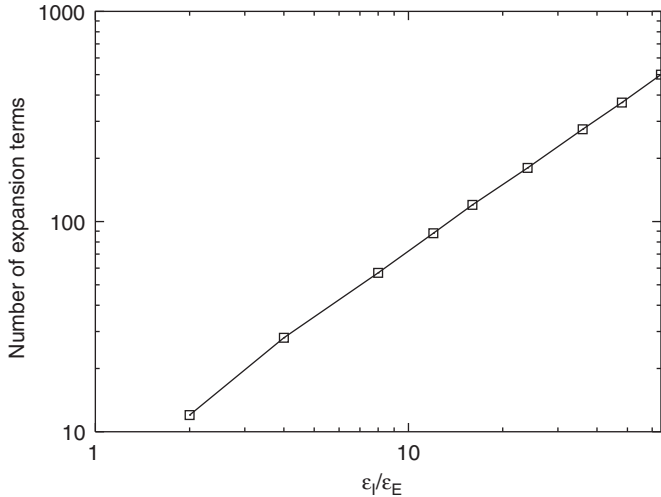
Fig. 12. Error of the electric field E_c at the contact point.Fig. 13. Number of expansion terms required in order that the error of E_c is smaller than 1%.

This arrangement is chosen as an example here because the analytical solution of the electric field is available. The electric field E_c at the contact point between the cylinder and the ground plane has been derived by the classical method of images using only line-dipole images [17]. The solution of E_c in the ϵ_E -side is

$$\frac{E_c}{E_0} = \frac{\epsilon_I}{\epsilon_E}. \quad (81)$$

When $\epsilon_I/\epsilon_E \gg 1$, the electric field near the contact point is highly non-uniform, thus making it difficult for numerical field-calculation methods. We use this arrangement to investigate the accuracy attained by the calculation method. Fig. 12 shows the error of the field E_c , compared with the analytical solution in Eq. (81), in relation to the permittivity ratio ϵ_I/ϵ_E when $N_j = 500$. As can be seen from the figure, we attain the error of the contact-point

electric field, E_c , smaller than 1% for ϵ_I/ϵ_E up to 64. Fig. 13 shows the minimum order N_j of expansion required to achieve an error of E_c smaller than 1% as a function of the permittivity ratio. We can see that N_j increases approximately as an exponential function with the permittivity ratio.

4. Conclusions

This paper describes the method of multipole images for calculating the electric field in two-dimensional arrangements. The method is similar to the classical method of images, but it uses multipoles of higher order than line or dipole charges. Three kinds of multipole re-expansion are utilized to extend the application of the method for more general arrangements. Calculation examples are given to demonstrate the use of this method in various arrangements. In addition to the application to fundamental arrangements, the method can be used for evaluating the accuracy of results obtained by numerical field-calculation methods.

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Calculation of the electric field in arrangements of intersecting spheres

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The application of the method of images to the calculation of electric field in arrangements of intersecting conducting spheres is presented. The multipole re-expansions are utilized to determine the multipole images and the potential coefficients. The method is applicable for cases of electrically floating, grounded, or energized conducting spheres that intersect each other. Examples are given for two grounded intersecting spheres of equal radii and those of different radii, respectively, under an external uniform field. The results for the spheres of equal radii agree well with the analytical solutions for the case of completely overlapped spheres and touching spheres, respectively. For the spheres of different radii, the numerical results show that the method may also be used when the degree of intersection is not too high. © 2006 American Institute of Physics.

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I. INTRODUCTION

The electrostatics of two intersecting spheres have been studied in many works related to applications in fusing and merging particles. The existing analytical solutions based on the method of images utilize the geometrical inversion^{1,2} or the Kelvin transformation.³ However, these solutions were restricted to cases in which the spheres must intersect each other at an angle π/n (n is an integer) or must have equal radii. The solutions have also been proposed for an arbitrary angle of intersection,^{4–8} but the calculation, involving integrals of infinite series, is complicated even where the spheres are under a uniform field. Felderhof *et al.*⁹ also derived the solutions based on integrals for any arbitrary angle of intersection.

This paper describes the application of the method of multipole images using re-expansion techniques to the general case of intersecting conducting spheres under an external potential. The method is extended from the one proposed for configurations of separated spheres.^{10,11} It is more general than that using the inversion or transformation,^{1–3} as it can treat spheres of unequal radii that intersect each other at an arbitrary angle. Compared with the analysis by Radchik *et al.*,^{4–8} the method presented in this paper is simpler in that integrals are not involved directly in the main calculation.

II. PHYSICAL CONFIGURATION

Figure 1 shows a schematic of two intersecting conducting spheres lying along the z axis. The lower and the upper spheres are centered at $\mathbf{p}(0,0,z_p)$ and $\mathbf{q}(0,0,z_q)$, and have radii of a and b , respectively. The separation between the sphere centers is denoted by $D=z_q-z_p$. We consider a case in which the spheres are grounded at zero potential in this pa-

per. The case of electrically floating spheres can be treated in a similar manner with an additional condition of zero net charge of both spheres.

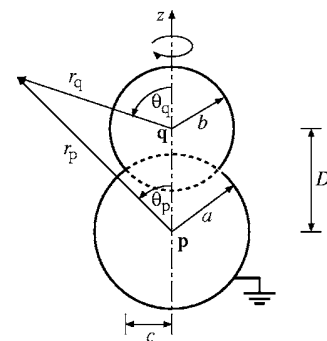
Let the upper sphere be smaller than or equal to the lower one. It can be shown that, the coordinate z_i on the line (circle) of intersection between the sphere surfaces is given by

$$z_i - z_p = \frac{D^2 + (a^2 - b^2)}{2D}, \quad (1)$$

and the radius c of the circle of intersection is determined by

$$c = \sqrt{a^2 - (z_i - z_p)^2}. \quad (2)$$

Note that, in the following sections, symbols (r_x, θ_x) are used to denote the first two spherical coordinates when a point \mathbf{x} is treated as the origin. See (r_p, θ_p) and (r_q, θ_q) in Fig. 1, for example. If the potential is axisymmetrical in the arrangement, the third coordinate (azimuthal angle) shall not be involved in calculation.

FIG. 1. Intersecting spheres of radii a and b .^{a)}Electronic mail: boonchai.t@chula.ac.th

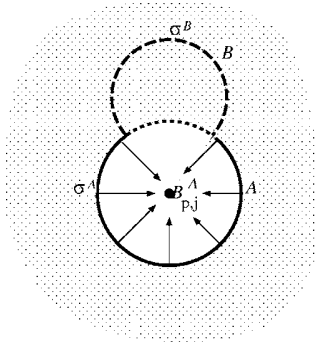


FIG. 2. Potential due to surface charge σ^A on the surface A is represented by multipoles $B_{p,j}^A$ at \mathbf{p} .

III. CALCULATION METHOD

When the spheres in the arrangement of Fig. 1 are subject to an external potential ϕ^E , surface charges are induced on them. The external potential is assumed to be axisymmetrical. These induced charges deviate the resultant potential from ϕ^E . This section describes various mathematical operations on the potential due to these surface charges that are necessary for determining the resultant potential.

A. Expression of potentials for the method of multipole images

For simplicity, let first consider only the surface charge σ^A on the contour A of the lower sphere, indicated by the solid line in Fig. 2. (The charge σ^B on the upper sphere, denoted by contour B, can also be treated in a similar way.) The potential due to σ^A may be written as the multipole potential ψ_p^A expanded about the sphere center \mathbf{p} as

$$\psi_p^A = \sum_{j=0}^{\infty} \frac{B_{p,j}^A}{r_p^{j+1}} P_j(\cos \theta_p), \quad (3)$$

where P_j is the j th-order Legendre polynomial. Note that, in expressions of a potential, we shall consistently use the subscript and superscript to indicate the center of expansion and the source of potential, respectively. In Eq. (3), $B_{p,j}^A$ is considered to be the j th-order multipole moment at \mathbf{p} . The definition of multipole moment here is slightly different from the conventional definition to simplify the potential expression. The multipole potential given by Eq. (3) converges in the shaded open region shown in Fig. 2, i.e., outside the sphere of center \mathbf{p} and radius a .

On the other hand, the potential due to σ^A can also be expanded about the center \mathbf{q} of the upper sphere (see Fig. 3).

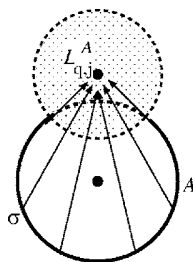


FIG. 3. Potential due to surface charge σ^A on A is locally expanded about \mathbf{q} .

The potential is in the form of local or Taylor expansion as

$$\varphi_q^A = \sum_{j=0}^{\infty} L_{q,j}^A r_q^j P_j(\cos \theta_q), \quad (4)$$

where φ_q^A is the potential locally expanded about \mathbf{q} , and $L_{p,j}^A$ is the j th-order potential coefficient. The potential φ_q^A converges in the shaded, closed region shown in Fig. 3, i.e., the sphere of center \mathbf{q} and radius b . The symbol φ is used for the potential locally expanded in order to mark the difference from the multipole potential ψ in Eq. (3).

B. Reexpansion of potential

Potential in the form of multipole [Eq. (3)] or local expression [Eq. (4)] can be further reexpanded about other points. Here we restrict the re-expansions only to points on the z axis, as they are sufficient for the solutions in the axisymmetrical case. Two kinds of re-expansions involved in the calculation are described as follows.

1. Multipole-to-local re-expansion

By this kind of re-expansion, the multipole potential in Eq. (3) is expanded locally about a different point in the form similar to Eq. (4). Let $\mathbf{u}(0,0,z_u)$ be the new center of expansion. An n th-order multipole potential of Eq. (3) can be re-written as

$$\psi_{p,n}^A = \frac{B_{p,n}^A}{r_p^{n+1}} P_n(\cos \theta_p) = \sum_{j=0}^{\infty} \varphi_{u,j}^A = \sum_{j=0}^{\infty} L_{u,j}^A r_u^j P_j(\cos \theta_u). \quad (5)$$

$L_{u,j}^A$ is related to $B_{p,n}^A$ by the following equations^{10,12}:

$$L_{u,j}^A = (-1)^j \frac{(n+j)!}{n! j!} \frac{B_{p,n}^A}{d^{n+j+1}} \quad \text{for } z_u > z_p \quad (6)$$

$$= (-1)^n \frac{(n+j)!}{n! j!} \frac{B_{p,n}^A}{d^{n+j+1}} \quad \text{for } z_u < z_p, \quad (7)$$

where $d = |z_u - z_p|$. After re-expansion, the radius of convergence is equal to the smallest distance between \mathbf{u} and the boundary of the shaded region in Fig. 2. It is obvious that the re-expansion shall be invalid if \mathbf{u} is located in the unshaded region.

2. Local-to-local re-expansion

Consider an n th-order potential in Eq. (4) and the new center of expansion at \mathbf{u} . This kind of re-expansion expresses the potential as¹³

$$\varphi_{q,n}^A = L_{q,n}^A r_q^n P_n(\cos \theta_q) = \sum_{j=0}^n \varphi_{u,j}^A = \sum_{j=0}^n L_{u,j}^A r_u^j P_j(\cos \theta_u), \quad (8)$$

where

$$L_{u,j}^A = \frac{n!}{(n-j)! j!} d^{n-j} L_{q,n}^A \quad \text{for } z_u > z_p, \quad (9)$$

and

$$L_{u,j}^A = (-1)^{n-j} \frac{n!}{(n-j)! j!} d^{n-j} L_{q,n}^A \quad \text{for } z_u < z_p. \quad (10)$$

After re-expansion, the radius of convergence is equal to the smallest distance between **u** and the boundary of the shaded region in Fig. 3.

C. Solutions of the intersecting spheres

We are now in a position to obtain the solutions for the arrangement of Fig. 1. Considering the surface charges σ^A and σ^B on both spheres induced by an external potential φ^E , the potential φ at any point outside the spheres is the result of superposition as

$$\phi = \varphi^E + \psi_p^A + \psi_q^B, \quad (11)$$

where ψ_p^A and ψ_q^B are the multipole potentials due to σ^A and σ^B expanded about **p** and **q**, respectively. To satisfy the zero-potential condition on both spheres, we express the potential separately for each sphere as functions expanded about its center:

$$\phi = \varphi_p^E + \psi_p^A + \varphi_p^B \quad \text{for the lower sphere} \quad (12)$$

and

$$\phi = \varphi_q^E + \varphi_q^A + \psi_q^B \quad \text{for the upper sphere}, \quad (13)$$

where φ_p^B and φ_q^A are the potentials due to σ^B and σ^A locally expanded about **p** and **q**, respectively. Note that the external potential is also written in the form of local expansion in Eqs. (12) and (13). If the external potential is due to a uniform field E_0 in the z direction and the potential is referenced to be zero at $z=0$, for example, then we have

$$\varphi_p^E = -z_p E_0 - E_0 r_p P_1(\cos \theta_p) \quad (14)$$

and

$$\varphi_q^E = -z_q E_0 - E_0 r_q P_1(\cos \theta_q), \quad (15)$$

which are expanded locally about **p** and **q**, respectively.

Suppose we want to obtain the solution of potentials for the order j from 0 up to N . Then, the total number of potential coefficients ($B_{p,j}^A$, $B_{q,j}^B$, $L_{q,j}^A$, and $L_{p,j}^B$) to be determined is $4(N+1)$. The coefficients can be solved from a linear equation system, which is constructed by the following two procedures:

1. Satisfy the potential condition on the spheres. That is, for each order j , we have relations

$$\varphi_{p,j}^E + \psi_{p,j}^A + \varphi_{p,j}^B = 0 \quad (16)$$
 on the surface of the lower sphere ($r_p = a$), and

$$\varphi_{q,j}^E + \varphi_{q,j}^A + \psi_{q,j}^B = 0 \quad (17)$$
 on the surface of the upper sphere ($r_q = b$). Therefore, $2(N+1)$ equations are obtained by using Eqs. (16) and (17).
2. Equate the pairs of potentials that arise from the same charges, i.e., $\psi_p^A = \varphi_q^A$ and $\psi_q^B = \varphi_p^B$. In order to do so, each potential pair must be expanded about the same point. Then we can match the potentials for each order j . A natural choice seems to be multipole-to-local reexpanding ψ_p^A to **q** and ψ_q^B to **p** because the potentials can be

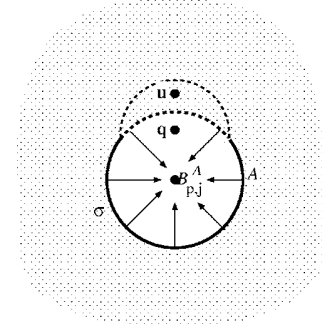


FIG. 4. Reexpanding the multipole potential ψ_p^A to **q** does not converge because **q** is not in the shaded region.

related directly to φ_q^A and φ_p^B , respectively. However, we notice that the potentials may not converge if $D < a$. Figure 4 presents a situation where the re-expansion of ψ_p^A to **q** fails to converge because **q** is located outside the shaded region. Hence, a better alternative is to reexpand both ψ_p^A and φ_q^A to a point in the shaded region of the upper sphere (e.g., **u** in Fig. 4). The same is also applied to the pair of ψ_q^B and φ_p^B . After re-expansion, we can match the potential coefficients for each j , and obtain other $2(N+1)$ equations.

After the linear equations are formed, they can be solved for the unknown coefficients $B_{p,j}^A$, $B_{q,j}^B$, $L_{q,j}^A$, and $L_{p,j}^B$. From our calculation experience, iterative methods are not recommended for solving the linear equations as they may not yield sufficiently accurate solutions in some cases.

IV. EXAMPLES

A. Spheres of equal radii

The case of two spheres of unit radii ($a=b=1$) under a uniform field E_0 in the z direction is taken as the first calculation example. The calculation has been performed for $D = 0.001a$ to $1.99a$. It is obvious that, for the spheres of equal radii, the calculation may be simplified by considering the vertical symmetry. However, we use this case to investigate the validity of the method as the spheres become nearly completely overlapped with D close to zero. Figure 5 shows the residual potential evaluated on the surface of the upper sphere

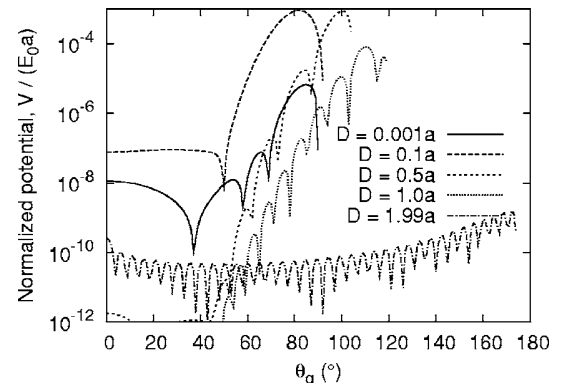
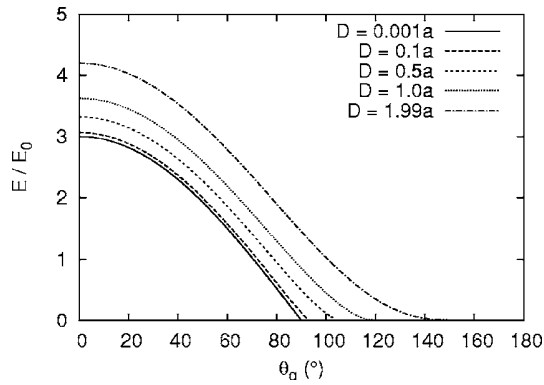


FIG. 5. Residual potential on the sphere surface for different values of D where $a=b$.

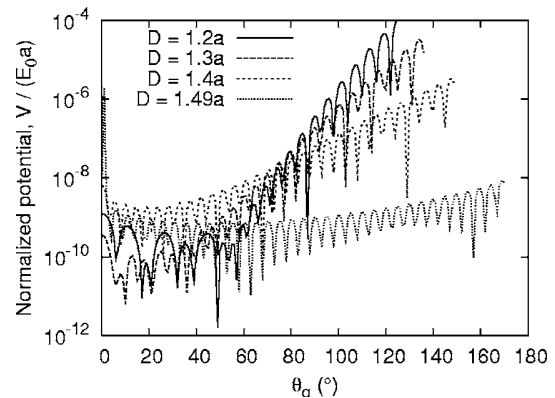
FIG. 6. Electric field on the sphere surface for different values of D .

sphere by Eq. (11). The figure confirms the validity of the proposed method for these values of D . The maximum orders N of the potential expansions used to obtain all the results in Fig. 5 are equal to 35. Although higher N should give better accuracy in the potential expansions, we have found that the linear equation systems are more difficult to be accurately solved. Note that for a special case in which equal-sized spheres are almost completely overlapped each other, i.e., very small D/a , a small N may be used in the calculation. We have found from the numerical results that $B_{p,j}^A = (-1)^{j+1} B_{q,j}^B$ and $B_{p,j}^A = B_{q,j}^B \approx 0$ for odd $j \neq 1$ when $D/a \rightarrow 0$, while the geometrical symmetry due to $a=b$ guarantees that the calculation always gives $B_{p,j}^A = (-1)^{j+1} B_{q,j}^B$. On the other hand, difficulty in numerical computation arises when D is smaller than a but not close to zero. This can be seen from the higher residual potentials in Fig. 5 for $D=0.1a$ and $D=0.5a$. The numerical errors may be caused by the use of finite N in the numerical calculation and also by the small region of convergence of the potentials after re-expansion, making the accurate solution of the linear equation system difficult. In addition, every value of D in Fig. 5 exhibits relatively high residual potential near the position of intersection (at the right end of each line).

Figure 6 shows the electric field on the surface of the upper sphere. As can be seen from the figure, the electric field is maximal at $\theta_q=0$, and decreases to zero at the line of intersection for all D . The maximum electric field decreases with decreasing D (increasing the overlapped portion). It should also be noted that, for D close to zero, the electric field at $\theta_q=0$ approaches $3E_0$, which is the solution for the case of a single conducting sphere. For $D=2a$, the arrangement is identical in electric field to that of a conducting sphere in contact with a grounded plane under a uniform field, and the analytical value of the maximum electric field is $4.207E_0$.¹⁴ The maximum field obtained from the numerical calculation is $4.202E_0$ for $D=1.99a$. It is also known that the charge acquired by the conducting sphere in this condition is equal to $\frac{2}{3}\pi^3\epsilon a^2 E_0^2$.¹⁵ With $D \rightarrow 2a$, we obtain the amount of charges that agrees well with the analytical solution. (The difference of the charges is 0.03% when $D=1.99a$ by $N=35$.)

B. Spheres of different radii

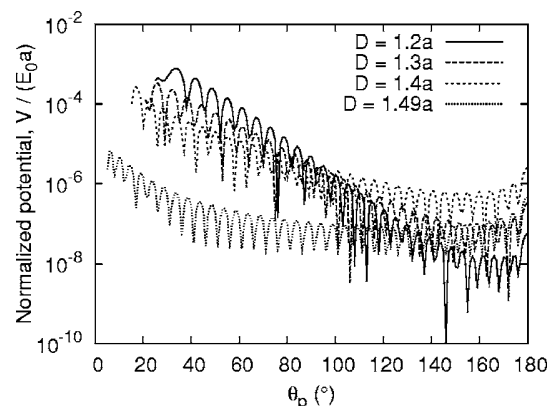
The arrangement of intersecting spheres of different radii has been calculated for $a=2b$ by using $N=35$. The residual

FIG. 7. Residual potential on the upper sphere for different values of D , where $a=2b$.

potential calculated from Eq. (11) on the upper and the lower spheres is shown, respectively, in Figs. 7 and 8 for $D/a=1.2-1.49$. From the figures we can see the same tendency as Fig. 5 that the errors are high near the position of intersection. For this case, the residual potential consistently becomes higher with decreasing D/a , i.e., increasing the degree of intersection. The normalized residual potential smaller than 10^{-3} can be attained by the method for $D/a \geq 1.2$. Note that we cannot obtain results of acceptable accuracy for smaller D/a by decreasing N as in the first calculation example. In the absence of the vertical symmetry $a=b$, a large N must be used for the potential expansion in Eq. (5) when D/a is small. Figures 7 and 8 also show that the error is greater on the larger (lower) sphere. We have also confirmed that the electric field from the calculation decreases properly to zero at the line of intersection for $D/a \geq 1.2$, although the field plot is not shown here.

V. CONCLUSIONS

We have explained the application of the method of multipole images to the potential problems of intersecting spheres. The solution is made possible by considering the convergence of the expressions of potential and by utilizing appropriate kinds of multipole re-expansion. The multipole images and potential coefficients are then determined as the solution of a linear equation system to satisfy the potential

FIG. 8. Residual potential on the lower sphere for different values of D , where $a=2b$.

condition of the spheres. Numerical results are given for the spheres of equal radii and those of different radii. Theoretically, the method can be used for cases of unequal-sized spheres. The external potential is also arbitrary, although it is axisymmetrical here. The numerical results in the case of equal radii have been confirmed that they agree with the case of completely-overlapped spheres and that of touching spheres. In the case of unequal radii, the method works well for slightly-overlapped spheres. For general three-dimensional cases, the approach described in this paper is still applicable. However, the expressions of potential and the re-expansions become more complicated in three dimensions.

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Polarization and Membrane Voltage of Ellipsoidal Particle with a Constant Membrane Thickness – a Series Expansion Approach

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Polarization and Membrane Voltage of Ellipsoidal Particle with a Constant Membrane Thickness
– a Series Expansion Approach

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Abstract

The estimation of the membrane voltage and the polarization factor of biological cell provide a base for the study of bio-manipulation techniques, such as dielectrophoresis, electroporation or electrofusion. To model a biological cell, an ellipsoidal particle with an insulating membrane is sometimes employed, but due to the limitation of the confocal nature of the coordinate system, the membrane thickness is assumed to vary with the position, despite the fact that the lipid bilayer membrane has a uniform thickness. We in this paper present a method to rigorously treat the uniform-thickness condition in a system having an axial symmetry. The method is based on the harmonic expansion of the field, to include the condition of the uniform membrane thickness as a series expansion of the geometrical factor, and to solve the field problem as an interaction of the harmonic components. It is identified that the conventional variable thickness model is equivalent to neglecting the harmonic interactions in the uniform thickness model. Numerical calculations of the membrane voltage and the polarization factor are made, and it is found that the discrepancy between our rigorous model and the conventional variable thickness model becomes significant where the field deformation is large due to the high axial ratio of the ellipsoid.

Key words:

biological cell, ellipsoid, membrane voltage, polarization, harmonic expansion

1. Introduction

The electrostatic polarization of a particle having an insulating membrane, as well as the voltage induced on the membrane, has been studied as a model of biological cells in relation to such applications for the characterization of cells through impedance analysis, dielectrophoresis, electrorotation, electroporation or electrofusion [1-5]. In particular, on-chip applications of these electrokinetic phenomena often require the design of the device and the interpretation of the result based on a precise estimation of the polarization.

An exact solution for the field problem is known for a spherical particle, however not for an ellipsoidal particle with a uniform membrane thickness. This is due to the confocal nature of the coordinate system employed in the analysis. In spheroidal or ellipsoidal coordinate system, the surface of constant radial parameter, (hereafter denoted as η) becomes a spheroid or an ellipsoid, so that the membrane thickness, i.e. the distance between the inner surface $\eta = \eta_0$ and outer surface $\eta = \eta_0 + d\eta$ of the membrane varies as depicted in Fig.1 b) for a spheroid having axis lengths b and c . The ratio of the thinnest to the thickest part is equal to the aspect ratio of the particle b/c , not precisely representing the biological membrane consisting of a uniform lipid bilayer. Gimsa and Wachner employed an equivalent circuit approach to tackle this problem [6-7]; however, it was based on a simplification in exchange for mathematical exactness. There are also numerical approaches based on integral equations [8-10], but they still have problems in calculation time and accuracy assessments.

In this paper, we confine our interest to the particles having rotational symmetry, i.e. prolate and oblate spheroid, and present a new method for a rigorous analysis of the field problem based on the harmonic expansion, which can provide an exact solution for general 3D external field distributions.

2. Principle

2.1 Harmonic expansion of the field

We shall first consider the case of a prolate spheroid. The prolate spheroidal coordinate system (η, θ, φ) is related to the Cartesian coordinate system by

$$\begin{aligned} x &= a \sinh \eta \sin \theta \cos \varphi \\ y &= a \sinh \eta \sin \theta \sin \varphi \\ z &= a \cosh \eta \cos \theta \end{aligned} \quad (1)$$

The surface $\eta = \eta_0$ is a prolate spheroid having a rotational symmetry on the z -axis and focal points at $z = \pm a$:

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (2)$$

where

$$b = a \sinh \eta_0 \quad \text{and} \quad c = a \cosh \eta_0 \quad (3)$$

and

$$a = \sqrt{c^2 - b^2} \quad (4)$$

With the use of the eccentricity of the ellipsoid e

$$e = \sqrt{1 - (b/c)^2} = a/c \quad (5)$$

where $0 \leq e \leq 1$, eq.(3) can be rewritten as

$$\cosh \eta_0 = 1/e \quad \text{and} \quad \sinh \eta_0 = \sqrt{(1/e)^2 - 1} \quad (6)$$

The solution of Laplace equation in the prolate spheroidal coordinate system is [11]

$$\phi = \left\{ \begin{matrix} P_n^m(\cosh \eta) \\ Q_n^m(\cosh \eta) \end{matrix} \right\} \left\{ \begin{matrix} P_n^m(\cos \theta) \\ Q_n^m(\cos \theta) \end{matrix} \right\} \text{cs } m\varphi \quad (7)$$

where cs is either of cos or sin, italicized functions $P_n^m(u)$ and $Q_n^m(u)$ are the Legendre functions of the first and the second kind for the argument $u \geq 1$. The former has the singularity at $u = \infty$,

while the latter at $u = \pm 1$, same as in the case the Legendre functions $P_n^m(u)$ and $Q_n^m(u)$ for the argument $-1 \leq u \leq 1$.

Under the influence of the externally applied potential

$$\phi_{app} = - \sum_{n,m} E_{n,m} a^n P_n^m(\cosh \eta) P_n^m(\cos \theta) \cos m\varphi \quad (8)$$

the potential outside and inside of the spheroid can be written as

$$\begin{aligned} \phi_{out} &= \phi_{app} + \phi_{ind} \\ &= - \sum_{n,m} E_{n,m} a^n P_n^m(\cosh \eta) P_n^m(\cos \theta) \cos m\varphi + \sum_{n,m} C_{n,m} Q_n^m(\cosh \eta) P_n^m(\cos \theta) \cos m\varphi \end{aligned} \quad (9)$$

$$\phi_{in} = \sum_{n,m} A_{n,m} P_n^m(\cosh \eta) P_n^m(\cos \theta) \cos m\varphi \quad (10)$$

2.2 Boundary conditions

The complex permittivity ε^* of a medium is defined by

$$\varepsilon^* = \varepsilon + \frac{\mu}{j\omega} = \frac{\mu}{j\omega} (1 + j\omega\tau_s) \quad (11)$$

where ε and μ are the permittivity and conductivity, ω the angular frequency, j the imaginary unit, and τ_s is the time constant of the medium

$$\tau_s = \varepsilon / \mu \quad (12)$$

τ_s is often very short for the medium of biological interest, as short as 50 ns even for 1mM KCl solution ($\mu = 13$ mS/m, 1/150 of physiological condition). The pulse duration used for electroporation or electrofusion is far longer than this value, so $\omega\tau_s \ll 1$ holds for the media inside and outside of cells and the contribution of ε in ε^* can be neglected. On the other hand, the biological membrane is a good insulator, and μ makes little contribution to ε^* . For simplicity, we shall use these approximations throughout the paper, as we did in [12-13].

