



รายงานวิจัยฉบับสมบูรณ์

ความสัมพันธ์เชิงฟังก์ชันนัลและเชิงพลศาสตร์ของฟังก์ชันนิวตัน,
ฟังก์ชันการประมาณสืบเนื่อง และฟังก์ชันฮัลเลย์

Functional relation and dynamical relation of Newton's functions,
Successive approximations and Halley's functions

โดย ดร.ปิยะพงศ์ เนียมทรัพย์

*30 มิถุนายน 2543

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คณะผู้วิจัย

สังกัด

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มหาวิทยาลัยเชียงใหม่

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

ชุดโครงการ ทุนวิจัยหลังปริญญาเอก

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กิตติกรรมประกาศ

งานวิจัยนี้สำเร็จได้ด้วยดีโดยความร่วมมือจากคณาจารย์และนักศึกษา และหน่วยงานที่เกี่ยวข้อง โดยเฉพาะอย่างยิ่งข้าพเจ้าใคร่ขอแสดงความขอบคุณ ศ.ดร.ยงศิริมล เลณบุรี, ภาควิชาคณิตศาสตร์, มหาวิทยาลัยมหิดล ซึ่งเป็นอาจารย์ที่ปรึกษาโครงการ สำหรับการให้ข้อคิดและข้อเสนอแนะ ในการทำวิจัยในระยะเริ่มต้น ข้าพเจ้าขอแสดงความขอบคุณ Prof.Dr.Julian Palmore, Department of Mathematics, University of Illinois, USA สำหรับการให้คำปรึกษาเกี่ยวกับเนื้อหาของการวิจัย ข้าพเจ้าขอแสดงความขอบคุณ สำนักงานกองทุนสนับสนุนการวิจัย ที่ได้ให้การสนับสนุนทุนวิจัยหลังปริญญาเอก ในการทำวิจัยนี้ ข้าพเจ้าหวังเป็นอย่างยิ่งว่าผลวิจัยที่ได้จะเป็นประโยชน์ต่อนักคณิตศาสตร์ที่สนใจ และทำให้ผู้ที่ศึกษาคณิตศาสตร์ได้เริ่มทำวิจัยกันอย่างจริงจัง

Abstract

Project Code: PDF/29/2541

Project Title: Functional relation and dynamical relation of Newton's functions, Successive approximations and Halley's functions

Investigator: Piyapong Niamsup, Department of Mathematics, Chiangmai University

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Project Period: July 1, 1998 – June 30, 2000

Objectives: 1. To find functional relations for Newton's functions, Successive approximations and Halley's functions

2. To find dynamical relations for Newton's functions, Successive approximations and Halley's functions

3. To obtain numerical results to verify results in 1 and 2

Methodology: 1. Collect related papers and books

2. Study in details materials in 1 to obtain new topics and techniques

3. Do research to obtain new results

4. Submit papers for publications in international journals

5. Conclude and submit the final reports

Results: 1. We obtain some new relations between Newton's functions, Successive approximations and Halley's functions for quadratic polynomials

2. We obtain the roll of Schwarzian derivative in controlling the rate of convergence of the composition of Halley's functions and Newton's functions

3. We obtain rational solutions for certain functional equations related to Newton's functions, Successive approximations and Halley's functions

Discussion Conclusion:

The results obtained in this research can be used in finding roots of quadratic polynomials with our desired of the order of convergence and accuracy. Moreover, we obtain new results on rational solutions of certain functional equations. Some of our results extended results obtained by Prof.Dr.Julian Palmore and some results are truly new

Suggestions/Further Implication/Implementation: We propose to study similar results obtained to higher degree polynomials

Keywords: Newton's functions, Successive approximations and Halley's functions

บทคัดย่อ

รหัสโครงการ: PDF/29/2541

ชื่อโครงการ: ความสัมพันธ์เชิงฟังก์ชันนัลและเชิงพลศาสตร์ของฟังก์ชันนิวตัน, ฟังก์ชันการประมาณสี่เหลี่ยม และฟังก์ชันฮัลเลย์

ผู้วิจัย: ดร.ปิยะพงศ์ เนียมทรัพย์, ภาควิชาคณิตศาสตร์ มหาวิทยาลัยเชียงใหม่

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ระยะเวลาการวิจัย: 1 กรกฎาคม 2541 – 30 มิถุนายน 2543

วัตถุประสงค์:

1. หาความสัมพันธ์เชิงฟังก์ชันนัลของการประมาณสี่เหลี่ยม, ฟังก์ชันนิวตัน และฟังก์ชันฮัลเลย์
2. หาความสัมพันธ์เชิงพลศาสตร์ของการประมาณสี่เหลี่ยม, ฟังก์ชันนิวตัน และฟังก์ชันฮัลเลย์
3. หาคำตอบเชิงตัวเลขเพื่อทดสอบผลเชิงทฤษฎีที่ได้

วิธีการวิจัย:

1. รวบรวมเอกสารที่เกี่ยวข้องกับการวิจัยเพิ่มเติม
2. ศึกษาเอกสารเพื่อหาเทคนิคและหัวข้อวิจัยใหม่ๆ ที่เกี่ยวข้อง
3. ทำวิจัยเพื่อให้ได้ผลเชิงทฤษฎีใหม่ๆ
4. เสนอผลงานเพื่อการตีพิมพ์ในวารสารวิชาการระดับนานาชาติ
5. สรุปผลและจัดทำรายงานฉบับสมบูรณ์

ผลการวิจัยที่ได้รับ:

1. ได้ความสัมพันธ์ใหม่ๆ ระหว่างฟังก์ชันฮัลเลย์, ฟังก์ชันนิวตัน, และฟังก์ชันการประมาณสี่เหลี่ยม ของฟังก์ชันพหุนาม
2. ได้ลักษณะการควบคุมการลู่เข้าของรากของฟังก์ชันคอมโพสิตระหว่างฟังก์ชันฮัลเลย์และฟังก์ชันนิวตัน โดยอนุพันธ์ซ้ำๆ
3. ได้คำตอบที่เป็นฟังก์ชันตรรกยะของสมการเชิงฟังก์ชันที่เกี่ยวข้องกับฟังก์ชันฮัลเลย์, ฟังก์ชันนิวตัน, และฟังก์ชันการประมาณสี่เหลี่ยม

สรุปผลและอภิปรายผล:

ผลการวิจัยที่ได้สามารถนำไปใช้ในการหารากของฟังก์ชันพหุนามที่มีดีกรีเป็นสอง โดยได้อัตราการลู่เข้าตามที่เรากำลังต้องการ และมีค่าความถูกต้องสูง อีกทั้งยังได้ความสัมพันธ์ในเชิงทฤษฎีของ ฟังก์ชันฮัลเลย์, ฟังก์ชันนิวตัน, และฟังก์ชันการประมาณสี่เหลี่ยม โดยผลที่ได้มีทั้งในส่วนที่ขยายผลจากงานวิจัยของ Prof.Dr.Julian Palmore และ ทั้งส่วนที่คิดใหม่ทั้งหมด

ข้อเสนอแนะ: ศึกษาผลที่ได้กับฟังก์ชันพหุนามที่มีดีกรีมากกว่าสองขึ้นไป

คำหลักของการวิจัย: ฟังก์ชันฮัลเลย์, ฟังก์ชันนิวตัน, และฟังก์ชันการประมาณสี่เหลี่ยม

หน้าสรุปโครงการ (Executive Summary)

1. ชื่อโครงการ

(ภาษาไทย) ความสัมพันธ์เชิงฟังก์ชันนัลและเชิงพลศาสตร์ของฟังก์ชันนิวตัน, ฟังก์ชันการประมาณสืบเนื่อง และฟังก์ชันฮัลเลย์

(ภาษาอังกฤษ) Functional relation and dynamical relation of Newton's functions, Successive approximations and Halley's functions

2. ชื่อหัวหน้าโครงการ หน่วยงานที่สังกัด ที่อยู่ หมายเลขโทรศัพท์ โทรสาร และ e-mail

นายปิยะพงศ์ เนียมทรัพย์

ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ จ.เชียงใหม่ 50200

โทรศัพท์ (053)943327 โทรสาร (053)892280

e-mail:scipnmsp@chiangmai.ac.th

3. สาขาวิชาที่ทำการวิจัย

พลศาสตร์เชิงดิสครีต (Discrete dynamical systems)

