สัญญาเลขที่ RMU4980013

รายงานวิจัยฉบับสมบูรณ์

โครงการวิจัยพลังงานมืดและพลวัตรของจักรวาล

บุรินทร์ กำจัดภัย

สถาบันสำนักเรียนท่าโพธิ์สำหรับฟิสิกส์ทฤษฎีและจักรวาลวิทยา
หน่วยวิจัยฟิสิกส์รากฐานและจักรวาลวิทยา
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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา
และสำนักงานกองทุนสนับสนุนการวิจัย

31 สิงหาคม พ.ศ. 2552

กิตติกรรมประกาศ

ผู้วิจัยขอขอบคุณสำนักงานกองทุนสนับสนุนการวิจัย สำนักงานคณะกรรมการการอุดมศึกษา สำหรับงบประมาณในการทำวิจัยและค่าตอบแทนนักวิจัยขอขอบคุณมหาวิทยาลัยนเรศวรสำหรับการสนับสนุนด้านสถานที่ในการทำการวิจัย ขอขอบคุณ Dr. Tapan Naskar (Inter-University Centre for Astronomy and Astrophysics, Pune, India) Dr. John Ward (University of Victoria (Canada); Queen Mary, University of London (UK) & CERN) สำหรับความร่วมมือในการทำวิจัย

บทคัดย่อ

รหัสโครงการ: **RMU4980013**
ชื่อโครงการ: **พลังงานมืดและพลวัตรของจักรวาล**

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เราสนใจศึกษาแบบจำลองพลังงานมืดหลายแบบในโครงการนี้ เราได้ศึกษาในประเด็นที่เกี่ยวข้องกับผลเชิงจักรวาลวิทยา พลวัตร ทางเลือกของรูปแบบเชิงคณิตศาสตร์ที่ใช้ และการสร้างแบบจำลองจากผลเชิงการสังเกตการณ์

ส่วนแรกของโครงการนี้ศึกษาเอกภพที่พรรณนาโดยทฤษฎีสตริงที่มีแผ่นเบรน $D3$ เคลื่อนที่ในพื้นหลังทางเรขาคณิตที่เป็นวงของเบรนที่วางตัวเป็นวงกลมรัศมีR. การเคลื่อนที่ตามขวางกับระนาบวงโดยแผ่นเบรน $D3$ นี้ทำให้เกิดสนามเรดิออนซึ่งสามารถเชื่อมโยงเป็นสนามสเกลาร์ที่มีมวลและเป็นแบบ Born-Infeld ที่ไม่เป็น BPS ภายใต้ศักย์แบบ $\cosh$ การพองตัวของเอกภพสามารถเกิดขึ้นได้สำหรับค่าความตึงของเบรนในขอบเขตหนึ่งโดยที่เรากำหนดให้ระดับพลังงานของสตริงอยู่ในระดับเดียวกับระดับพลังงานพลังค์สภาพการรบกวนที่เกี่ยวข้องและค่าดัชนีสเปคตรัมอยู่ในขอบเขตของค่าจากการสังเกตการณ์ดังคาดในที่สุดแล้วเราพบว่าการวิวัฒนาการของเอกภพจะมีองค์ประกอบของพลังงานโดดเด่นดังเช่นการสังเกตการณ์ซึ่งเป็นตังดึงดูด (attractor) ของแบบจำลองนี้เรายังพบว่าสมการสภาวะนั้นขึ้นกับเวลาซึ่งนำไปสู่ควินเทสเซนซ์ที่ระยะปลายอายุของเอกภพ

ในบริบทที่แตกต่างอีกอันหนึ่งเราได้ศึกษาระบบพลวัตรของสนามสเกลาร์แบบแฟนธอม (ปีศาจ) ภายใต้ศักย์แบบเอกโปเนนเชียลและพื้นหลังทางเรขาคณิตของเอกภพแบบลูปควอนตัม ผลการวิเคราะห์พบว่าไม่ปรากฏทั้งโนดเสถียรและโนดไม่เสถียรที่ผลักออกแต่ปรากฏจุดอานม้าขึ้นสองจุด ดังนั้นจึงไม่มีภาวะเอกฐานแบบบิ๊กริบ (Big Rip) ผลเฉลยที่สอดคล้องกับฟิสิกส์มีพลังงานศักย์มากกว่าขนาดของพลังงานจลย์ที่มีค่าลบเสมอ Physical solutions always เราพบว่าเอกภพเกิดการเด้งหลังจากยุคที่เกิดการเร่งแม้ว่าจะมีสนามแฟนธอมโดดเด่นหลังจากเด้งแล้วเอกภพในที่สุดจะเข้าสู่สภาพการสั่นแกว่งคือขยายและหดสลับกันไปเราได้ศึกษากรณีที่มีการควบคู่เปลี่ยนสภาพจากพลังงานมืดสนามแฟนธอมเป็นสสารมืด และสสารมืดกลับมาเป็นพลังงานมืดสนามแฟนธอมด้วย

เราเข้าไปสำรวจรูปแบบชโรงดิงเจอร์แบบไม่เชิงเส้น (NLS) ของจักรวาลวิทยาซึ่งเป็นรูปแบบทางคณิตศาสตร์ทางเลือกอีกประการหนึ่งในการใช้พรรณนาจักรวาลแบบมาตรฐานที่มีสนามสเกลาร์แบบคาโนนิคัลและสสารบาร์โรโทรปิคอยู่ด้วยกันเราได้ให้แนวทางมาตรฐานในการแปลงจากรูปแบบฟรีดมานน์ไปยังรูปแบบ NLS ไว้เป็นครั้งแรก เราได้รายงานการใช้งานและรายงานข้อด้อยของรูปแบบ NLS ไว้ ณ ที่นี้โดยเราได้ขยายความรูปแบบ NLS ให้รวมเอากรณีสนามแฟนธอมไว้ด้วย เราพิจารณากฎการขยายตัวสามแบบคือ แบบกฎกำลัง แบบเดอซิทเตอร์และแบบแฟนธอมเราได้พบว่าฟังก์ชันคลื่นของรูปแบบ NLS นั้นเป็นแบบนอร์มัลไลซ์ไม่ได้เราได้หาผลเฉลยแบบแม่นตรงและตรวจสอบพฤติกรรมของสัมประสิทธิ์สมการสภาวะยังผลของของไหลจักรวาลในทั้งสามกรณีเราพบว่าในกรณีกฎการขยายตัวแบบแฟนธอมถ้าเอกภพมีเรขาคณิตไม่แบนแล้วจะไม่มีการแบ่งเขตแฟนธอม (phantom divide) ที่แน่นอนและแม้ว่าค่าสัมประสิทธิ์สมการสภาวะยังผลจะมีค่ามากกว่า 1 การขยายตัวก็ยังเป็นแบบแฟนธอมได้ยิ่งไปกว่านั้นก็คือในเอกภพแบบเปิดการขยายตัวแบบแฟนธอมสามารถเกิดขึ้นได้แม้ว่าค่าสัมประสิทธิ์สมการสภาวะยังผลมีค่ามากกว่า 0 เรายังได้ศึกษาเงื่อนไขการกลิ้งช้าของสนาม เงื่อนไขการเร่ง การประมาณ WKB และภาวะเอกฐานบิ๊กริบในรูปแบบ NLS อีกด้วยเราพิจารณาการประมาณ WKB (กรณีเชิงเส้น) ในรูปแบบ NLS ศักย์ชโรงดิงเจอร์มีช่วงที่เปลี่ยนแปลง

ช้ามาก (very slowly-varying) เป็นพิสัยกว้างดังนั้นการประมาณ WKB จึงใช้ได้ในพิสัยดังกล่าวสำหรับภาวะเอกฐาน บิ๊กริบด้วยรูปแบบ NLS นั้น มีปริมาณเพียงสองปริมาณที่มีค่าเข้าสู่อนันต์แทนที่จะเป็นสามปริมาณดังในรูปแบบฟรีดมานน์เราพบว่าเอกภพที่ไม่แบนเมื่อเข้าใกล้บิ๊กริบ $w_{\rm eff} \to -1 + 2/3q$, $(q < 0)$ ซึ่งเป็นการลู่เข้าค่าเดียวกันกับในกรณีเอกภพแบน

เนื่องจากสนามสเกลาร์แบบ DBI มีข้อดีในการอธิบายการเกิด non-Gaussianity ที่พบในแบบลายของรังสีคอสมิคไมโครเวฟพื้นหลัง (CMB) เราจึงตรวจสอบเฟสสเปซของสนามควิเทสเซนซ์ที่ได้จากรูปแบบทั่วไปของแอคชันแบบ DBI เราได้ตรวจสอบระบบนี้ด้วยวิธีทางการวิเคราะห์และวิธีการเชิงตัวเลขของศาแห่งความเสรีที่เพิ่มขึ้นทำให้โครงสร้างเฟสสเปซซับซ้อนมากขึ้น โดยสนามสเกลาร์ที่ได้มีค่าสัมประสิทธิ์สมการสภาวะอยู่ในเต็มช่วง$-1 \leq w \leq 1$ เราพบว่ามีผลเฉลยของสมการการเคลื่อนที่ของสนามที่ไม่เป็นศูนย์หลายผลเฉลยชี้ว่าควินเทสเซนซ์แบบ DBI น่าจะเป็นแบบจำลอง k-essence ที่สมเหตุสมผลได้

ส่วนสุดท้ายของโครงการนี้คือการระบุรูปแบบศักย์ของสนามสเกลาร์แบบคาโนนิคัลในเอกภพแบบ FLRW ที่มีของไหลสัมบูรณ์บาร์โรโทรปิคและขยายตัวแบบกฎกำลังข้อมูลร่วมจาก WMAP5+BAO+SN dataset และข้อมูลจากดาวเทียม WMAP5 dataset ได้ถูกนำมาใช้ในการระบุค่าของศักย์การวิเคราะห์ข้อมูลได้ชี้ว่าเอกภพน่าจะมีเรขาคณิตแบบปิดเล็กน้อย ถ้าเอกภพปิดดังที่ข้อมูลระบุแล้วตัวเลขกำลังของกฎกำลัง $a \propto t^q$ คือ$q = 1.01$ (ใช้ข้อมูลของ WMAP5) และ $q = 0.985$ (ใช้ข้อมูลร่วม) ขอบเขตล่างของค่าa_0 (เรขาคณิตปิด) คือเท่ากับ 5.1×10^{26} เมื่อใช้ WMAP5 dataset และเท่ากับ 9.85×10^{26} สำหรับข้อมูลร่วมการโดดเด่นของพจน์กฎกำลังเหนือพจน์ความโค้งและพจน์ความหนาแน่นบาร์โรโทรปิคจะถูกระบุได้โดยการเปลี่ยนค่าอัตราการเปลี่ยนความชัน (inflection) ของเส้นกราฟศักย์ ซึ่งทั้งสองชนิดของข้อมูลระบุว่าการโดดเด่นนี้จะเกิดขึ้นเมื่อเอกภพมีอายุได้ 5.3 Gyr

คำหลัก: ทฤษฎีพลังงานมืด ระบบพลวัตร จักรวาลวิทยาแบบสนามสเกลาร์ สมการชโรดิงเจอร์แบบไม่เชิงเส้น การขยายตัวของเอกภพแบบกฎกำลัง แบบเดอซิทเตอร์และแบบแฟนธอม

Abstract

Project Code: **RMU4980013**

Project Title: **Dark Energy and Cosmological Dynamics**

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Project Period: 3 years (20 July 2006 - 5 August 2009)

We consider various models of dark energy and study them in cosmological aspects of dynamics, mathematical alternatives and modeling from obeservational data. The first part consider string inspired cosmology on a solitary $D3$ brane moving in the background of a ring of branes located on a circle of radius R. The motion of the $D3$ brane transverse to the plane of the ring gives rise to a radion field which can be mapped to a massive non-BPS Born-Infeld type field with a $\cosh$ potential. For certain bounds of the brane tension we find an inflationary phase is possible, with the string scale relatively close to the Planck scale. The relevant perturbations and spectral indices are all well within the expected observational bounds. The evolution of the universe eventually comes to be dominated by dark energy, which we show is a late time attractor of the model. However we also find that the equation of state is time dependent, and will lead to late time quintessence.

Next scenario, a dynamical system of phantom scalar field under exponential potential in background of loop quantum cosmology is investigated. In our analysis, there is neither stable node nor repeller unstable node but only two saddle points, hence no Big Rip singularity. Physical solutions always possess potential energy greater than magnitude of the negative kinetic energy. We found that the universe bounces after accelerating even in the domination of the phantom field. After bouncing, the universe finally enters oscillatory regime. We also study this system when phantom scalar field dark energy under exponential potential is coupled to barotropic dark matter fluid.

We investigate non-linear Schrödinger-type (NLS) formulation of cosmology. This is a mathematical alternative to the Friedmann formulation. We set up the procedure of transforming the Friedmann formulation to the NLS formulation. We also reports all usages and disadvantage of the NLS in the cases of power-law expansion, de-Sitter expansion and phantom expansion. We extend the formulation to include phantom field case and we have found that Schrödinger wave function in this formulation is generally non-normalizable. We also find exact solutions for the scalar field in various cases and analyzing the effective equation of state for these three cases. In the phantom expansion case, we found that, in a non-flat universe, there is no fixed w_{eff} value for the phantom divide. In a non-flat universe, even $w_{\text{eff}} > -1$, the expansion can be phantom. Moreover, in open universe, phantom expansion can happen even with $w_{\text{eff}} > 0$. We also study NLS-formulation of slow-roll approximation, acceleration condition, WKB approximation and Big Rip singularity. We reexpress all slow-roll parameters, slow-roll conditions and acceleration condition in NLS form. WKB approximation in the NLS formulation is also discussed when simplifying to linear case. Most of the Schrödinger potentials in NLS formulation are very slowly-varying, hence WKB approximation is valid in the ranges. In the NLS form of Big Rip singularity, two quantities are infinity in stead of three. We also found that approaching the Big Rip, $w_{\rm eff} \rightarrow -1 + 2/3q$, $(q < 0)$ which is the same as effective phantom equation of state in the flat case.

Since the DBI scalar field can describe non-Gaussianity found in the CMB. We investigate the phase space of a quintessence theory governed by a generalised version of the DBI action, using a combination of numeric and analytic methods. The additional degrees of freedom lead to a vastly richer phase space structure, where the field covers the full equation of state parameter space; $-1 \leq w \leq 1$. We find many non-trivial solution curves to the equations of motion which indicate that DBI quintessence is an interesting candidate for a viable k-essence model.

Last part of the project is to determine potential function of a canonical scalar field in FLRW universe in presence of barotropic perfect fluid expanding with power-law. The combined WMAP5+BAO+SN dataset and WMAP5 dataset are used here to determine the value of the potential. The datasets suggest slightly closed universe. If the universe is closed, the exponents of the power-law cosmology are $q = 1.01$ (WMAP5 dataset) and $q = 0.985$ (combined dataset). The lower limits of a_0 (closed geometry) are 5.1×10^{26} for WMAP5 dataset and 9.85×10^{26} for the combined dataset. The domination of the power-law term over the curvature and barotropic density terms is characterised by the inflection of the potential curve. This happens when the universe is 5.3 Gyr old for both datasets.

Keywords: Dark energy theory, dynamical system, scalar field cosmology, non-linear Schrödinger-type equation, power-law expansion, de-Sitter expansion, phantom expansion

Executive Summary

รหัสโครงการ: **RMU4980013**

ชื่อโครงการ: **พลังงานมืดและพลวัตรของจักรวาล**

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งบประมาณโครงการ: 1,200,000 บาท

โครงการวิจัยนี้เป็นโครงการเกี่ยวกับแบบจำลองพลังงานมืดที่ทำให้เกิดการเร่งออกของเอกภพหลายๆแบบ โดยเราศึกษาในประเด็นที่เกี่ยวข้องกับผลเชิงจักรวาลวิทยา พลวัตร ทางเลือกของรูปแบบเชิงคณิตศาสตร์ที่ใช้ และการสร้างแบบจำลองจากผลเชิงการสังเกตการณ์

ส่วนแรกของโครงการนี้ศึกษาเอกภพที่พรรณนาโดยทฤษฎีสตริงที่มีแผ่นเบรน $D3$ เคลื่อนที่ในพื้นหลังทางเรขาคณิตที่เป็นวงของเบรนที่วางตัวเป็นวงกลมรัศมีR. การเคลื่อนที่ตามขวางกับระนาบวงโดยแผ่นเบรน $D3$ นี้ทำให้เกิดสนามเรดิออนซึ่งสามารถเชื่อมโยงเป็นสนามสเกลาร์ที่มีมวลและเป็นแบบ Born-Infeld ที่ไม่เป็น BPS ภายใต้ศักย์แบบ $\cosh$ เราได้เสนอแบบจำลองนี้และตรวจสอบได้ว่าแบบจำลองนี้สามารถที่จะพรรณนาเอกภพทั้งที่ปลายอายุขัยและระยะแรกเริ่มได้อย่างสอดคล้องกับความรู้ในปัจจุบัน

จากนั้นเราได้ศึกษาพลังงานมืดแบบแฟนธอม (ปิศาจ) ซึ่งทำให้เกิดจุดจบแบบเอกฐานฉีกขาดออกของอวกาศ เราได้ตรวจสอบการฉีกขาดออกภายใต้ศักย์แบบเอกโปเนนเชียลและพื้นหลังทางเรขาคณิตของเอกภพแบบลูปควอนตัมซึ่งสามารถทำให้เกิดการเด้งของเอกภพได้เราพบว่าในเอกภพแบบลูปควอนตัมแม้จะมีสนามแฟนธอมก็ตามเอกภพก็สามารถเด้งได้ ในที่สุดเอกภพจะเข้าสู่สภาพการสั่นแกว่งคือขยายและหดสลับกันไปโดยที่ไม่เกิดการฉีกขาดของอวกาศขึ้นแต่อย่างใด เรายังได้ขยายผลไปยังพลังงานมืดแบบคู่ควบอีกด้วย

หลังจากนั้นเราได้ตรวจสอบประเด็นต่างๆ และสร้างกระบวนวิธีมาตรฐานในการใช้งานรูปแบบชโรงดิงเจอร์แบบไม่เชิงเส้น (NLS) ของจักรวาลวิทยาที่มีสนามสเกลาร์แบบคาโนนิคัลและสสารบาร์โรโทรปิคอยู่ด้วยกัน โดยศึกษารูปแบบการขยายตัวสามแบบคือแบบกฎกำลัง แบบ de Sitter และแบบแฟนธอมเราพบว่ารูปแบบ NLS จะทำให้เราสามารถหาผลเฉลยแบบแม่นตรงได้ง่ายขึ้น

เรายังได้พบว่าถ้าเรขาคณิตไม่แบนแล้วจะไม่มีการแบ่งเขตแฟนธอม (phantom divide) สำหรับเอกภพที่ขยายตัวแบบแฟนธอมและพบว่าการขยายตัวแบบแฟนธอมอาจเกิดขึ้นได้แม้ว่าค่าสัมประสิทธิ์สมการสภาวะยังผลจะมีค่าอยู่ในช่วงอื่นๆเรายังได้ศึกษาประเด็นทางจักรวาลวิทยาอื่นๆ สำหรับรูปแบบ NLS ไว้อีกด้วย

นอกจากนั้นเราได้ตรวจสอบเฟสสเปซของสนามควิเทสเซนซ์ที่ได้จากรูปแบบทั่วไปของแอคชันแบบ DBI และพบว่าทำให้โครงสร้างเฟสสเปซซับซ้อนมากขึ้น

ส่วนสุดท้ายของโครงการนี้คือการระบุรูปแบบศักย์ของสนามสเกลาร์แบบคาโนนิคัลในเอกภพแบบ FLRW ที่มีของไหลสัมบูรณ์บาร์โรโทรปิคและขยายตัวแบบกฎกำลังโดยใช้ข้อมูลข้อมูลร่วมจาก WMAP5+BAO+SN dataset และข้อมูลจากดาวเทียม WMAP5 dataset เราได้คำนวณหาตำแหน่งเวลาที่พลังงานมืดเริ่มโดดเด่นในเอกภพเหนือพจน์อื่นๆได้

- **การตีพิมพ์เผยแพร่**

ผลผลิตจากโครงการมีบทความวิจัยตีพิมพ์ในวารสารวิชาการระดับนานาชาติจำนวน 7 บทความโดย[มีค่า Journal Impact Factor หนึ่งปีย้อนหลังของแต่ละวารสารที่บทความได้รับการตีพิมพ์รวมเท่ากับ **29.841**] และวารสารวิชาการระดับชาติเป็นภาษาอังกฤษจำนวน 1 บทความ นอกจากนี้ยังมีมี 1 บทความปริทรรศน์ที่ได้รับการเผยแพร่ในหนังสือคือDark Energy-Current Advances and Ideas, Research Signpost (to appear in 2009) และยังมีอีก 1 บทความวิจัยที่อยู่ระหว่างการรอการพิจารณาจากวารสารระดับนานาชาติ

- **การสร้างนักวิจัยใหม่**
  โครงการนี้เกี่ยวข้องกับการสนับสนุนการวิจัยของนิสิตระดับปริญญาตรี 4 คนและปริญญาโท 2 คน

- **การพัฒนาการเรียนการสอนและการสร้างกลุ่มวิจัย**
  การดำเนินกิจกรรมวิจัยในโครงการนี้ได้ยกระดับโครงงานวิจัยระดับปริญญาตรีและระดับปริญญาโทของนิสิตในหน่วยวิจัยให้เข้าสู่ระดับสากลและยกระดับความเข้มข้นของเนื้อหาวิชาในรายวิชาทฤษฎีสัมพัทธภาพทั่วไปและรายวิชาจักรวาลวิทยาและทำให้มีการเปิดสอนจริงในสองวิชาทั้งขั้นต้นและขั้นสูงรวม 4 รายวิชาที่มหาวิทยาลัยนเรศวร โครงการนี้ทำให้กิจกรรมทางการวิจัยของหน่วยวิจัยยกระดับขึ้นและทำให้มีผู้สนใจเรียนสาขาวิชานี้มากขึ้น

- **การเสนอผลงานในที่ประชุมวิชาการและการบรรยายสัมมนาภายนอก**
  ผลการสนับสนุนไม่ว่าโดยทางตรงหรือทางอ้อมจากโครงการนี้ ได้ทำให้หัวหน้าโครงการวิจัยนี้ได้รับเชิญบรรยายพิเศษ ให้สัมมนารับเชิญและได้ไปนำเสนอผลงานแบบบรรยายในโอกาสต่างๆ 26 ครั้ง

- **การจัดการประชุมวิชาการ อบรมและสัมมนา**

  - การประชุมวิชาการ สัมพัทธภาพทั่วไป ฟิสิกส์พลังงานสูง และจักรวาลวิทยา แห่งชาติครั้งที่ 4
    ในวันที่ 26-28 กรกฎาคม พ.ศ. 2552 ที่โรงแรมรัตนาปาร์ค พิษณุโลกและที่มหาวิทยาลัยนเรศวรโดยในส่วนการประชุมวิชาการและ shortcourse มีผู้เข้าร่วมการประชุมรวม 61 คน จากกลุ่มวิจัยชั้นนำของมหาวิทยาลัยทั่วประเทศโดยกิจกรรมทางวิชาการทั้งหมดใช้ภาษาอังกฤษเป็นสื่อในการนำเสนอ ในส่วนฟอรัมเสวนาสาธารณะฯ มีผู้เข้าร่วมซึ่งรวมประชาชน ครู และนักเรียน ผู้สนใจทั่วไปรวม 117 คน
  - กิจกรรมวิชาการของสถาบันสำนักเรียนท่าโพธิ์ฯที่เกี่ยวข้องกับโครงการวิจัยนี้
    โครงการวิจัยนี้ได้สนับสนุนการจัดการสอนกระบวนวิชา Introduction to General Relativity ของสถาบันสำนักเรียนท่าโพธิ์ฯ ปี พ.ศ. 2552 และสัมมนาท่าโพธิ์อนุกรมที่ 7, 8, 9, 10, 11, 12 และ 13

- **การสร้างเครือข่ายนักวิจัย**
  ผลทางอ้อมจากการสนับสนุนตามโครงการนี้ได้ทำให้มีการประกาศตั้ง Thai Theoretical Astrophysics and Cosmology Network ขึ้นที่การประชุมวิชาการ สัมพัทธภาพทั่วไป ฟิสิกส์พลังงานสูง และจักรวาลวิทยา แห่งชาติครั้งที่ 4 ที่ได้รับการสนับสนุนจาก สกว.

- **การได้รับเชิญให้เข้าร่วมเป็นคณะกรรมการทางวิชาการและสมาชิกสมาคมวิชาการจากภายนอก**
  หัวหน้าโครงการนี้ได้รับเชิญเป็นกรรมการวิชาการของศูนย์สื่อสารวิทยาศาสตร์ไทย ของ สวทช. ได้รับเลือกเป็น Chair ของ Theoretical Astrophysics and Cosmology Working Group ของ South East Asia Astronomy Network ได้รับเชิญเป็น Reviewer ของ European Physical Journal C (I.F.(2007)=3.255) ตั้งแต่

ปี 2551 ได้รับเชิญเป็น Reviewer ของ Thai Journal of Physics: Proc. of the SIAM Physics Congress (ตั้งแต่ปี 2550 เป็นต้นมา) และได้รับเชิญเป็นภาคีสมาชิกของบัณฑิตยสภาวิทยาศาสตร์และเทคโนโลยีแห่งประเทศไทย (ปี 2551)

- **การเชิดชูเกียรติที่เป็นผลจากการสนับสนุนจากเฉพาะโครงการนี้**
  หัวหน้าโครงการนี้ถูกคัดเลือกให้ได้รับ **รางวัลนักวิทยาศาสตร์รุ่นใหม่ ประจำปี พ.ศ. 2551** ของ มูลนิธิส่งเสริมวิทยาศาสตร์และเทคโนโลยีในพระบรมราชูปถัมภ์และรางวัลนักวิจัยดีเด่นประเภทต่างๆ ของของมหาวิทยาลัยนเรศวรในปี 2551 และ 2550 อีก 3 รางวัล

สารบัญ

บทที่ 1

Introduction

Recently, present accelerating expansion of the universe has been confirmed with observations via cosmic microwave background anisotropies [1, 2], large scale galaxy surveys [3] and type Ia supernovae [4, 5]. However, the problem is that the acceleration can not be understood in standard cosmology. This motivates many groups of cosmologists to find out the answers. Proposals to explain this acceleration made till today could be, in general, categorized into three ways of approach [6]. In the first approach, in order to achieve acceleration, we need some form of scalar fluid so called *dark energy* with equation of state $p = w\rho$ where $w < -1/3$. Various types of model in this category have been proposed and classified (for a recent review see Ref. [132, 8]). The other two ways are that accelerating expansion is an effect of backreaction of cosmological perturbations [9] or late acceleration is an effect of modification in action of general relativity. This modified gravity approach includes braneworld models (for review, see [10]). Till today there has not yet been true satisfied explanation of the present acceleration expansion.

Considering dark energy models, a previous first year WMAP data analysis combined with 2dF galaxy survey and SN-Ia data and even a previous SN-Ia analysis alone favor $w < -1$ than cosmological constant or quintessence [11, 12]. Precise observational data analysis (combining CMB, Hubble Space Telescope, type Ia Supernovae and 2dF datasets) allows equation of state $p = w\rho$ with constant w value between -1.38 and -0.82 at the 95 % of confident level [13]. The recent WMAP three year results combined with Supernova Legacy Survey (SNLS) data when assuming flat universe yields $-1.06 < w < -0.90$. However without assumption of flat universe but only combined WMAP, large scale structure and supernova data implies strong constraint, $w = -1.06^{+0.13}_{-0.08}$ [14]. While assuming flat universe, the first result from ESSENCE Supernova Survey Ia combined with SuperNova Legacy Survey Ia gives a constraint of $w = -1.07 \pm 0.09$ [15]. Interpretation of various data brings about a possibility that dark energy could be in a form of phantom field-a fluid with $w < -1$ (which violates dominant energy condition, $\rho \geq |p|$) rather than quintessence field [16, 17, 18]. The phantom equation of state $p < -\rho$ can be attained by negative kinetic energy term of the phantom field. However there are some types of braneworld model

[19] as well as Brans-Dicke scalar-tensor theory [20] and gravitational theory with higher derivatives of scalar field [21] that can also yield phantom energy. There has been investigation on dynamical properties of the phantom field in the standard FRW background with exponential and inverse-power law potentials by [22, 23, 24, 25] and with other forms of potential by [26, 27, 25]. These studies describe fates of the phantom dominated universe with different steepness of the potentials.

A problem for phantom field dark energy in standard FRW cosmology is that it leads to singularity. Fluid with w less than -1 can end up with future singularity so called the Big Rip [28] which is of type I singularity according to classification by [30, 29]. The Big Rip singularity corresponds to $a \to \infty, \rho \to \infty$ and $|p| \to \infty$ at finite time $t \to t_s$ in future. Choosing particular class of potential for phantom field enables us to avoid future singularity. However, the avoidance does not cover general classes of potential [26]. In addition, alternative model, in which two scalar fields appear with inverse power-law and exponential potentials, can as well avoid the Big Rip singularity [31]. The higher-order string curvature correction terms can also show possibility that the Big-Rip singularity can be absent [32].

In this report, our circa is to investigate dark energy and its dynamics in various models motivated by string theory. Situations depend on the set-up system relevant to observational data. The second chapter consider a situation where geometrical properties of objects in string theory give rise to a tachyon field which plays a role of dark energy [33]. Then with the phantom favor of observation recently, we study the dynamics of the canonical phantom field with aim to solve the Big Rip singularity problem [34]. It is also interesting to consider when the phantom field is coupled to dark matter [35]. Exact solution of the scalar field is hard to solve. Maybe exploring the scalar field cosmology in alternative mathematical context could reveal some sense of mathematical manipulation to the scalar field solution. This bought us to study the *non=linear Schrödinger-type formulation* [36, 37, 38, 39] and this is reviewed by the author in [40]. Lately, the string-inspired kinetic term called *DBI* is believed to responsible for non=Gaussianity in the Cosmic Microwave Background (CMB). We investigate a very generalised class of DBI action which renders DBI scalar field playing a role of quintessence dark energy [41]. At the end of this report, we discuss about the situation that the universe had been undergoing power-law expansion caused by a single canonical scalar field. The potential of this scalar field is determined using the latest CMB data [42].

บทที่ 2

Dark energy from geometrical tachyon in string theory

It was recently suggested that the rolling open string tachyon, inspired by a class of string theories, can have important cosmological implications. The decay of a non-BPS $D3$-brane filling four dimensional space time leads to a pressureless dust phase which we identify with the closed string vacuum. The rolling tachyon has an interesting equation of state whose parameter ranges from 0 to -1. It was therefore thought to be a candidate of inflation and dark matter, or a model of transient dark energy [43]. However if we rigorously stick to string theory, the effective tachyon potential contains no free parameter. A viable inflationary scenario should lead to enough number of $e-$folding, and the correct level of density perturbations. The latter requires a free parameter in the effective potential which could be tuned to give rise to an adequate amount of primordial density perturbations. One also requires an adjustable free parameter in the effective potential to account for the late time acceleration.

Recently a time dependent configuration in a string theory was investigated and was shown to have interesting cosmological application [54]. In this scenario a BPS $D3$-brane is placed in the background of several coincident, static $NS5$-branes which are extremely heavy compared to the $D3$-brane and form an infinite throat in the space time. This system is inherently non-supersymmetric because the two different kinds of branes preserve different halves of the bulk supersymmetries. As a result the $D3$ brane can be regarded as a probe of the warped background and is gravitationally attracted toward the $NS5$-branes. Furthermore there exists an exact conformal field theory description of this background where the number of five-branes determines the level of the WZW current algebra [55], which allows for exact string based calculations. Despite the fact that the string coupling diverges as we approach the fivebranes, it was shown that we can trust our effective Dirac-Born-Infeld (DBI) action to late times in the evolution provided that the energy of the probe brane is sufficiently high. In any event, as the probe $D3$-brane approaches the background branes the spatial components of the energy-momentum tensor tend to zero in exactly the same way as in the effective action description of the open string tachyon. Thus it was anticipated that the dynamics of branes in these backgrounds had remarkably similar properties to rolling tachyon solutions. This relationship was further developed by Kutasov who showed that it was possible to mimic the open string tachyon potential by considering brane motion in a specific kind of 10D geometry. In order to do this one must take the action of the BPS probe brane in the gravitational background and map it to a non-trivial scalar field solution described by the non-BPS action [44]. The new field is essentially a holographic field living on the world-volume of the brane, but encodes all the physics of the bulk background. This is known as the geometrical tachyon construction. Another particularly interesting solution considered the background branes distributed around a ring of radius R, which was analysed in ref [57, 58], and whose geometry is described by a coset model [56], again potentially opening the way for an exact string calculation.

It seems natural to enquire as to whether these geometrical tachyon solutions have any relevance for cosmology, since they neatly avoid the problems associated with open string tachyon inflation [46] by having a significantly different mass scale. This change in scale is due to the motion of the probe brane in a gravitationally warped background, provided by the branes in the bulk geometry. In essence, this is an alternative formulation of the simple Randall-Sundrum model [74]. More recently, flux compactification has opened up the possibility of realising these models in a purely four-dimensional string theory context [51]. The fluxes form a throat which is glued onto a compact manifold in the UV end of the geometry. The warp factor in the metric has explicit dependence on the fluxes, and so provides us with a varying energy scale. The recent approaches to brane cosmology [49] are based on the motion of $D3$-branes in these compactifications. Typically we find $\bar{D}3$-branes located at some point in the IR end of the throat, which provide a potential for a solitary probe brane, with the inflaton being the inter-brane distance. In this context we can obtain slow roll inflation, and also the so-called DBI inflation [52], which relies heavily on the red-shifting of energy scales. However flux compactification models have an unacceptably large number of vacua, characterised by the string landscape. They are also low energy models, where the string scale is significantly lower than the Planck scale and so there is no attempt to deal with the initial singularity. In addition, we require multiple throats attached to the compact manifold where the standard model is supposed to live, however there is no explanation for the decoupling of the inflaton sector. These problems need to be addressed if we are to fully understand early universe cosmology in a string theory context. The alternative approach is to consider cosmology in the full ten dimensional string theory. Although these models are plagued by their own problems there is a definite sense of where the standard model is assumed to live, and a natural realisation of inflation. Furthermore we can invoke a Brandenberger-Vafa type mechanism to explain the origin of our $D3$-brane, arising from the mutual cascade annihilation of a gas of $D9$-$\bar{D}9$-branes [79].

An alternative approach is compactify our theory on a compact manifold, where some mechanism is employed to stabilise the various moduli fields. This will naturally induce an Einstein-Hilbert term into the four dimensional action [72]. However this is a highly non-trivial problem whose precise details remain unknown. Despite being unable to embed this into String Theory, we can still learn a great deal about the physics of the model - as emphasised by recent works [60].

A specific case of interest has been to study inflation in the ring solution ref [61]. Due to the unusual nature of the harmonic function we find decoupled scalar modes, one transverse to the ring plane and the other inside the ring. The cosmology of modes inside the ring have been studied in ref [59]. In this note we will consider the situation in which the $D3$-brane moves in the transverse direction to the ring. Performing the tachyon map in this instance yields a $\cosh$ type potential implying that the resulting scalar field in the dual picture is massive. It is interesting that in this setting we do not have to worry about the continuity condition around the ring. And unlike the longitudinal motion, we have an analytic expression for the effective potential every where in the transverse directions. We study the cosmological application of the resulting scenario and show that the model leads to an ever accelerating universe. We study the autonomous form of field evolution equation in the presence of matter and radiation and show that the de-Sitter solution is a late time attractor of the model. We also demonstrate the viability of the geometrical tachyon for dark energy in the setting under consideration, arising in a natural way due to the non-linearity of the DBI action. In the next section we will introduce the string theory inspired model, and discuss how we an relate it to four dimensional cosmology. In section III we will consider the more phenomenological aspects of our model by comparing our results with experimental observation. Section IV shows how we have a natural realisation of reheating in our model, whilst section V discusses the final stage of dark energy domination. Our model predicts that the equation of state parameter will tend to $w \sim -1$, but on even larger timescales we expect it to increase toward zero as in models of quintessence [83]. We will conclude with some remarks and a discussion of possible future directions.

2.1 Geometrical scalar field and coupling to gravity.

We begin with the string frame CHS solution for k parallel, static $NS5$ branes in type IIB String Theory [62, 63]. The metric is given by:

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + F(x^n)dx^m dx^m, \tag{2.1}$$

where χ is the dilaton field define as $e^{2(\chi-\chi_0)} = F(x^n)$, and there exists the three form field strength of the NS B-field $H_{mnp} = -\varepsilon^q_{mnp}\partial_q\phi$. Here $F(x^n)$ is the harmonic function describing the position of branes. For a large number of branes we can consider the throat approximation, which amounts to dropping the factor of unity in the function. Inherently we are decoupling Minkowski space time from the theory, and therefore only interested in the region around the $NS5$-branes. The harmonic function is given by:

$$F = 1 + \frac{kl_s^2 \ \sinh(ky)}{2R\rho \ \sinh(ky) \ (\cosh(ky) - \cos(k\theta))}$$
$$\approx \frac{kl_s^2 \ \sinh(ky)}{2R\rho \ \sinh(ky) \ (\cosh(ky) - \cos(k\theta))}, \tag{2.2}$$

where ρ, θ parameterise polar coordinates in the ring plane, and the factor y is given by:

$$\cosh(y) = \frac{R^2 + \rho^2}{2R\rho}. \tag{2.3}$$

We put a probe $D3$ brane at the centre of $NS5$ branes, as mention in the introduction this brane will move toward the circumference due to gravitational interaction if it shifted a little from the centre keeping the brane in the plane of the ring; the cosmology in this case is described elsewhere. We consider the case where the probe brane lies in the centre of the ring but shifted a little from the plane. In this case the probe brane shows transverse motion. Note that because of the form of the DBI action, the configuration here is actually S-dual to the $D5$-brane ring solution. The only difference is the shift of $k \to 2g_s k$ in the harmonic function. The physics however are very different as we know that F-strings cannot end on the $NS5$-branes, but can end on the $D5$-branes. This implies that in the case of the $D5$-brane ring we can have additional open string tachyonic modes once the probe brane starts to resolve distances of order of the string scale. The cosmological implications for this extra field were discussed in [60].

For the brane at the center ($\rho = 0$) moving transverse to the ring ($\dot{\rho} = 0$), the harmonic function is given by:

$$F(\sigma) = \frac{kl_s^2}{R^2 + \sigma^2}, \tag{2.4}$$

and the DBI action for the probe brane can be written in the following form, in static gauge

$$S = -\tau_3 \int d^4\xi \sqrt{F^{-1} - \dot{\sigma}^2}. \tag{2.5}$$

The tachyon map in this instance arises via field redefinition. We define the following scalar field, which has dimensions of length

$$\phi(\sigma) = \int \sqrt{F} d\sigma, \tag{2.6}$$

which maps the BPS action to a form commonly used in the non-BPS case [44]

$$S = -\int d^4\xi V(\phi) \sqrt{1 - \dot{\phi}^2}, \tag{2.7}$$

where $V(\phi)$ is the potential for the scalar field which describes the changing tension of the D-brane.

From the above mapping we get the solution of field as:

$$\begin{aligned}\phi(\sigma) &= \int_0^{\sigma} \sqrt{F(\sigma')}d\sigma' \\ &= \sqrt{kl_s^2}\ln\left(\frac{\sigma}{R}+\sqrt{1+\frac{\sigma^2}{R^2}}\right) \\ &= \sqrt{kl_s^2}\text{arcsinh}\left(\frac{\sigma}{R}\right) \qquad (2.8) \\ V(\phi) &= \frac{\tau_3}{\sqrt{F}} \\ &= \frac{\tau_3 R}{\sqrt{kl_s^2}}\cosh\left(\frac{\phi}{\sqrt{kl_s^2}}\right). \qquad (2.9)\end{aligned}$$

Clearly we see that $\phi \to \pm\infty$ as $\sigma \to \pm\infty$, and that at the minimum of the potential we have $\phi = 0$ [1]. The potential of the field suggests that the mass is given by $1/kl_s^2$, corresponding to a massive scalar fluctuation. One may ask if there is a known string mode exhibiting this profile. In fact the fluctuations of a massive scalar were computed in [71] using a similar approach to the construction of the open string tachyon mode in boundary conformal field theory [43]. This field was then used in ref. [50, 73] as a candidate for the inflaton living on a $\bar{D}3$-brane in the KKLT scenario ref. [76]. The potential for the scalar is known to fourth order and was been assumed to be exponential in profile, although globally it may be hyperbolic.

In order to discuss the cosmological evolution of our scalar field we need to couple our effective action to four dimensional Einstein gravity. There are several ways we can accomplish this. Firstly we can consider the Mirage Cosmology scenario [75]. This requires us to re-write the induced metric on the $D3$-brane world-volume in a Friedmann-Robertson-Walker (FRW) form. The universe will automatically be flat, or closed if we imagine the D-brane to be spherical. The problem here is that there is no natural way to couple gravity to the brane action and therefore we must insert it by hand, however the cosmological dynamics are expected to be reliable virtually all the way to the string scale. The second option is a slight modification of the first. We imagine that the bulk is infinite in extent, and that the $D3$-brane is again coupled to gravity through some unknown mechanism. However rather than writing the induced metric in FRW form, we switch to the holographic theory. Now, the tachyon mapping in this case is only concerned with time-dependent quantities, and in particular only with the temporal component of the Minkowski metric. Therefore we choose to include a scale factor component in the spatial directions. This means that we have a cosmological coupling for the holographic scalar field, and the universe lives on the $D3$-brane world-volume. The final approach would be to compactify the theory down to four dimensions. In order to do this we need to truncate the background to ensure the space is compact [51]. In our case the ring can naturally impose a cut-off in the planar direction, however we must still impose some constraint in the transverse direction to the ring plane. Our solution simplifies somewhat if we can consider the $R \to 0$ limit, or equivalently the $\sigma \gg 1$ limit, as the background will appear point like.

[1] We must bear in mind that our approximation of the harmonic function prevents us from taking the $\sigma \to \infty$ limit.

Smoothly gluing the truncated space to a proper compact manifold will now automatically include an Einstein-Hilbert term in the effective action [72]. However, although we now have a natural coupling to gravity, the compactification itself is far from trivial as we also need to wrap two of the world-volume directions of the $NS5$-branes on a compact cycle. In order to proceed we must first uplift the full solution to M-theory [2], where we now have a ring of $M5$-branes magnetically charged under the three-form $C_{(3)}$. Compactification demands that the magnetic directions of the three-form are wrapped on toroidal cycles, which is further complicated by the ring geometry and will generally result in large corrections to the potential once reduced down to four-dimensions. So, although we have a natural gravitational coupling we may have large corrections to the theory. The complete description of this compactification is interesting, but well beyond the scope of this note and should be tackled as a future problem. However we could also assume a large volume toroidal compactification, where again all the relevant moduli have been stabilised. Provided we introduce some 'sink' for the five-brane charge, located at the some distant point in the compact space, and also only concentrate on the region close to the branes so that the harmonic function remains valid and we will have an induced gravitational coupling in the low energy theory. The corrections to the scalar potential in this region of moduli space may well be sub-leading with respect to the scalar field dynamics and thus we can treat our model as the leading order behaviour.

Recent work in this direction has been concerned with the compactification approach [60, 61], where it was assumed all the relevant moduli are fixed along the lines of the KKLT model [76] and that all corrections to the potential are sub dominant. We will tentatively assume that this will also hold in our toy model.

We can now analyse our four dimensional minimally coupled action, where we find the following solutions to the Einstein equations

$$H^2 = \frac{V(\phi)}{3M_p^2\sqrt{1-\dot{\phi}^2}} \tag{2.10}$$

$$\frac{\ddot{a}}{a} = \frac{V(\phi)}{3M_p^2\sqrt{1-\dot{\phi}^2}}\left(1-\frac{3\dot{\phi}^2}{2}\right). \tag{2.11}$$

These expressions are different to those associated with a traditional canonical scalar field. In particular we see that inflation will automatically end once $\dot{\phi}^2 \sim 2/3$ as in the tachyon cosmology models [45, 47, 53]. For completeness we write the equation of motion for the inflaton derived from the non-BPS action as follows

$$\frac{V(\phi)\ddot{\phi}}{1-\dot{\phi}^2} + 3HV(\phi)\dot{\phi} + V'(\phi) = 0, \tag{2.12}$$

where dots are derivatives with respect to time and primes are derivatives with respect to the field. Note that we are suppressing all delta functions in the expressions. We can now proceed with the analysis of our theory in the usual manner. It must be noted that this model corresponds to large field inflation,

[2]This was discussed by Ghodsi et al in [60]. We refer the interested reader there for more details.

where the initial value of the scalar field must satisfy the following condition

$$\phi_0 \ll \sqrt{kl_s^2}\,\text{arccosh}\left(\frac{\sqrt{kl_s^2}}{R}\right), \tag{2.13}$$

according to our truncation of the harmonic function.

Note that in what follows we will frequently switch between the field theory and the bulk geometry. The latter is more geometrical and so provides us with extra intuition about the physics of the solution, however both are equivalent - at least in this simplified model.

Using the slow-roll approximation, $H^2 \simeq V(\phi)/3M_p^2$ and $3H\dot{\phi} \simeq -V_\phi/V$, the e-folding

$$\begin{aligned} N &= \int_t^{t_f} H dt \\ &= \frac{\tau_3 R\sqrt{kl_s^2}}{M_p^2}\int_{x(\phi_f)}^{x(\phi)} \frac{\cosh^2 x}{\sinh x} dx \\ &= s\left[-\cosh(x_f) + \cosh(x) - \ln\left(\frac{\tanh(x_f/2)}{\tanh(x/2)}\right)\right]. \end{aligned} \tag{2.14}$$

Where we have introduced the dimensionless quantities $x = \phi/\sqrt{kl_s^2}$ and $s = \tau_3 R\sqrt{kl_s^2}/M_p^2$. Further defining the new quantity: $y \equiv \cosh x$ we can write the number of e-folds as follows:

$$N = s\left[-y_f + y - \frac{1}{2}\ln\left(\frac{(y_f-1)(y+1)}{(y_f+1)(y-1)}\right)\right] \tag{2.15}$$

Now, the relevant slow-roll parameter is defined as $\epsilon \equiv -\dot{H}/H$ which in our solution reduces to

$$\epsilon = \frac{y^2-1}{2sy^3}. \tag{2.16}$$

Note that our model is explicitly non-supersymmetric, and therefore we don't need to calculate the second slow roll parameter η since we anticipate that this will be trivially satisfied if ϵ is. At the end of inflation $\epsilon = 1$, then $y_f \equiv f(s)$ is given by the root of above equation, setting $\epsilon = 1$

$$f(s) = \frac{1}{6s}\left[g(s) + \frac{1}{g(s)} + 1\right] \tag{2.17}$$

where $g(s) = \left(-54s^2 + 1 + 6s\sqrt{3(27s^2-1)}\right)^{1/3}$ From eqn(2.15) the equation for y is:

$$\ln\left(\frac{y+1}{y-1}\right) - 2y = -\frac{2N}{s} - 2f(s) - \ln\left(\frac{f(s)-1}{f(s)+1}\right) \tag{2.18}$$

For $s > 1$ and as $y_{\min} = 1$, ϵ always remains less than one leading to an ever accelerating universe. Thus, in this case the geometrical scalar field in the present setting is not suitable to describe inflation but can become a possible candidate of dark energy. However if τ_3 is small enough so that $s < 1$, then we will find that inflation is possible as the slow roll parameter will naturally tend toward unity. There is a critical bound $s \leq 1/(3\sqrt{3})$, which must be satisfied if we are to consider inflation in this context.

2.2 Inflationary constraints.

To know the observational constraint on s we have to calculate the density perturbations. In the slow-roll approximation, the power spectrum of curvature perturbation is given by [64, 65, 66]:

$$\begin{aligned} P_S &= \frac{1}{12\pi^2 M_p^6}\left(\frac{V^2}{V_\phi}\right)^2 \\ &= \frac{\tau_3^2 R^2}{12\pi^2 M_p^6}\left(\frac{\cosh^2(\phi/\sqrt{kl_s^2})}{\sinh(\phi/\sqrt{kl_s^2})}\right)^2 \end{aligned} \tag{2.19}$$

The COBE normalisation corresponds to $P_S \simeq 2\times 10^{-9}$ for modes which crossed $N = 60$ before the end of inflation [48] which gives the following constraint:

$$k(l_s M_p)^2 \simeq \frac{10^9}{12\pi^2}\frac{s^2\cosh^4(\phi/\sqrt{kl_s^2})}{\cosh^2(\phi/\sqrt{kl_s^2})-1} \tag{2.20}$$

From the numerics using eqn(2.18) and eqn(2.19), we find that

$$k(l_s M_p)^2 \geq 3\times 10^{10} \tag{2.21}$$

which corresponds to $s \sim 10^{-3}$ when we impose the constraints $\tau_3 = 10^{-10}M_p^4$ and $R = 10^2/M_p$ which we regard as being typical values. The constraint on the tension in fact implies the following relationship

$$\frac{M_p}{M_s} \sim \frac{10^2}{g_s^{1/4}}, \tag{2.22}$$

which we need to be consistently satisfied. However, note that because of our basic assumptions about the theory we will generally obtain the bound

$$\frac{\tau_3 R}{M_p^3} \leq \frac{1}{9\times 10^5}. \tag{2.23}$$

If we write the tension of the brane in terms of fundamental parameters we can estimate the relationship between the String and Planck scales using the fact that we require $R > M_s^{-1}$ for the action to be valid

$$\frac{M_p}{M_s} \geq \frac{15}{g_s^{1/3}}, \tag{2.24}$$

where g_s is the string coupling constant. Note that this potentially constrains the String scale to be close to the Planck scale, as even if we demand weak coupling with $g_s = 0.001$ this gives us $M_p \geq 10^2 M_s$. Of course this is only a bound, and in our model we are treating this as a free parameter. In any event our typical values are consistent and thus we feel free to proceed. We should note that from a string theoretic point of view we should not take s as being a variable in this model. However our earlier analysis has shown that if we wish to consider non-eternal inflation, there exists a maximum bound on this parameter which is quite small. Thus we can make the assumption that s will always be small, with appropriate tuning of the ratio of the string and Planck scales.. In the following analysis we will always be assuming that this is satisfied so as to avoid en eternal inflation scenario. Of course, in the string theory picture we have a probe brane moving in a non-trivial background geometry, and we would expect that the RR charge on the brane will be radiated away in the form of closed string modes. This effectively means that there is an additional decay constant in the definition of the field ϕ, which we have neglected in this note. Thus what we have here is a first-order approximation to the behaviour of the solution. It remains an open question as to whether we can define a tachyon map in this instance - and how this changes the inflationary scenario described here.

At leading order in our solutions, where s is assumed to be small and making sure our effective action remains valid, we obtain

$$k(l_s M_p^2)^2 \simeq \frac{10^9}{48\pi^2}(2N+1)^2 \tag{2.25}$$

which corresponds to $s \sim 10^{-5}(2N+1)$ and $y \sim \frac{(2N+1)}{2s}$, when $\tau_3 = 10^{-10}M_p^4$ and $R = 10^2/M_p$. Again, more generally we would find the following upper limit on the solution

$$s \leq 10^{-3}(2N+1), \tag{2.26}$$

which is easily satisfied by our typical values. In fact our results remain robust when compared to the WMAPII and SDSS results combined [69]. The new data constrains $n_s = 0.98 \pm 0.02$ at the 68 confidence level, and $r < 0.24$ at the 95 confidence level.

The spectral index of scalar perturbations is defined as [64, 65, 66]:

$$\begin{aligned} n_S - 1 &\equiv -4\frac{M_p^2 V_\phi^2}{V^3} + 2\frac{M_p^2 V_{\phi\phi}}{V^2} \\ &= \frac{2}{s}\left(\frac{2-y^2}{y^3}\right) \end{aligned} \tag{2.27}$$

The spectral index of tensor perturbations is defined as:

$$\begin{aligned} n_T &= -\frac{M_p^2 V_\phi}{V^3} \\ &= -\frac{1}{s}\left(\frac{y^2-1}{y^3}\right) \end{aligned} \tag{2.28}$$

The tensor-to-scalar ratio is:

$$\begin{aligned} r &\equiv 8\frac{M_p^2 V_\phi^2}{V^3} \\ &= \frac{8}{s}\left(\frac{y^2-1}{y^3}\right) \end{aligned} \tag{2.29}$$

With the limit $s \to 0$ we get

$$n_S = 1 - \frac{4}{(2N+1)}, \quad n_T = -\frac{2}{(2N+1)}, \quad r = \frac{16}{(2N+1)} \tag{2.30}$$

For $N = 60$, we get $n_S = 0.96694$ and $r = 0.13223$; for $N = 50$, we get $n_S = 0.96040$ and $r = 0.15842$. We know from observations that the constraint on the tensor-to-scalar ratio is $r < 0.36$ [67, 68], and so our model appears to be well within this bound.

2.3 Reheating

We see that the potential is a symmetric potential with a minima. In terms of the bulk field σ it can be written as:

$$V(\sigma) = \frac{\tau_3 R}{2\sqrt{kl_s^2}}\left[\left(\frac{\sigma}{R}+\sqrt{1+\frac{\sigma^2}{R^2}}\right) + \left(\frac{\sigma}{R}+\sqrt{1+\frac{\sigma^2}{R^2}}\right)^{-1}\right] \tag{2.31}$$

Now the question is would the brane oscillate back and forth through the ring, and if so what are the necessary conditions for oscillation? In the bulk picture we would naturally anticipate oscillation with a decaying amplitude due to RR-emission. Moreover the minimum of the potential in this case is actually metastable. However this has not been verified as we need to calculate the energy emission in the coset model description [56], which we leave as future work. This will alter the dynamics of the inflaton field as discussed in the previous section.

In any event we may also expect similar behaviour once our field is coupled to gravity, with the damping being provided by the Hubble term. This is particularly important because we may find inflation occurring in the phase space region beyond $s \geq s_{\mathrm{crit}}$, once enough damping has occurred. The relevant dynamical equations are the inflaton field equation (2.12) and the Friedmann equation. We repeat them below for convenience.

$$\ddot{\phi} + 3H\dot{\phi}(1-\dot{\phi}^2) + \frac{V_\phi}{V}(1-\dot{\phi}^2) = 0 \tag{2.32}$$

$$H^2 = \frac{1}{3M_p^2}\left(\frac{V}{\sqrt{1-\dot{\phi}^2}} + \rho_B\right) \tag{2.33}$$

where the terms inside the curly brackets cause damping. For an easy treatment let us first consider the slow-roll approximation, then in the damping equation only the $\dot{\phi}$ term remains and all other powers of $\dot{\phi}$ can be ignored. That is to say we are considering the case near the stable point. Then $H^2 \sim \frac{\tau_3 R}{3M_p^2\sqrt{kl_s^2}} + \frac{\rho_B}{3M_p^3}$ is constant and $\frac{V_\phi}{V} \sim \frac{2}{kl_s^2}\phi$. The equation of motion is then:

$$\ddot{\phi} + 3\dot{\phi}\sqrt{\left(\frac{\tau_3 R}{3M_p^2\sqrt{kl_s^2}} + \frac{\rho_B}{3M_p^2}\right)} + \frac{2}{kl_s^2}\phi = 0 \tag{2.34}$$

for critically damped motion we need:

$$\left(\frac{\tau_3 R}{3M_p^2\sqrt{kl_s^2}} + \frac{\rho_B}{3M_p^2}\right) = \frac{8}{9kl_s^2} \tag{2.35}$$

If the RHS of (2.35) is greater than the LHS we will find oscillations but it is reduced by damping which depends on the size of the damping factor ($= \frac{3}{2}\sqrt{\left(\frac{\tau_3 R}{3M_p^2\sqrt{kl_s^2}} + \frac{\rho_B}{3M_p^2}\right)}$), compared to the oscillation frequency $(= \sqrt{\frac{8}{kl_s^2} - 9\left(\frac{\tau_3 R}{3M_p^2\sqrt{kl_s^2}} + \frac{\rho_B}{3M_p^2}\right)}$).

From the definition of Ω_B setting it to 0.3, we get $\rho_B = \frac{3\tau_3 R}{7\sqrt{kl_s^2}}$, then from eqn(2.35) we obtain

$$
\begin{aligned}
s &> \frac{168}{90} \quad \text{Over damped} \\
&= \frac{168}{90} \quad \text{Critically damped} \\
&< \frac{168}{90} \quad \text{Oscillatory with a decaying amplitude.}
\end{aligned}
\tag{2.36}
$$

Recall from the previous section that for us to have non-eternal inflation there is a maximal bound for s, and so only the last solution can be considered physical. From the constraint we get $\sqrt{kl_s^2} \sim 10^5 M_p^{-1}$, $\tau_3 \sim 10^{-10} M_p^4$ and $R \sim 10^2 M_p^{-1}$. Hence it is oscillatory near the critical point. The energy of the decaying scalar field is used in expansion and particle production. If the rate of expansion of universe is much less than the decaying rate of the amplitude of the field then most of the energy released by the scalar field goes to reheating. The explicit solution of eqn (2.34) is:

$$
\phi(t) \quad = \phi_0 e^{\left[-\frac{3}{2}t\sqrt{\frac{10\tau_3 R}{21M_p^2\sqrt{kl_s^2}}}\right]} e^{\left[\pm\beth t\sqrt{\frac{2}{kl_s^2} - \frac{15\tau_3 R}{14M_p^2\sqrt{kl_s^2}}}\right]} \tag{2.37}
$$

The ratio of rate of field decay to the rate of expansion of universe is is defined to be:

$$
\Theta \quad \equiv |\frac{\dot{\phi}}{H\phi}| \tag{2.38}
$$

For this case we find:

$$
\Theta \quad = \quad \sqrt{\frac{21M_p^2}{5\tau_3 R\sqrt{kl_s^2}}} \tag{2.39}
$$

The above quantity can be made to be less than one by adjusting the various parameters.

Using eqn(2.25) we obtain:

$$
\Theta \quad \sim \quad \sqrt{\frac{21 \times 10^5}{5(2N+1)}} \tag{2.40}
$$

which allows us to write the parameter as a function of the number of e-foldings, provided we can trust our small s expansion. We know that reheating ends when $\Theta = 1$, thus the minimal number of e-foldings we require for this to be satisfied is

$$
N_{\text{end}} \quad \sim \quad 10^5. \tag{2.41}
$$

Clearly this is a large number of e-foldings, and this should motivate us to do a more thorough analysis. For now it would appear that unless there is a large amount of fine tuning, reheating would not end in this scenario. The difficulty is that we cannot use the WKB approximation in this case due to rapid fluctuations in the variation of the potential. Moreover, the analysis will be incomplete without specifying the exact form the gravitational coupling - as there will be corrections to the effective action arising from any compactification. For these reasons we will postpone the analysis and return to it in a later publication.