We further assume that the membrane located at $\eta = \eta_0$ is infinitesimally thin, and the membrane charging is modeled by a dipole sheet. Under these assumptions, the boundary conditions to be met at $\eta = \eta_0$ become [12-13]

$$\begin{aligned}\phi_{out} - \phi_{in} &= \sigma / C_m \\ \mu_{out} E_{out} &= \mu_{in} E_{in}\end{aligned}\quad (13)$$

and the charging of the membrane is governed by

$$\frac{d\sigma}{dt} = -\mu_{out} E_{out} = -\mu_{in} E_{in} \quad (14)$$

where C_m is the membrane capacitance per unit area, σ is the dipolar charge density on the membrane ($+\sigma$ on the outer and $-\sigma$ on the inner surface respectively),

$$E_{out} = -\frac{1}{g_\eta} \frac{\partial \phi_{out}}{\partial \eta} \Big|_{\eta=\eta_0}, \quad E_{in} = -\frac{1}{g_\eta} \frac{\partial \phi_{in}}{\partial \eta} \Big|_{\eta=\eta_0} \quad (15)$$

g_η is the geometric coefficient

$$g_\eta = a\sqrt{\sinh^2 \eta_0 + \sin^2 \theta} = a\sqrt{\cosh^2 \eta_0 - \cos^2 \theta} = c\sqrt{1 - (\cos \theta / \cosh \eta_0)^2} \quad (16)$$

It should be noted that, from these equations,

$$\phi_{out} - \phi_{in} = -\frac{1}{C_m} \int \frac{1}{g_\eta} \frac{\partial \phi_{out}}{\partial \eta} \Big|_{\eta=\eta_0} dt \quad (17)$$

so, for constant C_m , θ dependence of g_η in this equation yields the interaction of the harmonic components between different degree n in eqs.(9) and (10), but not that between different order m as to be explained in the following sections. Even under a uniform external field ($n = 1$), harmonics with higher degree are evoked.

For the model where the capacitance varies with θ in such a way that

$$C_m = \varepsilon_m / (g_\eta d\eta) \quad (18)$$

where ε_m is the permittivity of the membrane and $g_\eta d\eta$ is the thickness of the membrane (refer to fig.1 b), the θ -dependent factor g_η cancels out in eq.(17), and there will be no interaction between

different n . In this case, the surface defined by $\eta = \eta_0 + d\eta$ satisfies

$$\frac{x^2}{b^2 + \xi} + \frac{y^2}{b^2 + \xi} + \frac{z^2}{c^2 + \xi} = 1 \quad (19)$$

with

$$\xi = 2bc d\eta \quad (20)$$

and the thickness on x and z axes respectively becomes

$$\delta_x = c d\eta \quad \text{and} \quad \delta_z = b d\eta \quad (21)$$

Hence, the conventional variable thickness model is equivalent to neglecting g_η in eq.(15), and the solution for a uniform external field is obtained just by considering $n = 1$ only.

2.3 The solution of the boundary value problem

The solution can be obtained by expanding σ in Legendre harmonics as

$$\sigma = \sum_{n,m} S_{n,m} P_n^m(\cos \theta) \cos m\varphi \quad (22)$$

and we have the potential coefficients in eqs.(9) and (10) in terms of the surface charge $S_{n,m}$ and the external potential $E_{n,m}$ as

$$A_{n,m} = \frac{d_\eta Q_n^m(\cosh \eta_0)}{G_n^m(\kappa_\mu, \cosh \eta_0)} \frac{S_{n,m}}{C_m} - \frac{H_n^m(\cosh \eta_0)}{G_n^m(\kappa_\mu, \cosh \eta_0)} a^n E_{n,m} \quad (23)$$

$$C_{n,m} = \frac{\kappa_\mu d_\eta P_n^m(\cosh \eta_0)}{G_n^m(\kappa_\mu, \cosh \eta_0)} \frac{S_{n,m}}{C_m} + \frac{(\kappa_\mu - 1) P_n^m(\cosh \eta_0) d_\eta P_n^m(\cosh \eta_0)}{G_n^m(\kappa_\mu, \cosh \eta_0)} a^n E_{n,m} \quad (24)$$

where we wrote

$$\kappa_\mu = \mu_{in} / \mu_{out} \quad (25)$$

$$d_\eta P_n^m(\cosh \eta_0) = \left. \frac{d}{d\eta} P_n^m(\cosh \eta) \right|_{\eta=\eta_0}, \quad d_\eta Q_n^m(\cosh \eta_0) = \left. \frac{d}{d\eta} Q_n^m(\cosh \eta) \right|_{\eta=\eta_0} \quad (26)$$

$$G_n^m(\kappa_\mu, \cosh \eta_0) = \kappa_\mu Q_n^m(\cosh \eta_0) d_\eta P_n^m(\cosh \eta_0) - P_n^m(\cosh \eta_0) d_\eta Q_n^m(\cosh \eta_0) \quad (27)$$

$$H_n^m(\cosh \eta_0) = Q_n^m(\cosh \eta_0) d_\eta P_n^m(\cosh \eta_0) - P_n^m(\cosh \eta_0) d_\eta Q_n^m(\cosh \eta_0) \quad (28)$$

The time evolution of the membrane charge is governed by

$$\frac{d\sigma}{dt} = \mu_{in} \frac{1}{g_\eta} \frac{\partial \phi_{in}}{\partial \eta} \bigg|_{\eta=\eta_0} = \frac{\mu_{in}}{c \sqrt{1 - (\cos \theta / \cosh \eta_0)^2}} \sum_{n,m} A_{n,m} d_\eta P_n^m(\cosh \eta_0) P_n^m(\cos \theta) \cos m\varphi \quad (29)$$

2.4 The treatment of g_η in series expansion

To solve the interaction between different harmonic degree n , we first expand c/g_η in power series of $\cos \theta$ in such a way that

$$\frac{1}{\sqrt{1 - (\cos \theta / \cosh \eta_0)^2}} = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \left(\frac{\cos \theta}{\cosh \eta_0} \right)^{2k} \quad (30)$$

The series converges because

$$-1 \leq \cos \theta \leq 1 \quad \text{and} \quad \cosh \eta_0 \geq 1 \quad (31)$$

Eq.(29) then becomes

$$\frac{d\sigma}{dt} = \frac{\mu_{in}}{c} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \left(\frac{\cos \theta}{\cosh \eta_0} \right)^{2k} \sum_{n,m} A_{n,m} d_\eta P_n^m(\cosh \eta_0) P_n^m(\cos \theta) \cos m\varphi \quad (32)$$

This equation contains the components of $(\cos \theta)^{2k} P_n^m(\cos \theta)$, which can be re-expanded in harmonic series using an identity of Legendre function

$$x P_n^m(u) = \frac{n+m}{2n+1} P_{n-1}^m(u) + \frac{n-m+1}{2n+1} P_{n+1}^m(u) \quad (33)$$

Note that this re-expansion yields $P_n^m(u)$ of different degree n but same order m . Hence, we can confine the following arguments for a fixed m .

In matrix form, eq.(33) is written as

$$x \begin{bmatrix} P_0^m(x) \\ P_1^m(x) \\ \vdots \\ P_{n-1}^m(x) \\ P_n^m(x) \\ P_{n+1}^m(x) \\ \vdots \end{bmatrix} = [\mathbf{D}_1] \begin{bmatrix} P_0^m(x) \\ P_1^m(x) \\ \vdots \\ P_{n-1}^m(x) \\ P_n^m(x) \\ P_{n+1}^m(x) \\ \vdots \end{bmatrix}, \quad x^k \begin{bmatrix} P_0^m(x) \\ P_1^m(x) \\ \vdots \\ P_{n-1}^m(x) \\ P_n^m(x) \\ P_{n+1}^m(x) \\ \vdots \end{bmatrix} = [\mathbf{D}_1]^k \begin{bmatrix} P_0^m(x) \\ P_1^m(x) \\ \vdots \\ P_{n-1}^m(x) \\ P_n^m(x) \\ P_{n+1}^m(x) \\ \vdots \end{bmatrix} \quad (34)$$

where

$$[\mathbf{D}_1] = \begin{bmatrix} 0 & 1-m & 0 & 0 & 0 & 0 & 0 \\ \frac{m+1}{3} & 0 & \frac{(2-m)}{3} & 0 & 0 & 0 & 0 \\ 0 & \cdot & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \frac{n+m}{2n+1} & 0 & \frac{n-m+1}{2n+1} & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & 0 \end{bmatrix} \quad (35)$$

Writing

$$[\mathbf{s}] = \begin{bmatrix} S_{0,m} \\ S_{1,m} \\ \vdots \\ S_{n,m} \\ \vdots \end{bmatrix}, \quad [\mathbf{P}_{nm}] = \begin{bmatrix} P_0^m(\cos \theta) \cos m\varphi \\ P_1^m(\cos \theta) \cos m\varphi \\ \vdots \\ P_n^m(\cos \theta) \cos m\varphi \\ \vdots \end{bmatrix}, \quad [\mathbf{a}] = \begin{bmatrix} A_{0,m} \\ A_{1,m} \\ \vdots \\ A_{n,n} \\ \vdots \end{bmatrix} \quad (36)$$

we have from eq.(32) that

$$\frac{d}{dt}[\mathbf{s}] = \frac{\mu_{in}}{c} [\mathbf{D}^p] [\mathbf{a}] \quad (37)$$

where

$$[\mathbf{D}^p] = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \frac{d_n P_n^m(\cosh \eta_0)}{\cosh \eta_0^{2k}} [\mathbf{D}_1^T]^{2k} \quad (38)$$

and the superscript P denotes that the quantity is for a prolate spheroid, and T denotes the transpose of the matrix.

On the other hand, from eq.(23),

$$[\mathbf{a}] = \frac{1}{C_m} [\mathbf{M}^P][\mathbf{s}] - [\mathbf{N}^P][\mathbf{e}] \quad (39)$$

where $[\mathbf{e}]$ is an array of the external-field coefficient $E_{n,m}$, \mathbf{M}^P and \mathbf{N}^P are diagonal matrices whose components respectively are

$$M_n^P = \frac{d_\eta Q_n^m(\cosh \eta_0)}{G_n^m(\kappa_\mu, \cosh \eta_0)}, \quad N_n^P = a^n \frac{H_n^m(\cosh \eta_0)}{G_n^m(\kappa_\mu, \cosh \eta_0)} \quad (40)$$

Substitution of eq.(39) into eq.(37) yields

$$\frac{d}{dt}[\mathbf{s}] = \frac{\mu_{in}}{c C_m} [\mathbf{D}^P][\mathbf{M}^P][\mathbf{s}] - \frac{\mu_{in}}{c} [\mathbf{D}^P][\mathbf{N}^P][\mathbf{e}] \quad (41)$$

This equation governs the temporal evolution of each harmonic component of the membrane charge.

For $n=1$ and $m=0$ or 1 , and first order external field, i.e.

$$E_{n,m} = E_1 \text{ for } n=1, \quad E_{n,m} = 0 \text{ for } n>1 \quad (42)$$

eq.(41) at the limit of $e \rightarrow 0$ becomes

$$\frac{d}{dt}s_1 = \frac{2}{c C_m} \frac{\mu_{in} \mu_{out}}{\mu_{in} + 2\mu_{out}} s_1 - \frac{3\mu_{in} \mu_{out}}{\mu_{in} + 2\mu_{out}} E_1 \quad (43)$$

which reduces to the well-known membrane voltage, V_m , for a spherical cell with the radius c

$$V_m = \frac{s_1}{C_m} = \frac{3}{2} c E_1 (1 - \exp[-t/\tau_1]) \quad (44)$$

where

$$\tau_1 = c C_m (\mu_{in}^{-1} + \mu_{out}^{-1}/2) \quad (45)$$

The case of an oblate spheroid can be formulated in a similar manner, which is shown in the appendix.

2.5 Polarization factor under an a.c. field

Assuming the time variation with $\exp[j\omega t]$, we have from eq.(41) that

$$j\omega [\mathbf{s}] = \frac{\mu_{out}}{cC_m} [\mathbf{D}^i] [\kappa_\mu \mathbf{M}^i] [\mathbf{s}] - \frac{\mu_{out}}{c} [\mathbf{D}^i] [\kappa_\mu \mathbf{N}^i] [\mathbf{e}] \quad (46)$$

where the superscript i is either P (prolate) or O (oblate). Hence, the dipole charge density can be determined from the external field by

$$[\mathbf{s}] = C_m \left([\mathbf{D}^i] [\kappa_\mu \mathbf{M}^i] - I\omega\tau [\mathbf{U}] \right)^{-1} [\mathbf{D}^i] [\kappa_\mu \mathbf{N}^i] [\mathbf{e}] \stackrel{\text{write}}{=} C_m [\mathbf{L}^i] [\mathbf{D}^i] [\kappa_\mu \mathbf{N}^i] [\mathbf{e}] \quad (47)$$

where the time constant is

$$\tau = cC_m / \mu_{out}. \quad (48)$$

On the other hand, eq.(24) (or (A-9) in the Appendix) can be written as

$$[\mathbf{c}] = \frac{\kappa_\mu}{C_m} [\mathbf{S}^i] [\mathbf{s}] + (\kappa_\mu - 1) [\mathbf{T}^i] [\mathbf{e}] \quad (49)$$

where $[\mathbf{c}]$ is an array of $C_{n,m}$, \mathbf{S}^i and \mathbf{T}^i are diagonal matrices whose components respectively are

$$\text{prolate : } S_n^P = \frac{d_\eta P_n^m(\cosh \eta_0)}{G_n^m(\kappa_\mu, \cosh \eta_0)}, \quad T_n^P = a^n \frac{P_n^m(\cosh \eta_0) d_\eta P_n^m(\cosh \eta_0)}{G_n^m(\kappa_\mu, \cosh \eta_0)} \quad (50)$$

$$\text{oblate : } S_n^O = \frac{(-1)^n j^{n+1} d_\eta P_n^m(j \sinh \eta_0)}{G_n^m(\kappa_\mu, j \sinh \eta_0)}, \quad T_n^O = a^n \frac{j P_n^m(j \sinh \eta_0) d_\eta P_n^m(j \sinh \eta_0)}{G_n^m(\kappa_\mu, j \sinh \eta_0)} \quad (51)$$

Substitution of eq.(47) into eq.(49) yields

$$[\mathbf{c}] = \left(\kappa_\mu [\mathbf{S}^i] [\mathbf{L}^i] [\mathbf{D}^i] [\mathbf{N}^i] + (\kappa_\mu - 1) [\mathbf{T}^i] \right) [\mathbf{e}] \quad (52)$$

This equation gives the multipolar polarization of the ellipsoid.

In this paper, we shall confine our interest to dipolar moments. In this case, at the limit of $r \rightarrow \infty$,

$$Q_1^0(\cosh \eta) \approx \frac{1}{3} \left(\frac{a}{r} \right)^2, \quad Q_1^1(\cosh \eta) \approx -\frac{2}{3} \left(\frac{a}{r} \right)^2 \quad (53)$$

$$Q_1^0(j \sinh \eta) \approx -\frac{1}{3} \left(\frac{a}{r} \right)^2, \quad Q_1^1(j \sinh \eta) \approx \frac{2}{3} \left(\frac{a}{r} \right)^2 \quad (54)$$

Therefore, the dipolar moments are

$$\text{prolate : } d_z^P = \frac{4}{3} \pi \varepsilon_{out} a^2 C_{1,0}^P, \quad d_x^P = -\frac{8}{3} \pi \varepsilon_{out} a^2 C_{1,1}^P \quad (55)$$

$$\text{oblate : } d_z^O = -\frac{4}{3} \pi \varepsilon_{out} a^2 C_{1,0}^O, \quad d_x^O = \frac{8}{3} \pi \varepsilon_{out} a^2 C_{1,1}^O \quad (56)$$

The polarization factor per unit volume P_m^i is then given as

$$P_m^i = \frac{\frac{4}{3} \pi \varepsilon_{out} a^2 f_m C_{1,m}^i}{\frac{4}{3} \pi b^2 c} \quad (57)$$

where $m = 0$ for z -dipole and $m = 1$ for x -dipole, $f_0 = 1$ and $f_1 = 2$.

3. Example of numerical calculations

We shall show some numerical solutions of the equation

$$\frac{d}{dt}[\mathbf{s}] = \frac{\mu_{in}}{c C_m} [\mathbf{D}^i] [\mathbf{M}^i] [\mathbf{s}] - \frac{\mu_{in}}{c} [\mathbf{D}^i] [\mathbf{N}^i] [\mathbf{e}] \quad (58)$$

where i is either P (prolate) or O (oblate). Because in most practical cases, $\mu_{in} \geq \mu_{out}$, we normalize time t with the time constant τ in eq.(48), i.e.

$$t = \tau t_n$$

Then the membrane voltage

$$[v_m] = \frac{1}{C_m} [\mathbf{s}] \quad (59)$$

is governed by

$$\frac{d}{dt_n} [v_m] = [\mathbf{D}^i] [\kappa_\mu \mathbf{M}^i] [v_m] - [\mathbf{D}^i] [\kappa_\mu \mathbf{N}^i] [\mathbf{e}] \quad (60)$$

Numerical calculations are made for a uniform applied field E_0 in z - and x -directions, for the case of $R=5$, i.e. $c/b=5$ (prolate) and $b/c=5$ (oblate) and $\kappa_\mu=10$, taking the harmonic order n up to $n_{max} = 200$. With $n_{max} = 200$, the error of the series expansion of eq.(30) is less than 0.4%.

Fig.2 shows the time evolution of each harmonic component v_n normalized by cE_0 , from $t_n = 0$ to t_{max} . t_{max} is chosen to cover about $4 \tau_e$ where τ_e is the time constant of nearly exponential growth of the first harmonic. It is seen that higher harmonic components decay very quickly as n is increased, and come into play only at the beginning of the transient.

Fig.3 shows the time evolution of the membrane voltage V_m (total of harmonic components) plotted as a function of θ , with the time division of $t_{max}/20$. In Fig.4 is plotted the same quantity calculated by the variable thickness approximation model, i.e. taking only $n=1$ in eq.(29). With our rigorous model, due to the effect of higher harmonics, the membrane voltage builds up rather quickly at the locations where field becomes stronger ($\theta=0$ in prolate spheroid with E_z , or $\theta=\pi/2$ in oblate spheroid with E_x) at the beginning. As time passes, the membrane voltage distribution approaches sinusoidal shape, and as expected, both models give the same value of steady-state V_m .

The fact that the higher harmonics come into play at the beginning of the transient suggests that they should as well affect the polarizability at higher frequencies. Calculated polarization factor (divided by ϵ_{out}) for $\kappa_\mu=10$ and $R=5$ is shown in Fig.5 in comparison to that calculated by variable thickness model. In fact the shift in spectra is seen (between the thick and thin lines in the figures). In particular for the case where large field non-uniformity is induced by the existence of the particle (prolate spheroid with E_z or oblate spheroid with E_x), the large difference is observed between our rigorous model and the variable thickness model, which amounts to 2.4 times frequency change in the case of oblate spheroid with E_x .

It should be noted that the present method is far faster than numerical methods like that based on BEM [13], especially for the cases the field is perpendicular to the major axis where the problem is

3-dimensional.

4. Conclusions

A method for rigorous calculation for the membrane voltage and the polarization factor of an ellipsoidal particle with an insulating membrane is developed. The method is based on the harmonic expansion of the field, to include the condition of the uniform membrane thickness as a series expansion of the geometrical factor, and to solve the field problem as an interaction of the harmonic components. With the use of the harmonic expansion, the method enables the analysis under arbitrary external field. It is identified that the conventional variable thickness model is equivalent to neglecting the harmonic interactions. Numerical calculation of the membrane voltage and the polarization factors are made, and the discrepancy between our rigorous model and the conventional variable thickness model becomes eminent where the field deformation is large due to the high axial ratio of the ellipsoid.

An approximation that the field is dominated by conduction current in the media is employed in the present analysis, but the method of the series expansion itself does not depend upon the assumption, and is applicable for the cases where the capacitive current in the media is significant.

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Appendix: Solution for the case of an oblate spheroid

The oblate spheroidal coordinate system (η, θ, φ) is related to the Cartesian coordinate system by

$$\begin{aligned} x &= a \cosh \eta \sin \theta \cos \varphi \\ y &= a \cosh \eta \sin \theta \sin \varphi \\ z &= a \sinh \eta \cos \theta \end{aligned} \quad (\text{A-1})$$

The surface of $\eta = \eta_0$, is an oblate spheroid,

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (\text{A-2})$$

where

$$b = a \cosh \eta_0 \quad \text{and} \quad c = a \sinh \eta_0$$

and

$$a = \sqrt{b^2 - c^2} \quad (\text{A-3})$$

With the use of the eccentricity of the ellipsoid e

$$e = \sqrt{1 - (c/b)^2} = a/b \quad (\text{A-4})$$

where $0 \leq e \leq 1$, we have the same expressions for $\cosh \eta_0$ and $\sinh \eta_0$ that

$$\cosh \eta_0 = 1/e \quad \text{and} \quad \sinh \eta_0 = \sqrt{(1/e)^2 - 1}$$

The solution of Laplace equation in the oblate spheroidal coordinate system is

$$\phi = \left\{ \begin{matrix} P_n^m(j \sinh \eta) \\ Q_n^m(j \sinh \eta) \end{matrix} \right\} \left\{ \begin{matrix} P_n^m(\cos \theta) \\ Q_n^m(\cos \theta) \end{matrix} \right\} \text{cs } m\varphi \quad (\text{A-5})$$

The potentials inside and outside the spheroid respectively are

$$\begin{aligned} \phi_{out} &= \phi_{app} + \phi_{ind} \\ &= -\sum_{n,m} E_{n,m} a^n j^n P_n^m(j \cosh \eta) P_n^m(\cos \theta) \text{cs } m\varphi + \sum_{n,m} C_{n,m} j^{n-1} Q_n^m(j \cosh \eta) P_n^m(\cos \theta) \text{cs } m\varphi \end{aligned} \quad (\text{A-6})$$

$$\phi_{in} = \sum_{n,m} A_{n,m} j^n P_n^m(j \sinh \eta) P_n^m(\cos \theta) \cos m\varphi \quad (A-7)$$

In the above equations the powers of j are introduced to make all the coefficients real values.

If we expand σ in Legendre harmonics as in eq.(22) and apply the boundary conditions, we have

$$A_{n,m} = \frac{(-j)^n d_\eta Q_n^m(j \sinh \eta_0) S_{n,m}}{G_n^m(\kappa_\mu, j \sinh \eta_0) C_m} - \frac{H_n^m(j \sinh \eta_0)}{G_n^m(\kappa_\mu, j \sinh \eta_0)} a^n E_{n,m} \quad (A-8)$$

$$C_{n,m} = \frac{(-1)^n j^{n+1} \kappa_\mu d_\eta P_n^m(j \sinh \eta_0) S_{n,m}}{G_n^m(\kappa_\mu, j \sinh \eta_0) C_m} + \frac{j(\kappa_\mu - 1) P_n^m(j \sinh \eta_0) d_\eta P_n^m(j \sinh \eta_0)}{G_n^m(\kappa_\mu, j \sinh \eta_0)} a^n E_{n,m} \quad (A-9)$$

where

$$d_\eta P_n^m(j \sinh \eta_0) = \left. \frac{d}{d\eta} P_n^m(j \sinh \eta) \right|_{\eta=\eta_0}, \quad d_\eta Q_n^m(j \sinh \eta_0) = \left. \frac{d}{d\eta} Q_n^m(j \sinh \eta) \right|_{\eta=\eta_0} \quad (A-10)$$

$$G_n^m(\kappa_\mu, j \sinh \eta_0) = \kappa_\mu Q_n^m(j \sinh \eta_0) d_\eta P_n^m(j \sinh \eta_0) - P_n^m(j \sinh \eta_0) d_\eta Q_n^m(j \sinh \eta_0) \quad (A-11)$$

$$H_n^m(j \sinh \eta_0) = Q_n^m(j \sinh \eta_0) d_\eta P_n^m(j \sinh \eta_0) - P_n^m(j \sinh \eta_0) d_\eta Q_n^m(j \sinh \eta_0) \quad (A-12)$$

The time evolution of the membrane charge is governed by

$$\frac{d\sigma}{dt} = \mu_{in} \frac{1}{g_\eta} \left. \frac{\partial \phi_{in}}{\partial \eta} \right|_{\eta=\eta_0} = \frac{\mu_{in}}{c \sqrt{1 - (\sin \theta / \cosh \eta_0)^2}} \sum_{n,m} A_{n,m} \tanh \eta_0 d_\eta P_n^m(j \sinh \eta_0) P_n^m(\cos \theta) \cos m\varphi \quad (A-13)$$

where the geometric coefficient g_η in this coordinate system is

$$g_\eta = a \sqrt{\sinh^2 \eta_0 + \cos^2 \theta} = a \sqrt{\cosh^2 \eta_0 - \sin^2 \theta} = \frac{c}{\tanh \eta_0} \sqrt{1 - (\sin \theta / \cosh \eta_0)^2} \quad (A-14)$$

The expansion of c/g_η is

$$\frac{1}{\sqrt{1 - (\sin \theta / \cosh \eta_0)^2}} = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \left(\frac{\sin^2 \theta}{\cosh^2 \eta_0} \right)^k = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \left(\frac{1 - \cos^2 \theta}{\cosh^2 \eta_0} \right)^k \quad (A-15)$$

so that eq.(A-13) becomes

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{\mu_{in}}{c \sqrt{1 - (\sin \theta / \cosh \eta_0)^2}} \sum_{n,m} A_{n,m} \tanh \eta_0 d_\eta P_n^m(j \sinh \eta_0) P_n^m(\cos \theta) \cos m\varphi \\ &= \frac{\mu_{in}}{c} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \left(\frac{1 - \cos^2 \theta}{\cosh^2 \eta_0} \right)^k \sum_{n,m} A_{n,m} \tanh \eta_0 d_\eta P_n^m(j \sinh \eta_0) P_n^m(\cos \theta) \cos m\varphi \end{aligned} \quad (A-16)$$

This equation again contains the components of $(\cos \theta)^{2k} P_n^m(\cos \theta)$, which can be re-expanded in Legendre harmonic series as before, i.e. using

$$[\mathbf{D}_2^T] = [\mathbf{D}_1^T]^2 \quad (\text{A-17})$$

and

$$[\mathbf{D}^O] = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \frac{\tanh \eta_0 d_\eta P_n^m(j \sinh \eta_0)}{\cosh \eta_0^{2k}} [\mathbf{U} - \mathbf{D}_2^T]^k \quad (\text{A-18})$$

where the superscript O denotes that the quantity is for an oblate spheroid and \mathbf{U} is the identity matrix. Then we have

$$\frac{d}{dt}[\mathbf{s}] = \frac{\mu_{in}}{c C_m} [\mathbf{D}^O][\mathbf{M}^O][\mathbf{s}] - \frac{\mu_{in}}{c} [\mathbf{D}^O][\mathbf{N}^O][\mathbf{e}] \quad (\text{A-19})$$

where \mathbf{M}^O and \mathbf{N}^O are diagonal matrices whose components respectively are

$$M_n^O = \frac{(-j)^n d_\eta Q_n^m(j \sinh \eta_0)}{G_n^m(\kappa_\mu, j \sinh \eta_0)}, \quad N_n^O = a^n \frac{H_n^m(j \sinh \eta_0)}{G_n^m(\kappa_\mu, j \sinh \eta_0)} \quad (\text{A-20})$$

For the first order external field and at the limit of $\epsilon \rightarrow 0$, the same expression as eq.(44) results.

[Figure Captions]

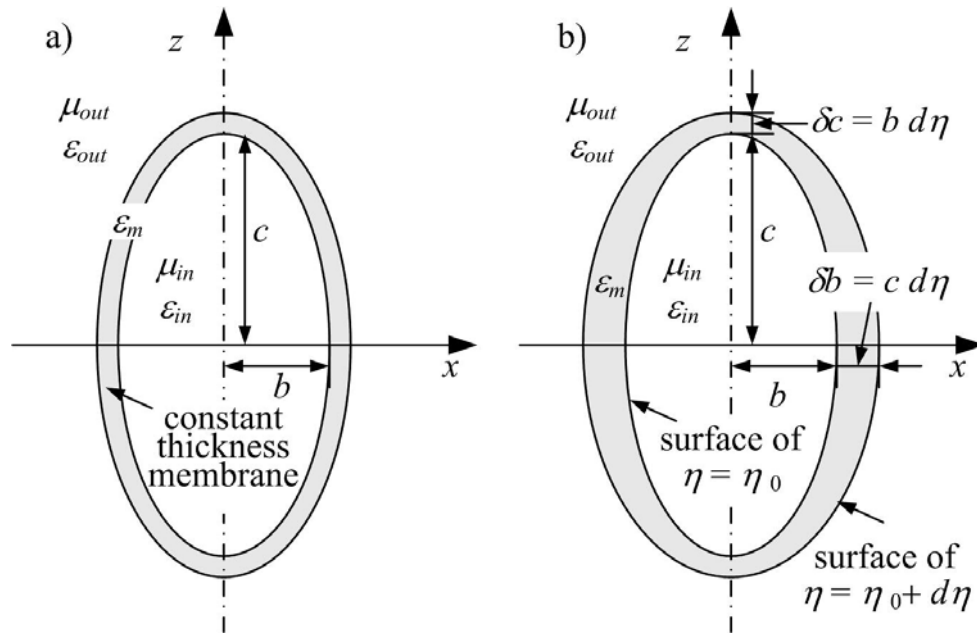
Fig.1 Mathematical modeling of a spheroidal cell. a) Actual cell with a uniform membrane thickness, b) Confocal modeling with variable thickness.

