รายงานฉบับสมบูรณ์

โครงการ วิธีเชิงคำนวณสำหรับปัญหาผกผันของการนำความร้อน

(Computational Methods for Inverse Heat Conduction Problems)

โดย นายสมชาติ ฉันทศิริวรรณ

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สัญญาเลขที่ PDF/07/2541

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

ชุดโครงการทุนวิจัยหลังปริญญาเอก

กิตติกรรมประกาศ

ผู้วิจัยขอขอบคุณสำนักงานกองทุนสนับสนุนการวิจัยที่ให้ทุนอุดหนุนโครงการวิจัยนี้ ขอขอบคุณ รศ. ดร. สมชาย วงศ์วิเศษ ที่กรุณาให้คำแนะนำปรึกษาด้วยดีตลอดมา นอกจากนี้ ขอขอบคุณ ภาควิชาวิศวกรรมเครื่องกลของมหาวิทยาลัยเทคโนโลยีพระจอมเกล้าธนบุรี และ ภาควิชาวิศวกรรมเครื่องกลของมหาวิทยาลัยธรรมศาสตร์ที่สนับสนุนการดำเนินงานวิจัยของผู้วิจัยในโครงการนี้

บทคัดย่อ

รหัสโครงการ : PDF/07/2541
ชื่อโครงการ : วิธีเชิงคำนวณสำหรับปัญหาผกผันของการนำความร้อน
ผู้วิจัย : นายสมชาติ ฉันทศิริวรรณ
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ปัญหาที่วิจัยเป็นปัญหาผกผันของการนำความร้อนในวัตถุหนึ่งมิติหรือหลายมิติ ซึ่งโดยทั่วไปหมายถึงการคำนวณหาฟลักซ์ความร้อนที่ขอบเขตของวัตถุ จากอุณหภูมิที่วัดได้ภายในหรือบนขอบเขตของวัตถุ โดยสมมติว่าคุณสมบัติทางกายภาพความร้อนของวัตถุมีค่าคงที่ จุดประสงค์หลักของโครงการวิจัยนี้คือการพัฒนาการวิธีเชิงคำนวณสำหรับแก้ปัญหาผกผันที่มีประสิทธิภาพ และสะดวกต่อการใช้งาน โดยที่การวิจัยนี้เริ่มต้นจากการศึกษาวิธีที่มีผู้เคยคิดขึ้นมาแล้ว จากนั้นก็เลือกวิธีที่มีศักยภาพสำหรับการพัฒนาให้ดีขึ้นซึ่งได้แก่ sequential function specification method จากการพัฒนาวิธีดังกล่าวพบว่าถ้ามีการแก้ไข basis function และ stabilization scheme ผลเฉลยที่ได้จากวิธีใหม่จะมีความแม่นยำและเสถียรภาพสูงกว่าผลเฉลยจากวิธีเดิม ผลจากโครงการวิจัยนี้อีกประการหนึ่งก็คือการหาผลเฉลยของปัญหาการคำนวณหาสัมประสิทธิ์การถ่ายเทความร้อนจากการวัดอุณหภูมิ การวิเคราะห์เชิงทฤษฎีของผลเฉลยที่ได้แสดงให้เห็นว่าถ้าอุณหภูมิที่วัดได้มีความแปรปรวน (variance) ผลเฉลยที่ได้นอกจากจะมีความแปรปรวนแล้ว ยังมี nonlinear bias อีกด้วย สำหรับการแก้ปัญหาหลายมิตินั้นก็ได้มีการประยุกต์ใช้วิธี boundary element ทำให้ได้วิธีเชิงคำนวณที่มีประสิทธิภาพสูงและสะดวกสำหรับการนำไปใช้งานจริง งานวิจัยที่ต่อเนื่องควรจะมุ่งไปที่การทำการทดลองตามวิธีที่พัฒนาขึ้นมาในงานวิจัยนี้ ในปัจจุบันการวัดสัมประสิทธิ์การถ่ายเทความร้อนมีความซับซ้อนค่อนข้างมาก เนื่องจากเทคนิคการหาสัมประสิทธิ์การถ่ายเทความร้อนจากอุณหภูมิที่วัดได้บนผิวของวัตถุมีความเป็นไปได้ในเชิงคำนวณ จึงควรมีการทดลองจริงเพื่อทดสอบความเป็นไปได้ในเชิงปฏิบัติของเทคนิคนี้ ซึ่งมีความยืดหยุ่นมากกว่าและต้องการอุปกรณ์ที่ซับซ้อนน้อยกว่าเทคนิคที่ใช้กันอยู่

Keywords : Inverse heat conduction, Boundary element method

Abstract

Project Code : PDF/07/2541

Project Title : Computational Methods for Inverse Heat Conduction Problems

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Project Period : 1 July 1998 – 30 June 1999

The problem of interest is the inverse time-dependent linear heat conduction problem in one or more dimensions, which generally means the problem of determining boundary heat flux from boundary or interior temperature measurements. The main objective of this research is to develop a computational method for solving the inverse heat conduction problem that is both efficient and easy to implement. This research began with the review of previous relevant research efforts. A method with the potential for further development was found to be the sequential function specification method. It was found that by modifying the basis function and the stabilization scheme, the solutions by the resulting method were more accurate and more stable than the solutions by the original method. Another result from this research is the solution to the problem of determining time-dependent heat transfer coefficient from temperature measurements. The theoretical analysis of the solution showed that if there was variance in temperature measurements, there would be not only variance but also nonlinear bias in the solution. For multidimensional problems, the time-dependent formulation of the boundary element method was developed, which resulted in an efficient method that was easy to implement. Future researches should focus on the experimental implementation of the method developed in this research. At the moment, conventional measurements of heat transfer coefficient are quite complicated. The technique of determining heat transfer coefficient from boundary temperature measurements has already been shown to be computationally feasible in this research, experiments should be carried out to verify the viability of this technique, which is more flexible and requires a simpler setup than conventional techniques.

Keywords : Inverse heat conduction, Boundary element method

Executive Summary

The inverse heat conduction problem generally means a problem in which the geometry of the problem, thermophysical properties, initial condition, and part of the boundary condition are known. In addition, temperature measurement data are available. However, part of the boundary condition is unknown and to be solved for. Since temperature measurements always contain random errors, the solution will contain some errors too. The inverse heat conduction problem is ill-posed because small errors in measured temperatures can cause large errors in estimated heat flux. There have been several research works on finding methods that produce accurate and stable solutions. The main objective of research is to develop an efficient computational method.

After reviewing most of the methods currently in use, it was found that the sequential function specification method appeared to have the best potential to be developed into an efficient method that is also convenient to implement in computer programs. The improvement of this method was accomplished by changing the basis function from the piecewise constant function to the piecewise linear function and using the assumption of linearly varying estimated components over future times instead of the assumption of constant estimated components over future times. Comparison between the new method and the existing method in solving a one-dimensional inverse heat conduction problem was made using computer programs. The new method was found to be superior to the existing method because it was able to give a more accurate and more stable solution without requiring more programming effort.

The method was used to solve the one-dimensional problem of determining time-dependent heat transfer coefficient from temperature measurements. The solution can be shown to be a nonlinear function of temperatures. If there were errors in temperature measurements, the theoretical analysis indicated that the solution contained not only variance but also nonlinear bias, which was the difference between the correct estimate and the expected value of the estimate. The relations between both quantities and variance in temperature measurements were given for the case when errors in temperature measurements were normally distributed and uncorrelated. It was found that both quantities increase monotonically with variance in temperature measurements. However, it was possible to reduce the variance and the nonlinear bias by increasing the ratio of the time step for heat transfer coefficient components to the time step for temperature measurements.

