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รายงานวิจัยฉบับสมบูรณ์

โครงการนัยทั่วไปของการเป็นอินเจคทีฟ
(On the generalization of injectivity)

โดย รองศาสตราจารย์ ดร. สมยศ พลับเที่ยง
และคณะ

มิถุนายน 2546

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

กิตติกรรมประกาศ

ขอขอบคุณสำนักงานกองทุนสนับสนุนการวิจัยเป็นอย่างสูงที่สนับสนุนทุนวิจัยหลังปริญญาเอก ประจำปี 2544 ให้แก่ข้าพเจ้า และขอกราบขอบพระคุณ ศาสตราจารย์ ดร. สมพงษ์ ธรรมพงษา (นักวิจัยพี่เลี้ยง) เป็นอย่างสูงที่เสียสละเวลาในการให้คำปรึกษาและตรวจแก้ไขผลงานที่ส่งไปลงตีพิมพ์ในวารสารระดับนานาชาติ

สมยศ พลับเที่ยง

25 มิถุนายน 2546

บทคัดย่อ

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ชื่อนักวิจัย : รองศาสตราจารย์ ดร. สมยศ พลับเที่ยง

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งานวิจัยนี้มีวัตถุประสงค์ต้องการพิสูจน์ทฤษฎีบทใหม่เกี่ยวกับการอธิบายลักษณะเฉพาะของริงที่เป็นแบบฉบับโดยใช้กลุ่มของมอดูลที่เป็นนัยทั่วไปของมอดูลอินเจคทีฟ ผลงานหลักของงานวิจัยนี้มีดังต่อไปนี้

1. ริง R จะเป็นริงเนอแทเรียนทางขวาก็ต่อเมื่อ ทุก ๆ R– มอดูลก่อกำเนิด 2 ทางขวา เป็นผลบวกตรงของมอดูลโพรเจคทีฟและมอดูลซีเอส หรือมอดูลเนอแทเรียน
2. ริง R จะเป็นริงอาร์ทิเนียนทางขวาก็ต่อเมื่อทุก ๆ (R–) มอดูลก่อกำเนิด –2 ทางขวาเป็นมอดูลควอซี-คอนทินิวอัส หรือมอดูลที่มีความยาวจำกัด
3. ริงซิมเปิล R จะเป็นริงเนอแทเรียนทางขวาก็ต่อเมื่อทุก ๆ (R-) มอดูลซิคลิกซิงกูลาร์ทางขวาเป็นมอดูลซีเอสหรือมอดูลเนอแทเรียน
4. ให้ R เป็นริงไพรมถ้าทุก ๆ (R–) มอดูลซิคลิกแท้ทางขวาเป็นผลบวกตรงของมอดูลควอซี-อินเจคทีฟ และมอดูลไฟไนท์โคเจเนอเรเตด แล้ว R จะเป็นริงเซมิซิมเปิลอาร์ทิเนียนหรือโดเมนออทางขวา
5. ริงไพรม R จะเป็นริงเนอแทเรียนทางขวาก็ต่อเมื่อทุก ๆ (R–) มอดูลซิคลิกทางขวาเป็นผลบวกตรงของมอดูลควอซี-อินเจคทีฟและมอดูลเนอแทเรียน
6. ริง R จะเป็นริงเซมิเพอร์เฟคท์ถ้าทุก ๆ (R-) มอดูลซิคลิกทางขวาเป็นผลบวกตรงของมอดูลคอนทินิวอัสและมอดูลเซมิเพอร์เฟคท์
7. ริง R จะเป็นเนอแทเรียนทางขวาก็ต่อเมื่อทุก ๆ (R–) มอดูลก่อกำเนิดแบบนับได้ทางขวาเป็นผลบวกตรง ของมอดูลคอนทินิวอัส
8. ริง R จะเป็นริงอาร์ทิเนียนทางขวาก็ต่อเมื่อทุก ๆ (R–) มอดูลก่อกำเนิดแบบนับได้ทางขวาเป็นผลบวกตรงของมอดูลคอนทินิวอัสและมอดูลโลคัลลีอาร์ทิเนียน
9. ริง R จะเป็นริงอาร์ทิเนียนทางขวาถ้าทุก ๆ (R–) มอดูลก่อกำเนิดแบบนับได้ทางขวาเป็นผลบวกตรงของมอดูลควอซี-คอนทินิวอัสและมอดูลเซมิซิมเปิล
10. ริง R จะเป็นริงอาร์ทิเนียนทางขวาถ้าทุก ๆ (R–) มอดูลเป็นผลบวกตรงของมอดูลซีเอสและมอดูลเซมิซิมเปิล

คำหลัก : Noetherian rings, Artinian rings, Injective modules, Continuous modules, CS-modules

Abstract

Project Code : PDF/91/2544

Project Title : On the generalization of injectivity

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The purpose of this research was establish some characterizations of the classical rings through the class of generalized of injective modules. Our maim results, among any others, are list as follow:

1. A ring R is right noetherian if and only if every 2-generated right R-module is a direct sum of a projective module and a module that is either CS or noetherian.
2. A ring R is right artinian if and only if every 2-generated right R-module is either quasi-continuous or finite length.
3. A simple ring R is right noetherian if and only if every cyclic singular right R-module is either a CS-module or a noetherian module.
4. Let R be a prime ring. If every proper cyclic right R-module is a direct sum of a quasi-injective module and a finitely cogenerated module, then R is either semisimple artinian or right Ore domain.
5. A prime ring R is right noetherian if and only if every cyclic right R-module is a direct sum of a quasi-injective module and a noetherian module.
6. A ring R is semiperfect if every cyclic right R-module is a direct sum of a continuous module and a semiperfect module.
7. A ring R is right noetherian if and only if every countably generated right R-module is a direct sum of a continuous module and a locally noetherian module.
8. A ring R is right artinian if and only if every countably generated right R-module is a direct sum of a continuous module and a locally artinian module.
9. A ring R is right artinian if every countably generated right R-module is a direct sum of a quasi-continuous module and a semisimple module.
10. A ring R is right artinian if every right R-module is a direct sum of a CS-module and a semisimple module.

Keywords : Noetherian rings, Artinian rings, Injective modules, Continuous modules, CS-modules

เนื้อหางานวิจัย

1. ความสำคัญและที่มาของปัญหาที่ทำการวิจัย

การศึกษาทฤษฎีมอดูลมีแนวทางหลักที่ทำการศึกษาวิจัยอยู่ 2 แนวทาง คือ แนวทางแรกจะศึกษาคุณสมบัติของมอดูลบนริงใด ๆ หรือศึกษาคุณสมบัติของมอดูลบนเฉพาะริง (เช่น บนริงแบบเนอแทร์(noetherian ringa), ริงแบบเซมิเปอร์เฟคต์(semiperfect rings) เป็นต้น) อีกแนวทางหนึ่งจะศึกษาเพื่ออธิบายลักษณะเฉพาะของริงที่เป็นแบบฉบับ (classical rings) โดยใช้กลุ่ม(class)ของมอดูลบนริงดังกล่าวนั้น

มอดูลอินเจคทีฟ (injective module) ถือว่าเป็นมอดูลชนิดหนึ่งที่มีความสำคัญมากในการค้นพบและพัฒนาทฤษฎีบทมอดูลเองและทฤษฎีบทริงด้วยซึ่งจะพบว่ามีงานวิจัยจำนวนมากที่ศึกษาคุณสมบัติของมอดูลอินเจคทีฟ เช่น คุณสมบัติการมีอินเจคทีฟฮูล (infective hull) คุณสมบัติการแยกเป็นผลบวกตรงของมอดูลอินดิคอมเพสเซเบิล(indecomposable modules) เป็นต้น และอีกด้านหนึ่งมีนักคณิตศาสตร์จำนวนมากที่ศึกษาทำงานวิจัยในการอธิบายลักษณะเฉพาะของริงที่เป็นแบบฉบับ (classical rings) ซึ่งได้แก่ ริงแบบเซมิซิมเปิล(semisimple rings) ริงแบบคิวเอฟ (QF- rings) ริงแบบอาร์ทิน (Artinian ring) เป็นต้น โดยที่เราทราบว่าริงที่เป็นแบบฉบับมีความสัมพันธ์กันดังนี้

$$
\begin{array}{ccccccccc}
\text{เซมิซิมเปิล} & \Rightarrow & \text{อาร์ทิน} & \Rightarrow & \text{เซมิพรีมิทีฟ} & \Rightarrow & \text{เปอร์เฟคต์} & \Rightarrow & \text{เซมิเปอร์เฟคต์} \\
 & & \downarrow & & \text{(semiprimitive)} & & \text{(Perfect)} & & \text{(semiperfect)} \\
 & & \text{เนอแทร์} & & & & & &
\end{array}
$$

เนื่องจากในช่วง 30 ปีที่ผ่านมามีนักคณิตศาสตร์ที่ศึกษานัยทั่วไปของมอดูลอินเจคทีฟเป็นจำนวนมาก ซึ่งทราบกันว่ามอดูลที่เป็นนัยทั่วไปของมอดูลอินเจคทีฟมีการศึกษากันในหลาย ๆ ทิศทางและนัยทั่วไปของมอดูลอินเจคทีฟที่นิยมศึกษากันมากที่สุด คือ มอดูลควอซี-อินเจคทีฟ (quasi-injvetive modules) มอดูลคอนทินิวอัส (continuous modules) มอดูลควอซี-คอนทินิวอัส (quasi-continuous modules) และมอดูลซีเอส (CS-modules) โดยที่มอดูลดังกล่าวทั้งหลายข้างต้นมีความสัมพันธ์ดังนี้

$$
\begin{array}{ccccccccc}
\text{อินเจคทีฟ} & \Rightarrow & \text{ควอซี - อินเจคทีฟ} & \Rightarrow & \text{คอนทินิวอัส} & \Rightarrow & \text{ควอซี-คอนทินิวอัส} & \Rightarrow & \text{ซีเอส} \\
\text{(injective)} & & \text{(quasi-injective)} & & \text{(continuous)} & & \text{(quasi-continuous)} & & \text{(CS)}
\end{array}
$$

การพัฒนาทฤษฎีบทเกี่ยวกับคุณสมบัติของมอดูลดังกล่าวข้างบนนี้ จะทำให้การศึกษาลักษณะเฉพาะของริงที่เป็นแบบฉบับมีความสมบูรณ์และชัดเจนยิ่งขึ้น ซึ่งจะเป็นพื้นฐานในการสร้างทฤษฎีใหม่ทางคณิตศาสตร์บริสุทธิ์ และคณิตศาสตร์ประยุกต์ต่อไป

2. วัตถุประสงค์ของโครงการ

เป้าหมายหลักของโครงการอยู่ที่การสร้างองค์ความรู้ใหม่และการขยายขอบเขตขององค์ความรู้เดิม ให้กว้างขวางยิ่งขึ้น ตามรายละเอียด ดังนี้

1. เพื่อสร้างทฤษฎีใหม่เกี่ยวกับคุณสมบัติของมอดูลที่เป็นนัยทั่วไปของมอดูลอินเจคทีฟเช่นคุณสมบัติ Exchange property และคุณสมบัติ Decomposition complements direct summands ตามแนวความคิดของ R.B. Warfield และ E. Matlis ที่ได้ทำไว้กับมอดูลอินเจคทีฟ
2. เพื่อสร้างทฤษฎีบทใหม่วกับการอธิบายลักษณะเฉพาะของริงที่เป็นแบบฉบับ(เช่น ริงแบบเซมิซิมเปิล ริงแบบเนอแทร์ ริงแบบอาร์ทิน) โดยใช้กลุ่มของมอดูล (บนริงดังกล่าว) ที่เป็นนัยทั่วไปของมอดูลอินเจคทีฟ ทั้งนี้จะอาศัยแนวความคิดของ B.L. Osofsky, P.F. Smith และ D.V. Huynh ที่ได้อธิบายลักษณะเฉพาะของริงดังกล่าวด้วยมอดูลอินเจคทีฟ

3. เอกสารและผลงานวิจัยที่เกี่ยวข้อง

การศึกษาเกี่ยวกับมอดูลที่เป็นนัยทั่วไปของมอดูลอินเจคทีฟได้ทำกันอย่างแพร่หลายตั้งแต่ปี ค.ศ. 1960 เป็นต้นมา โดยมีผลงานวิจัยจำนวนมากที่ได้ถูกรวบรวมไว้ในหนังสือ Continuous and Discrete Modules โดย S.H. Mohammed and B.J. Muller [11] ในปี ค.ศ. 1990 และในหนังสือ Extending modules ซึ่งรวบรวมโดย N.V. Dung, และคณะ [5] ในปี ค.ศ. 1994 นอกจากนี้ยังมีผลงานวิจัยที่ยังไม่ได้รวมรวมไว้ในหนังสือดังกล่าว ซึ่งในที่นี้จะยกเอามาบางประเด็นเฉพาะที่ผู้เสนอโครงการสนใจดังนี้

3.1 คุณสมบัติที่เกี่ยวกับทฤษฎีบทการแยก(decomposition)เป็นผลบวกตรง(direct sum)ของมอดูล ซึ่งจะเกี่ยวข้องกับคุณสมบัติ complements direct summands, locally semi-T-nilpotent และ exchange property ซึ่งมีผลงานของผู้เชี่ยวชาญที่เด่น ๆ ดังนี้

- ในปี ค.ศ. 1969 R.B. Warfield [18] ได้พิสูจน์ว่าทุก ๆ มอดูลควอซี-อินเจคทีฟมีคุณสมบัติ exchange property
- ในปี ค.ศ. 1988 S.H. Mohamed B.J. Muller [11] ได้ขยายแนวความคิดของ Warfield พิสูจน์ว่าทุก ๆ มอดูลคอนทินิวอัสมีคุณสมบัติ exchange property
- ในปี ค.ศ. 1996 K.Oshiro และ S.T. Rizvi [13] ได้พบเงื่อนไขที่ทำให้ทุก ๆ มอดูลควอซี-คอนทินิวอัสมีคุณสมบัติ exchange property ซึ่งเงื่อนไขดังกล่าวคือ คุณสมบัติ finite exchange property.

