \frac{1}{2} d^{ace} + \frac{i}{3} f^{ace} \right) \left( \frac{1}{2} d^{bde} + \frac{i}{3} f^{bde} \right) \end{aligned} \quad (60)$$

เมื่อเราทำการเบรียบเทียบโครงสร้างสปินและเฟลเวอร์ของศักย์ที่ได้จากโครงสร้างเจียนกับตัวดำเนินการ  $1/N_c$  ในสมการที่ 47, 52, 53 และ 59,60 ตามลำดับ เราจะสามารถหาความสัมพันธ์ของค่าคงที่คู่ควร ณ อันดับการแก้ไขที่  $\mathcal{O}(1/N_c^2)$  ได้ดังนี้

$$\begin{aligned} C_1^{(1)} = C_1^{(2)} &= -3 C_1^{(3)} - 2 C_4^{(2)} - 6 C_4^{(3)}, & C_2^{(1)} = C_2^{(2)} &= -3 C_2^{(3)} - 2 C_4^{(2)} - 6 C_4^{(3)}, \\ C_3^{(1)} = C_3^{(2)} &= -3 C_3^{(3)} + C_4^{(2)} + 3 C_4^{(3)}, & C_4^{(1)} = C_4^{(2)}, & C_5^{(1)} = C_5^{(2)} = -3 C_5^{(3)} \end{aligned} \quad (61)$$

	$C_{1S0}^{\Lambda\Lambda}$	$C_{1S0}^{\Sigma\Sigma}$	$C_{3S1}^{\Lambda\Lambda}$	$C_{3S1}^{\Sigma\Sigma}$	$C_{3S1}^{\Lambda\Sigma}$
EG	-0.04795(151)	-0.07546(81)	-0.01727(124)	0.36367(30310)	0.01271(471)
HB	-0.03894(1)	-0.07657(1)	-0.01629(13)	0.20029(14050)	-0.00176(304)

Table 4: ค่าที่หาได้จากการเปรียบเทียบกับผลการทดลองจากการกระเจิงไฮเปอรอนกับนิวเคลื่อนที่ s-wave (ในหน่วยของ  $10^4 \text{ GeV}^{-2}$ ) ที่ค่าของ  $\Lambda = 600 \text{ MeV}$  ของทั้งวีซี EG และ HB [22]

ลำดับต่อไปเราจะนำผลจากการวิเคราะห์ศักย์แบบริอนด้วยการกระจาย  $1/N_c$  ในการกระเจิงของไฮเปอรอนและนิวเคลื่อนที่อันดับแรกเริ่มในการกระจายแบบไครัลซึ่งจำนวนของพารามิเตอร์อิสระในแบบจำลองนี้มีอยู่ 6 ตัวดังต่อไปนี้ [20]

$$C_S^{(1)}, \quad C_S^{(2)}, \quad C_S^{(3)}, \quad C_T^{(1)}, \quad C_T^{(2)}, \quad C_T^{(3)} \quad (62)$$

และ  $C_S^{(1,2,3)}$  และ  $C_T^{(1,2,3)}$  สามารถเขียนให้อยู่ในรูปของผลรวมเชิงเส้นจากไครัลลากrang เจียนในสมการที่ 44 ได้ว่า

$$C_S^{(1,2,3)} = C_1^{(1,2,3)} + C_2^{(1,2,3)}, \quad C_T^{(1,2,3)} = C_3^{(1,2,3)} - C_4^{(1,2,3)} \quad (63)$$

เมื่อใช้ความสัมพันธ์ของค่าคงที่คู่ควบในสมการที่ 61 เราจะได้ว่า

$$C_S^{(1)} = C_S^{(2)}, \quad C_T^{(1)} = C_T^{(2)} = -3C_T^{(3)} \quad (64)$$

ผลการคำนวณการกระจายคลื่นย่ออยู่ s-wave ศักย์ของการกระเจิงไฮเปอรอนกับนิวเคลื่อนสามารถเขียนได้ว่า [20, 22]

$$\begin{aligned} V_{1S0}^{\Lambda\Lambda} &= 4\pi \left[ \frac{1}{6} \left( C_S^{(1)} - 3C_T^{(1)} \right) + \frac{5}{3} \left( C_S^{(2)} - 3C_T^{(2)} \right) + 2 \left( C_S^{(3)} - 3C_T^{(3)} \right) \right], \\ V_{3S1}^{\Lambda\Lambda} &= 4\pi \left[ \frac{3}{2} \left( C_S^{(1)} + C_T^{(1)} \right) + \left( C_S^{(2)} + C_T^{(2)} \right) + 2 \left( C_S^{(3)} + C_T^{(3)} \right) \right], \\ V_{1S0}^{\Sigma\Sigma} &= 4\pi \left[ 2 \left( C_S^{(2)} - 3C_T^{(2)} \right) + 2 \left( C_S^{(3)} - 3C_T^{(3)} \right) \right], \\ V_{3S1}^{\Sigma\Sigma} &= 4\pi \left[ -2 \left( C_S^{(2)} + C_T^{(2)} \right) + 2 \left( C_S^{(3)} + C_T^{(3)} \right) \right], \\ V_{3S1}^{\Lambda\Sigma} &= 4\pi \left[ -\frac{3}{2} \left( C_S^{(1)} + C_T^{(1)} \right) + \left( C_S^{(2)} + C_T^{(2)} \right) \right] \end{aligned} \quad (65)$$

เนื่องจากศักย์ที่ได้ดังกล่าวเป็นค่าคงที่เพื่อความสะดวกเราจะกำหนดให้

$$C_{1S0}^{\Lambda\Lambda} \equiv V_{1S0}^{\Lambda\Lambda}, \quad C_{3S1}^{\Lambda\Lambda} \equiv V_{3S1}^{\Lambda\Lambda}, \quad C_{1S0}^{\Sigma\Sigma} \equiv V_{1S0}^{\Sigma\Sigma}, \quad C_{3S1}^{\Sigma\Sigma} \equiv V_{3S1}^{\Sigma\Sigma}, \quad C_{3S1}^{\Lambda\Sigma} \equiv V_{3S1}^{\Lambda\Sigma}, \quad (66)$$

เมื่อนำความสัมพันธ์จากสมการที่ 64 แทนค่าในสมการ 66 เราจะได้ความสัมพันธ์สุดท้ายของค่าพารามิเตอร์อิสระในการศึกษาการกระเจิงของไฮเปอรอนกับนิวเคลื่อนได้ว่า

$$C_{1S0}^{\Sigma\Sigma} = \frac{8}{7} C_{1S0}^{\Lambda\Lambda} - \frac{1}{7} C_{3S1}^{\Lambda\Lambda} - \frac{11}{21} C_{3S1}^{\Lambda\Sigma}, \quad C_{3S1}^{\Sigma\Sigma} = C_{3S1}^{\Lambda\Lambda} + 9C_{3S1}^{\Lambda\Sigma}. \quad (67)$$

เราจะเห็นได้ว่าในแบบจำลองการกระเจิงของไฮเปอรอนกับนิวเคลื่อนจากทฤษฎีสนา�ยังผลไครัลมีพารามิเตอร์อิสระอยู่ 5 ตัว [20] เมื่อให้การวิเคราะห์การกระจาย  $1/N_c$  จะทำให้เราสามารถลดรูปความอิสระของพารามิเตอร์ในแบบจำลองนี้ได้เหลือเพียง 3 ตัวเท่านั้น

เพื่อเป็นการทดสอบความถูกต้องของการกระจาย  $1/N_c$  เราจะทดสอบผลจากสมการ 67 กับผลการหาค่าคงที่คู่คุบเชิงตัวเลขกับผลการทดลองจาก [22] โดยที่งานวิจัยดังกล่าวได้ทำการหาค่าเชิงตัวเลขจากแบบจำลองใน [20] ซึ่งมีวิธีในการหาค่าสองแบบคือ วิธีการโควารีเยนท์เชิงสัมพัทธภาพ (EG) และวิธีการแบริโอนหนัก (HB) ซึ่งผลที่ได้ถูกแสดงไว้ในตารางที่ 4 และผลลัพท์ที่ได้จากการแทนค่าตัวเลขจากตารางที่ 4 ในแต่ละวิธีการจะได้ว่า

$$\begin{aligned} C_{1S0,EG}^{\Sigma\Sigma} &= -0.06327, & C_{3S1,EG}^{\Sigma\Sigma} &= 0.1271, \\ C_{1S0,HB}^{\Sigma\Sigma} &= -0.04333, & C_{3S1,HB}^{\Sigma\Sigma} &= -0.0176 \end{aligned} \quad (68)$$

เราจะเห็นได้ว่าผลจากการวิเคราะห์ด้วยการกระจาย  $1/N_c$  จะเห็นได้ว่าค่า  $C_{1S0}^{\Sigma\Sigma}$  และ  $C_{3S1}^{\Sigma\Sigma}$  จะมีความใกล้เคียงกับวิธี EG มากกว่าวิธี HB

## สรุปและวิจารณ์ผลการทดลอง

บทสรุปและวิจารณ์ผลการดำเนินโครงการวิจัยนี้สามารถแบ่งได้เป็นสองหัวข้อหลัก ดังนี้

1. การศึกษาการสันนิพ้องของ  $N^*$ (1875) จากภาวะเอกสารสามเหลี่ยมด้วยวิธีการยนิทรรศ์ปรัชญาคู่คี่

การเกิดการสั่นพ้องของอนุภาค  $N^*(1875)$  ที่มีเลขค่าอนตัมที่เกี่ยวข้องเป็น  $J^P = 3/2^-$ ,  $I = 1/2$  และ  $S = 0$  ด้วยวิธีการยูนิทารีปฏิกิริยาคู่คิวบ์โดยพิจารณาแผนภาพไฟย์น์แมนแบบสามเหลี่ยมพบว่า การสั่นพ้องของอนุภาคดังกล่าวเกิดจากภาวะเอกฐานสามเหลี่ยมในช่องปฏิกิริยาของคู่อนุภาค  $\Sigma^* K$  เป็นหลัก และช่องปฏิกิริยาดังกล่าวมีอันตรกิริยาคู่คิวบ์กับ  $\Delta \pi$ ,  $N^*(1535) \pi$  และ  $N \sigma$  สำหรับช่องปฏิกิริยาหลัก คือ  $\Sigma^* K$  ที่ทำให้เกิดการสั่นพ้องของ  $N^*(1875)$  ซึ่งมีกลไกการเกิดขึ้นดังนี้คือเมื่ออนุภาค  $\Sigma^*$  slavery ตัวไปเป็น  $\Lambda \pi$  และ อนุภาค  $\Lambda$  วิ่งเข้าไปรวมตัวกับอนุภาค  $K$  จนเกิดเป็นอนุภาค  $N^*(1535)$  โดยโมเมนตัมสี่มิติของอนุภาค  $\Sigma^* \Lambda$  และ  $K$  มีค่าเท่ากับมวลนิ่งของตัวพวkmันเอง ดังแสดงไว้ในรูปที่ 1 อย่างไรก็ตามค่าคงที่ความกว้างของการ slavery ตัวในช่องการ slavery ตัวย่อยของปฏิกิริยาคู่คิวบ์ไม่พอกับผลจากการทดลองเมื่อพิจารณาช่องหลัก  $\Sigma^* K$  เพียงอย่างเดียวแต่ช่องปฏิกิริยาหลักที่ผลต่อความกว้างของการ slavery ตัวคือช่องปฏิกิริยา  $N \sigma$  ซึ่งสามารถทำการตรวจสอบได้โดยพิจารณาการ slavery ตัวของ  $N^*(1875)$  ไปเป็น  $p \pi^+ \pi^-$  เมื่อเราสังเกตไปที่การกระจายตัวเชิงมวลของ  $\pi^+ \pi^-$  แล้วจะเห็นได้ว่าเกิดจุดสูงสุดที่ตำแหน่งของมวลอนุภาค  $\sigma$  พอดีนั่นก็แสดงว่าอันตรกิริยาคู่คิวบ์ของ  $N \sigma$  มีส่วนต่อการทำให้เกิดการสั่นพ้องของ  $N^*(1875)$  นั่นเอง

2. วิธีการกระจายส่วนกลับเลขค่อนต้มสีในทฤษฎีคิวชีดีมีความสำคัญอย่างยิ่งในการศึกษาการกระจายของอุคเตท-อุคเตทเบริอัน

ถึงแม้ว่าทฤษฎีสามารถยังผลให้รัลจะประสบความสำเร็จในการศึกษาการกระเจิงนิวเคลียร์อนเป็นอย่างมาก แต่เมื่อนำมาใช้ศึกษาเกี่ยวกับการกระเจิงไฮเปอรอนกับนิวเคลียร์อนกลับพบว่ามีข้อด้อยคือจำนวนพารามิเตอร์อิสระจากโครงสร้างทางเคมีมีเยอะเกินไปเนื่องจากข้อมูลการทดลองที่อยู่ไม่มากและมีอย่างจำนวนจำกัดด้วยเหตุนี้การวิเคราะห์ด้วยการกระจาย  $1/N_c$  จะช่วยให้เกิดการลดรูปของจำนวนพารามิเตอร์อิสระ ซึ่งในการศึกษานี้เราสามารถลดจำนวนพารามิเตอร์อิสระของโครงสร้างทางเคมีลงจาก 15 เหลือ 9 ณ อันดับการแก้ไขที่  $\mathcal{O}(1/N_c^2)$  และเมื่อนำไปประยุกต์กับการกระเจิงไฮเปอรอนกับนิวเคลียร์อนที่อันดับการกระจายโครงสร้างที่อันดับแรกเริ่ม ความสามารถลดจำนวนพารามิเตอร์อิสระในการกระจายคลื่นย่ออย่างที่ r-wave ได้เหลือ 3 ตัวจาก 5 นอกจากนั้นรายังได้ทำการทดสอบค่าเชิงตัวเลขการลดรูปของจำนวนพารามิเตอร์ด้วยการกระจาย  $1/N_c$  ยังคงชี้ว่า วิธีการคำนวณการกระเจิงไฮเปอรอนกับนิวเคลียร์อนด้วยวิธีการโครงเรียนที่ใช้สัมพัทธภาพมีความเข้ากันได้กับการกระจาย  $1/N_c$  มากกว่าวิธีการแบบริโอนหนักอีกด้วย

## ข้อเสนอแนะสำหรับงานวิจัยในอนาคต

1. ในอนาคตอันใกล้นี้หัวหน้าโครงการวิจัยจะทำการคำนวณค่าภาคตัวของสำหรับการกระจายโดยไฟฟ้ากับอนุภาค  $N^*(1875)$  เพื่อศึกษาถึงกลไกภาวะเอกฐานสามเหลี่ยมที่จะสามารถทำให้เกิดสถานะกระแสตันของอนุภาค  $N^*$  ที่มีมวลและเลขคุณตัมสปินที่สูงขึ้นได้หรือไม่
2. หัวหน้าโครงการวิจัยมีแผนที่จะใช้วิธีการกระจาย  $1/N_c$  เพื่อขยายขอบเขตการศึกษาอันตรกิริยาระหว่างอุคเตห(สปิน-1/2)กับดีคับเลท(สปิน-3/2)แบบรีอันและดีคับเลทกับดีคับเลทแบบรีอันเพื่อสร้างกรอบการอธิบายแบบรีอันที่มีเลขคุณตัมสปินที่สูงขึ้นให้ได้โดยใช้กรอบทฤษฎีเดียวด้วยวิธีการกระจาย  $1/N_c$

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# Triangle mechanisms in the build up and decay of the $N^*(1875)$

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We studied the  $N^*(1875)(3/2^-)$  resonance with a multichannel unitary scheme, considering the  $\Delta\pi$  and  $\Sigma^*K$ , with their interaction extracted from chiral Lagrangians, and then added two more channels, the  $N^*(1535)\pi$  and  $N\sigma$ , which proceed via triangle diagrams involving the  $\Sigma^*K$  and  $\Delta\pi$  respectively in the intermediate states. The triangle diagram in the  $N^*(1535)\pi$  case develops a singularity at the same energy as the resonance mass. We determined the couplings of the resonance to the different channels and the partial decay widths. We found a very large decay width to  $\Sigma^*K$ , and also observed that, due to interference with other terms, the  $N\sigma$  channel has an important role in the  $\pi\pi$  mass distributions at low invariant masses, leading to an apparently large  $N\sigma$  decay width. We discuss justifying the convenience of an experimental reanalysis of this resonance, in light of the findings of the paper, using multichannel unitary schemes.

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## I. INTRODUCTION

The  $N^*(1875)(3/2^-)$  is relatively new in the Review of Particle Physics by the Particle Data Group (PDG) [1]. Quoting from the latest PDG edition, “Before the 2012 Review, all evidence for a  $J^P = 3/2^-$  state with a mass above 1800 MeV was filed under a two-star  $N(2080)$ . There is now evidence from Ref. [2] for two  $3/2^-$  states in this region, so we have split the older data (according to mass) between a three-star  $N(1875)$  and a two-star  $N(2120)$ .” The mass according to the PDG is 1820–1920 MeV (1875 MeV PDG estimate) and the width  $250 \pm 70$  MeV. Quoting directly from Ref. [2], the mass is  $1880 \pm 20$  MeV and the width  $200 \pm 70$  MeV. A more recent experiment [3] agrees with these values, with  $1875 \pm 20$  MeV for the mass and  $200 \pm 25$  MeV for the width. The most important decay modes are  $N\omega$  (15–25%),  $\Delta(1232)\pi$  (10–35%), mostly in the  $s$  wave, and  $N\sigma$  [ $Nf_0(500)$ ] (30–60%).

It is interesting to recall that, prior to its acceptance as a new resonance, a peak in the amplitudes was observed around 1875 MeV from the study of the pseudoscalar meson-baryon decuplet interaction in Ref. [4]. For the case of strangeness  $S = 0$  and isospin  $I = \frac{1}{2}$ , the coupled channels  $\Delta\pi$  and  $\Sigma^*K$  were used, and the interaction was obtained from the meson-baryon Lagrangians of Ref. [5]. The peak appears at the  $\Sigma^*K$  threshold and it was identified as a threshold effect, not a genuine resonance. One should note that the identification of threshold effects with resonances is quite common, and one has a good example with the  $a_0(980)$  which is catalogued as a resonance, but it shows both theoretically [6] and experimentally [7] as a cusp effect with no clear pole associated with it.

In the present paper we take the work of Ref. [4] and include triangle mechanisms associated with the main building channels  $\Delta\pi$  and  $\Sigma^*K$ , which lead to new channels  $N^*(1535)\pi$  and  $Nf_0(500)$ . The first channel has not been measured yet, but the second channel has, together with  $\Delta\pi$  the main decay channel of the resonance. An effective transition potential is constructed from the  $\Delta\pi$ ,  $\Sigma^*K$  channels to the  $N^*(1535)\pi$  and  $Nf_0(500)$ , and a four-channel problem is then solved with a unitary coupled channel scheme, leading to a resonant peak around 1875 MeV in the amplitudes, from which we extract the coupling of the  $N^*(1875)$  resonance to the different channels and the partial decay widths to these channels.

Triangle diagrams have long been part of hadron physics, but of particular interest are those that lead to singularities in the amplitudes, known as triangle singularities. The concept and detailed study was introduced by Landau [8], but now, after much information on the hadron spectrum and reactions has been accumulated, many examples of triangle singularities have appeared [9]. A triangle diagram stems from a particle  $A$  decaying into  $1 + 2$ , particle  $2$  decaying to  $B + 3$ , and particles  $1 + 3$  merging into another particle  $C$ . In some cases, when the process can occur at the classical level, a singularity appears in the corresponding Feynman diagram, per the Coleman-Norton theorem [10], and the field-theoretical amplitude becomes infinity if the intermediate particles are stable. In practice, some of these particles have a finite width and the infinity becomes a peak, with important experimental consequences.

An alternative formulation to the standard method to deal with the triangle singularities is done in Ref. [11], with a different method to perform the integrals and an easy and intuitive rule to determine where the singularities appear.

Recent examples of processes where the triangle singularities are relevant can be seen in  $\eta(1405) \rightarrow \pi a_0(980)$  and  $\eta(1405) \rightarrow \pi f_0(980)$  [12–14]. The latter process is isospin forbidden and its results are largely enhanced due to a triangle singularity involving  $\eta(1405) \rightarrow K^* \bar{K}$  followed by

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$K^* \rightarrow K\pi$  and fusion of  $K\bar{K}$  to give the  $f_0(980)$ . A more recent example can be seen work by in the COMPASS Collaboration [15], associated with a new resonance,  $a_1(1420)$ , which, as hinted in Ref. [9] and proved in Refs. [16,17], comes from the  $\pi f_0(980)$  decay of the  $a_1(1260)$  via a triangle singularity proceeding through  $a_1 \rightarrow K^*\bar{K}$ ,  $K^* \rightarrow K\pi$ , and  $K\bar{K} \rightarrow f_0(980)$ . Related to this is the recent interpretation of the  $f_1(1420)$  resonance as a decay mode of the  $f_1(1285)$  into  $\pi a_0(980)$  and  $K^*\bar{K}$  [18]. Another interesting example is the role played by a triangle singularity in the  $\gamma p \rightarrow K^+\Lambda(1405)$  reaction [19]. The process  $\gamma p \rightarrow K^*\Sigma$ ,  $K^* \rightarrow K\pi$ ,  $\Sigma\pi \rightarrow \Lambda(1405)$  leads to a peak in the cross section around  $\sqrt{s} = 2120$  MeV that solved a standing problem in that reaction.

Similarly, the  $f_2(1810)$  is also explained as a consequence of the  $f_2(1650) \rightarrow K^*\bar{K}^*$ ,  $K^* \rightarrow \pi K$ , and  $K\bar{K}^*$  merging into the  $a_1(1260)$  [20]. Other examples can be found in Refs. [21–25]. Renewed interest in the triangular singularities came from the suggestion that the narrow peak of the  $J/\psi p$  invariant mass at 4450 MeV seen by the LHCb Collaboration [26,27], and interpreted there as a pentaquark state, could be due to a triangle singularity with  $\Lambda_b \rightarrow \Lambda(1890)\chi_{c1}$ ,  $\Lambda(1890) \rightarrow \bar{K}p$ ,  $p\chi_{c1} \rightarrow J/\psi p$  [28,29]. However, as shown in Ref. [11], for the preferred experimental quantum numbers of this peak,  $3/2^-, 5/2^+$ , the  $\chi_{c1}p \rightarrow J/\psi p$  proceeds with  $\chi_{c1}p$  in the  $p$  wave or  $d$  wave and the  $\chi_{c1}p$  threshold is exactly 4450 MeV, hence this amplitude vanishes there on shell and the suggested process cannot be responsible for the observed peak.

In the present work we will show that the process  $N^*(1875) \rightarrow \Sigma^*K$ ,  $\Sigma^* \rightarrow \pi\Lambda$ ,  $\Lambda K \rightarrow N^*(1535)$  develops a triangle singularity precisely at the same mass of resonance and reinforces it. The other interesting finding of this work is that there can also be triangle mechanisms, which, without developing a singularity, can be very important. This is actually the case with  $N^*(1875) \rightarrow \Delta\pi$ ,  $\Delta \rightarrow \pi N$ ,  $\pi\pi \rightarrow f_0(500)$ . We shall see that, because of the large strengths of all the couplings involved, this process becomes even more important than the  $N^*(1875) \rightarrow \pi N^*(1535)$  and leads to a sizable partial decay width of  $N^*(1875) \rightarrow N\sigma(f_0(500))$ .

## II. FORMALISM

### A. Brief review of the pseudoscalar meson-baryon decuplet interaction

Following Ref. [4], the sector with  $S = 0$ ,  $I = \frac{1}{2}$  is reached with the channels  $\Delta\pi$ ,  $\Sigma^*K$ . In the  $s$  wave the interaction leads to  $J^P = 3/2^-$  states. The interaction is given by

$$V_{ij} = -C_{ij} \frac{1}{4f^2} (k^0 + k'^0), \quad (1)$$

where  $k^0, k'^0$  are energies of the initial and final mesons respectively and the coefficients  $C_{ij}$  are given in Table I. The scattering matrix is given via the Bethe-Salpeter equation in matrix form by

$$T = [1 - VG]^{-1} V, \quad (2)$$

where  $G$  is the ordinary meson-baryon loop function. The  $\Delta\pi \rightarrow \Delta\pi$  amplitude develops a strong peak around

TABLE I. The  $C_{ij}$  coefficients of Eq. (1).

	$\Delta\pi$	$\Sigma^*K$
$\Delta\pi$	5	2
$\Sigma^*K$	2	2

1500 MeV that was associated in Ref. [4] with the  $N^*(1520)$  resonance. By contrast, this amplitude is very small around 1875 MeV, as a consequence of interference of terms, and it is the  $\Sigma^*K \rightarrow \Sigma^*K$  amplitude that shows up as a clear peak around 1875 MeV. In the next subsection we shall include the  $N^*(1535)\pi$  and  $Nf_0(500)$  channels.

### B. The $N^*(1535)\pi$ channel

We shall look into the diagram of Fig. 1, where the state  $i$  stands for  $\Delta\pi$  and  $\Sigma^*K$ . Since we are looking into the states with isospin  $I = \frac{1}{2}$ , we must consider in detail the different charge combinations that enter the evaluation of Fig. 1. This is shown in Figs. 2 and 3, where the state  $i$  is  $\Delta\pi$  or  $\Sigma^*K$ , respectively.

We must project all of them into  $I = \frac{1}{2}$  and sum the diagrams. This is done in Appendix A, and the full contribution of the loop is given by

$$\begin{aligned} -it_{\Delta\pi, \pi N^*} &= V_{\Delta\pi, \Sigma^*K}^{(I=1/2)} g_{\Sigma^*, \pi\Lambda} g_{N^*, K\Lambda} (\vec{S} \cdot \vec{k}) 2M_\Lambda 2M_{\Sigma^*} \\ &\times \int \frac{d^4 q}{(2\pi)^4} \frac{i}{(P - q)^2 - M_{\Sigma^*}^2 + i\epsilon} \\ &\times \frac{i}{(P - q - k)^2 - M_\Lambda^2 + i\epsilon} \frac{i}{q^2 - m_K^2 + i\epsilon} \\ &\equiv V_{\Delta\pi, \Sigma^*K}^{(I=1/2)} g_{\Sigma^*, \pi\Lambda} g_{N^*, K\Lambda} (\vec{S} \cdot \vec{k}) 2M_\Lambda 2M_{\Sigma^*} t_T, \end{aligned} \quad (3)$$

where the last line defines the triangle integral  $t_T$ . The factors  $2M_\Lambda, 2M_{\Sigma^*}$  are consequence of using the Mandl and Shaw normalization for the Fermion fields [30]. This integral is performed by doing analytically the  $q^0$  integration, and we obtain [11,31]

$$\begin{aligned} t_T &= \int \frac{d^3 q}{(2\pi)^3} \frac{1}{8\omega_K E_{\Sigma^*} E_\Lambda} \frac{1}{k^0 - E_\Lambda - E_{\Sigma^*}} \frac{1}{P^0 + \omega_K + E_\Lambda - k^0} \\ &\times \frac{1}{P^0 - \omega_K - E_\Lambda - k^0 + i\epsilon} \frac{1}{P^0 - E_{\Sigma^*} - \omega_K + i\epsilon} \\ &\times \{2P^0 \omega_K + 2k^0 E_\Lambda - 2(\omega_K + E_\Lambda)(\omega_K + E_\Lambda + E_{\Sigma^*})\}, \end{aligned} \quad (4)$$

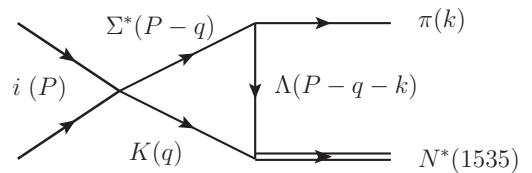
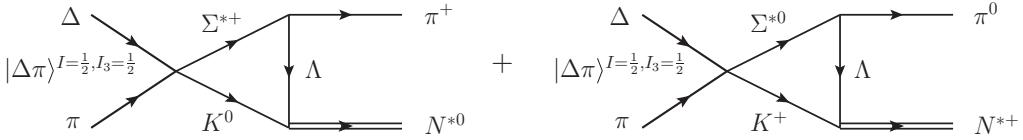


FIG. 1. The triangle diagram for the  $i(\Delta\pi, \Sigma^*K) \rightarrow N^*(1535)\pi$  transition. The momenta of the lines are in parentheses.

FIG. 2. Different diagrams leading to  $\Delta\pi \rightarrow N^*\pi$  ( $I = \frac{1}{2}$ ,  $I_3 = \frac{1}{2}$ ).

where

$$\begin{aligned}\omega_K &= \sqrt{m_K^2 + \vec{q}^2}, \quad E_{\Sigma^*} = \sqrt{M_{\Sigma^*}^2 + \vec{q}^2} + \frac{i\Gamma_{\Sigma^*}}{2}, \\ E_\Lambda &= \sqrt{M_\Lambda^2 + (\vec{q} + \vec{k})^2}, \quad k^0 = \frac{s + m_\pi^2 - M_{N^*}^2}{2\sqrt{s}}, \\ |\vec{k}| &= \frac{\lambda^{\frac{1}{2}}(s, m_\pi^2, M_{N^*}^2)}{2\sqrt{s}},\end{aligned}$$

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz. \quad (5)$$

We must note that originally the  $\vec{S} \cdot \vec{k}$  operator appeared for the  $\Sigma^* \rightarrow \pi \Lambda$  transition (see Appendix A), but upon summing the intermediate  $\Sigma^*$  spin components it becomes now in Eq. (3) the spin transition operator from  $\Delta$  to  $N^*$  because the  $s$ -wave potentials  $V_{\Delta\pi, \Sigma^*K}^{(I=1/2)}$  and  $V_{\Sigma^*K, \Sigma^*K}^{(I=1/2)}$  are independent of the  $\Delta$  and  $\Sigma^*$  spins.