Fig.2 Time evolution of harmonic components, $t = 0$ to t_{max} .

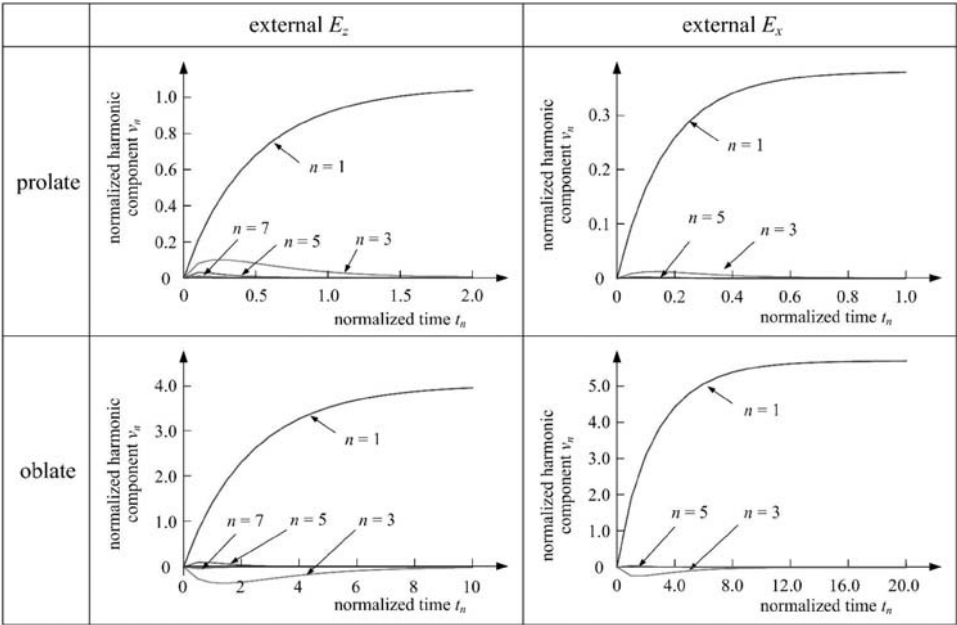
Fig.3 Time evolution of the membrane voltage as a function of θ , at $t = i/20 t_{max}$ ($i = 1$ to 20) .

Fig.4 Time evolution of the membrane voltage with the variable capacitance model, all parameters same as that of fig.3.

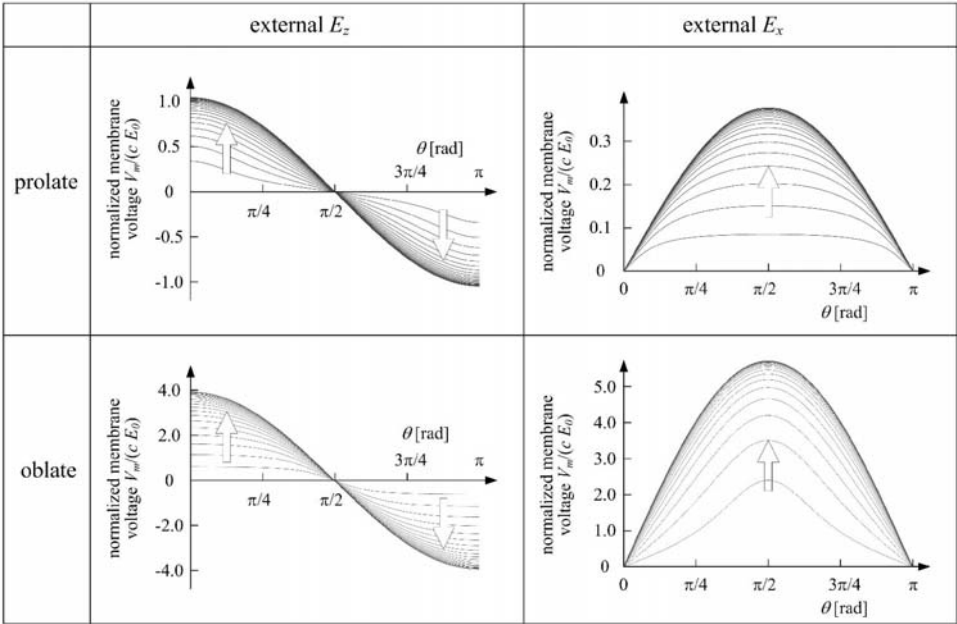
Fig.5 Dipolar polarization factors for $R=5$. Thick lines for the present rigorous model, thin lines for the variable thickness model, solid lines for the real part, and dashed lines for the imaginary part. The abscissa is the log frequency normalized by $1/\tau$, where τ is given by eq.(48)



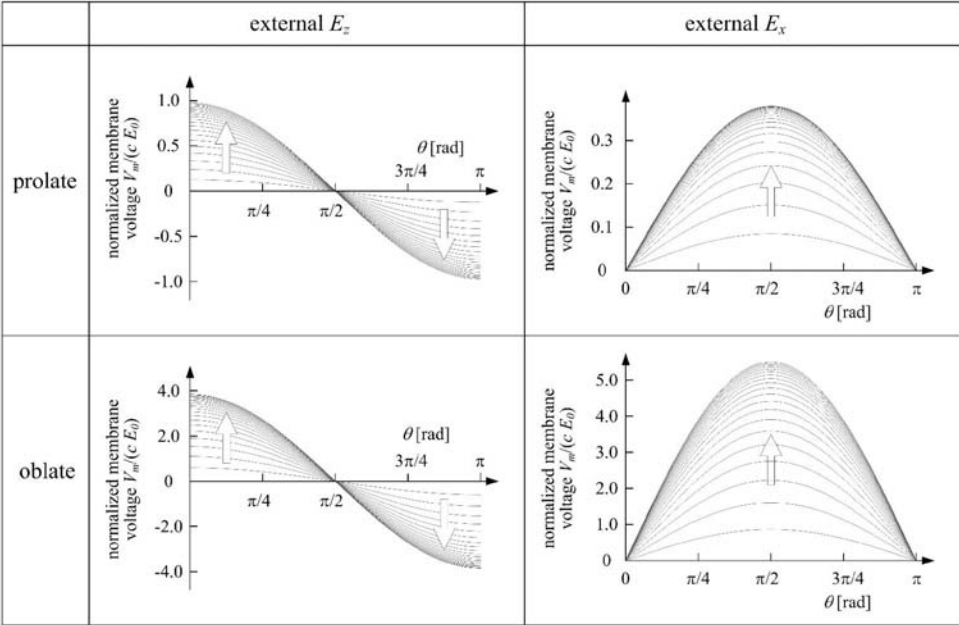
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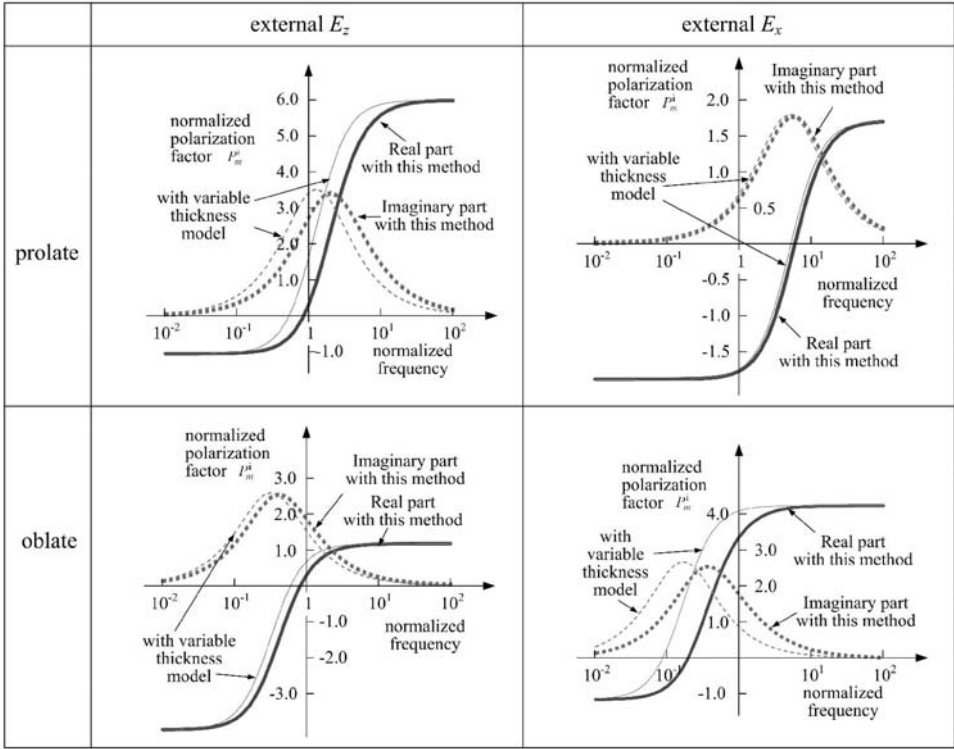
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Electric field and force on a conducting sphere in contact with a dielectric solid

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Abstract

This paper presents the analysis of electric field and force on a conducting sphere lying on a dielectric solid under a uniform field. To achieve high accuracy, we have applied the analytical method of successively placing three infinite sequences of point and dipole charges (zero- or first-order multipoles). The electric field is highest at the contact point, called the triple junction, where the conductor, the dielectric solid, and the surrounding medium (gas or vacuum) meet together. Both the contact-point field and the force increase with the permittivity ratio of the solid to that of the surrounding medium. The resulting force always attracts the sphere to the solid, in contrast to the repulsive force in the case of a conducting sphere lying on a plane conductor under an external field. We have given very simple formulae for approximating the contact-point field and the force which agree with the precise values within a difference of 3% for permittivity ratios up to 32 and 64, respectively.

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1. Introduction

Electric fields and resulting forces are of particular importance in many fields of electrostatics. One important case concerns the behavior of particles present under an electric field. In a gas insulated system, for example, the force induced by an electric field may lift a foreign conducting particle against the gravitational force, thus possibly affecting the insulation characteristics of the system. In photocopying, the electrostatic behavior of toners under an external field is significant to the overall performance of the process. Therefore, the electric field distribution and force have been reported in detail for the case of a conducting particle lying on a conducting plate under an external field [1].

On the other hand, the electric field and force have not been fully investigated for the case of a conducting particle lying on a dielectric solid. In high-voltage insulation systems, dielectric solids are always used for mechanical support and separation. There exist basically two different cases for such arrangements if we consider general cases including a conductor not necessarily small like an electrode:

- (1) the conductor is charged to a fixed potential, or
- (2) the conductor resides at a floating potential under an applied field.

For the former case (1), the electric field has been computed for a spherical conductor at a fixed potential on a dielectric solid by the method of placing an infinite number of point charges (images) [2,3]. This arrangement represents a fundamental case for studying the field behavior at a triple junction where the conductor,

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the dielectric solid, and the surrounding medium (gas or vacuum) meet together.

This paper presents an analysis of the electric field and resulting force for case (2) wherein a conducting sphere lies on a dielectric solid of semi-infinite thickness under an external (uniform) field. This is the most fundamental arrangement for considering the electric field and force acting on a conducting particle lying on a dielectric solid, such as an insulating spacer or support in gas insulated systems. We assume that the conducting sphere is not charged (i.e., has no net charge) here. However, the analytical method used here can also treat a general case of a charged, electrically floating conductor under an external field. For this case, the electric field is obtained by a linear superposition of the field on the uncharged conductor with the external field, which is treated here, and that on a charged conductor without the external field.

It is very difficult to obtain high accuracy by any numerical method for an arrangement with a triple junction where a rounded conductor meets a dielectric solid. This is because an infinitesimally narrow gap is involved and the field is intensified, particularly when the permittivity of the solid is much greater than that of the surrounding medium. In order to achieve high accuracy, we have analyzed the field on the conducting sphere using the method of images for point and dipole charges (zero- and first-order multipoles). Although the image for an arrangement of a dielectric solid and a point charge has been well known [4], in this paper we generalize the image for the case of a dielectric solid and a multipole of any order, e.g., dipole or quadrupole.

2. Arrangement

The calculation arrangement is shown in Fig. 1. We consider an uncharged conducting sphere of radius R

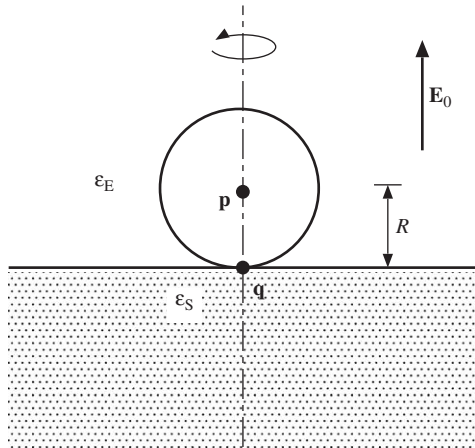


Fig. 1. A conducting sphere with a radius R lying on a dielectric solid of semi-infinite thickness under a uniform field E_0 .

lying on a dielectric solid of permittivity ϵ_S . The sphere, the dielectric solid, and the background medium form a triple junction at the point q in Fig. 1. We assume that the sphere radius is much smaller than the thickness of the dielectric solid that the solid can be treated as semi-infinite. The conducting sphere and the dielectric solid are subjected to a uniform electric field E_0 in the vertical direction. The permittivity of the exterior is denoted by ϵ_E . We have carried out the calculation for $1 \leq \epsilon_S/\epsilon_E \leq 64$.

In the absence of the dielectric solid, the maximum field is located at the top and the bottom of the conducting sphere. The dielectric solid further increases the electric field at the bottom point, q . Therefore, we are interested in the contact-point electric field E_c (referred to the ϵ_E -side) at q , which is the highest field in this arrangement. The electric field on the ϵ_S -side of the contact point is simply given by $\epsilon_E E_c / \epsilon_S$.

3. Calculation method

As already mentioned, we calculate the electric field distribution by using a method of images that utilizes point and dipole charges. The calculation is principally similar to the case of a conducting sphere at a fixed potential and a dielectric solid [2,3]. However, the situation is more complicated here because the net charge on the conducting sphere must remain zero. We express the potential and electric field in the terms of zero-order multipoles (i.e., monopoles) B_0 instead of point charges Q and first-order multipoles B_1 instead of point dipoles P in the $+z$ -direction to simplify the expression. The potential ϕ at any point r due to B_0 or B_1 located at r_B is defined as follows:

$$\phi = \frac{B_0}{\|r - r_B\|}, \quad (1)$$

$$\phi = \frac{B_1 \cos \theta}{\|r - r_B\|^2}, \quad (2)$$

where θ is the angle between $r - r_B$ and the dipole direction. It is clear from Eqs. (1) and (2) that the magnitudes of the multipoles are given by $Q = 4\pi\epsilon_E B_0$ and $P = 4\pi\epsilon_E B_1$.

3.1. Images for a multipole and a conducting sphere

Consider a grounded conducting sphere and a zero-order multipole B_0 located at a distance d from its center. The zero-potential condition on the sphere can be satisfied by placing an image B'_0 at a distance R^2/d from the center (Fig. 2(a)). The image magnitude is related to B_0 by [4]

$$B'_0 = -\frac{R}{d} B_0. \quad (3)$$

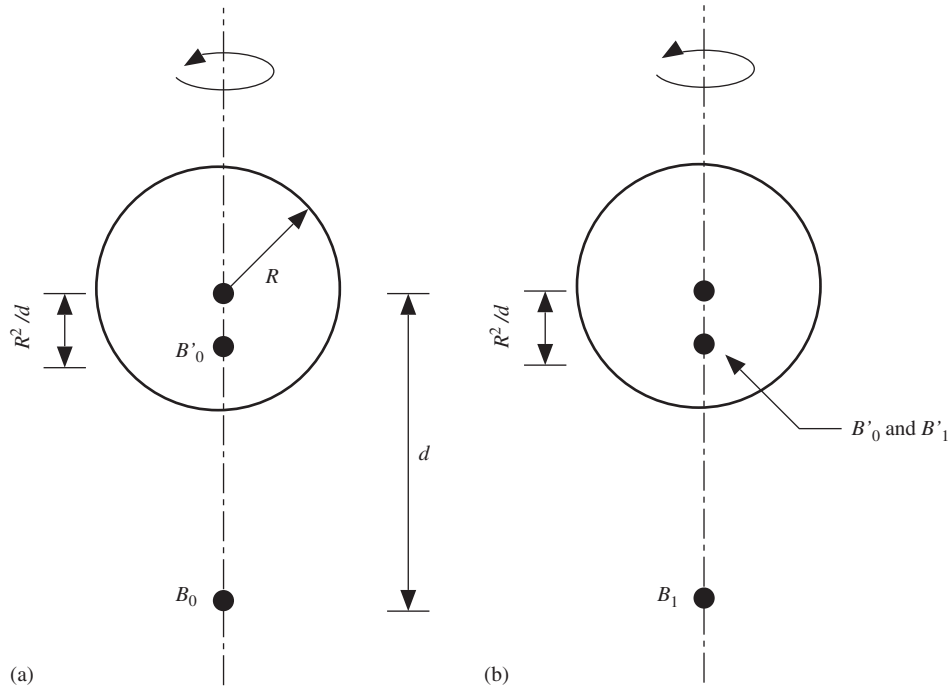


Fig. 2. Images in a grounded conducting-sphere for a point charge (a) or dipole (b).

For an electrically floating conducting sphere under an external field, an additional image $B'_0 = -B'_0$ must be placed at the sphere center to fulfill the zero-charge condition while maintaining the equipotential condition on the sphere.

Images for a grounded conducting sphere and a first-order multipole B_1 are shown in Fig. 2 (b). Two images, B'_1 and B'_0 , are required in this case [5]. The images are related to B_1 by

$$B'_1 = \left(\frac{R}{d}\right)^3 B_1, \quad (4)$$

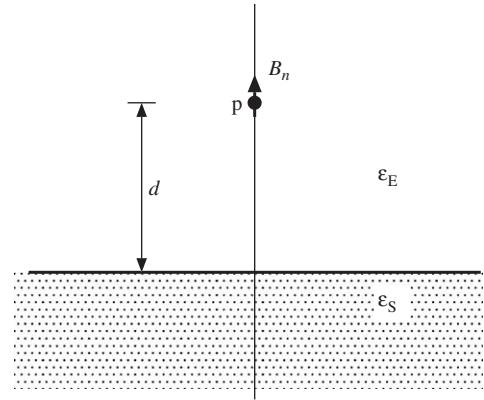
$$B'_0 = -\frac{R}{d^2} B_1. \quad (5)$$

Similarly, an additional image $B'_0 = -B'_0$ is needed at the sphere center for an electrically floating conducting sphere under an external field. Note that there also exist images for linear multipoles of higher orders [6].

3.2. Images for a multipole and a dielectric solid

Consider an n th-order multipole B_n located at a point \mathbf{p} above a dielectric solid of semi-infinite thickness at a height d , as shown in Fig. 3. The potential due to this multipole is expressed by [7]

$$\phi_B = \frac{B_n}{r_p^{n+1}} P_n(\cos \theta_p), \quad (6)$$


 Fig. 3. An n th-order multipole B_n and a dielectric solid.

where r_p and θ_p are the spherical coordinates in which \mathbf{p} is the origin, and P_n is the Legendre function.

Similar to the case of a point charge and a dielectric solid in [4], we place two multipole images of B_n to satisfy the boundary conditions of potential and electric field on the dielectric solid. The images are B'_n at \mathbf{p}' in Fig. 4(a) and B''_n at \mathbf{p} in Fig. 4(b). By the method of multipole images, we separately express the resultant potentials ϕ_S in the dielectric solid and ϕ_E in the surrounding medium as follows:

$$\phi_S = \frac{B''_n}{r_p^{n+1}} P_n(\cos \theta_p), \quad (7)$$

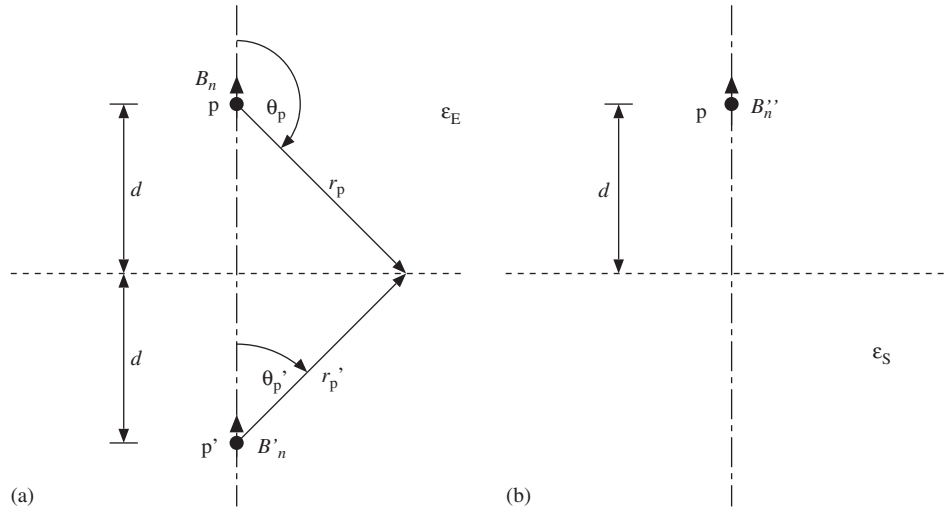


Fig. 4. Multipole images for the (a) exterior and (b) interior of the dielectric solid.

$$\phi_E = \frac{B'_n}{r_{p'}^{n+1}} P_n(\cos \theta_{p'}) + \phi_B, \quad (8)$$

where $r_{p'}$ and $\theta_{p'}$ are defined in the similar manner as r_p and θ_p , respectively.

The boundary conditions on the surface of the dielectric solid are

$$\phi_S = \phi_E, \quad (9)$$

$$\varepsilon_S \frac{\partial \phi_S}{\partial n} = \varepsilon_E \frac{\partial \phi_E}{\partial n}, \quad (10)$$

where $\partial \phi / \partial n$ is the normal potential gradient. At any point on the dielectric interface, we have $r_{p'} = r_p$ and $\theta_{p'} = \pi - \theta_p$ (see Fig. 4(a)). Noting that $P_n(\cos \theta_{p'}) = (-1)^n P_n(\cos \theta_p)$ and $\partial \phi / \partial n = \cos \theta \partial \phi / \partial r + \sin \theta \partial \phi / \partial \theta$, we obtain the following equations from the boundary conditions.

$$B''_n = B_n + (-1)^n B'_n, \quad (11)$$

$$\varepsilon_S B''_n = \varepsilon_E [B_n + (-1)^{n+1} B'_n], \quad (12)$$

From Eqs. (11) and (12), the magnitudes of B'_n and B''_n are related to B_n by

$$B'_n = (-1)^{n+1} K B_n, \quad (13)$$

$$B''_n = (1 - K) B_n, \quad (14)$$

where K is defined as a function of the permittivities,

$$K = \frac{\varepsilon_S - \varepsilon_E}{\varepsilon_E + \varepsilon_S}. \quad (15)$$

Note that $0 \leq K < 1$ for $\varepsilon_E \leq \varepsilon_S < \infty$. The images described in this section can also be utilized in electric field calculations by the method of multipole images presented in Refs. [7,8].

3.3. Calculation by the method of images

We calculate the electric field by the method of images in three main steps that can be briefly explained as follows.

In the first step, the calculation starts from the external field E_0 and the corresponding multipole $B_1^{(1)}$, at the sphere center, that satisfies the boundary conditions on the sphere surface under E_0 . The multipole magnitude is related to E_0 by the equation

$$B_1^{(1)} = R^3 E_0. \quad (16)$$

With the dielectric solid present, $B_1^{(1)}$ induces its image in the solid via Eq. (13), which further interacts with the conducting sphere via Eqs. (4) and (5). To simplify the calculation in this step, we temporarily treat the sphere as grounded, and also neglect the zero-order multipoles (point charges), which are to be taken into account later. We focus only on the first-order multipoles (dipole charges), and obtain an infinite sequence of dipole images in the sphere and the dielectric solid. See Appendix A for the positions and magnitudes of the first set of images.

In the second step, we treat the point charges induced as the images of dipole charges with respect to the conducting sphere in the first step. The sphere is still treated as if grounded. Each point charge leads to an infinite sequence of image charges in the sphere and the dielectric solid. This second set of images is given in detail in Appendix B.

In the last step, we fulfill the zero-charge condition of the conducting sphere under an external field. To negate all the charges previously applied, we place a point charge at the sphere center. Similar to the second step, this negating charge results in sequences of images required to satisfy the conditions of potential and field

with the existence of the dielectric solid. However, the insertion of these images differs from the other steps because, in every repetitive step, we maintain zero net charge by adding an appropriate image at the sphere center. Appendix C discusses the third set of images in more detail.

All the images described in the Appendices can be grouped by their positions. At each height $h = R/i$ ($i = 1, 2, \dots$) above the surface of the solid dielectric, From Eqs. (A.4), (B.2), and (C.5) in the Appendices, we obtain the sums $(B_0)_i$ and $(B_1)_i$ of the zero-order and first-order multipoles, respectively, as follows:

$$\frac{(B_0)_i}{B_1^{(1)}} = -\frac{K^{i-1}}{R} \frac{1}{i} \left\{ \frac{i-1}{i} - \frac{\alpha_K}{K} \left[\ln \left(\frac{1}{1-K} \right) - \beta_K \right] \right\}, \quad (17)$$

$$\frac{(B_1)_i}{B_1^{(1)}} = \left(\frac{1}{i^3} \right) K^{i-1}. \quad (18)$$

The functions α_K and β_K in Eq. (17) originate from terms in the summation. They are defined as

$$\alpha_K = \sum_{k=1}^{\infty} S_k, \quad (19)$$

$$\beta_K = \sum_{j=1}^{\infty} \frac{K^j}{j^2} = \int_0^K \frac{1}{t} \ln \left(\frac{1}{1-t} \right) dt, \quad (20)$$

where S_k is a sequence in which $S_0 = 1.0$ and

$$S_k = \sum_{l=0}^{k-1} -\frac{K^{k-l}}{k-l+1} S_l. \quad (21)$$

The magnitudes $(B'_0)_i$ and $(B'_1)_i$ of the zero-order and first-order multipoles at each $h = -R/i$ ($i = 1, 2, \dots$) are simply related to $(B_0)_i$ and $(B_1)_i$ by

$$(B'_0)_i = -K(B_0)_i, \quad (22)$$

$$(B'_1)_i = K(B_1)_i. \quad (23)$$

Finally, the contact-point electric field E_c is the sum of the applied field and the field due to all images.

$$E_c = E_0 + E_{c1} + E_{c2} + E_{c3}, \quad (24)$$

where E_{c1} , E_{c2} , and E_{c3} are the fields due to the images obtained in the first, second, and last steps, respectively. They are given by Eqs. (A.5), (B.4), and (C.7). Thus, we can write the contact-point electric field E_c at \mathbf{q} as

$$\frac{E_c}{E_0} = 1 + \frac{1+K}{1-K} \left\{ 2 + \frac{K}{1-K} - \frac{\alpha_K}{K(1-K)} \left[\ln \left(\frac{1}{1-K} \right) - \beta_K \right] \right\}. \quad (25)$$

Alternatively, as a function of the permittivities (except the terms α_K and β_K), the contact-point electric field can

be expressed as

$$\frac{E_c}{E_0} = 1 + \frac{\epsilon_S}{\epsilon_E} \left\{ 2 + \frac{\epsilon_S - \epsilon_E}{2\epsilon_E} - \alpha_K \frac{(\epsilon_S + \epsilon_E)^2}{2\epsilon_E(\epsilon_S - \epsilon_E)} \times \left[\ln \frac{\epsilon_S + \epsilon_E}{2\epsilon_E} - \beta_K \right] \right\}. \quad (26)$$

4. Results and discussion

4.1. Numerical values of α_K and β_K

The contact-point electric field has been expressed using α_K and β_K as parameters. Although we cannot write these parameters in a closed form, Eqs. (19) and (20) converge well if K is not close to unity. This condition is true for most dielectrics in practical use. For example, K ranges from 0 to 0.818 for $1 \leq \epsilon_S/\epsilon_E \leq 10$. Figs. 5 and 6 provide the values of α_K and β_K for calculating the contact-point electric field using Eq. (25) or (26). Fig. 5 shows the variation of α_K for $K = 0$ to 0.999. It can be seen from the figure that $\alpha_K = 1$ at $K = 0$ and α_K decreases with increasing K . The decrease of α_K with K is roughly linear for small K , but becomes nonlinear at larger K .

