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spaces of constant curvature.

Section 2 gives the definition and basic properties of total curvature. These include, for example, rectifiability of curves of finite total curvature. The following length estimate in terms of total curvature and chordlength (the distance between endpoints of a curve) is due to Schmidt [22] for regular curves in Euclidean space, and to Alexandrov and Reshetnyak [8, Theorem 5.8.1] for arbitrary curves in Euclidean space. The example of a cylinder (also mentioned in [3]) shows that this estimate, proved in Sections 3, fails in general. Hence we require in Section 3 that the space is CAT(K). Sufficient conditions that guarantee this are given in [3] for $K > 0$ (see [14, 19] for the smooth case); and [2, 17, 18] for $K \leq 0$ (see also [9, 10, 12]).

Theorem 1.1 *Let γ be a curve in a CAT(K) space, s its arclength, r its chordlength, and κ its total curvature. Suppose that $s < \frac{\pi}{\sqrt{K}}$ and that $\kappa < \pi$ if $K \leq 0$ and $\kappa + r\sqrt{K} < \pi$ if $K > 0$. Then $s \leq s(r, \kappa)$, where the maximum length $s(r,\kappa)$ is realized by isosceles bisegments (curves consisting of two geodesic segments equal in length) in spaces of constant curvature K.*

The following comparison theorem and majorization theorem are due respectively to Alexandrov and Reshetnyak. These are powerful tools to convert problems in CAT(K) spaces into ones in corresponding model spaces. A *nonexpanding* map is a map between metric spaces that never increases the distance between points. A convex domain D in S_K *majorizes* a rectifiable closed curve γ in a metric space if a nonexpanding map exists from D to the space, with its restriction to the boundary of D an arclength preserving map onto the image

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of γ. We shall also say that the boundary of D majorizes γ.

Theorem 1.2 (Alexandrov [5]) *In a CAT(K) space, the angle between any two geodesics at their common endpoints exists. If α_1, α_2 and α_3 are angles of a triangle in a CAT(K) space corresponding respectively to angles $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ of a triangle in S_K with the same sidelengths, then $\alpha_i \leq \tilde{\alpha}_i$ for $i = 1, 2, 3$. An equality holds for some i if and only if the two triangles bound totally geodesic surfaces isometric to each other.*

Theorem 1.3 (Reshetnyak [21]) *If the length of a rectifiable closed curve in a CAT(K) space is less than $\frac{2\pi}{\sqrt{K}}$ then there is a convex domain in S_K that majorizes it.*

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2 Total Curvature

Let X be a CAT(K) space. By a *polysegment* in X with ordered vertices p_0, $p_1, \ldots, p_k$ corresponding respectively to parameter values $t_0 < t_1 < \ldots < t_k$, we mean a curve $\sigma : [t_0, t_k] \to X$ with the property that the restriction $\sigma_i = \sigma|_{[t_{i-1}, t_i]}$ of σ on each subinterval $[t_{i-1}, t_i]$ is a nonconstant minimizing geodesic. A bisegment is a polysegment with two geodesic segments and an n-segment is one with n geodesic segments. If σ is a polysegment with ordered vertices p_0, $p_1, \ldots, p_k$, then the *angle* $\widehat{p_i}$ of σ at an interior vertex p_i is the angle subtended by the two geodesics $p_{i-1}p_i$ and p_ip_{i+1} ([6], see [13]), and the *total rotation* $\kappa^*(\sigma)$ of σ is the sum of its *rotations* $\pi - \widehat{p_i}$ at p_i:

$$\kappa^*(\sigma) = \sum_{i=1}^{k-1}(\pi - \widehat{p_i}).$$

Let $\gamma : [a, b] \to X$ be a curve. A polysegment σ is *inscribed* in γ if there are a partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a, b]$ and a parametrization of σ on $[a, b]$ such that $\sigma(t_i) = \gamma(t_i)$ for $0 \leq i \leq k$, and $\sigma|_{[t_{i-1}, t_i]}$ is a minimizing geodesic for $1 \leq i \leq k$. Unless otherwise specified, any polysegment inscribed in a curve is parametrized in this way. Following the terminology used in [8], for each polysegment σ inscribed in γ the *modulus* of σ associated with γ is

$$\mu_\gamma(\sigma) = \max_{1 \leq i \leq k} \operatorname{diam}\left(\gamma|_{[t_{i-1}, t_i]}\right),$$

where $a = t_0 < t_1 < \cdots < t_k = b$ is the partition of $[a, b]$ associated with σ as above. Finally, the *total curvature* of γ is