Neglecting the width of the  $\Sigma^*$  in Eq. (4), the integrand in  $t_T$  will have poles when

$$P^0 - k^0 - \omega_K - E_\Lambda = 0 \quad \text{and} \quad P^0 - E_{\Sigma^*} - \omega_K = 0. \quad (6)$$

In principle, the integral will give rise to imaginary parts and principal values, via  $i\epsilon$ . However, the cancellations in the principal values will not occur when we are at the extremes of  $\cos\theta(\hat{k} \cdot \hat{q})$  when  $\cos\theta = \pm 1$ . Then a singularity can appear in the integral, triangle singularity (which, occurs for  $\cos\theta = -1$  [11]), which, however, is rendered finite when the width of the  $\Sigma^*$  is explicitly considered [11]. The integral in  $t_T$  is then convergent, but we perform a cutoff in  $q$  in the rest mass of the  $N^*$ , when the chiral unitary approach is taken, and we use  $q_{\max}^{\text{cm}} \equiv 1000$  MeV, suited to the results of Ref. [32].

Let us elaborate further on the singularities. The second of Eqs. (6) has a solution (keeping the  $i\epsilon$  of the propagators and neglecting the  $\Sigma^*$  width)  $q_{\text{on}} + i\epsilon$  (see Eq. (13) of Ref. [11]; the negative solution is irrelevant). The first of the Eqs. (6) for  $\cos\theta = -1$  has solution  $q_{a+} + i\epsilon$ ,  $q_{a-} - i\epsilon$  (see Eq. (15) of Ref. [11]) and, for  $\cos\theta = +1$ ,  $q_{b+} + i\epsilon$ ,  $q_{b-} - i\epsilon$  ( $q_{b-}$  is negative and irrelevant). With this pole structure it is clear that we have an unavoidable singularity when  $q_{a+} = q_{a-}$  or  $q_{\text{on}} = q_{a-}$ , since in both cases in the  $q$  integration we must pass between the two poles and we cannot deform the path

integral in the complex plane to avoid them. In the first case we have a threshold singularity ( $q_{a+} = q_{a-}$ ) and in the second case we have a triangle singularity ( $q_{\text{on}} = q_{a-}$ ).

We should note that the singularity arises from placing all intermediated particles of the loop on shell plus having  $\cos\theta = -1$ , and these conditions lead to the Coleman-Norton theorem [10] as discussed in Ref. [11]. In order to see if a triangle diagram develops a triangle singularity, we then search for a value of  $\sqrt{s}$  at which the following relationship is fulfilled:

$$q_{\text{on}+} - q_{a-} = 0. \quad (7)$$

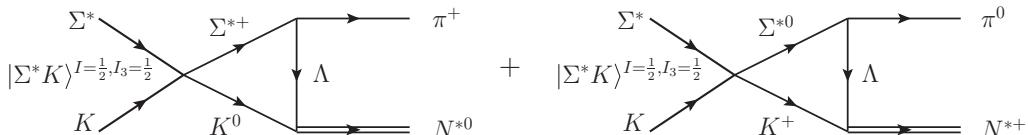
One must check Eq. (7) for a mass of the  $N^*(1535)$  bigger than  $m_\Lambda + m_K$ . At a mass about 1615 MeV, which is in the range of the  $N^*(1535)$  mass considering the width (150 MeV), Eq. (7) shows a solution at 1878 MeV. But a peak in the amplitude develops for smaller  $N^*$  masses within the range of the  $N^*(1535)$  spectral function, which we shall take into account in the evaluation of the diagram of Fig. 1.

We would like to include now the  $\pi N^*$  in the coupled channels, together with  $\Delta\pi$  and  $\Sigma^*K$ . However, we can see that while the interaction between  $\Delta\pi$  and  $\Sigma^*K$  proceeds via the  $s$  wave, the transition  $\Delta\pi \rightarrow \pi N^*$  proceeds via the  $p$  wave with the  $\vec{S} \cdot \vec{k}$  operator. This is a consequence from the transition of a  $\Delta(3/2^+)$  to  $N^*(1/2^-)$  which requires change of parity. Yet, it is possible to mix the channels via an effective  $s$ -wave potential, as done in Refs. [33–36]. In order to define this effective potential we look at the diagram of Fig. 4, which makes transitions  $\Delta\pi(\Sigma^*K) \rightarrow \Delta\pi(\Sigma^*K)$  via an intermediate  $\pi N^*$  state. We can write for the transition amplitude

$$-it_{ij} = -it_{i,\pi N^*} iG_{\pi N^*} (-i)t_{\pi N^*,j}. \quad (8)$$

In the chiral unitary approach the transition potentials are evaluated for the external lines on shell and we wish to do the same with the new channel  $\pi N^*$ . For this purpose, we take the imaginary part of  $G_{\pi N^*}$  in Eq. (8), which places  $\pi N^*$  on shell. Considering that in the  $\pi N^* \rightarrow \Delta\pi(\Sigma^*K)$  transition the pion momentum is ingoing instead of outgoing as in  $\Delta\pi(\Sigma^*K) \rightarrow \pi N^*$ , we have

$$\begin{aligned}\text{Im } t_{i,j} &= -V_{j,\Sigma^*K} V_{i,\Sigma^*K} (g_{\Sigma^*,\pi\Lambda} g_{N^*,K\Lambda} 2M_\Lambda 2M_{\Sigma^*})^2 \\ &\times (-)(\vec{S}^\dagger \cdot \vec{k})(\vec{S} \cdot \vec{k}) t_T t_T^* \text{Im } G_{\pi N^*}.\end{aligned}$$

FIG. 3. Different diagrams leading to  $\Sigma^*K \rightarrow N^*\pi$  ( $I = \frac{1}{2}$ ,  $I_3 = \frac{1}{2}$ ).

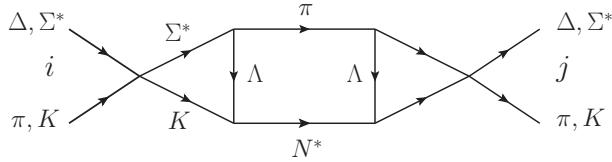


FIG. 4. Transition diagram from  $\Delta\pi(\Sigma^*K) \rightarrow \Delta\pi(\Sigma^*K)$  via the intermediate  $N^*\pi$ .

Now we have<sup>1</sup>

$$\begin{aligned} & \sum_m \langle M' | \vec{S}^\dagger \cdot \vec{k} | m \rangle \langle m | \vec{S} \cdot \vec{k} | M \rangle \\ &= \frac{1}{3} \vec{k}^2 \delta_{MM'} - \frac{1}{3} \vec{q}^2 \sqrt{4\pi} \mathcal{C} \left( \frac{3}{2}, 2, \frac{3}{2}; M', M - M' \right) \\ & \times Y_{2,M-M'}(\hat{q}), \end{aligned} \quad (9)$$

which indicates that we can have transitions in the  $s$  and  $d$  waves. But we are only interested in the  $s$  wave and hence we keep the  $\frac{1}{3} \vec{k}^2 \delta_{MM'}$  factor in Eq. (9). Thus, effectively we can take

$$\begin{aligned} \text{Im } \tilde{t}_{i,j} &= V_{j,\Sigma^*K} V_{i,\Sigma^*K} (g_{\Sigma^*,\pi\Lambda} g_{N^*,K\Lambda} 2M_\Lambda 2M_{\Sigma^*})^2 \\ & \times t_T t_T^* \frac{1}{3} \vec{k}^2 \text{Im } G_{\pi N^*} \end{aligned} \quad (10)$$

with

$$\text{Im } G_{\pi N^*} = -\frac{1}{4\pi} \frac{M_{N^*}}{\sqrt{s}} |\vec{k}|, \quad |\vec{k}| = \frac{\lambda^{\frac{1}{2}}(s, m_\pi^2, M_{N^*}^2)}{2\sqrt{s}}. \quad (11)$$

However, since the triangle singularity is sensitive to the external masses and the  $N^*$  has a width of 150 MeV, we make a convolution of Eq. (10) with the spectral function of the  $N^*$ , such that

$$\begin{aligned} \text{Im } \tilde{t}_{i,j} &= V_{j,\Sigma^*K} V_{i,\Sigma^*K} (g_{\Sigma^*,\pi\Lambda} g_{N^*,K\Lambda} 2M_\Lambda 2M_{\Sigma^*})^2 \\ & \times \frac{1}{N_{N^*}} \int d\tilde{m} \frac{1}{3} \vec{k}^2 S_{N^*}(\tilde{m}) \text{Im } G_{\pi N^*}(s, \tilde{m}) |t_T(s, \tilde{m})|^2, \end{aligned} \quad (12)$$

where  $\text{Im } G_{\pi N^*}(s, \tilde{m})$ ,  $\vec{k}$ , and  $t_T$  are obtained substituting  $M_{N^*} \rightarrow \tilde{m}$  in Eqs. (11) and (4). The spectral function of the  $N^*$  is given by

$$S_{N^*}(\tilde{m}) = -\frac{1}{\pi} \text{Im} \frac{1}{\tilde{m} - M_{N^*} + i\frac{\Gamma_{N^*}}{2}}, \quad (13)$$

and the factor  $N_{N^*}$  in Eq. (12) is normalized:

$$N_{N^*} = \int S_{N^*}(\tilde{m}) d\tilde{m}. \quad (14)$$

The limits of  $\tilde{m}$  in Eqs. (12) and (14) are taken from  $M_{N^*} - \alpha\Gamma_{N^*}$  to  $M_{N^*} + \alpha\Gamma_{N^*}$  with  $\alpha$  around 1 or 2. The  $\tilde{m}$  dependence in Eq. (12) does not affect  $V_{j,\Sigma^*K} V_{i,\Sigma^*K}$ , hence we can define

$$\text{Im } \tilde{t} = \frac{1}{N} \int d\tilde{m} \frac{1}{3} \vec{k}^2 S_{N^*}(\tilde{m}) \text{Im } G_{\pi N^*}(s, \tilde{m}) |t_T(s, \tilde{m})|^2 \quad (15)$$

<sup>1</sup>Note the order of  $\vec{S}^\dagger$  and  $\vec{S}$  and the sum over  $m$ , the spin of the 1/2 baryon.

and a function  $\tilde{V}$  such that

$$\text{Im } \tilde{t} = \tilde{V} \text{Im } G_{\pi N^*} \tilde{V}. \quad (16)$$

Then we can construct an effective  $s$ -wave transition potential

$$\tilde{V}_{i,\pi N^*} = V_{i,\Sigma^*K}^{(I=1/2)} g_{\Sigma^*,\pi\Lambda} g_{N^*,K\Lambda} 2M_\Lambda 2M_{\Sigma^*} \tilde{V} \quad (17)$$

such that

$$\tilde{V}_{i,\pi N^*} \text{Im } G_{\pi N^*}(M_{N^*}) \tilde{V}_{j,\pi N^*} = \text{Im } \tilde{t}_{i,j}. \quad (18)$$

This means that, using  $\tilde{V}_{i,\pi N^*}$  in coupled channels with the extra  $\pi N^*$  channel we can effectively incorporate the mechanism of Fig. 4, and when the resonance shows up in the amplitudes we can evaluate the coupling of the resonance to the  $\pi N^*$  channel and then the partial decay width into this channel. We will have now a new  $V$  matrix, containing the  $\Delta\pi(\Sigma^*K) \rightarrow \Delta\pi(\Sigma^*K)$  of Eq. (1) plus the  $\Delta\pi(\Sigma^*K) \rightarrow \pi N^*$  transition of Eq. (17). We do not include a direct  $\pi N^* \rightarrow \pi N^*$  transition, assuming such transition would occur via the  $\Delta\pi(\Sigma^*K)$  intermediate states involving the  $\Delta\pi(\Sigma^*K) \rightarrow \pi N^*$  transition which contains the triangle diagram.

In order to take into account the  $\Delta$  and  $\Sigma^*$  widths in the  $G$  functions of Eq. (2) we also do a convolution, as done in Ref. [4], with the spectral function of the baryons  $S_B(\tilde{M})$ :

$$G \rightarrow \tilde{G} = \frac{1}{N} \int d\tilde{M} G(\tilde{M}) S_B(\tilde{M}). \quad (19)$$

### C. The $Nf_0(500)(\sigma)$ channel

We can now consider a triangle diagram which involves  $\Delta\pi$  instead of  $\Sigma^*K$  in the intermediate states. This is depicted in Fig. 5. The states  $\Delta\pi$ ,  $\Sigma^*K$  can now make transition to the  $\Delta\pi$ , the  $\Delta$  decays into  $\pi N$ , and then the two pions fuse to give the  $f_0(500)(\sigma)$ . The first thing one must check is if this diagram can develop a singularity at some energy  $\sqrt{s}$ . Application of Eq. (7) immediately tells us that this is not the case, and  $q_{\text{on}+} - q_{\text{a}-}$  does not vanish for any energy of the original system. However, we have now other elements to make this mechanism particularly relevant. First, we can have now  $\Delta\pi \rightarrow \Delta\pi$  transitions that have a weight of a factor 5 (see Table I) instead of 2, as we had before. Second, the  $\Delta \rightarrow \pi N$  coupling is very large and so is the coupling of the  $\pi\pi$  to the  $\sigma$ . The evaluation of the  $\Delta\pi(\Sigma^*K) \rightarrow N\sigma$  transition proceeds in a way analogous to that in the previous subsection. First, in analogy to Figs. 2 and 3 we have now Figs. 6 and 7. The details are given in Appendix B. The new potentials  $\tilde{V}_{i,\pi N^*}$  and  $\tilde{V}_{i,N\sigma}$  are incorporated into the full  $T$  matrix for scattering between the  $\Delta\pi$ ,  $\Sigma^*K$ ,  $N^*\pi$ , and  $N\sigma$  channels using Eq. (2).

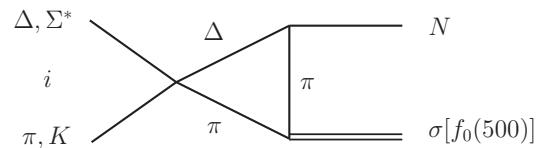
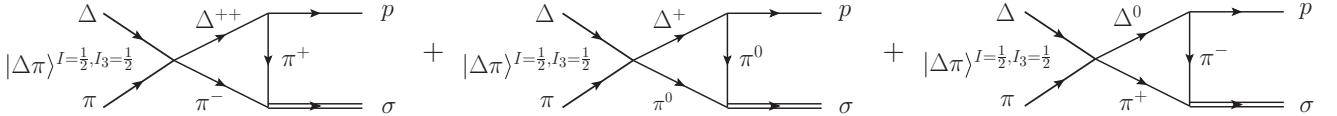


FIG. 5. Triangle diagram for the transition of  $i(\Delta\pi, \Sigma^*K) \rightarrow N\sigma$ .

FIG. 6. Diagrams for the  $|\Delta\pi, I = \frac{1}{2}, I_3 = \frac{1}{2}\rangle$  transition to  $p\sigma$ .

#### D. Couplings and partial decay widths

In order to obtain the couplings, we look at the amplitudes  $T_{ij}$  in Eq. (2), with  $i, j = \Delta\pi, \Sigma^*K, N^*\pi, N\sigma$ , and plot  $|T_{ij}|^2$ . We define the mass and width of the resonance, the position of the peak, and the width of the  $|T_{ij}|^2$  distribution as a function of  $\sqrt{s}$  close to the peak. In that region we have

$$T_{ij} = \frac{g_i g_j}{\sqrt{s} - M_R + i \frac{\Gamma_R}{2}}. \quad (20)$$

We take the  $\Sigma^*K$  channel as reference and then have

$$T_{22}|_{\text{peak}} = \frac{g_2^2}{i \frac{\Gamma_R}{2}}, \quad g_2^2 = i \frac{\Gamma_R}{2} T_{22}|_{\text{peak}}. \quad (21)$$

This defines  $g_2$  up to an arbitrary sign, but then the rest of the couplings are defined relative to this via

$$\frac{g_i}{g_2} = \frac{T_{i2}}{T_{22}}|_{\text{peak}}. \quad (22)$$

Once we have the coupling, the partial decay widths are given by

$$\Gamma_i = \frac{1}{2\pi} \frac{M_B}{M_R} |g_i|^2 p_i, \quad (23)$$

where  $M_B$  is the mass of the final baryon and  $M_R$  the mass of the resonance, and

$$p_i = \frac{\lambda^{1/2}(M_R^2, M_m^2, M_B^2)}{2M_R} \quad (24)$$

with  $M_m$  the mass of the final meson in the channels  $\Delta\pi, \Sigma^*K, N^*\pi, N\sigma$ .

The width of all channels is well defined except for the  $\Sigma^*K$ , since the resonance is close to threshold and both theoretically and experimentally the determination of  $\Sigma^*K$  in width is uncertain. With this caveat, we shall check that the sum of all partial decay widths is close to the total width determined from the shape of  $|T_{ij}|^2$ .

### III. RESULTS

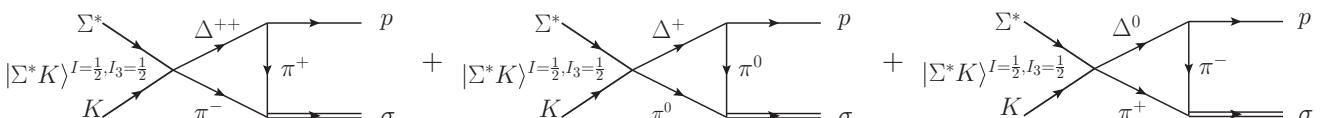
In Fig. 8 we show results for  $t_T$  for the loop function of Figs. 2 and 3 neglecting the width of  $\Sigma^*$ . Actually the evaluation is done by taking  $\Gamma_{\Sigma^*}/2 = 1$  MeV. The evaluation is done for different masses of the  $N^*(1535)$ , 1515, 1565,

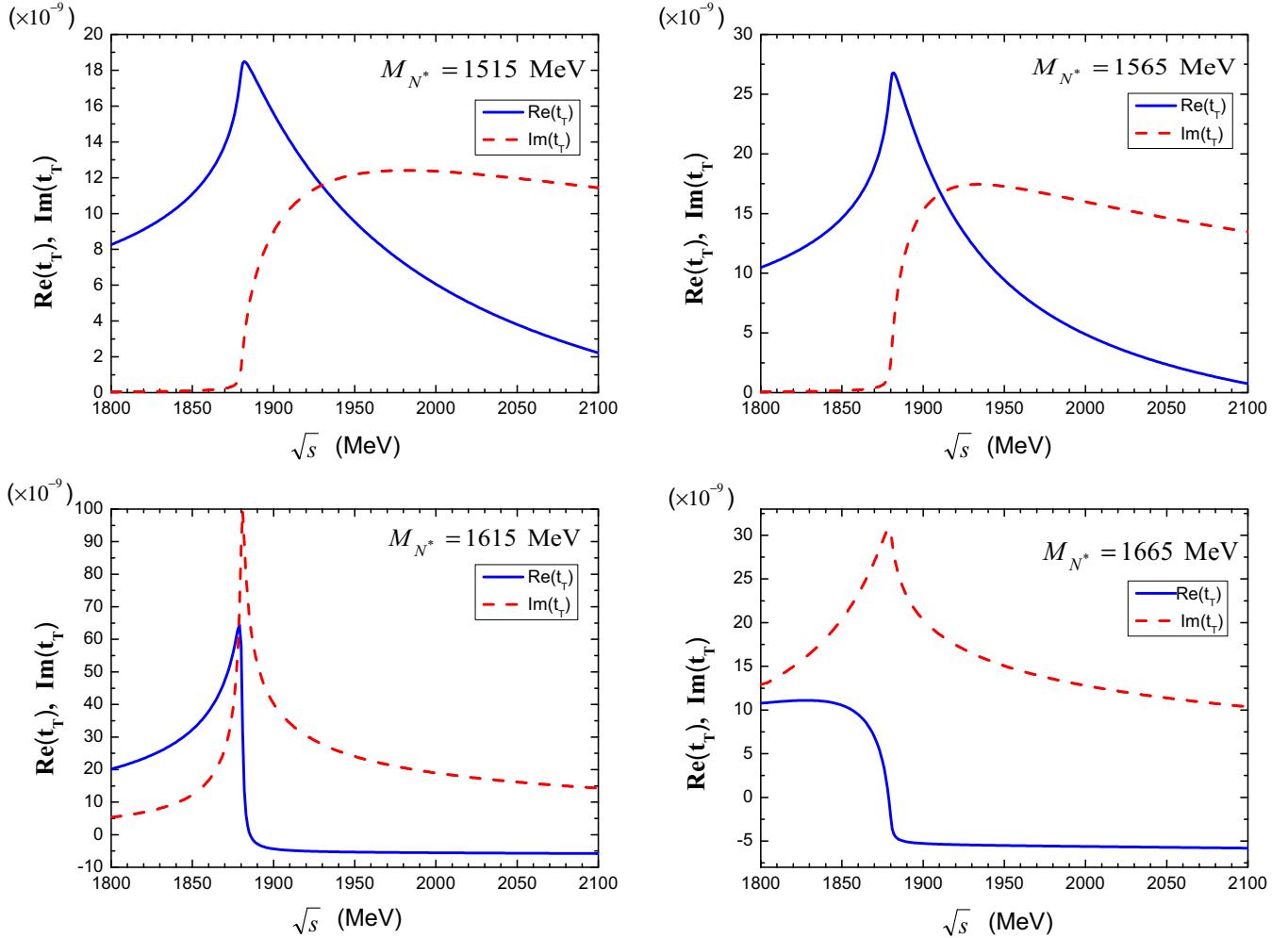
1615, 1665 MeV, in order to span the masses of the  $N^*(1535)$  given by its spectral function with a  $N^*$  width of 150 MeV. It is interesting to see how we get a narrow peak in all cases in the real part of  $t_T$ . The modulus square  $|t_T|^2$  is similar to  $\text{Re}(t_T)$  and peaks at the same energy. It is worth noting the fact that for  $M_{N^*} = 1615$  MeV the peak becomes sharper. This is because now this energy is very close to the  $\Delta K$  threshold 1608 MeV, such that, as discussed in Sec. II B, the triangle singularity and the threshold singularity superpose, leading to this sharp peak. The figure also shows how the peak of the singularity changes with the mass of the  $N^*(1535)$ . When the  $\Sigma^*$  width is considered, the sharp peaks become finite broader peaks, as we show below. We can see that  $\text{Re}(t_T)$  peaks at lower energies for the lower  $M_{N^*}$  masses, but in all the span of 150 MeV for  $M_{N^*}$  the position of the peak varies only by about 10 MeV.

In Fig. 9, we show the results for  $t_T$  as a function of  $\sqrt{s}$  for  $M_{N^*} \simeq 1535$  MeV when the  $\Sigma^*$  width is considered. We can see that  $\text{Re}(t_T)$  has a peak structure with a peak around 1885 MeV. The imaginary part has a different behavior, and does not show any peak. Actually,  $-it_T$  would resemble a Breit-Wigner amplitude with a constant magnitude added to the real part, which does not go through zero. The peak observed in  $\text{Re}(t_T)$  is tied to the triangle singularity that one would have in the case where  $\Gamma_{\Sigma^*} \rightarrow 0$ .

In Fig. 10 we show  $\tilde{V}_{i,\pi N^*}/V_{i,\Sigma^*K}^{(I=1/2)}$  from Eq. (17). This magnitude provides the relative strength of the effective transition potential  $i \rightarrow \pi N^*$ , with respect to  $i \rightarrow \Sigma^*K$ . We observe that the effective potential rises rapidly up to  $\sqrt{s} = 1900$  MeV and stabilizes there. The relative strength with respect to  $V_{i,\Sigma^*K}^{(I=1/2)}$  is of the order of 0.22 at the peak, which anticipates a moderate effect of this channel. However, the added strength around 1880 MeV helps stabilize the molecule that builds up around this energy from the interaction of the  $\Delta\pi$  and  $\Sigma^*K$  channels.

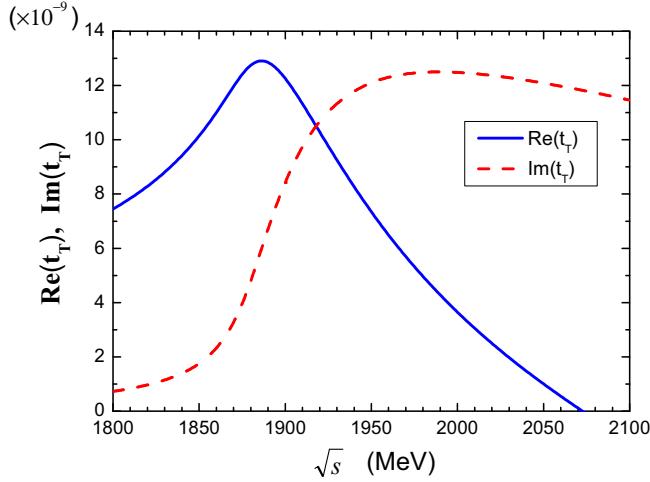
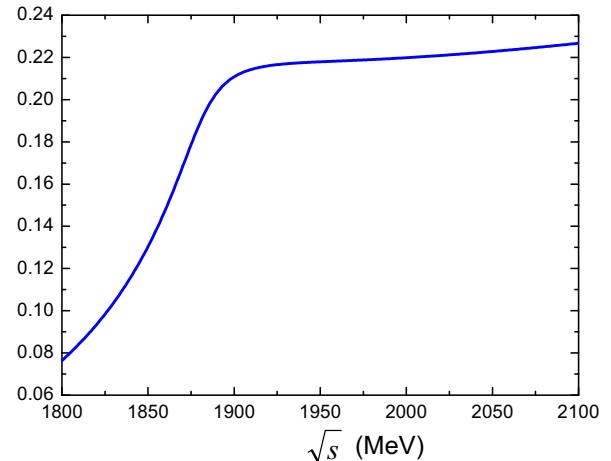
Next we show in Fig. 11 the results for  $t'_T$  of Sec. II C for Fig. 5 (see Appendix B). The convolution of Eq. (B11) over the  $\sigma$  mass is done between the masses  $2m_\pi$  and 800 MeV, and in Fig. 11 we plot  $t'_T$  in the middle of the range at  $\tilde{m}_\sigma = 540$  MeV. We can see that now we do not have any peak, as anticipated, since Eq. (7), which shows when there is a triangle singularity, is not fulfilled in this case. Yet we see that  $\text{Re}(t'_T)$  is of the same order of magnitude as  $\text{Re}(t_T)$  at the peak. However, since the effective transition potential contains different couplings now,

FIG. 7. Diagrams for the  $|\Sigma^*K, I = \frac{1}{2}, I_3 = \frac{1}{2}\rangle$  transition to  $p\sigma$ .

FIG. 8.  $\text{Re}(t_T), \text{Im}(t_T)$  for Figs. 2 and 3 as a function of  $\sqrt{s}$  for different values of  $M_{N^*}$ .

its strength becomes bigger than the one of the  $\Sigma^* K$  in the loop, as we show below.

In Fig. 12 we plot  $\tilde{V}_{i,\sigma N}/V_{i,\Delta\pi}^{(I=1/2)}$ . We can see that this magnitude is relatively constant, and from 1800 to 2100 MeV it changes from 0.73 to 0.69. However, we can see now that

FIG. 9. Real and imaginary parts of  $t_T$  of the triangle diagram, Eq. (4), with  $\Sigma^* K$  intermediate state.FIG. 10.  $\tilde{V}_{i,\pi N^*}/V_{i,\Sigma^* K}^{(I=1/2)}$  of Eq. (17).

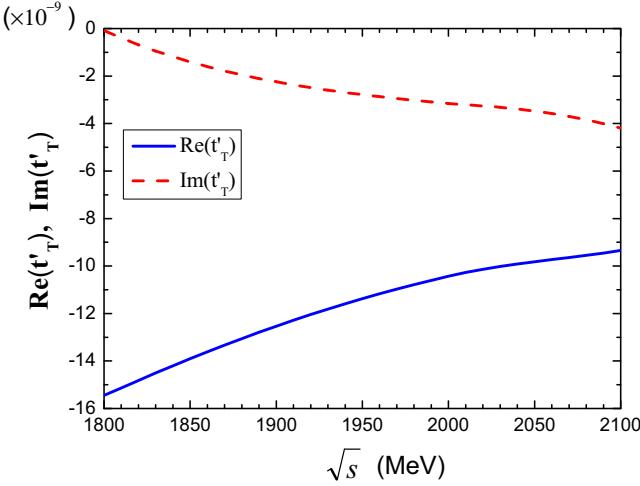


FIG. 11. Real and imaginary parts of  $t'_T$  corresponding to the triangle diagram of Fig. 5.

the strength is bigger than the one obtained from the  $\Sigma^* K$  loop at its peak ( $\sim 0.22$ ), in spite of the fact that we do not have a singularity now. As mentioned before, the different couplings in the mechanism are responsible for this relatively large strength. We see that the strength of  $\tilde{V}_{i,\sigma N}$  is of the same order of magnitude as the  $V_{i,\Delta\pi}^{(I=1/2)}$  transitions, and one anticipates an important role for this channel.