3.2 การอธิบายลักษณะของริงแบบเนอร์แทร์มีผลงานวิจัยที่สำคัญ ๆ ดังนี้

- ในปี ค.ศ. 1958 E. Matlis [10] ได้พิสูจน์ว่า R จะเป็นริงแบบเนอแทร์ทางขวาก็ต่อเมื่อทุก ๆ มอดูลอินเจคทีฟทางขวา (บนริง R) เป็นผลบวกตรงของมอดูลอินดิคอมโพสเซเบิล (indecomposable modules)

- ในปี ค.ศ. 1964 B.L. Osofsky [14] ได้พิสูจน์ว่า R จะเป็นริงแบบเซมิซิมเปิล(semisimple rings) ก็ต่อเมื่อทุก ๆ มอดูลซิคลิก(cyclic module)ทางขวา (บนริง R) เป็นมอดูลอินเจคทีฟ
- ในปี ค.ศ. 1984 M.Okado [12] ได้ขยายแนวคิดของ Matlis มาใช้มอดูลซีเอสในการอธิบายลักษณะเฉพาะของริงแบบเนอแทร์ โดยพิสูจน์ว่า R จะเป็นริงแบบเนอแทร์ทางขวาก็ต่อเมื่อทุก ๆ มอดูลซีเอสทางขวา (บนริง R) เป็นผลบวกตรงของมอดูลอินดิคอมโพสเซเบิล
- ในปี ค.ศ. 1982 A.W.Chatters [3] ได้พิสูจน์ว่า R จะเป็นริงแบบเนอแทร์ทางขวาก็ต่อเมื่อทุก ๆ มอดูลซิคลิก (cyclic modulde) ทางขวา (บนริง R) เป็นผลบวกตรงของมอดูลโพรเจคทีฟ (projective modules) และ มอดูลเนอร์แทร์ (noetherian module)
- ในปี ค.ศ. 1991 B.L. Osofsky and P.F. Smith [16] ได้พิสูจน์ว่า R จะเป็นริงแบบเนอแทร์ ทางขวาถ้าทุก ๆ มอดูลซิคลิกทางขวา (บนริง R) เป็นผลบวกตรงของมอดูลโพรเจคทีฟ และมอดูลอินเจคทีฟ
- ในปี ค.ศ. 1996 D.V. Huynh [7] ได้ขยายแนวความคิดของ Chatters [3] และ Osofsky and Smith [16] โดยพิสูจน์ว่า R จะเป็นริงแบบเนอแทร์ ทางขวาก็ต่อเมื่อทุก ๆ มอดูลอินเจคทีฟทางขวา (บนริง R) เป็นมอดูลอินเจคทีฟหรือเป็นผลบวกตรงของมอดูลโพรเจคทีฟและมอดูลเนอร์แทร์
- ในปี ค.ศ. 1996 D.V. Huynh , S.T. Rizvi and M.F.Y.ousif [8] ได้ใช้มอดูลซีเอสในการอธิบายลักษณะเฉพาะของริงแบบเนอแทร์โดยพิสูจน์ว่า R เป็นริงแบบเนอร์แทร์ทางขวาถ้าทุก ๆ มอดูลไฟไนลีเจเนอเรเตด (finitiely generated modules) เป็นมอดูลซีเอส

3.3 การอธิบายลักษณะเฉพาะของริงแบบคิวเอฟ (QF-rings)

- ในปี ค.ศ. 1951 M. Ikeda [9] ได้นิยามริงแบบคิวเอฟว่า ถ้า R เป็นริงแบบอาร์ทินทางขวา (หรือทางซ้าย) และเป็นริงอินเจคทีฟ ทางขวา (หรือทางซ้าย) แล้วเราจะกล่าวว่า R เป็นริงแบบคิวเอฟ พร้อมทั้งพิสูจน์ได้ว่าเงื่อนไขการเป็นริงแบบอาร์ทินสามารถแทนด้วยริงแบบเนอแทร์
- ในปี ค.ศ. 1966 B.L. Osofsky [15] ได้พิสูจน์ว่า R เป็นริงแบบคิวเอฟ ก็ต่อเมื่อ R เป็นริงแบบเพอเฟคต์ทางขวา และเป็นริงอินเจคทีฟทั้งทางซ้ายและทางขวา
- ในปี ค.ศ. 1966 C. Faith [6] ได้พิสูจน์เงื่อนไขที่เป็นนัยทั่วไปยิ่งขึ้นของริงแบบคิวเอฟโดยแทนเงื่อนไขการเป็นริงแบบเนอแทร์ด้วยเงื่อนไขที่อ่อนกว่า พร้อมทั้งเสนอข้อความคาดการณ์ที่มีชื่อว่า Faith Conjecture ดังนี้

"ริงแบบเซมิพริมิทีฟอินเจคทีฟทางขวา R เป็นริงแบบคิวเอฟ"

ซึ่งปัญหานี้ยังไม่สามารถหาคำตอบได้จนถึงปัจจุบันนี้

- ในปี ค.ศ. 1965 Y. Utumi [17] ได้พิสูจน์ว่า R เป็นริงแบบคิวเอฟ ก็ต่อ

เมื่อ R เป็นริงแบบอาร์ทินทั้งทางขวาและทางซ้าย และเป็นริงคอนทินิวอัสทั้งทางขวาและทางซ้าย

- ในปี ค.ศ. 1994 J. Clark and D.V. Huynh [14] ได้ศึกษาว่าเมื่อไร ริงแบบเซมิเพฟเฟคต์อินเจคทีฟทางขวา (บนริง R) จะเป็นริงแบบคิวเอฟ ซึ่งพวกเขาพิสูจน์ได้ว่า R เป็นริงแบบคิวเอฟก็ต่อเมื่อ R เป็นริงแบบเซมิเพอเฟคต์อินเจคทีฟทางขวา และทุก ๆ ยูนิฟอร์มสับมอดูล (uniform submodule)ของแต่ละมอดูลโพรเจคทีฟ P (บนริง R) ถูกบรรจุอยู่ในไฟไนต์เจเนอเรเตด (finite generated) สับมอดูลของ P

4. วิธีการดำเนินวิจัย

1. รวบรวมศึกษาเอกสารงานวิจัยที่เกี่ยวข้องกับเรื่อง A characterization of Noetherian rings, Artinain rings and QF-rings
2. กำหนดสมมติฐานและพิสูจน์ทฤษฎีบทที่กำหนดลักษณะเฉพาะของริงเนอแทเรียน (Noetherian rings)และเขียนเป็น paper ในชื่อ "Conditions for a ring to be Noetherian and Artinian" ให้นักวิจัยพี่เลี้ยงตรวจสอบ
3. ส่งมอบ paper ในหัวข้อ 2) เพื่อตีพิมพ์ในวารสาร The Communication in Algebra
4. แก้ไข paper ของงานวิจัยในข้อ 3) ที่ได้รับการตอบรับเพื่อลงตีพิมพ์แล้ว ตามข้อเสนอแนะของ Referee ให้นักวิจัยพี่เลี้ยงตรวจสอบและส่งมอบให้ Editor
5. กำหนดสมมติฐานและพิสูจน์ทฤษฎีบทลักษณะเฉพาะของริงเนอแทเรียนบนริงซิมเปิล และเขียน paper ของงานวิจัยชื่อ " On Simple Noetherian rings" ให้นักวิจัยพี่เลี้ยงตรวจสอบ
6. ส่งมอบ paper ในหัวข้อ 5) เพื่อตีพิมพ์ในวารสาร Algebra Colloquium
7. แก้ไข paper ของงานวิจัยในข้อ 6) ที่ได้รับการตอบรับเพื่อลงตีพิมพ์แล้ว ตามข้อเสนอแนะของ Referee ให้นักวิจัยพี่เลี้ยงตรวจสอบและส่งมอบให้ Editor
8. กำหนดสมมติฐานและพิสูจน์ทฤษฎีบทลักษณะเฉพาะของริงเนอแทเรียนและริงอาร์ทิเนียน และเขียน paper ของงานวิจัยชื่อ " Rings with many direct summands" ให้นักวิจัยพี่เลี้ยงตรวจสอบ
9. ส่งมอบ paper ในหัวข้อ 8) เพื่อตีพิมพ์ในวารสาร The Mathematical Journal of Okayama University
10. แก้ไข paper ของงานวิจัยในข้อ 9) ที่ได้รับการตอบรับเพื่อลงตีพิมพ์แล้ว ตามข้อเสนอแนะของ Referee ให้นักวิจัยพี่เลี้ยงตรวจสอบและส่งมอบให้ Editor
11. กำหนดนิยามของมอดูลที่เป็นนัยทั่วไปของมอดูลคอนทินิวอัส กำหนดสมมติฐานและพิสูจน์ทฤษฎีบทลักษณะเฉพาะมอดูล ec-continuous และริงคิวเอฟ และเขียน

paper ของงานวิจัยชื่อ " A generalization of continuous modules and their application to QF-rings " ให้นักวิจัยพี่เลี้ยงตรวจสอบ

12. ส่งมอบ paper ในหัวข้อ 11) เพื่อตีพิมพ์ในวารสาร The Kyungpook Mathematical Journal
13. แก้ไข paper ของงานวิจัยในข้อ 12) ที่ได้รับการตอบรับเพื่อลงตีพิมพ์แล้ว ตามข้อเสนอแนะของ Referee ให้นักวิจัยพี่เลี้ยงตรวจสอบและส่งมอบให้ Editor
14. กำหนดสมมติฐานและพิสูจน์ทฤษฎีบทลักษณะเฉพาะของริงเนอแทเรียนและริงอาร์ทิเนียน และเขียน paper ของงานวิจัยชื่อ "Modules characterized by their proper cyclic subfactors"ให้นักวิจัยพี่เลี้ยงตรวจสอบ
15. ส่งมอบ paper ในหัวข้อ 14) เพื่อตีพิมพ์ในวารสาร Southeast Asian Bulletin of Mathematics
16. แก้ไข paper ของงานวิจัยในข้อ 15) ที่ได้รับการตอบรับเพื่อลงตีพิมพ์แล้ว ตามข้อเสนอแนะของ Referee ให้นักวิจัยพี่เลี้ยงตรวจสอบและส่งมอบให้ Editor

5. ผลงานวิจัยที่ได้รับ

ผลงานหลักของงานวิจัยนี้มีดังต่อไปนี้ กำหนดให้ M เป็นมอดูลก่อกำเนิดจำกัด

1. ลักษณะเฉพาะของริงเนอแทเรียน และอาร์ทีเนียน

1.1 มอดูล M เป็นเนอแทเรียนก็ต่อเมื่อทุก ๆ มอดูลก่อกำเนิดจำกัดใดน $\sigma[M]$ เป็นผลบวกตรงของมอดูลโพรเจคทีฟ และมอดูลซีเอสหรือมอดูลเนอแทเรียน

1.2 ริง R จะเป็นจริงเนอแทเรียนทางขวาก็ต่อเมื่อ ทุก ๆ R– มอดูลก่อกำเนิด 2 ทางขวาเป็นผลบวกตรงของมอดูลโพรเจคทีฟและมอดูลซีเอสหรือมอดูลเนอแทเรียน

1.3 ริง R จะเป็นริงอาร์ทิเนียนทางขวาก็ต่อเมื่อทุก ๆ (R–) มอดูลก่อกำเนิด –2 ทางขวาเป็นมอดูลควอซี-คอนทินิวอัส หรือมอดูลที่มีความยาวจำกัด

1.4 ริงซิมเปิล R จะเป็นริงเนอแทเรียนทางขวาก็ต่อเมื่อทุก ๆ (R-) มอดูลซิคลิกซิงกูลาร์ทางขวาเป็นมอดูลซีเอสหรือมอดูลเนอแทเรียน