$$\kappa(\gamma) = \limsup_{\mu_\gamma(\sigma) \to 0} \kappa^*(\sigma) = \lim_{\epsilon \to 0+} \sup_{\sigma \in \Sigma_\epsilon(\gamma)} \kappa^*(\sigma),$$

where for each $\epsilon > 0$, $\Sigma_\epsilon(\gamma)$ is the set of polysegments σ inscribed in γ such that $\mu_\gamma(\sigma) < \epsilon$. Since angles subtended by pairs of geodesics are independent of the choice of model spaces [8], it follows that the total curvature of a curve depends only locally on the metric, and not on the bound K.

In [4], Alexander and Bishop defined total curvature for curves in CAT(0) spaces, thereby generalized the concept from the Euclidean case [8]. On the other hand, generalization to S_K for positive K has also been done in [8]. The definitions in these settings agree with ours. In fact, in CAT(0) spaces (and hence in all CAT(K) spaces for $K \leq 0$), the results in this section can be deduced from monotonic increase in total curvature under refinement of inscribed polysegments [4]. For $K > 0$, on the contrary, no monotonic property holds in general, or even in S_K. However, Proposition 2.4 can be proved in S_K for $K > 0$ using integral-geometric methods [8, Theorem 6.3.2], which are not applicable in singular spaces. Yet, total curvature of the inscribed polysegments can be controlled in arbitrary CAT(K) spaces. We begin by verifying the equivalence of total rotation and total curvature for polysegments.

Proposition 2.1 *For a polysegment in a CAT(K) space, its total curvature and its total rotation coincide.*

Proof. Without loss of generality, we let η be a polysegment with at least two minimizing geodesic segments. That $\kappa(\eta) \geq \kappa^*(\eta)$ is easily verified. To show that $\kappa(\eta) \leq \kappa^*(\eta)$, we let $p_0, p_1 \dots, p_k$ ($k \geq 2$) be ordered vertices of η. Fix a small positive number ϵ. Let σ be an arbitrary polysegment inscribed in η with modulus less than ϵ. For each i, $1 \leq i \leq k-1$, we let m_i and q_i be the unique consecutive vertices of σ such that, as vertices of η, $m_i \leq p_i < q_i$. Since ϵ is small, we assume that there are at least three vertices of σ on each segment of η. If $m_i < p_i$, we let α_i, β_i and γ_i be the interior angles at m_i, p_i and q_i, respectively, of the geodesic triangle Δ_i defined by these three points, with $\widetilde{\alpha}_i$, $\widetilde{\beta}_i$ and $\widetilde{\gamma}_i$ the corresponding angles of a comparison triangle $\widetilde{\Delta}_i$ of Δ_i in S_K. Denote by $\widehat{m}_i$ and $\widehat{q}_i$ the angles of σ at m_i and q_i, respectively, and by $\widehat{p}_i = \beta_i$ the angle of η at p_i. Now, by the combined use of the triangle inequality for angles, Alexandrov's angle comparison, and the Gauss-Bonnet formula,

$$\begin{aligned}
\kappa^*(\sigma) - \kappa^*(\eta) &= \sum_{\substack{1 \leq i \leq k-1 \\ m_i < p_i}} [(\pi - \widehat{m}_i) + (\pi - \widehat{q}_i) - (\pi - \widehat{p}_i)] \\
&\leq \sum_{\substack{1 \leq i \leq k-1 \\ m_i < p_i}} [\alpha_i + \gamma_i - (\pi - \beta_i)] \\
&\leq \sum_{\substack{1 \leq i \leq k-1 \\ m_i < p_i}} [\widetilde{\alpha}_i + \widetilde{\beta}_i + \widetilde{\gamma}_i - \pi] \\
&= K \sum_{\substack{1 \leq i \leq k-1 \\ m_i < p_i}} a_i,
\end{aligned}$$

where a_i is the area of the triangle $\widetilde{\Delta}_i$. If $K \leq 0$ then we have $\kappa^*(\sigma) \leq \kappa^*(\eta)$, and thus $\kappa(\eta) \leq \kappa^*(\eta)$ as required. Suppose $K > 0$. Since each a_i is bounded above by the area of a disk with circumference 3ϵ in S_K, which tends to zero as ϵ tends to zero, it follows that

$$\kappa(\eta) = \lim_{\epsilon \to 0^+} \sup_{\sigma \in \Sigma_\epsilon(\eta)} \kappa^*(\sigma) \leq \kappa^*(\eta)$$

