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ON SIMPLE NOETHERIAN RINGS

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On Simple Noetherian Rings*

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Abstract. A module M is called a CS-module (or extending module [5]) if every submodule of M is essential in a direct summand of M . It is shown that (i) a simple ring R is right noetherian if and only if every cyclic singular right R -module is either a CS-module or a noetherian module; (ii) for a prime ring R , if every proper cyclic right R -module is a direct sum of a quasi-injective module and a finitely cogenerated module, then R is either semisimple artinian or a right Ore domain; and (iii) a prime ring R is right noetherian if and only if every cyclic right R -module is a direct sum of a quasi-injective module and a noetherian module.

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1 Introduction

The class of simple noetherian rings is a topic of considerable interest in ring theory and has been extensively studied by many authors (see, for example, [2, 3, 10]). From a theorem of Osofsky and Smith [14], it follows that if every cyclic right module over a ring R is CS, then every cyclic right R -module has finite uniform dimension. However, in general, such a ring need not be right noetherian. Furthermore, Huynh, Jain, and Lopez-Permouth [10] considered the problem when a simple ring is noetherian, and proved that a simple ring R is right noetherian if every cyclic singular right R -module is CS.

Rings over which proper cyclics are injective (called *PCI-rings*) have been studied by many authors, including Cozzens, Damiano, Faith, Boyle,

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Goodearl, and others. According to Cozzens and Faith [3, Theorem 6.13], it is proved that a right PCI-ring is either semisimple or a right semihereditary simple domain. In [4], Damiano showed that a right PCI-ring is either semisimple, or a simple right noetherian, right hereditary, right Ore V-domain. A ring R is called a *right PCQI-ring* if every proper cyclic right R -module is quasi-injective. It is known that a right PCQI-ring is either semiperfect or prime (see, for example, [13]). It is shown in [10] that a prime PCQI-ring is either artinian or a right Ore domain. Consequently, they proved that a simple PCQI-ring is either artinian or a right noetherian hereditary domain. On the other hand, Huynh and Dung [7] proved that a ring R is right artinian if and only if every cyclic right R -module is a direct sum of an injective module and a finitely cogenerated module.

In this paper, we follow this investigation and aim to show a similar result under weaker sufficient conditions. First, we show that a simple ring R is right noetherian if and only if every cyclic singular right R -module is either a CS-module or a noetherian module; and a prime ring R is either semisimple artinian or a right Ore domain if every proper cyclic right R -module is a direct sum of a quasi-injective module and a finitely cogenerated module. Finally, we prove that a prime ring R is right noetherian if and only if every cyclic right R -module is a direct sum of a quasi-injective module and a noetherian module; and a prime ring R is right noetherian if and only if every 2-generated right R -module is a direct sum of a continuous module and a noetherian module.

2 Preliminaries

Throughout this paper, we consider associative rings R with identity and unitary right R -modules. For a module M , we denote by $\text{Soc}(M)$ and $E(M)$ the socle and the injective hull of M , respectively. If $M = \text{Soc}(M)$, then M is called a *semisimple module*. For a ring R and $x \in R$, we set $r_R(x) = \{a \in R \mid xa = 0\}$. The set

$$Z(R_R) = \{x \in R \mid r_R(x) \text{ is essential in } R_R\}$$

is an ideal of R , called the *right singular ideal* of R . In case $Z(R_R) = 0$, R is called *right non-singular*. For a module M , the Krull dimension of M is defined in [6].

We will refer to [1, 5] for undefined notions used in the text, and also for basic facts concerning CS-modules, simple rings, noetherian rings, and artinian rings. We record here some known results which will be used repeatedly in the sequel.

Lemma 2.1. [15, Proposition 4.3] *Let U be a uniform right R -module and S a simple right R -module such that $U \oplus S$ is CS. Then S is $U/\text{Soc}(U)$ -injective.*

Lemma 2.2. [8, Corollary 4] *For a right ideal A of a semiprime right Goldie ring R , the following conditions are equivalent:*

- (1) A_R is semisimple.
- (2) A_R is quasi-injective.
- (3) A_R is injective.

3 The Main Results

We start our investigation by proving the following result. The proof below is inspired by some ideas in the proof of [10, Theorem A].

Theorem 3.1. *A simple ring R is right noetherian if and only if every cyclic singular right R -module is either a CS-module or a noetherian module.*

Proof. If $\text{Soc}(R_R) \neq 0$, then $R = \text{Soc}(R_R)$, and we are done. Hence, we consider the case $\text{Soc}(R_R) = 0$. Let $M = R/E$, where E is an essential right ideal of R . Then by hypothesis and by [5, Corollary 9.4], M has finite uniform dimension. By [5, Lemma 5.14], $R/\text{Soc}(R_R)$, and hence, R has finite uniform dimension. This implies that R is right Goldie.

Assume there is an essential right ideal $E \subseteq R$ such that $M = R/E$ is not noetherian. Note that each factor of M has finite uniform dimension. Let S be a singular simple module. Then by [10, Lemma 3.1], $S \oplus M$ is cyclic. By hypothesis, $S \oplus M$ must be CS. Hence, by Lemma 2.1, S is $M/\text{Soc}(M)$ -injective. Therefore, $M/\text{Soc}(M)$ is a V-module. By [12], $M/\text{Soc}(M)$ is noetherian, hence M is noetherian, a contradiction. Therefore, for each essential right ideal $E \subseteq R$, R/E is noetherian. By [5, 5.15], $R/\text{Soc}(R_R)$ is right noetherian. But $\text{Soc}(R_R) = 0$. Thus, R is right noetherian, as desired. $\square$

From Theorem 3.1, we immediately obtain the following.

