



รายงานวิจัยฉบับสมบูรณ์

โครงการ คุณสมบัติของกราฟที่มีความต่อเนื่อง 4
แบบภายในที่ไม่มีกราฟลูกบาศก์เป็นกราฟย่อย

โดย ชนันท ลีวเฉลิมวงศ์

เมษายน 2562

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ชนันท์ ลีเฉลิมวงศ์

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มหาวิทยาลัยมหิดล

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและต้นสังกัด

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ ต้นสังกัดไม่จำเป็นต้องเห็นด้วยเสมอไป)

Abstract

Project Code: MRG6080201

Project Title: Properties of All Internally 4-Connected Graphs with no Cube Minor
(คุณสมบัติของกราฟที่มีความต่อเนื่อง 4 แบบภายในที่ไม่มีกราฟลูกบาศก์เป็นกราฟย่อย)

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A Simple graph is a minor of another if the first is obtained from the second by deleting vertices, deleting edges, contracting edges, and deleting loops and parallel edges that are created when we contract edges. A cube is an internally 4-connected planar graph with eight vertices and twelve edges corresponding to the skeleton of the cube in the platonic solid. A complete characterization of graphs with no cube minor is given only for the case of 3-connected graphs but not for the case of internally 4-connected graphs. In this research, we want to investigate properties of such graphs. We determine all internally 4-connected graphs that contain neither cube nor V_8 as minors, where V_8 or Wagner graph is an internally 4-connected nonplanar graph obtained from a cube by introducing a twist. Then we investigate 4-connected graphs with no cube minor. These results provide a step closer to a complete characterization of all internally 4-connected graphs with no cube minor.

กราฟอย่างง่ายกราฟหนึ่งเป็นกราฟย่อยของกราฟอย่างง่ายอีกกราฟถ้ากราฟแรกสามารถถูกสร้างจากกราฟหลังด้วยการลบจุดยอด การลบเส้นเชื่อม การหดเส้นเชื่อม และการลบห่วงและเส้นเชื่อมขนานในกราฟที่เกิดจากการหดเส้นเชื่อม กราฟลูกบาศก์เป็นกราฟระนาบที่เป็นโครงสร้างของทรงลูกบาศก์ซึ่งมี 8 จุดยอด 12 เส้นเชื่อม และมีความต่อเนื่อง 4 แบบภายใน การอธิบายลักษณะอย่างสมบูรณ์ของกราฟที่ไม่มีกราฟลูกบาศก์เป็นกราฟย่อยมีเฉพาะในกราฟที่มีความต่อเนื่อง 3 เท่านั้น แต่ยังไม่มีการศึกษากราฟที่มีความต่อเนื่อง 4 แบบภายใน ในงานวิจัยนี้เราต้องการศึกษาคุณสมบัติของกราฟประเภทนี้ เราอธิบายลักษณะของกราฟที่ไม่มีทั้งกราฟลูกบาศก์และกราฟ V_8 เป็นกราฟย่อย โดยกราฟ V_8 หรือ กราฟ Wagner เป็นกราฟไม่ระนาบที่มีความต่อเนื่อง 4 แบบภายในที่ได้จากการบิดเส้นเชื่อมคู่หนึ่งในกราฟลูกบาศก์ จากนั้นเราศึกษากราฟความต่อเนื่อง 4 ที่ไม่มีกราฟลูกบาศก์เป็นกราฟย่อย ผลที่ได้ทำให้เข้าใจการอธิบายลักษณะของกราฟความต่อเนื่อง 4 แบบภายในที่ไม่มีกราฟลูกบาศก์เป็นกราฟย่อย

Keywords: internally 4-connected, minor, cube graph, V_8 graph

รายงานสรุปการนำผลงานวิจัยไปใช้ประโยชน์

สัญญาเลขที่ MRG6080201 ชื่อโครงการ คุณสมบัติของกราฟที่มีความต่อเนื่อง 4 แบบภายในที่ไม่มีกราฟลูกบาศก์เป็นกราฟย่อย
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ความสำคัญ / ความเป็นมา

ทฤษฎีกราฟ (graph theory) เป็นสาขาหนึ่งในคณิตศาสตร์ที่ศึกษาเกี่ยวกับกราฟ (graph) ซึ่งในที่นี้คือสิ่งที่ประกอบไปด้วยจุดยอด (vertex) และเส้นเชื่อม (edge) กราฟอย่างง่าย (simple graph) คือกราฟที่ไม่มีห่วง (loop) และเส้นเชื่อมขนาน (parallel edges) กราฟอย่างง่ายกราฟหนึ่งเป็นกราฟย่อย (minor) ของกราฟอย่างง่ายอีกกราฟถ้ากราฟแรกสามารถถูกสร้างจากกราฟหลังด้วยการลบจุดยอด (vertex deletion) การลบเส้นเชื่อม (edge deletion) การหดเส้นเชื่อม (edge contraction) และการลบห่วงและเส้นเชื่อมขนานในกราฟที่เกิดจากการหดเส้นเชื่อม กำหนดให้ G และ H เป็นกราฟอย่างง่าย เราเรียก G ว่า H -free ถ้า H ไม่ได้เป็นกราฟย่อยของ G มีปัญหาสำคัญหลายปัญหาที่สามารถถามในรูปของกราฟแบบ H -free ตัวอย่างเช่น ทฤษฎีสี่สี (the four color theorem) กล่าวไว้ว่า ทุกกราฟที่เป็น K_5 -free สามารถระบายสีจุดยอดได้ด้วยสี่สีได้ (4-colorable) โดยที่ K_5 เป็นกราฟสมบูรณ์ (complete graph) ที่มีห้าจุดยอดหรือสำหรับ Hadwiger's Conjecture สามารถกล่าวว่า ทุกกราฟที่เป็น K_n -free สามารถระบายสีจุดยอดได้ด้วย $n-1$ สีได้ ($n-1$ colorable) ซึ่งปัญหานี้ยังคงไม่สามารถพิสูจน์ได้สำหรับกรณีจุดยอดมีจำนวนตั้งแต่ 7 ขึ้นไป ความยากของปัญหานี้คือเราไม่มีข้อมูลเกี่ยวกับโครงสร้างของกราฟที่เป็น K_n -free การอธิบายลักษณะอย่างสมบูรณ์ของ K_4 -free และ K_5 -free ได้ถูกศึกษาเรียบร้อยแล้วแต่ K_6 -free ยังคงไม่สมบูรณ์ สังเกตว่า K_6 มีเส้นเชื่อมทั้งหมด 15 เส้น ดังนั้นจึงมีการพยายามศึกษากราฟที่มีเส้นเชื่อมน้อยกว่า 15 เส้น กราฟลูกบาศก์ (cube) เป็นกราฟระนาบ (planar graph) ที่เป็นโครงสร้างของทรงลูกบาศก์ซึ่งมี 8 จุดยอด 12 เส้นเชื่อม และมีความต่อเนื่อง 4 แบบภายใน (internally 4-connected) การอธิบายลักษณะอย่างสมบูรณ์ (complete characterization) ของกราฟที่เป็น cube-free มีเฉพาะในกราฟที่มีความต่อเนื่อง 3 (3-connected) เท่านั้น แต่ยังไม่มีการศึกษากราฟที่มีความต่อเนื่อง 4 แบบภายใน การเข้าใจคุณสมบัติของกราฟประเภทนี้จะทำให้ได้การอธิบายลักษณะอย่างสมบูรณ์ของกราฟประเภทนี้ได้

วัตถุประสงค์ของโครงการ

1. ศึกษาโครงสร้างของกราฟที่มีความต่อเนื่อง 4 แบบภายในที่เป็น cube-free
2. ศึกษากราฟ V_8 ที่มีความสัมพันธ์กับกราฟลูกบาศก์ และคุณสมบัติของกราฟที่มีความต่อเนื่อง 4 แบบภายในที่เป็น $\{cube, V_8\}$ -free
3. ศึกษาคุณสมบัติของกราฟที่มีความต่อเนื่อง 4 แบบภายในที่เป็น cube-free โดยพิจารณาจากจำนวนเส้นเชื่อม

ผลการวิจัย (สั้น ๆ ที่ปึงชีประเด็นข้อค้นพบ กระบวนการ ผลผลิต และการเรียนรู้)

1. การอธิบายลักษณะอย่างสมบูรณ์ของกราฟที่มีความต่อเนื่อง 4 แบบภายในที่เป็น $\{cube, V_8\}$ -free โดยศึกษาแยกตามคุณสมบัติความระนาบของกราฟ
2. ศึกษาคุณสมบัติของกราฟที่มีความต่อเนื่อง 4 ที่เป็น cube-free

คำสืบค้น (Keywords)

internally 4-connected, minor, cube graph, V_8 graph

การนำผลงานวิจัยไปใช้ประโยชน์ (ดูคำจำกัดความ และตัวอย่างด้านหลังแบบฟอร์ม)

☐ ด้านนโยบาย โดยใคร (กรุณาให้ข้อมูลเจาะจง).....

มีการนำไปใช้อย่างไร

☐ ด้านสาธารณะ โดยใคร (กรุณาให้ข้อมูลเจาะจง)

มีการนำไปใช้อย่างไร

☐ ด้านชุมชนและพื้นที่ โดยใคร (กรุณาให้ข้อมูลเจาะจง)

มีการนำไปใช้อย่างไร

☐ ด้านพาณิชย์ โดยใคร (กรุณาให้ข้อมูลเจาะจง)

มีการนำไปใช้อย่างไร

☐ ด้านวิชาการ โดยใคร (กรุณาให้ข้อมูลเจาะจง)

มีการนำไปใช้อย่างไร (กรุณาให้ข้อมูลเจาะจง)

☒ ยังไม่มีการนำไปใช้ (โปรดกรอกในกรอบถัดไป)

(กรณีที่ยังไม่มีการใช้ประโยชน์) ผลงานวิจัยมีศักยภาพในการนำไปใช้ประโยชน์

☐ ด้านนโยบาย ☐ ด้านสาธารณะ ☐ ด้านชุมชนและพื้นที่ ☐ ด้านพาณิชย์ ☒ ด้านวิชาการ

ข้อเสนอแนะเพื่อให้ผลงานถูกนำไปใช้ประโยชน์

นำความรู้ด้านโครงสร้างของกราฟที่มีความต่อเนื่อง 4 แบบภายในมาบวกรวมกับคุณสมบัติของกราฟความต่อเนื่อง 4 แบบภายในที่ไม่มีกราฟลูกบาศก์เป็นกราฟย่อย สามารถนำมาซึ่งการอธิบายลักษณะอย่างสมบูรณ์ของกราฟความต่อเนื่อง 4 แบบภายในที่ไม่มีกราฟลูกบาศก์เป็นกราฟย่อย

การเผยแพร่/ประชาสัมพันธ์ (กรุณาให้รายละเอียด พร้อมแนบหลักฐาน)

1. สิ่งพิมพ์ หรือสื่อทั่วไป

☐ หนังสือพิมพ์ ☐ วารสาร ☐ โทรทัศน์ ☐ วิทยุ ☐ เว็บไซต์ ☐ คู่มือ/แผ่นพับ ☐ จัดประชุม/อบรม ☐ อื่น ๆ

2. สิ่งพิมพ์ทางวิชาการ (วารสาร, การประชุม ให้ระบุรายละเอียดแบบการเขียนเอกสารอ้างอิง เพื่อการค้นหาซึ่งควรประกอบด้วย ชื่อผู้แต่ง ชื่อเรื่อง แหล่งพิมพ์ ปี พ.ศ. (ค.ศ.) ฉบับที่ หน้า)

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คำอธิบายและตัวอย่างการนำไปใช้ประโยชน์ในแต่ละด้าน

1. การใช้ประโยชน์ด้านนโยบาย

คำจำกัดความ : การนำความรู้จากงานวิจัยไปใช้ในกระบวนการกำหนดนโยบาย ซึ่งนโยบายหมายถึง หลักการ แนวทาง กลยุทธ์ ในการดำเนินงานเพื่อให้บรรลุวัตถุประสงค์ อาจเป็นนโยบายระดับประเทศ ระดับภูมิภาค ระดับจังหวัด ระดับท้องถิ่น หรือระดับหน่วยงาน นโยบายที่ดีจะต้องประกอบด้วยวัตถุประสงค์ แนวทาง และกลไกในการดำเนินงานที่ชัดเจน สอดคล้องกับปัญหาและความต้องการ การใช้ประโยชน์ด้านนโยบายจะรวมทั้งการนำองค์ความรู้ไปสังเคราะห์เป็นนโยบายหรือทางเลือกเชิงนโยบาย (policy options) แล้วนำนโยบายนั้นไปสู่ผู้ใช้ประโยชน์

2. การใช้ประโยชน์ด้านสาธารณะ

คำจำกัดความ : การดำเนินงานเพื่อนำผลงานวิจัยและนวัตกรรม ไปใช้ในวงกว้างเพื่อประโยชน์ของสังคม และประชาชนทั่วไป ให้มีความรู้ความเข้าใจ เกิดความตระหนัก รู้เท่าทันการเปลี่ยนแปลง ซึ่งนำไปสู่ การเปลี่ยนวิธีคิด พฤติกรรม เพื่อเพิ่มคุณภาพชีวิตของประชาชน สร้างสังคมคุณภาพ และส่งเสริมคุณภาพสิ่งแวดล้อม

3. การใช้ประโยชน์ด้านพาณิชย์

คำจำกัดความ : การนำนวัตกรรม เทคโนโลยี ผลิตภัณฑ์ใหม่ พันธุ์พืช พันธุ์สัตว์ ไปสู่การผลิตในเชิงพาณิชย์ การสร้างมูลค่าเพิ่มของผลิตภัณฑ์ การแปรรูป การสร้างตราสินค้า การเพิ่มประสิทธิภาพในกระบวนการผลิต และการลดต้นทุนการผลิต การสร้างอาชีพ และทางเลือกให้กับผู้ประกอบการ เกษตรกรหรือผู้ประกอบการอาชีพอื่น ๆ

4. การใช้ประโยชน์ด้านชุมชนและพื้นที่

คำจำกัดความ : การนำกระบวนการ วิธีการ องค์ความรู้ การเปลี่ยนแปลง การเสริมพลัง อันเป็นผลกระทบที่เกิดจากการวิจัยและพัฒนาชุมชน ท้องถิ่น พื้นที่ ไปใช้ให้เกิดประโยชน์การขยายผลต่อชุมชน ท้องถิ่นและสังคมอื่น

5. การใช้ประโยชน์ด้านวิชาการ

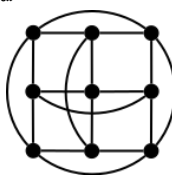
คำจำกัดความ : การนำองค์ความรู้จากผลงานวิจัยที่ตีพิมพ์ในรูปแบบต่าง ๆ เช่น ผลงานตีพิมพ์ในวารสารระดับนานาชาติ ระดับชาติ หนังสือ ตำรา บทเรียน ไปเป็นประโยชน์ด้านวิชาการ การเรียนรู้ การเรียน การสอน ในวงวิชาการ และผู้สนใจด้านวิชาการ รวมถึงการนำผลงานวิจัยไปวิจัยต่อยอด หรือ การนำไปสู่ product และ process ไปใช้ในการเสริมสร้างนวัตกรรม และเทคโนโลยี

เนื้อหางานวิจัย และ Output ที่ได้จากโครงการ

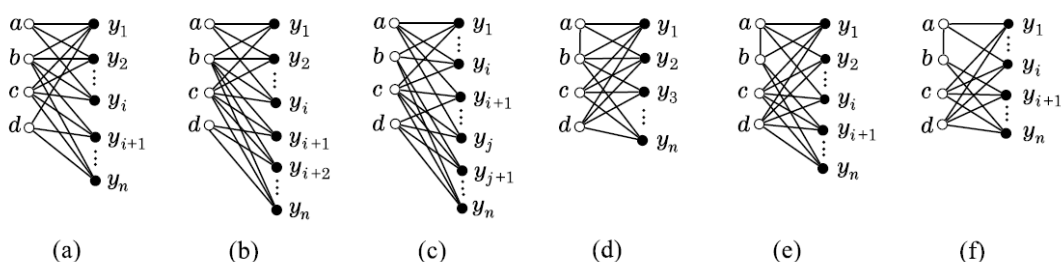
โครงการ คุณสมบัติของกราฟที่มีความต่อเนื่อง 4 แบบภายในที่ไม่มีกราฟลูกบาศก์เป็นกราฟย่อย มีนักวิจัยที่ปรึกษาคือ รศ. ดร. นวรัตน์ อนันต์ชื่น เราได้ศึกษาคุณสมบัติของกราฟความต่อเนื่อง 4 แบบภายในที่มีเส้นเชื่อมน้อยกว่า 21 เส้น หนึ่งในกราฟที่ศึกษาคือ กราฟ V_8 หรือ กราฟ Wagner ซึ่งเป็นกราฟไม่ระนาบที่มีความต่อเนื่อง 4 แบบภายในที่ได้จากการบิดเส้นเชื่อมคู่หนึ่งในกราฟลูกบาศก์ จะเห็นว่ากราฟ V_8 มีความใกล้เคียงกับกราฟลูกบาศก์มาก ทั้งจำนวนจุดยอด จำนวนเส้นเชื่อม ยกเว้นความระนาบ สำหรับกราฟ V_8 มีการอธิบายลักษณะอย่างสมบูรณ์ของกราฟความต่อเนื่อง 4 แบบภายในที่ไม่มีกราฟ V_8 เป็นกราฟย่อยแล้ว และเมื่อได้ศึกษาบทความดังกล่าวทำให้เห็นวิธีที่สามารถอธิบายลักษณะของกราฟที่ไม่มีทั้งกราฟลูกบาศก์และกราฟ V_8 เป็นกราฟย่อย โดยใช้คุณสมบัติความระนาบของกราฟทั้งสองเข้ามาช่วย ทำให้เราสามารถได้การอธิบายลักษณะอย่างสมบูรณ์ของกราฟประเภทดังกล่าว โดยการเขียนโปรแกรมเพื่อช่วยในการตรวจสอบกราฟที่คุณสมบัติที่ต้องการ และได้ตีพิมพ์ผลงาน Internally 4-connected graphs with no $\{cube, V_8\}$ -minor ใน *Discussiones Mathematicae Graph Theory* (เอกสารแนบ 1)

ในการพิจารณากราฟความต่อเนื่อง 4 แบบภายในที่ไม่มี $cube$ และ V_8 เป็นกราฟย่อย นั้น สิ่งที่เราทราบแน่นอน คือ กราฟที่มีจุดยอดน้อยกว่า 7 จุด ไม่มีกราฟทั้งสองเป็นกราฟย่อยอย่างแน่นอน จากนั้นเราพิจารณากราฟไม่ระนาบที่ไม่มี $cube$ และ V_8 เป็นกราฟย่อยเราได้ผลว่ากราฟดังกล่าวมีลักษณะดังนี้