4. งบประมาณทั้งโครงการ

400,000 บาท

5. ระยะเวลาดำเนินงาน

2 ปี

6. ได้เสนอโครงการนี้ หรือโครงการที่มีส่วนเหมือนกับเรื่องนี้บางส่วนเพื่อขอทุนต่อแหล่งทุนอื่นที่ใดบ้าง

ไม่ได้เสนอ

7. ปัญหาที่ทำการวิจัย และความสำคัญของปัญหา

ในการศึกษาระบบพลศาสตร์เชิงดิสครีต ตัวอย่างของระบบที่ได้รับการศึกษาอย่างมากได้แก่

1. พลศาสตร์เชิงดิสครีตของฟังก์ชันนิวตัน (Newton's functions) ของฟังก์ชันเชิงซ้อน (Complex functions) โดยเฉพาะฟังก์ชันโพลิโนเมียล (Polynomials), ฟังก์ชันเอนไทเมอร์ (Entire functions), หรือ ฟังก์ชันมีโรมอร์ฟิก (Meromorphic functions) โดยประโยชน์ของการศึกษานี้คือ ทราบโครงสร้างของเซตของจำนวนเชิงซ้อนที่ลู่เข้าสู่รากของฟังก์ชันภายใต้การทำซ้ำของฟังก์ชันนิวตัน

2. พลศาสตร์เชิงดิสครีตของการประมาณสืบเนื่อง (Successive Approximations) ของฟังก์ชันโพลิโนเมียล โดยประโยชน์ที่ได้คือการประมาณสืบเนื่องแก่วิธีที่ง่ายในการหารากของโพลิโนเมียล
3. พลศาสตร์เชิงดิสครีตของฟังก์ชันแฮลเลย์ (Halley's functions) ของฟังก์ชันเชิงซ้อน ซึ่งเป็นฟังก์ชันที่ดัดแปลงมาจากฟังก์ชันนิวตันเพื่อเพิ่มอัตราการลู่เข้าหารากของฟังก์ชันที่ศึกษา

อย่างไรก็ตาม การศึกษาความสัมพันธ์ของฟังก์ชันทั้งสามยังมีน้อย แต่ประโยชน์ของฟังก์ชันเหล่านี้สามารถใช้ได้อย่างกว้างขวาง จึงนำที่จะศึกษาความสัมพันธ์ของฟังก์ชันเหล่านี้ทั้งในเชิงฟังก์ชันนัล (Functional relation) และเชิงพลศาสตร์ (Dynamical relation) รวมถึงอธิบายถึงพลศาสตร์ของฟังก์ชันเหล่านี้

8. วัตถุประสงค์

- 8.1 หาความสัมพันธ์เชิงฟังก์ชันนัลของการประมาณสืบเนื่อง, ฟังก์ชันนิวตัน และฟังก์ชันแฮลเลย์
- 8.2 หาความสัมพันธ์เชิงพลศาสตร์ของการประมาณสืบเนื่อง, ฟังก์ชันนิวตัน และฟังก์ชันแฮลเลย์
- 8.3 หาคำตอบเชิงตัวเลขเพื่อทดสอบผลเชิงทฤษฎีที่ได้

9. ระเบียบวิธีวิจัย

- 9.1 รวบรวมเอกสารที่เกี่ยวข้องกับการวิจัยเพิ่มเติม
- 9.2 ศึกษาเอกสารเพื่อหาเทคนิคและหัวข้อวิจัยใหม่ๆ ที่เกี่ยวข้อง
- 9.3 ทำวิจัยเพื่อให้ได้ผลเชิงทฤษฎีใหม่ๆ
- 9.4 เสนอผลงานเพื่อการตีพิมพ์ในวารสารวิชาการระดับนานาชาติ
- 9.5 สรุปผลและจัดทำรายงานฉบับสมบูรณ์

10. จำนวนโครงการที่ผู้สมัครกำลังดำเนินการอยู่

เนื้อหางานวิจัย และ Output

การศึกษาทฤษฎีการทำซ้ำ (Iteration theory) ของฟังก์ชันวิเคราะห์ (Analytic function) $f(z)$ บนระนาบเชิงซ้อนขยาย (Extended complex plane) โดยทั่วไปเรามักจะให้ความสนใจในพฤติกรรมของเซต $\{f^k\}_{k \geq 0}$ ของฟังก์ชันการทำซ้ำ (Iterating functions) ของ $f(z)$ เช่น การลู่เข้าหาจุดตรึงดึงดูด (Attracting fixed point) หรือการมีพฤติกรรมแบบอลวน (Chaotic behavior) ของฟังก์ชัน $f(z)$ บนเซตจูเลีย (Julia set)

ประโยชน์ที่เห็นได้ชัดข้อหนึ่งของการศึกษาทฤษฎีการทำซ้ำ คือการหารากของฟังก์ชันวิเคราะห์บนระนาบเชิงซ้อนขยาย ซึ่งวิธีการหารากนั้นมีหลายวิธีด้วยกัน เช่น การประมาณสืบเนื่อง (Successive approximation), ฟังก์ชันนิวตัน (Newton function) และ ฟังก์ชันฮัลเลย์ (Halley function) โดยมีงานวิจัยมากมายได้ศึกษาการลู่เข้าสู่รากของฟังก์ชันโดยการทำซ้ำของวิธีการเหล่านี้

วัตถุประสงค์ของงานวิจัยนี้คือการศึกษาความสัมพันธ์เชิงฟังก์ชันนัล และเชิงพลศาสตร์ (Functional and dynamical relations) ของวิธีการดังกล่าวข้างต้น โดยผลที่ได้สามารถนำไปเพิ่มความมีประสิทธิภาพของการลู่เข้าของวิธีการดังกล่าวได้

ผลงานวิจัยในรอบ 18 เดือน ที่ได้คือ ความสัมพันธ์ที่สำคัญระหว่าง การประมาณสืบเนื่อง และฟังก์ชันนิวตัน , ความสัมพันธ์ระหว่างการประมาณสืบเนื่องและฟังก์ชันฮัลเลย์ และคำตอบ f ของสมการเชิงฟังก์ชัน $f \circ s = s^k \circ f$ โดยที่ s คือการประมาณสืบเนื่องของโพลิโนเมียลดีกรีสองที่มีรากไม่ซ้ำกัน และ k คือจำนวนเต็มบวก โดยได้ทำการส่งผลงานวิจัยเพื่อการตีพิมพ์จำนวน 3 ผลงาน โดยผลงานวิจัยที่ได้รับระหว่างการศึกษาวิจัยมีดังนี้

1. Some relations among Halley's method, Newton's method, and successive approximations, by Piyapong Niamsup and Yongwimon Lenbury (submitted to SEAMS Bulletin)
2. The composition of Halley's and Newton's functions and its Schwarzian derivative, by Piyapong Niamsup, Julian Palmore and Yongwimon Lenbury (submitted to Complex Variables)
3. A relation between successive approximation and certain root finding methods, by Piyapong Niamsup and Julian Palmore (in preparation)
4. Rational solutions for certain functional equation, by Piyapong Niamsup and Julian Palmore (accepted for publication in Complex Variables)

ภาคผนวก

ผลงานวิจัยที่ได้รับ

piyapong

From: Esther Tam <esthertam@cuhk.edu.hk>
To: <scipnmsp@chiangmai.ac.th>
Sent: 11 มีนาคม 2543 10:06
Subject: Fw: 2914

> Dear Prof. Miamsup:

>

> Thank you for your jointed paper "some relations among halley's method,
newton's method and successive approximations". we have received on Feb.
2000. We will let you know

> the outcome of your paper when the referee's report has been obtained.

If

> you have any further correspondence of the above paper, please let me know

> your Ref no: 2914.

>

> Thank you.

