รายงานวิจัยฉบับสมบูรณ์

โครงการ สมบัติบางประการของกราฟจำนวนควบคุมวิกฤติ

โดย ภวธน เขมะวิชานุรัตน์

มกราคม 2562

สัญญาเลขที่ MRG6080139

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
และต้นสังกัด

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โครงการวิจัยต้องการศึกษาสมบัติของกราฟดังต่อไปนี้

1. หาขอบเขตบนของจำนวนจุดตัดของกราฟจำนวนควบคุมต่อเนื่องวิกฤติที่มีจำนวนควบคุมต่อเนื่องอย่างน้อย 5
2. ศึกษาสมบัติของกราฟจำนวนควบคุมต่อเนื่องวิกฤติที่มีจำนวนจุดตัดเท่ากับขอบเขตบนที่หาได้จากข้อที่ 1
3. ศึกษาการมีอยู่จริงของกราฟจำนวนควบคุมต่อเนื่องวิกฤติที่มีจุดตัด a จุดเมื่อ a มีค่าตั้งแต่ 0 จนถึงค่าขอบเขตบน
4. ศึกษาว่ากราฟจำนวนควบคุมต่อเนื่องวิกฤติที่มีจำนวนจุดตัดเท่ากับขอบเขตบนมีการจับคู่สมบูรณ์หรือไม่

วิธีวิจัย :

1. ศึกษางานที่เกี่ยวของกับกราฟจำนวนควบคุมวิกฤติทั้งหมด
2. พยายามสร้างกราฟจำนวนควบคุมต่อเนื่องวิกฤติที่มีจำนวนควบคุมต่อเนื่องอย่างน้อย 5 ที่มีจำนวนจุดตัดมากที่สุด จากนั้นพิสูจน์ขอบเขตบนที่ได้จากตัวอย่างกราฟเหล่านี้ แล้วพยายามหาลักษณะเฉพาะของกราฟเหล่านี้เมื่อมีจำนวนจุดตัดเท่ากับหรือใกล้เคียงกับขอบเขตบน
3. พยายามสร้างกราฟจำนวนควบคุมต่อเนื่องวิกฤติที่มีจำนวนควบคุมต่อเนื่องอย่างน้อย 5 ที่มีจำนวนจุดตัดต่างๆกัน
4. ศึกษาคุณสมบัติการจับคู่ของกราฟจำนวนควบคุมต่อเนื่องวิกฤติที่มีจำนวนควบคุมต่อเนื่องอย่างน้อย 5 ที่มีจำนวนจุดตัดเท่ากับของเขตบนหรือใกล้เคียง โดยใช้ผลลัพธ์ที่ได้จากข้อ 2.

สรุปผล :

1. เราพิสูจน์ว่ากราฟจำนวนควบคุมต่อเนื่องวิกฤติที่มีจำนวนควบคุมต่อเนื่องเท่ากับ k มีจุดตัดไม่เกิน k – 2 โดยให้บทพิสูจน์ไว้ในบทที่ 3 นอกจากนี้เราได้หาลักษณะเฉพาะของกราฟจำนวนควบคุมต่อเนื่องวิกฤติที่มีจุดตัด k - 2 และ k – 3 จุดอีกด้วย โดยใส่ผลลัพธ์ไว้ในบทที่ 4
2. ในการหากราฟจำนวนควบคุมต่อเนื่องวิกฤติที่มีจุดตัด a จุดเมื่อ a มีค่าตั้งแต่ 0 จนถึง k – 2 สามารถทำได้โดยได้ให้คำอธิบายตอนท้ายบทที่ 4

3. จากการหาลักษณะเฉพาะของกราฟจำนวนควบคุมต่อเนื่องวิกฤติที่มีจุดตัด k - 2 และ k – 3 จุด เราได้ผลลัพธ์เกี่ยวกับสมบัติการจับคู่โดยมีคำอธิบายตอนท้ายบทที่ 4
4. ในระหว่างการทำวิจัยเราได้ผลลัพธ์ที่น่าสนใจที่นอกเหนือจากวัตถุประสงค์ โดยผลลัพธ์นี้เป็นของกราฟจำนวนควบคุมวิกฤติอีกประเภทหนึ่ง ทางผู้วิจัยจึงใส่ไว้ในบทที่ 5 แล้วเปลี่ยนชื่อโครงการวิจัยจาก "กราฟจำนวนควบคุมต่อนื่องวิกฤติที่มีจุดตัด" เป็น "สมบัติบางประการของกราฟจำนวนควบคุมวิกฤติ" เพื่อให้สอดคล้องกับเนื้อหาและมีความเป็นทั่วไปมากกว่า

Keywords : จำนวนควบคุม;จุดตัด;กราฟวิกฤติ

Abstract

Project Code : MRG 6080139

Project Title : Some Results in Critical Graphs with Respect to Connected Domination and Double Domination

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Project Period : 1 April 2017 - 31 March 2019

Objectives :

1. Find the upper bound of the number of cut vertices of k-γ_c-critical graphs for $k \geq 5$.
2. For $k \geq 5$, study the properties of the k-γ_c-critical graphs having the number of cut vertices equal or closed to the upper bound.
3. Solve the realizable problem to find the existence of k-γ_c-critical graphs, for $k \geq 5$, of prescribe the number of cut vertices.
4. For $k \geq 5$, determine whether or not every k-γ_c-critical graph having the number of cut vertices equal to the upper bound contains a perfect matching.

Methodology :

1. Study all related works not only in connected domination concept but also study on various types of domination numbers.
2. For $k \geq 5$, we proceed the research as the followings : Construct a number of k-γ_c-critical graphs having as many cut vertices as possible. Establish the upper bound of the number of cut vertices. We can use some results from another types of domination critical graphs to guideline. Then investigate the properties of all k-γ_c-critical graphs achieving the upper bound. Try to characterize these graphs if possible.
3. Find a collection of k-γ_c-critical graphs containing cut vertices. Try to construct such graphs having a different number of cut vertices from 0 to the upper bounds.
4. Study related works on matching theory and use this knowledge to determine the matching properties of k-γ_c-critical graphs having the number of cut vertices equal or closed to the upper bound.

Output :

1. We prove that the upper bound of the number of cut vertices of k-γ_c-critical graphs for k ≥ 5 is at most k – 2. The proof of which is established in Chapter 3. We further characterized such graphs having k – 3 and k – 2 cut vertices. We provide the characterizations in Chapter 4.
2. We construct k-γ_c-critical graphs with prescribed cut vertices at the end of Chapter 4.
3. According to the characterizations, we can established matching properties of k-γ_c-critical graphs with k – 3 and k – 2 cut vertices at the end of Chapter 4.
4. When we review related works in domination critical graphs, we can prove some new results in critical graphs but on another type of domination number which is called double domination. We can get one publication beside our main purpose but still under sponsorship by TRF. We put this work in Chapter 5. Further, we change the name of our project from "Connected Domination Critical Graphs" to be "Some Results in Critical Graphs with Respect to Connected Domination and Double Domination" which is more general and appropriate to the contents of this report.

Keywords : domination;cut vertex;critical

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Contents

CHAPTER 1

Introduction and Background

1.1 Notations

All graphs in this paper are finite, undirected and simple (no loops or multiple edges). For a graph G, let $V(G)$ denote the set of all vertices of G and let $E(G)$ denote the set of all edges of G. The *complement* $\overline{G}$ of G is the graph having the same set of vertices as G but the edge e is in $E(\overline{G})$ if and only if $e \notin E(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. The *open neighborhood* $N_G(v)$ of a vertex v in G is $\{u \in V(G) : uv \in E(G)\}$. Further, the *closed neighborhood* $N_G[v]$ of a vertex v in G is $N_G(v) \cup \{v\}$. For subsets X and Y of $V(G)$, $N_Y(X)$ is the set $\{y \in Y : yx \in E(G)$ for some $x \in X\}$. For a subgraph H of G, we use $N_Y(H)$ instead of $N_Y(V(H))$ and we use $N_H(X)$ instead of $N_{V(H)}(X)$. If $X = \{x\}$, we use $N_Y(x)$ instead of $N_Y(\{x\})$. The *degree* $deg(x)$ of a vertex x in G is $|N_G(x)|$. When no ambiguity occur, we write $N(x)$ and $N(X)$ instead of $N_G(x)$ and $N_G(X)$, respectively. An *end vertex* is a vertex of degree one and a *support vertex* is the vertex which is adjacent to an end vertex.

We let $\omega(G)$ to denote the number of components of G. A *cut set* S is a vertex subset for which $\omega(G-S) > \omega(G)$. The *connectivity* κ is the minimum cardinality of a cut set. A graph G is *l-connected* if $\kappa \geq l$. Moreover, if $S = \{v\}$, then v is called a *cut vertex* if $G - v$ is not connected. The number of cut vertices of G is denoted by $\zeta(G)$. A *block* B of a graph G is a maximal connected subgraph such that B has no cut vertex. An *end block* of G is a block containing exactly one cut vertex of G. For a connected graph G, a *bridge* xy of G is an edge such that $G - xy$ is not connected.

An *independent set* is a set of pairwise non-adjacent vertices. A graph G is *bipartite* if there exists a bipartition X and Y of $V(G)$ such that X and Y are independent sets. A *complete bipartite graph* $K_{m,n}$ is a bipartite graph with the partite sets X and Y such that $|X| = m$ and $|Y| = n$ containing all edges joining the vertices between X and Y. A *star* $K_{1,n}$ is a complete bipartite graph when $m = 1$, in particular if $n = 3$, a star $K_{1,3}$ is called a *claw*. The support vertex of a star is called the *center*. For integers

$s_1, s_2, s_3 \geq 1$, let $u_1, u_2, ..., u_{s_1+1}; v_1, v_2, ..., v_{s_2+1}$ and $w_1, w_2, ..., w_{s_3+1}$ be three disjoint paths of length s_1, s_2 and s_3, respectively. A *net* N_{s_1,s_2,s_3} is constructed by adding edges $u_{s_1+1}v_{s_2+1}, v_{s_2+1}w_{s_3+1}$ and $w_{s_3+1}u_{s_1+1}$. For a family of graphs $\mathscr{F}$, a graph G is said to be *$\mathscr{F}$-free* if there is no induced subgraph of G isomorphic to H for all $H \in \mathscr{F}$. A *Hamiltonian path(cycle)* is a path(cycle) containing all vertices of the graph. A graph G is *Hamiltonian* if G contains a Hamiltonian cycle. The *distance* $d(u,v)$ between vertices u and v of G is the length of a shortest (u,v)-path in G. The diameter of G $diam(G)$ is the maximum distance of any two vertices of G.

For a finite sequence of graphs $G_1, ..., G_l$ for $l \geq 2$, the *joins* $G_1 \vee \cdots \vee G_l$ is the graph consisting of the disjoint union of $G_1, ..., G_l$ and each vertex in G_i is joined to all vertices in G_{i+1} for $1 \leq i \leq l-1$ by edges. If $V(G_i) = \{x\}$, then we simply write $G_1 \vee \cdots \vee G_{i-1} \vee x \vee G_{i+1} \vee \cdots \vee G_l$. Moreover, for a subgraph H of G_2, the *join* $G_1 \vee_H G_2$ is the graph consisting of the disjoint union of G_1 and G_2 and edges that join each vertex in G_1 to each vertex in H.

For subsets D and X of $V(G)$, D *dominates* X if every vertex in X is either in D or adjacent to a vertex in D. If D dominates X, then we write $D \succ X$. We also write $a \succ X$ when $D = \{a\}$ and $D \succ x$ when $X = \{x\}$. Moreover, if $X = V(G)$, then D is a *dominating set* of G and we write $D \succ G$ instead of $D \succ V(G)$. A *connected dominating set* of a graph G is a dominating set D of G such that $G[D]$ is connected. If D is a connected dominating set of G, we then write $D \succ_c G$. A smallest connected dominating set is called a *γ_c-set*. The cardinality of a γ_c-set is called the *connected domination number* of G and is denoted by $\gamma_c(G)$. Moreover, we say that D *doubly dominates* X if $|N_D[X]| \geq 2$ for all $x \in X$. We write $D \succ_{\times 2} X$ if D doubly dominates X. Moreover, if $X = V(G)$, then D is a *double dominating set* of G. A smallest double dominating set of G is called a *$\gamma_{\times 2}$-set* of G. The *double domination number* of G is the cardinality of a $\gamma_{\times 2}$-set of G and is denoted by $\gamma_{\times 2}(G)$.

A graph G is said to be *k-γ_c-critical* if $\gamma_c(G) = k$ and $\gamma_c(G + uv) < k$ for any pair of non-adjacent vertices u and v of G. Similarly, a graph G is said to be *k double domination critical*, or *k-$\gamma_{\times 2}$-critical*, if $\gamma_{\times 2}(G) = k$ and $\gamma(G + uv) < k$ for all $uv \notin E(G)$. On the other hand, a graph G is said to be *double domination edge addition stable*, or *k-$\gamma^+_{\times 2}$-stable*, if $\gamma_{\times 2}(G) = k$ and $\gamma(G + uv) = k$ for all $uv \notin E(G)$ and a graph G is said to be *double domination edge removal stable*, or *k-$\gamma^-_{\times 2}$-stable*, if $\gamma_{\times 2}(G) = k$ and $\gamma(G - uv) = k$ for all $uv \in E(G)$. A graph which is either k-$\gamma^+_{\times 2}$-stable or k-$\gamma^-_{\times 2}$-stable is called *double domination stable*.

1.2 Background

1.2.1 Connected Domination

For related results on k-γ_c-critical graphs, Chen et al. [6] completely characterized these graphs when $1 \leq k \leq 2$.

Theorem 1.1. *[6] A graph G is 1-γ_c-critical if and only if G is a complete graph. Moreover, a graph G is 2-γ_c-critical if and only if $\overline{G} = \cup_{i=1}^{l} K_{1,n_i}$ where $l \geq 2$ and $n_i \geq 1$ for all $1 \leq i \leq l$.*

By Theorem 1.1, we observe that a k-γ_c-critical graph does not contain a cut vertex when $1 \leq k \leq 2$.

Observation 1.2.1. Let G be a k-γ_c-critical graph with $1 \leq k \leq 2$. Then G has no cut vertex.

For $k \geq 3$, there is no complete characterization of these graphs so far. However, there are some structural characterizations of k-γ_c-critical graphs when $3 \leq k \leq 4$ by focusing on the maximum number of cut vertices of the graphs. Ananchuen [1] proved that the number of cut vertices of a 3-γ_c-critical graph does not exceed one.

Theorem 1.2. *[1] Let G be a 3-γ_c-critical graph. Then G contains at most one cut vertex.*

In our previous work in [12], we established the maximum number of cut vertices that 4-γ_c-critical graphs can have.

Theorem 1.3. *[12] Let G be a 4-γ_c-critical graph. Then G contains at most two cut vertices.*

By these results, we generalize that, for $k \geq 5$, every k-γ_c-critical graph contains at most $k-2$ cut vertices. The proof of this theorem is given in Chapter 3.

1.2.2 Double Domination

This project also focuses on the Hamiltonicity of double domination critical graphs and double domination stable graphs. It is worth noting that there are some results concerning Hamiltonicities of critical graph with respect to other types of domination numbers. For example, see [2, 8, 9, 13, 15, 17, 18, 21, 25, 27]. For related results in k-$\gamma_{\times 2}$-critical graphs, Thacker [20] first studied these graphs. He characterized 3-$\gamma_{\times 2}$-critical graphs and 4-$\gamma_{\times 2}$-critical graphs with maximum diameter. It is easy to see that

2-$\gamma_{\times 2}$-critical graphs are complete graphs of order at least two. When $k = 4$, Wang and Kang [22] showed that G is factor-critical if G is a connected 4-$\gamma_{\times 2}$-critical $K_{1,4}$-free graph of odd order with minimum degree two. Wang and Shan [23] showed further that if the order is even and at least six then the connected 4-$\gamma_{\times 2}$-critical $K_{1,4}$-free graph has a perfect matching except one family of graphs. Moreover, if G is a 2-connected 4-$\gamma_{\times 2}$-critical claw-free of even order with minimum degree three or G is a 3-connected 4-$\gamma_{\times 2}$-critical $K_{1,4}$-free of even order with minimum degree four, then G is bi-critical. Recently, Wang et al. [24] established that if a graph G is a 3-connected 4-$\gamma_{\times 2}$-critical claw-free graph of odd order with minimum degree at least four, then G is 3-factor-critical except one family of graphs. All the related results have not been done when $k \geq 5$. In double domination stable graphs, we introduce a new concept in k-$\gamma^{+}_{\times 2}$-stable graphs and investigate their Hamiltonian property in this paper. For k-$\gamma^{-}_{\times 2}$-stable graphs, Chellali and Haynes [5] established fundamental properties of these graphs. All our reults related to double domination are given in Chapter 5.

CHAPTER 2

Preliminaries

In this chapter, we give a number of results that we use of in establishing our works.

2.1 Hamitonicities

We begin with a result of Chvátal [7] which is a well known property of a Hamiltonian graph.

Proposition 2.1. *[7] If G is a Hamiltonian graph, then $\frac{|S|}{\omega(G-S)} \geq 1$ for every cut set $S \subseteq V(G)$.*

In the following, we introduce the technique in Ryjáček [16] so called *local completion* to study Hamiltonian properties of claw-free graphs. Let G be a claw-free graph. A vertex x in G is *eligible* if $G[N_G(x)]$ is connected and non-complete. Further, let G_x be the graph such that $V(G_x) = V(G)$ and $E(G_x) = E(G) \cup \{uv$: for a pair of non-adjacent vertices $u, v \in N_G(x)\}$. Then, we repeat this process until there is no eligible vertex in the graph. That is, we will have a finite sequence of graphs $G_0, G_1, ..., G_{n_0}$ such that $G = G_0$ and, for $1 \leq i \leq n_0$, we have $G_i = (G_{i-1})_y$ where y is an eligible vertex of G_{i-1}. The process finishes at G_{n_0} which contains no eligible vertex. Here G_{n_0} is the *closure* of G and is denoted by $cl(G)$. Brousek et al. [3] use this operation to establish the Hamiltonicities of $\{K_{1,3}, N_{s_1,s_2,s_3}\}$-free graphs. Before we state this theorem, we need to provide some classes of graphs from [3].

The Class $\mathscr{H}_1$

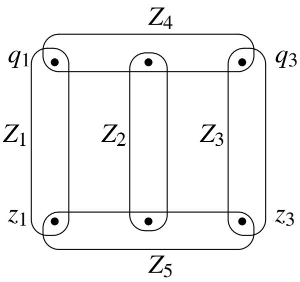
Let $Z_1, ..., Z_5$ be complete graphs of order at least three. For $1 \leq i \leq 3$, let q_i, z_i be two different vertices of Z_i. Moreover, let q'_1, q'_2, q'_3 be three different vertices of Z_4 and z'_1, z'_2, z'_3 be three different vertices of Z_5. A graph in this class is constructed from $Z_1, ..., Z_5$ by identifying q'_i with q_i and z'_i with z_i for $1 \leq i \leq 3$. A graph in this class is given Figure 2.1(a).

The Class $\mathscr{H}_2$

Let c_1, c_2, c_3, c_1 and d_1, d_2, d_3, d_1 be two disjoint triangles. We also let T_1 and T_2 be two complete graphs of order at least three and T_3 a complete graph of order at least two. Let c_i', d_i' be two different vertices of T_i for $1 \leq i \leq 2$ and let c_3', r be two different vertices of T_3. A graph in this class is obtained by identifying c_i' with c_i and d_i' with d_i for $1 \leq i \leq 2$ and identifying c_3' with c_3 and adding an edge rd_3. A graph in this class is illustrated by Figure 2.1(b).

The Class $\mathscr{H}_3$

Let $h_1, h_2, \ldots, h_6, h_1$ be a cycle of six vertices and K a complete graph of order at least three. Let s and s' be two different vertices of K. We define a graph G in the class $\mathscr{H}_3$ by adding edges $sh_1, sh_6, s'h_3, s'h_4$. A graph in this class is illustrated by Figure 2.1(c).



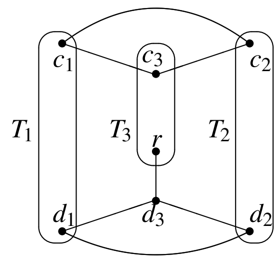


Figure 2.1(a) : The Class $\mathscr{H}_1$ **Figure 2.1(b) :** The Class $\mathscr{H}_2$

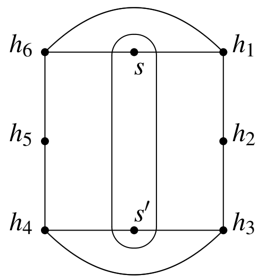


Figure 2.1(c) : The Class $\mathscr{H}_3$

Let $P = p_1, p_2, p_3, P' = p_1', p_2', p_3'$ and $P'' = p_1'', p_2'', p_3''$ be three paths of length two. The graph $P_{3,3,3}$ is constructed from P, P' and P'' by adding edges so that $\{p_1, p_1', p_1''\}$ and $\{p_3, p_3', p_3''\}$ form two complete graphs of order three. Brousek et al. [3] proved :

Theorem 2.1. *[3] Let G be a* 2*-connected* $\{K_{1,3}, N_{1,2,2}, N_{1,1,3}\}$*-free graph. Then either G is Hamiltonian, or G is isomorphic to* $P_{3,3,3}$ *or* $cl(G) \in \mathscr{H}_1 \cup \mathscr{H}_2 \cup \mathscr{H}_3$.

Recently, Xiong et al. [26] established this following theorem.

Theorem 2.2. *[26] Let G be a* 3*-connected* $\{K_{1,3}, N_{s_1,s_2,s_3}\}$*-free graph. If* $s_1 + s_2 + s_3 \leq 9$ *and* $s_i \geq 1$*, then G is Hamiltonian.*

2.2 Connected Domination

In this section, we state a number of results in connected domination that we make use of in establishing our theorems. At the end of this section, we also prove some crucial results which will be used to settle the maximum number of cut vertices of k-γ_c-critical graphs in Chapter 3. We begin with a result of Chartrand and Oellermann [4] which gives the relationship between the numbers of end blocks and cut vertices.

Lemma 2.1. *(see [4], Page 24) Let G be a connected graph with at least one cut vertex. Then G has at least two end blocks.*

In [6], Chen et al. established fundamental properties of k-γ_c-critical graphs.

Lemma 2.2. *[6] Let G be a k-γ_c-critical graph and let x and y be a pair of non-adjacent vertices of G. Further, let D_{xy} be a γ_c-set of $G+xy$. Then*

(1) $k-2 \le |D_{xy}| \le k-1$,

(2) $D_{xy} \cap \{x,y\} \ne \emptyset$ *and*

(3) if $\{x\} = \{x,y\} \cap D_{xy}$, *then* $N_G(y) \cap D_{xy} = \emptyset$.

In [13], we further observed some structure 2.of the subgraph of G (not $G+xy$) induced by D_{xy}. For completeness, we provide the proof.

Observation 2.2.1. If $\{x,y\} \subseteq D_{xy}$, then $G[D_{xy}]$ consists of 2 components and each of which contains exactly one vertex of $\{x,y\}$.

Proof. If $G[D_{xy}]$ is connected, then D_{xy} is a connected dominating set of G. It follows by Lemma 2.2(1) that $\gamma_c(G) \le k-1$, contradiction. Thus $G[D_{xy}]$ is not connected. As $(G+xy)[D_{xy}]$ is connected and xy is the only one edge which is added to G, it follows that xy is a bridge of $(G+xy)[D_{xy}]$. Therefore, $G[D_{xy}]$ has exactly 2 components and each of which contains exactly one vertex of $\{x,y\}$. This completes the proof. □

When $k \ge 3$, Ananchuen [1] established structures of k-γ_c-critical graphs with a cut vertex.

Lemma 2.3. *[1] For $k \ge 3$, let G be a k-γ_c-critical graph with a cut vertex c and let D be a connected dominating set. Then*

(1) $G-c$ *contains exactly two components,*

(2) if C_1 and C_2 are the components of $G-c$, then $G[N_{C_1}(c)]$ and $G[N_{C_2}(c)]$ are complete and

(3) $c \in D$.

In our previous work in [12], we established the diameter of k-γ_c-critical graphs.

Lemma 2.4. *[12] Let G be a k-γ_c-critical graph. Then $diam(G) \leq k$.*

2.3 Double Domination

We conclude this chapter by giving some results on double domination. Thacker [20] established some observations of this parameter.

Observation 2.3.1. [20] For $k \geq 2$, let G be a k-$\gamma_{\times 2}$-critical graph. Moreover, for a pair of non-adjacent vertices u and v of G, we let D_{uv} be a $\gamma_{\times 2}$-set of $G+uv$. Then $D_{uv} \cap \{u,v\} \neq \emptyset$.

The following proposition is a special case of a result of Thacker [20] by restricting the original result to connected graphs.

Proposition 2.2. *[20] For any connected graph G, let u and v be a pair of non-adjacent vertices of G. Then*

$$\gamma_{\times 2}(G) - 2 \leq \gamma_{\times 2}(G+uv) \leq \gamma_{\times 2}(G).$$

The following result, from [5], gives the double domination number of a graph when any edge is removed.

Observation 2.3.2. [20] For a graph G and edge $uv \in E(G)$ such that $G-uv$ have no isolated vertex, $\gamma_{\times 2}(G) \leq \gamma_{\times 2}(G-uv)$.

CHAPTER 3

Connected Domination Critical Graphs with Many Cut Vertices

This chapter, we prove that every k-γ_c-critical graphs when $k \geq 3$ has at most $k-2$ cut vertices. In Sections 3.1 and 3.2, we may need to establish related results that we use to prove our main theorem. Our main theorem of this chapter is proved in Section 3.3.

3.1 Forbidden Subgraphs

We start this section by establishing a forbidden subgraph of k-γ_c-critical graphs when $k \geq 3$ which is Lemma 3.2. We also need to prove the following lemma.

Lemma 3.1. *Let G be a k-γ_c-critical graph and let x and y be a pair of non-adjacent vertices of G such that $|D_{xy} \cap \{x,y\}| = 1$. Then, for a pair of vertices a and b in D_{xy}, we have that $N(a) \nsubseteq N[b]$.*

Proof. Suppose to the contrary that $N(a) \subseteq N[b]$ for some $a, b \in D_{xy}$.

Claim : $D_{xy} - \{a\} \succ_c a$.

As $|D_{xy} \cap \{x,y\}| = 1$, we must have $G[D_{xy}]$ is connected. Because $N(a) \subseteq N[b]$ and $b \in D_{xy}$, it follows that $G[D_{xy} - \{a\}]$ is connected. We next show that $D_{xy} - \{a\} \succ a$. As $G[D_{xy}]$ is connected, a must be adjacent to a vertex in D_{xy}. That is $D_{xy} - \{a\} \succ a$. Therefore $D_{xy} - \{a\} \succ_c a$. This settles the claim.

Since $|D_{xy} \cap \{x,y\}| = 1$, we may assume without loss of generality that $\{x\} = D_{xy} \cap \{x,y\}$. We distinguish 2 cases.

Case 1 : $a \neq x$.

Because $N(a) \subseteq N[b]$ and $b \in D_{xy}$, it follows that $D_{xy} - \{a\} \succ V(G+xy) - \{a\}$. Thus, by the claim, we have $D_{xy} - \{a\} \succ_c G + xy$. This contradicts the minimality of D_{xy}. So Case 1 cannot occur.

Case 2 : $a = x$.

As $N(a) \subseteq N[b]$, we must have $D_{xy} - \{a\} \succ V(G+xy) - \{y,a\}$. By the claim, $D_{xy} - \{a\} \succ_c V(G) - \{y\}$. Because G is connected, it follows that $N(y) \neq \emptyset$. Let $z \in N(y)$. By Lemma 2.2(3), $z \notin D_{xy}$. As $D_{xy} \succ_c G+xy$, we must have that z is adjacent to a vertex in D_{xy}. If $za \notin E(G)$, then $(D_{xy} - \{a\}) \cup \{z\} \succ_c G$. Lemma 2.2(1) implies that $|(D_{xy} - \{a\}) \cup \{z\}| \leq k-1$ contradicting the minimality of $\gamma_c(G)$. Therefore, $za \in E(G)$. As $N(a) \subseteq N[b]$, we must have $zb \in E(G)$. Since $b \in D_{xy}$, it follows that $(D_{xy} - \{a\}) \cup \{z\} \succ_c G$. Similarly, $|(D_{xy} - \{a\}) \cup \{z\}| \leq k-1$, a contradiction. So Case 2 cannot occur and this completes the proof. □

We are ready to provide the construction of a forbidden subgraph of k-γ_c-critical graphs. For a connected graph G, let X, Y, X_1 and Y_1 be disjoint vertex subsets of $V(G)$. We, further, let $Z = X \cup X_1 \cup Y \cup Y_1$ and $\overline{Z} = V(G) - Z$. The induced subgraph $G[Z]$ is called a *bad subgraph* if

(i) $x_1 \succ X \cup X_1$ for any vertex $x_1 \in X_1$,

(ii) $N[x] \subseteq X \cup X_1$ for any vertex $x \in X$,

(iii) $y_1 \succ Y \cup Y_1$ for any vertex $y_1 \in Y_1$ and

(iv) $N[y] \subseteq Y \cup Y_1$ for any vertex $y \in Y$.

Figure 3.1 illustrates our set up.

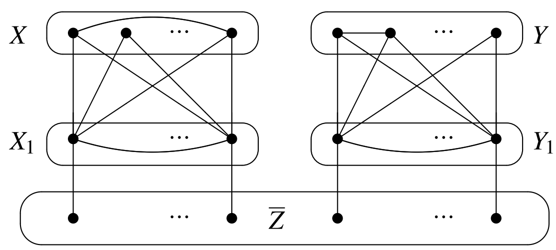


Figure 3.1 : The induced subgraph $G[Z]$

Observe that $G[X_1]$ and $G[Y_1]$ are complete subgraphs. Further, if $\overline{Z} = \emptyset$, then there exists an edge x_1y_1 where $x_1 \in X_1$and $y_1 \in Y_1$ because G is connected. Thus $\{x_1, y_1\} \succ_c G$. This implies that $\gamma_c(G) \leq 2$. Therefore, if $\gamma_c(G) \geq 3$, then $\overline{Z} \neq \emptyset$. The next lemma gives that every k-γ_c-critical graph has no bad subgraph as an induced subgraph.

Lemma 3.2. *For $k \geq 3$, let G be a k-γ_c-critical graph. Then G does not contain a bad subgraph as an induced subgraph.*

Proof. Suppose to the contrary that G contains $G[Z]$ as a bad subgraph. Let $x \in X$ and $y \in Y$. Consider $G+xy$. Lemma 2.2(2) implies that $D_{xy} \cap \{x,y\} \neq \emptyset$.

We first show that $\{x,y\} \subseteq D_{xy}$. Suppose to the contrary that $|D_{xy} \cap \{x,y\}| = 1$. Without loss of generality let $\{x\} = D_{xy} \cap \{x,y\}$. Since x is not adjacent to any vertex in Y_1, in order to dominate Y_1, $D_{xy} \cap (V(G)-X) \neq \emptyset$. Because $N[x] \subseteq X \cup X_1$, by the connectedness of $(G+xy)[D_{xy}]$, $D_{xy} \cap X_1 \neq \emptyset$. Let $x_1 \in D_{xy} \cap X_1$. Thus $N(x) \subseteq N[x_1]$ contradicting Lemma 3.1. Hence $\{x,y\} \subseteq D_{xy}$.

By Observation 2.2.1, $G[D_{xy}]$ has exactly two components H_1 and H_2 containing x and y, respectively. Let

$$U_1 = N(H_1) - V(H_1) \text{ and } U_2 = N(H_2) - V(H_2).$$

Thus $V(G) = U_1 \cup U_2 \cup V(H_1) \cup V(H_2)$ because $D_{xy} = V(H_1) \cup V(H_2)$ and $D_{xy} \succ_c G+xy$. We next establish the following claim.

Claim : For a vertex $u \in V(H_1) \cup U_1$, if $(V(H_1) \cup \{u\}) \cap X_1 \neq \emptyset$, then $V(H_1 - x) \cup \{u\} \succ_c U_1 \cup \{x\}$. Similarly, for a vertex $v \in V(H_2) \cup U_2$, if $(V(H_2) \cup \{v\}) \cap Y_1 \neq \emptyset$, then $V(H_2 - y) \cup \{v\} \succ_c U_2 \cup \{y\}$.

Suppose that there exists $x_1 \in (V(H_1) \cup \{u\}) \cap X_1$. By Property (i) of bad subgraph, $x_1 \succ X \cup X_1$. Hence, $N[x] \subseteq N[x_1]$. Clearly, $G[V(H_1) \cup \{u\}]$ is connected. Since $x_1 \in V(H_1 - x) \cup \{u\}$, it follows that $G[V(H_1 - x) \cup \{u\}]$ is connected. As $N[x] \subseteq N[x_1]$, we must have $V(H_1 - x) \cup \{u\} \succ_c x$. Thus, it remains to show that $V(H_1 - x) \cup \{u\} \succ U_1$. Let $w \in U_1$. So, w is adjacent to a vertex of H_1. If $wx \notin E(G)$, then w is adjacent to a vertex of $H_1 - x$. But, if $wx \in E(G)$, then $wx_1 \in E(G)$. These imply that w is adjacent to a vertex in $V(H_1 - x) \cup \{u\}$. So $V(H_1 - x) \cup \{u\} \succ U_1$. Therefore, $V(H_1 - x) \cup \{u\} \succ_c U_1 \cup \{x\}$. We can show that if $(V(H_2) \cup \{v\}) \cap Y_1 \neq \emptyset$, then $V(H_2 - y) \cup \{v\} \succ_c U_2 \cup \{y\}$ by the similar arguments. This settles the claim.