Corollary 3.2. *A simple ring R is right noetherian if and only if every proper cyclic right R -module is a direct sum of a projective module and a module Q , where Q is either a CS-module or a noetherian module.*

Proof. It follows by Theorem 3.1 since every cyclic singular right R -module is proper cyclic. $\square$

Proposition 3.3. *Let R be a ring such that every proper cyclic right R -module is a direct sum of a CS-module and a module of finite uniform dimension, or a direct sum of a projective module and a noetherian module. Then every cyclic right R -module has finite uniform dimension.*

Proof. Let X be a cyclic right R -module and E an essential submodule of X . Clearly, $X/E \neq R_R$. Moreover, we see that any cyclic subfactors of X/E cannot contain any projective submodule and so they are not isomorphic to R_R . Hence, by assumption, they are either a direct sum of a CS-module

and a module of finite uniform dimension, or noetherian. Therefore, by [5, Corollary 9.4], X/E has finite uniform dimension. Hence, $X/\text{Soc}(X_R)$ has finite uniform dimension by [5, Lemma 5.14]. To finish the proof, it suffices to show that $\text{Soc}(X)$ is finitely generated. Assume on the contrary that $\text{Soc}(X)$ is infinitely generated. Then we may write $\text{Soc}(X) = W \oplus V$, where W and V are infinite direct sums of simple modules. Since W cannot be a direct summand of X , it follows that X/W is not projective. In particular, $X/W \not\cong R_R$. Now we can apply the same argument preceding in the proof of [9, Theorem 2.10] to arrive at a contradiction. Hence, $\text{Soc}(X)$ must be of finite length, and so X has finite uniform dimension, completing the proof. $\square$

We are now in a position to prove the main result.

Theorem 3.4. *Let R be a prime ring. If every proper cyclic right R -module is a direct sum of a quasi-injective module and a finitely cogenerated module, then R is either semisimple artinian or a right Ore domain.*

Proof. Assume on the contrary that $Z(R_R) \neq 0$. Since R is prime, this implies that $\text{Soc}(R_R) = 0$. Therefore, by hypothesis, for any $0 \neq x \in Z(R_R)$, xR is quasi-injective. Thus, the argument in the proof of [10, Theorem 2.2] can be applied to get a contradiction. Hence, R is right non-singular.

Next, by [5, Corollary 9.4], R has finite right uniform dimension. Hence, R is right Goldie. On the other hand, by hypothesis, the right R -module R_R contains an essential submodule $E = S \oplus T$, where S is semisimple and T is quasi-injective. By Lemma 2.2, both S and T are quasi-injective. Therefore, E is injective. Thus, $R = E$, and so R is semisimple artinian. $\square$

In [8, Proposition 8], Huynh, Dung, and Smith proved that a semiprime ring R is right noetherian if every cyclic right R -module is a direct sum of an injective module and a noetherian module. A question arises naturally whether or not the same statement holds if “injective” is replaced by “quasi-injective”. Using [10, Theorem 2.2], we can answer the question positively in the case of prime rings.

Proposition 3.5. *A prime ring R is right noetherian if and only if every cyclic right R -module is a direct sum of a quasi-injective module and a noetherian module.*

Proof. Let R be a ring such that every cyclic right R -module is a direct sum of a quasi-injective module and a noetherian module. We first prove that R is right non-singular. By [5, Corollary 9.4], R_R has finite uniform dimension. Assume on the contrary that the right singular ideal $Z(R)$ of R is non-zero. If every cyclic submodule of $Z(R)$ is quasi-injective, then the argument in the proof of [10, Theorem 2.2] will produce a contradiction to the primeness of R . Hence, by hypothesis, $Z(R)$ contains a cyclic non-zero noetherian submodule N . Moreover, N has Krull dimension. Then we

can apply an argument presented in [8] to arrive at another contradiction. Thus, $Z(R) = 0$, i.e., R is right non-singular. Now $R_R = Q \oplus M$, where Q is quasi-injective and M is noetherian. By Lemma 2.2, Q_R is semisimple. Thus, R is right noetherian. $\square$

It is unknown if Proposition 3.5 holds true for semiprime rings. But we know that, in general, Proposition 3.5 is incorrect for non-semiprime rings. See an example in [11].

Proposition 3.6. *A prime ring R is right noetherian if and only if every 2-generated right R -module is a direct sum of a continuous module and a noetherian module.*

Proof. One direction is clear. Conversely, let R be a ring such that each 2-generated right module is a direct sum of a continuous module and a noetherian module. By [5, Corollary 9.4], R has finite uniform dimension. Assume the right singular ideal $Z(R)$ is non-zero. If $Z(R)$ contains a non-zero cyclic noetherian submodule C , then as done in the proof of Proposition 3.5, we get a contradiction. Hence, each cyclic submodule of $Z(R)$ must be continuous and contains no non-zero noetherian submodules. Let $0 \neq xR \subseteq Z(R)$ be a uniform module. Since $xR \oplus xR$ does not contain non-zero noetherian submodules, it must be continuous by the hypothesis. Hence, xR is quasi-injective. Now again by an argument in the proof of [10, Theorem 2.2], we obtain a contradiction. Hence, $Z(R) = 0$, i.e., R is right non-singular.

Assume R is not right noetherian. Then by using the hypothesis, we can prove that R contains a cyclic uniform right ideal U that is not noetherian. Hence, $U \oplus U = D \oplus N$, where D is continuous and N is noetherian. If $N \neq 0$, then either $(U, 0)$ or $(0, U)$ embeds in N because either of them has zero intersection with D . This is a contradiction. Hence, $N = 0$ and so U is quasi-injective. By Lemma 2.2, U is semisimple (of finite length), contradicting the assumption about U . Thus, R must be right noetherian. $\square$

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References

- [1] F.W. Anderson, K.R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1974.
- [2] A.W. Chatters, C.R. Hajarnavis, *Rings with Chain Conditions*, Pitman, London, 1980.
- [3] J.H. Cozzens, C. Faith, *Simple Noetherian Rings*, Cambridge University Press, Cambridge, 1975.

- [4] R.F. Damiano, A right PCI ring is right noetherian, *Proc. Amer. Math. Soc.* 77 (1979) 11–14.
- [5] N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, *Extending Modules*, Pitman Research Notes in Mathematics Series, Vol. 313, Longman, Harlow, 1994.
- [6] R. Gordon, J.C. Robson, *Krull Dimension*, Mem. Amer. Math. Soc., No. 133, Amer. Math. Soc., Providence, R.I., 1973.
- [7] D.V. Huynh, N.V. Dung, A characterization of artinian rings, *Glasgow Math. J.* 30 (1988) 67–73.
- [8] D.V. Huynh, N.V. Dung, P.F. Smith, Characterization of rings with Krull dimension, *J. Algebra* 132 (1990) 104–112.
- [9] D.V. Huynh, N.V. Dung, R. Wisbauer, On modules with finite uniform and Krull dimension, *Arch. Math.* 57 (1991) 122–132.
- [10] D.V. Huynh, S.K. Jain, S.R. Lopez-Permouth, When is simple ring noetherian? *J. Algebra* 184 (1996) 786–794.
- [11] D.V. Huynh, P.F. Smith, Some rings characterised by their modules, *Comm. Algebra* 18 (1990) 1971–1988.
- [12] D.V. Huynh, P.F. Smith, R. Wisbauer, A note on GV-modules with Krull dimension, *Glasgow Math. J.* 32 (1990) 389–390.
- [13] S.K. Jain, S. Singh, R.G. Symonds, Rings whose proper cyclic module are quasi-injective, *Pacific J. Math.* 67 (1976) 461–472.
- [14] B.L. Osofsky, P.F. Smith, Cyclic modules whose quotients have all complement submodules direct summands, *J. Algebra* 139 (1991) 342–354.
- [15] N. Vanaja, All finitely generated M -subgenerated modules are extending, *Comm. Algebra* 24 (1996) 543–572.

