รายงานวิจัยฉบับสมบูรณ์

โครงการ

การเลือกแบบต่อเนื่องที่อธิบายได้ในโครงสร้างขยายของฟีลด์ p-adic

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1 พฤษภาคม 2563

สัญญาเลขที่ MRG6180097

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและ
จุฬาลงกรณ์มหาวิทยาลัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย
สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

บทคัดย่อ

รหัสโครงการ: MRG6180097

ชื่อโครงการ: การเลือกแบบต่อเนื่องที่อธิบายได้ในโครงสร้างขยายของฟีลด์ p-adic

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บทคัดย่อ:

ให้ $E \subseteq Q_p^n$ และ T เป็นฟังก์ชันค่าเซตจาก E ไปยัง Q_p^m. เราพิสูจน์ว่าถ้า T เป็นฟังก์ชันกึ่งพีชคณิต p-adic และกึ่งต่อเนื่องจากด้านล่าง (นั่นคือ, สำหรับทุก $x_0 \in X$, $y_0 \in T(x_0)$ และย่านใกล้เคียง V ของ y_0, มีย่านใกล้เคียง U ของ x_0 ซึ่ง $\mathrm{T}(x) \cap V \neq \emptyset$ สำหรับทุก $x \in U$) และ $T(x)$ เป็นเซตปิดสำหรับทุก $x \in E$ แล้วจะมีฟังก์ชันกึ่งพีชคณิต p-adic ที่ต่อเนื่อง $f: E \to Q_p^m$ ซึ่งมีการเลือกของ T (นั่นคือ $f(x) \in T(x)$ สำหรับทุก $x \in E$.) ยิ่งไปกว่านั้นเราได้พัฒนาผลลัพธ์และได้ผลลัพธ์ที่ดีขึ้น คือ ถ้า T เป็นฟังก์ชันค่าเซตกึ่งพีชคณิต p-adic ซึ่งมีการเลือกแบบต่อเนื่องแล้ว T มีการเลือกกึ่งพีชคณิต p-adic แบบต่อเนื่อง นอกจากนี้เราสามารถนำผลลัพธ์นี้ไปประยุกต์ใช้ได้ดังนี้

อันดับแรก พิจารณาสมการ:

(*) $F(g_1, \ldots, g_k, y_1, \ldots, y_m) = 0$

โดยที่ $g_1, \ldots, g_k: E \to Q_p$ และ $F: Q_p^{k+m} \to Q_p$ เป็นฟังก์ชันกึ่งพีชคณิต p-adic แบบต่อเนื่อง และ $y_1, \ldots, y_m: E \to Q_p$ เป็นฟังก์ชันต่อเนื่องที่ไม่ทราบค่า เราพิสูจน์ว่าถ้ามีฟังก์ชันต่อเนื่อง $y_1, \ldots, y_m: E \to Q_p$ ที่เป็นคำตอบของสมการ (*) แล้วจะมีฟังก์ชันกึ่งพีชคณิต p-adic แบบต่อเนื่อง $y_1, \ldots, y_m: E \to Q_p$ ที่เป็นคำตอบของสมการ (*) ด้วย ในลำดับถัดไป ให้ E เป็นสับเซตปิดของ Q_p^n เราค้นพบการกำหนดลักษณะของการจำกัดของฟังก์ชันกึ่งพีชคณิต p-adic แบบต่อเนื่องลงไปที่ E นอกจากนี้ ให้ $(G,\circ)$ เป็นโมนอยด์ซึ่ง G เป็นเซตของฟังก์ชันกึ่งพีชคณิต p-adic แบบต่อเนื่องจาก Q_p^m ไปยัง Q_p^m ทั้งหมด และ $\circ$ เป็นการประกอบ ในที่นี้เราค้นพบการกำหนดลักษณะของฟังก์ชันที่มีตัวผกผันทางขวา และการกำหนดลักษณะของฟังก์ชันที่มีตัวผกผันทางซ้าย

คำหลัก : ฟีลด์ p-adic, ฟังก์ชันกึ่งพีชคณิต, การเลือก

Abstract

Project Code : MRG6180097

Project Title : Definable continuous selections in expansions of the p-adic field

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Project Period : May 2, 2018 – May 1, 2020

Abstract:

Let $E \subseteq Q_p^n$ and T be a set-valued map from E to Q_p^m. We prove that if T is p-adic semi-algebraic, lower semi-continuous (that is, for every $x_0 \in X$, $y_0 \in T(x_0)$ and a neighborhood V of y_0, there is a neighborhood U of x_0 such that for every $x \in U$, $\mathrm{T}(x) \cap V \neq \emptyset$) and $T(x)$ is closed for every $x \in E$, then there is a p-adic semi-algebraic continuous function $f: E \to Q_p^m$ that is a selection of T (that is, $f(x) \in T(x)$ for all $x \in E$.) In addition, we strengthen the result and obtain that if T is p-adic semi-algebraic and has a continuous selection, then T has p-adic semi-algebraic continuous selection. Moreover, we obtain three applications of this result.

First, consider the equation:

(*) $\qquad F(g_1, \dots, g_k, y_1, \dots, y_m) = 0,$

where $g_1, \dots, g_k: E \to Q_p$ and $F: Q_p^{k+m} \to Q_p$ are p-adic semi-algebraic and continuous and $y_1, \dots, y_m: E \to Q_p$ are unknown continuous functions. We prove that if there are continuous functions $y_1, \dots, y_m: E \to Q_p$ solving (*), then there are also p-adic semi-algebraic continuous functions $y_1, \dots, y_m: E \to Q_p$ that satisfy (*). Next, let E be closed in Q_p^n. We give a characterization of the restriction of p-adic semi-algebraic continuous functions to E. Finally, let $(G, \circ)$ be the monoid where G is the set of p-adic semi-algebraic continuous functions from Q_p^m to Q_p^m with the composition $\circ$. Here, we obtain a characterization of right invertible elements and a characterization of left invertible elements.

Keywords : p-adic fields, semi-algebraic functions, selections

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Executive Summary

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1. Introduction to Research

A set-valued map from a set X to another set Y is a map from X to the power of Y. For a set-valued map T from X to Y, a selection of T is a map f from X to Y such that $f(x) \in T(x)$ for every $x \in X$. E. Michael is one of pioneers on the question of the existence of continuous selections of set-valued maps. Michael's Selection Theorem [11], which is an important tool in many branches of mathematics (see e.g. [10] and [12]) asserts that:

Let X be a paracompact topological space, Y be a Banach space and T be a set-valued map from X to Y. If $T(x)$ is closed and convex for every $x \in X$ and T is lower semi-continuous (that is, for every $x_0 \in X$, $y_0 \in T(x_0)$ and a neighborhood V of y_0, there is a neighborhood U of x_0 such that for every $x \in U, T(x) \cap V \neq \emptyset$), then T has a continuous selection.

The give construction involves an infinitary process that can produce a continuous selection that is far removed from how T arises. This gives rise to the following question:

Let T be a set-valued map. Suppose we know that T has a continuous selection. If T is well behaved in some prescribed sense, does T has a continuous selection that is similarly well behaved?

To make this question precise, we employ notions from first-order logic: definability in expansions of the p-adic field. We now restate the question as follows:

Let T be a set-valued map from a subset of Q_p^n. to Q_p^m. If T has a continuous selection, does T have a continuous selection that is definable in $(Q_p, +, \cdot, T)$?

In this research, we study the above question in the context of p-adic semi-algebraic sets. In particular, if T is p-adic semi-algebraic and has a continuous selection, does T has a p-adic semi-algebraic continuous selection?

2. Literature Review

To answer the main question, we first ask whether there is a definable version of Michael's Selection Theorem in the p-adic semi-algebraic context. In [2], M. Aschenbrenner and A. Thamrongthanyalak prove analogues of Michael's Selection Theorem in o-minimal structures. The constructions involve the existence of the least norm selections, definable Tietze Extension Theorem and Cell Decomposition Theorem in o-minimal structures. The definable Tietze Extension Theorem asserted that every function on a closed set that is definable in a definably complete expansion of a real closed field has a continuous extension to the whole space that is also definable in the same structure (see [1]). Next, the Cell Decomposition Theorem is an important tool in the study of geometry of o-minimal sets. This theorem implies that every definable set in an o-minimal expansion of an ordered divisible abelian group has only finitely many connected components. We refer to [6] for more on o-minimal structures. In addition, the convexity of valued of maps also plays an important role in the construction in [2]. In [5], M. Czapla and W. Pawlucki relaxed the convexity condition when the dimension of the domain is 1. Another generalization in o-minimal expansions of the real field was studied in [14]. From [16], we know that a definable version of Michael's Selection Theorem also holds in d-minimal expansions of the real field (which is a generalization of o-minimal structures).

In [6] and [9], the model theory of the p-adic field were introduced. Definable sets in this context possess good geometric properties. A subset of Q_p^n is called p-adic semi-algebraic if it is definable in the p-adic field structure. One of important tools in the study of p-adic semi-algebraic sets is the p-adic Cell Decomposition Theorem. In particular, every p-adic semi-algebraic set can be decomposed into finitely many p-adic cells. This result can be considered as an analogue of Cell Decomposition Theorem in o-minimal structures. Later, R. Cluckers prove an p-adic analytic version of the p-adic Cell Decomposition Theorem in [3].

