รายงานวิจัยฉบับสมบูรณ์

โครงการ

การวิเคราะห์การลู่เข้าของอัลกอริทึมใหม่สำหรับการแก้ปัญหา
จุดตรึงร่วมแยกที่ไม่สอดคล้อง

โดย

ผู้ช่วยศาสตราจารย์ ดร.กนกวรรณ สิทธิเถกิงเกียรติ

เมษายน 2562

สัญญาเลขที่ MRG6080123

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ผู้วิจัย

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มหาวิทยาลัยเทคโนโลยีพระจอมเกล้าพระนครเหนือ

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและ
มหาวิทยาลัยเทคโนโลยีพระจอมเกล้าพระนครเหนือ

บทคัดย่อ

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ชื่อหัวหน้าโครงการ: ผู้ช่วยศาสตราจารย์ ดร.กนกวรรณ สิทธิเถกิงเกียรติ

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งานวิจัยนี้ศึกษาการหาคำตอบของปัญหาจุดตรึงร่วมแยกที่ไม่สอดคล้อง นั่นคือการหาจุดตรึงร่วมของคลาสของตัวดำเนินการบนปริภูมิฮิลเบิร์ตสนามจริงซึ่งภาพฉายภายใต้ตัวดำเนินการเชิงเส้นที่มีของเขตเป็นจุดตรึงร่วมของคลาสของตัวดำเนินการอีกชุดหนึ่งในปริภูมิที่ถูกส่งไป ซึ่งปัญหาดังกล่าวได้เกิดขึ้นจากการวิจัยด้านการรักษาด้วยรังสีแบบปรับความเข้มซึ่งพยายามอธิบายข้อจำกัดทางกายภาพและสอดคล้องกับข้อจำกัดปริมาณที่อยู่ในรูปแบบเดียว ในงานวิจัยชิ้นนี้ เราทำการออกแบบวิธีการหาคำตอบของปัญหาดังกล่าว พบว่าสามารถยืนยันการลู่เข้าสู่คำตอบของอัลกอริทึมที่ถูกสร้างขึ้นได้ และผู้วิจัยได้นำเสนอตัวอย่างของผลการจำลองเชิงตัวเลขสำหรับการหาคำตอบของปัญหาจุดตรึงร่วมดังกล่าว เพื่อแสดงให้เห็นถึงประสิทธิภาพของวิธีการหาคำตอบของอัลกอริทึมที่ได้พัฒนาขึ้นมา

Keywords : ทิศทางคอนจูเกตเกรเดียน, การวิเคราะห์การลู่เข้า, ปัญหาจุดตรึงร่วมแยก

Abstract

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Project Title: Convergence analysis of new algorithms for solving the inconsistent split common fixed point problem

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The split common fixed point problem (SCFP) is to find a common fixed point of a family of operators in one real Hilbert space, whose image under a bounded linear transformation is a common fixed point of another family of operators in the image space. Such problems arise in the field of intensity-modulated radiation therapy when one attempts to describe physical dose constraints and equivalent uniform dose constraints within a single model. If this problem has a solution, many iterative algorithms have been proposed for solving the problem. Then, in this research, we will devise new algorithm for solving the split common fixed point problem without assuming the existence of a solution of the (SCFP). In the framework of infinite-dimensional Hilbert spaces, we will prove the strong convergence theorem for the proposed algorithm and deduce many known convergence theorems. Moreover, we present some numerical examples to guarantee our result.

Keywords : Conjugate gradient direction; Convergence analysis; Split common fixed point problems

Executive Summary

To begin with, let us describe the split common fixed point problem (SCFP) (also called the multiple-sets split feasibility problem (MSFP)) is to find a common fixed point of a finite family of operators in one real Hilbert space, whose image under a bounded linear transformation is a common fixed point of another family of operators in the image space. That is to find a point x^* with the property:

$$x^* \in \bigcap_{i\in I} F(S_i) \text{ with } Ax^* \in \bigcap_{j\in I} F(T_j)$$

where $\{S_i\}$ and $\{T_j\}$ are two families of quasi-nonexpansive operators on Hilbert spaces H_1 and H_2, respectively, and A is a bounded linear operator. Note that F(T) is the fixed point set of operator T. The (SCFP) was introduced by Censor and Segal [1] in Euclidean spaces. Such problems arise in the field of intensity-modulated radiation therapy (IMRT) when one attempts to describe physical dose constraints and equivalent uniform dose (EUD) constraints within a single model, see [2]. The (SCFP) is a generalization of the convex feasibility problem (CFP)and of the split feasibility problem (SFP).

By assuming that the (SCFP) has a solution, the literature one can find many methods for solving this problem as well as for its special case. However, it would be difficult to verify whether the (SCFP) has a solution or not before executing the conventional algorithms. This implies the applications of the conventional algorithms are severely limited. Furthermore, in the setting of infinite dimensional space, many iterative algorithms to solving these problems guarantee only weak convergence.

Therefore, the purpose of this project is to modify the algorithms to obtain a strong convergence without assuming the existence of a solution to the (SCFP) in the setting of infinite-dimensional Hilbert spaces. In the framework of infinite-dimensional Hilbert spaces, we will prove the strong convergence theorem for the proposed algorithm and deduce many known convergence theorems. Moreover, we present some numerical examples to guarantee our result.

Contents

Chapter 1 Introduction

A convex minimization problem, a wide class of significant current mathematical applications, is to find a minizer x^* such that :

$$f(x^*) = \min f(x) := \min \sum_{i \in I} f_i(x) \quad \text{subject to} \quad x \in C := \bigcap_{i \in I} C_i, \tag{1.0.1}$$

where $f_i(i \in I := \{1, 2, \ldots, k\})$ is a convex functional of a real Hilbert space H and $C_i(i \in I)$ is nonempty closed and convex subset of H. The constrained term of this problem is a convex feasibility problem (CFP) formulated as the problem of finding a point $x^* \in \bigcap_{i=1}^k C_i$. Moreover, the well known special case of CFP is the problem to find a point $x^* \in C$ with $Ax^* \in Q$ if such x exist was call a split feasibility problem (SFP) introduced in Censor and Elfving [3]. However, we say the SFP that *the inverse problem* because it is transformable to find

$$x^* \in C \quad \text{and} \quad x^* \in A^{-1}Q. \tag{1.0.2}$$

where C and Q are two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. Then, we use Γ to denoted the solution set of the (SFP) , i.e.,

$$\Gamma = \{x^* \in C : Ax^* \in Q\} = C \cap A^{-1}Q,$$

and assume the consistency of (SEP) so that Γ is closed, convex and nonempty. It is not hard to see that if $C \cap A^{-1}(Q) \neq \emptyset$, then the fixed points of $P_C(I - \gamma A^*(I - P_Q)A)$ are exactly solutions of the (SFP) where $\gamma > 0$ is any positive constant, I is the identity operator, A^* denotes the adjoint of A, and P_C and P_Q denote the orthogonal projections onto C and Q, respectively.

The split feasibility problem (SFP) in the setting of finite-dimentional Hilbert spaces was introduce for modelling invers problem which arise from phase retrievals and in medical image reconstruction [4]. Since then, a lot of work has been done on finding a solution of (SFP) and (MSSFP). It has been found that the (SFP) can also be used to study the intensity-modulated radiation therapy. There are many algorithms invented to solve the (SFP), see e.g., [5, 6, 7] and references therein.

Various algorithms have been invented to solve the (SFP), see e.g., [3, 5, 8] and references therein. The well-known algorithm that solves the (SFP) due to Byrnes CQ-algorithm [4] that does not involve matrix inverses as follow;

$$x_{n+1} = P_C(x_n + \gamma A^T(P_Q - I)Ax_n) \tag{1.0.3}$$

where $\gamma \in (0, 2/L)$, with L being the largest eigenvalue of the matrix A^TA. When the (SFP) has no solutions, the CQ-algorithm converges to a minimizer of $\|P_Q(Ax) - Ax\|$ over all $x \in C$, whenever such a minimizer exists. A block-iterative CQ-algorithm, called the BICQ-method, is also available in [4], see also Byrne [9].

Recently, He and Zhao [6] introduced a new relaxed CQ algorithm which the strong convergence is guaranteed in infinite-dimentional Hilbert spaces:

$$x_{n+1} = P_{C_n}(\alpha_n u + (1 - \alpha_n)(x_n - \tau_n \nabla f_n(x_n))). \tag{1.0.4}$$

On the other hand, Kraikaew and Saejung [10] also guarantee the strong convergence of the following split common fixed point problem which is the generalization of (SEP):

$$\text{Find } \ x^* \in F(S) \ \text{ with } \ Ax^* \in F(T), \tag{1.0.5}$$

where S and T are two of quasi-nonexpansive operators with nonempty fixed point sets on Hilbert spaces H_1 and H_2, respectively.

Let us describe the split common fixed point problem (SCFP) (also called the multiple-sets split feasibility problem (MSFP)) is to find a common fixed point of a finite family of operators in one real Hilbert space, whose image under a bounded linear transformation is a common fixed point of another family of operators in the image space. That is to find a point x^* with the property:

$$x^* \in \bigcap_{i \in I} F(S_i) \ \text{ with } \ Ax^* \in \bigcap_{j \in J} F(T_j), \tag{1.0.6}$$

where $\{S_i\}$ and $\{T_j\}$ are two families of quasi-nonexpansive operators on Hilbert spaces H_1 and H_2, respectively, and A is a bounded linear operator. Note that $F(T)$ is the fixed point set of operator T. The (SCFP) was introduced by Censor and Segal [1] in Euclidean spaces. Such problems arise in the field of intensity-modulated radiation therapy (IMRT) when one attempts to describe physical dose constraints and equivalent uniform dose (EUD) constraints within a single model, see [2]. The (SCFP) is a generalization of the convex feasibility problem (CFP)and of the split feasibility problem (SFP).

Consider the multiple-sets split feasibility problem (MSFP) which is formulated as finding points x^* with the property:

$$x^* \in C := \bigcap_{i \in I} C_i \quad \text{such that} \quad Ax^* \in Q := \bigcap_{j \in J} Q_j, \tag{1.0.7}$$

where C_i and Q_j are two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is bounded linear operators. This can serve as a model for many inverse problems where constraints are imposed on the solutions in the domain of a linear operator as well as in the operators range. For solving the (MSSFP), Censor et al [11] propose a projection algorithm that minimizes a proximity function that measures the distance of a point from all sets. Moreover, many various methods were proposed for solving it see in [12, 13, 14, 15, 16].

In this research, we consider the split common fixed point problem (1.0.6) which is the generalization of the multiple-sets split feasibility problem (1.0.7) in the case $C_i = F(S_i)$ and $Q_j = F(T_j)$. Very recently, Tang et al [17] proposed new algorithm that converges strongly to a solution of the (SCFP) for two families of firmly nonexpansive mappings when the (SCFP) has at least one solution. On the other hand, Cegielski [18] proved the weak convergence of sequences generated by the method based on a block-iterative procedure with quasi-nonexpansive operators satisfying the demi-closedness principle and having a common fixed point.

The main objective of this research project is to devise algorithms for proving the strong convergence theorem of the split common fixed point problem for two families of quasi-nonexpansive mappings without having to assume the existence of a solution of the (SCFP) in the setting of infinite-dimensional Hilbert spaces. Moreover, we present some numerical examples to guarantee our result.

Purpose of the research The purpose of this research is given below.

1. To devise new algorithms based on conjugate gradient direction for solving the split common fixed point problem for two families of quasi-nonexpansive mappings without assuming the existence of a solution of the (SCFP).

2. To provide sufficient conditions for proving the strong convergence theorem of the split common fixed point problem.

3. To present numerical results for guarantee our result and for compare previous known algorithms that solving the split common fixed point problem.

Scope of the research We restrict ourselves in the setting of infinite-dimensional Hilbert spaces.

Methodology

1. Construct new algorithms based on conjugate gradient direction for solving the split common fixed point problem for two families of quasi-nonexpansive mappings without assuming the existence of a solution of the (SCFP).

2. Develop or investigate sufficient conditions for proving the strong convergence theorem of the split common fixed point problem.

3. Present numerical examples for guarantee our result.

4. Compare previous known algorithms that solving the split common fixed point problem to our algorithm obtained in 7.1.

5. Write research papers for publication in the ISI international journals.

6. Conclude the research project and submit the full report to the granter.

Chapter 2 Mathematical Backgrounds

2.1 Notations, Definitions and Useful Lemmas

Throughout this paper unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. In this paper, we use $Fix(S)$ to denote the fixed points set of the mapping S. For each x and y in H_1, we recall that a mapping $B : H_1 \to H_1$ is said to be

(i) *nonexpansive* if

$$\|Bx - By\| \leq \|x - y\|; \tag{2.1.1}$$

(ii) *monotone*, if

$$\langle Bx - By, x - y \rangle \geq 0;$$

(iii) *α-strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \alpha \|x - y\|^2;$$

(iv) *β-inverse strongly monotone* (for short; *β-ism*), if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \beta \|Bx - By\|^2;$$

(v) *firmly nonexpansive*, if

$$\langle Bx - By, x - y \rangle \geq \|Bx - By\|^2.$$

Also, we recall that a mapping D on H_1 is said to be *strongly positive*, if there is a constant $\bar{\xi} > 0$ such that

$$\langle Dx, x \rangle \geq \bar{\xi} \|x\|^2, \forall x \in H_1. \tag{2.1.2}$$

We recall some concepts and results which are needed in sequel. Let H_1 be a Hilbert space and let symbols " $\to$ " and " $\rightharpoonup$ " denote by strong and weak convergence, respectively. In Hilbert spaces, it is well known that,

$$\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda (1 - \lambda) \|x - y\|^2,$$

for all $x, y \in H_1$ and $\lambda \in [0, 1]$.

For any $x \in H_1$, there exists a unique nearest point in a nonempty closed convex subset C denoted by $P_C(x)$ such that $\|x - P_C(x)\| \leq \|x - y\|$, for all $y \in C$. The mapping P_C is called the *metric projection* of H_1 onto C. We know that P_C is a nonexpansive mapping from H_1 onto C. The metric projection can be characterised by $P_C(x) \in C$ and satisfied

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H_1. \tag{2.1.3}$$

Moreover,for all $x \in H_1$ and $y \in C$, $P_C x$ is characterized by

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0. \tag{2.1.4}$$

It is easy to see that (2.1.4) is equivalent to the following inequality:

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \tag{2.1.5}$$

for all $x \in H_1$ and $y \in C$. For more details, see [42].

Let $S : H_1 \to H_1$ be a mapping, Then, we have the following equality:

$$\|(x-y)-(S(x)-S(y))\|^2 = \|x-y\|^2+\|S(x)-S(y)\|^2-2\langle x-y, S(x)-S(y)\rangle, \forall (x, y) \in H_1\times H_1 \tag{2.1.6}$$

Further, if S is a nonexpansive operator, we have also that

$$\begin{aligned} \|(x - y) - (S(x) - S(y))\|^2 &\geq 2\|S(x) - S(y)\|^2 - 2\langle x - y, S(x) - S(y)\rangle \\ &= 2\langle S(x) - S(y), S(x) - S(y)\rangle - 2\langle x - y, S(x) - S(y)\rangle \\ &= 2\langle (S(x) - S(y)) - (x - y), S(x) - S(y)\rangle. \end{aligned} \tag{2.1.7}$$

Then we obtain that every nonexpansive operator $S : H_1 \to H_1$ satisfies the following inequality

$$\langle (x-S(x))-(y-S(y)), S(y)-S(x)\rangle \leq \frac{1}{2}\|(S(x)-x)-(S(y)-y)\|^2, \forall (x, y) \in H_1\times H_1 \tag{2.1.8}$$

and therefore, we get

$$\langle x - S(x), y - S(x)\rangle \leq \frac{1}{2}\|S(x) - x\|^2, \forall (x, y) \in H_1 \times Fix(S). \tag{2.1.9}$$

A mapping $T : H_1 \to H_1$ is said to be be *averaged* mapping if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T \equiv (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0,1)$ and $S : H_1 \to H_1$ is nonexpansive. We note that averaged mappings are nonexpansive. Further, firmly nonexpansive mappings (in particular, projections on nonempty closed and convex subsets and resolvent operators of maximal monotone operators) are averaged. Some important properties of averaged mappings are gathered in the following proposition (see [41, 43, 44]).

(i) If $T = (1-\alpha)S + \alpha V$, where $S : H_1 \to H_1$ is averaged, $V : H_1 \to H_1$ is nonexpansive and $\alpha \in (0,1)$, then T is averaged.

(ii) The composite of finitely many averaged mappings is averaged.

(iii) If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then $\bigcap_{i=1}^N Fix(T_i) = Fix(T_1, T_2, \ldots, T_N)$.

(iv) If T is τ-ism, then for $\gamma > 0, \gamma T$ is $\frac{\tau}{\gamma}$-ism.

