



รายงานวิจัยฉบับสมบูรณ์

โครงการ

การควบคุมแบบ H_∞ สำหรับระบบโครงข่ายประสาท
ที่มีตัวหน่วงขึ้นกับเวลา

H_∞ state estimation control for neural networks
with time-varying delay

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หัวหน้าโครงการวิจัย

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ในงานวิจัยนี้ได้ศึกษา การควบคุม H_∞ แบบจำกัดเวลา สำหรับระบบไม่เชิงเส้นที่มีตัวหน่วงขึ้นกับเวลา ซึ่งได้อาศัยการปรับปรุงฟังก์ชันนัลของไลปูนอฟรวมกับเทอมของการอินทิกรัล การประยุกต์ใช้เมตริกซ์บนเงื่อนไขแบบนูนกำลังสอง รวมกับอสมการของเวอร์ทิงเจอร์ และการใช้ประโยชน์จากอสมการอินทิกรัล เราได้สร้างเงื่อนไขที่พอเพียงสำหรับขอบเขตของการจำกัดเวลา โดยงานวิจัยนี้เป็นการปรับปรุงทำให้ผลลัพธ์ที่ได้เกิดการหยึดหยุ่นของเงื่อนไขมากขึ้น กล่าวคือ ขอบเขตบนของอนุพันธ์ของตัวหน่วงกว้างขึ้นและไม่จำเป็นต้องน้อยกว่า 1 หลังจากนั้นเราได้ออกแบบตัวควบคุมแบบ H_∞ ย้อนกลับสำหรับระบบ ซึ่งแสดงในเทอมของอสมการเมตริกซ์เชิงเส้น และในตอนท้ายของการวิจัยเราได้แสดงตัวอย่างการนำไปใช้ประโยชน์และประสิทธิภาพของผลลัพธ์ที่ได้

คำหลัก: การควบคุม H_∞ แบบจำกัดเวลา, ระบบไม่เชิงเส้น, ตัวหน่วงขึ้นกับเวลา, อสมการเมตริกซ์เชิงเส้น, ฟังก์ชันนัลไลปูนอฟ

Abstract

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In this manuscript, we consider the finite-time H_∞ control for nonlinear systems with time-varying delay. With the assistance of a novel Lyapunov-Krasovskii functional which includes some integral terms, a matrix-based on quadratic convex approach, combined with Wirtinger inequalities and some useful integral inequalities, a sufficient condition of finite-time boundedness is established. A novel feature presents in this paper is that the restriction which is necessary for the upper bound derivative is not restricted to less than 1. Further a H_∞ controller is designed via memoryless state feedback control and a new sufficient conditions for the existence of finite-time H_∞ state feedback for the system are given in terms of linear matrix inequalities (LMIs). At the end, some numerical examples with simulations are given to illustrate the effectiveness of the obtained result.

Keywords: finite-time H_∞ control; nonlinear system; time-varying delay; linear matrix inequalities (LMIs); Lyapunov-Krasovskii functional (LKF)

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CHAPTER 1

Executive Summary

1 Introduction

The occurrence of time delays is an important fact in many of the networking and processing control systems. Such delays can have the capacity to destabilize the control systems and also make some crucial disintegration in the performance of the closed-loop systems, see the references cited therein [1-7]. While modeling a real control system, the existence of time delays is always taken to be a time-varying one that satisfies the condition $d_1 \leq d(t) \leq d_2$ and d_1 which is not necessarily restricted to be 0. In recent years, the study on finite-time stability (FTS) has increased the research interest from various researches around the world due to the wider applications in mathematical control theory, which has been studied by different approaches in various kinds of systems, see for instance [8-15]. To this extent, the author Dorato in [8], explained the fundamental concepts of stability theory of dynamical systems in finite-time sense. Generally, the given system leads to be finite-time stable if the considered state of the system should be within the bounded limit for a fixed interval of time. From this, one can observe that the concept of finite-time stability mainly attracts the boundedness of a system during a fixed interval of time period. Some of the exciting results for finite-time stability and stabilization with the existence of time-delay have been obtained in [8-12]. Moreover, in some practical systems, there is a need to outline the system that guarantees a maximum H_∞ performance rather than the finite-time stability. Hence this motivates us to concentrate on the present study of research.

On the other hand, the study on H_∞ control problem will make a sense in reducing the consequences of the external disturbances from both inside and outside the system. The main theme of the H_∞ problem is to design a controller from outside the system and to obtain the robust stability (i.e. to minimize the errors). Also this will result in minimizing the guaranteeing disturbance attenuation level γ in the H_∞ sense for the system. Hence this finite-time H_∞ control concerns in the design of feedback controller which ensures the FTS of the closed-loop system and guarantees a maximum H_∞ performance bound.

Recently, the authors [10, 11, 16, 17] have enhanced the results on finite time stability and H_∞ performance analysis. In [10], finite-time H_∞ control for a continuous system with norm-bounded disturbance has been studied but a continuous system is not a nonlinear system. Robust finite-time H_∞ control of linear time-varying delay systems with bounded control has been considered in [11] based on Riccati Equations. In [12], the problem of robust finite-time stabilization with guaranteed cost control was studied based on the Lyapunov functional method and generalized Jensen integral inequality. These techniques allow us to design the state feedback controllers which stabilize the closed-loop system in the finite time. In [18], authors used an improved Lyapunov-Krasovskii functional (LKF) with triple-integral terms, augment terms and convex

combination technique to show the effectiveness of the obtained results [19] developed a novel problem on time-varying delayed nonlinear systems with finite-time stability and stabilization by employing the integral inequality and some free fuzzy by weighting metrics, which are less conservative than other existing ones. In [20], delay-dependent finite-time stability criteria for an uncertain continuous-time system with time varying delays has been studied but a continuous-time system is not a nonlinear system and without H_∞ performance analysis. In [21], improved results on delay-dependent H_∞ control for uncertain systems with time-varying delays have been considered by using bounding techniques for some cross-term of the LKF method and the free-weighting matrix method.

Several approaches that reduce the conservatism for the system with time delay have been reported in the literature. They are namely an appropriate Lyapunov-Krasovskii functional method by using bounding techniques while finding the time-derivative, delay decomposition approach; free weighting matrices approach and reciprocally convex optimization techniques, see [21-24]. Of all the above mentioned approaches, a novel method to reduce the conservatism is matrix-based quadratic convex approach. This approach will gives a better maximum allowable upper bound for time-varying delay over some existing ones, see for references [25-27].

So with the intuition from the above evidences, in this paper, we have followed a matrix-based quadratic convex approach to obtain a better maximum bound value. This is the first time that we have incorporated this method to study the finite-time H_∞ problem for the considered control system with time-varying delay. Further, the purposes of this paper are given as follows:

- I. We consider some new Lyapunov-Krasovskii functional which has not been considered yet in stability analysis of finite-time H_∞ control. The new Lyapunov-Krasovskii functional includes some integral terms of the form $\int_{t-h}^t (h-t-s)^j \dot{x}^T(s) R_j \dot{x}(s) ds$ ($j = 1, 2$) which the integrands are polynomial multiplied by $\dot{x}^T(s) R_j \dot{x}(s) ds$ ($j = 1, 2$) and one may estimate an upper bound of the integral by employing some techniques from [22,25], the matrix based quadratic convex approach, the use of a tighter bounding technique and useful integral inequality such as Wirtinger inequality.
- II. Lyapunov-Krasovskii with the matrix based quadratic convex approach is introduced to formulate finite-time stability criteria and H_∞ performance level where the time-varying delay satisfies $0 \leq d_1 \leq d(t) \leq d_2$, $\mu_1 \leq \dot{d}(t) \leq \mu_2$. Moreover, the restriction of upper bound derivative is not necessary restricted less than 1 compared with [20]
- III. Two numerical examples are given to demonstrate the effectiveness of theoretical result.

2 Problem statement

In this section, we consider a system with time-varying delay and control input as

$$\left. \begin{aligned} \dot{x}(t) &= Ax(t) + Dx(t-d(t)) + Bu(t) + Cw(t) + f(x(t), t) + g(x(t-d(t)), t) \\ z(t) &= Ex(t) + Gx(t-d(t)) + Fu(t) + f(x(t), t) + g(x(t-d(t)), t), \\ x(t_0 + \theta) &= \phi(\theta), \theta \in [-d_2, 0], (t_0, \phi) \in \mathcal{R}^+ \times \mathcal{C}([-d_2, t_0], \mathcal{R}^n), \end{aligned} \right\} \quad (1)$$

where $x(t) \in \mathcal{R}^n$ is the state; $u(t) \in \mathcal{R}^m$ is the control input, $w(t) \in \mathcal{L}_2([0, \infty], \mathcal{R}^r)$ is a disturbance input and $z(t) \in \mathcal{R}^s$ is the observation output. The delay $d(t)$ is time-varying

continuous function which satisfies

$$0 \leq d_1 \leq d(t) \leq d_2, \quad \mu_1 \leq \dot{d}(t) \leq \mu_2.$$

In this paper, we consider the nonlinear functions satisfying

$$\begin{aligned} f(x(t), t) &= (f_1(x_1(t), t), f_2(x_2(t), t), \dots, f_n(x_n(t), t))^T \in \mathcal{R}^n \\ g(x(t - d(t)), t) &= (g_1(x_1(t - d(t)), t), g_2(x_2(t - d(t)), t), \dots, g_n(x_n(t - d(t)), t))^T \in \mathcal{R}^n, \end{aligned}$$

$f, g : \mathcal{R}^n \times [-d_M, \infty) \rightarrow \mathcal{R}^n$ are nonlinear function satisfying the Lipschitz conditions; namely, there exist positive constants $\beta_1, \beta_2, \forall x, y \in \mathcal{R}^n$, such that

$$\begin{aligned} \|f(y, t) - f(x, t)\|^2 &\leq \beta_1 \|y(t) - x(t)\|^2, \\ \|g(y, t) - g(x, t)\|^2 &\leq \beta_2 \|y(t) - x(t)\|^2. \end{aligned}$$

We assume the following restrictions on the nonlinear perturbations

$$f^T(x(t), t)f(x(t), t) \leq \beta_1 x^T(t)x(t), \quad (2)$$

$$g^T(x(t - d(t)), t)g(x(t - d(t)), t) \leq \beta_2 x^T(t - d(t))x(t - d(t)). \quad (3)$$

The initial condition, $\phi(\cdot) := \sup_{t \in [-d_M, 0]} \{\|\phi(t)\|, \|\dot{\phi}(t)\|\}$. The disturbance is a continuous function satisfying

$$\exists k > 0 : \int_0^T w^T(t)w(t)dt \leq k. \quad (4)$$

Under the above assumptions on $d(\cdot), f(\cdot), g(\cdot)$ and the initial function $\phi(t)$, the system (2) has a unique solution $x(t, \phi)$ on $[0, T]$. For a prescribed scalar $\gamma > 0$, we define the performance index as

$$J(t) = \int_0^\infty [z^T(s)z(s) - \gamma^2 w^T(s)w(s)]ds. \quad (5)$$

The objective of this paper is to design a memoryless state feedback controller $u(t) = Kx(t)$.

3 Preliminaries

The following definition and lemma are necessary in the proof of the main results:

[?] The nonlinear system (2) where $w(t)$ is a perturbation satisfying (4). The system (2) is said to be finite-time bounded with respect to (c_1, c_2, T, R, d) with $0 < c_1 < c_2$, and $R > 0$, if

$$\sup_{-\tau_2 \leq s \leq 0} \{\phi^T(s)R\phi(s)\dot{\phi}^T(s)R\dot{\phi}(s)\} \leq c_1 \Rightarrow x^T(t)Rx(t) < c_2, \forall t \in [0, T]. \quad (1)$$

[?] The nonlinear system (2) is said to be finite-time H_∞ bounded with respect to $(c_1, c_2, T, R, d, \gamma)$ with $0 < c_1 < c_2, d \geq 0, \gamma > 0, R > 0$ and a memoryless state feedback controller $u(t) = Kx(t)$, following conditions should be satisfied:

(i) The zero solution of the closed-loop system, where $w(t) = 0$,

$$\dot{x}(t) = -(A - BK)x(t) + f(x(t)) + g(x(t - \tau(t))) + Cw(t), \quad (2)$$

is finite-time bounded.

(ii) Under zero-initial condition $\phi(t) = 0, \forall t \in [-d_2, 0]$ the output $z(t)$ satisfies

$$\int_0^T z^T(t)z(t)dt < \gamma^2 w^T(t)w(t)dt. \quad (3)$$

We introduce the following technical lemmas, which will be used in the proof of our results. [?] Let $P \in M^{n \times n}, R \in M^{n \times n}$ be symmetric positive definite matrices. We have

- (i) $\lambda_{\min}(P)(R) > 0, \lambda_{\max}(P)(R) > 0$ and $\lambda_{\min}(P)x^T x \leq x^T P x \leq \lambda_{\max}(P)x^T x, \forall x \in \mathcal{R}^n$
- (ii) $x^T x \leq \lambda_{\max}(R^{-1})x^T R x, \forall x \in \mathcal{R}^n$
- (iii) $x^T P x \leq \lambda_{\max}(P)\lambda_{\max}(R^{-1})x^T R x, \forall x \in \mathcal{R}^n$.