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ภาคผนวก 4

**A GENERALIZATION OF CONTINUOUS MODULES AND
THEIR APPLICATION TO QF-RINGS**

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A Generalization of Continuous Modules and Their
Application to QF-rings

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A Generalization of Continuous Modules and Their Application to QF-rings

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ABSTRACT. A ring R is called quasi-Frobenius, briefly QF , if R is right or left artinian and right or left self-injective. In this note we study the new class of modules which generalize the concept of continuous (quasi-continuous) modules, that is, ec-continuous (ec-quasi-continuous) modules. We prove the decomposition theorems for ec-quasi-continuous modules, and we also give a sufficient condition for ec-continuous to be continuous. Moreover, we give a characterization of QF -rings with the injectivity condition replaced by ec-continuity.

1. Introduction

A well-known result of Faith [6] asserts that a right self-injective rings with ACC on right or left annihilators is QF . On the other hand, it is shown in Clark and Huynh [4] that a right self-injective semiperfect ring R is QF if and only if every uniform submodule of any projective right R -module is contained in a finitely generated submodule. In Camillo and Yousif [3], motivated by a result of Faith [6] on self-injective rings, it was shown that a two-sided continuous ring with ACC on left annihilators is QF . Moreover, it was shown in Nicholson and Yousif [9] that a two-sided quasi continuous ring with DCC on essential left ideals is QF .

In this paper we introduce some new notions which generalize the concept of continuous modules (rings) and quasi-continuous modules (rings) (that is, ec-continuous modules (rings) and ec-quasi-continuous modules(rings), respectively). We prove that a right ec-continuous module M_R is a direct sum of uniform modules if one of the following holds: (i) R satisfies ACC on right ideals of the form $r_R(m)$, for all $m \in M$, or (ii) M has ACC (or DCC) on essential submodules and $Soc(M)$ is essential in a direct summand of M .

Moreover, we will consider the following property for a given ring R ;

(*) Every uniform submodule of $R^{(N)}$ is contained in a finitely generated sub-

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module.

(**) $(eR)^{(2)}$ is a *CS*-module for each primitive idempotent $e \in R$.

We also show that a right *ec*-continuous semiperfect ring R is *QF* if and only if R satisfies the conditions (*) and (**). Finally, we prove that a right *ec*-continuous ring R which satisfies (**) is *QF* if one of the following holds: (i) R has *ACC* on right annihilators, or (ii) $\text{Soc}(R)$ is essential in a direct summand of R and R has *ACC* (or *DCC*) on essential right ideals.

2. Definitions and preliminaries

Throughout this paper all rings are associative with identity and all modules are unitary right R -modules. Consider the following conditions on a module M_R :

(C1) Every submodule of M is essential in a direct summand of M .

(C2) Every submodule isomorphic to a direct summand of M is itself a direct summand.

(C3) If M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M .

The module M is called *continuous* if it satisfies conditions (C1) and (C2), *quasi-continuous* if it satisfies (C1) and (C3), and a *CS*-module (or *extending module*) if it satisfies condition (C1) only. Recall that a module which has a cyclic essential submodule is said to be *essentially cyclic*. A right R -module M is called *ec-CS* if it satisfies the condition:

(C1') Every essentially cyclic submodule of M is essential in a direct summand of M (see [12]).

The module M is called *ec-continuous* if it satisfies condition (C1') and (C2), and an *ec-quasi-continuous module* if it satisfies conditions (C1') and (C3). A ring R is called *right ec-continuous* (*ec-quasi-continuous*) if the module R_R is *ec-continuous* (*ec-quasi continuous*).

A ring R is called *right finitely continuous* if any finitely generated right ideal is essential in a direct summand and satisfying the condition (C2) (see [2]). In [14], Utumi proved that left *PF* rings are right finitely continuous. We notice that if A, B and C are submodules of a non-singular right R -module M with $A \subseteq^{ess} C \subseteq^{\oplus} M$ and $A \subseteq^{ess} B$, then B is contained in C (see, for example, [11, Remark 1, p.693]). Hence left nonsingular *PF*-rings are right *ec-continuous*.

Lemma 1. *A right R -module M has (C1') if and only if every closed essentially cyclic submodule of M is a direct summand of M .*

Proof. The necessity is clear. For the sufficiency, let N be an essentially cyclic submodule of M . By Zorn's lemma, N has a maximal essential extension L in M . Clearly L is closed and essentially cyclic, hence it is a direct summand. $\square$

Lemma 2. *Let M be an *ec-continuous* right R -module. Then*

- (1) every direct summand of M is also ec-continuous, and
- (2) if M is indecomposable then the endomorphism ring $\text{End}(M)$ is local.

Proof. (1) This is clear by Lemma 1 and the modular law.

(2) It is immediate since every indecomposable ec-continuous module is continuous. □

Lemma 3. *Let M be an ec-quasi continuous right R -module. If M has an indecomposable decomposition $M = \bigoplus_{\alpha \in I} M_\alpha$, then that decomposition complements direct summands.*

Proof. See in [8, Theorem 2.22 (1) $\Rightarrow$ (2)]. □

3. On the ec-quasi continuous modules

Next, we have the decomposition theorems for an ec-quasi-continuous module with chain conditions. In this result, we replace the conditions (C1) in [10, Lemma 3] by the conditions (C1') and (C3). For any $m \in M_R$, $\tau_R(m)$ will denote $\{r \in R \mid mr = 0\}$.