>

> Esther Tam

>

>

>

Some Relations Among Halley's Method, Newton's Method and Successive Approximations

Piyapong Niamsup and Yongwimon Lenbury

June 30, 2000

1 Introduction

Let $P(z) = (z - a)(z - b)$, where $a, b \in \mathbb{C}$ such that $0 < |a| < |b|$. Let $H(z) = z - \frac{P(z)}{P'(z) - \frac{P(z)P''(z)}{2P'(z)}}$ and $N(z) = z - \frac{P(z)}{P'(z)}$ be the Halley's method and Newton's method of P and let $S(z) = \frac{-ab}{z - (a+b)}$ be a successive approximation of P . The following properties of S , N and H are well-known:

Proposition 2.1 *From $0 < |a| < |b|$, we have*

- (a) *b is the repelling fixed point of S .*
- (b) *a is the attracting fixed point of S . Moreover, a is the global attractor of S . In other words, $\lim_{k \rightarrow \infty} S^k(z) = a$ for all $z \in \mathbb{C} - \{b\}$.*

Proposition 2.2 *The following hold true for N :*

- (a) *N has two superattracting fixed points at a and b and has one repelling fixed point at ∞ .*
- (b) *The basin of attraction of a for N contains $z = 0$. In other words, $\lim_{k \rightarrow \infty} N^k(0) = a$. This is because $0 < |a| < |b|$ so that $z = 0$ is closer to a than to b .*

Proposition 2.3 *The following hold true for H :*

- (a) *H has two superattracting fixed points at a and b and has two repelling fixed point at $\frac{a+b}{2}$ and ∞ .*

(b) The basin of attraction of a for H contains $z = 0$. In other words, $\lim_{k \rightarrow \infty} H^k(0) = a$.

In general, H , N and S have orders of convergence equal to three, two and one respectively. In this note, we will derive some interesting relations among H , N and S and show an example on how the results can be used to get faster convergence. In what follows let $P(z)$, $H(z)$, $N(z)$ and $S(z)$ be as above. The i^{th} iterate of a function f will be denoted by f^i . We will prove the following relation between iterations of N and S :

Theorem 3.1 $N^i(S^j(0)) = S^{(j+1)2^i-1}(0)$ where $i, j \geq 0$.

For this purpose we need the following lemmata:

Lemma 3.2 $(N \circ S)(z) = (S^2 \circ N)(z)$ for all $z \in \mathbb{C} \cup \{\infty\}$.

Proof For the quadratic polynomial $P(z) = (z-a)(z-b)$ we have $N(z) = \frac{z^2-ab}{2z-(a+b)}$. It is straightforward to check that

$$(N \circ S)(z) = \frac{(ab)z^2 - 2ab(a+b)z + ab(a+b)^2 - (ab)^2}{(a+b)z^2 - 2(a^2+ab+b^2)z - (a+b)(a^2+b^2)} = (S^2 \circ N)(z). \quad \square$$

Lemma 3.3 $(N \circ S^i)(z) = (S^{2^i} \circ N)(z)$ for all $z \in \mathbb{C} \cup \{\infty\}$.

Proof We prove by induction on i . For $i = 1$ the result follows from Lemma 3.2. Assume the result is true for $i > 1$. Then

$$\begin{aligned} N \circ S^{i+1} &= (N \circ S^i) \circ S \\ &= (S^{2^i} \circ N) \circ S \quad \text{by Lemma 3.2} \\ &= S^{2^i} \circ (N \circ S) \\ &= S^{2^i} \circ (S^2 \circ N) \\ &= S^{2(i+1)} \circ N. \end{aligned}$$

The proof is complete. $\square$

The following lemma can be proved by induction as in Lemma 3.3:

Lemma 3.4 $(N^i \circ S^j)(z) = (S^{j \cdot 2^i} \circ N^i)(z)$ for all $z \in \mathbb{C} \cup \{\infty\}$, $i, j \geq 0$.

Proposition 3.5 $N^i(0) = S^{2^i-1}(0)$, $i \geq 0$.

Proof From Lemma 3.4 we have, for $i \geq 1$, that

$$N^i(0) = (S^{2^i} \circ N^i \circ S^{-1})(0)$$

$$\begin{aligned}
&= (S^{2^i} \circ N^i)(S^{-1}(0)) \\
&= (S^{2^i} \circ N^i)(\infty) \\
&= S^{2^i}(N^i(\infty)) \\
&= S^{2^i}(\infty) \\
&= S^{2^i}(S^{-1}(0)) \\
&= S^{2^i-1}(0).
\end{aligned}$$

This is valid because S is one-to-one and $S^{-1}(0) = \infty$. This finishes the proof. $\square$

Proof of theorem 3.1 From Lemma 3.4 we have $(N^i \circ S^j)(0) = (S^{j \cdot 2^i} \circ N^i)(0)$. Thus

$$\begin{aligned}
N^i(S^j(0)) &= S^{j \cdot 2^i}(N^i(0)) \\
&= S^{j \cdot 2^i}(S^{2^i-1}(0)) \text{ by Proposition 3.5} \\
&= S^{(j+1) \cdot 2^i-1}(0).
\end{aligned}$$

This completes the proof of the theorem. $\square$

Remark 1. Proposition 3.5 is proved in [2] by using the technique of topological conjugacy which can also be used to prove our result (Theorem 3.1). In this work, we present a simpler proof by observing that one iteration of N followed by two iterations of S is equivalent to one iteration of S followed by one iteration of N (Lemma 3.2).

Remark 2. If we replace $S(z)$ with $T(z) = \frac{(a+b)z-ab}{z}$, the inverse of S , which is also a Successive approximation of P , then we have similar results as follows:

Proposition 3.6 *The following hold true for T :*

- (a) a is the repulsive fixed point of T .
- (b) b is the attractive fixed point of T . Moreover, b is the global attractor of T .

Proposition 3.8 *The following hold true:*

- (a) $(N^i \circ T^j)(z) = (T^{j \cdot 2^i} \circ N^i)(z)$ for all $z \in \mathbb{C} \cup \{\infty\}$ and $i, j \geq 0$.
- (b) $N^i(T^j(\frac{ab}{a+b})) = T^{(j+1) \cdot 2^i-1}(\frac{ab}{a+b})$ where $i, j \geq 0$.

With the same argument as above, we obtain the following relations among H , N and S :

Theorem 3.9 *The following hold true:*

- (a) $(H^i \circ S^j)(z) = (S^{j \cdot 3^i} \circ H^i)(z)$ for all $z \in \mathbb{C} \cup \{\infty\}$, $i, j \geq 0$.
- (b) $H^i(S^j(0)) = S^{(j+1)3^i-1}(0)$ where $i, j \geq 0$.
- (c) $H^i(S^{j \cdot 2^i-1}(0)) = N^i(S^{j \cdot 3^i-1}(0))$ where $i, j \geq 0$.
- (d) $(H \circ N)^i(z) = (H^i \circ N^i)(z) = (N \circ H)^i(z)$ for all $z \in \mathbb{C} \cup \{\infty\}$, $i \geq 0$.
- (e) $(H^i \circ N^j \circ S^k)(z) = (S^{k \cdot 2^j \cdot 3^i} \circ H^i \circ N^j)(z)$ for all $z \in \mathbb{C} \cup \{\infty\}$, $i, j, k \geq 0$.
- (f) $(H^i \circ N^j)(S^k(0)) = S^{(k+1) \cdot 2^j \cdot 3^i-1}(0)$ where i, j and $k \geq 0$.

Remark 3. It is natural to ask if there are similar relations among Halley's method, Newton's method and successive approximations of polynomials of degree 3 or higher. The answer seems to be negative. One reason is because for polynomials of degree 3 (or higher), successive approximations of this polynomial have degree 2 (or higher). From which it follows that degree of $N \circ S$ is not equal to degree of $S^2 \circ N$ hence Lemma 3.2 does not hold.