Next we turn to the amplitudes obtained with the coupled channels problem.

In Fig. 13 we show the modulus square of amplitude  $T_{ii}$  (with the order of the channels being  $\Delta\pi, \Sigma^* K, N^*\pi, N\sigma$ ) with just the  $\Delta\pi$  and  $\Sigma^* K$  channels, both omitting the width of the  $\Delta$  and  $\Sigma^*$ , and taking it into account. The results are similar to those obtained in Ref. [4], though in Ref. [4] complex energies were used instead of the convolution in the evaluation of the  $G$  function of Eq. (2). We can see a clear peak around 1880 MeV and that the consideration of the width of the  $\Delta$  and  $\Sigma^*$  leads to a wider structure which has about 72 MeV, short

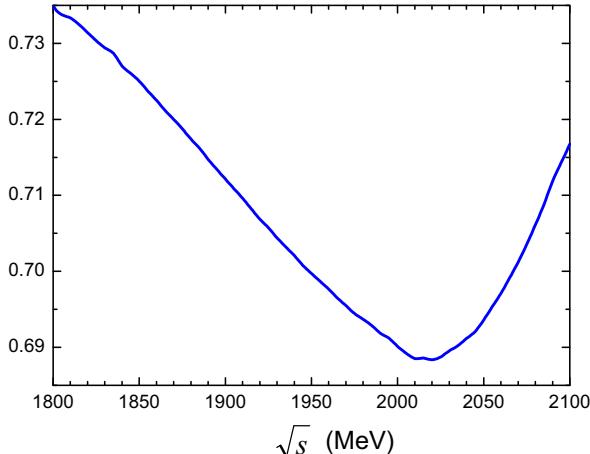


FIG. 12.  $\tilde{V}_{i,\sigma N}/V_{i,\Delta\pi}^{(I=1/2)}$  as a function of the energy corresponding to Eq. (B9).

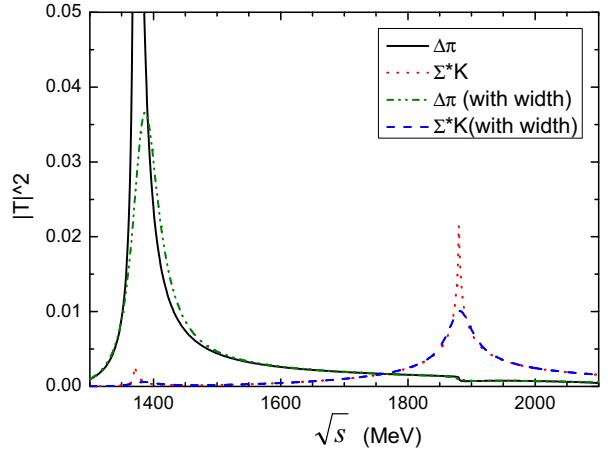


FIG. 13.  $|T_{ii}|^2$  for the  $\Delta\pi$  and  $\Sigma^* K$  channels alone as a function of the energy. The wider curve corresponds to considering the width of  $\Delta$  and  $\Sigma^*$ .

of the experimental central value of about 200 MeV, which, however, has large uncertainties. Note that the peak around 1880 MeV comes from the  $\Delta\pi$  and  $\Sigma^* K$  channels alone and it is not linked to the triangle singularity.

In Fig. 14 we show again the modulus square of amplitude  $T_{22}$  with two channels ( $\Delta\pi, \Sigma^* K$ ), three channels ( $\Delta\pi, \Sigma^* K, N^*\pi$ ), and four channels ( $\Delta\pi, \Sigma^* K, N^*\pi, N\sigma$ ). We can see that the introduction of the  $N^*\pi$  channel widens the peak a bit. The introduction of the  $N\sigma$  channel has not much effect on the width, but we shall see later that it has an important repercussion in the  $\pi\pi$  invariant mass distribution. From  $|T_{22}|^2$  with four channels we can get the mass and width of the  $N^*(1875)$  resonance:  $M_R = 1881.7$  MeV,  $\Gamma_R = 71.2$  MeV.

Next we look at the transition amplitudes from where we determine the couplings, via Eqs. (21) and (22). We show  $|T_{12}^2|$  in Fig. 15,  $|T_{32}^2|$  in Fig. 16, and  $|T_{42}^2|$  in Fig. 17, all of them evaluated with the four channels.

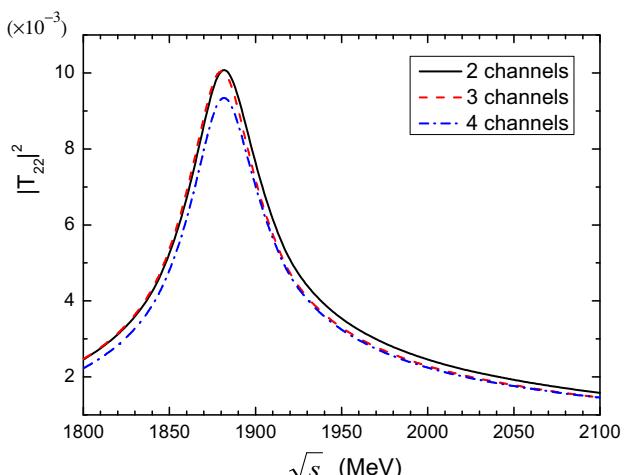


FIG. 14.  $|T_{22}|^2$  with two channels ( $\Delta\pi, \Sigma^* K$ ), three channels ( $\Delta\pi, \Sigma^* K, N^*\pi$ ), and four channels ( $\Delta\pi, \Sigma^* K, N^*\pi, N\sigma$ ).

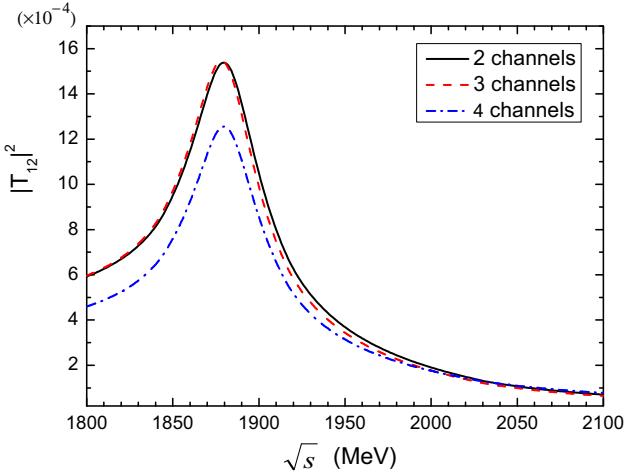


FIG. 15.  $|T_{12}|^2$  as a function of the energy, for the  $\Delta\pi \rightarrow \Sigma^*K$  transition.

The couplings that we get using Eqs. (21) and (22) are

$$\begin{aligned} g_{R,\Sigma^*K} &= 1.72 - 0.70i, & g_{R,\Delta\pi} &= 0.34 + 0.59i, \\ g_{R,N^*\pi} &= -0.29 + 0.17i, & g_{R,N\sigma} &= 0.22 - 0.01i. \end{aligned} \quad (25)$$

With these values and using  $\Gamma_i = \frac{1}{2\pi} \frac{M_i}{M_R} |g_i|^2 p_i$ , where  $M_i$  is the baryon mass for the final channel  $i$  and  $p_i$  its momentum, we obtain the partial decay widths

$$\begin{aligned} \Gamma_{\Delta\pi} &= 25.2 \text{ MeV}, & \Gamma_{\Sigma^*K} &= 13.1 \text{ MeV}, \\ \Gamma_{N^*\pi} &= 4.2 \text{ MeV}, & \Gamma_{N\sigma} &= 2.3 \text{ MeV}. \end{aligned} \quad (26)$$

We can see that  $\Gamma_{\Delta\pi}$  is quite large, but  $\Gamma_{N^*\pi}$  and  $\Gamma_{N\sigma}$  are much smaller.

The sum of  $\Gamma_i$  is 44.8 MeV, much smaller than the total width  $\Gamma_R = 71.2$  MeV. Yet, since the peak of the  $N^*(1875)$  has a mass distribution and the  $\Sigma^*$  has a width  $\Gamma_{\Sigma^*} = 36$  MeV [1], we should do a double convolution for the partial decay

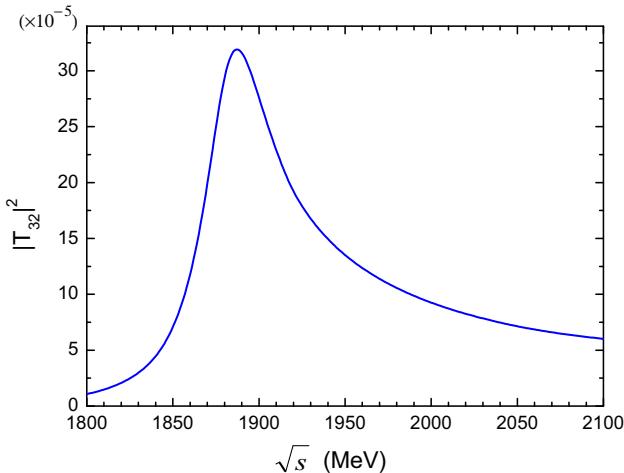


FIG. 16.  $|T_{32}|^2$  as a function of energy for the  $\Sigma^*K \rightarrow N^*\pi$  transition.

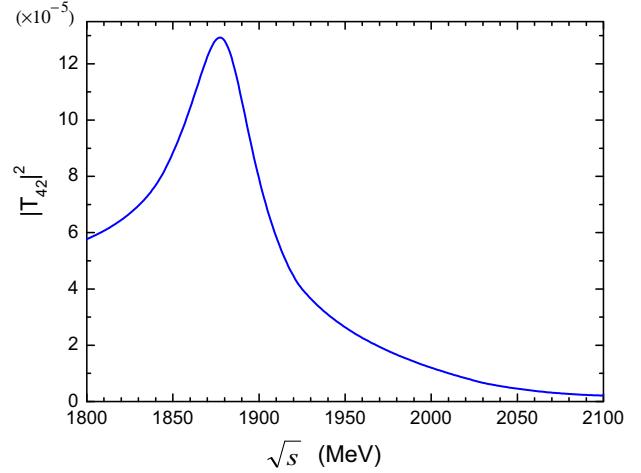


FIG. 17.  $|T_{42}|^2$  as a function of energy for the  $\Sigma^*K \rightarrow N\sigma$  transition.

width  $\Gamma_{\Sigma^*K}$ . This is done in Appendix C, and then we get

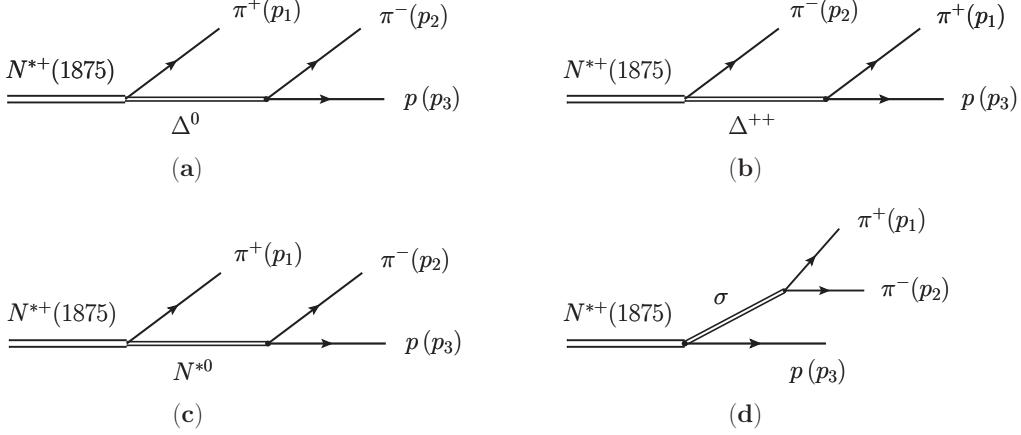
$$\tilde{\Gamma}_{\Sigma^*K} = 33.2 \text{ MeV}. \quad (27)$$

Then the sum of partial decay widths is 64.9 MeV, compatible with the total width.

For  $\pi^*\pi$  the prediction is new and should be observed in the  $\pi\eta N$  mode since  $N^*(1535)$  decays into  $\eta N$  with a branching fraction of 32–52%, and it would be a better channel than the  $\pi N$  that could be mixed with the  $\Delta\pi$  decay. Regarding  $\Sigma^*K$ , which is also not measured, there are certainly problems when one is close to threshold. However, a proper unitary multichannel analysis, as done in Ref. [37–39], should show the relevance of this channel. One similar case where this has been done is in the  $N^*(1700)(3/2^-)$  resonance, which in Ref. [40] is shown to appear from the interaction of a vector baryon, mostly from  $\rho N$ , which is at threshold there. This case has been revised in Ref. [41] to include the  $\Delta\pi$  channel, associated with another triangle singularity. The  $\rho N$  channel being around threshold is not an obstacle to obtain a  $(38 \pm 6)\%$  branching ratio for  $\rho N$  in the analysis of Ref. [37].

#### IV. MASS DISTRIBUTIONS

Now we wish to get the mass distributions for pairs of particles. We choose to study the  $\pi^+\pi^-p$  final state. We will have the contributions of Fig. 18. The first thing to observe is that the  $\pi^+\pi^-p$  channel does not exhaust all the width. Indeed, in the case of  $\pi\Delta$  decay we have three more cases:  $R^+ \rightarrow \pi^+\Delta^0 \rightarrow \pi^+\pi^0n$ ,  $R^+ \rightarrow \pi^0\Delta^+ \rightarrow \pi^0\pi^+n$ , and  $R^+ \rightarrow \pi^0\Delta^+ \rightarrow \pi^0\pi^0p$ , with  $R$  standing for the resonance  $N^*(1875)$ . Using the coefficients for the weights of the different  $\pi\Delta$  components in Eq. (A1) and those for  $\Delta \rightarrow \pi N$  in Eq. (B3), we find that the  $\pi^+\pi^-p$  mechanisms of Fig. 18 account for  $\frac{5}{9}$  of the  $\pi\Delta$  width, while the channels not considered account for  $\frac{4}{9}$  of the width. Similarly, for the  $\pi N^*$  we are missing  $R^+ \rightarrow \pi^+N^{*0} \rightarrow \pi^+\pi^0n$ ,  $R^+ \rightarrow \pi^0N^{*+} \rightarrow \pi^0\pi^+n$ , and  $R^+ \rightarrow \pi^0N^{*+} \rightarrow \pi^0\pi^0p$ . Taking into account the coefficients of Eq. (A1), we find again that with the  $\pi^+\pi^-p$  final state we take into account  $\frac{4}{9}$  of the  $\pi N^*$  width and

FIG. 18. Diagrams for the decay of the  $N^*(1875)$  resonance into  $\pi\Delta$  [(a) and (b)],  $\pi N^*$  (c), and  $\sigma p$  (d).

the missing channels account for  $\frac{5}{9}$  of it. Regarding the spin dependence of the diagrams of Fig. 18, the  $R^+(\frac{3}{2}^-)$  coupling to  $\pi\Delta$  goes as a constant, and the  $\Delta \rightarrow \pi N$  as  $\vec{S} \cdot \vec{p}_\pi$ . The  $N^* \rightarrow \pi N$  also goes as a constant but the  $R \rightarrow \pi N^*$  goes as  $\vec{S} \cdot \vec{p}_\pi$ . On the other hand the  $R^+(\frac{3}{2}^-)$  coupling to  $\sigma p$  also goes as  $\vec{S} \cdot \vec{p}_\pi$  [see Eq. (B3)]. Then considering Eqs. (A1) and (B3) and the isospin decomposition of the  $N^{*0} \rightarrow \pi N$  decay,  $|N^{*0}, \pi N, I=\frac{1}{2}, I_3=-\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|\pi^0 n\rangle - \sqrt{\frac{2}{3}}|\pi^- p\rangle$ , we have

$$\begin{aligned}
- i t_{R^+ \rightarrow \pi^+ \Delta^0 \rightarrow \pi^+ \pi^- p} &= g_{R, \pi\Delta} \left( -\sqrt{\frac{1}{6}} \right) \sqrt{\frac{1}{3}} \\
&\times \frac{1}{M_{23} - M_\Delta + i \frac{\Gamma_\Delta}{2}} \vec{S} \cdot \vec{p}_2 \frac{f_{\pi N\Delta}}{m_\pi}, \\
- i t_{R^+ \rightarrow \pi^- \Delta^{++} \rightarrow \pi^- \pi^+ p} &= g_{R, \pi\Delta} \sqrt{\frac{1}{2}} (-1) \vec{S} \cdot \vec{p}_1 \\
&\times \frac{1}{M_{13} - M_\Delta + i \frac{\Gamma_\Delta}{2}} \frac{f_{\pi N\Delta}}{m_\pi}, \\
- i t_{R^+ \rightarrow \pi^+ N^{*0} \rightarrow \pi^+ \pi^- p} &= g'_{R, \pi\Delta} \left( \sqrt{\frac{2}{3}} \right) \left( -\sqrt{\frac{2}{3}} \right) \vec{S} \cdot \vec{p}_1 \\
&\times \frac{1}{M_{23} - M_{N^*} + i \frac{\Gamma_{N^*}}{2}} g_{N^*, \pi N}, \\
- i t_{R^+ \rightarrow \sigma p \rightarrow \pi^+ \pi^- p} &= g'_{R, \sigma N} \left( -\sqrt{\frac{2}{3}} \right) \vec{S} \cdot (\vec{p}_1 + \vec{p}_2) (-1) \\
&\times \frac{1}{M_{12}^2 - m_\sigma^2 + i m_\sigma \Gamma_{N^*}} g_{\sigma, \pi\pi}, \tag{28}
\end{aligned}$$

where  $M_{ij}^2 = (p_i + p_j)^2$ ;  $i, j = 1, 2, 3$  for  $\pi^+, \pi^-, p$ . In the last equation we have considered that  $|\pi\pi, I=0\rangle = -\frac{1}{\sqrt{3}}(\pi^+\pi^- + \pi^-\pi^+ + \pi^0\pi^0)$ . The coupling of the  $\sigma$  to  $\pi^+\pi^-$  has a  $(-\frac{1}{\sqrt{3}})$  coefficient, and likewise for the  $\pi^-\pi^+$ , but considering the integrated  $\pi^+\pi^-$  and  $\pi^-\pi^+$  width one is

counting twice the contribution. All this is solved by taking the coefficient  $(-\sqrt{\frac{2}{3}})$ . The extra minus sign in the last of Eqs. (28) is because  $p_p = -\vec{p}_1 - \vec{p}_2$ .

We take  $g_{N^*, \pi N} = 0.70$  from Ref. [32]. In Eq. (28), we have used the couplings  $g'_{R, \pi N^*}$  and  $g'_{R, \sigma N}$  instead of  $g_{R, \pi N^*}$  and  $g_{R, \sigma N}$ . This is because the factors  $\vec{S} \cdot \vec{k}$  were already taken into account when we evaluated the effective transition potentials [see Eq. (9) and Eqs. (10) and (B11)], which already incorporate the factor  $\frac{1}{3}\vec{k}^2$  coming from this operator in the sum over  $\pi N^*$  and  $\sigma p$  intermediate states. To take this into account it is sufficient to write

$$g'_{R, \pi N^*} = \frac{\sqrt{3}}{p_1} g_{R, \pi N^*}, \quad g'_{R, \sigma p} = \frac{\sqrt{3}}{|\vec{p}_1 + \vec{p}_2|} g_{R, \sigma p}. \tag{29}$$

After this discussion, we can write the full amplitude for  $R^+ \rightarrow \pi^+ \pi^- p$  from the diagrams of Fig. 18 as

$$\begin{aligned}
- i t_{\text{tot}} &= (B + C + D) \vec{S} \cdot \vec{p}_1 + (A + D) \vec{S} \cdot \vec{p}_2 \\
&= A' \vec{S} \cdot \vec{p}_1 + B' \vec{S} \cdot \vec{p}_2, \tag{30}
\end{aligned}$$

where

$$\begin{aligned}
A &= -\frac{1}{3\sqrt{2}} \frac{g_{R, \pi\Delta} \cdot f_{\pi N\Delta}}{m_\pi} \frac{1}{M_{23} - M_\Delta + i \frac{\Gamma_\Delta}{2}}, \\
B &= -\frac{1}{\sqrt{2}} \frac{g_{R, \pi\Delta} \cdot f_{\pi N\Delta}}{m_\pi} \frac{1}{M_{13} - M_\Delta + i \frac{\Gamma_\Delta}{2}}, \\
C &= -\frac{2}{\sqrt{3}} \frac{g_{R, \pi N^*} \cdot g_{N^*, \pi N}}{p_1} \frac{1}{M_{23} - M_{N^*} + i \frac{\Gamma_{N^*}}{2}}, \\
D &= \sqrt{2} \frac{g_{R, \sigma N} \cdot g_{\sigma, \pi\pi}}{|\vec{p}_1 + \vec{p}_2|} \frac{1}{M_{12}^2 - m_\sigma^2 + i m_\sigma \Gamma_\sigma}, \\
A' &= B + C + D, \quad B' = A + D. \tag{31}
\end{aligned}$$

The differential mass distribution is given by [1]

$$\frac{d^2\Gamma}{dM_{12} dM_{23}} = \frac{1}{(2\pi)^3} \frac{4M_p M_R}{32M_R^3} \sum \sum |t_{\text{tot}}|^2 4M_{12} M_{23}, \tag{32}$$

where, using Eq. (9), we find

$$\sum \sum |t_{\text{tot}}|^2 = \frac{1}{3} [|A'|^2 \vec{p}_1^2 + |B'|^2 \vec{p}_2^2 + 2 \text{Re}(A'B'^*) \vec{p}_1 \cdot \vec{p}_2], \quad (33)$$

where  $\vec{p}_1 \cdot \vec{p}_2$  can be written in terms of  $M_{12}$  as

$$2\vec{p}_1 \cdot \vec{p}_2 = m_1^2 + m_2^2 + 2E_1 E_2 - M_{12}^2 \quad (34)$$

and  $E_1, E_2$  as

$$E_1 = \frac{M_R^2 + m_1^2 - M_{23}^2}{2M_R}, \quad E_2 = \frac{M_R^2 + m_2^2 - M_{13}^2}{2M_R}. \quad (35)$$

To obtain  $\frac{d\Gamma}{dM_{12}}$ , we integrate Eq. (32) over  $M_{23}$ , and the limits are given by the PDG [1]. In  $t_{\text{tot}}$  we need  $M_{12}, M_{13}, M_{23}$  as variables. To evaluate Eq. (32), we need  $M_{13}$ , which is given in terms of the other variables as

$$M_{13}^2 = M_R^2 + m_1^2 + m_2^2 + m_3^2 - M_{12}^2 - M_{23}^2. \quad (36)$$

If we wish to obtain  $\frac{d\Gamma}{dM_{23}}$ , we integrate Eq. (32) over  $M_{12}$ . The limits for  $M_{23}$  can be obtained from those for  $M_{12}$  by permuting the indices  $123 \rightarrow 321$ . Similarly we can obtain  $\frac{d^2\Gamma}{dM_{12}^2 dM_{13}}$  as in Eq. (32), substituting the factor  $2M_{23}$  by  $2M_{13}$ . Then we get  $\frac{d\Gamma}{dM_{13}}$  by integrating over  $M_{12}$ , and the limits for  $M_{12}$  are obtained from the standard formula of the PDG, permuting the indices  $123 \rightarrow 312$ .

## V. RESULTS FOR THE MASS DISTRIBUTIONS

In the limit of the small widths for the  $\Delta$ ,  $N^*$ , and  $\sigma$ , the different terms in Eq. (28) do not interfere since they correspond to different final states  $\pi\Delta$ ,  $\pi N^*$ ,  $\sigma N$ . However, if we look at  $\pi^+\pi^-p$  production and consider the widths, there can be interference. In particular there should be interference between  $\pi^-\Delta^{++}$  and  $\sigma p$  [B and D terms in Eqs. (31)]. Note that the B term is three times larger in strength than term A). The fact that these two terms have the same spin structure ( $\vec{S} \cdot \vec{p}_1$ ) helps for the interference.

In Fig. 19, we plot the  $\frac{d\Gamma}{dM_{12}}$ ,  $\frac{d\Gamma}{dM_{13}}$ , and  $\frac{d\Gamma}{dM_{23}}$  mass distributions for the  $N^{*+}(1875) \rightarrow \pi^+\pi^-p$  decay with 1,2,3 denoting  $\pi^+$ ,  $\pi^-$ , and  $p$ . Let us first look at the mass distributions considering only  $\pi\Delta$  production (red dashed lines in Fig. 19). We see that for  $M_{13}(\pi^+p)$  there is a large signal of the  $\Delta^{++}(1232)$  coming from term B. The  $\Delta(1232)$  is also seen in the  $M_{23}(\pi^-p)$  distribution ( $\pi^-p$ ) coming from term A. Removing a small background below the  $\Delta^0$  peak in the  $M_{23}$  distribution, we can see that the strength for  $\Delta^{++}$  in the  $M_{13}$  distribution is about nine times the one of the  $\Delta^0$ , as it corresponds to the coefficients in the terms A and B, squaring them. The rest of the strength in the  $M_{23}$  distribution peaks around  $M_{23} = 1500$  MeV, as a consequence of the phase space and the weight of the term B being nonresonant in this channel. The  $M_{12}(\pi^+\pi^-)$  mass distribution does not show any resonance and follows the phase space, weighted by the terms A and B.

Next we consider the  $\pi N^*$  term, including in addition the C term in Eq. (30). The results are shown in Fig. 19 as the blue dash-dotted lines. The results are shown in Fig. 19 as the blue dash-dotted lines. We do not see much change except

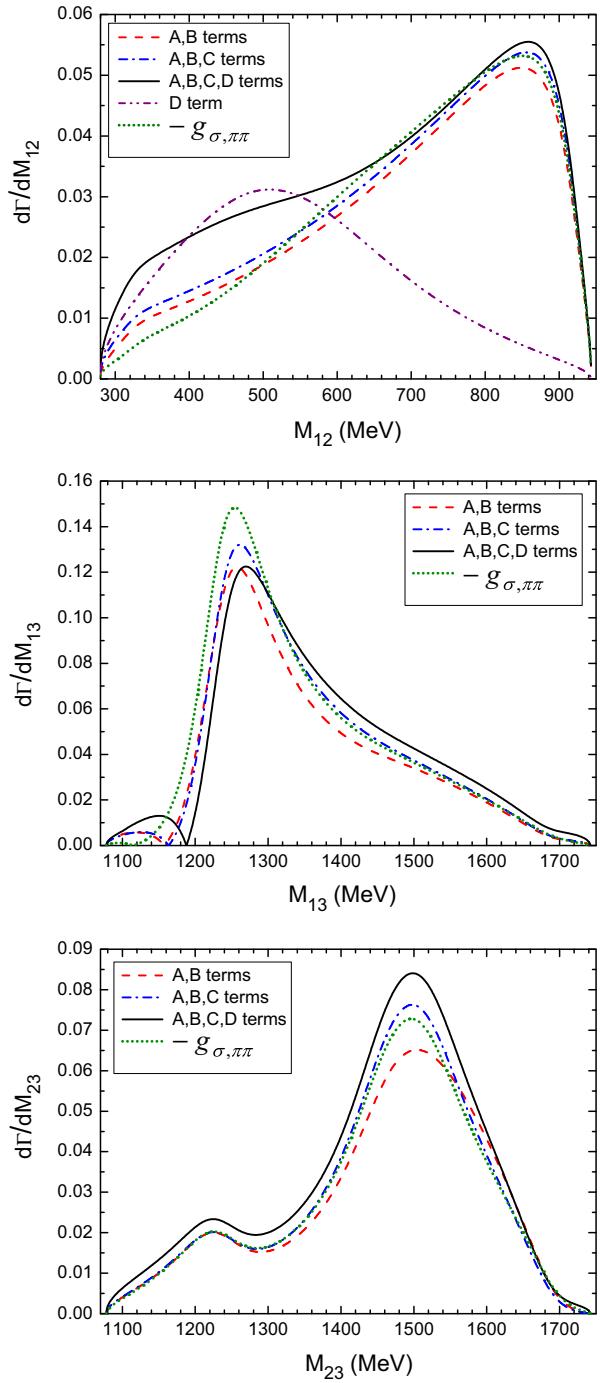


FIG. 19. The  $\frac{d\Gamma}{dM_{12}}$ ,  $\frac{d\Gamma}{dM_{13}}$ , and  $\frac{d\Gamma}{dM_{23}}$  mass distributions for the  $N^{*+}(1875) \rightarrow \pi^+\pi^-p$  decay with 1,2,3 denoting  $\pi^+$ ,  $\pi^-$ , and  $p$ , with the red dashed lines indicating the case considering only  $\pi\Delta$  production [i.e., A and B terms in Eq. (30)], the blue dash-dotted lines the case considering  $\pi\Delta$  and  $\pi N^*$  production [i.e., A, B, and C terms in Eq. (30)], the black solid lines the case considering  $\pi\Delta$ ,  $\pi N^*$ , and  $\sigma N$  production [i.e., all the A, B, C, and D terms in Eq. (30)], and the green dotted lines the case of changing the sign of  $g_{\sigma,\pi\pi}$  in Eq. (B7). The purple dash-dot-dotted line in the  $\frac{d\Gamma}{dM_{12}}$  distribution corresponds to the case considering only  $\sigma N$  production [i.e., the D term in Eq. (30)] fitted approximately to the  $M_{12}$  distribution at low masses.

for an enhancement of the peak in the  $M_{23}$  distribution ( $\pi^- p$ ) corresponding to the  $N^*$  excitation by the  $C$  term. However, the change is not large. Yet, here we see a possible reason why the  $\pi N^*$  channel has not been claimed experimentally. Indeed, the  $\pi \Delta$  mechanism alone already creates a peak in the  $M_{23}$  distribution in the region of 1500 MeV, which cannot be associated with  $\pi N^*$  production. Any  $\pi N^*$  production can be easily attributed to the  $\Delta$  production in the  $M_{13}(\pi^+ p)$  channel. This has also a consequence in terms of a message: To determine the  $\pi N^*$  production one should better look at  $\pi \eta N$  production.

