รายงานวิจัยฉบับสมบูรณ์

โครงการวิจัยเรื่อง สมบัติเชิงพีชคณิต อันดับ โครงสร้างและการวิเคราะห์ของผลคูณเทรซี-ซิงค์สำหรับตัวดำเนินการบนปริภูมิฮิลเบิร์ต

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มกราคม 2562

สัญญาเลขที่ MRG6080102

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สมบัติเชิงพีชคณิต อันดับ โครงสร้างและการวิเคราะห์ของผลคูณเทรซี-ซิงห์สำหรับตัวดำเนินการบนปริภูมิฮิลเบิร์ต

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บทคัดย่อ

เราขยายแนวคิดของผลคูณเทนเซอร์สำหรับตัวดำเนินการและผลคูณเทรซี-ซิงห์สำหรับเมทริกซ์ไปสู่ผลคูณเทรซี-ซิงห์สำหรับตัวดำเนินการที่กระทำบนผลบวกตรงของปริภูมิฮิลเบิร์ต ผลคูณดังกล่าวของตัวเนินการเข้ากันได้กับการดำเนินการต่าง ๆ เชิงพีชคณิตและความสัมพันธ์เชิงอันดับ ผลคูณเทรซี-ซิงห์ของสองตัวดำเนินการซึ่งแต่ละบล็อกไม่เป็นตัวดำเนินการศูนย์จะเป็นตัวดำเนินการกระชับก็ต่อเมื่อแต่ละตัวประกอบเป็นตัวดำเนินการกระชับ เราให้ขอบเขตบนและล่างสำหรับนอร์มแบบแชทเทน-พีต่าง ๆ ของผลคูณเทรซี-ซิงห์ของตัวดำเนินการ เราได้ว่าผลคูณดังกล่าวต่อเนื่องเทียบกับทอพอโลยีต่าง ๆ บนไอดีลแบบนอร์มของตัวดำเนินการกระชับ ตัวดำเนินการคลาสรอย และตัวดำเนินการคลาสฮิลเบิร์ต-ชมิดท์ ดังนั้นผลคูณเทรซี-ซิงห์จะคงสภาพคลาสต่าง ๆ เหล่านี้ของตัวดำเนินการ เรายังพิจารณาตรวจสอบความสัมพันธ์ระหว่างผลคูณเทรซี-ซิงห์กับคลาสอื่นๆ ของตัวดำเนินการ เราแสดงให้เห็นว่าความนอร์มอล ไฮโพนอร์มอล พารานอร์มอล และตัวดำเนินการคลาสเอจะถูกคงสภาพไว้โดยผลคูณเทรซี-ซิงห์ ยิ่งกว่านั้นเราได้เงื่อนไขที่จำเป็นและเพียงพอสำหรับผลคูณเทรซี-ซิงห์ของตัวดำเนินการที่จะมีสมบัตินอร์มอล ควอไซนอร์มอล (โค)ไอโซเมทรี และยูนิแทรี

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Abstract

We generalize the notions of tensor product for operators and Tracy-Singh product for matrices to the Tracy-Singh product for operators acting on the direct sum of Hilbert spaces. This kind of operator product is compatible with algebraic operations and order relations. The Tracy-Singh product of two operators such that each its block is nonzero is compact if and only if both factors are compact. We provide upper and lower bounds for certain Schatten p-norms of the Tracy-Singh product of operators. It turns out that this product is continuous with respect to the topologies on norm ideals of compact operators, trace class operators, and Hilbert-Schmidt class oper ators. Thus this product preserves such classes of operators. We also investigate relationship between Tracy-Singh products and another classes of operators. We show that the normality, hyponormality, paranormality, and operators of class-A type are preserved by Tracy-Singh products. are also preserved under Tracy-Singh products. Moreover, we obtain necessary and sufficient conditions for the Tracy-Singh product of two operators to be normal, quasinormal, (co)isometry, and unitary.

Keywords : tensor product; Tracy-Singh product; Hilbert space operator; operator inequality; compact operator

Executive Summary

We propose a natural definition of the Tracy-Singh product for bounded linear operators acting on the direct sum of Hilbert spaces. Then, we investigate the following:

- algebraic properties
- order properties
- analytic properties
- structural-preserving properties
- relations to various kinds of geometric means.

Our results generalize the results known so far in the literature for both Tracy-Singh products of matrices and tensor products of operators. Moreover, we obtain new properties.

In summary, the Tracy-Singh product for Hilbert space operators is compatible with algebraic operations and order relations. The Tracy-Singh product of two operators such that each its block is nonzero is compact if and only if both factors are compact. We provide upper and lower bounds for certain Schatten p-norms of the Tracy-Singh product of operators. It turns out that this product is continuous with respect to the topologies on norm ideals of compact operators, trace class operators, and Hilbert-Schmidt class oper ators. Thus this product preserves such classes of operators. We also investigate relationship between Tracy-Singh products and another classes of operators. We show that the normality, hyponormality, paranormality, and operators of class-A type are preserved by Tracy-Singh products. are also preserved under Tracy-Singh products. Moreover, we obtain necessary and sufficient conditions for the Tracy-Singh product of two operators to be normal, quasinormal, (co)isometry, and unitary.

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1) การเรียนการสอน

หัวข้อวิจัยนำมาสู่การจัดการเรียนการสอนรายวิชา Multilinear Algebra และรายวิชา Advanced Topics in Algebra ซึ่งเป็นวิชาเลือกในหลักสูตรปริญญาโทและปริญญาเอก สาขาคณิตศาสตร์ประยุกต์ ที่สจล. โดยหัวหน้าโครงการวิจัยได้ทำการสอนรายวิชาดังกล่าวในปีการศึกษา 2560 และปีการศึกษา 2561 มีเนื้อหาเกี่ยวกับ tensor product, Kronecker product, Hadamard product, Tracy-Singh product, Khatri-Rao product รายวิชาดังกล่าวนำมาสู่หัวข้อวิจัยเกี่ยวกับ Operator/matrix products

2) การสร้างนักวิจัยใหม่

โครงการวิจัยนี้ได้ทำให้ผู้ช่วยวิจัย นายอานนท์ พลอยมุกดา ซึ่งกำลังศึกษาระดับปริญญาเอก สาขาคณิตศาสตร์ประยุกต์ พัฒนาตนเองจนเป็นนักวิจัยใหม่ในสาขา Matrix and Operator Theory

3. อื่นๆ (เช่น ผลงานตีพิมพ์ในวารสารวิชาการในประเทศ การเสนอผลงานในที่ประชุมวิชาการ หนังสือ การจดสิทธิบัตร)

ไม่มี

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Chapter 1

Introduction

1.1 Research motivation

In scientific computing, we consider a matrix to be a two-dimensional array for stacking data. A processing of such data can be performed using matrix products. One of extremely useful matrix products is the Kronecker product. For any complex matrices $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, the Kronecker product of A and B is given by the block matrix

$$A \hat{\otimes} B \;=\; [a_{ij}B]_{ij} \in M_{mp,nq}(\mathbb{C}).$$

Equivalently, $A \hat{\otimes} B$ is the unique complex matrix of order $mp \times nq$ satisfying

$$(A \hat{\otimes} B)(x \hat{\otimes} y) \;=\; Ax \hat{\otimes} By \tag{1.1}$$

for all $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^q$. This matrix product has wide applications in mathematics, computer science, statistics, physics, system theory, signal processing, and related fields. See [2, 5, 6, 12] for more information.

Kronecker product was generalized to the Tracy-Singh product of partitioned matrices by Tracy and Singh [10]. Let $A = [A_{ij}] \in M_{m,n}(\mathbb{C})$ be a partitioned matrix with A_{ij} of order $m_i \times n_j$ as the (i,j)th submatrix where $\sum_i m_i = m$ and $\sum_j n_j = n$. Let $B = [B_{kl}] \in M_{p,q}(\mathbb{C})$ be a partitioned matrix with B_{kl} of order $p_k \times q_l$ as the (k,l)th submatrix where $\sum_k p_k = p$ and $\sum_l q_l = q$. The Tracy-Singh product of A and B is defined by

$$A \hat{\boxtimes} B \;=\; \left[\left[A_{ij} \hat{\otimes} B_{kl}\right]_{kl}\right]_{ij} \in M_{mp,nq}(\mathbb{C}),$$

where each block $A_{ij} \hat{\otimes} B_{kl}$ is of order $m_i p_k \times n_j q_l$. This kind of matrix product has several attractive properties in algebraic, order, and analytic points of views; see, *e.g.*, [3, 8, 9, 10]. The Tracy-Singh product can be applied widely in statistics, econometrics and related fields; see, *e.g.*, [9, 10].

As a natural generalization of a complex matrix, we consider a bounded linear operator between complex Hilbert spaces. The tensor product of Hilbert space operators can be viewed as an extension of the Kronecker product of complex matrices. Using the universal mapping property in the monoidal category of Hilbert spaces, the tensor product of $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ is the unique bounded linear operator from $\mathcal{H} \otimes \mathcal{K}$ into $\mathcal{H}' \otimes \mathcal{K}'$ such that for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$,

$$(A \otimes B)(x \otimes y) \;=\; Ax \otimes By. \tag{1.2}$$

A fundamental property of tensor product is the mixed product property:

$$(A \otimes B)(C \otimes D) \;=\; AC \otimes BD. \tag{1.3}$$

The theory of tensor product of operators has been continuously developed in the literature; see, *e.g.*, [4, 11].

From the previous discussion, it is natural to extend the notion of tensor product for operators to the "Tracy-Singh product"of operators. We shall propose a natural definition of such operator product. It turns out that this product is compatible with algebraic operations and order relations for operators. One of the most attractive properties, the mixed product property, also holds for Tracy-Singh products. It follows that this product preserves attractive properties of operators, such as being invertible, Hermitian, unitary, positive, and normal. Our results generalize the results known so far in the literature for both Tracy-Singh products of matrices and tensor products of operators.

1.2 Objectives and scopes

In this research, we develop theory of Hilbert space operators as follows.

1. Provide a natural definition of the Tracy-Singh product for operators on a Hilbert space.

2. Investigate algebraic, order, structure, and analytic properties for this product.

All Hilbert spaces considered here are complex Hilbert spaces. All operators considered here are bounded linear operators.

1.3 Research methodology

1. Review literature results about Kronecker products and Tracy-Singh products of matrices.

2. Review literature results about tensor products of Hilbert space operators.

3. Give a natural definition of Tracy-Singh product for operators acting on a Hilbert space.

4. Investigate the following algebraic-order properties of Tracy-Singh product:

   I. Bilinearity

   II. Compatibility with usual product, power, adjoint, direct sum

   III. Compatibility with ordinary inverses, left/right inverses, generalized inverses

   IV. Positivity and strict positivity preserving

   V. Monotonicity with respect to operator orderings

   VI. Certain inequalities

5. Investigate the following structural properties of Tracy-Singh product: Hermitian, unitary, isometry, co-isometry, partial isometry, idempotent, involuntary, projection, nilpotent, hyponormal, cohyponormal, semihyponormal, quasihyponormal, semi-quasihyponormal, posinormal

6. Investigate the following analytic properties of Tracy-Singh product:

   I. Lower/upper bounds for operator norm of the Tracy-Singh product.

   II. Continuity and convergence with respect to the operator norm topology

III. A necessary and sufficient condition for compactness of the Tracy-Singh product

IV. Lower/upper bounds for trace norm of the Tracy-Singh product.

V. Continuity and convergence with respect to the trace norm topology

VI. Lower/upper bounds for Hilbert-Schmidt norm of the Tracy-Singh product.

VII. Continuity and convergence with respect to the Hilbert-Schmidt norm topology.

VIII. Functions of operators.

7. In 4)-6), we use MatLab for computing Kronecker products, Tracy-Singh products, norms, Moore-Penrose inverses, ranks, eigenvalues, matrix decompositions, functions of matrices.

Chapter 2

Algebraic and Order Properties of Tracy-Singh Products for Operator Matrices

In this chapter, we generalize the tensor product for operators to the Tracy-Singh product for operator matrices acting on the direct sum of Hilbert spaces. This kind of operator product is compatible with algebraic operations and order relations for operators. It follows that this product preserves many structure properties of operators.

2.1 Defining the Tracy-Singh products for operators

In this section, we introduce the Tracy-Singh product of operators on a Hilbert space. Then we will show that this product is compatible with addition, scalar multiplication, adjoint operation, usual multiplication, power, and direct sum of operator inverses.

Throughout this paper, let $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{K}$ and $\mathcal{K}'$ be complex Hilbert spaces. When $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, denote by $\mathbb{B}(\mathcal{X},\mathcal{Y})$ the Banach space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, and abbreviate $\mathbb{B}(\mathcal{X},\mathcal{X})$ to $\mathbb{B}(\mathcal{X})$.

The projection theorem for Hilbert spaces allows us to decompose

$$\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^{m} \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{l=1}^{q} \mathcal{K}_l, \quad \mathcal{K}' = \bigoplus_{k=1}^{p} \mathcal{K}'_k$$

where each $\mathcal{H}_j, \mathcal{H}'_i, \mathcal{K}_l, \mathcal{K}'_k$ are Hilbert spaces. Such decompositions are fixed throughout the paper. For each $j = 1, \ldots, n$, let E_j be the canonical embedding from $\mathcal{H}_j$ into $\mathcal{H}$, defined by

$$x_j \mapsto (0, \ldots, 0, x_j, 0, \ldots, 0).$$

Similarly, let F_l be the canonical embedding from $\mathcal{K}_l$ into $\mathcal{K}$ for each $l = 1, \ldots, q$. For each $i = 1, \ldots, m$ and $k = 1, \ldots, p$, let $P'_i : \mathcal{H}' \to \mathcal{H}'_i$ and $Q'_k : \mathcal{K}' \to \mathcal{K}'_k$ be the orthogonal projections. Thus, each operator $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,n} \quad \text{and } B = [B_{kl}]_{k,l=1}^{p,q}$$

where $A_{ij} = P'_i A E_j$ and $B_{kl} = Q'_k B F_l$ for each i, j, k, l.

Definition 2.1. Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices defined as above. We define the *Tracy-Singh product* of A and B to be the operator matrix

$$A \boxtimes B = \big[[A_{ij} \otimes B_{kl}]_{kl}\big]_{ij} \tag{2.1}$$

which is a bounded linear operator from $\bigoplus_{j=1}^{n} \bigoplus_{l=1}^{q} \mathcal{H}_j \otimes \mathcal{K}_l$ to $\bigoplus_{i=1}^{m} \bigoplus_{k=1}^{p} \mathcal{H}'_i \otimes \mathcal{K}'_k$.

Note that if both A and B are 1×1 block operator matrices i.e. $m = n = p = q = 1$, then their Tracy-Singh product $A \boxtimes B$ is just the tensor product $A \otimes B$.

Next, we shall show that the Tracy-Singh product of two linear maps induced by two matrices is just the linear map induced by the Tracy-Singh product of these matrices. Recall that for each $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, the induced maps

$$L_A : \mathbb{C}^n \to \mathbb{C}^m,\ x \mapsto Ax \ \text{ and } \ L_B : \mathbb{C}^q \to \mathbb{C}^p,\ y \mapsto By$$

are bounded linear operators. Using the universal mapping property, we identify $\mathbb{C}^n \otimes \mathbb{C}^q$ with $\mathbb{C}^{nq} \cong M_{n,q}(\mathbb{C})$ together with the canonical bilinear map $(x, y) \mapsto x \hat{\otimes} y$ for each $(x, y) \in \mathbb{C}^n \times \mathbb{C}^q$. It is similar for $\mathbb{C}^m \otimes \mathbb{C}^p$.

Lemma 2.2. *For each $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, we have*

$$L_A \otimes L_B \;=\; L_{A \hat{\otimes} B}. \tag{2.2}$$

Proof. For any $x \otimes y \in \mathbb{C}^n \otimes \mathbb{C}^q$, we obtain from the mixed product property of the Kronecker product (1.1) that

$$\begin{aligned}(L_A \otimes L_B)(x \otimes y) \;&=\; L_A(x) \otimes L_B(y) \;=\; L_A(x) \hat{\otimes} L_B(y) \\ &=\; Ax \hat{\otimes} By \;=\; (A \hat{\otimes} B)(x \hat{\otimes} y) \\ &=\; (A \hat{\otimes} B)(x \otimes y) \;=\; L_{A \hat{\otimes} B}(x \otimes y).\end{aligned}$$

Thus, by the uniqueness of tensor product, $L_A \otimes L_B = L_{A \hat{\otimes} B}$. □

Proposition 2.3. *For any complex matrices $A = [A_{ij}]$ and $B = [B_{kl}]$ partitioned in block-matrix forms, we have*

$$L_A \boxtimes L_B \;=\; L_{A \hat{\boxtimes} B}. \tag{2.3}$$

Proof. Recall that the (i, j)th block of the matrix representation of L_A is the matrix A_{ij}. It follows from Lemma 5.4 that

$$L_A \boxtimes L_B \;=\; \big[\big[L_{A_{ij}} \otimes L_{B_{kl}}\big]_{kl}\big]_{ij} \;=\; \Big[\Big[L_{A_{ij} \hat{\otimes} B_{kl}}\Big]_{kl}\Big]_{ij} \;=\; L_{A \hat{\boxtimes} B}.$$

□

2.2 Tracy-Singh products and algebraic operations for operators

The next proposition shows that the Tracy-Singh product is compatible with the addition, the scalar multiplication and the adjoint operation of operators.

Proposition 2.4. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B, C \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices, and let $\alpha \in \mathbb{C}$. Then*

$$(\alpha A) \boxtimes B \;=\; \alpha(A \boxtimes B) \;=\; A \boxtimes (\alpha B), \tag{2.4}$$

$$(A \boxtimes B)^* \;=\; A^* \boxtimes B^*, \tag{2.5}$$

$$A \boxtimes (B + C) \;=\; A \boxtimes B + A \boxtimes C, \tag{2.6}$$

$$(B + C) \boxtimes A \;=\; B \boxtimes A + C \boxtimes A. \tag{2.7}$$

Proof. Since each (i,j)th block of αA is given by $(\alpha A)_{ij} = \alpha A_{ij}$, we get

$$(\alpha A) \boxtimes B \;=\; \big[[(\alpha A_{ij}) \otimes B_{kl}]_{kl}\big]_{ij} \;=\; \big[[\alpha(A_{ij} \otimes B_{kl})]_{kl}\big]_{ij} \;=\; \alpha(A \boxtimes B).$$

Similarly, $A \boxtimes (\alpha B) = \alpha(A \boxtimes B)$. Since $A^* = [A^*_{ji}]_{ij}$ and $B^* = [B^*_{lk}]_{kl}$ for all i,j,k,l, we obtain

$$(A \boxtimes B)^* \;=\; \big[[A_{ji} \otimes B_{kl}]^*_{kl}\big]_{ij} \;=\; \Big[\big[A^*_{ji} \otimes B^*_{lk}\big]_{kl}\Big]_{ij} \;=\; A^* \boxtimes B^*.$$

The proofs of (2.6) and (2.7) are done by using the fact that $(B + C)_{kl} = B_{kl} + C_{kl}$ for all k,l together with the left/right distributivity of the tensor product over the addition. □

Properties (2.4), (2.6) and (2.7) say that the map $(A, B) \mapsto A \boxtimes B$ is bilinear.

Proposition 2.5. *Let $A = [A_{ij}] \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and let $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices. Then*

$$A \boxtimes B \;=\; [A_{ij} \boxtimes B]_{ij} \;=\; \begin{bmatrix} A_{11} \boxtimes B & \cdots & A_{1n} \boxtimes B \\ \vdots & \ddots & \vdots \\ A_{m1} \boxtimes B & \cdots & A_{mn} \boxtimes B \end{bmatrix}.$$

That is, the (i,j)th block of $A \boxtimes B$ is just $A_{ij} \boxtimes B$, regardless of how to partition B.

Proof. It follows directly from the definition of the Tracy-Singh product. □

Remark 2.6. *It is not true in general that the (k,l)th block of $A \boxtimes B$ is $A \boxtimes B_{kl}$.*

When $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, the direct sum of $A_1 \in \mathbb{B}(\mathcal{H}_1, \mathcal{K}_1)$ and $A_2 \in \mathbb{B}(\mathcal{H}_2, \mathcal{K}_2)$ is defined to be the operator

$$A_1 \oplus A_2 \;=\; \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \;\in\; \mathbb{B}(\mathcal{H}, \mathcal{K}).$$

The next result gives a relation between the direct sum and the Tracy-Singh product.

Proposition 2.7. *The Tracy-Singh product is right distributive over the direct sum of operators. That is, for any operator matrices A, B and C, we have*

$$(A \oplus B) \boxtimes C \;=\; (A \boxtimes C) \oplus (B \boxtimes C). \tag{2.8}$$

Proof. It follows from Proposition 2.5 that

$$\begin{aligned}(A \oplus B) \boxtimes C \;&=\; \begin{bmatrix} A \boxtimes C & 0 \boxtimes C \\ 0 \boxtimes C & B \boxtimes C \end{bmatrix} \;=\; \begin{bmatrix} A \boxtimes C & 0 \\ 0 & B \boxtimes C \end{bmatrix} \\ &=\; (A \boxtimes C) \oplus (B \boxtimes C).\end{aligned}$$

□

It is not true in general that the Tracy-Singh product is left distributive over the direct sum of operators.

The next theorem shows that the Tracy-Singh product is compatible with the ordinary product of operators. This fundamental property, called the *mixed product property*, will be used many times in later discussions.

Theorem 2.1. *Let $\mathcal{H}, \mathcal{H}', \mathcal{H}'', \mathcal{K}, \mathcal{K}'$ and $\mathcal{K}''$ be complex Hilbert spaces. Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}', \mathcal{H}'')$, $C = [C_{ij}]_{i,j=1}^{n,r} \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$, $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K}', \mathcal{K}'')$ and $D = [D_{kl}]_{k,l=1}^{q,s} \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices partitioned so that they are compatible with the decompositions of the corresponding Hilbert spaces. Then*

$$(A \boxtimes B)(C \boxtimes D) \;=\; AC \boxtimes BD. \tag{2.9}$$

Proof. Using block multiplication of operators and the mixed product property of the tensor product (1.3), we have

$$\begin{aligned}(A \boxtimes B)(C \boxtimes D) \;&=\; \big[[A_{ij} \otimes B_{kl}]_{kl}\big]_{ij}\,\big[[C_{ij} \otimes D_{kl}]_{kl}\big]_{ij} \\ &=\; \left[\left[\sum_{\alpha=1}^{n}\sum_{\beta=1}^{q}(A_{i\alpha} \otimes B_{k\beta})(C_{\alpha j} \otimes D_{\beta l})\right]_{kl}\right]_{ij} \\ &=\; \left[\left[\sum_{\alpha=1}^{n}\sum_{\beta=1}^{q}(A_{i\alpha}C_{\alpha j} \otimes B_{k\beta}D_{\beta l})\right]_{kl}\right]_{ij} \\ &=\; \left[\sum_{\alpha=1}^{n} A_{i\alpha}C_{\alpha j}\right]_{ij} \boxtimes \left[\sum_{\beta=1}^{q} B_{k\beta}D_{\beta l}\right]_{kl} \\ &=\; AC \boxtimes BD.\end{aligned}$$

□

Corollary 2.8. *For any operator matrices $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$, we have*

$$(A \boxtimes B)^r = A^r \boxtimes B^r \tag{2.10}$$

for any $r \in \mathbb{N}$.

In the rest of section, we investigate structure properties of operators under taking Tracy-Singh products. Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is said to be *involutary* if $T^2 = I$, *idempotent* if $T^2 = T$, an *isometry* if $T^*T = I$, a *partial isometry* if the restriction of T to a closed subspace is an isometry, or equivalently, $TT^*T = T$.

Corollary 2.9. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. If both A and B satisfy one of the following properties, then the same property holds for $A \boxtimes B$: Hermitian, unitary, isometry, co-isometry, partial isometry, idempotent, involutary, projection.*

Proof. Applying Theorem 2.1 and Proposition 2.4, we get the results. □

If A and B are skew-Hermitian operators, then $A \boxtimes B$ is Hermitian. Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is said to be *nilpotent* if there is a positive integer k such that $T^k = 0$. The smallest such integer k is called the degree of nilpotency of T. If $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ are nilpotent operators with degrees of nilpotency r and s, respectively, then $A \boxtimes B$ is also nilpotent with degree of nilpotency not exceed $\min\{r, s\}$.

2.3 Tracy-Singh products and operator inverses

Next, we discuss the invertibility of the Tracy-Singh product of operators. Recall that an operator $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ is said to be *regular* if there is an operator $A^- \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ such that $AA^-A = A$. The operator A^- is called an *inner inverse* of A. An operator $X \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ is said to be an *outer inverse* of A if $XAX = X$.

Proposition 2.10. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$.*

(i) If A and B are left invertible with left inverses $\hat{A}$ and $\hat{B}$ respectively, then $A \boxtimes B$ is left invertible and $\hat{A} \boxtimes \hat{B}$ is its left inverse.

(ii) If A and B are right invertible with right inverses $\hat{A}$ and $\hat{B}$ respectively, then $A \boxtimes B$ is right invertible and $\hat{A} \boxtimes \hat{B}$ is its right inverse.

(iii) If A and B are regular with inner inverses A^- and B^- respectively, then $A \boxtimes B$ is regular with $A^- \boxtimes B^-$ as its inner inverse.

(iv) If A and B have A^- and B^- as their outer inverses respectively, then $A \boxtimes B$ has $A^- \boxtimes B^-$ as its outer inverse.

Proof. It follows from Theorem 2.1 and the facts that $I_{\mathcal{X}} \boxtimes I_{\mathcal{Y}} = I_{\mathcal{X} \otimes \mathcal{Y}}$ for any Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$. □

As a consequence of (i) and (ii) in Proposition 2.10, we obtain the following result.

Corollary 2.11. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. If A and B are invertible, then $A \boxtimes B$ is invertible and*

$$(A \boxtimes B)^{-1} \;=\; A^{-1} \boxtimes B^{-1}. \tag{2.11}$$

Next, we consider a kind of operator inverse, called Moore-Penrose inverse. Recall that a *Moore-Penrose inverse* of $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ is an operator $A^\dagger \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ satisfying the following Penrose conditions ([7])

(i) $A^\dagger$ is an inner inverse of A ;

(ii) $A^\dagger$ is an outer inverse of A ;

(iii) $AA^\dagger$ is Hermitian ;

(iv) $A^\dagger A$ is Hermitian.

It is well known that the following statements are equivalent for $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ (see *e.g.* [1]):

(i) a Moore-Penrose inverse of A exists ;

(ii) a Moore-Penrose inverse of A is unique ;

(iii) the range of A is closed.

Theorem 2.2. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. If A and B have closed ranges, then*

1. *the range of* $A \boxtimes B$ *is closed ;*

2. $(A \boxtimes B)^\dagger = A^\dagger \boxtimes B^\dagger$.

Proof. Since the ranges of A and B are closed, the Moore-Penrose inverses $A^\dagger$ and $B^\dagger$ exist and are unique. Making use of Theorem 2.1 and Proposition 2.4, we can verify that $A^\dagger \boxtimes B^\dagger$ satisfies the following Penrose equations:

(i) $(A \boxtimes B)(A^\dagger \boxtimes B^\dagger)(A \boxtimes B) = A \boxtimes B$

(ii) $(A^\dagger \boxtimes B^\dagger)(A \boxtimes B)(A^\dagger \boxtimes B^\dagger) = A^\dagger \boxtimes B^\dagger$

(iii) $\big((A \boxtimes B)(A^\dagger \boxtimes B^\dagger)\big)^* = (A \boxtimes B)(A^\dagger \boxtimes B^\dagger)$

(iv) $\big((A^\dagger \boxtimes B^\dagger)(A \boxtimes B)\big)^* = (A^\dagger \boxtimes B^\dagger)(A \boxtimes B)$.

Hence, a Moore-Penrose inverse of $A \boxtimes B$ exists and it is uniquely determined by $A^\dagger \boxtimes B^\dagger$. It follows that $A \boxtimes B$ has a closed range. □

The results in this section indicate that the Tracy-Singh product is compatible with various kinds of operator inverses.

2.4 Tracy-Singh products and operator orderings

Now, we focus on order properties of Tracy-Singh products related to algebraic properties.

Theorem 2.3. *Let* $A \in \mathbb{B}(\mathcal{H})$ *and* $B \in \mathbb{B}(\mathcal{K})$.

(i) If $A, B \geqslant 0$, *then* $A \boxtimes B \geqslant 0$.

(ii) If $A, B > 0$, *then* $A \boxtimes B > 0$.

Proof. Assume $A, B \geqslant 0$. Using Theorem 2.1 and property (2.5), we obtain

$$\begin{aligned} A \boxtimes B &= A^{\frac{1}{2}} A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}} B^{\frac{1}{2}} = \left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right)\left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right) \\ &= \left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right)^* \left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right) \geqslant 0. \end{aligned}$$

Consider the case $A, B > 0$. We have immediately by (i) that $A \boxtimes B \geqslant 0$. By Corollary 2.11, $A \boxtimes B$ is invertible. This implies that $A \boxtimes B > 0$. □

The next result provides the monotonicity of Tracy-Singh product.

Corollary 2.12. *Let $A_1, A_2 \in \mathbb{B}(\mathcal{H})$ and $B_1, B_2 \in \mathbb{B}(\mathcal{K})$.*

(i) If $A_1 \geqslant A_2 \geqslant 0$ and $B_1 \geqslant B_2 \geqslant 0$, then $A_1 \boxtimes B_1 \ \geqslant\ A_2 \boxtimes B_2$.

(ii) If $A_1 > A_2 > 0$ and $B_1 > B_2 > 0$, then $A_1 \boxtimes B_1 \ >\ A_2 \boxtimes B_2$.

Proof. Suppose that $A_1 \geqslant A_2 \geqslant 0$ and $B_1 \geqslant B_2 \geqslant 0$. Applying Proposition 2.4 and Theorem 2.3 yields

$$\begin{aligned} A_1 \boxtimes B_1 - A_2 \boxtimes B_2 &= A_1 \boxtimes B_1 - A_2 \boxtimes B_1 + A_2 \boxtimes B_1 - A_2 \boxtimes B_2 \\ &= (A_1 - A_2) \boxtimes B_1 + A_2 \boxtimes (B_1 - B_2) \\ &\geqslant 0. \end{aligned}$$

The proof of (ii) is similar to that of (i). □

Chapter 3

Analytic Properties of Tracy-Singh Products for Operator Matrices

In this chapter, we show that the Tracy-Singh product of Hilbert space operators is continuous with respect to the operator-norm topology. The Tracy-Singh product of two nonzero operators is compact if and only if both factors are compact. We provide upper and lower bounds for certain Schatten p-norms of the Tracy-Singh product of operators. It turns out that this product is continuous with respect to the topologies on norm ideals of compact operators, trace class operators, and Hilbert-Schmidt class operators. Thus the Tracy-Singh product preserves such classes of operators.

3.1 Introduction

In matrix theory, one of useful matrix products is the Kronecker product. Recall that the Kronecker product of two complex matrices $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$ is given by the block matrix

$$A \hat{\otimes} B \;=\; [a_{ij}B]_{ij} \in M_{mp,nq}(\mathbb{C}).$$

This matrix product was generalized to the Tracy-Singh product by Tracy and Singh [3]. Let $A = [A_{ij}] \in M_{m,n}(\mathbb{C})$ be a partitioned matrix with A_{ij} as the (i,j)th submatrix. Let $B = [B_{kl}] \in M_{p,q}(\mathbb{C})$ be a partitioned matrix with B_{kl} as the (k,l)th submatrix. The Tracy-Singh product of A and B is defined by

$$A \hat{\boxtimes} B \;=\; \big[\big[A_{ij} \hat{\otimes} B_{kl}\big]_{kl}\big]_{ij} \in M_{mp,nq}(\mathbb{C}).$$

This kind of matrix product has several attractive properties and can be applied widely in statistics, econometrics and related fields; see *e.g.*, [3, 5, 8, 9, 10].

The tensor product of Hilbert space operators is a natural extension of the Kronecker product to infinite-dimensional setting. Theory of Hilbert tensor product has been continuously investigated in the literature; see, *e.g.*, [14, 4, 11]. It is well known that the tensor product is continuous with respect to the operator-norm topology. Moreover, on the norm ideals of compact operators generated by Schatten p-norm for $p = 1, 2, \infty$, the tensor product are also continuous. Recently, the tensor product for operators was generalized to the Tracy-Singh product for operator matrices acting on the direct sum of Hilbert spaces in [15]. This kind of operator product satisfies certain pleasing algebraic and order properties.

In this chapter, we discuss continuity, convergence, and compactness of the Tracy-Singh product for operators in the operator-norm topology. Then we obtain relations between Tracy-Singh product and certain analytic functions. We also investigate the Tracy-Singh product on norm ideals of compact operators generated by certain Schatten p-norms. In fact, this product is continuous with respect to the Schatten p-norm for $p = 1, 2, \infty$. Estimations by such norms for Tracy-Singh products are provided. It follows that trace class operators and Hilbert-Schmidt class operators are preserved under this product.

3.2 Reviews of Tracy-Singh products for operator matrices

Throughout, let $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{K}$ and $\mathcal{K}'$ be complex Hilbert spaces . When X and Y are Hilbert spaces, denote by $\mathbb{B}(X, Y)$ the Banach space of bounded linear operators from X into Y, and abbreviate $\mathbb{B}(X, X)$ to $\mathbb{B}(X)$.

In order to define the Tracy-Singh product, we have to fix the decompositions of Hilbert spaces, namely,

$$\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^{m} \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{l=1}^{q} \mathcal{K}_l, \quad \mathcal{K}' = \bigoplus_{k=1}^{p} \mathcal{K}'_k$$

where each $\mathcal{H}_j, \mathcal{H}'_i, \mathcal{K}_l, \mathcal{K}'_k$ are Hilbert spaces. For each $j = 1, \dots, n$ and $l = 1, \dots, q$, let $E_j : \mathcal{H}_j \to \mathcal{H}$ and $F_l : \mathcal{K}_l \to \mathcal{K}$ be the canonical embeddings. For each $i = 1, \dots, m$ and $k = 1, \dots, p$, let P'_i and Q'_k be the orthogonal projections. Thus, each operator $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and B in $\mathbb{B}(\mathcal{K}, \mathcal{K}')$ can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,n} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{p,q}$$

where $A_{ij} = P'_i A E_j : \mathcal{H}_j \to \mathcal{H}'_i$ and $B_{kl} = Q'_k B F_l : \mathcal{K}_l \to \mathcal{K}'_k$ for each i, j, k, l. We define the *Tracy-Singh product* of A and B to be a bounded linear operator from $\bigoplus_{j,l=1}^{n,q} \mathcal{H}_j \otimes \mathcal{K}_l$ to $\bigoplus_{i,k=1}^{m,p} \mathcal{H}'_i \otimes \mathcal{K}'_k$ represented in the block-matrix form as follows:

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij}.$$

When $m = n = p = q = 1$, the Tracy-Singh product $A \boxtimes B$ becomes the tensor product $A \otimes B$.

Lemma 3.1 ([15]). *Fundamental properties of the Tracy-Singh product for operators are listed below (provided that each term is well-defined):*

1. *The map $(A, B) \mapsto A \boxtimes B$ is bilinear.*
2. *Compatibility with adjoints: $(A \boxtimes B)^* = A^* \boxtimes B^*$.*
3. *Mixed-product property: $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$.*
4. *Compatibility with powers: $(A \boxtimes B)^r = A^r \boxtimes B^r$ for any $r \in \mathbb{N}$.*
5. *Compatibility with inverses: if A and B are invertible, then $A \boxtimes B$ is invertible with $(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}$.*
6. *Positivity: if $A \geqslant 0$ and $B \geqslant 0$, then $A \boxtimes B \geqslant 0$.*
7. *Strictly positivity: if $A > 0$ and $B > 0$, then $A \boxtimes B > 0$.*
8. *If A and B are partial isometries, then so is $A \boxtimes B$. Recall that an operator T is a partial isometry if and only if the restriction of T to a closed subspace is an isometry.*

3.3 Analytic properties of the Tracy-Singh product

In this section, we establish some analytic properties of the Tracy-Singh product involving operator norms. These properties involve continuity, convergence, norm estimates, and certain analytic functions. We denote the operator norm by $\|\cdot\|_\infty$.

In order to discuss the continuity of the Tracy-Singh product, recall the following bounds for the operator norm of operator matrices.

Lemma 3.2 ([13]). *Let $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ be an operator matrix. Then*

$$n^{-2} \sum_{i,j=1}^{n} \|A_{ij}\|_\infty^2 \leqslant \|A\|_\infty^2 \leqslant \sum_{i,j=1}^{n} \|A_{ij}\|_\infty^2. \tag{3.1}$$

Lemma 3.3. *Let $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ be an operator matrix and let $(A_r)_{r=1}^\infty$ be a sequence in $\mathbb{B}(\mathcal{H})$ where $A_r = [A_{ij}^{(r)}]_{i,j=1}^{n,n}$ for each $r \in \mathbb{N}$. Then $A_r \to A$ if and only if $A_{ij}^{(r)} \to A_{ij}$ for all $i,j = 1,\ldots,n$.*

Proof. It is a direct consequence of Lemma 3.2. □

The next theorem explains that the Tracy-Singh product is (jointly) continuous with respect to the topology induced by the operator norm.

Theorem 3.1. *Let $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$ and $B = [B_{kl}] \in \mathbb{B}(\mathcal{K})$ be operator matrices, and let $(A_r)_{r=1}^\infty$ and $(B_r)_{r=1}^\infty$ be sequences in $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$, respectively. If $A_r \to A$ and $B_r \to B$, then $A_r \boxtimes B_r \to A \boxtimes B$.*

Proof. Suppose that $A_r \to A$ and $B_r \to B$. By Lemma 3.3, we have $A_{ij}^{(r)} \to A_{ij}$ and $B_{kl}^{(r)} \to B_{kl}$ for each i,j,k,l. Since the tensor product is continuous, we have

$$A_{ij}^{(r)} \otimes B_{kl}^{(r)} \to A_{ij} \otimes B_{kl}$$

for each i,j,k,l. It follows that $A_r \boxtimes B_r \to A \boxtimes B$ by Lemma 3.3. □

The next theorem provides upper/lower bounds for the operator norm of the Tracy-Singh product.

Theorem 3.2. *For any operator matrices* $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ *and* $B = [A_{kl}]_{k,l=1}^{q,q} \in \mathbb{B}(\mathcal{K})$, *we have*

$$\frac{1}{nq}\|A\|_\infty \|B\|_\infty \;\leqslant\; \|A \boxtimes B\|_\infty \;\leqslant\; nq\|A\|_\infty \|B\|_\infty. \tag{3.2}$$

Proof. It follows from Lemma 3.2 that

$$\begin{aligned}\|A \boxtimes B\|_\infty^2 \;&\leqslant\; \sum_{k,l}\sum_{i,j}\|A_{ij} \otimes B_{kl}\|_\infty^2 \;=\; \sum_{k,l}\sum_{i,j}\|A_{ij}\|_\infty^2 \|B_{kl}\|_\infty^2 \\ &=\; \Big(\sum_{i,j}\|A_{ij}\|_\infty^2\Big)\Big(\sum_{k,l}\|B_{kl}\|_\infty^2\Big) \;\leqslant\; (nq)^2\|A\|_\infty^2\|B\|_\infty^2.\end{aligned}$$

We also have

$$\begin{aligned}\|A \boxtimes B\|_\infty^2 \;&\geqslant\; (nq)^{-2}\sum_{k,l}\sum_{i,j}\|A_{ij} \otimes B_{kl}\|_\infty^2 \;=\; (nq)^{-2}\sum_{k,l}\sum_{i,j}\|A_{ij}\|_\infty^2 \|B_{kl}\|_\infty^2 \\ &=\; (nq)^{-2}\Big(\sum_{i,j}\|A_{ij}\|_\infty^2\Big)\Big(\sum_{k,l}\|B_{kl}\|_\infty^2\Big) \;\geqslant\; (nq)^{-2}\|A\|_\infty^2\|B\|_\infty^2.\end{aligned}$$

Hence, we obtain the bound (3.2). □

Theorem 3.3. *Let* $A \in \mathbb{B}(\mathcal{H})$.

(i) If f *is an analytic function on a region containing the spectra of* A *and* $I \boxtimes A$, *then*

$$f(I \boxtimes A) \;=\; I \boxtimes f(A). \tag{3.3}$$

(ii) If f *is an analytic function on a region containing the spectra of* A *and* $A \boxtimes I$, *then*

$$f(A \boxtimes I) \;=\; f(A) \boxtimes I. \tag{3.4}$$

Proof. (i) Since f is analytic on spectra of A and $I \boxtimes A$, we have the Taylor series expansion

$$f(z) \;=\; \sum_{r=0}^{\infty} \alpha_r z^r.$$

It follows that

$$f(A) \;=\; \sum_{r=0}^{\infty} \alpha_r A^r \quad \text{and} \quad f(I \boxtimes A) \;=\; \sum_{r=0}^{\infty} \alpha_r (I \boxtimes A)^r.$$

Making use of the bilinearity of Tracy-Singh product and Theorem 3.1 yields

$$\begin{aligned} f(I \boxtimes A) &= \sum_{r=0}^{\infty} \alpha_r (I \boxtimes A^r) = \sum_{r=0}^{\infty} (I \boxtimes \alpha_r A^r) \\ &= I \boxtimes \sum_{r=0}^{\infty} \alpha_r A^r = I \boxtimes f(A). \end{aligned}$$

Similarly, we obtain the assertion (ii). □

Theorem 3.4. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive operators. For any $\alpha > 0$, we have*

$$(A \boxtimes B)^{\alpha} = A^{\alpha} \boxtimes B^{\alpha}. \tag{3.5}$$

Proof. First, note that $A \boxtimes B$ is positive by property (6) of Lemma 3.1. It follows from the property (4) in Lemma 3.1 that for any $r, s \in \mathbb{N}$,

$$\left(A^{\frac{r}{s}} \boxtimes B^{\frac{r}{s}}\right)^{s} = A^{r} \boxtimes B^{r} = (A \boxtimes B)^{r},$$

and thus $(A \boxtimes B)^{\frac{r}{s}} = A^{\frac{r}{s}} \boxtimes B^{\frac{r}{s}}$. Now, for $\alpha > 0$, there is a sequence (q_n) of positive rational numbers such that $q_n \to \alpha$. It follows from the previous claim and the continuity of Tracy-Singh product (Theorem 3.1) that

$$\begin{aligned} (A \boxtimes B)^{\alpha} &= \lim_{n\to\infty} (A \boxtimes B)^{q_n} = \lim_{n\to\infty} A^{q_n} \boxtimes B^{q_n} \\ &= \lim_{n\to\infty} A^{q_n} \boxtimes \lim_{n\to\infty} B^{q_n} = A^{\alpha} \boxtimes B^{\alpha}. \end{aligned}$$ □

Corollary 3.4. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be strictly positive operators. For any real number α, we have*

$$(A \boxtimes B)^{\alpha} = A^{\alpha} \boxtimes B^{\alpha}. \tag{3.6}$$

Proof. Note that $A \boxtimes B$ is strictly positive by property (7) of Lemma 3.1. For $\alpha < 0$, it follows from Theorem 3.4 and the property (5) in Lemma 3.1 that

$$\begin{aligned} (A \boxtimes B)^{\alpha} &= [(A \boxtimes B)^{-1}]^{-\alpha} = (A^{-1} \boxtimes B^{-1})^{-\alpha} \\ &= (A^{-1})^{-\alpha} \boxtimes (B^{-1})^{-\alpha} = A^{\alpha} \boxtimes B^{\alpha}. \end{aligned}$$ □

Corollary 3.5. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. Then*

$$|A \boxtimes B| = |A| \boxtimes |B|. \tag{3.7}$$

Proof. Applying Lemma 3.1 and property (3.5), we get

$$\begin{aligned} |A \boxtimes B| &= [(A \boxtimes B)^*(A \boxtimes B)]^{\frac{1}{2}} = [(A^* \boxtimes B^*)(A \boxtimes B)]^{\frac{1}{2}} \\ &= (A^*A \boxtimes B^*B)^{\frac{1}{2}} = (A^*A)^{\frac{1}{2}} \boxtimes (B^*B)^{\frac{1}{2}} = |A| \boxtimes |B|. \end{aligned}$$ □

Recall the polar decomposition theorem: for any $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$, there exists a partial isometry U such that $A = U|A|$. The next result is a polar decomposition for the Tracy-Singh product of operators.

Corollary 3.6. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. If $A = U|A|$ and $B = V|B|$ are polar decompositions of A and B, respectively, then a polar decomposition of $A \boxtimes B$ is given by*

$$A \boxtimes B = (U \boxtimes V)|A \boxtimes B|. \tag{3.8}$$

Proof. Let U and V be partial isometries such that $A = U|A|$ and $B = V|B|$. It follows from Lemma 3.1(3) and Corollary 3.5 that

$$A \boxtimes B = U|A| \boxtimes V|B| = (U \boxtimes V)(|A| \boxtimes |B|) = (U \boxtimes V)|A \boxtimes B|.$$

Note that $U \boxtimes V$ is also a partial isometry, according to property (8) in Lemma 3.1. Hence, the decomposition (3.8) is a polar one. □

3.4 Tracy-Singh products on norm ideals of compact operators

In this section, we investigate the Tracy-Singh product on norm ideals of $\mathbb{B}(\mathcal{H})$. Recall that any proper ideal of $\mathbb{B}(\mathcal{H})$ is contained in the ideal $\mathcal{S}_\infty$ of compact operators. For any compact operator $A \in \mathbb{B}(\mathcal{H})$, let $(s_i(A))_{i=1}^\infty$ be the sequence of decreasingly-ordered singular values of A (i.e. eigenvalues of $|A|$). For each $1 \leqslant p < \infty$, the *Schatten p-norm* of A is defined by

$$\|A\|_p = \left(\sum_{i=1}^\infty s_i^p(A) \right)^{1/p}.$$

If $\|A\|_p$ is finite, we say that A is a *Schatten p-class operator*. The Schatten ∞-norm is just the operator norm. For each $1 \leqslant p \leqslant \infty$, let $\mathcal{S}_p$ be the Schatten p-class operators . In particular, $\mathcal{S}_1$ and $\mathcal{S}_2$ are the trace class and the Hilbert-Schmidt class, respectively. Each Schatten p-norm induces a norm ideal of $\mathbb{B}(\mathcal{H})$ and this ideal is closed under the topology generated by this norm.

