DETERMINATION OF TIME-DEPENDENT BIOT NUMBER

Once $q^{(i)}$ and $\theta^{(i)}$ have been determined, $B^{(i)}$ can be obtained from

$$B^{(i)} \quad = \quad \frac{q^{(i)}}{1-\theta^{(i)}} \tag{24}$$

It is interesting to note that $B^{(i)}$ is a nonlinear function of measured temperatures,

$$B^{(i)} \quad = \quad \frac{\sum_{k=1}^{(n+r-1)p} \alpha_k^{(i)} T_k}{1 \; - \; \sum_{k=1}^{(n+r-1)p} \beta_k^{(i)} T_k} \tag{25}$$

whereas $q^{(i)}$ and $\theta^{(i)}$ are linear functions of temperatures. Hence, the evaluation of statistical errors in $B^{(i)}$, resulting from errors in temperature measurement, is more complicated than the evaluation of statistical errors in $q^{(i)}$ and $\theta^{(i)}$.

It is useful to make the following statistical assumptions regarding temperature measurement errors [15]:

1. Additive errors: $T_j = \overline{T}_j + \varepsilon_j$
2. Zero mean errors: $E(\varepsilon_j) = 0$
3. Constant variance: $Var(\varepsilon_j) = \sigma^2$
4. Uncorrelated errors: $E(\varepsilon_j \varepsilon_k) = 0$ if $j \neq k$
5. Normal probability distribution for errors
6. Nonstochastic independent variable

As a result of these assumptions, the probability density function for errors is

$$f(\varepsilon_j) \quad = \quad \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\varepsilon_j}{\sigma}\right)^2\right], \qquad -\infty < \varepsilon_j < \infty \tag{26}$$

In inverse heat conduction problem of estimating boundary heat flux or boundary temperature, the quality of the solution is determined by two measures: the deterministic bias and the variance of the solution. Deterministic bias represents the difference between the estimated solution and exact solution when temperature measurements are error-free. The deterministic biases for $q^{(i)}$ and $\theta^{(i)}$ may be defined, respectively, as

$$\Delta_{d,q} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(q(ip\Delta t) - \mathrm{E}\left(q^{(i)}\right)\right)^2} \tag{27}$$

and

$$\Delta_{d,\theta} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\theta(ip\Delta t) - \mathrm{E}\left(\theta^{(i)}\right)\right)^2} \tag{28}$$

where

$$\mathrm{E}(q^{(i)}) = \sum_{k=1}^{(n+r-1)p}\alpha_k^{(i)}\overline{T}_k \tag{29}$$

and

$$\mathrm{E}(\theta^{(i)}) = \sum_{k=1}^{(n+r-1)p}\beta_k^{(i)}\overline{T}_k \tag{30}$$

are the expected values of the estimated boundary heat flux component and the estimated boundary temperature component. Thus, the deterministic bias depends on the solution algorithm, but not the statistical errors present in actual input data. On the other hand, the variance of the solution is a function of the variance of input data. As a consequence of the above assumptions regarding temperature measurement errors, the variances of boundary heat flux and boundary temperature are, respectively,

$$\mathrm{Var}(q^{(i)}) = \sigma^2\sum_{k=1}^{(n+r-1)p}\left(\alpha_k^{(i)}\right)^2 \tag{31}$$

and

$$\mathrm{Var}(\theta^{(i)}) = \sigma^2\sum_{k=1}^{(n+r-1)p}\left(\beta_k^{(i)}\right)^2 \tag{32}$$

STATISTICAL ERRORS IN ESTIMATED BIOT NUMBER

Define average Biot number component as

$$\bar{B}^{(i)} \quad = \quad \frac{\sum_{k=1}^{(n+r-1)p} \alpha_k^{(i)} \bar{T}_k}{1 - \sum_{k=1}^{(n+r-1)p} \beta_k^{(i)} \bar{T}_k} \tag{33}$$

Taylor series expansion for $B^{(i)}$ yields

$$B^{(i)} \quad = \quad \bar{B}^{(i)} + \sum_j \left.\frac{\partial B^{(i)}}{\partial T_j}\right|_{\bar{T}_j} \varepsilon_j + \frac{1}{2}\sum_{j,k} \left.\frac{\partial^2 B^{(i)}}{\partial T_j \partial T_k}\right|_{\bar{T}_j,\bar{T}_k} \varepsilon_j \varepsilon_k +$$

$$\frac{1}{6}\sum_{j,k,l} \left.\frac{\partial^3 B^{(i)}}{\partial T_j \partial T_k \partial T_l}\right|_{\bar{T}_j,\bar{T}_k,\bar{T}_l} \varepsilon_j \varepsilon_k \varepsilon_l + \frac{1}{24}\sum_{j,k,l,m} \left.\frac{\partial^4 B^{(i)}}{\partial T_j \partial T_k \partial T_l \partial T_m}\right|_{\bar{T}_j,\bar{T}_k,\bar{T}_l,\bar{T}_m} \varepsilon_j \varepsilon_k \varepsilon_l \varepsilon_m + \ldots \tag{34}$$

where each index in above summations and summations to follow runs from 1 to $(n+r-1)p$. The expressions for derivatives of $B^{(i)}$ are given below.

$$\frac{\partial B^{(i)}}{\partial T_j} \quad = \quad \frac{\alpha_j^{(i)} + B^{(i)}\beta_j^{(i)}}{\left(1 - \sum_k \beta_k^{(i)} T_k\right)} \tag{35}$$

$$\frac{\partial^2 B^{(i)}}{\partial T_j \partial T_k} \quad = \quad \frac{\beta_j^{(i)}\alpha_k^{(i)} + \beta_k^{(i)}\alpha_j^{(i)} + 2B^{(i)}\beta_j^{(i)}\beta_k^{(i)}}{\left(1 - \sum_k \beta_k^{(i)} T_k\right)^2} \tag{36}$$

$$\frac{\partial^3 B^{(i)}}{\partial T_j \partial T_k \partial T_l} \quad = \quad \frac{1}{\left(1 - \sum_k \beta_k^{(i)} T_k\right)^3} \Big[\beta_j^{(i)}\beta_k^{(i)}\alpha_l^{(i)} + \beta_j^{(i)}\beta_l^{(i)}\alpha_k^{(i)} + \beta_k^{(i)}\beta_l^{(i)}\alpha_j^{(i)} +$$

$$\beta_k^{(i)}\beta_j^{(i)}\alpha_l^{(i)} + \beta_l^{(i)}\beta_j^{(i)}\alpha_k^{(i)} + \beta_l^{(i)}\beta_k^{(i)}\alpha_j^{(i)} + 6B^{(i)}\beta_j^{(i)}\beta_k^{(i)}\beta_l^{(i)} \Big] \tag{37}$$

$$\frac{\partial^4 B^{(i)}}{\partial T_j \partial T_k \partial T_l \partial T_m} \quad = \quad \frac{1}{\left(1 - \sum_k \beta_k^{(i)} T_k\right)^4} \Big[\beta_j^{(i)}\beta_k^{(i)}\beta_l^{(i)}\alpha_m^{(i)} + \beta_j^{(i)}\beta_k^{(i)}\beta_m^{(i)}\alpha_l^{(i)} + \beta_j^{(i)}\beta_l^{(i)}\beta_k^{(i)}\alpha_m^{(i)} +$$

$$\beta_j^{(i)}\beta_l^{(i)}\beta_m^{(i)}\alpha_k^{(i)} + \beta_j^{(i)}\beta_m^{(i)}\beta_k^{(i)}\alpha_l^{(i)} + \beta_j^{(i)}\beta_m^{(i)}\beta_l^{(i)}\alpha_k^{(i)} + \beta_k^{(i)}\beta_j^{(i)}\beta_l^{(i)}\alpha_m^{(i)} + \beta_k^{(i)}\beta_j^{(i)}\beta_m^{(i)}\alpha_l^{(i)} +$$

$$\beta_k^{(i)}\beta_l^{(i)}\beta_j^{(i)}\alpha_m^{(i)} + \beta_k^{(i)}\beta_l^{(i)}\beta_m^{(i)}\alpha_j^{(i)} + \beta_k^{(i)}\beta_m^{(i)}\beta_j^{(i)}\alpha_l^{(i)} + \beta_k^{(i)}\beta_m^{(i)}\beta_l^{(i)}\alpha_j^{(i)} + \beta_l^{(i)}\beta_j^{(i)}\beta_k^{(i)}\alpha_m^{(i)} +$$

$$\beta_l^{(i)}\beta_j^{(i)}\beta_m^{(i)}\alpha_k^{(i)} + \beta_l^{(i)}\beta_k^{(i)}\beta_j^{(i)}\alpha_m^{(i)} + \beta_l^{(i)}\beta_k^{(i)}\beta_m^{(i)}\alpha_j^{(i)} + \beta_l^{(i)}\beta_m^{(i)}\beta_j^{(i)}\alpha_k^{(i)} + \beta_l^{(i)}\beta_m^{(i)}\beta_k^{(i)}\alpha_j^{(i)} +$$

$$\beta_m^{(i)}\beta_j^{(i)}\beta_k^{(i)}\alpha_l^{(i)} + \beta_m^{(i)}\beta_j^{(i)}\beta_l^{(i)}\alpha_k^{(i)} + \beta_m^{(i)}\beta_k^{(i)}\beta_j^{(i)}\alpha_l^{(i)} + \beta_m^{(i)}\beta_k^{(i)}\beta_l^{(i)}\alpha_j^{(i)} + \beta_m^{(i)}\beta_l^{(i)}\beta_j^{(i)}\alpha_k^{(i)} +$$

$$\beta_m^{(i)}\beta_l^{(i)}\beta_k^{(i)}\alpha_j^{(i)} + 24\beta_j^{(i)}\beta_k^{(i)}\beta_l^{(i)}\beta_m^{(i)} \Bigg] \tag{38}$$

Expected value for $B^{(i)}$ is given by

$$\mathrm{E}(B^{(i)}) = \mathrm{E}(\overline{B}^{(i)}) + \sum_j \frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\overline{T}_j} \mathrm{E}(\varepsilon_j) + \frac{1}{2}\sum_{j,k}\frac{\partial^2 B^{(i)}}{\partial T_j \partial T_k}\bigg|_{\overline{T}_j,\overline{T}_k} \mathrm{E}(\varepsilon_j\varepsilon_k) +$$

$$\frac{1}{6}\sum_{j,k,l}\frac{\partial^3 B^{(i)}}{\partial T_j \partial T_k \partial T_l}\bigg|_{\overline{T}_j,\overline{T}_k,\overline{T}_l} \mathrm{E}(\varepsilon_j\varepsilon_k\varepsilon_l) + \frac{1}{24}\sum_{j,k,l,m}\frac{\partial^4 B^{(i)}}{\partial T_j \partial T_k \partial T_l \partial T_m}\bigg|_{\overline{T}_j,\overline{T}_k,\overline{T}_l,\overline{T}_m} \mathrm{E}(\varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m) + \ldots \tag{39}$$

Note that, in addition to the first moment of ε_j, which is the zero mean, and the second moment of ε_j, which is the variance, the right hand side contains the third moment and the fourth moment of ε_j. With the density distribution function $f(\varepsilon_j)$ given in equation (26), they can be simply evaluated.

$$\mathrm{E}(\varepsilon_j^3) = \int_{-\infty}^{\infty} \varepsilon_j^3 f(\varepsilon_j) d\varepsilon_j = 0 \tag{40}$$

$$\mathrm{E}(\varepsilon_j^4) = \int_{-\infty}^{\infty} \varepsilon_j^4 f(\varepsilon_j) d\varepsilon_j = 3\sigma^4 \tag{41}$$

Hence, the second term on right hand side of equation (39) vanishes. As a consequence of the zero correlation between measurement errors, the fourth term also vanishes. The third and fifth terms can be rewritten as

$$\frac{1}{2}\sum_{j,k}\frac{\partial^2 B^{(i)}}{\partial T_j \partial T_k}\bigg|_{\overline{T}_j,\overline{T}_k}. \mathrm{E}(\varepsilon_j\varepsilon_k) = \frac{1}{2}\sum_j \frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\overline{T}_j} \mathrm{E}(\varepsilon_j^2)$$

$$= \frac{\sigma^2}{2}\sum_j \frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\overline{T}_j} \tag{42}$$

and

$$\frac{1}{24}\sum_{j,k,l,m}\left.\frac{\partial^4 B^{(i)}}{\partial T_j \partial T_k \partial T_l \partial T_m}\right|_{\bar{T}_j,\bar{T}_k,\bar{T}_l,\bar{T}_m} \mathrm{E}\left(\varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m\right)$$

$$= \frac{1}{24}\left[3\sum_{j\neq k}\left.\frac{\partial^4 B^{(i)}}{\partial T_j^2 \partial T_k^2}\right|_{\bar{T}_j,\bar{T}_k} \mathrm{E}\left(\varepsilon_j^2\right)\mathrm{E}\left(\varepsilon_k^2\right) + \sum_j \left.\frac{\partial^4 B^{(i)}}{\partial T_j^4}\right|_{\bar{T}_j} \mathrm{E}\left(\varepsilon_j^4\right)\right]$$

$$+ \frac{\sigma^4}{8}\left[\sum_{j\neq k}\left.\frac{\partial^4 B^{(i)}}{\partial T_j^2 \partial T_k^2}\right|_{\bar{T}_j,\bar{T}_k} + \sum_j \left.\frac{\partial^4 B^{(i)}}{\partial T_j^4}\right|_{\bar{T}_j}\right] \tag{43}$$

Expected value for $B^{(i)}$ can now be expressed in terms of temperature measurement variance σ^2.

$$\mathrm{E}(B^{(i)}) = \bar{B}^{(i)} + \frac{\sigma^2}{2}\sum_j \left.\frac{\partial^2 B^{(i)}}{\partial T_j^2}\right|_{\bar{T}_j} + \frac{\sigma^4}{8}\left[\sum_{j\neq k}\left.\frac{\partial^4 B^{(i)}}{\partial T_j^2 \partial T_k^2}\right|_{\bar{T}_j,\bar{T}_k} + \sum_j \left.\frac{\partial^4 B^{(i)}}{\partial T_j^4}\right|_{\bar{T}_j}\right]$$

$$+ \ O(\sigma^6) \tag{44}$$

It is interesting to note that variance in measurement errors will cause the expected value, $\mathrm{E}(B^{(i)})$, of the estimated Biot number component to deviate from the true average value, $\bar{B}^{(i)}$, of the estimated Biot number component. The difference between $\mathrm{E}(B^{(i)})$ and $\bar{B}^{(i)}$ may be called the nonlinear bias $\Delta_{\mathrm{n}}^{(i)}$. If terms of order $O(\sigma^6)$ and higher are neglected,

$$\Delta_{\mathrm{n}}^{(i)} = \mathrm{E}(B^{(i)}) - \bar{B}^{(i)}$$

$$= \frac{\sigma^2}{2}\sum_j \left.\frac{\partial^2 B^{(i)}}{\partial T_j^2}\right|_{\bar{T}_j} + \frac{\sigma^4}{8}\left[\sum_{j\neq k}\left.\frac{\partial^4 B^{(i)}}{\partial T_j^2 \partial T_k^2}\right|_{\bar{T}_j,\bar{T}_k} + \sum_j \left.\frac{\partial^4 B^{(i)}}{\partial T_j^4}\right|_{\bar{T}_j}\right] \tag{45}$$

In the estimation of boundary heat flux or boundary temperature, where the dependence on measured temperatures is linear, nonlinear bias is zero. However, in a nonlinear estimation such as the problem considered here, nonlinear bias cannot be ignored unless variance of measurement errors is negligible.