Proposition 4. *Let M_R be an ec-quasi continuous module and R satisfies ACC on right ideals of the form $\tau_R(m)$, $m \in M$. Then M is a direct sum of uniform modules.*

Proof. First, we show that M contain a maximal local direct summand $N = \bigoplus_{\alpha \in I} N_\alpha$, with N_α uniform for each $\alpha \in I$. Let m be a non-zero element of M such that $\tau_R(m)$ is maximal in $\{\tau_R(m) \mid 0 \neq m \in M\}$. There exists a direct summand K of M such that $mR \subseteq^{ess} K$. Suppose that K is not indecomposable. Then there exist non-zero submodules K_1 and K_2 of K such that $K = K_1 \oplus K_2$. Since $m \in K = K_1 \oplus K_2$, $m = m_1 + m_2$ for some $m_i \in K_i$ ($i = 1, 2$). If $m_1 = 0$ then $m = m_2 \in K_2$, and $mR \cap K_1 = 0$ giving $K_1 = 0$, a contradiction. Thus $m_1 \neq 0$. Clearly $\tau_R(m) \subseteq \tau_R(m_1)$. Hence $\tau_R(m) = \tau_R(m_1)$, by the choice of m . Similarly $m_2 \neq 0$, and $\tau_R(m) = \tau_R(m_2)$. Because $m_1 \neq 0$ there exist $r_1, r_2 \in R$ such that $0 \neq m_1 r_1 = m r_2 = (m_1 + m_2) r_2 = m_1 r_2 + m_2 r_2$. Thus $m_2 r_2 = 0$, and hence $r_2 \in \tau_R(m_2) \setminus \tau_R(m)$, a contradiction. Thus, because K is ec-CS, K is uniform. By Zorn's Lemma M contain a maximal local direct summand $N = \bigoplus_{\alpha \in I} N_\alpha$ where N_α is an uniform submodule of M for each $\alpha \in I$. Claim that $N \subseteq^{ess} M$. Assume that there exists a non-zero element $m \in M$ such that $mR \cap N = 0$. Let $0 \neq y \in M$ such that $\tau_R(y)$ is maximal in the set of the form $\{\tau_R(m) \mid 0 \neq m \in M \text{ and } mR \cap N = 0\}$. Note that $yR \subseteq^{ess} N'$ for some $N' \subseteq^\oplus M$. By the above argument N' is a uniform module. Then by the condition (C3), $N \oplus N'$ is a local direct summand, a contradiction to the choice of N . Hence $N \subseteq^{ess} M$ and by [5, Lemma 8.1], we have $N = M$. Therefore M is a direct sum of uniform modules. □

Now, we extend Theorem 25.6 in [1] for self-injective rings to ec-quasi continuous rings.

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Corollary 5. For a ring R the following statements are equivalent:

- (a) R is right noetherian;
- (b) Every ec-quasi continuous right R -module is a direct sum of indecomposable modules;
- (c) Every ec-quasi continuous right R -module has a decomposition that complements maximal direct summands.

Proposition 6. Let M be a finitely generated ec-quasi continuous module with $\text{Soc}(M)$ essential in a direct summand of M . If M has ACC (or DCC) on essential submodules then M is a direct sum of uniform modules.

Proof. It follows from [5, Theorem 5.15], that $M/\text{Soc}(M)$ is noetherian (or artinian, respectively). Hence $M/\text{Soc}(M)$ has finite uniform dimension and so it has ACC on direct summands. Let $S = \text{Soc}(M)$. Then $M = M_1 \oplus M_2$, where $S \subseteq^{\text{ess}} M_1$.

We now show that S is finitely generated, which would imply that M has finite uniform dimension. Assume on the contrary that S is infinitely generated. Then we may write $S = \bigoplus_{i=1}^{\infty} S_i$, where each S_i is a simple module. Since M_1 is ec-quasi-continuous, there are submodules $N_1, N_2, \dots, N_n, \dots$ of M_1 such that $S_i \subseteq^{\text{ess}} N_i$ and $\bigoplus_{i=1}^n N_i$ is a direct summand of M_1 for each $n \geq 1$. Clearly M_1 is finitely generated and so $S_i \neq N_i$ for infinite many i . For all $i \geq 1$, set $K_i = N_i$ whenever $N_i \neq S_i$. By [5, Lemma 18.4], $(K_1 + S)/S \oplus (K_2 + S)/S \oplus \dots$ is a direct sum of submodules of M_1/S and $\bigoplus_{i=1}^n (K_i + S)/S$ is a direct summand of M_1/S for each $n \geq 1$. By ACC on direct summands, this process must stop. Hence there exists a natural number m such that $K_{m+j} \subseteq S$, and therefore $K_{m+j} = S_{m+j}$ for all $j \geq 1$, which is a contradiction. This implies that S is finitely generated and hence M_1 has finite uniform dimension. Therefore M is a finite direct sum of uniform modules. $\square$

Proposition 7. Let $M = \bigoplus_{\alpha \in \Omega} M_{\alpha}$ be an ec-continuous module where each M_{α} is uniform. Then the following condition are equivalent:

- (a) every uniform submodule of M^N is essential in a direct summand of M^N ;
- (b) M^N is quasi-injective.

Proof. (b) $\Rightarrow$ (a) is trivial.

(a) $\Rightarrow$ (b) Assume (a). It follows by Lemma 2 that each M_{α} is continuous and $\text{End}(M_{\alpha})$ is local. For convenience we write

$$Q = M^N = \bigoplus_{\gamma \in \Gamma} Q_{\gamma},$$

where each Q_{γ} is isomorphic to some M_{α} in the set $\{M_{\alpha}/\alpha \in \Omega\}$. We notice further that, for any $\alpha, \beta \in \Omega$, every monomorphism from Q_{α} to Q_{β} is an isomorphism. Hence, by [5, Corollary 8.9], we obtain that $\bigoplus_{\gamma \in \Gamma - \{\alpha_0\}} Q_{\gamma}$ is Q_{α_0} -injective for all $\alpha_0 \in \Gamma$. Observe that $Q_{\alpha_0} \oplus Q_{\alpha_0}$ is a direct summand of $Q \oplus Q$ which is isomorphic to Q . Hence every uniform submodule of $Q_{\alpha_0} \oplus Q_{\alpha_0}$ is essential in a direct summand

of $Q_{\alpha_0} \oplus Q_{\alpha_0}$. By [5, Corollary 8.9], Q_{α_0} is Q_{α_0} -injective. Hence Q is a quasi-injective module, as desired. $\square$

Proposition 8. *Let M be an ec-continuous right R -module. If M has finite uniform dimension then M is continuous.*

Proof. By hypothesis, we obtain that

$$M = M_1 \oplus \dots \oplus M_n$$

, where each M_i is uniform ec-continuous. It follows by Lemma 2 that each M_i is continuous and hence $\text{End}(M_i)$ is local. Note that, for any $i \neq j$, $M_i \oplus M_j$ is ec-continuous. This implies that every uniform submodule of $M_i \oplus M_j$ is essential in a direct summand of $M_i \oplus M_j$. We note that every monomorphism from M_i to M_j is an isomorphism. Hence M_i is M_j -injective by [5, Corollary 8.9] and so $\bigoplus_{i \neq j} M_i$ is M_j -injective. Therefore, by [8, Theorem 2.13 and Theorem 3.16], M is continuous. The proof of proposition is complete. $\square$

4. Application: quasi-Frobenius rings

First we obtain a condition for ec-continuous ring to be self-injective.