On the other hand, it is obvious from Eq. (20) that β_K varies from 0 to the Riemann zeta function $\zeta(2)$ for

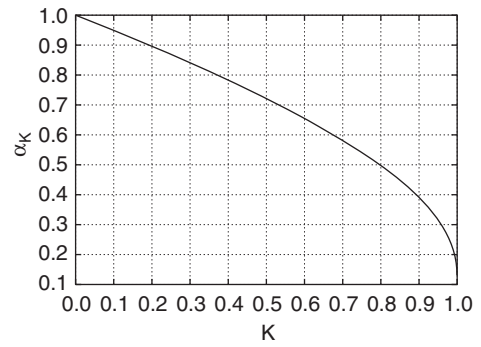


Fig. 5. Variation of α_K in Eq. (19) for $K = 0$ to 0.999.

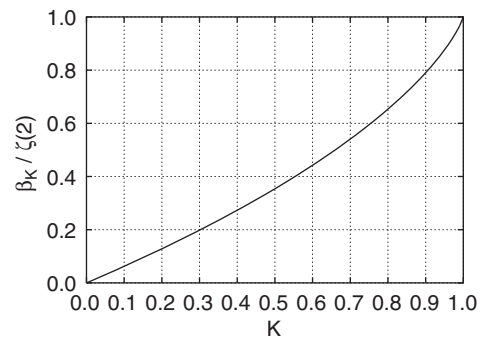


Fig. 6. Variation of β_K in Eq. (20) for $K = 0$ to 1.

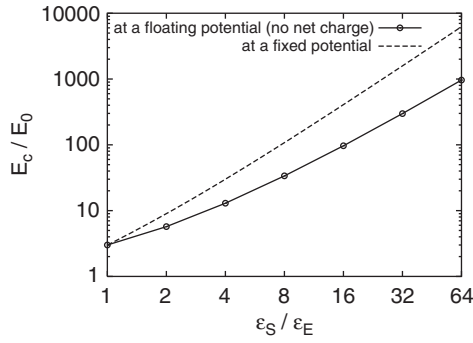


Fig. 7. Contact-point electric field as a function of the permittivity ratio $\varepsilon_S/\varepsilon_E$.

$K = 0$ to 1 , where $\zeta(2) = \sum_{j=1}^{\infty} (1/j^2) = \pi^2/6$. Fig. 6 shows the variation of β_K between $K = 0$ to 1.0 .

4.2. Contact-point electric field

Fig. 7 shows the contact-point electric field E_c as a function of the permittivity ratio $\varepsilon_S/\varepsilon_E$. Note that there exists no gap at the contact point; however, E_c represents the upper bound of electric field in the small gap near the contact point. On the solid side, the field is equal to $\varepsilon_E E_c/\varepsilon_S$ at \mathbf{q} , as already mentioned. It can be clearly seen from Fig. 7 that the ratio E_c/E_0 increases as the permittivity ratio increases. The figure also implies that the field is infinitely high if $\varepsilon_S = \infty$. In usual electric field distributions, the field of $\varepsilon_S = \infty$ corresponds to that of the conducting solid. However, in the contact field problems, $\varepsilon_S = \infty$ is not equivalent to the conducting solid. The solid has still no conductivity, so that no charge transfer occurs between the sphere and the solid. This corresponds to the situation that an infinitesimally small gap exists between them.

For comparison, the dotted line in Fig. 7 presents E_c on a conducting sphere at a fixed potential in contact with a dielectric solid (without the external field). E_c on the sphere is given by [2,3]

$$\frac{E_c}{E_0} = \frac{3}{2} \left(\frac{\varepsilon_S}{\varepsilon_E} \right) \left(\frac{\varepsilon_S}{\varepsilon_E} + 1 \right). \quad (27)$$

The charge on the conductor is chosen so that the magnitude of E_c for $\varepsilon_S = \varepsilon_E$ is the same as that on the sphere under E_0 . That is to say, the magnitude of E_c is equal to $3E_0$. We can see from Fig. 7 that, for $\varepsilon_S > \varepsilon_E$, E_c on the conducting sphere at the fixed potential is higher than that on the sphere under E_0 . The difference in E_c becomes greater as the permittivity ratio increases.

Although the contact-point electric field, E_c , can be computed from Eq. (25) or (26) by using α_K and β_K , we obtain the following simple formula, based on curve

Table 1

Comparison of the contact-point electric field with the approximation by Eq. (28)

$\varepsilon_S/\varepsilon_E$	E_c/E_0		Difference (%)
	Calculation	Eq. (28)	
1	3.000	2.996	0.133
2	5.710	5.571	2.435
4	12.94	12.60	2.640
8	33.68	33.26	1.237
16	96.80	97.27	0.478
32	298.2	302.0	1.286
64	964.3	969.2	0.504

fitting, for E_c :

$$\frac{E_c}{E_0} \approx 0.696 \left(\frac{\varepsilon_S}{\varepsilon_E} + \sqrt{\frac{7}{4}} \right)^{\sqrt{3}}. \quad (28)$$

Eq. (28) adequately approximates the contact-point electric field over the range $\varepsilon_S = \varepsilon_E$ to $64\varepsilon_E$. Table 1 compares the field magnitude approximated by Eq. (28) with our calculation results. Table 1 shows that the error by the approximation is smaller than 3% for permittivity ratios up to 64.

4.3. Force on the conducting sphere

Force on the conducting sphere may be calculated from the Maxwell stress tensor [9]. Alternatively, it is possible to determine the force as the sum of forces acting on the image charges and dipoles in the sphere. The force F_z in the $+z$ -direction is determined by

$$\begin{aligned} \frac{F_z}{4\pi\varepsilon_E} = & -\frac{K}{R^2} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (B_0)_j (B_0)_k \left(\frac{jk}{j+k} \right)^2 \\ & - \frac{6K}{R^4} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (B_1)_j (B_1)_k \left(\frac{jk}{j+k} \right)^4 \\ & + \frac{4K}{R^3} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (B_0)_j (B_1)_k \left(\frac{jk}{j+k} \right)^3. \end{aligned} \quad (29)$$

From Eqs. (17), (18), and (29), we obtain a negative F_z signifying a sphere that is attracted to the dielectric solid. This force direction is opposite to that on a conducting sphere in contact with a conducting solid under an external field, which repels the sphere from the solid [1,10]. This is because, for the case of the conducting solid, charges are transferred between the solid and the sphere. After the charge transfer, the Coulomb force acting on the sphere changes its direction from attractive to repulsive. In contrast, for the case of a dielectric solid, no charge is transferred between the objects. Thus, the force direction remains attractive.

Fig. 8 shows the variation of the force magnitude with the permittivity ratio $\varepsilon_S/\varepsilon_E$. To generalize the force F_z in Fig. 8 for any values of ε_E , E_0 , and R , F_z is normalized by a factor $F_0 = \varepsilon_E E_0^2 R^2$, and thus becomes dimensionless. The indices j and k required to compute the force in Eq. (29) increase with increasing $\varepsilon_S/\varepsilon_E$. (Up to 850 terms were used for $\varepsilon_S/\varepsilon_E = 64$.) The force is equal to zero for $\varepsilon_S = \varepsilon_E$, and becomes stronger with larger $\varepsilon_S/\varepsilon_E$.

The following formula, obtained by curve fitting, may be used to roughly approximate the force magnitude:

$$\left| \frac{F_z}{F_0} \right| \approx 1.8 \left(\frac{\varepsilon_S}{\varepsilon_E} - 1 \right) \left(\frac{\varepsilon_S}{\varepsilon_E} + 1 \right)^{0.31}. \quad (30)$$

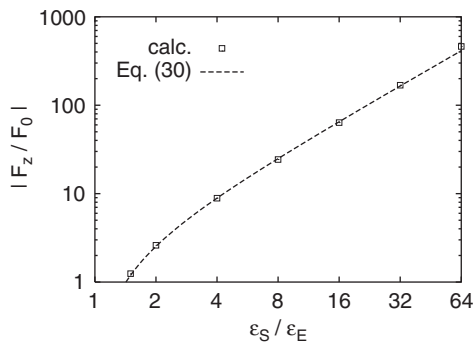


Fig. 8. Force on the conducting sphere as a function of the permittivity ratio $\varepsilon_S/\varepsilon_E$.

Table 2
Comparison of the force on the conducting sphere with the approximation by Eq. (30)

$\varepsilon_S/\varepsilon_E$	F_z/F_0		Difference (%)
	Calculation	Eq. (30)	
2	2.603	2.530	2.791
4	8.880	8.894	0.152
8	24.32	24.90	2.382
16	63.65	64.98	2.095
32	168.5	165.0	2.102
64	461.7	413.6	10.41

The force magnitude given by Eq. (30) is presented by the dotted line in Fig. 8. Table 2 shows the difference between the forces by the method of images and those obtained using Eq. (30). It can be seen from the table that the error is smaller than 3% for permittivity ratios of 2–32. For $\varepsilon_S/\varepsilon_E > 32$, the error in the approximation is considerably high (e.g., 10.4% at $\varepsilon_S/\varepsilon_E = 64$); thus, the force should be directly computed from Eq. (29).

Finally, it is worth noting that we can apply the method described here to numerically calculate the field and force even if the sphere is not in contact with the solid. For example, Fig. 9 presents the maximum field and force on the conducting sphere having its center located above the solid surface by a distance d . Fig. 9(a) shows that the enhancement of the maximum field above the value $3E_0$ (the field where $d/R = \infty$) decreases with increasing d , and is not so significant when $d/R \geq 2$. The force also becomes consistently weaker with increasing d . At sufficiently large d/R , the force varies with the separation roughly as $F \propto d^{-k}$, where k is between 4.0 and 4.2 in Fig. 9(b).

5. Conclusions

This paper presents the analytical solutions of the electric field and force on a conducting sphere in contact with a dielectric solid under a uniform field. The solutions are based on the method of images using point and dipole charges. Fundamental solutions (images) for a general case of a multipole and a dielectric solid are also given here. The contact-point electric field is obtained as a function of K , where $K = (\varepsilon_S - \varepsilon_E)/(\varepsilon_S + \varepsilon_E)$. The numerical results show that the electric field and force increase with increasing the permittivity of the solid dielectric. We propose simple formulae based on curve fitting for estimating the electric field and force. The errors realized by using the formulae are smaller than 3% for permittivity ratios of solid to surrounding medium ranging from 1 to 64 for the field and 2 to 32 for the force.

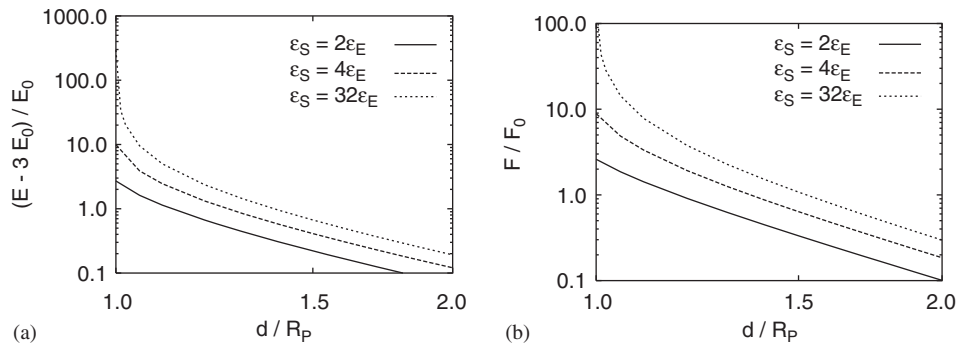


Fig. 9. Variation of the field enhancement and force with separation: (a) electric field, (b) force.

Acknowledgements

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Appendix A. First set of images

This appendix describes the first set of images mentioned in Section 3.3. The images begin with the multipole $B_1^{(1)}$ in Eq. (16). As the image of $B_1^{(1)}$ induced by the dielectric solid, we insert $(B_1')^{(1)}$ in the solid below its surface by a distance R . From Eq. (13),

$$(B_1')^{(1)} = KB_1^{(1)}. \quad (\text{A.1})$$

Next, we apply the images of $(B_1')^{(1)}$ to maintain the equipotential condition on the conducting sphere (see Section 3.1). To simplify the calculation procedure, we treat the sphere as grounded here. The total charge of the sphere must be set to zero in the final step after all the point charges have been applied. Because the distance $d^{(1)}$ between $(B_1')^{(1)}$ and the center of the sphere is equal to $2R$, Eqs. (4) and (5) yield:

$$(B_1)^{(2)} = \left(\frac{1}{2}\right)^3 KB_1^{(1)}, \quad (\text{A.2})$$

$$(B_0)^{(2)} = -\left(\frac{1}{2}\right)^2 \frac{KB_1^{(1)}}{R}, \quad (\text{A.3})$$

at a height $h^{(2)} = R/2$ above the dielectric solid. Fig. 10 shows the positions of these images.

The multipoles $(B_1)^{(2)}$ and $(B_0)^{(2)}$ induce their images in the dielectric solid, which further interact with the conducting sphere. However, for the first set of images, we neglect the further contribution of all $(B_0)^{(j)}$ (to be treated as the second set of images), and focus only on $(B_1)^{(i)}$, where the superscript (i) denotes the step of repetition. As a result, the images shown in Table 3 are obtained from the repetition. Note that the multipole $(B_1)_i$ at a height R/i for $i = 1, 2, \dots$ is

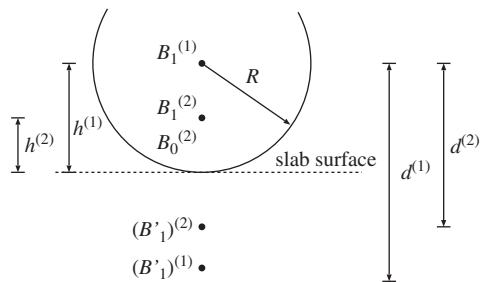


Fig. 10. Positions of the first set of images corresponding to a first-order multipole (dipole) $B_1^{(1)}$.

expressed by

$$(B_1)_i = \frac{1}{i^3} K^{i-1} B_1^{(1)}. \quad (\text{A.4})$$

At the contact point \mathbf{q} in Fig. 1, the electric field E_{c1} due to all $B_1^{(i)}$ and $(B_1')^{(i)}$ is equal to the sum of the field due to each image. Accordingly, E_{c1} , which is in the direction of E_0 , can be written as

$$E_{c1} = \sum_{i=1}^{\infty} \left\{ \frac{2B_1^{(i)}}{[h^{(i)}]^3} + \frac{2(B_1')^{(i)}}{[h^{(i)}]^3} \right\} = 2E_0 \frac{1+K}{1-K}. \quad (\text{A.5})$$

Appendix B. Second set of images

The second set of images includes all the images corresponding to $B_0^{(i)}$ ($i = 2, 3, \dots$) described in Appendix A. Consider a zero-order multipole B_{0m} located at a height R/m above the solid surface, where m is an integer and $m \geq 2$. The sphere is still considered to be grounded. Using the proper images described in Sections 3.1 and 3.2, we obtain the sequence of images shown in Table 4, where the superscript (j) denotes the step of repetition. Fig. 11 shows the positions of these images for $j = 1$ and 2.

From Table 4, if we focus on a height R/i for $i = 2, 3, \dots$, the sum $(B_0)_i$ of all the images at this position can be determined as

$$(B_0)_i = \sum_{m=2}^i \frac{m}{i} K^{i-m} B_{0m}. \quad (\text{B.1})$$

Substituting $B_0^{(m)} = -K^{m-1}/[(m-1)m^2R]$ (Table 3) for B_{0m} , we yield

$$\begin{aligned} (B_0)_i &= -\frac{K^{i-1}}{R} \left(\frac{1}{i}\right) \sum_{m=2}^i \frac{1}{m(m-1)} \\ &= -\frac{K^{i-1}}{R} \left(\frac{i-1}{i^2}\right). \end{aligned} \quad (\text{B.2})$$

Note that Eq. (B.2) also valids for $i = 1$ because it gives $(B_0)_1 = 0$ as in Table 3.

The electric field $(E_c)_{B_{0m}}$ due to B_{0m} and all of its images is calculated from the formula

$$\begin{aligned} (E_c)_{B_{0m}} &= -(1+K) \sum_{j=1}^{\infty} \frac{B_{0m}}{R^2} K^{j-1} m(m+j-1) \\ &= -\frac{B_{0m}}{R^2} \frac{1+K}{1-K} \left(m + \frac{K}{1-K}\right) m. \end{aligned} \quad (\text{B.3})$$

Let E_{c2} denote the electric field at \mathbf{q} due to all $B_{0m}^{(j)}$ and $(B_0')^{(j)}$ for $m = 2, 3, \dots$ and $j = 1, 2, \dots$. Substituting $B_0^{(m)}$ for B_{0m} in the first set of images (shown in

Table 3
Positions and magnitudes of the first set of images in each repetitive step

(i)	$h^{(i)}/R$	$B_0^{(i)}/B_1^{(1)}$	$B_1^{(i)}/B_1^{(1)}$	$(B_1')^{(i)}/B_1^{(1)}$	$d^{(i)}/R$
1	1	—	1	K	2
2	1/2	$-(1/2)^2 K/R$	$(1/2)^3 K$	$(1/2)^3 K^2$	3/2
3	1/3	$-(1/2)(1/3)^2 K^2/R$	$(1/3)^3 K^2$	$(1/3)^3 K^3$	4/3
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
i	1/i	$-[1/(i-1)](1/i)^2 K^{(i-1)}/R$	$(1/i)^3 K^{(i-1)}$	$(1/i)^3 K^i$	$(i+1)/i$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots

Table 4
Positions and magnitudes of the second set of images in each repetitive step

(j)	$h^{(j)}/R$	$B_{0m}^{(j)}/B_{0m}$	$(B_{0m}')^{(j)}/B_{0m}$	$d^{(j)}/R$
1	1/m	1.0	$-K$	$(m+1)/m$
2	1/(m+1)	$[m/(m+1)]K$	$-[m/(m+1)]K^2$	$(m+2)/(m+1)$
3	1/(m+2)	$[m/(m+2)]K^2$	$-[m/(m+2)]K^3$	$(m+3)/(m+2)$
\vdots	\vdots	\vdots	\vdots	\vdots
j	1/(m+j-1)	$[m/(m+j-1)]K^{j-1}$	$-[m/(m+j-1)]K^j$	$(m+j)/(m+j-1)$
\vdots	\vdots	\vdots	\vdots	\vdots

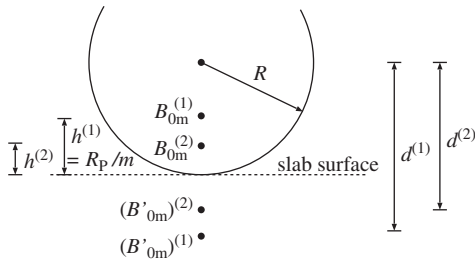


Fig. 11. Positions of the images corresponding to a zero-order multipole B_{0m} .

Table 3) for $m = 2, 3, \dots$, we obtain the following expression for E_{c2} :

$$\begin{aligned}
 E_{c2} &= \sum_{m=2}^{\infty} (E_c)_{B_{0m}} \\
 &= \sum_{m=2}^{\infty} \left\{ \left[\frac{1}{m-1} \left(\frac{1}{m^2} \right) K^{m-1} E_0 R^2 \right] \left[\frac{1}{R^2} \left(\frac{1+K}{1-K} \right) \right. \right. \\
 &\quad \left. \left. \times \left(m + \frac{K}{1-K} \right) m \right] \right\} \\
 &= E_0 \frac{1+K}{1-K} \sum_{m=1}^{\infty} \left(\frac{K^m}{m} + \frac{K}{1-K} \frac{K^m}{(m+1)m} \right).
 \end{aligned}$$

Because $\sum_{m=1}^{\infty} x^m/m = -\ln(1-x)$ and $\sum_{m=1}^{\infty} x^m/[m(m+1)] = \frac{1-x}{x} \ln(1-x) + 1$, we can write E_{c2} in a

simple form:

$$E_{c2} = E_0 \frac{(1+K)K}{(1-K)^2}. \quad (\text{B.4})$$

It is worth noting that this set of images can also be used to calculate the electric field in the case of a charged conducting sphere lying on a dielectric solid (without an external field). From Table 4, if we take $m = 1$ and $B_{0m} = 1$, then the sum of the images inside the sphere is expressed by

$$\sum_{j=1}^{\infty} \frac{K^{j-1}}{j} = \frac{1}{K} \ln \frac{1}{1-K}, \quad (\text{B.5})$$

for $0 < K < 1$. Consequently, all the images in Table 4 for $m = 1$ and $B_{0m} = 1$ are the solution for the sphere having a net charge equal to $(4\pi\epsilon_E/K) \ln[1/(1-K)]$. These images can be used to obtain the electric field in a general case of a charged, floating sphere, as mentioned in Section 1.

Appendix C. Third set of images

All the zero-order multipoles previously applied contribute to net charge on the conducting sphere. The third set of images is inserted to fulfill the zero-charge condition of the conducting sphere. To neutralize all the charges applied in Appendix B, we place a zero-order

Table 5

Image multipoles due to a unit multipole $B_0'' = 1$

(k)	Images inside the sphere				Images inside the dielectric solid			
	$h = R$	$R/2$	$R/3$	$R/4$	$h = -R$	$-R/2$	$-R/3$	$-R/4$
1	1				$-K$			
2	$-K/2$	$K/2$			$K^2/2$	$-K^2/2$		
3	$-K^2/12$	$-K^2/4$	$K^2/3$		$K^3/12$	$K^3/4$	$-K^3/3$	
4	$-K^3/24$	$-K^3/24$	$-K^3/6$	$K^3/4$	$K^4/24$	$K^4/24$	$K^4/6$	$-K^4/4$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots

multipole at the sphere center:

$$B_0'' = \sum_{m=2}^{\infty} B_{0m}'' = - \sum_{m=2}^{\infty} \sum_{j=1}^{\infty} B_{0m}^{(j)}, \quad (\text{C.1})$$

where $B_{0m}^{(j)}$ are the images listed in Table 4, and B_{0m}'' is the multipole that balances all $B_{0m}^{(j)}$. From $B_0^{(m)}$ in Table 3 and Eq. (C.1), we get

$$\begin{aligned} B_{0m}'' &= - \sum_{j=1}^{\infty} \frac{m}{m+j-1} K^{j-1} B_{0m} \\ &= \frac{B_1^{(1)}}{R} \frac{1}{K} \left[\frac{1}{m(m-1)} \right] \left[\ln \left(\frac{1}{1-K} \right) - \sum_{j=1}^{m-1} \frac{K^j}{j} \right]. \end{aligned} \quad (\text{C.2})$$

B_0'' can be evaluated from Eqs. (C.1) and (C.2):

$$\begin{aligned} B_0'' &= \sum_{m=2}^{\infty} \left\{ \frac{B_1^{(1)}}{R} \frac{1}{K} \left[\frac{1}{m(m-1)} \right] \left[\ln \left(\frac{1}{1-K} \right) - \sum_{j=1}^{m-1} \frac{K^j}{j} \right] \right\} \\ &= \frac{B_1^{(1)}}{R} \frac{1}{K} \left\{ \ln \left(\frac{1}{1-K} \right) - \sum_{m=2}^{\infty} \left[\frac{1}{m(m-1)} \sum_{j=1}^{m-1} \frac{K^j}{j} \right] \right\}. \end{aligned} \quad (\text{C.3})$$

Rearranging the summations of j and m in Eq. (C.3), we obtain

$$\begin{aligned} B_0'' &= \frac{B_1^{(1)}}{R} \frac{1}{K} \left\{ \ln \left(\frac{1}{1-K} \right) - \sum_{j=1}^{\infty} \left[\frac{K^j}{j} \sum_{m=j+1}^{\infty} \frac{1}{m(m-1)} \right] \right\} \\ &= \frac{B_1^{(1)}}{R} \frac{1}{K} \left\{ \ln \left(\frac{1}{1-K} \right) - \sum_{j=1}^{\infty} \frac{K^j}{j^2} \right\}. \end{aligned}$$

Then, B_0'' can be written in a simple form as

$$B_0'' = \frac{B_1^{(1)}}{R} \frac{1}{K} \left[\ln \left(\frac{1}{1-K} \right) - \beta_K \right], \quad (\text{C.4})$$

where β_K is a function of K , $\beta_K = \sum_{j=1}^{\infty} K^j/j^2$, as defined in Eq. (20).

As explained in Appendix B, B_0'' leads to another sequence of images because of the existence of the dielectric solid. However, for this set of images, we maintain zero net charge by adding an appropriate image

at the sphere center in every repetitive step. Table 5 summarizes the positions and magnitudes of images for an initial multipole $B_0'' = 1.0$ at the sphere center.

In Table 5, the magnitudes of the images inside the sphere can be expressed by the sequence S_k defined in Eq. (21). Consider the images at $h = R$ in the table. The magnitude of the multipole for each k is equal to S_k . Thus, the sum of the multipoles at $h = R$ for all k is denoted by α_K defined in Eq. (19). In general, for a height $h = R/i$ ($i = 1, 2, \dots$) in Table 5, it can be further concluded that the sum of the multipoles at this position is equal to $\alpha_K K^{i-1}/i$. Hence, the sum $(B_0)_i$ of all the images at a height R/i is expressed as

$$(B_0)_i = \frac{\alpha_K K^{i-2}}{i} \frac{B_1^{(1)}}{R} \left[\ln \left(\frac{1}{1-K} \right) - \beta_K \right]. \quad (\text{C.5})$$

We can write the electric field E'_{c3} at \mathbf{q} due to all the multipoles induced by a unit B_0'' in Table 5 as

$$\begin{aligned} E'_{c3} &= -(1+K) \sum_{l=1}^{\infty} \frac{\alpha_K K^{l-1}}{l} \left(\frac{l}{R} \right)^2 \\ &= -(1+K) \frac{\alpha_K}{R^2} \sum_{l=0}^{\infty} K^l (l+1) \\ &= -\frac{\alpha_K}{R^2} \frac{1+K}{(1-K)^2}. \end{aligned} \quad (\text{C.6})$$

Let E_{c3} denote the field at \mathbf{q} due to B_0'' . From Eqs. (C.2) and (C.6), we have

$$\begin{aligned} E_{c3} &= E'_{c3} \sum_{m=2}^{\infty} B_{0m}'' \\ &= -E_0 \frac{\alpha_K}{K} \frac{1+K}{(1-K)^2} \left[\ln \left(\frac{1}{1-K} \right) - \beta_K \right]. \end{aligned} \quad (\text{C.7})$$

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Analysis of the Electric Field and Force in an Arrangement of a Conducting Sphere and a Plane Electrode with a Dielectric Barrier

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ABSTRACT

This paper presents an analysis of the electric field and dielectrophoretic force in an arrangement consisting of an uncharged conducting sphere and a plane electrode with a dielectric barrier. The electric field is calculated by using the method of multipole images using an iterative algorithm proposed for calculating the images of the dielectric barrier of finite thickness. The calculation results show electric field intensification due to the presence of the dielectric barrier having higher permittivity, ϵ_s , than that of the surrounding medium, ϵ_e ; however, if the barrier is separated from the conducting sphere by at least the sphere radius, its influence is negligible. Inside the dielectric barrier, the electric field on the axis of symmetry becomes more uniform and the average field significantly increases with decreasing its thickness. For a case where dielectric barrier is sufficiently thin, the electric field at the contact point and the force on the conducting sphere vary approximately as power functions of ϵ_s / ϵ_e .

Index Terms — electric fields, force, Laplace equation, spheres, electrostatic analysis.

1 INTRODUCTION

ELECTRIC field and force on particles are important in many applications of dielectric materials. In gas insulation systems, for example, particles tend to be attracted to a stressed electrode by the force, resulting in field enhancement that may cause partial discharge or breakdown [1]. The behavior of the field and the force on a particle decisively depends on whether the particle is dielectric or conducting. For the dielectric particle, the electric field and force have been analyzed under various conditions for high-voltage insulation equipment or electrorheological fluid [2-6]. For the conducting particle, most of the works reported deal with configurations of two particles [7-10] or those of a particle and a plane electrode [11, 12]. Interaction between a conducting sphere and a solid dielectric of semi-infinite thickness [13, 14] or finite thickness [14, 15] has been already reported. However, the analysis in [14, 15] has considered only the

electric field in the cases where the thickness of the solid dielectric is larger than or equal to the radius of the sphere.

This paper focuses on an arrangement consisting of an uncharged conducting sphere and a plane electrode under an external field where there exists a dielectric barrier preventing the electrode from making direct contact with the sphere. This arrangement is a fundamental one for analyzing various applications. For example, such a dielectric barrier may be coated on an electrode in gas-insulated switchgear [16,17] or may be used in experiments on particle motion [18]. A dielectric barrier also appears as the medium between two conductors in printing systems. Another example is coated plate electrodes used in the application of an electrorheological fluid with metal particles [19]. The electric field and force are of interest here because the field is related to the occurrence of partial discharge, by which the sphere can acquire charges, and the force may attract the particle to the electrode, thus having an influence on the particle motion and

indirectly enhancing the field. By numerical field-calculation methods, an accurate analysis of the electric field and force in this arrangement is difficult because: (a) the field is highly non-uniform if the sphere is very close to the electrode, and (b) significant numerical errors may occur if the dielectric barrier is very thin. Although a thin dielectric layer may be approximated by a zero-thickness element in numerical field calculations, the representation is not valid unless its thickness is sufficiently small or its permittivity is great enough [20]. In this work, to attain results of high accuracy, we present an iterative algorithm for calculating the electric field in the arrangement by the method of multipole images, an analytical method. The algorithm has been applied to obtain the electric field and force for a wider range of the solid-dielectric thickness than that in [14, 15]. It is suggested here that the method of multipole images can be easily modified to deal with the arrangement of sphere-sphere or sphere-plane gaps used for the breakdown tests of solid dielectrics where the sphere is not exposed to an external field but stressed to a fixed voltage [21].