The variance of $B^{(i)}$ can be determined from the following definition.

$$\mathrm{Var}(B^{(i)}) \quad = \quad \mathrm{E}((B^{(i)})^2) - (\mathrm{E}(B^{(i)}))^2 \tag{46}$$

The right hand side of equation (46) will now be evaluated term by term.

$$\begin{aligned}
(B^{(i)})^2 \;=\;& \left(\overline{B}^{(i)}\right)^2 + 2\overline{B}^{(i)}\sum_j \left.\frac{\partial B^{(i)}}{\partial T_j}\right|_{\bar{T}_j} \varepsilon_j + \overline{B}^{(i)}\sum_{j,k} \left.\frac{\partial^2 B^{(i)}}{\partial T_j \partial T_k}\right|_{\bar{T}_j,\bar{T}_k} \varepsilon_j\varepsilon_k + \\
& \frac{\overline{B}^{(i)}}{3}\sum_{j,k,l} \left.\frac{\partial^3 B^{(i)}}{\partial T_j \partial T_k \partial T_l}\right|_{\bar{T}_j,\bar{T}_k,\bar{T}_l} \varepsilon_j\varepsilon_k\varepsilon_l + \frac{\overline{B}^{(i)}}{12}\sum_{j,k,l,m} \left.\frac{\partial^4 B^{(i)}}{\partial T_j \partial T_k \partial T_l \partial T_m}\right|_{\bar{T}_j,\bar{T}_k,\bar{T}_l,\bar{T}_m} \varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m + \\
& \left(\sum_j \left.\frac{\partial B^{(i)}}{\partial T_j}\right|_{\bar{T}_j} \varepsilon_j\right)^2 + \frac{1}{4}\left(\sum_{j,k} \left.\frac{\partial^2 B^{(i)}}{\partial T_j \partial T_k}\right|_{\bar{T}_j,\bar{T}_k} \varepsilon_j\varepsilon_k\right)^2 + \left(\sum_j \left.\frac{\partial B^{(i)}}{\partial T_j}\right|_{\bar{T}_j} \varepsilon_j\right)\left(\sum_{j,k} \left.\frac{\partial^2 B^{(i)}}{\partial T_j \partial T_k}\right|_{\bar{T}_j,\bar{T}_k} \varepsilon_j\varepsilon_k\right) \\
& + \frac{1}{3}\left(\sum_j \left.\frac{\partial B^{(i)}}{\partial T_j}\right|_{\bar{T}_j} \varepsilon_j\right)\left(\sum_{j,k,l} \left.\frac{\partial^3 B^{(i)}}{\partial T_j \partial T_k \partial T_l}\right|_{\bar{T}_j,\bar{T}_k,\bar{T}_l} \varepsilon_j\varepsilon_k\varepsilon_l\right) + \ldots
\end{aligned} \tag{47}$$

$$\begin{aligned}
\mathrm{E}((B^{(i)})^2) \;=\;& \mathrm{E}\left(\left(\overline{B}^{(i)}\right)^2\right) + \overline{B}^{(i)}\sum_j \left.\frac{\partial^2 B^{(i)}}{\partial T_j^2}\right|_{\bar{T}_j} \mathrm{E}\left(\varepsilon_j^2\right) + \sum_j \left(\left.\frac{\partial B^{(i)}}{\partial T_j}\right|_{\bar{T}_j}\right)^2 \mathrm{E}\left(\varepsilon_j^2\right) + \\
& \frac{\overline{B}^{(i)}}{12}\sum_j \left.\frac{\partial^4 B^{(i)}}{\partial T_j^4}\right|_{\bar{T}_j} \mathrm{E}\left(\varepsilon_j^4\right) + \frac{1}{4}\sum_j \left(\left.\frac{\partial^2 B^{(i)}}{\partial T_j^2}\right|_{\bar{T}_j}\right)^2 \mathrm{E}\left(\varepsilon_j^4\right) + \frac{1}{3}\sum_j \left.\frac{\partial B^{(i)}}{\partial T_j}\right|_{\bar{T}_j} \left.\frac{\partial^3 B^{(i)}}{\partial T_j^3}\right|_{\bar{T}_j} \mathrm{E}\left(\varepsilon_j^4\right) + \\
& \frac{\overline{B}^{(i)}}{2}\sum_{j\neq k} \left.\frac{\partial^4 B^{(i)}}{\partial T_j^2 \partial T_k^2}\right|_{\bar{T}_j,\bar{T}_k} \mathrm{E}\left(\varepsilon_j^2\right)\mathrm{E}\left(\varepsilon_k^2\right) + \frac{1}{4}\sum_{j\neq k} \left.\frac{\partial^2 B^{(i)}}{\partial T_j^2}\right|_{\bar{T}_j} \left.\frac{\partial^2 B^{(i)}}{\partial T_k^2}\right|_{\bar{T}_k} \mathrm{E}\left(\varepsilon_j^2\right)\mathrm{E}\left(\varepsilon_k^2\right) + \\
& \frac{1}{2}\sum_{j\neq k} \left(\left.\frac{\partial^2 B^{(i)}}{\partial T_j \partial T_k}\right|_{\bar{T}_j,\bar{T}_k}\right)^2 \mathrm{E}\left(\varepsilon_j^2\right)\mathrm{E}\left(\varepsilon_k^2\right) + \sum_{j\neq k} \left.\frac{\partial B^{(i)}}{\partial T_j}\right|_{\bar{T}_j} \left.\frac{\partial^3 B^{(i)}}{\partial T_j \partial T_k^2}\right|_{\bar{T}_j,\bar{T}_k} \mathrm{E}\left(\varepsilon_j^2\right)\mathrm{E}\left(\varepsilon_k^2\right) + \ldots \\
=\;& \left(\overline{B}^{(i)}\right)^2 + \sigma^2\left[\overline{B}^{(i)}\sum_j \left.\frac{\partial^2 B^{(i)}}{\partial T_j^2}\right|_{\bar{T}_j} + \sum_j \left(\left.\frac{\partial^2 B^{(i)}}{\partial T_j^2}\right|_{\bar{T}_j}\right)^2\right] +
\end{aligned}$$

$$\sigma^4\left[\frac{3}{4}\sum_j\left(\left.\frac{\partial^2 B^{(i)}}{\partial T_j^2}\right|_{\bar{T}_j}\right)^2 + \sum_j\left.\frac{\partial B^{(i)}}{\partial T_j}\right|_{\bar{T}_j}\left.\frac{\partial^3 B^{(i)}}{\partial T_j^3}\right|_{\bar{T}_j} + \right.$$

$$\frac{\bar{B}^{(i)}}{4}\sum_{j\neq k}\left.\frac{\partial^4 B^{(i)}}{\partial T_j^2\partial T_k^2}\right|_{\bar{T}_j,\bar{T}_k} + \frac{\bar{B}^{(i)}}{4}\sum_j\left.\frac{\partial^4 B^{(i)}}{\partial T_j^4}\right|_{\bar{T}_j} + \frac{1}{4}\sum_{j\neq k}\left.\frac{\partial^2 B^{(i)}}{\partial T_j^2}\right|_{\bar{T}_j}\left.\frac{\partial^2 B^{(i)}}{\partial T_k^2}\right|_{\bar{T}_k} +$$

$$\left.\frac{1}{2}\sum_{j\neq k}\left(\left.\frac{\partial^2 B^{(i)}}{\partial T_j\partial T_k}\right|_{\bar{T}_j,\bar{T}_k}\right)^2 + \sum_{j\neq k}\left.\frac{\partial B^{(i)}}{\partial T_j}\right|_{\bar{T}_j}\left.\frac{\partial^3 B^{(i)}}{\partial T_j\partial T_k^2}\right|_{\bar{T}_j,\bar{T}_k}\right] + O(\sigma^6) \qquad (48)$$

$$(E(B^{(i)}))^2 = \left(\bar{B}^{(i)}\right)^2 + \sigma^2\bar{B}^{(i)}\sum_j\left.\frac{\partial^2 B^{(i)}}{\partial T_j^2}\right|_{\bar{T}_j} + \sigma^4\left[\frac{\bar{B}^{(i)}}{4}\sum_{j\neq k}\left.\frac{\partial^4 B^{(i)}}{\partial T_j^2\partial T_k^2}\right|_{\bar{T}_j,\bar{T}_k} + \right.$$

$$\left.\frac{\bar{B}^{(i)}}{4}\sum_j\left.\frac{\partial^4 B^{(i)}}{\partial T_j^4}\right|_{\bar{T}_j} + \frac{1}{4}\sum_{j\neq k}\left.\frac{\partial^2 B^{(i)}}{\partial T_j^2}\right|_{\bar{T}_j}\left.\frac{\partial^2 B^{(i)}}{\partial T_k^2}\right|_{\bar{T}_k} + \frac{1}{4}\sum_j\left(\left.\frac{\partial^2 B^{(i)}}{\partial T_j^2}\right|_{\bar{T}_j}\right)^2\right] + O(\sigma^6)$$

(49)

Substitute equations (48) and (49) into (46), and retain terms of order $O(\sigma^2)$ and $O(\sigma^4)$.

$$\mathrm{Var}(B^{(i)}) = \sigma^2\sum_j\left(\left.\frac{\partial B^{(i)}}{\partial T_j}\right|_{\bar{T}_j}\right)^2 + \sigma^4\left[\frac{1}{2}\sum_j\left(\left.\frac{\partial^2 B^{(i)}}{\partial T_j^2}\right|_{\bar{T}_j}\right)^2 + \sum_j\left.\frac{\partial B^{(i)}}{\partial T_j}\right|_{\bar{T}_j}\left.\frac{\partial^3 B^{(i)}}{\partial T_j^3}\right|_{\bar{T}_j}\right.$$

$$\left. + \frac{1}{2}\sum_{j\neq k}\left(\left.\frac{\partial^2 B^{(i)}}{\partial T_j\partial T_k}\right|_{\bar{T}_j,\bar{T}_k}\right)^2 + \sum_{j\neq k}\left.\frac{\partial B^{(i)}}{\partial T_j}\right|_{\bar{T}_j}\left.\frac{\partial^3 B^{(i)}}{\partial T_j\partial T_k^2}\right|_{\bar{T}_j,\bar{T}_k}\right] \qquad (50)$$

RESULTS AND DISCUSSION

Let the Biot number distribution be described by the following function:

$$B(t) = \begin{cases} 2t, & 0 \le t < 0.5 \\ 2(1-t), & 0.5 \le t < 0.75 \\ 0.5, & 0.75 \le t \le 1 \end{cases} \qquad (51)$$

The direct problem, described by equations (6)-(8) and (10), is solved by the explicit finite-difference method with uniform grid, $\Delta x = 0.01$, and $\Delta t = \Delta x^2/6$. This choice of Δt results in the solution that is accurate to fourth order in Δx. An inverse heat conduction problem can be constructed using equations (6)-(9). The temperature measurements at $x_0 = 1$ in equation (9) are obtained from the solution to the direct problem. The inverse problem is then solved for Biot number components using the algorithm described above. The quality of the solution can readily be determined since the exact solution is known.

The quality of the estimation depends on $\Delta_n^{(i)}$ (nonlinear bias), $\mathrm{Var}(B^{(i)})$ (variance), and $\Delta_{d,B}$ (deterministic bias). The deterministic bias may be defined as

$$\Delta_{d,B} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(B(ip\Delta t) - \overline{B}^{(i)}\right)^2} \tag{52}$$

where $B(t)$ is given by equation (51), and $\overline{B}^{(i)}$ is the expected value of estimated Biot number at time $ip\Delta t$ when the variance (σ^2) of input data is zero. There are three tunable parameters in the present method, n, r, and p. The effects of n on the quality of the solution are quite predictable. Hence, the number of n is set at 50, and only the effects of r and p on the solution will be considered.

Figure 3 shows the variations of $\Delta_n^{(i)}$ and $\mathrm{Var}(B^{(i)})$ with i for $p = 1$, $r = 20$, and $\sigma^2 = 0.01$. In general, both nonlinear bias and variance vary from component to component. For this particular form of $B(t)$, both $\Delta_n^{(i)}$ and $\mathrm{Var}(B^{(i)})$ reach maximum at $i = n$. To compare results obtained with different p and r, it is sufficient to compare maximum $\Delta_n^{(i)}$ and $\mathrm{Var}(B^{(i)})$.