In [15], A. Thamrongthanyalak proved that every p-adic semi-algebraic continuous function on a closed subset of Q_p^n has a p-adic semi-algebraic continuous extension to the ambient space Q_p^n. This result can be considered as a p-adic semi-algebraic version of Tietze Extension Theorem.

Equipping Q_p by the usual ultrametric, we obtain that every point in a ball is its center (see [13]). Therefore, the least norm selection does not exist in the p-adic context. Note that in the reals, the least norm selection of convex sets is a 1-Lipschitz map. In [4],

R. Cluckers and F. Martin prove that every p-adic semi-algebraic Lipschitz function on a subset of Q_p^n is the restriction of a p-adic semi-algebraic Lipschitz function on Q_p^n with the same Lipschitz constant. This provides controls on oscillations of p-adic semi-algebraic functions.

In [8], C. Fefferman and J. Kollar raised the following question:

Let $g_1, \dots, g_k: R^n \to R$ be polynomials in n indeterminates. Consider Suppose that there are continuous functions $f, y_1, \dots, y_k$ such that $f = g_1 y_1 + \cdots + g_k y_k$. Are there polynomials $f, y_1, \dots, y_k$ that solves this equation?

They found that the answer is `no'. However, this equation admits a solution that are rational functions.

3. Objectives

3.1 To prove that if T is a p-adic semi-algebraic set valued-map from a subset of Q_p^n to Q_p^m that has a continuous selection, then T has a p-adic semi-algebraic continuous selection.

3.2 To find other criterions that guarantee the existence of definable continuous selections.

3.3 To use the positive answer to the main question to solve Fefferman and Kollar's question on continuous solutions of linear equations.

4. Research Methodology

4.1 Review related literatures.

4.2 Modify techniques used in the proof of definable Michael's Selection Theorem in o-minimal structures.

4.3 Prove that if T is a p-adic semi-algebraic set valued-map from a subset of Q_p^n to Q_p^m that has a continuous selection, then T has a p-adic semi-algebraic continuous selection.

4.4 Use the answer to the main question to solve C. Fefferman and J. Kollar's question on continuous solutions of linear equations.

Result and Conclusion

Throughout, let p be a fixed, but arbitrary, prime number. We equip the p-adic field with the usual p-adic valuation v. A *p-adic semi-algebraic set* is a subset of Q_p^n that is a finite boolean combination of sets of the forms $\{(x_1,\dots,x_n) \in Q_p^n : q(x_1,\dots,x_n) = 0\}$ and $\{(x_1,\dots,x_n) \in Q_p^n : x_i = \lambda y^k \text{ for some } y \in Q_p\}$ where is a polynomial over Q_p, $\lambda \in Q_p$, $k \in N$ and $i = 1,\dots,n$. Let $E \subseteq Q_p^n$. We say that a function $f: E \to Q_p^m$ is *p-adic semi-algebraic* if the set $\{(x,y) \in E \times Q_p^m : f(x) = y\}$ is p-adic semi-algebraic. Similarly, a set-valued map T from E to Q_p^m is *p-adic semi-algebraic* if the set $\{(x,y) \in E \times Q_p^m : y \in T(x)\}$ is p-adic semi-algebraic.

From now, we fix $\mathrm{E} \subseteq \mathrm{Q}_\mathrm{p}^\mathrm{n}$ and a set-valued map T from E to Q_p^m. As consequences of the p-adic Cell Decompostion Theorem, we have:

Lemma 1 *Let $f: E \to Q_p^m$ be semi-algebraic. Then there is a semi-algebraic set $X \subseteq E$ such that $dim(E \setminus X) < dim(E)$ and the restriction $f \restriction X$ is continuous.*

Lemma 2 *Let $A \subseteq Q_p^{n+1}$ be semi-algebraic and $\pi: Q_p^{n+m} \to Q_p^n$ be the projection onto the first n coordinates. Then there exists a semi-algebraic function $f: \pi A \to Q_p^n$ such that the graph of f is contained in A.*

By these two lemmas, we obtain that:

Lemma 3 *Let $A \subseteq Q_p^{n+1}$ be semi-algebraic and $\pi: Q_p^{n+m} \to Q_p^n$ be the projection onto the first n coordinates. Then there exist a semi-algebraic set $X \subseteq \pi A$ and a semi-algebraic continuous map $f: X \to Q_p^m$ such that $dim(\pi A \setminus X) < dim(\pi A)$ and the graph of f is contained in A.*

Let $Y \subseteq Q_p^\mathrm{m}$. A map $f: Q_p^m \to Q_p^m$ is a *retraction* from Q_p^m to Y if f is continuous, the range of f is Y and the restriction of f to Y is the identity map on Y; and a map $g: Y \to Y$ is *nonexpansive* if $v(g(x) - g(y) \geq v(x-y)$ for all $x, y \in Y$.

Lemma 4 *Let $r: E \times Q_p^m \to Q_p^m$ be a semi-algebraic map such that $r(x,-)$ is nonexpansive for every $x \in E$. There is a semi-algebraic set $E_0 \subseteq E$ such that $dim(E \setminus E_0) < dimE$ and $r \restriction (E_0 \times Q_p^m)$ is continuous.*

Recall that a *selection* of T is a map $f: E \to Q_p^m$ such that $f(x) \in T(x)$ for all $x \in E$. Now we obtain the following theorem.

Theorem 5 *If T is p-adic semi-algebraic and lower semi-continuous (that is, for every $x_0 \in X, y_0 \in T(x_0)$ and a neighborhood V of y_0, there is a neighborhood U of x_0 such that for every $x \in U, T(x) \cap V \neq \emptyset$) and $T(x)$ is closed for every $x \in E$, then T has a p-adic semi-algebraic continuous selection.*

In addition, we study generalization of this theorem.

The *Glaeser refinement* of T is a set-valued map T' from E to Q_p^n defined by

$$T'(x_0) := \{y \in T(x_0): v\big(y, T(x)\big) \to \infty \text{ as } E \ni x \to x_0\} \text{ for } x_0 \in E.$$

Next we define a sequence $\big(T^{(k)}\big)_{k \in N}$ inductively by $T^{(0)} := T$ and $T^{(k+1)} := \big(T^{(k)}\big)'$. We found that:

Lemma 6 *If T is p-adic semi-algebraic, then $T^{(n)}$ is p-adic semi-algebraic and lower semi-continuous.*

Lemma 7 *If T has a continuous selection, then $T^{(n)} \neq \emptyset$ for all $x \in E$.*

We obtain the following characterization.

Theorem 8 *Suppose T is p-adic semi-algebraic and $T(x)$ is closed for every $x \in E$. Then the following are equivalent:*

1. *T has a continuous selection;*
2. *$T^{(n)} \neq \emptyset$ for all $x \in E$;*
3. *T has a p-adic semi-algebraic continuous selection.*

When $dim\, E = 1$, we can show that:

Theorem 9 *Suppose $dim\, E = 1$ and T is p-adic semi-algebraic. Then T has a continuous selection if and only if T has a p-adic semi-algebraic continuous selection.*

In addition to these main theorems, we found three applications of these results.

First, consider the equation:

(*) $\quad F(g_1, \dots, g_k, y_1, \dots, y_m) = 0,$

where $g_1, \dots, g_k: E \to Q_p$ and $F: Q_p^{k+m} \to Q_p$ are p-adic semi-algebraic and continuous and $y_1, \dots, y_m: E \to Q_p$ are unknown continuous functions. By Theorem 9, we can show that:

Proposition 10 *If there are continuous functions* $y_1, \dots, y_m : E \to Q_p$ *solving (*), then there are also p-adic semi-algebraic continuous functions* $y_1, \dots, y_m : E \to Q_p$ *that satisfy (*).*

Next, let E be closed in Q_p^n and $f: E \to Q_p^m$ be p-adic semi-algebraic. Define a set-valued map T_f from Q_p^n to Q_p^m by $T_f(x) := \{f(x)\}$ if $x \in E$; and $T_f(x) := Q_p^m$ if $x \in Q_p^m \setminus E$. We obtain that:

Proposition 11 *The function* f *is the restriction of a continuous function from* Q_p^n *to* $\mathrm{Q}_\mathrm{p}^\mathrm{m}$ *if and only if* $\left(T_f\right)^{(n)}(x) \neq \emptyset$ *for every* $\mathrm{x} \in \mathrm{Q}_\mathrm{p}^\mathrm{n}$.

Finally, let $(G,\circ)$ be the monoid where G is the set of p-adic semi-algebraic continuous functions from Q_p^m to Q_p^m and the group operation $\circ$ is the composition operation. Let $\mathrm{h} \in \mathrm{G}$. Define the set-valued map h^{-1} from $\mathrm{Q}_\mathrm{p}^\mathrm{m}$ to $\mathrm{Q}_\mathrm{p}^\mathrm{m}$ by $\mathrm{h}^{-1}(x)$ is the pre-image of $\{\mathrm{x}\}$ under h. Here, we give a characterization of right invertible elements and a characterization of left invertible elements.