Lemma 3.7. *Let $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$ be an operator matrix. Then A is compact if and only if A_{ij} is compact for all i, j.*

Proof. If A is compact, then $A_{ij} = P_i' A E_j$ is also compact for each i, j due to the fact that $\mathcal{S}_\infty$ is an ideal of $\mathbb{B}(\mathcal{H})$. Conversely, suppose that A_{ij} is compact for all i, j. Recall that a bounded linear operator is compact if and only if it maps a bounded sequence into a sequence having a convergent subsequence. Let $(x_r)_{r=1}^\infty$ be a bounded sequence in $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$. Write $x_r = [x_r^{(1)}\ x_r^{(2)}\ \dots\ x_r^{(n)}]^T \in \bigoplus_{i=1}^n \mathcal{H}_i$ for each $r \in \mathbb{N}$. Consider

$$Ax_r = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_r^{(1)} \\ \vdots \\ x_r^{(n)} \end{bmatrix} = \begin{bmatrix} A_{11}x_r^{(1)} + \cdots + A_{1n}x_r^{(n)} \\ \vdots \\ A_{n1}x_r^{(1)} + \cdots + A_{nn}x_r^{(n)} \end{bmatrix}.$$

For each $l = 1, 2, \dots, n$, since $(x_r^{(l)})_{r=1}^\infty$ is bounded, the sequence $(A_{ij}x_r^{(l)})_{r=1}^\infty$ has a convergent subsequence, namely, $(A_{ij}x_{r_k}^{(l)})_{k=1}^\infty$. Hence,

$$\begin{bmatrix} A_{11}x_{r_k}^{(1)} + \cdots + A_{1n}x_{r_k}^{(n)} \\ \vdots \\ A_{n1}x_{r_k}^{(1)} + \cdots + A_{nn}x_{r_k}^{(n)} \end{bmatrix}$$

is a desired convergent subsequence of $(Ax_r)_{r=1}^\infty$. □

Lemma 3.8 ([13])**.** *Let $A = [A_{ij}]_{i,j=1}^{n,n}$ be an operator matrix in the Schatten p-class.*

(i) For $1 \leqslant p \leqslant 2$, we have

$$\sum_{i,j=1}^n \|A_{ij}\|_p^2 \leqslant \|A\|_p^2 \leqslant n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}\|_p^2. \tag{3.9}$$

(ii) For $2 \leqslant p < \infty$, we have

$$n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}\|_p^2 \leqslant \|A\|_p^2 \leqslant \sum_{i,j=1}^n \|A_{ij}\|_p^2. \tag{3.10}$$

Lemma 3.9. *Let $1 \leqslant p < \infty$. An operator matrix $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$ is a Schatten p-class operator if and only if A_{ij} is a Schatten p-class operator for all i, j.*

Proof. This is a direct consequence of the norm estimations in Lemma 3.8. $\square$

Lemma 3.10. *Let $1 \leqslant p \leqslant \infty$. Let $A = [A_{ij}]_{i,j=1}^{n,n}$ be an operator matrix in the class $\mathcal{S}_p$ and let $(A_r)_{r=1}^{\infty}$ be a sequence in $\mathcal{S}_p$ where $A_r = \left[A_{ij}^{(r)}\right]_{i,j=1}^{n,n}$ for each $r \in \mathbb{N}$. Then $A_r \to A$ in $\mathcal{S}_p$ if and only if $A_{ij}^{(r)} \to A_{ij}$ in $\mathcal{S}_p$ for all $i, j = 1, \ldots, n$.*

Proof. Lemma 3.9 assures that A_{ij} and $A_{ij}^{(r)}$ belong to $\mathcal{S}_p$ for any $i, j = 1, \ldots, n$ and $r \in \mathbb{N}$. Consider the case $1 \leqslant p \leqslant 2$. Suppose that $A_r \to A$ in $\mathcal{S}_p$. For any fixed $i, j \in \{1, ..., n\}$, we have from the estimation (3.9) that

$$\|A_{ij}^{(r)} - A_{ij}\|_p^2 \;\leqslant\; \sum_{i,j=1}^{n} \|A_{ij}^{(r)} - A_{ij}\|_p^2 \;\leqslant\; \|A_r - A\|_p^2.$$

Hence, $A_{ij}^{(r)} \to A_{ij}$ in $\mathcal{S}_p$. Conversely, suppose $A_{ij}^{(r)} \to A_{ij}$ in $\mathcal{S}_p$ for each i, j. Lemma 3.8 implies that

$$\|A_r - A\|_p^2 \;\leqslant\; n^{4/p-2} \sum_{i,j=1}^{n} \|A_{ij}^{(r)} - A_{ij}\|_p^2.$$

Hence, $A_r \to A$ in $\mathcal{S}_p$. The case $2 < p < \infty$ and the case $p = \infty$ are done by using the norm estimations (3.10) and (3.1), respectively. $\square$

Next, we discuss compactness of Tracy-Singh product of operators.

Lemma 3.11 ([11])**.** *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be nonzero operators. Then $A \otimes B$ is compact if and only if both A and B are compact.*

Theorem 3.5. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be nonzero operator matrices. Then $A \boxtimes B$ is compact if and only if both A and B are compact.*

Proof. Write $A = [A_{ij}]$ and $B = [B_{kl}]$. For sufficiency, suppose that A and B are compact. By Lemma 3.7, we deduce that A_{ij} and B_{kl} are compact for all i, j, k, l. It follows from Lemma 3.11 that $A_{ij} \otimes B_{kl}$ is compact for all i, j, k, l. Lemma 3.7 ensures the compactness of $A \boxtimes B$. For necessity part, reverse the previous procedure. $\square$

The following theorem supplies bounds for Schatten 1-norm of the Tracy-Singh product of operators.

Theorem 3.6. *For any nonzero compact operator* $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ *and* $B = [A_{kl}]_{k,l=1}^{q,q} \in \mathbb{B}(\mathcal{K})$*, we have*

$$\frac{1}{nq}\|A\|_1\|B\|_1 \;\leqslant\; \|A \boxtimes B\|_1 \;\leqslant\; nq\|A\|_1\|B\|_1. \tag{3.11}$$

Hence, $A \boxtimes B$ *is trace-class if and only if both* A *and* B *are trace-class.*

Proof. Suppose that both A and B are nonzero and compact. Then the operator $A \boxtimes B$ is compact by Theorem 3.5. It follows from the norm bound (3.9) that

$$\begin{aligned}\|A \boxtimes B\|_1^2 \;&\leqslant\; (nq)^2 \sum_{k,l}\sum_{i,j}\|A_{ij} \otimes B_{kl}\|_1^2 \;=\; (nq)^2 \sum_{k,l}\sum_{i,j}\|A_{ij}\|_1^2\|B_{kl}\|_1^2 \\ &=\; (nq)^2\Big(\sum_{i,j}\|A_{ij}\|_1^2\Big)\Big(\sum_{k,l}\|B_{kl}\|_1^2\Big) \;\leqslant\; (nq)^2\|A\|_1^2\|B\|_1^2.\end{aligned}$$

We also have

$$\begin{aligned}\|A \boxtimes B\|_1^2 \;&\geqslant\; \sum_{k,l}\sum_{i,j}\|A_{ij} \otimes B_{kl}\|_1^2 \;=\; \sum_{k,l}\sum_{i,j}\|A_{ij}\|_1^2\|B_{kl}\|_1^2 \\ &=\; \Big(\sum_{i,j}\|A_{ij}\|_1^2\Big)\Big(\sum_{k,l}\|B_{kl}\|_1^2\Big) \;\geqslant\; (nq)^{-2}\|A\|_1^2\|B\|_1^2.\end{aligned}$$

Hence, we obtain the bound (3.11). □

Theorem 3.7. *For any nonzero compact operator matrices* $A \in \mathbb{B}(\mathcal{H})$ *and* $B \in \mathbb{B}(\mathcal{K})$*, we have*

$$\|A \boxtimes B\|_2 \;=\; \|A\|_2\,\|B\|_2. \tag{3.12}$$

Hence, $A \boxtimes B$ *is a Hilbert-Schmidt operator if and only if both* A *and* B *are Hilbert-Schmidt operators.*

Proof. Since both A and B are nonzero and compact, the operator $A \boxtimes B$ is compact by Theorem 3.5. Write $A = [A_{ij}]$ and $B = [B_{kl}]$. Then by Lemma

3.8(ii), we have

$$\begin{aligned}
\|A \boxtimes B\|_2^2 &= \sum_{k,l}\sum_{i,j} \|A_{ij} \otimes B_{kl}\|_2^2 = \sum_{k,l}\sum_{i,j} \|A_{ij}\|_2^2 \, \|B_{kl}\|_2^2 \\
&= \Big(\sum_{i,j} \|A_{ij}\|_2^2\Big)\Big(\sum_{k,l} \|B_{kl}\|_2^2\Big) = \|A\|_2^2 \, \|B\|_2^2.
\end{aligned}$$

Hence, we get the multiplicative property (3.12). □

The final result asserts that the Tracy-Singh product is continuous with respect to the topology induced by the Schatten p-norm for each $p \in \{1, 2, \infty\}$.

Theorem 3.8. *Let $p \in \{1, 2, \infty\}$. If a sequence $(A_r)_{r=1}^{\infty}$ converges to A and a sequence $(B_r)_{r=1}^{\infty}$ converges to B in the norm ideal $\mathcal{S}_p$, then $A_r \boxtimes B_r$ converges to $A \boxtimes B$ in $\mathcal{S}_p$.*

Proof. Write $A = [A_{ij}]$ and $B = [B_{kl}]$. In the viewpoint of Lemma 3.10, it suffices to show that $A_{ij}^{(r)} \otimes B_{kl}^{(r)} \to A_{ij} \otimes B_{kl}$ in $\mathcal{S}_p$ for all i, j, k, l. Since $A_r \to A$ and $B_r \to B$ in $\mathcal{S}_p$, we have by Lemma 3.10 that $A_{ij}^{(r)} \to A_{ij}$ and $B_{kl}^{(r)} \to B_{kl}$ for all i, j, k, l. It follows that

$$\begin{aligned}
\|A_{ij}^{(r)} \otimes B_{kl}^{(r)} - A_{ij} \otimes B_{kl}\|_p &= \|A_{ij}^{(r)} \otimes B_{kl}^{(r)} - A_{ij}^{(r)} \otimes B_{kl} + A_{ij}^{(r)} \otimes B_{kl} - A_{ij} \otimes B_{kl}\|_p \\
&\leqslant \|A_{ij}^{(r)} \otimes (B_{kl}^{(r)} - B_{kl})\|_p + \|(A_{ij}^{(r)} - A_{ij}) \otimes B_{kl}\|_p \\
&= \|A_{ij}^{(r)}\|_p \, \|B_{kl}^{(r)} - B_{kl}\|_p + \|A_{ij}^{(r)} - A_{ij}\|_p \, \|B_{kl}\|_p \\
&\to \|A_{ij}\|_p \cdot 0 + 0 \cdot \|B_{kl}\|_p = 0.
\end{aligned}$$

Hence, $A_{ij}^{(r)} \otimes B_{kl}^{(r)} \to A_{ij} \otimes B_{kl}$ in $\mathcal{S}_p$ for all i, j, k, l. □

Chapter 4

Tracy-Singh Products and Classes of Operators

In this chapter, we investigate relationship between Tracy-Singh products and certain classes of Hilbert space operators. We show that the normality, hyponormality, paranormality of operators are preserved by Tracy-Singh products. Operators of class-$B(\mathcal{H})$ type are also preserved under Tracy-Singh products. Moreover, we obtain necessary and sufficient conditions for the Tracy-Singh product of two operators to be normal, quasinormal, (co)isometry, and unitary.

4.1 Introduction

Tensor product of bounded linear operators plays a crucial role in functional analysis and operator theory. Many algebraic-order-analytic properties of operators are preserved under taking tensor products, but by no means all of them. Importance results on tensor product involving certain classes of operators (e.g. positive, unitary, normal, compact) have been noticed by many mathematicians from the beginning of the theory to nowadays (e.g. [37]). In the last two decades, the concepts of normality, hyponormality, and paranormality have been introduced and investigated by many authors, see e.g., [20, 28, 36]. Relations between tensor products and class-$\mathcal{A}$ type operators also have received much attention, e.g., [25, 26, 27, 34, 35]. See more information about classes of operators in the monograph [22].

Recently, the notion of tensor product was extended to the Tracy-Singh product for Hilbert space operators in [49]. It was shown that compactness, positivity and strict-positivity of operators are preserved under Tracy-Singh products [49, 50].

In this paper, we investigate relationship between Tracy-Singh products and certain classes of operators. We divide such classes into three categories. The first category consists of nilpotent, (skew)-Hermitian, (co)isometry, and unitary operators. The second one contains operator normality, hyponormality, and paranormality. The last one is the class-$\mathcal{A}$ type operators, which includes class $B(\mathcal{H})(k)$, class $B(\mathcal{H})$, quasi-class $(B(\mathcal{H}), k)$, quasi-class $B(\mathcal{H})$, $*$-class $B(\mathcal{H})$, quasi-$*$-class $B(\mathcal{H})$, and quasi-$*$-class $(B(\mathcal{H}), k)$ operators. We will show that the mentioned properties of operators are preserved under taking Tracy-Singh products. Moreover, we obtain necessary and sufficient conditions for the Tracy-Singh product of two operators to be normal, quasinormal, (co)isometry, and unitary operators.

The paper is structured as follows. The next section supplies some prerequisites about the tensor product and the Tracy-Singh product of operators. Next, we discuss relationship between Tracy-Singh products and the normality, hyponormality, and paranormality of operators. Then we consider Tracy-Singh products and certain properties of operators–being nilpotent, (skew)-Hermitian, (co)isometry, and unitary. The last section deals with class $\mathcal{A}$ type operators.

4.2 Preliminaries

In what follows, $\mathcal{H}$ and $\mathcal{K}$ denote complex separable Hilbert spaces. When X and Y are Hilbert spaces, denote by $\mathbb{B}(X, Y)$ the Banach space of bounded linear operators from X into Y, equipped with the operator norm $\|\cdot\|$ and abbreviate $\mathbb{B}(X, X)$ to $\mathbb{B}(X)$. For Hermitian operators A and B on the same Hilbert space, we use the notation $A \geqslant B$ to mean that $A - B$ is a positive operator.

In order to define the Tracy-Singh product, we have to fix the orthogonal decompositions of Hilbert spaces, namely,

$$\mathcal{H} = \bigoplus_{i=1}^{m} \mathcal{H}_i, \quad \mathcal{K} = \bigoplus_{l=1}^{n} \mathcal{K}_l$$

where all $\mathcal{H}_i$'s and $\mathcal{K}_k$'s are Hilbert spaces. Any operator $A \in \mathbb{B}(\mathcal{H})$ and

$B \in \mathbb{B}(\mathcal{K})$ thus can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,m} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{n,n}$$

where $A_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$ and $B_{kl} \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}_k)$ for each i, j, k, l. Then the *Tracy-Singh product* of A and B is defined to be

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij}, \tag{4.1}$$

which is a bounded linear operator from $\bigoplus_{i,k=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_k$ into itself. Note that when $m = n = 1$, the Tracy-Singh product $A \boxtimes B$ reduces to the tensor product $A \otimes B$.

Lemma 4.1 ([49])**.** *Algebraic and order properties of the Tracy-Singh product for operators are listed here (provided that every operation is well-defined):*

1. *The map $(A, B) \mapsto A \boxtimes B$ is bilinear.*
2. *Compatibility with adjoints: $(A \boxtimes B)^* = A^* \boxtimes B^*$.*
3. *Compatibility with ordinary products: $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$.*
4. *Compatibility with powers: $(A \boxtimes B)^r = A^r \boxtimes B^r$ for any $r \in \mathbb{N}$.*
5. *Compatibility with inverses: if A and B are invertible, then $A \boxtimes B$ is invertible with $(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}$.*
6. *Positivity: if $A \geqslant 0$ and $B \geqslant 0$, then $A \boxtimes B \geqslant 0$.*
7. *Monotonicity: if $A_1 \geqslant B_1$ and $A_2 \geqslant B_2$, then $A_1 \boxtimes A_2 \geqslant B_1 \boxtimes B_2$.*

Lemma 4.2 ([49])**.** *Let $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be operator matrices. Then each (i, j)-block of $A \boxtimes B$ is $A_{ij} \boxtimes B$.*

Analytic properties of the Tracy-Singh product for operators are listed below.

Lemma 4.3 ([50])**.** *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$. Then we have*

(i) $\dfrac{1}{mn}\|A\|\|B\| \leqslant \|A \boxtimes B\| \leqslant mn\|A\|\|B\|$.

(ii) $|A \boxtimes B| = |A| \boxtimes |B|$, *here the absolute value of* A *is defined by* $|A| = (A^*A)^{\frac{1}{2}}$.

(iii) *If* A *and* B *are positive operators, then* $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$ *for any nonnegative real* α.

Lemma 4.4. *Let* $A \in \mathbb{B}(\mathcal{H})$ *and* $B \in \mathbb{B}(\mathcal{K})$.

(i) *The condition* $A \boxtimes B = 0$ *holds if and only if* $A = 0$ *or* $B = 0$.

(ii) *If* $A \boxtimes B = A \boxtimes C$ *and* $A \neq 0$, *then* $B = C$.

(iii) *If* $B \boxtimes A = C \boxtimes A$ *and* $A \neq 0$, *then* $B = C$.

Proof. From the norm estimation in Lemma 4.3(i), one can deduce property (i). Properties (ii) and (iii) follow from (i) and the bilinearity of Tracy-Singh product in Lemma 4.1. □

Lemma 4.5 ([36])**.** *Let* $A, C \in \mathbb{B}(\mathcal{H})$ *and* $B, D \in \mathbb{B}(\mathcal{K})$ *be nonzero operators. Then* $A \otimes B = C \otimes D$ *if and only if there exists* $\alpha \in \mathbb{C} \setminus \{0\}$ *such that* $C = \alpha A$ *and* $D = \alpha^{-1} B$.

Proposition 4.6. *Let* $A = [A_{ij}]_{i,j=1}^{m,m}, C = [C_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$ *and* $B = [B_{kl}]_{k,l=1}^{n,n}, D = [D_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ *be operator matrices such that* A_{ij}, B_{kl}, C_{ij} *and* D_{kl} *are nonzero operators for all* $i, j = 1, \ldots, m$ *and* $k, l = 1, \ldots, n$. *Then* $A \boxtimes B = C \boxtimes D$ *if and only if there exists* $\alpha \in \mathbb{C} \setminus \{0\}$ *such that* $C = \alpha A$ *and* $D = \alpha^{-1} B$.

Proof. If $C = \alpha A$ and $D = \alpha^{-1} B$ for some $\alpha \in \mathbb{C} \setminus \{0\}$, then by Lemma 4.1,

$$C \boxtimes D = (\alpha A) \boxtimes (\alpha^{-1} B) = \alpha \alpha^{-1} (A \boxtimes B) = A \boxtimes B.$$

Assume that $A \boxtimes B = C \boxtimes D$. By using Lemma 4.2, we get $A_{ij} \boxtimes B = C_{ij} \boxtimes D$ for all $i, j = 1, \ldots, m$. For any fixed $i, j \in \{1, \ldots, m\}$, we have $A_{ij} \otimes B_{kl} = C_{ij} \otimes D_{kl}$ for all $k, l = 1, \ldots, n$. For each $i, j \in \{1, \ldots, m\}$ and $k, l \in \{1, \ldots, n\}$, by applying Lemma 4.5, there exists $\alpha_{ij,kl} \in \mathbb{C} \setminus \{0\}$ such that $C_{ij} = \alpha_{ij,kl} A_{ij}$ and $D_{kl} = \alpha_{ij,kl}^{-1} B_{kl}$. For any fixed $i, j \in \{1, \ldots, m\}$, we have $C_{ij} = \alpha_{ij,kl} A_{ij}$ for all $k, l = 1, \ldots, n$. This implies that $\alpha_{ij,11} = \cdots = \alpha_{ij,nn} = \alpha_{ij}$. For any fixed $k, l \in \{1, \ldots, n\}$, we have $D_{kl} = \alpha_{ij}^{-1} B_{kl}$ for all $i, j = 1, \ldots, m$. It follows that $\alpha_{11} = \cdots = \alpha_{mm} = \alpha$. Thus $C_{ij} = \alpha A_{ij}$ and $D_{kl} = \alpha^{-1} B_{kl}$ for all $i, j = 1, \ldots, m$ and $k, l = 1, \ldots, n$. Therefore $C = \alpha A$ and $D = \alpha^{-1} B$. □

Recall that the commutator of A and B in $\mathbb{B}(\mathcal{H})$ is defined by

$$[A, B] \;=\; AB - BA.$$

Proposition 4.7. *Let $A, C \in \mathbb{B}(\mathcal{H})$ and $B, D \in \mathbb{B}(\mathcal{K})$.*

(i) If $[A, C] \geqslant 0$ and $[B, D] \geqslant 0$, then $[A \boxtimes B, C \boxtimes D] \geqslant 0$.

(ii) If $[A, C] \leqslant 0$ and $[B, D] \leqslant 0$, then $[A \boxtimes B, C \boxtimes D] \leqslant 0$.

(iii) If $[A, C] = 0$ and $[B, D] = 0$, then $[A \boxtimes B, C \boxtimes D] = 0$.

Proof. (i) Since $AC \geqslant CA$ and $BD \geqslant DB$, we have $AC \boxtimes BD \geqslant CA \boxtimes DB$ by Lemma 4.1. Then

$$[A \boxtimes B, C \boxtimes D] \;=\; AC \boxtimes BD - CA \boxtimes DB \geqslant 0.$$

The assertion (ii) follows from (i) and the fact that $-[X, Y] = [Y, X]$ for any operators X and Y. The assertion (iii) follows from (i) and (ii). □

4.3 Tracy-Singh products and operator normality

In this section, we discuss normality of Tracy-Singh products of operators. The contents can be divided into three parts. The first part deals with general properties of normality, the second one concerns hyponormality, and the last one consists of paranormality.

4.3.1 Normality

Recall the following types of operator normality; see e.g. [22, Chapter 2] and [32] for more details.

Definition 4.8. An operator $T \in \mathbb{B}(\mathcal{H})$ is said to be

- *normal* if $[T^*, T] = 0$;
- *binormal* if $[T^*T, TT^*] = 0$;
- *quasinormal* if $[T, T^*T] = 0$;

- *posinormal* if $TT^* = T^*PT$ for some positive operator P.

Stochel [36] showed that for non-zero $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$, the tensor product $A \otimes B$ is normal (resp. quasinormal) if and only if A and B are normal (resp. quasinormal). Now, we will extend this result to the case of Tracy-Singh products.

Theorem 4.1. *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ be operator matrices such that A_{ij} and B_{kl} are nonzero operators for all $i, j = 1, \dots, m$ and $k, l = 1, \dots, n$. Then $A \boxtimes B$ is normal if and only if so are A and B.*

Proof. If A and B are normal, then by Lemma 4.1 and Proposition 4.7 we have

$$[(A \boxtimes B)^*, A \boxtimes B] \;=\; [A^* \boxtimes B^*, A \boxtimes B] \;=\; [A^*, A] \boxtimes [B^*, B] \;=\; 0,$$

i.e., $A \boxtimes B$ is also normal. Conversely, suppose that $A \boxtimes B$ is normal. Note that

$$A^*A \boxtimes B^*B \;=\; (A \boxtimes B)^*(A \boxtimes B) \;=\; (A \boxtimes B)(A \boxtimes B)^* \;=\; AA^* \boxtimes BB^*.$$

By Proposition 4.6, there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $AA^* = \alpha A^*A$ and $BB^* = \alpha^{-1}B^*B$. Since AA^* and A^*A are positive, we have $\alpha > 0$. Then

$$\begin{aligned} \|A\|^2 \;&=\; \|AA^*\| \;=\; \|\alpha A^*A\| \;=\; \alpha\|A\|^2, \\ \|B\|^2 \;&=\; \|BB^*\| \;=\; \|\alpha^{-1}B^*B\| \;=\; \alpha^{-1}\|B\|^2. \end{aligned}$$

We arrive at $\alpha = 1$, meaning that both A and B are normal. □

Theorem 4.2. *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ be operator matrices such that A_{ij} and B_{kl} are nonzero operators for all $i, j = 1, \dots, m$ and $k, l = 1, \dots, n$. Then $A \boxtimes B$ is quasinormal if and only if so are A and B.*

Proof. Assume that A and B are quasinormal. Since $[A, A^*A] = 0$ and $[B, B^*B] = 0$, we have

$$[A \boxtimes B, (A \boxtimes B)^*(A \boxtimes B)] \;=\; [A \boxtimes B, A^*A \boxtimes B^*B] \;=\; 0.$$

Hence, $A \boxtimes B$ is quasinormal. Suppose that $A \boxtimes B$ is quasinormal. Note that

$$\begin{aligned} AA^*A \boxtimes BB^*B &= (A \boxtimes B)(A \boxtimes B)^*(A \boxtimes B) \\ &= (A \boxtimes B)^*(A \boxtimes B)^2 \\ &= A^*A^2 \boxtimes B^*B^2. \end{aligned}$$

Then there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $A^*A^2 = \alpha AA^*A$ and $B^*B^2 = \alpha^{-1}BB^*B$. This in turn implies that

$$\begin{aligned} (A^2)^*A^2 &= A^*(A^*A^2) = \alpha(A^*A)^2, \\ (B^2)^*B^2 &= B^*(B^*B^2) = \alpha^{-1}(B^*B)^2. \end{aligned}$$

Since $(A^2)^*A^2$ and $\alpha(A^*A)^2$ are positive, we conclude $\alpha > 0$. We have

$$\alpha\|A\|^4 = \alpha\|(A^*A)^2\|^2 = \|(A^2)^*A^2\| = \|A^2\|^2 \leqslant \|A\|^4$$

and, similarly, $\alpha^{-1}\|B\|^4 \leqslant \|B\|^4$. This forces $\alpha = 1$ and, thus, both A and B are quasinormal. □

Proposition 4.9. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. If both A and B satisfy one of the following properties, then the same property holds for $A \boxtimes B$: binormal, posinormal.*

Proof. The assertion for binormality follows from Lemma 4.1 and Proposition 4.7. Now, suppose that $AA^* = A^*PA$ and $BB^* = B^*QB$ for some positive operators P and Q. By Lemma 4.1, we get

$$\begin{aligned} (A \boxtimes B)(A \boxtimes B)^* &= AA^* \boxtimes BB^* = A^*PA \boxtimes B^*QB \\ &= (A \boxtimes B)^*(P \boxtimes Q)(A \boxtimes B). \end{aligned}$$

According to Lemma 4.1, $P \boxtimes Q$ is positive. Therefore $A \boxtimes B$ is posinormal. □

4.3.2 Hyponormality

Recall the following hyponormal structures of operators; see e.g. [16, 19, 28] and [22, Chapter 2] for more information.

Definition 4.10. Let $p > 0$ be a constant. An operator $T \in \mathbb{B}(\mathcal{H})$ is said to be

- *hyponormal* if $[T^*, T]$ is positive ;
- *p-hyponormal* if $(T^*T)^p \geqslant (TT^*)^p$;
- *quasihyponormal* if $T^*[T^*, T]T$ is positive ;
- *p-quasihyponormal* if $T^*(T^*T)^pT \geqslant T^*(TT^*)^pT$;
- *cohyponormal* if T^* is hyponormal ;
- *log-hyponormal* if T is invertible and $\log(T^*T) \geqslant \log(TT^*)$.

Definition 4.11. Let $T \in \mathbb{B}(\mathcal{H})$ have the polar decomposition $T = U|T|$ where U is a unitary operator. The *Aluthge transformation* of T is defined by

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

Then T is said to be

- *w-hyponormal* if $|\tilde{T}| \geqslant |T| \geqslant |\tilde{T}^*|$;
- *iw-hyponormal* if T is invertible and $|T| \geqslant \left(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}\right)^{\frac{1}{2}}$.

Theorem 4.3. *Let $A \in \mathbb{B}(\mathcal{H})$, $B \in \mathbb{B}(\mathcal{K})$, and let $p > 0$ be a constant. If both A and B satisfy one of the following properties, then the same property holds for $A \boxtimes B$: hyponormal, p-hyponormal, cohyponormal, quasihyponormal, p-quasihyponormal.*

Proof. The assertions for hyponormality and cohyponormality follow from Lemma 4.1 and Proposition 4.7. The assertion for p-hyponormality is done by applying Lemmas 4.1 and 4.3. Now, suppose that A and B are quasihyponormal. By Lemma 4.1, we obtain

$$\begin{aligned}
&(A \boxtimes B)^* \left[(A \boxtimes B)^*, A \boxtimes B\right] (A \boxtimes B) \\
&\quad = (A^* \boxtimes B^*)((A^* \boxtimes B^*)(A \boxtimes B) - (A \boxtimes B)(A^* \boxtimes B^*))(A \boxtimes B) \\
&\quad = (A^* \boxtimes B^*)(A^* \boxtimes B^*)(A \boxtimes B)(A \boxtimes B) - (A^* \boxtimes B^*)(A \boxtimes B)(A^* \boxtimes B^*)(A \boxtimes B) \\
&\quad = A^*A^*AA \boxtimes B^*B^*BB - A^*AA^*A \boxtimes B^*BB^*B.
\end{aligned}$$

Since $A^*A^*AA - A^*AA^*A = A^*[A^*, A]A \geqslant 0$ and $B^*B^*BB - B^*BB^*B = B^*[B^*, B]B \geqslant 0$, we have by Lemma 4.1 that

$$A^*A^*AA \boxtimes B^*B^*BB - A^*AA^*A \boxtimes B^*BB^*B \;\geqslant\; 0.$$

Hence, $(A \boxtimes B)^* [(A \boxtimes B)^*, A \boxtimes B] (A \boxtimes B) \geqslant 0$. This means that $A \boxtimes B$ is quasihyponormal.

Assume that A and B are p-quasihyponormal. Lemmas 4.1 and 4.3 together imply that

$$\begin{aligned}
&(A \boxtimes B)^* ((A \boxtimes B)^*(A \boxtimes B))^p (A \boxtimes B) \\
&\quad = (A^* \boxtimes B^*) (A^*A \boxtimes B^*B)^p (A \boxtimes B) \\
&\quad = A^*(A^*A)^p A \boxtimes B^*(B^*B)^p B \\
&\quad \geqslant A^*(AA^*)^p A \boxtimes B^*(BB^*)^p B \\
&\quad = (A \boxtimes B)^* (AA^* \boxtimes BB^*)^p (A \boxtimes B) \\
&\quad = (A \boxtimes B)^* ((A \boxtimes B)(A \boxtimes B)^*)^p (A \boxtimes B).
\end{aligned}$$

This show that $A \boxtimes B$ is p-quasihyponormal. □

Kim [28] investigated the tensor product of log-hyponormal (reps. w-hyponormal, iw-hyponormal) operators. Now, we consider the case of Tracy-Singh products.

Lemma 4.12 ([21])**.** *Let S and T be positive invertible operators. Then $\log T \geqslant \log S$ if and only if $T^p \geqslant \left(T^{\frac{p}{2}} S^p T^{\frac{p}{2}}\right)^{\frac{1}{2}}$ for all $p \geqslant 0$.*

Theorem 4.4. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive invertible operators. If A and B are log-hyponormal, then $A \boxtimes B$ is also log-hyponormal.*

Proof. Assume that A and B are log-hyponormal operators. Since A and B are invertible, Lemma 4.1 implies that $A \boxtimes B$ is invertible. Using Lemmas 4.1 and 4.3, we obtain that for any $p \geqslant 0$,

$$\begin{aligned}
&[(A \boxtimes B)^*(A \boxtimes B)]^p \\
&\quad = (A^*A \boxtimes B^*B)^p \\
&\quad = (A^*A)^p \boxtimes (B^*B)^p \\
&\quad \geqslant [(A^*A)^{\frac{p}{2}} (AA^*)^p (A^*A)^{\frac{p}{2}}]^{\frac{1}{2}} \boxtimes [(B^*B)^{\frac{p}{2}} (BB^*)^p (B^*B)^{\frac{p}{2}}]^{\frac{1}{2}} \\
&\quad = [(A^*A)^{\frac{p}{2}} (AA^*)^p (A^*A)^{\frac{p}{2}} \boxtimes (B^*B)^{\frac{p}{2}} (BB^*)^p (B^*B)^{\frac{p}{2}}]^{\frac{1}{2}} \\
&\quad = [(A^*A \boxtimes B^*B)^{\frac{p}{2}} (AA^* \boxtimes BB^*)^p (A^*A \boxtimes B^*B)^{\frac{p}{2}}]^{\frac{1}{2}} \\
&\quad = [((A \boxtimes B)^*(A \boxtimes B))^{\frac{p}{2}} ((A \boxtimes B)(A \boxtimes B)^*)^p ((A \boxtimes B)^*(A \boxtimes B))^{\frac{p}{2}}]^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 4.12, we have $\log(A \boxtimes B)^*(A \boxtimes B) \geqslant \log(A \boxtimes B)(A \boxtimes B)^*$. This means that $A \boxtimes B$ is log-hyponormal. □

Lemma 4.13 ([16])**.** *An operator $T \in \mathbb{B}(\mathcal{H})$ is w-hyponormal if and only if $|T| \geqslant \left(|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}}\right)^{\frac{1}{2}}$ and $|T^*| \leqslant \left(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}}\right)^{\frac{1}{2}}$.*

Theorem 4.5. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. If A and B are w-hyponormal, then $A \boxtimes B$ is also w-hyponormal.*

Proof. Assume that A and B are w-hyponormal. By applying Lemmas 4.1 and 4.3, we have

$$\begin{aligned}
|A \boxtimes B| &= |A| \boxtimes |B| \\
&\geqslant \left(|A|^{\frac{1}{2}}|A^*||A|^{\frac{1}{2}}\right)^{\frac{1}{2}} \boxtimes \left(|B|^{\frac{1}{2}}|B^*||B|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left(|A|^{\frac{1}{2}}|A^*||A|^{\frac{1}{2}} \boxtimes |B|^{\frac{1}{2}}|B^*||B|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left[\left(|A|^{\frac{1}{2}} \boxtimes |B|^{\frac{1}{2}}\right)(|A^*| \boxtimes |B^*|)\left(|A|^{\frac{1}{2}} \boxtimes |B|^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \\
&= \left(|A \boxtimes B|^{\frac{1}{2}}|(A \boxtimes B)^*||A \boxtimes B|^{\frac{1}{2}}\right)^{\frac{1}{2}}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
|(A \boxtimes B)^*| &= |A^*| \boxtimes |B^*| \\
&\leqslant \left(|A^*|^{\frac{1}{2}}|A||A^*|^{\frac{1}{2}}\right)^{\frac{1}{2}} \boxtimes \left(|B^*|^{\frac{1}{2}}|B||B^*|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left(|A^*|^{\frac{1}{2}}|A||A^*|^{\frac{1}{2}} \boxtimes |B^*|^{\frac{1}{2}}|B||B^*|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left[\left(|A^*|^{\frac{1}{2}} \boxtimes |B^*|^{\frac{1}{2}}\right)(|A| \boxtimes |B|)\left(|A^*|^{\frac{1}{2}} \boxtimes |B^*|^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \\
&= \left(|(A \boxtimes B)^*|^{\frac{1}{2}}|A \boxtimes B||(A \boxtimes B)^*|^{\frac{1}{2}}\right)^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 4.13, the operator $A \boxtimes B$ is w-hyponormal. ☐

Corollary 4.14. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be invertible operators. If A and B are iw-hyponormal, then $A \boxtimes B$ is also iw-hyponormal.*

Proof. It follows from Lemma 4.1, Proposition 4.5 and the fact that every iw-hyponormal operator is w-hyponormal and every invertible w-hyponormal operator is iw-hyponormal ([28]). ☐

4.3.3 Paranormality

Consider the following paranormality of operators; see [17, 18, 29, 33].

Definition 4.15. Let $M \geqslant 1$ be a constant. An operator $T \in \mathbb{B}(\mathcal{H})$ is said to be

- *M-paranormal* if $M^2 T^{*2} T^2 - 2\alpha T^* T + \alpha^2 I \ \geqslant\ 0$ for all $\alpha > 0$;
- *paranormal* if $T^{*2} T^2 - 2\alpha T^* T + \alpha^2 I \ \geqslant\ 0$ for all $\alpha > 0$;
- *M^*-paranormal* if $M^2 T^{*2} T^2 - 2\alpha T T^* + \alpha^2 I \ \geqslant\ 0$ for all $\alpha > 0$;
- *$*$-paranormal* if $T^{*2} T^2 - 2\alpha T T^* + \alpha^2 I \ \geqslant\ 0$ for all $\alpha > 0$.

Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is an isometry if $T^*T = I$; it is called an involution if $T^2 = I$.

Proposition 4.16. *Let $A \in \mathbb{B}(\mathcal{H})$, $X \in \mathbb{B}(\mathcal{K})$ and let $M \geqslant 1$ be a constant. If X is an isometry and A is M-paranormal (resp. paranormal), then $A \boxtimes X$ and $X \boxtimes A$ are M-paranormal (resp. paranormal).*

Proof. Assume that A is M-paranormal and X is an isometry. It follows that for any $\alpha > 0$ we have

$$\begin{aligned} M^2(A \boxtimes X)^{*2}(A \boxtimes X)^2 - 2\alpha(A \boxtimes X)^*(A \boxtimes X) + \alpha^2(I \boxtimes I) \\ &= M^2 A^{*2} A^2 \boxtimes X^{*2} X^2 - 2\alpha A^* A \boxtimes X^* X + \alpha^2 I \boxtimes I \\ &= M^2 A^{*2} A^2 \boxtimes I - 2\alpha A^* A \boxtimes I + \alpha^2 I \boxtimes I \\ &= \big(M^2 A^{*2} A^2 - 2\alpha A^* A + \alpha^2 I\big) \boxtimes I \\ &\geqslant 0. \end{aligned}$$

Thus $A \boxtimes X$ is M-paranormal. Similarly, the operator $X \boxtimes A$ is M-paranormal. The case of paranormality is just the case of M-paranormality when $M = 1$. □

Proposition 4.17. *Let $A \in \mathbb{B}(\mathcal{H})$, $X \in \mathbb{B}(\mathcal{K})$ and let $M \geqslant 1$ be a constant. If X is a self-adjoint involution and A is an M^*-paranormal (resp. $*$-paranormal) operator, then $A \boxtimes X$ and $X \boxtimes A$ are M-paranormal (resp. $*$-paranormal).*

Proof. The proof is similar to that of Proposition 4.16. □

Ando [17] showed that for any paranormal operator A, the tensor products $A \otimes I$ and $I \otimes A$ are paranormal. The next result is an extension of this fact to the case of Tracy-Singh products.

Corollary 4.18. *Let $A \in \mathbb{B}(\mathcal{H})$ and let $M \geqslant 1$ be a constant. If A satisfies one of the following properties, then the same property hold for $A \boxtimes I$ and $I \boxtimes A$: paranormal, M-paranormal, $*$-paranormal, M^*-paranormal.*

4.4 Tracy-Singh products and operators of type nilpotent, Hermitian, and isometry

In this section, we discuss relationship between Tracy-Singh products and certain classes of operators, namely, nilpotent operators, (skew)-Hermitian operators, (co)isometry operators, and unitary operators. Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is said to be nilpotent if $T^k = 0$ for some natural number k.

Proposition 4.19. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. Then $A \boxtimes B$ is nilpotent if and only if A or B is nilpotent.*

Proof. It follows directly from Lemmas 4.1 and 4.4. □

Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is Hermitian if $T^* = T$, and T is skew-Hermitian if $T^* = -T$. It follows from Lemma 4.1 that the Tracy-Singh product of Hermitian operators is also Hermitian. The Tracy-Singh product of two skew-Hermitian operators is Hermitian. The Tracy-Singh product between a Hermitian operator and a skew-Hermitian operator is skew-Hermitian.

Proposition 4.20. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be nonzero operators.*

1. *Assume $A \boxtimes B$ is Hermitian. Then A is Hermitian (resp. skew-Hermitian) if and only if B is Hermitian (resp. skew-Hermitian).*

2. *Assume $A \boxtimes B$ is skew-Hermitian. Then A is Hermitian (resp. skew-Hermitian) if and only if B is skew-Hermitian (resp. Hermitian).*

Proof. It follows directly from Lemmas 4.1 and 4.4. □

Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is a coisometry if $TT^* = I$. A unitary operator is an operator which is both an isometry and a coisometry. Stochel [36] gave a necessary and sufficient condition for $A \otimes B$ to be an isometry (resp. a coisometry, unitary). Now, we will extend this result to the case of Tracy-Singh products.

Proposition 4.21. *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ be operator matrices such that A_{ij} and B_{kl} are nonzero operators for all $i, j = 1, \ldots, m$ and $k, l = 1, \ldots, n$. Then $A \boxtimes B$ is an isometry (resp. a coisometry) if and only if so are αA and $\alpha^{-1}B$ for some $\alpha \in \mathbb{C} \setminus \{0\}$.*

Proof. If αA and $\alpha^{-1}B$ are isometries, then by Lemma 4.1,

$$\begin{aligned}(A \boxtimes B)^*(A \boxtimes B) &= A^*A \boxtimes B^*B \\ &= (\alpha A)^*(\alpha A) \boxtimes (\alpha^{-1}B)^*(\alpha^{-1}B) \\ &= I \boxtimes I.\end{aligned}$$

Suppose that $A \boxtimes B$ is an isometry. Then $A^*A \boxtimes B^*B = I \boxtimes I$. Thus, by Proposition 4.6, there exists $\beta \in \mathbb{C} \setminus \{0\}$ such that $\beta A^*A = I$ and $\beta^{-1}B^*B = I$. Setting $\alpha = \sqrt{\beta}$, we obtain $(\alpha A)^*(\alpha A) = I$ and $(\alpha^{-1}B)^*(\alpha^{-1}B) = I$. Hence αA and $\alpha^{-1}B$ are isometries. The proof for the case of coisometry is similar to that of isometry. □

Theorem 4.6. *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ be operator matrices such that A_{ij} and B_{kl} are nonzero operators for all $i, j = 1, \ldots, m$ and $k, l = 1, \ldots, n$. Then $A \boxtimes B$ is unitary if and only if so are αA and $\alpha^{-1}B$ for some $\alpha \in \mathbb{C} \setminus \{0\}$.*

Proof. If αA and $\alpha^{-1}B$ are unitary, then Lemma 4.1 implies

$$(A \boxtimes B)^*(A \boxtimes B) = A^*A \boxtimes B^*B = (\alpha A)^*(\alpha A) \boxtimes (\alpha^{-1}B)^*(\alpha^{-1}B) = I.$$

Similarly, we have $(A \boxtimes B)(A \boxtimes B)^* = I$. Conversely, suppose that $A \boxtimes B$ is unitary. We know that $A \boxtimes B$ is both an isometry and a coisometry. By Proposition 4.21, there exist $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ such that αA and $\alpha^{-1}B$ are isometries, and βA and $\beta^{-1}B$ are coisometries. We have $(\alpha A)^*(\alpha A) = I = (\beta A)(\beta A)^*$ and

$$(\alpha^{-1}B)^*(\alpha^{-1}B) = I = (\beta^{-1}B)(\beta^{-1}B)^*.$$

Since $A \boxtimes B$ is normal, so are A and B (Theorem 4.1). Then $\alpha^2 AA^* = \alpha^2 A^*A = \beta^2 AA^*$ and $\alpha^{-2}BB^* = \alpha^{-2}B^*B = \beta^{-2}BB^*$. Since $\alpha, \beta > 0$, it comes to the conclusion that $\alpha = \beta$. Hence αA and $\alpha^{-1}B$ are unitary. □

4.5 Tracy-Singh products and class-$\mathcal{A}$ type operators

The following classes of operators bring attention to operator theorists; see more information in [23, 24, 26, 27, 35].

Definition 4.22. Let $k \in \mathbb{N}$. An operator $T \in \mathbb{B}(\mathcal{H})$ is said to be

- *class* $B(\mathcal{H})$ if $|T^2| \geqslant |T|^2$;
- *class* $B(\mathcal{H})(k)$ if $\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geqslant |T|^2$;
- *quasi-class* $B(\mathcal{H})$ if $T^*|T^2|T \geqslant T^*|T|^2T$;
- *quasi-class* $(B(\mathcal{H}), k)$ if $T^{*k}|T^2|T^k \geqslant T^{*k}|T|^2T^k$;
- $*$-*class* $B(\mathcal{H})$ if $|T^2| \geqslant |T^*|^2$;
- *quasi-$*$-class* $B(\mathcal{H})$ if $T^*|T^2|T \geqslant T^*|T^*|^2T$;
- *quasi-$*$-class* $(B(\mathcal{H}), k)$ if $T^{*k}|T^2|T^k \geqslant T^{*k}|T^*|^2T^k$.

The next theorem shows that such classes of operators are preserved under Tracy-Singh products.

Theorem 4.7. *Let $A \in \mathbb{B}(\mathcal{H})$, $B \in \mathbb{B}(\mathcal{K})$, and let $k \in \mathbb{N}$. If both A and B satisfy one of the following properties, then the same property holds for $A \boxtimes B$: class $B(\mathcal{H})(k)$, class $B(\mathcal{H})$, quasi-class $(B(\mathcal{H}), k)$, quasi-class $B(\mathcal{H})$, $*$-class $B(\mathcal{H})$, quasi-$*$-class $B(\mathcal{H})$, quasi-$*$-class $(B(\mathcal{H}), k)$.*

Proof. Assume that A and B are class $B(\mathcal{H})(k)$. By Lemmas 4.1 and 4.3, we get

$$\begin{aligned}
[(A \boxtimes B)^*|A \boxtimes B|^{2k}(A \boxtimes B)]^{\frac{1}{k+1}} &= \left[(A^* \boxtimes B^*)\left(|A|^{2k} \boxtimes |B|^{2k}\right)(A \boxtimes B)\right]^{\frac{1}{k+1}} \\
&= \left(A^*|A|^{2k}A \boxtimes B^*|B|^{2k}B\right)^{\frac{1}{k+1}} \\
&= \left(A^*|A|^{2k}A\right)^{\frac{1}{k+1}} \boxtimes \left(B^*|B|^{2k}B\right)^{\frac{1}{k+1}} \\
&\geqslant |A|^2 \boxtimes |B|^2 \\
&= |A \boxtimes B|^2.
\end{aligned}$$

Hence $A \boxtimes B$ is a class $B(\mathcal{H})(k)$ operator. Now, assume that A and B are quasi-class $B(\mathcal{H})(k)$. Applying Lemmas 4.1 and 4.3, we get

$$\begin{aligned}(A \boxtimes B)^{k*}|(A \boxtimes B)^2|(A \boxtimes B)^k &= (A^{k*} \boxtimes B^{k*})(|A^2| \boxtimes |B^2|)(A^k \boxtimes B^k) \\ &= A^{k*}|A^2|A^k \boxtimes B^{k*}|B^2|B^k \\ &\geqslant A^{k*}|A|^2A^k \boxtimes B^{k*}|B|^2B^k \\ &= (A^{k*} \boxtimes B^{k*})(|A|^2 \boxtimes |B|^2)(A^k \boxtimes B^k) \\ &= (A \boxtimes B)^{k*}|A \boxtimes B|^2(A \boxtimes B)^k.\end{aligned}$$

Hence, $A \boxtimes B$ is a quasi-class $B(\mathcal{H})(k)$ operator. The proof for class $B(\mathcal{H})$ (resp. quasi-class $B(\mathcal{H})$) is done by replacing $k = 1$ in the case of class $B(\mathcal{H})(k)$ (resp. quasi-class $(B(\mathcal{H}), k)$). The proof for the case of quasi $*$-class $(B(\mathcal{H}), k)$ is similar to that of quasi-class $(B(\mathcal{H}), k)$. Similarly, the proof for $*$-class $B(\mathcal{H})$ (resp. quasi-$*$-class A) is done by replacing $k = 0$ (resp. $k = 1$) in the case of quasi-$*$-class $(B(\mathcal{H}), k)$. □

Chapter 5

Tracy-Singh Products and Geometric Means for Positive Operators

In this chapter, we investigate relationship between metric/spectral/Sagae-Tanabe geometric means for several positive operators and Tracy-Singh products in terms of identities and inequalities. In particular, we obtain various generalizations of arithmetic-geometric-harmonic means inequality and its reverse. Moreover, we introduce the weighted Sagae-Tanabe spectral geometric mean for several positive operators and deduce its properties related to Tracy-Singh products.