1.5 ให้ R เป็นริงไพรมถ้าทุก ๆ (R–) มอดูลซิคลิกแท้ทางขวาเป็นผลบวกตรงของมอดูลควอซี-อินเจคทีฟ และมอดูลไพไนท์โคเจเนอเรเตด แล้ว R จะเป็นริงเซมิซิมเปิลอาร์ทิเนียนหรือโดเมนออทางขวา

1.6 ริงไพรม R จะเป็นริงเนอแทเรียนทางขวาก็ต่อเมื่อทุก ๆ (R–) มอดูลซิคลิกทางขวาเป็นผลบวกตรงของมอดูลควอซี-อินเจคทีฟและมอดูลเนอแทเรียน

1.7 ริง R จะเป็นริงเซมิเพอร์เฟคท์ถ้าทุก ๆ (R-) มอดูลซิคลิกทางขวาเป็นผลบวกตรงของมอดูลคอนทินิวอัสและมอดูลเซมิเพอร์เฟคท์

1.8 ริง R จะเป็นเนอแทเรียนทางขวาก็ต่อเมื่อทุก ๆ (R–) มอดูลก่อกำเนิดแบบนับได้ทางขวาเป็นผลบวกตรง ของมอดูลคอนทินิวอัส

1.9 ริง R จะเป็นริงอาร์ทิเนียนทางขวาก็ต่อเมื่อทุก ๆ (R–) มอดูลก่อกำเนิดแบบนับได้ทางขวาเป็นผลบวกตรงของมอดูลคอนทินิวอัสและมอดูลโลคัลลีอาร์ทิเนียน

1.10 ริง R จะเป็นริงอาร์ทิเนียนทางขวาถ้าทุก ๆ (R–) มอดูลก่อกำเนิดแบบนับได้ทางขวาเป็นผลบวกตรงของมอดูลควอซี-คอนทินิวอัสและมอดูลเซมิซิมเปิล

1.11 ริง R จะเป็นริงอาร์ทิเนียนทางขวาถ้าทุก ๆ (R–) มอดูลเป็นผลบวกตรงของมอดูลซีเอสและมอดูลเซมิซิมเปิล

1.12 ถ้าทุก ๆ ซิคลิกลับแฟกเตอร์แท้ของ M เป็นผลบวกตรงของมอดูลซีเอสและมอดูลที่มีมิติยูนิฟอร์มจำกัด หรือเป็นผลบวกตรงของมอดูล M– โพรเจคทีฟกับมอดูล Q หรือ Q เป็นมอดูลซีเอสหรือเนอแทเรียนแล้วทุก ๆ มอดูลแฟกเตอร์ของ M จะมีมิติยูนิฟอร์มจำกัด

1.13 ริง R จะเป็นริงเนอแทเรียนทางขวาก็ต่อเมื่อทุก ๆ (R–) มอดูลซิคลิกแท้ทางขวาเป็นผลบวกตรงของมอดูลโพรเจคทีฟและมอดูลอินเจคทีฟหรือมอดูลเนอแทเรียน

1.14 ริง R จะเป็นริงเนอแทเรียนทางขวาก็ต่อเมื่อทุก ๆ (R–) มอดูลก่อกำเนิดแบบจำกัดแท้เป็นผลบวกตรงของมอดูลโพรเจคทีฟ และมอดูลซีเอสหรือมอดูลเนอแทเรียน

2 นัยทั่วไปของมอดูลคอนทินิวอัสและริงคิวเอฟ

2.1 งานวิจัยที่ได้ศึกษากลุ่มของมอดูลที่เป็นนัยทั่วไปของมอดูลคอนทินิวอัสและมอดูลควอซี-คอนทินิวอัส ซึ่งได้แก่มอดูลอีซี-คอนทินิวอัส และมอดูลอีซี-ควอซี-คอนทินิวอัส

2.2 งานวิจัยนี้ได้พิสูจน์ทฤษฎีของดีคอมโพลิชันสำหรับมอดูลอีซี-ควอซี-คอนทินิวอัส และพิสูจน์เงื่อนไขเพียงพอที่ทำให้มอดูลอีซี-คอนทินิวอัสเป็นมอดูลคอนทินิวอัส

2.3 งานวิจัยนี้ได้พิสูจน์ทฤษฎีบทที่อธิบายลักษณะเฉพาะของริงคิวเอฟโดยใช้เงื่อนไข "อีซี-คอนทินิวอัส" แทนเงื่อนไข "อินเจคทีฟ"

ผลการวิจัยดังกล่าวเขียนเป็น paper จำนวน 5 paper ซึ่งสรุปบทคัดย่อแต่ละ paper ดังกล่าวได้ดังนี้

1) Conditions for a rings to be noetherian and artinian

Abstract. Let M be a finitely generated module. It is shown that M is noetherian if and only if every finitely generated module in σ[M] is a direct sum of a projective module and a module Q, where Q is either CS or noetherian. Consequently, (i) a ring R is right noetherian if and only if every 2-generated right R-module is a direct sum of a projective module and a module that is either CS or noetherian, and (ii) a ring R is right artinian if and only if every 2-generated right R-module is either quasi-continuous r finite length.(ดูรายละเอียดในภาคผนวก 1)

2) On simple noetherian rings

Abstract. A module M is called a CS-module (or extending module) if every submodule of M$ is essential in a direct summand of M. It is shown that (i) a simple ring R is right noetherian if and only if every cyclic singular right R-module is either a CS-module or a noetherian module, (ii) over a prime ring R if every proper cyclic right R-module is a direct sum of a quasi-injective module and a finitely cogenerated module, then R is either semisimple artinian or right Ore domain, and (iii) a prime ring R is right noetherian if and only if every cyclic right R-module is a direct sum of a quasi-injective module and a noetherian module. (ดูรายละเอียดในภาคผนวก 2)

3) Rings with many direct summands

Abstract. It is shown that (i) a ring R is semiperfect if every cyclic right R-module is a direct sum of a continuous module and a semiperfect module, (ii) a ring R is right noetherian if and only if every countably generated right R-module is a direct sum of a continuous module and a locally noetherian module, (iii) a ring R is right artinian if and only if every countably generated right R-module is a direct sum of a continuous module and a locally artinian module,(iv) a ring R is right artinian if every countably generated right R-module is a direct sum of a quasi-continuous module and a semisimple module, and (v) a ring R is artinian if every right R-module is a direct sum of a CS-module and a semisimple module. (ดูรายละเอียดในภาคผนวก 3)

4) A generalization of continuous modules and their application to QF-rings

Abstract. A ring R is called quasi-Frobenius, briefly QF, if R is right or left artinian and right or left self-injective. In this note we study the new class of modules which generalize the concept of continuous (quasi-continuous) modules, that is, ec-continuous (ec-quasi-continuous) modules. We prove the decomposition theorems for ec-quasi-continuous modules, and we also give a sufficient condition for ec-continuous to be continuous. Moreover, we give a characterization of QF-rings with the injectivity condition replaced by ec-continuity. (ดูรายละเอียดในภาคผนวก 4)

5) Modules charaterized by their proper cyclic subfactors

Abstract. For a finitely generated self-projective right R-module M, we show that if every proper cyclic subfactor of M is a direct sum of a CS-module and a module of finite uniform dimension, or a direct sum of an M-projective module and a module Q,

where Q is either CS or noetherian, then every factor module of M has finite uniform dimension. Consequently, (i) a ring R is right noetherian if and only if every proper cyclic right R-module is a direct sum of a projective module and a module Q, where Q is either injective or noetherian, and (ii) a ring R is right noetherian if and only if every proper finitely generated right R-module is a direct sum of a projective module and a module Q, where Q is either CS or noetherian. (ดูรายละเอียดในภาคผนวก 5)

6. ผลงานวิจัยที่ตีพิมพ์ในวารสารวิชาการระดับนานาชาติ

6.1) S. Plubtieng and H., Tansee, *Conditions for a ring to be Noetherian and Artinian*, Comm. Algebra 30 (2002), 305-308.

6.2) S. Plubtieng, *On simple Noetherian rings*, To appear in Algebra Colloquium, (2003).

6.3) S. Plubtieng, *Rings with many direct summands*, Math. J. Okayama. Univ. 44 (2002), 29-35.

6.4) S. Plubtieng *A generalization of continuous modules and their application to QF-rings*, Kyungpook Math J. 43 (2003), 11-18.

6.5) S. Plubtieng, *Modules charaterized by their proper cyclic subfactors*, To appear in Southeast Asian Bulletin of Mathematics (2004).

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Output จากโครงการวิจัยที่ได้รับทุนจาก สกว.

1. ผลงานวิจัยที่ตีพิมพ์ในวารสารวิชาการระดับนานาชาติ

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2. การนำผลงานวิจัยไปใช้ประโยชน์

2.1) ทำให้นักคณิตศาสตร์ได้ค้นพบทฤษฎีบทที่อธิบายลักษณะเฉพาะของริงแบบเนอแทเรียนซึ่งครอบคลุมทั่วไปกว่าของเดิมซึ่ง Huynh , Rizvi and Yousif ได้ทำการพิสูจน์ไว้

2.2) ทำให้นักคณิตศาสตร์ได้ค้นพบทฤษฎีบททีอธิบายลักษณะเฉพาะของริงแบบอาร์ทีเนียนขึ้นใหม่โดยใช้มอดูซีเอส และมอดูลควอซี-คอนทินิวอัส

2.3) ได้ค้นพบองค์ความรู้ใหม่ที่จะเป็นประโยชน์ในการศึกษาอ้างอิงในสาขา Ring and Module theory เพิ่มขึ้น

2.4) ได้ค้นพบองค์ความรู้พื้นฐานใหม่ที่เป็นประโยชน์ในการประยุกต์เพื่อการค้นคว้าวิจัยในสาขาอื่น ๆ

3. กิจกรรมอื่น ๆ ที่เกียวข้อง ได้แก่

3.1) ผลงานอื่น ๆ

เข้าร่วมการเสนอผลงานวิชาการทางคณิตศาสตร์ ที่มหาวิทยาลัยสงขลานครินทร์ ระหว่างวันที่ 13-14 พฤษภาคม 2545 โดยเสนอผลงานวิจัยเรื่อง " On simple noetherian rings"

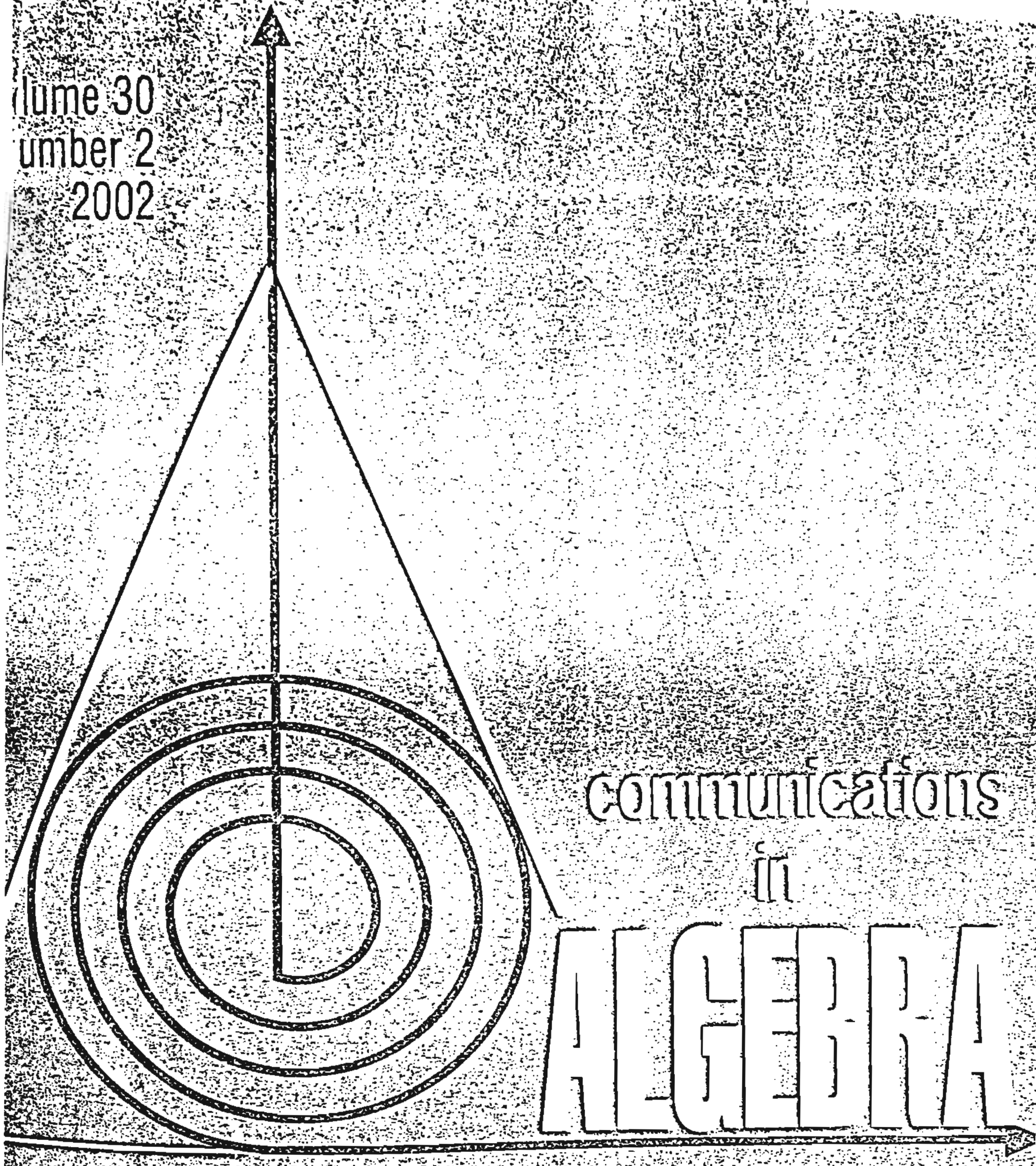
ภาคผนวก 1

CONDITION FOR A RING TO BE NOETHERIAN AND ARTINIAN

Somyot Plubtieng and Hansuk Tansee

COMMUNICATIONS IN ALGEBRA, 30(2), 78-786(2002)

ภาคผนวก 1/2



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COMMUNICATIONS IN ALGEBRA, 30(2), 783–786 (2002)

CONDITIONS FOR A RING TO BE NOETHERIAN OR ARTINIAN

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ABSTRACT

Let M be a finitely generated module. It is shown that M is noetherian iff every finitely generated module in $\sigma[M]$ is a direct sum of a projective module and a module Q, where Q is either CS or noetherian.