[?] For a given matrix $R > 0$, the following inequality holds for any continuously differentiable function $w : [a, b] \rightarrow \mathcal{R}^n$

$$\int_a^b \dot{w}^T(u)R\dot{w}(u)du \geq \frac{1}{b-a}(\Gamma_1^T R \Gamma_1 + 3\Gamma_2^T R \Gamma_2) \quad (4)$$

where $\Gamma_1 := \omega(b) - \omega(a), \Gamma_2 := \omega(b) + \omega(a) - \frac{2}{b-a} \int_a^b \omega(u)du$. : From the above inequality, it can be observed that the inequality in Lemma 3 gives a firm lower bound for $\int_a^b \dot{w}^T(u)R\dot{w}(u)du$ than Jensen's inequality since $3\Gamma_2^T R \Gamma_2 > 0$ for $\Gamma_2 \neq 0$. Hence it shows that the inequality (4) is improved than the Jensen's inequality. Before we introduce some useful integral inequalities, we denote

$$\left. \begin{aligned} \nu_1(t) &:= \frac{1}{d_2-d(t)} \int_{t-d_2}^{t-d(t)} y(s)ds \\ \nu_2(t) &:= \frac{1}{d(t)-d_1} \int_{t-d(t)}^{t-d_1} y(s)ds \\ \nu_3(t) &:= \frac{1}{d_1} \int_{t-d_1}^t y(s)ds. \end{aligned} \right\} \quad (5)$$

[?] For a given scalar $d_1 \geq 0$ and any $n \times n$ real matrices $Y_1 > 0$ and $Y_2 > 0$ and a vector $\dot{y} : [-d_1, 0] \rightarrow \mathcal{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any vector-valued function $\pi_1(t) : [0, \infty) \rightarrow \mathcal{R}^k$ and matrices $M_1 \in \mathcal{R}^{k \times k}$ and $N_1 \in \mathcal{R}^{k \times n}$ satisfying $\begin{bmatrix} M_1 & N_1 \\ N_1^T & Y_1 \end{bmatrix} \geq 0$,

$$\begin{aligned} \wp_1 &:= \int_{t-d_1}^t (d_1 - t + s) \dot{y}^T(s) Y_1 \dot{y}(s) ds \\ &\geq -\frac{d_1^2}{2} \pi_1^T(t) M_1 \pi_1(t) - 2d_1 \pi_1^T(t) N_1 [y(t) - \nu_3(t)], \\ \wp_2 &:= \int_{t-d_1}^t (d_1 - t + s)^2 \dot{y}^T(s) Y_2 \dot{y}(s) ds \end{aligned}$$

$$\geq d_1[y(t) - \nu_3(t)]^T Y_2[y(t) - \nu_3(t)]$$

where $\nu_3(t)$ is defined in (5). [?] Let $d(t)$ be a continuous function satisfying $0 \leq d_1 \leq d(t) \leq d_2$. For any $n \times n$ real matrix $R_2 > 0$ and a vector $\dot{y} : [-d_2, 0] \rightarrow \mathcal{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any $\phi_{i1} \in \mathcal{R}^q$ and real matrices $Z_i \in \mathcal{R}^{q \times q}$, $F_i \in \mathcal{R}^{q \times n}$ satisfying $\begin{bmatrix} Z_i & F_i \\ F_i^T & R_2 \end{bmatrix} \geq 0, (i = 1, 2)$,

$$\begin{aligned} - \int_{t-d_2}^{t-d_1} (d_2 - t + s) \dot{y}^T(s) R_2 \dot{y}(s) ds &\leq \frac{1}{2} (d_2 - d(t))^2 \phi_{11}^T Z_1 \phi_{11} + 2(d_2 - d(t)) \phi_{11}^T F_1 \phi_{12} \\ &\quad + \frac{1}{2} [(d_2 - d_1)^2 - (d_2 - d(t))^2] \phi_{21}^T Z_2 \phi_{21} \\ &\quad + 2\phi_{21}^T F_2 [(d_2 - d(t)) \phi_{22} + (d(t) - d_1) \phi_{23}] \end{aligned}$$

where $\phi_{12} := y(t - d(t)) - \nu_1(t)$, $\phi_{22} := y(t - d_1) - x(t - d(t))$, $\phi_{23} := y(t - d_1) - \nu_2(t)$ with $\nu_i(t) (i = 1, 2)$ being defined in (5). [?] Let $d(t)$ be a continuous function satisfying $0 \leq d_1 \leq d(t) \leq d_2$. For any $n \times n$ real matrix $R_1 > 0$ and a vector $\dot{y} : [-d_2, 0] \rightarrow \mathcal{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any $2n \times 2n$ real matrix S_1 satisfying $\begin{bmatrix} \tilde{R}_1 & S_1 \\ S_1^T & \tilde{R}_1 \end{bmatrix} \geq 0$,

$$-(d_2 - d_1) \int_{t-d_2}^{t-d_1} \dot{y}^T(s) R_1 \dot{y}(s) ds \leq 2\psi_{11}^T S_1 \psi_{21} - \psi_{11}^T \tilde{R}_1 \psi_{11} - \psi_{21}^T \tilde{R}_1 \psi_{21},$$

where $\tilde{R}_1 := \text{diag}\{R_1, 3R_1\}$; and

$$\psi_{11} := \begin{bmatrix} y(t - d(t)) - y(t - d_2) \\ y(t - d(t)) + y(t - d_2) - 2\nu_1(t) \end{bmatrix}, \quad \psi_{21} := \begin{bmatrix} y(t - d_1) - y(t - d(t)) \\ y(t - d_1) + y(t - d(t)) - 2\nu_2(t) \end{bmatrix}.$$

If we substitute $d_1 = 0$, in Lemma 3, then the inequality can be reduced and it is similar to that of the one in [?]. Also, the dimensions of the slack matrix variables of S_1 is $2n \times 2n$ compared to the dimension $2n \times 5n$ introduced in [?]. [?] Let χ_0, χ_1 and χ_2 be $m \times m$ real symmetric matrices and a continuous function d satisfy $d_1 \leq d \leq d_2$, where d_1 and d_2 are constants satisfying $0 \leq d_1 \leq d_2$. If $\chi_0 \geq 0$, then

$$\begin{aligned} &d^2 \chi_0 + d \chi_1 + \chi_2 < 0 (\leq 0), \forall d \in [d_1, d_2] \\ &\Leftrightarrow d_i^2 \chi_0 + d_i \chi_1 + \chi_2 < 0 (\leq 0), (i = 1, 2) \\ \text{or} \quad &d^2 \chi_0 + d \chi_1 + \chi_2 > 0 (\geq 0), \forall d \in [d_1, d_2] \\ &\Leftrightarrow d_i^2 \chi_0 + d_i \chi_1 + \chi_2 > 0 (\geq 0), (i = 1, 2). \end{aligned}$$

CHAPTER 2

Main result

1 Main Results

In this section, we firstly design a memoryless H_∞ feedback control for the addressed system (??) with the inclusion of time-varying delays and then obtain the finite-time stabilizability analysis conditions. Here we derive a novel finite-time stability for the system (??) by using the matrix-based quadratic convex approach with some integral inequalities in [?]. To achieve this status, we choose the following Lyapunov-Krasovskii functional:

$$V(t, x_t, \dot{x}_t) = V_1(t) + V_2(t) + V_3(t)$$

where x_t denotes the function $x(t)$ defined on the interval $[t - d_2, t]$. Setting $P_1 = P^{-1}$, $y(t) = P_1 x(t)$, $d_{21} := d_2 - d_1$ and

$$\begin{aligned} V_1(t) &:= e^{\alpha t} y^T(t) P y(t) + e^{\alpha t} \int_{t-d_1}^t \dot{y}^T(s) Q_0 \dot{y}(s) ds, \\ V_2(t) &:= e^{\alpha t} \int_{t-d_1}^t [y^T(t) \ y^T(s)] Q_1 [y^T(t) \ y^T(s)]^T ds + e^{\alpha t} \int_{t-d(t)}^{t-d_1} [y^T(t) \ y^T(s)] Q_2 [y^T(t) \ y^T(s)]^T ds \\ &\quad + e^{\alpha t} \int_{t-d_2}^{t-d(t)} [y^T(t) \ y^T(s)] Q_3 [y^T(t) \ y^T(s)]^T ds, \\ V_3(t) &:= e^{\alpha t} \int_{t-d_1}^t [(d_1 - t + s) \dot{y}^T(s) d_1 W_1 \dot{y}(s) + (d_1 - t + s)^2 \dot{y}^T(s) W_2 \dot{y}(s)] ds \\ &\quad + d_{21} e^{\alpha t} \int_{t-d_2}^{t-d_1} [(d_1 - t + s) \dot{y}^T(s) R_1 \dot{y}(s) + (d_1 - t + s)^2 \dot{y}^T(s) R_2 \dot{y}(s)] ds \end{aligned}$$

where $Q_j > 0$, ($j = 0, 1, 2, 3$), $W_1 > 0$, $W_2 > 0$, $R_1 > 0$, $R_2 > 0$ and P are real matrices to be determined. Before introducing the main result, the following notations of several matrix

variables are defined for simplicity: $\tilde{R}_1 = \text{diag}\{R_1, 3R_1\}$;

$$\Xi_2(d(t), \dot{d}(t)) := \Xi_{20} + [d(t) - d_1]\Xi_{21} + [d_2 - d(t)]\Xi_{22}, \quad (1)$$

$$\begin{aligned} \Xi_3(d(t)) &:= \tilde{\varphi}_1^T S_1 \tilde{\varphi}_2 + \tilde{\varphi}_2^T S_1^T \tilde{\varphi}_1 - \tilde{\varphi}_1^T \tilde{R}_1^T \tilde{\varphi}_1 + (d_2 - d(t))^2 (Z_1 - Z_2) + (d_2 - d(t))\Xi_{31} \\ &\quad + (d(t) - d_1)\Xi_{32} + d_{21}^2 Z_2 - \tilde{\varphi}_2^T \tilde{R}_1 \tilde{\varphi}_2, \end{aligned} \quad (2)$$

$$\begin{aligned} \hat{\Xi}_4 &:= d_1^2 Z_3 - \tilde{\varphi}_3^T \tilde{W}_1 \tilde{\varphi}_3 + e_9^T (d_1^2 W_1 + d_1^2 W_2) e_9 + 2d_1 N_3 (e_1 - e_7) + e_8^T (d_{21}^T R_1 + d_{21}^T R_2) e_8 \\ &\quad + 2d_1 (e_1 - e_7)^T N_3^T + e_1^T (AP + PA^T + (BY + Y^T B^T) + \varepsilon I) e_1 \\ &\quad + e_1^T (PD) e_2 + e_2^T (D^T P) e_1 + e_1^T (AP + YB^T) e_9 + e_9^T (PA^T + Y^T B) e_1 \\ &\quad + e_2^T (DP) e_9 + e_9^T (PD^T) e_2 + e_8^T (Q_0) e_8 + e_9^T (-2P + \varepsilon I + Q_0) e_9 \\ &\quad + e_{10}^T (I) e_{10} + e_{11}^T (I) e_{11} + e_1^T (PE^T) e_{12} + e_{12}^T (EP) e_1 \\ &\quad + e_2^T (PG^T) e_{13} + e_{13}^T (GP) e_2 + e_{14}^T (YF^T) e_{14} + e_{11}^T (Y) e_1 + e_{15}^T (-\gamma^2 I) e_{15}, \end{aligned} \quad (3)$$

with $e_i \in \mathcal{R}^{n \times 15n}$ ($i = 1, 2, \dots, 15$) denoting the i -th row-block vector of the $14n \times 14n$ identity matrix $\tilde{W}_1 = \text{diag}\{W_1, 3W_1\}$; and

$$\begin{aligned} \Xi_{20} &:= [e_1^T \ e_3^T] (Q_2 - Q_1) [e_1^T \ e_3^T]^T \\ &\quad + \tau_1 [e_9^T \ 0] Q_1 [e_1^T \ e_7^T]^T + \tau_1 [e_1^T \ e_7^T] Q_1 [e_9^T \ 0]^T \\ &\quad - (1 - \dot{\tau}(t)) [e_1^T \ e_2^T] (Q_2 - Q_3) [e_1^T \ e_2^T]^T \\ &\quad - [e_1^T \ e_4^T] Q_3 [e_1^T \ e_4^T]^T + [e_1^T \ e_1^T] Q_1 [e_1^T \ e_1^T]^T \\ \Xi_{21} &:= [e_1^T \ e_6^T] Q_2 [e_9^T \ 0]^T + [e_9^T \ 0] Q_2 [e_1^T \ e_6^T]^T \\ \Xi_{22} &:= [e_1^T \ e_5^T] Q_3 [e_9^T \ 0]^T + [e_9^T \ 0] Q_3 [e_1^T \ e_5^T]^T \\ \Xi_{31} &:= 2N_1 (e_2 - e_5) + 2N_2 (e_3 - e_2) + 2(e_3 - e_2)^T N_2^T + 2(e_2 - e_5)^T N_1^T \\ \Xi_{32} &:= 2N_1 (e_3 - e_6) + 2(e_3 - e_6)^T N_1^T \\ \tilde{\varphi}_1 &:= \text{col}\{e_2 - e_4, e_2 + e_4 - 2e_5\} \\ \tilde{\varphi}_2 &:= \text{col}\{e_3 - e_2, e_3 + e_2 - 2e_6\} \\ \tilde{\varphi}_3 &:= \text{col}\{e_1 - e_3, e_1 + e_3 - 2e_7\}. \end{aligned}$$

Consider $\gamma > 0$. Then system (??) is finite-time H_∞ control with respect to $(c_1, c_2, T, R, d, \gamma)$ and satisfies $\|z(t)\|_2 < \gamma \|w(t)\|_2$ for all nonzero $w \in \mathcal{L}_2[0, \infty)$ if there exist positive definite matrices $P, Q_j > 0$, ($j = 0, 1, 2, 3$), W_1, W_2, R_1, R_2 , $S_1, Z_1, Z_2, Z_3, N_1, N_2, N_3$ and Y such that the following linear matrix inequalities (LMIs) hold

$$\begin{aligned} \begin{bmatrix} \tilde{R}_1 & S_1 \\ S_1^T & \tilde{R}_1 \end{bmatrix} &\geq 0, \quad \begin{bmatrix} Z_i & N_i \\ N_i^T & R_2 \end{bmatrix} \geq 0, \quad (i = 1, 2) \\ \begin{bmatrix} Z_3 & N_2 \\ N_3^T & W_2 \end{bmatrix} &\geq 0, \quad Z_1 \geq Z_2, \\ \left. \begin{aligned} \Xi_2(d_1, \mu_1) + \Xi_3(d_1) + \hat{\Xi}_4 &< 0 \\ \Xi_2(d_1, \mu_2) + \Xi_3(d_1) + \hat{\Xi}_4 &< 0 \\ \Xi_2(d_2, \mu_1) + \Xi_3(d_2) + \hat{\Xi}_4 &< 0 \\ \Xi_2(d_2, \mu_2) + \Xi_3(d_2) + \hat{\Xi}_4 &< 0 \end{aligned} \right\} &\quad (4) \end{aligned}$$

and

$$\frac{\alpha_2 c_1 + \gamma^2 k}{\alpha_1} \leq e^{-\alpha T} c_2 \quad (5)$$

For this problem, the feedback control is taken to be of

$$u(t) = YP^{-1}x(t), t \geq 0. \quad (6)$$

[Proof] By finding the time-derivative of V for the considered system (??), we obtain

$$\dot{V}_1 = 2e^{\alpha t} y^T(t) P \dot{y}(t) + \alpha e^{\alpha t} y^T(t) P y(t) + e^{\alpha t} \{ \dot{y}^T(t) Q_0 \dot{y}(t) - \dot{y}^T(t - d_1) Q_0 \dot{y}(t - d_1) \} + \alpha V_1(\cdot) \quad (7)$$

$$\begin{aligned} \dot{V}_2 = & e^{\alpha t} \{ [y^T(t) \ y^T(t)] Q_1 [y^T(t) \ y^T(t)]^T - [y^T(t) \ y^T(t - d_1)] Q_1 [y^T(t) \ y^T(t - d_1)]^T \\ & + \int_{t-d_1}^t 2[y^T(t) \ y^T(s)] Q_1 [\dot{y}(t)^T \ 0]^T ds + [y^T(t) \ y^T(t - d_1)] Q_2 [y^T(t) \ y^T(t - d_1)]^T \\ & - (1 - d(\dot{t})) [y^T(t) \ y^T(t - d(t))] Q_2 [y^T(t) \ y^T(t - d(t))]^T \\ & + 2 \int_{t-d(t)}^{t-d_1} [y^T(t) \ y^T(s)] Q_2 [\dot{y}(t)^T \ 0]^T ds - [y^T(t) \ y^T(t - d_2)] Q_3 [y^T(t) \ y^T(t - d_2)]^T \\ & + (1 - d(\dot{t})) [y^T(t) \ y^T(t - d(t))] Q_3 [y^T(t) \ y^T(t - d(t))]^T \\ & + \int_{t-d_2}^{t-d(t)} 2[y^T(t) \ y^T(s)] Q_3 [\dot{y}(t)^T \ 0]^T ds \} + \alpha V_2(\cdot) \end{aligned} \quad (8)$$

$$\begin{aligned} \dot{V}_3 = & e^{\alpha t} \{ \dot{y}^T(t) d_1 W_1 \dot{y}(t) + d_1^2 \dot{y}^T(t) W_2 \dot{y}(t) - \int_{t-d_1}^t \dot{y}^T(s) d_1 W_1 \dot{y}(s) ds \\ & - 2 \int_{t-d_1}^t (d_1 - t + s) \dot{y}^T(s) W_2 \dot{y}(s) ds + (d_{21})^2 \dot{y}^T(t - d_1) R_1 \dot{y}(t - d_1) \\ & + (d_{21})^2 \dot{y}^T(t - d_1) R_2 \dot{y}(t - d_1) - d_{21} \int_{t-d_2}^{t-d_1} \dot{y}^T(s) R_1 \dot{y}(s) ds \\ & - 2 \int_{t-d_2}^{t-d_1} (d_2 - t + s) \dot{y}^T(s) R_2 \dot{y}(s) ds \} + \alpha V_3(\cdot). \end{aligned} \quad (9)$$