5.1 Introduction

It is well known that the tensor product (or Kronecker product) plays a fundamental role in linear algebra, functional analysis and related fields. Nowadays, theory of tensor products of operators is still developing, see [55] for instance. Recently, the authors of [49] introduced the Tracy-Singh product of operators which generalizes both tensor products of operators and Tracy-Singh products of complex matrices [54].

On the other hand, geometric means of positive definite matrices arise naturally in several areas of pure and applied mathematics. There are at least two types of geometric means. The first one is the metric geometric mean, introduced by Ando [40]. Recall that the set of n-by-n positive definite matrices is a Riemannian manifold, in which the Riemannian metric between two matrices A and B is given by

$$\delta(A,B) \;=\; \|\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\|_2,$$

here, $\|\cdot\|_2$ denotes the Frobenius norm. The metric geometric mean $A\#B$ is indeed the unique metric midpoint of the geodesic line containing A and B (see, e.g., [46]). The second one is the spectral geometric mean $\natural$, introduced by Fiedler and Pták [43]. In fact, the square of $A\natural B$ is similar to the product AB, and in particular, its eigenvalues coincide with the positive square roots of the eigenvalues of AB. See more information about metric/spectral geometric means in [41, 47] and [42, Chapters 4 and 6].

These two kinds of geometric means can be extended to multiple matrices by iterative processes, see e.g. [45]). Another such iterative geometric mean is the Sagae-Tanabe geometric mean, introduced in [52]. One of the most interesting properties of geometric means is the arithmetic-geometric-harmonic means (AM-GM-HM) inequalities. Indeed, the AM-GM-HM inequality concerning the metric geometric mean was established in [41]. Another version concerning the Sagae-Tanabe geometric mean were discussed in [52, 38].

The Kronecker product of matrices turns out to be compatible with the metric geometric mean in the sense that

$$(A\#B)\otimes(C\#D) \;=\; (A\otimes C)\#(B\otimes D) \tag{5.1}$$

for positive semidefinite matrices A, B, C, D of appropriate sizes (see [41]). Of course, the similar result for the spectral geometric mean is also true ([45]). Mooreover, Kilicman and Al Zhour [45] discussed relations between Tracy-Singh products and metric geometric means, spectral geometric means, and Sagae-Tanabe geometric means of several positive definite matrices.

In this paper, we develop further theory for geometric means of Hilbert space operators. We investigate relationship between metric/spectral/Sagae-Tanabe geometric means and Tracy-Singh products in terms of operator identities and inequalities. In particular, we obtain various generalizations of the property (5.1) and the AM-GM-HM inequality and its reverse. Moreover, we introduce the weighted Sagae-Tanabe spectral geometric mean for several positive operators and deduce its properties related to Tracy-Singh products.

This paper is organized as follows. Section 4.2 consists of prerequisites on Tracy-Singh and Khatri-Rao products for Hilbert space operators. In Section

5.3, we establish certain identities and inequalities between metric geometric means and Tracy-Singh products of several positive operators. Sections 5.4 and 5.5 deal with spectral geometric means and Sagae-Tanabe metric geometric means, respectively. In Section 5.6, we define the weighted Sagae-Tanabe spectral geometric mean for several positive operators and prove certain results related to Tracy-Singh products.

5.2 Preliminaries

Throughout this paper, let $\mathcal{H}$ and $\mathcal{K}$ be complex separable Hilbert spaces. When X and Y are Hilbert spaces, denote by $\mathbb{B}(X,Y)$ the Banach space of bounded linear operators from X into Y, and abbreviate $\mathbb{B}(X,X)$ to $\mathbb{B}(X)$. For Hermitian operators $A,B \in \mathbb{B}(\mathcal{H})$, the notation $A \geqslant B$ means that $A-B$ is a positive operator, while $A > 0$ indicates that A is positive and invertible.

5.2.1 Tracy-Singh products of operators

The projection theorem for Hilbert spaces allows us to decompose

$$\mathcal{H} \;=\; \bigoplus_{i=1}^{m} \mathcal{H}_i, \quad \mathcal{K} \;=\; \bigoplus_{l=1}^{n} \mathcal{K}_k$$

where all $\mathcal{H}_i$ and $\mathcal{K}_k$ are Hilbert spaces. Each operator $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ thus can be expressed uniquely as operator matrices

$$A \;=\; [A_{ij}]_{i,j=1}^{m,m} \text{ and } B \;=\; [B_{kl}]_{k,l=1}^{n,n}$$

where $A_{ij} \in \mathbb{B}(\mathcal{H}_j,\mathcal{H}_i)$ and $B_{kl} \in \mathbb{B}(\mathcal{K}_l,\mathcal{K}_k)$ for each $i,j = 1,\ldots,m$ and $k,l = 1,\ldots,n$.

Definition 5.1. Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ be operator matrices defined as above. The *Tracy-Singh product* of A and B is defined to be the operator matrix

$$A \boxtimes B \;=\; \left[[A_{ij} \otimes B_{kl}]_{kl}\right]_{ij}, \tag{5.2}$$

which is a bounded linear operator from $\bigoplus_{i,k=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_k$ into itself.

Note that if $m = n = 1$, the Tracy-Singh product $A \boxtimes B$ reduces to the tensor product $A \otimes B$.

Lemma 5.2 ([49, 50]). *Let $A, C \in \mathbb{B}(\mathcal{H})$ and $B, D \in \mathbb{B}(\mathcal{K})$ be compatible operator matrices.*

(i) The Tracy-Singh product is compatible with the usual product in the sense that

$$(A \boxtimes B)(C \boxtimes D) \;=\; AC \boxtimes BD.$$

(ii) If $A, B > 0$, then $(A \boxtimes B)^{\alpha} \;=\; A^{\alpha} \boxtimes B^{\alpha}$ for any real number α.

(iii) If $A, B \geqslant 0$, then $A \boxtimes B \geqslant 0$.

(iv) If $A, B > 0$, then $A \boxtimes B > 0$.

(v) If $A \geqslant C \geqslant 0$ and $B \geqslant D \geqslant 0$, then $A \boxtimes B \;\geqslant\; C \boxtimes D$.

5.2.2 Compatibility of Khatri-Rao products with algebraic operations

The notion of Khstri-Rao product for operators was introduced in [48]. We now recall the materials on that paper.

From now on, fix the following orthogonal decompositions of Hilbert spaces:

$$\mathcal{H} \;=\; \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \mathcal{H}' \;=\; \bigoplus_{i=1}^{m} \mathcal{H}'_i, \quad \mathcal{K} \;=\; \bigoplus_{j=1}^{n} \mathcal{K}_j, \quad \mathcal{K}' \;=\; \bigoplus_{i=1}^{m} \mathcal{K}'_i. \tag{5.3}$$

That is, we fix how to partition any operator matrix in $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $\mathbb{B}(\mathcal{K}, \mathcal{K}')$. We now extend the Khatri-Rao product of matrices ([56]) to that of operators on a Hilbert space.

Definition 5.3. Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B = [B_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operators partitioned into matrices according to the decomposition (5.3). We define the *Khatri-Rao product* of A and B to be the bounded linear operator from $\bigoplus_{j=1}^{n} \mathcal{H}_j \otimes \mathcal{K}_j$ to $\bigoplus_{i=1}^{m} \mathcal{H}'_i \otimes \mathcal{K}'_i$ represented by the block-matrix

$$A \boxdot B \;=\; [A_{ij} \otimes B_{ij}]_{i,j=1}^{m,n}. \tag{5.4}$$

If both A and B are 1×1 block operator matrices, then $A \boxdot B$ is $A \otimes B$. When $\mathcal{H}_i = \mathcal{K}_i = \mathbb{C}$ and $\mathcal{H}'_j = \mathcal{K}'_j = \mathbb{C}$ for all i, j, the Khatri-Rao product is the Hadamard product of complex matrices.

Note that if $m = 1$, then $A \boxdot B = A \otimes B$. We set $\boxtimes_{i=1}^{1} A_i = A_1 = \boxdot_{i=1}^{1} A_i$. For $r \in \mathbb{N} - \{1\}$ and a finite number of operator matrices $A_i \in \mathbb{B}(\mathcal{H}_i)$ $(i = 1, \dots, r)$, denote

$$\begin{aligned} \boxtimes_{i=1}^{r} A_i &= ((A_1 \boxtimes A_2) \boxtimes \cdots \boxtimes A_{r-1}) \boxtimes A_r, \\ \boxdot_{i=1}^{r} A_i &= ((A_1 \boxdot A_2) \boxdot \cdots \boxdot A_{r-1}) \boxdot A_r. \end{aligned}$$

Next we shall show that the Khatri-Rao product of two linear maps induced by matrices is just the linear map induced by the Khatri-Rao product of these matrices. Recall that, for each $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, the induced maps

$$L_A : \mathbb{C}^n \to \mathbb{C}^m, \; x \mapsto Ax, \quad \text{and} \quad L_B : \mathbb{C}^q \to \mathbb{C}^p, \; y \mapsto By,$$

are bounded linear operators. We identify $\mathbb{C}^n \otimes \mathbb{C}^q$ with $\mathbb{C}^{nq}$ together with the canonical bilinear map $(x, y) \mapsto x \hat{\otimes} y$ for each $(x, y) \in \mathbb{C}^n \times \mathbb{C}^q$.

Lemma 5.4. *For any $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, we have*

$$L_A \otimes L_B = L_{A \hat{\otimes} B}. \tag{5.5}$$

Proof. Recall that the Kronecker product of matrices has the following property (see, *e.g.*, [12]):

$$(A \hat{\otimes} B)(C \hat{\otimes} D) = AC \hat{\otimes} BD$$

provided that all matrix products are well-defined. It follows that for any $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^q$,

$$\begin{aligned} (L_A \otimes L_B)(x \otimes y) &= L_A(x) \otimes L_B(y) = L_A(x) \hat{\otimes} L_B(y) \\ &= Ax \hat{\otimes} By = (A \hat{\otimes} B)(x \hat{\otimes} y) \\ &= (A \hat{\otimes} B)(x \otimes y) = L_{A \hat{\otimes} B}(x \otimes y). \end{aligned}$$

The uniqueness of tensor products implies that $L_A \otimes L_B = L_{A \hat{\otimes} B}$. $\square$

Proposition 5.5. *For any complex matrices $A = [A_{ij}]$ and $B = [B_{ij}]$ partitioned in block-matrix form, we have*

$$L_A \boxdot L_B \;=\; L_{A \hat{\boxdot} B}. \tag{5.6}$$

Proof. Recall that the (i,j)th block of the matrix representation of L_A is $L_{A_{ij}}$. By Lemma 5.4, we obtain $L_A \boxdot L_B = \left[L_{A_{ij}} \otimes L_{B_{ij}}\right]_{ij} = \left[L_{A_{ij} \hat{\otimes} B_{ij}}\right]_{ij} = L_{A \hat{\boxdot} B}$. □

The next result states that the Khatri-Rao product is bilinear and compatible with the adjoint operation.

Proposition 5.6. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B, C \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices, and let $\alpha \in \mathbb{C}$. Then*

$$(A \boxdot B)^* \;=\; A^* \boxdot B^*, \tag{5.7}$$
$$A \boxdot (B + C) \;=\; A \boxdot B + A \boxdot C, \tag{5.8}$$
$$(B + C) \boxdot A \;=\; B \boxdot A + C \boxdot A, \tag{5.9}$$
$$(\alpha A) \boxdot B \;=\; \alpha(A \boxdot B) \;=\; A \boxdot (\alpha B). \tag{5.10}$$

Proof. Since $A^* = \left[A^*_{ji}\right]_{ij}$ and $B^* = \left[B^*_{ji}\right]_{ij}$, we obtain

$$(A \boxdot B)^* \;=\; \left[(A_{ij} \otimes B_{ij})^*\right]_{ij} \;=\; \left[A^*_{ji} \otimes B^*_{ji}\right]_{ij} \;=\; A^* \boxdot B^*.$$

The fact that $(B+C)_{ij} = B_{ij} + C_{ij}$ for all i, j together with the left distributivity of the tensor product over the addition imply

$$\begin{aligned} A \boxdot (B + C) \;&=\; \left[A_{ij} \otimes (B_{ij} + C_{ij})\right]_{ij} \\ &=\; \left[(A_{ij} \otimes B_{ij}) + (A_{ij} \otimes C_{ij})\right]_{ij} \\ &=\; A \boxdot B + A \boxdot C. \end{aligned}$$

Similarly, we obtain the property (5.9). Since $(\alpha A)_{ij} = \alpha A_{ij}$ for all i, j, we get

$$(\alpha A) \boxdot B \;=\; \left[(\alpha A_{ij}) \otimes B_{ij}\right]_{ij} \;=\; \left[\alpha\,(A_{ij} \boxdot B_{ij})\right]_{ij} \;=\; \alpha(A \boxdot B).$$

Similarly, $A \boxdot (\alpha B) = \alpha(A \boxdot B)$. □

By property (5.7), the selfadjointness of operators is closed under taking Khatri-Rao products, *i.e.*, if A and B are selfadjoint, then so is $A \boxdot B$. The next proposition shows that in order to compute the Khatri-Rao of operator matrices, we can freely merges the partition of each operator.

Proposition 5.7. *Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B = [B_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices represented according to the decomposition (5.3). We merge the partition of A to be $A = \left[A^{kl}\right]_{k,l=1}^{r,s}$ where r, s are given natural numbers such that $r \leqslant m$ and $s \leqslant n$. Here, each operator A^{kl} is of $m_k \times n_l$ block in which the (g,h)th block of A^{kl} is the (u,v)th block of A, where*

$$u = \begin{cases} g, & k = 1 \\ \sum_{i=1}^{k-1} m_i + g, & k > 1 \end{cases}, \quad \sum_{k=1}^{r} m_k = m,$$

$$v = \begin{cases} h, & l = 1 \\ \sum_{j=1}^{l-1} n_j + h, & l > 1 \end{cases}, \quad \sum_{l=1}^{s} n_l = n.$$

Similarly, we repartition $B = \left[B^{kl}\right]_{k,l=1}^{r,s}$ where each operator B^{kl} is of $m_k \times n_l$ block in which the (g,h)th block of B^{kl} is the (u,v)th block of B. Then

$$A \boxdot B = \left[A^{kl} \boxdot B^{kl}\right]_{kl} = \begin{bmatrix} A^{11} \boxdot B^{11} & \cdots & A^{1s} \boxdot B^{1s} \\ \vdots & \ddots & \vdots \\ A^{r1} \boxdot B^{r1} & \cdots & A^{rs} \boxdot B^{rs} \end{bmatrix}.$$

That is, each (k,l)th block of $A \boxdot B$ is just $A^{kl} \boxdot B^{kl}$.

Proof. Write $A \boxdot B = \left[C^{kl}\right]_{k,l=1}^{r,s}$ where C^{kl} is $m_k \times n_l$ block operator matrix such that the (g,h)th block of C^{kl} is the (u,v)th block of $A \boxdot B$. We know that the (u,v)th block of $A \boxdot B$ is $A_{uv} \otimes B_{uv}$. Then

$$\begin{aligned} C^{11} &= \begin{bmatrix} A_{11} \otimes B_{11} & \cdots & A_{1n_1} \otimes B_{1n_1} \\ \vdots & \ddots & \vdots \\ A_{m_1 1} \otimes B_{m_1 1} & \cdots & A_{m_1 n_1} \otimes B_{m_1 n_1} \end{bmatrix} \\ &= \begin{bmatrix} A_{11} & \cdots & A_{1n_1} \\ \vdots & \ddots & \vdots \\ A_{m_1 1} & \cdots & A_{m_1 n_1} \end{bmatrix} \boxdot \begin{bmatrix} B_{11} & \cdots & B_{1n_1} \\ \vdots & \ddots & \vdots \\ B_{m_1 1} & \cdots & B_{m_1 n_1} \end{bmatrix} \\ &= A^{11} \boxdot B^{11}. \end{aligned}$$

Similarly, we have $C^{kl} = A^{kl} \boxdot B^{kl}$ for all $k = 1, \dots, r$ and $l = 1, \dots, s$. $\square$

Recall that the direct sum of $A_i \in \mathbb{B}(\mathcal{H}_i, \mathcal{H}_i')$, $i = 1, \ldots, n$, is defined to be the operator matrix

$$A_1 \oplus \cdots \oplus A_n \;=\; \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}.$$

The next result shows that the Khatri-Rao product is compatible with the direct sum of operators.

Proposition 5.8. *For each $i = 1, \ldots, n$, let $A_i \in \mathbb{B}(\mathcal{H}_i, \mathcal{H}_i')$ and $B_i \in \mathbb{B}(\mathcal{K}_i, \mathcal{K}_i')$ be compatible operator matrices. Then*

$$\left(\bigoplus_{i=1}^{n} A_i \right) \boxdot \left(\bigoplus_{i=1}^{n} B_i \right) \;=\; \bigoplus_{i=1}^{n} (A_i \boxdot B_i). \tag{5.11}$$

Proof. It follows directly from Proposition 5.7. $\square$

In summary, the Khatri-Rao product is compatible with fundamental algebraic operations for operators.

5.2.3 The Khatri-Rao product as a generalization of the Hadamard product

In this section, we explain how the Khatri-Rao product can be viewed as a generalization of the Hadamard product. To do this, we construct two isometries which identify which blocks of the Tracy-Singh product we need to get the Khatri-Rao product.

Fix a countable orthonormal basis $\mathcal{E}$ for $\mathcal{H}$. Recall that the Hadamard product of A and B in $\mathbb{B}(\mathcal{H})$ is defined to be the operator $A \odot B$ in $\mathbb{B}(\mathcal{H})$ such that

$$\langle (A \odot B)e, e \rangle \;=\; \langle Ae, e \rangle \langle Be, e \rangle$$

for all $e \in \mathcal{E}$. More explicitly, it was shown in [67] that

$$A \odot B \;=\; U^*(A \otimes B)U \tag{5.12}$$

where $U : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is the isometry defined by $Ue = e \otimes e$ for all $e \in \mathcal{E}$. When $\mathcal{H} = \mathbb{C}^n$ and $\mathcal{E}$ is the standard ordered basis of $\mathbb{C}^n$, the Hadamard product of two matrices reduces to the entrywise product (**??**).

We now extend selection matrices in [61] to selection operators. Fix an ordered 4-tuple $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ of Hilbert spaces endowed with decompositions (5.3). For each $r = 1, \dots, m$, consider the operator matrix

$$E_r = \left[E_{gh}^{(r)}\right]_{g,h=1}^{m,m} : \bigoplus_{i=1}^{m} \mathcal{H}_i' \otimes \mathcal{K}_i' \to \bigoplus_{i=1}^{m} \mathcal{H}_r' \otimes \mathcal{K}_i',$$

where $E_{gh}^{(r)}$ is the identity operator if $g = h = r$ and the others are zero operators. Similarly, for $s = 1, ..., n$, we define the operator matrix

$$F_s = \left[F_{gh}^{(s)}\right]_{g,h=1}^{n,n} : \bigoplus_{j=1}^{n} \mathcal{H}_j \otimes \mathcal{K}_j \to \bigoplus_{j=1}^{n} \mathcal{H}_s \otimes \mathcal{K}_j,$$

where $F_{gh}^{(s)}$ is the identity operator if $g = h = s$ and the others are zero operators. Now construct

$$Z_1 = \begin{bmatrix} E_1 \\ \vdots \\ E_m \end{bmatrix} \quad \text{and} \quad Z_2 = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}. \tag{5.13}$$

We call Z_1 and Z_2 *selection operators* associated with the ordered tuple $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$. Notice that Z_1 depends only on the ordered tuple $(\mathcal{H}', \mathcal{K}')$ and how we decomposed $\mathcal{H}'$ and $\mathcal{K}'$. The operator Z_2 depends on $(\mathcal{H}, \mathcal{K})$ and how we decomposed $\mathcal{H}$ and $\mathcal{K}$. For instance, an ordered tuple $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ with decompositions

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3, \;\; \mathcal{H}' = \mathcal{H}_1' \oplus \mathcal{H}_2', \;\; \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3, \;\; \mathcal{K}' = \mathcal{K}_1' \oplus \mathcal{K}_2'$$

has the following selection operators:

$$Z_1 = \begin{bmatrix} I_{\mathcal{H}_1' \otimes \mathcal{K}_1'} \oplus 0 \\ 0 \oplus I_{\mathcal{H}_2' \otimes \mathcal{K}_2'} \end{bmatrix}, \quad Z_2 = \begin{bmatrix} I_{\mathcal{H}_1 \otimes \mathcal{K}_1} \oplus 0 \oplus 0 \\ 0 \oplus I_{\mathcal{H}_2 \otimes \mathcal{K}_2} \oplus 0 \\ 0 \oplus 0 \oplus I_{\mathcal{H}_3 \otimes \mathcal{K}_3} \end{bmatrix}.$$

In the case of $\mathcal{H} = \mathcal{H}'$ and $\mathcal{K} = \mathcal{K}'$, the construction (5.13) gives

$$Z_1 = Z_2 = Z. \tag{5.14}$$

If (Z_1, Z_2) is the ordered pair of selection operators associated with the ordered tuple $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ with decompositions given by (5.3), then (Z_2, Z_1) is the ordered pair of selection operators associated with the ordered collection $(\mathcal{H}', \mathcal{H}, \mathcal{K}', \mathcal{K})$ with the same decompositions.

Lemma 5.9. *Let Z_1 and Z_2 be selection operators defined by* (5.13)*. Then for $i = 1, 2$,*

(i) $Z_i^* Z_i = I$, *i.e., Z_i is an isometry,*

(ii) $0 \leqslant Z_i Z_i^* \leqslant I$.

Proof. A direct computation shows that $Z_1^* Z_1 = I$ and $Z_2^* Z_2 = I$. We know that $E_i E_i^*$ is an $m \times m$ block operator matrix which consists only of zero and identity operators. More precisely, the (i, i)th block of $E_i E_i^*$ is the identity operator and $E_i E_j^* = 0$ for $i \neq j$. Then

$$Z_1 Z_1^* = \begin{bmatrix} E_1E_1^* & E_1E_2^* & \cdots & E_1E_m^* \\ E_2E_1^* & E_2E_2^* & \cdots & E_2E_m^* \\ \vdots & \vdots & \ddots & \vdots \\ E_mE_1^* & E_mE_2^* & \cdots & E_mE_m^* \end{bmatrix} = \begin{bmatrix} E_1E_1^* & 0 & \cdots & 0 \\ 0 & E_2E_2^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_mE_m^* \end{bmatrix}.$$

Since $E_i E_i^* \leqslant I$ for all $i = 1, \dots, m$, we have $Z_1 Z_1^* \leqslant I$. Similarly, $Z_2 Z_2^* \leqslant I$. □

Next we relate the Khatri-Rao and the Tracy-Singh product of operators.

Theorem 5.1. *For any operator matrices $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$, we have*

$$A \boxdot B = Z_1^*(A \boxtimes B) Z_2, \tag{5.15}$$

where Z_1 and Z_2 are the selection operators defined by (5.13)*. If $\mathcal{H} = \mathcal{H}'$ and $\mathcal{K} = \mathcal{K}'$, $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$, then*

$$A \boxdot B = Z^*(A \boxtimes B) Z, \tag{5.16}$$

where Z is the selection operator defined by (5.14)*.*

Proof. Let $B(i)$ denote the ith column of B for $i = 1, \ldots, n$. Then we have

$$\begin{aligned}
Z_1^*(A \boxtimes B)Z_2 &= \begin{bmatrix} E_1^* & \cdots & E_m^* \end{bmatrix} \begin{bmatrix} A_{11} \boxtimes B & \cdots & A_{1n} \boxtimes B \\ \vdots & \ddots & \vdots \\ A_{m1} \boxtimes B & \cdots & A_{mn} \boxtimes B \end{bmatrix} \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix} \\
&= \begin{bmatrix} E_1^* & \cdots & E_m^* \end{bmatrix} \begin{bmatrix} (A_{11} \boxtimes B)F_1 + \ldots + (A_{1n} \boxtimes B)F_n \\ \vdots \\ (A_{m1} \boxtimes B)F_1 + \ldots + (A_{mn} \boxtimes B)F_n \end{bmatrix} \\
&= \begin{bmatrix} E_1^* & \cdots & E_m^* \end{bmatrix} \begin{bmatrix} A_{11} \boxtimes B(1) & \cdots & A_{1n} \boxtimes B(n) \\ & \vdots & \\ A_{m1} \boxtimes B(1) & \cdots & A_{mn} \boxtimes B(n) \end{bmatrix} \\
&= \begin{bmatrix} A_{11} \otimes B_{11} & \cdots & A_{1n} \otimes B_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} \otimes B_{m1} & \cdots & A_{mn} \otimes B_{mn} \end{bmatrix} \\
&= A \boxdot B.
\end{aligned}$$

If $\mathcal{H} = \mathcal{H}'$ and $\mathcal{K} = \mathcal{K}'$, then $Z_1 = Z_2$ and (5.15) becomes (5.16). □

We mention that Theorem 5.1 is an extension of both [61, Theorem 3] and the result (5.12) in [67].

Remark 5.10. *If we partition A and B into row operator matrices, we have*

$$A \boxdot B \;=\; (A \boxtimes B)Z_2.$$

If both A and B are column operator matrices, then

$$A \boxdot B \;=\; Z_1^*(A \boxtimes B).$$

Comparing (5.12) and (5.16), Theorem 5.1 shows that the Khatri-Rao product can be regarded as a generalization of the Hadamard product.

Recall that a map Φ between two C*-algebras is said to be positive if Φ preserves positive elements. The map Φ is unital if Φ preserves the multiplicative identity.

Corollary 5.11. *There is a unital positive linear map*

$$\Phi \;:\; \mathbb{B}\left(\bigoplus_{i,j=1}^{n,n} \mathcal{H}_i \otimes \mathcal{K}_j \right) \to \mathbb{B}\left(\bigoplus_{i=1}^{n} \mathcal{H}_i \otimes \mathcal{K}_i \right)$$

such that $\Phi(A \boxtimes B) = A \boxdot B$ for any $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$.

Proof. Define $\Phi(X) = Z^*XZ$, where Z is the selection operator defined by (5.16) in Theorem 5.1. The map Φ is clearly linear and positive. The map Φ is unital since Z is an isometry (Lemma 5.9). $\square$

Corollary 5.11 provides a natural way to deriving operator inequalities concerning Khatri-Rao products from existing inequalities for Tracy-Singh products.

The next result extends [68, Corollary 3] to the case of Khatri-Rao and Tracy-Singh products of operators.

Corollary 5.12. *Let $A = A_1 \oplus \cdots \oplus A_n$ and $B = B_1 \oplus \cdots \oplus B_n$ be operators in $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$, respectively. Then*

$$\begin{aligned} Z^*(A \boxtimes B) &= (A \boxdot B)Z^*, \\ (A \boxtimes B)Z &= Z(A \boxdot B). \end{aligned}$$

Proof. Using the fact that $E_iE_i^*X = X_i = XE_iE_i^*$ and $E_iE_j^*X = 0 = XE_iE_j^*$ if $i \neq j$, where $X = X_1 \oplus \cdots \oplus X_n$, we compute

$$\begin{aligned} ZZ^*(A \boxtimes B) &= \begin{bmatrix} E_1E_1^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_nE_n^* \end{bmatrix} \begin{bmatrix} A_1 \boxtimes B & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \boxtimes B \end{bmatrix} \\ &= \begin{bmatrix} E_1E_1^*(A_1 \boxtimes B) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_nE_n^*(A_n \boxtimes B) \end{bmatrix} \\ &= \begin{bmatrix} (A_1 \boxtimes B)E_1E_1^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (A_n \boxtimes B)E_nE_n^* \end{bmatrix} \\ &= \begin{bmatrix} A_1 \boxtimes B & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \boxtimes B \end{bmatrix} \begin{bmatrix} E_1E_1^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_nE_n^* \end{bmatrix} \\ &= (A \boxtimes B)ZZ^*. \end{aligned}$$

By applying Theorem 5.1, we get

$$Z^*(A \boxtimes B) = Z^*ZZ^*(A \boxtimes B) = Z^*(A \boxtimes B)ZZ^* = (A \boxdot B)Z^*.$$

Similarly, $(A \boxtimes B)Z = Z(A \boxdot B)$. $\square$

5.2.4 Positivity and monotonicity of Khatri-Rao products

In this subsection, we shows that the Khatri-Rao product preserves positivity and strict positivity. It follows that operator orderings are preserved under Khatri-Rao products.

Theorem 5.2. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be operator matrices. If $A \geqslant 0$ and $B \geqslant 0$, then $A \boxdot B \geqslant 0$.*

Proof. It follows from the positivity of the Tracy-Singh product (Lemma **??**) and Theorem 5.1. □

The next result provides the monotonicity of Khatri-Rao product which is an extension of [61, Theorem 5] to the case of operators.

Corollary 5.13. *Let $A_1, A_2 \in \mathbb{B}(\mathcal{H})$ and $B_1, B_2 \in \mathbb{B}(\mathcal{K})$. If $A_1 \geqslant A_2 \geqslant 0$ and $B_1 \geqslant B_2 \geqslant 0$, then $A_1 \boxdot B_1 \geqslant A_2 \boxdot B_2$.*

Proof. Applying Proposition 5.6 and Theorem 5.2 yields

$$\begin{aligned}(A_1 \boxdot B_1) - (A_2 \boxdot B_2) &= A_1 \boxdot B_1 - A_2 \boxdot B_1 + A_2 \boxdot B_1 - A_2 \boxdot B_2 \\ &= (A_1 - A_2) \boxdot B_1 + A_2 \boxdot (B_1 - B_2) \\ &\geqslant 0.\end{aligned}$$

Thus $A_1 \boxdot B_1 \geqslant A_2 \boxdot B_2$. □

Now, we will develop the result of [61, Theorem 6] to the case of Khatri-Rao product of operators.

Theorem 5.3. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be operator matrices. If $A > 0$ and $B > 0$, then $A \boxdot B > 0$.*

Proof. The strictly positivity of A and the spectral theorem imply the existence of an increasing sequence $(\mathcal{H}_n)_{n=1}^{\infty}$ of closed subspaces of $\mathcal{H}$ such that for each $n \in \mathbb{N}$,

$$\langle Ax, x \rangle \geqslant \frac{1}{n} \|x\|^2$$

for each $x \in \mathcal{H}_n$. Let P_n be the orthogonal projection onto $\mathcal{H}_n$ for each $n \in \mathbb{N}$. There are similar subspaces $\mathcal{K}_n$ and orthogonal projections Q_n for the operator B. Then for each $n \in \mathbb{N}$, we have $A \geqslant (1/n)P_n$ and $B \geqslant (1/n)Q_n$ and hence

$$A \boxdot B \;\geqslant\; \frac{1}{n^2} P_n \boxdot Q_n$$

by Corollary 5.13. Since the union of the subspaces $\mathcal{H}_n$ in $\mathcal{H}$ and of the subspaces $\mathcal{K}_n$ in $\mathcal{K}$ is dense, it follows that for any $z \in \mathcal{H} \otimes \mathcal{K}$, there is an $m \in \mathbb{N}$ for which $\langle (P_m \boxdot Q_m)z,\, z\rangle > 0$. Hence

$$\langle (A \boxdot B)z,\, z\rangle \;\geqslant\; \frac{1}{m^2}\langle (P_m \boxdot Q_m)z,\, z\rangle \;>\; 0.$$

This shows that $A \boxdot B > 0$. □

Corollary 5.14. *Let $A_1, A_2 \in \mathbb{B}(\mathcal{H})$ and $B_1, B_2 \in \mathbb{B}(\mathcal{K})$. If $A_1 > A_2 > 0$ and $B_1 > B_2 > 0$, then $A_1 \boxdot B_1 > A_2 \boxdot B_2$.*

Proof. The proof is similar to that of Corollary 5.13. Instead of Theorem 5.2, we apply Theorem 5.3. □

Finally, we mention that by using the results in this paper, we can develop further operator identities/inequalities parallel to matrix results for Khatri-Rao products.

Lemma 5.15 ([51])**.** *Let $r \in \mathbb{N} - \{1\}$. There exists an isometry Z such that*

$$\mathop{\boxdot}_{i=1}^{r} A_i \;=\; Z^* \left(\mathop{\boxtimes}_{i=1}^{r} A_i \right) Z \tag{5.17}$$

for any $A_i \in \mathbb{B}(\mathcal{H}_i)$, $i = 1, \ldots, r$.

5.3 Metric geometric mean

In this section, we establish certain operator identities and inequalities involving metric geometric means and Tracy-Singh products. First of all, we recall some background about metric geometric means.

The metric geometric mean for matrices/operators was firstly defined by Ando [40]:

$$A \,\#\, B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}, \quad A, B > 0. \tag{5.18}$$

This formula comes from two natural requirements. First, it should coincide with the usual geometric mean for positive real numbers: $A \,\#\, B = (AB)^{1/2}$ provided that $AB = BA$. The second condition is the congruent invariance

$$T^*(A \,\#\, B)T = (T^*AT) \,\#\, (T^*BT)$$

for any invertible $T \in \mathbb{B}(\mathcal{H})$. Now, consider positive invertible operators A and B in $\mathbb{B}(\mathcal{H})$ and let $w \in [0,1]$. The w-weighted geometric mean of A and B is defined by

$$A \#_w B \;=\; A^{1/2}(A^{-1/2}BA^{-1/2})^w A^{1/2}.$$

For arbitrary positive operators A and B, we define the w-weighted geometric mean of A and B to be

$$A \#_w B \;=\; \lim_{\varepsilon \to 0^+} (A + \varepsilon I) \#_w (B + \varepsilon I).$$

Here, the limit is taken in the strong-operator topology. For briefly, we write $A \# B$ for $A \#_{1/2} B$.

Theorem 5.4. *Let A_1, A_2, B_1 and B_2 be positive operators in $\mathbb{B}(\mathcal{H})$ and $w \in [0,1]$. Then*

$$(A_1 \#_w A_2) \boxtimes (B_1 \#_w B_2) \;=\; (A_1 \boxtimes B_1) \#_w (A_2 \boxtimes B_2), \tag{5.19}$$

$$(A_1 \#_w A_2) \boxdot (B_1 \#_w B_2) \;\leqslant\; (A_1 \boxdot B_1) \#_w (A_2 \boxdot B_2). \tag{5.20}$$

Proof. First, consider the case $A_1, A_2, B_1, B_2 > 0$. By using Lemma 5.2, we get

$$\begin{aligned}
&(A_1 \boxtimes B_1) \#_w (A_2 \boxtimes B_2) \\
&\quad = (A_1 \boxtimes B_1)^{1/2} \left[(A_1 \boxtimes B_1)^{-1/2} (A_2 \boxtimes B_2)(A_1 \boxtimes B_1)^{-1/2} \right]^w (A_1 \boxtimes B_1)^{1/2} \\
&\quad = \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \left[\left(A_1^{-1/2} \boxtimes B_1^{-1/2} \right) (A_2 \boxtimes B_2) \left(A_1^{-1/2} \boxtimes B_1^{-1/2} \right) \right]^w \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \\
&\quad = \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \left[\left(A_1^{-1/2} A_2 A_1^{-1/2} \right) \boxtimes \left(B_1^{-1/2} B_2 B_1^{-1/2} \right) \right]^w \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \\
&\quad = \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \left[\left(A_1^{-1/2} A_2 A_1^{-1/2} \right)^w \boxtimes \left(B_1^{-1/2} B_2 B_1^{-1/2} \right)^w \right] \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \\
&\quad = \left[A_1^{1/2} \left(A_1^{-1/2} A_2 A_1^{-1/2} \right)^w A_1^{1/2} \right] \boxtimes \left[B^{1/2} \left(B_1^{-1/2} B_2 B_1^{-1/2} \right)^w B_1^{1/2} \right] \\
&\quad = (A_1 \#_w A_2) \boxtimes (B_1 \#_w B_2).
\end{aligned}$$

Ando's result [41] states that if Φ is a positive linear map, then for all $A, B \geqslant 0$,

$$\Phi(A \#_w B) \leqslant \Phi(A) \#_w \Phi(B). \tag{5.21}$$

By applying Lemma 5.15 and inequality (5.21), we have

$$\begin{aligned}(A_1 \#_w A_2) \boxdot (B_1 \#_w B_2) &= Z^* \left[(A_1 \#_w A_2) \boxtimes (B_1 \#_w B_2)\right] Z \\ &= Z^* \left[(A_1 \boxtimes A_2) \#_w (A_2 \boxtimes B_2)\right] Z \\ &\leqslant [Z^*(A_1 \boxtimes B_1)Z] \#_w [Z^*(A_2 \boxtimes B_2)Z] \\ &= (A_1 \boxdot B_1) \#_w (A_2 \boxdot B_2).\end{aligned}$$

For arbitrary $A_1, A_2, B_1, B_2 \geqslant 0$, perturb each of them with εI and then take limit as $\varepsilon \to 0^+$. □

Corollary 5.16. *Let $r \in \mathbb{N}$ and $w \in [0,1]$. For each $1 \leqslant i \leqslant r$, let $A_i, B_i \in \mathbb{B}(\mathcal{H})$ be positive operators. Then*

$$\mathop{\boxtimes}_{i=1}^{r} (A_i \#_w B_i) = \left(\mathop{\boxtimes}_{i=1}^{r} A_i \right) \#_w \left(\mathop{\boxtimes}_{i=1}^{r} B_i \right). \tag{5.22}$$

Proof. The proof is by induction on r. □

In [45], Kilicman and Al Zhour invesigated weighted metric geometric means of any finite number of positive definite matrices. Now, we will extend this geometric mean to the case of finite number of positive operators.

Definition 5.17. Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathbb{B}(\mathcal{H})$ be a positive operator. Let $\alpha_i \in [0,1]$ for each $1 \leqslant i \leqslant r-1$ and denote $\alpha = (\alpha_1, \dots, \alpha_{r-1})$. We define

$$\mathcal{G}_{\alpha_1}(A_1, A_2) = A_2 \#_{\alpha_1} A_1.$$

Now continue recurrently, setting

$$\mathcal{G}_{\alpha}(A_1, \dots, A_r) = \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(A_1, \dots, A_{r-1}), A_r)$$

where $\tilde{\alpha} = (\alpha_1, \dots, \alpha_{r-2})$. We call $\mathcal{G}_{\alpha}(A_1, \dots, A_r)$ the *iterative α-weighted metric geometric mean* of $A_1, \dots, A_r$.

The next two results asserts the compatibility between Tracy-Singh products and iterative weighted metric geometric means.

Theorem 5.5. *Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i, B_i \in \mathbb{B}(\mathcal{H})$ be positive operators. Let $\alpha_i \in [0,1]$ for each $1 \leqslant i \leqslant r-1$. Then*

$$\mathcal{G}_\alpha(A_1 \boxtimes B_1, \dots, A_r \boxtimes B_r) \;=\; \mathcal{G}_\alpha(A_1, \dots, A_r) \boxtimes \mathcal{G}_\alpha(B_1, \dots, B_r) \tag{5.23}$$

Proof. We use induction on r. By continuity, we may assume that $A_i, B_i > 0$ for all $i = 1, \dots, r$. When $r = 2$, we have by Proposition 5.4 that

$$\begin{aligned}
\mathcal{G}_\alpha(A_1 \boxtimes B_1, A_2 \boxtimes B_2) &= (A_2 \boxtimes B_2) \#_\alpha (A_1 \boxtimes B_1) \\
&= (A_2 \#_\alpha A_1) \boxtimes (B_2 \#_\alpha B_1) \\
&= \mathcal{G}_\alpha(A_1, A_2) \boxtimes \mathcal{G}_\alpha(B_1, B_2)
\end{aligned}$$

where $\alpha \in [0,1]$. This gives the claim when $r = 2$. Suppose that the property (5.23) holds for $r-1$ ($r \geqslant 3$). Let $\alpha = (\alpha_1, \dots, \alpha_{r-1})$ and $\tilde{\alpha} = (\alpha_1, \dots, \alpha_{r-2})$ where $\alpha_i \in [0,1]$ for any $1 \leqslant i \leqslant r-1$. Using Theorem 5.4, we have

$$\begin{aligned}
&\mathcal{G}_\alpha(A_1 \boxtimes B_1, \dots, A_r \boxtimes B_r) \\
&\quad= \mathcal{G}_{\alpha_{r-1}}\left(\mathcal{G}_{\tilde{\alpha}}(A_1 \boxtimes B_1, \dots, A_{r-1} \boxtimes B_{r-1}), A_r \boxtimes B_r\right) \\
&\quad= \mathcal{G}_{\alpha_{r-1}}\left(\mathcal{G}_{\tilde{\alpha}}(A_1, \dots, A_{r-1}) \boxtimes \mathcal{G}_{\tilde{\alpha}}(B_1, \dots, B_{r-1}), A_r \boxtimes B_r\right) \\
&\quad= \mathcal{G}_{\alpha_{r-1}}\left(\mathcal{G}_{\tilde{\alpha}}(A_1, \dots, A_{r-1}), A_r\right) \boxtimes \mathcal{G}_{\alpha_{r-1}}\left(\mathcal{G}_{\tilde{\alpha}}(B_1, \dots, B_{r-1}), B_r\right) \\
&\quad= \mathcal{G}_\alpha(A_1, \dots, A_r) \boxtimes \mathcal{G}_\alpha(B_1, \dots, B_r).
\end{aligned}$$

□

Corollary 5.18. *Let $r \in \mathbb{N} - \{1\}$ and $s \in \mathbb{N}$. For each $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, let $A_{ij} \in \mathbb{B}(\mathcal{H})$ be a positive operator. Let $\alpha_i \in [0,1]$ for each $1 \leqslant i \leqslant r-1$. Then*

$$\mathcal{G}_\alpha\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j}, \dots, \mathop{\boxtimes}_{j=1}^{s} A_{rj}\right) \;=\; \mathop{\boxtimes}_{j=1}^{s} \mathcal{G}_\alpha(A_{1j}, \dots, A_{rj}). \tag{5.24}$$

The *Thompson metric* [53] on the open convex cone of positive invertible operators is defined for each $A, B > 0$ by

$$d(A,B) \;=\; \max\{\log M(A/B), \log M(B/A)\},$$

where $M(A/B) = \inf\{\lambda > 0 : A \leqslant \lambda B\}$. The *diameter* of $\{A_1, \dots, A_r\}$ with respect to the Thompson metric d is defined by

$$\triangle(A_1, \dots, A_r) \;=\; \max\{d(A_i, A_j) : 1 \leqslant i, j \leqslant r\}.$$

Lemma 5.19. *Let $r \in \mathbb{N} - \{1\}$. Let A_i for each $1 \leqslant i \leqslant r$ and B be positive invertible operators on $\mathcal{H}$. Let $\alpha_i \in [0,1]$ for each $1 \leqslant i \leqslant r-1$. Then*

$$d(\mathcal{G}_\alpha(A_1, \ldots, A_r), B) \;\leqslant\; \triangle(A_1, \ldots, A_r, B). \tag{5.25}$$

Proof. See [38, Proposition 3.1]. □

The next result is a generalization of inequality (5.25).

Proposition 5.20. *Let $r \in \mathbb{N} - \{1\}$ and $s \in \mathbb{N}$. For each $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, let $A_{ij}, B_j \in \mathbb{B}(\mathcal{H})$ be positive invertible operators. Let $\alpha_i \in [0,1]$ for each $1 \leqslant i \leqslant r-1$. Then*

$$d\left(\mathop{\boxtimes}_{j=1}^{s} \mathcal{G}_\alpha(A_{1j}, \ldots, A_{rj}), \mathop{\boxtimes}_{j=1}^{s} B_j\right) \;\leqslant\; \triangle\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j}, \ldots, \mathop{\boxtimes}_{j=1}^{s} A_{rj}, \mathop{\boxtimes}_{j=1}^{s} B_j\right). \tag{5.26}$$

Proof. This proposition follows from Lemma 5.19 and Corollary 5.18. □

5.4 Spectral geometric mean

Recall that for positive definite matrices A and B of the same size, its spectral geometric mean [43] is defined by

$$A \natural B \;=\; (A^{-1} \# B)^{\frac{1}{2}} A (A^{-1} \# B)^{\frac{1}{2}}.$$

Now, let A and B be positive invertible operators in $\mathbb{B}(\mathcal{H})$ and $w \in [0,1]$. The w-weighted spectral geometric mean of A and B is defined by

$$A \natural_w B \;=\; (A^{-1} \# B)^{w} A (A^{-1} \# B)^{w}.$$

For arbitrary positive operators A and B, we define the w-weighted spectral geometric mean of A and B to be

$$A \natural_w B \;=\; \lim_{\varepsilon \to 0^+} (A + \varepsilon I) \natural_w (B + \varepsilon I).$$

Theorem 5.6. *Let A_1, A_2, B_1 and B_2 be positive operators in $\mathbb{B}(\mathcal{H})$. Then*

$$(A_1 \boxtimes B_1) \natural_w (A_2 \boxtimes B_2) \;=\; (A_1 \natural_w A_2) \boxtimes (B_1 \natural_w B_2). \tag{5.27}$$

Proof. By continuity, we may assume that $A_1, A_2, B_1, B_2 > 0$. It follows from Lemma 5.2 and Proposition 5.4 that

$$\begin{aligned}
&(A_1 \boxtimes B_1) \natural_w (A_2 \boxtimes B_2) \\
&\quad = \left[(A_1 \boxtimes B_1)^{-1} \# (A_2 \boxtimes B_2)\right]^w (A_1 \boxtimes B_1) \left[(A_1 \boxtimes B_1)^{-1} \# (A_2 \boxtimes B_2)\right]^w \\
&\quad = \left[(A_1^{-1} \boxtimes B_1^{-1}) \# (A_2 \boxtimes B_2)\right]^w (A_1 \boxtimes B_1) \left[(A_1^{-1} \boxtimes B_1^{-1}) \# (A_2 \boxtimes B_2)\right]^w \\
&\quad = \left[(A_1^{-1} \# A_2) \boxtimes (B_1^{-1} \# B_2)\right]^w (A_1 \boxtimes B_1) \left[(A_1^{-1} \# A_2) \boxtimes (B_1^{-1} \# B_2)\right]^w \\
&\quad = \left[(A_1^{-1} \# A_2)^w \boxtimes (B_1^{-1} \# B_2)^w\right] (A_1 \boxtimes B_1) \left[(A_1^{-1} \# A_2)^w \boxtimes (B_1^{-1} \# B_2)^w\right] \\
&\quad = \left[(A_1^{-1} \# A_2)^w A_1 (A_1^{-1} \# A_2)^w\right] \boxtimes \left[(B_1^{-1} \# B_2)^w B_1 (B_1^{-1} \# B_2)^w\right] \\
&\quad = (A_1 \natural_w A_2) \boxtimes (B_1 \natural_w B_2). \qquad \square
\end{aligned}$$

Corollary 5.21. *Let $r \in \mathbb{N} - \{1\}$ and $w \in [0, 1]$. For each $1 \leqslant i \leqslant r$, let $A_i, B_i \in \mathbb{B}(\mathcal{H})$ be positive operators. Then*

$$\left(\mathop{\boxtimes}_{i=1}^{r} A_i\right) \natural_w \left(\mathop{\boxtimes}_{i=1}^{r} B_i\right) = \mathop{\boxtimes}_{i=1}^{r} (A_i \natural_w B_i). \tag{5.28}$$

Proof. The proof is by induction on r. We have that the property (5.28) holds for $r = 2$ by Lemma 5.6. Suppose that the property (5.28) holds for $r - 1$ ($r \geqslant 3$). By using Lemma 5.6, we get

$$\begin{aligned}
\left(\mathop{\boxtimes}_{i=1}^{r} A_i\right) \natural_w \left(\mathop{\boxtimes}_{i=1}^{r} B_i\right) &= \left[\left(\mathop{\boxtimes}_{i=1}^{r-1} A_i\right) \boxtimes A_r\right] \natural_w \left[\left(\mathop{\boxtimes}_{i=1}^{r-1} B_i\right) \boxtimes B_r\right] \\
&= \left[\left(\mathop{\boxtimes}_{i=1}^{r-1} A_i\right) \natural_w \left(\mathop{\boxtimes}_{i=1}^{r-1} B_i\right)\right] \boxtimes (A_r \natural_w B_r) \\
&= \left(\mathop{\boxtimes}_{i=1}^{r-1} (A_i \natural_w B_i)\right) \boxtimes (A_r \natural_w B_r) \\
&= \mathop{\boxtimes}_{i=1}^{r} (A_i \natural_w B_i). \qquad \square
\end{aligned}$$

In [45], Kilicman and Al Zhour studied weighted spectral geometric means of any finite number of positive definite matrices and proved several properties related to Tracy-Singh products. Now, we will extend this geometric mean to the case of any finite number of positive operators.