2.4 Dark energy

What are the implications of our model for dark energy? It is well known that the non-linear form of the DBI action admits an unusual equation of state, which is of the form

$$\begin{aligned} w &= \frac{P}{\rho} \\ &= \dot{\phi}^2 - 1 \end{aligned} \tag{2.42}$$

where P and ρ are the pressure and energy densities respectively. In tachyon models the field is moving relativistically near the vacuum and the equation of state will tend to $w \sim 0$, which is problematic for reheating. However our model has significantly different late time behaviour because our scalar field will oscillate about the minimum of its potential, eventually coming to a halt at the minimum. Therefore we expect the equation of state to become $w \sim -1$, corresponding to the vacuum energy of the universe. This motivates us to analyse our system as a potential candidate for dark matter. One problem, however, is that the reheating phase doesn't seem to have a natural termination point. Rather, reheating of the universe continues whilst the brane oscillates around the minimum of the potential, and then terminates in what appears to be a dark energy dominated phase. From the perspective of model building this is obviously a difficult problem. For now let us assume that there is some ad hoc mechanism which ends inflation, and look at the evolution of the system in this dark matter dominated phase. The corresponding evolution equations of interest are:

$$\frac{\ddot{\phi}}{1-\dot{\phi}^2} + 3H\dot{\phi} + \frac{V_\phi}{V} = 0 \tag{2.43}$$

$$\dot{H} + \frac{V(\phi)\dot{\phi}^2}{2M_p^2\sqrt{1-\dot{\phi}^2}} + \frac{\gamma\rho_B}{2M_p^2} = 0 \tag{2.44}$$

where we have included contribution from a barotropic fluid in the second equation. Defining the following dimensionless quantities:

$$\begin{aligned} Y_1 &= \frac{\phi}{\sqrt{kl_s^2}} \\ Y_2 &= \dot{\phi}, \end{aligned} \tag{2.45}$$

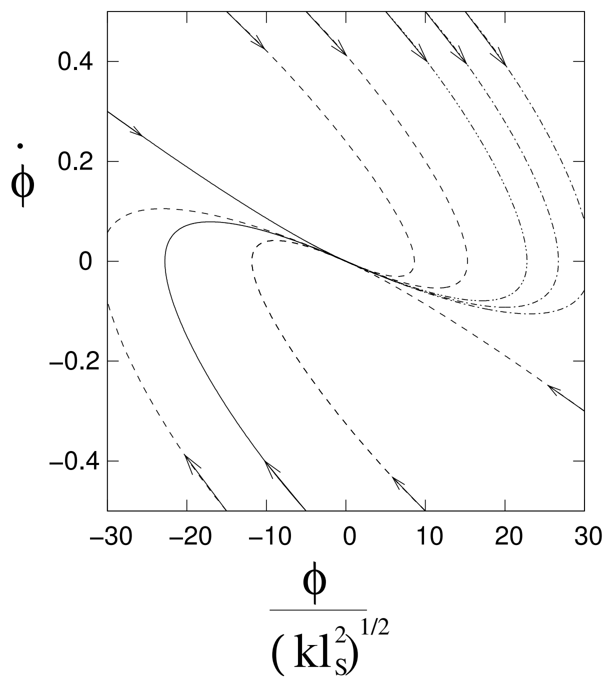
and using eqn(2.43) and eqn(2.45) we get the autonomous equations:

$$Y_1' = \frac{1}{\sqrt{kl_s^2}H} Y_2 \tag{2.46}$$

$$Y_2' = -\left(1 - Y_2^2\right)\left(3Y_2 + \frac{1}{H}\frac{dY_3}{dY_1}\right) \tag{2.47}$$

Where we have switched to using the number of e-folds as the time parameter, and now primes denote derivatives with respect to N. The final expressions we require can be read off as

$$Y_3 = \ln\left(\frac{V(\phi)}{3M_p^2}\right)$$



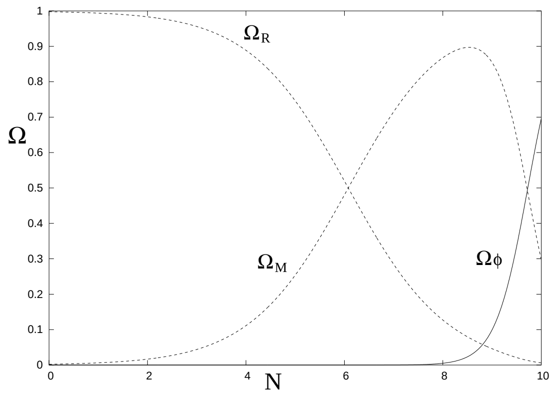
รูปที่ 2.1: Plot of the phase space solution with a variety of initial conditions. Here we see the presence of global attractor at $(\phi = 0, \dot{\phi} = 0)$

$$H^2 = \frac{e^{Y_3}}{\sqrt{1 - Y_2^2}} + \frac{\rho_B}{3M_p^2}. \tag{2.48}$$

Simple analysis shows us that critical point is at $Y_1 = 0$ and $Y_2 = 0$ which is a global attractor. This agrees with our physical intuition since it implies the probe brane will slow down, eventually coming to rest at the origin of the transverse space. In terms of our critical ratios we find

$$\Omega_\phi = \frac{e^{Y_3}}{e^{Y_3} + \frac{\rho_B}{3M_p^2}\sqrt{1-Y_2^2}} \tag{2.49}$$

$$\Omega_B = \frac{\rho_B}{\frac{3M_p^2 e^{Y_3}}{\sqrt{1-Y_2^2}} + \rho_B} \tag{2.50}$$



รูปที่ 2.2: Here we have taken $\rho_m^0 = 4.58 \times 10^6$, $\rho_R^0 = 10^{10}$ and $V_0 = 10^{-6}$. The dark line is for Ω_R, dotted line is for Ω_ϕ and light line is for Ω_m

Note that they are constrained by $\Omega_\phi + \Omega_B = 1$. We also have $\Omega_B = \Omega_M + \Omega_R$, where M and R denote matter and radiation respectively, whilst ϕ is associated with our scalar field.
From the plots fig(2.2) we see that the Ω_ϕ goes to 0.7 and Ω_M goes to 0.3 and Ω_R goes to 0 in the presence epoch. We see that at late times, the field settles at the potential minimum leading to de-Sitter solution with energy scale $V_0 = \tau_3 R/\sqrt{kl_s^2}$. Using the numerical data from the preceding sections we can write this an upper bound on the energy density as follows

$$V_0 \leq 10^{-12} M_p^4. \tag{2.51}$$

Although this is several orders of magnitude higher than the observed value, we note that this value is heavily dependent on the scales in the theory, and with appropriate tuning could be substantially smaller. Since there exists no realistic scaling solution (which could mimic matter/radiation), the model also requires the fine tuning of the initial value of the scalar field. The field should remain sub dominant for most of the cosmic evolution and become comparable to the background at late times. It would then evolve to dominate the background energy density ultimately settling down in the de-Sitter phase.

However, recall from the bulk picture that the point $\sigma = 0, \rho = 0$ will be gravitationally unstable and the probe brane will eventually be attracted toward the ring. In terms of our cosmological theory we see that this de-Sitter point will actually be only quasi-stable and that a tachyonic field will eventually condense forcing the vacuum energy down toward zero. This suggests that the vacuum energy will not be constant, but will slowly varying. Furthermore our equation of state should be modified to incorporate the dynamics of this additional field. It is trivial to see that the inflationary phase will terminate and give way to a dark energy phase where $w \sim -1$. Once the tachyon field starts to roll, w will increase toward zero from below giving rise to a phase of quintessence [83]. Eventually we will begin to probe the strong coupling regime and our effective action will break down.

let us return to the bulk picture to understand this in more detail. We introduce a complex field $\xi = \rho + i\sigma$ which can actually be globally defined in the target space. The harmonic function factorises in this coordinate system into holomorphic and anti-holomorphic parts $F(\xi, \bar{\xi}) = f(\xi) f(\bar{\xi})$. Thus the tachyon map will also split accordingly

$$\partial_t \phi = f(\xi) \partial_t \xi, \qquad \partial_t \bar{\phi} = f(\bar{\xi}) \partial_t \bar{\xi}. \tag{2.52}$$

These expressions are exactly solvable provided we continue them into the complex plane. If we now re-construct the potential for these fields in terms of our holographic theory we obtain the general solution

$$V(\phi, \bar{\phi}) = \frac{R\tau_3}{\sqrt{kl_s^2}} \left[\cos\left(\frac{\phi}{\sqrt{kl_s^2}} \right) \cos\left(\frac{\bar{\phi}}{kl_s^2} \right) \right]^{1/2}. \tag{2.53}$$

Clearly when ϕ is real we recover our $cosine$ potential, whilst if it is purely imaginary we recover the $cosh$ solution. These correspond to motion inside the ring and motion transverse to the ring respectively. The tachyonic instability forces the field from the false vacuum state toward the true ground state. Therefore we expect the dark energy potential to be

$$V(\phi, \bar{\phi}) \sim \frac{R\tau_3}{\sqrt{kl_s^2}} \cos\left(\frac{\phi}{\sqrt{kl_s^2}} \right), \tag{2.54}$$

and so the true minimum will occur when $V \sim 0$ at $\phi = \pm\pi\sqrt{kl_s^2}/2$ corresponding to the location of the ring in the bulk picture. The cosmological dynamics in this particular phase are well described by [61, 59], where it was shown to be possible for the true vacuum to be non-zero, provided the trajectory of the probe brane is sufficiently fine tuned.

We finally comment on the instability for the field fluctuations for potential with a minimum [70]. In a flat FRW background each Fourier mode of ϕ satisfies the following equation

$$\frac{\delta\ddot{\phi}_{\tilde{k}}}{1-\dot{\phi}^2} + \left[3H + \frac{2\dot{\phi}\ddot{\phi}}{(1-\dot{\phi}^2)^2} \right] \delta\dot{\phi}_{\tilde{k}} + \left[\frac{\tilde{k}^2}{a^2} + (\ln V)_{\phi,\phi} \right] \delta\phi_{\tilde{k}} = 0 \tag{2.55}$$

Where $\tilde{k}$ is the comoving wavenumber. We now compute the second derivatives of the potential and obtain

$$(\ln V)_{\phi,\phi} = \frac{1}{kl_s^2} \left(1 - \tanh\left[\frac{\phi}{\sqrt{kl_s^2}} \right] \right). \tag{2.56}$$

Here we see that $(\ln V)_{\phi,\phi}$ is never divergent for any value of ϕ, and is always non-negative i.e that $(\ln V)_{\phi,\phi} \in [0,1]$. Thus we do not have any instability associated with the perturbation $\delta\phi_k$ with our potential (2.9). This is to be contrasted with the result obtained for the open string tachyon. which has rapid fluctuations and instabilities associated with its evolution.

2.5 Conclusion and comment

In this note we have examined the time dependant configuration of a single $D3$ brane in the background of $NS5$ branes distributed on a ring of radius R, taking the near horizon approximation. We then studied the cosmological implications of the effective potential which arises due to the transverse motion of $D3$ with respect to the plane of the ring. The model appears to describe an inflationary phase giving way to a natural reheating mechanism, and then a further phase of dark energy driven expansion. Although we cannot accurately predict the scale of the energy density at this point, we do obtain an upper bound. In this case the dark energy phase is a late time attractor of our model, and we predict that the vacuum energy will eventually decay to zero - although on extremely large time-scales [3]. In fact our results will be dramatically improved by keeping the full structure of the harmonic function, because at large distances the potential is even flatter yielding even more e-foldings of inflation. Due to the absence of scaling solutions in our field theory, we need to tune the initial value of the scalar field such that it can become relevant only at late times. With these described fine tunings, the geometrical field is a potential candidate for dark energy. The model is free from tachyon instabilities, and the field perturbations behave in a similar manner to those of the canonical scalar field.

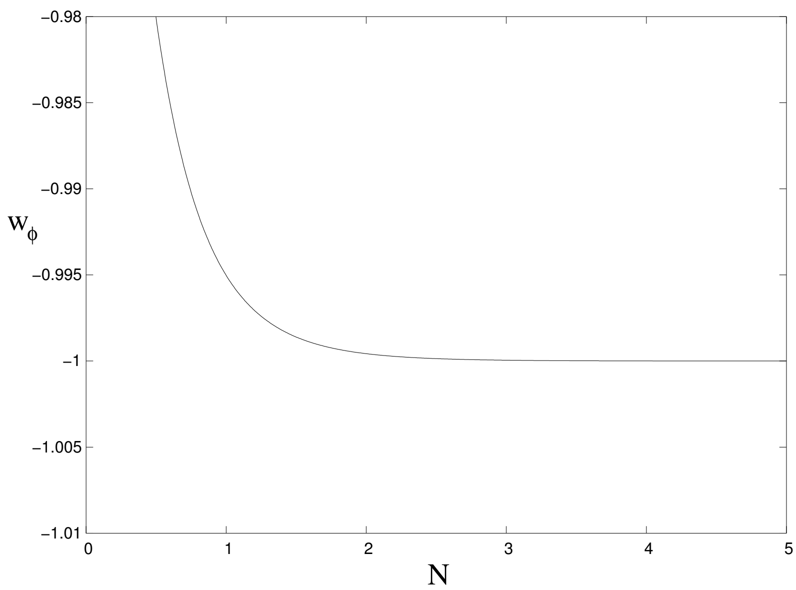
Of the model we have several potential problems. Firstly our assumption about the coupling of the DBI to four-dimensional gravity, although as we have pointed out this can be resolved by a full string theory compactification. However there will generally be large corrections, potentially destroying the simplicity of the solution. Secondly the trajectory of the brane in the bulk space is particularly special. In the most generic case we would anticipate a general spiralling trajectory toward the ring. In this case there would be no simple decoupling of the modes and we would need to consider the full form of the potential. This amounts to a certain amount of fine tuning of the initial conditions. Another problem is that we have not turned on any standard model fields which would be expected to couple to the inflaton on the world-volume. However the inclusion of U(1) gauge fields on the brane will act to reduce the velocity of the field by a factor of $\sqrt{1-E^2}$, where E is our dimensionless electric field. More importantly however is that we have neglected the induced two-form field strength, which can have important applications in cosmology as seen in the recent paper [82]. Despite these problems, we know there is a coset model describing the background which opens the way for exact string theory calculations. Furthermore the

[3] However we must be careful since the DBI action will not be valid once it coalesces with the $NS5$-branes so we must assume that it passes between the branes. This requires fine tuning of the initial trajectory which is not realistic. This problem may be resolved by switching to the description of the model in terms of Little String Theory [77].

relationship between the two energy scales in the theory means it is possible to talk about long-standing problems such as the Transplanckian issue [80]. One further problem is the termination of reheating in this model. We have emphasised that this is indeed difficult to tackle in this model due to its analytic simplicity. One may hope that a careful analysis of the tachyon mapping will lead to more realistic behaviour for the inflaton field, and thus a possible exit from reheating. In fact this may also be possible by considering more general trajectories of the probe brane in the bulk picture. We hope to return to this issue in a future publication.

One thing that emerges though is the relationship between a dark energy dominated phase and the 'fast rolling' DBI action [52]. Although our proposal is far from rigorous, it does capture the majority of the same physics as in the flux compactification scenario. We know that D-branes moving in non-trivial backgrounds have sub-luminal velocities as measured by observers in the far UV of the geometry, due to the gravitational red-shifting. In fact the branes are decelerating and for late times will have negligible velocities. This in turn implies that the equation of state parameter will tend to $w \sim -1$ at late times. A concrete example where this could be examined is in the case of the warped deformed conifold [81]. The RR flux will wrap the S^3 in the IR end of the geometry, and we can imagine a solitary $D3$-brane probing this part of the conifold after an inflationary phase. To an observer in the compact space the brane will slow down as it reaches the origin of the S^3 yielding a dark matter dominated phase [78].

However our model opens up the possibility that non-trivial background configurations may have important implications for brane cosmology, as we have seen how to combine inflation, reheating and dark energy in a single model. Furthermore this is not subject to the same landscape problems as the flux compactification models, and we can try and tackle higher energy issues in a clear formalism [51]. Although we acknowledge the simplicity of our solution we hope that this will encourage more research in this direction.



รูปที่ 2.3: Evolution of the equation of state parameter with the number of e-folds. Note that w rapidly approaches -1 as expected.

บทที่ 3

Phantom field in loop quantum cosmological background

Phantom dominated FRW universe possesses singularity problem as stated in Chapter 1. Then instead of using standard FRW cosmology, the fundamental background theory in which we are interested is Loop Quantum Gravity-LQG. This theory is a non-perturbative type of quantization of gravity and is background-independent [84, 85]. It has been applied in cosmological context as seen in various literatures where it is known as Loop Quantum Cosmology-LQC (for review, see Ref. [86]). Effective loop quantum modifies standard Friedmann equation by adding a correction term $-\rho^2/\rho_{\rm lc}$ into the Friedmann equation [88, 89, 90, 87, 91]. When this term becomes dominant, the universe begins to bounce and then expands backwards. LQG can resolve of singularity problem in various situations [85, 92, 93, 88]. However, derivation of the modified term is under a condition that there is no matter potential otherwise, in presence of a potential, quantum correction would be more complicated [94]. Nice feature of LQC is avoidance of the future singularity from the correction quadratic term $-\rho^2/\rho_{\rm lc}$ in the modified LQC Friedmann equation [95] as well as the singularity avoidance at semi-classical regime [96]. The early-universe inflation has also been studied in the context of LQC at semi-classical limit [98, 97, 91, 99, 100, 101]. We aim to investigate dynamics of the phantom field and its late time behavior in the loop quantum cosmological context, and to check if the loop quantum effect could remove Big Rip singularity from the phantom dominated universe. The study could also reveal some other interesting features of the model.

We organize this article as follows: in section 3.1, we introduce LQC Friedmann equation, after that we briefly present relevant features of the phantom scalar field in section 3.2. Section 3.3 contains dynamical analysis of the phantom field in LQC background with exponential potential. The potential is a simplest case due to constancy of its steepness variable λ. Two real fixed points are found in this section. Stability analysis yields that both fixed points are saddle points. Numerical results and analysis of solutions can be seen in section 3.4 where we give conditions for physical solutions. Finally, conclusion is in section 3.5.

3.1 Loop quantum cosmology

LQC naturally gives rise to inflationary phase of the early universe with graceful exit, however the same mechanism leads to a prediction that present-day acceleration must be very small [97]. At late time and at large scale, the semi-classical approximation in LQC formalisms can be validly used [102]. The effective Friedmann equation can be obtained by using an effective Hamiltonian with loop quantum modifications [89, 95, 103]:

$$\mathcal{C}_{\rm eff} \;=\; -\frac{3M_{\rm P}^2}{\gamma^2\bar{\mu}^2}\,a\sin^2(\bar{\mu}\mathfrak{c}) + \mathcal{C}_{\rm m}\,. \tag{3.1}$$

The effective constraint (3.1) is valid for isotropic model and if there is scalar field, the field must be free, massless scalar field. The equation (3.1), when including field potential, must have some additional correction terms [94]. In this scenario, the Hamilton's equation is

$$\dot{\mathfrak{p}} \;=\; \{\mathfrak{p}, \mathcal{C}_{\rm eff}\} = -\frac{\gamma}{3M_{\rm P}^2}\frac{\partial \mathcal{C}_{\rm eff}}{\partial \mathfrak{c}}\,, \tag{3.2}$$

where $\mathfrak{c}$ and $\mathfrak{p}$ are respectively conjugate connection and triad satisfying $\{\mathfrak{c}, \mathfrak{p}\} = \gamma/3M_{\rm P}^2$. Dot symbol denotes time derivative. These are two variables in the simplified phase space structure under FRW symmetries [86]. Here $M_{\rm P}^2 = (8\pi G)^{-1}$ is square of reduced Planck mass, G is Newton's gravitational constant and γ is Barbero-Immirzi dimensionless parameter. There are relations between the two variables to scale factor as $\mathfrak{p} = a^2$ and $\mathfrak{c} = \gamma\dot{a}$. The parameter $\bar{\mu}$ is inferred as kinematical length of the square loop since its order of magnitude is similar to that of length. The area of the loop is given by minimum eigenvalue of LQG area operator. $\mathcal{C}_{\rm m}$ is the corresponding matter Hamiltonian. Using the Eq. (3.2) with constraint from realization that loop quantum correction of effective Hamiltonian $\mathcal{C}_{\rm eff}$ is small at large scale, $\mathcal{C}_{\rm eff} \approx 0$ [86, 90, 89, 95], one can obtain (effective) modified Friedmann equation in flat universe:

$$H^2 \;=\; \frac{\rho_{\rm t}}{3M_{\rm P}^2}\left(1 - \frac{\rho_{\rm t}}{\rho_{\rm lc}}\right)\,, \tag{3.3}$$

where $\rho_{\rm lc} = \sqrt{3}/(16\pi\gamma^3G^2\hbar)$ is critical loop quantum density, $\hbar$ is Planck constant and $\rho_{\rm t}$ is total density.

3.2 Phantom canonical scalar field

The energy density ρ and the pressure p of the phantom field contain negative kinetic term. They are given as [16]

$$\rho \;=\; -\frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{3.4}$$

$$p \;=\; -\frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{3.5}$$

The conservation law is

$$\dot{\rho} + 3H(\rho + p) = 0\,. \tag{3.6}$$

Using the Eqs. (3.4), (3.5) and (3.6), we obtain Klein-Gordon equation:

$$\ddot{\phi} + 3H\dot{\phi} - V' = 0\,, \tag{3.7}$$

where $V' \equiv \mathrm{d}V/\mathrm{d}\phi$ and the negative sign comes from the negative kinetic terms. The phantom equation of state is therefore given by

$$w = \frac{p}{\rho} = \frac{\dot{\phi}^2 + 2V}{\dot{\phi}^2 - 2V}\,. \tag{3.8}$$

From the Eq. (3.8), when the field is slowly rolling, as long as the approximation, $\dot{\phi}^2 \sim 0$ holds, the approximated value of w is -1. When the bound, $\dot{\phi}^2 < 2V$ holds, w is always less than -1.

There has not yet been a derivation of effective LQC Friedmann equation in consistence with a presence of potential. Even though, the Friedmann background is valid only in absence of field potential, however, investigation of a phantom field evolving under a potential is a challenged task. Here we also neglect loop quantum correction effect in the classical expression of Eqs. (3.4) and (3.5) (see Refs. [94] and [104] for discussion).

3.3 Dynamical analysis

Differentiating the Eq. (5.1) and using the fluid Eq. (3.6), we obtain

$$\dot{H} = -\frac{(\rho + p)}{2M_{\mathrm{P}}^2}\left(1 - \frac{2\rho}{\rho_{\mathrm{lc}}}\right)\,. \tag{3.9}$$

The Eqs. (5.1) , (3.6) and (3.9), in domination of the phantom field, become

$$H^2 = \frac{1}{3M_{\mathrm{P}}^2}\left(-\frac{\dot{\phi}^2}{2} + V\right)\left(1 - \frac{\rho}{\rho_{\mathrm{lc}}}\right)\,, \tag{3.10}$$

$$\dot{\rho} = -3H\rho\left(1 + \frac{\dot{\phi}^2 + 2V}{\dot{\phi}^2 - 2V}\right)\,, \tag{3.11}$$

$$\dot{H} = \frac{\dot{\phi}^2}{2M_{\mathrm{P}}^2}\left(1 - \frac{2\rho}{\rho_{\mathrm{lc}}}\right)\,. \tag{3.12}$$

We define dimensionless variables following the style of [114]

$$X \equiv \frac{\dot{\phi}}{\sqrt{6}M_{\mathrm{P}}H}\,,\ \ Y \equiv \frac{\sqrt{V}}{\sqrt{3}M_{\mathrm{P}}H}\,,\ \ Z \equiv \frac{\rho}{\rho_{\mathrm{lc}}}\,, \tag{3.13}$$

$$\lambda \equiv -\frac{M_{\mathrm{P}}V'}{V}\,,\ \ \Gamma \equiv \frac{V\,V''}{(V')^2}\,,\ \ \frac{\mathrm{d}}{\mathrm{d}N} \equiv \frac{1}{H}\frac{\mathrm{d}}{\mathrm{d}t}\,, \tag{3.14}$$

where $N \equiv \ln a$ is e-folding number. Using new variables in Eqs. (3.8) and (3.10), the equation of state is rewritten as[1]

$$w = \frac{X^2 + Y^2}{X^2 - Y^2}\,, \tag{3.15}$$

where $|X| \neq |Y|$ and the Friedmann constraint is reexpressed as

$$(-X^2 + Y^2)(1 - Z) = 1\,. \tag{3.16}$$

Clearly, if $|X| \neq |Y|$, following the Eq. (3.16), then $Z \neq 1$. Using the new defined variables above, Eq. (3.12) becomes

$$\frac{\dot{H}}{H^2} = 3X^2(1 - 2Z)\,. \tag{3.17}$$

The acceleration condition,

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0, \tag{3.18}$$

in expression of the new variables, is therefore

$$3X^2(2Z - 1) < 1\,. \tag{3.19}$$

Divided by the Eq. (3.16), the acceleration condition under the constraint is

$$\frac{3}{1 - (Y^2/X^2)}\left(\frac{1 - 2Z}{1 - Z}\right) < 1\,, \tag{3.20}$$

where the conditions $|X| \neq |Y|$ and $Z \neq 1$ must hold. As we consider $Z = \rho/\rho_{\rm lc}$ with $\rho = -(\dot{\phi}^2/2) + V$, we can write

$$\frac{\rho_{\rm lc} Z}{3M_{\rm P}^2 H^2} \quad = \quad -X^2 + Y^2\,. \tag{3.21}$$

With the condition $|X| \neq |Y|$, clearly from Eq. (3.21), we have one additional condition, $Z \neq 0$.

3.3.1 Autonomous system

Differential equations in autonomous system are

$$\frac{{\rm d}X}{{\rm d}N} \quad = \quad -3X - \sqrt{\frac{3}{2}}\,\lambda Y^2 - 3X^3\,(1 - 2Z)\,, \tag{3.22}$$

$$\frac{{\rm d}Y}{{\rm d}N} \quad = \quad -\sqrt{\frac{3}{2}}\,\lambda XY - 3X^2Y\,(1 - 2Z)\,, \tag{3.23}$$

$$\frac{{\rm d}Z}{{\rm d}N} \quad = \quad -3Z\left(1 + \frac{X^2 + Y^2}{X^2 - Y^2}\right)\,, \tag{3.24}$$

$$\frac{{\rm d}\lambda}{{\rm d}N} \quad = \quad -\sqrt{6}(\Gamma - 1)\lambda^2 X\,. \tag{3.25}$$

[1]The relation $\Omega_\phi = \rho/3H^2M_{\rm P}^2 = -X^2 + Y^2 = 1$ can not be applied here since it is valid only for standard cosmology with flat geometry.

Name	X	Y	Z	Existence	Stability	w	Acceleration
(a)	$-\frac{\lambda}{\sqrt{6}}$	$\sqrt{1+\frac{\lambda^2}{6}}$	0	All λ	Saddle point for all λ	$-1-\frac{\lambda^2}{3}$	For all λ
(b)	$-\frac{\lambda}{\sqrt{6}}$	$-\sqrt{1+\frac{\lambda^2}{6}}$	0	All λ	Saddle point for all λ	$-1-\frac{\lambda^2}{3}$	For all λ

ตารางที่ 3.1: Properties of fixed points of phantom field dynamics in LQC background under the exponential potential.

Here we will apply exponential potential,

$$V(\phi) = V_0 \exp\left(-\frac{\lambda}{M_{\rm P}}\phi\right), \tag{3.26}$$

to this system. The potential is known to yield power-law inflation in standard cosmology with canonical scalar field. Its slow-roll parameters are related as $\epsilon = \eta/2 = 1/P$ where $\lambda = \sqrt{2/P}$ and $P > 1$ [165, 107]. Although the potential has been applied to the quintessence scalar field with tracking behavior in standard cosmology [108], the quintessence field can not dominate the universe due to constancy of the ratio between densities of matter and quintessence field (see discussion in Ref. [132]). In case of phantom field in standard cosmology under this potential, a stable node is a scalar-field dominated solution with the equation of state, $w = -1 - \lambda^2/3$ [27, 24, 109]. In our LQC phantom domination context, from Eq. (3.25), we can see that for the exponential potential, $\Gamma = 1$. This yields trivial value of $\mathrm{d}\lambda/\mathrm{d}N$ and therefore λ is a non-zero constant otherwise the potential is flat.

3.3.2 Fixed points

Let $f \equiv \mathrm{d}X/\mathrm{d}N, g \equiv \mathrm{d}Y/\mathrm{d}N$ and $h \equiv \mathrm{d}Z/\mathrm{d}N$. We can find fixed points of the autonomous system under condition:

$$(\,f\,,\ g\,,\ h\,)\,|_{(X_{\rm c}\,,\,Y_{\rm c}\,,\,Z_{\rm c})} = 0\,. \tag{3.27}$$

The are two real fixed points of this system: [2]

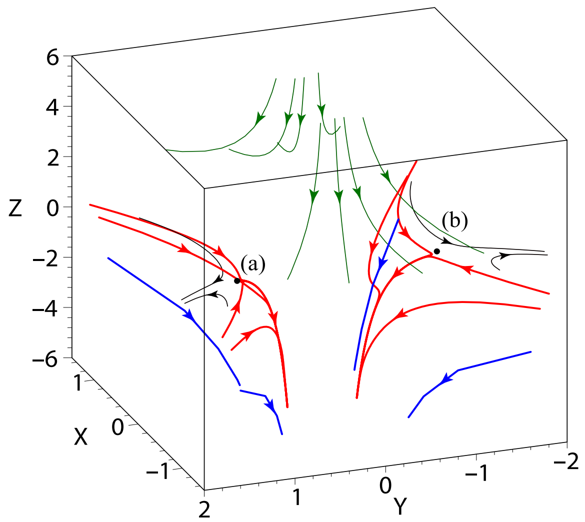
- $$\text{Point (a)} : \left(\frac{-\lambda}{\sqrt{6}},\quad \sqrt{1+\frac{\lambda^2}{6}}\,,\ 0\right), \tag{3.28}$$
- $$\text{Point (b)} : \left(\frac{-\lambda}{\sqrt{6}}, -\sqrt{1+\frac{\lambda^2}{6}}\,,\ 0\right). \tag{3.29}$$

3.3.3 Stability analysis

Suppose that there is a small perturbation δX, δY and δZ about the fixed point $(X_{\rm c}, Y_{\rm c}, Z_{\rm c})$, i.e.,

$$X = X_{\rm c} + \delta X\,, \quad Y = Y_{\rm c} + \delta Y\,, \quad Z = Z_{\rm c} + \delta Z\,. \tag{3.30}$$

[2]The other two imaginary fixed points $(i, 0, 0)$ and $(-i, 0, 0)$ also exist. However they are not interesting here since we do not consider model that includes complex scalar field.



รูปที่ 3.1: Three-dimensional phase space of X, Y and Z. The saddle points (a) (-0.40825, 1.0801, 0) and (b) (-0.40825, -1.0801, 0) appear in the figure. λ is set to 1. In region $Z < 0$, the solutions (red and blue lines) are non physical. In this region, $Z \to -\infty$ when $(X, Y) \to (0, 0)$. The green lines (class I) are in region $|X| > |Y|$ and $Z > 1$ but they are also non physical since they correspond to imaginary H values. The only set of physical solutions (class II) is presented with black lines. They are in region $|Y| > |X|$ and range $0 < Z < 1$. This is the region above (a) and (b) of which H takes real value. There are separatices $|X| = |Y|$, $Z = 0$ and $Z = 1$ in the system (see section 3.4.2).

From Eqs. (3.22), (3.23) and (3.24), neglecting higher order terms in the perturbations, we obtain first-order differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}N}\begin{pmatrix}\delta X\\ \delta Y\\ \delta Z\end{pmatrix} = \mathcal{M}\begin{pmatrix}\delta X\\ \delta Y\\ \delta Z\end{pmatrix}. \tag{3.31}$$

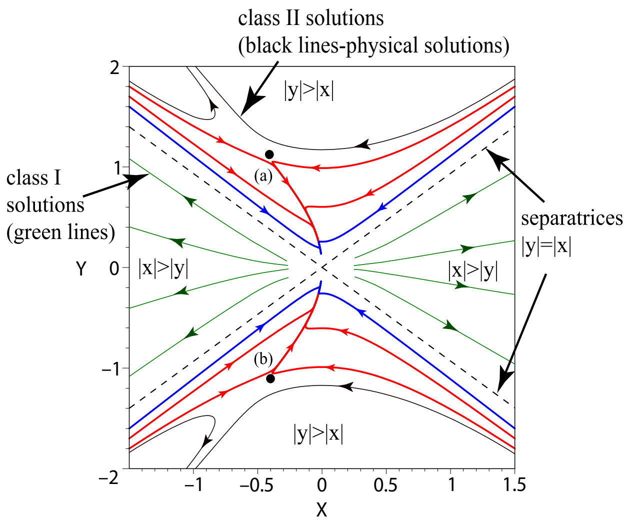
The matrix $\mathcal{M}$ defined at a fixed point (X_c, Y_c, Z_c) is given by

$$\mathcal{M} = \begin{pmatrix}\frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} & \frac{\partial f}{\partial Z}\\ \frac{\partial g}{\partial X} & \frac{\partial g}{\partial Y} & \frac{\partial g}{\partial Z}\\ \frac{\partial h}{\partial X} & \frac{\partial h}{\partial Y} & \frac{\partial h}{\partial Z}\end{pmatrix}_{(X=X_c, Y=Y_c, Z=Z_c)}. \tag{3.32}$$

We find eigenvalues of the matrix $\mathcal{M}$ for each fixed point:

- At point (a):

$$\mu_1 = \lambda^2, \quad \mu_2 = -\lambda^2, \quad \mu_3 = -3 - \frac{\lambda^2}{2}. \tag{3.33}$$



รูปที่ 3.2: Phase space of the kinetic part X and potential part Y (top view). The saddle points (a) (-0.40825, 1.0801) and (b) (-0.40825, -1.0801) are shown here. The blue lines and red lines are in the region $Z < 0$ which is non physical. Green lines are of class I solutions which yields imaginary H. Only class II solutions shown as black lines are physical with real H value.

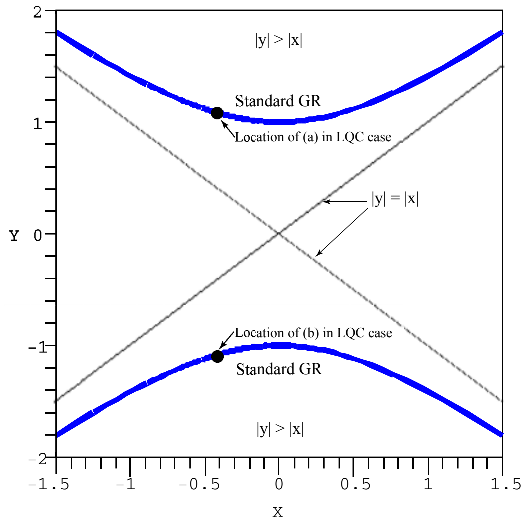
- At point (b):

$$\mu_1 = \lambda^2\,, \quad \mu_2 = -\lambda^2\,, \quad \mu_3 = -3 - \frac{\lambda^2}{2}\,. \tag{3.34}$$

From the above analysis, each point possesses eigenvalues with opposite signs, therefore both point (a) and (b) are saddle. Results from our analysis are concluded in TABLE 3.1. Location of the points depends only on λ and the points exist for all values of λ. Both points correspond to the equation of state $-1 - \lambda^2/3$, that is to say, it has phantom equation of state for all values of $\lambda \neq 0$. Since there is no any attractor in the system, a phase trajectory is very sensitive to initial conditions given to the system. The stable node (the Big Rip) of the standard general relativistic case in presence of phantom field and a barotropic fluid, disappears here (see [23]).

3.4 Numerical results

Numerical results from the autonomous set (3.22), (3.23) and (3.24) are presented in Figs. 3.1 and 3.2 where we set $\lambda = 1$. Locations of the two saddle points are: point (a) $(X_c = -0.40825, Y_c = 1.0801, Z_c = 0)$ and point (b) $(X_c = -0.40825, Y_c = -1.0801, Z_c = 0)$ which match our analytical results. In Fig. 3.3, we present a trajectory solution of a phantom field evolving in standard cosmological background for comparing with the trajectories in Fig. 3.2 when including loop quantum effects. The



รูปที่ 3.3: Phase space of the kinetic part X and potential part Y in standard general relativistic case. The location of points (a) and (b) in Fig. 3.2 are on the trajectory solutions here. This plot shows dynamics of phantom field in standard cosmological background without any other fluids. In presence of a barotropic fluid with any equation of state, the point (a) and (b) correspond to the Big Rip [23, 25].

standard case has only simple two trajectories corresponding to a constraint $-X^2 + Y^2 = 1$. This is attained when taking classical limit, $Z = 0$. In loop quantum case, since there is no any stable node and the solutions are sensitive to initial conditions, we need to classify solutions according to each domain region separated by separatrices $|X| = |Y|$, $Z = 0$ and $Z = 1$ so that we can analyze them separately. Note that the condition, $Z > 0$ must hold for physical solutions since the density can not be negative or zero, i.e. $\rho > 0$. The blue lines and red lines in Figs. 3.1 and 3.2 are solutions in the region $Z < 0$ hence are not physical and will no longer be of our interest. From now on we consider only the region $Z > 0$. In regions with $|X| > |Y|$, the solutions therein are green lines (hereafter classified as class I). The other regions with $|Y| > |X|$ contain solutions seen as black line (classified as class II). Note that all solutions can not cross the separatices due to conditions in Eqs. (3.16), (3.20) and (3.21).)

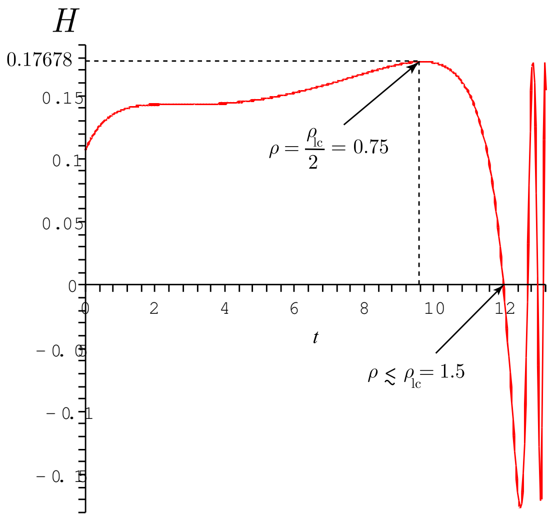
3.4.1 Class I solutions

Consider the Friedmann equation (3.10), the Hubble parameter, H takes real value only if

$$\frac{1}{3M_{\rm P}^2}\left(-\frac{\dot{\phi}^2}{2} + V\right)\left(1 - \frac{\rho}{\rho_{\rm lc}}\right) \quad \geq \quad 0\,. \tag{3.35}$$

Divided by H^2 on both sides, the expression above becomes

$$(-X^2 + Y^2)(1 - Z) \quad \geq \quad 0\,. \tag{3.36}$$



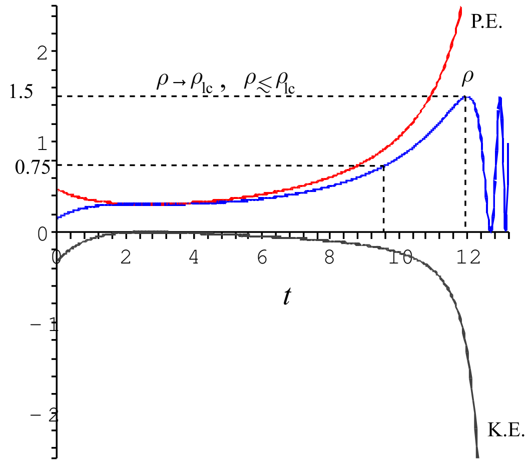
รูปที่ 3.4: Evolution of H with time of a class II solution. Set values are $\lambda = 1, \rho_{\rm lc} = 1.5, V_0 = 1$ and $M_{\rm P} = 2$. The universe undergoes acceleration from the beginning until reaching turning point at $\rho = \rho_{\rm lc}/2 = 0.75$ where $H = H_{\rm max} = 0.17678$. Beyond this point, the universe expands with deceleration until halting ($H = 0$) at $\rho \approx \rho_{\rm lc} = 1.5$. After halting, it undergoes contraction until H bounces. The oscillating in H goes on forever.

It is clear from (3.36) that, in order to obtain real value of H, class I solutions (green line) must obey both conditions $|X| > |Y|$ and $Z > 1$ together. However, when imposing $|X| > |Y|$ to the Eq. (3.21), we obtain $Z < 0$ instead. This contradicts to the required range $Z > 1$. Therefore this class of solutions does not possess any real values of H and hence not physical solutions.

3.4.2 Class II solutions

Proceeding the same analysis done for class I, we found that in order for H to be real, class II solutions must obey both $|Y| > |X|$ and $0 < Z < 1$ together. Moreover when imposing $|Y| > |X|$ into Eq. (3.21), we obtain $Z > 0$. Therefore as we combine both results, it can be concluded that class II solutions can possess real H value in the region $|Y| > |X|$ and $0 < Z < 1$, i.e. $0 < \rho < \rho_{\rm lc}$. The bound is slightly different from the case of canonical scalar field in LQC (see Ref. [110]) of which the bound is $0 \leq \rho \leq \rho_{\rm lc}$. The class II is therefore the only class of physical solutions.

For class II solutions, we consider another set of autonomous equations from which the evolution of cosmological variables are conveniently obtained by using numerical approach. In the new autonomous set, instead of using N, which could decrease after the bounce from LQC effect, time is taken as independent



รูปที่ 3.5: Time evolution of potential energy density (P.E.), kinetic energy density (K.E.) and $\rho =$ K.E. + P.E. of the field for a class II solution. K.E. is always negative and, at late time, it goes to $-\infty$ while P.E. is always positive. ρ is maximum when $\rho \approx \rho_{\rm lc} = 1.5$. Other features are discussed as in Fig. 3.4.

variable. We define new variable as

$$\dot{\phi} \;=\; S\,. \tag{3.37}$$

The Eqs. (3.7) and (3.12) are therefore rewritten as

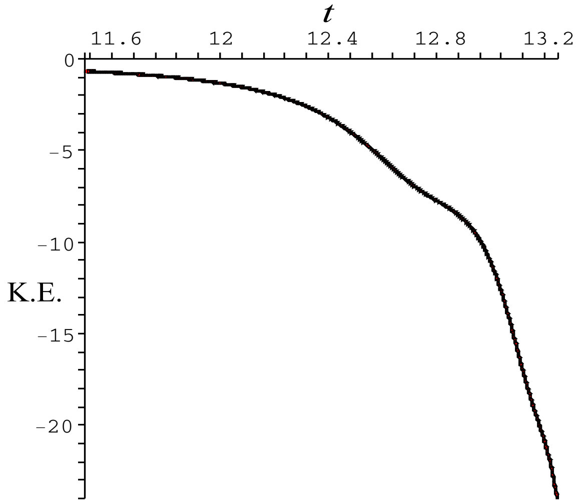
$$\dot{H} \;=\; \frac{S^2}{2M_{\rm P}^2}\left[1-\frac{2}{\rho_{\rm lc}}\left(-\frac{S^2}{2}+V(\phi)\right)\right]\,, \tag{3.38}$$

$$\dot{S} \;=\; -3HS+V'\,. \tag{3.39}$$

The Eqs. (3.37), (3.38) and (3.39) form another closed autonomous system. Numerical integrations from the new system yield result plotted in Figs. 3.4 and 3.5 in which set values are $\lambda = 1, \rho_{\rm lc} = 1.5, V_0 = 1$ and $M_{\rm P} = 2$. From Eq. (5.1) the slope of H with respect to ρ, ${\rm d}H/{\rm d}\rho$, is flat when $\rho = \rho_{\rm lc}/2$ [110]. Another fact is

$$\left(\frac{{\rm d}^2 H}{{\rm d}\rho^2}\right)_{\rho=\rho_{\rm lc}/2} = \frac{-2}{M_{\rm P}\sqrt{3\rho_{\rm lc}^3}} \;<\; 0\,, \tag{3.40}$$

hence, as $\rho = \rho_{\rm lc}/2$, H takes maximum value, $H_{\rm max} = \sqrt{\rho_{\rm lc}/12M_{\rm P}^2}$. This result is valid in LQC scenario regardless of types of fluid. In Figs. 3.4 and 3.5, with set parameters given above, as $\rho = \rho_{\rm lc}/2 = 0.75$, H is maximum, $H_{\rm max} = 0.17678$. When $H \approx 0$, i.e. ρ is approximately $\rho_{\rm lc} = 1.5$, the expansion halts and then bounces. At this bouncing point, the dynamics enters loop quantum regime which is a quantum gravity limit. Beyond the bounce, H turns negative, i.e. contracting of scale factor. The



รูปที่ 3.6: Oscillation in kinetic energy density (K.E.) that contributes to oscillation in ρ. This is a zoom-in portion of the Fig. 3.5.

universe undergoes accelerating contraction until reaching $H_{\rm min}$. After that it contracts deceleratingly until bouncing at $H \approx 0$. The universe goes on faster bouncing forward and backward. The faster bounce in H is an effect from the faster bounce in ρ as illustrated in Fig. 3.5 where the red line represents potential energy density $V(\phi)$, the black line represents kinetic energy density $-\dot{\phi}^2/2$ and the blue line is total energy density ρ. Oscillation in ρ is from oscillation in the field speed $\dot{\phi}$ and therefore oscillation in K.E. as shown in Fig. 3.6. This hence contributes to oscillation in ρ. The negative magnitude of kinetic energy density becomes larger and larger as the field rolling faster and faster up the potential. The exponential potential energy density therefore becomes larger and larger. However, magnitude of potential part is always greater or equal to that of the kinetic part. Therefore sum of them, ρ, is positive. Numerical result shows oscillation in ρ. This results in oscillation of ρ and affects in oscillation of H about the bounce $H = 0$. With a different approach, recently a similar result in H oscillation is also obtained by Naskar and Ward [111].

3.5 Conclusion and comment

A dynamical system of phantom canonical scalar field evolving in background of loop quantum cosmology is considered and analyzed in this work. Exponential potential is used in this system. Dynamical analysis of autonomous system renders two real fixed points $(-\lambda/\sqrt{6},\ \sqrt{1+\lambda^2/6}\,,\ 0\,)$ and $(-\lambda/\sqrt{6}, -\sqrt{1+\lambda^2/6}\,,\ 0\,)$, both of which are saddle points corresponding to equation of state, $w = -1-\lambda^2/3$. Note that in case of standard cosmology, the fixed point $(X_{\rm c}, Y_{\rm c})$ = $(-\lambda/\sqrt{6},\ \sqrt{1+\lambda^2/6})$ is the Big Rip attractor with the same equation of state, $w = -1-\lambda^2/3$ [24]. Due to absence of stable node, the late time behavior depends on initial conditions given. Therefore we do numerical plots to investigate solutions of the system and then classify the solutions. Separatrix conditions $|X| \neq |Y|$, $Z \neq 1$ and $Z \neq 0$ arise from equation of state (3.15), Friedmann constraint (3.16) and definition of Z in Eq. (3.21). At first, we consider solutions in region $Z > 0$, i.e. $\rho > 0$ for physical solutions. Secondly, within this $Z > 0$ region, we classify solutions into class I & II. Solutions in region $|X| > |Y|$ and $Z > 1$ are of class I. However, in order to obtain real value of H in class I, Z must be negative which contradicts to $Z > 1$. Therefore the class I solutions are non physical. Class II set is identified by $|Y| > |X|$ and $0 < Z < 1$. It is an only set of physical solutions since it yields real value of H. In class II set, the universe undergoes accelerating expansion from the beginning until $\rho = \rho_{\rm lc}/2$ where $H = H_{\rm max} = \sqrt{\rho_{\rm lc}/12M_{\rm P}^2}$. After that the universe expands deceleratingly until it bounces, i.e. stops expansion $H \approx 0$ at $\rho \approx \rho_{\rm lc}$. At the bounce the universe enters quantum gravity regime. Contraction with backward acceleration happens right after the bounce, however the contraction does not go on forever. When the universe reaches minimum value of negative H, the contraction turns decelerated, i.e. contracts slower and slower down. The universe, after undergoing contraction to minimum spatial size, bounces again and starts to expand acceleratingly. Our numerical results yield that oscillation in H becomes faster as time passes.

บทที่ 4

Dynamics of phantom scalar field coupling to matter in loop quantum cosmological background

Here we consider both components coupling to each other. Fluid equations for coupled scalar fields proposed by [112] assuming flat standard FRW universe are

$$\dot{\rho}_\phi + 3H(1+w_\phi)\rho_\phi = -Q\rho_{\rm m}\dot{\phi}\,, \tag{4.1}$$

$$\dot{\rho}_{\rm m} + 3H(1+w_{\rm m})\rho_{\rm m} = +Q\rho_{\rm m}\dot{\phi}\,. \tag{4.2}$$

These fluid equations contain a constant coupling between dark matter (the barotropic fluid) and dark energy (the phantom scalar field) as in [113]. Eqs. (4.1) and (4.2) can also be assumed as conservation equations of fluids in the LQC. Total action for matter and phantom scalar field is [112]

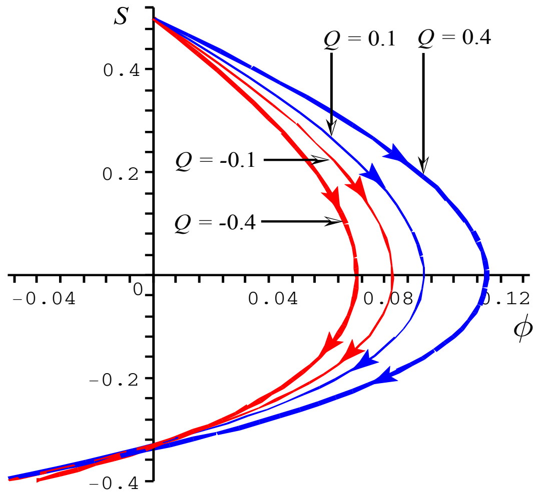
$$S = \int {\rm d}^4x\sqrt{-g}\left[\frac{M_{\rm P}^2}{2}R + p(X,\phi)\right] + S_{\rm m}(\phi)\,. \tag{4.3}$$

Assuming scaling solution of the dark energy in standard cosmology, the pressure Lagrangian density is written as

$$p(X,\phi) = -X - c\exp(-\lambda\phi/M_{\rm P}^2)\,, \tag{4.4}$$

where X is the kinetic term, $-g^{ab}\partial_a\phi\partial_b\phi/2$ of the Lagrangian density (**??**) and (4.4). The second term on the right of Eq. (4.4) is exponential potential, $V(\phi) = c\exp(-\lambda\phi/M_{\rm P}^2)$ which gives scaling solution for canonical and phantom ordinary scalar field in standard general relativistic cosmology when steepness of the potential, λ is fine tuned as

$$\lambda = Q\frac{1+w_{\rm m} - \Omega_\phi(w_{\rm m}-w_\phi)}{\Omega_\phi(w_{\rm m}-w_\phi)}\,. \tag{4.5}$$



รูปที่ 4.1: Phase portrait of $S(t)$ versus $\phi(t)$ for $Q = -0.4, -0.1, 0.1$ and 0.4 from left to right. All trajectories have the same initial conditions $S(0) = 0.5$ and $\phi(0) = 0$.

The steepness (4.5) is, in standard cosmological circumstance, constant in the scaling regime due to constancy of w_ϕ and Ω_ϕ [112, 25]. However, in LQC case, there has been a report recently that the scaling solution does not exist for phantom field evolving in LQC [?]. Therefore our spirit to consider constant λ is the same as in [114] not a motivation from scaling solution as in [112]. The exponential potential is also originated from fundamental physics theories such as higher-order gravity [115] or higher dimensional gravity [116].

4.1 Cosmological dynamics

Time derivative of the effective LQC Friedmann equation LQC (5.1) is

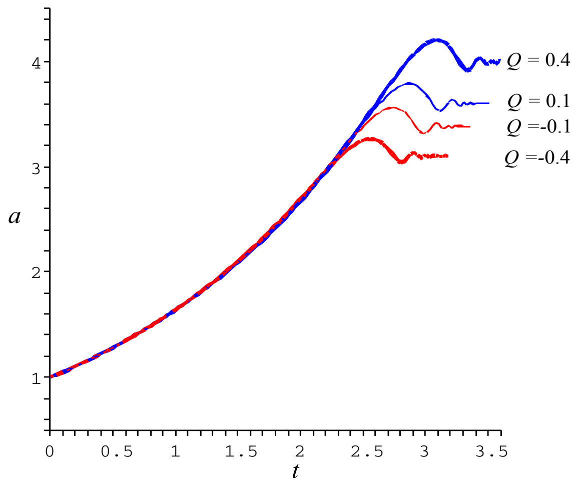
$$\dot{H} = -\frac{(\rho + p)}{2M_{\rm P}^2}\left(1 - \frac{2\rho}{\rho_{\rm lc}}\right), \tag{4.6}$$

$$= -\frac{(1+w_\phi)\rho_\phi + (1+w_{\rm m})\rho_{\rm m}}{2M_{\rm P}^2}\left[1 - \frac{2}{\rho_{\rm lc}}(\rho_\phi + \rho_{\rm m})\right], \tag{4.7}$$

$$= -\frac{\left[-S^2 + (1+w_{\rm m})\rho_{\rm m}\right]}{2M_{\rm P}^2} \times \left[1 - \frac{2}{\rho_{\rm lc}}\left(-\frac{S^2}{2} + ce^{-\lambda\phi/M_{\rm P}^2} + \rho_{\rm m}\right)\right]. \tag{4.8}$$

In above equations we define new variable

$$S \equiv \dot{\phi}. \tag{4.9}$$



รูปที่ 4.2: Scale factor plotted versus time for $Q = -0.4, -0.1, 0.1$ and 0.4 (from bottom to top).

The coupled fluid equations (4.1) and (4.2) are re-expressed in term of S as

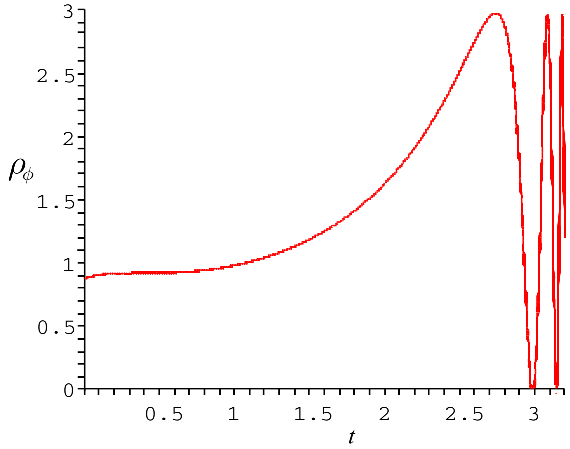
$$\dot{S} \;=\; -3HS + \frac{\mathrm{d}V}{\mathrm{d}\phi} + Q\rho_{\mathrm{m}}\,, \tag{4.10}$$

$$\dot{\rho}_{\mathrm{m}} \;=\; -3H(1+w_{\mathrm{m}})\rho_{\mathrm{m}} + Q\rho_{\mathrm{m}}S\,. \tag{4.11}$$

The Eqs. (4.8), (4.9), (4.10) and (4.11) form a closed autonomous set of four equations. The variables here are ρ_{m}, S, ϕ and H. The autonomous set recovers standard general relativistic cosmology in the limit $\rho_{\mathrm{lc}} \to \infty$. The general relativistic limit affects only the equation involving H. From the above autonomous set, one can do a qualitative analysis with numerical integration similar to phase plane presented in different situation [117]. Another approach of analysis is to consider a quantitative analysis [**?**].

4.2 Numerical solutions

Here we present some numerical solution for a positive and negative coupling between the phantom field and barotropic fluid. The solutions presented here are physically valid solutions corresponding to Class II solutions characterized in [**?**]. For non-minimally coupled scalar field in Einstein frame [118], the coupling Q lies in a range $-1/\sqrt{6} < Q < 1/\sqrt{6}$. Here we set $Q = -0.4, -0.1, 0.1$ and 0.4 which lie in the range. Effect of the coupling can be seen from Eqs. (4.1) and (4.2). Negative Q enhances decay rate of scalar field to matter while giving higher matter creation rate. On the other hand, positive Q yields opposite result. Greater magnitude of $Q < 0$ gives higher decay rate of the field to matter. Greater magnitude of $Q > 0$ will result in higher production rate of phantom field from matter.



รูปที่ 4.3: Phantom field density plotted versus time for $Q = -0.1$. The other values of Q also yield bouncing and oscillation.

4.2.1 Phase portrait

The greater Q value results in greater value of the field turning point (see ϕ-intercept in Fig. 4.1). The kinetic term $S(t)$ turns negative at the turning points corresponding to the field rolling down and then halting before rolling up the hill of exponential potential. When Q is greater, the field can fall down further, hence gaining more total energy. The result agrees with the prediction of Eqs. (4.1) and (4.2).

4.2.2 Scale factor

From Fig. 4.2, the bounce in scale factor occurs later for greater Q value of which the phantom field production rate is higher. The field has more phantom energy to accelerate the universe in counteracting the effect of loop quantum (the bounce). For less positive Q, the phantom production rate is smaller, and for negative Q, the phantom decays. Therefore it has less energy for accelerating the expansion in counteracting with the loop quantum effect. This makes the bounce occurs sooner.

4.2.3 Energy density

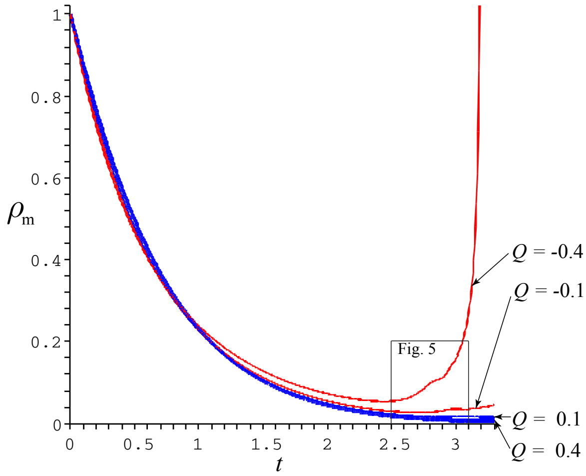
Time evolutions of energy density of the matter and the phantom field are presented in Figs. 4.3 and 4.4. If $Q > 0$, the matter decays to phantom. This reduces density of matter. While for $Q < 0$, the matter gains its density from decaying of phantom field. In Fig. 4.3 there is a bounce of phantom density before undergoing oscillation. For a non-coupled case, it has recently been reported that the phantom density also undergoes oscillation [34]. As seen in Figs. 4.4 and 4.5, the oscillation in phantom density of the phantom decay case ($Q < 0$) affects in oscillation in matter density while for the case of matter decay

($Q > 0$), the matter density is reduced for stronger coupling. The oscillation in the phantom density comes from oscillation of the kinetic term $\dot{\phi}$, i.e. $S(t)$ as shown in [**?**].

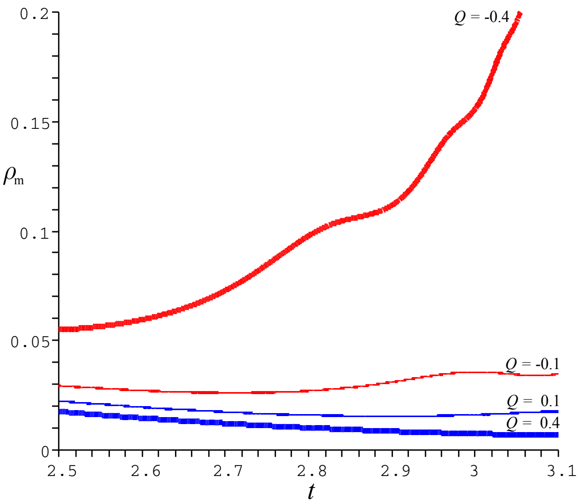
4.3 Conclusion and comment

In this letter, we have derived an autonomous system of a loop quantum cosmological equations in presence of phantom scalar field coupling to barotropic matter fluid. We choose constant coupling Q between matter and the phantom field to positive and negative values and check numerically the effect of Q values on (1) phase portrait, (2) scale factor and (3) energy density of phantom field and matter. We found that field value tends to roll up the hill of potential due to phantom nature. With greater Q, the field can fall down on the potential further. This increases total energy of the field. For canonical scalar field either standard or phantom, LQC yields a bounce. The bounce is useful since it is able to avoid Big Bang singularity in the early universe. Here our numerical result shows a bouncing in scale factor at late time. This is a Type I singularity avoidance even in presence of phantom energy. The greater coupling results in more and more phantom density. Greater phantom effect therefore delays the bounce, which is LQC effect, to later time. In the case of matter decay to phantom ($Q > 0$), oscillation in phantom energy density does not affect matter density. On the other hand, when $Q < 0$, phantom decays to matter, oscillation in phantom density results in oscillation in the increasing matter density.