A module M is called a *CS-module* (or an *extending module* [1]) if every submodule of M is essential in a direct summand of M. The study of rings over which finitely generated right modules are CS was initiated by Huynh, Rizvi and Yousif [2]. They were shown that such rings must be right noetherian. On the other hand, Huynh showed in [3] that a ring R is right noetherian if and only if every cyclic right R-module is injective or a direct sum of a projective module and a noetherian module.

In this note we consider some properties of 2-generated modules in Mod-R which make R to be right noetherian or right artinian. For the basic definitions and properties of rings, modules and categories we refer to Wisbauer [4].

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Lemma 1. *If U_R (resp. S_R) is a uniform (resp. simple) right R-module such that $U \oplus S$ is CS, then S is $(U/Soc(U))$-injective.*

Proof: See [5]. □

Lemma 2. *Let M_R be a finitely generated module such that $M/Soc(M)$ is noetherian. If for any direct summand N of M and for any submodule $H \subseteq Soc(N)$, N/H is a direct sum of a projective module and a module of finite uniform dimension, then M is noetherian.*

Proof. Using the same argument presented in the last part of the proof of [6, Theorem 1]. □

Lemma 3. *Let M_R be a finitely generated module. If every finitely generated module in $\sigma[M]$ is either CS or noetherian, then M is noetherian.*

Proof. By [1, Corollary 9.4], every factor module of M has finite uniform dimension. If M is not noetherian, then M must be CS. Hence there is a non-noetherian uniform direct summand U of M. It implies that U is finitely generated, $U/Soc(U)$ is also non-noetherian, and every factor module of U has finite uniform dimension. For any simple module $S \in \sigma[M]$, $S \oplus U$ is CS because it is finitely generated and non-noetherian. By Lemma 1, S is $U/Soc(U)$-injective. This shows that $U/Soc(U)$ is a V-module. Hence $U/Soc(U)$ is noetherian by [7], a contradiction. Thus M must be noetherian. □

A module M is said to satisfy the condition (*) if every finitely generated module in $\sigma[M]$ is a direct sum of a projective module and a module Q, where Q is either CS or noetherian.

The condition (*) was first considered in [8] for cyclic modules in Mod-R, where "CS" is replaced by "injective".

Theorem 4. *If M is a finitely generated module satisfying (*), then M is noetherian.*

Proof. Let E be an essential submodule of M, then $N = M/E$ is a singular module. It implies that each finitely generated module in $\sigma[N]$ is either CS or noetherian. Hence N is noetherian by Lemma 3. Thus $M/Soc(M)$ is noetherian by [1, 5.15(1)].

Next, if K is a direct summand of M and N be a submodule of $Soc(K)$, then by hypothesis, K/N is a direct sum of a projective module P and a module Q, where Q is noetherian or CS. Clearly, Q is finitely generated, and $Q/Soc(Q)$ is noetherian. Hence, $Soc(Q)$ must have finite length also in case Q being CS (see [1, 9.1]). We have shown that, in any case, the uniform dimension of Q is finite. Now applying Lemma 2 we get that M is noetherian. □

From the proofs of Lemma 3 and Theorem 4 we see that, if we put $M_R = R$, then Lemma 3 and Theorem 4 remain true for R even if we make the same assumption only on 2-generated right R-modules. That means we obtain the following.

Corollary 5. *A ring R is right noetherian if and only if every 2-generated right R-module is a direct sum of a projective module and a module that is either CS or noetherian.*

We remark that if we assume the same only for cyclic right R-modules then the situation is much more complicated, and we don't know yet if R is right noetherian or not. It was shown in [8] by Huynh-Rizvi, that a ring R is right noetherian iff every *cyclic* right R-module is a direct sum of a projective module and a module that is either injective or noetherian.

Corollary 6. *A ring R is right artinian if and only if every 2-generated right R-module is either quasi-continuous or of finite length.*

Proof. One direction is clear. Now consider R with every 2-generated right R-module quasi-continuous or of finite length. By Corollary 5, R is right noetherian. If R is not right artinian, then $(R \oplus R)_R$ is quasi-continuous. Hence R is right self-injective. Then, it is known that R is right (and left) artinian, a contradiction. Hence R must be right artinian. □

Proposition 7. *Let R be a ring such that every 2-generated right R-module is a direct sum of a projective module and a quasi-continuous module. If R/J is von Neumann regular, then R is a right artinian ring with $J^2 = 0$, where J is the Jacobson radical of R.*

Proof. From the assumption it follows that every singular 2-generated right R-module is quasi-continuous. Hence by [9], $R/Soc(R_R)$ is right SI with $J^2 = 0$. Moreover, by Corollary 5, R is right noetherian. As R/J is von Neumann regular, R is semiprimary. Thus R is right artinian, as desired. □

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ภาคผนวก 2

RINGS WITH MANY DIRECT SUMMANDS

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RINGS WITH MANY DIRECT SUMMANDS

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RINGS WITH MANY DIRECT SUMMANDS

SOMYOT PLUBTIENG

ABSTRACT. It is shown that (i) a ring R is semiperfect if every cyclic right R-module is a direct sum of a continuous module and a semiperfect module, (ii) a ring R is right noetherian if and only if every countably generated right R-module is a direct sum of a continuous module and a locally noetherian module, (iii) a ring R is right artinian if and only if every countably generated right R-module is a direct sum of a continuous module and a locally artinian module, (iv) a ring R is right artinian if every countably generated right R-module is a direct sum of a quasi-continuous module and a semisimple module, and (v) a ring R is artinian if every right R-module is a direct sum of a CS-module and a semisimple module.

1. INTRODUCTION

A module M is called a *CS-module* (or an *extending module* [6]) if every submodule of M is essential in a direct summand of M. CS-modules generalize quasi-continuous modules, which in turn generalize continuous, quasi-injective and injective modules, in this order (see example [11]). It was proved by Jain and Mohamed [10] that a ring R is semiperfect if every cyclic right R-module is a continuous module. The study of rings, via decomposition properties of cyclic or finitely generated modules, was originally initiated by P. F. Smith in [14]. A famous theorem of Chatters stated that a ring R is right noetherian if and only if every cyclic right R-module is a direct sum of a projective module and a noetherian module (see [2, Theorem 3]). Moreover, Dung and Smith showed in [5, Theorem 7 and Theorem 11] that a ring R is right and left artinian, right and left serial with $J(R)^2 = 0$ if and only if every right R-module is a CS-module, and if and only if every cyclic right R-module is a direct sum of an injective module and a semisimple module. On the other hand, it was shown in [6, 20.9] that there is a non-right noetherian ring R for which every cyclic right R-module is a direct sum of an injective module and a noetherian module. However, we notice that a ring R is right noetherian if and only if every right R-module is a direct sum of an injective module and a locally noetherian module (see [6, 9.15]).

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In this paper, we consider the related properties in more general settings. First, we shows that a ring R is semiperfect if every cyclic right R-module is a direct sum of a continuous module and a semiperfect module. Further we proves that (i) a ring R is right noetherian if and only if every countably generated right R-module is a direct sum of a continuous module and a locally noetherian module, (ii) a ring R is right artinian if and only if every countably generated right R-module is a direct sum of a continuous module and a locally artinian module, (iii) a ring R is right artinian if every countably generated right R-module is a direct sum of a quasi-continuous module and a semisimple module, and (iv) a ring R is artinian if every right R-module is a direct sum of a CS-module and a semisimple module.

2. Preliminaries

Throughout this paper we consider associative rings R with identity and unitary right R-modules. For a module M we denote by $\mathrm{Soc}(M)$ and $E(M)$ the socle and the injective hull of M, respectively. If $M = \mathrm{Soc}(M)$, then M is called a *semisimple module*. A right R-module M is called *semiperfect* if it is a projective module and every its homomorphic image has a projective cover.

We will refer to Anderson and Fuller [1], Dung, Huynh, Smith and Wisbauer [6] and Wisbauer [15] for undefined notions used in the text, and also for basic facts concerning CS-modules, noetherian rings and artinian rings. We begin with the following lemmas.

Lemma 1 ([3, Theorem 1.24]). *Let R be a prime right Goldie ring and assume that the socle $\mathrm{Soc}(R_R)$ of R is non-zero, then R is semisimple artinian.*

Lemma 2 ([9, Lemma 1]). *Let R be a semiprime right Goldie ring. If every uniform right ideal of R is countably Σ-CS, then R is semisimple artinian.*

Lemma 3. *Let R be a ring which every cyclic right R-module is a direct sum of a projective module and a CS-module. Then every cyclic right R-module has finite uniform dimension.*

Proof. Let X be a cyclic right R-module. We first aim to show that $X/\mathrm{Soc}(X)$ has finite uniform dimension. Let E be an essential submodule of X, and set $M = X/E$. Let $\sigma[M]$ denote the full subcategory of Mod-R whose objects are submodules of M-generated modules. By our assumption, every cyclic module in $\sigma[M]$ is CS. Hence, M has finite uniform dimension by [6, Corollary 9.4]. Therefore, by [6, Lemma 5.14], $X/\mathrm{Soc}(X)$ has finite uniform dimension.

Now we shall use an argument in [13. Theorem 1, p. 160] to show that $\mathrm{Soc}(X)$ is finitely generated. Assume on the contrary that $\mathrm{Soc}(X)$ is infinitely generated. Then we may write $\mathrm{Soc}(X) = H_1 \oplus H_2$, where H_1 and H_2 are infinite direct sums of simple modules. By hypothesis, we have $X/H_1 = \bar{P}_1 \oplus \bar{Q}_1$ where $\bar{P}_1$ is a projective module and $\bar{Q}_1$ is a CS-module. Let Q_1 be the inverse image of $\bar{Q}_1$ in X. Then clearly $\bar{P}_1 \simeq X/Q_1$, and Q_1/H_1 (being isomorphic to $\bar{Q}_1$) is a CS-module. Since $\bar{P}_1$ is projective, $X = Q_1 \oplus Q_2$ for some submodule Q_2 of X. Then $\mathrm{Soc}(X) = \mathrm{Soc}(Q_1) \oplus \mathrm{Soc}(Q_2)$.

Observe that, because $\bar{Q}_1/\mathrm{Soc}(\bar{Q}_1)$ has finite uniform dimension by the above argument, and $\bar{Q}_1$ is a finitely generated CS-module, it follows from [6, Lemma 9.1] that $\mathrm{Soc}(\bar{Q}_1)$ is finitely generated. Hence $\bar{Q}_1$, and so Q_1/H_1, has finite uniform dimension. Since $H_2 \simeq \mathrm{Soc}(X)/H_1$ has infinite uniform dimension, it follows that X/H_1 has infinite uniform dimension. Therefore, this clearly implies that $\mathrm{Soc}(Q_2)$ is infinitely generated since $Q_2 \simeq \bar{P}_1$ has infinite uniform dimension but $Q_2/\mathrm{Soc}(Q_2)$ has finite uniform dimension. Note that

$$X/\mathrm{Soc}(X) \simeq (Q_1/\mathrm{Soc}(Q_1)) \oplus (Q_2/\mathrm{Soc}(Q_2)),$$

where $Q_1 \neq \mathrm{Soc}(Q_1)$ and $Q_2 \neq \mathrm{Soc}(Q_2)$. Hence, $X/\mathrm{Soc}(X)$ has uniform dimension at least 2. Applying the same arguments to the module Q_2, and continuing the process in a similar manner, an obvious induction shows that $X/\mathrm{Soc}(X)$ has infinite uniform dimension, a contradiction. This shows that $\mathrm{Soc}(X)$ is finitely generated, and therefore X has finite uniform dimension, completing our proof. □

3. The Main Results

We start our investigation by proving the following result.