We show the results including all the production terms,  $A + B + C + D$ , in Fig. 19 as the black solid lines. This includes  $\sigma p$  production in addition to the former channels. The results are interesting. Apart from the basic features that we have observed in the former cases, now the  $M_{12}(\pi^+ \pi^-)$  mass distribution contains a large bump in the region of low invariant  $\pi\pi$  masses corresponding to the  $\sigma$  production. A smooth extrapolation of the low energy  $M_{12}$  distribution with a wide  $\sigma$  shape would tell us that about  $\frac{1}{2}$  of the width could be attributed to  $\sigma N$  production. To quantify this we have used the  $D$  term of Eq. (31) alone, and roughly adjusted its strength to the low mass region of the  $M_{12}$  distribution. This is telling us that an analysis of the mass distributions, due to interference of terms, would provide an apparently larger strength for the  $\sigma p$  channel than one would induce from the coupling of the resonance to the different channels, as done in Eqs. (26). Actually, since we are only considering  $\frac{5}{9}$  of the  $\pi \Delta$  production in these figures, taking into account the results of Eq. (26), we would be extracting a width of around 7 MeV from this analysis, which would turn into  $\frac{3}{2} \times 7 \sim 11$  MeV if one considers the  $\sigma \rightarrow \pi^0 \pi^0$  decay also. This is bigger than the 2.3 MeV that we obtained in Eq. (26), and would correspond to a branching fraction of about 15%.

There is another issue worth considering. In the determination of the couplings there is always a global sign which is arbitrary. The result of the couplings in a coupled channel problem have the relative phase well determined with respect to this one. But the  $\pi\pi$  channel is not coupled to  $\pi N$  or  $\pi \Delta$ . We would like to see what happens if we change the sign of the  $g_{\sigma, \pi\pi}$  in Eq. (B7). The results are shown in Fig. 19 as the green dotted lines, and we see that the effects are moderate. One should note that it is precisely in observables that involve interference of the terms that the signs of couplings relative to other signs of, in principle, unrelated couplings can be determined.

## VI. CONCLUSIONS

In this work we have complemented the developments of Ref. [4] in which a  $3/2^-$  resonance appears around 1875 MeV from the interaction of the  $\Sigma^* K$  and  $\Delta \pi$  channels. In a first step we introduced the  $N^*(1535)\pi$  channel, which is produced via a triangle singularity in which  $\Sigma^* K$  is produced, then the  $\Sigma^*$  decays to  $\Lambda \pi$  and finally the  $\Lambda K$  merge to produce the  $N^*(1535)$ . The interesting observation is that the singularity appears at the same energy as the resonance, and then it shows at the same peak and helps stabilize the resonance in the sense

that even with a weaker  $\Sigma^* K$  interaction the singularity always appears at the same energy. The other decay channel that we introduced is the  $N\sigma$  channel. We also used a triangle diagram to take it into account, taking the  $\Delta \pi$  intermediate state, letting the  $\Delta$  decay to  $N\pi$  and then merging the two pions into a  $\sigma$  meson. Then we take the scheme of [4] adding the two new channels to the original  $\Sigma^* K$  and  $\Delta \pi$  ones, and with the four coupled channels we study again the resonance, the couplings to the different channels, and its decay into these channels. We observe that the partial decay widths of the resonance to  $N^*(1535)\pi$  and  $N\sigma$  are not large but are measurable. In particular, we observed that the  $N\sigma$  channel was much smaller than what is determined experimentally from some experiments. Yet, since channel separation is done from mass distributions, we showed that due to interference with other terms the  $\pi\pi$  mass distribution showed an important enhancement at low invariant masses, from which one could extract an appreciably larger fraction of  $N\sigma$  than one gets from the couplings yet smaller than the experimental claims.

An important part of the work was the study of the  $\Sigma^* K$  decay channel. This channel is not easy to separate in an analysis because the resonance has its mass at the threshold of the channel. In fact no experiment has made claims about this channel. However, we see that the channel is very important in the building of the resonance, and that taking into account the width of the resonance and the width of the  $\Sigma^*$  we obtained a branching fraction of about 45%. It is clear that if this channel is omitted in the analysis, its strength can easily be attributed to another channel. So, in view of the unavoidable large strength of this channel, we suggest that modern multichannel analyses implementing unitarity in coupled channels are used to revise this resonance. There is a clear example in a related case, where the multichannel analysis provides also a sizable contribution of a threshold channel—the  $N\rho$  in the case of the  $N^*(1700)(3/2^-)$  [37]—where other analyses [2] neglect it.

The determination of the  $N^*(1535)\pi$  channel is also relevant since it will evidence the role of a triangle singularity peaking at the resonance position. Yet, the discussion of the mass distributions in the  $\pi^+ \pi^- p$  final state showed that the mass distribution for  $N^*(1535)\pi^+$ ,  $N^*(1535) \rightarrow \pi^- p$  had the same signature in the  $\pi^- p$  mass distribution as that coming from the  $\Delta^{++} \pi^-$  excitation mechanisms, where the  $\pi^- p$  is not forming the  $\Delta$ . This is why if one wishes to determine this channel, the ideal final state should be  $\pi \eta N$  not  $\pi\pi N$ .

The thorough work conducted here on the building up of the resonance, its decay channels, and the mass distributions in the  $\pi\pi N$  channel, together with the discussion above, clearly indicate that a reanalysis of this resonance to the light of the present findings should be most welcome.

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## APPENDIX A: EVALUATION OF THE AMPLITUDES OF FIGS. 2 AND 3.

We follow the sign convention of Ref. [4],  $|\pi^+\rangle = -|1,1\rangle$ , of isospin. Hence we have

$$\begin{aligned} \left| \Delta\pi, I = \frac{1}{2}, I_3 = \frac{1}{2} \right\rangle &= \sqrt{\frac{1}{2}} |\Delta^{++}\pi^-\rangle - \sqrt{\frac{1}{3}} |\Delta^+\pi^0\rangle - \sqrt{\frac{1}{6}} |\Delta^0\pi^+\rangle, \\ \left| N^*(1535)\pi, I = \frac{1}{2}, I_3 = \frac{1}{2} \right\rangle &= \sqrt{\frac{2}{3}} |N^{*0}\pi^+\rangle + \sqrt{\frac{1}{3}} |N^{*+}\pi^0\rangle, \\ \left| \Sigma^*K, I = \frac{1}{2}, I_3 = \frac{1}{2} \right\rangle &= \sqrt{\frac{2}{3}} |\Sigma^{*+}K^0\rangle - \sqrt{\frac{1}{3}} |\Sigma^{*0}K^+\rangle. \end{aligned} \quad (\text{A1})$$

We need the  $\Sigma^* \rightarrow \Lambda\pi$  coupling and the  $K\Lambda \rightarrow N^*(1535)$  coupling. The first one is of the type

$$-it_{\Sigma^*,\pi\Lambda} = g_{\Sigma^*,\pi\Lambda} \vec{S} \cdot \vec{k}, \quad (\text{A2})$$

where  $\vec{S}$  is the spin transition operator from 3/2 to 1/2. The width for  $\Sigma^* \rightarrow \pi\Lambda$  is given by ( $\vec{k} \equiv \vec{p}_\pi$ )

$$\Gamma = \frac{2M_\Lambda 2M_{\Sigma^*}}{8\pi} \frac{1}{M_{\Sigma^*}^2} \sum \sum |t|^2 |\vec{k}|, \quad (\text{A3})$$

with

$$\begin{aligned} &\sum \sum |t|^2 \\ &= \frac{1}{4} g_{\Sigma^*,\pi\Lambda}^2 \sum_M \sum_m \langle m | S_i | M \rangle \langle M | S_j^\dagger | m \rangle k_i k_j \\ &= \frac{1}{4} g_{\Sigma^*,\pi\Lambda}^2 \sum_m \langle m | \frac{2}{3} \delta_{ij} - \frac{i}{3} \epsilon_{ijl} \sigma_l | m \rangle k_i k_j \\ &= \frac{1}{3} g_{\Sigma^*,\pi\Lambda}^2 \vec{k}^2. \end{aligned} \quad (\text{A4})$$

Hence,

$$\Gamma = \frac{1}{2\pi} \frac{M_\Lambda}{M_{\Sigma^*}} \frac{1}{3} g_{\Sigma^*,\pi\Lambda}^2 k^3, \quad (\text{A5})$$

and using the experimental value for the  $\Sigma^* \rightarrow \pi\Lambda$  width we obtain

$$g_{\Sigma^*,\pi\Lambda} = 0.0090 \text{ MeV}^{-1}. \quad (\text{A6})$$

The coupling of  $N^*$  to  $K\Lambda$  we get from Ref. [32], where the chiral unitary approach has been used to obtain  $\pi N$  scattering in the region of the  $N^*(1535)$ . One has

$$g_{N^*,K\Lambda} = -1.28. \quad (\text{A7})$$

With these ingredients we can already evaluate the triangle diagrams of Figs. 2 and 3. Considering the isospin coefficients, the sum of the diagrams in Fig. 2, for  $I = \frac{1}{2}$ ,  $I_3 = \frac{1}{2}$ , up to the propagators, is given by

$$\begin{aligned} &(-i) \sqrt{\frac{2}{3}} V_{\Delta\pi,\Sigma^*K}^{(I=1/2)} (-1) g_{\Sigma^*,\pi\Lambda} (-i) g_{N^*,K\Lambda} \sqrt{\frac{2}{3}} \\ &+ (-i) (-1) \sqrt{\frac{1}{3}} V_{\Delta\pi,\Sigma^*K}^{(I=1/2)} g_{\Sigma^*,\pi\Lambda} (-i) g_{N^*,K\Lambda} \sqrt{\frac{1}{3}} \\ &= V_{\Delta\pi,\Sigma^*K}^{(I=1/2)} g_{\Sigma^*,\pi\Lambda} g_{N^*,K\Lambda}, \end{aligned} \quad (\text{A8})$$

where  $V_{\Delta\pi,\Sigma^*K}^{(I=1/2)}$  is given by Eq. (1) and Table I.

The case of the transition  $\Sigma^*K \rightarrow N^*\pi$  in Fig. 3 proceeds in an identical way, and the only difference with respect to the results of Eq. (3) is that we must substitute  $V_{\Delta\pi,\Sigma^*K}^{(I=1/2)}$  by  $V_{\Sigma^*K,\Sigma^*K}^{(I=1/2)}$ .

## APPENDIX B: EVALUATION OF THE AMPLITUDES OF FIGS. 6 AND 7.

We need now the state

$$|\pi\pi, I = 0\rangle = -\frac{1}{\sqrt{3}} (\pi^+\pi^- + \pi^-\pi^+ + \pi^0\pi^0) \quad (\text{B1})$$

and the  $\Delta \rightarrow N\pi$  coupling, similar to Eq. (A2):

$$-it = \frac{f_{\pi N\Delta}}{m_\pi} \vec{S} \cdot \vec{p}_\pi \mathcal{C}(i), \quad (\text{B2})$$

with  $\mathcal{C}(i)$  the corresponding isospin Clebsch-Gordan coefficient,

$$\mathcal{C}(i) = \begin{cases} -1 & \text{for } \Delta^{++} \rightarrow p\pi^+, \\ \sqrt{\frac{2}{3}} & \text{for } \Delta^+ \rightarrow p\pi^0, \\ \sqrt{\frac{1}{3}} & \text{for } \Delta^0 \rightarrow p\pi^-. \end{cases} \quad (\text{B3})$$

The coupling  $f_{\pi N\Delta}$ , taken to obtain the  $\Delta$  width, is given by

$$f_{\pi N\Delta} = 2.2 \quad (\text{B4})$$

corresponding to  $f_{\pi NN}^2/4\pi = 0.38$ , very close to standard value used in pion physics, 0.36 [42]. The isospin combination of vertices corresponding to Eq. (A8) for Fig. 6 is now given, taking into account Eq. (A1), by

$$\begin{aligned} &-i V_{\Delta\pi,\Delta\pi}^{(I=1/2)} (-) \sqrt{\frac{1}{2}} \frac{f_{\pi N\Delta}}{m_\pi} (-) \frac{1}{\sqrt{3}} (-i) g_{\sigma,\pi\pi} \\ &-i V_{\Delta\pi,\Delta\pi}^{(I=1/2)} (-) \sqrt{\frac{1}{3}} \sqrt{\frac{2}{3}} \frac{f_{\pi N\Delta}}{m_\pi} (-) \frac{1}{\sqrt{3}} (-i) g_{\sigma,\pi\pi} \\ &-i V_{\Delta\pi,\Delta\pi}^{(I=1/2)} (-) \sqrt{\frac{1}{6}} \sqrt{\frac{1}{3}} \frac{f_{\pi N\Delta}}{m_\pi} (-) \frac{1}{\sqrt{3}} (-i) g_{\sigma,\pi\pi} \\ &= -\sqrt{\frac{2}{3}} V_{\Delta\pi,\Delta\pi}^{(I=1/2)} \frac{f_{\pi N\Delta}}{m_\pi} g_{\sigma,\pi\pi}. \end{aligned} \quad (\text{B5})$$

For the coupling of the  $\sigma$  to  $\pi\pi$  obtained from the unitary matrix, and unitary normalization ( $\frac{1}{\sqrt{2}}$  extra in the wave function of  $\pi\pi$  as identical particles), we take

$$g'_{\sigma,\pi\pi} = 3.6 \text{ GeV}, \quad (\text{B6})$$

where we have taken an average between the results of the chiral unitary approach [43] and different results using an analysis of data implementing Roy's equations [44,45] (see Table 4 of Ref. [46]). With the good normalization to be used in Eq. (B5), we have

$$g_{\sigma,\pi\pi} = \sqrt{2} \times 3.6 \text{ GeV}. \quad (\text{B7})$$

Following the argumentation of Eq. (3), we obtain now

$$-it_{\Delta\pi,N\sigma} = -V_{\Delta\pi,\Delta\pi}^{(I=1/2)} \sqrt{\frac{2}{3}} \frac{f_{\pi N\Delta}}{m_\pi} g_{\sigma,\pi\pi} (\vec{S} \cdot \vec{k}) 2M_\Delta t'_T, \quad (\text{B8})$$

with  $\vec{k}$  the nucleon momentum, where  $t'_T$  is obtained from Eq. (4) by simply changing the masses of the intermediate particles,  $\Sigma^* \rightarrow \Delta$ ,  $K \rightarrow \pi$ ,  $\Lambda \rightarrow \pi$ , and multiplying the integrand by  $(1 + \frac{\vec{q} \cdot \vec{k}}{|\vec{k}|^2})$ . The reason for this latter factor is that before in Eq. (3) the factor  $(\vec{S} \cdot \vec{k})$  was factorized outside the integral. Here we have  $\vec{S} \cdot \vec{p}_\pi \equiv \vec{S} \cdot (-\vec{q} - \vec{k})$  and

$$\int d^3q q_i \cdots \equiv k_i \int d^3q \frac{\vec{q} \cdot \vec{k}}{\vec{k}^2},$$

since  $\vec{q}$  is an integration variable and  $\vec{k}$  is the only vector in the integrand which is not integrated.

For the transition of  $\Sigma^* K \rightarrow N\sigma$ , we will have the same expression as in Eq. (B8), changing  $V_{\Delta\pi,\Delta\pi}^{(I=1/2)}$  to  $V_{\Sigma^* K,\Delta\pi}^{(I=1/2)}$ .

Finally, in analogy to Eq. (17) we will now have the effective transition potential

$$\tilde{V}_{i,N\sigma} = V_{i,\Delta\pi}^{I=1/2} \sqrt{\frac{2}{3}} \frac{f_{\pi N\Delta}}{m_\pi} g_{\sigma,\pi\pi} 2M_\Delta \tilde{V}'_{N\sigma}(s), \quad (\text{B9})$$

where  $\tilde{V}'_{N\sigma}$  is defined such that

$$\text{Im } \tilde{t}'(s) = \tilde{V}'_{N\sigma}(s) \text{ Im } G_{N\sigma}(s, m_\sigma) \tilde{V}'_{N\sigma}(s), \quad (\text{B10})$$

with

$$\text{Im } \tilde{t}'(s) = \frac{1}{N_\sigma} \int d\tilde{m}_\sigma^2 \frac{1}{3} \tilde{k}^2 \text{ Im } G_{N\sigma}(s, \tilde{m}_\sigma) S_\sigma(\tilde{m}_\sigma^2) |t'_T(s, \tilde{m}_\sigma)|^2, \quad (\text{B11})$$

with the  $\sigma$  spectral function,

$$\begin{aligned} S_\sigma(\tilde{m}_\sigma^2) &= -\frac{1}{\pi} \text{ Im } \left[ \frac{1}{\tilde{m}_\sigma^2 - m_\sigma^2 + im_\sigma \Gamma_\sigma} \right], \\ N_\sigma &= \int S_\sigma(\tilde{m}_\sigma^2) d\tilde{m}_\sigma^2, \\ \text{Im } G_{N\sigma}(s, m_\sigma) &= -\frac{1}{4\pi} \frac{M_N}{\sqrt{s}} k(s, m_\sigma), \\ k(s, m_\sigma) &= \frac{\lambda^{1/2}(s, M_N^2, m_\sigma^2)}{2\sqrt{s}}, \end{aligned} \quad (\text{B12})$$

and  $\text{Im } G_{N\sigma}(s, \tilde{m}_\sigma)$ ,  $\tilde{k}(s, \tilde{m}_\sigma)$  given by the same expressions, changing  $m_\sigma$  to  $\tilde{m}_\sigma$ . For  $m_\sigma$  and  $\Gamma_\sigma$  we take values from Ref. [46]:

$$m_\sigma = 445 \text{ MeV}, \quad \frac{\Gamma_\sigma}{2} = 275 \text{ MeV}.$$

Now  $\tilde{V}_{i,N\sigma}$  of Eq. (B9) provides transitions from  $\Delta\pi(\Sigma K)$  to  $N\sigma$ . As before, we introduce the  $N\sigma$  channel into the coupled channels and have now a  $4 \times 4$  matrix for  $V$ , allowing the  $\Delta\pi \rightarrow N\sigma$ ,  $\Sigma^* K \rightarrow N\sigma$  transitions and neglecting direct transitions  $N\sigma \rightarrow N\sigma$  and  $N\sigma \rightarrow N^*\pi$ . For cutoff in the integral of  $d^3q$  in  $t'_T$  we take now  $q_{\text{max}} = 700$  MeV, suited for the study of the  $\pi\pi$  interaction [6,47].

## APPENDIX C: DOUBLE CONVOLUTION TO OBTAIN THE DECAY WIDTH IN THE $\Sigma^* K$ CHANNEL.

We perform the double convolution with the spectral functions of  $\Sigma^*$  and the resonance  $N^*(1875)$ . The convolution width is given by

$$\begin{aligned} \tilde{\Gamma}_{\Sigma^* K} &= \frac{1}{N_R} \int_{M_R - 2\Gamma_R}^{M_R + 2\Gamma_R} d\tilde{M}_R \int_{M_{\Sigma^*} - 2\Gamma_{\Sigma^*}}^{M_{\Sigma^*} + 2\Gamma_{\Sigma^*}} d\tilde{M}_{\Sigma^*} \\ &\quad \times S_R(\tilde{M}_R) S_{\Sigma^*}(\tilde{M}_{\Sigma^*}) \Gamma_{\Sigma^* K}(\tilde{M}_R, \tilde{M}_{\Sigma^*}, m_K), \end{aligned} \quad (\text{C1})$$

where  $S_R(\tilde{M}_R)$  [or  $S_{\Sigma^*}(\tilde{M}_{\Sigma^*})$ ] is the spectral function of  $N^*(1875)$  (or  $\Sigma^*$ ), taking the same form as Eq. (13) with proper mass and width for the resonance; and

$$\begin{aligned} N_R &= \int_{M_R - 2\Gamma_R}^{M_R + 2\Gamma_R} d\tilde{M}_R \int_{M_{\Sigma^*} - 2\Gamma_{\Sigma^*}}^{M_{\Sigma^*} + 2\Gamma_{\Sigma^*}} d\tilde{M}_{\Sigma^*} S_R(\tilde{M}_R) S_{\Sigma^*}(\tilde{M}_{\Sigma^*}), \\ \Gamma_{\Sigma^* K}(\tilde{M}_R, \tilde{M}_{\Sigma^*}, m_K) &= \frac{1}{2\pi} \frac{\tilde{M}_{\Sigma^*}}{\tilde{M}_R} g_{R,\Sigma^* K}^2 \tilde{p}, \end{aligned} \quad (\text{C2})$$

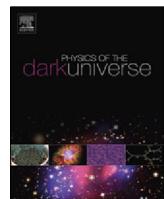
with

$$\tilde{p} = \frac{\lambda^{1/2}(\tilde{M}_R^2, \tilde{M}_{\Sigma^*}^2, m_K^2)}{2\tilde{M}_R} \theta(\tilde{M}_R - \tilde{M}_{\Sigma^*} - m_K).$$

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# Cosmological dynamics of magnetic Bianchi I in viable $f(R)$ models of gravity



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## ABSTRACT

Standard dynamical system analysis of Einstein–Maxwell equation in  $f(R)$  theories is considered in this work. We investigate cosmological dynamics of a uniform magnetic field in the Orthogonal Spatially Homogeneous (OSH) Bianchi I universe with viable  $f(R)$  models of gravity. In this work, the  $f(R) = R - \alpha R^n$  and  $f(R) = (R^b - \Lambda)^c$  models are examined by using our dynamical system analysis. Our results show that both of two  $f(R)$  models have a viable cosmological consequence identical to the analysis present in Amendola and Tsujikawa (2008) for the FLRW background. Contrary to Amendola and Tsujikawa (2008), we discover in our models that there is an additional anisotropic and non-zero cosmological magnetic fields fixed point emerging before the present of the standard matter epoch. This means that the universe has initially isotropic stage with the intermediated epoch as the anisotropic background and it ends up with the isotropic late-time acceleration. The primordial magnetic fields play a crucial role of the shear evolutions obtained from these two models which have the same scaling of the cosmic time as  $\sigma \sim t^{-\frac{1}{3}}$ , instead of  $\sigma \sim t^{-1}$  for the absence of the primordial magnetic cases.

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## 1. Introduction

An alternative way of explaining the accelerating expansion of the universe [1,2] or the Dark Energy (DE) problem [3] (for review see [4]) is to modify the Einstein's theory of gravity (General Relativity (GR)) as the source of accelerating behavior of the universe (for review see [5]). The simple versions of such modification, i.e.  $f(R)$  gravity, were proposed by Capozziello et al. [6] and Carroll et al. [7]. The main idea of modified gravity is that, on the one hand, one considers gravitational Lagrangian as a function of the Ricci scalar, i.e.  $\mathcal{L} = \sqrt{-g}f(R)$  [8]. On the other hand, the inverse power of the Ricci scalar ( $1/R$ ) is placed into the Einstein–Hilbert action which gives a deviation from GR at small curvature and causes the present acceleration of the universe at very large scale. This happens either with de Sitter and anti-de Sitter solutions in the vacuum case which provides a purely gravitation driving the acceleration universe. The modified  $f(R)$  gravity gives good explanation for the cosmic acceleration without introducing the dark energy component that implies from the cosmological data [9]. In

addition, the modified  $f(R)$  gravity also has been shown that it can be derived from string/M-theory [10]. However,  $f(R)$  gravity encounters a number of problems, for example, the unstable problem of the scalar degree of freedom, especially in the  $1/R$  model, [11]: incompatible with the constraints of local gravity [11,12]: the instability of cosmological perturbations in the large scale structure [13] and the wrong sequence of the universe's evolution [14]. The necessary conditions in both of local gravity and cosmological observations for viable  $f(R)$  and its detailed implications are explicitly demonstrated and given in Refs. [15–19] and see Ref. [5] for review.

The dynamics of anisotropic models with  $f(R)$  gravity have been less studied compared with their isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) counterparts. As a result, it is not known how the behavior of the shear is modified in these theories of gravity. The dynamical systems methods for analyzing the qualitative properties of cosmological models have proven very useful. It has been successfully used to study and to understand a number of cosmological models such as the standard GR cosmology [20], the scalar fields models of dark energy [21], the scalar–tensor theories of gravity [22] and the brane-world models [23]. Moreover, the cosmological dynamics of  $f(R)$  gravity was extensively studied in [7,24–29] by using the dynamical system analysis frameworks in homogeneous and isotropic universe (a.k.a. the FLRW model)

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and in the less anisotropic counterpart (Bianchi types and the others) [30–33]. However, a huge number of  $f(R)$  models in the FLRW universe is ruled out by the corrected cosmological expansion sequences [16]. A few models survives from both cosmological and local gravity constraints. Those models are called viable  $f(R)$  models of gravity. The interesting models among them are  $f(R) = R - \alpha R^n$  and  $f(R) = (R^b - \Lambda)^c$  where they were purposed by Refs. [16] and [17] respectively. An extensive analysis of these viable  $f(R)$  models in the anisotropic universe should be a very worth study to quantify some interesting features in this scenario of the  $f(R)$  theories. Therefore, we will perform the dynamical system study for the cosmological solutions of  $f(R)$  gravity in anisotropic Bianchi I universe with the existence of the uniform magnetic fields in this work.

The cosmological magnetic field is one of yet unsolved problems in cosmology. The magnetic fields seem observationally emerge at large scale of the universe [34]. There are many theoretical explanations to disentangle the origin of primordial cosmological magnetic field. One of the compelling explanations is that the cosmological magnetic field has a primordial origin and this idea was purposed by Ref. [35]. The explanation is that it has been created from the Big Bang like all matters populating the universe. From this assumption, we therefore include the primordial magnetic fields into the energy-momentum tensor in the RHS of the Einstein field equation directly. This idea inspired us trying to understand its evolution by finding the exact solutions [36–38] in the wide classes of the Bianchi universe and its existences via the cosmological perturbation theory approach (see for the Bianchi I background [39] and for a review [40]). The cosmological magnetic fields will naturally appear in the universe when the anisotropic cosmological models are taken into account. More importantly, the (primordial) cosmological magnetic fields also might play some roles on the cosmic microwave background (CMB) radiation and might be relics of the existence of the magnetic fields from the early universe.

Previously, the cosmological magnetic fields have been studied in the homogeneous anisotropic universe (Bianchi models) context. It was mentioned in Refs. [36] that a universe with a primordial magnetic field is necessary anisotropy. The exact solutions of magnetic fields in many classes of the Bianchi models were discovered [36–38]. The first significant study of cosmological magnetism in the dynamical system approach was performed by the authors of Ref. [41]. The authors of [42] was systematically performed the standard technique of dynamical system of the magnetic fields evolutions with the perfect fluids in the Bianchi IV<sub>0</sub> with the orthogonal frame formalisms and Hubble normalization variables. Later, there were many works on the dynamical system approach to study the magnetic fields in several classes of the Bianchi models [43–45] (see also reference therein) with the GR and the brane-world cosmology [46].

In this work, we will perform the standard dynamical system analysis on the dynamics of the Orthogonally Spatially Homogeneous (OSH) Bianchi I model in  $f(R) = R - \alpha R^n$  and  $f(R) = (R^b - \Lambda)^c$  models of gravity with the existence of the primordial magnetic fields and the standard barotropic perfect fluid matter ( $p_m = w\rho_m$ ) for  $w = 1$ . The two selected viable  $f(R)$  models in this work have advantages in both local gravity constraints and cosmological viabilities which are demonstrated in Refs. [16,17] for the FLRW case. The Bianchi I is reasonable to be examined because such models are anisotropic generalization of the flat FLRW model and its mathematical simplicity as well. The geometrical property of the spacetime in the Bianchi I, in this work, is assumed to exhibit a property of “Locally Rotational Symmetry”(LRS) [31]. The LRS is rotational symmetry with a preferred spatial direction of the space-like hypersurface. Physical quantities and also observations are rotationally invariant about this direction [47–49]. Here the  $x$ -direction is chosen to be the axis of this symmetry. Therefore, we

impose the uniform magnetic field aligned along the axis of the LRS (in the  $x$ -direction). The aim of this work is also to study the cosmological dynamics of anisotropic universe with the magnetic fields in  $f(R)$  gravity via the dynamical system technique. The  $f(R)$  gravity might have some interesting features on the anisotropies in the shear evolutions of the early universe till the present observed universe. The results from this study might help to understand more about the alternative effects of the viable  $f(R)$  gravity DE models on the small anisotropic effect and contributions of magnetic fields in CMB and its evolution epoch of the universe. In addition, this work is the first systematic dynamical system analysis of the magnetic Bianchi I in  $f(R)$  models of gravity which has never been studied.

