of f, respectively. If f has a simple zero ζ such that $f'(\zeta) \neq 0$, then the values of the first five derivatives of $N \circ H$ evaluated at ζ are zero, and $(N \circ H)^{(6)}(\zeta) = 10\left(\frac{f''(\zeta)}{f'(\zeta)}\right)(H'''(\zeta))^2 = -\frac{15}{2}\left(\frac{f''(\zeta)}{f'(\zeta)}\right)^3 \cdot S(\zeta)$. Therefore, $N \circ H$ has order of convergence to ζ equal to 6 or greater and the order of convergence is controlled by value of the second derivative and value of Schwarzian derivative of f evaluated at ζ.

Proof The proof is similar to that of Theorem 3.1 and will be omitted. □

Remark It is natural to further consider the roles of Schwarzian derivative and derivatives of f in controlling order of convergence of $H^i \circ N^j$ where $i, j \geq 0$.

4 Example

Let $f(x) = (x-1)(x+2) = x^2 + x - 2$. In this example, we have $N(x) = \frac{x^2+2}{2x+1}$, $H(x) = \frac{x^3+6x+2}{3x^2+3x+3}$, and $S(x) = -3\left(\frac{2}{2x+1}\right)^2$. By computation, we get $(N \circ H)(x) = (H \circ N)(x) = \frac{x^6+30x^4+40x^3+90x^2+60x+22}{(2x+1)(3x^4+6x^3+27x^2+24x+21)}$. Since $S(1) = S(-2) = -\frac{4}{3}$, $\frac{f''(1)}{f'(1)} = \frac{2}{3}$, and $\frac{f''(-2)}{f'(-2)} = -\frac{2}{3}$, we get $(N \circ H)^{(6)}(1) = (H \circ N)^{(6)}(1) = -\frac{15}{2} \cdot \left(\frac{2}{3}\right)^3 \cdot \left(-\frac{4}{3}\right) = \frac{80}{27}$ and $(N \circ H)^{(6)}(-2) = (H \circ N)^{(6)}(-2) = -\frac{15}{2} \cdot \left(-\frac{2}{3}\right)^3 \cdot \left(-\frac{4}{3}\right) = -\frac{80}{27}$. Hence, the order of convergence of $N \circ H$ and of $H \circ N$ to the zeros of f is 6.

5 Acknowledgement

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A RELATION BETWEEN SUCCESSIVE APPROXIMATION AND CERTAIN ROOTS FINDING METHODS

PIYAPONG NIAMSUP AND JULIAN PALMORE

ABSTRACT. In this paper, we give some interesting relation between successive approximation and certain roots finding method for quadratic polynomials.

1. INTRODUCTION

For a function $f(z)$, the Halley's method and Newton's method of f are defined as follows:

$$\begin{aligned} H(z) &= z - \frac{f(z)}{f'(z) - \frac{f(z)f''(z)}{2f'(z)}}, \text{ and} \\ N(z) &= z - \frac{f(z)}{f'(z)} \end{aligned}$$

respectively. A successive approximation of $f(z)$ can be obtained by letting $f(z) = 0$ and write this equation in a form $z = S(z)$ where $S(z)$ will be called a successive approximation of f. For example, if $f(z) = (z-a)(z-b)$ where , $a, b \in \mathbb{C} - \{0\}$, $a \neq b$, then we obtain a successive approximation $S(z) = \frac{-ab}{z-(a+b)}$. Halley's method, Newton's method and successive approximation are iterative method which are used to find certain roots of f in which we choose an initial guest x_0 and then iterate these method to x_0 to obtain a sequence $\{x_n\}$ which usually converge to certain roots of f (the choice of initial guest x_0 is essential for the convergence of the sequence, see [1], [9] and references cited therein). In general, the orders of convergence of Halley's method, Newton's method and successive approximation are three, two and one respectively. In [3], an interesting relation between Newton's method and successive approximation was obtained as follows:

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Theorem 1.1 *Let* $\zeta^+ = \frac{a+\sqrt{b}}{c}$, $\zeta^- = \frac{a-\sqrt{b}}{c}$ *be quadratic irrationals where* a, b, *and* c *are integers satisfying* $a < 0$, $b > 0$, b *is not the square of an integer, and* $c \neq 0$. *Let* $P(z) = (z-\zeta^+)(z-\zeta^-)$, $g(z)$ *be the Newton's method of* P *and* $h(z) = \frac{-\zeta^+\zeta^-}{z-(\zeta^+ + \zeta^-)}$ *be a sucessive approximation of* P. *Then for all* $n \geq 0$, *we have*

$$g^n(0) = h^{2^n-1}(0)$$

where f^i *denotes the* i^{th} *iterate of function* f.

