



รายงานวิจัยฉบับสมบูรณ์

โครงการ: การค้นหาซิงกูลาริตี้โดยใช้ลำดับของฟังก์ชันของ
การประมาณค่าแบบเอร์มิต-พาร์เตย์, การประมาณค่าแบบ
พาร์เตย์-พาร์เบอร์ในรูปแบบเวกเตอร์, และ การประมาณค่า
แบบพาร์เตย์แอลฟาเบต้าในรูปแบบเวกเตอร์

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและต้น
สังกัด

(ความเห็นในรายงานนี้เป็นของผู้วิจัย
สกว.และต้นสังกัดไม่จำเป็นต้องเห็นด้วยเสมอไป)

Abstract (บทคัดย่อ)

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Project Title: Determining singularities using row sequences of Hermite-Padé approximants, simultaneous Padé-Faber approximants, and simultaneous Padé (α, β) -approximants

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โครงการวิจัยโครงการนี้ประกอบด้วยสองส่วน

ในส่วนแรกจะเกี่ยวข้องกับผลลัพธ์โดยตรงและผกผันของบทขยายของการประมาณค่าแบบพาร์เตย์บนลำดับแบบแถว ทฤษฎีของการลู่เข้าของแถวลำดับของการประมาณค่าแบบพาร์เตย์-พาร์เบอร์ ในรูปแบบเวกเตอร์และการประมาณค่าแบบมัลติพอยท์ เอร์มิต-พาร์เตย์ ได้ถูกพิสูจน์ นอกจากนั้นเราได้พิสูจน์ว่าโพลของลำดับของเวกเตอร์ของฟังก์ชันตรรกยะดังกล่าวลู่เข้าไปยังโพลระบบของเวกเตอร์ของฟังก์ชันที่ถูกประมาณค่าด้วยอัตราเร็วเรขาคณิต

ในส่วนที่สองเกี่ยวกับการศึกษาตำแหน่งของจุด N จุดที่ทำให้ฟังก์ชันศักยะแบบริสซ์มีค่าคงที่บนวงกลมในระนาบยูคลิด $\mathbb{R}^2$ และได้นำผลลัพธ์ดังกล่าวไปใช้ในการแก้ปัญหาค่าเหมาะสมโพลาริเซชัน

This project contains two parts. The first part concerns direct and inverse results on row sequences of generalized Padé approximations. Convergence theorems for these approximants were proved. In particular, we gave necessary and sufficient conditions for the convergence with geometric rate of the denominators of simultaneous Padé-Faber approximants and multipoint Hermite-Padé approximants (multipoint Hermite-Padé approximation is a special case of simultaneous Padé (α, β) -approximation). The second part focuses on generalized Riesz potentials on the Euclidean space $\mathbb{R}^2$. We gave a characterization for a generalized Riesz s -potential function of a multiset of N points in $\mathbb{R}^2$

when $s = 2 - 2N$ and the potential function is constant on a circle in $\mathbb{R}^2$. The characterization allows us to partially prove a conjecture of Nikolov and Rafailov about the relation between the potential functions being constant on a circle in $\mathbb{R}^2$ and locations of N points. Moreover, we find all maximal and minimal polarization constants and configurations of two concentric circles in $\mathbb{R}^2$ using the above the generalized potential function for certain values of s .

Keywords: Montessus de Ballore's Theorem; Padé-Faber approximation; Simultaneous approximation; Multipoint Padé approximation; Inverse problems; Polarization; Potentials; Roots of unity; Max-min and min-max problems.

Executive Summary

Our project consists of two parts:

(i) Generalized Padé Approximations

(i.1) Vector Cases

(i.1.1) We proved two convergence theorems for classical simultaneous Padé-Faber approximants on row sequences in (2) and (3).

(i.1.2) Analogues of the Montessus de Ballore-Gonchar theorem for new simultaneous Padé-Faber approximants and multipoint Hermite-Padé approximants were obtained in (6) and (7). These are the main results of this project.

(i.1.3) Convergences of (classical and new) simultaneous Padé-Faber approximants and (classical and new) simultaneous orthogonal Padé approximants in Hausdorff content in the maximal canonical domain in which each function of an approximated vector can be continued to a meromorphic function were proved in (8).

(i.2) Scalar Cases

(i.2.1) We showed in (4) that each pole of the approximated function in the maximal canonical domain in which the approximated function can be extended as a meromorphic function with at most m poles attracts as many poles of new orthogonal Padé and new Padé-Faber approximants on the m row sequences as its order at a geometric rate.

(i.2.2) In (4), we also proved that if all m poles of new orthogonal Padé or new Padé-Faber approximants on the m row sequences converge to m limit points, then m limit points are singularities of the approximated function.

(i.2.3) We proved in (5) the convergence of new Padé-Faber approximants in Hausdorff content in the maximal canonical domain in which the approximated function can be extended to a meromorphic function. This is a special case of (i.1.3).

(ii) Constant Riesz Potentials and Polarization Optimality Problems

(ii.1) A characterization for a Riesz s -potential function of a multiset ω_N of N points in $\mathbb{R}^2$ was given when $s = 2 - 2N$ and the potential function is constant on a circle in $\mathbb{R}^2$. The characterization allows us to partially prove a conjecture of Nikolov and Rafailov, namely if the potential function is constant on a circle Γ , then the points in ω_N should be equally spaced on a circle concentric to Γ . As an application of constant Riesz s -potential functions, we also located all maximal and minimal polarization constants and configurations of two concentric circles in $\mathbb{R}^2$ for certain values of s . All results here were proved in (1).

(ii.2) The results in (9) are similar to the results in (1). In (9), we considered more general Riesz potential functions.

Here is the list of 7 accepted papers.

- (1) **N. Bosuwan** and P. Ruengrot, Constant Riesz potential on a circle in a plane with an application to polarization optimality problems, *ScienceAsia*, 43(4) (2017), 267-274. (Impact Factor 2017: 0.447)
- (2) **N. Bosuwan**, Convergence of row sequences of simultaneous Padé-Faber approximants, *Math. Notes*, 103(5) (2018), 643-656. (Impact Factor 2017: 0.577)
- (3) **N. Bosuwan**, On Montessus de Ballore's theorem for simultaneous Padé-Faber approximants, *Demonstr. Math.*, 51 (2018), 45-61. (SJR 2017: Q3)

- (4) **N. Bosuwan**, Direct and inverse results on row sequences of generalized Padé approximants to polynomial expansions, *Acta Math. Hungar.*, 157(1) (2019), 191–219. (Impact Factor 2017: 0.481)
- (5) W. Chonlapap and **N. Bosuwan**, Convergence in Hausdorff content of Padé-Faber approximants and its applications, *Thai J. Math.*, 17 (2019), 272-287 (SJR 2017: Q3)
- (6) **N. Bosuwan** and G. López Lagomasino, Direct and inverse results on row sequences of simultaneous Padé-Faber approximants, *Mediterr. J. Math.* 16(36) (2019) <https://doi.org/10.1007/s00009-019-1307-0>. (Impact Factor 2017: 1.000)
- (7) **N. Bosuwan**, G. López Lagomasino, and Y. Zaldivar Gerpe, Direct and inverse results for multipoint Hermite-Padé approximants, *Anal. Math. Phys.*, accepted. (Impact Factor 2017: 1.381)

Here is the list of 3 submitted papers. We are still waiting for the results.

- (8) M. Wajasat and **N. Bosuwan**, Convergences in Hausdorff content of generalized Hermite-Padé approximants to polynomial expansions, submitted.
- (9) **N. Bosuwan**, Constant generalized Riesz potential functions and polarization optimality problems, submitted.
- (10) **N. Bosuwan**, On the boundedness of poles of generalized Padé approximants, submitted.

Our works were presented at both national and international conferences. Here is the list of topics and conference locations.

- Constant Riesz Potentials on a circle in a plane and symmetric equations, The 13th Conference of Young Algebraists in Thailand, December 6-9, 2017, Nakhon Pathom, Thailand.
- Convergence of new Padé-Faber approximants, The 23rd Annual Meeting in Mathematics 2018, May 3-5, 2018, KMUTT, Bangkok, Thailand.
- Convergence of classical Padé-Faber approximants, Annual Pure and Applied Mathematics Conference 2018, May 30-June 1, 2018, Chulalongkorn University, Bangkok, Thailand.
- Convergences of row sequences of simultaneous Padé-Faber approximants, International Conference on Complex Analysis, Potential Theory, and Applications, June 11-15, 2018, University College of Dublin, Dublin, Ireland.
- Convergences in Hausdorff content of generalized Hermite-Padé approximants to polynomial expansions, XIX Conference on Analytic Functions and Related Topics, June 25-29, 2018, University of Rzeszow, Poland.
- Inverse results on row sequences of multipoint Padé approximants, International Conference in Mathematics and Applications 2018, December 16-18, 2018, Bangkok, Thailand.

Under this project, we produced two undergraduate students and one master student. Here is the list of my students.

- Nitipon Chattrakul, Bachelor's Degree in Mathematics
- Waraporn Chonlapap, Bachelor's Degree in Mathematics
- Methawee Wajasat, Master's Degree in Mathematics

1. บทนำ (Introduction)

เนื้อหาของเรื่องที่เคยมีผู้ทำการวิจัยมาก่อน ความสำคัญ ที่มาของปัญหา

1.1. Pade Approximations

Rational approximation theory has been a mainstay of approximation theory from the beginning. This subject occupies a large place in the literature. Especially, Padé approximation, one type of rational approximations, can be classified as an independent branch of complex analysis and approximation theory. Although polynomials seem to be more familiar and comfortable, they are not such a good class of functions if one wants to approximate functions with singularities because polynomials are entire functions without singularities. Rational functions are the simplest functions with singularities.

The concept of (classical) Padé approximants generalizes the idea of Taylor polynomials to rational approximants. Given a formal Taylor series at the origin

$$F(z) = \sum_{k=0}^{\infty} f_k z^k,$$

for any integers $n, m \geq 0$, we can find polynomials $P_{n,m} \in \mathbb{P}_n$ and polynomials $Q_{n,m} \in \mathbb{P}_m$, $Q_{n,m} \neq 0$, such that

$$(Q_{n,m}F - P_{n,m})(z) = \mathcal{O}(z^{n+m+1}), \quad \text{as } z \rightarrow 0 \quad (1)$$

($\mathbb{P}_n$ is the set of all polynomials of degree at most n). The rational function

$$[n/m]_F \stackrel{\text{def}}{=} \frac{P_{n,m}}{Q_{n,m}}$$

is uniquely defined and is called the *Padé approximant of type (n, m) to F* .

Padé approximants are more powerful than polynomials at approximating functions *near singularities, with jumps, and on unbounded domains* (see, e.g., [1, 2, 3, 4, 5]). The idea is that under appropriate conditions, the poles of the classical Padé approximant $[n/m]_F$ will move to the singularities of the function F (as n and m increase). Hence, the domain of convergence could be enlarged and this allows classical Padé approximants to approximate functions beyond their singularities. Padé approximation and its generalizations have applications not only in approximation theory itself but also in many other areas such as Numerical Analysis, Number Theory, Integral Equations, Spectral Theory of Operators, Random Matrix Theory, Quantum Mechanics, Quantum Field Theory, Brownian Motion, Toda lattices, Fluid Dynamics, Volatility Modelling, Multiple Orthogonal Polynomials, Quadrature Formulas, Control and Identification of Linear Systems, Inverse Problems for the Laplacian, and so on (see e.g., [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]).

One of the goals of this project is to investigate the relation between the convergence of the zeros of the common denominators of generalized Hermite-Padé approximants on row sequences, and the singularities of a vector of approximated functions. To clearly state known results and our goals, let us define Hermite-Padé approximation.

Let $\mathbf{F} = (F_1, \dots, F_d)$ be a system of d formal Taylor expansions at the origin; that is, for each $i = 1, \dots, d$, we have

$$F_i(z) = \sum_{n=0}^{\infty} f_{n,i} z^n, \quad f_{n,i} \in \mathbb{C}. \quad (2)$$

Definition 1. Given a system of d formal Taylor expansions $\mathbf{F} = (F_1, \dots, F_d)$ as in (2) and a multi-index $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$. Set $|\mathbf{m}| = m_1 + m_2 + \dots + m_d$. Then, for each

$n \geq \max\{m_1, \dots, m_d\}$, there exist polynomials $Q, P_i, i = 1, \dots, d$, such that

$$\deg(P_i) \leq n - m_i, \quad i = 1, \dots, d, \quad \deg(Q) \leq |\mathbf{m}|, \quad Q \neq 0,$$

$$Q(z)F_i(z) - P_i(z) = A_i z^{n+1} + \dots.$$

The vector rational function $\mathbf{R}_{n,\mathbf{m}} = (P_1/Q, \dots, P_d/Q)$ is called an $(n, \mathbf{m})$ (type II) Hermite-Padé approximant of $\mathbf{F}$.

Therefore, Hermite-Padé approximation is an approximation of each function F_i separately by rational functions with a common denominator Q . For any fixed $(n, \mathbf{m}) \in \mathbb{N} \times \mathbb{N}^d$, in general, $\mathbf{R}_{n,\mathbf{m}}$, may not be unique and we will assume that given $(n, \mathbf{m})$, one particular solution is taken. For that solution, we write

$$\mathbf{R}_{n,\mathbf{m}} = (R_{n,\mathbf{m},1}, \dots, R_{n,\mathbf{m},d}) = (P_{n,\mathbf{m},1}, \dots, P_{n,\mathbf{m},d})/Q_{n,\mathbf{m}},$$

where $Q_{n,\mathbf{m}}$ is a monic polynomial that has no common zero with all the $P_{n,\mathbf{m},i}$. Sequences $\{\mathbf{R}_{n,\mathbf{m}}\}_{n \geq 1}$ when $\mathbf{m}$ is fixed are called *row sequences* and sequences $\{\mathbf{R}_{n,\mathbf{m}}\}_{n \geq 1}$ when $m_1 = \dots = m_d = m, n = (d+1)m, m \in \mathbb{N}$ (or nearby configurations of multi-indices) are called *diagonal sequences*. We would like to emphasize that there is another construction called type I Hermite-Padé approximants which has very close relation to the type II Hermite-Padé approximants (see [19, Chap. 4] for more details). In this work, we studied just the type II Hermite-Padé approximants. For convenience, we will omit the word “type II” when we refer to the type II Hermite-Padé approximants.

Most of the studies of Hermite-Padé approximants were concentrated on diagonal sequences (for more information, see some important papers [27, 28, 29, 30] and a book by E.M. Nikishin [19] in this direction). There are very few papers [31, 32, 34, 35, 36]

dedicated to the study of row sequences. The pioneering one in this direction is the work of Graves-Morris and Saff [31] where they proved an analogue of the Montessus de Ballore theorem. The other significant work in this direction is due to Cacoq, de la Calle, and López [32] where they proved some results on the inverse problem of row sequences.

Hermite-Padé approximation and their relatives have various applications in many areas, for example, in Number Theory (see [37, 38, 39]), in Numerical Analysis (see [13, 40, 41, 42, 43, 44, 45, 46, 50]), in Multiple Orthogonal Polynomials (see [13, 17, 23, 47, 48]), in Linear Algebraic Equations (see [49]), in Nonlinear Dynamical Systems (see [7]), in Brownian Motion (see [12]), in Random matrices (see [9, 17, 18]), and so on. In particular, Charles Hermite introduced the subject of Hermite-Padé approximation in his proof of the transcendence of e . Hermite-Padé approximants were also used in various irrationality proofs and transcendence proofs of important numbers (see the discussions in [24, Section 3.2] and [51, Section 4.3] for more details). Moreover, one can say that the origin of the definition of multiple orthogonal polynomials came from Hermite-Padé approximants (see for example in [24, Section 2.2] for the explanation). See also [51] for the applications of multiple orthogonal polynomials.

Let us continue introducing a definition and notations. The following definition is an intrinsic definition of pole when we study a system of functions.

Definition 2. Given $\mathbf{F} = (F_1, F_2, \dots, F_d)$ as in (2) and $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$, we say that $\xi \in \mathbb{C} \setminus \{0\}$ is a *system pole of order τ of $\mathbf{F}$ with respect to $\mathbf{m}$* if τ is the largest positive integer such that for each $t = 1, 2, \dots, \tau$, there exists at least one polynomial combination of the form

$$\sum_{i=1}^d v_i F_i, \quad \deg v_i < m_i, \quad i = 1, 2, \dots, d, \quad (3)$$

which is holomorphic on a neighborhood of $\{z \in \mathbb{C} : |z| \leq |\xi|\}$ except for a pole at $z = \xi$ of exact order t .

Let ξ be a system pole of order τ of $\mathbf{F}$ with respect to $\mathbf{m}$. Let $\mathcal{G}(\mathbf{F}, \mathbf{m}, \xi, k)$ be the space of all functions g of type (3) that are analytic on a set $\{z \in \mathbb{C} : |z| \leq |\xi|\}$ except for a pole at $z = \xi$ of order k . For each $k = 1, \dots, \tau$, we define

$$r_{\xi,k}(\mathbf{F}, \mathbf{m}) := \min\{R_k(g) : g \in \mathcal{G}(\mathbf{F}, \mathbf{m}, \xi, k)\},$$

where $R_k(g)$ is the radius of the largest disk centered at 0 containing at most k poles of g . Set

$$R_\xi(\mathbf{F}, \mathbf{m}) := \min_{k=1, \dots, \tau} r_{\xi,k}(\mathbf{F}, \mathbf{m}).$$

We denote by $Q_{\mathbf{m}}^{\mathbf{F}}$ the monic polynomial whose zeros are the system poles of $\mathbf{F}$ with respect to $\mathbf{m}$ taking account of their order. The set of distinct zeros of $Q_{\mathbf{m}}^{\mathbf{F}}$ is denoted by $\mathcal{P}(\mathbf{F}, \mathbf{m})$. Moreover, we define $R_0(\mathbf{F})$ to be the radius of the largest disk centered at 0 in which all the expansions F_i , $i = 1, \dots, d$ correspond to analytic functions.

The following is the main result in [32].

Theorem 1. *Let $\mathbf{F}$ be a vector of formal Taylor expansions at the origin and fix a multi-index $\mathbf{m} \in \mathbb{N}^d$. The following two assertions are equivalent:*

- (a) $R_0(\mathbf{F}) > 0$ and $\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ counting multiplicities.
- (b) The denominators $Q_{n,\mathbf{m}}$, $n \geq |\mathbf{m}|$, of the Hermite-Padé approximants of $\mathbf{F}$ are uniquely determined for all sufficiently large n , and there exists a polynomial $Q_{|\mathbf{m}|}$

of degree $|\mathbf{m}|$, $Q_{|\mathbf{m}|}(0) \neq 0$, such that

$$\limsup_{n \rightarrow \infty} \|Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}}\|^{1/n} = \theta < 1, \quad (4)$$

where $\|\cdot\|$ denotes the coefficient norm in the space of polynomials.

Moreover, if either (a) or (b) takes place, then $Q_{|\mathbf{m}|} \equiv Q_{\mathbf{m}}^{\mathbf{F}}$ and

$$\theta = \max \left\{ \frac{|\xi|}{R_{\xi}(\mathbf{F}, \mathbf{m})} : \mathcal{P}(\mathbf{F}, \mathbf{m}) \right\}.$$

In [32, Theorem 3.7], the rate of convergence of row sequences of the Hermite-Padé approximants to the system of functions $\mathbf{F}$ is given. Theorem 1 characterizes the situation when $\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ (counting multiplicities) in terms of the exact rate of convergence in (4). The (b) $\Rightarrow$ (a) statement means that if $Q_{n,\mathbf{m}}$ has $|\mathbf{m}|$ zeros for n sufficiently large and all the zeros of $Q_{n,\mathbf{m}}$ converge to $|\mathbf{m}|$ limit points at the rate of a geometric progression, then all the limit points are the system poles of $\mathbf{F}$ with respect to $\mathbf{m}$.

In this project, we studied generalizations of the classical construction of Hermite-Padé approximation, namely classical and new simultaneous Padé-Faber approximations, classical and new simultaneous orthogonal Padé approximations, and multipoint Hermite-Padé approximation. Before we state our interested problems and the significance of this project, we need to introduce some more notation.

Let E be an infinite compact subset of the complex plane $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus E$ is simply connected. Note that we consider $\overline{\mathbb{C}} \setminus E$ as a domain. Then, there exists a unique exterior conformal mapping Φ from $\overline{\mathbb{C}} \setminus E$ onto $\overline{\mathbb{C}} \setminus \{w \in \mathbb{C} : |w| \leq 1\}$ satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. We assume that E is such that the inverse function $\Psi = \Phi^{-1}$

can be extended continuously to $\overline{\mathbb{C}} \setminus \{w \in \mathbb{C} : |w| < 1\}$. Note that the closure of a bounded Jordan region and a finite interval satisfy the above conditions. For the rest of this report, E is as described above.

The first and second approximations are constructed from orthogonal polynomials on E . Let μ be a finite positive Borel measure with infinite support $\text{supp}(\mu)$ contained in E . We write $\mu \in \mathcal{M}(E)$ and define the associated inner product,

$$\langle g, h \rangle_\mu := \int g(\zeta) \overline{h(\zeta)} d\mu(\zeta), \quad g, h \in L_2(\mu).$$

Let

$$p_n(z) := \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, 2, \dots,$$

be the orthonormal polynomial of degree n with respect to μ having positive leading coefficient; that is, $\langle p_n, p_m \rangle_\mu = \delta_{n,m}$. Denote by $\mathcal{H}(E)$ the space of all functions holomorphic in some neighborhood of E . We define

$$\mathcal{H}(E)^d := \{(F_1, F_2, \dots, F_d) : F_i \in \mathcal{H}(E) \text{ for all } i = 1, 2, \dots, d\}$$

and the set of all nonnegative integers is denoted by $\mathbb{N}_0$.

Definition 3. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ and $\mu \in \mathcal{M}(E)$. Fix a multi-index $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector in $\mathbb{N}_0^d$. Set $|\mathbf{m}| = m_1 + m_2 + \dots + m_d$. Then, for each $n \geq \max\{m_1, m_2, \dots, m_d\}$, there exist polynomials $Q_{n,\mathbf{m}}^\mu$ and $P_{n,\mathbf{m},i}^\mu$, $i = 1, 2, \dots, d$ such that

$$\deg(P_{n,\mathbf{m},i}^\mu) \leq n - m_i, \quad \deg(Q_{n,\mathbf{m}}^\mu) \leq |\mathbf{m}|, \quad Q_{n,\mathbf{m}}^\mu \not\equiv 0,$$

$$\langle Q_{n,m}^\mu F_i - P_{n,m,i}^\mu, p_j \rangle_\mu = 0, \quad j = 0, 1, \dots, n,$$

for all $i = 1, 2, \dots, d$. The vector of rational functions

$$\mathbf{R}_{n,m}^\mu := (R_{n,m,1}^\mu, R_{n,m,2}^\mu, \dots, R_{n,m,d}^\mu) = (P_{n,m,1}^\mu/Q_{n,m}^\mu, P_{n,m,2}^\mu/Q_{n,m}^\mu, \dots, P_{n,m,d}^\mu/Q_{n,m}^\mu)$$

is called an $(n, \mathbf{m})$ *classical simultaneous orthogonal Padé approximant of $\mathbf{F}$ corresponding to a measure μ* .

This approximation was first introduced in [63]. In [63], the author proved two convergence theorems for this approximation on row sequences. These two theorems are direct results. It turns out, that the correct way to extend the notion of Hermite-Padé approximation to the case of vector orthogonal expansions in order to obtain both direct and inverse type results is through the following definition.

Definition 4. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ and $\mu \in \mathcal{M}(E)$. Fix a multi-index $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}$. Then, there exist polynomials $\tilde{Q}_{n,m}^\mu, \tilde{P}_{n,m,k,i}^\mu$, $k = 0, 1, \dots, m_i - 1, i = 1, 2, \dots, d$ such that for all $i = 1, 2, \dots, d$,

$$\deg(\tilde{P}_{n,m,k,i}^\mu) \leq n - 1, \quad k = 0, 1, \dots, m_i - 1, \quad \deg(\tilde{Q}_{n,m}^\mu) \leq |\mathbf{m}|, \quad \tilde{Q}_{n,m}^\mu \not\equiv 0, \quad (5)$$

$$\langle \tilde{Q}_{n,m}^\mu z^k F_i - \tilde{P}_{n,m,k,i}^\mu, p_j \rangle_\mu = 0, \quad k = 0, 1, \dots, m_i - 1 \quad j = 0, 1, \dots, n. \quad (6)$$

The vector rational function

$$\tilde{\mathbf{R}}_{n,m}^\mu := (\tilde{R}_{n,m,1}^\mu, \dots, \tilde{R}_{n,m,d}^\mu) = (\tilde{P}_{n,m,0,1}^\mu, \dots, \tilde{P}_{n,m,0,d}^\mu) / \tilde{Q}_{n,m}^\mu$$

is called an $(n, \mathbf{m})$ *new simultaneous orthogonal Padé approximant of $\mathbf{F}$ with respect to μ* .

Because this was recently introduced in [64], there is only one paper studying this approximation. In [64], the authors proved an analogue of (b) $\Leftrightarrow$ (a) in Theorem 1.

Next, let us define both classical and new simultaneous Padé-Faber approximations. In order to do that, we need to state some more definitions.

Let us clarify what we mean by a pole of a vector function and its order.

Definition 5. Let $\Omega := (\Omega_1, \Omega_2, \dots, \Omega_d)$ be a system of domains such that, for each $\alpha = 1, 2, \dots, d$, F_α is meromorphic in Ω_α . We say that the point λ is a pole of $\mathbf{F} := (F_1, F_2, \dots, F_d)$ in Ω of order τ if there exists an index $\alpha \in \{1, 2, \dots, d\}$ such that $\lambda \in \Omega_\alpha$ and it is a pole of F_α of order τ , and for $\beta \neq \alpha$ either λ is a pole of F_β of order less than or equal to τ or $\lambda \notin \Omega_\beta$. When $\Omega = (\Omega, \Omega, \dots, \Omega)$, we say that λ is a pole of $\mathbf{F}$ in Ω .

Denote by $\rho_0(\mathbf{F})$ the index ρ of the largest canonical domain D_ρ to which all F_α , $\alpha = 1, \dots, d$, can be extended as holomorphic functions and by $\rho_m(\mathbf{F})$ the index ρ of the largest canonical domain D_ρ to which all F_α , $\alpha = 1, \dots, d$ can be extended so that $\mathbf{F}$ has at most m poles counting multiplicities.

The *Faber polynomial* of E of degree n is defined as the polynomial part of the Laurent expansion of Φ^n at infinity. The n -th *Faber coefficient* of $G \in \mathcal{H}(E)$ with respect to Φ_n is given by

$$[G]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{G(t)\Phi'(t)}{\Phi^{n+1}(t)} dt,$$

where $\rho \in (1, \rho_0(G))$ and $\rho_0(G)$ denotes the index of the largest canonical region to which G can be extended as a holomorphic function.

The following is the definition of classical simultaneous Padé-Faber approximation.

Definition 6. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. Fix a multi-index $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector in $\mathbb{N}_0^d$. Set $|\mathbf{m}| = m_1 + m_2 + \dots + m_d$. Then, for each

$n \geq \max\{m_1, m_2, \dots, m_d\}$, there exist polynomials $Q_{n,\mathbf{m}}^E$ and $P_{n,\mathbf{m},i}^E$, $i = 1, 2, \dots, d$ such that

$$\deg(P_{n,\mathbf{m},i}^E) \leq n - m_i, \quad \deg(Q_{n,\mathbf{m}}^E) \leq |\mathbf{m}|, \quad Q_{n,\mathbf{m}}^E \neq 0,$$

$$[Q_{n,\mathbf{m}}^E F_i - P_{n,\mathbf{m},i}^E]_j = 0, \quad j = 0, 1, \dots, n,$$

for all $i = 1, 2, \dots, d$. The vector of rational functions

$$\mathbf{R}_{n,\mathbf{m}}^E := (R_{n,\mathbf{m},1}^E, R_{n,\mathbf{m},2}^E, \dots, R_{n,\mathbf{m},d}^E) = (P_{n,\mathbf{m},1}^E/Q_{n,\mathbf{m}}^E, P_{n,\mathbf{m},2}^E/Q_{n,\mathbf{m}}^E, \dots, P_{n,\mathbf{m},d}^E/Q_{n,\mathbf{m}}^E)$$

is called an $(n, \mathbf{m})$ *classical simultaneous Padé-Faber approximant of $\mathbf{F}$ corresponding to E* .

This definition was first introduced in [60] for the scalar case (when $d = 1$). This scalar case was completely studied. We are the first who extended this scalar case to the vector case in [66] and studied the convergence properties of this vector case.

The following is a new simultaneous Padé-Faber approximation. We need this definition in order to prove an analogue of (b) $\Leftrightarrow$ (a) in Theorem 1.

Definition 7. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. Fix a multi-index $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}$. Then, there exist polynomials $\tilde{Q}_{n,\mathbf{m}}^E$, $\tilde{P}_{n,\mathbf{m},k,i}^E$, $k = 0, 1, \dots, m_i - 1$, $i = 1, 2, \dots, d$ such that for all $i = 1, 2, \dots, d$,

$$\deg(\tilde{P}_{n,\mathbf{m},k,i}^E) \leq n - 1, \quad k = 0, 1, \dots, m_i - 1, \quad \deg(\tilde{Q}_{n,\mathbf{m}}^E) \leq |\mathbf{m}|, \quad \tilde{Q}_{n,\mathbf{m}}^E \neq 0, \quad (7)$$

$$[\tilde{Q}_{n,\mathbf{m}}^E z^k F_i - \tilde{P}_{n,\mathbf{m},k,i}^E]_j = 0, \quad k = 0, 1, \dots, m_i - 1 \quad j = 0, 1, \dots, n. \quad (8)$$

The vector rational function

$$\tilde{\mathbf{R}}_{n,\mathbf{m}}^E := (\tilde{R}_{n,\mathbf{m},1}^E, \dots, \tilde{R}_{n,\mathbf{m},d}^E) = (\tilde{P}_{n,\mathbf{m},0,1}^E, \dots, \tilde{P}_{n,\mathbf{m},0,d}^E) / \tilde{Q}_{n,\mathbf{m}}^E$$

is called an $(n, \mathbf{m})$ *new simultaneous Padé-Faber approximant of $\mathbf{F}$ corresponding to E* .

The above definition is a brand new definition. Nobody studied this before (even the scalar case).

The final approximation is multipoint Hermite-Padé approximation defined as follows. Let $\alpha \subset E$ be a table of points; more precisely, $\alpha = \{\alpha_{n,k}\}$, $k = 1, \dots, n$, $n = 1, 2, \dots$

Definition 8. Let $\mathbf{F} \in \mathcal{H}(E)^d$. Fix a multi-index $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}$. Set $|\mathbf{m}| = m_1 + \dots + m_d$. Then, there exist polynomials $Q_{n,\mathbf{m}}^\alpha$, $P_{n,\mathbf{m},k}^\alpha$, $k = 1, \dots, d$ such that

$$\deg P_{n,\mathbf{m},k}^\alpha \leq n - m_k, \quad \deg Q_{n,\mathbf{m}}^\alpha \leq |\mathbf{m}|, \quad Q_{n,\mathbf{m}}^\alpha \not\equiv 0,$$

$$(Q_{n,\mathbf{m}}^\alpha F_k - P_{n,\mathbf{m},k}^\alpha) / a_{n+1} \in \mathcal{H}(E),$$

where $a_n(z) = \prod_{k=1}^n (z - \alpha_{n,k})$. The vector rational function

$$\mathbf{R}_{n,\mathbf{m}}^\alpha = (R_{n,\mathbf{m},1}^\alpha, \dots, R_{n,\mathbf{m},d}^\alpha) = (P_{n,\mathbf{m},1}^\alpha, \dots, P_{n,\mathbf{m},d}^\alpha) / Q_{n,\mathbf{m}}^\alpha$$

is called an $(n, \mathbf{m})$ *multipoint Hermite-Padé approximant of $\mathbf{F}$ with respect to α* .

There are many papers studying multipoint Hermite-Padé approximation when $d = 1$. We are the first who introduced this vector multipoint Padé approximation and studied the convergence of this approximation.

Note that for given $(n, \mathbf{m}) \in \mathbb{N} \times \mathbb{N}^d$, $\mathbf{R}_{n,\mathbf{m}}^\mu$, $\tilde{\mathbf{R}}_{n,\mathbf{m}}^\mu$, $\mathbf{R}_{n,\mathbf{m}}^E$, $\tilde{\mathbf{R}}_{n,\mathbf{m}}^E$, and $\mathbf{R}_{n,\mathbf{m}}^\alpha$ always exist but they may not be unique.

1.2. Constant Riesz Potentials and Polarization Optimality Problems

1.2.1. Constant Riesz Potentials

For a fixed multiset of N points $\omega_N := \{x_1, x_2, \dots, x_N\}$ in $\mathbb{R}^2$ and a given constant $s \in \mathbb{R}$, we define the Riesz potential function $U^s(\cdot; \omega_N) : \mathbb{R}^2 \rightarrow [0, \infty]$ as the following

$$U^s(x; \omega_N) := \sum_{j=1}^N |x - x_j|^{-s},$$

where $x \in \mathbb{R}^2$ and $|\cdot|$ is the 2-dimensional Euclidean norm in $\mathbb{R}^2$. We call $U^s(\cdot; \omega_N)$ a *Riesz s -potential function of ω_N* . We refer the reader to [52] for more information on Riesz s -potential functions in a d -dimensional Euclidean space $\mathbb{R}^d$.

Let ω_N be a fixed set of distinct equally spaced points on a circle $T \subseteq \mathbb{R}^2$ and Γ be a circle concentric to T . In [53], Nikolov and Rafailov showed in Theorem 1 that $U^s(x; \omega_N)$ is constant as a function of x on Γ if and only if $s \in \{0, -2, -4, \dots, 4 - 2N, 2 - 2N\}$. They also showed in Theorem 2 that this actually gives a characteristic property of distinct equally spaced points on a circle. More precisely, given a set ω_N of N distinct points such that $U^s(x; \omega_N)$ is constant on a circle Γ for every $s \in \{-2, -4, \dots, 2 - 2N\}$ (the constant may depend on s), then the points in ω_N are equally spaced on some circle concentric to Γ . In the same paper, it was conjectured (Conjecture 1) that only $s = 2 - 2N$ should be sufficient. We state the conjecture below.

Conjecture 1. *Given a set of N distinct points $\omega_N := \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{R}^2$ such that*

$$U^{2-2N}(x; \omega_N) = \sum_{j=1}^N |x - x_j|^{2N-2}$$

is constant as a function of x on $\mathbb{S}^1$. Then, ω_N forms a set of distinct equally spaced

points on $\mathbb{S}_R^1 := \{x \in \mathbb{R}^2 : |x| = R\}$ for some R .

This conjecture is still open. We are able to solve many special cases of this conjecture.

1.2.2. Constant Generalized Riesz Potentials

After one year studying 1.2.1., we found that Conjecture 1 in 1.2.1. can be generalized by considering more general Riesz potential functions defined as follows. For a fixed multiset of N points $\omega_N := \{x_1, x_2, \dots, x_N\} \subset \mathbb{R}^2$, a given constant $s \in \mathbb{R}$, and a given constant $h \geq 0$, we define the potential function $U^{s,h}(\cdot; \omega_N) : \mathbb{R}^2 \rightarrow [0, \infty]$ as the following:

$$U^{s,h}(x; \omega_N) := \sum_{j=1}^N (|x - x_j|^2 + h)^{-s/2}, \quad (9)$$

where $x \in \mathbb{R}^2$ and $|\cdot|$ is the 2-dimensional Euclidean norm in $\mathbb{R}^2$. We call $U^{s,h}(\cdot, \omega_N)$ a *Riesz (s, h) -potential function of ω_N* . The geometric interpretation of the function $U^{s,h}(\cdot; \omega_N)$ is as follows. Let us consider two parallel planes in $\mathbb{R}^3$: one is $\mathbb{R}^2 \times \{0\}$ and the other is $\mathbb{R}^2 \times \{\sqrt{h}\}$. Basically, the potential function $U^{s,h}(x, \omega_N)$ is the Riesz s -potential function in the 3-dimensional Euclidean space $\mathbb{R}^3$ of $\omega'_N \subset \mathbb{R}^2 \times \{0\}$ at $x' \in \mathbb{R}^2 \times \{\sqrt{h}\}$, where the projection from $\mathbb{R}^2 \times \{0\}$ to $\mathbb{R}^2$ of ω'_N is ω_N and the projection from $\mathbb{R}^2 \times \{\sqrt{h}\}$ to $\mathbb{R}^2$ of x' is x . Moreover, if $h = 0$, then $U^{s,h}(\cdot; \omega_N)$ is the Riesz s -potential function in the 2-dimensional Euclidean space $\mathbb{R}^2$ of ω_N .

We propose the following conjecture (see [73, Conjecture 2.1]).

Conjecture 2. Let $h \geq 0$. Given a set of N distinct points $\omega_N := \{x_1, x_2, \dots, x_N\} \subset \mathbb{R}^2$ such that

$$U^{2-2N,h}(x; \omega_N) = \sum_{j=1}^N (|x - x_j|^2 + h)^{N-1}$$

is constant as a function of x on $\mathbb{S}^1$. Then, $\{x_1, x_2, \dots, x_N\}$ forms a set of distinct equally spaced points on a circle centered at 0.

Again, Conjecture 2 is still open but we can solve many special cases of this conjecture.

1.2.3. Polarization Optimality Problems

The next problems considered in this project are polarization optimality problems. Let $\omega_N = \{x_1, \dots, x_N\}$ denote a configuration of N (not necessarily distinct) points in $\mathbb{R}^2$. Denote by

$$\mathbb{S}_R^1 := \{x \in \mathbb{R}^2 : |x| = R\}$$

the circle centered at the origin of radius R . When $R = 1$, we simply use the notation $\mathbb{S}^1$. Given $s \in \mathbb{R}$, $R > 0$, and $r > 0$, we define polarization constants

$$M_N^s(\mathbb{S}_r^1; \mathbb{S}_R^1) := \begin{cases} \max_{\substack{\omega_N \subseteq \mathbb{S}_r^1 \\ \#\omega_N = N}} \min_{y \in \mathbb{S}_R^1} U^s(y; \omega_N) & \text{if } s \neq 0 \\ N & \text{if } s = 0 \end{cases} \quad (10)$$

$$m_N^s(\mathbb{S}_r^1; \mathbb{S}_R^1) := \begin{cases} \min_{\substack{\omega_N \subseteq \mathbb{S}_r^1 \\ \#\omega_N = N}} \max_{y \in \mathbb{S}_R^1} U^s(y; \omega_N) & \text{if } s \neq 0 \\ N & \text{if } s = 0 \end{cases} \quad (11)$$

where $\#\omega_N$ denotes the cardinality of the multiset ω_N . We will call ω_N a *maximal (minimal) N -point Riesz s -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$* if ω_N attains the maximum in (12) (minimum in (13)). We give a brief history of such polarization optimality problems below.

Farkas and Révész [54] were the first to introduce two-plate polarization constants in general sense. However, all previous results [55, 56, 57] on polarization optimality

problems in (12) and (13) were considered for the case when $R = r = 1$. The maximality of N distinct equally spaced points on the unit circle for the maximal Riesz s -polarization problem of $(\mathbb{S}^1; \mathbb{S}^1)$ in (12) was proved by Ambrus, Ball, and Erdélyi in [55] for $s = 2$. Erdélyi and Saff [52] established this for $s = 4$. For arbitrary $s > 0$, this result was proved by Hardin, Kendall, and Saff [56]. In [56], they also showed the minimality of N distinct equally spaced points on the unit circle for the minimal Riesz s -polarization problem of $(\mathbb{S}^1; \mathbb{S}^1)$ in (13) for $-1 \leq s < 0$. Note that minimal N -point Riesz s -polarization problems of $(\mathbb{S}^1; \mathbb{S}^1)$ when $s > 0$ are not interesting because $m_N^s(\mathbb{S}^1; \mathbb{S}^1) = \infty$ for all $s > 0$.

In [65], we gave a characterization of all maximal and minimal N -point Riesz s -polarization configurations of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$ when $s = -2, -4, \dots, 2 - 2N$.

1.2.4. Generalized Polarization Optimality Problems

Considering more general potential functions in (9), we define generalized polarization constants and configurations as follows. Given $s \in \mathbb{R}$, $h \geq 0$, $R > 0$, and $r > 0$, we define polarization constants

$$M_N^{s,h}(\mathbb{S}_r^1; \mathbb{S}_R^1) := \max_{\substack{\omega_N \subset \mathbb{S}_r^1 \\ \#\omega_N = N}} \min_{y \in \mathbb{S}_R^1} U^{s,h}(y; \omega_N), \quad M_N^{0,h}(\mathbb{S}_r^1; \mathbb{S}_R^1) := N, \quad (12)$$

$$m_N^{s,h}(\mathbb{S}_r^1; \mathbb{S}_R^1) := \min_{\substack{\omega_N \subset \mathbb{S}_r^1 \\ \#\omega_N = N}} \max_{y \in \mathbb{S}_R^1} U^{s,h}(y; \omega_N), \quad m_N^{0,h}(\mathbb{S}_r^1; \mathbb{S}_R^1) := N, \quad (13)$$

where $\#\omega_N$ stands for the cardinality of the multiset ω_N . We will call ω_N a *maximal (minimal) N -point Riesz (s, h) -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$* if ω_N attains the maximum in (12) (minimum in (13)).

In [73], we gave a characterization of all maximal and minimal N -point Riesz (s, h) -polarization configurations of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$ when $s = -2, -4, \dots, 2 - 2N$.

2. วัตถุประสงค์และขอบเขตการวิจัย

We try to do the following:

- 2.1. Prove the analogue of Montessus de Ballore-Gonchar theorem for simultaneous Padé-Faber approximants.
- 2.2. Prove the analogue of Montessus de Ballore-Gonchar theorem for multipoint Hermite-Padé approximants.

We promised to have at least 2 accepted papers at the end of the second year. These two papers must be published in Q1 mathematics journals indexed by JCR (impact factors at least 1.000).

3. ผลลัพธ์ (Main Results)

3.1. Convergence Theorems for Classical Simultaneous Padé-Faber Approximants

We proved two convergence theorems for classical simultaneous Padé-Faber approximants in [66, 67] on row sequences.

3.1.1. Convergence theorem under polewise independence

The first one is based on the the notion of polewise independence defined as follows.

Definition 9. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ be a vector of functions meromorphic in some canonical domain D_ρ and let $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$ be the multi-index. Then the function $\mathbf{F}$ is said to be *polewise independent with respect to the multi-index*

$\mathbf{m}$ in D_ρ if and only if there do not exist polynomials $v_1, v_2, \dots, v_d$ at least one of which is non-null, satisfying

$$(i) \deg v_i \leq m_i - 1, i = 1, 2, \dots, d, \text{ if } m_i \geq 1,$$

$$(ii) v_i \equiv 0 \text{ if } m_i = 0,$$

$$(iii) \sum_{i=1}^d (v_i \circ \Phi) \cdot F_i \in \mathcal{H}(D_\rho \setminus E),$$

where $\mathcal{H}(D_\rho \setminus E)$ is the space of all holomorphic functions in $D_\rho \setminus E$.

Denote by $Q_{|\mathbf{m}|}^{\mathbf{F}}$ the monic polynomial whose zeros are the poles of $\mathbf{F}$ in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$ counting multiplicities.

Our first result [66, Theorem 1] under the concept of polewise independence is stated as follows.

Theorem 2. *Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ be a vector of functions meromorphic in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$ and $\mathbf{m} \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$ be a fixed multi-index. Suppose that $\mathbf{F}$ is polewise independent with respect to the multi-index $\mathbf{m}$ in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$. Then, $\mathbf{R}_{n,\mathbf{m}}^E$ is uniquely determined for all sufficiently large n and for each $i = 1, 2, \dots, d$, $R_{n,\mathbf{m},i}^E$ converges uniformly to F_i inside $D_{\rho_{|\mathbf{m}|}(\mathbf{F})} \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{F})$. Moreover, for each $i = 1, 2, \dots, d$ and for any compact set $K \subset D_{\rho_{|\mathbf{m}|}(\mathbf{F})} \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{F})$,*

$$\limsup_{n \rightarrow \infty} \|F_i - R_{n,\mathbf{m},i}^E\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{|\mathbf{m}|}(\mathbf{F})}, \quad (14)$$

where $\|\cdot\|_K$ denotes the sup-norm on K and if $K \subset E$, then $\|\Phi\|_K$ is replaced by 1. Additionally,

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}}^E - Q_{|\mathbf{m}|}^{\mathbf{F}}\|^{1/n} \leq \frac{\max_{\lambda \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{F})} |\Phi(\lambda)|}{\rho_{|\mathbf{m}|}(\mathbf{F})}, \quad (15)$$

where $\|\cdot\|$ denotes (for example) the norm induced in the space of polynomials of degree at most $|\mathbf{m}|$ by the maximum of the absolute value of the coefficients.

3.1.2. Convergence theorem under incomplete Padé-Faber approximants

Without the concept of polewise independence, we are able to prove another convergence theorem for classical simultaneous Padé-Faber approximants. Before stating this result, we need some more notation.

Given a vector $\mathbf{F} = (F_1, F_2, \dots, F_d)$ and a multi-index $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$, we define

$$\mathbf{D}_{\mathbf{m}}(\mathbf{F}) := (D_{\rho_{m_1}(F_1)}, D_{\rho_{m_2}(F_2)}, \dots, D_{\rho_{m_d}(F_d)}).$$

By $\widehat{Q}_{\mathbf{m}}^{\mathbf{F}}$, we denote the monic polynomial whose zeros are the poles of $\mathbf{F}$ in $\mathbf{D}_{\mathbf{m}}(\mathbf{F})$ counting multiplicities. This set of poles is denoted by $\mathcal{P}_{\mathbf{m}}(\mathbf{F})$. For $i = 1, 2, \dots, d$, set $\mathcal{P}_{\mathbf{m},i}(\mathbf{F}) = \mathcal{P}_{\mathbf{m}}(\mathbf{F}) \cap D_{\rho_{m_i}(F_i)}$.

To each pole λ of $\mathbf{F}$ in this system of domains

$$\mathbf{D}_{\mathbf{m}}(\mathbf{F}) := (D_{\rho_{m_1}(F_1)}, D_{\rho_{m_2}(F_2)}, \dots, D_{\rho_{m_d}(F_d)}),$$

we associate an index $i(\lambda) \in \{1, 2, \dots, d\}$ as follows. The index $i(\lambda)$ verifies that $\lambda \in D_{\rho_{m_{i(\lambda)}}(F_{i(\lambda)})}$ and λ is a pole of $F_{i(\lambda)}$ of the same order as is a pole of $\mathbf{F}$ in $\mathbf{D}_{\mathbf{m}}(\mathbf{F})$. If there are several indices i satisfying this condition, then we choose one among those with greatest $\rho_{m_i}(F_i)$.

Making use of the concept of incomplete Padé-Faber approximation, we proved another Montessus de Ballore type theorem for classical simultaneous Padé-Faber approximants in [67, Theorem 1] stated as follows.

Theorem 3. *Let $\mathcal{P}_{\mathbf{m}}(\mathbf{F}) = \{\lambda_1, \lambda_2, \dots, \lambda_q\}$. Suppose that $\mathbf{F} \in \mathcal{H}(E)^d$ has exactly $|\mathbf{m}|$*

poles in $D_{\mathbf{m}}(\mathbf{F})$. Then, $R_{n,\mathbf{m}}^E$ is uniquely determined for all sufficiently large n and for each $i = 1, 2, \dots, d$, $R_{n,\mathbf{m},i}^E$ converges uniformly to F_i inside $D_{\rho_{m_i}(F_i)} \setminus \mathcal{P}_{\mathbf{m},i}(\mathbf{F})$. Moreover,

$$\limsup_{n \rightarrow \infty} \|F_i - R_{n,\mathbf{m},i}^E\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{m_i}(F_i)}, \quad i = 1, 2, \dots, d, \quad (16)$$

where K is any compact subset of $D_{\rho_{m_i}(F_i)} \setminus \mathcal{P}_{\mathbf{m},i}(\mathbf{F})$ and if $K \subset E$, then $\|\Phi\|_K$ is replaced by 1. Additionally, we have

$$\limsup_{n \rightarrow \infty} \|\widehat{Q}_{\mathbf{m}}^{\mathbf{F}} - Q_{n,\mathbf{m}}^E\|^{1/n} \leq \max_{j=1,2,\dots,q} \left\{ \frac{|\Phi(\lambda_j)|}{\rho_{m_{i(\lambda_j)}}(F_{i(\lambda_j)})} \right\}. \quad (17)$$

3.2. Analogues of Theorem 1

It turns out that we need a concept of system pole and want to redefine the definitions of vector Padé-Faber approximants and vector multipoint Padé approximants (see Definitions 7 and 8) in order to prove analogues of Gonchar's converse statement to the Montessus de Ballore theorem.

Definition 10. Given $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ and $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$, we say that $\xi \in \mathbb{C}$ is a *system pole of order τ of $\mathbf{F}$ with respect to $\mathbf{m}$* if τ is the largest positive integer such that for each $t = 1, 2, \dots, \tau$, there exists at least one polynomial combination of the form

$$\sum_{i=1}^d v_i F_i, \quad \deg v_i < m_i, \quad i = 1, 2, \dots, d, \quad (18)$$

which is holomorphic on a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole at $z = \xi$ of exact order t .

Let τ be the order of ξ as a system pole of $\mathbf{F}$. For each $t = 1, \dots, \tau$, denote by

$\rho_{\xi,t}(\mathbf{F}, \mathbf{m})$ the largest of all the numbers $\rho_t(G)$ (the index of the largest canonical domain containing at most t poles of G), where G is a polynomial combination of type (18) that is holomorphic on a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole at $z = \xi$ of order t . Then, we define

$$\rho_{\xi,t}(\mathbf{F}, \mathbf{m}) := \min_{k=1,\dots,t} \rho_{\xi,k}(\mathbf{F}, \mathbf{m}),$$

$$\rho_{\xi}(\mathbf{F}, \mathbf{m}) := \rho_{\xi,\tau}(\mathbf{F}, \mathbf{m}) = \min_{t=1,\dots,\tau} \rho_{\xi,t}(\mathbf{F}, \mathbf{m}).$$

Fix $i \in \{1, \dots, d\}$ and $k \in \{0, 1, \dots, m_i - 1\}$. Let $D_{i,k}(\mathbf{F}, \mathbf{m})$ be the largest canonical domain in which all the poles of $z^k F_i$ are system poles of $\mathbf{F}$ with respect to $\mathbf{m}$, their order as poles of $z^k F_i$ does not exceed their order as system poles, and $z^k F_i$ has no other singularity. By $\rho_{i,k}(\mathbf{F}, \mathbf{m})$, we denote the index of this canonical domain. Let $\xi_1, \dots, \xi_N$ be the poles of $z^k F_i$ in $D_{i,k}(\mathbf{F}, \mathbf{m})$. For each $j = 1, \dots, N$, let $\hat{\tau}_j$ be the order of ξ_j as pole of $z^k F_i$ and τ_j its order as a system pole. By assumption, $\hat{\tau}_j \leq \tau_j$. Set

$$\rho_{i,k}^*(\mathbf{F}, \mathbf{m}) := \min\{\rho_{i,k}(\mathbf{F}, \mathbf{m}), \min_{j=1,\dots,N} \rho_{\xi_j, \hat{\tau}_j}(\mathbf{F}, \mathbf{m})\}$$

and let $D_{i,k}^*(\mathbf{F}, \mathbf{m})$ be the canonical domain with this index.

Let $Q_{\mathbf{m}}^{\mathbf{F}}$ denote the monic polynomial whose zeros are the system poles (in the sense of Definition 10) of $\mathbf{F}$ with respect to $\mathbf{m}$ taking account of their order. The set of distinct zeros of $Q_{\mathbf{m}}^{\mathbf{F}}$ is denoted by $\mathcal{P}(\mathbf{F}, \mathbf{m})$.

3.2.1. New Simultaneous Padé-Faber Approximants

We proved an analogue of the Montessus de Ballore-Gonchar theorem for simultaneous Padé-Faber approximants in [70, Theorem 1.4] stated as follows.

Theorem 4. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ and $\mathbf{m} \in \mathbb{N}^d$ be a fixed multi-index. Then, the following two assertions are equivalent:

- (a) $\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ counting multiplicities.
- (b) The polynomials $\tilde{Q}_{n,\mathbf{m}}^E$ of $\mathbf{F}$ are uniquely determined for all sufficiently large n , and there exists a polynomial $Q_{|\mathbf{m}|}$ of degree $|\mathbf{m}|$ such that

$$\limsup_{n \rightarrow \infty} \|\tilde{Q}_{n,\mathbf{m}}^E - Q_{|\mathbf{m}|}\|^{1/n} = \theta < 1.$$

Moreover, if either (a) or (b) takes place, then $Q_{|\mathbf{m}|} = Q_{\mathbf{m}}^{\mathbf{F}}$,

$$\theta = \max \left\{ \frac{|\Phi(\xi)|}{\rho_{\xi}(\mathbf{F}, \mathbf{m})} : \xi \in \mathcal{P}(\mathbf{F}, \mathbf{m}) \right\},$$

and for any compact subset K of $D_{i,0}^*(\mathbf{F}, \mathbf{m}) \setminus \mathcal{P}(\mathbf{F}, \mathbf{m})$,

$$\limsup_{n \rightarrow \infty} \|\tilde{R}_{n,\mathbf{m},i}^E - F_i\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{i,0}^*(\mathbf{F}, \mathbf{m})},$$

where $\|\cdot\|_K$ denotes the sup-norm on K and if $K \subset E$, then $\|\Phi\|_K$ is replaced by 1.

3.2.2. Multipoint Hermite-Padé Approximants

We proved an analogue of the Montessus de Ballore-Gonchar theorem for multipoint Hermite-Padé approximants in [70, Theorem 1.4] stated as follows.

Theorem 5. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$, $\mathbf{m} \in \mathbb{N}^d$ be a fixed multi-index and $\alpha \subset E$ be a table of interpolation points satisfying

$$\lim_{n \rightarrow \infty} a_n(z)/c^n \Phi^n(z) = G(z) \neq 0, \tag{19}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus E$, where c denotes some positive constant. Then, the following two assertions are equivalent:

- (a) $\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ counting multiplicities.
- (b) The polynomials $\tilde{Q}_{n,\mathbf{m}}^E$ of $\mathbf{F}$ are uniquely determined for all sufficiently large n , and there exists a polynomial $Q_{|\mathbf{m}|}$ of degree $|\mathbf{m}|$ such that

$$\limsup_{n \rightarrow \infty} \|\tilde{Q}_{n,\mathbf{m}}^E - Q_{|\mathbf{m}|}\|^{1/n} = \theta < 1.$$

Moreover, if either (a) or (b) takes place, then $Q_{|\mathbf{m}|} = Q_{\mathbf{m}}^{\mathbf{F}}$,

$$\theta = \max \left\{ \frac{|\Phi(\xi)|}{\rho_{\xi}(\mathbf{F}, \mathbf{m})} : \xi \in \mathcal{P}(\mathbf{F}, \mathbf{m}) \right\},$$

and for any compact subset K of $D_{i,0}^*(\mathbf{F}, \mathbf{m}) \setminus \mathcal{P}(\mathbf{F}, \mathbf{m})$,

$$\limsup_{n \rightarrow \infty} \|\tilde{R}_{n,\mathbf{m},i}^E - F_i\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{i,0}^*(\mathbf{F}, \mathbf{m})},$$

where $\|\cdot\|_K$ denotes the sup-norm on K and if $K \subset E$, then $\|\Phi\|_K$ is replaced by 1.

3.3. Behaviors when $m_j \rightarrow \infty$ for all j

Let us introduce a concept of convergence in Hausdorff content. Let B be a subset of the complex plane $\mathbb{C}$. By $\mathcal{U}(B)$, we denote the class of all coverings of B by at most a numerable set of disks. Let $\beta > 0$ and set

$$h_{\beta}(B) := \inf \left\{ \sum_{i=1}^{\infty} |U_i|^{\beta} : \{U_i\} \in \mathcal{U}(B) \right\},$$

where $|U_i|$ stands for the radius of the disk U_i . The quantity $h_\beta(B)$ is called the β -dimensional Hausdorff content of the set B . This set function is not a measure but it is subadditive and monotonic. Clearly, if B is a disk, then $h_\beta(B) = |B|^\beta$.

Definition 11. Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of complex valued functions defined on a domain $D \subset \mathbb{C}$ and g another complex function defined on D . We say that $\{g_n\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to the function g inside D if for every compact subset K of D and for each $\varepsilon > 0$, we have

$$\lim_{n \rightarrow \infty} h_\beta\{z \in K : |g_n(z) - g(z)| > \varepsilon\} = 0.$$

Such a convergence will be denoted by $h_\beta\text{-}\lim_{n \rightarrow \infty} g_n = g$ in D .

We proved convergences in Hausdorff content of (classical and new) simultaneous orthogonal Padé approximants and (classical and new) simultaneous Padé-Faber approximants in [74].

3.3.1. Classical Simultaneous Orthogonal Padé Approximants

We need to define three classes of measures contained in $\mathcal{M}(E)$. The measure $\mu \in \text{Reg}_{1,2}^m(E)$ iff the corresponding sequences of p_n and s_n satisfy

$$\lim_{n \rightarrow \infty} |p_n(z)|^{1/n} = |\Phi(z)|, \quad (20)$$

$$\lim_{n \rightarrow \infty} |s_n(z)|^{1/n} = |\Phi(z)|^{-1}, \quad (21)$$

and there exists a positive constant c such that

$$\frac{\kappa_{n-m}}{\kappa_n} \geq c, \quad n \geq n_0. \quad (22)$$

We write $\mu \in \mathcal{R}(E)$ when the corresponding sequence of orthonormal polynomials has ratio asymptotics; that is,

$$\lim_{n \rightarrow \infty} \frac{p_n(z)}{p_{n+1}(z)} = \frac{1}{\Phi(z)}. \quad (23)$$

We say that Szegő or strong asymptotics takes place, and write $\mu \in \mathcal{S}(E)$, if

$$\lim_{n \rightarrow \infty} \frac{p_n(z)}{c_n \Phi^n(z)} = S(z) \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} = 1. \quad (24)$$

The first limit in (24) and the ones in (20), (21), and (23) are assumed to hold uniformly inside $\overline{\mathbb{C}} \setminus E$, the c_n 's are positive constants, and $S(z)$ is some holomorphic and non-vanishing function on $\overline{\mathbb{C}} \setminus E$. It is not difficult to check that $\mathcal{S}(E) \subset \mathcal{R}(E) \subset \mathbf{Reg}_{1,2}^m(E)$. The above three classes of measures are very common when we study asymptotic properties of orthogonal polynomials (see the book [62] for more information about orthogonal polynomials).

Convergence in Hausdorff content of classical simultaneous orthogonal Padé approximants was proved in [74].

Theorem 6. *Let $E \in \mathcal{K}_1$, $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$, and $\mu \in \mathcal{R}(E)$. For each $i = 1, 2, \dots, d$, denote by $D_{\rho_\infty(F_i)}$ the maximal canonical domain in which F_i can be continued to a meromorphic function. Assume that*

$$\liminf_{n \rightarrow \infty} m_{n,j} = \infty, \quad j = 1, 2, \dots, d$$

and

$$\lim_{n \rightarrow \infty} \frac{|\mathbf{m}_n| \ln n}{n} = 0.$$

Then for fixed numbers $\beta > 0$ and $i = 1, 2, \dots, d$, each sequence $\{R_{n, \mathbf{m}_n, i}^\mu\}_{n \in \mathbb{N}}$ converges

in β -dimensional Hausdorff content to F_i inside $D_{\rho_\infty(F_i)}$ as $n \rightarrow \infty$.

3.3.2. New Simultaneous Orthogonal Padé Approximants

Convergence in Hausdorff content of new simultaneous orthogonal Padé approximants was proved in [74].

Theorem 7. *Let $E \in \mathcal{K}$, $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$, and $\mu \in \mathbf{Reg}_{1,2}(E)$. For each $i = 1, 2, \dots, d$, denote by $D_{\rho_\infty(F_i)}$ the maximal canonical domain in which F_i can be continued to a meromorphic function. Assume that*

$$\liminf_{n \rightarrow \infty} m_{n,j} = \infty, \quad j = 1, 2, \dots, d$$

and

$$\lim_{n \rightarrow \infty} \frac{|\mathbf{m}_n| \ln n}{n} = 0.$$

Then for fixed numbers $\beta > 0$ and $i = 1, 2, \dots, d$, each sequence $\{\tilde{R}_{n, \mathbf{m}_n, i}^\mu\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to F_i inside $D_{\rho_\infty(F_i)}$ as $n \rightarrow \infty$.

3.3.3. Classical Simultaneous Padé-Faber Approximants

Convergence in Hausdorff content of classical simultaneous Padé-Faber approximants was proved in [74].

Theorem 8. *Let $E \in \mathcal{K}_1$ and $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. For each $i = 1, 2, \dots, d$, denote by $D_{\rho_\infty(F_i)}$ the maximal canonical domain in which F_i can be continued to a meromorphic function. Assume that*

$$\liminf_{n \rightarrow \infty} m_{n,j} = \infty, \quad j = 1, 2, \dots, d$$

and

$$\lim_{n \rightarrow \infty} \frac{|\mathbf{m}_n| \ln n}{n} = 0.$$

Then for fixed numbers $\beta > 0$ and $i = 1, 2, \dots, d$, each sequence $\{R_{n, \mathbf{m}_n, i}^E\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to F_i inside $D_{\rho_\infty(F_i)}$ as $n \rightarrow \infty$.

3.3.4. New Simultaneous Padé-Faber Approximants

Convergence in Hausdorff content of new simultaneous Padé-Faber approximants was proved in [74].

Theorem 9. Let $E \in \mathcal{K}$ and $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. For each $i = 1, 2, \dots, d$, denote by $D_{\rho_\infty(F_i)}$ the maximal canonical domain in which F_i can be continued to a meromorphic function. Assume that

$$\liminf_{n \rightarrow \infty} m_{n,j} = \infty, \quad j = 1, 2, \dots, d$$

and

$$\lim_{n \rightarrow \infty} \frac{|\mathbf{m}_n| \ln n}{n} = 0.$$

Then for fixed numbers $\beta > 0$ and $i = 1, 2, \dots, d$, each sequence $\{\tilde{R}_{n, \mathbf{m}_n, i}^E\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to F_i inside $D_{\rho_\infty(F_i)}$ as $n \rightarrow \infty$.

3.4. Scalar Cases

in this section, because we limit ourselves to the scalar case ($d = 1$), we write $\mathbf{F} = F$, $|\mathbf{m}| = \mathbf{m} = m \in \mathbb{N}$, $\rho_{|\mathbf{m}|}(\mathbf{F}) = \rho_m(F)$, $\tilde{\mathbf{R}}_{n, \mathbf{m}}^\mu = \tilde{R}_{n, m}^\mu$, and $\tilde{\mathbf{R}}_{n, \mathbf{m}}^E = \tilde{R}_{n, m}^E$.

3.4.1. Rate of Attraction

Let us define two indicators of the asymptotic behavior of the zeros of $\tilde{Q}_{n,m}^E$ and $\tilde{Q}_{n,m}^\mu$

Fix $m \in \mathbb{N}$. Let

$$\mathcal{P}_{n,m}^\mu := \{\lambda_{n,1}^\mu, \lambda_{n,2}^\mu, \dots, \lambda_{n,m_n}^\mu\}, \quad m_n \leq m, \quad n \in \mathbb{N}_0,$$

$$\mathcal{P}_{n,m}^E := \{\lambda_{n,1}^E, \lambda_{n,2}^E, \dots, \lambda_{n,m_n}^E\}, \quad m_n \leq m, \quad n \in \mathbb{N}_0$$

denote the collections of zeros of $\tilde{Q}_{n,m}^\mu$ and $\tilde{Q}_{n,m}^E$ (repeated according to their multiplicity), respectively. Define

$$|z - w|_1 := \min\{1, |z - w|\}, \quad z, w \in \mathbb{C}.$$

Fix $\lambda \in \mathbb{C}$. The first indicators are defined by

$$\Delta^\mu(\lambda) := \limsup_{n \rightarrow \infty} \prod_{j=1}^{m_n} |\lambda_{n,j}^\mu - \lambda|_1^{1/n} = \limsup_{n \rightarrow \infty} \prod_{|\lambda_{n,j}^\mu - \lambda| < 1} |\lambda_{n,j}^\mu - \lambda|^{1/n},$$

$$\Delta^E(\lambda) := \limsup_{n \rightarrow \infty} \prod_{j=1}^{m_n} |\lambda_{n,j}^E - \lambda|_1^{1/n} = \limsup_{n \rightarrow \infty} \prod_{|\lambda_{n,j}^E - \lambda| < 1} |\lambda_{n,j}^E - \lambda|^{1/n}.$$

Clearly, $0 \leq \Delta^\mu(\lambda) \leq 1$ and $0 \leq \Delta^E(\lambda) \leq 1$ (when $m_n = 0$, the product is taken to be 1). The second indicators, nonnegative integers $\sigma^\mu(\lambda)$ and $\sigma^E(\lambda)$, are defined as follows.

We suppose that for each n , the points in

$$\mathcal{P}_{n,m}^\mu := \{\lambda_{n,1}^\mu, \lambda_{n,2}^\mu, \dots, \lambda_{n,m_n}^\mu\}, \quad (25)$$

$$\mathcal{P}_{n,m}^E := \{\lambda_{n,1}^E, \lambda_{n,2}^E, \dots, \lambda_{n,m_n}^E\} \quad (26)$$

are enumerated in nondecreasing distance to the point λ . We set

$$\delta_j^\mu(\lambda) := \limsup_{n \rightarrow \infty} |\lambda_{n,j}^\mu - \lambda|_1^{1/n}, \quad (27)$$

$$\delta_j^E(\lambda) := \limsup_{n \rightarrow \infty} |\lambda_{n,j}^E - \lambda|_1^{1/n}. \quad (28)$$

These numbers are defined by (27) for $j = 1, 2, \dots, m'$, $m' = \liminf_{n \rightarrow \infty} m_n$; for $j = m' + 1, \dots, n$, we define $\delta_j^\mu(\lambda) = 1$. We have $0 \leq \delta_j^\mu(\lambda) \leq 1$. If $\Delta^\mu(\lambda) = 1$ (in that case all $\delta_j^\mu(\lambda) = 1$), then $\sigma^\mu(\lambda) = 0$. If $\Delta^\mu(\lambda) < 1$, then for some ν , $1 \leq \nu \leq m$, we have that $\delta_1^\mu(\lambda) \leq \dots \leq \delta_\nu^\mu(\lambda) < 1$ and $\delta_{\nu+1}^\mu(\lambda) = 1$ or $\nu = m$; in this case we take $\sigma^\mu(\lambda) = \nu$. The same rules are applied for $\sigma^E(\lambda)$.

3.4.1.1. New Orthogonal Padé Approximants

Main results [68, Theorem 2.1 and Corollary 2.2] about the rate of attraction of poles of the approximated function to poles of new orthogonal Padé approximants are stated below.

Theorem 10. *Let $E \in \mathcal{K}$, $F \in \mathcal{H}(E)$, and $\mu \in \mathbf{Reg}_2(E)$. Fix $m \in \mathbb{N}$. If F has a pole of order ν at ξ in $D_{\rho_m(F)}$, then*

$$\Delta^\mu(\xi) \leq \frac{|\Phi(\xi)|}{\rho_m(F)} \quad \text{and} \quad \sigma^\mu(\xi) \geq \nu.$$

Corollary 1. *Let $F \in \mathcal{H}(E)$, $\mu \in \mathbf{Reg}_2(E)$, and ξ be a pole of F in $D_{\rho_m(F)}$ of order ν . Assume that $\liminf_{n \rightarrow \infty} |\xi - \tilde{\xi}_{n,\nu+1}^\mu| > 0$, where $\{\tilde{\xi}_{n,1}^\mu, \tilde{\xi}_{n,2}^\mu, \dots, \tilde{\xi}_{n,m_n}^\mu\}$ are enumerated in nondecreasing distance to the point ξ . Then,*

$$\delta_1^\mu(\xi) \leq \delta_2^\mu(\xi) \leq \dots \leq \delta_\nu^\mu(\xi) \leq \left(\frac{|\Phi(\xi)|}{\rho_m(F)} \right)^{1/\nu}. \quad (29)$$

In particular, $\delta_1^\mu(\xi) = \delta_2^\mu(\xi) = \dots = \delta_\nu^\mu(\xi) = (|\Phi(\xi)|/\rho_m(F))^{1/\nu}$ if and only if $\Delta^\mu(\xi) = |\Phi(\xi)|/\rho_m(F)$.

3.4.1.2. New Padé-Faber Approximants

Main results [68, Theorem 2.3 and Corollary 2.4] about the rate of attraction of poles of the approximated function to poles of new Padé-Faber approximants are stated below.

Theorem 11. *Let $F \in \mathcal{H}(E)$. Fix $m \in \mathbb{N}$. If F has a pole of order ν at ξ in $D_{\rho_m(F)}$, then*

$$\Delta^E(\xi) \leq \frac{|\Phi(\xi)|}{\rho_m(F)} \quad \text{and} \quad \sigma^E(\xi) \geq \nu.$$

Corollary 2. *Let $E \in \mathcal{K}$, $F \in \mathcal{H}(E)$, and ξ be a pole of F in $D_{\rho_m(F)}$ of order ν . Assume that $\liminf_{n \rightarrow \infty} |\xi - \xi_{n,\nu+1}^E| > 0$, where $\{\xi_{n,1}^E, \xi_{n,2}^E, \dots, \xi_{n,m_n}^E\}$ are enumerated in nondecreasing distance to the point ξ . Then,*

$$\delta_1^E(\xi) \leq \delta_2^E(\xi) \leq \dots \leq \delta_\nu^E(\xi) \leq \left(\frac{|\Phi(\xi)|}{\rho_m(F)} \right)^{1/\nu}.$$

In particular, $\delta_1^E(\xi) = \delta_2^E(\xi) = \dots = \delta_\nu^E(\xi) = (|\Phi(\xi)|/\rho_m(F))^{1/\nu}$ if and only if $\Delta^E(\xi) = |\Phi(\xi)|/\rho_m(F)$.

3.4.2. Inverse Results

3.4.2.1. New Orthogonal Padé Approximants

Inverse result on new orthogonal Padé approximants [68, Theorem 2.5] is stated as follows.

Theorem 12. Let $F \in \mathcal{H}(E)$ and $\mu \in \mathcal{S}(E)$. Fix $m \in \mathbb{N}$. If for all n sufficiently large, $\tilde{Q}_{n,m}^\mu$ (for F) has precisely m zeros and the zeros of $\tilde{Q}_{n,m}^\mu$ have limits $\xi_1, \dots, \xi_m$, as $n \rightarrow \infty$, then

- (i) F is holomorphic in $D_{\rho_{\min}}$, where $\rho_{\min} := \min_{1 \leq j \leq m} |\Phi(\xi_j)|$;
- (ii) $\rho_{m-1}(F) = \max_{1 \leq j \leq m} |\Phi(\xi_j)|$;
- (iii) $\xi_1, \dots, \xi_m$ are singularities of F ; those lying in $D_{\rho_{m-1}(F)}$ are poles (counting multiplicities), and F has no other poles in $D_{\rho_{m-1}(F)}$.

3.4.2.2. New Padé-Faber Approximants

Inverse result on new Padé-Faber approximants [68, Theorem 2.6] is stated as follows.

Theorem 13. Let $F \in \mathcal{H}(E)$. Fix $m \in \mathbb{N}$. If for all n sufficiently large, $\tilde{Q}_{n,m}^E$ (for F) has precisely m zeros and the zeros of $\tilde{Q}_{n,m}^E$ have limits $\xi_1, \dots, \xi_m$, as $n \rightarrow \infty$, then

- (i) F is holomorphic in $D_{\rho_{\min}}$, where $\rho_{\min} := \min_{1 \leq j \leq m} |\Phi(\xi_j)|$;
- (ii) $\rho_{m-1}(F) = \max_{1 \leq j \leq m} |\Phi(\xi_j)|$;
- (iii) $\xi_1, \dots, \xi_m$ are singularities of F ; those lying in $D_{\rho_{m-1}(F)}$ are poles (counting multiplicities), and F has no other poles in $D_{\rho_{m-1}(F)}$.

3.4.3. On the Boundedness of Poles of Generalized Padé

Approximants to Polynomial Expansions

3.4.3.1. Classical Orthogonal Padé Approximants

The following result was proved in [72, Theorem 2.1].

Theorem 14. Let $F \in \mathcal{H}(E)$ and $\mu \in \mathbf{Reg}_1^1$. Fix $m \in \mathbb{N}$ and denote by $\mathcal{P}_n^\mu$ the set of all zeros of a polynomial $Q_{n,m}^\mu$. Assume that the cardinality of $\mathcal{P}_n^\mu$ is at least 1 for all n sufficiently large and

$$\inf_{N \geq m} \sup_{n \geq N} \{|\zeta| : \zeta \in \mathcal{P}_n^\mu\} < \infty. \quad (30)$$

Then, either F is a polynomial or $\rho_0(F) < \infty$.

3.4.3.2. Classical Padé-Faber Approximants

The following result was proved in [72, Theorem 2.2].

Theorem 15. Let $F \in \mathcal{H}(E)$. Fix $m \in \mathbb{N}$ and denote by $\mathcal{P}_n^E$ the set of all zeros of a polynomial $Q_{n,m}^E$. Assume that the cardinality of $\mathcal{P}_n^E$ is at least 1 for all n sufficiently large and

$$\inf_{N \geq m} \sup_{n \geq N} \{|\zeta| : \zeta \in \mathcal{P}_n^E\} < \infty. \quad (31)$$

Then, either F is a polynomial or $\rho_0(F) < \infty$.

3.4.3.3. New Orthogonal Padé Approximants

The following result was proved in [72, Theorem 2.3].

Theorem 16. Let $F \in \mathcal{H}(E)$ and $\mu \in \mathbf{Reg}_1^1$. Fix $m \in \mathbb{N}$ and denote by $\tilde{\mathcal{P}}_n^\mu$ the set of all zeros of a polynomial $\tilde{Q}_{n,m}^\mu$. Assume that the cardinality of $\tilde{\mathcal{P}}_n^\mu$ is at least 1 for all n sufficiently large and

$$\inf_{N \geq m} \sup_{n \geq N} \{|\zeta| : \zeta \in \tilde{\mathcal{P}}_n^\mu\} < \infty. \quad (32)$$

Then, either F is a polynomial or $\rho_0(F) < \infty$.

3.4.3.4. New Padé-Faber Approximants

The following result was proved in [72, Theorem 2.4].

Theorem 17. *Let $F \in \mathcal{H}(E)$. Fix $m \in \mathbb{N}$ and denote by $\tilde{\mathcal{P}}_n^E$ the set of all zeros of a polynomial $\tilde{Q}_{n,m}^E$. Assume that the cardinality of $\tilde{\mathcal{P}}_n^E$ is at least 1 for all n sufficiently large and*

$$\inf_{N \geq m} \sup_{n \geq N} \{|\zeta| : \zeta \in \tilde{\mathcal{P}}_n^E\} < \infty. \quad (33)$$

Then, either F is a polynomial or $\rho_0(F) < \infty$.

3.5. Constant Riesz Potentials and Polarization Optimality Problems

3.5.1. Constant Riesz Potentials on $\mathbb{R}^2$

A characterization of ω_N when $U^{2-2N}(\cdot; \omega_N)$ is constant on the unit circle was proved in [65, Theorem 1].

Theorem 18. *Let $\omega_N = \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{C}$ be a set of N distinct points. Then, the function*

$$U^{2-2N}(x; \omega_N) = \sum_{j=1}^N |x - x_j|^{2N-2}$$

is constant on the circle $\mathbb{S}^1$ if and only if

$$\sum_{j=1}^N \sum_{q=0}^{N-k-1} \binom{N-1}{q} \binom{N-1}{k+q} |x_j|^{2N-2k-2q-2} x_j^k = 0, \quad \text{for all } k = 1, \dots, N-1, \quad (34)$$

where

$$x^k := (r^k \cos(kt), r^k \sin(kt))$$

if $x = (r \cos(t), r \sin(t)) \in \mathbb{R}^2$.

Using the characterization given in Theorem 18, we can verify Conjecture 1 in various cases. Our first result [65, Proposition 1] asserts that Conjecture 1 holds if the points $x_1, x_2, \dots, x_N$ lie on the same circle centered at the origin (that is, they have the same norm).

Theorem 19. *Let $\omega_N = \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{R}^2$ be a set of N distinct nonzero points lying on some circle centered at the origin. If $U^{2-2N}(\cdot; \omega_N)$ is constant on $\mathbb{S}^1$, then $x_1, x_2, \dots, x_N$ are equally spaced.*

Now we will consider another special case. Instead of assuming that all points have the same norm, we will assume that they have equal angle distribution around the origin. More precisely, let $\zeta = e^{2\pi i/N}$ and, without loss of generality, assume that

$$x_1 = r_1 \zeta^1, x_2 = r_2 \zeta^2, \dots, x_N = r_N \zeta^N \quad (35)$$

for some positive real numbers $r_1, r_2, \dots, r_N$.

If we further assume that all norms are rational, then Conjecture 1 holds for all prime N (see [65, Proposition 3]).

Theorem 20. *Let N be a prime number. Let $x_1, x_2, \dots, x_N$ be as in (35) where all $r_j \in \mathbb{Q}$. Suppose that $U^{2-2N}(\cdot; \omega_N)$ is constant on $\mathbb{S}^1$. Then $x_1, x_2, \dots, x_N$ are equally spaced on a circle centered at the origin.*

3.5.2. Constant Generalized Riesz Potentials on $\mathbb{R}^2$

Considering more general Riesz potentials, we gave a generalization of Theorem 1 in [53]. The following theorem was proved in [73, Theorem 2.1].

Theorem 21. Let $h \geq 0$. Given a set of N distinct points $\omega_N := \{x_1, x_2, \dots, x_N\} \subset \mathbb{R}^2$ such that for each $s = -2, -4, \dots, 2 - 2N$,

$$U^{s,h}(x; \omega_N) = \sum_{j=1}^N (|x - x_j|^2 + h)^{-s/2}$$

is independent of the position of $x \in \mathbb{S}^1$. Then, ω_N forms a set of distinct equally spaced points on a circle centered at 0. Moreover, if $|x_1| = |x_2| = \dots = |x_N| = r$, then for each $p = 1, 2, \dots, N - 1$,

$$U^{-2p,h}(x; \omega_N) = \frac{N}{2^p} \sum_{q=0}^p \binom{p}{q}^2 (2r)^{2q} \left(r^2 + 1 + h + \sqrt{((r-1)^2 + h)((r+1)^2 + h)} \right)^{p-2q}$$

for all $x \in \mathbb{S}^1$.

A characterization of sets of N distinct points ω_N such that $U^{2-2N,h}(\cdot, \omega_N)$ is constant on $\mathbb{S}^1$ is the following (see [73, Theorem 2.2]):

Theorem 22. Let $h \geq 0$ and $\omega_N = \{x_1, x_2, \dots, x_N\} \subset \mathbb{R}^2$ be a set of N distinct points. Then, the function

$$U^{2-2N,h}(x; \omega_N) = \sum_{j=1}^N (|x - x_j|^2 + h)^{N-1}$$

is constant on $\mathbb{S}^1$ if and only if

$$\sum_{j=1}^N B_{k,j} x_j^k = 0, \quad k = 1, \dots, N - 1, \quad (36)$$

where

$$x^k := (r^k \cos(kt), r^k \sin(kt))$$

if $x = (r \cos(t), r \sin(t)) \in \mathbb{R}^2$ and

$$B_{k,j} := \sum_{q=0}^{N-k-1} \left[\binom{N-1}{q} \binom{N-1}{k+q} (2|x_j|)^{2q} \times \left(|x_j|^2 + 1 + h + \sqrt{((|x_j| - 1)^2 + h)((|x_j| + 1)^2 + h)} \right)^{N-2q-k-1} \right]. \quad (37)$$

As a consequence of this characterization, we obtain the following corollary (see [73, Corollary 2.2]).

Corollary 3. *Let $h \geq 0$ and let $\omega_N := \{x_1, x_2, \dots, x_N\}$ be a set of N distinct points in $\mathbb{R}^2$, which belong to a circle $\mathbb{S}_r^1 \subset \mathbb{R}^2$. Assume that*

$$U^{2-2N,h}(x; \omega_N) = \sum_{j=1}^N (|x - x_j|^2 + h)^{N-1}$$

is constant on $\mathbb{S}_r^1$. Then, $\{x_1, x_2, \dots, x_N\}$ forms a set of distinct equally spaced points on $\mathbb{S}_r^1$.

Applying Theorem 22 and Corollary 3, we proved Conjecture 2 when $N = 3$. The following corollary was proved in [73, Corollary 2.3].

Corollary 4. *Let $h \geq 0$ and $\{x_1, x_2, x_3\} \subset \mathbb{R}^2$ be a set of 3 distinct points. If the function $U^{-4,h}(x, \{x_1, x_2, x_3\})$ is constant on $\mathbb{S}^1$, then $\{x_1, x_2, x_3\}$ forms a set of distinct equally spaced points on a circle centered at 0.*

3.5.3. Polarization Optimality Problems

A complete characterization of all maximal and minimal N -point Riesz s -polarization configurations of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$ when $s = -2, -4, \dots, 2-2N$ is the following (see [65, Theorem

2]):

Theorem 23. *Let $N \in \mathbb{N}$, $p \in \{1, 2, \dots, N-1\}$, $R > 0$, $r > 0$, and $\{x_1, x_2, \dots, x_N\} \subseteq \mathbb{S}_r^1$. The following statements are equivalent:*

- (a) $\{x_1, x_2, \dots, x_N\}$ is a maximal N -point Riesz $-2p$ -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$;
- (b) $\{x_1, x_2, \dots, x_N\}$ is a minimal N -point Riesz $-2p$ -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$;
- (c) $\sum_{j=1}^N x_j = \sum_{j=1}^N x_j^2 = \dots = \sum_{j=1}^N x_j^p = 0$, where

$$x^k := (r^k \cos(kt), r^k \sin(kt))$$

$$\text{if } x = (r \cos(t), r \sin(t)) \in \mathbb{R}^2$$

Furthermore,

$$\begin{aligned} M_N^{-2p}(\mathbb{S}_r^1; \mathbb{S}_R^1) &= m_N^{-2p}(\mathbb{S}_r^1; \mathbb{S}_R^1) \\ &= \frac{N}{2^p} \sum_{j=0}^p \binom{p}{j}^2 (2rR)^{2j} (r^2 + R^2 + |r^2 - R^2|)^{p-2j}. \end{aligned} \quad (38)$$

3.5.4. Generalized Polarization Optimality Problems

A complete characterization of all maximal and minimal N -point Riesz (s, h) -polarization configurations of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$ when $s = -2, -4, \dots, 2 - 2N$ and $h \geq 0$ is the following (see [73, Theorem 2]):

Theorem 24. *Let $N \in \mathbb{N}$, $p \in \{1, 2, \dots, N-1\}$, $R > 0$, $r > 0$, $h \geq 0$, and $\{x_1, x_2, \dots, x_N\} \subset \mathbb{S}_r^1$. The following statements are equivalent:*

- (a) $\{x_1, x_2, \dots, x_N\}$ is a maximal N -point Riesz $(-2p, h)$ -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$;

(b) $\{x_1, x_2, \dots, x_N\}$ is a minimal N -point Riesz $(-2p, h)$ -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$;

(c) $\sum_{j=1}^N x_j = \sum_{j=1}^N x_j^2 = \dots = \sum_{j=1}^N x_j^p = 0$, where $x^k := (r^k \cos(kt), r^k \sin(kt))$ if $x = (r \cos(t), r \sin(t)) \in \mathbb{R}^2$.

Furthermore,

$$M_N^{-2p,h}(\mathbb{S}_r^1; \mathbb{S}_R^1) = m_N^{-2p,h}(\mathbb{S}_r^1; \mathbb{S}_R^1) \\ = \frac{N}{2^p} \sum_{j=0}^p \binom{p}{j}^2 (2rR)^{2j} \left(r^2 + R^2 + h + \sqrt{((r-R)^2 + h)(r+R)^2 + h} \right)^{p-2j}. \quad (39)$$

4. Outputs

4.1. Papers

4.1.1. Accepted Papers

- (1) **N. Bosuwan** and P. Ruengrot, Constant Riesz potential on a circle in a plane with an application to polarization optimality problems, *ScienceAsia*, 43(4) (2017), 267-274. (Impact Factor 2017: 0.447)
- (2) **N. Bosuwan**, Convergence of row sequences of simultaneous Padé-Faber approximants, *Math. Notes*, 103(5) (2018), 643-656. (Impact Factor 2017: 0.577)
- (3) **N. Bosuwan**, On Montessus de Ballore's theorem for simultaneous Padé-Faber approximants, *Demonstr. Math.*, 51 (2018), 45-61. (SJR 2017: Q3)
- (4) **N. Bosuwan**, Direct and inverse results on row sequences of generalized Padé approximants to polynomial expansions, *Acta Math. Hungar.*, 157(1) (2019), 191-219. (Impact Factor 2017: 0.481)

- (5) W. Chonlapap and **N. Bosuwan**, Convergence in Hausdorff content of Padé-Faber approximants and its applications, Thai J. Math., 17 (2019), 272-287 (SJR 2017: Q3)
- (6) **N. Bosuwan** and G. López Lagomasino, Direct and inverse results on row sequences of simultaneous Padé-Faber approximants, Mediterr. J. Math. 16(36) (2019) <https://doi.org/10.1007/s00009-019-1307-0>. (Impact Factor 2017: 1.000)
- (7) **N. Bosuwan**, G. López Lagomasino, and Y. Zaldivar Gerpe, Direct and inverse results for multipoint Hermite-Padé approximants, Anal. Math. Phys., accepted. (Impact Factor 2017: 1.381)

4.1.2. Submitted Papers (Waiting for the results)

- (8) M. Wajarat and **N. Bosuwan**, Convergences in Hausdorff content of generalized Hermite-Padé approximants to polynomial expansions, submitted.
- (9) **N. Bosuwan**, Constant generalized Riesz potential functions and polarization optimality problems, submitted.
- (10) **N. Bosuwan**, On the boundedness of poles of generalized Padé approximants, submitted.

4.2. Conferences

- Constant Riesz Potentials on a circle in a plane and symmetric equations, The 13th Conference of Young Algebraists in Thailand, December 6-9, 2017, Nakhon Pathom, Thailand.
- Convergence of new Padé-Faber approximants, The 23rd Annual Meeting in Mathematics 2018, May 3-5, 2018, KMUTT, Bangkok, Thailand.

- Convergence of classical Padé-Faber approximants, Annual Pure and Applied Mathematics Conference 2018, May 30-June 1, 2018, Chulalongkorn University, Bangkok, Thailand.
- Convergences of row sequences of simultaneous Padé-Faber approximants, International Conference on Complex Analysis, Potential Theory, and Applications, June 11-15, 2018, University College of Dublin, Dublin, Ireland.
- Convergences in Hausdorff content of generalized Hermite-Padé approximants to polynomial expansions, XIX Conference on Analytic Functions and Related Topics, June 25-29, 2018, University of Rzeszow, Poland.
- Inverse results on row sequences of multipoint Padé approximants, International Conference in Mathematics and Applications 2018, December 16-18, 2018, Bangkok, Thailand.

4.3. Students under this project

- Nitipon Chattrakul, Bachelor's Degree in Mathematics
- Waraporn Chonlapapl, Bachelor's Degree in Mathematics
- Methawee Wajasat, Master's Degree in Mathematics

5. ภาคผนวก

ตาม papers ที่แนบมา

Constant Riesz potentials on a circle in a plane with an application to polarization optimality problems

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ABSTRACT: A characterization for a Riesz s -potential function of a multiset ω_N of N points in $\mathbb{R}^2$ is given when $s = 2 - 2N$ and the potential function is constant on a circle in $\mathbb{R}^2$. The characterization allows us to partially prove a conjecture of Nikolov and Rafailov that if the potential function is constant on a circle Γ then the points in ω_N should be equally spaced on a circle concentric to Γ . As an application of constant Riesz s -potential functions, we also find all maximal and minimal polarization constants and configurations of two concentric circles in $\mathbb{R}^2$ for certain values of s .

KEYWORDS: roots of unity, max-min and min-max problems

MSC2010: 52A40

INTRODUCTION

For a fixed multiset of N points $\omega_N := \{x_1, x_2, \dots, x_N\}$ in $\mathbb{R}^2$ and a given constant $s \in \mathbb{R}$, we define the Riesz potential function $U^s(\cdot; \omega_N) : \mathbb{R}^2 \rightarrow [0, \infty]$ as

$$U^s(x; \omega_N) := \sum_{j=1}^N |x - x_j|^{-s},$$

where $x \in \mathbb{R}^2$ and $|\cdot|$ is the 2-dimensional Euclidean norm in $\mathbb{R}^2$. We call $U^s(\cdot; \omega_N)$ a *Riesz s -potential function* of ω_N . See Ref. 1 for more information on Riesz s -potential functions in a d -dimensional Euclidean space $\mathbb{R}^d$.