- (1) กราฟที่มีคุณสมบัติดังกล่าวจะมีรูปร่างพื้นฐานเหมือนกับ $L(K_{3,3})$, ดูรูปด้านล่างประกอบ



- (2) กราฟที่มีคุณสมบัติดังกล่าวจะมีรูปร่างพื้นฐานเหมือนกับ $K_{3,n}$ สำหรับบาง $n \geq 5$
- (3) กราฟความต่อเนื่อง 4 แบบภายในที่ไม่ระนาบและได้จากขับกราฟแผ่ตัวของ $K_{n,n}$ ($n \geq 4$) โดยการเพิ่มเส้นเชื่อมไปยังกลุ่มของจุดยอดที่อยู่ในชั้นเดียวกันที่มี 4 จุดยอด ซึ่งเป็นกราฟในกลุ่มต่อไปนี้



สำหรับกราฟความต่อเนื่อง 4 แบบภายในที่ไม่มี $cube$ และ V_8 เป็นกราฟย่อย เราพิสูจน์ได้ว่า กราฟระนาบที่มีความต่อเนื่อง 4 แบบภายในซึ่งมีอย่างน้อย 8 จุดยอดจะมี $cube$ เป็นกราฟย่อย เราพิสูจน์โดยใช้การแยกลักษณะของกราฟความต่อเนื่อง 4 แบบภายใน ผลที่ได้ถูกนำไปตีพิมพ์ดังในเอกสารแนบ 1 ซึ่งผลที่ได้ทำให้เราเข้าใจโครงสร้างของกราฟความต่อเนื่อง 4 แบบภายในที่ไม่มีกราฟลูกบาศก์เป็นกราฟย่อยมากขึ้น จากนั้นเราได้ศึกษากราฟความต่อเนื่อง 4 ที่ไม่มีกราฟลูกบาศก์เป็นกราฟย่อย (เอกสารแนบ 2) เพื่อขยายการอธิบายลักษณะไปยังกราฟที่ใหญ่ที่สุดที่

จะไม่มีกราฟลูกบาศก์เป็นกราฟย่อยต่อไป จากนั้นเราจะนำผลที่ได้มาใช้ในการอธิบายลักษณะอย่างสมบูรณ์ของกราฟความต่อเนื่อง 4 แบบภายในที่ไม่มีกราฟลูกบาศก์เป็นกราฟย่อยต่อไป

ในระหว่างการทำโครงการ ได้นำโครงสร้างของกราฟต้นไม้ไปศึกษาตัวเลขเกมการครอบครอง (domination game number) ของกราฟต้นไม้กับ ชัชวาล วีรานุกุลจิต นักศึกษาของภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยมหิดล ผลที่ได้ถูกนำไปเสนอใน การประชุมวิชาการทางคณิตศาสตร์ ครั้งที่ 23 ประจำปี 2561 (The 23th Annual Meeting in Mathematics, AMM2018) ชื่อผลงานคือ Domination Game Played on a Graph Constructed from 1-Sum of Paths และได้ตีพิมพ์ผลงานวิจัยใน Thai Journal of Mathematics (เอกสารแนบ 3)

นอกจากนี้ยังได้นำความรู้ทางทฤษฎีกราฟประยุกต์ใช้ในการศึกษาระบบสัญญาณไฟจราจรที่อนุสาวรีย์ชัยสมรภูมิ โดยศึกษาพร้อมกับ สุวิทย์ แก้วนิมิต นักศึกษาของภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยมหิดล เราได้ตรวจสอบประสิทธิภาพของการเปิดปิดไฟจราจร และนำเสนอวิธีแก้ปัญหาการติดที่เกิดจากการปล่อยให้คนข้ามถนนที่ไม่สัมพันธ์กับไฟจราจร ผลที่ได้ถูกนำไปเสนอใน ICMA-MU 2018: International Conference in Mathematics and Applications ชื่อผลงานคือ Analyzing the Efficiency of the Traffic Light Phasing System of the Traffic Circle at Victory Monument (Thailand) with Graph Theory (เอกสารแนบ 4) ซึ่งจะมีการขยายงานชิ้นนี้เพื่อตีพิมพ์ในวารสารระดับนานาชาติต่อไป

สรุป Output ที่ได้ดังนี้

- 1) Lewchalermvongs C, Ananchuen N. Internally 4 - connected graphs with no {cube, V_8 }-minor. *Discussiones Mathematicae Graph Theory*. Accepted: 10-01-2019, DOI: 10.7151/dmgt.2205.
- 2) Weeranukujit C, Lewchalermvongs C. Domination Game Played on a Graph Constructed from 1 -Sum of Paths. *Thai Journal of Mathematics* 2019;17(1):34-45.
- 3) Kaewnimit S, Lewchalermvongs C. Analyzing the Efficiency of the Traffic Light Phasing System of the Traffic Circle at Victory Monument (Thailand) with Graph Theory. *ICMA-MU 2018 Book on the Conference Proceedings*. 2018 Dec:79-87.

ภาคผนวก

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INTERNALLY 4-CONNECTED GRAPHS WITH NO $\{\text{CUBE}, V_8\}$ -MINOR

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Abstract

A simple graph is a minor of another if the first is obtained from the second by deleting vertices, deleting edges, contracting edges, and deleting loops and parallel edges that are created when we contract edges. A cube is an internally 4-connected planar graph with eight vertices and twelve edges corresponding to the skeleton of the cube in the platonic solid, and the Wagner graph V_8 is an internally 4-connected nonplanar graph obtained from a cube by introducing a twist. A complete characterization of all internally 4-connected graphs with no V_8 minor is given in J. Maharry and N. Robertson, *The structure of graphs not topologically containing the Wagner graph*, J. Combin. Theory Ser. B 121 (2016) 398–420; on the other hand, only a characterization of 3-connected graphs with no cube minor is given in J. Maharry, *A characterization of graphs with no cube minor*, J. Combin. Theory Ser. B 80 (2008) 179–201. In this paper we determine all internally 4-connected graphs that contain neither cube nor V_8 as minors. This result provides a step closer to a complete characterization of all internally 4-connected graphs with no cube minor.

Keywords: internally 4-connected, minor, cube graph, V_8 graph.

2010 Mathematics Subject Classification: 05C83.

1. INTRODUCTION

A graph G is called H -free, where H is a graph, if no minor of G is isomorphic to H . The structure of H -free graphs can be used to studied other properties of the class of graphs; in addition, many important problems in graph theory can be formulated in terms of H -free graphs. For example, the four color theorem can be equivalently stated as: all K_5 -free graphs are 4-colorable, where K_5 is a complete graph on five vertices. Hadwiger's Conjecture states that every K_n -free graph is $n - 1$ colorable, where K_n is a complete graph on n vertices. This conjecture is still open for $n \geq 7$ and the main difficulty for proving the conjecture is the lack of structural information on K_n -free graphs. Determining K_6 -free graphs is one of the two most famous problems in this area, and another problem is to determine Petersen-free graphs, see Figure 1. Notice that both graphs have fifteen edges. As an attempt to better understand these graphs, we try to exclude 3-connected graphs H with at most fifteen edges. The complement of a path on seven vertices, $\overline{P}_7$, also has 15 edges and it is the largest graph H for which 4-connected H -free graphs are completely determined, see [5]. The octahedron with an additional edge is a graph with 13 edges and its characterization problem is solved in [8]. The octahedron, the cube, and V_8 are graphs H with twelve edges and their characterizations can be found in [3, 6, 7], and [9], respectively. For H with at most eleven edges, all H -free graphs have been determined and their results are surveyed in [4].

Let k be a non-negative integer. A k -separation of a graph G is an unordered pair $\{G_1, G_2\}$ of induced subgraphs of G such that $V(G_1) \cup V(G_2) = V(G)$, $E(G_1) \cup E(G_2) = E(G)$, $V(G_1) - V(G_2) \neq \emptyset$, $V(G_2) - V(G_1) \neq \emptyset$, and $|V(G_1) \cap V(G_2)| = k$. If G has a k -separation, then there is $X \subseteq V(G)$ such that $|X| = k$ and $G \setminus X$ has at least two components. A 3-connected graph G on at least five vertices is said to be *internally 4-connected* if for every 3-separation $\{G_1, G_2\}$ of G , one of them is isomorphic to $K_{1,3}$. The characterization of 3-connected cube-free graphs is solved in [7]; however, the result does not completely determine all the internally 4-connected cube-free graphs, see the theorem below. For each integer $n \geq 3$, let V_{2n} denote a Möbius ladder, which is a graph obtained from a cycle on $2n$ vertices by joining the n pairs of opposite vertices. Notice that V_6 is $K_{3,3}$. For any graph G , the line graph of G , denoted by $L(G)$, is a graph such that each vertex of $L(G)$ represents an edge of G , and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common end vertex in G . The 3-sum is an operation of combining two graphs by identifying a triangle (C_3) of one graph with a triangle of the other graph to produce a new graph.

Theorem 1 [7]. *A 3-connected graph G is cube-free if and only if G is a minor of a graph constructed from $L(\text{Petersen})$, $L(V_{2n})$ for each integer $n \geq 3$ (Figure 1), and the ten graphs in Figure 2, of order ≤ 8 , by 3-sums over the triangles*

shaded or the vertices of the triangle circled.

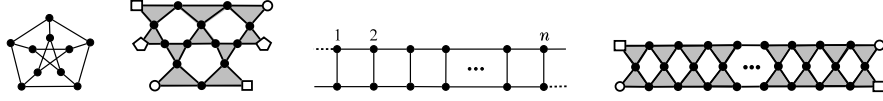


Figure 1. Petersen graph, $L(\text{Petersen})$, V_{2n} and $L(V_{2n})$.

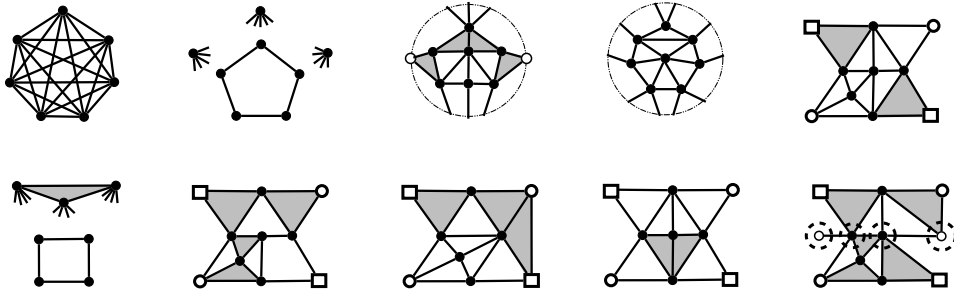


Figure 2. The ten graphs of order ≤ 8 in [7].

This theorem contains an (possibly printing) error in the second last graph, that contains two triangles shaded. Performing two 3-sums of K_4 's over these triangles results in a cube-minor.

By a graph we mean a finite, simple, undirected graph. All undefined terminology can be found in [2]. In this paper, we consider internally 4-connected $\{\text{cube}, V_8\}$ -free graphs. To state our main result we need a few definitions. Let $K_{m,n}$ be a complete bipartite graph with partitions of m and n vertices. Let $\mathcal{K}$ consist of internally 4-connected nonplanar graphs that are obtained from spanning subgraphs of some $K_{4,n}$ ($n \geq 4$) by adding edges to the color class of size four.

Theorem 2. *Let G be an internally 4-connected $\{\text{cube}, V_8\}$ -free graph. Then G satisfies one of the following:*

- (i) G has at most seven vertices,
- (ii) G is isomorphic to $L(K_{3,3})$,
- (iii) G is isomorphic to $K_{3,n}$ for some $n \geq 5$,
- (iv) G is a graph in $\mathcal{K}$, which is one of the six types of graph shown in Figure 3.

We close this section by providing an outline of the rest of the paper. In the next section, we introduce a characterization of internally 4-connected V_8 -free graphs and a chain theorem for internally 4-connected graphs. Our proof of Theorem 2 will be divided into two parts, Sections 3 and 4. First, we determine

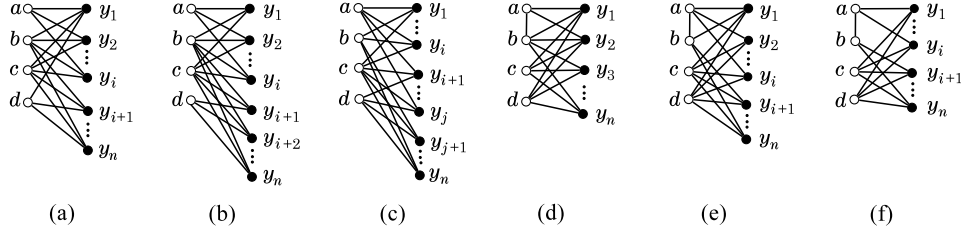


Figure 3. The six-type of graph in Theorem 2(iv).

all internally 4-connected nonplanar $\{\text{cube}, V_8\}$ -free graphs. Then we prove Theorem 2 by showing that all internally 4-connected planar graphs on at least eight vertices contains a cube-minor.

2. BASIC LEMMAS

All internally 4-connected V_8 -free graphs are determined in [9]. To state the theorem we need to define a few classes of graphs. For each integer $n \geq 3$, a *double-wheel*, DW_n ($n \geq 3$), is a graph on $n + 2$ vertices obtained from a cycle C_n by adding two nonadjacent vertices u, v and joining them to all vertices on the cycle. An *alternating double-wheel* AW_{2n} is a subgraph of DW_{2n} ($n \geq 3$) such that u and v are alternately adjacent to every vertex in C_{2n} . Notice that AW_6 is a cube, see Figure 4. For each integer $n \geq 3$, let DW_n^+ and AW_{2n}^+ be graphs obtained from DW_n and AW_{2n} , respectively, by joining u and v . Let $\mathcal{D}^+ = \{DW_n^+ : n \geq 3\} \cup \{AW_{2n}^+ : n \geq 3\}$. Then every graph in $\mathcal{D}^+$ is nonplanar.

Theorem 3 [9]. *Every internally 4-connected V_8 -free graph G satisfies one of the following conditions:*

- (i) G is planar,
- (ii) G has at most seven vertices,
- (iii) G is isomorphic to $L(K_{3,3})$,
- (iv) $G \setminus \{w, x, y, z\}$ has no edges for some $w, x, y, z \in V(G)$, or G is in $\mathcal{D}^+$.

This result suggests a process for determining all internally 4-connected nonplanar $\{\text{cube}, V_8\}$ -free graphs. We also need the following lemma from [9].

Lemma 4 [9]. *If G is an internally-4-connected graph, then either G contains two disjoint cycles, each of which contains at least four edges, or G has at most seven vertices, or G is isomorphic to $L(K_{3,3})$.*

This lemma implies that $L(K_{3,3})$ is $\{\text{cube}, V_8\}$ -free. Another main tool is a chain theorem for internally 4-connected graphs. To explain this result we need

a few definitions. For each integer $n \geq 5$, let C_n^2 be a graph obtained from a cycle C_n by joining all pairs of vertices of distance two on the cycle. Notice that $C_5^2 = DW_3^+ = K_5$, see Figure 4. Let *terrahawk* be the graph shown in Figure 4, which can be obtained from a cube by adding a new vertex and joining it to four vertices in the same C_4 . We denote the number of edges of a graph G by $\|G\|$.

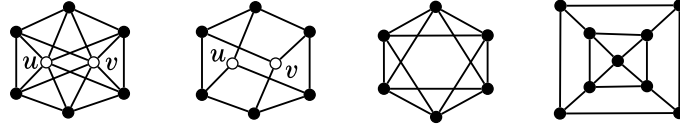


Figure 4. Graphs DW_6 , AW_6 , C_6^2 , and terrahawk.

Let $G \setminus e$ denote the graph obtained from G by deleting an edge e . The reverse operation of deleting an edge is *adding* an edge, that is G obtained from $G \setminus e$ by adding edge e . We use G/e denote the graph obtained from G by first contracting an edge e then deleting all but one edge from each parallel family. The reverse operation of contracting an edge is *splitting* a vertex. To be precise, suppose v is a vertex with degree at least four in a graph G . Let $N_G(v)$ denote the set of *neighbors* of v , which are vertices adjacent to v . Let $X, Y \subseteq N_G(v)$ such that $X \cup Y = N_G(v)$ and $|X|, |Y| \geq 2$. The splitting v results in the new graph G' obtained from $G \setminus v$ by adding two new adjacent vertices x, y then joining x to all vertices in X and y to all vertices in Y . We call G' a *split* of G , v a *predecessor* of x and y , and the other vertex in G a *predecessor of itself* in G' . Note that $G'/xy = G$ and G' is 3-connected as long as G is. To investigate internally 4-connected graphs, the following chain theorem of Chun, Mayhew and Oxley [1] will be useful in creating an algorithm that generates all internally 4-connected graphs.

Theorem 5 [1]. *Let G be an internally 4-connected graph such that G is not $K_{3,3}$, terrahawk, C_n^2 ($n \geq 5$), or AW_{2n} ($n \geq 3$). Then G has an internally 4-connected minor H with $1 \leq \|G\| - \|H\| \leq 3$.*

This theorem says that every internally 4-connected can be obtained from $K_{3,3}$, terrahawk, C_n^2 ($n \geq 5$), or AW_{2n} ($n \geq 3$) by repeatedly adding edges and splitting vertices. Equivalently, for every internally 4-connected graph G , there exists a sequence of internally 4-connected graphs $G_0, G_1, G_2, \dots, G_k$ such that

- (i) $G_k \cong G$ and G_0 is $K_{3,3}$, terrahawk, C_n^2 ($n \geq 5$), or AW_{2n} ($n \geq 3$), and
- (ii) G_i ($i = 2, \dots, k$) is obtained from G_{i-1} by adding edges or splitting vertices at most three times.

3. NONPLANAR $\{\text{cube}, V_8\}$ -FREE GRAPHS

The cube and V_8 can be obtained from two disjoint cycles C_4 by connecting them with four edges that preserves the ordering of the cycles; however, V_8 is nonplanar. To determine all internally 4-connected nonplanar $\{\text{cube}, V_8\}$ -free graphs, we will follow the characterization in Theorem 3. All graphs with at most seven vertices have no cube and V_8 minors.