Definition 5.22. Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathbb{B}(\mathcal{H})$ be positive operators. Let $\alpha_i \in [0,1]$ for each $1 \leqslant i \leqslant r-1$. We define

$$\mathcal{G}^{sp}_{\alpha_1}(A_1, A_2) \ = \ A_1 \natural_{\alpha_1} A_2.$$

Now continue recurrently, setting for each $r \geqslant 3$,

$$\mathcal{G}^{sp}_{\alpha}(A_1, \ldots, A_r) \ = \ \mathcal{G}^{sp}_{\alpha_{r-1}}\left(\mathcal{G}^{sp}_{\tilde{\alpha}}(A_1, \ldots, A_{r-1}), A_r\right)$$

where $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_{r-2})$. We call $\mathcal{G}^{sp}_{\alpha}(A_1, \ldots, A_r)$ the *iterated α-weighted spectral geometric mean* of $A_1, \ldots, A_r$.

From Definition 5.22, we can rewrite (5.28) in Corollary 5.21 to be

$$\mathcal{G}^{sp}_{\alpha}\left(\boxtimes_{i=1}^{r} A_i, \boxtimes_{i=1}^{r} B_i\right) \ = \ \boxtimes_{i=1}^{r} \mathcal{G}^{sp}_{\alpha}(A_i, B_i)$$

where $\alpha = w$.

Corollary 5.23. *Let $r \in \mathbb{N} - \{1\}$. Let A_i and B_i be compatible positive operators in $\mathbb{B}(\mathcal{H})$ for each $i = 1, \ldots, r$. Then*

$$\mathcal{G}^{sp}_{\alpha}(A_1 \boxtimes B_1, \ldots, A_r \boxtimes B_r) \ = \ \mathcal{G}^{sp}_{\alpha}(A_1, \ldots, A_r) \boxtimes \mathcal{G}^{sp}_{\alpha}(B_1, \ldots, B_r). \quad (5.29)$$

Proof. The proof is by induction on r. By Theorem 5.6, we have that the property (5.29) is true for $r = 2$. Suppose that the property (5.29) is true for $r - 1$. By Theorem 5.6, we obtain

$$\begin{aligned}
&\mathcal{G}^{sp}_{\alpha}(A_1 \boxtimes B_1, \ldots, A_r \boxtimes B_r) \\
&\quad = \ \mathcal{G}^{sp}_{\alpha_{r-1}}\left(\mathcal{G}^{sp}_{\tilde{\alpha}}(A_1 \boxtimes B_1, \ldots, A_{r-1} \boxtimes B_{r-1}), A_r \boxtimes B_r\right) \\
&\quad = \ \mathcal{G}^{sp}_{\alpha_{r-1}}\left(\mathcal{G}^{sp}_{\tilde{\alpha}}(A_1, \ldots, A_{r-1}) \boxtimes \mathcal{G}^{sp}_{\tilde{\alpha}}(B_1, \ldots, B_{r-1}), A_r \boxtimes B_r\right) \\
&\quad = \ \mathcal{G}^{sp}_{\alpha_{r-1}}\left(\mathcal{G}^{sp}_{\tilde{\alpha}}(A_1, \ldots, A_{r-1}), A_r\right) \boxtimes \mathcal{G}^{sp}_{\alpha_{r-1}}\left(\mathcal{G}^{sp}_{\tilde{\alpha}}(B_1, \ldots, B_{r-1}), B_r\right) \\
&\quad = \ \mathcal{G}^{sp}_{\alpha}(A_1, \ldots, A_r) \boxtimes \mathcal{G}^{sp}_{\alpha}(B_1, \ldots, B_r).
\end{aligned}$$

□

Corollary 5.24. *Let $r \in \mathbb{N} - \{1\}$ and $s \in \mathbb{N}$. Let $A_{ij} \in \mathbb{B}(\mathcal{H})$ be a positive invertible operator for each $i = 1, \ldots, r,\ j = 1, \ldots, s$. Then*

$$\mathcal{G}^{sp}_{\alpha}\left(\boxtimes_{j=1}^{s} A_{1j}, \ldots, \boxtimes_{j=1}^{s} A_{rj}\right) \ = \ \boxtimes_{j=1}^{s} \mathcal{G}^{sp}_{\alpha}(A_{1j}, \ldots, A_{rj}). \quad (5.30)$$

Proof. The proof is by induction on s. □

5.5 Sagae-Tanabe metric geometric mean

Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathbb{B}(\mathcal{H})$ be a positive invertible operator. Let $t = (t_1, \dots, t_r)$ where $t_i \in [0,1]$ for each $1 \leqslant i \leqslant r$ and $\sum_{i=1}^r t_i = 1$. The *weighted arithmetic* and *harmonic means* of $A_1, \dots, A_r$ are defined by

$$\begin{aligned} B(\mathcal{H})_t(A_1, \dots .A_r) &= \sum_{i=1}^r t_i A_i, \\ \mathcal{H}_t(A_1, \dots .A_r) &= \left(\sum_{i=1}^r t_i A_i^{-1} \right)^{-1}. \end{aligned}$$

Sagae and Tanabe [52] proposed weighted geometric means of severable positive definite matrices as follows.

Definition 5.25. Let A and B be positive invertible operators in $\mathbb{B}(\mathcal{H})$ and let $v = (v_1, v_2)$ where $v_1,\ v_2 \in [0,1]$ and $v_1 + v_2 = 1$. We define

$$\mathcal{G}_v(A, B) = A \#_\alpha B$$

where $\alpha = 1 - v_2$. Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathbb{B}(\mathcal{H})$ be a positive invertible operator. Let $t = (t_1, \dots, t_r)$ where $t_i \in [0,1]$ for each $1 \leqslant i \leqslant r$ and $\sum_{i=1}^r t_i = 1$. For each $1 \leqslant i \leqslant r-1$, let

$$\alpha_i = 1 - (t_{i+1} / \sum_{j=1}^{i+1} t_j).$$

The *Sagae-Tanabe weighted geometric mean* of $A_1, \dots, A_r$ is defined by

$$\mathcal{G}_t(A_1, \dots .A_r) = \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{t}}(A_1, \dots, A_{r-1}), A_r)$$

where $\mathcal{G}_{\tilde{t}}(A_1, \dots, A_{r-1})$ is the Sagae-Tanabe weighted geometric mean of $A_1, \dots, A_{r-1}$ with weighted $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{r-1})$ where $\tilde{t}_i = t_i / \sum_{j=1}^{r-1} t_j$ for each $1 \leqslant i \leqslant r-1$. Note that

$$\mathcal{G}_t(A_1, \dots, A_r) = \mathcal{G}_\alpha(A_1, \dots, A_r)$$

where $\mathcal{G}_\alpha(A_1, \dots, A_r)$ is the weighted metric geometric mean of $A_1, \dots, A_r$ in Definition 5.17 with weight $\alpha = (\alpha_1, \dots, \alpha_{r-1})$.

Theorem 5.7. *Let $r \in \mathbb{N} - \{1\}$ and $s \in \mathbb{N}$. For each $1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s$, let $A_{ij} \in \mathbb{B}(\mathcal{H})$ be a positive invertible operator. Let $t = (t_1, \dots, t_r)$ where $t_i \in [0,1]$ for each $1 \leqslant i \leqslant r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\mathcal{G}_t \left(\boxtimes_{j=1}^{s} A_{1j}, \dots, \boxtimes_{j=1}^{s} A_{rj} \right) = \boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j}, \dots, A_{rj}). \tag{5.31}$$

Proof. Let $\alpha_i = 1 - (t_{i+1} / \sum_{j=1}^{i+1} t_j)$ for each $1 \leqslant i \leqslant r-1$ and denote $\alpha = (\alpha_1, \dots, \alpha_{r-1})$. By Definition 5.25, we have

$$\mathcal{G}_t(A_{11} \boxtimes A_{12}, \dots, A_{r1} \boxtimes A_{r2}) = \mathcal{G}_\alpha(A_{11} \boxtimes A_{12}, \dots, A_{rj} \boxtimes B_{rj}).$$

Applying Theorem 5.5, we obtain

$$\mathcal{G}_\alpha(A_{11} \boxtimes A_{12}, \dots, A_{rj} \boxtimes B_{rj}) = \mathcal{G}_\alpha(A_{11}, \dots, A_{r1}) \boxtimes \mathcal{G}_\alpha(A_{12}, \dots, A_{r2}).$$

This implies that

$$\mathcal{G}_t(A_{11} \boxtimes A_{12}, \dots, A_{r1} \boxtimes A_{r2}) = \mathcal{G}_t(A_{11}, \dots, A_{r1}) \boxtimes \mathcal{G}_t(A_{12}, \dots, A_{r2}).$$

We get the result by using induction on s. □

Lemma 5.26. *Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathbb{B}(\mathcal{H})$ be a positive invertible operator and let $t = (t_1, \dots, t_r)$ where $t_i \in [0,1]$ for $i = 1, \dots, r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\mathcal{H}_t(A_1, \dots. A_r) \leqslant \mathcal{G}_t(A_1, \dots. A_r) \leqslant B(\mathcal{H})_t(A_1, \dots, A_r). \tag{5.32}$$

Proof. See [38, Proposition 2.4]. □

We extend [45, Theorem 4.6] to AM-GM-HM inequalities involving Tracy-Singh product of positive invertible operators as in the next two results.

Corollary 5.27. *Let $r \in \mathbb{N} - \{1\}$ and $s \in \mathbb{N}$. For each $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s$, let $A_{ij} \in \mathbb{B}(\mathcal{H})$ be a positive invertible operator and let $t = (t_1, \dots, t_r)$ where $t_i \in [0,1]$ for $i = 1, \dots, r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\boxtimes_{j=1}^{s} \mathcal{H}_t(A_{1j}, \dots, A_{rj}) \leqslant \mathcal{G}_t \left(\boxtimes_{j=1}^{s} A_{1j}, \dots, \boxtimes_{j=1}^{s} A_{rj} \right) \leqslant \boxtimes_{j=1}^{s} B(\mathcal{H})_t(A_{1j}, \dots, A_{rj}). \tag{5.33}$$

Proof. Lemma 5.26 tells us that

$$\mathcal{H}_t(A_{1j},\ldots,A_{rj}) \leqslant \mathcal{G}_t\,(A_{1j},\ldots,A_{rj}) \leqslant B(\mathcal{H})_t(A_{1j},\ldots,A_{rj})$$

for each $1 \leqslant j \leqslant s$. By using Lemma 5.2, we get

$$\boxtimes_{j=1}^{s} \mathcal{H}_t(A_{1j},\ldots,A_{rj}) \leqslant \boxtimes_{j=1}^{s} \mathcal{G}_t\,(A_{1j},\ldots,A_{rj}) \leqslant \boxtimes_{j=1}^{s} B(\mathcal{H})_t(A_{1j},\ldots,A_{rj}).$$

Applying Theorem 5.7, we obtain

$$\boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j},\ldots,A_{rj}) = \mathcal{G}_t\left(\boxtimes_{j=1}^{s} A_{1j},\ldots,\boxtimes_{j=1}^{s} A_{rj}\right)$$

and the inequality (5.33) follows. □

Corollary 5.28. *Let $r \in \mathbb{N}-\{1\}$ and $s \in \mathbb{N}$. For each $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s$, let $A_{ij} \in \mathbb{B}(\mathcal{H})$ be a positive invertible operator and let $t=(t_1,\ldots,t_r)$ where $t_i \in [0,1]$ for $i=1,\ldots,r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\mathcal{H}_t\left(\boxtimes_{j=1}^{s} A_{1j},\ldots,\boxtimes_{j=1}^{s} A_{rj}\right) \leqslant \boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j},\ldots,A_{rj}) \leqslant B(\mathcal{H})_t\left(\boxtimes_{j=1}^{s} A_{1j},\ldots,\boxtimes_{j=1}^{s} A_{rj}\right). \tag{5.34}$$

Proof. It follows directly from the AM-GM-HM inequality (5.32) and Theorem 5.7. □

We now turn to the AM-GM-HM inequality involving Khatri-Rao products.

Corollary 5.29. *Let $A_{ij} \in \mathbb{B}(\mathcal{H})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ be compatible positive invertible operators and let $t=(t_1,\ldots,t_r)$ where $t_i \in [0,1]$ for $i=1,\ldots,r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\boxdot_{j=1}^{s} \mathcal{H}_t(A_{1j},\ldots,A_{rj}) \leqslant \boxdot_{j=1}^{s} \mathcal{G}_t(A_{1j},\ldots,A_{rj}) \leqslant B(\mathcal{H})_t\left(\boxdot_{j=1}^{s} A_{1j},\ldots,\boxdot_{j=1}^{s} A_{rj}\right). \tag{5.35}$$

Proof. We have by Lemmas 5.15 and 5.26 that

$$\begin{aligned}
\boxdot_{j=1}^{s} \mathcal{H}_t(A_{1j},\ldots,A_{rj}) &= Z^* \left(\boxtimes_{j=1}^{s} \mathcal{H}_t(A_{1j},\ldots,A_{rj}) \right) Z \\
&\leqslant Z^* \left(\boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j},\ldots,A_{rj}) \right) Z \\
&= \boxdot_{j=1}^{s} \mathcal{G}_t(A_{1j},\ldots,A_{rj}).
\end{aligned}$$

Using Lemma 5.15, we get

$$\begin{aligned}
& Z^* \left[B(\mathcal{H})_t \left(\boxtimes_{j=1}^{s} A_{1j},\ldots,\boxtimes_{j=1}^{s} A_{rj} \right) \right] Z \\
&= Z^* \left[\sum_{i=1}^{r} t_i \left(\boxtimes_{j=1}^{s} A_{ij} \right) \right] Z = \sum_{i=1}^{r} t_i \left[Z^* \left(\boxtimes_{j=1}^{s} A_{ij} \right) Z \right] \\
&= \sum_{i=1}^{r} t_i \left(\boxdot_{j=1}^{s} A_{ij} \right) = B(\mathcal{H})_t \left(\boxdot_{j=1}^{s} A_{1j},\ldots,\boxdot_{j=1}^{s} A_{rj} \right).
\end{aligned}$$

Applying Lemma 5.15 and Corollary 5.28, we obtain

$$\begin{aligned}
\boxdot_{j=1}^{s} \mathcal{G}_t(A_{1j},\ldots,A_{rj}) &= Z^* \left(\boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j},\ldots,A_{rj}) \right) Z \\
&\leqslant Z^* \left[B(\mathcal{H})_t \left(\boxtimes_{j=1}^{s} A_{1j},\ldots,\boxtimes_{j=1}^{s} A_{rj} \right) \right] Z \\
&= B(\mathcal{H})_t \left(\boxdot_{j=1}^{s} A_{1j},\ldots,\boxdot_{j=1}^{s} A_{rj} \right). \qquad \square
\end{aligned}$$

The next result is a generalization of Lemma 5.19.

Proposition 5.30. *Let A_{ij} and B_j $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ be compatible positive invertible operators and let $t = (t_1,\ldots,t_r)$ where* $t_i \in$

$[0,1]$ *for* $i = 1, \ldots, r$ *and* $\sum_{i=1}^{r} t_i = 1$. *Then*

$$d\left(\boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j}, \ldots, A_{rj}), \boxtimes_{j=1}^{s} B_j\right) \leqslant \triangle\left(\boxtimes_{j=1}^{s} A_{1j}, \ldots, \boxtimes_{j=1}^{s} A_{rj}, \boxtimes_{j=1}^{s} B_j\right). \tag{5.36}$$

Proof. The desire result follows from Lemma 5.19 and Corollary 5.18. □

For $h, x \geqslant 1$, the *(generalized) Specht ratio* is defined by

$$S_h(x) = \frac{(h^x - 1)h^{x(h^x-1)^{-1}}}{e \log h^x} \quad \text{for } h \neq 1 \text{ and } S_1(x) = 1.$$

We denote $S_h(1)$ by S_h. See [38, 67] for more information. The next result is a reverse version of AM-GM-HM inequality involving Tracy-Singh products via Specht ratio.

Proposition 5.31. *Let* A_{ij} $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ *be compatible positive invertible operators and let* $t = (t_1, \ldots, t_r)$ *where* $t_i \in [0,1]$ *for* $i = 1, \ldots, r$ *and* $\sum_{i=1}^{r} t_i = 1$. *Then*

$$B(\mathcal{H})_t\left(\boxtimes_{j=1}^{s} A_{1j}, \ldots, \boxtimes_{j=1}^{s} A_{rj}\right) \leqslant S_h^{r-1} \cdot \left(\boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j}, \ldots, A_{rj})\right) \tag{5.37}$$

where $h = e^{\triangle(\boxtimes_{j=1}^{s} A_{1j}, \ldots, \boxtimes_{j=1}^{s} A_{rj})}$.

Proof. By using Lemma 5.19 and Corollary 5.18, we get the result. □

Lemma 5.32. *Let* $A_i \in \mathbb{B}(\mathcal{H})$ $(1 \leqslant i \leqslant r, r \geqslant 2)$ *be positive invertible operators and* t_i $(1 \leqslant i \leqslant r)$ *be real numbers such that* $t_1 > 0, t_i < 0$ $(2 \leqslant i \leqslant r)$ *and* $\sum_{i=1}^{r} t_i = 1$. *Then*

$$B(\mathcal{H})_t(A_1, \ldots, A_r) \leqslant \mathcal{G}_t(A_1, \ldots, A_r). \tag{5.38}$$

If $\sum_{i=1}^{r} t_i A_i^{-1} > 0$, *then*

$$\mathcal{G}_t(A_1, \ldots, A_r) \leqslant \mathcal{H}_t(A_1, \ldots, A_r). \tag{5.39}$$

Proof. The proof is similar to the case of matrices, given in [39, Theorem 2.1]. □

We now obtain reverse AM-GM-HM inequalities involving Tracy-Singh products as follows.

Theorem 5.8. *Let $A_{ij} \in \mathbb{B}(\mathcal{H})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ be compatible positive invertible operators and t_i $(1 \leqslant i \leqslant r)$ be real numbers such that $t_1 > 0, t_i < 0$ $(2 \leqslant i \leqslant r)$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\boxtimes_{j=1}^{s} B(\mathcal{H})_t(A_{1j}, \ldots, A_{rj}) \leqslant \mathcal{G}_t\left(\boxtimes_{j=1}^{s} A_{1j}, \ldots, \boxtimes_{j=1}^{s} A_{rj}\right). \tag{5.40}$$

If $\mathcal{H}_t(A_{1j}, \ldots, A_{rj}) > 0$ for all $j = 1, \ldots, s$, then

$$\mathcal{G}_t\left(\boxtimes_{j=1}^{s} A_{1j}, \ldots, \boxtimes_{j=1}^{s} A_{rj}\right) \leqslant \boxtimes_{j=1}^{s} \mathcal{H}_t(A_{1j}, \ldots, A_{rj}). \tag{5.41}$$

Proof. It follows from Lemma 5.32 that

$$A_t(A_{1j}, \ldots, A_{rj}) \leqslant \mathcal{G}_t\,(A_{1j}, \ldots, A_{rj}) \leqslant \mathcal{H}_t(A_{1j}, \ldots, A_{rj})$$

for each $j = 1, \ldots, s$. Since $A_t(A_{1j}, \ldots, A_{rj}) \geqslant 0$ for all $j = 1, \ldots, s$, we have by Lemmas 5.2 and 5.32 that

$$\mathcal{G}_t\left(\boxtimes_{j=1}^{s} A_{1j}, \ldots, \boxtimes_{j=1}^{s} A_{rj}\right) = \boxtimes_{j=1}^{s} \mathcal{G}_t\,(A_{1j}, \ldots, A_{rj}) \geqslant \boxtimes_{j=1}^{s} B(\mathcal{H})_t(A_{1j}, \ldots, A_{rj}).$$

Since $\mathcal{H}_t(A_{1j}, \ldots, A_{rj}) > 0$ for all $j = 1, \ldots, s$, we obtain by Lemma 5.2 that

$$\boxtimes_{j=1}^{s} \mathcal{H}_t(A_{1j}, \ldots, A_{rj}) > 0.$$

The proof is complete by applying Lemma 5.32 and Corollary 5.7. □

Theorem 5.9. *Let $A_{ij} \in \mathbb{B}(\mathcal{H})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ be compatible positive invertible operators and t_i $(1 \leqslant i \leqslant r)$ be real numbers such that $t_1 > 0, t_i < 0$ $(2 \leqslant i \leqslant r)$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$B(\mathcal{H})_t\left(\boxtimes_{j=1}^{s} A_{1j}, \ldots, \boxtimes_{j=1}^{s} A_{rj}\right) \leqslant \boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j}, \ldots, A_{rs}). \tag{5.42}$$

If $\mathcal{H}_t\left(\boxtimes_{j=1}^s A_{1j},\ldots,\boxtimes_{j=1}^s A_{rj}\right) > 0$, *then*

$$\mathop{\boxtimes}_{j=1}^{s} \mathcal{G}_t(A_{1j},\ldots,A_{rs}) \;\leqslant\; \mathcal{H}_t\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j},\ldots,\mathop{\boxtimes}_{j=1}^{s} A_{rj}\right). \tag{5.43}$$

Proof. By applying Lemma 5.32 and Corollary 5.7, we get the results. □

Corollary 5.33. *Let* $A_{ij} \in \mathbb{B}(\mathcal{H})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ *be compatible positive invertible operators and* t_i $(1 \leqslant i \leqslant r)$ *be real numbers such that* $t_1 > 0, t_i < 0$ $(2 \leqslant i \leqslant r)$ *and* $\sum_{i=1}^r t_i = 1$. *Then*

$$B(\mathcal{H})_t\left(\mathop{\boxdot}_{j=1}^{s} A_{1j},\ldots,\mathop{\boxdot}_{j=1}^{s} A_{rj}\right) \;\leqslant\; \mathop{\boxdot}_{j=1}^{s} \mathcal{G}_t(A_{1j},\ldots,A_{rj}) \;\leqslant\; \mathop{\boxdot}_{j=1}^{s} \mathcal{H}_t(A_{1j},\ldots,A_{rj}). \tag{5.44}$$

Proof. This result is a direct consequence of Theorem 5.9 and Lemmas 5.15 and 5.26. □

5.6 Sagae-Tanabe spectral geometric mean

We introduce the following definition:

Definition 5.34. Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathbb{B}(\mathcal{H})$ be a positive invertible operator. Let $t = (t_1,\ldots,t_r)$ where $t_i \in [0,1]$ for each $1 \leqslant i \leqslant r$ and $\sum_{i=1}^r t_i = 1$. Let $\alpha_i = 1 - (t_{i+1}/\sum_{j=1}^{i+1} t_j)$ for each $1 \leqslant i \leqslant r-1$. The *Sagae-Tanabe spectral geometric mean* of $A_1,\ldots,A_r$ is defined by

$$\mathcal{G}_t^{sp}(A_1,\ldots,A_r) \;=\; \mathcal{G}_\alpha^{sp}(A_1,\ldots,A_r)$$

where $\alpha = (\alpha_1,\ldots,\alpha_{r-1})$.

Proposition 5.35. *Let* A_i *and* B_i $(1 \leqslant i \leqslant r, r \geqslant 2)$ *be compatible positive operators in* $\mathbb{B}(\mathcal{H})$ *and let* $t = (t_1,\ldots,t_r)$ *where* $t_i \in [0,1]$ *for* $i = 1,\ldots,r-1$ *and* $\sum_{i=1}^r t_i = 1$. *Then*

$$\mathcal{G}_t^{sp}(A_1 \boxtimes B_1,\ldots,A_r \boxtimes B_r) \;=\; \mathcal{G}_t^{sp}(A_1,\ldots,A_r) \boxtimes \mathcal{G}_t^{sp}(B_1,\ldots,B_r) \tag{5.45}$$

$$\mathcal{G}_t^{sp}\left(\mathop{\boxtimes}_{i=1}^{r} A_i, \mathop{\boxtimes}_{i=1}^{r} B_i\right) \;=\; \mathop{\boxtimes}_{i=1}^{r} \mathcal{G}_t^{sp}(A_i,B_i). \tag{5.46}$$

Proof. Let $\alpha_i = 1 - (t_{i+1}/\sum_{j=1}^{i+1} t_j)$ for each $1 \leqslant i \leqslant r-1$ and denote $\alpha = (\alpha_1, \dots, \alpha_{r-1})$. By Definition 5.34, we have

$$\begin{aligned}\mathcal{G}_t^{sp}(A_1 \boxtimes B_1, \dots, A_r \boxtimes B_r) &= \mathcal{G}_\alpha^{sp}(A_1 \boxtimes B_1, \dots, A_r \boxtimes B_r) \\ \mathcal{G}_t^{sp}\left(\bigboxtimes_{i=1}^{r} A_i, \bigboxtimes_{i=1}^{r} B_i\right) &= \mathcal{G}_\alpha^{sp}\left(\bigboxtimes_{i=1}^{r} A_i, \bigboxtimes_{i=1}^{r} B_i\right).\end{aligned}$$

By Corollary 5.23, we get (5.45). Applying Corollary 5.21, we obtain (5.46). ☐

Corollary 5.36. *Let $A_{ij} \in \mathbb{B}(\mathcal{H})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ be compatible positive invertible operators and let $t = (t_1, \dots, t_r)$ where $t_i \in [0,1]$ for $i = 1, \dots, r-1$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\mathcal{G}_t^{sp}\left(\bigboxtimes_{j=1}^{s} A_{1j}, \dots, \bigboxtimes_{j=1}^{s} A_{rj}\right) = \bigboxtimes_{j=1}^{s} \mathcal{G}_t^{sp}(A_{1j}, \dots, A_{rj}). \tag{5.47}$$

Proof. From (5.45), we have

$$\mathcal{G}_t^{sp}(A_{11} \boxtimes A_{12}, \dots, A_{r1} \boxtimes A_{r2}) = \mathcal{G}_t^{sp}(A_{11}, \dots, A_{r1}) \boxtimes \mathcal{G}_t^{sp}(A_(12), \dots, A_{r2}).$$

We obtain (5.47) by induction on s. ☐

Chapter 6

Refinements and Reverses of Operator Callebaut Inequality Involving Tracy-Singh Products and Khatri-Rao Products

In this chapter, we establish certain refinements and reverses of Callebaut-type inequality for bounded continuous fields of Hilbert space operators, parametrized by a locally compact Hausdorff space equipped with a finite Radon measure. These inequalities involve Tracy-Singh products, Khatri-Rao products and weighted geometric means. In addition, we obtain integral Callebaut-type inequalities for tensor products and Hadamard products. Our results extend Callebaut-type inequalities for real numbers, matrices and operators.

6.1 Introduction

In mathematics, the Cauchy-Schwarz inequality is an important inequality which can be applied in many fields, e.g. operator theory, linear algebra, analysis, probability and statistics (see [71, 74, 77, 80, 86]). This inequality states that for vectors $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ of real numbers, we have

$$\left(\sum_{i=1}^{k} a_i b_i \right)^2 \leqslant \left(\sum_{i=1}^{k} a_i^2 \right) \left(\sum_{i=1}^{k} b_i^2 \right). \tag{6.1}$$

In 1965, Callebaut [73] published a refinement of the Cauchy-Schwarz inequality (6.1). For each $\alpha \in [0,1]$ and for any tuples $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ of positive real numbers, let us denote $\mathcal{I}^k_\alpha(x,y) = \sum_{i=1}^k x_i \sharp_\alpha y_i$, where $\sharp_\alpha$ is the α-weighted geometric mean. For either $0 \leqslant \beta \leqslant \alpha \leqslant \frac{1}{2}$ or $\frac{1}{2} \leqslant \alpha \leqslant \beta \leqslant 1$, the classical Callebaut inequality [73] can be stated as

$$\left(\mathcal{I}^k_{1/2}(x,y)\right)^2 \leqslant \mathcal{I}^k_\alpha(x,y) \cdot \mathcal{I}^k_{1-\alpha}(x,y) \leqslant \mathcal{I}^k_\beta(x,y) \cdot \mathcal{I}^k_{1-\beta}(x,y) \leqslant \mathcal{I}^k_0(x,y) \cdot \mathcal{I}^k_1(x,y). \tag{6.2}$$

There have been several investigations and generalizations on the Callebaut inequality; see [69, 70, 76, 79, 87] and references therein. Hiai and Zhan [76] gave a matrix analog of the Callebaut inequality (6.2) by considering the convexity of a certain norm function. The paper [79] presented a matrix version of (6.2) involving the tensor product,the Hadamard product and weighted geometric means. Operator versions of (6.2) associated with a Kubo-Ando mean and an interpolational path were also established in that paper. Wada [87] provided a simple form of (6.2) for positive operators involving an operator mean its dual.. Some refinements and reverses of (6.2) for operators concerning the Hadamard product and weighted geometric means were presented in [69, 70]. In [85], the authors establish integral versions of the Callebaut inequality and its refinements for bounded continuous fields of Hilbert space operators concerning the Tracy-Singh product, Khatri-Rao product and weighted geometric means.

In this paper, we investigate refinements and reverses of the operator Callebaut inequalities for bounded continuous fields of positive operators parametrized by a locally compact Hausdorff space endowed with a finite Radon measure. Such integral inequalities involves Tracy-Singh products, Khatri-Rao products, tensor products, Hadamard products and weighted geometric means. In particular, our results are refinements and reverses of Callebaut-type inequalities obtained in the previous works [73, 79, 85].

This paper is organized as follows. In Section 6.2, we give preliminaries on operator products and Bochner integration of continuous fields of operators on a locally compact Housdorff space. In Section 6.3, we provide certain refinements of integral Callebaut inequalities for bounded continuous fields of operators involving some kind of operator products and weighted geometric means. Some reversed Callebaut-type inequalities for bounded continuous fields of operators are presented in Section 6.4. The conclusion is given in the last section.

6.2 Preliminaries

Throughout this paper, let $\mathcal{H}$ be a complex Hilbert space. When $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, denote by $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ the Banach space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, and abbreviate $\mathbb{B}(\mathcal{X}, \mathcal{X})$ to $\mathbb{B}(\mathcal{X})$. For self-adjoint operators $A, B \in \mathbb{B}(\mathcal{X})$, the notation $A \geqslant B$ means that $A - B$ is a positive operator. The set of all positive invertible operators on $\mathcal{X}$ is denoted by $\mathbb{B}(\mathcal{X})^+$.

The projection theorem for Hilbert spaces allows us to decompose $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$ where all $\mathcal{H}_i$ are Hilbert spaces. For each $i = 1, \ldots, n$, let P_i be the natural projection from $\mathcal{H}$ onto $\mathcal{H}_i$ and E_i the canonical embedding from $\mathcal{H}_i$ into $\mathcal{H}$. Note that $P_i^* = E_i$. Each operator $A \in \mathbb{B}(\mathcal{H})$ can be uniquely determined by an operator matrix

$$A \;=\; [A_{ij}]_{i,j=1}^{n,n}\,,$$

where $A_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$ is defined by $A_{ij} = P_i A E_j$ for each $i, j = 1, \ldots, n$.

6.2.1 Operator products

Recall that tensor product of $A, B \in \mathbb{B}(\mathcal{H})$ is a unique bounded linear operator from $\mathcal{H} \otimes \mathcal{H}$ into itself such that for all $x, y \in \mathcal{H}$,

$$(A \otimes B)(x \otimes y) = Ax \otimes By.$$

Fix a countable orthonormal basis $\mathbb{E}$ on $\mathcal{H}$. Recall that the Hadamard product of $A, B \in \mathbb{B}(\mathcal{H})$ is defined to be bounded linear operator $A \odot B$ from $\mathcal{H}$ into itself such that for all $e \in \mathbb{E}$,

$$\langle (A \odot B)e, e\rangle \;=\; \langle Ae\rangle\langle Be, e\rangle.$$

Following [75], the Hadamard product can be expressed as

$$A \odot B \;=\; U^*(A \otimes B)U, \tag{6.3}$$

where $U : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is the isometry defined by $Ue = e \otimes e$ for all $e \in \mathcal{E}$. In the case of matrices, the Hadamard product of $A = [a_{ij}]_{i,j=1}^{n,n}$ and $B = [b_{ij}]_{i,j=1}^{n,n}$ is $A \odot B = [a_{ij} b_{ij}]$ which is a principal submatrix of the Kronecker (tensor) product $A \otimes B = [a_{ij} B]_{ij}$ (see [78] for more information).

Definition 6.1. Let $A = [A_{ij}]_{i,j=1}^{n,n}$ and $B = [B_{ij}]_{i,j=1}^{n,n}$ be operator matrices in $\mathbb{B}(\mathcal{H})$. The Tracy-Singh product of A and B is defined to be the operator matrix

$$A \boxtimes B \;=\; \big[[A_{ij} \otimes B_{kl}]_{kl}\big]_{ij}\,, \tag{6.4}$$

which is a bounded linear operator from $\bigoplus_{i,j=1}^{n,n} \mathcal{H}_i \otimes \mathcal{H}_j$ into itself. The Khatri-Rao product of A and B is defined to be the operator matrix

$$A \boxdot B \;=\; [A_{ij} \otimes B_{ij}]_{i,j} \tag{6.5}$$

which is a bounded linear operator from $\bigoplus_{i=1}^{n} \mathcal{H}_i \otimes \mathcal{H}_i$ into itself.

Lemma 6.2 ([81, 82])**.** *Let $A, B, C, D \in \mathbb{B}(\mathcal{H})$.*

1. $\alpha(A \boxtimes B) = (\alpha A) \boxtimes B = A \boxtimes (\alpha B)$ *for any* $\alpha \in \mathbb{C}$.
2. $(A + B) \boxtimes (C + D) = A \boxtimes B + A \boxtimes D + B \boxtimes C + B \boxtimes D$.
3. $(A \boxtimes B)^* = A^* \boxtimes B^*$.
4. *If* $A \geqslant C \geqslant 0$ *and* $B \geqslant D \geqslant 0$, *then* $A \boxtimes B \;\geqslant\; C \boxtimes D \geqslant 0$.
5. $(A \boxtimes B)(C \boxtimes D) \;=\; AC \boxtimes BD$.
6. *If* $A, B \in \mathbb{B}(\mathcal{H})^+$, *then* $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$ *for any* $\alpha \in \mathbb{R}$.

Lemma 6.3 ([84])**.** *There is a unital positive linear map*

$$\Phi : \mathbb{B}\Big(\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{n} \mathcal{H}_i \otimes \mathcal{H}_j\Big) \;\to\; \mathbb{B}\Big(\bigoplus_{i=1}^{n} \mathcal{H}_i \otimes \mathcal{H}_i\Big) \tag{6.6}$$

such that $\Phi(A \boxtimes B) = A \boxdot B$ *for any* $A, B \in \mathbb{B}(\mathcal{H})$.

6.2.2 Bochner integration

Let Ω be a locally compact Hausdorff space endowed with a finite Radon measure μ. A family $(A_t)_{t\in\Omega}$ of operators in $\mathbb{B}(\mathcal{H})$ is said to be a continuous field if the parametrization $t \mapsto A_t$ is norm-continuous on Ω. If, in addition, the function $t \mapsto \|A_t\|$ is Lebesgue integrable on Ω, then we can form the Bochner integral $\int_\Omega A_t d\mu(t)$ as a unique element in $\mathbb{B}(\mathcal{H})$ such that

$$T\left(\int_\Omega A_t d\mu(t)\right) = \int_\Omega T(A_t)\, d\mu(t) \tag{6.7}$$

for every T in the dual of $\mathbb{B}(\mathcal{H})$. A field $(A_t)_{t\in\Omega}$ is said to be bounded if there is a positive constant M such that $\|A_t\| \leqslant M$ for all $t \in \Omega$. In particular, every bounded continuous field of operators on Ω is always Bochner integrable.

Lemma 6.4 ([83]). *Let $(A_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathbb{B}(\mathcal{H})$. Then for any $X \in \mathbb{B}(\mathcal{H})$, we have*

$$\left(\int_\Omega A_t d\mu(t)\right) \boxtimes X = \int_\Omega (A_t \boxtimes X) d\mu(t). \tag{6.8}$$

6.3 Refined Callebaut-type inequalities for operators

From now on, let Ω be a locally compact Hausdorff space endowed with a finite Radon measure μ.

Lemma 6.5 ([70]). *Let $x, y > 0$ and $r \in (0,1)$. Then*

$$x^r y^{1-r} + x^{1-r} y^r + 2p\left(\sqrt{x} - \sqrt{y}\right)^2 + q\left(2\sqrt{xy} + x + y - 2x^{\frac{1}{4}} y^{\frac{3}{4}} - 2x^{\frac{3}{4}} y^{\frac{1}{4}}\right) \leqslant x + y, \tag{6.9}$$

where $p = \min\{r, 1-r\}$ and $q = \min\{2p, 1-2p\}$.

Lemma 6.6. *Let $A, B \in \mathbb{B}(\mathcal{H})^+$ and either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$. Then*

$$\begin{aligned} A^\beta \boxtimes B^{1-\beta} &+ A^{1-\beta} \boxtimes B^\beta \\ &\geqslant A^\alpha \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^\alpha + \delta\Big(A^\beta \boxtimes B^{1-\beta} + A^{1-\beta} \boxtimes B^\beta - 2A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\Big) \\ &\quad + \eta\Big(A^\beta \boxtimes B^{1-\beta} + A^{1-\beta} \boxtimes B^\beta + 2A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}} - 2A^\gamma \boxtimes B^{1-\gamma} - 2A^{1-\gamma} \boxtimes B^\gamma\Big), \end{aligned} \tag{6.10}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

Proof. If we replace y by x^{-1} and r by $\frac{1-u}{2}$ in (6.9), then we get

$$x^u + x^{-u} + 2p\left(x + x^{-1} - 2\right) + q\left(x + x^{-1} + 2 - 2x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}\right) \leqslant x + x^{-1}, \tag{6.11}$$

where $p = \min\left\{\frac{1-u}{2}, \frac{1+u}{2}\right\}$ and $q = \min\{2p, 1-2p\}$. Consider $v, w \in \mathbb{R}$ such that $v \leqslant w$. Applying the functional calculus on the spectrum of $A \boxtimes B$ with $u := \frac{v}{w}$ in (6.11), then we get

$$\begin{aligned} A^w \boxtimes B^{-w} &+ A^{-w} \boxtimes B^w \\ &\geqslant A^v \boxtimes B^{-v} + A^{-v} \boxtimes B^v + \left(\frac{w-v}{w}\right)\left(A^w \boxtimes B^{-w} + A^{-w} \boxtimes B^w - 2I \boxtimes I\right) \\ &\quad + \eta\left(A^w \boxtimes B^{-w} + A^{-w} \boxtimes B^w - 2I \boxtimes I - 2A^{\frac{w}{2}} \boxtimes B^{-\frac{w}{2}} - 2A^{-\frac{w}{2}} \boxtimes B^{\frac{w}{2}}\right), \end{aligned} \tag{6.12}$$

where $\eta = \min\left\{\frac{w-v}{w}, \frac{v}{w}\right\}$. Multiplying both sides of (6.12) by $A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}$ we reach

$$\begin{aligned} & A^{1+w} \boxtimes B^{1-w} + A^{1-w} \boxtimes B^{1+w} \\ & \geqslant A^{1+v} \boxtimes B^{1-v} + A^{1-v} \boxtimes B^{1+v} + \left(\frac{w-v}{w-1/2}\right)\left(A^{1+w} \boxtimes B^{1-w} + A^{1-w} \boxtimes B^{1+w} - 2A \boxtimes B\right) \\ & \quad + \eta\left(A^{1+w} \boxtimes B^{1-w} + A^{1-w} \boxtimes B^{1+w} + 2A \boxtimes B - 2A^{1+\frac{w}{2}} \boxtimes B^{1-\frac{w}{2}} - 2A^{1-\frac{w}{2}} \boxtimes B^{1+\frac{w}{2}}\right). \end{aligned} \tag{6.13}$$

Replacing v, w, A, B by $2\alpha - 1, 2\beta - 1, A^{\frac{1}{2}}, B^{\frac{1}{2}}$, respectively, in (6.13), we obtain the result. ☐

For any bounded continuous fields $\mathcal{X} = (X_t)_{t\in\Omega}$ and $\mathcal{W} = (W_t)_{t\in\Omega}$ of operators in $\mathbb{B}(\mathcal{H})$, we set

$$\mathcal{F}_{\mathcal{W}}(\mathcal{X}) = \int_{\Omega} W_t^* X_t W_t \, d\mu(t).$$

For any bounded continuous field $\mathcal{X} = (X_t)_{t\in\Omega}$ of operators in $\mathbb{B}(\mathcal{H})^+$ and $\alpha \in [0,1]$, we set $\mathcal{X}^\alpha = (X_t^\alpha)_{t\in\Omega}$.

Lemma 6.7. *Let $\mathcal{X} = (X_t)_{t\in\Omega}$ and $\mathcal{W} = (W_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathbb{B}(\mathcal{H})^+$ and $\mathbb{B}(\mathcal{H})$, respectively. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, then*

$$\begin{aligned}
&\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \\
&\quad \geqslant \mathcal{F}_{\mathcal{W}}(\mathcal{X})^{\alpha} \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\alpha}) \\
&\qquad + \delta\Big[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) - 2\mathbb{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\Big] \\
&\qquad + \eta\Big[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) + 2\mathbb{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \\
&\qquad - 2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma}) - 2\mathbb{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma})\Big],
\end{aligned} \tag{6.14}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

Proof. By using Lemmas 6.2 and 6.4, and Fubini's theorem for Bochner integrals [72], we get

$$\begin{aligned}
\mathcal{F}_{\mathcal{W}}(\mathcal{X})^{\alpha} \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) &= \int_{\Omega} W_t^* A_t^{\alpha} W_t d\mu(t) \boxtimes \int_{\Omega} W_s^* X_s^{1-\alpha} W_s d\mu(s) \\
&= \iint_{\Omega^2} \left(W_t^* X_t^{\alpha} W_t\right) \boxtimes \left(W_s^* X_s^{1-\alpha} W_s\right)\, d\mu(t)\mu(s) \\
&= \iint_{\Omega^2} (W_t \boxtimes W_s)^* \left(X_t^{\alpha} \boxtimes X_s^{1-\alpha}\right) (W_t \boxtimes W_s)\, d\mu(t)\mu(s).
\end{aligned}$$

We have by applying Lemma 6.6 that

$$
\begin{aligned}
&\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \\
&\quad= \iint_{\Omega^2} (W_t \boxtimes W_s)^* \left(X_t^{\beta} \boxtimes X_s^{1-\beta} + X_t^{1-\beta} \boxtimes X_s^{\beta} \right) (W_t \boxtimes W_s)\ d\mu(t)\mu(s) \\
&\quad\geqslant \iint_{\Omega^2} (W_t \boxtimes W_s)^* \Big[X_t^{\alpha} \boxtimes X_s^{1-\alpha} + X_t^{1-\alpha} \boxtimes X_s^{\alpha} + \delta \Big(X_t^{\beta} \boxtimes X_s^{1-\beta} + X_t^{1-\beta} \boxtimes X_s^{\beta} \\
&\qquad - 2X_t^{\frac{1}{2}} \boxtimes X_s^{\frac{1}{2}} \Big) + \eta \Big(X_t^{\beta} \boxtimes X_s^{1-\beta} + X_t^{1-\beta} \boxtimes X_s^{\beta} + 2X_t^{\frac{1}{2}} \boxtimes X_s^{\frac{1}{2}} - 2X_t^{\gamma} \boxtimes X_t^{1-\gamma} \\
&\qquad - 2X_t^{1-\gamma} \boxtimes X_t^{\gamma} \Big) \Big] (W_t \boxtimes W_s)\ d\mu(t)\mu(s) \\
&\quad= \mathcal{F}_{\mathcal{W}}(\mathcal{X})^{\alpha} \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\alpha}) \\
&\qquad + \delta \Big[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) - 2\mathbb{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \Big] \\
&\qquad + \eta \Big[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) + 2\mathbb{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \\
&\qquad - 2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma}) - 2\mathbb{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma}) \Big]. \qquad \square
\end{aligned}
$$

For two continuous fields $A = (A_t)_{t\in\Omega}, B = (B_t)_{t\in\Omega}$ of operators in $\mathbb{B}(\mathcal{H})^+$ and any $\alpha \in [0, 1]$, we set

$$\mathcal{I}_{\alpha}(\mathcal{A}, \mathcal{B}) \;=\; \int_{\Omega} A_t \sharp_{\alpha} B_t d\mu(t).$$

Here, $\sharp_{\alpha}$ denotes the α-weighted geometric mean of operators X and Y in $\mathbb{B}(\mathcal{H})^+$ which is defined as

$$X \sharp_{\alpha} Y \;=\; X^{\frac{1}{2}} \big(X^{-\frac{1}{2}} Y X^{-\frac{1}{2}} \big)^{\alpha} X^{\frac{1}{2}}.$$

In particular,

$$\mathcal{I}_0(\mathcal{A}, \mathcal{B}) \;=\; \int_{\Omega} A_t d\mu(t), \quad \mathcal{I}_1(\mathcal{A}, \mathcal{B}) \;=\; \int_{\Omega} B_t d\mu(t).$$

Consider bounded continuous fields $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ of operators in $\mathbb{B}(\mathcal{H})^+$. From the result in [85], the integral Callebaut inequality states that for either $0 \leqslant \beta \leqslant \alpha \leqslant \frac{1}{2}$ or $\frac{1}{2} \leqslant \alpha \leqslant \beta \leqslant 1$,

$$
\begin{aligned}
2\mathcal{I}_{1/2}(\mathcal{A}, \mathcal{B}) \boxtimes \mathcal{I}_{1/2}(\mathcal{A}, \mathcal{B}) &\leqslant \mathcal{I}_{\alpha}(\mathcal{A}, \mathcal{B}) \boxtimes \mathcal{I}_{1-\alpha}(\mathcal{A}, \mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A}, \mathcal{B}) \boxtimes \mathcal{I}_{\alpha}(\mathcal{A}, \mathcal{B}) \\
&\leqslant \mathcal{I}_{\beta}(\mathcal{A}, \mathcal{B}) \boxtimes \mathcal{I}_{1-\beta}(\mathcal{A}, \mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A}, \mathcal{B}) \boxtimes \mathcal{I}_{\beta}(\mathcal{A}, \mathcal{B}) \\
&\leqslant \mathcal{I}_0(\mathcal{A}, \mathcal{B}) \boxtimes \mathcal{I}_1(\mathcal{A}, \mathcal{B}) + \mathcal{I}_1(\mathcal{A}, \mathcal{B}) \boxtimes \mathcal{I}_0(\mathcal{A}, \mathcal{B}).
\end{aligned} \tag{6.15}
$$

Now, we provide a refinement of the integral Callebaut inequality (6.15) and as a consequence give an operator Callebaut type inequality for Khatri-Rao products.