The future-time parameter r acts as a stabilizing parameter in the sequential function specification method. This is apparent from Table 1, where it is shown that increasing r, while keeping p constant, results in a more stable solution (lower maximum variance), but a less accurate one (higher deterministic bias). It is interesting to note that a more stable solution also has lower maximum nonlinear bias. When p is increased, and r is kept constant, Table 2 shows that variance, nonlinear bias, and deterministic bias all decrease. This means that, with the same number of Biot number components to be estimated, taking more measurements at one sensor location can lead to a more stable and accurate solution. Although the accuracy of the solution does not appear to improve much with p, the solution becomes noticeably more stable when p is increased. However, one should be cautioned that when p becomes too large, the time step for temperature measurements may be too small, causing correlation among different measurements, which will probably invalidate the above conclusion. Nevertheless, stabilizing the solution without deteriorating its accuracy by letting p equal to 2 or 3 is worth taking into consideration when designing an experiment since it is less costly and more convenient than increasing the number of sensors.

In Fig. 4, three different plots of $\mathrm{E}(B^{(i)})$ obtained with $\sigma^2 = 0$, 0.005, and 0.01 are compared with exact Biot number function. The parameters used in obtaining these results are $n = 50$, $r = 12$, $p = 1$, and $x_0 = 1$. It can be seen that, without taking nonlinear bias into consideration, the quality of estimated Biot number components is expected to worsen as the variance of temperature measurements increase. This is in contrast with the estimation of boundary heat flux or boundary temperature, where the expected value of the solution does not depend on the variance of temperature measurements.

Obviously, it is desirable to have as small σ^2 as possible. From the relation between T_i and T_i', one can see that the variance of actual temperature measurement

T_i can be related to the variance of dimensionless temperature T_i' via the relation

$$\mathrm{Var}(T_i) \quad = \quad \sigma^2 \quad = \quad \frac{\mathrm{Var}(T_i')}{(T_\infty - T_0)^2} \tag{53}$$

Thus, besides decreasing the variance of actual temperature measurement, increasing the difference between the ambient and the initial temperatures will also result in less variance in estimated $B^{(i)}$.

CONCLUSIONS

The solution to the one-dimensional inverse heat conduction problem of estimating time-dependent heat transfer coefficient has been presented. Estimations of boundary heat flux and boundary temperature are performed by using the sequential function specification method with piecewise linear basis functions and the assumption of linearly varying boundary heat flux or boundary temperature components. They are then used to obtain the solution for Biot number. It is found that, in addition to variance and deterministic bias, the solution is characterized by nonlinear bias, which results from the nonlinear dependence of the solution on measured temperatures. If certain statistical assumptions regarding the measurement errors are made, it has been shown that variance and nonlinear bias can be expressed as functions of variance of temperature measurements. For a given number of Biot number components to be estimated, the method of solution offers two tunable parameters. Whereas an increase in parameter r results in decreasing variance, decreasing nonlinear bias, and increasing deterministic bias, an increase in parameter p results in decreasing variance, decreasing nonlinear bias, and slightly decreasing deterministic bias.

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Table 1. Variations of maximum variance, maximum nonlinear bias, and deterministic bias with parameter r. ($n = 50, p = 1, x_0 = 1.0, \sigma^2 = 0.01$)

r	$[\mathrm{Var}(B^{(i)})]_{\max}$	$[\Delta_n^{(i)}]_{\max}$	$\Delta_{d,B}$
10	7.02139	0.83771	0.01775
11	3.07827	0.43058	0.02202
12	1.57819	0.25020	0.02676
13	0.90504	0.15850	0.03178
14	0.56385	0.10705	0.03696
15	0.37418	0.07595	0.04232
16	0.26093	0.05603	0.04792
17	0.18936	0.04266	0.05381
18	0.14195	0.03334	0.06005
19	0.10929	0.02664	0.06663
20	0.08605	0.02168	0.07352

Table 2. Variations of maximum variance, maximum nonlinear bias, and deterministic bias with parameter p. ($n = 50$, $r = 12$, $x_0 = 1.0$, $\sigma^2 = 0.01$)

p	$[\mathrm{Var}(B^{(i)})]_{\max}$	$[\Delta_{\mathrm{n}}^{(i)}]_{\max}$	$\Delta_{\mathrm{d,B}}$
1	1.57819	0.25020	0.02676
2	0.78634	0.13100	0.02573
3	0.51912	0.08846	0.02536
4	0.38661	0.06674	0.02518
5	0.30775	0.05356	0.02506

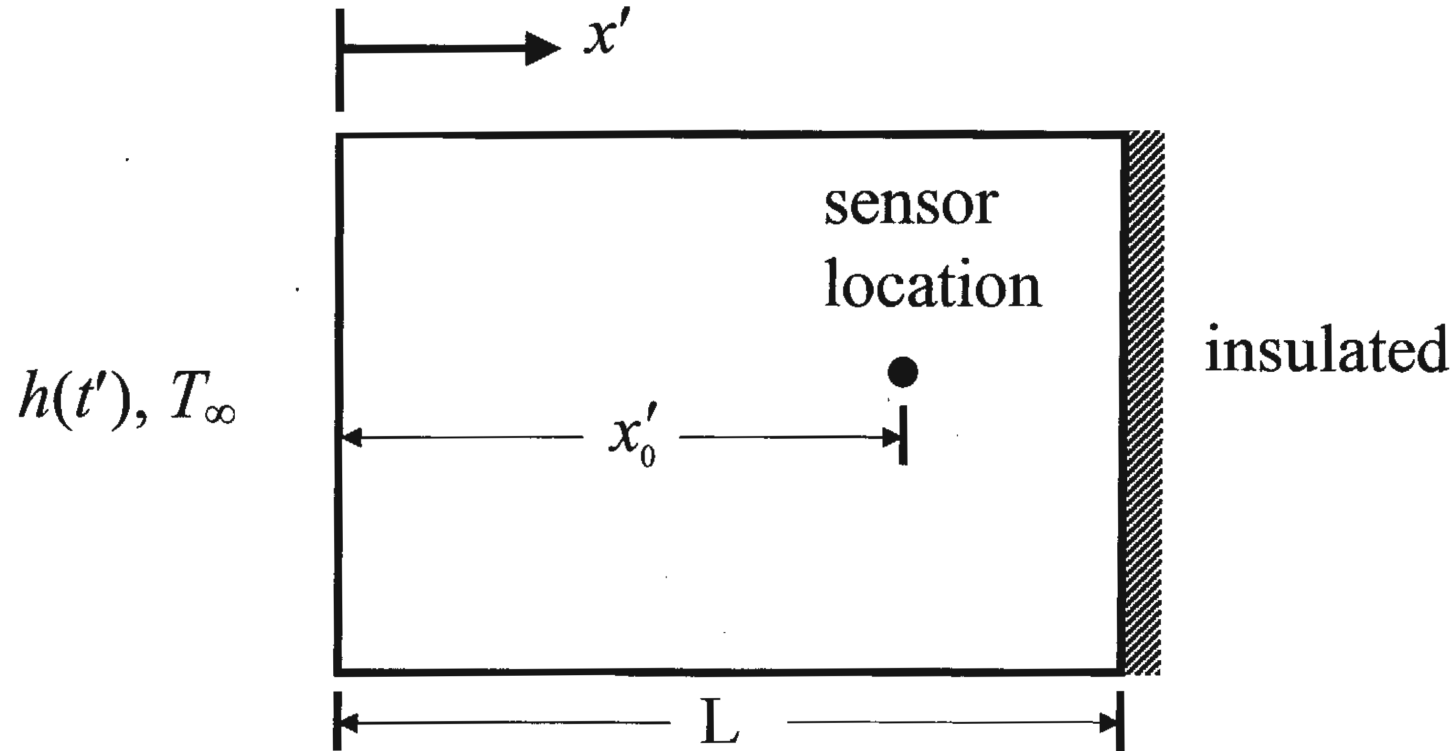
FIGURE CAPTIONS

Fig. 1 One-dimensional inverse heat conduction problem to be solved for $h(t')$

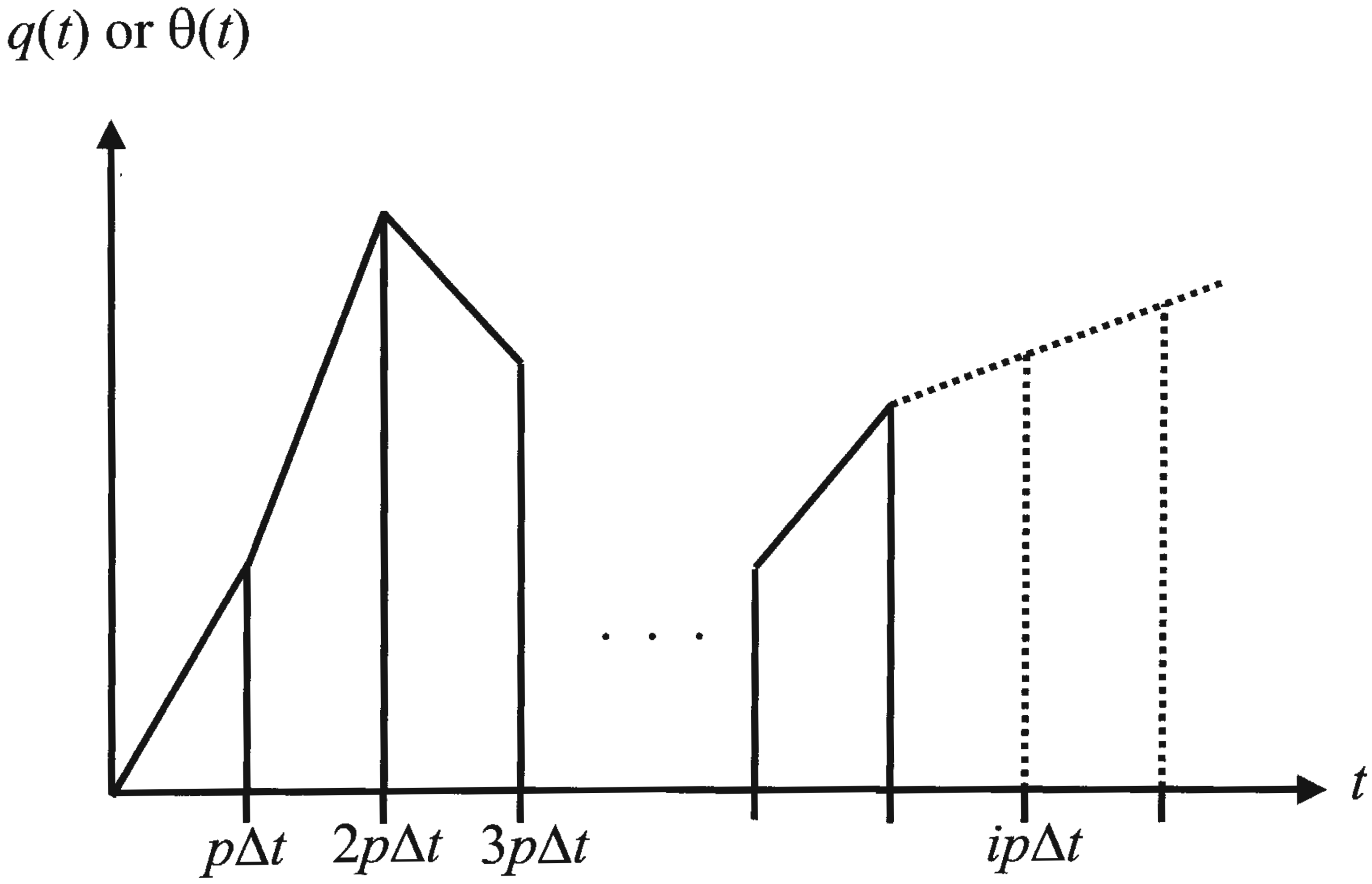
Fig. 2 Pictorial representation of the sequential function specification algorithm

Fig. 3 Variations of variance and nonlinear bias of estimated Biot number components. Calculations were performed using $x_0 = 1.0$, $n = 50$, $r = 12$, $p = 1$, and $\sigma^2 = 0.01$.

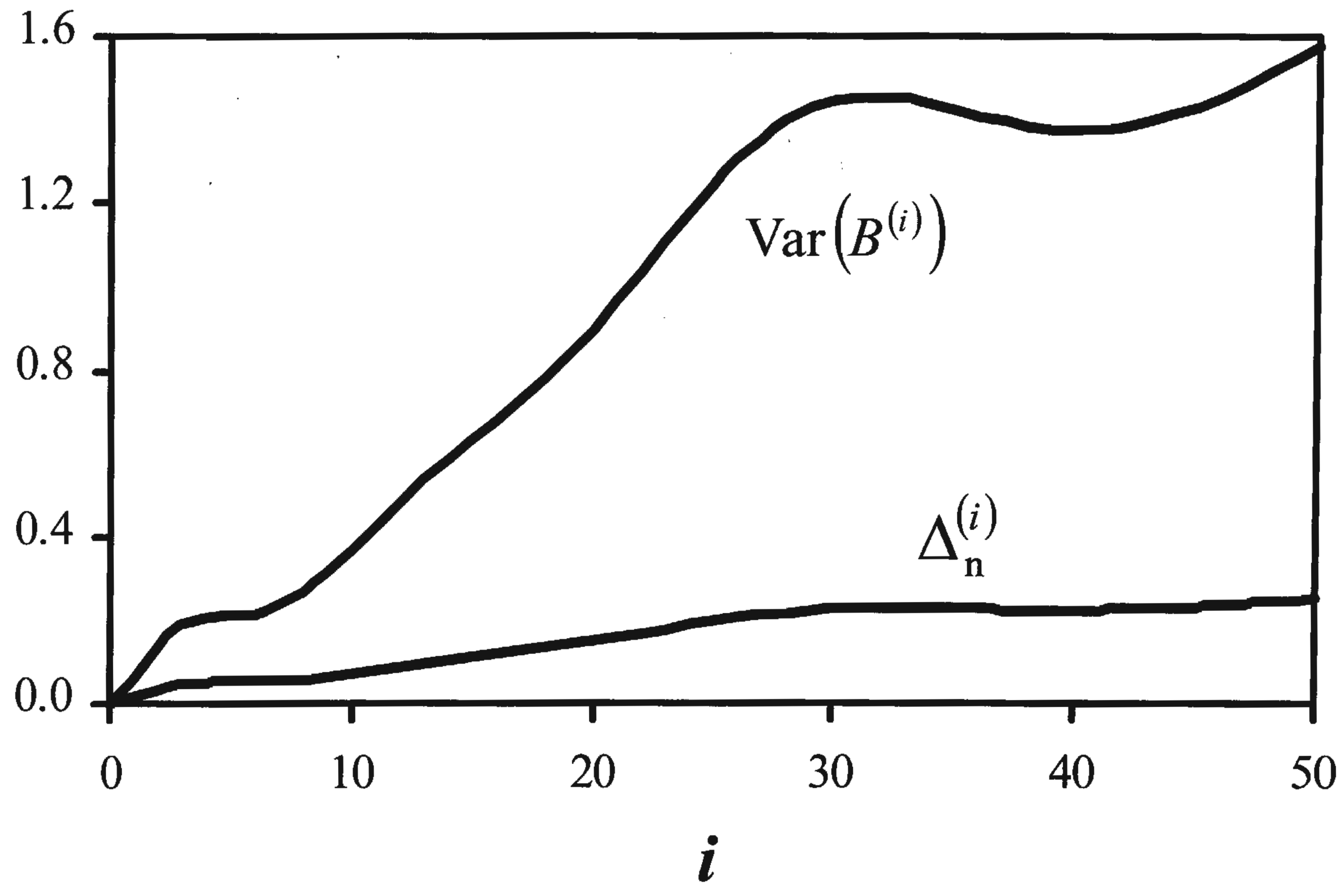
Fig. 4 Comparison between the expected values of estimated Biot number components at three different σ^2 and the exact Biot number distribution. Calculations were performed using $x_0 = 1.0$, $n = 50$, $r = 12$, and $p = 1$