Lemma 9. *Let R be a right ec-continuous semiperfect ring. Then R is right continuous. In addition, if R satisfies the condition $(**)$ then R is self-injective.*

Proof. By [1, Theorem 27.6], there is a complete set of orthogonal primitive idempotents $\{e_1, \dots, e_n\}$ in R such that

$$R = e_1 R \oplus \dots \oplus e_n R,$$

where each $\text{End}(e_i R)$ is a local ring. Note that each $e_i R$ is indecomposable and hence is uniform by $(C1')$. Clearly, $e_i R \simeq f(e_i R) \subseteq e_j R$ for any $i \neq j$ in I and every monomorphism f from $e_i R$ to $e_j R$. By $(C2)$, $f(e_i R)$ must be a direct summand of R and also of $e_j R$. Hence $f(e_i R) = e_j R$ and therefore f is an isomorphism. Since R_R is a CS module, by [5, Corollary 8.9], $\bigoplus_{j \in J} e_j R$ is $e_{i_0} R$ -injective for every $i_0 \in I$, where $J = I \setminus \{i_0\}$. By Lemma 2, each $e_i R$ is continuous. This implies that R is continuous (see, for example, [8, Theorem 3.16]). Then, by previous argument, every monomorphism from $e_{i_0} R$ to itself is an isomorphism. Since $e_{i_0} R_R^{(2)} = e_{i_0} R \oplus e_{i_0} R$ is CS, $e_{i_0} R$ is $e_{i_0} R$ -injective by [5, Corollary 8.9]. This implies that R is $e_{i_0} R$ -injective for all $i_0 \in I$. Hence R is self-injective. $\square$

We now extend a result of Clark and Huynh [4, Theorem 1] for a right self-injective ring to an ec-continuous ring.

Theorem 10. *For a ring R the following statements are equivalent:*

- (a) R is QF;
- (b) R is a right ec-continuous semiperfect ring satisfying $(*)$ and $(**)$;

(c) R is a right ec-continuous semiperfect ring satisfying $(*)$ and $R_R^{(n)}$ is a CS-module for each natural number $n \in \mathbb{N}$.

Proof. (a) $\Rightarrow$ (b) This is clear by Clark and Huynh [4, Theorem 1].

(b) $\Rightarrow$ (c) Assume (b). By Lemma 9, R is a right self-injective and hence $R_R^{(n)}$ is an injective module for each $n \in \mathbb{N}$. Therefore $R_R^{(n)}$ is a CS-module for each $n \in \mathbb{N}$.

(c) $\Rightarrow$ (a) Assume (c). There is a complete set of orthogonal primitive idempotents $\{e_1, \dots, e_n\}$ in R such that $R = e_1 R \oplus \dots \oplus e_n R$, and each endomorphism ring $\text{End}(e_i R)$ is local. Since e_i is primitive, $e_i R$ is indecomposable. By Lemma 9, R_R is continuous and hence each $e_i R$ is a uniform continuous module. Note that

$$R_R^{(\mathbb{N})} \simeq (e_1 R)^{(\mathbb{N})} \oplus \dots \oplus (e_n R)^{(\mathbb{N})}.$$

We write $R_R^{(\mathbb{N})}$ in the form $R_R^{(\mathbb{N})} = \bigoplus_{\alpha \in A} I_\alpha$, where each I_α is isomorphic to some $e_i R$ in $\{e_1 R, \dots, e_n R\}$ and A is an index set. Now we shall use an argument in the proof of Clark and Huynh [4, Lemma 5(i)] to show that every closed uniform submodule of $R^{(\mathbb{N})}$ is essential in a direct summand of $R^{(\mathbb{N})}$. Let V be a closed uniform submodule of $R^{(\mathbb{N})}$. Then, by $(*)$, there exists a finite subset F of A such that $V \subseteq \bigoplus_{\alpha \in F} I_\alpha$. Since R_R^F is a CS-module by our assumption, it follows that $\bigoplus_{\alpha \in F} I_\alpha$ must be also CS. Hence V is a direct summand of $\bigoplus_{\alpha \in F} I_\alpha$ and also of $R^{(\mathbb{N})}$.

Now, it follows by Proposition 7 that $R^{(\mathbb{N})}$ is self-injective. Hence R is a Σ -CS module by [5, Corollary 11.13]. Therefore, by [4, Theorem 1], R is a QF ring. Thus our proof is complete. $\square$

Next, we improved the results of Camillo and Yousif [3, Theorem 1] for a two-sided continuous ring and Faith [6, Theorem 2] for a self-injective ring to a one-sided ec-continuous ring.

Proposition 11. *A ring R is QF if and only if R is a right ec-continuous ring which satisfies $(**)$ and has ACC on right annihilators.*

Proof. By Proposition 4, R is a direct sum of indecomposable modules. Then, there exists a complete set of orthogonal primitive idempotents $\{e_1, \dots, e_n\}$ such that $R = e_1 R \oplus \dots \oplus e_n R$. Clearly each $e_i R$ is indecomposable ec-continuous. By Lemma 2 each of endomorphism $\text{End}(e_i R) \simeq e_i R e_i$ is local. Hence, by [1, Theorem 27.6], R is semiperfect and so it is self-injective, by Lemma 9. Therefore, by Faith [6, Corollary 4], R is a QF ring. $\square$

Proposition 12. *Let R be an ec-continuous ring with $\text{Soc}(R)$ essential in a direct summand of R . If R satisfies $(**)$ and has ACC on essential right ideals then R is QF.*

Proof. By Proposition 6, R is a direct sum of uniform modules. Thus, there is a complete set of orthogonal primitive idempotents $\{e_1, \dots, e_n\}$ in R such that $R =$

$e_1R \oplus \dots \oplus e_nR$. By Lemma 2, we note that each endomorphism ring $\text{End}(e_iR) \simeq e_iRe_i$ is a local ring. Hence R is a semiperfect ring and it implies that R_R is self-injective. Therefore R is QF. $\square$

Finally, we improve Nicholson and Yousif's result [9, Corollary 2] for a two sided quasi-continuous ring to a one sided quasi-continuous rings.