From (??) and Cauchy inequality, we get the following equality:

$$-2\dot{x}^T(t) P_1 [\dot{x}(t) - Ax(t) - Dx(t - d(t)) - Bu(t) - Cw(t) - f(x(t), t) - g(x(t - d(t)), t)] = 0$$

we obtain the following

$$\begin{aligned} 0 = & -2\dot{x}^T(t) P_1 [\dot{x}(t) - Ax(t) - Dx(t - \tau(t)) - Bu(t) - Cw(t) - f(x(t), t) - g(x(t - d(t)), t)] \\ = & -2\dot{y}^T(t) P \dot{y}(t) + 2\dot{y}^T(t) A P y(t) + 2\dot{y}^T(t) D P y(t - d(t)) + 2\dot{y}^T(t) 2B Y y(t) + 2\dot{y}^T(t) C P w(t) \\ & + 2\dot{y}^T(t) P f(y(t)) + 2\dot{y}^T(t) P g(y(t - d(t))) \end{aligned} \quad (10)$$

From (??) and (??), we have $\dot{V}_1$ is

$$\begin{aligned} \dot{V}_1 = & 2e^{\alpha t} y^T(t) P \dot{y}(t) + \alpha e^{\alpha t} y^T(t) P y(t) + e^{\alpha t} \{ \dot{y}^T(t) Q_0 \dot{y}(t) - \dot{y}^T(t - d_1) Q_0 \dot{y}(t - d_1) \} \\ & - 2\dot{y}^T(t) P \dot{y}(t) + 2\dot{y}^T(t) A P y(t) + 2\dot{y}^T(t) D P y(t - d(t)) + 2\dot{y}^T(t) 2B Y y(t) + 2\dot{y}^T(t) C P w(t) \\ & + 2\dot{y}^T(t) P f(y(t)) + 2\dot{y}^T(t) P g(y(t - d(t))) \\ = & e^{\alpha t} \xi^T(t) \Xi_1 \xi(t) + \alpha V_1(\cdot). \end{aligned} \quad (11)$$

where $\xi(t) := \text{col}\{y(t), y(t-d(t)), y(t-d_1), y(t-d_2), \nu_1(t), \nu_2(t), \nu_3(t), \dot{y}(t-d_1), \dot{y}(t), f(y(t)), g(y(t-d(t)))\}$,

$$\begin{aligned} \Xi_1 := & 2e_1^T(P)e_9 + \alpha e_1^T(P)e_1 + e_9^T(Q_0)e_9 - e_8^T(Q_0)e_8 - 2e_9^T(P)e_9 + 2e_9^T(AP + 4BY)e_1 + 2e_9^T DPe_2 \\ & + 2e_9^T CPe_{12} + 2e_9Pe_{10} + 2e_9Pe_{11}. \end{aligned}$$

With the consideration of the three terms of $\dot{V}_2(t)$, we obtained the following inequalities:

$$\begin{aligned} \int_{t-d_1}^t 2[y^T(t) \ y^T(s)]Q_1[\dot{y}(t)^T \ 0]^T ds &\leq 2d_1[y^T(t) \ \nu_3^T]Q_1[\dot{y}^T(t) \ 0]^T, \\ \int_{t-d(t)}^{t-d_1} 2[y^T(t) \ y^T(s)]Q_2[\dot{y}(t)^T \ 0]^T ds &\leq 2(d(t) - d_1)[y^T(t) \ \nu_2^T]Q_2[\dot{y}^T(t) \ 0]^T \end{aligned}$$

and

$$\int_{t-d_2}^{t-d(t)} 2[y^T(t) \ y^T(s)]Q_3[\dot{y}(t)^T \ 0]^T ds \leq 2(d_2 - d(t))[y^T(t) \ \nu_1^T]Q_3[\dot{y}^T(t) \ 0]^T.$$

Therefore, the estimation of $\dot{V}_2(t)$ is estimated as

$$\begin{aligned} \dot{V}_2(t) &\leq \Xi_{20} + (d(t) - d_1)\Xi_{21} + (d_2 - d(t))\Xi_{22} + \alpha V_2 \\ &= e^{\alpha t} \xi^T(t) \Xi_2(d(t), \dot{d}(t)) \xi(t) + \alpha V_2. \end{aligned} \tag{12}$$

where Ξ_2 is given as the same as that of in (1). Further, $\dot{V}_3(t)$ is estimated as

$$\dot{V}_3(t) = e^{\alpha t} \{\xi^T(t) \Xi_{30} \xi^T(t) + \delta_1(t) + \delta_2(t)\} + \alpha V_3$$

where

$$\begin{aligned} \Xi_{30} &:= e_9^T(d_1^2 W_1 + d_1^2 W_2)e_9 + e_8^T(d_{21}^2 R_1 + d_{21}^2 R_2)e_8, \\ \delta_1(t) &= - \int_{t-d_2}^{t-d_1} \dot{y}^T(s) d_{21} R_1 \dot{y}(s) ds - 2 \int_{t-d_2}^{t-d_1} (d_2 - t + s) \dot{y}^T(s) R_2 \dot{y}(s) ds, \\ \delta_2(t) &= - \int_{t-d_1}^t \dot{y}^T(s) d_1 W_1 \dot{y}(s) ds - 2 \int_{t-d_1}^t (d_1 - t + s) \dot{y}^T(s) W_2 \dot{y}(s) ds. \end{aligned}$$

By Lemma ?? and Lemma ??, we obtain the following

$$-(d_2 - d_1) \int_{t-d_2}^{t-d_1} \dot{y}^T(s) R_1 \dot{y}(s) ds \leq 2\psi_{11}^T S_1 \psi_{21} - \psi_{11}^T \tilde{R}_1 \psi_{11} - \psi_{21}^T \tilde{R}_1 \psi_{21},$$

where $\tilde{R}_1 := \text{diag}\{R_1, 3R_1\}$; and

$$\begin{aligned} \psi_{11} &:= \begin{bmatrix} y(t-d(t)) - y(t-d_2) \\ y(t-d(t)) + y(t-d_2) - 2\nu_1(t) \end{bmatrix}, \\ \psi_{21} &:= \begin{bmatrix} y(t-d_1) - y(t-d(t)) \\ y(t-d_1) + y(t-d(t)) - 2\nu_2(t) \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned}
-2 \int_{t-d_2}^{t-d_1} (d_2 - t + s) \dot{y}^T(s) R_2 \dot{y}(s) ds &\leq -2 \left\{ \frac{1}{2} (d_2 - d(t))^2 \xi^T(t) Z_1 \xi(t) \right. \\
&\quad + 2(d_2 - d(t)) \xi^T(t) N_1 [y(t - d(t)) - \nu_3] \\
&\quad + \frac{1}{2} [(d_2 - d_1)^2 (d_2 - d(t))^2] \xi^T(t) Z_2 \xi(t) \\
&\quad + 2 \xi^T(t) N_2 [(d_2 - d(t)) [y(t - d_1) - y(t - d(t))]] \\
&\quad \left. + (d(t) - d_1) [y(t - d_1) - \nu_2] \right\}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\delta_1(t) &\leq [2\psi_{11}^T S_1 \psi_{21} - \psi_{11}^T \tilde{R}_1 \psi_{11} - \psi_{21}^T \tilde{R}_1 \psi_{21} - 2 \left\{ \frac{1}{2} (d_2 - d(t))^2 \xi^T(t) Z_1 \xi(t) \right. \\
&\quad + 2(d_2 - d(t)) \xi^T(t) N_1 [y(t - d(t)) - \nu_3] + \frac{1}{2} [(d_2 - d_1)^2 (d_2 - d(t))^2] \xi^T(t) Z_2 \xi(t) \\
&\quad + 2 \xi^T(t) N_2 [(d_2 - d(t)) [y(t - d_1) - y(t - d(t))]] + (d(t) - d_1) [y(t - d_1) - \nu_2] \left. \right\} \\
&= \xi^T(t) \Xi_3(d(t)) \xi(t),
\end{aligned} \tag{13}$$

where $\Xi_3(d(t))$ is given in (2). From Lemma ?? and Lemma ??, we obtain

$$- \int_{t-d_1}^t \dot{y}^T(s) d_1 W_1 \dot{y}(s) ds \leq [y(t) - y(t - d_1)]^T W_1 [y(t) - y(t - d_1)] + 3\tilde{\Omega}_1^T W_1 \tilde{\Omega}_1$$

and

$$-2 \int_{t-d_1}^t (d_1 - t + s) \dot{y}^T(s) W_2 \dot{y}(s) ds \leq -d_1^2 \xi^T(t) M_1 \xi(t) - 2d_1 \xi^T(t) N_3 [y(t) - \nu_3].$$

From which it follows that

$$\begin{aligned}
\delta_2(t) &\leq [y(t) - y(t - d_1)]^T W_1 [y(t) - y(t - d_1)] \\
&\quad + 3\tilde{\Omega}_1^T W_1 \tilde{\Omega}_1 - d_1^2 \xi^T(t) M_1 \xi(t) - 2d_1 \xi^T(t) N_3 [y(t) - \nu_3], \\
&= \xi^T(t) \Xi_{33} \xi(t)
\end{aligned} \tag{14}$$

where $\tilde{\Omega}_1 = y(t) + y(t - d_1) - 2\nu_3$, $\tilde{\Omega}_2 = y(t - d(t)) + y(t + d_2) - \nu_1$, $\tilde{\Omega}_3 = y(t - d_1) + y(t - d(t)) - \nu_2$,

$$\Xi_{33} := -\tilde{\varphi}_3^T \text{diag}\{W_1, 3W_1\} \tilde{\varphi}_3 + d_1^2 Z_3 + 2d_1 N_3 (e_1 - e_7) + 2d_1 (e_1 - e_7)^T N_3^T$$

Hence, from (??) and (??), we obtain

$$\dot{V}_3 \leq e^{\alpha t} \{ \xi^T(t) [\Xi_3(d(t)) + \Xi_4] \xi(t) \} + \alpha V_3 \tag{15}$$

where $\Xi_4 := \Xi_{30} + \Xi_{33}$. From (??), (??) and (??), we obtain $\dot{V}(t, y_t, \dot{y}_t)$ along the solution of the system (??) as

$$\dot{V}(t, x_t) \leq \alpha V(t, x_t) + e^{\alpha t} \xi^T \Delta(d(t), \dot{d}(t)) \xi(t)$$

where

$$\Delta(d(t), \dot{d}(t)) = \Xi_2(d(t), \dot{d}(t)) + \Xi_3(d(t)) + (\Xi_1 + \Xi_4).$$

Therefore, we have

$$\begin{aligned} \dot{V}(t, x_t) \leq & \alpha V(\cdot) + \xi^T(t) \hat{\Delta}(d(t), \dot{d}(t)) \xi(t) + \gamma \|w(t)\|^2 - y^T(t) [PE^T EP + YF^T FY] y(t) \\ & - y^T(t - d(t)) [PG^T GP] y(t - d(t)) - f^T(x(t)) f(x(t)) - g^T(x(t - d(t))) g(x(t - d(t))) \end{aligned} \quad (16)$$

where

$$\begin{aligned} \hat{\Delta}(d(t), \dot{d}(t)) &= \Xi_2(d(t), \dot{d}(t)) + \Xi_3(d(t)) + (\hat{\Xi}_1 + \Xi_4) \\ &= \Xi_2(d(t), \dot{d}(t)) + \Xi_3(d(t)) + \hat{\Xi}_4, \end{aligned}$$

$$\hat{\Xi}_1 = \Xi_1 + e_1^T [PE^T EP + YF^T FY] e_1 + e_2^T [PG^T GP] e_2 + e_{10}^T [I] e_{10} + e_{11}^T [I] e_{11},$$

and $\hat{\Xi}_4 = \hat{\Xi}_1 + \Xi_4$ is defined in ([?]). $\hat{\Delta}(d(t), \dot{d}(t))$ may be rewritten as

$$\hat{\Delta}(d(t), \dot{d}(t)) = d^2(t) \Delta_0 + d(t) \Delta_1 + \Delta_2 \quad (17)$$

where $\Delta_0 = Z_1 - Z_2$ and Δ_1, Δ_2 are $d(t)$ -independent real matrices. By Lemma (??), if $Z_1 - Z_2 \geq 0$ and the inequality in (4) holds, then $\hat{\Delta}(d(t), \dot{d}(t)) < 0$, $\forall d(t) \in [d_1, d_2]$, $\forall \dot{d}(t) \in [\mu_1, \mu_2]$. Moreover, the terms $\hat{\Delta}(d(t), \dot{d}(t))$ can be recast in the sense of convex combination of $\dot{d}(t)$ as follows:

$$\hat{\Delta}(d(t), \dot{d}(t)) = (1 - \dot{d}(t)) \diamond_0 + \dot{d}(t) \diamond_1 + \diamond_2 \quad (18)$$

where $\diamond_0 = Q_2 - Q_3$ and $\diamond_1, \diamond_2$ are $\dot{d}(t)$ -independent real matrices. Hence by making use of the Schur complement lemma, it follows from (??), (??) and (6) that the inequality $\hat{\Delta}(d(t), \dot{d}(t)) < 0$ holds. Therefore, we have from inequality (??) that

$$\dot{V}(t, x_t) \leq \alpha V(t, x_t) \gamma w(t)^T w(t) - z(t)^T z(t) \quad (19)$$

and hence

$$\dot{V}(t, x_t) \leq \alpha V(t, x_t) + \gamma^2 w^T(t) w(t).$$

because of $z(t)^T z(t) \geq 0$. Multiplying both sides with $e^{-\alpha t}$, we obtain

$$e^{-\alpha t} \dot{V}(t, x_t) - \alpha e^{-\alpha t} V(t, x_t) < e^{-\alpha t} \gamma^2 w^T(t) w(t).$$