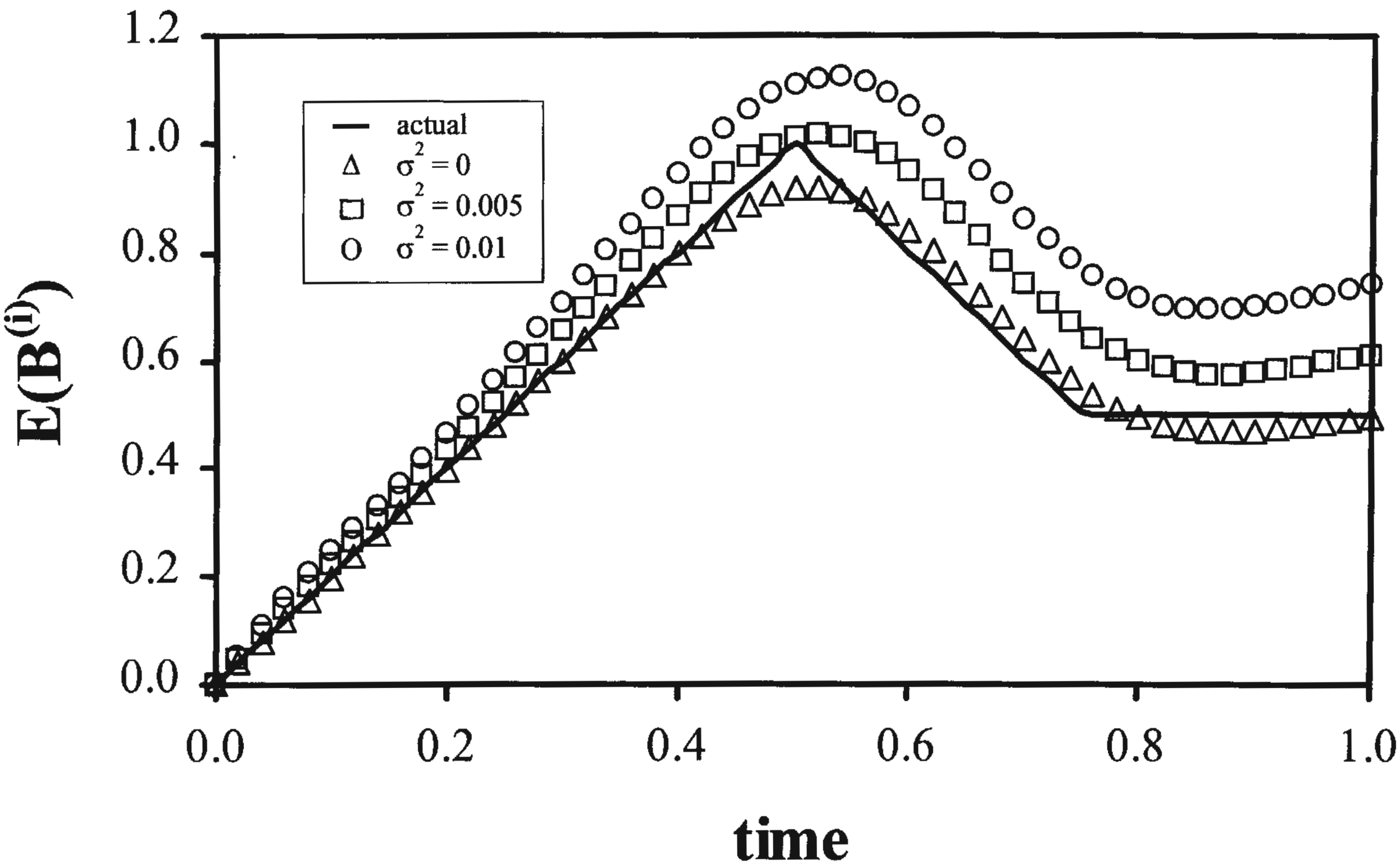
x'
sensor
location
insulated
$h(t')$, T_∞
x_0'
L



q(t) or θ(t)
t
pΔt
2pΔt
3pΔt
ipΔt



1.6
1.2
0.8
0.4
0.0
0
10
20
30
40
50
i
Var(B^(i))
Δn^(i)



actual
$\sigma^2 = 0$
$\sigma^2 = 0.005$
$\sigma^2 = 0.01$
1.2
1.0
0.8
0.6
0.4
0.2
0.0
$E(B^{(i)})$
0.0
0.2
0.4
0.6
0.8
1.0
time

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Title : Determination of Heat Transfer Coefficient From Surface Temperature Measurement

Dear Professor Chantasiriwan:

The above-mentioned manuscript has been received. We shall write to you as soon as the reviews have arrived.

W.J. Minkowycz
Editor

Determination of heat transfer coefficient from surface temperature measurements

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ABSTRACT

The unknown time-dependent heat transfer coefficient on the surface of a multidimensional body is determined from surface temperature measurements. Locating sensors at the surface causes the deterministic biases in the estimation of surface heat flux and the subsequent determination of heat transfer coefficient to be negligible. The boundary element method is used as the method of discretization since it provides an explicit expression between boundary heat flux and boundary temperature. The statistical analysis of the estimate is described and applied to a sample problem.

NOMENCLATURE

A	diagonal matrix
a	coefficient that depends on location of $\vec{\xi}$
$E(y)$	expected value of random variable y
G	fundamental solution
g	known boundary heat flux
h	heat transfer coefficient
i, j, k	dummy indices

L	the number of nodes in an element
M_s	the number of unknown heat flux components at given time
M_c	the number of additional heat flux components at corner or edge nodes
M_e	the number of boundary elements
M_n	the number of boundary nodes
m	dimension of the problem
N	number of time intervals
$\vec{n}$	outward pointing unit vector normal to boundary
P, R, S	coefficient matrices
q	boundary heat flux into the domain
$\vec{r}$	position vector
r_0	reference length
T	temperature
T_0	temperature
T_∞	temperature
t	time
X, Y, Z	coefficient matrices
Var(y)	variance of random variable y

Greek symbols

α	thermal diffusivity
Δ	nonlinear bias
ε	temperature measurement error
Φ	interpolating function
ϕ	function relating boundary temperature to boundary temperature

Γ	boundary
Γ_1	part of the boundary where heat flux is known
Γ_2	part of the boundary where heat flux is to be determined
κ	thermal conductivity
θ	surface temperature due to heat flux g alone
σ^2	variance in temperature measurement
ψ	function relating boundary temperature to boundary heat flux
$\vec{\xi}$	position vector

Subscripts and superscripts

2D	two-dimensional
3D	three-dimensional
i, j, k, l, m	indices

INTRODUCTION

The experimental determination of heat transfer coefficient can be accomplished by measuring surface heat flux or by employing techniques that make use of the analogy between convective mass transfer and convective heat transfer [1]. In comparison with temperature measurement, heat transfer coefficient measurement requires more expensive experimental equipment.

There exists an alternative approach to determining heat transfer coefficient, which requires only temperature measurements. However, it also requires the solution of an inverse heat conduction problem. Although this approach is more computationally intensive than conventional approaches, it has an advantage in a much simpler setup and less expensive equipment. Several previous works on the inverse problem of determining heat transfer coefficient have appeared in the literature [2-8]. Most recently, Chantasiriwan [8] studied the one-dimensional problem of estimating time-dependent Biot number from interior temperature measurements. He showed that Biot number could be expressed as a nonlinear function of temperature measurements. As a result, the inverse estimation of Biot number effected the nonlinear bias, in addition to deterministic bias inherent in the solution of a linear inverse heat conduction problem. Furthermore, the variance of the estimate depended on higher order terms of statistical errors in temperature measurements.

In this paper, the multidimensional inverse heat conduction problem of determining time-dependent heat transfer coefficient is considered. Typically, in an inverse problem, part of the boundary condition is unknown, and is to be determined from temperature measurements at interior locations. Such a problem is an ill-posed problem, causing the numerical solution to the problem to become unstable if the time step used is too low. However, if the sensors are located at the part of the boundary with unknown condition, the problem becomes well posed, meaning that a stable solution is possible at an arbitrarily small time step. Since, in many circumstances, placing sensors on the surface where heat transfer coefficient is to be determined are easier than placing sensors at subsurface locations, the problem of estimating heat transfer coefficient from surface temperature measurements both has practical applications and is much more amenable to numerical treatment.

Many different methods for solving an inverse heat conduction problem are available. The method used by Chantasiriwan [8] requires the determination of sensitivity coefficients of temperatures to boundary heat flux components. However, the problem considered by Chantasiriwan [8] is the one-dimensional problem, for which analytical expressions for sensitivity coefficients are readily available. For the multidimensional problem to be considered in this paper, sensitivity coefficients must be solved for numerically. The chosen numerical method will be the boundary element method. In addition to being able to handle a general problem of arbitrary geometry, this method is computationally efficient for the problem to be considered because it requires boundary mesh generation instead of domain mesh generation, and it provides an expression that relates boundary heat flux components to boundary temperatures explicitly. The following sections will present the statement of the problem, the boundary element formulation of the problem, and the expression relating heat transfer coefficient to surface temperature measurements. It will be found that such a relation is a nonlinear one. Hence, the method for analyzing the statistical errors in the estimated heat transfer coefficient, similar to the one given by Chantasiriwan [8], will be used. Finally, a sample problem will be discussed, and conclusions will be given.

STATEMENT OF THE PROBLEM

Consider a solid object with part of its boundary Γ_1 subjected to known heat flux and the remaining part of the boundary Γ_2 subjected to unknown convective heat flux. Suppose that the object has constant thermophysical properties, making the problem a linear one. Let $\vec{r}$, t, T, g, and q denote dimensionless quantities, whereas all other variables denote quantities with dimensions, and define $\vec{r} = \vec{r}'/r_0$, $t = \alpha t'/r_0^2$, and $T = (T' - T_0)/(T_\infty - T_0)$. The dimensionless heat conduction process can then be described by the following equations.

$$\frac{\partial T(\vec{r},t)}{\partial t} = \nabla^2 T(\vec{r},t) \qquad (1)$$

$$T(\vec{r},0) \quad = \quad 0 \tag{2}$$

$$\vec{n}\vec{\nabla}T(\vec{r},t)\Big|_{\Gamma_1} \quad = \quad g(\vec{r},t)\Big|_{\Gamma_1} \tag{3}$$

where $\vec{n}$ is the outward pointing unit vector normal to boundary and g is the known boundary heat flux. In order to render the problem solvable, the temperature measurements on Γ_2 are specified.

$$T(\vec{r}_i, j\Delta t) \quad = \quad T_i^{(j)} \tag{4}$$

where $\vec{r}_i$ is a sensor position vector, Δt is the measurement time step, and $T_i^{(j)}$ is surface temperature at the sensor position and time $j\Delta t$. Equations (1) – (4) are sufficient for determining heat flux $q_i^{(j)}$ at Γ_2, which will yield heat transfer coefficient from the Newton's law of cooling:

$$h_i^{(j)} \quad = \quad \left(\frac{q_i^{(j)}}{1 - T_i^{(j)}}\right)\frac{\kappa}{r_0} \tag{5}$$

BOUNDARY ELEMENT METHOD

The boundary element formulation for a time-dependent linear heat conduction problem is given by [9]

$$aT(\vec{\xi},t) \quad = \quad \int_{\Gamma}\int_0^t q(\vec{r},\tau)G(\vec{r}-\vec{\xi};t-\tau)\mathrm{d}\tau\mathrm{d}\vec{r} \;-\; \int_{\Gamma}\int_0^t T(\vec{r},\tau)\vec{n}\vec{\nabla}G(\vec{r}-\vec{\xi};t-\tau)\mathrm{d}\tau\mathrm{d}\vec{r} \tag{6}$$

where a depends on the location of $\vec{\xi}$, and the fundamental solution G is

$$G(\vec{r}-\vec{\xi};t-\tau) \quad = \quad \frac{\mathrm{e}^{-(\vec{r}-\vec{\xi})^2/4(t-\tau)}}{[4\pi(t-\tau)]^{m/2}} \tag{7}$$

and m is the dimension of the problem. Divide the boundary Γ into M_e boundary elements and time t into N equal time intervals. Equation (6) becomes

$$aT(\vec{\xi},t) = \sum_{i=1}^{M_e}\int_{\Gamma_i}\left[\sum_{j=1}^{N}\int_{(j-1)\Delta t}^{j\Delta t} {}_iq(\vec{r},\tau)G(\vec{r}-\vec{\xi};N\Delta t-\tau)\mathrm{d}\tau\right]\mathrm{d}\vec{r} \; -$$

$$\sum_{i=1}^{M_e}\int_{\Gamma_i}\left[\sum_{j=1}^{N}\int_{(j-1)\Delta t}^{j\Delta t} {}_iT(\vec{r},\tau)\vec{n}\vec{\nabla}G(\vec{r}-\vec{\xi};N\Delta t-\tau)\mathrm{d}\tau\right]\mathrm{d}\vec{r} \tag{8}$$

where front subscript denotes element index. Now, let's approximate ${}_iq$ and ${}_iT$ by piecewise linear functions in time.