2 An Example

Let $P(z) = (z-1)(z+2)$. Then $H(z) = \frac{z^3+6z+2}{3z^2+3z+3}$, $N(z) = \frac{z^2+2}{2z+1}$ and $S(z) = \frac{2}{z+1}$. The convergent sequences to the root $z = 1$ starting from initial point $z = 0$ produced by iteration of S is as follows:

$S : 0, 2, \frac{2}{3}, \frac{6}{5}, \frac{10}{11}, \frac{22}{21}, \frac{42}{43}, \frac{86}{85}, \frac{170}{171}, \frac{342}{341}, \frac{682}{683}, \frac{1366}{1365}, \frac{2730}{2731}, \frac{5462}{5461}, \frac{10922}{10923}, \frac{21846}{21845}, \frac{43690}{43691},$
 $\frac{87382}{87381}, \frac{174762}{174763}, \frac{349526}{349525}, \frac{699050}{699051}, \frac{1398102}{1398101}, \frac{2796202}{2796203}, \frac{5592406}{5592405}, \frac{11184810}{11184811}, \frac{22369622}{22369621}, \frac{44739242}{44739243},$
 $\frac{89478486}{89478485}, \frac{178956970}{178956971}, \frac{357913942}{357913941}, \frac{715827882}{715827883}, \frac{1431655766}{1431655765}, \dots$

For illustration, if we start from the initial points $z = 0$, $z = \frac{2}{3}$ and $z = \frac{6}{5}$ then the convergent subsequences (of sequences above) to the root $z = 1$ starting from these initial points produced by iterations of N and H are as follows:

$N : 0, 2, \frac{6}{5}, \frac{86}{85}, \frac{21846}{21845}, \frac{1431655766}{1431655765}, \dots$
 $N : \frac{2}{3}, \frac{22}{21}, \frac{1366}{1365}, \frac{5592406}{5592405}, \frac{93824992236886}{93824992236885}, \dots$
 $H : 0, \frac{2}{3}, \frac{170}{171}, \frac{44739242}{44739243}, \frac{805950546409752783137450}{805950546409752783137451}, \dots$
 $H : \frac{6}{5}, \frac{1366}{1365}, \frac{22906492246}{22906492245}, \dots$

As an example, we have the following correspondences between N and S :

$N^1(0) = 2 = S^1(0)$, $N^2(0) = \frac{6}{5} = S^3(0)$, $N^3(0) = \frac{86}{85} = S^7(0)$, $N^4(0) = \frac{21846}{21845} = S^{15}(0)$, ... and $N^0(\frac{2}{3}) = \frac{2}{3} = S^2(0)$, $N^1(\frac{2}{3}) = \frac{22}{21} = S^5(0)$, $N^2(\frac{2}{3}) = \frac{1366}{1365} = S^{11}(0)$, $N^3(\frac{2}{3}) = \frac{5592406}{5592405} = S^{24}(0)$, $N^4(\frac{2}{3}) = \frac{93824992236886}{93824992236885} = S^{17}(0)$, ... As we have seen, if we want a fast convergence to roots of P , then we select an initial point in the form $S^j(0)$ (or $T^j(\frac{ab}{a+b})$) and then apply either N or H successively.

3 Acknowledgement

The first author is supported by the Thailand Research Fund during the preparation of this work. He would like to thank Prof.Dr.Yongwimon Lenbury, Mahidol University, Thailand for many valuable suggestions.

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COMPLEX VARIABLES AN INTERNATIONAL JOURNAL

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June 1, 1999

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CV-1450 "The composition of Halley's and Newton's functions and its Schwarzian derivatives"

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The Composition of Halley's and Newton's Functions and its Schwarzian Derivatives

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June 30, 2000

keywords Newton's function, Halley's function, Schwarzian derivative

1991 *Mathematics Subject Classification*. Primary 26A18; Secondary 58F03.

1 abstract

The composition between Newton's function and Halley's function is studied. We show that the composition function has the order of convergence to a simple root of a function f equal to or exceed 6. We show that the Schwarzian derivative of f and the second derivative of f evaluated at the zero of f are the factors which determine the order of convergence.

2 Introduction

Let f be a differentiable function on $\mathbf{C}$. The Newton's function, Halley's functions and the Schwarzian derivative of f will be denoted by $N(z)$, $H(z)$, and $S(z)$ and are defined by

$$\begin{aligned} N(z) &= z - \frac{f(z)}{f'(z)} \\ H(z) &= z - \frac{f(z)}{f'(z) - \frac{f(z)f''(z)}{2f'(z)}} \end{aligned}$$

$$S(z) = 2 \frac{f'''(z)}{f'(z)} - 3 \left(\frac{f''(z)}{f'(z)} \right)^2.$$

Assume that f has a simple zero at ζ so that $f'(\zeta) \neq 0$. In [2] and [3], the role of Schwarzian derivatives of N and H in controlling the order of convergence of N and H to ζ are studied respectively. The main results in [2] and [3] are as follows:

Theorem 2.1 [3] *Let f be a differentiable function on $\mathbf{C}$. Let N and S be the Newton's function and the Schwarzian derivative of f . If f has a simple zero ζ such that $f'(\zeta) \neq 0$, then $N'''(\zeta) = S(\zeta)$. If $f''(\zeta) = 0$, then N has convergence to ζ of order 3 or greater. If $S(\zeta) = 0$, then N has convergence to ζ of order 4 or greater.*

Theorem 2.2 [2] *Let f be a differentiable function on $\mathbf{C}$. Let H and S be the Halley's function and the Schwarzian derivative of f . If f has a simple zero ζ such that $f'(\zeta) \neq 0$, then $H'''(\zeta) = -2S(\zeta)$. H has convergence to ζ of order 4 or greater if and only if $S(\zeta) = 0$.*

In theorem 1.2, the formula for $H'''(\zeta)$ is incorrect; the correct formula is $H'''(\zeta) = -\frac{S(\zeta)}{2}$. From theorem 1.1 it follows that the Schwarzian derivative of f at ζ controls the order of convergence of N in cases where the first two derivatives of f at ζ are nonzero. In this paper we study the role of Schwarzian derivatives of composition function between H and N , namely $H \circ N$ and $N \circ H$, in controlling the order of convergence to the simple root ζ of f .

3 Main result

In this section we will prove the following main result:

Theorem 3.1 *Let f be a differentiable function on $\mathbf{C}$. Let H , N and S be the Halley's function, Newton's function and Schwarzian derivative of f . If f has a simple zero ζ such that $f'(\zeta) \neq 0$, then the values of the first five derivatives of $H \circ N$ at ζ are zero, and $(H \circ N)^{(6)}(\zeta) = -\frac{15}{2} \left(\frac{f''(\zeta)}{f'(\zeta)} \right)^3 \cdot S(\zeta)$. Therefore, $H \circ N$ has order of convergence to ζ equal to 6 or greater. The order of convergence of $H \circ N$ is controlled by value of the second derivative and value of the Schwarzian derivative of f evaluated at ζ .*

Proof From [2], $H(z) = \frac{zN(z) - zN'(z)}{2 - N'(z)}$ and from theorem 2.2, $H(\zeta) = -\frac{S(\zeta)}{2}$. We know that $N(\zeta) = \zeta$, $N'(\zeta) = 0$, $H'(\zeta) = 0$, $H''(\zeta) = 0$ and $H'''(\zeta) = -\frac{S(\zeta)}{2}$. With these information and the chain rule, the first six derivatives of $H \circ N$ at ζ are