We note by the claim that u can be a vertex in H_1. Thus if $V(H_1) \cap X_1 \neq \emptyset$, then $V(H_1 - x) \succ_c U_1 \cup \{x\}$. Clearly $\overline{Z} \neq \emptyset$ because $k \geq 3$. To dominate $\overline{Z}$, we have $D_{xy} \cap (\overline{Z} \cup X_1 \cup Y_1) \neq \emptyset$ because $N[x] \subseteq X \cup X_1$ and $N[y] \subseteq Y \cup Y_1$. Thus, by the connectedness of H_1 and H_2, $V(H_1) \cap X_1 \neq \emptyset$ or $V(H_2) \cap Y_1 \neq \emptyset$. Suppose without loss of generality that $V(H_1) \cap X_1 \neq \emptyset$. By applying the claim, we have that

$$V(H_1 - x) \succ_c U_1 \cup \{x\}. \tag{3.1.1}$$

Case 1 : $U_1 \cap U_2 \neq \emptyset$.

Thus there is a vertex $v \in V(G) - (V(H_1) \cup V(H_2))$ such that v is adjacent to a vertex of H_1 and a vertex of H_2. That is $G[V(H_1) \cup \{v\} \cup V(H_2)]$ is connected.

We next show that $(V(H_2)\cup\{v\})\cap Y_1 \neq \emptyset$. Suppose that $(V(H_2)\cup\{v\})\cap Y_1 = \emptyset$. By the connectedness of H_2, $V(H_2)\subseteq Y$ because $y\in Y$. Moreover, $v\in Y$ because v is adjacent to a vertex of H_2. So, Property (iv) implies that $N[v]\subseteq Y\cup Y_1$. As v is adjacent to a vertex of H_1, we must have that $V(H_1)\cap(Y\cup Y_1)\neq\emptyset$. By the connectedness of H_1, $V(H_1)\cap Y_1\neq\emptyset$. Property (iii) yields that there exists a vertex of H_1 adjacent to a vertex of H_2. So H_1 and H_2 are the same component, a contradiction. Hence $(V(H_2)\cup\{v\})\cap Y_1\neq\emptyset$. By the claim, we have that

$$V(H_2-y)\cup\{v\}\succ_c U_2\cup\{y\}. \tag{3.1.2}$$

Since $V(G)=U_1\cup U_2\cup V(H_1)\cup V(H_2)$, by (3.1.1) and (3.1.2), $V(H_1-x)\cup V(H_2-y)\cup\{v\}\succ_c G$. Hence

$$(D_{xy}-\{x,y\})\cup\{v\}=V(H_1-x)\cup V(H_2-y)\cup\{v\}\succ_c G.$$

Lemma 2.2(1) yields that $|(D_{xy}-\{x,y\})\cup\{v\}|\leq k-1$ contradicting $\gamma_c(G)=k$. So Case 1 cannot occur.

Case 2 : $U_1\cap U_2=\emptyset$.

Since G is connected, there exist vertices u and v in $V(G)-(V(H_1)\cup V(H_2))$ such that $u\in U_1, v\in U_2$ and $uv\in E(G)$. Therefore $G[V(H_1)\cup\{u,v\}\cup V(H_2)]$ is connected.

We will show that $(V(H_1)\cup\{u\})\cap X_1\neq\emptyset$ and $(V(H_2)\cup\{v\})\cap Y_1\neq\emptyset$. Suppose to the contrary that $(V(H_1)\cup\{u\})\cap X_1=\emptyset$. So $V(H_1)\cup\{u\}\subseteq X$ by the connectedness of $G[V(H_1)\cup\{u\}]$. Since H_1 and H_2 are different components, by Property (i), $V(H_2)\cap X_1=\emptyset$. Thus $v\in X_1$ because $uv\in E(G)$ and $N[u]\subseteq X\cup X_1$. This implies by Property (i) that $v\succ H_1$, in particular $v\in U_1$. Thus $v\in U_1\cap U_2$. This contradicts $U_1\cap U_2=\emptyset$. Hence, $(V(H_1)\cup\{u\})\cap X_1\neq\emptyset$. By the same arguments, we have $(V(H_2)\cup\{v\})\cap Y_1\neq\emptyset$.

Hence, by the claim, we have that $V(H_1-x)\cup\{u\}\succ_c U_1\cup\{x\}$ and $V(H_2-y)\cup\{v\}\succ_c U_2\cup\{y\}$. As $V(G)=U_1\cup U_2\cup V(H_1)\cup V(H_2)$, we must have that $(D_{xy}-\{x,y\})\cup\{u,v\}\succ_c G$. Lemma 2.2(1) gives that $|(D_{xy}-\{x,y\})\cup\{u,v\}|\leq k-1$ contradicting $\gamma_c(G)=k$. So Case 2 cannot occur. Therefore G does not contain a bad subgraph as an induced subgraph. This completes the proof. □

By applying Lemma 3.2, we easily establish the maximum number of end vertices of k-γ_c-critical graphs.

Corollary 3.1.1. [19] For $k\geq 3$, every k-γ_c-critical graph has at most one end vertex.

Proof. Suppose to the contrary that G has x and y as two end vertices. Let x_1 and y_1 be the support vertices adjacent to x and y, respectively. Thus x_1 and y_1 are cut vertices.

Since $\gamma_c(G) \geq 3$, $V(G) - \{x, x_1, y, y_1\} \neq \emptyset$. Thus, Lemma 2.3(1) implies that $x_1 \neq y_1$. Choose $X_1 = \{x_1\}, Y_1 = \{y_1\}, X = \{x\}$ and $Y = \{y\}$. Clearly $G[X_1 \cup Y_1 \cup X \cup Y]$ is a bad subgraph contradicting Lemma 3.2. Hence, G has at most one end vertex and this completes the proof. □

It is worth noting that very recently Taylor and van der Merwe [19] proved Corollary 3.1.1 as well. They proved the corollary with contrapositive but did not apply the concept of a bad subgraph in their proof.

3.2 The Characterizations of Some End Blocks

In this section, we provide characterizations of some blocks of k-γ_c-critical graphs. For a connected graph G, we let

$\mathscr{A}(G)$ be the set of all cut vertices of G and

$\mathfrak{B}(G)$ be the family of all blocks of G.

When no ambiguity can occur, we use $\mathfrak{B}$ to denote $\mathfrak{B}(G)$. We first show that for a connected graph G and a pair of non-adjacent vertices x and y of G, $\mathscr{A}(G) = \mathscr{A}(G + xy)$ if x and y are in the same block of G.

Lemma 3.3. *For a connected graph G, let B be a block of G and $x, y \in V(B)$ such that $xy \notin E(G)$. Then $\mathscr{A}(G) = \mathscr{A}(G + xy)$.*

Proof. Since G is a subgraph of $G + xy$, $\mathscr{A}(G + xy) \subseteq \mathscr{A}(G)$. Suppose there exists c such that $c \in \mathscr{A}(G)$ but $c \notin \mathscr{A}(G + xy)$. Thus $(G + xy) - c$ is connected. Let C be the component of $G - c$ containing vertices of $V(B) - \{c\}$ and C' be a component of $G - c$ which is not C. Further, let $a \in N_{C'}(c)$ and $b \in N_C(c)$. Since c is a cut vertex of G, there is only one path a, c, b from a to b. But c is not a cut vertex in $G + xy$. This implies that $G - c$ has a path $P = p_1, p_2, ..., x, y, ..., p_r$ from b to a where $b = p_1, a = p_r, x = p_i$ and $y = p_{i+1}$ for some $1 \leq i \leq r - 1$ and $r \geq 2$. We see that P must contain an edge xy and $c \notin \{p_1, p_2, ..., p_r\}$. Since C and C' are the two different components of $G - c$, by the connectedness of the path P, $\{p_1, p_2, ..., p_i\} \subseteq V(C)$ and $\{p_{i+1}, ..., p_r\} \subseteq V(C')$. So $x \in V(C)$ and $y \in V(C')$ contradicting x and y are in the same block. Therefore $\mathscr{A}(G) \subseteq \mathscr{A}(G + xy)$ and thus, $\mathscr{A}(G) = \mathscr{A}(G + xy)$. This completes the proof. □

For a k-γ_c-critical graph G with a cut vertex, let B be an end block of G containing non-adjacent vertices x and y. Clearly, $V(B + xy) = V(B)$.

Lemma 3.4. *Let B be a block of G and $x, y \in V(B)$ such that $xy \notin E(G)$. Then $D \cap \mathscr{A} = D_{xy} \cap \mathscr{A}$, in particular, $D \cap \mathscr{A}(B') = D_{xy} \cap \mathscr{A}(B')$ for all $B' \in \mathfrak{B}(G + uv)$.*

Proof. We first show that $D \cap \mathscr{A} \subseteq D_{xy} \cap \mathscr{A}$. Let $c \in D \cap \mathscr{A}$. By Lemma 3.3, $c \in \mathscr{A}(G+xy)$. By the connectedness of $(G+xy)[D_{xy}]$, $c \in D_{xy}$. Thus $D \cap \mathscr{A} \subseteq D_{xy} \cap \mathscr{A}$. We now show that $D_{xy} \cap \mathscr{A} \subseteq D \cap \mathscr{A}$. Let $c \in D_{xy} \cap \mathscr{A}$. That is $c \in \mathscr{A}$. Lemma 2.3(3) yields that $c \in D$. So $c \in D \cap \mathscr{A}$ and thus, $D_{xy} \cap \mathscr{A} \subseteq D \cap \mathscr{A}$, as required.

In view of Lemma 3.3, $V(B') \cap \mathscr{A}(G+xy) = V(B') \cap \mathscr{A}$ for all $B' \in \mathfrak{B}(G+uv)$. Because $D \cap \mathscr{A} = D_{xy} \cap \mathscr{A}$, it follows that

$$D \cap \mathscr{A}(B') = D \cap \mathscr{A} \cap V(B') = D_{xy} \cap \mathscr{A} \cap V(B') = D_{xy} \cap \mathscr{A}(B').$$

This completes the proof. □

As we see in Lemma 3.3, if the two end vertices of the adding edge are in the same block, then the set of all cut vertices is still the same. For non-adjacent vertices x and y of the block B, the following lemma gives the number of vertices of a γ_c-set of $G+xy$ in B.

Lemma 3.5. *For all $x, y \in V(B)$ such that $xy \notin E(G)$, $|D_{xy} \cap V(B)| < |D \cap V(B)|$.*

Proof. We first establish the following claim.

Claim : For all block B' which is not B, $|D \cap V(B')| \leq |D_{xy} \cap V(B')|$.

Suppose to the contrary that $|D \cap V(B')| > |D_{xy} \cap V(B')|$. Lemma 3.4 gives that $D \cap \mathscr{A}(B') = D_{xy} \cap \mathscr{A}(B')$. Because $x, y \notin V(B')$, $G[(D - V(B')) \cup (D_{xy} \cap V(B'))]$ is connected. Moreover, $(D - V(B')) \cup (D_{xy} \cap V(B')) \succ_c G$. This implies that

$$\begin{aligned} k = |D| = |(D - V(B')) \cup (D \cap V(B'))| &= |D - V(B')| + |D \cap V(B')| \\ &> |D - V(B')| + |D_{xy} \cap V(B')| \\ &= |(D - V(B')) \cup (D_{xy} \cap V(B'))|, \end{aligned}$$

contradicting the minimality of D. Thus establishing the claim.

We now prove Lemma 3.5. Suppose to the contrary that $|D_{xy} \cap V(B)| \geq |D \cap V(B)|$. Lemma 3.4 yields that $D \cap \mathscr{A} = D_{xy} \cap \mathscr{A}$. Clearly $D = \cup_{\tilde{B} \in \mathfrak{B}}(D \cap V(\tilde{B}))$ and $D_{xy} = \cup_{\tilde{B} \in \mathfrak{B}(G+xy)}(D_{xy} \cap V(\tilde{B}))$. Lemma 2.3(1) yields, further, that each cut vertex is contained in exactly two blocks. Thus each cut vertex is counted twice in $\sum_{\tilde{B} \in \mathfrak{B}} |(D \cap V(\tilde{B}))|$ and $\sum_{\tilde{B} \in \mathfrak{B}(G+xy)} |(D_{xy} \cap V(\tilde{B}))|$. Therefore, $|D| = \Sigma_{\tilde{B} \in \mathfrak{B}} |D \cap V(\tilde{B})| - |\mathscr{A}|$ and $\Sigma_{\tilde{B} \in \mathfrak{B}(G+xy)} |D_{xy} \cap V(\tilde{B})| - |\mathscr{A}(G+xy)| = |D_{xy}|$. We note by Lemma 3.3 that $|\mathscr{A}| = |\mathscr{A}(G+xy)|$. By the claim and the assumption that $|D_{xy} \cap V(B)| \geq |D \cap V(B)|$, we have

$$\begin{aligned}k = |D| &= \Sigma_{\tilde{B}\in\mathfrak{B}}|D\cap V(\tilde{B})| - |\mathscr{A}|\\ &= |D\cap V(B)| + \Sigma_{\tilde{B}\in\mathfrak{B}-\{B\}}|D\cap V(\tilde{B})| - |\mathscr{A}|\\ &\leq |D_{xy}\cap V(B)| + \Sigma_{\tilde{B}\in\mathfrak{B}-\{B\}}|D\cap V(\tilde{B})| - |\mathscr{A}| \ \ (by\ the\ assumption)\\ &\leq |D_{xy}\cap V(B)| + \Sigma_{\tilde{B}\in\mathfrak{B}(G+xy)-\{B\}}|D_{xy}\cap V(\tilde{B})| - |\mathscr{A}| \ \ (by\ the\ claim)\\ &= \Sigma_{\tilde{B}\in\mathfrak{B}(G+xy)}|D_{xy}\cap V(\tilde{B})| - |\mathscr{A}(G+xy)| = |D_{xy}|.\end{aligned}$$

This contradicts Lemma 2.2(1). Thus $|D_{xy}\cap V(B)| < |D\cap V(B)|$ and this completes the proof. □

Corollary 3.2.1. For all block B of G and $x,y\in V(B)$ such that $xy\notin E(G)$, $|(D_{xy}\cap V(B)) - \mathscr{A}| < |(D\cap V(B)) - \mathscr{A}|$.

Proof. In view of Lemma 3.4, $D\cap V(B)\cap\mathscr{A} = D_{xy}\cap V(B)\cap\mathscr{A}$. Lemma 3.5 then implies that

$$\begin{aligned}|(D_{xy}\cap V(B)) - \mathscr{A}| &= |D_{xy}\cap V(B)| - |D_{xy}\cap V(B)\cap\mathscr{A}|\\ &< |D\cap V(B)| - |D\cap V(B)\cap\mathscr{A}|\\ &= |(D\cap V(B)) - \mathscr{A}|\end{aligned}$$

and this completes the proof. □

We now introduce four classes of graphs such that some graph in these classes is an end block of a k-γ_c-critical graph. For vertices c, z_1 and z_2, we let

$\mathscr{B}_0 = \{c\vee K_{t_1}$: for an integer $t_1\geq 1\}$,

$\mathscr{B}_1 = \{c\vee K_{t_2}\vee z_1$: for an integer $t_2\geq 1\}$ and

$\mathscr{B}_{2,1} = \{c\vee K_{t_3}\vee K_{t_4}\vee z_2$: for integers $t_3,t_4\geq 1\}$.

For a block $B\in\mathscr{B}_0\cup\mathscr{B}_1\cup\mathscr{B}_{2,1}$, we call c, the *head* of B. Before we construct the next class, it is worth to introduce graph T which occurs in the characterization of k-γ_c-critical graphs with a maximum number of cut vertices. For positive integers $l\geq 2, r$ and n_i, we let $\mathscr{S} = \cup_{i=1}^{l}K_{1,n_i}$ and

$$T = \begin{cases}\mathscr{S} & or\\ \mathscr{S}\cup\overline{K_r}.\end{cases}$$

Then, for $1\leq i\leq l$, we let $s_0^i, s_1^i, s_2^i, ..., s_{n_i}^i$ be the vertices of a star K_{1,n_i} centered at s_0^i. We, further, let $S = \cup_{i=1}^{l}\{s_1^i, s_2^i, ..., s_{n_i}^i\}$ and $S' = \cup_{i=1}^{l}\{s_0^i\}$, moreover, let $S'' = V(\overline{K_r})$ if

$T = \mathscr{S} \cup \overline{K_r}$ and $S'' = \emptyset$ if $T = \mathscr{S}$. We note that

$$\overline{T} = \begin{cases} \overline{\mathscr{S}} & or \\ \overline{\mathscr{S}} \vee K_r. \end{cases}$$

That is, $\overline{T}$ can be obtained by removing the edges in the stars of $\mathscr{S}$ from a complete graph on $S \cup S' \cup S''$. Throughout this paper, we are, in fact, using the complement of T. We are ready to define the next class. Recall that, for graphs G_1 and G_2 such that G_2 has H as a subgraph, the join $G_1 \vee_H G_2$ is the graph constructed from the disjoint union of G_1 and G_2 by joining each vertex in G_1 to each vertex in H with an edge.

$\mathscr{B}_{2,2} = \{c \vee_{\overline{T}[S]} \overline{T}$: for positive integers $l \geq 2, r$ and $n_i\}$.

We note by the construction that, in $\overline{T}$, every vertex in S is adjacent to exactly $|S' \cup S''| - 1$ vertices in $S' \cup S''$. A graph in this class is illustrated by Figure 3.2. According to the figure, an *oval* denotes a complete subgraph, *double lines* between subgraphs denote joining every vertex of one subgraph to every vertex of the other subgraph and a *dash line* denotes a removed edge.

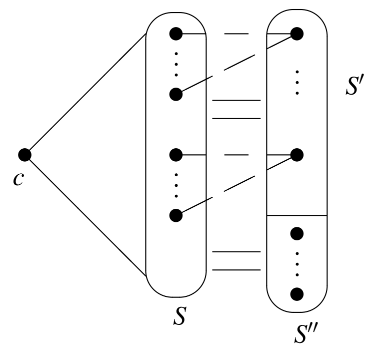


Figure 3.2 : A graph G in the class $\mathscr{B}_{2,2}$

It is worth noting that, for an end block B of a k-γ_c-critical graph having D as a γ_c-set, the number of vertices in $D \cap V(B)$ can be as large as k. We will give an example by using the graph $\overline{T}$. For an integer $k \geq 5$, let $K_{n_1}, ..., K_{n_{k-3}}$ be $k-3$ copies of complete graphs with $n_1, ..., n_{k-3} \geq 2$ and let a_1 and a_2 be two isolated vertices. It is not difficult to see that the graph

$$a_1 \vee a_2 \vee K_{n_1} \vee \cdots \vee K_{n_{k-3}} \vee_{\overline{T}[S]} \overline{T}$$

is a k-γ_c-critical graph having $R = a_2 \vee K_{n_1} \vee \cdots \vee K_{n_{k-3}} \vee_{\overline{T}[S]} \overline{T}$ as an end block. Clearly, $|D \cap V(R)| = k$.

In the following, we characterize an end block B such that $|D \cap V(B)| \leq 3$. Let c be the cut vertex of G in B and H be the component of $G-c$ such that $G[V(H) \cup \{c\}] = B$. We further let

$W = N_H(c)$,

$W' = \{w' \in V(H) - W : w'w \in E(G)$ for some $w \in W\}$ and

$W'' = V(H) - (W \cup W')$.

Note that W' or W'' can be empty. Since $c \in V(B)$, we have that $|D \cap V(H)| = i$ if and only if $|D \cap V(B)| = i+1$ for all $i \geq 0$. Thus, $|D \cap V(B)| \geq 1$.

Lemma 3.6. *Let G be a k-γ_c-critical graph with a γ_c-set D and let B be an end block of G. If $|D \cap V(B)| = 1$, then $B \in \mathscr{B}_0$.*

Proof. In view of Lemma 2.3(2), $G[W]$ is complete. Lemma 2.3(3) gives, further, that $D \cap V(B) = \{c\}$. Since $D \succ B$ and $|(D \cap V(B)) - \{c\}| = 0$, it follows that $W' \cup W'' = \emptyset$ and $c \succ W$. So $B \in \mathscr{B}_0$. This completes the proof. □

Lemma 3.7. *Let G be a k-γ_c-critical graph with a γ_c-set D and let B be an end block of G. If $|D \cap V(B)| = 2$, then $B \in \mathscr{B}_1$.*

Proof. Let $\{y\} = (D \cap V(B)) - \{c\}$. By the connectedness of $G[D]$, $y \in W$. Thus $W'' = \emptyset$ and $V(H) = W \cup W'$. Suppose that there exist $u, v \in V(H)$ such that $uv \notin V(G)$. Consider $G+uv$. Lemma 2.2(2) gives that $D_{uv} \cap \{u, v\} \neq \emptyset$. Lemma 3.3 gives also that $c \in D_{uv}$. Hence, $|D_{uv} \cap V(B+uv)| \geq 2$ contradicting Lemma 3.5. Thus $G[W \cup W']$ is complete. Let $z_1 \in W'$. Consider $G+cz_1$. Since $|D \cap V(B)| = 2$, by Lemma 3.5, $|D_{cz_1} \cap V(B+cz_1)| \leq 1$. Lemmas 2.3(3) and 3.3 yield that $c \in D_{cz_1}$. So $|D_{cz_1} \cap V(H)| = 0$. This implies that $c \succ B+cz_1$. Since $\{z_1\} = N_{G+cz_1}(c) \cap W'$, $W' = \{z_1\}$. So $B \in \mathscr{B}_1$ and this completes the proof. □

Lemma 3.8. *Let G be a k-γ_c-critical graph with a γ_c-set D and let B be an end block of G. Suppose that $|D \cap V(B)| = 3$. Then $B \in \mathscr{B}_{2,1}$ if $W'' \neq \emptyset$ and $B \in \mathscr{B}_{2,2}$ if $W'' = \emptyset$. Consequently, $B \in \mathscr{B}_{2,1} \cup \mathscr{B}_{2,2}$.*

Proof. Suppose that $|D \cap V(B)| = 3$. Lemma 2.3(2) implies that $G[W]$ is complete. We first establish the following claim.

Claim : For any non-adjacent vertices $u, v \in W \cup W' \cup W''$, we have $c \in D_{uv} \cap V(B+uv)$ and $|D_{uv} \cap W \cap \{u, v\}| = 1$.

Lemma 3.5 implies that $|D_{uv} \cap V(B+uv)| \leq 2$. In view of Lemmas 2.3(3) and

3.3, $c \in D_{uv} \cap V(B+uv)$. Thus $|D_{uv} \cap \{u,v\}| \le 1$. Lemma 2.2(2) then gives that $|D_{uv} \cap \{u,v\}| = 1$. So $|D_{uv} \cap W \cap \{u,v\}| = 1$ because $(G+uv)[D_{uv}]$ is connected. This settles the claim.

Suppose there exist $u,v \in W' \cup W''$ such that $uv \notin E(G)$. Consider $G+uv$. By the claim $|D_{uv} \cap W \cap \{u,v\}| = 1$ contradicting $W \cap \{u,v\} = \emptyset$. Thus $G[W' \cup W'']$ is complete.

We first consider the case when $W'' \neq \emptyset$. Let $w \in W$ and $z_2 \in W''$. Consider $G+wz_2$. By the claim, $D_{wz_2} \cap V(B+wz_2) = \{c,w\}$. Since $\{z_2\} = W'' \cap N_{G+wz_2}(w)$, it follows that $W'' = \{z_2\}$. Suppose there exists $w' \in W'$ such that $ww' \notin E(G)$. Consider $G+ww'$. By the claim, $D_{ww'} \cap V(B+ww') = \{c,w\}$. Thus $D_{ww'}$ does not dominate z_2, a contradiction. Therefore $G[W \cup W']$ is complete and $B \in \mathscr{B}_{2,1}$.

We finally consider the case when $W'' = \emptyset$. We will show that, for all $w \in W$, $|N_{W'}(w)| = |W'| - 1$. If $w \succ W'$, then $(D - V(H)) \cup \{w\} \succ_c G$. But $|(D - V(H)) \cup \{w\}| = k-1$ contradicting $\gamma_c(G) = k$. Thus $|N_{W'}(w)| \le |W'| - 1$. If w is not adjacent to x,y in W', then consider $G+wx$. By the claim, $D_{wx} \cap V(B+wx) = \{c,w\}$. Clearly D_{wx} does not dominate y, a contradiction. Hence, $|N_{W'}(w)| = |W'| - 1$ for all $w \in W$. We now have that $G[W \cup W']$ is the complement of disjoint union of isolated vertices in W' and stars whose centers are in W' and all of end vertices are in W. It remains to show that there are at least two stars in $\overline{G}[W \cup W']$. Suppose to the contrary that, in $\overline{G}[W \cup W']$, there is exactly one star centered at w'. Because $|N_{W'}(w)| = |W'| - 1$ for all $w \in W$, w' is not adjacent to any vertex in W. So $w' \in W''$ contradicting $W'' = \emptyset$. Hence, there are at least two stars in $\overline{G}[W \cup W']$. This completes the proof. □

3.3 The Upper Bound of The Number of Cut Vertices

In this section, we establish the maximum number of cut vertices of k-γ_c-critical graphs. In view of Observation 1.2.1, it suffices to restrict our attention to the case $k \ge 3$. We begin this section by showing that G does not have two end blocks in $\mathscr{B}_0 \cup \mathscr{B}_1$.

Lemma 3.9. *For $k \ge 3$, let G be a k-γ_c-critical graph. Then G contains at most one end block B such that $B \in \mathscr{B}_0 \cup \mathscr{B}_1$.*

Proof. Suppose to the contrary that there exist two different end blocks U and R which are, respectively, in the classes $\mathscr{B}_i$ and $\mathscr{B}_j$ where $\{i,j\} \subseteq \{0,1\}$. Let u be the cut vertex of G in U. If $U \in \mathscr{B}_0$, then $U = u \vee K_{t_1}$ for some integer $t_1 \ge 1$. If $U \in \mathscr{B}_1$, then there exist an integer $t_2 \geq 1$ and a vertex z_1 of U such that $U = u \vee K_{t_2} \vee z_1$. Then, we choose

$$X_1 = \begin{cases} \{u\} & if \;\; U \in \mathscr{B}_0, \\ V(K_{t_2}) & if \;\; U \in \mathscr{B}_1, \end{cases}$$

and we choose

$$X = \begin{cases} V(K_{t_1}) & if \;\; U \in \mathscr{B}_0 \;\; and \\ \{z_1\} & if \;\; U \in \mathscr{B}_1. \end{cases}$$

Clearly, U contains X and X_1 which satisfy the Properties (i) and (ii) respectively.

We now consider R. Let r be the cut vertex of G in R. If $R \in \mathscr{B}_0$, then $R = r \vee K_{t_1'}$ for some integer $t_1' \geq 1$. But, if $R \in \mathscr{B}_1$, then there exist an integer $t_2' \geq 1$ and a vertex w_1 of R such that $R = r \vee K_{t_2'} \vee w_1$. Then, we choose

$$Y_1 = \begin{cases} \{r\} & if \;\; R \in \mathscr{B}_0, \\ V(K_{t_2'}) & if \;\; R \in \mathscr{B}_1, \end{cases}$$

and we choose

$$Y = \begin{cases} V(K_{t_1'}) & if \;\; R \in \mathscr{B}_0 \;\; and \\ \{w_1\} & if \;\; R \in \mathscr{B}_1. \end{cases}$$

Clearly, R contains Y and Y_1 which satisfy the Properties (i) and (ii) respectively.

We observe that X, Y and Y_1 are pairwise disjoint because U and R are different blocks. Suppose that $Y_1 \cap X_1 \neq \emptyset$. By the choice of X_1 and Y_1, if $X_1 = V(K_{t_2})$ or $Y_1 = V(K_{t_2'})$, then $Y_1 \cap X_1 = \emptyset$ because U and R are different end blocks, contradicting the assumption that $Y_1 \cap X_1 \neq \emptyset$. Hence, $X_1 = \{u\}$ and $Y_1 = \{r\}$. This implies that $u = r$, moreover, U and R are both in $\mathscr{B}_0$. Thus $u \succ U$ and $u \succ R$. Lemma 2.3(1) yields that $G - u$ has $U - u$ and $R - u$ as the two components. We have that $G = K_{t_1} \vee u \vee K_{t_1'}$. Clearly, $u \succ_c G$ contradicting $\gamma_c(G) \geq 3$. Hence, $Y_1 \cap X_1 = \emptyset$. So, G contains a bad subgraph contradicting Lemma 3.2. This completes the proof. □

In the following, for a block B of G, we let

$\mathscr{A}(B) = V(B) \cap \mathscr{A}(G)$.

We also let

$\zeta(G) = |\mathscr{A}(G)|$, $\zeta(B) = |\mathscr{A}(B)|$ and

$\zeta_0(G) = max\{\zeta(B) : B$ is a block of $G\}$.

When no ambiguity can occur, we abbreviate $\zeta_0(G)$ to ζ_0. Clearly, $\zeta_0 \leq \zeta(G)$. In the following lemma, we establish the existence of ζ_0 end blocks.

Lemma 3.10. *For any k-γ_c-critical graph G, let B_0 be a block of G containing ζ_0 cut vertices $c_1, c_2, ..., c_{\zeta_0}$. Then there exist mutually disjoint end blocks $B_1, B_2, .., B_{\zeta_0}$.*

Proof. In view of Lemma 2.3(1), $G - c_i$ has only two components for $1 \leq i \leq \zeta_0$. Let C_i be the component of $G - c_i$ that does not contain any vertex of B_0. If graph $G[\{c_i\} \cup V(C_i)]$ does not contain any cut vertex, then $G[\{c_i\} \cup V(C_i)]$ is an end block and we let $B_i = G[\{c_i\} \cup V(C_i)]$. If graph $G[\{c_i\} \cup V(C_i)]$ contains a cut vertex, then, by Lemma 2.1, $G[\{c_i\} \cup V(C_i)]$ has at least two end blocks. Therefore, at least one end block of $G[\{c_i\} \cup V(C_i)]$ does not contain c_i and we let B_i be this end block. In both cases of the choice, B_i is an end block of G. Obviously, $B_1, B_2, .., B_{\zeta_0}$ are mutually disjoint and this completes the proof. □

Lemma 3.11. *For $k \geq 3$, let G be a k-γ_c-critical graph with a γ_c-set D and let $B_1, B_2, ...,$ B_{ζ_0} be the end blocks of G from Lemma 3.10. Moreover, for $1 \leq i \leq \zeta_0$, we let $x_i \in \mathscr{A}(G) \cap V(B_i)$. Then at least $\zeta_0 - 1$ of the end blocks $B_1, B_2, ..., B_{\zeta_0}$ satisfy $|(D \cap V(B_i)) - \{x_i\}| \geq 2$.*

Proof. Lemma 3.9 gives that at least $\zeta_0 - 1$ blocks of $\{B_i | 1 \leq i \leq \zeta_0\}$ are not in $\mathscr{B}_j$ where $j \in \{0, 1\}$. Without loss of generality let $B_1, B_2, ..., B_{\zeta_0 - 1}$ be such blocks. Hence

$$|(D \cap V(B_i)) - \{x_i\}| \geq 2$$

for $1 \leq i \leq \zeta_0 - 1$ and this completes the proof. □

We next let $\overline{\mathscr{A}} = \mathscr{A}(G) - \mathscr{A}(B_0)$ and $\overline{\zeta} = |\overline{\mathscr{A}}|$. That is, $\overline{\mathscr{A}}$ is the set of cut vertices which are not in B_0. Clearly,

$$\zeta(G) = \overline{\zeta} + \zeta_0. \tag{3.3.1}$$

Recall that, for $1 \leq i \leq \zeta_0$, C_i is the component of $G - c_i$ which does not contain any vertex of B_0. We also let

$$j_0 = min\{|D \cap V(C_i)| : \text{for all } 1 \leq i \leq \zeta_0\}.$$

The following theorem gives the relationship of $\zeta_0, \overline{\zeta}, j_0$ and k.

Theorem 3.1. *For $k \geq 3$, let G be a k-γ_c-critical graph. Then $3\zeta_0 - 2 + \overline{\zeta} + j_0 \leq k$.*

Proof. Lemma 2.3(3) yields that $\mathscr{A}(G) \subseteq D$. For each end block B_i of G which is a consequent of Lemma 3.10, $1 \leq i \leq \zeta_0$, let $x_i \in \mathscr{A}(G) \cap V(B_i)$. Clearly $(D \cap V(B_1)) - \{x_1\}, (D \cap V(B_2)) - \{x_2\}, ..., (D \cap V(B_{\zeta_0})) - \{x_{\zeta_0}\}$ and $\mathscr{A}(G)$ are pairwise disjoint. These imply that

$$\Sigma_{i=1}^{\zeta_0}|(D\cap V(B_i))-\{x_i\}|+\zeta(G)\leq k. \tag{3.3.2}$$

In view of Lemma 3.11, at least ζ_0-1 end blocks of $B_1,B_2,...,B_{\zeta_0}$ are not in $\mathscr{B}_0\cup\mathscr{B}_1$. Without loss of generality let $B_1,B_2,...,B_{\zeta_0-1}$ be such blocks. So $2\leq|(D\cap V(B_i))-\{x_i\}|$ for $1\leq i\leq\zeta_0-1$. Therefore

$$2(\zeta_0-1)\leq\Sigma_{i=1}^{\zeta_0-1}|(D\cap V(B_i))-\{x_i\}|. \tag{3.3.3}$$

By the minimality of j_0,

$$0\leq j_0\leq|(D\cap V(B_{\zeta_0}))-\{x_{\zeta_0}\}|. \tag{3.3.4}$$

Therefore

$$\begin{aligned}3\zeta_0-2+j_0+\overline{\zeta}&=2(\zeta_0-1)+j_0+\overline{\zeta}+\zeta_0\\&\leq\Sigma_{i=1}^{\zeta_0-1}|(D\cap V(B_i))-\{x_i\}|+j_0+\zeta(G)\ \ (by\ \ (3.3.1)\ \ and\ \ (3.3.3))\\&\leq\Sigma_{i=1}^{\zeta_0}|(D\cap V(B_i))-\{x_i\}|+\zeta(G)\ \ (by\ \ (3.3.4))\\&\leq k\ \ (by\ \ (3.3.2)),\end{aligned}$$

as required. □

Theorem 3.1 implies the following corollary.