Proposition 12 *Let* $\mathrm{h} \in \mathrm{G}$. *Then* h *is right invertible under* $\circ$ *if and only if* h *is surjective and* $(\mathrm{h}^{-1})^{(\mathrm{m})}(\mathrm{x}_0) \neq \emptyset$ *for every* $\mathrm{x}_0 \in \mathrm{Q}_\mathrm{p}^\mathrm{m}$.

Proposition 13 *Let* $\mathrm{h} \in \mathrm{G}$. *Then* h *is left invertible under* $\circ$ *if and only if* h *is injective and* $(T_{h^{-1}})^{(n)}(\mathrm{x}_0) \neq \emptyset$ *for every* $\mathrm{x}_0 \in \mathrm{Q}_\mathrm{p}^\mathrm{m}$.

Future Researches

In the future works, we know that the concept of p-adic semi-algebraic sets can be generalized to sets definable in P-minimal expansions of a p-adically closed field. Therefore, it is very interesting to know whether analogues of our results hold in this context.

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Output

Output จากโครงการวิจัยที่ได้รับทุนจาก สกว.

ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ

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Appendix

Math. Log. Quart. **66,** No. 1, 73–81 (2020) / **DOI** 10.1002/malq.201900024

On p-adic semi-algebraic continuous selections

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Received 30 March 2019, revised 11 November 2019, accepted 20 November 2019
Published online 25 February 2020

Let $E \subseteq \mathbb{Q}_p^n$ and T be a set-valued map from E to $\mathbb{Q}_p^m$. We prove that if T is p-adic semi-algebraic, lower semi-continuous and $T(x)$ is closed for every $x \in E$, then T has a p-adic semi-algebraic continuous selection. In addition, we include three applications of this result. The first one is related to Fefferman's and Kollár's question on existence of p-adic semi-algebraic continuous solution of linear equations with polynomial coefficients. The second one is about the existence of p-adic semi-algebraic continuous extensions of continuous functions. The other application is on the characterization of right invertible p-adic semi-algebraic continuous functions under the composition.

1 Introduction

In 1956, Michael presented a series of papers on the existence of continuous selections of set-valued maps [14–16]. For sets X and Y, a *set-valued map from X to Y* (denoted by $T : X \rightrightarrows Y$) is a map from X to the power set of Y. Suppose we equip X and Y with topologies and let $T : X \rightrightarrows Y$. A *continuous selection* of T is a continuous map $f : X \to Y$ such that $f(x) \in T(x)$ for every $x \in X$. We say that T is *lower semi-continuous* if for every $x_0 \in X$, $y_0 \in T(x_0)$ and a neighborhood V of y_0, there is a neighborhood U of x_0 such that for every $x \in U$, $T(x) \cap V \neq \varnothing$. In [15, Theorem 1.2], Michael asserted "if X is a zero-dimensional paracompact space, Y is a complete metric space, and $T(x)$ is closed and nonempty for every $x \in X$, then T has a continuous selection." The given construction involves an infinite iterated procedure which makes the selection far removed from how the set-valued map arose. Therefore, this gives rise to the following question:

Question 1.1 If T is well behaved in some prescribed sense, is it possible to find a continuous selection that is similarly well behaved?

This paper discusses the above problem in the p-adic semi-algebraic context. (Note that similar questions in the context of the reals were discussed in [2, 3, 7, 21].)

Throughout, let p be a fixed prime number, $\mathbb{Q}_p$ be the set of p-adic numbers with the p-adic valuation $v : \mathbb{Q}_p \to \mathbb{Z} \cup \{+\infty\}$ and E denote a subset of some $\mathbb{Q}_p^n$. We equip $\mathbb{Q}_p$ with the topology induced by the p-adic valuation. Let $T : E \rightrightarrows \mathbb{Q}_p^m$. To make the question precise, we employ first-order logic. Our language of valued fields is the language $\{+, \cdot, -, 0, 1\}$ of rings augmented by a binary relation symbol Div. For $x, y \in \mathbb{Q}_p$, we let x Div y if and only if $v(x) \geq v(y)$. The question can now be restated as follows.

Question 1.2 If T has a continuous selection, does it also have a continuous selection that is definable in $(\mathbb{Q}_p; +, \cdot, -, 0, 1, \mathrm{Div}, T)$ where $(\mathbb{Q}_p; +, \cdot, -, 0, 1, \mathrm{Div}, T)$ is the expansion of the p-adic valued field by a predicate T and the word "definable" means "definable possibly with parameters"?

In this paper, we restrict the class of set-valued maps under consideration to p-adic semi-algebraic set-valued maps. A subset of $\mathbb{Q}_p^n$ is called *p-adic semi-algebraic* (or *semi-algebraic* for short) if it is a boolean combination of sets of the form $\{x \in \mathbb{Q}_p^n :$ there exists $y \in \mathbb{Q}_p$ such that $f(x) = y^k\}$ where $f(X)$ is a polynomial with p-adic coefficients and indeterminates $X = (X_1, \dots, X_n)$ and $k \in \mathbb{N}$. It is known that the class of semi-algebraic sets is the same as the class of definable sets in $(\mathbb{Q}_p; +, \cdot, -, 0, 1, \mathrm{Div})$. We say that a set-valued map $T : E \rightrightarrows \mathbb{Q}_p^m$ is

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p-adic semi-algebraic if the graph of T, $\{(x, y) \in E \times \mathbb{Q}_p^m : y \in T(x)\}$, is semi-algebraic. Similarly, a function $f : E \to \mathbb{Q}_p^m$ is *p-adic semi-algebraic* if the graph of f, $\{(x, f(x)) : x \in E\}$, is semi-algebraic. The following is a main result.

Theorem 1.3 *If T is semi-algebraic and lower semi-continuous, and $T(x)$ is closed and nonempty for every $x \in E$, then T has a semi-algebraic continuous selection.*

In addition, by using the same techniques, we have an analogue of Theorem 1.3 in the p-adic subanalytic context, i.e., if T is subanalytic and and lower semi-continuous, and $T(x)$ is closed and nonempty for every $x \in E$, then T has a subanalytic continuous selection. We further investigate the main question when T is not necessarily lower semi-continuous and obtain the following.

Theorem 1.4 *If T is semi-algebraic and has a continuous selection, and $T(x)$ is closed for every $x \in E$, then T has a semi-algebraic continuous selection.*

In addition, we show that the closeness assumption is not necessary when $\dim E = 1$.

Theorem 1.5 *Suppose* $\dim E = 1$ *and T is semi-algebraic. Then T has a continuous selection if and only if T has a semi-algebraic continuous selection.*

As consequences of the proofs of the above theorems, we also know that analogous results hold for finite field extensions of $\mathbb{Q}_p$. We also include two applications to illustrate some uses of our main results. The first application is related to a result of Fefferman and Kollár. In [10], they showed (using algebraic-geometric techniques) that if the equation $f = g_1 y_1 + \cdots + g_m y_m$, where $f, g_1, \ldots, g_m$ are polynomials with p-adic coefficients and n indeterminates, has a continuous solution (i.e., there are continuous functions $y_1, \ldots, y_m : \mathbb{Q}_p^n \to \mathbb{Q}_p$ that satisfy the equation), then it also has a semi-algebraic continuous solution. Using our main results, we introduce a new approach and obtain the following generalization.

Corollary 1.6 *Let $g_1, \ldots, g_k : \mathbb{Q}_p^n \to \mathbb{Q}_p$ and $F : \mathbb{Q}_p^{k+m} \to \mathbb{Q}_p$ be semi-algebraic and continuous. If there are continuous functions $y_1, \ldots, y_m : \mathbb{Q}_p^n \to \mathbb{Q}_p$ such that*

$$F(g_1, \ldots, g_k, y_1, \ldots, y_m) = 0, \tag{*}$$

then there are semi-algebraic continuous functions $y_1, \ldots, y_m$ that satisfy (∗).

Observe that when $F(z_0, z_1, \ldots, z_m, y_1, \ldots, y_m) = z_0 - z_1 y_1 - z_2 y_2 - \cdots - z_m y_m$, the equation (∗) is equivalent to the linear equation under consideration in [10]. Next, let $E \subseteq E' \subseteq \mathbb{Q}_p^n$ be semi-algebraic and $f : E \to \mathbb{Q}_p^m$ be semi-algebraic. Recall that an *extension* of f to E' is a map $g : E' \to \mathbb{Q}_p^m$ such that $g(x) = f(x)$ for every $x \in E$. By [20], we know that if E is closed in E' and f is continuous, then f admits a semi-algebraic continuous extension to E'. Therefore, it is natural to ask how to determine whether f admits a semi-algebraic continuous extension to E' (when E is not necessarily closed in E').