We outline this work as follows: In Section 2 we set up the evolution equations of the  $f(R)$  gravity in OSH 1 + 3 covariant approach. In the next section, the dynamical system analysis of the magnetic Bianchi I in the  $f(R) = R - \alpha R^n$  and  $f(R) = (R^b - \Lambda)^c$  models is investigated. In Section 4, we discuss the cosmological implications stemming from our study. In the last section, we give the conclusion in this work. Unless otherwise specified, we will use natural units ( $c = 8\pi G = \hbar = 1$ ) and all conventions used in the present work are adopted from Refs. [30–32].

## 2. Evolution equations of $f(R)$ gravity in 1+3 covariant formalisms

In this section, we will briefly give a crucial setup for the  $f(R)$  gravity in terms of the 1+3 covariant formalisms. We mainly follow the notations and conventions from Refs. [31,50] and its detailed formalisms there in.

### 2.1. The Einstein field equation in $f(R)$ gravity

We begin at the gravitational action of  $f(R)$  gravity in the following form

$$S = \int \sqrt{-g} f(R) d^4x + \int \mathcal{L}_m d^4x \quad (1)$$

where  $\mathcal{L}_m$  is matter fields Lagrangian density. Varying above action with respect to metric  $g^{ab}$ , we obtain the Einstein field equation of  $f(R)$  as

$$F(R)R_{ab} - \frac{1}{2}f(R)g_{ab} + g_{ab}\nabla_c\nabla^c F(R) - \nabla_a\nabla_b F(R) = T_{ab}. \quad (2)$$

where  $F \equiv \partial f / \partial R$  and  $T_{ab} \equiv (2/\sqrt{-g})\delta\mathcal{L}_m/\delta g^{ab}$  and the Latin indices run from  $a, b, c = 0, \dots, 3$ . After some manipulations, the Ricci tensor takes form [31,50],

$$R_{ab} = \frac{1}{F} \left( \frac{1}{2}g_{ab}f - (g_{ab}g^{cd} - g_a^c g_b^d)S_{cd} + T_{ab} \right) \quad (3)$$

where  $S_{ab} = \nabla_a\nabla_b F$ . The energy-momentum tensor  $T_{ab}$  is defined by

$$T_{ab} = \rho u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab}. \quad (4)$$

Here  $h_{ab} = g_{ab} + u_a u_b$  is induce metric which associated with spatial hypersurface and  $u^a = (1, 0, 0, 0)$  is four-velocity which orthogonal to  $h_{ab}$  ( $h_{ab}u^a = 0$ ),  $q_a$  is the energy flux ( $q_a u^a = 0$ ) and  $\pi_{ab}$  is the symmetric trace-free anisotropic pressure, all relative to  $u^a$  ( $\pi_a^a = 0, \pi_{ab}u^a = 0$ ), [31]. The energy-momentum tensor in this work can be decomposed into two parts as

$$T_{ab} = T_{ab}^{PF} + T_{ab}^{EM} \quad (5)$$

where  $T_{ab}^{PF}$  is the energy-momentum tensor of the perfect fluid, given by

$$T_{ab}^{PF} = \rho_m u_a u_b + p_m h_{ab}. \quad (6)$$

$T_{ab}^{EM}$  is the energy-momentum tensor of Maxwell field, given by

$$T_{ab}^{EM} = F_{ac}F_b^c - \frac{1}{4}g_{ab}F_{cd}F^{cd} \quad (7)$$

the field strength  $F_{ab}$  is also defined by

$$F_{ab} = \frac{1}{2}u_{[a}E_{b]} + \eta_{abcd}H^c u^d \quad (8)$$

where  $E_a$  and  $H_a$  are electric and magnetic fields respectively. We will consider the energy-momentum tensor of Maxwell field in the pure magnetic case. The uniform magnetic fields are aligned in  $x$ -direction. This means the magnetic fields have component as  $H_a = (0, \tilde{H}, 0, 0)$  [42]. One can write the energy-momentum tensor of the Maxwell fields analogous with the perfect fluid as [46]

$$T_{ab}^{EM} = \rho_{EM}u_a u_b + p_{EM}h_{ab} + \pi_{ab} \quad (9)$$

where  $\rho_{EM} = \frac{1}{2}\tilde{H}^2$ ,  $p_{EM} = \frac{1}{6}\tilde{H}^2$  and  $\pi_{ab}$  is given by

$$\pi_{ab} = \frac{1}{3}H^2 h_{ab} - H_a H_b, \quad (10)$$

where  $H^2 \equiv H_a H^a = \tilde{H}^2$  is the magnitude of magnetic fields.

One notes that the energy-momentum tensor of Maxwell field is also trace-free ( $g^{ab}T_{ab}^{EM} = T^{EM} = 0$ ). Then the energy density  $\rho$  and pressure  $p$  can be decomposed to the fluid matter and the magnetic parts as

$$\begin{aligned} \rho &= \rho_{PF} + \rho_{EM}, \quad \rho_{PF} = \rho_m, \quad \rho_{EM} = \frac{1}{2}\tilde{H}^2, \\ p &= p_{PF} + p_{EM}, \quad p_{PF} = p_m, \quad p_{EM} = \frac{1}{6}\tilde{H}^2. \end{aligned} \quad (11)$$

Here we consider the energy-momentum tensor of matter part as a standard perfect fluid (the equation of states for perfect fluid and EM parts take form  $w = p_m/\rho_m$  and  $w_{EM} = 1/3$  respectively).

## 2.2. Propagation equations of kinematical quantities in $f(R)$ gravity

In the next step, we will set up the propagation equations for  $f(R)$  gravity in OSH 1 + 3 covariant formalism. The OSH formalism is used to describe the fluid velocity time-like vector fields which orthogonalize to the spatial vector (triad) fields. Having assumed the LRS of the spacetime, here the spatial vector fields span on the space-like hypersurface with one preferred spatial direction and they are invariant under the rotation of the preferred spatial axis (the  $x$ -direction) [47–49]. The main results have been done by [31,50,51]. Using Eq. (3), we can split  $R_{ab}$  in the following forms as [31,50]

$$R = F^{-1}(T + 2f - 3S) \quad (12)$$

$$R_{ab}u^a u^b = F^{-1}\left(T_{ab}u^a u^b - \frac{1}{2}f + h^{ab}S_{ab}\right) \quad (13)$$

$$R_{ab}u^a h_c^b = F^{-1}(S_{ab}u^a h_c^b - q_c) \quad (14)$$

$$R_{ab}h_c^a h_d^b = F^{-1}\left(\pi_{cd} - \left(p + \frac{1}{2}f + S\right)h_{cd} + S_{ab}h_c^a h_d^b\right), \quad (15)$$

and for the  $S_{ab}$ ,

$$S = -F(\ddot{R} + \Theta\dot{R}) - F''\dot{R}^2 \quad (16)$$

$$S_{ab}u^a u^b = F'\ddot{R} + F''\dot{R}^2 \quad (17)$$

$$S_{ab}h^{ab} = -F'\Theta\dot{R}. \quad (18)$$

According to Refs. [31] and [30], they were explicitly demonstrated the 1 + 3 covariant analysis of Bianchi I universe in the  $f(R)$

gravities, the Raychaudhuri equation is written by

$$\begin{aligned} \dot{\Theta} + \frac{1}{3}\Theta^2 + 2\sigma^2 + \frac{1}{F}\left(\rho - \frac{1}{2}f + h^{ab}S_{ab}\right) \\ = \dot{\Theta} + \frac{1}{3}\Theta^2 + 2\sigma^2 + \frac{1}{F}\left(\rho - \frac{1}{2}f - F'\Theta\dot{R}\right) = 0, \end{aligned} \quad (19)$$

and the Friedmann equation (the first integral of the above equation) is given by

$$\begin{aligned} \frac{1}{3}\Theta^2 - \sigma^2 - \frac{1}{F}(\rho + 3p + f - 3S + 2h^{ab}S_{ab}) \\ = \frac{1}{3}\Theta^2 - \sigma^2 - \frac{1}{F}\left(\rho + \frac{1}{2}(FR - f) - F'\Theta\dot{R}\right) = 0 \end{aligned} \quad (20)$$

The shear propagation equation is given by

$$\dot{\sigma}_{ab} + \Theta\sigma_{ab} = \frac{1}{F}(\pi_{ab} - F'\dot{R}\sigma_{ab}), \quad (21)$$

where  $\Theta \equiv \Theta_{ab}h^{ab}$  is the rate of volume expansion parameter (Hubble parameter),  $\sigma^2 \equiv \frac{1}{2}\sigma_{ab}\sigma^{ab}$  is magnitude of the shear tensor  $\sigma_{ab}$  ( $\sigma_{ab} = \Theta_{ab} - \frac{1}{3}h_{ab}\Theta$ ,  $\sigma_a^a = 0$  and  $\sigma_{ab}u^a = 0$ ). The tetrad field is decomposed into the orthonormal frame [49]. We restrict that the magnetic field is aligned along the shear eigenvector as were done in [42,43], then the shear tensor simultaneously diagonalize as<sup>1</sup>

$$\sigma_{ab} = \text{diag}(\sigma_{11}, \sigma_{22}, \sigma_{33}) \equiv \text{diag}(\sigma_1, \sigma_2, \sigma_3). \quad (22)$$

Therefore the shear propagation can be written in the following form

$$\dot{\sigma}_a + \Theta\sigma_a = \frac{1}{F}(\pi_a - F'\dot{R}\sigma_a), \quad (23)$$

where  $\pi_a \equiv \pi_{aa}$  and  $\pi_{aa}$  is the diagonal elements of  $\pi_{ab}$  tensor. By using the conservation of energy-momentum tensor with source-free of Maxwell field in the Bianchi I scenario, the propagation of matter parts is given by [42]

$$\dot{\rho}_m = -(1+w)\rho_m\Theta, \quad (24)$$

$$\dot{\tilde{H}} = -\frac{2}{3}\Theta\tilde{H} + \sigma_{11}\tilde{H} = -\frac{2}{3}\Theta\tilde{H} - 2(\sigma_2 + \sigma_3)\tilde{H}. \quad (25)$$

We close this section by introducing another helpful equation. With help of the Raychaudhuri and Friedmann equations, we come up with the following crucial relation:

$$R = 2\dot{\Theta} + \frac{4}{3}\Theta^2 + 2\sigma^2. \quad (26)$$

## 2.3. The autonomous system

According to Ref. [16], we introduce new dimensionless variables as follows:

$$\begin{aligned} \Sigma &= \frac{3\sigma_2 + \sigma_3}{2\Theta}, & M &= \sqrt{\frac{3}{2F}\frac{\tilde{H}}{\Theta}}, \\ x_1 &= -\frac{3}{\Theta}\frac{F'\dot{R}}{F}, & x_2 &= -\frac{3}{2}\frac{f}{F\Theta^2}, \\ x_3 &= \frac{3}{2}\frac{R}{\Theta^2}, & z &= \frac{3}{\Theta^2}\frac{\rho_m}{F}. \end{aligned} \quad (27)$$

<sup>1</sup> Alternatively, by using the LRS Bianchi I line element,  $ds^2 = -dt^2 + X(t)^2dx^2 + Y(t)^2(dy^2 + dz^2)$ , one can show that  $\sigma_{11} = \Theta_{11} - h_{11}\Theta = \frac{2}{3}\left(\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}\right)$ ,  $\sigma_{22} = \Theta_{22} - h_{22}\Theta = -\frac{1}{3}\left(\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}\right)$  and  $\sigma_{33} = \Theta_{33} - h_{33}\Theta = -\frac{1}{3}\left(\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}\right)$  where  $\Theta = \frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}$ ,  $\Theta_{11} = \frac{\dot{X}}{X}$ ,  $\Theta_{22} = \frac{\dot{Y}}{Y}$  and  $\Theta_{33} = \frac{\dot{Y}}{Y}$ . From such results, we obtain  $\sigma_{11} = \sigma_1 = -(\sigma_2 + \sigma_3)$ .

One notes that<sup>2</sup>  $\Sigma^2 = \left(\frac{3}{2} \frac{\sigma_2 + \sigma_3}{\Theta}\right)^2 = 3 \frac{\sigma^2}{\Theta^2}$ . The constraint equation written in terms of the new variables obeys

$$1 - \Sigma^2 - x_1 - x_2 - x_3 - z - M^2 = 0. \quad (28)$$

Using the time propagation equations of the kinematical quantities in the previous section, then the equations of autonomous system are given

$$\begin{aligned} \frac{d\Sigma}{d\tau} &= x_1 \Sigma - 3\Sigma + \Sigma (2 - x_3 + \Sigma^2) \\ &\quad + 2(1 - \Sigma^2 - x_1 - x_2 - x_3 - z), \\ \frac{dx_1}{d\tau} &= x_1^2 - 3x_1 - 4x_2 - 2x_3 - (1 - 3w)z + x_1(2 - x_3 + \Sigma^2), \\ \frac{dx_2}{d\tau} &= x_1 x_2 + \frac{x_1 x_3}{m} + 2x_2(2 - x_3 + \Sigma^2), \\ \frac{dx_3}{d\tau} &= -\frac{x_1 x_3}{m} + 2x_3(2 - x_3 + \Sigma^2), \\ \frac{dz}{d\tau} &= x_1 z - 3(1 + w)z + 2z(2 - x_3 + \Sigma^2), \end{aligned} \quad (29)$$

where  $m$  is the parameter which is very useful to study viable models of  $f(R)$  gravity and it is defined by [16]

$$m = \frac{RF'}{F}. \quad (30)$$

We note that the derivative with respect to the logarithm time scale is defined by  $\frac{d}{d\tau} = \frac{3}{\Theta} \frac{d}{dt}$ . In addition, one observes that by ignoring the  $\Sigma$  and  $M$  variables, the autonomous system will be identical to the FLRW case that have been done in [16]. For the general case of the evolution phase of the universe, one can be described by the sign of the rate of volume expansion  $\epsilon = \pm 1$ , where  $\epsilon \equiv |\Theta|/\Theta$ .  $\epsilon = 1$  for the expanding phase and  $\epsilon = -1$  for the collapsing phase, more detail discussion in this issue see [26,30,32]. In this work, we have focused our study in the future evolution of the expanding phase ( $\epsilon = 1$ ) of the universe only. The auxiliary of the autonomous system (evolution of magnetic term) is

$$\frac{dM}{d\tau} = \frac{x_1 M}{2} - 2M - 2\Sigma M + M(2 - x_3 + \Sigma^2). \quad (31)$$

The matter density and curvature density parameters of the universe are defined by  $\Omega_m \equiv z$  and  $\Omega_c \equiv x_1 + x_2 + x_3$ , respectively. In this work we consider  $\Omega_c$  as dark energy density parameter.

This section we note that “prime”denotes derivatives with respect to the logarithm time scale as  $\frac{d}{d\tau} = \frac{3}{\Theta} \frac{d}{dt}$ . In the general case of the evolution phase of the universe, it can be described by the sign of the rate of volume expansion  $\epsilon = \pm 1$ , where  $\epsilon \equiv |\Theta|/\Theta$ .  $\epsilon = 1$  for the expanding phase and  $\epsilon = -1$  for the collapsing phase, more detailed discussion in this issue see [26,30,32]. We also introduce the effective equation of state [16],

$$w_{\text{eff}} = -1 - 2 \frac{\dot{\Theta}}{\Theta^2}, \quad (32)$$

which is a useful parameter in this study. Using the definitions in Eq. (27), one rewrites the effective equation of states in terms of the dimensionless variables as

$$w_{\text{eff}} = -1 - \frac{2}{3} (x_3 - \Sigma^2 - 2). \quad (33)$$

More importantly, the associated solutions for each fixed points can be obtained by using Eqs. (26) and (27) to yield

$$\dot{\Theta} = (x_{3(i)} - \Sigma_{(i)}^2 - 2) \frac{\Theta^2}{3}, \quad (34)$$

<sup>2</sup> We can demonstrate that  $\left(\frac{3}{2} \frac{\sigma_2 + \sigma_3}{\Theta}\right)^2 = 3 \frac{\sigma^2}{\Theta^2} = \frac{3}{2} \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{\Theta^2}$  is valid by using the relation  $\sigma_1 = \frac{2}{3} \left(\frac{\dot{x}}{x} - \frac{\dot{y}}{y}\right)$  and  $\sigma_2 = \sigma_3 = -\frac{1}{3} \left(\frac{\dot{x}}{x} - \frac{\dot{y}}{y}\right)$ .

where  $x_{3(i)}$  and  $\Sigma_{(i)}$  are the associated “ $i$ ” fixed points. Performing the integration, one gets solutions of the scale factor for the associated fixed points

$$\begin{aligned} a(t) &= a_0(t - t_0)^\beta, \\ \beta &= \frac{1}{2 + \Sigma_{(i)}^2 - x_{3(i)}}. \end{aligned} \quad (35)$$

In order to see dynamical features of the anisotropic universe, one should consider the shear evolution in the model. With help of Eqs. (23) and (27), we obtain the shear evolution equation in terms of the dimensionless variables as

$$\frac{\dot{\sigma}}{\sigma} = -\eta\Theta, \quad \eta = \frac{1}{3} \left(3 - 2 \frac{M_{(i)}^2}{\Sigma_{(i)}} - x_{1(i)}\right). \quad (36)$$

Contrary to previous studies on the  $R^n$  gravity in Ref. [31], our parameter  $\eta$  depends on the magnetic field,  $M$ . This means has the magnetic fields play important role on the shear evolution. Integrating out Eq. (36), we find

$$\sigma = \sigma_0 a^{-\eta} = \sigma_0 a_0^{-\eta} (t - t_0)^{-\beta\eta}. \quad (37)$$

The exact solution of the shear evolution is very useful for understanding the behavior of the anisotropic effect in the universe. As discussed in Ref. [31], from the above equation the shear evolution for all points in the phase space that lie on the line  $\eta \equiv (3 + 2M_{(i)}^2/\Sigma_{(i)} + x_{1(i)})/3 = 1$  is the same as in the GR case. In order to deviate from the standard GR, the shear will dissipate faster than that in GR when  $\dot{\sigma}/\sigma < -\Theta$ , that is all points that lie in the region  $(3 + 2M_{(i)}^2/\Sigma_{(i)} + x_{1(i)})/3 > 1$ . This is called the fast shear dissipation (FSD) regime [31]. When  $\dot{\sigma}/\sigma > -\Theta$  and for all points in the region  $(3 + 2M_{(i)}^2/\Sigma_{(i)} + x_{1(i)})/3 < 1$ , the shear will dissipate slower than that in GR. This is named the slow shear dissipation (SSD) regime [31]. Notice that, however, the higher order terms of the  $f(R)$  gravity models, e.g. see Refs. [52], can play the same role as of the magnetic field investigated in this work.

In addition, exact solution of the magnetic fields are also obtained in terms of dimensionless variables. Using Eqs. (25) and (27), one gets,

$$\tilde{H} = \tilde{H}_0 a^{-\kappa} = \tilde{H}_0 a_0^{-\kappa} (t - t_0)^{-\beta\kappa}, \quad \kappa = \frac{2}{3} (1 + 2\Sigma_{(i)}). \quad (38)$$

In order to see how the shear parameter and magnetic fields evolve in cosmic time, we will substitute non-zero shear fixed points into the exact solutions of the shear and the magnetic fields evolutions, Eqs. (37) and (38). After outlining the autonomous system of the magnetic Bianchi I universe with the generic  $f(R)$  gravity and exact solutions in terms of dimensionless variables, we will consider such system of differential equations by using the standard dynamical system approach in next section.

### 3. Dynamics of magnetic Bianchi I universe in $f(R)$ models of gravity

This section is devoted to provide the dynamical system analysis for the  $f(R)$  models of gravity. We will define the dimensionless variables from the Friedmann equation given in Eq. (20) in the previous section and use these variables to set up an autonomous system of first-order non-linear differential equations. Next we will determine all fixed points of the autonomous system and analyze their stabilities for each of them. At the end of this section, the cosmological implications of the magnetic Bianchi I in  $f(R)$  models will be discussed in accord with the fixed points and its stabilities.

### 3.1. Dynamical system of the $f(R) = R - \alpha R^n$ gravity

We start with the  $f(R) = R - \alpha R^n$  gravity. This model has been studied extensively in several aspects, and it was shown that the model will be the viable  $f(R)$  DE if  $\alpha > 0$  and  $0 < n < 1$  [16]. Especially, the standard dynamical system method is used to analyze in the FLRW counterpart [27]. For the anisotropic cases, it was studied in the Kantowski–Sachs metric [29]. Here we will consider this model in the Bianchi I universe with the existence of the primordial magnetic fields. The  $f(R) = R - \alpha R^n$  model has the  $m$  function which can be practically written in terms of a variable  $r$  as

$$m = \frac{n(1+r)}{r}, \quad (39)$$

where  $r \equiv x_3/x_2$ . Substituting the  $m$  function into the autonomous system in Eq. (29), one obtains explicit dynamical system for the  $f(R) = R - \alpha R^n$  gravity. The dynamical system of this model is given as follows:

$$\begin{aligned} \frac{d\Sigma}{d\tau} &= x_1\Sigma - 3\Sigma + \Sigma(2 - x_3 + \Sigma^2) \\ &\quad + 2(1 - \Sigma^2 - x_1 - x_2 - x_3 - z), \\ \frac{dx_1}{d\tau} &= x_1^2 - 3x_1 - 4x_2 - 2x_3 - z + x_1(2 - x_3 + \Sigma^2), \\ \frac{dx_2}{d\tau} &= x_1x_2 + \frac{x_1x_3^2}{n(x_2 + x_3)} + 2x_2(2 - x_3 + \Sigma^2), \\ \frac{dx_3}{d\tau} &= -\frac{x_1x_3^2}{n(x_2 + x_3)} + 2x_3(2 - x_3 + \Sigma^2), \\ \frac{dz}{d\tau} &= x_1z - 3z + 2z(2 - x_3 + \Sigma^2). \end{aligned} \quad (40)$$

We will separately study the fixed points, their stabilities, the shear and the magnetic evolutions below.

#### 3.1.1. Fixed points and their stabilities

In what follows, we will consider the properties of each point in turn. There are 4 physical fixed points from the autonomous system of the  $f(R) = R - \alpha R^n$  gravity. We will classify into two cases: isotropic and anisotropic solutions, and the physically associated fixed points of this model are given below.

##### Isotropic solutions

- (1)  $P_1^{(1)}$ : de-Sitter fixed point In this case, we obtain the fixed point:

$$\Sigma = 0, x_1 = 0, x_2 = -1, x_3 = 2, z = 0. \quad (41)$$

Since  $w_{\text{eff}} = -1$ , the point  $P_1^{(1)}$  corresponds to de-Sitter solutions ( $\dot{\Theta} = 0$ ) and has eigenvalues

$$\left\{ -4, -3, -3, \frac{-3n - \sqrt{n\sqrt{-32 + 25n}}}{2n}, \frac{-3n + \sqrt{n\sqrt{-32 + 25n}}}{2n} \right\}.$$

Hence  $P_1^{(1)}$  is stable when  $0 \leq n < 2$  and saddle for  $n > 2$ . In this case, it is trivial to verify that,

$$a(t) = \exp(\lambda t), \quad \lambda = \text{arbitrary constant}, \quad M^2 = 0.$$

- $P_2^{(1)}$ : standard matter-like epoch fixed point.

In this case, we obtain the fixed point:

$$\begin{aligned} \Sigma &= 0, x_1 = 3 - \frac{3}{n}, x_2 = \frac{3 - 4n}{2n^2}, \\ x_3 &= 2 - \frac{3}{2n}, z = \frac{(13 - 8n)n - 3}{2n^2}. \end{aligned} \quad (42)$$

Employing Eqs. (33)–(35), we obtain in this case

$$w_{\text{eff}} = -1 + \frac{1}{n}, \quad a(t) = a_0(t - t_0)^{\frac{2n}{3}}, \quad M^2 = 0.$$

This point  $P_2^{(1)}$  corresponds to saddle solutions and has eigenvalues

$$\left\{ -1, -\frac{3}{2n}, \frac{3(-1+n)}{n}, N^-, N^+ \right\},$$

where

$$N^\pm = \frac{3n - 3n^2 \pm n\sqrt{81 - 498n + 1025n^2 - 864n^3 + 256n^4}}{4(-n^2 + n^3)}.$$

The solutions are a saddle point for  $\frac{1}{16}(13 - \sqrt{73}) < n < 3/4$ . We note that this fixed point becomes the standard matter epoch if  $n = 1$ .

- $P_3^{(1)}$ : curvature dominated fixed point.

In this case, we obtain the fixed point:

$$\begin{aligned} \Sigma &= 0, x_1 = \frac{3}{2n-1} - 1, x_2 = \frac{6}{1-2n} + \frac{1}{n-1}, \\ x_3 &= \frac{3}{2n-1} + \frac{1}{1-n} + 2, z = 0. \end{aligned} \quad (43)$$

Employing Eqs. (33)–(35), we obtain in this case

$$\begin{aligned} w_{\text{eff}} &= -1 + \frac{-10n^2 + 13n - 1}{6n^2 - 9n + 3}, \\ a(t) &= a_0(t - t_0)^{-\frac{(n-1)(2n-1)}{n-2}}, \quad M^2 = 0. \end{aligned}$$

This point  $P_3^{(1)}$  has eigenvalues

$$\left\{ \frac{5-4n}{-1+n}, \frac{5-4n}{-1+n}, -\frac{2(-2+n)}{-1+2n}, -\frac{2(2-8n+5n^2)}{1-3n+2n^2}, \right. \\ \left. -\frac{-3+13n-8n^2}{1-3n+2n^2} \right\}.$$

Regarding the above values, we find for unstable fixed points

$$1 < n \leq \frac{5}{4}, \quad (44)$$

and for stable ones

$$n < \frac{1}{16}(13 - \sqrt{73}) \vee n > 2. \quad (45)$$

##### Anisotropic solutions

- $P_4^{(1)}$ : Jacob magnetic-like (non-zero magnetic field with matter solution: the Jacobs magnetic field model in Bianchi I [37]) with curvature fixed point.

In this case, we obtain the fixed point:

$$\begin{aligned} \Sigma &= -\frac{2(n(5n-8)+2)}{n(7n-10)+4}, x_1 = -\frac{12(n-2)(n-1)}{n(7n-10)+4}, \\ x_2 &= -\frac{18(n-1)(n(11n-20)+8)}{(n(7n-10)+4)^2}, \\ x_3 &= \frac{18(n-1)(n(11n-20)+8)}{(n(7n-10)+4)^2}, z = 0. \end{aligned} \quad (46)$$

Employing Eqs. (33)–(35), we obtain in this case

$$\begin{aligned} w_{\text{eff}} &= -1 + \frac{-245n^4 + 1616n^3 - 492n^2 + 464n + 16}{3(7n^2 + 10n + 4)^2}, \\ a(t) &= a_0(t - t_0)^{\frac{(7n^2 + 10n + 4)^2}{6(193n^3 + 24n^2 + 72n + 8)}}, \end{aligned}$$

$$M^2 = -\frac{3(55n^4 - 188n^3 + 222n^2 - 104n + 16)}{(7n^2 - 10n + 4)^2}.$$

The existence of magnetic fields in this case satisfies

$$\begin{aligned} \frac{1}{5}(4 - \sqrt{6}) &< n < \frac{2}{11}(5 - \sqrt{3}) \vee \frac{2}{11}(5 + \sqrt{3}) \\ &< n < \frac{1}{5}(4 + \sqrt{6}). \end{aligned} \quad (47)$$

This point  $P_4^{(1)}$  has eigenvalues (see

$$\left\{ -\frac{12(2 - 3n + n^2)}{4 - 10n + 7n^2}, -\frac{3(8 - 20n + 11n^2)}{4 - 10n + 7n^2}, \right. \\ \left. -\frac{3(4 - 18n + 11n^2)}{4 - 10n + 7n^2}, P, Q \right\}$$

where

$$P \equiv -\frac{3(32 - 160n + 300n^2 - 250n^3 + 77n^4 + B)}{2(4 - 10n + 7n^2)^2}$$

and

$$Q \equiv \frac{3(-32 + 160n - 300n^2 + 250n^3 - 77n^4 + B)}{2(4 - 10n + 7n^2)^2}$$

with

$$B = \sqrt{(4 - 10n + 7n^2)^2 (320 - 1984n + 4128n^2 - 3448n^3 + 1001n^4)}.$$

Before going further to the next subsection, we will give some discussion towards these fixed points of the  $f(R) = R - \alpha R^n$  model. We find from this model of gravity that there are 4 fixed points for the isotropic and for the anisotropic cases. Interestingly, there is no Kasner fixed point contrary to that of the usual GR gravity. According to the fixed points in this model, it means no anisotropic singularity in this scenario. We classify the physical fixed points by considering the magnitude square of the magnetic fields which must be positive.