Corollary 3.3.1. For $k\geq3$, let G be a k-γ_c-critical graph. Then $\zeta_0\leq\lfloor\frac{k+2}{3}\rfloor$.

Proof. Theorem 3.1 implies that $3\zeta_0\leq k+2-\overline{\zeta}-j_0$. As $\overline{\zeta},j_0\geq0$, we must have that

$$\zeta_0\leq\lfloor\frac{k+2}{3}\rfloor$$

and this completes the proof. □

Note that Theorem 3.1 together with $\zeta(G)=\overline{\zeta}+\zeta_0$ give :

$$2\zeta_0\leq k-\zeta(G)-j_0+2. \tag{3.3.5}$$

We are now ready to establish our first main theorem.

Theorem 3.2. *For $k\geq5$, let G be a k-γ_c-critical graph with $\zeta(G)$ cut vertices. Then $\zeta(G)\leq k-2$.*

Proof. Suppose to the contrary that $|\mathscr{A}(G)| > k-2$. Lemma 2.3(3) gives that $|\mathscr{A}(G)| \leq k$. Thus either $|\mathscr{A}(G)| = k$ or $|\mathscr{A}(G)| = k-1$, in particular, $k - \zeta(G) \leq 1$. This implies by Equation 3.3.5 that

$$2\zeta_0 \leq k - \zeta(G) - j_0 + 2 \leq 3.$$

Therefore

$$\zeta_0 \leq 1.$$

If $\zeta(G) \geq 2$, then we always have a block containing more than one cut vertex. Thus $\zeta_0 \geq 2$, a contradiction. Therefore $\zeta(G) \leq 1$. As $k - \zeta(G) \leq 1$, we must have that

$$k \leq 2$$

contradicting $k \geq 3$. Hence $\zeta(G) \leq k-2$ and this completes the proof. □

3.4 Discussion

In this section, we discuss the related result on an another type of domination critical graphs. For a graph G, a vertex subset D of G is a *total dominating set* of G if every vertex of G is adjacent to a vertex in D. The minimum cardinality of a total dominating set of G is called the *total domination number* of G and is denoted by $\gamma_t(G)$. A graph G is said to be *k-γ_t-critical* if $\gamma_t(G) = k$ and $\gamma_t(G+uv) < k$ for any pair of non-adjacent vertices u and v of G. For $k = 3$, it was pointed out by Ananchuen in [1] that a graph G is 3-γ_t-critical if and only if G is 3-γ_c-critical. In [14], the authors established the similar result when $k = 4$. Therefore:

Theorem 3.3. *([1] and [14]) For $k \in \{3,4\}$, a connected graph G is k-γ_t-critical if and only if G is k-γ_c-critical.*

For related results on k-γ_t-critical graphs, Hattingh et al. [10] established the upper bound of the number of end vertices of k-γ_t-critical graphs. They proved that:

Theorem 3.4. *[10] For $k \geq 5$, every k-γ_t-critical graph has at most $k-2$ end vertices.*

They, further, established the existence of k-γ_t-critical graphs with prescribe end vertices according to the bound from Theorem 3.4.

Theorem 3.5. *[10] For integers $k \geq 3$ and $0 \leq h \leq k-2$ except only the case when $k = 4$ and $h = 2$, there exists a k-γ_t-critical graph with h end vertices.*

Hence, by Corollary 3.1.1 and Theorem 3.3, we can conclude that there is no 4-γ_t-critical graph with two end vertices. This fulfills Theorem 3.5 that :

Corollary 3.4.1. For integers $k \geq 3$ and $0 \leq h \leq k-2$, there exists a k-γ_t-critical graph with h end vertices if and only if $k \neq 4$ or $h \neq 2$.

CHAPTER 4

The Characterizations

In this chapter, our main results are the characterizations of k-γ_c-critical graphs when $\zeta(G) \in \{k-3, k-3\}$. When there is no danger of confusion, we write ζ for $\zeta(G)$.

4.1 The Critical Graphs with $\zeta(G) = k-2$

We first give the construction of k-γ_c-critical graphs with $k-2$ cut vertices.

The class $\mathscr{F}$

Let B be a graph in the class $\mathscr{B}_{2,2}$ containing c, S, S' and S'' which are defined by the same as in $\mathscr{B}_{2,2}$. We, further, let $P_{k-2} = x_1, x_2, ..., x_{k-2}$ be a path of order $k-2$. A graph G in the class $\mathscr{F}$ is constructed from the graphs B and P_{k-2} by joining x_{k-2} to c. A graph G in the class $\mathscr{F}$ is illustrated by Figure 4.1.

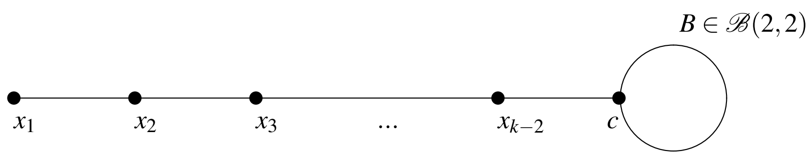


Figure 4.1 : A graph G in the class $\mathscr{F}$

Lemma 4.1. *Let G be a graph in the class $\mathscr{F}$, then G is a k-γ_c-critical graph with $k-2$ cut vertices.*

Proof. Clearly G has $x_2, x_3, ..., x_{k-2}$ and c as $k-2$ cut vertices. We observe that $\{x_2, x_3, ..., x_{k-2}, c, s_1^1, s_0^1\} \succ_c G$. Therefore $\gamma_c(G) \leq k$.

Let D be a γ_c-set of G. If $x_1 \notin D$, then $x_2 \in D$ to dominate x_1. If $x_1 \in D$, then $x_2 \in D$ because $G[D]$ is connected. In both cases, $x_2 \in D$.

We consider the case $D \cap S'' \neq \emptyset$. As $x_2 \in D$, by the connectedness of $G[D]$, we must have $\{x_3, x_4, ..., x_{k-2}, c, y\} \subseteq D$ where $y \in D \cap S'$. Thus $\gamma_c(G) = |D| \geq k$ and thus $\gamma_c(G) = k$.

We now consider the case $D \cap S'' = \emptyset$. To dominate B, $|D \cap (S \cup S')| \geq 2$. Similarly, by the connectedness of $G[D]$, we have $\{x_3, x_4, ..., x_{k-2}, c\} \subseteq D$ and thus $\gamma_c(G) \geq k$. Therefore $\gamma_c(G) = k$.

We establish the criticality. Let u and v be two non-adjacent vertices of G and $S_1 = S \cup S' \cup S''$. We first consider the case $\{u, v\} \subseteq S_1$. Thus $\{u, v\} = \{s^i_j, s^i_0\}$ for some $i \in \{1, 2, ..., |\mathscr{S}|\}$ and $j \in \{1, 2, ..., n_i\}$. Clearly $\{x_2, x_3, ..., c, s^i_j\} \succ_c G + uv$ and $\gamma_c(G + uv) \leq k-1$.

We now consider the case $|\{u, v\} \cap S_1| = 1$. If $\{u, v\} = \{c, s\}$ for some $s \in S_1$, then $s \notin S'$ and, clearly, $\{c, s\} \succ S_1$. Thus $\{x_2, x_3, ..., c, s_1\} \succ_c G + uv$. Therefore $\gamma_c(G + uv) \leq k-1$. Let $v \in S_1$. Since $|\mathscr{S}| \geq 2$, there exists $v' \in S' - \{v\}$ such that $\{v, v'\} \succ_c S_1$. Suppose that $u \in \{x_2, x_3, ..., x_{k-2}\}$. Thus $\{x_2, x_3, ..., u, ..., x_{k-2}, v, v'\} \succ_c G + uv$. Hence $\gamma_c(G + uv) \leq k-1$. If $u = x_1$, then $\{x_3, x_4, ..., u, ..., x_{k-2}, v, v'\} \succ_c G + uv$ and thus, $\gamma_c(G + uv) \leq k-1$.

We finally consider the case $|\{u, v\} \cap S_1| = 0$. Therefore $\{u, v\} \subseteq \{x_1, x_2, ..., x_{k-2}, c\}$. We may let $c = x_{k-1}$. Thus $u = x_i$ and $v = x_j$ for some $i, j \in \{1, 2, ..., k-1\}$. Without loss of generality let $i < j$. Clearly $i + 2 \leq j$. Hence

$$\{x_2, x_3, ..., x_i, x_{i+2}, x_{i+3}, ..., x_j, ..., x_{k-1}, s^l_1, s^{l'}_0\} \succ_c G + uv$$

where $1 \leq l \neq l' \leq |\mathscr{S}|$. So $\gamma_c(G + uv) \leq k-1$. Thus G is a k-γ_c-critical graph and this completes the proof. □

Let

$\mathscr{X}(k, \zeta)$: the class of k-γ_c-critical graphs containing ζ cut vertices.

Lemma 4.2. *Let $G \in \mathscr{X}(k, \zeta)$ where $\zeta \in \{k-3, k-2\}$. Then G has only two end blocks and another blocks contain two cut vertices.*

Proof. Clearly $\zeta \geq k-3$. We have by Equation 3.3.5 that

$$2h_0 \leq k - \zeta - j_0 + 2 \leq k - (k-3) - j_0 + 2 \leq 5.$$

That is $\zeta_0 \leq \frac{5}{2}$. Lemma 2.3(1) implies that G has only two end blocks and another blocks contain 2 cut vertices. This completes the proof. □

Lemma 4.3. *Let $G \in \mathscr{X}(k, h)$ where $\zeta \in \{k-3, k-2\}$ and B be a block of G containing two cut vertices c and c', moreover, $(D - \mathscr{C} \cap V(B)) = 0$. Then $B = cc'$.*

Proof. We prove for the case when $\zeta = k-2$ and thus, the case when $k = \zeta - 3$ can be proved by similar arguments. Suppose that $\zeta = k-2$. Lemma 4.2 implies that G has only two end blocks, B_1, B_{k-1} say, and another blocks $B_2, B_3, ..., B_{k-2}$ which contain two cut vertices. Without loss of generality let $c_1 \in V(B_1), c_{k-2} \in V(B_{k-1})$ and $c_{i-1}, c_i \in V(B_i)$ for $2 \le i \le k-2$(see Figure 4.2).

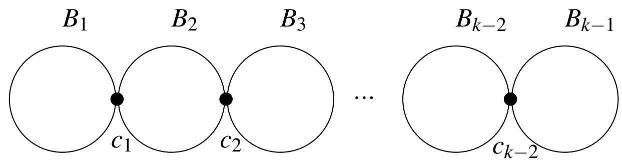


Figure 4.2 : The structure of G

We consider the case $i = k-2$. Let $z \in V(B_{k-1}) - \{c_{k-2}\}$. Suppose there exists $u \in V(B_{k-2}) - \{c_{k-3}, c_{k-2}\}$. Consider $G+uz$. We see that c_{k-3} is a cut vertex of $G+uz$. Lemma 2.3(3) implies that $c_{k-3} \in D_{uz}$. If $|D_{uz} \cap (\cup_{i=1}^{k-3} V(B_i))| \le k-2$, then $(D_{uz} \cap (\cup_{i=1}^{k-3} V(B_i))) \cup \{c_{k-2}\} \succ_c G$ contradicting $\gamma_c(G) = k$. Hence $|D_{uz} \cap (\cup_{i=1}^{k-3} V(B_i))| \ge k-1$. Lemma 2.2(2) yields also that $\{u, z\} \cap D_{uz} \ne \emptyset$. So $|D_{uz}| \ge k$ contradicting Lemma 2.2(1). Hence, $V(B_{k-2}) = \{c_{k-3}, c_{k-2}\}$.

We now consider the case $2 \le i \le k-3$. Suppose to the contrary that $B'_i = V(B_i) - \{c_{i-1}, c_i\} \ne \emptyset$. We see that $N_G[d] \subseteq B'_i \cup \{c_{i-1}, c_i\}$ for all $d \in B'_i$, $c_i \succ B'_i, c_{i-1} \succ B'_i$ and $G[\{c_i, c_{i-1}\}]$ is complete. We see also that $N_G[b] \subseteq (V(B_{k-1}) - \{c_{k-1}\}) \cup \{c_{k-2}\}$ for all $b \in V(B_{k-1})$ and $G[\{c_{k-2}\}]$ is complete.

Choose

$$X_1 = \{c_{k+2}\}, X = V(B_{k-1}) - \{c_{k-1}\}, Y = B'_i \text{ and } Y_1 = \{c_i, c_{i-1}\}.$$

Clearly X, X_1, Y and Y_1 form a bad subgraph. This contradicts Lemma 3.2. Hence, $B'_i = \emptyset$ for all $2 \le i \le k-3$. This completes the proof. □

The following theorem gives the characterization of the graphs in the class $\mathscr{X}(k, k-2)$.

Theorem 4.1. $\mathscr{X}(k, k-2) = \mathscr{F}$.

Proof. Lemma 4.4 implies that $\mathscr{F} \subseteq \mathscr{X}(k, k-2)$. It is sufficient to show that a k-γ_c-critical graph with $k-2$ cut vertices is in $\mathscr{F}$. Let $\mathscr{C} = \{c_1, c_2, ..., c_{k-2}\}$ be a set of cut vertices of G. Lemma 4.2 implies that G has only two end blocks, B_1, B_{k-1} say, and another blocks $B_2, B_3, ..., B_{k-2}$ which contain two cut vertices. Without loss of generality let $c_1 \in V(B_1), c_{k-2} \in V(B_{k-1})$ and $c_{i-1}, c_i \in V(B_i)$ for $2 \leq i \leq k-2$. Lemma 2.3(3) yields that $\mathscr{C} \subseteq D$. As $|\mathscr{C}| = k-2$, we must have $|D - \mathscr{C}| = 2$. Lemma 3.9 yields also that $(D - \mathscr{C}) \cap (V(B_1) \cup V(B_{k-1})) \neq \emptyset$. Without loss of generality let $y_1 \in (D - \mathscr{C}) \cup V(B_{k-1})$.

Claim 1 : $|(D - \mathscr{C}) \cap V(B_{k-1})| = 2$.

Suppose that $|(D - \mathscr{C}) \cap V(B_{k-1})| = 1$. Therefore $|(D - \mathscr{C}) \cap V(B_1)| \leq 1$ contradicting Lemma 3.9 thus establishing Claim 1.

As $\zeta = k-2$, by Claim 1, we must have $D \cap V(B_i) = \{c_{i-1}, c_i\}$ for $2 \leq i \leq k-2$ and $D \cap V(B_1) = \{c_1\}$. Therefore

$$\{c_{i-1}, c_i\} \succ_c B_i \text{ and } c_1 \succ B_1.$$

Let $z \in V(B_1) - \{c_1\}$. Clearly $d(c_1, z) = 1$.

Claim 2 : $B_{k-1} \in \mathscr{B}_{2,2}$.

By Claim 1, there exists $w \in V(B_{k-1}) - \{c_{k-2}\}$ such that $d(w, c_{k-2}) \geq 2$. Thus

$$d(z,w) \geq d(z,c_1) + d(c_1,c_2) + ... + d(c_{k-2},w) \geq k$$

Lemma 2.4 gives that $d(z,w) \leq k$. Hence $d(w', c_{k-2}) \leq 2$ for all $w' \in V(B_{k-1}) - \{c_{k-2}\}$. By Lemma 3.8, $B_{k-2} \in \mathscr{B}_{2,2}$ and thus establishing Claim 2.

Lemma 4.3 implies that, for all $i \in \{2,3,...,k-2\}$, $V(B_i) = \{c_{i-1}, c_i\}$. Moreover, in view of Claims 1 and 2, it suffices to show that $V(B_1) = \{c_1, z\}$. Consider $G + c_2z$. Since c_2 is a cut vertex of $G + c_2z$, $c_2 \in D_{c_2z}$ by Lemma 2.3(3). If $|D_{c_2z} \cap (\cup_{i=2}^{k-2} V(B_i))| \leq k-2$, then $(D_{c_2z} \cap (\cup_{i=2}^{k-2} V(B_i))) \cup \{c_1\} \succ_c G$ contradicting $\gamma_c(G) = k$. Therefore $|D_{c_2z} \cap (\cup_{i=2}^{k-2} V(B_i))| = k-1$ by Lemma 2.2(1). Thus $c_1, z \notin D_{c_2z}$. We have $V(B_1) = \{c_1, z\}$ and this completes the proof. □

4.2 The Critical Graphs with $\zeta(G) = k-3$

In this subsection, we characterize k-γ_c-critical graphs with $k-3$ cut vertices. Firstly, we give construction of two classes of such graphs. Let

$\boldsymbol{i} = (i_1, i_2, ..., i_{k-3})$

be a $k-3$ tuples such that $i_1, i_2, ..., i_{k-3} \in \{0,1\}$ and $\Sigma_{j=1}^{k-3} i_j = 1$ (there is exactly one $l \in \{1,2,...,k-3\}$ such that $i_l = 1$ and $i_{l'} = 0$ for all $l' \in \{1,2,...,k-3\} - \{l\}$).

The class $\mathscr{G}_1(i_1, i_2, ..., i_{k-3})$

For a $k-3$ tuples $\boldsymbol{i} = (0,0,...,i_l,...,0)$ where $i_l = 1$ and $i_{l'} = 0$ for $1 \leq i_l \neq i_{l'} \leq k-2$, a graph G in the class $\mathscr{G}_1\boldsymbol{i}$ can be constructed from paths $c_0, c_1, ..., c_{l-1}$ and

$c_l, c_{l+1}, ..., c_{k-4}$, a copy of a complete graph K_{n_l} and a block $B \in \mathscr{B}_{2,2}$ by adding edges according the join operations :

- $c_{l-1} \vee K_{n_l} \vee c_l$ and
- $c_{k-4} \vee c$

where c is the head of B. Examples of graphs in this case are illustrated by Figures 4.3 and 4.4.

Further, for a $k-3$ tuples $\boldsymbol{i} = (0,0,...,1)$ where $i_{k-3} = 1$ and $i_{l'} = 0$ for $1 \leq i_{l'} \leq k-2$, a graph G in the class $\mathscr{G}_1\boldsymbol{i}$ can be constructed from paths $c_0, c_1, ..., c_{k-4}$, a copy of a complete graph $K_{n_{k-3}}$ and a block $B \in \mathscr{B}_{2,2}$ by adding edges according the join operation :

- $c_{k-4} \vee K_{n_{k-3}} \vee c$

where c is the head of B. Example of a graph in this case is illustrated by Figures 4.5.

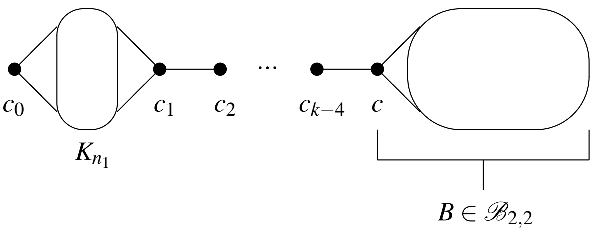


Figure 4.3 : A graph G in the class $\mathscr{G}_1(1,0,0,...,0)$

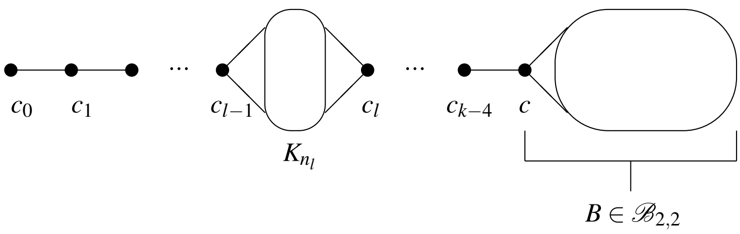


Figure 4.4 : A graph G in the class $\mathscr{G}_1(0,0,...,i_l = 1,0,...,0)$

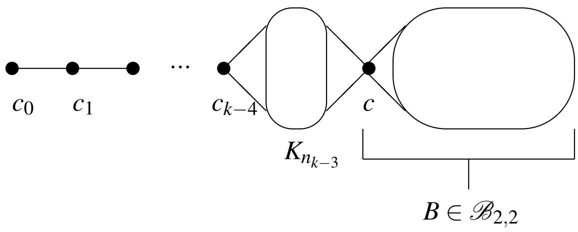


Figure 4.5 : A graph G in the class $\mathscr{G}_1(0,0,...,1)$

Lemma 4.4. *Let G be a graph in the class $\mathscr{G}_1(i_1, i_2, ..., i_{k-3})$, then G is a k-γ_c-critical graph with $k-3$ cut vertices.*

Proof. Clearly G has $c_1, c_2, ..., c_{k-4}$ and c as the $k-3$ cut vertices. Observe that, for any $\boldsymbol{i} = (i_1, i_2, ..., , i_{k-3})$, a graph $G \in \mathscr{G}_1\boldsymbol{i}$ has the path $P = c_0, c_1, ..., c_{l-1}, a, c_l, ..., c_{k-4}, c$ from c_0 to c where $a \in V(K_{n_l})$. To prove all cases of $\boldsymbol{i}$, we may relabel the path P to be $x_1, ..., x_{k-1}$. Hence, $c_0 = x_1, c_1 = x_2, ...c_{k-4} = x_{k-2}$ and $c = x_{k-1}$. We see that $\{x_2, x_3, ..., x_{k-2}, x_{k-1}, s_1^1, s_0^1\} \succ_c G$. Therefore $\gamma_c(G) \leq k$.

Let D be a γ_c-set of G. If $x_1 \notin D$, then, to dominate x_1, $x_2 \in D$. If $x_1 \in D$, then $x_2 \in D$ since $G[D]$ is connected. In both cases, $x_2 \in D$.

Suppose that $D \cap S'' = \emptyset$. Because $B \in \mathscr{B}_{2,2}$, to dominate B, $|D \cap (S \cup S')| \geq 2$. By the connectedness of $G[D]$, we have $\{x_3, x_4, ..., x_{k-2}, x_{k-1}\} \subseteq D$. Thus $\gamma_c(G) \geq k$ implying that $\gamma_c(G) = k$. Hence, suppose that $D \cap S'' \neq \emptyset$. Since $x_2 \in D$ and $G[D]$ is connected, it follows that $\{x_3, x_4, ..., x_{k-2}, x_{k-1}, y\} \subseteq D$ where $y \in D \cap S$. Thus $\gamma_c(G) = |D| \geq k$ implying that $\gamma_c(G) = k$.

Now, we will establish the criticality. Let u and v be a pair of non-adjacent vertices of G and let $S_1 = S \cup S' \cup S''$. We first assume that $|\{u, v\} \cap S_1| = 0$. Therefore $\{u, v\} \subseteq \{x_1, x_2, ..., x_{k-2}, x_{k-1}\}$. Thus $u = x_i$ and $v = x_j$ for some $i, j \in \{1, 2, ..., k-1\}$. Without loss of generality let $i < j$. Clearly $i+2 \leq j$. We see that

$$\{x_2, x_3, ..., x_i, x_{i+2}, x_{i+3}, ..., x_j, ..., x_{k-1}, s_1^1, s_0^2\} \succ_c G + uv.$$

So $\gamma_c(G + uv) \leq k-1$.

Hence, we assume that $|\{u, v\} \cap S_1| = 1$. If $\{u, v\} = \{x_{k-1}, s\}$ for some $s \in S_1$, then $s \notin S$ and, clearly, $\{x_{k-1}, s\} \succ S_1$. Thus $\{x_2, x_3, ..., x_{k-1}, s_1\} \succ_c G + uv$. Therefore $\gamma_c(G + uv) \leq k-1$. Let $v \in S_1$. Since $|\mathscr{S}| \geq 2$, there exists $v' \in S - \{v\}$ such that $\{v, v'\} \succ_c S_1$. Suppose that $u \in \{x_2, x_3, ..., x_{k-2}\}$. Thus $\{x_2, x_3, ..., u, ..., x_{k-2}, v, v'\} \succ_c G + uv$. Hence $\gamma_c(G + uv) \leq k-1$. If $u = x_1$, then $\{x_3, x_4, ..., x_{k-2}, v, v'\} \succ_c G + uv$ implying that $\gamma_c(G + uv) \leq k-1$.

Finally, we assume that $\{u, v\} \subseteq S_1$. Thus $\{u, v\} = \{s_j^i, s_0^i\}$ for some $i \in \{1, 2, ..., |\mathscr{S}|\}$ and $j \in \{1, 2, ..., m_i\}$. Clearly $\{x_2, x_3, ..., x_{k-1}, s_j^i\} \succ_c G + uv$ and $\gamma_c(G + uv) \leq k-1$. Thus G is a k-γ_c-critical graph and this completes the proof. □

We will construct another class of k-γ_c-critical graphs with $k-3$ cut vertices. Before giving the construction, we introduce the class of end blocks.

The class $\mathscr{B}_3$

An end block $B \in \mathscr{B}_3$ has b as the head. Let $N_B(b) = A$ and $\check{B} = G[V(B) - \{b\}]$. Moreover, B has the following properties

(1) Every vertex $v \in V(\check{B})$, there exists a γ_c-set D_v of B of size 3 such that $v \in D_v$.

(2) For every non-adjacent vertices x and y of $\check{B}$, there exists a γ_c-set D^B_{xy} of $B + xy$ such that $D^B_{xy} \cap \{x,y\} \neq \emptyset$, $|D^B_{xy}| = 2$ and $D^B_{xy} \cap A \neq \emptyset$.

It is worth noting that D_v in the property (1) satisfies $D_v \cap A \neq \emptyset$. We now ready to give the construction.

The class $\mathscr{G}_2(k)$ for $k \geq 5$

A graph G in this class can be constructed from a path $c_0, c_1, ..., c_{k-4}$ and an end block $B \in \mathscr{B}_3$ with the head b by adding the edge $c_{k-4}b$. For the sake of convenience, we may relabel b as c_{k-3}.

Lemma 4.5. *Let G be a graph in the class $\mathscr{G}_2(k)$. Then G is a k-γ_c-critical graph with $k-3$ cut vertices.*

Proof. Choose $v \in V(\check{B})$. By (1), there exists a γ_c-set D_v such that $|D_v| = 3$ and $D_v \cap A \neq \emptyset$. Thus $\{c_1, c_2, ..., c_{k-4}, c_{k-3}\} \cup D_v \succ_c G$. Therefore $\gamma_c(G) \leq k$.

Let D be a γ_c-set of G. As $c_1, c_2, ..., c_{k-3}$ are cut vertices, by Lemma 2.3(3), $c_1, c_2, ..., c_{k-3} \in D$. Let $v \in V(\check{B}) \cap D$. Observe that $V(\check{B}) \cap D$ is a connected dominating set of B containing v. By (1) and the minimality od D_v, $|V(\check{B}) \cap D| \geq |D_v| = 3$. So $\gamma_c(G) \geq k$ and this implies that $\gamma_c(G) = k$.

We will prove the criticality. Let u and v be non-adjacent vertices of G. Suppose first that $c_0 \in \{u,v\}$, $c_0 = u$ say. If $v \in \{c_2, c_3, ..., c_{k-3}\}$, then $\{c_2, c_3, ..., c_{k-3}\} \cup D_v \succ_c G + uv$. If $v \in V(\check{B})$, then, by (1), there exists a γ_c-set D_v of size 3 of B such that $v \in D_v$ and $A \cap D_v \neq \emptyset$. So $\{c_2, c_3, ..., c_{k-3}\} \cup D_v \succ_c G + uv$. These imply that $\gamma_c(G + uv) < \gamma_c(G)$.

We then suppose that $c_0 \notin \{u,v\}$. If $\{u,v\} \subseteq \{c_1, c_2, ..., c_{k-3}\}$, then there exists i and j such that $c_i = u$ and $c_j = v$. Without loss of generality let $i < j$. Clearly, $i + 2 \leq j$. So $\{c_1, c_2, ..., c_i, c_{i+2}, c_{i+3}, ..., c_{k-3}\} \cup D_v \succ_c G + uv$. If $|\{u,v\} \cap \{c_1, c_2, ..., c_{k-3}\}| = 1$, then $\{c_1, ..., c_i, ..., c_{k-4}\} \cup D_v \succ_c G + uv$. Finally, if $\{u,v\} \subseteq V(B)$, then, by (2), there exists a γ_c-set D^B_{uv} such that $D^B_{uv} \cap \{u,v\} \neq \emptyset$, $|D^B_{uv}| = 2$ and $D^B_{uv} \cap A \neq \emptyset$. Thus $\{c_1, c_2, ..., c_{k-3}\} \cup D^B_{uv} \succ_c G + uv$. This implies that $\gamma_c(G + uv) < \gamma_c(G)$. Clearly, $c_1, c_2, .., c_{k-3}$ are the $k-3$ cut vertices of G. This completes the proof. $\square$

In the following, we let $G \in \mathscr{X}(k, k-3)$ having a γ_c-set D. In view of Lemma 4.2, G has only two end blocks and another blocks contain two cut vertices. Thus, we let B_1 and B_{k-2} be the two end blocks and another blocks $B_2, B_3, ..., B_{k-3}$ contain two cut vertices. Without loss of generality let $c_1 \in V(B_1), c_{k-3} \in V(B_{k-2})$ and $c_{i-1}, c_i \in V(B_i)$

for $2 \le i \le k-3$. Moreover, let $C_i = V(B_i) - \mathscr{A}$ for all $1 \le i \le k-2$. Let D' be a γ_c-set of G such that $D' \neq D$, by the minimality of k, we have $|V(B_i) \cap D| = |V(B_i) \cap D'|$ for all i. Thus, we can let

$\mathscr{H}(b_1, b_2, b_3, ..., b_{k-2})$: the class of a graph $G \in \mathscr{Z}(k, k-3)$ such that $|V(C_i) \cap D| = b_i$ for $1 \le i \le k-2$.

Lemma 4.6. *For a γ_c-set of G, either $|V(C_1) \cap D| \ge 2$ or $|V(C_{k-2}) \cap D| \ge 2$.*

Proof. Suppose to the contrary that $|V(C_1) \cap D| \le 1$ and $|V(C_{k-2}) \cap D| \le 1$. Lemmas 3.6 and 3.7 imply that $B_1, B_{k-2} \in \mathscr{B}_0 \cup \mathscr{B}_1$. This contradicts Lemma 3.9. Thus either $|V(C_1) \cap D| \ge 2$ or $|V(C_{k-2}) \cap D| \ge 2$ and this completes the proof. □

By Lemma 4.6, we may suppose without loss of generality that $|V(C_{k-2}) \cap D| \ge |V(C_1) \cap D|$.

Lemma 4.7. *$\mathscr{Z}(k, k-3) = \mathscr{H}(0,0,0,...,3) \cup \mathscr{H}(b_1, b_2, ..., b_{k-3}, 2)$ where $b_i = 1$ for some $1 \le i \le k-3$ and $b_j = 0$ for all $1 \le j \neq i \le k-3$.*

Proof. By the definition, $\mathscr{H}(0,0,0,...,3) \cup \mathscr{H}(b_1, b_2, ..., b_{k-3}, 2) \subseteq \mathscr{Z}(k, k-3)$.

Conversely, let $G \in \mathscr{Z}(k, k-3)$. Thus, by Lemma 4.2, G has only two end blocks B_1 and B_{k-2} and another blocks $B_2, B_3, ..., B_{k-2}$ contain two cut vertices. Moreover, $c_1 \in V(B_1), c_{k-3} \in V(B_{k-1})$ and $c_{i-1}, c_i \in V(B_i)$ for $2 \le i \le k-3$. In view of Lemma 2.3(3), $c_1, c_2, ..., c_{k-3} \in D$. So $|D - \mathscr{A}| = 3$. Thus, $|V(C_i) \cap D| \le 3$ for all $i \in \{1, k-2\}$. Recall that $|V(C_{k-2}) \cap D| \ge |V(C_1) \cap D|$. Lemma 4.6 implies that either $|V(C_{k-2}) \cap D| = 3$ or $|V(C_{k-2}) \cap D| = 2$. That is $G \in \mathscr{H}(0,0,0,...,3) \cup \mathscr{H}(b_1, b_2, ..., b_{k-3}, 2)$. This completes the proof. □

By Lemma 4.7, to characterize a graph G in the class $\mathscr{Z}(k, k-3)$, it suffices to consider when G is either in $\mathscr{H}(0,0,0,...,3)$ or $\mathscr{H}(b_1, b_2, ..., b_{k-3}, 2)$. We first consider the case when $G \in \mathscr{H}(b_1, b_2, ..., b_{k-3}, 2)$. Let c_i and c_{i+1} be vertices and K_{n_i} a copy of a complete graph.

Lemma 4.8. *Let $G \in \mathscr{H}(b_1, b_2, ..., b_{k-3}, 2)$ with a block B_i containing two cut vertices c_{i-1} and c_i and $b_i = 1$. Then $B_i = c_{i-1} \vee K_{n_i} \vee c_i$ where $n_i \ge 2$.*

Proof. As $G \in \mathscr{H}(b_1, b_2, ..., b_{k-3}, 2)$ and $b_i = 1$ for some $1 \le i \le k-3$, we must have $b_j = 0$ for all $1 \le j \neq i \le k-3$. Because B_i contains two cut vertices, $i > 1$. Therefore, $b_1 = 0$. Lemma 3.6 then implies that

$B_1 = K_{n_1} \vee c_1$.