For all $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, let $v(x) = \min\{v(x_i) : i \in \{1, \ldots, n\}\}$. Note that v induces the usual topology on $\mathbb{Q}_p^n$ and satisfies the *ultrametric inequality*, i.e., for all $x, y, z \in \mathbb{Q}_p^n$, we have that $v(x - y) \geq \min\{v(x - z), v(z - y)\}$. For $Y \subseteq \mathbb{Q}_p^n$ and $x \in \mathbb{Q}_p^n$, let $v(x, Y) = \inf\{v(x - y) : y \in Y\}$. For each $\delta \in \mathbb{Z}$ and $x \in \mathbb{Q}_p^n$, let $B_\delta(x)$ denote the box $\{y : v(x - y) > \delta\}$. Let $T : E \rightrightarrows \mathbb{Q}_p^m$ be a set-valued map. Let $x_0 \in E$ and $y \in \mathbb{Q}_p^m$. We say that $v(y, T(x)) \to \infty$ as $E \ni x \to x_0$ if for every $t \in \mathbb{Q}_p \backslash \{0\}$ there is $s \in \mathbb{Q}_p \backslash \{0\}$ such that for all $x \in E \cap B_{v(s)}(x_0)$, $v(y, T(x)) > v(t)$. The Glaeser refinement of T is the set-valued map $T' : E \rightrightarrows \mathbb{Q}_p^m$ defined by

$$T'(x_0) := \{y \in T(x_0) : v(y, T(x)) \to \infty \text{ as } x \to x_0\}$$

for $x_0 \in E$. For each $k \in \mathbb{N}$, let $T^{(k)} : E \rightrightarrows \mathbb{Q}_p^m$ be the k-th time Glaeser refinement of T. Let $T_f : E' \rightrightarrows \mathbb{Q}_p^m$ be defined by

$$T_f(x) = \begin{cases} \{f(x)\}, & \text{if } x \in E, \\ \mathbb{Q}_p^m, & \text{if } x \in E' \backslash E, \end{cases}$$

for every $x \in E'$. We show the following consequence.

Corollary 1.7 *The function f admits a semi-algebraic continuous extension to E' if and only if $(T_f)^{(n)}(x) \neq \varnothing$ for every $x \in E'$.*

The other application is on the characterization of semi-algebraic continuous functions that are right invertible under the composition operation. Let n be a positive integer. Recall that the set of semi-algebraic continuous functions from $\mathbb{Q}_p^n$ to $\mathbb{Q}_p^n$ with the composition operation is a monoid but is not a group. For any map $h : \mathbb{Q}_p^n \to \mathbb{Q}_p^n$, let $h^{-1} : \mathbb{Q}_p^n \rightrightarrows \mathbb{Q}_p^n$ be a set-valued map defined by $h^{-1}(x)$ is the pre-image of $\{x\}$ under h. Now, we obtain the following result.

Corollary 1.8 *Let $h : \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ be semi-algebraic and continuous. Then h is right invertible under the composition operation $\circ$ if and only if h is surjective and $(h^{-1})^{(n)}(x) \neq \varnothing$ for every $x \in \mathbb{Q}_p^n$.*

We fix our conventions and notations: Throughout this paper, d, k, m, and n will range over the set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ of natural numbers. For a set $S \subseteq \mathbb{Q}_p^n$ we denote by $\operatorname{cl} S$ the closure of S.

2 p-adic semi-algebraic sets and the proof of Theorem 1.3

In this section, we recall some properties of p-adic semi-algebraic sets used in our proof of Theorem 1.3. For $E \subseteq \mathbb{Q}_p^n$, let $\dim E$ denote the largest k such that there is a coordinate projection from $\mathbb{Q}_p^n$ to $\mathbb{Q}_p^k$ where the image of E has nonempty interior. Obviously, we have $\dim E = n$ if and only if E has nonempty interior. It is known that (1) if $E_1, E_2 \subseteq \mathbb{Q}_p^n$ are semi-algebraic, then $\dim(E_1 \cup E_2) = \max\{\dim E_1, \dim E_2\}$; (2) if E is semi-algebraic and $\dim E = 0$, then E is finite; and (3) if E is semi-algebraic, then $\dim(\operatorname{cl} E \backslash E) < \dim E$ and $\dim E = \dim(\operatorname{cl} E)$; cf., e.g., [12].

Throughout this paper, we assume $E \subseteq \mathbb{Q}_p^n$ and $T : E \rightrightarrows \mathbb{Q}_p^m$ unless stated otherwise. The concept of cells is a corner stone of the study of semi-algebraic sets. Cell Decomposition Theorem (cf. [8, 18] for more information) provides that every semi-algebraic set is a finite disjoint union of cells. As consequences, we have:

Lemma 2.1 *Let $f : E \to \mathbb{Q}_p^m$ be semi-algebraic. Then there is a semi-algebraic set $X \subseteq E$ such that $\dim(E\backslash X) < \dim(E)$ and the restriction $f \restriction X$ is continuous.*

Lemma 2.2 *Let $A \subseteq \mathbb{Q}_p^{n+m}$ be semi-algebraic and $\pi : \mathbb{Q}_p^{n+m} \to \mathbb{Q}_p^n$ be the projection onto the first n coordinates. Then there exists a semi-algebraic map $f : \pi A \to \mathbb{Q}_p^m$ such that the graph of f is contained in A.*

The following corollary follows immediately from Lemmas 2.1 & 2.2.

Lemma 2.3 *Let $A \subseteq \mathbb{Q}_p^{n+m}$ be semi-algebraic and $\pi : \mathbb{Q}_p^{n+m} \to \mathbb{Q}_p^n$ be the projection onto the first n coordinates. Then there exist a semi-algebraic set $X \subseteq \pi A$ and a semi-algebraic continuous map $f : X \to \mathbb{Q}_p^m$ such that $\dim(\pi A\backslash X) < \dim(\pi A)$ and the graph of f is contained in A.*

Let $Y \subseteq \mathbb{Q}_p^m$. A map $f : \mathbb{Q}_p^m \to \mathbb{Q}_p^m$ is a *retraction* from $\mathbb{Q}_p^m$ to Y if f is continuous, the range of f is Y and the restriction of f to Y is the identity map on Y (i.e., $f(x) = x$ for every $x \in Y$); and a map $g : Y \to \mathbb{Q}_p^m$ is *nonexpansive* if $v(g(x) - g(y)) \geq v(x - y)$ for every $x, y \in Y$. Let $r : E \times \mathbb{Q}_p^m \to \mathbb{Q}_p^m$. For each $x \in E$, let $r(x, -)$ denote the map from $\mathbb{Q}_p^m$ to $\mathbb{Q}_p^m$: $y \mapsto r(x, y)$ for every $y \in \mathbb{Q}_p^m$.

In [6], Cluckers and Martin proved the following result.

Lemma 2.4 (Cluckers & Martin; [6, Theorem 20]) *If T is semi-algebraic and $T(x)$ is closed for every $x \in E$, then there exists a semi-algebraic map $r : E \times \mathbb{Q}_p^m \to \mathbb{Q}_p^m$ such that for each $x \in E$, $r(x, -)$ is a nonexpansive retraction from $\mathbb{Q}_p^m$ to $T(x)$.*

As a result, we obtain:

Lemma 2.5 *Let $r : E \times \mathbb{Q}_p^m \to \mathbb{Q}_p^m$ be a semi-algebraic map such that $r(x, -)$ is nonexpansive for every $x \in E$. There is a semi-algebraic set $E_0 \subseteq E$ such that $\dim(E\backslash E_0) < \dim E$ and $r \restriction (E_0 \times \mathbb{Q}_p^m)$ is continuous.*

Proof. Let $E' := \{x \in E : r$ is not continuous at (x, y) for some $y \in \mathbb{Q}_p^m\}$. It is enough to show that $\dim E' < \dim E$. Suppose to the contrary that $\dim E' = \dim E$. Since every coordinate projection is semi-algebraic, we may reduce to the case $\dim E = n$. By 2.3, there exist a semi-algebraic open set $U \subseteq E'$ and semi-algebraic continuous functions $g, G : U \to \mathbb{Q}_p^m$ such that $G(x) = r(x, g(x))$ and r is not continuous at

$(x, g(x))$ for every $x \in U$. Fix $x_0 \in U$. We shall show that G is not continuous at x_0. Since r is not continuous at $(x_0, g(x_0))$, there is $t \in \mathbb{Q}_p\backslash\{0\}$ such that $\varepsilon = v(t) \in \mathbb{Z}$ and

$$\forall s \in \mathbb{Q}_p\backslash\{0\}\, \exists (x, y) \in B_{v(s)}(x_0, g(x_0)),\ v(r(x, y) - r(x_0, g(x_0))) < \varepsilon.$$

Since g is continuous, there is $s_0 \in \mathbb{Q}_p\backslash\{0\}$ such that $v(g(x_0) - g(x)) > \varepsilon$ for every $x \in B_{v(s_0)}(x_0)$. Let $s \in \mathbb{Q}_p\backslash\{0\}$ such that $\delta = v(s) > \max\{\varepsilon, v(s_0)\}$. Then there exists $(x, y) \in B_\delta(x_0, g(x_0))$ such that

$$v(r(x, y) - r(x_0, g(x_0))) < \varepsilon.$$

Since $v(y - g(x_0)) > \varepsilon$ and $v(g(x) - g(x_0)) > \varepsilon$, by the ultrametric inequality, $v(y - g(x)) > \varepsilon$. Since $r(x, -)$ is nonexpansive, again by the ultrametric inequality, $v(r(x, y) - r(x, g(x))) > \varepsilon$. Therefore

$$v(G(x) - G(x_0)) = v(r(x, g(x)) - r(x_0, g(x_0))) = v(r(x, y) - r(x_0, g(x_0))) < \varepsilon.$$

Hence, G is not continuous at x_0 which is absurd. □

Observe that if $T(x)$ is a singleton on a closed subset A of T, i.e., the restriction of T to A canonically induces a function from A to $\mathbb{Q}_p^m$, then our main question becomes: "Is there a semi-algebraic continuous extension of $T\restriction A$ that is contained in the graph of T?" We can see that this problem on the existence of semi-algebraic continuous extensions has a connection with our main question.