Proposition 13. *A ring R is QF if and only if R is a right quasi-continuous ring which satisfies $(**)$ and has DCC on essential right ideals.*

Proof. The necessity is clear. Suppose that R_R is quasi-continuous ring satisfying $(**)$ and has DCC on essential right ideals. It follows by [5, Theorem 18.5] that R_R is artinian and hence it is semiprimary by [1, Theorem 15.20]. Then, by [9, Lemma 6], R is right continuous. It follows by Lemma 9 that R is self-injective and hence, by [5, Corollary 18.13], R is a QF-ring. $\square$

We conclude the paper with some remarks.

Remark.

(a) The results in this paper remain true (with similar arguments) if the condition $(**)$ is replaced by the weaker condition that "every uniform submodules of $(eR)^{(2)}$ is essential in a direct summand of $(eR)^{(2)}$, for each primitive idempotent $e \in R$ ".

(b) In Theorem 10, we can replace the condition " R is semiperfect" by the condition " R has finite Goldie dimension".

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References

- [1] F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, Springer-Verlag, (1974).
- [2] P. Ara and J. K. Park, *On continuous semiprimary rings*, Comm. Algebra, 19(1991), 1945-1957.
- [3] V. P. Camillo and M. F. Yousif, *Continuous rings with acc on annihilators*, Canad. Math. Bull., 34(1991), 462-464.
- [4] J. Clark and D. V. Huynh, *When is a self-injective semiperfect ring quasi-Frobenius?*, J. Algebra, 164(1994), 531-542.
- [5] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, *Extending module*, Research Note in Mathematics Series 313, Pitman London, (1994).

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- [6] C. Faith, *Rings with ascending chain condition on annihilators*, Nagoya Math. J., **27**(1966), 179-191.
- [7] D. V. Huynh, *A right countably sigma-CS ring with ACC or DCC on projective principal right ideals is left artinian and QF-3*, Trans. Amer. Math. Soc., **347**(1995), 3131-3139.
- [8] S. H. Mohamed and B. J. Müller, *Continuous and Discrete Modules*, London Math. Soc. Lecture Note Series 147, Cambridge Univ. Press., (1990).
- [9] W. K. Nicholson and M. F. Yousif, *On quasi-continuous rings*, Proc. Amer. Math. Soc., **120**(1994), 1049-1051.
- [10] M. Okado, *On the decomposition of extending modules*, Math. Japonica, **29**(1984), 939-941.
- [11] K. Oshiro, *Continuous modules and quasi-continuous modules*, Osaka J. Math., **20**(1983), 681-694.
- [12] L. V. Thuyet and R. Wisbauer, *Extending property for finitely generated submodules*, Vietnam J. Math., **25**(1997), 65-73.
- [13] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, (1991).
- [14] Y. Utumi, *Self injective rings*, J. Algebra, **6**(1967), 56-64.

ภาคผนวก 5

**MODULES CHARACTERIZED BY THEIR PROPER CYCLIC
SUBFACTORS**

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Modules characterized by their proper cyclic subfactors

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Abstract

For a finitely generated self-projective right R -module M , we show that if every proper cyclic subfactor of M is a direct sum of a CS-module and a module of finite uniform dimension, or a direct sum of an M -projective module and a module Q , where Q is either CS or noetherian, then every factor module of M has finite uniform dimension. Consequently, (i) a ring R is right noetherian if and only if every proper cyclic right R -module is a direct sum of a projective module and a module Q , where Q is either injective or noetherian, and (ii) a ring R is right noetherian if and only if every proper finitely generated right R -module is a direct sum of a projective module and a module Q , where Q is either CS or noetherian.

1. Introduction

The study of noetherian rings, via decomposition properties of cyclic or finitely generated modules, was initiated by P.F. Smith in [13]. On the other hand, A. Chatters state that a ring R is right noetherian if and only if every cyclic right R -module is a direct sum of a projective module and a noetherian module [2]. Later on, Osofsky and Smith proved in [11] that a ring R is right noetherian and hereditary if every cyclic right R -module is a direct sum of a projective module and an injective module. It was shown further by Huynh and Rizvi (see [9]) that a ring R is right noetherian if and only if every cyclic right R -module is a direct sum of a projective module and a module Q , where Q is either injective or noetherian. Rings over which proper cyclics are injective

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have been studied by many authors, including J.H. Cozzens, R.F. Damiano, C. Faith, and others (see [3],[4] and [6]).

In this paper, we use module-theoretic methods to consider the related properties in more general setting. We first show that for a finitely generated self-projective right R -module M , if every proper cyclic subfactor of M is a direct sum of a CS-module and a module of finite uniform dimension, or a direct sum of an M -projective module and a module Q , where Q is either CS or noetherian, then every factor module of M has finite uniform dimension. Consequently, a ring R is right noetherian if and only if every proper cyclic right R -module is a direct sum of a projective module and a module Q , where Q is either injective or noetherian. We also show that a ring R is right noetherian if and only if every proper finitely generated right R -module is a direct sum of a projective module and a module Q , where Q is either CS or noetherian.

Throughout this paper, all rings R are associative rings with identity and all modules are unitary right R -modules. For a module M we denote by $\sigma[M]$ and $Soc(M)$ the full subcategory of $Mod-R$, whose objects are submodules of M -generated modules, and the socle of M , respectively. If $M = Soc(M)$, then M is called a *semisimple module*. A submodule of a factor module of M is called *subfactor* of M . A cyclic subfactor of M which is not isomorphic to M is called *proper cyclic subfactor* of M . A cyclic right R module C is called a *proper cyclic right R -module* if $C \not\cong R_R$. A finitely generated right R -module X is called *proper* if X is not isomorphic to R_R .

The reader is referred to Anderson-Fuller [1] and Wisbauer [15] for undefined notions, and basic facts concerning injective modules, CS-modules, singular module, noetherian modules and noetherian rings. For the sake of convenience, we cite the following results which will be used in the sequel.