This work considers only the effects of sign and magnitude of the coupling constant to qualitative dynamics and evolution of the system. Studies of field dependent effects of coupling $Q(\phi)$ in some scalar-tensor theory of gravity and investigation of an evolution of effective equation of state could also yield further interesting features of the model. Quantitative dynamical analysis of the model under different types of potential is also motivated for future work. Frequency function of the oscillation in scale factor and phantom density are still unknown in coupled case. It looks like that the oscillation frequency tends to increase. This could lead to infinite frequency of oscillation which is another new singularity.



รูปที่ 4.4: Matter density plotted versus time for $Q = -0.4, -0.1, 0.1$ and 0.4 (from top to bottom).



รูปที่ 4.5: Zoom-in portion of Fig. 4.4. The phantom field decays to matter at highest rate for $Q = -0.4$ (top line). Oscillation in matter density due to oscillation in the phantom field density is seen clearly here.

บทที่ 5

Non-linear Schrödinger-type formulation of scalar field cosmology

In standard cosmology with Friedmann-Lemaître-Robertson-Walker (FLRW) background, major components of the late universe are mixture of dark matter which is a type of barotropic fliud and dark energy in form of scalar field. When assuming pure scalar fluid in flat universe, one can obtain analytical solutions otherwise the problem can also be solved numerically. However, considering arbitrary types of barotropic fluid and a non-flat universe, it is not always possible to solve the system analytically.

Apart from standard cosmological equations, there are few alternative mathematical formulations which are also equivalent to the scalar field cosmology with barotropic fluid. One is in form of non-linear Ermakov-attemping equation [119] and another idea proposed recently is in form of non-Ermakov-Milne-Pinney (non-EMP) equation. Cosmological equations in the latter proposal can be written in form of a non-linear Schrödinger-type equation when imposing relations between quantities in standard cosmological equations and Schrödinger-type equation [120]. In case of Bianchi I scalar field cosmology, recent work shows that it is possible to construct a corresponding linear Schrödinger-type equation by redefining cosmological quantities [121]. With the new representation, scalar field cosmology is reinterpreted in new way which might be able to give new methods of approaching mathematical problems in scalar field cosmology.

In this section, we investigate connection between standard cosmological equations and non-linear Schrödinger-type equation with a comment on normalization of the wave function. We modify the work of [120] to include phantom field case. A case of power-law expansion with scalar field and dark matter is considered as a toy model. We begin from Sec. 5.1 where we introduce our cosmological system. Afterward in Sec. 5.2, we discuss how non-linear Schrödinger-type formulation quantities are related to quantities in standard scalar field cosmology. In non-linear Schrödinger-type equation, one important quantity is wave function. We comment on normalization properties of the wave function in Sec. 5.3.

We consider a case of power-law expansion in Sec. 6 before deriving scalar field potential, Schrödinger potential and Schrödinger wave function. At last we conclude this work in Sec. **??**.

5.1 Cosmological equations

In a Friedmann-Lemaître-Robertson-Walker universe, the Einstein field equations are

$$H^2 = \frac{\kappa^2 \rho_{\rm t}}{3} - \frac{k}{a^2}\,, \tag{5.1}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}(\rho_{\rm t} + 3p_{\rm t})\,, \tag{5.2}$$

where $\kappa^2 \equiv 8\pi G = 1/M_{\rm P}^2$, G is Newton's gravitational constant, $M_{\rm P}$ is reduced Planck mass, k is spatial curvature, $\rho_{\rm t}$ and $p_{\rm t}$ are total density and total pressure, i.e. $\rho_{\rm t} = \rho_\gamma + \rho_\phi$ and $p_{\rm t} = p_\gamma + p_\phi$. The barotropic component is denoted by γ, while for scalar field, by ϕ. Equations of state for barotropic fluid and scalar field are $p_\gamma = w_\gamma \rho_\gamma$ and $p_\phi = w_\phi \rho_\phi$. We consider minimally couple scalar field with Lagrangian density,

$$\mathcal{L} = \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi)\,, \tag{5.3}$$

where $\epsilon = 1$ for non-phantom case and -1 for phantom case. Density and pressure of the field are given as

$$\rho_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi)\,, \tag{5.4}$$

$$p_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi)\,, \tag{5.5}$$

therefore

$$w_\phi = \frac{\epsilon\dot{\phi}^2 - 2V(\phi)}{\epsilon\dot{\phi}^2 + 2V(\phi)}\,. \tag{5.6}$$

The field obeys conservation equation

$$\epsilon\left[\ddot{\phi} + 3H\dot{\phi}\right] + \frac{{\rm d}V}{{\rm d}\phi} = 0\,. \tag{5.7}$$

For the barotropic fluid, we set $w_\gamma \equiv (n-3)/3$ so that $n = 3(1+w_\gamma)$. Hence for cosmological constant $n = 0$, for fluid at acceleration bound ($w_\gamma = -1/3$) $n = 2$, for dust $n = 3$, for radiation $n = 4$, and for stiff fluid $n = 6$. Solution of conservation equation for a barotropic fluid can be obtained directly by solving the conservation equation. The solution is

$$\rho_\gamma = \frac{D}{a^{3(1+w_\gamma)}} = \frac{D}{a^n}\,, \tag{5.8}$$

then

$$p_\gamma = w_\gamma \frac{D}{a^n} = \frac{(n-3)}{3}\frac{D}{a^n}\,, \tag{5.9}$$

where a proportional constant $D \geq 0$. Using Eqs. (5.1), (5.4), (7.2), (7.4) and (5.8), it is straightforward to show that

$$\epsilon\dot{\phi}(t)^2 = -\frac{2}{\kappa^2}\left[\dot{H} - \frac{k}{a^2}\right] - \frac{nD}{3a^n}\,, \tag{5.10}$$

$$V(\phi) = \frac{3}{\kappa^2}\left[H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right] + \left(\frac{n-6}{6}\right)\frac{D}{a^n}\,. \tag{5.11}$$

Therefore if one knows how the scale factor evolves with time, the scalar field velocity and potential can always be expressed as a function of time explicitly.

5.2 Non-linear Schrödinger-type equation

Non-linear Schrödinger-type equation corresponding to canonical scalar field cosmology with barotropic fluid is given by [120]

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + [E - P(x)]\,u(x) = -\frac{nk}{2}u(x)^{(4-n)/n}\,. \tag{5.12}$$

Quantities in the Schrödinger-type equation above, e.g. wave function $u(x)$, total energy E and Schrödinger potential $P(x)$ are related to the standard cosmology quantities as

$$u(x) \equiv a(t)^{-n/2}\,, \tag{5.13}$$

$$E \equiv -\frac{\kappa^2 n^2}{12}D\,, \tag{5.14}$$

$$P(x) \equiv \frac{\kappa^2 n}{4}a(t)^n\epsilon\dot{\phi}(t)^2\,. \tag{5.15}$$

The mapping from cosmic time t to the variable x is via

$$x = \sigma(t), \tag{5.16}$$

such that

$$\dot{\sigma}(t) = u(x)\,, \tag{5.17}$$

$$\phi(t) = \psi(x)\,. \tag{5.18}$$

We notice that relation

$$\psi'(x)^2 = \frac{4}{\kappa^2 n}P(x) \tag{5.19}$$

in Ref. [120] which gives $\psi(x) = \pm(2/\kappa\sqrt{n})\int\sqrt{P(x)}\,\mathrm{d}x$ does not include phantom field case. In order to include the phantom field case, we modify relation $\dot{\phi}(t) = \dot{x}\,\psi'(x)$ in [120] to $\epsilon\dot{\phi}(t)^2 = \dot{x}^2\,\epsilon\,\psi'(x)^2$ of which the field kinetic term ($\dot{\phi}^2$) is considered instead of the field velocity ($\dot{\phi}$) so that the parameter ϵ can be included. Therefore, to include the phantom field case, corrected relation to Eq. (5.19) is

$$\epsilon\,\psi'(x)^2 = \frac{4}{\kappa^2 n}\,P(x)\,, \tag{5.20}$$

and $\psi(x)$ should read

$$\psi(x) = \pm \frac{2}{\kappa\sqrt{n}} \int \sqrt{\frac{P(x)}{\epsilon}}\, \mathrm{d}x \,. \tag{5.21}$$

Inverse function of $\psi(x)$ exists if $P(x) \neq 0$ and $n \neq 0$. It is important for $\psi^{-1}(x)$ to exist as a function since existence of the relation $x = \sigma(t)$ (Eq. (5.16)) needs a condition,

$$x = \psi^{-1} \circ \phi(t) = \sigma(t)\,. \tag{5.22}$$

In case that $P(x) = 0$ and $n \neq 0$, then $\psi = C$, hence inverse of ψ is not a function since one x gives infinite values of ψ^{-1}. In this case the relation (5.22) is invalid. If the inverse function, ψ^{-1} exists (i.e. $P(x) \neq 0$ and $n \neq 0$), then the scalar field potential, $V \circ \sigma^{-1}(x)$ can be expressed as a function of time,

$$V(t) = \frac{12}{\kappa^2 n^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \frac{2u^2}{\kappa^2 n}P(x) + \frac{12u^2}{\kappa^2 n^2}E + \frac{3ku^{4/n}}{\kappa^2}\,. \tag{5.23}$$

Although the potential obtained is not expressed as function of ϕ, however if one can integrate Eq. (5.10) to obtain $\phi(t)$, the obtained solution can be inserted into a known function $V(\phi)$ motivated from fundamental physics. Then one can check which fundamental theories give a matched potential to $V(t)$. The Eqs. (5.11) and (11.9) are indeed equivalent. Both require only the knowledge of $a(t)$, D and k which can be constrained by observation. Therefore $V(t)$ in both Eqs. (5.11) and (11.9) can be constructed if knowing these observed parameters. To construct $V(t)$ in Eq. (11.9), one needs to know $a(t)$ as a function of time in order to find $u(x)$ and $P(x)$. However, in constructing $V(t)$ in Eq. (5.11), if knowing $a(t), D$ and k, one can directly use these quantities without employing Schrödinger-type quantities. The scalar field potential, $V \circ \sigma^{-1}(x)$ and $\epsilon\dot{\phi}(t)^2$ can be expressed in NLS formulation as

$$\epsilon\dot{\phi}(x)^2 = \frac{4}{\kappa^2 n}uu'' + \frac{2k}{\kappa^2}u^{4/n} + \frac{4E}{\kappa^2 n}u^2 = \frac{4P}{\kappa^2 n}u^2\,, \tag{5.24}$$

$$V(x) = \frac{12}{\kappa^2 n^2}(u')^2 - \frac{2P}{\kappa^2 n}u^2 + \frac{12E}{\kappa^2 n^2}u^2 + \frac{3k}{\kappa^2}u^{4/n}\,. \tag{5.25}$$

From Eqs. (5.24) and (11.9), we can find

$$\rho_\phi = \frac{12}{\kappa^2 n^2}(u')^2 + \frac{12E}{\kappa^2 n^2}u^2 + \frac{3k}{\kappa^2}u^{4/n}\,, \tag{5.26}$$

$$p_\phi = -\frac{12}{\kappa^2 n^2}(u')^2 + \frac{4P}{\kappa^2 n}u^2 - \frac{12E}{\kappa^2 n^2}u^2 - \frac{3k}{\kappa^2}u^{4/n}\,. \tag{5.27}$$

We know that $\rho_\gamma = Du^2 = -12Eu^2/(\kappa^2 n^2)$ from Eq. (5.14) and the barotropic pressure is $p_\gamma = [(n-3)/3]\rho_\gamma$, therefore

$$\rho_{\rm tot} = \frac{12}{\kappa^2 n^2}(u')^2 + \frac{3k}{\kappa^2}u^{4/n}\,, \tag{5.28}$$

$$p_{\rm tot} = -\frac{12}{\kappa^2 n^2}(u')^2 + \frac{4u^2}{\kappa^2 n}\left[P - E\right] - \frac{3k}{\kappa^2}u^{4/n}\,. \tag{5.29}$$

Using the Schrödinger-type equation (5.12), then

$$p_{\rm tot} = -\frac{12}{\kappa^2 n^2}(u')^2 + \frac{4}{\kappa^2 n}uu'' - \frac{k}{\kappa^2}u^{4/n}. \tag{5.30}$$

5.3 Normalization condition of wave function

Normalization condition for a wave function $u(x)$ in quantum mechanics is

$$\int_{-\infty}^{\infty} |u(x)|^2 \mathrm{d}x = 1 \,. \tag{5.31}$$

The wave function here expressed as $u(x) \equiv a^{-n/2} = \dot{x}(t)$, when applying to the normalization condition, reads

$$\int_{-\infty}^{\infty} \dot{x}^2 \mathrm{d}x = 1 \,. \tag{5.32}$$

In order to satisfy the condition, x must be constant and so is t. Since the form of the wave function must be $u(x) = \dot{x}(t)$ in order to connect equations of cosmology to the Schrödinger-type formulation, therefore $u(x)$ as defined is, in general, non-normalizable.

5.4 Conclusion and Comment

We consider Schrödinger-type formulation for a system of canonical scalar field and a barotropic fluid in standard FLRW cosmology with zero or non-zero spatial curvature. In the Schrödinger-type formulation, all quantities in cosmology are represented in Schrödinger-like quantities and the equation relating these Schrödinger-like quantities is written as a non-linear Schrödinger-type equation. If $a(t)$ is known as an exact function of time, a connection of two scale quantities, x and t can be found and then other Schrödinger-like quantities can be determined. We modified the formulation to include the phantom field case. The equation can be simplified to linear type if we consider the flat universe case $k = 0$ or the cases $n = 2$ or $n = 4$ [120]. However, even if the equation is linear, it can not be considered as an analog to non-relativistic time-independent quantum mechanics because in this work, the wave function of Schrödinger-type formulation is found to be, in general, non-normalizable.

Without knowledge of $a(t)$, one might wonder if we could start the calculation procedure from solving the Schrödinger-type equation (5.12) for example, the linear case as done in basic quantum mechanics. However, in order to do this, we must know the Schrödinger potential $P(x)$ (Eq. (5.15)) which depends explicitly on $a(t)$ and $\dot{\phi}$. Nevertheless, $\dot{\phi}$ (Eq. (5.10)) also depends on $a(t)$. Therefore we need to know the law of expansion $a(t)$ before proceeding the calculation. Knowing $a(t)$ enables us to know $u(x)$ directly (see Eq. (6.3)). Hence in Schrödinger-type formulation, we do not work as in basic quantum mechanics in which major task is to solve the Schrödinger equation for $u(x)$. There could be many solutions of a Schrödinger-type equation. In quantum mechanics valid solutions $u(x)$ must be only normalizable type. Here, unlike in quantum mechanics, our $u(x)$ must be non-normalizable.

บทที่ 6

Power-law expansion

Here in this section, we apply the method above to the power-law expansion in scalar field cosmology with barotropic fluid in a non-flat universe. The power-law expansion of the universe during inflation era,

$$a(t) = t^q \,, \tag{6.1}$$

with $q > 1$ was proposed by Lucchin and Matarrese [165] to give exponential potential

$$V(\phi) = \left[\frac{q(3q-1)}{\kappa^2 t_0^2}\right] \exp\left\{-\kappa\sqrt{\frac{2}{q}}\left[\phi(t) - \phi(t_0)\right]\right\} \,, \tag{6.2}$$

assuming domination of scalar field, negligible radiation density and negligible spatial curvature. Recent results from X-Ray gas of galaxy clusters put a constraint of $q \sim 2.3$ for $k = 0$, $q \sim 1.14$ for $k = -1$ and $q \sim 0.95$ for $k = 1$ [162]. Considering mixture of both fluids, we use effective equation of state, $w_{\rm eff} = (\rho_\phi w_\phi + \rho_\gamma w_\gamma)/\rho_{\rm t}$. For a flat universe, the power law expansion, $a = t^q$, is attained when $-1 < w_{\rm eff} < -1/3$ where $q = 2/[3(1+w_{\rm eff})]$. If using $q = 2.3$ as mentioned above, it gives $w_{\rm eff} = -0.71$.

6.1 Relating Friedmann quantities to NLS quantities

Assuming power-law expansion and using Eqs. (5.13) and (5.17), Schrödinger wave function is related to standard cosmological quantity as

$$u(x) = \dot{\sigma}(t) = t^{-qn/2} \,. \tag{6.3}$$

We can integrate the equation above so that the Schrödinger scale, x is related to cosmic time scale, t as

$$x = \sigma(t) = -\frac{t^{-\beta}}{\beta} + \tau \,, \tag{6.4}$$

where $\beta \equiv (qn-2)/2$ and τ is an integrating constant. The parameters x and t have the same dimension since β is only a number. Using Eq. (6.1), we can find $\epsilon\dot{\phi}(t)^2$ from Eq. (5.10):

$$\epsilon\dot{\phi}(t)^2 = \frac{2q}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} - \frac{nD}{3t^{qn}} \,. \tag{6.5}$$

We use Eqs. (6.1) and (6.22) in Eq. (5.15), therefore the Schrödinger potential is found to be

$$P(x) = \frac{qn}{2}t^{qn-2} + \frac{kn}{2}t^{q(n-2)} - \frac{\kappa^2 n^2 D}{12}\,. \tag{6.6}$$

With $E = -\kappa^2 n^2 D/12$, the Schrödinger kinetic energy is

$$T = -\frac{qn}{2}t^{qn-2} - \frac{kn}{2}t^{q(n-2)}\,. \tag{6.7}$$

6.1.1 Scalar field potential $V(t)$

In order to obtain $V(t)$ in Eq. (11.9), we need to know derivative of $u(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\,u(x) \quad = \quad -\frac{qn}{2t}\,. \tag{6.8}$$

At this step, using Eqs. (5.13), (5.14), (5.15) and (6.8) in Eq. (11.9), we finally obtain

$$V(t) = \frac{q(3q-1)}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} + \left(\frac{n-6}{6}\right)\frac{D}{t^{qn}}\,. \tag{6.9}$$

Assuming flat universe ($k = 0$) and $q = 2.3$, we show $V(t)$ in Fig. 6.1. Thickest line on top is of the case scalar field without barotropic fluid. The middle line is the case when the dust is presented with scalar field ($D \neq 0, n = 3$). The bottom line is the case of radiation ($D \neq 0, n = 4$). The $V(t)$ plots from the Schrödinger-type formulation matches the plots from standard cosmological equations. The result is independent of ϵ values. The solution $\phi(t)$ of Eq. (6.22) can not be integrated if $\epsilon = -1$ or if the integrand of Eq. (6.22) is imaginary. When $\epsilon = 1$ with dust ($D \neq 0, n = 3$) and $q = 2.3$, the integrand is imaginary. We therefore assume $q = 2$ to show numerical integrations in Fig. 6.2 for the case $D = 0, k = 0$ and the case $D \neq 0, n = 3, k = 0$. In the pure scalar field case $D = 0, k = 0$, numerical solution matches the analytical solution $\phi(t) = (\sqrt{2q}/\kappa)\ln(t)$. This solution can be substituted into Eq. (6.24) to obtain Eq. (6.2) as in [165] (setting $t_0 = 1$ and $\phi(t_0) = 0$). When considering cases of closed, flat and open universe containing dust matter, $V(t)$ of each case is presented in Fig. 6.3 where $q = 2$ is assumed in all cases so that we can see how the plots change their shapes when k is varied.

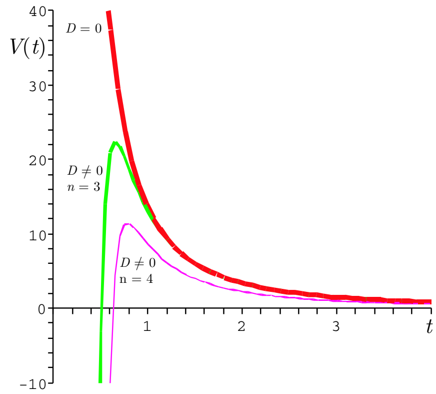
6.1.2 Schrödinger potential $P(x)$

We can find Schrödinger potential $P(x)$ from Eqs. (6.4) and (7.12) where time is expressed as a function of x as

$$t(x) = \frac{1}{[-\beta(x-\tau)]^{1/\beta}}\,. \tag{6.10}$$

Therefore

$$P(x) \quad = \quad \frac{2qn}{(qn-2)^2}\frac{1}{(x-\tau)^2} + \frac{kn}{2}\left[\frac{-2}{(qn-2)(x-\tau)}\right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12}\,. \tag{6.11}$$



รูปที่ 6.1: Potential $V(t)$ plots from non-linear Schrödinger-type formulation assuming $a \sim t^q$, $q = 2.3$ in flat universe ($k = 0$). The thickest line is when there is no barotropic fluid $D = 0$. The middle line is when there is dust fluid together with scalar field, i.e. $D \neq 0$ and $n = 3$. The small line is when the universe has scalar field with radiation fluid, i.e. $D \neq 0$ and $n = 4$. We set $\kappa = 1$ and in the last two plots, we set $D = 1$. All plots match results obtained from standard cosmological equations.

As in Eq. (6.7), the Schrödinger kinetic energy is

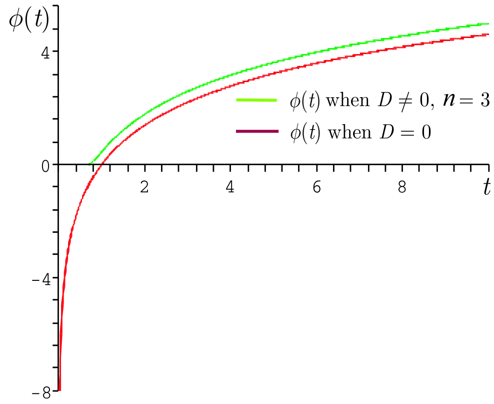
$$T(x) \quad = \quad -\frac{2qn}{(qn-2)^2}\frac{1}{(x-\tau)^2} - \frac{kn}{2}\left[\frac{-2}{(qn-2)(x-\tau)}\right]^{2q(n-2)/(qn-2)} . \tag{6.12}$$

The kinetic term has contribution only from the power q and spatial curvature k. A disadvantage of Eq. (6.11) is that we can not use it in the case of scalar field domination as in inflationary era. Dropping D term in Eq. (6.11) can not be considered as scalar field domination case since the barotropic fluid coefficient n still appears in the other terms. The non-linear Schrödinger-type formulation is therefore suitable when there are both scalar field and a barotropic fluid together such as the situation when dark matter and scalar field dark energy live together in the late universe. The Schrödinger potentials $P(x)$ plotted with x for power-law expansion with $q = 2$ in closed, flat and open universe are shown in Fig. 6.4. In the figure, the dust cases are shown on the right and radiation cases are on the left. We set $\kappa = 1, D = 1$ and $\tau = 0$.

6.1.3 Schrödinger wave function $u(x)$

The quantity analogous to Schrödinger wave function can be directly found from Eqs. (6.3) and (6.10) as

$$u(x) = \left[\left(-\frac{1}{2}qn + 1\right)(x-\tau)\right]^{qn/(qn-2)} , \tag{6.13}$$



รูปที่ 6.2: $\phi(t)$ for power-law expansion $a \sim t^q$, $q = 2$ in flat universe ($k = 0$). The red line is of the when the barotropic fluid density is negligible. The green line is in the presence of scalar field with dust ($D \neq 0$ and $n = 3$). In the figure, $\kappa = 1$ and $D = 1$.

which is independent of the spatial curvature k or the initial density D. However, coefficient n of the barotropic fluid equation of state and q must be expressed. Wave functions for a range of barotropic fluids are presented in Fig. 6.5. The result is confirmed by substituting Eq. (6.13) into Eq. (5.12).

6.2 Bound value of $\phi(t)$ from effective equation of state for $k = 0$ case

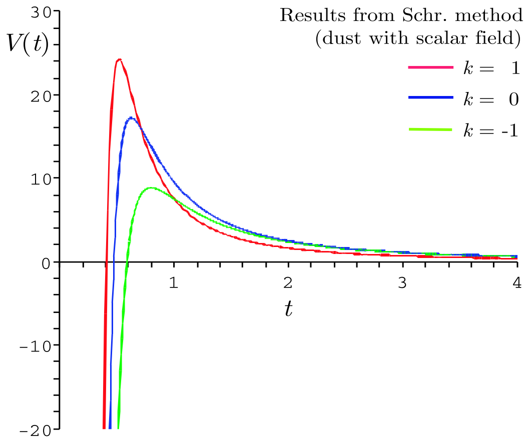
In power-law expanding universe, scale factor evolves with time as

$$a(t) = t^q \,. \tag{6.14}$$

where $q > 0$ is a constant. In flat ($k = 0$) universe, it is known that the power-law expansion, is attained when $-1 < w_{\text{eff}} < -1/3$ where $q = 2/[3(1 + w_{\text{eff}})]$. The effective equation of state, (Eq. (8.1)) hence is a condition

$$-\frac{n}{3}\rho_\gamma \;<\; \epsilon\dot{\phi}^2 \;<\; \frac{2}{3}\left(\frac{\epsilon\dot{\phi}^2}{2} + V\right) + \left(\frac{2-n}{3}\right)\rho_\gamma \,, \tag{6.15}$$

i.e. $0 < \epsilon\dot{\phi}^2 + \rho_\gamma n/3 < (2/3)\rho_{\text{tot}}$. Both values of ϵ can be assigned and the power-law expansion is sustained as long as the condition is satisfied.



รูปที่ 6.3: $V(t)$ obtained from non-linear Schrödinger-type formulation for closed, flat and open universe in presence of dust and scalar field.

6.3 Solution solved from Friedmann formulation

If we directly consider Eq. (5.10), the solution for power-law expansion is an integration:

$$\phi(t) = \pm \int \sqrt{\frac{1}{\epsilon}\left(\frac{2q}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} - \frac{nD}{3t^{qn}}\right)} \mathrm{d}t\,. \tag{6.16}$$

6.3.1 Simplest case

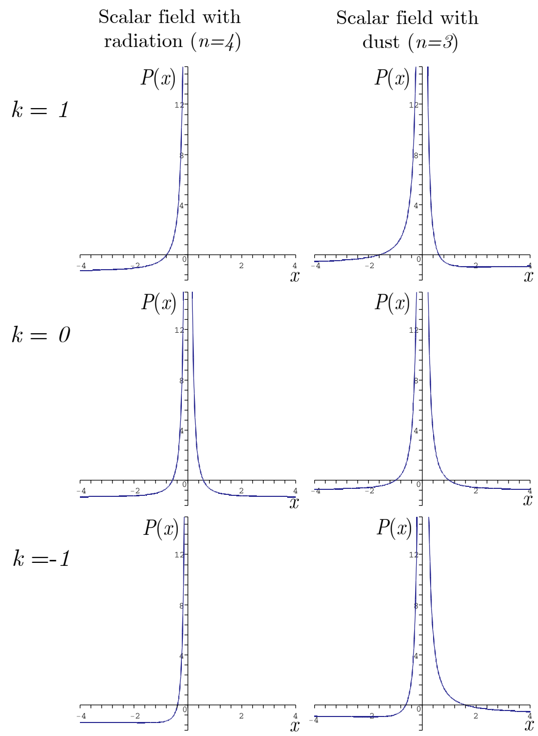
Simplest integration case is when $k = 0$ and $D = 0$. The solution of Eq. (6.16) is well known [165],

$$\phi(t) = \pm\sqrt{\frac{2q}{\epsilon\kappa^2}}\ln t + \phi_0\,, \tag{6.17}$$

provided that q and ϵ have the same sign. Considering power-law inflation, the WMAP five-year combined analysis based on flat and scalar field domination assumption yields $q > 60$ at more than 99 % of confident level otherwise excluded while $q \sim 120$ is at boundary of 68% confident level [?]. These results base on single field model which we can applied the above solution to. When assuming only $k = 0$ with $D \neq 0$, the solution of Eq. (6.16) is

$$\phi(t) = \pm\frac{1}{qn-2}\sqrt{\frac{2q}{\epsilon\kappa^2}}\left\{\ln\left[\frac{t^{-qn+2}}{\left(1+\sqrt{1-(nD\kappa^2/6q)\,t^{-qn+2}}\right)^2}\right] + 2\sqrt{1-\left(\frac{nD\kappa^2}{6q}\right)t^{-qn+2}} + \ln\left(-\frac{nD\kappa^2}{6q}\right)\right\} +$$

where, when $q = 2/n$, the field has infinite value. The last logarithmic term in the bracket is an integrating constant which is valid only when $q < 0$. To attain power-law expansion, q must be positive. Hence, this



รูปที่ 6.4: $P(x)$ plotted versus x for power-law expansion. We set $q = 2, \kappa = 1, D = 1$ and $\tau = 0$. The scalar field dominant case can not be plotted because even tough we set a condition $D = 0$, the coefficient n of the barotropic fluid equation of state still appears in the first and second terms of the Eq. (6.11). There is only a real-value $P(x)$ for the cases $k = \pm 1$ with $n = 4$ because, when $x > 0$, $P(x)$ becomes imaginary in these cases.

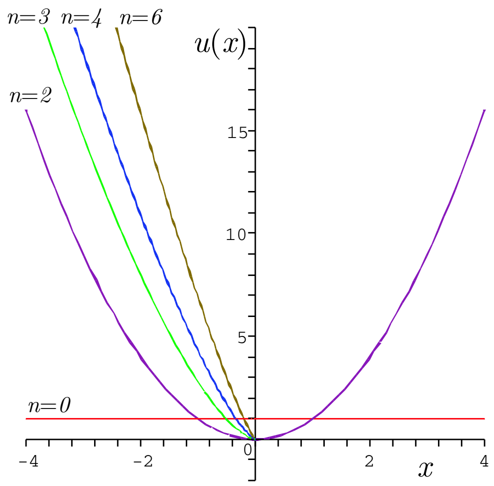
term is not defined for power-law expansion. We will see later that the NLS result does not have this problem. For the reverse case, $D = 0, k \neq 0$, the solution is

$$\phi(t) = \pm \frac{1}{q-1}\sqrt{\frac{2q}{\epsilon\kappa^2}}\left\{\ln\left[\frac{t^{q-1}}{\sqrt{k/q}}\left(1+\sqrt{\left(\frac{k}{q}\right)t^{-2q+2}+1}\right)\right] - \sqrt{\left(\frac{k}{q}\right)t^{-2q+2}+1}\right\} + \phi_0\,, \quad (6.19)$$

which becomes infinite when $q = 1$. The values of q, k and ϵ must have the same sign in all terms of the solution otherwise becoming imaginary. Hence, for $q > 0$, the condition for the solution to be valid is $k = 1$ and $\epsilon = 1$.

6.3.2 The case of non-zero k and non-zero D

When considering non negligible value of both k and D, the Eq. (6.16) can not be integrated analytically except when setting $n = 2$ $(w_\gamma = -1/3)$ which is not natural fluid. Hence it is not considered.



รูปที่ 6.5: $u(x)$ plotted versus x for power-law expansion with $q = 2$. We set $\tau = 0$. The wave function is plotted for $n = 0$ (cosmological constant), $n = 2$, $n = 3$ (dust), $n = 4$ (radiation) and $n = 6$ (stiff fluid). There is no real-value wave function for $n = 3$, $n = 4$ and $n = 6$ unless $x < 0$.

6.4 Solutions solved from NLS formulation

Power-law expansion cosmology in NLS-type formulation is presented and concluded in [?]. Important functions needed for evaluating the field exact solutions are

$$x = \sigma(t) = -\frac{t^{-\beta}}{\beta} + x_0\,, \tag{6.20}$$

$$t(x) = \frac{1}{[-\beta(x - x_0)]^{1/\beta}}\,, \tag{6.21}$$

$$\epsilon\dot{\phi}(t)^2 = \frac{2q}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} - \frac{nD}{3t^{qn}}\,, \tag{6.22}$$

$$P(x) = \frac{2qn}{(qn-2)^2}\frac{1}{(x-x_0)^2} + \frac{kn}{2}\left[\frac{-2}{(qn-2)(x-x_0)}\right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12}\,, \tag{6.23}$$

$$V(t) = \frac{q(3q-1)}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} + \left(\frac{n-6}{6}\right)\frac{D}{t^{qn}}\,. \tag{6.24}$$

We use Eq. (6.23) in Eq. (5.21), then

$$\psi(x) = \frac{\pm 2}{\kappa\sqrt{n}} \times \int \sqrt{\frac{2qn}{\epsilon(qn-2)^2}\frac{1}{(x-x_0)^2} + \frac{kn}{2\epsilon}\left[\frac{-2}{(qn-2)}\frac{1}{(x-x_0)}\right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12\epsilon}}\,\mathrm{d}x \tag{6.25}$$

We consider the solution in cases of $k = 0$ and $k \neq 0$. Recall that setting $D = 0$ can not be considered as an absence of barotropic fluid due to existence of n in the other terms.

6.4.1 The case $k = 0$

Solution to the integral (6.25) for $k = 0$ case is

$$\psi(x) = \pm\sqrt{\frac{8q}{\epsilon\kappa^2(qn-2)^2}} \times \left\{-\sqrt{1-\left[\frac{\kappa^2 Dn(qn-2)^2}{24q}(x-x_0)^2\right]} + \ln\left[\frac{1+\sqrt{1-[\kappa^2 Dn(qn-2)^2/24q]\,(x-x_0)^2}}{(x-x_0)}\frac{4qn}{\epsilon(qn-2)^2}\right]\right\} \tag{6.26}$$

Transforming to the t variable using Eq. (6.20), we obtain,

$$\phi(t) = \pm\frac{1}{qn-2}\sqrt{\frac{2q}{\epsilon\kappa^2}}\left\{\ln\left[\frac{t^{-qn+2}}{\left(1+\sqrt{1-(nD\kappa^2/6q)\,t^{-qn+2}}\right)^2}\right] + 2\sqrt{1-\left(\frac{nD\kappa^2}{6q}\right)t^{-qn+2}} + \ln\left(\frac{qn-2}{2qn}\right)^2\right\}$$

This solution differs from the solution (6.18) only the last logarithmic term in the bracket which is only an integrating constant. When $q = 2/n$ or $n = 0$, the field has infinite value. The last logarithmic term does not restrict the sign of q. Only q and ϵ must have the same sign for the solution to be real.

6.4.2 The case $k \neq 0$

In case of non-zero k and non-zero D, the integral (6.25) can not be integrated analytically even when assuming each n value except when $n = 2$ which is not natural fluid.

6.5 Analysis on effective equation of state coefficient

Similar to the analysis in Sec. 7.5, mixed effect of the two fluids and spatial curvature results in power-law expansion. The coefficient w_{eff}, with Eqs. (7.2), (6.22) and (6.24), reads

$$w_{\text{eff}} = \frac{(-3q^2+2q)\,t^{2q-2}-k}{3q^2\,t^{2q-2}+3k}\,, \tag{6.28}$$

which becomes infinity if

$$t = \left(\frac{-k}{q^2}\right)^{1/(2q-2)}. \tag{6.29}$$

We can also express w_ϕ in term of w_{eff} as

$$w_\phi = \frac{[(3q^2/\kappa^2)t^{-2}+(3k/\kappa^2)t^{-2q}]w_{\text{eff}} - [(n-3)/3]\,Dt^{-qn}}{(3q^2/\kappa^2)t^{-2}+(3k/\kappa^2)t^{-2q}-Dt^{-qn}}\,, \tag{6.30}$$

for power-law expansion. The Eq. (6.30), when $D = 0$ and $k = 0$, yields $w_\phi = w_{\text{eff}}$ as expected. Similar to the case of exponential expansion, setting $D = 0$ alone also yields $w_\phi = w_{\text{eff}}$. In flat universe, power-law expansion happens when w_{eff} lies in an interval $(-1, -1/3)$. But in $k \neq 0$ universe, it is no longer true. Considering flat universe, setting $k = 0$ in Eq. (6.28) yields $q = 2/[3(1 + w_{\rm eff})]$. The condition $-1 < w_{\rm eff} < -1/3$ therefore corresponds to $q > 0$ as known. The condition also yields

$$-1 - (1 + w_\gamma)\frac{\rho_\gamma}{\rho_\phi} \;<\; w_\phi \;<\; -\frac{1}{3} - \left(\frac{1}{3} + w_\gamma\right)\frac{\rho_\gamma}{\rho_\phi}\,. \tag{6.31}$$

If there is more non-negligible radiation fluid (with $w_\gamma = 1/3$), it is noticed that the interval shifts to the more left. For example, setting $\rho_\gamma = 0.1\rho_\phi$, the interval shifts to about $-1.133 < w_\phi < -0.4$. If we assume more realistic situation when dust (dark matter and other matter elements) is presented. The dust density and dark energy is about 28% and 72% of total density, therefore $\rho_\gamma \simeq (28/72)\rho_\phi \simeq 0.389\rho_\phi$, the interval is $-1.389 < w_\phi < -0.463$ which covers valid range of recent observational data, assuming dynamical w with flat universe, $-1.38 < w_{\phi,0} < -0.86$ at 95% confident level [**?**].

6.6 Conclusion and comment

We show relations between cosmological quantities in conventional form and in Schrödinger-like form for power-law expansion. We obtain scalar field potential $V(t)$, Schrödinger potential $P(x)$ and wave function $u(x)$. In the case of a scalar field dominant in flat universe, our analytical results $V(\phi)$ and $\dot{\phi}$ agree well with the well-known results in [165]. A range of plots in various cases of closed, flat or open geometries is presented. Wave functions for the power-law expansion case (seen in the Fig. 6.5) are found to be all non-normalizable as conjectured.

At late time the scalar field dark energy and cold dark matter (dust) are two major components of the universe while the others are negligible. For power-law expansion, the procedure is suitable for studying the system of scalar field dark energy and dark matter because it gives all real-value of $P(x)$ for any k. We need to know $a(t)$, k and D which are observable in order to find $V(t)$. Information of $V(\phi)$ is important because it is a link to fundamental physics. If one starts from fundamental physics with a particular potential $V(\phi)$ and if also knowing how ϕ evolves with t, then V could be expressed as function of t. Finally, the potential $V(t)$ obtained from observation and another $V(t)$ proposed by fundamental physics can be compared. The non-linear Schrödinger-type formulation might provide an alternative mathematical approach to problem solving in scalar field cosmology.

We consider phantom and non-phantom scalar field cases with exponential accelerating expansion. We expresses $w_{\rm eff}$ in term of q and k. In a flat universe, in order to have power-law expansion, the interval $(-1, -1/3)$ of the w_ϕ, is shifted leftward to more negative if more barotropic fluid density is presented.

Within framework of the standard Friedmann formulation, we obtained exact solution in various cases. Later we solved the problem using NLS formulation, in which the wave function is equivalent to the scalar field exact solution. NLS method is restricted by the fact that its scalar field solution is valid only when the barotropic fluid density is presented. Setting $D = 0$ does not imply the absence of barotropic fluid because the barotropic fluid parameter n still appears in the other terms of the Schrödinger potential. Therefore NLS formulation can not be applied to situation when the scalar field is dominant and $D \sim 0$.

Hence it is more suitable for a system of dark energy and dust dark matter fluid. This is a disadvantage point of the NLS formulation. Transforming from standard Friedmann formulation to NLS formulation makes n appear in all terms of the integrand and also changes fluid density term D from time-dependent term to a constant E. Hence the number of x(or equivalently t)-dependent terms is reduced by one. This is a good aspect of the NLS. In both Friedmann formulation and NLS formulation, the solutions when $k \neq 0$ and $D \neq 0$ are difficult or might be impossible to solve unless assuming values of q and n. Hence reduction number of x-dependent term helps simplifying the integration. There are also other good aspects of NLS formulation.

For power-law expansion with $k = 0$, the result (6.27) obtained from NLS formulation has integrating constant that does not restrict q value while (6.18) obtained from Friedmann formulation needs $q < 0$ which violates power-law expansion condition ($q > 0$). For power-law expansion, the most difficult case is when $k \neq 0$ with $D \neq 0$. The integral can not be integrated unless assuming $n = 2$ (equivalent to $w_\gamma = -1/3$) which is not a physical fluid. We introduce here alternative method to obtain scalar field exact solution with advantage over and disadvantage to standard Friedmann formulation.

บทที่ 7

Exponential expansion

Exponential expansion reads

$$a(t) = \exp(t/\tau)\,, \tag{7.1}$$

where τ is a positive constant. Density and pressure of the field are given as

$$\rho_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi)\,, \qquad p_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi)\,, \tag{7.2}$$

therefore

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\epsilon\dot{\phi}^2 - 2V(\phi)}{\epsilon\dot{\phi}^2 + 2V(\phi)}\,. \tag{7.3}$$

The field obeys conservation equation

$$\epsilon\left[\ddot{\phi} + 3H\dot{\phi}\right] + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0\,. \tag{7.4}$$

In presence of barotropic fluid therefore the effective equation of state is

$$w_{\text{eff}} = \frac{\rho_\phi w_\phi + \rho_\gamma w_\gamma}{\rho_{\text{tot}}}. \tag{7.5}$$

7.1 Relating Friedmann quantities to NLS quantities

For exponential expansion, following Eqs. (5.13) and (5.17), we get

$$u(x) = \dot{x}(t) = \exp(-nt/2\tau)\,. \tag{7.6}$$

Integrating the above equation, hence parameters x and t scale as

$$x(t) = -\frac{2\tau}{n}e^{-nt/2\tau} + x_0\,, \tag{7.7}$$

where x_0 is an integration constant. The reverse is

$$t(x) = -\frac{2\tau}{n}\ln\left[(-n/2\tau)(x - x_0)\right]\,, \tag{7.8}$$

where the condition $x < x_0$ must be imposed. Now we can write wave function as

$$u(x) = -\frac{n}{2\tau}(x - x_0)\,, \tag{7.9}$$

which is a linear function. Using Eq. (7.18), hence the Eq. (5.15) reads

$$P(t) \quad = \quad \frac{kn}{2}\, e^{(n-2)t/\tau} - \frac{\kappa^2 n^2 D}{12}\,. \tag{7.10}$$

Here the Schrödinger kinetic energy term is

$$T(t) \quad = \quad -\frac{kn}{2}\, e^{(n-2)t/\tau}\,. \tag{7.11}$$

Expressing in Schrödinger formulation, these functions are written in term of x,

$$P(x) \quad = \quad \frac{kn}{2}\left[-\frac{n}{2\tau}(x - x_0)\right]^{-2(n-2)/n} - \frac{\kappa^2 n^2 D}{12}\,, \tag{7.12}$$

$$T(x) \quad = \quad -\frac{kn}{2}\left[-\frac{n}{2\tau}(x - x_0)\right]^{-2(n-2)/n}\,. \tag{7.13}$$

In order to obtain the scalar field potential $V(t)$, we use Eqs. (5.13), (5.14), (5.15) in Eq. (11.9), we finally obtain

$$V(t) = \frac{3}{\kappa^2\tau^2} + \frac{2k}{\kappa^2}\, e^{-2t/\tau} + \left(\frac{n-6}{6}\right) D\, e^{-nt/\tau}\,. \tag{7.14}$$

which is checked by using Eq. (7.1) in standard formula (5.11). We use Eq. (7.12) in Eq. (5.21), then

$$\psi(x) = \frac{\pm 2}{\kappa\sqrt{n}} \times \int \sqrt{\frac{kn}{2\epsilon}\left[\frac{-n}{2\tau}(x - x_0)\right]^{-2(n-2)/n} - \frac{\kappa^2 n^2 D}{12\epsilon}}\, \mathrm{d}x\,. \tag{7.15}$$

We will integrate this equation in cases of $k = 0$ and $k \neq 0$.

7.2 Solution solved from effective equation of state for $k = 0$ case

Flat universe undergoes exponential expansion only when $w_{\mathrm{eff}} = -1$.

The effective equation of state, (Eq. (8.1)) with Eqs. (7.2) and (7.3) can therefore be written as[1]

$$\epsilon\dot{\phi}^2 \quad = \quad -\frac{n}{3}\rho_\gamma\,, \tag{7.16}$$

which can be integrated directly, using Eq. (5.8), to

$$\phi(t) = \pm 2\tau\sqrt{\frac{D}{3n}}\, e^{-nt/2\tau} + \phi_0\,. \tag{7.17}$$

The solution above is obtained when assuming phantom scalar field, i.e. $\epsilon = -1$. If the scalar field is not phantom, the solution is imaginary.

[1] We are not considering a cosmological constant but a dynamical scalar field and a barotropic fluid which together yield $w_{\mathrm{eff}} = -1$.

7.3 Solution solved from Friedmann formulation

Another way to find the exact solution is to use Eq. (7.1), in Eq. (5.10). Therefore

$$\epsilon\dot{\phi}(t)^2 = \frac{2k}{\kappa^2}\, e^{-2t/\tau} - \frac{nD}{3}\, e^{-nt/\tau}\,, \tag{7.18}$$

which gives an integration:

$$\phi(t) = \pm\int\sqrt{\frac{1}{\epsilon}\left(\frac{2k}{\kappa^2}e^{-2t/\tau} - \frac{nD}{3}e^{-nt/\tau}\right)}\mathrm{d}t\,. \tag{7.19}$$

7.3.1 Simplest case

In the case of $k = 0$ and $D = 0$, the integration yields a constant ϕ_0. Eq. (8.1) becomes $w_\phi = -1$. This is a cosmological constant as seen in simplest model of exponential expansion. When assuming only $k = 0$ and $\epsilon = -1$ but with $D \neq 0$, the solution of Eq. (7.19) is the same as the Eq. (7.17) previously. For a scalar field domination in a non-flat universe ($D = 0, k \neq 0$), the solution is

$$\phi(t) = \pm\frac{\tau}{\kappa}\sqrt{\frac{2k}{\epsilon}}\, e^{-t/\tau} + \phi_0\,. \tag{7.20}$$

where k and ϵ must have the same sign, otherwise the solution is imaginary.

7.3.2 The case of non-zero k and non-zero D

When k and D are both not negligible. Performing integration to the Eq. (7.19) is more complicated and could be impossible unless assumption of barotropic fluid type. When assuming a particular type of fluid in the integration, i.e. $n = 0, 2, 3, 4$ and 6, analytical solution can be found for all n vales in complicated forms. For example, the simplest among these is dust case ($n = 3$) which has solution:

$$\phi(t) = \pm\frac{2\tau}{3D\sqrt{\epsilon}}\left(\frac{2k}{\kappa^2} - De^{-t/\tau}\right)^{3/2} + \phi_0\,, \tag{7.21}$$

with additional rule that $k \geq 0$ and $\epsilon = 1$ otherwise it is imaginary. In the next section, we will show how to obtain solution in NLS formulation for dust and radiation cases.

7.4 Solutions solved with NLS formulation

7.4.1 The case $k = 0$

When $k = 0$ and $D \neq 0$ integrating Eq. (7.15) and transforming x to t with Eq. (7.7) yields same result as Eq. (7.17) obtained by solving effective equation of state equation or by integrating from the Friedmann formulation. Real solution exists only when the scalar field is phantom. With the solution (7.17), the scalar field potential in term of ϕ, reads

$$V(\phi) \;=\; \frac{3}{\kappa^2\tau^2} + \left(\frac{n-6}{6}\right)\frac{3n}{4\tau^2}(\phi - \phi_0)^2\,. \tag{7.22}$$

7.4.2 The case $k \neq 0$

When $k \neq 0$ and $D \neq 0$, the integral (7.15) can be integrated yielding complicated hypergeometric function even when n is not specified. The case $n = 0$ is excluded from our consideration by the reason mentioned in Sec **??**. For naturalness, we consider radiation ($n = 4$) and dust ($n = 3$).

Radiation case

Radiation fluid corresponds to $n = 4$, the integral (7.15) becomes

$$\psi(x) \quad = \quad \pm\frac{1}{\kappa}\int\sqrt{-\frac{k\tau}{\epsilon}\frac{1}{(x-x_0)} - \frac{4}{3}\frac{\kappa^2 D}{\epsilon}}\,\mathrm{d}x\,. \tag{7.23}$$

Here x could be negative, ϵ can possibly be either ± 1. The solution in radiation case is

$$\begin{aligned}\psi(x) \quad = \quad & \pm\sqrt{\frac{1}{\epsilon}\left[-\frac{4}{3}D(x-x_0)^2 - \frac{k\tau}{\kappa^2}(x-x_0)\right]} \\ & \pm\frac{k\tau}{4\kappa^2}\sqrt{\frac{3}{D\epsilon}}\arctan\left\{\frac{[8\kappa^2 D(x-x_0)/3\epsilon] + k\tau/\epsilon}{[4\kappa\sqrt{D}/(\epsilon\sqrt{3})]\sqrt{-[4\kappa^2 D(x-x_0)^2/3] - k\tau(x-x_0)}}\right\} + \psi_0\,,\end{aligned} \tag{7.24}$$

allowing only $\epsilon = 1$ case for the solution to be real. Transforming x scale to the t scale using Eq. (7.7), the solution therefore reads

$$\begin{aligned}\phi(t) \quad = \quad & \pm\sqrt{\frac{1}{\epsilon}\left(-\frac{D\tau^2}{3}\,e^{-4t/\tau} + \frac{k\tau^2}{2\kappa^2}\,e^{-2t/\tau}\right)} \\ & \pm\frac{k\tau}{4\kappa^2}\sqrt{\frac{3}{D\epsilon}} \times \arctan\left\{\frac{-[4\kappa^2 D\tau/(3\epsilon)]\,e^{-2t/\tau} + k\tau/\epsilon}{[4\kappa\sqrt{D}/(\epsilon\sqrt{3})]\sqrt{-(\kappa^2 D\tau^2/3)\,e^{-4t/\tau} + (k\tau^2/2)\,e^{-2t/\tau}}}\right\} + \phi_0.\end{aligned} \tag{7.25}$$

The solution above, when assuming $k = 0$, reduces to the solution (7.17) when $n = 4$, confirming the correctness of the result obtained.

Dust case

The integral (7.15) in the dust case $n = 3$ reads

$$\psi(x) = \pm\frac{2}{\kappa\sqrt{3}}\int\sqrt{\left(\frac{3}{2}\tau^2\right)^{1/3}\frac{k}{\epsilon}\frac{1}{(x-x_0)^{2/3}} - \frac{3}{4}\frac{\kappa^2 D}{\epsilon}}\,\mathrm{d}x\,, \tag{7.26}$$

with solutions

$$\psi(x) = \pm\sqrt{\frac{D}{\epsilon}}\left[\left(\frac{3\tau^2}{2}\right)^{1/3}\frac{4k}{3\kappa^2 D} - (x-x_0)^{2/3}\right]^{3/2} + \psi_0\,. \tag{7.27}$$

With similar procedure to the radiation case, using (7.7), the solution is therefore,

$$\phi(t) = \pm\frac{2\tau}{3D\sqrt{\epsilon}}\left(\frac{2k}{\kappa^2} - De^{-t/\tau}\right)^{3/2} + \phi_0\,, \tag{7.28}$$

which is the same as Eq. (7.21) derived from Friedmann formulation. This solution when assuming $k = 0$ is exactly the same as the solution (7.17) when $n = 3$ (dust fluid). This also confirms that our results from NLS formulation are correct. The NLS solution can solve the case when k and D are non-zero together without knowing n value while the standard procedures in Sec. 7.3 can not unless assuming a particular value $n = 0, 2, 3, 4, 6$. However, it must be noticed that one can not reduce the NLS solutions (7.25) and (7.28) to the $D = 0$ case directly since there are mixed multiplication term of n and k in the solution and also the value of n has already been put in. Hence setting $D = 0$ in (7.25) and (7.28) can not be considered as a pure scalar field dominant case.

7.5 Analysis on effective equation of state coefficient

The exponential expansion in our scenario is caused from mixed effect of fluids and spatial curvature. We discuss mixed effect on equation of state here. Definition of effective equation of state coefficient, $w_{\rm eff} = (\rho_\phi w_\phi + \rho_\gamma w_\gamma)/\rho_{\rm tot}$ together with Eqs. (7.2), (7.18) and (7.14) in context of exponential expansion becomes

$$w_{\rm eff} = \frac{-1 - (k\tau^2/3)e^{-2t/\tau}}{1 + k\tau^2 e^{-2t/\tau}}\,, \tag{7.29}$$

which is infinite when

$$t = \frac{\tau}{2}\ln(-k\tau^2)\,. \tag{7.30}$$

Infinity can possibly happen only when $k = -1$ because logarithm function forbids negative domain. In order to acquire exponential expansion in flat universe, one needs to have $w_{\rm eff} = -1$, but this is not true when k term is non-trivial. Therefore we can only express w_ϕ in term of $w_{\rm eff}$ as

$$w_\phi = \frac{[\,(3k/\kappa^2)e^{-2t/\tau} + 3/(\kappa^2\tau^2)\,]w_{\rm eff} \;-\; [(n-3)/3]\,De^{-nt/\tau}}{(3k/\kappa^2)e^{-2t/\tau} \;+\; 3/(\kappa^2\tau^2) \;-\; De^{-nt/\tau}}\,, \tag{7.31}$$

for exponential expansion. The Eq. (7.29) does not depend on properties (n) or amount (D) of the barotropic fluid. It reduces to $w_{\rm eff} = -1$ when $k = 0$ as expected. Considering Eq. (7.31), if $D = 0$ and $k = 0$, it yields $w_\phi = w_{\rm eff}$ while setting $D = 0$ alone also gives the same result.

7.6 Conclusion and comment

Firstly, in the case of exponential expansion with NLS formulation, the solution when $k \neq 0$ and $D \neq 0$ can be obtained without assuming n value while $n = 0, 2, 3, 4, 6$ must be given if working within Friedmann formulation. The integral can not be integrated unless assuming $n = 2$ (equivalent to $w_\gamma = -1/3$) which is not a physical fluid. The NLS formulation could render more interesting techniques for scalar field cosmology.

บทที่ 8

Phantom expansion

To attain accelerating expansion, one needs to have effective equation of state coefficient, $w_{\rm eff} < -1/3$ where

$$w_{\rm eff} = \frac{\rho_\phi w_\phi + \rho_\gamma w_\gamma}{\rho_{\rm tot}}\,, \tag{8.1}$$

ρ_γ is density of barotropic fluid, ρ_ϕ is density of the scalar field and $\rho_{\rm tot} = \rho_\phi + \rho_\gamma$. It has been known in standard cosmology that for flat universe ($k = 0$), if the expansion is $a \sim t^q$, then $-1 < w_{\rm eff} < -1/3$; if $a \sim \exp(t/\tau)$, then $w_{\rm eff} = -1$ and if $a \sim (t_{\rm a} - t)^q$, then $w_{\rm eff} < -1$. Here $q \equiv 2/[3(1 + w_{\rm eff})]$, $\tau, t_{\rm a}$ are finite characteristic times. In the last case, $w_{\rm eff} < -1$ corresponds to $q < 0$.

In this chapter, we consider phantom expansion

$$a \sim (t_{\rm a} - t)^q\,. \tag{8.2}$$

Expansion of this form is called phantom when $q < 0$ for a flat universe. Here in non-flat universe, q is considered to possess any negative value and the term phantom expansion also refers to expansion function of the form $a \sim (t_{\rm a} - t)^q$ as in the flat case.

8.1 Relating Friedmann quantities to NLS quantities

With the phantom expansion, $a \sim (t_{\rm a} - t)^q$, we use Eqs. (5.13) and (5.17) to relate Schrödinger wave function to standard cosmological quantity as

$$u(x) \;=\; \dot{x}(t) = (t_{\rm a} - t)^{-qn/2}\,. \tag{8.3}$$

Integrate the equation above so that the Schrödinger scale, x is related to cosmic time scale, t as

$$x(t) \;=\; \frac{1}{\beta}\,(t_{\rm a} - t)^{-\beta} + x_0\,, \tag{8.4}$$

where $\beta \equiv (qn-2)/2$ and x_0 is an integrating constant. Conversely,

$$t(x) \;=\; t_{\rm a} - \frac{1}{[\beta(x-x_0)]^{1/\beta}}\,. \tag{8.5}$$

The Schrödinger wave function can be directly found from Eqs. (8.3) and (8.5) as

$$u(x) \;=\; [\beta(x-x_0)]^{qn/(qn-2)}\,. \tag{8.6}$$

For $a \sim (t_{\rm a}-t)^q$, we can find $\epsilon\dot{\phi}(t)^2$ from Eq. (5.10):

$$\epsilon\dot{\phi}(t)^2 \;=\; \frac{2q}{\kappa^2(t_{\rm a}-t)^2} + \frac{2k}{\kappa^2(t_{\rm a}-t)^{2q}} - \frac{nD}{3(t_{\rm a}-t)^{qn}}\,. \tag{8.7}$$

Using (8.7) with phantom expansion in Eq. (5.15), therefore

$$P(t) \;=\; \frac{qn}{2}(t_{\rm a}-t)^{qn-2} + \frac{kn}{2}(t_{\rm a}-t)^{q(n-2)} - \frac{\kappa^2 n^2 D}{12}\,, \tag{8.8}$$

which can be expressed in term of x using Eq. (8.5) as

$$P(x) = \frac{2qn}{(qn-2)^2}\frac{1}{(x-x_0)^2} + \frac{kn}{2}\left[\frac{2}{(qn-2)(x-x_0)}\right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12}. \tag{8.9}$$

One might have a thought that all functions in phantom expansion case can be changed to those in power-law expansion case by interchanging $(t_{\rm a}-t) \Leftrightarrow t$. However when $(t_{\rm a}-t)$ is differentiated, there is an extra minus sign. The Eq. (8.9) slightly defers from that of the power-law expansion case because in the power-law case, the numerator of the second term is -2 instead of 2. The Schrödinger kinetic energy T is negative value of the first two terms of the Schrödinger potential. At last, the scalar field potential obtained from Eq. (11.9) is

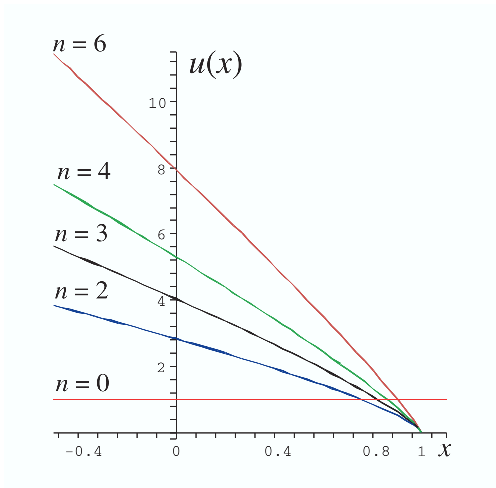
$$V(t) = \frac{q(3q-1)}{\kappa^2(t_{\rm a}-t)^2} + \frac{2k}{\kappa^2(t_{\rm a}-t)^{2q}} + \left(\frac{n-6}{6}\right)\frac{D}{(t_{\rm a}-t)^{qn}}\,. \tag{8.10}$$

which can be checked by using $a \sim (t_{\rm a}-t)^q$ in (5.11). Wave function of the NLS-formulation is found to be non-normalizable [**?**] as seen Fig. in 8.1 for case of phantom expansion with various types of barotropic fluid. Here q is chosen to -6.666. In flat universe $q = -6.666$ can be attained when $w_{\rm eff} = -1.1$. Fig. 8.2 shows $P(x)$ plots for three cases of k with dust and radiation. In there $x_0 = 1$, therefore $P(x)$ goes to negative infinity at $x = 1$.

8.2 Bound value of $\phi(t)$ from effective equation of state for $k = 0$ case

In flat universe, the phantom expansion occurs when $w_{\rm eff} < -1$. Using Eqs. (7.2), (7.3) in Eq. (8.1), we get a bound

$$\epsilon\dot{\phi}^2 \;<\; -\frac{n}{3}\rho_\gamma\,. \tag{8.11}$$



รูปที่ 8.1: Schrödinger wave function, $u(x)$ when assuming phantom expansion. $u(x)$ depends on only q, n and $t_{\rm a}$ but does not depend on k. Here we set $t_{\rm a} = 1.0$ and $q = -6.666$. If $k = 0$, $q = -6.666$ corresponds to $w_{\rm eff} = -1.1$.