(v) T is averaged if and only if, its complement $I - T$ is τ-ism for some $\tau > \frac{1}{2}$.

Lemma 2.1.1. *[41] SVIP (2.2.8)-(2.2.9) is equivalent to find $x^* \in H_1$ with $x^* = J_\lambda^{B_1}(x^*)$ such that*

$$y^* = Ax^* \in H_2 \quad \text{and} \quad y^* = J_\lambda^{B_2}(y^*),$$

for some $\lambda > 0$.

Definition 2.1.2. [45] Let H_1 be a real Hilbert space and $\{S_i\}$ be an infinite family of nonexpansive mappings and $\{\zeta_i\}$ be a nonnegative real sequence with $0 \le \zeta_i \le 1, \forall i \ge 1$. For $n \ge 1$, define a mapping W_n as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \zeta_n S_n U_{n,n+1} + (1-\zeta_n)I, \\ U_{n,n-1} = \zeta_{n-1} S_{n-1} U_{n,n} + (1-\zeta_{n-1})I, \\ \vdots \\ U_{n,k} = \zeta_k S_k U_{n,k+1} + (1-\zeta_k)I, \\ U_{n,k-1} = \zeta_{k-1} S_{k-1} U_{n,k} + (1-\zeta_{k-1})I, \\ \vdots \\ U_{n,2} = \zeta_2 S_2 U_{n,3} + (1-\zeta_2)I, \\ W_n = U_{n,1} = \zeta_1 S_1 U_{n,2} + (1-\zeta_1)I. \end{cases} \tag{2.1.10}$$

Such a mapping W_n is nonexpansive and it is called a W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\zeta_n, \zeta_{n-1}, \ldots, \zeta_1$.

Lemma 2.1.3. *[45] Let $\{S_i\}$ be an infinite family of nonexpansive mappings on a Hilbert space H_1 and with $\bigcap_{i=1}^{\infty} Fix(S_i) \neq \emptyset$ and $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \leq 1, \ \forall i \geq 1$. Then*

(1) W_n is nonexpansive and $Fix(W_n) = \bigcap_{i=1}^{n} Fix(S_i)$, for each $n \geq 1$;

(2) for each $x \in C$ and for each positive integer k, the $\lim_{n\to\infty} U_{n,k}x$ exists;

(3) the mapping W defined by

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in H_1, \tag{2.1.11}$$

is a nonexpansive mapping satisfying $Fix(W) = \bigcap_{i=1}^{\infty} Fix(S_i)$ and it is called the W-mapping generated by $S_1, S_2, \ldots$ and $\zeta_1, \zeta_2, \ldots$.

Lemma 2.1.4. *[47] Let $\{S_i\}$ be an infinite family of nonexpansive mappings on a Hilbert space H_1 and with $\bigcap_{i=1}^{\infty} Fix(S_i) \neq \emptyset$ and $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \leq l \leq 1, \ \forall i \geq 1$ where l is a positive real number. If C is any bounded subset of H_1, then*

$$\lim_{n\to\infty} \sup_{x\in C} \|Wx - W_n x\| = 0.$$

Throughout this paper, we always assume that $0 < \zeta_i \leq l < 1, \forall i \geq 1$.

Lemma 2.1.5. *[49] Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\longrightarrow\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then, $\lim_{n\longrightarrow\infty} \|z_n - x_n\| = 0$.

Lemma 2.1.6. *In a real Hilbert space H_1, there holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle,$$

for all $x, y \in H_1$.

Lemma 2.1.7. *[50] Each Hilbert space H_1 satisfies the Opial condition that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality* $\liminf_{n\longrightarrow\infty} \|x_n - x\| < \liminf_{n\longrightarrow\infty} \|x_n - y\|$ *holds for every $y \in H$ with $y \neq x$.*

Lemma 2.1.8. *[51] Assume that S is nonexpansive self mapping of a closed convex subset C of a Hilbert space H_1. If S has a fixed point, then $I - S$ is demiclosed, i.e., whenever $\{x_n\}$ converges weakly to some y, it follows that $(I - S)x = y$. Here I is the identity mapping on H_1.*

Lemma 2.1.9. *[28, 29] Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} \alpha_n = 0$.

Lemma 2.1.10. *[48] Assume D is a strongly positive bounded linear operator on Hilbert space H_1 with coefficient $\bar{\xi} > 0$ and $0 < \rho \leq \|D\|^{-1}$. Then, $\|I - \rho D\| \leq 1 - \rho\bar{\xi}$.*

2.2 Some Related Theorems

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [19, 20, 21, 22, 25, 26, 27, 28, 29] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H_1;

$$\min_{x \in Fix(S)} \frac{1}{2}\langle Dx, x\rangle - h(x), \tag{2.2.1}$$

where D is a bounded linear operator and h is a potential function for ξf (i.e., $h'(x) = \xi f(x)$ for $x \in H_1$).

In 2006, Marino and Xu [22] studied the following iterative scheme; put $x_1 \in H_1$ and

$$x_{n+1} = (I - \alpha_n D)Sx_n + \alpha_n \xi f(x_n), \quad n \geq 1, \tag{2.2.2}$$

where S is a nonexpansive mapping, f is a contraction mapping with the coefficient $\alpha \in (0,1)$, D is a strongly positive bounded linear self-adjoint operator with the coefficient $\bar{\xi}$ and ξ is a constant such that $0 < \xi < \frac{\bar{\xi}}{\alpha}$. They proved the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution x^* of the following variational inequality;

$$\langle (D - \xi f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in Fix(S),$$

which is the optimality condition for the minimization problem (2.2.1).

Let $B : C \to H_1$ be a nonlinear mapping. The *classical variational inequality problem* (VIP) is to find a $x^* \in C$ such that

$$\langle Bx^*, y - x^* \rangle \geq 0, \forall y \in C. \tag{2.2.3}$$

The solution set of VIP (2.2.3) is denoted by $VI(C, B)$.

In 2005, Iiduka and Takahashi [23] introduced and proved the theorem for finding a common element of the set of fixed points of a nonexpansive mapping and the solution sets of the variational inequality as follows:

Theorem IT. Let C be a closed convex subset of a real Hilbert space H_1. Let B be an α-inverse-strongly monotone mapping of C into H_1 and let S be a nonexpansive mapping of C into itself such that $Fix(S) \cap VI(C, B) \neq \emptyset$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S P_C(x_n - \lambda_n B x_n),$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[a, b]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so by $\{\lambda_n\} \in [a, b]$ for some real numbers a and b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{Fix(S)\cap VI(C,B)}x$.

Further results on this paper are due to Y. Yao and J. C. Yao [24]. More precisely, they proved the following result.

Theorem YY. Let C be a closed convex subset of a real Hilbert space H_1. Let B be an α-inverse-strongly monotone mapping of C into H_1 and let S be a nonexpansive mapping of C into itself such that $Fix(S) \cap \Omega \neq \emptyset$, where Ω denotes the set of solutions of a variational inequality for the α-inverse-strongly monotone mapping. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ are given by

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \lambda_n B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(I - \lambda_n B) y_n, \forall n \geq 1, \end{cases} \tag{2.2.4}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\alpha_n\}$ are three sequences in $[0, 1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2a]$. If $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are chosen by $\lambda_n \in [a, b]$ for some real numbers a and b with $0 < a < b < 2\alpha$ and the following hold;

(i) $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$;

(ii) $\lim_{n \longrightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$;

(iv) $\lim_{n \to \infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then the sequence $\{x_n\}$ defined by the iterative algorithm (2.2.4) converges strongly to $P_{Fix(S) \cap \Omega} u$.

Recall that a set-valued mapping $M : H_1 \to 2^{H_1}$ is called *monotone* if for all $x, y \in H_1, f \in Mx$ and $g \in My$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $M : H_1 \to 2^{H_1}$ is *maximal* if the graph of Graph(M) of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H_1 \times H_1, \langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in Graph(M)$ implies $f \in Mx$.

Let $M : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. For any positive number λ and identity operator I on H_1, the single-valued mapping $J_\lambda^M : H_1 \to H_1$ defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x), \forall x \in H_1,$$

is called the resolvent operator associated with M . We recall that the resolvent operator J_λ^M is firmly nonexpansive and hence in particular nonexpansive [30].

Let $C_1, C_2, \ldots, C_m$ be nonempty closed convex subsets of a Hilbert space H_1. The well known *convex feasibility problem* (CFP) is to find $x^* \in H_1$ such that

$$x^* \in C_1 \cap C_2 \cap \ldots C_m.$$

Convex feasibility problem has received a lot of attention due to its diverse applications in mathematics, approximation theory, communications, geophysics, control theory, biomedical engineering. One can refer to [31, 32]. When there are only two sets and constrains are imposed on the solutions in the domain of a linear operator as well as in this operator's ranges, the problem is said to be the *split feasibility problem* (SFP) which has the following formula:

$$x^* \in C \quad \text{such that} \quad Ax^* \in Q, \tag{2.2.5}$$

where C and Q is a nonempty closed convex subset of Hilbert space H_1 and H_2, respectively and $A : H_1 \to H_2$ is a bounded linear operator. The SFP in finite dimensional Hilbert spaces was first introduced by Censor and Elfving [33] in medical image reconstruction. Since then, the SFP has received much attention due to its applications in signal processing, image reconstruction, with particular progress in intensity-modulated radiation therapy, approximation theory, control theory, biomedical engineering, communications, and geophysics. For examples, one can refer to [33, 34, 35, 36] and related literatures.

Recently, Moudafi [37] introduced the following *split monotone variational inclusion problem* (SMVIP): find $x^* \in H_1$ such that

$$0 \in f_1(x^*) + B_1(x^*), \tag{2.2.6}$$

and

$$y^* = Ax^* \in H_2 \quad \text{solves} \quad 0 \in f_2(y^*) + B_2(y^*), \tag{2.2.7}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings.

Moudafi [37] introduced an iterative method for solving SMVIP (2.2.6)-(2.2.7), which can be seen as an important generalization of an iterative method given by Censor et al. [38] for split variational inequality problem. As Moudafi notes in [37], SMVIP (2.2.6)-(2.2.7) includes as special cases, the split common fixed point problem, split variational inequality problem, split zero problem and split feasibility problem [33, 35, 37, 38] which have already been studied and used in practice as a model in intensity-modulated radiation therapy treatment, see [33, 35]. This formalism is also at the core of modeling of many inverse problems arising from phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see e.g. [32, 39].

If $f_1 \equiv 0$ and $f_2 \equiv 0$ then SMVIP (2.2.6)-(2.2.7) reduces to the following the *split variational inclusion problem* (SVIP): Find $x^* \in H_1$ such that

$$0 \in B_1(x^*), \tag{2.2.8}$$

and

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in B_2(y^*). \tag{2.2.9}$$

When considered separately, (2.2.8) is the variational inclusion problem and we denoted its solution set by SOLVIP (B_1). The SVIP (2.2.8)-(2.2.9) constitutes a pair of variational inclusion problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator A of the solution x^* of SVIP (2.2.8) in H_1 is the solution of the other SVIP (2.2.9) in another space H_2, we denote the solution set of SVIP (2.2.9) by SOLVIP(B_2).

The solution set of SVIP (2.2.8)-(2.2.9) is denoted by $\Gamma = \{x^* \in H_1 : x^* \in SOLVIP(B_1)$ and $Ax^* \in SOLVIP(B_2)\}$.

Very recently, Byrne et al. [40] studied the weak and strong convergence of the following iterative method for SVIP (2.2.8)-(2.2.9): for given $x_1 \in H_1$, compute the iterative sequence $\{x_n\}$ generated by the following scheme:

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \forall n \geq 1, \ \exists \lambda > 0.$$

In 2014, Kazmi and Rizvi [41] introduced and studied an iterative method to approximate a common solution of SVIP and fixed point problem for a nonexpansive mapping in real Hilbert spaces.

Theorem KR. Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator and $f : H_1 \to H_1$ be a contraction mapping with constant $\alpha \in (0,1)$. Let $S : H_1 \to H_1$ be a nonexpansive mapping such that $Fix(S) \cap \Gamma \neq \emptyset$. For a given $x_0 \in H_1$ arbitrarily, let the iterative sequence $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n); \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \end{cases}$$

where $\lambda > 0$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A and α_n is a sequence in $(0,1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty.$$

Then the sequences $\{u_n\}$ and $\{x_n\}$ both converge strongly to $z \in Fix(S) \cap \Gamma$, where $z = P_{Fix(S)\cap\Gamma} f(z)$.

Chapter 3 Main Results

Throughout the rest of this paper, we always assume that D is a strongly positive bounded linear operator with coefficient $\bar{\xi} \in (0,1)$ and $0 < \xi < \bar{\xi}$. We consider the well known general iterative method as follow (see Xu 2003 [53]) with the initial guess $x_0 = u \in C$:

$$x_{n+1} = (I - \alpha_n D)Sx_n + \alpha_n u, \quad n \geq 0, \tag{3.0.1}$$

where D is a strongly positive bounded linear operator, S is nonexpansive self mapping on $C \subset H$ and α_n is so chosen that $T = I - \alpha_n D$ satisfies some conditions (see [52]). This iterative method converges to the unique solution x^* of the variational inequality

$$\langle Dx^* - \xi u, x^* - x \rangle \leq 0, \quad \forall x \in C,$$

which is the optimality for minimize a quadratic function of (2.2.1), that is

$$\min_{x \in Fix(S)} \frac{1}{2}\langle Dx, x \rangle - h(x),$$

where h is a potential function for ξf (i.e., $h^{'}(x) = \xi f(x)$) and the contraction mapping f is replaced by $f(x) = u$ for all $x \in H_1$ which is mentioned in the viscosity approximation method (2.2.2).

Also, we develop a new algorithm by mixing the algorithm (2.2.4) and (3.0.1) for finding a common solution of split variational inclusion of two Hilbert spaces H_1 and H_2 which is tool for solve the unique solution to the variational inequalities (3.1.2) below.

In this section, we first prove a strong conergence result on the general iterative algorithm for the fixed point problem and the split variational inclusion problem (SVIP) (2.2.8)-(2.2.9).

3.1 Strong Convergence Theorem

Theorem 3.1.1. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Assume that $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are* *maximal monotone mappings. Let* $\{S_i\}_{i=1}^{\infty}$ *be an infinite family of nonexpansive mappings from* H_1 *into itself. Let* D *be a strongly positive bounded linear operator with coefficient* $0 < \bar{\xi} < 1$ *and* $0 < \xi < \bar{\xi}$. *Assume that* $\Omega := (\bigcap_{i=1}^{\infty} Fix(S_i)) \cap \Gamma \neq \emptyset$. *Let* $x_1 \in H_1$ *arbitrarily, let the sequences* $\{y_n\}$ *and* $\{x_n\}$ *be generated by*

$$\begin{cases} y_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n); \\ x_{n+1} = \alpha_n \xi u + \beta_n x_n + [(1-\beta_n)I - \alpha_n D]W_n y_n, \forall n \geq 1, \end{cases} \tag{3.1.1}$$

where $u \in H_1$ *is a fixed element,* $\lambda > 0$ *and* $\gamma \in (0, \frac{1}{L}), L$ *is the spectral radius of the operator* A^*A *and* A^* *is the adjoint of* A, $\{W_n\}$ *is the sequence defined by (2.1.10),* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$. *Suppose the control consequences satisfy the following conditions:*

(C1) $0 < a \leq \beta_n \leq b < 1, \forall n \geq 1$, *for some two positive real numbers* a *and* b,

(C2) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to a point* $z \in \Omega$, *which is the unique solution to the variational inequalities*

$$\langle Dz - \xi u, z - p \rangle \leq 0, \quad \forall p \in \Omega. \tag{3.1.2}$$

Equivalently, we have $P_\Omega(z - Dz + \xi u) = z$.

Proof. **Step 1.** First, we will prove that the sequence $\{x_n\}$ is bounded.