Let $B' = B_i - c_{i-1} - c_i$. We first show that B' is complete. Let x and y be non-adjacent vertices of B'. Consider $G + xy$. Lemma 2.2(2) implies that $|D_{xy} \cap \{x,y\}| \geq 1$. As $x, y \in V(B')$, we must have $|D_{xy} \cap (V(B_i) \cap \mathscr{A})| \geq 1$ contradicting Corollary 3.2.1. So B' is complete.

We will show that $c_{i-1}c_i \notin E(G)$. Hence, we may assume to the contrary that $c_{i-1}c_i \in E(G)$. We let

$X_1 = N_{B_i}(\{c_{i-1}, c_i\})$ and

$X = V(B') - X_1$.

Since $|D \cap (V(B_i) - \{c_{i-1}, c_i\})| = 1$, it follows that $X \neq \emptyset$. Because B' is complete, $G[X_1 \cup X]$ is complete. In fact, X and X satisfy (i) and (ii) of bad subgraphs. We then let

$Y_1 = \{c_1\}$ and

$Y = V(K_{n_1})$.

Thus G has X, X, Y and Y_1 as a bad subgraph. This contradicts Lemma 3.2. Hence, $c_{i-1}c_i \notin E(G)$.

We finally show that $N_{B_i}(c_i) = N_{B_i}(c_{i-1}) = V(B')$. We may assume to the contrary that there exists a vertex u of B' which is not adjacent to c_{i-1}. Consider $G + uc_{i-1}$. Corollary 3.2.1 gives that $|D_{uc_{i-1}} \cap V(B')| = 0$. Lemma 3.3 gives further that $\{c_{i-1}, c_i\} \subseteq D_{uc_{i-1}}$. Since $c_{i-1}c_i \notin E(G)$, it follows that $(G + uc_{i-1})[D_{uc_{i-1}}]$ is not connected, a contradiction. Hence, $N_{B_i}(c_{i-1}) = V(B')$ and, similarly, $N_{B_i}(c_i) = V(B')$. Since B_i is a block, $n_i \geq 2$ and this completes the proof. □

Theorem 4.2. *For $k \geq 4$, $\mathscr{H}(b_1, b_2, ..., b_{k-3}, 2) = \mathscr{G}_1(i_1, i_2, ..., i_{k-3})$ where $b_j = i_j$ for all $1 \leq j \leq k-3$.*

Proof. Let $b_j = i_j$ for all $1 \leq j \leq k-3$. In views of Lemma 4.5, we have that $\mathscr{G}_1(i_1, i_2, ..., i_{k-3}) \subseteq \mathscr{H}(b_1, b_2, ..., b_{k-3}, 2)$. Thus, it suffces to show that $\mathscr{H}(b_1, b_2, ..., b_{k-3}, 2) \subseteq \mathscr{G}_1(i_1, i_2, ..., i_{k-3})$.

We will show that $B_{k-2} \in \mathscr{B}_{2,2}$. Clearly, b_1 is either 0 or 1. If $b_1 = 0$, then Lemma 3.6 implies that $B_1 = K_{n_1} \vee c_1$. But if $b_1 = 1$, then Lemma 3.7 implies that $B_1 = c_0 \vee K_{n_1} \vee c_1$. Thus, we let

$$X_1 = \begin{cases} \{c_1\} & if \;\; b_1 = 0 \;\; and \\ V(K_{n_1}) & if \;\; b_1 = 1. \end{cases}$$

and

$$X = \begin{cases} V(K_{n_1}) & if \;\; b_1 = 0 \;\; and \\ \{c_0\} & if \;\; b_1 = 1. \end{cases}$$

Since $b_{k-2} = 2$, by Lemma 3.8, $B_{k-2} \in \mathscr{B}_{2,1} \cup \mathscr{B}_{2,2}$. If $B_{k-2} \in \mathscr{B}_{2,1}$, then $B_{k-2} = c_{k-3} \vee K_{n_1} \vee K_{n_2} \vee z_2$ where z_2 is given at the definition of $\mathscr{B}_{2,1}$. We then let

$Y_1 = V(K_{n_2})$ and $Y = \{z_2\}$

Clearly, G has a bad subgraph, contradicting Lemma 3.2. Thus $B_{k-2} \in \mathscr{B}_{2,2}$.

We now consider the case when $b_1 = 1$. Thus $b_2 = b_3 = ... = b_{k-3} = 0$. By Lemma 3.7, $B_1 \in \mathscr{B}_1$ implying that $B_1 = c_0 \vee K_{n_1} \vee c_1$. Further, Lemma 4.3 implies also that $B_i = c_{i-1}c_i$ for $2 \leq i \leq k-3$. Thus $G \in \mathscr{G}_1(1,0,...,0)$.

We finally consider the case when $b_1 = 0$. Thus $b_j = 1$ for some $2 \leq j \leq k-3$ and $b_j = 0$ for $2 \leq j' \neq j \leq k-3$. Similarly, $B_{j'} = c_{j'-1}c_{j'}$ for all j' by Lemma 4.3. Moreover, Lemma 4.8 yields that $B_j = c_{j-1} \vee K_{n_j} \vee c_i$. We will show that $B_1 = c_0c_1$. We let a be a vertex in $V(B_2) - \{c_1, c_2\}$ if $j = 2$. Then, we let

$$x = \begin{cases} a & if \;\; j = 2 \;\; and \\ c_2 & if \;\; j > 2. \end{cases}$$

Consider $G + c_0x$. Since c_2 is a cut vertex of $G + c_0x$, $c_2 \in D_{c_0x}$ by Lemma 2.3(3). That is $x \in D_{c_0x}$ when $j > 2$. When $j = 2$, by Lemma 4.8, $xc_2 \in E(G)$. Since $c_2 \in D_{c_0x}$, by Lemma 2.2(3), $x \in D_{c_0x}$. In both cases, $x \in D_{c_0x}$. If $|D_{c_0x} \cap (\cup_{i=2}^{k-2} V(B_i))| \leq k-2$, then $(D_{c_0x} \cap (\cup_{i=2}^{k-2} V(B_i))) \cup \{c_1\} \succ_c G$ contradicting $\gamma_c(G) = k$. Therefore $|D_{c_0x} \cap (\cup_{i=2}^{k-2} V(B_i))| = k-1$ by Lemma 2.2(1). Thus $c_1, c_0 \notin D_{c_0x}$ implying that $B_1 = c_0c_1$. So $G \in \mathscr{G}_1(0,0,...,i_j = 1,...,0)$. This completes the proof. □

Theorem 4.3. *For $k \geq 4$, $\mathscr{H}(0,0,...,0,3) = \mathscr{G}_2(k)$.*

Proof. By Lemma 4.5, $\mathscr{G}_2(k) \subseteq \mathscr{H}(0,0,...,3)$. Thus, it is sufficient to show that $\mathscr{H}(0,0,...,3) \subseteq \mathscr{G}_2(k)$.

As $b_i = 0$ for all $2 \leq i \leq k-3$, by Lemma 4.3, $B_i = c_{i-1}c_i$. By Lemma 3.6 and similar arguments in Theorem 4.2, we have that $B_1 = c_0c_1$.

We will show that B_{k-2} satisfies (1). Let D' be a γ_c-set of B_{k-2}. Suppose that $|D'| \leq 2$. To dominate c_{k-3}, we have $D' \cap A \neq \emptyset$. Thus, $\{c_1, ..., c_{k-3}\} \cup D' \succ_c G$. But $|\{c_1, ..., c_{k-3}\} \cup D'| = k-1$ contradicting the minimality of k. Therefore, to prove that B_{k-2} satisfies (1), it suffices to give a γ_c-set of size 3 of B_{k-2} containing a chosen vertex from $\check{B}_{k-2}$. For a vertex v of $\check{B}_{k-2}$, consider $G + c_0v$. Lemma 2.2(2) implies that $\{c_0, v\} \cap D_{c_0v} \neq \emptyset$. Lemma 2.2(1) implies also that $|D_{c_0v}| \leq k-1$. We first show that $\{c_0\} \neq D_{c_0v} \cap \{c_0, v\}$. Suppose to the contrary that $\{c_0\} = D_{c_0v} \cap \{c_0, v\}$. Since $(G+uv)[D_{uv}]$ is connected there exists $w \in V(\check{B}_{k-2})$ which is adjacent to v. Because $D_{c_0v} \succ_c G + c_0v$, w is adjacent to a vertex of $D_{uv} \cap V(B_{k-2} - v)$. So

$$(D_{uv} - \{c_0\}) \cup \{w\} \succ_c G.$$

This contradicts the minimality of k. Thus, $\{c_0\} \neq D_{c_0v} \cap \{c_0, v\}$. Therefore $\{c_0, v\} \subseteq D_{c_0v}$ or $\{v\} = D_{c_0v} \cap \{c_0, v\}$.

Case 1 : $\{c_0, v\} \subseteq D_{c_0v}$

Let

$$i = max\{1 \leq j \leq k-3 : G[\{c_0, c_1, c_2, ..., c_j\} \cap D_{c_0y}] \text{ is connected }\}.$$

We first consider the case when $i = k-3$. Thus $\{c_1, c_2, ..., c_{k-3}\} \subseteq D_{c_0v}$. As $|D_{c_0v}| \leq k-1$ and $\{c_0, v\} \subseteq D_{c_0v}$, we must have

$$D_{c_0v} = \{c_0, c_1, ..., c_{k-3}, v\}.$$

So $v \succ V(B_{k-2}) - A$ and $N_A(v) = \emptyset$, otherwise $\{c_1, ..., c_{k-3}, w, v\} \succ_c G$ where $w \in N_A(v)$, contradicting the minimality of k. Let $u \in N_{B_{k-2}}(v)$ such that u is adjacent to a vertex a in A. Thus $\{v, u, a\} \succ_c B_{k-2}$ and so B_{k-2} satisfies (1).

We now consider the case when $i = k-4$. Let $D'_{c_0v} = D_{c_0v} \cap V(\check{B}_{k-2})$. Clearly, $v \in D'_{c_0v}$. Since $\{c_0, c_1, ..., c_{k-4}\} \subseteq D_{c_0v}$ and $|D_{c_0v}| \leq k-1$, it follows that $|D'_{c_0v}| \leq 2$. If $|D'_{c_0v}| = 1$, then $D'_{c_0v} = \{v\}$ implying that $v \succ \check{B}_{k-2}$, in particular, $v \succ A$. Thus, $\{c_1, ..., c_{k-3}, w, v\} \succ_c G$ where $w \in N_A(v)$ contradicting the minimality of D. Hence, we let $D'_{c_0v} = \{v, v'\}$. Since $D_{c_0v} \succ_c G + c_0v$, $D'_{c_0v} \succ_c \check{B}_{k-2}$. Hence, for a vertex a in A, $D'_{c_0v} \cup \{a\} \succ_c B_{k-2}$. Therefore, B_{k-2} satisfies (1).

We now consider the case when $i = k-5$. Thus $\{c_0, c_1, ..., c_{k-5}\} \subseteq D_{c_0v}$. So $|D'_{c_0v}| \leq 3$ and, $D'_{c_0v} \cap A \neq \emptyset$ to dominate c_{k-3}. So, B_{k-2} satisfies (1).

We finally consider the case when $i \leq k-6$. To dominate c_{i+2}, we have that $c_{i+3} \in D_{c_0v}$. By the connectedness of $(G + c_0v)[D_{c_0v}]$, $\{c_{i+3}, ..., c_{k-3}\} \subseteq D_{c_0v}$. Thus, $\{c_0, c_1, ..., c_i\} \cup \{c_{i+3}, ..., c_{k-3}\} \cup D'_{c_0v} \subseteq D_{c_0v}$ implying that

$$k-4 + |D'_{c_0v}| = (i+1) + ((k-3) - (i+3) + 1) + |D'_{c_0v}| \leq k-1.$$

Therefore, $|D'_{c_0v}| \leq 3$. To dominate c_{k-3}, $D'_{c_0v} \cap A \neq \emptyset$. Thus B_{k-2} satisfies (1) and this completes the proof of Case 1.

Case 2 : $\{v\} = D_{c_0v} \cap \{c_0, v\}$

To dominate c_1, we have that $c_2 \in D_{c_0v}$. By the connectedness of $(G + c_0v)[D_{c_0v}]$, $\{c_2, c_3, ..., c_{k-3}\} \subseteq D_{c_0v}$ and $D_{c_0v} \cap A \neq \emptyset$. As $|D_{c_0v}| \leq k-1$, we must have $|D_{c_0v} - \{c_2, c_3, ..., c_{k-3}\}| \leq 3$. So B_{k-2} satisfies (1). This completes the proof of Case 2.

We finally show that B_{k-2} satisfies (2). Let x and y be non-adjacent vertices of B_{k-2}. Lemma 2.2(2) implies that $\{x,y\} \cap D_{xy} \neq \emptyset$. Lemma 2.2(1) implies also that $|D_{xy}| \leq k-1$. To dominate c_0, we have that $c_1 \in D_{xy}$. Let $D^B_{xy} = D_{xy} \cap V(\check{B}_{k-2})$. By the connectedness of $(G+xy)[D_{xy}]$, $D^B_{xy} \cap A \neq \emptyset$ and $\{c_1, c_2, ..., c_{k-3}\} \subseteq D_{xy}$. As $|D_{xy}| \leq k-1$, we must have $|D^B_{xy}| = |D_{xy} \cap V(\check{B}_{k-2})| = |D_{xy} - \{c_1, c_2, ..., c_{k-3}\}| \leq 2$. Hence, B_{k-2} satisfies (2). Therefore $B_{k-2} \in \mathscr{B}_3$. This completes the proof. □

We conclude this paper by the following theorem.

Theorem 4.4. *For an integer $k \geq 4$, $\mathscr{Z}(k,k-3) = \mathscr{G}_1(i_1, i_2, ..., i_{k-3}) \cup \mathscr{G}_2(k)$.*

Proof. In view of Lemma 4.7, $\mathscr{Z}(k,k-3) = \mathscr{H}(0,0,0,...,3) \cup \mathscr{H}(b_1, b_2, ..., b_{k-3}, 2)$ where $b_i = 1$ for some $1 \leq i \leq k-3$ and $b_j = 0$ for all $1 \leq j \neq i \leq k-3$. Moreover, Theorems 4.2 and 4.3 imply that $\mathscr{Z}(k,k-3) = \mathscr{G}_1(i_1, i_2, ..., i_{k-3}) \cup \mathscr{G}_2(k)$. This completes the proof. □

4.3 Discussion

According to our Objectives 3 and 4, we would like to give a short proofs to answer these purposes. For realizability of k-γ_c-critical graphs with ζ cut vertices when $0 \leq \zeta \leq k-2$, we may use the characterization of the class $\mathscr{F}$. If $G \in \mathscr{F}$, then G is obtained by joining the vertex x_{k-2} of the path $x_1, x_2, ..., x_{k-2}$ to c, the head of a block $B \in \mathscr{B}_{2,2}$. For the sake of convenience, we relabel c to be x_{k-1}.

For fixing $h \in \{0, 1, ..., k-3\}$, a k-γ_c-critical graphs with $\zeta = h$ cut vertices is obtained by replacing the vertices $x_{2+h}, ..., x_{k-1}$ by completes cliques $K_{n_{2+h}}, ..., K_{n_{k-1}}$, respectively where $n_i \geq 2$ for all $2+h \leq i \leq k-1$ and, adding edges between x_{1+h} to every vertex in $K_{n_{2+h}}$, every vertex between K_{n_j} and $K_{n_{j+1}}$ for $2+h \leq j \leq k-2$, and every vertex between $K_{n_{k-1}}$ and the subset S of B. It can be observed that the resulting graph is k-γ_c-critical graph with $\zeta = h$ cut vertices. This answers Objective 3. That is:

Theorem 4.5. *For an integer $k \geq 5$, $\mathscr{Z}(k,\zeta) \neq \emptyset$ for all $0 \leq \zeta \leq k-2$.*

For the matching property of k-γ_c-critical graphs with $k-2$ cut vertices, by the characterization of G in the class $\mathscr{F}$, it is easy to observe that the block B has a Hamiltonian path P^B containing c as one of the two end vertices. The readers are encouraged to find this such Hamiltonian path of B. Thus, $x_1, x_2, ..., x_{k-2}, cP^B$ is a Hamiltonian path of G. Therefore, when G has even order, we can find a perfect matching of G from the Hamiltonian path. That is:

Theorem 4.6. *For an integer $k \geq 5$, if G is a k-γ_c-critical graph with $k-2$ cut vertices, then G has a perfect matching.*

Moreover, when G is in the class $\mathscr{G}_1\mathbf{i}$, we can prove that G has a perfect matching by similar arguments as that of Theorem 4.6. When $G \in \mathscr{G}_2(k)$, the problem turns out to be challenge as we cannot use a similar idea as above discussion. However, it would be a stronger result to find a Hamiltonian path of $G \in \mathscr{G}_2(k)$ instead of just finding a perfect matching. Hence, we do not only ask if every G in the class $\mathscr{G}_2(k)$ contains a perfect matching but we also ask whether or not every G in the class $\mathscr{G}_2(k)$ contains a Hamiltonian path. We conclude this chapter with the following problem which will be a part of our next project under sponsorship of Thailand Research Fund 2019.

For $k \geq 4$, does every k-γ_c-critical graph with $k-3$ cut vertices contain a Hamiltonian path?

CHAPTER 5

Hamiltonicity of Double Domination Critical and Stable Graphs

5.1 Double Domination Critical Graphs

In this section, we use the claw-free property to determine when 2-connected k-$\gamma_{\times 2}$-critical claw-free graphs are Hamiltonian. First of all, we give a construction of k-$\gamma_{\times 2}$-critical graphs when $k \geq 4$ which are non-Hamiltonian.

The class $\mathscr{D}(k)$

For $k \geq 4$, let $A = \{a_i b_i : 1 \leq i \leq k-1\}$ be a set of $k-1$ independent edges and let x be an isolated vertex. A graph G in the class $\mathscr{D}(k)$ is constructed by :

- joining x to every vertex in $V(A)$ and
- adding edges so that $b_1, b_2, ..., b_{k-1}$ form a clique.

A graph G in this class is illustrated by Figure 5.1.

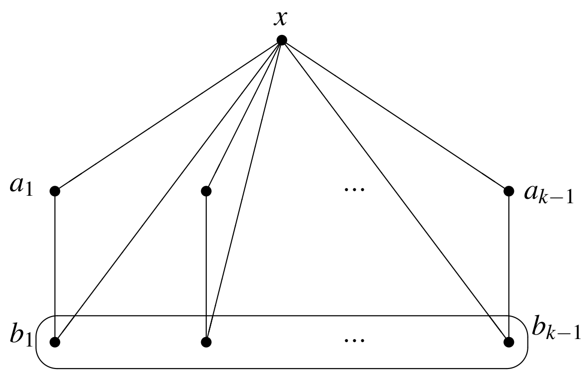


Figure 5.1 : A graph in the class $\mathscr{D}(k)$.

Lemma 5.1. *For an integer $k \geq 4$, if $G \in \mathscr{D}(k)$, then G is a 2-connected k-$\gamma_{\times 2}$-critical non-Hamiltonian graph.*

Proof. We first show that $\gamma_{\times 2}(G) = k$. Obviously, $\{x, a_1, a_2, ..., a_{k-1}\} \succ_{\times 2} G$. By the minimality of $\gamma_{\times 2}(G)$, we have $\gamma_{\times 2}(G) \leq k$. It remains to show that $\gamma_{\times 2}(G) \geq k$. Let D be a $\gamma_{\times 2}$-set of G. To doubly dominate $\{a_1\}$, we must have $|\{x, a_1, b_1\} \cap D| \geq 2$. Similarly, to doubly dominate $\{a_2, a_3, ..., a_{k-1}\}$, we have $|D \cap \{a_i, b_i\}| \geq 1$ for all $2 \leq i \leq k-1$. Thus $|D| \geq k$. This implies that $\gamma_{\times 2}(G) = k$.

We next establish the criticality. Let u and v be a pair of non-adjacent vertices of G. As x is adjacent to every vertex, we must have that $x \notin \{u, v\}$. Thus $\{u, v\} \subseteq \{a_1, b_1, a_2, b_2, ..., a_{k-1}, b_{k-1}\}$. By the construction, at least one of u and v is not in $\{b_1, b_2, ..., b_{k-1}\}$. Without loss of generality let $u = a_1$ and $v \in \{a_2, b_2\}$. Clearly, $\{x, v, a_3, a_4, ..., a_{k-2}, b_{k-1}\} \succ_{\times 2} G + uv$. That is $\gamma_{\times 2}(G + uv) \leq k - 1 < \gamma_{\times 2}(G)$. This establishes the criticality and hence, G is k-$\gamma_{\times 2}$-critical graph.

We finally show that G is non-Hamiltonian. Suppose to the contrary that G is Hamiltonian. Observe that $N_G(a_1) = \{x, b_1\}$. Thus, G is Hamiltonian if and only if $G - a_1$ has a Hamiltonian path P from x to b_1. Since $N_{G-a_1}(a_2) = \{x, b_2\}$, it follows that the path x, a_2, b_2 is a subgraph of P. Similarly, as $N_{G-a_1}(a_3) = \{x, b_3\}$, we must have that the path x, a_3, b_3 is a subgraph of P. Thus b_3, a_3, x, a_2, b_2 is a subgraph of P. This contradicts x is one of the two end vertices of P. Therefore, G is non-Hamiltonian. This completes the proof. □

In the following, we recall the classes $\mathscr{H}_1, \mathscr{H}_2$ and $\mathscr{H}_3$ and the graph $P_{3,3,3}$ from the previous section. We give an observation for the lower bound of the double domination numbers of graphs in these classes.

Observation 5.1.1. Let G be a 2-connected claw-free graph. If $cl(G) \in \mathscr{H}_1 \cup \mathscr{H}_2 \cup \mathscr{H}_3$ or G is isomorphic to $P_{3,3,3}$, then $\gamma_{\times 2}(G) \geq 6$.

Proof. We first consider the case when $cl(G) \in \mathscr{H}_1 \cup \mathscr{H}_2 \cup \mathscr{H}_3$. Let D be a $\gamma_{\times 2}$-set of $cl(G)$. In view of Proposition 2.2, it suffices to show that $\gamma_{\times 2}(cl(G)) \geq 6$. Suppose first that $cl(G) \in \mathscr{H}_1$. Since $|V(Z_i)| \geq 3$, there exist vertices $s_i \in V(Z_i) - \{q_i, z_i\}$ for all $i \in \{1, 2, 3\}$. To doubly dominate $\{s_1, s_2, s_3\}$, we have that $|D \cap V(Z_i)| \geq 2$. Thus $\gamma_{\times 2}(cl(G)) = |D| \geq 6$.

We now suppose that $cl(G) \in \mathscr{H}_2$. Because $|V(T_i)| \geq 3$, there exist vertices $r_i \in V(T_i) - \{c_i, d_i\}$ for all $i \in \{1, 2\}$. To doubly dominate $\{r, r_1, r_2\}$, we have that $|D \cap V(T_i)| \geq 2$ and $|D \cap (V(T_3) \cup \{d_3\})| \geq 2$. Thus $\gamma_{\times 2}(cl(G)) = |D| \geq 6$.

We now suppose that $cl(G) \in \mathscr{H}_3$. Because $|V(K)| \geq 3$, there exists a vertex $s'' \in V(K) - \{s, s'\}$. To doubly dominate $\{s'', h_2, h_5\}$, we have that $|D \cap V(K)| \geq 2, |D \cap \{h_1, h_2, h_3\}| \geq 2$ and $|D \cap \{h_4, h_5, h_6\}| \geq 2$. Thus $\gamma_{\times 2}(cl(G)) = |D| \geq 6$.

We finally consider the case when G is isomorphic to $P_{3,3,3}$. Let D' be a $\gamma_{\times 2}$-set of G. To doubly dominate $\{p_2, p_2', p_2''\}$, we have that $|D' \cap V(P)| \geq 2, |D' \cap V(P')| \geq 2$ and $|D' \cap V(P'')| \geq 2$. Thus $\gamma_{\times 2}(G) = |D'| \geq 6$. This completes the proof. □

We next establish the following lemma concerning the minimum number of vertices of a double dominating set when some independent set is given.

Lemma 5.2. *Let G be a claw-free graph, I be an independent set and X be a set of vertices such that $X \succ_{\times 2} I$. If there exists a vertex in $X - I$ adjacent to at most one vertex in I, then $|I| + 1 \leq |X|$.*

Proof. Let w be a vertex in $X - I$ which is adjacent to at most one vertex in I. Moreover, we let $I_1 = X \cap I, I_2 = I - I_1$ and $X' = X - (I_1 \cup \{w\})$. Clearly, $|X| = |X'| + |I_1| + 1$ and $|I| = |I_1| + |I_2|$. Let H be a subgraph of G such that $V(H) = X' \cup \{w\} \cup I$ and $E(H) = \{uv \in E(G) : u \in X' \cup \{w\}$ and $v \in I\}$. Clearly, H is bipartite with the bipartition sets $X' \cup \{w\}$ and I. Since $X \succ_{\times 2} I$, every vertex in I_1 is adjacent to at least one vertex in $X' \cup \{w\}$. Moreover, every vertex in I_2 is adjacent to at least two vertices in $X' \cup \{w\}$. Thus, $deg_H(v) \geq 1$ for all $v \in I_1$ and $deg_H(v) \geq 2$ for all $v \in I_2$. This gives the degree sum of vertices in I as the following:

$$|I_1| + 2|I_2| \leq \Sigma_{v \in I} deg_H(v). \tag{5.1.1}$$

Because G is claw-free, every vertex in X' is adjacent to at most two vertices in I. Therefore $deg_H(u) \leq 2$ for all $u \in X'$. Since w is adjacent to at most one vertex in I, it follows that

$$\Sigma_{u \in X' \cup \{w\}} deg_H(u) \leq 2|X'| + 1. \tag{5.1.2}$$

Because H is bipartite, $\Sigma_{v \in I} deg_H(v) = \Sigma_{u \in X' \cup \{w\}} deg_H(u)$. By (5.1.1) and (5.1.2), we have $|I_1| + 2|I_2| \leq 2|X'| + 1$. Hence,

$$|I| = |I_1| + |I_2| \leq (\frac{|I_1|}{2} + |I_2|) + \frac{|I_1|}{2} \leq (|X'| + \frac{1}{2}) + \frac{|I_1|}{2} < |X'| + |I_1| + 1 = |X|.$$

This completes the proof. □

We are now ready to establish our main theorems. We recall a net N_{s_1,s_2,s_3} from the first section.

Theorem 5.1. *Let G be a 2-connected k-$\gamma_{\times 2}$-critical claw-free graph. If $2 \leq k \leq 5$, then G is Hamiltonian.*

Proof. We first show that G is $\{K_{1,3}, N_{1,2,2}, N_{1,1,3}\}$-free. Suppose to the contrary that G contains $N_{1,2,2}$ or $N_{1,1,3}$ as an induced subgraph. We first consider the case when G contains $N_{1,2,2}$ as an induced subgraph. Consider $G+v_1w_1$. By Observation 2.3.1, $|D_{v_1w_1} \cap \{v_1,w_1\}| \geq 1$. Suppose that $|D_{v_1w_1} \cap \{v_1,w_1\}| = 1$. By symmetry, we let $w_1 \in D_{v_1w_1}$. Because $D_{v_1w_1}$ is a double dominating set, w_1 is adjacent to a vertex in $D_{v_1w_1}$, w say. Clearly, $\{u_1,v_2,w_1,w_3\}$ is an independent set. By claw-freeness, w is adjacent to at most one vertex in $\{u_1,v_2,w_3\}$. Let $X = D_{v_1w_1} - \{w_1\}$. Clearly, $X \succ_{\times 2} \{u_1,v_2,w_3\}$ and X contains w. In view of Lemma 5.2, $|X| \geq 4$. This implies that $|D_{v_1w_1}| \geq 5$ contradicting the criticality of G. Suppose that $\{v_1,w_1\} \subseteq D_{v_1w_1}$. Let $X' = D_{v_1w_1} - \{w_1\}$. Thus $X' \succ_{\times 2} \{u_1,v_2,w_3\}$ and X' contains v_1 which is adjacent to at most one vertex in $\{u_1,v_2,w_3\}$. This implies by Lemma 5.2 that $|X'| \geq 4$. Hence, $|D_{v_1w_1}| \geq 5$ contradicting the criticality of G. Therefore, G does not contains $N_{1,2,2}$ as an induced subgraph.

We now consider the case when G contains $N_{1,1,3}$ as an induced subgraph. Consider $G+u_1v_1$. By Observation 2.3.1, $|D_{u_1v_1} \cap \{u_1,v_1\}| \geq 1$. Suppose that $|D_{u_1v_1} \cap \{u_1,v_1\}| = 1$. By symmetry, we let $u_1 \in D_{u_1v_1}$. As $D_{u_1v_1}$ is a double dominating set, we must have that u_1 is adjacent to a vertex u in $D_{u_1v_1}$. Clearly, $\{u_1,v_2,w_1,w_3\}$ is an independent set. Since G is claw-free, u is adjacent to at most one vertex in $\{v_2,w_1,w_3\}$. Let $Y = D_{u_1v_1} - \{u_1\}$. Clearly, $Y \succ_{\times 2} \{v_2,w_1,w_3\}$ and Y contains u. In view of Lemma 5.2, $|Y| \geq 4$. Thus $|D_{u_1v_1}| \geq 5$ contradicting the criticality of G. We then suppose that $\{u_1,v_1\} \subseteq D_{u_1v_1}$. Let $Y' = D_{u_1v_1} - \{u_1\}$. Thus $Y' \succ_{\times 2} \{v_2,w_1,w_3\}$ and Y' contains v_1 which is adjacent to at most one vertex in $\{v_2,w_1,w_3\}$. This implies by Lemma 5.2 that $|Y'| \geq 4$. Hence, $|D_{u_1v_1}| \geq 5$ contradicting the criticality of G. Therefore, G does not contains $N_{1,1,3}$ as an induced subgraph. Hence, G is $\{K_{1,3},N_{1,2,2},N_{1,1,3}\}$-free.

Because $\gamma_{\times 2}(G) \leq 5$, by Observation 5.1.1, G is not isomorphic to $P_{3,3,3}$ and $cl(G) \notin \mathscr{H}_1 \cup \mathscr{H}_2 \cup \mathscr{H}_3$. In view of Theorem 2.1, G is Hamiltonian. This completes the proof. □

We can see that a graph G in the class $\mathscr{D}(k)$ when $4 \leq k \leq 5$ is non-Hamiltonian. Thus the condition claw-free in Theorem 5.1 is necessary. Moreover, when $k = 4$, the graph in the class $\mathscr{D}(k)$ is $K_{1,4}$-free. Hence, the condition claw-free is best possible for $k = 4$. We conclude this section with the following theorem which shows that a graph k-$\gamma_{\times 2}$-critical when $6 \leq k \leq 7$ is Hamiltonian if it is 3-connected and claw-free.

Theorem 5.2. *For an integer $2 \leq k \leq 7$, let G be a 3-connected k-$\gamma_{\times 2}$-critical claw-free graph. Then G is Hamiltonian.*

Proof. We will show that G is $N_{3,3,3}$-free. Suppose to the contrary that G contains $N_{3,3,3}$ as an induced subgraph. Consider $G+u_1v_1$. Observation 2.3.1 yields that, $|D_{u_1v_1}\cap\{u_1,v_1\}|\geq 1$. Suppose first that $|D_{u_1v_1}\cap\{u_1,v_1\}|=1$. By symmetry, we may let $u_1\in D_{u_1v_1}$. It is easy to see that $\{u_1,u_3,v_2,v_4,w_1,w_3\}$ is an independent set. As $D_{u_1v_1}$ is a double dominating set, we must have that u_1 is adjacent to a vertex u in $D_{u_1v_1}$. Since G is claw-free, u is adjacent to at most one vertex in $\{u_3,v_2,v_4,w_1,w_3\}$. Let $X=D_{u_1v_1}-\{u_1\}$. Clearly, $X\succ_{\times 2}\{u_3,v_2,v_4,w_1,w_3\}$ and X contains u. Lemma 5.2 implies that $|X|\geq 6$. Thus $|D_{u_1v_1}|\geq 7$ contradicting the criticality of G. We then suppose that $\{u_1,v_1\}\subseteq D_{u_1v_1}$. Let $X'=D_{u_1v_1}-\{u_1\}$. Thus $X'\succ_{\times 2}\{u_3,v_2,v_4,w_1,w_3\}$ and X' contains v_1 which is adjacent to at most one vertex in $\{u_3,v_2,v_4,w_1,w_3\}$. This implies by Lemma 5.2 that $|X'|\geq 6$. Hence, $|D_{u_1v_1}|\geq 7$ contradicting the criticality of G. Therefore, G does not contains $N_{3,3,3}$ as an induced subgraph. Theorem 2.2 implies that G is Hamiltonian. This completes the proof. □

We see that the graphs in the class $\mathscr{D}(k)$ when $6\leq k\leq 7$ are non-Hamiltonian. Thus, the condition claw-free together with 3-connected is necessary in Theorem 5.2.

5.2 Double Domination Stable Graphs

In this section, we use the claw-free property to determine when 2-connected k-$\gamma^{+}_{\times 2}$-stable claw-free graphs and 2-connected k-$\gamma^{-}_{\times 2}$-stable claw-free graphs are Hamiltonian. We first establish the following lemma concerning the minimum number of vertices of a double dominating set when some independent set is given. The proof of which is similar to Lemma 5.2. For completeness, we provide the proof.