as required. □

For the purpose of studying total curvature, it is worth rephrasing the following fact, which appeared in [21] as part of the proof of Reshetnyak's majorization theorem. Let σ be a minimizing geodesic segment of an n-segment γ in S_K. A *supporting half space* of γ corresponding to σ is a closed half space of S_K containing all segments of γ adjacent to σ, with boundary containing σ. Two supporting half spaces corresponding to adjacent segments are *compatible* if one can be deformed to the other by rotation about the common vertex in such a way that the two segments always lie in the intermediate half spaces. The polysegment γ is said to be *weakly convex* with respect to a point $O \in S_K$ if there corresponds to each segment of γ a supporting half space containing O such that each pair of supporting half spaces corresponding to adjacent segments are compatible. Then Reshetnyak's fan construction results in the following

Theorem 2.2 (Reshetnyak [21]) *For any n-segment γ in a closed ball of radius $R < \frac{\pi}{2\sqrt{K}}$ centered at a point O in a CAT(K) space, there exist a closed disk D of radius R centered at some point O' in S_K, and an n-segment η in D that is weakly convex with respect to O' with geodesic segments of the same sequence of lengths and with an angle at each interior vertex no smaller than the corresponding angle of γ.*

To define total curvature for curves in a space of curvature bounded above, we need the additive property of total curvature in CAT(K) spaces. This follows from an expression of total curvature of a curve as the limit of the total curvature of a polysegment inscribed in the curve as its modulus goes to zero. To achieve this, we need an appropriate curve length estimate in S_K. This estimate is due to Dekster, and a short version of it is given below. A more general result appears in [20].

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Theorem 2.3 (Dekster [16]) *For each real number K, there exists a positive number $\theta_K < \frac{\pi}{2}$ such that if $0 \leq \theta \leq \theta_K$ then the maximum length among piecewise C^2 curves in a closed disk of radius less than $\frac{\pi}{2\sqrt{K}}$ in S_K with total curvature at most θ is finite and attained by a curve with total curvature θ.*

Proposition 2.4 *Let τ_n be any sequence of polysegments inscribed in a curve γ in a CAT(K) space such that $\mu_\gamma(\tau_n) \to 0$. Then $\kappa(\tau_n) \to \kappa(\gamma)$. Furthermore, if $\kappa(\gamma)$ is finite then γ is rectifiable.*

Proof. Fix a curve $\gamma : [a, b] \to X$ in a CAT(K) space. We consider two cases.

Case I. γ is not rectifiable. Without loss of generality, we assume that γ is contained in a closed ball of radius $R < \frac{\pi}{2\sqrt{K}}$. Let τ_n be a sequence of polysegments inscribed in γ with $\mu_\gamma(\tau_n) \to 0$. Let us denote the length of any curve η by $\ell(\eta)$. Then $\ell(\tau_n) \to \infty$. Given $k > 0$, we choose a positive integer N such that $\ell(\tau_n) > (M+1)^2 L$ for $n \geq N$, where M is the greatest integer not exceeding $\frac{k}{\frac{1}{2}\theta_K}$ and L is the maximum length referred to in Theorem 2.3. We shall show that $\kappa(\tau_n) > k$ for $n \geq N$, from which it follows that $\kappa(\gamma) = \infty$, that $\kappa(\tau_n) \to \kappa(\gamma)$, and that γ must be rectifiable if $\kappa(\gamma)$ is finite. Suppose on the contrary that $\kappa(\tau_n) \leq k$ for some $n \geq N$. By Theorem 2.2, there exists a polysegment η in a closed disk of radius R in S_K with $\ell(\eta) = \ell(\tau_n)$ and $\kappa(\eta) \leq \kappa(\tau_n) \leq k$. But then there are at most M vertices of η with rotation more than $\frac{1}{2}\theta_K$. These vertices cut η into at most $M + 1$ subarcs, each with rotation at most $\frac{1}{2}\theta_K$ at its vertices and with total curvature at most k. Now it is possible to choose, on each of these subarcs, at most M points that cut it into subarcs of total curvature at most θ_K, with at most one having total curvature less than $\frac{1}{2}\theta_K$. Thus we end up with a decomposition of η into at most $(M+1)^2$ subarcs, each of length at most L, a contradiction.