An important quantity needed in solving the one-dimensional problem of determining heat transfer coefficient is the analytical relation between boundary heat flux and temperature. For the multidimensional problem, the analytical relation is available only if the geometry of the problem is very simple. Generally, the numerical determination of the relation between boundary heat flux and temperature is required. If temperature measurements are taken at the boundary, the numerical method that should be used to obtain this relation is the boundary element method because it relates boundary temperature to boundary heat flux explicitly. The time-dependent formulation of the time-dependent boundary element method was developed in this research, and used to solve a sample problem.

One possible application of the results of this research is in the employment of the experimental technique for determining heat transfer coefficient from surface temperature measurements. This technique requires the computational solution of the related problem, which can be obtained by using the proposed computational method. Heat transfer coefficient is an important quantity in thermal design and analysis. Currently, its determination usually requires complicated experimental setup and expensive instruments. Since the new technique has the potential to become a better alternative for determining heat transfer coefficient, it deserves more research and development.

Background

A direct time-dependent heat conduction problem is characterized by known thermophysical properties, initial condition, and boundary condition. The solution to the problem yields temperature distribution. On the other hand, when part of the boundary condition is unknown, and temperature measurement data are available, the problem is known as the inverse heat conduction problem. Such a problem arises when the boundary condition, which may be boundary temperature or boundary heat flux, is to be determined, but direct measurements of boundary temperature or boundary heat flux using sensors are difficult or impossible. Examples of situations when the physical characteristics of the surface make it unsuitable for attaching a sensor or when the accuracy of a surface measurement may be severely impaired by the presence of a sensor include measurements at the inner surface of a heat pipe, at the inside of a combustion chamber, at the outer surface of a re-entry space vehicle, and at the tool-work interface of a cutting machine. In these situations, it is better to select locations accessible to sensors from which temperature measurements are taken, and determine the desired boundary condition by solving the inverse heat conduction problem.

Whereas the analytical solutions of the direct heat conduction problem of various geometries exist in the literature, there are very few analytical solutions for the inverse heat conduction problem. The well-known analytical solution, given by Burggraf [1], is for one-dimensional problem of determining boundary heat flux from temperature measurement at one interior location. It reveals the dependence of boundary heat flux on all orders of time derivatives of interior temperature and related heat flux. One disadvantage of the analytical solution is that it requires the numerical evaluation of these derivatives. Moreover, it is not applicable when the problem is overspecified; i.e. when there are more than one temperature measurement. For such a problem and for multidimensional problems of complicated geometry, the numerical method is generally appropriate. Since the numerical method often requires much computation, the interests in the solution techniques have only emerged after the recent advances in computer technology.

In applying numerical treatment to the inverse heat conduction problem, certain characteristics of the problem must be considered. The heat conduction process is normally described by a parabolic differential equation. The nature of the equation is such that effects from disturbances at the boundary are damped at an interior location. Conversely, effects of disturbances at an interior location are magnified as the unknown boundary condition is calculated. This behavior of the inverse heat conduction problem makes it an ill-posed problem. Since, in practice, disturbances in interior temperature measurements always occur as a result of measurement errors, the numerical method must be able to handle the ill-posed nature of the inverse heat conduction problem. This means that the principal aims of the numerical method consist of not only accuracy (the small difference between the calculated solution and the analytical solution) but also stability (the small sensitivity of the solution to errors in input data). Early attempts at solving the problem resulted in numerical methods that yielded the solution close to the exact solution if there were no errors in input data [2]. However, when the input data were corrupted with errors like those resulting from statistical fluctuation, the solution was found to be unstable. In other words, the accuracy of the solution diminished quickly as the errors in input data increased. Modern numerical methods are therefore designed to produce stable solutions. Unfortunately, the two aims are often in conflict [3]. A good numerical method must therefore allow a trade-off between accuracy and stability via the adjustment of one or more parameters in the method.

Problem Description

Consider a solid object with part of its boundary Γ_1 subjected to known heat flux g' and the remaining part of the boundary Γ_2 subjected to unknown heat flux. Let ρ denote density of the object, c denote heat capacity, and κ denote thermal conductivity. Suppose that all thermophysical properties are constant, making the problem a linear one. The heat conduction process can be described by the following equations.

$$\rho c \frac{\partial T'(\vec{r}\,',t')}{\partial t'} = \vec{\nabla}\left(\kappa \vec{\nabla} T'(\vec{r}\,',t')\right) \tag{1}$$

$$T'(\vec{r}\,',0) = T_0 \tag{2}$$

$$\vec{n}\kappa\vec{\nabla}T'(\vec{r}',t')\Big|_{\Gamma_1} = g'(\vec{r}',t')\Big|_{\Gamma_1} \tag{3}$$

where $\vec{n}$ is the outward pointing unit vector normal to boundary Γ_1, and T_0 is the initial temperature. Since the problem is linear, it is advantageous to nondimensional-ize the problem. Let r_0 be the reference length and T_r be the reference temperature (which differs from T_0). Define $\vec{r} = \vec{r}'/r_0$, $t = \kappa t'/\rho c r_0^2$, $T = (T' - T_0)/(T_r - T_0)$, and $g = g'/\kappa(T_r - T_0)$. The heat conduction process can then be described by the following nondimensionalized equations.

$$\frac{\partial T(\vec{r},t)}{\partial t} = \nabla^2 T(\vec{r},t) \tag{4}$$

$$T(\vec{r},0) = 0 \tag{5}$$

$$\vec{n}\vec{\nabla}T(\vec{r},t)\Big|_{\Gamma_1} = g(\vec{r},t)\Big|_{\Gamma_1} \tag{6}$$

In order to render the problem solvable, the temperature measurements on Γ_2 are specified.

$$T(\vec{r}_i, j\Delta t) = Y_i^{(j)} \tag{7}$$

where $\vec{r}_i$ is a sensor position vector, Δt is the measurement time step, and $Y_i^{(j)}$ is measured surface temperature at the sensor position and time $j\Delta t$. Equations (4) – (7) constitute the inverse heat conduction problem, which may be solved for unknown heat flux components $q_i^{(j)}$ at Γ_2.

Previous Research Works

Although the inverse heat conduction problem has been known for quite some time, its significance was brought into attention by Beck et al. [3]. Since the appearance of Ref. 3, there has been continuing progress in the inverse heat conduction research. Some of the important previous works are reviewed below.

- Errors in temperature measurement can be divided into systematic errors, which are due to calibration, physical presence of sensor, and conduction and convection losses, and random errors, which are due to human errors, disturbances to the instrument, and fluctuating experimental conditions. For the purpose of analyzing the solution, it may be assumed that systematic errors are well controlled to be negligible so that only random errors are significant. Furthermore, Beck et al. [3] proposed the following statistical descriptions of the errors:

1. Errors are additive.
2. Errors have constant variance, which is known.
3. Errors are uncorrelated.
4. Distribution of errors can be represented by the normal probability density function.