Assuming $a(t) = (t_{\rm a} - t)^q$ and phantom scalar field, i.e. $\epsilon = -1$ with using Eq. (5.8), the solution is found to be in the region,

$$\phi(t) > \frac{1}{\beta}\sqrt{\frac{Dn}{3}}\,(t_{\rm a} - t)^{-\beta} + \phi_0\,. \tag{8.12}$$

where $\beta \equiv (qn - 2)/2\,.$

8.3 Solution solved from Friedmann equation

8.3.1 Scalar field potential in flat and scalar field dominated case

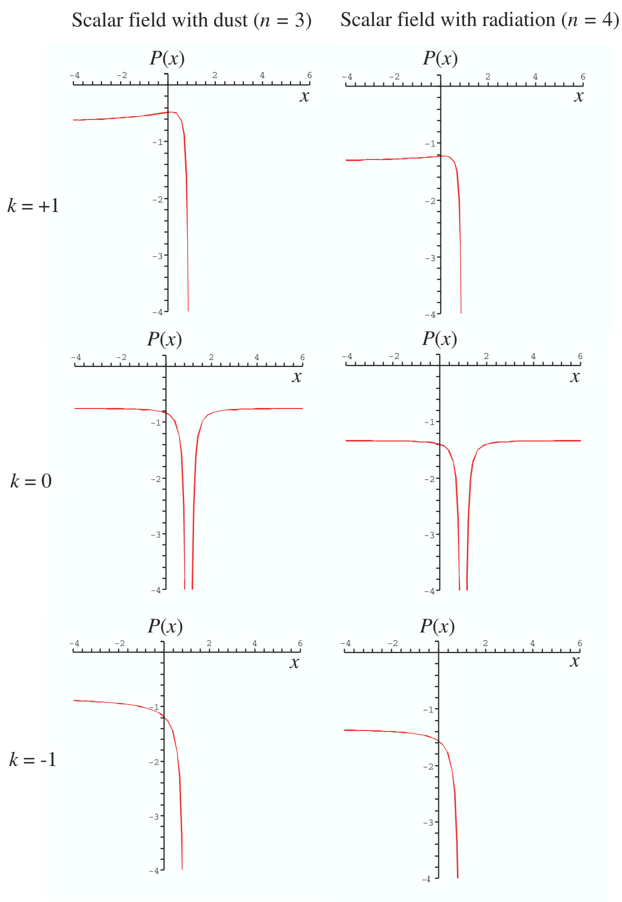
A simplest case for analysis is when considering flat universe ($k = 0$) with negligible amount of barotropic fluid ($D = 0$). The Eq. (8.7) is hence simply integrated out. The solution is

$$\phi(t) = \pm\frac{1}{\kappa}\sqrt{\frac{2q}{\epsilon}}\,\ln(t_{\rm a} - t)\, + \,\phi_0 \tag{8.13}$$

Insert this result into Eq. (8.10), we obtain the scalar field potential,

$$V(\phi) = \frac{q(3q-1)}{\kappa^2}\exp\left\{\pm\kappa\sqrt{\frac{2\epsilon}{q}}[\phi(t) - \phi_0]\right\}\,. \tag{8.14}$$

The solutions above are real only when q and ϵ have the same sign, i.e. when $\epsilon = 1, q > 0$ and $\epsilon = -1, q < 0$. This looks similar to potential that gives power-law expansion as well-known [165]. It is not surprised since in our case ($q < 0$) it has been known that phantom field, when rolling up the hill of slope-varying exponential potential (varying q), results in phantom expansion $a \sim (t_{\rm a} - t)^q$ [28].



รูปที่ 8.2: Schrödinger potential in phantom expansion case for dust and radiation fluids with $k = 0, \pm 1$. Numerical parameters are as in the $u(x)$ plots (Fig. 8.1). x_0 is set to 1. For non-zero k, there is only one real branch of $P(x)$.

8.3.2 Solution for $k = 0$, $D \neq 0$ case

For the case $k = 0$ with $D \neq 0$, the solution of Eq. (8.7) is

$$\begin{aligned}\phi(t) &= \pm\frac{1}{qn-2}\sqrt{\frac{2q}{\epsilon\kappa^2}}\times \\ &\left\{\ln\left[\frac{(t_a-t)^{-qn+2}}{\left(1+\sqrt{1-(nD\kappa^2/6q)(t_a-t)^{-qn+2}}\right)^2}\right]+2\sqrt{1-\frac{nD\kappa^2(t_a-t)^{-qn+2}}{6q}}+\ln\left(\frac{-nD\kappa^2}{6q}\right)\right\}+\phi_0\,,\end{aligned} \tag{8.15}$$

which is infinite when $q = 2/n$. The last logarithmic term in the bracket is an integrating constant. Logarithmic function is valid only when $q < 0$.

8.3.3 Solution for $k \neq 0$, $D = 0$ case

For the reverse case, $k \neq 0, D = 0$, the solution is

$$\begin{aligned}\phi(t) &= \pm\frac{1}{q-1}\sqrt{\frac{2q}{\epsilon\kappa^2}}\times \\ &\left\{\ln\left[\frac{(t_{\rm a}-t)^{q-1}}{\sqrt{k/q}}\left(1+\sqrt{\left(\frac{k}{q}\right)(t_{\rm a}-t)^{-2q+2}+1}\right)\right]-\sqrt{\left(\frac{k}{q}\right)(t_{\rm a}-t)^{-2q+2}+1}\right\}+\phi_0\,,\end{aligned} \tag{8.16}$$

which becomes infinite when $q = 1$. The values q and ϵ must have the same sign for it to be real-value function. The case $k \neq 0$ with $D \neq 0$ can not be found analytically except when setting $n = 2$ ($w_\gamma = -1/3$) which is not natural fluid.

8.4 Solution solved with NLS-type formulation

One can obtain exact solution of Eq. (8.7) indirectly via NLS-type formulation. Consider Eq. (8.9), we notice that setting $D = 0$ does not make sense in NLS-formulation since even D vanishes, n (barotropic fluid parameter) still appears in other terms. Therefore we can only consider non-zero D case. Assuming $k = 0$ with $D \neq 0$ and using Eq. (8.9) in Eq. (**??**), the solution is

$$\begin{aligned}\psi(x) &= \pm\sqrt{\frac{8q}{\epsilon\kappa^2(qn-2)^2}}\times \\ &\left\{-\sqrt{1-\left[\frac{\kappa^2 Dn(qn-2)^2}{24q}(x-x_0)^2\right]}+\ln\left[\frac{1+\sqrt{1-[\kappa^2 Dn(qn-2)^2/24q]\,(x-x_0)^2}}{(x-x_0)}\frac{4qn}{\epsilon(qn-2)^2}\right]\right\}\end{aligned} \tag{8.17}$$

Transforming to t variable using Eq. (8.4),

$$\begin{aligned}\phi(t) &= \pm\frac{1}{qn-2}\sqrt{\frac{2q}{\epsilon\kappa^2}}\times \\ &\left\{\ln\left[\frac{(t_{\rm a}-t)^{-qn+2}}{\left(1+\sqrt{1-(nD\kappa^2/6q)(t_{\rm a}-t)^{-qn+2}}\right)^2}\right]+2\sqrt{1-\frac{nD\kappa^2(t_{\rm a}-t)^{-qn+2}}{6q}}+\ln\left(\frac{qn-2}{2qn}\right)^2\right\}+\phi_0\,.\end{aligned} \tag{8.18}$$

The only difference from the solution (8.15) obtained from standard method is the logarithmic integrating constant term in the bracket. In case of $k \neq 0$ with $D \neq 0$, the integral (**??**) can not be integrated analytically even when assuming n value except for $n = 2$ which is integrable. However $n = 2$ is not natural fluid. This is similar to using standard method in Sec. 8.3.3.

8.5 Analysis on effective equation of state coefficient

The definition of effective equation of state coefficient, $w_{\rm eff} = (\rho_\phi w_\phi + \rho_\gamma w_\gamma)/\rho_{\rm tot}$ together with Eq. (7.2) and the results in Eqs. (8.7) and (8.10) in context of phantom expansion $a \sim (t_{\rm a} - t)^q$, we can derive

$$w_{\rm eff} = \frac{(-3q^2 + 2q)(t_{\rm a} - t)^{2q-2} - k}{3q^2(t_{\rm a} - t)^{2q-2} + 3k}\,. \tag{8.19}$$

There is a locus,

$$t = t_{\rm a} - \left(\frac{-k}{q^2}\right)^{1/(2q-2)}\,, \tag{8.20}$$

where $w_{\rm eff}$ becomes infinite along the locus. Hence for $k = -1$ the locus is $t = t_{\rm a} - q^{-1/(q-1)}$ (in term of x, it is $x = [2/(qn-2)]q^{(qn-2)/2(q-1)} + x_0$). Hence for $k = 0$, the coefficient $w_{\rm eff}$ is infinite at $q = 0$ or $t = t_{\rm a}$. It seems from the equation above that $w_{\rm eff}$ does not depend on properties, n or amount of the barotropic fluid, D. Indeed $w_{\rm eff}$ implicitly depends on D and n since time variable and q are related to ρ_γ in the Friedmann equation. If $k = 0$, it reduces to $q = 2/3(1 + w_{\rm eff})$ and therefore the phantom condition $w_{\rm eff} < -1$ implies $q < 0$ as it is known. This corresponds to a condition,

$$w_\phi < -1 - (1 + w_\gamma)\frac{\rho_\gamma}{\rho_\phi}\,. \tag{8.21}$$

Therefore for a fluid with $w_\gamma > -1$, w_ϕ is always less than -1 in a flat universe. In order to have the expansion $a \sim (t_{\rm a} - t)^q$ in $k = 0$ universe, we must have $w_{\rm eff} < -1$, i.e. in phantom region. We can rewrite w_ϕ in term of $w_{\rm eff}$ as

$$w_\phi = \frac{[\frac{3q^2}{\kappa^2}(t_{\rm a} - t)^{-2} + \frac{3k}{\kappa^2}(t_{\rm a} - t)^{-2q}]w_{\rm eff} - \frac{n-3}{3}D(t_{\rm a} - t)^{-qn}}{\frac{3q^2}{\kappa^2}(t_{\rm a} - t)^{-2} + \frac{3k}{\kappa^2}(t_{\rm a} - t)^{-2q} - D(t_{\rm a} - t)^{-qn}}\,. \tag{8.22}$$

Eq. (8.22), when $D = 0$ and $k = 0$, yields $w_\phi = w_{\rm eff}$. Albeit we set only $D = 0$, it gives the same result since w_ϕ is independent of geometrical background. However, since the expansion law is fixed, w_ϕ is tied up with D implicitly via Eq. (8.1). Note that w_ϕ has value in the range $(-\infty, -1]$ and $[1, \infty)$ so that the phantom crossing can not happen when the scalar field is dominant. However, presence of the dust barotropic fluid in the system gives a multiplication factor that is less than 1 to the equation of state, i.e.

$$w_{\rm eff} = \left(\frac{\rho_\phi}{\rho_\phi + \rho_\gamma}\right) w_\phi\,. \tag{8.23}$$

We can see that the phantom crossing from $w_{\rm eff} > -1$ to $w_{\rm eff} < -1$ can happen in this situation. Fig. 8.3 presents parametric plots of the $w_{\rm eff}, q, t$ diagram for various k values. From the figure, we see the locus in Eq. (8.20) where $w_{\rm eff}$ blows up. In the parametric plots, the value of $w_{\rm eff}$ at any instance can be obtained if we know the value of q. We need to know q from observation in order to know the realistic value of $w_{\rm eff}$ or the other way around. Fig. 8.4 plotted from Eq. (8.19) setting $t_a = 1$ and $t = 0.7$ shows that if $k = \pm 1$, q could be negative, i.e. phantom accelerated expansion, even when $w_{\rm eff} > -1$. Regardless of t_a and t,

$$\lim_{q\to -\infty} w_{\rm eff}(q) = -1 \quad \text{and} \quad \lim_{q \to +\infty} w_{\rm eff}(q) = -\frac{1}{3}\,, \tag{8.24}$$

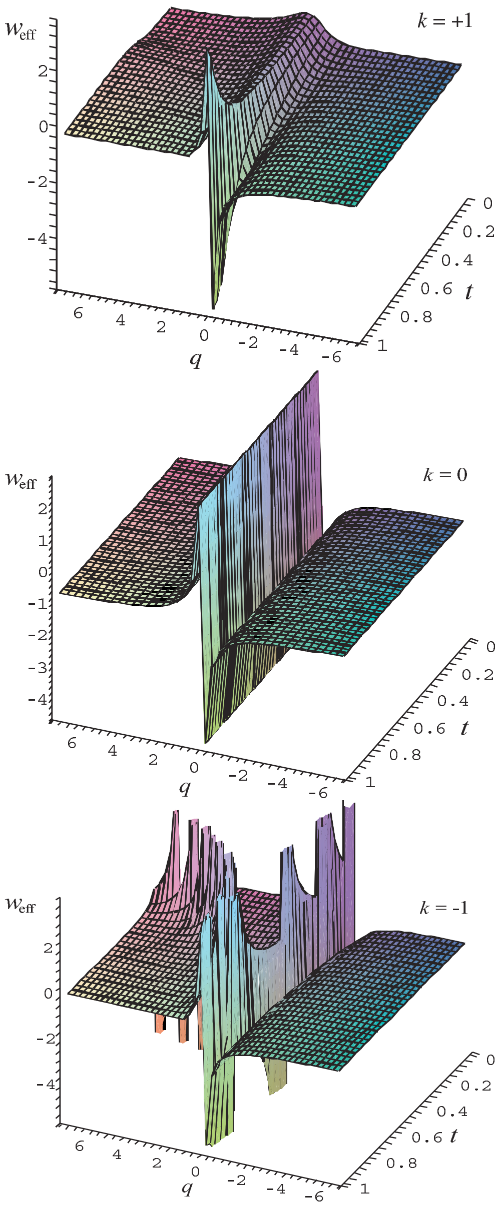
for phantom expansion. In particular, for $k = -1$, $w_{\text{eff}} > 0$ could give $q < 0$ and w_{eff} is infinite when $\ln q / \ln (t_{\text{a}} - t) + q = 1$ (see Eq. (8.20)).

8.6 Conclusion and comment

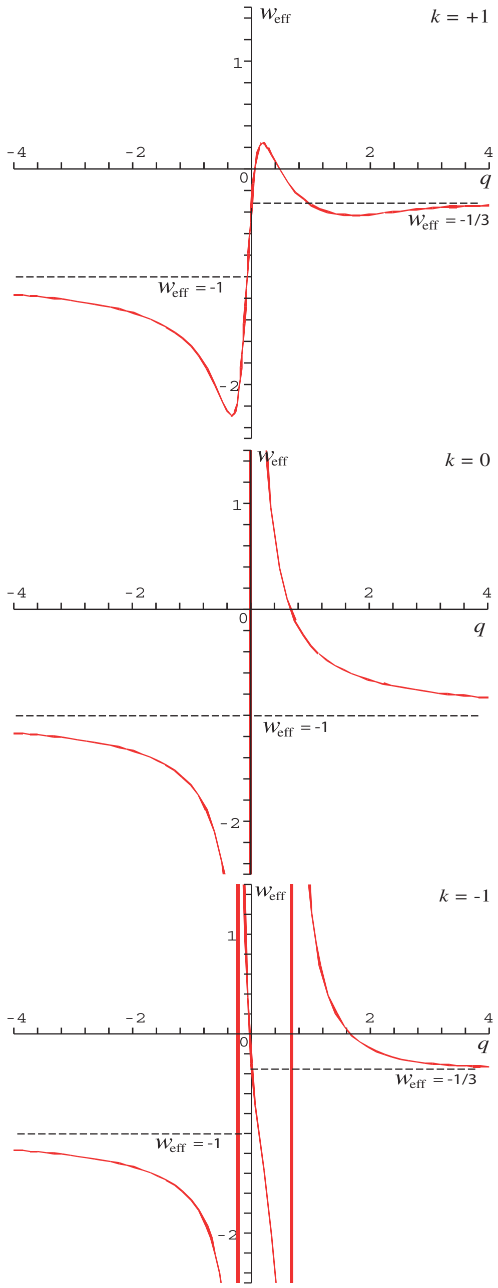
We consider a system of FLRW cosmology of scalar field and barotropic fluid assuming phantom acceleration. We have worked out cosmological quantities in the NLS-formulation of the system for flat and non-flat curvature. The Schrödinger wave functions are illustrated in Fig. 8.1 for various types of barotropic fluid. These wave functions are non-normalizable. We show Schrödinger potential plots for dust and radiation cases in closed, flat and open universe. The procedure considered here is reverse to a problem solving in quantum mechanics in which the Schrödinger potential must be known before solving for wave function. In NLS formulation, the Schrödinger equation is non-linear (reducible to linear in some cases) and the wave function is expressed first by the expansion function, $a(t)$. Afterward the Schrödinger potential is worked out based on expansion function assumed. Moreover, the NLS total energy E is negative (see Eq. (5.14)). We also perform analysis on effective equation of state. We expresses w_{eff} in term of q and k. In a non-flat universe, there is no fixed w_{eff} value for a phantom divide. We show this by analyzing Eq. (8.19) and by presenting illustrations in Figs. (8.3) and (8.4). In these plots, even $w_{\text{eff}} > -1$, the expansion can still be phantom, i.e. q can be negative. Especially, in $k = -1$ case, positive w_{eff} could also give $q < 0$. The value of w_{eff} approaches -1 when $q \to -\infty$ and $-1/3$ when $q \to +\infty$. In open universe, w_{eff} blows up when $\ln q / \ln (t_{\text{a}} - t) + q = 1$.

The last part of this work is to solve for scalar field exact solution for phantom expansion. Within framework of the standard Friedmann formulation, we obtained exact solution in simplest case where scalar field is dominated in flat universe. Apart from that we also obtained exact solutions in the cases of non-flat universe with scalar field domination and flat universe with mixture of barotropic fluid and scalar field. Afterward, we use NLS formulation, in which the wave function is equivalent to the scalar field exact solution, to solve for the exact solutions. We can apply the NLS method to solve for the solution only when the barotropic fluid density is non-negligible. Setting $D = 0$ in NLS framework is not sensible because even if D term vanishes, the barotropic fluid parameter n still appears in other terms of the wave function. This is a disadvantage point of the NLS formulation.

Transforming standard Friedmann formulation to NLS formulation renders a few effects to the integration. In standard form (Eq. (8.7)), n appears in only D-term and all terms are t-dependent. In NLS-form (Eq. (8.9) when inserted in Eq. (5.21)), D-term becomes a constant (E), hence the number of x(or equivalently t)-dependent terms is reduced by one. This is a good aspect of the NLS. In both Friedmann-form and NLS-form, the solutions when $k \neq 0$ and $D \neq 0$ are difficult or might be impossible to integrate unless assuming values of q and n. Therefore reduction number of x-dependent term helps simplifying the integration.



รูปที่ 8.3: Parametric plots of w_{eff} for the expansion $a \sim (t_{\text{a}} - t)^q$ in closed, flat and open universe. Here t_{a} is set to 1.



รูปที่ 8.4: $w_{\rm eff}$ for the expansion $a \sim (t_{\rm a} - t)^q$ in closed, flat and open universe. Here $t_{\rm a}$ is set to 1 and t is 0.7.

บทที่ 9

Other aspects of NLS formulation

It is interesting to see the other features of field velocity, $\dot{\phi}$, e.g. acceleration condition, slow-roll approximation, written in NLS formulation. Mathematical tools such as WKB approximation in quantum mechanics might also be interesting for application in standard scalar field cosmology. It is worthwhile to investigate this possibility. It is worth noting that Schrödinger-type equation in scalar field cosmology was previously considered in different procedure to study inflation and phantom field problems [124].

9.1 Slow-roll conditions

9.1.1 Slow-roll conditions: flat geometry and scalar field domination

In flat universe with scalar field domination ($k = 0, \rho_\gamma = 0$), the Friedmann equation $H^2 = \kappa^2 \rho_\phi / 3$, together with the Eq. (7.4) yield $\dot{H} = -\kappa^2 \dot{\phi}^2 \epsilon / 2$. For $\epsilon = -1$, we get $\dot{H} > 0$ and

$$0 < aH^2 < \ddot{a}\,, \tag{9.1}$$

i.e. the acceleration is greater than speed of expansion per Hubble radius, $\dot{a}/cH^{-1}$. On the other hand, for $\epsilon = 1$, we get $\dot{H} < 0$ and

$$0 < \ddot{a} < aH^2\,. \tag{9.2}$$

Slow-roll condition in [128, 127] assumes negligible kinetic term hence $|\epsilon\dot{\phi}^2/2| \ll V(\phi)$, therefore $\rho_\phi \simeq V(\phi)$ hence $H^2 \simeq \kappa^2 V/3$. With this approximation,

$$H^2 = -\frac{\dot{H}}{3} + \frac{\kappa^2}{3}V\,, \;\; \Rightarrow \;\; H^2 \simeq -\frac{\dot{H}}{3} + H^2\,. \tag{9.3}$$

This results in an approximation $|\dot{H}| \ll H^2$ from which the slow-roll parameter,

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} \tag{9.4}$$

is defined. Then the condition $|\epsilon\dot{\phi}^2/2| \ll V(\phi)$ is equivalent to $|\varepsilon| \ll 1$, i.e. $-1 \ll \varepsilon < 0$ for phantom field case and $0 < \varepsilon \ll 1$ for non-phantom field case. For the non-phantom field, this condition is necessary for inflation to happen (though not sufficient) [128] but for the phantom field case, the slow-roll condition is not needed because the negative kinetic term results in acceleration with $w_\phi \leq -1$. The other slow-roll parameter is defined by balancing magnitude of the field friction and acceleration terms in Eq. (7.4). This is independent of k or ρ_γ. When friction dominates $|\ddot{\phi}| \ll |3H\dot{\phi}|$, then

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{9.5}$$

is defined [128]. The condition is then $|\eta| \ll 1$ and the fluid equation is approximated to $\dot{\phi} \simeq -V_\phi/3\epsilon H$ which allows the field to roll up the hill when $\epsilon = -1$. Using both conditions, e.g. $|\epsilon\dot{\phi}^2/2| \ll V$ and $|\ddot{\phi}| \ll |3H\dot{\phi}|$ together, one can derive $\varepsilon = (1/2\kappa^2\epsilon)(V_\phi/V)^2$ and $\eta = (1/\kappa^2)(V_{\phi\phi}/V)$ as well-known.

9.1.2 Slow-roll conditions: non-flat geometry and non-negligible barotropic density

Friedmann formulation

When considering the case of $k \neq 0$ and $\rho_\gamma \neq 0$, then

$$\dot{H} = -\frac{\kappa^2}{2}\dot{\phi}^2\epsilon + \frac{k}{a^2} - \frac{n\kappa^2}{6}\frac{D}{a^n}\,. \tag{9.6}$$

We can then write slow-roll condition as: $|\kappa^2\epsilon\dot{\phi}^2/6| \ll (\kappa^2 V/3) - (k/a^2) + (\kappa^2 D/3a^n)$ and hence $H^2 \simeq (\kappa^2 V/3) + (\kappa^2 D/3a^n) - (k/a^2)$. Using this approximation and Eq. (9.6) in (5.1),

$$H^2 \simeq -\frac{\dot{H}}{3} + \frac{k}{3a^2} - \frac{n\kappa^2}{18}\frac{D}{a^n} + H^2\,, \tag{9.7}$$

which implies $|-(\dot{H}/3) + (k/3a^2) - (n\kappa^2 D/18a^n)| \ll H^2$. We can reexpress this slow-roll condition as

$$|\varepsilon + \varepsilon_k + \varepsilon_D| \;\ll\; 1\,, \tag{9.8}$$

where $\varepsilon_k \equiv k/a^2H^2$ and $\varepsilon_D \equiv -n\kappa^2 D/6a^nH^2$. Another slow-roll parameter η is defined as $\eta \equiv -\ddot{\phi}/H\dot{\phi}$, i.e. the same as the flat scalar field dominated case since the condition $|\ddot{\phi}| \ll |3H\dot{\phi}|$ is derived from fluid equation of the field which is independent of k and ρ_γ.

NLS formulation

In NLS formulation, the Hubble parameter takes the form

$$H = -\frac{2}{n}u'\,, \tag{9.9}$$

with

$$\dot{H} \;=\; -\frac{2}{n}uu'' \;=\; \frac{2}{n}u^2\left[E - P(x)\right] + ku^{4/n}\,. \tag{9.10}$$

The slow-roll condition $|\epsilon\dot{\phi}^2/2| \ll V$ using Eqs. (5.15) and (11.9) in NLS form, is then

$$|P(x)| \ll \frac{3}{n}\left[\left(\frac{u'}{u}\right)^2 + E\right] + \frac{3}{4}k\,n\,u^{(4-2n)/n}\,. \tag{9.11}$$

If the absolute sign is not used, the condition is then $\epsilon\dot{\phi}^2/2 \ll V$, allowing fast-roll negative kinetic energy. Then Eq. (9.11), when combined with the NLS equation (5.12), yields

$$u'' \ll \frac{3}{n}\frac{u'^{\,2}}{u} + \left(\frac{3}{n} - 1\right)Eu + \frac{kn}{4}u^{(4-n)/n}\,. \tag{9.12}$$

Friedmann formulation analog of this condition can be obtained simply by using Eqs (5.10) and (5.11) in the condition. Consider another aspect of slow-roll in the fluid equation, the field acceleration can be written in NLS form:

$$\ddot{\phi} = \frac{2Puu' + P'u^2}{\kappa\sqrt{P\epsilon n}}\,, \tag{9.13}$$

while the friction term in NLS form is

$$3H\dot{\phi} = -\frac{12u'u}{n\kappa}\sqrt{\frac{P}{\epsilon n}}\,. \tag{9.14}$$

The second slow-roll condition, $|\ddot{\phi}| \ll |3H\dot{\phi}|$ hence corresponds to

$$\left|\frac{P'}{P}\right| \ll \left|-2\left(\frac{6+n}{n}\right)\frac{u'}{u}\right|\,. \tag{9.15}$$

This condition yields the approximation $3H\epsilon\dot{\phi}^2 \simeq -\mathrm{d}V/\mathrm{d}\phi$. Using Eqs. (5.24), (11.9), (9.9) and (9.10), one can express the approximation, $3H\epsilon\dot{\phi}^2 \simeq -\mathrm{d}V/\mathrm{d}\phi$, in NLS form as

$$\frac{P'}{P} \simeq -\frac{2u'}{u} = nHa^{n/2}\,. \tag{9.16}$$

and finally the slow-roll parameters ε, ε_k and ε_D, introduced previously, become

$$\varepsilon = \frac{nuu''}{2u'^2}\,, \quad \varepsilon_k = \frac{n^2ku^{4/n}}{4u'^2}\,, \quad \varepsilon_D = \frac{nE}{2}\left(\frac{u}{u'}\right)^2\,, \tag{9.17}$$

in NLS form. With help of NLS equation (5.12), summation of the slow-roll parameters takes simple form,

$$\varepsilon_{\rm tot} = \varepsilon + \varepsilon_k + \varepsilon_D = \frac{n}{2}\left(\frac{u}{u'}\right)^2 P(x)\,. \tag{9.18}$$

Finally the slow-roll condition, $|\varepsilon_{\rm tot}| \ll 1$ (Eq. (9.8)), in NLS form, is

$$\left|\left(\frac{u}{u'}\right)^2 P(x)\right| \ll 1\,. \tag{9.19}$$

Another slow-roll parameter $\eta = -\ddot{\phi}/H\dot{\phi}$ can be found as follow. First considering $\psi(x) = \phi(t)$ (Eq. (**??**)), using relation $\mathrm{d}/\mathrm{d}t = \dot{x}\,\mathrm{d}/\mathrm{d}x$ and Eq. (9.9), we obtain

$$\eta = \frac{n}{2}\left(\frac{u}{u'}\frac{\psi''}{\psi'} + 1\right)\,. \tag{9.20}$$

The Eq. (5.21) yields

$$\psi' = \pm\frac{2}{\kappa}\sqrt{\frac{P}{n\epsilon}} \quad \text{and} \quad \psi'' = \pm\frac{P'}{\kappa\sqrt{nP\epsilon}}\,. \tag{9.21}$$

Hence

$$\eta = \frac{n}{2}\left(\frac{u}{u'}\frac{P'}{2P} + 1\right). \tag{9.22}$$

At last, the slow-roll condition $|\eta| \ll 1$ then reads

$$\left|\frac{u}{u'}\frac{P'}{2P} + 1\right| \ll 1\,. \tag{9.23}$$

9.2 Acceleration condition

The slow-roll condition is useful for non-phantom field because it is a necessary condition for inflating acceleration. However, in case of phantom field, the kinetic term is always negative and could take any large negative values hence slow-roll condition is not necessary for acceleration condition. More generally, to ensure acceleration, the Eq. (5.2) must be positive. It is straightforward to show that, obeying acceleration condition, $\ddot{a} > 0$, the Eq. (**??**), takes the form,

$$\epsilon\dot{\phi}(x)^2 \;<\; -\left(\frac{n-2}{2}\right)\frac{D}{a^n} + V\,. \tag{9.24}$$

With Eqs. (5.13), (5.14), (5.15) and (11.9)), the acceleration condition (9.24) in NLS-type formulation is

$$E - P \;>\; -\frac{2}{n}\left(\frac{u'}{u}\right)^2 - \frac{nk}{2}\left(\frac{u^{2/n}}{u}\right)^2. \tag{9.25}$$

With help of non-linear Schrödinger-type equation (5.12), it is simplified to

$$u'' < \frac{2}{n}\frac{u'^2}{u}\,. \tag{9.26}$$

Using Eqs. (9.9) and (9.10), the acceleration condition is just $\varepsilon < 1$ without using any slow-roll assumptions.

9.3 WKB approximation

WKB approximation can be assumed when the coefficient of highest-order derivative term in the Schrödinger equation is small or when the potential is very slowly-varying. The Eq. (5.12), when $k = 0$, is linear. It is then

$$-\frac{1}{n}u'' + \left[\tilde{P}(x) - \tilde{E}\right]u = 0\,. \tag{9.27}$$

where $\tilde{P}(x) \equiv P(x)/n$ and $\tilde{E} \equiv E/n$. For a slowly-varying $P(x)$ with assumption of $n \gg 1$, the solution of Eq. (9.27) can be written as $u(x) \simeq A\exp[\pm i n W_0(x)]$, where $W_0(x) = W(x_0)$ is the lowest-order term in Taylor expansion of the function $W(x)$ in $(1/n)$ about $x = x_0$,

$$W(x) = W(x_0) + W'(x_0)\frac{(x - x_0)}{n} + \ldots . \tag{9.28}$$

Then an approximation

$$W(x) = \pm\frac{1}{n}\int_{x_1}^{x_2} k(x)\,\mathrm{d}x \simeq W_0(x)\,, \tag{9.29}$$

is made in analogous to the method in time-independent quantum mechanics. The Schrödinger wave number is hence

$$k(x) = \frac{2\pi}{\lambda(x)} = \sqrt{n\left[\tilde{E} - \tilde{P}(x)\right]}\,, \tag{9.30}$$

and small variation in $\lambda(x)$ is

$$\frac{\delta\lambda}{\lambda(x)} = \left|\frac{\pi\tilde{P}'}{\sqrt{n}\left[\tilde{E} - \tilde{P}(x)\right]^{3/2}}\right| = \left|\frac{\pi P'}{\left[E - P(x)\right]^{3/2}}\right| . \tag{9.31}$$

For WKB approximation, $\delta\lambda/\lambda(x) \ll 1$. In real universe, we have $n = 3$ (dust) or $n = 4$ (radiation) which is not much greater than one. However, if considering a range of very slowly-varying potential, $P' \simeq 0$ implying $\delta P/\delta x \sim 0$, hence $\delta k/\delta x \sim 0 \sim W'(x)$. Therefore $W(x) \simeq W_0(x)$ still holds in this range. Since $u(x) = a^{-n/2}$, using WKB approximation, we get

$$a \sim A\exp\left[\pm(2/n)i\int_{x_1}^{x_2}\sqrt{E - P(x)}\,\mathrm{d}x\right]\,, \tag{9.32}$$

where A is a constant. Examples of Schrödinger potentials for exponential, power-law and phantom expansions are presented in previous chapters. These potentials are steep only in some small particular region but very slowly-varying in most regions, especially at large value of $|x|$ which are WKB-well valid.

9.4 Big Rip singularity

When the field becomes phantom, i.e. $\epsilon = -1$, in a flat FRLW universe it leads to future Big Rip singularity[**?**, 28]. In flat universe, when $w_{\rm eff} < -1$, i.e. being phantom, the expansion obeys $a(t) \sim (t_{\rm a} - t)^q$, where $q = 2/3(1 + w_{\rm eff}) < 0$ is a constant in time and $t_{\rm a}$ is a finite time[1]. The NLS phantom expansion was studied in Ref. [**?**] with inclusion of non-zero k case. Therein, the same expansion function is assumed with constant $q < 0$ and x is related to cosmic time scale, t as $x(t) = (1/\beta)\,(t_{\rm a} - t)^{-\beta} + x_0\,,$

[1]The relation $q = 2/3(1 + w_{\rm eff}) < 0$ holds only when $k = 0$.

so that $u(x) = [\beta(x - x_0)]^{\alpha}$. Here $\alpha \equiv qn/(qn-2)$ and $\beta \equiv (qn-2)/2$ with conditions $0 < \alpha < 1$ and $\beta < -1$ since $n > 0$ always. The first and second x-derivative of u are [2]

$$u'(x) = \alpha\beta[\beta(x - x_0)]^{\alpha-1}, \tag{9.33}$$

$$u''(x) = \alpha(\alpha-1)\beta^2[\beta(x - x_0)]^{\alpha-2}, \tag{9.34}$$

where exponents $\alpha - 1$ and $\alpha - 2$ are always negative. Using Eqs. (5.28) and (5.30), then

$$\rho_{\rm tot} = \frac{12\alpha^2\beta^2}{\kappa^2 n^2}[\beta(x - x_0)]^{2(\alpha-1)} + \frac{3k}{\kappa^2}[\beta(x - x_0)]^{4\alpha/n}, \tag{9.35}$$

$$\begin{aligned} p_{\rm tot} &= \frac{4\beta^2}{\kappa^2 n}[\beta(x - x_0)]^{2(\alpha-1)}\left[\left(1 - \frac{3}{n}\right)\alpha^2 - \alpha\right] \\ &\quad -\frac{k}{\kappa^2}[\beta(x - x_0)]^{4\alpha/n}. \end{aligned} \tag{9.36}$$

$$= \frac{4u'^2}{\kappa^2 n}\left[\left(1 - \frac{3}{n}\right) - \frac{1}{\alpha}\right] - \frac{k}{\kappa^2}[\beta(x - x_0)]^{4\alpha/n}. \tag{9.37}$$

The Big Rip: $(a, \rho_{\rm tot}, |p_{\rm tot}|) \rightarrow \infty$ happens when $t \rightarrow t_{\rm a}^-$. In NLS formulation, if $a \rightarrow \infty$, then $u \rightarrow 0^+$ (Eq. (5.13)). From above, we see that conditions of the Big Rip singularity are

$$\begin{aligned} t \rightarrow t_{\rm a}^- &\Leftrightarrow x \rightarrow x_0^-, \\ a \rightarrow \infty &\Leftrightarrow u(x) \rightarrow 0^+, \\ \rho_{\rm tot} \rightarrow \infty &\Leftrightarrow u'(x) \rightarrow \infty, \\ |p_{\rm tot}| \rightarrow \infty &\Leftrightarrow u'(x) \rightarrow \infty. \end{aligned} \tag{9.38}$$

The effective equation of state $w_{\rm eff} = p_{\rm tot}/\rho_{\rm tot}$ can also be stated in NLS language as a function of x. Approaching the Big Rip, $x \rightarrow x_0^-$ and the effective equation of state approaches a value

$$\lim_{x\rightarrow x_0^-} w_{\rm eff} = \frac{n}{3}\left(1 - \frac{1}{\alpha}\right) - 1 = -1 + \frac{2}{3q}, \tag{9.39}$$

which is similar to the equation of state in flat case.

9.5 Conclusion and comment

We feature cosmological aspects of NLS formulation of scalar field cosmology such as slow-roll condition-s, acceleration condition and the Big Rip. We conclude these aspects in standard Friedmann formulation before deriving them in the NLS formulation. We consider a non-flat FRLW universe filled with s-calar (phantom) field and barotropic fluid because, in presence of barotropic fluid density, the NLS-type formulation is consistent. We obtain all NLS version of slow-roll parameters, slow-roll conditions and

[2]Note that $(x - x_0)$ and β are negative hence $(x - x_0)^{\alpha}$, β^{α}, $(x - x_0)^{\alpha-1}$ and $\beta^{\alpha-1}$ are imaginary.

acceleration condition. This provides such analytical tools in the NLS formulation. For phantom field, due to its negative kinetic term, the slow-roll condition is not needed. When the NLS system is simplified to linear equation (this happens when $k = 0$.) as in time-independent quantum mechanics, we can apply WKB approximation to the problem. When $n \gg 1$, the wave function is semi-classical which is suitable for the WKB approximation. However, this does not work since physically n can not be much greater than unity, i.e. $n = 3$ for dust and $n = 4$ for radiation. However, the WKB approximation can still be well-valid in a range of very slowly-varying Schrödinger potentials $P(x)$ which were illustrated in [36, 38, 39]. Using the WKB approximation, we obtain approximated scale factor function (Eq. (9.32)). In a flat universe with phantom expansion, the Big Rip singularity is its final fate. When the Big Rip happens, three quantities $(a(t), p(t) \text{and } \rho(\text{t}))$ become infinity. Rewriting the singularity in NLS form (Eq. (9.38)), we can remove one infinite (see Eq. 9.38). We found that at near the Big Rip, $w_{\text{eff}} \to -1 + 2/3q$ where $q < 0$ is a constant exponent of the expansion $a(t) \sim (t_{\text{a}} - t)^q$. This limit is the same as the effective phantom equation of state in the case $k = 0$.

บทที่ 10

Generalised DBI quintessence

The dark energy problem continues to be a sticking point for theoretical physicists. The simplest solution to this problem is to postulate the existence of a vacuum energy or cosmological constant, which agrees with all the current observational bounds [14, ?, 4, 129]. However we are then left with a secondary problem, namely explaining why the vacuum energy is tuned to such a small value without some obvious symmetry to protect it. For many years we have hoped that UV complete theories of gravity would shed light on this issue, which is in effect an extremely embarrassing IR problem from this perspective. However despite much effort, neither string theory nor loop quantum gravity has shed any compelling light on this issue - although there have been many interesting proposals.

An alternative approach is to assume that the cosmological constant is exactly zero - since supersymmetry can then be invoked as the regulating symmetry in this case. However one then has to account for the fact that low-energy supersymmetry must be broken and an alternative explanation for the current expansion must be sought. One way to deal with the latter problem is to assume that the dark energy phase is driven by a dynamical field, implying that the equation of state is an explicit function of time [132, 133]. Currently this cannot be ruled out by our best observations and therefore remains a possible solution to the dark energy problem. However one cannot just consider ad hoc scalar fields coupling to gravity, since the low energy theory will still be sensitive to high energy physics. In particular we must ensure that any additional scalars are neutral under all the standard model symmetries, and that they do not introduce additional fifth forces. Therefore one must search for viable models of dynamical dark energy within UV sensitive theories.

Phenomenological models of our universe have proven difficult to construct within string theory, due to technical difficulties arising from moduli stabilization, whereby we assume that the extra dimensions of the theory are compactified on manifolds with $SU(3) \times SU(3)$ structure (in the type IIB case) [130], and orientifolded to preserve the minimal amount of supersymmetry in four-dimensions. As a result, embedding realistic cosmology into string theory has proven difficult. One area which has been well explored in recent years, is inflation driven by the open string sector through dynamical Dp-branes. This is the so-called DBI inflation [136, 140] - which lies in a special class of K-inflation models. It was originally thought that such models yielded large levels of non-Gaussian perturbations which could be used as a falsifiable signature of string theory [137]. However subsequent work has shown that this is *not* in fact the case, and that the simplest DBI models are essentially indistinguishable from standard field theoretic slow roll models [145, 144, 142] [1] The problem is that the WMAP5 data set [**?**] imposes very tight constraints on the allowed tuning of the free parameters in the theory. We are then left with the choice of either having large non-gaussianities but with vanishing tensors, or assume that the tensor spectrum will be visible - in which case there is no non-Gaussian signature. The models are only falsifiable once these conditions are relaxed. One can get around these conditions by considering more complicated models such as multi-field, multiple branes [143], wrapped branes [146] or monodromies [147] - but even here there are still problems with fine tuning, backreaction and the apparent breakdown of perturbation theory in the inflationary regime [148].

In models of dynamical dark energy, on the other hand [132, 133, 134], the WMAP constraints can be relaxed and therefore DBI models may still have some use as an explanation for a dynamical equation of state. Moreover this fits in nicely with several intuitive ideas from string theory. Namely that inflation can still occur, albeit only through the closed string sector - where one (or more) of the geometric pseudo-moduli are actually responsible for the initial inflationary epoch (see [131] for the phenomenologically most viable proposals). After inflation the universe lives on branes that wrap various cycles within the compact space and are extended along the large Minkowski directions. In this sense we see that a GUT or electroweak (EW) phase transition can manifest through a geometrical fashion - namely the Higgsing of branes in the compact bulk space. This suggests that dark energy may well be a dynamical process, and moreover in the light of these open string constructions, retains a sense of being geometric in nature.

With this in mind, various authors have begun to explore the phase space of DBI-driven dark energy [138, 139]. The initial works have dealt with the dynamics of a solitary $D3$-brane moving through a particular warped compactification of type IIB. In this note we wish to generalise this further to a more phenomenological class of models that include multiple and partially wrapped branes. We believe that this may be a more generic situation to consider, since typically one should expect branes of varying degrees to be wrapped on non-trivial cycles of the compact space. Our work is a first step into considerations of a more general set-up for quintessence in IIB (open)-string theory, and we hope will be a valuable starting point for further endeavour.

[1] Note however, that the models proposed in [145, 141] evade such problems.

10.1 Dynamics of the effective theory

To begin let us assume that the the universe at such late times can be adequately described by a flat FRW metric and the matter sector consists of a dynamical scalar field and a perfect fluid, which are both separately conserved. The usual cosmological equations of motion are therefore independent of any particular model and can be written as

$$\begin{aligned} H^2 &= \frac{(\rho + \rho_\phi)}{3M_p^2}\,, \\ \dot{\rho}_i &= -3H(P_i + \rho_i)\,, \end{aligned} \tag{10.1}$$

where i runs over the contributing components. The equation of state is given by $w_i = P_i/\rho_i$, however if w of the fluid component is assumed to be constant then we can integrate the appropriate conservation equation exactly to obtain

$$\rho \propto a^{-3(1+w)}\,, \tag{10.2}$$

where the scale factor varies as a function of time such that $a(t) \sim t^{2/(3[1+w])}$.

The model dependence arises in the parameterisation of the scalar field sector. In our case we are assuming that the dark energy is driven by open string modes, which at low energies are described by fluctuations of a Dp-brane whose dynamics are governed by the Dirac-Born-Infeld action (DBI) - which is a generalisation of non-linear electrodynamics [136, 140]. Typically one assumes that the standard model is localised on an intersecting brane stack, in one of the many warped throats that are attached to the internal space. For consistency reasons in the simplest cases, these are taken to be either $D3$ or $D7$-branes. In this note we will consider a bottom up approach therefore we shall not worry too much about the geometric deformations of the compact space, nor any constraints imposed by Orientifold Op-planes - aside from those that ensure that all tadpoles are consistently canceled so that we can trust the low energy supergravity theory.

The action we consider is a generalised form of the DBI one coupled to Einstein-Hilbert gravity, which can be embedded into this background and takes the following generalised form [2]

$$S = -\int d^4x a^3(t)\left(T(\phi)W(\phi)\sqrt{1-\frac{\dot{\phi}^2}{T(\phi)}} - T(\phi) + \tilde{V}(\phi)\right) + S_M\,, \tag{10.3}$$

where $T(\phi)$ is the warped tension of the brane and S_M is the action for matter localised in the Standard Model (SM) sector. Thus our assumption here is that our dynamical open string sector is coupled only gravitationally to the SM sector and so we do not have to worry about additional forces or particle production. There are two potential terms for the scalar field which are denoted by $W(\phi)$ and $\tilde{V}(\phi)$. The first of these terms can arise in different places within the theory. Firstly if the brane is actually a non-BPS one [135], then the scalar field mode is actually tachyonic and the potential is therefore of the

[2] We refer the more interested readers to [143] for more details on the precise structure of this action.

usual runaway form. If there are N multiple coincident branes, then the world-volume field theory is a $U(N)$ non-Abelian gauge theory and the potential term is simply a reflection of the additional degrees of freedom [149]. Through the dielectric effect, one can also see that this configuration is related to a $D5$-brane wrapping a two-cycle within the compact space and carrying a non-zero magnetic flux along this cycle. Both of these configurations lead to an additional potential multiplying the usual DBI kinetic term.

The origin of the $\tilde{V}(\phi)$ term is less explicit - but is a sum of terms. One expects open or closed string interactions to generate a scalar potential $V(\phi)$, however the precise form of such an interaction depends upon many factors such as the number of additional branes and geometric moduli, the number of non-trivial cycles in the compact space, and the choice of embedding for branes on these cycles. Typically one can only compute this in special cases in the full string theory. There are also additional terms coming from coupling of the brane to any background RR form fields. The action above is assumed to be that of $D3$-brane(s) filling the space-time directions, which naturally couple to the field $C^{(4)}$ through the Chern-Simons part of the action. However for wrapped $D5$-branes there is also the possibility of a coupling $C^{(4)} \wedge F$, where F is the magnetic field through the two-cycle. For example in the warped deformed conifold one can see that $dC^{(6)} = \star dC^{(2)}$ and therefore there is an additional term in the DBI action

$$S \sim \int d^4x a^3(t) g_s^{-1} M\alpha' T(\phi)\,, \tag{10.4}$$

up to a normalisation factor of order one. Terms such as this have been added to the interaction potential to define the *full* scalar potential $\tilde{V}(\phi)$.

The corresponding equations for the energy density and pressure of the DBI can then be written succinctly as

$$P_\phi = \frac{T(\phi)}{\gamma}\left[\gamma - W(\phi)\right] - \tilde{V}(\phi)\,, \qquad \rho_\phi = T(\phi)\left[W(\phi)\gamma - 1\right] + \tilde{V}(\phi)\,, \tag{10.5}$$

where $\gamma = [1 - \dot{\phi}^2/T(\phi)]^{-1/2}$ is the usual generalisation of the relativistic factor. The subscript $_\phi$ denotes the scalar field component here. We can also immediately define the equation of state parameter for the quintessence field to be

$$w_\phi = \frac{T(\phi)\left[\gamma - W(\phi)\right] - \tilde{V}(\phi)\gamma}{T(\phi)\gamma\left[W(\phi)\gamma - 1\right] + \tilde{V}(\phi)\gamma}\,, \tag{10.6}$$

from which one clearly sees that it is dynamically sensitive and can take a wide range of values. For instance we only recover $w_\phi \sim -1$ in the limit that the field is non-relativistic and the entire solution is dominated by the $\tilde{V}(\phi)$ terms - which will clearly require large amounts of fine tuning to accomplish. There are clearly several regions of parameter space that are of interest. First let us assume that the potential term is zero, either because it is suppressed or there is an unlikely cancellation between the contributing terms. The more general case with non-zero $\tilde{V}$ leads to a wide variety of complex behaviour. We can therefore identify several limits of interest - focusing on the behaviour of W:

- $W(\phi) = 1$ - which reduces the action back to the usual DBI case which has $w_\phi = 1/\gamma$ as discussed in [138].

- $W(\phi) = \alpha\gamma$ - which leads to constant γ if α is constant, since the two are related via $\gamma^2 w_\phi \alpha = 1 - \alpha + w_\phi$. Moreover this again means that $\dot{\phi} \propto t^{-(1+w_\phi)/(1+w)}$ as in the case where $W = 1$.

- $W(\phi) \to 0$ - as could occur in the case of a tachyonic theory, which mimics a dark energy dominates phase with $w_\phi = -1$. However one must be careful if this is to be representative of non-BPS D-brane actions, since the coupling to the form field is non canonical in this instance. In fact the coupling term will typically be of the form $d\phi \wedge C$. This means that there is no solitary $T(\phi)$ term in the action and therefore the equation of state in this instance will vary like $-1/\gamma^2$.

- $W(\phi) \gg \gamma$ - which can occur in the multi-brane/wrapped brane case and yields $w_\phi \sim -1/\gamma^2$.

Note that in all cases the equation of state parameter remains bounded between $-1 \le w_\phi \le 1$.

One can combine the expressions for the energy-momentum tensor components, and together with the continuity equation we obtain the following equation of motion - assuming that the scalar field follows a monotonic path

$$\ddot{\phi} + \frac{3H\dot{\phi}}{\gamma^2} + \frac{3T_\phi}{2\gamma^2} + \frac{1}{W\gamma^3}(\tilde{V}_\phi - T_\phi) - \frac{T_\phi}{2} + \frac{TW_\phi}{W\gamma^2} = 0\,, \tag{10.7}$$

which is a generalisation of the Klein-Gordon equation for the DBI Lagrangian. The subscript $_\phi$ of T, W and $\tilde{V}$ denotes derivative with respect to the field value. The other dynamical equation of motion for the Hubble parameter can be written as

$$\dot{H} = -\frac{1}{2M_p^2}\left[\rho(1+w) + \gamma W(\phi)\dot{\phi}^2\right]\,, \tag{10.8}$$

where we have defined the pressure of the barotropic fluid to be $P = w\rho$ and that it is non-interacting. We leave the interesting case of interacting pressure for future endeavour.

Let us consider, as an example solution, the case where there is a scaling solution with $W = 1$, which has been reviewed elsewhere [139]. We will find it convenient to define the quantity

$$X = \frac{1+w_\phi}{1+w}\,, \tag{10.9}$$

in which case we see that $\dot{\phi} \sim t^{-X}$. This allows us to reconstruct the tension of the brane as follows

$$\begin{aligned} T(\phi) &= \mathcal{M}^4 e^{-\lambda\phi}\,, && X = 1 \\ &= \mathcal{M}^{4+\alpha}\phi^{-\alpha}\,, && X \neq 1 \end{aligned} \tag{10.10}$$

where $\mathcal{M}$ is a dimensionful mass scale, λ is a constant and $\alpha = 2X/(1-X)$. Using the fact that $w_\phi = 1/\gamma$ we can then see that for $X \neq 1$ the solution is physically valid only when $w > 2/\alpha$ since we define γ to be the positive root. Let us now consider the phase-space dynamics of the theory in more detail following along the lines of [132]. It is initially convenient to define the following new variables

$$\begin{aligned} x &= \sqrt{\frac{T(\phi)W(\phi)\gamma}{3}}\frac{1}{HM_p}, \qquad \mu_1 = \frac{\sqrt{T}M_p\tilde{V}_\phi}{\tilde{V}^{3/2}}, \\ y &= \sqrt{W(\phi)\gamma}\frac{\dot{\phi}}{HM_p}, \qquad \mu_2 = -\frac{\sqrt{T}M_pT_\phi}{\tilde{V}^{3/2}}, \\ z &= \sqrt{\frac{\tilde{V}}{3}}\frac{1}{HM_p}, \qquad \mu_3 = \frac{W_\phi M_p}{W^{3/2}\gamma^{5/2}}, \end{aligned} \tag{10.11}$$

in terms of which we can see that $\gamma = [1 - y^2/(3x^2)]^{-1/2}$ and the fluid density parameter can be written as

$$\Omega = 1 - \Omega_\phi = 1 - \left(z^2 + x^2\left[1 - \frac{1}{W(\phi)\gamma}\right]\right), \tag{10.12}$$

whilst the equation of state in dimensionless variables will become

$$w_\phi = \frac{1}{\gamma}\left(\frac{x^2[\gamma - W(\phi)] - z^2W(\phi)\gamma^2}{x^2[W(\phi)\gamma - 1] + z^2W(\phi)\gamma}\right). \tag{10.13}$$

As is customary we will now switch to dimensionless derivatives, denoted by a prime, replacing time derivatives by derivatives with respect to the e-folding number, $\mathcal{N}$. Therefore we can easily determine

$$\frac{H'}{H} = -\frac{y^2}{2} - \frac{3(1+w)}{2}\left(1 - z^2 - x^2\left[1 - \frac{1}{W(\phi)\gamma}\right]\right). \tag{10.14}$$

A useful quantity to calculate is the variation of the kinetic function, which we can write in the following manner using the equation of motion

$$\frac{\dot{\gamma}}{\gamma} = -\frac{3H\dot{\phi}^2}{T} - \frac{W_\phi\dot{\phi}}{W} - \frac{T_\phi\dot{\phi}}{T} - \frac{\dot{\phi}}{\gamma WT}(\tilde{V}_\phi - T_\phi). \tag{10.15}$$

We can then determine the dynamical equations for the dimensionless fields as derivatives with respect to $\mathcal{N}$

$$\begin{aligned} x' &= -\frac{1}{2}(\mu_1 + \mu_2)\frac{yz^3}{x^2} - \frac{y^2}{2x} - x\frac{H'}{H}, \\ y' &= -3y\left(1 - \frac{y^2}{6x^2}\right)\left(1 + \frac{z^3}{xy}[\mu_1 + \mu_2]\right) + \frac{3\mu_2z^3W}{\gamma x} - 3x^2\mu_3 - y\frac{H'}{H}, \\ z' &= \frac{z^2y\mu_1}{2x} - z\frac{H'}{H}, \end{aligned} \tag{10.16}$$

and the remaining parametric solutions are

$$\begin{aligned} \mu_1' &= \frac{\mu_1^2yz}{x}\left(-\frac{3}{2} + \frac{\tilde{V}_{\phi\phi}\tilde{V}}{\tilde{V}_\phi^2} + \frac{T_\phi\tilde{V}}{\tilde{V}\tilde{V}_\phi}\right), \\ \mu_2' &= \frac{\mu_1\mu_2yz}{x}\left(-\frac{3}{2} + \frac{T_\phi\tilde{V}}{2T\tilde{V}_\phi} + \frac{T_{\phi\phi}\tilde{V}}{T_\phi\tilde{V}_\phi}\right), \\ \mu_3' &= y\mu_3^2\gamma^{3/2}\left(1 + \frac{W_{\phi\phi}W}{W_\phi^2} + \frac{5T_\phi W}{2TW_\phi} + \frac{5}{2T\gamma W_\phi}[\tilde{V}_\phi - T_\phi]\right) + \frac{5\mu_3y^2}{2x^2}. \end{aligned} \tag{10.17}$$

Note that if the μ_i are constants, then the previous three equations form an autonomous set and should uniquely specify the dynamics of the quintessence field. We will consider this case as the simplest (canonical) example. If we wish to appeal to string theoretic constructions then we restrict the parameter space of solutions. It is more interesting to consider the above equations in the context of a phenomenological model and see what kind of functions yield the correct behaviour. Explicit constructions of string backgrounds are typically difficult and there are only a few well known examples that are ritually invoked, however if we take string theory seriously then there are undoubtedly other non-trivial backgrounds that are cosmologically interesting but not yet constructed. Since an analytic analysis of this generalised system is highly complicated, it is convenient to use a combination of analytic and numerical methods to understand the dynamics of the system. For a numeric analysis it is necessary to re-write the fluid equation in terms of more useful variables. It turns out that the simplest variables to use are the following

$$\phi' = \Phi\,, \tag{10.18}$$

$$\Phi' = -\frac{3\Phi}{\gamma^2} + \frac{3M_p z^3}{x}\left(\frac{\sqrt{W\gamma}\mu_2}{2}\left[\frac{3}{\gamma^2}-1\right] - \frac{(\mu_1+\mu_2)}{\sqrt{W}\gamma^{5/2}}\right) - \frac{3M_p x^2\mu_3}{\sqrt{W\gamma}} - \Phi\frac{H'}{H}\,, \tag{10.19}$$

which are easily derivable from the terms written above. The equations (10.14), (10.16), (10.17), (10.18) and (10.19) together with barotropic fluid equation: $\rho' = -3\rho(\mathcal{N})(1+w)$, hence form a closed ten-dimensional autonomous system if T, W or $\tilde{V}$ are given as explicit functions of ϕ or as constants.

10.1.1 Case I

Let us take the canonical string theoretic example arising when the local geometry can be approximated by an AdS space. This geometry typically arises in the near horizon limit of coincident $D3$-branes (or flux). In this case we see that (at leading order)

$$T(\phi) = \frac{\phi^4}{\lambda^4}\,, \qquad \tilde{V}(\phi) = \frac{m^2\phi^2}{2}\,, \qquad W(\phi) = W\,, \tag{10.20}$$

where we have also included an effective ϕ^2 potential for the system. This means that $\mu_3 = 0$ and we also have a constant μ_1 which allows us to write the remaining μ terms as

$$\mu_1 = \frac{2\sqrt{2}M_p}{m\lambda^2}\,, \qquad \mu_2 = -\frac{2x^2\mu_1}{W\gamma z^2}\,, \tag{10.21}$$

and therefore the dynamical equations reduce to

$$\begin{aligned} x' &= -\frac{\mu_1 y z^3}{2x^2}\left(1-\frac{2x^2}{W\gamma z^2}\right) - \frac{y^2}{2x} - \frac{xH'}{H}\,, \\ y' &= -3y\left(1-\frac{y^2}{6x^2}\right)\left(1+\frac{z^3\mu_1}{xy}\left[1-\frac{2x^2}{W\gamma z^2}\right]\right) - \frac{6\mu_1 z x}{\gamma^2} - \frac{yH'}{H}\,, \\ z' &= \frac{z^2 y\mu_1}{2x} - \frac{zH'}{H}\,. \end{aligned} \tag{10.22}$$

The simplest way to proceed with the analysis is to consider the final equation above, since this splits the solution space neatly into two components. Thus we search for solutions where either $z = 0$ or $z = (2x/y\mu_1)H'/H$ as initial conditions.

The first sub-set of solutions admits $(0, 0, 0)$ as a (trivial) fixed point, which is a fluid dominated solution since $\Omega = 1$ in this instance. Let us remark here that this fixed point solution will occur for *all* the cases we consider, however since this implies a vanishing of the action, causality implies that this fixed point must be unstable - i.e. phase space trajectories will flow away from it. By making this field a phantom scalar, one can evade this causal bound and the point can become a stable fixed point. This behaviour arises in many places in the literature, so we will not discuss it further here.

There is also a critical point at $(1, \sqrt{3}, 0)$ which is a kinetic dominated solution. This solution actually exists as solutions to the quadratic expression $y^2 = 3x^2$ which corresponds to the limit $\gamma \to \infty$. In terms of the density parameter, a quick calculation shows that along the general curve (parameterised by y_0 and x_0) we find $\Omega = 1 - x_0^2$. Thus at the trivial fixed point we see $\Omega \to 1$, however for $x_0 \to 1$ we see that $\Omega \to 0$ corresponding to non-relativistic matter, i.e. dust. In this instance we also find $a(t) \sim t^{2/3}$ as expected from the cosmological evolution equations. Again due to the special algebraic properties of the DBI action, we anticipate that this solution will also be found for the other cases of interest.

The second sub-set of solutions are more interesting, as initially one can solve the system by slicing the phase space at $y = 0$ [3]. One can use the condition on H' to fix z through $z^2 = 1 - x^2(W - 1)/W$. Combining this with the equations of motion gives us the following fixed point (taking positive signs of all roots for simplicity)

$$x = \sqrt{\frac{W}{1-W}}\,, \qquad y = 0\,, \qquad z = 1\,, \tag{10.23}$$

which is valid for all $W < 1$ in order for these points to be real and at finite distance in phase space. If we then compute the density of the fluid we find $\Omega = 0$, since $\Omega_\phi = 1$, which corresponds to a purely dust-like solution. Note that this class of solutions does not exist for the simple $D3$-brane analysis as in [139], since it arises from additional degrees of freedom which are neglected in these models. The remaining solutions in this sub-set are difficult to find analytically.