In this paper, we consider two problems concerning the Riesz s -potential functions $U^s(\cdot; \omega_N)$. The first problem is to prove, in parts, Nikolov and Rafailov's conjecture about points in ω_N being equally spaced on some circle when a Riesz s -potential function is constant. The second problem is to solve polarization optimality problems when this Riesz s -potential function is constant.

Let ω_N be a fixed set of distinct equally spaced points on a circle $T \subseteq \mathbb{R}^2$ and Γ be a circle concentric to T . In Ref. 2, Nikolov and Rafailov show in Theorem 1 that $U^s(x; \omega_N)$ is constant as a function of x on Γ if and only if $s \in \{0, -2, -4, \dots, 4 - 2N, 2 -$

$2N\}$. They also show in Theorem 2 that this gives a characteristic property of distinct equally spaced points on a circle. More precisely, given a set ω_N of N distinct points such that $U^s(x; \omega_N)$ is constant on a circle Γ for every $s \in \{-2, -4, \dots, 2 - 2N\}$ (the constant may depend on s), the points in ω_N are equally spaced on some circle concentric to Γ . In the same paper, it was conjectured (Conjecture 2) that only $s = 2 - 2N$ should be sufficient. We state the conjecture below.

Conjecture 1 Given a set of N distinct points $\omega_N := \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{R}^2$ and a circle $\Gamma \subseteq \mathbb{R}^2$ such that

$$U^{2-2N}(x; \omega_N) = \sum_{j=1}^N |x - x_j|^{2N-2}$$

is constant as a function of x on Γ . Then ω_N forms a set of distinct equally spaced points on a circle concentric to Γ .

The conjecture was verified in the case $N = 3$ (see Ref. 2, Proposition 2). In this paper, we prove Conjecture 1 in the following cases (after translating the centre of Γ to the origin):

(i) when all points $x_1, x_2, \dots, x_N$ have the same norm (Proposition 1);

- (ii) when $N = 4$ and x_1, x_2, x_3, x_4 have an equal angle distribution (Proposition 2);
- (iii) when N is prime and $x_1, x_2, \dots, x_N$ have an equal angle distribution and rational norms (Proposition 3).

The above results are based on a characterization of ω_N when $U^{2-2N}(\cdot; \omega_N)$ is constant on the unit circle (Theorem 1).

The next problems considered in this paper are polarization optimality problems. Let $\omega_N = \{x_1, \dots, x_N\}$ denote a configuration of N (not necessarily distinct) points in $\mathbb{R}^2$. Denote by

$$\mathbb{S}_R^1 := \{x \in \mathbb{R}^2 : |x| = R\}$$

the circle centred at the origin of radius R . When $R = 1$, we simply use the notation $\mathbb{S}^1$. Given $s \in \mathbb{R}$, $R > 0$, and $r > 0$, we define polarization constants

$$M_N^s(\mathbb{S}_r^1; \mathbb{S}_R^1) := \max_{\substack{\omega_N \subseteq \mathbb{S}_r^1 \\ \#\omega_N = N}} \min_{y \in \mathbb{S}_R^1} U^s(y; \omega_N), \quad (1)$$

$$M_N^0(\mathbb{S}_r^1; \mathbb{S}_R^1) := N,$$

$$m_N^s(\mathbb{S}_r^1; \mathbb{S}_R^1) := \min_{\substack{\omega_N \subseteq \mathbb{S}_r^1 \\ \#\omega_N = N}} \max_{y \in \mathbb{S}_R^1} U^s(y; \omega_N), \quad (2)$$

$$m_N^0(\mathbb{S}_r^1; \mathbb{S}_R^1) := N,$$

where $\#\omega_N$ denotes the cardinality of the multiset ω_N . We will call ω_N a *maximal (minimal) N -point Riesz s -polarization configuration* of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$ if ω_N attains the maximum in (1) (minimum in (2)). We give a brief history of such polarization optimality problems below.

Farkas and Révész³ were the first to introduce two-plate polarization constants in a general sense. However, all previous results⁴⁻⁶ on polarization optimality problems related to Riesz potentials were considered for the case when $R = r = 1$. The maximality of N distinct equally spaced points on the unit circle for the maximal Riesz s -polarization problem of $(\mathbb{S}^1; \mathbb{S}^1)$ in (1) was proved in Ref. 4 for $s = 2$. Erdélyi and Saff¹ established this for $s = 4$. For arbitrary $s > 0$, this result was proved in Ref. 5 where they also showed the minimality of N distinct equally spaced points on the unit circle for the minimal Riesz s -polarization problem of $(\mathbb{S}^1; \mathbb{S}^1)$ in (2) for $-1 \leq s < 0$. Note that minimal N -point Riesz s -polarization problems of $(\mathbb{S}^1; \mathbb{S}^1)$ when $s > 0$ are not interesting because $m_N^s(\mathbb{S}^1; \mathbb{S}^1) = \infty$ for all $s > 0$.

Up to the present, there are no results on polarization optimality problems in (1) and (2) for

$R \neq r$. In this paper, we give a characterization of all maximal and minimal N -point Riesz s -polarization configurations of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$ when $s = -2, -4, \dots, 2 - 2N$.

Although the asymptotic properties of polarization constants are not our main interest in this paper, it is worth mentioning the asymptotic types of behaviour of $M_N^s(\mathbb{S}^1; \mathbb{S}^1)$ as $N \rightarrow \infty$ ⁵:

$$M_N^s(\mathbb{S}^1; \mathbb{S}^1) \sim \begin{cases} \frac{2\zeta(s)}{(2\pi)^s} (2^s - 1) N^s, & s > 1, \\ (1/\pi) N \log N, & s = 1, \\ \frac{2^{-s}}{\sqrt{\pi}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(1-\frac{s}{2})} N, & 0 \leq s < 1, \end{cases}$$

where $\zeta(s)$ denotes the classical Riemann zeta function and $a_N \sim b_N$ means that $\lim_{N \rightarrow \infty} a_N/b_N = 1$. The reader is referred to Refs. 1, 7, 8 for asymptotic results of polarization constants and configurations of general subsets of $\mathbb{R}^d$ as $N \rightarrow \infty$ when $s > 0$.

CONSTANT RIESZ s -POTENTIAL FUNCTIONS

The Euclidean space $\mathbb{R}^2$ and the complex space $\mathbb{C}$ over $\mathbb{R}$ have the same dimension and the same norm. However, the complex space $\mathbb{C}$ has a richer algebraic structure; for example, $\mathbb{C}$ is a field. Hence when we prove all theorems in this and the next section, any element $x \in \mathbb{R}^2$ will be replaced by $x \in \mathbb{C}$, the 2-dimensional Euclidean norm $|\cdot|$ is replaced by the modulus in $\mathbb{C}$, and the notation xy is adopted from the multiplication in $\mathbb{C}$ and the notation x/y is adopted from the division in $\mathbb{C}$. We recall that the usual dot product in $\mathbb{C}$ is defined by

$$(a_1 + a_2 i) \cdot (b_1 + b_2 i) := a_1 b_1 + a_2 b_2.$$

Now let $\omega_N := \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{C}$ be a set of N distinct points. In this section, we will assume that $U^{2-2N}(x; \omega_N) = \sum_{j=1}^N |x - x_j|^{2N-2}$ is constant (as a function of x) on a circle $\Gamma \subset \mathbb{C}$ and prove that, under various conditions, the points $x_1, x_2, \dots, x_N$ are equally spaced on some circle concentric to Γ . By translating and scaling the circle Γ , we can assume without loss of generality that Γ is the unit circle $\mathbb{S}^1$. The following conjecture is equivalent to Conjecture 1.

Conjecture 2 Given a set of N distinct points $\omega_N := \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{C}$ such that

$$U^{2-2N}(x; \omega_N) = \sum_{j=1}^N |x - x_j|^{2N-2}$$

is constant as a function of x on $\mathbb{S}^1$, then ω_N forms a set of distinct equally spaced points on $\mathbb{S}_R^1$ for some R .

We begin with our main theorem which gives a characterization of ω_N when $U^{2-2N}(\cdot; \omega_N)$ is constant on the unit circle.

Theorem 1 Let $\omega_N = \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{C}$ be a set of N distinct points. Then the function

$$U^{2-2N}(x; \omega_N) = \sum_{j=1}^N |x - x_j|^{2N-2}$$

is constant on the circle $\mathbb{S}^1$ if and only if

$$\sum_{j=1}^N \sum_{q=0}^{N-k-1} \binom{N-1}{q} \binom{N-1}{k+q} |x_j|^{2N-2k-2q-2} x_j^k = 0, \quad \text{for all } k = 1, \dots, N-1. \quad (3)$$

Note that (3) gives a system of $N-1$ equations in terms of elements in the set ω_N . The proof of Theorem 1 requires a technical lemma which involves a lot of calculations, and so we will postpone it to the end of this section.

Example 1 Suppose $U^{2-2N}(x; \omega_N)$ is constant on $\mathbb{S}^1$. We list the systems of equations (3) that the x_j must satisfy for small values of N below.

(i) Let $N = 3$. Then x_1, x_2, x_3 must satisfy

$$\sum_{j=1}^3 x_j^2 = 0, \quad \sum_{j=1}^3 (1 + |x_j|^2) x_j = 0.$$

(ii) Let $N = 4$. Then x_1, x_2, x_3, x_4 must satisfy

$$\begin{aligned} \sum_{j=1}^4 x_j^3 &= 0, & \sum_{j=1}^4 (1 + |x_j|^2) x_j^2 &= 0, \\ \sum_{j=1}^4 (1 + 3|x_j|^2 + |x_j|^4) x_j &= 0. \end{aligned}$$

(iii) Let $N = 5$. Then x_1, x_2, x_3, x_4, x_5 must satisfy

$$\begin{aligned} \sum_{j=1}^5 x_j^4 &= 0, & \sum_{j=1}^5 (1 + |x_j|^2) x_j^3 &= 0, \\ \sum_{j=1}^5 (3 + 8|x_j|^2 + 3|x_j|^4) x_j^2 &= 0, \\ \sum_{j=1}^5 (1 + 5|x_j|^2 + |x_j|^4) (1 + |x_j|^2) x_j &= 0. \end{aligned}$$

Using the characterization given in Theorem 1, we can verify Conjecture 2 in various cases. Our first result asserts that Conjecture 2 holds if the points $x_1, x_2, \dots, x_N$ already lie on the same circle centred at the origin (i.e., they have the same norm).

Proposition 1 Let $\omega_N = \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{C}$ be a set of N distinct non-zero points lying on some circle centred at the origin. If $U^{2-2N}(\cdot; \omega_N)$ is constant on $\mathbb{S}^1$, then $x_1, x_2, \dots, x_N$ are equally spaced.

Proof: It suffices to show that $x_1, x_2, \dots, x_N$ are the N th roots of some complex number. Suppose $|x_1| = |x_2| = \dots = |x_N| = R$. From (3) we deduce that

$$\sum_{j=1}^N x_j^k = 0,$$

for all $k = 1, 2, \dots, N-1$. By Newton's identities,

$$e_k(x_1, x_2, \dots, x_N) = 0, \quad k = 1, 2, \dots, N-1,$$

where the e_k are elementary symmetric polynomials. Thus $x_1, x_2, \dots, x_N$ are distinct roots of the polynomial

$$\prod_{k=1}^N (X - x_k) = X^N - \mu$$

for some $\mu \in \mathbb{C}$. $\square$

Now we will consider another special case. Instead of assuming that all points have the same norm, we will assume that they have an equal angle distribution around the origin. More precisely, let $\zeta = e^{2\pi i/N}$ and, without loss of generality, we assume that

$$x_1 = r_1 \zeta^1, x_2 = r_2 \zeta^2, \dots, x_N = r_N \zeta^N \quad (4)$$

for some positive real numbers $r_1, r_2, \dots, r_N$. Our next result proves Conjecture 2 when $N = 4$ and x_1, x_2, x_3, x_4 have an equal angle distribution.

Proposition 2 Let x_1, x_2, x_3, x_4 be as in (4). Suppose that

$$U^{-6}(x; \omega_N) := \sum_{j=1}^N |x - x_j|^6$$

is constant as a function of x on $\mathbb{S}^1$. Then x_1, x_2, x_3, x_4 are equally spaced on a circle centred at the origin.

Proof: By Proposition 1, it suffices to show that $|x_1| = |x_2| = |x_3| = |x_4|$. From Example 1, the points

x_1, x_2, x_3, x_4 must satisfy

$$\begin{aligned}\sum_{j=1}^4 x_j^3 &= \sum_{j=1}^4 (1 + |x_j|^2) x_j^2 \\ &= \sum_{j=1}^4 (1 + 3|x_j|^2 + |x_j|^4) x_j = 0.\end{aligned}$$

With $x_j = r_j \zeta^j$, the equation $\sum_{j=1}^4 x_j^3 = 0$ becomes

$$r_1^3 \zeta^3 + r_2^3 \zeta^2 + r_3^3 \zeta + r_4^3 = 0.$$

Let $P(X) = r_1^3 X^3 + r_2^3 X^2 + r_3^3 X + r_4^3 \in \mathbb{R}[X]$. Since $\zeta = i, \bar{\zeta} = -i$ are roots of $P(X)$, we have

$$P(X) = C(X^2 + 1)(X + b) = C(X^3 + bX^2 + X + b),$$

for some non-zero $C \in \mathbb{R}$. Comparing the coefficients, we have $r_1 = r_3, r_2 = r_4$.

The equation $\sum_{j=1}^4 (1 + |x_j|^2) x_j^2 = 0$ becomes $\sum_{j=1}^4 (1 + r_j^2) r_j^2 \zeta^{2j} = 0$. Expanding the sum and using $r_1 = r_3, r_2 = r_4$ we have

$$2(1 + r_1^2) r_1^2 \zeta^2 + 2(1 + r_2^2) r_2^2 = 0.$$

Since $\zeta^2 = -1$ we obtain $(1 + r_1^2) r_1^2 = (1 + r_2^2) r_2^2$. Let $t = r_2/r_1$ and $a = 1/r_1^2$. We have

$$(a + 1) = (a + t^2) t^2 \implies t^4 + at^2 - (a + 1) = 0.$$

Thus

$$t^2 = \frac{-a \pm \sqrt{a^2 + 4a + 4}}{2} = \frac{-a \pm (a + 2)}{2}.$$

The only possible case is $t^2 = \frac{1}{2}(-a + (a + 2)) = 1$. Since $t > 0$ we have $t = 1$. Hence $r_2 = r_1$. We have shown that $r_1 = r_2 = r_3 = r_4$. $\square$

Actually, if we further assume that all norms are rational, then Conjecture 2 holds for all prime N .

Proposition 3 Let N be a prime number. Let $x_1, x_2, \dots, x_N$ be as in (4) where all $r_j \in \mathbb{Q}$. Suppose that $U^{2-2N}(\cdot; \omega_N)$ is constant on $\mathbb{S}^1$. Then $x_1, x_2, \dots, x_N$ are equally spaced on a circle centred at the origin.

Proof: By Proposition 1, it suffices to show that $|x_1| = |x_2| = \dots = |x_N|$. Applying the condition (3) with $k = N - 1$ gives $\sum_{j=1}^N x_j^{N-1} = 0$. Thus

$$\sum_{j=1}^N r_j^{N-1} \zeta^{-j} = \sum_{j=1}^N r_{N-j}^{N-1} \zeta^j = 0.$$

Let A be a positive integer so that $Ar_{N-j}^{N-1} \in \mathbb{Z}_{>0}$ for every j . Then $\sum_{j=1}^N (Ar_{N-j}^{N-1}) \zeta^k = 0$. This is a vanishing linear combination of $1, \zeta, \dots, \zeta^{N-1}$ with positive-integer coefficients. Since the minimal polynomial of ζ is $1 + X + \dots + X^{N-1}$ (N is prime), this implies that all coefficients are equal. Thus $Ar_1^{N-1} = Ar_2^{N-1} = \dots = Ar_N^{N-1}$ and hence $r_1 = r_2 = \dots = r_N$. $\square$

Proof of Theorem 1

The following technical lemma is needed for the proofs of Theorem 1 and Theorem 2.

Lemma 1 Let $N \in \mathbb{N}$ and $p \in \{1, 2, \dots, N-1\}$ be fixed. If $x_j := |x_j| \cos t_j + i|x_j| \sin t_j$ for all $j = 1, 2, \dots, N$, then for all $y := \cos t + i \sin t \in \mathbb{S}^1$,

$$\begin{aligned}\sum_{j=1}^N |y - x_j|^{2p} &= E_0 + \sum_{k=1}^p \sum_{j=1}^N [E_{k,j} \cos kt_j \cos kt + E_{k,j} \sin kt_j \sin kt],\end{aligned}\tag{5}$$

where

$$\begin{aligned}E_0 &= \sum_{j=1}^N \sum_{q=0}^p \binom{p}{q}^2 |x_j|^{2p-2q}, \\ E_{k,j} &= (-1)^k 2 \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} |x_j|^{2p-k-2q}.\end{aligned}$$

Proof: Let $y := \cos t + i \sin t \in \mathbb{S}^1$ and $x_j := |x_j| \cos t_j + i|x_j| \sin t_j$ for all $j = 1, 2, \dots, N$. A simple calculation shows that

$$f_j(t) := |y - x_j|^{2p} = (|x_j|^2 + 1 - 2|x_j| \cos(t - t_j))^p.$$

Since $A := \{1, \cos(t - t_j), \dots, \cos p(t - t_j)\}$ forms an orthogonal system with respect to the inner product

$$\langle f, g \rangle := \int_0^{2\pi} f(t) g(t) dt$$

and $f_j \in \text{span}(A)$, we have

$$\begin{aligned}f_j(t) &= \sum_{k=0}^p E_{k,j} \cos k(t - t_j) = E_{0,j} \\ &\quad + \sum_{k=1}^p E_{k,j} (\cos kt_j \cos kt + \sin kt_j \sin kt).\end{aligned}$$

Now,

$$\begin{aligned} \sum_{j=1}^N |y - x_j|^{2p} &= \sum_{j=1}^N f_j(t) \\ &= E_0 + \sum_{k=1}^p \sum_{j=1}^N [E_{k,j} \cos kt_j \cos kt + E_{k,j} \sin kt_j \sin kt], \end{aligned}$$

where $E_0 = \sum_{j=1}^N E_{0,j}$. By the orthogonality of the elements in the set A and the calculation in Lemma 3 in the last section, we have

$$E_0 = \sum_{j=1}^N \frac{\langle f_j, 1 \rangle}{2\pi} = \sum_{j=1}^N \sum_{q=0}^p \binom{p}{q}^2 |x_j|^{2p-2q}$$

and

$$\begin{aligned} E_{k,j} &= \frac{\langle f_j, \cos k(t - t_j) \rangle}{\pi} \\ &= (-1)^k 2 \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} |x_j|^{2p-k-2q}, \end{aligned}$$

for all $k \in \{1, 2, \dots, p\}$ and $j \in \{1, 2, \dots, N\}$. $\square$

Proof of Theorem 1: For each $j = 1, 2, \dots, N$, set

$$x_j := |x_j| \cos t_j + i |x_j| \sin t_j.$$

($\Rightarrow$) By our assumption, $f(y) := \sum_{j=1}^N |y - x_j|^{2N-2}$ is constant on $\mathbb{S}^1$, say $f(y) = C$ on $\mathbb{S}^1$. Set

$$y = \cos t + i \sin t \in \mathbb{S}^1.$$

By (5) for all $t \in [0, 2\pi]$,

$$\begin{aligned} C = f(y) &= E_0 \\ &+ \sum_{k=1}^{N-1} \sum_{j=1}^N [E_{k,j} \cos kt_j \cos kt + E_{k,j} \sin kt_j \sin kt]. \end{aligned}$$

(6)

Because the set $\{1, \cos t, \sin t, \dots, \cos(N-1)t, \sin(N-1)t\}$ is linearly independent over $\mathbb{R}$, we deduce

$$C - E_0 = 0$$

and for all $k = 1, 2, \dots, N-1$,

$$\sum_{j=1}^N E_{k,j} \cos kt_j = 0 \quad \text{and} \quad \sum_{j=1}^N E_{k,j} \sin kt_j = 0. \quad (7)$$

Using the formulae of $E_{k,j}$ from Lemma 1, it follows

from (7) that for all $k = 1, 2, \dots, N-1$,

$$\begin{aligned} 0 &= \sum_{j=1}^N E_{k,j} (\cos kt_j + i \sin kt_j) = \sum_{j=1}^N \frac{E_{k,j}}{|x_j|^k} x_j^k \\ &= (-1)^k 2 \sum_{j=1}^N \sum_{q=0}^{N-k-1} \binom{N-1}{q} \binom{N-1}{k+q} \\ &\quad \times |x_j|^{2N-2k-2q-2} x_j^k, \end{aligned} \quad (8)$$

which implies (3).

($\Leftarrow$) Assume that (3) holds. Then we have (8) and (7). Combining (7) and the second identity in (6), we have for all $y \in \mathbb{S}^1$,

$$\sum_{j=1}^N |y - x_j|^{2N-2} = E_0,$$

which implies that $U^{2-2N, h}(\cdot; \omega_N)$ is constant on $\mathbb{S}^1$. $\square$

AN APPLICATION TO POLARIZATION OPTIMALITY PROBLEMS

We remind the reader that we will consider polarization optimality problems in the complex plane. A complete characterization of all maximal and minimal N -point Riesz s -polarization configurations of $(\mathbb{S}_r^1, \mathbb{S}_R^1)$ when $s = -2, -4, \dots, 2-2N$ is given as follows.

Theorem 2 Let $N \in \mathbb{N}$, $p \in \{1, 2, \dots, N-1\}$, $R > 0$, $r > 0$, and $\{x_1, x_2, \dots, x_N\} \subseteq \mathbb{S}_r^1$. The following statements are equivalent:

- (a) $\{x_1, x_2, \dots, x_N\}$ is a maximal N -point Riesz $-2p$ -polarization configuration of $(\mathbb{S}_r^1, \mathbb{S}_R^1)$;
- (b) $\{x_1, x_2, \dots, x_N\}$ is a minimal N -point Riesz $-2p$ -polarization configuration of $(\mathbb{S}_r^1, \mathbb{S}_R^1)$;
- (c) $\sum_{j=1}^N x_j = \sum_{j=1}^N x_j^2 = \dots = \sum_{j=1}^N x_j^p = 0$.

Furthermore,

$$\begin{aligned} M_N^{-2p}(\mathbb{S}_r^1, \mathbb{S}_R^1) &= m_N^{-2p}(\mathbb{S}_r^1, \mathbb{S}_R^1) \\ &= \frac{N}{2^p} \sum_{j=0}^p \binom{p}{j}^2 (2rR)^{2j} (r^2 + R^2 + |r^2 - R^2|)^{p-2j}. \end{aligned} \quad (9)$$

Unlike the case when $R = r = 1$ and $s > 0$, optimal configurations for the cases in Theorem 2 may not be unique up to rotation. For example, when $p = 1$ and $N = 4$, our characterization of optimal configurations is $\sum_{j=1}^4 x_j = 0$. Hence there are infinitely many optimal configurations that are not rotations of one another. The proof of Theorem 2 relies on

the fact that if ω_N is a configuration of N distinct equally spaced points on $\mathbb{S}_r^1$, then for each $s = -2, -4, \dots, 2-2N$, $U^s(\cdot, \omega_N)$ is constant on $\mathbb{S}_r^1$. This special property allows the problems to have more than one solution (up to rotation). Furthermore, our experimental study suggests that for the cases when $s \in \mathbb{R}^2 \setminus \{0, -2, -4, \dots, 2-2N\}$, any maximal and minimal N -point Riesz s -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$ is unique up to rotation, namely, it is a configuration of distinct equally spaced points on $\mathbb{S}_r^1$. We make the following conjecture.

Conjecture 3 Let $N \in \mathbb{N}$, $s \in \mathbb{R} \setminus \{0, -2, -4, \dots, 2-2N\}$, $R > 0$, $r > 0$, and $\{x_1, x_2, \dots, x_N\} \subseteq \mathbb{S}_r^1$. The following statements are equivalent:

- (a) $\{x_1, x_2, \dots, x_N\}$ is a maximal N -point Riesz s -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$;
- (b) $\{x_1, x_2, \dots, x_N\}$ is a minimal N -point Riesz s -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$;
- (c) $\{x_1, x_2, \dots, x_N\}$ is a configuration of distinct equally spaced points on $\mathbb{S}_r^1$.

Proof of Theorem 2

We need the following lemma.

Lemma 2 Let $N \in \mathbb{N}$, $p \in \{1, 2, \dots, N-1\}$, $R > 0$, and $r > 0$. Then any configuration of N distinct equally spaced points on $\mathbb{S}_r^1$ is both a maximal and a minimal N -point Riesz $-2p$ -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$.

Proof: Let $\omega_N := \{x_1, x_2, \dots, x_N\}$ be a configuration of N distinct equally spaced points on $\mathbb{S}_r^1$ and $p \in \{1, 2, \dots, N-1\}$ be fixed. By Theorem 1 in Ref. 2, we know that $f(x) := \sum_{j=1}^N |x - x_j|^{2p}$ is constant as a function of x on $\mathbb{S}_R^1$, say $f(x) = C$ for all $x \in \mathbb{S}_R^1$. Thus

$$\max_{x \in \mathbb{S}_R^1} \sum_{i=1}^N |x_i - x|^{2p} = C = \min_{x \in \mathbb{S}_R^1} \sum_{i=1}^N |x_i - x|^{2p}. \quad (10)$$

Let $\{y_1, y_2, \dots, y_N\}$ be any N -point configuration on $\mathbb{S}_r^1$. To show that ω_N is a minimal N -point Riesz $-2p$ -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$, we will show that

$$\max_{x \in \mathbb{S}_R^1} \sum_{i=1}^N |x_i - x|^{2p} \leq \max_{x \in \mathbb{S}_R^1} \sum_{i=1}^N |y_i - x|^{2p}. \quad (11)$$

Consider

$$\left| x_j - \frac{R}{y_i/r} \right| = \left| \frac{x_j}{y_i} \left(y_i - \frac{R}{x_j/r} \right) \right| = \left| y_i - \frac{R}{x_j/r} \right|.$$

As $R/(y_i/r) \in \mathbb{S}_R^1$ for all i , we have

$$\begin{aligned} NC &= \sum_{i=1}^N f\left(\frac{R}{y_i/r}\right) = \sum_{i=1}^N \sum_{j=1}^N \left| x_j - \frac{R}{y_i/r} \right|^{2p} \\ &= \sum_{j=1}^N \sum_{i=1}^N \left| y_i - \frac{R}{x_j/r} \right|^{2p}. \end{aligned} \quad (12)$$

It follows from (12) that there is $j_0 \in \{1, 2, \dots, N\}$ such that

$$C \leq \sum_{i=1}^N \left| y_i - \frac{R}{x_{j_0}/r} \right|^{2p} \leq \max_{x \in \mathbb{S}_R^1} \sum_{i=1}^N |y_i - x|^{2p}.$$

But $C = \max_{x \in \mathbb{S}_R^1} \sum_{i=1}^N |x_i - x|^{2p}$ from (10). Hence we have (11) as required.

To show that ω_N is a maximal N -point Riesz $-2p$ -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$, we will show that

$$\min_{x \in \mathbb{S}_R^1} \sum_{i=1}^N |y_i - x|^{2p} \leq \min_{x \in \mathbb{S}_R^1} \sum_{i=1}^N |x_i - x|^{2p}. \quad (13)$$

It follows from (12) that there is $j'_0 \in \{1, 2, \dots, N\}$ such that

$$\min_{x \in \mathbb{S}_R^1} \sum_{i=1}^N |y_i - x|^{2p} \leq \sum_{i=1}^N \left| y_i - \frac{R}{x_{j'_0}/r} \right|^{2p} \leq C.$$

But $C = \min_{x \in \mathbb{S}_R^1} \sum_{i=1}^N |x_i - x|^{2p}$ from (10). Hence we have (13) as required. $\square$

Proof of Theorem 2: Because the proof of (a) $\Leftrightarrow$ (c) is similar to the proof of (b) $\Leftrightarrow$ (c), we will show only (b) $\Leftrightarrow$ (c) and skip the proof of (a) $\Leftrightarrow$ (c). Without loss of generality, we can assume that $R = 1$.

Let $N \in \mathbb{N}$, $p \in \{1, 2, \dots, N-1\}$, and $r > 0$ be fixed and $\{x_1, x_2, \dots, x_N\}$ be any configuration in $\mathbb{S}_r^1$. We recall from Lemma 1 that for $x_j := r \cos t_j + ir \sin t_j$ for all $j = 1, 2, \dots, N$ and for all $y := \cos t + i \sin t \in \mathbb{S}^1$,

$$\begin{aligned} \sum_{j=1}^N |y - x_j|^{2p} &= E_0 \\ &+ \sum_{k=1}^p \sum_{j=1}^N [E_{k,j} \cos kt_j \cos kt + E_{k,j} \sin kt_j \sin kt], \end{aligned} \quad (14)$$

$$= E_0 + \sum_{k=1}^p \sum_{j=1}^N \left[\frac{E_{k,j}}{r^k} (y^k \cdot x_j^k) \right], \quad (15)$$

where

$$E_0 = \sum_{j=1}^N \sum_{q=0}^p \binom{p}{q}^2 r^{2p-2q},$$

$$\frac{E_{k,j}}{r^k} = (-1)^k 2 \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} r^{2p-2k-2q}. \quad (16)$$

Notice that the constant E_0 does not depend on a configuration on $\mathbb{S}_r^1$ and the constants $E_{k,j}/r^k$ do not depend on a configuration on $\mathbb{S}_r^1$ and j . For convenience for all configurations $\{x_1, x_2, \dots, x_N\} \subseteq \mathbb{S}_r^1$, we set

$$E_k := \frac{E_{k,j}}{r^k}, \quad \text{for all } k = 1, 2, \dots, p. \quad (17)$$

First of all, we will show that

$$m_N^{-2p}(\mathbb{S}_r^1, \mathbb{S}^1) = E_0.$$

Let $\omega'_N := \{x'_1, x'_2, \dots, x'_N\}$ be a configuration of distinct equally spaced points on $\mathbb{S}_r^1$. Using (15), we have for all $y \in \mathbb{S}^1$,

$$\begin{aligned} \sum_{j=1}^N |y - x'_j|^{2p} &= E_0 + \sum_{k=1}^p \sum_{j=1}^N E_k (y^k \cdot (x'_j)^k) \\ &= E_0 + \sum_{k=1}^p E_k \left(y^k \cdot \sum_{j=1}^N (x'_j)^k \right) = E_0 \end{aligned} \quad (18)$$

where the last equality follows from the fact that $\sum_{j=1}^N (x'_j)^k = 0$ for all $k = 1, 2, \dots, p$. Since ω'_N is a minimal N -point Riesz $-2p$ -polarization configuration of $(\mathbb{S}_r^1, \mathbb{S}^1)$ (by Lemma 2), we obtain

$$m_N^{-2p}(\mathbb{S}_r^1, \mathbb{S}^1) = \max_{y \in \mathbb{S}^1} U^{-2p}(y; \omega'_N) = E_0.$$

We now prove (c) $\Rightarrow$ (b). Assume that $\omega_N = \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{S}_r^1$ such that $\sum_{j=1}^N x_j^k = 0$ for all $k = 1, 2, \dots, p$. Applying the same argument as in (18), we have for all $y \in \mathbb{S}^1$,

$$U^{-2p}(y; \omega_N) = E_0 + \sum_{k=1}^p E_k \left(y^k \cdot \sum_{j=1}^N x_j^k \right) = E_0,$$

which implies that ω_N is a minimal N -point Riesz $-2p$ -polarization configuration of $(\mathbb{S}_r^1, \mathbb{S}^1)$.

Next, we show (b) $\Rightarrow$ (c). Assume that $\omega_N = \{x_1, x_2, \dots, x_N\}$ is a minimal N -point Riesz $-2p$ -polarization configuration of $(\mathbb{S}_r^1, \mathbb{S}^1)$. Then for all $y \in \mathbb{S}^1$,

$$U^{-2p}(y; \omega_N) = \sum_{j=1}^N |y - x_j|^{2p} \leq m_N^{-2p}(\mathbb{S}_r^1, \mathbb{S}^1) = E_0.$$

Then, by (14) and (17) for all $t \in [0, 2\pi]$,

$$E_0 \geq U^{-2p}(y; \omega_N) = E_0 + \sum_{k=1}^p (\mathcal{C} \cos kt + \mathcal{S} \sin kt),$$

where $\mathcal{C} = \sum_{j=1}^N E_k \cos kt_j$ and $\mathcal{S} = \sum_{j=1}^N E_k \sin kt_j$. Thus for all $t \in [0, 2\pi]$,

$$0 \geq \sum_{k=1}^p (\mathcal{C} \cos kt + \mathcal{S} \sin kt).$$

Hence for all $t \in [0, 2\pi]$,

$$\sum_{k=1}^p (\mathcal{C} \cos kt + \mathcal{S} \sin kt) = 0.$$

Because $\{\cos t, \sin t, \cos 2t, \sin 2t, \dots, \cos pt, \sin pt\}$ is a linearly independent set over $\mathbb{R}$ for all $k = 1, 2, \dots, p$,

$$\sum_{j=1}^N E_k \cos kt_j = \sum_{j=1}^N E_k \sin kt_j = 0.$$

Since for all $k = 1, 2, \dots, p$, $E_k \neq 0$ ((16)),

$$\sum_{j=1}^N \cos kt_j = \sum_{j=1}^N \sin kt_j = 0, \quad k = 1, 2, \dots, p,$$

which implies that $\sum_{j=1}^N x_j^k = \sum_{j=1}^N r^k (\cos kt_j + i \sin kt_j) = 0$ for all $k = 1, 2, \dots, p$.

To compute $M_N^{-2p}(\mathbb{S}_r^1, \mathbb{S}_R^1)$ and $m_N^{-2p}(\mathbb{S}_r^1, \mathbb{S}_R^1)$ in (9), we can use a similar argument in Lemma 1 by replacing $y = R \cos t + iR \sin t$ and $f_j(t) = |y - x_j|^{2p} = (r^2 + R^2 - 2Rr \cos(t - t_j))^p$. Applying the calculations as in Lemma 4, it is not difficult to check that if ω_N is a configuration of N distinct equally spaced points on $\mathbb{S}_r^1$, then for all $y \in \mathbb{S}_R^1$,

$$\begin{aligned} U^{-2p}(y; \omega_N) &= \frac{N}{2^p} \sum_{j=0}^p \binom{p}{j}^2 (2rR)^{2j} (r^2 + R^2 + |r^2 - R^2|)^{p-2j}. \end{aligned}$$

□

COMPUTATIONS OF INTEGRALS

We collect our computations of all integrals in this section.

Lemma 3 Let $p \in \mathbb{N}$, $k \in \{0, 1, \dots, p\}$, and $z \in \mathbb{C}$. Then

$$\begin{aligned} \int_0^{2\pi} (z^2 + 1 - 2z \cos t)^p \cos kt \, dt \\ = (-1)^k 2\pi \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} z^{2p-k-2q}. \end{aligned} \quad (19)$$

Proof: Let $p \in \mathbb{N}$ and $k \in \{0, 1, \dots, p\}$. First, we prove the equality (19) for $z \in \mathbb{R}$. Let $z \in \mathbb{R}$. Then, for $\zeta = e^{it}$,

$$\begin{aligned} & \int_0^{2\pi} (z^2 + 1 - 2z \cos t)^p \cos kt \, dt \\ &= \int_0^{2\pi} (z^2 + 1 - z(e^{it} + e^{-it}))^p e^{ikt} \, dt \\ &= \int_0^{2\pi} (z - e^{it})^p (z - e^{-it})^p e^{ikt} \, dt \\ &= \frac{1}{i} \int_{\mathbb{S}^1} (z - \zeta)^p (z - 1/\zeta)^p \zeta^{k-1} \, d\zeta \\ &= 2\pi \operatorname{Res} \left(\frac{(z - \zeta)^p (z\zeta - 1)^p}{\zeta^{p-k+1}}; \zeta = 0 \right) \\ &= (-1)^k 2\pi \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} z^{2p-k-2q}, \end{aligned}$$

where the first equality follows from the fact that the last expression is a real number. Notice that the left-hand side and the right-hand side of (19) are polynomials as functions of z . Thus both functions are analytic on $\mathbb{C}$ and we have (19) for all $z \in \mathbb{C}$. $\square$

Lemma 4 Let $p \in \mathbb{N}$ and $k \in \{0, 1, \dots, p\}$. For $a, b \in \mathbb{C}$,

$$\begin{aligned} & \int_0^{2\pi} (a - b \cos t)^p \cos kt \, dt \\ &= \frac{(-1)^k \pi}{2^{p-1}} \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} C_{a,b,p,q,k}, \quad (20) \end{aligned}$$

where $C_{a,b,p,q,k} = b^{2q+k} (a \pm \sqrt{a^2 - b^2})^{p-k-2q}$ and the square root function in (20) can be selected to be both branches of the complex square root function.

Proof: Clearly, if $b = 0$, then the equation in (20) is $0 = 0$. Assume that $b \in \mathbb{C} \setminus \{0\}$ and $a \in \mathbb{C}$. To reduce (20) to (19), we consider

$$(\lambda a - \lambda b \cos t)^p,$$

where λ is chosen to satisfy the equations

$$2z = b\lambda, \quad z^2 + 1 = a\lambda,$$

for some $z \in \mathbb{C}$. From the above equations,

$$z = \frac{a \pm \sqrt{a^2 - b^2}}{b}$$

and

$$\lambda = \frac{2z}{b} = \frac{2a \pm 2\sqrt{a^2 - b^2}}{b^2}.$$

Furthermore, $\lambda \neq 0$ because if $\lambda = 0$, then $z = 0$ which implies that $b = 0$. Hence by Lemma 3,

$$\begin{aligned} & \int_0^{2\pi} (a - b \cos t)^p \cos kt \, dt \\ &= \frac{1}{\lambda^p} \int_0^{2\pi} (\lambda a - \lambda b \cos t)^p \cos kt \, dt \\ &= \frac{1}{\lambda^p} \int_0^{2\pi} (z^2 + 1 - 2z \cos t)^p \cos kt \, dt \\ &= \frac{(-1)^k \pi}{2^{p-1}} \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} C_{a,b,p,q,k}. \end{aligned}$$

$\square$

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Convergence of Row Sequences of Simultaneous Padé–Faber Approximants*

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Abstract—We consider row sequences of vector valued Padé–Faber approximants (simultaneous Padé–Faber approximants) and prove a Montessus de Ballore-type theorem.

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Keywords: Montessus de Ballore theorem, Padé–Faber approximants, simultaneous Padé approximants, Hermite–Padé approximants.

1. INTRODUCTION

Let E be a compact set in the complex plane $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus E$ is simply connected and E contains more than one point. There exists a unique exterior conformal mapping Φ from $\overline{\mathbb{C}} \setminus E$ onto $\overline{\mathbb{C}} \setminus \{w \in \mathbb{C} : |w| \leq 1\}$ satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. For any $\rho > 1$, by

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\} \quad \text{and} \quad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\}$$

we denote the *level curve with respect to E of index ρ* and the *canonical domain with respect to E of index ρ* , respectively. The Faber polynomials (see [1]) for E are defined by the formulas

$$\Phi_n(z) := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\Phi^n(t)}{t-z} dt, \quad z \in D_\rho, \quad n = 0, 1, 2, \dots \quad (1)$$

Let $\mathcal{H}(E)$ denote the space of all functions holomorphic in some neighborhood of E . We set

$$\mathcal{H}(E)^d := \{(F_1, F_2, \dots, F_d) : F_\alpha \in \mathcal{H}(E) \text{ for all } \alpha = 1, 2, \dots, d\}$$

and denote the set of all nonnegative integers by $\mathbb{N}$.

Definition 1. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. Fix a multi-index

$$\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d \setminus \{\mathbf{0}\},$$

where $\mathbf{0}$ is the zero vector in $\mathbb{N}^d$. Let $|\mathbf{m}| = m_1 + m_2 + \dots + m_d$. Then, for each

$$n \geq \max\{m_1, m_2, \dots, m_d\},$$

there exist polynomials $Q_{n,\mathbf{m}}$ and $P_{n,\mathbf{m},\alpha}$, $\alpha = 1, 2, \dots, d$, such that

$$\deg(P_{n,\mathbf{m},\alpha}) \leq n - m_\alpha, \quad \deg(Q_{n,\mathbf{m}}) \leq |\mathbf{m}|, \quad Q_{n,\mathbf{m}} \neq 0,$$

$$Q_{n,\mathbf{m}} F_\alpha - P_{n,\mathbf{m},\alpha} = a_{n+1,n}^{(\alpha)} \Phi_{n+1}(z) + a_{n+2,n}^{(\alpha)} \Phi_{n+2}(z) + \dots,$$

for all $\alpha = 1, 2, \dots, d$. The vector of rational functions

$$\mathbf{R}_{n,\mathbf{m}} := (R_{n,\mathbf{m},1}, R_{n,\mathbf{m},2}, \dots, R_{n,\mathbf{m},d}) = \left(\frac{P_{n,\mathbf{m},1}}{Q_{n,\mathbf{m}}}, \frac{P_{n,\mathbf{m},2}}{Q_{n,\mathbf{m}}}, \dots, \frac{P_{n,\mathbf{m},d}}{Q_{n,\mathbf{m}}} \right)$$

is called the $(n, \mathbf{m})$ (linear) simultaneous Padé–Faber approximant of $\mathbf{F}$.

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In fact, the numbers $a_{k,n}^{(\alpha)}$ depend on $\mathbf{m}$, but to simplify notation, we do not indicate this dependence. It is easy to see that if $d = 1$, then the linear simultaneous Padé–Faber approximants reduce to the linear Padé–Faber approximants with a slight modification of the index n (see, e.g., [2] for the definition of linear Padé–Faber approximants). Moreover, let us mention that, for the case where $d = 1$, there is another related construction, called nonlinear Padé–Faber approximants (see [3]). Unlike the classical ones, these linear and nonlinear Padé–Faber approximants generally lead to different rational functions (see examples in [3] and [4]). In this paper, we restrict our attention to linear simultaneous Padé–Faber approximants; therefore, in the sequel, we omit the word “linear” when we refer to them.

For any pair $(n, \mathbf{m})$, a vector of rational functions $\mathbf{R}_{n,\mathbf{m}}$, always exists, but may be nonunique in the general case. In what follows, we assume that, given $(n, \mathbf{m})$, only one solution is taken.

Now, let us introduce the definition of a pole and its order for a vector of functions.

Definition 2. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ be a vector of functions meromorphic in some domain D . We say that λ is a pole of $\mathbf{F}$ in D of order τ if there exists an index $\alpha \in \{1, 2, \dots, d\}$ such that λ is a pole of F_α in D of order τ and, for the rest of the indices $j \neq \alpha$, either λ is not a pole of F_j or λ is a pole of F_j with order less than or equal to τ .

Let $\mathbf{F} \in \mathcal{H}(E)^d$, and let $\rho_{|\mathbf{m}|}(\mathbf{F})$ denote the index $\rho > 1$ of the largest canonical domain D_ρ inside which $\mathbf{F}$ has at most $|\mathbf{m}|$ poles. Let $\lambda_1, \lambda_2, \dots, \lambda_q$ be distinct poles of $\mathbf{F}$ in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$, and let

$$L := \frac{(1 + \min_{j=1,2,\dots,q} |\Phi(\lambda_j)|)}{2}.$$

The set of these poles is denoted by $\mathcal{P}_{|\mathbf{m}|}(\mathbf{F})$. The normalization of $Q_{n,\mathbf{m}}$ used in this paper in terms of its zeros $\lambda_{n,j}$ is as follows:

$$Q_{n,\mathbf{m}}(z) := \prod_{|\Phi(\lambda_{n,j})| \leq L} (z - \lambda_{n,j}) \prod_{|\Phi(\lambda_{n,j})| > L} \left(1 - \frac{z}{\lambda_{n,j}}\right). \quad (2)$$

By $Q_{|\mathbf{m}|}^{\mathbf{F}}$ we denote the polynomial whose zeros are the poles of $\mathbf{F}$ in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$, counting multiplicities, normalized as in (2).

Before going into details, we describe the convergence of row sequences of Padé–Faber approximants corresponding to the simultaneous Padé–Faber approximants in the scalar case ($d = 1$). For $d = 1$ we write

$$\begin{aligned} \mathbf{F} &= F, & |\mathbf{m}| &= \mathbf{m} = m \in \mathbb{N}, & \mathcal{P}_{|\mathbf{m}|}(\mathbf{F}) &= \mathcal{P}_m(F), \\ \rho_{|\mathbf{m}|}(\mathbf{F}) &= \rho_m(F), & \mathbf{R}_{n,\mathbf{m}} &= R_{n,m}. \end{aligned}$$

Suetin [2] proved the following analog of Montessus de Ballore’s theorem for Padé–Faber approximants.

Theorem A. Suppose that $F \in \mathcal{H}(E)$ has poles of total multiplicity exactly m in $D_{\rho_m(F)}$. Then $R_{n,m}$ is uniquely determined for all sufficiently large n , and the sequence $R_{n,m}$ converges uniformly to F inside $D_{\rho_m(F)} \setminus \mathcal{P}_m(F)$ as $n \rightarrow \infty$. Moreover, for any compact subset K of $D_{\rho_m(F)} \setminus \mathcal{P}_m(F)$,

$$\limsup_{n \rightarrow \infty} \|F - R_{n,m}\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_m(F)}, \quad (3)$$

where $\|\cdot\|_K$ denotes the sup-norm on K ; if $K \subset E$, then $\|\Phi\|_K$ is replaced by 1.

Here and in what follows, the phrase “uniformly inside a domain” means “uniformly on each compact subset of the domain.” The goal of this paper is to extend the above result from the scalar case to the vector case.

In [5], Graves-Morris and Saff proved a Montessus de Ballore-type theorem for simultaneous Padé approximants (in the context of Taylor expansions) using the concept of polewise independence of a vector of functions. We adapt their notion to fit our type of regions.

Definition 3. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ be a vector of functions meromorphic in some canonical domain D_ρ and let $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d \setminus \{\mathbf{0}\}$ be a multi-index. Then the function $\mathbf{F}$ is said to be *polewise independent with respect to the multi-index $\mathbf{m}$ in D_ρ* if there do not exist polynomials $v_1, v_2, \dots, v_d$, at least one of which is not identically vanishing, that satisfy the following conditions:

- (i) $\deg v_\alpha \leq m_\alpha - 1$, $\alpha = 1, 2, \dots, d$, for $m_\alpha \geq 1$;
- (ii) $v_\alpha \equiv 0$ for $m_\alpha = 0$;
- (iii) $\sum_{\alpha=1}^d (v_\alpha \circ \Phi) \cdot F_\alpha \in \mathcal{H}(D_\rho \setminus E)$,

where $\mathcal{H}(D_\rho \setminus E)$ is the space of all holomorphic functions on $D_\rho \setminus E$.

Our main result, which extends Theorem A, is as follows.

Theorem 1. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ be a vector of functions meromorphic in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$, and let $\mathbf{m} \in \mathbb{N}^d \setminus \{\mathbf{0}\}$ be a fixed multi-index. Suppose that $\mathbf{F}$ is polewise independent with respect to the multi-index $\mathbf{m}$ in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$. Then $\mathbf{R}_{n,\mathbf{m}}$ is uniquely determined for all sufficiently large n , and for each $\alpha = 1, 2, \dots, d$, the sequence $R_{n,\mathbf{m},\alpha}$ converges uniformly to F_α inside $D_{\rho_{|\mathbf{m}|}(\mathbf{F})} \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{F})$. Moreover, for each $\alpha = 1, 2, \dots, d$ and any compact set $K \subset D_{\rho_{|\mathbf{m}|}(\mathbf{F})} \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{F})$,

$$\limsup_{n \rightarrow \infty} \|F_\alpha - R_{n,\mathbf{m},\alpha}\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{|\mathbf{m}|}(\mathbf{F})}, \quad (4)$$

where $\|\cdot\|_K$ denotes the sup-norm on K ; if $K \subset E$, then $\|\Phi\|_K$ is replaced by 1. Moreover,

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}} - Q_{|\mathbf{m}|}^{\mathbf{F}}\|^{1/n} \leq \frac{\max_{\lambda \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{F})} |\Phi(\lambda)|}{\rho_{|\mathbf{m}|}(\mathbf{F})}, \quad (5)$$

where $\|\cdot\|$ denotes (for example) the norm induced on the space of polynomials of degree at most $|\mathbf{m}|$ by the maximum absolute value of the coefficients.

Since the space of polynomials of degree at most $|\mathbf{m}|$ has finite dimension, all of its norms are equivalent, so that we can take any norm in (5).

This paper is organized as follows. In Sec. 2, we introduce more notation and auxiliary lemmas. The proof of the main result is given in Sec. 3.

2. NOTATION AND AUXILIARY RESULTS

First, we discuss some properties of Faber polynomial expansions of holomorphic functions, which play a major role in our proof. The *Faber coefficient* of $G \in \mathcal{H}(E)$ with respect to Φ_n is given by

$$[G]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{G(t)\Phi'(t)}{\Phi^{n+1}(t)} dt,$$

where $\rho \in (1, \rho_0(G))$. The following lemma (see, e.g., [6]) is proved in the same way as similar statements for Taylor series.

Lemma 1. Let $G \in \mathcal{H}(E)$. Then

$$\rho_0(G) = \left(\limsup_{n \rightarrow \infty} |[G]_n|^{1/n} \right)^{-1}.$$

Moreover, the series $\sum_{n=0}^{\infty} [G]_n \Phi_n$ converges to G uniformly inside $D_{\rho_0(G)}$.

As a consequence of Lemma 1, if $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$, then, for each $\alpha = 1, 2, \dots, d$,

$$Q_{n,\mathbf{m}}(z)F_\alpha(z) - P_{n,\mathbf{m},\alpha}(z) = \sum_{k=n+1}^{\infty} [Q_{n,\mathbf{m}}F_\alpha]_k \Phi_k(z), \quad z \in D_{\rho_0(F_\alpha)}, \quad (6)$$

and $P_{n,\mathbf{m},\alpha} = \sum_{k=0}^{n-m_\alpha} [Q_{n,\mathbf{m}}F_\alpha]_k \Phi_k$ is uniquely determined by $Q_{n,\mathbf{m}}$.

Next, let us introduce the concept of convergence in h -content. Let B be a subset of the complex plane $\mathbb{C}$. By $\mathcal{U}(B)$ we denote the class of all coverings of B by at most countably many disks. We set

$$h(B) := \inf \left\{ \sum_{j=1}^{\infty} |U_j| : \{U_j\} \in \mathcal{U}(B) \right\},$$

where $|U_j|$ stands for the radius of the disk U_j . The quantity $h(B)$ is called the *1-dimensional Hausdorff content* of the set B . This set function is not a measure, but it is semi-additive and monotonic.

Definition 4. Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of complex-valued functions defined on a domain $D \subset \mathbb{C}$, and let g be another complex function defined on D . We say that $\{g_n\}_{n \in \mathbb{N}}$ *converges in h -content to the function g on compact subsets of D* if, for every compact subset K of D and each $\varepsilon > 0$, we have

$$\lim_{n \rightarrow \infty} h\{z \in K : |g_n(z) - g(z)| > \varepsilon\} = 0.$$

We denote such a convergence by $h\text{-}\lim_{n \rightarrow \infty} g_n = g$ in D .

The next lemma, which is due to Gonchar (see [7, Lemma 1] or [8, Sec. 2, Subsec. 2, b]) allows us to derive uniform convergence on compact subsets of the region under consideration from convergence in h -content.

Lemma 2. Suppose that $h\text{-}\lim_{n \rightarrow \infty} g_n = g$ in D . If each of the functions g_n is meromorphic in D and has no more than $k < +\infty$ poles in this domain, then the limit function g is meromorphic (except on a set of h -content zero) and has no more than k poles in D . Hence, in particular, if g has a pole of order ν at a point $\lambda \in D$, then at least ν poles of g_n tend to λ as $n \rightarrow \infty$.

Now, we discuss upper and lower bounds for normalized $Q_{n,\mathbf{m}}$ in (2). We take an arbitrary $\varepsilon > 0$ and define an open set $J_\varepsilon := J_\varepsilon(\mathbf{F})$ as follows. For $n \geq |\mathbf{m}|$, let $J_{n,\varepsilon}$ denote the $(\varepsilon/6)|\mathbf{m}|n^2$ -neighborhood of the set of zeros of $Q_{n,\mathbf{m}}$, and let $J_{|\mathbf{m}|-1,\varepsilon}$ denote the $(\varepsilon/6)|\mathbf{m}|$ -neighborhood of the set of poles of $\mathbf{F}$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$. We set

$$J_\varepsilon = \bigcup_{n \geq |\mathbf{m}|-1} J_{n,\varepsilon}.$$

It easily follows from monotonicity and subadditivity that $h(J_\varepsilon) < \varepsilon$ and $J_{\varepsilon_1} \subset J_{\varepsilon_2}$ for $\varepsilon_1 < \varepsilon_2$. For any set $B \subset \mathbb{C}$, we put $B(\varepsilon) := B \setminus J_\varepsilon$. Clearly, if $\{g_n\}_{n \in \mathbb{N}}$ converges uniformly to g on $K(\varepsilon)$ for every compact set $K \subset D$ and any $\varepsilon > 0$, then $h\text{-}\lim_{n \rightarrow \infty} g_n = g$ in D .

The normalization of $Q_{n,\mathbf{m}}$ provides the following useful upper and lower bounds for $Q_{n,\mathbf{m}}$.

Lemma 3. Let $K \subset \mathbb{C}$ be a compact set, and let $\varepsilon > 0$ be arbitrary. Then there exist constants $C_1, C_2 > 0$ independent of n and such that

$$\|Q_{n,\mathbf{m}}\|_K \leq C_1, \quad \min_{z \in K(\varepsilon)} |Q_{n,\mathbf{m}}(z)| \geq C_2 n^{-2|\mathbf{m}|}, \quad (7)$$

where the second inequality makes sense when $K(\varepsilon)$ is a nonempty set.

3. PROOF OF THEOREM 1

Proof of Theorem 1. It follows from (6) that, for each $\alpha = 1, 2, \dots, d$,

$$Q_{n,\mathbf{m}}(z)F_\alpha(z) - P_{n,\mathbf{m},\alpha}(z) = \sum_{k=n+1}^{\infty} a_{k,n}^{(\alpha)} \Phi_k(z), \quad z \in D_{\rho_0(F_\alpha)}, \quad (8)$$

where

$$a_{k,n}^{(\alpha)} := [Q_{n,\mathbf{m}}F_\alpha]_k = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{Q_{n,\mathbf{m}}(t)F_\alpha(t)\Phi'_k(t)}{\Phi^{k+1}(t)} dt, \quad \rho \in (1, \rho_0(F_\alpha)).$$

Let

$$Q_{|\mathbf{m}|}^{\mathbf{F}}(z) := \prod_{j=1}^q \left(1 - \frac{z}{\lambda_j}\right)^{\tau_j},$$

where $\lambda_1, \lambda_2, \dots, \lambda_q$ are distinct poles of $\mathbf{F}$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ and $\tau_1, \tau_2, \dots, \tau_q$ are their respective multiplicities. Since $\mathbf{F}$ is polewise independent with respect to $\mathbf{m}$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$, it follows that the vector function $\mathbf{F}$ has exactly $|\mathbf{m}|$ poles in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ and $\sum_{j=1}^q \tau_j = |\mathbf{m}|$. Multiplying (8) by $Q_{|\mathbf{m}|}^{\mathbf{F}}$ and expanding

$$Q_{|\mathbf{m}|}^{\mathbf{F}}Q_{n,\mathbf{m}}F_\alpha - Q_{|\mathbf{m}|}^{\mathbf{F}}P_{n,\mathbf{m},\alpha} \in \mathcal{H}(D_{\rho_{|\mathbf{m}|}}(\mathbf{F}))$$

in the Faber polynomial system $\{\Phi_\nu\}_{\nu=0}^\infty$, we see that, for each $\alpha = 1, 2, \dots, d$ and any $z \in D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$,

$$\begin{aligned} Q_{|\mathbf{m}|}^{\mathbf{F}}(z)Q_{n,\mathbf{m}}(z)F_\alpha(z) - Q_{|\mathbf{m}|}^{\mathbf{F}}(z)P_{n,\mathbf{m},\alpha}(z) &= \sum_{k=n+1}^{\infty} a_{k,n}^{(\alpha)} Q_{|\mathbf{m}|}^{\mathbf{F}}(z)\Phi_k(z) = \sum_{\nu=0}^{\infty} b_{\nu,n}^{(\alpha)} \Phi_\nu(z) \\ &= \sum_{\nu=0}^{n+|\mathbf{m}|-m_\alpha} b_{\nu,n}^{(\alpha)} \Phi_\nu(z) + \sum_{\nu=n+|\mathbf{m}|-m_\alpha+1}^{\infty} b_{\nu,n}^{(\alpha)} \Phi_\nu(z). \end{aligned} \quad (9)$$

Note that the constants $b_{\nu,n}^{(\alpha)}$ can be calculated in the form

$$\begin{aligned} b_{\nu,n}^{(\alpha)} &:= \sum_{k=n+1}^{\infty} a_{k,n}^{(\alpha)} [Q_{|\mathbf{m}|}^{\mathbf{F}}\Phi_k]_\nu, & \nu = 0, 1, \dots, n+|\mathbf{m}|-m_\alpha, \\ b_{\nu,n}^{(\alpha)} &:= [Q_{|\mathbf{m}|}^{\mathbf{F}}Q_{n,\mathbf{m}}F_\alpha]_\nu = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{Q_{|\mathbf{m}|}^{\mathbf{F}}(t)Q_{n,\mathbf{m}}(t)F_\alpha(t)\Phi'_\nu(t)}{\Phi^{\nu+1}(t)} dt, & \nu \geq n+|\mathbf{m}|-m_\alpha+1, \end{aligned} \quad (10)$$

where $\rho \in (1, \rho_{|\mathbf{m}|}(\mathbf{F}))$. We want to show that, given any $\alpha = 1, 2, \dots, d$,

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=0}^{\infty} b_{\nu,n}^{(\alpha)} \Phi_\nu \right\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{|\mathbf{m}|}(\mathbf{F})} \quad (11)$$

for any compact set K such that $E \subset K \subset D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$. Let K be a fixed compact set such that $E \subset K \subset D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$. Let $\rho_1 \in (1, \rho_{|\mathbf{m}|}(\mathbf{F}))$ satisfy

$$K \cup \{\lambda_1, \lambda_2, \dots, \lambda_q\} \subset D_{\rho_1}. \quad (12)$$

Choose $\delta > 0$ sufficiently small, so that

$$\|\Phi\|_K + \delta < \rho_1 - \delta. \quad (13)$$

We first prove that, for each $\alpha = 1, 2, \dots, d$,

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=n+|\mathbf{m}|-m_\alpha+1}^{\infty} b_{\nu,n}^{(\alpha)} \Phi_\nu \right\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{|\mathbf{m}|}(\mathbf{F})}. \quad (14)$$

Using the normalization of $Q_{n,\mathbf{m}}$ (the first inequality in (7)) and (10) when $\rho = \rho_1$, we see that there exists an $n_0 \in \mathbb{N}$ such that, for each $\alpha = 1, 2, \dots, d$,

$$|b_{\nu,n}^{(\alpha)}| \leq \frac{c_1}{(\rho_1 - \delta)^\nu}, \quad \nu \geq n_0, \quad (15)$$

where c_1 does not depend on n (from now on, we will denote constants not depending on n by $c_2, c_3, \dots$). Using (1), we obtain

$$\|\Phi_\nu\|_K \leq c_2(\|\Phi\|_K + \delta)^\nu, \quad \nu \geq 0. \quad (16)$$

Therefore, by (15) and (16), for $n \geq n_0$, we have

$$\begin{aligned} \left\| \sum_{\nu=n+|\mathbf{m}|-m_\alpha+1}^{\infty} b_{\nu,n}^{(\alpha)} \Phi_\nu \right\|_K &\leq \sum_{\nu=n+|\mathbf{m}|-m_\alpha+1}^{\infty} |b_{\nu,n}^{(\alpha)}| \|\Phi_\nu\|_K \\ &\leq \sum_{\nu=n+|\mathbf{m}|-m_\alpha+1}^{\infty} c_3 \left(\frac{\|\Phi\|_K + \delta}{\rho_1 - \delta} \right)^\nu \leq c_4 \left(\frac{\|\Phi\|_K + \delta}{\rho_1 - \delta} \right)^n. \end{aligned}$$

Thus, for each $\alpha = 1, 2, \dots, d$, we have

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=n+|\mathbf{m}|-m_\alpha+1}^{\infty} b_{\nu,n}^{(\alpha)} \Phi_\nu \right\|_K^{1/n} \leq \frac{\|\Phi\|_K + \delta}{\rho_1 - \delta}.$$

Letting $\delta \rightarrow 0$ and $\rho_1 \rightarrow \rho_{|\mathbf{m}|}(\mathbf{F})$, we obtain (14), as required.

Secondly, we show that

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=0}^{n+|\mathbf{m}|-m_\alpha} b_{\nu,n}^{(\alpha)} \Phi_\nu \right\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{|\mathbf{m}|}(\mathbf{F})}. \quad (17)$$

Recall that $b_{\nu,n}^{(\alpha)} = \sum_{k=n+1}^{\infty} a_{k,n}^{(\alpha)} [Q_{|\mathbf{m}|}^{\mathbf{F}} \Phi_k]_\nu$. Therefore, to approximate $b_{\nu,n}^{(\alpha)}$, we need to approximate $a_{k,n}^{(\alpha)}$ first. We will adapt the technique used in [2] to approximate $a_{k,n}^{(\alpha)}$. Take $\rho_1 \in (1, \rho_{|\mathbf{m}|}(\mathbf{F}))$ satisfying (12), as above. Choose $\rho_2 \in (1, \rho_0(\mathbf{F}))$. We have

$$a_{k,n}^{(\alpha)} [Q_{n,\mathbf{m}} F_\alpha]_k = \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{Q_{n,\mathbf{m}}(t) F_\alpha(t) \Phi'(t)}{\Phi^{k+1}(t)} dt, \quad \alpha = 1, 2, \dots, d.$$

We set

$$\gamma_{k,n}^{(\alpha)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{Q_{n,\mathbf{m}}(t) F_\alpha(t) \Phi'(t)}{\Phi^{k+1}(t)} dt, \quad \alpha = 1, 2, \dots, d. \quad (18)$$

By virtue of our choice of ρ_1 and ρ_2 , for each $k \geq 0$ and each $\alpha = 1, 2, \dots, d$, the polynomial $Q_{n,\mathbf{m}} F_\alpha \Phi' / \Phi^{k+1}$ is meromorphic in

$$\overline{D}_{\rho_1} \setminus D_{\rho_2} = \{z \in \mathbb{C} : \rho_2 \leq |\Phi(z)| \leq \rho_1\}$$

and has poles at $\lambda_1, \lambda_2, \dots, \lambda_q$ with multiplicities at most $\tau_1, \tau_2, \dots, \tau_q$. Applying Cauchy's residue theorem, we obtain

$$\gamma_{k,n}^{(\alpha)} - a_{k,n}^{(\alpha)} = \sum_{j=1}^q \operatorname{res} \left(\frac{Q_{n,\mathbf{m}} F_\alpha \Phi'}{\Phi^{k+1}}, \lambda_j \right), \quad \alpha = 1, 2, \dots, d, \quad k \geq 0. \quad (19)$$

The limit residue formula gives

$$\operatorname{res}(Q_{n,\mathbf{m}} F_\alpha \Phi' / \Phi^{k+1}, \lambda_j) = \frac{1}{(\tau_j - 1)!} \lim_{z \rightarrow \lambda_j} \left(\frac{(z - \lambda_j)^{\tau_j} Q_{n,\mathbf{m}}(z) F_\alpha(z) \Phi'(z)}{\Phi^{k+1}(z)} \right)^{(\tau_j - 1)}.$$

Leibnitz's formula allows us to write

$$\begin{aligned} & \left(\frac{(z - \lambda_j)^{\tau_j} Q_{n,\mathbf{m}}(z) F_\alpha(z) \Phi'(z)}{\Phi^{k+1}(z)} \right)^{(\tau_j-1)} \\ &= \sum_{t=0}^{\tau_j-1} \binom{\tau_j-1}{t} \left(\frac{Q_{n,\mathbf{m}}(z) \Phi'(z)}{\Phi^{n+1}(z)} \right)^{(\tau_j-1-t)} ((z - \lambda_j)^{\tau_j} F_\alpha(z) \Phi^{n-k}(z))^{(t)}. \end{aligned}$$

For $j = 1, 2, \dots, q$ and $t = 0, 1, \dots, \tau_j - 1$, we set

$$\beta_n(j, t) := \frac{1}{(\tau_j - 1)!} \binom{\tau_j - 1}{t} \lim_{z \rightarrow \lambda_j} \left(\frac{Q_{n,\mathbf{m}}(z) \Phi'(z)}{\Phi^{n+1}(z)} \right)^{(\tau_j-1-t)}$$

(note that the $\beta_n(j, t)$ do not depend on k and α). Thus, we can rewrite (19) as

$$\gamma_{k,n}^{(\alpha)} - a_{k,n}^{(\alpha)} = \sum_{j=1}^q \sum_{t=0}^{\tau_j-1} \beta_n(j, t) ((z - \lambda_j)^{\tau_j} F_\alpha(z) \Phi^{n-k}(z))_{z=\lambda_j}^{(t)}, \quad \alpha = 1, 2, \dots, d, \quad k \geq 0. \quad (20)$$

By the definition of simultaneous Padé–Faber approximants, we have

$$a_{k,n}^{(\alpha)} = 0, \quad \alpha = 1, 2, \dots, d, \quad k = n - m_\alpha + 1, n - m_\alpha + 2, \dots, n,$$

which implies

$$\gamma_{k,n}^{(\alpha)} = \sum_{j=1}^q \sum_{t=0}^{\tau_j-1} \beta_n(j, t) ((z - \lambda_j)^{\tau_j} F_\alpha(z) \Phi^{n-k}(z))_{z=\lambda_j}^{(t)} \quad (21)$$

for all $\alpha = 1, 2, \dots, d$ and $k = n - m_\alpha + 1, n - m_\alpha + 2, \dots, n$. We regard (21) as a system of $|\mathbf{m}|$ equations for the $|\mathbf{m}|$ unknowns $\beta_n(j, t)$. The determinant of this system is

$$\Delta := \begin{vmatrix} [(z - \lambda_j)^{\tau_j} F_\alpha(z) \Phi^{m_\alpha-1}(z)]_{z=\lambda_j} & \cdots & [(z - \lambda_j)^{\tau_j} F_\alpha(z) \Phi^{m_\alpha-1}(z)]_{z=\lambda_j}^{(\tau_j-1)} \\ [(z - \lambda_j)^{\tau_j} F_\alpha(z) \Phi^{m_\alpha-2}(z)]_{z=\lambda_j} & \cdots & [(z - \lambda_j)^{\tau_j} F_\alpha(z) \Phi^{m_\alpha-2}(z)]_{z=\lambda_j}^{(\tau_j-1)} \\ \vdots & \ddots & \vdots \\ [(z - \lambda_j)^{\tau_j} F_\alpha(z)]_{z=\lambda_j} & \cdots & [(z - \lambda_j)^{\tau_j} F_\alpha(z)]_{z=\lambda_j}^{(\tau_j-1)} \end{vmatrix}_{\substack{j=1, \dots, q, \\ \alpha=1, \dots, d}},$$

where the subscript means that the indicated columns are successively written for $j = 1, 2, \dots, q$, and the rows are repeated for $\alpha = 1, 2, \dots, d$.

If $\Delta = 0$, then there exists a linear combination of rows giving the zero vector. This means that there exist polynomials $v_1(z), v_2(z), \dots, v_d(z)$ such that $\deg v_\alpha \leq m_\alpha - 1$ and

$$\sum_{\alpha=1}^d [(z - \lambda_j)^{\tau_j} v_\alpha(\Phi(z)) F_\alpha(z)]_{z=\lambda_j}^{(l)} = 0, \quad j = 1, 2, \dots, q, \quad l = 0, 1, \dots, \tau_j - 1.$$

Equivalently, $\sum_{\alpha=1}^d v_\alpha(\Phi(z)) F_\alpha(z) \in \mathcal{H}(D_{\rho_{|\mathbf{m}|}(\mathbf{F})} \setminus E)$. This is impossible, because $\mathbf{F}$ is polewise independent with respect to $\mathbf{m}$ in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$. Therefore, $\Delta \neq 0$ and $|\Delta| \geq c_5 > 0$.

In order to avoid long expressions, for all $w = 1, 2, \dots, d$, $y = 1, 2, \dots, m_w$, $j = 1, 2, \dots, q$, $t = 0, 1, \dots, \tau_j - 1$, we set

$$g_{w,y} := \left(\sum_{r=0}^{w-1} m_r \right) + y \quad \text{and} \quad h_{j,t} := \left(\sum_{l=0}^{j-1} \tau_l \right) + t + 1,$$

where $m_0 = \tau_0 = 0$. Applying Cramer's rule to (21), we obtain

$$\beta_n(j, t) = \frac{\Delta_n(j, t)}{\Delta} = \frac{1}{\Delta} \sum_{w=1}^d \sum_{y=1}^{m_w} \gamma_{n-m_w+y, n}^{(w)} C[g_{w, y}, h_{j, t}], \quad (22)$$

where $\Delta_n(j, t)$ is the determinant obtained from Δ by replacing the $h_{j, t}$ th column by the column

$$\left[\gamma_{n-m_w+1, n}^{(w)} \quad \gamma_{n-m_w+2, n}^{(w)} \quad \cdots \quad \gamma_{n, n}^{(w)} \right]_{w=1, 2, \dots, d}^T$$

and $C[g, h]$ is the determinant of the (g, h) th cofactor matrix of $\Delta_n(j, t)$. Substituting the expression (22) for $\beta_n(j, t)$ into (20) for $\alpha = 1, 2, \dots, d$ and $k \geq n + 1$, we obtain

$$\gamma_{k, n}^{(\alpha)} - a_{k, n}^{(\alpha)} = \frac{1}{\Delta} \sum_{j=1}^q \sum_{t=0}^{\tau_j-1} \sum_{w=1}^d \sum_{y=1}^{m_w} \gamma_{n-m_w+y, n}^{(w)} C[g_{w, y}, h_{j, t}] ((z - \lambda_j)^{\tau_j} F_\alpha(z) \Phi^{n-k}(z))_{z=\lambda_j}^{(t)}. \quad (23)$$

We set $\mathbb{B}(\lambda, r) := \{z \in \mathbb{C} : |z - \lambda| < r\}$. Let $\varepsilon > 0$ be sufficiently small, so that

$$\{z \in \mathbb{C} : |z - \lambda_j| = \varepsilon\} \subset \{z \in \mathbb{C} : |\Phi(z)| > \rho_2\} \quad \text{for all } j = 1, 2, \dots, q,$$

and let $\overline{\mathbb{B}(\lambda_j, \varepsilon)} \cap \overline{\mathbb{B}(\lambda_\alpha, \varepsilon)} = \emptyset$ for all $\alpha \neq j$. Using Cauchy's integral formula, we obtain

$$((z - \lambda_j)^{\tau_j} F_\alpha(z) \Phi^{n-k}(z))_{z=\lambda_j}^{(l)} = \frac{l!}{2\pi i} \int_{|z-\lambda_j|=\varepsilon} \frac{(z - \lambda_j)^{\tau_j} F_\alpha(z) \Phi^{n-k}(z) dz}{(z - \lambda_j)^{l+1}}. \quad (24)$$

We can easily check that there exists a constant c_6 such that, for all $j = 1, 2, \dots, q$, $l = 0, 1, \dots, \tau_j - 1$, $\alpha = 1, 2, \dots, d$, and $k \geq n + 1$, we have

$$|((z - \lambda_j)^{\tau_j} F_\alpha(z) \Phi^{n-k}(z))_{z=\lambda_j}^{(l)}| \leq \frac{c_6}{\rho_2^{k-n}} \quad (25)$$

for sufficiently large n . Similarly, there exists a constant c_7 such that, for all $j = 1, 2, \dots, q$, $l = 0, 1, \dots, \tau_j - 1$, $\alpha = 1, 2, \dots, d$, and $k = n - m_\alpha + 1, n - m_\alpha + 2, \dots, n$, we have

$$|((z - \lambda_j)^{\tau_j} F_\alpha(z) \Phi^{n-k}(z))_{z=\lambda_j}^{(l)}| \leq c_7, \quad (26)$$

for sufficiently large n . It follows from (26) that

$$|C(g, h)| \leq c_8, \quad g, h = 1, 2, \dots, |\mathbf{m}|. \quad (27)$$

Using (25), (27), and $\Delta \geq c_5 > 0$ and applying (23), we see that

$$|a_{k, n}^{(\alpha)}| \leq |\gamma_{k, n}^{(\alpha)}| + \frac{c_9}{\rho_2^{k-n}} \sum_{w=1}^d \sum_{y=1}^{m_w} |\gamma_{n-m_w+y, n}^{(w)}|, \quad \alpha = 1, 2, \dots, d, \quad k \geq n + 1. \quad (28)$$

By the definition of $\gamma_{k, n}^{(\alpha)}$ (see (18)), for all sufficiently large n , we have

$$|\gamma_{k, n}^{(\alpha)}| \leq \frac{c_{10}}{(\rho_1 - \delta)^k}, \quad \alpha = 1, 2, \dots, d, \quad k \geq n - |\mathbf{m}| + 1,$$

where δ is sufficiently small, so that δ satisfies (13) and $\rho_2 < \rho_1 - \delta$. This and equality (28) imply

$$|a_{k, n}^{(\alpha)}| \leq \frac{c_{11}}{\rho_2^{k-n}(\rho_1 - \delta)^n}, \quad \alpha = 1, 2, \dots, d, \quad k \geq n + 1. \quad (29)$$

Moreover, for all $\nu \geq 0$ and $k \geq n + 1$, we have

$$|[Q_{|\mathbf{m}|}^{\mathbf{F}} \Phi_k]_\nu| = \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_2-2\delta}} \frac{Q_{|\mathbf{m}|}^{\mathbf{F}}(t) \Phi_k(t) \Phi'(t)}{\Phi^{\nu+1}(t)} dt \right| \leq c_{12} \frac{(\rho_2 - \delta)^k}{(\rho_2 - 3\delta)^\nu}, \quad (30)$$

where δ is sufficiently small, so that $\rho_2 - 3\delta > 1$. Combining (29) and (30), we obtain the following inequalities for all $\alpha = 1, 2, \dots, d$:

$$\begin{aligned} |b_{\nu,n}^{(\alpha)}| &= \left| \sum_{k=n+1}^{\infty} a_{k,n}^{(\alpha)} [Q_{|\mathbf{m}|}^{\mathbf{F}} \Phi_k]_{\nu} \right| \leq \sum_{k=n+1}^{\infty} |a_{k,n}^{(\alpha)}| |[Q_{|\mathbf{m}|}^{\mathbf{F}} \Phi_k]_{\nu}| \\ &\leq \frac{c_{13}}{(\rho_2 - 3\delta)^{\nu}} \left(\frac{\rho_2}{\rho_1 - \delta} \right)^n \sum_{k=n+1}^{\infty} \left(\frac{\rho_2 - \delta}{\rho_2} \right)^k \\ &\leq \frac{c_{14}}{(\rho_2 - 3\delta)^{\nu}} \left(\frac{\rho_2}{\rho_1 - \delta} \right)^n \left(\frac{\rho_2 - \delta}{\rho_2} \right)^n = \frac{c_{14}}{(\rho_2 - 3\delta)^{\nu}} \left(\frac{\rho_2 - \delta}{\rho_1 - \delta} \right)^n. \end{aligned}$$

Now, let us prove (17). Recall that ρ_1 and δ were chosen so that (see (12) and (13), respectively)

$$\|\Phi\|_K + \delta < \rho_1 - \delta.$$

Moreover, $\|\Phi_{\nu}\|_K \leq c_2(\|\Phi\|_K + \delta)^{\nu}$ for all $\nu \geq 0$. Therefore, for each $\alpha = 1, 2, \dots, d$, we have

$$\begin{aligned} \left\| \sum_{\nu=0}^{n+|\mathbf{m}|-m_{\alpha}} b_{\nu,n}^{(\alpha)} \Phi_{\nu} \right\|_K &\leq \sum_{\nu=0}^{n+|\mathbf{m}|-m_{\alpha}} |b_{\nu,n}^{(\alpha)}| \|\Phi_{\nu}\|_K \\ &\leq c_{15} \left(\frac{\rho_2 - \delta}{\rho_1 - \delta} \right)^n \sum_{\nu=0}^{n+|\mathbf{m}|-m_{\alpha}} \left(\frac{\|\Phi\|_K + \delta}{\rho_2 - 3\delta} \right)^{\nu} \\ &\leq c_{15} (n + |\mathbf{m}| - m_{\alpha} + 1) \left(\frac{\rho_2 - \delta}{\rho_1 - \delta} \right)^n \left(\frac{\|\Phi\|_K + \delta}{\rho_2 - 3\delta} \right)^{n+|\mathbf{m}|-m_{\alpha}}. \end{aligned}$$

Hence, for each $\alpha = 1, 2, \dots, d$,

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=0}^{n+|\mathbf{m}|-m_{\alpha}} b_{\nu,n}^{(\alpha)} \Phi_{\nu} \right\|_K^{1/n} \leq \left(\frac{\|\Phi\|_K + \delta}{\rho_1 - \delta} \right) \left(\frac{\rho_2 - \delta}{\rho_2 - 3\delta} \right).$$

Letting $\delta \rightarrow 0$ and $\rho_1 \rightarrow \rho_{|\mathbf{m}|}(\mathbf{F})$, we see that, for each $\alpha = 1, 2, \dots, d$, we have

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=0}^{n+|\mathbf{m}|-m_{\alpha}} b_{\nu,n}^{(\alpha)} \Phi_{\nu} \right\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{|\mathbf{m}|}(\mathbf{F})}. \quad (31)$$

Combining (14) and (31), we obtain (11). Therefore, from (9) we have

$$\limsup_{n \rightarrow \infty} \|Q_{|\mathbf{m}|}^{\mathbf{F}} Q_{n,\mathbf{m}} F_{\alpha} - Q_{|\mathbf{m}|}^{\mathbf{F}} P_{n,\mathbf{m},\alpha}\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{|\mathbf{m}|}(\mathbf{F})}, \quad \alpha = 1, 2, \dots, d, \quad (32)$$

where K is any compact set such that $E \subset K \subset D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$. To show that (32) is true for any compact subset K of $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$, we let K be any compact subset of $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$. If $K \subset E$, then, clearly, $\|\Phi\|_K$ on the right-hand side of (32) can be replaced by 1. If $K \cap (D_{\rho_{|\mathbf{m}|}(\mathbf{F})} \setminus E) \neq \emptyset$, then, for any $\alpha = 1, 2, \dots, d$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|Q_{|\mathbf{m}|}^{\mathbf{F}} Q_{n,\mathbf{m}} F_{\alpha} - Q_{|\mathbf{m}|}^{\mathbf{F}} P_{n,\mathbf{m},\alpha}\|_K^{1/n} \\ \leq \limsup_{n \rightarrow \infty} \|Q_{|\mathbf{m}|}^{\mathbf{F}} Q_{n,\mathbf{m}} F_{\alpha} - Q_{|\mathbf{m}|}^{\mathbf{F}} P_{n,\mathbf{m},\alpha}\|_{K \cup E}^{1/n} \leq \frac{\|\Phi\|_{K \cup E}}{\rho_{|\mathbf{m}|}(\mathbf{F})} = \frac{\|\Phi\|_K}{\rho_{|\mathbf{m}|}(\mathbf{F})}. \end{aligned}$$

Therefore, (32) holds for any compact set $K \subset D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$.

Let $\varepsilon > 0$. From the second inequality of (7), we obtain

$$\|F_{\alpha} - R_{n,\mathbf{m},\alpha}\|_{K(\varepsilon)} \leq c_{16} n^{2|\mathbf{m}|} \|Q_{|\mathbf{m}|}^{\mathbf{F}} Q_{n,\mathbf{m}} F_{\alpha} - Q_{|\mathbf{m}|}^{\mathbf{F}} P_{n,\mathbf{m},\alpha}\|_K, \quad \alpha = 1, 2, \dots, d.$$

Using (32), we see that

$$\limsup_{n \rightarrow \infty} \|F_\alpha - R_{n,\mathbf{m},\alpha}\|_{K(\varepsilon)}^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{|\mathbf{m}|}(\mathbf{F})}, \quad \alpha = 1, 2, \dots, d, \quad (33)$$

for any compact subset K of $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$. This implies that, for each $\alpha = 1, 2, \dots, d$,

$$h\text{-}\lim_{n \rightarrow \infty} R_{n,\mathbf{m},\alpha} = F_\alpha$$

in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$. By Lemma 2, each pole of $\mathbf{F}$ attracts zeros of $Q_{n,\mathbf{m}}$ according to its multiplicity. Since $\deg Q_{n,\mathbf{m}} \leq |\mathbf{m}|$, we have $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$ for sufficiently large n . For such n , $R_{n,\mathbf{m}}$ is unique. In fact, if this were not the case, we could find an infinite subsequence of indices for which Definition 1 has solutions with $\deg Q_{n,\mathbf{m}} < |\mathbf{m}|$, which contradicts the above considerations. In what follows, we consider only such n . Moreover, for sufficiently large n , we have

$$Q_{n,\mathbf{m}}(z) = \prod_{k=1}^{|\mathbf{m}|} \left(1 - \frac{z}{\lambda_{n,k}}\right) \quad \text{and} \quad \lim_{n \rightarrow \infty} Q_{n,\mathbf{m}}(z) = Q_{|\mathbf{m}|}^{\mathbf{F}}(z).$$

Since the set of limit points of the zeros of $Q_{n,\mathbf{m}}$ is $\mathcal{P}_{|\mathbf{m}|}(\mathbf{F})$, the inequality (33) implies (4).

Finally, we prove (5). We first need to show that, for $j = 1, 2, \dots, q$,

$$\limsup_{n \rightarrow \infty} |(Q_{n,\mathbf{m}})^{(k)}(\lambda_j)|^{1/n} \leq \frac{|\Phi(\lambda_j)|}{\rho_{|\mathbf{m}|}(\mathbf{F})}, \quad k = 0, 1, \dots, \tau_j - 1. \quad (34)$$

We argue by induction on k . Let $\varepsilon > 0$ be sufficiently small, so that $\overline{\mathbb{B}(\lambda_j, \varepsilon)} \subset D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ for all $j = 1, 2, \dots, q$ and the disks $\overline{\mathbb{B}(\lambda_j, \varepsilon)}$, $j = 1, 2, \dots, q$, are pairwise disjoint. Let $j \in \{1, 2, \dots, q\}$. There exists an $\alpha := \alpha(j) \in \{1, 2, \dots, d\}$ such that λ_j is a pole of F_α of order τ_j . As a consequence of (32), we have

$$\limsup_{n \rightarrow \infty} \|(z - \lambda_j)^{\tau_j} F_\alpha Q_{n,\mathbf{m}} - (z - \lambda_j)^{\tau_j} P_{n,\mathbf{m},\alpha}\|_{\overline{\mathbb{B}(\lambda_j, \varepsilon)}}^{1/n} \leq \frac{\|\Phi\|_{\overline{\mathbb{B}(\lambda_j, \varepsilon)}}}{\rho_{|\mathbf{m}|}(\mathbf{F})}, \quad (35)$$

so by Cauchy's integral formula for the derivative, we obtain

$$\limsup_{n \rightarrow \infty} \|[(z - \lambda_j)^{\tau_j} F_\alpha Q_{n,\mathbf{m}} - (z - \lambda_j)^{\tau_j} P_{n,\mathbf{m},\alpha}]^{(k)}\|_{\overline{\mathbb{B}(\lambda_j, \varepsilon)}}^{1/n} \leq \frac{\|\Phi\|_{\overline{\mathbb{B}(\lambda_j, \varepsilon)}}}{\rho_{|\mathbf{m}|}(\mathbf{F})} \quad (36)$$

for all $k \geq 0$. Letting $\varepsilon \rightarrow 0^+$, we see that inequality (35) implies

$$\limsup_{n \rightarrow \infty} |L_j Q_{n,\mathbf{m}}(\lambda_j)|^{1/n} \leq \frac{|\Phi(\lambda_j)|}{\rho_{|\mathbf{m}|}(\mathbf{F})},$$

where $L_j := \lim_{z \rightarrow \lambda_j} (z - \lambda_j)^{\tau_j} F_\alpha(z) \neq 0$ (because F_α has a pole of order τ_j at λ_j). Therefore,

$$\limsup_{n \rightarrow \infty} |Q_{n,\mathbf{m}}(\lambda_j)|^{1/n} \leq \frac{|\Phi(\lambda_j)|}{\rho_{|\mathbf{m}|}(\mathbf{F})}.$$

This is the base case. Now, let $r \leq \tau_j - 1$ and assume that

$$\limsup_{n \rightarrow \infty} |(Q_{n,\mathbf{m}})^{(k)}(\lambda_j)| \leq \frac{|\Phi(\lambda_j)|}{\rho_{|\mathbf{m}|}(\mathbf{F})}, \quad k = 0, 1, \dots, r - 1. \quad (37)$$

Let us show that the above inequality also holds for $k = r$. Since $r < \tau_j$, it follows from (36) that

$$\limsup_{n \rightarrow \infty} |[(z - \lambda_j)^{\tau_j} F_\alpha Q_{n,\mathbf{m}}]^{(r)}(\lambda_j)|^{1/n} \leq \frac{|\Phi(\lambda_j)|}{\rho_{|\mathbf{m}|}(\mathbf{F})}. \quad (38)$$

By Leibnitz' formula, we have

$$[(z - \lambda_j)^{\tau_j} F_\alpha Q_{n,\mathbf{m}}]^{(r)}(\lambda_j) = \sum_{l=0}^r \binom{r}{l} [(z - \lambda_j)^{\tau_j} F_\alpha]^{(l)}(\lambda_j) (Q_{n,\mathbf{m}})^{(r-l)}(\lambda_j).$$

Therefore, relations (37) and (38) and the fact that $L_j \neq 0$ imply the inequality

$$\lim_{n \rightarrow \infty} |(Q_{n,\mathbf{m}})^{(r)}(\lambda_j)|^{1/n} \leq \frac{|\Phi(\lambda_j)|}{\rho_{|\mathbf{m}|}(\mathbf{F})},$$

which completes the induction construction and the proof of (34).

Using Hermite interpolation, it is easy to construct a basis $\{e_{j,t}\}_{j=1,2,\dots,q, t=0,1,\dots,\tau_j-1}$ in the space of polynomials of degree at most $|\mathbf{m}| - 1$ so that

$$e_{j,t}^{(k)}(\lambda_\ell) = \delta_{\ell,j} \delta_{k,t}, \quad 1 \leq \ell \leq q, \quad 0 \leq k \leq \tau_\ell - 1.$$

We have

$$Q_{n,\mathbf{m}}(z) = \sum_{j=1}^q \sum_{t=0}^{\tau_j-1} (Q_{n,\mathbf{m}})^{(t)}(\lambda_j) e_{j,t}(z) + C_n Q_{|\mathbf{m}|}^{\mathbf{F}}(z),$$

where $C_n = \prod_{j=1}^q \lambda_j^{\tau_j} / \prod_{k=1}^{|\mathbf{m}|} \lambda_{n,k}$. Using (34), we obtain

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}} - C_n Q_{|\mathbf{m}|}^{\mathbf{F}}\|^{1/n} \leq \frac{\max_{\lambda \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{F})} |\Phi(\lambda)|}{\rho_{|\mathbf{m}|}(\mathbf{F})}.$$

Evaluation at zero yields

$$\limsup_{n \rightarrow \infty} |1 - C_n|^{1/n} \leq \frac{\max_{\lambda \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{F})} |\Phi(\lambda)|}{\rho_{|\mathbf{m}|}(\mathbf{F})}.$$

This implies (5), which completes the proof. $\square$

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Research Article

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On Montessus de Ballore's theorem for simultaneous Padé-Faber approximants

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Abstract: We consider row sequences of vector valued Padé-Faber approximants (simultaneous Padé-Faber approximants) and prove a Montessus de Ballore type theorem. Moreover, we provide a comparison between this new Montessus de Ballore type theorem and the Montessus de Ballore type theorem in Bosuwan (Math. Notes, 2018, 103).

Keywords: Montessus de Ballore's theorem, Padé-Faber approximants, simultaneous Padé approximants, Hermite-Padé approximants

MSC: Primary 30E10, Secondary 41A21

1 Introduction

Let E be a compact subset of the complex plane $\mathbb{C}$, such that $\overline{\mathbb{C}} \setminus E$ is simply connected and E contains more than one point. By the Riemann mapping theorem, there exists a unique exterior conformal mapping Φ from $\overline{\mathbb{C}} \setminus E$ onto $\overline{\mathbb{C}} \setminus \{w \in \mathbb{C} : |w| \leq 1\}$ satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. For any $\rho > 1$, we denote by

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\}, \quad \text{and} \quad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\},$$

a level curve with respect to E of index ρ and a canonical domain with respect to E of index ρ , respectively. The Faber polynomials (see [1]) for E are defined by the formulas

$$\Phi_n(z) := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\Phi^n(t)}{t-z} dt, \quad z \in D_\rho, \quad n = 0, 1, 2, \dots \quad (1)$$

(that is, the polynomial parts of the functions Φ^n). Denote by $\mathcal{H}(E)$ the space of all functions holomorphic in some neighborhood of E . We define

$$\mathcal{H}(E)^d := \{(F_1, F_2, \dots, F_d) : F_\alpha \in \mathcal{H}(E) \text{ for all } \alpha = 1, 2, \dots, d\}$$

and the set of all nonnegative integers is denoted by $\mathbb{N}_0$.