- The solution to the inverse heat conduction problem can be considered as the optimization problem with the objective function f being defined as

$$f \quad = \quad \sum_{j=1}^{n}\sum_{i=1}^{M_s}\left(T_i^{(j)} - Y_i^{(j)}\right)^2 \tag{8}$$

where $T_i^{(j)}$ is calculated temperature, n is the number of time levels, and M_s is the number of temperature sensors. Note that since $T_i^{(j)}$ is a function of heat flux components, f is also a function of heat flux components. The optimization of f will therefore give heat flux components, which are the desired solution.

For one-dimensional problem with one sensor and unknown heat flux at one end of the domain, Eq. (8) can be rewritten as

$$f \quad = \quad \sum_{j=1}^{n}\left(T^{(j)} - Y^{(j)}\right)^2 \tag{9}$$

of which optimization yields $q^{(1)}, q^{(2)}, \ldots, q^{(n)}$. This method gives the same solution as the Stolz method [2]. Hence, the solution is very sensitive to errors in temperature measurements $Y_i^{(j)}$. In other words, the method produces unstable solutions.

Beck et al. [3] argued that one reason why the Stolz method yielded unstable solutions was that it did not take into account the effects of temperature measurements taken in the future on the estimate of the heat flux component at the current time level. They presented the sequential function specification method, which used the following objective function.

$$f \quad = \quad \sum_{j=1}^{n+r-1}\left(T^{(j)} - Y^{(j)}\right)^2 \tag{10}$$

where r is known as the future time parameter, and $r > 1$. The optimization of f also gives n estimated heat flux components. The calculation of $T_i^{(j)}$ requires a certain assumption regarding the unknown heat flux component $q^{(n+r-1)}$. Beck et al. [3] assumed that

$$q^{(n+k)} = q^{(n)} \qquad (k = 1, 2, \ldots, r-1) \tag{11}$$

It was shown that although a larger r results in a more stable solution, it also results in a less accurate solution. Hence, the sequential function specification method permits the trade-off between stability and accuracy via the future-time parameter. An additional advantage of this method is that it is applicable when more than one sensor is used.

- Instead of fixing parameter r in the sequential function specification method, Blanc et al. [4] allowed r to vary during the course of heat flux estimation. They showed that a better solution might be obtained this way.
- In Tikhonov regularization method [5], a penalty term is added to the objective function in Eq. (10). This penalty term will become large when the estimated heat flux, the derivative of the heat flux, or the second derivative of the heat flux is large. The optimization of the modified objective function will therefore give a stable solution. However, the solution will also be less accurate because of the modification to the objective function.
- Osman et al. [6] presented a method for solving the two-dimensional inverse heat conduction problem. Their method combined the sequential function specification method with the Tikhonov regularization method. The discretization scheme used was the finite element method.
- A different approach to solving the inverse heat conduction is provided by the iterative regularization method [5]. This method seeks to optimize f in Eq. (9) iteratively using the conjugate gradient method. The gradients of f, which are necessary in the iterative scheme, are determined by solving the adjoint problem. Since the fully optimized solution is unstable, a partially optimized solution is solved for instead. The number of iteration will determine how close the solution is to the fully optimized solution. Thus, it can be viewed as the adjustable parameter that influences the stability of the solution.
- There are also methods that solve Eqs. (4) – (7) without formulating the optimization problem as the above methods do. Among them, the well-known method is the space-marching method [7, 8]. This method divides the domain into direct re-

gion, for which all boundary conditions are known, and inverse region, from which part of the boundary condition is to be determined. The solution in the direct region can be relatively easily obtained. The temperature measurement data at sensor locations, along with the calculated heat flux at those locations from the direct solution, are used as the starting input for the algorithm that calculates temperature field along the front moving from the part of the boundary where sensors are located to the part of boundary where heat flux is to be determined. The space-marching method has been used to solve one-dimensional problems [3] and two-dimensional problems [7]. In order to stabilize the solution, the digital filter technique may be used to smooth the temperature data and their derivatives [7].

- Other methods for solving the inverse heat conduction problem include the Newton's method [8], the mollification method [9], the method of lines [10], the dynamic programming method [11], and control theory method [12], the genetic algorithm [13].
- Apart from the above methods, other contributions to the inverse heat conduction research include the determination of heat transfer coefficients from solving inverse heat conduction problems [14-19].

New Developments from This Research

- In Ref. 20, the sequential function specification method was generalized. The structure of this method was shown to consist of the basis function and the assumption regarding the variation of heat flux over future time. In the classical sequential function specification method [3], the basis function is a piecewise step function, and heat flux is assumed to remain constant over future time. A comparison among three algorithms having different basis functions and/or assumptions regarding the variation of heat flux over future time was carried out. It was found that the algorithm, in which the basis function was piecewise linear, and heat flux was assumed to vary linearly over future time, gave a more accurate and more stable solution than the classical algorithm.
- In Ref. 21, the one-dimensional inverse heat conduction problem was solved for time-dependent heat transfer coefficient. The method used was the sequential

function specification method similar to the method described in Ref. 20. In addition to the future-time parameter, the method presented the ratio between the time step of estimated heat transfer coefficient components and the time step of temperature measurements as another tunable parameter. An increase in the latter parameter was shown to increase both the accuracy and the stability of the solution. The estimated heat transfer coefficient was found to be a nonlinear function of measured temperatures. As a result, when there were statistical errors in temperature measurements, the estimate would contain not only deterministic bias but also nonlinear bias.

- In Ref. 22, the inverse heat conduction problem of determining time-dependent heat transfer coefficient on the surface of a multidimensional body was considered. Since temperature sensors were located on the surface, the problem was well-posed. The discretization scheme used was the boundary element method, for which the time-dependent formulation was constructed. This problem has a practical significance because of the difficulty in measuring heat transfer coefficient by conventional means.

Suggestion for Future Research

- It was shown in Ref. 21 that an efficient numerical method was available for solving the inverse heat conduction problem of determining heat transfer coefficient. Future research should be directed toward testing this new method for estimating heat transfer coefficient from surface temperature measurements with experiments and comparing the results with those obtained by conventional methods.
- The inverse heat conduction problem described so far is also known as the boundary inverse heat conduction problem. Another important problem that has a high potential for applications and deserves more attention is the coefficient inverse heat conduction problem. Such a problem is characterized by known initial and boundary conditions but unknown thermophysical properties. There have been very few published works related to this problem [23,24]. One reason that makes this problem a highly challenging one is the fact that the problem is nonlinear, and an efficient numerical scheme must be devised to handle it.

Conclusion

The inverse heat conduction problem has been under continuing interest. Many of the contributions in this research area have been new and modified methods of solving the problem. The nature of this problem is such that there is a trade-off between the accuracy and the stability of the solution. Each method usually offers at least one parameter that can be adjusted to improve the solution. However, it is possible that one method may yield a more accurate and more stable solution than another method. This was shown in Ref. 20. Therefore, the challenge of the inverse heat conduction research is to keep finding the method that yields a more accurate and more stable solution. Nevertheless, the computational efficiency of the method should also be taken into consideration since it is almost certain that the method must be implemented in computer programs.

With a repertoire of solution methods available, the applications of the research knowledge to real-world problems should be facilitated considerably. One such application is the determination of heat transfer coefficient from temperature measurements. Unlike conventional experimental methods, which require expensive and complicated setup, the inverse method requires only temperature measurements, which are less expensive and more flexible. The analysis performed in Ref. 21 showed that care must be exerted when solving this type of problem because of the existence of nonlinear bias. Further development of this technique should include experimental verifications and comparisons with previous results. It will make this technique a reliable alternative way of determining heat transfer coefficient.