More generally we can see that the above solution corresponds is a special case of the more general Case I behavior, which we paramaterise by

$$T(\phi) = \frac{\phi^\alpha}{\lambda^\alpha}\,, \qquad \tilde{V}(\phi) = \frac{m^\beta \phi^\beta}{\beta}\,, \qquad W(\phi) = W\,, \tag{10.24}$$

where we can then explicitly write

$$\mu_1 = A\left(\frac{x}{z\gamma^{1/2}}\right)^{(\alpha-\beta-2)/(\alpha-\beta)}\,, \qquad \mu_2 = -\frac{\alpha}{\beta}\frac{\mu_1}{W\gamma}\frac{x^2}{z^2}\,, \qquad \mu_3 = 0\,, \tag{10.25}$$

where A is a (real, positive) constant provided that $\beta > 0$.

$$A = \frac{M_p \beta^{3/2}}{\lambda^{\alpha/2} m^{\beta/2}}\left(\frac{\lambda^\alpha m^\beta}{\beta W}\right)^{(\alpha-\beta-2)/2(\alpha-\beta)}\,, \tag{10.26}$$

[3]Note that one cannot do this for $x = 0$ since the action becomes singular and ill-defined.

but which simplifies in the limit $\alpha = \beta + 2$. As before, the solution space splits into two disconnected sub-sets, therefore in the first instance where we take slices through $z = 0$, we find the following bound

$$\frac{2}{(\alpha - \beta)} > 0 \,, \tag{10.27}$$

which implies that $\alpha > \beta$ and so the brane tension should dominate the dynamics (in the large field regime). Let us therefore assume that α, β are chosen such that this condition is satisfied - then we find the solution branch is governed again by the relation $y^2 = 3x^2$ as expected - which contains the solution $(0, 0, 0)$ as a special case. Moreover this is valid for all values of α, β satisfying the above constraint. The secondary solution branch occurs when we find solutions to

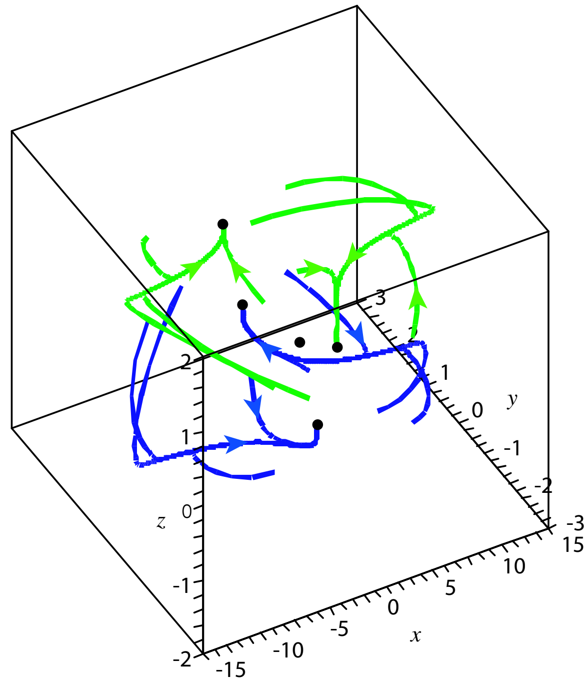
$$\frac{zy\mu_1}{2x} = \frac{H'}{H} \,, \tag{10.28}$$

which is generally very complicated. A simple set of solutions do arise when we consider slices at $y = 0$, since the fixed points are localised along the curve

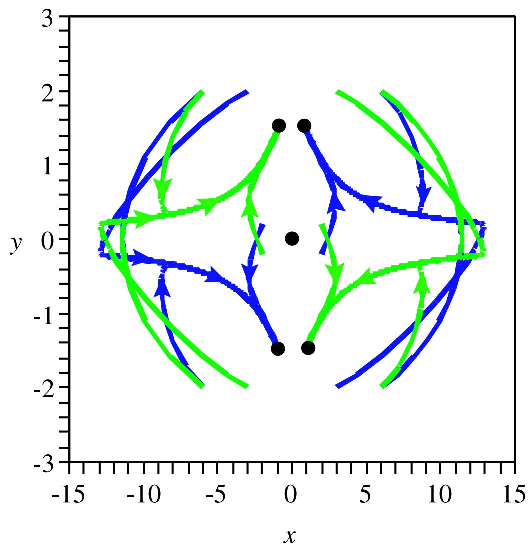
$$x = \pm\sqrt{\frac{\beta W}{(\alpha - \beta)(1 - W)}}, \qquad y = 0, \qquad z = \pm\sqrt{\frac{\alpha}{(\alpha - \beta)}} \,, \tag{10.29}$$

which corresponds to a dust-like solution $\Omega = 0 \;\forall \alpha, \beta$. The reality constraint here demands that $\alpha > \beta$ which in turn fixes $W < 1$. However there are also additional solutions where $\beta < 0$ and positive α - provided that $W > 1$. Explicit realisations of this scenario within a string theory context can arise through potentials arising from brane/anti-brane interactions and is therefore a non-trivial and interesting solution.

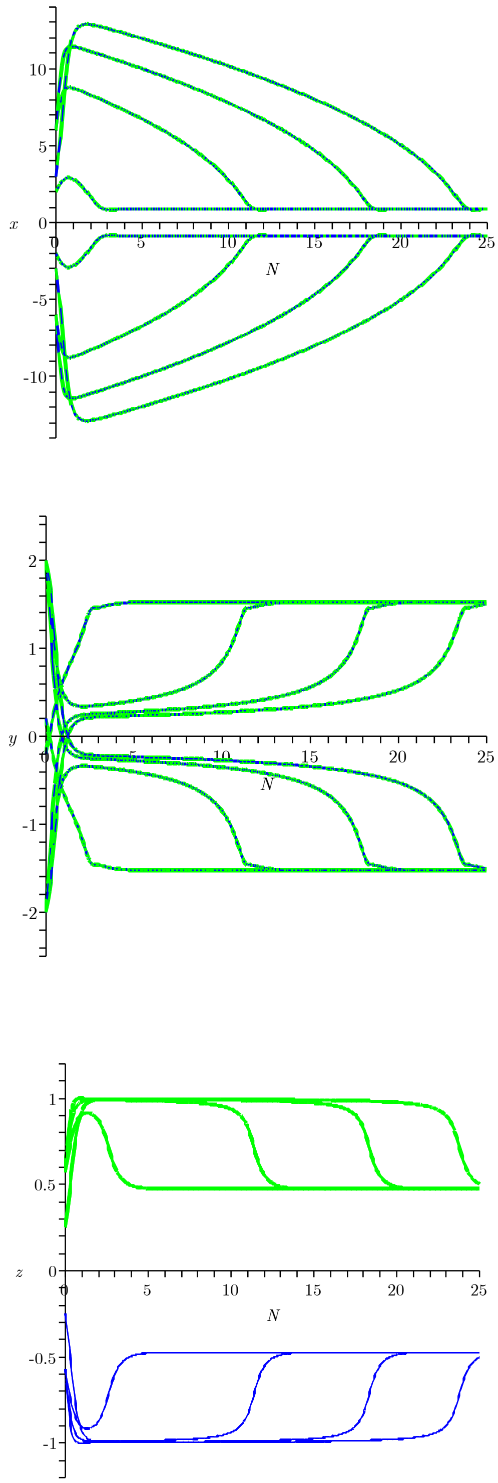
Figs. 10.1 and 10.2 show the numerical solutions in phase space. For $W = 1$ case, the numerical constants are given as $M_p = 1, m = 1, \lambda = 1$ and $w = 0$ (dust case). Other parameters are $\alpha = 4, \beta = 2$ and $A = 2\sqrt{2}$. As expected the (five) fixed points all lie along the curve $y^2 = 3x^2$. We also plot the evolution of each parameter (x, y, z) as a function of the e-folding number in Fig. 10.3 where each of the coordinates tends to its critical value. As expected the phase space dynamics are $\mathbb{Z}_2$ symmetric about the origin. Note that in the case of $y(N)$ one can keep y suppressed for a few e-foldings with enough tuning, before eventually it evolved towards the points $\pm\sqrt{3}$ at late times. The full numerical solution of the case $W < 1$ is illustrated in Fig. 10.4 where $W = 0.95$, which uniquely fixes the critical points to be $x = \pm\sqrt{20}, y = 0, z = 1$. As one can see from the resulting plot, this is an unstable node because the general behaviour is divergent. Note that $x \to \infty$ in this regime effectively solves all the dynamical equations trivially.



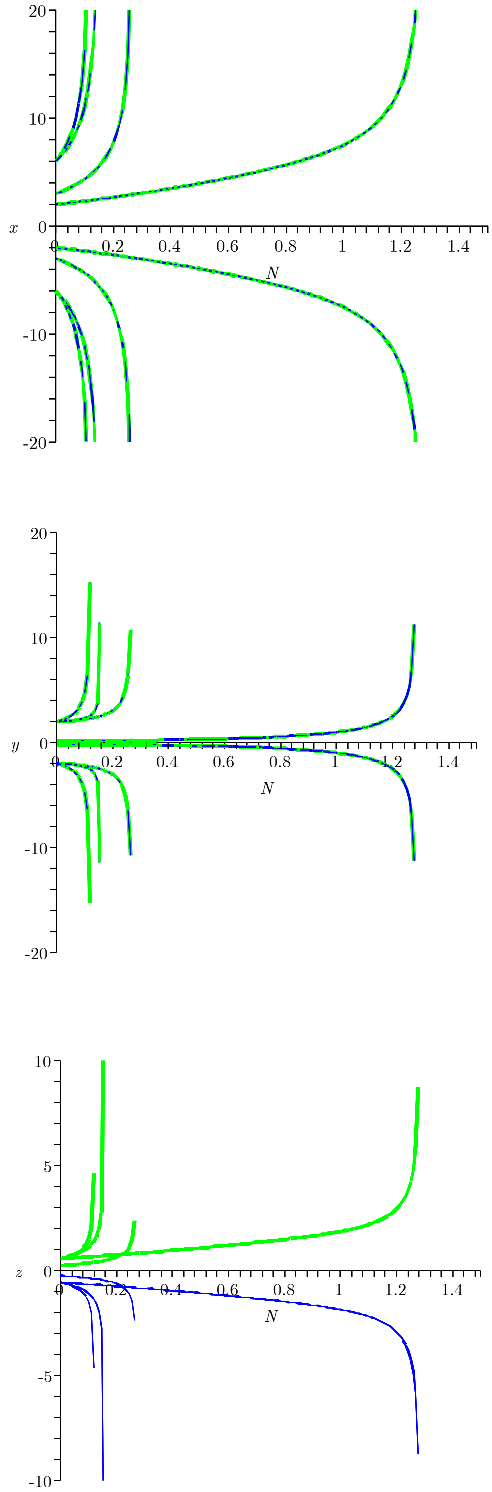
รูปที่ 10.1: (Case I) 3-D xyz phase space trajectories for $T(\phi) = \phi^4/\lambda^4, \tilde{V}(\phi) = m^2\phi^2/2$ and $W(\phi) = W$. We have set here, $M_p = 1, W = 1, m = 1, \lambda = 1$ and $w = 0$ (dust case).



รูปที่ 10.2: (Case I) Phase space trajectories in xy plane. Four attractors $(\pm 1, \pm\sqrt{3}, 0)$ and one unstable node (0, 0, 0) can be seen here. z is bounded within $(-1, 1)$ range.



รูปที่ 10.3: (Case I) Evolution of x, y, z versus e-folding number setting $W = 1$.



รูปที่ 10.4: (Case I) Evolution of x, y, z versus e-folding number setting $W = 0.95$. All solutions diverge from the origin

10.1.2 Case II

Analagous to the first case, let us now consider another branch of solutions where this time the tension of the brane is taken to be constant. This dramatically alters the relativistic rolling of the scalar field since the γ factor is no longer warped. Initially let us consider the ansatz

$$\tilde{V}(\phi) = \frac{m^2\phi^2}{2}, \qquad T(\phi) = T, \qquad W(\phi) = \frac{\phi^4}{\lambda^4}, \tag{10.30}$$

which implies that

$$\mu_1 = \left(\frac{4\sqrt{2T^3}M_p}{\lambda^4 m^3}\right)\frac{z^2\gamma}{x^2}, \qquad \mu_2 = 0, \qquad \mu_3 = \frac{2z\mu_1}{\gamma^2 x}, \tag{10.31}$$

and the corresponding field equations become

$$\begin{aligned} x' &= -\frac{\alpha\gamma y z^5}{x^4} - \frac{y^2}{2x} - \frac{xH'}{H}, \\ y' &= -3y\left(1-\frac{y^2}{6x^2}\right)\left(1+\frac{\alpha\gamma z^5}{2x^3 y}\right) - \frac{6\alpha z^3}{\gamma x} - \frac{yH'}{H}, \\ z' &= \frac{\alpha\gamma z^4}{2x^3} - \frac{zH'}{H}, \end{aligned} \tag{10.32}$$

where we have defined α as the constant prefactor in the definition of μ_1.

As before we separate the solution space into two - first finding solutions to $z = 0$ and then solutions to $H'/H = \alpha\gamma z^3/(2x^3)$. In the first case is is straightforward to see that there are the usual fixed point solutions at $(0,0,0)$ and $(1,\sqrt{3},0)$ (with their respective partner solutions) respectively coming from the usual condition that $y^2 = 3x^2$. The secondary branch of solutions also admit fixed points when $y = 0$, however the condition on z is that $z = 0, -4x^2$. Since we want real solutions we are forced to set $z = 0$ as a secondary constraint. This forces W to diverge and therefore in the limit that $z \to 0$ we find that $x^2 \to \pm 1$ which is a unique solution. Again the density parameter vanishes identically in this limit as one would expect. The remaining solutions are actually extremely difficult to solve analytically as they correspond to high order polynomials. As a result we are forced to sketch their behaviour numerically.

Phenomenologically we see that the ansatz presented above is a special class of the more general solution

$$T(\phi) = T, \quad \tilde{V}(\phi) = \frac{m^\beta\phi^\beta}{\beta}, \quad W(\phi) = \frac{\phi^\alpha}{\lambda^\alpha}, \tag{10.33}$$

which has the parameterisation constraints

$$\mu_1 = A\left(\frac{z\gamma^{1/2}}{x}\right)^{(2+\beta)/(\alpha-\beta)}, \quad \mu_2 = 0, \quad \mu_3 = B\gamma^{(2-4\alpha+5\beta)/2(\alpha-\beta)}\left(\frac{z}{x}\right)^{(2+\alpha)/(\alpha-\beta)}, \tag{10.34}$$

where A, B are both constants. One can see from the dynamical equations that fixed points with $z = 0$ can only occur when the following condition is met

$$\frac{2(1+\alpha)-\beta}{\alpha-\beta} > 0, \tag{10.35}$$

which is trivially satisfied for cases where $\alpha > \beta$ (which we assume as an initial constraint).

More generically we see that, provided $\alpha > -2$, we recover the usual fixed point equation $y^2 = 3x^2$. However we need to be careful here because if this condition is satisfied then W becomes undefined. Since this is the overall prefactor multiplying the DBI action, the action is undefined in this limit and it should therefore correspond to a point of instability in the phase space. In the limit where $\alpha = -2$, which implies that $\beta > -2$, the fixed point solution now lives on the zeros of the polynomial

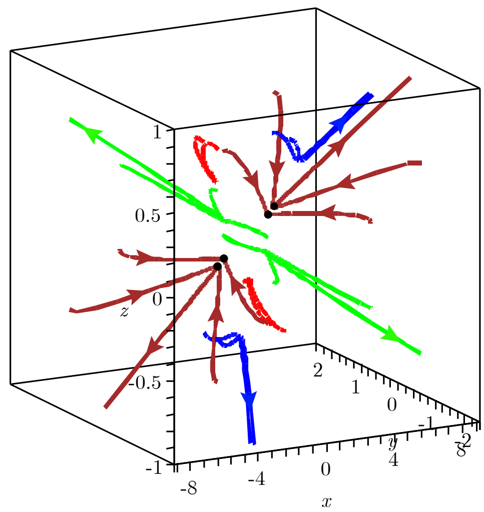
$$3x^4 B\gamma^{-5(2+\beta)/2(1+\beta)} + 3x^2 y - y^3 = 0\,, \tag{10.36}$$

which can be used to fix $x = x(y)$ or vice-versa. This solution is actually indicative of a more general branch of physical solutions where we take $\beta > 2(1+\alpha)$. The resulting fixed point equation (provided $\alpha \neq 2$) is trivially calculated to be $y^2 = 3x^2$ as before, but now we see that W vanishes identically. In turn this means that the kinetic terms also vanish and the solution is dominated solely by the potential interaction. One could imagine a situation such as this occurring in the condensation of an open string tachyon mode on a non-BPS brane, where the vanishing of W indicates that we are living in the closed string vacuum. For dynamic solutions it seems reasonable to consider this particular case as the late time attractor for the solution $z \to 0$.

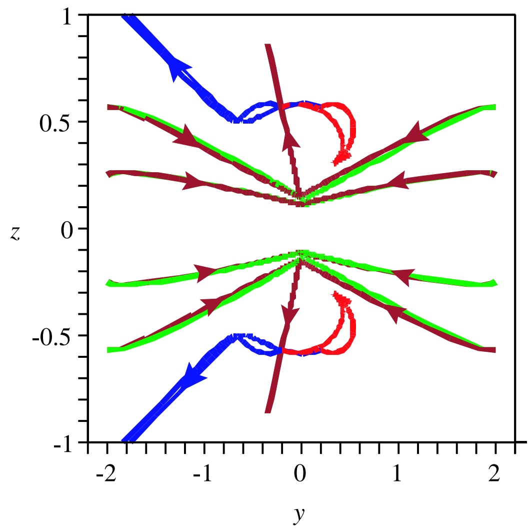
The second sub-set of solutions is again complicated, but again we can analytically understand the plane at $y = 0$, which gives us the fixed point solutions

$$x = \pm\sqrt{-\frac{\beta z^2}{\alpha}}\,, \qquad y = 0\,, \qquad z = \pm\left(1 - \frac{\beta}{\alpha}\left(1 - \left[-\frac{\alpha T}{\lambda^\alpha m^\beta}\right]^{\alpha/(\alpha-\beta)}\right)\right)^{-1/2}\,. \tag{10.37}$$

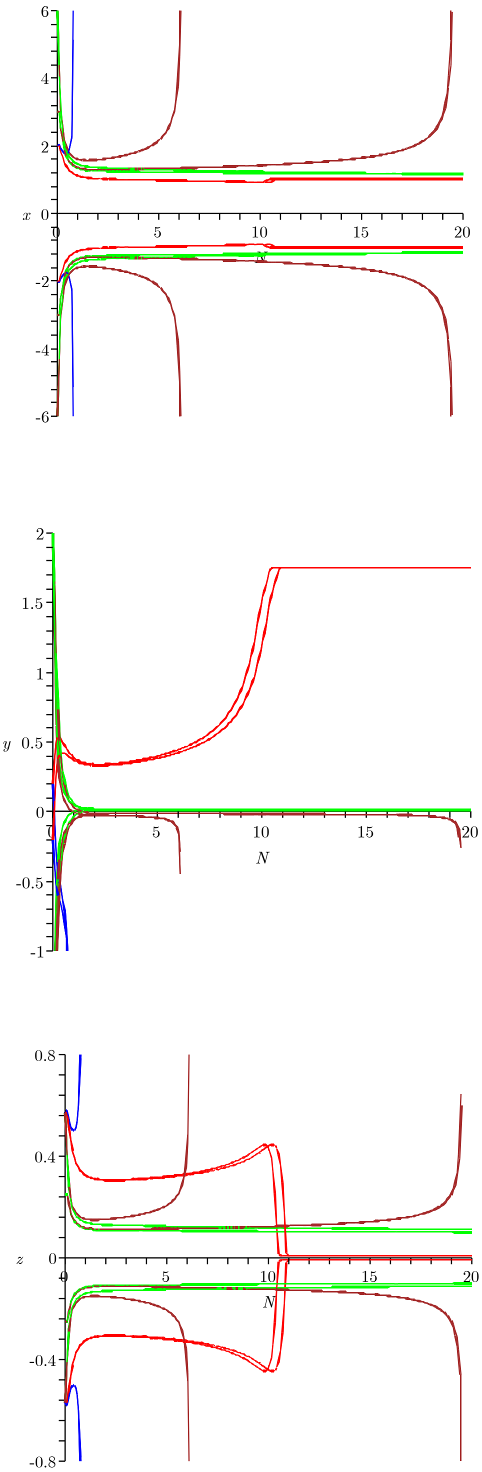
Clearly for the solution to be real we require that α, β have opposite signs. This satisfies our primary constraint, therefore is a physical possibility. Moreover in the limit where we set $\beta = -\alpha$, we find that $\Omega = 0$ which is again the dust solution. Illustrations of numerical solutions for the case II are in Figs. 10.5, 10.6 and 10.7. Constants are set as $M_p = 1, T = 1, m = 1, \lambda = 1$ and $w = 0$ (dust case). Other parameters are $\alpha = 4, \beta = 2$. From the numerical analysis one sees that there are six saddle nodes, only two attractors and one repulsive point which is the origin $(0, 0, 0)$ as expected. The dynamical trajectories are particularly interesting due to their apparent lack of monoticity as a function of e-fold number. The z term in particular appears to have a large variation in trajectory, diverging in some instances whilst rapidly reaching zero in other instances. Conversely the y variable displays very uniform (physical) trajectory behaviour, with several curves almost on top of one another at $y = 0$ and the remainder smoothly driven to the (unstable) critical point $y_c \sim 1.8$ in the example given.



รูปที่ 10.5: (Case II) 3-D xyz phase space trajectories for $T(\phi) = T, \tilde{V}(\phi) = m^2\phi^2/2$ and $W(\phi) = \phi^4/\lambda^4$. We have set here, $M_p = 1, T = 1, m = 1, \lambda = 1$ and $w = 0$ (dust case).



รูปที่ 10.6: (Case II) Trajectory slice through the yz plane.



รูปที่ 10.7: (Case II) Evolution of x, y, z versus e-folding number.

10.1.3 Case III

Let us now consider a new case where only $W = W(\phi)$, with all the other terms being constant. We will take $W = \phi^\alpha/\lambda^\alpha$ for generality - which in turn should impose a constraint on the allowed values of α. In this case we see that

$$\mu_1 = 0 \qquad \mu_2 = 0 \qquad \mu_3 = \alpha A \left(\frac{z}{x}\right)^{(\alpha+2)/\alpha} \frac{1}{\gamma^{(2\alpha-1)/\alpha}} \tag{10.38}$$

where A is a function of the constant parameters $A = M_p/\lambda(T/\tilde{V})^{(\alpha+2)/2\alpha}$. Because only μ_3 is non-zero the resulting dynamical expressions are considerably easy to work with

$$\begin{aligned} x' &= -\frac{y^2}{2x} - \frac{xH'}{H} \\ y' &= -3y\left(1 - \frac{y^2}{6x^2}\right) - 3\alpha A z^{(\alpha+2)/\alpha} x^{(\alpha-2)/\alpha} - \frac{yH'}{H} \\ z' &= -\frac{zH'}{H} \end{aligned} \tag{10.39}$$

Considering the slice again through $z = 0$, we see that the solutions split into two types depending upon the integer α. We recover the usual $y^2 = 3x^2$ curve only when $\alpha > 0$ or when $\alpha < -2$. If $\alpha = -2$ then the corresponding polynomial equation becomes

$$y\gamma^{9/2} = 2Ax^2 \tag{10.40}$$

which is difficult to solve analytically due to the dependence of γ on both x, y. This expression does not admit anything but the trivial solution if we set y to zero[4]. Again we see that there is a potential problem here since the potential W goes like $1/z^2$, and is therefore divergent in this limit. Solutions to this expression are possible, but complicated. Interestingly there does exist a solution curve given by

$$y^2 = ax_c^2, \qquad x_c = \frac{81}{2A}\frac{\sqrt{3a}}{(9-3a)^{9/4}} \tag{10.41}$$

where the parameter a factor must satisfy $0 \leq a < 3$ for this solution to be physical. Since a need not be integer, there are essentially a continuum of curves giving rise to fixed points in this theory.

The secondary branch of solutions again admit fixed point behaviour for $y = 0$, however things are more complicated since the fixed points are now obtained by solving more non-linear expressions. There are two cases of immediate interest however. Firstly if we have $\alpha = 2$ then we see that $z^2 = -1/(2A)$ which is only real when A is negative. Since we have chosen our parameterisation such that this quantity is positive, this particular branch of solutions is ruled out. Interestingly when $\alpha = -2$ there is a unique fixed point located at

$$x = \pm\frac{1}{2A}, \qquad y = 0, \qquad z = \pm\frac{1}{\sqrt{T/\tilde{V} - 1}}\sqrt{\frac{1}{2A} - 1}. \tag{10.42}$$

[4]By trivial we mean the point $(0, 0, 0)$

which corresponds to a positive definite equation of state parameter

$$\Omega = \frac{2T^2 A(A-1) + \tilde{V}^2(T/\tilde{V}-1)}{AT\tilde{V}(T/\tilde{V}-1)(2A-1)}. \tag{10.43}$$

Note that we must require $T > \tilde{V}$ for this solution to be non-singular, which means (again) that the tension term dominates the energetics of the theory. What is also obvious is that demanding $A = 1/2$ leads to a novel fixed point at $(\pm 1, 0, 0)$ regardless of the ratio $T/\tilde{V}$. Using the definition of A this fixes $\lambda = 2M_p$ and therefore W is vanishingly small unless the scalar takes is trans-Planckian. This is manifest in a divergence in the equation of state parameter and is therefore unphysical. Therefore we must ensure that $A < 1/2$ implying that $\lambda > 2M_p$. Since this is the largest scale in our theory, one again expects this to be unphysical.

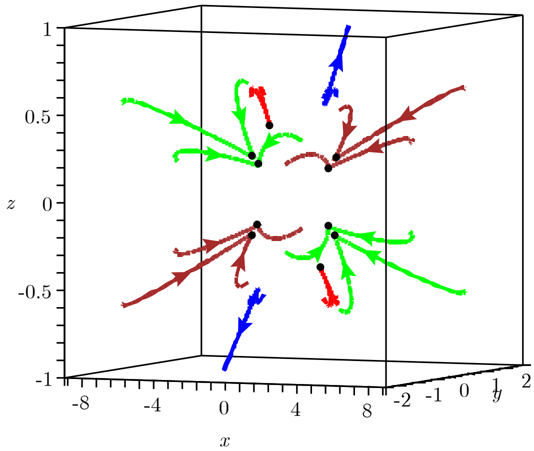
The more general solutions can be found numerically and correspond to $x_0^2 = 1 + z_0^2(T/\tilde{V}-1)$ where z_0^2 are the characteristic solutions to the non-linear equation

$$1 + \alpha A z^{(2+\alpha)/2}\left(1 + z^2(T/\tilde{V}-1)\right)^{(\alpha-2)/2\alpha} = 0. \tag{10.44}$$

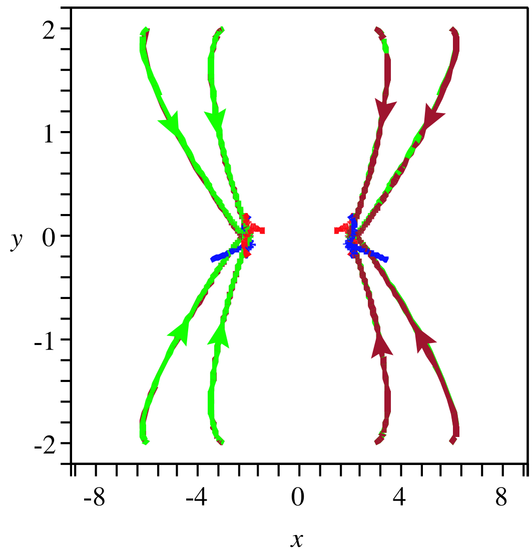
In this more general case we can set $T = \tilde{V}$ without the solution diverging, and we therefore find the corresponding fixed point solution is thus given by

$$x = \pm 1, \quad y = 0, \quad z = \left(-\frac{1}{A\alpha}\right)^{2/(2+\alpha)} \tag{10.45}$$

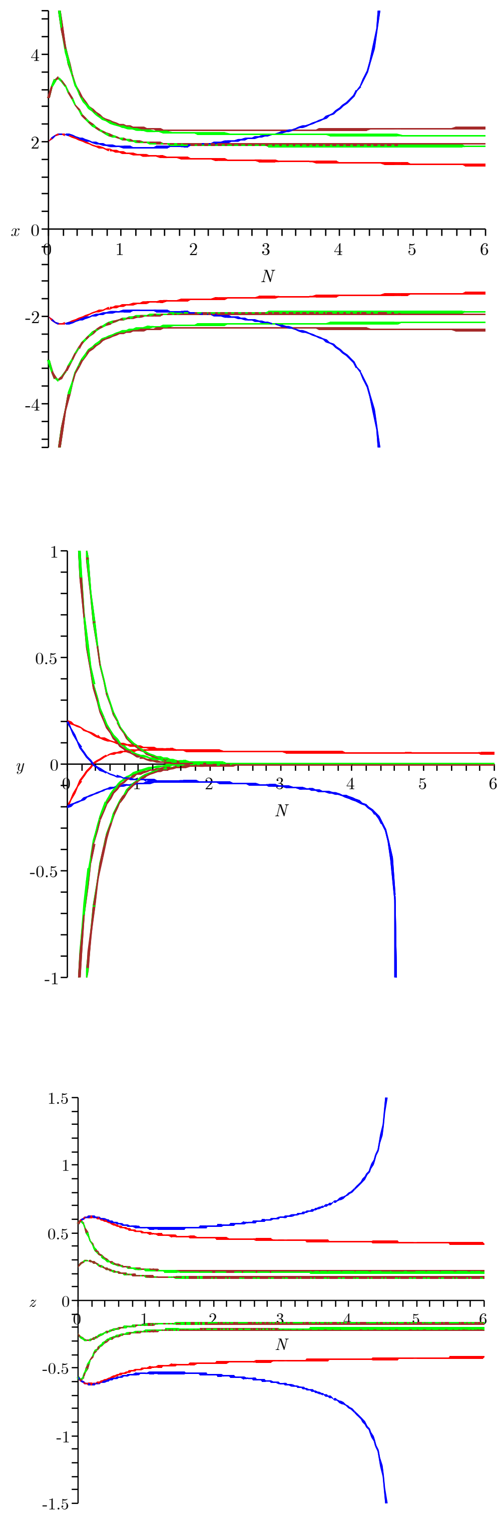
which implies that α is negative. Moreover we see that Ω is again zero here for all physical values of α, although there is no additional constraint upon the magnitude of A. Now, we see numerical solutions in Figs. 10.8, 10.9 and 10.10. Constants are set as $M_p = 1, T = 1, \tilde{V} = 1, \lambda = 1$ and $w = 0$ (dust case). Other parameters are $\alpha = 1$ and $A = 1$.



รูปที่ 10.8: (Case III) 3-D xyz phase space trajectories for $T(\phi) = T, \tilde{V}(\phi) = V$ and $W(\phi) = \phi^{\alpha}/\lambda^{\alpha}$. We have set here, $M_p = 1, T = V = 1, m = 1, \lambda = 1$ and $w = 0$ (dust case), $\alpha = 1$. Green lines approach an attractor



รูปที่ 10.9: (Case III) Phase space trajectories in xy plane.



รูปที่ 10.10: (Case III) Evolution of x, y, z versus e-folding number.

10.1.4 Case IV

Following on from the previous class of models, we can find solutions where the scalar potential is now constant, using the ansatz.

$$\tilde{V} = V, \qquad T(\phi) = \left(\frac{\phi}{\lambda}\right)^{\alpha}, \qquad W(\phi) = \left(\frac{\phi}{\delta}\right)^{\beta} \tag{10.46}$$

where λ, δ are terms of the requisite dimensionality. From this expression we see that μ_1 is identically zero. It will be convenient to define the following function $Q = V\lambda^{\alpha}\delta^{\beta}$ which in turn can be used in the definitions of the remaining μ_i functions

$$\begin{aligned} \mu_2 &= -\frac{\alpha M_p}{\lambda^{\alpha/2}V}\left(\frac{Qx^2}{\gamma z^2}\right)^{n_1} \\ \mu_3 &= \frac{\beta M_p \delta^{\beta/2}}{\gamma^{4/2}}\left(-\frac{\mu_2\lambda^{\alpha/2}V^{3/2}}{\alpha M_p}\right)^{-n_2} \\ n_1 &= \frac{3\alpha-2}{2(\alpha+\beta)}, \qquad n_2 = \frac{1+\beta}{3\alpha-2} \end{aligned} \tag{10.47}$$

and now the dynamical equations simplify to become

$$\begin{aligned} x' &= -\frac{\mu_2 y z^3}{2x^2} - \frac{y^2}{2x} - \frac{xH'}{H} \\ y' &= -3y\left(1-\frac{y^2}{6x^2}\right)\left(1+\frac{z^3\mu_2}{xy}\right) + 3\mu_2\left(\frac{z^{3\alpha+\beta}}{x^{\beta-\alpha}\gamma^{2\beta+\alpha}}\left[\frac{Q}{\delta}\right]^{\beta}\right)^{1/(\alpha+\beta)} - 3x^2\mu_3 - \frac{yH'}{H} \\ z' &= -\frac{zH'}{H}. \end{aligned} \tag{10.48}$$

The resulting analysis is far more complicated than in the previous cases. Let us again start with the simplest solution slices at $z = 0$. The expressions for x' and z' readily simplify in this instance, however the equation for y' requires us to be more careful. We see that in order for the $z^3\mu_2$ term to vanish in this limit we require $(2+3\beta)/(\alpha+\beta) > 0$. The remaining μ_2 term only vanishes if this condition is tightened to $(2+\beta)/(\alpha+\beta) > 0$ and the term coming from μ_3 only vanishes if $(1+\beta)/(\alpha+\beta) > 0$. If these inequalities are reversed, for example, then these terms diverge in the $z \to 0$ limit. If we restrict ourselves to well-behaved solutions such that α, β satisfy the above bounds (either by both $\alpha, \beta \geq 0$ or by $\alpha \geq 0, \beta \leq 0$ with $|\beta| > |\alpha|$), then we obtain the solution curve $y^2 = 3x^2$ as usual. If the parameters α, β do not satisfy at least the minimal bound, then one can only solve these expressions numerically.

The only other solution branch occurs when $H'/H = 0$. This is again a complicated solution, however things simplify somewhat when we slice through $y = 0$, but also tune the solution such that $\alpha = \beta$, which gives us

$$z = \frac{x\delta^{\alpha}}{2\sqrt{Q}}\left(1 \pm \sqrt{1 - \frac{4Q(x^2-1)}{x^2\delta^{2\alpha}}}\right) \tag{10.49}$$

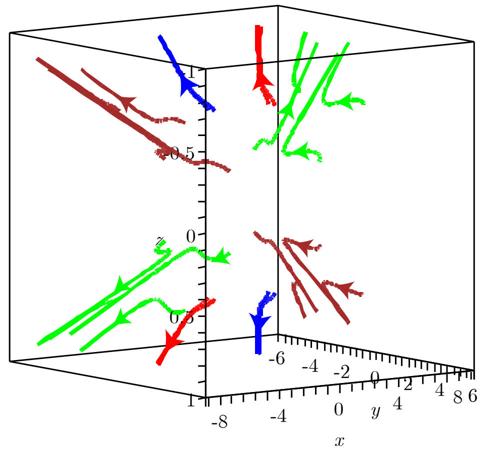
and therefore the fixed point solution in this instance is given by solutions of the polynomial

$$x\sqrt{\frac{Q}{\delta}} + \left(\frac{\sqrt{Q}}{z}\right)^{1/(2\alpha)} \frac{x^{(1+8\alpha)/(4\alpha)}}{\lambda^{\alpha/2}\delta^{\alpha/2}} = 1. \tag{10.50}$$

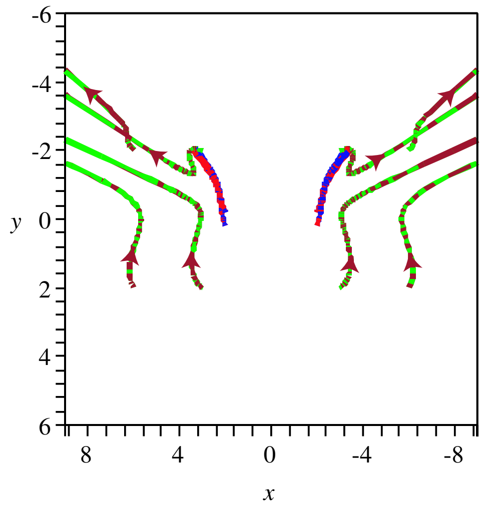
This can actually be solved exactly when $\alpha = -1$, but numerically for more general α. The exact case gives us the following solution

$$\begin{aligned} x_0 &= \frac{Q\delta - 2\sqrt{Q} + \delta^2 \pm \delta\sqrt{F(\lambda,\delta)}}{2\lambda\delta^4} \\ F(\lambda,\delta) &= Q^2 + \delta^2 - 4\sqrt{Q^3\delta} + 6Q\delta - 4\sqrt{Q\delta^3} + 16\sqrt{Q^3\delta^7} - 4Q^3\delta^2 \\ &+ 16\sqrt{Q^{5/2}\delta^{5/2}} - 24Q^2\delta^3 - 4Q\delta^4 + 4\delta^6\lambda \end{aligned} \tag{10.51}$$

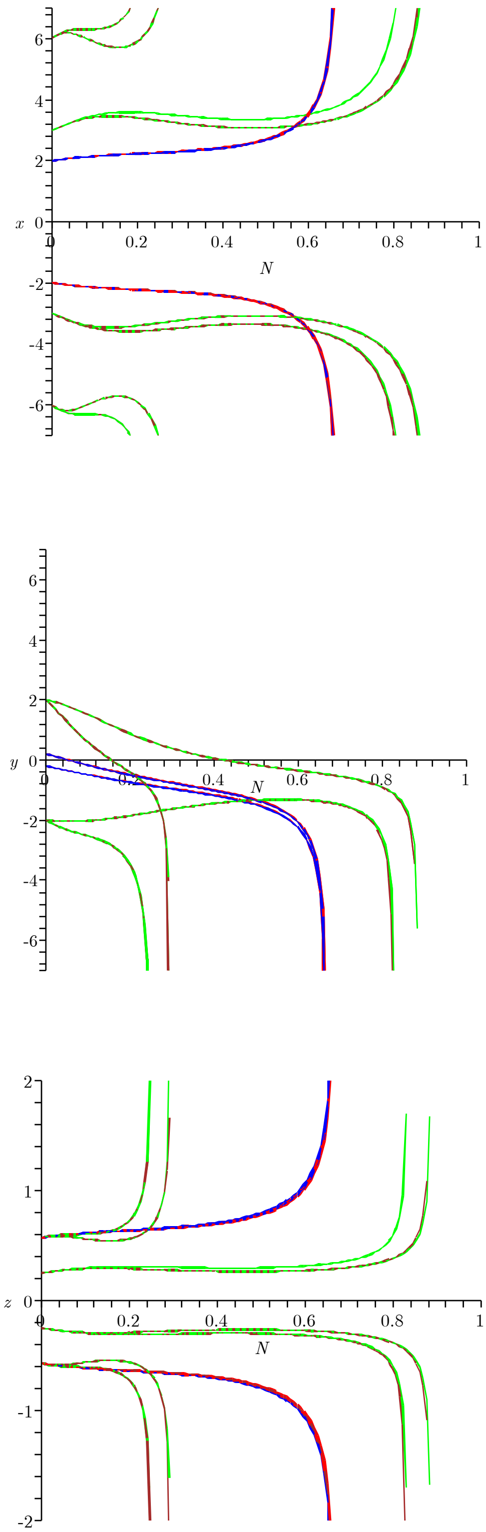
where z_0 is given by the term written above. This is a highly complicated solution, but one sees that in principle there are many fixed points along the plane $(x_0, 0, z_0)$ depending on the constants λ, δ. One also sees that there is a simple solution when $x = 1$, since this implies that $z_0 = \delta^\alpha/\sqrt{Q}$ or $z_0 = 0$, the latter again giving rise to the point $(1, 0, 0)$ which corresponds to the non-propagating end point of the brane dynamics.



รูปที่ 10.11: (Case IV) 3-D xyz phase space trajectories for $T(\phi) = (\phi/\lambda)^\alpha, \tilde{V}(\phi) = V$ and $W(\phi) = (\phi/\delta)^\beta$. Here, $M_p = 1, V = 1, m = 1, \lambda = 1, \alpha = 1, \beta = 1, \delta = 1$ and $w = 0$ (dust case)



รูปที่ 10.12: (Case IV) Phase space trajectories in xy plane.



รูปที่ 10.13: (Case IV) Evolution of x, y, z versus e-folding number.

10.1.5 Case V

Finally let us comment on perhaps the most general form of the solution one could obtain from this model, namely that corresponding to turning on all the relevant degrees of freedom. One can therefore see that Cases $I - IV$ are actually slices through the full phase space described in this section. We will take the following parameterisation for simplicity

$$T = \left(\frac{\phi}{\lambda}\right)^{\alpha}, \quad W = \left(\frac{\phi}{\delta}\right)^{\beta}, \quad \tilde{V} = \frac{m^{\xi}\phi^{\xi}}{\xi}. \tag{10.52}$$

In this case we will have all three μ_i non zero which complicates the analysis somewhat, and reality again imposes the condition that $\xi > 0$. Let us initially search for the fixed points around $z = 0$. The primary constraint equation for this becomes

$$\frac{\alpha - \xi + 2(1-\beta)}{(\alpha + \beta - \xi)} > 0 \tag{10.53}$$

Let us initially assume that the denominator is positive definite. Going through the same analysis as before yields the usual solution curve $y^2 = 3x^2$ provided that we tune $\beta > 0$ and $\alpha + \beta > \xi$. However with reference to the action, we see that this situation leads to both W, T diverging and therefore we should be wary of this part of the solution. Returning to the constraint equation let us therefore assume that $\xi > \alpha + \beta$ and re-do the analysis. We then find that the $y^2 = 3x^2$ is perfectly valid, and moreover the parameters W, T are not divergent provided that the parameters satisfy $\alpha + \beta - \xi < -(2 + \beta)$. Moreover we also see that β is bounded from above such that $\beta < -2/3$ - thus severely restricting the form of the variable phase space.

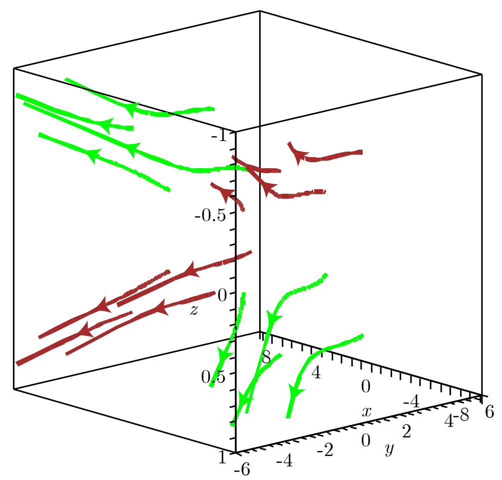
If we search for solutions along the $y = 0$ slicing things are again complicated. However we can simplify things by identifying $\alpha = \xi$, since we can then solve explicitly for x via

$$x^2 = 1 + z^2 \left(\frac{\xi^2}{\lambda^{\xi} m^{\xi}} - 1\right). \tag{10.54}$$

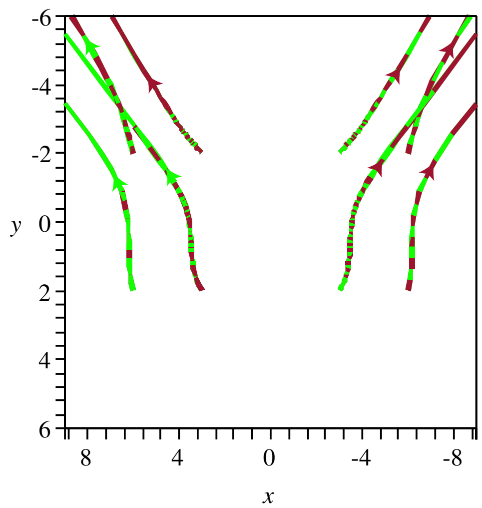
The remaining equation coming from $y' = 0$ has several solutions. The simplest being $z^2 = 0, (\lambda^{-\xi} m^{-\xi} \xi^2 - 1)^{-1}$ which give rise to the points

$$\begin{aligned} x_0 &= \pm\sqrt{2}, \quad y_0 = 0, \quad z_0 = \frac{1}{\sqrt{\lambda^{-\xi} m^{-\xi} \xi^2 - 1}} \\ x_0 &= \pm 1, \quad y_0 = 0, \quad z_0 = 0 \end{aligned} \tag{10.55}$$

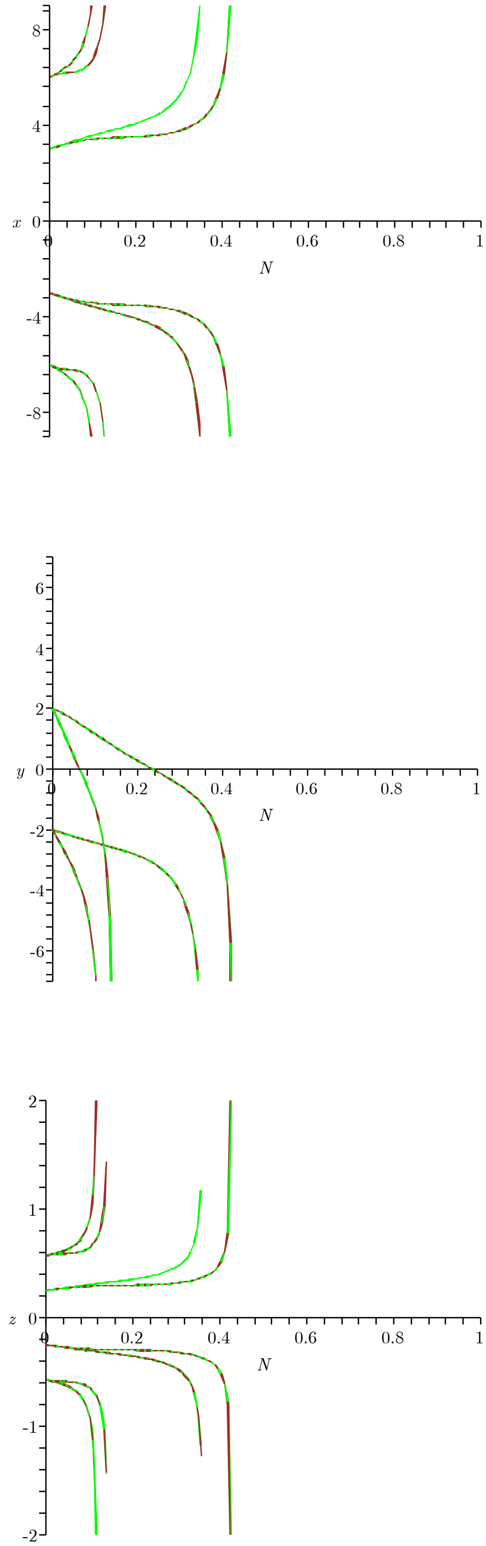
however the first of these conditions also requires that $\xi^{2/\xi} > \lambda m$ for the solution to be real. The maximal value of $\xi^{2/\xi}$ is actually given by $\xi = e^1$ which imposes a tight constraint on the background parameters which can only be satisfied through substantial fine-tuning. Again more general solutions are only available through numeric methods.



รูปที่ 10.14: (Case V) 3-D xyz phase space trajectories for $T(\phi) = (\phi/\lambda)^{\alpha}, \tilde{V}(\phi) = (m\phi)^{\xi}/\xi$ and $W(\phi) = (\phi/\delta)^{\beta}$. Here, $M_p = 1, V = 1, m = 1, \lambda = 1, \alpha = 1, \beta = 1, \delta = 1, \xi = 2$ and $w = 0$ (dust case)



รูปที่ 10.15: (Case V) Phase space trajectories in xy plane.



รูปที่ 10.16: (Case V) Evolution of x, y, z versus e-folding number.

10.2 Perturbations and fixed point stability.

We now need to evaluate the stability of these fixed point solutions. Clearly one may anticipate that solutions such as $(0,0,0)$ may well be unstable. We must perturb the field equations about small values; therefore we need

$$x \to x_0 + \delta x, \qquad y \to y_0 + \delta y, \qquad z \to z_0 + \delta z. \tag{10.56}$$

Now the analysis is more complicated than in standard models due to the complexity of the DBI action and the general (unknown) phase space dependence of the variables $T, W, \tilde{V}$. Since γ is independent of any particular parameterisation, we can calculate the general result.

$$\gamma \to \gamma \left(1 + \frac{\gamma^2 y_0 \delta y}{3x_0^2} - \frac{\gamma^2 y_0^2 \delta x}{3x_0^3} + \dots\right) \tag{10.57}$$

Using this we can write the perturbation in H'/H. In general we can Taylor expand the function W such that we have $W(x^i + \epsilon^i) \sim W(x_0^i) + \partial_i W \epsilon^i$ and therefore the general result is true

$$\delta\left(\frac{H'}{H}\right) = -y_0\delta y - \frac{3(1+w)}{2}\left(-2z_0\delta z - 2x_0\delta x\left[1 - \frac{1}{W\gamma}\right] - \frac{x_0^2}{\gamma W}\left\{-\frac{\gamma^2 y_0 \delta y}{3x_0^2} + \frac{\gamma^2 y_0^2 \delta x}{3x_0^3} - \frac{\partial_i W \epsilon^i}{W}\right\}\right)$$

where all terms such as γ, W are evaluated on the classical solution and there is a summation over latin indices.

The general equations even for the linear perturbation, are shown below for Case V - which encompasses all the other solutions in the relevant limit:

$$\begin{aligned}
\delta x' &= -\frac{yz^3}{2x^2}(\mu_1+\mu_2)\left(\frac{\delta y}{y} + 3\frac{\delta z}{z} - 2\frac{\delta x}{x}\right) - \frac{yz^3}{2x}(\mu_1\delta\mu_1 + \mu_2\delta\mu_2) - \frac{y^2}{2x}\left(2\frac{\delta y}{y} - \frac{\delta x}{x}\right) \\
&- \delta x\frac{H_0'}{H_0} - x\delta\left(\frac{H'}{H}\right) \\
\delta y' &= -\frac{3z^3}{x}\left\{\mu_1\delta\mu_1 + \mu_2\delta\mu_2 + [\mu_1+\mu_2]\left(3\frac{\delta z}{z} - \frac{\delta x}{x} - \frac{\delta y}{y}\right)\right\} - 3x^2\mu_3\left(2\frac{\delta x}{x} + \delta\mu_3\right) \\
&+ \left(1 + \frac{z^3}{xy}[\mu_1+\mu_2]\right)\left(\frac{y^3}{x^2}[\frac{\delta y}{y} - \frac{\delta x}{x}] - 3\delta y\right) - \delta y\frac{H_0'}{H_0} - y\delta\left(\frac{H'}{H}\right) \\
&= \frac{3z^3\mu_2 W}{\gamma x}\left(\delta\mu_2 + 3\frac{\delta z}{z}\left(1 - \frac{2\beta}{3n}\right) - \frac{\gamma^2 y\delta y}{3x^2}\left(1 + \frac{\beta}{n} + \frac{\delta x}{x}\left\{\frac{2\beta}{n} - 1 + \frac{\gamma^2 y^2}{3x^2}\left(1 + \frac{\beta}{n}\right)\right\}\right)\right) \\
\delta z' &= \frac{z^2 y\mu_1}{2x}\left(2\frac{\delta z}{z} + \frac{\delta y}{y} - \frac{\delta x}{x} + \delta\mu_1\right) - \delta z\frac{H_0'}{H_0} - z\delta\left(\frac{H'}{H}\right)
\end{aligned} \tag{10.58}$$

where we have defined $n = \alpha + \beta - \rho$ for simplicity and also the following terms

$$\begin{aligned}
\delta\mu_1 &= -\frac{2(\alpha-2\rho)}{n}\frac{\delta z}{z} - \frac{(\alpha-2-\rho)}{2n}\frac{\gamma^2 y\delta y}{3x^2} + \frac{2(\alpha-2-\rho)}{n}\frac{\delta x}{x}\left(1 + \frac{\gamma^2 y^2}{12x^2}\right) \qquad (10.59) \\
\delta\mu_2 &= -\frac{4(\alpha-1-\rho)}{n}\frac{\delta z}{z} - \frac{(3\alpha-3\rho-2)}{n}\frac{\gamma^2 y\delta y}{6x^2} + \frac{\delta x}{nx}\left(4(\alpha-1-\rho) + \frac{(3\alpha-3\rho-2)\gamma^2 y^2}{6x^2}\right) \\
\delta\mu_3 &= \frac{2(\alpha+2+3\rho+2\beta)}{n}\frac{\delta z}{z} + \frac{(2\alpha-2-10\rho+\beta)\gamma^2 y\delta y}{6nx^2} + \frac{\delta x}{nx}\frac{(-2\alpha+2+10\rho-\beta)\gamma^2 y^2}{6x^2} \\
&- \frac{\delta x}{x}\frac{(2\alpha+4+6\rho+4\beta)}{n}
\end{aligned}$$

We will work through an explicit example to illustrate the formalism, namely the Case I solutions. Firstly we can calculate the following expression

$$\delta\left(\frac{H'}{H}\right) \sim -y_0\delta y + \frac{3(1+w)}{2}\left(2z_0\delta z + 2x_0\delta x\left[1-\frac{1}{W\gamma}\right] + \frac{x_0^2\gamma y_0}{3Wx_0^2}\left(\delta y - \frac{y\delta x}{x_0}\right)\right) \tag{10.60}$$

which will allow us to calculate the perturbed phase space variables. The perturbed dynamic expressions then take the following form

$$\begin{aligned}
\delta x' &= \frac{yz^3\mu_1}{2x^2}\left(\frac{\alpha x^2}{\beta W\gamma^2}[\frac{2\delta x}{x}(1+\frac{\gamma^2y^2}{6x^3}) - 2\frac{\delta z}{z} - \frac{\gamma^2 y\delta y}{3x^2}]\right) - \frac{y^2}{2x}\left(\frac{2\delta y}{y} - \frac{\delta x}{x}\right) \\
&- \frac{yz^3\mu_1}{2x^2}\left(1-\frac{\alpha x^2}{\beta W\gamma^2}\right)\left(\frac{\delta y}{y}(1-\frac{n\gamma^2 y}{6x^2}) + (3-n)\frac{\delta z}{z} + \frac{\delta x}{x}(n-2+\frac{n\gamma^2y^2}{6x^2})\right) \\
&- \delta x\frac{H_0'}{H_0} - x\delta\left(\frac{H'}{H}\right) \\
\delta y' &= 3z^3\mu_1\left(1-\frac{y^2}{6x^2}\right)\frac{\alpha x}{\beta W\gamma^2}\left[2\frac{\delta x}{x}(1+\frac{\gamma^2y^2}{6x^2}) - 2\frac{\delta z}{z} - \frac{\gamma^2y\delta y}{3x^2}\right] \\
&- \frac{3z^3\mu_1}{2x}(1-\frac{\alpha x^2}{\beta W\gamma^2})(1-\frac{y^2}{6x^2})\left((3-n)\frac{\delta z}{z} + \frac{\delta x}{x}(n-1+\frac{n\gamma^2y^2}{6x^2}) - \frac{\delta y}{y}(1+\frac{n\gamma^2y^2}{6x^2})\right) \\
&+ \left(1+\frac{z^3\mu_1}{xy}[1-\frac{\alpha x^2}{\beta W\gamma^2}]\right)\left\{\frac{y^3}{x^2}[\frac{\delta y}{y} - \frac{\delta x}{x}] - 3\delta y(1-\frac{y^2}{6x^2})\right\} \\
&- \frac{3xz\alpha\mu_1}{\beta\gamma^2}\left(\frac{\delta x}{x}[1+n+\frac{2\gamma^2y^2}{3x^2}(1+\frac{n}{4})] + (1-n)\frac{\delta z}{z} - \frac{\delta y}{y}\frac{2\gamma^2y^2}{3x^2}(1+\frac{n}{4})\right) \\
&- y\delta\left(\frac{H'}{H}\right) - \delta y\frac{H_0'}{H_0} \\
\delta z' &= \frac{z^2y\mu_1}{2x}\left((2-n)\frac{\delta z}{z} + \frac{\delta x}{x}[n-1+\frac{n\gamma^2y^2}{6x^2}] + \frac{\delta y}{y}[1-\frac{n\gamma^2y^2}{6x^2}]\right) - \delta z\frac{H_0'}{H_0} - z\delta\left(\frac{H'}{H}\right)
\end{aligned} \tag{10.61}$$

where the notation H_0'/H_0 implies that we take this function evaluated at the critical points, and we have defined $n = (\alpha-\beta-2)/(\alpha-\beta)$ for simplicity. Note that these are the leading order solutions only, and that all terms proportional to δ^2 have been neglected.

The stability of the fixed point solutions is therefore determined by the eigenvalues of the resulting perturbation matrix. A lengthy calculation which we will omit here shows that the point $(0,0,0)$ leads to the eigenvalues

$$\lambda_1 = \frac{3(w-1)}{2}, \qquad \lambda_2 = \frac{3(1+w)}{2}, \qquad \lambda_3 = \frac{3(1+w)}{2} \tag{10.62}$$

which indicates that this is never a point of stability for the theory unless the equation of state is phantom ie $w < -1$. In fact this statement will be true for all the various cases we have considered in the physical limit, since the dynamical equations of motion all reduce to the exact same form in this instance.

Another relatively simple case to consider is that in Case III. For slices through the (x, y) plane at

$z = 0$ we find the eigenvalues

$$\begin{aligned}
\lambda &= \frac{1}{2}(x^2+y^2) + \frac{3}{2}\left(w(x^2-1)-1\right) \qquad (10.63)\\
\lambda_{\pm} &= \frac{1}{4x^2}\left(-6x^4(1+w) - 6x^2 + 2y^2x^2 + 5y^2 \pm F(x,y)\right)\\
F(x,y) &= \sqrt{12y^2x^2w + 96y^2x^4w - 48y^2x^6w - 8y^4x^2 + 48y^2x^4 + 16y^4x^4 - 48y^2x^6 + 36vw^2x^4 + 17y^4}\,.
\end{aligned}$$

If one now slices this through $y = 0$ we see that we are left with the same situation discussed above (as expected), indicative of a phantom equation of state.

On the other hand, through the $y = 0$ plane we see that the eigenvalues become

$$\begin{aligned}
\lambda &= \frac{3}{2}(1+w)\left(1 - z^2 - x^2\left(1 - \frac{Qz^2}{x^2}\right)\right) \qquad (10.64)\\
\lambda_{\pm} &= -\frac{3}{2x}\left(-Qz^2 - x + 2xz^2 + x^3 - xz^2Q + x^2 \pm F(x,y)\right)
\end{aligned}$$

where F is another polynomial in x, z and we have defined $Q = T/\tilde{V}$ for simplicity. In the limit that $z \to 0$, we find that these simplify to yield

$$\begin{aligned}
\lambda &\to \frac{3}{2}(1+w)(1-x^2)\,,\\
\lambda_{\pm} &\to \frac{3}{2}(1+w)\left(x^2 - 1 - x^2(1 \pm 1)\right). \qquad (10.65)
\end{aligned}$$

Note that two of the eigenvalues are therefore degenerate as before, requiring a phantom equation of state, however the final eigenvalue has the opposite sign and therefore this fixed point is always unstable.

The remaining fixed points can be analyzed in precisely the same manner, although the analysis is somewhat awkward. We will postpone the relevant discussion here and return to it in a follow-up publication.

10.3 Discussion

We have initiated an alternate approach to the problem of k-essence, or DBI quintessence, using a more generalised form of the DBI action. Since this has more degrees of freedom, the resulting analysis is typically complicated, but the phase space structure is far richer. We have attempted to make some headway by restricting the phase space volume to various two-dimensional slices, and attempting to identify the relevant solution curves upon which the fixed points may lie. Our ansatz for each of the unknown functions is also potentially restrictive, however we are confident that it represents the leading semi-classical contributions which may (or may-not) be derivable from a full string theory embedding of out model.