Definition 1. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. Fix a multi-index $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector in $\mathbb{N}_0^d$. Set $|\mathbf{m}| := m_1 + m_2 + \dots + m_d$. Then, for each $n \geq \max\{m_1, m_2, \dots, m_d\}$, there exist polynomials $Q_{n,\mathbf{m}}$ and $P_{n,\mathbf{m},\alpha}$, $\alpha = 1, 2, \dots, d$ such that

$$\deg(P_{n,\mathbf{m},\alpha}) \leq n - m_\alpha, \quad \deg(Q_{n,\mathbf{m}}) \leq |\mathbf{m}|, \quad Q_{n,\mathbf{m}} \not\equiv 0, \quad (2)$$

$$Q_{n,\mathbf{m}} F_\alpha - P_{n,\mathbf{m},\alpha} = a_{n+1,n}^{(\alpha)} \Phi_{n+1}(z) + a_{n+2,n}^{(\alpha)} \Phi_{n+2}(z) + \dots, \quad (3)$$

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for all $\alpha = 1, 2, \dots, d$. The vector of rational functions

$$\mathbf{R}_{n,\mathbf{m}} := (R_{n,\mathbf{m},1}, R_{n,\mathbf{m},2}, \dots, R_{n,\mathbf{m},d}) = (P_{n,\mathbf{m},1}/Q_{n,\mathbf{m}}, P_{n,\mathbf{m},2}/Q_{n,\mathbf{m}}, \dots, P_{n,\mathbf{m},d}/Q_{n,\mathbf{m}})$$

is called an $(n, \mathbf{m})$ (linear) simultaneous Padé-Faber approximant of $\mathbf{F}$.

In fact, the numbers $a_{k,n}^{(\alpha)}$ depend on $\mathbf{m}$, but to simplify the notation we will not indicate it. Finding a solution of (2)-(3) reduces to solving a homogeneous system of $d(n+1)$ linear equations on $d(n+1)+1$ coefficients of $Q_{n,\mathbf{m}}$ and $P_{n,\mathbf{m},\alpha}$, $\alpha = 1, 2, \dots, d$. This is where the word “linear” in Definition 1 comes from. Note that if $d = 1$, then (n, m) linear simultaneous Padé-Faber approximants are the classical $(n-m, m)$ linear Padé-Faber approximants (see, e.g., [2] for the definition of classical linear Padé-Faber approximants). Moreover, for the case when $d = 1$, there is another related construction called nonlinear Padé-Faber approximants (see [3]). Unlike the classical case, these linear and nonlinear Padé-Faber approximants lead, in general, to different rational functions (see the examples in [3] and [4]). Because in this paper we will restrict our attention to linear simultaneous Padé-Faber approximants, in the rest of the paper we will omit the word “linear” when we refer to them.

Furthermore, if E is the closed unit disk, then the corresponding Faber polynomials are $\Phi_n(z) = z^n$, which implies that these simultaneous Padé-Faber approximants are exactly simultaneous Padé approximants or type II Hermite-Padé approximants (see, e.g., [5, 6] for the definition of simultaneous Padé approximants or type II Hermite-Padé approximants). So, simultaneous Padé-Faber approximation serves as one of the generalizations of type II Hermite-Padé approximation.

For any pair $(n, \mathbf{m})$, a vector of rational functions $\mathbf{R}_{n,\mathbf{m}}$ always exists but, in general, it may not be unique. In what follows, we assume that given $(n, \mathbf{m})$, one solution is taken. Moreover, since $Q_{n,\mathbf{m}} \neq 0$, we normalize $Q_{n,\mathbf{m}}$ to be a “monic” polynomial.

Because the studies of simultaneous Padé-Faber approximants and simultaneous Padé-orthogonal approximants are quite similar, we will follow many definitions from [7]. The following is a definition of poles for a vector of functions.

Definition 2. Let $\Omega := (\Omega_1, \Omega_2, \dots, \Omega_d)$ be a system of domains, such that, for each $\alpha = 1, 2, \dots, d$, F_α is meromorphic in Ω_α . We say that the point λ is a pole of $\mathbf{F}$ in Ω of order τ if there exists an index $\alpha \in \{1, 2, \dots, d\}$ such that $\lambda \in \Omega_\alpha$ and it is a pole of F_α of order τ , and for $\beta \neq \alpha$ either λ is a pole of F_β of order less than or equal to τ or $\lambda \notin \Omega_\beta$. When $\Omega = (\Omega, \Omega, \dots, \Omega)$, we say that λ is a pole of $\mathbf{F}$ in Ω .

Let $\mathbf{F} \in \mathcal{H}(E)^d$. Denote by $\rho_{|\mathbf{m}|}(\mathbf{F})$ the index $\rho > 1$ of the largest canonical domain D_ρ inside which $\mathbf{F}$ has at most $|\mathbf{m}|$ poles. Denote by $Q_{|\mathbf{m}|}^{\mathbf{F}}$ the monic polynomial whose zeros are the poles of $\mathbf{F}$ in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$ counting multiplicities. The set of these poles is denoted by $\mathcal{P}_{|\mathbf{m}|}(\mathbf{F})$.

In [8], the author proved a Montessus de Ballore type theorem for simultaneous Padé-Faber approximants using the following concept of polewise independence of a vector of functions adapted for our type of regions.

Definition 3. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ be a vector of functions meromorphic in some canonical domain D_ρ and let $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$ be the multi-index. Then the function $\mathbf{F}$ is said to be polewise independent with respect to the multi-index $\mathbf{m}$ in D_ρ if and only if there do not exist polynomials $v_1, v_2, \dots, v_d$ at least one of which is non-null, satisfying

- (i) $\deg v_\alpha \leq m_\alpha - 1$, $\alpha = 1, 2, \dots, d$, if $m_\alpha \geq 1$,
- (ii) $v_\alpha \equiv 0$ if $m_\alpha = 0$,
- (iii) $\sum_{\alpha=1}^d (v_\alpha \circ \Phi) \cdot F_\alpha \in \mathcal{H}(D_\rho \setminus E)$,

where $\mathcal{H}(D_\rho \setminus E)$ is the space of all holomorphic functions in $D_\rho \setminus E$.

Note that if E is the closed unit disk, then Definition 3 is equivalent to the classical definition of polewise independence (see, e.g., [5, Definition 1.3] or [6, Definition 1] for the classical definition of polewise independence).

A Montessus de Ballore type theorem in [8, Theorem 1] is the following:

Theorem A. Let $\mathbf{m} \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$ be a fixed multi-index and $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ be a vector of functions meromorphic in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$. Suppose that $\mathbf{F}$ is polewise independent with respect to the multi-index $\mathbf{m}$ in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$. Then, $\mathbf{R}_{n,\mathbf{m}}$ is uniquely determined for all sufficiently large n and for each $\alpha = 1, 2, \dots, d$, $R_{n,\mathbf{m},\alpha}$ converges uniformly to F_α on compact subsets of $D_{\rho_{|\mathbf{m}|}(\mathbf{F})} \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{F})$. Moreover, for each $\alpha = 1, 2, \dots, d$ and for any compact set $K \subset D_{\rho_{|\mathbf{m}|}(\mathbf{F})} \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{F})$,

$$\limsup_{n \rightarrow \infty} \|F_\alpha - R_{n,\mathbf{m},\alpha}\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{|\mathbf{m}|}(\mathbf{F})}, \quad (4)$$

where $\|\cdot\|_K$ denotes the sup-norm on K , and if $K \subset E$, then $\|\Phi\|_K$ is replaced by 1. Additionally,

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}} - Q_{|\mathbf{m}|}^{\mathbf{F}}\|^{1/n} \leq \frac{\max_{\lambda \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{F})} |\Phi(\lambda)|}{\rho_{|\mathbf{m}|}(\mathbf{F})}, \quad (5)$$

where $\|\cdot\|$ denotes (for example) the norm induced in the space of polynomials of degree at most $|\mathbf{m}|$ by the maximum of the absolute value of the coefficients.

Because the space of polynomials of degree at most $|\mathbf{m}|$ has a finite dimension, all of its norms are equivalent and therefore we can use any norm in (5). In [8, Theorem 1], the author used a different normalization for $Q_{n,\mathbf{m}}$ and $Q_{|\mathbf{m}|}^{\mathbf{F}}$ to state inequality (5). However, it is not difficult to see that inequality (5) above and the equation (5) in [8, Theorem 1] are equivalent.

The goal of this paper is to give a new Montessus de Ballore type theorem for simultaneous Padé-Faber approximants. Importantly, we provide some examples to show that in many cases, the new theorem offers many advantages over Theorem A. For example, there are many simple examples of vectors of functions that are not polewise independent yet satisfy our sufficient conditions in the new theorem. Moreover, for some examples of vectors of functions (where both theorems can be applied), our new estimates for the limits in (4) and (5) on the new theorem are better. We will present these examples and discussions in the next section.

Before stating this new result, we need some more terminology.

Given a vector $\mathbf{F} = (F_1, F_2, \dots, F_d)$ and a multi-index $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$, we define

$$\mathbf{D}_{\mathbf{m}}(\mathbf{F}) := (D_{\rho_{m_1}(F_1)}, D_{\rho_{m_2}(F_2)}, \dots, D_{\rho_{m_d}(F_d)}).$$

Denote by $Q_{\mathbf{m}}^{\mathbf{F}}$ the monic polynomial whose zeros are the poles of $\mathbf{F}$ in $\mathbf{D}_{\mathbf{m}}(\mathbf{F})$ counting multiplicities. The set of these poles is denoted by $\mathcal{P}_{\mathbf{m}}(\mathbf{F})$. For $\alpha = 1, 2, \dots, d$, we define $\mathcal{P}_{\mathbf{m},\alpha}(\mathbf{F}) := \mathcal{P}_{\mathbf{m}}(\mathbf{F}) \cap D_{\rho_{m_\alpha}(F_\alpha)}$.

For each pole λ of $\mathbf{F}$ in this system of domains

$$\mathbf{D}_{\mathbf{m}}(\mathbf{F}) = (D_{\rho_{m_1}(F_1)}, D_{\rho_{m_2}(F_2)}, \dots, D_{\rho_{m_d}(F_d)}),$$

we associate an index $\alpha(\lambda) \in \{1, 2, \dots, d\}$ as follows. The index $\alpha(\lambda)$ verifies that $\lambda \in D_{\rho_{m_{\alpha(\lambda)}}(F_{\alpha(\lambda)})}$ and λ is a pole of $F_{\alpha(\lambda)}$ of the same order as is a pole of $\mathbf{F}$ in $\mathbf{D}_{\mathbf{m}}(\mathbf{F})$. If there are several indices α satisfying this condition, then we choose the one with the greatest $\rho_{m_\alpha}(F_\alpha)$.

The following theorem is the main result of this paper.

Theorem 1. Let $\mathbf{m} \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$ be a fixed multi-index, $\mathbf{F} \in \mathcal{H}(E)^d$, and $\mathcal{P}_{\mathbf{m}}(\mathbf{F}) := \{\lambda_1, \lambda_2, \dots, \lambda_q\}$. Suppose that the vector of functions $\mathbf{F}$ has exactly $|\mathbf{m}|$ poles in $\mathbf{D}_{\mathbf{m}}(\mathbf{F})$. Then, $\mathbf{R}_{n,\mathbf{m}}$ is unique for all n sufficiently large and for each $\alpha = 1, 2, \dots, d$, $R_{n,\mathbf{m},\alpha}$ converges uniformly to F_α on compact subsets of $D_{\rho_{m_\alpha}(F_\alpha)} \setminus \mathcal{P}_{\mathbf{m},\alpha}(\mathbf{F})$. Moreover,

$$\limsup_{n \rightarrow \infty} \|F_\alpha - R_{n,\mathbf{m},\alpha}\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{m_\alpha}(F_\alpha)}, \quad \alpha = 1, 2, \dots, d, \quad (6)$$

where K is any compact subset of $D_{\rho_{m_\alpha}(F_\alpha)} \setminus \mathcal{P}_{\mathbf{m},\alpha}(\mathbf{F})$ and if $K \subset E$, then $\|\Phi\|_K$ is replaced by 1. Additionally, we obtain

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}} - Q_{\mathbf{m}}^{\mathbf{F}}\|^{1/n} \leq \max_{j=1,2,\dots,q} \left\{ \frac{|\Phi(\lambda_j)|}{\rho_{m_{\alpha(\lambda_j)}}(F_{\alpha(\lambda_j)})} \right\}. \quad (7)$$

Note that for $d = 1$, Theorem 1 reduces to a Montessus de Ballore type theorem for Padé-Faber approximants [2, Theorem 4]. Moreover, notice that the right hand side of (6) depends on α . Therefore, in many cases, this new estimate is better than the estimate in (4) (see an example in the next section).

An outline of this paper is as follows. Section 2 contains the comparison between Theorem A and Theorem 1. The proof of Theorem 1 is in Section 3.

2 Comparison between Theorem A and Theorem 1

In this section, let us show that in many cases Theorem 1 offers many advantages over Theorem A.

First of all, let us give some very simple vectors $\mathbf{F}$ that are not polewise independent in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ but have exactly $|\mathbf{m}|$ poles in $\mathbf{D}_{\mathbf{m}}(\mathbf{F})$. Define

$$\begin{aligned} E &:= \{z \in \mathbb{C} : |z| \leq 1\}, \\ G_1(z) &:= \frac{1}{z-2} + \frac{1}{z-3i}, \quad G_2(z) := \frac{1}{z-4}, \\ F_1(z) &:= \frac{1}{z-2} + \log(z-4), \quad F_2(z) := \frac{1}{z-4}, \end{aligned}$$

and fix the multi-index $\mathbf{m} := (1, 1)$. Let $\mathbf{G} := (G_1, G_2)$ and $\mathbf{F} := (F_1, F_2)$. Obviously, $\rho_{|\mathbf{m}|}(\mathbf{G}) = \rho_{|\mathbf{m}|}(\mathbf{F}) = 4$ and $D_{\rho_{|\mathbf{m}|}}(\mathbf{G}) = D_{\rho_{|\mathbf{m}|}}(\mathbf{F}) = \{z \in \mathbb{C} : |z| < 4\}$. If we set $v_1 \equiv 0$ and $v_2 \equiv 1$, then definitely, v_2 is non-null and it is easy to check that v_1 and v_2 verify (i), (ii), and (iii) in Definition 3, particularly

$$v_1 \cdot G_1 + v_2 \cdot G_2 = v_1 \cdot F_1 + v_2 \cdot F_2 = \frac{1}{z-4} \in \mathcal{H}(D_{\rho_{|\mathbf{m}|}}(\mathbf{F})) = \mathcal{H}(D_{\rho_{|\mathbf{m}|}}(\mathbf{G})).$$

This implies that $\mathbf{G}$ and $\mathbf{F}$ are not polewise independent with respect to $\mathbf{m}$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{G}) = D_{\rho_{|\mathbf{m}|}}(\mathbf{F}) = \{z \in \mathbb{C} : |z| < 4\}$. Therefore, using Theorem A, nothing can be said about the convergences of $\{R_{n,\mathbf{m},1}\}$, $\{R_{n,\mathbf{m},2}\}$, and $\{Q_{n,\mathbf{m}}\}$ as $n \rightarrow \infty$ (for both $\mathbf{G}$ and $\mathbf{F}$). However, $\mathbf{G}$ and $\mathbf{F}$ have exactly 2 poles in

$$\mathbf{D}_{\mathbf{m}}(\mathbf{G}) = (\{z \in \mathbb{C} : |z| < 3\}, \mathbb{C}),$$

$$\mathbf{D}_{\mathbf{m}}(\mathbf{F}) = (\{z \in \mathbb{C} : |z| < 4\}, \mathbb{C}),$$

respectively. Now, applying Theorem 1 to $\mathbf{F}$, we can conclude that for any $K \subset \{z \in \mathbb{C} : |z| < 4\} \setminus \{2\}$,

$$\limsup_{n \rightarrow \infty} \|F_1 - R_{n,\mathbf{m},1}\|_K^{1/n} \leq \frac{\|z\|_K}{4},$$

and for any $K \subset \mathbb{C} \setminus \{4\}$,

$$\limsup_{n \rightarrow \infty} \|F_2 - R_{n,\mathbf{m},2}\|_K^{1/n} = 0.$$

Additionally, the inequality (7) in Theorem 1 for $\mathbf{F}$ implies that

$$\limsup_{n \rightarrow \infty} \|Q_{\mathbf{m}}^{\mathbf{F}} - Q_{n,\mathbf{m}}\|^{1/n} \leq \max \left\{ \frac{1}{2}, 0 \right\} = \frac{1}{2}.$$

Next, let us give an example of a vector $\mathbf{F}$ and a multi-index $\mathbf{m}$ to show that the estimates on the rates of convergences of $\{R_{n,\mathbf{m},\alpha}\}$ for $\alpha = 1, 2, \dots, d$ in (6) and $\{Q_{n,\mathbf{m}}\}$ in (7) are better than the ones in (4) and in (5). Define

$$\begin{aligned} E &:= \{z \in \mathbb{C} : |z| \leq 1\}, \\ F_1(z) &:= \frac{1}{z-2} + \log(z-4), \quad F_2(z) := \frac{1}{z-3} + \log(z-5), \end{aligned}$$

and fix the multi-index $\mathbf{m} := (1, 1)$. Let $\mathbf{F} := (F_1, F_2)$. Clearly, $\rho_{|\mathbf{m}|}(\mathbf{F}) = 4$, $\mathbf{F}$ is polewise independent in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ and $\mathbf{F}$ has exactly 2 poles in

$$\mathbf{D}_{\mathbf{m}}(\mathbf{F}) = (\{z \in \mathbb{C} : |z| < 4\}, \{z \in \mathbb{C} : |z| < 5\}).$$

By (4), for any compact set $K \subset \{z \in \mathbb{C} : |z| < 4\} \setminus \{2, 3\}$,

$$\limsup_{n \rightarrow \infty} \|F_1 - R_{n,\mathbf{m},1}\|_K^{1/n} \leq \frac{\|z\|_K}{4}, \quad (8)$$

and

$$\limsup_{n \rightarrow \infty} \|F_2 - R_{n,\mathbf{m},2}\|_K^{1/n} \leq \frac{\|z\|_K}{4}. \quad (9)$$

Applying (6), for any compact set $K \subset \{z \in \mathbb{C} : |z| < 4\} \setminus \{2\}$,

$$\limsup_{n \rightarrow \infty} \|F_1 - R_{n,\mathbf{m},1}\|_K^{1/n} \leq \frac{\|z\|_K}{4}, \quad (10)$$

and for any compact set $K \subset \{z \in \mathbb{C} : |z| < 5\} \setminus \{3\}$,

$$\limsup_{n \rightarrow \infty} \|F_2 - R_{n,\mathbf{m},2}\|_K^{1/n} \leq \frac{\|z\|_K}{5}. \quad (11)$$

For this example, the estimates in (8) and (10) for the convergence of $\{R_{n,\mathbf{m},1}\}$ are the same and the estimate in (11) for the convergence of $\{R_{n,\mathbf{m},2}\}$ is better than the one in (9). Therefore, for this example, Theorem 1 provides the better estimates on the rates of convergences of $\{R_{n,\mathbf{m},1}\}$ and $\{R_{n,\mathbf{m},2}\}$ than Theorem A. Moreover, from (5) and (7), we have

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}} - Q_{|\mathbf{m}|}^{\mathbf{F}}\|^{1/n} \leq \frac{3}{4} \quad (12)$$

and

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}} - Q_{\mathbf{m}}^{\mathbf{F}}\|^{1/n} \leq \max \left\{ \frac{2}{4}, \frac{3}{5} \right\} = \frac{1}{2}, \quad (13)$$

respectively. Note that the equations above imply that $Q_{|\mathbf{m}|}^{\mathbf{F}}$ and $Q_{\mathbf{m}}^{\mathbf{F}}$ are the same. Moreover, the estimate on the rate of convergence of $Q_{n,\mathbf{m}}$ using Theorem 1 is better than the one using Theorem A.

3 Proof of Theorem 1

3.1 Notation and auxiliary results

First of all, we want to discuss the domain of convergence of Faber polynomial expansions of holomorphic functions. The *Faber coefficient* of $G \in \mathcal{H}(E)$ with respect to Φ_n is given by

$$[G]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{G(t)\Phi'(t)}{\Phi^{n+1}(t)} dt, \quad (14)$$

where $\rho \in (1, \rho_0(G))$. The following lemma (see, e.g., [9]) provides the formula for computing $\rho_0(G)$, where $G \in \mathcal{H}(E)$, and the domain of convergence of Faber polynomial expansions of holomorphic functions.

Lemma 1. *Let $G \in \mathcal{H}(E)$. Then,*

$$\rho_0(G) = \left(\limsup_{n \rightarrow \infty} |[G]_n|^{1/n} \right)^{-1}.$$

Moreover, the series $\sum_{n=0}^{\infty} [G]_n \Phi_n$ converges to G uniformly on compact subsets of $D_{\rho_0(G)}$.

As a consequence of Lemma 1, if $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$, then for each $\alpha = 1, 2, \dots, d$,

$$Q_{n,\mathbf{m}}(z)F_\alpha(z) - P_{n,\mathbf{m},\alpha}(z) = \sum_{k=n+1}^{\infty} [Q_{n,\mathbf{m}}F_\alpha]_k \Phi_k(z), \quad z \in D_{\rho_0(F_\alpha)}, \quad (15)$$

and $P_{n,\mathbf{m},\alpha} = \sum_{k=0}^{n-m_\alpha} [Q_{n,\mathbf{m}}F_\alpha]_k \Phi_k$ is uniquely determined by $Q_{n,\mathbf{m}}$.

The next lemma (see [1, p. 43] or [10, p. 583] for its proof) gives an estimate of Faber polynomials Φ_n on a level curve.

Lemma 2. Let $\rho > 1$ be fixed. Then, there exists $c > 0$, such that

$$\|\Phi_n\|_{\Gamma_\rho} \leq c\rho^n, \quad n \geq 0.$$

The proof of the main theorem is a consequence of convergence in h -content and Gonchar's lemma defined as follows. Let B be a subset of the complex plane $\mathbb{C}$. By $\mathcal{U}(B)$, we denote the class of all coverings of B by at most a numerable set of disks. Define

$$h(B) := \inf \left\{ \sum_{j=1}^{\infty} |U_j| : \{U_j\} \in \mathcal{U}(B) \right\},$$

where $|U_j|$ stands for the radius of the disk U_j . The quantity $h(B)$ is called the *1-dimensional Hausdorff content of the set B* . Note that this set function h is semi-additive and monotonic, but it is not a measure.

Definition 4. Let $\{g_n\}_{n \in \mathbb{N}_0}$ be a sequence of complex valued functions defined on a domain $\Omega \subset \mathbb{C}$ and g be another complex function defined on Ω . We say that $\{g_n\}_{n \in \mathbb{N}_0}$ *converges in h -content to g on compact subsets of Ω* if for each compact subset K of Ω and for any $\varepsilon > 0$, we have

$$\lim_{n \rightarrow \infty} h\{z \in K : |g_n(z) - g(z)| > \varepsilon\} = 0.$$

Such a convergence will be denoted by $h\text{-}\lim_{n \rightarrow \infty} g_n = g$ in Ω .

The following is Gonchar's lemma (see [11, Lemma 1] or [12, §2.2 (b)]) which allows us to derive uniform convergence on compact subsets of the region under consideration.

Lemma 3 (Gonchar's lemma). Assume that $h\text{-}\lim_{n \rightarrow \infty} g_n = g$ in Ω . If all functions g_n are meromorphic in Ω and have no more than $k < +\infty$ poles in this domain, then the limit function g is (except on a set of h -content zero) also meromorphic and has no more than k poles in Ω . Hence, in particular, if g has a pole of order v at the point $\lambda \in \Omega$, then at least v poles of g_n tend to λ as $n \rightarrow \infty$.

3.2 Incomplete Padé-Faber approximants

The following is a definition of incomplete Padé-Faber approximants which play a major role in the proof of the main theorem.

Definition 5. Let $F \in \mathcal{H}(E)$. Fix $m \geq m^*$. Let $n \geq m$. Then, there exist polynomials Q_{n,m,m^*} and P_{n,m,m^*} , such that

$$\deg(P_{n,m,m^*}) \leq n - m^*, \quad \deg(Q_{n,m,m^*}) \leq m, \quad Q_{n,m,m^*} \not\equiv 0, \\ [Q_{n,m,m^*}F - P_{n,m,m^*}]_j = 0, \quad j = 0, 1, \dots, n.$$

The rational function $R_{n,m,m^*} = P_{n,m,m^*}/Q_{n,m,m^*}$ is called an (n, m, m^*) *incomplete Padé-Faber approximant of F* .

Clearly, finding Q_{n,m,m^*} is equivalent to solving the following linear equations

$$[Q_{n,m,m^*}F]_j = 0, \quad j = n - m^* + 1, \dots, n. \quad (16)$$

Given $n \geq m \geq m^*$, R_{n,m,m^*} and Q_{n,m,m^*} are not unique, so we choose one candidate. Notice that for each $\alpha = 1, 2, \dots, d$, $R_{n,m,\alpha}$ in Definition 1 is an $(n, |\mathbf{m}|, m_\alpha)$ incomplete Padé-Faber approximant of F_α .

Now, we want to prove that $h\text{-}\lim_{n \rightarrow \infty} R_{n,m,m^*} = F$ in $D_{\rho_{m^*}(F)}$. Let $F \in \mathcal{H}(E)$. Let $\lambda_1, \lambda_2, \dots, \lambda_q$ be the distinct poles of F in $D_{\rho_{m^*}(F)}$ and let

$$L := \left(1 + \min_{j=1,2,\dots,q} |\Phi(\lambda_j)| \right) / 2 > 1.$$

The normalization of Q_{n,m,m^*} used in our proofs is the following:

$$Q_{n,m,m^*}(z) := \prod_{|\Phi(\lambda_{n,j})| \leq L} (z - \lambda_{n,j}) \prod_{|\Phi(\lambda_{n,j})| > L} \left(1 - \frac{z}{\lambda_{n,j}}\right). \quad (17)$$

Firstly, let us discuss some upper and lower estimates on the normalized polynomials Q_{n,m,m^*} in (17). Let $\varepsilon > 0$ and define an open set $J_\varepsilon := J_\varepsilon(F)$ as follows. For $n \geq m$, let $J_{n,\varepsilon}$ denote the $\varepsilon/6mn^2$ -neighborhood of the set of zeros of Q_{n,m,m^*} and let $J_{m-1,\varepsilon}$ denote the $\varepsilon/6m$ -neighborhood of the set of poles of F in $D_{\rho_{m^*}}(F)$. Define

$$J_\varepsilon := \bigcup_{n \geq m-1} J_{n,\varepsilon}.$$

By the monotonicity and subadditivity of h , we know that $h(J_\varepsilon) < \varepsilon$ and $h(J_{\varepsilon_1}) \leq h(J_{\varepsilon_2})$ for $\varepsilon_1 < \varepsilon_2$. For each set $B \subset \mathbb{C}$, we set $B(\varepsilon) := B \setminus J_\varepsilon$. It is easy to check that if $\{g_n\}_{n \in \mathbb{N}_0}$ converges uniformly to g on $K(\varepsilon)$ for every compact $K \subset D_{\rho_{m^*}}(F)$ and $\varepsilon > 0$, then $h\text{-}\lim_{n \rightarrow \infty} g_n = g$ in $D_{\rho_{m^*}}(F)$.

Since the following upper and lower bounds on the estimations of the normalized Q_{n,m,m^*} are used many times in our proofs, we keep them in the following lemma.

Lemma 4. *Let $K \subset \mathbb{C}$ be a compact set and $\varepsilon > 0$ be arbitrary. Then, there exist constants $C_1, C_2 > 0$ independent of n , such that*

$$\|Q_{n,m,m^*}\|_K \leq C_1 \quad \text{and} \quad \min_{z \in K(\varepsilon)} |Q_{n,m,m^*}(z)| \geq C_2 n^{-2m}, \quad (18)$$

where the second inequality has a meaning when $K(\varepsilon)$ is a non-empty set.

Theorem 2. *Let $F \in \mathcal{H}(E)$ and let m and m^* be fixed nonnegative integers, such that $m \geq m^*$. For each $n \geq m$, let R_{n,m,m^*} be an (n, m, m^*) incomplete Padé-Faber approximant of F . Then, for each $\varepsilon > 0$ and every compact subset K of $D_{\rho_{m^*}}(F)$,*

$$\limsup_{n \rightarrow \infty} \|F - R_{n,m,m^*}\|_{K(\varepsilon)}^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{m^*}(F)}, \quad (19)$$

where $\|\Phi\|_K$ should be replaced by 1 when $K \subset E$. In particular,

$$h\text{-}\lim_{n \rightarrow \infty} R_{n,m,m^*} = F \text{ in } D_{\rho_{m^*}}(F). \quad (20)$$

Moreover, for each pole λ of order τ of F in $D_{\rho_{m^*}}(F)$ and every $\varepsilon > 0$, the polynomials Q_{n,m,m^*} have at least τ zeros in the disk $\{z \in \mathbb{C} : |z - \lambda| < \varepsilon\}$ for all sufficiently large n .

Proof of Theorem 2. Let $Q_m^F(z) = \prod_{j=1}^q (z - \lambda_j)^{\tau_j}$ be the polynomial whose zeros are all the poles of F in $D_{\rho_{m^*}}(F)$ counting multiplicities and set

$$\hat{m} := \deg(Q_m^F). \quad (21)$$

By the definition of incomplete Padé-Faber approximants and Lemma 1, we have

$$Q_{n,m,m^*}(z)F(z) - P_{n,m,m^*}(z) = \sum_{k=n+1}^{\infty} a_{k,n} \Phi_k(z), \quad z \in D_{\rho_0(F)}, \quad (22)$$

where

$$a_{k,n} := [Q_{n,m,m^*}F]_k$$

and $a_{k,n} = 0$ for all $k = n - m^* + 1, \dots, n$. Multiplying equality (22) by Q_m^F and expanding $Q_m^F Q_{n,m,m^*} F - Q_m^F P_{n,m,m^*} \in \mathcal{H}(D_{\rho_{m^*}}(F))$ in terms of the system of Faber polynomials $\{\Phi_\nu\}_{\nu=0}^\infty$, we have for $z \in D_{\rho_{m^*}}(F)$,

$$\begin{aligned} Q_m^F(z)Q_{n,m,m^*}(z)F(z) - Q_m^F(z)P_{n,m,m^*}(z) &= \sum_{k=n+1}^{\infty} a_{k,n} Q_m^F(z) \Phi_k(z) = \sum_{\nu=0}^{\infty} b_{\nu,n} \Phi_\nu(z) \\ &= \sum_{\nu=0}^n b_{\nu,n} \Phi_\nu(z) + \sum_{\nu=n+1}^{\infty} b_{\nu,n} \Phi_\nu(z). \end{aligned} \quad (23)$$

Note that the constants $b_{v,n}$ can be calculated in two forms:

$$b_{v,n} := \sum_{k=n+1}^{\infty} a_{k,n} [Q_m^F \Phi_k]_v, \quad v = 0, 1, \dots$$

and

$$b_{v,n} := \begin{cases} [Q_m^F Q_{n,m,m^*} F - Q_m^F P_{n,m,m^*}]_v, & \text{if } v = 0, 1, 2, \dots, n \\ [Q_m^F Q_{n,m,m^*} F]_v, & \text{if } v \geq n+1. \end{cases}$$

Using (14) when $G = Q_m^F Q_{n,m,m^*} F - Q_m^F P_{n,m,m^*}$, we have for $\rho \in (1, \rho_{m^*}(F))$,

$$b_{v,n} := \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{(Q_m^F Q_{n,m,m^*} F - Q_m^F P_{n,m,m^*})(z) \Phi'(z)}{\Phi^{v+1}(z)} dz, & \text{if } v = 0, 1, \dots, n, \\ \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{(Q_m^F Q_{n,m,m^*} F)(z) \Phi'(z)}{\Phi^{v+1}(z)} dz, & \text{if } v \geq n+1. \end{cases} \quad (24)$$

Let K be a compact subset of $D_{\rho_{m^*}(F)}$ and set

$$\sigma := \max\{\|\Phi\|_K, 1\} \quad (25)$$

($\sigma = 1$ when $K \subset E$). We want to show that

$$\limsup_{n \rightarrow \infty} \left\| \sum_{v=0}^{\infty} b_{v,n} \Phi_v \right\|_K^{1/n} \leq \frac{\sigma}{\rho_{m^*}(F)}. \quad (26)$$

Let $\rho_1 \in (1, \rho_{m^*}(F))$ be such that

$$K \cup \{\lambda_1, \lambda_2, \dots, \lambda_q\} \subset D_{\rho_1}. \quad (27)$$

We first prove that

$$\limsup_{n \rightarrow \infty} \left\| \sum_{v=n+1}^{\infty} b_{v,n} \Phi_v \right\|_K^{1/n} \leq \frac{\sigma}{\rho_{m^*}(F)}. \quad (28)$$

Due to the normalization of Q_{n,m,m^*} (the upper estimate in (18)), the formula in (24) implies that for $\rho_1 \in (1, \rho_{m^*}(F))$,

$$|b_{v,n}| \leq \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{(Q_m^F Q_{n,m,m^*} F)(z) \Phi'(z)}{\Phi^{v+1}(z)} dz \right| \leq \frac{c_1}{\rho_1^v}, \quad v \geq n+1, \quad (29)$$

where c_1 does not depend on n (from now on, we will denote some constants that do not depend on n by $c_2, c_3, \dots$). Moreover, by Lemma 2, we have

$$\|\Phi_v\|_K \leq c_2 \sigma^v, \quad v \geq 0. \quad (30)$$

Therefore, by (29) and (30),

$$\begin{aligned} \left\| \sum_{v=n+1}^{\infty} b_{v,n} \Phi_v \right\|_K &\leq \sum_{v=n+1}^{\infty} |b_{v,n}| \|\Phi_v\|_K \\ &\leq \sum_{v=n+1}^{\infty} c_3 \left(\frac{\sigma}{\rho_1} \right)^v \leq c_4 \left(\frac{\sigma}{\rho_1} \right)^n. \end{aligned}$$

Then,

$$\limsup_{n \rightarrow \infty} \left\| \sum_{v=n+1}^{\infty} b_{v,n} \Phi_v \right\|_K^{1/n} \leq \frac{\sigma}{\rho_1}.$$

Letting $\rho_1 \rightarrow \rho_{m^*}(F)$, we have (28) as we wanted.

Secondly, we prove that

$$\limsup_{n \rightarrow \infty} \left\| \sum_{v=0}^n b_{v,n} \Phi_v \right\|_K^{1/n} \leq \frac{\sigma}{\rho_{m^*}(F)}. \quad (31)$$

Recall that $b_{v,n} = \sum_{k=n+1}^{\infty} a_{k,n} [Q_{m^*}^F \Phi_k]_v$. Hence, to estimate $b_{v,n}$, we need to estimate $a_{k,n}$ first. We will use the technique in [2] to estimate $a_{k,n}$. Let $\rho_1 \in (1, \rho_{m^*}(F))$ satisfying (27). Choose $\rho_2 \in (1, \rho_0(F))$. We have

$$a_{k,n} = [Q_{n,m,m^*} F]_k = \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{Q_{n,m,m^*}(t) F(t) \Phi'(t)}{\Phi^{k+1}(t)} dt.$$

Define

$$\gamma_{k,n} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{Q_{n,m,m^*}(t) F(t) \Phi'(t)}{\Phi^{k+1}(t)} dt. \quad (32)$$

By our choices of ρ_1 and ρ_2 , and for each $k \geq 0$, $Q_{n,m,m^*} F \Phi' / \Phi^{k+1}$ is meromorphic in $\overline{D_{\rho_1}} \setminus D_{\rho_2} = \{z \in \mathbb{C} : \rho_2 \leq |\Phi(z)| \leq \rho_1\}$ and has poles at $\lambda_1, \lambda_2, \dots, \lambda_q$ with multiplicities at most $\tau_1, \tau_2, \dots, \tau_q$. Cauchy's residue theorem implies that

$$\gamma_{k,n} - a_{k,n} = \sum_{j=1}^q \text{res}(Q_{n,m,m^*} F \Phi' / \Phi^{k+1}, \lambda_j), \quad k \geq 0. \quad (33)$$

Then, it follows from the limit formula for the residue that

$$\text{res}(Q_{n,m,m^*} F \Phi' / \Phi^{k+1}, \lambda_j) = \frac{1}{(\tau_j - 1)!} \lim_{z \rightarrow \lambda_j} \left(\frac{(z - \lambda_j)^{\tau_j} Q_{n,m,m^*}(z) F(z) \Phi'(z)}{\Phi^{k+1}(z)} \right)^{(\tau_j - 1)}.$$

Leibniz's formula allows us to write

$$\begin{aligned} & \left(\frac{(z - \lambda_j)^{\tau_j} Q_{n,m,m^*}(z) F(z) \Phi'(z)}{\Phi^{k+1}(z)} \right)^{(\tau_j - 1)} \\ &= \sum_{t=0}^{\tau_j - 1} \binom{\tau_j - 1}{t} \left(\frac{Q_{n,m,m^*}(z) \Phi'(z) (z - \lambda_j)^{\tau_j} F(z)}{\Phi^{n+1}(z)} \right)^{(\tau_j - 1 - t)} \left(\Phi^{n-k}(z) \right)^{(t)}. \end{aligned}$$

For $j = 1, 2, \dots, q$ and $t = 0, 1, \dots, \tau_j - 1$, set

$$\beta_n(j, t) := \frac{1}{(\tau_j - 1)!} \binom{\tau_j - 1}{t} \lim_{z \rightarrow \lambda_j} \left(\frac{Q_{n,m,m^*}(z) \Phi'(z) (z - \lambda_j)^{\tau_j} F(z)}{\Phi^{n+1}(z)} \right)^{(\tau_j - 1 - t)}$$

(notice that the $\beta_n(j, t)$ do not depend on k). Therefore, we rewrite (33) as

$$\gamma_{k,n} - a_{k,n} = \sum_{j=1}^q \sum_{t=0}^{\tau_j - 1} \beta_n(j, t) \left(\Phi^{n-k}(z) \right)_{z=\lambda_j}^{(t)}, \quad k \geq 0. \quad (34)$$

By the definition of incomplete Padé-Faber approximants,

$$a_{k,n} = 0, \quad k = n - m^* + 1, \dots, n,$$

which implies

$$\gamma_{k,n} = \sum_{j=1}^q \sum_{t=0}^{\tau_j - 1} \beta_n(j, t) \left(\Phi^{n-k}(z) \right)_{z=\lambda_j}^{(t)} \quad (35)$$

for all $k = n - \hat{m} + 1, \dots, n$ (we recall that $\hat{m}$ is defined in (21)). Equations (35) can be viewed as a system of $\hat{m}$ equations on $\hat{m}$ unknowns $\beta_n(j, t)$ and the corresponding determinant to this system is

$$\Delta := \begin{vmatrix} \left(\Phi^{\hat{m}-1}(z) \right)_{z=\lambda_j} & \left(\Phi^{\hat{m}-1}(z) \right)'_{z=\lambda_j} & \dots & \left(\Phi^{\hat{m}-1}(z) \right)^{(\tau_j-1)}_{z=\lambda_j} \\ \left(\Phi^{\hat{m}-2}(z) \right)_{z=\lambda_j} & \left(\Phi^{\hat{m}-2}(z) \right)'_{z=\lambda_j} & \dots & \left(\Phi^{\hat{m}-2}(z) \right)^{(\tau_j-1)}_{z=\lambda_j} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{vmatrix}_{j=1, \dots, q},$$

where the subindex on the determinant means that the indicated group of columns are successively written for $j = 1, 2, \dots, q$. It is not difficult to check that

$$|\Delta| = \prod_{j=1}^q (\tau_j - 1)!! |\Phi'(\lambda_j)|^{\tau_j(\tau_j-1)/2} \prod_{1 \leq \ell < k \leq q} |\Phi(\lambda_\ell) - \Phi(\lambda_k)|^{\tau_\ell \tau_k} =: c_5 > 0,$$

where $n!!$ stands for $0!1! \dots n!$ (use, e.g., [13, Theorem 1] for the calculation of the above equality). Therefore, the system of equations (35) has a unique solution.

To avoid a long expression, let us define: for all $j = 1, 2, \dots, q$ and $t = 0, 1, \dots, \tau_j - 1$,

$$h_{j,t} := \left(\sum_{l=0}^{j-1} \tau_l \right) + t + 1,$$

where $\tau_0 = 0$. Using Cramer's rule to (35), we have

$$\beta_n(j, t) = \frac{\Delta_n(j, t)}{\Delta} = \frac{1}{\Delta} \sum_{y=1}^{\hat{m}} \gamma_{n-\hat{m}+y, n} C(y, h_{j,t}), \quad (36)$$

where $\Delta_n(j, t)$ is the determinant obtained from Δ by replacing the $h_{j,t}^{\text{th}}$ column with the column

$$[\gamma_{n-\hat{m}+1, n} \quad \gamma_{n-\hat{m}+2, n} \quad \dots \quad \gamma_{n, n}]^T$$

and $C(y, h)$ is the determinant of the $(y, h)^{\text{th}}$ cofactor matrix of $\Delta_n(j, t)$. Replacing $\beta_n(j, t)$ in (34) by the expression in (36), we obtain for $k \geq n + 1$,

$$\gamma_{k, n} - a_{k, n} = \frac{1}{\Delta} \sum_{j=1}^q \sum_{t=0}^{\tau_j-1} \sum_{y=1}^{\hat{m}} \gamma_{n-\hat{m}+y, n} C(y, h_{j,t}) \left(\Phi^{n-k}(z) \right)_{z=\lambda_j}^{(t)}. \quad (37)$$

Define

$$\mathbb{B}(\lambda, r) := \{z \in \mathbb{C} : |z - \lambda| < r\}.$$

Let $\varepsilon > 0$ be sufficiently small so that $\{z \in \mathbb{C} : |z - \lambda_j| = \varepsilon\} \subset \{z \in \mathbb{C} : |\Phi(z)| > \rho_2\}$ for all $j = 1, 2, \dots, q$ and $\overline{\mathbb{B}(\lambda_j, \varepsilon)} \cap \overline{\mathbb{B}(\lambda_k, \varepsilon)} = \emptyset$ for all $k \neq j$. Using Cauchy's integral formula,

$$\left(\Phi^{n-k}(z) \right)_{z=\lambda_j}^{(t)} = \frac{t!}{2\pi i} \int_{|z-\lambda_j|=\varepsilon} \frac{\Phi^{n-k}(z) dz}{(z - \lambda_j)^{t+1}}. \quad (38)$$

Clearly, there exists a constant c_6 such that for all $j = 1, 2, \dots, q$, $t = 0, 1, \dots, \tau_j - 1$, and $k \geq n + 1$,

$$\left| \left(\Phi^{n-k}(z) \right)_{z=\lambda_j}^{(t)} \right| \leq \frac{c_6}{\rho_2^{k-n}}, \quad (39)$$

and there exists a constant c_7 such that for all $j = 1, 2, \dots, q$, $t = 0, 1, \dots, \tau_j - 1$, and $k = n - \hat{m} + 1, n - \hat{m} + 2, \dots, n$,

$$\left| \left(\Phi^{n-k}(z) \right)_{z=\lambda_j}^{(t)} \right| \leq c_7, \quad (40)$$

for sufficiently large n . From (40),

$$|C(y, h)| \leq c_8, \quad y, h = 1, 2, \dots, \hat{m}. \quad (41)$$

Using (39), (41), and $|\Delta| = c_5 > 0$, it follows from (37) that

$$|a_{k,n}| \leq |\gamma_{k,n}| + \frac{c_9}{\rho_2^{k-n}} \sum_{y=1}^{\hat{m}} |\gamma_{n-\hat{m}+y,n}|, \quad k \geq n+1. \quad (42)$$

By the definition of $\gamma_{k,n}$ (see (32)),

$$|\gamma_{k,n}| = \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{Q_{n,m,m^*}(t)F(t)\Phi'(t)}{\Phi^{k+1}(t)} dt \right| \leq \frac{c_{10}}{\rho_1^k},$$

where we recall that $\rho_2 < \rho_1$. This and equality (42) imply

$$|a_{k,n}| \leq \frac{c_{11}}{\rho_2^{k-n}\rho_1^n}, \quad k \geq n+1. \quad (43)$$

Moreover, by Lemma 2, we have for all $v \geq 0$ and for all $k \geq n+1$,

$$|[Q_{m^*}^F \Phi_k]_v| = \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_2-2\delta}} \frac{Q_{m^*}^F(t)\Phi_k(t)\Phi'(t)}{\Phi^{v+1}(t)} dt \right| \leq c_{12} \frac{(\rho_2 - 2\delta)^k}{(\rho_2 - 2\delta)^v}, \quad (44)$$

where $\delta > 0$ is sufficiently small and $\rho_2 \in (1, \rho_0(F))$ is chosen so that

$$\sigma + \delta \geq \rho_2 - 2\delta > 1 \quad (45)$$

(the reason for the first inequality will become clear later). Combining (43) and (44), we have

$$\begin{aligned} |b_{v,n}| &= \left| \sum_{k=n+1}^{\infty} a_{k,n} [Q_{\mathbf{m}}^F \Phi_k]_v \right| \leq \sum_{k=n+1}^{\infty} |a_{k,n}| |[Q_{\mathbf{m}}^F \Phi_k]_v| \\ &\leq \frac{c_{13}}{(\rho_2 - 2\delta)^v} \left(\frac{\rho_2}{\rho_1} \right)^n \sum_{k=n+1}^{\infty} \left(\frac{\rho_2 - 2\delta}{\rho_2} \right)^k \leq \frac{c_{14}}{(\rho_2 - 2\delta)^v} \left(\frac{\rho_2}{\rho_1} \right)^n \left(\frac{\rho_2 - 2\delta}{\rho_2} \right)^n \\ &= \frac{c_{14}}{(\rho_2 - 2\delta)^v} \left(\frac{\rho_2 - 2\delta}{\rho_1} \right)^n. \end{aligned}$$

Now, we show (31). Recall that by Lemma 2, we have $\|\Phi_v\|_K \leq c_2 \sigma^v$, for all $v \geq 0$. Therefore,

$$\begin{aligned} \left\| \sum_{v=0}^n b_{v,n} \Phi_v \right\|_K &\leq \sum_{v=0}^n |b_{v,n}| \|\Phi_v\|_K \leq c_{15} \left(\frac{\rho_2 - 2\delta}{\rho_1} \right)^n \sum_{v=0}^n \left(\frac{\sigma}{\rho_2 - 2\delta} \right)^v \\ &\leq c_{15} \left(\frac{\rho_2 - 2\delta}{\rho_1} \right)^n \sum_{v=0}^n \left(\frac{\sigma + \delta}{\rho_2 - 2\delta} \right)^v \leq c_{15}(n+1) \left(\frac{\rho_2 - 2\delta}{\rho_1} \right)^n \left(\frac{\sigma + \delta}{\rho_2 - 2\delta} \right)^n, \end{aligned}$$

where the last inequality follows from the first inequality in (45). Hence,

$$\limsup_{n \rightarrow \infty} \left\| \sum_{v=0}^n b_{v,n} \Phi_v \right\|_K^{1/n} \leq \left(\frac{\sigma + \delta}{\rho_1} \right).$$

Letting $\delta \rightarrow 0^+$ and $\rho_1 \rightarrow \rho_{m^*}(F)$,

$$\limsup_{n \rightarrow \infty} \left\| \sum_{v=0}^n b_{v,n} \Phi_v \right\|_K^{1/n} \leq \frac{\sigma}{\rho_{m^*}(F)}. \quad (46)$$

Combining (28) and (46), we have (26). Therefore, from (23), we obtain

$$\limsup_{n \rightarrow \infty} \|Q_{m^*}^F Q_{n,m,m^*} F - Q_{m^*}^F P_{n,m,m^*}\|_K^{1/n} \leq \frac{\sigma}{\rho_{m^*}(F)}, \quad (47)$$

where K is any compact subset of $D_{\rho_{m^*}(F)}$. Applying the second inequality of (18), we have

$$\|F - R_{n,m,m^*}\|_{K(\varepsilon)} \leq c_{16} n^{2m} \|Q_{m^*}^F Q_{n,m,m^*} F - Q_{m^*}^F P_{n,m,m^*}\|_K.$$

Hence,

$$\limsup_{n \rightarrow \infty} \|F - R_{n,m,m^*}\|_{K(\varepsilon)}^{1/n} \leq \frac{\sigma}{\rho_{m^*}(F)}, \quad (48)$$

which implies that $h\text{-}\lim_{n \rightarrow \infty} R_{n,m,m^*} = F$ in $D_{\rho_{m^*}(F)}$. As an immediate consequence of Lemma 3, the statement concerning the asymptotic behavior of some of the zeros of Q_{n,m,m^*} is proved. $\square$

Next, we study the rate of convergence of some of the zeros of Q_{n,m,m^*} to the poles of F in $D_{\rho_{m^*}(F)}$. Let us define two indicators related to the asymptotic behavior of the zeros of Q_{n,m,m^*} . These two indicators were first introduced in [12] and were also used in the study of incomplete Padé approximants in [5]. Let

$$\mathcal{P}_{n,m,m^*} := \{\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,v_n}\}, \quad v_n \leq m, \quad n \geq m,$$

be the set of zeros of Q_{n,m,m^*} (repeated according to their multiplicities). Define

$$|z - w|_1 := \min\{1, |z - w|\}, \quad z, w \in \mathbb{C}.$$

Choose a point $a \in \mathbb{C}$. The first indicator is defined by

$$\Delta(a) := \limsup_{n \rightarrow \infty} \prod_{j=1}^{v_n} |\lambda_{n,j} - a|_1^{1/n} = \limsup_{n \rightarrow \infty} \prod_{|\lambda_{n,j} - a| < 1} |\lambda_{n,j} - a|^{1/n}.$$

Clearly, $0 \leq \Delta(a) \leq 1$ (if $v_n = 0$, then the product is taken to be 1). The second indicator, a nonnegative integer $\gamma(a)$, is defined as follows. We suppose that for each n , all points in

$$\mathcal{P}_{n,m,m^*} = \{\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,v_n}\} \quad (49)$$

are enumerated in nondecreasing distance to the point a . Define

$$\delta_j(a) := \limsup_{n \rightarrow \infty} |\lambda_{n,j} - a|_1^{1/n}. \quad (50)$$

These numbers are defined by (50) for $j = 1, 2, \dots, m'$, where $m' := \liminf_{n \rightarrow \infty} v_n$ and for $j = m' + 1, \dots, n$, we define $\delta_j(a) = 1$. We have $0 \leq \delta_j(a) \leq 1$. If $\Delta(a) = 1$ (in that case all $\delta_j(a) = 1$), then $\gamma(a) = 0$. If $\Delta(a) < 1$, then for some γ , $1 \leq \gamma \leq m$, we have that $\delta_1(a) \leq \dots \leq \delta_\gamma(a) < 1$ and $\delta_{\gamma+1}(a) = 1$ or $\gamma = m$; in this case we take $\gamma(a) = \gamma$.

Note that $\Delta(a) < 1 \Leftrightarrow \gamma(a) \geq 1$ and $\sum_{a \in \mathbb{C}} \gamma(a) \leq m$. It is not difficult to check that

$$\Delta(a) = \limsup_{n \rightarrow \infty} |Q_{n,m,m^*}(a)|^{1/n}.$$

Moreover, we would like to emphasize that $\Delta(a)$ and $\gamma(a)$ depend on our choice of a sequence $\{Q_{n,m,m^*}\}_{n \geq m}$ (recall that for given $n \geq m \geq m^*$, Q_{n,m,m^*} may not be unique).

Theorem 3. Let $F \in \mathcal{H}(E)$ and let m and m^* be fixed nonnegative integers, such that $m \geq m^*$. For each $n \geq m$, let R_{n,m,m^*} be an (n, m, m^*) incomplete Padé-Faber approximant of F . Let λ be a pole of order τ of F in $D_{\rho_{m^*}(F)}$. Then,

$$\Delta(\lambda) \leq \frac{|\Phi(\lambda)|}{\rho_{m^*}(F)} \quad \text{and} \quad \gamma(\lambda) \geq \tau. \quad (51)$$

Proof of Theorem 3. Let λ be a pole of order τ of F in $D_{\rho_{m^*}(F)}$. Fix $\varepsilon > 0$ arbitrarily small. Let $r > 0$ be sufficiently small so that $\mathbb{B}(\lambda, r) \subset D_{\rho_{m^*}(F)}$ contains no other pole of F and

$$\mathbb{T}(\lambda, r) := \{z \in \mathbb{C} : |z - \lambda| = r\} \subset D_{\rho_{m^*}(F)} \setminus J_\varepsilon \quad (52)$$

(this is possible because $\varepsilon > 0$ is arbitrarily small and $h(J_\varepsilon) < \varepsilon$). By Theorem 2, Q_{n,m,m^*} have at least τ zeros in $\mathbb{B}(\lambda, r)$ for all n sufficiently large. Since r is arbitrarily small, each pole of F in $D_{\rho_{m^*}(F)}$ attracts at least as many zeros of Q_{n,m,m^*} as its order. Let $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,\gamma_n}$ be the zeros of Q_{n,m,m^*} in $\mathbb{B}(\lambda, r)$ indexed in non-decreasing distance from λ , i.e.,

$$|\lambda - \lambda_{n,1}| \leq |\lambda - \lambda_{n,2}| \leq \dots \leq |\lambda - \lambda_{n,\gamma_n}|.$$

Indeed, for all sufficiently large n , $\tau \leq \gamma_n \leq m$. From now, we will only consider such n 's. Set

$$Q_{n,\lambda}(z) := \prod_{j=1}^{\gamma_n} (z - \lambda_{n,j}).$$

Let $\eta > 0$ be sufficiently small so that

$$\frac{\|\Phi\|_{\mathbb{T}(\lambda,r)}}{\rho_{m^*}(F)} + \eta < 1.$$

Using (48) and (52), we obtain

$$\|F - R_{n,m,m^*}\|_{\mathbb{T}(\lambda,r)} \leq c_1 \left(\frac{\|\Phi\|_{\mathbb{T}(\lambda,r)}}{\rho_{m^*}(F)} + \eta \right)^n, \quad (53)$$

for n sufficiently large.

Let $q(z)/(z - \lambda)^\tau$ be the principal part of the function F at the point λ and $q_n/Q_{n,\lambda}$ be the sum of the principal parts of R_{n,m,m^*} corresponding to its poles in $\mathbb{B}(\lambda, r)$. Note that $\deg(q) < \tau$, $q(\lambda) \neq 0$, and $\deg(q_n) < \gamma_n$. It is known that the norm of the holomorphic component of a meromorphic function may be bounded in terms of the norm of the function and the number of poles (see [14, Theorem 1]). Therefore, using (53), we have

$$\left\| \frac{q(z)}{(z - \lambda)^\tau} - \frac{q_n(z)}{Q_{n,\lambda}(z)} \right\|_{\mathbb{T}(\lambda,r)} \leq c_2 \left(\frac{\|\Phi\|_{\mathbb{T}(\lambda,r)}}{\rho_{m^*}(F)} + \eta \right)^n,$$

for sufficiently large n . Multiplying the function in the norm by $(z - \lambda)^\tau Q_{n,\lambda}$ and applying the maximum principal, we obtain

$$\|q(z)Q_{n,\lambda}(z) - (z - \lambda)^\tau q_n(z)\|_{\overline{\mathbb{B}(\lambda,r)}} \leq c_3 \left(\frac{\|\Phi\|_{\mathbb{T}(\lambda,r)}}{\rho_{m^*}(F)} + \eta \right)^n, \quad (54)$$

for sufficiently large n . By (54), and the fact that $q(\lambda) \neq 0$,

$$\begin{aligned} \Delta(\lambda) &= \limsup_{n \rightarrow \infty} |Q_{n,m,m^*}(\lambda)|^{1/n} = \limsup_{n \rightarrow \infty} |Q_{n,\lambda}(\lambda)|^{1/n} \\ &\leq \limsup_{n \rightarrow \infty} \|q(z)Q_{n,\lambda}(z) - (z - \lambda)^\tau q_n(z)\|_{\overline{\mathbb{B}(\lambda,r)}}^{1/n} \leq \frac{\|\Phi\|_{\mathbb{T}(\lambda,r)}}{\rho_{m^*}(F)} + \eta. \end{aligned}$$

Letting $\eta, \varepsilon, r \rightarrow 0^+$, we have proved that

$$\Delta(\lambda) \leq \frac{|\Phi(\lambda)|}{\rho_{m^*}(F)}.$$

Now, we will verify that $\gamma(\lambda) \geq \tau$. Because $\Delta(\lambda) < 1$, we have $\delta_1(\lambda) < 1$. Proceeding by induction, we let

$$\delta_1(\lambda) \leq \delta_2(\lambda) \leq \dots \leq \delta_k(\lambda) < 1 \quad \text{and} \quad k < \tau.$$

We need to prove that $\delta_{k+1}(\lambda) < 1$. Note that $\deg(qQ_{n,\lambda} - (z - \lambda)^\tau q_n) < 2m$. We differentiate the polynomial inside the norm in (54) k times. Consequently, by Cauchy's integral formula, its k -th derivative satisfies an inequality like (54). Substituting $z = \lambda$ in the corresponding inequality, we have

$$\left| \left(q(z) \prod_{j=1}^{\gamma_n} (z - \lambda_{n,j}) \right)^{(k)} (\lambda) \right| \leq c_4 \left(\frac{\|\Phi\|_{\mathbb{T}(\lambda,r)}}{\rho_{m^*}(F)} + \eta \right)^n. \quad (55)$$

Furthermore, it is easy to check that $\left(q(z) \prod_{j=1}^{\gamma_n} (z - \lambda_{n,j})\right)_{z=\lambda}^{(k)}$ is different from $k!q(\lambda) \prod_{j=k+1}^{\gamma_n} (\lambda - \lambda_{n,j})$ by a group of terms, each of which has one of the terms $(\lambda - \lambda_{n,j})$, $j \in \{1, 2, \dots, k\}$, as a factor (the modulus of the products of the other factors are bounded at λ). Since we assumed that $\delta_j(\lambda) < 1$ for $j = 1, 2, \dots, k$, we obtain

$$\limsup_{n \rightarrow \infty} \prod_{j=k+1}^{\gamma_n} |\lambda_{n,j} - \lambda|^{1/n} < 1,$$

which implies $\delta_{k+1}(\lambda) = \limsup_{n \rightarrow \infty} |\lambda - \lambda_{n,k+1}|^{1/n} < 1$. Therefore, $\gamma(\lambda) \geq \tau$. The proof is complete. $\square$

Theorem 4. Let $F \in \mathcal{H}(E)$ and λ be a pole of order τ of F in $D_{\rho_{m^*}(F)}$. Suppose that $\liminf_{n \rightarrow \infty} |\lambda - \lambda_{n,\tau+1}| > 0$. Then,

$$\delta_1(\lambda) \leq \delta_2(\lambda) \leq \dots \leq \delta_\tau(\lambda) \leq \left(\frac{|\Phi(\lambda)|}{\rho_{m^*}(F)} \right)^{1/\tau}. \quad (56)$$

Particularly, $\delta_1(\lambda) = \delta_2(\lambda) = \dots = \delta_\tau(\lambda) = (|\Phi(\lambda)|/\rho_{m^*}(F))^{1/\tau}$ if and only if $\Delta(\lambda) = |\Phi(\lambda)|/\rho_{m^*}(F)$.

Proof of Theorem 4. In the proof of this theorem, we use the same notation defined in the proof of Theorem 3. By our assumption, we assume that

$$Q_{n,\lambda}(z) = \prod_{j=1}^{\tau} (z - \lambda_{n,j}).$$

Recall that $\deg(q) < \tau$ and $q(\lambda) \neq 0$. Substituting z with λ in (54), we have

$$|Q_{n,\lambda}(\lambda)| \leq c_1 \left(\frac{\|\Phi\|_{\mathbb{T}(\lambda,r)}}{\rho_{m^*}(F)} + \eta \right)^n,$$

which implies

$$\limsup_{n \rightarrow \infty} |Q_{n,\lambda}(\lambda)|^{1/n} \leq \frac{\|\Phi\|_{\mathbb{T}(\lambda,r)}}{\rho_{m^*}(F)}.$$

Using the Leibniz's formula, we obtain

$$(qQ_{n,\lambda})^{(k)}(\lambda) = q(\lambda)Q_{n,\lambda}^{(k)}(\lambda) + \sum_{j=0}^{k-1} \binom{k}{j} q^{(k-j)}(\lambda) Q_{n,\lambda}^{(j)}(\lambda).$$

Applying the equality above and (55), by induction,

$$\limsup_{n \rightarrow \infty} |Q_{n,\lambda}^{(k)}(\lambda)|^{1/n} \leq \frac{\|\Phi\|_{\mathbb{T}(\lambda,r)}}{\rho_{m^*}(F)}, \quad k = 0, 1, \dots, \tau - 1. \quad (57)$$

Combining the inequalities above and the following expression

$$Q_{n,\lambda}(z) = (z - \lambda)^\tau + \sum_{k=0}^{\tau-1} \frac{Q_{n,\lambda}^{(k)}(\lambda)}{k!} (z - \lambda)^k, \quad (58)$$

we have

$$\limsup_{n \rightarrow \infty} \|(z - \lambda)^\tau - Q_{n,\lambda}(z)\|_{\mathbb{B}(\lambda,r)}^{1/n} \leq \frac{\|\Phi\|_{\mathbb{T}(\lambda,r)}}{\rho_{m^*}(F)}.$$

If we substitute z with $\lambda_{n,\tau}$ in the inequality above, then

$$\limsup_{n \rightarrow \infty} |\lambda_{n,\tau} - \lambda|^{1/n} \leq \left(\frac{\|\Phi\|_{\mathbb{T}(\lambda,r)}}{\rho_{m^*}(F)} \right)^{1/\tau}.$$

Letting $r \rightarrow 0^+$, we obtain

$$\delta_\tau(\lambda) \leq \left(\frac{|\Phi(\lambda)|}{\rho_{m^*}(F)} \right)^{1/\tau},$$

which implies (56).

Furthermore, by Theorem 3, $\Delta(\lambda) \leq |\Phi(\lambda)|/\rho_{m^*}(F)$ always holds and the last equivalent conditions follow from (56), which we just proved. $\square$

3.3 Proof of Theorem 1

Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ and fix $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$. Let $\lambda_1, \lambda_2, \dots, \lambda_q$ be the distinct poles of $\mathbf{F}$ in $\mathbf{D}_{\mathbf{m}}(\mathbf{F})$ and set

$$L := \left(1 + \min_{j=1,2,\dots,q} |\Phi(\lambda_j)|\right) / 2 > 1.$$

Recall that the set of these poles is denoted by $\mathcal{P}_{\mathbf{m}}(\mathbf{F})$. Let $\tilde{Q}_{n,\mathbf{m}}$ be the polynomial $Q_{n,\mathbf{m}}$ normalized as in (17), namely

$$\tilde{Q}_{n,\mathbf{m}}(z) := \prod_{|\Phi(\lambda_{n,j})| \leq L} (z - \lambda_{n,j}) \prod_{|\Phi(\lambda_{n,j})| > L} \left(1 - \frac{z}{\lambda_{n,j}}\right).$$

From now on, Δ and γ are defined as in Section 3.2 taking $\mathcal{P}_{n,\mathbf{m},m^*}$ in (49) to be the collection of zeros of the denominator $\tilde{Q}_{n,\mathbf{m}}$. Clearly,

$$\Delta(a) = \limsup_{n \rightarrow \infty} |\tilde{Q}_{n,\mathbf{m}}(a)|^{1/n}.$$

Combining Theorems 2 and 3, we have the following.

Corollary 1. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ and let $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$. For any $\alpha = 1, 2, \dots, d$,

$$h\text{-}\lim_{n \rightarrow \infty} R_{n,\mathbf{m},\alpha} = F_\alpha \text{ in } D_{\rho_{m_\alpha}(F_\alpha)} \quad (59)$$

and if λ is a pole of order τ of F_α in $D_{\rho_{m_\alpha}(F_\alpha)}$, then

$$\Delta(\lambda) \leq \frac{|\Phi(\lambda)|}{\rho_{m_\alpha}(F_\alpha)}$$

and $\gamma(\lambda) \geq \tau$.

Proof of Corollary 1. Let $\alpha \in \{1, 2, \dots, d\}$ be fixed. Recall that $R_{n,\mathbf{m},\alpha}$ is an $(n, |\mathbf{m}|, m_\alpha)$ incomplete Padé-Faber approximant of F_α . Applying (20) in Theorem 2, we have (59). Let λ be a pole of order τ of F_α in $D_{\rho_{m_\alpha}(F_\alpha)}$. Since $\tilde{Q}_{n,\mathbf{m}}$ satisfies

$$[\tilde{Q}_{n,\mathbf{m}} F_\alpha]_j = 0, \quad j = n - m_\alpha + 1, \dots, n,$$

(see the equations (16)), $\tilde{Q}_{n,\mathbf{m}}$ is the same as $Q_{n,|\mathbf{m}|,m_\alpha}$ of F_α (up to a multiplication by some bounded constant). Using (51), we have

$$\Delta(\lambda) = \limsup_{n \rightarrow \infty} |\tilde{Q}_{n,\mathbf{m}}(\lambda)|^{1/n} = \limsup_{n \rightarrow \infty} |Q_{n,|\mathbf{m}|,m_\alpha}(\lambda)|^{1/n} \leq \frac{|\Phi(\lambda)|}{\rho_{m_\alpha}(F_\alpha)}.$$

Moreover, it is easy to see that $\gamma(\lambda) \geq \tau$. □

Now, we are ready to prove Theorem 1 which is the main theorem.

Proof of Theorem 1. Let λ be a pole of $\mathbf{F}$ in $\mathbf{D}_{\mathbf{m}}(\mathbf{F})$ and τ be its order. So, λ is a pole of order τ of $F_{\alpha(\lambda)}$ in $D_{\rho_{m_{\alpha(\lambda)}}(F_{\alpha(\lambda)})}$. By Corollary 1, we have $\gamma(\lambda) \geq \tau$. Because this holds for any other pole of $\mathbf{F}$ in $\mathbf{D}_{\mathbf{m}}(\mathbf{F})$ and $\deg(Q_{n,\mathbf{m}}) \leq |\mathbf{m}|$, we have $\deg(Q_{n,\mathbf{m}}) = |\mathbf{m}|$ for sufficiently large n , $\gamma(\lambda) = \tau$, and

$$\lim_{n \rightarrow \infty} \|Q_{\mathbf{m}}^{\mathbf{F}} - Q_{n,\mathbf{m}}\| = 0. \quad (60)$$

This means that $\mathbf{R}_{n,\mathbf{m}}$ is uniquely determined for all sufficiently large n . In fact, if this was not the case, we could find a sequence of $Q_{n,\mathbf{m}}$ with $\deg Q_{n,\mathbf{m}} < |\mathbf{m}|$, which contradicts (60).

Let $r > 0$ be sufficiently small so that $\mathbb{B}(\lambda, r)$ contains no other pole of $\mathbf{F}$. Let $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,\gamma_n}$ be the zeros of $Q_{n,\mathbf{m}}$ in $\mathbb{B}(\lambda, r)$ indexed in increasing distance from λ , i.e.,

$$|\lambda - \lambda_{n,1}| \leq |\lambda - \lambda_{n,2}| \leq \dots \leq |\lambda - \lambda_{n,\gamma_n}|.$$

Note that $\gamma_n \geq \tau$ and $\liminf_{n \rightarrow \infty} |\lambda - \lambda_{n,\tau+1}| > 0$, so we can use the arguments employed in Theorem 4. In particular, (57) and (58) imply that

$$\limsup_{n \rightarrow \infty} \|(z - \lambda)^\tau - Q_{n,\lambda}(z)\|^{1/n} \leq \frac{|\Phi(\lambda)|}{\rho_{m_{\alpha(\lambda)}(F_{\alpha(\lambda)})}}, \quad (61)$$

where

$$Q_{n,\lambda}(z) := \prod_{j=1}^{\tau} (z - \lambda_{n,j}).$$

Because inequality (61) is true for each of the poles of $\mathbf{F}$ in $\mathbf{D}_{\mathbf{m}}(\mathbf{F})$,

$$\limsup_{n \rightarrow \infty} \|(z - \lambda_j)^{\tau_j} - Q_{n,\lambda_j}\|^{1/n} \leq \frac{|\Phi(\lambda_j)|}{\rho_{m_{\alpha(\lambda_j)}(F_{\alpha(\lambda_j)})}}, \quad j = 1, \dots, q, \quad (62)$$

where τ_j is the order of λ_j as a pole of $\mathbf{F}$ in $\mathbf{D}_{\mathbf{m}}(\mathbf{F})$.

Therefore,

$$\begin{aligned} (Q_{\mathbf{m}}^{\mathbf{F}} - Q_{n,\mathbf{m}})(z) &= Q_{\mathbf{m}}^{\mathbf{F}}(z) - \frac{(Q_{\mathbf{m}}^{\mathbf{F}} Q_{n,\lambda_1})(z)}{(z - \lambda_1)^{\tau_1}} + \frac{(Q_{\mathbf{m}}^{\mathbf{F}} Q_{n,\lambda_1})(z)}{(z - \lambda_1)^{\tau_1}} - \dots \\ &\quad + \frac{(Q_{\mathbf{m}}^{\mathbf{F}} Q_{n,\lambda_1} \dots Q_{n,\lambda_{q-1}})(z)}{(z - \lambda_1)^{\tau_1} \dots (z - \lambda_{q-1})^{\tau_{q-1}}} - Q_{n,\mathbf{m}}(z). \end{aligned}$$

Hence,

$$\left| Q_{\mathbf{m}}^{\mathbf{F}} - Q_{n,\mathbf{m}} \right| (z) \leq \sum_{j=1}^q \left| \frac{(Q_{\mathbf{m}}^{\mathbf{F}} Q_{n,\lambda_1} \dots Q_{n,\lambda_{j-1}})(z)}{(z - \lambda_1)^{\tau_1} \dots (z - \lambda_{j-1})^{\tau_{j-1}}} [(z - \lambda_j)^{\tau_j} - Q_{n,\lambda_j}(z)] \right|.$$

Since

$$\lim_{n \rightarrow \infty} \frac{(Q_{\mathbf{m}}^{\mathbf{F}} Q_{n,\lambda_1} \dots Q_{n,\lambda_{j-1}})(z)}{(z - \lambda_1)^{\tau_1} \dots (z - \lambda_{j-1})^{\tau_{j-1}}} = \frac{Q_{\mathbf{m}}^{\mathbf{F}}(z)}{(z - \lambda_j)^{\tau_j}}, \quad j = 1, 2, \dots, q,$$

uniformly on compact subsets of $\mathbb{C}$ and (62), we obtain the inequality (7).

Finally, let $\alpha \in \{1, 2, \dots, d\}$ and let K be a compact subset of $D_{\rho_{m_{\alpha}}(F_{\alpha})} \setminus \mathcal{P}_{n,\alpha}(\mathbf{F})$. Using (60), if $\varepsilon > 0$ is sufficiently small, then $K(\varepsilon) = K$. By (19) applied to a sequence of incomplete Padé-Faber approximants $R_{n,\mathbf{m},\alpha}$, we obtain (6). This completes the proof. $\square$

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DIRECT AND INVERSE RESULTS ON ROW SEQUENCES OF GENERALIZED PADÉ APPROXIMANTS TO POLYNOMIAL EXPANSIONS

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Dedicated to Professor Guillermo López Lagomasino on the occasion of his 70th birthday

Abstract. Starting from the orthogonal and Faber polynomial expansions of a function F , we study the asymptotic behaviors of two generalized Padé approximations (orthogonal Padé approximation and Padé–Faber approximation). We obtain both direct and inverse results relating the convergence of the poles of these approximants and the singularities of F . Thereby, we obtain analogues of theorems by A. A. Gonchar and S. P. Suetin.

1. Introduction

Padé approximation theory contains two types of results, namely direct and inverse types. In the direct type results, we draw conclusions about the asymptotic behavior of the approximants and their poles from the analytic properties and location of singularities of the approximated function. In the inverse type results, the information is given in terms of the asymptotic behavior of the approximating functions from which the analyticity and location of the singularities of the approximated function can be deduced.

Let us recall the definition of classical Padé approximants and state some known direct and inverse type results on row sequences of classical Padé

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approximants related to our study in this paper. In the whole paper, $\mathbb{N}$ stands for the set of all positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

DEFINITION 1.1. Let $f(z) = \sum_{k=0}^{\infty} f_k z^k$ be a formal power series. Fix $(n, m) \in \mathbb{N}_0 \times \mathbb{N}_0$. Then, there exist polynomials P and Q such that

$$\deg(P) \leq n, \quad \deg(Q) \leq m, \quad Q \neq 0,$$

and

$$(Qf - P)(z) = \mathcal{O}(z^{n+m+1}), \quad \text{as } z \rightarrow 0.$$

The rational function $R_{n,m} := P/Q$ is called the (n, m) classical Padé approximant of f .

It is well-known that for any $(n, m) \in \mathbb{N}_0 \times \mathbb{N}_0$, $R_{n,m}$ always exists and is unique. For a given pair $(n, m) \in \mathbb{N}_0 \times \mathbb{N}_0$, we write

$$(1) \quad R_{n,m} = \frac{P_{n,m}}{Q_{n,m}},$$

where $Q_{n,m}$ is the monic polynomial that has no common zero with $P_{n,m}$.

Let $f(z) = \sum_{k=0}^{\infty} f_k z^k$ be a formal power series. Denote by $R_0(f)$ the radius of the largest disk centered at the origin to which f can be extended analytically and by $R_m(f)$ the radius of the largest disk centered at the origin to which f can be extended so that f has at most m poles counting multiplicities. Basically, $R_0(f)$ is the radius of convergence of f and $R_m(f)$ is the radius of m -meromorphy of f . Set

$$\mathbb{B}(a, R) := \{z \in \mathbb{C} : |z - a| < R\}.$$

Let us define two indicators of the asymptotic behavior of the zeros of $Q_{n,m}$. Fix $m \in \mathbb{N}$. Let

$$\mathcal{P}_{n,m} := \{\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,m_n}\}, \quad m_n \leq m, \quad n \in \mathbb{N}_0,$$

denote the collection of zeros of $Q_{n,m}$ (repeated according to their multiplicity). Define

$$|z - w|_1 := \min\{1, |z - w|\}, \quad z, w \in \mathbb{C}.$$

Fix $\lambda \in \mathbb{C}$. The first indicator is defined by

$$\Delta(\lambda) := \limsup_{n \rightarrow \infty} \prod_{j=1}^{m_n} |\lambda_{n,j} - \lambda|_1^{1/n} = \limsup_{n \rightarrow \infty} \prod_{|\lambda_{n,j} - \lambda| < 1} |\lambda_{n,j} - \lambda|^{1/n}.$$

Clearly, $0 \leq \Delta(\lambda) \leq 1$ (when $m_n = 0$ or $|\lambda_{n,j} - \lambda| \geq 1$ for all $j = 1, 2, \dots, m_n$, the product is taken to be 1). The second indicator, a nonnegative integer $\sigma(\lambda)$, is defined as follows. We suppose that for each n , the points in

$$(2) \quad \mathcal{P}_{n,m} = \{\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,m_n}\}$$

are enumerated in nondecreasing distance to the point λ . We set

$$(3) \quad \delta_j(\lambda) := \limsup_{n \rightarrow \infty} |\lambda_{n,j} - \lambda|_1^{1/n}.$$

These numbers are defined by (3) for $j = 1, 2, \dots, m'$, $m' = \liminf_{n \rightarrow \infty} m_n$; for $j = m' + 1, \dots, n$, we define $\delta_j(\lambda) = 1$. We have $0 \leq \delta_j(\lambda) \leq 1$. If $\Delta(\lambda) = 1$ (in that case all $\delta_j(\lambda) = 1$), then $\sigma(\lambda) = 0$. If $\Delta(\lambda) < 1$, then for some ν , $1 \leq \nu \leq m$, we have that $\delta_1(\lambda) \leq \dots \leq \delta_\nu(\lambda) < 1$ and $\delta_{\nu+1}(\lambda) = 1$ or $\nu = m$; in this case we take $\sigma(\lambda) = \nu$.

The first theorem proved by Gonchar [7, Theorem 1] serves as both direct and inverse type results.

GONCHAR'S THEOREM. *Let $f(z) = \sum_{k=0}^{\infty} f_k z^k$ be a formal power series, $m \in \mathbb{N}$, and let $\lambda \neq 0$ be a given point in $\mathbb{C}$. The following statements are equivalent:*

- (a) $\lambda \in \mathbb{B}(0, R_m(f))$ and f has a pole at λ .
- (b) $\Delta(\lambda) < 1$ (or equivalently $\sigma(\lambda) \geq 1$).

If either (a) or (b) holds, then

$$\Delta(\lambda) = \frac{|\lambda|}{R_m(f)} \quad \text{and} \quad \sigma(\lambda) = \nu,$$

where ν is the order of the pole at λ .

The direct part of this theorem refers to the statement: if f has a pole at $\lambda \in \mathbb{B}(0, R_m(f))$ of order ν , then

$$\Delta(\lambda) \leq \frac{|\lambda|}{R_m(f)} \quad \text{and} \quad \sigma(\lambda) \geq \nu.$$

On the other hand, the inverse result in this theorem is the statement: if $\Delta(\lambda) < 1$, then f has a pole at $\lambda \in \mathbb{B}(0, R_m(f))$,

$$R_m(f) \geq \frac{|\lambda|}{\Delta(\lambda)}, \quad \text{and} \quad \nu \geq \sigma(\lambda),$$

where ν is the order of the pole at λ .

In the same paper, Gonchar conjectured the following:

GONCHAR'S CONJECTURE. Fix $m \in \mathbb{N}$. Let $f(z) = \sum_{k=0}^{\infty} f_k z^k$ be a formal power series and let $Q_{n,m}$ be the denominator of the (n, m) classical Padé approximant of f . Assume that

$$\lim_{n \rightarrow \infty} \lambda_n = \lambda \neq 0,$$

where λ_n is a zero of $Q_{n,m}$. Then this series defines a function which is holomorphic at $z = 0$, $R_{m-1}(f) \geq |\lambda|$, and λ is a singularity of f .

Note that Gonchar's conjecture has not yet been proved. Some progress was made by Vasilov, López, Prokhorov, and Suetin (see [12–14]). In the final form, the following weaker version of Gonchar's conjecture was proved by Suetin [11].

SUETIN'S THEOREM. Assume that the formal power series

$$f(z) = \sum_{k=0}^{\infty} f_k z^k$$

has coefficients such that for fixed $m \in \mathbb{N}$ and sufficiently large $n \in \mathbb{N}$ the approximants $R_{n,m}$ have precisely m finite poles $\lambda_{n,1}, \dots, \lambda_{n,m}$, which are convergent:

$$\lim_{n \rightarrow \infty} \lambda_{n,j} = \lambda_j \neq 0, \quad j = 1, \dots, m.$$

Then

- (i) the power series defines a holomorphic function f in the disk $\mathbb{B}(0, R_{\min})$, where $R_{\min} := \min_{1 \leq j \leq m} |\lambda_j|$;
- (ii) $R_{m-1}(f) = \max_{1 \leq j \leq m} |\lambda_j|$;
- (iii) all the points $\lambda_1, \dots, \lambda_m$ are singularities of f , the ones lying in the disk $\mathbb{B}(0, R_{m-1}(f))$ are poles, and f has no other poles in this disk.

When $m = 1$, Gonchar's conjecture and Suetin's theorem reduce to the classical Fabry ratio theorem. However, since Suetin's theorem has the assumptions that for sufficiently large $n \in \mathbb{N}$, $R_{n,m}$ have precisely m finite poles and all m poles of $R_{n,m}$ are convergent, Suetin's theorem is much weaker than Gonchar's conjecture.

The aim of this paper is to prove the analogues of a direct part of Gonchar's theorem and Suetin's theorem for two generalized Padé approximations to polynomial expansions introduced in [3] and [4]. The proofs of analogues of Suetin's theorem for these two generalized Padé approximations are heavily relying on an extension of Poincaré's theorem on recurrence relations developed by Buslaev in [5].

The first approximation is called orthogonal Padé approximation defined as follows. Let E be an infinite compact subset of the complex plane $\mathbb{C}$

such that $\overline{\mathbb{C}} \setminus E$ is simply connected. Denote by $\mathcal{K}$ the collection of these compact sets. Let μ be a finite positive Borel measure with infinite support $\text{supp}(\mu)$ contained in E . We write $\mu \in \mathcal{M}(E)$ and define the associated inner product,

$$\langle g, h \rangle_\mu := \int g(\zeta) \overline{h(\zeta)} d\mu(\zeta), \quad g, h \in L_2(\mu).$$

Let

$$p_n(z) := \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, 2, \dots,$$

be the orthonormal polynomial of degree n with respect to μ with positive leading coefficient; that is, $\langle p_n, p_m \rangle_\mu = \delta_{n,m}$. Denote by $\mathcal{H}(E)$ the space of all functions holomorphic in some neighborhood of E .

DEFINITION 1.2. Let $E \in \mathcal{K}$, $F \in \mathcal{H}(E)$, and $\mu \in \mathcal{M}(E)$. Fix $(n, m) \in \mathbb{N} \times \mathbb{N}$. Then, there exists a polynomial $Q_{n,m}^\mu$ such that $\deg(Q_{n,m}^\mu) \leq m$, $Q_{n,m}^\mu \not\equiv 0$, and

$$(4) \quad \langle z^k Q_{n,m}^\mu F, p_n \rangle_\mu = 0, \quad k = 0, 1, \dots, m-1.$$

Define the corresponding polynomial

$$P_{n,m}^\mu(z) := \sum_{j=0}^{n-1} \langle Q_{n,m}^\mu F, p_j \rangle_\mu p_j(z).$$

The rational function

$$R_{n,m}^\mu := \frac{P_{n,m}^\mu}{Q_{n,m}^\mu}$$

is called an (n, m) orthogonal Padé approximant of F with respect to μ .

Finding $Q_{n,m}^\mu$ in (4) is equivalent to solving a homogeneous system of m linear equations on $m+1$ unknowns. Therefore, for any pair $(n, m) \in \mathbb{N} \times \mathbb{N}$, a polynomial $Q_{n,m}^\mu$ always exists but it may not be unique. It is not difficult to check, however, that the condition

$$(5) \quad \Lambda_{n,m}^\mu(F) := \begin{vmatrix} \langle F, p_n \rangle_\mu & \langle zF, p_n \rangle_\mu & \cdots & \langle z^{m-1}F, p_n \rangle_\mu \\ \langle zF, p_n \rangle_\mu & \langle z^2F, p_n \rangle_\mu & \cdots & \langle z^mF, p_n \rangle_\mu \\ \vdots & \vdots & \ddots & \vdots \\ \langle z^{m-1}F, p_n \rangle_\mu & \langle z^mF, p_n \rangle_\mu & \cdots & \langle z^{2m-2}F, p_n \rangle_\mu \end{vmatrix} \neq 0$$

and the condition that every solution of (4) has $\deg Q_{n,m}^\mu = m$ are equivalent. In turn, they imply the uniqueness of $R_{n,m}^\mu$. Since $Q_{n,m}^\mu \not\equiv 0$, we normalize

it to be a “monic” polynomial. We call $Q_{n,m}^\mu$ a *denominator of an (n, m) orthogonal Padé approximant of F with respect to μ* .

The second approximation is related to Faber polynomials defined as follows. Let $E \in \mathcal{K}$ and Φ be the exterior conformal mapping from $\overline{\mathbb{C}} \setminus E$ onto $\{w \in \overline{\mathbb{C}} : |w| > 1\}$ satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. For each $\rho > 1$, we introduce

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\} \quad \text{and} \quad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\},$$

as the *level curve of index ρ* and the *canonical domain of index ρ* , respectively. Let $F \in \mathcal{H}(E)$. Denote by $\rho_0(F)$ the index ρ of the largest canonical domain D_ρ to which F can be extended as a holomorphic function and by $\rho_m(F)$ the index of the largest canonical domain D_ρ to which F can be extended so that F has at most m poles counting multiplicities.

The *Faber polynomial* of $E \in \mathcal{K}$ of degree n is defined by the formula

$$(6) \quad \Phi_n(z) := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\Phi^n(t)}{t-z} dt, \quad z \in D_\rho, \quad n = 0, 1, 2, \dots$$

It equals the polynomial part of the Laurent expansion of Φ^n at infinity. The n -th *Faber coefficient* of $F \in \mathcal{H}(E)$ with respect to Φ_n is given by

$$[F]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{F(t)\Phi'(t)}{\Phi^{n+1}(t)} dt,$$

where $\rho \in (1, \rho_0(F))$.

DEFINITION 1.3. Let $E \in \mathcal{K}$ and $F \in \mathcal{H}(E)$. Fix $(n, m) \in \mathbb{N} \times \mathbb{N}$. Then, there exists a polynomial $Q_{n,m}^E$ such that $\deg(Q_{n,m}^E) \leq m$, $Q_{n,m}^E \neq 0$, and

$$(7) \quad [z^k Q_{n,m}^E F]_n = 0, \quad k = 0, 1, \dots, m-1.$$

Define the corresponding polynomial

$$P_{n,m}^E(z) := \sum_{j=0}^{n-1} [Q_{n,m}^E F]_j p_j(z).$$

The rational function

$$R_{n,m}^E := \frac{P_{n,m}^E}{Q_{n,m}^E}$$

is called an (n, m) *Padé–Faber approximant of F with respect to E* .

Similarly, for any pair $(n, m) \in \mathbb{N} \times \mathbb{N}$, a polynomial $Q_{n,m}^E$ always exists but it may not be unique. Moreover, the condition

$$(8) \quad \Lambda_{n,m}^E(F) := \begin{vmatrix} [F]_n & [zF]_n & \cdots & [z^{m-1}F]_n \\ [zF]_n & [z^2F]_n & \cdots & [z^mF]_n \\ \vdots & \vdots & \ddots & \vdots \\ [z^{m-1}F]_n & [z^mF]_n & \cdots & [z^{2m-2}F]_n \end{vmatrix} \neq 0$$

and the condition that every solution of (7) has $\deg Q_{n,m}^E = m$ are equivalent. In turn, they imply the uniqueness of $R_{n,m}^E$. Since $Q_{n,m}^E \neq 0$, we normalize it to be a “monic” polynomial. We call $Q_{n,m}^E$ a *denominator of an (n, m) Padé–Faber approximant of F with respect to E* .

We would like to emphasize that the approximations in Definitions 1.2 and 1.3 are different from Padé-orthogonal approximation in [1,2] and Padé–Faber approximation in [10] which are natural ways of extending the notion of classical Padé approximation. Moreover, the approximations in Definitions 1.2 and 1.3 were first introduced and studied in [3] and [4], respectively. Before stating the main results in [3] and [4], we need to define some classes of measures which are subsets of $\mathcal{M}(E)$. We say that $\mu \in \mathbf{Reg}_1(E)$ when

$$(9) \quad \lim_{n \rightarrow \infty} |p_n(z)|^{1/n} = |\Phi(z)|,$$

uniformly on compact subsets of $\mathbb{C} \setminus E$. Let us introduce the second kind function

$$s_n(z) := \int \frac{\overline{p_n(\zeta)}}{z - \zeta} d\mu(\zeta), \quad z \in \overline{\mathbb{C}} \setminus \text{supp}(\mu).$$

The measure $\mu \in \mathbf{Reg}_2(E)$ if and only if

$$(10) \quad \lim_{n \rightarrow \infty} |s_n(z)|^{1/n} = |\Phi(z)|^{-1},$$

uniformly on compact subsets of $\mathbb{C} \setminus E$. The classes $\mathbf{Reg}_1(E)$ and $\mathbf{Reg}_2(E)$ are more or less the same in some cases (see the details in [3, Section 1]). In particular, if E is convex, then $\mathbf{Reg}_1(E) = \mathbf{Reg}_2(E)$ and these two classes coincide with the regular class in the usual sense (see [8, Definition 3.1.2] for the definition of the regular class in the usual sense). Define

$$\mathbf{Reg}_{1,2}(E) := \mathbf{Reg}_1(E) \cap \mathbf{Reg}_2(E).$$

We say that $\mu \in \mathbf{Reg}_{1,2}^m(E)$ if it is in $\mathbf{Reg}_{1,2}(E)$ and there exists a positive constant c such that

$$\frac{\kappa_{n-m}}{\kappa_n} \geq c, \quad n \geq n_0.$$

Combining [3] and [4], the authors proved the following result.