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Output

1. S. Chantasiriwan, Comparison of three sequential function specification algorithms for the inverse heat conduction problem, *International Communications for Heat and Mass Transfer* **26**, 115-124 (1999).
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Appendix

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Int. Comm. Heat Mass Transfer, Vol. 26, No. 1, pp. 115–124, 1999

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PII S0735-1933(98)00127-4

COMPARISON OF THREE SEQUENTIAL FUNCTION SPECIFICATION ALGORITHMS FOR THE INVERSE HEAT CONDUCTION PROBLEM

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(Communicated by J.P. Hartnett and W.J. Minkowycz)

ABSTRACT

Three algorithms for implementing the sequential function specification method of estimating boundary heat flux in the inverse heat conduction problem are compared. They differ from one another in the type of piecewise function used to describe the heat flux and the assumed variation of heat flux over future time. The results of the comparison show that the algorithm that makes use of linear piecewise function for the heat flux and assumes linearly varying heat flux over future time performs slightly better than the other two algorithms. © 1999 Elsevier Science Ltd

Introduction

The determination of heat flux or temperature at the boundary of an object given sufficient temperature measurement data inside the object is known as the inverse heat conduction problem (IHCP). Since the solution to IHCP does not depend continuously on input data, IHCP is an ill-posed problem. Over the years, much research has focused on determining a stabilized solution of IHCP. Various methods for doing so have been proposed [1-3]. Among these, a simple and sufficiently effective method is the sequential function specification method [1]. For the determination of boundary heat flux, this method makes no assumption about the variation of the boundary heat flux over the entire time domain, unlike some other methods. Instead, it assumes that the heat flux is a piecewise function of time, and attempts to obtain the heat flux components sequentially at each time level. In doing so, it employs of measurement data at future time. When only one measurement at the next time level is used to estimate the heat flux component at the current time level, the solution is generally unstable. However, the solution will become more stable as the number of future measurements used in the estimation increases. Thus, the number of future data serves as a stabilizing parameter.

Although the sequential function specification method as described by Beck et al. [1] does not stipulate the piecewise function to be assumed for boundary heat flux and the relation between heat flux components at future time and the heat flux component at the current time, so far most applications of this method have used a piecewise constant function for heat flux and assumed that heat flux components at future time are constant. In this paper, the linearly varying piecewise function for heat flux and the linearly varying future heat flux will be considered in the implementation of the sequential function specification method. Three different algorithms will be compared to see whether a different algorithm will result in a better solution of IHCP than the commonly used algorithms

Sequential Function Specification Method

Consider a one-dimensional inverse heat conduction problem of the following dimensionless form:

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} \tag{1}$$

$$T(x, 0) = 0 \tag{2}$$

$$\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=1} = 0 \tag{3}$$

$$T(x_0, t) = f(t) \tag{4}$$

where $0 \le x_0 \le 1$. The solution to Eqs. (1)-(4) will yield heat flux at $x = 0$:

$$-\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=0} = g(t) \tag{5}$$

Although the analytical solution to Eqs. (1)-(4) exists [4]. The expression for $g(t)$ is quite complicated, requiring higher-order derivatives of $f(t)$. In practice, temperatures at x_0 are measured sequentially at time interval Δt from Δt to $m\Delta t$. Designate these temperature by Y_i. Equation (4) is replaced by

$$T(x_0, i\Delta t) = Y_i \tag{6}$$

for $1 \le i \le m$. As a consequence, the solution for heat flux at $x = 0$ becomes

$$-\left.\frac{\partial T(x,i\Delta t)}{\partial x}\right|_{x=0} = q(i\Delta t) \tag{7}$$

for $1 \le i \le n \le m$. Since different functions $f(t)$ can produce the same Y_i in Eq. (6), it follows that $q(t)$ in Eq. (7) are not unique. Furthermore, Y_i are likely to deviate from "true" temperatures due to measurement errors. The goal of a solution method for IHCP is to obtain $q(t)$ that is a good approximation to $g(t)$ and not sensitive to temperature measurement errors.

The heat flux may be assumed to be a piecewise function of time. From the Duhamel's theorem, it can be shown that [1]

$$Y_i = \sum_{j=1}^{i} q_j Z_{i-j+1} \tag{8}$$

where the subscripts denote time indices. Suppose that heat flux components q_1 to q_{k-1} are known from earlier calculations, and q_k is to be determined in the next calculation. Equation (8) can be rewritten as

$$Y_i = \sum_{j=1}^{k-1} q_j Z_{i-j+1} + \sum_{j=k}^{i} q_j Z_{i-j+1} \tag{9}$$

The sequential function specification method uses r future temperature measurements Y_i (where i ranges from k to $k+r-1$) to estimate q_k. Equation (9) actually represents r equations with r unknowns (q_k, q_{k+1}, ..., q_{k+r-1}). In order to reduce the number of unknowns to one, the relations between q_{k+1}, q_{k+2}, ..., q_{k+r-1} and q_1, q_2, ..., q_k must be specified:

$$q_{k+j} = \phi(q_1, q_2, \ldots, q_k) \qquad \text{for } j = 1, 2, \ldots, r-1 \tag{10}$$

Equation (9) becomes

$$q_k\, \alpha_{i-k+1} = Y_i - \sum_{j=1}^{k-1} q_j \beta_{i-k+1,k-j} \tag{11}$$

for $i = k, k+1, \ldots, k+r-1$. The coefficients α_j and $\beta_{i,j}$ depend on piecewise heat flux function and ϕ. Equation (11) may be solved by the linear least square method to obtain the heat flux components q_k ($k = 1, 2, \ldots, n$).

To evaluate the stability of the solution yielded by the sequential function specification method, it is useful to express q_k in terms of only temperatures.

$$q_k = \sum_{i=1}^{k+r-1} b_{k,i} Y_i \tag{12}$$

Straightforward algebraic manipulation of Eqs. (11) and (12) yields the following equation from which $b_{k,i}$ can be determined:

$$\sum_{i=1}^{k+r-1} b_{k,i} Y_i = \sum_{i=k}^{k+r-1} c_{i-k+1} Y_i - \sum_{i=1}^{k+r-2} \left(\sum_{j=1}^{\min(k-1,k+r-i-1)} d_j b_{k-j,i} \right) Y_i \tag{13}$$

with

$$c_i = \frac{\alpha_i}{\sum_{j=1}^{r} \alpha_j^2} \tag{14}$$

and

$$d_i = \frac{\sum_{j=1}^{r} \alpha_j \beta_{j,i}}{\sum_{j=1}^{r} \alpha_j^2} \tag{15}$$

Variance and Deterministic Bias

Two measures of the quality of a solution to IHCP are variance and deterministic bias. Variance measures the sensitivity of the solution to temperature measurement errors. Consider one such measurement at time $i\Delta t$ when Y_i is the measured temperature, T_i is the exact temperature, and ε_i is the random error.

$$Y_i = T_i + \varepsilon_i \tag{16}$$

In order to evaluate the sensitivity of calculated heat flux to measurement errors, it is useful to make the following statistical assumptions about the errors. Assume that measurement errors have a constant variance:

$$V(\varepsilon_i) = \sigma^2 \tag{17}$$

and that the errors are uncorrelated:

$$\operatorname{cov}(\varepsilon_i, \varepsilon_j) = 0 \qquad \text{for } i \neq j \tag{18}$$

It can then be shown that the variance in the estimated value of q_k is

$$V(q_k) = \sigma^2 \sum_{j=1}^{k+r-1} b_{k,j}^2 \tag{19}$$

When measurements are error-free, $Y_i = T_i$, and the calculated heat flux should closely approximate the true heat flux. Deterministic bias D indicates how good the approximation is.

$$D^2 = \frac{1}{n} \sum_{j=1}^{n} \left(q(t_j) - g(t_j) \right)^2 \tag{20}$$

where t_j is representative time between $(j-1)\Delta t$ and $j\Delta t$. The variance of an algorithm tells us how sensitive the calculated heat flux is to measurement errors, and the deterministic bias of the algorithm correlates with the accuracy of the algorithm. Both measures may be used to compare different algorithms for IHCP [5, 6].