$$\begin{aligned}
(H \circ N)'(\zeta) &= H'(N(\zeta)) \cdot N'(\zeta) = 0, \\
(H \circ N)''(\zeta) &= (N'(\zeta))^2 \cdot H''(N(\zeta)) + H'(N(\zeta)) \cdot N''(\zeta) = 0, \\
(H \circ N)'''(\zeta) &= 3N'(\zeta) \cdot N''(\zeta) \cdot H''(N(\zeta)) + (N'(\zeta))^3 \cdot H'''(N(\zeta)) + \\
&\quad H'(N(\zeta)) \cdot N'''(\zeta) \\
&= 0, \\
(H \circ N)^{(4)}(\zeta) &= 3H''(N(\zeta)) \cdot (N''(\zeta))^2 + 6(N'(\zeta))^2 \cdot N''(\zeta) \cdot H'''(N(\zeta)) + \\
&\quad 4N'(\zeta) \cdot H''(N(\zeta)) \cdot N'''(\zeta) + (N'(\zeta))^4 \cdot H^{(4)}(N(\zeta)) + \\
&\quad H'(N(\zeta)) \cdot N^{(4)}(\zeta) \\
&= 0, \\
(H \circ N)^{(5)}(\zeta) &= 15N'(\zeta) \cdot (N''(\zeta))^2 \cdot H'''(N(\zeta)) + 10H''(N(\zeta)) \cdot N''(\zeta) \cdot N'''(\zeta) + \\
&\quad 10(N'(\zeta))^2 \cdot H'''(N(\zeta)) \cdot N'''(\zeta) + 10(N'(\zeta))^3 \cdot N''(\zeta) \cdot H^{(4)}(N(\zeta)) + \\
&\quad 5N'(\zeta) \cdot H''(N(\zeta)) \cdot N^{(4)}(\zeta) + (N'(\zeta))^5 \cdot H^{(5)}(N(\zeta)) + \\
&\quad H'(N(\zeta)) \cdot N^{(5)}(\zeta) \\
&= 0, \text{ and} \\
(H \circ N)^{(6)}(\zeta) &= 15(N''(\zeta))^3 \cdot H'''(N(\zeta)) + 60N'(\zeta) \cdot N''(\zeta) \cdot H'''(N(\zeta)) \cdot N'''(\zeta) + \\
&\quad 10H''(N(\zeta)) \cdot (N'''(\zeta))^2 + 45(N'(\zeta))^2 \cdot (N''(\zeta))^2 \cdot H^{(4)}(N(\zeta)) + \\
&\quad 20(N'(\zeta))^3 \cdot N'''(\zeta) \cdot H^{(4)}(N(\zeta)) + 15H''(N(\zeta)) \cdot N''(\zeta) \cdot N^{(4)}(\zeta) + \\
&\quad 15(N'(\zeta))^2 \cdot H'''(N(\zeta)) \cdot N^{(4)}(\zeta) + 15(N'(\zeta))^4 \cdot N''(\zeta) \cdot H^{(5)}(N(\zeta)) + \\
&\quad 6N'(\zeta) \cdot H''(N(\zeta)) \cdot N^{(5)}(\zeta) + (N'(\zeta))^6 \cdot H^{(6)}(N(\zeta)) + \\
&\quad H'(N(\zeta)) \cdot N^{(6)}(\zeta) \\
&= 15(N''(\zeta))^3 \cdot H'''(N(\zeta)) = -\frac{15}{2} \left(\frac{f''(\zeta)}{f'(\zeta)} \right)^3 \cdot S(\zeta).
\end{aligned}$$

Therefore, $H \circ N$ has order of convergence to ζ equal to 6 or greater. Clearly, the order of convergence of $H \circ N$ is controlled by the second derivative and the Schwarzian derivative of f at ζ . This completes the proof. $\square$

Theorem 3.2 *Let f be a differentiable function on $\mathbb{C}$. Let H , N , and S be the Halley's function, Newton's function, and Schwarzian derivative*

of f , respectively. If f has a simple zero ζ such that $f'(\zeta) \neq 0$, then the values of the first five derivatives of $N \circ H$ evaluated at ζ are zero, and $(N \circ H)^{(6)}(\zeta) = 10 \left(\frac{f''(\zeta)}{f'(\zeta)} \right) (H'''(\zeta))^2 = -\frac{15}{2} \left(\frac{f''(\zeta)}{f'(\zeta)} \right)^3 \cdot S(\zeta)$. Therefore, $N \circ H$ has order of convergence to ζ equal to 6 or greater and the order of convergence is controlled by value of the second derivative and value of Schwarzian derivative of f evaluated at ζ .

Proof The proof is similar to that of Theorem 3.1 and will be omitted. $\square$

Remark It is natural to further consider the roles of Schwarzian derivative and derivatives of f in controlling order of convergence of $H^i \circ N^j$ where $i, j \geq 0$.

4 Example

Let $f(x) = (x-1)(x+2) = x^2 + x - 2$. In this example, we have $N(x) = \frac{x^2+2}{2x+1}$, $H(x) = \frac{x^3+6x+2}{3x^2+3x+3}$, and $S(x) = -3 \left(\frac{2}{2x+1} \right)^2$. By computation, we get $(N \circ H)(x) = (H \circ N)(x) = \frac{x^6+30x^4+40x^3+90x^2+60x+22}{(2x+1)(3x^4+6x^3+27x^2+24x+21)}$. Since $S(1) = S(-2) = -\frac{4}{3}$, $\frac{f''(1)}{f'(1)} = \frac{2}{3}$, and $\frac{f''(-2)}{f'(-2)} = -\frac{2}{3}$, we get $(N \circ H)^{(6)}(1) = (H \circ N)^{(6)}(1) = -\frac{15}{2} \cdot \left(\frac{2}{3} \right)^3 \cdot \left(-\frac{4}{3} \right) = \frac{80}{27}$ and $(N \circ H)^{(6)}(-2) = (H \circ N)^{(6)}(-2) = -\frac{15}{2} \cdot \left(-\frac{2}{3} \right)^3 \cdot \left(-\frac{4}{3} \right) = -\frac{80}{27}$. Hence, the order of convergence of $N \circ H$ and of $H \circ N$ to the zeros of f is 6.

5 Acknowledgement

This work is supported by Thailand Research Fund.

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A RELATION BETWEEN SUCCESSIVE APPROXIMATION AND CERTAIN ROOTS FINDING METHODS

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ABSTRACT. In this paper, we give some interesting relation between successive approximation and certain roots finding method for quadratic polynomials.

1. INTRODUCTION

For a function $f(z)$, the Halley's method and Newton's method of f are defined as follows:

$$H(z) = z - \frac{f(z)}{f'(z) - \frac{f(z)f''(z)}{2f'(z)}}, \text{ and}$$
$$N(z) = z - \frac{f(z)}{f'(z)}$$

respectively. A successive approximation of $f(z)$ can be obtained by letting $f(z) = 0$ and write this equation in a form $z = S(z)$ where $S(z)$ will be called a successive approximation of f . For example, if $f(z) = (z - a)(z - b)$ where $a, b \in \mathbb{C} - \{0\}$, $a \neq b$, then we obtain a successive approximation $S(z) = \frac{-ab}{z - (a+b)}$. Halley's method, Newton's method and successive approximation are iterative method which are used to find certain roots of f in which we choose an initial guest x_0 and then iterate these method to x_0 to obtain a sequence $\{x_n\}$ which usually converge to certain roots of f (the choice of initial guest x_0 is essential for the convergence of the sequence, see [1], [9] and references cited therein). In general, the orders of convergence of Halley's method, Newton's method and successive approximation are three, two and one respectively. In [3], an interesting relation between Newton's method and successive approximation was obtained as follows:

Date: June 30, 2000.

1991 Mathematics Subject Classification. 30D05, 58F08, 30B70.

Key words and phrases. Successive approximation, Halley's method, Newton's method.

Theorem 1.1 *Let $\zeta^+ = \frac{a+\sqrt{b}}{c}$, $\zeta^- = \frac{a-\sqrt{b}}{c}$ be quadratic irrationals where a , b , and c are integers satisfying $a < 0$, $b > 0$, b is not the square of an integer, and $c \neq 0$. Let $P(z) = (z - \zeta^+)(z - \zeta^-)$, $g(z)$ be the Newton's method of P and $h(z) = \frac{-\zeta^+\zeta^-}{z - (\zeta^+ + \zeta^-)}$ be a successive approximation of P . Then for all $n \geq 0$, we have*

$$g^n(0) = h^{2^n-1}(0)$$

where f^i denotes the i^{th} iterate of function f .

Theorem 1.1 was proved using the concept of topological conjugacy where we say f and g are topological conjugacy if there exists a Möbius transformation h such that $f \circ h = h \circ g$. If f and g are topological conjugate, then they have the same dynamical behavior such as they have the same type of periodic points, see [1] for more details. In this paper, we will give some interesting relation between successive approximations and certain root finding methods for quadratic polynomials which extend Theorem 1.1 to general root findings method of order of convergence equal to k , $k \geq 2$.

2. MAIN RESULTS

Let $P(z) = (z - a)(z - b)$ where $a, b \in \mathbb{C}$ satisfying $0 < |a| < |b|$ be a quadratic polynomial. We first introduce the following rational function

$$f_k(z) = \frac{a(z - b)^k - b(z - a)^k}{(z - b)^k - (z - a)^k}.$$

We can see that f_k is topological conjugate to the map $w \mapsto w^k$ by the Möbius transformation $M(z)$ that sends $w = 0$ to a and $w = \infty$ to b , where $M(z) = \frac{bz-a}{z-1}$ and $M^{-1}(z) = \frac{-z+a}{-z+b}$. In other words, we have

$$f_k(z) = M([M^{-1}(z)]^k).$$

Note that, when $k = 2$, f_k is the Newton's method of P and when $k = 3$, f_k is the Halley's method of P . In this section, we will prove the following main result:

Theorem 2.1 *The order of convergence of f_k to the roots a starting from $z_0 = 0$ is equal to k . More precisely, we have*

$$f_k^i(S^j(0)) = S^{(j+1)k^i-1}(0)$$

for all $i, j \geq 0$, where $S(z) = \frac{-ab}{z-(a+b)}$ is a successive approximation of P .