We now consider the case that an internally 4-connected graph G satisfies the condition (iv) in Theorem 3. Let X be a subset of $V(G)$ of at most four vertices such that $G \setminus X$ has no edges, and let $Y = V(G) - X$ consisting of $y_1, y_2, \dots, y_n$ for some $n \in \mathbb{N}$. Then all vertices in Y are nonadjacent. Since G is internally 4-connected, $|X| \geq 3$ and each y_i is adjacent to at least three vertices in X . Moreover, if $|X| = 3$, then G is $K_{3,n}$ for some $n \geq 5$. We will show that $K_{3,n}$ is cube-free. We denote the classes of graphs in Figure 5 as follows: $\mathcal{K}_I = \{K_{3,n} : n \geq 5\}$, $\mathcal{K}_{II} = \{K'_{3,n} : n \geq 5\}$, $\mathcal{K}_{III} = \{K''_{3,n} : n \geq 5\}$, $\mathcal{K}_{IV} = \{K'_{2,n} : n \geq 6\}$, and $\mathcal{K}_V = \{K_{1,n} : n \geq 7\}$. Let $\mathcal{K}_U = \mathcal{K}_I \cup \mathcal{K}_{II} \cup \mathcal{K}_{III} \cup \mathcal{K}_{IV} \cup \mathcal{K}_V$. To study a graph in these classes, a new vertex obtained from contracting an edge xy for $x \in X$ and $y \in Y$ will be put in the partition set X to keep the number of vertices in X .

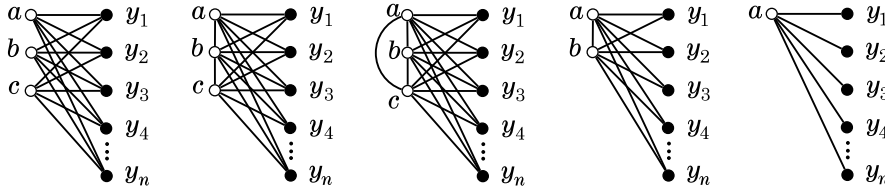


Figure 5. Graphs $K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, $K'_{2,n}$, and $K_{1,n}$.

Lemma 6. *For any $G \in \mathcal{K}_U$, G is cube-free.*

Proof. Let $G \in \mathcal{K}_U$. Since all vertices in Y are nonadjacent, if the cube is a subgraph of G , each disjoint C_4 of the cube must contain two vertices of X . However, $|X| \leq 3$, G does not contain a cube-subgraph. If the cube is a minor of G , then the minor can be obtained from G by a sequence of vertex deletions, edge deletions, and edge contractions, where the order of operations is irrelevant. Suppose that a sequence of edge contractions is performed on G first. Notice that there are two types of edge in G ; an edge connecting between X and Y , and an edge connecting vertices in X . Then for all $x \in X$ and $y \in Y$, $G/xy \in \mathcal{K}_U$, and for $x_i, x_j \in X$, $G/x_i x_j \in \mathcal{K}_U$. After performing the sequence of edge contractions on G , the resulting graph G^* is in $\mathcal{K}_U$. Then G^* does not contain a cube-subgraph. Hence, G is cube-free. ■

From Theorem 3 and Lemma 6, we obtain the following lemma.

Lemma 7. For $n \geq 5$, $K_{3,n}$ is $\{\text{cube}, V_8\}$ -free.

Next, we consider an internally 4-connected graph G satisfying the condition (iv) in Theorem 3 with $|X| = 4$. Then $G \in \mathcal{K}$ and $|Y| \geq 4$. Since G is internally 4-connected, at most one pair of vertices in X can be adjacent. Notice that for $x \in X$ and $y \in Y$, G/xy is not internally 4-connected. To study this type of graph, we relax the connectivity of G to 3-connected. Let $\mathcal{L}$ consist of 3-connected spanning subgraphs of some $K_{4,n}$, $n \geq 4$. Then $\mathcal{L} \subseteq \mathcal{K}$. The cube is also in $\mathcal{L}$, see Figure 6.

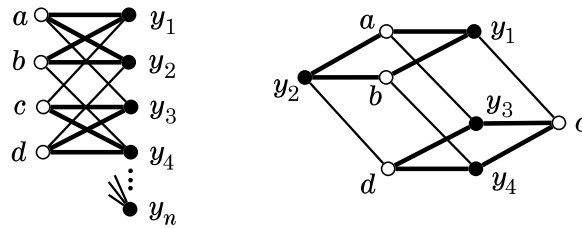


Figure 6. The cube in $\mathcal{L}$.

Lemma 8. A graph G in $\mathcal{L}$ contains a cube-subgraph if and only if there are $y_1, y_2, y_3, y_4 \in Y$ such that $\{a, b, c\} \subseteq N(y_1)$, $\{a, b, d\} \subseteq N(y_2)$, $\{a, c, d\} \subseteq N(y_3)$, and $\{b, c, d\} \subseteq N(y_4)$.

Proof. Let $G \in \mathcal{L}$. If there are $y_1, y_2, y_3, y_4 \in Y$ such that $\{a, b, c\} \subseteq N(y_1)$, $\{a, b, d\} \subseteq N(y_2)$, $\{a, c, d\} \subseteq N(y_3)$, and $\{b, c, d\} \subseteq N(y_4)$, then there are two disjoint C_4 's, $C_{4,1} : a, y_1, b, y_2$ and $C_{4,2} : y_3, c, y_4, d$, which four edges ay_3 , y_1c , by_4 , and y_2d preserve the ordering of the cycles. These form a cube-subgraph in G . Suppose that G contains a cube-subgraph. Since both X and Y consist of mutually nonadjacent vertices, each disjoint C_4 of the cube must contain exactly two vertices in X and two vertices in Y ; $C_{4,1} : a, y_1, b, y_2$ and $C_{4,2} : y_3, c, y_4, d$. We assume without loss of generality that edges ay_3 , y_1c , by_4 , and y_2d are edges in G which orderly join $C_{4,1}$ and $C_{4,2}$. Thus, $\{a, b, c\} \subseteq N(y_1)$, $\{a, b, d\} \subseteq N(y_2)$, $\{a, c, d\} \subseteq N(y_3)$, and $\{b, c, d\} \subseteq N(y_4)$. ■

Let $\mathcal{L}'$ be a class of 3-connected graphs that are obtained from spanning subgraphs of some $K_{4,n}$ ($n \geq 4$) by adding edges to the color class of size four. Then $\mathcal{L} \subseteq \mathcal{K} \subseteq \mathcal{L}'$.

Lemma 9. Let $G \in \mathcal{L}'$. Then the following statements are equivalent.

- (i) G contains a cube-subgraph.
- (ii) $G \setminus E(G[X])$ contains a cube-subgraph, where $G[X]$ is an induced subgraph of G with vertex set X , the color class of size four.

- (iii) *There are $y_1, y_2, y_3, y_4 \in Y$ such that $\{a, b, c\} \subseteq N(y_1)$, $\{a, b, d\} \subseteq N(y_2)$, $\{a, c, d\} \subseteq N(y_3)$, and $\{b, c, d\} \subseteq N(y_4)$.*

Proof. (i) $\Rightarrow$ (ii) Since G contains a cube-subgraph, if an edge uv joining two disjoint C_4 's of the cube is in $E(G[X])$, we have that $u, v \in X$, and there is another edge wz joining those two C_4 's such that $w, z \in Y$. This contradicts with the fact that all vertices in Y are nonadjacent. So $G \setminus E(G[X])$ contains a cube-subgraph.

(ii) $\Rightarrow$ (iii) Since $G \setminus E(G[X]) \in \mathcal{L}$, by Lemma 8, we obtain (iii).

(iii) $\Rightarrow$ (i) From Lemma 8, $G \setminus E(G[X])$ contains a cube-subgraph, so does G . ■

Lemma 10. *Let $G \in \mathcal{L}'$. Then G contains a cube-subgraph if and only if G contains a cube-minor.*

Proof. The forward direction is obvious. Suppose that G contains a cube-minor. We first perform all edge contractions in constructing the cube. Let G^* be the resulting graph. Then G^* contains a cube-subgraph. Note that contracting an edge in $G[X]$ leads to a graph in $\mathcal{K}_U$, by Lemma 6, it is cube-free. Thus, only edges connecting X and Y are contracted. By putting the new vertex obtaining from each edge contraction to the partite set X , G^* is in $\mathcal{L}'$. By Lemma 9, the cube is a subgraph $G^* \setminus E(G^*[X])$, which is a subgraph of $G \setminus E(G[X])$. So G contains a cube-subgraph. ■

From Lemma 10, to find an internally 4-connected cube-free graph G with the condition (iv), we have to find a graph with condition (iv) and no cube-subgraph.

Lemma 11. *An internally 4-connected graph $G \in \mathcal{K}$ with $X = \{a, b, c, d\}$ contains a cube-minor if and only if there are vertices $y_1, y_2, y_3, y_4 \in Y$ such that $\{a, b, c\} \subseteq N(y_1)$, $\{a, b, d\} \subseteq N(y_2)$, $\{a, c, d\} \subseteq N(y_3)$, and $\{b, c, d\} \subseteq N(y_4)$.*

Remark 12. Let G be an internally 4-connected cube-free graph in $\mathcal{K}$. Then G misses a neighbor set in Lemma 11. Since G is internally 4-connected, if X contains two pairs of adjacent vertices, G contains $K_{4,n}$ as a subgraph for some $n \geq 4$. So at most two vertices in X are adjacent. Then G can be classified as follows.

1. All vertices in X are nonadjacent and there is only one vertex y_1 in Y whose neighbor set is X . Then $G \setminus y_1$ misses two neighbor sets in Lemma 11. We may assume that there are no vertices in $Y \setminus y_1$ containing neighbor sets $\{a, b, d\}$ and $\{a, c, d\}$, see Figure 3(a).

2. All vertices in X are nonadjacent and $|N(y_i)| = 3$ for $i = 1, \dots, n$. Then G misses at most two neighbor sets in Lemma 11. We may assume that there are no vertices in Y containing neighbor sets $\{a, b, d\}$ or $\{a, c, d\}$, see Figures 3(b) and (c).

3. Two vertices in X are adjacent, say a and b . There are three different cases.

- (a) There are only two vertices in Y , say y_1 and y_2 , such that $N(y_1) = N(y_2) = X$. Then $G \setminus \{y_1, y_2\}$ misses three neighbor sets in Lemma 11, and all y_i 's, $3 \leq i \leq n$, have the same neighbor set. We may assume that $N(y_i) = \{b, c, d\}$ for $i = 3, \dots, n$, see Figure 3(d).
- (b) There is only one vertex in Y , say y_1 , such that $N(y_1) = X$. Then $G \setminus y_1$ misses two neighbor sets in Lemma 11. We may assume that there are no vertices in $Y \setminus y_1$ containing neighbor sets $\{a, b, c\}$ and $\{a, b, d\}$, see Figure 3(e).
- (c) For $i = 1, \dots, n$, $|N(y_i)| = 3$. Then G misses only two neighbor sets in Lemma 11. We may assume that there are no vertices in Y containing neighbor sets $\{a, b, c\}$ and $\{a, b, d\}$, see Figure 3(f).

We now consider graphs in $\mathcal{D}^+$. Notice that DW_6^+ contains a cube-minor by deleting edge uv , and AW_{2n}^+ is a subgraph of DW_{2n}^+ for each $n \geq 3$. The following lemma follows directly from the structure of cube-free graphs in Theorem 1.

Lemma 13. *For each integer $n \geq 3$, DW_{n+3}^+ and AW_{2n}^+ contain a cube-minor.*

4. PROOF OF THEOREM 2

To prove Theorem 2, we claim that all internally 4-connected planar graphs with at least eight vertices contain a cube-minor. From Theorem 5, this statement can be implied by the following lemma.

Lemma 14. *The only internally 4-connected planar cube-free graphs are C_6^2 and DW_5 .*

Proof. Let G be an internally 4-connected planar cube-free graph. Suppose, on contrary, that G is neither C_6^2 nor DW_5 . From Theorem 5, there is a sequence of internally 4-connected graphs $G_0, G_1, \dots, G_k$ satisfying the chain theorem such that G_k is isomorphic to G , and G_0 is isomorphic to $K_{3,3}$, terrahawk, C_n^2 ($n \geq 5$) or AW_{2n} ($n \geq 3$). Notice that G_i is a minor of G_j for all $i < j$. Then for each i , G_i is a planar cube-free graph. Since both terrahawk and AW_{2n} ($n \geq 3$) contain a cube-minor, G_0 is not isomorphic to these two graphs. From Kuratowski Theorem, a graph is planar if and only if it contains neither K_5 (or C_5^2) nor $K_{3,3}$ as a minor. So G_0 is not isomorphic to both C_5^2 and $K_{3,3}$. We now consider C_n^2 ($n > 5$). Let $\{v_1, v_2, \dots, v_n\}$ be the vertex set of C_n^2 such that for all $1 \leq i \leq n$, $N(v_i) = \{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}$, where the indices are taken modulo n . By contracting edges v_1v_3 and v_2v_4 , we obtain C_{n-2}^2 . For all odd $n > 5$, C_n^2 contains C_5^2 as a minor, so C_n^2 is nonplanar. Thus, G_0 is not isomorphic to C_n^2 , for all odd

$n > 5$. Since a cube can be obtained from C_8^2 by deleting edges v_1v_2 , v_3v_4 , v_5v_6 and v_7v_8 , C_8^2 contains a cube-minor, and so does C_n^2 for all even $n \geq 10$. So we only need to consider planar graphs constructed from C_6^2 by adding edges and splitting vertices.

Suppose G_0 is isomorphic to C_6^2 . Since adding an edge joining two nonadjacent vertices in C_6^2 gives a nonplanar graph with $K_{3,3}$ -subgraph, we assume that graph G_1 in the sequence is obtained from C_6^2 by splitting vertices at least one time. Up to symmetry, C_6^2 has ten splits, one of them is DW_5 and six of them are nonplanar, as illustrated in Figure 7.

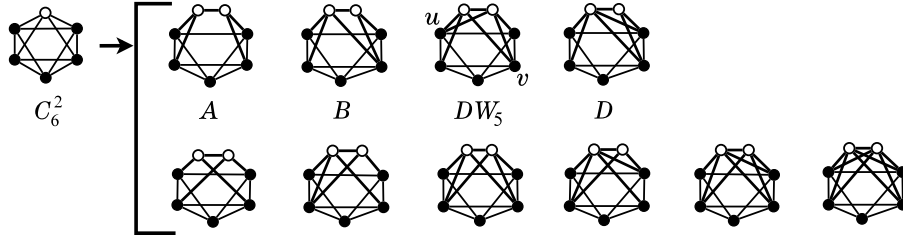


Figure 7. Ten splits of C_6^2 , where all graphs in the second row are nonplanar.

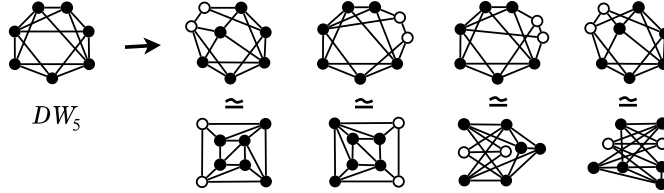
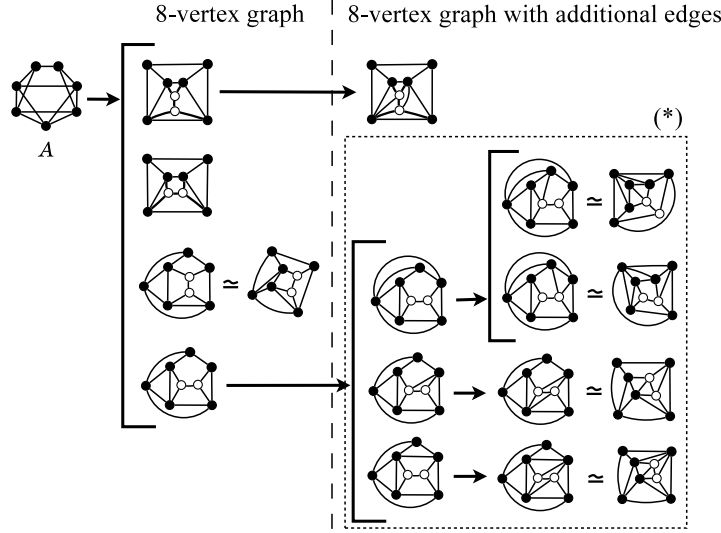


Figure 8. Splits of DW_5 where the first two graphs contain a cube-minor and the last two graphs contain a $K_{3,3}$ -minor.

From [9], DW_5 is the only internally 4-connected planar graph on seven vertices. If G_1 is DW_5 , then G_2 is a split of DW_5 . Up to symmetry, there are four splits of DW_5 as shown in Figure 8 such that all splits of DW_5 contain one of these graphs as a minor. In these four graphs, two of them contain a cube-minor and two of them are nonplanar. Since G is a planar cube-free graph, G_1 is not isomorphic to DW_5 . So G_1 is constructed from graph A , B , or D in Figure 7 by splitting vertices at least one times and adding edges. We claim that every internally 4-connected planar graph constructed from these three graphs by such methods contains a cube-minor.