By the conditions (C1) and (C2), without loss of generality we may assume $\alpha_n \leq (1-\beta_n)\|D\|^{-1}$ for all $n \geq 1$. By Lemma 2.1.10, we get that

$$\|I - \frac{\alpha_n}{1-\beta_n}D\| \leq 1 - \frac{\alpha_n}{1-\beta_n}\bar{\xi}.$$

It follows that

$$\|(1-\beta_n)I - \alpha_n D\| \leq 1 - \beta_n - \alpha_n \bar{\xi}. \tag{3.1.3}$$

Let $p \in \Omega$, we have $p = J_\lambda^{B_1} p, Ap = J_\lambda^{B_2}(Ap)$ and $S_n p = p$ for all $n \geq 1$. We estimate that

$$\begin{aligned} \|y_n - p\|^2 &= \|J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) - p\|^2 \\ &= \|J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) - J_\lambda^{B_1} p\|^2 \\ &\leq \|x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n - p\|^2 \\ &\leq \|x_n - p\|^2 + \gamma^2 \|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 \\ &\quad +2\gamma\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle. \end{aligned} \tag{3.1.4}$$

Thus, we have

$$\begin{aligned}\|y_n - p\|^2 &\leq \|x_n - p\|^2 + \gamma^2 \langle (J_\lambda^{B_2} - I)Ax_n, AA^*(J_\lambda^{B_2} - I)Ax_n\rangle \\ &\quad +2\gamma\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle. \end{aligned} \tag{3.1.5}$$

Note that

$$\begin{aligned}\gamma^2\langle (J_\lambda^{B_2} - I)Ax_n, AA^*(J_\lambda^{B_2} - I)Ax_n\rangle &\leq L\gamma^2\langle (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\ &= L\gamma^2\|(J_\lambda^{B_2} - I)Ax_n\|^2. \end{aligned} \tag{3.1.6}$$

Consider the term of $2\gamma\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle$ and using (2.1.9), we have

$$\begin{aligned}2\gamma\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle &= 2\gamma\langle A(x_n - p), (J_\lambda^{B_2} - I)Ax_n\rangle \\ &= 2\gamma\langle A(x_n - p) + (J_\lambda^{B_2} - I)Ax_n - (J_\lambda^{B_2} - I)Ax_n, \\ &\quad (J_\lambda^{B_2} - I)Ax_n\rangle \\ &= 2\gamma\Big\{\langle Ax_n - Ap + J_\lambda^{B_2}Ax_n - Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\ &\quad -\langle (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle\Big\} \\ &= 2\gamma\Big\{\langle Ap - J_\lambda^{B_2}Ax_n, Ax_n - J_\lambda^{B_2}Ax_n\rangle - \|(J_\lambda^{B_2} - I)Ax_n\|^2\Big\} \\ &\leq 2\gamma\Big\{\frac{1}{2}\|(J_\lambda^{B_2} - I)Ax_n\|^2 - \|(J_\lambda^{B_2} - I)Ax_n\|^2\Big\} \\ &= -\gamma\|(J_\lambda^{B_2} - I)Ax_n\|^2. \end{aligned} \tag{3.1.7}$$

Using (3.1.5), (3.1.6) and (3.1.7), we obtain

$$\begin{aligned}\|y_n - p\|^2 &\leq \|x_n - p\|^2 + L\gamma^2\|(J_\lambda^{B_2} - I)Ax_n\|^2 - \gamma\|(J_\lambda^{B_2} - I)Ax_n\|^2 \\ &= \|x_n - p\|^2 + \gamma(L\gamma - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2. \end{aligned} \tag{3.1.8}$$

Since $\gamma \in (0, \frac{1}{L})$, we obtain

$$\|y_n - p\|^2 \leq \|x_n - p\|^2. \tag{3.1.9}$$

We arrive that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n\xi u+\beta_n x_n+[(1-\beta_n)I-\alpha_n D]W_n y_n-p\| \\ &= \|\alpha_n\xi u-\alpha_n Dp-\beta_n p+\beta_n x_n+[(1-\beta_n)I-\alpha_n D]W_n y_n \\ &\quad +\alpha_n Dp+\beta_n p-p\| \\ &= \|\alpha_n(\xi u-Dp)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_n D)W_n y_n \\ &\quad -((1-\beta_n)I-\alpha_n D)p\| \\ &\leq \alpha_n\|\xi u-Dp\|+\beta_n\|x_n-p\|+\|(1-\beta_n)I \\ &\quad -\alpha_n D\|\|W_n y_n-p\| \\ &\leq \alpha_n\|\xi u-Dp\|+\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\bar{\xi})\|y_n-p\| \\ &\leq \alpha_n\|\xi u-Dp\|+\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\bar{\xi})\|x_n-p\| \\ &\leq \alpha_n\|\xi u-Dp\|+(1-\alpha_n\bar{\xi})\|x_n-p\|.\end{aligned}$$

By mathematical induction, we obtain

$$\|x_n-p\|\leq\max\left\{\|x_1-p\|,\frac{\|\xi u-Dp\|}{\bar{\xi}}\right\},$$

which gives that the sequence $\{x_n\}$ is bounded and also is $\{y_n\}$. Hence we can choose a bounded set $C\subset H_1$ such that

$$x_n, y_n\in C,\ \forall n\geq 1. \tag{3.1.10}$$

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Setting

$$v_n=\frac{1}{1-\beta_n}x_{n+1}-\frac{\beta_n}{1-\beta_n}x_n.$$

Then, we see that

$$x_{n+1}=(1-\beta_n)v_n+\beta_n x_n, \forall n\geq 1, \tag{3.1.11}$$

and

$$\begin{aligned}v_n &= \frac{1}{1-\beta_n}\{\alpha_n\xi u+\beta_n x_n+[(1-\beta_n)I-\alpha_n D]W_n y_n\}-\frac{\beta_n}{1-\beta_n}x_n \\ &= \frac{1}{1-\beta_n}\{\alpha_n\xi u+\beta_n x_n+W_n y_n-\beta_n W_n y_n-\alpha_n DW_n y_n\}-\frac{\beta_n}{1-\beta_n}x_n \\ &= \frac{1}{1-\beta_n}\{\alpha_n(\xi u-DW_n y_n)+(1-\beta_n)W_n y_n\} \\ &= \frac{\alpha_n}{1-\beta_n}(\xi u-DW_n y_n)+W_n y_n.\end{aligned}$$

Then,

$$\begin{aligned}\|v_{n+1}-v_n\| &= \left\|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\xi u - DW_{n+1}y_{n+1}) + W_{n+1}y_{n+1}\right.\\ &\quad \left.-[\frac{\alpha_n}{1-\beta_n}(\xi u - DW_n y_n) + W_n y_n]\right\|\\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\xi u - DW_{n+1}y_{n+1}\|\\ &\quad +\frac{\alpha_n}{1-\beta_n}\|\xi u - DW_n y_n\| + \|W_{n+1}y_{n+1} - W_n y_n\|. \quad (3.1.12)\end{aligned}$$

Since $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ both are firmly nonexpansive, they are averaged. For $\gamma \in (0, \frac{1}{L})$, the mapping $(I + \gamma A^*(J_\lambda^{B_2} - I)A)$ is averaged, see [37]. It follows from Proposition 2.1 (ii) that the mapping $J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)$ is averaged and hence nonexpansive. So, we obtain that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|J_\lambda^{B_1}(x_{n+1} + \gamma A^*(J_\lambda^{B_2} - I)Ax_{n+1}) - J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n)\|\\ &= \|J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)x_{n+1} - J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)x_n\|\\ &\leq \|x_{n+1} - x_n\|. \quad (3.1.13)\end{aligned}$$

On the other hand, one has

$$\begin{aligned}\|W_{n+1}y_{n+1} - W_n y_n\| &= \|W_{n+1}y_{n+1} - Wy_{n+1} + Wy_{n+1} - Wy_n + Wy_n - W_n y_n\|\\ &\leq \|W_{n+1}y_{n+1} - Wy_{n+1}\| + \|Wy_{n+1} - Wy_n\| + \|Wy_n - W_n y_n\|\\ &\leq \sup_{x\in C}\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\} + \|y_{n+1} - y_n\|(3.1.14)\end{aligned}$$

where C is the bounded subset of H_1 defined by (3.1.10). Substituting (3.1.13) into (3.1.14), one arrive that

$$\|W_{n+1}y_{n+1} - W_n y_n\| \leq \sup_{x\in C}\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\} + \|x_{n+1} - x_n\|. \quad (3.1.15)$$

From (3.1.12) combine with (3.1.15), one obtains

$$\begin{aligned}\|v_{n+1}-v_n\| - \|x_{n+1}-x_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\xi u + DW_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\xi u - DW_n y_n\|\\ &\quad + \sup_{x\in C}\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\}.\end{aligned}$$

It follows from the conditions (C1), (C2) and Lemma 2.1.4 that

$$\limsup_{n\to\infty}(\|v_{n+1}-v_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

Hence, from Lemma 2.1.5 and (3.1.11), we obtain that

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{3.1.16}$$

From (3.1.11), we have that

$$\|x_{n+1} - x_n\| = \|(1-\beta_n)v_n + \beta_n x_n - [(1-\beta_n)x_n + \beta_n x_n]\| = (1-\beta_n)\|v_n - x_n\|.$$

By the condition (C1) and (3.1.16), we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.1.17}$$

Step 3. We will prove that three sequences $\{\|y_n - x_n\|\}$, $\{\|W_n y_n - x_n\|\}$ and $\{\|W_n y_n - y_n\|\}$ converge to 0.

We set $f_n = \xi u - DW_n y_n$, for all $n \geq 1$. For any $p \in \Omega$ and by Lemma 2.1.6, we see that,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n \xi u + \beta_n x_n + [(1-\beta_n)I - \alpha_n D]W_n y_n - p\|^2 \\
&= \|\alpha_n \xi u + \beta_n x_n + W_n y_n - \beta_n W_n y_n - \alpha_n D W_n y_n - p\|^2 \\
&= \|\alpha_n(\xi u - DW_n y_n) + \beta_n x_n + (1-\beta_n)W_n y_n - p\|^2 \\
&= \|\alpha_n f_n + \beta_n x_n + (1-\beta_n)W_n y_n - \beta_n p - (1-\beta_n)p\|^2 \\
&= \|\alpha_n f_n + \beta_n(x_n - p) + (1-\beta_n)(W_n y_n - p)\|^2 \\
&\leq \|\beta_n(x_n - p) + (1-\beta_n)(W_n y_n - p)\|^2 + 2\langle \alpha_n f_n, x_{n+1} - p\rangle \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|W_n y_n - p\|^2 - \beta_n(1-\beta_n)\|W_n y_n - x_n\|^2 \\
&\quad +2\alpha_n\|f_n\|\|x_{n+1} - p\| \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 + 2\alpha_n M^2,
\end{aligned} \tag{3.1.18}$$

where $M = \max\{\sup_{n\geq 1}\|f_n\|, \sup_{n\geq 1}\|x_n - p\|\}$.

Observe that from (3.1.8) substituting into (3.1.18), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)[\|x_n - p\|^2 \\
&\quad +\gamma(L\gamma - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2] + 2\alpha_n M^2 \\
&= \|x_n - p\|^2 - \gamma(1-\beta_n)(1-L\gamma)\|(J_\lambda^{B_2} - I)Ax_n\|^2 + 2\alpha_n M^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma(1-\beta_n)(1-L\gamma)\|(J_\lambda^{B_2} - I)Ax_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n M^2 \\
&\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + 2\alpha_n M^2,
\end{aligned}$$

and from the condition (C1), (C2), $\gamma(1-\beta_n)(1-L\gamma)>0$ and (3.1.17), we get

$$\lim_{n\to\infty}\|(J_\lambda^{B_2}-I)Ax_n\|=0. \quad (3.1.19)$$

Since $J_\lambda^{B_1}$ is firmly nonexpansive mapping and $\gamma\in(0,\frac{1}{L})$, by the inequalities (3.1.6) and (3.1.7), then we have

$$\begin{aligned}
\|y_n-p\|^2 &= \|J_\lambda^{B_1}(x_n+\gamma A^*(J_\lambda^{B_2}-I)Ax_n)-p\|^2\\
&= \|J_\lambda^{B_1}(x_n+\gamma A^*(J_\lambda^{B_2}-I)Ax_n)-J_\lambda^{B_1}p\|^2\\
&\le \langle y_n-p, x_n+\gamma A^*(J_\lambda^{B_2}-I)Ax_n-p\rangle\\
&\le \frac{1}{2}\{\|y_n-p\|^2+\|x_n+\gamma A^*(J_\lambda^{B_2}-I)Ax_n-p\|^2\\
&\quad -\|(y_n-p)-[x_n+\gamma A^*(J_\lambda^{B_2}-I)Ax_n-p]\|^2\}\\
&= \frac{1}{2}\{\|y_n-p\|^2+\|x_n-p\|^2+2\gamma\langle x_n-p, A^*(J_\lambda^{B_2}-I)Ax_n\rangle\\
&\quad +\gamma^2\|A^*(J_\lambda^{B_2}-I)Ax_n\|^2\\
&\quad -\|y_n-x_n-\gamma A^*(J_\lambda^{B_2}-I)Ax_n\|^2\}\\
&\le \frac{1}{2}\{\|y_n-p\|^2+\|x_n-p\|^2-\gamma\|(J_\lambda^{B_2}-I)Ax_n\|^2+L\gamma^2\|(J_\lambda^{B_2}-I)Ax_n\|^2\\
&\quad -[\|y_n-x_n\|^2+\gamma^2\|A^*(J_\lambda^{B_2}-I)Ax_n\|^2-2\gamma\langle y_n-x_n, A^*(J_\lambda^{B_2}-I)Ax_n\rangle]\}\\
&= \frac{1}{2}\{\|y_n-p\|^2+\|x_n-p\|^2+\gamma(L\gamma-1)\|(J_\lambda^{B_2}-I)Ax_n\|^2\\
&\quad -\|y_n-x_n\|^2-\gamma^2\|A^*(J_\lambda^{B_2}-I)Ax_n\|^2\\
&\quad +2\gamma\langle y_n-x_n, A^*(J_\lambda^{B_2}-I)Ax_n\rangle\}\\
&\le \frac{1}{2}\{\|y_n-p\|^2+\|x_n-p\|^2-\|y_n-x_n\|^2+2\gamma\|A(y_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|\}.
\end{aligned}$$

Hence, we obtain

$$\|y_n-p\|^2\le\|x_n-p\|^2-\|y_n-x_n\|^2+2\gamma\|A(y_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|. \quad (3.1.20)$$

Substituting (3.1.20) into (3.1.18), one arrive that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)[\|x_n-p\|^2-\|y_n-x_n\|^2\\
&\quad +2\gamma\|A(y_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|]+2\alpha_nM^2\\
&= \|x_n-p\|^2-(1-\beta_n)\|y_n-x_n\|^2\\
&\quad +2\gamma(1-\beta_n)\|A(y_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|+2\alpha_nM^2.
\end{aligned}$$

So, we get

$$\begin{aligned}(1-\beta_n)\|y_n - x_n\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad +2\gamma(1-\beta_n)\|A(y_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| + 2\alpha_n M^2 \\ &\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\ &\quad +2\gamma(1-\beta_n)\|A(y_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| + 2\alpha_n M^2.\end{aligned}$$

From the condition (C1), (C2), (3.1.17) and (3.1.19), we obtain that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.1.21}$$

Observe that

$$\begin{aligned}\|W_n y_n - x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - W_n y_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n \xi u + \beta_n x_n + [(1-\beta_n)I - \alpha_n D]W_n y_n - W_n y_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n \xi u + \beta_n x_n + W_n y_n - \beta_n W_n y_n \\ &\quad -\alpha_n D W_n y_n - W_n y_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n(\xi u - D W_n y_n) + \beta_n(x_n - W_n y_n)\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|\xi u - D W_n y_n\| + \beta_n\|x_n - W_n y_n\|.\end{aligned}$$

This implies that

$$(1-\beta_n)\|W_n y_n - x_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|\xi u - D W_n y_n\|.$$

From the conditions (C1), (C2) and (3.1.17), we get

$$\lim_{n\to\infty} \|W_n y_n - x_n\| = 0. \tag{3.1.22}$$

Note that

$$\|W_n y_n - y_n\| \le \|W_n y_n - x_n\| + \|x_n - y_n\|,$$

from (3.1.21) and (3.1.22), we get

$$\lim_{n\to\infty} \|W_n y_n - y_n\| = 0. \tag{3.1.23}$$

Step 4. We will show that $\limsup_{n\to\infty}\langle \xi u - Dz, x_n - z\rangle \leq 0$, where $z = P_\Omega[(I - D)z + \xi u]$ or $z = P_\Omega(z - Dz + \xi u)$.

To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \xi u - Dz, x_n - z\rangle = \lim_{i\to\infty}\langle \xi u - Dz, x_{n_i} - z\rangle. \tag{3.1.24}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. From (3.1.21), we also see that $y_{n_i} \rightharpoonup w$.

Next, we will show that $w \in \Omega$.

Step 4.1 We will show that $w \in \cap_{i=1}^{\infty} F(S_i) = F(W)$.

Suppose to the contrary that, $w \notin F(W)$, i.e., $Ww \neq w$ and by Lemma 2.1.7, we see that

$$\begin{aligned}
\liminf_{i\to\infty}\|y_{n_i} - w\| &< \liminf_{i\to\infty}\|y_{n_i} - Ww\| \\
&\leq \liminf_{i\to\infty}\{\|y_{n_i} - Wy_{n_i}\| + \|Wy_{n_i} - Ww\|\} \\
&\leq \liminf_{i\to\infty}\{\|y_{n_i} - Wy_{n_i}\| + \|y_{n_i} - w\|\}.
\end{aligned} \tag{3.1.25}$$

On the other hand, we have

$$\|Wy_n - y_n\| \leq \|Wy_n - W_n y_n\| + \|W_n y_n - y_n\| \leq \sup_{x\in C}\|Wx - W_n x\| + \|W_n y_n - y_n\|.$$

By using Lemma 2.1.4 and (3.1.23), we obtain that $\lim_{n\longrightarrow\infty}\|Wy_n - y_n\| = 0$, which combines with (3.1.25) yields that

$$\liminf_{i\to\infty}\|y_{n_i} - w\| < \liminf_{i\to\infty}\|y_{n_i} - w\|.$$

Which is a contradiction, so we have $w \in F(W) = \cap_{i=1}^{\infty} F(S_i)$.