In order to prove this theorem, we need the following lemma:

Lemma 2.2 *For the successive approximation S , the point b is the repulsive fixed point of S and a is the global attractor of S , that is*

$\lim_{k \rightarrow \infty} S^k(z) = a$ for all $z \in \mathbb{C} - \{b\}$. Moreover, S is topological conjugate to the map $w \mapsto \frac{a}{b}w$ by the Möbius transformation $M(z) = \frac{bz-a}{z-1}$, that is

$$\begin{aligned} S^k(z) &= M \left(\left(\frac{a}{b} \right)^k M^{-1}(z) \right) \\ &= \frac{ba^k(-z+a) - ab^k(-z+b)}{a^k(-z+a) - b^k(-z+b)}. \end{aligned}$$

Proof It is routine to see that b is the repulsive fixed point of S , a is the attractive fixed point of S and that

$$S(z) = M \left(\left(\frac{a}{b} \right) M^{-1}(z) \right)$$

that is S is topological conjugate to the map $w \mapsto \frac{a}{b}w$ by the Möbius transformation $M(z) = \frac{bz-a}{z-1}$. Since $0 < |a| < |b|$, it follows that $\lim_{i \rightarrow \infty} w^i(z) = \lim_{i \rightarrow \infty} \left(\frac{a}{b} \right)^i z = 0$ for all $z \in \mathbb{C}$. Hence, 0 is the global attractor of the map $w \mapsto \frac{a}{b}w$. Since S is topological conjugate to the map $w \mapsto \frac{a}{b}w$ and $M(0) = a$, it follows that a is the global attractor of S . $\square$

Lemma 2.3 For all $z \in \mathbb{C} \cup \{\infty\}$ and $i \geq 0$, we have

$$(f_k^i \circ S)(z) = (S^{k^i} \circ f_k^i)(z).$$

Proof From (2.1) and Lemma 2.2, it follows that $f_k \circ S = S^k \circ f_k$, hence (2.3) holds true for $k = 1$. Assume that (2.3) holds true for $0 \leq i \leq N-1$. Then we have

$$\begin{aligned} f_k^N \circ S &= f_k \circ (f_k^{N-1} \circ S) \\ &= f_k \circ (S^{k^{N-1}} \circ f_k^{N-1}) \text{ by induction argument} \\ &= (f_k \circ S^{k^{N-1}}) \circ f_k^{N-1} \\ &= (f_k \circ S) \circ S^{k^{N-1}-1} \circ f_k^{N-1} \\ &= S^k \circ (f_k \circ S^{k^{N-1}-1}) \circ f_k^{N-1} \text{ since (2.3) holds true for } i = 1. \end{aligned}$$

Continue this process inductively, we obtain $f_k^N \circ S = S^{k^N} \circ f_k^N$. This completes the proof. $\square$

Proof of Theorem We will prove this by induction on i and j . First, we let $j = 0$ and show that

$$f_k^i(0) = S^{k^i-1}(0)$$

for all $i \geq 0$. The equation (2.4) is trivially true for $i = 0$. Suppose that (2.4) holds true for $0 \leq i \leq N - 1$. Let $\alpha = k^{N-1} - 1$. Then we have

$$\begin{aligned}
f_k^N(0) &= f_k(f_k^{N-1}(0)) \\
&= f_k(S^\alpha(0)) \text{ by induction argument} \\
&= \frac{a(S^\alpha(0) - b)^k - b(S^\alpha(0) - a)^k}{(S^\alpha(0) - b)^k - (S^\alpha(0) - a)^k} \\
&= \frac{a \left(\frac{ba^{\alpha+1} - ab^{\alpha+1}}{a^{\alpha+1} - b^{\alpha+1}} - b \right)^k - b \left(\frac{ba^{\alpha+1} - ab^{\alpha+1}}{a^{\alpha+1} - b^{\alpha+1}} - a \right)^k}{\left(\frac{ba^{\alpha+1} - ab^{\alpha+1}}{a^{\alpha+1} - b^{\alpha+1}} - b \right)^k - \left(\frac{ba^{\alpha+1} - ab^{\alpha+1}}{a^{\alpha+1} - b^{\alpha+1}} - a \right)^k} \text{ by Lemma 2.2} \\
&= \frac{a(b^{\alpha+1}(b - a))^k - b(a^{\alpha+1}(b - a))^k}{(b^{\alpha+1}(b - a))^k - (a^{\alpha+1}(b - a))^k} \\
&= \frac{ab^{k(\alpha+1)} - ba^{k(\alpha+1)}}{b^{k(\alpha+1)} - a^{k(\alpha+1)}} \\
&= \frac{ab^{k^N} - ba^{k^N}}{b^{k^N} - a^{k^N}} = S^{k^N-1}(0).
\end{aligned}$$

Therefore, (2.4) holds true for all $i \geq 0$. Next, we suppose that (2.2) holds true for $0 \leq j \leq N - 1$ and for all $i \geq 0$. Then, for $i \geq 0$, we have

$$\begin{aligned}
f_k^i(S^N(0)) &= (f_k^i \circ S)(S^{N-1}(0)) \\
&= S^{k^i}(f_k^i(S^{N-1}(0))) \text{ by Lemma 2.3} \\
&= S^{k^i}(S^{Nk^i-1}(0)) \text{ by induction argument} \\
&= S^{(N+1)k^i-1}(0)
\end{aligned}$$

which implies that (2.2) holds true for $j = N$ and for all $i \geq 0$. This completes the proof. $\square$

Remark 2.1 When a and b are quadratic irrationals corresponding to $a = \frac{u+\sqrt{v}}{w}$ and $b = \frac{u-\sqrt{v}}{w}$ where u , v , and w are integers, $u < 0$ and $v > 0$, b is not the square of an integer, and $w \neq 0$, it was shown in [6] that f_k , $k \geq 2$, is the rational function of integers u , v , and w and that $\lim_{i \rightarrow \infty} f_k^i(0) = a$. This is important because we can generate a computable orbit that converges to a .

Remark 2.2 If we replace $S(z)$ with $T(z) = \frac{(a+b)z-ab}{z}$, the inverse of S , which is also a Successive approximation of P , then we have similar results as follows:

Theorem 2.4 *The order of convergence of f_k to the roots b starting from $z_0 = \frac{ab}{a+b}$ is equal to k . More precisely, we have*

$$f_k^i(T^j(\frac{ab}{a+b})) = T^{(j+1)k^i-1}(\frac{ab}{a+b})$$

for all $i, j \geq 0$, where $T(z) = S^{-1}(z) = \frac{(a+b)z-ab}{z}$ is a successive approximation of P .