What is clear is that the ratio of the (warped) brane tension to the potential is an important factor in the dynamics of the theory, where we found $T \geq \tilde{V}$ in several cases. Moreover the additional multiplicative factor $W(\phi)$ plays a crucial role, even when it is a constant, since it comes into the field equations non-trivially in the expression for H'/H. In the usual DBI analysis, $W = 1$ and the tension is the sole term responsible for the interesting quintessence behaviour. In some string compactifications, where the warp factor has no cut-off at small distances, we typically find W is constant and greater than unity. However there may be entire classes of solution where $W \leq 1$, which can lead to novel phase space trajectories. Since our approach has been phenomenological, and that there may be additional string backgrounds of interest that have yet to be fully explored, we cannot rule out $W < 1$ - which is vital for obtaining fixed point solutions in Case I for example.

Our numerical results have shown that there is indeed a rich phase space structure present due to the increased number of degrees of freedom. We expect many of these to yield highly non-trivial stable fixed points in the full analysis, which is beyond the scope of the current note. We have classified the nature of as many of the fixed points as is feasible within the current analysis. Ultimately we hope that this will lead to a renewed interest in dynamical dark energy models driven by a more generalised approach to $D3$-brane dynamics.

In light of the recent developments in holographic dark energy [150, 151] and the apparent relation to agegraphic [152, 153] dark energy, we hope that it may be possible to reconstruct the various potentials in our generalised model along the lines of [154].

บทที่ 11

Scalar field potential in a power-law expansion universe

Power-law cosmology, where $a \propto t^q$, describes an acceleration phase if $q > 1$. Modelling the present expansion with a power-law function where $q \sim 1$ was found to be consistent with nucleosynthesis [156, 155], the age of high-redshift objects such as globular clusters [155, 157, 159, 158], SNe Ia data [158], SNe Ia with $H(z)$ data [160], and X-ray gas mass fraction measurement of galaxy clusters [161, 162]. In the context of the power-law model, other aspects such as gravitational lensing statistics [159], angular size-redshift data of compact radio sources [163], and SN Ia magnitude-redshift relation [164, 158] have also been studied. Originally, the power-law expansion has its motivation from the simplest inflationary model that can remove the flatness and horizon problems with simple spectrum [165]. For the present universe, the idea of linear coasting cosmology ($a \propto t$) [166] can resolve the age problem of the CDM model [157] while as well agreeing with the nucleosynthesis constraint. The coasting model arises from non-minimally coupled scalar-tensor theory in which the scalar field couples to the curvature to contribute to the energy density that cancels out the vacuum energy [156, 167]. The model could also be a result of the domination of an SU(2) cosmological instanton [168].

Here our assumption is that the universe is expanding in the form of the power law function. Two major ingredients are scalar field dark energy evolving under the scalar field potential $V(\phi)$, and barotropic fluid consisting of cold dark matter and baryons. We derive the potential, and use the combined WMAP5 data [169] as well as the WMAP5 data alone to determine the values of q and other relevant parameters of the potential. The numerical results are subsequently compared and discussed.

11.1 Cosmological system with power-law expansion

Two perfect fluids, the cold dark matter and scalar field $\phi \equiv \phi(t)$, in the late FLRW universe of the simplest CDM model with zero cosmological constant are considered. The time evolution of the barotropic fluid is governed by the fluid equation $\dot{\rho}_\gamma = -3H\rho_\gamma$, with a solution

$$\rho_\gamma = \frac{D}{a^n}, \tag{11.1}$$

where $n \equiv 3(1 + w_\gamma)$ and $D \geq 0$ is a proportional constant. For the scalar field, supposed that it is minimally coupled to gravity, its Lagrangian density is $\mathcal{L} = \dot{\phi}^2/2 - V(\phi)$. The energy density and pressure are

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{11.2}$$

The fluid equation of the field describing its energy conservation as the universe expands is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}}{\mathrm{d}\phi}V = 0. \tag{11.3}$$

Total energy density ρ_{tot} and total pressure p_{tot} of the mixture are simply the sums of those contributed by each fluid, for which the Friedmann equation is

$$H^2 = \frac{8\pi G}{3}\rho_{\mathrm{tot}} - \frac{kc^2}{a^2}. \tag{11.4}$$

It is straightforward to show that

$$V(\phi) = \frac{3}{8\pi G}\left(H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right) + \left(\frac{n-6}{6}\right)\frac{D}{a^n}, \tag{11.5}$$

where $8\pi G$ is related to the reduced Planck mass M_{P} by $8\pi G = M_{\mathrm{P}}^{-2}$. The power-law scale factor is

$$a(t) = a_0\left(\frac{t}{t_0}\right)^q, \tag{11.6}$$

without fixing $a_0 = 1$ at the present time because we have implicitly rescaled it to allow for k taking only either one of the three discrete values $0, \pm 1$. The Hubble parameter is

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{q}{t}. \tag{11.7}$$

Our goal is to construct $V(t)$ using recent observational data, as far as the simplest CDM model is concerned.

11.2 Scalar field potential

We will work with observational data in SI units. Restoring the physical constants in place, we obtain

$$V(\phi) = \frac{3M_{\mathrm{P}}^2 c}{\hbar}\left(H^2 + \frac{\dot{H}}{3} + \frac{2kc^2}{3a^2}\right) - \frac{Dc^2}{2a^3}, \tag{11.8}$$

where $M_{\rm P}^2 = \hbar c/8\pi G$ and we have set $n = 3$ ($w_\gamma = 0$ for dust). Incorporating (11.6) and (11.7) into the above equation, we obtain

$$V(t) = \frac{M_{\rm P}^2 c}{\hbar}\left(\frac{3q^2 - q}{t^2} + \frac{2kc^2 t_0^{2q}}{a_0^2 t^{2q}}\right) - \frac{Dc^2}{2}\frac{t_0^{3q}}{a_0^3 t^{3q}}. \qquad (11.9)$$

We shall consider contribution of the first term alone in comparison to total contribution when including the second (the curvature) and the third (density) terms. It is worth noting that reconstruction of scalar field potential $V(\phi)$ was considered previously in context of flat universe with non-specified expansion law and using luminosity function of redshift z from SNe Ia observation [**?**].

11.2.1 Cosmological parameters

Using the equation for the Hubble parameter (11.7) at the present time, we have

$$q = H_0 t_0. \qquad (11.10)$$

The sign of k depends on the sign of the density parameter $\Omega_k \equiv -kc^2/a^2H^2$. In our convention here, $k = 1$ ($\Omega_k < 0$) for a closed universe, $k = 0$ for a flat one, and $k = -1$ ($\Omega_k > 0$) for an open one. The present value of the scale factor can be found from the definition of $\Omega_{k,0}$, that is,

$$a_0 = \frac{c}{H_0}\sqrt{\frac{-k}{\Omega_{k,0}}}. \qquad (11.11)$$

The density constant D can be found from (11.1),

$$D = \rho_{\gamma,0} a_0^3 = \Omega_{\gamma,0}\rho_{c,0} a_0^3, \qquad (11.12)$$

where $\Omega_{\gamma,0} = \Omega_{\rm CDM,0} + \Omega_{b,0}$, i.e. the sum of the present density parameters of the barotropic fluid components. $\rho_{c,0}$ is the present value of the critical density. The neutrino contribution is assumed to be negligible. The values of H_0, t_0, $\Omega_{k,0}, \Omega_{\rm CDM,0}$, and $\Omega_{b,0}$ are taken from observational data.

11.2.2 Observational data

We work on two sets of data provided by [169]. One comes solely from the WMAP5 data and the other is the WMAP5 data combined with distance measurements from Type Ia supernovae (SN) and the Baryon Acoustic Oscillations (BAO) in the distribution of galaxies. For t_0, H_0, $\Omega_{b,0}$, and $\Omega_{\rm CDM,0}$, we take their maximum likelihood values. The curvature density parameter $\Omega_{k,0}$ comes as a range with 95% confidence level on deviation from the simplest ΛCDM model. The data are shown in Table 11.1.

11.3 Results and discussions

Using combined WMAP5+BAO+SN dataset, the potential is

$$V(t) = \frac{1.03\times 10^{26}}{t^2} + \frac{1.5\times 10^{23}}{t^{1.97}} - \frac{1.5\times 10^{42}}{t^{2.96}}, \qquad (11.13)$$

Parameter	WMAP5+BAO+SN	WMAP5
t_0	13.72 Gyr	13.69 Gyr
H_0	70.2 km/s/Mpc	72.4 km/s/Mpc
$\Omega_{b,0}$	0.0459	0.0432
$\Omega_{\text{CDM},0}$	0.231	0.206
$\Omega_{k,0}$	$-0.0179 < \Omega_{k,0} < 0.0081$	$-0.063 < \Omega_{k,0} < 0.017$

ตารางที่ 11.1: Observational data used in the construction of our scalar-field potentials [169]

	WMAP5+BAO+SN		WMAP5	
	$\bar{\Omega}_{k,0} = -0.0045$	$-0.0175 < \Omega_{k,0} < 0.0085$	$\bar{\Omega}_{k,0} = -0.023$	$-0.063 < \Omega_{k,0} < 0.017$
q	0.985	0.985	1.01	1.01
a_0	1.9×10^{27}	$a_0 > 9.85 \times 10^{26}$ (closed) $a_0 > 1.5 \times 10^{27}$ (open)	8.4×10^{26}	$a_0 > 5.1 \times 10^{26}$ (closed) $a_0 > 9.8 \times 10^{26}$ (open)
$t_{\text{intercept}}$	2.7 Gyr	2.62 Gyr $< t <$ 2.7 Gyr	2.7 Gyr	2.6 Gyr $< t <$ 2.8 Gyr
t_{max}	4.0 Gyr	3.94 Gyr $< t <$ 4.0 Gyr	4.0 Gyr	3.8 Gyr $< t <$ 4.1 Gyr
$t_{\text{inflection}}$	5.3 Gyr	5.26 Gyr $< t <$ 5.4 Gyr	5.3 Gyr	5.1 Gyr $< t <$ 5.5 Gyr

ตารางที่ 11.2: A summary of numerical results. Times are shown in Gyr for comprehensibility. Positive and negative Ω_k's correspond to open and closed universes, respectively.

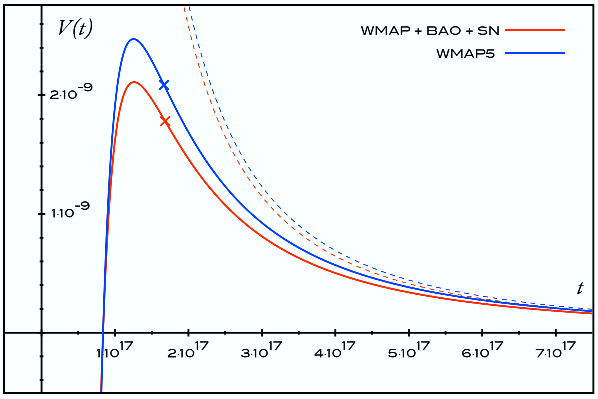
whereas, for WMAP5 dataset alone,

$$V(t) = \frac{1.11 \times 10^{26}}{t^2} + \frac{7.6 \times 10^{24}}{t^{2.03}} - \frac{4.6 \times 10^{43}}{t^{3.04}}. \tag{11.14}$$

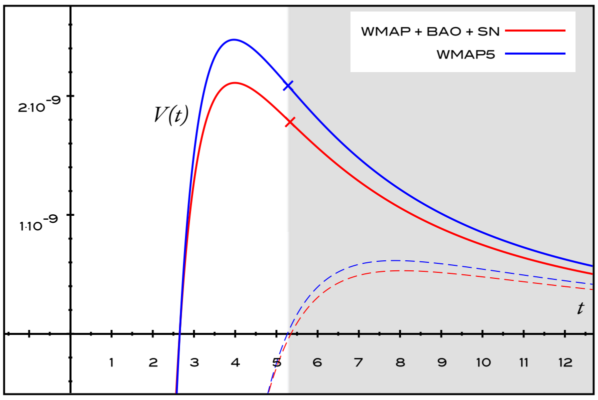
in SI units. We use the mean of each $\Omega_{k,0}$ interval to represent $\Omega_{k,0}$ in each of the above equations. Their plots are shown in Fig. 11.1. In both cases, $\bar{\Omega}_{k,0}$ is negative (a closed universe). The points at which the potential, its derivative, and its second-order derivative, are zero ($t_{\text{intercept}}$, t_{max}, and $t_{\text{inflection}}$, respectively) are also determined, for both $\bar{\Omega}_{k,0}$ and each end of the $\Omega_{k,0}$ interval. The results are summarised in Table 11.2.

The values of the exponent q from the two sets of data are only slightly different, but only the latter is an accelerated expansion as $q > 1$. The determination of q from X-Ray gas mass fractions in galaxy clusters favours open universe with $q > 1$ ($q = 1.14 \pm 0.05$) [162] and combined analysis from SNLS and $H(z)$ data (from Germini Deep Deep Survey) assuming open geometry yields $q = 1.31$ [160]. Note that, in the power-law regime, q only depends on the observed values of the Hubble constant and t_0. This may give an impression that the maximum likelihood values from the combined data has yet to be relied upon, but the power-law expansion has not been proven to be the case nonetheless.

After $t_{\text{inflection}}$, the potential from each data behaves like its first term, i.e. decreasing in its value while increasing in its slope (being less and less negative). The other terms quickly become weaker. This



รูปที่ 11.1: The potentials in (11.13) and (11.14). The units of the abscissa and ordinate axes are sec and J/m^3, respectively. The crosses mark their inflection points. Also plotted in dash lines are their first terms. Each potential does not actually converge to its first term, but later intersect with and deviate from it, though still very close together. However, this occurs much later (at $t = 2.8 \times 10^{84}$ sec $= 8.8 \times 10^{67}$ Gyr in both cases).



รูปที่ 11.2: The potentials in (11.13) and (11.14) along with the radicands of the integrands in (11.16) and (11.17) (dash line). The shaded region is the post-inflection phase. The unit of the abscissa axis is sec. After $t_{\text{inflection}}$, $\phi(t)$ is real.

can be seen in Fig. 11.1. Since the first term is contributed only by $H(t)$ (and its time derivative), it is dominant in the post-inflection phase. In fact, the convergence to zero of the potential is slower than its first term alone (see (11.13) and (11.14)), because the sum of the last two terms consequently becomes positive before converging to zero. This means that the plots of each potential and its first term in Fig. 11.1 eventually crosses, but it occurs much, much later at $t = 8.8 \times 10^{67}$ Gyr. Along with the potential function in (11.5), we also obtain the solution

$$\phi(t) = \int \sqrt{-\frac{2M_{\rm P}^2 c}{\hbar}\left(\dot{H} - \frac{kc^2}{a^2}\right) - \frac{Dc^2}{a^3}}\,{\rm d}t \tag{11.15}$$

in SI units. Using WMAP5+BAO+SN dataset,

$$\phi(t) = \int \sqrt{\frac{1.06 \times 10^{26}}{t^2} + \frac{1.5 \times 10^{23}}{t^{1.97}} - \frac{3.0 \times 10^{42}}{t^{2.96}}}\,{\rm d}t, \tag{11.16}$$

where, for WMAP5 dataset alone,

$$\phi(t) = \int \sqrt{\frac{1.09 \times 10^{26}}{t^2} + \frac{7.6 \times 10^{24}}{t^{2.03}} - \frac{9.3 \times 10^{43}}{t^{3.04}}}\,{\rm d}t. \tag{11.17}$$

In the late post-inflection phase, the first term is dominant over the k and D terms then the last two terms of the radicands are negligible (Fig. 11.2). The above two equations are approximated as

$$\phi(t) \approx 1.04 \times 10^{13} \ln t, \tag{11.18}$$

whereas, for WMAP5 dataset alone,

$$\phi(t) \approx 1.03 \times 10^{13} \ln t. \tag{11.19}$$

The radicand in (11.17) of the WMAP5 dataset is zero at approximately $t_{\text{inflection}} = 5.3$ Gyr (see Fig. 11.2), therefore so does $\phi(t)$. While the combined dataset has the zero radicand (then zero $\phi(t)$) in (11.16) later at approximately $t = 5.4$ Gyr. Scalar field exact solutions for the power-law cosmology with non-zero curvature and non-zero matter density are reported in [38]. It is also worth noting that the general exact form of the potential, that renders scaling solution, is some negative powers of a hyperbolic sine [170].

11.4 Conclusion

We consider a potential function of a homogeneous scalar field in late-time FLRW universe of the simplest CDM model with zero cosmological constant, assuming power-law expansion. The scalar field is minimally coupled to gravity and the other fluid is non-relativistic barotropic perfect fluid. We use two sets of observational data, combined WMAP5+BAO+SN dataset and WMAP5 dataset, as the inputs. Potential functions are obtained using numerical values from the observations. Mean values of both sets suggest slightly closed geometry. The WMAP5 dataset implies accelerated expansion ($q = 1.01$) while the combined dataset gives $q = 0.985$. This is slightly lower than the value obtained from SNLS and $H(z)$ data ($q = 1.31$) [160] and X-Ray gas mass fraction ($q = 1.14 \pm 0.05$) [162]. Our result is independent of the geometry unlike q obtained from [160] which assumes open geometry. For closed universe, the WMAP5 dataset puts the lower limit of 5.1×10^{26} for a_0 while the combined dataset puts the lower limit of 9.85×10^{26}. We characterise the domination of the first term of (11.9) by using the inflection of the potential plots from which the first term is found to be dominant to the potential 5.3 Gyr after the Big Bang in both datasets.

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ภาคผนวก ก

ผลผลิตจากโครงการ

ก.1 การตีพิมพ์เผยแพร่

ก.1.1 การตีพิมพ์ในวารสารวิชาการระดับนานาชาติ

ผลผลิตจากโครงการมีบทความวิจัยตีพิมพ์ในวารสารวิชาการระดับนานาชาติจำนวน 7 บทความได้แก่[มีค่า Journal Impact Factor หนึ่งปีย้อนหลังของแต่ละวารสารที่บทความได้รับการตีพิมพ์รวมเท่ากับ **29.841**][1]

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- Theerakarn Phetnora, Roongtum Sooksan (TPTP, Naresuan U.) and Burin Gumjudpai (TPTP, Naresuan U. and DAMTP, U. Cambridge) "Phantom expansion with non-linear Schrödinger-type formulation of scalar field cosmology" published in **online first** *General Relativity and Gravitation* [arXiv: 0805.3794 [gr-qc]] [Impact Factor (2008)=**1.803**] [Cited 2 time on SPIRES]

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[1] SPIRES is a research citation database in High Energy Physics at http://www.slac.stanford.edu/spires/hep/

- Burin Gumjudpai (TPTP, Naresuan U.) "Power-law expansion cosmology in Schrödinger-type formulation" *Astroparticle Physics* 30, 186 (2008) [arXiv: 0706.3674 [gr-qc]] [Impact Factor (2007)=**3.483**] [Cited 4 times on SPIRES]
- Daris Samart and Burin Gumjudpai (TPTP, Naresuan U.) "Phantom field dynamics in loop quantum cosmology" *Physical Review D* 76, 043514 (2007) arXiv:0704.3414 [gr-qc] Impact Factor (2006)=**4.896** Cited 30 times on SPIRES
- Burin Gumjudpai (TPTP, Naresuan U.), Tapan Naskar (IUCAA, India) & John Ward (Queen Mary, U. London) "A quintessentially geometric model" *Journal of Cosmology and Astroparticle Physics* 11, 006 (2006) [arXiv: hep-ph/0603210] [Impact Factor (2005)=**6.739**] [Cited 10 times on SPIRES]

ก.1.2 การตีพิมพ์ในวารสารวิชาการระดับชาติ

ผลผลิตจากโครงการมีบทความวิจัยตีพิมพ์ในวารสารวิชาการระดับชาติเป็นภาษาอังกฤษจำนวน 1 บทความ

- Burin Gumjudpai (TPTP, Naresuan U.) "Coupled phantom field in loop quantum cosmology" Thai Journal of Physics Series 3: Proc. of the SIAM Phys. Cong. 2007, (invited talk at the Congress) arXiv:0706.3467 [gr-qc] [Cited 11 times on SPIRES]

ก.1.3 การตีพิมพ์บทความปริทรรศน์ในหนังสือที่ได้รับการเผยแพร่ระดับนานาชาติ

มี 1 บทความปริทรรศน์ที่ได้รับการเผยแพร่ในหนังสือคือ

- Burin Gumjudpai (DAMTP, U. Cambridge and TPTP, Naresuan U.) "Scalar field cosmology: its non-linear Schrödinger-type formulation" Invited Review in Dark Energy-Current Advances and Ideas, Ed. Jeong Ryeol Choi, Research Signpost (to appear in 2009) [arXiv: 0904.2746 [gr-qc]][Cited n/a times on SPIRES]

ก.1.4 บทความวิจัยที่อยู่ระหว่างการรอการพิจารณาปริทรรศน์จากวารสารวิชาการระดับนานาชาติ

มี 1 บทความวิจัยที่อยู่ระหว่างการรอการพิจารณาคือ

- Kiattisak Tepsuriya (TPTP, Naresuan U. and NARIT) and Burin Gumjudpai (TPTP, Naresuan U.) "Determining scalar field potential in power-law cosmology with observational data" (Submitted) [arXiv: 0904.2743 [astro-ph.CO]] [Impact Factor (2008)= n/a] [Cited n/a time on SPIRES]

ก.2 การสร้างนักวิจัยใหม่

ในการดำเนินงานตามโครงการวิจัยนี้ได้ให้นิสิตของหน่วยวิจัยฟิสิกส์รากฐานและจักรวาลวิทยาร่วมคำนวณและช่วยตรวจสอบผลลัพธ์ในการคำนวณรวมทั้งได้มีการสอนให้นิสิตได้ใช้ software ในการช่วยคำนวณและยืนยันผลเชิงตัวเลขต่างๆ โดยมีนิสิตปริญญาตรีสำเร็จการศึกษาและกำลังศึกษาต่อ 3 คนคือ

- นายเกียรติศักดิ์ เทพสุริยะ นักศึกษาอาคันตุกะที่สังกัดสถาบันสำนักเรียนท่าโพธิ์ฯ ได้ทำวิจัยโดยใช้ความรู้ทางจักรวาลวิทยาที่เรียนรู้จากโครงการนี้ ผลจากการดำเนินงานดังกล่าวเกียรติศักดิ์ได้เขียนบทความวิจัยด้วยตนเอง (มีชื่อแรกใน [42]) และกำลังอยู่ระหว่างรอการพิจารณาจากผู้ปริทรรศน์ปัจจุบันกำลังรอศึกษาต่อระดับปริญญาโททางจักรวาลวิทยาที่ประเทศสหราชอาณาจักรด้วยทุนกระทรวงวิทยาศาสตร์ฯ

- นายธีรกานต์ เพ็ชรโนรา ได้ทำวิจัยโดยใช้ความรู้การคำนวณเชิงตัวเลขที่เรียนรู้จากโครงการนี้ ผลจากการดำเนินงานดังกล่าวนิสิตได้เขียนปริญญานิพนธ์ระดับปริญญาตรีเป็นภาษาอังกฤษและทำการสอบป้องกันปริญญานิพนธ์ด้วยวาจาเป็นภาษาอังกฤษ (มีชื่อแรกใน[39]) ปัจจุบันกำลังรอศึกษาต่อระดับปริญญาโททางจักรวาลวิทยา

- นาย จักรกฤษ แก้วนิคม ได้ถูกฝึกให้แก้สมการสนามของไอน์สไตน์และระบบพลวัตรพลังงานมืดได้เขียนปริญญานิพนธ์ระดับปริญญาตรีเป็นภาษาอังกฤษและทำการสอบป้องกันปริญญานิพนธ์ด้วยวาจาเป็นภาษาอังกฤษ ปัจจุบันกำลังศึกษาต่อระดับปริญญาโททางจักรวาลวิทยา

- นาย ดริศ สามารถ ได้รับการฝึกให้ศึกษาระบบพลวัตรผลจากการดำเนินงานดังกล่าวนิสิตได้เขียนปริญญานิพนธ์ระดับปริญญาโทเป็นภาษาอังกฤษและทำการสอบป้องกันปริญญานิพนธ์ด้วยวาจาเป็นภาษาอังกฤษ (มีชื่อแรกใน [34]) และกำลังศึกษาต่อระดับปริญญาเอก

- นาย เอกชัย อ่อนแก้ว ได้ถูกฝึกให้วิเคราะห์ระบบพลวัตรพลังงานมืดได้เขียนปริญญานิพนธ์ระดับปริญญาตรีเป็นภาษาอังกฤษและทำการสอบป้องกันปริญญานิพนธ์ด้วยวาจาเป็นภาษาอังกฤษปัจจุบันประกอบอาชีพอิสระ

- นาย สรายุทธ พานเทียน ได้ถูกฝึกให้วิเคราะห์ระบบพลวัตรพลังงานมืดได้เขียนปริญญานิพนธ์ระดับปริญญาโทเป็นภาษาอังกฤษและทำการสอบป้องกันปริญญานิพนธ์ด้วยวาจาเป็นภาษาอังกฤษ

ก.3 การพัฒนาการเรียนการสอนและการสร้างกลุ่มวิจัย

การดำเนินกิจกรรมวิจัยในโครงการนี้ได้ยกระดับโครงงานวิจัยระดับปริญญาตรีและระดับปริญญาโทของนิสิตในหน่วยวิจัยให้เข้าสู่ระดับสากลที่เป็นเช่นนี้เพราะว่าการกระตุ้นให้นิสิตร่วมทำงานวิจัยในโครงการที่มุ่งตีพิมพ์ผลงานวิจัยในวารสารระดับสากลเป็นภาษาอังกฤษย่อมต้องอาศัยมาตรฐานการสอนและความเข้มข้นของเนื้อหาวิชาในรายวิชาทฤษฎีสัมพัทธภาพทั่วไปและรายวิชาจักรวาลวิทยาระดับสากลที่มีระดับและปริมาณเนื้อหาไม่น้อยไปกว่าที่สอนในมหาวิทยาลัยในต่างประเทศเช่นกันโครงการนี้ให้ผลทางอ้อมที่ค่อนข้างชัดเจนคือการทำให้มีการเปิดสอนจริงในรายวิชาสัมพัทธภาพทั่วไปและรายวิชาจักรวาลวิทยาทั้งขั้นต้นและขั้นสูงรวม 4 รายวิชาที่มหาวิทยาลัยนเรศวร

การเรียนการสอนดังกล่าวได้ทำให้มีจำนวนนิสิตในหน่วยวิจัยเพิ่มขึ้น ทั้งนี้รวมถึงการมีนิสิตเข้าศึกษาต่อระดับปริญญาโทในสาขาวิชานี้ที่มหาวิทยาลัยนเรศวรเพิ่มขึ้นนิสิตวิจัยที่เข้าร่วมศึกษาแนวทางหลักของโครงการคือการศึกษาพลวัตรของพลังงานมืดและฟิสิกส์รากฐานของเอกภพได้รับการถ่ายทอดความรู้พื้นฐานและได้รับการฝึกฝนการอ่านและวิเคราะห์บทความที่เกี่ยวข้องในประเด็นที่เป็นที่สนใจของโครงการโครงการนี้ทำให้กิจกรรมทางการวิจัยของหน่วยวิจัยยกระดับขึ้นและทำให้เกิดความเข้มแข็งของหน่วยวิจัยขึ้นอย่างมาก

ก.4 การเสนอผลงานในที่ประชุมวิชาการและการบรรยายสัมมนาภายนอก

ผลการสนับสนุนไม่ว่าโดยทางตรงหรือทางอ้อมจากโครงการนี้ ได้ทำให้หัวหน้าโครงการวิจัยนี้ได้รับเชิญบรรยายพิเศษ ให้สัมมนารับเชิญและได้ไปนำเสนอผลงานแบบบรรยายในโอกาสต่างๆ 26 ครั้ง ดังนี้

- 3 สิงหาคม 2009 *Cosmology Research in Thailand* บุรินทร์ กำจัดภัยบรรยายพิเศษ (1 ชั่วโมง) ที่ ภาควิชาฟิสิกส์ มหาวิทยาลัยเชียงใหม่

- 9 ธันวาคม 2008 *Canonical Scalar Field Cosmology in Non-Linear Schrödinger Type Formulation* บุรินทร์ กำจัดภัยสัมมนารับเชิญ (1 ชั่วโมง) ที่ Cosmology Seminar at Astronomy Centre, University of Sussex, Brighton, U.K.

- 13 พฤศจิกายน 2008 *Dark Energy and Acceleration of the Universe* บุรินทร์ กำจัดภัยสัมมนารับเชิญ (3 ชั่วโมง) ที่ ภาควิชาฟิสิกส์ มหาวิทยาลัยอุบลราชธานี

- 1 พฤศจิกายน 2008 *Canonical Scalar Field Cosmology in Non-Linear Schrödinger Type Formulation* บุรินทร์ กำจัดภัยInvited Talk, Session D (Physics): Astrophysics, Cosmology & Electromagnetics Chair: ศ.ดร. สุทัศน์ ยกส้านการประชุมวิชาการวิทยาศาสตร์และเทคโนโลยีแห่งประเทศไทยครั้งที่ 34 ที่ศูนย์การประชุมแห่งชาติสิริกิติ์ (เจ้าภาพ สจล. ลาดกระบัง)

- 31 ตุลาคม 2008 *Dark Energy and Acceleration of the Universe* บุรินทร์ กำจัดภัยInvited Lecture, Thailand Outstanding Young Scientists Session Chair: ศ.ดร. อภิชาติ สุขสำราญการประชุมวิชาการวิทยาศาสตร์และเทคโนโลยีแห่งประเทศไทย ครั้งที่ 34 ที่ศูนย์การประชุมแห่งชาติสิริกิติ์ (เจ้าภาพ สจล. ลาดกระบัง)

- 24 ตุลาคม 2008 *การเสวนาประสบการณ์นักวิทยาศาสตร์รุ่นใหม่* บุรินทร์ กำจัดภัย (ฟิสิกส์) ม.นเรศวรผศ. สาธิต แซ่จึง (คณิตศาสตร์) ม.ขอนแก่น และ ผศ.ดร.นพ.นรัตถพล เจริญพันธ์ (สรีรวิทยา) ม.มหิดลเวทีเสวนารับเชิญ (1 ชั่วโมง) ค่ายโครงการพัฒนากำลังคนด้านวิทยาศาสตร์ (ทุนเรียนดีวิทยาศาสตร์แห่งประเทศไทย) ครั้งที่ 4 ระหว่างวันที่ 23-25 ตุลาคม 2551 ณ คณะวิทยาศาสตร์ มหาวิทยาลัยมหิดล และเวทีเสวนากลุ่มย่อยสาขาวิทยาศาศาสตร์กายภาพ(อีก 1 ชั่วโมง) ในงานเดียวกัน

- 9 สิงหาคม 2008 *บุรินทร์ GR: ประสบการณ์ชีวิตและการทำงานของนักวิทยาศาสตร์รุ่นใหม่* บุรินทร์ กำจัดภัยบรรยายพิเศษ (2 ชั่วโมง) แก่นิสิตและคณาจารย์ภาควิชาฟิสิกส์โดยการเชิญของภาควิชาฟิสิกส์ มหาวิทยาลัยนเรศวร ณ ตึกฟิสิกส์ มหาวิทยาลัยนเรศวร

- 9 มิถุนายน 2008 *Scalar Field Cosmology with NLS Formulation* บุรินทร์ กำจัดภัยสัมมนารับเชิญ (1 ชั่วโมง) ที่ Cosmology Lunch: the joint DAMTP-IoA-Cavendish Lab. Seminar, at Potter Room, Pavillion B, Centre for Mathematical Sciences, University of Cambridge, U.K. (ได้รับเกียรติจาก Prof. Stephen W. Hawking, F.R.S. และ Prof. Anne C. Davis เข้าร่วมฟังสัมมนานี้)

- 26 มกราคม 2008 *Cosmological Physics and Current Research Trends* บุรินทร์ กำจัดภัยการบรรยายพิเศษรับเชิญ (5 ชั่วโมง) โดยคณะวิทยาศาสตร์ มหาวิทยาลัยสงขลานครินทร์ ในงานเสวนาทางวิทยาศาสตร์

บริสุทธิ์ (ฟิสิกส์) ตามโครงการทุนเรียนดีวิทยาศาสตร์แห่งประเทศไทย ที่ศูนย์ปาฐกถาประดิษฐ เชยจิตร (ตึกฟักทอง) มหาวิทยาลัยสงขลานครินทร์ หาดใหญ่

- 16 มกราคม 2008 *Cosmology Research and Future Outreach Vision of the TPTP* บุรินทร์ กำจัดภัยการบรรยายเป็นภาษาอังกฤษ (10 นาที) ที่ THAI-U.K. Workshop: Towards the Future of Thai Astronomy ที่ Amari Rincome Hotel เชียงใหม่

- 18 ธันวาคม 2007 *Braneworld Effects on Cosmological Dynamics* บุรินทร์ กำจัดภัยการบรรยายหน้าโปสเตอร์เนื่องในงานแถลงข่าวต่อสื่อมวลชนเกี่ยวกับรางวัลสภาวิจัยแห่งชาติ 2550 ที่โรงแรมมิราเคิล แกรนด์ คอนเวนชัน กรุงเทพ

- 26 พฤศจิกายน 2007 *TPTP: About Knowing Us: What We Think, What We Do and What We Are.* บุรินทร์ กำจัดภัยการบรรยายสรุปเกี่ยวกับกิจการของสถาบันสำนักเรียนท่าโพธิ์ฯ เป็นภาษาอังกฤษ (20 นาที) เนื่องในวาระการขอเข้าดูงานของคณะผู้บริหารทางการศึกษาระดับสูงจากประเทศ Pakistan, Swaziland และ Iran และจากสถาบัน A.I.T. (Asian Institute of Technology) ที่สำนักงานอธิการบดี มหาวิทยาลัยนเรศวร

- 13 ตุลาคม 2007 *Phantom Field Dynamics in Loop Quantum Cosmology* บุรินทร์ กำจัดภัยการนำเสนอผลงานแบบโปสเตอร์เป็นภาษาอังกฤษ ที่ การประชุมนักวิจัยรุ่นใหม่พบเมธีวิจัยอาวุโส สกว. 2550 โรงแรมแอมบาสเดอร์ซิตี้ จอมเทียน พัทยา

- 24-29 กันยายน 2007 *Phantom Field Dynamics in Loop Quantum Cosmology* บุรินทร์ กำจัดภัยการนำเสนอผลงานแบบบรรยายเป็นภาษาอังกฤษ (20 นาที) ที่ The Forth Aegean School on Black Holes, 17-22 กันยายน 2550 ณ เมือง Mytilini, Island of Lesvos, Greece

- 7-8 กันยายน 2007 *Introduction to Cosmology and Dark Energy* บุรินทร์ กำจัดภัยการบรรยายพิเศษรับเชิญ (3 ชั่วโมง) ที่ภาควิชาฟิสิกส์ มหาวิทยาลัยอุบลราชธานี

- 29 กรกฎาคม 2007 *Phantom Field Dynamics in Loop Quantum Cosmology* บุรินทร์ กำจัดภัยการนำเสนอผลงานแบบบรรยาย (20 นาที) ที่การประชุมวิชาการนเรศวรวิจัยครั้งที่ 3 มหาวิทยาลัยนเรศวร

- 27-29 มิถุนายน 2007 *Introduction to Cosmology and Dark Energy* บุรินทร์ กำจัดภัยการบรรยายพิเศษรับเชิญเป็นภาษาอังกฤษ (4 ชั่วโมง) ที่สาขาวิชาฟิสิกส์ มหาวิทยาลัยเทคโนโลยีสุรนารี

- 19 เมษายน 2007 *การสร้างชุมชน KM กรณีศึกษาสถาบันสำนักเรียนท่าโพธิ์ฯ* บุรินทร์ กำจัดภัยการเสวนารับเชิญในการประชุมชุมชน KM และคนเขียน BLOG ของมหาวิทยาลัยนเรศวร ที่ห้อง Main Conference อาคาร CITSCOM มหาวิทยาลัยนเรศวร จัดโดยสถาบันบริหารการวิจัยและพัฒนา มหาวิทยาลัยนเรศวร

- 23 มีนาคม 2007 *Coupled Phantom Field in Cosmology* บุรินทร์ กำจัดภัยInvited talk [Session D3 Thai National Astronomy Meeting: TNAM 2007] การเสนอผลงานแบบรับเชิญ (เป็นภาษาอังกฤษ) ที่การประชุมวิชาการระดับชาติ SIAM Physics Congress 2007 ณ The Rose Garden Riverside จัดโดยภาควิชาฟิสิกส์ จุฬาลงกรณ์มหาวิทยาลัย และ สมาคมฟิสิกส์ไทย

- 13 กุมภาพันธ์ 2007 *Dark Energy Scaling Solution with Exponential Potential* บุรินทร์ กำจัดภัยสัมมนารับเชิญ ที่การสัมมนาฟิสิกส์ ภาควิชาฟิสิกส์ คณะวิทยาศาสตร์ มหาวิทยาลัยมหิดล กรุงเทพ
- 12 กุมภาพันธ์ 2007 *การเสวนานักวิจัยรุ่นใหม่พบนักวิจัยอาชีพ มหาวิทยาลัยนเรศวร* บุรินทร์ กำจัดภัย ในวงเสวนาของ วุฒิเมธีวิจัย สกว. และ เมธีวิจัย สกว. ของมหาวิทยาลัยนเรศวรเสวนารับเชิญ ที่ห้องประชุมเอกาทศรถ 9 อาคารโรงพยาบาลมหาวิทยาลัยนเรศวร ดำเนินการเสวนาโดย ผศ.ดร. เสมอ ถาน้อย ผู้อำนวยการสถาบันบริหารการวิจัยและพัฒนา ม.น. ผู้เข้าฟังประกอบด้วยอาจารย์ นักวิจัยรุ่นใหม่และผู้บริหารการวิจัย ม.น. ประมาณ 80 คน
- 2 กุมภาพันธ์ 2007 *การทำงานของนักวิจัยรุ่นใหม่ในวัฒนธรรมวิทยาศาสตร์สากล (version ปรับปรุง)* บุรินทร์ กำจัดภัย การบรรยายพิเศษรับเชิญ ที่ การสัมมนาวิชาการวิทยาศาสตร์และเทคโนโลยีครั้งที่ 1 โดยสำนักวิทยาศาสตร์และเทคโนโลยี มหาวิทยาลัยนเรศวร พะเยา
- 21 มกราคม 2007 *ผลเฉลยปรับมาตรสำหรับการขยายตัวของเอกภพ* บุรินทร์ กำจัดภัยการเสนอผลงานแบบบรรยายใน การประชุมวิชาการวิทยาศาสตร์-คณิตศาสตร์ในโรงเรียนครั้งที่ 17 ณ มหาวิทยาลัยราชภัฏเพชรบูรณ์
- 24 ตุลาคม 2006 *จักรวาลวิทยาสำหรับคนกินข้าวมันไก่* บุรินทร์ กำจัดภัยการบรรยายรับเชิญและร่วมเสวนาร่วมกับ วินทร์ เลียววารินทร์ (นักเขียนรางวัลซีไรท์) ในหัวข้อ "จักรวาลวิทยา: จินตนาการ กับ อภิปรัชญา" โดยมี ดร. บัญชา ธนบุญสมบัติ เป็นผู้ดำเนินรายการเสวนา ที่งานมหกรรมหนังสือแห่งชาติครั้งที่ 11 ณ ศูนย์การประชุมแห่งชาติสิริกิติ์ [ดูกำหนดการจาก เวปของมหกรรมหนังสือแห่งชาติครั้งที่ 11] [ใบปลิวของ สวทช] [บทสัมภาษณ์ในผู้จัดการออนไลน์] [ข่าวจากกรุงเทพธุรกิจ]
- 13 ตุลาคม 2006 *Dynamics of Coupled Dark Energy* บุรินทร์ กำจัดภัยการเสนอผลงานแบบบรรยายเป็นภาษาอังกฤษ การประชุมนักวิจัยรุ่นใหม่พบเมธีวิจัยอาวุโส สกว. 12-14 ตุลาคม 2006 โรงแรมรีเจนท์ ชะอำ เพชรบุรี
- 6 กันยายน 2006 *Quintessence Field Dark Energy* บุรินทร์ กำจัดภัยสัมมนารับเชิญ ที่การสัมมนาฟิสิกส์บัณฑิตศึกษา ภาควิชาฟิสิกส์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่(ผู้ดำเนินรายการ ศ.ดร. ถิรพัฒน์ วิลัยทอง และ ศ.ดร. สมชาย ทองเต็ม)

ก.5 การจัดการประชุมวิชาการ อบรมและสัมมนา

มีการจัดการประชุมทางวิชาการดังนี้

- การประชุมวิชาการ สัมพัทธภาพทั่วไป ฟิสิกส์พลังงานสูง และจักรวาลวิทยา แห่งชาติครั้งที่ 4 ในวันที่ 26-28 กรกฎาคม พ.ศ. 2552 ที่โรงแรมรัตนาปาร์ค พิษณุโลกและที่มหาวิทยาลัยนเรศวร (ดูรายละเอียดจาก http://www.tptp.in.th) ซึ่งประกอบด้วย
  - การบรรยายลักษณะ short course สองเรื่องคือChamaleon Dark Energy (4 ชั่วโมง โดย Prof. Anne Christine-Davis (DAMTP, University of Cambridge, UK)) และ Introduction to Quantum Chromodynamics (3 ชั่วโมง โดย Prof. Yupeng Yan มหาวิทยาลัยเทคโนโลยีสุรนารี)

  - การปาฐกถานำ (plenary lectures) 5 sessions โดยมีความยาว session ละ 40 นาทีโดยศาสตราจารย์ทางฟิสิกส์ 5 ท่านคือ Prof. Anne Christine-Davis (DAMTP, University of Cambridge, UK), Prof. Douglas Singleton (California State University at Fresno, USA), Prof. Mike Bisset (Tsinghua University, P.R.China), Prof. Yupeng Yan (มหาวิทยาลัยเทคโนโลยีสุรนารี) และ Prof. David Ruffolo (มหาวิทยาลัยมหิดล)
  - การนำเสนอผลงานแบบด้วยวาจาโดยการรับเชิญ (invited talks) และ การนำเสนอผลงานด้วยวาจาโดยการสมัครเข้าร่วม (contributed talks) โดยนักวิจัยรุ่นใหม่ในสาขาที่เกี่ยวข้องรวม 20 ชิ้นผลงาน
  - ฟอรัมเสวนาสาธารณะวัฒนธรรมวิทยาศาสตร์ TPTP ครั้งที่ 3 (The Third TPTP Scientific Culture Public Forum) ในหัวข้อ ฟิสิกส์กับจักรวาลโดยเปิดโอกาสให้ครู นักเรียน และประชาชนทั่วไปได้เข้าร่วมและซักถามข้อสงสัยต่างๆ

  โดยในส่วนการประชุมวิชาการและ shortcourse มีนักฟิสิกส์วิจัยและนักศึกษาระดับปริญญาขั้นสูงเข้าร่วมการประชุมรวม 61 คน จากกลุ่มวิจัยชั้นนำของมหาวิทยาลัยทั่วประเทศโดยกิจกรรมทางวิชาการทั้งหมดใช้ภาษาอังกฤษเป็นสื่อในการนำเสนอ ในส่วนฟอรัมเสวนาสาธารณะฯ มีผู้เข้าร่วมซึ่งรวมประชาชนผู้สนใจทั่วไปรวม 117 คนกิจกรรมทางวิชาการนี้มีผู้สนับสนุนร่วมคือ สถาบันวิจัยดาราศาสตร์แห่งชาติ ภาควิชาฟิสิกส์ มหาวิทยาลัยนเรศวร และ สำนักงานคณะกรรมการวิจัยแห่งชาติในกิจกรรมนี้ได้ใช้งบประมาณ 40,000 บาทของ สกว. สำหรับค่าเดินทางของ Prof. Anne Christine-Davis ในการเดินทางจาก University of Cambridge มายังมหาวิทยาลัยนเรศวร

- การจัดการสอนกระบวนวิชา Introduction to General Relativity ของสถาบันสำนักเรียนท่าโพธิ์ฯ ปี พ.ศ. 2552 โดยได้ใช้เงินของ สกว. ในการสนับสนุนค่าเดินทางและค่าที่พักของ อาจารย์ ชาญกิจคันฉ่อง จากมหาวิทยาลัยเชียงใหม่
- สัมมนาท่าโพธิ์อนุกรมที่ 7, 8, 9, 10, 11, 12 และ 13 โดยได้ใช้เงินทุนเป็นค่าเดินทางและค่าที่พักผู้ให้สัมมนาบางส่วน

ก.6 การสร้างเครือข่ายนักวิจัย

ในการจัดประชุมสัมมนาทุกครั้งได้มีการพูดคุยกันระหว่างนักวิจัยที่เกี่ยวข้องโดยเฉพาะอย่างยิ่งนักวิจัยทางฟิสิกส์ทฤษฎีรุ่นใหม่จากมหาวิทยาลัยต่างๆโดยล่าสุดได้มีการประกาศตั้ง Thai Theoretical Astrophysics and Cosmology Network ขึ้นที่การประชุมวิชาการ สัมพัทธภาพทั่วไป ฟิสิกส์พลังงานสูง และจักรวาลวิทยา แห่งชาติครั้งที่ 4 ที่ได้รับการสนับสนุนจาก สกว.

ก.7 การได้รับเชิญให้เข้าร่วมเป็นคณะกรรมการทางวิชาการและสมาชิกสมาคมวิชาการจากภายนอก

- หัวหน้าโครงการนี้ได้รับเชิญเป็นกรรมการวิชาการของศูนย์สื่อสารวิทยาศาสตร์ไทย ของ สวทช.

- หัวหน้าโครงการนี้ได้รับเลือกเป็น Chair ของ Theoretical Astrophysics and Cosmology Working Group ของ South East Asia Astronomy Network
- หัวหน้าโครงการนี้ได้รับเชิญเป็น Reviewer ของ European Physical Journal C (I.F.(2007)=3.255) ตั้งแต่ปี 2551 เป็นต้นมา
- หัวหน้าโครงการนี้ได้รับเชิญเป็น Reviewer ของ Thai Journal of Physics: Proc. of the SIAM Physics Congress (ตั้งแต่ปี 2550 เป็นต้นมา)
- หัวหน้าโครงการนี้ได้รับเชิญเป็น **ภาคีสมาชิก** บัณฑิตยสภาวิทยาศาสตร์และเทคโนโลยีแห่งประเทศไทย (ปี 2551)

ก.8 รางวัลและการเชิดชูเกียรติที่เป็นผลจากการสนับสนุนจากเฉพาะโครงการนี้

- 1. **รางวัลนักวิทยาศาสตร์รุ่นใหม่ ประจำปี พ.ศ. 2551** ของ มูลนิธิส่งเสริมวิทยาศาสตร์และเทคโนโลยีในพระบรมราชูปถัมภ์
- 2. รางวัลนักวิจัยดีเด่นประเภทนักวิจัยที่มีผลงานวิจัยตีพิมพ์ที่มีค่า Impact Factor สูงสุด ของมหาวิทยาลัยนเรศวร ประจำปี 2551 (พร้อมเงินรางวัล 5,000 บาท)
- 3. รางวัลนักวิจัยดีเด่นประเภทนักวิจัยที่มีผลงานวิจัยตีพิมพ์ที่มีค่า Impact Factor สูงสุด ของมหาวิทยาลัยนเรศวร ประจำปี 2550 (พร้อมเงินรางวัล 5,000 บาท)
- 4. รางวัลรองชนะเลิศอันดับ 2 Oral Presentation (พร้อมเงินรางวัล 1,500 บาท) สาขาวิทยาศาสตร์เทคโนโลยีในการประชุมทางวิชาการนเรศวรวิจัยครั้งที่ 3 (29 กรกฎาคม 2550)

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A quintessentially geometric model

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Abstract. We consider string inspired cosmology on a solitary D3 brane moving in the background of a ring of branes located on a circle of radius R. The motion of the D3 brane transverse to the plane of the ring gives rise to a radion field which can be mapped to a massive non-BPS Born–Infeld type field with a cosh potential. For certain bounds of the brane tension we find an inflationary phase is possible, with the string scale relatively close to the Planck scale. The relevant perturbations and spectral indices are all well within the expected observational bounds. The evolution of the universe eventually comes to be dominated by dark energy, which we show is a late time attractor of the model. However we also find that the equation of state is time dependent, and will lead to late time quintessence.

Keywords: dark energy theory, string theory and cosmology, inflation, cosmology of theories beyond the SM

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Contents

1. Introduction

It was recently suggested that the rolling open string tachyon, inspired by a class of string theories, can have important cosmological implications. The decay of a non-BPS D3 brane filling four-dimensional space time leads to a pressureless dust phase which we identify with the closed string vacuum. The rolling tachyon has an interesting equation of state whose parameter ranges from 0 to -1. It was therefore thought to be an inflation and dark matter candidate, or a model of transient dark energy [1]. However if we rigorously stick to string theory, the effective tachyon potential contains no free parameter. A viable inflationary scenario should lead to enough numbers of e folds, and the correct level of density perturbations. The latter requires a free parameter in the effective potential which could be tuned to give rise to an adequate amount of primordial density perturbations. One also requires an adjustable free parameter in the effective potential to account for the late time acceleration.

Recently a time dependent configuration in a string theory was investigated and was shown to have interesting cosmological application [12]. In this scenario a BPS D3 brane is placed in the background of several coincident, static NS5 branes which are extremely heavy compared to the D3 brane and form an infinite throat in the space time. This system is inherently non-supersymmetric because the two different kinds of branes preserve different halves of the bulk supersymmetries. As a result the D3 brane can be regarded as a probe of the warped background and is gravitationally attracted toward the NS5 branes. Furthermore there exists an exact conformal field theory description of this background where the number of 5-branes determines the level of the WZW current algebra [13], which allows for exact string based calculations. Despite the fact that the string coupling diverges as we approach the 5-branes, it was shown that we can trust our effective Dirac–Born–Infeld (DBI) action to late times in the evolution provided that the energy of the probe brane is sufficiently high. In any event, as the probe D3 brane approaches the background branes the spatial components of the energy–momentum tensor tend to zero in exactly the same way as in the effective action description of the open string tachyon. Thus it was anticipated that the dynamics of branes in these backgrounds had remarkably similar properties to rolling tachyon solutions. This relationship was further developed by Kutasov who showed that it was possible to mimic the open string tachyon potential by considering brane motion in a specific kind of 10D geometry. In order to do this one must take the action of the BPS probe brane in the gravitational background and map it to a non-trivial scalar field solution described by the non-BPS action [2]. The new field is essentially a holographic field living on the world-volume of the brane, but encodes all the physics of the bulk background. This is known as the geometrical tachyon construction. Another particularly interesting solution considered the background branes distributed around a ring of radius R, which was analysed in [15, 16], and whose geometry is described by a coset model [14], again potentially opening the way for an exact string calculation.

It seems natural to enquire as to whether these geometrical tachyon solutions have any relevance for cosmology, since they neatly avoid the problems associated with open string tachyon inflation [4] by having a significantly different mass scale. This change in scale is due to the motion of the probe brane in a gravitationally warped background, provided by the branes in the bulk geometry. In essence, this is an alternative formulation of the simple Randall–Sundrum model [32]. More recently, flux compactification has opened up the possibility of realizing these models in a purely four-dimensional string theory context [9]. The fluxes form a throat which is glued onto a compact manifold in the UV end of the geometry. The warp factor in the metric has explicit dependence on the fluxes, and so provides us with a varying energy scale. The recent approaches to brane cosmology [7] are based on the motion of D3 branes in these compactifications. Typically we find $\bar{\text{D}}3$ branes located at some point in the IR end of the throat, which provide a potential for a solitary probe brane, with the inflaton being the inter-brane distance. In this context we can obtain slow roll inflation, and also the so-called DBI inflation [10], which relies heavily on the red-shifting of energy scales. However flux compactification models have an unacceptably large number of vacua, characterized by the string landscape. They are also low energy models, where the string scale is significantly lower than the Planck scale and so there is no attempt to deal with the initial singularity. In addition, we require multiple throats attached to the compact manifold where the standard model is supposed to live; however there is no explanation for the decoupling of the inflaton sector. These problems need to be addressed if we are to fully understand early universe cosmology in a string theory context. The alternative approach is to consider cosmology in the full ten-dimensional string theory. Although these models are plagued by their own problems there is a definite sense of where the standard model is assumed to live, and a natural realization of inflation. Furthermore we can invoke a Brandenberger–Vafa type mechanism to explain the origin of our D3 brane, arising from the mutual cascade annihilation of a gas of D9–$\bar{\text{D}}9$ branes [37].

An alternative approach is compactifying our theory on a compact manifold, where some mechanism is employed to stabilize the various moduli fields. This will naturally induce an Einstein–Hilbert term into the four-dimensional action [30]. However this is a highly non-trivial problem whose precise details remain unknown. Despite being unable to embed this into string theory, we can still learn a great deal about the physics of the model—as emphasized by recent works [18].

A specific case of interest has been studying inflation in the ring solution [19]. Due to the unusual nature of the harmonic function we find decoupled scalar modes, one transverse to the ring plane and the other inside the ring. The cosmology of modes inside the ring have been studied in [17]. In this note we will consider the situation in which the D3 brane moves in the direction transverse to the ring. Performing the tachyon map in this instance yields a cosh type potential implying that the resulting scalar field in the dual picture is massive. It is interesting that in this setting we do not have to worry about the continuity condition around the ring. And unlike for the longitudinal motion, we have an analytic expression for the effective potential every where in the transverse directions. We study the cosmological application of the resulting scenario and show that the model leads to an ever accelerating universe. We study the autonomous form of field evolution equation in the presence of matter and radiation and show that the de Sitter solution is a late time attractor of the model. We also demonstrate the viability of the geometrical tachyon for dark energy in the setting under consideration, arising in a natural way due to the non-linearity of the DBI action. In the next section we will introduce the string theory inspired model, and discuss how we an relate it to four-dimensional cosmology. In section 3 we will consider the more phenomenological aspects of our model by comparing our results with experimental observation. Section 4 shows how we have a natural realization of reheating in our model, whilst section 5 discusses the final stage of dark energy domination. Our model predicts that the equation of state parameter will tend to $\omega \sim -1$, but on even larger timescales we expect it to increase toward zero as in models of quintessence [41]. We will conclude with some remarks and a discussion of possible future directions.

2. Geometrical scalar field and coupling to gravity

We begin with the string frame CHS solution for k parallel, static NS5 branes in type IIB string theory [20, 21]. The metric is given by

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + F(x^n)\,\mathrm{d}x^m\,\mathrm{d}x^m, \tag{1}$$

where χ is the dilaton field defined as $\mathrm{e}^{2(\chi-\chi_0)} = F(x^n)$, and there exists a 3-form field strength of the NS B-field $H_{mnp} = -\varepsilon^q_{mnp}\partial_q\phi$. Here $F(x^n)$ is the harmonic function describing the position of the branes. For a large number of branes we can consider the throat approximation, which amounts to dropping the factor of unity in the function. Inherently we are decoupling Minkowski space time from the theory, and therefore only interested in the region around the NS5 branes. The harmonic function is given by

$$\begin{aligned} F &= 1 + \frac{kl_s^2\;\sinh(ky)}{2R\rho\;\sinh(ky)\;(\cosh(ky)-\cos(k\theta))} \\ &\approx \frac{kl_s^2\;\sinh(ky)}{2R\rho\;\sinh(ky)\;(\cosh(ky)-\cos(k\theta))}, \end{aligned} \tag{2}$$

where ρ, θ parametrize polar coordinates in the ring plane, and the factor y is given by

$$\cosh(y) = \frac{R^2+\rho^2}{2R\rho}. \tag{3}$$

We put a probe D3 brane at the centre of NS5 branes, as mention in the introduction this brane will move toward the circumference due to gravitational interaction if it shifted a little from the centre keeping the brane in the plane of the ring; the cosmology in

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this case is described elsewhere. We consider the case where the probe brane lies in the centre of the ring but shifted a little from the plane. In this case the probe brane shows transverse motion. Note that because of the form of the DBI action, the configuration here is actually S-dual to the D5 brane ring solution. The only difference is the shift of $k \longrightarrow 2g_{\mathrm{s}}k$ in the harmonic function. The physics however is very different, as we know that F-strings cannot end on the NS5 branes, but can end on the D5 branes. This implies that in the case of the D5 brane ring we can have additional open string tachyonic modes once the probe brane starts to resolve distances of order of the string scale. The cosmological implications for this extra field were discussed in [18].

For the brane at the centre ($\rho = 0$) moving transverse to the ring ($\dot{\rho} = 0$), the harmonic function is given by

$$F(\sigma) = \frac{kl_{\mathrm{s}}^2}{R^2 + \sigma^2} \tag{4}$$

and the DBI action for the probe brane can be written in the following form, in the static gauge:

$$S = -\tau_3 \int \mathrm{d}^4\xi \, \sqrt{F^{-1} - \dot{\sigma}^2}. \tag{5}$$

The tachyon map in this instance arises via field redefinition. We define the following scalar field, which has dimensions of length:

$$\phi(\sigma) = \int \sqrt{F} \, \mathrm{d}\sigma, \tag{6}$$

which maps the BPS action to a form commonly used in the non-BPS case [2]:

$$S = -\int \mathrm{d}^4\xi \, V(\phi)\sqrt{1 - \dot{\phi}^2}, \tag{7}$$

where $V(\phi)$ is the potential for the scalar field which describes the changing tension of the D3 brane.

From the above mapping we get the solution of field as

$$\begin{aligned}\phi(\sigma) &= \int_0^{\sigma} \sqrt{F(\sigma')} \, \mathrm{d}\sigma' \\ &= \sqrt{kl_{\mathrm{s}}^2} \, \ln\left(\frac{\sigma}{R} + \sqrt{1 + \frac{\sigma^2}{R^2}}\right) \\ &= \sqrt{kl_{\mathrm{s}}^2} \, \mathrm{arcsinh}\left(\frac{\sigma}{R}\right)\end{aligned} \tag{8}$$

$$\begin{aligned}V(\phi) &= \frac{\tau_3}{\sqrt{F}} \\ &= \frac{\tau_3 R}{\sqrt{kl_{\mathrm{s}}^2}} \cosh\left(\frac{\phi}{\sqrt{kl_{\mathrm{s}}^2}}\right).\end{aligned} \tag{9}$$

Clearly we see that $\phi \rightarrow \pm\infty$ as $\sigma \rightarrow \pm\infty$, and that at the minimum of the potential we have $\phi = 0$.[4] The potential of the field suggests that the mass is given by $1/kl_{\mathrm{s}}^2$,

[4] We must bear in mind that our approximation of the harmonic function prevents us from taking the $\sigma \rightarrow \infty$ limit.

corresponding to a massive scalar fluctuation. One may ask whether there is a known string mode exhibiting this profile. In fact the fluctuations of a massive scalar were computed in [29] using a similar approach to the construction of the open string tachyon mode in boundary conformal field theory [1]. This field was then used in [8, 31] as a candidate for the inflaton living on a $\bar{\text{D}}3$ brane in the KKLT scenario [34]. The potential for the scalar is known to fourth order and was assumed to be exponential in profile, although globally it may be hyperbolic.