Description of Three Algorithms

Different algorithms for implementing the sequential function specification method result from difference in (1) the function form of the piecewise heat flux $q(t)$ may differ from algorithm to algorithm, and (2) the assumed relations of future heat flux components $(q_{k+1}, q_{k+2}, \ldots, q_{k+r-1})$ to the heat flux component q_k that is being estimated. Three algorithms will now be described.

Algorithm 1

The simplest form for $q(t)$ is the piecewise constant function.

$$q(t) = q_i \qquad \text{for } t_{i-1} < t < t_i \tag{21}$$

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Algorithm 2

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This function gives the following expression for Z_i.

$$Z_i = U_1(x,i\Delta t) - U_1(x,(i-1)\Delta t) \tag{22}$$

where

$$U_1(x,t) = t + 2\sum_{j=1}^{\infty}\frac{\cos(j\pi x)}{(j\pi)^2}\left(1 - e^{-j^2\pi^2 t}\right) \tag{23}$$

Future heat flux components are assumed equal to the current heat flux component.

$$q_{k+r-1} = q_{k+r-2} = \dots = q_k \tag{24}$$

It can be shown that this assumption results in

$$\alpha_i = \sum_{j=1}^{i} Z_j \qquad \text{for } i = 1, 2, \dots, r \tag{25}$$

and

$$\beta_{i,j} = Z_{i+j} \qquad \text{for } i = 1, 2, \dots, r\text{, and } j = 1, 2, \dots, n-1 \tag{26}$$

The resulting algorithm is the classic sequential function specification algorithm [1]. The pictorial representation of this algorithm is shown in Fig. 1.

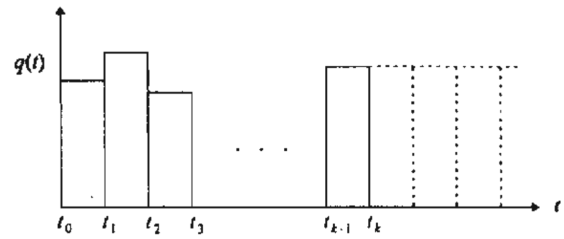

FIG. 1
Pictorial representation of Algorithm 1

Algorithm 2

Another simple form for $q(t)$ is the piecewise linear function.

$$q(t) = \left(q_i - q_{i-1}\right)\frac{\left(t - t_{i-1}\right)}{\Delta t} + q_{i-1} \qquad \text{for } t_{i-1} \le t \le t_i \tag{27}$$

where $q_0 = 0$ This function gives the following expression for Z_i.

$$Z_i = U_2(x,i\Delta t) - 2U_2(x,(i-1)\Delta t) + U_2(x,(i-2)\Delta t) \tag{28}$$

where

$$U_2(x,t) = \begin{cases} \dfrac{1}{\Delta t}\left[\dfrac{t^2}{2} + t\left(\dfrac{x^2}{2} - x + \dfrac{1}{3}\right) - 2\sum_{j=1}^{\infty}\dfrac{\cos(j\pi x)}{(j\pi)^4}\left(1 - e^{-j^2\pi^2 t}\right)\right], & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases} \quad (29)$$

Future heat flux components are assumed equal to the current heat flux component as in Algorithm 1. Consequently, Eqs. (25) and (26) give the expressions for α_i and $\beta_{i,j}$, respectively. The pictorial representation of this algorithm is shown in Fig. 2.

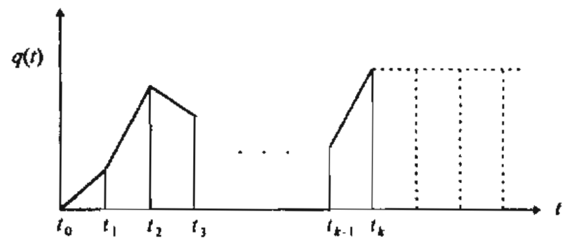

FIG. 2
Pictorial representation of Algorithm 2

__Algorithm 3__

The function form of $q(t)$ is also the piecewise linear function shown in Eq. (27). However, future heat flux components are assumed to vary linearly with a uniform slope. That is,

$$q_{k+j} = 2q_{k+j-1} - q_{k+j-2} \qquad \text{for } j = 1, 2, ..., r-1 \quad (30)$$

The pictorial representation of this algorithm is shown in Fig. 3.

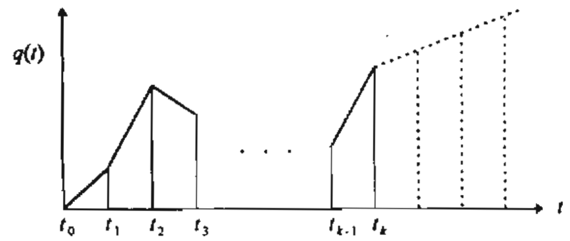

FIG. 3
Pictorial representation of Algorithm 3

The expression for Z_i in Algorithm 3 is identical with that in Algorithm 2. However, due to a different assumption regarding the future heat flux components, the expression for α_i and $\beta_{i,j}$ are different. Substitute

Eqs. (28) and (30) into Eq. (9), and rearrange the result in the form of Eq. (11) to obtain α_i and $\beta_{i,j}$.

$$\alpha_i = \sum_{j=1}^{i} jZ_{i-j+1} \qquad \text{for } i = 1, 2, \ldots, r \tag{31}$$

and, for $j = 1, 2, \ldots, n-1$,

$$\beta_{i,1} = Z_{i+1} - \sum_{k=1}^{i-1} kZ_{i-k} \tag{32}$$

$$\beta_{i,j} = Z_{i+j} \qquad (j > 1) \tag{33}$$

Numerical Results

The problem used to test the three algorithms is described by Eqs. (1)-(3), (5), and (6), with the applied heat flux $g(t)$ as illustrated in Fig. 4. The three algorithms are used to determine 100 heat flux components between $t = 0$ and $t = 1.2$. Thus, time interval $\Delta t = 0.012$. The only adjustable parameter in each algorithm is the number of future temperature measurements r. Therefore, the total number of measurements needed to determine 100 heat flux components is $99 + r$. Since small values of r result in unstable solutions, they are excluded from the presentation of the results.

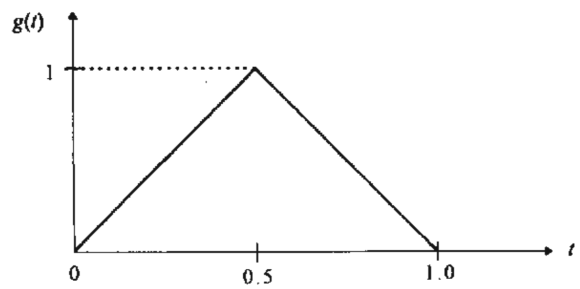

FIG. 4
Applied heat flux as a function of time in the test problem

Figure 5 compares variations of maximum variance, $V_{max} = \max_k(V(q_k))$, with parameter r from $r = 15$ to 26 of the three algorithms. It can be seen that V_{max} decreases monotically with r. Increasing r will therefore lead to a more stable solution in each algorithm. An interesting result from this study is that Algorithm 3 produces a significantly more stable solution than Algorithm 1 and Algorithm 2 at the same r.