Theorem 6.1. *Let $\mathcal{A}=(A_t)_{t\in\Omega}$ and $\mathcal{B}=(B_t)_{t\in\Omega}$ be two bounded continuous fields of operators in $\mathbb{B}(\mathcal{H})^+$. If either $0\leqslant\beta\leqslant\alpha<\frac{1}{2}$ or $\frac{1}{2}<\alpha\leqslant\beta\leqslant 1$, then*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\\
&\quad\geqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\\
&\qquad+\delta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\Big]\\
&\qquad+\eta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})+2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\\
&\qquad-2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\Big],
\end{aligned}\tag{6.16}$$

where $\gamma=\frac{1+2\beta}{4}$, $\delta=\frac{\beta-\alpha}{\beta-1/2}$ and $\eta=\min\{\delta,1-\delta\}$.

Proof. Setting $X_t=A_t^{-\frac{1}{2}}B_tA_t^{-\frac{1}{2}}$ and $W_t=A_t^{\frac{1}{2}}$ for all $t\in\Omega$, we have that for any $\alpha\in[0,1]$,

$$\mathcal{F}_\mathcal{W}(\mathcal{X}^\alpha)\;=\;\int_\Omega A_t^{\frac{1}{2}}\left(A_t^{-\frac{1}{2}}B_tA_t^{-\frac{1}{2}}\right)^\alpha A_t^{\frac{1}{2}}d\mu(t)\;=\;\int_\Omega A_t\sharp_\alpha B_t d\mu(t)\;=\;\mathcal{I}_\alpha(\mathcal{A},\mathcal{B}).$$

By using Lemma 6.7, we obtain the result. □

Corollary 6.8. *Let $\mathcal{A}=(A_t)_{t\in\Omega}$ and $\mathcal{B}=(B_t)_{t\in\Omega}$ be two bounded continuous fields of operators in $\mathbb{B}(\mathcal{H})^+$. If either $0\leqslant\beta\leqslant\alpha<\frac{1}{2}$ or $\frac{1}{2}<\alpha\leqslant\beta\leqslant 1$, then*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\\
&\quad\geqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\\
&\qquad+\delta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\beta(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\Big]\\
&\qquad+\eta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\beta(\mathcal{A},\mathcal{B})+2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\\
&\qquad-2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\Big],
\end{aligned}\tag{6.17}$$

where $\gamma=\frac{1+2\beta}{4}$, $\delta=\frac{\beta-\alpha}{\beta-1/2}$ and $\eta=\min\{\delta,1-\delta\}$

Proof. We have that for any $\alpha \in [0,1]$,

$$\Phi\left(\mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})\right) \;=\; \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}),$$

where Φ is the linear map defined in Lemma 6.3. The proof is done by using Theorem 6.1 and the fact that the map Φ is a positive unital linear map. □

The next result is an integral inequality involving tensor products which is a special case of Theorem 6.1 when $n = 1$. More precisely we obtain an integral inequality involving Hadamard products as a consequence.

Corollary 6.9. *Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be two bounded continuous fields of operators in $\mathbb{B}(\mathcal{H})^+$. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, then*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\beta(\mathcal{A},\mathcal{B})\\
&\quad\geqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\\
&\qquad + \delta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\Big]\\
&\qquad + \eta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) + 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\\
&\qquad - 2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\Big],
\end{aligned} \tag{6.18}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$

Corollary 6.10. *Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be two bounded continuous fields of operators in $\mathbb{B}(\mathcal{H})^+$. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, then*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_\beta(\mathcal{A},\mathcal{B})\\
&\quad\geqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\\
&\qquad + \delta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\Big]\\
&\qquad + \eta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) + 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\\
&\qquad - 2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\Big],
\end{aligned} \tag{6.19}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$

Proof. Using the fact that the Hadamard product is expressed as the deformation of the tensor product via the isometry U defined in (6.3), we get the result. $\square$

Remark 6.11. *When we set $\Omega = \{1, \ldots, k\}$ equiped with the counting measure, we have that $\mathcal{I}_\alpha(\mathcal{A}, \mathcal{B}) = \sum_{i=1}^{k} A_i \sharp_\alpha B_i$. From previous theorem and previous corollaries, we obtain discrete versions of refined Callebaut-type inequalities for Tracy-Singh products, Khatri-Rao products, tensor products and Hadamard products, respectively.*

Remark 6.12. *For a particular case of Theorem 6.1 when $\mathcal{H} = \mathbb{C}^n$ and $\Omega = \{1, \ldots, k\}$ equiped with the counting measure, we get a matrix inequality concerning Tracy-Singh products. In the same way, we get matrix versions of* (6.17) -(6.19) *for Khatri-Rao products, Kronecker products and Hadamard products, respectively. Matrix versions of Kronecker products and Hadamard products are refinements of matrix Callebaut inequalities in [79, Theorem 3.4 and Corollary 3.5].*

In the next corollary, we give a refined Callebaut-type inequality for real numbers.

Corollary 6.13. *Let $x = (x_t)_{t\in\Omega}$ and $y = (y_t)_{t\in\Omega}$ be two fields of positive real numbers. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, then*

$$\begin{aligned}\mathcal{I}_\beta(x,y)\cdot\mathcal{I}_{1-\beta}(x,y) \geqslant\ &\mathcal{I}_\alpha(x,y)\cdot\mathcal{I}_{1-\alpha}(x,y) + \delta\Big[\mathcal{I}_\beta(x,y)\cdot\mathcal{I}_{1-\beta}(x,y) - \big(\mathcal{I}_{1/2}(x,y)\big)^2\Big]\\ &+ \eta\Big[\mathcal{I}_\beta(x,y)\cdot\mathcal{I}_{1-\beta}(x,y) + \big(\mathcal{I}_{1/2}(x,y)\big)^2 - 2\mathcal{I}_\gamma(x,y)\cdot\mathcal{I}_{1-\gamma}(x,y)\Big],\end{aligned}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

Proof. Putting $A_t = x_t I$ and $B_t = y_t I$ for all $t \in \Omega$ in Theorem 6.1, we obtain the result. $\square$

We mention that if Ω is the finite set $\{1, \ldots, k\}$ equiped with the counting measure, we get a discrete version of (6.20) which is a refinement of the classical Callebaut inequality (6.2).

6.4 Reversed Callebaut-type inequalities for operators

In this section, we present reversed inequalities of Callebaut-type inequalities.

Lemma 6.14 ([88]). *Let $x, y \geqslant 0$ and $r \in (0,1)$.*

$$x+y \ \leqslant \ x^{1-r}y^r + x^r y^{1-r} + 2s(\sqrt{x}-\sqrt{y})^2 - q\left(x+y+2\sqrt{xy} - 2x^{\frac{1}{4}}y^{\frac{3}{4}} - 2x^{\frac{3}{4}}y^{\frac{1}{4}}\right),$$

where $p = \min\{r, 1-r\}$, $q = \min\{2p, 1-2p\}$ and $s = \max\{r, 1-r\}$.

Lemma 6.15. *Let $A, B \in \mathbb{B}(\mathcal{H})^+$ and either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$. Then*

$$\begin{aligned} A^{\beta}\boxtimes B^{1-\beta} + A^{1-\beta}\boxtimes B^{\beta} &\\ \leqslant \ A^{\alpha}\boxtimes B^{1-\alpha} + A^{1-\alpha}\boxtimes B^{\alpha} + (2-\delta)\left(A^{\beta}\boxtimes B^{1-\beta} + A^{1-\beta}\boxtimes B^{\beta} - 2A^{\frac{1}{2}}\boxtimes B^{\frac{1}{2}}\right) &\\ - \eta\left(A^{\beta}\boxtimes B^{1-\beta} + A^{1-\beta}\boxtimes B^{\beta} + 2A^{\frac{1}{2}}\boxtimes B^{\frac{1}{2}} - 2A^{\gamma}\boxtimes B^{1-\gamma} - 2A^{1-\gamma}\boxtimes B^{\gamma}\right), \end{aligned} \tag{6.20}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

Proof. In Lemma 6.14, we have by replacing y with x^{-1} that

$$x + x^{-1} \ \leqslant \ x^{1-2r} + x^{2r-1} + 2s(x+x^{-1}-2) - q(x+x^{-1}+2-2x^{-\frac{1}{2}} - 2x^{\frac{1}{2}}).$$

Let $u \in (0,1]$. Taking $r = \frac{1-u}{2}$, we obtain

$$x + x^{-1} \ \leqslant \ x^{u} + x^{-u} + (1+u)\left(x+x^{-1}-2\right) - q\left(x+x^{-1}+2-2x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}\right).$$

Consider real numbers v, w such that $\frac{v}{w} \in (0,1]$. Using the functional calculus on the spectrum of $A \boxtimes B$ and Lemma 6.2, and putting $u = \frac{v}{w}$, we get

$$\begin{aligned} A^{w}\boxtimes B^{-w} + A^{-w}\boxtimes B^{w} &\\ \leqslant \ A^{v}\boxtimes B^{-v} + A^{-v}\boxtimes B^{v} + \left(1+\frac{v}{w}\right)\left(A^{w}\boxtimes B^{-w} + A^{-w}\boxtimes B^{w} - 2I\boxtimes I\right) &\\ - \eta\left(A^{w}\boxtimes B^{-w} + A^{-w}\boxtimes B^{w} + 2I\boxtimes I - 2A^{\frac{w}{2}}\boxtimes B^{-\frac{w}{2}} - 2A^{-\frac{w}{2}}\boxtimes B^{\frac{w}{2}}\right). \end{aligned}$$

Multiplying both sides by $A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}$ and applying Lemma 6.2, we have

$$\begin{aligned}
&A^{1+w} \boxtimes B^{1-w} + A^{1-w} \boxtimes B^{1+w} \\
&\quad \leqslant A^{1+v} \boxtimes B^{1-v} + A^{1-v} \boxtimes B^{1+v} + \left(1 + \frac{v}{w}\right)\left(A^{1+w} \boxtimes B^{1-w} + A^{1-w} \boxtimes B^{1+w} - 2A \boxtimes B\right) \\
&\qquad - \eta\left(A^{1+w} \boxtimes B^{1-w} + A^{1-w} \boxtimes B^{1+w} + 2A \boxtimes B - 2A^{1+\frac{w}{2}} \boxtimes B^{1-\frac{w}{2}} - 2A^{1-\frac{w}{2}} \boxtimes B^{1+\frac{w}{2}}\right),
\end{aligned}$$

where $\gamma = \min\left\{\frac{v}{w}, 1 - \frac{v}{w}\right\}$. We reach the result by replace v, w, A, B with $2\alpha - 1, 2\beta - 1, A^{\frac{1}{2}}, B^{\frac{1}{2}}$, respectively. □

Lemma 6.16. *Let $\mathcal{X} = (X_t)_{t\in\Omega}$ and $\mathcal{W} = (W_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathbb{B}(\mathcal{H})^+$ and $\mathbb{B}(\mathcal{H})$, respectively. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, then*

$$\begin{aligned}
&\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}_t^{1-\beta}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}_t^{1-\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \\
&\quad \leqslant \mathcal{F}_{\mathcal{W}}(X_t^{\alpha}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\alpha}) \\
&\qquad + (2-\delta)\Big[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) - 2\mathbb{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\Big] \\
&\qquad - \eta\Big[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) + 2\mathbb{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \\
&\qquad - 2\mathbb{F}_{\mathcal{W}}(\mathcal{X}^{\gamma}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma}) - 2\mathbb{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma}) \boxtimes \mathbb{F}_{\mathcal{W}}(\mathcal{X}^{\gamma})\Big],
\end{aligned} \tag{6.21}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

Proof. The proof is similar to that of Lemma 6.7. Instead of using Lemma 6.6, we apply Lemma 6.15.

□

The next theorem is a reverse of the second inequality of (6.15) involving Tracy-Singh products. As a consequence, we get a reversed Callebaut-type inequality for Khatri-Rao products by using the unital positive linear map Φ in Lemma 6.3. For the case $n = 1$, this theorem reduces to the reversed Callebaut-type inequality for tensor products and consequently applies to Hadamard products.

Theorem 6.2. *Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be two bounded continuous fields of operators in $\mathbb{B}(\mathcal{H})^+$. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$,* *then*

$$
\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \\
&\quad\leqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \\
&\qquad + (2-\delta)\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\big] \\
&\qquad - \eta\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) + 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \\
&\qquad - 2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\big],
\end{aligned}
\tag{6.22}
$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ *and* $\eta = \min\{\delta, 1-\delta\}$.

Proof. The proof is similar to that of Theorem 6.1. Instead of using Lemma 6.7, we apply Lemma 6.16. □

Corollary 6.17. *Let* $\mathcal{A} = (A_t)_{t\in\Omega}$ *and* $\mathcal{B} = (B_t)_{t\in\Omega}$ *be two bounded continuous fields of operators in* $\mathbb{B}(\mathcal{H})^+$. *If either* $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ *or* $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, *then*

$$
\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \\
&\quad\leqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \\
&\qquad + (2-\delta)\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\big] \\
&\qquad - \eta\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) + 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \\
&\qquad - 2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\big],
\end{aligned}
\tag{6.23}
$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ *and* $\eta = \min\{\delta, 1-\delta\}$.

Corollary 6.18. *Let* $\mathcal{A} = (A_t)_{t\in\Omega}$ *and* $\mathcal{B} = (B_t)_{t\in\Omega}$ *be two bounded continuous fields of operators in* $\mathbb{B}(\mathcal{H})^+$. *If either* $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ *or* $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, *then*

$$
\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \\
&\quad\leqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \\
&\qquad + (2-\delta)\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\big] \\
&\qquad - \eta\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) + 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \\
&\qquad - 2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\big],
\end{aligned}
\tag{6.24}
$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ *and* $\eta = \min\{\delta, 1-\delta\}$.

Corollary 6.19. *Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be two bounded continuous fields of operators in $\mathbb{B}(\mathcal{H})^+$. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, then*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \\
&\leqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \\
&\quad + (2-\delta)\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\big] \\
&\quad - \eta\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) + 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \\
&\quad - 2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\big],
\end{aligned} \tag{6.25}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

Remark 6.20. *From previous results, we obtain discrete versions of reversed Callebaut-type inequalities for Tracy-Singh products, Khatri-Rao products, tensor products and Hadamard products, respectively, by setting $\Omega = \{1,\dots,k\}$ equiped with the counting measure. For a particular case, when $\mathcal{H} = \mathbb{C}^n$, we get matrix versions of* (6.22)-(6.25). *Matrix versions of Kronecker products and Hadamard products are reverses of matrix Callebaut inequalities in [79, Theorem 3.4 and Corollary 3.5].*

The following corollary is a reverse of the integral Callebaut inequality for real numbers. In particular, when Ω is the finite set $\{1,\dots,k\}$ equiped with the counting measure, we get a reversed inequality of the second inequality of (6.2).

Corollary 6.21. *Let $x = (x_t)_{t\in\Omega}$ and $y = (y_t)_{t\in\Omega}$ be two fields of positive real numbers. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, then*

$$\begin{aligned}
\mathcal{I}_\beta(x,y) \cdot \mathcal{I}_{1-\beta}(x,y) \leqslant\ & \mathcal{I}_\alpha(x,y) \cdot \mathcal{I}_{1-\alpha}(x,y) + (2-\delta)\big[\mathcal{I}_\beta(x,y) \cdot \mathcal{I}_{1-\beta}(x,y) - \big(\mathcal{I}_{1/2}(x,y)\big)^2\big] \\
& - \eta\big[\mathcal{I}_\beta(x,y) \cdot \mathcal{I}_{1-\beta}(x,y) + \big(\mathcal{I}_{1/2}(x,y)\big)^2 - 2\mathcal{I}_\gamma(x,y) \cdot \mathcal{I}_{1-\gamma}(x,y)\big],
\end{aligned}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

6.5 Conclusion

We establish certain refinements and reverses of Callebaut-type inequalities for bounded continuous fields of operators which are parametrized by a locally compact Hausdorff space Ω equipped with a finite Radon measure.

These inequalities involve Tracy-Singh products, Khatri-Rao products, tensor products, Hadamard products and weighted geometric means. When Ω is a finite space equipped with the counting measure, such integral inequalities reduce to discrete inequalities. Our results include matrix results concerning the Tracy-Singh product, the Khatri-Rao product, the Kronecker product, and the Hadamard product. In particular, we get a refinement and a reverse of the classical Callebaut inequality for real numbers.

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ภาคผนวก

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Algebraic and Order Properties of Tracy-Singh Products for Operator Matrices

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Abstract

We generalize the tensor product for operators to the Tracy-Singh product for operator matrices acting on the direct sum of Hilbert spaces. This kind of operator product is compatible with algebraic operations and order relations for operators. It follows that this product preserves many structure properties of operators.

Keywords: tensor product, Tracy-Singh product, operator matrix, Moore-Penrose inverse
Mathematics Subject Classifications 2010: 15A69, 47A05, 47A80.

1 Introduction

In scientific computing, we consider a matrix to be a two-dimensional array for stacking data. A processing of such data can be performed using matrix products. One of extremely useful matrix products is the Kronecker product. For any complex matrices $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, the Kronecker product of A and B is given by the block matrix

$$A \hat{\otimes} B = [a_{ij}B]_{ij} \in M_{mp,nq}(\mathbb{C}).$$

Equivalently, $A \hat{\otimes} B$ is the unique complex matrix of order $mp \times nq$ satisfying

$$(A \hat{\otimes} B)(x \hat{\otimes} y) = Ax \hat{\otimes} By \tag{1}$$

for all $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^q$. This matrix product has wide applications in mathematics, computer science, statistics, physics, system theory, signal processing, and related fields. See [2, 5, 6, 12] for more information.

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Kronecker product was generalized to the Tracy-Singh product of partitioned matrices by Tracy and Singh [10]. Let $A = [A_{ij}] \in M_{m,n}(\mathbb{C})$ be a partitioned matrix with A_{ij} of order $m_i \times n_j$ as the (i,j)th submatrix where $\sum_i m_i = m$ and $\sum_j n_j = n$. Let $B = [B_{kl}] \in M_{p,q}(\mathbb{C})$ be a partitioned matrix with B_{kl} of order $p_k \times q_l$ as the (k,l)th submatrix where $\sum_k p_k = p$ and $\sum_l q_l = q$. The Tracy-Singh product of A and B is defined by

$$A \hat{\boxtimes} B \;=\; \big[\big[A_{ij} \hat{\otimes} B_{kl}\big]_{kl}\big]_{ij} \in M_{mp,nq}(\mathbb{C}),$$

where each block $A_{ij} \hat{\otimes} B_{kl}$ is of order $m_i p_k \times n_j q_l$. This kind of matrix product has several attractive properties in algebraic, order, and analytic points of views; see, *e.g.*, [3, 8, 9, 10]. The Tracy-Singh product can be applied widely in statistics, econometrics and related fields; see, *e.g.*, [9, 10].

As a natural generalization of a complex matrix, we consider a bounded linear operator between complex Hilbert spaces. The tensor product of Hilbert space operators can be viewed as an extension of the Kronecker product of complex matrices. Using the universal mapping property in the monoidal category of Hilbert spaces, the tensor product of $A \in \mathbb{B}(\mathcal{H},\mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K},\mathcal{K}')$ is the unique bounded linear operator from $\mathcal{H} \otimes \mathcal{K}$ into $\mathcal{H}' \otimes \mathcal{K}'$ such that for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$,

$$(A \otimes B)(x \otimes y) \;=\; Ax \otimes By. \tag{2}$$

A fundamental property of tensor product is the mixed product property:

$$(A \otimes B)(C \otimes D) \;=\; AC \otimes BD. \tag{3}$$

The theory of tensor product of operators has been continuously developed in the literature; see, *e.g.*, [4, 11].

From the previous discussion, it is natural to extend the notion of tensor product for operators to the "Tracy-Singh product"of operators. We shall propose a natural definition of such operator product. It turns out that this product is compatible with algebraic operations and order relations for operators. One of the most attractive properties, the mixed product property, also holds for Tracy-Singh products. It follows that this product preserves attractive properties of operators, such as being invertible, Hermitian, unitary, positive, and normal. Our results generalize the results known so far in the literature for both Tracy-Singh products of matrices and tensor products of operators.

This paper is organized as follows. In section 2, we introduce the Tracy-Singh product for operator matrices and deduce its algebraic properties. In section 3, we show that the Tracy-Singh product is compatible with various kinds of operator inverses. We investigate the relationship between Tracy-Singh products and operator orderings in Section 4.

2 Tracy-Singh products and algebraic operations for operators

In this section, we introduce the Tracy-Singh product of operators on a Hilbert space. Then we will show that this product is compatible with addition, scalar multiplication, adjoint operation, usual multiplication, power, and direct sum of operator inverses.

Throughout this paper, let $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{K}$ and $\mathcal{K}'$ be complex Hilbert spaces. When $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, denote by $\mathbb{B}(\mathcal{X},\mathcal{Y})$ the Banach space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, and abbreviate $\mathbb{B}(\mathcal{X},\mathcal{X})$ to $\mathbb{B}(\mathcal{X})$.

The projection theorem for Hilbert spaces allows us to decompose

$$\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^{m} \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{l=1}^{q} \mathcal{K}_l, \quad \mathcal{K}' = \bigoplus_{k=1}^{p} \mathcal{K}'_k$$

where each $\mathcal{H}_j, \mathcal{H}'_i, \mathcal{K}_l, \mathcal{K}'_k$ are Hilbert spaces. Such decompositions are fixed throughout the paper. For each $j = 1, \dots, n$, let E_j be the canonical embedding from $\mathcal{H}_j$ into $\mathcal{H}$, defined by

$$x_j \mapsto (0, \dots, 0, x_j, 0, \dots, 0).$$

Similarly, let F_l be the canonical embedding from $\mathcal{K}_l$ into $\mathcal{K}$ for each $l = 1, \dots, q$. For each $i = 1, \dots, m$ and $k = 1, \dots, p$, let $P'_i : \mathcal{H}' \to \mathcal{H}'_i$ and $Q'_k : \mathcal{K}' \to \mathcal{K}'_k$ be the orthogonal projections. Thus, each operator $A \in \mathbb{B}(\mathcal{H},\mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K},\mathcal{K}')$ can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,n} \quad \text{and } B = [B_{kl}]_{k,l=1}^{p,q}$$

where $A_{ij} = P'_i A E_j$ and $B_{kl} = Q'_k B F_l$ for each i, j, k, l.

Definition 1. *Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H},\mathcal{H}')$ and $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K},\mathcal{K}')$ be operator matrices defined as above. We define the Tracy-Singh product of A and B to be the operator matrix*

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij} \tag{4}$$

which is a bounded linear operator from $\bigoplus_{j=1}^{n} \bigoplus_{l=1}^{q} \mathcal{H}_j \otimes \mathcal{K}_l$ to $\bigoplus_{i=1}^{m} \bigoplus_{k=1}^{p} \mathcal{H}'_i \otimes \mathcal{K}'_k$.

Note that if both A and B are 1×1 block operator matrices i.e. $m = n = p = q = 1$, then their Tracy-Singh product $A \boxtimes B$ is just the tensor product $A \otimes B$.

Next, we shall show that the Tracy-Singh product of two linear maps induced by two matrices is just the linear map induced by the Tracy-Singh product of these matrices. Recall that for each $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, the induced maps

$$L_A : \mathbb{C}^n \to \mathbb{C}^m, \; x \mapsto Ax \quad \text{and} \quad L_B : \mathbb{C}^q \to \mathbb{C}^p, \; y \mapsto By$$

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are bounded linear operators. Using the universal mapping property, we identify $\mathbb{C}^n \otimes \mathbb{C}^q$ with $\mathbb{C}^{nq} \cong M_{n,q}(\mathbb{C})$ together with the canonical bilinear map $(x, y) \mapsto x \,\hat{\otimes}\, y$ for each $(x, y) \in \mathbb{C}^n \times \mathbb{C}^q$. It is similar for $\mathbb{C}^m \otimes \mathbb{C}^p$.

Lemma 2. *For each $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, we have*

$$L_A \otimes L_B \;=\; L_{A \,\hat{\otimes}\, B}. \tag{5}$$

Proof. For any $x \otimes y \in \mathbb{C}^n \otimes \mathbb{C}^q$, we obtain from the mixed product property of the Kronecker product (1) that

$$\begin{aligned}(L_A \otimes L_B)(x \otimes y) \;&=\; L_A(x) \otimes L_B(y) \;=\; L_A(x) \,\hat{\otimes}\, L_B(y) \\ &=\; Ax \,\hat{\otimes}\, By \;=\; (A \,\hat{\otimes}\, B)(x \,\hat{\otimes}\, y) \\ &=\; (A \,\hat{\otimes}\, B)(x \otimes y) \;=\; L_{A \,\hat{\otimes}\, B}(x \otimes y).\end{aligned}$$

Thus, by the uniqueness of tensor product, $L_A \otimes L_B = L_{A \,\hat{\otimes}\, B}$. □

Proposition 3. *For any complex matrices $A = [A_{ij}]$ and $B = [B_{kl}]$ partitioned in block-matrix forms, we have*

$$L_A \boxtimes L_B \;=\; L_{A \,\hat{\boxtimes}\, B}. \tag{6}$$

Proof. Recall that the (i, j)th block of the matrix representation of L_A is the matrix A_{ij}. It follows from Lemma 2 that

$$L_A \boxtimes L_B \;=\; \left[\left[L_{A_{ij}} \otimes L_{B_{kl}} \right]_{kl} \right]_{ij} \;=\; \left[\left[L_{A_{ij} \,\hat{\otimes}\, B_{kl}} \right]_{kl} \right]_{ij} \;=\; L_{A \,\hat{\boxtimes}\, B}.$$

□

The next proposition shows that the Tracy-Singh product is compatible with the addition, the scalar multiplication and the adjoint operation of operators.

Proposition 4. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B, C \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices, and let $\alpha \in \mathbb{C}$. Then*

$$(\alpha A) \boxtimes B \;=\; \alpha(A \boxtimes B) \;=\; A \boxtimes (\alpha B), \tag{7}$$

$$(A \boxtimes B)^* \;=\; A^* \boxtimes B^*, \tag{8}$$

$$A \boxtimes (B + C) \;=\; A \boxtimes B + A \boxtimes C, \tag{9}$$

$$(B + C) \boxtimes A \;=\; B \boxtimes A + C \boxtimes A. \tag{10}$$

Proof. Since each (i, j)th block of αA is given by $(\alpha A)_{ij} = \alpha A_{ij}$, we get

$$(\alpha A) \boxtimes B \;=\; \left[[(\alpha A_{ij}) \otimes B_{kl}]_{kl} \right]_{ij} \;=\; \left[[\alpha (A_{ij} \otimes B_{kl})]_{kl} \right]_{ij} \;=\; \alpha(A \boxtimes B).$$

Similarly, $A \boxtimes (\alpha B) = \alpha(A \boxtimes B)$. Since $A^* = [A^*_{ji}]_{ij}$ and $B^* = [B^*_{lk}]_{kl}$ for all i, j, k, l, we obtain

$$(A \boxtimes B)^* \;=\; \left[[A_{ji} \otimes B_{kl}]^*_{kl} \right]_{ij} \;=\; \left[\left[A^*_{ji} \otimes B^*_{lk} \right]_{kl} \right]_{ij} \;=\; A^* \boxtimes B^*.$$

The proofs of (9) and (10) are done by using the fact that $(B+C)_{kl} = B_{kl} + C_{kl}$ for all k, l together with the left/right distributivity of the tensor product over the addition. □

Properties (7), (9) and (10) say that the map $(A, B) \mapsto A \boxtimes B$ is bilinear.

Proposition 5. *Let $A = [A_{ij}] \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and let $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices. Then*

$$A \boxtimes B \,=\, [A_{ij} \boxtimes B]_{ij} \,=\, \begin{bmatrix} A_{11} \boxtimes B & \cdots & A_{1n} \boxtimes B \\ \vdots & \ddots & \vdots \\ A_{m1} \boxtimes B & \cdots & A_{mn} \boxtimes B \end{bmatrix}.$$

That is, the (i, j)th block of $A \boxtimes B$ is just $A_{ij} \boxtimes B$, regardless of how to partition B.

Proof. It follows directly from the definition of the Tracy-Singh product. □

Remark 6. *It is not true in general that the (k, l)th block of $A \boxtimes B$ is $A \boxtimes B_{kl}$.*

When $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, the direct sum of $A_1 \in \mathbb{B}(\mathcal{H}_1, \mathcal{K}_1)$ and $A_2 \in \mathbb{B}(\mathcal{H}_2, \mathcal{K}_2)$ is defined to be the operator

$$A_1 \oplus A_2 \,=\, \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in \mathbb{B}(\mathcal{H}, \mathcal{K}).$$

The next result gives a relation between the direct sum and the Tracy-Singh product.

Proposition 7. *The Tracy-Singh product is right distributive over the direct sum of operators. That is, for any operator matrices A, B and C, we have*

$$(A \oplus B) \boxtimes C \,=\, (A \boxtimes C) \oplus (B \boxtimes C). \tag{11}$$

Proof. It follows from Proposition 5 that

$$\begin{aligned}(A \oplus B) \boxtimes C \,&=\, \begin{bmatrix} A \boxtimes C & 0 \boxtimes C \\ 0 \boxtimes C & B \boxtimes C \end{bmatrix} \,=\, \begin{bmatrix} A \boxtimes C & 0 \\ 0 & B \boxtimes C \end{bmatrix} \\ &=\, (A \boxtimes C) \oplus (B \boxtimes C).\end{aligned}$$ □

It is not true in general that the Tracy-Singh product is left distributive over the direct sum of operators.

The next theorem shows that the Tracy-Singh product is compatible with the ordinary product of operators. This fundamental property, called the *mixed product property*, will be used many times in later discussions.

Theorem 8. *Let $\mathcal{H}, \mathcal{H}', \mathcal{H}'', \mathcal{K}, \mathcal{K}'$ and $\mathcal{K}''$ be complex Hilbert spaces. Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}', \mathcal{H}'')$, $C = [C_{ij}]_{i,j=1}^{n,r} \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$, $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K}', \mathcal{K}'')$ and $D = [D_{kl}]_{k,l=1}^{q,s} \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices partitioned so that they are compatible with the decompositions of the corresponding Hilbert spaces. Then*

$$(A \boxtimes B)(C \boxtimes D) \,=\, AC \boxtimes BD. \tag{12}$$

Proof. Using block multiplication of operators and the mixed product property of the tensor product (3), we have

$$\begin{aligned}(A \boxtimes B)(C \boxtimes D) &= \left[[A_{ij} \otimes B_{kl}]_{kl}\right]_{ij} \left[[C_{ij} \otimes D_{kl}]_{kl}\right]_{ij} \\ &= \left[\left[\sum_{\alpha=1}^{n}\sum_{\beta=1}^{q}(A_{i\alpha} \otimes B_{k\beta})(C_{\alpha j} \otimes D_{\beta l})\right]_{kl}\right]_{ij} \\ &= \left[\left[\sum_{\alpha=1}^{n}\sum_{\beta=1}^{q}(A_{i\alpha}C_{\alpha j} \otimes B_{k\beta}D_{\beta l})\right]_{kl}\right]_{ij} \\ &= \left[\sum_{\alpha=1}^{n} A_{i\alpha}C_{\alpha j}\right]_{ij} \boxtimes \left[\sum_{\beta=1}^{q} B_{k\beta}D_{\beta l}\right]_{kl} \\ &= AC \boxtimes BD.\end{aligned}$$

□

Corollary 9. *For any operator matrices $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$, we have*

$$(A \boxtimes B)^r = A^r \boxtimes B^r \tag{13}$$

for any $r \in \mathbb{N}$.

In the rest of section, we investigate structure properties of operators under taking Tracy-Singh products. Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is said to be *involutary* if $T^2 = I$, *idempotent* if $T^2 = T$, an *isometry* if $T^*T = I$, a *partial isometry* if the restriction of T to a closed subspace is an isometry, or equivalently, $TT^*T = T$.

Corollary 10. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. If both A and B satisfy one of the following properties, then the same property holds for $A \boxtimes B$: Hermitian, unitary, isometry, co-isometry, partial isometry, idempotent, involutary, projection.*

Proof. Applying Theorem 8 and Proposition 4, we get the results. □

If A and B are skew-Hermitian operators, then $A \boxtimes B$ is Hermitian. Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is said to be *nilpotent* if there is a positive integer k such that $T^k = 0$. The smallest such integer k is called the degree of nilpotency of T. If $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ are nilpotent operators with degrees of nilpotency r and s, respectively, then $A \boxtimes B$ is also nilpotent with degree of nilpotency not exceed $\min\{r, s\}$.

3 Tracy-Singh products and operator inverses

Next, we discuss the invertibility of the Tracy-Singh product of operators. Recall that an operator $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ is said to be *regular* if there is an operator $A^- \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ such that $AA^-A = A$. The operator A^- is called an *inner inverse* of A. An operator $X \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ is said to be an *outer inverse* of A if $XAX = X$.

Proposition 11. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$.*

(i) If A and B are left invertible with left inverses $\hat{A}$ and $\hat{B}$ respectively, then $A \boxtimes B$ is left invertible and $\hat{A} \boxtimes \hat{B}$ is its left inverse.

(ii) If A and B are right invertible with right inverses $\hat{A}$ and $\hat{B}$ respectively, then $A \boxtimes B$ is right invertible and $\hat{A} \boxtimes \hat{B}$ is its right inverse.

(iii) If A and B are regular with inner inverses A^- and B^- respectively, then $A \boxtimes B$ is regular with $A^- \boxtimes B^-$ as its inner inverse.

(iv) If A and B have A^- and B^- as their outer inverses respectively, then $A \boxtimes B$ has $A^- \boxtimes B^-$ as its outer inverse.

Proof. It follows from Theorem 8 and the facts that $I_{\mathcal{X}} \boxtimes I_{\mathcal{Y}} = I_{\mathcal{X} \otimes \mathcal{Y}}$ for any Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$. □

As a consequence of (i) and (ii) in Proposition 11, we obtain the following result.

Corollary 12. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. If A and B are invertible, then $A \boxtimes B$ is invertible and*

$$(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}. \tag{14}$$

Next, we consider a kind of operator inverse, called Moore-Penrose inverse. Recall that a *Moore-Penrose inverse* of $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ is an operator $A^\dagger \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ satisfying the following Penrose conditions ([7])

(i) $A^\dagger$ is an inner inverse of A ;

(ii) $A^\dagger$ is an outer inverse of A ;

(iii) $AA^\dagger$ is Hermitian ;

(iv) $A^\dagger A$ is Hermitian.

It is well known that the following statements are equivalent for $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ (see *e.g.* [1]):

(i) a Moore-Penrose inverse of A exists ;

(ii) a Moore-Penrose inverse of A is unique ;

(iii) the range of A is closed.

Theorem 13. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. If A and B have closed ranges, then*

1. the range of $A \boxtimes B$ is closed ;

2. $(A \boxtimes B)^\dagger = A^\dagger \boxtimes B^\dagger$.

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Proof. Since the ranges of A and B are closed, the Moore-Penrose inverses $A^\dagger$ and $B^\dagger$ exist and are unique. Making use of Theorem 8 and Proposition 4, we can verify that $A^\dagger \boxtimes B^\dagger$ satisfies the following Penrose equations:

(i) $(A \boxtimes B)(A^\dagger \boxtimes B^\dagger)(A \boxtimes B) = A \boxtimes B$

(ii) $(A^\dagger \boxtimes B^\dagger)(A \boxtimes B)(A^\dagger \boxtimes B^\dagger) = A^\dagger \boxtimes B^\dagger$

(iii) $\big((A \boxtimes B)(A^\dagger \boxtimes B^\dagger)\big)^* = (A \boxtimes B)(A^\dagger \boxtimes B^\dagger)$

(iv) $\big((A^\dagger \boxtimes B^\dagger)(A \boxtimes B)\big)^* = (A^\dagger \boxtimes B^\dagger)(A \boxtimes B)$.

Hence, a Moore-Penrose inverse of $A \boxtimes B$ exists and it is uniquely determined by $A^\dagger \boxtimes B^\dagger$. It follows that $A \boxtimes B$ has a closed range. □

The results in this section indicate that the Tracy-Singh product is compatible with various kinds of operator inverses.

4 Tracy-Singh products and operator orderings

Now, we focus on order properties of Tracy-Singh products related to algebraic properties.

Theorem 14. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$.*

(i) If $A, B \geqslant 0$, then $A \boxtimes B \geqslant 0$.

(ii) If $A, B > 0$, then $A \boxtimes B > 0$.

Proof. Assume $A, B \geqslant 0$. Using Theorem 8 and property (8), we obtain

$$\begin{aligned} A \boxtimes B \;&=\; A^{\frac{1}{2}}A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}B^{\frac{1}{2}} \;=\; \left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right)\left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right) \\ &=\; \left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right)^*\left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right) \;\geqslant\; 0. \end{aligned}$$

Consider the case $A, B > 0$. We have immediately by (i) that $A \boxtimes B \geqslant 0$. By Corollary 12, $A \boxtimes B$ is invertible. This implies that $A \boxtimes B > 0$. □

The next result provides the monotonicity of Tracy-Singh product.

Corollary 15. *Let $A_1, A_2 \in \mathbb{B}(\mathcal{H})$ and $B_1, B_2 \in \mathbb{B}(\mathcal{K})$.*

(i) If $A_1 \geqslant A_2 \geqslant 0$ and $B_1 \geqslant B_2 \geqslant 0$, then $A_1 \boxtimes B_1 \;\geqslant\; A_2 \boxtimes B_2$.

(ii) If $A_1 > A_2 > 0$ and $B_1 > B_2 > 0$, then $A_1 \boxtimes B_1 \;>\; A_2 \boxtimes B_2$.

Proof. Suppose that $A_1 \geqslant A_2 \geqslant 0$ and $B_1 \geqslant B_2 \geqslant 0$. Applying Proposition 4 and Theorem 14 yields

$$\begin{aligned} A_1 \boxtimes B_1 - A_2 \boxtimes B_2 \;&=\; A_1 \boxtimes B_1 - A_2 \boxtimes B_1 + A_2 \boxtimes B_1 - A_2 \boxtimes B_2 \\ &=\; (A_1 - A_2) \boxtimes B_1 + A_2 \boxtimes (B_1 - B_2) \\ &\geqslant\; 0. \end{aligned}$$

The proof of (ii) is similar to that of (i). □

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Analytic Properties of Tracy-Singh Products for Operator Matrices

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Abstract

We show that the Tracy-Singh product of Hilbert space operators is continuous with respect to the operator-norm topology. The Tracy-Singh product of two nonzero operators is compact if and only if both factors are compact. We provide upper and lower bounds for certain Schatten p-norms of the Tracy-Singh product of operators. It turns out that this product is continuous with respect to the topologies on norm ideals of compact operators, trace class operators, and Hilbert-Schmidt class operators. Thus the Tracy-Singh product preserves such classes of operators.

Keywords: tensor product, Tracy-Singh product, operator matrix, compact operator, Schatten p-class operator
Mathematics Subject Classifications 2010: 47A80, 47A30, 47B10.

1 Introduction

In matrix theory, one of useful matrix products is the Kronecker product. Recall that the Kronecker product of two complex matrices $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$ is given by the block matrix

$$A \hat{\otimes} B = [a_{ij}B]_{ij} \in M_{mp,nq}(\mathbb{C}).$$

This matrix product was generalized to the Tracy-Singh product by Tracy and Singh [3]. Let $A = [A_{ij}] \in M_{m,n}(\mathbb{C})$ be a partitioned matrix with A_{ij} as the (i,j)th submatrix. Let $B = [B_{kl}] \in M_{p,q}(\mathbb{C})$ be a partitioned matrix with B_{kl} as the (k,l)th submatrix. The Tracy-Singh product of A and B is defined by

$$A \hat{\boxtimes} B = \left[\left[A_{ij} \hat{\otimes} B_{kl}\right]_{kl}\right]_{ij} \in M_{mp,nq}(\mathbb{C}).$$

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This kind of matrix product has several attractive properties and can be applied widely in statistics, econometrics and related fields; see *e.g.*, [3, 5, 7, 8, 9].

The tensor product of Hilbert space operators is a natural extension of the Kronecker product to infinite-dimensional setting. Theory of Hilbert tensor product has been continuously investigated in the literature; see, *e.g.*, [2, 4, 10]. It is well known that the tensor product is continuous with respect to the operator-norm topology. Moreover, on the norm ideals of compact operators generated by Schatten p-norm for $p = 1, 2, \infty$, the tensor product are also continuous. Recently, the tensor product for operators was generalized to the Tracy-Singh product for operator matrices acting on the direct sum of Hilbert spaces in [6]. This kind of operator product satisfies certain pleasing algebraic and order properties.

In this paper, we discuss continuity, convergence, and compactness of the Tracy-Singh product for operators in the operator-norm topology. Then we obtain relations between Tracy-Singh product and certain analytic functions. We also investigate the Tracy-Singh product on norm ideals of compact operators generated by certain Schatten p-norms. In fact, this product is continuous with respect to the Schatten p-norm for $p = 1, 2, \infty$. Estimations by such norms for Tracy-Singh products are provided. It follows that trace class operators and Hilbert-Schmidt class operators are preserved under this product.

This paper is organized as follows. In section 2, we give preliminaries on Tracy-Singh products for operators on a Hilbert space. In section 3, we establish analytic properties of the Tracy-Singh product in the operator-norm topology. We investigate the Tracy-Singh product on the norm ideals of compact operators generated by certain Schatten p-norms in Section 4.

2 Preliminaries on Tracy-Singh products for operator matrices

Throughout, let $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{K}$ and $\mathcal{K}'$ be complex Hilbert spaces . When X and Y are Hilbert spaces, denote by $\mathbb{B}(X, Y)$ the Banach space of bounded linear operators from X into Y, and abbreviate $\mathbb{B}(X, X)$ to $\mathbb{B}(X)$.

In order to define the Tracy-Singh product, we have to fix the decompositions of Hilbert spaces, namely,

$$\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^{m} \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{l=1}^{q} \mathcal{K}_l, \quad \mathcal{K}' = \bigoplus_{k=1}^{p} \mathcal{K}'_k$$

where each $\mathcal{H}_j, \mathcal{H}'_i, \mathcal{K}_l, \mathcal{K}'_k$ are Hilbert spaces. For each $j = 1, \ldots, n$ and $l = 1, \ldots, q$, let $E_j : \mathcal{H}_j \to \mathcal{H}$ and $F_l : \mathcal{K}_l \to \mathcal{K}$ be the canonical embeddings. For each $i = 1, \ldots, m$ and $k = 1, \ldots, p$, let P'_i and Q'_k be the orthogonal projections. Thus, each operator $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and B in $\mathbb{B}(\mathcal{K}, \mathcal{K}')$ can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,n} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{p,q}$$

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where $A_{ij} = P_i' A E_j : \mathcal{H}_j \to \mathcal{H}_i'$ and $B_{kl} = Q_k' B F_l : \mathcal{K}_l \to \mathcal{K}_k'$ for each i, j, k, l. We define the *Tracy-Singh product* of A and B to be a bounded linear operator from $\bigoplus_{j,l=1}^{n,q} \mathcal{H}_j \otimes \mathcal{K}_l$ to $\bigoplus_{i,k=1}^{m,p} \mathcal{H}_i' \otimes \mathcal{K}_k'$ represented in the block-matrix form as follows:

$$A \boxtimes B \;=\; \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij} .$$

When $m = n = p = q = 1$, the Tracy-Singh product $A \boxtimes B$ becomes the tensor product $A \otimes B$.

Lemma 1 ([6]). *Fundamental properties of the Tracy-Singh product for operators are listed below (provided that each term is well-defined):*

1. *The map $(A, B) \mapsto A \boxtimes B$ is bilinear.*
2. *Compatibility with adjoints: $(A \boxtimes B)^* \;=\; A^* \boxtimes B^*$.*
3. *Mixed-product property: $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$.*
4. *Compatibility with powers: $(A \boxtimes B)^r = A^r \boxtimes B^r$ for any $r \in \mathbb{N}$.*
5. *Compatibility with inverses: if A and B are invertible, then $A \boxtimes B$ is invertible with $(A \boxtimes B)^{-1} \;=\; A^{-1} \boxtimes B^{-1}$.*
6. *Positivity: if $A \geqslant 0$ and $B \geqslant 0$, then $A \boxtimes B \geqslant 0$.*
7. *Strictly positivity: if $A > 0$ and $B > 0$, then $A \boxtimes B > 0$.*
8. *If A and B are partial isometries, then so is $A \boxtimes B$. Recall that an operator T is a partial isometry if and only if the restriction of T to a closed subspace is an isometry.*

3 Analytic properties of the Tracy-Singh product

In this section, we establish some analytic properties of the Tracy-Singh product involving operator norms. These properties involve continuity, convergence, norm estimates, and certain analytic functions. We denote the operator norm by $\| \cdot \|_\infty$.

In order to discuss the continuity of the Tracy-Singh product, recall the following bounds for the operator norm of operator matrices.