Theorem 1.1 was proved using the concept of topological conjugacy where we say f and g are topological conjugacy if there exists a Möbius transformation h such that $f \circ h = h \circ g$. If f and g are topological conjugate, then they have the same dynamical behavior such as they have the same type of periodic points, see [1] for more details. In this paper, we will give some interesting relation between successive approximations and certain root findinig methods for quadratic polynomials which extend Theorem 1.1 to general root findings method of order of convergence equal to k, $k \geq 2$.

2. Main Results

Let $P(z) = (z-a)(z-b)$ where $a, b \in \mathbb{C}$ satisfying $0 < |a| < |b|$ be a quadratic polynomial. We first introduce the following rational function

$$f_k(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}.$$

We can see that f_k is topological conjugate to the map $w \mapsto w^k$ by the Möbius transformation $M(z)$ that sends $w = 0$ to a and $w = \infty$ to b, where $M(z) = \frac{bz-a}{z-1}$ and $M^{-1}(z) = \frac{-z+a}{-z+b}$. In other words, we have

$$f_k(z) = M([M^{-1}(z)]^k).$$

Note that, when $k = 2$, f_k is the Newton's method of P and when $k = 3$, f_k is the Halley's method of P. In this section, we will prove the following main result:

Theorem 2.1 *The order of convergence of* f_k *to the roots* a *starting from* $z_0 = 0$ *is equal to* k. *More precisely, we have*

$$f_k^i(S^j(0)) = S^{(j+1)k^i-1}(0)$$

for all $i, j \geq 0$, *where* $S(z) = \frac{-ab}{z-(a+b)}$ *is a successive approximation of* P.

In order to prove this theorem, we need the following lemma:

Lemma 2.2 *For the successive approximation* S, *the point* b *is the repulsive fixed point of* S *and* a *is the global attractor of* S, *that is*

$\lim_{k\to\infty} S^k(z) = a$ *for all* $z \in \mathbb{C} - \{b\}$*.Moreover,* S *is topological conjugate to the map* $w \mapsto \frac{a}{b}w$ *by the Möbius transformation* $M(z) = \frac{bz-a}{z-1}$*, that is*

$$\begin{aligned} S^k(z) &= M\left(\left(\frac{a}{b}\right)^k M^{-1}(z)\right) \\ &= \frac{ba^k(-z+a) - ab^k(-z+b)}{a^k(-z+a) - b^k(-z+b)}. \end{aligned}$$

Proof It is routine to see that b is the repulsive fixed point of S, a is the attractive fixed point of S and that

$$S(z) = M\left(\left(\frac{a}{b}\right) M^{-1}(z)\right)$$

that is S is topological conjugate to the map $w \mapsto \frac{a}{b}w$ by the Möbius transformation $M(z) = \frac{bz-a}{z-1}$. Since $0 < |a| < |b|$, it follows that $\lim_{i\to\infty} w^i(z) = \lim_{i\to\infty} \left(\frac{a}{b}\right)^i z = 0$ for all $z \in \mathbb{C}$. Hence, 0 is the global attractor of the map $w \mapsto \frac{a}{b}w$. Since S is topological conjugate to the map $w \mapsto \frac{a}{b}w$ and $M(0) = a$, it follows that a is the global attractor of S. □

Lemma 2.3 *For all* $z \in \mathbb{C} \cup \{\infty\}$ *and* $i \geq 0$*, we have*

$$(f_k^i \circ S)(z) = (S^{k^i} \circ f_k^i)(z).$$

Proof From (2.1) and Lemma 2.2, it follows that $f_k \circ S = S^k \circ f_k$, hence (2.3) holds true for $k = 1$. Assume that (2.3) holds true for $0 \leq i \leq N-1$. Then we have

$$\begin{aligned} f_k^N \circ S &= f_k \circ (f_k^{N-1} \circ S) \\ &= f_k \circ (S^{k^{N-1}} \circ f_k^{N-1}) \text{ by induction argument} \\ &= (f_k \circ S^{k^{N-1}}) \circ f_k^{N-1} \\ &= (f_k \circ S) \circ S^{k^{N-1}-1} \circ f_k^{N-1} \\ &= S^k \circ (f_k \circ S^{k^{N-1}-1}) \circ f_k^{N-1} \text{ since (2.3) holds true for } i = 1. \end{aligned}$$