Lemma 2.6 (Thamrongthanyalak; [20, Theorem 1.1]) *Let E and E' be semi-algebraic. Suppose $E \subseteq E'$, E is closed in E' and $f : E \to \mathbb{Q}_p^n$ is semi-algebraic and continuous. Then there is a semi-algebraic continuous map $g : E' \to \mathbb{Q}_p^n$ such that $g\restriction E = f$.*

P r o o f o f T h e o r e m 1.3. We proceed by induction on $d := \dim E$. The case $d = 0$ is clear. Suppose the result holds for every semi-algebraic set-valued map whose domain has dimension $< d$. By 2.4, let $r : E \times \mathbb{Q}_p^m \to \mathbb{Q}_p^m$ be semi-algebraic such that $r(x, -)$ is a nonexpansive retraction from $\mathbb{Q}_p^m \to T(x)$ for every $x \in E$. In addition, by 2.5, there is a semi-algebraic set $E_0 \subseteq E$ such that $\dim(E\backslash E_0) < \dim E$ and $r\restriction(E_0 \times \mathbb{Q}_p^m)$ is continuous. Replacing E_0 by $E_0\backslash \operatorname{cl}(E\backslash E_0)$ if necessary, we may assume that $E\backslash E_0$ is closed in E. By the inductive hypothesis, let $f : E\backslash E_0 \to \mathbb{Q}_p^m$ be a semi-algebraic continuous selection of $T\restriction(E\backslash E_0)$. By 2.6, let $g : E \to \mathbb{Q}_p^m$ be a semi-algebraic continuous extension of f.

Define $h : E \to \mathbb{Q}_p^m$ by $h(x) = r(x, g(x)) \in T(x)$. Obviously h is semi-algebraic and continuous on E_0, and $h\restriction(E\backslash E_0) = f$. It is enough to prove that h is continuous at x_0 for every $x_0 \in E\backslash E_0$. Let $x_0 \in E\backslash E_0$ and $t \in \mathbb{Q}_p\backslash\{0\}$. Set $\varepsilon = v(t) \in \mathbb{Z}$. Then there exists $s \in \mathbb{Q}_p\backslash\{0\}$ such that $\delta = v(s) < \varepsilon$ and for every $x \in B_\delta(x_0)$, $v(g(x_0) - g(x)) > \varepsilon$ and $T(x) \cap B_\varepsilon(g(x_0)) \neq \varnothing$. Let $x \in B_\delta(x_0)$ and $y \in T(x) \cap B_\varepsilon(g(x_0))$. By the ultrametric inequality, we have $v(y - g(x)) > \varepsilon$. Note that $y = r(x, y)$ because $y \in T(x)$ and $r(x, -)$ is a retraction from $\mathbb{Q}_p^m \to T(x)$. Therefore,

$$\begin{aligned} v(h(x_0) - h(x)) &= v(g(x_0) - r(x, g(x))) \\ &\geq \min\{v(g(x_0) - y), v(y - r(x, g(x)))\} \\ &\geq \min\{\varepsilon, v(r(x, y) - r(x, g(x)))\} \\ &\geq \min\{\varepsilon, v(y, g(x))\} \\ &\geq \varepsilon. \end{aligned}$$

Hence h is continuous at x_0. □

3 Glaeser refinement and the proof of Theorem 1.4

In this section, we introduce Glaeser refinements, which were first given by Glaeser [11]. This notion was used in the study of Whitney's extension problem (cf. [22] for the original question).

Let $T : E \rightrightarrows \mathbb{Q}_p^m$. Define $T' : E \rightrightarrows \mathbb{Q}_p^m$, the *Glaeser refinement* of T, by

$$T'(x_0) := \{y \in T(x_0) : v(y, T(x)) \to \infty \text{ as } E \ni x \to x_0\} \quad \text{for } x_0 \in E.$$

We say that T is *stable under Glaeser refinement* if $T' = T$.

Remark 3.1 If T is semi-algebraic, then so is T'. If $f : E \to \mathbb{Q}_p^m$ is continuous and $f(x) \in T(x)$ for every $x \in E$, then $f(x) \in T'(x)$ for every $x \in E$. Furthermore, T is stable under Glaeser refinement if and only if T is lower semi-continuous.

Lemma 3.2 *If $T(x)$ is closed for every $x \in E$, then $T'(x)$ is closed for every $x \in E$.*

Proof. Let $x_0 \in E$. To prove that $T'(x_0)$ is closed, let $y_0 \notin T'(x_0)$. Fix $t \in \mathbb{Q}_p\backslash\{0\}$ such that for every $s \in \mathbb{Q}_p\backslash\{0\}$ there is $x \in E \cap B_{v(s)}(x_0)$ with $v(y_0, T(x)) < v(t)$. It is routine to show that $B_{v(t)}(y_0) \cap T'(x_0) = \varnothing$. □

Next, we define a sequence $(T^{(k)})_{k\in\mathbb{N}}$ inductively by $T^{(0)} := T$ and $T^{(k+1)} := (T^{(k)})'$. It is easy to see that for each $k \in \mathbb{N}$, $T^{(k+1)} = (T^{(k)})' = (T')^{(k)}$. In o-minimal expansions of the real field, we know that this sequence of iterated Glaeser refinements of set-valued maps is eventually stable (cf., e.g., [2 19]). Here, we shall show that the same result also holds in the p-adic semi-algebraic context. To prove this, we first show that every p-adic semi-algebraic set-valued map is almost lower semi-continuous.

Lemma 3.3 *If T is semi-algebraic, then there is a semi-algebraic set $E' \subseteq E$ such that* $\dim(E\backslash E') < \dim E$ *and $T\restriction E'$ is lower semi-continuous.*

Proof. Let $K = \{(x, y) : x \in E, y \in T(x)$ and $\exists t \in \mathbb{Q}_p\backslash\{0\}\forall s \in \mathbb{Q}_p\backslash\{0\}\exists x' \in B_{v(s)}(x) \cap E,\ v(y, T(x')) < v(t)\}$ be the set of witnesses of lower semi-discontinuity of T. Let $\pi : \mathbb{Q}_p^n \times \mathbb{Q}_p^m \to \mathbb{Q}_p^n$ be the coordinate projection onto the first n coordinates. Obviously, $T\restriction(E\backslash\pi K)$ is lower semi-continuous. Therefore, it is enough to show that $\dim(\pi K) < \dim E$. Suppose to the contrary that $\dim(\pi K) = \dim E$. By 2.3, there exist a semi-algebraic open subset U of $\mathbb{Q}_p^n$ and a continuous semi-algebraic map $f : U \cap E \to \mathbb{Q}_p^m$ such that $(x, f(x)) \in K$ for every $x \in U \cap E$. Let $x \in U \cap E$. Then there is $t \in \mathbb{Q}_p\backslash\{0\}$ such that for every $s \in \mathbb{Q}_p\backslash\{0\}$ there is $x' \in B_{v(s)}(x) \cap U \cap E$ with $v(f(x) - f(x')) < v(t)$. This contradicts the continuity of f at x. □

Lemma 3.4 *If T is semi-algebraic, then $T^{(\dim E)}(x_0) = T^{(k)}(x_0)$ for some $x_0 \in E$ and $k \geq \dim E$.*

Proof. We proceed by induction on $d = \dim E$. If $d = 0$, then T is lower semi-continuous and this case is done by Remark 3.1. Suppose the result holds for every semi-algebraic set-valued map whose domain has dimension $< d$. By 3.3, let $E_0 \subseteq E$ such that $\dim(E\backslash E_0) < \dim E$ and $T\restriction E_0$ is lower semi-continuous. We may assume further that E_0 is open in E. Therefore $T(x_0) = T^{(k)}(x_0)$ for every $x_0 \in E_0$ and $k \geq 0$; so we have $(T')^{(k)}(x) = (T'\restriction E\backslash E_0)^{(k)}(x)$ for every $x \in E\backslash E_0$ and $k \geq 0$. Since $\dim(E\backslash E_0) < d$ and E_0 is open in E, by the inductive hypothesis, we have $(T'\restriction E\backslash E_0)^{(d-1)}(x) = (T'\restriction E\backslash E_0)^{(l)}(x)$ for every $x \in E$ and $l \geq d - 1$. Let $x_0 \in E\backslash E_0$ and $k \geq d$. Therefore

$$\begin{aligned} T^{(d)}(x_0) &= (T')^{(d-1)}(x_0) \\ &= (T'\restriction E\backslash E_0)^{(d-1)}(x_0) \\ &= (T'\restriction E\backslash E_0)^{(k-1)}(x_0) \\ &= (T')^{(k-1)}(x_0) \\ &= T^{(k)}(x_0). \end{aligned}$$