Theorem 4. *Let R be a ring. If every cyclic right R-module is a direct sum of a continuous module and a semiperfect module, then R is semiperfect.*

Proof. Note that every semiperfect module is projective. By our assumption, we see that every cyclic right R-module is a direct sum of a projective module and a continuous module. Hence, by Lemma 3, R has finite uniform dimension. Therefore

$$R = (C_1 \oplus \cdots \oplus C_m) \oplus (S),$$

where each C_i are continuous indecomposable and S is semiperfect. Notice that each C_i has a local endomorphism ring. Since each C_i is finitely generated self-projective, it follows by [15, 41.4] that each C_i is local and $\mathrm{Rad}(C_i) \ll C_i$. Moreover, it follows by [15, 42.5] that $S = S_1 \oplus \cdots \oplus S_n$, where each S_i is local and $\mathrm{Rad}(S) \ll S$. Hence R is a direct sum of local modules and $\mathrm{Rad}(R) \ll R$. Therefore R is semiperfect by [15, 42.6]. □

Corollary 5. *A ring R is semiperfect if every cyclic right R-module is a direct sum of a continuous module and a module of finite length.*

Proof. It follows [6, Corollary 9.4] that R has finite uniform dimension. Then as done in the proof of Theorem 4, we get R is semiperfect since every indecomposable module of finite length has a local endomorphism ring. □

Lemma 6. *If $R^{(\mathbb{N})}$ is quasi-continuous, then R is QF.*

Proof. First we observe that $R^{(\mathbb{N})} \oplus R^{(\mathbb{N})}$ is isomorphic to $R^{(\mathbb{N})}$. Since $R^{(\mathbb{N})}$ is quasi-continuous, it follows that $R^{(\mathbb{N})}$ is self-injective. This implies that $R^{(\mathbb{N})}$ is injective and hence, by [7. Proposition 20.3A], R is Σ-injective. Therefore R is a QF-ring by [6, 18.1]. □

We are now in position to prove our main results.

Theorem 7. *A ring R is noetherian if and only if every countably generated right R-module is a direct sum of a continuous module and a locally noetherian module.*

Proof. The necessity is clear since every right module over a noetherian ring is locally noetherian. Conversely, we first observe that $R_R = C \oplus N$, where C is continuous and N is noetherian. By [6, Corollary 9.4], it follows that R_R has finite uniform dimension. Hence

$$R_R = (\bigoplus_{\alpha\in I} C_\alpha) \oplus (\bigoplus_{\beta\in J} N_\beta),$$

where each C_α is indecomposable continuous and each N_β is indecomposable noetherian. Without loss of generality we clearly may assume that each C_α is non-noetherian. Set $\bar{R} = R/N \cong (\bigoplus_{\alpha\in I} C_\alpha)$. We aim to show that $\bar{R}$ is noetherian. By our assumption, we obtain that

$$\bar{R}^{(\mathbb{N})} = \bar{C} \oplus \bar{N},$$

where $\bar{C}$ is continuous and $\bar{N}$ is locally noetherian. Since $\bar{R}^{(\mathbb{N})} \cong \bigoplus_{\alpha\in I} C_\alpha{}^{(\mathbb{N})}$ and each C_α has a local endomorphism ring, it follows by the Azumaya theorem that every non-zero direct summand of $\bar{R}^{(\mathbb{N})}$ has an indecomposable direct summand K which is isomorphic to some $C_\gamma \in \{C_\alpha \mid \alpha \in I\}$. If $\bar{N} \neq 0$, then there exists $C_\gamma \in \{C_\alpha \mid \alpha \in I\}$ which is isomorphic to a direct summand of $\bar{N}$. Hence C_γ is noetherian since it is finitely generated, a contradiction. Then $\bar{N} = 0$ and hence $\bar{R}^{(\mathbb{N})}$ is a continuous module. It follows by Lemma 6 that $\bar{R}^{(\mathbb{N})}$ is a QF-ring and hence it is right noetherian. Therefore R is right noetherian, as desired. □

As a consequence of Theorem 7 we obtain the following characterization of artinian rings.

Corollary 8. *A ring R is right artinian if and only if every countably generated module is a direct sum of a continuous module and a locally artinian module.*

Proof. The necessity is clear since every right module over an artinian ring is locally artinian. Conversely, we first see that $R_R = C \oplus A$, where C is continuous and A is artinian. Without loss of generally we may write $C = \bigoplus_{i=1}^{n} C_i$, where each C_i is an indecomposable continuous non-artinian module. By using an argument similar to that of the proof of Theorem 7 we can verify that R/A is a QF-ring. Therefore R is right artinian. □

Lemma 9. *Let R be a prime right noetherian ring. If every countably generated right R-module is a direct sum of a CS-module and a semisimple module, then R is semisimple artinian.*

Proof. If $\mathrm{Soc}(R_R) \neq 0$, then it follows by Lemma 1 that R is semisimple artinian. We now suppose that $\mathrm{Soc}(R_R) = 0$. Let U be any uniform right ideal of R. By our assumption, $U^{(\mathbb{N})} = C \oplus S$, where C is CS and S is semisimple. Since $\mathrm{Soc}(U^{(\mathbb{N})}) = 0$, it follows that U is countably Σ-CS. This implies that every uniform right ideal of R is countably Σ-CS. Hence R is semisimple artinian by Lemma 2, which is a contradiction to our assumption. Therefore R is right semisimple artinian, completing our proof. □

Theorem 10. *A ring R is right artinian if every countably generated right R-module is a direct sum of a quasi-continuous module and a semisimple module.*

Proof. We first show that R is right noetherian. Without loss of generality we may assume that R is quasi-continuous and $R = \bigoplus_{\alpha\in I} C_\alpha$, where each C_α is an indecomposable non-simple quasi-continuous module. For convenience, we write

$$Q = R^{(\mathbb{N})} = \bigoplus_{\gamma\in\Gamma} C_\gamma,$$

where each C_γ is isomorphic to some C_α in $\{C_\alpha \mid \alpha \in I\}$. By assumption, $\bigoplus_{\gamma\in\Gamma} C_\gamma = R^{(\mathbb{N})} = C \oplus S$ where C is a CS-module and S is a semisimple module. Since every quasi-injective module has the exchange property, it follows that $\bigoplus_{\gamma\in\Gamma} C_\gamma = S \oplus (\bigoplus_{\gamma\in\Gamma_0} C_\gamma)$ for some $\Gamma_0 \subseteq \Gamma$. Assume $\Gamma_0 \subset \Gamma$. Then $S \cong \bigoplus_{\gamma\in(\Gamma\setminus\Gamma_0)} C_\gamma$ and hence each C_γ is simple, a contradiction. Thus $S = 0$ and therefore R is right countably Σ-CS.

For any $\alpha \in I$, we see that $C_\alpha \oplus C_\alpha = Q \oplus S_1$ where Q is quasi-continuous and S_1 is semisimple. If $S_1 \neq 0$, $C_\alpha \cong S_1$ by the exchange property of S_1, a contradiction. Hence $S_1 = 0$ and therefore $C_\alpha \oplus C_\alpha$ is a quasi-continuous module for each $\alpha \in I$. It follows by [11, Theorem 2.13] that each C_α is quasi-injective and hence each C_α has a local endomorphism ring. Therefore

R is a semiperfect ring. By [8, Theorem 1], we obtain that R is right Σ-CS and hence R has ACC on right annihilators by [4, Corollary 2.9]. It implies by [12, Lemma 3] that every CS right R-module is a direct sum of uniform submodules. Therefore R is right noetherian by [12, Theorem 4].

Next, we aim to show that R is right artinian. Assume on contrary that R is not right artinian. Then, by [7, 18.34B], there exist a prime ideal P of R such that R/P is not right artinian. Put $\bar{R} = R/P$. Hence $\bar{R}$ is a prime right noetherian ring such that every countably generated right module over factor ring $\bar{R}$ is a direct sum of a CS module and a semisimple module. It follows by Lemma 9 that $\bar{R}$ is right artinian, a contradiction. Therefore R is right artinian, completing our proof. □

Corollary 11. *A ring R is right artinian if every right R-module is a direct sum of a CS-module and a semisimple module.*

Proof. Using the same argument presented in the proof of Theorem 10. □

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ภาคผนวก 3

ON SIMPLE NOETHERIAN RINGS

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On Simple Noetherian Rings*

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Abstract. A module M is called a CS-module (or extending module [5]) if every submodule of M is essential in a direct summand of M. It is shown that (i) a simple ring R is right noetherian if and only if every cyclic singular right R-module is either a CS-module or a noetherian module; (ii) for a prime ring R, if every proper cyclic right R-module is a direct sum of a quasi-injective module and a finitely cogenerated module, then R is either semisimple artinian or a right Ore domain; and (iii) a prime ring R is right noetherian if and only if every cyclic right R-module is a direct sum of a quasi-injective module and a noetherian module.

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Keywords: simple rings, prime rings, noetherian rings, artinian rings

1 Introduction

The class of simple noetherian rings is a topic of considerable interest in ring theory and has been extensively studied by many authors (see, for example, [2, 3, 10]). From a theorem of Osofsky and Smith [14], it follows that if every cyclic right module over a ring R is CS, then every cyclic right R-module has finite uniform dimension. However, in general, such a ring need not be right noetherian. Furthermore, Huynh, Jain, and Lopez-Permouth [10] considered the problem when a simple ring is noetherian, and proved that a simple ring R is right noetherian if every cyclic singular right R-module is CS.

Rings over which proper cyclics are injective (called *PCI-rings*) have been studied by many authors, including Cozzens, Damiano, Faith, Boyle,

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Goodearl, and others. According to Cozzens and Faith [3, Theorem 6.13], it is proved that a right PCI-ring is either semisimple or a right semihereditary simple domain. In [4], Damiano showed that a right PCI-ring is either semisimple, or a simple right noetherian, right hereditary, right Ore V-domain. A ring R is called a *right PCQI-ring* if every proper cyclic right R-module is quasi-injective. It is known that a right PCQI-ring is either semiperfect or prime (see, for example, [13]). It is shown in [10] that a prime PCQI-ring is either artinian or a right Ore domain. Consequently, they proved that a simple PCQI-ring is either artinian or a right noetherian hereditary domain. On the other hand, Huynh and Dung [7] proved that a ring R is right artinian if and only if every cyclic right R-module is a direct sum of an injective module and a finitely cogenerated module.

In this paper, we follow this investigation and aim to show a similar result under weaker sufficient conditions. First, we show that a simple ring R is right noetherian if and only if every cyclic singular right R-module is either a CS-module or a noetherian module; and a prime ring R is either semisimple artinian or a right Ore domain if every proper cyclic right R-module is a direct sum of a quasi-injective module and a finitely cogenerated module. Finally, we prove that a prime ring R is right noetherian if and only if every cyclic right R-module is a direct sum of a quasi-injective module and a noetherian module; and a prime ring R is right noetherian if and only if every 2-generated right R-module is a direct sum of a continuous module and a noetherian module.

2 Preliminaries

Throughout this paper, we consider associative rings R with identity and unitary right R-modules. For a module M, we denote by $\mathrm{Soc}(M)$ and $E(M)$ the socle and the injective hull of M, respectively. If $M = \mathrm{Soc}(M)$, then M is called a *semisimple module*. For a ring R and $x \in R$, we set $r_R(x) = \{a \in R \mid xa = 0\}$. The set

$$Z(R_R) = \{x \in R \mid r_R(x) \text{ is essential in } R_R\}$$

is an ideal of R, called *the right singular ideal* of R. In case $Z(R_R) = 0$, R is called *right non-singular*. For a module M, the Krull dimension of M is defined in [6].

We will refer to [1, 5] for undefined notions used in the text, and also for basic facts concerning CS-modules, simple rings, noetherian rings, and artinian rings. We record here some known results which will be used repeatedly in the sequel.

Lemma 2.1. [15, Proposition 4.3] *Let U be a uniform right R-module and S a simple right R-module such that $U \oplus S$ is CS. Then S is $U/\mathrm{Soc}(U)$-injective.*

Lemma 2.2. [8, Corollary 4] *For a right ideal A of a semiprime right Goldie ring R, the following conditions are equivalent*:
(1) A_R *is semisimple.*
(2) A_R *is quasi-injective.*
(3) A_R *is injective.*

3 The Main Results

We start our investigation by proving the following result. The proof below is inspired by some ideas in the proof of [10, Theorem A].

Theorem 3.1. *A simple ring R is right noetherian if and only if every cyclic singular right R-module is either a CS-module or a noetherian module.*

Proof. If $\mathrm{Soc}(R_R) \neq 0$, then $R = \mathrm{Soc}(R_R)$, and we are done. Hence, we consider the case $\mathrm{Soc}(R_R) = 0$. Let $M = R/E$, where E is an essential right ideal of R. Then by hypothesis and by [5, Corollary 9.4], M has finite uniform dimension. By [5, Lemma 5.14], $R/\mathrm{Soc}(R_R)$, and hence, R has finite uniform dimension. This implies that R is right Goldie.