Step 4.2 We will show that $w \in \Gamma$. Note that $y_{n_i} = J_\lambda^{B_1}(x_{n_i} + \gamma A^*(J_\lambda^{B_2} - I)Ax_{n_i})$ can be rewritten as

$$\frac{(x_{n_i} - y_{n_i}) + \gamma A^*(J_\lambda^{B_2} - I)Ax_{n_i}}{\lambda} \in B_1 y_{n_i}. \tag{3.1.26}$$

By passing to limit $i \to \infty$ in (3.1.26) and by taking into account (3.1.19) and (3.1.21) and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(w)$, i.e., $w \in \mathrm{SOLVIP}(B_1)$. Furthermore, since

$\{x_n\}$ and $\{y_n\}$ have the same asymptotical behavior, $\{Ax_{n_i}\}$ weakly converges to Aw. Again, by (3.1.19) and the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive and Lemma 2.1.8, we obtain that $Aw \in B_2(Aw)$, i.e., $Aw \in \text{SOLVIP}(B_2)$. Thus, $w \in \Omega$.

Since $z = P_\Omega(z - Dz + \xi u)$ and $w \in \Omega$, by (3.1.24) and the property of metric projection, we get that

$$\begin{aligned}
\limsup_{n\to\infty}\langle \xi u - Dz, x_n - z\rangle &= \lim_{i\to\infty}\langle \xi u - Dz, x_{n_i} - z\rangle \\
&= \langle \xi u - Dz, w - z\rangle \\
&= \langle (z - Dz + \xi u) - z, w - z\rangle \\
&\leq 0. \qquad (3.1.27)
\end{aligned}$$

Step 5. Finally, we will show that $x_n \to z$, as $n \to \infty$. By the inequality (3.1.3), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n \xi u + \beta_n x_n + [(1-\beta_n)I - \alpha_n D]W_n y_n - z\|^2 \\
&= \langle \alpha_n \xi u + \beta_n x_n + [(1-\beta_n)I - \alpha_n D]W_n y_n - z, x_{n+1} - z\rangle \\
&= \alpha_n\langle \xi u - Dz, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle \\
&\quad + \langle ((1-\beta_n)I - \alpha_n D)(W_n y_n - z), x_{n+1} - z\rangle \\
&\leq \alpha_n\langle \xi u - Dz, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle \\
&\quad + \|((1-\beta_n)I - \alpha_n D)(W_n y_n - z)\|\|x_{n+1} - z\| \\
&\leq \alpha_n\langle \xi u - Dz, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle \\
&\quad + (1 - \beta_n - \alpha_n\bar{\xi})\|W_n y_n - z\|\|x_{n+1} - z\| \\
&\leq \alpha_n\langle \xi u - Dz, x_{n+1} - z\rangle + \frac{1}{2}\beta_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\quad + (1 - \beta_n - \alpha_n\bar{\xi})\|y_n - z\|\|x_{n+1} - z\| \\
&\leq \alpha_n\langle \xi u - Dz, x_{n+1} - z\rangle + \frac{1}{2}\beta_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\quad + (1 - \beta_n - \alpha_n\bar{\xi})\|x_n - z\|\|x_{n+1} - z\| \\
&\leq \alpha_n\langle \xi u - Dz, x_{n+1} - z\rangle + \frac{1}{2}\beta_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\quad + \frac{1}{2}(1 - \beta_n - \alpha_n\bar{\xi})(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&= \alpha_n\langle \xi u - Dz, x_{n+1} - z\rangle + \frac{1}{2}(1 - \alpha_n\bar{\xi})(\|x_n - z\|^2 + \|x_{n+1} - z\|^2).
\end{aligned}$$

This implies that

$$\begin{aligned} 2\|x_{n+1}-z\|^2 &\leq 2\alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + (1-\alpha_n\bar{\xi})(\|x_n-z\|^2+\|x_{n+1}-z\|^2) \\ &= 2\alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + (1-\alpha_n\bar{\xi})\|x_n-z\|^2 + (1-\alpha_n\bar{\xi})\|x_{n+1}-z\|^2 \\ &\leq 2\alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + (1-\alpha_n\bar{\xi})\|x_n-z\|^2 + \|x_{n+1}-z\|^2, \end{aligned}$$

and so we have

$$\|x_{n+1}-z\|^2 \leq 2\alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + (1-\alpha_n\bar{\xi})\|x_n-z\|^2.$$

From the condition (C2), (3.1.27) and Lemma 2.1.9, we see that $\lim_{n\longrightarrow\infty}\|x_n-z\| = 0$. This completes the proof. □

3.2 Corollaries

Corollary 3.2.1. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Assume that $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are maximal monotone mappings. Let $\{S_i\}_{i=1}^{\infty}$ be an infinitely family of nonexpansive mappings from H_1 into itself. Assume that $\Omega := (\bigcap_{i=1}^{\infty} Fix(S_i)) \cap \Gamma \neq \emptyset$. Let $x_1 = u \in H_1$ arbitrarily, let the sequences $\{y_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n); \\ x_{n+1} = \alpha_n u + \beta_n x_n + (1-\beta_n-\alpha_n)W_n y_n, \forall n \geq 1, \end{cases} \tag{3.2.1}$$

where $\lambda > 0$ and $\gamma \in (0, \frac{1}{L}), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A, $\{W_n\}$ is the sequence defined by (2.1.10), $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Suppose the control consequences satisfy the following conditions:

(C1) $0 < a \leq \beta_n \leq b < 1, \forall n \geq 1,$

(C2) $\lim_{n\longrightarrow\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty.$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to a point $z \in \Omega$, which solves uniquely solution of the variational inequalities

$$\langle (I-u)z, z-p\rangle \leq 0, \quad \forall p \in \Omega. \tag{3.2.2}$$

Equivalently, we have $P_\Omega u = z$.

Proof. Taking $\xi = 1$ and $D = I$ in Theorem 3.1.1, then the conclusion of Corollary 3.2.1 is obtained. □

Corollary 3.2.2. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Assume that $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are maximal monotone mappings. Let $S : C \to C$ be a nonexpansive mapping. Assume that $\Omega := Fix(S) \cap \Gamma \neq \emptyset$. Let $x_1 = u \in C$ and the sequences $\{y_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n); \\ x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \beta_n - \alpha_n)Sy_n, \forall n \geq 1, \end{cases} \tag{3.2.3}$$

where $\lambda > 0$ and $\gamma \in (0, \frac{1}{L}), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A. If the control consequences satisfying the following:

(C1) $0 < a \leq \beta_n \leq b < 1, \forall n \geq 1,$

(C2) $\lim_{n \longrightarrow \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty,$

then, $\{x_n\}$ and $\{y_n\}$ converge strongly to a point $z \in \Omega$, which solves uniquely solution of the variational inequalities

$$\langle (I - u)z, z - p \rangle \leq 0, \quad \forall p \in \Omega. \tag{3.2.4}$$

Equivalently, we have $P_\Omega u = z$.

Proof. Taking $\xi = D = 1$ and $S_n = S$ for all $n \geq 1$ in Theorem 3.1.1, then the conclusion of Corollary 3.2.3 is obtained. □

Chapter 4 Numerical examples

In this section, let us present the following common fixed point optimization algorithm by using $W-$mapping and discuss some examples to verify the theoretical results.

Algorithm 4.0.1. *(Common fixed point optimization algorithm by using $W-$mapping)*
Step 1. *Choose the initial pooint $x_1 \in H_1$, the parameters $\lambda > 0$ and $\gamma \in (0, \frac{1}{L})$ and $0 < \xi < 1$ arbitrarily real numbers. Fixed the element $u \in H_1$ and let $n = 1$.*
Step 2. *Given $x_n \in H_1$ and compute $x_{n+1} \in H_1$ as follows;*

$$
\begin{aligned}
& y_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) \\
& W_n = U_{n,1} \quad \textit{where} \begin{cases} U_{n,1} = \zeta_1 S_1 U_{n,2} + (1-\zeta_1)I, \\ U_{n,2} = \zeta_2 S_2 U_{n,3} + (1-\zeta_2)I, \\ \vdots \\ U_{n,n} = \zeta_n S_n U_{n,n+1} + (1-\zeta_n)I, \\ U_{n,n+1} = I, \end{cases} \\
& x_{n+1} = \alpha_n \xi u + \beta_n x_n + (1-\beta_n-\alpha_n)W_n y_n.
\end{aligned}
\tag{4.0.1}
$$

Step 3. *Put $n := n+1$ and go to step 2.*

Example 4.1 For $n \geq 1$, let $W_n : \mathbb{R} \longrightarrow \mathbb{R}$ be a mapping generated by an infinite family of nonexpansive mapping $\{\frac{1}{2^n}\}$ and a nonnegative real sequence $\{\frac{n}{n+1}\}$. Define three operators A, B_1 and B_2 on a real line by $Ax = 3x, \quad B_1 x = 2x \quad and \quad B_2 x = \frac{3}{4}x$ for all $x \in \mathbb{R}$. In this example, we set the parameters on algorithm (4.0.1) by $\xi = 0.5,\ D = 1,\ \alpha_n = \frac{10^{-3}}{n}$ and $\beta_n = 0.5 - \frac{1}{10n+2}$ for all $n \in \mathbb{N}$ and fix the element $u = 5$.

First, we take $\lambda = 0.99$, $\gamma = 0.5$ and three initial points randomly generated by Matlab. In this way, Figure 4.1 indicates the behavior of x_n for algorithm (4.0.1) that converges to the same solution, i.e., $0 \in (\cap_{i=1}^{\infty} Fix(S_i)) \cap \Gamma$ as a solution of

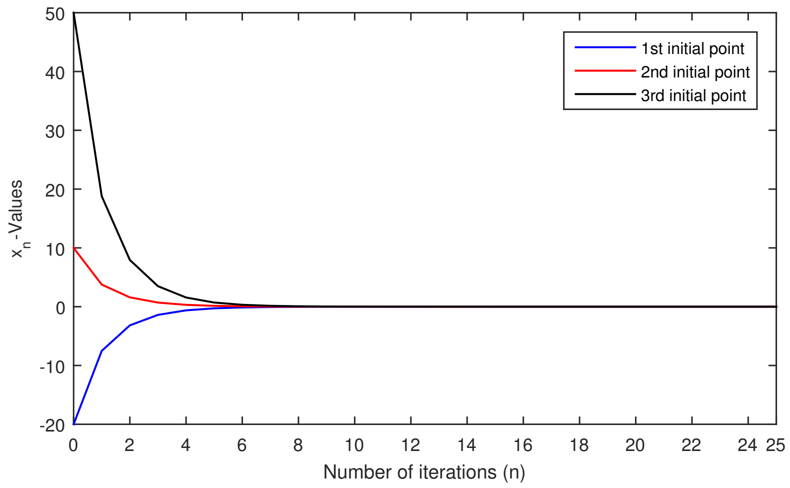


Figure 4.1 Behavior of x_n for the different initial points

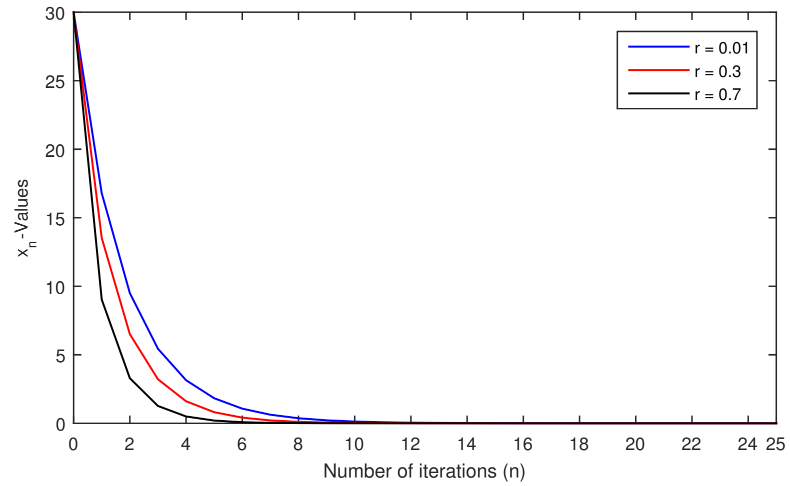


Figure 4.2 Behavior of x_n for the different parameters γ

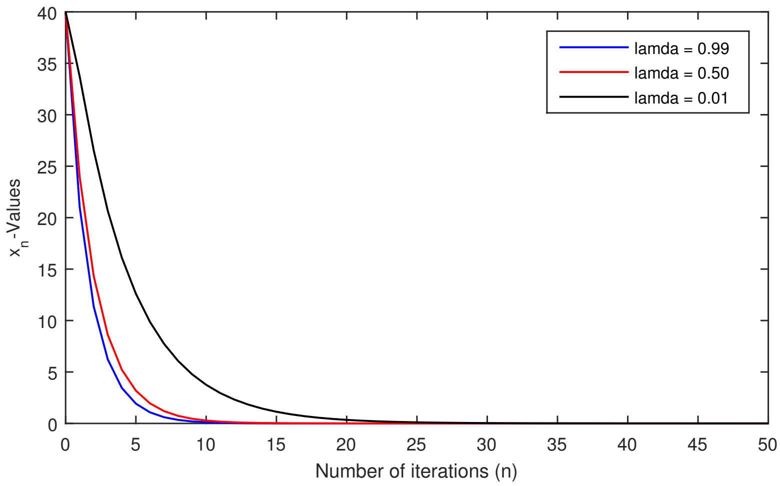


Figure 4.3 Behavior of x_n for the different parameters λ

this example. Next, we test the effect of the parameter γ to rate of convergence by choosing $\gamma = 0.01,\ 0.3$ and 0.7 where the initial point $x_1 = 30$ and the parameter $\lambda = 0.99$ are fixed. In this test, it shows by Figure 4.2. Finally, we fixed the initial point $x_1 = 40$ and the parameter $\gamma = 0.1$ and choosing the different parameters $\lambda = 0.99,\ 0.5$ and 0.01. Figure 4.3 indicates the behavior of x_n generated by algorithm (4.0.1) with $\lambda = 0.01$ decreases slowly.

Example 4.2 Define an infinite family of nonexpansive mapping $S_n : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $S_n = \{\frac{1}{2n}\}$ and a nonnegative real sequence $\zeta_n = \{\frac{n}{n+1}\}$ for all $n \in \mathbb{N}$. Let W_n be a mapping generated by $\{S_n\}$ and $\{\zeta_n\}$. Setting $\gamma = 0.01,\ \lambda = 0.09,\ \xi = 0.2,\ D = I,\ A = \begin{bmatrix} 6 & 3 & 1 \\ 8 & 7 & 5 \\ 3 & 6 & 2 \end{bmatrix},\ B_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix},\ \alpha_n = \frac{10^{-3}}{n}$ and $\beta_n = 0.5 - \frac{1}{10n+2}$ for all $n \in \mathbb{N}$.

Firstly, the experiment used random vector u in $\mathbb{R}^3$ and fixed initial vector $x_1 = (13, -12, 25)$. Using algorithm (4.0.1), the test results are reported in Table 1 and the size of the increment of $\{x_n\}$ and $\{y_n\}$ are presented in Figure 4.4. It's easy to see that $(0,0,0) \in (\cap_{i=1}^{\infty} Fix(S_i)) \cap \Gamma$ is a solution of this experiment.