Lemma 1 [14, Proposition 4.3]. *Let U be a uniform right R -module and S a simple right R -module such that $U \oplus S$ is CS then S is $(U/Soc(U))$ -injective.*

Lemma 2 [7, p.254]. *A ring R is right noetherian if and only if every cyclic right R -module is injective or a direct sum of a projective module and a noetherian module.*

2. Main Results

We consider the following two conditions for a proper factor module C of a right R -module M :

(*) C is a direct sum of a CS-module and a module of finite uniform dimension;

(**) C is a direct sum of an M -projective module and a module Q , where Q is either a CS-module or a noetherian module.

The condition (**) was first considered in [9] for cyclic modules in $\text{Mod-}R$, where “CS” is replaced by “injective”.

The following is our main result:

Theorem 3. *Let M be a finitely generated self-projective right R -module. Assume that every proper cyclic subfactor of M satisfies (*) or (**). Then every factor module of M has finite uniform dimension.*

Proof. Let X be a factor module of M and let E be an essential submodule of X . Then, clearly, X/E is M -singular and hence $X/E \neq M_R$ since M is self-projective. Moreover, we see that any cyclic subfactor of X/E can not contain any non-zero M -projective submodule and so they are not isomorphic to M_R . Hence, by assumption, X/E is a direct sum of a CS-module and a module of finite uniform dimension or X/E is noetherian. Therefore, by [5, Corollary 9.4], we see that X/E has finite uniform dimension. Hence $X/\text{Soc}(X_R)$ has finite uniform dimension by [5, Lemma 5.14].

To finish our proof it suffices to show that $\text{Soc}(X)$ is finitely generated. Assume on contrary that $\text{Soc}(X)$ is infinitely generated. Then we may write $\text{Soc}(X) = W \oplus V$, where W and V are infinite direct sums of simple modules. Since W can not be a direct summand of X , it follows that $X/W \neq M_R$. By hypothesis, we have

(1)

$$X/W = \bar{Q} \oplus \bar{F},$$

where $\bar{Q}$ is CS and $\bar{F}$ has finite uniform dimension.

By using the same argument to the module $\bar{Q}$, we see that $\bar{Q}/\text{Soc}(\bar{Q})$ has finite uniform dimension. If $\text{Soc}(\bar{Q})$ is infinitely generated, then, by [5, Lemma 9.1], $\bar{Q}/\text{Soc}(\bar{Q})$ has infinite uniform dimension, a contradiction. Hence $\text{Soc}(\bar{Q})$ and therefore X/W has finite uniform dimension. This contradicts X/W has infinite uniform dimension. Therefore the decomposition (1) is not possible and we have

(2)

$$X/W = \bar{P}_1 \oplus \bar{Q}_1,$$

where $\bar{P}_1$ is a projective module and $\bar{Q}_1 (\neq 0)$ is either a CS-module or a noetherian module. Let Q_1 be the inverse image of $\bar{Q}_1$ in X . Then clearly $\bar{P}_1 \simeq X/Q_1$, and Q_1/W (being isomorphic to $\bar{Q}_1$) is either a CS-module or a noetherian module. Since $\bar{P}_1$ is projective, $X = Q_1 \oplus Q_2$ for some submodule Q_2 of X . Then $\text{Soc}(X) = \text{Soc}(Q_1) \oplus \text{Soc}(Q_2)$. Note that $\bar{Q}_1$ is a finitely generated right R -module. Thus by the argument as above, $\bar{Q}_1/\text{Soc}(\bar{Q}_1)$ has finite uniform dimension. If $\bar{Q}_1$ is CS then, by [5, Lemma 9.1], $\text{Soc}(\bar{Q}_1)$ is finitely generated. This implies that $\bar{Q}_1$, and so Q_1/W , has finite uniform dimension. Hence, in any case, $\bar{Q}_1$, and so Q_1/W , has finite uniform dimension. Therefore, $\text{Soc}(Q_2)$ is clearly infinitely generated since $Q_2 \simeq \bar{P}_1$ has infinite uniform dimension but $Q_2/\text{Soc}(Q_2)$ has finite uniform dimension.

Because $W \subseteq Q_1$ and W is an infinite direct sum of simple modules, Q_1 has an infinitely generated socle. Note that

$$X/\text{Soc}(X) \simeq (Q_1/\text{Soc}(Q_1)) \oplus (Q_2/\text{Soc}(Q_2)),$$

where $Q_1 \neq \text{Soc}(Q_1)$ and $Q_2 \neq \text{Soc}(Q_2)$. Hence, $X/\text{Soc}(X)$ has uniform dimension at least 2. Applying the same arguments to the module Q_2 , and continue the process in a similar manner, an obvious induction shows that $X/\text{Soc}(X)$ has infinite uniform dimension, which is a contradiction to the fact that $X/\text{Soc}(X)$ has finite uniform dimension. This shows that $\text{Soc}(X)$ is finitely generated and so X has finite uniform dimension. Therefore every factor module of M has finite uniform dimension. $\square$

Theorem 3 gives immediately the following result.

Corollary 4. *Let R be a ring which every proper cyclic right R -module is a direct sum of a CS-module and a module of finite uniform dimension, or a*

direct sum of a projective module and a module Q , where Q is either CS or noetherian. Then every cyclic right R -module has finite uniform dimension.

As an application, we have the following corollary.

Corollary 5. *A ring R is right noetherian if and only if every proper cyclic right R -module is a direct sum of a projective module and a module Q , where Q is either injective or noetherian.*

Proof. Let E be an essential right ideal of R . Then R/E is clearly singular and $R/E \neq R_R$. Moreover, every cyclic right (R/E) -module can not contain any projective submodule and so they are not isomorphic to R_R . By hypothesis, they are either an injective module or a noetherian module. Hence, by Lemma 2, R/E is right noetherian and therefore $R/\text{Soc}(R_R)$ is right noetherian by [5, 5.15]. By Corollary 4, R has finite uniform dimension and hence $\text{Soc}(R_R)$ is finitely generated. Therefore R is right noetherian as desired. $\square$

Proposition 6. *Let M be a finitely generated self-projective right R -module. If every proper cyclic subfactor of M is a direct sum of a projective and a noetherian module then M is a noetherian module.*

Proof. Let E be an essential submodule of M . Then M/E is M -singular and $M/E \neq M$. By hypothesis, M/E is noetherian so that $M/\text{Soc}(M)$ is noetherian by [5, 5.15]. Moreover, by Theorem 3, M has finite uniform dimension so that $\text{Soc}(M)$ is finitely generated. Hence M is noetherian. $\square$

Proposition 7. *Let M be a cyclic self-projective right R -module. If every proper cyclic subfactor of M is a direct sum of a projective module and an M -injective module then M is a noetherian module.*

Proof. Let E be an essential submodule of M . Then M/E is M -singular and $M/E \neq M_R$. This implies that every cyclic subfactor of $[M/E]$ is a proper cyclic subfactor of M . By hypothesis, it is M -injective and so M/E is a semisimple module. Consequently, by [5, 5.15], $M/\text{Soc}(M)$ is noetherian. Moreover, M has finite uniform dimension by Theorem 3 and hence $\text{Soc}(M)$ is a finitely generated right R -module. This shows that M is noetherian. $\square$

Corollary 8 *If every proper cyclic right R -module is a direct sum of a projec-*

tive module and an injective module then R is right noetherian and hereditary.