Lemma 5.3. *Let G be a claw-free graph, I be an independent set and X be a set of vertices such that $X\succ_{\times 2} I$. Then $|I|\leq|X|$.*

Proof. Let $I_1=X\cap I, I_2=I-I_1$ and $X'=X-I_1$. Clearly, $|X|=|X'|+|I_1|$ and $|I|=|I_1|+|I_2|$. Let H be a subgraph of G such that $V(H)=X'\cup I$ and $E(H)=\{uv\in E(G): u\in X'$ and $v\in I\}$. Clearly, H is bipartite with the bipartition sets X' and I. Since $X\succ_{\times 2} I$, every vertex in I_1 is adjacent to at least one vertex in X' and, every vertex in I_2 is adjacent to at least two vertices in X'. Thus, $deg_H(v)\geq 1$ for all $v\in I_1$ and $deg_H(v)\geq 2$ for all $v\in I_2$. This gives the degree sum of vertices in I as the following:

$$|I_1|+2|I_2|\leq\Sigma_{v\in I}deg_H(v). \tag{5.2.1}$$

Because G is claw-free, every vertex in X' is adjacent to at most two vertices in I.

Therefore $deg_H(u) \leq 2$ for all $u \in X'$. Thus,

$$\Sigma_{u\in X'} deg_H(u) \leq 2|X'| = 2|X| - 2|I_1|. \quad (5.2.2)$$

Because H is bipartite, $\Sigma_{v\in I} deg_H(v) = \Sigma_{u\in X'} deg_H(u)$. By (5.2.1) and (5.2.2), we have $|I_1| + 2|I_2| \leq 2|X| - 2|I_1|$. Hence,

$$|I| = |I_1| + |I_2| \leq 3|I_1|/2 + |I_2| \leq |X|.$$

This completes the proof. □

By Lemma 5.3 and Theorems 2.1 and 2.2, we easily establish the following corollaries.

Corollary 5.2.1. Let G be a 2-connected claw-free graph with $\gamma_{\times 2}(G) \leq 3$. Then G is Hamiltonian.

Proof. By Observation 5.1.1, $cl(G) \notin \mathscr{H}_1 \cup \mathscr{H}_2 \cup \mathscr{H}_3$ and G is not isomorphic to $P_{3,3,3}$. Thus, by Theorem 2.1, it suffices to show that G is $\{N_{1,2,2}, N_{1,1,3}\}$-free. Suppose to the contrary that G contains $N_{1,2,2}$ as an induced subgraph. Clearly, $\{u_1, v_1, v_3, w_2\}$ is an independent set of four vertices. Lemma 5.3 yields that $4 \leq \gamma_{\times 2}(G) \leq 3$, a contradiction. Thus, G is $N_{1,2,2}$-free. We can prove that G is $N_{1,1,3}$ by the same arguments. Thus, by Theorem 2.1, G is Hamiltonian. □

Corollary 5.2.2. Let G be a 3-connected claw-free graph with $\gamma_{\times 2}(G) \leq 5$. Then G is Hamiltonian.

Proof. By Theorem 2.2, it suffices to show that G is $N_{3,3,3}$-free. Suppose to the contrary that G contains $N_{3,3,3}$ as an induced subgraph. Clearly, $\{u_1, u_3, v_1, v_3, w_1, w_3\}$ is an independent set of six vertices. Lemma 5.3 gives that $6 \leq \gamma_{\times 2}(G) \leq 5$, a contradiction. Thus, G is $N_{3,3,3}$-free. By Theorem 2.2, G is Hamiltonian. □

5.2.1 k-$\gamma^+_{\times 2}$-Stable Claw-Free Graphs

In this subsection, we study Hamiltonian property of k-$\gamma^+_{\times 2}$-stable claw-free graphs. Although all 2-$\gamma_{\times 2}$-critical graphs of order at leat three are Hamiltonian because they are complete graphs, this is not always true for 2-$\gamma^+_{\times 2}$-stable graphs. That is there exist 2-$\gamma^+_{\times 2}$-stable graphs which are non-Hamiltonian. We first give a construction of k-$\gamma^+_{\times 2}$-stable graphs when $k \geq 2$ which are non-Hamiltonian.

The class $\mathscr{S}^+(k)$

For $k \geq 2$, let K_k be a complete graph of order k with the vertices $x_1, x_2, ..., x_k$ and for $1 \le i \neq j \le k$, we let $a_{\{i,j\}}, b_{\{i,j\}}, c_{\{i,j\}}$ be $3 \cdot \binom{k}{2}$ isolated vertices. The graph G in this class is obtained from K_k and all the $3 \cdot \binom{k}{2}$ isolated vertices by adding the edges $a_{\{i,j\}}x_p, b_{\{i,j\}}x_p, c_{\{i,j\}}x_p$ for all $1 \le i \neq j \le k$ and for all $p \in \{i,j\}$. The following lemma establishes the properties of the graphs in the class $\mathscr{S}^+(k)$.

Lemma 5.4. *For an integer $k \ge 2$, if $G \in \mathscr{S}^+(k)$, then G is a 2-connected k-$\gamma^+_{\times 2}$-stable non-Hamiltonian graph.*

Proof. Clearly, G is 2-connected. Moreover, it is easy to see that G is k-$\gamma^+_{\times 2}$-stable when $k = 2$. Hence, we assume that $k \ge 3$. We first show that $\gamma_{\times 2}(G) = k$. Obviously, $V(K_k) \succ_{\times 2} G$. By the minimality of $\gamma_{\times 2}(G)$, we have $\gamma_{\times 2}(G) \le k$. Let D be a $\gamma_{\times 2}$-set of G. So, $|D| \le k$. We will show that $V(K_k) \subseteq D$. Suppose to the contrary that $\{x_1, x_2, ..., x_k\} \nsubseteq D$. Without loss of generality, let $x_1 \notin D$. To doubly dominate $A_1 = \{a_{\{1,j\}}, b_{\{1,j\}}, c_{\{1,j\}} : 1 < j \le k\}$, we must have that $A_1 \subseteq D$. Since $k \ge 3$, it follows that $k \ge |D| \ge |A_1| = 3k - 3 > k$, a contradiction. Thus $V(K_k) \subseteq D$ and $|D| \ge k$. This implies that $\gamma_{\times 2}(G) = k$.

We next establish the stability. Let u and v be a pair of non-adjacent vertices of G and let D_{uv} be a $\gamma_{\times 2}$-set of $G + uv$. Because $V(K_k) \succ_{\gamma_{\times 2}} G + uv$, it follows that $|D_{uv}| \le |V(K_k)| = k$. It suffice to show that $|D_{uv}| \ge k$. By the construction, $|\{u,v\} \cap V(K_k)| \le 1$. We first consider the case when $|\{u,v\} \cap V(K_k)| = 1$. Without loss of generality let $u = x_1$ and $v = a_{\{2,3\}}$. Suppose that there exists $x_i \notin D_{uv}$. If $i \neq \{1,2,3\}$, then, to doubly dominate $A_i = \{a_{\{i,j\}}, b_{\{i,j\}}, c_{\{i,j\}} : 1 \le j \le k$ and $j \neq i\}$, we must have that $A_i \subseteq D_{uv}$. This implies that $|D_{uv}| \ge 3k - 3 > k$ contradicting $|D_{uv}| \le k$. Thus, $i \in \{1,2,3\}$. To doubly dominate $A_i - \{a_{\{2,3\}}\}$, we must have that $(A_i - \{a_{\{2,3\}}\}) \subseteq D_{uv}$. This implies that $k \ge |D_{uv}| \ge 3k - 4 > k$, a contradiction. Thus, $V(K_k) \subseteq D_{uv}$. This implies that $|D_{uv}| \ge k$.

We now consider the case when $|\{u,v\} \cap V(K_k)| = 0$. Similarly, suppose that there exists $x_i \notin D_{uv}$. To doubly dominate $A_i - \{u,v\}$, we must have that $(A_i - \{u,v\}) \subseteq D_{uv}$. This implies that $k \ge |D_{uv}| \ge (3k-3) - 2 > k$, a contradiction. Thus, $V(K_k) \subseteq D_{uv}$. This implies that $|D_{uv}| \ge k$. This establishes the stability and hence, G is k-$\gamma^+_{\times 2}$-stable graph.

We finally show that G is non-Hamiltonian. Clearly, $V(K_k)$ is a cut set of G such that $G - V(K_k)$ has $3 \cdot \binom{k}{2}$ isolated vertices as the components. Thus, $\frac{|V(K_k)|}{\omega(G - V(K_k))} < 1$. By Proposition 2.1, G is non-Hamiltonian. This completes the proof. □

In views of Lemma 5.4, there exist 2-connected k-$\gamma^+_{\times 2}$-stable graphs which are non-Hamiltonian for all $k \ge 2$. However, by using Corollaries 5.2.1 and 5.2.2, we easily obtain that all 2-connected k-$\gamma^+_{\times 2}$-stable graphs are Hamiltonian when k is small. The proof are omitted as the class of 2-connected k-$\gamma^{+}_{\times 2}$-stable graphs is a subclass of graphs with $\gamma_{\times 2}(G) = k$.

Corollary 5.2.3. Let G be a 2-connected k-$\gamma^{+}_{\times 2}$-stable claw-free graph. If $2 \leq k \leq 3$, then G is Hamiltonian.

Corollary 5.2.4. For an integer $2 \leq k \leq 5$, let G be a 3-connected k-$\gamma^{+}_{\times 2}$-stable claw-free graph. Then G is Hamiltonian.

Observe that a graph $G \in \mathscr{S}^{+}(2)$ is $K_{1,4}$-free. Hence, the condition claw-free in Corollaries 5.2.1 and 5.2.3 is best possible when $k = 2$. For a graph $G \in \mathscr{S}^{+}(3)$, it is easy to see that the graph $G' = G - c_{\{1,2\}} - b_{\{1,3\}} - c_{\{1,3\}} - b_{\{2,3\}} - c_{\{2,3\}}$ is 3-$\gamma^{+}_{\times 2}$-stable $K_{1,4}$-free graph. Hence, the condition claw-free in Corollaries 5.2.1 and 5.2.3 is best possible when $k = 3$.

5.2.2 k-$\gamma^{-}_{\times 2}$-Stable Claw-Free Graphs

First of all, we give a construction of k-$\gamma^{-}_{\times 2}$-stable graphs when $k \geq 2$ which are non-Hamiltonian.

The class $\mathscr{S}^{-}(k)$

For $k \geq 2$, we let K_{2k} be a complete graph of order $2k$ with the vertices $x_1, y_1, x_2, y_2, \ldots, x_k, y_k$ and for $1 \leq i \neq j \leq k$, we let $a_{\{i,j\}}, b_{\{i,j\}}, c_{\{i,j\}}, d_{\{i,j\}}, e_{\{i,j\}}$ be $5 \cdot \binom{k}{2}$ isolated vertices. The graph G in this class is obtained from K_{2k} and all the $5 \cdot \binom{k}{2}$ isolated vertices by joining the vertices x_p and y_p to $a_{\{i,j\}}, b_{\{i,j\}}, c_{\{i,j\}}$, $d_{\{i,j\}}, e_{\{i,j\}}$ for all $1 \leq i \neq j \leq k$ and for all $p \in \{i, j\}$. The following lemma establishes the properties of the graphs in the class $\mathscr{S}^{-}(k)$.

Lemma 5.5. *For an integer $k \geq 2$, if $G \in \mathscr{S}^{-}(k)$, then G is a 2-connected k-$\gamma^{-}_{\times 2}$-stable non-Hamiltonian graph.*

Proof. Clearly, G is 2-connected. Moreover, it is easy to see that G is k-$\gamma^{-}_{\times 2}$-stable when $k = 2$. Hence, we assume that $k \geq 3$. We first show that $\gamma_{\times 2}(G) = k$. Since $\{x_1, x_2, \ldots, x_k\} \succ_{\times 2} G$, it follows that $\gamma_{\times 2}(G) \leq k$. It remains to show that $\gamma_{\times 2}(G) \geq k$. Let D be a $\gamma_{\times 2}$-set of G. By the minimality of D, we have $|D| \leq k$. If $|\{x_i, y_i\} \cap D| \geq 1$ for all $1 \leq i \leq k$, then $|D| \geq k$ as required. We may suppose that there exists $i \in \{1, 2, \ldots, k\}$ such that $\{x_i, y_i\} \cap D = \emptyset$. To doubly dominate $A_i = \{a_{\{i,j\}}, b_{\{i,j\}}, c_{\{i,j\}}, d_{\{i,j\}}, e_{\{i,j\}} : 1 \leq j \leq k$ and $j \neq i\}$, we must have that $|D \cap \{a_{\{i,j\}}, b_{\{i,j\}}, c_{\{i,j\}}, d_{\{i,j\}}, e_{\{i,j\}}, x_j, y_j\}| \geq 2$ for all $1 \leq j \leq k$ and $j \neq i$. This implies that $k \geq |D| \geq 2(k-1) > k$, a contradiction. Thus, $\gamma_{\times 2}(G) = |D| \geq k$. Therefore $\gamma_{\times 2}(G) = k$.

We next establish the stability. Let u and v be a pair of adjacent vertices of G. By Observation 2.3.2, we have $\gamma_{\times 2}(G-uv) \geq k$. Hence, it suffices to show that there exists a $\gamma_{\times 2}$-set of $G-uv$ containing k vertices. Clearly, $|\{u,v\} \cap V(K_{2k})| \geq 1$. We first suppose that $|\{u,v\} \cap V(K_{2k})| = 1$. Without loss of generality let $u \in V(K_{2k})$. If $u \in \{x_1,x_2,...,x_k\}$, then $\{y_1,y_2,...,y_k\} \succ_{\times 2} G-uv$. If $u \in \{y_1,y_2,...,y_k\}$, then $\{x_1,x_2,...,x_k\} \succ_{\times 2} G-uv$. Hence, we now suppose that $u,v \in V(K_{2k})$. We consider the case when $\{u,v\} = \{x_i,y_j\}$ for some $i,j \in \{1,2,...,k\}$. Clearly $\{x_1,x_2,...,x_k\} \succ_{\times 2} G-uv$. We now consider the case when $\{u,v\} = \{x_i,x_j\}$. Thus, $\{y_1,y_2,...,y_k\} \succ_{\times 2} G-uv$. Similarly, $\{x_1,x_2,...,x_k\} \succ_{\times 2} G-uv$ when $\{u,v\} = \{y_i,y_j\}$. Therefore, G is k-$\gamma_{\times 2}^{-}$-stable graph.

We finally show that G is non-Hamiltonian. Clearly, $V(K_{2k})$ is a cut set of G such that $G-V(K_{2k})$ has $5 \cdot \binom{k}{2}$ isolated vertices as the components. Thus, $\frac{|V(K_{2k})|}{\omega(G-V(K_k))} < 1$. By Proposition 2.1, G is non-Hamiltonian. This completes the proof. □

We next establish the following theorems.

Theorem 5.3. *Let G be a 2-connected k-$\gamma_{\times 2}^{-}$-stable claw-free graph. If $2 \leq k \leq 4$, then G is Hamiltonian.*

Proof. We first show that G is $\{N_{1,2,2}, N_{1,1,3}\}$-free. Suppose to the contrary that G contains $N_{1,2,2}$ or $N_{1,1,3}$ as an induced subgraph. We first consider the case when G contains $N_{1,2,2}$ as an induced subgraph. Consider $G-v_1v_2$. By stability of G, $\gamma_{\times 2}(G-v_1v_2) = k$. Clearly, $\{u_1,v_1,v_2,w_1,w_3\}$ is an independent set of $G-v_1v_2$ containing 5 vertices. Lemma 5.3 implies that $4 \geq \gamma_{\times 2}(G-v_1v_2) \geq 5$, a contradiction. Thus, G is $N_{1,2,2}$-free.

We now consider the case when G contains $N_{1,1,3}$ as an induced subgraph. Consider $G-u_1u_2$. By stability of G, $\gamma_{\times 2}(G-u_1u_2) = k$. Clearly, $\{u_1,u_2,v_1,w_1, w_3\}$ is an independent set of $G-u_1u_2$ containing 5 vertices. Lemma 5.3 implies that $4 \geq \gamma_{\times 2}(G-v_1v_2) \geq 5$, a contradiction. Thus, G is $N_{1,1,3}$-free. Hence, G is $\{K_{1,3}, N_{1,2,2}, N_{1,1,3}\}$-free.

Because $\gamma_{\times 2}(G) \leq 4$, by Observation 5.1.1, G is not isomorphic to $P_{3,3,3}$ and $cl(G) \notin \mathscr{H}_1 \cup \mathscr{H}_2 \cup \mathscr{H}_3$. In view of Theorem 2.1, G is Hamiltonian. This completes the proof. □

Theorem 5.4. *Let G be a 3-connected k-$\gamma_{\times 2}^{-}$-stable claw-free graph. If $2 \leq k \leq 6$, then G is Hamiltonian.*

Proof. We will show that G is $N_{3,3,3}$-free. Suppose to the contrary that G contains $N_{3,3,3}$ as an induced subgraph. Consider $G-u_1u_2$. By stability of G, $\gamma_{\times 2}(G-u_1u_2) =$

k. We see that $\{u_1, u_2, u_4, v_1, v_3, w_1, w_3\}$ is an independent set of $G - u_1u_2$ containing 7 vertices. By Lemma 5.3, $6 \geq \gamma_{\times 2}(G - v_1v_2) \geq 7$, a contradiction. Therefore, G does not contain $N_{3,3,3}$ as an induced subgraph. Theorem 2.2 implies that G is Hamiltonian. This completes the proof. □

5.3 Discussion

On double domination critical graphs. For $6 \leq k \leq 7$, we have seen neither 2-connected k-$\gamma_{\times 2}$-critical claw-free graphs which are non-Hamiltonian nor 3-connected k-$\gamma_{\times 2}$-critical graphs which are non-Hamiltonian. Hence, the questions that arise are, for an integer $6 \leq k \leq 7$,

is every 2-connected k-$\gamma_{\times 2}$-critical claw-free graph Hamiltonian?

and

is every 3-connected k-$\gamma_{\times 2}$-critical graph Hamiltonian?

On double domination stable graphs. We have seen neither 2-connected k-$\gamma^{+}_{\times 2}$-stable claw-free graphs which are non-Hamiltonian for $4 \leq k \leq 5$ nor 2-connected k-$\gamma^{-}_{\times 2}$-stable graphs which are non-Hamiltonian for $5 \leq k \leq 6$. Hence, the questions that arise are, for $4 \leq k \leq 5$,

is every 2-connected k-$\gamma^{+}_{\times 2}$-stable claw-free graph Hamiltonian?

and, for $5 \leq k \leq 6$,

is every 2-connected k-$\gamma^{-}_{\times 2}$-stable claw-free graph Hamiltonian?

CHAPTER 6

Outputs and Publications

This research has 3 manuscripts as the output. Our manuscript entitled Connected Domination Critical Graphs with Cut Vertices is accepted by the journal Discussiones Mathematicae Graph Theory which is in Quartile 3 in Web of Science and has Impact Factor 0.302. Now, we are preparing to submit another work which is the result concerning the characterization of k-γ_c-critical graphs having $k-3$ cut vertices.

Further, when we reviewed related results in domination critical graphs, we have got an idea and proved some results related to critical graphs but with different domination number. That is we established Hamiltonian property of double domination critical graphs. So, this work is also considered a part of our project that we have done under Thailand Research Fund sponsorship. The title of work is Hamiltonicities of Double Domination Critical and Stable Claw-Free Graphs. This work is also accepted by the journal Discussiones Mathematicae Graph Theory. The content of this part is given in Chapter 5. So, the following are our publications and conference presentation.

1. Publications

- P. Kaemawichanurat. "Hamiltonicities of Double Domination Critical and Stable Claw-Free Graphs", accepted by Discussiones Mathematicae Graph Theory.
- P. Kaemawichanurat and N. Ananchuen. "Connected Domination Critical Graphs with Cut Vertices", accepted by Discussiones Mathematicae Graph Theory.
- P. Kaemawichanurat. "The Characterization of k-γ_c-Critical Graphs with $k-3$ Cut Vertices", in preparation.

2. Conference Presentations

- Connected Domination Critical Graphs with Cut Vertices, 7th Polish Combinatorial Conference(7PCC), Bedlewo, Poland, 2018.
- Connected Domination Critical Graphs with Cut Vertices, TRF-OHEC Annual Congress 2019, Phetchaburi, Thailand, 2019.

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Appendix

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HAMILTONICITIES OF DOUBLE DOMINATION CRITICAL AND STABLE CLAW-FREE GRAPHS

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Abstract

A graph G with the double domination number $\gamma_{\times 2}(G) = k$ is said to be k-$\gamma_{\times 2}$-critical if $\gamma_{\times 2}(G + uv) < k$ for any $uv \notin E(G)$. On the other hand, a graph G with $\gamma_{\times 2}(G) = k$ is said to be k-$\gamma^{+}_{\times 2}$-stable if $\gamma_{\times 2}(G + uv) = k$ for any $uv \notin E(G)$ and is said to be k-$\gamma^{-}_{\times 2}$-stable if $\gamma_{\times 2}(G - uv) = k$ for any $uv \in E(G)$. The problem of interest is to determine whether or not 2-connected k-$\gamma_{\times 2}$-critical graphs are Hamiltonian. In this paper, for $k \geq 4$, we provide a 2-connected k-$\gamma_{\times 2}$-critical graph which is non-Hamiltonian. We prove that all 2-connected k-$\gamma_{\times 2}$-critical claw-free graphs are Hamiltonian when $2 \leq k \leq 5$. We show that the condition claw-free when $k = 4$ is best possible. We further show that every 3-connected k-$\gamma_{\times 2}$-critical claw-free graph is Hamiltonian when $2 \leq k \leq 7$. We also investigate Hamiltonian properties of k-$\gamma^{+}_{\times 2}$-stable graphs and k-$\gamma^{-}_{\times 2}$-stable graphs.

Keywords: double domination, critical, stable, Hamiltonian.

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1. Introduction

All graphs in this paper are connected and simple (i.e., no loops or multiple edges). We let G denote a finite graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex subset S of G, $\langle S \rangle$ denotes the subgraph of G induced by S. The *neighborhood* $N_G(x)$ of a vertex x in G is the set of vertices of G which are adjacent to x. The *degree* $\deg_G(v)$ of a vertex v in G is $|N_G(v)|$. For a vertex subset X and a vertex y of G, we let $N_X[y] = (N_G(y) \cap X) \cup \{y\}$. For a graph G, $\omega(G)$ denotes the number of components of G. A *cut set* S is a vertex subset for which $\omega(G-S) > \omega(G)$. The *connectivity* κ is the minimum cardinality of a cut set. A graph G is *l-connected* if $\kappa \geq l$. An *independent set* is a set of pairwise non-adjacent vertices. A graph G is *bipartite* if there exists a bipartition X and Y of $V(G)$ such that X and Y are independent sets. A *complete bipartite graph* $K_{m,n}$ is a bipartite graph with the partite sets X and Y such that $|X| = m$ and $|Y| = n$ containing all edges joining the vertices between X and Y. A *star* $K_{1,n}$ is a complete bipartite graph when $m = 1$, in particular if $n = 3$, a star $K_{1,3}$ is called a *claw*. For integers $s_1, s_2, s_3 \geq 1$, let $u_1, u_2, \ldots, u_{s_1+1}$; $v_1, v_2, \ldots, v_{s_2+1}$ and $w_1, w_2, \ldots, w_{s_3+1}$ be three disjoint paths of length s_1, s_2 and s_3, respectively. A *net* N_{s_1,s_2,s_3} is constructed by adding edges $u_{s_1+1}v_{s_2+1}, v_{s_2+1}w_{s_3+1}$ and $w_{s_3+1}u_{s_1+1}$. For a family of graphs $\mathcal{F}$, a graph G is said to be *$\mathcal{F}$-free* if there is no induced subgraph of G isomorphic to H for all $H \in \mathcal{F}$.

For vertex subsets X and Y of G, we say that X *doubly dominates* Y if $|N_X[y]| \geq 2$ for all $y \in Y$. We write $X \succ_{\times 2} Y$ if X doubly dominates Y. Moreover, if $Y = V(G)$, then X is a *double dominating set* of G. A smallest double dominating set of G is called a *$\gamma_{\times 2}$-set* of G. The *double domination number* of G is the cardinality of a $\gamma_{\times 2}$-set of G and is denoted by $\gamma_{\times 2}(G)$. A graph G is said to be *k double domination critical*, or *k-$\gamma_{\times 2}$-critical*, if $\gamma_{\times 2}(G) = k$ and $\gamma(G + uv) < k$ for all $uv \notin E(G)$. On the other hand, a graph G is said to be *double domination edge addition stable*, or *k-$\gamma^+_{\times 2}$-stable*, if $\gamma_{\times 2}(G) = k$ and $\gamma(G + uv) = k$ for all $uv \notin E(G)$ and a graph G is said to be *double domination edge removal stable*, or *k-$\gamma^-_{\times 2}$-stable*, if $\gamma_{\times 2}(G) = k$ and $\gamma(G - uv) = k$ for all $uv \in E(G)$. A graph which is either k-$\gamma^+_{\times 2}$-stable or k-$\gamma^-_{\times 2}$-stable is called *double domination stable*.

This paper focuses on the Hamiltonicity of double domination critical graphs and double domination stable graphs. It is worth noting that there are some results concerning Hamiltonicities of critical graph with respect to other types of domination numbers. For example, see [1,5,6,8–11,13,17,19]. For related results in k-$\gamma_{\times 2}$-critical graphs, Thacker [12] first studied these graphs. He characterized 3-$\gamma_{\times 2}$-critical graphs and 4-$\gamma_{\times 2}$-critical graphs with maximum diameter. It is easy to see that 2-$\gamma_{\times 2}$-critical graphs are complete graphs of order at least two. When $k = 4$, Wang and Kang [14] showed that G is factor-critical if G is a connected 4-$\gamma_{\times 2}$-critical $K_{1,4}$-free graph of odd order with minimum degree two. Wang and Shan [15] showed further that if the order is even and at least six then the connected 4-$\gamma_{\times 2}$-critical $K_{1,4}$-free graph has a perfect matching except one family of graphs. Moreover, if G is a 2-connected 4-$\gamma_{\times 2}$-critical claw-free of even order with minimum degree three or G is a 3-connected 4-$\gamma_{\times 2}$-critical $K_{1,4}$-free of even order with minimum degree four, then G is bi-critical. Recently, Wang *et al.* [16] established that if a graph G is a 3-connected 4-$\gamma_{\times 2}$-critical claw-free graph of odd order with minimum degree at least four, then G is 3-factor-critical except one family of graphs. All the related results have not been done when $k \geq 5$. In double domination stable graphs, we introduce a new concept in k-$\gamma^+_{\times 2}$-stable graphs and investigate their Hamiltonian property in this paper. For k-$\gamma^-_{\times 2}$-stable graphs, Chellali and Haynes [4] established fundamental properties of these graphs.

In this paper, we proceed as follows. In Section 2, we provide some results that we use in our proofs. In Section 3, for $k \geq 4$, we give a construction of a 2-connected k-$\gamma_{\times 2}$-critical graph which is non-Hamiltonian. We prove that 2-connected k-$\gamma_{\times 2}$-critical claw-free graphs are Hamiltonian when $2 \leq k \leq 5$. By the construction, we have that the condition claw-free is sharp when $k = 4$. We show further that every 3-connected k-$\gamma_{\times 2}$-critical claw-free graph is Hamiltonian when $2 \leq k \leq 7$. In Section 4, for $k \geq 2$, we give constructions of a class of k-$\gamma^+_{\times 2}$-stable non-Hamiltonian graphs and a class of k-$\gamma^-_{\times 2}$-stable non-Hamiltonian graphs. We prove that 2-connected k-$\gamma^+_{\times 2}$-stable claw-free graphs are Hamiltonian when $2 \leq k \leq 3$. We also prove that 3-connected k-$\gamma^+_{\times 2}$-stable claw-free graphs are Hamiltonian when $2 \leq k \leq 5$. For k-$\gamma^-_{\times 2}$-stable graphs, we prove that 2-connected k-$\gamma^-_{\times 2}$-stable claw-free graphs are Hamiltonian when $2 \leq k \leq 4$. We also prove that 3-connected k-$\gamma^-_{\times 2}$-stable claw-free graphs are Hamiltonian when $2 \leq k \leq 6$.

2. PRELIMINARIES

In this section, we state a number of results from the literature that we make use of in our work. We begin with a result of Chvátal [3] which is a well known property of a Hamiltonian graph.

Proposition 1 [3]**.** *If G is a Hamiltonian graph, then $\frac{|S|}{\omega(G-S)} \geq 1$ for every cut set $S \subseteq V(G)$.*

In the following, we introduce the technique in Ryjáček [7] so called *local completion* to study Hamiltonian properties of claw-free graphs. Let G be a claw-free graph. A vertex x in G is *eligible* if $\langle N_G(x) \rangle$ is connected and non-complete. Further, let G_x be the graph such that $V(G_x) = V(G)$ and $E(G_x) = E(G) \cup \{uv :$ for a pair of non-adjacent vertices $u, v \in N_G(x)\}$. Then, we repeat this process until there is no eligible vertex in the graph. That is, we will have a finite sequence of graphs $G_0, G_1, \ldots, G_{n_0}$ such that $G = G_0$ and, for $1 \leq i \leq n_0$, we have $G_i = (G_{i-1})_y$ where y is an eligible vertex of G_{i-1}. The process finishes at G_{n_0} which contains no eligible vertex. Here G_{n_0} is the *closure* of G and is denoted by $cl(G)$. Brousek *et al.* [2] use this operation to establish the Hamiltonicities of $\{K_{1,3}, N_{s_1,s_2,s_3}\}$-free graphs. Before we state this theorem, we need to provide some classes of graphs from [2].

The Class $\mathcal{H}_1$. Let $Z_1, \ldots, Z_5$ be complete graphs of order at least three. For $1 \le i \le 3$, let q_i, z_i be two different vertices of Z_i. Moreover, let q'_1, q'_2, q'_3 be three different vertices of Z_4 and z'_1, z'_2, z'_3 be three different vertices of Z_5. A graph in this class is constructed from $Z_1, \ldots, Z_5$ by identifying q'_i with q_i and z'_i with z_i for $1 \le i \le 3$. A graph in this class is given in Figure 1(a).

The Class $\mathcal{H}_2$. Let c_1, c_2, c_3, c_1 and d_1, d_2, d_3, d_1 be two disjoint triangles. We also let T_1 and T_2 be two complete graphs of order at least three and T_3 a complete graph of order at least two. Let c'_i, d'_i be two different vertices of T_i for $1 \le i \le 2$ and let c'_3, r be two different vertices of T_3. A graph in this class is obtained by identifying c'_i with c_i and d'_i with d_i for $1 \le i \le 2$ and identifying c'_3 with c_3 and adding an edge rd_3. A graph in this class is illustrated by Figure 1(b).

The Class $\mathcal{H}_3$. Let $h_1, h_2, \ldots, h_6, h_1$ be a cycle of six vertices and K a complete graph of order at least three. Let s and s' be two different vertices of K. We define a graph G in the class $\mathcal{H}_3$ by adding edges $sh_1, sh_6, s'h_3, s'h_4$. A graph in this class is illustrated by Figure 1(c).

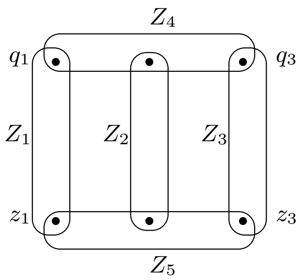

Figure 1(a). The Class $\mathcal{H}_1$.

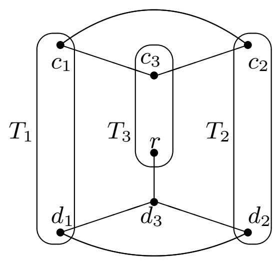

Figure 1(b). The Class $\mathcal{H}_2$.

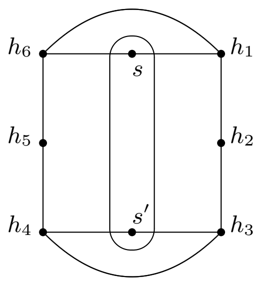

Figure 1(c). The Class $\mathcal{H}_3$.

Let $P = p_1, p_2, p_3$, $P' = p'_1, p'_2, p'_3$ and $P'' = p''_1, p''_2, p''_3$ be three paths of length two. The graph $P_{3,3,3}$ is constructed from P, P' and P'' by adding edges so that

$\{p_1, p_1', p_1''\}$ and $\{p_3, p_3', p_3''\}$ form two complete graphs of order three. Brousek *et al.* [2] proved the following.

Theorem 2 [2]**.** *Let G be a 2-connected $\{K_{1,3}, N_{1,2,2}, N_{1,1,3}\}$-free graph. Then either G is Hamiltonian, or G is isomorphic to $P_{3,3,3}$ or $cl(G) \in \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$.*

Recently, Xiong *et al.* [18] established the following theorem.

Theorem 3 [18]**.** *Let G be a 3-connected $\{K_{1,3}, N_{s_1,s_2,s_3}\}$-free graph. If $s_1 + s_2 + s_3 \leq 9$ and $s_i \geq 1$, then G is Hamiltonian.*

We conclude this section by giving some results on double domination. Thacker [12] established some observations of this parameter.

Observation 4 [12]**.** *For $k \geq 2$, let G be a k-$\gamma_{\times 2}$-critical graph. Moreover, for a pair of non-adjacent vertices u and v of G, we let D_{uv} be a $\gamma_{\times 2}$-set of $G + uv$. Then $D_{uv} \cap \{u, v\} \neq \emptyset$.*

The following proposition is a special case of a result of Thacker [12] by restricting the original result to connected graphs.

Proposition 5 [12]**.** *For any connected graph G, let u and v be a pair of non-adjacent vertices of G. Then*

$$\gamma_{\times 2}(G) - 2 \leq \gamma_{\times 2}(G + uv) \leq \gamma_{\times 2}(G).$$

The following result, from [4], gives the double domination number of a graph when any edge is removed.