Hence, we have

$$\frac{d}{dt} (e^{-\alpha t} V(t, x_t)) = e^{-\alpha t} \dot{V}(t, x_t) - \alpha e^{-\alpha t} V(t, x_t).$$

So,

$$\frac{d}{dt} (e^{-\alpha t} V(t, x_t)) < e^{-\alpha t} \gamma^2 w^T(t) w(t).$$

Integrating both sides from 0 to t , we get

$$(e^{-\alpha t}V(t, x_t)) - V(0, x_0) < \gamma^2 \int_0^t e^{-\alpha s} w^T(s)w(s)ds,$$

which can be reformulated as

$$e^{-\alpha t}V(t, x_t) < V(0, x_0) + \gamma^2 \int_0^t e^{-\alpha s} w^T(s)w(s)ds,$$

note that

$$\alpha_1 x^T(t)Rx(t) \leq V(t, x_t), \forall : 0 \leq T. \quad (20)$$

Hence, we have

$$\begin{aligned} V_1(t, x_t) &= e^{\alpha t} x^T(t) P^{-1} x(t) \\ &= e^{\alpha t} x^T(t) R \frac{1}{2} R^{-1} \frac{-1}{2} P^{-1} R \frac{-1}{2} \frac{1}{2} x(t) \\ &= e^{\alpha t} x^T(t) R \frac{1}{2} \hat{P}^{-1} R \frac{1}{2} x(t) \\ &\geq x^T(t) R \frac{1}{2} \hat{P}^{-1} R \frac{1}{2} x(t) \\ &\geq \lambda_{\min}(\hat{P})^{-1} x^T(t) R x(t) \end{aligned} \quad (21)$$

and $\alpha_1 = \lambda_{\min}(\hat{P})^{-1}$.

Consider

$$\begin{aligned} V(0, x_0) &= y^T(0)Py(0) + \int_{-d_1}^0 \dot{y}^T(s)Q_0\dot{y}(s)ds, + \int_{-d_1}^0 [y^T(0)y^T(s)]Q_1[y^T(0) \ y^T(s)]^T ds \\ &\quad + \int_{-d(t)}^{-d_1} [y^T(0)y^T(s)]Q_2[y^T(0) \ y^T(s)]^T ds + \int_{-d_2}^{-d(t)} [y^T(0) \ y^T(s)]Q_3[y^T(0) \ y^T(s)]^T ds, \\ &\quad + \int_{-d_1}^0 [(d_1 + s)\dot{y}^T(s)d_1W_1\dot{y}(s) + (d_1 + s)^2\dot{y}^T(s)W_2\dot{y}(s)]ds \\ &\quad + d_{21} \int_{-d_2}^{-d_1} [(d_1 + s)\dot{y}^T(s)R_1\dot{y}(s) + (d_1 + s)^2\dot{y}^T(s)R_2\dot{y}(s)]ds \\ &\leq \lambda_{\max}(P)\|\phi\|^2 + \lambda_{\max}(Q_0)\|\phi\|^2 + \lambda_{\max}(Q_1)\|i\|^2 + \lambda_{\max}(Q_2)\|\phi\|^2 + \lambda_{\max}(Q_3)\|\phi\|^2 \\ &\quad + \lambda_{\max}(W_1)d_1^2\|\phi\|^2 + \lambda_{\max}(W_2)d_1^2\|\phi\|^2 + \lambda_{\max}(R_1)d_{21}^2\|\phi\|^2 + \lambda_{\max}(R_2)d_{21}\|\phi\|^2, \end{aligned}$$

so we have

$$V(0, x_0) \leq (\lambda_{\max}(P) + \lambda_{\max}(Q_0) + \lambda_{\max}(Q_1) + \lambda_{\max}(Q_2) + \lambda_{\max}(Q_3)) \quad (22)$$

$$\begin{aligned} &+ \lambda_{\max}(W_1)d_1^2 + \lambda_{\max}(W_2)\pi_6 + \lambda_{\max}(R_1)d_{21}^2 + \lambda_{\max}(R_2)d_{21}^3)c_1, \\ &= \alpha_2 c_1 \end{aligned} \quad (23)$$

where $\alpha_2 = \lambda_{\max}(P) + \lambda_{\max}(Q_0) + \lambda_{\max}(Q_1) + \lambda_{\max}(Q_2) + \lambda_{\max}(Q_3) + \lambda_{\max}(W_1)d_1^2 + \lambda_{\max}(W_2)\pi_6 + \lambda_{\max}(R_1)d_{21}^2 + \lambda_{\max}(R_2)d_{21}^3$.

Therefore, from (??), (??), it follows that

$$\alpha_1 e^{-\alpha t} x^T(t) R x(t) < e^{-\alpha t} V(t, x(t)) \leq \alpha_2 c_1 + \gamma^2 k, \forall t \in [0, T].$$

and hence from (??), we have

$$x^T(t) R x(t) < \left(\frac{\alpha_2 c_1 + \gamma^2 \beta k}{\alpha_1} \right) e^{\alpha t} \leq c_2, \forall t \in [0, T],$$

which implies that the closed-loop system is finite-time stable w.r.t $(c_1, c_2, T, R, d, \gamma)$. To complete the proof of the theorem, it remains to show the γ optimal level condition (??). For this, we consider the following relation

$$\dot{V}(t, x_t) + z^T(t) z(t) - \gamma^2 w^T(t) w(t) < 0.$$

Integrating both sides of above equation from t_0 to t , we get

$$\int_{t_0}^t \left[\dot{V}(t, x_t) + z^T(t) z(t) - \gamma^2 w^T(t) w(t) \right] dt < 0.$$

It follows that

$$\begin{aligned} \int_{t_0}^t \left[z^T(t) z(t) - \gamma^2 w^T(t) w(t) \right] dt &\leq V(t_0, x_{t_0}) - V(t, x_t) \\ &\leq 0. \end{aligned} \tag{24}$$

Therefore, under zero initial condition $x(t) = 0, t \in [-\tau_2, t_0]$, by letting $t \rightarrow +\infty$ in (??), we get

$$\int_{t_0}^{\infty} z^T(t) z(t) dt < \gamma^2 \int_{t_0}^{\infty} w^T(t) w(t) dt$$

which gives $\|z\|_2 < \gamma \|w\|_2$. This completes the proof.

ผลงานวิจัยที่ตีพิมพ์

- ผู้แต่ง: Chanikan Emharuethai^{*}, Piyapong Niamsup, Raja Ramachandran
Wajaree Weera
- ชื่อเรื่องงานวิจัยที่ได้: Time-Varying Delayed H_∞ Control Problem for
Nonlinear Systems: A Finite Time Study Using
Quadratic Convex Approach
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การเสนอผลงานในที่ประชุมวิชาการนานาชาติ

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| 2.1 ชื่อการจัดการประชุม | ICMA-MU 2018: International Conference
in Mathematics and Applications |
| สถานที่จัดประชุม | The Century Park Hotel, Bangkok, Thailand
ระหว่างวันที่ 16 - 18 ธันวาคม 2561 |
| ชื่อเรื่องที่น่าสนใจ | H_∞ control problem for neural networks with time-
varying delay based on an improved reciprocally
convex inequality |
| | |
| 2.2 ชื่อการจัดการประชุม | NUMTA 2018: International Conference
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| ชื่อเรื่องที่น่าสนใจ | Finite-time stability for neural networks with
time varying delay based on an improved
reciprocally convex inequality |
| | |
| 2.3 ชื่อการจัดการประชุม | 2 nd International Conference on Mathematical
Modeling and Computational Methods in Science
and Engineering (ICMMCMSE-2020) |
| สถานที่จัดประชุม | ณ สาขาวิชาคณิตศาสตร์ มหาวิทยาลัยอัลกาปปา
รัฐทมิฬนาฑู ประเทศอินเดีย
ระหว่างวันที่ 22-24 มกราคม 2563 |
| ชื่อเรื่องที่น่าสนใจ | Finite-time stability for neural networks with
time varying delay based on an improved
reciprocally convex inequality |

Appendix

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ (ระบุชื่อผู้แต่ง ชื่อเรื่อง ชื่อวารสาร ปี เล่มที่ เลขที่ และหน้า) หรือผลงานที่คาดว่าจะได้ในสัญญาโครงการ

Chanikan Emharuethai*, Piyapong Niamsup, RajaRamachandran, Wajaree Weera,:
Time-Varying Delayed H_∞ Control Problem for Nonlinear Systems: A Finite Time Study Using Quadratic Convex Approach, *Symmetry* **2020**, 12, 713, 1-16.

หมายเหตุ * Corresponding author

2. อื่นๆ (เช่น ผลงานตีพิมพ์ในวารสารวิชาการในประเทศ การเสนอผลงานในที่ประชุมวิชาการ หนังสือ การจดสิทธิบัตร)

การเสนอผลงานในที่ประชุมวิชาการนานาชาติ	
2.1 ชื่อการจัดการประชุม	ICMA-MU 2018: International Conference in Mathematics and Applications
สถานที่จัดประชุม	The Century Park Hotel, Bangkok, Thailand ระหว่างวันที่ 16 - 18 ธันวาคม 2561
ชื่อเรื่องที่น่าสนใจ	H_∞ control problem for neural networks with time-varying delay based on an improved reciprocally convex inequality
2.2 ชื่อการจัดการประชุม	Numerical Computations :Theory and Algorithms
สถานที่จัดประชุม	ณ Crotone ประเทศอิตาลี ระหว่างวันที่ 15-21 มิถุนายน 2562
ชื่อเรื่องที่น่าสนใจ	H_∞ control problem for neural networks with time-varying delay
2.3 ชื่อการจัดการประชุม	2 nd International Conference on Mathematical Modeling and Computational Methods in Science and Engineering (ICMMCMSE-2020)
สถานที่จัดประชุม	ณ สาขาวิชาคณิตศาสตร์ มหาวิทยาลัยอัลกาปปา รัฐทมิฬนาฑู ประเทศอินเดีย ระหว่างวันที่ 22-24 มกราคม 2563
ชื่อเรื่องที่น่าสนใจ	Finite-time stability for neural networks with the time-varying delay based on an improved reciprocally convex inequality

3. การนำผลงานวิจัยไปใช้ประโยชน์

การศึกษาโครงการ การควบคุมแบบ H_∞ สำหรับระบบโครงข่ายประสาทที่มีตัวหน่วงขึ้นกับเวลา ซึ่งคณะผู้วิจัยได้มีการรวมกลุ่มวิจัย ชื่อกลุ่ม Stability Theory and Application Research Group ประกอบด้วย นักศึกษาระดับปริญญา โท-เอก อาจารย์ และนักวิจัย ทั้งในและต่างประเทศ ผลที่ได้จากการศึกษาโครงการ คือ ได้นำองค์ความรู้มาต่อยอด นำไปใช้ประโยชน์ในเชิงวิชาการ โดยการพูดคุย สัมมนา ทำให้มีการพัฒนาการเรียนการสอน เกิดแนวทางการทำงานวิจัยใหม่ ๆ มีเครือข่ายความร่วมมือการวิจัยในวงกว้าง อันเป็นประโยชน์ต่อการพัฒนางานวิจัยต่อไป

Article

Time-Varying Delayed H_∞ Control Problem for Nonlinear Systems: A Finite Time Study Using Quadratic Convex Approach

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Abstract: In this manuscript, we consider the finite-time H_∞ control for nonlinear systems with time-varying delay. With the assistance of a novel Lyapunov-Krasovskii functional which includes some integral terms, a matrix-based on quadratic convex approach, combined with Wirtinger inequalities and some useful integral inequalities, a sufficient condition of finite-time boundedness is established. A novel feature presents in this paper is that the restriction which is necessary for the upper bound derivative is not restricted to less than 1. Further a H_∞ controller is designed via memoryless state feedback control and a new sufficient conditions for the existence of finite-time H_∞ state feedback for the system are given in terms of linear matrix inequalities (LMIs). At the end, some numerical examples with simulations are given to illustrate the effectiveness of the obtained result.

Keywords: finite-time H_∞ control; nonlinear system; time-varying delay; linear matrix inequalities (LMIs); Lyapunov-Krasovskii functional (LKF)

1. Introduction

The occurrence of time delays is an important fact in many of the networking and processing control systems. Such delays can have the capacity to destabilize the control systems and also make some crucial disintegration in the performance of the closed-loop systems, see the references cited therein [1–7]. While modeling a real control system, the existence of time delays is always taken to be a time-varying one that satisfies the condition $d_1 \leq d(t) \leq d_2$ and d_1 which is not necessarily restricted to be 0. In recent years, the study on finite-time stability (FTS) has increased the research interest from various researches around the world due to the wider applications in mathematical control theory, which has been studied by different approaches in various kinds of systems, see for instance [8–15]. To this extent, the author Dorato in [8], explained the fundamental concepts of stability theory of dynamical systems in finite-time sense. Generally, the given system leads to be finite-time stable if the considered state of the system should be within the bounded limit for a fixed interval of time. From this, one can observe that the concept of finite-time stability mainly attracts the boundedness of a system during a fixed interval of time period. Some of the exciting results for finite-time stability and stabilization with the existence of time-delay have been obtained in [8–12]. Moreover, in some practical

systems, there is a need to outline the system that guarantees a maximum H_∞ performance rather than the finite-time stability. Hence this motivates us to concentrate on the present study of research.

On the other hand, the study on H_∞ control problem will make a sense in reducing the consequences of the external disturbances from both inside and outside the system. The main theme of the H_∞ problem is to design a controller from outside the system and to obtain the robust stability (i.e., to minimize the errors). Also this will result in minimizing the guaranteeing disturbance attenuation level γ in the H_∞ sense for the system. Hence this finite-time H_∞ control concerns in the design of feedback controller which ensures the FTS of the closed-loop system and guarantees a maximum H_∞ performance bound.