Assume there is an essential right ideal $E \subseteq R$ such that $M = R/E$ is not noetherian. Note that each factor of M has finite uniform dimension. Let S be a singular simple module. Then by [10, Lemma 3.1], $S \oplus M$ is cyclic. By hypothesis, $S \oplus M$ must be CS. Hence, by Lemma 2.1, S is $M/\mathrm{Soc}(M)$-injective. Therefore, $M/\mathrm{Soc}(M)$ is a V-module. By [12], $M/\mathrm{Soc}(M)$ is noetherian, hence M is noetherian, a contradiction. Therefore, for each essential right ideal $E \subseteq R$, R/E is noetherian. By [5, 5.15], $R/\mathrm{Soc}(R_R)$ is right noetherian. But $\mathrm{Soc}(R_R) = 0$. Thus, R is right noetherian, as desired. □

From Theorem 3.1, we immediately obtain the following.

Corollary 3.2. *A simple ring R is right noetherian if and only if every proper cyclic right R-module is a direct sum of a projective module and a module Q, where Q is either a CS-module or a noetherian module.*

Proof. It follows by Theorem 3.1 since every cyclic singular right R-module is proper cyclic. □

Proposition 3.3. *Let R be a ring such that every proper cyclic right R-module is a direct sum of a CS-module and a module of finite uniform dimension, or a direct sum of a projective module and a noetherian module. Then every cyclic right R-module has finite uniform dimension.*

Proof. Let X be a cyclic right R-module and E an essential submodule of X. Clearly, $X/E \not\cong R_R$. Moreover, we see that any cyclic subfactors of X/E cannot contain any projective submodule and so they are not isomorphic to R_R. Hence, by assumption, they are either a direct sum of a CS-module and a module of finite uniform dimension, or noetherian. Therefore, by [5, Corollary 9.4], X/E has finite uniform dimension. Hence, $X/\mathrm{Soc}(X_R)$ has finite uniform dimension by [5, Lemma 5.14]. To finish the proof, it suffices to show that $\mathrm{Soc}(X)$ is finitely generated. Assume on the contrary that $\mathrm{Soc}(X)$ is infinitely generated. Then we may write $\mathrm{Soc}(X) = W \oplus V$, where W and V are infinite direct sums of simple modules. Since W cannot be a direct summand of X, it follows that X/W is not projective. In particular, $X/W \not\cong R_R$. Now we can apply the same argument preceding in the proof of [9, Theorem 2.10] to arrive at a contradiction. Hence, $\mathrm{Soc}(X)$ must be of finite length, and so X has finite uniform dimension, completing the proof. □

We are now in a position to prove the main result.

Theorem 3.4. *Let R be a prime ring. If every proper cyclic right R-module is a direct sum of a quasi-injective module and a finitely cogenerated module, then R is either semisimple artinian or a right Ore domain.*

Proof. Assume on the contrary that $Z(R_R) \neq 0$. Since R is prime, this implies that $\mathrm{Soc}(R_R) = 0$. Therefore, by hypothesis, for any $0 \neq x \in Z(R_R)$, xR is quasi-injective. Thus, the argument in the proof of [10, Theorem 2.2] can be applied to get a contradiction. Hence, R is right non-singular.

Next, by [5, Corollary 9.4], R has finite right uniform dimension. Hence, R is right Goldie. On the other hand, by hypothesis, the right R-module R_R contains an essential submodule $E = S \oplus T$, where S is semisimple and T is quasi-injective. By Lemma 2.2, both S and T are quasi-injective. Therefore, E is injective. Thus, $R = E$, and so R is semisimple artinian. □

In [8, Proposition 8], Huynh, Dung, and Smith proved that a semiprime ring R is right noetherian if every cyclic right R-module is a direct sum of an injective module and a noetherian module. A question arises naturally whether or not the same statement holds if "injective" is replace by "quasi-injective". Using [10, Theorem 2.2], we can answer the question positively in the case of prime rings.

Proposition 3.5. *A prime ring R is right noetherian if and only if every cyclic right R-module is a direct sum of a quasi-injective module and a noetherian module.*

Proof. Let R be a ring such that every cyclic right R-module is a direct sum of a quasi-injective module and a noetherian module. We first prove that R is right non-singular. By [5, Corollary 9.4], R_R has finite uniform dimension. Assume on the contrary that the right singular ideal $Z(R)$ of R is non-zero. If every cyclic submodule of $Z(R)$ is quasi-injective, then the argument in the proof of [10, Theorem 2.2] will produce a contradiction to the primeness of R. Hence, by hypothesis, $Z(R)$ contains a cyclic non-zero noetherian submodule N. Moreover, N has Krull dimension. Then we can apply an argument presented in [8] to arrive at another contradiction. Thus, $Z(R) = 0$, i.e., R is right non-singular. Now $R_R = Q \oplus M$, where Q is quasi-injective and M is noetherian. By Lemma 2.2, Q_R is semisimple. Thus, R is right noetherian. □

It is unknown if Proposition 3.5 holds true for semiprime rings. But we know that, in general, Proposition 3.5 is incorrect for non-semiprime rings. See an example in [11].

Proposition 3.6. *A prime ring R is right noetherian if and only if every 2-generated right R-module is a direct sum of a continuous module and a noetherian module.*

Proof. One direction is clear. Conversely, let R be a ring such that each 2-generated right module is a direct sum of a continuous module and a noetherian module. By [5, Corollary 9.4], R has finite uniform dimension. Assume the right singular ideal $Z(R)$ is non-zero. If $Z(R)$ contains a non-zero cyclic noetherian submodule C, then as done in the proof of Proposition 3.5, we get a contradiction. Hence, each cyclic submodule of $Z(R)$ must be continuous and contains no non-zero noetherian submodules. Let $0 \neq xR \subseteq Z(R)$ be a uniform module. Since $xR \oplus xR$ does not contain non-zero noetherian submodules, it must be continuous by the hypothesis. Hence, xR is quasi-injective. Now again by an argument in the proof of [10, Theorem 2.2], we obtain a contradiction. Hence, $Z(R) = 0$, i.e., R is right non-singular.

Assume R is not right noethrian. Then by using the hypothesis, we can prove that R contains a cyclic uniform right ideal U that is not noetherian. Hence, $U \oplus U = D \oplus N$, where D is continuous and N is noetherian. If $N \neq 0$, then either $(U, 0)$ or $(0, U)$ embeds in N because either of them has zero intersection with D. This is a contradiction. Hence, $N = 0$ and so U is quasi-injective. By Lemma 2.2, U is semsimple (of finite length), contradicting the assumption about U. Thus, R must be right noetherian. □

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ภาคผนวก 4

A GENERALIZATION OF CONTINUOUS MODULES AND THEIR APPLICATION TO QF-RINGS

Somyot Plubtieng

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A Generalization of Continuous Modules and Their Application to QF-rings

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A Generalization of Continuous Modules and Their Application to QF-rings

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ABSTRACT. A ring R is called quasi-Frobenius, briefly QF, if R is right or left artinian and right or left self-injective. In this note we study the new class of modules which generalize the concept of continuous (quasi-continuous) modules, that is, ec-continuous (ec-quasi-continuous) modules. We prove the decomposition theorems for ec-quasi-continuous modules, and we also give a sufficient condition for ec-continuous to be continuous. Moreover, we give a characterization of QF-rings with the injectivity condition replaced by ec-continuity.

1. Introduction

A well-known result of Faith [6] asserts that a right self-injective rings with ACC on right or left annihilators is QF. On the other hand, it is shown in Clark and Huynh [4] that a right self-injective semiperfect ring R is QF if and only if every uniform submodule of any projective right R-module is contained in a finitely generated submodule. In Camillo and Yousif [3], motivated by a result of Faith [6] on self-injective rings, it was shown that a two-sided continuous ring with ACC on left annihilators is QF. Moreover, it was shown in Nicholson and Yousif [9] that a two-sided quasi continuous ring with DCC on essential left ideals is QF.

In this paper we introduce some new notions which generalize the concept of continuous modules (rings) and quasi-continuous modules (rings) (that is, ec-continuous modules (rings) and ec-quasi-continuous modules(rings), respectively). We prove that a right ec-continuous module M_R is a direct sum of uniform modules if one of the following holds: (i) R satisfies ACC on right ideals of the form $r_R(m)$, for all $m \in M$, or (ii) M has ACC (or DCC) on essential submodules and $Soc(M)$ is essential in a direct summand of M.

Moreover, we will consider the following property for a given ring R;
(*) Every uniform submodule of $R^{(\mathbb{N})}$ is contained in a finitely generated sub-

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module.

(**) $(eR)^{(2)}$ is a CS-module for each primitive idempotent $e \in R$.

We also show that a right ec-continuous semiperfect ring R is QF if and only if R satisfies the conditions (*) and (**). Finally, we prove that a right ec-continuous ring R which satisfies (**) is QF if one of the following holds: (i) R has ACC on right annihilators, or (ii) $Soc(R)$ is essential in a direct summand of R and R has ACC (or DCC) on essential right ideals.

2. Definitions and preliminaries

Throughout this paper all rings are associative with identity and all modules are unitary right R-modules. Consider the following conditions on a module M_R:

$(C1)$ Every submodule of M is essential in a direct summand of M.

$(C2)$ Every submodule isomorphic to a direct summand of M is itself a direct summand.

$(C3)$ If M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.

The module M is called *continuous* if it satisfies conditions $(C1)$ and $(C2)$, *quasi-continuous* if it satisfies $(C1)$ and $(C3)$, and a *CS-module* (or *extending module*) if it satisfies condition $(C1)$ only. Recall that a module which has a cyclic essential submodule is said to be *essentially cyclic*. A right R-module M is called *ec-CS* if it satisfies the condition:

$(C1')$ Every essentially cyclic submodule of M is essential in a direct summand of M (see [12]).

The module M is called *ec-continuous* if it satisfies condition $(C1')$ and $(C2)$, and *an ec-quasi-continuous module* if it satisfies conditions $(C1')$ and $(C3)$. A ring R is called *right ec-continuous (ec-quasi-continuous)* if the module R_R is ec-continuous (ec-quasi continuous).

A ring R is called *right finitely continuous* if any finitely generated right ideal is essential in a direct summand and satisfying the condition $(C2)$ (see [2]). In [14], Utumi proved that left PF rings are right finitely continuous. We notice that if A, B and C are submodules of a non-singular right R-module M with $A \subseteq^{ess} C \subseteq^{\oplus} M$ and $A \subseteq^{ess} B$, then B is contained in C (see, for example, [11, Remark 1, p.693]). Hence left nonsingular PF-rings are right ec-continuous.

Lemma 1. *A right R-module M has $(C1')$ if and only if every closed essentially cyclic submodule of M is a direct summand of M.*

Proof. The necessity is clear. For the sufficiency, let N be an essentially cyclic submodule of M. By Zorn's lemma, N has a maximal essential extension L in M. Clearly L is closed and essentially cyclic, hence it is a direct summand. □

Lemma 2. *Let M be an ec-continuous right R-module. Then*

(1) *every direct summand of M is also ec-continuous, and*

(2) *if M is indecomposable then the endomorphism ring $End(M)$ is local.*

Proof. (1) This is clear by Lemma 1 and the modular law.

(2) It is immediate since every indecomposable ec-continuous module is continuous. □

Lemma 3. *Let M be an ec-quasi continuous right R-module. If M has an indecomposable decomposition $M = \oplus_{\alpha \in I} M_\alpha$, then that decomposition complements direct summands.*

Proof. See in [8, Theorem 2.22 (1)⇒ (2)]. □

3. On the ec-quasi continuous modules

Next, we have the decomposition theorems for an ec-quasi-continuous module with chain conditions. In this result, we replace the conditions $(C1)$ in [10, Lemma 3] by the conditions $(C1')$ and $(C3)$. For any $m \in M_R, r_R(m)$ will denote $\{r \in R \mid mr = 0\}$.