Case II. γ is rectifiable. Let us define for each polysegment σ inscribed in γ the *mesh* of σ associated with γ, denoted by $\widetilde{\mu}_\gamma(\sigma)$, as

$$\widetilde{\mu}_\gamma(\sigma) = \max_{1\le i\le k} \ell\left(\gamma|_{[t_{i-1},t_i]}\right),$$

where $a = t_0 < t_1 < \cdots < t_k = b$ is the partition of $[a,b]$ associated with the inscribed polysegment σ. Let σ_n be a sequence of polysegments inscribed in γ such that $\mu_\gamma(\sigma_n) \to 0$ and $\kappa(\sigma_n) \to \kappa(\gamma)$. Let τ_m be an arbitrary sequence of polysegments inscribed in γ such that $\mu_\gamma(\tau_m) \to 0$. Then $\widetilde{\mu}_\gamma(\sigma_n) \to 0$ and $\widetilde{\mu}_\gamma(\tau_m) \to 0$ as well [8, p. 30].

Fix n and $\sigma = \sigma_n$. Let $\sigma(a) = p_0, p_1, \ldots, p_k = \sigma(b)$ be ordered vertices of σ on γ. Because $\tau_m \to \gamma$ (see also [8, p. 23]), it is possible to find for each m a finite sequence of points $\tau_m(a) = p_0 = p_0^{(m)}, p_1^{(m)}, \ldots, p_k^{(m)} = p_k = \tau_m(b)$ on τ_m such that $p_i^{(m)} \to p_i$ for each i. For each m let $\tau'_m = \tau'_m(n)$ be a polysegment inscribed in τ_m with ordered vertices $p_0^{(m)}, p_1^{(m)}, \ldots, p_k^{(m)}$. Then $\tau'_m \to \sigma$ (see Figure 1), and each τ'_m cuts τ_m into k polysegments $\tau''_i = \tau''_i(m)$, each with ordered vertices $p_{i-1}^{(m)} = q_0^{(i)}, q_1^{(i)}, \ldots, q_{k_i}^{(i)} = p_i^{(m)}$. Putting $q_{-1}^{(i)} = q_{k_i}^{(i)}$ and $q_{k_i+1}^{(i)} = q_0^{(i)}$, we now write for each j, $0 \le j \le k_i$,

$$\alpha_j^{(i)} = \angle q_{j-1}^{(i)} q_j^{(i)} q_{j+1}^{(i)},$$

and for each i,

$$\begin{aligned} \beta_i = \beta_i^{(m)} &= \angle p_{i-1}^{(m)} p_i^{(m)} p_{i+1}^{(m)}, \\ \text{and } \delta_i = \delta_i^{(m)} &= \angle q_{k_i-1}^{(i)} p_i^{(m)} q_1^{(i+1)}. \end{aligned}$$

See Figure 2.

Figure 1. Construction of τ'_m.

Figure 2. The angles $\alpha_j^{(i)}, \beta_i$ and δ_i.

For each i, let $\widetilde{\alpha}_j^{(i)}$, $0 \le j \le k_i$, be the angle corresponding to $\alpha_j^{(i)}$ of a convex polygon $P_i = P_i(m,n)$ in S_K that majorizes the closed curve formed by τ_i'' and its chord. Then

$$\kappa(\sigma_n) \le \liminf_{m \to \infty} \kappa(\tau'_m(n))$$

(see [10, p. 18]), and

$$\begin{aligned}
\kappa(\tau_m) - \kappa(\tau'_m) &= \sum_{i=1}^{k-1}(\pi - \beta_i) - \sum_{i=1}^{k}\sum_{j=1}^{k_i-1}(\pi - \alpha_j^{(i)}) - \sum_{i=1}^{k-1}(\pi - \delta_i) \\
&\geq \sum_{i=1}^{k}\sum_{j=1}^{k_i-1}(\pi - \alpha_j^{(i)}) - \sum_{i=1}^{k-1}(\alpha_0^{(i)} + \alpha_{k_i}^{(i)}) \\
&\geq \sum_{i=1}^{k}\left((k_i - 1)\pi - \sum_{j=0}^{k_i}\widetilde{\alpha}_j^{(i)}\right) \\
&= -K\sum_{i=1}^{k} a_i,
\end{aligned}$$

where $a_i = a_i(m,n)$ is the area of the convex region in S_K bounded by the polygon P_i. Let

$$L(m,n) = \sum_{i=1}^{k} l_i(m,n),$$

where $l_i(m,n)$ is the perimeter of the convex polygon P_i, which exists for sufficiently large m and n, i.e., for m and n such that $\mu(m,n) = \widetilde{\mu}_{\tau_m}(\tau'_m(n)) < \frac{\pi}{\sqrt{K}}$. Dropping the parameters m and n for simplification and letting $A(l)$ denote the area enclosed by a circle of circumference l in S_K, since the maximum of the real-valued function