Hence, $T^{(k)}(x_0) = T^{(d)}(x_0)$ for every $x_0 \in E$ and $k \geq d$. □

Therefore, we have:

Lemma 3.5 *If T is semi-algebraic, then $T^{(\dim E)}$ is stable under Glaeser refinement.*

Let $T^{(*)} = T^{(\dim E)}$. By Remark 3.1, we obtain

Lemma 3.6 *If T is semi-algebraic, then $T^{(*)}$ is semi-algebraic and $T^{(*)}$ is lower semi-continuous.*

Lemma 3.7 *If T has a continuous selection, then $T^{(*)}(x) \neq \varnothing$ for all $x \in E$.*

To prove Theorem 1.4 , it is enough to prove

Lemma 3.8 *Suppose T is semi-algebraic and $T(x)$ is closed for every $x \in E$. Then the following are equivalent:*

1. *T has a continuous selection;*
2. *$T^{(*)}(x) \neq \varnothing$ for all $x \in E$;*
3. *T has a semi-algebraic continuous selection.*

Proof. We need only to prove (2) $\Rightarrow$ (3). Assume $T^{(*)}(x_0) \neq \varnothing$ for all $x_0 \in E$. By 3.5, we have $T^{(*)}$ is stable under Glaeser refinement and so lower semi-continuous. In addition, we also have that $T^{(*)}(x)$ is closed for every $x \in E$ by 3.2. Hence, by Theorem 1.3 , let $f : E \to \mathbb{Q}_p^m$ be a semi-algebraic continuous selection of $T^{(*)}$. Since $T^{(*)}(x) \subseteq T(x)$ for every $x \in E$, f is also a semi-algebraic continuous selection of T. □

This completes the proof of Theorem 1.4.

Next, we consider the case $n = 1$.

Lemma 3.9 *Let $a \in E$ and $b \in T(a)$. If T is lower semi-continuous and semi-algebraic, then there exist a semi-algebraic open neighborhood B of a and a semi-algebraic continuous function $f : E \cap B \to \mathbb{Q}_p^m$ such that $f(a) = b$ and $f(x) \in T(x)$ for all $x \in E \cap B$.*

Proof. It is trivial if a is an isolated point. Suppose a is not an isolated point. Let

$$E_1 = \{x \in E : \text{ there exists } y \in T(x) \text{ such that } v(y - b) > v(x - a)\}$$

and $E_2 = (E\backslash\{a\})\backslash E_1$. By [12, Lemma 4.4], if $x \in E_2$, then there exists $y \in T(x)$ such that $v(y - b) \geq v(z - b)$ for every $z \in T(x)$. Let

$$\begin{aligned} A = \{(x, y) \in E_1 \times \mathbb{Q}_p^m &: y \in T(x) \,\&\, v(y - b) > v(x - a)\} \\ &\cup \{(x, y) \in E_2 \times \mathbb{Q}_p^m : y \in T(x) \,\&\, v(y - b) \geq v(z - b) \text{ for every } z \in T(x)\}. \end{aligned}$$

Obviously, A is semi-algebraic. Therefore, there exist a finite set $Z \subseteq E$ and a semi-algebraic continuous function $g : E\backslash Z \to \mathbb{Q}_p^m$ such that $a \in Z$ and $(x, g(x)) \in A$ for every $x \in E\backslash Z$. Since Z is finite, there exists a semi-algebraic open neighborhood B of a such that $B \cap Z = \{a\}$. Define $f : E \cap B \to \mathbb{Q}_p^m$ by

$$f(x) = \begin{cases} g(x), & \text{if } x \neq a, \\ b, & \text{if } x = a. \end{cases}$$

We show that $g(x) \to b$ as $E \ni x \to a$. Let $t \in \mathbb{Q}_p\backslash\{0\}$ and set $\varepsilon = v(t)$. Since T is lower semi-continuous, there is $s \in \mathbb{Q}_p\backslash\{0\}$ such that $\delta = v(s) > \varepsilon$ and $B_\delta(a) \subseteq B$ and $T(x) \cap B_\varepsilon(b) \neq \varnothing$ for all $x \in B_\delta(a)$. Let $x \in B_\delta(a)\backslash\{a\}$. If $x \in E_1$, then $v(g(x) - b) > v(x - a) > \delta > \varepsilon$. Suppose $x \in E_2$. Since $T(x) \cap B_\varepsilon(b) \neq \varnothing$, there is $z \in T(x)$ such that $v(z - b) > \varepsilon$. Hence, we have $v(g(x) - b) \geq v(z - b) > \varepsilon$. This completes the proof. □

Proof of Theorem 1.5. Let $f : E \to \mathbb{Q}_p^m$ be a continuous selection of T. Then we know that $T^{(*)}(x_0) \neq \varnothing$ for all $x_0 \in E$. Since $\dim E = 1$, by 2.3, we have a semi-algebraic set $X \subseteq E$ and a semi-algebraic function $g : E \to \mathbb{Q}_p^m$ such that $E\backslash X$ is finite, $g{\restriction}X$ is continuous and $g(x) \in T^{(*)}(x)$ for every $x \in E$. If $E\backslash X = \varnothing$, then we're done. Suppose that $E\backslash X \neq \varnothing$. Write $E\backslash X = \{a_1, \dots, a_N\}$. Let $i \in \{1, \dots, N\}$. Observe that $f(a_i) \in T^{(*)}(a_i)$. By lower semi-continuity of $T^{(*)}$ and 3.9, there exist $s \in \mathbb{Q}_p\backslash\{0\}$ and a semi-algebraic continuous function $f_i : E \cap B_{v(s)}(a_i) \to \mathbb{Q}_p^m$ such that $f_i(a_i) = f(a_i)$ and $f_i(x) \in T^{(*)}(x)$ for all $x \in E \cap B_{v(s)}(a_i)$.

Set $\Delta = \max(\{v(a_i - a_j) : 1 \leq i < j \leq N\} \cup \{v(s)\})$. Define $h : E \to \mathbb{Q}_p^m$ by

$$h(x) = \begin{cases} f_i(x), & \text{if } v(x - a_i) > \Delta \text{ for some } i \in \{1, \dots, N\}; \\ g(x), & \text{otherwise.} \end{cases}$$

Note that $\{B_\Delta(a_i) : 1 \leq i \leq N\}$ is a finite pairwise disjoint collection of clopen sets. Then h is continuous. Therefore, we can easily see that h is a semi-algebraic continuous selection of T. □

Observe that by tracking the parameters throughout the proofs and applying semi-algebraic Skolem functions, we have a version of Theorem 1.3 for definable families. In addition, the definition of Glaeser refinement can be extended to definable families of set-valued maps. Therefore, we now know that Theorem 1.4 is independent of parameters.

Lemma 3.10 *Let $(T_y)_{y\in Y}$ be a semi-algebraic family of set-valued maps $T_y : E_y \rightrightarrows \mathbb{Q}_p^m$. Suppose for every $y \in Y$, T_y is lower semi-continuous and $T_y(x)$ is closed and nonempty for every $x \in E_y$. Then there exists a semi-algebraic family $(f_y)_{y\in Y}$ of functions $f_y : E_y \to \mathbb{Q}_p^m$ such that f_y is a continuous selection of T_y.*

Lemma 3.11 *Let $(T_y)_{y\in Y}$ be a semi-algebraic family of set-valued maps $T_y : E_y \rightrightarrows \mathbb{Q}_p^m$. Suppose for every $y \in Y$, T_y has a continuous selection and $T_y(x)$ is closed for every $x \in E_y$. Then there exists a semi-algebraic family $(f_y)_{y\in Y}$ of functions $f_y : E_y \to \mathbb{Q}_p^m$ such that f_y is a continuous selection of T_y.*

4 Applications

In this section, we provide three applications of the main theorems.