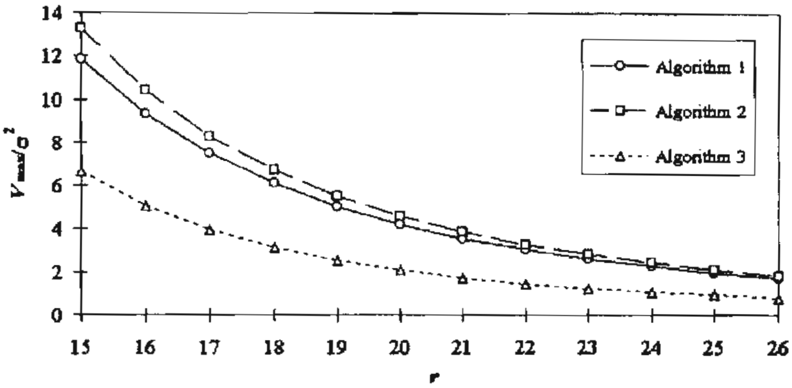

FIG. 5
Variation of maximum variance (V_{max}/σ^2) with r at x_0 = 0.5 for the three algorithms

Figure 6 compares deterministic biases (D), which are calculated using Eq. (20) with t_j equal to $(j - 0.5)\Delta t$. As expected, D increases with r, meaning that the solution becomes less accurate as it becomes more stable in each algorithm. Among the three algorithms, Algorithm 3 appears to produce the most accurate solution.

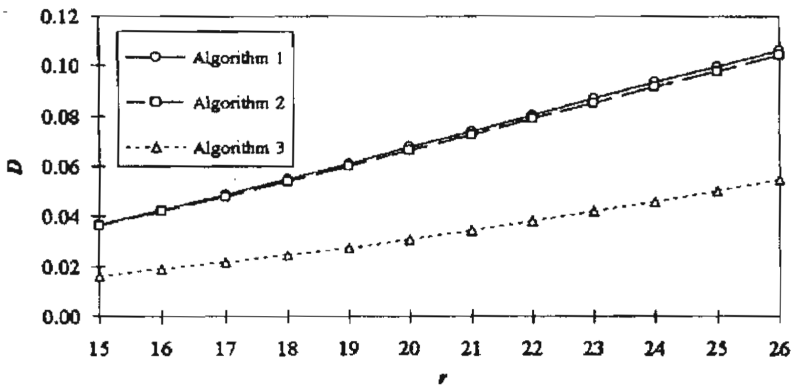

FIG. 6
Variation of deterministic bias (D) with r at x_0 = 0.5 for the three algorithms

Finally, the calculated heat flux components obtained by the three algorithms at r = 20 are compared with the actual heat flux in Fig. 7. It should be noted that all algorithms predicted a peak in the

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heat flux function before it actually occurs. The solution yielded by Algorithm 3 clearly resembles the actual heat flux more than the solution by Algorithm 1 or Algorithm 2.

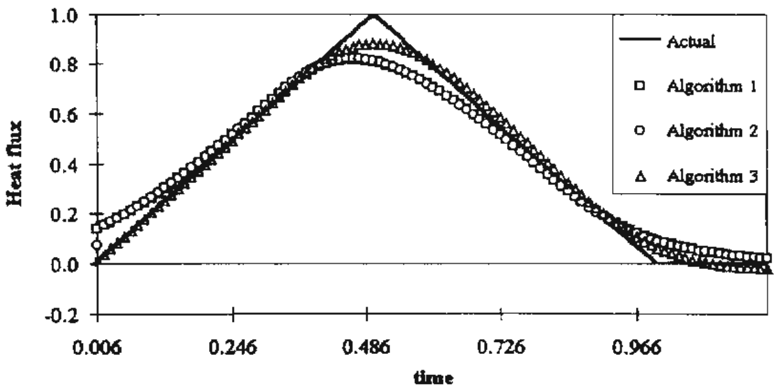

FIG. 7
Comparison between actual heat flux and calculated heat fluxes

Discussion

For the same accuracy, Algorithm 3 produces a more stable solution than Algorithm 1 or 2; and for the same variance, Algorithm 3 also produces a more accurate solution than Algorithm 1 or 2. We may therefore conclude that Algorithm 3 is a better algorithm. Beck et al. [6] showed that when the constant piecewise function was used, the assumption of linearly varying future heat flux components did not give a much better solution than the assumption of constant future heat flux components. The comparison between Algorithm 1 and Algorithm 3 in the present study, however, shows that changing both the piecewise heat flux function and the assumed relation between future heat flux components and the current heat flux component can lead to an improved algorithm. Despite a slightly more complicated formulation, Algorithm 3 is computationally efficient and easy to use. Thus, it should be considered as alternative algorithm for implementing the sequential function specification method. It is conceivable that an even better sequential function specification algorithm may be obtained by using a more complicated piecewise heat flux function and a more complicated assumption regarding the relation between future heat flux components and the current heat flux component.

Acknowledgements

The author would like to acknowledge the financial support from the Thailand Research Fund.

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February 19, 1999

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Subj: Ms. #IJ98/7097 - "Inverse Heat Conduction Problem of Determining Time-Dependent Heat Transfer Coefficient" by S. Chantasiriwan

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Inverse heat conduction problem of determining time-dependent heat transfer coefficient

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Abstract

The time-dependent Biot number in a one-dimensional linear heat conduction problem is obtained from the solutions of the inverse heat conduction problems of determining boundary heat flux and boundary temperature. The sequential function specification method with the linear basis function and the assumption of linearly varying future boundary heat flux or temperature components is used to solve the inverse problem. The expression for Biot number is found to be a nonlinear function of measured temperatures. The variance in input data is shown to cause variance and nonlinear bias in estimated Biot number. The method presented offers three tunable parameters that may be used to improve the quality of the solution.

NOMENCLATURE

A	response vector
B	Biot number
C	response vector
c_p	heat capacity
D	coefficient matrix
$E(y)$	expected value of random variable y
f	probability density distribution function
h	heat transfer coefficient
L	length

l, m	dummy indices
n	the number of Biot number components to be determined
p	ratio of time step for estimated Biot number components to time step for temperature measurements
q	boundary heat flux
r	future-time parameter
$\boldsymbol{S}$	transformation matrix
T	temperature
T_0	initial temperature
T_∞	ambient temperature
t	time
$\mathrm{Var}(y)$	variance of random variable y
x	location
x_0	sensor location

Greek symbols

$\alpha_j^{(i)}$	coefficient that relates estimated boundary heat flux component to measured temperature
$\beta_j^{(i)}$	coefficient that relates estimated boundary temperature to measured temperature
Δ_d	deterministic bias
Δ_n	nonlinear bias
ε	temperature measurement error
ϕ	response function
κ	thermal conductivity
θ	boundary temperature
ρ	density
σ^2	variance of temperature measurements

ψ response function

Subscripts and superscripts

i, j, k, l, m indices

INTRODUCTION

A heat conduction problem in a solid with the initial and boundary conditions completely specified is a well-posed problem that can be solved by various analytical and numerical methods. On the other hand, when the boundary condition is to be determined from temperature measurement data inside the solid, the problem is an ill-posed one known as the inverse heat conduction problem (IHCP) [1]. Although the analytical solution of IHCP exists for the case of one-dimensional problem [2], a numerical method is generally preferable since it offers control over the accuracy and the stability of the solution. Among the well-known numerical methods are the space-marching technique [3], the frequency domain adjoint method [4], the mollification method [5], the iterative regularization method [6], the direct sensitivity coefficient method [7], and the sequential function specification method [1]. The aims of these methods are to obtain a solution that is accurate and not very sensitive to changes in input temperature data.