Observation 6 [12]**.** *For a graph G and edge $uv \in E(G)$ such that $G - uv$ have no isolated vertex, $\gamma_{\times 2}(G) \leq \gamma_{\times 2}(G - uv)$.*

3. Double Domination Critical Graphs

In this section, we use the claw-free property to determine when 2-connected k-$\gamma_{\times 2}$-critical claw-free graphs are Hamiltonian. First of all, we give a construction of k-$\gamma_{\times 2}$-critical graphs when $k \geq 4$ which are non-Hamiltonian.

The class $\mathcal{D}(k)$. For $k \geq 4$, let $A = \{a_i b_i : 1 \leq i \leq k - 1\}$ be a set of $k - 1$ independent edges and let x be an isolated vertex. A graph G in the class $\mathcal{D}(k)$ is constructed by:

- joining x to every vertex in $V(A)$, and
- adding edges so that $b_1, b_2, \ldots, b_{k-1}$ form a clique.

A graph G in this class is illustrated by Figure 2.

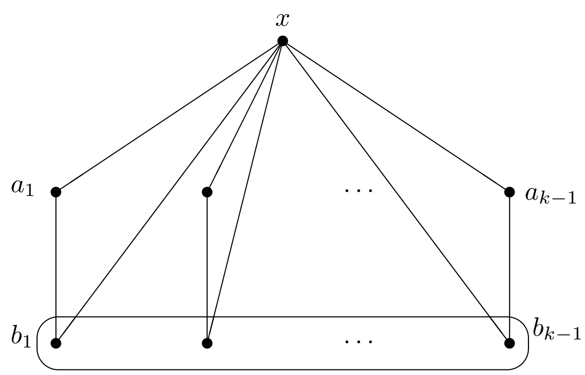

Figure 2. A graph in the class $\mathcal{D}(k)$.

Lemma 7. *For an integer $k \geq 4$, if $G \in \mathcal{D}(k)$, then G is a 2-connected k-$\gamma_{\times 2}$-critical non-Hamiltonian graph.*

Proof. We first show that $\gamma_{\times 2}(G) = k$. Obviously, $\{x, a_1, a_2, \dots, a_{k-1}\} \succ_{\times 2} G$. By the minimality of $\gamma_{\times 2}(G)$, we have $\gamma_{\times 2}(G) \leq k$. It remains to show that $\gamma_{\times 2}(G) \geq k$. Let D be a $\gamma_{\times 2}$-set of G. To doubly dominate $\{a_1\}$, we must have $|\{x, a_1, b_1\} \cap D| \geq 2$. Similarly, to doubly dominate $\{a_2, a_3, \dots, a_{k-1}\}$, we have $|D \cap \{a_i, b_i\}| \geq 1$ for all $2 \leq i \leq k-1$. Thus $|D| \geq k$. This implies that $\gamma_{\times 2}(G) = k$.

We next establish the criticality. Let u and v be a pair of non-adjacent vertices of G. As x is adjacent to every vertex, we must have that $x \notin \{u, v\}$. Thus $\{u, v\} \subseteq \{a_1, b_1, a_2, b_2, \dots, a_{k-1}, b_{k-1}\}$. By the construction, at least one of u and v is not in $\{b_1, b_2, \dots, b_{k-1}\}$. Without loss of generality let $u = a_1$ and $v \in \{a_2, b_2\}$. Clearly, $\{x, v, a_3, a_4, \dots, a_{k-2}, b_{k-1}\} \succ_{\times 2} G + uv$. That is $\gamma_{\times 2}(G + uv) \leq k - 1 < \gamma_{\times 2}(G)$. This establishes the criticality and hence, G is a k-$\gamma_{\times 2}$-critical graph.

We finally show that G is non-Hamiltonian. Suppose to the contrary that G is Hamiltonian. Observe that $N_G(a_1) = \{x, b_1\}$. Thus, G is Hamiltonian if and only if $G - a_1$ has a Hamiltonian path P from x to b_1. Since $N_{G-a_1}(a_2) = \{x, b_2\}$, it follows that the path x, a_2, b_2 is a subgraph of P. Similarly, as $N_{G-a_1}(a_3) = \{x, b_3\}$, we must have that the path x, a_3, b_3 is a subgraph of P. Thus b_3, a_3, x, a_2, b_2 is a subgraph of P. This contradicts x is one of the two end vertices of P. Therefore, G is non-Hamiltonian. This completes the proof. ∎

In the following, we recall the classes $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ and the graph $P_{3,3,3}$ from the previous section. We give an observation for the lower bound of the double domination numbers of graphs in these classes.

Observation 8. *Let G be a 2-connected claw-free graph. If $cl(G) \in \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$ or G is isomorphic to $P_{3,3,3}$, then $\gamma_{\times 2}(G) \geq 6$.*

Proof. We first consider the case when $cl(G) \in \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$. Let D be a $\gamma_{\times 2}$-set of $cl(G)$. In view of Proposition 5, it suffices to show that $\gamma_{\times 2}(cl(G)) \geq 6$. Suppose first that $cl(G) \in \mathcal{H}_1$. Since $|V(Z_i)| \geq 3$, there exist vertices $s_i \in V(Z_i) - \{q_i, z_i\}$ for all $i \in \{1, 2, 3\}$. To doubly dominate $\{s_1, s_2, s_3\}$, we have that $|D \cap V(Z_i)| \geq 2$. Thus $\gamma_{\times 2}(cl(G)) = |D| \geq 6$.

We now suppose that $cl(G) \in \mathcal{H}_2$. Because $|V(T_i)| \geq 3$, there exist vertices $r_i \in V(T_i) - \{c_i, d_i\}$ for all $i \in \{1, 2\}$. To doubly dominate $\{r, r_1, r_2\}$, we have that $|D \cap V(T_i)| \geq 2$ and $|D \cap (V(T_3) \cup \{d_3\})| \geq 2$. Thus $\gamma_{\times 2}(cl(G)) = |D| \geq 6$.

We now suppose that $cl(G) \in \mathcal{H}_3$. Because $|V(K)| \geq 3$, there exists a vertex $s'' \in V(K) - \{s, s'\}$. To doubly dominate $\{s'', h_2, h_5\}$, we have that $|D \cap V(K)| \geq 2, |D \cap \{h_1, h_2, h_3\}| \geq 2$ and $|D \cap \{h_4, h_5, h_6\}| \geq 2$. Thus $\gamma_{\times 2}(cl(G)) = |D| \geq 6$.

We finally consider the case when G is isomorphic to $P_{3,3,3}$. Let D' be a $\gamma_{\times 2}$-set of G. To doubly dominate $\{p_2, p'_2, p''_2\}$, we have that $|D' \cap V(P)| \geq 2, |D' \cap V(P')| \geq 2$ and $|D' \cap V(P'')| \geq 2$. Thus $\gamma_{\times 2}(G) = |D'| \geq 6$. This completes the proof. ■

We next establish the following lemma concerning the minimum number of vertices of a double dominating set when some independent set is given.

Lemma 9. *Let G be a claw-free graph, I be an independent set and X be a set of vertices such that $X \succ_{\times 2} I$. If there exists a vertex in $X - I$ adjacent to at most one vertex in I, then $|I| + 1 \leq |X|$.*

Proof. Let w be a vertex in $X - I$ which is adjacent to at most one vertex in I. Moreover, we let $I_1 = X \cap I, I_2 = I - I_1$ and $X' = X - (I_1 \cup \{w\})$. Clearly, $|X| = |X'| + |I_1| + 1$ and $|I| = |I_1| + |I_2|$. Let H be a subgraph of G such that $V(H) = X' \cup \{w\} \cup I$ and $E(H) = \{uv \in E(G) : u \in X' \cup \{w\}$ and $v \in I\}$. Clearly, H is bipartite with the bipartition sets $X' \cup \{w\}$ and I. Since $X \succ_{\times 2} I$, every vertex in I_1 is adjacent to at least one vertex in $X' \cup \{w\}$. Moreover, every vertex in I_2 is adjacent to at least two vertices in $X' \cup \{w\}$. Thus, $\deg_H(v) \geq 1$ for all $v \in I_1$ and $\deg_H(v) \geq 2$ for all $v \in I_2$. This gives the degree sum of vertices in I as the following

$$|I_1| + 2|I_2| \leq \Sigma_{v \in I} \deg_H(v). \tag{1}$$

Because G is claw-free, every vertex in X' is adjacent to at most two vertices in I. Therefore $\deg_H(u) \leq 2$ for all $u \in X'$. Since w is adjacent to at most one vertex in I, it follows that

$$\Sigma_{u \in X' \cup \{w\}} \deg_H(u) \leq 2\,|X'| + 1. \tag{2}$$

Because H is bipartite, $\Sigma_{v\in I}\deg_H(v) = \Sigma_{u\in X'\cup\{w\}}\deg_H(u)$. By (1) and (2), we have $|I_1| + 2|I_2| \le 2|X'| + 1$. Hence,

$$\begin{aligned}|I| = {} & |I_1| + |I_2| \le \left(\frac{|I_1|}{2} + |I_2|\right) + \frac{|I_1|}{2} \le \left(|X'| + \frac{1}{2}\right) + \frac{|I_1|}{2} \\ < {} & |X'| + |I_1| + 1 = |X|.\end{aligned}$$

This completes the proof. ■

We are now ready to establish our main theorems. We recall a net N_{s_1,s_2,s_3} from the first section.

Theorem 10. *Let G be a 2-connected k-$\gamma_{\times 2}$-critical claw-free graph. If $2 \le k \le 5$, then G is Hamiltonian.*

Proof. We first show that G is $\{K_{1,3}, N_{1,2,2}, N_{1,1,3}\}$-free. Suppose to the contrary that G contains $N_{1,2,2}$ or $N_{1,1,3}$ as an induced subgraph. We first consider the case when G contains $N_{1,2,2}$ as an induced subgraph. Consider $G + v_1w_1$. By Observation 4, $|D_{v_1w_1} \cap \{v_1, w_1\}| \ge 1$. Suppose that $|D_{v_1w_1} \cap \{v_1, w_1\}| = 1$. By symmetry, we let $w_1 \in D_{v_1w_1}$. Because $D_{v_1w_1}$ is a double dominating set, w_1 is adjacent to a vertex in $D_{v_1w_1}$, w say. Clearly, $\{u_1, v_2, w_1, w_3\}$ is an independent set. By claw-freeness, w is adjacent to at most one vertex in $\{u_1, v_2, w_3\}$. Let $X = D_{v_1w_1} - \{w_1\}$. Clearly, $X \succ_{\times 2} \{u_1, v_2, w_3\}$ and X contains w. In view of Lemma 9, $|X| \ge 4$. This implies that $|D_{v_1w_1}| \ge 5$ contradicting the criticality of G. Suppose that $\{v_1, w_1\} \subseteq D_{v_1w_1}$. Let $X' = D_{v_1w_1} - \{w_1\}$. Thus $X' \succ_{\times 2} \{u_1, v_2, w_3\}$ and X' contains v_1 which is adjacent to at most one vertex in $\{u_1, v_2, w_3\}$. This implies by Lemma 9 that $|X'| \ge 4$. Hence, $|D_{v_1w_1}| \ge 5$ contradicting the criticality of G. Therefore, G does not contains $N_{1,2,2}$ as an induced subgraph.

We now consider the case when G contains $N_{1,1,3}$ as an induced subgraph. Consider $G + u_1v_1$. By Observation 4, $|D_{u_1v_1} \cap \{u_1, v_1\}| \ge 1$. Suppose that $|D_{u_1v_1} \cap \{u_1, v_1\}| = 1$. By symmetry, we let $u_1 \in D_{u_1v_1}$. As $D_{u_1v_1}$ is a double dominating set, we must have that u_1 is adjacent to a vertex u in $D_{u_1v_1}$. Clearly, $\{u_1, v_2, w_1, w_3\}$ is an independent set. Since G is claw-free, u is adjacent to at most one vertex in $\{v_2, w_1, w_3\}$. Let $Y = D_{u_1v_1} - \{u_1\}$. Clearly, $Y \succ_{\times 2} \{v_2, w_1, w_3\}$ and Y contains u. In view of Lemma 9, $|Y| \ge 4$. Thus $|D_{u_1v_1}| \ge 5$ contradicting the criticality of G. We then suppose that $\{u_1, v_1\} \subseteq D_{u_1v_1}$. Let $Y' = D_{u_1v_1} - \{u_1\}$. Thus $Y' \succ_{\times 2} \{v_2, w_1, w_3\}$ and Y' contains v_1 which is adjacent to at most one vertex in $\{v_2, w_1, w_3\}$. This implies by Lemma 9 that $|Y'| \ge 4$. Hence, $|D_{u_1v_1}| \ge 5$ contradicting the criticality of G. Therefore, G does not contain $N_{1,1,3}$ as an induced subgraph. Hence, G is $\{K_{1,3}, N_{1,2,2}, N_{1,1,3}\}$-free.

Because $\gamma_{\times 2}(G) \le 5$, by Observation 8, G is not isomorphic to $P_{3,3,3}$ and $cl(G) \notin \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$. In view of Theorem 2, G is Hamiltonian. This completes the proof. ■

We can see that a graph G in the class $\mathcal{D}(k)$ when $4 \leq k \leq 5$ is non-Hamiltonian. Thus the condition claw-free in Theorem 10 is necessary. Moreover, when $k = 4$, the graph in the class $\mathcal{D}(k)$ is $K_{1,4}$-free. Hence, the condition claw-free is best possible for $k = 4$. We conclude this section with the following theorem which shows that a graph k-$\gamma_{\times 2}$-critical when $6 \leq k \leq 7$ is Hamiltonian if it is 3-connected and claw-free.

Theorem 11. *For an integer $2 \leq k \leq 7$, let G be a 3-connected k-$\gamma_{\times 2}$-critical claw-free graph. Then G is Hamiltonian.*

Proof. We will show that G is $N_{3,3,3}$-free. Suppose to the contrary that G contains $N_{3,3,3}$ as an induced subgraph. Consider $G + u_1v_1$. Observation 4 yields that $|D_{u_1v_1} \cap \{u_1, v_1\}| \geq 1$. Suppose first that $|D_{u_1v_1} \cap \{u_1, v_1\}| = 1$. By symmetry, we may let $u_1 \in D_{u_1v_1}$. It is easy to see that $\{u_1, u_3, v_2, v_4, w_1, w_3\}$ is an independent set. As $D_{u_1v_1}$ is a double dominating set, we must have that u_1 is adjacent to a vertex u in $D_{u_1v_1}$. Since G is claw-free, u is adjacent to at most one vertex in $\{u_3, v_2, v_4, w_1, w_3\}$. Let $X = D_{u_1v_1} - \{u_1\}$. Clearly, $X \succ_{\times 2} \{u_3, v_2, v_4, w_1, w_3\}$ and X contains u. Lemma 9 implies that $|X| \geq 6$. Thus $|D_{u_1v_1}| \geq 7$ contradicting the criticality of G. We then suppose that $\{u_1, v_1\} \subseteq D_{u_1v_1}$. Let $X' = D_{u_1v_1} - \{u_1\}$. Thus $X' \succ_{\times 2} \{u_3, v_2, v_4, w_1, w_3\}$ and X' contains v_1 which is adjacent to at most one vertex in $\{u_3, v_2, v_4, w_1, w_3\}$. This implies by Lemma 9 that $|X'| \geq 6$. Hence, $|D_{u_1v_1}| \geq 7$ contradicting the criticality of G. Therefore, G does not contain $N_{3,3,3}$ as an induced subgraph. Theorem 3 implies that G is Hamiltonian. This completes the proof. ■

We see that the graphs in the class $\mathcal{D}(k)$ when $6 \leq k \leq 7$ are non-Hamiltonian. Thus, the condition claw-free together with 3-connected is necessary in Theorem 11.

4. DOUBLE DOMINATION STABLE GRAPHS

In this section, we use the claw-free property to determine when 2-connected k-$\gamma^+_{\times 2}$-stable claw-free graphs and 2-connected k-$\gamma^-_{\times 2}$-stable claw-free graphs are Hamiltonian. We first establish the following lemma concerning the minimum number of vertices of a double dominating set when some independent set is given. The proof of which is similar to Lemma 9. For completeness, we provide the proof.

Lemma 12. *Let G be a claw-free graph, I be an independent set and X be a set of vertices such that $X \succ_{\times 2} I$. Then $|I| \leq |X|$.*

Proof. Let $I_1 = X \cap I, I_2 = I - I_1$ and $X' = X - I_1$. Clearly, $|X| = |X'| + |I_1|$ and $|I| = |I_1| + |I_2|$. Let H be a subgraph of G such that $V(H) = X' \cup I$ and $E(H) = \{uv \in E(G) : u \in X'$ and $v \in I\}$. Clearly, H is bipartite with the bipartition sets X' and I. Since $X \succ_{\times 2} I$, every vertex in I_1 is adjacent to at least one vertex in X' and, every vertex in I_2 is adjacent to at least two vertices in X'. Thus, $\deg_H(v) \geq 1$ for all $v \in I_1$ and $\deg_H(v) \geq 2$ for all $v \in I_2$. This gives the degree sum of vertices in I as the following

$$|I_1| + 2|I_2| \leq \Sigma_{v\in I} \deg_H(v). \tag{3}$$

Because G is claw-free, every vertex in X' is adjacent to at most two vertices in I. Therefore $\deg_H(u) \leq 2$ for all $u \in X'$. Thus,

$$\Sigma_{u\in X'} \deg_H(u) \leq 2|X'| = 2|X| - 2|I_1|. \tag{4}$$

Because H is bipartite, $\Sigma_{v\in I} \deg_H(v) = \Sigma_{u\in X'} \deg_H(u)$. By (3) and (4), we have $|I_1| + 2|I_2| \leq 2|X| - 2|I_1|$. Hence,

$$|I| = |I_1| + |I_2| \leq 3|I_1|/2 + |I_2| \leq |X|.$$

This completes the proof. ■

By Lemma 12 and Theorems 2 and 3, we easily establish the following corollaries.

Corollary 13. *Let G be a 2-connected claw-free graph with $\gamma_{\times 2}(G) \leq 3$. Then G is Hamiltonian.*

Proof. By Observation 8, $cl(G) \notin \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$ and G is not isomorphic to $P_{3,3,3}$. Thus, by Theorem 2, it suffices to show that G is $\{N_{1,2,2}, N_{1,1,3}\}$-free. Suppose to the contrary that G contains $N_{1,2,2}$ as an induced subgraph. Clearly, $\{u_1, v_1, v_3, w_2\}$ is an independent set of four vertices. Lemma 12 yields that $4 \leq \gamma_{\times 2}(G) \leq 3$, a contradiction. Thus, G is $N_{1,2,2}$-free. We can prove that G is $N_{1,1,3}$-free by the same arguments. Thus, by Theorem 2, G is Hamiltonian. ■

Corollary 14. *Let G be a 3-connected claw-free graph with $\gamma_{\times 2}(G) \leq 5$. Then G is Hamiltonian.*

Proof. By Theorem 3, it suffices to show that G is $N_{3,3,3}$-free. Suppose to the contrary that G contains $N_{3,3,3}$ as an induced subgraph. Clearly, $\{u_1, u_3, v_1, v_3, w_1, w_3\}$ is an independent set of six vertices. Lemma 12 gives that $6 \leq \gamma_{\times 2}(G) \leq 5$, a contradiction. Thus, G is $N_{3,3,3}$-free. By Theorem 3, G is Hamiltonian. ■

4.1. k-$\gamma^+_{\times 2}$-stable claw-free graphs

In this subsection, we study Hamiltonian property of k-$\gamma^+_{\times 2}$-stable claw-free graphs. Although all 2-$\gamma_{\times 2}$-critical graphs of order at least three are Hamiltonian because they are complete graphs, this is not always true for 2-$\gamma^+_{\times 2}$-stable graphs. That is there exist 2-$\gamma^+_{\times 2}$-stable graphs which are non-Hamiltonian. We first give a construction of k-$\gamma^+_{\times 2}$-stable graphs when $k \geq 2$ which are non-Hamiltonian.

The class $\mathcal{S}^+(k)$. For $k \geq 2$, let K_k be a complete graph of order k with the vertices $x_1, x_2, \dots, x_k$ and for $1 \leq i \neq j \leq k$, we let $a_{\{i,j\}}, b_{\{i,j\}}, c_{\{i,j\}}$ be $3 \cdot \binom{k}{2}$ isolated vertices. The graph G in this class is obtained from K_k and all the $3 \cdot \binom{k}{2}$ isolated vertices by adding the edges $a_{\{i,j\}}x_p, b_{\{i,j\}}x_p, c_{\{i,j\}}x_p$ for all $1 \leq i \neq j \leq k$ and for all $p \in \{i, j\}$. The following lemma establishes the properties of the graphs in the class $\mathcal{S}^+(k)$.

Lemma 15. *For an integer $k \geq 2$, if $G \in \mathcal{S}^+(k)$, then G is a 2-connected k-$\gamma^+_{\times 2}$-stable non-Hamiltonian graph.*

Proof. Clearly, G is 2-connected. Moreover, it is easy to see that G is k-$\gamma^+_{\times 2}$-stable when $k = 2$. Hence, we assume that $k \geq 3$. We first show that $\gamma_{\times 2}(G) = k$. Obviously, $V(K_k) \succ_{\times 2} G$. By the minimality of $\gamma_{\times 2}(G)$, we have $\gamma_{\times 2}(G) \leq k$. Let D be a $\gamma_{\times 2}$-set of G. So, $|D| \leq k$. We will show that $V(K_k) \subseteq D$. Suppose to the contrary that $\{x_1, x_2, \dots, x_k\} \nsubseteq D$. Without loss of generality, let $x_1 \notin D$. To doubly dominate $A_1 = \{a_{\{1,j\}}, b_{\{1,j\}}, c_{\{1,j\}} : 1 < j \leq k\}$, we must have that $A_1 \subseteq D$. Since $k \geq 3$, it follows that $k \geq |D| \geq |A_1| = 3k-3 > k$, a contradiction. Thus $V(K_k) \subseteq D$ and $|D| \geq k$. This implies that $\gamma_{\times 2}(G) = k$.

We next establish the stability. Let u and v be a pair of non-adjacent vertices of G and let D_{uv} be a $\gamma_{\times 2}$-set of $G + uv$. Because $V(K_k) \succ_{\gamma_{\times 2}} G + uv$, it follows that $|D_{uv}| \leq |V(K_k)| = k$. It suffice to show that $|D_{uv}| \geq k$. By the construction, $|\{u, v\} \cap V(K_k)| \leq 1$. We first consider the case when $|\{u, v\} \cap V(K_k)| = 1$. Without loss of generality let $u = x_1$ and $v = a_{\{2,3\}}$. Suppose that there exists $x_i \notin D_{uv}$. If $i \neq \{1, 2, 3\}$, then, to doubly dominate $A_i = \{a_{\{i,j\}}, b_{\{i,j\}}, c_{\{i,j\}} : 1 \leq j \leq k \text{ and } j \neq i\}$, we must have that $A_i \subseteq D_{uv}$. This implies that $|D_{uv}| \geq 3k - 3 > k$ contradicting $|D_{uv}| \leq k$. Thus, $i \in \{1, 2, 3\}$. To doubly dominate $A_i - \{a_{\{2,3\}}\}$, we must have that $\big(A_i - \{a_{\{2,3\}}\}\big) \subseteq D_{uv}$. This implies that $k \geq |D_{uv}| \geq 3k - 4 > k$, a contradiction. Thus, $V(K_k) \subseteq D_{uv}$. This implies that $|D_{uv}| \geq k$.

We now consider the case when $|\{u, v\} \cap V(K_k)| = 0$. Similarly, suppose that there exists $x_i \notin D_{uv}$. To doubly dominate $A_i - \{u, v\}$, we must have that $(A_i - \{u, v\}) \subseteq D_{uv}$. This implies that $k \geq |D_{uv}| \geq (3k - 3) - 2 > k$, a contradiction. Thus, $V(K_k) \subseteq D_{uv}$. This implies that $|D_{uv}| \geq k$. This establishes the stability and hence, G is a k-$\gamma^+_{\times 2}$-stable graph.

We finally show that G is non-Hamiltonian. Clearly, $V(K_k)$ is a cut set of G such that $G - V(K_k)$ has $3 \cdot \binom{k}{2}$ isolated vertices as the components. Thus, $\frac{|V(K_k)|}{\omega(G - V(K_k))} < 1$. By Proposition 1, G is non-Hamiltonian. This completes the proof. ■

By Lemma 15, there exist 2-connected k-$\gamma^+_{\times 2}$-stable graphs which are non-Hamiltonian for all $k \geq 2$. However, by using Corollaries 13 and 14, we easily obtain that all 2-connected k-$\gamma^+_{\times 2}$-stable graphs are Hamiltonian when k is small. The proofs are omitted as the class of 2-connected k-$\gamma^+_{\times 2}$-stable graphs is a subclass of graphs with $\gamma_{\times 2}(G) = k$.

Corollary 16. *Let G be a 2-connected k-$\gamma^+_{\times 2}$-stable claw-free graph. If $2 \leq k \leq 3$, then G is Hamiltonian.*

Corollary 17. *For an integer $2 \leq k \leq 5$, let G be a 3-connected k-$\gamma^+_{\times 2}$-stable claw-free graph. Then G is Hamiltonian.*

Observe that a graph $G \in \mathcal{S}^+(2)$ is $K_{1,4}$-free. Hence, the condition claw-free in Corollaries 13 and 16 is best possible when $k = 2$. For a graph $G \in \mathcal{S}^+(3)$, it is easy to see that the graph $G' = G - c_{\{1,2\}} - b_{\{1,3\}} - c_{\{1,3\}} - b_{\{2,3\}} - c_{\{2,3\}}$ is 3-$\gamma^+_{\times 2}$-stable $K_{1,4}$-free graph. Hence, the condition claw-free in Corollaries 13 and 16 is best possible when $k = 3$.

4.2. k-$\gamma^-_{\times 2}$-stable claw-free graphs

We first give a construction of k-$\gamma^-_{\times 2}$-stable graphs when $k \geq 2$ which are non-Hamiltonian.

The class $\mathcal{S}^-(k)$. For $k \geq 2$, we let K_{2k} be a complete graph of order $2k$ with the vertices $x_1, y_1, x_2, y_2, \ldots, x_k, y_k$ and for $1 \leq i \neq j \leq k$, we let $a_{\{i,j\}}, b_{\{i,j\}}, c_{\{i,j\}}, d_{\{i,j\}}, e_{\{i,j\}}$ be $5 \cdot \binom{k}{2}$ isolated vertices. The graph G in this class is obtained from K_{2k} and all the $5 \cdot \binom{k}{2}$ isolated vertices by joining the vertices x_p and y_p to $a_{\{i,j\}}, b_{\{i,j\}}, c_{\{i,j\}}, d_{\{i,j\}}, e_{\{i,j\}}$ for all $1 \leq i \neq j \leq k$ and for all $p \in \{i, j\}$. The following lemma establishes the properties of the graphs in the class $\mathcal{S}^-(k)$.

Lemma 18. *For an integer $k \geq 2$, if $G \in \mathcal{S}^-(k)$, then G is a 2-connected k-$\gamma^-_{\times 2}$-stable non-Hamiltonian graph.*

Proof. Clearly, G is 2-connected. Moreover, it is easy to see that G is k-$\gamma^-_{\times 2}$-stable when $k = 2$. Hence, we assume that $k \geq 3$. We first show that $\gamma_{\times 2}(G) = k$. Since $\{x_1, x_2, \ldots, x_k\} \succ_{\times 2} G$, it follows that $\gamma_{\times 2}(G) \leq k$. It remains to show that $\gamma_{\times 2}(G) \geq k$. Let D be a $\gamma_{\times 2}$-set of G. By the minimality of D, we have $|D| \leq k$. If $|\{x_i, y_i\} \cap D| \geq 1$ for all $1 \leq i \leq k$, then $|D| \geq k$ as required. We may suppose that there exists $i \in \{1, 2, \ldots, k\}$ such that $\{x_i, y_i\} \cap D = \emptyset$. To doubly dominate $A_i = \big\{a_{\{i,j\}}, b_{\{i,j\}}, c_{\{i,j\}}, d_{\{i,j\}}, e_{\{i,j\}} : 1 \leq j \leq k \text{ and } j \neq i\big\}$, we must have that $\big|D \cap \big\{a_{\{i,j\}}, b_{\{i,j\}}, c_{\{i,j\}}, d_{\{i,j\}}, e_{\{i,j\}}, x_j, y_j\big\}\big| \geq 2$ for all $1 \leq j \leq k$ and $j \neq i$. This implies that $k \geq |D| \geq 2(k-1) > k$, a contradiction. Thus, $\gamma_{\times 2}(G) = |D| \geq k$. Therefore $\gamma_{\times 2}(G) = k$.

We next establish the stability. Let u and v be a pair of adjacent vertices of G. By Observation 6, we have $\gamma_{\times 2}(G-uv) \geq k$. Hence, it suffices to show that there exists a $\gamma_{\times 2}$-set of $G-uv$ containing k vertices. Clearly, $|\{u,v\} \cap V(K_{2k})| \geq 1$. We first suppose that $|\{u,v\} \cap V(K_{2k})| = 1$. Without loss of generality let $u \in V(K_{2k})$. If $u \in \{x_1, x_2, \ldots, x_k\}$, then $\{y_1, y_2, \ldots, y_k\} \succ_{\times 2} G-uv$. If $u \in \{y_1, y_2, \ldots, y_k\}$, then $\{x_1, x_2, \ldots, x_k\} \succ_{\times 2} G-uv$. Hence, we now suppose that $u, v \in V(K_{2k})$. We consider the case when $\{u,v\} = \{x_i, y_j\}$ for some $i, j \in \{1, 2, \ldots, k\}$. Clearly $\{x_1, x_2, \ldots, x_k\} \succ_{\times 2} G-uv$. We now consider the case when $\{u,v\} = \{x_i, x_j\}$. Thus, $\{y_1, y_2, \ldots, y_k\} \succ_{\times 2} G-uv$. Similarly, $\{x_1, x_2, \ldots, x_k\} \succ_{\times 2} G-uv$ when $\{u,v\} = \{y_i, y_j\}$. Therefore, G is k-$\gamma^{-}_{\times 2}$-stable graph.

We finally show that G is non-Hamiltonian. Clearly, $V(K_{2k})$ is a cut set of G such that $G-V(K_{2k})$ has $5 \cdot \binom{k}{2}$ isolated vertices as the components. Thus, $\frac{|V(K_{2k})|}{\omega(G-V(K_k))} < 1$. By Proposition 1, G is non-Hamiltonian. This completes the proof. ∎

We next establish the following theorems.

Theorem 19. *Let G be a 2-connected k-$\gamma^{-}_{\times 2}$-stable claw-free graph. If $2 \leq k \leq 4$, then G is Hamiltonian.*

Proof. We first show that G is $\{N_{1,2,2}, N_{1,1,3}\}$-free. Suppose to the contrary that G contains $N_{1,2,2}$ or $N_{1,1,3}$ as an induced subgraph. We first consider the case when G contains $N_{1,2,2}$ as an induced subgraph. Consider $G - v_1v_2$. By stability of G, $\gamma_{\times 2}(G - v_1v_2) = k$. Clearly, $\{u_1, v_1, v_2, w_1, w_3\}$ is an independent set of $G - v_1v_2$ containing 5 vertices. Lemma 12 implies that $4 \geq \gamma_{\times 2}(G - v_1v_2) \geq 5$, a contradiction. Thus, G is $N_{1,2,2}$-free.

We now consider the case when G contains $N_{1,1,3}$ as an induced subgraph. Consider $G - u_1u_2$. By stability of G, $\gamma_{\times 2}(G - u_1u_2) = k$. Clearly, $\{u_1, u_2, v_1, w_1, w_3\}$ is an independent set of $G - u_1u_2$ containing 5 vertices. Lemma 12 implies that $4 \geq \gamma_{\times 2}(G - v_1v_2) \geq 5$, a contradiction. Thus, G is $N_{1,1,3}$-free. Hence, G is $\{K_{1,3}, N_{1,2,2}, N_{1,1,3}\}$-free.

Because $\gamma_{\times 2}(G) \leq 4$, by Observation 8, G is not isomorphic to $P_{3,3,3}$ and $cl(G) \notin \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$. In view of Theorem 2, G is Hamiltonian. This completes the proof. ∎

Theorem 20. *Let G be a 3-connected k-$\gamma^{-}_{\times 2}$-stable claw-free graph. If $2 \leq k \leq 6$, then G is Hamiltonian.*

Proof. We will show that G is $N_{3,3,3}$-free. Suppose to the contrary that G contains $N_{3,3,3}$ as an induced subgraph. Consider $G - u_1u_2$. By stability of G, $\gamma_{\times 2}(G - u_1u_2) = k$. We see that $\{u_1, u_2, u_4, v_1, v_3, w_1, w_3\}$ is an independent set of $G - u_1u_2$ containing 7 vertices. By Lemma 12, $6 \geq \gamma_{\times 2}(G - u_1u_2) \geq 7$, a contradiction. Therefore, G does not contain $N_{3,3,3}$ as an induced subgraph. Theorem 3 implies that G is Hamiltonian. This completes the proof. ■

5. Discussion

On double domination critical graphs. For $6 \leq k \leq 7$, we have seen neither 2-connected k-$\gamma_{\times 2}$-critical claw-free graphs which are non-Hamiltonian nor 3-connected k-$\gamma_{\times 2}$-critical graphs which are non-Hamiltonian. Hence, the questions that arise are, for an integer $6 \leq k \leq 7$, is every 2-connected k-$\gamma_{\times 2}$-critical claw-free graph Hamiltonian? and is every 3-connected k-$\gamma_{\times 2}$-critical graph Hamiltonian?