For graph A , up to symmetry, every planar one-time split of A contains an 8-vertex graph in Figure 9 as a subgraph. We can construct all planar internally 4-connected graphs on eight vertices from A by adding edges to 8-vertex graphs in Figure 9 and preserving the planar and the internally 4-connected properties. To

Figure 9. Planar splits of graph A on eight vertices (set A1).

preserve such properties, if a split has a 3-separation $\{G_1, G_2\}$ such that neither G_1 nor G_2 is isomorphic to $K_{1,3}$, then we can add an edge joining two vertices in G_1 and G_2 in which their predecessors are adjacent, see 8-vertex graphs with additional edges in Figure 9. So every planar internally 4-connected graph on eight vertices constructed from A contains a non-cube-free graph in Figure 9 as a subgraph. Thus, all of these graphs contain a cube-minor. For splitting A two times, up to symmetry, every planar two-time split of A contains a 9-vertex graph in Figures 10, 11, and 12 as a subgraph. By the same argument, every planar internally 4-connected graph on nine vertices constructed from A contains a cube-minor. So every internally 4-connected planar graph constructed from A contains a cube-minor. The proof for splits of graphs B is of the same flavor, see Figures 13, 14, 15, 16, and 17. For graph D , up to symmetry, every planar one-time split of D contains an 8-vertex graph in Figure 18 as a subgraph. Since every internally 4-connected graph with K_4 -subgraph is nonplanar, there are no internally 4-connected planar graphs on eight and nine vertices that can be constructed from a graph in the dotted rectangles in Figures 18 and 19 by adding edges. Then, for the same reason, every internally 4-connected planar graph constructed from D contains a cube-minor, see Figures 18, 19, and 20. So G_1 contains a cube-minor, and so does G . This is a contradiction since G is cube-free. Hence, G is isomorphic to either C_6^2 or DW_5 . ■

Proof of Theorem 2. From Lemmas 4, 7, 11, and 13, we obtain a characterization of internally 4-connected nonplanar $\{\text{cube}, V_8\}$ -free graphs. The result of Theorem 2 follows from this characterization and Lemma 14. ■

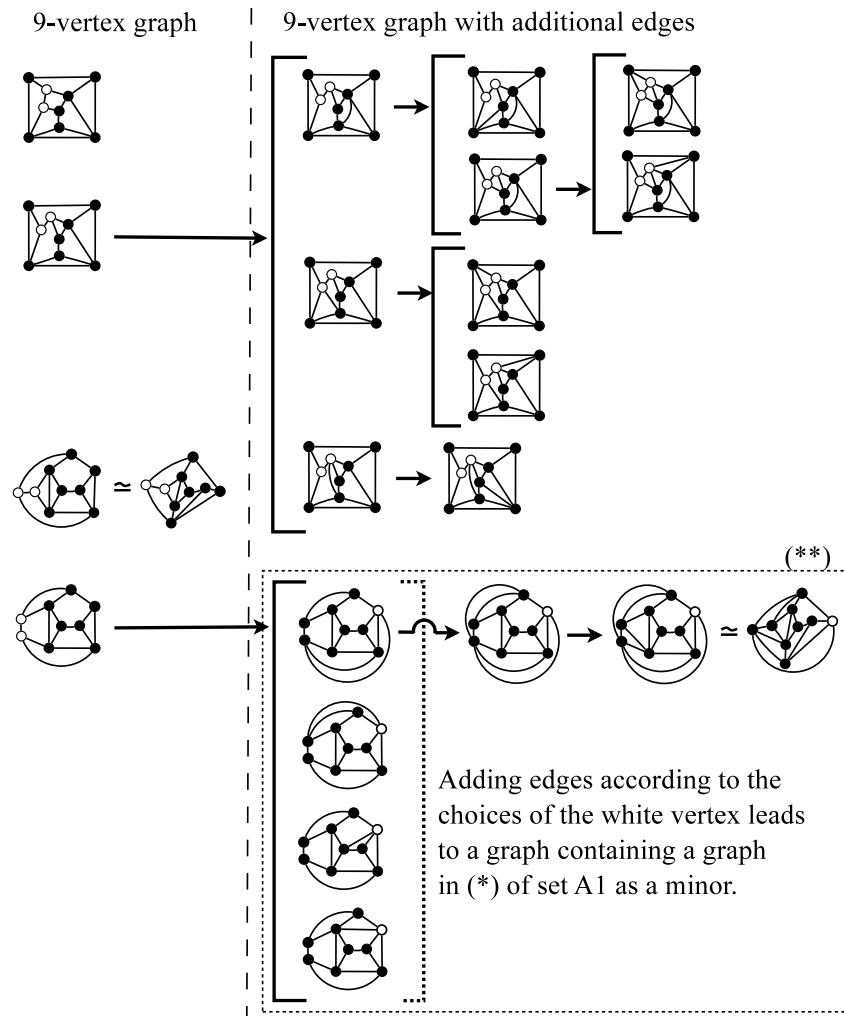


Figure 10. Planar splits of graph A on nine vertices (set A2).

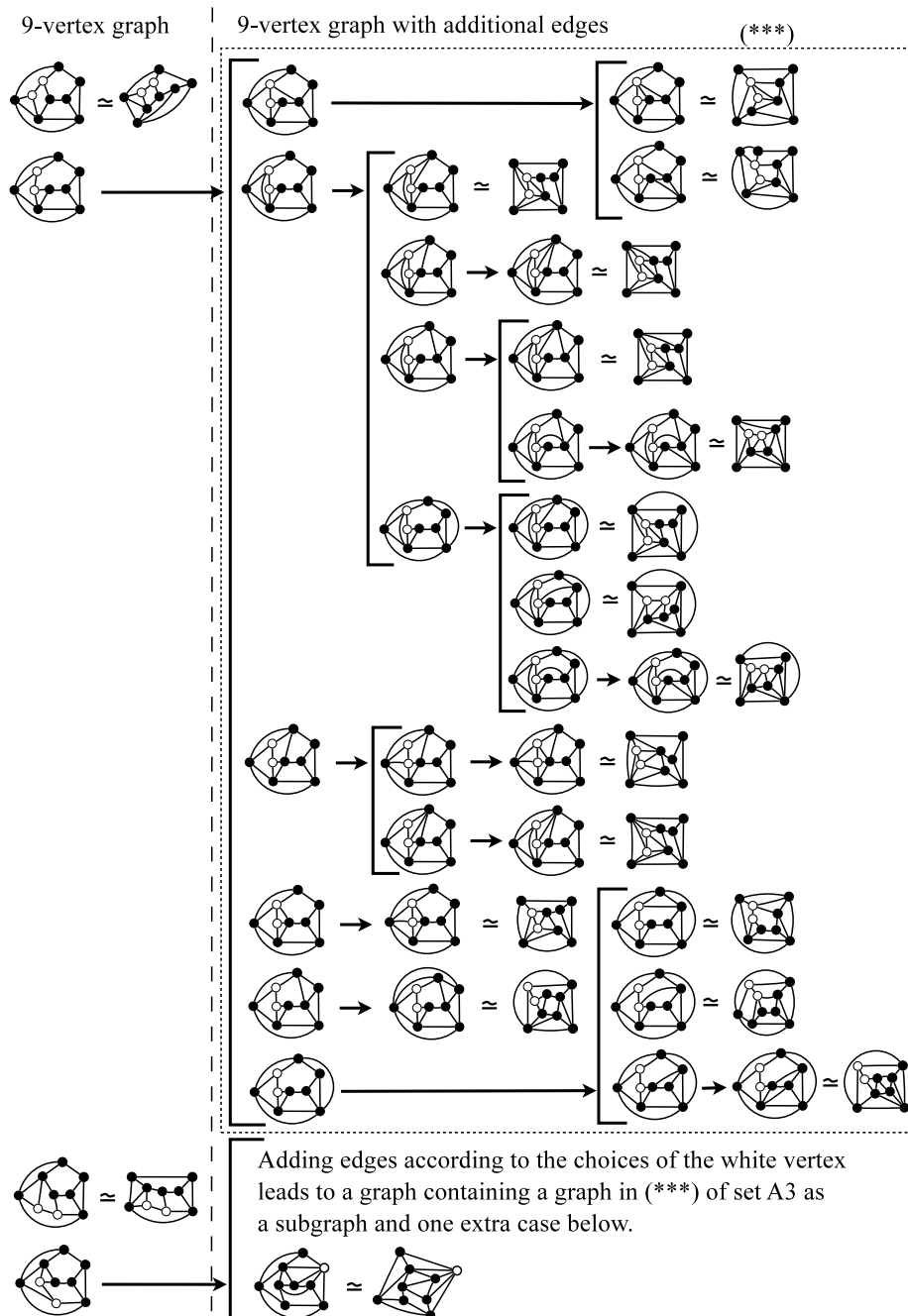


Figure 11. Planar splits of graph A on nine vertices (set A3).

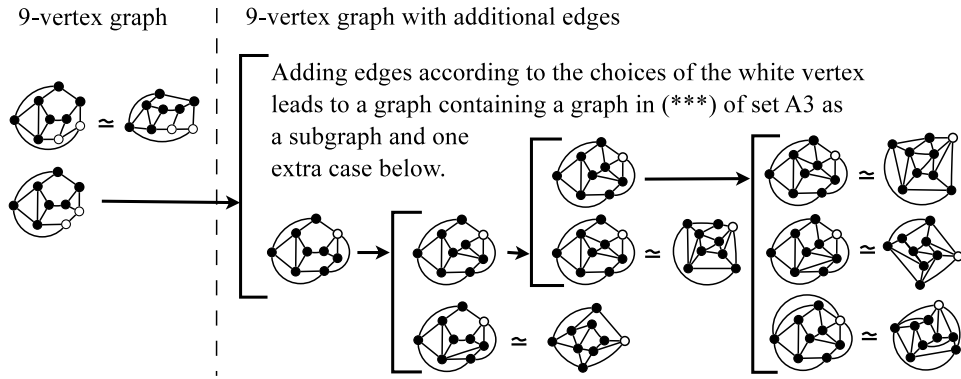


Figure 12. Planar splits of graph A on nine vertices (set A4).

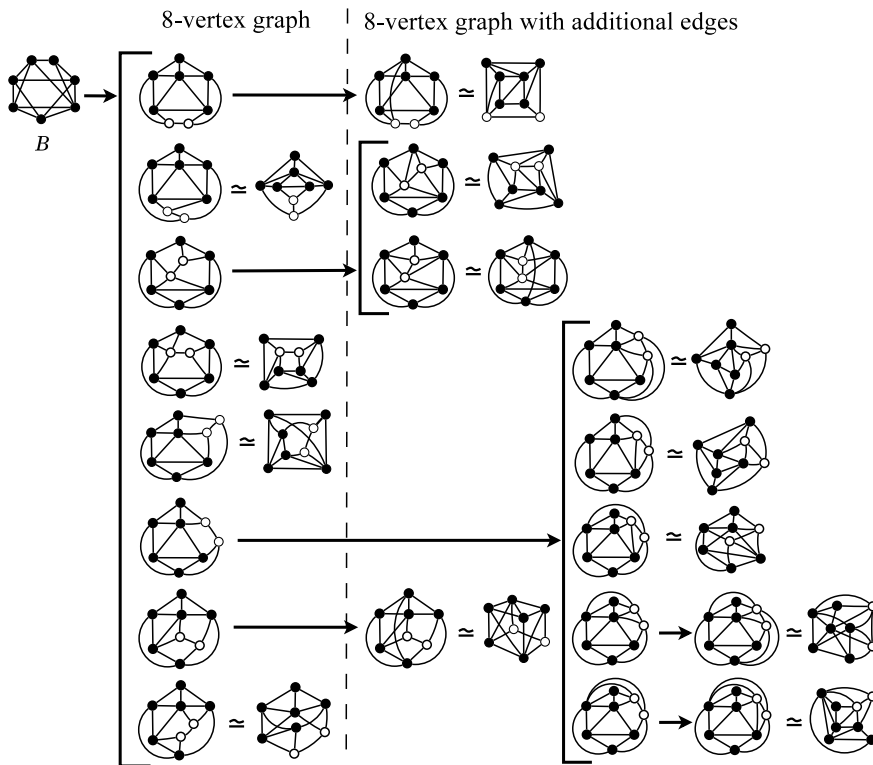


Figure 13. Planar splits of graph B on eight vertices.

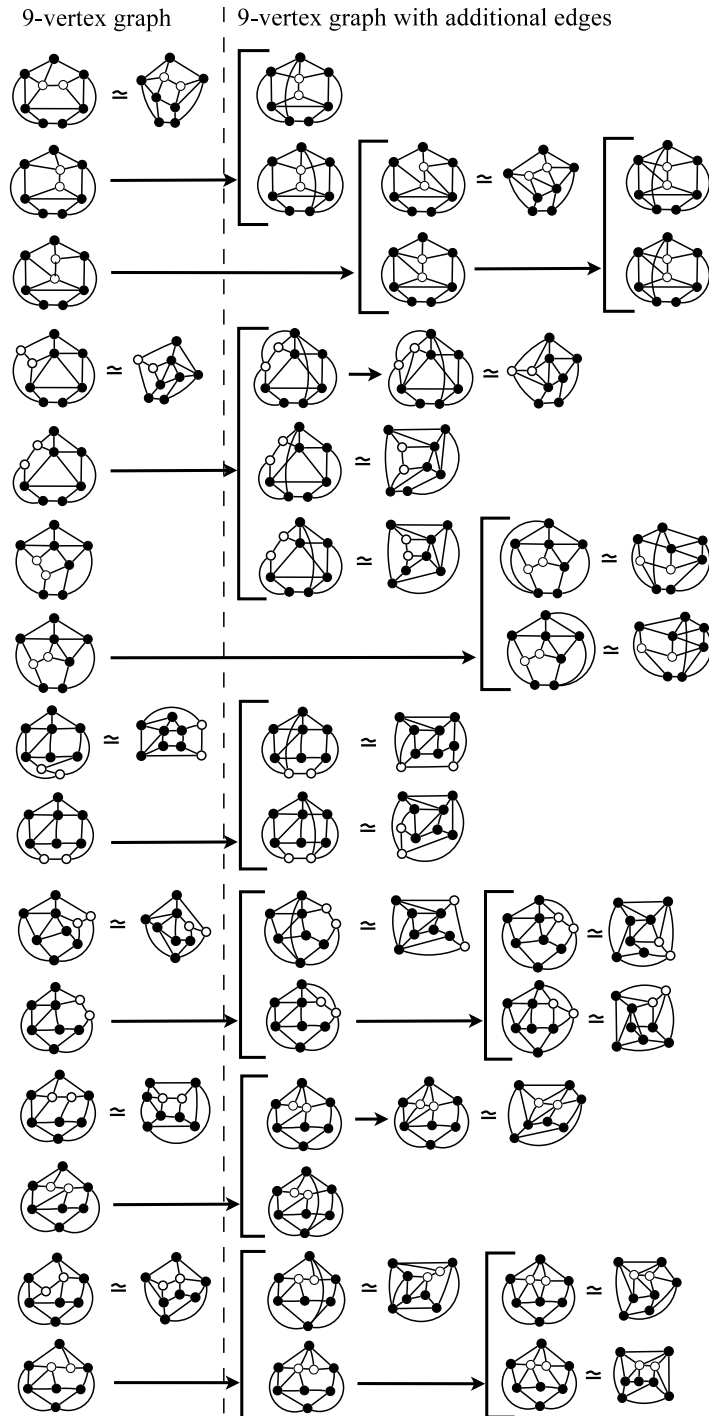


Figure 14. Planar splits of graph B on nine vertices.

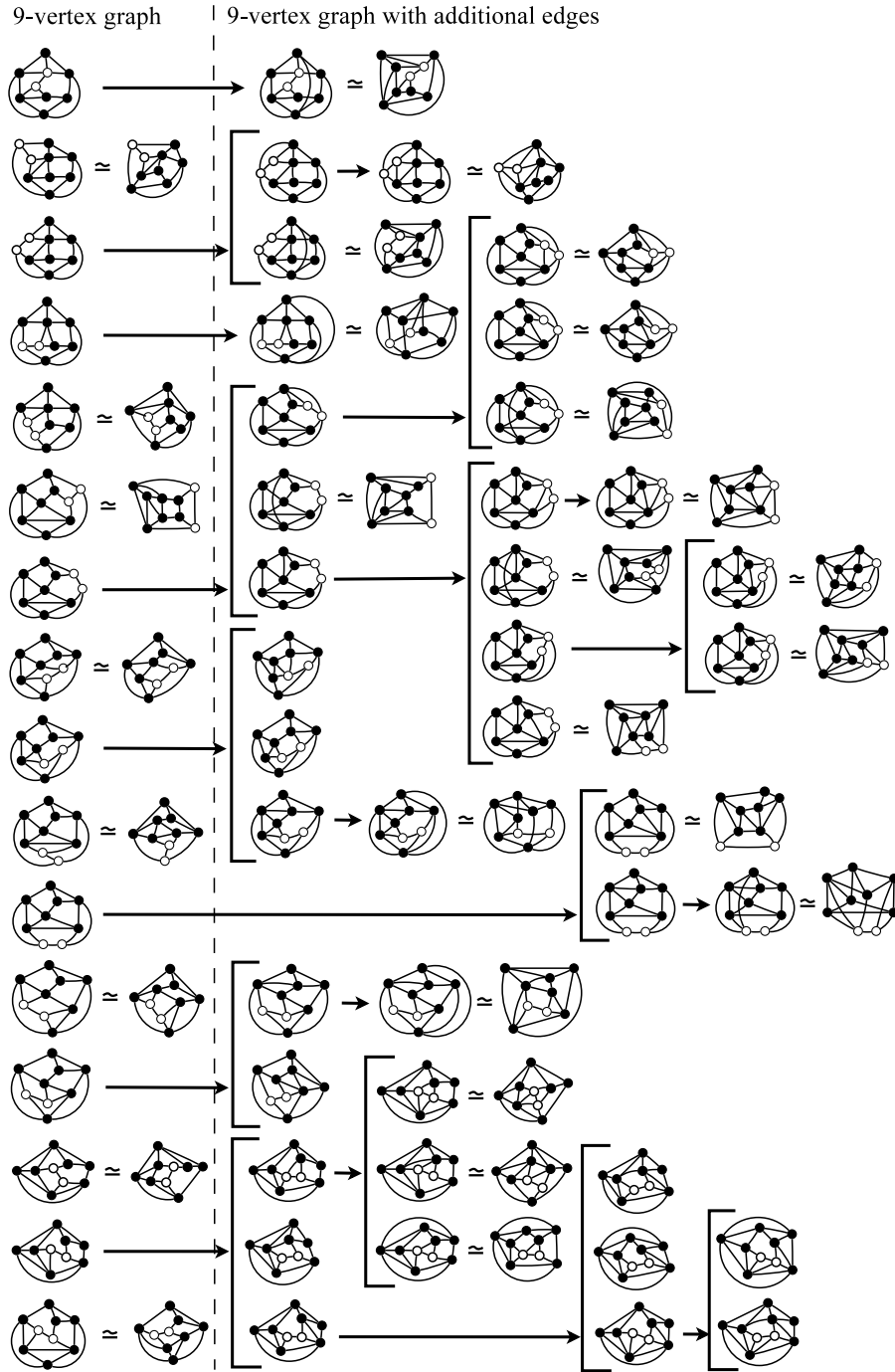


Figure 15. Planar splits of graph B on nine vertices.