Remark 2.3 *It is natural to ask if there are similar relations between f_k and successive approximations of polynomials of degree 3 or higher. The answer seems to be negative. One reason is because for polynomials of degree 3 (or higher), successive approximations of this polynomial have degree 2 (or higher). From which it follows that degree of $f_k \circ S$ is not equal to degree of $S^k \circ f_k$ hence Lemma 2.3 does not hold.*

3. AN EXAMPLE

Let $P(z) = (z-1)(z+2)$. Then $f_3(z) = \frac{z^3+6z+2}{3z^2+3z+3}$, $f_2(z) = \frac{z^2+2}{2z+1}$ and $S(z) = \frac{2}{z+1}$. The convergent sequences to the root $z = 1$ starting from initial point $z = 0$ produced by iteration of S is as follows:

$S : 0, 2, \frac{2}{3}, \frac{6}{5}, \frac{10}{11}, \frac{22}{21}, \frac{42}{43}, \frac{86}{85}, \frac{170}{171}, \frac{342}{341}, \frac{682}{683}, \frac{1366}{1365}, \frac{2730}{2731}, \frac{5462}{5461}, \frac{10922}{10923}, \frac{21846}{21845}, \frac{43690}{43691}, \frac{87382}{87381}, \frac{174762}{174763}, \frac{349526}{349525}, \frac{699050}{699051}, \frac{1398102}{1398101}, \frac{2796202}{2796203}, \frac{5592406}{5592405}, \frac{11184810}{11184811}, \frac{22369622}{22369621}, \frac{44739242}{44739243}, \frac{89478486}{89478485}, \frac{178956970}{178956971}, \frac{357913942}{357913941}, \frac{715827882}{715827883}, \frac{1431655766}{1431655765}, \dots$

For illustration, if we start from the initial points $z = 0$, $z = \frac{2}{3}$ and $z = \frac{6}{5}$ then the convergent subsequences (of sequences above) to the root $z = 1$ starting from these initial points produced by iterations of f_2 and f_3 are as follows:

$f_2 : 0, 2, \frac{6}{5}, \frac{86}{85}, \frac{21846}{21845}, \frac{1431655766}{1431655765}, \dots$
 $f_2 : \frac{2}{3}, \frac{22}{21}, \frac{1366}{1365}, \frac{5592406}{5592405}, \frac{93824992236886}{93824992236885}, \dots$
 $f_3 : 0, \frac{2}{3}, \frac{170}{171}, \frac{44739242}{44739243}, \frac{805950546409752783137450}{805950546409752783137451}, \dots$
 $f_3 : \frac{6}{5}, \frac{1366}{1365}, \frac{22906492246}{22906492245}, \dots$

As an example, we have the following correspondences between N and S :

$f_2^1(0) = 2 = S^1(0)$, $f_2^2(0) = \frac{6}{5} = S^3(0)$, $f_2^3(0) = \frac{86}{85} = S^7(0)$, $f_2^4(0) = \frac{21846}{21845} = S^{15}(0)$, ... and $f_2^0(\frac{2}{3}) = \frac{2}{3} = S^2(0)$, $f_2^1(\frac{2}{3}) = \frac{22}{21} = S^5(0)$, $f_2^2(\frac{2}{3}) = \frac{1366}{1365} = S^{11}(0)$, $f_2^3(\frac{2}{3}) = \frac{5592406}{5592405} = S^{24}(0)$, $f_2^4(\frac{2}{3}) = \frac{93824992236886}{93824992236885} = S^{47}(0)$, ... As we have seen, if we want a fast convergence to roots of P , then we select an initial point in the form $S^j(0)$ (or $T^j(\frac{ab}{a+b})$) and then apply either f_2 or f_3 successively.

4. ACKNOWLEDGEMENT

The first author is supported by the Thailand Research Fund and a research grant from the Faculty of Science, Chiangmai University during the preparation of this paper. He would like to thank Prof. Dr. Yongwimon Lenbury, Mahidol University, Thailand for many valuable suggestions.

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Ratioanl Solutions of Certain Functional Equation

Piyapong Niamsup and Julian Palmore

June 30, 2000

Abstract

In this paper we study a functional equation of the form $f \circ S = S^k \circ f$ where $S(z) = -\frac{ab}{z-(a+b)}$ is a Successive approximation of a quadratic polynomial with roots a and b such that $0 < |a| < |b|$ and f is a rational function of degree k of the form $f(z) = \frac{a_k z^k + a_{k-1} z^{k-1} + \dots + a_1 z + a_0}{b_k z^k + b_{k-1} z^{k-1} + \dots + b_1 z + b_0}$, $a_i, b_j \in C$, $(a_0, b_0) \neq (0, 0)$.

1 Introduction

Halley's method, Newton's method of a given function $P(z)$ are defined respectively as follows

$$H(z) = z - \frac{P(z)}{P'(z) - \frac{P(z)P''(z)}{2P'(z)}},$$
$$N(z) = z - \frac{P(z)}{P'(z)}.$$

A successive approximation of $P(z)$, $S(z)$ can be obtained by setting $P(z) = 0$ then write this equation as $z = S(z)$. For example, if $P(z)$ is a quadratic polynomial with roots a and b such that $0 < |a| < |b|$ In general, Halley's method, Newton's method and successive approximation are iterative methods which can be used to locate roots of functions where the order of convergence of these methods are three, two and one, respectively. In [5] and [6], the roles of the Schwarzian derivative of Halley's method, Newton's

method and the composition between these two methods were studied in controlling the order of convergence of these methods. In [2] and [4], the following relations between Halley's method, Newton's method and successive approximation for $P(z) = (z - a)(z - b)$, $a, b \in \mathbb{C}$ such that $0 < |a| < |b|$ were given. The following are some of these relations:

1. $H \circ S = S^3 \circ H$,
2. $N \circ S = S^2 \circ N$,
3. $(H \circ N) \circ S = S^6 \circ (H \circ N)$,
4. $H^i(S^j(0)) = S^{(j+1)3^i-1}(0)$, $i, j \geq 0$,
5. $N^i(S^j(0)) = S^{(j+1)2^i-1}(0)$, $i, j \geq 0$,
6. $(H \circ N)^i(S^j(0)) = S^{(j+1)6^i-1}(0)$, $i, j \geq 0$.

In [7] and [8], a rational function of the following form was studied:

$$f_k(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}. \quad (1)$$

It was shown that when a and b are quadratic irrational numbers of the form

$$a = \frac{u + v^{1/2}}{w} \text{ and } b = \frac{u - v^{1/2}}{w}$$

where u , v , and w are integers such that $v > 0$, v is not the square of an integer and $w \neq 0$, then $f_k(z)$ is a rational function of integers u , v , and w . This is important when we study a computable orbit converging to a under the iteration of f_k . It was also shown that

$$f_k^{(i)}(0) = S^{k^i-1}(0)$$

where $k \geq 2$ and $i \geq 1$. That is, the order of convergence of f_k to a is equal to k . Note that f_2 is the usual Newton's method of P and f_3 is the Halley's method of P . Motivating by these results, we propose to study a functional equation

$$f \circ S = S^k \circ f \quad (2)$$

where $k \geq 2$, S is defined as above and f is a rational function of degree k of the form

$$f(z) = \frac{a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0}{b_k z^k + b_{k-1} z^{k-1} + \cdots + b_1 z + b_0}, \quad (3)$$

where $a_i, b_j \in \mathbb{C}$, $(a_0, b_0) \neq (0, 0)$. In the next section, we will find all rational solutions f of (2).

2 Main Results

We begin by showing that (2) has a rational solution.

Theorem 1 *The functional equation (2) has a solution, namely*

$$f_k(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}.$$

Proof For $k \geq 2$, the function defined above is conjugate to the map $w \mapsto w^k$ by the Möbius transformation that sends $w = 0$ to a and $w = \infty$ to b , namely

$$M(w) = \frac{bw - a}{w - 1}.$$

The inverse, M^{-1} , of M is given by

$$M^{-1}(z) = \frac{-z + a}{-z + b}.$$

We have the following commutative diagram

$$\begin{array}{ccc} z & \longrightarrow & f_k(z) \\ M \uparrow & & \uparrow M \\ w & \longrightarrow & w^k \end{array}$$

Therefore, for $k \geq 2$, we have $f_k(z) = M([M^{-1}(z)]^k)$. The map S is conjugate to the map $w \mapsto \frac{a}{b}w$ by the same Möbius transformation M as above. Therefore, we obtain

$$\begin{aligned} S^k(z) &= M\left(\left(\frac{a}{b}\right)^k M^{-1}(z)\right) \\ &= \frac{ba^k(-z+a) - ab^k(-z+b)}{a^k(-z+a) - b^k(-z+b)}. \end{aligned}$$

Hence,

$$\begin{aligned} S^k(f_k(z)) &= \frac{ba^k \left(\frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k} + a \right) - ab^k \left(\frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k} + b \right)}{a^k \left(a - \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k} \right) - b^k \left(b - \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k} \right)} \\ &= \frac{ab^k(z-b)^k - ba^k(z-a)^k}{b^k(z-b)^k - a^k(z-a)^k} \\ &= f_k(S(z)). \end{aligned}$$

That is f_k is a solution of (1) as desired. $\square$

From this theorem, we obtain the following result which is more general than the result in [4].