Secondly, we suppose that $\{x_n(j)\}_{j=1}^{m}$ and $\{y_n(j)\}_{j=1}^{m}$ are the sequences generated

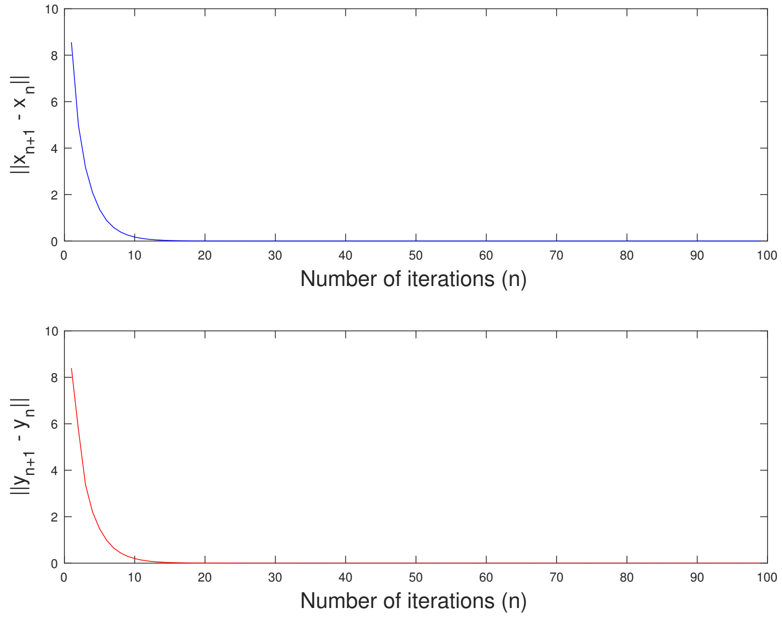


Figure 4.4 The values of $\|x_{n+1} - x_n\|$ and $\|y_{n+1} - y_n\|$

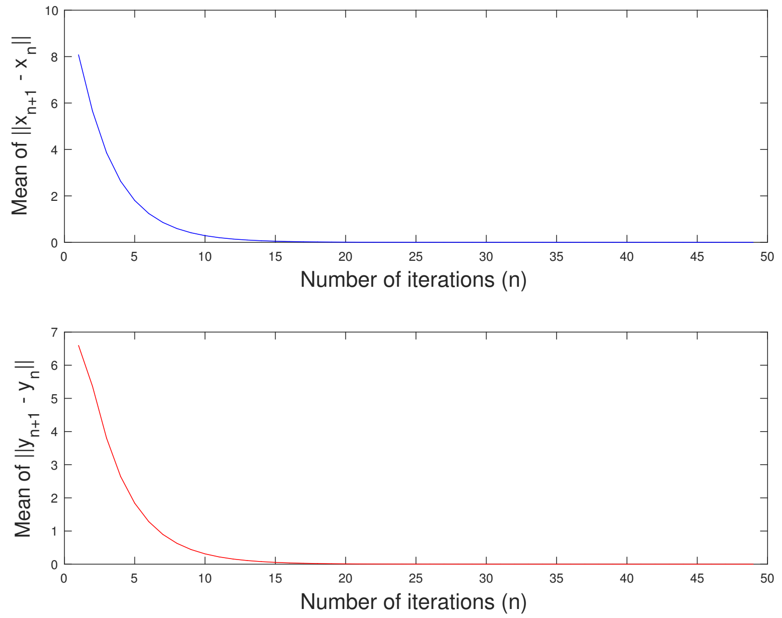


Figure 4.5 The values of $MS(x_n)$ and $MS(y_n)$

n	x_n	y_n	$\|x_n\|$	$\|y_n\|$
1	(13, -12, 25)	(5.1998, -11.3381, 20.3501)	30.6268	23.8687
2	(7.4737, -12.7600, 16.8046)	(2.9885, -6.8856, 13.6536)	22.3846	15.5809
3	(4.1953, -7.5669, 10.9574)	(1.6769, -4.0810, 8.8892)	13.9615	9.9239
4	(2.3823, -4.5946, 7.4465)	(0.9514, -2.4760, 6.0316)	9.0684	6.5891
5	(1.3411, -2.7814, 5.0707)	(0.5349, -1.4974, 4.1016)	5.9369	4.3990
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	(0.0000, 0.0003, 0.0078)	(0.0000, 0.0001, 0.0094)	0.0078	0.0094
25	(0.0001, 0.0002, 0.0011)	(0.0000, 0.0001, 0.0013)	0.0011	0.0013
30	(0.0001, 0.0001, 0.0003)	(0.0001, 0.0001, 0.0003)	3.3166e-04	3.3166e-04
35	(0.0000, 0.0000, 0.0000)	(0.0000, 0.0000, 0.0000)	0.0000	0.0000

Table 4.1 The convergece of sequences $\{x_n\}$ and $\{y_n\}$

by $\{x_n\}$ and $\{y_n\}$ in algorithm (4.0.1), respectively. We performed 50 sampling ($m = 50$ different random initial points) and averaged their size of the increment by using 2-norm. Define the mean size of the increment of $\{x_n(j)\}_{j=1}^m$ and $\{y_n(j)\}_{j=1}^m$ by

$$MS(x_n) := \frac{1}{m}\sum_{j=1}^{m}\|x_{n+1}(j) - x_n(j)\| \quad \text{and} \quad MS(y_n) := \frac{1}{m}\sum_{j=1}^{m}\|y_{n+1}(j) - y_n(j)\|.$$

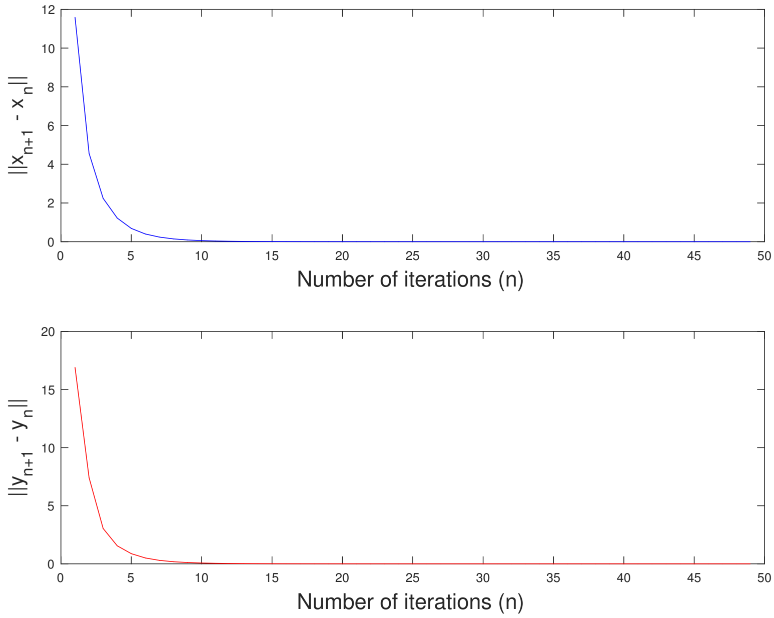
Figure 4.5 shows that the mean size of the increment of $\{x_n\}$ and $\{y_n\}$ converge to 0 which imply that $\{x_n\}$ and $\{y_n\}$ converge to a solution.

Example 4.3 In this example, we replace an infinite family of nonexpansive mapping S_n in Example 5.2 by

$$S_n = \begin{bmatrix} 1/n & 0 & 0 \\ 0 & 1/2^n & 0 \\ 0 & 0 & 1/2^{n+1} \end{bmatrix}$$

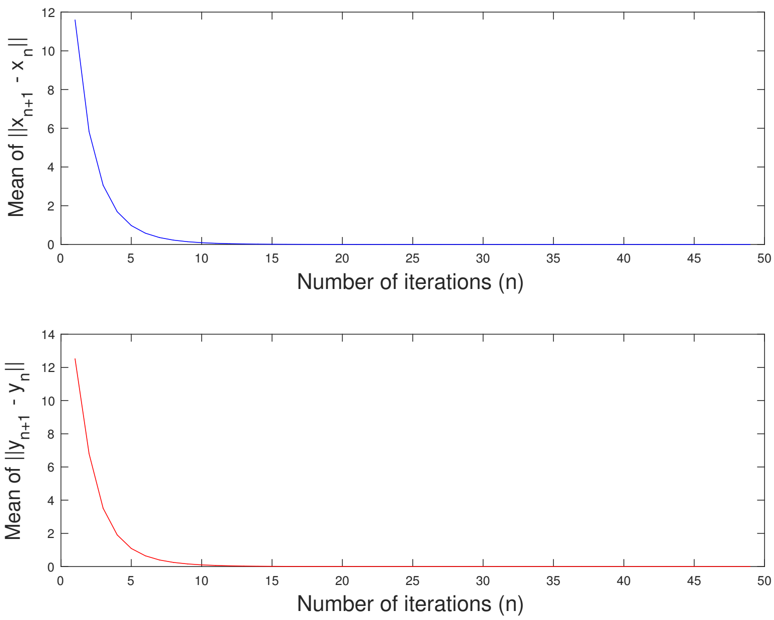
and others are still the same.

Figure 4.6 and 4.7 plot the behavior of sequences of $\|x_{n+1} - x_n\|$, $\|y_{n+1} - y_n\|$, $MS(x_n)$ and $MS(y_n)$ that converge to 0. This concludes that $\{x_n\}$ and $\{y_n\}$ converge to a solution.



0.45

Figure 4.6 The values of $\|x_{n+1} - x_n\|$ and $\|y_{n+1} - y_n\|$



0.45

Figure 4.7 The values of $MS(x_n)$ and $MS(y_n)$

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Appendix

Outputs of the research project

Numer Algor
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ORIGINAL PAPER

Convergence analysis of a general iterative algorithm for finding a common solution of split variational inclusion and optimization problems

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Abstract The purpose of this paper is to introduce a general iterative method for finding a common element of the set of common fixed points of an infinite family

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of nonexpansive mappings and the set of split variational inclusion problem in the framework Hilbert spaces. Strong convergence theorem of the sequences generated by the purpose iterative scheme is obtained. In the last section, we present some computational examples to illustrate the assumptions of the proposed algorithms.

Keywords Split variational inclusion problem · Fixed-point problem · Convex minimization problems · Nonexpansive mapping · Iterative method

Mathematics Subject Classification (2010) 47H09 · 47H10

1 Introduction

A popular general iterative algorithm method for finding the fixed points of nonexpansive mappings was first proposed by Marino and Xu [30]. They considered a general iterative method and proved that the sequence generated by the method converges strongly to a unique solution of a certain variational inequality problem which is the optimality condition for a particular minimization problem. They proved the convergence of the sequence generated by the proposed method. Viscosity approximation method for finding the fixed points of nonexpansive mappings was first proposed by Moudafi [28] in 2000. He proved the convergence of the sequence generated by the proposed method. Yamada [36] introduced the good method the so-called hybrid steepest-descent method for solving the variational inequality problem and also studied the convergence of the sequence generated by the proposed method. In 2010, Tian [37] combined the iterative methods of [28, 30, 36] in order to propose implicit and explicit schemes for constructing a fixed point of a nonexpansive mapping defined on a real Hilbert space. He also proved the strong convergence of these two schemes to a fixed point of under appropriate conditions. Related iterative methods for solving fixed-point problems, variational inequalities, split variational inclusion problem, and optimization problems can be found in the references therein.

Throughout this paper, unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. In this paper, we use $Fix(S)$ to denote the fixed points set of the mapping S. For each x and y in H_1, we recall that a mapping $B : H_1 \to H_1$ is said to be

(i) *nonexpansive* if

$$\|Bx - By\| \leq \|x - y\|; \tag{1.1}$$

(ii) *monotone*, if

$$\langle Bx - By, x - y \rangle \geq 0;$$

(iii) α-*strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \alpha \|x - y\|^2;$$

(iv) β-*inverse-strongly monotone* (for short; β-*ism*), if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \beta \|Bx - By\|^2;$$

(v) *firmly nonexpansive*, if

$$\langle Bx - By, x - y\rangle \geq \|Bx - By\|^2.$$

Also, we recall that a mapping D on H_1 is said to be *strongly positiv*e, if there is a constant $\bar{\xi} > 0$ such that

$$\langle Dx, x\rangle \geq \bar{\xi}\|x\|^2, \forall x \in H_1. \tag{1.2}$$

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [1–4, 7–11] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H_1;

$$\min_{x \in Fix(S)} \frac{1}{2}\langle Dx, x\rangle - h(x), \tag{1.3}$$

where D is a bounded linear operator and h is a potential function for ξf (i.e., $h^{'}(x) = \xi f(x)$ for $x \in H_1$).

In 2006, Marino and Xu [4] studied the following iterative scheme; put $x_1 \in H_1$ and

$$x_{n+1} = (I - \alpha_n D)Sx_n + \alpha_n \xi f(x_n), \quad n \geq 1, \tag{1.4}$$

where S is a nonexpansive mapping, f is a contraction mapping with the coefficient $\alpha \in (0, 1)$, D is a strongly positive bounded linear self-adjoint operator with the coefficient $\bar{\xi}$ and ξ is a constant such that $0 < \xi < \frac{\bar{\xi}}{\alpha}$. They proved the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution x^* of the following variational inequality;

$$\langle (D - \xi f)x^*, x - x^*\rangle \geq 0, \quad \forall x \in Fix(S),$$

which is the optimality condition for the minimization problem (1.3).

Let $B : C \to H_1$ be a nonlinear mapping. The *classical variational inequality problem* (VIP) is to find a $x^* \in C$ such that

$$\langle Bx^*, y - x^*\rangle \geq 0, \forall y \in C. \tag{1.5}$$

The solution set of VIP (1.5) is denoted by $VI(C, B)$.

In 2005, and Takahashi [5] introduced and proved the theorem for finding a common element of the set of fixed points of a nonexpansive mapping and the solution sets of the variational inequality as follows:

Theorem IT Let C be a closed convex subset of a real Hilbert space H_1. Let B be an α-inverse-strongly monotone mapping of C into H_1 and let S be a nonexpansive mapping of C into itself such that $Fix(S) \cap VI(C, B) \neq \emptyset$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)SP_C(x_n - \lambda_n Bx_n),$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[a, b]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so by $\{\lambda_n\} \in [a, b]$ for some real numbers a and b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{Fix(S)\cap VI(C,B)}x$.

Further results on this paper are due to Yao and Yao [6]. More precisely, they proved the following result.

Theorem YY Let C be a closed convex subset of a real Hilbert space H_1. Let B be an α-inverse-strongly monotone mapping of C into H_1 and let S be a nonexpansive mapping of C into itself such that $Fix(S) \cap \Omega \neq \emptyset$, where Ω denotes the set of solutions of a variational inequality for the α-inverse-strongly monotone mapping. Suppose $x_1 = u \in C$ and $\{x_n\}$, $\{y_n\}$ are given by

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \lambda_n Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(I - \lambda_n B)y_n, \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\alpha_n\}$ are three sequences in $[0, 1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2a]$. If $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are chosen by $\lambda_n \in [a, b]$ for some real numbers a and b with $0 < a < b < 2\alpha$ and the following hold;

(i) $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$;
(ii) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iv) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$. Then the sequence $\{x_n\}$ defined by the iterative algorithm (1.6) converges strongly to $P_{Fix(S)\cap\Omega}u$.

Recall that a set-valued mapping $M : H_1 \to 2^{H_1}$ is called *monotone* if for all $x, y \in H_1$, $f \in Mx$ and $g \in My$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $M : H_1 \to 2^{H_1}$ is *maximal* if the graph of Graph(M) of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H_1 \times H_1$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in Graph(M)$ implies $f \in Mx$.

Let $M : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. For any positive number λ and identity operator I on H_1, the single-valued mapping $J_\lambda^M : H_1 \to H_1$ defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x), \forall x \in H_1,$$

is called the resolvent operator associated with M . We recall that the resolvent operator J_λ^M is firmly nonexpansive and hence in particular nonexpansive [12].

Let $C_1, C_2, \ldots, C_m$ be nonempty closed convex subsets of a Hilbert space H_1. The well known *convex feasibility problem* (CFP) is to find $x^* \in H_1$ such that

$$x^* \in C_1 \cap C_2 \cap \ldots C_m.$$

Convex feasibility problem has received a lot of attention due to its diverse applications in mathematics, approximation theory, communications, geophysics, control theory, biomedical engineering. One can refer to [13, 14]. When there are only two sets and constrains are imposed on the solutions in the domain of a linear operator as well as in this operator's ranges, the problem is said to be the *split feasibility problem* (SFP) which has the following formula:

$$x^* \in C \;\text{ such that }\; Ax^* \in Q, \tag{1.7}$$

where C and Q is a nonempty closed convex subset of Hilbert space H_1 and H_2, respectively and $A : H_1 \to H_2$ is a bounded linear operator. The SFP in finite dimensional Hilbert spaces was first introduced by Censor and Elfving [15] in medical image reconstruction. Since then, the SFP has received much attention due to its applications in signal processing, image reconstruction, with particular progress in intensity-modulated radiation therapy, approximation theory, control theory, biomedical engineering, communications, and geophysics. For examples, one can refer to [15–18] and related literatures.

Recently, Moudafi [19] introduced the following *split monotone variational inclusion problem* (SMVIP): find $x^* \in H_1$ such that

$$0 \in f_1(x^*) + B_1(x^*), \tag{1.8}$$

and

$$y^* = Ax^* \in H_2 \;\text{ solves }\; 0 \in f_2(y^*) + B_2(y^*), \tag{1.9}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings.