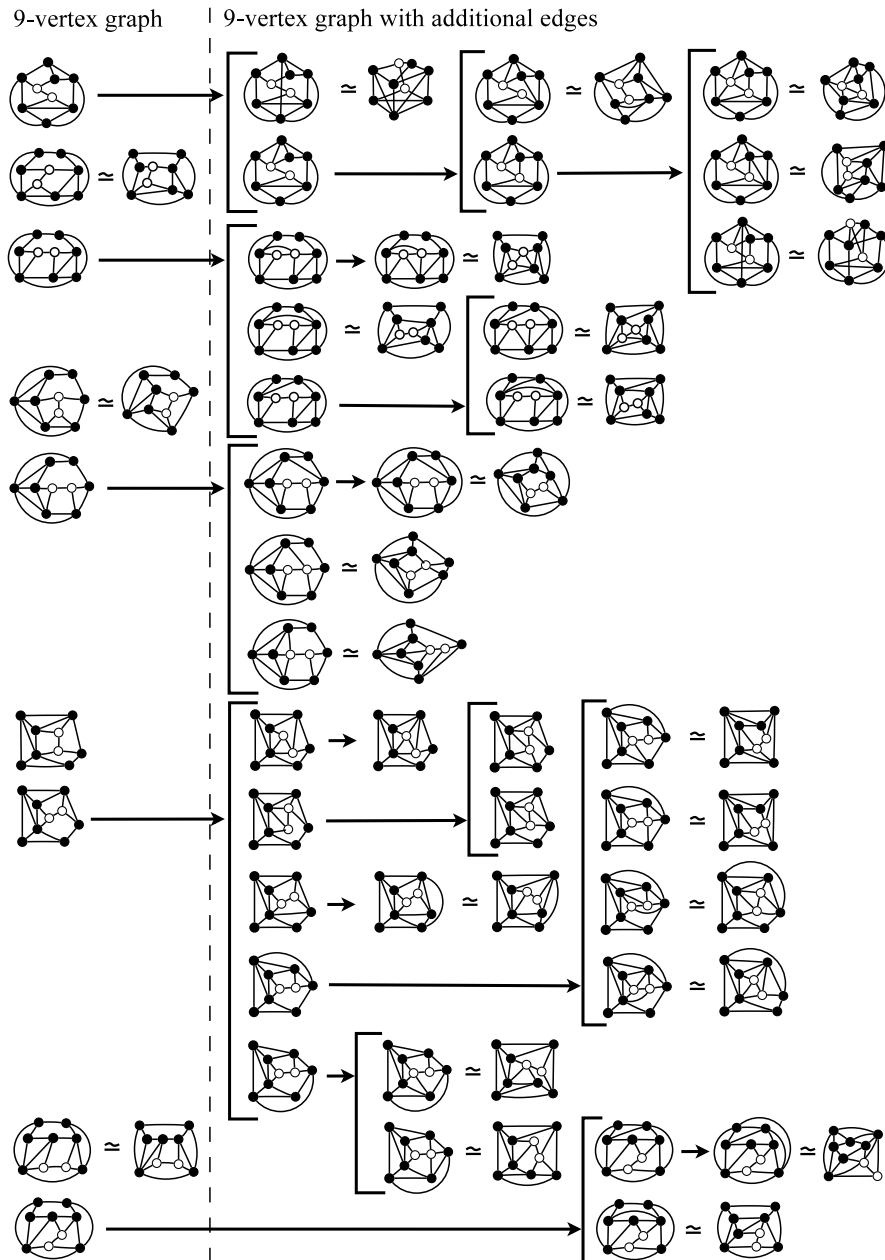


Figure 16. Planar splits of graph B on nine vertices.

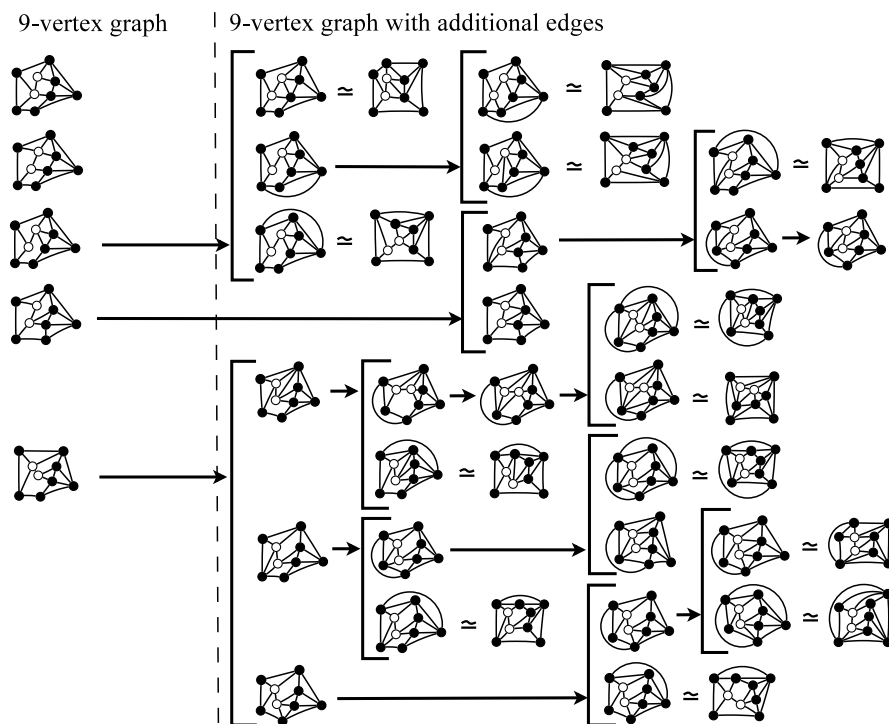


Figure 17. Planar splits of graph B on nine vertices.

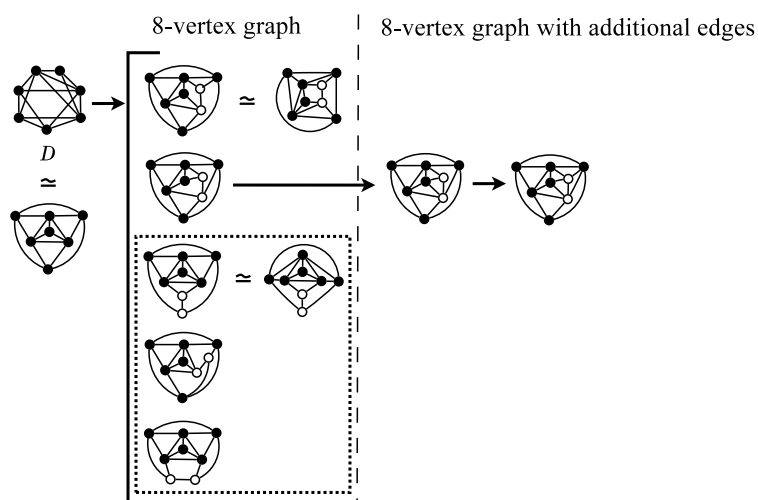


Figure 18. Splits of graph D on eight vertices, where graphs in the dotted rectangle contain a K_4 -subgraph.

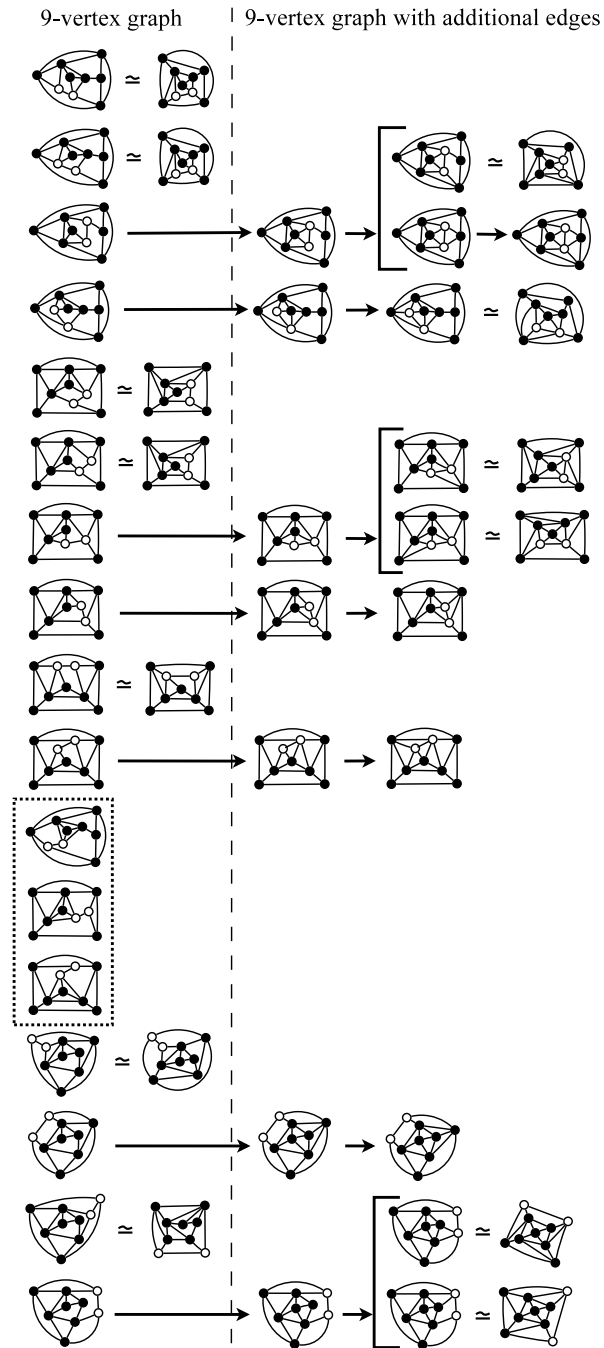


Figure 19. Splits of graph D on nine vertices, where graphs in the dotted rectangle contain a K_4 -subgraph.

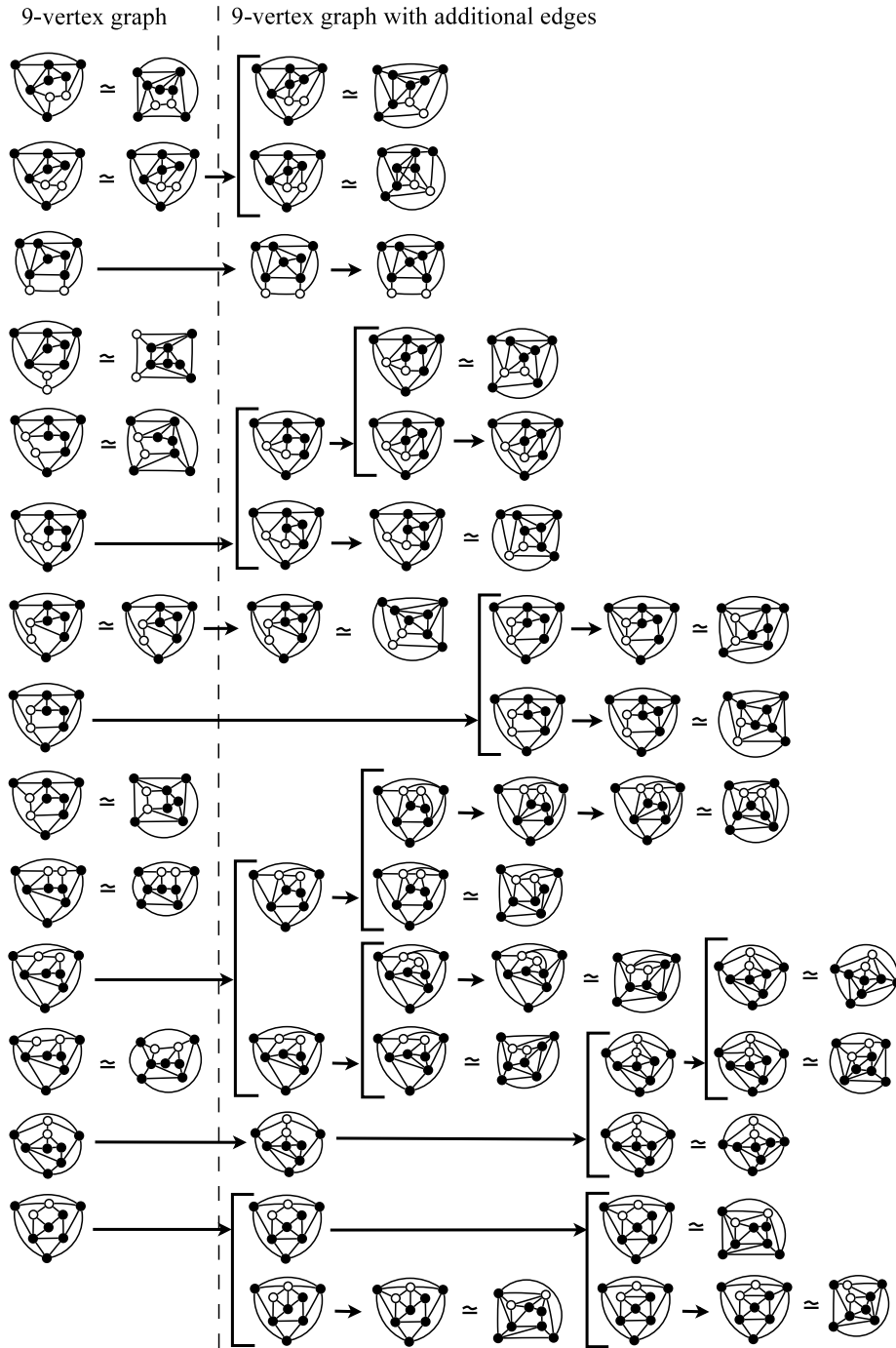


Figure 20. Splits of graph D on nine vertices.

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1 4-connected cube-free graphs

All graphs considered in this paper are simple. In particular, we use G/e to denote the graph obtained from G by contracting edges, and deleting loops and parallel edges that are created when we contract edges. A graph G is called H -free, where H is a graph, if no minor of G is isomorphic to H . Maharry [2] characterized all maximal cube-free graphs; however, the result does not provide all the internally 4-connected cube-free graphs. Our goal is to characterize all cube-free graphs. Equivalently, we determine all internally 4-connected cube-free graphs. We will adopt Maharry's main idea and use a theorem from [1], and then we verify more facts by computer. So we can eliminate the lengthy case analysis by hands in [2].

2 Basic lemmas

Lemma 2.1. [2] *Let H be a cubic internally 4-connected graph.*

1. H is cube-free if and only if H is isomorphic to P_{10} or isomorphic to some V_n where $n \geq 6$.
2. H is cube-free if and only if $L(H)$ is cube-free.

Theorem 2.2. [2] *All 4-connected cube-free graphs can be generated from the following families of graphs by repeatedly performing all possible vertex splits which do not introduce a cube minor;*

1. the octahedron,
2. the Möbius zigzag graphs for $k \geq 2$,
3. $L(P_{10})$,
4. $L(V_n)$ for $n \geq 6$.

Theorem 2.3. [1] *All 4-connected graph G can be constructed from octahedron, K_5 , unless G is a line graph of a cubic internally 4-connected graph H or C_n^2 , for some n .*

3 4-Connected cube-free graphs

To explain our result we need a few definitions. A *weak X -ladder* is a graph G that admits an embedding on the Möbius strip such that the following are satisfied. A *special cycle* C of G is embedded on the boundary of the strip and C has two vertices such that if C_1 and C_2 are the two paths of C between these two vertices, then every chord of C has at least one end in each of C_1 and C_2 , and every vertex outside C has exactly four neighbors, two of which are adjacent in C_1 and the other two are adjacent in C_2 . If G has the minimum degree $\delta(G) \geq 4$ and C is Hamiltonian, then G is isomorphic to C_{2n+1}^2 for some n . An *X -ladder* is a 4-connected weak X -ladder.

From Lemma 2.1, and Theorem 2.3, other than C_{2n+1}^2 , $L(V_n)$, and $L(P_{10})$, every 4-connected cube-free graphs can be generated from octahedron or K_5 . We can obtain all such cube-free graphs with at most 13 vertices; in addition, we verified that among all of those with 13 vertices, only one graph is not an X -ladder

and this graph cannot be extended into any other cube-free graphs. So all cube-free graphs with at least 14 vertices are generated from X -ladders on 13 vertices. We prove that they are all X -ladders. To do so, we prove the following theorem.

Theorem 3.1. *If a 4-connected cube-free graph G with at least 12 vertices is obtained from an X -ladder H by splitting a vertex x , then G is an X -ladder. Moreover, G can be embedded with a special cycle C such that either C is a special cycle of H or C/e is a special cycle of H for some edge e .*

To prove this theorem, we need the lemma below.

Lemma 3.2. *Let H be an X -ladder with at least 13 vertices and let C be a special cycle of H . Then for any vertex x of H , one of the following statement is true.*

- (i) $C \setminus N(x)$ has an edge e such that H/e is 4-connected.
- (ii) $C \setminus N(x)$ has two edges e, f such that there are three chords between them and $H/\{e, f\}$ is 4-connected.
- (iii) $H \setminus N(x)$ has a triangle with vertex set $\{u, v, w\}$, where u is not on C but the edge vw is, such that $H/\{uv, vw, uw\}$ is 4-connected.

Proof. From the structure of the X -ladder, $4 \leq |N(x)| \leq 6$. We can assume that there is y in $N(x)$ such that y is in C_1 . Then there is an edge vw in C_1 such that $v, w \notin N(x)$ and $y \in N(v)$. If $N(v) \cap N(w)$ is empty, then H/vw is 4 connected. Now, let $u \in N(v) \cap N(w)$. If u is outside C , then $\{u, v, w\}$ forms a triangle in $H \setminus N(x)$ that satisfies (iii). If u is in C and $\deg(u) = 4$, then there is $s \in N(u) \cap N(w)$. So vw and us forms two edges satisfying (ii). If u is in C and $\deg(u) \geq 5$, then vw forms an edge satisfying (i). $\square$

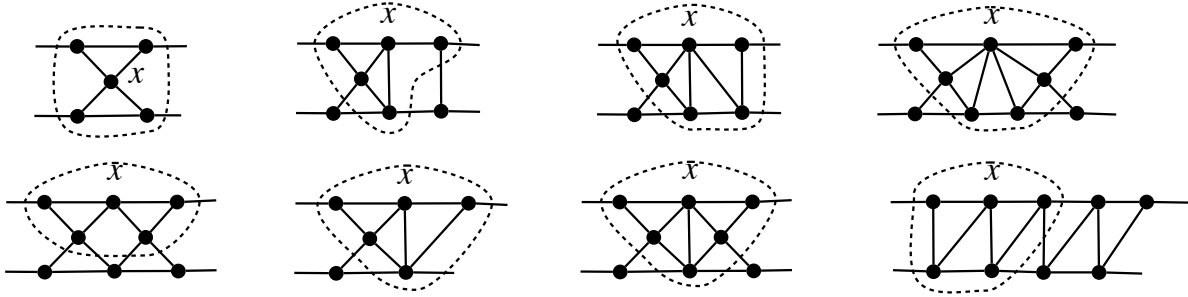


Figure 3.1: I use this picture to think about the lemma. We may not need this picture.

Proof of Theorem 3.1. We prove by induction on $|G|$. If $|G| = 12$ or 13 , the result is verified by computer as follows. Let $\mathcal{L}_k$ denote the set of all X -ladders on k vertices and let $\mathcal{Q}_k$ denote the set of all 4-connected cube-free graphs on k vertices.

1. $\mathcal{Q}_5 = \{K_5\}$
2. $\mathcal{Q}_6 = \{K_6, K_6 \setminus e, oct, oct^+\}$
3. $|\mathcal{Q}_7| = 24$: all these and the above are minors of K_7 .

4. $|Q_8| = 19$: consists of A, B, C, D (see Figure 3.2) and their 4-connected spanning subgraphs. Graph D is a minor of $L(P_{10})$ but $D \notin \mathcal{L}_8$.

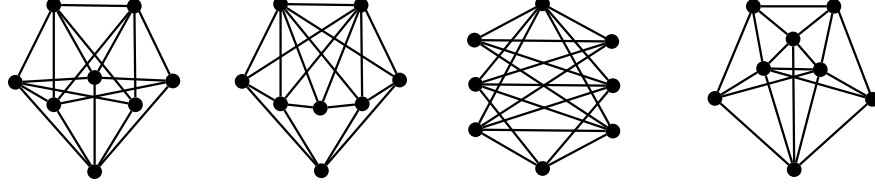


Figure 3.2: Graphs A, B, C , and D .