Recently, the authors [10,11,16,17] have enhanced the results on finite time stability and H_∞ performance analysis. In [10], finite-time H_∞ control for a continuous system with norm-bounded disturbance has been studied but a continuous system is not a nonlinear system. Robust finite-time H_∞ control of linear time-varying delay systems with bounded control has been considered in [11] based on Riccati Equations. In [12], the problem of robust finite-time stabilization with guaranteed cost control was studied based on the Lyapunov functional method and generalized Jensen integral inequality. These techniques allow us to design the state feedback controllers which stabilize the closed-loop system in the finite time. In [18], authors used an improved Lyapunov-Krasovskii functional (LKF) with triple-integral terms, augment terms and convex combination technique to show the effectiveness of the obtained results. Hao et al. [19] developed a novel problem on time-varying delayed nonlinear systems with finite-time stability and stabilization by employing the integral inequality and some free fuzzy by weighting metrices, which are less conservative than other existing ones. In [20], delay-dependent finite-time stability criteria for an uncertain continuous-time system with time varying delays has been studied but a continuous-time system is not a nonlinear system and without H_∞ performance analysis. In [21], improved results on delay-dependent H_∞ control for uncertain systems with time-varying delays have been considered by using bounding techniques for some cross-term of the LKF method and the free-weighting matrix method.

Several approaches that reduce the conservatism for the system with time delay have been reported in the literature. They are namely an appropriate Lyapunov-Krasovskii functional method by using bounding techniques while finding the time-derivative, delay decomposition approach; free weighting matrices approach and reciprocally convex optimization techniques, see [21–24]. Of all the above mentioned approaches, a novel method to reduce the conservatism is matrix-based quadratic convex approach. This approach will gives a better maximum allowable upper bound for time-varying delay over some existing ones, see for references [25–27].

So with the intuition from the above evidences, in this paper, we have followed a matrix-based quadratic convex approach to obtain a better maximum bound value. This is the first time that we have incorporated this method to study the finite-time H_∞ problem for the considered control system with time-varying delay. Further, the purposes of this paper are given as follows:

- I. We consider some new Lyapunov-Krasovskii functional which has not been considered yet in stability analysis of finite-time H_∞ control. The new Lyapunov-Krasovskii functional includes some integral terms of the form $\int_{t-h}^t (h-t-s)^j \dot{x}^T(s) R_j \dot{x}(s) ds$ ($j = 1, 2$) which the integrands are polynomial multiplied by $\dot{x}^T(s) R_j \dot{x}(s) ds$ ($j = 1, 2$) and one may estimate an upper bound of the integral by employing some techniques from [22,25], the matrix based quadratic convex approach, the use of a tighter bounding technique and useful integral inequality such as Wirtinger inequality.
- II. Lyapunov-Krasovskii with the matrix based quadratic convex approach is introduced to formulate finite-time stability criteria and H_∞ performance level where the time-varying delay satisfies $0 \leq d_1 \leq d(t) \leq d_2$, $\mu_1 \leq \dot{d}(t) \leq \mu_2$. Moreover, the restriction of upper bound derivative is not necessary restricted less than 1 compared with [20]
- III. Two numerical examples are given to demonstrate the effectiveness of theoretical result.

2. Problem statement

In this section, we consider a system with time-varying delay and control input as

$$\left. \begin{aligned} \dot{x}(t) &= Ax(t) + Dx(t-d(t)) + Bu(t) + Cw(t) + f(x(t), t) + g(x(t-d(t)), t) \\ z(t) &= Ex(t) + Gx(t-d(t)) + Fu(t) + f(x(t), t) + g(x(t-d(t)), t), \\ x(t_0 + \theta) &= \phi(\theta), \theta \in [-d_2, 0], (t_0, \phi) \in \mathcal{R}^+ \times \mathcal{C}([-d_2, t_0], \mathcal{R}^n), \end{aligned} \right\} \quad (1)$$

where $x(t) \in \mathcal{R}^n$ is the state; $u(t) \in \mathcal{R}^m$ is the control input, $w(t) \in \mathcal{L}_2([0, \infty], \mathcal{R}^r)$ is a disturbance input and $z(t) \in \mathcal{R}^s$ is the observation output. The delay $d(t)$ is time-varying continuous function which satisfies

$$0 \leq d_1 \leq d(t) \leq d_2, \mu_1 \leq \dot{d}(t) \leq \mu_2.$$

In this paper, we consider the nonlinear functions satisfying

$$\begin{aligned} f(x(t), t) &= (f_1(x_1(t), t), f_2(x_2(t), t), \dots, f_n(x_n(t), t))^T \in \mathcal{R}^n \\ g(x(t-d(t)), t) &= (g_1(x_1(t-d(t)), t), g_2(x_2(t-d(t)), t), \dots, g_n(x_n(t-d(t)), t))^T \in \mathcal{R}^n, \end{aligned}$$

$f, g : \mathcal{R}^n \times [-d_M, \infty) \rightarrow \mathcal{R}^n$ are nonlinear function satisfying the Lipschitz conditions; namely, there exist positive constants $\beta_1, \beta_2, \forall x, y \in \mathcal{R}^n$, such that

$$\begin{aligned} \|f(y, t) - f(x, t)\|^2 &\leq \beta_1 \|y(t) - x(t)\|^2, \\ \|g(y, t) - g(x, t)\|^2 &\leq \beta_2 \|y(t) - x(t)\|^2. \end{aligned}$$

We assume the following restrictions on the nonlinear perturbations

$$f^T(x(t), t)f(x(t), t) \leq \beta_1 x^T(t)x(t), \quad (2)$$

$$g^T(x(t-d(t)), t)g(x(t-d(t)), t) \leq \beta_2 x^T(t-d(t))x(t-d(t)). \quad (3)$$

The initial condition, $\phi(\cdot) := \sup_{t \in [-d_M, 0]} \{\|\phi(t)\|, \|\dot{\phi}(t)\|\}$. The disturbance is a continuous function satisfying

$$\exists k > 0 : \int_0^T w^T(t)w(t)dt \leq k. \quad (4)$$

Under the above assumptions on $d(\cdot), f(\cdot), g(\cdot)$ and the initial function $\phi(t)$, the system (2) has a unique solution $x(t, \phi)$ on $[0, T]$. For a prescribed scalar $\gamma > 0$, we define the performance index as

$$J(t) = \int_0^\infty [z^T(s)z(s) - \gamma^2 w^T(s)w(s)]ds. \quad (5)$$

The objective of this paper is to design a memoryless state feedback controller $u(t) = Kx(t)$.

3. Preliminaries

The following definition and lemma are necessary in the proof of the main results:

Definition 1. [9] The nonlinear system (2) where $w(t)$ is a perturbation satisfying (4). The system (2) is said to be finite-time bounded with respect to (c_1, c_2, T, R, d) with $0 < c_1 < c_2$, and $R > 0$, if

$$\sup_{-\tau_2 \leq s \leq 0} \{\phi^T(s)R\phi(s)\dot{\phi}^T(s)R\dot{\phi}(s)\} \leq c_1 \Rightarrow x^T(t)Rx(t) < c_2, \forall t \in [0, T]. \quad (6)$$

Definition 2. [9] The nonlinear system (2) is said to be finite-time H_∞ bounded with respect to $(c_1, c_2, T, R, d, \gamma)$ with $0 < c_1 < c_2, d \geq 0, \gamma > 0, R > 0$ and a memoryless state feedback controller $u(t) = Kx(t)$, following conditions should be satisfied:

(i) The zero solution of the closed-loop system, where $w(t) = 0$,

$$\dot{x}(t) = -(A - BK)x(t) + f(x(t)) + g(x(t - \tau(t))) + Cw(t), \quad (7)$$

is finite-time bounded.

(ii) Under zero-initial condition $\phi(t) = 0, \forall t \in [-d_2, 0]$ the output $z(t)$ satisfies

$$\int_0^T z^T(t)z(t)dt < \gamma^2 w^T(t)w(t)dt. \quad (8)$$

We introduce the following technical lemmas, which will be used in the proof of our results.

Proposition 1. [11] Let $P \in M^{n \times n}, R \in M^{n \times n}$ be symmetric positive definite matrices. We have

- (i) $\lambda_{\min}(P)(R) > 0, \lambda_{\max}(P)(R) > 0$ and $\lambda_{\min}(P)x^T x \leq x^T P x \leq \lambda_{\max}(P)x^T x, \forall x \in \mathcal{R}^n$
- (ii) $x^T x \leq \lambda_{\max}(R^{-1})x^T R x, \forall x \in \mathcal{R}^n$
- (iii) $x^T P x \leq \lambda_{\max}(P)\lambda_{\max}(R^{-1})x^T R x, \forall x \in \mathcal{R}^n$.

Lemma 1. [22] For a given matrix $R > 0$, the following inequality holds for any continuously differentiable function $w : [a, b] \rightarrow \mathcal{R}^n$

$$\int_a^b \dot{w}^T(u)R\dot{w}du \geq \frac{1}{b-a}(\Gamma_1^T R \Gamma_1 + 3\Gamma_2^T R \Gamma_2) \quad (9)$$

where $\Gamma_1 := w(b) - w(a), \Gamma_2 := w(b) + w(a) - \frac{2}{b-a} \int_a^b w(u)du$.

Remark 1. : From the above inequality, it can be observed that the inequality in Lemma 1 gives a firm lower bound for $\int_a^b \dot{w}^T(u)R\dot{w}(u)du$ than Jensen's inequality since $3\Gamma_2^T R \Gamma_2 > 0$ for $\Gamma_2 \neq 0$. Hence it shows that the inequality (9) is improved than the Jensen's inequality.

Before we introduce some useful integral inequalities, we denote

$$\left. \begin{aligned} v_1(t) &:= \frac{1}{d_2 - d(t)} \int_{t-d_2}^{t-d(t)} y(s)ds \\ v_2(t) &:= \frac{1}{d(t) - d_1} \int_{t-d(t)}^{t-d_1} y(s)ds \\ v_3(t) &:= \frac{1}{d_1} \int_{t-d_1}^t y(s)ds. \end{aligned} \right\} \quad (10)$$

Lemma 2. [25] For a given scalar $d_1 \geq 0$ and any $n \times n$ real matrices $Y_1 > 0$ and $Y_2 > 0$ and a vector $\dot{y} : [-d_1, 0] \rightarrow \mathcal{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any vector-valued function $\pi_1(t) : [0, \infty) \rightarrow \mathcal{R}^k$ and matrices $M_1 \in \mathcal{R}^{k \times k}$ and $N_1 \in \mathcal{R}^{k \times n}$ satisfying

$$\begin{bmatrix} M_1 & N_1 \\ N_1^T & Y_1 \end{bmatrix} \geq 0,$$

$$\begin{aligned} \wp_1 &:= \int_{t-d_1}^t (d_1 - t + s) \dot{y}^T(s) Y_1 \dot{y}(s) ds \\ &\geq -\frac{d_1^2}{2} \pi_1^T(t) M_1 \pi_1(t) - 2d_1 \pi_1^T(t) N_1 [y(t) - v_3(t)], \\ \wp_2 &:= \int_{t-d_1}^t (d_1 - t + s)^2 \dot{y}^T(s) Y_2 \dot{y}(s) ds \end{aligned}$$

$$\geq d_1[y(t) - v_3(t)]^T Y_2[y(t) - v_3(t)]$$

where $v_3(t)$ is defined in (10).

Lemma 3. [26] Let $d(t)$ be a continuous function satisfying $0 \leq d_1 \leq d(t) \leq d_2$. For any $n \times n$ real matrix $R_2 > 0$ and a vector $\dot{y} : [-d_2, 0] \rightarrow \mathcal{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any $\phi_{i1} \in \mathcal{R}^q$ and real matrices $Z_i \in \mathcal{R}^{q \times q}$, $F_i \in \mathcal{R}^{q \times n}$ satisfying

$$\begin{bmatrix} Z_i & F_i \\ F_i^T & R_2 \end{bmatrix} \geq 0, (i = 1, 2),$$

$$\begin{aligned} - \int_{t-d_2}^{t-d_1} (d_2 - t + s) \dot{y}^T(s) R_2 \dot{y}(s) ds &\leq \frac{1}{2} (d_2 - d(t))^2 \phi_{11}^T Z_1 \phi_{11} + 2(d_2 - d(t)) \phi_{11}^T F_1 \phi_{12} \\ &\quad + \frac{1}{2} [(d_2 - d_1)^2 - (d_2 - d(t))^2] \phi_{21}^T Z_2 \phi_{21} \\ &\quad + 2\phi_{21}^T F_2 [(d_2 - d(t)) \phi_{22} + (d(t) - d_1) \phi_{23}] \end{aligned}$$

where $\phi_{12} := y(t - d(t)) - v_1(t)$, $\phi_{22} := y(t - d_1) - x(t - d(t))$, $\phi_{23} := y(t - d_1) - v_2(t)$ with $v_i(t)$ ($i = 1, 2$) being defined in (10).

Lemma 4. [25] Let $d(t)$ be a continuous function satisfying $0 \leq d_1 \leq d(t) \leq d_2$. For any $n \times n$ real matrix $R_1 > 0$ and a vector $\dot{y} : [-d_2, 0] \rightarrow \mathcal{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any $2n \times 2n$ real matrix S_1 satisfying

$$\begin{bmatrix} \tilde{R}_1 & S_1 \\ S_1^T & \tilde{R}_1 \end{bmatrix} \geq 0,$$

$$-(d_2 - d_1) \int_{t-d_2}^{t-d_1} \dot{y}^T(s) R_1 \dot{y}(s) ds \leq 2\psi_{11}^T S_1 \psi_{21} - \psi_{11}^T \tilde{R}_1 \psi_{11} - \psi_{21}^T \tilde{R}_1 \psi_{21},$$

where $\tilde{R}_1 := \text{diag}\{R_1, 3R_1\}$; and

$$\psi_{11} := \begin{bmatrix} y(t - d(t)) - y(t - d_2) \\ y(t - d(t)) + y(t - d_2) - 2v_1(t) \end{bmatrix}, \quad \psi_{21} := \begin{bmatrix} y(t - d_1) - y(t - d(t)) \\ y(t - d_1) + y(t - d(t)) - 2v_2(t) \end{bmatrix}.$$

Remark 2. If we substitute $d_1 = 0$, in Lemma 4, then the inequality can be reduced and it is similar to that of the one in [22]. Also, the dimensions of the slack matrix variables of S_1 is $2n \times 2n$ compared to the dimension $2n \times 5n$ introduced in [23].