Moudafi [19] introduced an iterative method for solving SMVIP (1.8)–(1.9), which can be seen as an important generalization of an iterative method given by Censor et al. [20] for split variational inequality problem. As Moudafi notes in [19], SMVIP (1.8)–(1.9) includes as special cases, the split common fixed-point problem, split variational inequality problem, split zero problem, and split feasibility problem [15, 17, 19, 20] which have already been studied and used in practice as a model in intensity-modulated radiation therapy treatment, see [15, 17]. This formalism is also at the core of modeling of many inverse problems arising from phase retrieval and other real-world problems, for instance, in sensor networks in computerized tomography and data compression; see, e.g., [14, 21].

If $f_1 \equiv 0$ and $f_2 \equiv 0$, then SMVIP (1.8)–(1.9) reduces to the following the *split variational inclusion problem* (SVIP): Find $x^* \in H_1$ such that

$$0 \in B_1(x^*), \tag{1.10}$$

and

$$y^* = Ax^* \in H_2 \;\text{ solves }\; 0 \in B_2(y^*). \tag{1.11}$$

When considered separately, (1.10) is the variational inclusion problem and we denoted its solution set by SOLVIP (B_1). The SVIP (1.10)–(1.11) constitutes a pair of variational inclusion problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator A of the solution x^* of SVIP (1.10) in H_1 is the solution of the other SVIP (1.11) in another space H_2, we denote the solution set of SVIP (1.11) by SOLVIP(B_2).

The solution set of SVIP (1.10)–(1.11) is denoted by $\Gamma = \{x^* \in H_1 : x^* \in SOLVIP(B_1) \text{ and } Ax^* \in SOLVIP(B_2)\}$.

Very recently, Byrne et al. [22] studied the weak and strong convergence of the following iterative method for SVIP (1.10)–(1.11): for given $x_1 \in H_1$, compute the iterative sequence $\{x_n\}$ generated by the following scheme:

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \forall n \geq 1, \ \exists \lambda > 0.$$

In 2014, Kazmi and Rizvi [23] introduced and studied an iterative method to approximate a common solution of SVIP and fixed-point problem for a nonexpansive mapping in real Hilbert spaces.

Theorem KR Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator and $f : H_1 \to H_1$ be a contraction mapping with constant $\alpha \in (0, 1)$. Let $S : H_1 \to H_1$ be a nonexpansive mapping such that $Fix(S) \cap \Gamma \neq \emptyset$. For a given $x_0 \in H_1$ arbitrarily, let the iterative sequence $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n); \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \end{cases}$$

where $\lambda > 0$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A and α_n is a sequence in (0, 1) such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty.$$

Then, the sequences $\{u_n\}$ and $\{x_n\}$ both converge strongly to $z \in Fix(S) \cap \Gamma$, where $z = P_{Fix(S)\cap\Gamma} f(z)$.

In this paper, motivated by Marino and Xu [4], Iiduka and Takahashi [5], Y. Yao and J. C. Yao [6], Moudafi [19], Byrne et al. [22], and Kazmi and Rizvi [23], we introduce an iterative method for finding a common element of the set of common fixed points of an infinite family of nonexpansive mappings and the set of SVIP (1.10)–(1.11). Strong convergence theorem is established in the framework of Hilbert spaces. Finally, the numerical results are presented for confirming the main theorem. Moreover, the last section presents some numerical examples to illustrate the behavior of the proposed algorithm.

2 Preliminaries

We recall some concepts and results which are needed in sequel. Let H_1 be a Hilbert space and let symbols " $\to$" and " $\rightharpoonup$" denote by strong and weak convergence, respectively. In Hilbert spaces, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in H_1$ and $\lambda \in [0, 1]$.

For any $x \in H_1$, there exists a unique nearest point in a nonempty closed convex subset C denoted by $P_C(x)$ such that $\|x - P_C(x)\| \leq \|x - y\|$, for all $y \in C$. The mapping P_C is called the *metric projection* of H_1 onto C. We know that P_C is a nonexpansive mapping from H_1 onto C. The metric projection can be characterized by $P_C(x) \in C$ and satisfied

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H_1. \tag{2.1}$$

Moreover, for all $x \in H_1$ and $y \in C$, $P_C x$ is characterized by

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0. \tag{2.2}$$

It is easy to see that (2.2) is equivalent to the following inequality:

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \tag{2.3}$$

for all $x \in H_1$ and $y \in C$. For more details, see [24].

Let $S : H_1 \to H_1$ be a mapping, Then, we have the following equality:

$$\|(x-y)-(S(x)-S(y))\|^2 = \|x-y\|^2+\|S(x)-S(y)\|^2-2\langle x-y, S(x)-S(y)\rangle, \forall(x, y) \in H_1\times H_1 \tag{2.4}$$

Further, if S is a nonexpansive operator, we have also that

$$\begin{aligned}\|(x - y) - (S(x) - S(y))\|^2 &\geq 2\|S(x) - S(y)\|^2 - 2\langle x - y, S(x) - S(y)\rangle \\ &= 2\langle S(x) - S(y), S(x) - S(y)\rangle - 2\langle x - y, S(x) - S(y)\rangle \\ &= 2\langle (S(x) - S(y)) - (x - y), S(x) - S(y)\rangle. \end{aligned} \tag{2.5}$$

Then, we obtain that every nonexpansive operator $S : H_1 \to H_1$ satisfies the following inequality

$$\langle (x-S(x))-(y-S(y)), S(y)-S(x)\rangle \leq \frac{1}{2}\|(S(x)-x)-(S(y)-y)\|^2, \forall(x, y) \in H_1\times H_1 \tag{2.6}$$

and therefore, we get

$$\langle x - S(x), y - S(x)\rangle \leq \frac{1}{2}\|S(x) - x\|^2, \forall(x, y) \in H_1 \times Fix(S). \tag{2.7}$$

A mapping $T : H_1 \to H_1$ is said to be *averaged* mapping if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T \equiv (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0, 1)$ and $S : H_1 \to H_1$ is nonexpansive. We note that averaged mappings are nonexpansive. Further, firmly nonexpansive mappings (in particular, projections on nonempty closed and convex subsets and resolvent operators of maximal monotone operators) are averaged. Some important properties of averaged mappings are gathered in the following proposition (see [23, 25, 26]).

Proposition 2.1 *(i) If $T = (1 - \alpha)S + \alpha V$, where $S : H_1 \to H_1$ is averaged, $V : H_1 \to H_1$ is nonexpansive and $\alpha \in (0, 1)$, then T is averaged.*

(ii) The composite of finitely many averaged mappings is averaged.

(iii) If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then $\bigcap_{i=1}^N Fix(T_i) = Fix(T_1, T_2, \ldots, T_N)$.

(iv) *If T is τ-ism, then for $\gamma > 0$, γT is $\frac{\tau}{\gamma}$-ism.*

(v) *T is averaged if and only if, its complement $I - T$ is τ-ism for some $\tau > \frac{1}{2}$.*

Lemma 2.2 ([23]) *SVIP (1.10)–(1.11) is equivalent to find $x^* \in H_1$ with $x^* = J_\lambda^{B_1}(x^*)$ such that*

$$y^* = Ax^* \in H_2 \text{ and } y^* = J_\lambda^{B_2}(y^*),$$

for some $\lambda > 0$.

Definition 2.3 ([27]) Let H_1 be a real Hilbert space and $\{S_i\}$ be an infinite family of nonexpansive mappings and $\{\zeta_i\}$ be a nonnegative real sequence with $0 \leq \zeta_i \leq 1, \forall i \geq 1$. For $n \geq 1$, define a mapping W_n as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \zeta_n S_n U_{n,n+1} + (1-\zeta_n)I, \\ U_{n,n-1} = \zeta_{n-1} S_{n-1} U_{n,n} + (1-\zeta_{n-1})I, \\ \vdots \\ U_{n,k} = \zeta_k S_k U_{n,k+1} + (1-\zeta_k)I, \\ U_{n,k-1} = \zeta_{k-1} S_{k-1} U_{n,k} + (1-\zeta_{k-1})I, \\ \vdots \\ U_{n,2} = \zeta_2 S_2 U_{n,3} + (1-\zeta_2)I, \\ W_n = U_{n,1} = \zeta_1 S_1 U_{n,2} + (1-\zeta_1)I. \end{cases} \tag{2.8}$$

Such a mapping W_n is nonexpansive and it is called a W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\zeta_n, \zeta_{n-1}, \ldots, \zeta_1$.

Lemma 2.4 ([27]) *Let $\{S_i\}$ be an infinite family of nonexpansive mappings on a Hilbert space H_1 and with $\bigcap_{i=1}^{\infty} Fix(S_i) \neq \emptyset$ and $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \leq 1,\ \forall i \geq 1$. Then*

(1) *W_n is nonexpansive and $Fix(W_n) = \bigcap_{i=1}^{n} Fix(S_i)$, for each $n \geq 1$;*
(2) *for each $x \in C$ and for each positive integer k, the $\lim_{n\to\infty} U_{n,k}x$ exists;*
(3) *the mapping W defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in H_1, \tag{2.9}$$

is a nonexpansive mapping satisfying $Fix(W) = \bigcap_{i=1}^{\infty} Fix(S_i)$ and it is called the W-mapping generated by $S_1, S_2, \ldots$ and $\zeta_1, \zeta_2, \ldots$.

Lemma 2.5 ([29]) *Let $\{S_i\}$ be an infinite family of nonexpansive mappings on a Hilbert space H_1 and with $\bigcap_{i=1}^{\infty} Fix(S_i) \neq \emptyset$ and $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \leq l \leq 1,\ \forall i \geq 1$ where l is a positive real number. If C is any bounded subset of H_1, then*

$$\lim_{n\to\infty} \sup_{x\in C} \|Wx - W_n x\| = 0.$$

Throughout this paper, we always assume that $0 < \zeta_i \leq l < 1, \forall i \geq 1$.

Lemma 2.6 ([31]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\longrightarrow\infty}\beta_n \le \limsup_{n\longrightarrow\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \ge 0$ and*

$$\limsup_{n\longrightarrow\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \le 0.$$

Then, $\lim_{n\longrightarrow\infty}\|z_n - x_n\| = 0$.

Lemma 2.7 *In a real Hilbert space H_1, there holds the following inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle,$$

for all $x, y \in H_1$.

Lemma 2.8 ([32]) *Each Hilbert space H_1 satisfies the Opial condition that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\longrightarrow\infty}\|x_n - x\| < \liminf_{n\longrightarrow\infty}\|x_n - y\|$ holds for every $y \in H$ with $y \ne x$.*

Lemma 2.9 ([33]) *Assume that S is nonexpansive self mapping of a closed convex subset C of a Hilbert space H_1. If S has a fixed point, then $I - S$ is demiclosed, i.e., whenever $\{x_n\}$ converges weakly to some y, it follows that $(I-S)x = y$. Here I is the identity mapping on H_1.*

Lemma 2.10 ([10, 11]) *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$;
(ii) $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\longrightarrow\infty}\alpha_n = 0$.

Lemma 2.11 [30] *Assume D is a strongly positive bounded linear operator on Hilbert space H_1 with coefficient $\bar{\xi} > 0$ and $0 < \rho \le \|D\|^{-1}$. Then, $\|I - \rho D\| \le 1 - \rho\bar{\xi}$.*

3 Main result

Throughout the rest of this paper, we always assume that D is a strongly positive bounded linear operator with coefficient $\bar{\xi} \in (0,1)$ and $0 < \xi < \bar{\xi}$. We consider the well-known general iterative method as follows (see Xu 2003 [35]) with the initial guess $x_0 = u \in C$:

$$x_{n+1} = (I - \alpha_n D)Sx_n + \alpha_n u, \quad n \ge 0, \tag{3.1}$$

where D is a strongly positive bounded linear operator, S is nonexpansive self mapping on $C \subset H$ and α_n is so chosen that $T = I - \alpha_n D$ satisfies some conditions (see

[34]). This iterative method converges to the unique solution x^* of the variational inequality

$$\langle Dx^* - \xi u, x^* - x\rangle \leq 0, \quad \forall x \in C,$$

which is the optimality for minimize a quadratic function of (1.3), that is

$$\min_{x\in Fix(S)} \frac{1}{2}\langle Dx, x\rangle - h(x),$$

where h is a potential function for ξf (i.e., $h^{'}(x) = \xi f(x)$) and the contraction mapping f is replaced by $f(x) = u$ for all $x \in H_1$ which is mentioned in the viscosity approximation method (1.4).

Also, we develop a new algorithm by mixing the algorithm (1.6) and (3.1) for finding a common solution of split variational inclusion of two Hilbert spaces H_1 and H_2 which is a tool for solving the unique solution to the variational inequalities (3.3) below.

In this section, we first prove a strong convergence result on the general iterative algorithm for the fixed-point problem and the split variational inclusion problem (SVIP) (1.10)–(1.11).

Theorem 3.1 *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Assume that $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are maximal monotone mappings. Let $\{S_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings from H_1 into itself. Let D be a strongly positive bounded linear operator with coefficient $0 < \bar{\xi} < 1$ and $0 < \xi < \bar{\xi}$. Assume that $\Omega := (\bigcap_{i=1}^{\infty} Fix(S_i)) \cap \Gamma \neq \emptyset$. Let $x_1 \in H_1$ arbitrarily, let the sequences $\{y_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n); \\ x_{n+1} = \alpha_n \xi u + \beta_n x_n + [(1-\beta_n)I - \alpha_n D]W_n y_n, \forall n \geq 1, \end{cases} \tag{3.2}$$

where $u \in H_1$ is a fixed element, $\lambda > 0$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A, $\{W_n\}$ is the sequence defined by (2.8), $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. Suppose the control consequences satisfy the following conditions:

(C1) $0 < a \leq \beta_n \leq b < 1, \forall n \geq 1,$ *for some two positive real numbers a and b,*
(C2) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to a point $z \in \Omega$, which is the unique solution to the variational inequalities

$$\langle Dz - \xi u, z - p\rangle \leq 0, \quad \forall p \in \Omega. \tag{3.3}$$

Equivalently, we have $P_\Omega(z - Dz + \xi u) = z$.

Proof **Step 1.** First, we will prove that the sequence $\{x_n\}$ is bounded.

By the conditions (C1) and (C2), without loss of generality we may assume $\alpha_n \leq (1-\beta_n)\|D\|^{-1}$ for all $n \geq 1$. By Lemma 2.11, we get that

$$\|I - \frac{\alpha_n}{1-\beta_n} D\| \leq 1 - \frac{\alpha_n}{1-\beta_n}\bar{\xi}.$$

It follows that

$$\|(1-\beta_n)I - \alpha_n D\| \leq 1 - \beta_n - \alpha_n \bar{\xi}. \tag{3.4}$$

Let $p \in \Omega$, we have $p = J_\lambda^{B_1} p$, $Ap = J_\lambda^{B_2}(Ap)$ and $S_n p = p$ for all $n \geq 1$. We estimate that

$$\begin{aligned} \|y_n - p\|^2 &= \|J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) - p\|^2 \\ &= \|J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) - J_\lambda^{B_1} p\|^2 \\ &\leq \|x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n - p\|^2 \\ &\leq \|x_n - p\|^2 + \gamma^2 \|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 + 2\gamma\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle. \end{aligned} \tag{3.5}$$

Thus, we have

$$\begin{aligned} \|y_n - p\|^2 \leq \|x_n - p\|^2 &+ \gamma^2\langle (J_\lambda^{B_2} - I)Ax_n, AA^*(J_\lambda^{B_2} - I)Ax_n\rangle \\ &+ 2\gamma\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle. \end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned} \gamma^2\langle (J_\lambda^{B_2} - I)Ax_n, AA^*(J_\lambda^{B_2} - I)Ax_n\rangle &\leq L\gamma^2\langle (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\ &= L\gamma^2\|(J_\lambda^{B_2} - I)Ax_n\|^2. \end{aligned} \tag{3.7}$$

Consider the term of $2\gamma\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle$ and using (2.7), we have

$$\begin{aligned} 2\gamma\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle &= 2\gamma\langle A(x_n - p), (J_\lambda^{B_2} - I)Ax_n\rangle \\ &= 2\gamma\langle A(x_n - p) + (J_\lambda^{B_2} - I)Ax_n - (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\ &= 2\gamma\Big\{\langle Ax_n - Ap + J_\lambda^{B_2}Ax_n - Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle - \langle (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle\Big\} \\ &= 2\gamma\Big\{\langle Ap - J_\lambda^{B_2}Ax_n, Ax_n - J_\lambda^{B_2}Ax_n\rangle - \|(J_\lambda^{B_2} - I)Ax_n\|^2\Big\} \\ &\leq 2\gamma\Big\{\frac{1}{2}\|(J_\lambda^{B_2} - I)Ax_n\|^2 - \|(J_\lambda^{B_2} - I)Ax_n\|^2\Big\} \\ &= -\gamma\|(J_\lambda^{B_2} - I)Ax_n\|^2. \end{aligned} \tag{3.8}$$