THEOREM A 1. *Let $F \in \mathcal{H}(E)$, $m \in \mathbb{N}$ be fixed, and $\mu \in \mathbf{Reg}_{1,2}^m(E)$. Denote by $\mathcal{P}(F, m)$ the set of all poles of F in $D_{\rho_m(F)}$ and Q_m^F the monic polynomial whose zeros are these poles counting multiplicities. Then, the following assertions are equivalent:*

- (a) *F has exactly m poles in $D_{\rho_m(F)}$.*
- (b) *The polynomials $Q_{n,m}^\mu$ for F are uniquely determined for all sufficiently large n , and there exists a polynomial $\tilde{Q}_m$ of degree m such that*

$$\limsup_{n \rightarrow \infty} \|Q_{n,m}^\mu - \tilde{Q}_m\|^{1/n} = \tilde{\theta} < 1.$$

- (c) *The polynomials $Q_{n,m}^E$ for F are uniquely determined for all sufficiently large n , and there exists a polynomial $\hat{Q}_m$ of degree m such that*

$$\limsup_{n \rightarrow \infty} \|Q_{n,m}^E - \hat{Q}_m\|^{1/n} = \hat{\theta} < 1.$$

The norm $\|\cdot\|$ in (b) and (c) denotes (for example) the norm induced in the space of polynomials of degree at most m by the maximum of the absolute value of the coefficients. Moreover, if one of the assertions (a), (b), or (c) takes place, then $\tilde{Q}_m = \hat{Q}_m = Q_m^F$,

$$\tilde{\theta} = \hat{\theta} = \max \left\{ \frac{|\Phi(\xi)|}{\rho_m(F)} : \xi \in \mathcal{P}(F, m) \right\},$$

and for any compact subset K of $D_{\rho_m(F)} \setminus \mathcal{P}(F, m)$,

$$\limsup_{n \rightarrow \infty} \|R_{n,m}^\mu - F\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_m(F)} \quad \text{and} \quad \limsup_{n \rightarrow \infty} \|R_{n,m}^E - F\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_m(F)},$$

where $\|\cdot\|_K$ denotes the sup-norm on K and if $K \subset E$, then $\|\Phi\|_K$ is replaced by 1.

In the current paper, we continue studying the relation between the convergences of zeros of $Q_{n,m}^\mu$ and $Q_{n,m}^E$ (when m is fixed and $n \rightarrow \infty$) and the singularities of $F \in \mathcal{H}(E)$.

An outline of this paper is as follows. The main results in this paper are stated in Section 2. The proofs of the main results are in Section 3.

2. Main results

2.1. Direct type results. Theorems 2.1 and 2.3 stated below are the analogues of the direct part of Gonchar's theorem for orthogonal Padé ap-

proximants and Padé–Faber approximants, respectively. Note that the notations Δ , σ , and δ_j in Theorem 2.1 and Corollary 2.2 are defined as Section 1 taking

$$\mathcal{P}_{n,m} := \{\tilde{\xi}_{n,1}, \tilde{\xi}_{n,2}, \dots, \tilde{\xi}_{n,m_n}\}, \quad m_n \leq m, \quad n \in \mathbb{N},$$

to be the collection of zeros of $Q_{n,m}^\mu$.

THEOREM 2.1. *Let $E \in \mathcal{K}$, $F \in \mathcal{H}(E)$, and $\mu \in \mathbf{Reg}_2(E)$. Fix $m \in \mathbb{N}$. If F has a pole of order ν at ξ in $D_{\rho_m(F)}$, then*

$$\Delta(\xi) \leq \frac{|\Phi(\xi)|}{\rho_m(F)} \quad \text{and} \quad \sigma(\xi) \geq \nu.$$

The following corollary concerns the numbers $\delta_j(\xi)$, $j = 1, \dots, \nu$, where ξ is a pole of F in $D_{\rho_m(F)}$ and ν is an order of ξ .

COROLLARY 2.2. *Let $E \in \mathcal{K}$, $F \in \mathcal{H}(E)$, $\mu \in \mathbf{Reg}_2(E)$, and ξ be a pole of F in $D_{\rho_m(F)}$ of order ν . Assume that $\liminf_{n \rightarrow \infty} |\xi - \tilde{\xi}_{n,\nu+1}| > 0$, where $\{\tilde{\xi}_{n,1}, \tilde{\xi}_{n,2}, \dots, \tilde{\xi}_{n,m_n}\}$ are enumerated in nondecreasing distance to the point ξ . Then,*

$$(11) \quad \delta_1(\xi) \leq \delta_2(\xi) \leq \dots \leq \delta_\nu(\xi) \leq \left(\frac{|\Phi(\xi)|}{\rho_m(F)} \right)^{1/\nu}.$$

In particular, $\delta_1(\xi) = \delta_2(\xi) = \dots = \delta_\nu(\xi) = (|\Phi(\xi)|/\rho_m(F))^{1/\nu}$ if and only if $\Delta(\xi) = |\Phi(\xi)|/\rho_m(F)$.

For Theorem 2.3 and Corollary 2.4, the notations Δ , σ , and δ_j are defined as Section 1 taking

$$\mathcal{P}_{n,m} := \{\hat{\xi}_{n,1}, \hat{\xi}_{n,2}, \dots, \hat{\xi}_{n,m_n}\}, \quad m_n \leq m, \quad n \in \mathbb{N},$$

to be the collection of zeros of $Q_{n,m}^E$. We also obtain similar results to Theorem 2.1 and Corollary 2.2 for Padé–Faber approximants.

THEOREM 2.3. *Let $E \in \mathcal{K}$ and $F \in \mathcal{H}(E)$. Fix $m \in \mathbb{N}$. If F has a pole of order ν at ξ in $D_{\rho_m(F)}$, then*

$$\Delta(\xi) \leq \frac{|\Phi(\xi)|}{\rho_m(F)} \quad \text{and} \quad \sigma(\xi) \geq \nu.$$

COROLLARY 2.4. *Let $E \in \mathcal{K}$, $F \in \mathcal{H}(E)$, and ξ be a pole of F in $D_{\rho_m(F)}$ of order ν . Assume that*

$$\liminf_{n \rightarrow \infty} |\xi - \hat{\xi}_{n,\nu+1}| > 0,$$

where $\{\hat{\xi}_{n,1}, \hat{\xi}_{n,2}, \dots, \hat{\xi}_{n,m_n}\}$ are enumerated in nondecreasing distance to the point ξ . Then,

$$\delta_1(\xi) \leq \delta_2(\xi) \leq \dots \leq \delta_\nu(\xi) \leq \left(\frac{|\Phi(\xi)|}{\rho_m(F)} \right)^{1/\nu}.$$

In particular, $\delta_1(\xi) = \delta_2(\xi) = \dots = \delta_\nu(\xi) = (|\Phi(\xi)|/\rho_m(F))^{1/\nu}$ if and only if $\Delta(\xi) = |\Phi(\xi)|/\rho_m(F)$.

2.2. Inverse type results. We obtain inverse type results (Theorems 2.5 and 2.6) analogous to Suetin's theorem for orthogonal Padé approximants and Padé–Faber approximants, respectively. In the results for orthogonal Padé approximants, we need to restrict ourselves to a smaller collection of compact sets E and a smaller collection of measures μ defined as follows. Denote by $\mathcal{K}_1$ the collection of all sets $E \in \mathcal{K}$ that E is such that the inverse function $\Psi = \Phi^{-1}$ can be extended continuously to $\overline{\mathbb{C}} \setminus \{w \in \mathbb{C} : |w| < 1\}$. We say that Szegő or strong asymptotics takes place, and write $\mu \in \mathcal{S}(E)$, if

$$(12) \quad \lim_{n \rightarrow \infty} \frac{p_n(z)}{c_n \Phi^n(z)} = S(z) \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} = 1,$$

where the first limit in (12) is assumed to hold uniformly on compact subsets of $\overline{\mathbb{C}} \setminus E$, the c_n 's are positive constants, and $S(z)$ is some holomorphic and non-vanishing function on $\overline{\mathbb{C}} \setminus E$. It is not difficult to check that (12) $\Rightarrow$ (9).

THEOREM 2.5. *Let $E \in \mathcal{K}_1$, $F \in \mathcal{H}(E)$, and $\mu \in \mathcal{S}(E)$. Fix $m \in \mathbb{N}$. If for all n sufficiently large, $Q_{n,m}^\mu$ (for F) has precisely m zeros and the zeros of $Q_{n,m}^\mu$ have limits $\xi_1, \dots, \xi_m$, as $n \rightarrow \infty$, then*

- (i) F is holomorphic in $D_{\rho_{\min}}$, where $\rho_{\min} := \min_{1 \leq j \leq m} |\Phi(\xi_j)|$;
- (ii) $\rho_{m-1}(F) = \max_{1 \leq j \leq m} |\Phi(\xi_j)|$;
- (iii) $\xi_1, \dots, \xi_m$ are singularities of F ; those lying in $D_{\rho_{m-1}(F)}$ are poles (counting multiplicities), and F has no other poles in $D_{\rho_{m-1}(F)}$.

THEOREM 2.6. *Let $E \in \mathcal{K}$ and $F \in \mathcal{H}(E)$. Fix $m \in \mathbb{N}$. If for all n sufficiently large, $Q_{n,m}^E$ (for F) has precisely m zeros and the zeros of $Q_{n,m}^E$ have limits $\xi_1, \dots, \xi_m$, as $n \rightarrow \infty$, then*

- (i) F is holomorphic in $D_{\rho_{\min}}$, where $\rho_{\min} := \min_{1 \leq j \leq m} |\Phi(\xi_j)|$;
- (ii) $\rho_{m-1}(F) = \max_{1 \leq j \leq m} |\Phi(\xi_j)|$;
- (iii) $\xi_1, \dots, \xi_m$ are singularities of F ; those lying in $D_{\rho_{m-1}(F)}$ are poles (counting multiplicities), and F has no other poles in $D_{\rho_{m-1}(F)}$.

Applying Suetin's theorem, Theorems 2.5 and 2.6 are direct consequences of Theorems 2.7 and 2.8, respectively.

THEOREM 2.7. *Let $E \in \mathcal{K}_1$, $F \in \mathcal{H}(E)$ and $\mu \in \mathcal{S}(E)$. Define $f(w) := F(\Psi(w))$. Let $R > 1$ be such that $f \in \mathcal{H}(\{w \in \mathbb{C} : 1 < |w| < R\})$. Denote by*

$\sum_{k=-\infty}^{\infty} f_k w^k$ the Laurent series of f in the annulus $\{w \in \mathbb{C} : 1 < |w| < R\}$ and by $\hat{f}(w) := \sum_{k=0}^{\infty} f_k w^k$ the regular part of f . For each fixed $m \geq 1$, the following conditions are equivalent:

- (a) The zeros of $Q_{n,m}$ for $\hat{f}$ have finite limits $\lambda_1, \dots, \lambda_m$, as $n \rightarrow \infty$.
- (b) The zeros of $Q_{n,m}^\mu$ for F have finite limits $\xi_1, \dots, \xi_m$, as $n \rightarrow \infty$.

Under appropriate enumeration of the sub-indices, the values λ_j and ξ_j , $j = 1, \dots, m$, are related by the formula $\Phi(\xi_j) = \lambda_j$ for all $j = 1, \dots, m$.

THEOREM 2.8. Let $E \in \mathcal{K}$ and $F \in \mathcal{H}(E)$. Define $f(w) := F(\Psi(w))$. Let $R > 1$ be such that $f \in \mathcal{H}(\{w \in \mathbb{C} : 1 < |w| < R\})$. Denote by $\sum_{k=-\infty}^{\infty} f_k w^k$ the Laurent series of f in the annulus $\{w \in \mathbb{C} : 1 < |w| < R\}$ and by $\hat{f}(w) := \sum_{k=0}^{\infty} f_k w^k$ the regular part of f . For each fixed $m \geq 1$, the following conditions are equivalent:

- (a) The zeros of $Q_{n,m}$ for $\hat{f}$ have finite limits $\lambda_1, \dots, \lambda_m$, as $n \rightarrow \infty$.
- (b) The zeros of $Q_{n,m}^E$ for F have finite limits $\xi_1, \dots, \xi_m$, as $n \rightarrow \infty$.

Under appropriate enumeration of the sub-indices, the values λ_j and ξ_j , $j = 1, \dots, m$, are related by the formula $\Phi(\xi_j) = \lambda_j$ for all $j = 1, \dots, m$.

REMARK 2.9. Under the assumptions on Theorem 2.7, the condition (b) in Theorem 2.7 is equivalent to the condition (b) in Theorem 2.8.

3. Proofs of main results

3.1. Proofs of direct type results. We begin this section by defining some more notation and stating some needed lemmas.

3.1.1. Auxiliary Lemmas. For a given compact set $E \in \mathcal{K}$, the n -th Fourier coefficient of $F \in \mathcal{H}(E)$ with respect to p_n is given by

$$\langle F \rangle_n := \langle F, p_n \rangle_\mu = \int F(z) \overline{p_n(z)} d\mu(z).$$

Using Cauchy's integral formula and Fubini's theorem, it is easy to check the following relation used frequently in this paper.

LEMMA 3.1. Let $E \in \mathcal{K}$, $F \in \mathcal{H}(E)$, $n \in \mathbb{N}_0$, and $\rho \in (1, \rho_0(F))$. Then,

$$(13) \quad \langle F \rangle_n = \frac{1}{2\pi i} \int_{\Gamma_\rho} F(w) s_n(w) dw.$$

The next lemma (see [6, p. 583] or [9, p. 43] for its proof) gives an estimate of Faber polynomials Φ_n (defined in (6)) on a level curve.

LEMMA 3.2. Let $E \in \mathcal{K}$ and $\rho > 1$ be fixed. Then, there exists $c > 0$ such that

$$(14) \quad \|\Phi_n\|_{\Gamma_\rho} \leq c\rho^n, \quad n \in \mathbb{N}_0.$$

3.1.2. Proofs of the direct type results.

PROOF OF THEOREM 2.1. For each $n \in \mathbb{N}$, let $q_{n,m}^\mu$ be the polynomial $Q_{n,m}^\mu$ normalized so that

$$(15) \quad \sum_{k=0}^m |\alpha_{n,k}| = 1, \quad q_{n,m}^\mu(z) = \sum_{k=0}^m \alpha_{n,k} z^k.$$

This normalization implies that the polynomials $q_{n,m}^\mu$ are uniformly bounded on each compact subset of $\mathbb{C}$.

First of all, we wish to show that for each pole ξ of order ν of F in $D_{\rho_m(F)}$,

$$(16) \quad \limsup_{n \rightarrow \infty} |(q_{n,m}^\mu)^{(j)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_m(F)}, \quad j = 0, 1, \dots, \nu - 1.$$

Denote by Q_m the monic polynomial whose zeros are all the poles of F in $D_{\rho_m(F)}$. Define

$$G_\ell(z) := \frac{Q_m(z)F(z)}{(z - \xi)^\ell}, \quad \ell = 1, 2, \dots, \nu.$$

Now, we consider a function G_1 . Notice that G_1 is holomorphic on a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a simple pole at $z = \xi$. Define

$$H_1(z) := (z - \xi)G_1(z) \quad \text{and} \quad a_{n,n}^{(1)} := \langle q_{n,m}^\mu G_1 \rangle_n.$$

By the definition of $q_{n,m}^\mu$, since $\deg(Q_m/(z - \xi)) < m$, it is easy to check that $a_{n,n}^{(1)} = 0$. Moreover, using (13), we have

$$a_{n,n}^{(1)} = \langle q_{n,m}^\mu G_1 \rangle_n = \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} q_{n,m}^\mu(z) G_1(z) s_n(z) dz,$$

where $1 < \rho_1 < |\Phi(\xi)|$. Define

$$\tau_{n,n}^{(1)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} q_{n,m}^\mu(z) G_1(z) s_n(z) dz,$$

where $|\Phi(\xi)| < \rho_2 < \rho_m(F)$. The function $q_{n,m}^\mu G_1 s_n$ is meromorphic on $\overline{D}_{\rho_2} \setminus D_{\rho_1} = \{z \in \mathbb{C} : \rho_1 \leq |\Phi(z)| \leq \rho_2\}$ and has a pole at ξ of order at most 1. Applying Cauchy's residue theorem to the function $q_{n,m}^\mu G_1 s_n$, we have

$$(17) \quad \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} q_{n,m}^\mu(t) G_1(t) s_n(t) dt$$

$$-\frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} q_{n,m}^{\mu}(t) G_1(t) s_n(t) dt = \text{res}(q_{n,m}^{\mu} G_1 s_n, \xi).$$

The limit formula for the residue of $q_{n,m}^{\mu} G_1 s_n$ at ξ is

$$\text{res}(q_{n,m}^{\mu} G_1 s_n, \xi) = \lim_{z \rightarrow \xi} (z - \xi) q_{n,m}^{\mu}(z) G_1(z) s_n(z) = H_1(\xi) q_{n,m}^{\mu}(\xi) s_n(\xi).$$

We can rewrite (17) as

$$\tau_{n,n}^{(1)} = \tau_{n,n}^{(1)} - a_{n,n}^{(1)} = H_1(\xi) q_{n,m}^{\mu}(\xi) s_n(\xi)$$

(recall that $a_{n,n}^{(1)} = 0$) which implies

$$(18) \quad q_{n,m}^{\mu}(\xi) = \frac{\tau_{n,n}^{(1)}}{H_1(\xi) s_n(\xi)}.$$

Choose $\delta > 0$ sufficiently small so that

$$(19) \quad \rho_2 := \rho_m(F) - \delta > |\Phi(\xi)| \quad \text{and} \quad |\Phi(\xi)| - \delta > 1.$$

Using (10), there exist $n_0 \in \mathbb{N}$ and $c_1 > 0$, $c_2 > 0$ such that

$$(20) \quad \frac{c_1}{(\rho + \delta)^n} \leq \|s_n\|_{\Gamma_{\rho}} \leq \frac{c_2}{(\rho - \delta)^n}, \quad n \geq n_0,$$

where c_1 and c_2 do not depend on n (from now on, $c_3, c_4, \dots$ denote constants that do not depend on n). From (20), we have

$$(21) \quad |\tau_{n,n}^{(1)}| = \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} q_{n,m}^{\mu}(z) G_1(z) s_n(z) dz \right| \leq \frac{c_3}{(\rho_2 - \delta)^n}$$

and

$$(22) \quad |s_n(\xi)| \geq \frac{c_1}{(|\Phi(\xi)| + \delta)^n}.$$

Combining (21) and (22), it follows from (18) that

$$(23) \quad |q_{n,m}^{\mu}(\xi)| \leq c_4 \left(\frac{|\Phi(\xi)| + \delta}{\rho_2 - \delta} \right)^n.$$

which means that

$$\limsup_{n \rightarrow \infty} |q_{n,m}^{\mu}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)| + \delta}{\rho_2 - \delta}.$$

Letting $\delta \rightarrow 0$, we obtain $\rho_2 \rightarrow \rho_m(F)$ and

$$\limsup_{n \rightarrow \infty} |q_{n,m}^\mu(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_m(F)}.$$

Now we employ induction. Suppose that

$$(24) \quad \limsup_{n \rightarrow \infty} |(q_{n,m}^\mu)^{(j)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_m(F)}, \quad j = 0, 1, \dots, \ell - 2,$$

with $\ell \leq \nu$, and let us prove that the formula (24) holds for $j = \ell - 1$. Consider a function G_ℓ that is holomorphic on a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole of order ℓ at $z = \xi$. Set

$$H_\ell(z) := (x - \xi)^\ell G_\ell(z) \quad \text{and} \quad a_{n,n}^{(\ell)} := \langle q_{n,m}^\mu G_\ell \rangle_n.$$

By the definition of $Q_{n,m}^\mu$, since $\deg(Q_m/(z - \xi)^\ell) < m$, it is easy to check that $a_{n,n}^{(\ell)} = 0$. Using (13), we have

$$a_{n,n}^{(\ell)} = \langle q_{n,m}^\mu G_\ell \rangle_n = \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} q_{n,m}^\mu(z) G_\ell(z) s_n(z) dz,$$

where $1 < \rho_1 < |\Phi(\xi)|$. Define

$$\tau_{n,n}^{(\ell)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} q_{n,m}^\mu(z) G_\ell(z) s_n(z) dz,$$

where $|\Phi(\xi)| < \rho_2 < \rho_m(F)$. The function $q_{n,m}^\mu G_\ell s_n$ is meromorphic on $\overline{D}_{\rho_2} \setminus D_{\rho_1} = \{z \in \mathbb{C} : \rho_1 \leq |\Phi(z)| \leq \rho_2\}$ and has a pole at ξ of order at most ℓ . Applying Cauchy's residue theorem to the function $q_{n,m}^\mu G_\ell s_n$, we have

$$(25) \quad \begin{aligned} \tau_{n,n}^{(\ell)} - a_{n,n}^{(\ell)} &= \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} q_{n,m}^\mu(t) G_\ell(t) s_n(t) dt \\ &\quad - \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} q_{n,m}^\mu(t) G_\ell(t) s_n(t) dt = \text{res}(q_{n,m}^\mu G_\ell s_n, \xi). \end{aligned}$$

The limit formula for the residue of $q_{n,m}^\mu G_\ell s_n$ at ξ is

$$(26) \quad \begin{aligned} \text{res}(q_{n,m}^\mu G_\ell s_n, \xi) &= \frac{1}{(\ell - 1)!} \lim_{z \rightarrow \xi} ((z - \xi)^\ell G_\ell(z) s_n(z) q_{n,m}^\mu(z))^{(\ell-1)} \\ &= \frac{1}{(\ell - 1)!} \sum_{t=0}^{\ell-1} \binom{\ell-1}{t} (H_\ell s_n)^{(\ell-1-t)}(\xi) (q_{n,m}^\mu)^{(t)}(\xi), \end{aligned}$$

where the last equality follows from Leibniz's formula. Since $a_{n,n}^{(\ell)} = 0$, the equation (25) becomes

$$(\ell-1)! \tau_{n,n}^{(\ell)} = \sum_{t=0}^{\ell-2} \binom{\ell-1}{t} (H_\ell s_n)^{(\ell-1-t)}(\xi) (q_{n,m}^\mu)^{(t)}(\xi) \\ + H_\ell(\xi) s_n(\xi) (q_{n,m}^\mu)^{(\ell-1)}(\xi),$$

which implies that

$$(27) \quad (q_{n,m}^\mu)^{(\ell-1)}(\xi) = \frac{(\ell-1)! \tau_{n,n}^{(\ell)}}{H_\ell(\xi) s_n(\xi)} - \sum_{t=0}^{\ell-2} \binom{\ell-1}{t} \frac{(H_\ell s_n)^{(\ell-1-t)}(\xi) (q_{n,m}^\mu)^{(t)}(\xi)}{H_\ell(\xi) s_n(\xi)}.$$

Choosing $\delta > 0$ and ρ_2 as in (19) and applying (20), we have

$$(28) \quad |\tau_{n,n}^{(\ell)}| = \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} q_{n,m}^\mu(z) G_\ell(z) s_n(z) dz \right| \leq \frac{c_5}{(\rho_2 - \delta)^n}$$

and for all $t = 0, 1, \dots, \ell-2$,

$$(29) \quad |(H_\ell s_n)^{(\ell-1-t)}(\xi)| = \left| \frac{(\ell-1-t)!}{2\pi i} \int_{|z-\xi|=\varepsilon} \frac{H_\ell(z) s_n(z)}{(z-\xi)^{\ell-t}} dz \right| \leq \frac{c_6}{(|\Phi(\xi)| - \delta)^n},$$

where $\{z \in \mathbb{C} : |z - \xi| = \varepsilon\} \subset \{z \in \mathbb{C} : |\Phi(z)| > |\Phi(\xi)| - \delta\}$. Combining (22), (24), (28), and (29), it follows from (27) that

$$(30) \quad \limsup_{n \rightarrow \infty} |(q_{n,m}^\mu)^{(\ell-1)}(\xi)|^{1/n} \\ = \limsup_{n \rightarrow \infty} \left| \frac{(\ell-1)! \tau_{n,n}^{(\ell)}}{H_\ell(\xi) s_n(\xi)} - \sum_{t=0}^{\ell-2} \binom{\ell-1}{t} \frac{(H_\ell s_n)^{(\ell-1-t)}(\xi) (q_{n,m}^\mu)^{(t)}(\xi)}{H_\ell(\xi) s_n(\xi)} \right|^{1/n} \\ \leq \max \left\{ \frac{|\Phi(\xi)| + \delta}{\rho_2 - \delta}, \left(\frac{|\Phi(\xi)| + \delta}{|\Phi(\xi)| - \delta} \right) \left(\frac{|\Phi(\xi)|}{\rho_m(F)} \right) \right\}.$$

Letting $\delta \rightarrow 0$, we have $\rho_2 \rightarrow \rho_m(F)$ and from (30), we obtain

$$\limsup_{n \rightarrow \infty} |(q_{n,m}^\mu)^{(\ell-1)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_m(F)}.$$

This completes the induction proof.

Next, we want to show that

$$(31) \quad \lim_{n \rightarrow \infty} \tilde{\xi}_{n,j} = \xi, \quad j = 1, 2, \dots, \nu,$$

i.e., there exist at least ν zeros of $Q_{n,m}^\mu$ converge to ξ . By the normalization of $q_{n,m}^\mu$ (see (15)), it suffices to show that for any subsequence of indices Ω such that

$$\lim_{n \in \Omega} q_{n,m}^\mu = q_\Omega,$$

q_Ω is a non-null polynomial with a zero of order at least ν at ξ . Due to the normalization of $q_{n,m}^\mu$, $q_\Omega \not\equiv 0$. Computing Taylor's expansion of $q_{n,m}^\mu$ around ξ , we obtain

$$q_{n,m}^\mu(z) = \sum_{k=0}^m \frac{(q_{n,m}^\mu)^{(k)}(\xi)}{k!} (z - \xi)^k.$$

Applying (16) and the Weierstrass approximation theorem for derivatives, we have

$$q_\Omega(z) = \lim_{n \in \Omega} q_{n,m}^\mu(z) = \lim_{n \in \Omega} \sum_{k=0}^m \frac{(q_{n,m}^\mu)^{(k)}(\xi)}{k!} (z - \xi)^k = \sum_{k=\nu}^m \frac{(q_\Omega)^{(k)}(\xi)}{k!} (z - \xi)^k,$$

which implies what we wanted.

Let $\varepsilon > 0$ be sufficiently small so that $\mathbb{B}(\xi, 2\varepsilon)$ contains no other poles of F in $D_{\rho_m(F)}$ except ξ . Let $\tilde{\xi}_{n,1}, \dots, \tilde{\xi}_{n,\sigma_n}$ be the zeros of $q_{n,m}^\mu$ contained in $\mathbb{B}(\xi, 2\varepsilon)$. By (31), we have $\nu \leq \sigma_n \leq m$ for all sufficiently large n . In the sequel, we only consider such values of n . Set

$$\tilde{Q}_n(z) := \prod_{j=1}^{\sigma_n} (z - \tilde{\xi}_{n,j}).$$

It is easy to check that the functions $\tilde{Q}_n/q_{n,m}^\mu$ are holomorphic in $\mathbb{B}(\xi, 2\varepsilon)$ and uniformly bounded on any compact subset of $\mathbb{B}(\xi, 2\varepsilon)$, in particular on $\overline{\mathbb{B}(\xi, \varepsilon)}$. Therefore, by Cauchy's integral formula, for any $j = 0, 1, \dots, \nu - 1$, the sequence $(\tilde{Q}_n/q_{n,m}^\mu)^{(j)}$ is uniformly bounded on $\overline{\mathbb{B}(\xi, \varepsilon)}$. Using Leibniz's formula and the inequalities in (16), we obtain

$$\begin{aligned} (32) \quad \limsup_{n \rightarrow \infty} |\tilde{Q}_n^{(j)}(\xi)|^{1/n} &= \limsup_{n \rightarrow \infty} \left| \left(q_{n,m}^\mu \frac{\tilde{Q}_n}{q_{n,m}^\mu} \right)^{(j)}(\xi) \right|^{1/n} \\ &= \limsup_{n \rightarrow \infty} \left| \sum_{k=0}^j \binom{j}{k} (q_{n,m}^\mu)^{(k)}(\xi) \left(\frac{\tilde{Q}_n}{q_{n,m}^\mu} \right)^{(j-k)}(\xi) \right|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_m(F)} < 1, \end{aligned}$$

for each $j = 0, \dots, \nu - 1$.

Finally, we want to show that

$$(33) \quad \Delta(\xi) \leq \frac{|\Phi(\xi)|}{\rho_m(F)} \quad \text{and} \quad \sigma(\xi) \geq \nu.$$

Using (32) for $j = 0$ and the ordering imposed on the indexing of zeros of $Q_{n,m}^\mu$, it follows that

$$\Delta(\xi) = \limsup_{n \rightarrow \infty} |Q_{n,m}^\mu(\xi)|_1^{1/n} = \limsup_{n \rightarrow \infty} |\tilde{Q}_n(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_m(F)} < 1$$

and $\limsup_{n \rightarrow \infty} |\xi - \tilde{\xi}_{n,1}|^{1/n} < 1$ so that $\sigma(\xi) \geq 1$. Assume that for each $j = 1, \dots, k$, where $k \leq \nu - 1$,

$$(34) \quad \limsup_{n \rightarrow \infty} |\xi - \tilde{\xi}_{n,j}|^{1/n} < 1,$$

and let us show that it is also true for $k + 1$. Consider $\tilde{Q}_n^{(k)}(\xi)$. Notice that one of the terms thus obtained is $\prod_{j=k+1}^{\sigma_n} (\xi - \tilde{\xi}_{n,j})$ and each one of the other terms contains at least one factor of the form $(\xi - \tilde{\xi}_{n,j})$ for some $j = 1, \dots, k$. Combining (32) for $j = k$ and (34), it follows that

$$\limsup_{n \rightarrow \infty} \left| \prod_{j=k+1}^{\sigma_n} (\xi - \tilde{\xi}_{n,j}) \right|^{1/n} < 1,$$

and due to the ordering of the indices, we get

$$\limsup_{n \rightarrow \infty} |\xi - \tilde{\xi}_{n,k+1}|^{1/n} < 1.$$

Therefore, $\sigma(\xi) \geq \nu$. $\square$

PROOF OF COROLLARY 2.2. Let us use the same notation defined in the proof of Theorem 2.1. By our assumption, we can assume that

$$\tilde{Q}_n(z) = \prod_{j=1}^{\nu} (z - \xi_{n,j}).$$

Recall that for each $j = 0, 1, \dots, \nu$,

$$\limsup_{n \rightarrow \infty} |\tilde{Q}_n^{(j)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_m(F)}.$$

Combining these inequalities and the expression,

$$\tilde{Q}_n(z) = (z - \xi)^\nu + \sum_{k=0}^{\nu-1} \frac{\tilde{Q}_n^{(k)}(\xi)}{k!} (z - \xi)^k,$$

we have

$$\limsup_{n \rightarrow \infty} \left\| (z - \xi)^\nu - \tilde{Q}_n(z) \right\|_{\mathbb{B}(\xi, 2\varepsilon)}^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_m(F)}.$$

In particular, if we replace z by $\xi_{n,\nu}$, then

$$\delta_\nu(\xi) = \limsup_{n \rightarrow \infty} |\xi_{n,\nu} - \xi|^{1/n} \leq \left(\frac{|\Phi(\xi)|}{\rho_m(F)} \right)^{1/\nu}.$$

This clearly implies (11).

Moreover, by Theorem 2.1, $\Delta(\xi) \leq |\Phi(\xi)|/\rho_m(F)$ is always true and the last statement readily follows. $\square$

PROOF OF THEOREM 2.3. Since the structure of the proof of Theorem 2.3 is similar to that of Theorem 2.1, we will skip some steps. For each $n \in \mathbb{N}$, let $q_{n,m}^E$ be the polynomial $Q_{n,m}^E$ normalized so that

$$(35) \quad \sum_{k=0}^m |\alpha_{n,k}| = 1, \quad q_{n,m}^E(z) = \sum_{k=0}^m \alpha_{n,k} z^k.$$

The polynomials $q_{n,m}^E$ are uniformly bounded on each compact subset of $\mathbb{C}$.

First of all, we show that for each pole ξ of order ν of F in $D_{\rho_m(F)}$,

$$(36) \quad \limsup_{n \rightarrow \infty} |(q_{n,m}^E)^{(j)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_m(F)}, \quad j = 0, 1, \dots, \nu - 1.$$

Denote by Q_m the monic polynomial whose zeros are all the poles of F in $D_{\rho_m(F)}$. Fix $\ell \in \{1, \dots, \nu\}$. Define

$$G_\ell(z) := \frac{Q_m(z)F(z)}{(z - \xi)^\ell}, \quad \ell = 1, 2, \dots, \nu$$

Set

$$H_\ell(z) := (x - \xi)^\ell G_\ell(z) \quad \text{and} \quad a_{n,n}^{(\ell)} := [q_{n,m}^E G_\ell]_n.$$

By the definition of $Q_{n,m}^E$, it follows that $a_{n,n}^{(\ell)} = 0$. Therefore,

$$a_{n,n}^{(\ell)} = [q_{n,m}^E G_\ell]_n = \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{q_{n,m}^E(z) G_\ell(z) \Phi'(z)}{\Phi^{n+1}(z)} dz = 0,$$

where $1 < \rho_1 < |\Phi(\xi)|$. Set

$$\tau_{n,n}^{(\ell)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{q_{n,m}^E(z) G_\ell(z) \Phi'(z)}{\Phi^{n+1}(z)} dz,$$

where $|\Phi(\xi)| < \rho_2 < \rho_m(F)$. Using Cauchy's residue theorem on the function $(q_{n,m}^E G_\ell \Phi')/\Phi^{n+1}$, we obtain

$$\begin{aligned} (37) \quad \tau_{n,n}^{(\ell)} &= \tau_{n,n}^{(\ell)} - a_{n,n}^{(\ell)} \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{q_{n,m}^E(t) G_\ell(t) \Phi'(t)}{\Phi^{n+1}(t)} dt - \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{q_{n,m}^E(t) G_\ell(t) \Phi'(t)}{\Phi^{n+1}(t)} dt \\ &= \text{res}((q_{n,m}^E G_\ell \Phi')/\Phi^{n+1}, \xi). \end{aligned}$$

Arguing as in (26),

$$\begin{aligned} &\text{res}((q_{n,m}^E G_\ell \Phi')/\Phi^{n+1}, \xi) \\ &= \frac{1}{(\ell-1)!} \sum_{t=0}^{\ell-1} \binom{\ell-1}{t} \left(\frac{H_\ell \Phi'}{\Phi^{n+1}} \right)^{(\ell-1-t)} (\xi) (q_{n,m}^E)^{(t)}(\xi). \end{aligned}$$

The above equality and the relation (37) imply

$$\begin{aligned} (38) \quad (q_{n,m}^E)^{(\ell-1)}(\xi) &= \frac{(\ell-1)! \tau_{n,n}^{(\ell)} \Phi^{n+1}(\xi)}{H_\ell(\xi) \Phi'(\xi)} \\ &- \sum_{t=0}^{\ell-2} \binom{\ell-1}{t} \left(\frac{H_\ell \Phi'}{\Phi^{n+1}} \right)^{(\ell-1-t)} (\xi) \frac{\Phi^{n+1}(\xi) (q_{n,m}^E)^{(t)}(\xi)}{H_\ell(\xi) \Phi'(\xi)}, \end{aligned}$$

where the sum is empty when $\ell = 1$.

Choose $\delta > 0$ small enough so that

$$(39) \quad \rho_2 := \rho_m(F) - \delta > |\Phi(\xi)|.$$

We have

$$(40) \quad |\tau_{n,n}^{(\ell)}| = \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{q_{n,m}^E(z) G_\ell(z) \Phi'(z)}{\Phi^{n+1}(z)} dz \right| \leq \frac{c_1}{\rho_2^n}.$$

If $\ell = 1$, combining (38) and (40), we obtain

$$|q_{n,m}^E(\xi)| \leq c_2 \left(\frac{|\Phi(\xi)|}{\rho_2} \right)^n$$

which implies that

$$\limsup_{n \rightarrow \infty} |q_{n,m}^E(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_2}.$$

Letting $\delta \rightarrow 0$, (36) readily follows for $j = 0$. For the remaining values of j , we use induction.

Suppose that (36) is true for $j = 0, \dots, \ell - 2$, $2 \leq \ell \leq \tau$ and let us prove that it is also valid for $j = \ell - 1$. Choosing $\delta > 0$ as in (39) and applying Cauchy's integral formula as in (29), we obtain for $t = 0, 1, \dots, \ell - 2$,

$$(41) \quad \left| \left(\frac{H_\ell \Phi'}{\Phi^{n+1}} \right)^{(\ell-1-t)}(\xi) \right| \leq \frac{c_3}{(|\Phi(\xi)| - \delta)^n}.$$

Combining the induction hypothesis, (38), (40), and (41), we have

$$(42) \quad \begin{aligned} \limsup_{n \rightarrow \infty} |(q_{n,m}^E)^{(\ell-1)}(\xi)|^{1/n} &= \limsup_{n \rightarrow \infty} \left| \frac{(\ell-1)! \tau_{n,n}^{(\ell)} \Phi^{n+1}(\xi)}{H_\ell(\xi) \Phi'(\xi)} \right. \\ &\quad \left. - \sum_{t=0}^{\ell-2} \binom{\ell-1}{t} \left(\frac{H_\ell \Phi'}{\Phi^{n+1}} \right)^{(\ell-1-t)}(\xi) \frac{\Phi^{n+1}(\xi) (q_{n,m}^E)^{(t)}(\xi)}{H_\ell(\xi) \Phi'(\xi)} \right|^{1/n} \\ &\leq \max \left\{ \frac{|\Phi(\xi)|}{\rho_2}, \left(\frac{|\Phi(\xi)|}{|\Phi(\xi)| - \delta} \right) \left(\frac{|\Phi(\xi)|}{\rho_m(F)} \right) \right\}. \end{aligned}$$

Letting $\delta \rightarrow 0$, we have $\rho_2 \rightarrow \rho_m(F)$ and from (42), we obtain

$$\limsup_{n \rightarrow \infty} |(q_{n,m}^E)^{(\ell-1)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_m(F)},$$

which completes the induction.

Arguing as in the proofs of (31) and (33) by replacing $q_{n,m}^\mu$ by $q_{n,m}^E$, $Q_{n,m}^\mu$ by $Q_{n,m}^E$, and

$$\{\tilde{\xi}_{n,1}, \tilde{\xi}_{n,2}, \dots, \tilde{\xi}_{n,\sigma_n}\} \quad \text{by} \quad \{\hat{\xi}_{n,1}, \hat{\xi}_{n,2}, \dots, \hat{\xi}_{n,\sigma_n}\},$$

we have

$$\lim_{n \rightarrow \infty} \hat{\xi}_{n,j} = \xi, \quad j = 1, 2, \dots, \nu$$

and

$$\Delta(\xi) \leq \frac{|\Phi(\xi)|}{\rho_m(F)} \quad \text{and} \quad \sigma(\xi) \geq \nu,$$

as we wanted. $\square$

3.2. Proofs of the inverse type results. We keep all needed lemmas used in the proofs of the inverse type results in Section 3.2.1.

3.2.1. Auxiliary Lemmas. We begin by stating two lemmas due to Buslaev (see [5, Theorems 5–6]). These results constitute the main tools for proving our inverse type results. We make use of the following notation. Let $f(w) = \sum_{k=-\infty}^{\infty} f_k w^k$ be a Laurent series. We denote the regular part of $f(w)$ by $\hat{f}(w) := \sum_{k=0}^{\infty} f_k w^k$. Define the closed annulus

$$T_{\delta,m}(f) := \{w \in \mathbb{C} : e^{-\delta} R_0(\hat{f}) \leq |w| \leq e^{\delta} R_{m-1}(\hat{f})\},$$

where $m \in \mathbb{N}$ and $\delta \geq 0$. We will use $(\cdot)_n$ to denote the coefficient of w^n in the Laurent series expansion of the function in the parentheses. Set

$$U := \overline{\mathbb{C}} \setminus \overline{\mathbb{B}(0,1)}.$$

LEMMA 3.3 (Buslaev [5]). *Let $m \in \mathbb{N}$, $\delta > 0$, and let $f(w) = \sum_{n=-\infty}^{\infty} f_n w^n$ be a Laurent series such that*

$$0 < R_0(\hat{f}) \leq R_{m-1}(\hat{f}) < \infty, \quad \text{and} \quad \limsup_{n \rightarrow \infty} |f_{-n}|^{1/n} \leq R_0(\hat{f}).$$

Assume further that

$$(43) \quad \lim_{n \rightarrow \infty} (f \alpha_n \eta_{n,j})_n R_{m-1}^n(\hat{f}) e^{\delta n} = 0, \quad j = 0, \dots, m-1,$$

where the functions $\alpha_n, \eta_{n,j} \in \mathcal{H}(T_{\delta,m}(f))$ have the limits

$$\alpha(w) := \lim_{n \rightarrow \infty} \alpha_n(w) \not\equiv 0, \quad \eta_j(w) := \lim_{n \rightarrow \infty} \eta_{n,j}(w) = \eta^j(w), \quad j = 0, \dots, m-1,$$

uniformly on $T_{\delta,m}(f)$, $\eta(w)$ is a univalent function in $T_{\delta,m}(f)$, and $\alpha(w)$ has at most m zeros in the annulus $T_{0,m}(f)$. Then the function $\alpha(w)$ has precisely m zeros $\lambda_1, \dots, \lambda_m$ in $T_{0,m}(f)$ and $\lim_{n \rightarrow \infty} \lambda_{n,j} = \lambda_j$, where the $\lambda_{n,j}$, $j = 1, \dots, m$, are poles of the classical approximants $R_{n,m}$ of $\hat{f}$. Moreover, for any functions $K_{n,1}, \dots, K_{n,m}, L_{n,1}, \dots, L_{n,m} \in \mathcal{H}(T_{\nu,m}(f))$, $\nu > 0$, that converge to $K_1, \dots, K_m, L_1, \dots, L_m$ uniformly on $T_{\nu,m}(f)$,

$$(44) \quad \lim_{n \rightarrow \infty} \frac{\det[(f K_{n,i} L_{n,j})_n]_{i,j=1,\dots,m}}{\det[f_{n-i-j}]_{i,j=0,\dots,m-1}} = \frac{\det[K_r(\lambda_s)]_{s,r=1,\dots,m} \det[L_r(\lambda_s)]_{s,r=1,\dots,m}}{W^2(\lambda_1, \dots, \lambda_m)},$$

where $W(\lambda_1, \dots, \lambda_m) = \det[\lambda_s^{r-1}]_{s,r=1,\dots,m}$ is the Vandermonde determinant of the numbers $\lambda_1, \dots, \lambda_m$ (for multiple zeros the right-hand side of (44) is defined by continuity).

LEMMA 3.4 (Buslaev [5]). *Let $m \in \mathbb{N}$, $\sigma > 1$, and $f(w) = \sum_{n=-\infty}^{\infty} f_n w^n$ be a holomorphic function in the annulus $\{1 < |w| < \sigma\}$. Assume further that*

$$(45) \quad (f\alpha_n\eta_{n,j})_n = 0, \quad j = 0, \dots, m-1, \quad n \geq n_0,$$

hold, where $\alpha_n(w)$ and $w^{-j}\eta_{n,j}(w)$ are holomorphic functions in U , the limits

$$\alpha(w) := \lim_{n \rightarrow \infty} \alpha_n(w) \not\equiv 0, \quad \eta_j(w) := \lim_{n \rightarrow \infty} \eta_{n,j}(w) = \eta^j(w), \quad j = 0, \dots, m-1,$$

exist uniformly on each compact subset of $U \setminus \{\infty\}$, the function $\alpha(w)$ has at most m zeros in $U \setminus \{\infty\}$, and $\eta(w)$ is a univalent function in U such that $\eta(\infty) = \infty$. Then, only one of the following assertions takes place:

- (i) $\hat{f}(w)$ is a rational function with at most $m-1$ poles;
- (ii) $\alpha(w)$ has precisely m zeros $\lambda_1, \dots, \lambda_m$ in $U \setminus \{\infty\}$, these zeros are singularities of $f(w)$, with an appropriate ordering $|\lambda_1| = R_0(\hat{f}), \dots, |\lambda_m| = R_{m-1}(\hat{f})$, and the limits $\lim_{n \rightarrow \infty} \lambda_{n,j} = \lambda_j$ exist, where the $\lambda_{n,j}, j = 1, \dots, m$, are the poles of the classical Padé approximants $R_{n,m}$ of $\hat{f}$.

Define

$$h_n(w) := c_n w^{n+1} s_n(\Psi(w)) \Psi'(w), \quad w \in U.$$

Properties of h_n are stated the following lemma (see [1, Lemma 4.3] for its proof).

LEMMA 3.5. *Let $E \in \mathcal{K}_1$. Then, the functions $h_n(w)$ are holomorphic in U . Moreover, if $\mu \in \mathcal{S}(E)$, then the sequence $h_n(w)$ converges to some non-vanishing function $h(w)$ uniformly on each compact subset of U .*

Note that the restriction condition that $E \in \mathcal{K}_1$ is used to prove this lemma (see [1, Lemma 3.1 and Lemma 4.3] for more details).

3.2.2. Proofs of Theorems 2.7 and 2.8.

PROOF OF THEOREM 2.7. First of all, we prove that (b) implies (a) using Lemma 3.4. We assume that the zeros of $Q_{n,m}^\mu(z)$ have limits $\xi_1, \dots, \xi_m$, as $n \rightarrow \infty$. For $w \in U$, we define

$$\alpha_n(w) := w^{-m} h(w) Q_{n+m,m}^\mu(\Psi(w)),$$

$$\eta_{n,j}(w) := \frac{c_{n+m} \Psi^j(w) w^{n+m+1} s_{n+m}(\Psi(w)) \Psi'(w)}{h(w)}, \quad j = 0, \dots, m-1.$$

The functions $\alpha_n(w)$ and $w^{-j}\eta_{n,j}(w) = w^{-j}\Psi^j(w)h_{n+m}(w)/h(w)$, $j = 1, \dots, m-1$, are holomorphic in U , and

$$\alpha(w) := \lim_{n \rightarrow \infty} \alpha_n(w) = w^{-m}h(w) \prod_{j=1}^m (\Psi(w) - \xi_j),$$

$$\eta_j(w) := \lim_{n \rightarrow \infty} \eta_{n,j}(w) = \Psi^j(w), \quad j = 0, 1, \dots, m-1,$$

uniformly on each compact subset of $U \setminus \{\infty\}$. Since $h(w)$ is never zero in U , $\alpha(w)$ has at most m zeros in $U \setminus \{\infty\}$. Set

$$\gamma_r := \{w \in \mathbb{C} : |w| = r\}.$$

By Cauchy's integral formula, Fubini's theorem, and the definition of $Q_{n,m}^\mu$, we have, for $\varepsilon > 0$ sufficiently small so that $F(z)$ is analytic on $\overline{D_{1+\varepsilon}}$, and for $j = 0, \dots, m-1$,

$$\begin{aligned} (f\alpha_n\eta_{n,j})_n &= \frac{c_{n+m}}{2\pi i} \int_{\gamma_{1+\varepsilon}} \Psi^j(w) F(\Psi(w)) Q_{n+m,m}^\mu(\Psi(w)) s_{n+m}(\Psi(w)) \Psi'(w) dw \\ &= \frac{c_{n+m}}{2\pi i} \int_{\Gamma_{1+\varepsilon}} t^j F(t) Q_{n+m,m}^\mu(t) s_{n+m}(t) dt \\ &= \frac{c_{n+m}}{2\pi i} \int_{\Gamma_{1+\varepsilon}} t^j F(t) Q_{n+m,m}^\mu(t) \int \frac{\overline{p_{n+m}(z)}}{t-z} d\mu(z) dt \\ &= c_{n+m} \int \frac{1}{2\pi i} \int_{\Gamma_{1+\varepsilon}} \frac{t^j F(t) Q_{n+m,m}^\mu(t)}{t-z} dt \overline{p_{n+m}(z)} d\mu(z) \\ &= c_{n+m} \int z^j F(z) Q_{n+m,m}^\mu(z) \overline{p_{n+m}(z)} d\mu(z) = 0. \end{aligned}$$

Therefore, the assumptions of Lemma 3.4 are satisfied. If the regular part of $f(w)$ is a rational function with at most $m-1$ poles, then $F(z)$ is a rational function with at most $m-1$ poles which implies that $\Lambda_{n,m}^\mu(F) = 0$ for n sufficiently large. This is impossible because $\deg(Q_{n,m}^\mu) = m$, for n sufficiently large. Therefore, by Lemma 3.4, $\alpha(w)$ has precisely m zeros $\lambda_1, \dots, \lambda_m$ in $U \setminus \{\infty\}$ and the limits of the poles of the classical Padé approximants $R_{n,m}$ of $\hat{f}$ are $\lambda_1, \dots, \lambda_m$, as $n \rightarrow \infty$. This implies (a).

Now, we prove that (a) implies (b) using Lemma 3.3. Assume that the zeros of $Q_{n,m}$ (where $\hat{f}$ is the approximated function) have limits $\lambda_1, \dots, \lambda_m$, as $n \rightarrow \infty$. Define, for $w \in U$,

$$(46) \quad \tilde{\alpha}_n(w) := w^{-m} Q_{n,m}(w),$$

$$(47) \quad \tilde{\eta}_{n,j}(w) := w^j, \quad j = 0, \dots, m-1.$$

Then,

$$\tilde{\alpha}(w) := \lim_{n \rightarrow \infty} \tilde{\alpha}_n(z) = w^{-m} \prod_{j=1}^m (w - \lambda_j),$$

$$\tilde{\eta}_j(w) = w^j, \quad j = 0, \dots, m-1,$$

uniformly on each compact subset of $U \setminus \{\infty\}$. By the definition of $Q_{n,m}(w)$, it follows that, for $\varepsilon > 0$ sufficiently small so that $f(w)$ is holomorphic on $\gamma_{1+\varepsilon}$ and for n sufficiently large,

$$(f\tilde{\alpha}_n\tilde{\eta}_{n,j})_n = (\hat{f}\tilde{\alpha}_n\tilde{\eta}_{n,j})_n = \frac{1}{2\pi i} \int_{\gamma_{1+\varepsilon}} \frac{\hat{f}(w)Q_{n,m}(w)}{w^{m-j+n+1}} dw = 0, \quad j = 0, \dots, m-1.$$

We can easily check the rest of the conditions required in Lemma 3.3 for $\tilde{\alpha}_n(w)$ and $\tilde{\eta}_{n,j}(w)$, so we can apply the equality (44) in Lemma 3.3.

Next, set

$$(48) \quad \tilde{Q}_{n,m}(z) := \begin{vmatrix} c_n \langle F, p_n \rangle_\mu & c_n \langle zF, p_n \rangle_\mu & \cdots & c_n \langle z^m F, p_n \rangle_\mu \\ c_n \langle zF, p_n \rangle_\mu & c_n \langle z^2 F, p_n \rangle_\mu & \cdots & c_n \langle z^{m+1} F, p_n \rangle_\mu \\ \vdots & \vdots & \cdots & \vdots \\ c_n \langle z^{m-1} F, p_n \rangle_\mu & c_n \langle z^m F, p_n \rangle_\mu & \cdots & c_n \langle z^{2m-1} F, p_n \rangle_\mu \\ 1 & z & \cdots & z^m \end{vmatrix}.$$

Note that the polynomials $\tilde{Q}_{n,m}(z)$ satisfy

$$(49) \quad \langle z^k \tilde{Q}_{n,m} F, p_n \rangle_\mu = 0, \quad k = 0, \dots, m-1,$$

and if we show that $\Lambda_{n,m}^\mu(F) \neq 0$ (the leading coefficient of $\tilde{Q}_{n,m}(z)/c_n^m$), which will be verified at the end of this proof, then $Q_{n,m}^\mu(z)$ is unique and

$$Q_{n,m}^\mu(z) = \frac{\tilde{Q}_{n,m}(z)}{\Lambda_{n,m}^\mu(F)c_n^m}.$$

Using Cauchy's integral formula and Fubini's theorem, for $\varepsilon > 0$ sufficiently small so that $F(z)$ is holomorphic on $\overline{D_{1+\varepsilon}}$, for $\ell = 0, 1, \dots, 2m-1$, we have

$$\begin{aligned} c_n \langle z^\ell F, p_n \rangle_\mu &= c_n \int \frac{1}{2\pi i} \int_{\Gamma_{1+\varepsilon}} \frac{\zeta^\ell F(\zeta)}{\zeta - z} d\zeta \overline{p_n(z)} d\mu(z) \\ &= \frac{c_n}{2\pi i} \int_{\Gamma_{1+\varepsilon}} \zeta^\ell F(\zeta) \int \frac{\overline{p_n(z)}}{\zeta - z} d\mu(z) d\zeta = \frac{c_n}{2\pi i} \int_{\Gamma_{1+\varepsilon}} \zeta^\ell F(\zeta) s_n(\zeta) d\zeta \end{aligned}$$

$$= \frac{c_n}{2\pi i} \int_{\gamma_{1+\varepsilon}} \Psi^\ell(w) f(w) s_n(\Psi(w)) \Psi'(w) dw = (fh_n \Psi^\ell)_n.$$

Computing the determinant in (48) by expanding along the last row and applying the previous formula, we obtain

$$(50) \quad \tilde{Q}_{n,m}(z) = \sum_{k=0}^m (-1)^{m+k} z^k \det[(fK_{n,t}L_{n,r})_n]_{t=1,\dots,m, r=1,\dots,k,k+2,\dots,m+1},$$

where

$$K_{n,t}(w) := \Psi^{t-1}(w)h_n(w), \quad t = 1, \dots, m,$$

$$L_{n,r}(w) := \Psi^{r-1}(w), \quad r = 1, \dots, m+1.$$

Moreover, all the functions $K_{n,t}(w)$ and $L_{n,r}(w)$, are holomorphic in $U \setminus \{\infty\}$, and

$$K_t(w) := \lim_{n \rightarrow \infty} K_{n,t}(w) = \Psi^{t-1}(w)h(w), \quad t = 1, \dots, m,$$

$$L_r(w) := \Psi^{r-1}(w), \quad r = 1, \dots, m+1,$$

uniformly on each compact subset of $U \setminus \{\infty\}$. By Lemma 3.3 and (50), we have that $\lambda_1, \dots, \lambda_m \in T_{0,m}(f)$ and

$$\begin{aligned} (51) \quad & \lim_{n \rightarrow \infty} \frac{\tilde{Q}_{n,m}(z)}{\det[f_{n-i-j}]_{i,j=0,1,\dots,m-1}} \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^m (-1)^{m+k} z^k \frac{\det[(fK_{n,t}L_{n,r})_n]_{t=1,\dots,m, r=1,\dots,k,k+2,\dots,m+1}}{\det[f_{n-i-j}]_{i,j=0,1,\dots,m-1}} \\ &= \sum_{k=0}^m (-1)^{m+k} z^k \frac{\det[K_r(\lambda_t)]_{t,r=1,\dots,m} \det[L_r(\lambda_t)]_{t=1,\dots,m, r=1,\dots,k,k+2,\dots,m+1}}{W^2(\lambda_1, \lambda_2, \dots, \lambda_m)} \\ &= \frac{\det[K_r(\lambda_t)]_{r,t=1,2,\dots,m}}{W^2(\lambda_1, \lambda_2, \dots, \lambda_m)} \begin{vmatrix} 1 & \Psi(\lambda_1) & \dots & \Psi^m(\lambda_1) \\ \vdots & \vdots & & \vdots \\ 1 & \Psi(\lambda_m) & \dots & \Psi^m(\lambda_m) \\ 1 & z & \dots & z^m \end{vmatrix} \\ &= \prod_{t=1}^m h(\lambda_t) \prod_{1 \leq i < j \leq m} \left(\frac{\Psi(\lambda_j) - \Psi(\lambda_i)}{\lambda_j - \lambda_i} \right)^2 z^m + \dots, \end{aligned}$$

where $W(\lambda_1, \lambda_2, \dots, \lambda_m) = \det[\lambda_t^{r-1}]_{t,r=1,\dots,m}$ is the Vandermonde determinant of the numbers $\lambda_1, \dots, \lambda_m$. Since the degree of the polynomial in the

last expression is m , the degree of $\tilde{Q}_{n,m}(z)$ is m for all n sufficiently large. Thus $\Lambda_{n,m}^\mu(F) \neq 0$ and $Q_{n,m}^\mu(z) = \tilde{Q}_{n,m}(z)/(\Lambda_{n,m}^\mu(F)c_n^m)$. Moreover, the zeros of the polynomial in the second last equality are $\xi_1, \dots, \xi_m$, so the zeros of $\tilde{Q}_{n,m}(z)$ (and $Q_{n,m}^\mu(z)$) converge to $\xi_1, \dots, \xi_m$, as $n \rightarrow \infty$. $\square$

PROOF OF THEOREM 2.8. First of all, we prove that (b) implies (a) using Lemma 3.4. We assume that the zeros of $Q_{n,m}^E(z)$ have limits $\xi_1, \dots, \xi_m$, as $n \rightarrow \infty$. For $w \in U$, we define

$$\alpha_n(w) := w^{-m} Q_{n+m,m}^E(\Psi(w)), \quad \eta_{n,j}(w) := \Psi^j(w), \quad j = 0, \dots, m-1.$$

The functions $\alpha_n(w)$ and $w^{-j} \eta_{n,j}(w) = w^{-j} \Psi^j(w)$, $j = 1, \dots, m-1$, are holomorphic in U , and

$$\alpha(w) := \lim_{n \rightarrow \infty} \alpha_n(w) = w^{-m} \prod_{j=1}^m (\Psi(w) - \xi_j),$$

$$\eta_j(w) := \lim_{n \rightarrow \infty} \eta_{n,j}(w) = \Psi^j(w), \quad j = 0, 1, \dots, m-1,$$

uniformly on each compact subset of $U \setminus \{\infty\}$. Then, $\alpha(w)$ has at most m zeros in $U \setminus \{\infty\}$. By the definition of $Q_{n,m}^E$, we have, for $\varepsilon > 0$ sufficiently small so that $F(z)$ is analytic on $\overline{D_{1+\varepsilon}}$, and for $j = 0, \dots, m-1$,

$$\begin{aligned} (f \alpha_n \eta_{n,j})_n &= \frac{1}{2\pi i} \int_{\gamma_{1+\varepsilon}} \frac{\Psi^j(w) F(\Psi(w)) Q_{n+m,m}^E(\Psi(w))}{w^{n+m+1}} dw \\ &= \frac{1}{2\pi i} \int_{\Gamma_{1+\varepsilon}} \frac{t^j F(t) Q_{n+m,m}^E(t) \Phi'(t)}{\Phi^{n+m+1}(t)} dt = [z^j F Q_{n+m,m}^E]_{n+m} = 0, \end{aligned}$$

where we recall that $\gamma_r := \{w \in \mathbb{C} : |w| = r\}$. Therefore, the assumptions of Lemma 3.4 are satisfied. If the regular part of $f(w)$ is a rational function with at most $m-1$ poles, then $F(z)$ is a rational function with at most $m-1$ poles which implies that $\Lambda_{n,m}^E(F) = 0$ for n sufficiently large. This is impossible because $\deg(Q_{n,m}^E) = m$, for n sufficiently large. Therefore, by Lemma 3.4, $\alpha(w)$ has precisely m zeros $\lambda_1, \dots, \lambda_m$ in $U \setminus \{\infty\}$ and the limits of the poles of the classical Padé approximants $R_{n,m}$ of $\hat{f}$ are $\lambda_1, \dots, \lambda_m$, as $n \rightarrow \infty$. This implies (a).

Now, we prove that (a) implies (b) using Lemma 3.3. Assume that the zeros of $Q_{n,m}$ (where $\hat{f}$ is the approximated function) have limits $\lambda_1, \dots, \lambda_m$, as $n \rightarrow \infty$. Define $\tilde{\alpha}_n(w)$ and $\tilde{\eta}_{n,j}(w)$, $j = 0, \dots, m-1$, as in (46) and (47), respectively. Applying exactly the same argument, we can check the conditions required in Lemma 3.3 for $\tilde{\alpha}_n(w)$ and $\tilde{\eta}_{n,j}(w)$, so we can apply the equality (44) in Lemma 3.3.

Now, set

$$(52) \quad \tilde{Q}_{n,m}(z) := \begin{vmatrix} [F]_n & [zF]_n & \cdots & [z^m F]_n \\ [zF]_n & [z^2 F]_n & \cdots & [z^{m+1} F]_n \\ \vdots & \vdots & \cdots & \vdots \\ [z^{m-1} F]_n & [z^m F]_n & \cdots & [z^{2m-1} F]_n \\ 1 & z & \cdots & z^m \end{vmatrix}.$$

Note that the polynomials $\tilde{Q}_{n,m}(z)$ satisfy

$$(53) \quad [z^k \tilde{Q}_{n,m} F]_n = 0, \quad k = 0, \dots, m-1,$$

and if we show that $\Lambda_{n,m}^E(F) \neq 0$ (the leading coefficient of $\tilde{Q}_{n,m}(z)$), which will be verified at the end of this proof, then $Q_{n,m}^E(z)$ is unique and

$$Q_{n,m}^E(z) = \frac{\tilde{Q}_{n,m}(z)}{\Lambda_{n,m}^E(F)}.$$

For $\varepsilon > 0$ sufficiently small so that $F(z)$ is holomorphic on $\overline{D_{1+\varepsilon}}$, for $\ell = 0, 1, \dots, 2m-1$, we have

$$[z^\ell F]_n = \frac{1}{2\pi i} \int_{\Gamma_{1+\varepsilon}} \frac{t^\ell F(t) \Phi'(t)}{\Phi^{n+1}(t)} dt = \frac{1}{2\pi i} \int_{\gamma_{1+\varepsilon}} \frac{\Psi^\ell(w) f(w)}{w^{n+1}} dw = (\Psi^\ell f)_n.$$

Computing the determinant in (52) by expanding along the last row and applying the previous formula, we obtain

$$(54) \quad \tilde{Q}_{n,m}(z) = \sum_{k=0}^m (-1)^{m+k} z^k \det[(f K_{n,t} L_{n,r})_n]_{t=1, \dots, m, r=1, \dots, k, k+2, \dots, m+1},$$

where

$$\begin{aligned} K_{n,t}(w) &:= \Psi^{t-1}(w), \quad t = 1, \dots, m, \\ L_{n,r}(w) &:= \Psi^{r-1}(w), \quad r = 1, \dots, m+1. \end{aligned}$$

Clearly, all the functions $K_{n,t}(w)$ and $L_{n,r}(w)$, are holomorphic in $U \setminus \{\infty\}$, and

$$\begin{aligned} K_t(w) &:= \Psi^{t-1}(w), \quad t = 1, \dots, m, \\ L_r(w) &:= \Psi^{r-1}(w), \quad r = 1, \dots, m+1, \end{aligned}$$

for all $w \in U \setminus \{\infty\}$. Applying Lemma 3.3 and (54), we have that $\lambda_1, \dots, \lambda_m \in T_{0,m}(f)$ and the same line of reasoning used to derive (51) implies that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\tilde{Q}_{n,m}(z)}{\det[f_{n-i-j}]_{i,j=0,1,\dots,m-1}} \\ &= \frac{W(\Psi(\lambda_1), \Psi(\lambda_2), \dots, \Psi(\lambda_m))}{W^2(\lambda_1, \lambda_2, \dots, \lambda_m)} \begin{vmatrix} 1 & \Psi(\lambda_1) & \dots & \Psi^m(\lambda_1) \\ \vdots & \vdots & & \vdots \\ 1 & \Psi(\lambda_m) & \dots & \Psi^m(\lambda_m) \\ 1 & z & \dots & z^m \end{vmatrix} \\ &= \prod_{1 \leq i < j \leq m} \left(\frac{\Psi(\lambda_j) - \Psi(\lambda_i)}{\lambda_j - \lambda_i} \right)^2 z^m + \dots \end{aligned}$$

Since the degree of the polynomial in the last expression is m , the degree of $\tilde{Q}_{n,m}(z)$ is m for all n sufficiently large. Thus $\Lambda_{n,m}^E(F) \neq 0$ and $Q_{n,m}^E(z) = \tilde{Q}_{n,m}(z)/\Lambda_{n,m}^E(F)$. Moreover, the zeros of the polynomial in the second equality are $\xi_1, \dots, \xi_m$, so the zeros of $\tilde{Q}_{n,m}(z)$ (and $Q_{n,m}^E(z)$) converge to $\xi_1, \dots, \xi_m$, as $n \rightarrow \infty$. $\square$

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Convergence in Hausdorff Content of Padé-Faber Approximants and Its Applications

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Abstract : A convergence in Hausdorff content of Padé-Faber approximants (recently introduced in [1]) on some certain sequences is proved. As applications of this result, we give an alternate proof of a Montessus de Ballore type theorem for these Padé-Faber approximants and a proof of a convergence of Padé-Faber approximants in the maximal canonical domain in which the approximated function can be continued to a meromorphic function.

Keywords : Padé approximation; Faber polynomials; Montessus de Ballore's theorem; Hausdorff content

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1 Introduction

Let E be a compact subset of the complex plane $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus E$ is simply connected and E contains more than one point. It is convenient to assume that $0 \in E$ and this can be done, if necessary, without loss of generality making a change of variables. By the Riemann mapping theorem, there exists a unique exterior conformal mapping Φ from $\overline{\mathbb{C}} \setminus E$ onto $\overline{\mathbb{C}} \setminus \{w \in \mathbb{C} : |w| \leq 1\}$ satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. For any $\rho > 1$, we define

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\} \quad \text{and} \quad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\},$$

as the *level curve of index ρ* and the *canonical domain of index ρ* , respectively. We denote by $\rho_0(F)$ the index $\rho > 1$ of the largest canonical domain D_ρ to which F can be extended as a holomorphic function, and by $\rho_m(F)$ the index $\rho > 1$ of the largest canonical domain D_ρ to which F can be extended as a meromorphic function with at most m poles (counting multiplicities). We denote by

$$D_{\rho_\infty(F)} := \bigcup_{m=0}^{\infty} D_{\rho_m(F)}$$

the maximum canonical domain in which F can be continued to a meromorphic function.

The *Faber polynomial of E of degree n* is defined by the formula

$$\Phi_n(z) := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\Phi^n(t)}{t - z} dt, \quad z \in D_\rho, \quad n = 0, 1, 2, \dots$$

Denote by $\mathcal{H}(E)$ the space of all functions holomorphic in some neighborhood of E . The n -th *Faber coefficient* of $F \in \mathcal{H}(E)$ with respect to Φ_n is given by

$$[F]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{F(t)\Phi'(t)}{\Phi^{n+1}(t)} dt,$$

where $1 < \rho < \rho_0(F)$. Denote by $\mathbb{N}$ the set of all positive integers. Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The definition of Padé-Faber approximants (first introduced in [1]) is stated below.

Definition 1.1. Let $F \in \mathcal{H}(E)$ and $(n, m) \in \mathbb{N} \times \mathbb{N}$ be fixed. Then, there exist polynomials $q_{n,m}^E, p_{n,m,k}^E$, $k = 0, 1, \dots, m-1$ such that

$$\deg(p_{n,m,k}^E) \leq n-1, \quad \deg(q_{n,m}^E) \leq m, \quad q_{n,m}^E \neq 0, \quad (1.1)$$

$$[z^k q_{n,m}^E F - p_{n,m,k}^E]_j = 0, \quad j = 0, 1, 2, \dots, n. \quad (1.2)$$

For each $k = 0, 1, \dots, m-1$, the rational function

$$R_{n,m,k}^E := \frac{p_{n,m,k}^E}{q_{n,m}^E}$$

is called an (n, m, k) *Padé-Faber approximant* of F .

To solve for ordered pairs $(p_{n,m,k}^E, q_{n,m}^E)$, we need to find $nm + m + 1$ unknown coefficients in (1.1) from $nm + m$ linear equations in (1.2). Then, $R_{n,m,k}^E$ always exist but they may not be unique. Moreover, since $q_{n,m}^E \neq 0$, we normalize it to have leading coefficient equal to 1. Note that the definition of Padé-Faber approximants in Definition 1.1 is totally different from the definition of “classical” Padé-Faber approximants (see, e.g. [2]). Since this new definition of Padé-Faber approximants was recently introduced, there are only two publications [1, 3] studying this approximation. In [1], Bosuwan and López gave necessary and sufficient conditions for the convergence with geometric rate of $\{q_{n,m}^E\}_{n \in \mathbb{N}}$ (when m is fixed), namely, proving the analogue of the Montessus de Ballore-Gonchar theorem for Padé-Faber approximants on row sequences (see [1, Corollary 1.6]). Later, Bosuwan [3] further studied the convergence of zeros of $\{q_{n,m}^E\}_{n \in \mathbb{N}}$ (when m is fixed). These two results show that the zeros of $\{q_{n,m}^E\}_{n \in \mathbb{N}}$ can be used to detect the location of the poles of the approximated function $F \in \mathcal{H}(E)$.

Next, let us introduce a concept of convergence in Hausdorff content. Let B be a subset of the complex plane $\mathbb{C}$. By $\mathcal{U}(B)$, we denote the class of all coverings of B by at most a numerable set of disks. Let $\beta > 0$ and set

$$h_\beta(B) := \inf \left\{ \sum_{j=1}^{\infty} |U_j|^\beta : \{U_j\} \in \mathcal{U}(B) \right\},$$

where $|U_j|$ stands for the radius of the disk U_j . The quantity $h_\beta(B)$ is called the β -dimensional Hausdorff content of the set B . This set function is not a measure but it is subadditive and monotonic. Clearly, if B is a disk, then $h_\beta(B) = |B|^\beta$.

Definition 1.2. Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of complex valued functions defined on a domain $D \subset \mathbb{C}$ and g be another complex function defined on D . We say that $\{g_n\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to the function g inside D if for every compact subset K of D and for each $\varepsilon > 0$, we have

$$\lim_{n \rightarrow \infty} h_\beta\{z \in K : |g_n(z) - g(z)| > \varepsilon\} = 0.$$

Such a convergence will be denoted by $h_\beta\text{-}\lim_{n \rightarrow \infty} g_n = g$ in D .

The objective of this paper is to investigate a convergence in Hausdorff content of the sequences of Padé-Faber approximants $R_{n,m_n,k}^E$ as $n \rightarrow \infty$ when the sequences $\{m_n\}_{n \in \mathbb{N}}$ satisfy

$$\lim_{n \rightarrow \infty} \frac{m_n \ln n}{n} = 0. \quad (1.3)$$

This type of sequences of indices $\{(n, m_n)\}_{n \in \mathbb{N}}$ when $\{m_n\}_{n \in \mathbb{N}}$ satisfy the limit (1.3) was first considered by Gonchar [4] for Padé (α, β) -approximants. In the current paper, we prove many results analogous to those in the paper by Gonchar (see Theorem 2, Corollary 1, and Corollary 2 in [4]). As a consequence of our main theorem in this paper, we give an alternative proof of a Montessus de Ballore type theorem for row sequences of Padé-Faber approximants which was originally

proved in [1]. Note that the normalization of $q_{n,m}^E$ introduced in the next section is different from the one in [1].

An outline of the paper is as follows. In section 2, we state the main theorem and its corollaries. All auxiliary lemmas are in section 3. Section 4 is devoted to the proofs of all results in section 2.

2 Main Results

An analogue of Theorem 2 in [4] is the following theorem. This theorem constitutes our main result.

Theorem 2.1. Let $\rho > 1$, $F \in \mathcal{H}(E)$ be meromorphic in D_ρ . Assume that

$$m^* := \liminf_{n \rightarrow \infty} m_n \geq d_k \quad (2.1)$$

and

$$\lim_{n \rightarrow \infty} \frac{m_n \ln n}{n} = 0, \quad (2.2)$$

where k is a fixed number in $\{0, 1, \dots, m^* - 1\}$ and d_k denotes the number of poles of $z^k F$ in D_ρ . Then, for any $\beta > 0$, each sequence $\{R_{n,m_n,k}^E\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to $z^k F$ inside D_ρ as $n \rightarrow \infty$.

One of the consequences of Theorem 2.1 is a Montessus de Ballore type theorem for Padé-Faber approximants stated below.

Corollary 2.2. Let $k \in \{0, 1, \dots, m - 1\}$ be fixed. Suppose that $z^k F \in \mathcal{H}(E)$ has poles of total multiplicity exactly m in $D_{\rho_m(z^k F)}$ at the (not necessarily distinct) points $\lambda_1, \lambda_2, \dots, \lambda_m$. Then, $R_{n,m,k}^E$ is uniquely determined for all sufficiently large n and the sequence $\{R_{n,m,k}^E\}_{n \in \mathbb{N}}$ converges uniformly to $z^k F$ inside $D_{\rho_m(z^k F)} \setminus \{\lambda_1, \lambda_2, \dots, \lambda_m\}$ as $n \rightarrow \infty$. Moreover, for any compact subset K of $D_{\rho_m(z^k F)} \setminus \{\lambda_1, \lambda_2, \dots, \lambda_m\}$,

$$\limsup_{n \rightarrow \infty} \|z^k F - R_{n,m,k}^E\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_m(z^k F)},$$

where $\|\cdot\|_K$ denotes the sup-norm on K and if $K \subset E$, then $\|\Phi\|_K$ is replaced by 1.

Here and in what follows, the phrase “uniformly inside a domain” means “uniformly on each compact subset of the domain”.

The following corollary is an analogue of Corollary 2 in [4].

Corollary 2.3. Let $k \in \mathbb{N}_0$ be fixed and $F \in \mathcal{H}(E)$. Denote by $D_{\rho_\infty(z^k F)}$ the maximal canonical domain in which $z^k F$ can be continued to a meromorphic function. Assume that

$$\lim_{n \rightarrow \infty} m_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{m_n \ln n}{n} = 0.$$

Then, for any $\beta > 0$, each sequence $\{R_{n,m_n,k}^E\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to $z^k F$ inside $D_{\rho_\infty(z^k F)}$ as $n \rightarrow \infty$.

3 Notation and Auxiliary Results

For each $n \in \mathbb{N}$, let Q_{n,m_n}^E be the polynomial q_{n,m_n}^E normalized in terms of its zeros $\lambda_{n,j}$ so that

$$Q_{n,m_n}^E(z) := \prod_{|\lambda_{n,j}| \leq 1} (z - \lambda_{n,j}) \prod_{|\lambda_{n,j}| > 1} \left(1 - \frac{z}{\lambda_{n,j}}\right) \quad (3.1)$$

and for all $k = 0, 1, \dots, m_n - 1$,

$$R_{n,m_n,k}^E = \frac{p_{n,m_n,k}^E}{q_{n,m_n}^E} = \frac{P_{n,m_n,k}^E}{Q_{n,m_n}^E}.$$

Now, we discuss some upper and lower estimates on the normalized Q_{n,m_n}^E in (3.1). Let $\varepsilon > 0$, $d \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $F \in \mathcal{H}(E)$ be fixed. Suppose that the poles of $z^k F$ in $D_{\rho_d(z^k F)}$ are $\lambda_1, \lambda_2, \dots, \lambda_{d'}$ (they are not necessarily distinct and $d' \leq d$) and the zeros of Q_{n,m_n}^E for F are $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,l_{m_n}}$ (they are not necessarily distinct and $l_{m_n} \leq m_n$). We would like to emphasize that since $0 \in E$, for any $k \in \mathbb{N}_0$, $D_{\rho_d(z^k F)} = D_{\rho_d(F)}$ and $\lambda_1, \lambda_2, \dots, \lambda_{d'}$ are exactly all the poles of F in $D_{\rho_d(F)}$. We cover each pole of $z^k F$ in $D_{\rho_d(z^k F)}$ with an open disk of radius $(\varepsilon/(6d))^{1/\beta}$ and denote by $J_{0,\varepsilon}^\beta(F, d)$ the union of these disks. For each $n \in \mathbb{N}$, we cover each zero of Q_{n,m_n}^E with an open disk of radius $(\varepsilon/(6m_n n^2))^{1/\beta}$ and denote by $J_{n,\varepsilon}^\beta(F)$ the union of these disks. Set for each $\ell \in \mathbb{N}$,

$$J_\varepsilon^\beta(F, d; \ell) := J_{0,\varepsilon}^\beta(F, d) \cup \left(\bigcup_{n=\ell}^{\infty} J_{n,\varepsilon}^\beta(F) \right) \quad (3.2)$$

and

$$J_\varepsilon^\beta(F, d) := J_\varepsilon^\beta(F, d; 1).$$

Using the monotonicity and subadditivity of h_β , we have

$$\begin{aligned} h_\beta(J_\varepsilon^\beta(F, d)) &\leq h_\beta(J_{0,\varepsilon}^\beta(F, d)) + \sum_{n=1}^{\infty} h_\beta(J_{n,\varepsilon}^\beta(F)) \\ &\leq \frac{\varepsilon}{6} + \sum_{n=1}^{\infty} \frac{\varepsilon}{6n^2} = \varepsilon \left(\frac{1}{6} + \frac{\pi^2}{6^2} \right) < \varepsilon. \end{aligned}$$

Note that $J_{\varepsilon_1}^\beta(F, d) \subset J_{\varepsilon_2}^\beta(F, d)$ for $\varepsilon_1 < \varepsilon_2$. For any set $B \subset D_{\rho_d(z^k F)}$, we put $B(\varepsilon) := B \setminus J_\varepsilon^\beta(F, d)$. Clearly, if $\{g_n\}_{n \in \mathbb{N}}$ converges uniformly to g on $K(\varepsilon)$ for any compact $K \subset D_{\rho_d(F)}$ and $\varepsilon > 0$, then $h_\beta\text{-}\lim_{n \rightarrow \infty} g_n = g$ in $D_{\rho_d(z^k F)}$.

The normalization of Q_{n,m_n}^E provides the following useful upper and lower bounds on the estimation of Q_{n,m_n}^E .

Lemma 3.1. Fix $k \in \mathbb{N}_0$ and $d \in \mathbb{N}$. Let $F \in \mathcal{H}(E)$, $K \subset D_{\rho_d(z^k F)}$ be a compact set, $\varepsilon > 0$ be fixed, and $\ell \in \mathbb{N}$ be fixed. Suppose that

$$\liminf_{n \rightarrow \infty} m_n \geq d',$$

where d' is the total multiplicity of poles of $z^k F$ in $D_{\rho_d(z^k F)}$, and

$$\lim_{n \rightarrow \infty} \frac{m_n \ln n}{n} = 0.$$

Then, there exist constants $C_1 > 0$ and $C_2 > 0$ independent of n such that for all sufficiently large n ,

$$\|Q_{n,m_n}^E\|_K \leq C_1^{m_n}, \quad (3.3)$$

where $\|\cdot\|_K$ is the sup-norm on K and

$$\min_{z \in K \setminus J_\varepsilon^\beta(F, d; \ell)} |Q_{n,m_n}^E(z)| \geq (C_2 m_n n^2)^{-2m_n/\beta}, \quad (3.4)$$

where the above inequality is meaningful when $K \setminus J_\varepsilon^\beta(F, d; \ell)$ is a nonempty set.

Proof of Lemma 3.1. Without loss of generality, we assume that K is a nonempty compact subset of $D_{\rho_d(z^k F)}$. Moreover, it is easy to check that if $K = \{0\}$, the inequalities (3.3) and (3.4) hold. Then, we can assume further that $K \neq \{0\}$ and set $M := \|z\|_K > 0$. Therefore, there exists $S \in \mathbb{N}$ such that $SM > 1$. From the normalization of Q_{n,m_n}^E ,

$$\|Q_{n,m_n}^E\|_K = \max_{z \in K} \left| \prod_{|\lambda_{n,j}| \leq 1} (z - \lambda_{n,j}) \prod_{|\lambda_{n,j}| > 1} \left(1 - \frac{z}{\lambda_{n,j}}\right) \right| \leq (M+1)^{m_n}$$

and for $z \in K \setminus J_\varepsilon^\beta(F, d; \ell)$ and $n \geq \ell$,

$$\begin{aligned} |Q_{n,m_n}^E(z)| &= \left| \prod_{|\lambda_{n,j}| \leq 1} (z - \lambda_{n,j}) \prod_{|\lambda_{n,j}| > 1} \left(1 - \frac{z}{\lambda_{n,j}}\right) \right| \\ &= \left| \prod_{|\lambda_{n,j}| \leq 1} (z - \lambda_{n,j}) \prod_{1 < |\lambda_{n,j}| \leq SM} \left(1 - \frac{z}{\lambda_{n,j}}\right) \prod_{|\lambda_{n,j}| > SM} \left(1 - \frac{z}{\lambda_{n,j}}\right) \right| \\ &= \left| \prod_{|\lambda_{n,j}| \leq 1} (z - \lambda_{n,j}) \prod_{1 < |\lambda_{n,j}| \leq SM} \left(\frac{\lambda_{n,j} - z}{\lambda_{n,j}}\right) \prod_{|\lambda_{n,j}| > SM} \left(1 - \frac{z}{\lambda_{n,j}}\right) \right| \\ &\geq \prod_{|\lambda_{n,j}| \leq 1} \left(\frac{\varepsilon}{6m_n n^2}\right)^{1/\beta} \prod_{1 < |\lambda_{n,j}| \leq SM} \left[\left(\frac{\varepsilon}{6m_n n^2}\right)^{1/\beta} \frac{1}{SM}\right] \prod_{|\lambda_{n,j}| > SM} \left(1 - \frac{1}{S}\right). \end{aligned} \quad (3.5)$$

Since $(\varepsilon/(6m_n n^2))^{1/\beta} \rightarrow 0$ as $n \rightarrow \infty$, it is easy to see that for n sufficiently large,

$$\left(1 - \frac{1}{S}\right) \geq \left(\frac{\varepsilon}{6m_n n^2}\right)^{1/\beta} \quad \text{and} \quad \frac{1}{SM} \geq \left(\frac{\varepsilon}{6m_n n^2}\right)^{1/\beta}.$$

Therefore, there exists a constant $C_2 > 0$ such that the expression in (3.5) is greater than $(C_2 m_n n^2)^{-(2m_n/\beta)}$. This completes the proof. $\square$

Next, the following lemma (see, e.g., [5]) concerns the formula for computing $\rho_0(F)$ and the domain of convergence of Faber polynomial expansions of holomorphic functions.

Lemma 3.2. *Let $F \in \mathcal{H}(E)$. Then,*

$$\rho_0(F) = \left(\limsup_{n \rightarrow \infty} |[F]_n|^{1/n} \right)^{-1}.$$

Moreover, the series $\sum_{n=0}^{\infty} [F]_n \Phi_n$ converges to F uniformly inside $D_{\rho_0(F)}$.

As a consequence of Lemma 3.2 and Definition 1.1, if $F \in \mathcal{H}(E)$, then for any $k = 0, 1, \dots, m_n$,

$$z^k Q_{n,m_n}^E(z) F(z) - P_{n,m_n,k}^E(z) = \sum_{\ell=n+1}^{\infty} [z^k Q_{n,m_n}^E F]_{\ell} \Phi_{\ell}(z), \quad z \in D_{\rho_0(z^k F)}, \quad (3.6)$$

and $P_{n,m_n,k}^E = \sum_{\ell=0}^{n-1} [z^k Q_{n,m_n}^E F]_{\ell} \Phi_{\ell}$ are uniquely determined by Q_{n,m_n}^E .

The next lemma (see [6, p. 43] or [7, p. 583] for its proof) gives an estimate of Faber polynomials Φ_n on a level curve.

Lemma 3.3. *Let $\rho > 1$ be fixed. Then, there exists $c > 0$ such that*

$$\|\Phi_n\|_{\Gamma_{\rho}} \leq c\rho^n, \quad n \geq 0. \quad (3.7)$$

Indeed, by the maximum modulus principle, the inequalities in (3.7) can be replaced by the inequalities

$$\|\Phi_n\|_{\overline{D}_{\rho}} \leq c\rho^n, \quad n \geq 0, \quad (3.8)$$

which are used frequently in this paper.

The following lemma is about the uniqueness of $Q_{n,m}^E$ (and $q_{n,m}^E$).

Lemma 3.4. *Let $(n, m) \in \mathbb{N} \times \mathbb{N}$ be fixed. Assume that for all $q_{n,m}^E$ in Definition 1.1, $\deg(q_{n,m}^E) = m$. Then, $q_{n,m}^E$ is unique.*

Proof of Lemma 3.4. Let $(n, m) \in \mathbb{N} \times \mathbb{N}$ be fixed. From (1.1) and (1.2) in Definition 1.1, it is easy to check that a polynomial $c_m z^m + c_{m-1} z^{m-1} + \dots + c_0$ is $q_{n,m}^E$ if

and only if $c_m z^m + c_{m-1} z^{m-1} + \dots + c_0$ is monic and the constants $c_m, c_{m-1}, \dots, c_0$ must satisfy the following equation

$$\begin{bmatrix} [z^m F]_n & [z^{m-1} F]_n & \dots & [F]_n \\ [z^{m+1} F]_n & [z^m F]_n & \dots & [zF]_n \\ \vdots & \vdots & \dots & \vdots \\ [z^{2m-1} F]_n & [z^{2m-2} F]_n & \dots & [z^{m-1} F]_n \end{bmatrix} \begin{bmatrix} c_m \\ c_{m-1} \\ \vdots \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (3.9)$$

For contradiction, let us suppose that there are distinct polynomials $\hat{q} = z^m + \hat{c}_{m-1} z^{m-1} + \hat{c}_{m-2} z^{m-2} + \dots + \hat{c}_0$ and $\tilde{q} = z^m + \tilde{c}_{m-1} z^{m-1} + \tilde{c}_{m-2} z^{m-2} + \dots + \tilde{c}_0$ satisfying (3.9). Let $\tilde{q}$ be the polynomial $\hat{q} - \tilde{q}$ normalized to be monic. Clearly, $\deg(\tilde{q}) < m$ and $\tilde{q} \not\equiv 0$ is a monic polynomial where all coefficients satisfying (3.9). Therefore, $\tilde{q}$ is $q_{n,m}^E$. This contradicts with the assumption that for all $q_{n,m}^E$, $\deg(q_{n,m}^E) = m$. $\square$

The final lemma proved by Gonchar (see [4, Lemma 1]) allows us to derive uniform convergence on compact subsets of the region under consideration from convergence in h_1 -content under appropriate assumptions.

Lemma 3.5. *Suppose that $h_1\text{-}\lim_{n \rightarrow \infty} g_n = g$ in D . Then the following assertions hold true:*

- (i) *If the functions $g_n, n \in \mathbb{N}$, are holomorphic in D , then the sequence $\{g_n\}_{n \in \mathbb{N}}$ converges uniformly inside D and g is holomorphic in D .*
- (ii) *If each of the functions g_n is meromorphic in D and has no more than $k < +\infty$ poles in this domain, then the limit function g is also meromorphic and has no more than k poles in D .*
- (iii) *If each function g_n is meromorphic and has no more than $k < +\infty$ poles in D and the function g is meromorphic and has exactly k poles in D , then all $g_n, n \geq N$, also have k poles in D ; the poles of g_n tend to the poles $\lambda_1, \lambda_2, \dots, \lambda_k$ of g (taking account of their orders) and the sequence $\{g_n\}_{n \in \mathbb{N}}$ tends to g uniformly inside the domain $D' = D \setminus \{\lambda_1, \lambda_2, \dots, \lambda_k\}$.*

4 Proofs of main results

Proof of Theorem 2.1. Let $k \in \{0, 1, \dots, m^* - 1\}$ be fixed and d be the number of poles of $z^k F$ (counting multiplicities) in D_ρ (particularly, in $D_{\rho_d(z^k F)}$). For $j = 1, 2, \dots, \gamma$, let α_j be a distinct pole of $z^k F$ in $D_{\rho_d(z^k F)}$, and τ_j be the order of α_j . Note that since $0 \in E$, $D_{\rho_d(z^k F)} = D_{\rho_d(F)}$ and $\alpha_1, \alpha_2, \dots, \alpha_\gamma$ are all the poles of F in $D_{\rho_d(F)}$ with orders $\tau_1, \tau_2, \dots, \tau_\gamma$, respectively.

In the first step, we want to show that for each $j = 1, 2, \dots, \gamma$,

$$\limsup_{n \rightarrow \infty} |(Q_{n,m_n}^E)^{(u)}(\alpha_j)|^{1/n} \leq \frac{|\Phi(\alpha_j)|}{\rho_d(F)}, \quad (4.1)$$

where $u = 0, 1, \dots, \tau_j - 1$. This can be done by induction. Let $j \in \{1, 2, \dots, \gamma\}$ be fixed. Define

$$\omega_d(z) := \prod_{j=1}^{\gamma} (z - \alpha_j)^{\tau_j},$$

where $d = \sum_{j=1}^{\gamma} \tau_j$,

$$G_\ell(z) := \frac{\omega_d(z)F(z)}{(z - \alpha_j)^\ell}, \quad \text{and} \quad H_\ell(z) := (z - \alpha_j)^\ell G_\ell(z),$$

where $\ell = 1, 2, \dots, \tau_j$. Note that $H_\ell(\alpha_j) \neq 0$ for all $\ell = 1, 2, \dots, \tau_j$. By Definition 1.1, since $\deg(\omega_d/(z - \alpha_j)^\ell) = d - \ell \leq m_n - 1$, it is not difficult to check that

$$a_{n,n}^{(\ell)} := [G_\ell Q_{n,m_n}^E]_n = \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{G_\ell(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz = 0, \quad (4.2)$$

where $1 < \rho_1 < |\Phi(\alpha_j)|$. Define

$$\tau_{n,n}^{(\ell)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{G_\ell(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz,$$

where $|\Phi(\alpha_j)| < \rho_2 < \rho_d(F)$.