### 3.1.2. The shear and magnetic fields evolutions

We turn to consider the shear and magnetic fields evolutions for the  $f(R) = R - \alpha R^n$  model. According to the existence of the 4 physical fixed points above. There is only one fixed point with non-zero shear and magnetic solution. To see how the shear and magnetic fields dissipate, we recall the exact solution of the shear evolution from Eq. (37) and substitute the anisotropic fixed point in the solution to yield

$$\begin{aligned} \sigma &= \sigma_0 a^{-\eta} = \sigma_0 a_0^{\left(\frac{2n-4}{7n^2-10n+4}\right)} (t - t_0)^{-\frac{1}{3}}, \\ \eta &= \frac{4 - 2n}{7n^2 - 10n + 4}. \end{aligned} \quad (48)$$

In the above results from the shear evolution, we find the shear dissipation scale in cosmic time as  $\sigma \sim t^{-\frac{1}{3}}$ . The  $\eta$  parameter can be classified into FSD and SSD as

$$\begin{aligned} \frac{4 - 2n}{7n^2 - 10n + 4} &> 1, \quad \xrightarrow{\text{FSD}} 0 < n < \frac{8}{7}, \\ \frac{4 - 2n}{7n^2 - 10n + 4} &< 1, \quad \xrightarrow{\text{SSD}} n < 0 \vee n > \frac{8}{7}. \end{aligned} \quad (49)$$

The exact solution of the magnetic fields is given by,

$$\begin{aligned} \tilde{H} &= \tilde{H}_0 a^{-\kappa} = \tilde{H}_0 a_0^{\left(\frac{26n^2-44n+8}{21n^2-30n+12}\right)} (t - t_0)^{\left(\frac{13n^2-22n+4}{18-9n}\right)}, \\ \kappa &= -\frac{26n^2 - 44n + 8}{21n^2 - 30n + 12}. \end{aligned} \quad (50)$$

We will see the numeric results of the shear and the magnetic fields evolutions in the Section 4.

### 3.2. Dynamical system of the $f(R) = (R^b - \Lambda)^c$ gravity

Next, we will consider another viable  $f(R)$  DE model. The  $f(R) = (R^b - \Lambda)^c$  was proposed by Ref. [17]. This model has original idea from a generalized  $\Lambda$ CDM model by parameterizing the power of the Ricci scalar and a whole term of the power of the Ricci scalar with the cosmological constant. This model is viable for the  $f(R)$  DE. The model was studied by using the dynamical system method and constrained by data from local gravity and cosmology in the FLRW case [17]. More importantly, this model will be the viable  $f(R)$  DE with the conditions  $c \geq 1$  and  $bc \approx 1$  [17]. Therefore, it is interesting to extend the study of this model to the anisotropic universe counterpart. The  $m$  function of the model can be written in the following form

$$m = \frac{(1 - c)r}{c} + b - 1. \quad (51)$$

The dynamical system of this model is given as follows:

$$\begin{aligned} \frac{d\Sigma}{d\tau} &= x_1\Sigma - 3\Sigma + \Sigma(2 - x_3 + \Sigma^2) \\ &\quad + 2(1 - \Sigma^2 - x_1 - x_2 - x_3 - z), \\ \frac{dx_1}{d\tau} &= x_1^2 - 3x_1 - 4x_2 - 2x_3 - z + x_1(2 - x_3 + \Sigma^2), \\ \frac{dx_2}{d\tau} &= x_1x_2 + \frac{c x_1 x_2 x_3}{c(b-1)x_2 + (1-c)x_3} + 2x_2(2 - x_3 + \Sigma^2), \\ \frac{dx_3}{d\tau} &= -\frac{c x_1 x_2 x_3}{c(b-1)x_2 + (1-c)x_3} + 2x_3(2 - x_3 + \Sigma^2), \\ \frac{dz}{d\tau} &= x_1z - 3z + 2z(2 - x_3 + \Sigma^2). \end{aligned} \quad (52)$$

We will discuss the fixed points, their stabilities, the shear and the magnetic evolutions below.

#### 3.2.1. Fixed points and their stabilities

As of the study in previous subsection, we will consider the properties of each point in turn. There are 4 physical fixed points in this model. We will organize by two cases: isotropic and anisotropic solutions, and the associated fixed points of this model are given below.

##### Isotropic solutions

- $P_1^{(2)}$ : de-Sitter fixed point.

In this case, we obtain the fixed point:

$$\Sigma = 0, x_1 = 0, x_2 = -1, x_3 = 2, z = 0. \quad (53)$$

Since  $w_{\text{eff}} = -1$ , the point  $P_1^{(2)}$  corresponds to de-Sitter solutions ( $\dot{\Theta} = 0$ ) and has eigenvalues

$$\left\{ -4, -3, -3, -\frac{-6 + 3(1+b)c + A}{2(-2 + c + bc)}, \frac{6 - 3(1+b)c + A}{2(-2 + c + bc)} \right\}.$$

where  $A \equiv \sqrt{100 - 4(17 + 25b)c + (9 + 34b + 25b^2)c^2}$ . In this case, it can verify that,

$$a(t) = \exp(\lambda t), \quad \lambda = \text{arbitrary constant}, \quad M^2 = 0.$$

Notice that the stability conditions of the fixed points satisfy

$$\begin{aligned} &\left( b < -\frac{9}{25} \& \frac{50}{9+25b} \leq c < \frac{2}{b} \right) \\ &\vee \left( b = -\frac{9}{25} \& c < -\frac{50}{9} \right) \\ &\vee \left( -\frac{9}{25} < b < 0 \& \left( c < \frac{2}{b} \vee c \geq \frac{50}{9+25b} \right) \right), \end{aligned} \quad (54)$$

$$\vee \left( b = 0 \& c \geq \frac{50}{9} \right) \vee \left( b > 0 \& \frac{50}{9 + 25b} \leq c < \frac{2}{b} \right).$$

- $P_2^{(2)}$ : standard matter-like epoch fixed point.  
In this case, we obtain the fixed point:

$$\begin{aligned} \Sigma = 0, x_1 &= 3 - \frac{3}{bc}, x_2 = \frac{3 - 4bc}{2b^2c^2}, \\ x_3 &= 2 - \frac{3}{2bc}, z = \frac{bc(13 - 8bc) - 3}{2b^2c^2}, \\ \xrightarrow{b \rightarrow \frac{1}{c}} \Sigma &= 0, x_1 = 0, x_2 = -\frac{1}{2}, x_3 = \frac{1}{2}, z = 1. \end{aligned} \quad (55)$$

Employing Eqs. (33)–(35), we obtain in this case

$$w_{\text{eff}} = -1 + \frac{1}{bc}, \quad a(t) = a_0(t - t_0)^{\frac{2bc}{3}}, \quad M^2 = 0.$$

This point  $P_2^{(2)}$  has eigenvalues

$$\left\{ -\frac{3}{2bc}, \frac{3}{c}, -1, -\frac{-3 + 3bc + B}{4bc(-1 + bc)}, -\frac{3 - 3bc + B}{4bc - 4b^2c^2} \right\}.$$

where  $B \equiv \sqrt{81 - 498bc + 1025b^2c^2 - 864b^3c^3 + 256b^4c^4}$ .

This fixed point is always saddle.

- $P_3^{(2)}$ : curvature dominate fixed point.

In this case, we obtain the fixed point:

$$\begin{aligned} \Sigma = 0, x_1 &= \frac{3}{2bc - 1} - 1, x_2 = \frac{6}{1 - 2bc} + \frac{1}{bc - 1}, \\ x_3 &= \frac{1}{1 - bc} + \frac{3}{2bc - 1} + 2, z = 0, \\ \xrightarrow{b \rightarrow \frac{1}{c}} \Sigma &= 0, x_1 = 2, x_2 = \text{undefined}, \\ x_3 &= \text{undefined}, z = 0. \end{aligned} \quad (56)$$

Employing Eqs. (33)–(35), we obtain in this case

$$\begin{aligned} w_{\text{eff}} &= -1 + \frac{4 - 2bc}{3 - 9bc + 6b^2c^2}, \\ a(t) &= a_0(t - t_0)^{-\frac{(-1+bc)(-1+2bc)}{-2+bc}}, \quad M^2 = 0. \end{aligned}$$

This point  $P_3^{(2)}$  has eigenvalues

$$\left\{ -\frac{2b(-2 + bc)}{(-1 + bc)(-1 + 2bc)}, \frac{5 - 4bc}{-1 + bc}, \frac{5 - 4bc}{-1 + bc}, -\frac{2(2 - 8bc + 5b^2c^2)}{(-1 + bc)(-1 + 2bc)}, \frac{-3 + 13bc - 8b^2c^2}{1 - 3bc + 2b^2c^2} \right\}.$$

Regarding the above values, we find for unstable fixed points

$$b > 0 \& \left( \frac{1}{b} < c < \frac{5}{4b} \right), \quad (57)$$

and for stable ones

$$\left( b < 0 \& \left( \frac{2}{b} < c < -\frac{1}{16}\sqrt{\frac{73}{b^2}} + \frac{13}{16b} \vee c > \frac{1}{16}\sqrt{\frac{73}{b^2}} + \frac{13}{16b} \right) \right) \vee \left( b > 0 \& \left( \frac{1}{2b} < c < \frac{1}{b} \vee c > \frac{2}{b} \right) \right).$$

Anisotropic solutions

- $P_4^{(2)}$ : Jacob magnetic-like (non-zero magnetic field with matter solution: the Jacobs magnetic field model in Bianchi I [37]) with curvature fixed point.

In this case, we obtain the fixed point:

$$\begin{aligned} \Sigma &= -\frac{2(bc(5bc - 8) + 2)}{bc(7bc - 10) + 4}, x_1 = -\frac{12(bc - 2)(bc - 1)}{bc(7bc - 10) + 4}, \\ x_2 &= -\frac{18(bc - 1)(bc(11bc - 20) + 8)}{(bc(7bc - 10) + 4)^2}, \\ x_3 &= \frac{18bc(bc - 1)(bc(11bc - 20) + 8)}{(bc(7bc - 10) + 4)^2}, z = 0, \\ \xrightarrow{b \rightarrow \frac{1}{c}} \Sigma &= 2, x_1 = 0, x_2 = 0, x_3 = 0, z = 0. \end{aligned} \quad (58)$$

Employing Eqs. (33)–(35), we obtain in this case

$$\begin{aligned} w_{\text{eff}} &= \frac{4 + 6bc - 7b^2c^2}{4 - 10bc + 7b^2c^2}, \quad a(t) = a_0(t - t_0)^{\frac{4 - 10bc + 7b^2c^2}{12 - 6bc}}, \\ M^2 &= -\frac{3(16 - 104bc + 222b^2c^2 - 188b^3c^3 + 55b^4c^4)}{(4 - 10bc + 7b^2c^2)^2}. \end{aligned}$$

The existence of the magnetic fields is given by

$$\begin{aligned} \left( b < 0 \wedge \left( \frac{4}{5b} - \frac{1}{5}\sqrt{\frac{6}{b^2}} < c < \frac{10}{11b} - \frac{2}{11}\sqrt{\frac{3}{b^2}} \vee \right. \right. \\ \left. \left. \frac{1}{11}2\sqrt{\frac{3}{b^2}} + \frac{10}{11b} < c < \frac{1}{5}\sqrt{\frac{6}{b^2}} + \frac{4}{5b} \right) \right) \\ \vee \left( b > 0 \wedge \left( \frac{4}{5b} - \frac{1}{5}\sqrt{\frac{6}{b^2}} < c < \frac{10}{11b} - \frac{2}{11}\sqrt{\frac{3}{b^2}} \vee \right. \right. \\ \left. \left. \frac{1}{11}2\sqrt{\frac{3}{b^2}} + \frac{10}{11b} < c < \frac{1}{5}\sqrt{\frac{6}{b^2}} + \frac{4}{5b} \right) \right). \end{aligned}$$

This point  $P_4^{(2)}$  has eigenvalues given in Box I.

We turn to discuss the physical fixed points from the autonomous system in the  $f(R) = (R^b - \Lambda)^c$  model. First of all, it is interesting to see all of fixed points in the limits of  $c \geq 1$  and  $b \rightarrow 1/c$  due to the cosmological viability that pointed out in the literature. At the point  $P_2^{(2)}$ , there is an existence of the standard matter epoch in this model at the limits of  $c \geq 1$  and  $bc \approx 1$  but the point  $P_3^{(2)}$  is an undefined point in this limit. We do keep the general form of the parameters in this model because the autonomous system suffers from the singularity when the fixed points have  $x_2 = -x_3$  in the function  $1/m$  at the limits of  $c \geq 1$  and  $b \rightarrow 1/c$ . We also classify the physical fixed points by considering the magnitude square of the magnetic fields which must be positive.

### 3.2.2. The shear and magnetic fields evolutions

Here we will see behaviors of the shear and magnetic fields evolutions for the  $f(R) = (R^b - \Lambda)^c$  model. In this model, we find the 4 physical fixed points.  $P_4^{(2)}$  has non-zero shear and magnetic fixed point. Substituting the anisotropic fixed point to the exact solution of the shear evolution from Eq. (37), we find,

$$\begin{aligned} \sigma &= \sigma_0 a^{-\eta} = \sigma_0 a_0^{\left(\frac{2bc-4}{7b^2c^2-10bc+4}\right)} (t - t_0)^{-\frac{1}{3}}, \\ \eta &= \frac{4 - 2bc}{7b^2c^2 - 10bc + 4}. \end{aligned} \quad (59)$$

Surprisingly, the shear dissipation in this model has the same cosmic time scale as the  $f(R) = R - \alpha R^n$  gravity i.e.  $\sigma \sim t^{-\frac{1}{3}}$ . The conditions for the FSD and SSD regions from the  $\eta$  parameter

$$\left\{ -\frac{12b(-2+bc)}{4-10bc+7b^2c^2}, -\frac{3(8-20bc+11b^2c^2)}{4-10bc+7b^2c^2}, 3P^-, 3P^+, -\frac{3(4-18bc+11b^2c^2)}{4-10bc+7b^2c^2} \right\}.$$

where

$$P^\mp \equiv \frac{(128-1088bc+3984b^2c^2-8144b^3c^3+10028b^4c^4-7428b^5c^5+3059b^6c^6-539b^7c^7 \mp Q)}{(2(-1+bc)(4-10bc+7b^2c^2)^3)}$$

and

$$Q \equiv \sqrt{(-1+bc)^2(4-10bc+7b^2c^2)^4(320-1984bc+4128b^2c^2-3448b^3c^3+1001b^4c^4)}.$$

#### Box I.

are given by

$$\begin{aligned} \frac{4-2bc}{7b^2c^2-10bc+4} > 1, & \xrightarrow{\text{FSD}} \left( b < 0 \wedge \frac{8}{7b} < c < 0 \right) \\ & \vee \left( b > 0 \wedge 0 < c < \frac{8}{7b} \right), \\ \frac{4-2bc}{7b^2c^2-10bc+4} < 1, & \xrightarrow{\text{SSD}} \left( b < 0 \wedge \left( c < \frac{8}{7b} \vee c > 0 \right) \right) \\ & \vee \left( b > 0 \wedge \left( c < 0 \vee c > \frac{8}{7b} \right) \right). \end{aligned} \quad (60)$$

The magnetic fields evolution has the exact solution in terms of cosmic time in the following form:

$$\begin{aligned} \tilde{H} = \tilde{H}_0 a^{-\kappa} &= \tilde{H}_0 a_0^{-\frac{2}{3}(1-\frac{4(bc(5bc-8)+2)}{bc(7bc-10)+4})} (t-t_0)^{\left(\frac{13b^2c^2-22bc+4}{18-9bc}\right)}, \\ \kappa &= \frac{2}{3} \left( 1 - \frac{4(bc(5bc-8)+2)}{bc(7bc-10)+4} \right). \end{aligned} \quad (61)$$

As the same procedure in the previous  $f(R)$  model, the numeric results of the shear and the magnetic fields evolutions will be given in the section IV. Noting that the invariant submanifold issues in the phase space of the dynamical system have been so far discussed in details in Ref. [26]. Regarding our chosen (physical) fixed points, they do not admit any singularity or even generate invariant submanifolds.

#### 4. Cosmological implications

In this section, we will discuss some relevant cosmological implications of our models. The cosmological implications of magnetic Bianchi I in viable  $f(R)$  models of gravity in this present investigation are of great interest to be highlighted. In the following two subsections we discuss the two models:  $f(R) = R - \alpha R^n$  and  $f(R) = (R^b - \Lambda)^c$ , separately.

##### 4.1. The $f(R) = R - \alpha R^n$ gravity

In this model, we study the  $f(R) = R - \alpha R^n$  model of gravity in the presence of a uniform magnetic field. We investigate the influence of the primordial magnetic field on the dynamics of the Bianchi I universe. The physical fixed points from the autonomous system in this model provide physical interest. The general conditions for a successful  $f(R) = R - \alpha R^n$  model can be summarized as follows:

- The point  $P_1^{(1)}$  is a stable fixed point when  $0 \leq n < 2$ . It behaves like a de-Sitter fixed point featuring a late-time de-Sitter acceleration. It can also be the saddle if  $n > 2$ . At late

time, the universe in this model can be described by the de-Sitter acceleration solution given by

$$a(t) = \exp(\lambda t), \quad \lambda = \text{arbitrary constant}, \quad M^2 = 0. \quad (62)$$

- The fixed point  $P_2^{(1)}$  is always saddle point. The standard matter-dominated epoch with the non existence of the magnetic field might be represented by this fixed point. It is controlled by the following parameters:

$$w_{\text{eff}} = -1 + \frac{1}{n}, \quad a(t) = a_0(t-t_0)^{\frac{2n}{3}}, \quad M^2 = 0.$$

We note that this fixed point becomes the standard matter epoch if  $n = 1$ .

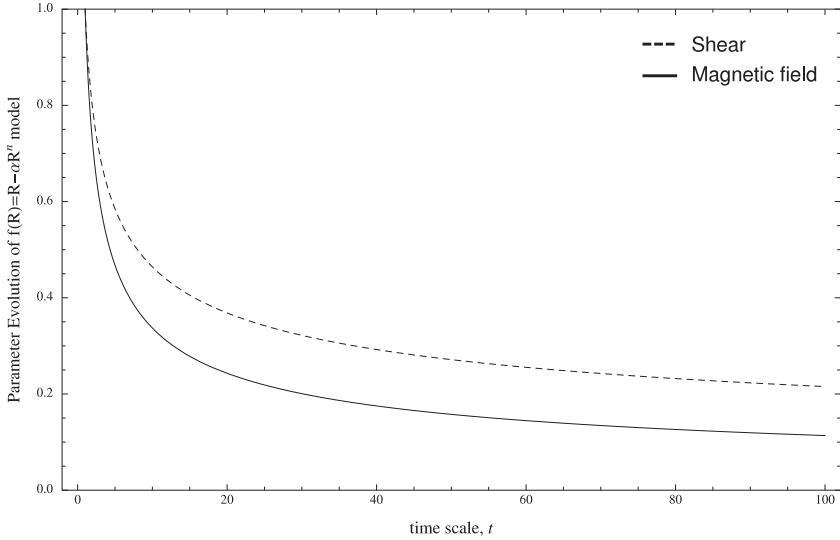
- For the fixed point  $P_3^{(1)}$  in the  $f(R) = R - \alpha R^n$  model, this fixed point might be presented as the beginning of the universe with the curvature-dominated epoch if it is unstable node with the condition in Eq. (44). In this epoch, the curvature may drive cosmic inflation. However, the fixed point is stable if it is satisfied the condition in Eq. (45).
- The fixed point  $P_4^{(1)}$  is called the Jacobs magnetic-like (with curvature) fixed point stemming from the fact that it has the analogous solution to the Jacobs magnetic field solution in Bianchi I in GR theory [37]. This fixed point is always saddle point. Here at this stage the universe is anisotropic with the existence of the magnetic field. Its existence satisfies the condition given in Eq. (47). Interestingly, this would also be compelling since the universe with a primordial magnetic field is necessary anisotropic.

Previously, we have already discussed about the shear evolution for all points in the phase space. The conditions of these points to fall whether into the SSD or FSD regions are given in Eq. (49). With the given number  $n = 1.25$  of the parameter in this model, the shear evolution falls into SSD regime when  $n > 8/7$ . We will end this subsection by examining how shear and magnetic field in this model evolve with time. We find that the shear and the magnetic field will be diluted as illustrated in Fig. 1. Moreover, by comparing the magnetic field decays a bit faster than the shear.

##### 4.2. The $f(R) = (R^b - \Lambda)^c$ gravity

In this model, we study the  $f(R) = (R^b - \Lambda)^c$  model of gravity in the presence of a uniform magnetic field. We investigate the influence of the magnetic field on the dynamics of the Bianchi I universe. The physical fixed points from the autonomous system in this model provide physical interest. The general conditions for a successful  $f(R) = (R^b - \Lambda)^c$  model can be summarized as follows:

- The  $P_1^{(2)}$  point can be the late-time de-Sitter acceleration if it is stable fixed point. Its stable condition of this point



**Fig. 1.** The plot shows the parameter evolution of the  $f(R) = R - \alpha R^n$  model. The dashed line shows how shear evolves with the cosmic-time scale; whilst the solid line represents how the magnetic field evolves with the cosmic-time scale for  $\sigma_0 = a_0 = \tilde{H}_0 = 1$ .

is shown in Eq. (54). On the other hand, this fixed point might be represented as the early epoch of the universe with the condition in Eq. (54). The saddle point condition is otherwise. The universe at late time can be described by the de-Sitter acceleration parametrized by

$$a(t) = \exp(\lambda t), \quad \lambda = \text{arbitrary constant}, \quad M^2 = 0. \quad (63)$$

- The fixed point  $P_2^{(2)}$  is always saddle point. The standard matter-dominated epoch with the non existence of the magnetic field might be represented by this fixed point. It is controlled by the following parameters:

$$w_{\text{eff}} = -1 + \frac{1}{bc}, \quad a(t) = a_0(t - t_0)^{\left(\frac{2bc}{3}\right)}, \quad M^2 = 0. \quad (64)$$

Note that this fixed point exactly becomes the standard matter epoch if  $bc = 1$ .

- For the fixed point  $P_3^{(2)}$  in the  $f(R) = (R^b - \Lambda)^c$  model, this fixed point might be presented as the beginning of the universe with the curvature-dominated epoch if it is unstable node with the condition in Eq. (57). Note that in this epoch the curvature drives cosmic inflation in agreement with the Starobinski model of inflation. To be more concrete, in the following discussion, we will use the specific values of  $b$  and  $c$  such that  $bc \rightarrow 1$  with  $c \geq 1$ . These special values are given by the local-gravity constraints of the viable  $f(R)$  DE model for the standard flat-FLRW spacetime (with an isotropic universe) [17]. For example, using  $b \sim 0.50$ , we obtain  $2.0 < c < 2.5$ . For our purpose, we select  $b \sim 0.50$  and  $c \sim 2.33$ . Using these values, we come up with only 15% deviation from  $bc \rightarrow 1$ . However, there are many other choices for their values.
- The fixed point  $P_4^{(2)}$  is called the Jacobs magnetic-like (with curvature) fixed point, i.e. it has the analogous solution to the Jacobs magnetic field solution in Bianchi I in GR theory [37]. This fixed point is always saddle point. Here at this stage the universe is anisotropic with the existence of the magnetic field. Its existence satisfies the condition given in Eq. (59). As of the preceding model, this would also be compelling since the universe with a primordial magnetic field is necessary anisotropic.

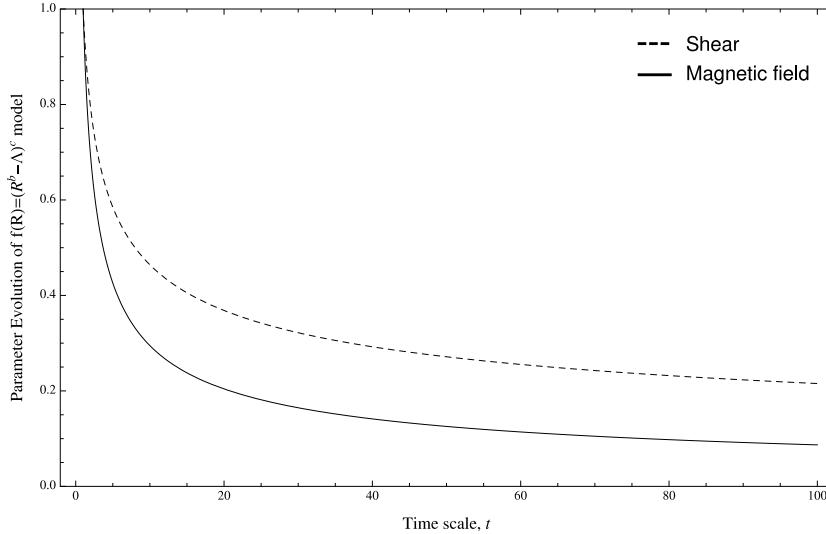
As we already discussed about the shear evolution for all points in the phase space, we then obtain the conditions in which these points fall either into the SSD or FSD regions given in Eq. (60). In contrast to the previous  $f(R)$  model, the shear evolution of the present model falls into FSD regime with the given number of the parameters  $b = 0.5$  and  $c = 2.33$ . Here we will furnish this subsection by examining how shear and magnetic field in this model evolve with time. We find that the shear and the magnetic field will be diluted as illustrated in Fig. 2. Similarly to the preceding model, the magnetic field also decays a bit faster than the shear.

It was noticed that the class of  $f(R)$  gravity models based on the isotropic manner which have a viable cosmological expansion chronology, i.e. a matter dominated epoch followed by a late-time acceleration, was classified in Ref. [17]. Here they provided a common value of the parameters of the model for which one can assume the presence of a relevant cosmological orbit. However, in the present investigation, we extended the selected models of  $f(R)$  gravity by considering the anisotropic counterpart of flat FLRW metric. Hence, the value of the parameters for the presence of the chronology of a cosmological orbit in our work may deviate from those present in Ref. [17].

## 5. Conclusions

In this work, we study the cosmological dynamics of the magnetic Bianchi I with viable  $f(R)$  model of gravity. The dynamical system analysis are utilized to examine the viable  $f(R) = R - \alpha R^n$  and  $(R^b - \Lambda)^c$  models. In summary, we can highlight our study into 2 distinct cases:

- For the  $f(R) = R - \alpha R^n$  model, we found 4 physical fixed points. There are 3 isotropic solutions and 1 anisotropic case with the presence of primordial cosmological magnetic fields. Based on the viable cosmological sequence, by taking  $n = 1.25$ , the universe starts with the isotropic spacetime with curvature-dominated epoch ( $P_3^{(1)}$ ) and it develops to the anisotropic universe with the presence of the primordial cosmological magnetic fields ( $P_4^{(1)}$ ). After that, the universe isotropizes with the standard matter epoch ( $P_2^{(1)}$ ) and evolves to the de-Sitter late-time acceleration scenario ( $P_1^{(1)}$ ). Even though, the given number  $n = 1.25$  violates the local gravity and cosmological constraints but its constraint is viable only in the FLRW counterpart. Our



**Fig. 2.** The plot shows the parameter evolution of the  $f(R) = (R^b - \Lambda)^c$  model. The dashed line shows how shear evolves with the cosmic-time scale; whilst the solid line represents how the magnetic field evolves with the cosmic-time scale for  $\sigma_0 = a_0 = \tilde{H}_0 = 1$ .

numerical value,  $n = 1.25$ , might be correct in the magnetic Bianchi I background. The shear evolution of this model has the scale in the cosmic-time as  $t^{-\frac{1}{3}}$  which is slower than the standard GR. The magnetic fields play an important role on the shear dissipation as shown in Eq. (37). The primordial cosmological magnetic fields decay a little bit faster than the shear with almost the same scale.

- The  $f(R) = (R^b - \Lambda)^c$  gravity has 4 physical fixed points obtained from the autonomous system. It has the same cosmological chronology as the previous model. The sequence of the universe is  $P_3^{(2)} \rightarrow P_4^{(2)} \rightarrow P_2^{(2)} \rightarrow P_1^{(2)}$  which gives the reasonable evolution of the universe history with the parameters  $b = 0.5$  and  $c = 2.33$ . In addition, these parameter values are compatible with the conditions of the viable  $f(R)$  DE of this model, i.e.  $c \geq 1$  and  $bc \approx 1$ . Surprisingly, the  $f(R) = (R^b - \Lambda)^c$  model has the same cosmic-time scale of the shear evolution as the previous model, a.k.a.  $t^{-\frac{1}{3}}$ .

One notes that these 2 models fall into the class A1 of the  $f(R)$  model, i.e. they have the de-Sitter stable point at late-time. The explicit treatment of these two models have been carried out in Ref. [17] on the FLRW background. Our present study is extended to the anisotropic counterpart of flat FLRW metric. We found that the presence of the anisotropic geometry with LSR from Bianchi I background and the cosmological magnetic fields give an additional fixed point before the emergence of the standard matter epoch. This fixed point shows the existence of the primordial magnetic fields and the anisotropy of spacetime before the universe expands to become the isotropic geometry. The shear evolution modifies dissipative behavior by the primordial cosmological magnetic fields significantly as  $\sigma \sim t^{-\frac{1}{3}}$  (in both of two  $f(R)$  models). While for the absence of the magnetic fields case, it gives  $\sigma \sim t^{-1}$ . In addition, the shear dissipation of the  $f(R) = R - \alpha R^n$  gravity is in the SSD regime with the given number  $n = 1.25$ ; whilst the shear evolution in the  $f(R) = (R^b - \Lambda)^c$  model is in the FSD regime with  $b = 0.5$  and  $c = 2.33$ .