Using (3.6), (3.7), and (3.8), we obtain

$$\begin{aligned} \|y_n - p\|^2 &\leq \|x_n - p\|^2 + L\gamma^2\|(J_\lambda^{B_2} - I)Ax_n\|^2 - \gamma\|(J_\lambda^{B_2} - I)Ax_n\|^2 \\ &= \|x_n - p\|^2 + \gamma(L\gamma - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2. \end{aligned} \tag{3.9}$$

Since $\gamma \in (0, \frac{1}{L})$, we obtain

$$\|y_n - p\|^2 \leq \|x_n - p\|^2. \tag{3.10}$$

We arrive that

$$
\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n\xi u+\beta_n x_n+[(1-\beta_n)I-\alpha_n D]W_n y_n-p\| \\
&= \|\alpha_n\xi u-\alpha_n Dp-\beta_n p+\beta_n x_n+[(1-\beta_n)I-\alpha_n D]W_n y_n+\alpha_n Dp+\beta_n p-p\| \\
&= \|\alpha_n(\xi u-Dp)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_n D)W_n y_n-((1-\beta_n)I-\alpha_n D)p\| \\
&\le \alpha_n\|\xi u-Dp\|+\beta_n\|x_n-p\|+\|(1-\beta_n)I-\alpha_n D\|\|W_n y_n-p\| \\
&\le \alpha_n\|\xi u-Dp\|+\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\bar{\xi})\|y_n-p\| \\
&\le \alpha_n\|\xi u-Dp\|+\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\bar{\xi})\|x_n-p\| \\
&\le \alpha_n\|\xi u-Dp\|+(1-\alpha_n\bar{\xi})\|x_n-p\|.
\end{aligned}
$$

By mathematical induction, we obtain

$$
\|x_n-p\|\le \max\left\{\|x_1-p\|,\frac{\|\xi u-Dp\|}{\bar{\xi}}\right\},
$$

which gives that the sequence $\{x_n\}$ is bounded and also is $\{y_n\}$. Hence, we can choose a bounded set $C\subset H_1$ such that

$$
x_n, y_n\in C,\ \forall n\ge 1. \tag{3.11}
$$

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Setting

$$
v_n=\frac{1}{1-\beta_n}x_{n+1}-\frac{\beta_n}{1-\beta_n}x_n.
$$

Then, we see that

$$
x_{n+1}=(1-\beta_n)v_n+\beta_n x_n, \forall n\ge 1, \tag{3.12}
$$

and

$$
\begin{aligned}
v_n &= \frac{1}{1-\beta_n}\{\alpha_n\xi u+\beta_n x_n+[(1-\beta_n)I-\alpha_n D]W_n y_n\}-\frac{\beta_n}{1-\beta_n}x_n \\
&= \frac{1}{1-\beta_n}\{\alpha_n\xi u+\beta_n x_n+W_n y_n-\beta_n W_n y_n-\alpha_n DW_n y_n\}-\frac{\beta_n}{1-\beta_n}x_n \\
&= \frac{1}{1-\beta_n}\{\alpha_n(\xi u-DW_n y_n)+(1-\beta_n)W_n y_n\} \\
&= \frac{\alpha_n}{1-\beta_n}(\xi u-DW_n y_n)+W_n y_n.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\|v_{n+1}-v_n\| &= \left\|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\xi u-DW_{n+1}y_{n+1})+W_{n+1}y_{n+1}-[\frac{\alpha_n}{1-\beta_n}(\xi u-DW_n y_n)+W_n y_n]\right\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\xi u-DW_{n+1}y_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|\xi u-DW_n y_n\|+\|W_{n+1}y_{n+1}-W_n y_n\|.
\end{aligned} \tag{3.13}
$$

Since $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ both are firmly nonexpansive, they are averaged. For $\gamma \in (0, \frac{1}{L})$, the mapping $(I + \gamma A^*(J_\lambda^{B_2} - I)A)$ is averaged, see [19]. It follows from Proposition 2.1 (ii) that the mapping $J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)$ is averaged and hence nonexpansive. So, we obtain that

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|J_\lambda^{B_1}(x_{n+1} + \gamma A^*(J_\lambda^{B_2} - I)Ax_{n+1}) - J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n)\| \\ &= \|J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)x_{n+1} - J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)x_n\| \\ &\leq \|x_{n+1} - x_n\|. \end{aligned} \tag{3.14}$$

On the other hand, one has

$$\begin{aligned}\|W_{n+1}y_{n+1} - W_n y_n\| &= \|W_{n+1}y_{n+1} - Wy_{n+1} + Wy_{n+1} - Wy_n + Wy_n - W_n y_n\| \\ &\leq \|W_{n+1}y_{n+1} - Wy_{n+1}\| + \|Wy_{n+1} - Wy_n\| + \|Wy_n - W_n y_n\| \\ &\leq \sup_{x\in C}\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\} + \|y_{n+1} - y_n\|, \end{aligned} \tag{3.15}$$

where C is the bounded subset of H_1 defined by (3.11). Substituting (3.14) into (3.15), one arrive that

$$\|W_{n+1}y_{n+1} - W_n y_n\| \leq \sup_{x\in C}\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\} + \|x_{n+1} - x_n\|. \tag{3.16}$$

From (3.13) combine with (3.16), one obtains

$$\begin{aligned}\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\| \leq\ & \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|\xi u + DW_{n+1}y_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|\xi u - DW_n y_n\| \\ & + \sup_{x\in C}\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\}. \end{aligned}$$

It follows from the conditions (C1), (C2), and Lemma 2.5 that

$$\limsup_{n\to\infty}(\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, from Lemma 2.6 and (3.12), we obtain that

$$\lim_{n\to\infty}\|v_n - x_n\| = 0. \tag{3.17}$$

From (3.12), we have that

$$\|x_{n+1} - x_n\| = \|(1 - \beta_n)v_n + \beta_n x_n - [(1 - \beta_n)x_n + \beta_n x_n]\| = (1 - \beta_n)\|v_n - x_n\|.$$

By the condition (C1) and (3.17), we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.18}$$

Step 3. We will prove that three sequences $\{\|y_n - x_n\|\}$, $\{\|W_n y_n - x_n\|\}$, and $\{\|W_n y_n - y_n\|\}$ converge to 0.

We set $f_n = \xi u - DW_n y_n$, for all $n \geq 1$. For any $p \in \Omega$ and by Lemma 2.7, we see that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n \xi u + \beta_n x_n + [(1-\beta_n)I - \alpha_n D]W_n y_n - p\|^2 \\
&= \|\alpha_n \xi u + \beta_n x_n + W_n y_n - \beta_n W_n y_n - \alpha_n D W_n y_n - p\|^2 \\
&= \|\alpha_n(\xi u - DW_n y_n) + \beta_n x_n + (1-\beta_n)W_n y_n - p\|^2 \\
&= \|\alpha_n f_n + \beta_n x_n + (1-\beta_n)W_n y_n - \beta_n p - (1-\beta_n)p\|^2 \\
&= \|\alpha_n f_n + \beta_n(x_n - p) + (1-\beta_n)(W_n y_n - p)\|^2 \\
&\leq \|\beta_n(x_n - p) + (1-\beta_n)(W_n y_n - p)\|^2 + 2\langle \alpha_n f_n, x_{n+1} - p\rangle \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|W_n y_n - p\|^2 - \beta_n(1-\beta_n)\|W_n y_n \\
&\qquad - x_n\|^2 + 2\alpha_n\|f_n\|\|x_{n+1} - p\| \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 + 2\alpha_n M^2,
\end{aligned} \tag{3.19}$$

where $M = \max\{\sup_{n\geq 1}\|f_n\|, \sup_{n\geq 1}\|x_n - p\|\}$.

Observe that from (3.9) substituting into (3.19), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)[\|x_n - p\|^2 + \gamma(L\gamma - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2] + 2\alpha_n M^2 \\
&= \|x_n - p\|^2 - \gamma(1-\beta_n)(1-L\gamma)\|(J_\lambda^{B_2} - I)Ax_n\|^2 + 2\alpha_n M^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma(1-\beta_n)(1-L\gamma)\|(J_\lambda^{B_2} - I)Ax_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n M^2 \\
&\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + 2\alpha_n M^2,
\end{aligned}$$

and from the condition (C1), (C2), $\gamma(1-\beta_n)(1-L\gamma) > 0$, and (3.18), we get

$$\lim_{n\to\infty} \|(J_\lambda^{B_2} - I)Ax_n\| = 0. \tag{3.20}$$

Since $J_\lambda^{B_1}$ is firmly nonexpansive mapping and by using the inequalities (3.8) and Cauchy-Schwarz inequality, then we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) - p\|^2 \\
&= \|J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) - J_\lambda^{B_1} p\|^2 \\
&\leq \langle y_n - p, x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n - p\rangle \\
&\leq \frac{1}{2}\{\|y_n - p\|^2 + \|x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n - p\|^2 - \| \\
&\quad -[x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n - p]\|^2\} \\
&= \frac{1}{2}\{\|y_n - p\|^2 + \|x_n - p\|^2 + 2\gamma\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle \\
&\quad +\gamma^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 - \|y_n - x_n - \gamma A^*(J_\lambda^{B_2} - I)Ax_n\|^2\} \\
&\leq \frac{1}{2}\{\|y_n - p\|^2 + \|x_n - p\|^2 - \gamma\|(J_\lambda^{B_2} - I)Ax_n\|^2 + \gamma^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 \\
&\quad -\|y_n - x_n - \gamma A^*(J_\lambda^{B_2} - I)Ax_n\|^2\} \\
&= \frac{1}{2}\{\|y_n - p\|^2 + \|x_n - p\|^2 - \gamma\|(J_\lambda^{B_2} - I)Ax_n\|^2 + \gamma^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 \\
&\quad -\|y_n - x_n\|^2 - \gamma^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 + 2\gamma\langle y_n - x_n, A^*(J_\lambda^{B_2} - I)Ax_n\rangle\} \\
&\leq \frac{1}{2}\{\|y_n - p\|^2 + \|x_n - p\|^2 - \|y_n - x_n\|^2 + 2\gamma\langle y_n - x_n, A^*(J_\lambda^{B_2} - I)Ax_n\rangle\} \\
&= \frac{1}{2}\{\|y_n - p\|^2 + \|x_n - p\|^2 - \|y_n - x_n\|^2 + 2\gamma\langle A(y_n - x_n), (J_\lambda^{B_2} - I)Ax_n\rangle\} \\
&\leq \frac{1}{2}\{\|y_n - p\|^2 + \|x_n - p\|^2 - \|y_n - x_n\|^2 + 2\gamma\|A(y_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\|\}.
\end{aligned}$$

Hence, we obtain

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|y_n - x_n\|^2 + 2\gamma\|A(y_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\|. \tag{3.21}$$

Substituting (3.21) into (3.19), one arrive that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)[\|x_n - p\|^2 - \|y_n - x_n\|^2 \\
&\quad +2\gamma\|A(y_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\|] + 2\alpha_n M^2 \\
&= \|x_n - p\|^2 - (1 - \beta_n)\|y_n - x_n\|^2 \\
&\quad +2\gamma(1 - \beta_n)\|A(y_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| + 2\alpha_n M^2.
\end{aligned}$$

So, we get

$$\begin{aligned}
(1 - \beta_n)\|y_n - x_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad +2\gamma(1 - \beta_n)\|A(y_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| + 2\alpha_n M^2 \\
&\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\
&\quad +2\gamma(1 - \beta_n)\|A(y_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| + 2\alpha_n M^2.
\end{aligned}$$

From the condition (C1), (C2), (3.18), and (3.20), we obtain that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.22}$$

Observe that

$$\begin{aligned}
\|W_n y_n - x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - W_n y_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n \xi u + \beta_n x_n + [(1-\beta_n)I - \alpha_n D]W_n y_n - W_n y_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n \xi u + \beta_n x_n + W_n y_n - \beta_n W_n y_n - \alpha_n D W_n y_n - W_n y_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n(\xi u - D W_n y_n) + \beta_n(x_n - W_n y_n)\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n \|\xi u - D W_n y_n\| + \beta_n \|x_n - W_n y_n\|.
\end{aligned}$$

This implies that

$$(1-\beta_n)\|W_n y_n - x_n\| \leq \|x_n - x_{n+1}\| + \alpha_n \|\xi u - D W_n y_n\|.$$

From the conditions (C1), (C2), and (3.18), we get

$$\lim_{n\to\infty} \|W_n y_n - x_n\| = 0. \tag{3.23}$$

Note that

$$\|W_n y_n - y_n\| \leq \|W_n y_n - x_n\| + \|x_n - y_n\|,$$

from (3.22) and (3.23), we get

$$\lim_{n\to\infty} \|W_n y_n - y_n\| = 0. \tag{3.24}$$

Step 4. We will show that $\limsup_{n\to\infty}\langle \xi u - Dz, x_n - z\rangle \leq 0$, where $z = P_\Omega[(I - D)z + \xi u]$ or $z = P_\Omega(z - Dz + \xi u)$.

To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \xi u - Dz, x_n - z\rangle = \lim_{i\to\infty}\langle \xi u - Dz, x_{n_i} - z\rangle. \tag{3.25}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. From (3.22), we also see that $y_{n_i} \rightharpoonup w$.

Next, we will show that $w \in \Omega$.

Step 4.1 We will show that $w \in \cap_{i=1}^{\infty} F(S_i) = F(W)$.

Suppose to the contrary that, $w \notin F(W)$, i.e., $Ww \neq w$ and by Lemma 2.8, we see that

$$\begin{aligned}\liminf_{i\to\infty} \|y_{n_i} - w\| &< \liminf_{i\to\infty} \|y_{n_i} - Ww\| \\ &\leq \liminf_{i\to\infty} \{\|y_{n_i} - Wy_{n_i}\| + \|Wy_{n_i} - Ww\|\} \\ &\leq \liminf_{i\to\infty} \{\|y_{n_i} - Wy_{n_i}\| + \|y_{n_i} - w\|\}. \end{aligned} \tag{3.26}$$

On the other hand, we have

$$\|Wy_n - y_n\| \leq \|Wy_n - W_n y_n\| + \|W_n y_n - y_n\| \leq \sup_{x\in C} \|Wx - W_n x\| + \|W_n y_n - y_n\|.$$

By using Lemma 2.5 and (3.24), we obtain that $\lim_{n\longrightarrow\infty} \|Wy_n - y_n\| = 0$, which combines with (3.26) yields that

$$\liminf_{i\to\infty} \|y_{n_i} - w\| < \liminf_{i\to\infty} \|y_{n_i} - w\|.$$

Which is a contradiction, so we have $w \in F(W) = \cap_{i=1}^{\infty} F(S_i)$.

Step 4.2 We will show that $w \in \Gamma$. Note that $y_{n_i} = J_\lambda^{B_1}(x_{n_i} + \gamma A^*(J_\lambda^{B_2} - I)Ax_{n_i})$ can be rewritten as

$$\frac{(x_{n_i} - y_{n_i}) + \gamma A^*(J_\lambda^{B_2} - I)Ax_{n_i}}{\lambda} \in B_1 y_{n_i}. \tag{3.27}$$

By passing to limit $i \to \infty$ in (3.27) and by taking into account (3.20) and (3.22) and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(w)$, i.e., $w \in \text{SOLVIP}(B_1)$. Furthermore, since $\{x_n\}$ and $\{y_n\}$ have the same asymptotical behavior, $\{Ax_{n_i}\}$ weakly converges to Aw. Again, by (3.20) and the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive and Lemma 2.9, we obtain that $Aw \in B_2(Aw)$, i.e., $Aw \in \text{SOLVIP}(B_2)$. Thus, $w \in \Omega$.