Because $G_1 Q_{n,m_n}^E \Phi' / \Phi^{n+1}$ is meromorphic on $\{z \in \mathbb{C} : \rho_1 \leq |z| \leq \rho_2\}$ and has a pole at α_j of order at most 1, it follows from Cauchy's Residue theorem to $G_1 Q_{n,m_n}^E \Phi' / \Phi^{n+1}$ at α_j that

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{G_1(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz - \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{G_1(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz \\ &= \text{res} (G_1 Q_{n,m_n}^E \Phi' / \Phi^{n+1}, \alpha_j) \\ &= \lim_{z \rightarrow \alpha_j} \frac{(z - \alpha_j) G_1(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} \\ &= \frac{H_1(\alpha_j) Q_{n,m_n}^E(\alpha_j) \Phi'(\alpha_j)}{\Phi^{n+1}(\alpha_j)}. \end{aligned} \quad (4.3)$$

From (4.2) and (4.3), we have

$$\frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{G_1(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz = \frac{H_1(\alpha_j) Q_{n,m_n}^E(\alpha_j) \Phi'(\alpha_j)}{\Phi^{n+1}(\alpha_j)}, \quad (4.4)$$

and by Lemma 3.1, we know that for all $\ell = 1, 2, \dots, \tau_j$,

$$\left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{G_\ell(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz \right| \leq \frac{c_1 c^{m_n}}{\rho_2^n}, \quad (4.5)$$

where the numbers c and c_1 do not depend on n (from now on, we will denote some constants that do not depend on n by $c_2, c_3, c_4, \dots$). By (4.4) and (4.5), we obtain

$$|Q_{n,m_n}^E(\alpha_j)| \leq \frac{c_2 c^{m_n} |\Phi(\alpha_j)|^n}{\rho_2^n}.$$

Letting $\rho_2 \rightarrow \rho_d(F)$, it is easy to check that

$$\limsup_{n \rightarrow \infty} |Q_{n,m_n}^E(\alpha_j)|^{1/n} \leq \frac{|\Phi(\alpha_j)|}{\rho_d(F)}.$$

Next, we suppose that the inequality (4.1) is true for $u = 0, 1, \dots, \ell - 2$, where $\ell = 2, 3, \dots, \tau_j$, and we will show that the inequality (4.1) holds for $\ell - 1$. Since $G_\ell Q_{n,m_n}^E \Phi' / \Phi^{n+1}$ is meromorphic on $\{z \in \mathbb{C} : \rho_1 \leq |z| \leq \rho_2\}$ and has poles at α_j of order at most ℓ , it follows from Cauchy's Residue theorem to $G_\ell Q_{n,m_n}^E \Phi' / \Phi^{n+1}$ at α_j that

$$\begin{aligned} \tau_{n,n}^{(\ell)} - a_{n,n}^{(\ell)} &= \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{G_\ell(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz - \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{G_\ell(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz \\ &= \text{res} \left(G_\ell Q_{n,m_n}^E \Phi' / \Phi^{n+1}, \alpha_j \right) \\ &= \frac{1}{(\ell-1)!} \lim_{z \rightarrow \alpha_j} \left(\frac{(z - \alpha_j)^\ell G_\ell(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} \right)^{(\ell-1)}. \end{aligned}$$

Using (4.2) and the Leibniz formula, we have

$$\tau_{n,n}^{(\ell)} = \frac{1}{(\ell-1)!} \sum_{t=0}^{\ell-1} \binom{\ell-1}{t} \left(\frac{H_\ell \Phi'}{\Phi^{n+1}} \right)^{(\ell-1-t)} (\alpha_j) (Q_{n,m_n}^E)^{(t)} (\alpha_j).$$

Consequently,

$$\begin{aligned} (Q_{n,m_n}^E)^{(\ell-1)} (\alpha_j) &= (\ell-1)! \tau_{n,n}^{(\ell)} \left(\frac{\Phi^{n+1}}{H_\ell \Phi'} \right) (\alpha_j) \\ &\quad - \sum_{t=0}^{\ell-2} \binom{\ell-1}{t} \left(\frac{H_\ell \Phi'}{\Phi^{n+1}} \right)^{(\ell-1-t)} (\alpha_j) (Q_{n,m_n}^E)^{(t)} (\alpha_j) \left(\frac{\Phi^{n+1}}{H_\ell \Phi'} \right) (\alpha_j). \end{aligned} \quad (4.6)$$

Let $\delta > 0$ such that $\rho_2 := \rho_d(F) - \delta > |\Phi(\alpha_j)|$. Moreover, by (4.5),

$$|\tau_{n,n}^{(\ell)}| = \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{G_\ell(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz \right| \leq \frac{c_1 c^{m_n}}{\rho_2^n}, \quad (4.7)$$

and by Cauchy's integral formula, for all $t = 0, 1, \dots, \ell - 2$,

$$\begin{aligned} \left| \left(\frac{H_\ell \Phi'}{\Phi^{n+1}} \right)^{(\ell-1-t)} (\alpha_j) \right| &= \left| \frac{(\ell-1-t)!}{2\pi i} \int_{|z-\alpha_j|=\varepsilon} \frac{H_\ell(z) \Phi'(z)}{(z - \alpha_j)^{\ell-t} \Phi^{n+1}(z)} dz \right| \\ &\leq \frac{c_2}{(|\Phi(\alpha_j)| - \delta)^n}, \end{aligned} \quad (4.8)$$

where $\{z \in \mathbb{C} : |z - \alpha_j| = \varepsilon\} \subset \{z \in \mathbb{C} : |\Phi(z)| > |\Phi(\alpha_j)| - \delta\}$. From (4.7) and (4.8), the equality (4.6) implies that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} |(Q_{n,m_n}^E)^{(\ell-1)}(\alpha_j)|^{1/n} \\ &= \limsup_{n \rightarrow \infty} \left| (\ell-1)! \tau_{n,n}^{(\ell)} \left(\frac{\Phi^{n+1}}{H_\ell \Phi'} \right) (\alpha_j) \right. \\ & \quad \left. - \sum_{t=0}^{\ell-2} \binom{\ell-1}{t} \left(\frac{H_\ell \Phi'}{\Phi^{n+1}} \right)^{(\ell-1-t)} (\alpha_j) (Q_{n,m_n}^E)^{(t)}(\alpha_j) \left(\frac{\Phi^{n+1}}{H_\ell \Phi'} \right) (\alpha_j) \right|^{1/n} \\ & \leq \max \left\{ \frac{|\Phi(\alpha_j)|}{\rho_2}, \left(\frac{|\Phi(\alpha_j)|}{\rho_d(F)} \right) \left(\frac{|\Phi(\alpha_j)|}{|\Phi(\alpha_j)| - \delta} \right) \right\}. \end{aligned}$$

Letting $\delta \rightarrow 0$, we obtain the inequality

$$\limsup_{n \rightarrow \infty} |(Q_{n,m_n}^E)^{(\ell-1)}(\alpha_j)|^{1/n} \leq \frac{|\Phi(\alpha_j)|}{\rho_d(F)}.$$

Therefore, we have the inequality (4.1) for all $u = 0, 1, \dots, \tau_j - 1$.

From (3.6), we obtain

$$z^k Q_{n,m_n}^E F - P_{n,m_n,k}^E = \sum_{\ell=n+1}^{\infty} a_{\ell,n}^{(k)} \Phi_\ell, \quad (4.9)$$

where

$$a_{\ell,n}^{(k)} := [z^k Q_{n,m_n}^E F]_\ell.$$

Multiplying the equation (4.9) by ω_d and expanding the result in terms of Faber polynomial expansion, we have

$$\begin{aligned} z^k \omega_d Q_{n,m_n}^E F - \omega_d P_{n,m_n,k}^E &= \sum_{\ell=n+1}^{\infty} a_{\ell,n}^{(k)} \omega_d \Phi_\ell = \sum_{\nu=0}^{\infty} b_{\nu,n}^{(k)} \Phi_\nu \\ &= \sum_{\nu=0}^{n+d} b_{\nu,n}^{(k)} \Phi_\nu + \sum_{\nu=n+d+1}^{\infty} b_{\nu,n}^{(k)} \Phi_\nu, \end{aligned} \quad (4.10)$$

where $b_{\nu,n}^{(k)} := \sum_{\ell=n+1}^{\infty} a_{\ell,n}^{(k)} [\omega_d \Phi_\ell]_\nu$ or $b_{\nu,n}^{(k)} := [z^k \omega_d Q_{n,m_n}^E F - \omega_d P_{n,m_n,k}^E]_\nu$.

Let K be a compact subset of $D_{\rho_d(z^k F)}$ and set

$$\sigma := \max\{|\Phi|_K, 1\}$$

($\sigma = 1$ when $K \subset E$). Next, we will estimate $\sum_{\nu=n+d+1}^{\infty} |b_{\nu,n}^{(k)}| |\Phi_\nu(z)|$ on $\overline{D}_\sigma$. Since $\deg(\omega_d P_{n,m_n,k}^E) < d + n$, for all $\nu \geq n + d + 1$,

$$b_{\nu,n}^{(k)} := [z^k \omega_d Q_{n,m_n}^E F - \omega_d P_{n,m_n,k}^E]_\nu = [z^k \omega_d Q_{n,m_n}^E F]_\nu$$

$$= \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{z^k \omega_d(z) Q_{n,m_n}^E(z) F(z) \Phi'(z)}{\Phi^{\nu+1}(z)} dz,$$

where $\sigma < \rho_2 < \rho_d(z^k F)$. From Lemma 3.1, for sufficiently large n , it is easy to see that

$$|b_{\nu,n}^{(k)}| \leq \frac{c_3 c^{m_n}}{\rho_2^\nu}. \quad (4.11)$$

By (3.8) and (4.11), we get

$$\left\| \sum_{\nu=n+d+1}^{\infty} |b_{\nu,n}^{(k)}| |\Phi_\nu| \right\|_{\overline{D}_\sigma} \leq \sum_{\nu=n+d+1}^{\infty} \left(\frac{c_3 c^{m_n}}{\rho_2^\nu} \right) (c_4 \sigma^\nu) = c_5 c^{m_n} \left(\frac{\sigma}{\rho_2} \right)^n. \quad (4.12)$$

Consequently, as $\rho_2 \rightarrow \rho_d(z^k F)$, we have

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=n+d+1}^{\infty} |b_{\nu,n}^{(k)}| |\Phi_\nu| \right\|_{\overline{D}_\sigma}^{1/n} \leq \frac{\sigma}{\rho_d(z^k F)}. \quad (4.13)$$

Now, we find the estimate of $\sum_{\nu=0}^{n+d} |b_{\nu,n}^{(k)}| |\Phi_\nu(z)|$ on $\overline{D}_\sigma$. By Definition 1.1, we know

$$a_{\ell,n}^{(k)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{z^k Q_{n,m_n}^E(z) F(z) \Phi'(z)}{\Phi^{\ell+1}(z)} dz,$$

where $1 < \rho_1 < \rho_0(z^k F)$, and we define

$$\tau_{\ell,n}^{(k)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{z^k Q_{n,m_n}^E(z) F(z) \Phi'(z)}{\Phi^{\ell+1}(z)} dz, \quad (4.14)$$

where $\rho_{d-1}(z^k F) < \rho_2 < \rho_d(z^k F)$. Because $z^k Q_{n,m_n}^E F \Phi' / \Phi^{\ell+1}$ is meromorphic on $\{z \in \mathbb{C} : \rho_1 \leq |z| \leq \rho_2\}$ and has poles at $\alpha_1, \alpha_2, \dots, \alpha_d$ of orders at most $\tau_1, \tau_2, \dots, \tau_d$, respectively, it follows from Cauchy's Residue theorem that

$$\begin{aligned} \tau_{\ell,n}^{(k)} - a_{\ell,n}^{(k)} &= \sum_{j=1}^{\gamma} \operatorname{res} \left(\frac{z^k Q_{n,m_n}^E(z) F(z) \Phi'(z)}{\Phi^{\ell+1}(z)}, \alpha_j \right) \\ &= \sum_{j=1}^{\gamma} \frac{1}{(\tau_j - 1)!} \lim_{z \rightarrow \alpha_j} \left(\frac{(z - \alpha_j)^{\tau_j} z^k Q_{n,m_n}^E(z) F(z) \Phi'(z)}{\Phi^{\ell+1}(z)} \right)^{(\tau_j - 1)} \\ &= \sum_{j=1}^{\gamma} \frac{1}{(\tau_j - 1)!} \sum_{u=0}^{\tau_j - 1} \binom{\tau_j - 1}{u} \left(\frac{(z - \alpha_j)^{\tau_j} z^k F \Phi'}{\Phi^{\ell+1}} \right)^{(\tau_j - 1 - u)} (\alpha_j) (Q_{n,m_n}^E)^{(u)}(\alpha_j). \end{aligned} \quad (4.15)$$

Let $\delta > 0$. By computations similar to (4.7) and (4.8), we have

$$|\tau_{\ell,n}^{(k)}| \leq \frac{c_6 c^{m_n}}{\rho_2^\ell} \quad (4.16)$$

and

$$\left| \left(\frac{(z - \alpha_j)^{\tau_j} z^k F \Phi'}{\Phi^{\ell+1}} \right)^{(\tau_j-1-u)} (\alpha_j) \right| \leq \frac{c_7}{(|\Phi(\alpha_j)| - \delta)^\ell}. \quad (4.17)$$

Moreover, the inequalities (4.1) imply that for all $u = 0, 1, \dots, \tau_j - 1$,

$$|(Q_{n,m_n}^E)^{(u)}(\alpha_j)| \leq c_8 \left(\frac{|\Phi(\alpha_j)| + \delta}{\rho_d(z^k F) + \delta} \right)^n \quad (4.18)$$

(recall that $D_{\rho_d(z^k F)} = D_{\rho_d(F)}$). From (4.15), (4.16), (4.17), and (4.18), we obtain

$$\begin{aligned} |a_{\ell,n}^{(k)}| &\leq |\tau_{\ell,n}^{(k)}| \\ &+ \left| \sum_{j=1}^{\gamma} \frac{1}{(\tau_j - 1)!} \sum_{u=0}^{\tau_j-1} \binom{\tau_j-1}{u} \left(\frac{(z - \alpha_j)^{\tau_j} z^k F \Phi'}{\Phi^{\ell+1}} \right)^{(\tau_j-1-u)} (\alpha_j) (Q_{n,m_n}^E)^{(u)}(\alpha_j) \right| \\ &\leq \frac{c_6 c^{m_n}}{\rho_2^\ell} + \frac{c_9}{(\rho_d(z^k F) + \delta)^n} \sum_{j=1}^{\gamma} \frac{(|\Phi(\alpha_j)| + \delta)^n}{(|\Phi(\alpha_j)| - \delta)^\ell}. \end{aligned}$$

Next, we estimate $|\omega_d \Phi_\ell|_\nu$. Suppose that $\delta > 0$ is sufficiently small so that $\rho_1 - \delta > 1$. Then, by (3.7),

$$|\omega_d \Phi_\ell|_\nu = \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_1-\delta}} \frac{\omega_d(z) \Phi_\ell(z) \Phi'(z)}{\Phi^{\nu+1}(z)} dz \right| \leq \frac{c_{10}(\rho_1 - \delta)^\ell}{(\rho_1 - \delta)^\nu}.$$

Consequently, we get

$$\begin{aligned} |b_{\nu,n}^{(k)}| &\leq \sum_{\ell=n+1}^{\infty} |a_{\ell,n}^{(k)}| |\omega_d \Phi_\ell|_\nu \\ &\leq \sum_{\ell=n+1}^{\infty} \left(\frac{c_6 c^{m_n}}{\rho_2^\ell} + \frac{c_9}{(\rho_d(z^k F) + \delta)^n} \sum_{j=1}^{\gamma} \frac{(|\Phi(\alpha_j)| + \delta)^n}{(|\Phi(\alpha_j)| - \delta)^\ell} \right) \left(\frac{c_{10}(\rho_1 - \delta)^\ell}{(\rho_1 - \delta)^\nu} \right) \\ &= \frac{c_{11} c^{m_n}}{(\rho_1 - \delta)^\nu} \left(\frac{\rho_1 - \delta}{\rho_2} \right)^n + \frac{c_{12}(\rho_1 - \delta)^n}{(\rho_d(z^k F) + \delta)^n (\rho_1 - \delta)^\nu} \sum_{j=1}^{\gamma} \left(\frac{|\Phi(\alpha_j)| + \delta}{|\Phi(\alpha_j)| - \delta} \right)^n. \end{aligned} \quad (4.19)$$

Applying (3.8) and (4.19), we have

$$\begin{aligned} &\sum_{\nu=0}^{n+d} |b_{\nu,n}^{(k)}| \|\Phi_\nu\|_{\overline{D}_\sigma} \\ &\leq \left(c_{13} c^{m_n} \left(\frac{\rho_1 - \delta}{\rho_2} \right)^n + \frac{c_{14}(\rho_1 - \delta)^n}{(\rho_d(z^k F) + \delta)^n} \sum_{j=1}^{\gamma} \left(\frac{|\Phi(\alpha_j)| + \delta}{|\Phi(\alpha_j)| - \delta} \right)^n \right) \sum_{\nu=0}^{n+d} \left(\frac{\sigma}{(\rho_1 - \delta)} \right)^\nu \end{aligned}$$

$$\begin{aligned}
&\leq \left(c_{13} c^{m_n} \left(\frac{\rho_1 - \delta}{\rho_2} \right)^n + \frac{c_{14} (\rho_1 - \delta)^n}{(\rho_d(z^k F) + \delta)^n} \sum_{j=1}^{\gamma} \left(\frac{|\Phi(\alpha_j)| + \delta}{|\Phi(\alpha_j)| - \delta} \right)^n \right) \sum_{\nu=0}^{n+d} \sigma^\nu \\
&\leq \left(c_{13} c^{m_n} \left(\frac{\rho_1 - \delta}{\rho_2} \right)^n + \frac{c_{14} (\rho_1 - \delta)^n}{(\rho_d(z^k F) + \delta)^n} \sum_{j=1}^{\gamma} \left(\frac{|\Phi(\alpha_j)| + \delta}{|\Phi(\alpha_j)| - \delta} \right)^n \right) (n + d + 1) \sigma^{n+d}.
\end{aligned} \tag{4.20}$$

This implies that

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=0}^{n+d} |b_{\nu,n}^{(k)}| |\Phi_\nu| \right\|_{\overline{D}_\sigma}^{1/n} \leq \max \left\{ \frac{\sigma(\rho_1 - \delta)}{\rho_2}, \frac{\sigma(\rho_1 - \delta)}{\rho_d(z^k F) + \delta} \max_{j=1, \dots, \gamma} \left(\frac{|\Phi(\alpha_j)| + \delta}{|\Phi(\alpha_j)| - \delta} \right) \right\}.$$

Letting $\delta \rightarrow 0$, $\rho_1 \rightarrow 1^+$, and $\rho_2 \rightarrow \rho_d(z^k F)$, we have

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=0}^{n+d} |b_{\nu,n}^{(k)}| |\Phi_\nu| \right\|_{\overline{D}_\sigma}^{1/n} \leq \frac{\sigma}{\rho_d(z^k F)}. \tag{4.21}$$

Finally, by (3.4), (4.10), (4.13) and (4.21), we obtain for sufficiently large ℓ ,

$$\begin{aligned}
&\limsup_{n \rightarrow \infty} \|z^k F - R_{n,m_n,k}^E\|_{\overline{D}_\sigma \setminus J_\varepsilon^\beta(F,d;\ell)}^{1/n} \\
&\leq \limsup_{n \rightarrow \infty} \left\| \sum_{\nu=0}^{n+d} \frac{b_{\nu,n}^{(k)} \Phi_\nu}{w_d Q_{n,m_n}^E} + \sum_{\nu=n+d+1}^{\infty} \frac{b_{\nu,n}^{(k)} \Phi_\nu}{w_d Q_{n,m_n}^E} \right\|_{\overline{D}_\sigma \setminus J_\varepsilon^\beta(F,d;\ell)}^{1/n}, \\
&\leq \frac{\sigma}{\rho_d(z^k F)} \cdot \limsup_{n \rightarrow \infty} \left(\frac{1}{\min_{z \in K \setminus J_\varepsilon^\beta(F,d;\ell)} |Q_{n,m_n}^E(z)|} \right)^{1/n} \\
&\leq \frac{\sigma}{\rho_d(z^k F)} \cdot \limsup_{n \rightarrow \infty} (c_{15} m_n n^2)^{\frac{2m_n}{n\beta}} = \frac{\sigma}{\rho_d(z^k F)},
\end{aligned} \tag{4.22}$$

where $c_{15} > 0$ and the last equality follows from the limit condition (2.2). Therefore, for any $\beta > 0$, $h_\beta\text{-}\lim_{n \rightarrow \infty} R_{n,m_n,k}^E = z^k F$ in $D_{\rho_d(z^k F)}$. Since $D_\rho \subset D_{\rho_d(z^k F)}$, $h_\beta\text{-}\lim_{n \rightarrow \infty} R_{n,m_n,k}^E = z^k F$ in D_ρ . $\square$

Proof of Corollary 2.2. Let $k \in \{0, 1, \dots, m-1\}$ be fixed. By the assumption of Corollary 2.2, we have $m_n = m$. Then, the conditions (2.1) and (2.2) in Theorem 2.1 are obtained. By Theorem 2.1, we get $h_1\text{-}\lim_{n \rightarrow \infty} R_{n,m_n,k}^E = z^k F$ in $D_{\rho_d(z^k F)}$. Applying (iii) in Lemma 3.5, we get that each pole of $z^k F$ in $D_{\rho_m(z^k F)}$ attracts as many zeros of $Q_{n,m}^E$ as its order. Therefore, since $z^k F$ has m poles in $D_{\rho_m(z^k F)}$, $\deg Q_{n,m}^E = m$ for all sufficiently large n . Applying Lemma 3.4, $Q_{n,m}^E$ is unique for

all such n . From the discussion below (3.6), since $P_{n,m,k}^E$ is uniquely determined by $Q_{n,m}^E$, $R_{n,m,k}^E$ is also unique for all such n .

Let $K \subset D_{\rho_d(z^k F)} \setminus \{\lambda_1, \lambda_2, \dots, \lambda_m\}$ be a compact set. Choose $\sigma := \max\{\|\Phi\|_K, 1\}$. Since all points $\lambda_1, \lambda_2, \dots, \lambda_m$ attract all zeros of $Q_{n,m}^E$, for sufficiently small $\epsilon > 0$ and large ℓ ,

$$K \subset \overline{D}_\sigma \setminus J_\epsilon^\beta(F, d; \ell).$$

By the inequality (4.22), we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|z^k F - R_{n,m,k}^E\|_K^{1/n} &\leq \limsup_{n \rightarrow \infty} \|z^k F - R_{n,m,k}^E\|_{\overline{D}_\sigma \setminus J_\epsilon^\beta(F, d; \ell)}^{1/n} \\ &\leq \frac{\sigma}{\rho_d(z^k F)}. \end{aligned}$$

This implies that the sequence $\{R_{n,m,k}^E\}_{n \in \mathbb{N}}$ converges uniformly to $z^k F$ inside $D_{\rho_m(z^k F)} \setminus \{\lambda_1, \lambda_2, \dots, \lambda_m\}$ as $n \rightarrow \infty$. The proof is completed. $\square$

Proof of Corollary 2.3. Let K be a compact subset of $D_{\rho_\infty(z^k F)}$, and let $\varepsilon > 0$, $\beta > 0$, and $k \in \mathbb{N}_0$ be fixed. Then, since K is compact, $K \subset D_{\rho_d(z^k F)}$ for some $d \in \mathbb{N}$. Clearly, $\lim_{n \rightarrow \infty} m_n \geq d$. Applying Theorem 2.1, because $h_\beta\text{-}\lim_{n \rightarrow \infty} R_{n,m_n,k}^E = z^k F$ in $D_{\rho_d(z^k F)}$,

$$\lim_{n \rightarrow \infty} h_\beta\{z \in K : |R_{n,m_n,k}^E(z) - z^k F(z)| > \varepsilon\} = 0.$$

This completes the proof. $\square$

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Direct and Inverse Results on Row Sequences of Simultaneous Padé–Faber Approximants

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Abstract. Given a vector function $\mathbf{F} = (F_1, \dots, F_d)$, analytic on a neighborhood of some compact subset E of the complex plane with simply connected complement, we define a sequence of vector rational functions with common denominator in terms of the expansions of the components F_k , $k = 1, \dots, d$, with respect to the sequence of Faber polynomials associated with E . Such sequences of vector rational functions are analogous to row sequences of type II Hermite–Padé approximation. We give necessary and sufficient conditions for the convergence with geometric rate of the common denominators of the sequence of vector rational functions so constructed. The exact rate of convergence of these denominators is provided and the rate of convergence of the approximants is estimated. It is shown that the common denominators of the approximants detect the poles of the system of functions “closest” to E and their order.

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1. Introduction

Throughout the paper, E denotes a compact subset of the complex plane $\mathbb{C}$, which does not reduce to one point, such that $\mathbb{C} \setminus E$ is simply connected. There exists a unique conformal representation Φ from $\mathbb{C} \setminus E$ onto the complement of the unit circle, such that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. It is well known

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that $\Phi'(\infty) = 1/\text{cap}(E)$, where $\text{cap}(E)$ is the logarithmic capacity of E . The *Faber polynomial* of E of degree n is defined as the polynomial part of the Laurent expansion of Φ^n at infinity.

Faber polynomials have been a subject of major interest in complex analysis due to their close connection with conformal mappings and the logarithmic capacity. Their general properties and use in the approximation of functions analytic on a neighborhood of E are well covered in [13]. For compact sets with special configurations (m-cusped hypocycloid, starlike sets, circular lunes, and arcs), they are known and numerical methods for their computation have been proposed (see [12]). For recent results on their asymptotic behavior, see also [1, 10].

The object of this paper is to prove a Montessus de Ballore–Gonchar type theorem for simultaneous Padé–Faber approximants analogous to the one obtained in [7] in the context of Hermite–Padé approximation. Such results, motivated in [9], include a direct part where convergence of the approximants and their poles is derived provided that the functions being approximated have convenient analytic properties, and an inverse statement in which, starting out from the asymptotic properties of the poles of the approximants, some important analytic properties of the functions being approximated are determined. For scalar functions, several approximating models have been explored which, in one way or another, extend the notion of Padé approximation; for example, see [5, 9, 14]. In the introduction of [5, 7, 11], you can find an account on the history of the problem. In [4], we studied a similar problem when the approximants are built on the basis of orthogonal expansions. Here, we expand in terms of Faber polynomials which, as mentioned above, are well known for a wide range of compact sets.

Let us clarify what we understand as a pole of a vector function and its order.

Definition 1.1. Let $\Omega := (\Omega_1, \Omega_2, \dots, \Omega_d)$ be a system of domains, such that, for each $\alpha = 1, 2, \dots, d$, F_α is meromorphic in Ω_α . We say that the point λ is a pole of $\mathbf{F} := (F_1, F_2, \dots, F_d)$ in Ω of order τ if there exists an index $\alpha \in \{1, 2, \dots, d\}$, such that $\lambda \in \Omega_\alpha$ and it is a pole of F_α of order τ , and for $\beta \neq \alpha$, either λ is a pole of F_β of order less than or equal to τ or $\lambda \notin \Omega_\beta$. When $\Omega = (\Omega, \Omega, \dots, \Omega)$, we say that λ is a pole of $\mathbf{F}$ in Ω .

Let E be a compact subset as described above and Φ the associated conformal map. For each $\rho > 1$, we define

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\} \quad \text{and} \quad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\},$$

as the *level curve of index ρ* and the *canonical domain of index ρ* , respectively. Denote by $\mathcal{H}(E)$ the space of all functions holomorphic in some neighborhood of E and

$$\mathcal{H}(E)^d := \{(F_1, F_2, \dots, F_d) : F_\alpha \in \mathcal{H}(E) \text{ for all } \alpha = 1, 2, \dots, d\}.$$

Let $\mathbf{F} \in \mathcal{H}(E)^d$. Denote by $\rho_0(\mathbf{F})$ the index ρ of the largest canonical domain D_ρ to which all F_α , $\alpha = 1, \dots, d$, can be extended as holomorphic functions and by $\rho_m(\mathbf{F})$ the index ρ of the largest canonical domain D_ρ to

which all F_α , $\alpha = 1, \dots, d$ can be extended, so that $\mathbf{F}$ has at most m poles counting multiplicities.

From the definition given above, it is easy to see that the *Faber polynomial* of E of degree n is given by the formula:

$$\Phi_n(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\Phi^n(t)}{t-z} dt, \quad z \in D_\rho, \quad n = 0, 1, 2, \dots \quad (1.1)$$

Notice that

$$\Phi_n(z) = (z/\text{cap}(E))^n + \text{lower degree terms.} \quad (1.2)$$

The n th *Faber coefficient* of $G \in \mathcal{H}(E)$ with respect to Φ_n is given by the following:

$$[G]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{G(t)\Phi'(t)}{\Phi^{n+1}(t)} dt,$$

where $\rho \in (1, \rho_0(G))$ and $\rho_0(G)$ denotes the index of the largest canonical region to which G can be extended as a holomorphic function. For an account on Faber polynomials and its properties, see [13]. In particular, it is well known that

$$\lim_{n \rightarrow \infty} |\Phi_n(z)|^{1/n} = |\Phi(z)|, \quad (1.3)$$

uniformly on compact subsets of $\mathbb{C} \setminus E$.

Let us introduce simultaneous Padé–Faber approximants.

Definition 1.2. Let $\mathbf{F} = (F_1, \dots, F_d) \in \mathcal{H}(E)^d$. Fix $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}$. Set $|\mathbf{m}| := m_1 + m_2 + \dots + m_d$. Then, there exist polynomials $Q_{n,\mathbf{m}}$, $P_{n,\mathbf{m},k,\alpha}$, such that

$$\deg P_{n,\mathbf{m},k,\alpha} \leq n-1, \quad \deg(Q_{n,\mathbf{m}}) \leq |\mathbf{m}|, \quad Q_{n,\mathbf{m}} \not\equiv 0, \quad (1.4)$$

$$[Q_{n,\mathbf{m}} z^k F_\alpha - P_{n,\mathbf{m},k,\alpha}]_j = 0, \quad j = 0, 1, \dots, n \quad (1.5)$$

for all $k = 0, 1, \dots, m_\alpha - 1$ and $\alpha = 1, 2, \dots, d$. The vector of rational functions

$$\mathbf{R}_{n,\mathbf{m}} := (R_{n,\mathbf{m},1}, \dots, R_{n,\mathbf{m},d}) = (P_{n,\mathbf{m},0,1}/Q_{n,\mathbf{m}}, \dots, P_{n,\mathbf{m},0,d}/Q_{n,\mathbf{m}})$$

is called an $(n, \mathbf{m})$ *simultaneous Padé–Faber approximant* of $\mathbf{F}$.

Clearly:

$$[Q_{n,\mathbf{m}} z^k F_\alpha]_n = 0, \quad \alpha = 1, \dots, d, \quad k = 0, 1, \dots, m_\alpha - 1. \quad (1.6)$$

Since $Q_{n,\mathbf{m}} \not\equiv 0$, we normalize it with leading coefficient equal 1. We call $Q_{n,\mathbf{m}}$ *the denominator of the $(n, \mathbf{m})$ simultaneous Padé–Faber approximant of $\mathbf{F}$* .

We would like to emphasize that the solutions of the homogeneous linear system of $|\mathbf{m}|$ Eq. (1.6) on the $|\mathbf{m}|+1$ unknown coefficients of $Q_{n,\mathbf{m}}$ determine the possible denominators of the Padé–Faber approximants. Once a $Q_{n,\mathbf{m}}$ is given, for each $k = 0, 1, \dots, m_\alpha - 1$ and $\alpha = 1, 2, \dots, d$, the polynomial $P_{n,\mathbf{m},k,\alpha}$ is uniquely determined. However, the linear system (1.6) can have linearly independent solutions which may give rise to distinct Padé–Faber approximants for a given n and $\mathbf{m}$. This effect is well known already in

the context of Hermite–Padé approximation (but does not occur in Padé approximation).

Finding a solution of (1.4)–(1.5) reduces to solving a homogeneous system of $(n+1)|\mathbf{m}|$ linear equations on $(n+1)|\mathbf{m}|+1$ coefficients of $Q_{n,\mathbf{m}}$ and $P_{n,\mathbf{m},k,\alpha}$. Therefore, for any pair $(n, \mathbf{m}) \in \mathbb{N} \times \mathbb{N}^d$, a vector of rational functions $\mathbf{R}_{n,\mathbf{m}}$ always exists. As already mentioned, it may not be unique. For each n , we choose one solution. The definition of simultaneous Padé–Faber approximants employed here differs from the one used in [2] which may seem more natural, but has serious inconveniences for proving inverse-type results.

Notice that (1.5) implies that linear combinations of the functions $z^k F_\alpha$, $0 \leq k < m_\alpha$, $\alpha = 1, \dots, d$ also verify (1.5) (with the same $Q_{n,\mathbf{m}}$ and convenient polynomial P , $\deg P < n$). This motivates the concept of system pole. System poles may not coincide with the poles of the individual functions F_α (see examples in [7]).

Definition 1.3. Given $\mathbf{F} = (F_1, \dots, F_d) \in \mathcal{H}(E)^d$ and $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$, we say that $\xi \in \mathbb{C}$ is a *system pole of order τ of $\mathbf{F}$ with respect to $\mathbf{m}$* if τ is the largest positive integer, such that, for each $t = 1, 2, \dots, \tau$, there exists at least one polynomial combination of the form:

$$\sum_{\alpha=1}^d v_\alpha F_\alpha, \quad \deg(v_\alpha) < m_\alpha, \quad \alpha = 1, 2, \dots, d, \quad (1.7)$$

which is holomorphic in a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole at $z = \xi$ of exact order t .

To each system pole ξ of $\mathbf{F}$ with respect to $\mathbf{m}$, we associate several characteristic values. Let τ be the order of ξ as a system pole of $\mathbf{F}$. For each $t = 1, \dots, \tau$, denote by $\rho_{\xi,t}(\mathbf{F}, \mathbf{m})$ the largest of all the numbers $\rho_t(G)$ (the index of the largest canonical domain containing at most t poles of G), where G is a polynomial combination of type (1.7) that is holomorphic in a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole at $z = \xi$ of order t . There is only a finite number of such possible values, so the maximum is indeed attained. Then, we define the following:

$$\begin{aligned} \rho_{\xi,t}(\mathbf{F}, \mathbf{m}) &:= \min_{k=1,\dots,t} \rho_{\xi,k}(\mathbf{F}, \mathbf{m}), \\ \rho_\xi(\mathbf{F}, \mathbf{m}) &:= \rho_{\xi,\tau}(\mathbf{F}, \mathbf{m}) = \min_{k=1,\dots,\tau} \rho_{\xi,k}(\mathbf{F}, \mathbf{m}). \end{aligned}$$

Fix $\alpha \in \{1, \dots, d\}$. Let $D_\alpha(\mathbf{F}, \mathbf{m})$ be the largest canonical domain in which all the poles of F_α are system poles of $\mathbf{F}$ with respect to $\mathbf{m}$, their order as poles of F_α does not exceed their order as system poles, and F_α has no other singularity. By $\rho_\alpha(\mathbf{F}, \mathbf{m})$, we denote the index of this canonical domain. Let $\xi_1, \dots, \xi_N$ be the poles of F_α in $D_\alpha(\mathbf{F}, \mathbf{m})$. For each $j = 1, \dots, N$, let $\hat{\tau}_j$ be the order of ξ_j as a pole of F_α and τ_j its order as a system pole. By assumption, $\hat{\tau}_j \leq \tau_j$. Set

$$\rho_\alpha^*(\mathbf{F}, \mathbf{m}) := \min \left\{ \rho_\alpha(\mathbf{F}, \mathbf{m}), \min_{j=1,\dots,N} \rho_{\xi_j, \hat{\tau}_j}(\mathbf{F}, \mathbf{m}) \right\},$$

and let $D_\alpha^*(\mathbf{F}, \mathbf{m})$ be the canonical domain with this index. For convenience, in the sequel, we assume that $0 \in E$ (E is the set where all the functions F_α are holomorphic); consequently, for a fixed α if we were to define an analogous quantity for $z^k F_\alpha$, we would obtain the same number $\rho_\alpha^*(\mathbf{F}, \mathbf{m})$ independently of k .

By $Q_{\mathbf{m}}^{\mathbf{F}}$, we denote the monic polynomial whose zeros are the system poles of $\mathbf{F}$ with respect to $\mathbf{m}$ taking into account their order. The set of distinct zeros of $Q_{\mathbf{m}}^{\mathbf{F}}$ is denoted by $\mathcal{P}(\mathbf{F}, \mathbf{m})$.

We are ready to state the direct result.

Theorem 1.4. *Let $\mathbf{F} = (F_1, \dots, F_d) \in \mathcal{H}(E)^d$ and let $\mathbf{m} \in \mathbb{N}^d$ be a fixed multi-index. Suppose that $\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ counting multiplicities. Then, for all sufficiently large n , the polynomials $Q_{n,\mathbf{m}}$ and the approximants $R_{n,\mathbf{m},\alpha}$ are uniquely determined:*

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}} - Q_{\mathbf{m}}^{\mathbf{F}}\|^{1/n} = \max \left\{ \frac{|\Phi(\xi)|}{\rho_\xi^*(\mathbf{F}, \mathbf{m})} : \xi \in \mathcal{P}(\mathbf{F}, \mathbf{m}) \right\}, \quad (1.8)$$

where $\|\cdot\|$ denotes the coefficient norm in the space of polynomials. For any $\alpha = 1, \dots, d$, $k = 1, \dots, m_\alpha - 1$, and any compact subset K of $D_\alpha^*(\mathbf{F}, \mathbf{m}) \setminus \mathcal{P}(\mathbf{F}, \mathbf{m})$:

$$\limsup_{n \rightarrow \infty} \left\| \frac{P_{n,\mathbf{m},k,\alpha}}{Q_{n,\mathbf{m}}} - z^k F_\alpha \right\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_\alpha^*(\mathbf{F}, \mathbf{m})}, \quad (1.9)$$

where $\|\cdot\|_K$ denotes the sup-norm on K . If $K \subset E$, $\|\Phi\|_K$ is replaced by 1.

In the inverse direction, we have the following.

Theorem 1.5. *Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ and $\mathbf{m} \in \mathbb{N}^d$ be a fixed multi-index. Suppose that the polynomials $Q_{n,\mathbf{m}}$ are uniquely determined for all sufficiently large n and there exists a polynomial $Q_{|\mathbf{m}|}$ of degree $|\mathbf{m}|$, such that*

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}} - Q_{|\mathbf{m}|}\|^{1/n} = \theta < 1.$$

Then, $\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ counting multiplicities and $Q_{|\mathbf{m}|} = Q_{\mathbf{m}}^{\mathbf{F}}$.

An immediate consequence of Theorems 1.4 and 1.5 is the following corollary which is the analog of the Montessus de Ballore–Gonchar theorem for simultaneous Padé–Faber approximation.

Corollary 1.6. *Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ and $\mathbf{m} \in \mathbb{N}^d$ be a fixed multi-index. Then, the following assertions are equivalent:*

- (a) $\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ counting multiplicities.
- (b) The polynomials $Q_{n,\mathbf{m}}$ of $\mathbf{F}$ are uniquely determined for all sufficiently large n and there exists a polynomial $Q_{|\mathbf{m}|}$ of degree $|\mathbf{m}|$, such that

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}} - Q_{|\mathbf{m}|}\|^{1/n} = \theta < 1.$$

Consequently, if either (a) or (b) takes place, then $Q_{|\mathbf{m}|} = Q_{\mathbf{m}}^{\mathbf{F}}$, and (1.8)–(1.9) hold.

The outline of this paper is as follows. Section 2 contains the proof of Theorem 1.4. The proof of Theorem 1.5 is in Sect. 3.

2. Proof of Theorem 1.4

2.1. Auxiliary Lemmas

The following lemma (see [13]) is obtained using (1.3) the same way as similar statements are proved for Taylor series.

Lemma 2.1. *Let $G \in \mathcal{H}(E)$. Then:*

$$\rho_0(G) = \left(\limsup_{n \rightarrow \infty} |[G]_n|^{1/n} \right)^{-1}. \quad (2.1)$$

Moreover, $\sum_{n=0}^{\infty} [G]_n \Phi_n(z)$ converges to $G(z)$ uniformly inside $D_{\rho_0(G)}$.

Here and in what follows, the phrase “uniformly inside a domain” means “uniformly on each compact subset of the domain”.

As a consequence of Lemma 2.1, if $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$, then for each $\alpha = 1, 2, \dots, d$ and $k = 0, 1, \dots, m_\alpha - 1$ fixed:

$$z^k Q_{n,\mathbf{m}}(z) F_\alpha(z) - P_{n,\mathbf{m},k,\alpha}(z) = \sum_{\ell=n+1}^{\infty} [z^k Q_{n,\mathbf{m}} F_\alpha]_\ell \Phi_\ell(z), \quad z \in D_{\rho_0(F_\alpha)}, \quad (2.2)$$

and $P_{n,\mathbf{m},k,\alpha} = \sum_{\ell=0}^{n-1} [z^k Q_{n,\mathbf{m}} F_\alpha]_\ell \Phi_\ell$ is uniquely determined by $Q_{n,\mathbf{m}}$.

The next lemma (see [8, p. 583] or [13, p. 43] for its proof) gives an estimate of Faber polynomials Φ_n on a level curve.

Lemma 2.2. *Let $\rho > 1$ be fixed. Then, there exists $c > 0$, such that*

$$\|\Phi_n\|_{\Gamma_\rho} \leq c\rho^n, \quad n \geq 0. \quad (2.3)$$

2.2. Proof of Theorem 1.4

Proof of Theorem 1.4. For each $n \in \mathbb{N}$, let $q_{n,\mathbf{m}}$ be the polynomial $Q_{n,\mathbf{m}}$ normalized, so that

$$q_{n,\mathbf{m}}(z) = \sum_{k=0}^{|\mathbf{m}|} \lambda_{n,k} z^k, \quad \sum_{k=0}^{|\mathbf{m}|} |\lambda_{n,k}| = 1. \quad (2.4)$$

With this normalization, the polynomials $q_{n,\mathbf{m}}$ are uniformly bounded on each compact subset of $\mathbb{C}$.

Let ξ be a system pole of order τ of $\mathbf{F}$ with respect to $\mathbf{m}$. We show that

$$\limsup_{n \rightarrow \infty} |q_{n,\mathbf{m}}^{(j)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_{\xi,j+1}(\mathbf{F}, \mathbf{m})}, \quad j = 0, 1, \dots, \tau - 1. \quad (2.5)$$

Fix $\ell \in \{1, \dots, \tau\}$. Consider a polynomial combination of G_ℓ of the type (1.7) that is holomorphic in a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole of order ℓ at $z = \xi$ and verifies that $\rho_\ell(G_\ell) = \rho_{\xi,\ell}(\mathbf{F}, \mathbf{m})$. Then, we have the following:

$$G_\ell = \sum_{\alpha=1}^d v_{\alpha,\ell} F_\alpha, \quad \deg v_{\alpha,\ell} < m_\alpha, \quad \alpha = 1, 2, \dots, d.$$

Set

$$H_\ell(z) := (x - \xi)^\ell G_\ell(z) \quad \text{and} \quad a_{n,n}^{(\ell)} := [q_{n,\mathbf{m}} G_\ell]_n.$$

By the definition of $Q_{n,\mathbf{m}}$, it follows that $a_{n,n}^{(\ell)} = 0$. Therefore:

$$a_{n,n}^{(\ell)} = [q_{n,\mathbf{m}} G_\ell]_n = \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{q_{n,\mathbf{m}}(z) G_\ell(z) \Phi'(z)}{\Phi^{n+1}(z)} dz = 0,$$

where $1 < \rho_1 < |\Phi(\xi)|$. Set

$$\tau_{n,n}^{(\ell)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{q_{n,\mathbf{m}}(z) G_\ell(z) \Phi'(z)}{\Phi^{n+1}(z)} dz,$$

where $|\Phi(\xi)| < \rho_2 < \rho_{\xi,\ell}(\mathbf{F}, \mathbf{m})$. Using Cauchy's residue theorem on the function $(q_{n,\mathbf{m}} G_\ell \Phi')/\Phi^{n+1}$, we obtain the following:

$$\begin{aligned} \tau_{n,n}^{(\ell)} &= \tau_{n,n}^{(\ell)} - a_{n,n}^{(\ell)} = \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{q_{n,\mathbf{m}}(t) G_\ell(t) \Phi'(t)}{\Phi^{n+1}(t)} dt \\ &\quad - \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{q_{n,\mathbf{m}}(t) G_\ell(t) \Phi'(t)}{\Phi^{n+1}(t)} dt \\ &= \text{res}((q_{n,\mathbf{m}} G_\ell \Phi')/\Phi^{n+1}, \xi). \end{aligned} \quad (2.6)$$

Now:

$$\begin{aligned} \text{res}((q_{n,\mathbf{m}} G_\ell \Phi')/\Phi^{n+1}, \xi) &= \frac{1}{(\ell-1)!} \lim_{z \rightarrow \xi} \left(\frac{(z - \xi)^\ell G_\ell(z) \Phi'(z) q_{n,\mathbf{m}}(z)}{\Phi^{n+1}(z)} \right)^{(\ell-1)} \\ &= \frac{1}{(\ell-1)!} \sum_{t=0}^{\ell-1} \binom{\ell-1}{t} \left(\frac{H_\ell \Phi'}{\Phi^{n+1}} \right)^{(\ell-1-t)} (\xi) q_{n,\mathbf{m}}^{(t)}(\xi). \end{aligned} \quad (2.7)$$

Consequently:

$$\begin{aligned} (\ell-1)! \tau_{n,n}^{(\ell)} &= \left(\frac{H_\ell(\xi) \Phi'(\xi)}{\Phi^{n+1}(\xi)} \right) q_{n,\mathbf{m}}^{(\ell-1)}(\xi) \\ &\quad + \sum_{t=0}^{\ell-2} \binom{\ell-1}{t} \left(\frac{H_\ell \Phi'}{\Phi^{n+1}} \right)^{(\ell-1-t)} (\xi) q_{n,\mathbf{m}}^{(t)}(\xi), \end{aligned}$$

where the sum is empty when $\ell = 1$. Therefore:

$$\begin{aligned} q_{n,\mathbf{m}}^{(\ell-1)}(\xi) &= \frac{(\ell-1)! \tau_{n,n}^{(\ell)} \Phi^{n+1}(\xi)}{H_\ell(\xi) \Phi'(\xi)} \\ &\quad - \sum_{t=0}^{\ell-2} \binom{\ell-1}{t} \left(\frac{H_\ell \Phi'}{\Phi^{n+1}} \right)^{(\ell-1-t)} (\xi) \frac{\Phi^{n+1}(\xi) q_{n,\mathbf{m}}^{(t)}(\xi)}{H_\ell(\xi) \Phi'(\xi)}. \end{aligned} \quad (2.8)$$

Choose $\delta > 0$ small enough, so that

$$\rho_2 := \rho_{\xi,\ell}(\mathbf{F}, \mathbf{m}) - \delta > |\Phi(\xi)|. \quad (2.9)$$

We have

$$|\tau_{n,n}^{(\ell)}| = \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{q_{n,\mathbf{m}}(z) G_\ell(z) \Phi'(z)}{\Phi^{n+1}(z)} dz \right| \leq \frac{c_1}{\rho_2^n}. \quad (2.10)$$

If $\ell = 1$, from (2.8) to (2.10), we obtain

$$|q_{n,\mathbf{m}}(\xi)| \leq c_2 \left(\frac{|\Phi(\xi)|}{\rho_2} \right)^n,$$

which implies that

$$\limsup_{n \rightarrow \infty} |q_{n,\mathbf{m}}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_2}.$$

Letting $\delta \rightarrow 0$, (2.5) readily follows for $j = 0$. For the remaining values of j , we use induction.

Suppose that (2.5) is true for $j = 0, \dots, \ell - 2, 2 \leq \ell \leq \tau$ and let us prove that it is also valid for $j = \ell - 1$. Choosing $\delta > 0$ as in (2.9), for $t = 0, 1, \dots, \ell - 2$, we obtain the following:

$$\begin{aligned} \left| \left(\frac{H_\ell \Phi'}{\Phi^{n+1}} \right)^{(\ell-1-t)}(\xi) \right| &= \left| \frac{(\ell-1-t)!}{2\pi i} \int_{|z-\xi|=\varepsilon} \frac{H_\ell(z) \Phi'(z)}{(z-\xi)^{\ell-t} \Phi^{n+1}(z)} dz \right| \\ &\leq \frac{c_3}{(|\Phi(\xi)| - \delta)^n}, \end{aligned} \quad (2.11)$$

where $\{z \in \mathbb{C} : |z - \xi| = \varepsilon\} \subset \{z \in \mathbb{C} : |\Phi(z)| > |\Phi(\xi)| - \delta\}$. Combining the induction hypothesis, (2.8), (2.10), and (2.11), it follows from (2.8) that:

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \left| (q_{n,\mathbf{m}})^{(\ell-1)}(\xi) \right|^{1/n} \\ &= \limsup_{n \rightarrow \infty} \left| \frac{(\ell-1)! \tau_{n,n}^{(\ell)} \Phi^{n+1}(\xi)}{H_\ell(\xi) \Phi'(\xi)} \right. \\ &\quad \left. - \sum_{t=0}^{\ell-2} \binom{\ell-1}{t} \left(\frac{H_\ell \Phi'}{\Phi^{n+1}} \right)^{(\ell-1-t)}(\xi) \frac{\Phi^{n+1}(\xi) (q_{n,\mathbf{m}})^{(t)}(\xi)}{H_\ell(\xi) \Phi'(\xi)} \right|^{1/n} \\ &\leq \max \left\{ \frac{|\Phi(\xi)|}{\rho_2}, \left(\frac{|\Phi(\xi)|}{|\Phi(\xi)| - \delta} \right) \left(\frac{|\Phi(\xi)|}{\rho_{\xi,\ell-1}(\mathbf{F}, \mathbf{m})} \right) \right\}. \end{aligned} \quad (2.12)$$

Letting $\delta \rightarrow 0$, we have $\rho_2 \rightarrow \rho_{\xi,\ell}(\mathbf{F}, \mathbf{m})$ and from (2.12), we obtain the following:

$$\limsup_{n \rightarrow \infty} \left| (q_{n,\mathbf{m}})^{(\ell-1)}(\xi) \right|^{1/n} \leq \max \left\{ \frac{|\Phi(\xi)|}{\rho_{\xi,\ell}(\mathbf{F}, \mathbf{m})}, \frac{|\Phi(\xi)|}{\rho_{\xi,\ell-1}(\mathbf{F}, \mathbf{m})} \right\} \leq \frac{|\Phi(\xi)|}{\rho_{\xi,\ell}(\mathbf{F}, \mathbf{m})},$$

which completes the induction.

Let $\xi_1, \dots, \xi_w$ be the distinct system poles of $\mathbf{F}$ and let τ_j be the order of ξ_j as a system pole, $j = 1, \dots, w$. By assumption, $\tau_1 + \dots + \tau_w = |\mathbf{m}|$. We have proved that, for $j = 1, \dots, w$ and $t = 0, 1, \dots, \tau_j - 1$:

$$\limsup_{n \rightarrow \infty} |q_{n,\mathbf{m}}^{(t)}(\xi_j)|^{1/n} \leq \frac{|\Phi(\xi_j)|}{\rho_{\xi_j,t+1}(\mathbf{F}, \mathbf{m})} \leq \frac{|\Phi(\xi_j)|}{\rho_{\xi_j}(\mathbf{F}, \mathbf{m})}. \quad (2.13)$$

Let $L_{j,t}, j = 1, \dots, w, t = 0, 1, \dots, \tau_j - 1$, be the basis of polynomials of degree $\leq |\mathbf{m}| - 1$ defined by the interpolation conditions:

$$L_{j,t}^{(s)}(\xi_k) = \delta_{j,k} \delta_{t,s}, \quad k = 1, \dots, w, \quad s = 0, \dots, \tau_k - 1.$$

Then

$$q_{n,\mathbf{m}} - \lambda_{n,|\mathbf{m}|} Q_{\mathbf{m}}^{\mathbf{F}} = \sum_{j=1}^w \sum_{t=0}^{\tau_j-1} q_{n,\mathbf{m}}^{(t)}(\xi_j) L_{j,t},$$

where $\lambda_{n,|\mathbf{m}|}$ is the leading coefficient of $q_{n,\mathbf{m}}$. From (2.13), it follows that

$$\limsup_{n \rightarrow \infty} \|q_{n,\mathbf{m}} - \lambda_{n,|\mathbf{m}|} Q_{\mathbf{m}}^{\mathbf{F}}\|_K^{1/n} \leq \max \left\{ \frac{|\Phi(\xi)|}{\rho_{\xi}(\mathbf{F}, \mathbf{m})} : \xi \in \mathcal{P}(\mathbf{F}, \mathbf{m}) \right\}$$

for every compact set $K \subset \mathbb{C}$. In finite-dimensional spaces, all norms are equivalent; therefore:

$$\limsup_{n \rightarrow \infty} \|q_{n,\mathbf{m}} - \lambda_{n,|\mathbf{m}|} Q_{\mathbf{m}}^{\mathbf{F}}\|^{1/n} \leq \max \left\{ \frac{|\Phi(\xi)|}{\rho_{\xi}(\mathbf{F}, \mathbf{m})} : \xi \in \mathcal{P}(\mathbf{F}, \mathbf{m}) \right\}. \quad (2.14)$$

In turn, this implies that

$$\liminf_{n \rightarrow \infty} |\lambda_{n,|\mathbf{m}|}| > 0, \quad (2.15)$$

since, otherwise, for a subsequence of indices Λ , we would have $\lim_{n \in \Lambda} \|q_{n,\mathbf{m}}\| = 0$ which contradicts the normalization imposed on the polynomials $Q_{n,\mathbf{m}}$ (see (2.4)). Combining (2.14) and (2.15), we get (1.8) with $\leq$ in place of $=$.

Now, we know that $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|, n \geq n_0$, since these polynomials converge to a polynomial of degree $|\mathbf{m}|$. In turn, this implies that $Q_{n,\mathbf{m}}$ is uniquely determined for all sufficiently large n , because the difference of any two distinct monic polynomials satisfying Definition 1.2 with the same degree produces a new solution of degree strictly less than $|\mathbf{m}|$, but we have proved that any solution must have degree $|\mathbf{m}|$ for all sufficiently large n . Definition 1.2 implies that $P_{n,\mathbf{m},0,\alpha}$ is determined uniquely through $Q_{n,\mathbf{m}}$; consequently, $R_{n,\mathbf{m},\alpha}$ is uniquely determined for all large enough n .

Now, we prove the equality in (1.8). To the contrary, suppose that

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}} - Q_{\mathbf{m}}^{\mathbf{F}}\|^{1/n} = \theta < \max \left\{ \frac{|\Phi(\xi)|}{\rho_{\xi}(\mathbf{F}, \mathbf{m})} : \xi \in \mathcal{P}(\mathbf{F}, \mathbf{m}) \right\}. \quad (2.16)$$

Let ζ be a system pole of $\mathbf{F}$, such that

$$\frac{|\Phi(\zeta)|}{\rho_{\zeta}(\mathbf{F}, \mathbf{m})} = \max \left\{ \frac{|\Phi(\xi)|}{\rho_{\xi}(\mathbf{F}, \mathbf{m})} : \xi \in \mathcal{P}(\mathbf{F}, \mathbf{m}) \right\}.$$

Clearly, the inequality (2.16) implies that $\rho_{\zeta}(\mathbf{F}, \mathbf{m}) < \infty$.

Choose a polynomial combination

$$G = \sum_{\alpha=1}^d v_{\alpha} F_{\alpha}, \quad \deg v_{\alpha} < m_{\alpha}, \quad \alpha = 1, 2, \dots, d, \quad (2.17)$$

that is holomorphic in a neighborhood of $\overline{D}_{|\Phi(\zeta)|}$ except for a pole of order s at $z = \zeta$ with $\rho_s(G) = \rho_\zeta(\mathbf{F}, \mathbf{m})$. Notice that $Q_{\mathbf{m}}^{\mathbf{F}}G$ must have a singularity on the boundary of $D_{\rho_s}(G)$ which implies that

$$\limsup_{n \rightarrow \infty} |[Q_{\mathbf{m}}^{\mathbf{F}}G]_n|^{1/n} = \frac{1}{\rho_\zeta(\mathbf{F}, \mathbf{m})}. \quad (2.18)$$

In fact, if $Q_{\mathbf{m}}^{\mathbf{F}}G$ had no singularity on the boundary of $D_{\rho_s}(G)$, then all singularities of G on the boundary of $D_{\rho_s}(G)$ would be at most poles and their order as poles of G would be smaller than their order as system poles of $\mathbf{F}$. In this case, we could find a different polynomial combination G_1 of type (2.17) for which $\rho_s(G_1) > \rho_s(G) = \rho_\zeta(\mathbf{F}, \mathbf{m})$ which contradicts the definition of $\rho_\zeta(\mathbf{F}, \mathbf{m})$. Therefore, $Q_{\mathbf{m}}^{\mathbf{F}}G$ has a singularity on the boundary of $D_{\rho_s}(G)$ and the equality (2.18) holds.

Choose $1 < \rho < |\Phi(\zeta)|$. Then, by the definition of $Q_{n,\mathbf{m}}$, (2.16), and (2.18):

$$\begin{aligned} \frac{1}{\rho_\zeta(\mathbf{F}, \mathbf{m})} &= \limsup_{n \rightarrow \infty} |[Q_{\mathbf{m}}^{\mathbf{F}}G]_n|^{1/n} = \limsup_{n \rightarrow \infty} |[Q_{\mathbf{m}}^{\mathbf{F}}G - Q_{n,\mathbf{m}}G]_n|^{1/n} \\ &= \limsup_{n \rightarrow \infty} \left| \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{(Q_{\mathbf{m}}^{\mathbf{F}} - Q_{n,\mathbf{m}})(z)G(z)\Phi'(z)}{\Phi^{n+1}(z)} dz \right|^{1/n} \leq \frac{\theta}{\rho}. \end{aligned}$$

Letting $\rho \rightarrow |\Phi(\zeta)|$ in the above inequality, we obtain the contradiction:

$$\frac{1}{\rho_\zeta(\mathbf{F}, \mathbf{m})} \leq \frac{\theta}{|\Phi(\zeta)|} < \frac{|\Phi(\zeta)|/\rho_\zeta(\mathbf{F}, \mathbf{m})}{|\Phi(\zeta)|} = \frac{1}{\rho_\zeta(\mathbf{F}, \mathbf{m})}.$$

Let us prove the inequality (1.9). Let $\alpha \in \{1, \dots, d\}$ and $k \in \{0, 1, \dots, m_\alpha - 1\}$ be fixed and let $\xi_1, \dots, \xi_N$ be the poles of $z^k F_\alpha$ in $D_\alpha(\mathbf{F}, \mathbf{m})$. For each $j = 1, \dots, N$, let $\hat{\tau}_j$ be the order of ξ_j as a pole of $z^k F_\alpha$ and $\tilde{\tau}_j$ its order as a system pole of $\mathbf{F}$. Recall that, by assumption, $\hat{\tau}_j \leq \tilde{\tau}_j$. From Eq. (2.2), we have the following:

$$Q_{n,\mathbf{m}} z^k F_\alpha - P_{n,\mathbf{m},k,\alpha} = \sum_{\ell=n+1}^{\infty} a_{\ell,n} \Phi_\ell.$$

Multiplying the above equality by $\omega(z) := \prod_{j=1}^N (z - \xi_j)^{\hat{\tau}_j}$ and expanding the result in terms of the Faber polynomial expansion, we obtain the following:

$$\begin{aligned} \omega Q_{n,\mathbf{m}} z^k F_\alpha - \omega P_{n,\mathbf{m},k,\alpha} &= \sum_{\ell=n+1}^{\infty} a_{\ell,n} \omega \Phi_\ell \\ &= \sum_{\nu=0}^{\infty} b_{\nu,n} \Phi_\nu = \sum_{\nu=0}^{n+|\mathbf{m}|} b_{\nu,n} \Phi_\nu + \sum_{\nu=n+|\mathbf{m}|+1}^{\infty} b_{\nu,n} \Phi_\nu. \end{aligned} \quad (2.19)$$

Let K be a compact subset of $D_\alpha^*(\mathbf{F}, \mathbf{m}) \setminus \mathcal{P}(\mathbf{F}, \mathbf{m})$ and set

$$\sigma := \max\{\|\Phi\|_K, 1\} \quad (2.20)$$

($\sigma = 1$ when $K \subset E$). Choose $\delta > 0$ so small that

$$\rho_2 := \rho_\alpha^*(\mathbf{F}, \mathbf{m}) - \delta > \sigma. \quad (2.21)$$

Let us estimate $\sum_{\nu=n+|\mathbf{m}|+1}^{\infty} |b_{\nu,n}| |\Phi_{\nu}|$ on $\overline{D}_{\sigma}$. For $\nu \geq n + |\mathbf{m}| + 1$:

$$\begin{aligned} b_{\nu,n} &:= [\omega Q_{n,\mathbf{m}} z^k F_{\alpha} - \omega P_{n,\mathbf{m},k,\alpha}]_{\nu} = [\omega Q_{n,\mathbf{m}} z^k F_{\alpha}]_{\nu} \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{z^k \omega(z) Q_{n,\mathbf{m}}(z) F_{\alpha}(z) \Phi'(z)}{\Phi^{\nu+1}(z)} dz, \end{aligned}$$

where $1 < \rho_2 < \rho_{\alpha}^*(\mathbf{F}, \mathbf{m})$. By a computation similar to (2.10), we obtain the following:

$$|b_{\nu,n}| \leq \frac{c_4}{\rho_2^{\nu}}. \quad (2.22)$$

Combining (2.21), (2.22), and Lemma 2.2, we have, for $z \in \overline{D}_{\sigma}$:

$$\sum_{\nu=n+|\mathbf{m}|+1}^{\infty} |b_{\nu,n}| |\Phi_{\nu}(z)| \leq c_5 \sum_{\nu=n+|\mathbf{m}|+1}^{\infty} \left(\frac{\sigma}{\rho_2} \right)^{\nu} = c_6 \left(\frac{\sigma}{\rho_2} \right)^n,$$

which implies that

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=n+|\mathbf{m}|+1}^{\infty} |b_{\nu,n}| |\Phi_{\nu}| \right\|_{\overline{D}_{\sigma}}^{1/n} \leq \frac{\sigma}{\rho_2}.$$

Letting $\delta \rightarrow 0$, we have $\rho_2 \rightarrow \rho_{\alpha}^*(\mathbf{F}, \mathbf{m})$ and

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=n+|\mathbf{m}|+1}^{\infty} |b_{\nu,n}| |\Phi_{\nu}| \right\|_{\overline{D}_{\sigma}}^{1/n} \leq \frac{\sigma}{\rho_{\alpha}^*(\mathbf{F}, \mathbf{m})}. \quad (2.23)$$

Now, we wish to estimate $\sum_{\nu=0}^{n+|\mathbf{m}|} |b_{\nu,n}| |\Phi_{\nu}|$ on $\overline{D}_{\sigma}$. Notice that

$$b_{\nu,n} = \sum_{\ell=n+1}^{\infty} a_{\ell,n} [\omega \Phi_{\ell}]_{\nu}.$$

Therefore, we need to estimate both $|a_{\ell,n}|$ and $|\omega \Phi_{\ell}|_{\nu}$.

First, we work on $|a_{\ell,n}|$. Combining (2.13) and (2.15), it follows that, for the system poles $\xi_1, \dots, \xi_w$ of $\mathbf{F}$, if τ_j is the order (as a system pole) of ξ_j , then

$$\limsup_{n \rightarrow \infty} |Q_{n,\mathbf{m}}^{(u)}(\xi_j)|^{1/n} \leq \frac{|\Phi(\xi_j)|}{\rho_{\xi_j, u+1}(\mathbf{F}, \mathbf{m})}, \quad u = 0, 1, \dots, \tau_j - 1. \quad (2.24)$$

We have

$$a_{\ell,n} = [Q_{n,\mathbf{m}} z^k F_{\alpha}]_{\ell} = \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{Q_{n,\mathbf{m}}(z) z^k F_{\alpha}(z) \Phi'(z)}{\Phi^{\ell+1}(z)} dz,$$

where $1 < \rho_1 < \rho_0(z^k F_{\alpha})$ and define

$$\tau_{\ell,n} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{Q_{n,\mathbf{m}}(z) z^k F_{\alpha}(z) \Phi'(z)}{\Phi^{\ell+1}(z)} dz,$$

where $\max\{|\Phi(\tilde{\xi}_j)| : j = 1, \dots, N\} < \rho_2 < \rho_\alpha^*(\mathbf{F}, \mathbf{m})$. Arguing as in (2.6) and (2.7), we obtain the following:

$$\begin{aligned} \tau_{\ell,n} - a_{\ell,n} &= \sum_{j=1}^N \operatorname{res}(Q_{n,\mathbf{m}} z^k F_\alpha \Phi' / \Phi^{\ell+1}, \tilde{\xi}_j) \\ &= \sum_{j=1}^N \frac{1}{(\hat{\tau}_j - 1)!} \\ &\quad \times \sum_{u=0}^{\hat{\tau}_j-1} \binom{\hat{\tau}_j - 1}{u} \left(\frac{(z - \tilde{\xi}_j)^{\hat{\tau}_j} z^k F_\alpha \Phi'}{\Phi^{\ell+1}} \right)^{(\hat{\tau}_j-1-u)} (\tilde{\xi}_j) Q_{n,\mathbf{m}}^{(u)}(\tilde{\xi}_j). \end{aligned} \quad (2.25)$$

Notice that $(z - \tilde{\xi}_j)^{\hat{\tau}_j} z^k F_\alpha$ is holomorphic at $\tilde{\xi}_j$. Let $\delta > 0$ be such that

$$|\Phi(\tilde{\xi}_j)| - 2\delta > 1, \quad j = 1, \dots, N.$$

Computations similar to (2.10) and (2.11) give us

$$\begin{aligned} |\tau_{\ell,n}| &\leq \frac{c_7}{\rho_2^\ell} \\ \text{and} \quad \left| \left(\frac{(z - \tilde{\xi}_j)^{\hat{\tau}_j} z^k F_\alpha \Phi'}{\Phi^{\ell+1}} \right)^{(\hat{\tau}_j-1-u)} (\tilde{\xi}_j) \right| &\leq \frac{c_8}{(|\Phi(\tilde{\xi}_j)| - \delta)^\ell}, \end{aligned} \quad (2.26)$$

respectively. Take $\varepsilon > 0$. From (2.24), it follows that for all $j = 1, \dots, N$,

$$|Q_{n,\mathbf{m}}^{(u)}(\tilde{\xi}_j)| \leq c_9 \left(\frac{|\Phi(\tilde{\xi}_j)| + \varepsilon}{\rho_{\tilde{\xi}_j, \hat{\tau}_j}(\mathbf{F}, \mathbf{m}) + \varepsilon} \right)^n.$$

Using (2.25), (2.26) and the previous inequalities, we obtain

$$\begin{aligned} |a_{\ell,n}| &\leq |\tau_{\ell,n}| \\ &\quad + \sum_{j=1}^N \sum_{u=0}^{\hat{\tau}_j-1} \frac{1}{(\hat{\tau}_j - 1)!} \binom{\hat{\tau}_j - 1}{u} \left| \left(\frac{(z - \tilde{\xi}_j)^{\hat{\tau}_j} z^k F_\alpha \Phi'}{\Phi^{\ell+1}} \right)^{(\hat{\tau}_j-1-u)} (\tilde{\xi}_j) \right| \\ &\quad \times |Q_{n,\mathbf{m}}^{(u)}(\tilde{\xi}_j)| \\ &\leq \frac{c_7}{\rho_2^\ell} + c_{10} \sum_{j=1}^N \frac{(|\Phi(\tilde{\xi}_j)| + \varepsilon)^n}{(\rho_{\tilde{\xi}_j, \hat{\tau}_j}(\mathbf{F}, \mathbf{m}) + \varepsilon)^n (|\Phi(\tilde{\xi}_j)| - \delta)^\ell} \\ &\leq \frac{c_7}{\rho_2^\ell} + \frac{c_{10}}{(\rho_\alpha^*(\mathbf{F}, \mathbf{m}) + \varepsilon)^n} \sum_{j=1}^N \frac{(|\Phi(\tilde{\xi}_j)| + \varepsilon)^n}{(|\Phi(\tilde{\xi}_j)| - \delta)^\ell}. \end{aligned} \quad (2.27)$$

Next, we estimate $|\omega \Phi_\ell|_\nu$. We can assume that $\rho_1 - \delta > 1$. By Lemma 2.2:

$$|\omega \Phi_\ell|_\nu \leq \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_1-\delta}} \frac{\omega(z) \Phi_\ell(z) \Phi'(z)}{\Phi^{\nu+1}(z)} dz \right| \leq c_{11} \frac{(\rho_1 - \delta)^\ell}{(\rho_1 - \delta)^\nu}. \quad (2.28)$$

By (2.27) and (2.28), we have the following:

$$\begin{aligned}
 |b_{\nu,n}| &\leq \sum_{\ell=n+1}^{\infty} |a_{\ell,n}| |\omega \Phi_{\ell}|_{\nu} \\
 &\leq \frac{c_{12}}{(\rho_1 - \delta)^{\nu}} \left(\frac{\rho_1 - \delta}{\rho_2} \right)^n \\
 &\quad + \frac{c_{13}(\rho_1 - \delta)^n}{(\rho_{\alpha}^*(\mathbf{F}, \mathbf{m}) + \varepsilon)^n (\rho_1 - \delta)^{\nu}} \sum_{j=1}^N \left(\frac{|\Phi(\tilde{\xi}_j)| + \varepsilon}{|\Phi(\tilde{\xi}_j) - \delta|} \right)^n. \quad (2.29)
 \end{aligned}$$

Combining (2.29) and Lemma 2.2, for $z \in \overline{D}_{\sigma}$, we obtain the following:

$$\begin{aligned}
 &\sum_{\nu=0}^{n+|\mathbf{m}|} |b_{\nu,n}| |\Phi_{\nu}(z)| \\
 &\leq \left(c_{14} \left(\frac{\rho_1 - \delta}{\rho_2} \right)^n + \frac{c_{15}(\rho_1 - \delta)^n}{(\rho_{\alpha}^*(\mathbf{F}, \mathbf{m}) + \varepsilon)^n} \sum_{j=1}^N \left(\frac{|\Phi(\tilde{\xi}_j)| + \varepsilon}{|\Phi(\tilde{\xi}_j) - \delta|} \right)^n \right) \\
 &\quad \times \sum_{\nu=0}^{n+|\mathbf{m}|} \left(\frac{\sigma}{\rho_1 - \delta} \right)^{\nu} \\
 &\leq \left(c_{14} \left(\frac{\rho_1 - \delta}{\rho_2} \right)^n + \frac{c_{15}(\rho_1 - \delta)^n}{(\rho_{\alpha}^*(\mathbf{F}, \mathbf{m}) + \varepsilon)^n} \sum_{j=1}^N \left(\frac{|\Phi(\tilde{\xi}_j)| + \varepsilon}{|\Phi(\tilde{\xi}_j) - \delta|} \right)^n \right) \\
 &\quad \times (n + |\mathbf{m}| + 1) \sigma^{n+|\mathbf{m}|}.
 \end{aligned}$$

This implies that

$$\begin{aligned}
 &\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=0}^{n+|\mathbf{m}|} |b_{\nu,n}| |\Phi_{\nu}| \right\|_{\overline{D}_{\sigma}}^{1/n} \\
 &\leq \max \left\{ \frac{\sigma(\rho_1 - \delta)}{\rho_2}, \frac{\sigma(\rho_1 - \delta)}{\rho_{\alpha}^*(\mathbf{F}, \mathbf{m}) + \varepsilon} \max_{j=1, \dots, N} \left(\frac{|\Phi(\tilde{\xi}_j)| + \varepsilon}{|\Phi(\tilde{\xi}_j) - \delta|} \right) \right\}.
 \end{aligned}$$

Letting $\varepsilon, \delta \rightarrow 0$, and $\rho_1 \rightarrow 1$, we have $\rho_2 \rightarrow \rho_{\alpha}^*(\mathbf{F}, \mathbf{m})$ and we obtain the following:

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=0}^{n+|\mathbf{m}|} |b_{\nu,n}| |\Phi_{\nu}| \right\|_{\overline{D}_{\sigma}}^{1/n} \leq \frac{\sigma}{\rho_{\alpha}^*(\mathbf{F}, \mathbf{m})}. \quad (2.30)$$

Using (1.8), (2.19), (2.23), and (2.30), we obtain (1.9) and the proof is complete. $\square$

3. Proof of Theorem 1.5

3.1. Incomplete Padé–Faber Approximation

Let us introduce the notion of incomplete Padé–Faber approximation. Similar concepts proved to be effective in the study of Hermite–Padé approximation and orthogonal Hermite–Padé approximation (see [4, 6]).

Definition 3.1. Let $F \in \mathcal{H}(E)$. Fix $m \geq m^* \geq 1$ and $n \in \mathbb{N}$. Then, there exist polynomials Q_{n,m,m^*} and $P_{n,m,m^*,k}$, $k = 0, 1, \dots, m^* - 1$, such that

$$\deg(P_{n,m,m^*,k}) \leq n-1, \quad \deg(Q_{n,m,m^*}) \leq m, \quad Q_{n,m,m^*} \neq 0, \\ [Q_{n,m,m^*} z^k F - P_{n,m,m^*,k}]_j = 0, \quad j = 0, 1, \dots, n.$$

The rational function $R_{n,m,m^*} := P_{n,m,m^*,0}/Q_{n,m,m^*}$ is called an (n, m, m^*) incomplete Padé–Faber approximant of F .

Clearly:

$$[z^k Q_{n,m,m^*} F]_n = 0, \quad k = 0, 1, \dots, m^* - 1,$$

and Q_{n,m,m^*} may not be unique. For each $m \geq m^* \geq 1$ and $n \in \mathbb{N}$, we choose one candidate of Q_{n,m,m^*} . Since $Q_{n,m,m^*} \neq 0$, we normalize it to have leading coefficient equal to 1. We call Q_{n,m,m^*} a denominator of an (n, m, m^*) incomplete Padé–Faber approximant of F . Notice that, for each $\alpha = 1, \dots, d$, $Q_{n,m}$ [from (1.6)] is a denominator of an $(n, |\mathbf{m}|, m_\alpha)$ incomplete Padé–Faber approximant of F_α .

Let $D_{\rho_{m^*}(F)}$ be the largest canonical region in which F can be extended as a meromorphic function having at most m^* poles and $\rho_{m^*}(F)$ be the index of this region.

Lemma 3.2. Let $F \in \mathcal{H}(E)$. Fix $m \geq m^* \geq 1$. Suppose that there exists a polynomial Q_m of degree m , such that

$$\lim_{n \rightarrow \infty} Q_{n,m,m^*} = Q_m. \quad (3.1)$$

Then, $\rho_0(Q_m F) \geq \rho_{m^*}(F)$.

Proof. Let q_{n,m,m^*} be the polynomial Q_{n,m,m^*} normalized, so that

$$q_{n,m,m^*}(z) = \sum_{k=0}^m \lambda_{n,k} z^k, \quad \sum_{k=0}^m |\lambda_{n,k}| = 1. \quad (3.2)$$

Let ξ be a pole of order τ of F in $D_{\rho_{m^*}(F)}$. Modifying conveniently the proof of (2.5), one can show that

$$\limsup_{n \rightarrow \infty} |q_{n,m,m^*}^{(j)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_{m^*}(F)}, \quad j = 0, 1, \dots, \tau - 1. \quad (3.3)$$

Since the sequence of polynomials Q_{n,m,m^*} converges to Q_m , (3.3) entails that ξ is a zero of Q_m of multiplicity at least τ . Being this the case for each pole of F in $D_{\rho_{m^*}(F)}$, the thesis readily follows. $\square$

The following technical lemma, whose proof may be found in [3, Lemma 3], is used for proving Lemma 3.4.

Lemma 3.3. *If a sequence of complex numbers $\{A_N\}_{N \in \mathbb{N}}$ has the following properties:*

- (i) $\lim_{N \rightarrow \infty} |A_N|^{1/N} = 0$;
- (ii) *there exists $N_0 \in \mathbb{N}$ and $C > 0$, such that $|A_N| \leq C \sum_{k=N+1}^{\infty} |A_k|$, for all $N \geq N_0$,*

then there exists $N_1 \in \mathbb{N}$, such that $A_N = 0$ for all $N \geq N_1$.

Lemma 3.4 below is the cornerstone for the proof of Theorem 1.5.

Lemma 3.4. *Let $F \in \mathcal{H}(E)$. Fix $m \geq m^* \geq 1$. Suppose that F is not a rational function with at most $m^* - 1$ poles and there exists a polynomial Q_m of degree m , such that*

$$\limsup_{n \rightarrow \infty} \|Q_{n,m,m^*} - Q_m\|^{1/n} = \theta < 1. \quad (3.4)$$

Then, the poles of F in $D_{\rho_{m^}(F)}$ are zeros of Q_m counting multiplicities and either F has exactly m^* poles in $D_{\rho_{m^*}(F)}$ or $\rho_0(Q_m F) > \rho_{m^*}(F)$.*

Proof. From Lemma 3.2, we know that the poles of F in $D_{\rho_{m^*}(F)}$ are zeros of Q_m counting multiplicities and $\rho_0(Q_m F) \geq \rho_{m^*}(F)$. Assume that $\rho_0(Q_m F) = \rho_{m^*}(F)$. Let us show that F has exactly m^* poles in $D_{\rho_{m^*}(F)}$. To the contrary, suppose that F has in $D_{\rho_{m^*}(F)}$ at most $m^* - 1$ poles. Then, there exists a polynomial q_{m^*} with $\deg q_{m^*} < m^*$, such that

$$\rho_0(q_{m^*} F) = \rho_{m^*}(F) = \rho_0(Q_m q_{m^*} F).$$

Since $\deg q_{m^*} < m^*$, by the definition of Q_{n,m,m^*} , $[Q_{n,m,m^*} q_{m^*} F]_n = 0$. Take $1 < \rho < \rho_{m^*}(F)$. Then, by Lemma 2.1:

$$\begin{aligned} \frac{1}{\rho_{m^*}(F)} &= \limsup_{n \rightarrow \infty} |[Q_m q_{m^*} F]_n|^{1/n} \\ &= \limsup_{n \rightarrow \infty} |[Q_m q_{m^*} F - Q_{n,m,m^*} q_{m^*} F]_n|^{1/n} \\ &= \limsup_{n \rightarrow \infty} \left| \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{(Q_m - Q_{n,m,m^*})(z) q_{m^*}(z) F(z) \Phi'(z)}{\Phi^{n+1}(z)} dz \right|^{1/n}. \end{aligned}$$

From the equation above, using (3.4), it is easy to show that

$$\frac{1}{\rho_{m^*}(F)} \leq \frac{\theta}{\rho_{m^*}(F)},$$

which is possible only if $\rho_{m^*}(F) = \rho_0(q_{m^*} F) = \infty$. Let us show that this is not so.

From (3.4), without loss of generality, we can assume that $\deg Q_{n,m,m^*} = m$. Set

$$q_{m^*}(z) F(z) = \sum_{k=0}^{\infty} a_k \Phi_k(z)$$

and

$$Q_{n,m,m^*}(z) = \sum_{j=0}^m b_{n,j} z^j,$$

where $b_{n,m} = 1$. From (3.4), we have the following:

$$\sup\{|b_{n,j}| : 0 \leq j \leq m, n \in \mathbb{N}\} \leq c_1. \quad (3.5)$$

Since $[Q_{n,m,m^*} q_{m^*} F]_n = 0$, $[z^j \Phi_k]_n = 0$ whenever $\deg(z^j \Phi_k) < n$ and $[z^m \Phi_{n-m}]_n = \text{cap}^m(E)$ (see (1.2)), we obtain the following:

$$\begin{aligned} 0 &= [Q_{n,m,m^*} q_{m^*} F]_n = \sum_{k=0}^{\infty} \sum_{j=0}^m a_k b_{n,j} [z^j \Phi_k]_n \\ &= \sum_{k=n-m}^{\infty} \sum_{j=0}^m a_k b_{n,j} [z^j \Phi_k]_n \\ &= \text{cap}^m(E) a_{n-m} + \sum_{k=n-m+1}^{\infty} \sum_{j=0}^m a_k b_{n,j} [z^j \Phi_k]_n. \end{aligned} \quad (3.6)$$

Take $\rho > 1$. Using Lemma 2.2, for $j = 0, 1, \dots, m$, and $k \geq n - m + 1$, we obtain the following:

$$|[z^j \Phi_k]_n| = \left| \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{z^j \Phi_k(z) \Phi'(z)}{\Phi^{n+1}(z)} dz \right| \leq c_2 \frac{\rho^k}{\rho^n}. \quad (3.7)$$

Combining (3.5), (3.6), and (3.7), it follows that

$$|a_{n-m}| \rho^{n-m} \leq c_3 \sum_{k=n-m+1}^{\infty} |a_k| \rho^k. \quad (3.8)$$

Taking $n - m = N$ and $|a_k| \rho^k = A_k$, (3.8) is (ii) of Lemma 3.3 and we also have (i), because

$$\lim_{N \rightarrow \infty} |A_N|^{1/N} = \lim_{N \rightarrow \infty} (|a_N| \rho^N)^{1/N} = \rho / \rho_0 (q_{m^*} F) = 0.$$

Consequently, there exists $N_1 \in \mathbb{N}$, such that $a_N = 0$ for all $N \geq N_1$. Thus, $q_{m^*} F$ is a polynomial and F is a rational function with at most $m^* - 1$ poles contradicting the assumption that F is not a rational function with at most $m^* - 1$ poles. Therefore, F has exactly m^* poles in $D_{\rho_{m^*}(F)}$ as we wanted to prove. $\square$

3.2. Proof of Theorem 1.5

Before proving the main result, let us point out several important ingredients.

Given a system of functions $\mathbf{F} \in \mathcal{H}(E)^d$ and a multi-index $\mathbf{m} \in \mathbb{N}^d$, the space generated through polynomial combinations of the form (1.7) has dimension $\leq |\mathbf{m}|$. Therefore, $\mathbf{F}$ can have at most $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ counting multiplicities, since the functions which determine the system poles and their order are of the form (1.7) and they are obviously linearly independent. For more details, see [7, Lemma 3.5].

The concept of polynomial independence of a vector of functions was introduced in [7] and is also useful in this context.

Definition 3.5. A vector $\mathbf{F} = (F_1, \dots, F_d) \in \mathcal{H}(E)^d$ is said to be *polynomially independent with respect to* $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ if there do not exist polynomials $p_1, \dots, p_d$, at least one of which is non-null, such that

- (i) $\deg p_\alpha < m_\alpha$, $\alpha = 1, \dots, d$;
- (ii) $\sum_{\alpha=1}^d p_\alpha F_\alpha$ is a polynomial.

According to the assumptions of Theorem 1.5, for all $n \geq n_0$, the polynomial $Q_{n,\mathbf{m}}$ is unique and $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$. This implies that $\mathbf{F}$ is polynomially independent with respect to $\mathbf{m}$ for, otherwise, it is easy to see that for all sufficiently large n , we can construct $(n, \mathbf{m})$ simultaneous Padé–Faber approximants of $\mathbf{F}$ with $\deg Q_{n,\mathbf{m}} < |\mathbf{m}|$ (see [7, Lemma 3.2]). Notice that if $\mathbf{F}$ is polynomially independent with respect to $\mathbf{m}$, then, for each $\alpha = 1, \dots, d$, F_α is not a rational function with at most $m_\alpha - 1$ poles. As we pointed out in Sect. 3.1, for each $\alpha = 1, \dots, d$, $Q_{n,\mathbf{m}}$ is a denominator of an $(n, |\mathbf{m}|, m_\alpha)$ incomplete Padé–Faber approximant of F_α . Consequently, the assumptions of Theorem 1.5 allow us to make use of Lemma 3.4 in its proof.

Finally, one can reduce the proof of Theorem 1.5 to the case when the multi-index $\mathbf{m}$ has all its components equal to 1. Indeed, given $\mathbf{F} \in \mathcal{H}(E)^d$ and $\mathbf{m} \in \mathbb{N}^d$, define

$$\bar{\mathbf{F}} := (F_1, \dots, z^{m_1-1}F_1, F_2, \dots, z^{m_d-1}F_d) = (f_1, f_2, \dots, f_{|\mathbf{m}|}) \quad (3.9)$$

and

$$\bar{\mathbf{m}} := (1, 1, \dots, 1) \quad (3.10)$$

with $|\bar{\mathbf{m}}| = |\mathbf{m}|$. The following assertions are easy to verify:

- (i) the systems of equations that define $Q_{n,\mathbf{m}}$ for $\mathbf{F}$ and $\mathbf{m}$, and $Q_{n,\bar{\mathbf{m}}}$ for $\bar{\mathbf{F}}$ and $\bar{\mathbf{m}}$ are the same.
- (ii) $\mathbf{F}$ is polynomially independent with respect to $\mathbf{m}$ if and only if $\bar{\mathbf{F}}$ is polynomially independent with respect to $\bar{\mathbf{m}}$.
- (iii) the poles and system poles of $(\mathbf{F}, \mathbf{m})$ and $(\bar{\mathbf{F}}, \bar{\mathbf{m}})$, as well as their orders, coincide.
- (iv) $\rho_m(\mathbf{F}) = \rho_m(\bar{\mathbf{F}})$, for all $m \in \mathbb{N} \cup \{0\}$.

Proof of Theorem 1.5. As shown above, without loss of generality, we can restrict our attention to the analysis of $(\bar{\mathbf{F}}, \bar{\mathbf{m}})$ defined in (3.9) and (3.10). Notice that (1.7) reduces to taking linear combinations of the components of $\bar{\mathbf{F}}$. We also have that $Q_{n,\mathbf{m}} = Q_{n,\bar{\mathbf{m}}}$ and $\bar{\mathbf{F}}$ is polynomially independent with respect to $\bar{\mathbf{m}}$.

The arguments used in the proof follow closely those employed in proving the inverse part of [7, Theorem 1.4];

Choose $\beta = 1, \dots, |\mathbf{m}|$. From Lemma 3.4, either $D_{\rho_1(f_\beta)}$ contains exactly one pole of f_β and it is a zero of $Q_{|\mathbf{m}|}$, or $\rho_0(Q_{|\mathbf{m}|}f_\beta) > \rho_1(f_\beta)$. Hence, $D_{\rho_0(\bar{\mathbf{F}})} \neq \mathbb{C}$ and the zeros of $Q_{|\mathbf{m}|}$ contain all the poles of f_β on the boundary of $D_{\rho_0(f_\beta)}$ counting their order. Moreover, the function f_β cannot have on the boundary of $D_{\rho_0(f_\beta)}$ singularities other than poles. Thus, the poles of $\bar{\mathbf{F}}$ on the boundary of $D_{\rho_0(\bar{\mathbf{F}})}$ are zeros of $Q_{|\mathbf{m}|}$ counting multiplicities and the boundary contains no other singularity but poles. Let us call them candidate system poles of $\bar{\mathbf{F}}$ and denote them by $a_1, \dots, a_{n_1}$ taking into account their order. They constitute a first layer of candidate system poles of $\bar{\mathbf{F}}$.

Since $\deg Q_{|\mathbf{m}|} = |\mathbf{m}|$, $n_1 \leq |\mathbf{m}|$. If $n_1 = |\mathbf{m}|$, we are done finding candidate system poles. Let us assume that $n_1 < |\mathbf{m}|$ and let us find coefficients

$c_1, \dots, c_{|\mathbf{m}|}$, such that $\sum_{\beta=1}^{|\mathbf{m}|} c_\beta f_\beta$ is holomorphic in a neighborhood of $\overline{D}_{\rho_0}(\overline{\mathbf{F}})$. For this purpose, we solve a homogeneous system of n_1 linear equations with $|\mathbf{m}|$ unknowns. In fact, if $z = a$ is a candidate system pole of $\overline{\mathbf{F}}$ with multiplicity τ , we obtain τ equations choosing the coefficients c_β , so that

$$\int_{|w-a|=\delta} (w-a)^k \left(\sum_{\beta=1}^{|\mathbf{m}|} c_\beta f_\beta(w) \right) dw = 0, \quad k = 0, \dots, \tau-1. \quad (3.11)$$

We write the equations for each distinct candidate system pole on the boundary of $D_{\rho_0}(\overline{\mathbf{F}})$. This homogeneous system of linear equations has at least $|\mathbf{m}| - n_1$ linearly independent solutions, which we denote by $\mathbf{c}_j^1$, $j = 1, \dots, |\mathbf{m}| - n_1^*$, where $n_1^* \leq n_1$ denotes the rank of the system of equations.

Let

$$\mathbf{c}_j^1 := (c_{j,1}^1, \dots, c_{j,|\mathbf{m}|}^1), \quad j = 1, \dots, |\mathbf{m}| - n_1^*.$$

Define the $(|\mathbf{m}| - n_1^*) \times |\mathbf{m}|$ dimensional matrix:

$$C^1 := \begin{pmatrix} \mathbf{c}_1^1 \\ \vdots \\ \mathbf{c}_{|\mathbf{m}| - n_1^*}^1 \end{pmatrix}.$$

Define the vector $\mathbf{g}_1$ of $|\mathbf{m}| - n_1^*$ functions given by the following:

$$\mathbf{g}_1^t := C^1 \overline{\mathbf{F}}^t = (g_{1,1}, \dots, g_{1,|\mathbf{m}| - n_1^*})^t,$$

where A^t denotes the transpose of the matrix A . Since all the rows of C^1 are non-null and $\overline{\mathbf{F}}$ is polynomially independent with respect to $\overline{\mathbf{m}}$, none of the functions $g_{1,j} = \sum_{\beta=1}^{|\mathbf{m}|} c_{j,\beta}^1 f_\beta$, $j = 1, \dots, |\mathbf{m}| - n_1^*$, are polynomials.