4.1 Semi-algebraic continuous solutions of semi-algebraic equations

Let $g_1, \dots, g_k : E \to \mathbb{Q}_p$ and $F : \mathbb{Q}_p^{k+m} \to \mathbb{Q}_p$ be semi-algebraic and continuous. We consider the equation:

$$F(g_1, \dots, g_k, y_1, \dots, y_m) = 0, \tag{*}$$

in unknown continuous functions $y_1, \dots, y_m : E \to \mathbb{Q}_p$. It is clear that the set

$$\{(z_1, \dots, z_m) \in \mathbb{Q}_p^m : F(g_1(x), \dots, g_k(x), z_1, \dots, z_m) = 0\}$$

is a closed subset of $\mathbb{Q}_p^m$ for every $x \in E$. Applying Theorem 1.4 to the semi-algebraic set-valued map $T : E \rightrightarrows \mathbb{Q}_p^m$ where

$$T(x) := \{(z_1, \dots, z_m) \in \mathbb{Q}_p^m : F(g_1(x), \dots, g_k(x), z_1, \dots, z_m) = 0\},$$

we have:

Lemma 4.1 *If there are continuous functions $y_1, \dots, y_m : E \to \mathbb{Q}_p$ solving (∗), then there are also semi-algebraic continuous functions $y_1, \dots, y_m$ that satisfy (∗).*

We can see that Corollary 1.6 still holds when we replace the equation (∗) by a finite system of equations of the same kind as (∗). In addition, by Lemma 3.11, we obtain:

Lemma 4.2 *Let $g_1, \dots, g_k : \mathbb{Q}_p^{N+n}$ and $F : \mathbb{Q}_p^{N+k+m} \to \mathbb{Q}_p$ be semi-algebraic. Suppose for every $c \in \mathbb{Q}_p^N$, the functions $g_1(c, -), \dots, g_k(c, -) : \mathbb{Q}_p^n \to \mathbb{Q}_p$ and $F(c, -) : \mathbb{Q}_p^{k+m} \to \mathbb{Q}_p$ are continuous. Assume that for every $c \in \mathbb{Q}_p^N$ there are continuous functions $y_1, \dots, y_m : \mathbb{Q}_p^n \to \mathbb{Q}_p$ such that*

$$F(c, g_1(c, x), \dots, g_k(c, x), y_1(x), \dots, y_m(x)) = 0$$

for every $x \in \mathbb{Q}_p^n$. Then there exist semi-algebraic functions $y_1, \dots, y_m : \mathbb{Q}_p^{N+n} \to \mathbb{Q}_p$ such that for every $c \in \mathbb{Q}_p^N$, the functions $y_1(c, -), \dots, y_m(c, -) : \mathbb{Q}_p^n \to \mathbb{Q}_p$ are continuous and

$$F(c, g_1(c, x), \dots, g_k(c, x), y_1(c, x), \dots, y_m(c, x)) = 0$$

for every $x \in \mathbb{Q}_p^n$.

4.2 Semi-algebraic continuous extensions

The extension problem, one of classic problems in topology and analysis, asks: "Let $A \subseteq X$ and $f : A \to Y$. Is it possible to find an extension of f to X that satisfies some prescribed properties?" (Readers can find more studies of extension problems in, e.g., [4]).

Let $E \subseteq E' \subseteq \mathbb{Q}_p^n$ and $f : E \to \mathbb{Q}_p^m$ be semi-algebraic. By 2.6, we know that if E is closed in E' and f is continuous, then f has a semi-algebraic continuous extension to E'. Observe that $\mathbb{Q}_p\backslash\{0\}$ is not closed in $\mathbb{Q}_p$ and the map from $\mathbb{Q}_p\setminus\{0\}$ to $\mathbb{Q}_p$ $x \mapsto 1/x$ has no continuous extension to $\mathbb{Q}_p$. Therefore, this gives rise to the question: "How do we determine whether f admits semi-algebraic continuous extensions to E'?" Let $T_f : E' \rightrightarrows \mathbb{Q}_p^m$ be defined by, for each $x \in E'$,

$$T_f(x) = \begin{cases} \{f(x)\}, & \text{if } x \in E, \\ \mathbb{Q}_p^m, & \text{if } x \in E'\setminus E. \end{cases}$$

We obtain that:

Lemma 4.3 *The function f admits a semi-algebraic continuous extension to E' if and only if $(T_f)^{(*)}(x) \neq \varnothing$ for every $x \in E'$*

Proof. This follows immediately from 3.8 and the fact that a continuous function $g : E' \to \mathbb{Q}_p^m$ is an extension of f if and only if g is a continuous selection of T_f. □

4.3 Characterization of right invertible elements

For any map $h : E \to \mathbb{Q}_p^m$, let $h^{-1} : \mathbb{Q}_p^m \rightrightarrows E$ be the set-valued map defined by $h^{-1}(x)$ is the pre-image of $\{x\}$ under h. Observe that if $h : E \to \mathbb{Q}_p^m$ is an open map, then $h^{-1} : \mathbb{Q}_p^m \rightrightarrows E$ is lower semi-continuous. Therefore, we have:

Lemma 4.4 *If $h : \mathbb{Q}_p^n \to \mathbb{Q}_p^m$ is semi-algebraic, surjective, continuous, and open, then there is a semi-algebraic continuous map $f : \mathbb{Q}_p^m \to \mathbb{Q}_p^n$ such that $h \circ f$ is the identity map on $\mathbb{Q}_p^m$.*

Let $(G, \circ)$ be the monoid where G is the set of semi-algebraic continuous function from $\mathbb{Q}_p^n$ to $\mathbb{Q}_p^n$ and the group operation $\circ$ is the composition operation. Obviously, the identity map on $\mathbb{Q}_p^n$ is the identity element of this monoid. The result 4.4 implies that every member of G that is surjective and open is right invertible in $(G, \circ)$. Therefore, a question arose naturally: "What is the characterization of right invertible element in $(G, \circ)$?"

We now give an answer to the above question.

Lemma 4.5 *Let $h \in G$. Then h is right invertible under $\circ$ if and only if h is surjective and $(h^{-1})^{(*)}(x_0) \neq \varnothing$ for every $x_0 \in \mathbb{Q}_p^n$.*

Proof. Suppose h is surjective and h^{-1} is lower semi-continuous. Since h is continuous, $h^{-1}(x)$ is closed for every $x \in \mathbb{Q}_p^n$. By 3.8, h^{-1} has a semi-algebraic continuous selection $f : \mathbb{Q}_p^n \to \mathbb{Q}_p^n$. Hence, we have $h(f(x)) = x$ for every $x \in \mathbb{Q}_p^n$, that is $h \circ f$ is the identity map on $\mathbb{Q}_p^n$.

Conversely, suppose h is right invertible. Then there exists semi-algebraic and continuous $f : \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ such that $h \circ f$ is the identity map on $\mathbb{Q}_p^n$. Since $h \circ f$ is surjective, h is also surjective. Observe that the function f is contained in h^{-1} (as sets). Since f is continuous, $(h^{-1})^{(*)}$ contains f still. Therefore, $(h^{-1})^{(*)}(x_0) \neq \varnothing$ for every $x_0 \in \mathbb{Q}_p^n$. □

Let $h \in G$. It is clear to see that h is left invertible under $\circ$ if and only if h is injective and h^{-1} admits a semi-algebraic continuous extension to $\mathbb{Q}_p^n$. By Corollary 1.7 , we obtain:

Lemma 4.6 *Let $h \in G$. Then h is left invertible under $\circ$ if and only if h is injective and $(T_{h^{-1}})^{(*)}(x_0) \neq \varnothing$ for every $x_0 \in \mathbb{Q}_p^n$.*

5 Concluding remarks

As mentioned in § 1, we also know that Theorem 1.3 holds in the p-adic subanalytic context. Therefore, a natural question arises: To what extent can we generalize Theorem 1.3? Recall that for every P-minimal expansion $\mathscr{K}$ of a p-adically closed field, $\mathscr{K}$ admits Cell Decomposition Theorem if and only if $\mathscr{K}$ admits definable Skolem functions (cf. [18] for more details). In addition, by [13], there exist P-minimal expansions of $(\mathbb{Q}_p; +, \cdot, -, 0, 1, \mathrm{Div})$ that does not admit definable Skolem functions. We may ask whether Theorem 1.3 holds for all P-minimal expansions of p-adically closed fields that admit definable Skolem functions. From the above proof, we know that Theorem 1.3 holds for every P-minimal structure that satisfies analogs of Theorems 2.4 & 2.6.

Remark 5.1 Let $E \subseteq E' \subseteq \mathbb{Q}_p^n$ and $f : E \to \mathbb{Q}_p^m$. Let T_f be defined as in 4.2. Note that a variant of Corollary 1.7 also holds when f is not semi-algebraic. Observe that for every $x \in E'$, we have $T_f(x)$ is an affine subspace of $\mathbb{Q}_p^m$. By the same argument as the proof of [9, Lemma 2.2], we have $(T_f)^{(2m+1)}$ is stable under Glaeser refinement; therefore, it is lower semi-continuous. Michael's Selection Theorem (mentioned in the introduction) implies that f admits a continuous extension to E' if and only if $(T_f)^{(2m+1)}(x_0) \neq \varnothing$ for every $x_0 \in E'$.

Let $(H, \circ)$ be the monoid where H is the set of continuous functions $\mathbb{Q}_p^n \to \mathbb{Q}_p^n$ and $\circ$ is the composition operation. Let $h \in H$. By Remark 5.1, we obtain that h is left invertible under $\circ$ if and only if h is injective and $(T_{h^{-1}})^{(2n+1)}(x_0) \neq \varnothing$ for every $x_0 \in E'$. Observe that if h is right invertible under $\circ$, then h is surjective and $(h^{-1})^{(k)}(x_0) \neq \varnothing$ for every $x_0 \in \mathbb{Q}_p^n$ and $k \geq 0$. However, we don't know whether the converse is true or not. The difference here follows from the fact that we now do not know whether every set-valued map from $\mathbb{Q}_p^n$ to $\mathbb{Q}_p^n$ is eventually stable under Glaeser refinement.