$$F(x_1, x_2, \ldots, x_k) = \sum_{i=1}^{k} A(x_i)$$

on the set $P = \{(x_1, x_2, \ldots, x_k) \in [0, 2\mu]^k : x_1 + x_2 + \cdots + x_k = L\}$ is attained at a point $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k)$ only if the inequalities $0 < \overline{x}_i < 2\mu$ hold simultaneously for at most one i, we have for any $x \in P$,

$$F(x) \leq (q+1)A(2\mu) < (L + 2\mu)\frac{A(2\mu)}{2\mu},$$

where $L = q \cdot 2\mu + r$ with q an integer and $0 < r \leq 2\mu$. As a consequence, the sum $a(m,n) = \sum_{i=1}^{k} a_i$ is bounded above by $\sum_{i=1}^{k} A(l_i) \le (L+2\mu)\frac{A(2\mu)}{2\mu}$, and

$$\limsup_{m\to\infty} Ka(m,n) \le K(2\ell(\gamma) + 2\tilde{\mu}_\gamma(\sigma_n))\frac{A(2\tilde{\mu}_\gamma(\sigma_n))}{2\tilde{\mu}_\gamma(\sigma_n)},$$

which has a trivial limit as $n \to \infty$. Thus as $n \to \infty$,

$$\kappa(\sigma_n) \le \liminf_{m\to\infty} \kappa(\tau'_m(n)) \le \limsup_{m\to\infty} Ka(m,n) + \liminf_{m\to\infty} \kappa(\tau_m)$$

implies

$$\kappa(\gamma) = \lim_{n\to\infty} \kappa(\sigma_n) \le \liminf_{m\to\infty} \kappa(\tau_m) \le \limsup_{m\to\infty} \kappa(\tau_m) \le \kappa(\gamma).$$

This completes the proof of Proposition 2.4. $\square$

Existence of directions and angles between curves also follows from Proposition 2.4. According to Alexandrov's definition, an arc *has a direction* if the angle with itself exists. An *angle* at an interior point of a curve is generalized from the case of polysegments above in an obvious manner.

Proposition 2.5 *Let γ_1 and γ_2 be curves in a CAT(K) space originating from the same point. If $\kappa(\gamma_1)$ and $\kappa(\gamma_2)$ are finite then the angle between the two curves exists.*

Proof. Let $[a,b]$ be a common closed interval on which γ_1 and γ_2 are both defined with $\gamma_1(a) = \gamma_2(a) = p$. We want to show that $\angle\gamma_1(s)\, p\, \gamma_2(t)$ has a limit as s and t approach a from above. To see this let $s_n \to a$ and $t_n \to a$ be convergent sequences of points in (a,b), whose images under γ_1 and γ_2, respectively, are different from p. For each n let σ_n^1 be a polysegment inscribed in γ_1 with p and $\gamma_1(s_n)$ its first two vertices, and with $\mu_{\gamma_1}(\sigma_n^1) = \operatorname{diam}\left(\gamma_1|_{[a,s_n]}\right)$.

Likewise, let σ_n^2 be a polysegment inscribed in γ_2 with p and $\gamma_2(t_n)$ its first two vertices, and with $\mu_{\gamma_2}(\sigma_n^2) = \operatorname{diam}\left(\gamma_2|_{[a,t_n]}\right)$. Let $\gamma = (-\gamma_1) * \gamma_2$ and let $\sigma_n = (-\sigma_n^1) * \sigma_n^2$ for every n. Then σ_n is a polysegment inscribed in γ, $\mu_{\gamma_1}(\sigma_n^1) \to 0$, $\mu_{\gamma_2}(\sigma_n^2) \to 0$ and $\mu_\gamma(\sigma_n) \to 0$. It follows that

$$\kappa(\sigma_n) = \kappa(\sigma_n^1) + \kappa(\sigma_n^2) + [\pi - \angle\ \gamma_1(s_n)\, p\, \gamma_2(t_n)] \leq \kappa(\sigma_n^1) + \kappa(\sigma_n^2) + \pi.$$

Now because $\kappa(\sigma_n^1) \to \kappa(\gamma_1) < \infty$ and $\kappa(\sigma_n^2) \to \kappa(\gamma_2) < \infty$ as $n \to \infty$, we see that $\kappa(\sigma_n) \to \kappa(\gamma) < \infty$ and

$$\lim_{n\to\infty} \angle\gamma_1(s_n)\, p\, \gamma_2(t_n) = \pi - \kappa(\gamma) + \kappa(\gamma_1) + \kappa(\gamma_2).$$