5. $|Q_9| = 19$: twelve of these graphs are minors of $L(P_{10})$ and nine of these graphs are members of $\mathcal{L}_9$. (two are both)
6. $|Q_{10}| = 14$: five of these graphs are minors of $L(P_{10})$ and ten of these graphs are members of $\mathcal{L}_{10}$. (one is both)
7. $|Q_{11}| = 21$: three of these graphs are minors of $L(P_{10})$ and 18 of these graphs are members of $\mathcal{L}_{11}$.
8. $|Q_{12}| = 24$: one of these graphs is a minor of $L(P_{10})$ and 23 of these graphs are members of $\mathcal{L}_{12}$.
9. $|Q_{13}| = 37$: one of these graphs is a minor of $L(P_{10})$ and 36 of these graphs are members of $\mathcal{L}_{12}$.

Suppose $|G| > 13$. Then $|H| > 12$. Let e be the edge of G with $H = G/e$, and let H be embedded with special cycle C . We need to find an embedding of G with special cycle D such that $D = C$ or $D/e = C$. Let F be the set of edges in H determined in Lemma 3.2. Then H/F is an X -ladder. In addition, H/F can be embedded with special cycle $C_0 = C/F$. Let U be the set of vertices created when F is contracted. Note that G/F is obtained from H/F by splitting x , this is because F is contained in $H \setminus N(x)$. By the induction hypothesis implies that G/F is an X -ladder and it can be embedded with special cycle C_1 , where $C_1 = C_0$ or $C_1/e = C_0$. Notice that when we obtain the embedding of G/F from the embedding H/F , only x and its incident edges are affected; however, the rest of the embedding, including U and its neighbors, are not affected. Therefore, the contraction C_0 can be reversed which gives us an embedding of G . For this embedding, the special cycle D satisfies $D = C$ or $D/e = C$ because $C_1 = C/F$ or $C_1/e = C/F$. This completes our induction and the theorem is proved. $\square$

4 Maximal good sets of triangles

In this section, we consider X -ladder that which triangles we can add triads to generate internally 4-connected graphs.

5 Internally 4-connected cube-free graphs

Finally, we determine all internally 4-connected cube-free graphs.



Domination Game Played on a Graph Constructed from 1-Sum of Paths

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Abstract : The domination game consists of two players, Dominator and Staller, who construct a dominating set in a given graph G by alternately choosing a vertex from G , with the restriction that in each turn at least one new vertex must be dominated. Dominator wants to minimize the size of the dominating set, while Staller wants to maximize it. In the game, both play optimally. The game domination number $\gamma_g(G)$ is the number of vertices chosen in the game which Dominator starts, and $\gamma'_g(G)$ is the number of vertices chosen in the game which Staller starts. In this paper these two numbers are analyzed when the game is played on a graph constructed from paths on n vertices, P_n , and on two vertices, P_2 , by gluing them together at a vertex. This type of operation is called 1-sum. The motivation behind our research is to study the game domination number of a tree that can be constructed from 1-sum of paths.

Keywords : domination game; dominating set; game domination number; 1-sum; paths.

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1 Introduction

In this paper, the domination game is played on a finite simple graph G . The domination game was first introduced by Brešar, Klavžar and Rall in 2010 [1]. It is basically different from the domination number of a graph G (the minimum size of its dominating set), $\gamma(G)$, although $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$, see [1]. The game domination numbers, γ_g and γ'_g , of some simple graphs such as paths and cycles are determined in [2, 3]. For a tree T , a connected graph with no cycles, the problem of determining its game domination numbers are non-trivial and the lower bound of $\gamma_g(T)$ is given in terms of the number of vertices and maximum degree of T [4]. To explain the relationship between $\gamma_g(G)$ and $\gamma'_g(G)$ of a graph G , they use imagination strategy, which compares the moves in a real game with an imaginary game both played on G . It is showed in [7] that these two numbers can differ only by 1, $|\gamma_g(G) - \gamma'_g(G)| \leq 1$. We call a pair (k, l) is *realized* by G if $\gamma_g(G) = k$ and $\gamma'_g(G) = l$. Some possible realizable pairs are studied in [1, 4]. All possible realizable pairs are given in [5]. For example, for every k , $(k, k + 1)$ can be realized by a tree [4], and for all $k \geq 2$, $(2k, 2k - 1)$ can be realized by a class of 2-connected graphs[5]. One way to study the game domination numbers of a graph is by considering graph operations such as deletion of a vertex or of an edge. As proved in [6], for a graph G and an edge e in G , the game domination numbers of G and G deleted e can deffer only by 2, $|\gamma_g(G) - \gamma_g(G - e)| \leq 2$ and $|\gamma'_g(G) - \gamma'_g(G - e)| \leq 2$. The same result holds for deleting a vertex in G .

We can think of a tree as joining paths together at vertices. The operation of combining two graphs by identifying a vertex of one graph with a vertex of another is called the *1-sum*. Then a tree can be constructed from 1-sum of paths. In our paper, we consider the game domination numbers of a tree constructed from 1-sum of a path on n vertices, P_n , and a path on two vertices, P_2 . To state our main result we need to define a few graphs. Let $x_1, x_2, \dots, x_n$ be vertices of P_n , and let v_1, v_2 be vertices of P_2 . We define a graph Q_{n+1} , $n \geq 4$, to be a 1-sum of P_{n+4} and P_2 at x_2 and v_1 , see Figure 1.

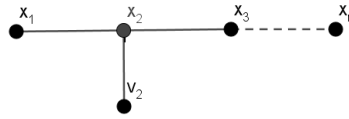


Figure 1: Graph Q_{n+1}

In a graph G , vertices u and v in G are *neighbours* if uv is an edge in G . Let $N[u]$ be the set consisting of u and all its neighbours. Note that a vertex in a graph is called *dominated* if it is chosen or it is a neighbour of the vertex chosen. Let S be a subset of the vertex set of G , $V(G)$. Then a *partially dominated graph* $G|S$ is a subgraph of G where the vertices of S are already dominated. So these vertices do not need to be dominated in the course of the game. The *residual graph* corresponding to $G|S$ is a graph obtained from G by deleting all edges between

dominated vertices and all vertices u that cannot be chosen any more, $N[u] \subseteq S$. Our main results are as follows.

Theorem 1.1. $\gamma(Q_{n+1}) \leq 1 + \gamma'_g(Q_{n+1}|N[x_2]) < 1 + \gamma'_g(Q_{n+1}|N[x_3])$.

Theorem 1.2. $\gamma'_g(Q_{n+1}) \geq 1 + \gamma_g(Q_{n+1}|N[x_3])$.

Theorem 1.3. *In a Staller-start game played on Q_{n+1} , for $n \equiv 3 \pmod{4}$, if the Staller first move is v_2 , then Dominator cannot choose x_4 .*

For the rest of the paper, we start with introducing our tools used in our proofs. Then we analyze domination games played on Q_{n+1} . Finally, we consider a Dominator-start game played on 1-sum of P_n and P_2 .

2 Basic Lemmas

In this section, we introduce our main tools, which are the continuation principal, properties of realization, and formulas involving the game domination numbers of a path P_n .

Theorem 2.1 (Continuation Principle). *[7] Let G be a graph and $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma'_g(G|A) \leq \gamma'_g(G|B)$.*

The next theorem shows the relation between the game domination numbers.

Theorem 2.2. *[7] For any graph G , $|\gamma_g(G) - \gamma'_g(G)| \leq 1$.*

Suppose that $\gamma_g(G) = k$ and $\gamma'_g(G) = m$. Theorem 2.2 implies that the realization of G is $(k, k), (k, k+1), (k, k-1)$, where $m = \{k-1, k, k+1\}$. We call *equal* for the case (k, k) , *plus* for the case $(k, k+1)$, and *minus* for the case $(k, k-1)$. If G is a family of forests, then the realization is (k, k) or $(k, k+1)$.

Theorem 2.3. *[1, 7] Forests are no-minus graphs.*

If the disjoint union of no-minus graphs has at least one equal graph (component), then the following holds.

Theorem 2.4. *[8] Let $G_1|S_1$ and $G_2|S_2$ be partially dominated no-minus graphs. If $G_1|S_1$ realizes (k, k) and $G_2|S_2$ realizes (l, m) (where $m \in \{l, l+1\}$), then the disjoint union $(G_1 \cup G_2)|(S_1 \cup S_2)$ realizes $(k+l, k+m)$.*

In the case that both components of a no-minus graph are plus, the following statement holds.

Theorem 2.5. *[8] Let $G_1|S_1$ and $G_2|S_2$ be partially dominated no-minus graphs such that $G_1|S_1$ realizes $(k, k+1)$ and $G_2|S_2$ realizes $(l, l+1)$. Then*

$$\begin{aligned} k+l &\leq \gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq k+l+1, \\ k+l+1 &\leq \gamma'_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq k+l+2. \end{aligned}$$

Let P''_n denote the partially dominated path of order $n+2$, which its endpoints are dominated, see Figure 2, and let P'_n denote the partially dominated path of order $n+1$, which only one of its endpoint is dominated, see Figure 2. In both cases, n vertices are not dominated. The following is an important lemma involving the proof of the game domination numbers of a path.



Figure 2: Partially dominated paths of P''_n (left) and P'_n (right)

Lemma 2.6. [3] For every $n \geq 0$, we have

$$\gamma_g(P''_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1; & n \equiv 3 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil; & \text{otherwise,} \end{cases}$$

$$\gamma'_g(P''_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1; & n \equiv 2 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil; & \text{otherwise.} \end{cases}$$

Moreover, for every $i, j \geq 0$ such that $i + j = n$, $i_r = (i \bmod 4)$ and $j_r = (j \bmod 4)$, we also have

$$\gamma_g(P''_i \cup P''_j) = \begin{cases} \gamma_g(P''_i) + \gamma_g(P''_j); & (i_r, j_r) \in \{0, 1\} \times \{0, 1, 2, 3\} \cup \{0, 1, 2, 3\} \times \{0, 1\}, \\ \gamma_g(P''_i) + \gamma_g(P''_j) + 1; & (i_r, j_r) \in \{2, 3\} \times \{2, 3\}, \end{cases}$$

$$\gamma'_g(P''_i \cup P''_j) = \begin{cases} \gamma_g(P''_i) + \gamma_g(P''_j); & (i_r, j_r) \in \{0, 1\} \times \{0, 1\}, \\ \gamma_g(P''_i) + \gamma_g(P''_j) + 1; & (i_r, j_r) \in \{0, 1\} \times \{2, 3\} \cup \{2, 3\} \times \{0, 1\} \cup \{(2, 2)\}, \\ \gamma_g(P''_i) + \gamma_g(P''_j) + 2; & (i_r, j_r) \in \{(2, 3), (3, 2), (3, 3)\}. \end{cases}$$

This lemma shows the optimal first move of both players playing on a partially dominated graph P''_n . Dominator always chooses a vertex distance two from the dominated endpoint, but Staller always choose dominated endpoint. Hence, both players play the same way in P'_n . The following statement holds.

Lemma 2.7. [3] For every $n, m \geq 0$, we have

$$\gamma_g(P'_n \cup P'_m) = \gamma_g(P''_n \cup P'_m) = \gamma_g(P''_n \cup P''_m) \text{ and}$$

$$\gamma'_g(P'_n \cup P'_m) = \gamma'_g(P''_n \cup P'_m) = \gamma'_g(P''_n \cup P''_m)$$

We can apply Lemmas 2.6 and 2.7 to determine the game domination number of paths.

Theorem 2.8. [3] For every $n \geq 0$, we have

$$\gamma_g(P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1; & n \equiv 3 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil; & \text{otherwise,} \end{cases}$$

$$\gamma'_g(P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

3 A Dominator-Start Game Played on Q_{n+1}

In this section, we analyze $\gamma_g(Q_{n+1})$. First, we study the game when the Dominator first move is vertex x_3 .

Lemma 3.1. Suppose the Dominator first move is x_3 . Then

$$\gamma'_g(Q_{n+1}|N[x_3]) \geq \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1; & n \equiv 3 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil; & \text{otherwise.} \end{cases}$$

Proof. After the Dominator first move at x_3 , the residual graph is a disjoint union between graph $P_{x_1x_2v_2}$ and $P'_{n-4 \geq 0}$, where $P_{x_1x_2v_2}$ is a path in P_n with the vertex set $\{x_1, x_2, v_2\}$. Notice that $\gamma'_g(Q_{n+1}|N[x_3]) = \gamma'_g(P_{x_1x_2v_2} \cup P'_{n-4})$. We calculate the game domination number directly, and obtain that $\gamma_g(P_{x_1x_2v_2}) = 1$ and $\gamma'_g(P_{x_1x_2v_2}) = 2$. So $P_{x_1x_2v_2}$ is a plus graph. We now consider $\gamma'_g(P_{x_1x_2v_2} \cup P'_{n-4})$. If P'_{n-4} is a plus graph where $n - 4 \equiv 2, 3 \pmod{4}$, then the residual graph is a disjoint union between plus graphs $P_{x_1x_2v_2}$ and P'_{n-4} . By Theorem 2.5, we have

$$\begin{aligned} \gamma'_g(P_{x_1x_2v_2} \cup P'_{n-4}) &\geq \gamma_g(P_{x_1x_2v_2}) + \gamma_g(P'_{n-4}) + 1 \\ &\geq 2 + \gamma_g(P'_{n-4}). \end{aligned}$$

If P'_{n-4} is an equal graph where $n - 4 \equiv 0, 1 \pmod{4}$, then the residual graph is a disjoint union between plus and equal graphs. By Theorem 2.4, we have

$$\begin{aligned} \gamma'_g(P_{x_1x_2v_2} \cup P'_{n-4}) &= \gamma'_g(P_{x_1x_2v_2}) + \gamma'_g(P'_{n-4}) \\ &= \gamma_g(P_{x_1x_2v_2}) + 1 + \gamma_g(P'_{n-4}) \\ &= 2 + \gamma_g(P'_{n-4}). \end{aligned}$$

We can easily check by hand for the case $n = 4$. Suppose that $n \geq 5$, we consider four cases according to the value of $n \pmod{4}$. We apply Lemmas 2.6 and 2.7 to obtain the solution for all $k \geq 1$ as follows.

$$\begin{aligned} \gamma'_g(P_{x_1x_2v_2} \cup P'_{4(k-1)}) &= 2 + \gamma_g(P'_{4(k-1)}) \\ &= 2 + \gamma_g(P''_{4(k-1)}) \\ &= 2 + 2k - 2 = 2k, \end{aligned}$$

$$\begin{aligned}
 \gamma'_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+1}) &= 2 + \gamma_g(P'_{4(k-1)+1}) \\
 &= 2 + \gamma_g(P''_{4(k-1)+1}) \\
 &= 2 + 2k - 2 + 1 = 2k + 1,
 \end{aligned}$$

$$\begin{aligned}
 \gamma'_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+2}) &\geq 2 + \gamma_g(P'_{4(k-1)+2}) \\
 &\geq 2 + 2(k-1) + 1 = 2k + 1,
 \end{aligned}$$

$$\begin{aligned}
 \gamma'_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+3}) &\geq 2 + \gamma_g(P'_{4(k-1)+3}) \\
 &\geq 2 + 2(k-1) + 2 - 1 = 2k + 1.
 \end{aligned}$$

□

Notice that $\gamma_g(Q_{n+1}) = 1 + \min_{x \in Q_{n+1}} \{\gamma'_g(Q_{n+1}|N[x])\}$. We obtain this equality when x is the Dominator first move in an optimal strategy. Since it does not guarantee that the Dominator first move at x_3 is an optimal strategy, we obtain the following corollary.

Corollary 3.1. $\gamma(Q_{n+1}) \leq 1 + \gamma'_g(Q_{n+1}|N[x_3])$.

Next, we consider the game domination number on graph Q_{n+1} after the Dominator first move choosing vertex x_2 .

Lemma 3.2. *If the Dominator first move is x_2 , then*

$$\gamma'_g(Q_{n+1}|N[x_2]) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 2; & n \equiv 3 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil - 1; & \text{otherwise.} \end{cases}$$

Proof. Suppose that the Dominator first move is x_2 . Then vertices x_1, x_2, x_3, v_2 are all dominated, and the residual graph is P'_{n-3} . We consider four cases according to the value of $n \bmod 4$. Let $k \geq 1$. By Lemmas 2.6 and 2.7, we obtain that

$$\begin{aligned}
 \gamma'_g(P'_{4(k-1)+1}) &= \gamma'_g(P''_{4(k-1)+1}) \\
 &= \gamma_g(P''_{4(k-1)+1}) \\
 &= 2(k-1) + 1 = 2k - 1,
 \end{aligned}$$

$$\begin{aligned}
 \gamma'_g(P'_{4(k-1)+2}) &= \gamma'_g(P''_{4(k-1)+2}) \\
 &= \gamma_g(P''_{4(k-1)+2}) + 1 = 2k,
 \end{aligned}$$

$$\begin{aligned}
 \gamma'_g(P'_{4(k-1)+3}) &= \gamma'_g(P''_{4(k-1)+3}) \\
 &= \gamma_g(P''_{4(k-1)+3}) + 1 = 2k,
 \end{aligned}$$

$$\begin{aligned}\gamma'_g(P'_{4k}) &= \gamma'_g(P''_{4k}) \\ &= \gamma_g(P''_{4k}) = 2k.\end{aligned}$$

□

Proof of Theorem 1.1. We compare $\gamma'_g(Q_{n+1}|N[x_2])$ and $\gamma'_g(Q_{n+1}|N[x_3])$. Since $\gamma'_g(Q_{n+1}|N[x_2]) < \gamma'_g(Q_{n+1}|N[x_3])$, we obtain the result. □

We now consider some vertices which are not the Dominator first move in an optimal strategy.

Lemma 3.3. *In an optimal strategy of the Dominator-start game played on Q_{n+1} , the Dominator first move cannot be x_1, x_3, x_n and v_2 .*

Proof. We know that $N[x_1]$ and $N[v_2]$ are subsets of $N[x_2]$, and $\{x_n\}$ is a subset of $N[x_{n-1}]$. By the continuation principle and Theorem 1.1, the result follows. □

From our analysis, we propose the following conjecture. In an optimal strategy of the Dominator-start game played on Q_{n+1} , the Dominator first move is x_2 . Then

$$\begin{aligned}\gamma_g(Q_{n+1}) &= 1 + \gamma'_g(Q_{n+1}|N[x_2]) \\ &= \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1; & n \equiv 3 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil; & \text{otherwise.} \end{cases}\end{aligned}$$

4 A Staller-Start Game Played on Q_{n+1}

In this part, we consider the Staller-start game domination number on graph Q_{n+1} .