2 ARRANGEMENT OF CALCULATION

Figure 1 shows the physical arrangement considered in this paper. The arrangement consists of an uncharged conducting sphere with a radius r_p centered above a grounded plane electrode by a height h . A dielectric barrier of thickness t exists on the electrode and prevents direct contact between the sphere and the electrode. The conducting sphere is subjected to an external uniform field \mathbf{E}_0 in the vertical direction. The permittivities of the dielectric barrier and the surrounding medium are denoted by ϵ_S and ϵ_E , respectively. The representation by the plane electrode should be justified in most practical cases that the sphere is much smaller than the plane electrode involved. For generality, the sphere radius and the external field are taken as unit value. We vary the ratios t/r_p from 0.001 to 100 and ϵ_S/ϵ_E from 1 to 10 in the analysis to clarify their effects on the electric field and the force on the sphere. For arbitrary values of r_p , h , t , and \mathbf{E}_0 , the field and force can be readily obtained by scaling the normalized results presented in this paper.

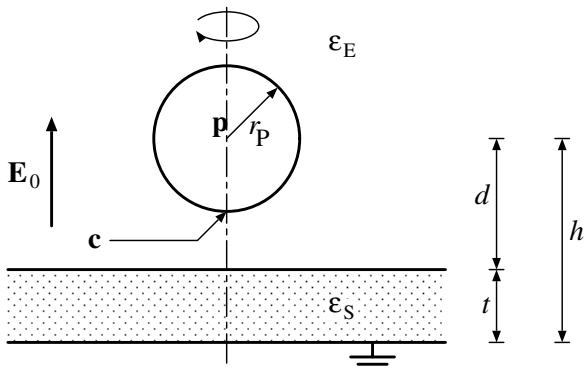


Figure 1. Calculation arrangement.

3 ANALYTICAL METHOD

The method of multipole images [5, 6] is used to calculate the electric field in the arrangement of Figure 1. The method is principally the classical method of images [22] using multipoles of higher orders than point or dipole charges, e.g. quadrupoles or octopoles. Since the potential due to multipoles naturally satisfies the Laplace equation, the analytical solutions are obtained by placing the multipoles to match the boundary conditions given. The potential, axisymmetrical in the arrangement, is expanded around the sphere center as the sum of spherical harmonics. The existence of the dielectric barrier on the electrode is taken into account by placing multipole images repetitively, as described in the following sections.

3.1 IMAGES FOR A MULTIPOLE AND A DIELECTRIC SOLID WITH SEMI-INFINITE THICKNESS

Consider an n th-order multipole B_n located at a point \mathbf{p} above a dielectric solid with semi-infinite thickness by a distance d , as shown in Figure 2. The potential, ϕ_B , due to this multipole is expressed by

$$\phi_B = \frac{B_n}{r_p^{n+1}} P_n(\cos \theta_p), \quad (1)$$

where (r_p, θ_p) are the spherical coordinates in which \mathbf{p} is the origin, and P_n is the Legendre function. Two multipole images of B_n are placed to satisfy the boundary conditions of potential and electric field on the dielectric interface. The images are B'_n at \mathbf{p}' in Figure 3(a) and B''_n at \mathbf{p} in Figure 3(b).

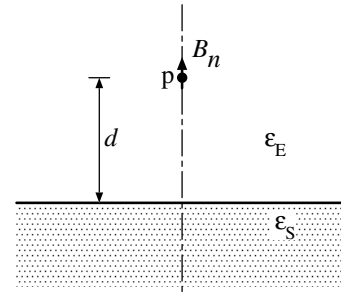


Figure 2. Multipole B_n and a dielectric solid of semi-infinite thickness.

By the method of images, we express separately the resultant potentials ϕ_S inside and ϕ_E outside the dielectric solid as follows.

$$\phi_S = \frac{B''_n}{r_p^{n+1}} P_n(\cos \theta_p) \quad (2)$$

$$\phi_E = \frac{B'_n}{r_{p'}^{n+1}} P_n(\cos \theta_{p'}) + \phi_B \quad (3)$$

where $(r_{p'}, \theta_{p'})$ are defined in the similar manner as (r_p, θ_p) . On the dielectric interface, the potentials ϕ_S and ϕ_E must satisfy the following boundary conditions:

$$\phi_S = \phi_E \quad (4)$$

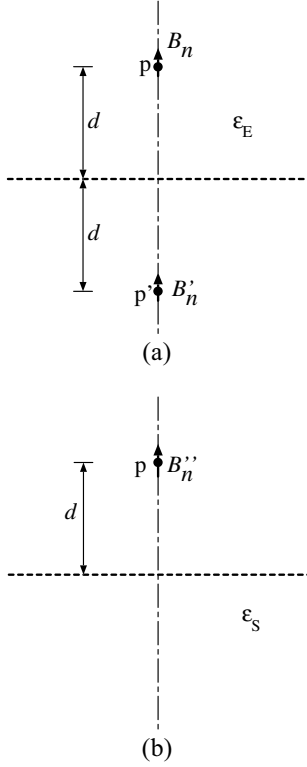


Figure 3. Multipole images for (a) the exterior and (b) the interior of the dielectric solid in Figure 2.

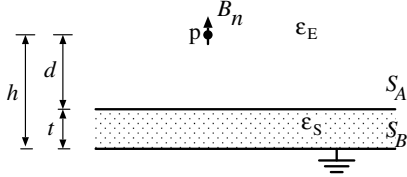


Figure 4. Multipole B_n and a dielectric barrier of finite-thickness.

$$\epsilon_S \frac{\partial \phi_S}{\partial n} = \epsilon_E \frac{\partial \phi_E}{\partial n} \quad (5)$$

where $\partial/\partial n$ is the partial derivative in the direction normal to the interface. From equations (4) and (5), it can be derived that B'_n and B''_n are related to B_n by

$$B'_n = (-1)^{n+1} K B_n \quad (6)$$

$$B''_n = (1 - K) B_n \quad (7)$$

where K is defined as a function of the permittivities,

$$K = \frac{\epsilon_S - \epsilon_E}{\epsilon_S + \epsilon_E} \quad (8)$$

Note that $0 \leq K < 1$ for $\epsilon_E \leq \epsilon_S < \infty$.

3.2 IMAGES FOR A MULTIPOLE AND A GROUNDED PLANE ELECTRODE WITH A DIELECTRIC BARRIER

Consider an n th-order multipole B_n at a point \mathbf{p} above a grounded plane electrode with a dielectric barrier, as shown in Figure 4. The multipole is separated from the electrode surface S_B by a distance h and from the dielectric interface S_A by a distance $d = h - t$, where t is the barrier thickness. In this

case, image multipoles must be iteratively inserted to satisfy the boundary conditions (4) and (5) on the interface S_A and the zero-potential condition on the surface S_B .

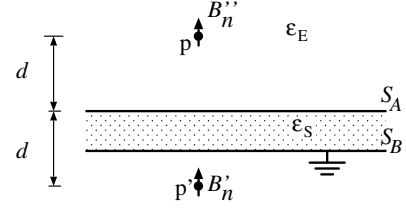


Figure 5. First set of images, B'_n and B''_n , induced by the dielectric barrier.

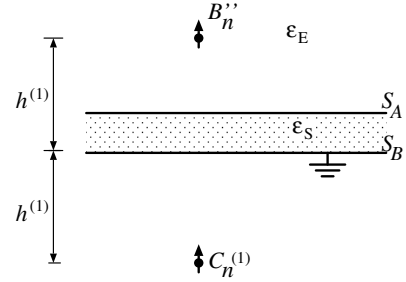


Figure 6. Image $C_n^{(1)}$ induced by the plane electrode.

The repetitive procedure is carried out in the following steps to fulfill the boundary conditions on S_A and S_B , where $d^{(i)}$ and $h^{(i)}$ denote the distances from S_A and S_B of an image multipole at the i -th repetition, respectively.

1. Place B'_n and B''_n as the images of B_n using equations (6) and (7) for the dielectric barrier (Figure 5) to satisfy the boundary conditions on the interface S_A . The potentials ϕ_E in the exterior and ϕ_S in the dielectric barrier are expressed by

$$\phi_E = \phi_B + \phi_{B'} \quad (9)$$

$$= \frac{B_n}{r_p^{n+1}} P_n(\cos \theta_p) + \frac{B'_n}{r_{p'}^{n+1}} P_n(\cos \theta_{p'}) \quad (10)$$

$$\phi_S = \phi_{B''} \quad (11)$$

$$= \frac{B''_n}{r_p^{n+1}} P_n(\cos \theta_p) \quad (12)$$

2. Place a multipole $C_n^{(1)}$ at a distance $h^{(1)} = h$ inside the plane electrode as the image of B''_n with respect to the electrode surface S_B (see Figure 6). $C_n^{(1)}$ is related to B''_n by [5]

$$C_n^{(1)} = (-1)^{n+1} B''_n \quad (13)$$

The potential within the dielectric barrier then becomes

$$\phi_S = \phi_{B''} + \phi_{C_n^{(1)}} \quad (14)$$

3. Place $C_n^{(1)}$ and $C_n^{(1)}$ as the images of $C_n^{(1)}$ induced by the surrounding medium to satisfy the boundary conditions on the dielectric interface S_A (Figure 7). Note that the constant K in this step is equal to $(\epsilon_E - \epsilon_S)/(\epsilon_E + \epsilon_S)$, different from that in Step 1, because $C_n^{(1)}$ contributes to the

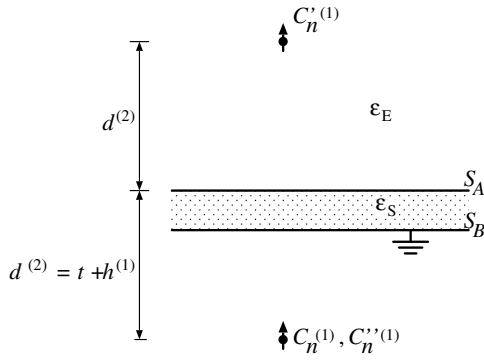


Figure 7. Image $C_n^{(1)}$ induced by the plane electrode.

potential in the dielectric barrier. The potentials are then rewritten as

$$\phi_E = \phi_B + \phi_{B'} + \phi_{C_n^{(1)}} \quad (15)$$

$$\phi_S = \phi_{B''} + \phi_{C_n^{(1)}} + \phi_{C_n^{(1)'}} \quad (16)$$

4. Because $C_n^{(1)}$ perturbs the potential in the dielectric barrier, we insert $C_n^{(2)}$ below the electrode surface S_B by $h^{(2)} = d^{(2)} + t$ as the image of $C_n^{(1)}$ with respect to S_B , where

$$C_n^{(2)} = (-1)^{n+1} C_n^{(1)} \quad (17)$$

5. Repeat Steps 3 and 4 until the potentials ϕ_E and ϕ_S converge.

The convergence of the solutions depends on the thickness t and the permittivity ratio ϵ_S/ϵ_E as the potentials converge faster for larger t and smaller ϵ_S/ϵ_E . Note that we may consider all the multipoles B'_n , B''_n , $C_n^{(i)}$, $C_n^{(i)'}$, and $C_n^{(i)''}$ as a set of the images of B_n induced by the plane electrode with the dielectric barrier.

3.3 CONDUCTING SPHERE AND A GROUNDED PLANE ELECTRODE WITH A DIELECTRIC BARRIER

The calculation procedure for the arrangement in Figure 1 is more complicated than that in the previous section because of the presence of the conducting sphere. The arrangement is treated as composed of two objects: (1) the sphere and (2) the plane electrode with the dielectric barrier. Two additional conditions on the conducting sphere to be fulfilled are that the potential must be constant and the net charge must be zero. The calculation begins with an initial multipole B (dipole) at the sphere center, \mathbf{p} , induced by the external field. B satisfies the boundary conditions on the former object (sphere surface) in the absence of the latter one. Since the potential ϕ_{ext} due to the external uniform field can be expressed as

$$\phi_{ext} = -E_0 h - E_0 r_p \cos \theta_p \quad (18)$$

we get from the boundary conditions on the sphere that [5]

$$B_1 = E_0 r_p^3 \quad (19)$$

Using the procedure described in the previous section, we obtain the set of the images of B that satisfy the conditions on

the surfaces of the dielectric barrier. The first few multipoles are expressed as follows:

$$B'_1 = K E_0 r_p^3 \quad (20)$$

$$B''_1 = (1-K) E_0 r_p^3 \quad (21)$$

$$C_1^{(1)} = (1-K) E_0 r_p^3 \quad (22)$$

$$C_1^{(1)'} = -(1-K) K E_0 r_p^3, \text{ and} \quad (23)$$

$$C_1^{(1)''} = (1-K)(1+K) E_0 r_p^3 \quad (24)$$

In the above expressions, $K = (\epsilon_S - \epsilon_E)/(\epsilon_S + \epsilon_E)$.

Then, the potentials due to all B' and C'' are re-expanded to \mathbf{p} , and the next set of multipoles B is inserted at \mathbf{p} to satisfy the boundary conditions on the sphere surface again. Details of the multipole re-expansion and the image for a conducting sphere are described in [5, 6]. Note that, starting from the second iteration, multipoles of higher degrees (B_2, B_3, \dots) appear as a result of the multipole re-expansion. The images are repetitively applied in this manner until ϕ_E and ϕ_S converge. Figure 8 shows the schematic description of the calculation process. For all the calculations carried out in this paper, we use the multipole order up to 400 and the numbers of repetitions lie between 5 and 110, depending on the dielectric-barrier thickness and the permittivity ratio. It is worth noting that, although we treat the uncharged conducting sphere in this paper, the same calculation process can also be used for a charged conducting sphere or a dielectric sphere. Only the determination of B_n from M_n is different, depending on the boundary conditions of the sphere.

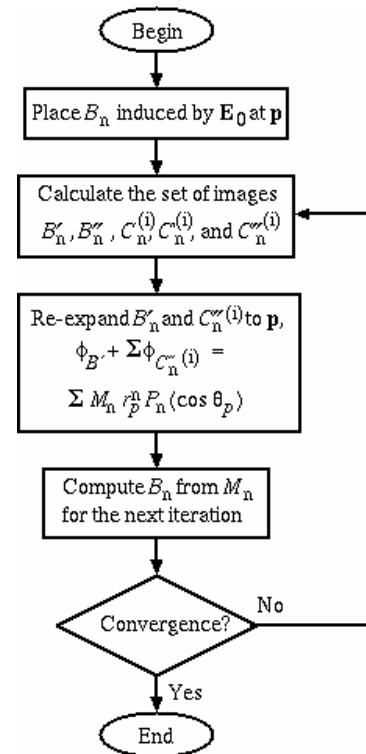


Figure 8. Schematic description of the repetitive process.

4 CALCULATION RESULTS AND DISCUSSIONS

4.1 ELECTRIC FIELD AND FORCE IN RELATION TO DISTANCE d

Figure 9 displays the potential distribution near the conducting sphere for $t/r_p = 1.0$, $d/r_p = 1.2$ and $\varepsilon_s/\varepsilon_E = 4$. The potential normalized by $E_0 r_p$ is represented in grey scale in the lower horizontal bar. From the equipotential lines, field intensification is clearly noticeable near the sphere surface. The field intensification is particularly high in the gap between the conducting sphere and the dielectric barrier. The maximal field, located at the point **c** in Figure 1, is equal to $3E_0$ in the absence of the grounded plane and the dielectric barrier. With the grounded plane but without the dielectric barrier (or $\varepsilon_s/\varepsilon_E = 1$), the maximal field becomes $3.14E_0$. The presence of the dielectric barrier having $\varepsilon_s/\varepsilon_E > 1$ further enhances the maximal electric field, which is equal to $4.44E_0$ in this case.

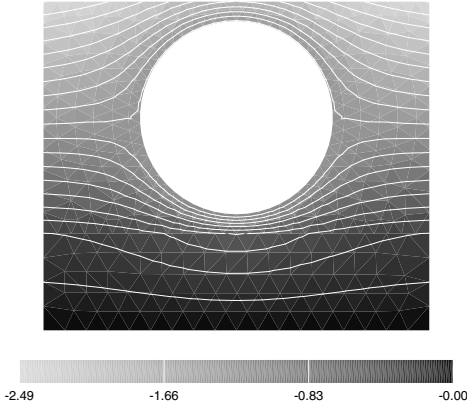


Figure 9. Potential distribution for $t/r_p = 1.0$, $d/r_p = 1.2$, and $\varepsilon_s/\varepsilon_E = 4$.

Figure 10 gives the maximal electric field relative to the normalized distance, $d/r_p - 1$, between the sphere surface and the dielectric barrier for $t/r_p = 1$ and 0.01 . It is clear from the figure that the influence of the dielectric barrier is negligible for $d/r_p \geq 2$. For smaller d/r_p , the maximal field increases with decreasing distance. Increase in the permittivity ratio also enhances the maximal field. If the distance is sufficiently small, the field reaches its saturation value depending on $\varepsilon_s/\varepsilon_E$. Note that the field varies with the distance even for $\varepsilon_s/\varepsilon_E = 1$ because of the effect of the plane electrode as explained already.

Because the arrangement is axisymmetrical, the force acting on the conducting sphere is in the vertical direction. The vertical force F_z is downward, attracting the sphere to the

electrode. Figure 11 shows F_z normalized by $F_0 = \varepsilon_E E_0^2 r_p^2 / 2$ as a function of the distance. The force behavior is obviously similar to that of the maximal field in Figure 10; however, the force magnitude cannot be estimated from E_{\max} because the electric field is highly non-uniform on the sphere. The force magnitude is zero where $d/r_p \rightarrow \infty$ because the influences of the dielectric barrier and the plane electrode diminish in this condition. With decreasing d/r_p , the force becomes stronger, and then approaches to the saturation value corresponding to $d/r_p = 1$.

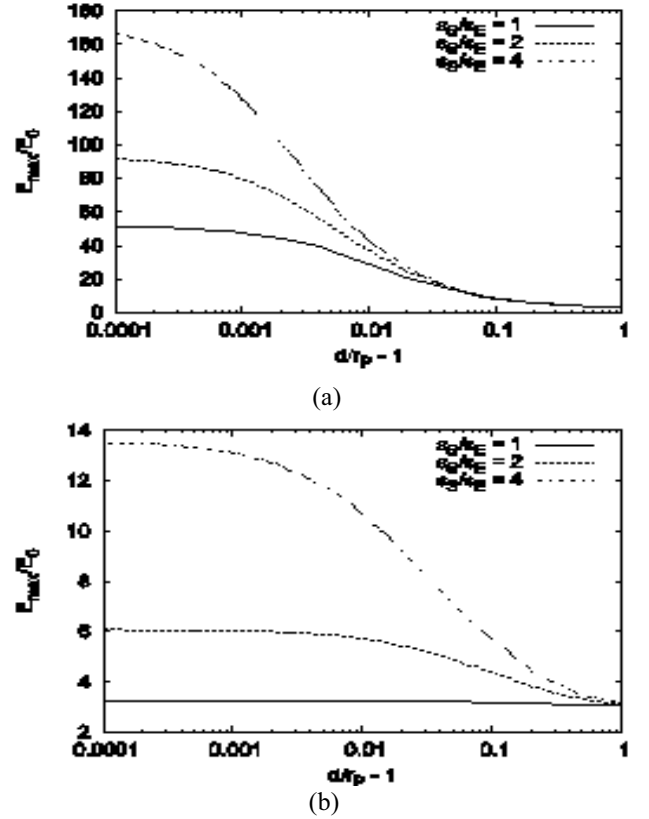


Figure 10. Maximal electric field in relation to the distance from the dielectric barrier for (a) $t/r_p = 1.0$ and (b) $t/r_p = 0.01$.

4.2 ELECTRIC FIELD AND FORCE WHEN THE SPHERE IS IN CONTACT WITH THE BARRIER

4.2.1 ELECTRIC FIELD DISTRIBUTION INSIDE THE DIELECTRIC BARRIER

Figure 12 shows the normalized potential distribution from the upper to the lower surface of the dielectric barrier in contact ($d/r_p = 1$) with the conducting sphere for $t/r_p = 1$, 0.1 , and 0.01 . Figures 12b and 12c show the enlarged $4t \times t$ view of the dielectric barrier for $d/r_p = 0.1$ and 0.01 , respectively, so that we can more clearly see the distribution. The contact point **c** is located at the middle of the upper boundary. In Figure 12, the potential is minimal at **c** and

maximal (zero) on the bottom boundary. We shall focus on the average electric field along the axis of symmetry, the vertical line passing \mathbf{c} , as it is the shortest path between the sphere and the grounded plane. From the minimal potentials shown in Figure 12, it is clear that the average field on this line increases with decreasing t .

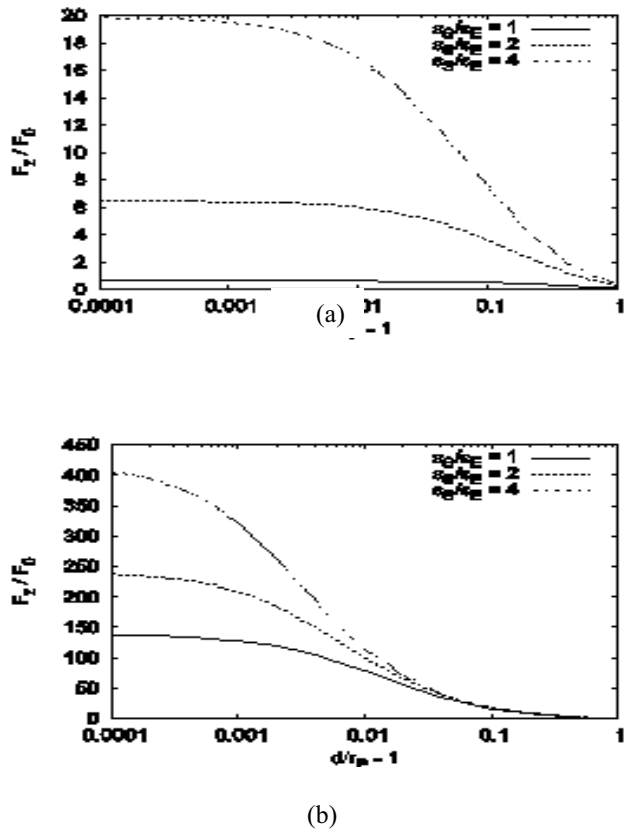


Figure 11. Attractive force acting on the conducting sphere in relation to the distance from the dielectric barrier for (a) $t/r_p = 1.0$ and (b) $t/r_p = 0.01$.

Figure 13 gives the average electric field, E_{av} , on the axis in the barrier as a function of t/r_p for various values of ϵ_S/ϵ_E . The figure shows that E_{av} is close to $\epsilon_E E_0 / \epsilon_S$ for a thick dielectric barrier. E_{av} increases with decreasing t/r_p . The increase of E_{av} is slow at large t/r_p , but becomes much faster at smaller values of t/r_p (less than about ten). For a thin dielectric barrier, the effect of the permittivity ratio becomes less significant as the values of E_{av} in Figure 13 for all ϵ_S/ϵ_E are very close to each other (although E_{av} is still lower for higher ϵ_S/ϵ_E). The average field becomes infinitely high as $t/r_p \rightarrow 0$, i.e. in the absence of the barrier.

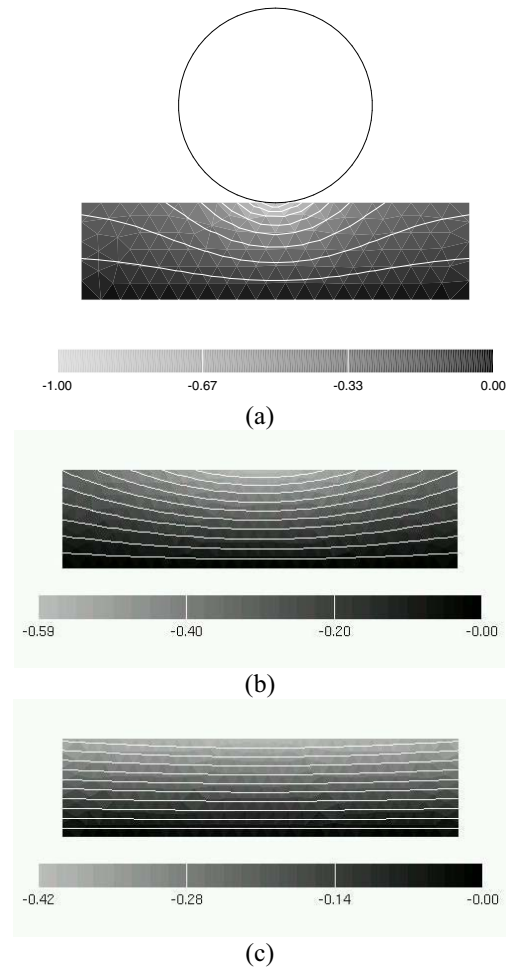


Figure 12. Potential distribution inside the dielectric barrier in contact with the conducting sphere for (a) $t/r_p = 1.0$, (b) $t/r_p = 0.1$, and (c) $t/r_p = 0.01$.

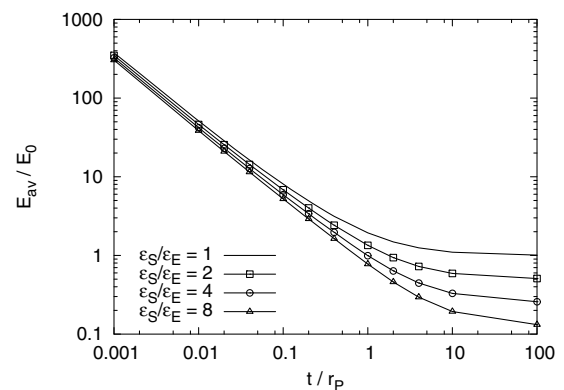


Figure 13. Average electric field on the axis of symmetry in the dielectric barrier.

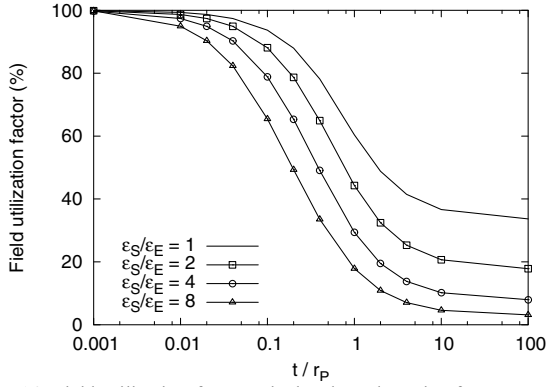


Figure 14. Field utilization factor calculated on the axis of symmetry in the dielectric barrier.

Another important parameter in the design of the dielectric barrier is the field utilization factor, η , defined as the ratio of the average field to the maximal field. Figure 14 gives the field utilization factor in relation to t/r_p for various values of $\varepsilon_s/\varepsilon_E$. The figure shows that, for a same value of t/r_p , η is smaller for a greater $\varepsilon_s/\varepsilon_E$ ratio. With decreasing t/r_p , η increases and converges to unity as $t/r_p \rightarrow 0$. For the range of $\varepsilon_s/\varepsilon_E$ considered here, η is higher than 90% if $t/r_p \leq 0.02$. This means that, within a sufficiently thin dielectric barrier, the field distribution along the axis of symmetry becomes approximately uniform.

4.2.2 ELECTRIC FIELD AND FORCE ON THE CONDUCTING SPHERE

This section discusses the electric field in the ε_E -side and the force on the conducting sphere in contact with the dielectric barrier. A triple junction or a contact point, the point where three media meet together, is formed at **c**. Figures 15a and 15b show the triple-junction field, E_c at **c** (in the ε_E -side), and the force F_z as functions of t/r_p . For sufficiently large t/r_p , the arrangement approximates to that of a conducting sphere in contact with a solid dielectric of semi-infinite thickness. The triple-junction field and the force for this case depend only on $\varepsilon_s/\varepsilon_E$, and can be simply approximated by [13]

$$\frac{E_c}{E_0} = 0.696 \left(\frac{\varepsilon_s}{\varepsilon_E} + 1.32 \right)^{1.73} \quad (25)$$

$$\frac{F_z}{F_0} = 1.8 \left(\frac{\varepsilon_s}{\varepsilon_E} - 1 \right) \left(\frac{\varepsilon_s}{\varepsilon_E} + 1 \right)^{0.31} \quad (26)$$

In Figure 15, the magnitudes of E_c and F_z are clearly enhanced by the effect of the plane electrode when t/r_p is smaller than unity. The figure also implies that, without the dielectric barrier ($t/r_p \rightarrow 0$), both E_c and F_z become infinity.