Proof. By Proposition 7, R is clearly right noetherian. Let L be an injective right R -module and $N \subseteq L$. Then, there is an injective hull $E(N)$ of N in L such that $L = E(N) \oplus K$ for some injective submodule K of L . By our assumption we see that every cyclic singular right R -module is an injective module. Since $L/N \simeq ((E(N)/N) \oplus K)$, it follows by [11, Corollary 5] that the singular module $E(N)/N$ is injective. This implies that L/N is also injective. Hence R is right hereditary by [5, 3.9]. $\square$

It was proved in [12, Theorem 1] that a ring R is noetherian if every finitely generated right R -module is a direct sum of a projective module and a CS-module. We consider below a similar question for proper finitely generated modules using Theorem 3.

Lemma 9. *A ring R is right noetherian if every proper finitely generated right R -module is either CS or noetherian.*

Proof. By Theorem 3, every factor module of R_R has finite uniform dimension. In particular, R_R has finite uniform dimension and hence it is a finite direct sum of uniform modules. Assume that R_R is not noetherian. Then R must be CS. Hence there exists a (uniform) direct summand U of R which is not noetherian. Clearly, U is a CS-module and $U/\text{Soc}(U)$ has finite uniform dimension.

We claim that $U/\text{Soc}(U)$ is a V -module. Suppose that S is a simple right R -module. Note that $S \oplus U$ is not noetherian and by our assumption, $S \oplus U$ must be CS. Then, by Lemma 1, S is $(U/\text{Soc}(U))$ -injective. This shows that $U/\text{Soc}(U)$ is a V -module. If $\text{Soc}(U) = 0$ then U is a V -module. By [8, Lemma 2], U is noetherian, a contradiction. Hence $\text{Soc}(U) \neq 0$ and so $\text{Soc}(U)$ is an essential submodule of U . By the argument as above we obtain that $U/\text{Soc}(U)$ has finite uniform dimension. Hence, by [8, Lemma 2], $U/\text{Soc}(U)$ is noetherian and therefore U is noetherian since $\text{Soc}(U)$ is simple. This contradict U is not noetherian. Thus, M is a noetherian module, completing our proof. $\square$

Proposition 10. *A ring R is right noetherian if and only if every proper finitely generated right R -module is a direct sum of a projective module and a module Q , where Q is either CS or noetherian.*

Proof. We first show that $R/\text{Soc}(R)$ is noetherian. Let E be an essential right ideal of R . Set $N = R/E$. Then N_R is a singular module. Clearly, every finitely generated module in $\sigma[N]$ can not contain nonzero projective submodules. By our assumption, every finitely generated right R -module in $\sigma[N]$ is either CS or noetherian. Then, by Lemma 9, N is noetherian and hence R has ACC on essential submodules. This shows that $R/\text{Soc}(R_R)$ is noetherian by [5, 5.15(1)]. Moreover, by Theorem 3, R_R has finite uniform dimension. Hence $\text{Soc}(M)$ is finitely generated and therefore M is noetherian as desired. $\square$

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References

- [1] F.W. Anderson and K.R. Fuller, "Rings and Categories of Modules", Springer-Verlag, 1974.
- [2] A.W. Chatters, *A characterization of right noetherian rings*, Quart. J. Math. Oxford **33** (1982), 65-69.
- [3] J.H. Cozzens and C. Faith, "Simple Noetherian Rings", Cambridge Univ. Press, Cambridge, UK, 1975.
- [4] R.F. Damiano, *A right PCI ring is right Noetherian*, Pro. Amer. Math. Soc. **77** (1979), 11-14.
- [5] N.V. Dung, D.V. Huynh, P.F. Smith and R. Wisbauer, "Extending Modules", Pitman Research Notes in Mathematics Series 313, Longman, Harlow, 1994.
- [6] C. Faith, *When are all proper cyclics injective?*, Pacific. J. Math. **45** (1973), 97-112.
- [7] D.V. Huynh, *A characterization of noetherian rings by cyclic modules*, Pro. Edinburgh Math. Soc. **39**(1996), 253-262.
- [8] D.V. Huynh, S.T. Rizvi and M.F. Yousif, *Rings whose finitely generated modules are extending*, J. Pure Appl. Algebra **111** (1996), 325-328.

- [9] D.V. Huynh, S.T. Rizvi , *A new characterization of noetherian rings by cyclic modules*, Preprint.
- [10] S.H. Mohamed and B.J. Müller, "Continuous and Discrete Modules", London Math. Soc. LN 147, Cambridge Univ. Press., 1990.
- [11] B.L. Osofsky and P.F. Smith, *Cyclic modules whose quotients have all complement submodules direct summands*, J. Algebra **139** (1991), 342-354.
- [12] S. Plubtieng, *Decompositions of modules into projective modules and CS-modules*, Bull. Austral. Math. Soc. **62** (2000), 159-164.
- [13] P.F. Smith, *Rings characterized by their cyclic modules*, Comm. Algebra **24** (1996), 543-572.
- [14] N. Vanaja, *All finitely generated M -subgenerated modules are extending*, Comm. Algebra, **24** (1996), 543-572.
- [15] R. Wisbauer, "Foundations of Module and Ring Theory", Gordon and Breach, 1991.

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Date: 2003.02.08

Dear Professor Somyot Plubtieng,

I am pleased to inform you that your paper "**Modules Characterized by Their Proper Cyclic Subfactors**" has been recommended by the referee for publication.

We are writing you that the editorial office of the SEAMS Bull Mathematics has been moved to the Department of Mathematics, South China Normal University, Guangzhou 510631, China and the Department of Mathematics, Yunnan University, Kunming 650091, China. The Publisher is still Springer-verlag.

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