Continue this process inductively, we obtain $f_k^N \circ S = S^{k^N} \circ f_k^N$. This completes the proof. □

Proof of Theorem We will prove this by induction on i and j. First, we let $j = 0$ ad show that

$$f_k^i(0) = S^{k^i-1}(0)$$

for all $i \geq 0$. The equation (2.4) is trivially true for $i = 0$. Suppose that (2.4) holds true for $0 \leq i \leq N-1$. Let $\alpha = k^{N-1} - 1$. Then we have

$$\begin{aligned}
f_k^N(0) &= f_k(f_k^{N-1}(0)) \\
&= f_k(S^\alpha(0)) \text{ by induction argument} \\
&= \frac{a(S^\alpha(0)-b)^k - b(S^\alpha(0)-a)^k}{(S^\alpha(0)-b)^k - (S^\alpha(0)-a)^k} \\
&= \frac{a\left(\frac{ba^{\alpha+1}-ab^{\alpha+1}}{a^{\alpha+1}-b^{\alpha+1}} - b\right)^k - b\left(\frac{ba^{\alpha+1}-ab^{\alpha+1}}{a^{\alpha+1}-b^{\alpha+1}} - a\right)^k}{\left(\frac{ba^{\alpha+1}-ab^{\alpha+1}}{a^{\alpha+1}-b^{\alpha+1}} - b\right)^k - \left(\frac{ba^{\alpha+1}-ab^{\alpha+1}}{a^{\alpha+1}-b^{\alpha+1}} - a\right)^k} \text{ by Lemma 2.2} \\
&= \frac{a\left(b^{\alpha+1}(b-a)\right)^k - b\left(a^{\alpha+1}(b-a)\right)^k}{\left(b^{\alpha+1}(b-a)\right)^k - \left(a^{\alpha+1}(b-a)\right)^k} \\
&= \frac{ab^{k(\alpha+1)} - ba^{k(\alpha+1)}}{b^{k(\alpha+1)} - a^{k(\alpha+1)}} \\
&= \frac{ab^{k^N} - ba^{k^N}}{b^{k^N} - a^{k^N}} = S^{k^{N}-1}(0).
\end{aligned}$$

Therefore, (2.4) holds true for all $i \geq 0$. Next, we suppose that (2.2) holds true for $0 \leq j \leq N-1$ and for all $i \geq 0$. Then, for $i \geq 0$, we have

$$\begin{aligned}
f_k^i(S^N(0)) &= (f_k^i \circ S)(S^{N-1}(0)) \\
&= S^{k^i}(f_k^i(S^{N-1}(0))) \text{ by Lemma 2.3} \\
&= S^{k^i}(S^{Nk^i-1}(0)) \text{ by induction argument} \\
&= S^{(N+1)k^i-1}(0)
\end{aligned}$$

which implies that (2.2) holds true for $j = N$ and for all $i \geq 0$. This completes the proof. □

Remark 2.1 *When* a and b are quadratic irrationals corresponding to $a = \frac{u+\sqrt{v}}{w}$ and $b = \frac{u-\sqrt{v}}{w}$ where u, v, and w are integers, $u < 0$ and $v > 0$, b is not the square of an integer, and $w \neq 0$, it was shown in [6] that f_k, $k \geq 2$, is the rational function of integers u, v, and w and that $\lim_{i\to\infty} f_k^i(0) = a$. This is important because we can generate a computable orbit that converges to a.

Remark 2.2 *If we replace* $S(z)$ *with* $T(z) = \frac{(a+b)z-ab}{z}$, *the inverse of* S, *which is also a Successive approximation of* P, *then we have similar results as follows:*

Theorem 2.4 *The order of convergence of f_k to the roots b starting from $z_0 = \frac{ab}{a+b}$ is equal to k. More precisely, we have*

$$f_k^i(T^j(\frac{ab}{a+b})) = T^{(j+1)k^i-1}(\frac{ab}{a+b})$$

for all $i, j \geq 0$, where $T(z) = S^{-1}(z) = \frac{(a+b)z-ab}{z}$ is a successive approximation of P.