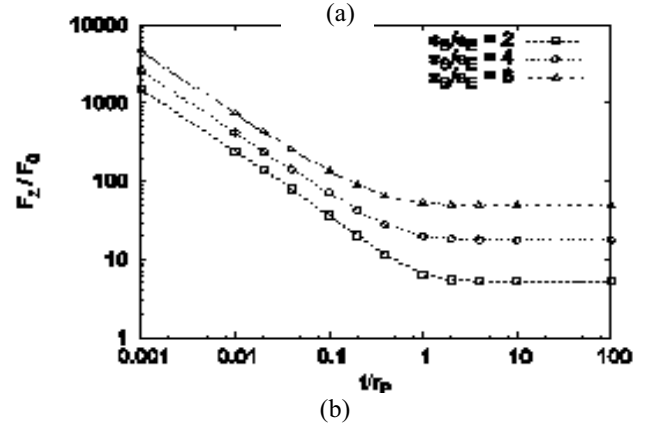
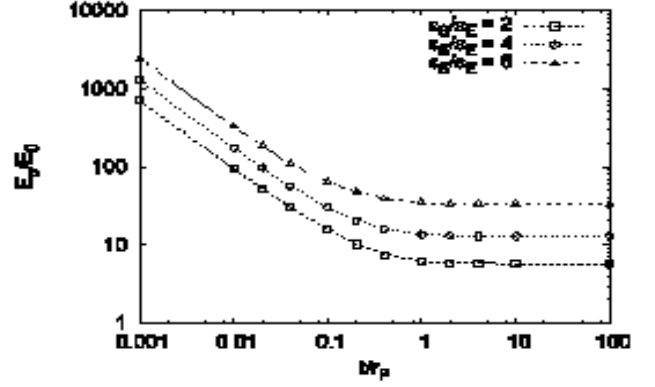


Figure 15. Contact-point electric field (a) and force (b) as functions of the thickness of the dielectric barrier.

Figures 16a and 16b show the contact-point field and the force, respectively, as functions of the permittivity ratio for $t/r_p = 0.1, 0.01$, and 0.001 . The field and the force for $t/r_p \leq 0.1$ vary with the permittivity ratio approximately by the following relations:

$$E_c = E_{c1} \left(\frac{\varepsilon_s}{\varepsilon_E} \right)^a \quad (27)$$

$$F_z = F_{z1} \left(\frac{\varepsilon_s}{\varepsilon_E} \right)^b \quad (28)$$

where E_{c1} and F_{z1} are, respectively, the field and the force where $\varepsilon_s/\varepsilon_E = 1$, a and b are appropriate constants. In the case where the conducting sphere in Figure 1 possesses a fixed potential ϕ_0 instead of the external field E_0 , it has been deduced that $a=1$ for a very thin dielectric barrier [14]. However, in our results of the uncharged conducting sphere, the values of a are between 0.88 and 0.95 for $0.001 \leq t/r_p \leq 0.1$. The values of b are between 0.80 and 0.97 for the same range of t/r_p . Because a differs from unity, it is implied that the electric field in the dielectric barrier (ε_s -side) at the contact point still exhibits small dependence on the permittivity ratio. The approximations by equations

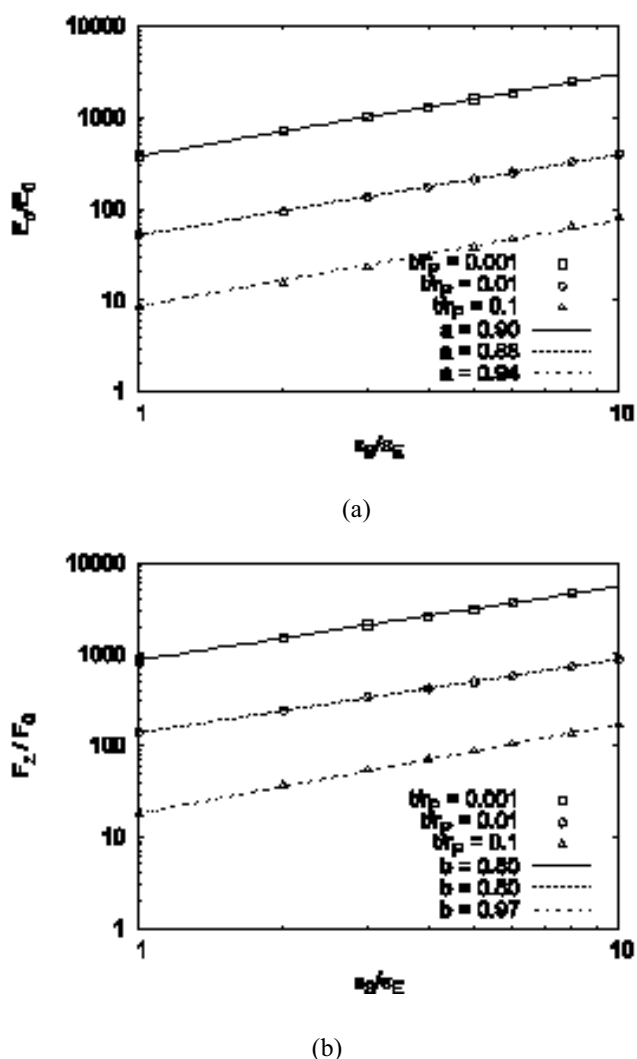


Figure 16. Contact-point electric field (a) and force (b) as functions of the permittivity ratio.

indicates the values of a and b computed by using the method of least squares.

5 CONCLUSIONS

The electric field and force have been analyzed by the method of multipole images for an uncharged conducting sphere on the grounded plane covered with a dielectric barrier under an external field. The results can be summarized as follows for the sphere radius r_p , barrier thickness t , and separation d .

1. The dielectric barrier mitigates the electric field and force when compared with those in the absence of the barrier ($t = 0$). However, the field and force are still enhanced if the permittivity of the barrier is greater than that of the surrounding medium.
2. If the separation $d > 2r_p$, the field will approximate to that on a single sphere without the grounded plane and barrier, and the force will be very small.

3. Thinner barriers cause higher contact field, which increases with the barrier permittivity.
4. For a sphere having $r_p < t$, the analysis can be simplified by treating the barrier as one with semi-infinite thickness.
5. The maximal field and force approximate to power functions of ϵ_s / ϵ_e if $r_p > 10t$.

ACKNOWLEDGMENT

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Dynamic simulation using a multipolar model of particles under dielectrophoretic force

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Abstract

This article presents the simulation of an electrorheological (ER) fluid system by using a multipole model that includes multipolar interactions between particles. The model uses the multipole re-expansion and the method of images for calculating electric field and force. The highest order of multipoles (N_{mp}) and the number of iterations (N_{iter}) used in the method of images can be chosen for the accuracy of the force approximation and the simulation time required. Study of a two-particle configuration shows that the force does not increase linearly with increasing N_{mp} and N_{iter} . The specific case $N_{mp} = 4$ and $N_{iter} = 2$ is chosen for dynamic simulation. We have performed the simulation of a system of 20 particles, and compared the formulation of particle chains with that obtained using the dipole model. The results imply that the response time for the change in viscosity of real-ER fluids is significantly shorter than that predicted by the dipole model.

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1. Introduction

Systems of dielectric particles are used in various applications. For example, an electrorheological (ER) fluid is an application that utilizes the dielectrophoretic (DEP) force on dielectric particles. An ER fluid is a suspension of fine, insulating particles in a liquid of low dielectric constant. In the presence of an applied electric field strong enough for aggregating the particles, the apparent viscosity of the fluid increases, and the fluid turns to a gel-like phase. For the typical ER fluid, this transition occurs on the order of milliseconds and is reversible. Practical devices designed to use ER fluids include electrically controlled clutches and shock absorbers.

Computer simulation has become an important tool for investigating the properties of ER fluid systems. By using simulations, the complicated effects of physical parameters that may be simultaneously present in the fluid can be

considered separately for better understanding. Prior studies on the dynamics of ER fluids often employ a dipole model for computing the interparticle force induced by an electric field [1–4]. However, the force predicted using the dipole model is significantly smaller than the true force, especially when the particles are close to each other. A more accurate model uses an empirical force function that includes the effects of higher-order multipolar terms between a pair of particles [5]. A method that considers a full set of multipolar interactions for a system of multiparticles has also been proposed in [6]. However, the method requires setting up and solving a system of linear equations and is thus not suitable for dynamic simulation if a large number of particles are involved.

This paper presents the application of a multipole model to approximate the interparticle force. The method of multipole images and the multipole re-expansion method [7] are used to calculate the electric field and the multipole images. The force is then computed from the multipole images by using analytical formulae. The main advantage over the existing dipole models is that multipolar interactions

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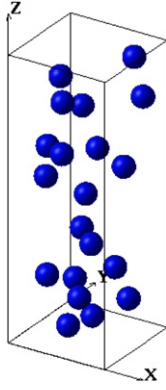


Fig. 1. Initial state of the randomly distributed particles.

between all particles are taken into account. In addition, the accuracy of the force approximation and the time required for calculation can be varied by choosing the highest order of the multipoles and the number of iterations used for the method of images. As an example, we have performed a dynamic simulation using the multipole model for a simplified system of an ER fluid, and we have compared the results to those obtained using the dipole model.

2. Physical configuration

The system of particles in this study is represented by a monodisperse suspension of dielectric particles in a host fluid confined between two $L_x \times L_y$ parallel plate electrodes at $z = 0$ and $z = L_z$. The particles have a dielectric constant $\epsilon_p = 8$, mass density $\rho = 1 \text{ g/cm}^3$, and diameter $\sigma = 10 \mu\text{m}$, and are treated as spheres. The dielectric constant of the fluid is $\epsilon_f = 2$. The viscosity η of the host fluid is 0.02 Pa s . These simulation parameters describe a real-ER fluid consisting of alumina particles in silicone oil [2,4,5]. A uniform external field $= 3 \text{ kV/mm}$ is applied to the fluid via the parallel electrodes. (This value was reported in [4] that it can solidify the ER fluid.) The electrode dimensions are $L_x = L_y = 5\sigma$, $L_z = 14\sigma$, and the number of particles between the electrodes is 20. Note that the dimensions and the particle number have been chosen in order that a single chain (not thick columns) could be formed inside the system when the applied field is sufficiently strong. The initial state of the ER fluid is shown in Fig. 1, where the particle positions are random.

3. Methods

3.1. Calculation of DEP force

3.1.1. Point-dipole force

DEP force is the force exerted by a nonuniform electric field on a polarized but uncharged particle. Consider an interaction force between particle a at \vec{r}_a and particle b at \vec{r}_b , as shown in Fig. 2. The dipole model gives the simplest

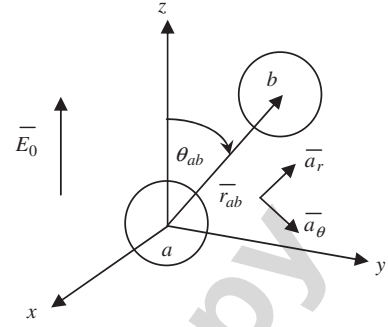


Fig. 2. Coordinate system of the spherical particle pair.

approximation of the DEP force between these two particles as [1]

$$\vec{F}_{ab}^{\text{dip}} = \left(\frac{3|\vec{p}|^2}{4\pi\epsilon_0\epsilon_f|\vec{r}_{ab}|^4} \right) \times [(1 - 3\cos^2\theta_{ab})\vec{a}_r - \sin(2\theta_{ab})\vec{a}_\theta], \quad (1)$$

where

$$\vec{p} = \frac{1}{2}\pi\epsilon_0\epsilon_f\sigma^3 \left(\frac{\epsilon_p - \epsilon_f}{\epsilon_p + 2\epsilon_f} \right) \vec{E}_0$$

is the induced dipole moment of each particle due to external field \vec{E}_0 , $\vec{r}_{ab} = \vec{r}_a - \vec{r}_b$, θ_{ab} the angle between \vec{r}_{ab} and the z -axis, $\vec{a}_r = \vec{r}_{ab}/|\vec{r}_{ab}|$, $\vec{a}_\theta = \vec{a}_r \times (\vec{a}_r \times \vec{a}_z)/|\vec{a}_r \times (\vec{a}_r \times \vec{a}_z)|$, and \vec{a}_z is the unit vector in the z -direction, as shown in Fig. 2.

3.1.2. Multipolar force

As already mentioned, the dipole model does not yield a good approximation to the DEP force when particles are close to each other. The reason is that the effects of higher-order multipoles are not negligible in this case. The DEP force, including the multipolar effects, on particle a can be determined by

$$\vec{F}_a^{\text{dep}} = \int_{r=\sigma/2} \epsilon_f q_{\text{total}} [-\nabla(\psi_a)_{\text{ext}}] dS_a, \quad (2)$$

where q_{total} , is the total effective charge density, which is the equivalent real-charge distribution that generates the same induced potential as the spherical dielectric particle [8], appearing on the surface of particle a ; S_a , the surface area of the particle, and; $(\psi_a)_{\text{ext}}$, is the electrostatic potential on particle a due to all sources exterior to particle a .

Note that Eq. (2) is expressed in general for any dielectric particle. In the case of two particles in close proximity, as in Fig. 2, $(\psi_a)_{\text{ext}}$ is the potential around particle a due to the externally applied field and the disturbance produced by particle b . To rewrite Eq. (2) in terms of multipoles, we take the center of particle a as the origin, and express the potentials $(\psi_a)_I$ inside and $(\psi_a)_E$ outside the particle as [7]

$$(\psi_a)_I = \sum_{j=0}^{\infty} \sum_{k=-j}^j L_{j,k} r^j \tilde{P}_{j,k}(\cos\theta) e^{ik\varphi}, \quad (3)$$

$$(\psi_a)_E = \sum_{j=0}^{\infty} \sum_{k=-j}^j \left[M_{j,k} r^j + \frac{B_{j,k}}{r^{j+1}} \right] \bar{P}_{j,|k|}(\cos \theta) e^{ik\varphi}, \quad (4)$$

where (r, θ, φ) are the spherical coordinates; $L_{j,k}$, $M_{j,k}$, and $B_{j,k}$ the coefficients to be determined, and $\bar{P}_{j,|k|}$ is the associated Legendre function normalized by $\sqrt{\frac{(j-|k|)!}{(j+|k|)!}}$.

The three potential coefficients are not independent of each other. Rather, $L_{j,k}$ and $B_{j,k}$ can be determined from $M_{j,k}$ so as to satisfy the boundary conditions of the potential and field on the particle surface [7]. From Eqs. (3) and (4), the total effective charge density and the exterior potential on the surface of particle a can be written, respectively as

$$q_{\text{total}} = \sum_{j=0}^{\infty} \sum_{k=-j}^j \left[\frac{\varepsilon_0(2j+1)}{(\sigma/2)^{j+2}} \right] B_{j,k} \bar{P}_{j,|k|}(\cos \theta) e^{ik\varphi} \quad (5)$$

and

$$(\psi_a)_{\text{ext}} = \sum_{j=0}^{\infty} \sum_{k=-j}^j M_{j,k} r^j \bar{P}_{j,|k|}(\cos \theta) e^{ik\varphi}. \quad (6)$$

Using Eqs. (2), (5) and (6), the Cartesian components of \bar{F}_a^{dep} can be determined from $B_{j,k}$ and $M_{j,k}$ as follows. (The superscript dep means “of DEP origin”)

$$\begin{aligned} \frac{F_{a,x}^{\text{dep}}}{2\pi\varepsilon_0\varepsilon_f} &= \sum_{j=0}^{\infty} 2\text{Re}[M_{j+1,1}](B_{j,0})\beta(j+1) \\ &\quad - \sum_{j=1}^{\infty} 2\text{Re}[B_{j,1}](M_{j+1,0})\beta(j) \\ &\quad + \sum_{j=1}^{\infty} 2\text{Re}[(B_{j,1})(M_{j+1,-2})]\beta(j+2) \\ &\quad + \sum_{j=2}^{\infty} \sum_{k=2}^j 2\text{Re}[B_{j,k}(M_{j+1,-(k+1)} - M_{j+1,-(k-1)})]\beta(j+k+1), \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{F_{a,y}^{\text{dep}}}{2\pi\varepsilon_0\varepsilon_f} &= - \sum_{j=0}^{\infty} 2\text{Im}[M_{j+1,1}](B_{j,0})\beta(j+1) \\ &\quad + \sum_{j=1}^{\infty} 2\text{Im}[B_{j,1}](M_{j+1,0})\beta(j) \\ &\quad + \sum_{j=1}^{\infty} 2\text{Im}[(B_{j,1})(M_{j+1,-2})]\beta(j+2) \\ &\quad + \sum_{j=2}^{\infty} \sum_{k=2}^j 2\text{Im}[B_{j,k}(M_{j+1,-(k+1)} + M_{j+1,-(k-1)})] \\ &\quad \times \beta(j+k+1), \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{F_{a,z}^{\text{dep}}}{4\pi\varepsilon_0\varepsilon_f} &= - \sum_{j=0}^{\infty} (B_{j,0})(M_{j+1,0})(j+1) \\ &\quad - \sum_{j=1}^{\infty} \sum_{k=1}^j 2\text{Re}[(B_{j,k})(M_{j+1,-k})] \sqrt{(j+1)^2 - k^2} \end{aligned} \quad (9)$$

where $\beta(j) = \sqrt{j(j+1)}$, and $\text{Re}[\cdot]$ and $\text{Im}[\cdot]$ designate, respectively, the real and imaginary parts of a complex number.

3.1.3. Calculation of the coefficients $B_{j,k}$ and $M_{j,k}$

Using the method of images, we calculate the coefficients $B_{j,k}$ and $M_{j,k}$ in an iterative manner similar to the schemes reported in [9,10]. The latter are proposed for systems of plate electrodes. For each particle in the system of N particles, the calculation procedure may be briefly described as follows. (The superscript (n) is used to indicate the number of the repetition step.)

1. Determine $M_{j,k}^{(0)}$ to present the initial potential ψ_{E_0} due to the plate electrodes. For our arrangement, if the particle is at \bar{c} , ψ_{E_0} at \bar{p} is written as

$$\psi_{E_0} = M_{0,0}^{(0)} + M_{1,0}^{(0)} r \bar{P}_{1,0}(\cos \theta), \quad (10)$$

where $M_{0,0}^{(0)}$ is the potential at \bar{c} due to \bar{E}_0 and $M_{1,0}^{(0)} = E_0$.

2. Compute $B_{j,k}^{(n)}$ from $M_{j,k}^{(n)}$ by Techaumant et al. [7] as

$$B_{j,k}^{(n)} = \left[\frac{(\varepsilon_f - \varepsilon_p)}{(\varepsilon_f + \varepsilon_p)j + \varepsilon_f} \right] \left(\frac{\sigma}{2} \right)^{2j+1} M_{j,k}^{(n)}. \quad (11)$$

3. Place the multipole images $\tilde{B}_{j,k}^{(n)}$ with respect to both plate electrodes to satisfy the potential conditions on them. Details of the image placement are explained in Appendix A.

4. Re-expand all of the images $\tilde{B}_{j,k}^{(n)}$ induced by the plate electrodes and $B_{j,k}^{(n)}$ of the other $N-1$ particles to \bar{c} . That is, the external potential, to which the current particle is subject in the next step, is

$$\psi_{\text{ext}} = \sum_{j=0}^{\infty} \sum_{k=-j}^j M_{j,k}^{(n+1)} r^j \bar{P}_{j,|k|}(\cos \theta) e^{ik\varphi}, \quad (12)$$

where $M_{j,k}^{(n+1)}$ is obtained from the re-expansions.

5. Repeat Steps 2–4. The potential can then be written in the final form as

$$\psi_E = \sum_{j=1}^{N_{\text{mp}}} \sum_{k=-j}^j \sum_{n=1}^{N_{\text{iter}}} \left[M_{j,k}^{(n+1)} r^j + \frac{B_{j,k}^{(n+1)}}{r^{j+1}} \right] \bar{P}_{j,|k|}(\cos \theta) e^{ik\varphi}, \quad (13)$$

where N_{mp} and N_{iter} are the highest order of multipoles and the largest number of iterations, respectively.

The accurate value of force can be obtained by using sufficiently high order N_{mp} for the multipoles (i.e., $j = 0, 1, \dots, N_{\text{mp}}$) and carrying out this procedure until the solutions converge. However, with higher N_{mp} the calculation time shall be very long, and not suitable for dynamic simulations. Here, we use the order N_{mp} of the multipoles up to four and perform the image placement (Steps 2–4) twice ($N_{\text{iter}} = 2$). The contributions of N_{mp} and N_{iter} to the calculation results will be presented later in Section 4.1. Because the interactions between all particles are included in the calculations, we expect the computed force to be similar to the accurate value calculated by using $N_{\text{mp}} = 100$.

3.2. Calculation of particle motion

Let particle a , having mass m , be at \bar{r}_a in the host fluid at time t . The motion of this particle is determined by

$$m \frac{d^2 \bar{r}_a}{dt^2} = \bar{F}_a^{\text{dep}} + \bar{F}_a^{\text{rep}} + \bar{F}_a^{\text{hyd}}, \quad (14)$$

where \bar{F}_a^{dep} is the DEP force as described in Section 3.1, \bar{F}_a^{rep} the repulsive force, and \bar{F}_a^{hyd} is the hydrodynamic force on the particle.

The force \bar{F}_a^{rep} is an artificial force applied in simulations of particles [1–4] to avoid overlapping with the other particles and with the walls. Details of \bar{F}_a^{rep} are explained in Appendix B. The hydrodynamic force is given for a spherical particle by Stoke's equation

$$\bar{F}_a^{\text{hyd}} = -3\pi\sigma\eta \frac{d\bar{r}_a}{dt}. \quad (15)$$

For typical ER fluids, we can neglect the acceleration term on the left-hand side of Eq. (14) as the viscosity is predominant. Substituting (15) into (14), we get

$$3\pi\sigma\eta \frac{d\bar{r}_a}{dt} = \bar{F}_a^{\text{dep}} + \bar{F}_a^{\text{rep}}. \quad (16)$$

Applying the following scaling factors

$$F_0 = \frac{1}{16}\epsilon_0\epsilon_f\alpha^2\sigma^2 E_0^2,$$

$$t_0 = 3\pi\eta\sigma^2/F_0,$$

$$r_0 = \sigma,$$

where $\alpha = (\epsilon_p - \epsilon_f)/(\epsilon_p + 2\epsilon_f)$, we rewrite Eq. (16) in the dimensionless form as

$$\frac{d\bar{r}_a^*}{dt^*} = \bar{F}_a^{\text{dep}*} + \bar{F}_a^{\text{rep}*}. \quad (17)$$

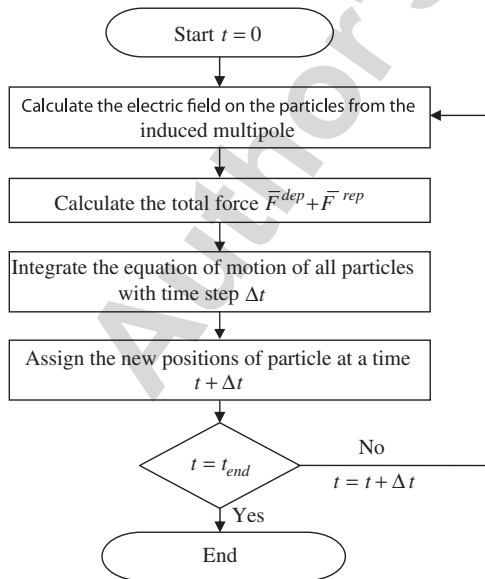


Fig. 3. Procedure for computing the particle position.

The symbol $*$ in Eq. (17) denotes dimensionless variables. Using Eq. (17), we compute the position of each particle at time t^* using the fourth-order Runge–Kutta algorithm. The procedure for computing the particle position is shown in Fig. 3. We follow the motion of the particles to monitor how pairs and chains of the particles are formed and determine the aggregation time of the first single chain that spans electrodes by using the dipole and the multipole models. This is because the formation of particle chain is directly related to the change in viscosity of ER fluids.

4. Results

4.1. Force on two touching particles

This section discusses the difference between the force predicted by the dipole model and that predicted by the multipole model. We first consider the case of two particles in contact under an external field \bar{E}_0 in the vertical direction (i.e., two plate electrodes are located so far from the particles), as in Fig. 4. The angle θ is measured between the z -axis and the line joining two-particle centers (lines a–b in Fig. 4.). The particles and the host fluid have dielectric constants $\epsilon_p = 8$, and $\epsilon_f = 2$, respectively. The case is considered here, as it is known that the DEP force becomes maximal when particles are in contact [11]. The force computed by the multipole model depends on two main parameters: the number of iterations N_{iter} and the maximum order of multipoles N_{mp} . To study the effects of N_{iter} and N_{mp} , we compute the multipolar forces on the particles in Fig. 4 for various N_{iter} and N_{mp} with $\sigma = 10 \mu\text{m}$ and $E_0 = 3 \text{ kV/mm}$.

The effect of N_{iter} is shown in Fig. 5. The horizontal force F_{horiz} and vertical force F_{vert} are calculated by using $N_{\text{mp}} = 4$ for $N_{\text{iter}} = 2, 3$, and 4. Compared to the dipolar forces and the accurate solutions (computed by $N_{\text{mp}} = 100$) in the figure the multipolar forces vary with θ similarly to them. F_{horiz} and F_{vert} change from the attractive to the repulsive forces as θ varies from 0° to 90° (the sign changes from positive to negative.). F_{horiz} and F_{vert} increase with increasing N_{iter} , while F_{horiz} reaches its peak at $\theta = 33^\circ$ for all value of N_{iter} and is repulsively maximum at $\theta = 90^\circ$.

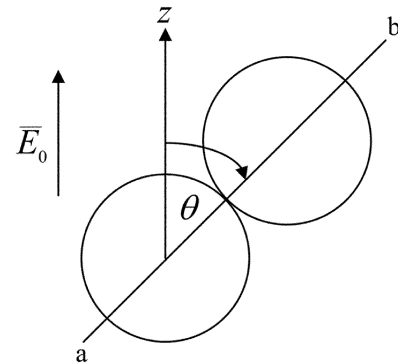


Fig. 4. Two touching particles.

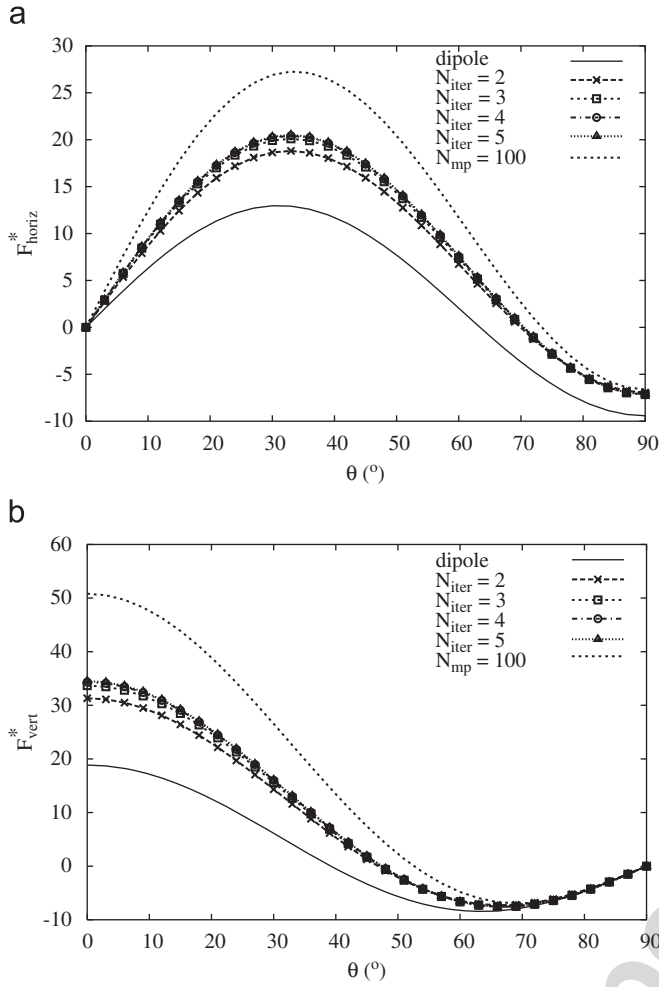


Fig. 5. The effect of N_{iter} on the forces on the lower particle in Fig. 4 for $N_{\text{mp}} = 4$: (a) F_{horiz} and (b) F_{vert} .