On double domination stable graphs. We have seen neither 2-connected k-$\gamma^{+}_{\times 2}$-stable claw-free graphs which are non-Hamiltonian for $4 \leq k \leq 5$ nor 2-connected k-$\gamma^{-}_{\times 2}$-stable graphs which are non-Hamiltonian for $5 \leq k \leq 6$. Hence, the questions that arise are, for $4 \leq k \leq 5$, is every 2-connected k-$\gamma^{+}_{\times 2}$-stable claw-free graph Hamiltonian? and, for $5 \leq k \leq 6$, is every 2-connected k-$\gamma^{-}_{\times 2}$-stable claw-free graph Hamiltonian?

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CONNECTED DOMINATION CRITICAL GRAPHS WITH CUT VERTICES

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Abstract

A graph G is said to be k-γ_c-critical if the connected domination number of G, $\gamma_c(G)$, is k and $\gamma_c(G + uv) < k$ for any pair of non-adjacent vertices u and v of G. Let G be a k-γ_c-critical graph and $\zeta(G)$ the number of cut vertices of G. It was proved, in [1, 6], that, for $3 \leq k \leq 4$, every k-γ_c-critical graph satisfies $\zeta(G) \leq k - 2$. In this paper, we generalize that every k-γ_c-critical graph satisfies $\zeta(G) \leq k - 2$ for all $k \geq 5$. We also characterize all k-γ_c-critical graphs when $\zeta(G)$ is achieving the upper bound.

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1. Introduction

All graphs in this paper are finite, undirected and simple (no loops or multiple edges). For a graph G, let $V(G)$ denote the set of all vertices of G and let $E(G)$ denote the set of all edges of G. The *complement* $\overline{G}$ of G is the graph having the same set of vertices as G but the edge e is in $E(\overline{G})$ if and only if $e \notin E(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. The *open neighborhood* $N_G(v)$ of a vertex v in G is $\{u \in V(G) : uv \in E(G)\}$. Further, the *closed neighborhood* $N_G[v]$ of a vertex v in G is $N_G(v) \cup \{v\}$. For subsets X and Y of $V(G)$, $N_Y(X)$ is the set $\{y \in Y : yx \in E(G)$ for some $x \in X\}$. For a subgraph H of G, we use $N_Y(H)$ instead of $N_Y(V(H))$ and we use $N_H(X)$ instead of $N_{V(H)}(X)$. If $X = \{x\}$, we use $N_Y(x)$ instead of $N_Y(\{x\})$. The *degree* $deg(x)$ of a vertex x in G is $|N_G(x)|$. When no ambiguity occur, we write $N(x)$ and $N(X)$ instead of $N_G(x)$ and $N_G(X)$, respectively. An *end vertex* is a vertex of degree one and a *support vertex* is the vertex which is adjacent to an end vertex. A *star* $K_{1,n}$ is a graph of order $n + 1$ containing one support vertex and n end vertices. The support vertex of a star is called the *center*. For a connected graph G, a vertex v of G is called a *cut vertex* if $G - v$ is not connected. The number of cut vertices of G is denoted by $\zeta(G)$. A *block* B of a graph G is a maximal connected subgraph such that B has no cut vertex. An *end block* of G is a block containing exactly one cut vertex of G. The *distance* $d(u, v)$ between vertices u and v of G is the length of a shortest (u, v)-path in G. The diameter of G *diam*(G) is the maximum distance of any two vertices of G. For a connected graph G, a *bridge* xy of G is an edge such that $G - xy$ is not connected.

For a finite sequence of graphs $G_1, \ldots, G_l$ for $l \geq 2$, the *joins* $G_1 \vee \cdots \vee G_l$ is the graph consisting of the disjoint union of $G_1, \ldots, G_l$ and each vertex in G_i is joined to all vertices in G_{i+1} for $1 \leq i \leq l - 1$ by edges. If $V(G_i) = \{x\}$, then we simply write $G_1 \vee \cdots \vee G_{i-1} \vee x \vee G_{i+1} \vee \cdots \vee G_l$. Moreover, for a subgraph H of G_2, the *join* $G_1 \vee_H G_2$ is the graph consisting of the disjoint union of G_1 and G_2 and edges that join each vertex in G_1 to each vertex in H.

For subsets D and X of $V(G)$, D *dominates* X if every vertex in X is either in D or adjacent to a vertex in D. If D dominates X, then we write $D \succ X$. We also write $a \succ X$ when $D = \{a\}$ and $D \succ x$ when $X = \{x\}$. Moreover, if $X = V(G)$, then D is a *dominating set* of G and we write $D \succ G$ instead of $D \succ V(G)$. A *connected dominating set* of a graph G is a dominating set D of G such that $G[D]$ is connected. If D is a connected dominating set of G, we then write $D \succ_c G$. A smallest connected dominating set is called a *γ_c-set*. The cardinality of a γ_c-set is called the *connected domination number* of G and is denoted by $\gamma_c(G)$. A graph G is said to be *k-γ_c-critical* if $\gamma_c(G) = k$ and $\gamma_c(G + uv) < k$ for any pair of non-adjacent vertices u and v of G.

For related results on k-γ_c-critical graphs, Chen *et al.* [3] completely characterized these graphs when $1 \leq k \leq 2$.

Theorem 1 [3]**.** *A graph G is 1-γ_c-critical if and only if G is a complete graph. Moreover, a graph G is 2-γ_c-critical if and only if $\overline{G} = \bigcup_{i=1}^{l} K_{1,n_i}$, where $l \geq 2$ and $n_i \geq 1$ for all $1 \leq i \leq l$.*

By Theorem 1, we observe that a k-γ_c-critical graph does not contain a cut vertex when $1 \leq k \leq 2$.

Observation 2. *Let G be a k-γ_c-critical graph with $1 \leq k \leq 2$. Then G has no cut vertex.*

For $k \geq 3$, there is no complete characterization of these graphs so far. However, there are some structural characterizations of k-γ_c-critical graphs when $3 \leq k \leq 4$ by focusing on the maximum number of cut vertices of the graphs. Ananchuen [1] proved that the number of cut vertices of a 3-γ_c-critical graph does not exceed one.

Theorem 3 [1]**.** *Let G be a 3-γ_c-critical graph. Then G contains at most one cut vertex.*

In our previous work in [6], we established the maximum number of cut vertices that 4-γ_c-critical graphs can have.

Theorem 4 [6]**.** *Let G be a 4-γ_c-critical graph. Then G contains at most two cut vertices.*

By these results, we naturally, ask for $k \geq 5$, whether every k-γ_c-critical graph contains at most $k-2$ cut vertices. It turns out affirmatively as we shall see in the following theorem.

Theorem 5. *For $k \geq 5$, let G be a k-γ_c-critical graph with $\zeta(G)$ cut vertices. Then $\zeta(G) \leq k-2$.*

The proof of this theorem is presented in Section 4. In this paper, we also characterize all k-γ_c-critical graphs when the number of cut vertices is achieving the upper bound.

For the outline of this paper, we provide related results and prove that there exists a forbidden subgraph of k-γ_c-critical graphs in Section 2. In Section 3, we characterize some end blocks of G. We then use the results from Sections 2 and 3 to establish the upper bound of the number of cut vertices of k-γ_c-critical graphs in Section 4. We also characterize all k-γ_c-critical graphs when $\zeta(G) = k-2$ in Section 5. Finally, we discuss our result with some related result in another type of domination critical graphs in Section 6.

2. RELATED RESULTS

In this section, we state a number of results that we make use of in establishing our theorems. At the end of this section, we also prove some crucial results which will be used to settle the maximum number of cut vertices of k-γ_c-critical graphs in Section 4. We begin with a result of Chartrand and Oellermann [2] which gives the relationship between the numbers of end blocks and cut vertices.

Lemma 6 (see [2], page 24). *Let G be a connected graph with at least one cut vertex. Then G has at least two end blocks.*

In [3], Chen *et al.* established fundamental properties of k-γ_c-critical graphs.

Lemma 7 [3]. *Let G be a k-γ_c-critical graph and let x and y be a pair of non-adjacent vertices of G. Further, let D_{xy} be a γ_c-set of $G + xy$. Then*

(1) $k - 2 \leq |D_{xy}| \leq k - 1$,

(2) $D_{xy} \cap \{x, y\} \neq \emptyset$, *and*

(3) *if* $\{x\} = \{x, y\} \cap D_{xy}$, *then* $N_G(y) \cap D_{xy} = \emptyset$.

In [5], we further observed some structure of the subgraph of G (not $G + xy$) induced by D_{xy}. For completeness, we provide the proof.

Observation 8. *If $\{x, y\} \subseteq D_{xy}$, then $G[D_{xy}]$ consists of 2 components and each of which contains exactly one vertex of $\{x, y\}$.*

Proof. If $G[D_{xy}]$ is connected, then D_{xy} is a connected dominating set of G. It follows by Lemma 7(1) that $\gamma_c(G) \leq k - 1$, contradiction. Thus $G[D_{xy}]$ is not connected. As $(G + xy)[D_{xy}]$ is connected and xy is the only one edge which is added to G, it follows that xy is a bridge of $(G+xy)[D_{xy}]$. Therefore, $G[D_{xy}]$ has exactly 2 components and each of which contains exactly one vertex of $\{x, y\}$. This completes the proof. ∎

When $k \geq 3$, Ananchuen [1] established structures of k-γ_c-critical graphs with a cut vertex.

Lemma 9 [1]. *For $k \geq 3$, let G be a k-γ_c-critical graph with a cut vertex c and let D be a connected dominating set. Then*

(1) $G - c$ *contains exactly two components,*

(2) *if C_1 and C_2 are the components of $G - c$, then $G[N_{C_1}(c)]$ and $G[N_{C_2}(c)]$ are complete and*

(3) $c \in D$.

In our previous work in [6], we established the diameter of k-γ_c-critical graphs.

Lemma 10 [6]. *Let G be a k-γ_c-critical graph. Then $diam(G) \leq k$.*

We conclude this section by establishing a forbidden subgraph of k-γ_c-critical graphs when $k \geq 3$ in Lemma 12. We also need to prove the following lemma.

Lemma 11. *Let G be a k-γ_c-critical graph and let x and y be a pair of non-adjacent vertices of G such that $|D_{xy} \cap \{x, y\}| = 1$. Then, for a pair of vertices a and b in D_{xy}, we have that $N(a) \nsubseteq N[b]$.*

Proof. Suppose to the contrary that $N(a) \subseteq N[b]$ for some $a, b \in D_{xy}$.

Claim. $D_{xy} - \{a\} \succ_c a$.

Proof. As $|D_{xy} \cap \{x, y\}| = 1$, we must have $G[D_{xy}]$ is connected. Because $N(a) \subseteq N[b]$ and $b \in D_{xy}$, it follows that $G[D_{xy} - \{a\}]$ is connected. We next show that $D_{xy} - \{a\} \succ a$. As $G[D_{xy}]$ is connected, a must be adjacent to a vertex in D_{xy}. That is $D_{xy} - \{a\} \succ a$. Therefore $D_{xy} - \{a\} \succ_c a$. This settles the claim. □

Since $|D_{xy} \cap \{x, y\}| = 1$, we may assume without loss of generality that $\{x\} = D_{xy} \cap \{x, y\}$. We distinguish two cases.

Case 1. $a \neq x$. Because $N(a) \subseteq N[b]$ and $b \in D_{xy}$, it follows that $D_{xy} - \{a\} \succ V(G + xy) - \{a\}$. Thus, by the claim, we have $D_{xy} - \{a\} \succ_c G + xy$. This contradicts the minimality of D_{xy}. So Case 1 cannot occur.

Case 2. $a = x$. As $N(a) \subseteq N[b]$, we must have $D_{xy} - \{a\} \succ V(G+xy) - \{y, a\}$. By the claim, $D_{xy} - \{a\} \succ_c V(G) - \{y\}$. Because G is connected, it follows that $N(y) \neq \emptyset$. Let $z \in N(y)$. By Lemma 7(3), $z \notin D_{xy}$. As $D_{xy} \succ_c G + xy$, we must have that z is adjacent to a vertex in D_{xy}. If $za \notin E(G)$, then $(D_{xy} - \{a\}) \cup \{z\} \succ_c G$. Lemma 7(1) implies that $|(D_{xy} - \{a\}) \cup \{z\}| \leq k - 1$ contradicting the minimality of $\gamma_c(G)$. Therefore, $za \in E(G)$. As $N(a) \subseteq N[b]$, we must have $zb \in E(G)$. Since $b \in D_{xy}$, it follows that $(D_{xy} - \{a\}) \cup \{z\} \succ_c G$. Similarly, $|(D_{xy} - \{a\}) \cup \{z\}| \leq k - 1$, a contradiction. So Case 2 cannot occur and this completes the proof. ■

We are ready to provide the construction of a forbidden subgraph of k-γ_c-critical graphs. For a connected graph G, let X, Y, X_1 and Y_1 be disjoint vertex subsets of $V(G)$. We, further, let $Z = X \cup X_1 \cup Y \cup Y_1$ and $\overline{Z} = V(G) - Z$. The induced subgraph $G[Z]$ is called a *bad subgraph* if

(i) $x_1 \succ X \cup X_1$ for any vertex $x_1 \in X_1$,

(ii) $N[x] \subseteq X \cup X_1$ for any vertex $x \in X$,

(iii) $y_1 \succ Y \cup Y_1$ for any vertex $y_1 \in Y_1$, and

(iv) $N[y] \subseteq Y \cup Y_1$ for any vertex $y \in Y$.

Figure 1 illustrates our set up.

Observe that $G[X_1]$ and $G[Y_1]$ are complete subgraphs. Further, if $\overline{Z} = \emptyset$, then there exists an edge x_1y_1 where $x_1 \in X_1$and $y_1 \in Y_1$ because G is connected. Thus $\{x_1, y_1\} \succ_c G$. This implies that $\gamma_c(G) \leq 2$. Therefore, if $\gamma_c(G) \geq 3$, then

$\overline{Z} \neq \emptyset$. The next lemma gives that every k-γ_c-critical graph has no bad subgraph as an induced subgraph.

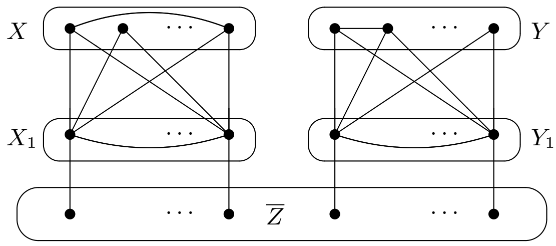

Figure 1. The induced subgraph $G[Z]$.

Lemma 12. *For $k \geq 3$, let G be a k-γ_c-critical graph. Then G does not contain a bad subgraph as an induced subgraph.*

Proof. Suppose to the contrary that G contains $G[Z]$ as a bad subgraph. Let $x \in X$ and $y \in Y$. Consider $G + xy$. Lemma 7(2) implies that $D_{xy} \cap \{x, y\} \neq \emptyset$.

We first show that $\{x, y\} \subseteq D_{xy}$. Suppose to the contrary that $|D_{xy} \cap \{x, y\}| = 1$. Without loss of generality let $\{x\} = D_{xy} \cap \{x, y\}$. Since x is not adjacent to any vertex in Y_1, in order to dominate Y_1, $D_{xy} \cap (V(G) - X) \neq \emptyset$. Because $N[x] \subseteq X \cup X_1$, by the connectedness of $(G + xy)[D_{xy}]$, $D_{xy} \cap X_1 \neq \emptyset$. Let $x_1 \in D_{xy} \cap X_1$. Thus $N(x) \subseteq N[x_1]$ contradicting Lemma 11. Hence $\{x, y\} \subseteq D_{xy}$.

By Observation 8, $G[D_{xy}]$ has exactly two components H_1 and H_2 containing x and y, respectively. Let

$$U_1 = N(H_1) - V(H_1) \text{ and } U_2 = N(H_2) - V(H_2).$$

Thus $V(G) = U_1 \cup U_2 \cup V(H_1) \cup V(H_2)$ because $D_{xy} = V(H_1) \cup V(H_2)$ and $D_{xy} \succ_c G + xy$. We next establish the following claim.

Claim. For a vertex $u \in V(H_1) \cup U_1$, if $(V(H_1) \cup \{u\}) \cap X_1 \neq \emptyset$, then $V(H_1 - x) \cup \{u\} \succ_c U_1 \cup \{x\}$. Similarly, for a vertex $v \in V(H_2) \cup U_2$, if $(V(H_2) \cup \{v\}) \cap Y_1 \neq \emptyset$, then $V(H_2 - y) \cup \{v\} \succ_c U_2 \cup \{y\}$.

Proof. Suppose that there exists $x_1 \in (V(H_1) \cup \{u\}) \cap X_1$. By Property (i) of bad subgraph, $x_1 \succ X \cup X_1$. Hence, $N[x] \subseteq N[x_1]$. Clearly, $G[V(H_1) \cup \{u\}]$ is connected. Since $x_1 \in V(H_1 - x) \cup \{u\}$, it follows that $G[V(H_1 - x) \cup \{u\}]$ is connected. As $N[x] \subseteq N[x_1]$, we must have $V(H_1 - x) \cup \{u\} \succ_c x$. Thus, it remains to show that $V(H_1 - x) \cup \{u\} \succ U_1$. Let $w \in U_1$. So, w is adjacent to a vertex of H_1. If $wx \notin E(G)$, then w is adjacent to a vertex of $H_1 - x$. But, if $wx \in E(G)$, then $wx_1 \in E(G)$. These imply that w is adjacent to a vertex in

$V(H_1-x)\cup\{u\}$. So $V(H_1-x)\cup\{u\} \succ U_1$. Therefore, $V(H_1-x)\cup\{u\} \succ_c U_1\cup\{x\}$. We can show that if $(V(H_2)\cup\{v\})\cap Y_1 \neq \emptyset$, then $V(H_2-y)\cup\{v\} \succ_c U_2\cup\{y\}$ by the similar arguments. This settles the claim. □

We note by the claim that u can be a vertex in H_1. Thus if $V(H_1)\cap X_1 \neq \emptyset$, then $V(H_1-x) \succ_c U_1\cup\{x\}$. Clearly $\overline{Z} \neq \emptyset$ because $k \geq 3$. To dominate $\overline{Z}$, we have $D_{xy}\cap(\overline{Z}\cup X_1\cup Y_1) \neq \emptyset$ because $N[x] \subseteq X\cup X_1$ and $N[y] \subseteq Y\cup Y_1$. Thus, by the connectedness of H_1 and H_2, $V(H_1)\cap X_1 \neq \emptyset$ or $V(H_2)\cap Y_1 \neq \emptyset$. Suppose without loss of generality that $V(H_1)\cap X_1 \neq \emptyset$. By applying the claim, we have that

$$V(H_1-x) \succ_c U_1\cup\{x\}. \tag{1}$$

Case 1. $U_1\cap U_2 \neq \emptyset$. Thus there is a vertex $v \in V(G)-(V(H_1)\cup V(H_2))$ such that v is adjacent to a vertex of H_1 and a vertex of H_2. That is $G[V(H_1)\cup\{v\}\cup V(H_2)]$ is connected.

We next show that $(V(H_2)\cup\{v\})\cap Y_1 \neq \emptyset$. Suppose that $(V(H_2)\cup\{v\})\cap Y_1 = \emptyset$. By the connectedness of H_2, $V(H_2) \subseteq Y$ because $y \in Y$. Moreover, $v \in Y$ because v is adjacent to a vertex of H_2. So, Property (iv) implies that $N[v] \subseteq Y\cup Y_1$. As v is adjacent to a vertex of H_1, we must have that $V(H_1)\cap(Y\cup Y_1) \neq \emptyset$. By the connectedness of H_1, $V(H_1)\cap Y_1 \neq \emptyset$. Property (iii) yields that there exists a vertex of H_1 adjacent to a vertex of H_2. So H_1 and H_2 are the same component, a contradiction. Hence $(V(H_2)\cup\{v\})\cap Y_1 \neq \emptyset$. By the claim, we have that

$$V(H_2-y)\cup\{v\} \succ_c U_2\cup\{y\}. \tag{2}$$

Since $V(G) = U_1\cup U_2\cup V(H_1)\cup V(H_2)$, by (1) and (2), $V(H_1-x)\cup V(H_2-y)\cup\{v\} \succ_c G$. Hence

$$(D_{xy}-\{x,y\})\cup\{v\} = V(H_1-x)\cup V(H_2-y)\cup\{v\} \succ_c G.$$

Lemma 7(1) yields that $|(D_{xy}-\{x,y\})\cup\{v\}| \leq k-1$ contradicting $\gamma_c(G) = k$. So Case 1 cannot occur.

Case 2. $U_1\cap U_2 = \emptyset$. Since G is connected, there exist vertices u and v in $V(G)-(V(H_1)\cup V(H_2))$ such that $u \in U_1, v \in U_2$ and $uv \in E(G)$. Therefore $G[V(H_1)\cup\{u,v\}\cup V(H_2)]$ is connected.

We will show that $(V(H_1)\cup\{u\})\cap X_1 \neq \emptyset$ and $(V(H_2)\cup\{v\})\cap Y_1 \neq \emptyset$. Suppose to the contrary that $(V(H_1)\cup\{u\})\cap X_1 = \emptyset$. So $V(H_1)\cup\{u\} \subseteq X$ by the connectedness of $G[V(H_1)\cup\{u\}]$. Since H_1 and H_2 are different components, by Property (i), $V(H_2)\cap X_1 = \emptyset$. Thus $v \in X_1$ because $uv \in E(G)$ and $N[u] \subseteq X\cup X_1$. This implies by Property (i) that $v \succ H_1$, in particular $v \in U_1$. Thus $v \in U_1\cap U_2$. This contradicts $U_1\cap U_2 = \emptyset$. Hence, $(V(H_1)\cup\{u\})\cap X_1 \neq \emptyset$. By the same arguments, we have $(V(H_2)\cup\{v\})\cap Y_1 \neq \emptyset$.

Hence, by the claim, we have that $V(H_1 - x) \cup \{u\} \succ_c U_1 \cup \{x\}$ and $V(H_2 - y) \cup \{v\} \succ_c U_2 \cup \{y\}$. As $V(G) = U_1 \cup U_2 \cup V(H_1) \cup V(H_2)$, we must have that $(D_{xy} - \{x, y\}) \cup \{u, v\} \succ_c G$. Lemma 7(1) gives that $|(D_{xy} - \{x, y\}) \cup \{u, v\}| \leq k-1$ contradicting $\gamma_c(G) = k$. So Case 2 cannot occur. Therefore G does not contain a bad subgraph as an induced subgraph. This completes the proof. ■

By applying Lemma 12, we easily establish the maximum number of end vertices of k-γ_c-critical graphs.

Corollary 13 [8]**.** *For $k \geq 3$, every k-γ_c-critical graph has at most one end vertex.*

Proof. Suppose to the contrary that G has x and y as two end vertices. Let x_1 and y_1 be the support vertices adjacent to x and y, respectively. Thus x_1 and y_1 are cut vertices. Since $\gamma_c(G) \geq 3$, $V(G) - \{x, x_1, y, y_1\} \neq \emptyset$. Thus, Lemma 9(1) implies that $x_1 \neq y_1$. Choose $X_1 = \{x_1\}, Y_1 = \{y_1\}, X = \{x\}$ and $Y = \{y\}$. Clearly $G[X_1 \cup Y_1 \cup X \cup Y]$ is a bad subgraph contradicting Lemma 12. Hence, G has at most one end vertex and this completes the proof. ■

It is worth noting that very recently Taylor and van der Merwe [8] proved Corollary 13 as well. They proved the corollary with contrapositive but did not apply the concept of a bad subgraph in their proof.

3. The Characterizations of Some end Blocks

In this section, we provide characterizations of some blocks of k-γ_c-critical graphs. For a connected graph G, we let $\mathcal{A}(G)$ be the set of all cut vertices of G.

We first show that for a connected graph G and a pair of non-adjacent vertices x and y of G, $\mathcal{A}(G) = \mathcal{A}(G + xy)$ if x and y are in the same block of G.

Lemma 14. *For a connected graph G, let B be a block of G and $x, y \in V(B)$ such that $xy \notin E(G)$. Then $\mathcal{A}(G) = \mathcal{A}(G + xy)$.*

Proof. Since G is a subgraph of $G+xy$, $\mathcal{A}(G+xy) \subseteq \mathcal{A}(G)$. Suppose there exists c such that $c \in \mathcal{A}(G)$ but $c \notin \mathcal{A}(G+xy)$. Thus $(G+xy)-c$ is connected. Let C be the component of $G-c$ containing vertices of $V(B) - \{c\}$ and C' be a component of $G - c$ which is not C. Further, let $a \in N_{C'}(c)$ and $b \in N_C(c)$. Since c is a cut vertex of G, there is only one path a, c, b from a to b. But c is not a cut vertex in $G + xy$. This implies that $G - c$ has a path $P = p_1, p_2, \ldots, x, y, \ldots, p_r$ from b to a where $b = p_1, a = p_r, x = p_i$ and $y = p_{i+1}$ for some $1 \leq i \leq r - 1$ and $r \geq 2$. We see that P must contain an edge xy and $c \notin \{p_1, p_2, \ldots, p_r\}$. Since C and C' are the two different components of $G - c$, by the connectedness of the path P, $\{p_1, p_2, \ldots, p_i\} \subseteq V(C)$ and $\{p_{i+1}, \ldots, p_r\} \subseteq V(C')$. So $x \in V(C)$ and $y \in V(C')$

contradicting x and y are in the same block. Therefore $\mathcal{A}(G) \subseteq \mathcal{A}(G + xy)$ and thus, $\mathcal{A}(G) = \mathcal{A}(G + xy)$. This completes the proof. ■

For a k-γ_c-critical graph G with a cut vertex, let B be an end block of G containing non-adjacent vertices x and y. Clearly, $V(B + xy) = V(B)$.

Lemma 15. *For an integer $k \geq 3$, let G be a k-γ_c-critical graph with a γ_c-set D and let B be an end block of G. For all $x, y \in V(B)$ such that $xy \notin E(G)$, $|D_{xy} \cap V(B + xy)| < |D \cap V(B)|$.*

Proof. Let c be the cut vertex of G such that $\mathcal{A}(G) \cap V(B) = \{c\}$. Note that $D - V(B)$ and $D \cap V(B)$ are disjoint as well as $D_{xy} - V(B + xy)$ and $D_{xy} \cap V(B + xy)$. We first establish the following claim.

Claim. $|D_{xy} - V(B + xy)| \geq |D - V(B)|$.

Proof. Suppose to the contrary that $|D_{xy} - V(B + xy)| < |D - V(B)|$. Clearly, $|D_{xy} - V(B + xy)| = |D_{xy} - V(B)|$. Thus $|D_{xy} - V(B)| < |D - V(B)|$. We will show that $(D_{xy} - V(B)) \cup (D \cap V(B)) \succ_c G$. Firstly, we show that $G[(D_{xy} - V(B)) \cup (D \cap V(B))]$ is connected. As D_{xy} is a γ_c-set of $G + xy$, we must have that $(G + xy)[D_{xy}]$ is connected. Since $xy \in E(B + xy)$, we have that $G[(D_{xy} - V(B + xy)) \cup \{c\}]$ is connected. Hence, $G[(D_{xy} - V(B)) \cup \{c\}]$ is connected. Clearly, $G[D \cap V(B)]$ is connected. Moreover, $c \in D \cap V(B)$ by Lemma 9(3). Thus $G[(D_{xy} - V(B)) \cup (D \cap V(B))]$ is connected.

We next show that $(D_{xy} - V(B)) \cup (D \cap V(B)) \succ G$. Because $D_{xy} \succ_c G + xy$ and $xy \in E(B + xy)$, it follows that $(D_{xy} - V(B)) \cup \{c\} \succ V(G) - V(B)$. It is easy to see that $D \cap V(B) \succ V(B)$. So $(D_{xy} - V(B)) \cup (D \cap V(B)) \succ G$. This implies that $(D_{xy} - V(B)) \cup (D \cap V(B)) \succ_c G$. But

$$\begin{aligned} |(D_{xy} - V(B)) \cup (D \cap V(B))| &\leq |(D_{xy} - V(B))| + |(D \cap V(B))| \\ &< |(D - V(B))| + |(D \cap V(B))| \\ &= |(D - V(B)) \cup (D \cap V(B))| = |D|, \end{aligned}$$

contradicting the minimality of D. Therefore $|D_{xy} - V(B + xy)| \geq |D - V(B)|$ and this settles the claim. □

We are now ready to prove this lemma. Suppose to the contrary that $|D_{xy} \cap V(B + xy)| \geq |D \cap V(B)|$. Thus

$$\begin{aligned} |D_{xy}| &= |(D_{xy} - V(B + xy)) \cup (D_{xy} \cap V(B + xy))| \\ &= |(D_{xy} - V(B + xy))| + |(D_{xy} \cap V(B + xy))| \\ &\geq |(D - V(B))| + |(D_{xy} \cap V(B + xy))| \quad \text{(by the claim)} \\ &\geq |(D - V(B))| + |(D \cap V(B))| \\ &= |(D - V(B)) \cup (D \cap V(B))| = |D|, \end{aligned}$$

contradicting Lemma 7(1). Therefore, $|D_{xy} \cap V(B + xy)| < |D \cap V(B)|$ and this completes the proof. ∎

We now introduce four classes of graphs such that some graph in these classes is an end block of a k-γ_c-critical graph. For vertices c, z_1 and z_2, we let

$$\mathcal{B}_0 = \{c \vee K_{t_1} : \text{for an integer } t_1 \geq 1\},$$
$$\mathcal{B}_1 = \{c \vee K_{t_2} \vee z_1 : \text{for an integer } t_2 \geq 1\}, \text{ and}$$
$$\mathcal{B}_{2,1} = \{c \vee K_{t_3} \vee K_{t_4} \vee z_2 : \text{for integers } t_3, t_4 \geq 1\}.$$

Before we construct the next class, it is worth to introduce a graph T which occurs in the characterization of k-γ_c-critical graphs with a maximum number of cut vertices. For positive integers $l \geq 2, r$ and n_i, we let $\mathcal{S} = \bigcup_{i=1}^{l} K_{1,n_i}$ and

$$T = \mathcal{S} \text{ or}$$
$$T = \mathcal{S} \cup \overline{K_r}.$$

Then, for $1 \leq i \leq l$, we let $s_0^i, s_1^i, s_2^i, \ldots, s_{n_i}^i$ be the vertices of a star K_{1,n_i} centered at s_0^i. We, further, let $S = \bigcup_{i=1}^{l} \{s_1^i, s_2^i, \ldots, s_{n_i}^i\}$ and $S' = \bigcup_{i=1}^{l} \{s_0^i\}$, moreover, let $S'' = V(\overline{K_r})$ if $T = \mathcal{S} \cup \overline{K_r}$ and $S'' = \emptyset$ if $T = \mathcal{S}$. We note that

$$\overline{T} = \overline{\mathcal{S}} \text{ or}$$
$$\overline{T} = \overline{\mathcal{S}} \vee K_r.$$

That is, $\overline{T}$ can be obtained by removing the edges in the stars of $\mathcal{S}$ from a complete graph on $S \cup S' \cup S''$. Throughout this paper, we are, in fact, using the complement of T. We are ready to define the next class. Recall that, for graphs G_1 and G_2 such that G_2 has H as a subgraph, the join $G_1 \vee_H G_2$ is the graph constructed from the disjoint union of G_1 and G_2 by joining each vertex in G_1 to each vertex in H with an edge.

$$\mathcal{B}_{2,2} = \left\{c \vee_{\overline{T}[S]} \overline{T} : \text{for positive integers } l \geq 2, r \text{ and } n_i\right\}.$$

We note by the construction that, in $\overline{T}$, every vertex in S is adjacent to exactly $|S' \cup S''| - 1$ vertices in $S' \cup S''$. A graph in this class is illustrated in Figure 2. According to the figure, an *oval* denotes a complete subgraph, *double lines* between subgraphs denote joining every vertex of one subgraph to every vertex of the other subgraph and a *dash line* denotes a removed edge.

It is worth noting that, for an end block B of a k-γ_c-critical graph having D as a γ_c-set, the number of vertices in $D \cap V(B)$ can be as large as k. We will give an example by using the graph $\overline{T}$. For an integer $k \geq 5$, let $K_{n_1}, \ldots, K_{n_{k-3}}$ be $k - 3$ copies of complete graphs with $n_1, \ldots, n_{k-3} \geq 2$ and let a_1 and a_2 be two isolated vertices. It is not difficult to see that the graph

$$a_1 \vee a_2 \vee K_{n_1} \vee \cdots \vee K_{n_{k-3}} \vee_{\overline{T}[S]} \overline{T}$$

is a k-γ_c-critical graph having $R = a_2 \vee K_{n_1} \vee \cdots \vee K_{n_{k-3}} \vee_{\overline{T}[S]} \overline{T}$ as an end block. Clearly, $|D \cap V(R)| = k$.

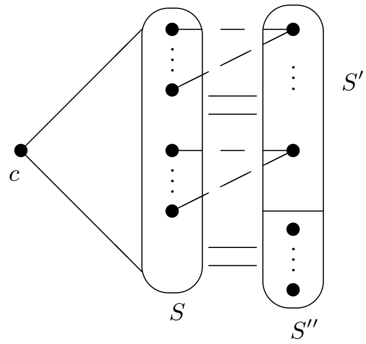

Figure 2. A graph G in the class $\mathcal{B}_{2,2}$.

In the following, we characterize an end block B such that $|D \cap V(B)| \leq 3$. Let c be the cut vertex of G in B and H be the component of $G - c$ such that $G[V(H) \cup \{c\}] = B$. We further let

$W = N_H(c)$,

$W' = \{w' \in V(H) - W : w'w \in E(G)$ for some $w \in W\}$ and

$W'' = V(H) - (W \cup W')$.