Since s_n and t_n are arbitrarily chosen, we conclude that an angle between γ_1 and γ_2 exists. $\square$

Corollary 2.6 *In a CAT(K) space X, if p is an interior point of a curve γ identified by a fixed parameter value, and γ_1 and γ_2 are the two subarcs which p cuts γ into, then $\kappa(\gamma)$ is finite if and only if $\kappa(\gamma_1)$ and $\kappa(\gamma_2)$ are both finite. Furthermore, if $\kappa(\gamma)$ is finite then an angle α of γ at p exists and*

$$\kappa(\gamma) = \kappa(\gamma_1) + \kappa(\gamma_2) + (\pi - \alpha).$$

Proof. Immediate. $\square$

Corollary 2.7 *Any curve of finite total curvature has left and right directions at each of its interior points. The directions exist at the ends as well.*

Proof. Immediate. $\square$

Now suppose γ is a curve in a space X of curvature bounded above by K. We shall now define the total curvature of γ. Since (the image of) γ is covered by a family of CAT(K) domains, by compactness of the parametrizing interval the total curvature of γ can be defined by first subdividing γ into finitely many subarcs so that each subarc lies entirely in one of these CAT(K) domains. If one of these subarcs has infinite total curvature then we let $\kappa(\gamma)$ be infinity. Otherwise, using Corollary 2.6, we define $\kappa(\gamma)$ to be the sum of the total curvatures of the subarcs and the supplementary angles of the angles at the subdividing points. Since both angle and total curvature depend only on the metric, the sum so obtained is well-defined and hence agrees with the previous definition of total curvature if X is itself a CAT(K) space.

3 Chord-curvature Length Estimate

In this section, we constrain ourself to work in a CAT(K) space. Putting

$$\lambda = \begin{cases} \sqrt{|K|}, & \text{if } K \neq 0, \\ 1, & \text{if } K = 0, \end{cases}$$

it is easily observed that if the length of an isosceles bisegment with minimizing segments in S_K does not exceed $\frac{\pi}{\sqrt{K}}$ then it is given by

$$s(r,\kappa) = \begin{cases} \frac{2}{\lambda}\operatorname{arcsinh}\left(\frac{\sinh\frac{\lambda r}{2}}{\cos\frac{\kappa}{2}}\right) & \text{if } K < 0, \\ \frac{r}{\cos\frac{\kappa}{2}} & \text{if } K = 0, \\ \frac{2}{\lambda}\arcsin\left(\frac{\sin\frac{\lambda r}{2}}{\cos\frac{\kappa}{2}}\right) & \text{if } K > 0, \end{cases}$$

where $r < \frac{\pi}{\sqrt{K}}$ and $\kappa < \pi$ are respectively the chordlength and the total curvature of the bisegment. This and Theorem 3.3 below imply that a sharp upper estimate of the length of a curve exists for any given pair of small chordlength and total curvature, and that an isosceles bisegment in S_K is indeed an optimizing curve. First we need the following deformation lemmas.

Lemma 3.1 *In S_K, $K \leq 0$, let σ be a polysegment with at least four vertices. Let A, B, C and D be four consecutive vertices of σ ordered according to their parameter values, with A an endpoint. Assume that the segments AB and CD both lie on the same closed halfspace whose boundary contains the segment BC. Let the polysegment deform by fixing all the vertices except B, which moves in such a way that the length of the segment AB increases but the total length of the polysegment remains unchanged. Then the deformation can be carried out without increasing the total curvature $\kappa(\sigma)$ until the segments BC and CD form a geodesic.*

If A is not an endpoint of σ, the above result is still valid under additional assumptions that the segment AB is initially no shorter than BC, and that the segments XA and BC both lie on the same halfspace whose boundary contains the segment AB, where $X \neq B$ is the other vertex of σ adjacent to A.