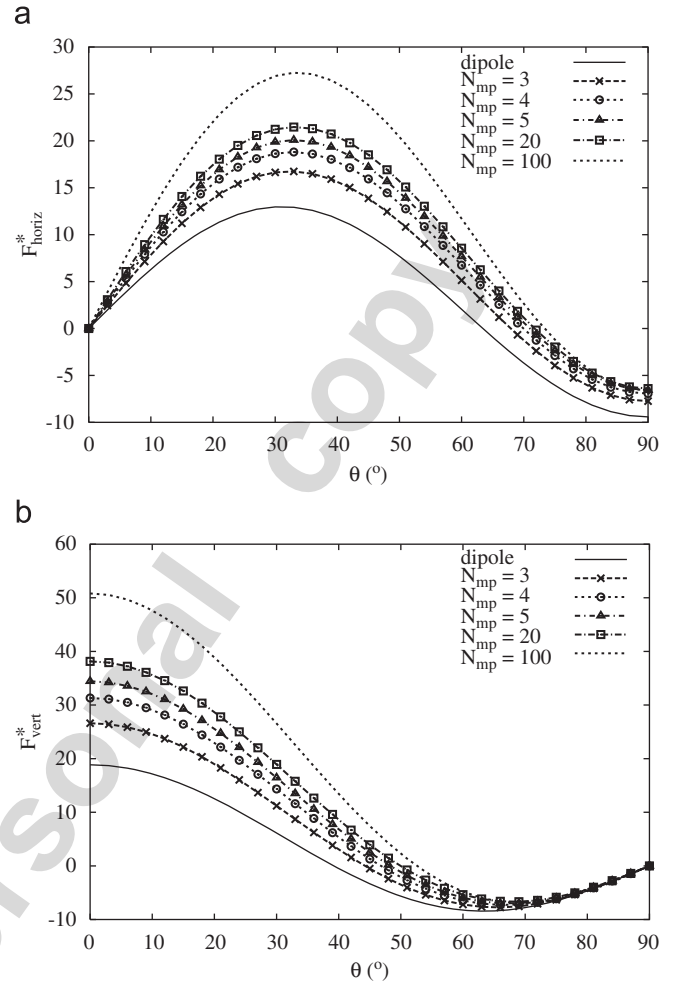


Fig. 6. The effect of N_{mp} on the forces on the lower particle in Fig. 5 for $N_{\text{iter}} = 2$: (a) F_{horiz} and (b) F_{vert} .

The force F_{vert} reaches a maximum attractive value when the particles are aligned in the direction of with \vec{E}_0 . The maximum magnitudes of F_{horiz} and F_{vert} for $N_{\text{iter}} = 2$ are 1.45 and 1.66 times the dipolar forces, respectively. For $N_{\text{iter}} = 3$ and 4, the force magnitudes are slightly higher than those for $N_{\text{iter}} = 2$. Although, the results are not shown here, we have found that, at higher N_{iter} , the multipolar force increases with N_{iter} by a smaller degree.

In Fig. 6, F_{horiz} and F_{vert} are calculated by using different values of N_{mp} ($N_{\text{iter}} = 2$, constant). From Fig. 6, we can clearly see the improvement of the force magnitude by using the multipolar model. For example, the maximum magnitudes of F_{horiz} calculated with $N_{\text{mp}} = 3, 4, 5$, and 20 are greater than that of the dipolar forces by 1.29, 1.45, 1.55, and 1.65 times, respectively. The increasing rate of F_{horiz} and F_{vert} with N_{mp} is smaller at higher value of N_{mp} .

4.2. ER fluid simulation

For the ER fluid system in Section 2, the time base, which is used for normalization in Eqs. (14) and (16), was $t_0 = 75$ ms. At $t = 0$, an external field is turned on and the

particles begin to move. We want to calculate the evolution of the particle trajectories with $\Delta t^* = 8 \times 10^{-5}$ for 50,000 time steps. First, we consider the calculation time required when various values of N_{iter} and N_{mp} are used. The calculation time per time step on a PC with 1.92 GHz CPU and GNU/Linux operating system is shown in Fig. 7. The figure shows that, by using $N_{\text{iter}} = 3$ and 4, the computation times are much longer than that by using $N_{\text{iter}} = 2$, but the multipolar forces do not considerably increase (as explained in the previous section). Because the simulations take about 15 h for $N_{\text{mp}} = 4$ but 45 h for $N_{\text{mp}} = 5$, we choose $N_{\text{iter}} = 2$ and $N_{\text{mp}} = 4$ for the calculations.

The evolution of the particle positions obtained from the dipole and multipole models are shown in Figs. 8 and 9, respectively. In Fig. 8(a), at $t = 30$ ms, the particles begin to form small chains but these are not arranged well in the direction parallel to the electric field. In Fig. 8(b), at $t = 180$ ms, the particles form longer chains and pairs, and only one particle is isolated. After $t = 180$ ms, the chains in Figs. 8(c) and (d) are only slightly different from those in Fig. 8(b).

For the particles simulated by the multipole model in Fig. 9, at $t = 30$ ms, the particles begin to form chains in

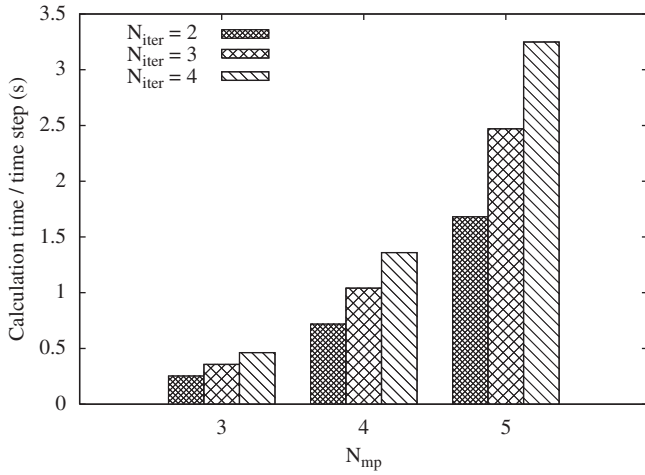
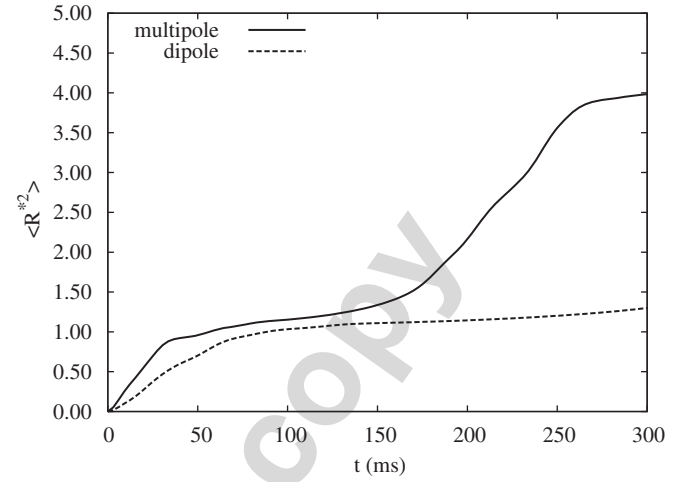
Fig. 7. Calculation time per time step for various values of N_{iter} and N_{mp} .

Fig. 10. Mean square displacement of particles simulated by the dipole and the multipole models.

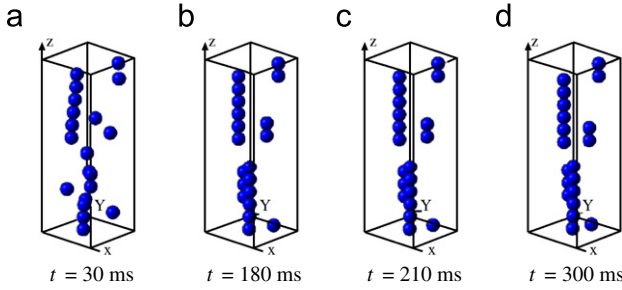


Fig. 8. The positions of particles at different times simulated by the dipole model.

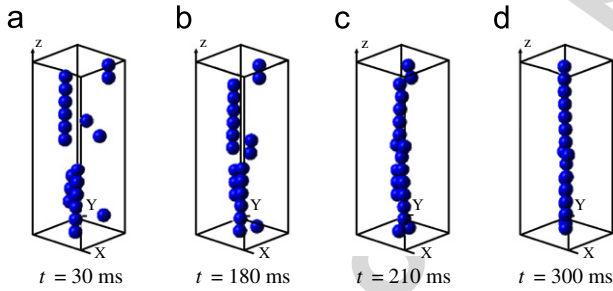


Fig. 9. The positions of particles at different times simulated by the multipole model.

the direction of the field. The chains are more parallel to the electric field than those by the dipole model in Fig. 8(a). In Fig. 9(b), the alignment of chains is improved and chains move close to each other as time increases. In Fig. 9(c), chains merge together, and a long chain spanning the electrodes is formed. Note that at this time ($t = 210$ ms) the particles simulated by the dipole model have just formed into arranged chains. In Fig. 9(d), the ordering of chains in the field direction is almost perfect. Thus, the aggregation time of the particles simulated by the multipole model is much shorter than when the dipole model is used.

In addition to the visual inspection of Figs. 8 and 9, we monitor the mean square displacement $\langle R^2 \rangle$ [1,12] to identify the particle aggregation. The definition of the square displacement of particle a at time t is

$$[R_a^*(t)]^2 = [x_a^*(t) - x_a^*(0)]^2 + [y_a^*(t) - y_a^*(0)]^2 + [z_a^*(t) - z_a^*(0)]^2, \quad (18)$$

where $x_a^*(t)$, $y_a^*(t)$, and $z_a^*(t)$ are the position of particle a at time t .

The mean square displacement is determined as

$$\langle R^2 \rangle = \frac{1}{N} \sum_{a=1}^N [R_a^*(t)]^2, \quad (19)$$

where N is the number of particles.

The value of $\langle R^2 \rangle$ is plotted against time in Fig. 10. The figure shows that $\langle R^2 \rangle$ computed via the multipole model has a rapid change at the initial state, because the particles begin to form chains and their positions change considerably. After the particles attach to each other, they move more gradually. At $t \approx 175$ ms, the chains merge together and rearrange, resulting in rapid change in $\langle R^2 \rangle$ again. After the chains merge and the long chain spans the electrodes, the transition of particles slows down at $t \approx 250$ ms. The value of $\langle R^2 \rangle$ in the dipole model gradually approaches saturation after $t \approx 150$ ms. As the particles do not span the electrodes, $\langle R^2 \rangle$ is still much smaller than that of the multipole. These results are consistent with the visual aggregation of particles in Figs. 8 and 9.

5. Conclusions

We have presented a multipole model for the simulation of particle systems, and performed the simulation of a system of 20 particles with parameters referred to a real-ER fluid. When the multipole model is used, the force approximation and the simulation time vary with the

number of iterations N_{iter} and the highest order of multipoles N_{mp} . Increasing N_{iter} and N_{mp} does not improve the multipolar force linearly; rather, the force increases only slightly at high values of N_{iter} and N_{mp} . The values $N_{\text{iter}} = 2$ and $N_{\text{mp}} = 4$ are chosen based on the force approximation and the simulation time. The simulation results obtained via the multipole model have been compared to those found using the dipole model. From the multipole model, the time by which a particle chain spans the electrodes is about 200 ms. Conversely, using the dipole model, there is no chain spanning the electrodes after 300 ms. These results imply that in a real-ER fluid, the response time for the change in viscosity is significantly shorter than that predicted by the dipole model.

Acknowledgments

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Appendix A

Placement of the multipole images

A particle is located between plate electrodes separated by a distance L_z in Fig. A1. Consider a multipole $B_{j,k}$ at the center of the particle at a distance h above the lower plate electrode. The multipole images $\tilde{B}_{j,k}$ with respect to the plate electrodes are placed in a sequential manner. The placement is divided into two sequences. The first one is that $\tilde{B}_{j,k}$ is placed with respect to the lower electrode prior to the upper electrode. The second one is that $\tilde{B}_{j,k}$ is placed with respect to the upper electrode prior to the lower electrode. The m th images ($m = 1, 2, 3$, etc.) of $\tilde{B}_{j,k}$ of the first and the second sequences are denoted by $[\tilde{B}_{j,k}]_{1(m)}$ and $[\tilde{B}_{j,k}]_{2(m)}$, respectively. Fig. A1 shows an example of

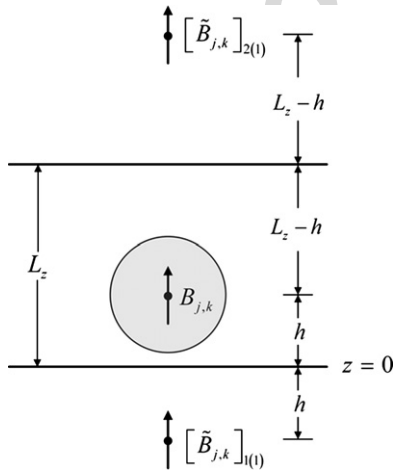


Fig. A1. Placement of the multipole images with respect to the parallel plate electrodes.

$[\tilde{B}_{j,k}]_{1(1)}$ and $[\tilde{B}_{j,k}]_{2(1)}$. The positions of $[\tilde{B}_{j,k}]_{1(m)}$ and $[\tilde{B}_{j,k}]_{2(m)}$ can be determined by

$$z_{1(m)} = (-1)^m \{[(m-1)L_z] + L_z[(m-1)\%2] + h\}, \quad (\text{A.1})$$

$$z_{2(m)} = (-1)^{m-1} \{[(m-1)L_z] + [(m\%2)L_z] + (L_z - h)\}, \quad (\text{A.2})$$

where $z_{1(m)}$ is the position of $[\tilde{B}_{j,k}]_{1(m)}$, $z_{2(m)}$ the position of $[\tilde{B}_{j,k}]_{2(m)}$, and $\%$ is the operator that provides the remainder after division. ($z = 0$ at the lower plate electrode).

In Fig. A1, $[\tilde{B}_{j,k}]_{1(1)}$ and $[\tilde{B}_{j,k}]_{2(1)}$ are located at $z_{1(1)} = -h$ and $z_{2(1)} = 2L_z - h$.

The magnitude of $\tilde{B}_{j,k}$ that fulfills the potential conditions on the plate electrodes is [7]

$$\tilde{B}_{j,k} = (-1)^{j+k+1} B_{j,k}. \quad (\text{A.3})$$

Appendix B

Repulsive force

To prevent the particles from overlapping each other and the electrodes, we use an artificial, short-range repulsive force between two particles (a and b) [2,4]

$$\bar{F}_{ab}^{\text{rep}} = 2F_{\text{max}} \exp \left[-100 \left(\frac{|\bar{r}_{ab}|}{\sigma} - 1.0 \right) \right] \bar{a}_r, \quad (\text{B.1})$$

where \bar{a}_r and \bar{r}_{ab} are shown in Fig. 2, and F_{max} is taken as the maximum DEP force determined in the case of two particles for the number of $N_{\text{iter}} = 2$ and $N_{\text{mp}} = 4$, as shown in Fig. 5. A short-range repulsive force between particle a and both electrodes is defined as

$$\bar{F}_a^{\text{wall}} = 2F_{\text{max}} \left\{ \exp \left[-100 \left(\frac{z_a}{\sigma} - 0.5 \right) \right] - \exp \left[-100 \left(\frac{(L_z - z_a)}{\sigma} - 0.5 \right) \right] \right\} \bar{a}_z. \quad (\text{B.2})$$

The first exponential term on the right-hand side of Eq. (B.2) is the repulsive force between particle a and the bottom electrode, and the second term is that between particle a and the top electrode. The total repulsive force \bar{F}_a^{rep} on particle a is the sum of the force in Eq. (B.2) and the forces in (B.1) for all the particles b , includes the image particles.

That is,

$$\bar{F}_a^{\text{rep}} = \sum_{b, b \neq a}^N \bar{F}_{ab}^{\text{rep}} + \bar{F}_a^{\text{wall}}, \quad (\text{B.3})$$

where N is the number of particles.

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High-yield electrofusion of biological cells based on field tailoring by microfabricated structures

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High-yield electrofusion of biological cells based on field tailoring by microfabricated structures

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Abstract

This paper presents the use of electric-field constriction created by a microfabricated structure to realize high-yield electrofusion of biological cells. The method uses an orifice on an electrically insulating wall (orifice plate), whose diameter is as small as that of the cells. Due to the field constriction created by the orifice, we can induce controlled magnitude of membrane voltage selectively around the contact point, regardless of the cell size. The field constriction also ensures 1:1 fusion even when more than two cells are forming a chain at the orifice. A device for electrofusion has been made with a standard SU-8 lithography and PDMS molding, and real-time observation of the electrofusion process is made. Experiments using plant protoplasts or mammalian cells show that the process is highly reproducible, and the yield higher than 90% is achieved.

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I. Introduction

Electrofusion is a process for creating hybrid among different types of cells with the use of electrical pulses[1]. The process is based on the reversible breakdown of the membrane. When a cell is placed under the influence of a pulsed external field, electrical charging of the membrane occurs and the potential difference between inside and outside of the membrane (hereafter called membrane voltage V_m) develops. When V_m reaches the breakdown voltage (denoted as V_b), which is about 1 V for most biological cells [1-3], the electrical breakdown of the membrane takes place. If the pulse is moderate and $V_m \approx V_b$, the breakdown is reversible; the membrane is locally and temporarily destroyed, but the lipid bilayer structure is restored spontaneously due to the fluidity of the membrane. Such a reversible breakdown at the contact point of two cells lead to the reconnection of the membrane and the cytoplasm are merged, which is the principle of the electrofusion.

Obviously, nothing happens if V_m is too low, and the cells are ruptured permanently if V_m is too high, and successful fusion is expected in a narrow range of V_m , say $\pm 15\%$ of V_b [2]. Therefore, a precise control over the magnitude of the induced membrane voltage is an essential key for a high-yield fusion.

In conventional electrofusion, a mixture of cell suspensions is placed under the influence of ac field first, and the cells are dielectrophoretically (DEP) aligned to form chains, and then a high-intensity electric pulse is applied to fuse the cells. However, the process is essentially low yield due to following reasons:

- 1) An analytical solution for the membrane voltage, when a stepwise uniform external field E_0 is applied, is given by [1]

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$$V_m(t) = \frac{3}{2} a E_0 \cos \theta [1 - \exp(-t/\tau_c)] \quad (1)$$

where θ is the zenith angle (as in Fig. 2), and τ_c is the time constant of membrane charging which is

$$\tau_c = a C_m (\rho_{in} + \rho_{out} / 2) \quad (2)$$

Its steady-state value V_m , i.e. after time t adequately longer than τ_c and the membrane charging is completed, becomes

$$V_m(\theta) = \frac{3}{2} a E_0 \cos \theta \quad (3)$$

This equation shows that V_m is proportional to the cell radius a , so that for a cell suspension with size distribution, successful fusion should occur only for the cells with an appropriate size.

- 2) For non-spherical cells, V_m depends on their orientation.
- 3) A prerequisite for the fusion is that more than two cells are in contact. Our analysis for two contacting cells [4-5] shows that V_m is more than 20% larger at the top or bottom of the cell chains ($\theta = 180^\circ$ in Fig. 2) than that the contact point ($\theta = 0^\circ$). It is likely that the membrane at the end of the chain ruptures before fusion occurs.
- 4) Whereas biological hybrids results in high-yield only when fusion between two cells is predominant [1], there is no way to guarantee 1:1 fusion by the conventional procedure based on DEP chain formation in cell suspension.

In the past, a group including some of the authors achieved electrofusion of yeast cells in a fluid integrated circuit, in which field nonuniformity is utilized to position two cells to be in contact to each other before applying fusing electric pulse [6-7]. In the current work, we solve the aforementioned problems by tailoring the field pattern, so that the reversible breakdown is restricted to occur exclusively at the contact point, thus improving the

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fusion yield. The principle, the device and the experimental results are presented in this paper.

II. Principle

Our group has already reported on the electroporation using a field constriction at a microorifice [8-9]. Unlike other on-chip electroporation methods [10-11], our method uses radio-frequency modulated pulse [9] with the voltage as low as 1.5V to restrict pore formation to take place exclusively at the orifice, and hence low-invasive and high-yield electroporation is achieved.

The electrofusion method presented in this paper is based on a similar principle, as depicted in Fig. 1. The device consists of an insulator wall, having an orifice which is comparable to the cells in size, or an array of such orifices, and the electrodes on both sides. When a voltage is applied between the electrodes, the field lines, being unable to penetrate through the insulator wall, converge into the orifice opening to create field constriction there.

First, cells are fed, say of type A on one side and type B on the other side, and the electrodes are energized with a.c. voltage. Due to the field non-uniformity created by the constriction, cells are dielectrophoretically moved towards the orifice, forming a cell pair A-B. This process up to here is identical to what is presented in refs. [6-7], but the important difference appears when the pulse voltage is applied. Because the orifice diameter in this work is comparable to that of the cells; most voltage drop occurs around the orifice, and the membrane breakdown takes place nowhere but in the orifice.

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This situation can be looked into in more detail with an aid of a field analysis using a model shown in Fig.2. Two cells of radii a and a' are in contact at the orifice of radius b on an insulating barrier located at the middle of two electrodes separated by the distance $2h$. Because our purpose here is to see the effect of field constriction, we assume an axial symmetry for simplicity, that the cells are aligned along the center line, despite that this position is unstable and the cell pair will be dielectrophoretically attached to the barrier as experimentally observed (to be later shown in Fig. 7). We employ a boundary element method formulation [4], and calculate the membrane voltage V_m as the function of position expressed by the zenith angle θ (see fig.2), taking the cell radius as a parameter. For the numerical analysis, a fictitious boundary of radius d is assumed and all electrical flux is confined within the boundary, but it is chose far enough that it does not affect the result. As in [4], we solved the temporal evolution of V_m for a stepwise applied voltage, but in the following part of the paper, we shall confine our interest to its steady-state value, which is of importance for practical applications.

Figure 3 plots the calculated steady-state voltage V_m normalized by the applied voltage V_0 for a case of two equal-sized cells ($a = a'$). It is seen from the figure that, when the cells are large enough that the orifice is almost plugged (line a), V_m takes maximum on the membrane within the orifice ($\theta < 12^\circ$), where $V_m/V_0 \approx 1/2$, and V_m is very small elsewhere. $V_m/V_0 \approx 1/2$ means that the applied voltage is borne by the two membranes contacting in the orifice. This is the effect of field constriction, where the voltage drop is concentrated in the vicinity of the orifice, and the drop in the medium is negligible. In this case, the reversible membrane breakdown can be expected to take place only in the orifice by applying a voltage $2V_b$ to the electrodes, regardless of the electrode spacing h . This effect

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remains so far as the cells are large enough compared with the orifice, so that the induction of precisely controlled membrane voltage becomes possible irrespective of the cell size or electrode position.

As the orifice becomes wider (lines b, c and d in Fig. 3), the peak of V_m broadens, and V_m/V_0 there becomes smaller. The electrical current can flow through the gap between the cells and the insulating barrier in this case, which creates a voltage drop in the medium. This results in the reduction of the voltage applied to the membrane. Voltage higher than $2V_b$ is required for the reversible breakdown, but still in this case V_m is higher at $\theta = 0^\circ$ than $\theta = 180^\circ$, so the breakdown occurs selectively around the contact point. The magnitude of V_m at $\theta = 0^\circ$ and 180° becomes the same when the gap is increased to about $1.6a$.

The figure also shows the case without orifice by circular dots. V_m is higher at $\theta = 180^\circ$ than $\theta = 0^\circ$ by 28%. Therefore, upper and lower ends ($\theta = 180^\circ$) of the cell pair are likely to breakdown first, which we believe is one of the reasons why conventional cell-suspension based electrofusion is low yield. Note that the membrane potential in this case depends on the distance h between the electrodes as indicated by equation (3); however, the distribution of V_m on the cells is independent of h , provided it is wide enough.

When cells are dielectrophoretically pulled and made into contact in the orifice as in Fig. 2, they deform slightly. The contact is locally flat, and when the breakdown occurs at a certain point on one of the membranes, the voltage drop borne by the membrane is added to the other membrane. It is like breakdown propagation of two series capacitors, and such a process will lead to the membrane reconnection and the cell fusion.

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Figure 4 presents the case of two different sized cells ($a' = a/2$). It is clear that V_m is larger for the larger cell. The maximum V_m , which does not conform to equation (3), reflects the different degrees of leak between the cells and the orifice. However, the tendency that V_m is largest at the contact point is still there. If the membrane of the larger cell breaks down, it will cause domino-like breakdown of the opposing membrane of the smaller cell, initiating the fusion.

A particular case of interest is when more than one cell is attached to the orifice. The effect of the chain length on the membrane voltage is illustrated by using an arrangement of a $2N$ -cell chain having N (equal-sized) cells located above and under the orifice, respectively. The membrane potential on the cells above the orifice is presented for $N = 1, 2$ and 3 in Fig. 5A, in which the first cell is the one closest to the orifice. It is obvious from the figure that, in the steady-state, 1) maximum membrane voltage V_m occurs in the orifice (small θ on the first cell) and V_m is much smaller on other locations or on other cells, and 2) the maximum V_m virtually unaffected by the number of cells on the chain. Hence, appropriate magnitude of electric pulse can be readily selected to generate one-to-one fusion between the two cells at the orifice.

Discontinuities, for example between that at $\theta=180^\circ$ of the first cell and that at $\theta=0^\circ$ of the second cell, are seen. Its physical interpretation is as follows: because in steady-state where the membrane charging is completed, no current flows across the membrane, and the field is zero inside the cell. The potential drop per cell along the symmetry axis is borne by the membrane at $\theta=0^\circ$ and $\theta=180^\circ$, and the ratio between these two is determined by the field non-uniformity. On the first cell, both the field itself and the field

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non-uniformity is large, so large V_m at $\theta=0^\circ$ results. On the other hand, on the second or third cell, which is located far from the orifice, field is weaker and more uniform, so that V_m at $\theta=0^\circ$ and $\theta=180^\circ$ are small and more or less equal in magnitude.

For comparison, Fig. 5B gives the membrane voltage on the same cells in the absence of the orifice plate. V_m maxima is located at the chain end, and changes only slightly with N . The membrane potential at all the contact points is more or less the same. This would be the reason why the typical procedure results in many-cell fusion.

III. Device

For the cell fusion based on the field constriction, we need an orifice with the diameter of several μm , which is within the capability of microfabrication technique based on photolithography. However, to make the device low cost and make it disposable for each biological experiment, complex three-dimensional structure is not desirable. Another design consideration is to facilitate the observation of the fusion process under a short working distance of high-magnification optical microscopes.

We have the options among a) making an orifice array in a thin film as is done in our electroporation device, and observe the cell pair from the top, or b) making the orifice on the side wall, and view the cell pair from the side. In the former design, circular and smaller orifice can be made easily, but did not provide a good view for observation. Better observation can be made with the latter design, however, making a circular orifice on the side-wall in the microchannel requires complex 3D fabrication. In this paper, we made a compromise by employing the latter design, and use a rectangular orifice, which can be

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made with a single-mask process. This does not precisely represent the model used in the numerical analysis, but still adequately create the field constriction effect.

The device is shown schematically in Fig. 6. Two channels run horizontally as in Fig. 6A, which are separated by a wall, that has periodical openings (orifices). The device is fabricated by single-mask lithography using SU-8 and PDMS (poly-dimethyl siloxane) molding, so the structure is two dimensional, and the orifices' cross-section is rectangular. The cover lid has a pair of vacuum-evaporated Al electrodes for energization.

The dimensions of the device are varied for the type of cells to be used for experiments. Typical values are listed in Table 1. It is worth noting here, owing to the design that gives a high degree of field constriction in membrane charging, we can use wider electrode gap than that in the past [6-7] and still achieve cell fusion by smaller applied voltage.

IV. Material and Method

In the following, experimental demonstrations of fusion among the same type of cell are made. Two type of cells, plant and mammalian, are used.

Plant protoplasts are obtained from the stems of radish sprouts. A solution of 1.5% cellulase R10, 0.75% macerozyme, 0.6 M mannitol, and 10 mM MES (pH 5.7) is used for digestion and isolation. After isolation, cells are washed twice with a solution of 0.6 M mannitol, and the conductivity of cell suspension is controlled to be about 7 mS/m or lower.

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As a mammalian cell, Jurkat cells (human T cell lymphoblast-like cell line) are used. The cells are cultured in RPMI 1640 medium supplemented with 10% FBS under 37°C, 5% CO₂, and they were suspended in 0.25M glucose solution for fusion. The conductivity of the cell suspension is 6 mS/m.

The cells are fed into the device, the electrodes are energized to collect cells into opening by dielectrophoresis, and radio-frequency (r.f.) modulated pulse voltage is applied for fusion. Typical electrical parameters are listed in Table II. With the solution conductivity and the cell size used in experiment, the charging time constant of a single cell under a uniform field (τ_c in eq.(2)) is about 20 and 6 μ s for the protoplast and lymphoblast, respectively [3]. (1- μ F/cm² membrane capacitance and 1-S/m inner conductivity are assumed.) In order to generate positive DEP to attract cells into the orifice, the time constant of the applied frequency should be shorter than these charging time constant τ_c , while in order to reach the steady-state V_m they should be longer than τ_c . Thus, frequencies above 1 MHz are used for DEP, while those in the range of kHz are applied to initiate fusion. The reason why the modulated pulse is used is to prevent secondary breakdown of the membrane [9], and is found to be very effective in preventing the breakdown at other locations than at the contact point.

V. Results and discussion

Fig. 7 shows a temporal sequence of cell fusion of the plant protoplasts. First two cells, one about twice larger than the other, are dielectrophoretically made in contact at the orifice in Fig. 7A. Then, r.f. modulated pulse is applied, by which the cell fusion is initiated, as shown in Fig. 7B. The voltage for DEP is still there, so the fused cell is

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elongated due to the electrical stress. In fact, such an elongation has been used to confirm that the membrane is reconnected in our experiments. By removing the voltage, the fused cell is released from the orifice, and becomes spherical due to surface tension [Fig. 7C].