Most of the inverse heat conduction problems that have been investigated so far are concerned with the estimation of boundary heat flux. Another interesting problem that has not yet received as much interest is the estimation of heat transfer coefficient. Osman and Beck [8] treated the problem of estimating time-dependent heat transfer coefficient in the quenching of a sphere as a nonlinear parameter estimation problem. Heat transfer coefficient was assumed to be a piecewise constant function of time. The unknown heat transfer coefficient parameters were estimated one by one using the sequential function specification method. Naylor and Oosthuizen [9] employed the temperature-time data measured at subsurface locations to determine the heat transfer coefficient in a forced convective flow over a square prism. They expressed the distribution of heat transfer coefficient in terms of several piecewise constant functions. The coefficients were computed using an iterative algorithm. Hernandez-Morales et al. [10] studied the one-dimensional problem of estimating the transient heat transfer coefficient at the surface of steel bars subjected to quenching using the sequential function specification method. They found that filtering the input data led to improved estimation. Mehrotra et al. [11] estimated interfacial heat transfer coefficient in solidification of a molten metal on a metal substrate. Their transient one-dimensional problem was divided into a direct region and an indirect region. The solution for the direct region was obtained using a conventional method. The Burggraf solution [2] was then used to compute the temperature and heat flux at the interface between the molten metal and the substrate, from which the heat transfer coefficient could be determined in a straightforward manner. Xu and Chen [12] studied the steady-state nonlinear problem of determining the heat transfer coefficient in two-phase mixture flow in an inclined tube. Their algorithm was a simple iterative procedure. Most recently, Martin and Dulikravich [13] employed the boundary element method to set up the inverse problem of determining boundary heat flux and boundary temperature simultaneously in a steady-state multidimensional problem. The single value decomposition method was then used to obtain stabilized solutions for boundary heat flux and boundary temperature, from which heat transfer coefficient was determined.

In this paper, an algorithm for estimating time-dependent heat transfer coefficient for a one-dimensional linear inverse heat conduction problem is proposed. The method used is the sequential function specification method with the linear basis function and the assumption of linearly varying future boundary heat flux (or temperature) components. Recent results by Chantasiriwan [14] showed that this method yielded better estimates of boundary condition than the well-known sequential function specification method [1]. Hence, it is expected that the estimation of heat transfer coefficient should perform better with the new method as well. The following sections will describe the matrix formulation of the algorithm, which will facilitate computer implementation. The method for analyzing the accuracy and stability of the estimate will then be described. Sample results and discussion of how to improve the estimate will follow. Finally, the conclusions that can be drawn from this paper will be given.

MATHEMATICAL FORMULATION OF THE PROBLEM

The problem to be considered is shown in Fig. 1. A one-dimensional object is subjected to unknown time-dependent heat transfer coefficient at one end whereas the other end is insulated. The ambient temperature is assumed constant throughout the time period considered. The temperature measurement is made at a distance from the boundary of unknown heat transfer coefficient. The measurement data along with the known geometrical and thermophysical data give rise to the inverse heat conduction problem, which can be mathematically described by the following governing equation, initial condition, and boundary condition.

$$\rho c_p \frac{\partial T'(x',t')}{\partial t'} = \kappa \frac{\partial^2 T'(x',t')}{\partial x'^2} \quad (1)$$

$$T'(x',0) = T_0 \quad (2)$$

$$\left.\frac{\partial T'(x',t')}{\partial x'}\right|_{x'=L} = 0 \tag{3}$$

Temperature measurements at sensor location x_0', which is between 0 and L, are available at a regular time interval.

$$T'(x_0', i\Delta t') = T_i' \tag{4}$$

The heat transfer coefficient at the convective boundary $x' = 0$ is to be determined. The definition of $h(t')$ is given by

$$-\kappa\left.\frac{\partial T'(x',t')}{\partial x'}\right|_{x'=0} = h(t')(T_\infty - T') \tag{5}$$

Define dimensionless variables $x = x'/L$, $t = \kappa t'/\rho c_p L^2$, $T = (T' - T_0)/(T_\infty - T_0)$, and $B(t) = h(t')L/\kappa$. Equations (1)-(5) can be rewritten in dimensionless forms.

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} \tag{6}$$

$$T(x, 0) = 0 \tag{7}$$

$$\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=1} = 0 \tag{8}$$

$$T(x_0, i\Delta t) = T_i \tag{9}$$

$$-\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=0} = B(t)(1 - T) \tag{10}$$

The goal of the IHCP solution is to determine B(t). Instead of devising an algorithm to determine $B(t)$ directly, it is more convenient to estimate boundary heat flux $q(t)$ and boundary temperature $\theta(t)$ at $x = 0$ from equations (6)–(9), and use them to obtain the expression for $B(t)$.

DETERMINATION OF BOUNDARY HEAT FLUX

The expression for boundary heat flux into the object is

$$q(t) = -\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=0} \tag{11}$$

Chantasiriwan [14] described a sequential function specification algorithm for estimating $q(t)$. The algorithm makes use of piecewise linear function in estimating boundary heat flux component by component. The solution is stabilized by employing future-time temperature measurements as input data, and assuming that heat flux components vary linearly. A distinctive feature of that algorithm and the conventional sequential function specification algorithm [1] is the presence of future-time parameter r, which is related to the number of future-time measurements used as input data. This parameter acts as a stabilizing parameter in that the solution becomes more stable as r increases. However, that algorithm is limited to cases in which the time step of temperature measurements equals the time step of the estimated heat flux components. Because more input data will probably lead to a more stable solution, it may be advantageous to allow the former to be smaller than the latter. In the algorithm to be used in this paper, the time step of temperature measurement is Δt, and the time step of the estimated heat flux components is $p\Delta t$, where p is a positive integer. The revised formulation for the sequential function specification algorithm will be now described.