Lemma 2 ([1]). *Let $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ be an operator matrix. Then*

$$n^{-2} \sum_{i,j=1}^{n} \|A_{ij}\|_\infty^2 \;\leqslant\; \|A\|_\infty^2 \;\leqslant\; \sum_{i,j=1}^{n} \|A_{ij}\|_\infty^2. \tag{1}$$

Lemma 3. *Let $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ be an operator matrix and let $(A_r)_{r=1}^\infty$ be a sequence in $\mathbb{B}(\mathcal{H})$ where $A_r = [A_{ij}^{(r)}]_{i,j=1}^{n,n}$ for each $r \in \mathbb{N}$. Then $A_r \to A$ if and only if $A_{ij}^{(r)} \to A_{ij}$ for all $i, j = 1, \ldots, n$.*

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Proof. It is a direct consequence of Lemma 2. □

The next theorem explains that the Tracy-Singh product is (jointly) continuous with respect to the topology induced by the operator norm.

Theorem 4. *Let $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$ and $B = [B_{kl}] \in \mathbb{B}(\mathcal{K})$ be operator matrices, and let $(A_r)_{r=1}^{\infty}$ and $(B_r)_{r=1}^{\infty}$ be sequences in $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$, respectively. If $A_r \to A$ and $B_r \to B$, then $A_r \boxtimes B_r \to A \boxtimes B$.*

Proof. Suppose that $A_r \to A$ and $B_r \to B$. By Lemma 3, we have $A_{ij}^{(r)} \to A_{ij}$ and $B_{kl}^{(r)} \to B_{kl}$ for each i, j, k, l. Since the tensor product is continuous, we have

$$A_{ij}^{(r)} \otimes B_{kl}^{(r)} \;\to\; A_{ij} \otimes B_{kl}$$

for each i, j, k, l. It follows that $A_r \boxtimes B_r \to A \boxtimes B$ by Lemma 3. □

The next theorem provides upper/lower bounds for the operator norm of the Tracy-Singh product.

Theorem 5. *For any operator matrices $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ and $B = [A_{kl}]_{k,l=1}^{q,q} \in \mathbb{B}(\mathcal{K})$, we have*

$$\frac{1}{nq}\|A\|_\infty \|B\|_\infty \;\leqslant\; \|A \boxtimes B\|_\infty \;\leqslant\; nq\|A\|_\infty \|B\|_\infty. \tag{2}$$

Proof. It follows from Lemma 2 that

$$\begin{aligned}\|A \boxtimes B\|_\infty^2 \;&\leqslant\; \sum_{k,l}\sum_{i,j}\|A_{ij} \otimes B_{kl}\|_\infty^2 \;=\; \sum_{k,l}\sum_{i,j}\|A_{ij}\|_\infty^2 \|B_{kl}\|_\infty^2 \\ &=\; \Big(\sum_{i,j}\|A_{ij}\|_\infty^2\Big)\Big(\sum_{k,l}\|B_{kl}\|_\infty^2\Big) \;\leqslant\; (nq)^2\|A\|_\infty^2 \|B\|_\infty^2.\end{aligned}$$

We also have

$$\begin{aligned}\|A \boxtimes B\|_\infty^2 \;&\geqslant\; (nq)^{-2}\sum_{k,l}\sum_{i,j}\|A_{ij} \otimes B_{kl}\|_\infty^2 \;=\; (nq)^{-2}\sum_{k,l}\sum_{i,j}\|A_{ij}\|_\infty^2 \|B_{kl}\|_\infty^2 \\ &=\; (nq)^{-2}\Big(\sum_{i,j}\|A_{ij}\|_\infty^2\Big)\Big(\sum_{k,l}\|B_{kl}\|_\infty^2\Big) \;\geqslant\; (nq)^{-2}\|A\|_\infty^2 \|B\|_\infty^2.\end{aligned}$$

Hence, we obtain the bound (2). □

Theorem 6. *Let $A \in \mathbb{B}(\mathcal{H})$.*

(i) If f is an analytic function on a region containing the spectra of A and $I \boxtimes A$, then

$$f(I \boxtimes A) \;=\; I \boxtimes f(A). \tag{3}$$

(ii) If f is an analytic function on a region containing the spectra of A and $A \boxtimes I$, then

$$f(A \boxtimes I) \;=\; f(A) \boxtimes I. \tag{4}$$

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Proof. (i) Since f is analytic on spectra of A and $I \boxtimes A$, we have the Taylor series expansion

$$f(z) \;=\; \sum_{r=0}^{\infty} \alpha_r z^r.$$

It follows that

$$f(A) \;=\; \sum_{r=0}^{\infty} \alpha_r A^r \quad \text{and} \quad f(I \boxtimes A) \;=\; \sum_{r=0}^{\infty} \alpha_r (I \boxtimes A)^r.$$

Making use of the bilinearity of Tracy-Singh product and Theorem 4 yields

$$\begin{aligned} f(I \boxtimes A) \;&=\; \sum_{r=0}^{\infty} \alpha_r \left(I \boxtimes A^r\right) \;=\; \sum_{r=0}^{\infty} \left(I \boxtimes \alpha_r A^r\right) \\ &=\; I \boxtimes \sum_{r=0}^{\infty} \alpha_r A^r \;=\; I \boxtimes f(A). \end{aligned}$$

Similarly, we obtain the assertion (ii). □

Theorem 7. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive operators. For any $\alpha > 0$, we have*

$$(A \boxtimes B)^{\alpha} \;=\; A^{\alpha} \boxtimes B^{\alpha}. \tag{5}$$

Proof. First, note that $A \boxtimes B$ is positive by property (6) of Lemma 1. It follows from the property (4) in Lemma 1 that for any $r, s \in \mathbb{N}$,

$$\left(A^{\frac{r}{s}} \boxtimes B^{\frac{r}{s}}\right)^{s} \;=\; A^r \boxtimes B^r \;=\; (A \boxtimes B)^r,$$

and thus $(A \boxtimes B)^{\frac{r}{s}} = A^{\frac{r}{s}} \boxtimes B^{\frac{r}{s}}$. Now, for $\alpha > 0$, there is a sequence (q_n) of positive rational numbers such that $q_n \to \alpha$. It follows from the previous claim and the continuity of Tracy-Singh product (Theorem 4) that

$$\begin{aligned} (A \boxtimes B)^{\alpha} \;&=\; \lim_{n\to\infty} (A \boxtimes B)^{q_n} \;=\; \lim_{n\to\infty} A^{q_n} \boxtimes B^{q_n} \\ &=\; \lim_{n\to\infty} A^{q_n} \boxtimes \lim_{n\to\infty} B^{q_n} \;=\; A^{\alpha} \boxtimes B^{\alpha}. \end{aligned}$$ □

Corollary 8. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be strictly positive operators. For any real number α, we have*

$$(A \boxtimes B)^{\alpha} \;=\; A^{\alpha} \boxtimes B^{\alpha}. \tag{6}$$

Proof. Note that $A \boxtimes B$ is strictly positive by property (7) of Lemma 1. For $\alpha < 0$, it follows from Theorem 7 and the property (5) in Lemma 1 that

$$\begin{aligned} (A \boxtimes B)^{\alpha} \;&=\; [(A \boxtimes B)^{-1}]^{-\alpha} \;=\; (A^{-1} \boxtimes B^{-1})^{-\alpha} \\ &=\; (A^{-1})^{-\alpha} \boxtimes (B^{-1})^{-\alpha} \;=\; A^{\alpha} \boxtimes B^{\alpha}. \end{aligned}$$ □

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Corollary 9. *Let* $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ *and* $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. *Then*

$$|A \boxtimes B| = |A| \boxtimes |B|. \tag{7}$$

Proof. Applying Lemma 1 and property (5), we get

$$\begin{aligned} |A \boxtimes B| &= [(A \boxtimes B)^*(A \boxtimes B)]^{\frac{1}{2}} = [(A^* \boxtimes B^*)(A \boxtimes B)]^{\frac{1}{2}} \\ &= (A^*A \boxtimes B^*B)^{\frac{1}{2}} = (A^*A)^{\frac{1}{2}} \boxtimes (B^*B)^{\frac{1}{2}} = |A| \boxtimes |B|. \end{aligned}$$ □

Recall the polar decomposition theorem: for any $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$, there exists a partial isometry U such that $A = U|A|$. The next result is a polar decomposition for the Tracy-Singh product of operators.

Corollary 10. *Let* $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ *and* $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. *If* $A = U|A|$ *and* $B = V|B|$ *are polar decompositions of* A *and* B*, respectively, then a polar decomposition of* $A \boxtimes B$ *is given by*

$$A \boxtimes B = (U \boxtimes V)|A \boxtimes B|. \tag{8}$$

Proof. Let U and V be partial isometries such that $A = U|A|$ and $B = V|B|$. It follows from Lemma 1(3) and Corollary 9 that

$$A \boxtimes B = U|A| \boxtimes V|B| = (U \boxtimes V)(|A| \boxtimes |B|) = (U \boxtimes V)|A \boxtimes B|.$$

Note that $U \boxtimes V$ is also a partial isometry, according to property (8) in Lemma 1. Hence, the decomposition (8) is a polar one. □

4 Tracy-Singh products on norm ideals of compact operators

In this section, we investigate the Tracy-Singh product on norm ideals of $\mathbb{B}(\mathcal{H})$. Recall that any proper ideal of $\mathbb{B}(\mathcal{H})$ is contained in the ideal $\mathcal{S}_\infty$ of compact operators. For any compact operator $A \in \mathbb{B}(\mathcal{H})$, let $(s_i(A))_{i=1}^\infty$ be the sequence of decreasingly-ordered singular values of A (i.e. eigenvalues of $|A|$). For each $1 \leqslant p < \infty$, the *Schatten p-norm* of A is defined by

$$\|A\|_p = \left(\sum_{i=1}^{\infty} s_i^p(A) \right)^{1/p}.$$

If $\|A\|_p$ is finite, we say that A is a *Schatten p-class operator*. The Schatten ∞-norm is just the operator norm. For each $1 \leqslant p \leqslant \infty$, let $\mathcal{S}_p$ be the Schatten p-class operators . In particular, $\mathcal{S}_1$ and $\mathcal{S}_2$ are the trace class and the Hilbert-Schmidt class, respectively. Each Schatten p-norm induces a norm ideal of $\mathbb{B}(\mathcal{H})$ and this ideal is closed under the topology generated by this norm.

Lemma 11. *Let* $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$ *be an operator matrix. Then* A *is compact if and only if* A_{ij} *is compact for all* i, j.

Proof. If A is compact, then $A_{ij} = P_i' A E_j$ is also compact for each i, j due to the fact that $\mathcal{S}_\infty$ is an ideal of $\mathbb{B}(\mathcal{H})$. Conversely, suppose that A_{ij} is compact for all i, j. Recall that a bounded linear operator is compact if and only if it maps a bounded sequence into a sequence having a convergent subsequence. Let $(x_r)_{r=1}^\infty$ be a bounded sequence in $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$. Write $x_r = [x_r^{(1)}\ x_r^{(2)}\ \ldots\ x_r^{(n)}]^T \in \bigoplus_{i=1}^n \mathcal{H}_i$ for each $r \in \mathbb{N}$. Consider

$$Ax_r = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_r^{(1)} \\ \vdots \\ x_r^{(n)} \end{bmatrix} = \begin{bmatrix} A_{11}x_r^{(1)} + \cdots + A_{1n}x_r^{(n)} \\ \vdots \\ A_{n1}x_r^{(1)} + \cdots + A_{nn}x_r^{(n)} \end{bmatrix}.$$

For each $l = 1, 2, \ldots, n$, since $(x_r^{(l)})_{r=1}^\infty$ is bounded, the sequence $(A_{ij}x_r^{(l)})_{r=1}^\infty$ has a convergent subsequence, namely, $(A_{ij}x_{r_k}^{(l)})_{k=1}^\infty$. Hence,

$$\begin{bmatrix} A_{11}x_{r_k}^{(1)} + \cdots + A_{1n}x_{r_k}^{(n)} \\ \vdots \\ A_{n1}x_{r_k}^{(1)} + \cdots + A_{nn}x_{r_k}^{(n)} \end{bmatrix}$$

is a desired convergent subsequence of $(Ax_r)_{r=1}^\infty$. □

Lemma 12 ([1]). *Let $A = [A_{ij}]_{i,j=1}^{n,n}$ be an operator matrix in the Schatten p-class.*

(i) For $1 \leqslant p \leqslant 2$, we have

$$\sum_{i,j=1}^n \|A_{ij}\|_p^2 \leqslant \|A\|_p^2 \leqslant n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}\|_p^2. \tag{9}$$

(ii) For $2 \leqslant p < \infty$, we have

$$n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}\|_p^2 \leqslant \|A\|_p^2 \leqslant \sum_{i,j=1}^n \|A_{ij}\|_p^2. \tag{10}$$

Lemma 13. *Let $1 \leqslant p < \infty$. An operator matrix $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$ is a Schatten p-class operator if and only if A_{ij} is a Schatten p-class operator for all i, j.*

Proof. This is a direct consequence of the norm estimations in Lemma 12. □

Lemma 14. *Let $1 \leqslant p \leqslant \infty$. Let $A = [A_{ij}]_{i,j=1}^{n,n}$ be an operator matrix in the class $\mathcal{S}_p$ and let $(A_r)_{r=1}^\infty$ be a sequence in $\mathcal{S}_p$ where $A_r = \left[A_{ij}^{(r)}\right]_{i,j=1}^{n,n}$ for each $r \in \mathbb{N}$. Then $A_r \to A$ in $\mathcal{S}_p$ if and only if $A_{ij}^{(r)} \to A_{ij}$ in $\mathcal{S}_p$ for all $i, j = 1, \ldots, n$.*

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Proof. Lemma 13 assures that A_{ij} and $A_{ij}^{(r)}$ belong to $\mathcal{S}_p$ for any $i, j = 1, \dots, n$ and $r \in \mathbb{N}$. Consider the case $1 \leqslant p \leqslant 2$. Suppose that $A_r \to A$ in $\mathcal{S}_p$. For any fixed $i, j \in \{1, ..., n\}$, we have from the estimation (9) that

$$\|A_{ij}^{(r)} - A_{ij}\|_p^2 \leqslant \sum_{i,j=1}^{n} \|A_{ij}^{(r)} - A_{ij}\|_p^2 \leqslant \|A_r - A\|_p^2.$$

Hence, $A_{ij}^{(r)} \to A_{ij}$ in $\mathcal{S}_p$. Conversely, suppose $A_{ij}^{(r)} \to A_{ij}$ in $\mathcal{S}_p$ for each i, j. Lemma 12 implies that

$$\|A_r - A\|_p^2 \leqslant n^{4/p-2} \sum_{i,j=1}^{n} \|A_{ij}^{(r)} - A_{ij}\|_p^2.$$

Hence, $A_r \to A$ in $\mathcal{S}_p$. The case $2 < p < \infty$ and the case $p = \infty$ are done by using the norm estimations (10) and (1), respectively. □

Next, we discuss compactness of Tracy-Singh product of operators.

Lemma 15 ([10]). *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be nonzero operators. Then $A \otimes B$ is compact if and only if both A and B are compact.*

Theorem 16. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be nonzero operator matrices. Then $A \boxtimes B$ is compact if and only if both A and B are compact.*

Proof. Write $A = [A_{ij}]$ and $B = [B_{kl}]$. For sufficiency, suppose that A and B are compact. By Lemma 11, we deduce that A_{ij} and B_{kl} are compact for all i, j, k, l. It follows from Lemma 15 that $A_{ij} \otimes B_{kl}$ is compact for all i, j, k, l. Lemma 11 ensures the compactness of $A \boxtimes B$. For necessity part, reverse the previous procedure. □

The following theorem supplies bounds for Schatten 1-norm of the Tracy-Singh product of operators.

Theorem 17. *For any nonzero compact operator $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ and $B = [A_{kl}]_{k,l=1}^{q,q} \in \mathbb{B}(\mathcal{K})$, we have*

$$\frac{1}{nq}\|A\|_1\|B\|_1 \leqslant \|A \boxtimes B\|_1 \leqslant nq\|A\|_1\|B\|_1. \tag{11}$$

Hence, $A \boxtimes B$ is trace-class if and only if both A and B are trace-class.

Proof. Suppose that both A and B are nonzero and compact. Then the operator $A \boxtimes B$ is compact by Theorem 16. It follows from the norm bound (9) that

$$\begin{aligned} \|A \boxtimes B\|_1^2 &\leqslant (nq)^2 \sum_{k,l} \sum_{i,j} \|A_{ij} \otimes B_{kl}\|_1^2 = (nq)^2 \sum_{k,l} \sum_{i,j} \|A_{ij}\|_1^2 \|B_{kl}\|_1^2 \\ &= (nq)^2 \Big(\sum_{i,j} \|A_{ij}\|_1^2 \Big) \Big(\sum_{k,l} \|B_{kl}\|_1^2 \Big) \leqslant (nq)^2 \|A\|_1^2 \|B\|_1^2. \end{aligned}$$

We also have

$$\begin{aligned}\|A \boxtimes B\|_1^2 &\geqslant \sum_{k,l}\sum_{i,j}\|A_{ij}\otimes B_{kl}\|_1^2 = \sum_{k,l}\sum_{i,j}\|A_{ij}\|_1^2\|B_{kl}\|_1^2 \\ &= \Big(\sum_{i,j}\|A_{ij}\|_1^2\Big)\Big(\sum_{k,l}\|B_{kl}\|_1^2\Big) \geqslant (nq)^{-2}\|A\|_1^2\|B\|_1^2.\end{aligned}$$

Hence, we obtain the bound (11). □

Theorem 18. *For any nonzero compact operator matrices $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$, we have*

$$\|A \boxtimes B\|_2 = \|A\|_2\,\|B\|_2. \tag{12}$$

Hence, $A \boxtimes B$ is a Hilbert-Schmidt operator if and only if both A and B are Hilbert-Schmidt operators.

Proof. Since both A and B are nonzero and compact, the operator $A \boxtimes B$ is compact by Theorem 16. Write $A = [A_{ij}]$ and $B = [B_{kl}]$. Then by Lemma 12(ii), we have

$$\begin{aligned}\|A \boxtimes B\|_2^2 &= \sum_{k,l}\sum_{i,j}\|A_{ij}\otimes B_{kl}\|_2^2 = \sum_{k,l}\sum_{i,j}\|A_{ij}\|_2^2\,\|B_{kl}\|_2^2 \\ &= \Big(\sum_{i,j}\|A_{ij}\|_2^2\Big)\Big(\sum_{k,l}\|B_{kl}\|_2^2\Big) = \|A\|_2^2\,\|B\|_2^2.\end{aligned}$$

Hence, we get the multiplicative property (12). □

The final result asserts that the Tracy-Singh product is continuous with respect to the topology induced by the Schatten p-norm for each $p \in \{1, 2, \infty\}$.

Theorem 19. *Let $p \in \{1, 2, \infty\}$. If a sequence $(A_r)_{r=1}^{\infty}$ converges to A and a sequence $(B_r)_{r=1}^{\infty}$ converges to B in the norm ideal $\mathcal{S}_p$, then $A_r \boxtimes B_r$ converges to $A \boxtimes B$ in $\mathcal{S}_p$.*

Proof. Write $A = [A_{ij}]$ and $B = [B_{kl}]$. In the viewpoint of Lemma 14, it suffices to show that $A_{ij}^{(r)} \otimes B_{kl}^{(r)} \to A_{ij} \otimes B_{kl}$ in $\mathcal{S}_p$ for all i, j, k, l. Since $A_r \to A$ and $B_r \to B$ in $\mathcal{S}_p$, we have by Lemma 14 that $A_{ij}^{(r)} \to A_{ij}$ and $B_{kl}^{(r)} \to B_{kl}$ for all i, j, k, l. It follows that

$$\begin{aligned}\|A_{ij}^{(r)} \otimes B_{kl}^{(r)} - A_{ij} \otimes B_{kl}\|_p &= \|A_{ij}^{(r)} \otimes B_{kl}^{(r)} - A_{ij}^{(r)} \otimes B_{kl} + A_{ij}^{(r)} \otimes B_{kl} - A_{ij} \otimes B_{kl}\|_p \\ &\leqslant \|A_{ij}^{(r)} \otimes (B_{kl}^{(r)} - B_{kl})\|_p + \|(A_{ij}^{(r)} - A_{ij}) \otimes B_{kl}\|_p \\ &= \|A_{ij}^{(r)}\|_p\,\|B_{kl}^{(r)} - B_{kl}\|_p + \|A_{ij}^{(r)} - A_{ij}\|_p\,\|B_{kl}\|_p \\ &\to \|A_{ij}\|_p \cdot 0 + 0 \cdot \|B_{kl}\|_p = 0.\end{aligned}$$

Hence, $A_{ij}^{(r)} \otimes B_{kl}^{(r)} \to A_{ij} \otimes B_{kl}$ in $\mathcal{S}_p$ for all i, j, k, l. □

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Tracy-Singh Products and Classes of Operators

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Abstract

We investigate relationship between Tracy-Singh products and certain classes of Hilbert space operators. We show that the normality, hyponormality, paranormality of operators are preserved by Tracy-Singh products. Operators of class-$\mathcal{A}$ type are also preserved under Tracy-Singh products. Moreover, we obtain necessary and sufficient conditions for the Tracy-Singh product of two operators to be normal, quasinormal, (co)isometry, and unitary.

Keywords: Tracy-Singh product, tensor product, normality, class $\mathcal{A}$ operator
Mathematics Subject Classifications 2010: 47A05, 47A80, 47B20, 47B47.

1 Introduction

Tensor product of bounded linear operators plays a crucial role in functional analysis and operator theory. Many algebraic-order-analytic properties of operators are preserved under taking tensor products, but by no means all of them. Importance results on tensor product involving certain classes of operators (e.g. positive, unitary, normal, compact) have been noticed by many mathematicians from the beginning of the theory to nowadays (e.g. [22]). In the last two decades, the concepts of normality, hyponormality, and paranormality have been introduced and investigated by many authors, see e.g., [5, 13, 21]. Relations between tensor products and class-$\mathcal{A}$ type operators also have received much attention, e.g., [10, 11, 12, 19, 20]. See more information about classes of operators in the monograph [7].

Recently, the notion of tensor product was extended to the Tracy-Singh product for Hilbert space operators in [15]. It was shown that compactness,

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positivity and strict-positivity of operators are preserved under Tracy-Singh products [15, 16].

In this paper, we investigate relationship between Tracy-Singh products and certain classes of operators. We divide such classes into three categories. The first category consists of nilpotent, (skew)-Hermitian, (co)isometry, and unitary operators. The second one contains operator normality, hyponormality, and paranormality. The last one is the class-$\mathcal{A}$ type operators, which includes class $\mathcal{A}(k)$, class $\mathcal{A}$, quasi-class $(\mathcal{A}, k)$, quasi-class $\mathcal{A}$, $*$-class $\mathcal{A}$, quasi-$*$-class $\mathcal{A}$, and quasi-$*$-class $(\mathcal{A}, k)$ operators. We will show that the mentioned properties of operators are preserved under taking Tracy-Singh products. Moreover, we obtain necessary and sufficient conditions for the Tracy-Singh product of two operators to be normal, quasinormal, (co)isometry, and unitary operators.

The paper is structured as follows. The next section supplies some prerequisites about the tensor product and the Tracy-Singh product of operators. Next, we discuss relationship between Tracy-Singh products and the normality, hyponormality, and paranormality of operators. Then we consider Tracy-Singh products and certain properties of operators–being nilpotent, (skew)-Hermitian, (co)isometry, and unitary. The last section deals with class $\mathcal{A}$ type operators.

2 Preliminaries

In what follows, $\mathbb{H}$ and $\mathbb{K}$ denote complex separable Hilbert spaces. When X and Y are Hilbert spaces, denote by $\mathcal{B}(X, Y)$ the Banach space of bounded linear operators from X into Y, equipped with the operator norm $\|\cdot\|$ and abbreviate $\mathcal{B}(X, X)$ to $\mathcal{B}(X)$. For Hermitian operators A and B on the same Hilbert space, we use the notation $A \geqslant B$ to mean that $A - B$ is a positive operator.

In order to define the Tracy-Singh product, we have to fix the orthogonal decompositions of Hilbert spaces, namely,

$$\mathbb{H} = \bigoplus_{i=1}^{m} \mathbb{H}_i, \quad \mathbb{K} = \bigoplus_{l=1}^{n} \mathbb{K}_l$$

where all $\mathbb{H}_i$'s and $\mathbb{K}_k$'s are Hilbert spaces. Any operator $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ thus can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,m} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{n,n}$$

where $A_{ij} \in \mathcal{B}(\mathbb{H}_j, \mathbb{H}_i)$ and $B_{kl} \in \mathcal{B}(\mathbb{K}_l, \mathbb{K}_k)$ for each i, j, k, l. Then the *Tracy-Singh product* of A and B is defined to be

$$A \boxtimes B = \big[[A_{ij} \otimes B_{kl}]_{kl}\big]_{ij}\,, \tag{1}$$

which is a bounded linear operator from $\bigoplus_{i,k=1}^{m,n} \mathbb{H}_i \otimes \mathbb{K}_k$ into itself. Note that when $m = n = 1$, the Tracy-Singh product $A \boxtimes B$ reduces to the tensor product $A \otimes B$.

Lemma 1 ([15]). *Algebraic and order properties of the Tracy-Singh product for operators are listed here (provided that every operation is well-defined):*

1. *The map $(A, B) \mapsto A \boxtimes B$ is bilinear.*
2. *Compatibility with adjoints: $(A \boxtimes B)^* = A^* \boxtimes B^*$.*
3. *Compatibility with ordinary products: $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$.*
4. *Compatibility with powers: $(A \boxtimes B)^r = A^r \boxtimes B^r$ for any $r \in \mathbb{N}$.*
5. *Compatibility with inverses: if A and B are invertible, then $A \boxtimes B$ is invertible with $(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}$.*
6. *Positivity: if $A \geqslant 0$ and $B \geqslant 0$, then $A \boxtimes B \geqslant 0$.*
7. *Monotonicity: if $A_1 \geqslant B_1$ and $A_2 \geqslant B_2$, then $A_1 \boxtimes A_2 \geqslant B_1 \boxtimes B_2$.*

Lemma 2 ([15]). *Let $A = [A_{ij}] \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ be operator matrices. Then each (i, j)-block of $A \boxtimes B$ is $A_{ij} \boxtimes B$.*

Analytic properties of the Tracy-Singh product for operators are listed below.

Lemma 3 ([16]). *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$. Then we have*

(i) $\dfrac{1}{mn}\|A\|\|B\| \leqslant \|A \boxtimes B\| \leqslant mn\|A\|\|B\|$.

(ii) $|A \boxtimes B| = |A| \boxtimes |B|$, *here the absolute value of A is defined by $|A| = (A^*A)^{\frac{1}{2}}$.*

(iii) If A and B are positive operators, then $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$ for any nonnegative real α.

Lemma 4. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$.*

(i) The condition $A \boxtimes B = 0$ holds if and only if $A = 0$ or $B = 0$.

(ii) If $A \boxtimes B = A \boxtimes C$ and $A \neq 0$, then $B = C$.

(iii) If $B \boxtimes A = C \boxtimes A$ and $A \neq 0$, then $B = C$.

Proof. From the norm estimation in Lemma 3(i), one can deduce property (i). Properties (ii) and (iii) follow from (i) and the bilinearity of Tracy-Singh product in Lemma 1. □

Lemma 5 ([21]). *Let $A, C \in \mathcal{B}(\mathbb{H})$ and $B, D \in \mathcal{B}(\mathbb{K})$ be nonzero operators. Then $A \otimes B = C \otimes D$ if and only if there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $C = \alpha A$ and $D = \alpha^{-1} B$.*

Proposition 6. *Let $A = [A_{ij}]_{i,j=1}^{m,m}, C = [C_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n}, D = [D_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ be operator matrices such that A_{ij}, B_{kl}, C_{ij} and D_{kl} are nonzero operators for all $i, j = 1, \ldots, m$ and $k, l = 1, \ldots, n$. Then $A \boxtimes B = C \boxtimes D$ if and only if there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $C = \alpha A$ and $D = \alpha^{-1} B$.*

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Proof. If $C = \alpha A$ and $D = \alpha^{-1}B$ for some $\alpha \in \mathbb{C} \setminus \{0\}$, then by Lemma 1,

$$C \boxtimes D = (\alpha A) \boxtimes (\alpha^{-1}B) = \alpha\alpha^{-1}(A \boxtimes B) = A \boxtimes B.$$

Assume that $A \boxtimes B = C \boxtimes D$. By using Lemma 2, we get $A_{ij} \boxtimes B = C_{ij} \boxtimes D$ for all $i, j = 1, \ldots, m$. For any fixed $i, j \in \{1, \ldots, m\}$, we have $A_{ij} \otimes B_{kl} = C_{ij} \otimes D_{kl}$ for all $k, l = 1, \ldots, n$. For each $i, j \in \{1, \ldots, m\}$ and $k, l \in \{1, \ldots, n\}$, by applying Lemma 5, there exists $\alpha_{ij,kl} \in \mathbb{C} \setminus \{0\}$ such that $C_{ij} = \alpha_{ij,kl}A_{ij}$ and $D_{kl} = \alpha_{ij,kl}^{-1}B_{kl}$. For any fixed $i, j \in \{1, \ldots, m\}$, we have $C_{ij} = \alpha_{ij,kl}A_{ij}$ for all $k, l = 1, \ldots, n$. This implies that $\alpha_{ij,11} = \cdots = \alpha_{ij,nn} = \alpha_{ij}$. For any fixed $k, l \in \{1, \ldots, n\}$, we have $D_{kl} = \alpha_{ij}^{-1}B_{kl}$ for all $i, j = 1, \ldots, m$. It follows that $\alpha_{11} = \cdots = \alpha_{mm} = \alpha$. Thus $C_{ij} = \alpha A_{ij}$ and $D_{kl} = \alpha^{-1}B_{kl}$ for all $i, j = 1, \ldots, m$ and $k, l = 1, \ldots, n$. Therefore $C = \alpha A$ and $D = \alpha^{-1}B$. □

Recall that the commutator of A and B in $\mathcal{B}(\mathbb{H})$ is defined by

$$[A, B] \;=\; AB - BA.$$

Proposition 7. *Let $A, C \in \mathcal{B}(\mathbb{H})$ and $B, D \in \mathcal{B}(\mathbb{K})$.*

(i) If $[A, C] \geqslant 0$ and $[B, D] \geqslant 0$, then $[A \boxtimes B, C \boxtimes D] \geqslant 0$.

(ii) If $[A, C] \leqslant 0$ and $[B, D] \leqslant 0$, then $[A \boxtimes B, C \boxtimes D] \leqslant 0$.

(iii) If $[A, C] = 0$ and $[B, D] = 0$, then $[A \boxtimes B, C \boxtimes D] = 0$.

Proof. (i) Since $AC \geqslant CA$ and $BD \geqslant DB$, we have $AC \boxtimes BD \geqslant CA \boxtimes DB$ by Lemma 1. Then

$$[A \boxtimes B, C \boxtimes D] \;=\; AC \boxtimes BD - CA \boxtimes DB \geqslant 0.$$

The assertion (ii) follows from (i) and the fact that $-[X, Y] = [Y, X]$ for any operators X and Y. The assertion (iii) follows from (i) and (ii). □

3 Tracy-Singh products and operator normality

In this section, we discuss normality of Tracy-Singh products of operators. The contents can be divided into three parts. The first part deals with general properties of normality, the second one concerns hyponormality, and the last one consists of paranormality.

3.1 Normality

Recall the following types of operator normality; see e.g. [7, Chapter 2] and [17] for more details.

Definition 8. *An operator $T \in \mathcal{B}(\mathbb{H})$ is said to be*

- normal *if $[T^*, T] = 0$;*

- binormal *if* $[T^*T, TT^*] = 0$ *;*
- quasinormal *if* $[T, T^*T] = 0$ *;*
- posinormal *if* $TT^* = T^*PT$ *for some positive operator* P.

Stochel [21] showed that for non-zero $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$, the tensor product $A \otimes B$ is normal (resp. quasinormal) if and only if A and B are normal (resp. quasinormal). Now, we will extend this result to the case of Tracy-Singh products.

Theorem 9. *Let* $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ *and* $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ *be operator matrices such that* A_{ij} *and* B_{kl} *are nonzero operators for all* $i, j = 1, \ldots, m$ *and* $k, l = 1, \ldots, n$. *Then* $A \boxtimes B$ *is normal if and only if so are* A *and* B.

Proof. If A and B are normal, then by Lemma 1 and Proposition 7 we have

$$[(A \boxtimes B)^*, A \boxtimes B] \;=\; [A^* \boxtimes B^*, A \boxtimes B] \;=\; [A^*, A] \boxtimes [B^*, B] \;=\; 0,$$

i.e., $A \boxtimes B$ is also normal. Conversely, suppose that $A \boxtimes B$ is normal. Note that

$$A^*A \boxtimes B^*B \;=\; (A \boxtimes B)^*(A \boxtimes B) \;=\; (A \boxtimes B)(A \boxtimes B)^* \;=\; AA^* \boxtimes BB^*.$$

By Proposition 6, there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $AA^* = \alpha A^*A$ and $BB^* = \alpha^{-1}B^*B$. Since AA^* and A^*A are positive, we have $\alpha > 0$. Then

$$\begin{aligned} \|A\|^2 \;&=\; \|AA^*\| \;=\; \|\alpha A^*A\| \;=\; \alpha\|A\|^2, \\ \|B\|^2 \;&=\; \|BB^*\| \;=\; \|\alpha^{-1}B^*B\| \;=\; \alpha^{-1}\|B\|^2. \end{aligned}$$

We arrive at $\alpha = 1$, meaning that both A and B are normal. □

Theorem 10. *Let* $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ *and* $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ *be operator matrices such that* A_{ij} *and* B_{kl} *are nonzero operators for all* $i, j = 1, \ldots, m$ *and* $k, l = 1, \ldots, n$. *Then* $A \boxtimes B$ *is quasinormal if and only if so are* A *and* B.

Proof. Assume that A and B are quasinormal. Since $[A, A^*A] = 0$ and $[B, B^*B] = 0$, we have

$$[A \boxtimes B, (A \boxtimes B)^*(A \boxtimes B)] \;=\; [A \boxtimes B, A^*A \boxtimes B^*B] \;=\; 0.$$

Hence, $A \boxtimes B$ is quasinormal. Suppose that $A \boxtimes B$ is quasinormal. Note that

$$\begin{aligned} AA^*A \boxtimes BB^*B \;&=\; (A \boxtimes B)(A \boxtimes B)^*(A \boxtimes B) \\ &=\; (A \boxtimes B)^*(A \boxtimes B)^2 \\ &=\; A^*A^2 \boxtimes B^*B^2. \end{aligned}$$

Then there exists $\alpha \in \mathbb{C}\setminus\{0\}$ such that $A^*A^2 = \alpha AA^*A$ and $B^*B^2 = \alpha^{-1}BB^*B$. This in turn implies that

$$\begin{aligned} (A^2)^*A^2 \;&=\; A^*(A^*A^2) \;=\; \alpha(A^*A)^2, \\ (B^2)^*B^2 \;&=\; B^*(B^*B^2) \;=\; \alpha^{-1}(B^*B)^2. \end{aligned}$$

Since $(A^2)^*A^2$ and $\alpha(A^*A)^2$ are positive, we conclude $\alpha > 0$. We have

$$\alpha\|A\|^4 = \alpha\|(A^*A)^2\|^2 = \|(A^2)^*A^2\| = \|A^2\|^2 \leqslant \|A\|^4$$

and, similarly, $\alpha^{-1}\|B\|^4 \leqslant \|B\|^4$. This forces $\alpha = 1$ and, thus, both A and B are quasinormal. □

Proposition 11. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$. If both A and B satisfy one of the following properties, then the same property holds for $A \boxtimes B$: binormal, posinormal.*

Proof. The assertion for binormality follows from Lemma 1 and Proposition 7. Now, suppose that $AA^* = A^*PA$ and $BB^* = B^*QB$ for some positive operators P and Q. By Lemma 1, we get

$$\begin{aligned}(A \boxtimes B)(A \boxtimes B)^* &= AA^* \boxtimes BB^* = A^*PA \boxtimes B^*QB \\ &= (A \boxtimes B)^*(P \boxtimes Q)(A \boxtimes B).\end{aligned}$$

According to Lemma 1, $P \boxtimes Q$ is positive. Therefore $A \boxtimes B$ is posinormal. □

3.2 Hyponormality

Recall the following hyponormal structures of operators; see e.g. [1, 4, 13] and [7, Chapter 2] for more information.

Definition 12. *Let $p > 0$ be a constant. An operator $T \in \mathcal{B}(\mathbb{H})$ is said to be*

- hyponormal *if $[T^*, T]$ is positive ;*
- p-hyponormal *if $(T^*T)^p \geqslant (TT^*)^p$;*
- quasihyponormal *if $T^*[T^*, T]T$ is positive ;*
- p-quasihyponormal *if $T^*(T^*T)^pT \geqslant T^*(TT^*)^pT$;*
- cohyponormal *if T^* is hyponormal ;*
- log-hyponormal *if T is invertible and $\log(T^*T) \geqslant \log(TT^*)$.*

Definition 13. *Let $T \in \mathcal{B}(\mathbb{H})$ have the polar decomposition $T = U|T|$ where U is a unitary operator. The* Aluthge transformation *of T is defined by*

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

Then T is said to be

- w-hyponormal *if $|\tilde{T}| \geqslant |T| \geqslant |\tilde{T}^*|$;*
- iw-hyponormal *if T is invertible and $|T| \geqslant \left(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}\right)^{\frac{1}{2}}$.*

Theorem 14. *Let $A \in \mathcal{B}(\mathbb{H})$, $B \in \mathcal{B}(\mathbb{K})$, and let $p > 0$ be a constant. If both A and B satisfy one of the following properties, then the same property holds for $A \boxtimes B$: hyponormal, p-hyponormal, cohyponormal, quasihyponormal, p-quasihyponormal.*

Proof. The assertions for hyponormality and cohyponormality follow from Lemma 1 and Proposition 7. The assertion for p-hyponormality is done by applying Lemmas 1 and 3. Now, suppose that A and B are quasihyponormal. By Lemma 1, we obtain

$$\begin{aligned}
&(A \boxtimes B)^* \left[(A \boxtimes B)^*, A \boxtimes B\right] (A \boxtimes B) \\
&\quad = (A^* \boxtimes B^*)((A^* \boxtimes B^*)(A \boxtimes B) - (A \boxtimes B)(A^* \boxtimes B^*))(A \boxtimes B) \\
&\quad = (A^* \boxtimes B^*)(A^* \boxtimes B^*)(A \boxtimes B)(A \boxtimes B) - (A^* \boxtimes B^*)(A \boxtimes B)(A^* \boxtimes B^*)(A \boxtimes B) \\
&\quad = A^*A^*AA \boxtimes B^*B^*BB - A^*AA^*A \boxtimes B^*BB^*B.
\end{aligned}$$

Since $A^*A^*AA - A^*AA^*A = A^*[A^*, A]A \geqslant 0$ and $B^*B^*BB - B^*BB^*B = B^*[B^*, B]B \geqslant 0$, we have by Lemma 1 that

$$A^*A^*AA \boxtimes B^*B^*BB - A^*AA^*A \boxtimes B^*BB^*B \ \geqslant\ 0.$$

Hence, $(A \boxtimes B)^* \left[(A \boxtimes B)^*, A \boxtimes B\right] (A \boxtimes B) \geqslant 0$. This means that $A \boxtimes B$ is quasihyponormal.

Assume that A and B are p-quasihyponormal. Lemmas 1 and 3 together imply that

$$\begin{aligned}
&(A \boxtimes B)^* \left((A \boxtimes B)^*(A \boxtimes B)\right)^p (A \boxtimes B) \\
&\quad = (A^* \boxtimes B^*) \left(A^*A \boxtimes B^*B\right)^p (A \boxtimes B) \\
&\quad = A^*(A^*A)^p A \boxtimes B^*(B^*B)^p B \\
&\quad \geqslant A^*(AA^*)^p A \boxtimes B^*(BB^*)^p B \\
&\quad = (A \boxtimes B)^* \left(AA^* \boxtimes BB^*\right)^p (A \boxtimes B) \\
&\quad = (A \boxtimes B)^* \left((A \boxtimes B)(A \boxtimes B)^*\right)^p (A \boxtimes B).
\end{aligned}$$

This show that $A \boxtimes B$ is p-quasihyponormal. □

Kim [13] investigated the tensor product of log-hyponormal (reps. w-hyponormal, iw-hyponormal) operators. Now, we consider the case of Tracy-Singh products.

Lemma 15 ([6])**.** *Let S and T be positive invertible operators. Then $\log T \geqslant \log S$ if and only if $T^p \geqslant \left(T^{\frac{p}{2}} S^p T^{\frac{p}{2}}\right)^{\frac{1}{2}}$ for all $p \geqslant 0$.*

Theorem 16. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ be positive invertible operators. If A and B are log-hyponormal, then $A \boxtimes B$ is also log-hyponormal.*

Proof. Assume that A and B are log-hyponormal operators. Since A and B are invertible, Lemma 1 implies that $A \boxtimes B$ is invertible. Using Lemmas 1 and 3,

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we obtain that for any $p \geqslant 0$,

$$\begin{aligned}
&[(A\boxtimes B)^*(A\boxtimes B)]^p \\
&\quad= (A^*A\boxtimes B^*B)^p \\
&\quad= (A^*A)^p\boxtimes (B^*B)^p \\
&\quad\geqslant [(A^*A)^{\frac{p}{2}}(AA^*)^p(A^*A)^{\frac{p}{2}}]^{\frac{1}{2}}\boxtimes [(B^*B)^{\frac{p}{2}}(BB^*)^p(B^*B)^{\frac{p}{2}}]^{\frac{1}{2}} \\
&\quad= [(A^*A)^{\frac{p}{2}}(AA^*)^p(A^*A)^{\frac{p}{2}}\boxtimes (B^*B)^{\frac{p}{2}}(BB^*)^p(B^*B)^{\frac{p}{2}}]^{\frac{1}{2}} \\
&\quad= [(A^*A\boxtimes B^*B)^{\frac{p}{2}}\,(AA^*\boxtimes BB^*)^p\,(A^*A\boxtimes B^*B)^{\frac{p}{2}}]^{\frac{1}{2}} \\
&\quad= [((A\boxtimes B)^*(A\boxtimes B))^{\frac{p}{2}}\,((A\boxtimes B)(A\boxtimes B)^*)^p\,((A\boxtimes B)^*(A\boxtimes B))^{\frac{p}{2}}]^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 15, we have $\log(A\boxtimes B)^*(A\boxtimes B) \geqslant \log(A\boxtimes B)(A\boxtimes B)^*$. This means that $A\boxtimes B$ is log-hyponormal. □

Lemma 17 ([1]). *An operator $T \in \mathcal{B}(\mathbb{H})$ is w-hyponormal if and only if* $|T| \geqslant \left(|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}}\right)^{\frac{1}{2}}$ *and* $|T^*| \leqslant \left(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}}\right)^{\frac{1}{2}}$.

Theorem 18. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$. If A and B are w-hyponormal, then $A\boxtimes B$ is also w-hyponormal.*

Proof. Assume that A and B are w-hyponormal. By applying Lemmas 1 and 3, we have

$$\begin{aligned}
|A\boxtimes B| &= |A|\boxtimes |B| \\
&\geqslant \left(|A|^{\frac{1}{2}}|A^*||A|^{\frac{1}{2}}\right)^{\frac{1}{2}}\boxtimes \left(|B|^{\frac{1}{2}}|B^*||B|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left(|A|^{\frac{1}{2}}|A^*||A|^{\frac{1}{2}}\boxtimes |B|^{\frac{1}{2}}|B^*||B|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left[\left(|A|^{\frac{1}{2}}\boxtimes |B|^{\frac{1}{2}}\right)(|A^*|\boxtimes |B^*|)\left(|A|^{\frac{1}{2}}\boxtimes |B|^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \\
&= \left(|A\boxtimes B|^{\frac{1}{2}}|(A\boxtimes B)^*||A\boxtimes B|^{\frac{1}{2}}\right)^{\frac{1}{2}}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
|(A\boxtimes B)^*| &= |A^*|\boxtimes |B^*| \\
&\leqslant \left(|A^*|^{\frac{1}{2}}|A||A^*|^{\frac{1}{2}}\right)^{\frac{1}{2}}\boxtimes \left(|B^*|^{\frac{1}{2}}|B||B^*|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left(|A^*|^{\frac{1}{2}}|A||A^*|^{\frac{1}{2}}\boxtimes |B^*|^{\frac{1}{2}}|B||B^*|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left[\left(|A^*|^{\frac{1}{2}}\boxtimes |B^*|^{\frac{1}{2}}\right)(|A|\boxtimes |B|)\left(|A^*|^{\frac{1}{2}}\boxtimes |B^*|^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \\
&= \left(|(A\boxtimes B)^*|^{\frac{1}{2}}|A\boxtimes B||(A\boxtimes B)^*|^{\frac{1}{2}}\right)^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 17, the operator $A\boxtimes B$ is w-hyponormal. □

Corollary 19. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ be invertible operators. If A and B are iw-hyponormal, then $A \boxtimes B$ is also iw-hyponormal.*

Proof. It follows from Lemma 1, Proposition 18 and the fact that every *iw*-hyponormal operator is w-hyponormal and every invertible w-hyponormal operator is *iw*-hyponormal ([13]). □

3.3 Paranormality

Consider the following paranormality of operators; see [2, 3, 14, 18].

Definition 20. *Let $M \geqslant 1$ be a constant. An operator $T \in \mathcal{B}(\mathbb{H})$ is said to be*

- M-paranormal *if* $M^2T^{*2}T^2 - 2\alpha T^*T + \alpha^2 I \geqslant 0$ *for all* $\alpha > 0$ *;*
- paranormal *if* $T^{*2}T^2 - 2\alpha T^*T + \alpha^2 I \geqslant 0$ *for all* $\alpha > 0$ *;*
- M^*-paranormal *if* $M^2T^{*2}T^2 - 2\alpha TT^* + \alpha^2 I \geqslant 0$ *for all* $\alpha > 0$ *;*
- $*$-paranormal *if* $T^{*2}T^2 - 2\alpha TT^* + \alpha^2 I \geqslant 0$ *for all* $\alpha > 0$.

Recall that an operator $T \in \mathcal{B}(\mathbb{H})$ is an isometry if $T^*T = I$; it is called an involution if $T^2 = I$.

Proposition 21. *Let $A \in \mathcal{B}(\mathbb{H})$, $X \in \mathcal{B}(\mathbb{K})$ and let $M \geqslant 1$ be a constant. If X is an isometry and A is M-paranormal (resp. paranormal), then $A \boxtimes X$ and $X \boxtimes A$ are M-paranormal (resp. paranormal).*

Proof. Assume that A is M-paranormal and X is an isometry. It follows that for any $\alpha > 0$ we have

$$\begin{aligned} M^2(A\boxtimes X)^{*2}(A\boxtimes X)^2 - 2\alpha(A\boxtimes X)^*(A\boxtimes X) + \alpha^2(I\boxtimes I) \\ &= M^2A^{*2}A^2 \boxtimes X^{*2}X^2 - 2\alpha A^*A\boxtimes X^*X + \alpha^2 I\boxtimes I \\ &= M^2A^{*2}A^2\boxtimes I - 2\alpha A^*A\boxtimes I + \alpha^2 I\boxtimes I \\ &= \big(M^2A^{*2}A^2 - 2\alpha A^*A + \alpha^2 I\big)\boxtimes I \\ &\geqslant 0. \end{aligned}$$

Thus $A\boxtimes X$ is M-paranormal. Similarly, the operator $X\boxtimes A$ is M-paranormal. The case of paranormality is just the case of M-paranormality when $M = 1$. □

Proposition 22. *Let $A \in \mathcal{B}(\mathbb{H})$, $X \in \mathcal{B}(\mathbb{K})$ and let $M \geqslant 1$ be a constant. If X is a self-adjoint involution and A is an M^*-paranormal (resp. $*$-paranormal) operator, then $A \boxtimes X$ and $X \boxtimes A$ are M-paranormal (resp. $*$-paranormal).*

Proof. The proof is similar to that of Proposition 21. □

Ando [2] showed that for any paranormal operator A, the tensor products $A \otimes I$ and $I \otimes A$ are paranormal. The next result is an extension of this fact to the case of Tracy-Singh products.