Corollary 2 For $k \geq 2$, we have $f_k^{(i)}(S^j(-\frac{ab}{z_k-(a+b)})) = S^{(j+1)k^i-1}(-\frac{ab}{z_k-(a+b)})$ for $i, j \geq 0$, where z_k is a fixed point of f_k . In particular, for $b_k = 0$, f_k has a fixed point at ∞ and hence $f_k^{(i)}(S^j(0)) = S^{(j+1)k^i-1}(0)$ for $i, j \geq 0$.

Proof. See [2] for the detail of the proof where mathematical induction was used to prove this corollary. See [4] for the detail of the proof where the concept of topological conjugacy was used to prove this corollary. $\square$

We now consider all rational solutions of (2) when $k = 2$. We obtain the following result.

Theorem 3 Let f_2 be a solution of (2), then f_2 is of the following form

(a) If $a_2 \neq 0$, then

$$f_2(z) = \frac{z^2 + (-2abb_2)z + (-ab + ab(a+b)b_2)}{b_2z^2 + (2 - 2(a+b)b_2)z + (-abb_2 - (a+b) + (a+b)^2b_2)}$$

where b_2 is any complex number. Moreover, if $b_2 = 0$ then f_2 is the Newton's method for P and if b_2 is a nonzero complex number, then we obtain $f_2(z) = T_2(N(z))$ where $T_2(z) = \frac{z-abb_2}{b_2z+(1-(a+b)b_2)}$.

(b) If $a_2 = 0$, and $a_1 \neq 0$, then

$$f_2(z) = \frac{z - \frac{a+b}{2}}{\left(-\frac{1}{2ab}\right)z^2 + \left(\frac{a+b}{ab}\right)z - \left(\frac{a^2+ab+b^2}{2ab}\right)}$$

Note that $f_2(z) = (S^{-1} \circ N \circ S)(z)$.

(c) If $a_2 = a_1 = 0$ and $a_0 \neq 0$, then there are no rational solutions for (2) of this form.

Conversely, if T is any mapping such that $T \circ S = S \circ T$, then $N \circ T$ and $T \circ N$ are solutions of (2).

Proof. We first assume that $a_2 \neq 0$. In this case we can assume that $f_2(z) = \frac{z^2+a_1z+a_0}{b_2z^2+b_1z+b_0}$. From $S(z) = -\frac{ab}{z-(a+b)}$ we substitute f_2 and S into (2) and solve for the coefficients a_1 , a_0 , b_2 , b_1 , and b_0 we obtain

$$\begin{aligned} a_1 &= -2abb_2, \\ a_0 &= -ab + ab(a+b)b_2, \\ b_1 &= 2 - 2(a+b)b_2, \text{ and} \\ b_0 &= -abb_2 - (a+b) + (a+b)^2b_2 \end{aligned}$$

where b_2 is any complex number. That is,

$$f_2(z) = \frac{z^2 + (-2abb_2)z + (-ab + ab(a+b)b_2)}{b_2z^2 + (2 - 2(a+b)b_2)z + (-abb_2 - (a+b) + (a+b)^2b_2)}$$

where $b_2 \in C$ is the set of all solutions of (2) for $k = 2$ when $a_2 \neq 0$. Now assume that $a_2 = 0$ and $a_1 \neq 0$. In this case we can assume that $f_2(z) = \frac{z+a_0}{b_2z^2+b_1z+b_0}$. Substitute f_2 and S and solve for a_0 , b_0 , b_1 , and b_2 we obtain

$$f_2(z) = \frac{z - \frac{a+b}{2}}{\left(-\frac{1}{2ab}\right)z^2 + \left(\frac{a+b}{ab}\right)z - \left(\frac{a^2+ab+b^2}{2ab}\right)}.$$

For the cases, $a_2 = a_1 = 0$ and $a_0 \neq 0$, we can show similarly to previous two cases that (2) doesn't have rational solutions in this case. $\square$

For general positive integer k , we have the following result

Theorem 4 Let f_k be a rational solution of (2) of the form

(a) If $a_k \neq 0$, then

$$f_k = T_k \circ f_{0,k}$$

where $f_{0,k}(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}$, $T_k(z) = \frac{z - abb_k}{b_kz + (1 - (a+b)b_k)}$ and $b_k \in C$.

(b) If $a_k = 0$, and $a_{k-1} \neq 0$, then there is only one solution in this form for (2) and we can explicitly find such the solution.

(c) If $a_k = a_{k-1} = 0$, then there are no nonzero rational solutions for (2) of this form.

Conversely, if T is any mapping such that $T \circ S = S \circ T$, then $f_0 \circ T$ and $T \circ f_0$ are solutions of (1).

Proof We first assume that $a_k \neq 0$. Without loss of generality, we assume that

$$f(z) = \frac{z^k + a_{k-1}z^{k-1} + \cdots + a_1z + a_0}{b_kz^k + b_{k-1}z^{k-1} + \cdots + b_1z + b_0}.$$

From Theorem 1, $S^k(z) = \frac{ba^k(-z+a) - ab^k(-z+b)}{a^k(-z+a) - b^k(-z+b)}$, and we obtain

$$S^k(z) = \frac{abT_{k-2}z + abS_{k-1}}{T_{k-1}z + S_k}$$

where $S_j = \sum_{i=0}^j a^i b^{j-i}$, $T_j = -S_j$ where $j \geq 1$. It follows that

$$S^k(f(z)) = \frac{abT_{k-2}(z^k + a_{k-1}z^{k-1} + \cdots + a_1z + a_0) + abS_{k-1}(b_kz^k + b_{k-1}z^{k-1} + \cdots + b_1z + b_0)}{T_{k-1}(z^k + a_{k-1}z^{k-1} + \cdots + a_1z + a_0) + S_k(b_kz^k + b_{k-1}z^{k-1} + \cdots + b_1z + b_0)}.$$

Now we have

$$f_k(S(z)) = \frac{\sum_{i=0}^k \left\{ \sum_{j=i}^k (-1)^{j-i} \binom{j}{j-1} (-ab)^{k-j} (a+b)^{j-i} a_{k-j} \right\} z^i}{\sum_{i=0}^k \left\{ \sum_{j=i}^k (-1)^{j-i} \binom{j}{j-1} (-ab)^{k-j} (a+b)^{j-i} b_{k-j} \right\} z^i}$$

where $a_k = 1$. By comparing first the coefficients of z^i in numerators of $S^k(f(z))$ and $f_k(S(z))$ and then the coefficients of z^i in denominators of $S^k(f(z))$ and $f_k(S(z))$ we have a system of equations which can be solved explicitly to obtain the results of the Theorem in case (a). Case (b) and (c) can be proved similarly to case (a). $\square$

Remark 1. When $k = 3$, $f_{0,3}(z)$ is the Halley's method for P .

Remark 2. We have proved Theorem 4 by solving directly a linear system of equations. It would be more interesting if we could prove Theorem 4 more analytically.

Remark 3. If $P(z)$ is a polynomials of degree 3 or more with distinct roots, then any successive approximation of $P(z)$ would have degree 2 or more. From which it follows that (1) does not hold (since degree of S is not equal to degree of S^k).

Remark 4. From [7], [8] and Theorem 4, f_k is a rational function with integer coefficients if and only if $b_k \in \mathbb{Z}$, $a = \frac{s(u-\sqrt{v})}{w}$ and $b = \frac{s(u+\sqrt{v})}{w}$ where $u, v, w, s \in \mathbb{Z} - \{0\}$, $u > 0$, $v > 0$, $v^2 \notin \mathbb{Z}^+$.

Remark 5. Consider $R(z) = \frac{(a+b)z-ab}{z} = S^{-1}(z)$ which is also a successive approximation of P . By replacing S with R in (2), we obtain similar results with the same argument of proof where we leave the details to interested readers.

Remark 6. If T is a Möbius transformation which commutes with S , then $T(z) = \frac{Az-abC}{Cz+(A-(a+b)C)}$, where A and C are complex numbers. When $A = 1$ and $C = b_k$ we obtain $T = T_k$.

3 Acknowledgement

The first author is supported by the Thailand Research Fund and a research grant from the Faculty of Science, Chiangmai University during the preparation of this paper. He would like to thank Prof.Dr.Yongwimon Lenbury, Mahidol University, Thailand for many valuable suggestions.

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