Proposition 4. *Let M_R be an ec-quasi continuous module and R satisfies ACC on right ideals of the form $r_R(m)$, $m \in M$. Then M is a direct sum of uniform modules.*

Proof. First, we show that M contain a maximal local direct summand $N = \oplus_{\alpha \in I} N_\alpha$, with N_α uniform for each $\alpha \in I$. Let m be a non-zero element of M such that $r_R(m)$ is maximal in $\{r_R(m) \mid 0 \neq m \in M\}$. There exists a direct summand K of M such that $mR \subseteq^{ess} K$. Suppose that K is not indecomposable. Then there exist non-zero submodules K_1 and K_2 of K such that $K = K_1 \oplus K_2$. Since $m \in K = K_1 \oplus K_2, m = m_1 + m_2$ for some $m_i \in K_i$ $(i = 1, 2)$. If $m_1 = 0$ then $m = m_2 \in K_2$, and $mR \cap K_1 = 0$ giving $K_1 = 0$, a contradiction. Thus $m_1 \neq 0$. Clearly $r_R(m) \subseteq r_R(m_1)$. Hence $r_R(m) = r_R(m_1)$, by the choice of m. Similarly $m_2 \neq 0$, and $r_R(m) = r_R(m_2)$. Because $m_1 \neq 0$ there exist $r_1, r_2 \in R$ such that $0 \neq m_1 r_1 = m r_2 = (m_1 + m_2) r_2 = m_1 r_2 + m_2 r_2$. Thus $m_2 r_2 = 0$, and hence $r_2 \in r_R(m_2) \backslash r_R(m)$, a contradiction. Thus, because K is ec-CS, K is uniform. By Zorn's Lemma M contain a maximal local direct summand $N = \oplus_{\alpha \in I} N_\alpha$ where N_α is an uniform submodule of M for each $\alpha \in I$. Claim that $N \subseteq^{ess} M$. Assume that there exists a non-zero element $m \in M$ such that $mR \cap N = 0$. Let $0 \neq y \in M$ such that $r_R(y)$ is maximal in the set of the form $\{r_R(m) \mid 0 \neq m \in M \text{ and } mR \cap N = 0\}$. Note that $yR \subseteq^{ess} N'$ for some $N' \subseteq^{\oplus} M$. By the above argument N' is a uniform module. Then by the condition $(C3)$, $N \oplus N'$ is a local direct summand, a contradiction to the choice of N. Hence $N \subseteq^{ess} M$ and by [5, Lemma 8.1], we have $N = M$. Therefore M is a direct sum of uniform modules. □

Now, we extend Theorem 25.6 in [1] for self-injective rings to ec-quasi continuous rings.

orollary 5. *For a ring R the following statements are equivalent:*

(a) *R is right noetherian;*

(b) *Every ec-quasi continuous right R-module is a direct sum of indecomposable modules;*

(c) *Every ec-quasi continuous right R-module has a decomposition that complements maximal direct summands.*

Proposition 6. *Let M be a finitely generated ec-quasi continuous module with $Soc(M)$ essential in a direct summand of M. If M has ACC (or DCC) on essential submodules then M is a direct sum of uniform modules.*

Proof. It follows from [5, Theorem 5.15], that $M/Soc(M)$ is noetherian (or artinian, respectively). Hence $M/Soc(M)$ has finite uniform dimension and so it has ACC on direct summands. Let $S = Soc(M)$. Then $M = M_1 \oplus M_2$, where $S \subseteq^{ess} M_1$.

We now show that S is finitely generated, which would imply that M has finite uniform dimension. Assume on the contrary that S is infinitely generated. Then we may write $S = \oplus_{i=1}^{\infty} S_i$, where each S_i is a simple module. Since M_1 is ec-quasi-continuous, there are submodules $N_1, N_2, ..., N_n, ...$ of M_1 such that $S_i \subseteq^{ess} N_i$ and $\oplus_{i=1}^{n} N_i$ is a direct summand of M_1 for each $n \geq 1$. Clearly M_1 is finitely generated and so $S_i \neq N_i$ for infinite many i. For all $i \geq 1$, set $K_i = N_i$ whenever $N_i \neq S_i$. By [5, Lemma 18.4], $(K_1 + S)/S \oplus (K_2 + S)/S \oplus ...$ is a direct sum of submodules of M_1/S and $\bigoplus_{i=1}^{n}(K_i + S)/S$ is a direct summand of M_1/S for each $n \geq 1$. By ACC on direct summands, this process must stop. Hence there exists a natural number m such that $K_{m+j} \subseteq S$, and therefore $K_{m+j} = S_{m+j}$ for all $j \geq 1$, which is a contradiction. This implies that S is finitely generated and hence M_1 has finite uniform dimension. Therefore M is a finite direct sum of uniform modules. □

Proposition 7. *Let $M = \oplus_{\alpha\in\Omega} M_\alpha$ be an ec-continuous module where each M_α is uniform. Then the following condition are equivalent:*

(a) *every uniform submodule of $M^{\mathbb{N}}$ is essential in a direct summand of $M^{\mathbb{N}}$;*

(b) *$M^{\mathbb{N}}$ is quasi-injective.*

Proof. (b)$\Rightarrow$ (a) is trivial.

(a)$\Rightarrow$ (b) Assume (a). It follows by Lemma 2 that each M_α is continuous and End(M_α) is local. For convenience we write

$$Q = M^{\mathbb{N}} = \oplus_{\gamma\in\Gamma} Q_\gamma,$$

where each Q_γ is isomorphic to some M_α in the set $\{M_\alpha/\alpha \in \Omega\}$. We notice further that, for any $\alpha, \beta \in \Omega$, every monomorphism from Q_α to Q_β is an isomorphism. Hence, by [5, Corollary 8.9], we obtain that $\bigoplus_{\Gamma-\{\alpha_0\}} Q_\alpha$ is Q_{α_0}-injective for all $\alpha_0 \in \Gamma$. Observe that $Q_{\alpha_0} \oplus Q_{\alpha_0}$ is a direct summand of $Q \oplus Q$ which is isomorphic to Q. Hence every uniform submodule of $Q_{\alpha_0} \oplus Q_{\alpha_0}$ is essential in a direct summand of $Q_{\alpha_0} \oplus Q_{\alpha_0}$. By [5, Corollary 8.9], Q_{α_0} is Q_{α_0}-injective. Hence Q is a quasi-injective module, as desired. □

Proposition 8. *Let M be an ec-continuous right R-module. If M has finite uniform dimension then M is continuous.*

Proof. By hypothesis, we obtain that

$$M = M_1 \oplus \ldots \oplus M_n$$

, where each M_i is uniform ec-continuous. It follows by Lemma 2 that each M_i is continuous and hence $End(M_i)$ is local. Note that, for any $i \neq j$, $M_i \oplus M_j$ is ec-continuous. This implies that every uniform submordule of $M_i \oplus M_j$ is essential in a direct summand of $M_i \oplus M_j$. We note that every monomorphism from M_i to M_j is an isomorphism. Hence M_i is M_j-injective by [5, Corollary 8.9] and so $\oplus_{i \neq j} M_i$ is M_j-injective. Therefore, by [8, Theorem 2.13 and Theorem 3.16], M is continuous. The proof of proposition is complete. □

4. Application: quasi-Frobenius rings

First we obtain a condition for ec-continuous ring to be self-injective.

Lemma 9. *Let R be a right ec-continuous semiperfect ring. Then R is right continuous. In addition, if R satisfies the condition (**) then R is self-injective.*

Proof. By [1, Theorem 27.6], there is a complete set of orthogonal primitive idempotents $\{e_1, \ldots, e_n\}$ in R such that

$$R = e_1 R \oplus \ldots \oplus e_n R,$$

where each End$(e_i R)$ is a local ring. Note that each $e_i R$ is indecomposable and hence is uniform by $(C1')$. Clearly, $e_i R \simeq f(e_i R) \subseteq e_j R$ for any $i \neq j$ in I and every monomorphism f from $e_i R$ to $e_j R$. By $(C2)$, $f(e_i R)$ must be a direct summand of R and also of $e_j R$. Hence $f(e_i R) = e_j R$ and therefore f is an isomorphism. Since R_R is a CS module, by [5, Corollary 8.9], $\oplus_{j \in J} e_j R$ is $e_{i_o} R$-injective for every $i_o \in I$, where $J = I \backslash \{i_o\}$. By Lemma 2, each $e_i R$ is continuous. This implies that R is continuous (see, for example, [8, Theorem 3.16]). Then, by previous argument, every monomorphism from $e_{i_o} R$ to itself is an isomorphism. Since $e_{i_o} R_R^{(2)} = e_{i_o} R \oplus e_{i_o} R$ is CS, $e_{i_o} R$ is $e_{i_o} R$-injective by [5, Corollary 8.9]. This implies that R is $e_{i_o} R$-injective for all $i_o \in I$. Hence R is self-injective. □

We now extend a result of Clark and Huynh [4, Theorem 1] for a right self-injective ring to an ec-continuous ring.

Theorem 10. *For a ring R the following statements are equivalent:*

(a) *R is QF;*

(b) *R is a right ec-continuous semiperfect ring satisfying (*) and (**);*

(c) *R is a right ec-continuous semiperfect ring satisfying* $(*)$ *and* $R_R^{(n)}$ *is a CS-module for each natural number* $n \in \mathbb{N}$.

Proof. (a) $\Rightarrow$ (b) This is clear by Clark and Huynh [4, Theorem 1].

(b) $\Rightarrow$ (c) Assume (b). By Lemma 9, R is a right self-injective and hence $R_R^{(n)}$ is an injective module for each $n \in \mathbb{N}$. Therefore $R_R^{(n)}$ is a CS-module for each $n \in \mathbb{N}$.

(c) $\Rightarrow$ (a) Assume (c). There is a complete set of orthogonal primitive idempotents $\{e_1, \ldots, e_n\}$ in R such that $R = e_1R \oplus \ldots \oplus e_nR$, and each endomorphism ring End(e_iR) is local. Since e_i is primitive, e_iR is indecomposable. By Lemma 9, R_R is continuous and hence each e_iR is a uniform continuous module. Note that

$$R_R^{(\mathbb{N})} \simeq (e_1R)^{(\mathbb{N})} \oplus \ldots \oplus (e_nR)^{(\mathbb{N})}.$$

We write $R_R^{(\mathbb{N})}$ in the form $R_R^{(\mathbb{N})} = \oplus_{\alpha\in A} I_\alpha$, where each I_α is isomorphic to some e_iR in $\{e_1R, \ldots, e_nR\}$ and A is an index set. Now we shall use an argument in the proof of Clark and Huynh [4, Lemma 5(i)] to show that every closed uniform submodule of $R^{(\mathbb{N})}$ is essential in a direct summand of $R^{(\mathbb{N})}$. Let V be a closed uniform submodule of $R^{\mathbb{N}}$. Then, by $(*)$, there exists a finite subset F of A such that $V \subseteq \oplus_{\alpha\in F} I_\alpha$. Since R_R^F is a CS-module by our assumption, it follows that $\oplus_{\alpha\in F} I_\alpha$ must be also CS. Hence V is a direct summand of $\oplus_{\alpha\in F} I_\alpha$ and also of $R^{(\mathbb{N})}$.

Now, it follows by Proposition 7 that $R^{(\mathbb{N})}$ is self- injective. Hence R is a Σ-CS module by [5, Corollary 11.13]. Therefore, by [4, Theorem 1], R is a QF ring. Thus our proof is complete. □

Next, we improved the results of Camillo and Yousif [3, Theorem 1] for a two-sided continuous ring and Faith [6, Theorem 2] for a self-injective ring to a one sided ec-continuous ring.

Proposition 11. *A ring R is QF if and only if R is a right ec-continuous ring which satisfies* $(**)$ *and has ACC on right annihilators.*

Proof. By Proposition 4, R is a direct sum of indecomposable modules. Then, there exists a complete set of orthogonal primitive idempotents $\{e_1, \ldots, e_n\}$ such that $R = e_1R\oplus\ldots\oplus e_nR$. Clearly each e_iR is indecomposable ec-continuous. By Lemma 2 each of endomorphism End$(e_iR) \simeq e_iRe_i$ is local. Hence, by [1, Theorem 27.6], R is semiperfect and so it is self-injective, by Lemma 9. Therefore, by Faith [6, Corollary 4], R is a QF ring. □

Proposition 12. *Let R be an ec-continuous ring with Soc(R) essential in a direct summand of R. If R satisfies* $(**)$ *and has ACC on essential right ideals then R is QF.*

Proof. By Proposition 6, R is a direct sum of uniform modules. Thus, there is a complete set of orthogonal primitive idempotents $\{e_1, \ldots, e_n\}$ in R such that $R =$

$e_1R \oplus \ldots \oplus e_nR$. By Lemma 2, we note that each endomorphism ring $\mathrm{End}(e_iR) \simeq e_iRe_i$ is a local ring. Hence R is a semiperfect ring and it implies that R_R is self-injective. Therefore R is QF. □

Finally, we improve Nicholson and Yousif's result [9, Corollary 2] for a two sided quasi-continuous ring to a one sided quasi-continuous rings.

Proposition 13. *A ring R is QF if and only if R is a right quasi-continuous ring which satisfies (**) and has DCC on essential right ideals.*

Proof. The necessity is clear. Suppose that R_R is quasi-continuous ring satisfying (**) and has DCC on essential right ideals. It follows by [5, Theorem 18.5] that R_R is artinian and hence it is semiprimary by [1, Theorem 15.20]. Then, by [9, Lemma 6], R is right continuous. It follows by Lemma 9 that R is self-injective and hence, by [5, Corollary 18.13], R is a QF-ring. □

We conclude the paper with some remarks.

Remark.

(a) The results in this paper remain true (with similar arguments) if the condition "(**)" is replaced by the weaker condition that "every uniform submodules of $(eR)^{(2)}$ is essential in a direct summand of $(eR)^{(2)}$, for each primitive idempotent $e \in R$".