Lemma 4.1. *If the Staller first move is x_3 , then*

$$\gamma_g(Q_{n+1}|N[x_3]) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1; & n \equiv 0, 1 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil - 1 \text{ or } \left\lceil \frac{n}{2} \right\rceil; & n \equiv 2 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil - 1 \text{ or } \left\lceil \frac{n}{2} \right\rceil - 2; & n \equiv 3 \pmod{4}. \end{cases}$$

Proof. Suppose that the Staller first move is x_3 . Then the residual graph is a disjoint union between $P_{x_1x_2v_2}$ and $P'_{n-4 \geq 0}$, where $P_{x_1x_2v_2}$ is a path in P_n with the vertex set $\{x_1, x_2, v_2\}$. Notice that $\gamma_g(Q_{n+1}|N[x_3]) = \gamma_g(P_{x_1x_2v_2} \cup P'_{n-4})$. We can find the game domination number directly from the graph; $\gamma_g(P_{x_1x_2v_2}) = 1$ and $\gamma'_g(P_{x_1x_2v_2}) = 2$. So $P_{x_1x_2v_2}$ is a plus graph. Next we find $\gamma_g(P_{x_1x_2v_2} \cup P'_{n-4})$. If P'_{n-4} is a plus graph, where $n - 4 \equiv 2, 3 \pmod{4}$, then the residual graph is a disjoint union between plus graphs $P_{x_1x_2v_2}$ and P'_{n-4} . By Theorem 2.5, we have that

$$\begin{aligned}\gamma_g(P_{x_1x_2v_2}) + \gamma_g(P'_{n-4}) &\leq \gamma_g(P_{x_1x_2v_2} \cup P'_{n-4}) \leq \gamma_g(P_{x_1x_2v_2}) + \gamma_g(P'_{n-4}) + 1 \\ 1 + \gamma_g(P'_{n-4}) &\leq \gamma_g(P_{x_1x_2v_2} \cup P'_{n-4}) \leq 2 + \gamma_g(P'_{n-4}).\end{aligned}$$

If P'_{n-4} is an equal graph, where $n-4 \equiv 0, 1 \pmod{4}$, then the residual graph is a disjoint union between plus graph and equal graph. By Theorem 2.4, we have that

$$\begin{aligned}\gamma_g(P_{x_1x_2v_2} \cup P'_{n-4}) &= \gamma_g(P_{x_1x_2v_2}) + \gamma_g(P'_{n-4}) \\ &= 1 + \gamma_g(P'_{n-4}).\end{aligned}$$

It can be easily checked for $n = 4$. Assume that $n \geq 5$. There are four cases according to the value of $n \pmod{4}$. Then we apply Lemmas 2.6 and 2.7 to obtain the solution for all $k \geq 1$.

$$\begin{aligned}\gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)}) &= 1 + \gamma_g(P'_{4(k-1)}) \\ &= 1 + \gamma_g(P''_{4(k-1)}) \\ &= 1 + 2k - 2 = 2k - 1,\end{aligned}$$

$$\begin{aligned}\gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+1}) &= 1 + \gamma_g(P'_{4(k-1)+1}) \\ &= 1 + \gamma_g(P'_{4(k-1)+1}) \\ &= 1 + 2k - 2 + 1 = 2k,\end{aligned}$$

$$\begin{aligned}1 + \gamma_g(P'_{4(k-1)+2}) &\leq \gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+2}) \leq 2 + \gamma_g(P'_{4(k-1)+2}) \\ 1 + 2k - 2 + 1 &\leq \gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+2}) \leq 2 + 2k - 2 + 1 \\ 2k &\leq \gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+2}) \leq 2k + 1,\end{aligned}$$

$$\begin{aligned}1 + \gamma_g(P'_{4(k-1)+3}) &\leq \gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+3}) \leq 2 + \gamma_g(P'_{4(k-1)+3}) \\ 1 + 2k - 2 + 1 &\leq \gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+3}) \leq 2 + 2k - 2 + 1 \\ 2k &\leq \gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+3}) \leq 2k + 1.\end{aligned}$$

□

Proof of Theorem 1.2. We know that

$$\gamma_g(Q_{n+1}) = 1 + \max_{x \in V(Q_{n+1})} \{\gamma'_g(Q_{n+1}|N[x])\}.$$

We can obtain this equality when x is the Staller first move in an optimal strategy. Since it does not guarantee that the Staller first move at x_3 is an optimal strategy, we obtain the result. □

We next consider the Staller-start game domination number when Staller chooses v_2 and Dominator chooses x_2 .

Lemma 4.2. *If the Staller first move is v_2 and the next move by Dominator is x_2 , then*

$$\gamma'_g(Q_{n+1}|N[v_2, x_2]) = \begin{cases} \lceil \frac{n}{2} \rceil - 2; & n \equiv 3 \pmod{4}, \\ \lceil \frac{n}{2} \rceil - 1; & \text{otherwise.} \end{cases}$$

Proof. Suppose that the first move of Staller is v_2 , then x_2 is dominated. If Dominator chooses x_2 , then $\gamma_g(Q_{n+1}|N[v_2]) = 1 + \gamma'_g(Q_{n+1}|N[v_2, x_2])$. The corresponding residual graph is P'_r , where $r \geq 0$ and $r = n - 3$. We have that $\gamma'_g(P'_{n-3}) = \gamma'_g(Q_{n+1}|N[v_2, x_2])$. There are four cases according to the value of $n \bmod 4$. For $k \geq 1$, by Lemmas 2.6 and 2.7, we have that

$$\begin{aligned} \gamma'_g(P'_{4(k-1)+1}) &= \gamma'_g(P''_{4(k-1)+1}) \\ &= \gamma_g(P''_{4(k-1)+1}) \\ &= 2(k-1) + 1 = 2k - 1, \end{aligned}$$

$$\begin{aligned} \gamma'_g(P'_{4(k-1)+2}) &= \gamma'_g(P''_{4(k-1)+2}) \\ &= \gamma_g(P''_{4(k-1)+2}) + 1 = 2k, \end{aligned}$$

$$\begin{aligned} \gamma'_g(P'_{4(k-1)+3}) &= \gamma'_g(P''_{4(k-1)+3}) \\ &= \gamma_g(P''_{4(k-1)+3}) + 1 = 2k, \end{aligned}$$

$$\begin{aligned} \gamma'_g(P'_{4k}) &= \gamma'_g(P''_{4k}) \\ &= \gamma_g(P''_{4k}) = 2k. \end{aligned}$$

□

We assume that the Dominator first move is x_4 in the Staller-start game.

Lemma 4.3. *In the Staller-start game, if the Staller first move is v_2 and the Dominator first move is x_4 , then*

$$\gamma'_g(Q_{n+1}|N[v_2, x_4]) = \begin{cases} \lceil \frac{n}{2} \rceil - 2; & n \equiv 1 \pmod{4}, \\ \lceil \frac{n}{2} \rceil - 1; & \text{otherwise.} \end{cases}$$

Proof. Suppose that the Staller first move is v_2 and the Dominator first move is x_4 . Then $\gamma_g(Q_{n+1}|N[x_4]) = 1 + \gamma'_g(Q_{n+1}|N[v_2, x_4])$. The corresponding residual graph is $P'_1 \cup P'_r$, where $r \geq 0$ and $r + 1 = n - 4$. We have that $\gamma'_g(P'_1 \cup P'_{n-3}) = \gamma'_g(Q_{n+1}|N[v_2, x_4])$. There are four cases according to the value of $n \bmod 4$. For $k \geq 1$, by Lemmas 2.6 and 2.7, we have that

$$\begin{aligned} \gamma'_g(P'_1 \cup P'_{4(k-2)+3}) &= \gamma_g(P''_1) + \gamma_g(P''_{4(k-2)+3}) + 1 \\ &= 2 + \gamma_g(P''_{4(k-2)+3}) \\ &= 2 + 2(k-2) + 2 - 1 \\ &= 2 + 2k - 4 + 1 = 2k - 1, \end{aligned}$$

$$\begin{aligned}
\gamma'_g(P'_1 \cup P'_{4(k-1)}) &= \gamma_g(P''_1) + \gamma_g(P''_{4(k-1)}) \\
&= 1 + \gamma_g(P''_{4(k-1)}) \\
&= 1 + 2k - 2 = 2k - 1,
\end{aligned}$$

$$\begin{aligned}
\gamma'_g(P'_1 \cup P'_{4(k-1)+1}) &= \gamma_g(P''_1) + \gamma_g(P''_{4(k-1)+1}) \\
&= 1 + \gamma_g(P''_{4(k-1)+1}) \\
&= 1 + 2k - 2 + 1 = 2k,
\end{aligned}$$

$$\begin{aligned}
\gamma'_g(P'_1 \cup P'_{4(k-1)+2}) &= \gamma_g(P''_1) + \gamma_g(P''_{4(k-1)+2}) + 1 \\
&= 2 + \gamma_g(P''_{4(k-1)+2}) \\
&= 2 + 2k - 2 + 1 = 2k + 1.
\end{aligned}$$

□

Proof of Theorem 1.3. Note that for $u \in Q_{n+1}$,

$$\gamma_g(Q_{n+1}|N[u]) = 1 + \min_{v \in V(Q_{n+1})-u} \{\gamma'_g(Q_{n+1}|N[u, v])\}.$$

From Lemmas 4.2 and 4.3, for $n \equiv 3 \pmod{4}$,

$$\gamma'_g(Q_{n-1}|N[v_2, x_2]) = \left\lceil \frac{n}{2} \right\rceil - 2 < \left\lceil \frac{n}{2} \right\rceil - 1 = \gamma'_g(Q_{n-1}|N[v_2, x_4]).$$

So for Dominator, choosing x_2 is better than choosing x_4 .

□

From our analysis, we propose the following conjecture. In an optimal strategy of the Staller-start game, if the Staller first move is v_2 and the Dominator first move is x_{n-1} , then

$$\gamma'_g(Q_{n+1}) = 1 + \begin{cases} \left\lceil \frac{n}{2} \right\rceil; & n \equiv 1, 3 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil + 1; & \text{otherwise.} \end{cases}$$

5 A Dominator-Start Game Played on 1-sum of P_n and P_2

In this section, we analyze the game domination number on a graph T_{n+1} , which is a graph constructed from 1-sum of P_n and P_2 at x_k , for some $k = 2, \dots, n-1$, and v_1 , see figure 3. Then we find the upper bound of $\gamma_g(T_{n+1})$ by assuming that the Dominator first move is x_k . By applying Lemmas 2.6 and 2.7, we obtain the following lemma.

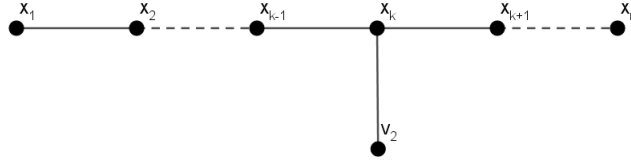


Figure 3: Graph T_{n+1}

Lemma 5.1. *If $k \equiv 0 \pmod{4}$, then*

$$\gamma_g(T_{n+1}) \leq \begin{cases} \left\lceil \frac{n}{2} \right\rceil; & n \equiv 0 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil + 1; & n \equiv 1 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil + 2; & n \equiv 2, 3 \pmod{4}. \end{cases}$$

If $k \equiv 1 \pmod{4}$, then

$$\gamma_g(T_{n+1}) \leq \begin{cases} \left\lceil \frac{n}{2} \right\rceil; & n \equiv 0 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil + 1; & n \equiv 1, 2 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil + 2; & n \equiv 3 \pmod{4}. \end{cases}$$

If $k \equiv 2 \pmod{4}$, then $\gamma_g(T_{n+1}) \leq \left\lceil \frac{n}{2} \right\rceil + 1$.

If $k \equiv 3 \pmod{4}$, then

$$\gamma_g(T_{n+1}) \leq \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1; & n \equiv 0, 1 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil + 2; & n \equiv 2, 3 \pmod{4}. \end{cases}$$

Proof. We can easily check by hand for the case $n = 4$. Assume that $n \geq 5$. Suppose that Dominator chooses x_k in the first move, then the residual graph is $P'_r \cup P'_s$, where $r + s = n - 3$. We now consider the following cases of the residual graph according to the value of $n \pmod{4}$.

If $n \equiv 0 \pmod{4}$ or $n = 4j$, where $j > 0$, there are two cases: 1) $P'_{4l} \cup P'_{4m+1}$ where $l + m + 1 = j$ and $l, m \geq 0$; and 2) $P'_{4l+2} \cup P'_{4m+3}$ where $l + m + 2 = j$ and $l, m \geq 0$.

If $n \equiv 1 \pmod{4}$ or $n = 4j + 1$, where $j > 0$, there are three cases: 1) $P'_{4l+3} \cup P'_{4m+3}$ where $l + m + 2 = j$ and $l, m \geq 0$; 2) $P'_{4l} \cup P'_{4m+2}$ where $l + m + 1 = j$ and $l, m \geq 0$; and 3) $P'_{4l+1} \cup P'_{4m+1}$ where $l + m + 1 = j$ and $l, m \geq 0$.

If $n \equiv 2 \pmod{4}$ or $n = 4j + 2$, where $j > 0$, there are two cases: 1) $P'_{4l} \cup P'_{4m+3}$ where $l + m + 1 = j$ and $l, m \geq 0$; and 2) $P'_{4l+1} \cup P'_{4m+2}$ where $l + m + 1 = j$ and $l, m \geq 0$.

If $n \equiv 3 \pmod{4}$ or $n = 4j + 3$, where $j > 0$, there are three cases: 1) $P'_{4l} \cup P'_{4m}$ where $l + m = j$ and $l, m \geq 0$; 2) $P'_{4l+1} \cup P'_{4m+3}$ where $l + m + 1 = j$ and $l, m \geq 0$; and 3) $P'_{4l+2} \cup P'_{4m+2}$ where $l + m + 1 = j$ and $l, m \geq 0$.

By applying Lemmas 2.6 and 2.7 to consider each cases, the result follows. $\square$

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ANALYZING THE EFFICIENCY OF THE TRAFFIC LIGHT PHASING SYSTEM OF THE TRAFFIC CIRCLE AT VICTORY MONUMENT (THAILAND) WITH GRAPH THEORY

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Abstract

The traffic circle at Victory Monument (Thailand) is the center of the traffic network and one of the most important transportation hub in Bangkok. There are three roads intersecting at the monument, Phaya Thai Road, Ratchawithi Road, and Phahon Yothin Road; and traffic congestion in rush hours is the most serious problem of this area. Especially on Ratchawithi Road, an incompatible between the phasing traffic lights at the traffic circle and the crosswalk can causes a traffic jam. Graph theory can be applied to formulate the system of traffic streams approaching this traffic circle into a compatibility graph. To investigate the efficiency of the traffic light phasing system at this traffic circle, a corresponding linear programming problem is constructed to find an optimal green light assignment that minimizes the total amount of waiting time. A method to relieve the traffic jam on Ratchawithi Road is also provided.

Key words: graph theory, compatibility graph, traffic light phasing system, linear programming problem.

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1 Introduction

A traffic light is one of the most effective and flexible active control of traffic and is widely used in several cities worldwide. The benefits of traffic light are reducing the various type of accidents, increasing the capacity of the intersection and providing the orderly movement of traffic flows. However, it also has the disadvantage. For example, it may cause the excessive delays or the large stopped delays because of the waiting for a traffic light. A complexity in the design and implementation make not only the cost of traffic light but also the cost of installing and maintaining. Traffic lights are installed when the traffic engineers guarantee that they are as beneficial as possible. In Thailand, we always encounter the traffic problems or traffic jams, especially in the city areas. The traffic congestion statistical history in year 2016 by TomTom traffic index [6] shows that Bangkok was the second place of the traffic congestion in the world rank and the first place in the Asia rank. From this data, in two years the Bangkok drivers spent an average 64.1 hours stuck in congestion. This is a direct effect on increasing of fuel consumption and air pollution.

Recently, the surrounding area of the Victory Monument is the biggest and the most important transportation hub in Bangkok. This area consists of four junctions, and three roads intersecting: Phaya Thai Road, Ratchawithi Road, and Phahon Yothin Road, see Figure 1. There are more than ten thousand people use these roads because of the fully located of many facilities, such as shopping malls, hospitals, and education institutions. So the traffic light phasing system is necessary to control the traffic circle at Victory monument. Nowadays there is a traffic jam although the traffic light system is used. Consequently, we are interested in using graph theoretical concepts to analyze feasible green light assignments of the traffic light phasing system at the monument. We investigate the efficiency of the assignments during rush hour in the morning from 6:16 to 10:00. We also find a suitable green light assignment in a situation that the phasing traffic lights at the traffic circle are incompatible with the crosswalk on Ratchawithi Road.

A *graph* G consists of a finite set of V of *vertices* and a set E of ordered pairs of vertices called *edges*. We shall represent a graph $G = (V, E)$ by drawing the vertices as points and drawing a curve from u to v if and only if the ordered pair (u, v) is an edge. Graph theory is a tool for formulating problems by defining fundamental interrelationships. The concepts of graph theory are used in many studies involving the traffic system. D. Dave [2] and S. M. Hosseini [3] study a method for solving traffic problems. The compatibility graph corresponding to the problem and circular arc graphs have been introduced. Then they apply the method to reduce traffic problem to the solution of LP problems. A.K. Baruah [1] conducts a research on the connectivity of compatibility graph of a traffic intersection that can be used to study the most efficient route or the traffic control system to direct the traffic flow to its maximum capacity by using the minimum number of edges or the minimum number of vertices. The

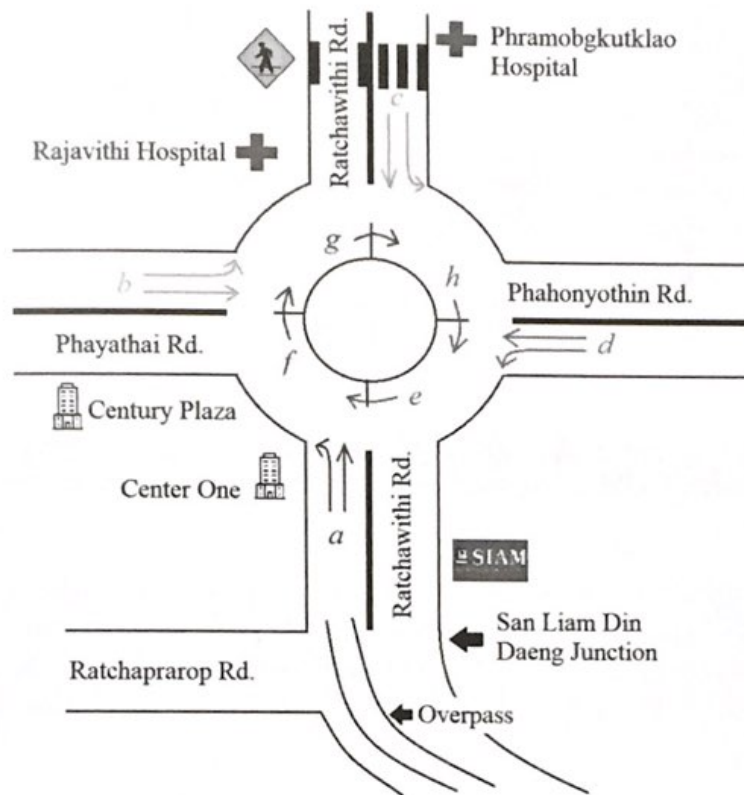


Figure 1: A diagram of the Victory Monument area.

application of this result in the traffic control problem at an arbitrary intersection is to minimize the waiting time of the traffic participants and the cost of locating the sensors in order to collect traffic data. S. Tanveer [5] proposes the application of graph connectivity (vertex or edge) in traffic management problems in an efficient way by minimizing the waiting time of the traffic participants and collecting traffic data by located sensors in an appropriate area. E.K. Setiawan [4] creates a mathematical model in the form of the total time of all flows function by determining conditions, such as minimizing green light of each flow affected by the volume and the weight of the traffic flow.