$${}_iq(\vec{r},\tau) = \frac{1}{\Delta t}\left[{}_iq^{(j)}(\vec{r}) - {}_iq^{(j-1)}(\vec{r})\right](\tau - N\Delta t) + {}_iq^{(j)}(\vec{r})(N-j+1) - {}_iq^{(j-1)}(\vec{r})(N-j) \tag{9}$$

$${}_iT(\vec{r},\tau) = \frac{1}{\Delta t}\left[{}_iT^{(j)}(\vec{r}) - {}_iT^{(j-1)}(\vec{r})\right](\tau - N\Delta t) + {}_iT^{(j)}(\vec{r})(N-j+1) - {}_iT^{(j-1)}(\vec{r})(N-j) \tag{10}$$

where superscript denotes time index. Next, approximate ${}_iq^{(j)}$ and ${}_iT^{(j)}$ over element i, making use of interpolating function Φ_k, as follows.

$${}_iq^{(j)}(\vec{r}) = \sum_{k=1}^{L}\left({}_{i,k}q^{(j)}\right)\Phi_k(\vec{r}) \tag{11}$$

$${}_iT^{(j)}(\vec{r}) = \sum_{k=1}^{L}\left({}_{i,k}T^{(j)}\right)\Phi_k(\vec{r}) \tag{12}$$

where k is local node index, and L is the number of nodes in an element. Substituting equations (9)-(12) into equation (8) yields

$$aT(\vec{\xi},t) = \sum_{i=1}^{M_e}\sum_{k=1}^{L}\left\{\int_{\Gamma_i}\left[\sum_{j=1}^{N}\int_{(j-1)\Delta t}^{j\Delta t}\left(\frac{(\tau-N\Delta t)}{\Delta t} + (N-j+1)\right)G\mathrm{d}\tau\right]\Phi_k(\vec{r})\mathrm{d}\vec{r}\right\}\left({}_{i,k}q^{(j)}\right) \; -$$

$$\sum_{i=1}^{M_e}\sum_{k=1}^{L}\left\{\int_{\Gamma_i}\left[\sum_{j=1}^{N}\int_{(j-1)\Delta t}^{j\Delta t}\left(\frac{(\tau-N\Delta t)}{\Delta t} + (N-j)\right)G\mathrm{d}\tau\right]\Phi_k(\vec{r})\mathrm{d}\vec{r}\right\}\left({}_{i,k}q^{(j-1)}\right) \; -$$

$$\sum_{i=1}^{M_e}\sum_{k=1}^{L}\left\{\int_{\Gamma_i}\left[\sum_{j=1}^{N}\int_{(j-1)\Delta t}^{j\Delta t}\left(\frac{(\tau-N\Delta t)}{\Delta t} + (N-j+1)\right)\vec{n}\vec{\nabla}G\mathrm{d}\tau\right]\Phi_k(\vec{r})\mathrm{d}\vec{r}\right\}\left({}_{i,k}T^{(j)}\right) \; +$$

$$\sum_{i=1}^{M_e}\sum_{k=1}^{L}\left\{\int_{\Gamma_i}\left[\sum_{j=1}^{N}\int_{(j-1)\Delta t}^{j\Delta t}\left(\frac{(\tau-N\Delta t)}{\Delta t} + (N-j)\right)\vec{n}\vec{\nabla}G\mathrm{d}\tau\right]\Phi_k(\vec{r})\mathrm{d}\vec{r}\right\}\left({}_{i,k}T^{(j-1)}\right) \tag{13}$$

If equation (13) is evaluated at a point $\vec{\xi}_k$ on the boundary or inside the object, the resulting equation after the assembly process can be written as

$$a_k T_k^{(N)} = \sum_{j=0}^{N}\sum_{i=1}^{M_n} \phi\left(\vec{\xi}_k, \vec{r}_i, (N-j)\Delta t\right) T_i^{(j)} + \sum_{j=0}^{N}\sum_{i=1}^{M_n+M_c} \psi\left(\vec{\xi}_k, \vec{r}_i, (N-j)\Delta t\right) q_i^{(j)} \quad (14)$$

where back subscript denotes global node index, M_n is the number of boundary nodes, and M_c is the number of additional heat flux components at corner or edge nodes. Note that coefficient a_k becomes unity if $\vec{\xi}_k$ is inside the object. For two-dimensional problems, each corner node can have two heat flux components; therefore, M_c is equal to the number of corners. For three-dimensional problems, each edge node can have two heat flux components, and each corner node can have three heat flux components. Functions ϕ and ψ are obtained from the evaluation of integrals shown in equation (13). The evaluation of time integrals can be done exactly as shown in the Appendix, whereas the evaluation of boundary integrals should be performed using the Gaussian quadrature.

Equation (14) is now written for M_n boundary node points, resulting in the following matrix equation.

$$A\vec{T}^{(N)} = \sum_{j=0}^{N} P^{(N-j)}\vec{T}^{(j)} + \sum_{j=0}^{N} R^{(N-j)}\vec{q}^{(j)} + \sum_{j=0}^{N} S^{(N-j)}\vec{g}^{(j)} \quad (15)$$

where A is diagonal matrix of coefficients a; $\vec{T}$ is the vector of temperatures on the boundary; $\vec{q}$ is the vector of boundary heat flux components that are to be determined; $\vec{g}$ is the vector of specified boundary heat flux components; and P, R, and S are coefficient matrices.

Let $\vec{\theta}$ be the boundary temperature responses when $\vec{q} = 0$.

$$A\vec{\theta}^{(N)} = \sum_{j=0}^{N} P^{(N-j)}\vec{\theta}^{(j)} + \sum_{j=0}^{N} S^{(N-j)}\vec{g}^{(j)} \quad (16)$$

If $\vec{g}$ is known as a function of time, dimensionless temperature $\vec{\theta}$ can be determined by a time-stepping procedure. Note that $\vec{\theta} \neq 0$ only if $\vec{g} \neq 0$. Subtracting equations (16) from (15) results in

$$A\left[\vec{T}^{(N)} - \vec{\theta}^{(N)}\right] = \sum_{j=0}^{N} P^{(N-j)}\left[\vec{T}^{(j)} - \vec{\theta}^{(j)}\right] + \sum_{j=0}^{N} R^{(N-j)}\vec{q}^{(j)} \tag{17}$$

Applying a time-stepping procedure to equation (17) yields the following relations between boundary temperatures and the unknown boundary heat flux.

$$\vec{T}^{(k)} - \vec{\theta}^{(k)} = \sum_{j=0}^{k} X^{(k-j)}\vec{q}^{(j)} \tag{18}$$

In order to make the computation of heat transfer coefficient straightforward, heat flux is expressed in terms of boundary temperatures as follows.

$$\vec{q}^{(k)} = \sum_{j=0}^{k} Y^{(k-j)}\left(\vec{T}^{(j)} - \vec{\theta}^{(j)}\right) \tag{19}$$

where $Y^{(0)} = \left(X^{(0)}\right)^{-1}$

$$Y^{(l)} = -\left(X^{(0)}\right)^{-1}\sum_{j=0}^{l-1} X^{(l-j)}Y^{(j)} \qquad (1 \leq l \leq k-1) \tag{20}$$

Heat transfer coefficient can now be expressed in terms of temperature measurements by substituting equation (19) into equation (5).

$$h_i^{(k)} = \frac{\sum_{j=1}^{M_s}\sum_{l=0}^{k} Y_{ij}^{(k-l)}\left(T_j^{(l)} - \theta_j^{(l)}\right)}{1 - T_i^{(k)}}\left(\frac{\kappa}{r_0}\right) \tag{21}$$

where Y_{ij} is a component of matrix Y, and M_s is the number of temperature sensors on the surface, which is equal to the number of unknown heat transfer coefficient components. Since Y scales with r_0, equation (21) can be rewritten as

$$h_i^{(k)} = \frac{\sum_{j=1}^{M_s}\sum_{l=0}^{k} \kappa Z_{ij}^{(k-l)}\left(T_j^{(l)} - \theta_j^{(l)}\right)}{1 - T_i^{(k)}} \tag{22}$$

where $Z = Y/r_0$.

STATISTICAL ERRORS IN THE ESTIMATE

The estimated heat transfer coefficient will contain a statistical error because even a well-calibrated temperature-measuring device will give a result that is subjected to statistical fluctuation. In order to evaluate this statistical error, it is expedient to model the probability density function for temperature measurement error as follows.

$$f\left(\varepsilon_i^{(k)}\right) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\varepsilon_i^{(k)}}{\sigma}\right)^2\right] \tag{23}$$

where error $\varepsilon_i^{(k)}$ is the difference between actual temperature measurement and the mean temperature at the same time and location.

$$\varepsilon_i^{(k)} = T_i'^{(k)} - \overline{T}_i'^{(k)} \tag{24}$$

This model results in the following implications.

1. Error has zero mean:

   $E\left(\varepsilon_i^{(k)}\right) = 0$

2. Variance of error is constant: $\mathrm{Var}\left(\varepsilon_i^{(k)}\right) = \sigma^2$

3. Temperature measurement errors at different time or location are uncorrelated:

   $E\left(\varepsilon_i^{(k)}\varepsilon_j^{(l)}\right) = 0$ if $i \neq j$ or $k \neq l$

Note that it has been assumed that other relevant parameters such as location of sensor, time of measurement, ambient temperature, and thermophysical properties do not contain statistical errors.

The analysis presented by Chantasiriwan [8] may be followed to show that the expected value and the variance of the estimate are given by

$$\mathrm{E}\left(h_i^{(k)}\right) = \bar{h}_i^{(k)} + \frac{\sigma^2\left(\kappa Z_{ii}^{(0)} + \bar{h}_i^{(k)}\right)}{\left(T_\infty - T_0\right)^2\left(1-\bar{T}_i^{(k)}\right)^2} + O\left(\sigma^4\right) \tag{25}$$

$$\mathrm{Var}\left(h_i^{(k)}\right) = \frac{\sigma^2}{\left(T_\infty - T_0\right)^2\left(1-\bar{T}_i^{(k)}\right)^2}\left[\sum_{j=1}^{M_s}\sum_{l=0}^{k}\left(\kappa Z_{ij}^{(k-l)}\right)^2 - \left(\kappa Z_{ii}^{(0)}\right)^2 + \left(\kappa Z_{ii}^{(0)} + \bar{h}_i^{(k)}\right)^2\right]$$

$$+ \frac{\sigma^4}{\left(T_\infty - T_0\right)^4\left(1-\bar{T}_i^{(k)}\right)^4}\left[3\sum_{j=1}^{M_s}\sum_{l=0}^{k}\left(\kappa Z_{ij}^{(k-l)}\right)^2 - 3\left(\kappa Z_{ii}^{(0)}\right)^2 + 8\left(\kappa Z_{ii}^{(0)} + \bar{h}_i^{(k)}\right)^2\right] + O\left(\sigma^6\right) \tag{26}$$

where

$$\bar{h}_i^{(k)} = \frac{\sum_{j=1}^{M_s}\sum_{l=1}^{k}\kappa Z_{ij}^{(k-l)}\left(\bar{T}_j^{(l)} - \theta_j^{(l)}\right)}{1-\bar{T}_i^{(k)}} \tag{27}$$

is the average heat transfer coefficient. Since it will be assumed that σ is small, higher-order terms in equations (25) and (26) may be neglected.

RESULTS AND DISCUSSION

The sample problem is illustrated in Fig. 1. A square object, which is insulated on three sides, is subjected to convective heat flux on the remaining side. For reference purpose, three positions along the side are denoted by letters A, O, and B. The heat transfer coefficient on that side is to be determined from temperature measurements taken on the same side. The object is made of aluminum, having κ = 229 W/m-K, $\alpha = 8.93 \times 10^{-5}$ m^2/s, and uniform initial temperature of 0 °C. The ambient temperature is 100 °C, and the initial temperature of the object is 25 °C. The actual heat transfer coefficient is a function of space and time, and shown in Fig. 2. The boundary is divided into 40 elements of equal length, and the boundary element method with linear interpolating function is used to calculate temperature distribution on the non-insulated surface, which is then used as input data for algorithm for determining heat transfer coefficient. It is found that the calculated heat transfer coefficient matches the actual heat transfer coefficient. In other words, the deterministic bias is negligible. This is to be expected from a problem in which the unknown boundary condition is determined from temperature measurements taken on the same boundary.

If the input temperature data are corrupted with statistical fluctuations, the calculated heat transfer coefficient will be different from the actual one. According to equation (23), the difference between the expected estimate and the desired estimate, or the average coefficient, is equal to

$$\Delta_i^{(k)} \quad = \quad \frac{\sigma^2\left(\kappa Z_{ii}^{(0)} + \bar{h}_i^{(k)}\right)}{\left(T_\infty - T_0\right)^2\left(1 - \bar{T}_i^{(k)}\right)^2} \tag{28}$$

Since $\Delta_i^{(k)}$ exists in only nonlinear estimate, it may be called nonlinear bias. Figure 3 shows the distribution of nonlinear bias in this sample problem, whereas Fig. 4 shows the distribution of variance. Both distributions increase monotonically with time due to their dependence on variance in earlier temperature measurements. The functional form of heat transfer coefficient also causes both to reach maximum at point O, which is the middle of the side.