Acknowledgements The author thanks Deirdre Haskell for discussions at the 2014 CMS winter meeting, which provided motivation to study these questions. The author was supported by the Thailand Research Fund (under grant MRG6180097) and Office of the Higher Education Commission.

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p-Adic Semialgebraic Sets and Trace Problems

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December 19, 2019

KTV Trilateral Workshop, KAIST

Let X, Y be metric spaces and $\mathcal{F}(X, Y)$ be the set of all maps from X to Y.
Suppose $\mathcal{F} \subseteq \mathcal{F}(X, Y)$ and $S \subseteq X$.

Definition

The **trace of $\mathcal{F}$ to S** is the set

$$\mathcal{F} \upharpoonright S = \{f \upharpoonright S : f \in \mathcal{F}\}.$$

Question

How can we completely describe the trace $\mathcal{F} \upharpoonright S$?

Trace Problems

Let X, Y be metric spaces and $\mathcal{F}(X, Y)$ be the set of all maps from X to Y.
Suppose $\mathcal{F} \subseteq \mathcal{F}(X, Y)$ and $S \subseteq X$.

Definition

The **trace of $\mathcal{F}$ to S** is the set

$$\mathcal{F} \restriction S = \{f \restriction S : f \in \mathcal{F}\}.$$

Question

How can we completely describe the trace $\mathcal{F} \restriction S$?

Let $f : S \to Y$.

Question

How to determine whether f is the restriction of some element in $\mathcal{F}$?

Examples

1. $\mathcal{F} = \mathcal{C}(X, Y)$ and S is a closed subset of X:

Attributed to H. Tietze

A function $f : S \to Y$ is in the trace $\mathcal{F} \restriction S$ if and only if f is continuous.

2. $\mathcal{F} = Lip_1(\mathbb{R}^n, \mathbb{R}^m)$ and $S \subseteq \mathbb{R}^n$:

Attributed to M.D. Kirszbraun

A function $f : S \to \mathbb{R}^m$ is in the trace $\mathcal{F} \restriction S$ if and only if f is 1-Lipschitz.

Examples

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A function $f : S \to \mathbb{R}^m$ is in the trace $\mathcal{F} \upharpoonright S$ if and only if f is 1-Lipschitz.

Let $T\colon X \to \mathcal{P}(Y)$.

Definition

A function $f\colon X \to Y$ is a **continuous selection of** T if f is continuous and $f(x) \in T(x)$ for all $x \in X$.

In 1956, E. Michael asked the following:

Question

How to determine whether T has a continuous selection of T that is in $\mathcal{F}$?

Selection Problems

Let $T\colon X \to \mathcal{P}(Y)$.

Definition

A function $f\colon X \to Y$ is a **continuous selection of** T if f is continuous and $f(x) \in T(x)$ for all $x \in X$.

In 1956, E. Michael asked the following:

Question

How to determine whether T has a continuous selection of T that is in $\mathcal{F}$?

p-adic Semialgebraic Sets

Definition

A **p-adic semialgebraic set** is a subset of $\mathbb{Q}_p^n$ that is a finite boolean combination of sets of the forms $\{(x_1, \ldots, x_n) \in \mathbb{Q}_p^n : q(x_1, \ldots, x_n) = 0\}$ and $\{(x_1, \ldots, x_n) \in \mathbb{Q}_p^n : x_i = \lambda y^k \text{ for some } y \in \mathbb{Q}_p\}$ where q is a polynomial over $\mathbb{Q}_p$, $\lambda \in \mathbb{Q}_p$, $k \in \mathbb{N}$ and $i = 1, \ldots, n$.

Let $S \subseteq \mathbb{Q}_p^n$.

Definition

A function $f \colon S \to \mathbb{Q}_p^m$ is **p-adic semialgebraic** if the graph of f, $\{(x, f(x)) \in \mathbb{Q}_p^{n+m} : x \in S\}$, is p-adic semialgebraic.

Definition

A p**-adic semialgebraic set** is a subset of $\mathbb{Q}_p^n$ that is a finite boolean combination of sets of the forms
$\{(x_1, \ldots, x_n) \in \mathbb{Q}_p^n : q(x_1, \ldots, x_n) = 0\}$ and
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Let $S \subseteq \mathbb{Q}_p^n$.

Definition

A function $f \colon S \to \mathbb{Q}_p^m$ is p**-adic semialgebraic** if the graph of f, $\{(x, f(x)) \in \mathbb{Q}_p^{n+m} : x \in S\}$, is p-adic semialgebraic.

Let $S \subseteq \mathbb{Q}_p^n$ be p-adic semialgebraic and $f \colon S \to \mathbb{Q}_p^m$ be p-adic semialgebraic.

Cell Decomposition Theorem (J. Denef)

There is a finite partition $\mathcal{C}$ of S such that $f \restriction C$ is continuous for all $C \in \mathcal{C}$.

Main Questions

Let $S \subseteq \mathbb{Q}_p^n$ be p-adic semialgebraic and $\mathcal{F}$ be the set of all continuous p-adic semialgebraic maps from $\mathbb{Q}_p^n$ to $\mathbb{Q}_p^m$

Question 1

How can we completely describe the trace $\mathcal{F} \restriction S$?

Let $T\colon \mathbb{Q}_p^n \to \mathcal{P}(\mathbb{Q}_p^m)$ be p-adic semialgebraic.

Question 2

How to determine whether T is a continuous selection that is p-adic semialgebraic?

Main Questions

Let $S \subseteq \mathbb{Q}_p^n$ be p-adic semialgebraic and $\mathcal{F}$ be the set of all continuous p-adic semialgebraic maps from $\mathbb{Q}_p^n$ to $\mathbb{Q}_p^m$

Question 1

How can we completely describe the trace $\mathcal{F} \upharpoonright S$?

Let $T\colon \mathbb{Q}_p^n \to \mathcal{P}(\mathbb{Q}_p^m)$ be p-adic semialgebraic.

Question 2

How to determine whether T is a continuous selection that is p-adic semialgebraic?

Results

Theorem 1 (T.)

If S is closed, then a function $f\colon S \to \mathbb{Q}_p^m$ is in $\mathcal{F} \restriction S$ if and only if f is continuous.

Results

The **Glaeser refinement of** T is the map $T' \colon \mathbb{Q}_p^n \to \mathcal{P}(\mathbb{Q}_p^m)$ defined by

$$T'(x_0) := \{y \in T(x_0) : d(y, T(x)) \to 0 \text{ as } x \to x_0\} \text{ for } x_0 \in \mathbb{Q}_p^n.$$

For each $k \in \mathbb{N}$, let $T^{(k)}$ be the k-th time Glaeser refinement of T.

Theorem 2 (T.)

The map T has a p-adic semialgebraic continuous selection if and only if $T^{(n)}(x_0) \neq \emptyset$ for all $x_0 \in \mathbb{Q}_p^n$.

Results

For each $f\colon S \to \mathbb{Q}_p^m$, let $T_f\colon \mathbb{Q}_p^n \to \mathcal{P}(\mathbb{Q}_p^m)$ be defined by

$$T_f(x) = \begin{cases} \{f(x)\}, & \text{if } x \in S; \\ \mathbb{Q}_p^m, & \text{if } x \in \mathbb{Q}_p^n \setminus S. \end{cases}$$

Theorem 3 (T.)

A function $f\colon S \to \mathbb{Q}_p^m$ is in $\mathcal{F} \restriction S$ if and only if $T_f^{(n)}(x) \neq \emptyset$ for all $x \in \mathbb{Q}_p^n$.

Let $g_1, \ldots, g_k \colon E \to \mathbb{Q}_p$ and $F \colon \mathbb{Q}_p^{k+m} \to \mathbb{Q}_p$ be p-adic semialgebraic and continuous. Consider

$$F(g_1, \ldots, g_k, y_1, \ldots, y_m) = 0, \qquad (*)$$

in unknown continuous functions $y_1, \ldots, y_m \colon E \to \mathbb{Q}_p$.

Corollary 1 (T.)

If there are continuous functions $y_1, \ldots, y_m \colon E \to \mathbb{Q}_p$ solving $(*)$, then there are also p-adic semialgebraic continuous functions $y_1, \ldots, y_m$ that satisfy $(*)$.

Let $(G, \circ)$ be the monoid where G is the set of p-adic semialgebraic continuous function from $\mathbb{Q}_p^n$ to $\mathbb{Q}_p^n$ and the group operation $\circ$ is the composition operation.

Corollary 2 (T.)

Let $h \in G$. Then h is right invertible under $\circ$ if and only if h is surjective and $(h^{-1})^{(n)}(x_0) \neq \emptyset$ for every $x_0 \in \mathbb{Q}_p^n$.

Corollary 3 (T.)

Let $h \in G$. Then h is left invertible under $\circ$ if and only if h is injective and $(T_{h^{-1}})^{(n)}(x_0) \neq \emptyset$ for every $x_0 \in \mathbb{Q}_p^n$.

Acknowledgements

This research was supported by The Thailand Research Fund and The Office of Higher Education Commission.