Since $z = P_\Omega(z - Dz + \xi u)$ and $w \in \Omega$, by (3.25) and the property of metric projection, we get that

$$\begin{aligned}\limsup_{n\to\infty} \langle \xi u - Dz, x_n - z\rangle &= \lim_{i\to\infty} \langle \xi u - Dz, x_{n_i} - z\rangle \\ &= \langle \xi u - Dz, w - z\rangle \\ &= \langle (z - Dz + \xi u) - z, w - z\rangle \\ &\leq 0. \end{aligned} \tag{3.28}$$

Step 5. Finally, we will show that $x_n \to z$, as $n \to \infty$. By the inequality (3.4), we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\alpha_n\xi u+\beta_n x_n+[(1-\beta_n)I-\alpha_n D]W_n y_n - z\|^2\\
&= \langle \alpha_n\xi u+\beta_n x_n+[(1-\beta_n)I-\alpha_n D]W_n y_n - z, x_{n+1}-z\rangle\\
&= \alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + \beta_n\langle x_n - z, x_{n+1}-z\rangle\\
&\quad +\langle((1-\beta_n)I-\alpha_n D)(W_n y_n - z), x_{n+1}-z\rangle\\
&\le \alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + \beta_n\langle x_n - z, x_{n+1}-z\rangle\\
&\quad +\|((1-\beta_n)I-\alpha_n D)(W_n y_n - z)\|\|x_{n+1}-z\|\\
&\le \alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + \beta_n\langle x_n - z, x_{n+1}-z\rangle\\
&\quad +(1-\beta_n-\alpha_n\bar{\xi})\|W_n y_n - z\|\|x_{n+1}-z\|\\
&\le \alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + \frac{1}{2}\beta_n(\|x_n-z\|^2+\|x_{n+1}-z\|^2)\\
&\quad +(1-\beta_n-\alpha_n\bar{\xi})\|y_n - z\|\|x_{n+1}-z\|\\
&\le \alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + \frac{1}{2}\beta_n(\|x_n-z\|^2+\|x_{n+1}-z\|^2)\\
&\quad +(1-\beta_n-\alpha_n\bar{\xi})\|x_n - z\|\|x_{n+1}-z\|\\
&\le \alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + \frac{1}{2}\beta_n(\|x_n-z\|^2+\|x_{n+1}-z\|^2)\\
&\quad +\frac{1}{2}(1-\beta_n-\alpha_n\bar{\xi})(\|x_n-z\|^2+\|x_{n+1}-z\|^2)\\
&= \alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + \frac{1}{2}(1-\alpha_n\bar{\xi})(\|x_n-z\|^2+\|x_{n+1}-z\|^2).
\end{aligned}$$

This implies that

$$\begin{aligned}
2\|x_{n+1}-z\|^2 &\le 2\alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + (1-\alpha_n\bar{\xi})(\|x_n-z\|^2+\|x_{n+1}-z\|^2)\\
&= 2\alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + (1-\alpha_n\bar{\xi})\|x_n-z\|^2+(1-\alpha_n\bar{\xi})\|x_{n+1}-z\|^2\\
&\le 2\alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + (1-\alpha_n\bar{\xi})\|x_n-z\|^2+\|x_{n+1}-z\|^2,
\end{aligned}$$

and so we have

$$\|x_{n+1}-z\|^2 \le 2\alpha_n\langle \xi u - Dz, x_{n+1}-z\rangle + (1-\alpha_n\bar{\xi})\|x_n-z\|^2.$$

From the condition (C2), (3.28), and Lemma 2.10, we see that $\lim_{n\longrightarrow\infty}\|x_n - z\| = 0$. This completes the proof. □

4 Corollaries

Corollary 4.1 *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Assume that $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are maximal monotone mappings. Let $\{S_i\}_{i=1}^{\infty}$ be an infinitely family of nonexpansive mappings from H_1 into itself. Assume that $\Omega := (\bigcap_{i=1}^{\infty} Fix(S_i)) \cap \Gamma \neq \emptyset$. Let $x_1 = u \in H_1$ arbitrarily, let the sequences $\{y_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J_{\lambda}^{B_1}(x_n + \gamma A^*(J_{\lambda}^{B_2} - I)Ax_n);\\ x_{n+1} = \alpha_n u + \beta_n x_n + (1-\beta_n-\alpha_n)W_n y_n, \forall n \ge 1, \end{cases} \tag{4.1}$$

where $\lambda > 0$ *and* $\gamma \in (0, \frac{1}{L})$, *L is the spectral radius of the operator* A^*A *and* A^* *is the adjoint of A,* $\{W_n\}$ *is the sequence defined by (2.8),* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0, 1]$. *Suppose the control consequences satisfy the following conditions:*

(C1) $0 < a \leq \beta_n \leq b < 1, \forall n \geq 1$,
(C2) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to a point* $z \in \Omega$, *which solves uniquely solution of the variational inequalities*

$$\langle (I - u)z, z - p\rangle \leq 0, \quad \forall p \in \Omega. \tag{4.2}$$

Equivalently, we have $P_\Omega u = z$.

Proof Taking $\xi = 1$ and $D = I$ in Theorem 3.1, then the conclusion of Corollary 4.1 is obtained. □

Corollary 4.2 *Let* H_1 *and* H_2 *be two real Hilbert spaces and* $A : H_1 \to H_2$ *be a bounded linear operator. Assume that* $B_1 : H_1 \to 2^{H_1}$ *and* $B_2 : H_2 \to 2^{H_2}$ *are maximal monotone mappings. Let* $S : C \to C$ *be a nonexpansive mapping. Assume that* $\Omega := Fix(S) \cap \Gamma \neq \emptyset$. *Let* $x_1 = u \in C$ *and the sequences* $\{y_n\}$ *and* $\{x_n\}$ *be generated by*

$$\begin{cases} y_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n); \\ x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \beta_n - \alpha_n)Sy_n, \forall n \geq 1, \end{cases} \tag{4.3}$$

where $\lambda > 0$ *and* $\gamma \in (0, \frac{1}{L})$, *L is the spectral radius of the operator* A^*A *and* A^* *is the adjoint of A. If the control consequences satisfying the following:*

(C1) $0 < a \leq \beta_n \leq b < 1, \forall n \geq 1$,
(C2) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then, $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to a point* $z \in \Omega$, *which solves uniquely solution of the variational inequalities*

$$\langle (I - u)z, z - p\rangle \leq 0, \quad \forall p \in \Omega. \tag{4.4}$$

Equivalently, we have $P_\Omega u = z$.

Proof Taking $\xi = D = 1$ and $S_n = S$ for all $n \geq 1$ in Theorem 3.1, then the conclusion of Corollary 4.1 is obtained. □

5 Numerical examples

In this section, let us present the following common fixed-point optimization algorithm by using $W-$mapping and discuss some examples to verify the theoretical results.

Algorithm 5.1 (Common fixed-point optimization algorithm by using $W-$mapping)
Step 1. Choose the initial point $x_1 \in H_1$, the parameters $\lambda > 0$ and $\gamma \in (0, \frac{1}{L})$ and $0 < \xi < 1$ arbitrarily real numbers. Fixed the element $u \in H_1$ and let $n = 1$.
Step 2. Given $x_n \in H_1$ and compute $x_{n+1} \in H_1$ as follows;

$$
\begin{aligned}
y_n &= J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) \\
W_n &= U_{n,1}, \quad \text{where} \begin{cases} U_{n,1} = \zeta_1 S_1 U_{n,2} + (1-\zeta_1)I, \\ U_{n,2} = \zeta_2 S_2 U_{n,3} + (1-\zeta_2)I, \\ \vdots \\ U_{n,n} = \zeta_n S_n U_{n,n+1} + (1-\zeta_n)I, \\ U_{n,n+1} = I, \end{cases} \\
x_{n+1} &= \alpha_n \xi u + \beta_n x_n + (1-\beta_n-\alpha_n)W_n y_n.
\end{aligned}
\tag{5.1}
$$

Step 3. Put $n := n+1$ and go to step 2.

Example 5.1 For $n \geq 1$, let $W_n : \mathbb{R} \longrightarrow \mathbb{R}$ be a mapping generated by an infinite family of nonexpansive mapping $\{\frac{1}{2^n}\}$ and a nonnegative real sequence $\{\frac{n}{n+1}\}$. Define three operators A, B_1, and B_2 on a real line by $Ax = 3x, \quad B_1x = 2x, \quad and \quad B_2x = \frac{3}{4}x$ for all $x \in \mathbb{R}$. In this example, we set the parameters on algorithm (5.1) by $\xi = 0.5, \ D = 1, \ \alpha_n = \frac{10^{-3}}{n}$, and $\beta_n = 0.5 - \frac{1}{10n+2}$ for all $n \in \mathbb{N}$ and fix the element $u = 5$.

First, we take $\lambda = 0.99$, $\gamma = 0.5$, and three initial points randomly generated by Matlab. In this way, Fig. 1 indicates the behavior of x_n for algorithm (5.1) that converges to the same solution, i.e., $0 \in (\cap_{i=1}^{\infty} Fix(S_i)) \cap \Gamma$ as a solution of this example. Next, we test the effect of the parameter γ to rate of convergence by choosing $\gamma = 0.01, \ 0.3$, and 0.7 where the initial point $x_1 = 30$ and the parameter $\lambda = 0.99$ are fixed. In this test, it shows by Fig. 2a. Finally, we fixed the initial point $x_1 = 40$ and the parameter $\gamma = 0.1$ and choosing the different parameters $\lambda = 0.99, \ 0.5,$

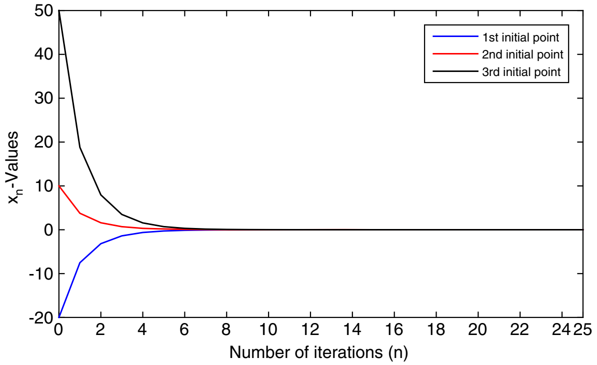
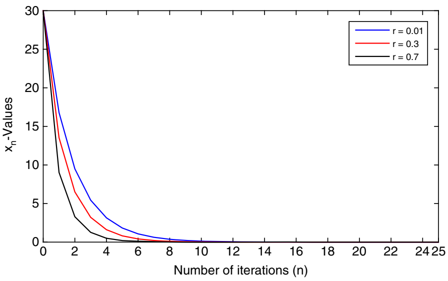
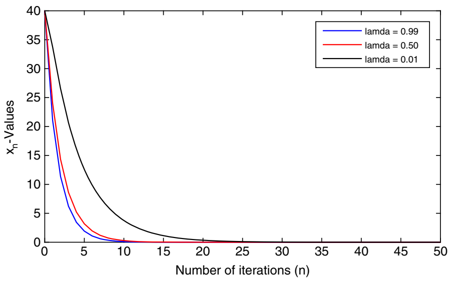


Fig. 1 Behavior of x_n for the different initial points



(a) the different parameters γ



(b) the different parameters λ

Fig. 2 Behavior of x_n for the different parameters γ and λ

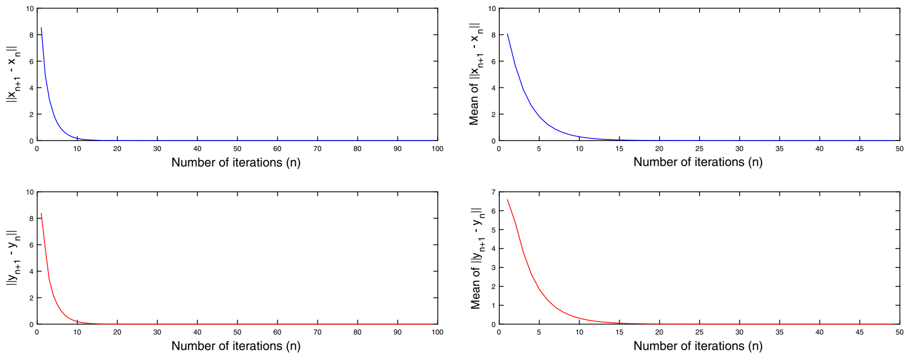
and 0.01. Figure 2b indicates the behavior of x_n generated by algorithm (5.1) with $\lambda = 0.01$ decreases slowly.

Example 5.2 Define an infinite family of nonexpansive mapping $S_n : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $S_n = \{\frac{1}{2n}\}$ and a nonnegative real sequence $\zeta_n = \{\frac{n}{n+1}\}$ for all $n \in \mathbb{N}$. Let W_n be a mapping generated by $\{S_n\}$ and $\{\zeta_n\}$. Setting $\gamma = 0.01$, $\lambda = 0.09$, $\xi = 0.2$, $D = I$, $A = \begin{bmatrix} 6 & 3 & 1 \\ 8 & 7 & 5 \\ 3 & 6 & 2 \end{bmatrix}$, $B_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$, $\alpha_n = \frac{10^{-3}}{n}$, and $\beta_n = 0.5 - \frac{1}{10n+2}$ for all $n \in \mathbb{N}$.

Firstly, the experiment used random vector u in $\mathbb{R}^3$ and fixed initial vector $x_1 = (13, -12, 25)$. Using algorithm (5.1), the test results are reported in Table 1 and the size of the increment of $\{x_n\}$ and $\{y_n\}$ are presented in Fig. 3a. It's easy to see that $(0, 0, 0) \in (\cap_{i=1}^{\infty} Fix(S_i)) \cap \Gamma$ is a solution of this experiment.

Table 1 The convergence of sequences $\{x_n\}$ and $\{y_n\}$

n	x_n	y_n	$\|x_n\|$	$\|y_n\|$
1	(13, −12, 25)	(5.1998, −11.3381, 20.3501)	30.6268	23.8687
2	(7.4737, −12.7600, 16.8046)	(2.9885, −6.8856, 13.6536)	22.3846	15.5809
3	(4.1953, −7.5669, 10.9574)	(1.6769, −4.0810, 8.8892)	13.9615	9.9239
4	(2.3823, −4.5946, 7.4465)	(0.9514, −2.4760, 6.0316)	9.0684	6.5891
5	(1.3411, −2.7814, 5.0707)	(0.5349, −1.4974, 4.1016)	5.9369	4.3990
⋮	⋮	⋮	⋮	⋮
20	(0.0000, 0.0003, 0.0078)	(0.0000, 0.0001, 0.0094)	0.0078	0.0094
25	(0.0001, 0.0002, 0.0011)	(0.0000, 0.0001, 0.0013)	0.0011	0.0013
30	(0.0001, 0.0001, 0.0003)	(0.0001, 0.0001, 0.0003)	3.3166e-04	3.3166e-04
35	(0.0000, 0.0000, 0.0000)	(0.0000, 0.0000, 0.0000)	0.0000	0.0000



(a) The values of $\|x_{n+1} - x_n\|$ and $\|y_{n+1} - y_n\|$ (b) The values of $MS(x_n)$ and $MS(y_n)$

Fig. 3 The size of the increment of $\{x_n\}$ and $\{y_n\}$

Secondly, we suppose that $\{x_n(j)\}_{j=1}^m$ and $\{y_n(j)\}_{j=1}^m$ are the sequences generated by $\{x_n\}$ and $\{y_n\}$ in algorithm (5.1), respectively. We performed 50 sampling ($m = 50$ different random initial points) and averaged their size of the increment by using 2-norm. Define the mean size of the increment of $\{x_n(j)\}_{j=1}^m$ and $\{y_n(j)\}_{j=1}^m$ by

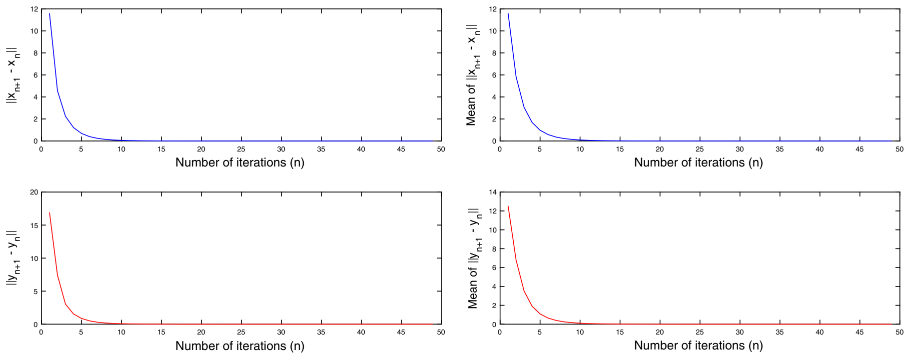
$$MS(x_n) := \frac{1}{m}\sum_{j=1}^{m} \|x_{n+1}(j) - x_n(j)\| \ \text{ and } \ MS(y_n) := \frac{1}{m}\sum_{j=1}^{m} \|y_{n+1}(j) - y_n(j)\|.$$

Figure 3b shows that the mean size of the increment of $\{x_n\}$ and $\{y_n\}$ converge to 0 which imply that $\{x_n\}$ and $\{y_n\}$ converge to a solution.

Example 5.3 In this example, we replace an infinite family of nonexpansive mapping S_n in Example 5.2 by

$$S_n = \begin{bmatrix} 1/n & 0 & 0 \\ 0 & 1/2^n & 0 \\ 0 & 0 & 1/2^{n+1} \end{bmatrix}$$

and others are still the same.



(a) The values of $\|x_{n+1} - x_n\|$ and $\|y_{n+1} - y_n\|$ (b) The values of $MS(x_n)$ and $MS(y_n)$

Fig. 4 The size of the increment of $\{x_n\}$ and $\{y_n\}$

Figure 4 plots the behavior of sequences of $\|x_{n+1} - x_n\|$, $\|y_{n+1} - y_n\|$, $MS(x_n)$ and $MS(y_n)$ that converge to 0. This concludes that $\{x_n\}$ and $\{y_n\}$ converge to a solution.

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