Consider the canonical domain:

$$D_{\rho_0(\mathbf{g}_1)} = \bigcap_{j=1}^{|\mathbf{m}| - n_1^*} D_{\rho_0(g_{1,j})}.$$

Clearly, $D_{\rho_0}(\overline{\mathbf{F}})$ is a proper subset of $D_{\rho_0(\mathbf{g}_1)}$ and $[Q_{n,\overline{\mathbf{m}}} g_{1,j}]_n = 0$ for all $j = 1, \dots, |\mathbf{m}| - n_1^*$. Therefore, for each $j = 1, \dots, |\mathbf{m}| - n_1^*$, $Q_{n,\overline{\mathbf{m}}}$ is a denominator of an $(n, |\overline{\mathbf{m}}|, 1)$ incomplete Padé-Faber approximant of $g_{1,j}$. Since the $g_{1,j}$ are not polynomials, by Lemma 3.4 with $m^* = 1$, for each $j = 1, \dots, |\mathbf{m}| - n_1^*$, either $D_{\rho_1(g_{1,j})}$ contains exactly one pole of $g_{1,j}$ and it is a zero of $Q_{|\mathbf{m}|}$, or $\rho_0(Q_{|\mathbf{m}|} g_{1,j}) > \rho_1(g_{1,j})$. In particular, $D_{\rho_0(\mathbf{g}_1)} \neq \mathbb{C}$ and all the singularities of $\mathbf{g}_1$ on the boundary of $D_{\rho_0(\mathbf{g}_1)}$ are poles which are zeros of $Q_{|\mathbf{m}|}$ counting their order. They form the next layer of candidate system poles of $\overline{\mathbf{F}}$.

Denote by $a_{n_1+1}, \dots, a_{n_1+n_2}$ these new candidate system poles. Again, if $n_1 + n_2 = |\mathbf{m}|$, we are done. Otherwise, $n_2 < |\mathbf{m}| - n_1 \leq |\mathbf{m}| - n_1^*$, and we repeat the same process eliminating the n_2 poles $a_{n_1+1}, \dots, a_{n_1+n_2}$. We have $|\mathbf{m}| - n_1^*$ functions which are holomorphic in $D_{\rho_0(\mathbf{g}_1)}$ and meromorphic in a neighborhood of $\overline{D}_{\rho_0(\mathbf{g}_1)}$. The corresponding homogeneous system of linear equations, similar to (3.11), has at least $|\mathbf{m}| - n_1^* - n_2$ linearly independent

solutions $\mathbf{c}_j^2$, $j = 1, \dots, |\mathbf{m}| - n_1^* - n_2^*$, where $n_2^* \leq n_2$ is the rank of the new system. Let

$$\mathbf{c}_j^2 := (c_{j,1}^2, \dots, c_{j,|\mathbf{m}|-n_1^*}^2), \quad j = 1, \dots, |\mathbf{m}| - n_1^* - n_2^*.$$

Define the $(|\mathbf{m}| - n_1^* - n_2^*) \times (|\mathbf{m}| - n_1^*)$ dimensional matrix:

$$C^2 := \begin{pmatrix} \mathbf{c}_1^2 \\ \vdots \\ \mathbf{c}_{|\mathbf{m}|-n_1^*-n_2^*}^2 \end{pmatrix}.$$

Define the vector $\mathbf{g}_2$ with $|\mathbf{m}| - n_1^* - n_2^*$ functions given by the following:

$$\mathbf{g}_2^t := C^2 \mathbf{g}_1^t = C^2 C^1 \bar{\mathbf{F}}^t = (g_{2,1}, \dots, g_{2,|\mathbf{m}|-n_1^*-n_2^*})^t.$$

As C^1 and C^2 have full rank, so does $C^2 C^1$. Therefore, the rows of $C^2 C^1$ are linearly independent; in particular, they are non-null. Thus, all the component functions of $\mathbf{g}_2$ are not polynomials, due to the polynomial independence of $\bar{\mathbf{F}}$ with respect to $\bar{\mathbf{m}}$, and we can apply again Lemma 3.4. The proof is completed using finite induction.

On each layer of system poles, $1 \leq n_k \leq |\mathbf{m}|$. Therefore, in a finite number of steps, say that $N - 1$, their sum equals $|\mathbf{m}|$. Consequently, the number of candidate system poles of $\bar{\mathbf{F}}$ in some canonical region, counting multiplicities, is exactly equal to $|\mathbf{m}|$ and they are precisely the zeros of $Q_{|\mathbf{m}|}$ as we wanted to prove.

Summarizing, in $N - 1$ steps, we have produced N layers of candidate system poles. Each layer contains n_k candidates, $k = 1, \dots, N$. At the same time, on each step k , $k = 1, \dots, N - 1$, we have solved a linear system of n_k equations, of rank n_k^* , with $|\mathbf{m}| - n_1^* - \dots - n_k^*$, $n_k^* \leq n_k$, linearly independent solutions. We find ourselves on the N th layer with n_N candidate system poles.

Let us try to eliminate the poles on the last layer. For that purpose, define the corresponding homogeneous system of linear equations as in (3.11), and we get

$$n_N = |\mathbf{m}| - n_1 - \dots - n_{N-1} \leq |\mathbf{m}| - n_1^* - \dots - n_{N-1}^* =: \bar{n}_N$$

equations with $\bar{n}_N$ unknowns. For each candidate system pole a of multiplicity τ on the N th layer, we impose the equations:

$$\int_{|w-a|=\delta} (w-a)^k \left(\sum_{\beta=1}^{\bar{n}_N} c_{\beta} g_{N-1,\beta}(w) \right) dw = 0, \quad k = 0, \dots, \tau - 1, \quad (3.12)$$

where δ is sufficiently small and the $g_{N-1,\beta}$, $\beta = 1, \dots, \bar{n}_N$, are the functions associated with the linearly independent solutions produced on step $N - 1$.

Let n_N^* be the rank of this last homogeneous system of linear equations. Assume that $n_k^* < n_k$ for some $k \in \{1, \dots, N\}$. Then, the rank of the last system of equations is strictly less than the number of unknowns, namely $n_N^* < \bar{n}_N$. Repeating the same process, there exists a vector of functions

$$\mathbf{g}_N := (g_{N,1}, \dots, g_{N,|\mathbf{m}|-n_1^*-\dots-n_N^*}),$$

such that none of $g_{N,\beta}$ is a polynomial because of the polynomial independence of $\bar{\mathbf{F}}$ with respect to $\bar{\mathbf{m}}$. Applying Lemma 3.4, each $g_{N,\beta}$ has on the boundary of its canonical domain of analyticity a pole which is a zero of $Q_{|\mathbf{m}|}$. However, this is impossible, because all the zeros of $Q_{|\mathbf{m}|}$ are strictly contained in a smaller domain. Consequently, $n_k = n_k^*$, $k = 1, \dots, N$.

We conclude that all the N homogeneous systems of linear equations that we have solved have full rank. This implies that if, in any one of those N systems of equations, we equate one equation to 1 instead of zero [see (3.11) or (3.12)], the corresponding nonhomogeneous system of linear equations has a solution. By the definition of a system pole, this implies that each candidate system pole is, indeed, a system pole of order at least equal to its multiplicity as zero of $Q_{|\mathbf{m}|}$. However, $\bar{\mathbf{F}}$ can have at most $|\mathbf{m}|$ system poles with respect to $\bar{\mathbf{m}}$; therefore, all the candidate system poles are system poles, and their order coincides with the multiplicity of that point as a zero of $Q_{|\mathbf{m}|}$. This also means that $Q_{|\mathbf{m}|} = Q_{\mathbf{m}}^{\mathbf{F}}$. We have completed the proof. $\square$

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To whom it may concern

We certify with this letter that the paper “Direct and inverse results for multipoint Hermite-Pade approximants”, submitted by the authors N. Bosuwan (Mahidol University, Thailand), G. Lopez Lagomasion and Y. Zaldivar Gerpe (Universidad Carlos III de Madrid, Spain) in October 2018 to the Conference Proceedings of the International Conference on Complex Analysis, Potential Theory and Application has been accepted by the editors Dr H. Render and M. Gerghu (University College Dublin, Ireland) in 24/02/2019.

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Direct and inverse results for multipoint Hermite-Padé approximants

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Dedicated to Professor Stephen J. Gardiner on the occasion of
his 60th Birthday

Abstract Given a system of functions $\mathbf{f} = (f_1, \dots, f_d)$ analytic on a neighborhood of some compact subset E of the complex plane with simply connected complement in the extended complex plane, we give necessary and sufficient conditions for the convergence with geometric rate of the common denominators of row sequences of multipoint Hermite-Padé approximants under a general extremal condition on the table of interpolation points. The exact rate of convergence of these denominators is provided and the rate of convergence of the simultaneous approximants is estimated. These results allow us to detect the location of the poles of the system of functions which are in some sense closest to E .

Keywords Montessus de Ballore theorem · multipoint Padé approximation · Hermite-Padé approximation · inverse type results

Mathematics Subject Classification 30E10 · 41A21 · 41A28

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1 Statement of the main result.

We shall consider a general interpolation scheme for constructing vector rational approximations to a given vector of analytic functions which generalizes the construction of the classical Hermite-Padé approximants.

Let E be a bounded continuum with connected complement in the complex plane $\mathbb{C}$. By $\mathcal{H}(E)$ we denote the space of all functions holomorphic in some neighborhood of E . Set

$$\mathcal{H}(E)^d := \{(f_1, \dots, f_d) : f_j \in \mathcal{H}(E), j = 1, \dots, d\}.$$

Let $\alpha \subset E$ be a table of points; more precisely, $\alpha = \{\alpha_{n,k}\}$, $k = 1, \dots, n$, $n = 1, 2, \dots$. We propose the following definition.

Definition 1.1 Let $\mathbf{f} \in \mathcal{H}(E)^d$. Fix a multi-index $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}$. Set $|\mathbf{m}| = m_1 + \dots + m_d$. Then, there exist polynomials $Q_{n,\mathbf{m}}$, $P_{n,\mathbf{m},k}$, $k = 1, \dots, d$ such that

- (i) $\deg P_{n,\mathbf{m},k} \leq n - m_k$, $\deg Q_{n,\mathbf{m}} \leq |\mathbf{m}|$, $Q_{n,\mathbf{m}} \neq 0$,
- (ii) $(Q_{n,\mathbf{m}} f_k - P_{n,\mathbf{m},k})/a_{n+1} \in \mathcal{H}(E)$,

where $a_n(z) = \prod_{k=1}^n (z - \alpha_{n,k})$. The vector rational function

$$\mathbf{R}_{n,\mathbf{m}} = (R_{n,\mathbf{m},1}, \dots, R_{n,\mathbf{m},d}) = (P_{n,\mathbf{m},1}, \dots, P_{n,\mathbf{m},d})/Q_{n,\mathbf{m}}$$

is called a *multipoint Hermite-Padé (MHP) approximant of $\mathbf{f}$ with respect to $\mathbf{m}$ and α* .

This vector rational approximation, in general, is not uniquely determined. Hereafter, we assume that given $(n, \mathbf{m})$, one particular solution is taken. Without loss of generality we can assume that $Q_{n,\mathbf{m}}$ is a monic polynomial that has no common zero simultaneously with all $P_{n,\mathbf{m},k}$. In all what follows $\mathbf{m}$ remains fixed and $\{\mathbf{R}_{n,\mathbf{m}}\}_{n \in \mathbb{N}}$ is called a row sequence of MHP of $\mathbf{f}$ with respect to $\mathbf{m}$.

Multipoint Hermite-Padé approximation reduces to classical Hermite-Padé approximation when E is a disk about the origin and $a_n(z) = z^n$. There are not many papers dealing with the convergence properties of row sequences of Hermite-Padé approximation. The first significant contribution in this direction is due to Graves-Morris and Saff in [9], where an analogue of the Montessus de Ballore theorem [10] was proved. In that paper, the authors studied the classical case and stated a result for multipoint interpolation. They assume that the system of approximated functions is, so called, polewise independent. More recently, the authors of [4] and [5] managed to weaken the assumption of polewise independence obtaining sharp estimates of the rate of convergence, improving the region of convergence, and giving an analogue of Gonchar's converse statement to the Montessus de Ballore theorem for row sequences of Padé approximants (see Remark in [6], also [7] and [8]). Here, we generalize the results in [5] to MHP approximants. Extensions in other directions using expansions in orthogonal and Faber polynomials of the vector function to produce the vector rational approximants of $\mathbf{f}$ were provided in [1, 2]. For other

approaches to the study of row sequences of vector rational approximation see [11] and [12].

In the study of the convergence of general interpolation schemes, it is common to impose on the table of interpolation nodes various restrictions which determine the asymptotic behavior of the sequence of polynomials a_n . Let Φ be a holomorphic univalent function mapping the complement of E onto the exterior of the closed unit disk with $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. It is well known that there exist tables of points α satisfying the condition

$$\lim_{n \rightarrow \infty} |a_n(z)|^{1/n} = c|\Phi(z)|, \quad (1.1)$$

or the stronger condition

$$\lim_{n \rightarrow \infty} a_n(z)/c^n \Phi^n(z) = G(z) \neq 0, \quad (1.2)$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus E$, where c denotes some positive constant, see [13, Chapters 8-9]. For each $\rho > 1$, we introduce

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\}, \quad \text{and} \quad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\}$$

as the *level curve of index ρ* and the *canonical domain of index ρ* , respectively. Let $\rho_0(\mathbf{f})$ be equal to the index ρ of the largest canonical domain D_ρ to which all f_k , $k = 1, \dots, d$ can be extended as holomorphic functions simultaneously.

Gonchar proved the following analogue of the Cauchy-Hadamard formula for $f \in \mathcal{H}(E)$ and interpolation tables satisfying (1.2):

$$\rho_0(f) = \left(c \cdot \limsup_{n \rightarrow \infty} \left| \int_\Gamma \frac{f(t)}{a_{n+1}(t)} dt \right|^{1/n} \right)^{-1}, \quad (1.3)$$

where Γ is a contour encircling E and lying in the domain of holomorphy of f . This formula is a special case of [3, Corollary 3]. (We point out that (1.3) is displayed as formula (17) in [3], but with the typo that c is missing.)

Definition 1.2 Given $\mathbf{f} = (f_1, \dots, f_d) \in \mathcal{H}(E)^d$ and $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ we say that $\xi \in \mathbb{C}$ is a *system pole of order τ of $(\mathbf{f}, \mathbf{m})$* if τ is the largest positive integer such that for each $s = 1, \dots, \tau$ there exists at least one polynomial combination of the form

$$\sum_{k=1}^d p_k f_k, \quad \deg p_k < m_k, \quad k = 1, \dots, d, \quad (1.4)$$

which is analytic in a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole at $z = \xi$ of exact order s .

The concept of system pole depends not only on the system of functions $\mathbf{f}$ but also on the multi-index $\mathbf{m}$. For example, poles of the individual functions f_k need not be system poles of $(\mathbf{f}, \mathbf{m})$ and system poles need not be poles of any of the functions f_k (see examples in [5]). It is easy to see that system

poles also depend on α , or more precisely on the geometry of the associated canonical regions. However, since $\mathbf{m}$ and α will remain fixed, occasionally we may simply refer to system poles of $\mathbf{f}$.

Let τ be the order of ξ as a system pole of $\mathbf{f}$. For each $s = 1, \dots, \tau$, let $\rho_{\xi,s}(\mathbf{f}, \mathbf{m})$ denote the largest of all the numbers $\rho_s(g)$ (the index of the largest canonical domain containing at most s poles of g), where g is a polynomial combination of type (1.4) that is holomorphic on a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole at $z = \xi$ of order s . Then, we define

$$R_{\xi,s}(\mathbf{f}, \mathbf{m}) := \min_{k=1,\dots,s} \rho_{\xi,k}(\mathbf{f}, \mathbf{m}),$$

and

$$R_{\xi}(\mathbf{f}, \mathbf{m}) := R_{\xi,\tau}(\mathbf{f}, \mathbf{m}) = \min_{k=1,\dots,\tau} \rho_{\xi,k}(\mathbf{f}, \mathbf{m}).$$

Fix $k = \{1, \dots, d\}$. Let $D_k(\mathbf{f}, \mathbf{m})$ be the largest canonical domain in which all the poles of f_k are system poles of $\mathbf{f}$ with respect to $\mathbf{m}$, their order as poles of f_k does not exceed their order as system poles, and f_k has no other singularity. By $R_k(\mathbf{f}, \mathbf{m})$, we denote the index of this canonical domain. Let $\xi_1, \dots, \xi_N$ be the poles of f_k in $D_k(\mathbf{f}, \mathbf{m})$. For each $j = 1, \dots, N$, let $\hat{\tau}_j$ be the order of ξ_j as pole of f_k and τ_j be its order as a system pole. By assumption, $\hat{\tau}_j \leq \tau_j$. Set

$$R_k^*(\mathbf{f}, \mathbf{m}) := \min \left\{ R_k(\mathbf{f}, \mathbf{m}), \min_{j=1,\dots,N} R_{\xi_j, \hat{\tau}_j}(\mathbf{f}, \mathbf{m}) \right\}$$

and let $D_k^*(\mathbf{f}, \mathbf{m})$ be the canonical domain with this index.

By $Q_{\mathbf{m}}^{\mathbf{f}}$ we denote the monic polynomial whose zeros are the system poles of $\mathbf{f}$ with respect to $\mathbf{m}$ taking account of their order. The set of distinct zeros of $Q_{\mathbf{m}}^{\mathbf{f}}$ is denoted by $\mathcal{P}_{\mathbf{m}}^{\mathbf{f}}$.

The following theorem constitutes our main result.

Theorem 1.3 *Suppose (1.2) takes place. Let $\mathbf{f} \in \mathcal{H}(E)^d$ and fix a multi-index $\mathbf{m} \in \mathbb{N}^d$. Then, the next two assertions are equivalent:*

- (a) $\mathbf{f}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ counting multiplicities.
- (b) For all sufficiently large n , the denominators $Q_{n,\mathbf{m}}$ of multipoint Hermite-Padé approximants of $\mathbf{f}$ are uniquely determined and there exists a polynomial $Q_{\mathbf{m}}$ of degree $|\mathbf{m}|$ such that

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}} - Q_{\mathbf{m}}\|^{1/n} = \theta < 1, \quad (1.5)$$

where $\|\cdot\|$ denotes the coefficient norm in the space of polynomials of degree $\leq |\mathbf{m}|$. Moreover, if either (a) or (b) takes place, then $Q_{\mathbf{m}} \equiv Q_{\mathbf{m}}^{\mathbf{f}}$,

$$\theta = \max \left\{ \frac{|\Phi(\xi)|}{R_{\xi}(\mathbf{f}, \mathbf{m})} : \xi \in \mathcal{P}_{\mathbf{m}}^{\mathbf{f}} \right\}, \quad (1.6)$$

and for any compact subset $\mathcal{K}$ of $D_k^*(\mathbf{f}, \mathbf{m}) \setminus \mathcal{P}_{\mathbf{m}}^{\mathbf{f}}$,

$$\limsup_{n \rightarrow \infty} \|R_{n,\mathbf{m},k} - f_k\|_{\mathcal{K}}^{1/n} \leq \frac{\|\Phi\|_{\mathcal{K}}}{R_k^*(\mathbf{f}, \mathbf{m})}, \quad (1.7)$$

where $\|\cdot\|_{\mathcal{K}}$ denotes the sup-norm on $\mathcal{K}$. If $\mathcal{K} \subset E$, $\|\Phi\|_{\mathcal{K}}$ is replaced by 1.

2 Direct statements

2.1 An auxiliary result

For each $n \geq |\mathbf{m}|$, let $q_{n,\mathbf{m}}$ be the polynomial $Q_{n,\mathbf{m}}$ normalized so that

$$\sum_{k=0}^{|\mathbf{m}|} |\lambda_{n,k}| = 1, \quad q_{n,\mathbf{m}}(z) = \sum_{k=0}^{|\mathbf{m}|} \lambda_{n,k} z^k. \quad (2.1)$$

This normalization implies that the polynomials $q_{n,\mathbf{m}}$ are uniformly bounded on each compact subset of $\mathbb{C}$.

Lemma 2.1 *Let $\mathbf{f} \in \mathcal{H}(E)^d$ and fix a multi-index $\mathbf{m} \in \mathbb{N}^d$. Assume that (1.2) takes place and ξ is a system pole of order τ of $\mathbf{f}$ with respect to $\mathbf{m}$. Then*

$$\limsup_{n \rightarrow \infty} |q_{n,\mathbf{m}}^{(s)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{R_{\xi,s+1}(\mathbf{f}, \mathbf{m})}, \quad s = 0, \dots, \tau - 1. \quad (2.2)$$

Proof. Consider a polynomial combination g_1 of type (1.4) that is analytic on a neighborhood of $\bar{D}_{|\Phi(\xi)|}$ except for a simple pole $z = \xi$ and verifies that $\rho_1(g_1) = R_{\xi,1}(\mathbf{f}, \mathbf{m}) = \rho_{\xi,1}(\mathbf{f}, \mathbf{m})$. Then, we have

$$g_1 = \sum_{k=1}^d p_{k,1} f_k, \quad \deg p_{k,1} < m_k, \quad k = 1, \dots, d.$$

Define $h_1(z) = (z - \xi)g_1(z)$. The function

$$\frac{q_{n,\mathbf{m}}(z)h_1(z)}{a_{n+1}(z)} - \frac{z - \xi}{a_{n+1}(z)} \sum_{k=1}^d p_{k,1}(z)P_{n,\mathbf{m},k}(z)$$

is analytic on $D_{\rho_1(g_1)}$. Take $1 < \rho < \rho_1(g_1)$, and set $\Gamma_\rho = \{z \in \mathbb{C} : |\Phi(z)| = \rho\}$.

Set $P_{n,1}(z) = \sum_{k=1}^d p_{k,1}(z)P_{n,\mathbf{m},k}(z)$. Since $\deg(z - \xi)P_{n,1}(z) \leq n$, we have

$$\frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{(t - \xi)P_{n,1}(t)}{(t - z)a_{n+1}(t)} dt = 0.$$

Using Hermite's interpolation formula (see [13]), we obtain

$$q_{n,\mathbf{m}}(z)h_1(z) - (z - \xi) \sum_{k=1}^d p_{k,1}P_{n,\mathbf{m},k}(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{q_{n,\mathbf{m}}(t)h_1(t)}{t - z} dt,$$

for all z with $|\Phi(z)| < \rho$. In particular, taking $z = \xi$ in the above formula, we arrive at

$$q_{n,\mathbf{m}}(\xi)h_1(\xi) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{a_{n+1}(\xi)}{a_{n+1}(t)} \frac{q_{n,\mathbf{m}}(t)h_1(t)}{t - \xi} dt. \quad (2.3)$$

Then, taking account of (1.2), it easily follows that

$$\limsup_{n \rightarrow \infty} |q_{n,\mathbf{m}}(\xi)h_1(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho}.$$

Using that $h_1(\xi) \neq 0$ and making ρ tend to $\rho_1(g_1)$, we obtain

$$\limsup_{n \rightarrow \infty} |q_{n,\mathbf{m}}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{R_{\xi,1}(\mathbf{f}, \mathbf{m})} < 1.$$

Now, we employ induction. Suppose that

$$\limsup_{n \rightarrow \infty} |q_{n,\mathbf{m}}^{(j)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{R_{\xi,j+1}(\mathbf{f}, \mathbf{m})}, \quad j = 0, 1, \dots, s-2, \quad (2.4)$$

where $s \leq \tau$. Let us prove that formula (2.4) holds for $j = s-1$. This will imply (2.2).

Consider a polynomial combination g_s of type (1.4) that is analytic on a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole of order s at $z = \xi$ and verifies that $\rho_s(g_s) = R_{\xi,s}(\mathbf{f}, \mathbf{m})$. Then,

$$g_s = \sum_{k=1}^d p_{k,s} f_k, \quad \deg p_{k,s} < m_k, \quad k = 1, \dots, d.$$

Set $h_s(z) = (z - \xi)^s g_s(z)$. The function

$$\frac{q_{n,\mathbf{m}}(z)h_s(z)}{a_{n+1}(z)(z - \xi)^{s-1}} - \frac{z - \xi}{a_{n+1}(z)} \sum_{k=1}^d p_{k,s}(z)P_{n,\mathbf{m},k}(z)$$

is analytic on $D_{\rho_s(g_s)} \setminus \{\xi\}$. Set $P_{n,s} = \sum_{k=1}^d p_{k,s} P_{n,\mathbf{m},k}$. Fix an arbitrary compact set $\mathcal{K} \subset D_{\rho_s(g_s)} \setminus \{\xi\}$. Take $\delta > 0$ sufficiently small so that $\{z \in \mathbb{C} : |z - \xi| \leq \delta\} \cap \mathcal{K} = \emptyset$ and $1 < \rho < \rho_s(g_s)$. Using Hermite's interpolation formula, for all $z \in \mathcal{K}$, we have

$$\frac{q_{n,\mathbf{m}}(z)h_s(z)}{(z - \xi)^{s-1}} - (z - \xi)P_{n,s}(z) = I_n(z) - J_n(z), \quad (2.5)$$

where

$$I_n(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{q_{n,\mathbf{m}}(t)h_s(t)}{(t - \xi)^{s-1}(t - z)} dt$$

and

$$J_n(z) = \frac{1}{2\pi i} \int_{|t - \xi| = \delta} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{q_{n,\mathbf{m}}(t)h_s(t)}{(t - \xi)^{s-1}(t - z)} dt.$$

The first integral I_n is estimated as in (2.3) to obtain

$$\limsup_{n \rightarrow \infty} \|I_n\|_{\mathcal{K}}^{1/n} \leq \frac{\|\Phi\|_{\mathcal{K}}}{\rho_s(g_s)}. \quad (2.6)$$

For J_n , as $\deg q_{n,\mathbf{m}} \leq |\mathbf{m}|$ write

$$q_{n,\mathbf{m}}(t) = \sum_{j=0}^{|\mathbf{m}|} \frac{q_{n,\mathbf{m}}^{(j)}(\xi)}{j!} (t - \xi)^j.$$

Then

$$J_n(z) = \sum_{j=0}^{s-2} \frac{1}{2\pi i} \int_{|t-\xi|=\delta} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{h_s(t)}{(t-\xi)^{s-1-j}} \frac{q_{n,\mathbf{m}}^{(j)}(\xi)}{j!(t-z)} dt. \quad (2.7)$$

Using the induction hypothesis (2.4), from (2.7) it easily follows that

$$\limsup_{n \rightarrow \infty} \|J_n\|_{\mathcal{K}}^{1/n} \leq \frac{\|\Phi\|_{\mathcal{K}}}{|\Phi(\xi)|} \frac{|\Phi(\xi)|}{R_{\xi,s-1}(\mathbf{f}, \mathbf{m})} = \frac{\|\Phi\|_{\mathcal{K}}}{R_{\xi,s-1}(\mathbf{f}, \mathbf{m})}. \quad (2.8)$$

Now, (2.5), (2.6), and (2.8) give

$$\limsup_{n \rightarrow \infty} \|q_{n,\mathbf{m}} h_s - (z - \xi)^s P_{n,s}\|_{\mathcal{K}}^{1/n} \leq \frac{\|\Phi\|_{\mathcal{K}}}{R_{\xi,s}(\mathbf{f}, \mathbf{m})}. \quad (2.9)$$

As the function inside the norm in (2.9) is analytic in $D_{\rho_l(g_l)}$, from the maximum principle it follows that (2.9) also holds for any compact set $\mathcal{K} \subset D_{\rho_l(g_l)}$. Using Cauchy's integral formula, from (2.9) we also obtain that

$$\limsup_{n \rightarrow \infty} \|(q_{n,\mathbf{m}} h_s - (z - \xi)^s P_{n,s})^{(s-1)}\|_{\mathcal{K}}^{1/n} \leq \frac{\|\Phi\|_{\mathcal{K}}}{R_{\xi,s}(\mathbf{f}, \mathbf{m})}. \quad (2.10)$$

Taking $z = \xi$ in (2.10), we have

$$\limsup_{n \rightarrow \infty} |(q_{n,\mathbf{m}} h_s)^{(s-1)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{R_{\xi,s}(\mathbf{f}, \mathbf{m})}.$$

Using the Leibniz formula for higher derivatives of a product of two functions, the induction hypothesis (2.4), and that $h_s(\xi) \neq 0$, it follows that

$$\limsup_{n \rightarrow \infty} |q_{n,\mathbf{m}}^{(s-1)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{R_{\xi,s}(\mathbf{f}, \mathbf{m})}.$$

This completes the induction and the proof. $\square$

2.2 Proof of (a) $\Rightarrow$ (b)

Let $\{\xi_1, \dots, \xi_p\}$ be the distinct system poles of $\mathbf{f}$ with respect to $\mathbf{m}$, and let τ_j be the order of ξ_j as a system pole, $j = 1, \dots, p$. By assumption, $\tau_1 + \dots + \tau_p = |\mathbf{m}|$. We have proved that, for $j = 1, \dots, p$ and $s = 0, 1, \dots, \tau_j - 1$,

$$\limsup_{n \rightarrow \infty} |q_{n,\mathbf{m}}^{(s)}(\xi_j)|^{1/n} \leq \frac{|\Phi(\xi_j)|}{R_{\xi_j,s+1}(\mathbf{f}, \mathbf{m})} \leq \frac{|\Phi(\xi_j)|}{R_{\xi_j}(\mathbf{f}, \mathbf{m})}, \quad (2.11)$$

where $R_{\xi_j}(\mathbf{f}, \mathbf{m}) := R_{\xi_j, \tau_j}(\mathbf{f}, \mathbf{m})$. Using the Hermite interpolation, it is easy to construct a basis $\{\ell_{j,s}\}$, $1 \leq j \leq p, 0 \leq s \leq \tau_j - 1$, in the space of polynomials of degree at most $|\mathbf{m}| - 1$ satisfying

$$\ell_{j,s}^{(k)}(\xi_i) = \delta_{i,j} \delta_{k,s}, \quad 1 \leq i \leq p, \quad 0 \leq k \leq \tau_i - 1.$$

Then,

$$q_{n,\mathbf{m}}(z) = \sum_{j=1}^p \sum_{s=0}^{\tau_j-1} q_{n,\mathbf{m}}^{(s)}(\xi_j) \ell_{j,s}(z) + \lambda_{n,|\mathbf{m}|} Q_{\mathbf{m}}^{\mathbf{f}}. \quad (2.12)$$

Using (2.11) and (2.12), we have for any compact set $\mathcal{K} \subset \mathbb{C}$,

$$\limsup_{n \rightarrow \infty} \|q_{n,\mathbf{m}} - \lambda_{n,|\mathbf{m}|} Q_{\mathbf{m}}^{\mathbf{f}}\|_{\mathcal{K}}^{1/n} \leq \theta, \quad (2.13)$$

where

$$\theta = \max \left\{ \frac{|\Phi(\xi)|}{R_{\xi}(\mathbf{f}, \mathbf{m})} : \xi \in \mathcal{P}_{\mathbf{m}}^{\mathbf{f}} \right\} < 1. \quad (2.14)$$

Now, necessarily

$$\liminf_{n \rightarrow \infty} |\lambda_{n,|\mathbf{m}|}| > 0. \quad (2.15)$$

Indeed, if there is a subsequence of indices $\Lambda \subset \mathbb{N}$ such that $\lim_{n \in \Lambda} |\lambda_{n,|\mathbf{m}|}| = 0$, then from (2.14), as the polynomials $q_{n,\mathbf{m}}$ converge, we would have that $\lim_{n \in \Lambda} q_{n,\mathbf{m}} = 0$ which contradicts (2.1). Since

$$q_{n,\mathbf{m}} = \lambda_{n,|\mathbf{m}|} Q_{n,\mathbf{m}},$$

from (2.13) and (2.15) it follows that

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}} - Q_{\mathbf{m}}^{\mathbf{f}}\|_{\mathcal{K}}^{1/n} \leq \theta. \quad (2.16)$$

In finite dimensional spaces all norms are equivalent; therefore, (2.16) is also true with the coefficient norm which means that (1.5) is satisfied with the equality replaced by the inequality $\leq$.

In particular, for all sufficiently large n necessarily $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$. The difference of any two distinct monic polynomials satisfying Definition 1.1 with the same degree produces a new solution of degree strictly less than $|\mathbf{m}|$, but we have proved that any solution must have degree $|\mathbf{m}|$ for all sufficiently large n . So, the polynomial $Q_{n,\mathbf{m}}$ is uniquely determined for all sufficiently large n .

Now, we prove the equality in (1.5). To the contrary, suppose that

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}} - Q_{\mathbf{m}}^{\mathbf{f}}\|^{1/n} < \max \left\{ \frac{|\Phi(\xi)|}{R_{\xi}(\mathbf{f}, \mathbf{m})} : \xi \in \mathcal{P}_{\mathbf{m}}^{\mathbf{f}} \right\}. \quad (2.17)$$

Let ζ be a system pole of $\mathbf{f}$ such that

$$\frac{|\Phi(\zeta)|}{R_{\zeta}(\mathbf{f}, \mathbf{m})} = \max \left\{ \frac{|\Phi(\xi)|}{R_{\xi}(\mathbf{f}, \mathbf{m})} : \xi \in \mathcal{P}_{\mathbf{m}}^{\mathbf{f}} \right\}. \quad (2.18)$$

Clearly, the inequality (2.17) implies that $R_\zeta(\mathbf{f}, \mathbf{m}) < \infty$.

Choose a polynomial combination

$$g = \sum_{k=1}^d p_k f_k, \quad \deg p_k < m_k, \quad k = 1, \dots, d, \quad (2.19)$$

that is holomorphic on a neighborhood of $\overline{D}_{|\Phi(\zeta)|}$ except for a pole of some order l at $z = \zeta$ with $\rho_l(g) = R_\zeta(\mathbf{f}, \mathbf{m})$. Notice that $Q_{\mathbf{m}}^{\mathbf{f}}g$ must have a singularity on the boundary of $D_{\rho_l(g)}$ which implies

$$\frac{1}{R_\zeta(\mathbf{f}, \mathbf{m})} = c \cdot \limsup_{n \rightarrow \infty} \left| \int_{\Gamma_\rho} \frac{Q_{\mathbf{m}}^{\mathbf{f}}(t)g(t)}{a_{n+1}(t)} dt \right|^{1/n}. \quad (2.20)$$

In fact, if $Q_{\mathbf{m}}^{\mathbf{f}}g$ had no singularity on the boundary of $D_{\rho_l(g)}$, then all singularities of g on the boundary of $D_{\rho_l(g)}$ would be at most poles and their order as poles of g would be smaller than their order as system poles of $\mathbf{f}$. In this case, we could find a different polynomial combination g_1 of type (2.19) for which $\rho_l(g_1) > \rho_l(g) = R_\zeta(\mathbf{f}, \mathbf{m})$ which contradicts the definition of $R_\zeta(\mathbf{f}, \mathbf{m})$. Therefore, $Q_{\mathbf{m}}^{\mathbf{f}}g$ has a singularity on the the boundary of $D_{\rho_l(g)}$ and the equality (2.20) holds.

Now,

$$\left(Q_{n,\mathbf{m}}(z)g(z) - \sum_{k=1}^d p_k(z)P_{n,\mathbf{m},k}(z) \right) / a_{n+1}(z)$$

is holomorphic in $D_{\rho_l(g)}$ and $\deg \sum_{k=1}^d p_k P_{n,\mathbf{m},k} < n$; therefore, from Cauchy's integral theorem we have that

$$0 = \int_{\Gamma_\rho} \frac{Q_{n,\mathbf{m}}(z)g(z) - \sum_{k=1}^d p_k(z)P_{n,\mathbf{m},k}(z)}{a_{n+1}(z)} dz = \int_{\Gamma_\rho} \frac{Q_{n,\mathbf{m}}(z)g(z)}{a_{n+1}(z)} dz, \quad (2.21)$$

where $1 < \rho < |\Phi(\zeta)|$. Combining (2.20) and (2.21), we get

$$\frac{1}{R_\zeta(\mathbf{f}, \mathbf{m})} = c \cdot \limsup_{n \rightarrow \infty} \left| \int_{\Gamma_\rho} \frac{g(t)}{a_{n+1}(t)} (Q_{\mathbf{m}}^{\mathbf{f}}(t) - Q_{n,\mathbf{m}}(t)) dt \right|^{1/n}. \quad (2.22)$$

This equality is impossible because from (1.2), (2.17), and (2.18) it follows that (2.22) is strictly less than $1/R_\zeta(\mathbf{f}, \mathbf{m})$. This proves the equality in (1.5) as well as (1.6).

If ξ is any one of the system poles of $\mathbf{f}$ and τ its order, from (2.11) and (2.15), we have

$$\max_{j=0,\dots,l} \limsup_{n \rightarrow \infty} |Q_{n,\mathbf{m}}^{(j)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{R_{\xi,l+1}(\mathbf{f}, \mathbf{m})}, \quad l = 0, 1, \dots, \tau - 1. \quad (2.23)$$

Now we are ready to prove (1.7). Let us fix $k \in \{1, \dots, d\}$. Let $\mathcal{K}$ be a compact subset contained in $D_k^*(\mathbf{f}, \mathbf{m}) \setminus \mathcal{P}_{\mathbf{m}}^{\mathbf{f}}$. Take $\delta > 0$ sufficiently small so that

$$1 < \rho := R_k^*(\mathbf{f}, \mathbf{m}) - \delta, \quad \mathcal{K} \subset D_\rho, \quad \bigcup_{j=1}^{N_k} \{z \in \mathbb{C} : |z - \xi_j| \leq \delta\} \subset D_\rho \setminus \mathcal{K},$$

where $\xi_1, \dots, \xi_{N_k}$ are the poles of f_k in $D_k^*(\mathbf{f}, \mathbf{m})$. Set

$$C_j := \{z \in \mathbb{C} : |z - \xi_j| = \delta\}.$$

Let $\Gamma_{\rho, \delta}$ be the positively oriented curve determined by Γ_ρ and those circles C_j . On account of Definition 1.1, using Hermite's formula, we have

$$(Q_{n, \mathbf{m}} f_k - P_{n, \mathbf{m}, k})(z) = \frac{1}{2\pi i} \int_{\Gamma_{\rho, \delta}} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{(Q_{n, \mathbf{m}} f_k)(t)}{t - z} dt. \quad (2.24)$$

From (1.2) it readily follows that for all $z \in \mathcal{K}$,

$$\limsup_{n \rightarrow \infty} \left| \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{(Q_{n, \mathbf{m}} f_k)(t)}{t - z} dt \right|^{1/n} \leq \frac{\|\Phi\|_{\mathcal{K}}}{R_k^*(\mathbf{f}, \mathbf{m})}. \quad (2.25)$$

Let $\hat{\tau}_j$ be the order of ξ_j as pole of f_k . Using the expansion

$$Q_{n, \mathbf{m}}(t) = \sum_{l=0}^{|\mathbf{m}|} \frac{Q_{n, \mathbf{m}}^{(l)}(\xi_j)}{l!} (t - \xi_j)^l,$$

for the circle C_j we have

$$\begin{aligned} & \frac{1}{2\pi i} \int_{C_j} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{(Q_{n, \mathbf{m}} f_k)(t)}{t - z} dt \\ &= \sum_{l=0}^{\hat{\tau}_j - 1} \frac{1}{2\pi i} \int_{C_j} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{(t - \xi_j)^{\hat{\tau}_j} f_k(t)}{(t - \xi_j)^{\hat{\tau}_j - l}} \frac{Q_{n, \mathbf{m}}^{(l)}(\xi_j)}{l! (t - z)} dt \end{aligned} \quad (2.26)$$

because the function under the integral sign is analytic inside C_j for $\hat{\tau}_j \leq l \leq |\mathbf{m}|$. Now, (1.2) and (2.23) allow us to deduce from (2.26) that for all $z \in \mathcal{K}$,

$$\limsup_{n \rightarrow \infty} \left| \frac{1}{2\pi i} \int_{C_j} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{(Q_{n, \mathbf{m}} f_k)(t)}{t - z} dt \right|^{1/n} \leq \frac{\|\Phi\|_{\mathcal{K}}}{|\Phi(\xi_j)| R_{\xi_j, \hat{\tau}_j}(\mathbf{f}, \mathbf{m})}. \quad (2.27)$$

Finally, (2.24), (2.25), and (2.27) give (1.7). $\square$

A slight variation of the arguments employed above allows us to deduce the following corollary of independent interest.

Corollary 2.2 *Let $\mathbf{f} \in \mathcal{H}(E)^d$ and fix a multi-index $\mathbf{m} \in \mathbb{N}^d$. Suppose that (1.2) takes place and $\mathbf{f}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$. Then, for every system pole ξ of $\mathbf{f}$,*

$$\max_{j=0,\dots,l} \limsup_{n \rightarrow \infty} |Q_{n,\mathbf{m}}^{(j)}(\xi)|^{1/n} = \frac{|\Phi(\xi)|}{R_{\xi,l+1}(\mathbf{f}, \mathbf{m})}, \quad l = 0, 1, \dots, \tau - 1. \quad (2.28)$$

where τ is the order of ξ .

Proof. If (2.28) fails, due to (2.23), there is a system pole ξ of $\mathbf{f}$ of order τ such that for some $l, 0 \leq l < \tau$

$$\max_{j=0,\dots,l} \limsup_{n \rightarrow \infty} |Q_{n,\mathbf{m}}^{(j)}(\xi)|^{1/n} < \frac{|\Phi(\xi)|}{R_{\xi,l+1}(\mathbf{f}, \mathbf{m})}. \quad (2.29)$$

Now, we argue by contradiction as in the proof of the equality in (1.5).

Choose a polynomial combination g as in (2.19) that is analytic on a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole of order $s(\leq l+1)$ at $z = \xi$ with $\rho_s(g) = R_{\xi,l+1}(\mathbf{f}, \mathbf{m})$. Set $Q_{\mathbf{m}}^{\mathbf{f}} = Q_{\mathbf{m}}$. Take $\delta > 0$ sufficiently small and $1 < \rho < \rho_s(g)$. Let $\Gamma_{\rho,\delta}$ be the positively oriented curve determined by Γ_ρ and $\{t \in \mathbb{C} : |t - \xi| = \delta\}$. Arguing as in (2.20), it follows from (1.3) that

$$\frac{1}{\rho_s(g)} = c \cdot \limsup_{n \rightarrow \infty} \left| \int_{\Gamma_{\rho,\delta}} \frac{Q_{\mathbf{m}}(t)g(t)}{a_{n+1}(t)} dt \right|^{1/n}. \quad (2.30)$$

The function

$$\frac{H_n(z)}{a_{n+1}(z)} = \frac{Q_{n,\mathbf{m}}(z)g(z) - \sum_{k=1}^d p_k(z)P_{n,\mathbf{m},k}(z)}{a_{n+1}(z)}$$

is analytic in $D_{\rho_s}(g) \setminus \{\xi\}$ and

$$\int_{\Gamma_{\rho,\delta}} \frac{H_n(t)}{a_{n+1}(t)} dt = 0.$$

Set $P_n := \sum_{k=1}^d p_k P_{n,\mathbf{m},k}$ and $h := (t - \xi)^s g$. Obviously,

$$Q_{\mathbf{m}}g = (Q_{\mathbf{m}} - Q_{n,\mathbf{m}})g + P_n + H_n,$$

and since $\deg P_n \leq n - 1$, we obtain

$$\begin{aligned} \int_{\Gamma_{\rho,\delta}} \frac{Q_{\mathbf{m}}(t)g(t)}{a_{n+1}(t)} dt &= \int_{\Gamma_{\rho,\delta}} \frac{[Q_{\mathbf{m}} - Q_{n,\mathbf{m}}](t)h(t)}{(t - \xi)^s a_{n+1}(t)} dt \\ &= \int_{\Gamma_\rho} \frac{[Q_{\mathbf{m}} - Q_{n,\mathbf{m}}](t)h(t)}{(t - \xi)^s a_{n+1}(t)} dt - \sum_{j=0}^{|\mathbf{m}|} \int_{|t-\xi|=\delta} \frac{[Q_{\mathbf{m}}^{(j)} - Q_{n,\mathbf{m}}^{(j)}](\xi)h(t)}{j!(t - \xi)^{s-j} a_{n+1}(t)} dt \\ &= \int_{\Gamma_\rho} \frac{[Q_{\mathbf{m}} - Q_{n,\mathbf{m}}](t)h(t)}{(t - \xi)^s a_{n+1}(t)} dt + \sum_{j=0}^{s-1} \int_{|t-\xi|=\delta} \frac{Q_{n,\mathbf{m}}^{(j)}(\xi)h(t)}{j!(t - \xi)^{s-j} a_{n+1}(t)} dt. \end{aligned}$$

Estimating these integrals, using (1.2), (1.5), and the assumption (2.29), it is easy to deduce that

$$c \cdot \limsup_{n \rightarrow \infty} \left| \int_{\Gamma_{\rho, \delta}} \frac{Q_{\mathbf{m}}(t)g(t)}{a_{n+1}(t)} dt \right|^{1/n} < \frac{1}{\rho_s(g)}$$

which contradicts (2.30). Therefore, (2.29) cannot occur and there is equality in (2.28). $\square$

Remark 2.1 We wish to underline that for the proof of the previous results, excluding the equality in (1.5) and (2.28), it would have been sufficient to assume that the table of points verifies (1.1) instead of (1.2). The condition (1.2) has only been used in order to have the Cauchy-Hadamard type formula (1.3). For the inverse type statement $(b) \Rightarrow (a)$ the stronger assumption (1.2) is much more substantial.

3 Inverse statements

3.1 Some auxiliary results

Let

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \quad (3.1)$$

be a power series convergent in some neighborhood of the point $z = 0$ whose radius of convergence we denote by $R_0(f)$. According to the Cauchy-Hadamard formula $R_0(f) = (\limsup_{n \rightarrow \infty} |f_n|^{1/n})^{-1}$.

The following theorem was proved by V.I. Buslaev in [3, Supplement of Theorem 2].

Buslaev's Theorem. Suppose that the power series (3.1) is not a polynomial, $R_0(f) = \infty$, and

$$\alpha_{n,0}f_n + \alpha_{n,-1}f_{n+1} + \cdots = 0 \quad (n = 1, 2, \dots) \quad (3.2)$$

where the $\alpha_n(z) = \sum_{p=0}^{\infty} \alpha_{n,-p} z^{-p}$ ($n = 1, 2, \dots$) are holomorphic and converge to $\alpha(z)$ in the exterior of some disk as $n \rightarrow \infty$. Then $\alpha(\infty) = 0$, and the coefficients $\{f_n\}$ of the series (3.1) satisfy

$$\epsilon_{n,0}f_n + \cdots + \epsilon_{n,-N+1}f_{n+N-1} + f_{n+N} = 0, \quad \lim_{n \rightarrow \infty} \epsilon_{n,p} = \epsilon_p,$$

for $p = 0, -1, \dots, -N+1$, N being the multiplicity of the zero of α at $z = \infty$.

This result will be useful in the next section to prove Lemma 3.2.

3.2 Incomplete multipoint Padé approximants

Let us introduce the notion of incomplete multipoint Padé approximants. A similar concept turned out to be effective in the study of Hermite-Padé approximation in [4] and [5] for proving results of inverse type.

Definition 3.1 Let $f \in \mathcal{H}(E)$. Fix $m \geq m^* \geq 1$ and $n \geq m$. We say that the rational function $R_{n,m}$ is an *incomplete multipoint Padé approximant of type (n, m, m^*) corresponding to f* if $R_{n,m}$ is the quotient of any two polynomials $P_{n,m}, Q_{n,m}$ that verify

- (i) $\deg P_{n,m} \leq n - m^*, \deg Q_{n,m} \leq m, Q_{n,m} \neq 0$,
- (ii) $\frac{Q_{n,m}f - P_{n,m}}{a_{n+1}} \in \mathcal{H}(E)$,

where $a_n(z) = \prod_{k=1}^n (z - \alpha_{n,k})$.

Since $Q_{n,m} \neq 0$, we normalize it to be monic. We call $Q_{n,m}$ the denominator of the corresponding (n, m, m^*) incomplete multipoint Padé approximant of f . Notice that for $k = 1, \dots, d$, $Q_{n,\mathbf{m}}$, given in Definition 1.1, is a denominator of an $(n, |\mathbf{m}|, m_k)$ incomplete multipoint Padé approximant of f_k .

In this section, we will study the relation between the convergence of $Q_{n,m}$ and some analytic properties of f .

Lemma 3.2 Let $f \in \mathcal{H}(E)$ and fix $m \geq m^* \geq 1$. Suppose that f is not a rational function with at most $m^* - 1$ poles and there exists a polynomial Q_m of degree m such that

$$\limsup_{n \rightarrow \infty} \|Q_{n,m} - Q_m\|^{1/n} \leq \theta < 1. \quad (3.3)$$

Then, either f has exactly m^* poles in $D_{\rho_{m^*}(f)}$ or $\rho_0(Q_m f) > \rho_{m^*}(f)$, where $\rho_{m^*}(f)$ is the index of the largest canonical region to which f can be extended as a meromorphic function with at most m^* poles counting multiplicities.

Proof. Let $\{\xi_1, \dots, \xi_\omega\}$ be the distinct poles of f in $D_{\rho_{m^*}(f)}$ and $\tau_1, \dots, \tau_\omega$ be their orders, respectively. Consequently,

$$\sum_{j=1}^{\omega} \tau_j \leq m^*.$$

Modifying conveniently the proof of (2.2), one can show that for $j = 1, \dots, \omega$

$$\limsup_{n \rightarrow \infty} |Q_{n,m}^{(\nu)}(\xi_j)|^{1/n} \leq \frac{|\Phi(\xi_j)|}{\rho_{m^*}(f)} < 1, \quad \nu = 0, 1, \dots, \tau_j - 1. \quad (3.4)$$

Since the sequence of polynomials $Q_{n,m}$ converges to Q_m , (3.4) entails that ξ_j is a zero of Q_m of multiplicity at least τ_j . Being this the case, we have

$$\rho_0(Q_m f) \geq \rho_{m^*}(f).$$

Suppose that $\rho_0(Q_m f) = \rho_{m^*}(f)$. To conclude the proof, let us show that in this situation f has exactly m^* poles in $D_{\rho_{m^*}(f)}$. To the contrary, suppose that f has in $D_{\rho_{m^*}(f)}$ at most $m^* - 1$ poles. Then, there exists a polynomial $\deg Q_{m^*} < m^*$ such that

$$\rho_0(Q_{m^*} f) = \rho_{m^*}(f) = \rho_0(Q_m Q_{m^*} f).$$

It follows from Definition 3.1 that

$$\frac{Q_{m^*}(Q_{n,m}f - P_{n,m})}{a_{n+1}} \in \mathcal{H}(E).$$

Then

$$\int_{\Gamma_\rho} \frac{Q_{m^*}(z)(Q_{n,m}f - P_{n,m})(z)}{a_{n+1}(z)} dz = 0,$$

where $1 < \rho < \rho_{m^*}(f)$. Since each one of the $n + 1$ zeros of the polynomial a_{n+1} lies on E and $\deg(Q_{m^*}P_{n,m}) \leq n - 1$, it follows that

$$\int_{\Gamma_\rho} \frac{Q_{m^*}(z)P_{n,m}(z)}{a_{n+1}(z)} dz = 0.$$

Therefore,

$$\int_{\Gamma_\rho} \frac{Q_{m^*}(z)Q_{n,m}(z)f(z)}{a_{n+1}(z)} dz = 0. \quad (3.5)$$

Then, by (1.3),

$$\begin{aligned} \frac{1}{\rho_{m^*}(f)} &= \frac{1}{\rho_0(Q_m Q_{m^*} f)} = c \cdot \limsup_{n \rightarrow \infty} \left| \int_{\Gamma_\rho} \frac{(Q_m Q_{m^*} f)(t)}{a_{n+1}(t)} dt \right|^{1/n} \\ &= c \cdot \limsup_{n \rightarrow \infty} \left| \int_{\Gamma_\rho} \frac{(Q_{m^*} f)(t)}{a_{n+1}(t)} (Q_{n,m} - Q_m)(t) dt \right|^{1/n}. \end{aligned}$$

Using (1.2) and (3.3) to estimate the last integral, it readily follows that

$$\frac{1}{\rho_{m^*}(f)} \leq \frac{\theta}{\rho_{m^*}(f)}, \quad \theta < 1,$$

which implies that $\rho_{m^*}(f) = \infty$. Now, let us show that this is not possible.

Take $F(w) := Q_{m^*}(\Psi(w))f(\Psi(w))$, where $\Psi = \Phi^{-1}$. Let γ be a contour encircling $\{w \in \mathbb{C} : |w| = 1\}$ lying in the domain of holomorphy of F . Using (3.5), we obtain

$$\begin{aligned} 0 &= \int_{\gamma} \frac{F(w)Q_{n,m}(\Psi(w))}{a_{n+1}(\Psi(w))} \Psi'(w) dw = \\ &= \int_{\gamma} F(w) \frac{Q_{n,m}(\Psi(w))}{w^m} \frac{w^{n+1}}{a_{n+1}(\Psi(w))} \Psi'(w) \frac{dw}{w^{n+1-m}} \end{aligned}$$

Setting

$$\alpha_n(w) = \frac{Q_{n,m}(\Psi(w))}{w^m} \frac{(cw)^{n+1}}{a_{n+1}(\Psi(w))} \Psi'(w),$$

the previous equality means that the equality in (3.2) holds. The functions α_n ($n = 1, 2, \dots$) are holomorphic in the exterior of the unit disk (including $w = \infty$) and, due to (1.2) and (3.3), converge as $n \rightarrow \infty$ to

$$\alpha(w) = \Psi'(w) \frac{Q_m(\Psi(w))}{w^m G(\Psi(w))} = \sum_{p=0}^{\infty} \alpha_{-p} w^{-p}, \quad \alpha_0 = \alpha(\infty) \neq 0.$$

Let $\sum_{n=-\infty}^{\infty} F_n w^n$ be the Laurent expansion of the function F outside the unit circle, i.e:

$$F(w) = \sum_{n=-\infty}^{\infty} F_n w^n = F_1(w) + F_2(w),$$

where $F_1(w) = \sum_{n=0}^{\infty} F_n w^n$. Then, $R_0(F_1) = \infty$ and (3.2) holds (for all sufficiently large n) replacing F with F_1 . According to Buslaev's Theorem and the fact that $\alpha(\infty) \neq 0$, we get that F_1 must be a polynomial. Consequently, F is either analytic or has a pole at ∞ . In turn this implies that $Q_{m^*} f$ is either analytic or has a pole at ∞ . However, $Q_{m^*} f$ is an entire function because it is holomorphic in $\mathbb{C}$ since $R_0(Q_{m^*} f) = \infty$. Therefore, $Q_{m^*} f$ is a polynomial, or what is the same f is a rational function with at most $m^* - 1$ poles against our hypothesis on f . This contradiction implies that the assumption that f had in $D_{\rho_{m^*}(f)}$ at most $m^* - 1$ poles is impossible. So the number of poles on f in $D_{\rho_{m^*}(f)}$ must equal m^* . $\square$

3.3 Polynomial independence

Let us introduce the concept of polynomial independence of a vector of functions.

Definition 3.3 A vector $\mathbf{f} = (f_1, \dots, f_d) \in \mathcal{H}(E)^d$ is said to be *polynomially independent with respect to* $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ if there do not exist polynomials $p_1, \dots, p_d$, at least one of which is non-null, such that

- (i) $\deg p_k < m_k$, $k = 1, \dots, d$,
- (ii) $\sum_{k=1}^d p_k f_k$ is a polynomial.

In particular, polynomial independence implies that for each $k = 1, \dots, d$, f_k is not a rational function with at most $m_k - 1$ poles.

Lemma 3.4 Let $\mathbf{f} \in \mathcal{H}(E)^d$ and fix a multi-index $\mathbf{m} \in \mathbb{N}^d$. Suppose that for all $n \geq n_0$, the polynomial $Q_{n,\mathbf{m}}$ is unique and $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$. Then the system $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$.

Proof. Except for a small detail, the proof coincides with that of [5, Lemma 3.2]. Given $\mathbf{f} := (f_1, \dots, f_d) \in \mathcal{H}(E)^d$ and $\mathbf{m} := (m_1, \dots, m_d) \in \mathbb{N}^d$, we consider the associated system

$$\bar{\mathbf{f}} := (f_1, \dots, z^{m_1-1}f_1, f_2, \dots, z^{m_d-1}f_d) = (\bar{f}_1, \dots, \bar{f}_{|\mathbf{m}|}).$$

We also define an associated multi-index $\bar{\mathbf{m}} := (1, \dots, 1)$ with $|\mathbf{m}| = |\bar{\mathbf{m}}|$. The systems $\mathbf{f}$ and $\bar{\mathbf{f}}$ share most properties. In particular, poles and system poles of $(\mathbf{f}, \mathbf{m})$ and $(\bar{\mathbf{f}}, \bar{\mathbf{m}})$ coincide and $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$ if and only if $\bar{\mathbf{f}}$ is polynomially independent with respect to $\bar{\mathbf{m}}$. Passing to $(\bar{\mathbf{f}}, \bar{\mathbf{m}})$ if necessary and relabeling the functions, we can assume without loss of generality that $\mathbf{m} = (1, \dots, 1)$ and $d = |\mathbf{m}|$.

Suppose that there exist constants c_k , $k = 1, \dots, d$, not all zero, such that $\sum_{k=1}^d c_k f_k$ is a polynomial. Without loss of generality, we can assume that $c_1 \neq 0$. Then,

$$f_1 = p - \sum_{k=2}^d c_k f_k,$$

where p is a polynomial of degree N .

On the other hand, for each $n \geq d - 1$, there exist polynomials Q_n , $P_{n,k}$, $k = 2, \dots, d$, such that for all $k = 2, \dots, d$,

$$\begin{aligned} & - \deg P_{n,k} \leq n - 1, \deg Q_n \leq d - 1, Q_n \not\equiv 0, \\ & - \frac{Q_n f_k - P_{n,k}}{a_{n+1}} \in \mathcal{H}(E). \end{aligned}$$

Therefore,

$$\frac{Q_n \left(p - \sum_{k=2}^d c_k f_k \right) - \left(Q_n p - \sum_{k=2}^d c_k P_{n,k} \right)}{a_{n+1}} \in \mathcal{H}(E)$$

and, for $n \geq d + N$, the polynomial $P_{n,1} = Q_n p - \sum_{k=2}^d c_k P_{n,k}$ verifies $\deg P_{n,1} \leq n - 1$. Thus, for all n sufficiently large, the polynomials $P_{n,k}$, $k = 1, \dots, d$ satisfy Definition 1.1 with respect to $\mathbf{f}$ and $\mathbf{m}$. Naturally, Q_n gives rise to a polynomial $Q_{n,\mathbf{m}}$ with $\deg Q_{n,\mathbf{m}} < d = |\mathbf{m}|$ against our assumption on $Q_{n,\mathbf{m}}$. $\square$

The following corollary is a straightforward consequence of Lemma 3.2.

Corollary 3.5 *Let $\mathbf{f} \in \mathcal{H}(E)^d$ and fix a multi-index $\mathbf{m} \in \mathbb{N}^d$. Assume that $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$ and there exists a polynomial $Q_{\mathbf{m}}$ of degree $|\mathbf{m}|$ such that*

$$\limsup_{n \rightarrow \infty} \|Q_{n,\mathbf{m}} - Q_{\mathbf{m}}\|^{1/n} \leq \theta < 1.$$

Then for each $k = 1, \dots, d$, either f_k has exactly m_k poles in $D_{\rho_{m_k}(f_k)}$ or $\rho_0(Q_{\mathbf{m}} f_k) > \rho_{m_k}(f_k)$.

An elementary dimensional analysis leads to the following property of system poles (for details see [5, Lemma 3.5]).

Lemma 3.6 *Let $\mathbf{f} \in \mathcal{H}(E)^d$ and $\mathbf{m} \in \mathbb{N}^d$. Then, $\mathbf{f}$ can have at most $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ (counting their order). Moreover, if the system $\mathbf{f}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ and ξ is a system pole of order τ , then for all $s > \tau$ there can be no polynomial combination of the form (1.4) holomorphic in a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole at $z = \xi$ of exact order s .*

3.4 Proof (b) $\Rightarrow$ (a)

The auxiliary results that we have established in this section allow us to adapt the proof used in [5] to obtain the inverse statement of [5, Theorem 1.4]. One simply has to follow step by step the arguments employed there and substitute the use of [5, Lemma 3.2] by Lemma 3.4, [5, Corollary 3.4] by Corollary 3.5, and [5, Lemma 3.5] by Lemma 3.6. The details are left to the reader.

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Research Article

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Convergences in Hausdorff content of generalized Padé approximants to polynomial expansions

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Abstract: Given a vector of approximated functions analytic on a neighborhood of some compact set of the complex plane with simply connected complement in the extended complex plane, we prove convergences in Hausdorff content of the corresponding four generalizations of type II Hermite-Padé approximants on some certain sequences. These four generalizations are based on orthogonal and Faber polynomial expansions. As consequences of these convergence results, we give alternate proofs of Montessus de Ballore type theorems for these generalizations.

Keywords: Padé approximation, orthogonal polynomials, Faber polynomials, Montessus de Ballore's theorem, Hausdorff content

MSC: Primary 30E10, 41A21, 41A18, Secondary 41A25

1 Introduction

In this paper, we study convergences of four generalizations of the construction of type II Hermite-Padé approximants. The first approximation is called simultaneous Padé-orthogonal approximation defined as follows. Let E be an infinite compact subset of the complex plane $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus E$ is simply connected. Denote by $\mathcal{K}$ the collection of these compact sets. Let μ be a finite positive Borel measure with an infinite support $\text{supp}(\mu)$ contained in E . We write $\mu \in \mathcal{M}(E)$ and define the associated inner product

$$\langle g, h \rangle_\mu := \int g(\zeta) \overline{h(\zeta)} d\mu(\zeta), \quad g, h \in L_2(\mu).$$

Let

$$p_n(z) := \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, 2, \dots,$$

be the orthonormal polynomial of degree n with respect to μ with positive leading coefficient; that is $\langle p_n, p_m \rangle_\mu = \delta_{n,m}$. Define

$$\mathcal{H}(E)^d := \{(F_1, F_2, \dots, F_d) : F_\alpha \in \mathcal{H}(E) \text{ for all } \alpha = 1, 2, \dots, d\},$$

where $\mathcal{H}(E)$ is the space of all functions holomorphic in some neighborhood of E .

Definition 1.1. Let $E \in \mathcal{K}$, $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ and $\mu \in \mathcal{M}(E)$. Fix a multi-index $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$ where $\mathbf{0}$ is the zero vector in $\mathbb{N}_0^d$. Set $|\mathbf{m}| := m_1 + m_2 + \cdots + m_d$. Then, for each $n \geq \max\{m_1, m_2, \dots, m_d\}$, there exist polynomials $q_{n,\mathbf{m}}^\mu, p_{n,\mathbf{m},\alpha}^\mu, \alpha = 1, 2, \dots, d$, such that

$$\deg(p_{n,\mathbf{m},\alpha}^\mu) \leq n - m_\alpha, \quad \deg(q_{n,\mathbf{m}}^\mu) \leq |\mathbf{m}|, \quad q_{n,\mathbf{m}}^\mu \neq 0,$$

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$$\langle q_{n,\mathbf{m}}^\mu F_\alpha - p_{n,\mathbf{m},\alpha}^\mu, p_j \rangle_\mu = 0, \quad j = 0, 1, \dots, n.$$

The vector of rational functions

$$\begin{aligned} \mathbf{R}_{n,\mathbf{m}}^\mu &= (R_{n,\mathbf{m},1}^\mu, R_{n,\mathbf{m},2}^\mu, \dots, R_{n,\mathbf{m},d}^\mu) \\ &:= (p_{n,\mathbf{m},1}^\mu/q_{n,\mathbf{m}}^\mu, p_{n,\mathbf{m},2}^\mu/q_{n,\mathbf{m}}^\mu, \dots, p_{n,\mathbf{m},d}^\mu/q_{n,\mathbf{m}}^\mu) \end{aligned}$$

is called an $(n, \mathbf{m})$ simultaneous Padé-orthogonal approximant of $\mathbf{F}$ with respect to μ .

Finding $q_{n,\mathbf{m}}^\mu$ is equivalent to solving a system of $|\mathbf{m}|$ homogeneous linear equations on $|\mathbf{m}| + 1$ unknown. Moreover, for each $\alpha = 1, 2, \dots, d$, $p_{n,\mathbf{m},\alpha}^\mu$ is uniquely determined by $q_{n,\mathbf{m}}^\mu$. Therefore, for any pair $(n, \mathbf{m})$, a vector of rational functions $\mathbf{R}_{n,\mathbf{m}}^\mu$ always exists but may not be unique.

The concept of simultaneous Padé-orthogonal approximation was first introduced by Cocoq and López in [6]. In their paper, those simultaneous Padé-orthogonal approximants are called simultaneous Fourier-Padé approximants and the set E is the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$. Their definition was extended to a general compact set $E \in \mathcal{K}$ with some restricted conditions in [1]. In [1] and [6], the authors proved convergences of row sequences of simultaneous Padé-orthogonal approximants, namely analogues of Montessus de Ballore's theorem.

Now, we introduce a definition of poles for a vector of functions.

Definition 1.2. Let $\Omega := (\Omega_1, \Omega_2, \dots, \Omega_d)$ be a system of domains such that for each $\alpha = 1, 2, \dots, d$, F_α is meromorphic in Ω_α . We say that the point λ is a pole of $\mathbf{F}$ in Ω of order τ if there exists an index $\alpha \in \{1, 2, \dots, d\}$ such that $\lambda \in \Omega_\alpha$ and it is a pole of F_α of order τ , and for $\beta \neq \alpha$ either λ is a pole of F_β of order less than or equal to τ or $\lambda \notin \Omega_\beta$. When $\Omega := (\Omega, \Omega, \dots, \Omega)$, we say that λ is a pole of $\mathbf{F}$ in Ω .

The second approximation is based on Faber polynomials defined as follows. Let $E \in \mathcal{K}$ and Φ be the exterior conformal mapping from $\overline{\mathbb{C}} \setminus E$ onto $\overline{\mathbb{C}} \setminus \{w \in \mathbb{C} : |w| \leq 1\}$ satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. For each $\rho > 1$, we define a level curve with respect to E of index ρ and a canonical domain with respect to E of index ρ by

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\} \quad \text{and} \quad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\},$$

respectively. Let $\mathbf{F} \in \mathcal{H}(E)^d$. Denote by $\rho_{|\mathbf{m}|}(\mathbf{F})$ the index $\rho > 1$ of the largest canonical domain D_ρ to which $\mathbf{F}$ has at most $|\mathbf{m}|$ poles. The Faber polynomial of degree n for E is defined by the formula

$$\Phi_n(z) := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\Phi^n(t)}{t - z} dt, \quad z \in D_\rho, \quad n = 0, 1, 2, \dots \quad (1)$$

and the Faber coefficient of $F \in \mathcal{H}(E)$ with respect to Φ_n is given by

$$[F]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{F(t)\Phi'(t)}{\Phi^{n+1}(t)} dt, \quad (2)$$

where $\rho \in (1, \rho_0(F))$.

Definition 1.3. Let $E \in \mathcal{K}$ and $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. Fix a multi-index $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$. Set $|\mathbf{m}| := m_1 + m_2 + \dots + m_d$. Then, for each $n \geq \max\{m_1, m_2, \dots, m_d\}$, there exist polynomials $q_{n,\mathbf{m}}^E, p_{n,\mathbf{m},\alpha}^E$, $\alpha = 1, 2, \dots, d$, such that

$$\begin{aligned} \deg(p_{n,\mathbf{m},\alpha}^E) &\leq n - m_\alpha, \quad \deg(q_{n,\mathbf{m}}^E) \leq |\mathbf{m}|, \quad q_{n,\mathbf{m}}^E \neq 0, \\ [q_{n,\mathbf{m}}^E F_\alpha - p_{n,\mathbf{m},\alpha}^E]_j &= 0, \quad j = 0, 1, \dots, n. \end{aligned}$$

The vector of rational functions

$$\begin{aligned} \mathbf{R}_{n,\mathbf{m}}^E &= (R_{n,\mathbf{m},1}^E, R_{n,\mathbf{m},2}^E, \dots, R_{n,\mathbf{m},d}^E) \\ &:= (p_{n,\mathbf{m},1}^E/q_{n,\mathbf{m}}^E, p_{n,\mathbf{m},2}^E/q_{n,\mathbf{m}}^E, \dots, p_{n,\mathbf{m},d}^E/q_{n,\mathbf{m}}^E) \end{aligned}$$

is called an $(n, \mathbf{m})$ simultaneous Padé-Faber approximant of $\mathbf{F}$ corresponding to E .

Note that for any pair $(n, \mathbf{m})$, a vector of rational $\mathbf{R}_{n,\mathbf{m}}^E$ always exists but may not be unique. In [2] and [4], the concept of simultaneous Padé-Faber approximants was introduced and analogues of Montessus de Ballore's theorem for simultaneous Padé-Faber approximants were proved.

The third approximation is called orthogonal Hermite-Padé approximation defined as follows.

Definition 1.4. Let $E \in \mathcal{K}$, $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$, and $\mu \in \mathcal{M}(E)$. Fix $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}$. Set $|\mathbf{m}| := m_1 + m_2 + \dots + m_d$. Then, there exists a polynomial $\tilde{q}_{n,\mathbf{m}}^\mu$ such that $\deg(\tilde{q}_{n,\mathbf{m}}^\mu) \leq |\mathbf{m}|$, $\tilde{q}_{n,\mathbf{m}}^\mu \neq 0$, and

$$\langle z^k \tilde{q}_{n,\mathbf{m}}^\mu F_\alpha, p_n \rangle_\mu = 0, \quad \alpha = 1, 2, \dots, d, \quad k = 0, 1, \dots, m_\alpha - 1.$$

Define the corresponding polynomials

$$\tilde{p}_{n,\mathbf{m},\alpha}^\mu(z) = \sum_{j=0}^{n-1} \langle \tilde{q}_{n,\mathbf{m}}^\mu F_\alpha, p_j \rangle_\mu p_j(z), \quad \alpha = 1, 2, \dots, d.$$

The vector of rational functions

$$\begin{aligned} \tilde{\mathbf{R}}_{n,\mathbf{m}}^\mu &= (\tilde{R}_{n,\mathbf{m},1}^\mu, \tilde{R}_{n,\mathbf{m},2}^\mu, \dots, \tilde{R}_{n,\mathbf{m},d}^\mu) \\ &:= (\tilde{p}_{n,\mathbf{m},1}^\mu / \tilde{q}_{n,\mathbf{m}}^\mu, \tilde{p}_{n,\mathbf{m},2}^\mu / \tilde{q}_{n,\mathbf{m}}^\mu, \dots, \tilde{p}_{n,\mathbf{m},d}^\mu / \tilde{q}_{n,\mathbf{m}}^\mu) \end{aligned}$$

is called an $(n, \mathbf{m})$ orthogonal Hermite-Padé approximant of $\mathbf{F}$ with respect to μ .

A vector of rational functions $\tilde{\mathbf{R}}_{n,\mathbf{m}}^\mu$ always exists but may not be unique.

The last approximation is Hermite-Padé-Faber approximation defined as follows.

Definition 1.5. Let $E \in \mathcal{K}$ and $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. Fix $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}$. Set $|\mathbf{m}| := m_1 + m_2 + \dots + m_d$. Then, there exists a polynomial $\tilde{q}_{n,\mathbf{m}}^E$ such that $\deg(\tilde{q}_{n,\mathbf{m}}^E) \leq |\mathbf{m}|$, $\tilde{q}_{n,\mathbf{m}}^E \neq 0$, and

$$[z^k \tilde{q}_{n,\mathbf{m}}^E F_\alpha]_n = 0, \quad \alpha = 1, 2, \dots, d, \quad k = 0, 1, \dots, m_\alpha - 1.$$

Define the corresponding polynomials

$$\tilde{p}_{n,\mathbf{m},\alpha}^E(z) = \sum_{j=0}^{n-1} [\tilde{q}_{n,\mathbf{m}}^E F_\alpha]_j \Phi_j(z), \quad \alpha = 1, 2, \dots, d.$$

The vector of rational functions

$$\begin{aligned} \tilde{\mathbf{R}}_{n,\mathbf{m}}^E &= (\tilde{R}_{n,\mathbf{m},1}^E, \tilde{R}_{n,\mathbf{m},2}^E, \dots, \tilde{R}_{n,\mathbf{m},d}^E) \\ &:= (\tilde{p}_{n,\mathbf{m},1}^E / \tilde{q}_{n,\mathbf{m}}^E, \tilde{p}_{n,\mathbf{m},2}^E / \tilde{q}_{n,\mathbf{m}}^E, \dots, \tilde{p}_{n,\mathbf{m},d}^E / \tilde{q}_{n,\mathbf{m}}^E) \end{aligned}$$

is called an $(n, \mathbf{m})$ Hermite-Padé-Faber approximant of $\mathbf{F}$ corresponding to E .

Again, a vector of rational functions $\tilde{\mathbf{R}}_{n,\mathbf{m}}^E$ always exists but may not be unique. The definitions of orthogonal Hermite-Padé approximants and Hermite-Padé-Faber approximants were recently introduced in [3] and [5]. In those papers, analogues of a Montessus de Ballore-Gonchar type theorem for both approximations were proved.

Next, let us introduce the concept of convergence in Hausdorff content. Let B be a subset of the complex plane $\mathbb{C}$. By $\mathcal{U}(B)$, we denote the class of all coverings of B by at most a numerable set of disks. Let $\beta > 0$ and set

$$h_\beta(B) := \inf \left\{ \sum_{j=1}^{\infty} |U_j|^\beta : \{U_j\} \in \mathcal{U}(B) \right\},$$

where $|U_j|$ is the radius of the disk U_j . This notation $h_\beta(B)$ is called the β -dimensional Hausdorff content of the set B .

Definition 1.6. Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of complex valued functions defined on a domain $D \subset \mathbb{C}$ and g another complex function defined on D . We say that $\{g_n\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to the function g inside D if for every compact subset K of D and for each $\epsilon > 0$, we have

$$\lim_{n \rightarrow \infty} h_\beta \{z \in K : |g_n(z) - g(z)| > \epsilon\} = 0.$$

Such a convergence will be denoted by $h_\beta - \lim_{n \rightarrow \infty} g_n = g$ in D .

The objective of this paper is to prove convergences in Hausdorff content of those four generalizations when the sequences of indices $\{(n, \mathbf{m}_n)\}_{n \in \mathbb{N}}$ satisfy

$$\lim_{n \rightarrow \infty} \frac{|\mathbf{m}_n| \ln n}{n} = 0. \quad (3)$$

This type of sequences of indices $\{(n, m_n)\}_{n \in \mathbb{N}}$ satisfying the limit (3) was first considered by Gonchar [7] for Padé (α, β) -approximants. In the current paper, we prove results analogous to Theorem 2 in [7] for four generalizations of Hermite-Padé approximants. As consequences of our main theorems, we give alternate proofs of the Montessus de Ballore type theorem for those generalizations.

The outline of this paper is as follows. Section 2 contains our main results. We collect needed auxiliary lemmas in Section 3. Section 4 is dedicated to the proofs of all results in Section 2.

2 Main Results

Before we state our results about the convergence of simultaneous Padé-orthogonal approximants, we need to define a class of measures and some more notation first. A class of measure that we are interested in the results of simultaneous Padé-orthogonal approximants is $\mathcal{R}(E) \subset \mathcal{M}(E)$. We write $\mu \in \mathcal{R}(E)$ when the corresponding sequence of orthonormal polynomials has *ratio asymptotics*; that is

$$\lim_{n \rightarrow \infty} \frac{p_n(z)}{p_{n+1}(z)} = \frac{1}{\Phi(z)},$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus E$. Moreover, we restrict ourselves to a smaller collection of compact sets E defined as follows. Denote by $\mathcal{K}_1$ the collection of all sets $E \in \mathcal{K}$ such that the inverse function of Φ can be extended continuously to $\overline{\mathbb{C}} \setminus \{w \in \mathbb{C} : |w| < 1\}$.

The definition of polewise independence is given below.

Definition 2.1. Let $E \in \mathcal{K}$, $\rho > 1$, $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ be a vector functions of meromorphic in D_ρ . Then $\mathbf{F}$ is said to be *polewise independent with respect to $\mathbf{m}$ in D_ρ* if and only if there do not exist polynomials $v_1, v_2, \dots, v_d$ at least one of which is non-null, satisfying

1. $\deg v_\alpha \leq m_\alpha - 1$, $\alpha = 1, \dots, d$, if $m_\alpha \geq 1$,
2. $v_\alpha \equiv 0$ if $m_\alpha = 0$,
3. $\sum_{\alpha=1}^d (v_\alpha \circ \Phi) \cdot F_\alpha \in \mathcal{H}(D_\rho \setminus E)$,

where $\mathcal{H}(D_\rho \setminus E)$ is the space of all holomorphic functions in $D_\rho \setminus E$.

The following theorem is our main result on simultaneous Padé-orthogonal approximants which is an analogue of Theorem 2 in [7].

Theorem 2.2. Let $E \in \mathcal{K}_1$, $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$, and $\mu \in \mathcal{R}(E)$. Suppose that $\mathbf{F}$ is polewise independent with respect to the multi-index $\mathbf{m} := (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$ in $D_{\rho|\mathbf{m}|}(\mathbf{F})$ and the sequence $\{\mathbf{m}_n\} := \{(m_{n,1}, m_{n,2}, \dots, m_{n,d})\}$ satisfies the following conditions

$$\liminf_{n \rightarrow \infty} m_{n,j} \geq m_j, \quad j = 1, 2, \dots, d$$

and

$$\lim_{n \rightarrow \infty} \frac{|\mathbf{m}_n| \ln n}{n} = 0.$$

Then for fixed numbers $\beta > 0$ and $\alpha = 1, 2, \dots, d$, each sequence $\{R_{n, \mathbf{m}_n, \alpha}^\mu\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to F_α inside $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ as $n \rightarrow \infty$.

As a consequence of Theorem 2.2, we also prove a Montessus de Ballore type theorem for simultaneous Padé-orthogonal approximants stated below.

Corollary 2.3. Let $E \in \mathcal{K}_1$, $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$, and $\mu \in \mathcal{R}(E)$. Suppose that $\mathbf{F}$ is polewise independent with respect to the multi-index $\mathbf{m} := (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ and has distinct poles at $\lambda_1, \lambda_2, \dots, \lambda_q$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$. Then, $\{\mathbf{R}_{n, \mathbf{m}}^\mu\}_{n \in \mathbb{N}}$ is uniquely determined for all sufficiently large n and for each $\alpha = 1, 2, \dots, d$, $\{R_{n, \mathbf{m}, \alpha}^\mu\}_{n \in \mathbb{N}}$ converges uniformly to F_α on each compact subset of $D_{\rho_{|\mathbf{m}|}}(\mathbf{F}) \setminus \{\lambda_1, \lambda_2, \dots, \lambda_q\}$ as $n \rightarrow \infty$. Moreover, for each $\alpha = 1, 2, \dots, d$ and for any compact subset K of $D_{\rho_{|\mathbf{m}|}}(\mathbf{F}) \setminus \{\lambda_1, \lambda_2, \dots, \lambda_q\}$,

$$\limsup_{n \rightarrow \infty} \|F_\alpha - R_{n, \mathbf{m}, \alpha}^\mu\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{|\mathbf{m}|}(\mathbf{F})},$$

where $\|\cdot\|_K$ denotes the sup-norm on K and if $K \subset E$; then $\|\Phi\|_K$ is replaced by 1.

Similar results for simultaneous Padé-Faber approximants are stated below.

Theorem 2.4. Let $E \in \mathcal{K}$ and $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. Suppose that $\mathbf{F}$ is polewise independent with respect to the multi-index $\mathbf{m} := (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ and the sequence $\{\mathbf{m}_n\} := \{(m_{n,1}, m_{n,2}, \dots, m_{n,d})\}$ satisfies the following conditions

$$\liminf_{n \rightarrow \infty} m_{n,j} \geq m_j, \quad j = 1, 2, \dots, d$$

and

$$\lim_{n \rightarrow \infty} \frac{|\mathbf{m}_n| \ln n}{n} = 0.$$

Then, for fixed numbers $\beta > 0$ and $\alpha = 1, 2, \dots, d$, each sequence $\{R_{n, \mathbf{m}_n, \alpha}^E\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to F_α inside $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ as $n \rightarrow \infty$.

Corollary 2.5. Let $E \in \mathcal{K}$ and $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. Suppose that $\mathbf{F}$ is polewise independent with respect to the multi-index $\mathbf{m} := (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ and has distinct poles at $\lambda_1, \lambda_2, \dots, \lambda_q$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$. Then, $\{\mathbf{R}_{n, \mathbf{m}}^E\}_{n \in \mathbb{N}}$ is uniquely determined for all sufficiently large n and for each $\alpha = 1, 2, \dots, d$, $\{R_{n, \mathbf{m}, \alpha}^E\}_{n \in \mathbb{N}}$ converges uniformly to F_α on each compact subset of $D_{\rho_{|\mathbf{m}|}}(\mathbf{F}) \setminus \{\lambda_1, \lambda_2, \dots, \lambda_q\}$ as $n \rightarrow \infty$. Moreover, for each $\alpha = 1, 2, \dots, d$ and for any compact subset K of $D_{\rho_{|\mathbf{m}|}}(\mathbf{F}) \setminus \{\lambda_1, \lambda_2, \dots, \lambda_q\}$,

$$\limsup_{n \rightarrow \infty} \|F_\alpha - R_{n, \mathbf{m}, \alpha}^E\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{|\mathbf{m}|}(\mathbf{F})}. \quad (4)$$

Before stating the similar results for orthogonal Hermite-Padé approximants, we need to define the class of measures $\mathbf{Reg}_{1,2}(E)$. We say that $\mu \in \mathbf{Reg}_{1,2}(E)$ if and only if

$$\lim_{n \rightarrow \infty} |p_n(z)|^{1/n} = |\Phi(z)|$$

and

$$\lim_{n \rightarrow \infty} |s_n(z)|^{1/n} = \frac{1}{|\Phi(z)|}$$

uniformly on each compact subset of $\mathbb{C} \setminus E$. Note that the above second type function s_n is defined as the following:

$$s_n(z) := \int \frac{\overline{p_n(\zeta)}}{z - \zeta} d\mu(\zeta), \quad z \in \overline{\mathbb{C}} \setminus \text{supp}(\mu).$$

Moreover, we need a definition of system pole and characteristic values.

Definition 2.6. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ and $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$. We say that ξ is a system pole of order τ of $\mathbf{F}$ with respect to $\mathbf{m}$ if τ is the largest positive integer such that for each $t = 1, 2, \dots, \tau$, there exists at least one polynomial combination of the form

$$\sum_{\alpha=1}^d v_\alpha F_\alpha, \quad \deg v_\alpha < m_\alpha, \quad \alpha = 1, 2, \dots, d, \quad (5)$$

which is holomorphic on a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole at $z = \xi$ of exact order t .

Let τ be the order of ξ as a system pole of $\mathbf{F}$. For each $t = 1, 2, \dots, \tau$, denote by $\rho_{\xi,t}(\mathbf{F}, \mathbf{m})$ the largest of all the numbers $\rho_t(G)$ (the index of the largest canonical domain containing at most t poles of G), where G is a polynomial combination of type (5) that is holomorphic on a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole at $z = \xi$ of order t . Then, we define

$$\begin{aligned} \rho_{\xi,t}(\mathbf{F}, \mathbf{m}) &:= \min_{k=1, \dots, t} \rho_{\xi,k}(\mathbf{F}, \mathbf{m}), \\ \rho_\xi(\mathbf{F}, \mathbf{m}) &:= \rho_{\xi,\tau}(\mathbf{F}, \mathbf{m}) = \min_{t=1, \dots, \tau} \rho_{\xi,t}(\mathbf{F}, \mathbf{m}). \end{aligned}$$

Fix $\alpha \in \{1, 2, \dots, d\}$. Let $D_\alpha(\mathbf{F}, \mathbf{m})$ be the largest canonical domain in which all the poles of F_α are system poles of $\mathbf{F}$ with respect to $\mathbf{m}$, their order as poles of F_α does not exceed their order as system poles, and F_α has no other singularity. By $\rho_\alpha(\mathbf{F}, \mathbf{m})$, we denote the index of this canonical domain. Let $\xi_1, \xi_2, \dots, \xi_N$ be the poles of F_α in $D_\alpha(\mathbf{F}, \mathbf{m})$. For each $j = 1, 2, \dots, N$, let $\hat{\tau}_j$ be the order of ξ_j as pole of F_α and τ_j its order as a system pole. By assumption, $\hat{\tau}_j \leq \tau_j$. Set

$$\rho_\alpha^*(\mathbf{F}, \mathbf{m}) := \min\{\rho_\alpha(\mathbf{F}, \mathbf{m}), \min_{j=1, \dots, N} \rho_{\xi_j, \hat{\tau}_j}(\mathbf{F}, \mathbf{m})\}$$

and let $D_\alpha^*(\mathbf{F}, \mathbf{m})$ be the canonical domain with this index.

Theorem 2.7. Let $E \in \mathcal{K}$, $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$, and $\mu \in \mathbf{Reg}_{1,2}(E)$. Suppose that $\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m} := (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$ and the sequence $\{\mathbf{m}_n\} := \{(m_{n,1}, m_{n,2}, \dots, m_{n,d})\}$ satisfies the following conditions

$$\liminf_{n \rightarrow \infty} m_{n,j} \geq m_j, \quad j = 1, 2, \dots, d$$

and

$$\lim_{n \rightarrow \infty} \frac{|\mathbf{m}_n| \ln n}{n} = 0.$$

Then, for fixed numbers $\beta > 0$ and $\alpha = 1, 2, \dots, d$, each sequence $\{\tilde{R}_{n, \mathbf{m}_n, \alpha}^\mu\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to F_α inside $D_\alpha^*(\mathbf{F}, \mathbf{m})$ as $n \rightarrow \infty$.

Corollary 2.8. Let $E \in \mathcal{K}$, $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$, and $\mu \in \mathbf{Reg}_{1,2}(E)$. Suppose that $\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m} := (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$ and the distinct system poles of $\mathbf{F}$ with respect to $\mathbf{m}$ are $\xi_1, \xi_2, \dots, \xi_q$. Then, $\{\tilde{\mathbf{R}}_{n, \mathbf{m}}^\mu\}_{n \in \mathbb{N}}$ is uniquely determined for all sufficiently large n and for any $\alpha = 1, 2, \dots, d$, $\{\tilde{R}_{n, \mathbf{m}, \alpha}^\mu\}_{n \in \mathbb{N}}$ converges uniformly to F_α on each compact subset of $D_\alpha^*(\mathbf{F}, \mathbf{m}) \setminus \{\xi_1, \xi_2, \dots, \xi_q\}$ as $n \rightarrow \infty$. Moreover, for any compact subset K of $D_\alpha^*(\mathbf{F}, \mathbf{m}) \setminus \{\xi_1, \xi_2, \dots, \xi_q\}$ and for any $\alpha = 1, 2, \dots, d$,

$$\limsup_{n \rightarrow \infty} \|F_\alpha - \tilde{R}_{n, \mathbf{m}, \alpha}^\mu\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_\alpha^*(\mathbf{F}, \mathbf{m})}. \quad (6)$$

With the same arguments used to prove Theorem 2.7 and Corollary 2.8, we prove

Theorem 2.9. Let $E \in \mathcal{K}$ and $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. Suppose that $\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m} := (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$ and the sequence $\{\mathbf{m}_n\} := \{(m_{n,1}, m_{n,2}, \dots, m_{n,d})\}$ satisfies the following conditions

$$\liminf_{n \rightarrow \infty} m_{n,j} \geq m_j, \quad j = 1, 2, \dots, d$$

and

$$\lim_{n \rightarrow \infty} \frac{|\mathbf{m}_n| \ln n}{n} = 0.$$

Then, for fixed numbers $\beta > 0$ and $\alpha = 1, 2, \dots, d$, each sequence $\{\tilde{R}_{n,\mathbf{m},\alpha}^E\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to F_α inside $D_\alpha^*(\mathbf{F}, \mathbf{m})$ as $n \rightarrow \infty$.