Lemma 5. [25] Let χ_0, χ_1 and χ_2 be $m \times m$ real symmetric matrices and a continuous function d satisfy $d_1 \leq d \leq d_2$, where d_1 and d_2 are constants satisfying $0 \leq d_1 \leq d_2$. If $\chi_0 \geq 0$, then

$$\begin{aligned} d^2 \chi_0 + d \chi_1 + \chi_2 &< 0 (\leq 0), \forall d \in [d_1, d_2] \\ \Leftrightarrow d_i^2 \chi_0 + d_i \chi_1 + \chi_2 &< 0 (\leq 0), (i = 1, 2) \\ \text{or} \quad d^2 \chi_0 + d \chi_1 + \chi_2 &> 0 (\geq 0), \forall d \in [d_1, d_2] \\ \Leftrightarrow d_i^2 \chi_0 + d_i \chi_1 + \chi_2 &> 0 (\geq 0), (i = 1, 2). \end{aligned}$$

4. Main Results

In this section, we firstly design a memoryless H_∞ feedback control for the addressed system (2) with the inclusion of time-varying delays and then obtain the finite-time stabilizability analysis conditions. Here we derive a novel finite-time stability for the system (2) by using the matrix-based

quadratic convex approach with some integral inequalities in [25]. To achieve this status, we choose the following Lyapunov-Krasovskii functional:

$$V(t, x_t, \dot{x}_t) = V_1(t) + V_2(t) + V_3(t)$$

where x_t denotes the function $x(t)$ defined on the interval $[t - d_2, t]$. Setting $P_1 = P^{-1}$, $y(t) = P_1 x(t)$, $d_{21} := d_2 - d_1$ and

$$\begin{aligned} V_1(t) &:= e^{\alpha t} y^T(t) P y(t) + e^{\alpha t} \int_{t-d_1}^t \dot{y}^T(s) Q_0 \dot{y}(s) ds, \\ V_2(t) &:= e^{\alpha t} \int_{t-d_1}^t [y^T(t) \ y^T(s)] Q_1 [y^T(t) \ y^T(s)]^T ds + e^{\alpha t} \int_{t-d(t)}^{t-d_1} [y^T(t) \ y^T(s)] Q_2 [y^T(t) \ y^T(s)]^T ds \\ &\quad + e^{\alpha t} \int_{t-d_2}^{t-d(t)} [y^T(t) \ y^T(s)] Q_3 [y^T(t) \ y^T(s)]^T ds, \\ V_3(t) &:= e^{\alpha t} \int_{t-d_1}^t [(d_1 - t + s) \dot{y}^T(s) d_1 W_1 \dot{y}(s) + (d_1 - t + s)^2 \dot{y}^T(s) W_2 \dot{y}(s)] ds \\ &\quad + d_{21} e^{\alpha t} \int_{t-d_2}^{t-d_1} [(d_1 - t + s) \dot{y}^T(s) R_1 \dot{y}(s) + (d_1 - t + s)^2 \dot{y}^T(s) R_2 \dot{y}(s)] ds \end{aligned}$$

where $Q_j > 0$, ($j = 0, 1, 2, 3$), $W_1 > 0$, $W_2 > 0$, $R_1 > 0$, $R_2 > 0$ and P are real matrices to be determined. Before introducing the main result, the following notations of several matrix variables are defined for simplicity: $\tilde{R}_1 = \text{diag}\{R_1, 3R_1\}$;

$$\Xi_2(d(t), \dot{d}(t)) := \Xi_{20} + [d(t) - d_1] \Xi_{21} + [d_2 - d(t)] \Xi_{22}, \quad (11)$$

$$\begin{aligned} \Xi_3(d(t)) &:= \tilde{\varphi}_1^T S_1 \tilde{\varphi}_2 + \tilde{\varphi}_2^T S_1^T \tilde{\varphi}_1 - \tilde{\varphi}_1^T \tilde{R}_1 \tilde{\varphi}_1 + (d_2 - d(t))^2 (Z_1 - Z_2) + (d_2 - d(t)) \Xi_{31} \\ &\quad + (d(t) - d_1) \Xi_{32} + d_{21}^2 Z_2 - \tilde{\varphi}_2^T \tilde{R}_1 \tilde{\varphi}_2, \end{aligned} \quad (12)$$

$$\begin{aligned} \hat{\Xi}_4 &:= d_1^2 Z_3 - \tilde{\varphi}_3^T \tilde{W}_1 \tilde{\varphi}_3 + e_9^T (d_1^2 W_1 + d_1^2 W_2) e_9 + 2d_1 N_3 (e_1 - e_7) + e_8^T (d_{21}^T R_1 + d_{21}^T R_2) e_8 \\ &\quad + 2d_1 (e_1 - e_7)^T N_3^T + e_1^T (AP + PA^T + (BY + Y^T B^T) + \varepsilon I) e_1 \\ &\quad + e_1^T (PD) e_2 + e_2^T (D^T P) e_1 + e_1^T (AP + YB^T) e_9 + e_9^T (PA^T + Y^T B) e_1 \\ &\quad + e_2^T (DP) e_9 + e_9^T (PD^T) e_2 + e_8^T (Q_0) e_8 + e_9^T (-2P + \varepsilon I + Q_0) e_9 \\ &\quad + e_{10}^T (I) e_{10} + e_{11}^T (I) e_{11} + e_1^T (PE^T) e_{12} + e_{12}^T (EP) e_1 \\ &\quad + e_2^T (PG^T) e_{13} + e_{13}^T (GP) e_2 + e_{14}^T (YF^T) e_{14} + e_{11}^T (Y) e_1 + e_{15}^T (-\gamma^2 I) e_{15}, \end{aligned} \quad (13)$$

with $e_i \in \mathcal{R}^{n \times 15n}$ ($i = 1, 2, \dots, 15$) denoting the i -th row-block vector of the $14n \times 14n$ identity matrix $\tilde{W}_1 = \text{diag}\{W_1, 3W_1\}$; and

$$\begin{aligned} \Xi_{20} &:= [e_1^T \ e_3^T] (Q_2 - Q_1) [e_1^T \ e_3^T]^T \\ &\quad + \tau_1 [e_9^T \ 0] Q_1 [e_1^T \ e_7^T]^T + \tau_1 [e_1^T \ e_7^T] Q_1 [e_9^T \ 0]^T \\ &\quad - (1 - \dot{\tau}(t)) [e_1^T \ e_2^T] (Q_2 - Q_3) [e_1^T \ e_2^T]^T \\ &\quad - [e_1^T \ e_4^T] Q_3 [e_1^T \ e_4^T]^T + [e_1^T \ e_1^T] Q_1 [e_1^T \ e_1^T]^T \\ \Xi_{21} &:= [e_1^T \ e_6^T] Q_2 [e_9^T \ 0]^T + [e_9^T \ 0] Q_2 [e_1^T \ e_6^T]^T \\ \Xi_{22} &:= [e_1^T \ e_5^T] Q_3 [e_9^T \ 0]^T + [e_9^T \ 0] Q_3 [e_1^T \ e_5^T]^T \\ \Xi_{31} &:= 2N_1 (e_2 - e_5) + 2N_2 (e_3 - e_2) + 2(e_3 - e_2)^T N_2^T + 2(e_2 - e_5)^T N_1^T \\ \Xi_{32} &:= 2N_1 (e_3 - e_6) + 2(e_3 - e_6)^T N_1^T \\ \tilde{\varphi}_1 &:= \text{col}\{e_2 - e_4, e_2 + e_4 - 2e_5\} \\ \tilde{\varphi}_2 &:= \text{col}\{e_3 - e_2, e_3 + e_2 - 2e_6\} \\ \tilde{\varphi}_3 &:= \text{col}\{e_1 - e_3, e_1 + e_3 - 2e_7\}. \end{aligned}$$

Theorem 1. Consider $\gamma > 0$. Then system (2) is finite-time H_∞ control with respect to $(c_1, c_2, T, R, d, \gamma)$ and satisfies $\|z(t)\|_2 < \gamma\|w(t)\|_2$ for all nonzero $w \in \mathcal{L}_2[0, \infty)$ if there exist positive definite matrices $P, Q_j > 0, (j = 0, 1, 2, 3), W_1, W_2, R_1, R_2, S_1, Z_1, Z_2, Z_3, N_1, N_2, N_3$ and Y such that the following linear matrix inequalities (LMIs) hold

$$\begin{bmatrix} \tilde{R}_1 & S_1 \\ S_1^T & \tilde{R}_1 \end{bmatrix} \geq 0, \begin{bmatrix} Z_i & N_i \\ N_i^T & R_2 \end{bmatrix} \geq 0, (i = 1, 2) \\ \begin{bmatrix} Z_3 & N_2 \\ N_3^T & W_2 \end{bmatrix} \geq 0, \quad Z_1 \geq Z_2,$$

$$\left. \begin{aligned} \Xi_2(d_1, \mu_1) + \Xi_3(d_1) + \hat{\Xi}_4 &< 0 \\ \Xi_2(d_1, \mu_2) + \Xi_3(d_1) + \hat{\Xi}_4 &< 0 \\ \Xi_2(d_2, \mu_1) + \Xi_3(d_2) + \hat{\Xi}_4 &< 0 \\ \Xi_2(d_2, \mu_2) + \Xi_3(d_2) + \hat{\Xi}_4 &< 0 \end{aligned} \right\} \quad (14)$$

and

$$\frac{\alpha_2 c_1 + \gamma^2 k}{\alpha_1} \leq e^{-\alpha T} c_2 \quad (15)$$

For this problem, the feedback control is taken to be of

$$u(t) = YP^{-1}x(t), t \geq 0. \quad (16)$$

Proof. By finding the time-derivative of V for the considered system (2), we obtain

$$\dot{V}_1 = 2e^{\alpha t} y^T(t) P \dot{y}(t) + \alpha e^{\alpha t} y^T(t) P y(t) + e^{\alpha t} \{ \dot{y}^T(t) Q_0 \dot{y}(t) - \dot{y}^T(t - d_1) Q_0 \dot{y}(t - d_1) \} + \alpha V_1(\cdot) \quad (17)$$

$$\begin{aligned} \dot{V}_2 = & e^{\alpha t} \{ [y^T(t) \ y^T(t)] Q_1 [y^T(t) \ y^T(t)]^T - [y^T(t) \ y^T(t - d_1)] Q_1 [y^T(t) \ y^T(t - d_1)]^T \\ & + \int_{t-d_1}^t 2[y^T(t) \ y^T(s)] Q_1 [\dot{y}^T(t) \ 0]^T ds + [y^T(t) \ y^T(t - d_1)] Q_2 [y^T(t) \ y^T(t - d_1)]^T \\ & - (1 - d'(t)) [y^T(t) \ y^T(t - d(t))] Q_2 [y^T(t) \ y^T(t - d(t))]^T \\ & + 2 \int_{t-d(t)}^{t-d_1} [y^T(t) \ y^T(s)] Q_2 [\dot{y}^T(t) \ 0]^T ds - [y^T(t) \ y^T(t - d_2)] Q_3 [y^T(t) \ y^T(t - d_2)]^T \\ & + (1 - d'(t)) [y^T(t) \ y^T(t - d(t))] Q_3 [y^T(t) \ y^T(t - d(t))]^T \\ & + \int_{t-d_2}^{t-d(t)} 2[y^T(t) \ y^T(s)] Q_3 [\dot{y}^T(t) \ 0]^T ds \} + \alpha V_2(\cdot) \end{aligned} \quad (18)$$

$$\begin{aligned} \dot{V}_3 = & e^{\alpha t} \{ \dot{y}^T(t) d_1 W_1 \dot{y}(t) + d_1^2 \dot{y}^T(t) W_2 \dot{y}(t) - \int_{t-d_1}^t \dot{y}^T(s) d_1 W_1 \dot{y}(s) ds \\ & - 2 \int_{t-d_1}^t (d_1 - t + s) \dot{y}^T(s) W_2 \dot{y}(s) ds + (d_{21})^2 \dot{y}^T(t - d_1) R_1 \dot{y}(t - d_1) \\ & + (d_{21})^2 \dot{y}^T(t - d_1) R_2 \dot{y}(t - d_1) - d_{21} \int_{t-d_2}^{t-d_1} \dot{y}^T(s) R_1 \dot{y}(s) ds \\ & - 2 \int_{t-d_2}^{t-d_1} (d_2 - t + s) \dot{y}^T(s) R_2 \dot{y}(s) ds \} + \alpha V_3(\cdot). \end{aligned} \quad (19)$$

From (2) and Cauchy inequality, we get the following equality:

$$-2\dot{x}^T(t) P_1 [\dot{x}(t) - Ax(t) - Dx(t - d(t)) - Bu(t) - Cw(t) - f(x(t), t) - g(x(t - d(t)), t)] = 0$$

we obtain the following

$$\begin{aligned} 0 &= -2\dot{x}^T(t)P_1[\dot{x}(t) - Ax(t) - Dx(t - \tau(t)) - Bu(t) - Cw(t) - f(x(t), t) - g(x(t - d(t)), t)] \\ &= -2\dot{y}^T(t)P\dot{y}(t) + 2\dot{y}^T(t)APy(t) + 2\dot{y}^T(t)DPy(t - d(t)) + 2\dot{y}^T(t)2BYy(t) + 2\dot{y}^T(t)CPw(t) \\ &\quad + 2\dot{y}^T(t)Pf(y(t)) + 2\dot{y}^T(t)Pg(y(t - d(t))) \end{aligned} \quad (20)$$

From (17) and (20), we have $\dot{V}_1$ is

$$\begin{aligned} \dot{V}_1 &= 2e^{\alpha t}y^T(t)P\dot{y}(t) + \alpha e^{\alpha t}y^T(t)Py(t) + e^{\alpha t}\{\dot{y}^T(t)Q_0\dot{y}(t) - \dot{y}^T(t - d_1)Q_0\dot{y}(t - d_1)\} \\ &\quad - 2\dot{y}^T(t)P\dot{y}(t) + 2\dot{y}^T(t)APy(t) + 2\dot{y}^T(t)DPy(t - d(t)) + 2\dot{y}^T(t)2BYy(t) + 2\dot{y}^T(t)CPw(t) \\ &\quad + 2\dot{y}^T(t)Pf(y(t)) + 2\dot{y}^T(t)Pg(y(t - d(t))) \\ &= e^{\alpha t}\xi^T(t)\Xi_1\xi(t) + \alpha V_1(.). \end{aligned} \quad (21)$$

where $\xi(t) := \text{col}\{y(t), y(t - d(t)), y(t - d_1), y(t - d_2), v_1(t), v_2(t), v_3(t), \dot{y}(t - d_1), \dot{y}(t), f(y(t)), g(y(t - d(t)))\}$,

$$\begin{aligned} \Xi_1 &:= 2e_1^T(P)e_9 + \alpha e_1^T(P)e_1 + e_9^T(Q_0)e_9 - e_8^T(Q_0)e_8 - 2e_9^T(P)e_9 + 2e_9^T(AP + 4BY)e_1 + 2e_9^T(DPe_2 \\ &\quad + 2e_9^T(CPe_{12} + 2e_9Pe_{10} + 2e_9Pe_{11}). \end{aligned}$$