Based on the viable  $f(R)$  DE models, moreover, the reasonable evolution of the universe history for the  $f(R) = (R^b - \Lambda)^c$  gravity with  $b = 0.5$  and  $c = 2.33$  is more compatible with the viable conditions ( $c \geq 1$  and  $bc \approx 1$ ) than the  $f(R) = R - \alpha R^n$  model with  $n = 1.25$  (the viable one is  $0 < n < 1$ ). It is worth noting that more complicated versions of viable  $f(R)$  models (e.g. Starobinski [18]

and Hu–Sawicki [19] models) have no close forms of the  $m$  function written in terms of the variable  $r = x_3/x_2$  by using the standard dynamical system approach. However, the authors of Refs. [53] proposed a new approach of the dynamical system to handle the problem. Our forthcoming work is to use such the new approach to tackle the Starobinski and Hu–Sawicki  $f(R)$  models.

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# Large- $N_c$ operator analysis of hyperon-nucleon interactions in SU(3) chiral effective field theory

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We study octet-octet baryon ( $J^P = \frac{1}{2}^+$ ) contact interactions in SU(3) chiral effective field theory by using large- $N_c$  operator analysis. Applying the  $1/N_c$  expansion of the Hartee Hamiltonian, we find 15 operators in the octet-octet baryon potential where 4 operators for leading order (LO) and 11 for and net-to-next-to-leading order (NNLO). The large- $N_c$  operator analysis of octet-octet baryon matrix elements reduces the number of free parameters from 15 to 6 at LO of the  $1/N_c$  expansion. The application of large- $N_c$  sum rules to the Jülich model of hyperon-nucleon (YN) interactions at the LO of the chiral expansion reduces the model parameters to 3 from 5 at the LO of  $1/N_c$  expansion. We find that the values of LECs fitted to YN scattering data in Ref. [20] in the relativistic covariant ChEFT (EG) approach is more consistent with the predictions of large- $N_c$  than in the heavy baryon (HB) formalism approach.

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## I. INTRODUCTION

Chiral effective field theory (ChEFT) [1, 2], based on the approximately and spontaneously broken chiral symmetry of QCD, allows for a systematic way of calculating low-energy hadronic observables. It is very efficient and convenient to use hadrons as basic degrees of freedom rather than quarks and gluons in the ChEFT. Chiral Lagrangian is required to include all possible interactions between hadrons which are constructed in terms of the relevant symmetries of QCD [3]. A number of low-energy properties in the strong interaction is very successfully described by using the ChEFT. The ChEFT is also utilized to shed light on the study of nuclear forces (see [4, 5] for reviews). It was demonstrated by Weinberg's seminal works [6, 7] that one can calculate the nuclear forces systematically by using appropriate power counting scheme. Therefore, loop-corrections and higher order terms can be included for the accuracy of the calculations. Nucleon-nucleon (NN) forces derived in the ChEFT successfully described a huge number of NN experimental data. The NN potentials are composed of the long and short range interactions, where the long range NN force is mainly contributed by the pion exchange while the short range part is encoded by contact term NN interactions with unknown low-energy constants (LECs) to be fitted to experimental data. The higher order contact terms of the NN potentials have been constructed in Refs. [8, 9] at next-to leading order (NLO) and in Refs. [10, 11] for next-to-next-to-leading order ( $N^3LO$ ) in terms of chiral expansions.

On the other hand, hyperon-nucleon (YN) and hyperon-hyperon (YY) forces have been less studied compared with the NN forces. YN interactions are keys for understanding hyper-nuclei and neutron stars [12, 13]. The contact and meson exchange terms of the YN interactions in the ChEFT were constructed by using the SU(3) flavor symmetry in Ref. [14] at leading order (LO) and extended to NLO in Ref. [15]. The most general SU(3) chiral Lagrangians of the octet-octet baryon contact term interactions have been worked out in Ref. [16]. The study of the YY interactions was performed in Refs. [17–19]. At the LO of the YN interactions [14, 20], the SU(3) chiral Lagrangian has 15 free parameters (LECs) and the partial-wave expansion analysis leads to 5 LECs which are fixed with YN data. In this work, we will use the large- $N_c$  operator analysis to explore the  $N_c$  scales and reduce the number of the unknown LECs in the SU(3) chiral Lagrangians and in the LO YN potential [14, 20].

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Large- $N_c$  is an approximate framework of QCD and very useful in the study of hadrons at low-energies. The basic idea is that one can consider the number of colors ( $N_c$ ) to be large and expand it in power of  $1/N_c$  [21, 22]. By using this framework, a number of simplifications of QCD occurs in the large- $N_c$  limit (see Refs. [23, 24] for reviews). The  $1/N_c$  expansion of QCD for the baryon [25–27] has been applied to the NN potential in [28–30] and three-nucleon potential in [31]. Moreover, the  $1/N_c$  expansion is used to study parity-violating NN potentials in [32, 33] as well as time-reversal violating NN potentials [34]. The study of the large- $N_c$  analysis in the NN system provides the understanding of the  $N_c$  scales of the LECs in the NN forces. In addition, the  $1/N_c$  expansion also helps us to reduce the independent number of the LECs [33]. However, the octet-octet baryon interactions in SU(3) flavor symmetry have not been investigated in the large- $N_c$  approach. In this work, we will extend the large- $N_c$  operator analysis in Refs. [29, 31] to the SU(3) chiral Lagrangian in Refs. [14, 20]. The large- $N_c$  octet-octet baryon potential is constructed up to NNLO in terms of the  $1/N_c$  expansion. We will apply large- $N_c$  sum rules to YN interactions at LO which has been recently investigated in Ref. [20]. Moreover, the results can be applied to the YN at NLO and YY sector.

We outline this work as follows: In section 2 we will setup the matrix elements of the octet-octet baryon potential from the SU(3) chiral Lagrangian. In the next section, the potential of the  $1/N_c$  expansion is constructed up to NNLO and large- $N_c$  sum rules for LECs are implied. In section 4, we apply results of the large- $N_c$  sum rules to the LO YN potential. In the last section, we give the conclusion in this work.

## II. THE POTENTIAL OF THE SU(3) OCTET-OCTET BARYON CONTACT TERM INTERACTIONS

We start with the SU(3) chiral Lagrangian of the octet-octet baryon interactions and it was proposed by Ref. [14]. The SU(3)-flavor symmetry is imposed and the chiral Lagrangian is Hermitian and invariant under Lorentz transformations and the CPT discrete symmetry is implied. The minimal SU(3) invariant chiral Lagrangian with non-derivative is given by,

$$\begin{aligned}\mathcal{L}^{(1)} &= C_i^{(1)} \langle \bar{B}_1 \bar{B}_2 (\Gamma_i B)_2 (\Gamma_i B)_1 \rangle , \\ \mathcal{L}^{(2)} &= C_i^{(2)} \langle \bar{B}_1 (\Gamma_i B)_1 \bar{B}_2 (\Gamma_i B)_2 \rangle , \\ \mathcal{L}^{(3)} &= C_i^{(3)} \langle \bar{B}_1 (\Gamma_i B)_1 \rangle \langle \bar{B}_2 (\Gamma_i B)_2 \rangle .\end{aligned}\quad (1)$$

Here 1 and 2 denote the label of the particles in the scattering process, the  $B$  is the usual irreducible octet representation of SU(3) given by

$$B = \frac{1}{\sqrt{2}} \sum_{a=1}^8 \lambda^a B^a = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & \frac{-\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ -\Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix}, \quad (2)$$

where the  $\langle \dots \rangle$  brackets denote taking the trace in the three-dimensional flavor space and the normalization of Gell-Mann matrices  $\langle \lambda^a \lambda^b \rangle = 2 \delta^{ab}$  is used. The  $\Gamma_i$  are the usual elements of the Clifford algebra

$$\Gamma_1 = 1, \quad \Gamma_2 = \gamma^\mu, \quad \Gamma_3 = \sigma^{\mu\nu}, \quad \Gamma_4 = \gamma^\mu \gamma_5, \quad \Gamma_5 = i \gamma_5. \quad (3)$$

By using the chiral power counting in Ref. [14], it has been shown that we have 15 LO non-derivative terms of the chiral Lagrangian. It has also been demonstrated in Ref. [14] that the above Lagrangians are the minimal set of the contact interaction terms in terms of flavor and spin structures by using Cayley-Hamilton identity and Fierz transformation.

To obtain the potentials, we follow approach in Refs. [37, 38] by imposing relativistic covariant constraints. Letting  $\mathcal{H} = -\mathcal{L}$  and taking the approach of the relativistic constraints in [37, 38] into account, one obtains the potential of the octet-octet baryon contact interactions up to the second order of the small momenta of the baryons and it reads,

$$\begin{aligned}V^{(1)} &= \langle \bar{\chi}_2, d; \bar{\chi}_1, c | \mathcal{H}^{(1)} | a, \chi_1; b, \chi_2 \rangle \\ &= \left\{ \frac{1}{3} \delta^{cd} \delta^{ba} + \frac{1}{2} (d^{cde} + i f^{cde}) (d^{eba} + i f^{eba}) \right\} \\ &\quad \times \left\{ c_S^{(1)} \tilde{O}_S + c_T^{(1)} \tilde{O}_T + \left( c_1^{(1)} p_-^2 + c_2^{(1)} p_+^2 \right) \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} + \left( c_3^{(1)} p_-^2 + c_4^{(1)} p_+^2 \right) \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right. \\ &\quad \left. + c_5^{(1)} \frac{i}{2} (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{p}_+ \times \vec{p}_-) + c_6^{(1)} (\vec{p}_- \cdot \vec{\sigma}_1) (\vec{p}_- \cdot \vec{\sigma}_2) + c_7^{(1)} (\vec{p}_+ \cdot \vec{\sigma}_1) (\vec{p}_+ \cdot \vec{\sigma}_2) \right\},\end{aligned}\quad (4)$$

where

$$\begin{aligned}\tilde{O}_S &= \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} + \frac{i}{2M^2} (\vec{p}_+ \times \vec{p}_-) \cdot (\vec{\sigma}_1 - \vec{\sigma}_2), \\ \tilde{O}_T &= \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{i}{2M^2} (\vec{p}_+ \times \vec{p}_-) \cdot (\vec{\sigma}_1 - \vec{\sigma}_2),\end{aligned}\quad (5)$$

and  $\vec{\sigma}_i \equiv \vec{\sigma}_{\bar{\chi}_i \chi_i}$  with  $i = 1, 2$ . The indices  $a(c)$ ,  $b(d)$ ,  $\chi_1(\bar{\chi}_1)$  and  $\chi_2(\bar{\chi}_2)$  are flavor and spin indices of incoming (outgoing) baryon number 1 and 2 respectively and  $M$  is the octet baryon mass in the SU(3) flavor symmetry limit. We note that the octet-octet baryon potentials agree with the heavy baryons formulation of CheFT in [39, 40] for the spin structures. By using the partial integrations and the baryon equation of motion to eliminate time derivative as shown in Refs. [37, 38], the potential in Eq. (4) is the minimal set of linearly independent operators and it consists of 2 LO and 7 NLO operators (see appendix A for the detail derivation of the potential). The LECs,  $c_i^{(1)}$  are linear combinations of the couplings  $C_i^{(1)}$  as,

$$\begin{aligned}c_S^{(1)} &= C_1^{(1)} + C_2^{(1)}, \quad c_T^{(1)} = C_3^{(1)} - C_4^{(1)}, \quad c_1^{(1)} = -\frac{1}{4M^2} (C_2^{(1)} + C_3^{(1)}), \quad c_2^{(1)} = -\frac{1}{2M^2} (C_1^{(1)} - C_2^{(1)}), \\ c_3^{(1)} &= -\frac{1}{4M^2} (C_2^{(1)} + C_3^{(1)}), \quad c_4^{(1)} = \frac{1}{4M^2} (C_3^{(1)} - C_4^{(1)}), \quad c_5^{(1)} = -\frac{1}{2M^2} (C_1^{(1)} - 3C_2^{(1)} - 3C_3^{(1)} - C_4^{(1)}), \\ c_6^{(1)} &= \frac{1}{4M^2} (C_2^{(1)} + C_3^{(1)} + C_4^{(1)} + C_5^{(1)}), \quad c_7^{(1)} = -\frac{1}{4M^2} (C_3^{(1)} + C_4^{(1)}).\end{aligned}\quad (6)$$

In addition, it is worth to discuss about the chiral power counting ( $Q/M$ ) where a  $Q$  is typical three momentum of the baryon. If we impose  $M \sim \Lambda$  where  $\Lambda$  is a chiral symmetry breaking scale. Therefore, our power counting rule in this work adopts  $Q/M \sim (Q/\Lambda)^2$  which has been used in Refs. [9, 10] for the NN potentials. The notations of the momentum in this work are defined below

$$\vec{p}_+ = \frac{1}{2}(\vec{p}' + \vec{p}), \quad p_+^2 = \vec{p}_+ \cdot \vec{p}_+, \quad \vec{p}_- = \vec{p}' - \vec{p}, \quad p_-^2 = \vec{p}_- \cdot \vec{p}_-, \quad \vec{n} = \vec{p} \times \vec{p}' = \vec{p}_+ \times \vec{p}_-, \quad (7)$$

where  $\vec{p}(\vec{p}')$  is incoming (outgoing) three-momentum in the c.m. frame and the on-shell condition of the external momenta is given by

$$\vec{p}_+ \cdot \vec{p}_- = 0. \quad (8)$$

With the same manner, the octet-octet baryon potentials for  $C_i^{(2)}$  and  $C_i^{(3)}$  are written by

$$\begin{aligned}V^{(2)} &= \langle \bar{\chi}_2, d; \bar{\chi}_1, c | \mathcal{H}^{(2)} | a, \chi_1; b, \chi_2 \rangle \\ &= \left\{ \frac{1}{3} \delta^{ca} \delta^{bd} + \frac{1}{2} (d^{cae} + i f^{cae}) (d^{edb} + i f^{edb}) \right\} \\ &\quad \times \left\{ c_S^{(2)} \tilde{O}_S + c_T^{(2)} \tilde{O}_T + \left( c_1^{(2)} p_-^2 + c_2^{(2)} p_+^2 \right) \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} + \left( c_3^{(2)} p_-^2 + c_4^{(2)} p_+^2 \right) \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right. \\ &\quad \left. + c_5^{(2)} \frac{i}{2} (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{p}_+ \times \vec{p}_-) + c_6^{(2)} (\vec{p}_- \cdot \vec{\sigma}_1) (\vec{p}_- \cdot \vec{\sigma}_2) + c_7^{(2)} (\vec{p}_+ \cdot \vec{\sigma}_1) (\vec{p}_+ \cdot \vec{\sigma}_2) \right\},\end{aligned}\quad (9)$$

and

$$\begin{aligned}V^{(3)} &= \langle \bar{\chi}_2, d; \bar{\chi}_1, c | \mathcal{H}^{(3)} | a, \chi_1; b, \chi_2 \rangle \\ &= \delta^{ca} \delta^{bd} \left\{ c_S^{(3)} \tilde{O}_S + c_T^{(3)} \tilde{O}_T + \left( c_1^{(3)} p_-^2 + c_2^{(3)} p_+^2 \right) \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} + \left( c_3^{(3)} p_-^2 + c_4^{(3)} p_+^2 \right) \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right. \\ &\quad \left. + c_5^{(3)} \frac{i}{2} (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{p}_+ \times \vec{p}_-) + c_6^{(3)} (\vec{p}_- \cdot \vec{\sigma}_1) (\vec{p}_- \cdot \vec{\sigma}_2) + c_7^{(3)} (\vec{p}_+ \cdot \vec{\sigma}_1) (\vec{p}_+ \cdot \vec{\sigma}_2) \right\},\end{aligned}\quad (10)$$

where the LECs in Eqs. (9) and (10) are the linear combinations of the couplings as in Eq. (6) by replacing  $c_i^{(1)} \rightarrow c_i^{(2,3)}$  and  $C_i^{(1)} \rightarrow C_i^{(2,3)}$ . By using relativistic reductions as in [37, 38], we obtain the minimal set of the SU(3) octet-octet baryon potentials and there are 27 operators totally. Moreover, Fierz identities for the Gell-mann matrices ( $\lambda^a$ ) are also taken into account for the calculations of the potentials in Eqs. (4), (9) and (10). We found that there is no the redundant terms of the SU(3) flavor structures. We obtain 6 and 21 operators at LO and NLO of the small momentum scale expansion ( $Q/M$ ). At the LO, the operators from the couplings  $C_{1,2,3,4}^{(1,2,3)}$  enter to contribute the potential but

the couplings  $C_5^{(1,2,3)}$  start at NLO. We will reduce the independent number of the LECs of the SU(3) octet-octet baryon interactions in the ChEFT by using the large- $N_c$  operator analysis in the next section.

### III. THE $1/N_c$ OPERATOR PRODUCT EXPANSION ANALYSIS OF THE TWO-BARYON MATRIX ELEMENTS

#### A. The $1/N_c$ expansion octet-octet baryon ansatz

In this section, we are going to study the  $1/N_c$  expansion for the octet-octet baryon matrix elements. According to Witten's conjecture [22], the matrix elements of baryon-baryon scattering should scale like  $N_c$ , i.e. [28, 29],

$$N_c \langle B_1 | \hat{O}_1^i | B_1 \rangle \langle B_2 | \hat{O}_2^{i'} | B_2 \rangle, \quad (11)$$

where  $\hat{O}_1^i$  and  $\hat{O}_2^{i'}$  operators are the  $i$ - and  $i'$ -quark current operators on the first and the second baryon. It has proven in the Ref. [27] that the matrix elements for one baryon in SU(3) flavor symmetry has the  $N_c$  scaling as,

$$\langle B_j | \hat{O}_j^i | B_j \rangle \lesssim N_c^0, \quad (12)$$

with  $j = 1, 2$ . This holds for the matrix elements of the second baryon as well. One can expand the matrix elements in terms of effective quark operator and effective spin-flavor baryon states in  $1/N_c$  expansion as [26, 27],

$$\langle B | \hat{O}^i | B \rangle = (B | \sum_r c_r^{(i)} \left( \frac{\mathcal{O}}{N_c} \right)^r | B), \quad (13)$$

where  $c_r^{(i)}$  is a function which contains dynamical properties of the system and  $|B\rangle$  is an effective baryon state composed of spin and flavor structures only [26, 27]. The  $\mathcal{O}^r$  are the  $r$ -body operators which comprises of the effective quark operators [28, 29],

$$\left( \frac{\mathcal{O}}{N_c} \right)^r = \left( \frac{J}{N_c} \right)^l \left( \frac{T}{N_c} \right)^m \left( \frac{G}{N_c} \right)^n, \quad \text{with } l + m + n = r. \quad (14)$$

The operators  $J$ ,  $T$  and  $G$  are spin, flavor and spin-flavor operators, respectively and they are defined by [26, 35],

$$\begin{aligned} \mathbb{1} &= q^\dagger (\mathbf{1} \otimes \mathbf{1}) q, & J_i &= q^\dagger \left( \frac{\sigma_i}{2} \otimes \mathbf{1} \right) q, \\ T^a &= q^\dagger \left( \mathbf{1} \otimes \frac{\lambda_a}{2} \right) q, & G_i^a &= q^\dagger \left( \frac{\sigma_i}{2} \otimes \frac{\lambda_a}{2} \right) q, \end{aligned} \quad (15)$$

where  $q$  and  $q^\dagger$  are quark annihilation and creation operators respectively. According to the fully antisymmetry and Fermi statistics of the  $SU(N_c)$  color group, the spin and flavor of baryonic ground state of the  $N_c$  quarks have to be completely symmetric representation. Therefore one can consider quark operators  $q$  and  $q^\dagger$  as bosonic operators with the commutation relation  $[q, q^\dagger] = 1$ . The  $N_c$  scaling of the  $r$ -body operator  $\mathcal{O}^r$  and the the coefficient  $c_r^{(i)}$  scale like [28, 29],

$$(B | \mathcal{O}^r | B) \lesssim N_c^r, \quad c_r^{(i)} \sim N_c^0. \quad (16)$$

In addition, The one-baryon matrix elements of the operators  $J$ ,  $T$  and  $G$  in SU(3) flavor symmetry framework have  $N_c$  scaling in the following way [26]

$$(B | J^i | B) \sim N_c^0, \quad (B | \mathbb{1} | B) \sim N_c, \quad (B | T^a | B) \lesssim N_c, \quad (B | G^{i a} | B) \lesssim N_c. \quad (17)$$

In contrast to the SU(2) flavor symmetry, there is only one operator that can suppress rising of the  $N_c$  for one-baryon matrix elements i.e. the  $J$  whereas all the rest of the effective operators rises the  $N_c$  factor. However, the symbol,  $\lesssim$  is used for saturating the maximum of the  $N_c$  scaling for the  $(B | T^a | B)$  and  $(B | G^{i a} | B)$  because the matrix elements of the  $T^a$  operator scales like  $N_c^0$  for  $a = 1, 2, 3$ , but as  $\sqrt{N_c}$  when  $a = 4, 5, 6, 7$  and as  $N_c$  when  $a = 8$ . On the other hand, the matrix elements of the  $G^{i a}$  scales like  $N_c$  for  $a = 1, 2, 3$ , as  $\sqrt{N_c}$  when  $a = 4, 5, 6, 7$  and as  $N_c^0$  when  $a = 8$  [26]. These are the differences of the effective operators between SU(2) and SU(3) flavor symmetries. Moreover, it is

worth to discuss about the  $N_c$  scaling of the external momentum variables. Here we consider all momentum in c.m. frame as we discussed in the previous section. One recalls the  $N_c$  scaling of the momentum variables in Eq. (7), it reads [29],

$$\vec{p}_+ \sim 1/N_c, \quad \vec{p}_- \sim N_c^0. \quad (18)$$

In a meson exchange picture, the  $\vec{p}_+$  can only appear in the baryon-baryon potential as a relativistic correction (i.e., a velocity dependent term). Therefore, the  $\vec{p}_+$  always come with the factor  $1/M$ . Since  $M \sim N_c$ , this gives  $\vec{p}_+ \sim 1/N_c$  (for more detail discussions see [29, 31, 33]). The baryon-baryon potential in terms of  $1/N_c$  expansion can be written in the Hartree Hamiltonian [29, 31, 33]. It takes the following form,

$$\hat{H} = N_c \sum_r \sum_{lm} c_{r,lm} \left( \frac{J}{N_c} \right)^l \left( \frac{T}{N_c} \right)^m \left( \frac{G}{N_c} \right)^{r-l-m}, \quad (19)$$

where again the  $c_{r,lm}$  coefficient function has scale  $N_c^0$ . It is well known that, at the large- $N_c$  limit, the spin-1/2 and 3/2 baryons are degeneracy states. In this work, we project the Hamiltonian  $\hat{H}$  to the octet (spin-1/2) baryon sector only. This has been discussed extensively in [29]. We will construct the Hamiltonian in order of  $1/N_c$  expansion. Then the leading-order (LO) is given by

$$\hat{H}_{\text{LO}} = U_1^{\text{LO}}(p_-^2) \mathbb{1}_1 \cdot \mathbb{1}_2 + U_2^{\text{LO}}(p_-^2) T_1 \cdot T_2 + U_3^{\text{LO}}(p_-^2) G_1 \cdot G_2 + U_4^{\text{LO}}(p_-^2) (p_+^i p_-^j)_{(2)} \cdot (G_1^{i,a} G_2^{j,a})_{(2)}, \quad (20)$$

where  $T_1 \cdot T_2 = T_1^a T_2^a$  and  $G_1 \cdot G_2 = G_1^{i,a} G_2^{i,a}$ .  $U_i^{\text{LO}}(p_-^2)$  is arbitrary function of the  $p_-^2$  and it has  $N_c^0$  scale. Here we also introduce the notation,

$$(A^i B^j)_{(2)} \equiv \frac{1}{2} \left( A^i B^j + A^j B^i - \frac{2}{3} \delta_{ij} A \cdot B \right), \quad (21)$$

and then

$$(p_+^i p_-^j)_{(2)} \cdot (\sigma_1^i \sigma_2^j)_{(2)} = (\vec{p}_+ \cdot \vec{\sigma}_1) (\vec{p}_- \cdot \vec{\sigma}_2) - \frac{1}{3} p_+^2 \sigma_1 \cdot \sigma_2. \quad (22)$$

In this work, we terminate the  $1/N_c$  expansion at the  $1/N_c^2$  order. Then, the octet-octet baryon Hamiltonian at NNLO takes the following form,

$$\begin{aligned} \hat{H}_{\text{NNLO}} = & U_1^{\text{NNLO}}(p_-^2) p_+^2 \mathbb{1}_1 \cdot \mathbb{1}_2 + U_2^{\text{NNLO}}(p_-^2) \vec{J}_1 \cdot \vec{J}_2 + U_3^{\text{NNLO}}(p_-^2) \vec{J}_1 \cdot \vec{J}_2 T_1 \cdot T_2 + U_4^{\text{NNLO}}(p_-^2) p_+^2 T_1 \cdot T_2 \\ & + U_5^{\text{NNLO}}(p_-^2) p_+^2 G_1 \cdot G_2 + U_6^{\text{NNLO}}(p_-^2) i(\vec{p}_+ \times \vec{p}_-) \cdot (\vec{J}_1 + \vec{J}_2) + U_7^{\text{NNLO}}(p_-^2) i(\vec{p}_+ \times \vec{p}_-) \cdot (T_1^a \tilde{G}_2^a + \tilde{G}_1^a T_2^a) \\ & + U_8^{\text{NNLO}}(p_-^2) i(\vec{p}_+ \times \vec{p}_-) \cdot (\vec{J}_1 + \vec{J}_2) T_1 \cdot T_2 + U_9^{\text{NNLO}}(p_-^2) (p_+^i p_-^j)_{(2)} \cdot (J_1^i J_2^j)_{(2)} \\ & + U_{10}^{\text{NNLO}}(p_-^2) (p_+^i p_-^j)_{(2)} \cdot (J_1^i J_2^j)_{(2)} T_1 \cdot T_2 + U_{11}^{\text{NNLO}}(p_-^2) (p_+^i p_-^j)_{(2)} \cdot (G_1^{i,a} G_2^{j,a})_{(2)}. \end{aligned} \quad (23)$$

Here the  $1/N_c$  scale factor is implied on each effective operators,  $\mathbb{1}$ ,  $J$ ,  $T$  and  $G$  implicitly. The functions  $U_i^{\text{LO}}(p_-^2)$  and  $U_i^{\text{NNLO}}(p_-^2)$  have  $N_c^0$  scale. Noting that there are no  $p_+^2 J_1 \cdot J_2$  and  $(p_+^i p_-^j)_{(2)} \cdot (J_1^i J_2^j)_{(2)}$  structures because these operators have a further suppression in order  $1/N_c^4$ .

Let's us discuss comparisons between the octet-octet baryon potential and the nucleon-nucleon potential in the  $1/N_c$  expansion. In the case of the SU(3) flavor symmetry, we find addition operator  $T_1 \cdot T_2$  at LO instead of NNLO because  $T^8 T^8 / N_c \sim N_c$  while there is no such operator in nucleon-nucleon potential. Superficially, the two-body operator,  $T^a G^{i,a} / N_c$  should scale like  $N_c$  by using the  $N_c$  scale counting rules in Eq. (17). But if we consider the operator more carefully then we find  $T^a G^{i,a} / N_c \sim N_c^0$  because  $T^{1,2,3} G^{i,1,2,3} / N_c \sim T^{4,5,6,7} G^{i,4,5,6,7} / N_c \sim T^8 G^{i,8} / N_c \sim N_c^0$ . Surprisingly, the SU(3) octet-octet potential has the same structures as the nucleon-nucleon potential in SU(2) flavor symmetry i.e. there is no NLO term in the  $1/N_c$  expansion. The extension of the flavor symmetry from SU(2) to SU(3) does not change the profile of the  $1/N_c$  potential. Before closing this section, we would like to summarize the  $1/N_c$  expansion octet-octet baryon Hamiltonian. There are 4 LO operators. At the NNLO of  $1/N_c$  expansion, we obtain 11 operators. We totally have 15 operators of  $1/N_c$  expansion for octet-octet baryon potential.