Note that W' or W'' can be empty. Since $c \in V(B)$, we have that $|D \cap V(H)| = i$ if and only if $|D \cap V(B)| = i + 1$ for all $i \geq 0$. Thus, $|D \cap V(B)| \geq 1$.

Lemma 16. *Let G be a k-γ_c-critical graph with a γ_c-set D and let B be an end block of G. If $|D \cap V(B)| = 1$, then $B \in \mathcal{B}_0$.*

Proof. In view of Lemma 9(2), $G[W]$ is complete. Lemma 9(3) gives, further, that $D \cap V(B) = \{c\}$. Since $D \succ B$ and $|(D \cap V(B)) - \{c\}| = 0$, it follows that $W' \cup W'' = \emptyset$ and $c \succ W$. So $B \in \mathcal{B}_0$. This completes the proof. ■

Lemma 17. *Let G be a k-γ_c-critical graph with a γ_c-set D and let B be an end block of G. If $|D \cap V(B)| = 2$, then $B \in \mathcal{B}_1$.*

Proof. Let $\{y\} = (D \cap V(B)) - \{c\}$. By the connectedness of $G[D]$, $y \in W$. Thus $W'' = \emptyset$ and $V(H) = W \cup W'$. Suppose that there exist $u, v \in V(H)$ such that $uv \notin V(G)$. Consider $G + uv$. Lemma 7(2) gives that $D_{uv} \cap \{u, v\} \neq \emptyset$. Lemma 14 gives also that $c \in D_{uv}$. Hence, $|D_{uv} \cap V(B + uv)| \geq 2$ contradicting Lemma 15. Thus $G[W \cup W']$ is complete. Let $z_1 \in W'$. Consider $G + cz_1$. Since $|D \cap V(B)| = 2$, by Lemma 15, $|D_{cz_1} \cap V(B + cz_1)| \leq 1$. Lemmas 9(3) and 14 yield that $c \in D_{cz_1}$. So $|D_{cz_1} \cap V(H)| = 0$. This implies that $c \succ B + cz_1$. Since $\{z_1\} = N_{G+cz_1}(c) \cap W'$, $W' = \{z_1\}$. So $B \in \mathcal{B}_1$ and this completes the proof. ■

Lemma 18. *Let G be a k-γ_c-critical graph with a γ_c-set D and let B be an end block of G. Suppose that $|D \cap V(B)| = 3$. Then $B \in \mathcal{B}_{2,1}$ if $W'' \neq \emptyset$ and $B \in \mathcal{B}_{2,2}$ if $W'' = \emptyset$. Consequently, $B \in \mathcal{B}_{2,1} \cup \mathcal{B}_{2,2}$.*

Proof. Suppose that $|D \cap V(B)| = 3$. Lemma 9(2) implies that $G[W]$ is complete. We first establish the following claim.

Claim. For any non-adjacent vertices $u, v \in W \cup W' \cup W''$, we have $c \in D_{uv} \cap V(B + uv)$ and $|D_{uv} \cap W \cap \{u, v\}| = 1$.

Proof. Lemma 15 implies that $|D_{uv} \cap V(B + uv)| \leq 2$. In view of Lemmas 9(3) and 14, $c \in D_{uv} \cap V(B + uv)$. Thus $|D_{uv} \cap \{u, v\}| \leq 1$. Lemma 7(2) then gives that $|D_{uv} \cap \{u, v\}| = 1$. So $|D_{uv} \cap W \cap \{u, v\}| = 1$ because $(G + uv)[D_{uv}]$ is connected. This settles the claim. □

Suppose there exist $u, v \in W' \cup W''$ such that $uv \notin E(G)$. Consider $G + uv$. By the claim $|D_{uv} \cap W \cap \{u, v\}| = 1$ contradicting $W \cap \{u, v\} = \emptyset$. Thus $G[W' \cup W'']$ is complete.

We first consider the case when $W'' \neq \emptyset$. Let $w \in W$ and $z_2 \in W''$. Consider $G + wz_2$. By the claim, $D_{wz_2} \cap V(B + wz_2) = \{c, w\}$. Since $\{z_2\} = W'' \cap N_{G+wz_2}(w)$, it follows that $W'' = \{z_2\}$. Suppose there exists $w' \in W'$ such that $ww' \notin E(G)$. Consider $G + ww'$. By the claim, $D_{ww'} \cap V(B + ww') = \{c, w\}$. Thus $D_{ww'}$ does not dominate z_2, a contradiction. Therefore $G[W \cup W']$ is complete and $B \in \mathcal{B}_{2,1}$.

We finally consider the case when $W'' = \emptyset$. We will show that, for all $w \in W$, $|N_{W'}(w)| = |W'| - 1$. If $w \succ W'$, then $(D - V(H)) \cup \{w\} \succ_c G$. But $|(D - V(H)) \cup \{w\}| = k - 1$ contradicting $\gamma_c(G) = k$. Thus $|N_{W'}(w)| \leq |W'| - 1$. If w is not adjacent to x, y in W', then consider $G + wx$. By the claim, $D_{wx} \cap V(B + wx) = \{c, w\}$. Clearly D_{wx} does not dominate y, a contradiction. Hence, $|N_{W'}(w)| = |W'| - 1$ for all $w \in W$. We now have that $G[W \cup W']$ is the complement of disjoint union of isolated vertices in W' and stars whose centers are in W' and all of end vertices are in W. It remains to show that there are at least two stars in $\overline{G}[W \cup W']$. Suppose to the contrary that, in $\overline{G}[W \cup W']$, there is exactly one star centered at w'. Because $|N_{W'}(w)| = |W'| - 1$ for all $w \in W$, w' is not adjacent to any vertex in W. So $w' \in W''$ contradicting $W'' = \emptyset$. Hence, there are at least two stars in $\overline{G}[W \cup W']$. This completes the proof. ■

4. The Upper Bound of the Number of Cut Vertices

In this section, we establish the maximum number of cut vertices of k-γ_c-critical graphs. In view of Observation 2, it suffices to restrict our attention to the case $k \geq 3$. We begin this section by showing that G does not have two end blocks in $\mathcal{B}_0 \cup \mathcal{B}_1$.

Lemma 19. *For $k \geq 3$, let G be a k-γ_c-critical graph. Then G contains at most one end block B such that $B \in \mathcal{B}_0 \cup \mathcal{B}_1$.*

Proof. Suppose to the contrary that there exist two different end blocks U and R which are, respectively, in the classes $\mathcal{B}_i$ and $\mathcal{B}_j$ where $\{i, j\} \subseteq \{0, 1\}$. Let u be the cut vertex of G in U. If $U \in \mathcal{B}_0$, then $U = u \vee K_{t_1}$ for some integer $t_1 \geq 1$. If $U \in \mathcal{B}_1$, then there exist an integer $t_2 \geq 1$ and a vertex z_1 of U such that $U = u \vee K_{t_2} \vee z_1$. Then, we choose

$$X_1 = \begin{cases} \{u\} & \text{if } U \in \mathcal{B}_0, \\ V(K_{t_2}) & \text{if } U \in \mathcal{B}_1, \end{cases}$$

and we choose

$$X = \begin{cases} V(K_{t_1}) & \text{if } U \in \mathcal{B}_0, \\ \{z_1\} & \text{if } U \in \mathcal{B}_1. \end{cases}$$

Clearly, U contains X and X_1 which satisfy the Properties (i) and (ii), respectively.

We now consider R. Let r be the cut vertex of G in R. If $R \in \mathcal{B}_0$, then $R = r \vee K_{t'_1}$ for some integer $t'_1 \geq 1$. But, if $R \in \mathcal{B}_1$, then there exist an integer $t'_2 \geq 1$ and a vertex w_1 of R such that $R = r \vee K_{t'_2} \vee w_1$. Then, we choose

$$Y_1 = \begin{cases} \{r\} & \text{if } R \in \mathcal{B}_0, \\ V(K_{t'_2}) & \text{if } R \in \mathcal{B}_1, \end{cases}$$

and we choose

$$Y = \begin{cases} V(K_{t'_1}) & \text{if } R \in \mathcal{B}_0, \\ \{w_1\} & \text{if } R \in \mathcal{B}_1. \end{cases}$$

Clearly, R contains Y and Y_1 which satisfy the Properties (i) and (ii), respectively.

We observe that X, Y and Y_1 are pairwise disjoint because U and R are different blocks. Suppose that $Y_1 \cap X_1 \neq \emptyset$. By the choice of X_1 and Y_1, if $X_1 = V(K_{t_2})$ or $Y_1 = V(K_{t'_2})$, then $Y_1 \cap X_1 = \emptyset$ because U and R are different end blocks, contradicting the assumption that $Y_1 \cap X_1 \neq \emptyset$. Hence, $X_1 = \{u\}$ and $Y_1 = \{r\}$. This implies that $u = r$, moreover, U and R are both in $\mathcal{B}_0$. Thus $u \succ U$ and $u \succ R$. Lemma 9(1) yields that $G - u$ has $U - u$ and $R - u$ as the two components. We have that $G = K_{t_1} \vee u \vee K_{t'_1}$. Clearly, $u \succ_c G$ contradicting $\gamma_c(G) \geq 3$. Hence, $Y_1 \cap X_1 = \emptyset$. So, G contains a bad subgraph contradicting Lemma 12. This completes the proof. ■

In the following, for a block B of G, we let

$$\mathcal{A}(B) = V(B) \cap \mathcal{A}(G).$$

We also let

$$\zeta(G) = |\mathcal{A}(G)|, \zeta(B) = |\mathcal{A}(B)| \text{ and}$$
$$\zeta_0(G) = \max\{\zeta(B) : B \text{ is a block of } G\}.$$

When no ambiguity can occur, we abbreviate $\zeta_0(G)$ to ζ_0. Clearly, $\zeta_0 \leq \zeta(G)$. In the following lemma, we establish the existence of ζ_0 end blocks.

Lemma 20. *For any k-γ_c-critical graph G, let B_0 be a block of G containing ζ_0 cut vertices $c_1, c_2, \ldots, c_{\zeta_0}$. Then there exist mutually disjoint end blocks $B_1, B_2, \ldots, B_{\zeta_0}$.*

Proof. In view of Lemma 9(1), $G - c_i$ has only two components for $1 \leq i \leq \zeta_0$. Let C_i be the component of $G - c_i$ that does not contain any vertex of B_0. If graph $G[\{c_i\} \cup V(C_i)]$ does not contain any cut vertex, then $G[\{c_i\} \cup V(C_i)]$ is an end block and we let $B_i = G[\{c_i\} \cup V(C_i)]$. If graph $G[\{c_i\} \cup V(C_i)]$ contains a cut vertex, then, by Lemma 6, $G[\{c_i\} \cup V(C_i)]$ has at least two end blocks. Therefore, at least one end block of $G[\{c_i\} \cup V(C_i)]$ does not contain c_i and we let B_i be this end block. In both cases of the choice, B_i is an end block of G. Obviously, $B_1, B_2, \ldots, B_{\zeta_0}$ are mutually disjoint and this completes the proof. ■

Lemma 21. *For $k \geq 3$, let G be a k-γ_c-critical graph with a γ_c-set D and let $B_1, B_2, \ldots, B_{\zeta_0}$ be the end blocks of G from Lemma 20. Moreover, for $1 \leq i \leq \zeta_0$, we let $x_i \in \mathcal{A}(G) \cap V(B_i)$. Then at least $\zeta_0 - 1$ of the end blocks $B_1, B_2, \ldots, B_{\zeta_0}$ satisfy $|(D \cap V(B_i)) - \{x_i\}| \geq 2$.*

Proof. Lemma 19 gives that at least $\zeta_0 - 1$ blocks of $\{B_i | 1 \leq i \leq \zeta_0\}$ are not in $\mathcal{B}_j$ where $j \in \{0, 1\}$. Without loss of generality let $B_1, B_2, \ldots, B_{\zeta_0 - 1}$ be such blocks. Hence

$$|(D \cap V(B_i)) - \{x_i\}| \geq 2$$

for $1 \leq i \leq \zeta_0 - 1$ and this completes the proof. ■

We next let $\overline{\mathcal{A}} = \mathcal{A}(G) - \mathcal{A}(B_0)$ and $\overline{\zeta} = |\overline{\mathcal{A}}|$. That is, $\overline{\mathcal{A}}$ is the set of cut vertices which are not in B_0. Clearly,

$$\zeta(G) = \overline{\zeta} + \zeta_0. \tag{3}$$

Recall that, for $1 \leq i \leq \zeta_0$, C_i is the component of $G - c_i$ which does not contain any vertex of B_0. We also let

$$j_0 = \min\left\{|D \cap V(C_i)| : \text{for all } 1 \leq i \leq \zeta_0\right\}.$$

The following theorem gives the relationship of $\zeta_0, \overline{\zeta}, j_0$ and k.

Theorem 22. *For $k \geq 3$, let G be a k-γ_c-critical graph. Then $3\zeta_0 - 2 + \overline{\zeta} + j_0 \leq k$.*

Proof. Lemma 9(3) yields that $\mathcal{A}(G) \subseteq D$. For each end block B_i of G which is a consequent of Lemma 20, $1 \leq i \leq \zeta_0$, let $x_i \in \mathcal{A}(G) \cap V(B_i)$. Clearly $(D \cap V(B_1)) - \{x_1\}, (D \cap V(B_2)) - \{x_2\}, \ldots, (D \cap V(B_{\zeta_0})) - \{x_{\zeta_0}\}$ and $\mathcal{A}(G)$ are pairwise disjoint. These imply that

$$\sum_{i=1}^{\zeta_0} |(D \cap V(B_i)) - \{x_i\}| + \zeta(G) \leq k. \tag{4}$$

In view of Lemma 21, at least $\zeta_0 - 1$ end blocks of $B_1, B_2, \ldots, B_{\zeta_0}$ are not in $\mathcal{B}_0 \cup \mathcal{B}_1$. Without loss of generality let $B_1, B_2, \ldots, B_{\zeta_0 - 1}$ be such blocks. So $2 \leq |(D \cap V(B_i)) - \{x_i\}|$ for $1 \leq i \leq \zeta_0 - 1$. Therefore

$$2(\zeta_0 - 1) \leq \sum_{i=1}^{\zeta_0 - 1} \big|(D \cap V(B_i)) - \{x_i\}\big|. \tag{5}$$

By the minimality of j_0,

$$0 \leq j_0 \leq \big|(D \cap V(B_{\zeta_0})) - \{x_{\zeta_0}\}\big|. \tag{6}$$

Therefore

$$\begin{aligned} 3\zeta_0 - 2 + j_0 + \overline{\zeta} &= 2(\zeta_0 - 1) + j_0 + \overline{\zeta} + \zeta_0 \\ &\leq \sum_{i=1}^{\zeta_0 - 1} |(D \cap V(B_i)) - \{x_i\}| + j_0 + \zeta(G) \quad \text{(by (3) and (5))} \\ &\leq \sum_{i=1}^{\zeta_0} |(D \cap V(B_i)) - \{x_i\}| + \zeta(G) \quad \text{(by (6))} \\ &\leq k \quad \text{(by (4))}, \end{aligned}$$

as required. ■

Theorem 22 implies the following corollary.

Corollary 23. *For $k \geq 3$, let G be a k-γ_c-critical graph. Then $\zeta_0 \leq \left\lfloor \frac{k+2}{3} \right\rfloor$.*

Proof. Theorem 22 implies that $3\zeta_0 \leq k + 2 - \overline{\zeta} - j_0$. As $\overline{\zeta}, j_0 \geq 0$, we must have that

$$\zeta_0 \leq \left\lfloor \frac{k+2}{3} \right\rfloor$$

and this completes the proof. ■

Note that Theorem 22 together with $\zeta(G) = \overline{\zeta} + \zeta_0$ give

$$2\zeta_0 \leq k - \zeta(G) - j_0 + 2. \tag{7}$$

We are now ready to establish Theorem 5. For completeness, we recall the statement of this theorem.

Theorem 5. *For $k \geq 3$, let G be a k-γ_c-critical graph with $\zeta(G)$ cut vertices. Then $\zeta(G) \leq k - 2$.*

Proof. Suppose to the contrary that $|\mathcal{A}(G)| > k - 2$. Lemma 9(3) gives that $|\mathcal{A}(G)| \leq k$. Thus either $|\mathcal{A}(G)| = k$ or $|\mathcal{A}(G)| = k-1$, in particular, $k-\zeta(G) \leq 1$. This implies by (7) that

$$2\zeta_0 \leq k - \zeta(G) - j_0 + 2 \leq 3.$$

Therefore

$$\zeta_0 \leq 1.$$

If $\zeta(G) \geq 2$, then we always have a block containing more than one cut vertex. Thus $\zeta_0 \geq 2$, a contradiction. Therefore $\zeta(G) \leq 1$. As $k - \zeta(G) \leq 1$, we must have that

$$k \leq 2,$$

contradicting $k \geq 3$. Hence $\zeta(G) \leq k - 2$ and this completes the proof. ∎

5. Characterizations

In this section, we characterize all k-γ_c-critical graphs G when $\zeta(G) = k - 2$. We first give the construction of a k-γ_c-critical graph with $k - 2$ cut vertices.

The class $\mathcal{F}(k)$

Let B be a graph in the class $\mathcal{B}_{2,2}$ containing c, S, S' and S'' which are defined in $\mathcal{B}_{2,2}$. We, further, let $P_{k-1} = z_0, z_1, \ldots, z_{k-2}$ be a path of order $k - 1$. A graph G in the class $\mathcal{F}(k)$ is constructed from the graphs B and P_{k-1} by identifying z_{k-2} with c. A graph G in the class $\mathcal{F}(k)$ is illustrated in Figure 3.

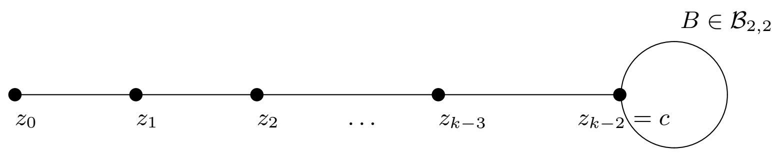

Figure 3. A graph G in the class $\mathcal{F}(k)$.

Lemma 24. *Let $G \in \mathcal{F}(k)$. Then G is a k-γ_c-critical graph with $k-2$ cut vertices.*

Proof. Clearly, $z_1, z_2, \ldots, z_{k-2}$ are the $k-2$ cut vertices of G. We observe that $\{z_1, z_2, \ldots, z_{k-2}, s_1^1, s_0^2\} \succ_c G$. Therefore $\gamma_c(G) \leq k$.

We next show that $\gamma_c(G) \geq k$. Let D be a γ_c-set of G. Since z_1 is a cut vertex, by the connectedness of $G[D]$, $z_1 \in D$. We first suppose that $D \cap S'' \neq \emptyset$. As $z_1 \in D$, by the connectedness of $G[D]$, we must have $\{z_2, z_3, \ldots, z_{k-2}, y\} \subseteq D$ where $y \in D \cap S$. Thus $\gamma_c(G) = |D| \geq k$ and $\gamma_c(G) = k$. We now suppose that $D \cap S'' = \emptyset$. To dominate B, $|D \cap (S \cup S')| \geq 2$. Similarly, by the connectedness of $G[D]$, we have $\{z_2, z_3, \ldots, z_{k-2}\} \subseteq D$ and thus $\gamma_c(G) \geq k$. Therefore $\gamma_c(G) = k$.

We next establish the criticality. Let u and v be two non-adjacent vertices of G and $S_1 = S \cup S' \cup S''$. We first consider the case when $\{u, v\} \subseteq S_1$. Thus $\{u, v\} = \{s_j^i, s_0^i\}$ for some $i \in \{1, 2, \ldots, l\}$ and $j \in \{1, 2, \ldots, n_i\}$. Clearly $\{z_1, z_2, \ldots, z_{k-2}, s_j^i\} \succ_c G + uv$ and $\gamma_c(G + uv) \leq k - 1$.

We now consider the case when $|\{u, v\} \cap S_1| = 1$. Without loss of generality let $\{v\} = \{u, v\} \cap S_1$. If $u = z_{k-2}$, then $v \notin S$. So $\{z_{k-2}, v\} \succ S_1$. Thus $\{z_1, z_2, \ldots, z_{k-2}, v\} \succ_c G + uv$. Therefore $\gamma_c(G + uv) \leq k - 1$. Since $l \geq 2$, there exists $v' \in S - \{v\}$ such that $\{v, v'\} \succ_c S_1$. Then, if $u \in \{z_1, z_2, \ldots, z_{k-3}\}$, we have $\{z_1, z_2, \ldots, u, \ldots, z_{k-3}, v, v'\} \succ_c G + uv$. Hence $\gamma_c(G + uv) \leq k - 1$. If $u = z_0$, then $\{z_2, z_3, \ldots, z_{k-2}, v, v'\} \succ_c G + uv$ and thus, $\gamma_c(G + uv) \leq k - 1$.

We finally consider the case when $|\{u, v\} \cap S_1| = 0$. Therefore $\{u, v\} \subseteq \{z_0, z_1, \ldots, z_{k-2}\}$. Thus $u = z_i$ and $v = z_j$ for some $i \neq j \in \{0, 1, 2, \ldots, k-2\}$. Without loss of generality let $i < j$. Clearly $i + 2 \leq j$. Hence

$$\big(\{z_1, \ldots, z_{k-2}\} - \{z_{i+1}\}\big) \cup \{s_1^1, s_0^2\} \succ_c G + uv.$$

So $\gamma_c(G + uv) \leq k - 1$. Thus G is a k-γ_c-critical graph and this completes the proof. ■

Let $\mathcal{Z}(k, \zeta)$ be the class of k-γ_c-critical graphs containing ζ cut vertices. As the graphs in these class have been characterized in [1] and [6] when $3 \leq k \leq 4$, we turn attention to the case when $k \geq 5$.

Lemma 25. *For $k \geq 5$, let $G \in \mathcal{Z}(k, \zeta)$ where $\zeta \in \{k-3, k-2\}$. Then G has only two end blocks and the remaining blocks contain two cut vertices.*

Proof. Clearly $\zeta(G) \geq k - 3$. We have by (7) that

$$2\zeta_0 \leq k - \zeta(G) - j_0 + 2 \leq k - (k-3) - j_0 + 2 \leq 5.$$

That is $\zeta_0 \leq 2$. Lemma 9(1) implies that G has only two end blocks and the other blocks contain two cut vertices. This completes the proof. ■

In view of Lemma 25, hereafter, G has exactly two end blocks, R_1, R_{k-1} say, and the other blocks $R_2, R_3, \ldots, R_{k-2}$ which contain two cut vertices. Without loss of generality let $z_1 \in V(R_1), z_{k-2} \in V(R_{k-1})$ and $z_{i-1}, z_i \in V(R_i)$ for $2 \leq i \leq k-2$ (see Figure 4).

Lemma 26. *For $k \geq 5$, let $G \in \mathcal{Z}(k, k-2)$ and R_1, R_{k-1} be two end blocks. Then $|(D \cap V(R_1)) - \{z_1\}| = 2$ or $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$.*

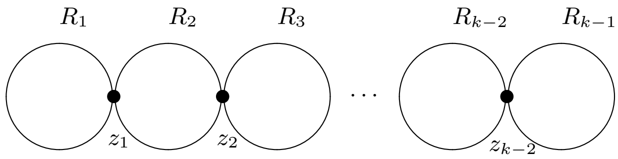

Figure 4. The structure of $G \in \mathcal{Z}(k, \zeta)$ where $\zeta \in \{k-3, k-2\}$.

Proof. Lemma 9(3) yields that $\mathcal{A}(G) \subseteq D$. As $\zeta(G) = k-2$, we must have $|D - \mathcal{A}(G)| = 2$. Clearly $(D - \mathcal{A}(G)) \cap (V(R_1) \cup V(R_{k-1})) \neq \emptyset$, otherwise $R_1, R_{k-1} \in \mathcal{B}_0$ contradicting Lemma 19.

Without loss of generality let $|(D \cap V(R_1)) - \{z_1\}| \leq |(D \cap V(R_{k-1})) - \{z_{k-2}\}|$. Suppose to the contrary that $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 1$. Thus $R_1, R_{k-1} \in \mathcal{B}_0 \cup \mathcal{B}_1$ contradicting Lemma 19. Hence $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$ and this completes the proof. ■

Lemma 27. *For $k \geq 5$, let $G \in \mathcal{Z}(k, k-2)$ and $R_2, R_3, \ldots, R_{k-2}$ be blocks which contain two cut vertices such that $z_{i-1}, z_i \in V(R_i)$ for $2 \leq i \leq k-2$. Then*

$$\{z_{i-1}, z_i\} \succ_c R_i \quad \textit{for} \quad 2 \leq i \leq k-1, \textit{ in particular,} \quad z_{i-1}z_i \in E(G).$$

Proof. As $\zeta(G) = k-2$, by Lemma 26, we must have $D \cap V(R_i) = \{z_{i-1}, z_i\}$ for $2 \leq i \leq k-2$. Therefore

$$\{z_{i-1}, z_i\} \succ_c R_i.$$

Clearly, $z_{i-1}z_i \in E(G)$ and this completes the proof. ■

Lemma 28. *For $k \geq 5$, let $G \in \mathcal{Z}(k, k-2)$ and R_i be a block of G containing two cut vertices z_{i-1} and z_i for $2 \leq i \leq k-2$. Then $V(R_i) = \{z_{i-1}, z_i\}$.*

Proof. By Lemma 26, $|(D \cap V(R_1)) - \{z_1\}| = 2$ or $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$. Without loss of generality let $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$. Since $|D \cap \mathcal{A}(G)| = k-2$, $|(D \cap V(R_1)) - \{z_1\}| = 0$ and thus, Lemma 16 gives that $R_1 \in \mathcal{B}_0$.

We consider the case when $i = 2$. Let $z \in V(R_1) - \{z_1\}$. Suppose there exists $u \in V(R_2) - \{z_1, z_2\}$. Consider $G + uz$. We see that z_2 is a cut vertex of $G + uz$.

Lemma 9(2) implies that $z_2 \in D_{uz}$. If $|D_{uz} - V(R_1)| \leq k-2$, then, by Lemma 27, $(D_{uz} - V(R_1)) \cup \{z_1\} \succ_c G$ contradicting $\gamma_c(G) = k$. Hence, by Lemma 7(1), $|D_{uz} - V(R_1)| = k-1$. Since $u \notin V(R_1)$, by Lemma 7(2), $\{u\} = D_{uz} \cap \{u, z\}$. Lemma 27 implies that $(D_{uz} - \{u\}) \cup \{z_1\} \succ_c G$. But $|(D_{uz} - \{u\}) \cup \{z_1\}| = k-1$ contradicting $\gamma_c(G) = k$. Hence, $V(R_2) = \{z_1, z_2\}$.

We consider the case when $3 \leq i \leq k-2$. Suppose to the contrary that $R'_i = V(R_i) - \{z_{i-1}, z_i\} \neq \emptyset$. Lemma 27 gives that $\{z_{i-1}, z_i\} \succ_c R'_i$ and $z_{i-1}z_i \in E(G)$. If there exists a vertex $b' \in R'_i$ which is not adjacent to z_j for some $j \in \{i, i-1\}$, then $b'z_{2i-1-j} \in E(G)$. Note that $b', z_j \in N_{R_i}(z_{2i-1-j})$. Thus $G[N_{R_i}(z_{2i-1-j})]$ is not a complete graph contradicting Lemma 9(2). Therefore, $z_i \succ R'_i$ and $z_{i-1} \succ R'_i$. We now have that $z_i \succ V(R_i), z_{i-1} \succ V(R_i)$ and $N[b'] \subseteq R'_i \cup \{z_{i-1}, z_i\}$ for all $b' \in R'_i$. Moreover, we have that $z_1 \succ R_1$ and $N[b] \subseteq V(R_1)$ for all $b \in V(R_1)$. Choose

$$X_1 = \{z_1\}, X = V(R_1) - \{z_1\}, Y = R'_i \text{ and } Y_1 = \{z_i, z_{i-1}\}.$$

Clearly X, X_1, Y and Y_1 form a bad subgraph. This contradicts Lemma 12. Hence, $R'_i = \emptyset$ for all $2 \leq i \leq k-3$. This completes the proof. ■

The following theorem gives the characterization of the graphs in the class $\mathcal{Z}(k, k-2)$.

Theorem 29. *For $k \geq 5$, we have that $\mathcal{Z}(k, k-2) = \mathcal{F}(k)$.*

Proof. Lemma 24 implies that $\mathcal{F}(k) \subseteq \mathcal{Z}(k, k-2)$. It suffices to show that a k-γ_c-critical graph with $k-2$ cut vertices is in $\mathcal{F}(k)$. Let G be a k-γ_c-critical graph with $k-2$ cut vertices. Lemma 25 implies that G has only two end blocks, R_1, R_{k-1} say, and the other blocks $R_2, R_3, \ldots, R_{k-2}$ which contain two cut vertices. Let $z_1 \in V(R_1), z_{k-2} \in V(R_{k-1})$ and $z_{i-1}, z_i \in V(R_i)$ for $2 \leq i \leq k-2$. Thus $\mathcal{A}(G) = \{z_1, z_2, \ldots, z_{k-2}\}$. Lemma 26 implies that $|(D \cap V(R_1)) - \{z_1\}| = 2$ or $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$. Without loss of generality let

$$|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2.$$

Thus $|(D \cap V(R_1)) - \{z_1\}| = 0$. By Lemma 16, $R_1 \in \mathcal{B}_0$. Clearly, $z_1 \succ R_1$. As $\zeta(G) = k-2$, we must have $D \cap V(R_i) = \{z_{i-1}, z_i\}$ for $2 \leq i \leq k-2$ and $D \cap V(R_1) = \{z_1\}$. By Lemma 27,

$$\{z_{i-1}, z_i\} \succ_c R_i.$$

Let $z_0 \in V(R_1) - \{z_1\}$. Clearly $d(z_1, z_0) = 1$. The following claim characterizes R_{k-1}.

Claim. $R_{k-1} \in \mathcal{B}_{2,2}$.

Proof. Since $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$, there exists $w \in V(R_{k-1}) - \{z_{k-2}\}$ such that $d(w, z_{k-2}) \geq 2$. Thus

$$d(z_0, w) \geq d(z_0, z_1) + d(z_1, z_2) + \cdots + d(z_{k-2}, w) \geq k.$$

Lemma 10 gives that $d(z_0, w) = k$. Hence $d(z_{k-2}, w') \leq 2$ for all $w' \in V(R_{k-1}) - \{z_{k-2}\}$. So $R_{k-1} \notin \mathcal{B}_{2,1}$. By Lemma 18, $R_{k-2} \in \mathcal{B}_{2,2}$ and thus establishing the claim. □

Lemma 28 implies that, for all $i \in \{2, 3, \ldots, k-2\}$, $V(R_i) = \{z_{i-1}, z_i\}$. So far, it remains to show that $V(R_1) = \{z_1, z_0\}$. Consider $G + z_2z_0$. Since z_2 is a cut vertex of $G + z_2z_0$, $z_2 \in D_{z_2z_0}$ by the connectedness of $(G + z_2z_0)[D_{z_2z_0}]$. We note by Lemma 27 that $z_1z_2 \in E(G)$. Then, if $|D_{z_2z_0} - V(R_1)| \leq k-2$, we have that $(D_{z_2z_0} - V(R_1)) \cup \{z_1\} \succ_c G$ contradicting $\gamma_c(G) = k$. Therefore, by Lemma 7(1), $|D_{z_2z_0} - V(R_1)| = k-1$. Thus $\{z_2\} = \{z_2, z_0\} \cap D_{z_2z_0}$ and this implies that $z_2 \succ R_1$ in $G + z_2z_0$. Since $V(R_1) \cap N_{G+z_2z_0}(z_2) = \{z_0\}$, $V(R_1) = \{z_1, z_0\}$ and this completes the proof. ■

6. DISCUSSION

In this section, we discuss the related result on an another type of domination critical graphs. For a graph G, a vertex subset D of G is a *total dominating set* of G if every vertex of G is adjacent to a vertex in D. The minimum cardinality of a total dominating set of G is called the *total domination number* of G and is denoted by $\gamma_t(G)$. A graph G is said to be *k-γ_t-critical* if $\gamma_t(G) = k$ and $\gamma_t(G + uv) < k$ for any pair of non-adjacent vertices u and v of G. For $k = 3$, it was pointed out by Ananchuen in [1] that a graph G is 3-γ_t-critical if and only if G is 3-γ_c-critical. In [7], the authors established the similar result when $k = 4$. Therefore we have the following result.

Theorem 30 ([1] and [7])**.** *For $k \in \{3, 4\}$, a connected graph G is k-γ_t-critical if and only if G is k-γ_c-critical.*

For related results on k-γ_t-critical graphs, Hattingh *et al.* [4] established the upper bound of the number of end vertices of k-γ_t-critical graphs. They proved the following.

Theorem 31 [4]**.** *For $k \geq 5$, every k-γ_t-critical graph has at most $k-2$ end vertices.*

They, further, established the existence of k-γ_t-critical graphs with prescribe end vertices according to the bound from Theorem 31.

Theorem 32 [4]**.** *For integers $k \geq 3$ and $0 \leq h \leq k-2$ except only the case when $k = 4$ and $h = 2$, there exists a k-γ_t-critical graph with h end vertices.*

Hence, by Corollary 13 and Theorem 30, we can conclude that there is no 4-γ_t-critical graph with two end vertices. This fulfills Theorem 32 in the following way.

Corollary 33. *For integers $k \geq 3$ and $0 \leq h \leq k-2$, there exists a k-γ_t-critical graph with h end vertices if and only if $k \neq 4$ or $h \neq 2$.*

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