Proof. Assume that A is an endpoint of σ. Let B' be a new position of B, and denote by σ' the new polysegment corresponding to B' obtained by the described deformation. In the triangle ABC, let α, β and γ be the interior angle at A, B and C, respectively. Likewise, in the triangle $AB'C$, let α', β' and γ' be the interior angle at A, B' and C, respectively. By the law of cosines and the variation of angles with respect to arclength, it is easily seen that $\alpha \geq \alpha'$ and $\gamma \leq \gamma'$. Furthermore,

$$\begin{aligned} \kappa(\sigma') - \kappa(\sigma) &= [(\pi - \beta') + (\pi - (\gamma' + \delta))] - [(\pi - \beta) + (\pi - (\gamma + \delta))] \\ &= (\beta - \beta') + (\gamma - \gamma'), \end{aligned} \tag{1}$$

where δ is a constant angle at C between the segments CA and CD, with a convention that δ is negative if CD intersects the interior of the triangle ABC. Now if $AB < BC$ then, by an elementary fact that an isosceles bisegment gives the smallest total curvature among bisegments with given chordlength and arclength in S_K, the deformation results in smaller total curvature as long as the inequality $AB' \leq B'C$ is satisfied. Thus we assume now that $AB \geq BC$. Let a be the area of the triangle ABC and a' that of $AB'C$. It is easy to verify that among triangles in S_K with perimeter and one sidelength fixed, an isosceles triangle has the largest area, from which it follows that $a \geq a'$. Now

$$\begin{aligned} K(a-a') &= (\alpha+\beta+\gamma-\pi)-(\alpha'+\beta'+\gamma'-\pi) \\ &= (\alpha-\alpha')+(\beta-\beta')+(\gamma-\gamma'), \end{aligned} \tag{2}$$

and thus $\kappa(\sigma')-\kappa(\sigma)=(\beta-\beta')+(\gamma-\gamma')=K(a-a')-(\alpha-\alpha')\leq 0$.

If A is not an endpoint of σ, then with the additional assumption on the position of XA and BC, Equation (1) becomes

$$\kappa(\sigma')-\kappa(\sigma) = (\alpha-\alpha')+(\beta-\beta')+(\gamma-\gamma'), \tag{3}$$

which is nonpositive as well, due to Equation (2). □

Lemma 3.2 *In S_K, $K \geq 0$, let σ be a polysegment with at least four vertices. Let A, B, C and D be four consecutive vertices of σ ordered according to their parameter values, with A an endpoint. Assume that $AB+BC \leq \frac{\pi}{2\sqrt{K}}$, and that the segments AB and CD both lie on the same closed halfspace whose boundary contains the segment BC. Then the deformation described in Lemma 3.1 can be carried out without increasing the total curvature of σ until the segments BC and CD form a geodesic.*

If $K > 0$, and if on the other hand $AB+BC > \frac{\pi}{2\sqrt{K}}$, then the deformation does not increase the total curvature as long as the new length AB does not exceed $\frac{\pi}{2\sqrt{K}}$.

Proof. We proceed using the same notations as we did for the case $K \leq 0$ in Lemma 3.1. To prove the first assertion, let us first note that Equations (1)–(3) are still valid here. Therefore, if $AB < BC$ then the correct monotonicity is obtained as the deformation continues, as long as $AB' \leq B'C$. Thus we consider the case $AB \geq BC$. Rearranging, Equation (1) becomes

$$\kappa(\sigma') - \kappa(\sigma) = (\beta + \gamma) - (\beta' + \gamma'),$$

where we note that β' is the angle opposite the larger of the unfixed sides of the triangle $AB'C$. It is elementary to show that in a class of triangles in S_K, $K \geq 0$, with perimeter and one sidelength fixed, the following statements hold.

(i) If the fixed perimeter is strictly less than $\frac{\pi}{\sqrt{K}}$, then the sum of the angle opposite the fixed side and the one opposite the larger of the unfixed sides is the smallest if the triangle is isosceles, and the sum increases as the difference between the unfixed sidelengths increases provided that the fixed sidelength is strictly less than the sum of the other two.

(ii) For $K > 0$, if the sum of the unfixed sidelengths is greater than $\frac{\pi}{2\sqrt{K}}$ and if the fixed sidelength is strictly less than the sum of the other two, then the angle sum also increases as the difference between the unfixed sidelengths increases, as long as the longer of them does not exceed $\frac{\pi}{2\sqrt{K}}$.