The 'current' heat flux component, $q^{(i)}$ ($1 \le i \le n$), is estimated using $r.p$ 'future' temperature measurements, $T_{(i-1)p+1}$, $T_{(i-1)p+2}$, …, $T_{(i+r-1)p}$. It is assumed that the basis function of the boundary heat flux components is the piecewise linear function and that future heat flux components vary linearly. (See Fig. 2.) For $i-1 \le k \le i+r-2$ and $1 \le m \le p$, temperatures at x_0 are related to boundary heat flux components as follows.

$$T_{kp+m} = \sum_{j=1}^{i-1} \frac{q^{(j)}}{p\Delta t} [\phi(x_0, ((k-j+1)p+m)\Delta t) - 2\phi(x_0, ((k-j)p+m)\Delta t)$$

$$+ \phi(x_0, ((k-j-1)p+m)\Delta t)] +$$

$$\sum_{j=0}^{k-i+1} \frac{q^{(i+j)}}{p\Delta t} [\phi(x_0, ((k-i-j+1)p+m)\Delta t) -$$

$$2\phi(x_0, ((k-i-j)p+m)\Delta t) + \phi(x_0, ((k-i-j-1)p+m)\Delta t)] \qquad (12)$$

where

$$\phi(x_0, t) = \begin{cases} \frac{t^2}{2} + t\left(\frac{x^2}{2} - x_0 + \frac{1}{3}\right) - 2\sum_{m=1}^{\infty} \frac{\cos(m\pi x_0)}{(m\pi)^4}\left(1 - e^{-m^2\pi^2 t}\right), & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases} \qquad (13)$$

is the temperature response at $x = x_0$ to the heat conduction problem described by equations (6)-(8) and the condition that the linearly increasing heat flux having unity slope is applied at $x = 0$. Since it is assumed that future heat flux components $q^{(i+1)}$, $q^{(i+2)}, \ldots, q^{(i+r-1)}$ vary linearly, $q^{(i+l)}$ can be expressed in terms of $q^{(i)}$ and $q^{(i-1)}$:

$$q^{(i+l)} = (l+1)\, q^{(i)} - l q^{(i-1)} \qquad (14)$$

for $1 \leq l \leq r-1$. Now, substitute equation (14) into (12), and simplify the result.

$$T_{kp+m} = \frac{q^{(i)}}{p\Delta t} \phi(x_0, ((k-j+1)p+m)\Delta t) +$$

$$\frac{q^{(i-1)}}{p\Delta t} [\phi(x_0, ((k-i+2)p+m)\Delta t) - 2\phi(x_0, ((k-i+1)p+m)\Delta t)]$$

$$+ \sum_{j=1}^{i-2} \frac{q^{(j)}}{p\Delta t} [\phi(x_0, ((k-j+1)p+m)\Delta t) -$$

$$2\phi(x_0, ((k-j)p+m)\Delta t) + \phi(x_0, ((k-j-1)p+m)\Delta t)] \qquad (15)$$

Because $q^{(i-1)}, q^{(i-2)}, \ldots, q^{(1)}$ are known from previous calculations, equation (15) represents an overdetermined system of linear algebraic equations with $q^{(i)}$ as the only unknown for the current calculation. Let's define the following vectors and matrices:

$$\boldsymbol{T} \quad = \quad \begin{bmatrix} T_1 & T_2 & \ldots & T_{(n+r-1)p} \end{bmatrix}^{\mathrm{T}}$$

$$\boldsymbol{A} \quad = \quad \frac{1}{p\Delta t}\begin{bmatrix} \phi(x_0,\Delta t) & \phi(x_0,2\Delta t) & \ldots & \phi(x_0,rp\Delta t) \end{bmatrix}^{\mathrm{T}}$$

$$\boldsymbol{C}^{(1)} \quad = \quad \frac{1}{p\Delta t}\begin{bmatrix} \phi(x_0,(p+1)\Delta t)-2\phi(x_0,\Delta t) \\ \phi(x_0,(p+2)\Delta t)-2\phi(x_0,2\Delta t) \\ \vdots \\ \phi(x_0,(r+1)p\Delta t)-2\phi(x_0,rp\Delta t) \end{bmatrix}$$

$$\boldsymbol{C}^{(i)} \quad = \quad \frac{1}{p\Delta t}\begin{bmatrix} \phi(x_0,(ip+1)\Delta t)-2\phi(x_0,((i-1)p+1)\Delta t)+\phi(x_0,((i-2)p+1)\Delta t) \\ \phi(x_0,(ip+2)\Delta t)-2\phi(x_0,((i-1)p+2)\Delta t)+\phi(x_0,((i-2)p+2)\Delta t) \\ \vdots \\ \phi(x_0,(i+r)p\Delta t)-2\phi(x_0,(i+r-1)p\Delta t)+\phi(x_0,(i+r-2)p\Delta t) \end{bmatrix}$$

for $2 \le i \le n-1$, and

$$\boldsymbol{S}^{(i)} \quad = \quad \left.\begin{bmatrix} 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underbrace{0 \quad 0 \quad \ldots \quad 0}_{(i-1)p \text{ columns}} & & & & \underbrace{0 \quad 0 \quad \ldots \quad 1}_{rp \text{ columns}} & & & & \underbrace{0 \quad 0 \quad \ldots \quad 0}_{(n-i)p \text{ columns}} \end{bmatrix}\right\} rp \text{ rows}$$

for $1 \le i \le n$. Equation (15) can now be rewritten as a matrix equation:

$$\boldsymbol{S}^{(i)}\boldsymbol{T} \quad = \quad \boldsymbol{A}q^{(i)} + \sum_{k=1}^{i-1}\boldsymbol{C}^{(i-k)}q^{(k)} \tag{16}$$

It is useful to express the unknown $q^{(i)}$ in terms of all other quantities.

$$q^{(i)} \quad = \quad \left(\boldsymbol{A}^T\boldsymbol{A}\right)^{-1}\boldsymbol{A}^T\left[\boldsymbol{S}^{(i)}\boldsymbol{T} - \sum_{k=1}^{i-1}\boldsymbol{C}^{(i-k)}q^{(k)}\right] \tag{17}$$

Furthermore, let

$$q^{(i)} = D^{(i)}T \quad (18)$$

The coefficient matrix $\boldsymbol{D}^{(i)}$, which relates the unknown heat flux component $q^{(i)}$ to known temperature measurement data, $T_1, T_2, \ldots, T_{(n+r-1)p}$, can be found from equations (17) and (18).

$$\boldsymbol{D}^{(i)} = \left(\boldsymbol{A}^T\boldsymbol{A}\right)^{-1}\boldsymbol{A}^T\left[\boldsymbol{S}^{(i)} - \sum_{k=1}^{i-1}\boldsymbol{C}^{(i-k)}\boldsymbol{D}^{(k)}\right] \quad (19)$$

Knowledge of $\boldsymbol{D}^{(i)}$ allows us to write $q^{(i)}$ in terms of $T_1, T_2, \ldots, T_{(n+r-1)p}$.

$$q^{(i)} = \sum_{k=1}^{(n+r-1)p}\alpha_k^{(i)}T_k \quad (20)$$

DETERMINATION OF BOUNDARY TEMPERATURE

The expression for boundary temperature is

$$\theta(t) = T(0,t) \quad (21)$$

Let $\psi(x_0,t)$ be the temperature response at $x = x_0$ to the heat conduction problem described by equations (6)-(8) and the condition that the linearly increasing temperature having unity slope is applied at $x = 0$. The expression for $\psi(x_0,t)$ is

$$\psi(x_0,t) = \begin{cases} t - 2\displaystyle\sum_{m=1}^{\infty}\frac{\sin\left((m-0.5)\pi x_0\right)}{\left((m-0.5)\pi\right)^3}\left(1-e^{-(m-0.5)^2\pi^2 t}\right), & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases} \quad (22)$$

With the replacement of $\phi(x_0,t)$ by $\psi(x_0,t)$ in the expressions for $\boldsymbol{A}$ and $\boldsymbol{C}^{(i)}$, the procedure described previously can be used to determine boundary temperature $\theta^{(i)}$ as a function of temperature measurement data.

$$\theta^{(i)} = \sum_{k=1}^{(n+r-1)p}\beta_k^{(i)}T_k \quad (23)$$