Inspection of equations (26) and (28) reveals that if the geometry of the problem, the heat transfer coefficient, and the variance in temperature measurements are given, factors influencing variance and nonlinear bias of the estimate are the difference between the ambient temperature and the initial temperature $T_\infty - T_0$ and thermal conductivity (κ). Figures 5 and 6 show the effects of $T_\infty - T_0$ on nonlinear bias and variance of the estimate at point O. It is clear that increasing $T_\infty - T_0$ reduces both quantities. This suggests that an experiment employing this technique of determining heat transfer coefficient should be designed so that $T_\infty - T_0$ is maximized. In one possible experimental design, the initial of the object may be kept at room temperature initially, and the temperature of the surrounding fluid should be set as high as practical. The other factor that influences variance and nonlinear bias of the estimate is thermal conductivity. Figures 7 and 8 show variations of nonlinear bias and variance of the estimate at point O for 5 hypothetical materials having different thermal conductivities. Although equations (26) and (28) seem to suggest that decreasing κ leads to decreases in $\mathrm{Var}\left(h_i^{(k)}\right)$ and $\Delta_i^{(k)}$, smaller κ actually causes $\overline{T}_i^{(k)}$ to increase. The net result is the slight decreases in $\mathrm{Var}\left(h_i^{(k)}\right)$ and $\Delta_i^{(k)}$ at small t and the progressive increases in both quantities at large t. It is apparent that a suitable material for determining heat transfer coefficient should have a high thermal conductivity.

CONCLUSIONS

Heat transfer coefficient may be determined by a technique employing only surface temperature measurements. Such a technique should require simpler setup and provide more flexibility than conventional techniques. However, it requires a numerical solution of a heat conduction problem. If the problem is linear, which means that the solid material to be used in the experiment has constant thermal conductivity and thermal diffusivity over a reasonable range of temperatures, the solution of the problem by the boundary element method is quite straightforward. It has been shown that estimated heat transfer coefficient can be expressed as a nonlinear function of surface temperature measurements. Although locating sensors at the surface causes the deterministic bias in the estimate to be negligible, the estimate still contains nonlinear bias and variance. Both quantities increase monotonically with time. Decreasing the difference between the ambient temperature and the initial temperature of the object also increases them, whereas decreasing the thermal conductivity of the material decreases them slightly when t is small, but increases them when t is large. Hence, the estimation of heat transfer coefficient should be performed with highly conductive material in a high-temperature environment.

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APPENDIX

Analytical evaluation of time integrals in equation (12) will now be given separately for two-dimensional and three-dimensional problems.

Two-dimensional problems

From equation (6),

$$G_{2D}(\vec{r}-\vec{\xi};t-\tau) = \frac{e^{-(\vec{r}-\vec{\xi})^2/4(t-\tau)}}{4\pi(t-\tau)} \tag{29}$$

$$\vec{\nabla}G_{2D}(\vec{r}-\vec{\xi};t-\tau) = -\frac{e^{-(\vec{r}-\vec{\xi})^2/4(t-\tau)}}{8\pi(t-\tau)^2}|\vec{r}-\vec{\xi}|\vec{\nabla}|\vec{r}-\vec{\xi}| \tag{30}$$

$$\int_{(j-1)\Delta t}^{j\Delta t} G_{2D}\mathrm{d}\tau = \frac{1}{4\pi}\left\{\mathrm{E}_1\left(\frac{(\vec{r}-\vec{\xi})^2}{4(N-j+1)\Delta t}\right) - \mathrm{E}_1\left(\frac{(\vec{r}-\vec{\xi})^2}{4(N-j)\Delta t}\right)\right\} \tag{31}$$

$$\begin{aligned}\int_{(j-1)\Delta t}^{j\Delta t}(N\Delta t-\tau)G_{2D}\mathrm{d}\tau = {} & \frac{(N-j+1)\Delta t}{4\pi}e^{-(\vec{r}-\vec{\xi})^2/4(N-j+1)\Delta t} - \frac{(N-j)\Delta t}{4\pi}e^{-(\vec{r}-\vec{\xi})^2/4(N-j)\Delta t} \\ & -\frac{(\vec{r}-\vec{\xi})^2}{16\pi}\left\{\mathrm{E}_1\left(\frac{(\vec{r}-\vec{\xi})^2}{4(N-j+1)\Delta t}\right) - \mathrm{E}_1\left(\frac{(\vec{r}-\vec{\xi})^2}{4(N-j)\Delta t}\right)\right\}\end{aligned} \tag{32}$$

$$\int_{(j-1)\Delta t}^{j\Delta t} \vec{\nabla} G_{2D}\mathrm{d}\tau = -\frac{|\vec{r}-\vec{\xi}|.\vec{\nabla}|\vec{r}-\vec{\xi}|}{2\pi(\vec{r}-\vec{\xi})^2}\left\{\mathrm{e}^{-(\vec{r}-\vec{\xi})^2/4(N-j+1)\Delta t} - \mathrm{e}^{-(\vec{r}-\vec{\xi})^2/4(N-j)\Delta t}\right\} \tag{33}$$

$$\int_{(j-1)\Delta t}^{j\Delta t} (N\Delta t-\tau)\vec{\nabla} G_{2D}\mathrm{d}\tau = -\frac{|\vec{r}-\vec{\xi}|.\vec{\nabla}|\vec{r}-\vec{\xi}|}{8\pi}\left\{\mathrm{E}_1\left(\frac{(\vec{r}-\vec{\xi})^2}{4(N-j+1)\Delta t}\right) - \mathrm{E}_1\left(\frac{(\vec{r}-\vec{\xi})^2}{4(N-j)\Delta t}\right)\right\} \tag{34}$$

where E_1 is exponential integral, defined as

$$\mathrm{E}_1(x) = \int_x^\infty \frac{\mathrm{e}^{-y}}{y}\mathrm{d}y$$

Three-dimensional problems

From equation (6),

$$G_{3D}(\vec{r}-\vec{\xi};t-\tau) = \frac{\mathrm{e}^{-(\vec{r}-\vec{\xi})^2/4(t-\tau)}}{[4\pi(t-\tau)]^{3/2}} \tag{35}$$

$$\vec{\nabla} G_{3D}(\vec{r}-\vec{\xi};t-\tau) = -\frac{\mathrm{e}^{-(\vec{r}-\vec{\xi})^2/4(t-\tau)}}{16\pi^{3/2}(t-\tau)^{5/2}}|\vec{r}-\vec{\xi}|.\vec{\nabla}|\vec{r}-\vec{\xi}| \tag{36}$$

$$\int_{(j-1)\Delta t}^{j\Delta t} G_{3D}\mathrm{d}\tau = \frac{1}{4\pi|\vec{r}-\vec{\xi}|}\left\{\mathrm{erfc}\left(\frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j+1)\Delta t}}\right) - \mathrm{erfc}\left(\frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j)\Delta t}}\right)\right\} \tag{37}$$

$$\begin{aligned}\int_{(j-1)\Delta t}^{j\Delta t} (N\Delta t-\tau)G_{3D}\mathrm{d}\tau = {} & \frac{\sqrt{(N-j+1)\Delta t}}{4\pi^{3/2}}\mathrm{e}^{-(\vec{r}-\vec{\xi})^2/4(N-j+1)\Delta t} - \\ & \frac{\sqrt{(N-j)\Delta t}}{4\pi^{3/2}}\mathrm{e}^{-(\vec{r}-\vec{\xi})^2/4(N-j)\Delta t} - \\ & \frac{|\vec{r}-\vec{\xi}|}{8\pi}\left\{\mathrm{erfc}\left(\frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j+1)\Delta t}}\right) - \mathrm{erfc}\left(\frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j)\Delta t}}\right)\right\}\end{aligned} \tag{38}$$

$$\int_{(j-1)\Delta t}^{j\Delta t} \vec{\nabla} G_{3D}\mathrm{d}\tau = \frac{\vec{\nabla}|\vec{r}-\vec{\xi}|}{2\pi^{3/2}(\vec{r}-\vec{\xi})^2}\left\{\frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j)\Delta t}}\mathrm{e}^{-(\vec{r}-\vec{\xi})^2/4(N-j)\Delta t} - \right.$$

$$\frac{\left|\vec{r}-\vec{\xi}\right|}{2\sqrt{(N-j+1)\Delta t}}\,\mathrm{e}^{-\left(\vec{r}-\vec{\xi}\right)^2/4(N-j+1)\Delta t} \quad -$$

$$\frac{2}{\sqrt{\pi}}\mathrm{erfc}\left(\frac{\left|\vec{r}-\vec{\xi}\right|}{2\sqrt{(N-j+1)\Delta t}}\right) + \frac{2}{\sqrt{\pi}}\mathrm{erfc}\left(\frac{\left|\vec{r}-\vec{\xi}\right|}{2\sqrt{(N-j)\Delta t}}\right)\Bigg\} \qquad (39)$$

$$\int_{(j-1)\Delta t}^{j\Delta t}(N\Delta t-\tau)\vec{\nabla}G_{3\mathrm{D}}\mathrm{d}\tau \quad = \quad -\frac{\vec{\nabla}\left|\vec{r}-\vec{\xi}\right|}{8\pi}\left\{\mathrm{erfc}\left(\frac{\left|\vec{r}-\vec{\xi}\right|}{2\sqrt{(N-j+1)\Delta t}}\right) - \right.$$

$$\left.\mathrm{erfc}\left(\frac{\left|\vec{r}-\vec{\xi}\right|}{2\sqrt{(N-j)\Delta t}}\right)\right\} \qquad (40)$$

where erfc is complementary error function, defined as

$$\mathrm{erfc}(x) \quad = \quad \frac{2}{\sqrt{\pi}}\int_x^{\infty}\mathrm{e}^{-y^2}\mathrm{d}y$$

FIGURE CAPTIONS

Fig. 1 A square object made of aluminum with the unknown heat transfer coefficient on one side. Temperature sensors are located on that side. Letters A, O, B are for reference purpose.

Fig. 2 Distribution of heat transfer coefficient for the sample problem

Fig. 3 Distribution of nonlinear bias of the estimated heat transfer coefficient

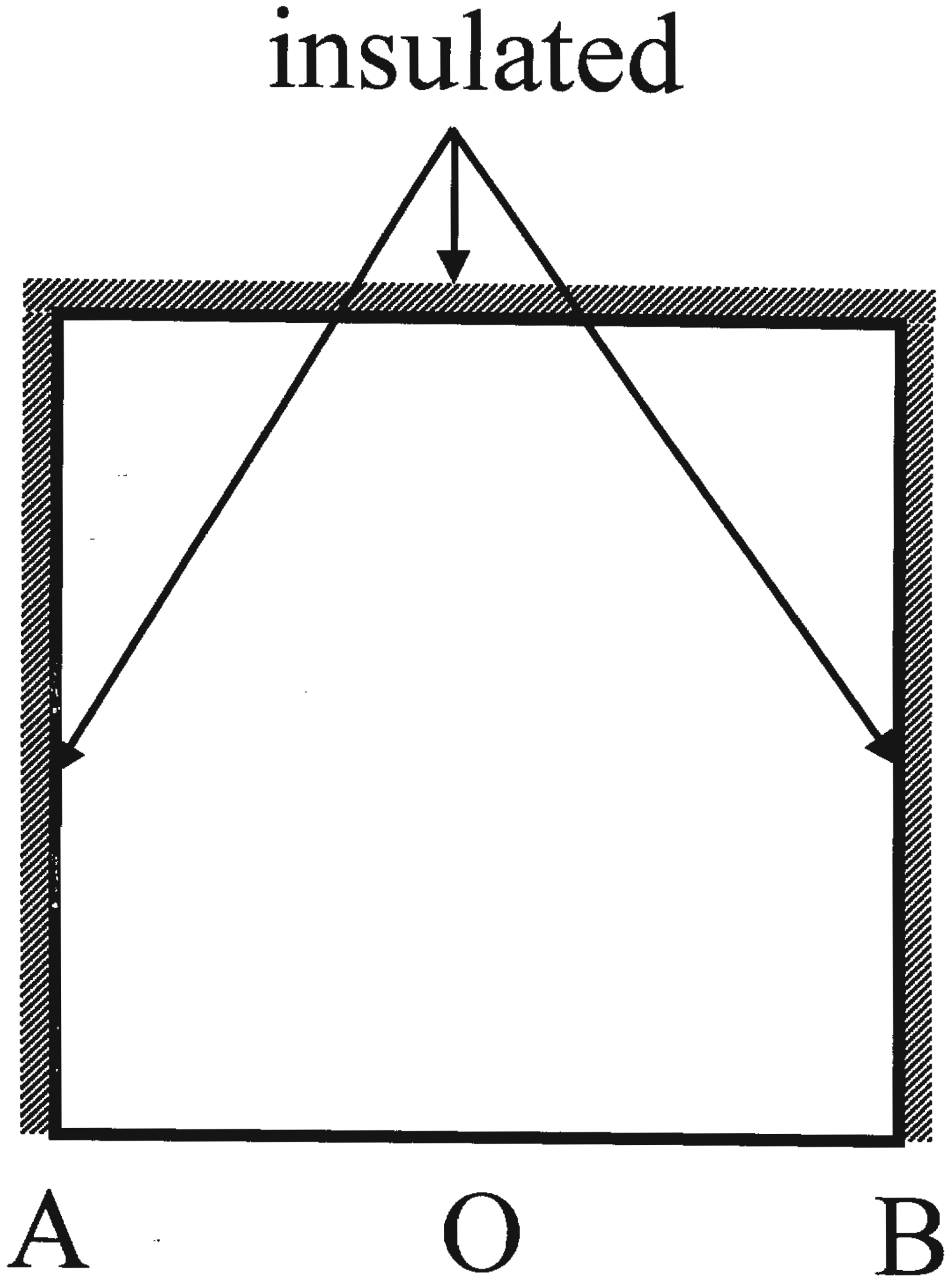
Fig. 4 Distribution of variance of the estimated heat transfer coefficient