Thus (i) implies the required assertion in the above case, while (ii) implies the assertion in the case $K > 0$ and $AB + BC > \frac{\pi}{2\sqrt{K}}$. This completes the proof. □

Theorem 3.3 *Let γ be a curve in a CAT(K) space, s its arclength, r its chordlength and κ its total curvature. Assume that $s < \frac{\pi}{\sqrt{K}}$ and that $\kappa < \pi$ if $K \leq 0$ and $\kappa + \lambda r < \pi$ if $K > 0$. Then*

$$s \leq s(r, \kappa).$$

Proof. Suppose first that γ is a polysegment. Let $\tilde{\gamma}$ be a convex polysegment in S_K which together with its chord defines a closed polysegment that majorizes the closed polysegment formed by γ and its chord. Then $\tilde{\gamma}$ has the same arclength and chordlength as γ does. By considering the triangle defined by any three consecutive vertices of γ, its comparison triangle and the triangle defined by the corresponding vertices of $\tilde{\gamma}$, it follows from the nonexpanding property, Alexandrov's angle comparison theorem and the classical hinge theorem that $\kappa(\tilde{\gamma}) \leq \kappa(\gamma)$. We claim that there is a deformation of $\tilde{\gamma}$ into an isosceles bisegment $\tilde{\sigma}$ in S_K with the same arclength and chordlength and with $\kappa(\tilde{\sigma}) \leq \kappa(\tilde{\gamma})$. Since $s(r, \kappa)$ is nondecreasing in κ, this implies the required inequality. Note that for $K > 0$ the condition $\kappa + \lambda r < \pi$ allows $s(r, \kappa)$ to be defined. It remains to prove the existence of $\tilde{\sigma}$. To see this, we first note that the case $K = 0$ is done in [8, pp. 151–152]. Moreover, the analogue of the following case $K < 0$ applies in this case as well. We perform induction on the number n of geodesic segments in $\tilde{\gamma}$. The case $n \leq 2$ is easy. Suppose $n \geq 3$. Note that the convexity of $\tilde{\gamma}$ implies that if AB, BC and CD are any consecutive segments of $\tilde{\gamma}$ then AB and CD lie on the same halfspace whose boundary contains BC. If $K < 0$ we apply Lemma 3.1 and get a new polysegment with the same arclength and chordlength as $\tilde{\gamma}$ but with smaller number of segments and no greater total curvature. Applying the induction hypothesis, the existence of $\tilde{\sigma}$ is obtained.

Suppose now that $K > 0$. We let $a_1, a_2, \ldots, a_n$ be the lengths of consecutive segments of $\tilde{\gamma}$. We consider three cases.

Case I. $a_1 + a_2 \leq \frac{\pi}{2\sqrt{K}}$. We apply the first part of Lemma 3.2 to get a new polysegment with smaller number of segments and with the same properties for arclength, chordlength and total curvature as in the above case $K < 0$.

Case II. $a_1 \geq \frac{\pi}{2\sqrt{K}}$. Then $n \geq 3$ implies $a_{n-1} + a_n \leq s - a_1 \leq \frac{\pi}{2\sqrt{K}}$. The first part of Lemma 3.2 applies at the other end of $\tilde{\gamma}$.

Case III. $a_1 < \frac{\pi}{2\sqrt{K}}$ *and* $a_1 + a_2 > \frac{\pi}{2\sqrt{K}}$. We apply the second part of Lemma 3.2 until either the number of segment is reduced or the length of the first segment reaches $\frac{\pi}{2\sqrt{K}}$. If the latter occurs, we apply case II above.

It is thus possible to apply the induction hypothesis to obtain a polysegment $\widetilde{\sigma}$ with the required properties.

Now we consider the general case. Let σ_n be a sequence of polysegments inscribed in γ such that $\mu_\gamma(\sigma_n) \to 0$ and $\kappa(\sigma_n) \to \kappa(\gamma) = \kappa$. Then σ_n has chordlength r for all n. Since $\ell(\sigma_n) \leq s$, it also follows that for every n we have $\ell(\sigma_n) < \frac{\pi}{\sqrt{K}}$. Now $\kappa < \pi$ implies that for sufficiently large n, the condition $\kappa(\sigma_n) < \pi$ is satisfied. Likewise, $\kappa + \lambda r < \pi$ implies that for sufficiently large n, the condition $\kappa(\sigma_n) + \lambda r < \pi$ holds. By the above result in the case of polysegments, $\ell(\sigma_n) \leq s(r, \kappa(\sigma_n))$ for large n. Taking into consideration the continuity of $s(r, \kappa)$ in κ and the fact that an isosceles bisegment gives the smallest total curvature among bisegments with given chordlength and arclength in S_K, the inequality in question is obtained by taking limits as $n \to \infty$. The theorem is proved. $\square$

Remark 3.4 Neither a global upper bound nor a lower bound exists for $K > 0$ and $s > \frac{\pi}{\sqrt{K}}$. Examples are easily constructed in an open half space of S_K.

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