Corollary 23. *Let $A \in \mathcal{B}(\mathbb{H})$ and let $M \geqslant 1$ be a constant. If A satisfies one of the following properties, then the same property hold for $A \boxtimes I$ and $I \boxtimes A$: paranormal, M-paranormal, $*$-paranormal, M^*-paranormal.*

4 Tracy-Singh products and operators of type nilpotent, Hermitian, and isometry

In this section, we discuss relationship between Tracy-Singh products and certain classes of operators, namely, nilpotent operators, (skew)-Hermitian operators, (co)isometry operators, and unitary operators. Recall that an operator $T \in \mathcal{B}(\mathbb{H})$ is said to be nilpotent if $T^k = 0$ for some natural number k.

Proposition 24. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$. Then $A \boxtimes B$ is nilpotent if and only if A or B is nilpotent.*

Proof. It follows directly from Lemmas 1 and 4. □

Recall that an operator $T \in \mathcal{B}(\mathbb{H})$ is Hermitian if $T^* = T$, and T is skew-Hermitian if $T^* = -T$. It follows from Lemma 1 that the Tracy-Singh product of Hermitian operators is also Hermitian. The Tracy-Singh product of two skew-Hermitian operators is Hermitian. The Tracy-Singh product between a Hermitian operator and a skew-Hermitian operator is skew-Hermitian.

Proposition 25. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ be nonzero operators.*

1. *Assume $A \boxtimes B$ is Hermitian. Then A is Hermitian (resp. skew-Hermitian) if and only if B is Hermitian (resp. skew-Hermitian).*
2. *Assume $A \boxtimes B$ is skew-Hermitian. Then A is Hermitian (resp. skew-Hermitian) if and only if B is skew-Hermitian (resp. Hermitian).*

Proof. It follows directly from Lemmas 1 and 4. □

Recall that an operator $T \in \mathcal{B}(\mathbb{H})$ is a coisometry if $TT^* = I$. A unitary operator is an operator which is both an isometry and a coisometry. Stochel [21] gave a necessary and sufficient condition for $A \otimes B$ to be an isometry (resp. a coisometry, unitary). Now, we will extend this result to the case of Tracy-Singh products.

Proposition 26. *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ be operator matrices such that A_{ij} and B_{kl} are nonzero operators for all $i, j = 1, \ldots, m$ and $k, l = 1, \ldots, n$. Then $A \boxtimes B$ is an isometry (resp. a coisometry) if and only if so are αA and $\alpha^{-1} B$ for some $\alpha \in \mathbb{C} \setminus \{0\}$.*

Proof. If αA and $\alpha^{-1} B$ are isometries, then by Lemma 1,

$$\begin{aligned}(A \boxtimes B)^*(A \boxtimes B) &= A^*A \boxtimes B^*B \\ &= (\alpha A)^*(\alpha A) \boxtimes (\alpha^{-1}B)^*(\alpha^{-1}B) \\ &= I \boxtimes I.\end{aligned}$$

Suppose that $A \boxtimes B$ is an isometry. Then $A^*A \boxtimes B^*B = I \boxtimes I$. Thus, by Proposition 6, there exists $\beta \in \mathbb{C} \setminus \{0\}$ such that $\beta A^*A = I$ and $\beta^{-1}B^*B = I$. Setting $\alpha = \sqrt{\beta}$, we obtain $(\alpha A)^*(\alpha A) = I$ and $(\alpha^{-1}B)^*(\alpha^{-1}B) = I$. Hence αA and $\alpha^{-1}B$ are isometries. The proof for the case of coisometry is similar to that of isometry. □

Theorem 27. *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ be operator matrices such that A_{ij} and B_{kl} are nonzero operators for all $i,j = 1,\dots,m$ and $k,l = 1,\dots,n$. Then $A \boxtimes B$ is unitary if and only if so are αA and $\alpha^{-1}B$ for some $\alpha \in \mathbb{C}\setminus\{0\}$.*

Proof. If αA and $\alpha^{-1}B$ are unitary, then Lemma 1 implies

$$(A\boxtimes B)^*(A\boxtimes B) \,=\, A^*A\boxtimes B^*B \,=\, (\alpha A)^*(\alpha A)\boxtimes(\alpha^{-1}B)^*(\alpha^{-1}B) \,=\, I.$$

Similarly, we have $(A\boxtimes B)(A\boxtimes B)^* = I$. Conversely, suppose that $A\boxtimes B$ is unitary. We know that $A\boxtimes B$ is both an isometry and a coisometry. By Proposition 26, there exist $\alpha,\beta \in \mathbb{C}\setminus\{0\}$ such that αA and $\alpha^{-1}B$ are isometries, and βA and $\beta^{-1}B$ are coisometries. We have $(\alpha A)^*(\alpha A) = I = (\beta A)(\beta A)^*$ and

$$(\alpha^{-1}B)^*(\alpha^{-1}B) \,=\, I \,=\, (\beta^{-1}B)(\beta^{-1}B)^*.$$

Since $A\boxtimes B$ is normal, so are A and B (Theorem 9). Then $\alpha^2AA^* = \alpha^2A^*A = \beta^2AA^*$ and $\alpha^{-2}BB^* = \alpha^{-2}B^*B = \beta^{-2}BB^*$. Since $\alpha,\beta > 0$, it comes to the conclusion that $\alpha = \beta$. Hence αA and $\alpha^{-1}B$ are unitary. ☐

5 Tracy-Singh products and class-$\mathcal{A}$ type operators

The following classes of operators bring attention to operator theorists; see more information in [8, 9, 11, 12, 20].

Definition 28. *Let $k \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathbb{H})$ is said to be*

- class $\mathcal{A}$ *if* $|T^2| \geqslant |T|^2$ *;*
- class $\mathcal{A}(k)$ *if* $\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geqslant |T|^2$ *;*
- quasi-class $\mathcal{A}$ *if* $T^*|T^2|T \geqslant T^*|T|^2T$ *;*
- quasi-class $(\mathcal{A},k)$ *if* $T^{*k}|T^2|T^k \geqslant T^{*k}|T|^2T^k$ *;*
- $*$-class $\mathcal{A}$ *if* $|T^2| \geqslant |T^*|^2$ *;*
- quasi-$*$-class $\mathcal{A}$ *if* $T^*|T^2|T \geqslant T^*|T^*|^2T$ *;*
- quasi-$*$-class $(\mathcal{A},k)$ *if* $T^{*k}|T^2|T^k \geqslant T^{*k}|T^*|^2T^k$.

The next theorem shows that such classes of operators are preserved under Tracy-Singh products.

Theorem 29. *Let $A \in \mathcal{B}(\mathbb{H})$, $B \in \mathcal{B}(\mathbb{K})$, and let $k \in \mathbb{N}$. If both A and B satisfy one of the following properties, then the same property holds for $A\boxtimes B$: class $\mathcal{A}(k)$, class $\mathcal{A}$, quasi-class $(\mathcal{A},k)$, quasi-class $\mathcal{A}$, $*$-class $\mathcal{A}$, quasi-$*$-class $\mathcal{A}$, quasi-$*$-class $(\mathcal{A},k)$.*

Proof. Assume that A and B are class $\mathcal{A}(k)$. By Lemmas 1 and 3, we get

$$\begin{aligned}
[(A \boxtimes B)^*|A \boxtimes B|^{2k}(A \boxtimes B)]^{\frac{1}{k+1}} &= \left[(A^* \boxtimes B^*)\left(|A|^{2k} \boxtimes |B|^{2k}\right)(A \boxtimes B)\right]^{\frac{1}{k+1}} \\
&= \left(A^*|A|^{2k}A \boxtimes B^*|B|^{2k}B\right)^{\frac{1}{k+1}} \\
&= \left(A^*|A|^{2k}A\right)^{\frac{1}{k+1}} \boxtimes \left(B^*|B|^{2k}B\right)^{\frac{1}{k+1}} \\
&\geqslant |A|^2 \boxtimes |B|^2 \\
&= |A \boxtimes B|^2.
\end{aligned}$$

Hence $A \boxtimes B$ is a class $\mathcal{A}(k)$ operator. Now, assume that A and B are quasi-class $\mathcal{A}(k)$. Applying Lemmas 1 and 3, we get

$$\begin{aligned}
(A \boxtimes B)^{k*}|(A \boxtimes B)^2|(A \boxtimes B)^k &= (A^{k*} \boxtimes B^{k*})(|A^2| \boxtimes |B^2|)(A^k \boxtimes B^k) \\
&= A^{k*}|A^2|A^k \boxtimes B^{k*}|B^2|B^k \\
&\geqslant A^{k*}|A|^2A^k \boxtimes B^{k*}|B|^2B^k \\
&= (A^{k*} \boxtimes B^{k*})(|A|^2 \boxtimes |B|^2)(A^k \boxtimes B^k) \\
&= (A \boxtimes B)^{k*}|A \boxtimes B|^2(A \boxtimes B)^k.
\end{aligned}$$

Hence, $A \boxtimes B$ is a quasi-class $\mathcal{A}(k)$ operator. The proof for class $\mathcal{A}$ (resp. quasi-class $\mathcal{A}$) is done by replacing $k = 1$ in the case of class $\mathcal{A}(k)$ (resp. quasi-class $(\mathcal{A}, k)$). The proof for the case of quasi $*$-class $(\mathcal{A}, k)$ is similar to that of quasi-class $(\mathcal{A}, k)$. Similarly, the proof for $*$-class $\mathcal{A}$ (resp. quasi-$*$-class A) is done by replacing $k = 0$ (resp. $k = 1$) in the case of quasi-$*$-class $(\mathcal{A}, k)$. ☐

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Research Article

Geometric Means and Tracy-Singh Products for Positive Operators

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Abstract. We investigate relationship between metric/spectral/Sagae-Tanabe geometric means for several positive operators and Tracy-Singh products in terms of identities and inequalities. In particular, we obtain various generalizations of arithmetic-geometric-harmonic means inequality and its reverse. Moreover, we introduce the weighted Sagae-Tanabe spectral geometric mean for several positive operators and deduce its properties related to Tracy-Singh products.

Keywords. Metric (spectral) geometric mean; Sagae-Tanabe metric (spectral) geometric mean; Tensor product; Tracy-Singh product; Khatri-Rao product

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1. Introduction

It is well known that the tensor product (or Kronecker product) plays a fundamental role in linear algebra, functional analysis and related fields. Nowadays, theory of tensor products of operators is still developing, see [18] for instance. Recently, the authors of [12] introduced the Tracy-Singh product of operators which generalizes both tensor products of operators and Tracy-Singh products of complex matrices [17].

On the other hand, geometric means of positive definite matrices arise naturally in several areas of pure and applied mathematics. There are at least two types of geometric means.

The first one is the metric geometric mean, introduced by Ando [3]. Recall that the set of n-by-n positive definite matrices is a Riemannian manifold, in which the Riemannian metric between two matrices A and B is given by

$$\delta(A,B) = \|\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\|_2,$$

here, $\|\cdot\|_2$ denotes the Frobenius norm. The metric geometric mean $A\#B$ is indeed the unique metric midpoint of the geodesic line containing A and B (see, e.g., [9]). The second one is the spectral geometric mean $\natural$, introduced by Fiedler and Pták [6]. In fact, the square of $A\natural B$ is similar to the product AB, and in particular, its eigenvalues coincide with the positive square roots of the eigenvalues of AB. See more information about metric/spectral geometric means in [4,10] and [5, Chapters 4 and 6].

These two kinds of geometric means can be extended to multiple matrices by iterative processes, see e.g. [8]. Another such iterative geometric mean is the Sagae-Tanabe geometric mean, introduced in [15]. One of the most interesting properties of geometric means is the arithmetic-geometric-harmonic means (AM-GM-HM) inequalities. Indeed, the AM-GM-HM inequality concerning the metric geometric mean was established in [4]. Another version concerning the Sagae-Tanabe geometric mean were discussed in [1,15].

The Kronecker product of matrices turns out to be compatible with the metric geometric mean in the sense that

$$(A\#B)\otimes(C\#D) = (A\otimes C)\#(B\otimes D) \tag{1}$$

for positive semidefinite matrices A,B,C,D of appropriate sizes (see [4]). Of course, the similar result for the spectral geometric mean is also true ([8]). Moreover, Kilicman and Al-Zhour [8] discussed relations between Tracy-Singh products and metric geometric means, spectral geometric means, and Sagae-Tanabe geometric means of several positive definite matrices.

In this paper, we develop further theory for geometric means of Hilbert space operators. We investigate relationship between metric/spectral/Sagae-Tanabe geometric means and Tracy-Singh products in terms of operator identities and inequalities. In particular, we obtain various generalizations of the property (1) and the AM-GM-HM inequality and its reverse. Moreover, we introduce the weighted Sagae-Tanabe spectral geometric mean for several positive operators and deduce its properties related to Tracy-Singh products.

This paper is organized as follows. Section 2 consists of prerequisites on Tracy-Singh and Khatri-Rao products for Hilbert space operators. In Section 3, we establish certain identities and inequalities between metric geometric means and Tracy-Singh products of several positive operators. Sections 4 and 5 deal with spectral geometric means and Sagae-Tanabe metric geometric means, respectively. In Section 6, we define the weighted Sagae-Tanabe spectral geometric mean for several positive operators and prove certain results related to Tracy-Singh products.

2. Preliminaries on Tracy-Singh and Khatri-Rao Products for Operators

Throughout this paper, let $\mathbb{H}$ and $\mathbb{K}$ be complex separable Hilbert spaces. When X and Y are Hilbert spaces, denote by $\mathcal{B}(X,Y)$ the Banach space of bounded linear operators from X into Y, and abbreviate $\mathcal{B}(X,X)$ to $\mathcal{B}(X)$. For Hermitian operators $A,B \in \mathcal{B}(\mathbb{H})$, the notation $A \geqslant B$ means that $A-B$ is a positive operator, while $A>0$ indicates that A is positive and invertible.

The projection theorem for Hilbert spaces allows us to decompose

$$\mathbb{H} = \bigoplus_{i=1}^{m} \mathbb{H}_i, \quad \mathbb{K} = \bigoplus_{l=1}^{n} \mathbb{K}_k$$

where all $\mathbb{H}_i$ and $\mathbb{K}_k$ are Hilbert spaces. Each operator $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ thus can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,m} \text{ and } B = [B_{kl}]_{k,l=1}^{n,n}$$

where $A_{ij} \in \mathcal{B}(\mathbb{H}_j,\mathbb{H}_i)$ and $B_{kl} \in \mathcal{B}(\mathbb{K}_l,\mathbb{K}_k)$ for each $i,j=1,\dots,m$ and $k,l=1,\dots,n$.

Definition 1. Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ be operator matrices defined as above. The *Tracy-Singh product* of A and B is defined to be the operator matrix

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij}, \tag{2}$$

which is a bounded linear operator from $\bigoplus_{i,k=1}^{m,n} \mathbb{H}_i \otimes \mathbb{K}_k$ into itself.

Note that if $m=n=1$, the Tracy-Singh product $A \boxtimes B$ reduces to the tensor product $A \otimes B$.

Lemma 1 ([12, 13]). *Let $A,C \in \mathcal{B}(\mathbb{H})$ and $B,D \in \mathcal{B}(\mathbb{K})$ be compatible operator matrices.*

(i) *The Tracy-Singh product is compatible with the usual product in the sense that*

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD.$$

(ii) *If $A,B>0$, then $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$ for any real number α.*

(iii) *If $A,B \geqslant 0$, then $A \boxtimes B \geqslant 0$.*

(iv) *If $A,B>0$, then $A \boxtimes B > 0$.*

(v) *If $A \geqslant C \geqslant 0$ and $B \geqslant D \geqslant 0$, then $A \boxtimes B \geqslant C \boxtimes D$.*

The notion of Khstri-Rao product for operators was introduced in [11].

Definition 2. Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{K})$ be operator matrices. The *Khatri-Rao product* of A and B is defined to be

$$A \boxdot B = [A_{ij} \otimes B_{ij}]_{i,j=1}^{m,m} \tag{3}$$

as a bounded linear operator from $\bigoplus_{i=1}^{m} \mathbb{H}_i \otimes \mathbb{K}_i$ into itself.

Note that if $m=1$, then $A \boxdot B = A \otimes B$. We set $\boxplus_{i=1}^{1} A_i = A_1 = \boxdot_{i=1}^{1} A_i$. For $r \in \mathbb{N}-\{1\}$ and a finite number of operator matrices $A_i \in \mathcal{B}(\mathbb{H}_i)$ $(i=1,\dots,r)$, denote

$$\mathop{\boxtimes}_{i=1}^{r} A_i = ((A_1 \boxtimes A_2) \boxtimes \cdots \boxtimes A_{r-1}) \boxtimes A_r, \qquad \mathop{\boxdot}_{i=1}^{r} A_i = ((A_1 \boxdot A_2) \boxdot \cdots \boxdot A_{r-1}) \boxdot A_r.$$

Lemma 2 ([14])**.** *Let* $r \in \mathbb{N} - \{1\}$*. There exists an isometry* Z *such that*

$$\underset{i=1}{\overset{r}{\boxdot}} A_i = Z^* \left(\underset{i=1}{\overset{r}{\boxtimes}} A_i \right) Z \tag{4}$$

for any $A_i \in \mathcal{B}(\mathbb{H}_i)$, $i = 1, \ldots, r$.

3. Metric Geometric Mean

In this section, we establish certain operator identities and inequalities involving metric geometric means and Tracy-Singh products. First of all, we recall some background about metric geometric means.

The metric geometric mean for matrices/operators was firstly defined by Ando [3]:

$$A \# B = A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}, \quad A, B > 0. \tag{5}$$

This formula comes from two natural requirements. First, it should coincide with the usual geometric mean for positive real numbers: $A \# B = (AB)^{1/2}$ provided that $AB = BA$. The second condition is the congruent invariance

$$T^*(A \# B)T = (T^* A T) \# (T^* B T)$$

for any invertible $T \in \mathcal{B}(\mathbb{H})$. Now, consider positive invertible operators A and B in $\mathcal{B}(\mathbb{H})$ and let $w \in [0,1]$. The w-weighted geometric mean of A and B is defined by

$$A \#_w B = A^{1/2}(A^{-1/2} B A^{-1/2})^w A^{1/2}.$$

For arbitrary positive operators A and B, we define the w-weighted geometric mean of A and B to be

$$A \#_w B = \lim_{\varepsilon \to 0^+} (A + \varepsilon I) \#_w (B + \varepsilon I).$$

Here, the limit is taken in the strong-operator topology. For briefly, we write $A \# B$ for $A \#_{1/2} B$.

Theorem 1. *Let* A_1, A_2, B_1 *and* B_2 *be positive operators in* $\mathcal{B}(\mathbb{H})$ *and* $w \in [0,1]$*. Then*

$$(A_1 \#_w A_2) \boxtimes (B_1 \#_w B_2) = (A_1 \boxtimes B_1) \#_w (A_2 \boxtimes B_2), \tag{6}$$

$$(A_1 \#_w A_2) \boxdot (B_1 \#_w B_2) \leqslant (A_1 \boxdot B_1) \#_w (A_2 \boxdot B_2). \tag{7}$$

Proof. First, consider the case $A_1, A_2, B_1, B_2 > 0$. By using Lemma 1, we get

$$\begin{aligned}
(A_1 \boxtimes B_1) \#_w (A_2 \boxtimes B_2) &= (A_1 \boxtimes B_1)^{1/2} \left[(A_1 \boxtimes B_1)^{-1/2} (A_2 \boxtimes B_2)(A_1 \boxtimes B_1)^{-1/2} \right]^w (A_1 \boxtimes B_1)^{1/2} \\
&= \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \left[\left(A_1^{-1/2} \boxtimes B_1^{-1/2} \right) (A_2 \boxtimes B_2) \left(A_1^{-1/2} \boxtimes B_1^{-1/2} \right) \right]^w \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \\
&= \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \left[\left(A_1^{-1/2} A_2 A_1^{-1/2} \right) \boxtimes \left(B_1^{-1/2} B_2 B_1^{-1/2} \right) \right]^w \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \\
&= \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \left[\left(A_1^{-1/2} A_2 A_1^{-1/2} \right)^w \boxtimes \left(B_1^{-1/2} B_2 B_1^{-1/2} \right)^w \right] \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \\
&= \left[A_1^{1/2} \left(A_1^{-1/2} A_2 A_1^{-1/2} \right)^w A_1^{1/2} \right] \boxtimes \left[B^{1/2} \left(B_1^{-1/2} B_2 B_1^{-1/2} \right)^w B_1^{1/2} \right] \\
&= (A_1 \#_w A_2) \boxtimes (B_1 \#_w B_2).
\end{aligned}$$

Ando's result [4] states that if Φ is a positive linear map, then for all $A, B \geqslant 0$,

$$\Phi(A \#_w B) \leqslant \Phi(A) \#_w \Phi(B). \tag{8}$$

By applying Lemma 2 and inequality (8), we have

$$\begin{aligned}(A_1\#_w A_2)\boxdot(B_1\#_w B_2) &= Z^*\,[(A_1\#_w A_2)\boxtimes(B_1\#_w B_2)]Z\\ &= Z^*\,[(A_1\boxtimes A_2)\#_w(A_2\boxtimes B_2)]Z\\ &\leqslant [Z^*(A_1\boxtimes B_1)Z]\#_w[Z^*(A_2\boxtimes B_2)Z]\\ &= (A_1\boxdot B_1)\#_w(A_2\boxdot B_2).\end{aligned}$$

For arbitrary $A_1, A_2, B_1, B_2 \geqslant 0$, perturb each of them with εI and then take limit as $\varepsilon \to 0^+$. □

Corollary 1. *Let $r \in \mathbb{N}$ and $w \in [0,1]$. For each $1 \leqslant i \leqslant r$, let $A_i, B_i \in \mathcal{B}(\mathbb{H})$ be positive operators. Then*

$$\mathop{\boxtimes}_{i=1}^{r}(A_i\#_w B_i) = \left(\mathop{\boxtimes}_{i=1}^{r} A_i\right)\#_w\left(\mathop{\boxtimes}_{i=1}^{r} B_i\right). \tag{9}$$

Proof. The proof is by induction on r. □

In [8], Kilicman and Al-Zhour invesigated weighted metric geometric means of any finite number of positive definite matrices. Now, we will extend this geometric mean to the case of finite number of positive operators.

Definition 3. Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathcal{B}(\mathbb{H})$ be a positive operator. Let $\alpha_i \in [0,1]$ for each $1 \leqslant i \leqslant r-1$ and denote $\alpha = (\alpha_1, \dots, \alpha_{r-1})$. We define

$$\mathcal{G}_{\alpha_1}(A_1, A_2) = A_2 \;\#_{\alpha_1}\; A_1.$$

Now continue recurrently, setting

$$\mathcal{G}_{\alpha}(A_1, \dots, A_r) = \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(A_1, \dots, A_{r-1}), A_r)$$

where $\tilde{\alpha} = (\alpha_1, \dots, \alpha_{r-2})$. We call $\mathcal{G}_{\alpha}(A_1, \dots, A_r)$ the *iterative α-weighted metric geometric mean* of $A_1, \dots, A_r$.

The next two results asserts the compatibility between Tracy-Singh products and iterative weighted metric geometric means.

Theorem 2. *Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i, B_i \in \mathcal{B}(\mathbb{H})$ be positive operators. Let $\alpha_i \in [0,1]$ for each $1 \leqslant i \leqslant r-1$. Then*

$$\mathcal{G}_{\alpha}(A_1\boxtimes B_1, \dots, A_r\boxtimes B_r) = \mathcal{G}_{\alpha}(A_1, \dots, A_r)\boxtimes\mathcal{G}_{\alpha}(B_1, \dots, B_r) \tag{10}$$

Proof. We use induction on r. By continuity, we may assume that $A_i, B_i > 0$ for all $i = 1, \dots, r$. When $r = 2$, we have by Proposition 1 that

$$\begin{aligned}\mathcal{G}_{\alpha}(A_1\boxtimes B_1, A_2\boxtimes B_2) &= (A_2\boxtimes B_2)\#_{\alpha}(A_1\boxtimes B_1)\\ &= (A_2\#_{\alpha}A_1)\boxtimes(B_2\#_{\alpha}B_1)\\ &= \mathcal{G}_{\alpha}(A_1, A_2)\boxtimes\mathcal{G}_{\alpha}(B_1, B_2)\end{aligned}$$

where $\alpha \in [0,1]$. This gives the claim when $r = 2$. Suppose that the property (10) holds for $r-1$ ($r \geqslant 3$). Let $\alpha = (\alpha_1, \dots, \alpha_{r-1})$ and $\tilde{\alpha} = (\alpha_1, \dots, \alpha_{r-2})$ where $\alpha_i \in [0,1]$ for any $1 \leqslant i \leqslant r-1$.

Using Theorem 1, we have

$$\begin{aligned}\mathcal{G}_\alpha(A_1\boxtimes B_1,\dots,A_r\boxtimes B_r) &= \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(A_1\boxtimes B_1,\dots,A_{r-1}\boxtimes B_{r-1}),A_r\boxtimes B_r)\\ &= \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(A_1,\dots,A_{r-1})\boxtimes \mathcal{G}_{\tilde{\alpha}}(B_1,\dots,B_{r-1}),A_r\boxtimes B_r)\\ &= \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(A_1,\dots,A_{r-1}),A_r)\boxtimes \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(B_1,\dots,B_{r-1}),B_r)\\ &= \mathcal{G}_\alpha(A_1,\dots,A_r)\boxtimes \mathcal{G}_\alpha(B_1,\dots,B_r).\end{aligned}$$

□

Corollary 2. *Let $r\in\mathbb{N}-\{1\}$ and $s\in\mathbb{N}$. For each $1\leqslant i\leqslant r$, $1\leqslant j\leqslant s$, let $A_{ij}\in\mathcal{B}(\mathbb{H})$ be a positive operator. Let $\alpha_i\in[0,1]$ for each $1\leqslant i\leqslant r-1$. Then*

$$\mathcal{G}_\alpha\left(\boxtimes_{j=1}^{s}A_{1j},\dots,\boxtimes_{j=1}^{s}A_{rj}\right)=\boxtimes_{j=1}^{s}\mathcal{G}_\alpha(A_{1j},\dots,A_{rj}). \tag{11}$$

The *Thompson metric* [16] on the open convex cone of positive invertible operators is defined for each $A,B>0$ by

$$d(A,B)=\max\{\log M(A/B),\log M(B/A)\},$$

where $M(A/B)=\inf\{\lambda>0:A\leqslant\lambda B\}$. The *diameter* of $\{A_1,\dots,A_r\}$ with respect to the Thompson metric d is defined by

$$\triangle(A_1,\dots,A_r)=\max\{d(A_i,A_j):1\leqslant i,j\leqslant r\}.$$

Lemma 3. *Let $r\in\mathbb{N}-\{1\}$. Let A_i for each $1\leqslant i\leqslant r$ and B be positive invertible operators on $\mathbb{H}$. Let $\alpha_i\in[0,1]$ for each $1\leqslant i\leqslant r-1$. Then*

$$d(\mathcal{G}_\alpha(A_1,\dots,A_r),B)\leqslant\triangle(A_1,\dots,A_r,B). \tag{12}$$

Proof. See [1, Proposition 3.1]. □

The next result is a generalization of inequality (12).

Proposition 1. *Let $r\in\mathbb{N}-\{1\}$ and $s\in\mathbb{N}$. For each $1\leqslant i\leqslant r$, $1\leqslant j\leqslant s$, let $A_{ij},B_j\in\mathcal{B}(\mathbb{H})$ be positive invertible operators. Let $\alpha_i\in[0,1]$ for each $1\leqslant i\leqslant r-1$. Then*

$$d\left(\boxtimes_{j=1}^{s}\mathcal{G}_\alpha(A_{1j},\dots,A_{rj}),\boxtimes_{j=1}^{s}B_j\right)\leqslant\triangle\left(\boxtimes_{j=1}^{s}A_{1j},\dots,\boxtimes_{j=1}^{s}A_{rj},\boxtimes_{j=1}^{s}B_j\right). \tag{13}$$

Proof. This proposition follows from Lemma 3 and Corollary 2. □

4. Spectral Geometric Mean

Recall that for positive definite matrices A and B of the same size, its spectral geometric mean [6] is defined by

$$A\natural B=(A^{-1}\#B)^{\frac{1}{2}}A(A^{-1}\#B)^{\frac{1}{2}}.$$

Now, let A and B be positive invertible operators in $\mathcal{B}(\mathbb{H})$ and $w\in[0,1]$. The w-weighted spectral geometric mean of A and B is defined by

$$A\natural_w B=(A^{-1}\#B)^{w}A(A^{-1}\#B)^{w}.$$

For arbitrary positive operators A and B, we define the w-weighted spectral geometric mean of A and B to be

$$A \natural_w B = \lim_{\varepsilon \to 0^+} (A + \varepsilon I) \natural_w (B + \varepsilon I).$$

Theorem 3. *Let A_1, A_2, B_1 and B_2 be positive operators in $\mathcal{B}(\mathbb{H})$. Then*

$$(A_1 \boxtimes B_1) \natural_w (A_2 \boxtimes B_2) = (A_1 \natural_w A_2) \boxtimes (B_1 \natural_w B_2). \tag{14}$$

Proof. By continuity, we may assume that $A_1, A_2, B_1, B_2 > 0$. It follows from Lemma 1 and Proposition 1 that

$$\begin{aligned}
(A_1 \boxtimes B_1) \natural_w (A_2 \boxtimes B_2) &= \left[(A_1 \boxtimes B_1)^{-1} \# (A_2 \boxtimes B_2)\right]^w (A_1 \boxtimes B_1) \left[(A_1 \boxtimes B_1)^{-1} \# (A_2 \boxtimes B_2)\right]^w \\
&= \left[(A_1^{-1} \boxtimes B_1^{-1}) \# (A_2 \boxtimes B_2)\right]^w (A_1 \boxtimes B_1) \left[(A_1^{-1} \boxtimes B_1^{-1}) \# (A_2 \boxtimes B_2)\right]^w \\
&= \left[(A_1^{-1} \# A_2) \boxtimes (B_1^{-1} \# B_2)\right]^w (A_1 \boxtimes B_1) \left[(A_1^{-1} \# A_2) \boxtimes (B_1^{-1} \# B_2)\right]^w \\
&= \left[(A_1^{-1} \# A_2)^w \boxtimes (B_1^{-1} \# B_2)^w\right] (A_1 \boxtimes B_1) \left[(A_1^{-1} \# A_2)^w \boxtimes (B_1^{-1} \# B_2)^w\right] \\
&= \left[(A_1^{-1} \# A_2)^w A_1 (A_1^{-1} \# A_2)^w\right] \boxtimes \left[(B_1^{-1} \# B_2)^w B_1 (B_1^{-1} \# B_2)^w\right] \\
&= (A_1 \natural_w A_2) \boxtimes (B_1 \natural_w B_2).
\end{aligned}$$

□

Corollary 3. *Let $r \in \mathbb{N} - \{1\}$ and $w \in [0,1]$. For each $1 \leqslant i \leqslant r$, let $A_i, B_i \in \mathcal{B}(\mathbb{H})$ be positive operators. Then*

$$\left(\boxtimes_{i=1}^{r} A_i\right) \natural_w \left(\boxtimes_{i=1}^{r} B_i\right) = \boxtimes_{i=1}^{r} (A_i \natural_w B_i). \tag{15}$$

Proof. The proof is by induction on r. We have that the property (15) holds for $r = 2$ by Lemma 3. Suppose that the property (15) holds for $r - 1$ ($r \geqslant 3$). By using Lemma 3, we get

$$\begin{aligned}
\left(\boxtimes_{i=1}^{r} A_i\right) \natural_w \left(\boxtimes_{i=1}^{r} B_i\right) &= \left[\left(\boxtimes_{i=1}^{r-1} A_i\right) \boxtimes A_r\right] \natural_w \left[\left(\boxtimes_{i=1}^{r-1} B_i\right) \boxtimes B_r\right] \\
&= \left[\left(\boxtimes_{i=1}^{r-1} A_i\right) \natural_w \left(\boxtimes_{i=1}^{r-1} B_i\right)\right] \boxtimes (A_r \natural_w B_r) \\
&= \left(\boxtimes_{i=1}^{r-1} (A_i \natural_w B_i)\right) \boxtimes (A_r \natural_w B_r) \\
&= \boxtimes_{i=1}^{r} (A_i \natural_w B_i).
\end{aligned}$$

□

In [8], Kilicman and Al-Zhour studied weighted spectral geometric means of any finite number of positive definite matrices and proved several properties related to Tracy-Singh products. Now, we will extend this geometric mean to the case of any finite number of positive operators.

Definition 4. Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathcal{B}(\mathbb{H})$ be positive operators. Let $\alpha_i \in [0,1]$ for each $1 \leqslant i \leqslant r - 1$. We define

$$\mathcal{G}_{\alpha_1}^{sp}(A_1, A_2) = A_1 \natural_{\alpha_1} A_2.$$

Now continue recurrently, setting for each $r \geqslant 3$,

$$\mathcal{G}_\alpha^{sp}(A_1,\dots,A_r) = \mathcal{G}_{\alpha_{r-1}}^{sp}\left(\mathcal{G}_{\tilde{\alpha}}^{sp}(A_1,\dots,A_{r-1}),A_r\right)$$

where $\tilde{\alpha} = (\alpha_1,\dots,\alpha_{r-2})$. We call $\mathcal{G}_\alpha^{sp}(A_1,\dots,A_r)$ the *iterated α-weighted spectral geometric mean* of $A_1,\dots,A_r$.

From Definition 4, we can rewrite (15) in Corollary 3 to be

$$\mathcal{G}_\alpha^{sp}\left(\boxtimes_{i=1}^{r} A_i, \boxtimes_{i=1}^{r} B_i\right) = \boxtimes_{i=1}^{r} \mathcal{G}_\alpha^{sp}(A_i,B_i)$$

where $\alpha = w$.

Corollary 4. *Let $r \in \mathbb{N}-\{1\}$. Let A_i and B_i be compatible positive operators in $\mathcal{B}(\mathbb{H})$ for each $i = 1,\dots,r$. Then*

$$\mathcal{G}_\alpha^{sp}(A_1 \boxtimes B_1,\dots,A_r \boxtimes B_r) = \mathcal{G}_\alpha^{sp}(A_1,\dots,A_r) \boxtimes \mathcal{G}_\alpha^{sp}(B_1,\dots,B_r). \tag{16}$$

Proof. The proof is by induction on r. By Theorem 3, we have that the property (16) is true for $r = 2$. Suppose that the property (16) is true for $r-1$. By Theorem 3, we obtain

$$\begin{aligned}\mathcal{G}_\alpha^{sp}(A_1 \boxtimes B_1,\dots,A_r \boxtimes B_r) &= \mathcal{G}_{\alpha_{r-1}}^{sp}\left(\mathcal{G}_{\tilde{\alpha}}^{sp}(A_1 \boxtimes B_1,\dots,A_{r-1} \boxtimes B_{r-1}),A_r \boxtimes B_r\right)\\ &= \mathcal{G}_{\alpha_{r-1}}^{sp}\left(\mathcal{G}_{\tilde{\alpha}}^{sp}(A_1,\dots,A_{r-1}) \boxtimes \mathcal{G}_{\tilde{\alpha}}^{sp}(B_1,\dots,B_{r-1}),A_r \boxtimes B_r\right)\\ &= \mathcal{G}_{\alpha_{r-1}}^{sp}\left(\mathcal{G}_{\tilde{\alpha}}^{sp}(A_1,\dots,A_{r-1}),A_r\right) \boxtimes \mathcal{G}_{\alpha_{r-1}}^{sp}\left(\mathcal{G}_{\tilde{\alpha}}^{sp}(B_1,\dots,B_{r-1}),B_r\right)\\ &= \mathcal{G}_\alpha^{sp}(A_1,\dots,A_r) \boxtimes \mathcal{G}_\alpha^{sp}(B_1,\dots,B_r).\end{aligned}$$

□

Corollary 5. *Let $r \in \mathbb{N}-\{1\}$ and $s \in \mathbb{N}$. Let $A_{ij} \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator for each $i = 1,\dots,r$, $j = 1,\dots,s$. Then*

$$\mathcal{G}_\alpha^{sp}\left(\boxtimes_{j=1}^{s} A_{1j},\dots,\boxtimes_{j=1}^{s} A_{rj}\right) = \boxtimes_{j=1}^{s} \mathcal{G}_\alpha^{sp}(A_{1j},\dots,A_{rj}). \tag{17}$$

Proof. The proof is by induction on s. □

5. Sagae-Tanabe Metric Geometric Mean

Let $r \in \mathbb{N}-\{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator. Let $t = (t_1,\dots,t_r)$ where $t_i \in [0,1]$ for each $1 \leqslant i \leqslant r$ and $\sum_{i=1}^{r} t_i = 1$. The *weighted arithmetic* and *harmonic means* of $A_1,\dots,A_r$ are defined by

$$\mathcal{A}_t(A_1,\dots,A_r) = \sum_{i=1}^{r} t_i A_i, \qquad \mathcal{H}_t(A_1,\dots,A_r) = \left(\sum_{i=1}^{r} t_i A_i^{-1}\right)^{-1}.$$

Sagae and Tanabe [15] proposed weighted geometric means of severable positive definite matrices as follows.

Definition 5. Let A and B be positive invertible operators in $\mathcal{B}(\mathbb{H})$ and let $v = (v_1,v_2)$ where v_1, $v_2 \in [0,1]$ and $v_1 + v_2 = 1$. We define

$$\mathcal{G}_v(A,B) = A \#_\alpha B$$

where $\alpha = 1 - v_2$. Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator. Let $t = (t_1, \dots, t_r)$ where $t_i \in [0,1]$ for each $1 \leqslant i \leqslant r$ and $\sum_{i=1}^{r} t_i = 1$. For each $1 \leqslant i \leqslant r-1$, let

$$\alpha_i = 1 - \left(t_{i+1} \Big/ \sum_{j=1}^{i+1} t_j \right).$$

The *Sagae-Tanabe weighted geometric mean* of $A_1, \dots, A_r$ is defined by

$$\mathcal{G}_t(A_1, \dots, A_r) = \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{t}}(A_1, \dots, A_{r-1}), A_r)$$

where $\mathcal{G}_{\tilde{t}}(A_1, \dots, A_{r-1})$ is the Sagae-Tanabe weighted geometric mean of $A_1, \dots, A_{r-1}$ with weighted $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{r-1})$ where $\tilde{t}_i = t_i / \sum_{j=1}^{r-1} t_j$ for each $1 \leqslant i \leqslant r-1$. Note that

$$\mathcal{G}_t(A_1, \dots, A_r) = \mathcal{G}_\alpha(A_1, \dots, A_r)$$

where $\mathcal{G}_\alpha(A_1, \dots, A_r)$ is the weighted metric geometric mean of $A_1, \dots, A_r$ in Definition 3 with weight $\alpha = (\alpha_1, \dots, \alpha_{r-1})$.

Theorem 4. *Let $r \in \mathbb{N} - \{1\}$ and $s \in \mathbb{N}$. For each $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, let $A_{ij} \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator. Let $t = (t_1, \dots, t_r)$ where $t_i \in [0,1]$ for each $1 \leqslant i \leqslant r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\mathcal{G}_t\left(\boxtimes_{j=1}^{s} A_{1j}, \dots, \boxtimes_{j=1}^{s} A_{rj} \right) = \boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j}, \dots, A_{rj}). \tag{18}$$

Proof. Let $\alpha_i = 1 - (t_{i+1} / \sum_{j=1}^{i+1} t_j)$ for each $1 \leqslant i \leqslant r-1$ and denote $\alpha = (\alpha_1, \dots, \alpha_{r-1})$. By Definition 5, we have

$$\mathcal{G}_t(A_{11} \boxtimes A_{12}, \dots, A_{r1} \boxtimes A_{r2}) = \mathcal{G}_\alpha(A_{11} \boxtimes A_{12}, \dots, A_{rj} \boxtimes B_{rj}).$$

Applying Theorem 2, we obtain

$$\mathcal{G}_\alpha(A_{11} \boxtimes A_{12}, \dots, A_{rj} \boxtimes B_{rj}) = \mathcal{G}_\alpha(A_{11}, \dots, A_{r1}) \boxtimes \mathcal{G}_\alpha(A_{12}, \dots, A_{r2}).$$

This implies that

$$\mathcal{G}_t(A_{11} \boxtimes A_{12}, \dots, A_{r1} \boxtimes A_{r2}) = \mathcal{G}_t(A_{11}, \dots, A_{r1}) \boxtimes \mathcal{G}_t(A_{12}, \dots, A_{r2}).$$

We get the result by using induction on s. □

Lemma 4. *Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator and let $t = (t_1, \dots, t_r)$ where $t_i \in [0,1]$ for $i = 1, \dots, r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\mathcal{H}_t(A_1, \dots, A_r) \leqslant \mathcal{G}_t(A_1, \dots, A_r) \leqslant \mathcal{A}_t(A_1, \dots, A_r). \tag{19}$$

Proof. See [1, Proposition 2.4]. □

We extend [8, Theorem 4.6] to AM-GM-HM inequalities involving Tracy-Singh product of positive invertible operators as in the next two results.

Corollary 6. *Let $r \in \mathbb{N} - \{1\}$ and $s \in \mathbb{N}$. For each $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s$, let $A_{ij} \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator and let $t = (t_1, \dots, t_r)$ where $t_i \in [0,1]$ for $i = 1, \dots, r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\boxtimes_{j=1}^{s} \mathcal{H}_t(A_{1j}, \dots, A_{rj}) \leqslant \mathcal{G}_t\left(\boxtimes_{j=1}^{s} A_{1j}, \dots, \boxtimes_{j=1}^{s} A_{rj} \right) \leqslant \boxtimes_{j=1}^{s} \mathcal{A}_t(A_{1j}, \dots, A_{rj}). \tag{20}$$

Proof. Lemma 4 tells us that

$$\mathcal{H}_t(A_{1j},\dots,A_{rj}) \leqslant \mathcal{G}_t\left(A_{1j},\dots,A_{rj}\right) \leqslant \mathcal{A}_t(A_{1j},\dots,A_{rj})$$

for each $1 \leqslant j \leqslant s$. By using Lemma 1, we get

$$\mathop{\boxtimes}_{j=1}^{s} \mathcal{H}_t(A_{1j},\dots,A_{rj}) \leqslant \mathop{\boxtimes}_{j=1}^{s} \mathcal{G}_t\left(A_{1j},\dots,A_{rj}\right) \leqslant \mathop{\boxtimes}_{j=1}^{s} \mathcal{A}_t(A_{1j},\dots,A_{rj}).$$

Applying Theorem 4, we obtain

$$\mathop{\boxtimes}_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj}) = \mathcal{G}_t\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j},\dots,\mathop{\boxtimes}_{j=1}^{s} A_{rj}\right)$$

and the inequality (20) follows. □

Corollary 7. *Let $r \in \mathbb{N} - \{1\}$ and $s \in \mathbb{N}$. For each $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s$, let $A_{ij} \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator and let $t = (t_1,\dots,t_r)$ where $t_i \in [0,1]$ for $i = 1,\dots,r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\mathcal{H}_t\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j},\dots,\mathop{\boxtimes}_{j=1}^{s} A_{rj}\right) \leqslant \mathop{\boxtimes}_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj}) \leqslant \mathcal{A}_t\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j},\dots,\mathop{\boxtimes}_{j=1}^{s} A_{rj}\right). \tag{21}$$

Proof. It follows directly from the AM-GM-HM inequality (19) and Theorem 4. □

We now turn to the AM-GM-HM inequality involving Khatri-Rao products.

Corollary 8. *Let $A_{ij} \in \mathcal{B}(\mathbb{H})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ be compatible positive invertible operators and let $t = (t_1,\dots,t_r)$ where $t_i \in [0,1]$ for $i = 1,\dots,r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\mathop{\boxdot}_{j=1}^{s} \mathcal{H}_t(A_{1j},\dots,A_{rj}) \leqslant \mathop{\boxdot}_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj}) \leqslant \mathcal{A}_t\left(\mathop{\boxdot}_{j=1}^{s} A_{1j},\dots,\mathop{\boxdot}_{j=1}^{s} A_{rj}\right). \tag{22}$$

Proof. We have by Lemmas 2 and 4 that

$$\begin{aligned}
\mathop{\boxdot}_{j=1}^{s} \mathcal{H}_t(A_{1j},\dots,A_{rj}) &= Z^*\left(\mathop{\boxtimes}_{j=1}^{s} \mathcal{H}_t(A_{1j},\dots,A_{rj})\right) Z \\
&\leqslant Z^*\left(\mathop{\boxtimes}_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj})\right) Z \\
&= \mathop{\boxdot}_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj}).
\end{aligned}$$

Using Lemma 2, we get

$$\begin{aligned}
Z^*\left[\mathcal{A}_t\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j},\dots,\mathop{\boxtimes}_{j=1}^{s} A_{rj}\right)\right] Z &= Z^*\left[\sum_{i=1}^{r} t_i \left(\mathop{\boxtimes}_{j=1}^{s} A_{ij}\right)\right] Z \\
&= \sum_{i=1}^{r} t_i \left[Z^*\left(\mathop{\boxtimes}_{j=1}^{s} A_{ij}\right) Z\right] \\
&= \sum_{i=1}^{r} t_i \left(\mathop{\boxdot}_{j=1}^{s} A_{ij}\right) \\
&= \mathcal{A}_t\left(\mathop{\boxdot}_{j=1}^{s} A_{1j},\dots,\mathop{\boxdot}_{j=1}^{s} A_{rj}\right).
\end{aligned}$$

Applying Lemma 2 and Corollary 7, we obtain

$$\boxdot_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj}) = Z^* \left(\boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj}) \right) Z$$
$$\leqslant Z^* \left[\mathcal{A}_t \left(\boxtimes_{j=1}^{s} A_{1j},\dots, \boxtimes_{j=1}^{s} A_{rj} \right) \right] Z$$
$$= \mathcal{A}_t \left(\boxdot_{j=1}^{s} A_{1j},\dots, \boxdot_{j=1}^{s} A_{rj} \right). \qquad \square$$

The next result is a generalization of Lemma 3.