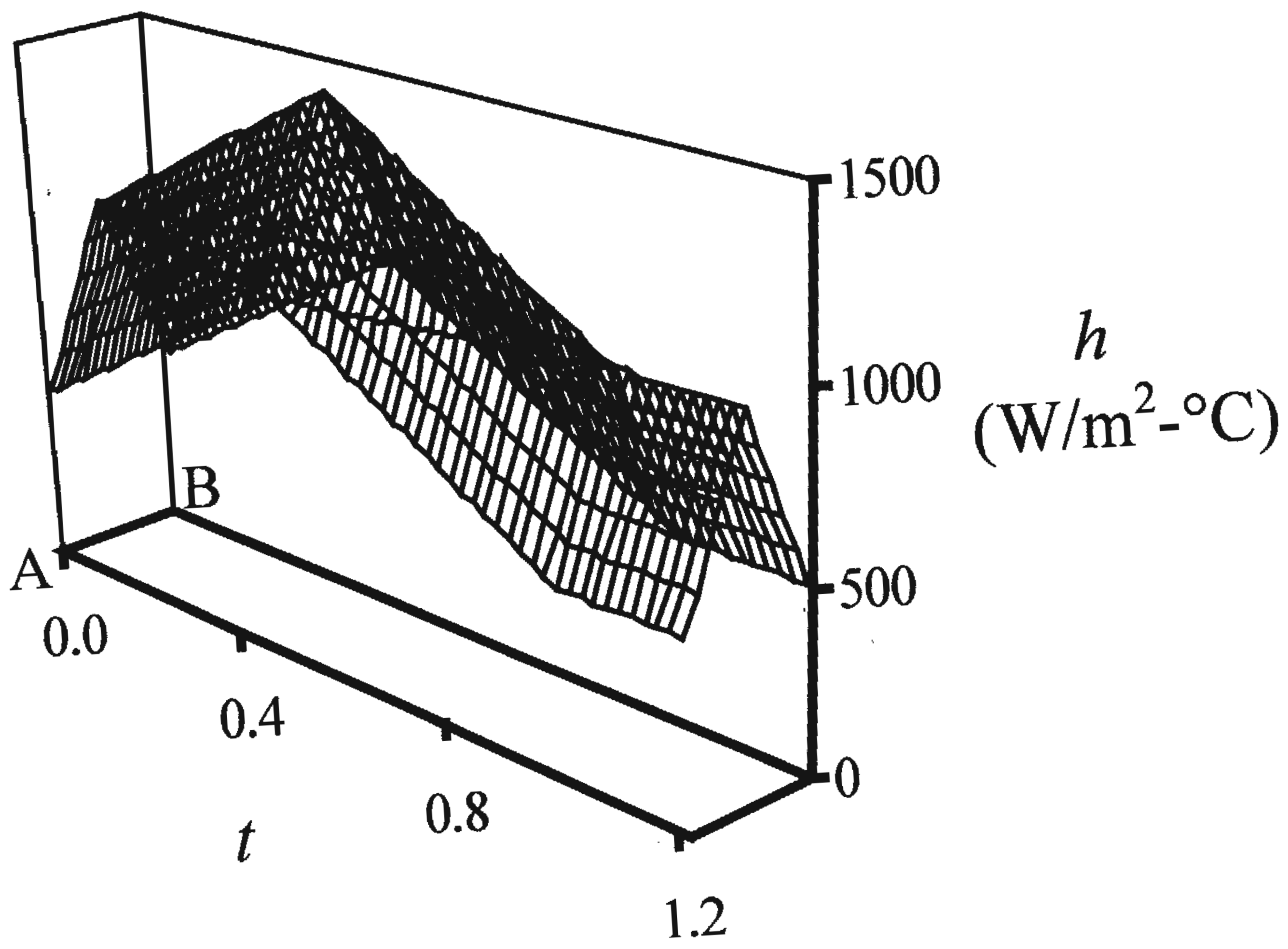
Fig. 5 Effect of decreasing T_∞ on nonlinear bias of the estimated heat transfer coefficient at point O. T_∞ is decreased from 150 °C to 100 °C, while T_0 is kept constant at 25 °C.

Fig. 6 Effect of decreasing T_∞ on variance of the estimated heat transfer coefficient at point O. T_∞ is decreased from 150 °C to 100 °C, while T_0 is kept constant at 25 °C.

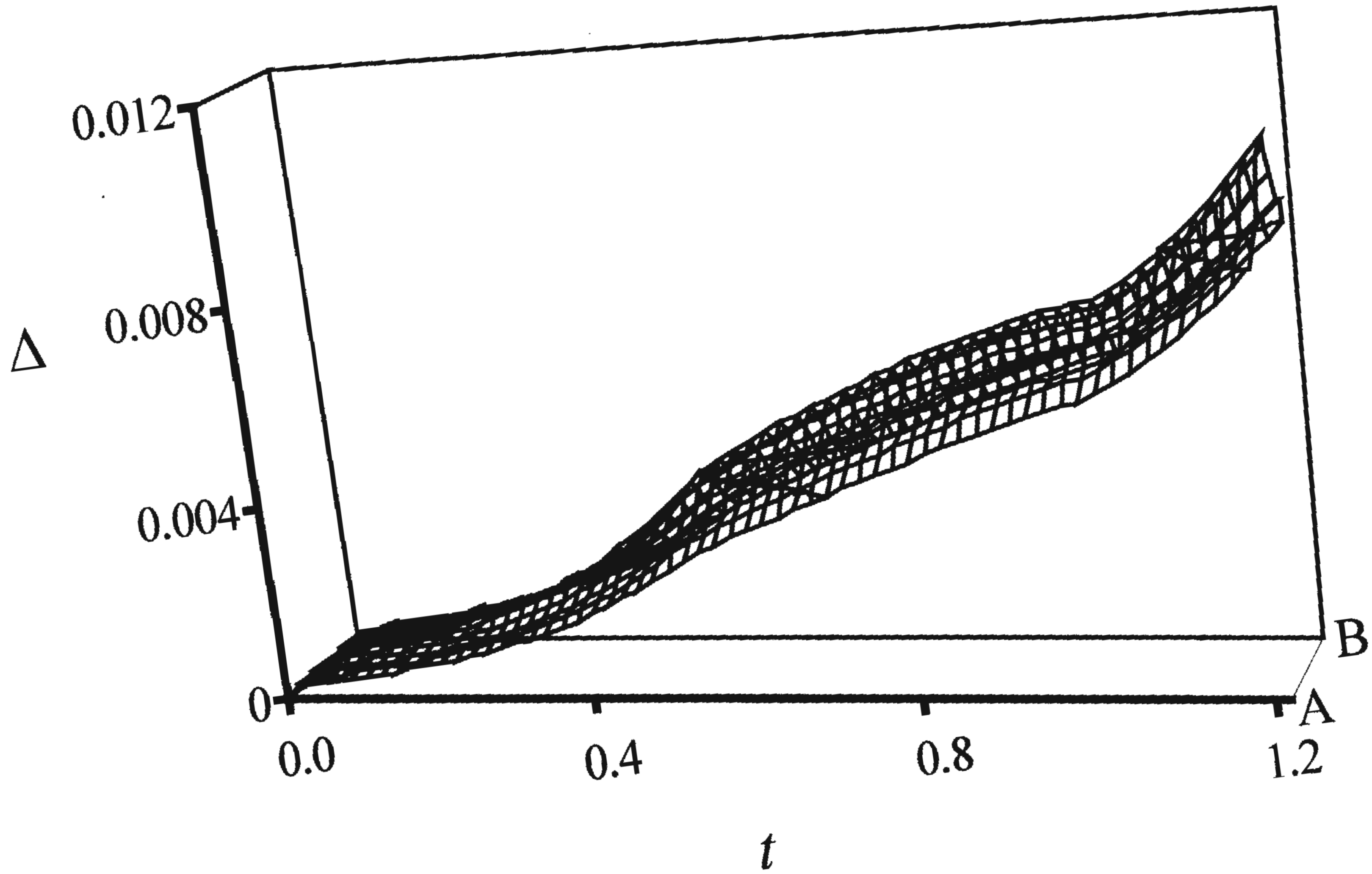
Fig. 7 Effect of decreasing κ on nonlinear bias of the estimated heat transfer coefficient at point O. κ is decreased from 200 W/m-°C to 100 W/m-°C.

Fig. 8 Effect of decreasing κ on variance of the estimated heat transfer coefficient at point O. κ is decreased from 200 W/m-°C to 100 W/m-°C.