Remark 2.3 *It is natural to ask if there are similar relations between f_k and successive approximations of polynomials of degree 3 or higher. The answer seems to be negative. One reason is because for polynomials of degree 3 (or higher), successive approximations of this polynomial have degree 2 (or higher). From which it follows that degree of $f_k \circ S$ is not equal to degree of $S^k \circ f_k$ hence Lemma 2.3 does not hold.*

3. An Example

Let $P(z) = (z-1)(z+2)$. Then $f_3(z) = \frac{z^3+6z+2}{3z^2+3z+3}$, $f_2(z) = \frac{z^2+2}{2z+1}$ and $S(z) = \frac{2}{z+1}$. The convergent sequences to the root $z = 1$ starting from initial point $z = 0$ produced by iteration of S is as follows:

$S: 0, 2, \frac{2}{3}, \frac{6}{5}, \frac{10}{11}, \frac{22}{21}, \frac{42}{43}, \frac{86}{85}, \frac{170}{171}, \frac{342}{341}, \frac{682}{683}, \frac{1366}{1365}, \frac{2730}{2731}, \frac{5462}{5461}, \frac{10922}{10923}, \frac{21846}{21845}, \frac{43690}{43691}, \frac{87382}{87381}, \frac{174762}{174763}, \frac{349526}{349525}, \frac{699050}{699051}, \frac{1398102}{1398101}, \frac{2796202}{2796203}, \frac{5592406}{5592405}, \frac{11184810}{11184811}, \frac{22369622}{22369621}, \frac{44739242}{44739243}, \frac{89478486}{89478485}, \frac{178956970}{178956971}, \frac{357913942}{357913941}, \frac{715827882}{715827883}, \frac{1431655766}{1431655765}, \ldots$

For illustration, if we start from the initial points $z = 0$, $z = \frac{2}{3}$ and $z = \frac{6}{5}$ then the convergent subsequences (of sequences above) to the root $z = 1$ starting from these initial points produced by iterations of f_2 and f_3 are as follows:

$f_2: 0, 2, \frac{6}{5}, \frac{86}{85}, \frac{21846}{21845}, \frac{1431655766}{1431655765}, \ldots$

$f_2: \frac{2}{3}, \frac{22}{21}, \frac{1366}{1365}, \frac{5592406}{5592405}, \frac{93824992236886}{93824992236885}, \ldots$

$f_3: 0, \frac{2}{3}, \frac{170}{171}, \frac{44739242}{44739243}, \frac{805950546409752783137450}{805950546409752783137451}, \ldots$

$f_3: \frac{6}{5}, \frac{1366}{1365}, \frac{22906492246}{22906492245}, \ldots$

As an example, we have the following correspondences between N and S:

$f_2^1(0) = 2 = S^1(0), f_2^2(0) = \frac{6}{5} = S^3(0), f_2^3(0) = \frac{86}{85} = S^7(0), f_2^4(0) = \frac{21846}{21845} = S^{15}(0)$, ... and $f_2^0\left(\frac{2}{3}\right) = \frac{2}{3} = S^2(0), f_2^1\left(\frac{2}{3}\right) = \frac{22}{21} = S^5(0), f_2^2\left(\frac{2}{3}\right) = \frac{1366}{1365} = S^{11}(0), f_2^3\left(\frac{2}{3}\right) = \frac{5592406}{5592405} = S^{24}(0)$, $f_2^4\left(\frac{2}{3}\right) = \frac{93824992236886}{93824992236885} = S^{47}(0)$, ... As we have seen, if we want a fast convergence to roots of P, then we select an initial point in the form $S^j(0)$ (or $T^j(\frac{ab}{a+b})$) and then apply either f_2 or f_3 successively.

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Ratioanl Solutions of Certain Functional Equation

Piyapong Niamsup and Julian Palmore

June 30, 2000

Abstract

In this paper we study a functional equation of the form $f \circ S = S^k \circ f$ where $S(z) = -\frac{ab}{z-(a+b)}$ is a Successive approximation of a quadratic polynomial with roots a and b such that $0 < |a| < |b|$ and f is a rational function of degree k of the form $f(z) = \frac{a_k z^k + a_{k-1} z^{z-1} + \cdots + a_1 z + a_0}{b_k z^k + b_{k-1} z^{k-1} + \cdots + b_1 z + b_0}$, $a_i, b_j \in C$, $(a_0, b_0) \neq (0, 0)$.