With the consideration of the three terms of $\dot{V}_2(t)$, we obtained the following inequalities:

$$\begin{aligned} \int_{t-d_1}^t 2[y^T(t) \ y^T(s)]Q_1[\dot{y}(t)^T \ 0]^T ds &\leq 2d_1[y^T(t) \ v_3^T]Q_1[\dot{y}^T(t) \ 0]^T, \\ \int_{t-d(t)}^{t-d_1} 2[y^T(t) \ y^T(s)]Q_2[\dot{y}(t)^T \ 0]^T ds &\leq 2(d(t) - d_1)[y^T(t) \ v_2^T]Q_2[\dot{y}^T(t) \ 0]^T \end{aligned}$$

and

$$\int_{t-d_2}^{t-d(t)} 2[y^T(t) \ y^T(s)]Q_3[\dot{y}(t)^T \ 0]^T ds \leq 2(d_2 - d(t))[y^T(t) \ v_1^T]Q_3[\dot{y}^T(t) \ 0]^T.$$

Therefore, the estimation of $\dot{V}_2(t)$ is estimated as

$$\begin{aligned} \dot{V}_2(t) &\leq \Xi_{20} + (d(t) - d_1)\Xi_{21} + (d_2 - d(t))\Xi_{22} + \alpha V_2 \\ &= e^{\alpha t}\xi^T(t)\Xi_2(d(t), \dot{d}(t))\xi(t) + \alpha V_2. \end{aligned} \quad (22)$$

where Ξ_2 is given as the same as that of in (11). Further, $\dot{V}_3(t)$ is estimated as

$$\dot{V}_3(t) = e^{\alpha t}\{\xi^T(t)\Xi_{30}\xi(t) + \delta_1(t) + \delta_2(t)\} + \alpha V_3$$

where

$$\begin{aligned} \Xi_{30} &:= e_9^T(d_1^2W_1 + d_1^2W_2)e_9 + e_8^T(d_{21}^2R_1 + d_{21}^2R_2)e_8, \\ \delta_1(t) &= -\int_{t-d_2}^{t-d_1} \dot{y}^T(s)d_{21}R_1\dot{y}(s)ds - 2\int_{t-d_2}^{t-d_1} (d_2 - t + s)\dot{y}^T(s)R_2\dot{y}(s)ds, \\ \delta_2(t) &= -\int_{t-d_1}^t \dot{y}^T(s)d_1W_1\dot{y}(s)ds - 2\int_{t-d_1}^t (d_1 - t + s)\dot{y}^T(s)W_2\dot{y}(s)ds. \end{aligned}$$

By Lemma 3 and Lemma 4, we obtain the following

$$-(d_2 - d_1) \int_{t-d_2}^{t-d_1} \dot{y}^T(s)R_1\dot{y}(s)ds \leq 2\psi_{11}^T S_1 \psi_{21} - \psi_{11}^T \tilde{R}_1 \psi_{11} - \psi_{21}^T \tilde{R}_1 \psi_{21},$$

where $\tilde{R}_1 := \text{diag}\{R_1, 3R_1\}$; and

$$\begin{aligned}\psi_{11} &:= \begin{bmatrix} y(t-d(t)) - y(t-d_2) \\ y(t-d(t)) + y(t-d_2) - 2v_1(t) \end{bmatrix}, \\ \psi_{21} &:= \begin{bmatrix} y(t-d_1) - y(t-d(t)) \\ y(t-d_1) + y(t-d(t)) - 2v_2(t) \end{bmatrix},\end{aligned}$$

and

$$\begin{aligned}-2 \int_{t-d_2}^{t-d_1} (d_2 - t + s) \dot{y}^T(s) R_2 \dot{y}(s) ds &\leq -2 \left\{ \frac{1}{2} (d_2 - d(t))^2 \tilde{\zeta}^T(t) Z_1 \tilde{\zeta}(t) \right. \\ &\quad + 2(d_2 - d(t)) \tilde{\zeta}^T(t) N_1 [y(t-d(t)) - v_3] \\ &\quad + \frac{1}{2} [(d_2 - d_1)^2 (d_2 - d(t))^2] \tilde{\zeta}^T(t) Z_2 \tilde{\zeta}(t) \\ &\quad + 2\tilde{\zeta}^T(t) N_2 [(d_2 - d(t)) [y(t-d_1) - y(t-d(t))]] \\ &\quad \left. + (d(t) - d_1) [y(t-d_1) - v_2] \right\}.\end{aligned}$$

Thus, we get

$$\begin{aligned}\delta_1(t) &\leq [2\psi_{11}^T S_1 \psi_{21} - \psi_{11}^T \tilde{R}_1 \psi_{11} - \psi_{21}^T \tilde{R}_1 \psi_{21} - 2 \left\{ \frac{1}{2} (d_2 - d(t))^2 \tilde{\zeta}^T(t) Z_1 \tilde{\zeta}(t) \right. \\ &\quad + 2(d_2 - d(t)) \tilde{\zeta}^T(t) N_1 [y(t-d(t)) - v_3] + \frac{1}{2} [(d_2 - d_1)^2 (d_2 - d(t))^2] \tilde{\zeta}^T(t) Z_2 \tilde{\zeta}(t) \\ &\quad + 2\tilde{\zeta}^T(t) N_2 [(d_2 - d(t)) [y(t-d_1) - y(t-d(t))]] + (d(t) - d_1) [y(t-d_1) - v_2] \left. \right\}] \\ &= \tilde{\zeta}^T(t) \Xi_3(d(t)) \tilde{\zeta}(t),\end{aligned}\quad (23)$$

where $\Xi_3(d(t))$ is given in (12). From Lemma 1 and Lemma 2, we obtain

$$- \int_{t-d_1}^t \dot{y}^T(s) d_1 W_1 \dot{y}(s) ds \leq [y(t) - y(t-d_1)]^T W_1 [y(t) - y(t-d_1)] + 3\tilde{\Omega}_1^T W_1 \tilde{\Omega}_1$$

and

$$-2 \int_{t-d_1}^t (d_1 - t + s) \dot{y}^T(s) W_2 \dot{y}(s) ds \leq -d_1^2 \tilde{\zeta}^T(t) M_1 \tilde{\zeta}(t) - 2d_1 \tilde{\zeta}^T(t) N_3 [y(t) - v_3].$$

From which it follows that

$$\begin{aligned}\delta_2(t) &\leq [y(t) - y(t-d_1)]^T W_1 [y(t) - y(t-d_1)] \\ &\quad + 3\tilde{\Omega}_1^T W_1 \tilde{\Omega}_1 - d_1^2 \tilde{\zeta}^T(t) M_1 \tilde{\zeta}(t) - 2d_1 \tilde{\zeta}^T(t) N_1 [y(t) - v_3], \\ &= \tilde{\zeta}^T(t) \Xi_{33} \tilde{\zeta}(t)\end{aligned}\quad (24)$$

where $\tilde{\Omega}_1 = y(t) + y(t-d_1) - 2v_3$, $\tilde{\Omega}_2 = y(t-d(t)) + y(t+d_2) - v_1$, $\tilde{\Omega}_3 = y(t-d_1) + y(t-d(t)) - v_2$,

$$\Xi_{33} := -\tilde{\varphi}_3^T \text{diag}\{W_1, 3W_1\} \tilde{\varphi}_3 + d_1^2 Z_3 + 2d_1 N_3 (e_1 - e_7) + 2d_1 (e_1 - e_7)^T N_3^T$$

Hence, from (23) and (24), we obtain

$$\dot{V}_3 \leq e^{\alpha t} \{ \tilde{\zeta}^T(t) [\Xi_3(d(t)) + \Xi_4] \tilde{\zeta}(t) \} + \alpha V_3 \quad (25)$$

where $\Xi_4 := \Xi_{30} + \Xi_{33}$. From (21), (22) and (25), we obtain $\dot{V}(t, y_t, \dot{y}_t)$ along the solution of the system (2) as

$$\dot{V}(t, x_t) \leq \alpha V(t, x_t) + e^{\alpha t} \xi^T \Delta(d(t), \dot{d}(t)) \xi(t)$$

where

$$\Delta(d(t), \dot{d}(t)) = \Xi_2(d(t), \dot{d}(t)) + \Xi_3(d(t)) + (\Xi_1 + \Xi_4).$$

Therefore, we have

$$\begin{aligned} \dot{V}(t, x_t) \leq & \alpha V(\cdot) + \xi^T(t) \hat{\Delta}(d(t), \dot{d}(t)) \xi(t) + \gamma \|w(t)\|^2 - y^T(t) [PE^T EP + YF^T FY] y(t) \\ & - y^T(t - d(t)) [PG^T GP] y(t - d(t)) - f^T(x(t)) f(x(t)) - g^T(x(t - d(t))) g(x(t - d(t))) \end{aligned} \quad (26)$$

where

$$\begin{aligned} \hat{\Delta}(d(t), \dot{d}(t)) &= \Xi_2(d(t), \dot{d}(t)) + \Xi_3(d(t)) + (\hat{\Xi}_1 + \Xi_4) \\ &= \Xi_2(d(t), \dot{d}(t)) + \Xi_3(d(t)) + \hat{\Xi}_4, \end{aligned}$$

$$\hat{\Xi}_1 = \Xi_1 + e_1^T [PE^T EP + YF^T FY] e_1 + e_2^T [PG^T GP] e_2 + e_{10}^T [I] e_{10} + e_{11}^T [I] e_{11},$$

and $\hat{\Xi}_4 = \hat{\Xi}_1 + \Xi_4$ is defined in ([23]). $\hat{\Delta}(d(t), \dot{d}(t))$ may be rewritten as

$$\hat{\Delta}(d(t), \dot{d}(t)) = d^2(t) \Delta_0 + d(t) \Delta_1 + \Delta_2 \quad (27)$$

where $\Delta_0 = Z_1 - Z_2$ and Δ_1, Δ_2 are $d(t)$ -independent real matrices. By Lemma (5), if $Z_1 - Z_2 \geq 0$ and the inequality in (14) holds, then $\hat{\Delta}(d(t), \dot{d}(t)) < 0, \forall d(t) \in [d_1, d_2], \forall \dot{d}(t) \in [\mu_1, \mu_2]$. Moreover, the terms $\hat{\Delta}(d(t), \dot{d}(t))$ can be recast in the sense of convex combination of $\dot{d}(t)$ as follows:

$$\hat{\Delta}(d(t), \dot{d}(t)) = (1 - \dot{d}(t)) \diamond_0 + \dot{d}(t) \diamond_1 + \diamond_2 \quad (28)$$

where $\diamond_0 = Q_2 - Q_3$ and $\diamond_1, \diamond_2$ are $\dot{d}(t)$ -independent real matrices. Hence by making use of the Schur complement lemma, it follows from (27), (28) and (6) that the inequality $\hat{\Delta}(d(t), \dot{d}(t)) < 0$ holds. Therefore, we have from inequality (26) that

$$\dot{V}(t, x_t) \leq \alpha V(t, x_t) \gamma w(t)^T w(t) - z(t)^T z(t) \quad (29)$$

and hence

$$\dot{V}(t, x_t) \leq \alpha V(t, x_t) + \gamma^2 w^T(t) w(t).$$

because of $z(t)^T z(t) \geq 0$. Multiplying both sides with $e^{-\alpha t}$, we obtain

$$e^{-\alpha t} \dot{V}(t, x_t) - \alpha e^{-\alpha t} V(t, x_t) < e^{-\alpha t} \gamma^2 w^T(t) w(t).$$

Hence, we have

$$\frac{d}{dt} (e^{-\alpha t} V(t, x_t)) = e^{-\alpha t} \dot{V}(t, x_t) - \alpha e^{-\alpha t} V(t, x_t).$$

So,

$$\frac{d}{dt}(e^{-\alpha t}V(t, x_t)) < e^{-\alpha t}\gamma^2 w^T(t)w(t).$$

Integrating both sides from 0 to t , we get

$$(e^{-\alpha t}V(t, x_t)) - V(0, x_0) < \gamma^2 \int_0^t e^{-\alpha s} w^T(s)w(s)ds,$$

which can be reformulated as

$$e^{-\alpha t}V(t, x_t) < V(0, x_0) + \gamma^2 \int_0^t e^{-\alpha s} w^T(s)w(s)ds,$$

note that

$$\alpha_1 x^T(t)Rx(t) \leq V(t, x_t), \forall : 0 \leq T. \quad (30)$$

Hence, we have

$$\begin{aligned} V_1(t, x_t) &= e^{\alpha t} x^T(t) P^{-1} x(t) \\ &= e^{\alpha t} x^T(t) R \bar{2} R \bar{2} P^{-1} R \bar{2} x(t) \\ &= e^{\alpha t} x^T(t) R \bar{2} \hat{P}^{-1} R \bar{2} x(t) \\ &\geq x^T(t) R \bar{2} \hat{P}^{-1} R \bar{2} x(t) \\ &\geq \lambda_{\min}(\hat{P})^{-1} x^T(t) R x(t) \end{aligned} \quad (31)$$

and $\alpha_1 = \lambda_{\min}(\hat{P})^{-1}$.