## B. Matching the octet-octet baryon potential of the SU(3) chiral Lagrangian with the $1/N_c$ operator product expansion

We will evaluate, in this section, the octet-octet baryon potential from the Hartee Hamiltonian in Eqs. (20) and (23). The  $1/N_c$  potential is given by

$$V = (\bar{\chi}_2, d; \bar{\chi}_1, c | \hat{H} | a, \chi_1; b, \chi_2), \quad (24)$$

where  $a(c)$ ,  $b(d)$ ,  $\chi_1(\bar{\chi}_1)$  and  $\chi_2(\bar{\chi}_2)$  are flavor and spin indices of incoming (outgoing) baryon number 1 and 2 respectively. After that we will do matching the octet-octet baryon potential and  $1/N_c$  operator product expansion to correlate the LECs from the chiral Lagrangian in Eq. (1). First of all, we recall the action of the effective operators on the effective baryon states at  $N_c = 3$  as [35],

$$\begin{aligned} \mathbb{1} |a, \chi\rangle &= 3 |a, \bar{\chi}\rangle, \\ J_i |a, \chi\rangle &= \frac{1}{2} \sigma_{\bar{\chi}\chi}^{(i)} |a, \bar{\chi}\rangle, \\ T^a |b, \chi\rangle &= i f^{bca} |c, \chi\rangle, \\ G_i^a |b, \chi\rangle &= \sigma_{\bar{\chi}\chi}^{(i)} \left( \frac{1}{2} d^{bca} + \frac{i}{3} f^{bca} \right) |c, \bar{\chi}\rangle + \dots, \end{aligned} \quad (25)$$

where  $\dots$  stands for a relevant structure of spin- $\frac{3}{2}$  baryons [35] but we do not consider the spin- $\frac{3}{2}$  baryons degree of freedom in this work. Before matching operators, we make ansatz for the arbitrary functions  $U_i^{LO}$  and  $U_i^{NNLO}$  that they are,

$$U_i^{LO}(p_-^2) = g_i, \quad U_i^{NNLO}(p_-^2) = h_i. \quad (26)$$

Using Eq. (25) in Eqs. (20) and (23), the potential in terms of the large- $N_c$  operators at the LO is given by,

$$\begin{aligned} V_{LO} &= 9 g_1 \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} \delta^{cd} \delta^{bd} + g_2 i^2 f^{ace} f^{bde} \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} + g_3 \vec{\sigma}_1 \cdot \vec{\sigma}_2 \left( \frac{1}{2} d^{ace} + \frac{i}{3} f^{ace} \right) \left( \frac{1}{2} d^{bde} + \frac{i}{3} f^{bde} \right) \\ &+ g_4 (p_-^i p_-^j)_{(2)} \cdot (\sigma_1^i \sigma_2^j)_{(2)} \left( \frac{1}{2} d^{ace} + \frac{i}{3} f^{ace} \right) \left( \frac{1}{2} d^{bde} + \frac{i}{3} f^{bde} \right), \end{aligned} \quad (27)$$

and at the NNLO of the  $1/N_c$  expansion takes form,

$$\begin{aligned} V_{NNLO} &= 9 h_1 p_+^2 \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} \delta^{cd} \delta^{bd} + \frac{1}{4} h_2 \vec{\sigma}_1 \cdot \vec{\sigma}_2 \delta^{cd} \delta^{bd} + \frac{1}{4} h_3 \vec{\sigma}_1 \cdot \vec{\sigma}_2 i^2 f^{ace} f^{bde} + h_4 p_+^2 i^2 f^{ace} f^{bde} \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} \\ &+ h_5 p_+^2 \vec{\sigma}_1 \cdot \vec{\sigma}_2 \left( \frac{1}{2} d^{ace} + \frac{i}{3} f^{ace} \right) \left( \frac{1}{2} d^{bde} + \frac{i}{3} f^{bde} \right) + \frac{3}{2} i h_6 (\vec{p}_+ \times \vec{p}_-) \cdot (\vec{\sigma}_1 + \vec{\sigma}_2) \delta^{cd} \delta^{bd} \\ &+ i h_7 (\vec{p}_+ \times \vec{p}_-) \cdot \left[ \vec{\sigma}_1 \left( \frac{1}{2} d^{ace} + \frac{i}{3} f^{ace} \right) i f^{bde} + \vec{\sigma}_2 i f^{ace} \left( \frac{1}{2} d^{bde} + \frac{i}{3} f^{bde} \right) \right] \\ &+ \frac{3}{2} i h_8 (\vec{p}_+ \times \vec{p}_-) \cdot (\vec{\sigma}_1 + \vec{\sigma}_2) i^2 f^{ace} f^{bde} + \frac{1}{4} h_9 (p_-^i p_-^j)_{(2)} \cdot (\sigma_1^i \sigma_2^j)_{(2)} \delta^{cd} \delta^{bd} \\ &+ \frac{1}{4} h_{10} (p_-^i p_-^j)_{(2)} \cdot (\sigma_1^i \sigma_2^j)_{(2)} i^2 f^{ace} f^{bde} + h_{11} (p_+^i p_+^j)_{(2)} \cdot (\sigma_1^i \sigma_2^j)_{(2)} \left( \frac{1}{2} d^{ace} + \frac{i}{3} f^{ace} \right) \left( \frac{1}{2} d^{bde} + \frac{i}{3} f^{bde} \right). \end{aligned} \quad (28)$$

We note that the  $N_c$  scales of the above potentials are  $V_{LO} \sim N_c$  and  $V_{NNLO} \sim N_c^{-1}$ .

By using Eqs. (4), (9), (10), (27) and (28), the  $N_c$  scaling relations of the LECs can be extracted,

$$C_{1,2}^{(1)} \sim C_{1,2}^{(2)} \sim C_{1,2}^{(3)} \sim N_c, \quad C_{3,4,5}^{(1)} \sim C_{3,4,5}^{(2)} \sim C_{3,4,5}^{(3)} \sim N_c^{-1}, \quad (29)$$

where  $\Lambda \sim N_c^0$  [29, 32, 33] is implied. Note that the couplings  $C_{1,2,3}^{(1)}$ ,  $C_{1,2,3,4,5}^{(2)}$ ,  $C_{1,2,3}^{(3)}$  are LO of order  $N_c$  while the  $N_c$  scaling of the  $C_{3,4,5}^{(1)}$ ,  $C_{3,4,5}^{(2)}$  and  $C_{3,4,5}^{(3)}$  are further suppressed by order  $1/N_c^2$ . We found that there is no NLO of the LECs in the  $1/N_c$  expansion.

Matching the spin and flavor structures between the octet-octet baryon potential of the SU(3) chiral Lagrangian and the  $1/N_c$  expansion up to NNLO, the large- $N_c$  operator analysis leads to the relations between the LECs of the SU(3) baryon contact interaction and we find the following results,

$$C_1^{(2)} = C_1^{(1)} + g_2 - 4 h_4 \Lambda^2,$$

$$\begin{aligned}
C_2^{(2)} &= C_2^{(1)} + g_2 + 4h_4\Lambda^2, \\
C_3^{(2)} &= C_3^{(1)} - \frac{1}{2}g_2 + \frac{1}{8}h_3 - 4h_4\Lambda^2 + 2h_+\Lambda^2, \\
C_4^{(2)} &= C_4^{(1)} - \frac{1}{2}g_2 - \frac{3}{8}h_3 - 4h_4\Lambda^2 + 2h_+\Lambda^2, \\
C_5^{(2)} &= C_5^{(1)} + \frac{1}{4}h_3 + 4h_4\Lambda^2 - 4h_+\Lambda^2 + 2h_{10}\Lambda^2, \\
C_1^{(3)} &= -\frac{1}{3}C_1^{(1)} + \frac{9}{2}g_1 - \frac{1}{3}g_2 - 18h_1\Lambda^2 + \frac{4}{3}h_4\Lambda^2, \\
C_2^{(3)} &= -\frac{1}{3}C_2^{(1)} + \frac{9}{2}g_1 - \frac{1}{3}g_2 + 18h_1\Lambda^2 - \frac{4}{3}h_4\Lambda^2, \\
C_3^{(3)} &= -\frac{1}{3}C_3^{(1)} - \frac{9}{4}g_1 + \frac{1}{6}g_2 - 18h_1\Lambda^2 + \frac{1}{16}h_2 - \frac{1}{24}h_3 + \frac{4}{3}h_4\Lambda^2 + \frac{3}{2}h_6\Lambda^2 - \frac{2}{3}h_+\Lambda^2, \\
C_4^{(3)} &= -\frac{1}{3}C_4^{(1)} - \frac{9}{4}g_1 + \frac{1}{6}g_2 - 18h_1\Lambda^2 - \frac{3}{16}h_2 + \frac{1}{8}h_3 + \frac{4}{3}h_4\Lambda^2 + \frac{3}{2}h_6\Lambda^2 - \frac{2}{3}h_+\Lambda^2, \\
C_5^{(3)} &= -\frac{1}{3}C_5^{(1)} + 18h_1\Lambda^2 + \frac{1}{8}h_2 - \frac{1}{12}h_3 - \frac{4}{3}h_4\Lambda^2 - 3h_6\Lambda^2 + h_9\Lambda^2 + \frac{4}{3}h_+\Lambda^2 - \frac{2}{3}h_{10}\Lambda^2,
\end{aligned} \tag{30}$$

where  $h_+ = 2h_7/3 + 3h_8$ . Note that the Jacobi identities for the  $f$  and  $d$  symbols,

$$\begin{aligned}
f^{abe}f^{ecd} + f^{bce}f^{ead} + f^{cae}f^{ebd} &= 0, \\
d^{abe}f^{ecd} + d^{bce}f^{ead} + d^{cae}f^{ebd} &= 0
\end{aligned} \tag{31}$$

have been used in the matching procedure.

To the LO contributions of the  $1/N_c$  expansion, one can reduce the number of the free parameters with  $\mathcal{O}(1/N_c^2) \equiv h_i$  corrections. 9 sum rules of the LECs of the SU(3) octet-octet baryon contact interactions in the ChEFT are derived

$$\begin{aligned}
C_1^{(1)} = C_1^{(2)} &= -3C_1^{(3)} - 2C_4^{(2)} - 6C_4^{(3)}, & C_2^{(1)} = C_2^{(2)} &= -3C_2^{(3)} - 2C_4^{(2)} - 6C_4^{(3)}, \\
C_3^{(1)} = C_3^{(2)} &= -3C_3^{(3)} + C_4^{(2)} + 3C_4^{(3)}, & C_4^{(1)} = C_4^{(2)}, & C_5^{(1)} = C_5^{(2)} = -3C_5^{(3)}.
\end{aligned} \tag{32}$$

We find that there are 6 free parameters of the SU(3) octet-octet baryon contact interactions in the ChEFT from the large- $N_c$  operator analysis. At  $N_c = 3$ , these sum rules are held up to corrections of the  $1/N_c^2 \approx 10\%$  approximately. In order to see the application of the 9 large- $N_c$  sum rules, we will apply our results to YN interactions in next section.

#### IV. APPLICATION OF THE LARGE- $N_c$ SUM RULES TO THE JÜLICH HYPERON-NUCLEON CONTACT INTERACTIONS AT THE LO

In this section, we will apply the large- $N_c$  sum rules to the Jülich hyperon-nucleon contact interactions at LO [14]. The LO contact terms of the chiral Lagrangians in Eq. (1) with the large component of the baryon spinors have 6 free parameters. They read, [14],

$$C_S^{(1)}, \quad C_S^{(2)}, \quad C_S^{(3)}, \quad C_T^{(1)}, \quad C_T^{(2)}, \quad C_T^{(3)}. \tag{33}$$

The  $C_{S,T}^{(1,2,3)}$  are linear combinations of the coupling constants in Eq. (1) as

$$C_S^{(1,2,3)} = C_1^{(1,2,3)} + C_2^{(1,2,3)}, \quad C_T^{(1,2,3)} = C_3^{(1,2,3)} - C_4^{(1,2,3)}. \tag{34}$$

The operator from the couplings,  $C_5^{(1,2,3)}$  does not contribute to the YN potentials at the LO of the chiral expansion. Applying the large- $N_c$  sum rules in Eq. (32), we find 3 sum rules i.e.,

$$C_S^{(1)} = C_S^{(2)}, \quad C_T^{(1)} = C_T^{(2)} = -3C_T^{(3)}. \tag{35}$$

Above sum rules give only 3 free parameters and the  $N_c$  scalings of those parameters are given by

$$C_S^{(1,2,3)} \sim N_c, \quad C_T^{(1,2,3)} \sim N_c^{-1}. \tag{36}$$

	$C_{1S0}^{\Lambda\Lambda}$	$C_{1S0}^{\Sigma\Sigma}$	$C_{3S1}^{\Lambda\Lambda}$	$C_{3S1}^{\Sigma\Sigma}$	$C_{3S1}^{\Lambda\Sigma}$
EG	-0.04795(151)	-0.07546(81)	-0.01727(124)	0.36367(30310)	0.01271(471)
HB	-0.03894(1)	-0.07657(1)	-0.01629(13)	0.20029(14050)	-0.00176(304)

TABLE I: Best-fitted values of  $YN$  s-wave LECs (in units of  $10^4$  GeV $^{-2}$ ) for cut-off,  $\Lambda = 600$  MeV in the EG and HB approaches [20].

It is interesting to note that  $N_c$  scalings of the  $C_{S,T}^{(1,2,3)}$  in Eq. (36) agree with the NN case [29, 31]. The sum rules in Eq. (35) are useful for calculating the partial wave potentials at the LO in the chiral expansion of the hyperon-nucleon scattering. The hyperon-nucleon partial wave potentials at LO have been constructed and studied in Ref. [14] and also re-investigated in [20]. According to the SU(3) flavor symmetry, the authors of the Ref. [14] find that there are only 5 parameters (potentials) which are used to fit the experimental data of the hyperon-nucleon scattering. The parameters are read

$$C_{1S0}^{\Lambda\Lambda} \equiv V_{1S0}^{\Lambda\Lambda}, \quad C_{3S1}^{\Lambda\Lambda} \equiv V_{3S1}^{\Lambda\Lambda}, \quad C_{1S0}^{\Sigma\Sigma} \equiv V_{1S0}^{\Sigma\Sigma}, \quad C_{3S1}^{\Sigma\Sigma} \equiv V_{3S1}^{\Sigma\Sigma}, \quad C_{3S1}^{\Lambda\Sigma} \equiv V_{3S1}^{\Lambda\Sigma}, \quad (37)$$

where the Jülich model of the LO hyperon-nucleon potentials are written in terms of the couplings  $C_{S,T}^{(1,2,3)}$  in the following forms [14],

$$\begin{aligned} V_{1S0}^{\Lambda\Lambda} &= 4\pi \left[ \frac{1}{6} \left( C_S^{(1)} - 3C_T^{(1)} \right) + \frac{5}{3} \left( C_S^{(2)} - 3C_T^{(2)} \right) + 2 \left( C_S^{(3)} - 3C_T^{(3)} \right) \right], \\ V_{3S1}^{\Lambda\Lambda} &= 4\pi \left[ \frac{3}{2} \left( C_S^{(1)} + C_T^{(1)} \right) + \left( C_S^{(2)} + C_T^{(2)} \right) + 2 \left( C_S^{(3)} + C_T^{(3)} \right) \right], \\ V_{1S0}^{\Sigma\Sigma} &= 4\pi \left[ 2 \left( C_S^{(2)} - 3C_T^{(2)} \right) + 2 \left( C_S^{(3)} - 3C_T^{(3)} \right) \right], \\ V_{3S1}^{\Sigma\Sigma} &= 4\pi \left[ -2 \left( C_S^{(2)} + C_T^{(2)} \right) + 2 \left( C_S^{(3)} + C_T^{(3)} \right) \right], \\ V_{3S1}^{\Lambda\Sigma} &= 4\pi \left[ -\frac{3}{2} \left( C_S^{(1)} + C_T^{(1)} \right) + \left( C_S^{(2)} + C_T^{(2)} \right) \right]. \end{aligned} \quad (38)$$

Using the sum rules in Eq. (35) to the 5 free parameters in Eq. (37), one finds at LO of the  $1/N_c$  expansion,

$$C_{1S0}^{\Sigma\Sigma} = \frac{8}{7} C_{1S0}^{\Lambda\Lambda} - \frac{1}{7} C_{3S1}^{\Lambda\Lambda} - \frac{11}{21} C_{3S1}^{\Lambda\Sigma}, \quad C_{3S1}^{\Sigma\Sigma} = C_{3S1}^{\Lambda\Lambda} + 9 C_{3S1}^{\Lambda\Sigma}. \quad (39)$$

Note that all of the LECs has the same  $N_c$  scaling as  $N_c$ . The large- $1/N_c$  analysis of the LO  $YN$  potentials predicts that there are 3 free parameters at the LO of  $1/N_c$  expansion with  $\mathcal{O}(1/N_c^2)$  corrections. With the same manner of the large- $N_c$  analysis of the LO  $YN$  potentials, one can apply the sum-rules in Eq. (32) for the partial-wave analysis in the  $YN$  potentials at NLO in Ref. [15] as well as for the  $YY$  sector in Refs. [15, 17, 18].

Next we will compare the prediction of the large- $N_c$  sum rules in Eq. (39) with the best fitted values of the LECs from  $YN$  scattering data in Ref. [20]. This reference has performed the partial wave analysis of the  $YN$  s-wave scattering by using the same chiral Lagrangian as in our work. Authors in Ref. [20] have used two approaches to solve scattering amplitudes via Kadyshevsky equation with the relativistic covariant ChEFT (referred as EG) and Lippmann-Schwinger equation with the heavy-baryon formalisms (referred as HB). The relativistic covariant ChEFT (EG) approach is also used to study NN interactions in [41]. The best fitted values of the LECs are shown in Tab. I. We will use the LECs,  $C_{1S0}^{\Lambda\Lambda}$ ,  $C_{3S1}^{\Lambda\Lambda}$  and  $C_{3S1}^{\Lambda\Sigma}$  as input values in Eq. (39) and the large- $N_c$  sum rules predict that

$$\begin{aligned} C_{1S0,EG}^{\Sigma\Sigma} &= -0.06327, & C_{3S1,EG}^{\Sigma\Sigma} &= 0.1271, \\ C_{1S0,HB}^{\Sigma\Sigma} &= -0.04333, & C_{3S1,HB}^{\Sigma\Sigma} &= -0.0176. \end{aligned} \quad (40)$$

Comparing the LECs,  $C_{1S0}^{\Sigma\Sigma}$  and  $C_{3S1}^{\Sigma\Sigma}$  from the large- $N_c$ 's predictions with the best fitted values in Tab. I, we found that  $C_{1S0}^{\Sigma\Sigma}$  and  $C_{3S1}^{\Sigma\Sigma}$  from large- $N_c$  are in the same order as the best fitted values and with the same relative sign in EG approach. On the other hand, for the HB formalisms, the  $C_{1S0}^{\Sigma\Sigma}$  is also in the same order as the large- $N_c$  value and with the same relative sign. But for the  $C_{3S1}^{\Sigma\Sigma}$  value in HB approach, it is different in order of magnitude of 1 with the large- $N_c$  prediction and with different relative sign. One notes that the LECs best fitted values from EG and HB approaches have statistical uncertainties at 68 % (one sigma) level. While Ref. [20] concluded that there is

not much difference between two approaches. But the large- $N_c$  sum rules in this work can show that the LECs from EG approach is more consistent with the predictions of large- $N_c$  than the HB formalism.

## V. CONCLUSIONS

In this work, we studied the large- $N_c$  operator analysis of the octet-octet baryon potential from the SU(3) ChEFT. The minimal set of the octet-octet baryon potential is derived by using the relativistic constraints as suggestion in Refs. [37, 38] as well as the Claley-Hamilton identity and Fierz rearrangement to eliminate the redundant operators as shown in Ref. [14]. Up to NLO of  $Q/\Lambda$  expansion, we found 27 operators for the octet-octet baryon potential in SU(3) flavor symmetry, 6 in LO and 21 in NLO of the small momentum scale.

The octet-octet baryon potential in the at LO in The  $1/N_c$  expansion is of order  $N_c$  and there are 4 operators while the NNLO potential is of order  $1/N_c$  and we found 11 operators. The LECs of the ChEFT have two  $N_c$  scalings, namely  $N_c$  and  $1/N_c$  orders as shown in Eq. (29). Interestingly, the extension of the flavor symmetry from SU(2) to SU(3) in the large- $N_c$  operator analysis does not change the profile of the potential in terms of the  $1/N_c$  expansion. There is no NLO for the SU(3) octet-octet baryon potential as for the NN potential [29, 31].

The matching between the octet-octet baryon potential and the  $1/N_c$  operator expansion leads to 6 free parameters of the LECs from the SU(3) chiral Lagrangian at the LO of the  $1/N_c$  expansion with  $\mathcal{O}(1/N_c^2) \approx 10\%$  correction. The application of the sum rules in Eqs. (32) from the large- $N_c$  constraint to the partial-wave potential of the YN interactions at LO of the chiral expansion reduces the LECs of the YN potential to 3 from 5.

The comparison of the large- $N_c$  predictions of the LECs with the best fitted values from the YN s-wave scattering reveals that the large- $N_c$  prediction of the LECs is more consistent with the EG results than the HB formalisms. Noted that The theoretical results from the EG and HB approaches in Ref. [20] are quantitatively similar in describing the YN scattering experimental data.

The large- $N_c$  sum rules in this work can also be applied to the NLO of the YN interactions and extended to the ChEFT potential of the YY sector. In addition, we expect that future lattice QCD calculations may check the hierarchy of the  $N_c$  scalings of the LECs and the large- $N_c$  sum rules predicted in this work.

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## Appendix A: The non-relativistic reductions of the chiral Lagrangian

In this appendix, we derive the non-relativistic reductions of the chiral Lagrangian in Eq. (1). Here we follow the derivation from Ref. [37, 38] and focus for the spin (Dirac) structures of the chiral Lagrangian only. The chiral Lagrangian can be re-written in terms of operator as

$$\begin{aligned}\tilde{O}_1 &\equiv (\bar{B}B)(\bar{B}B), \\ \tilde{O}_2 &\equiv (\bar{B}\gamma_\mu B)(\bar{B}\gamma^\mu B), \\ \tilde{O}_3 &\equiv (\bar{B}\sigma_{\mu\nu}B)(\bar{B}\sigma^{\mu\nu}B), \\ \tilde{O}_4 &\equiv (\bar{B}\gamma_\mu\gamma_5 B)(\bar{B}\gamma^\mu\gamma_5 B), \\ \tilde{O}_5 &\equiv (\bar{B}\gamma_5 B)(\bar{B}\gamma_5 B).\end{aligned}\tag{A1}$$

The relativistic fermion field  $B(x)$  can be expanded to the positive energy components  $\varphi_B(x)$  in the following from

$O_S$	$(\varphi_B^\dagger \varphi_B)(\varphi_B^\dagger \varphi_B)$
$O_T$	$(\varphi_B^\dagger \boldsymbol{\sigma} \varphi_B) \cdot (\varphi_B^\dagger \boldsymbol{\sigma} \varphi_B)$
$O_1$	$(\varphi_B^\dagger \vec{\nabla} \varphi_B)^2 + \text{h.c.}$
$O_2$	$(\varphi_B^\dagger \vec{\nabla} \varphi_B) \cdot (\varphi_B^\dagger \vec{\nabla} \varphi_B)$
$O_3$	$(\varphi_B^\dagger \varphi_B)(\varphi_B^\dagger \vec{\nabla}^2 \varphi_B) + \text{h.c.}$
$O_4$	$i(\varphi_B^\dagger \vec{\nabla} \varphi_B) \cdot (\varphi_B^\dagger \vec{\nabla} \times \boldsymbol{\sigma} \varphi_B) + \text{h.c.}$
$O_5$	$i(\varphi_B^\dagger \varphi_B)(\varphi_B^\dagger \vec{\nabla} \cdot \boldsymbol{\sigma} \times \vec{\nabla} \varphi_B)$
$O_6$	$i(\varphi_B^\dagger \boldsymbol{\sigma} \varphi_B) \cdot (\varphi_B^\dagger \vec{\nabla} \times \vec{\nabla} \varphi_B)$
$O_7$	$(\varphi_B^\dagger \boldsymbol{\sigma} \cdot \vec{\nabla} \varphi_B)(\varphi_B^\dagger \boldsymbol{\sigma} \cdot \vec{\nabla} \varphi_B) + \text{h.c.}$
$O_8$	$(\varphi_B^\dagger \sigma^j \vec{\nabla}^k \varphi_B)(\varphi_B^\dagger \sigma^k \vec{\nabla}^j \varphi_B) + \text{h.c.}$
$O_9$	$(\varphi_B^\dagger \sigma^j \vec{\nabla}^k \varphi_B)(\varphi_B^\dagger \sigma^j \vec{\nabla}^k \varphi_B) + \text{h.c.}$
$O_{10}$	$(\varphi_B^\dagger \boldsymbol{\sigma} \cdot \vec{\nabla} \varphi_B)(\varphi_B^\dagger \vec{\nabla} \cdot \boldsymbol{\sigma} \varphi_B)$
$O_{11}$	$(\varphi_B^\dagger \sigma^j \vec{\nabla}^k \varphi_B)(\varphi_B^\dagger \vec{\nabla}^j \sigma^k \varphi_B)$
$O_{12}$	$(\varphi_B^\dagger \sigma^j \vec{\nabla}^k \varphi_B)(\varphi_B^\dagger \vec{\nabla}^k \sigma^j \varphi_B)$
$O_{13}$	$(\varphi_B^\dagger \vec{\nabla} \cdot \boldsymbol{\sigma} \vec{\nabla}^j \varphi_B)(\varphi_B^\dagger \sigma^j \varphi_B) + \text{h.c.}$
$O_{14}$	$2(\varphi_B^\dagger \vec{\nabla} \sigma^j \cdot \vec{\nabla} \varphi_B)(\varphi_B^\dagger \sigma^j \varphi_B)$

TABLE II: Operators of the LO and NLO contact term interactions [9], the left (right) arrow on  $\nabla$  indicates that the gradient operates on the left (right) field. Normal-ordering of the field operator products is implied.

[37, 38],

$$B(x) = \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{i}{2M} \begin{pmatrix} 0 \\ \boldsymbol{\sigma} \cdot \nabla \end{pmatrix} + \frac{1}{8M^2} \begin{pmatrix} \nabla^2 \\ 0 \end{pmatrix} \right] \varphi_B(x) + \mathcal{O}(Q^3), \quad (\text{A2})$$

where  $M$  and  $Q$  are baryon mass in SU(3) flavor symmetry limit and small momentum scale respectively. Up to order  $Q^2$ , the non-relativistic reductions of the operators in Eq. (A1) are given by

$$\begin{aligned} \tilde{O}_1 &\stackrel{\text{NR}}{\simeq} O_S + \frac{1}{4M^2} (O_1 + 2O_2 + 2O_3 + 2O_5), \\ \tilde{O}_2 &\stackrel{\text{NR}}{\simeq} O_S + \frac{1}{4M^2} (-4O_2 - 2O_5 + 4O_6 + O_7 - O_9 + 2O_{10} - 2O_{12}), \\ \tilde{O}_3 &\stackrel{\text{NR}}{\simeq} O_T + \frac{1}{4M^2} (-O_1 - 2O_2 - 4O_5 + 2O_6 + O_7 - 2O_8 + 2O_{10} - 4O_{12} - 2O_{13}), \\ \tilde{O}_4 &\stackrel{\text{NR}}{\simeq} -O_T - \frac{1}{4M^2} (-2O_6 + O_7 - O_9 - 2O_{10} - 2O_{12} + 2O_{13} - 2O_{14}), \\ \tilde{O}_5 &\stackrel{\text{NR}}{\simeq} \frac{1}{4M^2} (O_7 + 2O_{10}), \end{aligned} \quad (\text{A3})$$

where we took the above results from Refs. [37, 38] and the operators  $O_i$  ( $i = 1, \dots, 14$ ) are listed in Tab. II.

By using partial integrations, Ref. [39] has been shown that there are only 12 operators are independent with the following constraints,

$$O_7 + 2O_{10} = O_8 + 2O_{11} \quad \text{and} \quad O_4 + O_5 = O_6. \quad (\text{A4})$$

Next step, one re-writes the non-relativistic reductions in Eq. (A3) in terms of the basis in Eqs. (4,9,10) as [37],

$$\begin{aligned} A_S &\equiv \tilde{O}_S = O_S + \frac{1}{4M^2} (O_1 + O_3 + O_5 + O_6), \\ A_T &\equiv \tilde{O}_T = O_T - \frac{1}{4M^2} (O_5 + O_6 - O_7 + O_8 + 2O_{12} + O_{14}), \\ A_1 &\equiv p_-^2 \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} = O_1 + 2O_2, \\ A_2 &\equiv p_+^2 \delta_{\bar{\chi}_1 \chi_1} \delta_{\bar{\chi}_2 \chi_2} = 2O_2 + O_3, \end{aligned}$$

$$\begin{aligned}
A_3 &\equiv p_-^2 \vec{\sigma}_1 \cdot \vec{\sigma}_2 = O_9 + 2O_{12}, \\
A_4 &\equiv p_+^2 \vec{\sigma}_1 \cdot \vec{\sigma}_2 = O_9 + O_{14}, \\
A_5 &\equiv i(\vec{p}_+ \times \vec{p}_-) \cdot (\vec{\sigma}_1 + \vec{\sigma}_2)/2 = O_5 - O_6, \\
A_6 &\equiv (\vec{p}_- \cdot \vec{\sigma}_1)(\vec{p}_- \cdot \vec{\sigma}_2) = O_7 + 2O_{10}, \\
A_7 &\equiv (\vec{p}_+ \cdot \vec{\sigma}_1)(\vec{p}_+ \cdot \vec{\sigma}_2) = O_7 + O_8 + 2O_{13}.
\end{aligned} \tag{A5}$$

By using above relations, we obtain the non-relativistic reductions of the chiral Lagrangian in Eq. (1) in terms of the operators  $A_i$  as,

$$\begin{aligned}
\tilde{O}_1 &\simeq A_S + \frac{1}{4M^2} (A_2 - A_5), \\
\tilde{O}_2 &\simeq A_S - \frac{1}{4M^2} (A_1 + A_2 + A_3 - 3A_5 - A_6), \\
\tilde{O}_3 &\simeq A_T - \frac{1}{4M^2} (A_1 + A_2 + A_3 - A_4 - 3A_5 - A_6 + A_7), \\
\tilde{O}_4 &\simeq -A_T + \frac{1}{4M^2} (A_4 + A_5 + A_6 - A_7), \\
\tilde{O}_5 &\simeq \frac{1}{4M^2} A_6.
\end{aligned} \tag{A6}$$

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