1 Introduction

Halley's method, Newton's method of a given function $P(z)$ are defined respectively as follows

$$\begin{aligned} H(z) &= z - \frac{P(z)}{P'(z) - \frac{P(z)P''(z)}{2P'(z)}}, \\ N(z) &= z - \frac{P(z)}{P'(z)}. \end{aligned}$$

A successive approximation of $P(z)$, $S(z)$ can be obtained by setting $P(z) = 0$ then write this equation as $z = S(z)$. For example, if $P(z)$ is a quadratic polynomial with roots a and b such that $0 < |a| < |b|$ In general, Halley's method, Newton's method and successive approximation are iterative methods which can be used to locate roots of functions where the order of convergence of these methods are three, two and one, respectively. In [5] and [6], the roles of the Schwarzian derivative of Halley's method, Newton's method and the composition between these two methods were studied in controlling the order of convergence of these methods. In [2] and [4], the following relations between Halley's method, Newton's method and successive approximation for $P(z) = (z - a)(z - b)$, $a, b \in C$ such that $0 < |a| < |b|$ were given. The following are some of these relations:

1. $H \circ S = S^3 \circ H$,
2. $N \circ S = S^2 \circ N$,
3. $(H \circ N) \circ S = S^6 \circ (H \circ N)$,
4. $H^i(S^j(0)) = S^{(j+1)3^i-1}(0)$, $i, j \geq 0$,
5. $N^i(S^j(0)) = S^{(j+1)2^i-1}(0)$, $i, j \geq 0$,
6. $(H \circ N)^i(S^j(0)) = S^{(j+1)6^i-1}(0)$, $i, j \geq 0$.

In [7] and [8], a rational function of the following form was studied:

$$f_k(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}. \tag{1}$$

It was shown that when a and b are quadratic irrational numbers of the form

$$a = \frac{u + v^{1/2}}{w} \text{ and } b = \frac{u - v^{1/2}}{w}$$

where u, v, and w are integers such that $v > 0$, v is not the square of an integer and $w \neq 0$, then $f_k(z)$ is a rational function of integers u, v, and w. This is important when we study a computable orbit converging to a under the iteration of f_k. It was also shown that

$$f_k^{(i)}(0) = S^{k^i-1}(0)$$

where $k \geq 2$ and $i \geq 1$. That is, the order of convergence of f_k to a is equal to k. Note that f_2 is the usual Newton's method of P and f_3 is the Halley's method of P. Motivating by these results, we propose to study a functional equation

$$f \circ S = S^k \circ f \tag{2}$$

where $k \geq 2$, S is defined as above and f is a rational function of degree k of the form

$$f(z) = \frac{a_k z^k + a_{k-1} z^{x-1} + \cdots + a_1 z + a_0}{b_k z^k + b_{k-1} z^{k-1} + \cdots + b_1 z + b_0}, \tag{3}$$

where a_i, $b_j \in C$, $(a_0, b_0) \neq (0, 0)$. In the next section, we will find all rational solutions f of (2).

2 Main Results

We begin by showing that (2) has a rational solution.

Theorem 1 *The functional equation (2) has a solution, namely*

$$f_k(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}.$$

Proof For $k \geq 2$, the function defined above is conjugate to the map $w \longmapsto w^k$ by the Möbius transformation that sends $w = 0$ to a and $w = \infty$ to b, namely

$$M(w) = \frac{bw - a}{w - 1}.$$

The inverse, M^{-1}, of M is given by

$$M^{-1}(z) = \frac{-z + a}{-z + b}.$$

We have the following commutative diagram

$$\begin{array}{ccc} z & \longrightarrow & f_k(z) \\ M \uparrow & & \uparrow M \\ w & \longrightarrow & w^k \end{array}$$

Therefore, for $k \geq 2$, we have $f_k(z) = M([M^{-1}(z)])^k$. The map S is conjugate to the map $w \longmapsto \frac{a}{b}w$ by the same Möbius transformation M as above. Therefore, we obtain

$$\begin{aligned} S^k(z) &= M(\left(\frac{a}{b}\right)^k M^{-1}(z)) \\ &= \frac{ba^k(-z+a) - ab^k(-z+b)}{a^k(-z+a) - b^k(-z+b)}. \end{aligned}$$