In order to discuss the cosmological evolution of our scalar field we need to couple our effective action to four-dimensional Einstein gravity. There are several ways we can accomplish this. Firstly we can consider the mirage cosmology scenario [33]. This requires us to rewrite the induced metric on the D3 brane world-volume in a Friedmann–Robertson–Walker (FRW) form. The universe will automatically be flat, or closed if we imagine the D brane to be spherical. The problem here is that there is no natural way to couple gravity to the brane action and therefore we must insert it by hand; however the cosmological dynamics is expected to be reliable virtually all the way to the string scale. The second option is a slight modification of the first. We imagine that the bulk is infinite in extent, and that the D3 brane is again coupled to gravity through some unknown mechanism. However rather than writing the induced metric in FRW form, we switch to holographic theory. Now, the tachyon mapping in this case is only concerned with time dependent quantities, and in particular only with the temporal component of the Minkowski metric. Therefore we choose to include a scale factor component in the spatial directions. This means that we have a cosmological coupling for the holographic scalar field, and the universe lives on the D3 brane world-volume. The final approach would be to compactify the theory down to four dimensions. In order to do this we need to truncate the background to ensure the space is compact [9]. In our case the ring can naturally impose a cut-off in the planar direction; however we must still impose some constraint in the transverse direction to the ring plane. Our solution simplifies somewhat if we can consider the $R \to 0$ limit, or equivalently the $\sigma \gg 1$ limit, as the background will appear point like. Smoothly gluing the truncated space to a proper compact manifold will now automatically include an Einstein–Hilbert term in the effective action [30]. However, although we now have a natural coupling to gravity, the compactification itself is far from trivial as we also need to wrap two of the world-volume directions of the NS5 branes on a compact cycle. In order to proceed we must first uplift the full solution to M-theory[5], where we now have a ring of M5 branes magnetically charged under the 3-form $C_{(3)}$. Compactification demands that the magnetic directions of the 3-form are wrapped on toroidal cycles, which is further complicated by the ring geometry and will generally result in large corrections to the potential once reduced down to four dimensions. So, although we have a natural gravitational coupling we may have large corrections to the theory. The complete description of this compactification is interesting, but well beyond the scope of this note and should be tackled as a future problem. However we could also assume a large volume toroidal compactification, where again all the relevant moduli have been stabilized. Provided we introduce some 'sink' for the 5-brane charge, located at the some distant point in the compact space, and also only concentrate on the region close to the branes so that the harmonic function remains valid, we will have an induced

[5] This was discussed by Ghodsi *et al* in [18]. We refer the interested reader there for more details.

gravitational coupling in the low energy theory. The corrections to the scalar potential in this region of moduli space may well be sub-leading with respect to the scalar field dynamics and thus we can treat our model as the leading order behaviour.

Recent work in this direction has been concerned with the compactification approach [18, 19], where it was assumed all the relevant moduli are fixed along the lines of the KKLT model [34] and that all corrections to the potential are sub-dominant. We will tentatively assume that this will also hold in our toy model.

We can now analyse our four-dimensional minimally coupled action, where we find the following solutions to the Einstein equations:

$$H^2 = \frac{V(\phi)}{3M_{\rm p}^2\sqrt{1-\dot{\phi}^2}} \tag{10}$$

$$\frac{\ddot{a}}{a} = \frac{V(\phi)}{3M_{\rm p}^2\sqrt{1-\dot{\phi}^2}}\left(1-\frac{3\dot{\phi}^2}{2}\right). \tag{11}$$

These expressions are different to those associated with a traditional canonical scalar field. In particular we see that inflation will automatically end once $\dot{\phi}^2 \sim 2/3$ as in the tachyon cosmology models [3, 5, 11]. For completeness we write the equation of motion for the inflaton derived from the non-BPS action as follows:

$$\frac{V(\phi)\ddot{\phi}}{1-\dot{\phi}^2} + 3HV(\phi)\dot{\phi} + V'(\phi) = 0, \tag{12}$$

where dots are derivatives with respect to time and primes are derivatives with respect to the field. Note that we are suppressing all delta functions in the expressions. We can now proceed with the analysis of our theory in the usual manner. It must be noted that this model corresponds to large field inflation, where the initial value of the scalar field must satisfy the following condition:

$$\phi_0 \ll \sqrt{kl_{\rm s}^2}\,\mathrm{arccosh}\left(\frac{\sqrt{kl_{\rm s}^2}}{R}\right), \tag{13}$$

according to our truncation of the harmonic function.

Note that in what follows we will frequently switch between the field theory and the bulk geometry. The latter is more geometrical and so provides us with extra intuition about the physics of the solution, however both are equivalent—at least in this simplified model.

Using the slow roll approximation, $H^2 \simeq V(\phi)/3M_{\rm p}^2$ and $3H\dot{\phi} \simeq -V_\phi/V$, the e folding

$$\begin{aligned} N &= \int_t^{t_f} H\,\mathrm{d}t \\ &= \frac{\tau_3 R\sqrt{kl_{\rm s}^2}}{M_{\rm p}^2}\int_{x(\phi_f)}^{x(\phi)} \frac{\cosh^2 x}{\sinh x}\,\mathrm{d}x \\ &= s\left[-\cosh(x_f) + \cosh(x) - \ln\left(\frac{\tanh(x_f/2)}{\tanh(x/2)}\right)\right] \end{aligned} \tag{14}$$

where we have introduced the dimensionless quantities $x = \phi/\sqrt{kl_{\rm s}^2}$ and $s = \tau_3 R\sqrt{kl_{\rm s}^2}/M_{\rm p}^2$.

Further defining the new quantity $y \equiv \cosh x$, we can write the number of e folds as follows:

$$N = s\left[-y_f + y - \frac{1}{2}\ln\left(\frac{(y_f-1)(y+1)}{(y_f+1)(y-1)}\right)\right]. \tag{15}$$

Now, the relevant slow roll parameter is defined as $\epsilon \equiv -\dot{H}/H$ which in our solution reduces to

$$\epsilon = \frac{y^2-1}{2sy^3}. \tag{16}$$

Note that our model is explicitly non-supersymmetric, and therefore we do not need to calculate the second slow roll parameter η since we anticipate that this will be trivially satisfied if ϵ is. At the end of inflation $\epsilon = 1$; then $y_f \equiv f(s)$ is given by the root of above equation, setting $\epsilon = 1$:

$$f(s) = \frac{1}{6s}\left[g(s) + \frac{1}{g(s)} + 1\right] \tag{17}$$

where $g(s) = (-54s^2 + 1 + 6s\sqrt{3(27s^2-1)})^{1/3}$. From equation (15) the equation for y is

$$\ln\left(\frac{y+1}{y-1}\right) - 2y = -\frac{2N}{s} - 2f(s) - \ln\left(\frac{f(s)-1}{f(s)+1}\right). \tag{18}$$

For $s > 1$ and as $y_{\text{min}} = 1$, ϵ always remains less than one leading to an ever accelerating universe. Thus, in this case the geometrical scalar field in the present setting is not suitable for describing inflation but can become a possible candidate of dark energy. However if τ_3 is small enough that $s < 1$, then we will find that inflation is possible as the slow roll parameter will naturally tend toward unity. There is a critical bound $s \leq 1/(3\sqrt{3})$, which must be satisfied if we are to consider inflation in this context.

3. Inflationary constraints

To know the observational constraint on s we have to calculate the density perturbations. In the slow roll approximation, the power spectrum of curvature perturbation is given by [22]–[24]

$$\begin{aligned} P_{\text{S}} &= \frac{1}{12\pi^2 M_{\text{p}}^6}\left(\frac{V^2}{V_\phi}\right)^2 \\ &= \frac{\tau_3^2 R^2}{12\pi^2 M_{\text{p}}^6}\left(\frac{\cosh^2(\phi/\sqrt{kl_{\text{s}}^2})}{\sinh(\phi/\sqrt{kl_{\text{s}}^2})}\right)^2. \end{aligned} \tag{19}$$

The COBE normalization corresponds to $P_{\text{S}} \simeq 2\times 10^{-9}$ for modes which crossed $N = 60$ before the end of inflation [6] which gives the following constraint:

$$k(l_{\text{s}}M_{\text{p}})^2 \simeq \frac{10^9}{12\pi^2}\frac{s^2\cosh^4(\phi/\sqrt{kl_{\text{s}}^2})}{\cosh^2(\phi/\sqrt{kl_{\text{s}}^2}) - 1}. \tag{20}$$

From the numerics using equations (18) and (19), we find that

$$k(l_s M_p)^2 \geq 3 \times 10^{10} \tag{21}$$

which corresponds to $s \sim 10^{-3}$ when we impose the constraints $\tau_3 = 10^{-10} M_p^4$ and $R = 10^2/M_p$ which we regard as being typical values. The constraint on the tension in fact implies the following relationship:

$$\frac{M_p}{M_s} \sim \frac{10^2}{g_s^{1/4}}, \tag{22}$$

which we need to be consistently satisfied. However, note that because of our basic assumptions about the theory we will generally obtain the bound

$$\frac{\tau_3 R}{M_p^3} \leq \frac{1}{9 \times 10^5}. \tag{23}$$

If we write the tension of the brane in terms of fundamental parameters we can estimate the relationship between the string and Planck scales using the fact that we require $R > M_s^{-1}$ for the action to be valid,

$$\frac{M_p}{M_s} \geq \frac{15}{g_s^{1/3}}, \tag{24}$$

where g_s is the string coupling constant. Note that this potentially constrains the string scale to be close to the Planck scale, as even if we demand weak coupling with $g_s = 0.001$ this gives us $M_p \geq 10^2 M_s$. Of course this is only a bound, and in our model we are treating this as a free parameter. In any event our typical values are consistent and thus we feel free to proceed. We should note that from a string theoretic point of view we should not take s as being a variable in this model. However our earlier analysis has shown that if we wish to consider non-eternal inflation, there exists a maximum bound on this parameter which is quite small. Thus we can make the assumption that s will always be small, with appropriate tuning of the ratio of the string and Planck scales. In the following analysis we will always be assuming that this is satisfied so as to avoid en eternal inflation scenario. Of course, in the string theory picture we have a probe brane moving in a non-trivial background geometry, and we would expect that the RR charge on the brane will be radiated away in the form of closed string modes. This effectively means that there is an additional decay constant in the definition of the field ϕ, which we have neglected in this note. Thus what we have here is a first-order approximation to the behaviour of the solution. It remains an open question whether we can define a tachyon map in this instance—and how this changes the inflationary scenario described here.

At leading order in our solutions, where s is assumed to be small and making sure our effective action remains valid, we obtain

$$k(l_s M_p^2)^2 \simeq \frac{10^9}{48\pi^2}(2N+1)^2 \tag{25}$$

which corresponds to $s \sim 10^{-5}(2N+1)$ and $y \sim (2N+1)/2s$, when $\tau_3 = 10^{-10} M_p^4$ and $R = 10^2/M_p$. Again, more generally we would find the following upper limit on the solution:

$$s \leq 10^{-3}(2N+1), \tag{26}$$

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which is easily satisfied by our typical values. In fact our results remain robust when compared to the WMAPII and SDSS results combined [27]. The new data constrain $n_s = 0.98 \pm 0.02$ at the 68% confidence level, and $r < 0.24$ at the 95% confidence level.

The spectral index of scalar perturbations is defined as [22]–[24]

$$\begin{aligned} n_S - 1 &\equiv -4\frac{M_p^2 V_\phi^2}{V^3} + 2\frac{M_p^2 V_{\phi\phi}}{V^2} \\ &= \frac{2}{s}\left(\frac{2-y^2}{y^3}\right). \end{aligned} \tag{27}$$

The spectral index of tensor perturbations is defined as

$$\begin{aligned} n_T &= -\frac{M_p^2 V_\phi}{V^3} \\ &= -\frac{1}{s}\left(\frac{y^2-1}{y^3}\right). \end{aligned} \tag{28}$$

The tensor-to-scalar ratio is

$$\begin{aligned} r &\equiv 8\frac{M_p^2 V_\phi^2}{V^3} \\ &= \frac{8}{s}\left(\frac{y^2-1}{y^3}\right). \end{aligned} \tag{29}$$

With the limit $s \to 0$ we get

$$n_S = 1 - \frac{4}{(2N+1)}, \qquad n_T = -\frac{2}{(2N+1)}, \qquad r = \frac{16}{(2N+1)}. \tag{30}$$

For $N = 60$, we get $n_S = 0.966\,94$ and $r = 0.132\,23$; for $N = 50$, we get $n_S = 0.960\,40$ and $r = 0.158\,42$. We know from observations that the constraint on the tensor-to-scalar ratio is $r < 0.36$ [25, 26], and so our model appears to be well within this bound.

4. Reheating

We see that the potential is a symmetric potential with a minima. In terms of the bulk field σ it can be written as

$$V(\sigma) = \frac{\tau_3 R}{2\sqrt{kl_s^2}}\left[\left(\frac{\sigma}{R} + \sqrt{1+\frac{\sigma^2}{R^2}}\right) + \left(\frac{\sigma}{R} + \sqrt{1+\frac{\sigma^2}{R^2}}\right)^{-1}\right]. \tag{31}$$

Now the question is would the brane oscillate back and forth through the ring, and if so what are the necessary conditions for oscillation? In the bulk picture we would naturally anticipate oscillation with a decaying amplitude due to RR emission. Moreover the minimum of the potential in this case is actually metastable. However this has not been verified as we need to calculate the energy emission in the coset model description [14], which we leave as future work. This will alter the dynamics of the inflaton field as discussed in the previous section.

In any event we may also expect similar behaviour once our field is coupled to gravity, with the damping being provided by the Hubble term. This is particularly important

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because we may find inflation occurring in the phase space region beyond $s \geq s_{\rm crit}$, once enough damping has occurred. The relevant dynamical equations are the inflaton field equation (12) and the Friedmann equation. We repeat them below for convenience.

$$\ddot{\phi} + 3H\dot{\phi}(1-\dot{\phi}^2) + \frac{V_\phi}{V}(1-\dot{\phi}^2) = 0 \tag{32}$$

$$H^2 = \frac{1}{3M_{\rm p}^2}\left(\frac{V}{\sqrt{1-\dot{\phi}^2}} + \rho_{\rm B}\right) \tag{33}$$

where the terms inside the curly brackets cause damping. For an easy treatment let us first consider the slow roll approximation; then in the damping equation only the $\dot{\phi}$ term remains and all other powers of $\dot{\phi}$ can be ignored. That is to say we are considering the case near the stable point. Then $H^2 \sim (\tau_3 R/3M_{\rm p}^2\sqrt{kl_{\rm s}^2}) + (\rho_{\rm B}/3M_{\rm p}^3)$ is constant and $(V_\phi/V) \sim (2/kl_{\rm s}^2)\phi$. The equation of motion is then

$$\ddot{\phi} + 3\dot{\phi}\sqrt{\left(\frac{\tau_3 R}{3M_{\rm p}^2\sqrt{kl_{\rm s}^2}} + \frac{\rho_{\rm B}}{3M_{\rm p}^2}\right)} + \frac{2}{kl_{\rm s}^2}\phi = 0 \tag{34}$$

and for critically damped motion we need

$$\left(\frac{\tau_3 R}{3M_{\rm p}^2\sqrt{kl_{\rm s}^2}} + \frac{\rho_{\rm B}}{3M_{\rm p}^2}\right) = \frac{8}{9kl_{\rm s}^2}. \tag{35}$$

If the RHS of (35) is greater than the LHS we will find oscillations, but it is reduced by damping which depends on the size of the damping factor $\left(=(3/2)\sqrt{((\tau_3 R/3M_{\rm p}^2\sqrt{kl_{\rm s}^2}) + (\rho_{\rm B}/3M_{\rm p}^2))}\right)$, compared to the oscillation frequency $\left(=\sqrt{(8/kl_{\rm s}^2) - 9((\tau_3 R)/(3M_{\rm p}^2\sqrt{kl_{\rm s}^2} + \rho_{\rm B}/3M_{\rm p}^2))}\right)$.

From the definition of $\Omega_{\rm B}$, setting it to 0.3, we get $\rho_{\rm B} = 3\tau_3 R/7\sqrt{kl_{\rm s}^2}$, and then from equation (35) we obtain

$$\begin{aligned} s &> \tfrac{168}{90} && \text{Over damped} \\ &= \tfrac{168}{90} && \text{Critically damped} \\ &< \tfrac{168}{90} && \text{Oscillatory with a decaying amplitude.} \end{aligned} \tag{36}$$

Recall from the previous section that for us to have non-eternal inflation there is a maximal bound for s, and so only the last solution can be considered physical. From the constraint we get $\sqrt{kl_{\rm s}^2} \sim 10^5 M_{\rm p}^{-1}$, $\tau_3 \sim 10^{-10} M_{\rm p}^4$ and $R \sim 10^2 M_{\rm p}^{-1}$. Hence it is oscillatory near the critical point. The energy of the decaying scalar field is used in expansion and particle production. If the rate of expansion of the universe is much less than the decaying rate of the amplitude of the field then most of the energy released by the scalar field goes to reheating. The explicit solution of equation (34) is

$$\phi(t) = \phi_0 \exp\left(\left[-\frac{3}{2}t\sqrt{\frac{10\tau_3 R}{21M_{\rm p}^2\sqrt{kl_{\rm s}^2}}}\right]\right)\exp\left(\left[\pm i t\sqrt{\frac{2}{kl_{\rm s}^2} - \frac{15\tau_3 R}{14M_{\rm p}^2\sqrt{kl_{\rm s}^2}}}\right]\right). \tag{37}$$

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The ratio of rate of field decay to the rate of expansion of universe is defined to be

$$\Theta \equiv \left| \frac{\dot{\phi}}{H\phi} \right|. \tag{38}$$

For this case we find:

$$\Theta = \sqrt{\frac{21 M_{\rm p}^2}{5\tau_3 R \sqrt{k l_{\rm s}^2}}}. \tag{39}$$

The above quantity can be made to be less than one by adjusting the various parameters.

Using equation (25) we obtain

$$\Theta \sim \sqrt{\frac{21 \times 10^5}{5(2N+1)}} \tag{40}$$

which allows us to write the parameter as a function of the number of e foldings, provided we can trust our small s expansion. We know that reheating ends when $\Theta = 1$; thus the minimal number of e foldings we require for this to be satisfied is

$$N_{\rm end} \sim 10^5. \tag{41}$$

Clearly this is a large number of e foldings, and this should motivate us to do a more thorough analysis. For now it would appear that unless there is a large amount of fine-tuning, reheating would not end in this scenario. The difficulty is that we cannot use the WKB approximation in this case due to rapid fluctuations in the variation of the potential. Moreover, the analysis will be incomplete without specifying the exact form the gravitational coupling—as there will be corrections to the effective action arising from any compactification. For these reasons we will postpone the analysis and return to it in a later publication.

5. Dark energy

What are the implications of our model for dark energy[6]? It is well known that the non-linear form of the DBI action admits an unusual equation of state, which is of the form

$$\begin{aligned} \omega &= \frac{P}{\rho} \\ &= \dot{\phi}^2 - 1 \end{aligned} \tag{42}$$

where P and ρ are the pressure and energy densities respectively. In tachyon models the field is moving relativistically near the vacuum and the equation of state will tend to $\omega \sim 0$, which is problematic for reheating. However our model has significantly different late time behaviour because our scalar field will oscillate about the minimum of its potential, eventually coming to a halt at the minimum. Therefore we expect the equation of state to become $\omega \sim -1$, corresponding to the vacuum energy of the universe. This motivates us to analyse our system as a potential dark matter candidate. One problem, however, is that

[6] See [42] for an excellent review.

the reheating phase does not seem to have a natural termination point. Rather, reheating of the universe continues whilst the brane oscillates around the minimum of the potential, and then terminates in what appears to be a dark energy dominated phase. From the perspective of model building this is obviously a difficult problem. For now let us assume that there is some ad hoc mechanism which ends inflation, and look at the evolution of the system in this dark matter dominated phase. The corresponding evolution equations of interest are

$$\frac{\ddot{\phi}}{1-\dot{\phi}^2} + 3H\dot{\phi} + \frac{V_\phi}{V} = 0 \tag{43}$$

$$\dot{H} + \frac{V(\phi)\dot{\phi}^2}{2M_{\rm p}^2\sqrt{1-\dot{\phi}^2}} + \frac{\gamma\rho_{\rm B}}{2M_{\rm p}^2} = 0 \tag{44}$$

where we have included contribution from a barotropic fluid in the second equation. Defining the following dimensionless quantities:

$$\begin{aligned} Y_1 &= \frac{\phi}{\sqrt{kl_{\rm s}^2}} \\ Y_2 &= \dot{\phi} \end{aligned} \tag{45}$$

and using equations (43) and (45) we get the autonomous equations

$$Y_1' = \frac{1}{\sqrt{kl_{\rm s}^2}H} Y_2 \tag{46}$$

$$Y_2' = -\left(1-Y_2^2\right)\left(3Y_2 + \frac{1}{H}\frac{{\rm d}Y_3}{{\rm d}Y_1}\right) \tag{47}$$

where we have switched to using the number of e folds as the time parameter, and now primes denote derivatives with respect to N. The final expressions we require can be read off as

$$\begin{aligned} Y_3 &= \ln\left(\frac{V(\phi)}{3M_{\rm p}^2}\right). \\ H^2 &= \frac{{\rm e}^{Y_3}}{\sqrt{1-Y_2^2}} + \frac{\rho_{\rm B}}{3M_{\rm p}^2}. \end{aligned} \tag{48}$$

Simple analysis shows us that critical point is at $Y_1 = 0$ and $Y_2 = 0$ which is a global attractor as shown in figure 1. This agrees with our physical intuition since it implies the probe brane will slow down, eventually coming to rest at the origin of the transverse space. In terms of our critical ratios we find

$$\Omega_\phi = \frac{{\rm e}^{Y_3}}{{\rm e}^{Y_3} + (\rho_{\rm B}/3M_{\rm p}^2)\sqrt{1-Y_2^2}} \tag{49}$$

$$\Omega_{\rm B} = \frac{\rho_{\rm B}}{(3M_{\rm p}^2{\rm e}^{Y_3}/\sqrt{1-Y_2^2}) + \rho_{\rm B}}. \tag{50}$$

Note that they are constrained by $\Omega_\phi + \Omega_{\rm B} = 1$. We also have $\Omega_{\rm B} = \Omega_M + \Omega_R$, where M and R denote matter and radiation respectively, whilst ϕ is associated with our scalar field.

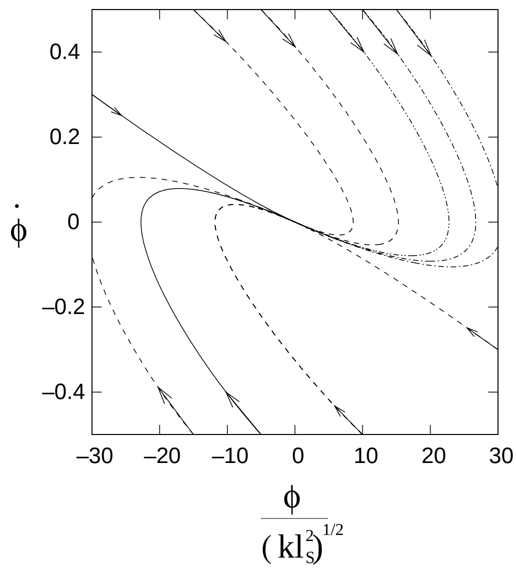

Figure 1. Plot of the phase space solution with a variety of initial conditions. Here we see the presence of a global attractor at $(\phi = 0, \dot{\phi} = 0)$.

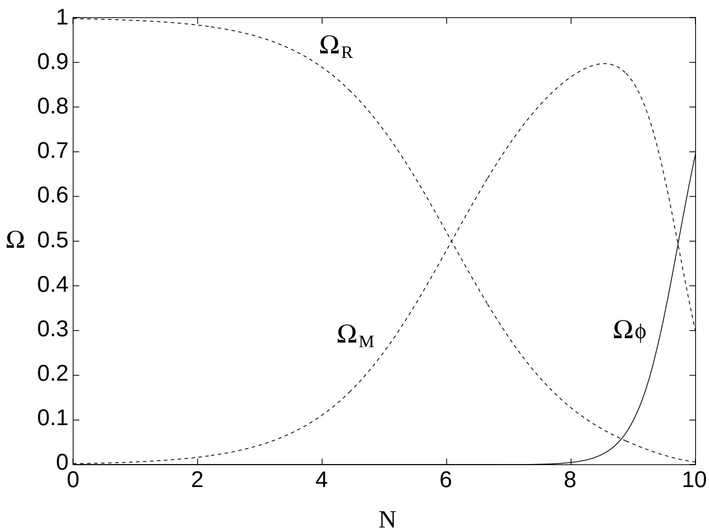

Figure 2. Illustration of the various behaviours for Ω_i. Here we have taken $\rho_m^0 = 4.58 \times 10^6$, $\rho_R^0 = 10^{10}$ and $V_0 = 10^{-6}$. The dark line is for Ω_R, the dotted line is for Ω_ϕ and the light line is for Ω_m.

From the plots of figure 2 we see that Ω_ϕ goes to 0.7, Ω_M goes to 0.3 and Ω_R goes to 0 in the presence epoch. We see that at late times, the field settles at the potential minimum leading to a de Sitter solution with energy scale $V_0 = \tau_3 R/\sqrt{kl_s^2}$. Using the numerical data from the preceding sections we can write this an upper bound on the energy density as follows:

$$V_0 \leq 10^{-12} M_p^4. \tag{51}$$

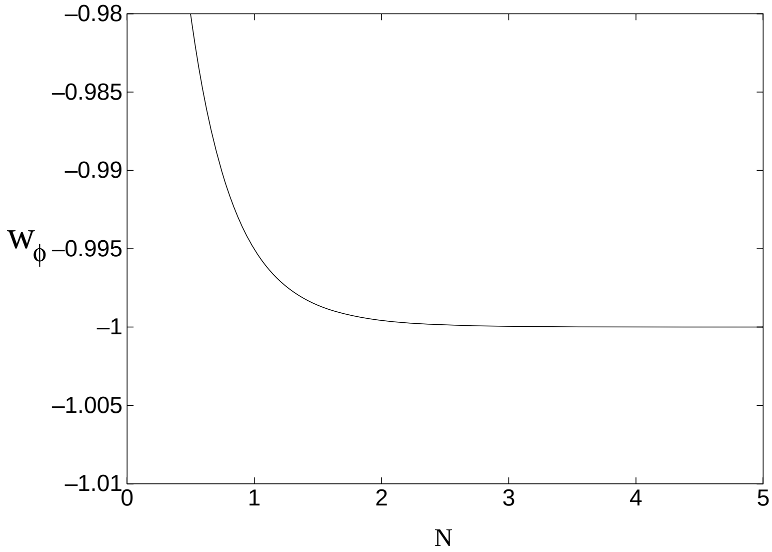

Figure 3. Evolution of the equation of state parameter with the number of e folds. Note that ω rapidly approaches -1 as expected.

Although this is several orders of magnitude higher than the observed value, we note that this value is heavily dependent on the scales in the theory, and with appropriate tuning could be substantially smaller. Since there exists no realistic scaling solution (which could mimic matter/radiation), the model also requires the fine-tuning of the initial value of the scalar field. The field should remain sub-dominant for most of the cosmic evolution and become comparable to the background at late times. It would then evolve to dominate the background energy density ultimately settling down in the de Sitter phase. Figure 3 shows the variation of the equation of state as a function of the number of e-folds.

However, recall from the bulk picture that the point $\sigma = 0, \rho = 0$ will be gravitationally unstable and the probe brane will eventually be attracted toward the ring. In terms of our cosmological theory we see that this de Sitter point will actually be only quasi-stable and that a tachyonic field will eventually condense forcing the vacuum energy down toward zero. This suggests that the vacuum energy will not be constant, but will be slowly varying. Furthermore our equation of state should be modified to incorporate the dynamics of this additional field. It is trivially apparent that the inflationary phase will terminate and give way to a dark energy phase where $\omega \sim -1$. Once the tachyon field starts to roll, ω will increase toward zero from below giving rise to a phase of quintessence [41]. Eventually we will begin to probe the strong coupling regime and our effective action will break down.

Let us return to the bulk picture to understand this in more detail. We introduce a complex field $\xi = \rho + i\sigma$ which can actually be globally defined in the target space. The harmonic function factorizes in this coordinate system into holomorphic and anti-holomorphic parts $F(\xi, \bar{\xi}) = f(\xi)f(\bar{\xi})$. Thus the tachyon map will also split accordingly:

$$\partial_t \phi = f(\xi)\partial_t \xi, \qquad \partial_t \bar{\phi} = f(\bar{\xi})\partial_t \bar{\xi}. \tag{52}$$

These expressions are exactly solvable provided we continue them into the complex plane. If we now reconstruct the potential for these fields in terms of our holographic theory we obtain the general solution

$$V(\phi,\bar{\phi}) = \frac{R\tau_3}{\sqrt{kl_s^2}}\left[\cos\left(\frac{\phi}{\sqrt{kl_s^2}}\right)\cos\left(\frac{\bar{\phi}}{kl_s^2}\right)\right]^{1/2}. \tag{53}$$

Clearly when ϕ is real we recover our cosine potential, whilst if it is purely imaginary we recover the cosh solution. These correspond to motion inside the ring and motion transverse to the ring respectively. The tachyonic instability forces the field from the false vacuum state toward the true ground state. Therefore we expect the dark energy potential to be

$$V(\phi,\bar{\phi}) \sim \frac{R\tau_3}{\sqrt{kl_s^2}}\cos\left(\frac{\phi}{\sqrt{kl_s^2}}\right) \tag{54}$$

and so the true minimum will occur when $V \sim 0$ at $\phi = \pm\pi\sqrt{kl_s^2}/2$, corresponding to the location of the ring in the bulk picture. The cosmological dynamics in this particular phase are well described by [17, 19], where it was shown to be possible for the true vacuum to be non-zero, provided the trajectory of the probe brane is sufficiently fine-tuned.

We finally comment on the instability for the field fluctuations for potential with a minimum [28]. In a flat FRW background each Fourier mode of ϕ satisfies the following equation:

$$\frac{\delta\ddot{\phi}_{\tilde{k}}}{1-\dot{\phi}^2} + \left[3H + \frac{2\dot{\phi}\ddot{\phi}}{(1-\dot{\phi}^2)^2}\right]\delta\dot{\phi}_{\tilde{k}} + \left[\frac{\tilde{k}^2}{a^2} + (\ln V)_{\phi,\phi}\right]\delta\phi_{\tilde{k}} = 0 \tag{55}$$

where $\tilde{k}$ is the comoving wavenumber. We now compute the second derivatives of the potential and obtain

$$(\ln V)_{\phi,\phi} = \frac{1}{kl_s^2}\left(1 - \tanh\left[\frac{\phi}{\sqrt{kl_s^2}}\right]\right). \tag{56}$$

Here we see that $(\ln V)_{\phi,\phi}$ is never divergent for any value of ϕ, and is always non-negative, i.e. that $(\ln V)_{\phi,\phi} \in [0,1]$. Thus we do not have any instability associated with the perturbation $\delta\phi_k$ with our potential (9). This is to be contrasted with the result obtained for the open string tachyon which has rapid fluctuations and instabilities associated with its evolution.

6. Conclusion

In this note we have examined the time dependent configuration of a single D3 brane in the background of NS5 branes distributed on a ring of radius R, taking the near horizon approximation. We then studied the cosmological implications of the effective potential which arises due to the transverse motion of D3 with respect to the plane of the ring. The model appears to describe an inflationary phase giving way to a natural reheating mechanism, and then a further phase of dark energy driven expansion. Although we cannot accurately predict the scale of the energy density at this point, we do obtain an upper bound. In this case the dark energy phase is a late time attractor of our model, and we predict that the vacuum energy will eventually decay to zero—although on extremely

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large timescales[7]. In fact our results will be dramatically improved by keeping the full structure of the harmonic function, because at large distances the potential is even flatter, yielding even more e foldings of inflation. Due to the absence of scaling solutions in our field theory, we need to tune the initial value of the scalar field such that it can become relevant only at late times. With these described fine-tunings, the geometrical field is a potential dark energy candidate. The model is free from tachyon instabilities, and the field perturbations behave in a similar manner to those of the canonical scalar field.

With the model we have several potential problems. Firstly there is our assumption about the coupling of the DBI to four-dimensional gravity, although as we have pointed out this can be resolved by a full string theory compactification. However there will generally be large corrections, potentially destroying the simplicity of the solution. Secondly the trajectory of the brane in the bulk space is particularly special. In the most generic case we would anticipate a general spiralling trajectory toward the ring. In this case there would be no simple decoupling of the modes and we would need to consider the full form of the potential. This amounts to a certain amount of fine-tuning of the initial conditions. Another problem is that we have not turned on any standard model fields which would be expected to couple to the inflaton on the world-volume. However the inclusion of $U(1)$ gauge fields on the brane will act to reduce the velocity of the field by a factor of $\sqrt{1-E^2}$, where E is our dimensionless electric field. More important however is that we have neglected the induced 2-form field strength, which can have important applications in cosmology as seen in the recent paper [40]. Despite these problems, we know there is a coset model describing the background which opens the way for exact string theory calculations. Furthermore the relationship between the two energy scales in the theory means it is possible to talk about long-standing problems such as the trans-Planckian issue [38]. One further problem is the termination of reheating in this model. We have emphasized that this is indeed difficult to tackle in this model due to its analytic simplicity. One may hope that a careful analysis of the tachyon mapping will lead to more realistic behaviour for the inflaton field, and thus a possible exit from reheating. In fact this may also be possible by considering more general trajectories of the probe brane in the bulk picture. We hope to return to this issue in a future publication.

One thing that emerges though is the relationship between a dark energy dominated phase and the 'fast rolling' DBI action [10]. Although our proposal is far from rigorous, it does capture the majority of the same physics as in the flux compactification scenario. We know that D3 branes moving in non-trivial backgrounds have sub-luminal velocities as measured by observers in the far UV of the geometry, due to the gravitational red-shifting. In fact the branes are decelerating and for late times will have negligible velocities. This in turn implies that the equation of state parameter will tend to $\omega \sim -1$ at late times. A concrete example where this could be examined is in the case of the warped deformed conifold [39]. The RR flux will wrap the S^3 in the IR end of the geometry, and we can imagine a solitary D3 brane probing this part of the conifold after an inflationary phase. To an observer in the compact space the brane will slow down as it reaches the origin of the S^3 yielding a dark matter dominated phase [36].

[7] However we must be careful since the DBI action will not be valid once it coalesces with the NS5 branes so we must assume that it passes between the branes. This requires fine-tuning of the initial trajectory which is not realistic. This problem may be resolved by switching to the description of the model in terms of the little string theory [35].

However our model opens up the possibility that non-trivial background configurations may have important implications for brane cosmology, as we have seen how to combine inflation, reheating and dark energy in a single model. Furthermore this is not subject to the same landscape problems as the flux compactification models, and we can try and tackle higher energy issues in a clear formalism [9]. Although we acknowledge the simplicity of our solution we hope that this will encourage more research in this direction.

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Phantom field dynamics in loop quantum cosmology

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We consider a dynamical system of phantom scalar field under exponential potential in the background of loop quantum cosmology. In our analysis, there is neither stable node nor repeller unstable node but only two saddle points, hence no big rip singularity. Physical solutions always possess potential energy greater than the magnitude of the negative kinetic energy. We found that the universe bounces after accelerating even in the domination of the phantom field. After bouncing, the universe finally enters the oscillatory regime.

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I. INTRODUCTION

Recently, present accelerating expansion of the universe has been confirmed with observations via cosmic microwave background anisotropies [1,2], large scale galaxy surveys [3], and type Ia supernovae [4,5]. However, the problem is that the acceleration can not be understood in standard cosmology. This motivates many groups of cosmologists to find out the answers. Proposals to explain this acceleration made previously could be, in general, categorized into three ways of approach [6]. In the first approach, in order to achieve acceleration, we need some form of scalar fluid called *dark energy* with equation of state $p = w\rho$, where $w < -1/3$. Various types of models in this category have been proposed and classified (for a recent review, see Refs. [7,8]). The other two ways are that accelerating expansion is an effect of backreaction of cosmological perturbations [9] or late acceleration is an effect of modification in the action of general relativity. This modified gravity approach includes braneworld models (for review, see [10]). Until today, there has not yet been a true satisfactory explanation of the present acceleration expansion.

Considering dark energy models, a previous first-year WMAP data analysis combined with 2dF galaxy survey and SN-Ia data and even a previous SN-Ia analysis alone favor $w < -1$ more than cosmological constant or quintessence [11,12]. A precise observational data analysis (combining CMB, Hubble Space Telescope, type Ia Supernovae, and 2dF data sets) allows the equation of state $p = w\rho$ with a constant w value between -1.38 and -0.82 at the 95% confident level [13]. The recent WMAP three-year results combined with Supernova Legacy Survey (SNLS) data when assuming flat universe yields $-1.06 < w < -0.90$. However, without assumption of flat universe but only combined WMAP, large scale structure and supernova data implies a strong constraint, $w = -1.06^{+0.13}_{-0.08}$ [14]. While assuming a flat universe, the first result from ESSENCE Supernova Survey Ia combined with SuperNova Legacy Survey Ia gives a constraint of $w = -1.07 \pm 0.09$ [15]. Interpretation of various data brings about a possibility that dark energy could be in a form of phantom field—a fluid with $w < -1$ (which violates dominant energy condition, $\rho \geq |p|$) rather than quintessence field [16–18]. The phantom equation of state $p < -\rho$ can be attained by the negative kinetic energy term of the phantom field. However, there are some types of braneworld model [19] as well as Brans-Dicke scalar-tensor theory [20] and gravitational theory with higher derivatives of scalar field [21] that can also yield phantom energy. There has been investigation on dynamical properties of the phantom field in the standard Friedmann-Robertson-Walker (FRW) background with exponential and inverse power law potentials by [22–25] and with other forms of potential by [25–27]. These studies describe fates of the phantom dominated universe with different steepness of the potentials.

A problem for phantom field dark energy in standard FRW cosmology is that it leads to singularity. Fluid with w less than -1 can end up with future singularity called the big rip [28], which is of type I singularity according to classification by [29,30]. The big rip singularity corresponds to $a \to \infty$, $\rho \to \infty$, and $|p| \to \infty$ at finite time $t \to t_s$ in the future. Choosing a particular class of potential for the phantom field enables us to avoid future singularity. However, the avoidance does not cover general classes of potential [26]. In addition, an alternative model, in which two scalar fields appear with inverse power law and exponential potentials, can as well avoid the big rip singularity [31]. The higher-order string curvature correction terms can also show the possibility that the big rip singularity can be absent [32].

Since the phantom dominated FRW universe possesses a singularity problem as stated above, in this work, instead of using standard FRW cosmology, the fundamental background theory in which we are interested is loop quantum gravity (LQG). This theory is a nonperturbative type of quantization of gravity and is background independent

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[33,34]. It has been applied in a cosmological context as seen in various literature where it is known as loop quantum cosmology (LQC) (for a review, see Ref. [35]). The effective loop quantum modifies the standard Friedmann equation by adding a correction term $-\rho^2/\rho_{\rm lc}$ into the Friedmann equation [36–40]. When this term becomes dominant, the universe begins to bounce and then expands backwards. LQG can resolve the singularity problem in various situations [34,37,41,42]. However, derivation of the modified term is under a condition that there is no matter potential otherwise; in the presence of a potential, quantum correction would be more complicated [43]. A nice feature of LQC is avoidance of the future singularity from the correction quadratic term $-\rho^2/\rho_{\rm lc}$ in the modified LQC Friedmann equation [44] as well as the singularity avoidance at the semiclassical regime [45]. The early-universe inflation has also been studied in the context of LQC at the semiclassical limit [40,46–50]. We aim to investigate the dynamics of the phantom field and its late time behavior in the loop quantum cosmological context, and to check if the loop quantum effect could remove big rip singularity from the phantom dominated universe. The study could also reveal some other interesting features of the model.

We organize this article as follows: in Sec. II, we introduce the LQC Friedmann equation; after that we briefly present relevant features of the phantom scalar field in Sec. III. Section IV contains dynamical analysis of the phantom field in LQC background with exponential potential. The potential is a simplest case due to constancy of its steepness variable λ. Two real fixed points are found in this section. Stability analysis yields that both fixed points are saddle points. Numerical results and analysis of solutions can be seen in Sec. V where we give conditions for physical solutions. Finally, the conclusion is in Sec. VI.

II. LOOP QUANTUM COSMOLOGY

LQC naturally gives rise to the inflationary phase of the early universe with graceful exit; however, the same mechanism leads to a prediction that present-day acceleration must be very small [46]. At late time and at large scale, the semiclassical approximation in LQC formalisms can be validly used [51]. The effective Friedmann equation can be obtained by using an effective Hamiltonian with loop quantum modifications [38,44,52]:

$$\mathcal{C}_{\rm eff} = -\frac{3M_{\rm P}^2}{\gamma^2\bar{\mu}^2} a \sin^2(\bar{\mu}\mathfrak{c}) + \mathcal{C}_{\rm m}. \tag{1}$$

The effective constraint (1) is valid for the isotropic model, and if there is scalar field, the field must be a free, massless scalar field. Equation (1), when including field potential, must have some additional correction terms [43]. In this scenario, Hamilton's equation is

$$\dot{\mathfrak{p}} = \{\mathfrak{p}, \mathcal{C}_{\rm eff}\} = -\frac{\gamma}{3M_{\rm P}^2}\frac{\partial \mathcal{C}_{\rm eff}}{\partial \mathfrak{c}}, \tag{2}$$

where $\mathfrak{c}$ and $\mathfrak{p}$ are, respectively, conjugate connection and triad satisfying $\{\mathfrak{c}, \mathfrak{p}\} = \gamma/3M_{\rm P}^2$. The dot symbol denotes time derivative. These are two variables in the simplified phase space structure under FRW symmetries [35]. Here $M_{\rm P}^2 = (8\pi G)^{-1}$ is the square of reduced Planck mass, G is Newton's gravitational constant, and γ is the Barbero-Immirzi dimensionless parameter. There are relations between the two variables to scale factor as $\mathfrak{p} = a^2$ and $\mathfrak{c} = \gamma\dot{a}$. The parameter $\bar{\mu}$ is inferred as kinematical length of the square loop since its order of magnitude is similar to that of length. The area of the loop is given by the minimum eigenvalue of the LQG area operator. $\mathcal{C}_{\rm m}$ is the corresponding matter Hamiltonian. Using Eq. (2) with the constraint from realization that the loop quantum correction of the effective Hamiltonian $\mathcal{C}_{\rm eff}$ is small at a large scale, $\mathcal{C}_{\rm eff} \approx 0$ [35,38,39,44], one can obtain the (effective) modified Friedmann equation in a flat universe:

$$H^2 = \frac{\rho_{\rm t}}{3M_{\rm P}^2}\left(1 - \frac{\rho_{\rm t}}{\rho_{\rm lc}}\right), \tag{3}$$

where $\rho_{\rm lc} = \sqrt{3}/(16\pi\gamma^3 G^2 \hbar)$ is the critical loop quantum density, $\hbar$ is the Planck constant, and $\rho_{\rm t}$ is the total density.

III. PHANTOM SCALAR FIELD

The energy density ρ and the pressure p of the phantom field contain a negative kinetic term. They are given as [16]

$$\rho = -\tfrac{1}{2}\dot{\phi}^2 + V(\phi), \tag{4}$$

$$p = -\tfrac{1}{2}\dot{\phi}^2 - V(\phi). \tag{5}$$

The conservation law is

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{6}$$

Using Eqs. (4)–(6), we obtain the Klein-Gordon equation:

$$\ddot{\phi} + 3H\dot{\phi} - V' = 0, \tag{7}$$

where $V' \equiv {\rm d}V/{\rm d}\phi$ and the negative sign comes from the negative kinetic terms. The phantom equation of state is therefore given by

$$w = \frac{p}{\rho} = \frac{\dot{\phi}^2 + 2V}{\dot{\phi}^2 - 2V}. \tag{8}$$

From Eq. (8), when the field is slowly rolling, as long as the approximation, $\dot{\phi}^2 \sim 0$ holds, the approximated value of w is -1. When the bound, $\dot{\phi}^2 < 2V$ holds, w is always less than -1.

As mentioned previously in Secs. I and II, there has not yet been a derivation of the effective LQC Friedmann equation consistent with a presence of potential. Even though, the Friedmann background is valid only in the absence of the field potential, however, investigation of a phantom field evolving under a potential is a challenged task. Here we also neglect the loop quantum correction effect in the classical expression of Eqs. (4) and (5) (see Refs. [43,53] for discussion).

IV. DYNAMICAL ANALYSIS

Differentiating Eq. (3) and using the fluid Eq. (6), we obtain

$$\dot{H} = -\frac{(\rho + p)}{2M_P^2}\left(1 - \frac{2\rho}{\rho_{lc}}\right). \tag{9}$$

Equations (3), (6), and (9), in domination of the phantom field, become

$$H^2 = \frac{1}{3M_P^2}\left(-\frac{\dot{\phi}^2}{2} + V\right)\left(1 - \frac{\rho}{\rho_{lc}}\right), \tag{10}$$

$$\dot{\rho} = -3H\rho\left(1 + \frac{\dot{\phi}^2 + 2V}{\dot{\phi}^2 - 2V}\right), \tag{11}$$

$$\dot{H} = \frac{\dot{\phi}^2}{2M_P^2}\left(1 - \frac{2\rho}{\rho_{lc}}\right). \tag{12}$$

We define dimensionless variables following the style of [54]

$$X \equiv \frac{\dot{\phi}}{\sqrt{6}M_P H}, \qquad Y \equiv \frac{\sqrt{V}}{\sqrt{3}M_P H}, \qquad Z \equiv \frac{\rho}{\rho_{lc}}, \tag{13}$$

$$\lambda \equiv -\frac{M_P V'}{V}, \qquad \Gamma \equiv \frac{VV''}{(V')^2}, \qquad \frac{d}{dN} \equiv \frac{1}{H}\frac{d}{dt}, \tag{14}$$

where $N \equiv \ln a$ is the e-folding number. Using new variables in Eqs. (8) and (10), the equation of state is rewritten as[1]

$$w = \frac{X^2 + Y^2}{X^2 - Y^2}, \tag{15}$$

where $|X| \neq |Y|$ and the Friedmann constraint is reexpressed as

$$(-X^2 + Y^2)(1 - Z) = 1. \tag{16}$$

Clearly, if $|X| \neq |Y|$, following Eq. (16), then $Z \neq 1$. Using the new defined variables above, Eq. (12) becomes

$$\frac{\dot{H}}{H^2} = 3X^2(1 - 2Z). \tag{17}$$

The acceleration condition,

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0, \tag{18}$$

in expression of the new variables, is therefore

$$3X^2(2Z - 1) < 1. \tag{19}$$

Divided by Eq. (16), the acceleration condition under the constraint is

$$\frac{3}{1 - (Y^2/X^2)}\left(\frac{1 - 2Z}{1 - Z}\right) < 1, \tag{20}$$

where the conditions $|X| \neq |Y|$ and $Z \neq 1$ must hold. As we consider $Z = \rho/\rho_{lc}$ with $\rho = -(\dot{\phi}^2/2) + V$, we can write

$$\frac{\rho_{lc} Z}{3M_P^2 H^2} = -X^2 + Y^2. \tag{21}$$

With the condition $|X| \neq |Y|$, clearly from Eq. (21), we have one additional condition, $Z \neq 0$.

A. Autonomous system

Differential equations in the autonomous system are

$$\frac{dX}{dN} = -3X - \sqrt{\frac{3}{2}}\lambda Y^2 - 3X^3(1 - 2Z), \tag{22}$$

$$\frac{dY}{dN} = -\sqrt{\frac{3}{2}}\lambda XY - 3X^2 Y(1 - 2Z), \tag{23}$$

$$\frac{dZ}{dN} = -3Z\left(1 + \frac{X^2 + Y^2}{X^2 - Y^2}\right), \tag{24}$$

$$\frac{d\lambda}{dN} = -\sqrt{6}(\Gamma - 1)\lambda^2 X. \tag{25}$$

Here we will apply the exponential potential,

$$V(\phi) = V_0 \exp\left(-\frac{\lambda}{M_P}\phi\right), \tag{26}$$

to this system. The potential is known to yield power-law inflation in standard cosmology with a canonical scalar field. Its slow-roll parameters are related as $\epsilon = \eta/2 = 1/P$, where $\lambda = \sqrt{2/P}$ and $P > 1$ [55,56]. Although the potential has been applied to the quintessence scalar field with tracking behavior in standard cosmology [57], the quintessence field cannot dominate the universe due to constancy of the ratio between densities of matter and quintessence field (see the discussion in Ref. [7]). In the case of a phantom field in standard cosmology under this potential, a stable node is a scalar-field dominated solution with the equation of state, $w = -1 - \lambda^2/3$ [24,27,58]. In our LQC phantom domination context, from Eq. (25), we can see that for the exponential potential, $\Gamma = 1$. This yields a trivial value of $d\lambda/dN$ and therefore λ is a nonzero constant; otherwise the potential is flat.

[1]The relation $\Omega_\phi = \rho/3H^2M_P^2 = -X^2 + Y^2 = 1$ cannot be applied here since it is valid only for standard cosmology with flat geometry.

TABLE I. Properties of fixed points of phantom field dynamics in LQC background under the exponential potential.

Name	X	Y	Z	Existence	Stability	w	Acceleration
(a)	$-\frac{\lambda}{\sqrt{6}}$	$\sqrt{1+\frac{\lambda^2}{6}}$	0	All λ	Saddle point for all λ	$-1-\frac{\lambda^2}{3}$	For all λ (i.e. $\lambda^2 > -2$)
(b)	$-\frac{\lambda}{\sqrt{6}}$	$-\sqrt{1+\frac{\lambda^2}{6}}$	0	All λ	Saddle point for all λ	$-1-\frac{\lambda^2}{3}$	For all λ (i.e. $\lambda^2 > -2$)

B. Fixed points

Let $f \equiv dX/dN$, $g \equiv dY/dN$, and $h \equiv dZ/dN$. We can find fixed points of the autonomous system under condition

$$(f, g, h)\,|_{(X_c, Y_c, Z_c)} = 0. \tag{27}$$

There are two real fixed points of this system:[2]

- Point (a): $$\left(\frac{-\lambda}{\sqrt{6}}, \sqrt{1+\frac{\lambda^2}{6}}, 0\right), \tag{28}$$

- Point (b): $$\left(\frac{-\lambda}{\sqrt{6}}, -\sqrt{1+\frac{\lambda^2}{6}}, 0\right). \tag{29}$$

C. Stability analysis

Suppose that there is a small perturbation δX, δY, and δZ about the fixed point (X_c, Y_c, Z_c), i.e.,

$$X = X_c + \delta X, \qquad Y = Y_c + \delta Y, \qquad Z = Z_c + \delta Z. \tag{30}$$

From Eqs. (22)–(24), neglecting higher-order terms in the perturbations, we obtain first-order differential equations:

$$\frac{d}{dN}\begin{pmatrix}\delta X\\ \delta Y\\ \delta Z\end{pmatrix} = \mathcal{M}\begin{pmatrix}\delta X\\ \delta Y\\ \delta Z\end{pmatrix}. \tag{31}$$

The matrix $\mathcal{M}$ defined at a fixed point (X_c, Y_c, Z_c) is given by

$$\mathcal{M} = \begin{pmatrix}\frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} & \frac{\partial f}{\partial Z}\\ \frac{\partial g}{\partial X} & \frac{\partial g}{\partial Y} & \frac{\partial g}{\partial Z}\\ \frac{\partial h}{\partial X} & \frac{\partial h}{\partial Y} & \frac{\partial h}{\partial Z}\end{pmatrix}_{(X=X_c, Y=Y_c, Z=Z_c)}. \tag{32}$$

We find eigenvalues of the matrix $\mathcal{M}$ for each fixed point:

(1) At point (a):

$$\mu_1 = \lambda^2, \qquad \mu_2 = -\lambda^2, \qquad \mu_3 = -3 - \frac{\lambda^2}{2}. \tag{33}$$

(2) At point (b):

$$\mu_1 = \lambda^2, \qquad \mu_2 = -\lambda^2, \qquad \mu_3 = -3 - \frac{\lambda^2}{2}. \tag{34}$$

From the above analysis, each point possesses eigenvalues with opposite signs; therefore both point (a) and point (b) are saddle. Results from our analysis are concluded in Table I. Location of the points depends only on λ and the points exist for all values of λ. Both points correspond to the equation of state $-1 - \lambda^2/3$, that is to say, it has a phantom equation of state for all values of $\lambda \neq 0$. Since there is not any attractor in the system, a phase trajectory is very sensitive to initial conditions given to the system. The stable node (the big rip) of the standard general relativistic case in the presence of a phantom field and a barotropic fluid disappears here (see [23]).

V. NUMERICAL RESULTS

Numerical results from the autonomous set (22)–(24) are presented in Figs. 1 and 2 where we set $\lambda = 1$. Locations of the two saddle points are: point (a) ($X_c = -0.408\,25$, $Y_c = 1.0801$, $Z_c = 0$) and point (b) ($X_c = -0.408\,25$, $Y_c = -1.0801$, $Z_c = 0$), which match our analytical results. In Fig. 3, we present a trajectory solution of a phantom field evolving in standard cosmological background for comparing with the trajectories in Fig. 2 when including loop quantum effects. The standard case has only two simple trajectories corresponding to a constraint $-X^2 + Y^2 = 1$. This is attained when taking the classical limit, $Z = 0$. In the loop quantum case, since there is not any stable node and the solutions are sensitive to initial conditions, we need to classify solutions according to each domain region separated by separatrices $|X| = |Y|$, $Z = 0$, and $Z = 1$, so that we can analyze them separately. Note that the condition, $Z > 0$ must hold for physical solutions since the density cannot be negative or zero, i.e. $\rho > 0$. The blue lines and red lines in Figs. 1 and 2 are solutions in the region $Z < 0$ hence are not physical and will no longer be of our interest. From now on, we consider only the region $Z > 0$. In regions with $|X| > |Y|$, the solutions therein are green lines (hereafter classified as class I). The other regions with $|Y| > |X|$ contain solutions seen as black lines (classified as class II). Note that all solutions cannot cross the separatrices due to conditions in Eqs. (16), (20), and (21).

[2]The other two imaginary fixed points $(i, 0, 0)$ and $(-i, 0, 0)$ also exist. However, they are not interesting here since we do not consider the model that includes a complex scalar field.

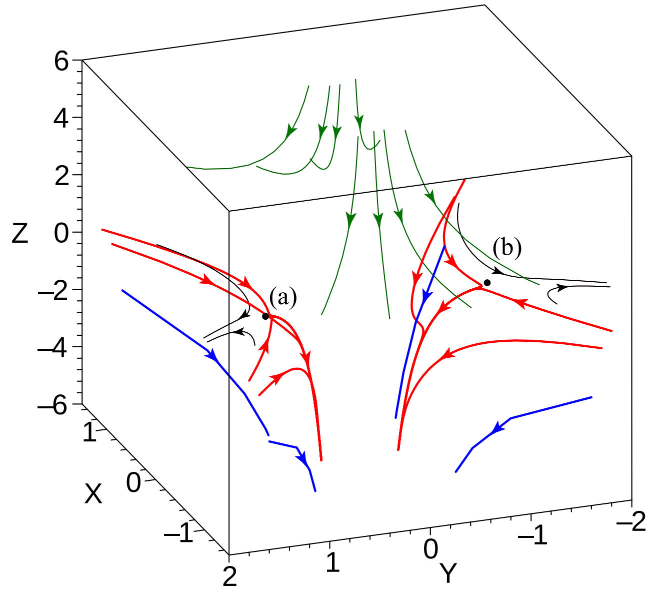

FIG. 1 (color online). Three-dimensional phase space of X, Y, and Z. The saddle points (a) $(-0.40825, 1.0801, 0)$ and (b) $(-0.40825, -1.0801, 0)$ appear in the figure. λ is set to 1. In region $Z < 0$, the solutions (red and blue lines) are nonphysical. In this region, $Z \to -\infty$ when $(X, Y) \to (0, 0)$. The green lines (class I) are in region $|X| > |Y|$ and $Z > 1$ but they are also nonphysical since they correspond to imaginary H values. The only set of physical solutions (class II) is presented with black lines. They are in region $|Y| > |X|$ and range from $0 < Z < 1$. This is the region above (a) and (b) of which H takes real value. There are separatrices $|X| = |Y|$, $Z = 0$, and $Z = 1$ in the system (see Sec. V B).

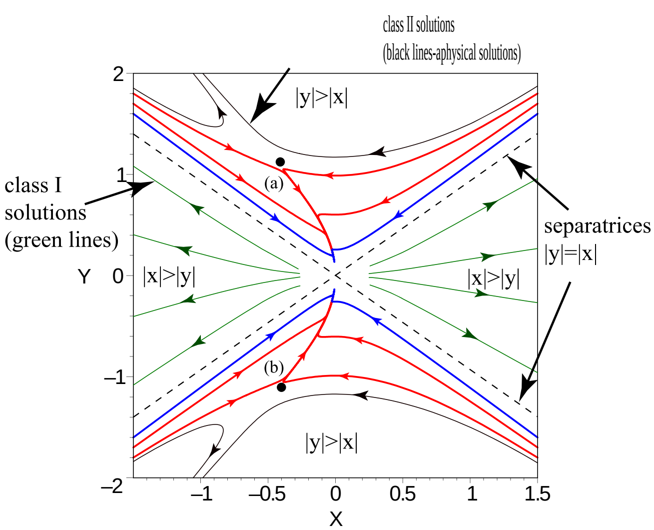

FIG. 2 (color online). Phase space of the kinetic part X and potential part Y (top view). The saddle points (a) $(-0.40825, 1.0801)$ and (b) $(-0.40825, -1.0801)$ are shown here. The blue lines and red lines are in the region $Z < 0$ which is nonphysical. Green lines are of class I solutions which yield imaginary H. Only class II solutions shown as black lines are physical with a real H value.

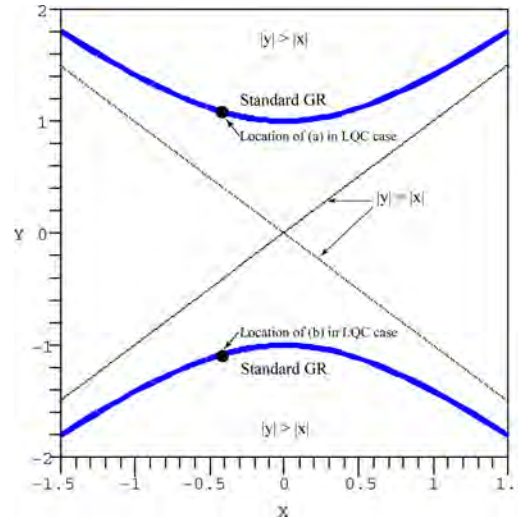

FIG. 3 (color online). Phase space of the kinetic part X and potential part Y in standard general relativistic case. The location of points (a) and (b) in Fig. 2 is on the trajectory solutions here. This plot shows the dynamics of a phantom field in standard cosmological background without any other fluids. In the presence of a barotropic fluid with any equation of state, points (a) and (b) correspond to the big rip [23,25].

A. Class I solutions

Consider the Friedmann equation (10); the Hubble parameter H takes real value only if

$$\frac{1}{3M_{\mathrm{P}}^2}\left(-\frac{\dot{\phi}^2}{2}+V\right)\left(1-\frac{\rho}{\rho_{\mathrm{lc}}}\right) \geq 0. \tag{35}$$

Divided by H^2 on both sides, the expression above becomes

$$(-X^2 + Y^2)(1 - Z) \geq 0. \tag{36}$$

It is clear from (36) that, in order to obtain a real value of H, class I solutions (green line) must obey both conditions $|X| > |Y|$ and $Z > 1$ together. However, when imposing $|X| > |Y|$ to Eq. (21), we obtain $Z < 0$ instead. This contradicts the required range $Z > 1$. Therefore this class of solutions does not possess any real values of H and hence not physical solutions.

B. Class II solutions

Proceeding with the same analysis done for class I, we found that in order for H to be real, class II solutions must obey both $|Y| > |X|$ and $0 < Z < 1$ together. Moreover, when imposing $|Y| > |X|$ into Eq. (21), we obtain $Z > 0$. Therefore, as we combine both results, it can be concluded that class II solutions can possess real H value in the region $|Y| > |X|$ and $0 < Z < 1$, i.e. $0 < \rho < \rho_{\mathrm{lc}}$. The bound is slightly different from the case of canonical scalar field in LQC (see Ref. [59]) of which the bound is $0 \leq \rho \leq \rho_{\mathrm{lc}}$. Class II is therefore the only class of physical solutions.