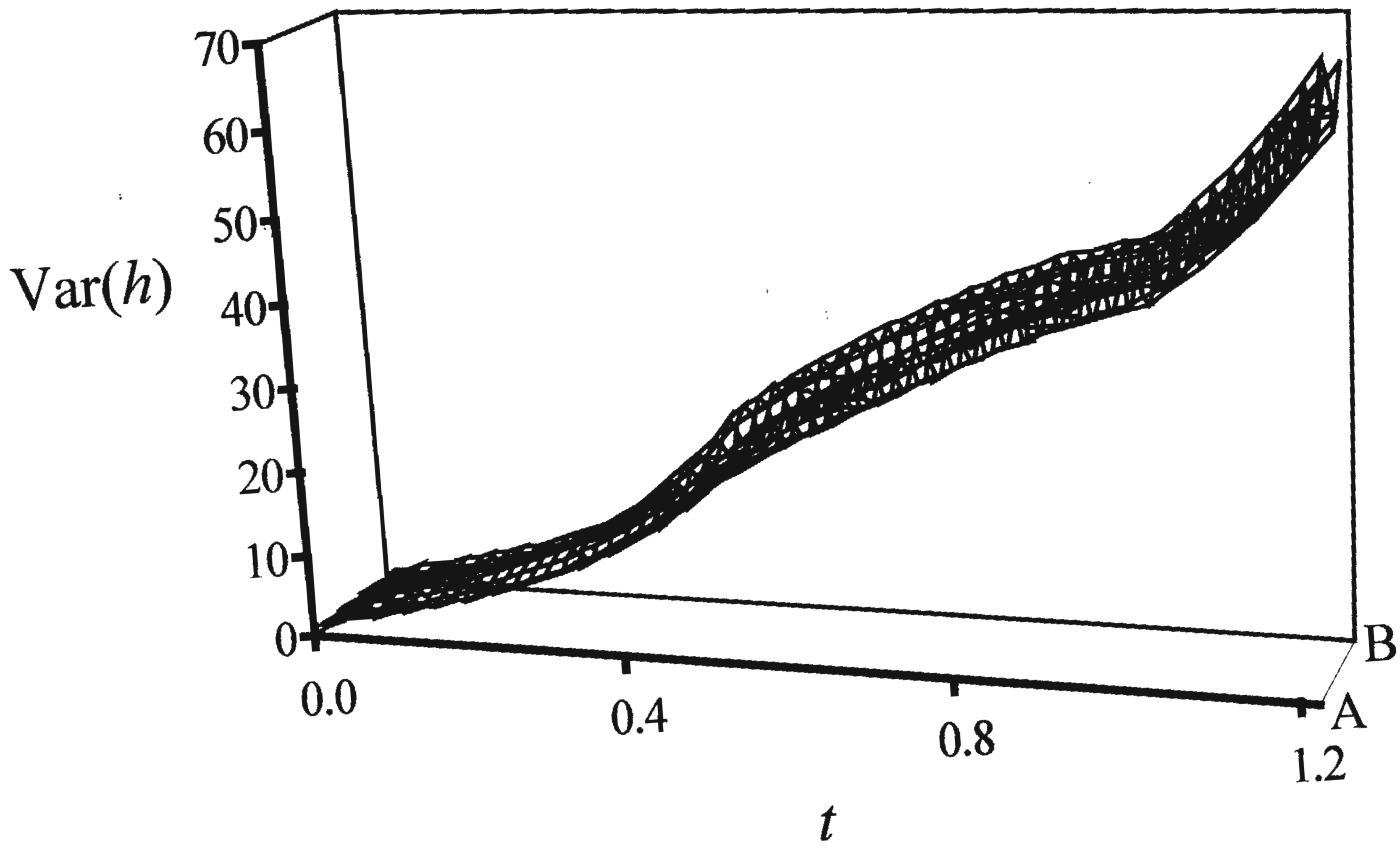

insulated
A
O
B

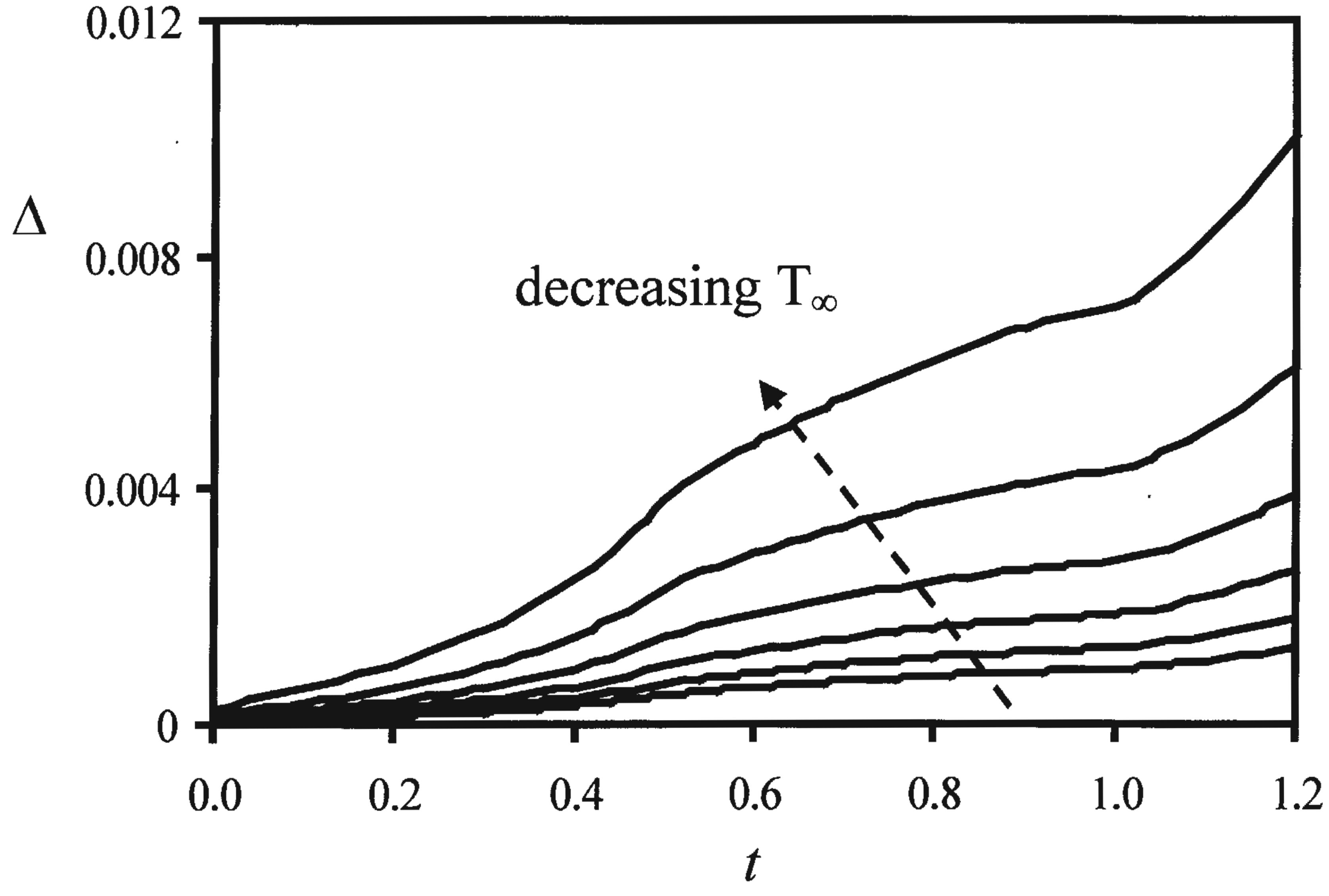


1500
1000
500
0
h
(W/m²-°C)
A
B
0.0
0.4
0.8
1.2
t

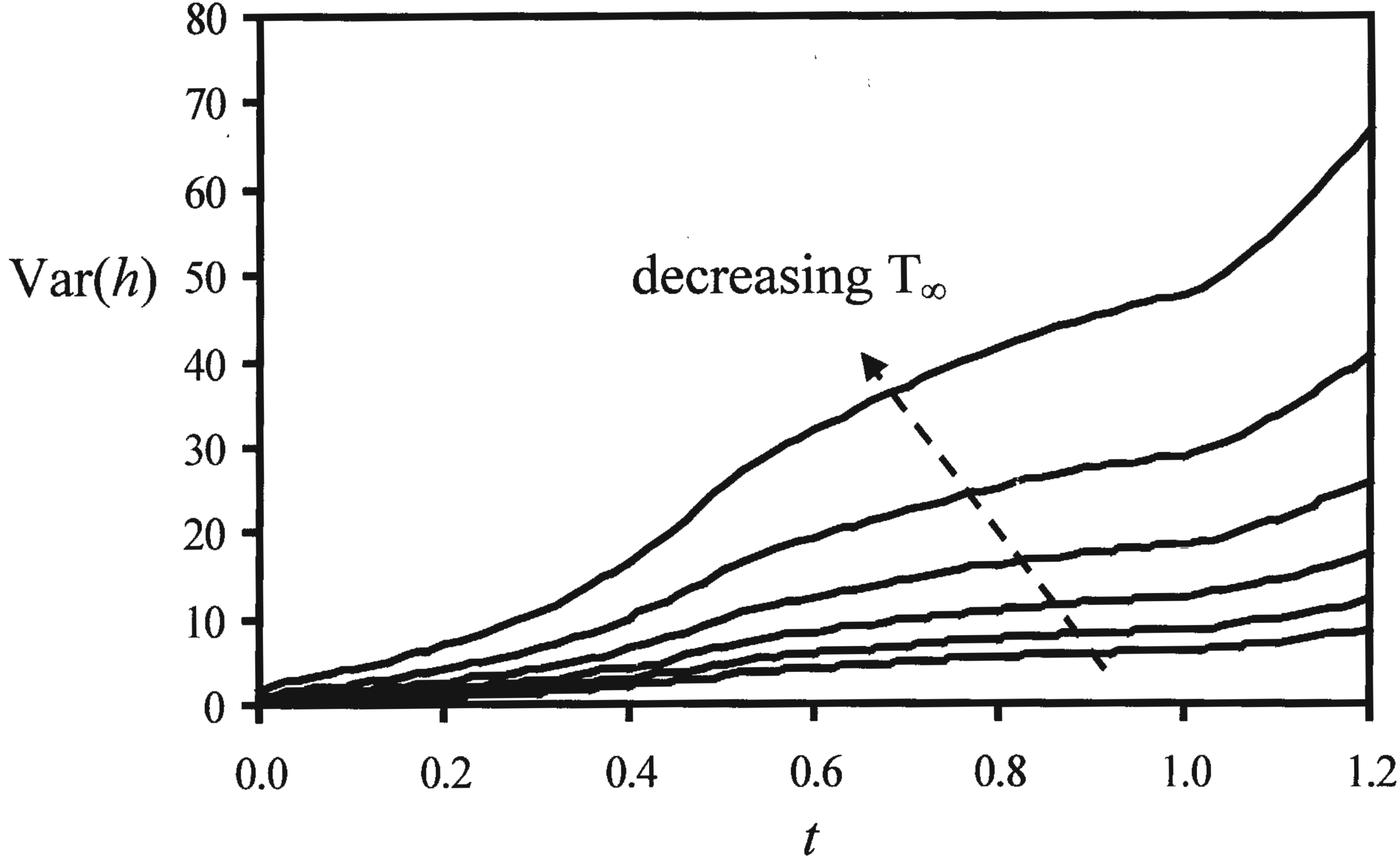


Δ
0.012
0.008
0.004
0
0.0
0.4
0.8
1.2
t
A
B

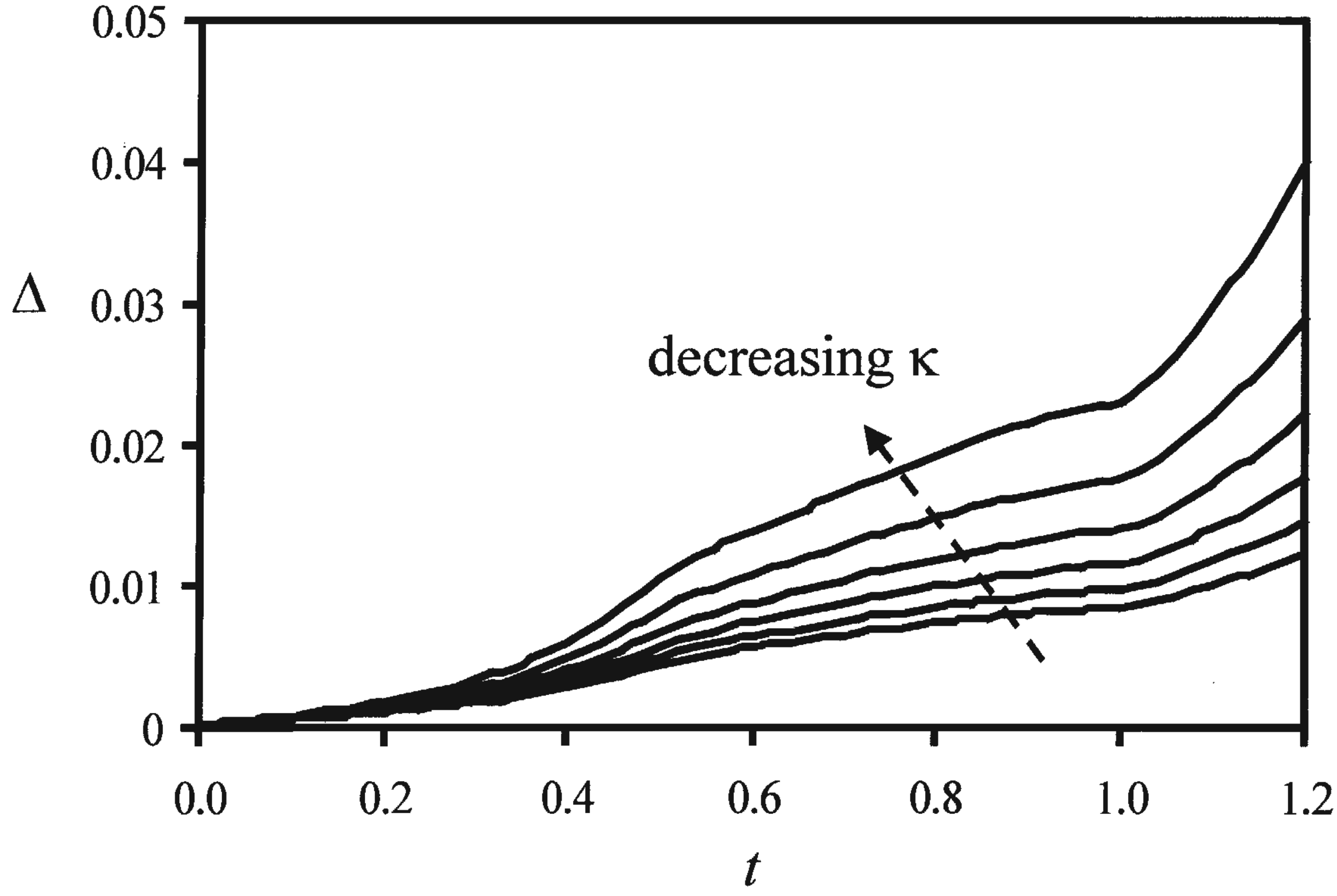

Var(h)
70
60
50
40
30
20
10
0
0.0
0.4
0.8
1.2
t
A
B

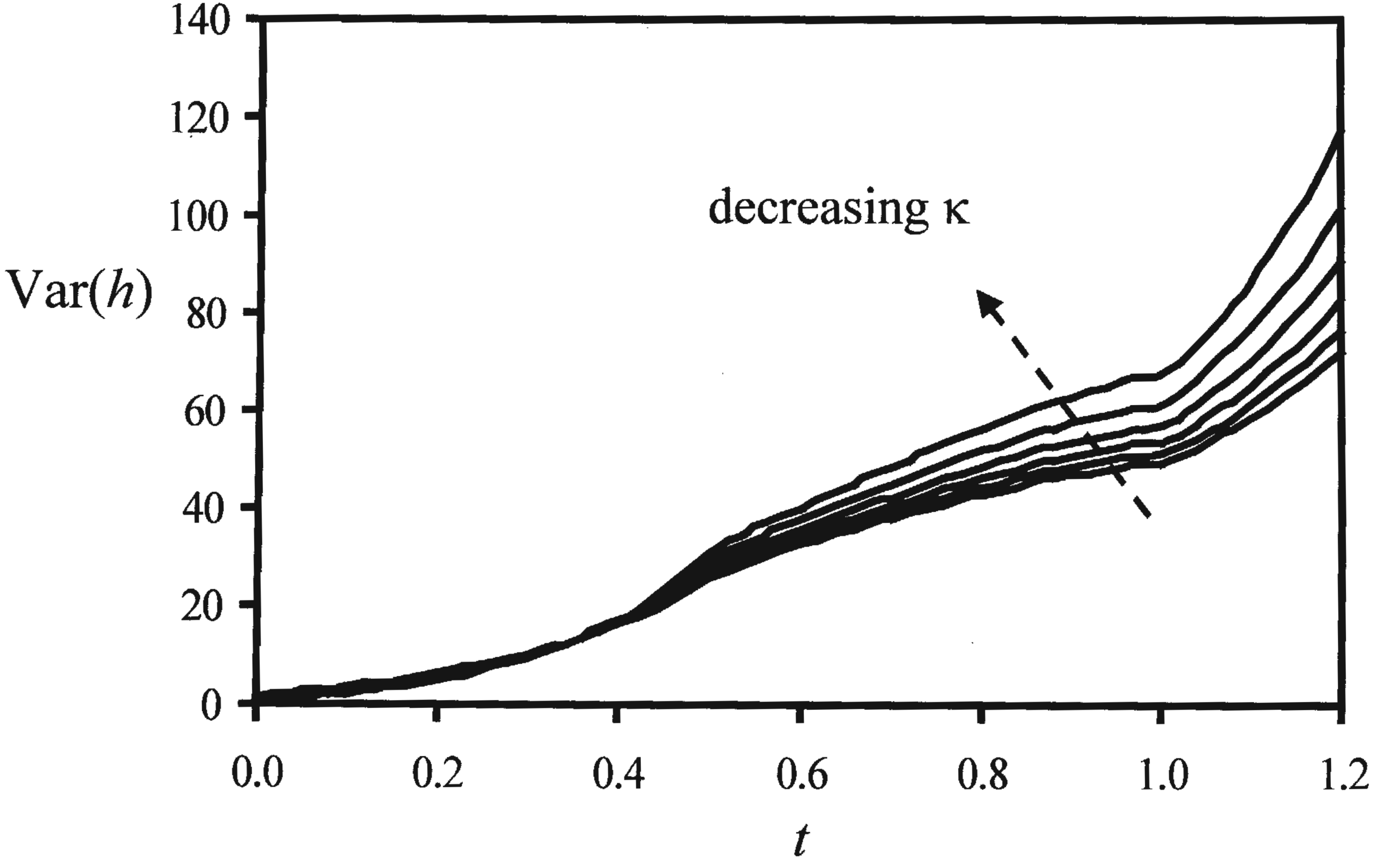


0.012
0.008
0.004
0
Δ
decreasing T∞
0.0
0.2
0.4
0.6
0.8
1.0
1.2
t



Var(h)
80
70
60
50
40
30
20
10
0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
t
decreasing T∞


0.05
0.04
0.03
0.02
0.01
0
Δ
decreasing κ
0.0
0.2
0.4
0.6
0.8
1.0
1.2
t



Var(h)
140
120
100
80
60
40
20
0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
t
decreasing κ