As is noted previously, if the orifice is ideally plugged by the cells, only $2 V_b \approx 2 \text{ V}$ shall be sufficient for initiating fusion. However, the orifice of the current device is rectangular with $60 \mu\text{m}$ -depth, and rather leaky. Because of this, we needed 6 V to induce the fusion in the figure. Still, the field constriction is found to be effective in selectively breaking down the contact point. This result confirms our prediction from fig. 3B that the orifice can be used to initiate fusion between two cells of unequal size. For simple comparison, we may consider the case of a single cell of $50\text{-}\mu\text{m}$ diameter, more or less the size of the larger cell in Fig. 7. In the absence of the orifice, to initiate the membrane breakdown in the electrode gap specified in Table 1, a potential difference of 11 V must be applied to the electrodes. (Needless to say that, without the orifice, this voltage still fails to induce cell fusion between the two cells in Fig. 7 because of higher membrane voltage at the opposite pole than at the contact-point pole.)

In Fig. 8 is shown the typical case where more than two cells are forming a chain. The application of the electric field for DEP usually results in chains much longer than two cells. This is because a cell chain attracts other cells located nearby to its terminals and the attractive DEP force is stronger when the chain becomes longer. In this case, the chain is composed of seven cells and bridges the electrode gap. Without the orifice, any of the contact points can breakdown, and the breakdown is often not limited to one as already explained in Section 2. However, with the highly concentrated field profile due to the orifice, we can restrict the breakdown only to the contact in the orifice, and realize

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one-to-one cell fusion. The applied voltage is 6.5V in this case. Compared to that of Fig. 7, it is clear that the voltage is independent of the number of cells on the chain. This also conforms to the analysis result in Fig. 5.

Figure 9 is the case of smaller cells. Size of the cells is about a half of those in Fig. 7, so, if the orifice is absent, the required voltage will be twice higher. However, due to the field constriction, such size-dependence is alleviated, and the fusion of Fig. 7 is achieved with a slightly higher voltage (7 V) than that in Fig. 7 (6 V). This radius dependence is not necessarily in quantitative agreement with the result of Fig. 3. This is rather due to the limitation of the analysis assuming axial symmetry. The cells in Fig. 7 are dielectrophoretically positioned at the tip of tapered insulating barrier, where higher non-field uniformity exists. Still, the result demonstrates the advantage of the use of the field constriction for a) having less size-dependence, and b) selective breakdown at the contact point.

Figure 10 shows a temporal sequence of the fusion between human Jurkat cells. The orifice is rectangular again, but is smaller than in the previous experiments, 8 μm (width) x 15 μm (depth). Since the DEP force acting on a cell is typically proportional to the cubic of cell size [12] and the Jurkat cell is about 3 times smaller than the protoplast, comparatively stronger electric field is required to attract the cells to the orifice in Fig. 10(1) and (2), as shown in Table 2. The membrane fusion is observed in (3) after the application of the electric pulse of 6 V. The cell is released from the orifice ((4) and (5)) and observed for several minutes after the fusion in (6). On average, the magnitude of applied voltage needed to induce fusion for Jurkat cells is just slightly higher than that for the protoplasts (see Table 2). Therefore, these results confirm the effectiveness of the

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orifice in field constriction, decreasing the degree of size dependence of V_m .

In all experiments with plant protoplasts as well as with Jurkat cells, cell fusion took place so far as the cells can be positioned in the orifice, and the membrane breakdown other than at the contact point was hardly observed. Including several ten fusion events not shown here, more than 90% fusion yield was obtained by the device, which is a significant improvement from the previous work. This is primarily due to the fact that, until membrane breakdown occurs, we can increase the magnitude of fusing pulses gradually without damaging cells, provided they are in good condition.

VI. Conclusions

A device has been designed and fabricated for high-yield electrofusion of cells, based on the use of field constriction at the orifice on an insulating barrier. Experiments performed on the device show that we can realize one-to-one cell fusion for cells. As the approach used here is in principle independent of cell size, it can be applied to the practical problems of fusion between cells of different size. The fusion is highly reproducible and more than 90% yield has been achieved.

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Table 1. Typical dimensions of the device shown in Fig. 6.

Cell type of the device	plant	human
	protoplast	lymphoblast
Cell size (μm)	20-60 ^ϕ	5-15 ^ϕ
Channel depth, D (μm)	60	15
Wall thickness, T (μm)	50	30
Orifice width, B (μm)	30-40	8-10
Electrode gap, G (μm)	400	500

Table 2. Typical electrical parameters for DEP and fusion.

Cell type	DEP	Fusion
plant protoplast	2MHz,	6-7V _p , 5ms,
(20-60 μm^{ϕ})	3-6V _p	50kHz-modulated
human	5 MHz,	6-10V _p , 10ms,
lymphoblast	7.5-10V _p	20kHz-modulated
(5-15 μm^{ϕ})		

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List of Figure Captions

1. Principle of electrofusion based on the field constriction.
2. Model used for analyzing membrane voltage in the presence of an orifice.
3. Membrane voltage V_m induced on the equal-sized cells in contact in an orifice as a function of position. δ is the minimum spacing between the cells and the orifice wall, $w = 0.1a$, $d = 5a$ and $h = 5a$ where a is the cell radius.
4. Membrane voltage on the unequal cells, of which the larger and smaller radii are a and $a/2$, respectively. Dimensions w , d and h are the same as those used for Fig. 3.
5. Membrane voltage V_m when N cells (of equal radius) exist on each side of the orifice: (a) V_m on the first, second and third cells from the orifice and (b) V_m on the same cells in the absence of the orifice. The analysis uses $w = 0.1a$, $d = 5a$, $h = 12a$ and $\delta = 0.1a$.
6. Structure of the device for electrofusion.
7. Temporal sequence of electrofusion between two unequal cells (plant protoplasts).
8. Membrane breakdown on a long cell chain occurs only at the contact point between two cells at the orifice when electric pulse is applied. As a result, 1:1 fusion is obtained. (a) Chain formed under DEP force. (b) Breakdown at the contact point by electric pulse.
9. Fusion between two cells smaller than the orifice. (a) Cells attracted to the orifice surface by DEP. (b) Breakdown of membrane at the contact point due to electric pulse. (c) Cell released from the orifice.
10. Fusion of mammalian cells in a temporal sequence from (1) to (6).

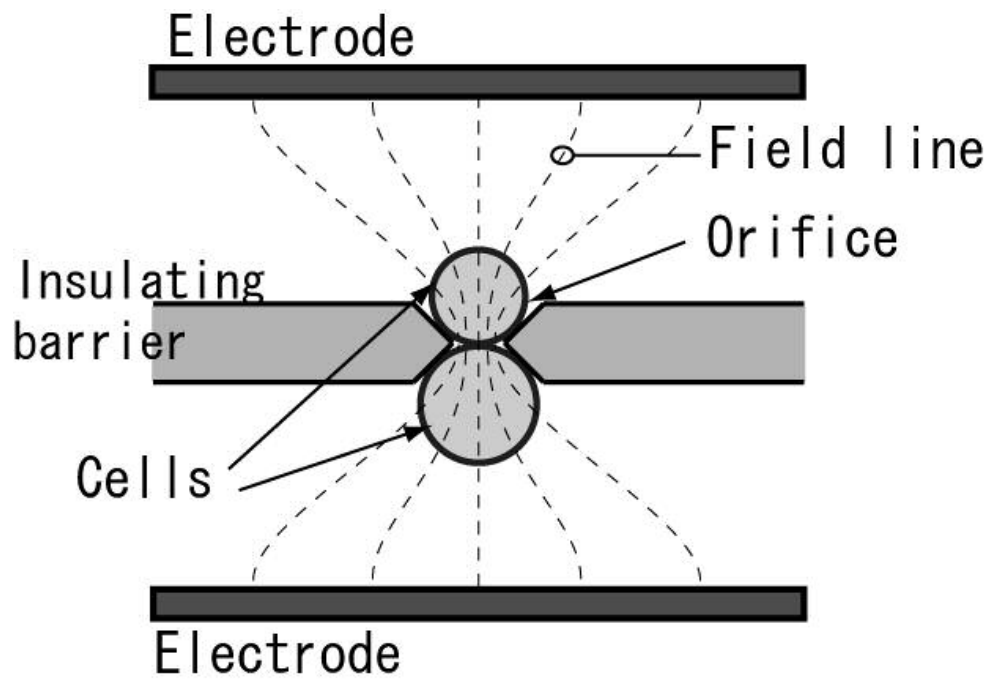


FIG. 1.

52x44mm (300 x 300 DPI)

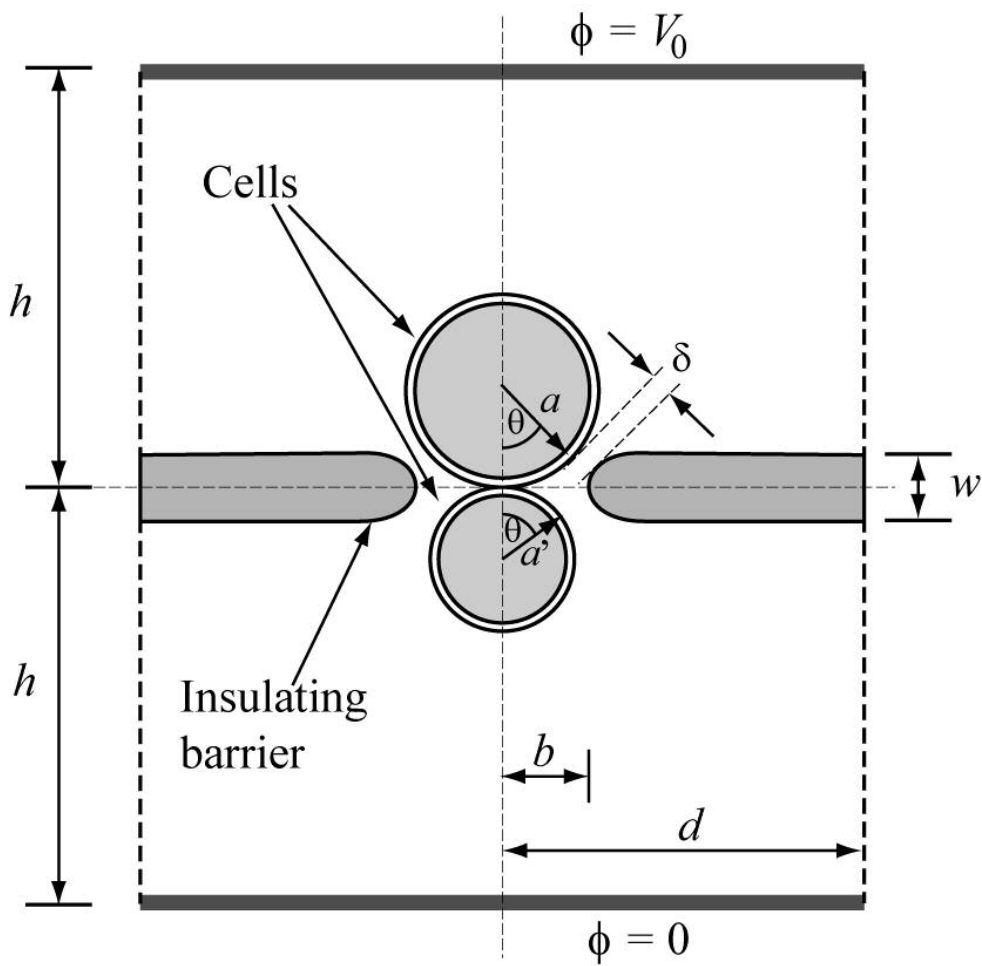


FIG. 2.

70x78mm (300 x 300 DPI)

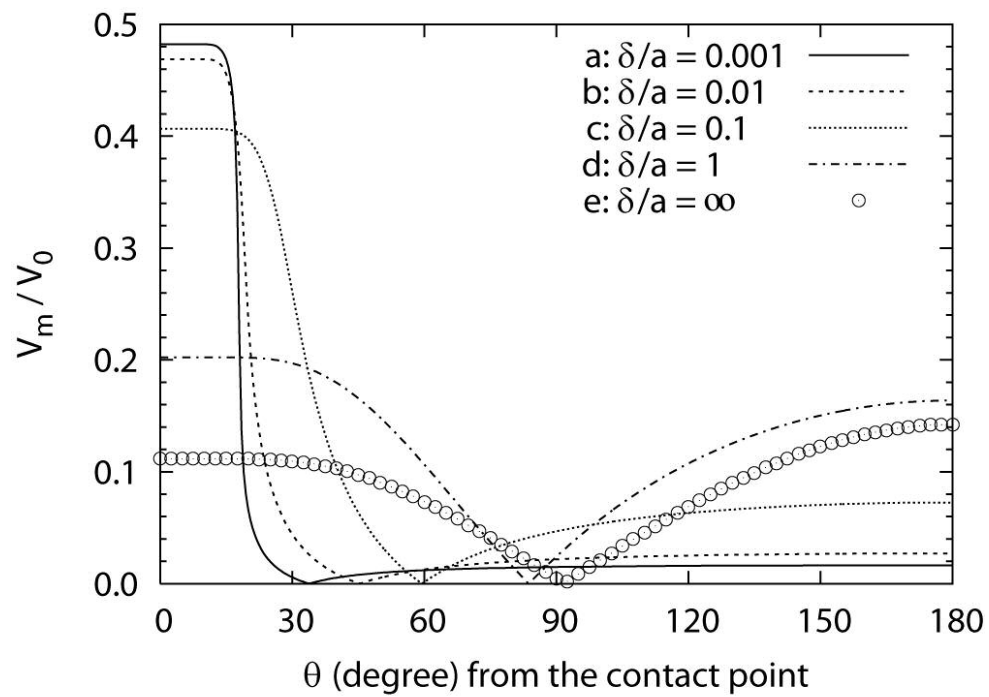


FIG. 3.

85x70mm (300 x 300 DPI)

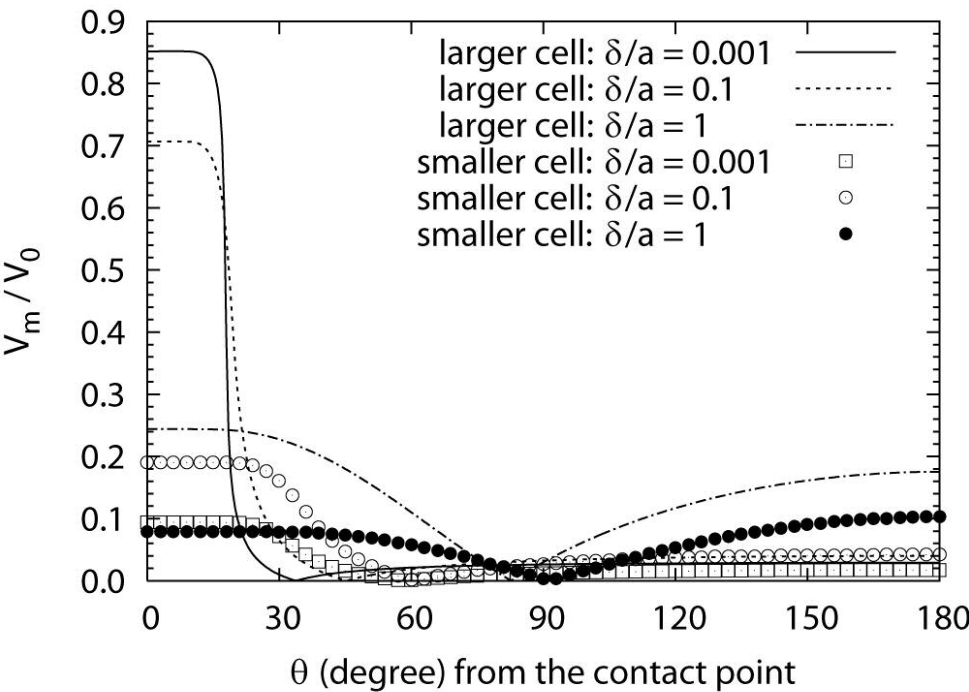


FIG. 4.

85x70mm (300 x 300 DPI)

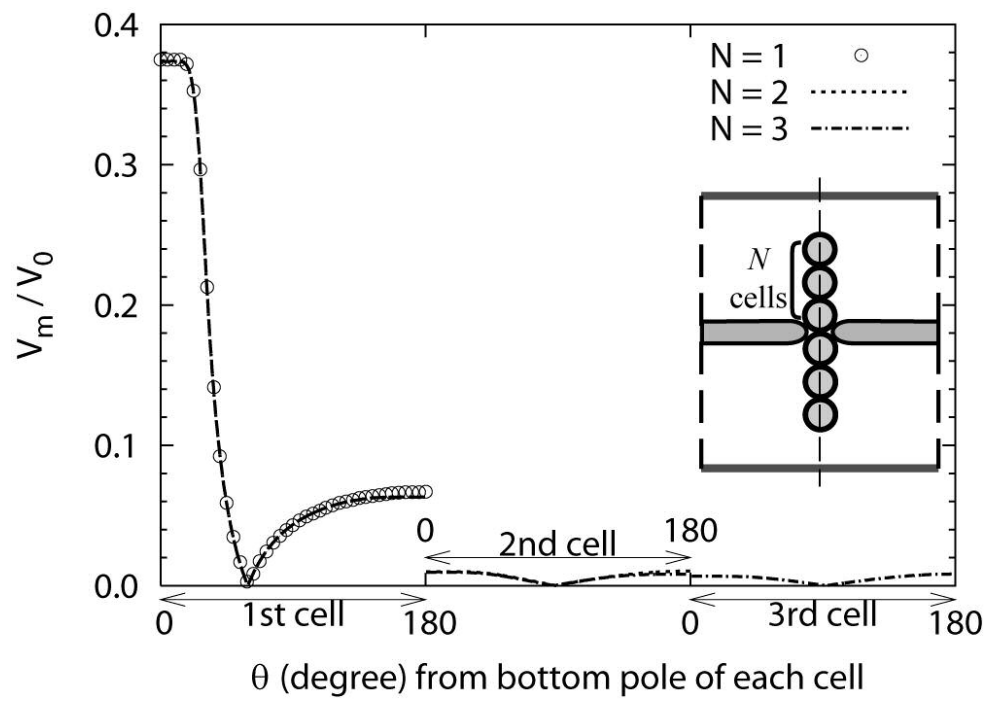


FIG. 5A.

84x69mm (300 x 300 DPI)

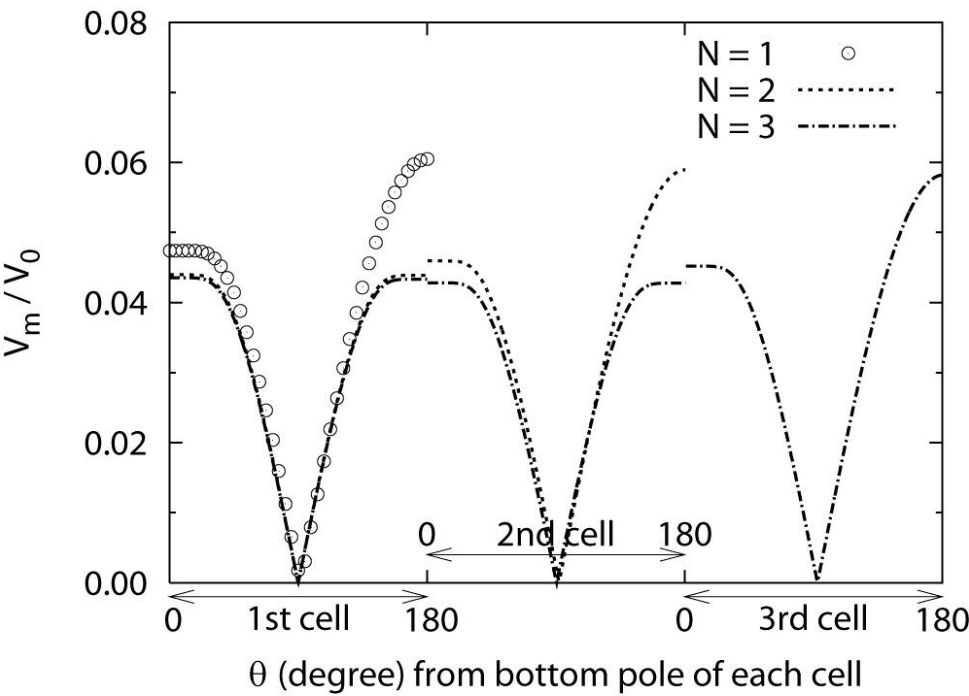


FIG. 5B.

84x68mm (300 x 300 DPI)

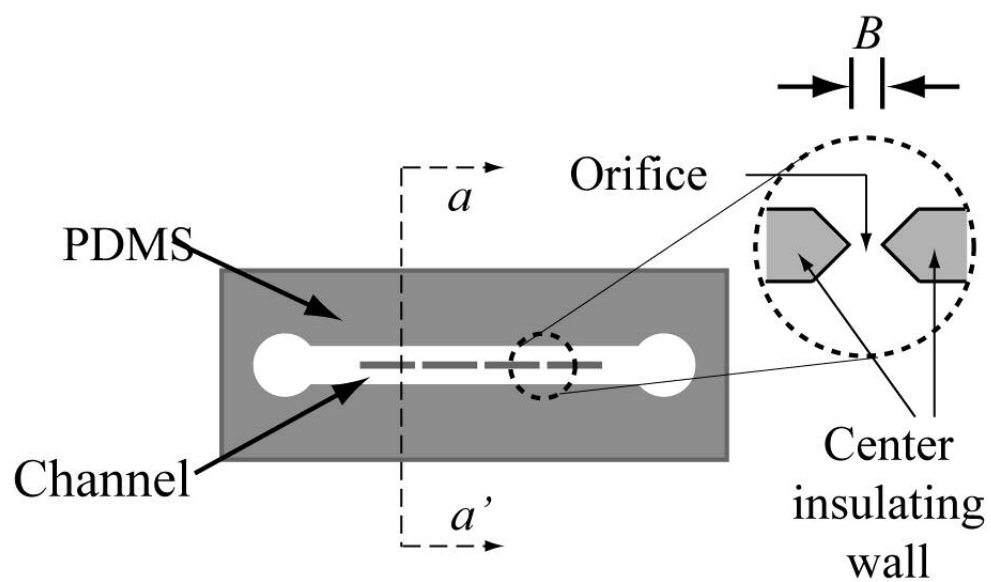


FIG. 6A.

77x56mm (300 x 300 DPI)

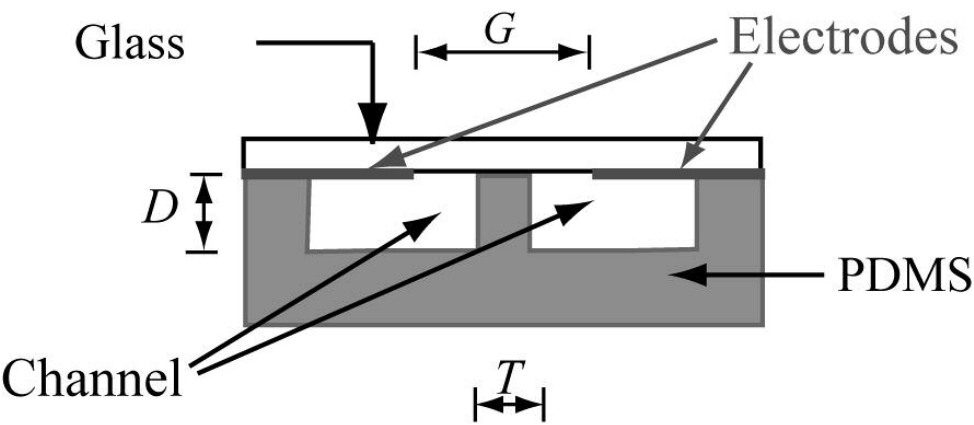
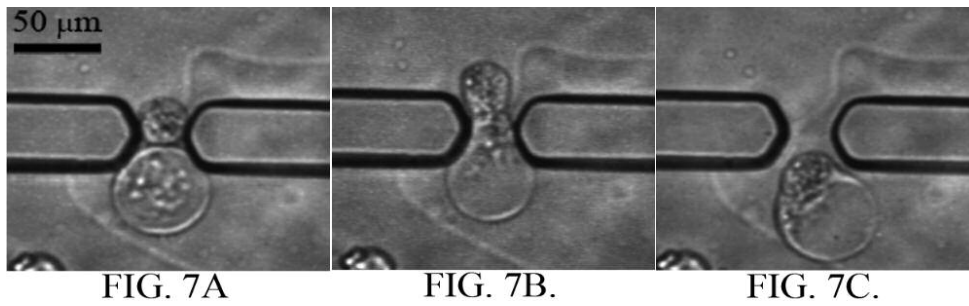


FIG. 6B

75x44mm (300 x 300 DPI)



96x29mm (264 x 264 DPI)

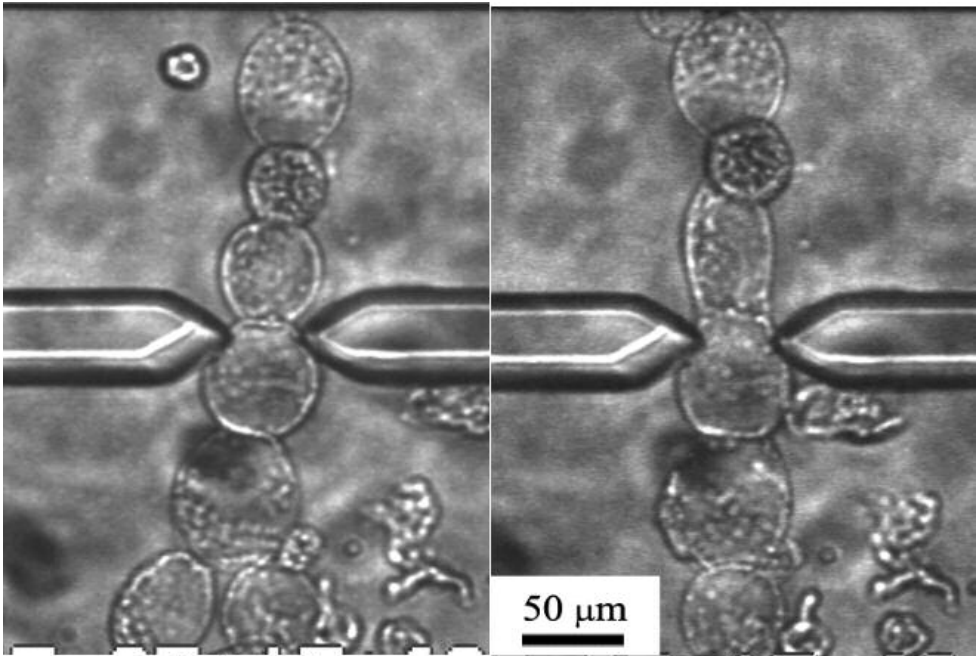
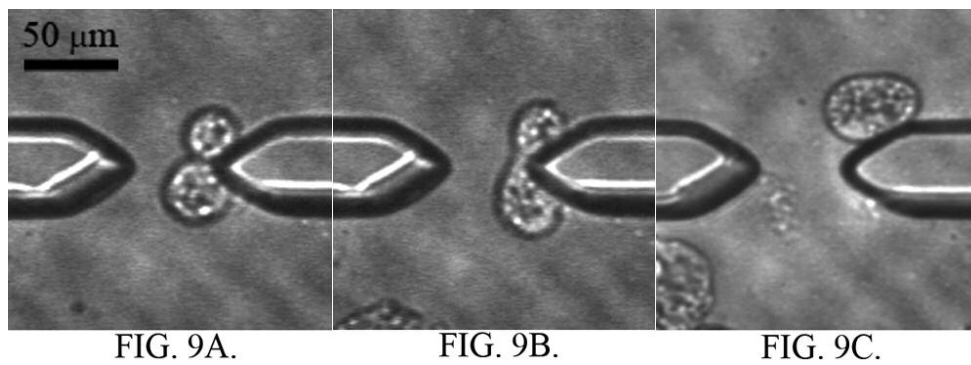


FIG. 8A.

FIG. 8B.

63x48mm (264 x 264 DPI)



95x35mm (264 x 264 DPI)

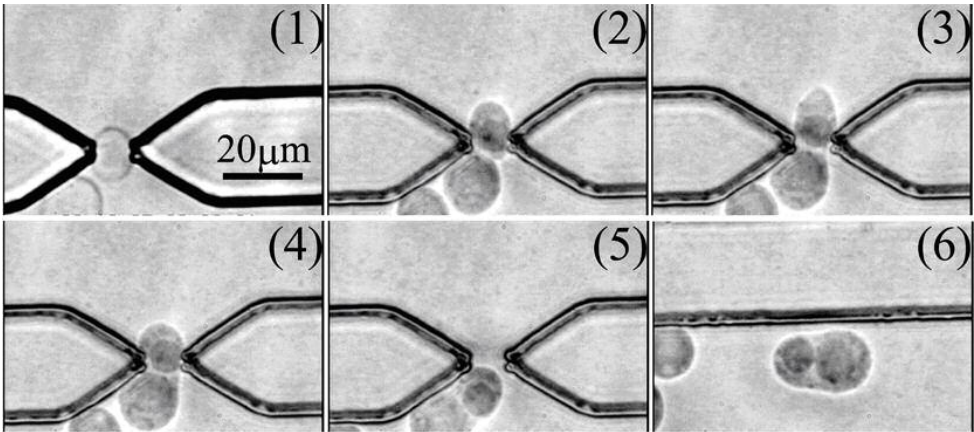


FIG. 10.

79x39mm (264 x 264 DPI)