Corollary 2.10. Let $E \in \mathcal{K}$ and $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. Suppose that $\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m} := (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$ and the distinct system poles of $\mathbf{F}$ with respect to $\mathbf{m}$ are $\xi_1, \xi_2, \dots, \xi_q$. Then, $\{\tilde{\mathbf{R}}_{n,\mathbf{m}}^E\}_{n \in \mathbb{N}}$ is uniquely determined for all sufficiently large n and for any $\alpha = 1, 2, \dots, d$, $\{\tilde{R}_{n,\mathbf{m},\alpha}^E\}_{n \in \mathbb{N}}$ converges uniformly to F_α on each compact subset of $D_\alpha^*(\mathbf{F}, \mathbf{m}) \setminus \{\xi_1, \xi_2, \dots, \xi_q\}$ as $n \rightarrow \infty$. Moreover, for any compact subset K of $D_\alpha^*(\mathbf{F}, \mathbf{m}) \setminus \{\xi_1, \xi_2, \dots, \xi_q\}$ and for any $\alpha = 1, 2, \dots, d$,

$$\limsup_{n \rightarrow \infty} \|F_\alpha - \tilde{R}_{n,\mathbf{m},\alpha}^E\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_\alpha^*(\mathbf{F}, \mathbf{m})}.$$

3 Auxiliary Lemmas

In this section we keep all needed notations and lemmas. Let $E \in \mathcal{K}$ and $\mu \in \mathcal{M}(E)$. We define the n -th Fourier coefficient of $G \in \mathcal{H}(E)$ with respect to p_n by

$$\langle G \rangle_n := \langle G, p_n \rangle_\mu = \int G(z) \overline{p_n(z)} d\mu(z).$$

We say that $\mu \in \mathbf{Reg}_1(E) \subset \mathcal{M}(E)$ when

$$\lim_{n \rightarrow \infty} |p_n(z)|^{1/n} = |\Phi(z)|, \quad (7)$$

uniformly on each compact subset of $\mathbb{C} \setminus E$. The following two lemmas (see [3, Lemma 2.1]) concern the formulas for computing $\rho_0(G)$ and the domain of convergence of orthogonal and Faber polynomial expansions of holomorphic functions.

Lemma 3.1. Let $E \in \mathcal{K}$, $G \in \mathcal{H}(E)$ and $\mu \in \mathbf{Reg}_1(E)$. Then,

$$\rho_0(G) = \left(\limsup_{n \rightarrow \infty} |\langle G \rangle_n|^{1/n} \right)^{-1}.$$

Moreover, the series $\sum_{n=0}^{\infty} \langle G \rangle_n p_n(z)$ converges to $G(z)$ uniformly on each compact subset of $D_{\rho_0(G)}$.

Lemma 3.2. Let $E \in \mathcal{K}$ and $G \in \mathcal{H}(E)$. Then,

$$\rho_0(G) = \left(\limsup_{n \rightarrow \infty} |[G]_n|^{1/n} \right)^{-1}.$$

Moreover, the series $\sum_{n=0}^{\infty} [G]_n \Phi_n(z)$ converges to $G(z)$ uniformly on each compact subset of $D_{\rho_0(G)}$.

Recall that the second type function s_n by

$$s_n(z) := \int \frac{\overline{p_n(\zeta)}}{z - \zeta} d\mu(\zeta) \quad z \in \overline{\mathbb{C}} \setminus \text{supp}(\mu).$$

The next lemma (see [8, Lemma 3.1]) is the asymptotic relation between the orthogonal polynomials p_n and the second type functions s_n .

Lemma 3.3. Let $E \in \mathcal{K}_1$. If $\mu \in \mathcal{R}(E)$, then

$$\lim_{n \rightarrow \infty} p_n(z) s_n(z) = \frac{\Phi'(z)}{\Phi(z)},$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus E$. Consequently, for any compact set $K \subset \overline{\mathbb{C}} \setminus E$, there exists $n_0 \in \mathbb{N}$ such that $s_n(z) \neq 0$ for all $z \in K$ and $n \geq n_0$.

Using Cauchy's integral formula and Fubini's theorem, one can easily check the following:

Lemma 3.4. *Let $E \in \mathcal{K}$, $G \in \mathcal{H}(E)$, $k \in \mathbb{N}_0$, and $\rho \in (1, \rho_0(G))$. Then,*

$$\langle G \rangle_k = \frac{1}{2\pi i} \int_{\Gamma_\rho} G(w) s_k(w) dw. \quad (8)$$

The following lemma (see [10], p. 43] or [11], p. 583] for its proof) gives an estimate of Faber polynomials on a level curve.

Lemma 3.5. *Let $\rho > 1$ be fixed. Then, there exists $c > 0$ such that*

$$\|\Phi_n\|_{\Gamma_\rho} \leq c\rho^n, \quad n \geq 0. \quad (9)$$

Indeed, by the maximum modulus principle, the inequalities (9) can be replaced by the inequalities

$$\|\Phi_n\|_{\overline{D}_\rho} \leq c\rho^n, \quad n \geq 0. \quad (10)$$

The following lemma is about the uniqueness of the common denominators of generalized Hermite-Padé approximants to polynomial expansions.

Lemma 3.6. *Let $(n, \mathbf{m})$ be a fixed index. Then the following assertions hold:*

- (a) *If for all $q_{n,\mathbf{m}}^\mu$ in Definition 1.1, $\deg q_{n,\mathbf{m}}^\mu = |\mathbf{m}|$, then $q_{n,\mathbf{m}}^\mu$ is unique.*
- (b) *If for all $q_{n,\mathbf{m}}^E$ in Definition 1.3, $\deg q_{n,\mathbf{m}}^E = |\mathbf{m}|$, then $q_{n,\mathbf{m}}^E$ is unique.*
- (c) *If for all $\tilde{q}_{n,\mathbf{m}}^\mu$ in Definition 1.4, $\deg \tilde{q}_{n,\mathbf{m}}^\mu = |\mathbf{m}|$, then $\tilde{q}_{n,\mathbf{m}}^\mu$ is unique.*
- (d) *If for all $\tilde{q}_{n,\mathbf{m}}^E$ in Definition 1.5, $\deg \tilde{q}_{n,\mathbf{m}}^E = |\mathbf{m}|$, then $\tilde{q}_{n,\mathbf{m}}^E$ is unique.*

The following lemma (see [1, Lemma 2.2]) provides the relation between the polewise independence of a vector of functions and the determinant Δ stated below.

Lemma 3.7. *Let $E \in \mathcal{K}$, and $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ be a vector of functions in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ and $\mathbf{m} := (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$ be a fixed multi-index. Suppose that $\mathbf{F}$ has exactly $|\mathbf{m}|$ poles in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$, $\lambda_1, \lambda_2, \dots, \lambda_q$ are distinct poles of $\mathbf{F}$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$, and $\tau_1, \tau_2, \dots, \tau_q$ are their multiplicities, respectively. Define*

$$\Delta := \begin{vmatrix} \left((z - \lambda_j)^{\tau_j} F_\alpha \Phi^{m_\alpha - 1} \right) (\lambda_j) & \dots & \left((z - \lambda_j)^{\tau_j} F_\alpha \Phi^{m_\alpha - 1} \right)^{(\tau_j - 1)} (\lambda_j) \\ \left((z - \lambda_j)^{\tau_j} F_\alpha \Phi^{m_\alpha - 2} \right) (\lambda_j) & \dots & \left((z - \lambda_j)^{\tau_j} F_\alpha \Phi^{m_\alpha - 2} \right)^{(\tau_j - 1)} (\lambda_j) \\ \vdots & \vdots & \vdots \\ \left((z - \lambda_j)^{\tau_j} F_\alpha \right) (\lambda_j) & \dots & \left((z - \lambda_j)^{\tau_j} F_\alpha \right)^{(\tau_j - 1)} (\lambda_j) \end{vmatrix}_{j=1,2,\dots,q, \alpha=1,2,\dots,d},$$

where the subindex on the determinant means that the indicated group of columns are successively written for $j = 1, 2, \dots, q$ and the rows repeated for $\alpha = 1, 2, \dots, d$. Then, $\mathbf{F}$ is polewise independent with respect to the multi-index $|\mathbf{m}|$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ if and only if $\Delta \neq 0$.

The final lemma proved by Gonchar (see [7, Lemma 1]) allows us to derive uniform convergence on compact subsets of the region under consideration from convergence in h_1 -content under appropriate assumptions.

Lemma 3.8. *Suppose that $h_1 - \lim_{n \rightarrow \infty} g_n = g$ in D . Then, if each function g_n is meromorphic and has no more than $k < +\infty$ poles in D and the function g is meromorphic and has exactly k poles in D , then all $g_n, n \geq N$, also have k poles in D ; the poles of g_n tend to the poles $\lambda_1, \lambda_2, \dots, \lambda_k$ of g (taking account of their orders) and the sequence $\{g_n\}_{n \in \mathbb{N}}$ tends to g uniformly on each compact subset of the domain $D' = D \setminus \{\lambda_1, \lambda_2, \dots, \lambda_k\}$.*

4 Proofs of Main Results

Proof of Theorem 2.2. For each $n \geq 0$, we normalize the polynomials $q_{n,\mathbf{m}_n}^\mu$ in terms of its zeros $\lambda_{n,j}$ such that

$$Q_{n,\mathbf{m}_n}^\mu(z) := \prod_{|\lambda_{n,j}| \leq 1} (z - \lambda_{n,j}) \prod_{|\lambda_{n,j}| > 1} \left(1 - \frac{z}{\lambda_{n,j}}\right) \quad (11)$$

and for each $\alpha = 1, 2, \dots, d$,

$$R_{n,\mathbf{m}_n,\alpha}^\mu = \frac{p_{n,\mathbf{m}_n,\alpha}^\mu}{q_{n,\mathbf{m}_n}^\mu} = \frac{P_{n,\mathbf{m}_n,\alpha}^\mu}{Q_{n,\mathbf{m}_n}^\mu}.$$

With this normalization, we can estimate upper and lower bounds on the normalized $Q_{n,\mathbf{m}_n}^\mu$. Let $\varepsilon > 0$ be fixed. Suppose that the poles of $\mathbf{F}$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ are $\lambda_1, \lambda_2, \dots, \lambda_q$ and the zeros of $Q_{n,\mathbf{m}_n}^\mu$ are $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,l_{\mathbf{m}_n}}$ (they are not necessary distinct and $l_{\mathbf{m}_n} \leq |\mathbf{m}_n|$). We cover each pole of $\mathbf{F}$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ with an open disk of radius $(\varepsilon/(6|\mathbf{m}|))^{1/\beta}$ and denote by $J_{-1,\varepsilon}^\beta(\mathbf{F}, \mathbf{m})$ the union of these disks. For each $n \geq 0$, we cover each zero of $Q_{n,\mathbf{m}_n}^\mu$ with an open disk of radius $(\varepsilon/(6|\mathbf{m}_n|(n+1)^2))^{1/\beta}$ and denote by $J_{n,\varepsilon}^\beta(\mathbf{F})$ the union of these disks. Set for each $k \geq 0$,

$$J_\varepsilon^\beta(\mathbf{F}, \mathbf{m}; k) := J_{-1,\varepsilon}^\beta(\mathbf{F}, \mathbf{m}) \bigcup \left(\bigcup_{n=k}^{\infty} J_{n,\varepsilon}^\beta(\mathbf{F}) \right) \quad \text{and} \quad J_\varepsilon^\beta(\mathbf{F}, \mathbf{m}) := J_\varepsilon^\beta(\mathbf{F}, \mathbf{m}; 0).$$

By using the monotonicity and subadditivity of h_β , it easy to check that $h_\beta(J_\varepsilon^\beta(\mathbf{F}, \mathbf{m})) < \varepsilon$ and $J_{\varepsilon_1}^\beta(\mathbf{F}, \mathbf{m}) \subset J_{\varepsilon_2}^\beta(\mathbf{F}, \mathbf{m})$ for $\varepsilon_1 < \varepsilon_2$. For any set $B \subset D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$, we put $B(\varepsilon) := B \setminus J_\varepsilon^\beta(\mathbf{F}, \mathbf{m})$. Clearly that if $\{g_n\}_{n \in \mathbb{N}}$ converges uniformly to g on $K(\varepsilon)$ for any compact subset $K \subset D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ and $\varepsilon > 0$, then $h_\beta - \lim_{n \rightarrow \infty} g_n = g$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$.

Due to the normalization in (11), for any compact subset $K \subset D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ and for any $\varepsilon > 0$, there exist positive constant $C_1 > 0$ and $C_2 > 0$ independent of n such that for all sufficiently large n ,

$$\|Q_{n,\mathbf{m}_n}^\mu\|_K \leq C_1^{|\mathbf{m}_n|}, \quad (12)$$

and

$$\min_{z \in K \setminus J_\varepsilon^\beta(\mathbf{F}, \mathbf{m}; k)} |Q_{n,\mathbf{m}_n}^\mu(z)| \geq (C_2 |\mathbf{m}_n| (n+1)^2)^{-2|\mathbf{m}_n|/\beta}. \quad (13)$$

By the assumption of μ , it follows that

$$\lim_{n \rightarrow \infty} \frac{p_n(z)}{p_{n+l}(z)} = \frac{1}{\Phi(z)^l}, \quad l = 0, 1, 2, \dots, \quad (14)$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus E$ which implies that $\mu \in \mathbf{Reg}_1(E)$. Then from (14) and Lemma 3.3, we obtain

$$\lim_{n \rightarrow \infty} \frac{s_{n+l}(z)}{s_n(z)} = \lim_{n \rightarrow \infty} \frac{p_n(z)}{p_{n+l}(z)} \frac{p_{n+l}(z)s_{n+l}(z)}{p_n(z)s_n(z)} = \frac{1}{\Phi(z)^l} \frac{\Phi(z)'/\Phi(z)}{\Phi(z)'/\Phi(z)} = \frac{1}{\Phi(z)^l}, \quad (15)$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus E$. Moreover, it follows from (14) and (15) that

$$\lim_{n \rightarrow \infty} |p_n(z)|^{1/n} = |\Phi(z)|, \quad (16)$$

and

$$\lim_{n \rightarrow \infty} |s_n(z)|^{1/n} = \frac{1}{|\Phi(z)|}, \quad (17)$$

uniformly on each compact subset of $\mathbb{C} \setminus E$, respectively.

Let

$$Q_{|\mathbf{m}|}^{\mathbf{F}}(z) := \prod_{j=1}^q (z - \lambda_j)^{\tau_j},$$

where $\lambda_1, \lambda_2, \dots, \lambda_q$ are distinct poles of $\mathbf{F}$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$. From the definition of simultaneous Padé-orthogonal approximants and Lemma 3.1, we have

$$Q_{n, \mathbf{m}_n}^\mu(z) F_\alpha(z) - P_{n, \mathbf{m}_n, \alpha}^\mu(z) = \sum_{k=n+1}^{\infty} a_{k, n}^{(\alpha)} p_k(z), \quad z \in D_{\rho_0(F_\alpha)}, \quad (18)$$

where

$$a_{k, n}^{(\alpha)} := \langle Q_{n, \mathbf{m}_n}^\mu F_\alpha \rangle_k, \quad k = 0, 1, 2, \dots,$$

and $a_{k, n}^{(\alpha)} = 0$, for all $k = n - m_{n, \alpha} + 1, n - m_{n, \alpha} + 2, \dots, n$. Since $\mathbf{F}$ is polewise independent with respect to $\mathbf{m}$ in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$, $\mathbf{F}$ has exactly $|\mathbf{m}|$ poles in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ and $\sum_{j=1}^q \tau_j = |\mathbf{m}|$. Multiplying (18) by $Q_{|\mathbf{m}|}^{\mathbf{F}}$ and expanding the result in terms of the orthogonal system $\{p_\nu\}_{\nu=0}^\infty$ such that for $z \in D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$,

$$\begin{aligned} Q_{|\mathbf{m}|}^{\mathbf{F}}(z) Q_{n, \mathbf{m}_n}^\mu(z) F_\alpha(z) - Q_{|\mathbf{m}|}^{\mathbf{F}}(z) P_{n, \mathbf{m}_n, \alpha}^\mu(z) &= \sum_{k=n+1}^{\infty} Q_{|\mathbf{m}|}^{\mathbf{F}}(z) a_{k, n}^{(\alpha)} p_k(z) \\ &= \sum_{\nu=0}^{\infty} b_{\nu, n}^{(\alpha)} p_\nu(z) = \sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n, \alpha}} b_{\nu, n}^{(\alpha)} p_\nu(z) + \sum_{\nu=n+|\mathbf{m}_n|-m_{n, \alpha}+1}^{\infty} b_{\nu, n}^{(\alpha)} p_\nu(z). \end{aligned} \quad (19)$$

Let K be a compact subset of $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ and set

$$\sigma := \max\{\|\Phi\|_K, 1\} \quad (20)$$

($\sigma = 1$ when $K \subset E$). Choose $\delta > 0$ sufficiently small such that

$$\rho_1 := \rho_{|\mathbf{m}|}(\mathbf{F}) - \delta > \rho_{|\mathbf{m}|-1}(\mathbf{F}), \quad \rho_1 - \delta > 1, \quad \text{and} \quad \frac{\sigma + \delta}{\rho_1 - \delta} < 1. \quad (21)$$

First, we approximate $\sum_{\nu=n+|\mathbf{m}_n|-m_{n, \alpha}+1}^{\infty} |b_{\nu, n}^{(\alpha)}| |p_\nu(z)|$ on $\overline{D}_\sigma$. Due to the normalization of $Q_{n, \mathbf{m}_n}^\mu$ (the upper estimate in (12)) and Lemma 3.4, it follows that for $\nu \geq n + |\mathbf{m}_n| - m_{n, \alpha} + 1$,

$$\begin{aligned} |b_{\nu, n}^{(\alpha)}| &= |\langle Q_{|\mathbf{m}|}^{\mathbf{F}} Q_{n, \mathbf{m}_n}^\mu F_\alpha - Q_{|\mathbf{m}|}^{\mathbf{F}} P_{n, \mathbf{m}_n, \alpha}^\mu \rangle_\nu| = |\langle Q_{|\mathbf{m}|}^{\mathbf{F}} Q_{n, \mathbf{m}_n}^\mu F_\alpha \rangle_\nu| \\ &= \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} Q_{|\mathbf{m}|}^{\mathbf{F}}(z) Q_{n, \mathbf{m}_n}^\mu(z) F_\alpha(z) s_\nu(z) dz \right| \leq c_1 C_1^{|\mathbf{m}_n|} \|s_\nu\|_{\Gamma_{\rho_1}}, \end{aligned} \quad (22)$$

where the constant c_1 does not depend on n (from now on, we will denote some constants that do not depend on n by $c_2, c_3, \dots$). By using (17), there exists $n_0 \in \mathbb{N}$ such that

$$\|s_\nu\|_{\Gamma_{\rho_1}} \leq \frac{c_2}{(\rho_1 - \delta)^\nu}, \quad \nu \geq n_0 \quad (23)$$

Moreover, from (16), it follows from maximum modulus principle that

$$\|p_\nu\|_{\overline{D}_\sigma} \leq c_3(\sigma + \delta)^\nu, \quad \nu \geq 0. \quad (24)$$

Therefore, by (22), (23), and (24), for $n_1 > n_0$,

$$\sum_{\nu=n+|\mathbf{m}_n|-m_{n, \alpha}+1}^{\infty} |b_{\nu, n}^{(\alpha)}| |p_\nu(z)| \leq \sum_{\nu=n+|\mathbf{m}_n|-m_{n, \alpha}+1}^{\infty} c_4 C_1^{|\mathbf{m}_n|} \left(\frac{\sigma + \delta}{\rho_1 - \delta} \right)^\nu \leq c_5 C_1^{|\mathbf{m}_n|} \left(\frac{\sigma + \delta}{\rho_1 - \delta} \right)^n, \quad z \in \overline{D}_\sigma. \quad (25)$$

Next, we approximate $\sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n, \alpha}} |b_{\nu, n}^{(\alpha)}| |p_\nu(z)|$ on $\overline{D}_\sigma$. To approximate $|b_{\nu, n}^{(\alpha)}|$, we need to approximate $|a_{k, n}^{(\alpha)}|$ first. Let $\rho_2 \in (1, \rho_0(\mathbf{F}))$. Using Lemma 3.4 when $G = Q_{n, \mathbf{m}_n}^\mu F_\alpha$, we have

$$a_{k, n}^{(\alpha)} = \langle Q_{n, \mathbf{m}_n}^\mu F_\alpha \rangle_k = \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} Q_{n, \mathbf{m}_n}^\mu(z) F_\alpha(z) s_k(z) dz.$$

Define

$$\gamma_{k,n}^{(\alpha)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} Q_{n,m_n}^{\mu}(z) F_{\alpha}(z) s_k(z) dz.$$

Notice that for each $k \geq 0$, $Q_{n,m_n}^{\mu} F_{\alpha}$ is meromorphic on $\overline{D}_{\rho_1} \setminus D_{\rho_2}$ and has poles at $\lambda_1, \lambda_2, \dots, \lambda_q$ with multiplicities at most $\tau_1, \tau_2, \dots, \tau_q$, respectively. Applying Cauchy's residue theorem, we obtain

$$\gamma_{k,n}^{(\alpha)} - a_{k,n}^{(\alpha)} = \sum_{j=1}^q \text{Res}(Q_{n,m_n}^{\mu} F_{\alpha} s_k, \lambda_j). \quad (26)$$

Recall that the limit formula for the residue of $Q_{n,m_n}^{\mu} F_{\alpha} s_k$ at λ_j is

$$\text{Res}(Q_{n,m_n}^{\mu} F_{\alpha} s_k, \lambda_j) = \frac{1}{(\tau_j - 1)!} \lim_{z \rightarrow \lambda_j} ((z - \lambda_j)^{\tau_j} Q_{n,m_n}^{\mu} F_{\alpha} s_k)^{(\tau_j - 1)}(z).$$

By using Leibniz's rule and the fact that for n sufficiently large $s_n(z) \neq 0$ for $z \in \mathbb{C} \setminus E$ (see Lemma 3.3), we can transform the expression under the limit sign as follow

$$((z - \lambda_j)^{\tau_j} Q_{n,m_n}^{\mu} F_{\alpha} s_k)^{(\tau_j - 1)}(z) = \sum_{t=0}^{\tau_j - 1} \binom{\tau_j - 1}{t} (Q_{n,m_n}^{\mu} s_n)^{(\tau_j - 1 - t)}(z) \left((z - \lambda_j)^{\tau_j} F_{\alpha} \frac{s_k}{s_n} \right)^{(t)}(z).$$

For $j = 1, 2, \dots, q$, and $t = 0, 1, \dots, \tau_j - 1$, set

$$\beta_n(j, t) := \frac{1}{(\tau_j - 1)!} \binom{\tau_j - 1}{t} \lim_{z \rightarrow \lambda_j} (Q_{n,m_n}^{\mu} s_n)^{(\tau_j - 1 - t)}(z)$$

(notice that $\beta_n(j, t)$ do not depend on k and α). Thus, we can rewrite (26) as

$$\gamma_{k,n}^{(\alpha)} - a_{k,n}^{(\alpha)} = \sum_{j=1}^q \sum_{t=0}^{\tau_j - 1} \beta_n(j, t) \left((z - \lambda_j)^{\tau_j} F_{\alpha} \frac{s_k}{s_n} \right)^{(t)}(\lambda_j). \quad (27)$$

Since $a_{k,n}^{(\alpha)} = 0$ for $\alpha = 1, 2, \dots, d$, $k = n - m_{n,\alpha} + 1, n - m_{n,\alpha} + 2, \dots, n$, it follows from (27) and the assumption that $m_{n,\alpha} \geq m_{\alpha}$,

$$\gamma_{k,n}^{(\alpha)} = \sum_{j=1}^q \sum_{t=0}^{\tau_j - 1} \beta_n(j, t) \left((z - \lambda_j)^{\tau_j} F_{\alpha} \frac{s_k}{s_n} \right)^{(t)}(\lambda_j), \quad \alpha = 1, 2, \dots, d, \quad k = n - m_{\alpha} + 1, n - m_{\alpha} + 2, \dots, n. \quad (28)$$

Now, we consider (28) as a system of $|\mathbf{m}|$ equations on the $|\mathbf{m}|$ unknowns $\beta_n(j, t)$ and the determinant Δ_n corresponding this system is

$$\begin{vmatrix} \left((z - \lambda_j)^{\tau_j} F_{\alpha} \frac{s_{n-m_{\alpha}+1}}{s_n} \right) (\lambda_j) & \cdots & \left((z - \lambda_j)^{\tau_j} F_{\alpha} \frac{s_{n-m_{\alpha}+1}}{s_n} \right)^{(\tau_j-1)} (\lambda_j) \\ \left((z - \lambda_j)^{\tau_j} F_{\alpha} \frac{s_{n-m_{\alpha}+2}}{s_n} \right) (\lambda_j) & \cdots & \left((z - \lambda_j)^{\tau_j} F_{\alpha} \frac{s_{n-m_{\alpha}+2}}{s_n} \right)^{(\tau_j-1)} (\lambda_j) \\ \vdots & \vdots & \vdots \\ \left((z - \lambda_j)^{\tau_j} F_{\alpha} \frac{s_n}{s_n} \right) (\lambda_j) & \cdots & \left((z - \lambda_j)^{\tau_j} F_{\alpha} \frac{s_n}{s_n} \right)^{(\tau_j-1)} (\lambda_j) \end{vmatrix}_{j=1,2,\dots,q, \alpha=1,2,\dots,d}$$

where the subindex on the determinant means that the indicated group of columns are successively written for $j = 1, 2, \dots, q$ and the rows repeated for $\alpha = 1, 2, \dots, d$. Using (15), we have

$$\lim_{n \rightarrow \infty} \Delta_n = \Delta := \begin{vmatrix} \left((z - \lambda_j)^{\tau_j} F_{\alpha} \Phi^{m_{\alpha}-1} \right) (\lambda_j) & \cdots & \left((z - \lambda_j)^{\tau_j} F_{\alpha} \Phi^{m_{\alpha}-1} \right)^{(\tau_j-1)} (\lambda_j) \\ \left((z - \lambda_j)^{\tau_j} F_{\alpha} \Phi^{m_{\alpha}-2} \right) (\lambda_j) & \cdots & \left((z - \lambda_j)^{\tau_j} F_{\alpha} \Phi^{m_{\alpha}-2} \right)^{(\tau_j-1)} (\lambda_j) \\ \vdots & \vdots & \vdots \\ \left((z - \lambda_j)^{\tau_j} F_{\alpha} \right) (\lambda_j) & \cdots & \left((z - \lambda_j)^{\tau_j} F_{\alpha} \right)^{(\tau_j-1)} (\lambda_j) \end{vmatrix}_{j=1,2,\dots,q, \alpha=1,2,\dots,d}.$$

Since $\mathbf{F}$ is polewise independent with respect to $\mathbf{m}$ in $D_{\rho|\mathbf{m}|}(\mathbf{F})$, it follows from Lemma 3.7 that $\Delta \neq 0$ and $|\Delta_n| \geq c_6 > 0$ for all sufficiently large n . From now on, we will only consider such n 's.

To avoid long expressions, we define for all $w = 1, 2, \dots, d$, $y = 1, 2, \dots, m_w$, $j = 1, 2, \dots, q$, and $t = 0, 1, \dots, \tau_j - 1$,

$$g_{w,y} := \left(\sum_{r=0}^{w-1} m_r \right) + y \quad \text{and} \quad h_{j,t} := \left(\sum_{l=0}^{j-1} \tau_l \right) + t + 1,$$

where $m_0 = \tau_0 = 0$. Applying Cramer's rule to (28), we have

$$\beta_n(j, t) = \frac{\Delta_n(j, t)}{\Delta_n} = \frac{1}{\Delta_n} \sum_{w=1}^d \sum_{y=1}^{m_w} \gamma_{n-m_w+y,n}^{(w)} C_n[g_{w,y}, h_{j,t}],$$

where $\Delta_n(j, t)$ is the determinant obtained from Δ_n by replacing $h_{j,t}^{\text{th}}$ column with the column

$$[\gamma_{n-m_w+1,n}^{(w)} + \gamma_{n-m_w+2,n}^{(w)}, \dots, \gamma_{n,n}^{(w)}]_{w=1,2,\dots,d}^T$$

and $C_n[g, h]$ is the determinant of the $(g, h)^{\text{th}}$ cofactor matrix of $\Delta_n(j, t)$. Substituting $\beta_n(j, t)$ in (27), we obtain for $\alpha = 1, 2, \dots, d$ and $k \geq n + 1$,

$$\begin{aligned} \gamma_{k,n}^{(\alpha)} - a_{k,n}^{(\alpha)} &= \frac{1}{\Delta_n} \sum_{j=1}^q \sum_{t=0}^{\tau_j-1} \sum_{w=1}^d \sum_{y=1}^{m_w} \gamma_{n-m_w+y,n}^{(w)} C_n[g_{w,y}, h_{j,t}] \left((z - \lambda_j)^{\tau_j} F_\alpha \frac{s_k}{s_n} \right)^{(t)} (\lambda_j) \\ &= \frac{1}{\Delta_n} \sum_{j=1}^q \sum_{t=0}^{\tau_j-1} \sum_{w=1}^d \sum_{y=1}^{m_w} \gamma_{n-m_w+y,n}^{(w)} C_n[g_{w,y}, h_{j,t}] \sum_{u=0}^t \binom{t}{u} ((z - \lambda_j)^{\tau_j} F_\alpha)^{(t-u)} (\lambda_j) \left(\frac{s_k}{s_n} \right)^{(u)} (\lambda_j) \\ &= \frac{1}{\Delta_n} \sum_{j=1}^q \sum_{t=0}^{\tau_j-1} \sum_{w=1}^d \sum_{y=1}^{m_w} \sum_{u=0}^t \binom{t}{u} \gamma_{n-m_w+y,n}^{(w)} C_n[g_{w,y}, h_{j,t}] ((z - \lambda_j)^{\tau_j} F_\alpha)^{(t-u)} (\lambda_j) \left(\frac{s_k}{s_n} \right)^{(u)} (\lambda_j). \end{aligned} \quad (29)$$

where the second identity follows from the Leibniz formula.

Define

$$\mathbb{B}(\lambda, r) := \{z \in \mathbb{C} : |z - \lambda| < r\}.$$

Let $\varepsilon > 0$ be sufficiently small such that $\{z \in \mathbb{C} : |z - \lambda_j| < \varepsilon\} \subset \{z \in \mathbb{C} : |\Phi(z)| > \rho_2\}$ for all $j = 1, 2, \dots, q$ and $\mathbb{B}(\lambda_j, \varepsilon) \cap \mathbb{B}(\lambda_k, \varepsilon) = \emptyset$ for all $k \neq j$. Using Cauchy's integral formula, we obtain

$$\left(\frac{s_k}{s_n} \right)^{(\ell)} (\lambda_j) = \frac{\ell!}{2\pi i} \int_{|z-\lambda_j|=\varepsilon} \frac{s_k(z)}{s_n(z)(z-\lambda_j)^{\ell+1}} dz. \quad (30)$$

Applying (15) and (30), we can see that there exists a constant c_7 such that for sufficiently large n ,

$$\left| \left(\frac{s_k}{s_n} \right)^{(\ell)} (\lambda_j) \right| \leq \frac{c_7}{\rho_2^{k-n}}, \quad j = 1, 2, \dots, q, \quad \ell = 0, 1, \dots, \tau_j - 1, \quad k \geq n + 1. \quad (31)$$

Moreover, by using Cauchy's integral formula as before, there exist constant c_8 and c_9 such that for all $\alpha = 1, 2, \dots, d$, $k = n - m_\alpha + 1, n - m_\alpha + 2, \dots, n$, $j = 1, 2, \dots, q$, and $\ell = 0, 1, \dots, \tau_j - 1$,

$$\left| \left((z - \lambda_j)^{\tau_j} F_\alpha \frac{s_k}{s_n} \right)^{(\ell)} (\lambda_j) \right| \leq c_8 \quad (32)$$

for sufficiently large n and

$$\left| ((z - \lambda_j)^{\tau_j} F_\alpha)^{(\ell)} (\lambda_j) \right| \leq c_9. \quad (33)$$

From (32),

$$|C_n(g, h)| \leq c_{10}, \quad g, h = 1, 2, \dots, |\mathbf{m}|. \quad (34)$$

Using (31), (33), (34), and $|\Delta_n| \geq c_{11} > 0$, it follows from (29) that

$$|a_{k,n}^{(\alpha)}| \leq |\gamma_{k,n}^{(\alpha)}| + \frac{c_{11}}{\rho_2^{k-n}} \sum_{w=1}^d \sum_{y=1}^{m_w} |\gamma_{n-m_w+y,n}^{(w)}|, \quad \alpha = 1, 2, \dots, d, \quad k \geq n+1. \quad (35)$$

By the definition of $\gamma_{k,n}^{(\alpha)}$ and (17), for all sufficiently large n , we obtain

$$|\gamma_{k,n}^{(\alpha)}| \leq \frac{c_{12}C_1^{|\mathbf{m}_n|}}{(\rho_1 - \delta)^k}, \quad \alpha = 1, 2, \dots, d, \quad k \geq n - |\mathbf{m}| + 1. \quad (36)$$

This implies that

$$|a_{k,n}^{(\alpha)}| \leq \frac{c_{13}C_1^{|\mathbf{m}_n|}}{\rho_2^{k-n}(\rho_1 - \delta)^n}, \quad \alpha = 1, 2, \dots, d, \quad k \geq n+1. \quad (37)$$

Recall that $b_{\nu,n}^{(\alpha)} = \sum_{k=n+1}^{\infty} a_{k,n}^{(\alpha)} \langle Q_{|\mathbf{m}|}^{\mathbf{F}} p_k \rangle_{\nu}$. By Cauchy-Schwarz inequality and the orthonormality of $\{p_k\}$, we have for all $\nu > 0$,

$$|\langle Q_{|\mathbf{m}|}^{\mathbf{F}} p_k \rangle_{\nu}| = |\langle Q_{|\mathbf{m}|}^{\mathbf{F}} p_k, p_{\nu} \rangle_{\mu}| \leq \|Q_{|\mathbf{m}|}^{\mathbf{F}}\|_E \langle p_k, p_k \rangle_{\mu}^{1/2} \langle p_{\nu}, p_{\nu} \rangle_{\mu}^{1/2} \leq \|Q_{|\mathbf{m}|}^{\mathbf{F}}\|_E \leq c_{14}. \quad (38)$$

Then,

$$|b_{\nu,n}^{(\alpha)}| \leq \sum_{k=n+1}^{\infty} |a_{k,n}^{(\alpha)}| |\langle Q_{|\mathbf{m}|}^{\mathbf{F}} p_k \rangle_{\nu}| \leq \frac{c_{15}C_1^{|\mathbf{m}_n|}}{(\rho_1 - \delta)^n}, \quad \alpha = 1, 2, \dots, d.$$

Therefore, for each $\alpha = 1, 2, \dots, d$ and sufficiently large n ,

$$\begin{aligned} \sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n,\alpha}} |b_{\nu,n}^{(\alpha)}| |p_{\nu}(z)| &\leq \sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n,\alpha}} c_{15}C_1^{|\mathbf{m}_n|} \frac{(\sigma + \delta)^{\nu}}{(\rho_1 - \delta)^n} \\ &\leq c_{15}(n + |\mathbf{m}_n| - m_{n,\alpha} + 1) \tilde{C}_1^{|\mathbf{m}_n|} \left(\frac{\sigma + \delta}{\rho_1 - \delta} \right)^n \end{aligned} \quad (39)$$

where $\tilde{C}_1 := C_1(\sigma + \delta)$ and $z \in \overline{D}_{\sigma}$.

Combining (25) and (39), it follows from (19) that for each $k \geq n_2$,

$$|Q_{|\mathbf{m}|}^{\mathbf{F}}(z) Q_{n,\mathbf{m}_n}^{\mu}(z) F_{\alpha}(z) - P_{n,\mathbf{m}_n,\alpha}^{\mu}(z)| \leq c_{16} \tilde{C}_1^{|\mathbf{m}_n|} \theta^n, \quad \alpha = 1, 2, \dots, d, \quad z \in \overline{D}_{\sigma}, \quad n \geq k, \quad (40)$$

where θ is an arbitrary constant which satisfies

$$\frac{\sigma + \delta}{\rho_1 - \delta} < \theta < 1.$$

Let $\beta > 0$ and $\varepsilon > 0$ be fixed. By the definition of $J_{\varepsilon}^{\beta}(F, \mathbf{m}; k)$ and (13), the inequality (40) implies that for each $k \geq n_2$,

$$|F_{\alpha}(z) - R_{n,\mathbf{m}_n,\alpha}^{\mu}(z)| \leq \frac{c_{16} \tilde{C}_1^{|\mathbf{m}_n|} \theta^n}{|Q_{|\mathbf{m}|}^{\mathbf{F}}(z) Q_{n,\mathbf{m}_n}^{\mu}(z)|} \leq c_{16} \tilde{C}_1^{|\mathbf{m}_n|} \theta^n \left(\frac{6|\mathbf{m}|}{\varepsilon} \right)^{|\mathbf{m}|/\beta} (C_2 |\mathbf{m}_n| (n+1)^2)^{2|\mathbf{m}_n|/\beta},$$

for all $z \in \overline{D}_{\sigma} \setminus J_{\varepsilon}^{\beta}(F, \mathbf{m}; k)$ and n sufficiently large. Then, for each $k \geq n_2$,

$$\begin{aligned} \|F_{\alpha} - R_{n,\mathbf{m}_n,\alpha}^{\mu}\|_{\overline{D}_{\sigma} \setminus J_{\varepsilon}^{\beta}(F, \mathbf{m}; k)}^{1/n} &\leq \left(c_{16} \left(\frac{6|\mathbf{m}|}{\varepsilon} \right)^{|\mathbf{m}|/\beta} \right)^{1/n} \theta (\tilde{C}_1^{1/2} C_2^{1/\beta} |\mathbf{m}_n|^{1/\beta} (n+1)^{2/\beta})^{2|\mathbf{m}_n|/n} \\ &\leq c_{17}^{1/n} \theta e^{(c_{18} + \frac{3}{\beta} \log(n+1))(2|\mathbf{m}_n|/n)}, \end{aligned}$$

for sufficiently large n , which implies that for each $k \geq n_2$,

$$\limsup_{n \rightarrow \infty} \|F_{\alpha} - R_{n,\mathbf{m}_n,\alpha}^{\mu}\|_{\overline{D}_{\sigma} \setminus J_{\varepsilon}^{\beta}(F, \mathbf{m}; k)}^{1/n} \leq \theta.$$

Letting $\delta \rightarrow 0$ and $\rho_1 \rightarrow \rho_{|\mathbf{m}|}(\mathbf{F})$, we have for each $k \geq n_2$,

$$\frac{\sigma}{\rho_{|\mathbf{m}|}(\mathbf{F})} < \theta < 1.$$

Since θ is arbitrary, we let $\theta \rightarrow \sigma/\rho_{|\mathbf{m}|}(\mathbf{F})$, then for $k \geq n_2$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|F_\alpha - R_{n, \mathbf{m}_n, \alpha}^\mu\|_{K(\varepsilon)}^{1/n} &\leq \limsup_{n \rightarrow \infty} \|F_\alpha - R_{n, \mathbf{m}_n, \alpha}^\mu\|_{\overline{D}_\sigma(\varepsilon)}^{1/n} \\ &\leq \limsup_{n \rightarrow \infty} \|F_\alpha - R_{n, \mathbf{m}_n, \alpha}^\mu\|_{\overline{D}_\sigma \setminus J_\varepsilon^\beta(\mathbf{F}, \mathbf{m}; k)}^{1/n} \leq \frac{\sigma}{\rho_{|\mathbf{m}|}(\mathbf{F})} < 1. \end{aligned} \quad (41)$$

This implies that for any $\beta > 0$ and $\alpha = 1, 2, \dots, d$, each sequence $\{R_{n, \mathbf{m}_n, \alpha}^\mu\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to F_α inside $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$, as $n \rightarrow \infty$. $\square$

Proof of Corollary 2.3. By the assumption of Corollary 2.3, $\mathbf{m}_n = \mathbf{m}$. Then, the conditions in Theorem 2.2 are obtained. By Theorem 2.2, we get $h_1 - \lim_{n \rightarrow \infty} R_{n, \mathbf{m}_n, \alpha}^\mu = F_\alpha$ in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$. Applying Lemma 3.8, each pole of F_α in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$ attracts as many zeros of $Q_{n, \mathbf{m}}^\mu$ as its order. Therefore, since $\mathbf{F}$ has $|\mathbf{m}|$ poles in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$, $\deg Q_{n, \mathbf{m}}^\mu = |\mathbf{m}|$ for all sufficiently large n . By Lemma 3.6, for such n , $Q_{n, \mathbf{m}}^\mu$ is unique. This implies that for sufficiently large n , $\mathbf{R}_{n, \mathbf{m}}^\mu$ is unique.

Let $K \subset D_{\rho_{|\mathbf{m}|}(\mathbf{F})} \setminus \{\lambda_1, \lambda_2, \dots, \lambda_q\}$ be a compact set. Choose $\sigma := \max\{\|\Phi\|_K, 1\}$. Since all points $\lambda_1, \lambda_2, \dots, \lambda_q$ attract all zeros of $Q_{n, \mathbf{m}}^\mu$, for sufficiently small $\varepsilon > 0$ and for sufficiently large k ,

$$K \in \overline{D}_\sigma \setminus J_\varepsilon^\beta(\mathbf{F}, \mathbf{m}, k).$$

By the inequality (41),

$$\limsup_{n \rightarrow \infty} \|F_\alpha - R_{n, \mathbf{m}_n, \alpha}^\mu\|_K^{1/n} \leq \limsup_{n \rightarrow \infty} \|F_\alpha - R_{n, \mathbf{m}_n, \alpha}^\mu\|_{\overline{D}_\sigma \setminus J_\varepsilon^\beta(\mathbf{F}, \mathbf{m}; k)}^{1/n} \leq \frac{\sigma}{\rho_{|\mathbf{m}|}(\mathbf{F})}. \quad (42)$$

This implies that the sequence $\{R_{n, \mathbf{m}_n, \alpha}^\mu\}_{n \in \mathbb{N}}$ converges uniformly to F_α on each compact subset of $D_{\rho_{|\mathbf{m}|}(\mathbf{F})} \setminus \{\lambda_1, \lambda_2, \dots, \lambda_q\}$ as $n \rightarrow \infty$. This completes the proof. $\square$

Proof of Theorem 2.4. Let $Q_{n, \mathbf{m}_n}^E$ be the polynomial $q_{n, \mathbf{m}_n}^E$ normalized as in (11) and we have for all $\alpha = 1, 2, \dots, d$,

$$R_{n, \mathbf{m}_n, \alpha}^E = \frac{p_{n, \mathbf{m}_n, \alpha}^E}{q_{n, \mathbf{m}_n}^E} = \frac{P_{n, \mathbf{m}_n, \alpha}^E}{Q_{n, \mathbf{m}_n}^E}.$$

Note that the notations $J_\varepsilon^\beta(\mathbf{F}, \mathbf{m}; k)$ and $B(\varepsilon)$ are defined as in the proof of Theorem 2.2 replacing $Q_{n, \mathbf{m}_n}^\mu$ by $Q_{n, \mathbf{m}_n}^E$. Then, for any compact subset $K \subset D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$ and for any $\varepsilon > 0$, there exist positive constants $C_1 > 0$ and $C_2 > 0$ independent of n such that for all sufficiently large n ,

$$\|Q_{n, \mathbf{m}_n}^E\|_K \leq C_1^{|\mathbf{m}_n|}, \quad (43)$$

and

$$\min_{z \in K \setminus J_\varepsilon^\beta(\mathbf{F}, \mathbf{m}; k)} |Q_{n, \mathbf{m}_n}^E(z)| \geq (C_2 |\mathbf{m}_n| (n+1)^2)^{-2|\mathbf{m}_n|/\beta}. \quad (44)$$

Let

$$Q_{|\mathbf{m}|}^{\mathbf{F}}(z) := \prod_{j=1}^q (z - \lambda_j)^{\tau_j},$$

where $\lambda_1, \lambda_2, \dots, \lambda_d$ are distinct poles of $\mathbf{F}$ in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$. From the definition of simultaneous Padé-Faber approximants and Lemma 3.2, we have,

$$Q_{n, \mathbf{m}_n}^E(z) F_\alpha(z) - P_{n, \mathbf{m}_n, \alpha}^E(z) = \sum_{k=n+1}^{\infty} a_{k, n}^{(\alpha)} \Phi_k(z), \quad z \in D_{\rho_0(F_\alpha)}, \quad (45)$$

where

$$a_{k, n}^{(\alpha)} := [Q_{n, \mathbf{m}_n}^E F_\alpha]_k, \quad k = 0, 1, 2, \dots,$$

and $a_{k,n}^{(\alpha)} = 0$, for all $k = n - m_{n,\alpha} + 1, n - m_{n,\alpha} + 2, \dots, n$. Since $\mathbf{F}$ is polewise independent with respect to $\mathbf{m}$ in $D_{\rho_{|\mathbf{m}|}}(F)$, $\mathbf{F}$ has exactly $|\mathbf{m}|$ poles in $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ and $\sum_{j=1}^q \tau_j = |\mathbf{m}|$. Multiplying (45) by $Q_{|\mathbf{m}|}^{\mathbf{F}}$ and expanding the result in terms of the Faber polynomial system $\{\Phi_\nu\}_{\nu=0}^\infty$ such that for $z \in D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$,

$$\begin{aligned} Q_{|\mathbf{m}|}^{\mathbf{F}}(z)Q_{n,\mathbf{m}_n}^E(z)F_\alpha(z) - Q_{|\mathbf{m}|}^{\mathbf{F}}(z)P_{n,\mathbf{m}_n,\alpha}^E(z) &= \sum_{k=n+1}^\infty Q_{|\mathbf{m}|}^{\mathbf{F}}(z)a_{k,n}^{(\alpha)}\Phi_k(z) \\ &= \sum_{\nu=0}^\infty b_{\nu,n}^{(\alpha)}\Phi_\nu(z) = \sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n,\alpha}} b_{\nu,n}^{(\alpha)}\Phi_\nu(z) + \sum_{\nu=n+|\mathbf{m}_n|-m_{n,\alpha}+1}^\infty b_{\nu,n}^{(\alpha)}\Phi_\nu(z). \end{aligned} \quad (46)$$

Let K be a compact subset of $D_{\rho_{|\mathbf{m}|}}(\mathbf{F})$ and set

$$\sigma := \max\{\|\Phi\|_K, 1\} \quad (47)$$

($\sigma = 1$ when $K \subset E$). Choose $\delta > 0$ sufficiently small such that

$$\rho_1 := \rho_{|\mathbf{m}|}(\mathbf{F}) - \delta > \rho_{|\mathbf{m}|-1}(\mathbf{F}), \quad \text{and} \quad \frac{\sigma + \delta}{\rho_1 - \delta} < 1. \quad (48)$$

First, we approximate $\sum_{\nu=n+|\mathbf{m}_n|-m_{n,\alpha}+1}^\infty |b_{\nu,n}^{(\alpha)}||\Phi_\nu(z)|$ on $\overline{D}_\sigma$. With the similar computation as (22), it follows from (2), (9), and (43) that for $\nu \geq n + |\mathbf{m}_n| - m_{n,\alpha} + 1$,

$$|b_{\nu,n}^{(\alpha)}| \leq \frac{c_1 C_1^{|\mathbf{m}_n|}}{\rho_1^\nu}. \quad (49)$$

Therefore, by (10) and (49),

$$\sum_{\nu=n+|\mathbf{m}_n|-m_{n,\alpha}+1}^\infty |b_{\nu,n}^{(\alpha)}||\Phi_\nu(z)| \leq \sum_{\nu=n+|\mathbf{m}_n|-m_{n,\alpha}+1}^\infty c_2 C_1^{|\mathbf{m}_n|} \left(\frac{\sigma}{\rho_1}\right)^\nu \leq c_3 C_1^{|\mathbf{m}_n|} \left(\frac{\sigma}{\rho_1}\right)^n, \quad z \in \overline{D}_\sigma. \quad (50)$$

Next, we approximate $\sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n,\alpha}} |b_{\nu,n}^{(\alpha)}||\Phi_\nu(z)|$ on $\overline{D}_\sigma$. Again, we begin by approximating $|a_{k,n}^{(\alpha)}|$. Choose $\rho_2 \in (1, \rho_0(\mathbf{F}))$, we have

$$a_{k,n}^{(\alpha)} = [Q_{n,\mathbf{m}_n}^E F_\alpha]_k = \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{Q_{n,\mathbf{m}_n}^E(z)F_\alpha(z)\Phi'(z)}{\Phi^{k+1}(z)} dz.$$

Define

$$\gamma_{k,n}^{(\alpha)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{Q_{n,\mathbf{m}_n}^E(z)F_\alpha(z)\Phi'(z)}{\Phi^{k+1}(z)} dz.$$

Arguing as (26), we obtain

$$\gamma_{k,n}^{(\alpha)} - a_{k,n}^{(\alpha)} = \sum_{j=1}^q \text{Res}(Q_{n,\mathbf{m}_n}^E F_\alpha \Phi' / \Phi^{k+1}, \lambda_j). \quad (51)$$

Recall that the limit formula for the residue of $Q_{n,\mathbf{m}_n}^E F_\alpha \Phi' / \Phi^{k+1}$ at λ_j is

$$\text{Res}(Q_{n,\mathbf{m}_n}^E F_\alpha \Phi' / \Phi^{k+1}, \lambda_j) = \frac{1}{(\tau_j - 1)!} \lim_{z \rightarrow \lambda_j} \left((z - \lambda_j)^{\tau_j} Q_{n,\mathbf{m}_n}^E F_\alpha \Phi' / \Phi^{k+1} \right)^{(\tau_j - 1)}(z).$$

Leibniz's formula allows us to write

$$\left((z - \lambda_j)^{\tau_j} Q_{n,\mathbf{m}_n}^E F_\alpha \frac{\Phi'}{\Phi^{k+1}} \right)^{(\tau_j - 1)}(z) = \sum_{t=0}^{\tau_j - 1} \binom{\tau_j - 1}{t} \left(Q_{n,\mathbf{m}_n}^E \frac{\Phi'}{\Phi^{n+1}} \right)^{(\tau_j - 1 - t)}(z) \left((z - \lambda_j)^{\tau_j} F_\alpha \Phi^{n-k} \right)^{(t)}(z).$$

For $j = 1, 2, \dots, q$, and $t = 0, 1, \dots, \tau_j - 1$, set

$$\beta_n(j, t) := \frac{1}{(\tau_j - 1)!} \binom{\tau_j - 1}{t} \lim_{z \rightarrow \lambda_j} \left(Q_{n, \mathbf{m}_n}^E \frac{\Phi'}{\Phi^{n+1}} \right)^{(\tau_j - 1 - t)}(z)$$

(notice that $\beta_n(j, t)$ do not depend on k and α). Then, we can rewrite (51) as

$$\gamma_{k, n}^{(\alpha)} - a_{k, n}^{(\alpha)} = \sum_{j=1}^q \sum_{t=0}^{\tau_j-1} \beta_n(j, t) \left((z - \lambda_j)^{\tau_j} F_\alpha \Phi^{n-k} \right)^{(t)}(\lambda_j). \quad (52)$$

Since $a_{k, n}^{(\alpha)} = 0$ for $\alpha = 1, 2, \dots, d$, $k = n - m_{n, \alpha} + 1, n - m_{n, \alpha} + 2, \dots, n$ and the assumption that $m_{n, \alpha} \geq m_\alpha$, we have

$$\gamma_{k, n}^{(\alpha)} = \sum_{j=1}^q \sum_{t=0}^{\tau_j-1} \beta_n(j, t) \left((z - \lambda_j)^{\tau_j} F_\alpha \Phi^{n-k} \right)^{(t)}(\lambda_j), \quad \alpha = 1, 2, \dots, d, \quad k = n - m_\alpha + 1, n - m_\alpha + 2, \dots, n. \quad (53)$$

Now, we use the same technique as for simultaneous Padé-orthogonal approximants to find $\beta_n(j, t)$ by replacing s_k/s_n with Φ^{n-k} in (28). Consider (53) as a system of $|\mathbf{m}|$ equations on the $|\mathbf{m}|$ unknowns $\beta_n(j, t)$, it follows that for $\alpha = 1, 2, \dots, d$ and $k \geq n + 1$,

$$\gamma_{k, n}^{(\alpha)} - a_{k, n}^{(\alpha)} = \frac{1}{\Delta_n} \sum_{j=1}^q \sum_{t=0}^{\tau_j-1} \sum_{w=1}^d \sum_{y=1}^{m_w} \gamma_{n-m_w+y, n}^{(w)} C_n[g_{w, y}, h_{j, t}] \left((z - \lambda_j)^{\tau_j} F_\alpha \Phi^{n-k} \right)^{(t)}(\lambda_j), \quad (54)$$

where Δ_n , $\gamma_{n-m_w+y, n}^{(w)}$, and $C_n[g_{w, y}, h_{j, t}]$ are defined in the same way as for simultaneous Padé-orthogonal approximants.

Arguing as (30) and (31) by replacing s_k/s_n with $(z - \lambda_j)^{\tau_j} F_\alpha \Phi^{n-k}$, (32) by replacing s_k/s_n with Φ^{n-k} , (34) and (35), we have for sufficiently large n , $\alpha = 1, 2, \dots, d$,

$$|a_{k, n}^{(\alpha)}| \leq |\gamma_{k, n}^{(\alpha)}| + \frac{c_4}{\rho_2^{k-n}} \sum_{w=1}^d \sum_{y=1}^{m_w} |\gamma_{n-m_w+y, n}^{(w)}|, \quad \alpha = 1, 2, \dots, d, \quad k \geq n + 1.$$

By the definition of $\gamma_{k, n}^{(\alpha)}$, for all sufficiently large n , we obtain

$$|\gamma_{k, n}^{(\alpha)}| \leq \frac{c_5 C_1^{|\mathbf{m}_n|}}{\rho_1^k}, \quad \alpha = 1, 2, \dots, d, \quad k \geq n - |\mathbf{m}| + 1.$$

This implies that

$$|a_{k, n}^{(\alpha)}| \leq \frac{c_6 C_1^{|\mathbf{m}_n|}}{\rho_2^{k-n} \rho_1^n}, \quad \alpha = 1, 2, \dots, d, \quad k \geq n + 1. \quad (55)$$

Now, we estimate $|[Q_{|\mathbf{m}|}^{\mathbf{F}} \Phi_k]_\nu|$. Suppose that $\delta > 0$ is sufficiently small such that $\rho_2 - \delta > 1$. then,

$$|[Q_{|\mathbf{m}|}^{\mathbf{F}} \Phi_k]_\nu| = \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_2-\delta}} \frac{Q_{|\mathbf{m}|}^{\mathbf{F}}(z) \Phi_k(z) \Phi'(z)}{\Phi^{\nu+1}(z)} dz \right| \leq c_7 \frac{(\rho_2 - \delta)^k}{(\rho_2 - \delta)^\nu}. \quad (56)$$

Consequently, we get

$$|b_{\nu, n}^{(\alpha)}| \leq \sum_{k=n+1}^{\infty} |a_{k, n}^{(\alpha)}| |[Q_{|\mathbf{m}|}^{\mathbf{F}} \Phi_k]_\nu| \leq \frac{c_8 C_1^{|\mathbf{m}_n|}}{(\rho_2 - \delta)^\nu} \left(\frac{\rho_2}{\rho_1} \right)^n \sum_{k=n+1}^{\infty} \left(\frac{\rho_2 - \delta}{\rho_2} \right)^k \leq \frac{c_9 C_1^{|\mathbf{m}_n|}}{(\rho_2 - \delta)^\nu} \left(\frac{\rho_2 - \delta}{\rho_1} \right)^n.$$

Therefore, for each $\alpha = 1, 2, \dots, d$ and sufficiently large n ,

$$\sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n, \alpha}} |b_{\nu, n}^{(\alpha)}| |\Phi_\nu(z)| \leq c_{10} C_1^{|\mathbf{m}_n|} \left(\frac{\rho_2 - \delta}{\rho_1} \right)^n \sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n, \alpha}} \left(\frac{\sigma}{\rho_2 - \delta} \right)^\nu$$

$$\leq c_{15}(n + |\mathbf{m}_n| - m_{n,\alpha} + 1)C_1^{|\mathbf{m}_n|} \left(\frac{\rho_2 - \delta}{\rho_1} \right)^n \left(\frac{\sigma}{\rho_2 - \delta} \right)^{n+|\mathbf{m}_n|-m_{n,\alpha}} \quad (57)$$

where $z \in \overline{D}_\sigma$.

Combining (50) and (57), it follows from (46) that for each $k \geq n_2$,

$$|Q_{|\mathbf{m}|}^{\mathbf{F}}(z)Q_{n,\mathbf{m}_n}^E(z)F_\alpha(z) - P_{n,\mathbf{m}_n,\alpha}^E(z)| \leq c_{11}C_1^{|\mathbf{m}_n|}\theta^n, \quad \alpha = 1, 2, \dots, d, \quad z \in \overline{D}_\sigma, \quad n \geq k, \quad (58)$$

where θ is an arbitrary constant which satisfies

$$\frac{\sigma}{\rho_1} < \theta < 1.$$

Repeating the lines of reasoning used after (40) in the proof of Theorem 2.2, we arrive

$$\limsup_{n \rightarrow \infty} \|F_\alpha - R_{n,\mathbf{m}_n,\alpha}^E\|_{K(\varepsilon)}^{1/n} \leq \frac{\sigma}{\rho_{|\mathbf{m}|}(\mathbf{F})} < 1.$$

This implies that for any $\beta > 0$ and $\alpha = 1, 2, \dots, d$, each sequence $\{R_{n,\mathbf{m}_n,\alpha}^E\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to F_α inside $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$, as $n \rightarrow \infty$. $\square$

Proof of Corollary 2.5. Arguing as the proof of Corollary 2.3 and replacing $R_{n,\mathbf{m}_n,\alpha}^\mu$ with $R_{n,\mathbf{m}_n,\alpha}^E$, we have (4). $\square$

Proof of Theorem 2.7. For each $n \in \mathbb{N}$, we normalize the polynomial $\tilde{q}_{n,\mathbf{m}_n}^\mu$ in terms of its zeros $\lambda_{n,j}$ so that

$$\tilde{Q}_{n,\mathbf{m}_n}^\mu(z) := \prod_{|\lambda_{n,j}| \leq 1} (z - \lambda_{n,j}) \prod_{|\lambda_{n,j}| > 1} \left(1 - \frac{z}{\lambda_{n,j}} \right) \quad (59)$$

and for each $\alpha = 1, 2, \dots, d$,

$$\tilde{R}_{n,\mathbf{m}_n,\alpha}^\mu = \frac{\tilde{p}_{n,\mathbf{m}_n,\alpha}^\mu}{\tilde{q}_{n,\mathbf{m}_n}^\mu} = \frac{\tilde{P}_{n,\mathbf{m}_n,\alpha}^\mu}{\tilde{Q}_{n,\mathbf{m}_n}^\mu}.$$

With this normalization, we can estimate upper and lower bounds on the normalized $\tilde{Q}_{n,\mathbf{m}_n}^\mu$. Let $\varepsilon > 0$ and α be fixed. Suppose that the poles of F_α in $D_\alpha^*(\mathbf{F}, \mathbf{m})$ are $\lambda_1, \lambda_2, \dots, \lambda_{d'}$ when $d' \leq |\mathbf{m}|$ and the zeros of $\tilde{Q}_{n,\mathbf{m}_n}^\mu$ are $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,l_{\mathbf{m}_n}}$ (they are not necessary distinct and $l_{\mathbf{m}_n} \leq |\mathbf{m}_n|$). We cover each pole of F_α in $D_\alpha^*(\mathbf{F}, \mathbf{m})$ with an open disk of radius $(\varepsilon/(6|\mathbf{m}|))^{1/\beta}$ and denote by $J_{0,\varepsilon}^\beta(F_\alpha, \mathbf{m})$ the union of these disks. For each $n \geq 1$, we cover each zero of $\tilde{Q}_{n,\mathbf{m}_n}^\mu$ with an open disk of radius $(\varepsilon/(6|\mathbf{m}_n|(n^2)))^{1/\beta}$ and denote by $J_{n,\varepsilon}^\beta(\mathbf{F})$ the union of these disks. Set for each $k \in \mathbb{N}$,

$$J_\varepsilon^\beta(F_\alpha, \mathbf{m}; k) := J_{0,\varepsilon}^\beta(F_\alpha, \mathbf{m}) \cup \left(\bigcup_{n=k}^{\infty} J_{n,\varepsilon}^\beta(\mathbf{F}) \right)$$

and

$$J_\varepsilon^\beta(F_\alpha, \mathbf{m}) := J_\varepsilon^\beta(F_\alpha, \mathbf{m}; 1).$$

By using the monotonicity and subadditivity of h_β , it easy to check that $h_\beta(J_\varepsilon^\beta(F_\alpha, \mathbf{m})) < \varepsilon$ and $J_{\varepsilon_1}^\beta(F_\alpha, \mathbf{m}) \subset J_{\varepsilon_2}^\beta(F_\alpha, \mathbf{m})$ for $\varepsilon_1 < \varepsilon_2$. For any set $B \subset D_\alpha^*(\mathbf{F}, \mathbf{m})$, we put $B(\varepsilon) := B \setminus J_\varepsilon^\beta(F_\alpha, \mathbf{m})$. Clearly that if $\{g_n\}_{n \in \mathbb{N}}$ converges uniformly to g on $K(\varepsilon)$ for any compact subset $K \subset D_\alpha^*(\mathbf{F}, \mathbf{m})$ and $\varepsilon > 0$, then $h_\beta - \lim_{n \rightarrow \infty} g_n = g$ in $D_\alpha^*(\mathbf{F}, \mathbf{m})$.

Note that for any compact subset $K \subset D_\alpha^*(\mathbf{F}, \mathbf{m})$ and for any $\varepsilon > 0$, there exist positive constants $C_1 > 0$ and $C_2 > 0$ independent of n such that for all sufficiently large n ,

$$\|\tilde{Q}_{n,\mathbf{m}_n}^\mu\|_K \leq C_1^{|\mathbf{m}_n|}, \quad (60)$$

and

$$\min_{z \in K \setminus J_\varepsilon^\beta(F_\alpha, \mathbf{m}; k)} |\tilde{Q}_{n,\mathbf{m}_n}^\mu(z)| \geq (C_2 |\mathbf{m}_n| n^2)^{-2|\mathbf{m}_n|/\beta}. \quad (61)$$

Let ξ be a system pole of order τ of $\mathbf{F}$ with respect to $\mathbf{m}$. We begin the proof by showing that

$$\limsup_{n \rightarrow \infty} |(\tilde{Q}_{n, \mathbf{m}_n}^\mu)^{(j)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_{\xi, j+1}(\mathbf{F}, \mathbf{m})}, \quad j = 0, 1, \dots, \tau - 1.$$

Consider polynomial combination G_1 of type (5) that is holomorphic on a neighborhood of $\bar{D}_{|\Phi(\xi)|}$ except for a simple pole at $z = \xi$ and verifies that $\rho_1(G_1) = \rho_{\xi, 1}(\mathbf{F}, \mathbf{m}) (= \rho_{\xi, 1}(\mathbf{F}, \mathbf{m}))$. Then, we have

$$G_1 = \sum_{\alpha=1}^d v_{\alpha, 1} F_\alpha, \quad \deg v_{\alpha, 1} < m_\alpha, \quad \alpha = 1, 2, \dots, d.$$

Define

$$H_1(z) := (z - \xi)G_1(z) \quad \text{and} \quad a_{n, n}^{(1)} := \langle \tilde{Q}_{n, \mathbf{m}_n}^\mu G_1 \rangle_n.$$

By the definition of $\tilde{Q}_{n, \mathbf{m}_n}^\mu$, it follows that $a_{n, n}^{(1)} = 0$. Then, we have

$$a_{n, n}^{(1)} = \langle \tilde{Q}_{n, \mathbf{m}_n}^\mu G_1 \rangle_n = \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \tilde{Q}_{n, \mathbf{m}_n}^\mu(z) G_1(z) s_n(z) dz = 0,$$

where $1 < \rho_1 < |\Phi(\xi)|$. Set

$$\gamma_{n, n}^{(1)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \tilde{Q}_{n, \mathbf{m}_n}^\mu(z) G_1(z) s_n(z) dz,$$

where $|\Phi(\xi)| < \rho_2 < \rho_{\xi, 1}(\mathbf{F}, \mathbf{m})$. Arguing as in (26), we have

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \tilde{Q}_{n, \mathbf{m}_n}^\mu(t) G_1(t) s_n(t) dt - \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \tilde{Q}_{n, \mathbf{m}_n}^\mu(t) G_1(t) s_n(t) dt \\ &= \text{Res}(\tilde{Q}_{n, \mathbf{m}_n}^\mu G_1 s_n, \xi) = \lim_{z \rightarrow \xi} (z - \xi) \tilde{Q}_{n, \mathbf{m}_n}^\mu(z) G_1(z) s_n(z) = H_1(\xi) \tilde{Q}_{n, \mathbf{m}_n}^\mu(\xi) s_n(\xi). \end{aligned} \quad (62)$$

Then, we can rewrite (62) as

$$\gamma_{n, n}^{(1)} = \gamma_{n, n}^{(1)} - a_{n, n}^{(1)} = H_1(\xi) \tilde{Q}_{n, \mathbf{m}_n}^\mu(\xi) s_n(\xi).$$

Since $a_{n, n}^{(1)} = 0$, it follows that

$$\tilde{Q}_{n, \mathbf{m}_n}^\mu(\xi) = \frac{\gamma_{n, n}^{(1)}}{H_1(\xi) s_n(\xi)}. \quad (63)$$

Choose $\delta > 0$ so small such that

$$\rho_2 := \rho_{\xi, 1}(\mathbf{F}, \mathbf{m}) - \delta > |\Phi(\xi)|, \quad |\Phi(\xi)| - \delta > 1, \quad \text{and} \quad \frac{|\Phi(\xi)| + \delta}{\rho_2 - \delta} < 1.$$

By using (??), there exists $n_0 \in \mathbb{N}$ and $c_2 > 0, c_3 > 0$ such that

$$\frac{c_1}{(\rho + \delta)^n} \leq \|s_n\|_{\Gamma_\rho} \leq \frac{c_2}{(\rho - \delta)^n}, \quad n \geq n_0. \quad (64)$$

Then, from the definition of $\gamma_{n, n}^{(1)}$ and (64), we have

$$|\gamma_{n, n}^{(1)}| \leq \frac{c_3 C_1^{|\mathbf{m}_n|}}{(\rho_2 - \delta)^n} \quad \text{and} \quad |s_n(\xi)| \geq \frac{c_1}{(|\Phi(\xi)| + \delta)^n}. \quad (65)$$

By (65), it follows from (63) that

$$|\tilde{Q}_{n, \mathbf{m}_n}^\mu(\xi)| \leq c_4 C_1^{|\mathbf{m}_n|} \left(\frac{|\Phi(\xi)| + \delta}{\rho_2 - \delta} \right)^n.$$

Letting $\delta \rightarrow 0$, we obtain $\rho_2 \rightarrow \rho_{\xi, 1}(\mathbf{F}, \mathbf{m})$ and

$$\limsup_{n \rightarrow \infty} |\tilde{Q}_{n, \mathbf{m}_n}^\mu(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_{\xi, 1}(\mathbf{F}, \mathbf{m})}.$$

Now we employ induction. Suppose that

$$\limsup_{n \rightarrow \infty} |(\tilde{Q}_{n, \mathbf{m}_n}^\mu)^{(j)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_{\xi, j+1}(\mathbf{F}, \mathbf{m})}, \quad j = 0, 1, \dots, \ell - 2, \quad \text{and} \quad \ell \leq \tau. \quad (66)$$

We will show that (66) holds for $j = \ell - 1$. Consider a polynomial combination G_ℓ of type (5) that is holomorphic on a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole of order ℓ at $z = \xi$ and verifies that $\rho_\ell(G_\ell) = \rho_{\xi, \ell}(\mathbf{F}, \mathbf{m})$. Then, we have

$$G_\ell = \sum_{\alpha=\ell}^d v_{\alpha, \ell} F_\alpha, \quad \deg v_{\alpha, \ell} < m_\alpha, \quad \alpha = 1, 2, \dots, d.$$

Define

$$H_\ell(z) := (z - \xi)^\ell G_\ell(z) \quad \text{and} \quad a_{n, n}^{(\ell)} := \langle \tilde{Q}_{n, \mathbf{m}_n}^\mu G_\ell \rangle_n.$$

By the definition of $\tilde{Q}_{n, \mathbf{m}_n}^\mu$, it follows that $a_{n, n}^{(\ell)} = 0$. Consider

$$a_{n, n}^{(\ell)} = [\tilde{Q}_{n, \mathbf{m}_n}^\mu G_\ell]_n = \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \tilde{Q}_{n, \mathbf{m}_n}^\mu(z) G_\ell(z) s_n(z) dz = 0,$$

where $1 < \rho_1 < |\Phi(\xi)|$. Set

$$\tau_{n, n}^{(\ell)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \tilde{Q}_{n, \mathbf{m}_n}^\mu(z) G_\ell(z) s_n(z) dz,$$

where $|\Phi(\xi)| < \rho_2 < \rho_{\xi, \ell}(\mathbf{F}, \mathbf{m})$. Again, arguing as in (26), we have

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\Gamma_{\rho_6}} \tilde{Q}_{n, \mathbf{m}_n}^\mu(t) G_\ell(t) s_n(t) dt - \frac{1}{2\pi i} \int_{\Gamma_{\rho_5}} \tilde{Q}_{n, \mathbf{m}_n}^\mu(t) G_\ell(t) s_n(t) dt \\ &= \text{Res}(\tilde{Q}_{n, \mathbf{m}_n}^\mu G_\ell s_n, \xi) = \frac{1}{(\ell - 1)!} \sum_{t=0}^{\ell-1} \binom{\ell-1}{t} (H_\ell s_n)^{(\ell-1-t)}(\xi) (\tilde{Q}_{n, \mathbf{m}_n}^\mu)^{(t)}(\xi). \end{aligned} \quad (67)$$

Since $a_{n, n}^{(\ell)} = 0$, equation (67) become

$$(\ell - 1)! \tau_{n, n}^{(\ell)} = \sum_{t=0}^{\ell-2} \binom{\ell-1}{t} (H_\ell s_n)^{(\ell-1-t)}(\xi) (\tilde{Q}_{n, \mathbf{m}_n}^\mu)^{(t)}(\xi) + H_\ell(\xi) s_n(\xi) (\tilde{Q}_{n, \mathbf{m}_n}^\mu)^{(\ell-1)}(\xi),$$

which implies that

$$(\tilde{Q}_{n, \mathbf{m}_n}^\mu)^{(\ell-1)}(\xi) = \frac{(\ell - 1)! \tau_{n, n}^{(\ell)}}{H_\ell(\xi) s_n(\xi)} - \sum_{t=0}^{\ell-2} \binom{\ell-1}{t} \frac{(H_\ell s_n)^{(\ell-1-t)}(\xi) (\tilde{Q}_{n, \mathbf{m}_n}^\mu)^{(t)}(\xi)}{H_\ell(\xi) s_n(\xi)}. \quad (68)$$

Choose $\delta > 0$ so small such that

$$\rho_2 := \rho_{\xi, \ell}(\mathbf{F}, \mathbf{m}) - \delta > |\Phi(\xi)|, \quad |\Phi(\xi)| - \delta > 1, \quad \text{and} \quad \frac{|\Phi(\xi)| + \delta}{\rho_2 - \delta} < 1.$$

Arguing as (65), we have

$$|\gamma_{n, n}^{(\ell)}| \leq \frac{c_5 C_1^{|\mathbf{m}_n|}}{(\rho_2 - \delta)^n}, \quad \text{and} \quad |s_n(\xi)| \geq \frac{c_1}{(|\Phi(\xi)| + \delta)^n}, \quad (69)$$

and for all $t = 0, 1, 2, \dots, \ell - 2$,

$$|(H_\ell s_n)^{(\ell-1-t)}(\xi)| = \left| \frac{(\ell - 1 - t)!}{2\pi i} \int_{|z - \xi| = \varepsilon} \frac{H_\ell(z) s_n(z)}{(z - \xi)^{\ell-t}} dz \right| \leq \frac{c_6}{(|\Phi(\xi)| - \delta_2)^n}, \quad (70)$$

where $\{z \in \mathbb{C} : |z - \xi| = \varepsilon\} \subset \{z \in \mathbb{C} : |\Phi(z)| > |\Phi(\xi)| - \delta\}$. Moreover, by (66), we have for all $j = 0, 2, \dots, \ell - 2$,

$$|(\tilde{Q}_{n, \mathbf{m}_n}^\mu)^{(j)}(\xi)| \leq c_7 \left(\frac{|\Phi(\xi)|}{\rho_{\xi, j+1}(\mathbf{F}, \mathbf{m})} \right)^n \leq c_7 \left(\frac{|\Phi(\xi)|}{\rho_{\xi, \ell-1}(\mathbf{F}, \mathbf{m})} \right)^n. \quad (71)$$

Combining (69), (70), and (71), it follows from (68) that

$$\begin{aligned} |(\tilde{Q}_{n, \mathbf{m}_n}^\mu)^{(\ell-1)}(\xi)| &= \left| \frac{(\ell-1)! \tau_{n, n}^{(\ell)}}{H_\ell(\xi) s_n(\xi)} - \sum_{t=0}^{\ell-2} \binom{\ell-1}{t} \frac{(H_\ell s_n)^{(\ell-1-t)}(\xi) (\tilde{Q}_{n, \mathbf{m}_n}^\mu)^{(t)}(\xi)}{H_\ell(\xi) s_n(\xi)} \right| \\ &\leq c_8 C_1^{|\mathbf{m}_n|} \left(\frac{|\Phi(\xi)| + \delta}{\rho_2 - \delta} \right)^n + c_9 \left(\frac{|\Phi(\xi)| + \delta}{|\Phi(\xi)| - \delta} \right)^n \left(\frac{|\Phi(\xi)|}{\rho_{\xi, \ell-1}(\mathbf{F}, \mathbf{m})} \right)^n, \end{aligned} \quad (72)$$

which implies that

$$\limsup_{n \rightarrow \infty} |(\tilde{Q}_{n, \mathbf{m}_n}^\mu)^{(\ell-1)}(\xi)|^{1/n} \leq \max \left\{ \frac{|\Phi(\xi)| + \delta}{\rho_2 - \delta}, \left(\frac{|\Phi(\xi)| + \delta}{|\Phi(\xi)| - \delta} \right) \left(\frac{|\Phi(\xi)|}{\rho_{\xi, \ell-1}(\mathbf{F}, \mathbf{m})} \right) \right\}.$$

Letting $\delta \rightarrow 0$, we obtain $\rho_2 \rightarrow \rho_{\xi, \ell}(\mathbf{F}, \mathbf{m})$ and

$$\begin{aligned} \limsup_{n \rightarrow \infty} |(\tilde{Q}_{n, \mathbf{m}_n}^\mu)^{(\ell-1)}(\xi)|^{1/n} &\leq \max \left\{ \frac{|\Phi(\xi)|}{\rho_{\xi, \ell}(\mathbf{F}, \mathbf{m})}, \left(\frac{|\Phi(\xi)|}{\rho_{\xi, \ell-1}(\mathbf{F}, \mathbf{m})} \right) \right\} \\ &\leq \frac{|\Phi(\xi)|}{\rho_{\xi, \ell}(\mathbf{F}, \mathbf{m})}, \end{aligned}$$

which completes the induction proof.

Let $\xi_1, \xi_2, \dots, \xi_w$ be the distinct system poles of $\mathbf{F}$ with respect to $\mathbf{m}$, and let τ_j be the order of ξ_j as a system pole, $j = 1, 2, \dots, w$. By the assumption that $\sum_{j=1}^w \tau_j = |\mathbf{m}|$, we have proved that, for $j = 1, 2, \dots, w$ and $t = 0, 1, \dots, \tau_j - 1$,

$$\limsup_{n \rightarrow \infty} |(\tilde{Q}_{n, \mathbf{m}_n}^\mu)^{(t)}(\xi_j)|^{1/n} \leq \frac{|\Phi(\xi_j)|}{\rho_{\xi_j, t+1}(\mathbf{F}, \mathbf{m})} \leq \frac{|\Phi(\xi_j)|}{\rho_{\xi_j}(\mathbf{F}, \mathbf{m})}. \quad (73)$$

Let $\alpha \in \{1, 2, \dots, d\}$ be fixed and let $\{\tilde{\xi}_1, \dots, \tilde{\xi}_N\}$ be the poles of F_α in $D_\alpha^*(\mathbf{F}, \mathbf{m})$. For each $j = 1, 2, \dots, N$, let $\hat{\tau}_j$ be the order of $\tilde{\xi}_j$ as a pole of F_α and $\tilde{\tau}_j$ its order as a system pole. Choose $\rho_1 \in (1, \rho_0(F_\alpha))$. Consider

$$a_{k, n}^{(\alpha)} = \langle \tilde{Q}_{n, \mathbf{m}_n}^\mu F_\alpha \rangle_k = \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \tilde{Q}_{n, \mathbf{m}_n}^\mu(z) F_\alpha(z) s_k(z) dz.$$

Define

$$\gamma_{k, n}^{(\alpha)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \tilde{Q}_{n, \mathbf{m}_n}^\mu(z) F_\alpha(z) s_k(z) dz,$$

where $\rho_2 \in (1, \rho_\alpha^*(\mathbf{F}, \mathbf{m}))$. Arguing as in (26), we have

$$\gamma_{k, n}^{(\alpha)} - a_{k, n}^{(\alpha)} = \sum_{j=1}^N \text{Res}(\tilde{Q}_{n, \mathbf{m}_n}^\mu F_\alpha s_k, \tilde{\xi}_j). \quad (74)$$

Recall that the limit formula for the residue of $Q_{n, \mathbf{m}_n}^\mu F_\alpha s_k$ at $\tilde{\xi}_j$ is

$$\begin{aligned} \text{Res}(\tilde{Q}_{n, \mathbf{m}_n}^\mu F_\alpha s_k, \tilde{\xi}_j) &= \frac{1}{(\hat{\tau}_j - 1)!} \lim_{z \rightarrow \tilde{\xi}_j} \left((z - \tilde{\xi}_j)^{\hat{\tau}_j} \tilde{Q}_{n, \mathbf{m}_n}^\mu F_\alpha s_k \right)^{(\hat{\tau}_j - 1)}(z) \\ &= \frac{1}{(\hat{\tau}_j - 1)!} \sum_{t=0}^{\hat{\tau}_j - 1} \binom{\hat{\tau}_j - 1}{t} ((z - \tilde{\xi}_j)^{\hat{\tau}_j} F_\alpha s_k)^{(\hat{\tau}_j - 1 - t)}(\tilde{\xi}_j) (\tilde{Q}_{n, \mathbf{m}_n}^\mu)^{(t)}(\tilde{\xi}_j). \end{aligned} \quad (75)$$

Let $\delta > 0$ be such that $\rho_2 - \delta > 1$ and $|\Phi(\tilde{\xi}_j)| - \delta > 1$. Arguing as (65) and the computation similar as (70),

$$|\gamma_{k,n}^{(\alpha)}| \leq \frac{c_{10}C_1^{|\mathbf{m}_n|}}{(\rho_2 - \delta)^k} \quad \text{and} \quad |((z - \tilde{\xi}_j)^{\hat{\tau}_j} F_\alpha s_k)^{(\hat{\tau}_j - 1 - t)}(\tilde{\xi}_j)| \leq \frac{c_{11}}{(|\Phi(\tilde{\xi}_j)| - \delta)^k}, \quad (76)$$

respectively. By (75) and (76), it follows from (74) that

$$|a_{k,n}^\alpha| \leq \frac{c_{10}C_1^{|\mathbf{m}_n|}}{(\rho_2 - \delta)^k} + c_{12} \sum_{j=1}^N \frac{|\Phi(\tilde{\xi}_j)|^n}{(\rho_{\tilde{\xi}_j}(\mathbf{F}, \mathbf{m})^n (|\Phi(\tilde{\xi}_j)| - \delta)^k)} \leq \frac{c_{10}C_1^{|\mathbf{m}_n|}}{(\rho_2 - \delta)^k} + \frac{c_{12}}{(\rho_\alpha^*(\mathbf{F}, \mathbf{m}))^n} \sum_{j=1}^N \frac{|\Phi(\tilde{\xi}_j)|^n}{(|\Phi(\tilde{\xi}_j)| - \delta)^k}. \quad (77)$$

Let

$$\omega(z) := \prod_{j=1}^N (z - \tilde{\xi}_j)^{\hat{\tau}_j}.$$

From the definition of orthogonal Hermite-Padé approximants and Lemma (3.1), we have,

$$\tilde{Q}_{n,\mathbf{m}_n}^\mu(z) F_\alpha(z) - \tilde{P}_{n,\mathbf{m},\alpha}^\mu(z) = \sum_{k=n+1}^{\infty} a_{k,n}^{(\alpha)} p_k, \quad z \in D_{\rho_0(F_\alpha)}, \quad (78)$$

where

$$a_{k,n}^{(\alpha)} := \langle \tilde{Q}_{n,\mathbf{m}_n}^\mu F_\alpha \rangle_k, \quad k = 0, 1, 2, \dots$$

Multiplying (94) by ω and expanding the result in terms of the orthogonal system $\{p_\nu\}_{\nu=0}^\infty$ such that for $z \in D_\alpha(\mathbf{F}, \mathbf{m})$,

$$\begin{aligned} \omega(z) \tilde{Q}_{n,\mathbf{m}_n}^\mu(z) F_\alpha(z) - \omega(z) \tilde{P}_{n,\mathbf{m},\alpha}^\mu(z) &= \sum_{k=n+1}^{\infty} \omega(z) a_{k,n}^{(\alpha)} p_k(z) \\ &= \sum_{\nu=0}^{\infty} b_{\nu,n}^{(\alpha)} p_\nu(z) = \sum_{\nu=0}^{n+|\mathbf{m}_n|} b_{\nu,n}^{(\alpha)} p_\nu(z) + \sum_{\nu=n+|\mathbf{m}_n|+1}^{\infty} b_{\nu,n}^{(\alpha)} p_\nu(z). \end{aligned} \quad (79)$$

Let K be a compact subset of $D_\alpha^*(\mathbf{F}, \mathbf{m})$ and set

$$\sigma := \max\{\|\Phi\|_K, 1\} \quad (80)$$

($\sigma = 1$ when $K \subset E$). Choose $\delta > 0$ sufficiently small such that

$$\rho_2 := \rho_\alpha^*(\mathbf{F}, \mathbf{m}) - \delta, \quad \rho_\alpha^*(\mathbf{F}, \mathbf{m}) - 2\delta > 1, \quad \text{and} \quad \frac{\sigma + \delta}{\rho_2 - \delta} < 1 \quad (81)$$

First, we approximate $\sum_{\nu=n+|\mathbf{m}_n|+1}^{\infty} |b_{\nu,n}^{(\alpha)}| |p_\nu(z)|$ on $\overline{D}_\sigma$. Auguring as in (22)-(25), we have

$$\sum_{\nu=n+|\mathbf{m}_n|+1}^{\infty} |b_{\nu,n}^{(\alpha)}| |p_\nu(z)| \leq c_{13} C_1^{|\mathbf{m}_n|} \left(\frac{\sigma + \delta}{\rho_2 - \delta} \right)^n, \quad z \in \overline{D}_\sigma. \quad (82)$$

Letting $\delta \rightarrow 0$, we have $\rho_1 \rightarrow \rho_\alpha^*(\mathbf{F}, \mathbf{m})$, then

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=n+|\mathbf{m}_n|+1}^{\infty} |b_{\nu,n}^{(\alpha)}| |p_\nu(z)| \right\|_{\overline{D}_\sigma}^{1/n} \leq \frac{\sigma}{\rho_\alpha^*(\mathbf{F}, \mathbf{m})}. \quad (83)$$

Next, we approximate $\sum_{\nu=0}^{n+|\mathbf{m}_n|} |b_{\nu,n}^{(\alpha)}| |p_\nu(z)|$ on $\overline{D}_\sigma$. By using the same technique as (38) and inequality (93), we have

$$|b_{\nu,n}^{(\alpha)}| \leq \sum_{k=n+1}^{\infty} |a_{k,n}^{(\alpha)}| |\langle Q_{|\mathbf{m}|}^{\mathbf{F}} p_k \rangle_\nu| \leq \frac{c_{14}C_1^{|\mathbf{m}_n|}}{(\rho_2 - \delta)^n} + \frac{c_{15}}{(\rho_\alpha^*(\mathbf{F}, \mathbf{m}))^n} \sum_{j=1}^N \frac{|\Phi(\lambda_j)|^n}{(|\Phi(\lambda_j)| - \delta)^n}. \quad (84)$$

Therefore, for $z \in \overline{D}_\sigma$,

$$\sum_{\nu=0}^{n+|\mathbf{m}_n|} |b_{\nu,n}^{(\alpha)}| |p_\nu(z)| \leq c_{16}(n+|\mathbf{m}_n|+1)(\sigma+\delta)^{n+|\mathbf{m}_n|} \left(\frac{C_1^{|\mathbf{m}_n|}}{(\rho_2 - \delta)^n} + \frac{1}{(\rho_\alpha^*(\mathbf{F}, \mathbf{m}))^n} \sum_{j=1}^q \frac{|\Phi(\lambda_j)|^n}{(|\Phi(\lambda_j)| - \delta)^n} \right), \quad (85)$$

which implies that

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=0}^{n+|\mathbf{m}_n|} |b_{\nu,n}^{(\alpha)}| |p_\nu(z)| \right\|_{\overline{D}_\sigma}^{1/n} \leq \max \left\{ \left(\frac{\sigma + \delta}{\rho_2 - \delta} \right), \frac{(\sigma + \delta)}{\rho_\alpha^*(\mathbf{F}, \mathbf{m})} \max_{j=1,2,\dots,q} \frac{|\Phi(\lambda_j)|}{(|\Phi(\lambda_j)| - \delta)} \right\}.$$

Letting $\delta \rightarrow 0$, we obtain

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=0}^{n+|\mathbf{m}_n|} |b_{\nu,n}^{(\alpha)}| |p_\nu(z)| \right\|_{\overline{D}_\sigma}^{1/n} \leq \frac{\sigma}{\rho_\alpha^*(\mathbf{F}, \mathbf{m})}. \quad (86)$$

By (61), (99) and (102), it follows from (94) that for sufficiently large k ,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left\| F_\alpha - \tilde{R}_{n,\mathbf{m}_n,\alpha}^\mu \right\|_{K(\varepsilon)}^{1/n} &\leq \limsup_{n \rightarrow \infty} \left\| F_\alpha - \tilde{R}_{n,\mathbf{m}_n,\alpha}^\mu \right\|_{\overline{D}_\sigma}^{1/n} \leq \limsup_{n \rightarrow \infty} \left\| F_\alpha - \tilde{R}_{n,\mathbf{m}_n,\alpha}^\mu \right\|_{\overline{D}_\sigma \setminus J_\varepsilon^\beta(F, |\mathbf{m}|; k)}^{1/n} \\ &\leq \limsup_{n \rightarrow \infty} \left\| \sum_{\nu=0}^{n+|\mathbf{m}_n|} \frac{|b_{\nu,n}^{(\alpha)}| |p_\nu(z)|}{\omega(z) \tilde{Q}_{n,\mathbf{m}_n}^\mu(z)} + \sum_{\nu=n+|\mathbf{m}_n|+1}^{\infty} \frac{|b_{\nu,n}^{(\alpha)}| |p_\nu(z)|}{\omega(z) \tilde{Q}_{n,\mathbf{m}_n}^\mu(z)} \right\|_{K(\varepsilon)}^{1/n} \\ &\leq \frac{\sigma}{\rho_\alpha^*(\mathbf{F}, \mathbf{m})} \limsup_{n \rightarrow \infty} \left(\frac{1}{\min_{z \in K \setminus J_\varepsilon^\beta(F_\alpha, \mathbf{m}; k)} |\tilde{Q}_{n,\mathbf{m}_n}^\mu(z)|} \right)^{1/n} \leq \frac{\sigma}{\rho_\alpha^*(\mathbf{F}, \mathbf{m})} \limsup_{n \rightarrow \infty} (C_2 |\mathbf{m}_n| n^2)^{\frac{2|\mathbf{m}_n|}{n\beta}} = \frac{\sigma}{\rho_\alpha^*(\mathbf{F}, \mathbf{m})}. \end{aligned} \quad (87)$$

This implies that for any $\beta > 0$ and $\alpha = 1, 2, \dots, d$, each sequence $\{\tilde{R}_{n,\mathbf{m}_n,\alpha}^E\}_{n \in \mathbb{N}}$ converges in β -dimensional Hausdorff content to F_α inside $D_\alpha^*(\mathbf{F}, \mathbf{m})$ as $n \rightarrow \infty$. $\square$

Proof of Corollary 2.5. Let $\xi_1, \xi_2, \dots, \xi_q$ be the distinct system poles of $\mathbf{F}$ with respect to $\mathbf{m}$, and let τ_j be the order of ξ_j as a system pole, $j = 1, 2, \dots, q$. By the assumption that $\sum_{j=1}^q \tau_j = |\mathbf{m}|$, we have proved that, for $j = 1, 2, \dots, q$ and $t = 0, 1, \dots, \tau_j - 1$,

$$\limsup_{n \rightarrow \infty} |(\tilde{Q}_{n,\mathbf{m}}^\mu)^{(t)}(\xi_j)|^{1/n} \leq \frac{|\Phi(\xi_j)|}{\rho_{\xi_j, t+1}(\mathbf{F}, \mathbf{m})} \leq \frac{|\Phi(\xi_j)|}{\rho_{\xi_j}(\mathbf{F}, \mathbf{m})}. \quad (88)$$

The above inequalities imply that

$$\limsup_{n \rightarrow \infty} \|\tilde{Q}_{n,\mathbf{m}}^\mu - Q_{\mathbf{m}}^{\mathbf{F}}\|^{1/n} \leq \max_{j=1,2,\dots,q} \frac{|\Phi(\xi_j)|}{\rho_{\xi_j}(\mathbf{F}, \mathbf{m})},$$

where $Q_{\mathbf{m}}^{\mathbf{F}}$ denotes the monic polynomial whose zeros are the system poles of $\mathbf{F}$ with respect to $\mathbf{m}$ and $\|\cdot\|$ is any norm in the space of polynomials of degree at most $|\mathbf{m}|$. This implies that for all n sufficiently large, $\deg \tilde{Q}_{n,\mathbf{m}}^\mu = |\mathbf{m}|$. By Lemma 3.6, for such n 's, $\tilde{Q}_{n,\mathbf{m}}^\mu$ is unique and $\tilde{\mathbf{R}}_{n,\mathbf{m}}^\mu$ is unique.