In what follow, it will be convenient to study some types of graphs which relate to our work. A *subgraph* of a graph $G = (V, E)$ is a graph such that its vertex set and edge set are subsets of V and E , respectively. A *spanning subgraph* of G is a subgraph with vertex set V . A *clique* in G is a subgraph which every vertex is joined to every other vertex. Let $F = \{S_1, S_2, \dots, S_p\}$ be a nonempty family of sets. The *intersection graph* of F is a graph with vertex set F such that two sets S_i and S_j , where $i \neq j$, are adjacent if and only if $S_i \cap S_j \neq \emptyset$. The *circular arc graph* is an intersection graph of a set of arcs on a given circle, where there is an edge joining two arcs if and only if they are overlapping, see Figure 2b.

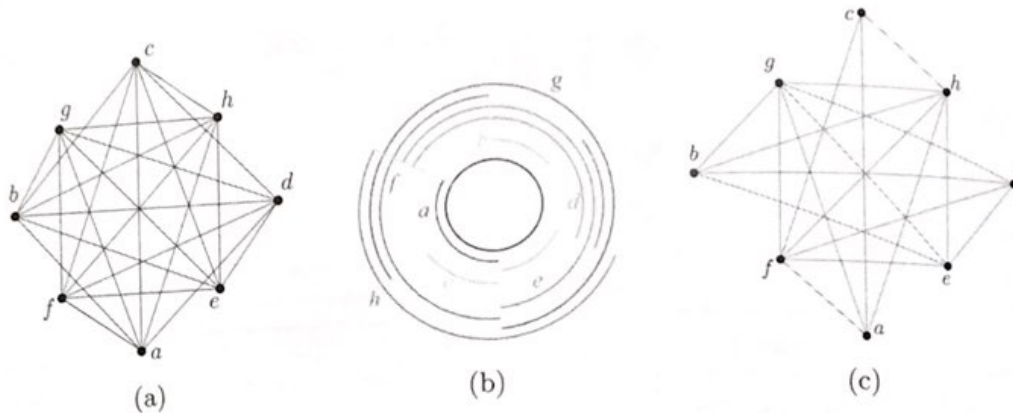


Figure 2: (a) a compatibility graph G , (b) a feasible green light assignment, and (c) the corresponding circular arc graph H of the assignment.

We close this section by providing an outline of the rest of the paper. In the next section we explain how to formulate a traffic light phasing system by using the graph theoretical concepts. Then the efficiency of the traffic light phasing system at Victory Monument is analyzed. Finally, we shall also discuss our result.

2 Traffic Light Phasing Problem Formulation

From the diagram of Victory Monument area in Figure 1, there are four traffic streams (a, b, c, d) approaching the traffic circle and four traffic streams (e, f, g, h) moving around the traffic circle. Two traffic streams are *compatible* with each other, if they can move at the same time without causing an accident. For example, traffic stream a from Sam Liam Din Daeng and f in the traffic circle can get green lights at the same time. To formulate a graph G by the compatibility relationship, each stream is represented by a vertex of G , and two vertices are joined by an edge if and only if their corresponding streams are compatible. We call G a *compatibility graph* for the traffic circle, see Figure 2a. For example, traffic stream a is compatible with traffic streams b, c, d, f, g, h , but not e ; so a is joined by edges to b, c, d, f, g, h .

Every traffic stream is assigned a period of time for a green light in such a way that only its compatible traffic streams can moved at the same time. This creates a cycle consisting of green lights and red lights. We consider a sequence of green lights of traffic streams in the cycle, which is called the *feasible green light assignment*. The cycle can be denoted by a circle and each period of green light is an arc around the circle. Two arcs overlap if their corresponding traffic streams get green light at the same time. Then we can represent the feasible green light assignment by a circular arc graph. For example, form our data one of feasible green light assignments that traffic polices use at the Victory

Monument traffic circle consists of a consecutive ordering of six phases: 1) only traffic streams d, e, g, f getting green lights, 2) only b, d, e, g getting green lights, 3) only d, e, f, g getting green lights, 4) only b, e, g, h getting green lights, 5) only a, c, f, h getting green lights, 6) only a, f, g, h getting green lights. This assignment is shown in Figure 2b and its corresponding circular arc graph is graph H in Figure 2c. Notice that H is a spanning subgraph of G , and each phase corresponds to a 4-vertex clique in G , which is the largest clique in G .

In this research, we consider feasible green light assignments that polices use at the Victory Monument traffic circle from 6:16 am - 10:00 am. There are forty different feasible green light assignments used.

3 Analyzing the Traffic Light Phasing System

One way to tell the efficiency of a traffic light phasing is by analyzing the feasible green light assignment with the criteria of minimizing the total amount of waiting time of each traffic stream. This is equivalent to minimize the total amount of red light time in the cycle. From our data, there are eight different phases that polices use to generate a feasible green light assignment, and each of them corresponds to a 4-vertex clique in G ; $K_1 = \{a, f, g, h\}$, $K_2 = \{b, e, g, h\}$, $K_3 = \{c, e, f, h\}$, $K_4 = \{d, e, f, g\}$, $K_5 = \{a, c, f, h\}$, $K_6 = \{b, d, e, g\}$, $K_7 = \{a, b, g, h\}$, $K_8 = \{e, f, g, h\}$, see Table 1.

To construct an objective function of total red light time in a cycle, we denote a duration of clique K_i by d_i . We consider the cycle at 8:11 am to show the method. This cycle is generated by cliques K_1, K_2, K_4, K_5, K_6 . Since traffic stream a gets a red light during phases K_2, K_4 and K_6 , the total red light time of a is $d_2 + d_4 + d_6$. Then the total red light time of each traffic stream is as follows; $b : d_1 + d_4 + d_5$, $c : d_1 + d_2 + d_4 + d_6$, $d : d_1 + d_2 + d_5$, $e : d_1 + d_5$, $f : d_2 + d_6$, $g : d_5$, $h : d_4 + d_6$. The total red light time of the cycle at 8:11 is given by

$$4d_1 + 4d_2 + 4d_4 + 4d_5 + 4d_6. \quad (3.1)$$

From our data, the green light time for each stream is at least 20 seconds, and the duration of the cycle is 275 seconds. Thus, we want to minimize equation (3.1) subject to the following constraints

$$\begin{aligned} d_1 + d_5 &\geq 20, & d_2 + d_6 &\geq 20, \\ d_5 &\geq 20, & d_4 + d_6 &\geq 20, \\ d_2 + d_4 + d_6 &\geq 20, & d_1 + d_4 + d_5 &\geq 20, \\ d_1 + d_2 + d_4 + d_6 &\geq 20, & d_1 + d_2 + d_5 &\geq 20, \\ d_1 + d_2 + d_4 + d_5 + d_6 &= 275, & d_1, d_2, d_4, d_5, d_6 &\geq 0. \end{aligned}$$

We solve this linear programming problem and obtain the minimum total red light time 1,100 seconds, where one of solutions is $d_1 = 0$, $d_2 = 0$, $d_4 = 0$, $d_5 = 20$, and $d_6 = 255$.

Table 1: Feasible green light assignments from 6:16 am to 10 am generated from cliques $K_1, K_2, \dots, K_8$.

Time	Sequence of cliques in each cycle	Duration of each cycle (sec)	Total waiting time (sec)
06:16	$K_4, K_6, K_2, K_1, K_5, K_7, K_5$	300	1200
06:20	K_4, K_6, K_2, K_5	305	1220
06:24	$K_4, K_6, K_2, K_5, K_7, K_5$	350	1400
06:30	$K_4, K_6, K_2, K_5, K_7, K_5$	260	1040
06:34	$K_4, K_6, K_2, K_5, K_1, K_5, K_7, K_1, K_5$	405	1620
06:42	$K_8, K_4, K_6, K_2, K_3, K_5, K_7, K_5$	375	1500
06:48	K_4, K_6, K_2, K_8, K_5	255	1020
06:53	$K_4, K_6, K_2, K_7, K_1, K_5$	256	1060
07:02	$K_2, K_6, K_3, K_5, K_7, K_1, K_5$	290	1160
07:07	$K_4, K_6, K_2, K_5, K_1, K_5, K_7, K_1, K_5$	290	1500
07:14	$K_4, K_6, K_2, K_5, K_1, K_5$	280	1120
07:19	K_6, K_2, K_1, K_5	160	640
07:22	$K_8, K_2, K_6, K_4, K_8, K_5, K_1, K_5$	360	1440
07:26	$K_8, K_2, K_6, K_4, K_8, K_5, K_1, K_7, K_5$	325	1300
07:31	$K_4, K_6, K_2, K_1, K_5, K_1, K_5$	305	1220
07:35	$K_4, K_6, K_2, K_5, K_1, K_5, K_1, K_5, K_7, K_1, K_5$	435	1740
07:42	$K_3, K_6, K_4, K_3, K_5, K_7, K_1, K_5$	360	1440
07:48	K_4, K_6, K_2, K_5, K_7	280	1120
07:50	K_2, K_1, K_5	120	480
07:53	K_6, K_2, K_3, K_5	185	740
07:56	K_4, K_6, K_2, K_1, K_5	180	720
08:02	K_4, K_6, K_2, K_1, K_5	225	900
08:05	$K_8, K_6, K_2, K_3, K_5, K_7, K_5$	380	1520
08:11	$K_4, K_6, K_4, K_2, K_5, K_1$	275	1100
08:15	K_2, K_5	150	600
08:18	K_6, K_2, K_3, K_5	220	880
08:22	$K_4, K_6, K_2, K_4, K_2, K_1, K_5, K_7, K_1, K_5$	535	2140
08:35	K_4, K_6, K_8, K_4, K_5	240	960
08:39	K_2, K_5	115	460
08:41	$K_6, K_4, K_3, K_5, K_7, K_1, K_5$	235	940
08:45	K_4, K_6, K_4, K_3, K_5	165	660
08:48	K_4, K_6, K_2, K_1, K_5	165	660
08:51	K_2, K_7, K_1, K_5	110	330
09:00	$K_4, K_6, K_4, K_8, K_5, K_7, K_1, K_5$	260	1040
09:04	K_4, K_6, K_2, K_1, K_5	225	900
09:08	K_4, K_6, K_4, K_3, K_5	160	640
09:11	K_4, K_6, K_8, K_4, K_5	220	880
09:15	K_2, K_5	105	460
09:25	K_4, K_6, K_4, K_3, K_5	160	640
09:33	K_2, K_7, K_1, K_5	120	480
09:35	$K_6, K_4, K_3, K_5, K_7, K_1, K_5$	280	1120
09:40	K_4, K_6, K_2, K_1, K_5	210	840
09:44	K_4, K_6, K_4, K_3, K_5	160	640
09:47	K_4, K_6, K_8, K_4, K_5	240	960
09:52	$K_4, K_6, K_2, K_1, K_5, K_7, K_5$	250	1000
10:00	$K_4, K_6, K_2, K_5, K_7, K_5$	320	1280

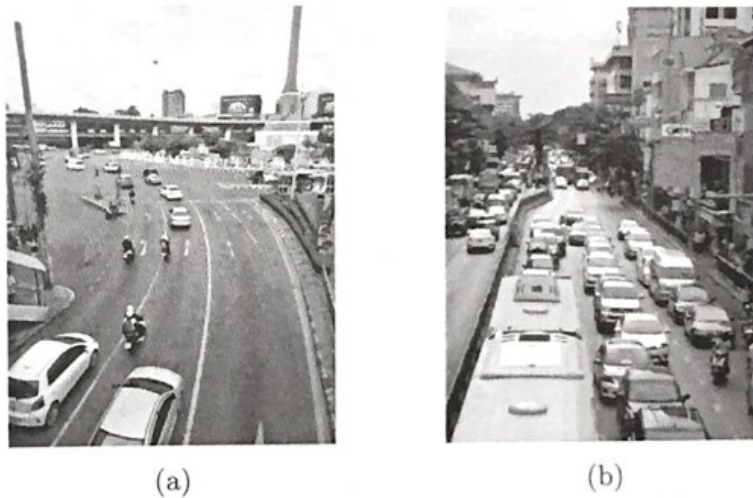


Figure 3: The real situation when traffic stream c gets a green light and polices allow people to cross Ratchawithi Road.

By applying this method to analyze other cycles and testing the duration polices use, we obtain the total red light time of each cycle shown in the last column of Table 1, and we found that each of these times is the minimum of its corresponding cycle.

4 A Special Case on Ratchawithi Road

From our observation at the Victory Monument area from 6:16 am - 8:11 am, there is an interesting problem about the crosswalk in front of the Rajavithi Hospital on Ratchawithi Road, Figure 1. Crossing the road depends on the polices signal, which is at the same time when traffic stream c gets a green light, see Figure 3a. This causes the delay on the traffic flow and leads to a traffic jam on the road. So, we propose a solution for the problem by considering the crosswalk as a new traffic stream and using the method we discussed to analyze.

Traffic stream i are compatible with almost every traffic streams, except b , c and f . The new compatibility graph is shown in Figure 4a. We obtain a new phase defined by clique $K_9 = \{i, d, e, g\}$. To show the method we consider the cycle at 8:11 am with the new circular arc graph in Figure 4b. We observe that cycle at 8:11 am with the new circular arc graph in Figure 4b. We observe that cliques $K_4 = \{d, e, f, g\}$ and $K_6 = \{b, d, e, g\}$ have nonempty intersections with K_9 . From our observation, traffic stream on Phayathai Road is not heavy in the morning, and it is safe for people using the crosswalk. So we choose K_6 to share the duration with K_9 by adding the constraint $d_6 + d_9 = 60$.

Then the total red light time of each traffic stream is as follows; $a : d_2 + d_4 + d_6 + d_9$, $b : d_1 + d_4 + d_5 + d_9$, $c : d_1 + d_2 + d_4 + d_6 + d_9$, $d : d_1 + d_2 + d_5$, $e : d_1 + d_5$, $f : d_2 + d_6 + d_9$, $g : d_5$, $h : d_4 + d_6 + d_9$. The total red light time of

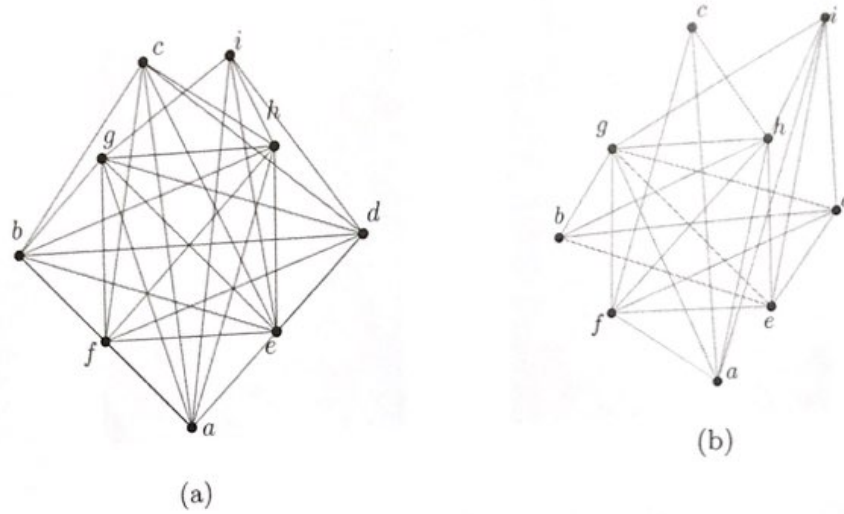


Figure 4: (a) the new compatibility graph adding crosswalk i , and (b) the corresponding intersection graph of the new cycle at 8:11 am.

the new cycle is given by

$$4d_1 + 4d_2 + 4d_4 + 4d_5 + 4d_6 + 5d_9. \quad (4.1)$$

We set the minimum green light time for each stream equal 20 seconds, and use the same duration for the cycle 275 seconds. Then we minimize equation (4.1) subject to the following constraints

$$\begin{aligned} d_1 + d_5 &\geq 20, & d_2 + d_6 &\geq 20, \\ d_5 &\geq 20, & d_4 + d_6 + d_9 &\geq 20, \\ d_2 + d_4 + d_6 + d_9 &\geq 20, & d_1 + d_4 + d_5 &\geq 20, \\ d_1 + d_2 + d_4 + d_6 + d_9 &\geq 20, & d_1 + d_2 + d_5 &\geq 20, \\ d_1 + d_2 + d_4 + d_5 + d_6 + d_9 &= 275, & d_1 &\geq 20, \\ d_2 &\geq 50, & d_4 &\geq 55, \\ d_5 &\geq 90, & d_6 &\geq 20, \\ d_9 &\geq 20, & d_6 + d_9 &= 60. \end{aligned}$$

The solution of this linear programming problem is 1,120 seconds, where one of solutions is $d_1 = 20$, $d_2 = 50$, $d_4 = 55$, $d_5 = 90$, $d_6 = 40$, and $d_9 = 20$. Notice that the duration of K_6 is decreased by 20 seconds; however, it is compensated by the safety of people using the crosswalk. From our analysis on all cycles from 6:16 am to 8:11 am, this method decreases the duration of K_6 and increases the total red light time by 20 seconds.

5 Concluding Remarks

The result shows that the traffic light phasing system of the Victory Monument traffic circle is efficient in the aspect of the total waiting time; however, a traffic jam still occurs seriously. From our observation, in the morning, a (from Sam Liam Din Daeng) is a heavily traveled traffic stream and has a traffic jam for a long period of time, while all traffic streams inside the traffic circle (e, f, g, h) and all traffic streams moving out from the traffic circle have light traffics with smoothly flowing. This reflects an inefficiency of phasing management. We now analyze this problem to come up with a solution that can relieve the problem with reasonable effect on other traffic streams.

For the safety reason of people who use the crosswalk on Ratchawithi Road, it is worth to study the possibility of installing a network linking the traffic light phasing system and the crosswalk light.

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