Proposition 2. *Let A_{ij} and B_j $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ be compatible positive invertible operators and let $t = (t_1,\dots,t_r)$ where $t_i \in [0,1]$ for $i = 1,\dots,r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$d\left(\boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj}), \boxtimes_{j=1}^{s} B_j \right) \leqslant \triangle \left(\boxtimes_{j=1}^{s} A_{1j},\dots, \boxtimes_{j=1}^{s} A_{rj}, \boxtimes_{j=1}^{s} B_j \right). \tag{23}$$

Proof. The desire result follows from Lemma 3 and Corollary 2. □

For $h,x \geqslant 1$, the *(generalized) Specht ratio* is defined by

$$S_h(x) = \frac{(h^x - 1)h^{x(h^x-1)^{-1}}}{e \log h^x} \quad \text{for } h \neq 1 \text{ and } S_1(x) = 1.$$

We denote $S_h(1)$ by S_h. See [1, 7] for more information. The next result is a reverse version of AM-GM-HM inequality involving Tracy-Singh products via Specht ratio.

Proposition 3. *Let A_{ij} $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ be compatible positive invertible operators and let $t = (t_1,\dots,t_r)$ where $t_i \in [0,1]$ for $i = 1,\dots,r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\mathcal{A}_t \left(\boxtimes_{j=1}^{s} A_{1j},\dots, \boxtimes_{j=1}^{s} A_{rj} \right) \leqslant S_h^{r-1} \cdot \left(\boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj}) \right) \tag{24}$$

where $h = e^{\triangle(\boxtimes_{j=1}^{s} A_{1j},\dots,\boxtimes_{j=1}^{s} A_{rj})}$.

Proof. By using Lemma 3 and Corollary 2, we get the result. □

Lemma 5. *Let $A_i \in \mathcal{B}(\mathbb{H})$ $(1 \leqslant i \leqslant r, r \geqslant 2)$ be positive invertible operators and t_i $(1 \leqslant i \leqslant r)$ be real numbers such that $t_1 > 0, t_i < 0$ $(2 \leqslant i \leqslant r)$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\mathcal{A}_t(A_1,\dots,A_r) \leqslant \mathcal{G}_t(A_1,\dots,A_r). \tag{25}$$

If $\sum_{i=1}^{r} t_i A_i^{-1} > 0$, then

$$\mathcal{G}_t(A_1,\dots,A_r) \leqslant \mathcal{H}_t(A_1,\dots,A_r). \tag{26}$$

Proof. The proof is similar to the case of matrices, given in [2, Theorem 2.1]. □

We now obtain reverse AM-GM-HM inequalities involving Tracy-Singh products as follows.

Theorem 5. *Let $A_{ij} \in \mathcal{B}(\mathbb{H})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ be compatible positive invertible operators and t_i $(1 \leqslant i \leqslant r)$ be real numbers such that $t_1 > 0, t_i < 0$ $(2 \leqslant i \leqslant r)$ and $\sum_{i=1}^{r} t_i = 1$.*

Then

$$\mathop{\boxtimes}_{j=1}^{s} \mathcal{A}_t(A_{1j},\dots,A_{rj}) \leqslant \mathcal{G}_t\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j},\dots,\mathop{\boxtimes}_{j=1}^{s} A_{rj}\right). \tag{27}$$

If $\mathcal{H}_t(A_{1j},\dots,A_{rj}) > 0$ *for all* $j = 1,\dots,s$, *then*

$$\mathcal{G}_t\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j},\dots,\mathop{\boxtimes}_{j=1}^{s} A_{rj}\right) \leqslant \mathop{\boxtimes}_{j=1}^{s} \mathcal{H}_t(A_{1j},\dots,A_{rj}). \tag{28}$$

Proof. It follows from Lemma 5 that

$$\mathcal{A}_t(A_{1j},\dots,A_{rj}) \leqslant \mathcal{G}_t\left(A_{1j},\dots,A_{rj}\right) \leqslant \mathcal{H}_t(A_{1j},\dots,A_{rj})$$

for each $j = 1,\dots,s$. Since $\mathcal{A}_t(A_{1j},\dots,A_{rj}) \geqslant 0$ for all $j = 1,\dots,s$, we have by Lemmas 1 and 5 that

$$\mathcal{G}_t\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j},\dots,\mathop{\boxtimes}_{j=1}^{s} A_{rj}\right) = \mathop{\boxtimes}_{j=1}^{s} \mathcal{G}_t\left(A_{1j},\dots,A_{rj}\right) \geqslant \mathop{\boxtimes}_{j=1}^{s} \mathcal{A}_t(A_{1j},\dots,A_{rj}).$$

Since $\mathcal{H}_t(A_{1j},\dots,A_{rj}) > 0$ for all $j = 1,\dots,s$, we obtain by Lemma 1 that

$$\mathop{\boxtimes}_{j=1}^{s} \mathcal{H}_t(A_{1j},\dots,A_{rj}) > 0.$$

The proof is complete by applying Lemma 5 and Corollary 4. □

Theorem 6. *Let* $A_{ij} \in \mathcal{B}(\mathbb{H})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ *be compatible positive invertible operators and* t_i $(1 \leqslant i \leqslant r)$ *be real numbers such that* $t_1 > 0, t_i < 0$ $(2 \leqslant i \leqslant r)$ *and* $\sum_{i=1}^{r} t_i = 1$. *Then*

$$\mathcal{A}_t\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j},\dots,\mathop{\boxtimes}_{j=1}^{s} A_{rj}\right) \leqslant \mathop{\boxtimes}_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rs}). \tag{29}$$

If $\mathcal{H}_t\left(\boxtimes_{j=1}^{s} A_{1j},\dots,\boxtimes_{j=1}^{s} A_{rj}\right) > 0$, *then*

$$\mathop{\boxtimes}_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rs}) \leqslant \mathcal{H}_t\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j},\dots,\mathop{\boxtimes}_{j=1}^{s} A_{rj}\right). \tag{30}$$

Proof. By applying Lemma 5 and Corollary 4, we get the results. □

Corollary 9. *Let* $A_{ij} \in \mathcal{B}(\mathbb{H})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ *be compatible positive invertible operators and* t_i $(1 \leqslant i \leqslant r)$ *be real numbers such that* $t_1 > 0, t_i < 0$ $(2 \leqslant i \leqslant r)$ *and* $\sum_{i=1}^{r} t_i = 1$. *Then*

$$\mathcal{A}_t\left(\mathop{\boxdot}_{j=1}^{s} A_{1j},\dots,\mathop{\boxdot}_{j=1}^{s} A_{rj}\right) \leqslant \mathop{\boxdot}_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj}) \leqslant \mathop{\boxdot}_{j=1}^{s} \mathcal{H}_t(A_{1j},\dots,A_{rj}). \tag{31}$$

Proof. This result is a direct consequence of Theorem 6 and Lemmas 2 and 4. □

6. Sagae-Tanabe Spectral Geometric Mean

We introduce the following definition:

Definition 6. Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator. Let $t = (t_1,\dots,t_r)$ where $t_i \in [0,1]$ for each $1 \leqslant i \leqslant r$ and $\sum_{i=1}^{r} t_i = 1$. Let $\alpha_i = 1 - (t_{i+1}/\sum_{j=1}^{i+1} t_j)$

for each $1 \leqslant i \leqslant r-1$. The *Sagae-Tanabe spectral geometric mean* of $A_1,\dots,A_r$ is defined by

$$\mathcal{G}_t^{sp}(A_1,\dots,A_r) = \mathcal{G}_\alpha^{sp}(A_1,\dots,A_r)$$

where $\alpha = (\alpha_1,\dots,\alpha_{r-1})$.

Proposition 4. *Let A_i and B_i $(1 \leqslant i \leqslant r, r \geqslant 2)$ be compatible positive operators in $\mathcal{B}(\mathbb{H})$ and let $t = (t_1,\dots,t_r)$ where $t_i \in [0,1]$ for $i = 1,\dots,r-1$ and $\sum_{i=1}^r t_i = 1$. Then*

$$\mathcal{G}_t^{sp}(A_1 \boxtimes B_1,\dots,A_r \boxtimes B_r) = \mathcal{G}_t^{sp}(A_1,\dots,A_r) \boxtimes \mathcal{G}_t^{sp}(B_1,\dots,B_r) \tag{32}$$

$$\mathcal{G}_t^{sp}\left(\boxtimes_{i=1}^r A_i, \boxtimes_{i=1}^r B_i\right) = \boxtimes_{i=1}^r \mathcal{G}_t^{sp}(A_i,B_i). \tag{33}$$

Proof. Let $\alpha_i = 1-(t_{i+1}/\sum_{j=1}^{i+1} t_j)$ for each $1 \leqslant i \leqslant r-1$ and denote $\alpha = (\alpha_1,\dots,\alpha_{r-1})$. By Definition 6, we have

$$\mathcal{G}_t^{sp}(A_1 \boxtimes B_1,\dots,A_r \boxtimes B_r) = \mathcal{G}_\alpha^{sp}(A_1 \boxtimes B_1,\dots,A_r \boxtimes B_r)$$

$$\mathcal{G}_t^{sp}\left(\boxtimes_{i=1}^r A_i, \boxtimes_{i=1}^r B_i\right) = \mathcal{G}_\alpha^{sp}\left(\boxtimes_{i=1}^r A_i, \boxtimes_{i=1}^r B_i\right).$$

By Corollary 4, we get (32). Applying Corollary 3, we obtain (33). □

Corollary 10. *Let $A_{ij} \in \mathcal{B}(\mathbb{H})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ be compatible positive invertible operators and let $t = (t_1,\dots,t_r)$ where $t_i \in [0,1]$ for $i = 1,\dots,r-1$ and $\sum_{i=1}^r t_i = 1$. Then*

$$\mathcal{G}_t^{sp}\left(\boxtimes_{j=1}^s A_{1j},\dots,\boxtimes_{j=1}^s A_{rj}\right) = \boxtimes_{j=1}^s \mathcal{G}_t^{sp}(A_{1j},\dots,A_{rj}). \tag{34}$$

Proof. From (32), we have

$$\mathcal{G}_t^{sp}(A_{11} \boxtimes A_{12},\dots,A_{r1} \boxtimes A_{r2}) = \mathcal{G}_t^{sp}(A_{11},\dots,A_{r1}) \boxtimes \mathcal{G}_t^{sp}(A_(12),\dots,A_{r2}).$$

We obtain (34) by induction on s. □

7. Conclusion

Several relations between metric/spectral/Sagae-Tanabe geometric means for several positive operators and Tracy-Singh products are established in terms of identities and inequalities. In particular, we obtain noncommutative arithmetic-geometric-harmonic means inequalities and their reverses. Moreover, we define the weighted Sagae-Tanabe spectral geometric mean for several positive operators and deduce its properties related to Tracy-Singh products.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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Research Article

Refinements and Reverses of Operator Callebaut Inequality Involving Tracy-Singh Products and Khatri-Rao Products

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Abstract. In this paper, we establish certain refinements and reverses of Callebaut-type inequality for bounded continuous fields of Hilbert space operators, parametrized by a locally compact Hausdorff space equipped with a finite Radon measure. These inequalities involve Tracy-Singh products, Khatri-Rao products and weighted geometric means. In addition, we obtain integral Callebaut-type inequalities for tensor products and Hadamard products. Our results extend Callebaut-type inequalities for real numbers, matrices and operators.

Keywords. Callebaut inequality; Tracy-Singh product; Khatri-Rao product; Weighted geometric mean; Continuous field of operators

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1. Introduction

In mathematics, the Cauchy-Schwarz inequality is an important inequality which can be applied in many fields, e.g. operator theory, linear algebra, analysis, probability and statistics. This inequality states that for vectors $(a_1,\dots,a_k)$ and $(b_1,\dots,b_k)$ of real numbers, we have

$$\left(\sum_{i=1}^{k} a_i b_i\right)^2 \leqslant \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right). \tag{1}$$

In 1965, Callebaut [4] published a refinement of the Cauchy-Schwarz inequality (1). For each $\alpha \in [0,1]$ and for any tuples $x = (x_1,\dots,x_k)$ and $y = (y_1,\dots,y_k)$ of positive real numbers, let us denote $\mathcal{J}_\alpha^k(x,y) = \sum_{i=1}^{k} x_i \sharp_\alpha y_i$, where $\sharp_\alpha$ is the α-weighted geometric mean. For either $0 \leqslant \beta \leqslant \alpha \leqslant \frac{1}{2}$ or $\frac{1}{2} \leqslant \alpha \leqslant \beta \leqslant 1$, the classical Callebaut inequality [4] can be stated as

$$\left(\mathcal{J}_{1/2}^k(x,y)\right)^2 \leqslant \mathcal{J}_\alpha^k(x,y)\cdot\mathcal{J}_{1-\alpha}^k(x,y) \leqslant \mathcal{J}_\beta^k(x,y)\cdot\mathcal{J}_{1-\beta}^k(x,y) \leqslant \mathcal{J}_0^k(x,y)\cdot\mathcal{J}_1^k(x,y). \tag{2}$$

There have been several investigations and generalizations on the Callebaut inequality; see [1, 2, 6, 7, 13] and references therein. Hiai and Zhan [6] gave a matrix analogue of the Callebaut inequality (2) by considering the convexity of a certain norm function. The paper [7] presented a matrix version of (2) associated to the tensor product, the Hadamard product, weighted geometric means and a Kubo-Ando mean. Wada [13] provided a simple form of (2) for positive operators involving an operator mean and its dual. Some refinements and reverses of (2) for operators concerning the Hadamard product and weighted geometric means were presented in [1, 2]. Recently in [12], the authors established integral versions of the Callebaut inequality and its refinements for bounded continuous fields of Hilbert space operators concerning the Tracy-Singh product, the Khatri-Rao product and weighted geometric means.

In this paper, we investigate refinements and reverses of the operator Callebaut inequalities for bounded continuous fields of positive operators parametrized by a locally compact Hausdorff space endowed with a finite Radon measure. Such integral inequalities involves Tracy-Singh products, Khatri-Rao products, tensor products, Hadamard products and weighted geometric means. In particular, our results are refinements and reverses of Callebaut-type inequalities obtained in the previous works [4, 7, 12].

This paper is organized as follows. In Section 2, we give preliminaries on operator products and Bochner integration of continuous fields of operators on a locally compact Hausdorff space. In Section 3, we provide certain refinements of integral Callebaut inequalities for bounded continuous fields of operators involving some kind of operator products and weighted geometric means. Some reversed Callebaut-type inequalities for bounded continuous fields of operators are presented in Section 4. The conclusion is given in the last section.

2. Preliminaries

Throughout this paper, let $\mathbb{H}$ be a complex Hilbert space. When $\mathbb{X}$ and $\mathbb{Y}$ are Hilbert spaces, denote by $\mathfrak{B}(\mathbb{X},\mathbb{Y})$ the Banach space of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$, and abbreviate $\mathfrak{B}(\mathbb{X},\mathbb{X})$ to $\mathfrak{B}(\mathbb{X})$. For self-adjoint operators $A, B \in \mathfrak{B}(\mathbb{X})$, the notation $A \geqslant B$ means that $A - B$ is a positive operator. The set of all positive invertible operators on $\mathbb{X}$ is denoted by $\mathfrak{B}(\mathbb{X})^+$.

The projection theorem for Hilbert spaces allows us to decompose

$$\mathbb{H} = \bigoplus_{i=1}^{n} \mathbb{H}_i \tag{3}$$

where all $\mathbb{H}_i$ are Hilbert spaces. For each $i = 1,\dots,n$, let P_i be the natural projection from $\mathbb{H}$ onto $\mathbb{H}_i$ and E_i the canonical embedding from $\mathbb{H}_i$ into $\mathbb{H}$. Note that $P_i^* = E_i$. Each operator

$A \in \mathfrak{B}(\mathbb{H})$ can be uniquely determined by an operator matrix

$$A = [A_{ij}]_{i,j=1}^{n,n},$$

where $A_{ij} \in \mathfrak{B}(\mathbb{H}_j, \mathbb{H}_i)$ is defined by $A_{ij} = P_i A E_j$ for each $i, j = 1, \dots, n$.

2.1 Operator Products

Recall that the tensor product of $A, B \in \mathfrak{B}(\mathbb{H})$ is a unique bounded linear operator from $\mathbb{H} \otimes \mathbb{H}$ into itself such that for all $x, y \in \mathbb{H}$,

$$(A \otimes B)(x \otimes y) = Ax \otimes By.$$

Fix a countable orthonormal basis $\mathbb{E}$ on $\mathbb{H}$. Recall that the Hadamard product of $A, B \in \mathfrak{B}(\mathbb{H})$ is defined to be bounded linear operator $A \odot B$ from $\mathbb{H}$ into itself such that for all $e \in \mathbb{E}$,

$$\langle (A \odot B)e, e \rangle = \langle Ae, e \rangle \langle Be, e \rangle.$$

Following [5], the Hadamard product can be expressed as

$$A \odot B = U^*(A \otimes B)U, \tag{4}$$

where $U : \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$ is the isometry defined by $Ue = e \otimes e$ for all $e \in \mathbb{E}$. In the case of matrices, the Hadamard product of $A = [a_{ij}]_{i,j=1}^{n,n}$ and $B = [b_{ij}]_{i,j=1}^{n,n}$ reduces to the entrywise product $A \odot B = [a_{ij} b_{ij}]$, which is a principal submatrix of the Kronecker (tensor) product $A \otimes B = [a_{ij} B]_{ij}$.

Definition 1. Let $A = [A_{ij}]_{i,j=1}^{n,n}$ and $B = [B_{ij}]_{i,j=1}^{n,n}$ be operator matrices in $\mathfrak{B}(\mathbb{H})$. The Tracy-Singh product of A and B is defined to be the operator matrix

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij}, \tag{5}$$

which is a bounded linear operator from $\bigoplus_{i,j=1}^{n,n} \mathbb{H}_i \otimes \mathbb{H}_j$ into itself. The Khatri-Rao product of A and B is defined to be the operator matrix

$$A \boxdot B = \left[A_{ij} \otimes B_{ij} \right]_{i,j} \tag{6}$$

which is a bounded linear operator from $\bigoplus_{i=1}^{n} \mathbb{H}_i \otimes \mathbb{H}_i$ into itself.

Lemma 1 ([8, 9]). *Let $A, B, C, D \in \mathfrak{B}(\mathbb{H})$.*

(1) $\alpha(A \boxtimes B) = (\alpha A) \boxtimes B = A \boxtimes (\alpha B)$ *for any* $\alpha \in \mathbb{C}$.

(2) $(A + B) \boxtimes (C + D) = A \boxtimes B + A \boxtimes D + B \boxtimes C + B \boxtimes D$.

(3) $(A \boxtimes B)^* = A^* \boxtimes B^*$.

(4) *If* $A \geqslant C \geqslant 0$ *and* $B \geqslant D$ [illegible] $\boxtimes D \geqslant 0$.

(5) $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes$ [illegible]

(6) *If* $A, B \in \mathfrak{B}(\mathbb{H})^+$, *then* $(A$ [illegible] *ny* $\alpha \in \mathbb{R}$.

Lemma 2 ([11]). *There is a u* [illegible] *ap*

$$\Phi : \mathfrak{B}\Big(\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} \mathbb{H}_i \otimes \mathbb{H}_j \Big) \to \mathfrak{B}\Big(\bigoplus_{i=1}^{n} \mathbb{H}_i \otimes \mathbb{H}_i \Big) \tag{7}$$

such that $\Phi(A \boxtimes B) = A \boxdot B$ *for any* $A, B \in \mathfrak{B}(\mathbb{H})$.

2.2 Bochner Integration

Let Ω be a locally compact Hausdorff space endowed with a finite Radon measure μ. A family $(A_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})$ is said to be a continuous field if the parametrization $t \mapsto A_t$ is norm-continuous on Ω. If, in addition, the function $t \mapsto \|A_t\|$ is Lebesgue integrable on Ω, then we can form the Bochner integral $\int_\Omega A_t d\mu(t)$ as a unique element in $\mathfrak{B}(\mathbb{H})$ such that

$$T\Big(\int_\Omega A_t d\mu(t)\Big) = \int_\Omega T(A_t)d\mu(t) \tag{8}$$

for every T in the dual of $\mathfrak{B}(\mathbb{H})$. A field $(A_t)_{t\in\Omega}$ is said to be bounded if there is a positive constant M such that $\|A_t\| \leqslant M$ for all $t \in \Omega$. In particular, every bounded continuous field of operators on Ω is always Bochner integrable.

Lemma 3 ([10]). *Let $(A_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathfrak{B}(\mathbb{H})$. Then for any $X \in \mathfrak{B}(\mathbb{H})$, we have*

$$\Big(\int_\Omega A_t d\mu(t)\Big) \boxtimes X = \int_\Omega (A_t \boxtimes X)d\mu(t). \tag{9}$$

3. Refined Callebaut-type Inequalities for Operators

In this section, we establish certain refined Callebaut-type inequalities for continuous fields of Hilbert space operators defined on a locally compact Hausdorff space Ω endowed with a finite Radon measure μ.

We start with recalling some auxiliary inequalities.

Lemma 4 ([2]). *Let $x, y > 0$ and $r \in (0,1)$. Then*

$$x^r y^{1-r} + x^{1-r}y^r + 2p\left(\sqrt{x}-\sqrt{y}\right)^2 + q\left(2\sqrt{xy}+x+y-2x^{\frac14}y^{\frac34}-2x^{\frac34}y^{\frac14}\right) \leqslant x+y, \tag{10}$$

where $p = \min\{r, 1-r\}$ and $q = \min\{2p, 1-2p\}$.

Lemma 5. *Decompose $\mathbb{H}$ as in (3). Let $A, B \in \mathfrak{B}(\mathbb{H})^+$. If either $0 \leqslant \beta \leqslant \alpha < \frac12$ or $\frac12 < \alpha \leqslant \beta \leqslant 1$, then*

$$\begin{aligned} &A^\beta \boxtimes B^{1-\beta} + A^{1-\beta}\boxtimes B^\beta \\ &\quad \geqslant A^\alpha \boxtimes B^{1-\alpha} + A^{1-\alpha}\boxtimes B^\alpha + \delta\Big(A^\beta \boxtimes B^{1-\beta} + A^{1-\beta}\boxtimes B^\beta - 2A^{\frac12}\boxtimes B^{\frac12}\Big) \\ &\qquad + \eta\Big(A^\beta \boxtimes B^{1-\beta} + A^{1-\beta}\boxtimes B^\beta + 2A^{\frac12}\boxtimes B^{\frac12} - 2A^\gamma \boxtimes B^{1-\gamma} - 2A^{1-\gamma}\boxtimes B^\gamma\Big), \end{aligned} \tag{11}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

Proof. If we replace y by x^{-1} and r by $\frac{1-u}{2}$ in (10), then we get

$$x^u + x^{-u} + 2p\left(x+x^{-1}-2\right) + q\left(x+x^{-1}+2-2x^{\frac12}-2x^{-\frac12}\right) \leqslant x + x^{-1}, \tag{12}$$

where $p = \min\{\frac{1-u}{2}, \frac{1+u}{2}\}$ and $q = \min\{2p, 1-2p\}$. Consider $v, w \in \mathbb{R}$ such that $v \leqslant w$. Applying the functional calculus on the spectrum of $A \boxtimes B$ with $u := \frac{v}{w}$ in (12), then we get

$$\begin{aligned} &A^w \boxtimes B^{-w} + A^{-w}\boxtimes B^w \\ &\quad \geqslant A^v \boxtimes B^{-v} + A^{-v}\boxtimes B^v + \Big(\frac{w-v}{w}\Big)\big(A^w \boxtimes B^{-w} + A^{-w}\boxtimes B^w - 2I\boxtimes I\big) \end{aligned}$$

$$+\eta\left(A^{w}\boxtimes B^{-w}+A^{-w}\boxtimes B^{w}-\text{[illegible]}\,B^{\frac{w}{2}}\right), \tag{13}$$

please correct "-" to "+"

where $\eta=\min\{\frac{w-v}{w},\frac{v}{w}\}$. Multiplying both sides [illegible]

$$\begin{aligned}
&A^{1+w}\boxtimes B^{1-w}+A^{1-w}\boxtimes B^{1+w}\\
&\geqslant A^{1+v}\boxtimes B^{1-v}+A^{1-v}\boxtimes B^{1+v}+\left(\frac{w-\text{[illegible]}}{w-\text{[illegible]}}\,\text{[illegible]}{}^{+w}-2A\boxtimes B\right)\\
&\quad+\eta\left(A^{1+w}\boxtimes B^{1-w}+A^{1-w}\boxtimes B^{1+w}+2A\boxtimes B-2A^{1+\frac{w}{2}}\boxtimes B^{1-\frac{w}{2}}-2A^{1-\frac{w}{2}}\boxtimes B^{1+\frac{w}{2}}\right).
\end{aligned}$$

Now, we have only to replace v,w,A,B by $2\alpha-1, 2\beta-1, A^{\frac{1}{2}}, B^{\frac{1}{2}}$, respectively. □

Definition 2. For any bounded continuous fields $\mathcal{X}=(X_t)_{t\in\Omega}$ and $\mathcal{W}=(W_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})$, we set

$$\mathcal{F}_{\mathcal{W}}(\mathcal{X})=\int_{\Omega} W_t^* X_t W_t \, d\mu(t).$$

For any bounded continuous field $\mathcal{X}=(X_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})^+$ and $\alpha\in[0,1]$, we set $\mathcal{X}^{\alpha}=\left(X_t^{\alpha}\right)_{t\in\Omega}$.

Lemma 6. *Decompose $\mathbb{H}$ as in (3). Let $\mathcal{X}=(X_t)_{t\in\Omega}$ and $\mathcal{W}=(W_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$ and $\mathfrak{B}(\mathbb{H})$, respectively. If either $0\leqslant\beta\leqslant\alpha<\frac{1}{2}$ or $\frac{1}{2}<\alpha\leqslant\beta\leqslant 1$, then*

$$\begin{aligned}
&\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\\
&\quad\geqslant \mathcal{F}_{\mathcal{W}}(\mathcal{X})^{\alpha}\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\alpha})\\
&\qquad+\delta\left[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})-2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\right]\\
&\qquad+\eta\left[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})+2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\right.\\
&\qquad\qquad\left.-2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma})-2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma})\right],
\end{aligned}\tag{14}$$

where $\gamma=\frac{1+2\beta}{4}$, $\delta=\frac{\beta-\alpha}{\beta-1/2}$ and $\eta=\min\{\delta,1-\delta\}$.

Proof. By using Lemmas 1 and 3, and Fubini's theorem for Bochner integrals [3], we get

$$\begin{aligned}
\mathcal{F}_{\mathcal{W}}(\mathcal{X})^{\alpha}\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})&=\int_{\Omega}W_t^*A_t^{\alpha}W_t d\mu(t)\boxtimes\int_{\Omega}W_s^*X_s^{1-\alpha}W_s d\mu(s)\\
&=\iint_{\Omega^2}\left(W_t^*X_t^{\alpha}W_t\right)\boxtimes\left(W_s^*X_s^{1-\alpha}W_s\right)\,d\mu(t)\mu(s)\\
&=\iint_{\Omega^2}(W_t\boxtimes W_s)^*\left(X_t^{\alpha}\boxtimes X_s^{1-\alpha}\right)(W_t\boxtimes W_s)\,d\mu(t)\mu(s).
\end{aligned}$$

We have by applying Lemma 5 that

$$\begin{aligned}
&\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\\
&=\iint_{\Omega^2}(W_t\boxtimes W_s)^*\left(X_t^{\beta}\boxtimes X_s^{1-\beta}+X_t^{1-\beta}\boxtimes X_s^{\beta}\right)(W_t\boxtimes W_s)\,d\mu(t)\mu(s)\\
&\geqslant\iint_{\Omega^2}(W_t\boxtimes W_s)^*\left[X_t^{\alpha}\boxtimes X_s^{1-\alpha}+X_t^{1-\alpha}\boxtimes X_s^{\alpha}+\delta\left(X_t^{\beta}\boxtimes X_s^{1-\beta}+X_t^{1-\beta}\boxtimes X_s^{\beta}-2X_t^{\frac{1}{2}}\boxtimes X_s^{\frac{1}{2}}\right)\right.\\
&\quad\left.+\eta\left(X_t^{\beta}\boxtimes X_s^{1-\beta}+X_t^{1-\beta}\boxtimes X_s^{\beta}+2X_t^{\frac{1}{2}}\boxtimes X_s^{\frac{1}{2}}-2X_t^{\gamma}\boxtimes X_t^{1-\gamma}-2X_t^{1-\gamma}\boxtimes X_t^{\gamma}\right)\right](W_t\boxtimes W_s)\,d\mu(t)\mu(s)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{F}_{\mathcal{W}}(\mathcal{X})^{\alpha} \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\alpha}) \\
&\quad + \delta\Big[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) - 2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\Big] \\
&\quad + \eta\Big[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) + 2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \\
&\qquad - 2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma}) - 2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma})\Big]. \qquad \square
\end{aligned}
$$

Recall that, for each $\alpha \in [0,1]$, the α-weighted geometric mean of operators $X, Y \in \mathfrak{B}(\mathbb{H})^{+}$ is defined as

$$X \sharp_{\alpha} Y = X^{\frac{1}{2}}\big(X^{-\frac{1}{2}} Y X^{-\frac{1}{2}}\big)^{\alpha} X^{\frac{1}{2}}.$$

Definition 3. For two continuous fields $A = (A_t)_{t\in\Omega}$, B = (must be "B-script" $)^{+}$ and any $\alpha \in [0,1]$, we set

$$\mathcal{I}_{\alpha}(\mathcal{A}, \mathcal{B}) = \int_{\Omega}$$ must be "A-script"

In particular, we have

$$\mathcal{I}_0(\mathcal{A}, \mathcal{B}) = \int_{\Omega} A_t \, d\mu(t), \quad \mathcal{I}_1(\mathcal{A}, \mathcal{B}) = \int_{\Omega} B_t \, d\mu(t).$$

Consider bounded continuous fields $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})^{+}$. An integral Callebaut inequality [12] states that for either $0 \leqslant \beta \leqslant \alpha \leqslant \frac{1}{2}$ or $\frac{1}{2} \leqslant \alpha \leqslant \beta \leqslant 1$, we have

$$
\begin{aligned}
2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) &\leqslant \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}) \\
&\leqslant \mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \\
&\leqslant \mathcal{I}_{0}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{0}(\mathcal{A},\mathcal{B}). \qquad (15)
\end{aligned}
$$

Now, we provide a refinement of the integral Callebaut inequality (15) and as a consequence give an operator Callebaut type inequality for Khatri-Rao products.

Theorem 1. *Decompose* $\mathbb{H}$ *as in* (3). *Let* $\mathcal{A} = (A_t)_{t\in\Omega}$ *and* $\mathcal{B} = (B_t)_{t\in\Omega}$ *be two bounded continuous fields of operators in* $\mathfrak{B}(\mathbb{H})^{+}$. *If either* $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ *or* $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, *then*

$$
\begin{aligned}
&\mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \\
&\quad \geqslant \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}) \\
&\qquad + \delta\Big[\mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\Big] \\
&\qquad + \eta\Big[\mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) + 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \\
&\qquad - 2\mathcal{I}_{\gamma}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\gamma}(\mathcal{A},\mathcal{B})\Big], \qquad (16)
\end{aligned}
$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ *and* $\eta = \min\{\delta, 1-\delta\}$.

Proof. Setting $X_t = A_t^{-\frac{1}{2}} B_t A_t^{-\frac{1}{2}}$ and $W_t = A_t^{\frac{1}{2}}$ for all $t \in \Omega$, we have that for any $\alpha \in [0,1]$,

$$\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\alpha}) = \int_{\Omega} A_t^{\frac{1}{2}}\Big(A_t^{-\frac{1}{2}} B_t A_t^{-\frac{1}{2}}\Big)^{\alpha} A_t^{\frac{1}{2}} \, d\mu(t) = \int_{\Omega} A_t \sharp_{\alpha} B_t \, d\mu(t) = \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}).$$

By using Lemma 6, we obtain the result. $\square$

Corollary 1. *Decompose* $\mathbb{H}$ *as in* (3)*. Let* $\mathcal{A} = (A_t)_{t\in\Omega}$ *and* $\mathcal{B} = (B_t)_{t\in\Omega}$ *be two bounded continuous fields of operators in* $\mathfrak{B}(\mathbb{H})^+$*. If either* $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ *or* $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$*, then*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\\
&\quad\geqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\\
&\qquad+\delta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\beta(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\Big]\\
&\qquad+\eta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\beta(\mathcal{A},\mathcal{B})+2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\\
&\qquad\qquad-2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\Big],
\end{aligned}\tag{17}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ *and* $\eta = \min\{\delta, 1-\delta\}$.

Proof. Let Φ be the linear map described in Lemma 2. We have that for any $\alpha \in [0,1]$,

$$\Phi(\mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})) = \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}).$$

The proof is done by using Theorem 1 and the fact that the map Φ is a positive unital linear map. □

The next result is an integral inequality involving tensor products which is a special case of Theorem 1 when $n = 1$.

Corollary 2. *Let* $\mathbb{H}$ *be a Hilbert space (not decomposed as in* (3)*). Let* $\mathcal{A} = (A_t)_{t\in\Omega}$ *and* $\mathcal{B} = (B_t)_{t\in\Omega}$ *be two bounded continuous fields of operators in* $\mathfrak{B}(\mathbb{H})^+$*. If either* $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ *or* $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$*, then*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\\
&\quad\geqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\\
&\qquad+\delta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\Big]\\
&\qquad+\eta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})+2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\\
&\qquad\qquad-2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\Big],
\end{aligned}\tag{18}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ *and* $\eta = \min\{\delta, 1-\delta\}$.

As a consequence, we obtain the following integral inequality concerning Hadamard products.

Corollary 3. *Let* $\mathbb{H}$ *be a Hilbert space. Let* $\mathcal{A} = (A_t)_{t\in\Omega}$ *and* $\mathcal{B} = (B_t)_{t\in\Omega}$ *be two bounded continuous fields of operators in* $\mathfrak{B}(\mathbb{H})^+$*. If either* $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ *or* $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$*, then*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\\
&\quad\geqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+\delta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})-\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\Big]\\
&\qquad+\eta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})-2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})\Big],
\end{aligned}\tag{19}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ *and* $\eta = \min\{\delta, 1-\delta\}$.

Proof. Using the fact that the Hadamard product is expressed as the deformation of the tensor product via the isometry U defined in (4), we get the result. □

Remark 1. When we set $\Omega = \{1,\dots,k\}$ equipped with the counting measure, we have that $\mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) = \sum_{i=1}^{k} A_i \sharp_\alpha B_i$. From previous theorem and previous corollaries, we obtain discrete versions of refined Callebaut-type inequalities for Tracy-Singh products, Khatri-Rao products, tensor products and Hadamard products, respectively.

Remark 2. For a particular case of Theorem 1 when $\mathbb{H} = \mathbb{C}^n$ and $\Omega = \{1,\dots,k\}$ equipped with the counting measure, we get a matrix inequality concerning Tracy-Singh products. In the same way, we get matrix versions of (17)-(19) for Khatri-Rao products, Kronecker products and Hadamard products, respectively. The matrix versions of Kronecker products and Hadamard products are refinements of matrix Callebaut inequalities in [7, Theorem 3.4 and Corollary 3.5].

In the next corollary, we get a refined Callebaut-type inequality for real numbers.

Corollary 4. *Let $x = (x_t)_{t\in\Omega}$ and $y = (y_t)_{t\in\Omega}$ be two fields of positive real numbers. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, then*

$$\mathcal{I}_\beta(x,y)\cdot\mathcal{I}_{1-\beta}(x,y) \geqslant \mathcal{I}_\alpha(x,y)\cdot\mathcal{I}_{1-\alpha}(x,y) + \delta\Big[\mathcal{I}_\beta(x,y)\cdot\mathcal{I}_{1-\beta}(x,y) - \big(\mathcal{I}_{1/2}(x,y)\big)^2\Big]$$
$$+\eta\Big[\mathcal{I}_\beta(x,y)\cdot\mathcal{I}_{1-\beta}(x,y) + \big(\mathcal{I}_{1/2}(x,y)\big)^2 - 2\mathcal{I}_\gamma(x,y)\cdot\mathcal{I}_{1-\gamma}(x,y)\Big],$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

Proof. Putting $A_t = x_t I$ and $B_t = y_t I$ for all $t \in \Omega$ in Theorem 1, we obtain the result. □

We mention that if Ω is the finite set $\{1,\dots,k\}$ equiped with the counting measure, we get a discrete version of (20) which is a refinement of the classical Callebaut inequality (2).

4. Reversed Callebaut-type Inequalities for Operators

In this section, we present reversed inequalities of Callebaut-type inequalities. We begin with recalling the following scalar inequality.

Lemma 7 ([14]). *Let $x, y \geqslant 0$ and $r \in (0,1)$.*

$$x+y \leqslant x^{1-r}y^r + x^r y^{1-r} + 2s(\sqrt{x}-\sqrt{y})^2 - q\left(x+y+2\sqrt{xy} - 2x^{\frac14}y^{\frac34} - 2x^{\frac34}y^{\frac14}\right),$$

where $p = \min\{r, 1-r\}$, $q = \min\{2p, 1-2p\}$ and $s = \max\{r, 1-r\}$.

This lemma is used to derive the following operator inequality.

Lemma 8. *Decompose $\mathbb{H}$ as in (3). Let $A, B \in \mathfrak{B}(\mathbb{H})^+$ and either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$. Then*

$$A^\beta \boxtimes B^{1-\beta} + A^{1-\beta} \boxtimes B^\beta$$
$$\leqslant A^\alpha \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^\alpha + (2-\delta)\left(A^\beta \boxtimes B^{1-\beta} + A^{1-\beta} \boxtimes B^\beta - 2A^{\frac12} \boxtimes B^{\frac12}\right)$$

$$-\eta\left(A^{\beta}\boxtimes B^{1-\beta}+A^{1-\beta}\boxtimes B^{\beta}+2A^{\frac{1}{2}}\boxtimes B^{\frac{1}{2}}-2A^{\gamma}\boxtimes B^{1-\gamma}-2A^{1-\gamma}\boxtimes B^{\gamma}\right), \qquad (20)$$

where $\gamma=\frac{1+2\beta}{4}$, $\delta=\frac{\beta-\alpha}{\beta-1/2}$ *and* $\eta=\min\{\delta,1-\delta\}$.

Proof. In Lemma 7, we have by replacing y with x^{-1} that

$$x+x^{-1}\leqslant x^{1-2r}+x^{2r-1}+2s(x+x^{-1}-2)-q(x+x^{-1}+2-2x^{-\frac{1}{2}}-2x^{\frac{1}{2}}).$$

Let $u\in(0,1]$. Taking $r=\frac{1-u}{2}$, we obtain

$$x+x^{-1}\leqslant x^{u}+x^{-u}+(1+u)(x+x^{-1}-2)-q\left(x+x^{-1}+2-2x^{\frac{1}{2}}-2x^{-\frac{1}{2}}\right).$$

Consider real numbers v,w such that $\frac{v}{w}\in(0,1]$. Using the functional calculus on the spectrum of $A\boxtimes B$ and Lemma 1, and putting $u=\frac{v}{w}$, we get

$$\begin{aligned}
&A^{w}\boxtimes B^{-w}+A^{-w}\boxtimes B^{w}\\
&\quad\leqslant A^{v}\boxtimes B^{-v}+A^{-v}\boxtimes B^{v}+\left(1+\frac{v}{w}\right)(A^{w}\boxtimes B^{-w}+A^{-w}\boxtimes B^{w}-2I\boxtimes I)\\
&\qquad-\eta\left(A^{w}\boxtimes B^{-w}+A^{-w}\boxtimes B^{w}+2I\boxtimes I-2A^{\frac{w}{2}}\boxtimes B^{-\frac{w}{2}}-2A^{-\frac{w}{2}}\boxtimes B^{\frac{w}{2}}\right).
\end{aligned}$$

Multiplying both sides by $A^{\frac{1}{2}}\boxtimes B^{\frac{1}{2}}$ and applying Lemma 1, we have

$$\begin{aligned}
&A^{1+w}\boxtimes B^{1-w}+A^{1-w}\boxtimes B^{1+w}\\
&\quad\leqslant A^{1+v}\boxtimes B^{1-v}+A^{1-v}\boxtimes B^{1+v}+\left(1+\frac{v}{w}\right)(A^{1+w}\boxtimes B^{1-w}+A^{1-w}\boxtimes B^{1+w}-2A\boxtimes B)\\
&\qquad-\eta\left(A^{1+w}\boxtimes B^{1-w}+A^{1-w}\boxtimes B^{1+w}+2A\boxtimes B-2A^{1+\frac{w}{2}}\boxtimes B^{1-\frac{w}{2}}-2A^{1-\frac{w}{2}}\boxtimes B^{1+\frac{w}{2}}\right),
\end{aligned}$$

where $\gamma=\min\left\{\frac{v}{w},1-\frac{v}{w}\right\}$. We reach the result by replace v,w,A,B with $2\alpha-1,2\beta-1,A^{\frac{1}{2}},B^{\frac{1}{2}}$, respectively. □

Lemma 9. *Decompose* $\mathbb{H}$ *as in* (3)*. Let* $\mathcal{X}=(X_t)_{t\in\Omega}$ *and* $\mathcal{W}=(W_t)_{t\in\Omega}$ *be bounded continuous fields of operators in* $\mathfrak{B}(\mathbb{H})^{+}$ *and* $\mathfrak{B}(\mathbb{H})$*, respectively. If either* $0\leqslant\beta\leqslant\alpha<\frac{1}{2}$ *or* $\frac{1}{2}<\alpha\leqslant\beta\leqslant1$*, then*

$$\begin{aligned}
&\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}_t^{1-\beta})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}_t^{1-\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\\
&\leqslant\mathcal{F}_{\mathcal{W}}(X_t^{\alpha})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\alpha})\\
&\quad+(2-\delta\,[illegible]\,\mathcal{X}^{1-\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})-2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\big]\\
&\quad-\eta\big[\mathcal{F}_{\mathcal{W}}\,[illegible]\,\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})+2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\\
&\quad-2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma})-2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma})\big], \qquad (21)
\end{aligned}$$

must be "X-script^{alpha}", (no "t")

where $\gamma=\frac{1+2\beta}{4}$, $\delta=\frac{\beta-\alpha}{\beta-1/2}$ *and* $\eta=\min\{\delta,1-\delta\}$.

Proof. The proof is similar to that of Lemma 6. Instead of using Lemma 5, we apply Lemma 8. □

The next theorem is a reverse of the second inequality of (15) involving Tracy-Singh products. As a consequence, we get a reversed Callebaut-type inequality for Khatri-Rao products by using the unital positive linear map Φ in Lemma 2.

Theorem 2. *Decompose* $\mathbb{H}$ *as in* (3). *Let* $\mathcal{A}=(A_t)_{t\in\Omega}$ *and* $\mathcal{B}=(B_t)_{t\in\Omega}$ *be two bounded continuous fields of operators in* $\mathfrak{B}(\mathbb{H})^+$. *If either* $0\leqslant\beta\leqslant\alpha<\frac{1}{2}$ *or* $\frac{1}{2}<\alpha\leqslant\beta\leqslant1$, *then*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\\
&\quad\leqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\\
&\qquad+(2-\delta)\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\big]\\
&\qquad-\eta\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})+2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\\
&\qquad-2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\big],
\end{aligned}$$

where $\gamma=\frac{1+2\beta}{4}$, $\delta=\frac{\beta-\alpha}{\beta-1/2}$ *and* $\eta=\min\{\delta,1-\delta\}$.

Proof. The proof is similar to that of Theorem 1. Instead of using Lemma 6, we apply Lemma 9. □

Corollary 5. *Under the same hypothesis and notation as in Theorem* (2)*, we have*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\\
&\quad\leqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\\
&\qquad+(2-\delta)\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\beta(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\big]\\
&\qquad-\eta\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\beta(\mathcal{A},\mathcal{B})+2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\\
&\qquad-2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\big].
\end{aligned}$$

For the case $n=1$ (i.e. $\mathbb{H}$ is not decomposed), Theorem 2 reduces to the reversed Callebaut-type inequality for tensor products and consequently applies to Hadamard products as follows.

Corollary 6. *Under the same hypothesis and notation as in Theorem* 2 *except that the Hilbert space* $\mathbb{H}$ *is not decomposed, we have*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\\
&\quad\leqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\\
&\qquad+(2-\delta)\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\big]\\
&\qquad-\eta\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})+2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\\
&\qquad-2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\big],\\
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\\
&\quad\leqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+(2-\delta)\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})-\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\big]\\
&\qquad-\eta\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})-2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})\big].
\end{aligned}$$

Remark 3. From previous results, we obtain discrete versions of reversed Callebaut-type inequalities for Tracy-Singh products, Khatri-Rao products, tensor products and Hadamard products, respectively, by setting $\Omega=\{1,\dots,k\}$ equiped with the counting measure. Matrix analogues of our results can be obtained particularly by setting $\mathbb{H}=\mathbb{C}^n$. Such matrix results of Kronecker products and Hadamard products are reverses of the matrix Callebaut inequalities in [7, Theorem 3.4 and Corollary 3.5].

The following corollary is a reverse of the integral Callebaut inequality for real numbers. In particular, when Ω is the finite set $\{1,\dots,k\}$ equiped with the counting measure, we get a reversed inequality of the second inequality of (2).

Corollary 7. *Let $x=(x_t)_{t\in\Omega}$ and $y=(y_t)_{t\in\Omega}$ be two fields of positive real numbers. If either $0\leqslant\beta\leqslant\alpha<\frac{1}{2}$ or $\frac{1}{2}<\alpha\leqslant\beta\leqslant 1$, then*

$$\mathcal{I}_\beta(x,y)\cdot\mathcal{I}_{1-\beta}(x,y)\leqslant \mathcal{I}_\alpha(x,y)\cdot\mathcal{I}_{1-\alpha}(x,y)+(2-\delta)\big[\mathcal{I}_\beta(x,y)\cdot\mathcal{I}_{1-\beta}(x,y)-\big(\mathcal{I}_{1/2}(x,y)\big)^2\big]$$
$$-\eta\big[\mathcal{I}_\beta(x,y)\cdot\mathcal{I}_{1-\beta}(x,y)+\big(\mathcal{I}_{1/2}(x,y)\big)^2-2\mathcal{I}_\gamma(x,y)\cdot\mathcal{I}_{1-\gamma}(x,y)\big],$$

where $\gamma=\frac{1+2\beta}{4}$, $\delta=\frac{\beta-\alpha}{\beta-1/2}$ and $\eta=\min\{\delta,1-\delta\}$.

5. Conclusion

We establish certain refinements and reverses of Callebaut-type inequalities for bounded continuous fields of operators which are parametrized by a locally compact Hausdorff space Ω equipped with a finite Radon measure. These inequalities involve Tracy-Singh products, Khatri-Rao products, tensor products, Hadamard products and weighted geometric means. When Ω is a finite space equipped with the counting measure, such integral inequalities reduce to discrete inequalities. Our results include matrix results concerning the Tracy-Singh product, the Khatri-Rao product, the Kronecker product, and the Hadamard product. In particular, we get a refinement and a reverse of the classical Callebaut inequality for real numbers.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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