Hence,

$$\begin{aligned} S^k(f_k(z)) &= \frac{ba^k\left(\frac{a(z-b)^k - b(z-a)^k}{(z-b)^k-(z-a)^k} + a\right) - ab^k\left(\frac{a(z-b)^k - b(z-a)^k}{(z-b)^k-(z-a)^k} + b\right)}{a^k\left(a - \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k-(z-a)^k}\right) - b^k\left(b - \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k-(z-a)^k}\right)} \\ &= \frac{ab^k(z-b)^k - ba^k(z-a)^k}{b^k(z-b)^k - a^k(z-a)^k} \\ &= f_k(S(z)). \end{aligned}$$

That is f_k is a solution of (1) as desired. $\square$

From this theorem, we obtain the following result which is more general than the result in [4].

Corollary 2 *For $k \geq 2$, we have $f_k^{(i)}(S^j(-\frac{ab}{z_k-(a+b)})) = S^{(j+1)k^i-1}(-\frac{ab}{z_k-(a+b)})$ for $i, j \geq 0$, where z_k is a fixed point of f_k. In particular, for $b_k = 0$, f_k has a fixed point at ∞ and hence $f_k^{(i)}(S^j(0)) = S^{(j+1)k^i-1}(0)$ for $i, j \geq 0$.*

Proof. See [2] for the detail of the proof where mathematical induction was used to prove this corollary. See [4] for the detail of the proof where the concept of topological conjugacy was used to prove this corollary. $\square$

We now consider all rational solutions of (2) when $k = 2$. We obtain the following result.

Theorem 3 *Let f_2 be a solution of (2), then f_2 is of the following form*

(a) If $a_2 \neq 0$, then

$$f_2(z) = \frac{z^2 + (-2abb_2)z + (-ab + ab(a+b)b_2)}{b_2z^2 + (2 - 2(a+b)b_2)z + (-abb_2 - (a+b) + (a+b)^2b_2)}$$

where b_2 is any complex number. Moreover, if $b_2 = 0$ then f_2 is the Newton's method for P and if b_2 is a nonzero complex number, then we obtain $f_2(z) = T_2(N(z))$ where $T_2(z) = \frac{z-abb_2}{b_2z+(1-(a+b)b_2)}$.

(b) If $a_2 = 0$, and $a_1 \neq 0$, then

$$f_2(z) = \frac{z - \frac{a+b}{2}}{\left(-\frac{1}{2ab}\right) z^2 + \left(\frac{a+b}{ab}\right) z - \left(\frac{a^2+ab+b^2}{2ab}\right)}$$

Note that $f_2(z) = (S^{-1} \circ N \circ S)(z)$.

(c) If $a_2 = a_1 = 0$ and $a_0 \neq 0$, then there are no rational solutions for (2) of this form.

Conversely, if T is any mapping such that $T \circ S = S \circ T$, then $N \circ T$ and $T \circ N$ are solutions of (2).

Proof. We first assume that $a_2 \neq 0$. In this case we can assume that $f_2(z) = \frac{z^2+a_1z+a_0}{b_2z^2+b_1z+b_0}$. From $S(z) = -\frac{ab}{z-(a+b)}$ we substitute f_2 and S into (2) and solve for the coefficients a_1, a_0, b_2, b_1, and b_0 we obtain

$$\begin{aligned} a_1 &= -2abb_2, \\ a_0 &= -ab + ab(a+b)b_2, \\ b_1 &= 2 - 2(a+b)b_2, \text{ and} \\ b_0 &= -abb_2 - (a+b) + (a+b)^2b_2 \end{aligned}$$

where b_2 is any complex number. That is,

$$f_2(z) = \frac{z^2 + (-2abb_2)z + (-ab + ab(a+b)b_2)}{b_2 z^2 + (2 - 2(a+b)b_2)z + (-abb_2 - (a+b) + (a+b)^2 b_2)}$$

where $b_2 \in C$ is the set of all solutions of (2) for $k = 2$ when $a_2 \neq 0$. Now assume that $a_2 = 0$ and $a_1 \neq 0$. In this case we can assume that $f_2(z) = \frac{z+a_0}{b_2 z^2 + b_1 z + b_0}$. Substitute f_2 and S and solve for a_0, b_0, b_1, and b_2 we obtain

$$f_2(z) = \frac{z - \frac{a+b}{2}}{\left(-\frac{1}{2ab}\right) z^2 + \left(\frac{a+b}{ab}\right) z - \left(\frac{a^2+ab+b^2}{2ab}\right)}.$$

For the cases, $a_2 = a_1 = 0$ and $a_0 \neq 0$. we can show similarly to previous two cases that (2) doesn't have rational solutions in this case. $\square$

For general positive integer k, we have the following result

Theorem 4 *Let f_k be a rational solution of (2) of the form*

(a) If $a_k \neq 0$, then

$$f_k = T_k \circ f_{0,k}$$

where $f_{0,k}(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}$, $T_k(z) = \frac{z - abb_k}{b_k z + (1-(a+b)b_k)}$ and $b_k \in C$.