Consider

$$\begin{aligned} V(0, x_0) &= y^T(0)Py(0) + \int_{-d_1}^0 \dot{y}^T(s)Q_0\dot{y}(s)ds + \int_{-d_1}^0 [y^T(0)y^T(s)]Q_1[y^T(0) \ y^T(s)]^T ds \\ &\quad + \int_{-d(t)}^{-d_1} [y^T(0)y^T(s)]Q_2[y^T(0) \ y^T(s)]^T ds + \int_{-d_2}^{-d(t)} [y^T(0) \ y^T(s)]Q_3[y^T(0) \ y^T(s)]^T ds, \\ &\quad + \int_{-d_1}^0 [(d_1 + s)\dot{y}^T(s)d_1W_1\dot{y}(s) + (d_1 + s)^2\dot{y}^T(s)W_2\dot{y}(s)]ds \\ &\quad + d_{21} \int_{-d_2}^{-d_1} [(d_1 + s)\dot{y}^T(s)R_1\dot{y}(s) + (d_1 + s)^2\dot{y}^T(s)R_2\dot{y}(s)]ds \\ &\leq \lambda_{\max}(P)\|\phi\|^2 + \lambda_{\max}(Q_0)\|\phi\|^2 + \lambda_{\max}(Q_1)\|i\|^2 + \lambda_{\max}(Q_2)\|\phi\|^2 + \lambda_{\max}(Q_3)\|\phi\|^2 \\ &\quad + \lambda_{\max}(W_1)d_1^2\|\phi\|^2 + \lambda_{\max}(W_2)d_1^2\|\phi\|^2 + \lambda_{\max}(R_1)d_{21}^2\|\phi\|^2 + \lambda_{\max}(R_2)d_{21}\|\phi\|^2, \end{aligned}$$

so we have

$$V(0, x_0) \leq (\lambda_{\max}(P) + \lambda_{\max}(Q_0) + \lambda_{\max}(Q_1) + \lambda_{\max}(Q_2) + \lambda_{\max}(Q_3)) \quad (32)$$

$$\begin{aligned} &+ \lambda_{\max}(W_1)d_1^2 + \lambda_{\max}(W_2)\pi_6 + \lambda_{\max}(R_1)d_{21}^2 + \lambda_{\max}(R_2)d_{21}^3)c_1, \\ &= \alpha_2 c_1 \end{aligned} \quad (33)$$

where $\alpha_2 = \lambda_{\max}(P) + \lambda_{\max}(Q_0) + \lambda_{\max}(Q_1) + \lambda_{\max}(Q_2) + \lambda_{\max}(Q_3) + \lambda_{\max}(W_1)d_1^2 + \lambda_{\max}(W_2)\pi_6 + \lambda_{\max}(R_1)d_{21}^2 + \lambda_{\max}(R_2)d_{21}^3$.

Therefore, from (31), (33), it follows that

$$\alpha_1 e^{-\alpha t} x^T(t) R x(t) < e^{-\alpha t} V(t, x(t)) \leq \alpha_2 c_1 + \gamma^2 k, \forall t \in [0, T].$$

and hence from (30), we have

$$x^T(t) R x(t) < \left(\frac{\alpha_2 c_1 + \gamma^2 \beta k}{\alpha_1} \right) e^{\alpha t} \leq c_2, \forall t \in [0, T],$$

which implies that the closed-loop system is finite-time stable w.r.t $(c_1, c_2, T, R, d, \gamma)$. To complete the proof of the theorem, it remains to show the γ optimal level condition (5). For this, we consider the following relation

$$\dot{V}(t, x_t) + z^T(t) z(t) - \gamma^2 w^T(t) w(t) < 0.$$

Integrating both sides of above equation from t_0 to t , we get

$$\int_{t_0}^t [\dot{V}(t, x_t) + z^T(t) z(t) - \gamma^2 w^T(t) w(t)] dt < 0.$$

It follows that

$$\begin{aligned} \int_{t_0}^t [z^T(t) z(t) - \gamma^2 w^T(t) w(t)] dt &\leq V(t_0, x_{t_0}) - V(t, x_t) \\ &\leq 0. \end{aligned} \quad (34)$$

Therefore, under zero initial condition $x(t) = 0, t \in [-\tau_2, t_0]$, by letting $t \rightarrow +\infty$ in (34), we get

$$\int_{t_0}^{\infty} z^T(t) z(t) dt < \gamma^2 \int_{t_0}^{\infty} w^T(t) w(t) dt$$

which gives $\|z\|_2 < \gamma \|w\|_2$. This completes the proof. $\square$

5. Numerical Examples

In this section, we provide two numerical examples with their simulations to demonstrate the effectiveness of our results.

Example 1. Consider the nonlinear system with interval time-varying delays which was considered in [7]

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Dx(t - d(t)) + Bu(t) + Cw(t) + f(x(t), t) + g(x(t - d(t)), t) \\ z(t) &= Ex(t) + Gx(t - d(t)) + Fu(t) + f(x(t), t) + g(x(t - d(t)), t). \end{aligned}$$

We have used theorem 1 to evaluated the value of minimum γ for H_∞ control condition. Where

$$\begin{aligned} A &= \begin{bmatrix} -1.3 & 0.3 \\ 0.5 & 0.1 \end{bmatrix}, & D &= \begin{bmatrix} -0.01 & 0.02 \\ 0.03 & -0.04 \end{bmatrix}, \\ B &= \begin{bmatrix} 0.2 & 0 \\ 0.3 & 0 \end{bmatrix}, & C &= \begin{bmatrix} -0.02 & 0.01 \\ 0.02 & -0.03 \end{bmatrix}, \\ E = G &= \begin{bmatrix} 0.06 & -0.06 \\ -0.08 & 0.08 \end{bmatrix}, & F &= \begin{bmatrix} 0.8 & 0 \\ 0.6 & 0 \end{bmatrix}, \end{aligned}$$

$$f(\cdot) = g(\cdot) = 0.01 \begin{bmatrix} \sqrt{x_1^2(t) + x_2^2(t - \tau(t))} \\ \sqrt{x_2^2(t) + x_1^2(t - \tau(t))} \end{bmatrix},$$

$\beta_1 = \beta_2 = 0.01$, $\tau_1 = 0.3$, $\tau_2 = 0.5$, $\mu_1 = -0.1$, $\mu_2 = 0.1$, $\gamma = 4$, $k = 2$. And the condition (15) is satisfied with $\alpha = 0.6$, $T = 10$, $c_1 = 1$, $c_2 = 50\gamma = 4$. By using LMI Toolbox in Matlab, it can be shown that the constructed LMI in Theorem (1) is feasible. Further the H_∞ controller feedback gain matrix is obtained as:

$$K = YP^{-1} = \begin{bmatrix} -3.4638 & -6.8069 \\ -4.3243 & -4.1846 \end{bmatrix}.$$

Table 1, shows the value of minimum γ with $\mu_1 = -0.1$ and $\mu_2 = 0.1$ by using Theorem (1). In Table 2, we show the value of minimum γ with $\mu_1 = 0.05$ and $\mu_2 = 0.1$ by using Theorem (1)

Table 1. The value of the minimum allowable disturbance attenuation γ with $\mu_1 = -0.1$ and $\mu_2 = 0.1$.

$\beta_1 = \beta_2 = 0.01$			
Method	τ_1	τ_2	γ_{min}
By Theorem 1	0.1	0.3	0.2377
	0.1	0.5	0.2474
$\beta_1 = \beta_2 = 0.05$			
Method	τ_1	τ_2	γ_{min}
By Theorem 1	0.1	0.3	0.8991
	0.1	0.5	0.9643

Table 2. The value of the minimum allowable disturbance attenuation γ with $\mu_1 = 0.05$ and $\mu_2 = 0.1$.

$\beta_1 = \beta_2 = 0.01$			
Method	τ_1	τ_2	γ_{min}
By Theorem 1	0.1	0.5	0.2487

Example 2. [20] Consider the following nonlinear system with interval time-varying delay which was considered in

$$\dot{x}(t) = Ax(t) + Dx(t - d(t)) + Bu(t) + Cw(t) + f(x(t), t) + g(x(t - d(t)), t)$$

with

$$A = \begin{bmatrix} 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.02 \\ 0.02 & 0.01 & 0.02 \end{bmatrix}, \quad D = \begin{bmatrix} 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.01 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$f(x(t), t) = 0.0025 \sin(0.05)x(t)[0 \ 0 \ 1]^T$$

$$g(x(t - d(t)), t) = 0.0025 \sin(0.05)x(t)[1 \ 0 \ 0]^T$$

we investigate delay function $d(t) = 0.5|\sin(3t)| + 0.1$. Consider the finite-time of nonlinear system with respect to (c_1, c_2, T, d, γ) with different fixed times ($T = 2, 4, 6, 8, 10$). The maximum values of the norm of state

vector are 1.1780, 0.5706, 0.8510, 0.3974, 4.8264, 0.2091 and 0.0558 for $T = 2, 4, 6, 8, 10$, respectively. We have Figure 1 to show the trajectories of $x^T(t)x(t)$ of the closed-loop system with the condition $\phi(t) = [0.5, -0.5]$. Moreover, we set $\gamma = 0.5, d = 2$ and the initial function $\phi^T = [0.1t + 0.2, 0.1t + 0.2, 0.1t + 0.2], \forall t \in [-0.6, 0]$, $d_1 = 1 \leq d_2 = 2, \beta_1 = \beta_2 = 1.25 \times 10^{-4} \phi^T(t)\phi(t) = 0.5 \leq 0.5 = c_1$. From (31) and (33), we obtain $\alpha_1 = 0.9227$ and $\alpha_2 = 8.0883$. By solving Theorem using MATLAB toolbox a feasible solution (show some solution) is

$$\begin{aligned}
 P &= \begin{bmatrix} 1.1153 & -0.0597 & -0.0872 \\ -0.0597 & 1.1062 & -0.0932 \\ -0.0872 & -0.0932 & 1.0498 \end{bmatrix}, & Q_0 &= \begin{bmatrix} 0.569877 & -0.0011 & -0.0013 \\ -0.0011 & 0.5690 & -0.0019 \\ -0.0013 & -0.0019 & 0.5682 \end{bmatrix}, \\
 R_1 &= \begin{bmatrix} 0.2088 & -0.0003 & -0.0004 \\ -0.0003 & 0.2087 & -0.0005 \\ -0.0004 & -0.0005 & 0.2085 \end{bmatrix}, & R_2 &= \begin{bmatrix} 0.0551 & -0.0006 & -0.0009 \\ -0.0006 & 0.0550 & -0.0010 \\ -0.0009 & -0.0010 & 0.0544 \end{bmatrix}, \\
 W_1 &= \begin{bmatrix} 0.3957 & 0.0005 & 0.0007 \\ 0.0005 & 0.3959 & 0.0008 \\ 0.0007 & 0.0008 & 0.3964 \end{bmatrix}, & W_2 &= \begin{bmatrix} 4.8208 & -0.0112 & -0.0110 \\ -0.0112 & 4.8039 & -0.0223 \\ -0.0110 & -0.0223 & 4.7988 \end{bmatrix}, \\
 Q_1 &= \begin{bmatrix} 0.5284 & -0.0012 & -0.0015 & -0.0000 & -0.0000 & -0.0000 \\ -0.0012 & 0.5276 & -0.0021 & -0.0000 & 0.0000 & -0.0000 \\ -0.0015 & -0.0021 & 0.5266 & -0.0000 & 0.0000 & 0.0000 \\ -0.0000 & -0.0000 & -0.0000 & 0.8473 & 0.0009 & 0.0010 \\ -0.0000 & 0.0000 & 0.0000 & 0.0009 & 0.8485 & 0.0018 \\ -0.0000 & -0.0000 & 0.0000 & 0.0010 & 0.0018 & 0.8490 \end{bmatrix}, \\
 Q_2 &= \begin{bmatrix} 0.4602 & -0.0017 & -0.0025 & 0.0000 & -0.0000 & -0.0000 \\ -0.0017 & 0.4598 & -0.0028 & 0.0000 & -0.0000 & -0.0000 \\ -0.0025 & -0.0028 & 0.4582 & -0.0000 & -0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & 0.8663 & 0.0146 & 0.0224 \\ -0.0000 & -0.0000 & -0.0000 & 0.0146 & 0.8649 & 0.0214 \\ -0.0000 & -0.0000 & 0.0000 & 0.0224 & 0.0214 & 0.8802 \end{bmatrix}, \\
 Q_3 &= \begin{bmatrix} 1.8321 & 0.0103 & 0.0167 & -0.0000 & -0.0000 & -0.0000 \\ 0.0103 & 1.8297 & 0.0151 & 0.0000 & 0.0000 & -0.0000 \\ 0.0167 & 0.0151 & 1.8413 & -0.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & -0.0000 & 2.0919 & 0.0091 & 0.0141 \\ -0.0000 & 0.0000 & 0.0000 & 0.0091 & 2.0907 & 0.0133 \\ -0.0000 & -0.0000 & -0.0000 & 0.0141 & 0.0133 & 2.1004 \end{bmatrix}.
 \end{aligned}$$

the H_∞ controller feedback gain can be computed as

$$K = YP^{-1} = \begin{bmatrix} -36.12 & 21.03 & -16.49 \\ -1.34 & -9.32 & -5.16 \\ 22.59 & -15.59 & 4.45 \end{bmatrix}.$$

Table 3 to shows the smallest value of c_2 with different $T = 2, 4, 6, 8, 10$

Table 3. Shows the smallest value of c_2 with different $T = 2, 4, 6, 8, 10$

T	2	4	6	8	10
By Theorem 1	5.5527	6.2606	7.0589	7.9589	8.9736
Stojanovic [20]	NF	NF	NF	NF	NF

Remark 3. From Table 3, the table lists the smallest values of c_2 with different $T = 2, 4, 6, 8, 10$. It is obvious that condition in [20] is not-feasible (NF) because $\dot{d}(t) = 1.5$ for all T . It not consistent with the conditions $\dot{d}(t) = 1.5 < \rho < 1$ in [20].

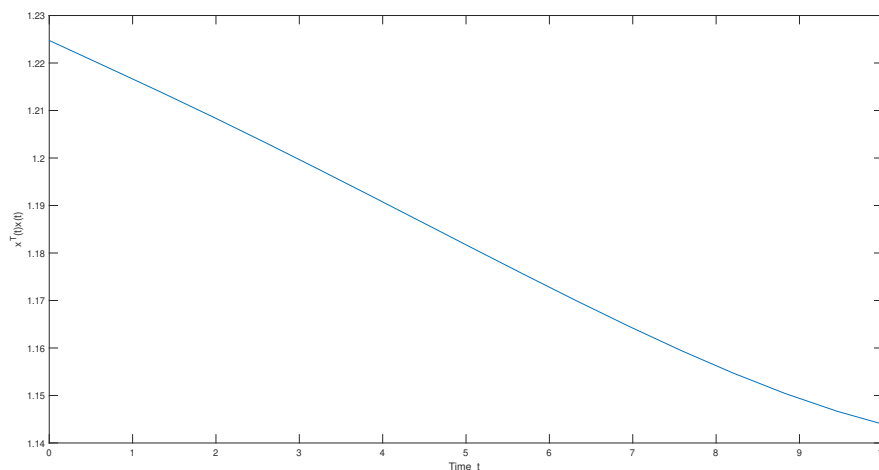


Figure 1. Trajectories of $x^T(t)x(t)$ in Example 2, the unit of T which is second.

6. Conclusions

In this paper, finite-time H_∞ control for nonlinear systems with time-varying delay is studied. By using a set of improved Lyapunov-Krasovskii functional including with some integral terms, a matrix-based on quadratic convex, combined with Wirtinger inequalities and some useful integral inequalities were proposed which illustrate the effectiveness of the obtained result in the numerical part. However, the improved method for the restriction on the upper bound of the delay derivative should be considered which means that a fast time-varying delay is allowed without any requirement on the derivative. New sufficient conditions of finite-time boundedness for above-mentioned class of system were given in term of linear matrix inequalities (LMIs).

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