Using the same lines of reasoning as in the proof of (42), we have (6). $\square$

Proof of Theorem 2.9. For each $\alpha = 1, 2, \dots, d$, we normalize $q_{n,\mathbf{m}_n}^E$ in the same way as orthogonal Hermite-Padé approximants (see (59)) such that

$$R_{n,\mathbf{m}_n,\alpha}^E = \frac{p_{n,\mathbf{m}_n,\alpha}^E}{q_{n,\mathbf{m}_n}^E} = \frac{P_{n,\mathbf{m}_n,\alpha}^E}{Q_{n,\mathbf{m}_n}^E}.$$

Note that the notations $J_\varepsilon^\beta(F_\alpha, \mathbf{m}; k)$ and $B(\varepsilon)$ are defined as in the proofs of Theorem 2.7. Then, for any compact subset $K \subset D_\alpha^*(\mathbf{F}, \mathbf{m})$ and for any $\varepsilon > 0$, there exist positive constant $C_1 > 0$ and $C_2 > 0$ independent of n such that for all sufficiently large n ,

$$\|\tilde{Q}_{n, \mathbf{m}_n}^E\|_K \leq C_1^{|\mathbf{m}_n|}, \quad (89)$$

and

$$\min_{z \in K \setminus J_\varepsilon^\beta(F_\alpha, \mathbf{m}; k)} |\tilde{Q}_{n, \mathbf{m}_n}^E(z)| \geq (C_2 |\mathbf{m}_n| n^2)^{-2|\mathbf{m}_n|/\beta}. \quad (90)$$

Let $\xi_1, \xi_2, \dots, \xi_w$ be the distinct system poles of $\mathbf{F}$ with respect to $\mathbf{m}$, and let τ_j be the order of ξ_j as a system pole, $j = 1, 2, \dots, w$. With the same idea as for orthogonal Hermite-Padé approximants, it is not difficult to check that for $j = 1, 2, \dots, w$ and $t = 0, 1, \dots, \tau_j - 1$,

$$\limsup_{n \rightarrow \infty} |(\tilde{Q}_{n, \mathbf{m}_n}^E)^{(t)}(\xi_j)|^{1/n} \leq \frac{|\Phi(\xi_j)|}{\rho_{\xi_j}(\mathbf{F}, \mathbf{m})}. \quad (91)$$

Let $\alpha \in \{1, 2, \dots, d\}$ be fixed and let $\{\tilde{\xi}_1, \dots, \tilde{\xi}_N\}$ be the poles of F_α in $D_\alpha^*(\mathbf{F}, \mathbf{m})$. For each $j = 1, 2, \dots, N$, let $\hat{\tau}_j$ be the order of $\tilde{\xi}_j$ as a pole of F_α and $\tilde{\tau}_j$ its order as a system pole. Choose $\rho_1 \in (1, \rho_0(F_\alpha))$. Consider

$$a_{k,n}^{(\alpha)} = [\tilde{Q}_{n, \mathbf{m}_n}^E F_\alpha]_k = \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{\tilde{Q}_{n, \mathbf{m}_n}^E(z) F_\alpha(z) \Phi'(z)}{\Phi^{k+1}(z)} dz.$$

Define

$$\gamma_{k,n}^{(\alpha)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{\tilde{Q}_{n, \mathbf{m}_n}^E(z) F_\alpha(z) \Phi'(z)}{\Phi^{k+1}(z)} dz,$$

where $\rho_2 \in (1, \rho_\alpha^*(\mathbf{F}, \mathbf{m}))$. Arguing as in (26), we have

$$\begin{aligned} \gamma_{k,n}^{(\alpha)} - a_{k,n}^{(\alpha)} &= \sum_{j=1}^q \text{Res}(\tilde{Q}_{n, \mathbf{m}_n}^E F_\alpha \Phi' / \Phi^{k+1}, \tilde{\xi}_j). \\ &= \sum_{j=1}^N \frac{1}{(\hat{\tau}_j - 1)!} \sum_{t=0}^{\hat{\tau}_j - 1} \binom{\hat{\tau}_j - 1}{t} \left(\frac{(z - \tilde{\xi}_j)^{\hat{\tau}_j} F_\alpha \Phi'}{\Phi^{k+1}} \right)^{(\hat{\tau}_j - 1 - t)} (\tilde{\xi}_j) (\tilde{Q}_{n, \mathbf{m}_n}^E)^{(t)}(\tilde{\xi}_j). \end{aligned}$$

Arguing as (65) and the computation similar as (70),

$$|\gamma_{k,n}^{(\alpha)}| \leq \frac{c_1 C_1^{|\mathbf{m}_n|}}{\rho_2^k} \quad \text{and} \quad \left| \left(\frac{(z - \tilde{\xi}_j)^{\hat{\tau}_j} F_\alpha \Phi'}{\Phi^{k+1}} \right)^{(\hat{\tau}_j - 1 - t)} (\tilde{\xi}_j) \right| \leq \frac{c_2}{(|\Phi(\tilde{\xi}_j)| - \delta)^k}, \quad (92)$$

respectively. By (92), it follows that

$$|a_{k,n}^{(\alpha)}| \leq \frac{c_1 C_1^{|\mathbf{m}_n|}}{\rho_2^k} + c_3 \sum_{j=1}^N \frac{|\Phi(\tilde{\xi}_j)|^n}{\rho_{\tilde{\xi}_j}(\mathbf{F}, \mathbf{m})^n (|\Phi(\tilde{\xi}_j)| - \delta)^k} \leq \frac{c_1 C_1^{|\mathbf{m}_n|}}{\rho_2^k} + \frac{c_3}{(\rho_\alpha^*(\mathbf{F}, \mathbf{m}))^n} \sum_{j=1}^N \frac{|\Phi(\tilde{\xi}_j)|^n}{(|\Phi(\tilde{\xi}_j)| - \delta)^k}. \quad (93)$$

Let

$$\omega(z) := \prod_{j=1}^N (z - \tilde{\xi}_j)^{\hat{\tau}_j}.$$

From the definition of Hermite-Padé-Faber approximants and Lemma (3.2), we have,

$$\tilde{Q}_{n, \mathbf{m}_n}^E(z) F_\alpha(z) - \tilde{P}_{n, \mathbf{m}, \alpha}^E(z) = \sum_{k=n+1}^{\infty} a_{k,n}^{(\alpha)} \Phi_k, \quad z \in D_{\rho_0(F_\alpha)}. \quad (94)$$

Multiplying (94) by ω and expanding the result in terms of Faber polynomial system $\{\Phi_\nu\}_{\nu=0}^\infty$ such that for $z \in D_\alpha(\mathbf{F}, \mathbf{m})$,

$$\begin{aligned} \omega(z)\tilde{Q}_{n,\mathbf{m}_n}^E(z)F_\alpha(z) - \omega(z)\tilde{P}_{n,\mathbf{m}_n,\alpha}^E(z) &= \sum_{k=n+1}^\infty \omega(z)a_{k,n}^{(\alpha)}\Phi_k(z) \\ &= \sum_{\nu=0}^\infty b_{\nu,n}^{(\alpha)}\Phi_\nu(z) = \sum_{\nu=0}^{n+|\mathbf{m}_n|} b_{\nu,n}^{(\alpha)}\Phi_\nu(z) + \sum_{\nu=n+|\mathbf{m}_n|+1}^\infty b_{\nu,n}^{(\alpha)}\Phi_\nu(z). \end{aligned} \quad (95)$$

Let K be a compact subset of $D_\alpha^*(\mathbf{F}, \mathbf{m})$ and set

$$\sigma := \max\{\|\Phi\|_K, 1\} \quad (96)$$

($\sigma = 1$ when $K \subset E$). Choose $\delta > 0$ sufficiently small such that

$$\rho_2 := \rho_\alpha^*(\mathbf{F}, \mathbf{m}) - \delta > \sigma. \quad (97)$$

First, we approximate $\sum_{\nu=n+|\mathbf{m}_n|+1}^\infty |b_{\nu,n}^{(\alpha)}||\Phi_\nu(z)|$ on $\overline{D}_\sigma$. Auguring as in (22)-(25), we have

$$\sum_{\nu=n+|\mathbf{m}_n|+1}^\infty |b_{\nu,n}^{(\alpha)}||\Phi_\nu(z)| \leq c_4 C_1^{|\mathbf{m}_n|} \left(\frac{\sigma}{\rho_2}\right)^n, \quad z \in \overline{D}_\sigma. \quad (98)$$

Letting $\delta \rightarrow 0$, we have $\rho_2 \rightarrow \rho_\alpha^*(\mathbf{F}, \mathbf{m})$, then

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=n+|\mathbf{m}_n|+1}^\infty |b_{\nu,n}^{(\alpha)}||\Phi_\nu(z)| \right\|_{\overline{D}_\sigma}^{1/n} \leq \frac{\sigma}{\rho_\alpha^*(\mathbf{F}, \mathbf{m})}. \quad (99)$$

Next, we approximate $\sum_{\nu=0}^{n+|\mathbf{m}_n|} |b_{\nu,n}^{(\alpha)}||\Phi_\nu(z)|$ on $\overline{D}_\sigma$. Suppose that $\delta > 0$ is sufficiently small such that $\rho_1 - \delta > 1$. Arguing as (56), we have

$$|[\omega\Phi_k]_\nu| \leq c_5 \frac{(\rho_1 - \delta)^k}{(\rho_1 - \delta)^\nu}.$$

Thus,

$$|b_{\nu,n}^{(\alpha)}| \leq \sum_{k=n+1}^\infty |a_{k,n}^{(\alpha)}| |[\omega\Phi_k]_\nu| \leq \frac{c_6 C_1^{|\mathbf{m}_n|}}{\rho_2^n} \frac{(\rho_1 - \delta)^n}{(\rho_1 - \delta)^\nu} + \frac{c_7}{\rho_\alpha^*(\mathbf{F}, \mathbf{m})^n} \sum_{j=1}^N \frac{|\Phi(\tilde{\xi}_j)|^n}{(|\Phi(\tilde{\xi}_j)| - \delta)^n} \frac{(\rho_1 - \delta)^n}{(\rho_1 - \delta)^\nu}. \quad (100)$$

Therefore, for $z \in \overline{D}_\sigma$,

$$\sum_{\nu=0}^{n+|\mathbf{m}_n|} |b_{\nu,n}^{(\alpha)}||\Phi_\nu(z)| \leq c_8(n + |\mathbf{m}_n| + 1)\sigma^{n+|\mathbf{m}_n|} \left(\frac{c_9 C_1^{|\mathbf{m}_n|}(\rho_1 - \delta)^n}{\rho_2^n} + \frac{c_{10}(\rho_1 - \delta)^n}{\rho_\alpha^*(\mathbf{F}, \mathbf{m})^n} \sum_{j=1}^N \frac{|\Phi(\tilde{\xi}_j)|^n}{(|\Phi(\tilde{\xi}_j)| - \delta)^n} \right), \quad (101)$$

which implies that

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=0}^{n+|\mathbf{m}_n|} |b_{\nu,n}^{(\alpha)}||\Phi_\nu(z)| \right\|_{\overline{D}_\sigma}^{1/n} \leq \max \left\{ \left(\frac{\sigma(\rho_1 - \delta)}{\rho_2} \right), \frac{\sigma(\rho_1 - \delta)}{\rho_\alpha^*(\mathbf{F}, \mathbf{m})} \max_{j=1,2,\dots,N} \frac{|\Phi(\tilde{\xi}_j)|}{(|\Phi(\tilde{\xi}_j)| - \delta)} \right\}.$$

Letting $\delta \rightarrow 0$, $\rho_1 \rightarrow 1^+$ and $\rho_2 \rightarrow \rho_\alpha^*(\mathbf{F}, \mathbf{m})$, we obtain

$$\limsup_{n \rightarrow \infty} \left\| \sum_{\nu=0}^{n+|\mathbf{m}_n|} |b_{\nu,n}^{(\alpha)}||\Phi_\nu(z)| \right\|_{\overline{D}_\sigma}^{1/n} \leq \frac{\sigma}{\rho_\alpha^*(\mathbf{F}, \mathbf{m})}. \quad (102)$$

Arguing as for orthogonal Hermite-Padé approximants, we are done. $\square$

Proof of Corollary 2.10. The proof of this corollary is identical to the one of Corollary 2.8. $\square$

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Constant Generalized Riesz Potential Functions and Polarization Optimality Problems

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Abstract : An extension of a conjecture of Nikolov and Rafailov [1, Conjecture 1] by considering the following potential function defined on $\mathbb{R}^2$:

$$f_s(x) = \sum_{j=1}^N (|x - x_j|^2 + h)^{-s/2}, \quad h \geq 0,$$

for $s = 2 - 2N$ is given. We obtain a characterization of sets of N distinct points $\{x_1, x_2, \dots, x_N\}$ such that f_{2-2N} is constant on some circle in $\mathbb{R}^2$. Using this characterization, we prove some special cases of this new conjecture. The other problems considered in this paper are polarization optimality problems. We find all maximal and minimal polarization constants and configurations of two concentric circles in $\mathbb{R}^2$ using the above potential function for certain values of s .

Keywords : polarization, potentials, roots of unity, max-min and min-max problems

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1 Introduction

For a fixed multiset of N points $\omega_N := \{x_1, x_2, \dots, x_N\} \subset \mathbb{R}^2$, a given constant $s \in \mathbb{R}$, and a given constant $h \geq 0$, we define the potential function $U^{s,h}(\cdot; \omega_N) : \mathbb{R}^2 \rightarrow [0, \infty]$ as the following:

$$U^{s,h}(x; \omega_N) := \sum_{j=1}^N (|x - x_j|^2 + h)^{-s/2}, \quad (1.1)$$

where $x \in \mathbb{R}^2$ and $|\cdot|$ is the 2-dimensional Euclidean norm in $\mathbb{R}^2$. In this paper, we call $U^{s,h}(\cdot, \omega_N)$ a *Riesz (s, h) -potential function of ω_N* . The geometric interpretation of the function $U^{s,h}(\cdot; \omega_N)$ is as follows. Let us consider two parallel planes in $\mathbb{R}^3$: one is $\mathbb{R}^2 \times \{0\}$ and the other is $\mathbb{R}^2 \times \{\sqrt{h}\}$. Basically, the potential function $U^{s,h}(x, \omega_N)$ is the Riesz s -potential function in the 3-dimensional Euclidean space $\mathbb{R}^3$ of $\omega'_N \subset \mathbb{R}^2 \times \{0\}$ at $x' \in \mathbb{R}^2 \times \{\sqrt{h}\}$, where the projection from $\mathbb{R}^2 \times \{0\}$ to $\mathbb{R}^2$ of ω'_N is ω_N and the projection from $\mathbb{R}^2 \times \{\sqrt{h}\}$ to $\mathbb{R}^2$ of x' is x . Moreover, if $h = 0$, then $U^{s,h}(\cdot; \omega_N)$ is the Riesz s -potential function in the 2-dimensional Euclidean space $\mathbb{R}^2$ of ω_N . We refer the reader to [2, 3, 4, 5] for more information on Riesz s -potential functions in a d -dimensional Euclidean space $\mathbb{R}^d$.

Now, let ω_N be a fixed set of distinct equally spaced points on a circle $T \subset \mathbb{R}^2$, Γ be a circle concentric to T , and $h \geq 0$ be fixed. In [1, Theorem 1], Nikolov and Rafailov showed that $f_s(x) := U^{s,h}(x; \omega_N)$ is constant as a function of x on Γ if and only if $s = 0, -2, -4, \dots, 4 - 2N$, or $2 - 2N$. Furthermore, for $s \in \mathbb{R} \setminus \{0, -2, -4, \dots, 2 - 2N\}$, they located extremum points of $U^{s,h}(\cdot; \omega_N)$ on Γ in [1, Theorem 1].

In the same paper, they also proved the following inverse type result (see [1, Theorem 2]) of what proceeds.

Theorem A. *Given a set of N distinct points $\omega_N := \{x_1, x_2, \dots, x_N\} \subset \mathbb{R}^2$ and a circle $\Gamma \subset \mathbb{R}^2$ such that for each $s = -2, -4, \dots, 2 - 2N$,*

$$U^{s,0}(x; \omega_N) = \sum_{j=1}^N |x - x_j|^{-s}$$

is independent of the position of $x \in \Gamma$. Then, ω_N forms a set of distinct equally spaced points on a circle concentric to Γ .

Moreover, they proposed the following conjecture (see [1, Conjecture 1]):

Conjecture B. *Given a set of N distinct points $\omega_N := \{x_1, x_2, \dots, x_N\} \subset \mathbb{R}^2$ and a circle $\Gamma \subset \mathbb{R}^2$ such that*

$$U^{2-2N,0}(x; \omega_N) = \sum_{j=1}^N |x - x_j|^{2N-2}$$

is constant as a function of x on Γ . Then, ω_N forms a set of distinct equally spaced points on a circle concentric to Γ .

Translating and scaling the circle Γ in the above conjecture, it is easy to check that Conjecture B is equivalent to the following conjecture.

Conjecture C. *Given a set of N distinct points $\omega_N := \{x_1, x_2, \dots, x_N\} \subset \mathbb{R}^2$ such that*

$$U^{2-2N,0}(x; \omega_N) = \sum_{j=1}^N |x - x_j|^{2N-2}$$

is constant as a function of x on the unit circle. Then, ω_N forms a set of distinct equally spaced points on a circle centered at 0.

In order to simplify further considerations, we shall study Conjecture C. This conjecture for the case when $N = 2$ is trivial. The proof of this conjecture when $N = 3$ is in [1, Proposition 2]. The one for the case when $x_1, x_2, \dots, x_N$ have the same norm is in [6, Proposition 1]. In the same paper, the authors also proved this conjecture for the case when N is prime and $x_1, x_2, \dots, x_N$ have an equal angle distribution and rational norms (see [6, Proposition 2]).

In this paper, we extend Theorem A to more general potential functions defined in (1.1). Moreover, the extension of Conjecture C is proposed (see Conjecture 2.1 in Section 2). A characterization of sets of N distinct points ω_N that $U^{2-2N,h}(\cdot, \omega_N)$ is constant on some circle in $\mathbb{R}^2$ is given. Using this characterization, we prove some special cases of this new extended conjecture.

The next problems considered in this paper are polarization optimality problems corresponding to the potential functions defined in (1.1). Let $\omega_N = \{x_1, \dots, x_N\}$ denote a configuration of N (not necessarily distinct) points in $\mathbb{R}^2$. Denote by

$$\mathbb{S}_R^1 := \{x \in \mathbb{R}^2 : |x| = R\}$$

the circle centered at 0 of radius R in $\mathbb{R}^2$. When $R = 1$, we simply use the notation $\mathbb{S}^1$. Given $s \in \mathbb{R}, h \geq 0, R > 0$, and $r > 0$, we define polarization constants

$$M_N^{s,h}(\mathbb{S}_r^1; \mathbb{S}_R^1) := \max_{\substack{\omega_N \subset \mathbb{S}_r^1 \\ \#\omega_N = N}} \min_{y \in \mathbb{S}_R^1} U^{s,h}(y; \omega_N), \quad M_N^{0,h}(\mathbb{S}_r^1; \mathbb{S}_R^1) := N, \quad (1.2)$$

$$m_N^{s,h}(\mathbb{S}_r^1; \mathbb{S}_R^1) := \min_{\substack{\omega_N \subset \mathbb{S}_r^1 \\ \#\omega_N = N}} \max_{y \in \mathbb{S}_R^1} U^{s,h}(y; \omega_N), \quad m_N^{0,h}(\mathbb{S}_r^1; \mathbb{S}_R^1) := N, \quad (1.3)$$

where $\#\omega_N$ stands for the cardinality of the multiset ω_N . We will call ω_N a *maximal (minimal) N -point Riesz (s, h) -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$* if ω_N attains the maximum in (1.2) (minimum in (1.3)). We give a brief history of such polarization optimality problems below.

The idea of two-plate polarization constants was introduced by Farkas and Révész [7] in general sense. However, almost all previous results on polarization optimality problems related to Riesz potentials [2, 3, 4, 5, 8, 9] were considered for the case when $R = r = 1$ and $h = 0$. The maximality of N distinct equally spaced points on the unit circle for the maximal Riesz $(s, 0)$ -polarization problem

of $(\mathbb{S}^1; \mathbb{S}^1)$ in (1.2) when $s > 0$ was proved by Hardin, Kendall, and Saff [5] (see also [4] and [8] for the history of this problem). In [5], they also showed the minimality of N distinct equally spaced points on the unit circle for the minimal Riesz $(s, 0)$ -polarization problem of $(\mathbb{S}^1; \mathbb{S}^1)$ in (1.3) for $-1 \leq s < 0$. Recently, a characterization of all maximal and minimal N -point Riesz $(s, 0)$ -polarization configurations of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$ when $s = -2, -4, \dots, 2 - 2N$ was given in [6, Theorem 2]. One of the aims of this paper is to provide a characterization analogous to Theorem 2 in [6] for the case when $h > 0$.

We would like call the reader's attention to papers [2, 3, 4, 5] that contain asymptotic results of polarization constants and configurations of subsets of $\mathbb{R}^d$ as $N \rightarrow \infty$ when $s > 0$ and $h = 0$.

An outline of this paper is as follows. In Section 2, we state the extension of Theorem A to more general potential functions in (1.1) and give an extension of Conjecture C. Some special cases of this new conjecture are considered. In Section 3, we state our results on polarization optimality problems. Section 4 and Section 5 are devoted to the proofs of all results in Section 2 and Section 3, respectively. Finally, we perform our auxiliary computations in Section 6.

2 Constant Riesz (s, h) -potential functions

The first theorem is a generalization of Theorem A.

Theorem 2.1. *Let $h \geq 0$. Given a set of N distinct points $\omega_N := \{x_1, x_2, \dots, x_N\} \subset \mathbb{R}^2$ such that for each $s = -2, -4, \dots, 2 - 2N$,*

$$U^{s,h}(x; \omega_N) = \sum_{j=1}^N (|x - x_j|^2 + h)^{-s/2}$$

is independent of the position of $x \in \mathbb{S}^1$. Then, ω_N forms a set of distinct equally spaced points on a circle centered at 0. Moreover, if $|x_1| = |x_2| = \dots = |x_N| = r$, then for each $p = 1, 2, \dots, N - 1$,

$$U^{-2p,h}(x; \omega_N) = \frac{N}{2^p} \sum_{q=0}^p \binom{p}{q}^2 (2r)^{2q} \left(r^2 + 1 + h + \sqrt{((r-1)^2 + h)((r+1)^2 + h)} \right)^{p-2q}$$

for all $x \in \mathbb{S}^1$.

This theorem brings us to the following conjecture which generalizes Conjecture C.

Conjecture 2.1. *Let $h \geq 0$. Given a set of N distinct points $\omega_N := \{x_1, x_2, \dots, x_N\} \subset \mathbb{R}^2$ such that*

$$U^{2-2N,h}(x; \omega_N) = \sum_{j=1}^N (|x - x_j|^2 + h)^{N-1}$$

is constant as a function of x on $\mathbb{S}^1$. Then, $\{x_1, x_2, \dots, x_N\}$ forms a set of distinct equally spaced points on a circle centered at 0.

A characterization of sets of N distinct points ω_N such that $U^{2-2N,h}(\cdot, \omega_N)$ is constant on $\mathbb{S}^1$ is the following:

Theorem 2.2. *Let $h \geq 0$ and $\omega_N = \{x_1, x_2, \dots, x_N\} \subset \mathbb{R}^2$ be a set of N distinct points. Then, the function*

$$U^{2-2N,h}(x; \omega_N) = \sum_{j=1}^N (|x - x_j|^2 + h)^{N-1}$$

is constant on $\mathbb{S}^1$ if and only if

$$\sum_{j=1}^N B_{k,j} x_j^k = 0, \quad k = 1, \dots, N-1, \quad (2.1)$$

where

$$x^k := (r^k \cos(kt), r^k \sin(kt))$$

if $x = (r \cos(t), r \sin(t)) \in \mathbb{R}^2$ and

$$B_{k,j} := \sum_{q=0}^{N-k-1} \left[\binom{N-1}{q} \binom{N-1}{k+q} (2|x_j|)^{2q} \right. \\ \left. \times \left(|x_j|^2 + 1 + h + \sqrt{((|x_j| - 1)^2 + h)((|x_j| + 1)^2 + h)} \right)^{N-2q-k-1} \right]. \quad (2.2)$$

As a consequence of this characterization, we obtain the following corollary.

Corollary 2.2. *Let $h \geq 0$ and let $\omega_N := \{x_1, x_2, \dots, x_N\}$ be a set of N distinct points in $\mathbb{R}^2$, which belong to a circle $\mathbb{S}_r^1 \subset \mathbb{R}^2$. Assume that*

$$U^{2-2N,h}(x; \omega_N) = \sum_{j=1}^N (|x - x_j|^2 + h)^{N-1}$$

is constant on $\mathbb{S}^1$. Then, $\{x_1, x_2, \dots, x_N\}$ forms a set of distinct equally spaced points on $\mathbb{S}_r^1$.

Applying Theorem 2.2 and Corollary 2.2, we prove Conjecture 2.1 when $N = 3$.

Corollary 2.3. *Let $h \geq 0$ and $\{x_1, x_2, x_3\} \subset \mathbb{R}^2$ be a set of 3 distinct points. If the function $U^{-4,h}(x, \{x_1, x_2, x_3\})$ is constant on $\mathbb{S}^1$, then $\{x_1, x_2, x_3\}$ forms a set of distinct equally spaced points on a circle centered at 0.*

3 Polarization optimality problems

A complete characterization of all maximal and minimal N -point Riesz (s, h) -polarization configurations of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$ when $s = -2, -4, \dots, 2 - 2N$ and $h \geq 0$ is the following:

Theorem 3.1. *Let $N \in \mathbb{N}$, $p \in \{1, 2, \dots, N - 1\}$, $R > 0$, $r > 0$, $h \geq 0$, and $\{x_1, x_2, \dots, x_N\} \subset \mathbb{S}_r^1$. The following statements are equivalent:*

- (a) $\{x_1, x_2, \dots, x_N\}$ is a maximal N -point Riesz $(-2p, h)$ -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$;
- (b) $\{x_1, x_2, \dots, x_N\}$ is a minimal N -point Riesz $(-2p, h)$ -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$;
- (c) $\sum_{j=1}^N x_j = \sum_{j=1}^N x_j^2 = \dots = \sum_{j=1}^N x_j^p = 0$, where $x^k := (r^k \cos(kt), r^k \sin(kt))$ if $x = (r \cos(t), r \sin(t)) \in \mathbb{R}^2$.

Furthermore,

$$\begin{aligned} M_N^{-2p, h}(\mathbb{S}_r^1; \mathbb{S}_R^1) &= m_N^{-2p, h}(\mathbb{S}_r^1; \mathbb{S}_R^1) \\ &= \frac{N}{2^p} \sum_{j=0}^p \binom{p}{j}^2 (2rR)^{2j} \left(r^2 + R^2 + h + \sqrt{((r-R)^2 + h)(r+R)^2 + h} \right)^{p-2j}. \end{aligned} \quad (3.1)$$

4 Proof of Section 2

The Euclidean space $\mathbb{R}^2$ and the complex space $\mathbb{C}$ have the same dimension and the same norm. However, the complex space $\mathbb{C}$ has a richer algebraic structure, for example, $\mathbb{C}$ is a field. Therefore, when we prove all results in Section 2 and 3, any element $x \in \mathbb{R}^2$ will be replaced by $x \in \mathbb{C}$, the 2-dimensional Euclidean norm $|\cdot|$ is replaced by the modulus in $\mathbb{C}$, and the notation xy is adopted from the multiplication in $\mathbb{C}$ and the notation x/y is adopted from the division in $\mathbb{C}$. We recall that the usual dot product in $\mathbb{C}$ is defined by

$$(a_1 + a_2 i) \cdot (b_1 + b_2 i) := a_1 b_1 + a_2 b_2.$$

Lemma 4.1. *Let $N \in \mathbb{N}$, $p \in \{1, 2, \dots, N - 1\}$, and $h \geq 0$. If $x_j := |x_j| \cos t_j + i|x_j| \sin t_j$ for all $j = 1, 2, \dots, N$, then for all $y := \cos t + i \sin t \in \mathbb{S}^1$,*

$$\sum_{j=1}^N (|y - x_j|^2 + h)^p = E_0^{(p)} + \sum_{k=1}^p \sum_{j=1}^N E_{k,j}^{(p)} \cos(kt_j - kt), \quad (4.1)$$

$$\sum_{j=1}^N (|y - x_j|^2 + h)^p = E_0^{(p)} + \sum_{k=1}^p \sum_{j=1}^N \frac{E_{k,j}^{(p)}}{|x_j|^k} (y^k \cdot x_j^k), \quad (4.2)$$

where

$$E_0^{(p)} := \frac{1}{2^p} \sum_{j=1}^N \sum_{q=0}^p \binom{p}{q}^2 (2|x_j|)^{2q} \left(|x_j|^2 + 1 + h + \sqrt{((|x_j| - 1)^2 + h)((|x_j| + 1)^2 + h)} \right)^{p-2q} \quad (4.3)$$

and for all $k = 1, 2, \dots, p$ and $j = 1, 2, \dots, N$,

$$E_{k,j}^{(p)} := \frac{(-1)^k}{2^{p-1}} \sum_{q=0}^{p-k} \left[\binom{p}{q} \binom{p}{k+q} (2|x_j|)^{2q+k} \times \left(|x_j|^2 + 1 + h + \sqrt{((|x_j| - 1)^2 + h)((|x_j| + 1)^2 + h)} \right)^{p-k-2q} \right]. \quad (4.4)$$

Proof of Lemma 4.1. Let $y := \cos t + i \sin t$ and $x_j := |x_j| \cos t_j + i|x_j| \sin t_j$ for all $j = 1, 2, \dots, N$. A simple calculation shows that

$$f_j(t) := (|y - x_j|^2 + h)^p = (|x_j|^2 + 1 + h - 2|x_j| \cos(t - t_j))^p.$$

We know that

$$A := \{1, \cos(t - t_j), \dots, \cos(p(t - t_j))\}$$

forms an orthogonal set with respect to the inner product

$$\langle f, g \rangle := \int_0^{2\pi} f(t)g(t)dt.$$

Moreover,

$$\begin{aligned} f_j &\in \text{span}\{1, \cos(t - t_j), \cos^2(t - t_j), \dots, \cos^p(t - t_j)\} \\ &= \text{span}\{1, \cos(t - t_j), \dots, \cos(p(t - t_j))\}. \end{aligned}$$

Therefore,

$$f_j(t) = \sum_{k=0}^p E_{k,j}^{(p)} \cos(kt_j - kt).$$

This implies that

$$\sum_{j=1}^N (|y - x_j|^2 + h)^p = \sum_{j=1}^N f_j(t) = E_0^{(p)} + \sum_{k=1}^p \sum_{j=1}^N E_{k,j}^{(p)} \cos(kt_j - kt),$$

where $E_0^{(p)} := \sum_{j=1}^N E_{0,j}^{(p)}$. By the orthogonality of the set A and the calculation in Lemma 6.2 (see Appendix), we have

$$E_0^{(p)} := \sum_{j=1}^N \frac{\langle f_j, 1 \rangle}{2\pi}$$

$$= \frac{1}{2^p} \sum_{j=1}^N \sum_{q=0}^p \binom{p}{q}^2 (2|x_j|)^{2q} \left(|x_j|^2 + 1 + h + \sqrt{((|x_j| - 1)^2 + h)((|x_j| + 1)^2 + h)} \right)^{p-2q}$$

and

$$E_{k,j}^{(p)} = \frac{\langle f_j, \cos k(t - t_j) \rangle}{\pi} = \frac{(-1)^k}{2^{p-1}} \sum_{q=0}^{p-k} \left[\binom{p}{q} \binom{p}{k+q} (2|x_j|)^{2q+k} \right. \\ \left. \times \left(|x_j|^2 + 1 + h + \sqrt{((|x_j| - 1)^2 + h)((|x_j| + 1)^2 + h)} \right)^{p-k-2q} \right],$$

for all $k \in \{0, 1, \dots, p\}$ and $j \in \{1, \dots, N\}$. Moreover, it is clear that the equations (4.1) and (4.2). $\square$

Proof of Theorem 2.1. Suppose that there exist constants C_p , $p = 1, 2, \dots, N-1$, such that

$$U^{-2p,h}(x; \omega_N) = C_p, \quad x \in \mathbb{S}^1, \quad p = 1, 2, \dots, N-1,$$

where $\omega_N = \{x_1, x_2, \dots, x_N\}$. If $x = \cos t + i \sin t$ and $x_j := |x_j| \cos t_j + i|x_j| \sin t_j$, then by (4.1), for each $p = 1, 2, \dots, N-1$, we have for all $t \in [0, 2\pi]$,

$$C_p = E_0^{(p)} + \sum_{k=1}^p \sum_{j=1}^N \left[E_{k,j}^{(p)} \cos(kt_j) \cos(kt) + E_{k,j}^{(p)} \sin(kt_j) \sin(kt) \right]$$

and

$$0 = (E_0^{(p)} - C_p) + \sum_{k=1}^p \left[\left(\sum_{j=1}^N E_{k,j}^{(p)} \cos(kt_j) \right) \cos(kt) + \left(\sum_{j=1}^N E_{k,j}^{(p)} \sin(kt_j) \right) \sin(kt) \right].$$

Since $\{1, \cos(t), \sin(t), \cos(2t), \sin(2t), \dots, \cos(pt), \sin(pt)\}$ is linearly independent over $\mathbb{R}$, for all $p = 1, 2, \dots, N-1$,

$$C_p = E_0^{(p)}, \quad (4.5)$$

$$\sum_{j=1}^N E_{p,j}^{(p)} \cos(kt_j) = 0 \quad \text{and} \quad \sum_{j=1}^N E_{p,j}^{(p)} \sin(kt_j) = 0. \quad (4.6)$$

Using (4.4), we can compute

$$E_{p,j}^{(p)} = \frac{(-1)^p}{2^{p-1}} (2|x_j|)^p. \quad (4.7)$$

Combining (4.6) and (4.7), we have for all $p = 1, 2, \dots, N-1$,

$$0 = \sum_{j=1}^N \frac{(-1)^p}{2^{p-1}} (2|x_j|)^p (\cos(pt_j) + i \sin(pt_j)) = (-1)^p 2 \sum_{j=1}^N x_j^p$$

which implies that $\sum_{j=1}^N x_j^p = 0$ for all $p = 1, 2, \dots, N-1$. Using Newton's identities, we have

$$e_p(x_1, x_2, \dots, x_N) = 0, \quad p = 1, 2, \dots, N-1.$$

Then,

$$\prod_{j=1}^N (x - x_j) = x^N + (-1)^N \prod_{j=1}^N x_j.$$

Hence, $|x_1| = |x_2| = \dots = |x_N| = r$ for some $r > 0$ and $\{x_1, x_2, \dots, x_N\}$ forms a set of distinct equally spaced points on $\mathbb{S}_r^1$. In turn, the equality (4.5) implies that for all $x \in \mathbb{S}^1$ and for all $p = 1, 2, \dots, N-1$,

$$\begin{aligned} U^{-2p,h}(x; \omega_N) &= C_p = E_0^{(p)} \\ &= \frac{N}{2^p} \sum_{q=0}^p \binom{p}{q}^2 (2r)^{2q} \left(r^2 + 1 + h + \sqrt{((r-1)^2 + h)((r+1)^2 + h)} \right)^{p-2q}. \end{aligned}$$

□

Proof of Theorem 2.2. Set

$$x_j := |x_j| \cos(t_j) + i|x_j| \sin(t_j)$$

for all $j = 1, 2, \dots, N$.

($\Rightarrow$) By our assumption, we assume that $f(y) := \sum_{j=1}^N (|y - x_j|^2 + h)^{N-1}$ is constant on $\mathbb{S}^1$, say $f(y) = C$ on $\mathbb{S}^1$. Set $y = \cos t + i \sin t \in \mathbb{S}^1$. By (4.1), for all $t \in [0, 2\pi]$,

$$\begin{aligned} C = f(y) &= \sum_{j=1}^N (|y - x_j|^2 + h)^{N-1} \\ &= E_0^{(N-1)} + \sum_{k=1}^{N-1} \sum_{j=1}^N \left[E_{k,j}^{(N-1)} \cos(kt_j) \cos(kt) + E_{k,j}^{(N-1)} \sin(kt_j) \sin(kt) \right] \\ &= E_0^{(N-1)} + \sum_{k=1}^{N-1} \left[\left(\sum_{j=1}^N E_{k,j}^{(N-1)} \cos(kt_j) \right) \cos(kt) + \left(\sum_{j=1}^N E_{k,j}^{(N-1)} \sin(kt_j) \right) \sin(kt) \right]. \end{aligned} \tag{4.8}$$

Because $\{1, \cos(t), \sin(t), \cos(2t), \sin(2t), \dots, \cos((N-1)t), \sin((N-1)t)\}$ is linearly independent over $\mathbb{R}$,

$$C - E_0^{(N-1)} = 0$$

and for all $k = 1, 2, \dots, N-1$,

$$\sum_{j=1}^N E_{k,j}^{(N-1)} \cos(kt_j) = 0 \quad \text{and} \quad \sum_{j=1}^N E_{k,j}^{(N-1)} \sin(kt_j) = 0. \tag{4.9}$$

Then, for all $k = 1, 2, \dots, N-1$,

$$0 = \sum_{j=1}^N E_{k,j}^{(N-1)} (\cos(kt_j) + i \sin(kt_j)) = \sum_{j=1}^N \frac{E_{k,j}^{(N-1)}}{|x_j|^k} x_j^k. \quad (4.10)$$

Using the calculation in (4.4), it is not difficult to check that the equations (4.10) imply the equations (2.1).

($\Leftarrow$) Assume that the equations (2.1) hold true. Then,

$$\sum_{j=1}^N \frac{E_{k,j}^{(N-1)}}{|x_j|^k} x_j^k = 0, \quad k = 1, \dots, N-1.$$

From (4.10), we have (4.9). Combining the relations (4.9) and the identity (4.8), we have for all $y \in \mathbb{S}^1$,

$$\sum_{j=1}^N (|y - x_j|^2 + h)^{N-1} = E_0^{(N-1)},$$

which implies that $U^{2-2N,h}(\cdot; \omega_N)$ is constant on $\mathbb{S}^1$. This completes the proof. $\square$

Proof of Corollary 2.2. Assume that $\{x_1, x_2, \dots, x_N\} \subset \mathbb{S}_r^1$. It is easy to check that the constants $B_{k,j} \neq 0$ do not depend on j . Therefore, by the system of equations (2.1), $\sum_{j=1}^N x_j^k = 0$ for all $k = 1, 2, \dots, N-1$. Using Newton's identities, we have

$$e_k(x_1, x_2, \dots, x_N) = 0, \quad k = 1, 2, \dots, N-1.$$

Then,

$$\prod_{j=1}^N (x - x_j) = x^N + (-1)^N \prod_{j=1}^N x_j.$$

Hence, $|x_1| = |x_2| = \dots = |x_N| = r$ and $\{x_1, x_2, \dots, x_N\}$ forms a set of distinct equally spaced points on $\mathbb{S}_r^1$. $\square$

Proof of Corollary 2.3. Using Theorem 2.2, we have

$$x_1^2 + x_2^2 + x_3^2 = 0, \quad (4.11)$$

$$E(|x_1|)x_1 + E(|x_2|)x_2 + E(|x_3|)x_3 = 0, \quad (4.12)$$

where

$$E(x) := \frac{\left(x^2 + 1 + h + \sqrt{((x-1)^2 + h)((x+1)^2 + h)}\right)^2 + 4x^2}{\left(x^2 + 1 + h + \sqrt{((x-1)^2 + h)((x+1)^2 + h)}\right)}.$$

Without loss of generality, we can assume that $|x_1| \geq |x_2| \geq |x_3|$. Moreover, it is easy to check that $E(x)$ is a positive increasing function on $[0, \infty)$. Therefore, $E(|x_1|) \geq E(|x_2|) \geq E(|x_3|) > 0$. From (4.12), we have

$$E(|x_3|)x_3 = -E(|x_1|)x_1 - E(|x_2|)x_2$$

and

$$E(|x_3|)\overline{x_3} = -E(|x_1|)\overline{x_1} - E(|x_2|)\overline{x_2},$$

which imply that

$$|x_3|^2 E(|x_3|)^2 = |x_1|^2 E(|x_1|)^2 + |x_2|^2 E(|x_2|)^2 + E(|x_1|)E(|x_2|)(x_1\overline{x_2} + x_2\overline{x_1}). \quad (4.13)$$

Note that since $|x_1| \geq |x_2| \geq |x_3|$ and x_1, x_2, x_3 are distinct,

$$x_1\overline{x_2} + x_2\overline{x_1} \in (-\infty, 0). \quad (4.14)$$

From (4.11), we have

$$x_3^2 = -x_1^2 - x_2^2 \quad \text{and} \quad \overline{x_3}^2 = -\overline{x_1}^2 - \overline{x_2}^2,$$

which imply that

$$|x_3|^4 = |x_1|^4 + |x_2|^4 + x_1^2\overline{x_2}^2 + x_2^2\overline{x_1}^2 = |x_1|^4 + |x_2|^4 - 2|x_1|^2|x_2|^2 + (x_1\overline{x_2} + x_2\overline{x_1})^2.$$

Therefore,

$$(x_1\overline{x_2} + x_2\overline{x_1})^2 = |x_3|^4 - (|x_1|^2 - |x_2|^2)^2.$$

By (4.14),

$$(x_1\overline{x_2} + x_2\overline{x_1}) = -\sqrt{|x_3|^4 - (|x_1|^2 - |x_2|^2)^2}$$

From (4.13), we obtain

$$|x_3|^2 E(|x_3|)^2 + E(|x_1|)E(|x_2|)\sqrt{|x_3|^4 - (|x_1|^2 - |x_2|^2)^2} = |x_1|^2 E(|x_1|)^2 + |x_2|^2 E(|x_2|)^2.$$

Since

$$E(|x_1|)E(|x_2|)\sqrt{|x_3|^4 - (|x_1|^2 - |x_2|^2)^2} \leq |x_1|^2 E(|x_1|)^2$$

and

$$|x_3|^2 E(|x_3|)^2 \leq |x_2|^2 E(|x_2|)^2,$$

$$\sqrt{|x_3|^4 - (|x_1|^2 - |x_2|^2)^2} = |x_1|^2,$$

which implies $|x_1| = |x_2| = |x_3|$. Applying Corollary 2.2, $\{x_1, x_2, x_3\}$ forms a set of distinct equally spaced points on a circle centered at 0.

□

5 Proof of Section 3

Recall that for the proofs in this section, we also consider our problems in the complex plane (see our discussion at the beginning of Section 4).

Lemma 5.1. *Let $N \in \mathbb{N}$, $p \in \{1, 2, \dots, N-1\}$, $R > 0$, $r > 0$, and $h \geq 0$. Then, any configuration of N distinct equally spaced points on $\mathbb{S}_r^1$ is both maximal and minimal N -point Riesz $(-2p, h)$ -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$.*

Proof of Lemma 5.1. Let $\omega_N := \{x_1, \dots, x_N\}$ be a configuration of N distinct equally spaced points on $\mathbb{S}_r^1$, $p \in \{1, 2, \dots, N-1\}$ be fixed, and $h \geq 0$ be fixed. By [1, Theorem 1], we know that $f(x) := \sum_{j=1}^N (|x - x_j|^2 + h)^p$ is constant as a function of x on $\mathbb{S}_R^1$, say $f(x) \equiv C$ for all $x \in \mathbb{S}_R^1$.

Let $\{y_1, \dots, y_N\}$ be any N -point configuration on $\mathbb{S}_r^1$. Clearly, $y_j/r, x_j/r \in \mathbb{S}^1$ for all $j = 1, 2, \dots, N$. Then,

$$\begin{aligned} NC &= \sum_{i=1}^N f\left(\frac{R}{y_i/r}\right) = \sum_{i=1}^N \sum_{j=1}^N \left(\left| x_j - \frac{R}{y_i/r} \right|^2 + h \right)^p \\ &= \sum_{i=1}^N \sum_{j=1}^N \left(\left| \frac{x_j/r}{y_i/r} \right|^2 \left| y_i - \frac{R}{x_j/r} \right|^2 + h \right)^p = \sum_{i=1}^N \sum_{j=1}^N \left(\left| y_i - \frac{R}{x_j/r} \right|^2 + h \right)^p \\ &= \sum_{j=1}^N \sum_{i=1}^N \left(\left| y_i - \frac{R}{x_j/r} \right|^2 + h \right)^p. \end{aligned}$$

Therefore, there exist $j_0, j'_0 \in \{1, 2, \dots, N\}$ such that

$$\sum_{i=1}^N \left(\left| y_i - \frac{R}{x_{j_0}/r} \right|^2 + h \right)^p \geq C \quad \text{and} \quad \sum_{i=1}^N \left(\left| y_i - \frac{R}{x_{j'_0}/r} \right|^2 + h \right)^p \leq C.$$

Then, we have

$$\max_{x \in \mathbb{S}_R^1} \sum_{i=1}^N (|y_i - x|^2 + h)^p \geq C = \max_{x \in \mathbb{S}_R^1} \sum_{i=1}^N (|x_i - x|^2 + h)^p$$

and

$$\min_{x \in \mathbb{S}_R^1} \sum_{i=1}^N (|y_i - x|^2 + h)^p \leq C = \min_{x \in \mathbb{S}_R^1} \sum_{i=1}^N (|x_i - x|^2 + h)^p$$

which imply

$$\max_{x \in \mathbb{S}_R^1} \sum_{i=1}^N (|x_i - x|^2 + h)^p = m_N^{-2p, h}(\mathbb{S}_r^1; \mathbb{S}_R^1)$$

and

$$\min_{x \in \mathbb{S}_R^1} \sum_{i=1}^N (|x_i - x|^2 + h)^p = M_N^{-2p, h}(\mathbb{S}_r^1; \mathbb{S}_R^1),$$

respectively. Therefore, ω_N is both maximal and minimal N -point Riesz $(-2p, h)$ -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$.

□

Proof of Theorem 3.1. Because the proof of (a) $\Leftrightarrow$ (c) is similar to the proof of (b) $\Leftrightarrow$ (c), we will show only (b) $\Leftrightarrow$ (c) and skip the proof of (a) $\Leftrightarrow$ (c). Moreover, without loss of generality, we can assume that $R = 1$.

Let $N \in \mathbb{N}$, $p \in \{1, 2, \dots, N-1\}$, $r > 0$, and $h \geq 0$ be fixed. Notice that for all configurations $\{x_1, x_2, \dots, x_N\} \subset \mathbb{S}_r^1$, the constants $E_0^{(p)}$ and $E_{k,j}^{(p)}$ in (4.3) and (4.4) depend only on k . For convenience, for all configurations $\{x_1, x_2, \dots, x_N\} \subset \mathbb{S}_r^1$, we set

$$E := E_0^{(p)} \quad \text{and} \quad E_k := \frac{E_{k,j}^{(p)}}{r^k}, \quad k = 1, 2, \dots, p.$$

First of all, we show that

$$m_N^{-2p,h}(\mathbb{S}_r^1; \mathbb{S}^1) = E. \quad (5.1)$$

Let $\omega'_N := \{x'_1, x'_2, \dots, x'_N\}$ be a configuration of distinct equally spaced points on $\mathbb{S}_r^1$. Using (4.2), we have for all $y \in \mathbb{S}^1$,

$$\sum_{j=1}^N (|y - x'_j|^2 + h)^p = E + \sum_{k=1}^p \sum_{j=1}^N E_k (y^k \cdot (x'_j)^k) = E + \sum_{k=1}^p E_k (y^k \cdot \sum_{j=1}^N (x'_j)^k) = E \quad (5.2)$$

where the last equality follows from the fact that $\sum_{j=1}^N (x'_j)^k = 0$ for all $k = 1, 2, \dots, N-1$. By Lemma 5.1, since ω'_N is a minimal N -point Riesz $(-2p, h)$ -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}^1)$,

$$m_N^{-2p,h}(\mathbb{S}_r^1; \mathbb{S}^1) = \max_{y \in \mathbb{S}^1} U^{-2p,h}(y; \omega'_N) = E \quad (5.3)$$

as we wanted.

Now, we prove (c) $\Rightarrow$ (b). Assume that $\omega_N = \{x_1, x_2, \dots, x_N\} \subset \mathbb{S}_r^1$ and $\sum_{j=1}^N x_j^k = 0$ for all $k = 1, 2, \dots, p$. Applying the same argument as in (5.2), we have for all $y \in \mathbb{S}^1$,

$$U^{-2p,h}(y; \omega_N) = E + \sum_{k=1}^p E_k (y^k \cdot \sum_{j=1}^N x_j^k) = E,$$

which implies that ω_N is a minimal N -point Riesz $(-2p, h)$ -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}^1)$.

Next, we show (b) $\Rightarrow$ (c). Assume that $\omega_N := \{x_1, x_2, \dots, x_N\}$ is a minimal N -point Riesz $(-2p, h)$ -polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}^1)$. Then, for all $y \in \mathbb{S}^1$,

$$U^{-2p,h}(y; \omega_N) = \sum_{j=1}^N (|y - x_j|^2 + h)^p \leq m_N^{-2p,h}(\mathbb{S}_r^1; \mathbb{S}^1) = E.$$

Set $y = \cos(t) + i \sin(t) \in \mathbb{S}^1$ and $x_j = r \cos(t_j) + ir \sin(t_j) \in \mathbb{S}_r^1$ for all $j = 1, 2, \dots, N$. Hence, by (4.1), for all $t \in [0, 2\pi]$,

$$E \geq U^{-2p,h}(y; \omega_N) = E + \sum_{k=1}^p \left[\left(\sum_{j=1}^N \frac{E_k}{r^k} \cos(kt_j) \right) \cos(kt) + \left(\sum_{j=1}^N \frac{E_k}{r^k} \sin(kt_j) \right) \sin(kt) \right].$$

Then, for all $t \in [0, 2\pi]$,

$$0 \geq \sum_{k=1}^p \left[\left(\sum_{j=1}^N \frac{E_k}{r^k} \cos(kt_j) \right) \cos(kt) + \left(\sum_{j=1}^N \frac{E_k}{r^k} \sin(kt_j) \right) \sin(kt) \right].$$

It is not difficult to check that for all $t \in [0, 2\pi]$,

$$\sum_{k=1}^p \left[\left(\sum_{j=1}^N \frac{E_k}{r^k} \cos(kt_j) \right) \cos(kt) + \left(\sum_{j=1}^N \frac{E_k}{r^k} \sin(kt_j) \right) \sin(kt) \right] = 0.$$

Because $\{\cos(t), \sin(t), \cos(2t), \sin(2t), \dots, \cos(pt), \sin(pt)\}$ is linearly independent over $\mathbb{R}$, for all $k = 1, 2, \dots, p$,

$$\sum_{j=1}^N \frac{E_k}{r^k} \cos(kt_j) = \sum_{j=1}^N \frac{E_k}{r^k} \sin(kt_j) = 0.$$

Since for all $k = 1, 2, \dots, p$, $E_k \neq 0$ (see the formula in (4.4)),

$$\sum_{j=1}^N \cos(kt_j) = \sum_{j=1}^N \sin(kt_j) = 0, \quad k = 1, 2, \dots, p,$$

which imply that $\sum_{j=1}^N x_j^k = 0$ for all $k = 1, 2, \dots, p$. Moreover, from (4.3), we have

$$\begin{aligned} M_N^{-2p,h}(\mathbb{S}_r^1; \mathbb{S}^1) &= m_N^{-2p,h}(\mathbb{S}_r^1; \mathbb{S}^1) \\ &= E = \frac{N}{2^p} \sum_{j=0}^p \binom{p}{j}^2 (2r)^{2j} \left(r^2 + 1 + h + \sqrt{((r-1)^2 + h)(r+1)^2 + h} \right)^{p-2j}. \end{aligned}$$

To compute $M_N^{-2p,h}(\mathbb{S}_r^1; \mathbb{S}_R^1) = m_N^{-2p,h}(\mathbb{S}_r^1; \mathbb{S}_R^1)$ in (9), we can use a similar argument as in the proof of Lemma 4.1 by replacing $y = R \cos t + iR \sin t$ and $f_j(t) := (|y - x_j|^2 + h)^p = (R_j^2 + R^2 + h - 2R_j R \cos(t - t_j))^p$. Applying the calculations as in Lemma 6.2, it is not difficult to check that if ω_N is a configuration of N distinct equally spaced points on $\mathbb{S}_r^1$, then for all $y \in \mathbb{S}_R^1$,

$$\begin{aligned} &U^{-2p,h}(y; \omega_N) \\ &= \frac{N}{2^p} \sum_{j=0}^p \binom{p}{j}^2 (2rR)^{2j} \left(r^2 + R^2 + h + \sqrt{((r-R)^2 + h)(r+R)^2 + h} \right)^{p-2j}. \end{aligned}$$

□

6 Appendix

We collect our computations of all integrals in this section.

Lemma 6.1. *Let $p \in \mathbb{N}$, $k \in \{0, 1, \dots, p\}$, and $z \in \mathbb{C}$. Then,*

$$\int_0^{2\pi} (z^2 + 1 - 2z \cos(t))^p \cos(kt) dt = (-1)^k 2\pi \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} z^{2p-k-2q}. \quad (6.1)$$

Proof of Lemma 6.1. Let $p \in \mathbb{N}$ and $k \in \{1, \dots, p\}$. First, we prove the equality (6.1) for $z \in \mathbb{R}$. Let $z \in \mathbb{R}$. Then, for $\zeta = e^{it}$,

$$\begin{aligned} \int_0^{2\pi} (z^2 + 1 - 2z \cos(t))^p \cos(kt) dt &= \int_0^{2\pi} (z^2 + 1 - z(e^{it} + e^{-it}))^p e^{ikt} dt \\ &= \int_0^{2\pi} (z - e^{it})^p (z - e^{-it})^p e^{ikt} dt = \frac{1}{i} \int_{\mathbb{S}^1} (z - \zeta)^p (z - 1/\zeta)^p \zeta^{k-1} d\zeta \\ &= 2\pi \cdot \text{res} \left(\frac{(z - \zeta)^p (z\zeta - 1)^p}{\zeta^{p-k+1}}; 0 \right) = (-1)^k 2\pi \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} z^{2p-k-2q}, \end{aligned}$$

where the first equality follows from the fact that the last expression is a real number. Notice that the left-hand side and the right-hand side of the equation (6.1) are polynomials as functions of z . Then, both functions are analytic on $\mathbb{C}$ and we have the equation (6.1) for all $z \in \mathbb{C}$. $\square$

Lemma 6.2. *Let $p \in \mathbb{N}$ and $k \in \{0, 1, \dots, p\}$. For $a, b \in \mathbb{C}$,*

$$\int_0^{2\pi} (a - b \cos(t))^p \cos(kt) dt = \frac{(-1)^k \pi}{2^{p-1}} \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} b^{2q+k} \left(a \pm \sqrt{a^2 - b^2} \right)^{p-k-2q}, \quad (6.2)$$

where the square root function in (6.2) can be selected to be both branches of the complex square root function.

Proof of Lemma 6.2. Clearly, if $b = 0$, then the equation in (6.2) is $0 = 0$. Assume that $b \in \mathbb{C} \setminus \{0\}$ and $a \in \mathbb{C}$. To reduce the equation (6.2) to the equation (6.1), we consider

$$(\lambda a - \lambda b \cos(t))^p,$$

where λ is chosen to satisfy the equations

$$2z = b\lambda \quad \text{and} \quad z^2 + 1 = a\lambda,$$

for some $z \in \mathbb{C}$. From above equations,

$$z = \frac{a \pm \sqrt{a^2 - b^2}}{b} \quad \text{and} \quad \lambda = \frac{2a \pm 2\sqrt{a^2 - b^2}}{b^2}.$$

Moreover, $\lambda \neq 0$ because if $\lambda = 0$, then $z = 0$ which implies that $b = 0$. Therefore, by Lemma 6.1,

$$\begin{aligned} \int_0^{2\pi} (a - b \cos(t))^p \cos(kt) dt &= \frac{1}{\lambda^p} \int_0^{2\pi} (\lambda a - \lambda b \cos(t))^p \cos(kt) dt \\ &= \frac{1}{\lambda^p} \int_0^{2\pi} (z^2 + 1 - 2z \cos(t))^p \cos(kt) dt \\ &= \frac{(-1)^k \pi}{2^{p-1}} \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} b^{2q+k} \left(a \pm \sqrt{a^2 - b^2} \right)^{p-k-2q}. \end{aligned}$$

□

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On the boundedness of poles of generalized Padé approximants

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Abstract

Given a function F holomorphic on a neighborhood of some compact subset of the complex plane, we prove that if zeros of denominators of generalized Padé approximants (orthogonal Padé approximants and Padé-Faber approximants) for some row sequence remain uniformly bounded away from ∞ , then either F is a polynomial or F has a singularity in the complex plane. The proofs of our main results rely, on the one hand, on difference equations where their coefficients relate to the coefficients of denominators of these generalized Padé approximants and, on the other hand, on an interesting property of Fourier and Faber coefficients of an entire function.

Keywords: Padé approximation, Orthogonal polynomials, Faber polynomials, Difference equations, Inverse results.

Mathematics Subject Classification: 30E10, 41A21, 41A27.

1 Introduction

Currently, Padé approximation theory emphasizes inverse-type problems where we want to describe the analytic properties of the approximated function from the knowledge of the asymptotic behavior of poles of the approximating functions. Moreover, the theory of higher order recurrence relations (difference equations) plays very important roles in solving recent inverse-type problems (see, e.g., [1, 3, 4, 5, 8]). The object of the present paper is to investigate the relation between the boundedness

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of zeros of denominators of orthogonal Padé approximants and Padé-Faber approximants on row sequences and an analyticity of the approximated function. Our results in this paper are considered as inverse-type results.

In order to state a known result related to our study, we need to remind the reader the definition of classical Padé approximants. In what follows, $\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{P}_n$ is the set of all polynomials of degree at most n .

The concept of (classical) Padé approximants generalizes the idea of Taylor polynomials to rational approximants. Given a formal Taylor series at the origin

$$F(z) = \sum_{k=0}^{\infty} f_k z^k,$$

for any integers $n, m \geq 0$, we can find polynomials $P_{n,m} \in \mathbb{P}_n$ and polynomials $Q_{n,m} \in \mathbb{P}_m$, $Q_{n,m} \not\equiv 0$, such that

$$(Q_{n,m}F - P_{n,m})(z) = \mathcal{O}(z^{n+m+1}), \quad \text{as } z \rightarrow 0.$$

The rational function

$$R_{n,m} := \frac{P_{n,m}}{Q_{n,m}}$$

is uniquely defined and is called the (n, m) *classical Padé approximant* of F . Here, $Q_{n,m}$ is normalized so that $Q_{n,m}(0) = 1$ and it does not share zeros with $P_{n,m}$. In order to find $Q_{n,m}(z) = 1 + \sum_{j=1}^m q_{n,j} z^j$, one has to solve for all $k = n+1, \dots, n+m$,

$$f_k + q_{n,1}f_{k-1} + \dots + q_{n,m}f_{k-m} = 0.$$

Indeed, the above recurrence relation has very strong connection to inverse-type problems (see e.g., [5, Propositions 1-3 and Theorems 4-6] and [8, Section 1.1]). In particular, a generalization of the Poincaré theorem for recurrence relations developing in [5] provides a bridge connecting an inverse result for classical Padé approximation in [11] and the ones for several generalized Padé approximations in [1, 3, 5].

Given a formal power series $F(z) = \sum_{k=0}^{\infty} f_k z^k$, we denote by $R_0(F)$ the radius of the largest disk centered at 0 to which F can be extended holomorphically. Basically, $R_0(F)$ is the same as the radius of convergence of F . In this paper, we are interested in proving analogues of the following theorem (see [8, Theorem 1.1] or [6, Corollary 2.4]) for orthogonal Padé approximants and Padé-Faber approximants.

Theorem A. *Let $m \in \mathbb{N}$ be fixed and let $\mathcal{P}_n$ be the set of all zeros of $Q_{n,m}$. Suppose that the cardinality of $\mathcal{P}_n$ is at least 1 for all n sufficiently large,*

$$\sup_{N \geq m} \inf_{n \geq N} \{|\zeta| : \zeta \in \mathcal{P}_n\} > 0,$$

and

$$\inf_{N \geq m} \sup_{n \geq N} \{|\zeta| : \zeta \in \mathcal{P}_n\} < \infty.$$

Then, F is a polynomial or $0 < R_0(F) < \infty$.

In other words, if F is not a polynomial and the poles of $R_{n,m}$ stay far from the origin and bounded for all n sufficiently large, then $0 < R_0(F) < \infty$. Up to my knowledge, this result is the first one of this sort. Moreover, the sequence $(R_{n,m})_{n \geq n_0}$, where m remains fixed, is called the *mth row sequence*.

Now, let us define four generalized Padé approximations. Let E be a bounded continuum with connected complement containing infinitely many points. From now on, the set E will verify the above condition. Let μ be a finite positive Borel measure with infinite support $\text{supp}(\mu)$ contained in E . We write $\mu \in \mathcal{M}(E)$ and the corresponding inner product is defined by

$$\langle g, h \rangle_\mu := \int g(z) \overline{h(z)} d\mu(z), \quad g, h \in L_2(\mu).$$

Using this inner product, one can generate a unique sequence of orthonormal polynomials

$$(p_n)_{n \geq 0} := (\kappa_n z^n + \cdots)_{n \geq 0}$$

with positive leading coefficients $\kappa_n > 0$. By $\mathcal{H}(E)$, we denote the space of all functions holomorphic in some neighborhood of E . The first two definitions are generalized Padé approximants constructed from the sequence of orthogonal polynomials $(p_n)_{n \geq 0}$.

Definition 1.1. Let $F \in \mathcal{H}(E)$ and $\mu \in \mathcal{M}(E)$. For any integers $n, m \geq 0$, there exists $Q_{n,m}^\mu \in \mathbb{P}_m$ such that $Q_{n,m}^\mu \not\equiv 0$ and $\langle Q_{n,m}^\mu F, p_{n+k} \rangle_\mu = 0$ for all $k = 1, \dots, m$. The associated rational function

$$R_{n,m}^\mu := \frac{\sum_{j=0}^n \langle Q_{n,m}^\mu F, p_j \rangle_\mu p_j}{Q_{n,m}^\mu}$$

is called an (n, m) *classical orthogonal Padé approximant of F with respect to μ* .

Definition 1.2. Let $F \in \mathcal{H}(E)$ and $\mu \in \mathcal{M}(E)$. For any integers $n \geq 0$ and $m \geq 1$, there exists $\tilde{Q}_{n,m}^\mu \in \mathbb{P}_m$ such that $\tilde{Q}_{n,m}^\mu \not\equiv 0$ and $\langle z^k \tilde{Q}_{n,m}^\mu F, p_{n+1} \rangle_\mu = 0$ for all $k = 0, \dots, m-1$. The associated rational function

$$\tilde{R}_{n,m}^\mu := \frac{\sum_{j=0}^n \langle \tilde{Q}_{n,m}^\mu F, p_j \rangle_\mu p_j}{\tilde{Q}_{n,m}^\mu}$$

is called an (n, m) *new orthogonal Padé approximant of F with respect to μ* .

Let Φ the unique Riemann mapping function from $\overline{\mathbb{C}} \setminus E$ to the exterior of the closed unit disk verifying $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$. For each $\rho > 1$, the *level curve of index ρ* and the *canonical domain of index ρ* are defined by

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\} \quad \text{and} \quad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\},$$

respectively. Given $F \in \mathcal{H}(E)$, we denote by $\rho_0(F)$ the largest index ρ of D_ρ to which F can be extended as a holomorphic function.

The *Faber polynomial* of E of degree n is

$$\Phi_n(z) := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\Phi^n(t)}{t - z} dt, \quad z \in D_\rho, \quad n = 0, 1, 2, \dots$$

One can check that

$$\Phi_n(z) = (z/\text{cap}(E))^n + \text{lower degree terms},$$

where $\text{cap}(E)$ is the logarithmic capacity of the set E . The n -th *Faber coefficient* of $F \in \mathcal{H}(E)$ with respect to Φ_n is defined by the formula

$$[F]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{F(t)\Phi'(t)}{\Phi^{n+1}(t)} dt,$$

where $\rho \in (1, \rho_0(F))$.

The next two definitions are definitions of generalized Padé approximants constructed from the sequence of Faber polynomials $(\Phi_n)_{n \geq 0}$.

Definition 1.3. Let $F \in \mathcal{H}(E)$. For any integers $n, m \geq 0$, there exists $Q_{n,m}^E \in \mathbb{P}_m$ such that $Q_{n,m}^E \not\equiv 0$ and $[Q_{n,m}^E F]_{n+k} = 0$ for all $k = 1, \dots, m$. The associated rational function

$$R_{n,m}^E := \frac{\sum_{j=0}^n [Q_{n,m}^E F]_j \Phi_j}{Q_{n,m}^E}$$

is called an (n, m) *classical Padé-Faber approximant of F with respect to E* .

Definition 1.4. Let $F \in \mathcal{H}(E)$. For any integers $n \geq 0$ and $m \geq 1$, there exists $\tilde{Q}_{n,m}^E \in \mathbb{P}_m$ such that $\tilde{Q}_{n,m}^E \not\equiv 0$ and $[z^k \tilde{Q}_{n,m}^E F]_{n+1} = 0$ for all $k = 0, \dots, m-1$. The associated rational function

$$\tilde{R}_{n,m}^E := \frac{\sum_{j=0}^n [\tilde{Q}_{n,m}^E F]_j \Phi_j}{\tilde{Q}_{n,m}^E}$$

is called an (n, m) *new Padé-Faber approximant of F with respect to E* .

In order to find $Q_{n,m}^\mu, \tilde{Q}_{n,m}^\mu, Q_{n,m}^E$, or $\tilde{Q}_{n,m}^E$ in Definitions 1.1-1.4, one has to solve for $m+1$ unknowns from a system of m homogeneous linear equations. Therefore, for any integers $n \geq 0$ and $m \geq 1$, polynomials $Q_{n,m}^\mu, \tilde{Q}_{n,m}^\mu, Q_{n,m}^E$, and $\tilde{Q}_{n,m}^E$ always exist but they may not be unique. Since $Q_{n,m}^\mu, \tilde{Q}_{n,m}^\mu, Q_{n,m}^E$, and $\tilde{Q}_{n,m}^E$ are not the zero function, we normalize them to be “monic” polynomials. Unlike the classical Padé approximants, for any integers $n \geq 0$ and $m \geq 1$, $R_{n,m}^\mu, \tilde{R}_{n,m}^\mu, R_{n,m}^E$, and $\tilde{R}_{n,m}^E$ may not be unique. The rational functions $R_{n,m}^\mu$ and $R_{n,m}^E$ are natural extensions of $R_{n,m}$ and were introduced by Maehly [9] in 1960. The rational functions $\tilde{R}_{n,m}^\mu$ and $\tilde{R}_{n,m}^E$ were recently introduced (in the vector forms) in order to solve some inverse-type problems about detecting poles of a vector of functions nearest the set E (see [2, 4] for more details). Note that in general, the approximations in Definitions 1.1 and 1.3 are not the same as the ones in Definitions 1.2 and 1.4, respectively.

An outline of this paper is as follows. In Section 2, we state analogues of Theorem A which are our main results. We keep all lemmas in Section 3. The proofs of the main results are in Section 4.

2 Main Results

Before stating the main results, we need to define two subclasses of $\mathcal{M}(E)$. The measure $\mu \in \mathbf{Reg}_1(E)$ if and only if

$$\lim_{n \rightarrow \infty} |p_n(z)|^{1/n} = |\Phi(z)|, \quad (1)$$

uniformly on each compact subset of $\mathbb{C} \setminus E$. This is the minimum requirement to have the limit formula for $\rho_0(F)$ and the convergence of the orthogonal polynomial expansion in Lemma 3.1 in Section 3. Moreover, the class $\mathbf{Reg}_1(E)$ is exactly the regular class in [10, Definition 3.1.2] when E is convex. The measure $\mu \in \mathbf{Reg}_1^*(E)$ when $\mu \in \mathbf{Reg}_1(E)$ and

$$\frac{\kappa_{n-1}}{\kappa_n} \geq c, \quad n \geq n_0, \quad (2)$$

for some $c > 0$ and $n_0 \in \mathbb{N}$.

The main results of this paper are the following.

Theorem 2.1. *Let $F \in \mathcal{H}(E)$ and $\mu \in \mathbf{Reg}_1^*$. Fix $m \in \mathbb{N}$ and denote by $\mathcal{P}_n^\mu$ the set of all zeros of a polynomial $Q_{n,m}^\mu$. Assume that the cardinality of $\mathcal{P}_n^\mu$ is at least 1 for all n sufficiently large and*

$$\inf_{N \geq m} \sup_{n \geq N} \{|\zeta| : \zeta \in \mathcal{P}_n^\mu\} < \infty. \quad (3)$$

Then, either F is a polynomial or $\rho_0(F) < \infty$.

Theorem 2.2. Let $F \in \mathcal{H}(E)$. Fix $m \in \mathbb{N}$ and denote by $\mathcal{P}_n^E$ the set of all zeros of a polynomial $Q_{n,m}^E$. Assume that the cardinality of $\mathcal{P}_n^E$ is at least 1 for all n sufficiently large and

$$\inf_{N \geq m} \sup_{n \geq N} \{|\zeta| : \zeta \in \mathcal{P}_n^E\} < \infty. \quad (4)$$

Then, either F is a polynomial or $\rho_0(F) < \infty$.

Theorem 2.3. Let $F \in \mathcal{H}(E)$ and $\mu \in \mathbf{Reg}_1^*$. Fix $m \in \mathbb{N}$ and denote by $\tilde{\mathcal{P}}_n^\mu$ the set of all zeros of a polynomial $Q_{n,m}^\mu$. Assume that the cardinality of $\tilde{\mathcal{P}}_n^\mu$ is at least 1 for all n sufficiently large and

$$\inf_{N \geq m} \sup_{n \geq N} \{|\zeta| : \zeta \in \tilde{\mathcal{P}}_n^\mu\} < \infty. \quad (5)$$

Then, either F is a polynomial or $\rho_0(F) < \infty$.

Theorem 2.4. Let $F \in \mathcal{H}(E)$. Fix $m \in \mathbb{N}$ and denote by $\tilde{\mathcal{P}}_n^E$ the set of all zeros of a polynomial $\tilde{Q}_{n,m}^E$. Assume that the cardinality of $\tilde{\mathcal{P}}_n^E$ is at least 1 for all n sufficiently large and

$$\inf_{N \geq m} \sup_{n \geq N} \{|\zeta| : \zeta \in \tilde{\mathcal{P}}_n^E\} < \infty. \quad (6)$$

Then, either F is a polynomial or $\rho_0(F) < \infty$.

Note that we are not interested in proving $\rho_0(F) > 1$ because this is a direct consequence of $F \in \mathcal{H}(E)$.

3 Auxiliary Lemmas

Recall that the n -th Fourier coefficient of $F \in \mathcal{H}(E)$ corresponding to p_n is defined as follows

$$\langle F \rangle_n := \langle F, p_n \rangle_\mu = \int F(z) \overline{p_n(z)} d\mu(z).$$

The first lemma (see, e.g., [2, Lemma 2.1.] and [12] for its proof) concerns the convergences of orthogonal and Faber polynomial expansions.

Lemma 3.1. Let $F \in \mathcal{H}(E)$ and $\mu \in \mathbf{Reg}_1(E)$. Then,

$$\rho_0(F) = \left(\limsup_{n \rightarrow \infty} |\langle F \rangle_n|^{1/n} \right)^{-1} = \left(\limsup_{n \rightarrow \infty} |[F]_n|^{1/n} \right)^{-1}$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \langle F \rangle_k p_k(z) = \lim_{n \rightarrow \infty} \sum_{k=0}^n [F]_k \Phi_k(z) = F(z),$$

uniformly on each compact subset of $D_{\rho_0(F)}$.

An estimation of $\|\Phi_n\|_{\Gamma_\rho}$ is given in the succeeding lemma (see [7, p. 583] or [12, p. 43]).

Lemma 3.2. *Fix $\rho > 1$. Then, there exists $c > 0$ such that for all $n \in \mathbb{N}_0$,*

$$\|\Phi_n\|_{\Gamma_\rho} \leq c\rho^n.$$

An interesting property of Fourier and Faber coefficients of an entire function which serves as the cornerstone for the proofs of our main results is the following.

Lemma 3.3. *Let $\mu \in \mathbf{Reg}_1(E)$. If G is an entire function and G satisfies one of the following properties:*

- (i) $|\langle G \rangle_n| \leq c \sum_{k=n+1}^{\infty} |\langle G \rangle_k|$, for all $n \geq n_0$ and for some constant $c > 0$;
- (ii) $|[G]_n| \leq c \sum_{k=n+1}^{\infty} |[G]_k|$, for all $n \geq n_0$ and for some constant $c > 0$,

then G is a polynomial.

Proof of Lemma 3.3. We will prove only the statement concerning Fourier coefficients of G . By Lemma 3.1, since G is an entire function,

$$\lim_{n \rightarrow \infty} |\langle G \rangle_n|^{1/n} = \frac{1}{\rho_0(G)} = 0,$$

which implies that

$$|\langle G \rangle_n| < \left(\frac{1}{c+2} \right)^n \quad (7)$$

for all $n \geq n_1$. By the assumption,

$$|\langle G \rangle_n| \leq c \sum_{k=n+1}^{\infty} |\langle G \rangle_k|, \quad (8)$$

for all $n \geq n_0$. Claim that for each $n \geq \max\{n_0, n_1\}$,

$$|\langle G \rangle_n| \leq \left(\frac{c}{c+1} \right)^\ell \left(\frac{1}{c+2} \right)^n, \quad (9)$$

for all $\ell \in \mathbb{N}_0$. We prove the above statement by mathematical induction on ℓ . The case when $\ell = 0$ is from (7). Now, we suppose that (9) holds for ℓ and show that (9) holds for $\ell + 1$. By (7), (8), and (9), for each $n \geq \max\{n_0, n_1\}$,

$$|\langle G \rangle_n| \leq c \sum_{k=n+1}^{\infty} |\langle G \rangle_k| \leq c \sum_{k=n+1}^{\infty} \left(\frac{c}{c+1} \right)^\ell \left(\frac{1}{c+2} \right)^k = \left(\frac{c}{c+1} \right)^{\ell+1} \left(\frac{1}{c+2} \right)^n,$$

which implies that (9) holds for $\ell + 1$. This completes the mathematical induction proof. Next, letting $\ell \rightarrow \infty$ in (9), we have for each $n \geq \max\{n_0, n_1\}$, $\langle G \rangle_n = 0$. Therefore, G is a polynomial. $\square$

Simple properties of $\langle \cdot \rangle_m$ and $[\cdot]_m$ used frequently in the proofs of the main results are in

Lemma 3.4. *Let $F \in \mathcal{H}(E)$ and $\mu \in \mathcal{M}(E)$. Then,*

$$[\Phi_n]_m = \langle p_n \rangle_m = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

4 Proofs of Main Results

Proof of Theorem 2.1. Let

$$Q_{n,m}^\mu(z) := \prod_{k=1}^{m_n} (z - \zeta_{n,k}) = \sum_{j=0}^{m_n} q_{n,j} z^j$$

and

$$F(z) := \sum_{\nu=0}^{\infty} \langle F \rangle_\nu p_\nu(z).$$

Note that $m_n \geq 1$ for all n sufficiently large and $q_{n,m_n} = 1$. We will show that if (3) holds and F is an entire function, then F is a polynomial.

From the definition of $Q_{n,m}^\mu$, we have for all $k = 1, \dots, m$,

$$\begin{aligned} 0 &= \langle Q_{n,m}^\mu F \rangle_{n+k} = \sum_{j=0}^{m_n} \sum_{\nu=0}^{\infty} \langle F \rangle_\nu q_{n,j} \langle z^j p_\nu \rangle_{n+k} = \sum_{j=0}^{m_n} \sum_{\nu=n+k-j}^{\infty} \langle F \rangle_\nu q_{n,j} \langle z^j p_\nu \rangle_{n+k} \\ &= \sum_{\nu=n+k-m_n}^{\infty} \langle F \rangle_\nu \langle z^{m_n} p_\nu \rangle_{n+k} + \sum_{j=0}^{m_n-1} \sum_{\nu=n+k-j}^{\infty} \langle F \rangle_\nu q_{n,j} \langle z^j p_\nu \rangle_{n+k} \\ &= \frac{\kappa_{n+k-m_n}}{\kappa_{n+k}} \langle F \rangle_{n+k-m_n} + \sum_{\nu=n+k-m_n+1}^{\infty} \langle F \rangle_\nu \langle z^{m_n} p_\nu \rangle_{n+k} + \sum_{j=0}^{m_n-1} \sum_{\nu=n+k-j}^{\infty} q_{n,j} \langle F \rangle_\nu \langle z^j p_\nu \rangle_{n+k}. \end{aligned} \tag{10}$$

Using the Vieta formulas, since

$$\inf_{N \geq m} \sup_{n \geq N} \{|\zeta| : \zeta \in \mathcal{P}_n^\mu\} < \infty,$$

there exists $c_1 > 0$ such that

$$\sup\{|q_{n,j}| : 0 \leq j \leq m_n, n \geq n_0\} \leq c_1. \quad (11)$$

From the Cauchy-Schwarz inequality and the orthonormality of p_ν , for all $n, \nu, k \in \mathbb{N}_0$ and $j = 0, \dots, m$, there exists $c_2 > 0$ such that

$$|\langle z^j p_\nu \rangle_{n+k}| = |\langle z^j p_\nu, p_{n+k} \rangle_\mu| \leq c_2. \quad (12)$$

Because $\mu \in \mathbf{Reg}_1^*$, there exists $c_3 > 0$ such that

$$\frac{\kappa_{n+k-m_n}}{\kappa_{n+k}} \geq c_3, \quad n \geq n_0, \quad (13)$$

where c_3 does not depend on k and m_n . Combining (11), (12), and (13), it is easy to check that (10) imply that for all $k = 1, \dots, m$, and for all $n \geq n_0$,

$$|\langle F \rangle_{n+k-m_n}| \leq c_4 \sum_{\nu=n+k-m_n+1}^{\infty} |\langle F \rangle_\nu|,$$

where c_4 is a positive constant that does not depend on n, k and m_n . For each $n \geq n_0$, we choose $k = m_n$ in the previous inequality and we obtain for all $n \geq n_0$,

$$|\langle F \rangle_n| \leq c_4 \sum_{\nu=n+1}^{\infty} |\langle F \rangle_\nu|.$$

Applying Lemma 3.3, F is a polynomial. □

Proof of Theorem 2.2. Let

$$Q_{n,m}^E(z) := \prod_{k=1}^{m_n} (z - \zeta_{n,k}) = \sum_{j=0}^{m_n} q_{n,j} z^j$$

and

$$F(z) := \sum_{\nu=0}^{\infty} [F]_\nu \Phi_\nu(z).$$

Note that $m_n \geq 1$ for all n sufficiently large and $q_{n,m_n} = 1$. We will follow the same plan by proving that if (4) holds and F is an entire function, then F is a polynomial.

The analogous arguments used to derive (10) implies that for all $k = 1, \dots, m$,

$$0 = (\text{cap}(E))^{m_n} [F]_{n+k-m_n} +$$

$$\sum_{\nu=n+k-m_n+1}^{\infty} [F]_{\nu} [z^{m_n} \Phi_{\nu}]_{n+k} + \sum_{j=0}^{m_n-1} \sum_{\nu=n+k-j}^{\infty} q_{n,j} [F]_{\nu} [z^j \Phi_{\nu}]_{n+k}. \quad (14)$$

Moreover, there exists $c_1 > 0$ such that

$$\sup\{|q_{n,j}| : 0 \leq j \leq m_n, n \geq n_0\} \leq c_1. \quad (15)$$

Take $\rho > 1$. Using Lemma 3.2, for $j = 0, 1, \dots, m$, $k = 1, \dots, m$, and $n, \nu \in \mathbb{N}_0$, we obtain

$$|[z^j \Phi_{\nu}]_{n+k}| = \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho}} \frac{z^j \Phi_{\nu}(z) \Phi'_{\nu}(z)}{\Phi^{n+k+1}(z)} dz \right| \leq c_2 \frac{\rho^{\nu}}{\rho^n}. \quad (16)$$

Combining (14), (15), and (16), it is easy to check that for all $k = 1, \dots, m$, and for all $n \geq n_0$,

$$|[F]_{n+k-m_n}| \rho^n \leq c_3 \sum_{\nu=n+k-m_n+1}^{\infty} |[F]_{\nu}| \rho^{\nu},$$

where c_3 is a positive constant that does not depend on n, k and m_n . For each $n \geq n_0$, we choose $k = m_n$ in the previous inequality and we obtain for all $n \geq n_0$,

$$|[F]_n| \rho^n \leq c_3 \sum_{\nu=n+1}^{\infty} |[F]_{\nu}| \rho^{\nu}. \quad (17)$$

Using Lemma 3.3 by setting

$$G = \sum_{n=0}^{\infty} ([F]_n \rho^n) \Phi_n,$$

since (17) is

$$|[G]_n| \leq c_3 \sum_{\nu=n+1}^{\infty} |[G]_{\nu}|$$

and

$$\lim_{n \rightarrow \infty} |[G]_n|^{1/n} = \frac{\rho}{\rho_0(F)} = 0$$

(which means G is an entire function), G is a polynomial. Consequently, F is a polynomial. $\square$

Proof of Theorem 2.3. Let

$$\tilde{Q}_{n,m}^{\mu}(z) := \prod_{k=1}^{m_n} (z - \zeta_{n,k}) = \sum_{j=0}^{m_n} q_{n,j} z^j$$

and

$$F(z) := \sum_{\nu=0}^{\infty} \langle F \rangle_{\nu} p_{\nu}(z).$$

Note that $m_n \geq 1$ for all n sufficiently large and $q_{n,m_n} = 1$. Our plan is to prove that F is a polynomial under the assumptions that (5) holds and F is an entire function.

From the definition of $\tilde{Q}_{n,m}^{\mu}$, we have for all $k = 0, \dots, m-1$,

$$\begin{aligned} 0 &= \langle z^k \tilde{Q}_{n,m}^{\mu} F \rangle_{n+1} = \sum_{j=0}^{m_n} \sum_{\nu=0}^{\infty} \langle F \rangle_{\nu} q_{n,j} \langle z^{k+j} p_{\nu} \rangle_{n+1} = \sum_{j=0}^{m_n} \sum_{\nu=n+1-k-j}^{\infty} \langle F \rangle_{\nu} q_{n,j} \langle z^{k+j} p_{\nu} \rangle_{n+1} \\ &= \sum_{\nu=n+1-k-m_n}^{\infty} \langle F \rangle_{\nu} \langle z^{k+m_n} p_{\nu} \rangle_{n+1} + \sum_{j=0}^{m_n-1} \sum_{\nu=n+1-k-j}^{\infty} \langle F \rangle_{\nu} q_{n,j} \langle z^{k+j} p_{\nu} \rangle_{n+1} \\ &= \frac{\kappa_{n+1-k-m_n}}{\kappa_{n+1}} \langle F \rangle_{n+1-k-m_n} + \sum_{\nu=n-k-m_n+2}^{\infty} \langle F \rangle_{\nu} \langle z^{k+m_n} p_{\nu} \rangle_{n+1} \\ &\quad + \sum_{j=0}^{m_n-1} \sum_{\nu=n+1-k-j}^{\infty} \langle F \rangle_{\nu} q_{n,j} \langle z^{k+j} p_{\nu} \rangle_{n+1}. \end{aligned} \tag{18}$$

Applying exactly the same arguments as in (11), (12), and (13), there exists $c_1 > 0$ such that

$$\sup\{|q_{n,j}| : 0 \leq j \leq m_n, n \geq n_0\} \leq c_1, \tag{19}$$

there exists $c_2 > 0$ such that for all $k = 0, \dots, m-1$, $j = 0, \dots, m$, and $n, \nu \in \mathbb{N}_0$,

$$|\langle z^{k+j} p_{\nu} \rangle_{n+1}| = |\langle z^j p_{\nu}, p_{n+k} \rangle_{\mu}| \leq c_2, \tag{20}$$

there exists $c_3 > 0$ such that for all $k = 0, \dots, m-1$, $m_n = 1, \dots, m$, and $n \geq n_0$,

$$\frac{\kappa_{n+1-k-m_n}}{\kappa_{n+1}} \geq c_3. \tag{21}$$

Using (18), (19), (20), and (21), we have for all $k = 0, \dots, m-1$ and $n \geq n_0$,

$$|\langle F \rangle_{n-k-m_n+1}| \leq c_4 \sum_{\nu=n-k-m_n+2}^{\infty} |\langle F \rangle_{\nu}|.$$

For each $n \geq n_0$, we choose $k = m - m_n$ in the previous inequality and we obtain for all $n \geq n_0$,

$$|\langle F \rangle_{n-m+1}| \leq c_4 \sum_{\nu=n-m+2}^{\infty} |\langle F \rangle_{\nu}|.$$

Setting $N = n - m + 1$, we have

$$|\langle F \rangle_N| \leq c_4 \sum_{\nu=N+1}^{\infty} |\langle F \rangle_{\nu}|, \quad N \geq N_0.$$

By Lemma 3.3, F is a polynomial. □

Proof of Theorem 2.4. Let

$$\tilde{Q}_{n,m}^E(z) := \prod_{k=1}^{m_n} (z - \zeta_{n,k}) = \sum_{j=0}^{m_n} q_{n,j} z^j$$

and

$$F(z) := \sum_{\nu=0}^{\infty} [F]_{\nu} \Phi_{\nu}(z).$$

Note that $m_n \geq 1$ for all n sufficiently large and $q_{n,m_n} = 1$. Following the same plan by assuming that (6) holds and F is an entire function, we show that F is a polynomial.

The analogous arguments used to derive (18) implies that for all $k = 0, \dots, m-1$,

$$\begin{aligned} 0 &= (\text{cap}(E))^{m_m+k} [F]_{n+1-k-m_n} + \sum_{\nu=n-k-m_n+2}^{\infty} [F]_{\nu} [z^{k+m_n} \Phi_{\nu}]_{n+1} \\ &\quad + \sum_{j=0}^{m_n-1} \sum_{\nu=n+1-k-j}^{\infty} [F]_{\nu} q_{n,j} [z^{k+j} \Phi_{\nu}]_{n+1}. \end{aligned} \quad (22)$$

Moreover, there exists $c_1 > 0$ such that

$$\sup\{|q_{n,j}| : 0 \leq j \leq m_n, n \geq n_0\} \leq c_1. \quad (23)$$

Take $\rho > 1$. Using Lemma 3.2, we obtain for all $j = 0, 1, \dots, m$, $k = 0, \dots, m-1$, and $n, \nu \in \mathbb{N}_0$,

$$|[z^{k+j} \Phi_{\nu}]_{n+1}| = \left| \frac{1}{2\pi i} \int_{\Gamma_{\rho}} \frac{z^{k+j} \Phi_{\nu}(z) \Phi'_{\nu}(z)}{\Phi^{n+2}(z)} dz \right| \leq c_2 \frac{\rho^{\nu}}{\rho^n}. \quad (24)$$

By (22), (23), and (24), we have for all $k = 0, \dots, m-1$ and $n \geq n_0$,

$$|[F]_{n+1-k-m_n}| \rho^{n-m+1} \leq c_3 \sum_{\nu=n-k-m_n+2}^{\infty} [F]_{\nu} \rho^{\nu}.$$

For each $n \geq n_0$, we choose $k = m - m_n$ in the previous inequality and we obtain for all $n \geq n_0$,

$$|[F]_{n-m+1}|\rho^{n-m+1} \leq c_3 \sum_{\nu=n-m+2}^{\infty} |[F]_{\nu}|.$$

Setting $N = n - m + 1$ and

$$G = \sum_{n=0}^{\infty} ([F]_n \rho^n) \Phi_n,$$

we have

$$|[G]_N| \leq c_3 \sum_{\nu=N+1}^{\infty} |[G]_{\nu}|, \quad N \geq N_0.$$

By Lemma 3.3, the above inequalities and

$$\lim_{N \rightarrow \infty} |[G]_N|^{1/N} = \frac{\rho}{\rho_0(F)} = 0$$

(G is an entire function) imply that G is a polynomial. Therefore, F is a polynomial. $\square$

5 Conclusion

We prove that if zeros of denominators of four generalized Padé approximations based on orthogonal and Faber polynomials stay uniformly bounded away from ∞ , then either the approximated function is a polynomial or it has a singularity in the complex plane.

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9 Competing interests

The authors declare that they have no competing interests.

10 Authors' contributions

All authors contributed equally to the writing of this paper. All authors conceived of the study, participated in its design and coordination, read and approved the final manuscript.

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