(b) If $a_k = 0$, and $a_{k-1} \neq 0$, then there is only one solution in this form for (2) and we can explicitly find such the solution.

(c) If $a_k = a_{k-1} = 0$, then there are no nonzero rational solutions for (2) of this form.

Conversely, if T is any mapping such that $T \circ S = S \circ T$, then $f_0 \circ T$ and $T \circ f_0$ are solutions of (1).

Proof We first assume that $a_k \neq 0$. Without loss of generality, we assume that

$$f(z) = \frac{z^k + a_{k-1} z^{x-1} + \cdots + a_1 z + a_0}{b_k z^k + b_{k-1} z^{k-1} + \cdots + b_1 z + b_0}.$$

From Theorem 1, $S^k(z) = \frac{ba^k(-z+a) - ab^k(-z+b)}{a^k(-z+a) - b^k(-z+b)}$, and we obtain

$$S^k(z) = \frac{abT_{k-2} z + abS_{k-1}}{T_{k-1} z + S_k}$$

where $S_j = \sum_{i=0}^{j} a^i b^{j-i}$, $T_j = -S_j$ where $j \geq 1$. It follows that

$$S^k(f(z)) = \frac{\begin{array}{c} abT_{k-2}(z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0) + \\ abS_{k-1}(b_k z^k + b_{k-1} z^{k-1} + \cdots + b_1 z + b_0) \end{array}}{\begin{array}{c} T_{k-1}(z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0) + \\ S_k(b_k z^k + b_{k-1} z^{k-1} + \cdots + b_1 z + b_0) \end{array}}.$$

Now we have

$$f_k(S(z)) = \frac{\sum_{i=0}^{k} \left\{ \sum_{j=i}^{k} (-1)^{j-i} \binom{j}{j-1} (-ab)^{k-j} (a+b)^{j-i} a_{k-j} \right\} z^i}{\sum_{i=0}^{k} \left\{ \sum_{j=i}^{k} (-1)^{j-i} \binom{j}{j-1} (-ab)^{k-j} (a+b)^{j-i} b_{k-j} \right\} z^i}$$

where $a_k = 1$. By comparing first the coefficients of z^i in numerators of $S^k(f(z))$ and $f_k(S(z))$ and then the coefficients of z^i in denominators of $S^k(f(z))$ and $f_k(S(z))$ we have a system of equations which can be solved explicitly to obtain the results of the Theorem in case (a). Case (b) and (c) can be proved similarly to case (a). $\square$

Remark 1. When $k = 3$, $f_{0,3}(z)$ is the Halley's method for P.

Remark 2. We have proved Theorem 4 by solving directly a linear system of equations. It would be more interesting if we could prove Theorem 4 more analytically.

Remark 3. If $P(z)$ is a polynomials of degree 3 or more with distinct roots, then any successive approximation of $P(z)$ would have degree 2 or more. From which it follows that (1) does not hold (since degree of S is not equal to degree of S^k).

Remark 4. From [7], [8] and Theorem 4, f_k is a rational function with integer coefficients if and only if $b_k \in Z$, $a = \frac{s(u-\sqrt{v})}{w}$ and $b = \frac{s(u+\sqrt{v})}{w}$ where $u, v, w, s \in Z - \{0\}$, $u > 0$, $v > 0$, $v^2 \notin Z^+$.

Remark 5. Consider $R(z) = \frac{(a+b)z-ab}{z} = S^{-1}(z)$ which is also a successive approximation of P. By replacing S with R in (2), we obtain similar results with the same argument of proof where we leave the details to interested readers.

Remark 6. If T is a Möbius transformation which commutes with S, then $T(z) = \frac{Az-abC}{Cz+(A-(a+b)C)}$, where A and C are complex numbers. When $A = 1$ and $C = b_k$ we obtain $T = T_k$.

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