(b) In Theorem 10, we can replace the condition "R is semiperfect" by the condition "R has finite Goldie dimension".

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MODULES CHARACTERIZED BY THEIR PROPER CYCLIC SUBFACTORS

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Modules characterized by their proper cyclic subfactors

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Abstract

For a finitely generated self-projective right R-module M, we show that if every proper cyclic subfactor of M is a direct sum of a CS-module and a module of finite uniform dimension, or a direct sum of an M-projective module and a module Q, where Q is either CS or noetherian, then every factor module of M has finite uniform dimension. Consequently, (i) a ring R is right noetherian if and only if every proper cyclic right R-module is a direct sum of a projective module and a module Q, where Q is either injective or noetherian, and (ii) a ring R is right noetherian if and only if every proper finitely generated right R-module is a direct sum of a projective module and a module Q, where Q is either CS or noetherian.

1. Introduction

The study of noetherian rings, via decomposition properties of cyclic or finitely generated modules, was initiated by P.F. Smith in [13]. On the other hand, A. Chatters state that a ring R is right noetherian if and only if every cyclic right R-module is a direct sum of a projective module and a noetherian module [2]. Later on, Osofsky and Smith proved in [11] that a ring R is right noetherian and hereditary if every cyclic right R-module is a direct sum of a projective module and an injective module. It was shown futher by Huynh and Rizvi (see [9]) that a ring R is right noetherian if and only if every cyclic right R-module is a direct sum of a projective module and a module Q, where Q is either injective or noetherian. Rings over which proper cyclics are injective

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have been studied by many authors, including J.H. Cozzens, R.F. Damiano, C. Faith, and others (see [3],[4] and [6]).

In this paper, we use module-theoretic methods to consider the related properties in more general setting. We first show that for a finitely generated self-projective right R-module M, if every proper cyclic subfactor of M is a direct sum of a CS-module and a module of finite uniform dimension, or a direct sum of an M-projective module and a module Q, where Q is either CS or noetherian, then every factor module of M has finite uniform dimension. Consequently, a ring R is right noetherian if and only if every proper cyclic right R-module is a direct sum of a projective module and a module Q, where Q is either injective or noetherian. We also show that a ring R is right noetherian if and only if every proper finitely generated right R-module is a direct sum of a projective module and a module Q, where Q is either CS or noetherian.

Throughout this paper, all rings R are associative rings with identity and all modules are unitary right R-modules. For a module M we denote by $\sigma[M]$ and $Soc(M)$ the full subcategory of *Mod-R*, whose objects are submodules of M-generated modules, and the socle of M, respectively. If $M = Soc(M)$, then M is called a *semisimple module*. A submodule of a factor module of M is called *subfactor* of M. A cyclic subfactor of M which is not isomorphic to M is called *proper cyclic subfactor of* M. A cyclic right R module C is called a *proper cyclic right R-module* if $C \not\cong R_R$. A finitely generated right R-module X is called *proper* if X is not isomorphic to R_R.

The reader is referred to Anderson-Fuller [1] and Wisbauer [15] for undefined notions, and basic facts concerning injective modules, CS-modules, singular module, noetherian modules and noetherian rings. For the sake of convenience, we cite the following results which will be used in the sequel.

Lemma 1 [14, Proposition 4.3]. *Let U be a uniform right R-module and S a simple right R-module such that $U \oplus S$ is CS then S is $(U/Soc(U))$-injective.*

Lemma 2 [7, p.254]. *A ring R is right noetherian if and only if every cyclic right R-module is injective or a direct sum of a projective module and a noetherian module.*

2. Main Results

We consider the following two conditions for a proper factor module C of a right R-module M:

(*) C is a direct sum of a CS-module and a module of finite uniform dimension;

(**) C is a direct sum of an M-projective module and a module Q, where Q is either a CS-module or a noetherian module.

The condition (**) was first considered in [9] for cyclic modules in Mod-R, where "CS" is replaced by "injective".

The following is our main result:

Theorem 3. *Let M be a finitely generated self-projective right R-module. Assume that every proper cyclic subfactor of M satisfies (*) or (**). Then every factor module of M has finite uniform dimension.*

Proof. Let X be a factor module of M and let E be an essential submodule of X. Then, clearly, X/E is M-singular and hence $X/E \not\simeq M_R$ since M is self-projective. Moreover, we see that any cyclic subfactor of X/E can not contain any non-zero M-projective submodule and so they are not isomorphic to M_R. Hence, by assumption, X/E is a direct sum of a CS-module and a module of finite uniform dimension or X/E is noetherian. Therefore, by [5, Corollary 9.4], we see that X/E has finite uniform dimension. Hence $X/Soc(X_R)$ has finite uniform dimension by [5, Lemma 5.14].

To finish our proof it suffices to show that $Soc(X)$ is finitely generated. Assume on contrary that $Soc(X)$ is infinitely generated. Then we may write $Soc(X) = W \oplus V$, where W and V are infinite direct sums of simple modules. Since W can not be a direct summand of X, it follows that $X/W \not\simeq M_R$. By hypothesis, we have

(1)

$$X/W = \bar{Q} \oplus \bar{F},$$

where $\bar{Q}$ is CS and $\bar{F}$ has finite uniform dimension.

By using the same argument to the module $\bar{Q}$, we see that $\bar{Q}/Soc(\bar{Q})$ has finite uniform dimension. If $Soc(\bar{Q})$ is infinitely generated, then, by [5, Lemma 9.1], $\bar{Q}/Soc(\bar{Q})$ has infinite uniform dimension, a contradiction. Hence $Soc(\bar{Q})$ and therefore X/W has finite uniform dimension. This contradicts X/W has infinite uniform dimension. Therefore the decomposition (1) is not possible and we have

(2)

$$X/W = \bar{P}_1 \oplus \bar{Q}_1,$$

where $\bar{P}_1$ is a projective module and $\bar{Q}_1(\neq 0)$ is either a CS-module or a noetherian module. Let Q_1 be the inverse image of $\bar{Q}_1$ in X. Then clearly $\bar{P}_1 \simeq X/Q_1$, and Q_1/W (being isomorphic to $\bar{Q}_1$) is either a CS-module or a noetherian module. Since $\bar{P}_1$ is projective, $X = Q_1 \oplus Q_2$ for some submodule Q_2 of X. Then $Soc(X) = Soc(Q_1) \oplus Soc(Q_2)$. Note that $\bar{Q}_1$ is a finitely generated right R-module. Thus by the argument as above, $\bar{Q}_1/Soc(\bar{Q}_1)$ has finite uniform dimension. If $\bar{Q}_1$ is CS then, by [5, Lemma 9.1], $Soc(\bar{Q}_1)$ is finitely generated. This implies that $\bar{Q}_1$, and so Q_1/W, has finite uniform dimension. Hence, in any case, $\bar{Q}_1$, and so Q_1/W, has finite uniform dimension. Therefore, $Soc(Q_2)$ is clearly infinitely generated since $Q_2 \simeq \bar{P}_1$ has infinite uniform dimension but $Q_2/Soc(Q_2)$ has finite uniform dimension.

Because $W \subseteq Q_1$ and W is an infinite direct sum of simple modules, Q_1 has an infinitely generated socle. Note that

$$X/Soc(X) \simeq (Q_1/Soc(Q_1)) \oplus (Q_2/Soc(Q_2)),$$

where $Q_1 \neq Soc(Q_1)$ and $Q_2 \neq Soc(Q_2)$. Hence, $X/Soc(X)$ has uniform dimension at least 2. Applying the same arguments to the module Q_2, and continue the process in a similar manner, an obvious induction shows that $X/Soc(X)$ has infinite uniform dimension, which is a contradiction to the fact that $X/Soc(X)$ has finite uniform dimension. This shows that $Soc(X)$ is finitely generated and so X has finite uniform dimension. Therefore every factor module of M has finite uniform dimension. □

Theorem 3 gives immediately the following result.

Corollary 4. *Let R be a ring which every proper cyclic right R-module is a direct sum of a CS-module and a module of finite uniform dimension , or a* *direct sum of a projective module and a module Q, where Q is either CS or noetherian. Then every cyclic right R-module has finite uniform dimension.*

As an application, we have the following corollary.

Corollary 5. *A ring R is right noetherian if and only if every proper cyclic right R-module is a direct sum of a projective module and a module Q, where Q is either injective or noetherian.*

Proof. Let E be an essential right ideal of R. Then R/E is clearly singular and $R/E \not\simeq R_R$. Moreover, every cyclic right (R/E)-module can not contain any projective submodule and so they are not isomorphic to R_R. By hypothesis, they are either an injective module or a noetherian module. Hence, by Lemma 2, R/E is right noetherian and therefore $R/Soc(R_R)$ is right noetherian by [5, 5.15]. By Corollary 4, R has finite uniform dimension and hence $Soc(R_R)$ is finitely generated. Therefore R is right noetherian as desried. □

Proposition 6. *Let M be a finitely generated self-projective right R-module. If every proper cyclic subfactor of M is a direct sum of a projective and a noetherian module then M is a noetherian module.*

Proof. Let E be an essential submodule of M. Then M/E is M-singular and $M/E \not\simeq M$. By hypothesis, M/E is noetherian so that $M/Soc(M)$ is noetherian by [5, 5.15]. Moreover, by Theorem 3, M has finite uniform dimension so that $Soc(M)$ is finitely generated. Hence M is notherian. □

Proposition 7. *Let M be a cyclic self-projective right R-module . If every proper cyclic subfactor of M is a direct sum of a projective module and an M-injective module then M is a noetherian module.*

Proof. Let E be an essentail submodule of M. Then M/E is M-singular and $M/E \not\simeq M_R$. This implies that every cyclic subfactor of $[M/E]$ is a proper cyclic subfactor of M. By hypothesis, it is M-injective and so M/E is a semisimple module. Consequently, by [5, 5.15], $M/Soc(M)$ is noetherian. Moreover, M has finite uniform dimension by Theoerem 3 and hence $Soc(M)$ is a finitely generated right R-module. This shows that M is noetherian. □

Corollary 8 *If every proper cyclic right R-module is a direct sum of a projec-*

tive module and an injective module then R is right noetherian and hereditary.

Proof. By Proposition 7, R is clearly right noetherian. Let L be an injective right R-module and $N \subseteq L$. Then, there is an injective hull $E(N)$ of N in L such that $L = E(N) \oplus K$ for some injective submodule K of L. By our assumption we see that every cyclic singular right R-module is an injective module. Since $L/N \simeq ((E(N)/N) \oplus K)$, it follows by [11, Corollary 5] that the singular module $E(N)/N$ is injective. This implies that L/N is also injective. Hence R is right hereditary by [5, 3.9]. □

It was proved in [12, Theorem 1] that a ring R is noetherian if every finitely generated right R-module is a direct sum of a projective module and a CS-module. We consider below a similar question for proper finitely generated modules using Theorem 3.

Lemma 9. *A ring R is right noetherian if every proper finitely generated right R-module is either CS or noetherian.*

Proof. By Theorem 3, every factor module of R_R has finite uniform dimension. In particular, R_R has finite uniform dimension and hence it is a finite direct sum of uniform modules. Assume that R_R is not noetherian. Then R must be CS. Hence there exists a (uniform) direct summand U of R which is not noetherian. Clearly, U is a CS-module and $U/Soc(U)$ has finite uniform dimension.

We claim that $U/Soc(U)$ is a V-module. Suppose that S is a simple right R-module. Note that $S \oplus U$ is not noetherian and by our assumption, $S \oplus U$ must be CS. Then, by Lemma 1, S is $(U/Soc(U))$-injective. This shows that $U/Soc(U)$ is a V-module. If $Soc(U) = 0$ then U is a V-module. By [8, Lemma 2], U is noetherian, a contradiction. Hence $Soc(U) \neq 0$ and so $Soc(U)$ is an essential submodule of U. By the argument as above we obtain that $U/Soc(U)$ has finite uniform dimension. Hence, by [8, Lemma 2], $U/Soc(U)$ is noetherian and therefore U is noetherian since $Soc(U)$ is simple. This contradict U is not noetherian. Thus, M is a noetherian module, completing our proof. □

Proposition 10. *A ring R is right noetherian if and only if every proper fintely generated right R-module is a direct sum of a projective module and a module Q, where Q is either CS or noetherian.*

Proof. We first show that $R/Soc(R)$ is noetherian. Let E be an essential right ideal of R. Set $N = R/E$. Then N_R is a singular module. Clearly, every finitely generated module in $\sigma[N]$ can not contain nonzero projective submodules. By our assumption, every finitely generated right R-module in $\sigma[N]$ is either CS or noetherian. Then, by Lemma 9, N is noetherian and hence R has ACC on essential submodules. This shows that $R/Soc(R_R)$ is noetherian by [5, 5.15(1)]. Moreover, by Theorem 3, R_R has finite uniform dimension. Hence $Soc(M)$ is finitely generated and therefore M is noetherian as desired. □

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