For class II solutions, we consider another set of autonomous equations from which the evolution of cosmological variables is conveniently obtained by using the numerical approach. In the new autonomous set, instead of using N, which could decrease after the bounce from LQC effect, time is taken as an independent variable. We define the new variable as

$$\dot{\phi} = S. \tag{37}$$

Equations (7) and (12) are therefore rewritten as

$$\dot{H} = \frac{S^2}{2M_{\mathrm{P}}^2}\left[1 - \frac{2}{\rho_{\mathrm{lc}}}\left(-\frac{S^2}{2} + V(\phi)\right)\right], \tag{38}$$

$$\dot{S} = -3HS + V'. \tag{39}$$

Equations (37)–(39) form another closed autonomous system. Numerical integrations from the new system yield the result plotted in Figs. 4 and 5 in which the set values are $\lambda = 1$, $\rho_{\mathrm{lc}} = 1.5$, $V_0 = 1$, and $M_{\mathrm{P}} = 2$. From Eq. (3) the slope of H with respect to ρ, $\mathrm{d}H/\mathrm{d}\rho$, is flat when $\rho = \rho_{\mathrm{lc}}/2$ [59]. Another fact is

$$\left(\frac{\mathrm{d}^2H}{\mathrm{d}\rho^2}\right)_{\rho=\rho_{\mathrm{lc}}/2} = \frac{-2}{M_{\mathrm{P}}\sqrt{3\rho_{\mathrm{lc}}^3}} < 0, \tag{40}$$

hence, as $\rho = \rho_{\mathrm{lc}}/2$, H takes the maximum value, $H_{\max} = \sqrt{\rho_{\mathrm{lc}}/12M_{\mathrm{P}}^2}$. This result is valid in the LQC scenario regardless of types of fluid. In Figs. 4 and 5, with set parameters given above, as $\rho = \rho_{\mathrm{lc}}/2 = 0.75$, H is maximum, $H_{\max} = 0.176\,78$. When $H \approx 0$, i.e. ρ is approximately $\rho_{\mathrm{lc}} = 1.5$, the expansion halts and then bounces. At this bouncing point, the dynamics enters the loop quantum regime which is a quantum gravity limit. Beyond the bounce, H turns negative, i.e. contracting of scale factor. The universe undergoes accelerating contraction until reaching $H_{\min}$. After that it contracts, decelerating until

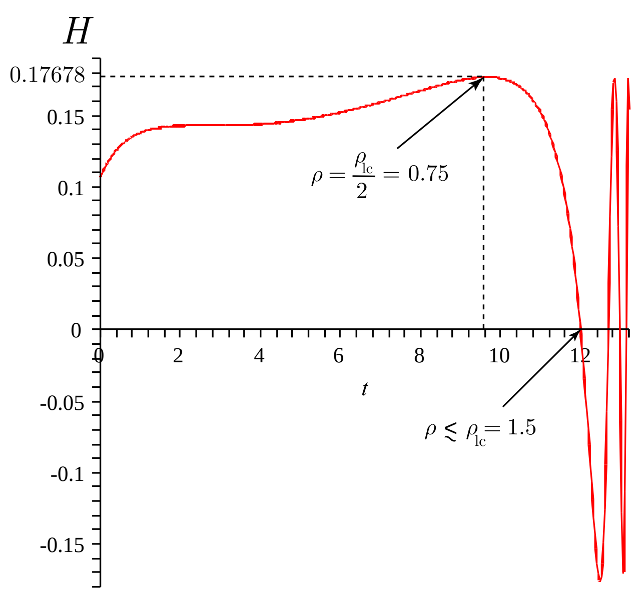

FIG. 4 (color online). Evolution of H with time of a class II solution. Set values are $\lambda = 1$, $\rho_{\mathrm{lc}} = 1.5$, $V_0 = 1$, and $M_{\mathrm{P}} = 2$. The universe undergoes acceleration from the beginning until reaching a turning point at $\rho = \rho_{\mathrm{lc}}/2 = 0.75$, where $H = H_{\max} = 0.176\,78$. Beyond this point, the universe expands with deceleration until halting ($H = 0$) at $\rho \approx \rho_{\mathrm{lc}} = 1.5$. After halting, it undergoes contraction until H bounces. The oscillating in H goes on forever.

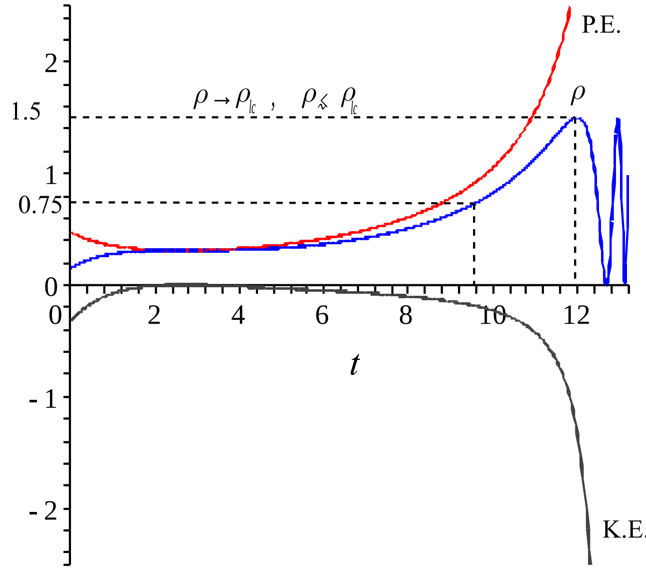

FIG. 5 (color online). Time evolution of potential energy density (P.E.), kinetic energy density (K.E.), and $\rho = $ K.E. + P.E. of the field for a class II solution. K.E. is always negative and, at late time, it goes to $-\infty$ while P.E. is always positive. ρ is maximum when $\rho \approx \rho_{\mathrm{lc}} = 1.5$. Other features are discussed as in Fig. 4.

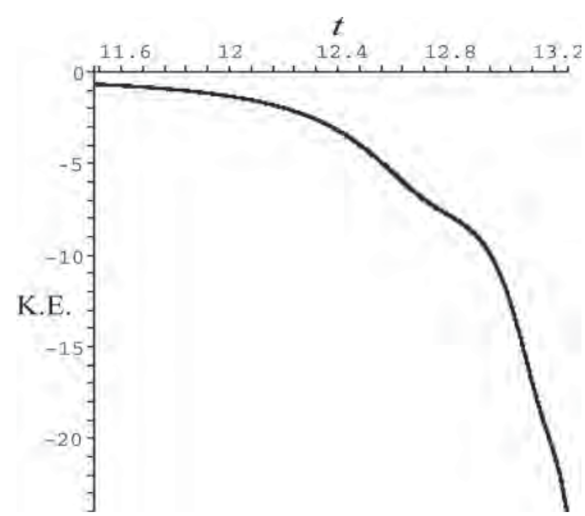

FIG. 6. Oscillation in kinetic energy density (K.E.) that contributes to oscillation in ρ. This is a zoomed-in portion of Fig. 5.

bouncing at $H \approx 0$. The universe goes on faster bouncing forward and backward. The faster bounce in H is an effect from the faster bounce in ρ as illustrated in Fig. 5 where the red line represents potential energy density $V(\phi)$, the black line represents kinetic energy density $-\dot{\phi}^2/2$, and the blue line is total energy density ρ. Oscillation in ρ is from oscillation in the field speed $\dot{\phi}$ and therefore oscillation in K.E. as shown in Fig. 6. This hence contributes to oscillation in ρ. The negative magnitude of kinetic energy density becomes larger and larger as the field rolling faster and faster up the potential. The exponential potential energy density therefore becomes larger and larger. This results in oscillation of ρ and affects in oscillation of H about the bounce $H = 0$. With a different approach, recently a similar result in H oscillation is also obtained by Naskar and Ward [60].

VI. CONCLUSION

A dynamical system of phantom canonical scalar field evolving in a background of loop quantum cosmology is considered and analyzed in this work. The exponential potential is used in this system. The dynamical analysis of the autonomous system renders two real fixed points $(-\lambda/\sqrt{6}, \sqrt{1+\lambda^2/6}, 0)$ and $(-\lambda/\sqrt{6}, -\sqrt{1+\lambda^2/6}, 0)$, both of which are saddle points corresponding to an equation of state, $w = -1 - \lambda^2/3$. Note that, in the case of standard cosmology, the fixed point $(X_c, Y_c) = (-\lambda/\sqrt{6}, \sqrt{1+\lambda^2/6})$ is the big rip attractor with the same equation of state, $w = -1 - \lambda^2/3$ [24]. Because of the absence of a stable node, the late time behavior depends on the initial conditions given. Therefore we do numerical plots to investigate solutions of the system and then classify the solutions. Separatrix conditions $|X| \neq |Y|$, $Z \neq 1$, and $Z \neq 0$ arise from the equation of state (15), Friedmann constraint (16), and definition of Z in Eq. (21). At first, we consider solutions in region $Z > 0$, i.e. $\rho > 0$ for physical solutions. Second, within this $Z > 0$ region, we classify solutions into class I and class II. Solutions in region $|X| > |Y|$ and $Z > 1$ are of class I. However, in order to obtain a real value of H in class I, Z must be negative which contradicts $Z > 1$. Therefore the class I solutions are nonphysical. The class II set is identified by $|Y| > |X|$ and $0 < Z < 1$. It is the only set of physical solutions since it yields a real value of H. In the class II set, the universe undergoes an accelerating expansion from the beginning until $\rho = \rho_{\rm lc}/2$, where $H = H_{\rm max} = \sqrt{\rho_{\rm lc}/12M_{\rm P}^2}$. After that the universe expands, decelerating until it bounces, i.e. stops expansion $H \approx 0$ at $\rho \approx \rho_{\rm lc}$. At the bounce the universe enters the quantum gravity regime. Contraction with backward acceleration happens right after the bounce; however, the contraction does not go on forever. When the universe reaches a minimum value of negative H, the contraction decelerates, i.e. contracts slower and slower down. The universe, after undergoing contraction to a minimum spatial size, bounces again and starts to expand, accelerating. Our numerical results yield that oscillation in H becomes faster as time passes.

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Coupled phantom field in loop quantum cosmology

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A model of phantom scalar field dark energy under exponential potential coupling to barotropic dark matter fluid in loop quantum cosmology is addressed here. We derive a closed-autonomous system for cosmological dynamics in this scenario. It is found that LQC can yield a bounce in scale factor even in presence of the phantom field. The greater decaying from dark matter to dark phantom energy results in greater energy storing in the phantom field. This results in further turning point of the field. Greater coupling also delays bouncing time. In the case of phantom decaying, oscillation in phantom density makes small oscillation in the increasing matter density.

1. INTRODUCTION

There has recently been evidence of present accelerating expansion of the universe from cosmic microwave background (CMB) anisotropies, large scale galaxy surveys and type Ia supernovae [1-3]. Dark energy (DE) in form of either cosmological constant or scalar field matter is a candidate answer to the acceleration expansion which could not be explained in the regime of standard big bang cosmology [4]. DE possesses equation of state $p = w\rho$ with $w < -1/3$ enabling it to give repulsive gravity and therefore accelerate the universe. Combination of observational data analysis of CMB, Hubble Space Telescope, type Ia Supernovae and 2dF datasets allows constant w value between -1.38 and -0.82 at the 95 % of confident level [5]. Meanwhile, assuming flat universe, the analysis result, $-1.06 < w < -0.90$ has been reported by [6] using WMAP three-year results combined with Supernova Legacy Survey (SNLS) data. Without assumption of flat universe, mean value of w is -1.06 (within a range of -1.14 to -0.93). Most recent data (flat geometry assumption) from ESSENCE Supernova Survey Ia combined with SuperNova Legacy Survey Ia gives a constraint of $w = -1.07 \pm 0.09$ [7]. Observations above show a possibility that a fluid with $w < -1$ could be allowed in the universe [8]. This type of cosmological fluid is dubbed “phantom”. Conventionally Phantom behavior arises from having negative kinetic energy term.

Dynamical properties of the phantom field in the standard FRW cosmology were studied before. However the scenario encounters singularity problems at late time [9]. While investigation of phantom in standard cosmological model is still ongoing, there is an alternative approach in order to resolve the singularity problem by considering Loop Quantum Cosmology (LQC) background instead of standard general relativistic background [9-10]. Loop Quantum Gravity-LQG is a non-perturbative type of quantization of gravity and is background-independent [11-12]. LQG provides cosmological background evolution for LQC. Effect from loop quantum modification gives an extra correction term $-\rho^2/\rho_c$ into the standard Friedmann equation [13-14]. This term, when dominant at late time, causes bouncing of expansion hence solving future singularity problem [15-16]. Recently, a general dynamics of scalar field including phantom scalar field coupled to barotropic fluid has been investigated in standard cosmological background. In this scenario, the scaling solution of the coupled phantom field is always unstable and it can not yield the observed value $\Omega_\varphi \approx 0.7$ [17].

Here, in this letter, we will investigate a case of coupled phantom field in LQC background alternative to the standard cosmology case. In Section 2, we introduce framework of cosmological equations before consider dynamical autonomous equations in Section 3. We show some numerical results in Section 4 where the coupling strength is adjusted and compared. Conclusion and comments are in Section 5.

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2 COSMOLOGICAL EQUATIONS

2.1 Loop Quantum Cosmology

The effective Friedmann equation from LQC is given as [18]

$$H^2 = \frac{\rho}{3M_{\rm P}^2}\left(1 - \frac{\rho}{\rho_{\rm c}}\right), \qquad (1)$$

where H is Hubble constant, $M_{\rm P}$ is reduced Planck mass, ρ is density of cosmic fluid, $\rho_{\rm c} = \sqrt{3}/(16\pi\varsigma^3 G^2\hbar)$. The parameter ς is Barbero-Immirzi dimensionless parameter and G is Newton’s gravitational constant.

2.2 Phantom Scalar Field

Nature of the phantom field can be extracted from action,

$$S = \int {\rm d}^4x\sqrt{-g}\left[\frac{1}{2}(\nabla\phi)^2 - V(\phi)\right], \qquad (2)$$

which, with variational principle, yields

$$\rho_\phi = -\frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad (3)$$

and

$$p_\phi = -\frac{1}{2}\dot{\phi}^2 - V(\phi). \qquad (4)$$

The phantom field possesses equation of state,

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 + 2V(\phi)}{\dot{\phi}^2 - 2V(\phi)}. \qquad (5)$$

When the field is slowly rolling, the approximate value of w is -1. As long as the approximation, $\dot{\phi}^2 \approx 0$ or the bound, $\dot{\phi}^2 < 2V$ holds, w is always less than -1. In our scenario, the universe contains two components which are barotropic fluid with equation of state $p_{\rm m} = \rho_{\rm m} w_{\rm m}$ and phantom scalar field fluid. The total energy density is $\rho = \rho_{\rm m} + \rho_\phi$.

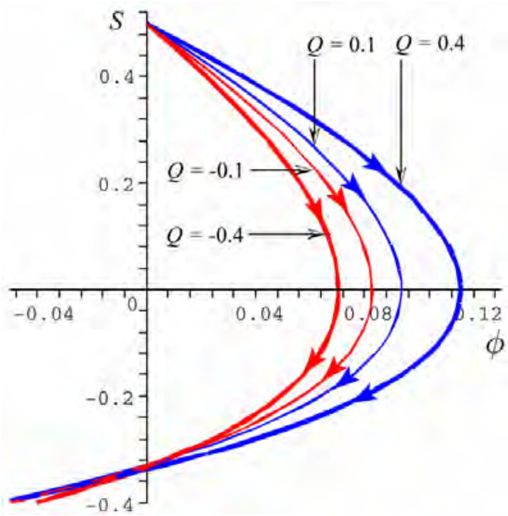

FIGURE 1. Phase portrait of $S(t)$ versus ϕ for Q = -0.4, -0.1, 0.1 and 0.4 from left to right. All trajectories have the same initial conditions $S(0)=0.5, \phi(0)=0$.

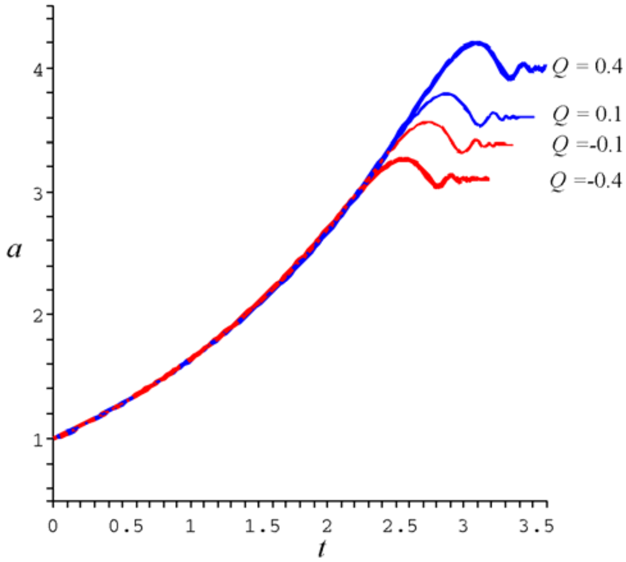

FIGURE 2. Scale factor plotted versus time for Q = -0.4, -0.1, 0.1 and 0.4 (from bottom to top).

2.3 Coupled Scalar Field

Here we consider both components coupling to each other. Fluid equations for couple scalar fields proposed by [19] assuming flat FRW universe are

$$\dot{\rho}_\phi + 3H(1+w_\phi)\rho_\phi = -Q\rho_{\rm m}\dot{\phi}, \tag{6}$$

and

$$\dot{\rho}_{\rm m} + 3H(1+w_{\rm m})\rho_{\rm m} = +Q\rho_{\rm m}\dot{\phi}. \tag{7}$$

These fluid equations contain a constant coupling Q between dark matter (the barotropic fluid) and dark energy (the phantom scalar field) as in [20]. Though Eqs. (6) and (7) are derived in FRW background, the LQC effective Friedmann equation, Eq. (1) is also obtained under flat and maximally symmetries. Discrete quantum effect of LQG shows up at high energy regimes. Therefore, Eqs. (6) and (7) can be used in the consideration. Total action for matter and phantom scalar field is [19]

$$S = \int {\rm d}^4x\sqrt{-g}\left[\frac{M_{\rm P}^2}{2}R + p(X,\phi)\right] + S_{\rm m}(\phi) \tag{8}$$

Assuming scaling solution of the dark energy, therefore the pressure is written as in [17, 19]

$$p(X,\phi) = -X - c\exp(-\lambda\phi/M_{\rm P}^2), \tag{9}$$

where X is the kinetic term, $-g^{ab}\partial_a\phi\partial_b\phi/2$ of the Lagrangian density (9) and (2). The second term on the right of Eq. (9) is exponential potential, $V(\phi)=c\exp(-\lambda\phi/M_{\rm P}^2)$ which gives scaling solution for canonical and phantom ordinary scalar field in standard general relativistic cosmology when steepness of the potential, λ is fine tuned as

$$\lambda = Q\frac{1+w_{\rm m}-\Omega_\phi(w_{\rm m}-w_\phi)}{\Omega_\phi(w_{\rm m}-w_\phi)}. \tag{10}$$

The steepness (10) is, in standard cosmological circumstance, constant in the scaling regime due to constancy of w_ϕ and Ω_ϕ [19]. However, in LQC case, there has been a report recently that the scaling solution does not exist for phantom field evolving in LQC [10]. Therefore, in our situation, our spirit to consider constant λ is a motivation from tracking behavior as in [21, 22], not a motivation from scaling solution as in [19]. The exponential potential is also originated from fundamental physics theories such as higher-order gravity [23] or higher dimensional gravity [24].

3 COSMOLOGICAL DYNAMICS

Time derivative of the effective LQC Friedmann equation LQC (1) is

$$\dot{H} = -\frac{(\rho+p)}{2M_{\rm P}^2}\left(1-\frac{2\rho}{\rho_{\rm c}}\right), \tag{11.1}$$

$$= -\frac{\left[(1+w_\phi)\rho_\phi + (1+w_{\rm m})\rho_{\rm m}\right]}{2M_{\rm P}^2}\left[1-\frac{2}{\rho_{\rm c}}\left(\rho_\phi+\rho_{\rm m}\right)\right], \tag{11.2}$$

$$= -\frac{\left[-S^2 + (1+w_{\rm m})\rho_{\rm m}\right]}{2M_{\rm P}^2}\left[1-\frac{2}{\rho_{\rm c}}\left(-\frac{S^2}{2} + c{\rm e}^{-\lambda\phi/M_{\rm P}^2} + \rho_{\rm m}\right)\right]. \tag{11.3}$$

In above equations we define new variable

$$\dot{\phi} \equiv S. \tag{12}$$

The coupled fluid equations (6) and (7) are re-expressed in term of S as

$$\dot{S} = -3HS + \frac{{\rm d}V}{{\rm d}\phi} + Q\rho_{\rm m}, \tag{13}$$

$$\dot{S} = -3HS + \frac{{\rm d}V}{{\rm d}\phi} + Q\rho_{\rm m} \tag{14}$$

The Eqs. (11.3), (12), (13) and (14) form a closed autonomous set of four equations. The variables here are $\rho_{\rm m}, S$, ϕ and H. The autonomous set recovers standard GR cosmology in the limit $\rho_{\rm c}\to\infty$. This GR limit affects only the equation involving H. From the above autonomous set, one can do a qualitative analysis with numerical integration similar to [25]. Another approach of analysis is to consider a quantitative analysis [26].

4. NUMERICAL SOLUTIONS

Here we present some numerical solution for a positive and negative coupling between the phantom and barotropic fluid. The solutions presented here are physically valid solutions corresponding to Class II solution as characterized in

[10]. For nonminimally coupled scalar field, in Einstein frame, the coupling Q lies in a range $-1/\sqrt{6} < Q < 1/\sqrt{6}$ (see [4]). Here we set Q = -0.4, -0.1, 0.1 and 0.4 which lie in the range. Effect of the coupling can be seen from Eqs. (6) and (7). Negative Q enhances decay rate of scalar field to matter while giving higher matter creation rate. On the other hand, positive Q yields opposite result. Greater magnitude of $Q < 0$ gives higher decay rate of the field to matter. Greater magnitude of $Q > 0$ will result in higher production rate of field from matter.

4.1 Phase Portrait

The greater Q value results in greater value of the field turning point (see S-intercept in both figures.). The kinetic term $S(t)$ turns negative at the turning points corresponding to the field rolls down and then stops before rolling up the hill of exponential potential. When Q is greater, the field can fall down further, therefore gaining more total energy. The result agrees with the prediction of Eqs. (6) and (7).

4.2 Scale Factor

From Figure 2, the bounce in scale factor occurs later for greater Q value of which the phantom field production rate is higher. The field has more phantom energy to accelerate the universe in counteracting the effect of loop quantum (the bounce). For less positive Q, the phantom production rate is smaller, and for negative Q, the phantom decays therefore it has less energy for accelerating the expansion in counteracting with the loop quantum effect. This makes the bounce occurs sooner.

4.3 Energy Density

Time evolutions of energy density of the matter and the phantom field are presented in Figs. 3 and 4. If $Q > 0$, the matter decays to phantom. This reduces density of matter. While for $Q < 0$, the matter gains its density from decaying of phantom field. In Fig. 3 there is a bounce of phantom density before undergoing oscillation. For a non-coupled case, it has recently been reported that the phantom density also undergoes expansion [10]. As seen in Figs. 4(a) and 4(b), the oscillation in phantom density of the phantom decay case ($Q < 0$) affects in small oscillation in matter density while for the case matter decay ($Q > 0$), the matter density is reduced for stronger coupling.

5. CONCLUSIONS AND COMMENTS

In this letter, we have derived an autonomous system of a loop quantum cosmological equation in presence of phantom scalar field coupling to barotropic matter fluid. We choose constant coupling Q between matter and the phantom field to positive and negative values and check numerically the effect of Q values on (1) phase portrait, (2) scale factor and (3) energy density of phantom field and matter. We found that field value tends to roll up the hill of potential due to phantom nature. With greater Q, the field can fall down on the potential further. This increases total energy of the field. For canonical scalar field either standard or phantom, LQC yields a bounce. The bounce is useful since it is able to avoid Big Bang singularity in the early universe [13]. Here our numerical result shows a bouncing in scale factor at late time. This is a Type I singularity avoidance even in presence of phantom energy. The greater coupling results in more and more phantom density. Greater phantom effect therefore delays the bounce, which is LQC effect, to later time. In the case of matter decay to phantom ($Q < 0$), oscillation in phantom energy density does

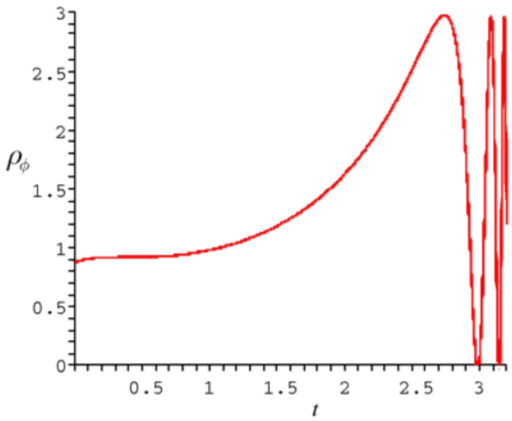

FIGURE 3. Phantom field density plotted versus time for Q = -0.1. The other values of Q also yield bouncing and oscillation.

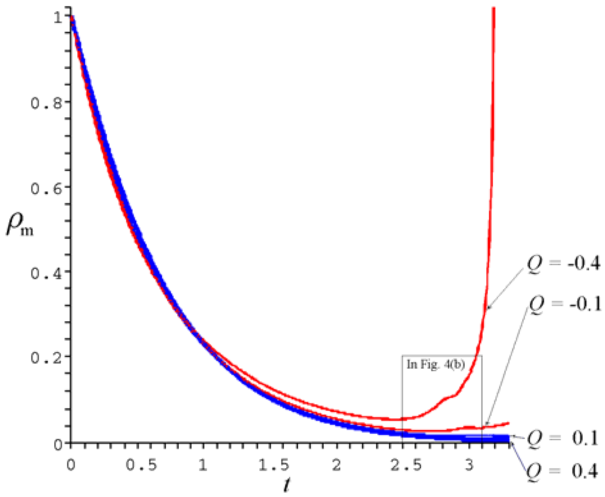

FIGURE 4(a). Matter density plotted versus time for Q = -0.4, -0.1, 0.1 and 0.4 (from top to bottom).

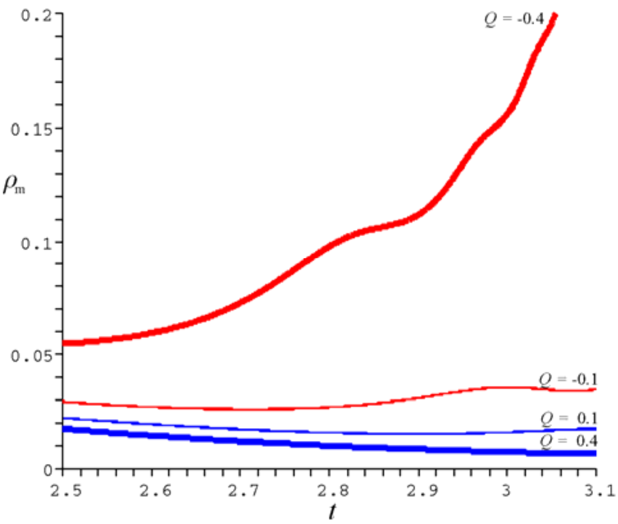

FIGURE 4(b). Zoom-in portion of Figure 4(a). The phantom field decays to matter at highest rate for Q = -0.4 (top line). Oscillation in matter density due to oscillation in the phantom field density is seen clearly here.

not affect matter density. On the other hand, when $Q > 0$, phantom decays to matter, oscillation in phantom density results in oscillation in the increasing matter density.

This work considers only the effects of sign and magnitude of the coupling constant to qualitative dynamics and evolution of the system. Studies of field dependent effects of coupling $Q(\phi)$ in some scalar-tensor theory of gravity and investigation of an evolution of effective equation of state could also yield further interesting features of the model. Quantitative dynamical analysis of the model under different types of potential is also motivated for future work. Although this work is to propose a way to resolve Type I future singularity, frequency function of the oscillation in scale factor and phantom density is still unknown and there might be a possibility that it might leads to infinite frequency of oscillation which is another new singularity.

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Power-law expansion cosmology in Schrödinger-type formulation

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ABSTRACT

We investigate non-linear Schrödinger-type formulation of cosmology of which our cosmological system is a general relativistic FRLW universe containing canonical scalar field under arbitrary potential and a barotropic fluid with arbitrary spatial curvatures. We extend the formulation to include phantom field case and we have found that Schrödinger wave function in this formulation is generally non-normalizable. Assuming power-law expansion, $a \sim t^q$, we obtain scalar field potential as function of time. The corresponding quantities in Schrödinger-type formulation such as Schrödinger total energy, Schrödinger potential and wave function are also presented.

1. Introduction

Canonical scalar field plays important role in inflationary phase in the early universe as well as acceleration in the late universe observed and confirmed by cosmic microwave background [1], large scale structure surveys [2] and supernovae type Ia [3]. The scalar field is considered as inflaton field in inflationary models [4]. It could also be considered as dark energy that drives the late acceleration described in review literatures [5] and references therein. In standard cosmology with Friedmann–Lemaître–Robertson–Walker (FLRW) background, major components of the late universe are mixture of dark matter which is a type of barotropic fliud and dark energy in form of scalar field. When assuming pure scalar fluid in flat universe, one can obtain analytical solutions otherwise the problem can also be solved numerically. However, considering arbitrary types of barotropic fluid and a non-flat universe, it is not always possible to solve the system analytically.

Apart from standard cosmological equations, there are few alternative mathematical formulations which are also equivalent to the scalar field cosmology with barotropic fluid. One is in form of non-linear Ermakov-attemping equation [6] and another idea proposed recently is in form of non-Ermakov–Milne–Pinney (non-EMP) equation. Cosmological equations in the latter proposal can be written in form of a non-linear Schrödinger-type equation when imposing relations between quantities in standard cosmological equations and Schrödinger-type equation [7]. In case of Bianchi I scalar field cosmology, recent work shows that it is possible to construct a corresponding linear Schrödinger-type equation by redefining cosmological quantities [8]. With the new representation, scalar field cosmology is reinterpreted in new way which might be able to give new methods of approaching mathematical problems in scalar field cosmology.

There are various observations allowing scalar field equation of state coefficient, w_ϕ to be less than -1 [9]. Recent data such as a combined WMAP, LSS and SN type Ia without assuming flat universe, puts a strong constraint, $w_\phi = -1.06^{+0.13}_{-0.08}$ [10]. Also the first result from ESSENCE Supernova Survey Ia combined with SuperNova Legacy Survey Ia assuming flat universe, gives a constraint of $w_\phi = -1.07 \pm 0.09$ [11]. Therefore, it is possible that the scalar field dark energy could be phantom, i.e $w_\phi < -1$ [12]. The phantom behavior, $w_\phi < -1$ can be attained by negative kinetic energy term of the scalar field density and pressure. In FLRW standard cosmology, the field can yield big rip singularity, i.e. $a, \rho, |p| \to \infty$ at finite time [13] with attempts of singularity avoidance in several ways [14].

In this work, we investigate connection between standard cosmological equations and non-linear Schrödinger-type equation with a comment on normalization of the wave function. We modify the work of [7] to include phantom field case. A case of power-law expansion with scalar field and dark matter is considered as a toy model. We begin from Section 2 where we introduce our cosmological system. Afterward in Section 3, we discuss how non-linear Schrödinger-type formulation quantities are related to quantities in standard scalar field cosmology. In non-linear Schrö-

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dinger-type equation, one important quantity is wave function. We comment on normalization properties of the wave function in Section 4. We consider a case of power-law expansion in Section 5 before deriving scalar field potential, Schrödinger potential and Schrödinger wave function. At last we conclude this work in Section 6.

2. Cosmological equations

In a Friedmann–Lemaître–Robertson–Walker universe, the Einstein field equations are

$$H^2 = \frac{\kappa^2\rho_t}{3} - \frac{k}{a^2}, \tag{1}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}(\rho_t + 3p_t), \tag{2}$$

where $\kappa^2 \equiv 8\pi G = 1/M_P^2$, G is Newton's gravitational constant, M_P is reduced Planck mass, k is spatial curvature, ρ_t and p_t are total density and total pressure, i.e. $\rho_t = \rho_\gamma + \rho_\phi$ and $p_t = p_\gamma + p_\phi$. The barotropic component is denoted by γ, while for scalar field, by ϕ. Equations of state for barotropic fluid and scalar field are $p_\gamma = w_\gamma\rho_\gamma$ and $p_\phi = w_\phi\rho_\phi$. We consider minimally couple scalar field with Lagrangian density,

$$\mathscr{L} = \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi), \tag{3}$$

where $\epsilon = 1$ for non-phantom case and -1 for phantom case. Density and pressure of the field are given as

$$\rho_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi), \tag{4}$$

$$p_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi), \tag{5}$$

therefore,

$$w_\phi = \frac{\epsilon\dot{\phi}^2 - 2V(\phi)}{\epsilon\dot{\phi}^2 + 2V(\phi)}. \tag{6}$$

The field obeys conservation equation

$$\epsilon\left[\ddot{\phi} + 3H\dot{\phi}\right] + \frac{dV}{d\phi} = 0. \tag{7}$$

For the barotropic fluid, we set $w_\gamma \equiv (n-3)/3$ so that $n = 3(1+w_\gamma)$. Hence, for cosmological constant $n = 0$, for fluid at acceleration bound $(w_\gamma = -1/3) n = 2$, for dust $n = 3$, for radiation $n = 4$, and for stiff fluid $n = 6$. Solution of conservation equation for a barotropic fluid can be obtained directly by solving the conservation equation. The solution is

$$\rho_\gamma = \frac{D}{a^{3(1+w_\gamma)}} = \frac{D}{a^n}, \tag{8}$$

then

$$p_\gamma = w_\gamma\frac{D}{a^n} = \frac{(n-3)}{3}\frac{D}{a^n}, \tag{9}$$

where a proportional constant $D \geqslant 0$. Using Eqs. (1), (4), (5), (7) and (8), it is straightforward to show that

$$\epsilon\dot{\phi}(t)^2 = -\frac{2}{\kappa^2}\left[\dot{H} - \frac{k}{a^2}\right] - \frac{nD}{3a^n}, \tag{10}$$

$$V(\phi) = \frac{3}{\kappa^2}\left[H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right] + \left(\frac{n-6}{6}\right)\frac{D}{a^n}. \tag{11}$$

Therefore, if one knows how the scale factor evolves with time, the scalar field velocity and potential can always be expressed as a function of time explicitly.

3. Non-linear Schrödinger-type equation

Non-linear Schrödinger-type equation corresponding to canonical scalar field cosmology with barotropic fluid is given by [7]

$$\frac{d^2}{dx^2}u(x) + [E - P(x)]u(x) = -\frac{nk}{2}u(x)^{(4-n)/n}. \tag{12}$$

Quantities in the Schrödinger-type equation above, e.g. wave function $u(x)$, total energy E and Schrödinger potential $P(x)$ are related to the standard cosmology quantities as

$$u(x) \equiv a(t)^{-n/2}, \tag{13}$$

$$E \equiv -\frac{\kappa^2 n^2}{12}D, \tag{14}$$

$$P(x) \equiv \frac{\kappa^2 n}{4}a(t)^n\epsilon\dot{\phi}(t)^2. \tag{15}$$

The mapping from cosmic time t to the variable x is via

$$x = \sigma(t), \tag{16}$$

such that

$$\dot{\sigma}(t) = u(x), \tag{17}$$

$$\phi(t) = \psi(x). \tag{18}$$

We notice that relation

$$\psi'(x)^2 = \frac{4}{\kappa^2 n}P(x) \tag{19}$$

in Ref. [7] which gives $\psi(x) = \pm(2/\kappa\sqrt{n})\int\sqrt{P(x)}dx$ does not include phantom field case. In order to include the phantom field case, we modify relation $\dot{\phi}(t) = \dot{x}\psi\prime(x)$ in [7] to $\epsilon\dot{\phi}(t)^2 = \dot{x}^2\epsilon\ \psi\prime(x)^2$ of which the field kinetic term ($\dot{\phi}^2$) is considered instead of the field velocity ($\dot{\phi}$) so that the parameter ϵ can be included. Therefore, to include the phantom field case, corrected relation to Eq. (19) is

$$\epsilon\psi'(x)^2 = \frac{4}{\kappa^2 n}P(x), \tag{20}$$

and $\psi(x)$ should read

$$\psi(x) = \pm\frac{2}{\kappa\sqrt{n}}\int\sqrt{\frac{P(x)}{\epsilon}}dx. \tag{21}$$

Inverse function of $\psi(x)$ exists if $P(x)\neq 0$ and $n\neq 0$. It is important for $\psi^{-1}(x)$ to exist as a function since existence of the relation $x = \sigma(t)$ (Eq. (16)) needs a condition,

$$x = \psi^{-1}\circ\phi(t) = \sigma(t). \tag{22}$$

In case that $P(x) = 0$ and $n\neq 0$, then $\psi = C$, hence inverse of ψ is not a function since one x gives infinite values of ψ^{-1}. In this case the relation (22) is invalid. If the inverse function, ψ^{-1} exists (i.e. $P(x)\neq 0$ and $n\neq 0$), then the scalar field potential, $V\circ\sigma^{-1}(x)$ can be expressed as a function of time,

$$V(t) = \frac{12}{\kappa^2 n^2}\left(\frac{du}{dx}\right)^2 - \frac{2u^2}{\kappa^2 n}P(x) + \frac{12u^2}{\kappa^2 n^2}E + \frac{3ku^{4/n}}{\kappa^2}. \tag{23}$$

Although the potential obtained is not expressed as function of ϕ, however if one can integrate Eq. (10) to obtain $\phi(t)$, the obtained solution can be inserted into a known function $V(\phi)$ motivated from fundamental physics. Then one can check which fundamental theories give a matched potential to $V(t)$. The Eqs. (11) and (23) are indeed equivalent. Both require only the knowledge of $a(t), D$ and k which can be constrained by observation. Therefore, $V(t)$ in both Eqs. (11) and (23) can be constructed if knowing these observed parameters. To construct $V(t)$ in Eq. (23), one needs to know $a(t)$ as a function of time in order to find $u(x)$ and $P(x)$. However, in constructing $V(t)$ in Eq. (11), if knowing $a(t), D$ and k, one can directly use these quantities without employing Schrödinger-type quantities.

4. Normalization condition of wave function

Normalization condition for a wave function $u(x)$ in quantum mechanics is

$$\int_{-\infty}^{\infty} |u(x)|^2 \mathrm{d}x = 1. \tag{24}$$

The wave function here expressed as $u(x) \equiv a^{-n/2} = \dot{x}(t)$, when applying to the normalization condition, reads

$$\int_{-\infty}^{\infty} \dot{x}^2 \mathrm{d}x = 1. \tag{25}$$

In order to satisfy the condition, x must be constant and so is t. Since the form of the wave function must be $u(x) = \dot{x}(t)$ in order to connect equations of cosmology to the Schrödinger-type formulation, therefore $u(x)$ as defined is, in general, non-normalizable.

5. Power-law expansion

Here in this section, we apply the method above to the power-law expansion in scalar field cosmology with barotropic fluid in a non-flat universe. The power-law expansion of the universe during inflation era,

$$a(t) = t^q, \tag{26}$$

with $q > 1$ was proposed by Lucchin and Matarrese [15] to give exponential potential

$$V(\phi) = \left[\frac{q(3q-1)}{\kappa^2 t_0^2}\right] \exp\left\{-\kappa\sqrt{\frac{2}{q}}[\phi(t) - \phi(t_0)]\right\}, \tag{27}$$

assuming domination of scalar field, negligible radiation density and negligible spatial curvature. Recent results from X-ray gas of galaxy clusters put a constraint of $q \sim 2.3$ for $k = 0, q \sim 1.14$ for $k = -1$ and $q \sim 0.95$ for $k = 1$ [16]. Considering mixture of both fluids, we use effective equation of state, $w_{\mathrm{eff}} = (\rho_\phi w_\phi + \rho_\gamma w_\gamma)/\rho_t$. For a flat universe, the power law expansion, $a = t^q$, is attained when $-1 < w_{\mathrm{eff}} < -1/3$ where $q = 2/[3(1 + w_{\mathrm{eff}})]$. If using $q = 2.3$ as mentioned above, it gives $w_{\mathrm{eff}} = -0.71$.

5.1. Relating Schrödinger quantities to standard cosmological quantities

Assuming power-law expansion and using Eqs. (13) and (17), Schrödinger wave function is related to standard cosmological quantity as

$$u(x) = \dot{\sigma}(t) = t^{-qn/2}. \tag{28}$$

We can integrate the equation above so that the Schrödinger scale, x is related to cosmic time scale, t as

$$x = \sigma(t) = -\frac{t^{-\beta}}{\beta} + \tau, \tag{29}$$

where $\beta \equiv (qn-2)/2$ and τ is an integrating constant. The parameters x and t have the same dimension since β is only a number. Using Eq. (26), we can find $\epsilon\dot{\phi}(t)^2$ from Eq. (10):

$$\epsilon\dot{\phi}(t)^2 = \frac{2q}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} - \frac{nD}{3t^{qn}}. \tag{30}$$

We use Eqs. (26) and (30) in Eq. (15), therefore the Schrödinger potential is found to be

$$P(x) = \frac{qn}{2} t^{qn-2} + \frac{kn}{2} t^{q(n-2)} - \frac{\kappa^2 n^2 D}{12}. \tag{31}$$

With $E = -\kappa^2 n^2 D/12$, the Schrödinger kinetic energy is

$$T = -\frac{qn}{2} t^{qn-2} - \frac{kn}{2} t^{q(n-2)}. \tag{32}$$

5.2. Scalar field potential $V(t)$

In order to obtain $V(t)$ in Eq. (23), we need to know derivative of $u(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x} u(x) = -\frac{qn}{2t}. \tag{33}$$

At this step, using Eqs. (13), (14), (15) and (33) in Eq. (23), we finally obtain

$$V(t) = \frac{q(3q-1)}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} + \left(\frac{n-6}{6}\right)\frac{D}{t^{qn}}. \tag{34}$$

Assuming flat universe ($k = 0$) and $q = 2.3$, we show $V(t)$ in Fig. 1. Thickest line on top is of the case scalar field without barotropic fluid. The middle line is the case when the dust is presented with scalar field ($D \neq 0, n = 3$). The bottom line is the case of radiation ($D \neq 0, n = 4$). The $V(t)$ plots from the Schrödinger-type formulation matches the plots from standard cosmological equations. The result is independent of ϵ values. The solution $\phi(t)$ of Eq. (30) cannot be integrated if $\epsilon = -1$ or if the integrand of Eq. (30) is imaginary. When $\epsilon = 1$ with dust ($D \neq 0, n = 3$) and $q = 2.3$, the integrand is imaginary. We therefore assume $q = 2$ to show numerical integrations in Fig. 2 for the case $D = 0, k = 0$ and the case $D \neq 0, n = 3, k = 0$. In the pure scalar field case $D = 0, k = 0$, numerical solution matches the analytical solution $\phi(t) = (\sqrt{2q}/\kappa)\ln(t)$. This solution can be substituted into Eq. (34) to obtain Eq. (27) as in [15] (setting $t_0 = 1$ and $\phi(t_0) = 0$). When considering cases of closed, flat and open universe containing dust matter, $V(t)$ of each case is presented in Fig. 3 where $q = 2$ is assumed in all cases so that we can see how the plots change their shapes when k is varied.

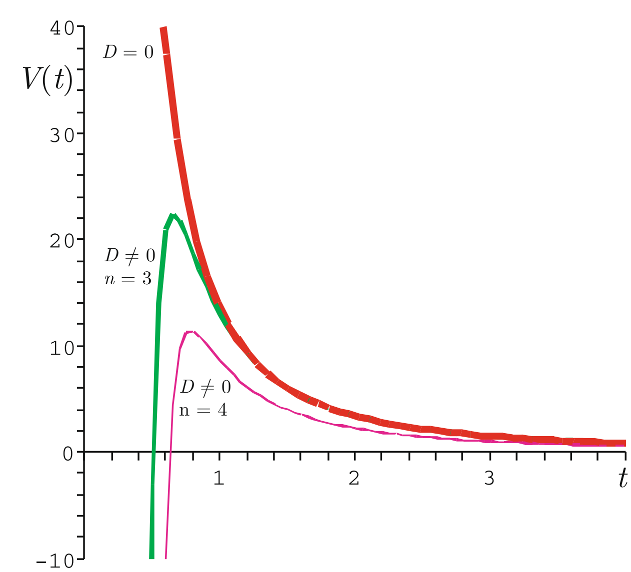

Fig. 1. Potential $V(t)$ plots from non-linear Schrödinger-type formulation assuming $a \sim t^q, q = 2.3$ in flat universe ($k = 0$). The thickest line is when there is no barotropic fluid $D = 0$. The middle line is when there is dust fluid together with scalar field, i.e. $D \neq 0$ and $n = 3$. The small line is when the universe has scalar field with radiation fluid, i.e. $D \neq 0$ and $n = 4$. We set $\kappa = 1$ and in the last two plots, we set $D = 1$. All plots match results obtained from standard cosmological equations.

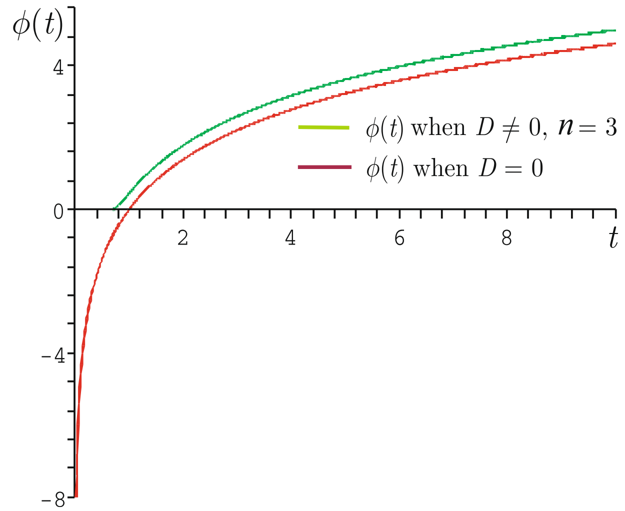

Fig. 2. $\phi(t)$ for power-law expansion $a \sim t^q, q = 2$ in flat universe ($k = 0$). The red line is of the when the barotropic fluid density is negligible. The green line is in the presence of scalar field with dust ($D \neq 0$ and $n = 3$). In the figure, $\kappa = 1$ and $D = 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

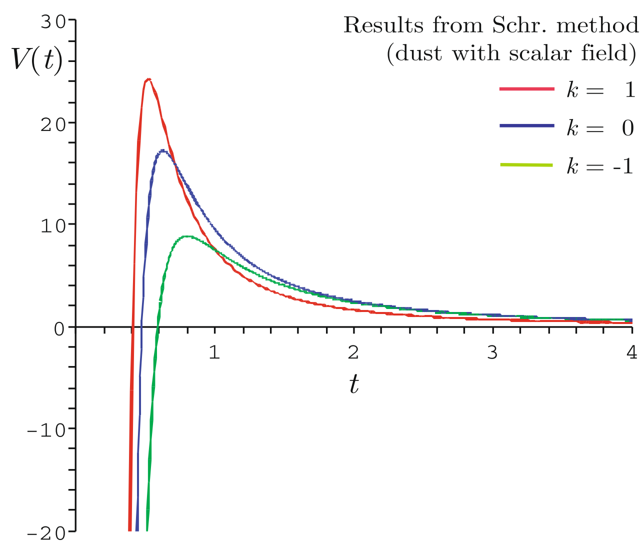

Fig. 3. $V(t)$ obtained from non-linear Schrödinger-type formulation for closed, flat and open universe in presence of dust and scalar field.

5.3. Schrödinger potential $P(x)$

We can find Schrödinger potential $P(x)$ from Eqs. (29) and (31) where time is expressed as a function of x as

$$t(x) = \frac{1}{[-\beta(x-\tau)]^{1/\beta}}. \tag{35}$$

Therefore,

$$P(x) = \frac{2qn}{(qn-2)^2}\frac{1}{(x-\tau)^2} + \frac{kn}{2}\left[\frac{-2}{(qn-2)(x-\tau)}\right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12}. \tag{36}$$

As in Eq. (32), the Schrödinger kinetic energy is

$$T(x) = -\frac{2qn}{(qn-2)^2}\frac{1}{(x-\tau)^2} - \frac{kn}{2}\left[\frac{-2}{(qn-2)(x-\tau)}\right]^{2q(n-2)/(qn-2)}. \tag{37}$$

The kinetic term has contribution only from the power q and spatial curvature k. A disadvantage of Eq. (36) is that we can not use it in the case of scalar field domination as in inflationary era. Dropping D term in Eq. (36) can not be considered as scalar field domination case since the barotropic fluid coefficient n still appears in the other terms. The non-linear Schrödinger-type formulation is therefore suitable when there are both scalar field and a barotropic fluid together such as the situation when dark matter and scalar field dark energy live together in the late universe. The Schrödinger potentials $P(x)$ plotted with x for power-law expansion with $q = 2$ in closed, flat and open universe are shown in Fig. 4. In the figure, the dust cases are shown on the right and radiation cases are on the left. We set $\kappa = 1, D = 1$ and $\tau = 0$.

5.4. Schrödinger wave function $u(x)$

The quantity analogous to Schrödinger wave function can be directly found from Eqs. (28) and (35) as

$$u(x) = \left[\left(-\frac{1}{2}qn + 1\right)(x-\tau)\right]^{qn/(qn-2)}, \tag{38}$$

which is independent of the spatial curvature k or the initial density D. However, coefficient n of the barotropic fluid equation of state and q must be expressed. Wave functions for a range of barotropic fluids are presented in Fig. 5. The result is confirmed by substituting Eq. (38) into Eq. (12).

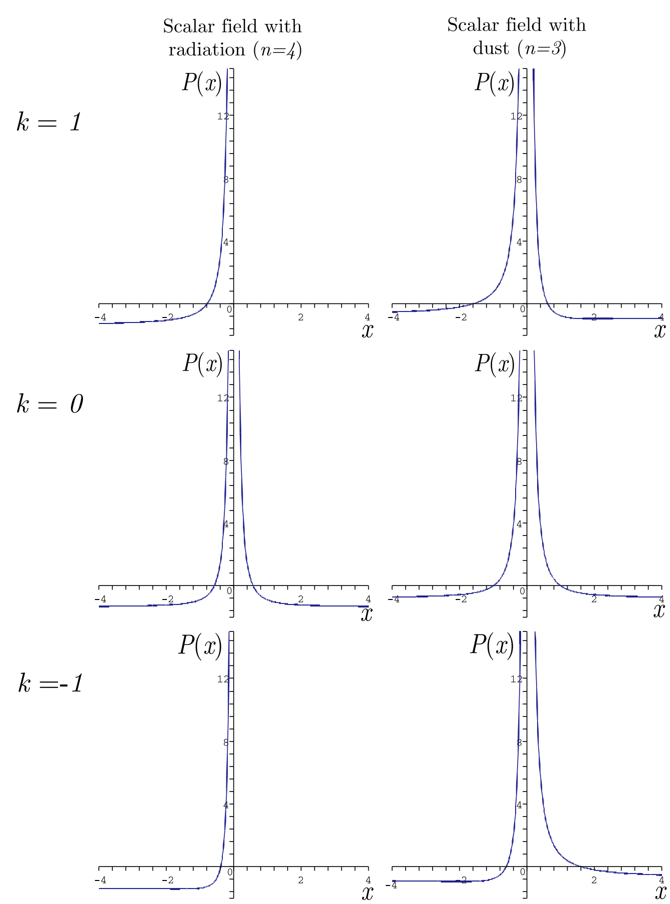

Fig. 4. $P(x)$ plotted versus x for power-law expansion. We set $q = 2, \kappa = 1, D = 1$ and $\tau = 0$. The scalar field dominant case can not be plotted because even tough we set a condition $D = 0$, the coefficient n of the barotropic fluid equation of state still appears in the first and second terms of the Eq. (36). There is only a real-value $P(x)$ for the cases $k = \pm 1$ with $n = 4$ because, when $x > 0, P(x)$ becomes imaginary in these cases.

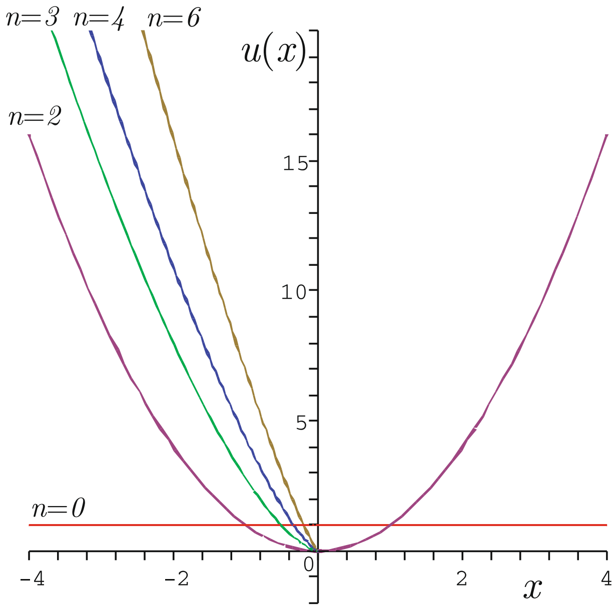

Fig. 5. $u(x)$ plotted versus x for power-law expansion with $q = 2$. We set $\tau = 0$. The wave function is plotted for $n = 0$ (cosmological constant), $n = 2, n = 3$ (dust), $n = 4$ (radiation) and $n = 6$ (stiff fluid). There is no real-value wave function for $n = 3, n = 4$ and $n = 6$ unless $x < 0$.

6. Conclusions and comments

We consider Schrödinger-type formulation for a system of canonical scalar field and a barotropic fluid in standard FLRW cosmology with zero or non-zero spatial curvature. In the Schrödinger-type formulation, all quantities in cosmology are represented in Schrödinger-like quantities and the equation relating these Schrödinger-like quantities is written as a non-linear Schrödinger-type equation. If $a(t)$ is known as an exact function of time, a connection of two scale quantities, x and t can be found and then other Schrödinger-like quantities can be determined. We modified the formulation to include the phantom field case. The equation can be simplified to linear type if we consider the flat universe case $k = 0$ or the cases $n = 2$ or $n = 4$ [7]. However, even if the equation is linear, it can not be considered as an analog to non-relativistic time-independent quantum mechanics because in this work, the wave function of Schrödinger-type formulation is found to be, in general, non-normalizable. Afterward, we consider a particular case of power-law expansion of scale factor. We show relations between cosmological quantities in conventional form and in Schrödinger-like form for power-law expansion. We obtain scalar field potential $V(t)$, Schrödinger potential $P(x)$ and wave function $u(x)$. In the case of a scalar field dominant in flat universe, our analytical results $V(\phi)$ and $\dot{\phi}$ agree well with the well-known results in [15]. A range of plots in various cases of closed, flat or open geometries is presented. Wave functions for the power-law expansion case (seen in the Fig. 5) are found to be all non-normalizable as conjectured.

Without knowledge of $a(t)$, one might wonder if we could start the calculation procedure from solving the Schrödinger-type Eq. (12) for example, the linear case as done in basic quantum mechanics. However, in order to do this, we must know the Schrödinger potential $P(x)$ (Eq. (15)) which depends explicitly on $a(t)$ and $\dot{\phi}$. Nevertheless, $\dot{\phi}$ (Eq. (10)) also depends on $a(t)$. Therefore, we need to know the law of expansion $a(t)$ before proceeding the calculation. Knowing $a(t)$ enables us to know $u(x)$ directly (see Eq. (28)). Hence, in Schrödinger-type formulation, we do not work as in basic quantum mechanics in which major task is to solve the Schrödinger equation for $u(x)$. There could be many solutions of a Schrödinger-type equation. In quantum mechanics valid solutions $u(x)$ must be only normalizable type. Here, unlike in quantum mechanics, our $u(x)$ must be non-normalizable.

At late time the scalar field dark energy and cold dark matter (dust) are two major components of the universe while the others are negligible. For power-law expansion, the procedure is suitable for studying the system of scalar field dark energy and dark matter because it gives all real-value of $P(x)$ for any k. We need to know $a(t), k$ and D which are observable in order to find $V(t)$. Information of $V(\phi)$ is important because it is a link to fundamental physics. If one starts from fundamental physics with a particular potential $V(\phi)$ and if also knowing how ϕ evolves with t, then V could be expressed as function of t. Finally, the potential $V(t)$ obtained from observation and another $V(t)$ proposed by fundamental physics can be compared. The non-linear Schrödinger-type formulation might provide an alternative mathematical approach to problem solving in scalar field cosmology.

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RESEARCH ARTICLE

Scalar field exact solutions for non-flat FLRW cosmology: a technique from non-linear Schrödinger-type formulation

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Abstract We report a method of solving for canonical scalar field exact solution in a non-flat FLRW universe with barotropic fluid using non-linear Schrödinger (NLS)-type formulation in comparison to the method in the standard Friedmann framework. We consider phantom and non-phantom scalar field cases with exponential and power-law accelerating expansion. Analysis on effective equation of state to both cases of expansion is also performed. We speculate and comment on some advantage and disadvantage of using the NLS formulation in solving for the exact solution.

Keywords Scalar field cosmology · Non-linear Schrödinger equation · Power-law expansion · Exponential expansion

1 Introduction

In the past decade, it has been observed that universe is now in acceleration phase [1–9] while inflationary scenario of the early universe [10–15] is strongly confirmed by cosmic microwave background data [16–19]. In both circumstances, the universe experiences accelerating expansion which can be attained by exploiting some

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dynamical scalar field with time-dependent equation of state coefficient $w_\phi(t) < -1/3$, or a cosmological constant with $w_\Lambda = -1$ [20–22]. It was further suggested that the scalar field could have phantom behavior with $w_\phi < -1$ as one considers kinetic energy term in its Lagrangian density to be negative [23–28]. Strong supports to the phantom idea are from observations previously, e.g. combined cosmic microwave background, large scale structure survey and supernovae type Ia without assuming flat universe yields $w_{\phi,0} = -1.06^{+0.13}_{-0.08}$ [16] while using supernovae data alone assuming flat universe, $w_{\phi,0} = -1.07 \pm 0.09$ [29]. The subscript 0 denotes the value at present. Moreover, most recent WMAP five-year result [17,18] combined with Baryon Acoustic Oscillation (BAO) of large scale structure survey from SDSS and 2dFGRS [30] and type Ia supernovae data from HST [7,8], SNLS [9] and ESSENCE [29] assuming dynamical w with flat universe yields $-1.38 < w_{\phi,0} < -0.86$ at 95% confident level and $w_{\phi,0} = -1.12 \pm 0.13$ at 68% confident level [19]. With additional BBN constraint of limit of expansion rate [31,32], $-1.32 < w_{\phi,0} < -0.86$ at 95% confident level and $w_{\phi,0} = -1.09 \pm 0.12$ at 68% confident level. This suggests that phantom field has firmed status in cosmology. However, phantom field does result in unwanted Big Rip singularity in a flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe [33–42]. This raises up many attempts to avoid the singularity based on both phenomenological and fundamental inspirations [43–50].

Recently, there are a few proposals for mathematical alternatives to the conventional Friedmann formulation of canonical scalar field cosmology, such as non-linear Ermakov–Pinney equation [51–56]. Moreover, a non-Ermakov–Pinney equation for the same system was also proposed in form of a non-linear Schrödinger-type equation (NLS) and it was found that solutions of the NLS-type equation are correspondent to solutions of the generalized Ermakov–Pinney equation [56,57].[1] Conclusion of how to relate NLS quantities to quantities of standard Friedmann formulation is shown in [59] which gives extension to phantom field case. It also shows that the NLS wave function is in general non-normalizable. Expressing cosmological quantities in form of NLS quantities may suggest an alternative way of solving problems in scalar field cosmology. In such method, presumed knowledge of scale factor function with time $a(t)$ must be given first and later one can evaluate NLS potential based on $a(t)$ assumed. Here we attempt to solve for scalar field exact solution within the NLS framework in various cases with flat and non-flat spatial geometries. We compare the results to the solutions obtained in standard Friedmann formulation. Exponential expansion $a \sim \exp(t/\tau)$ and power-law expansion $a \sim t^q$ are assumed where τ are finite characteristic time and q is a positive constant.

This article is organized as follow. We express our cosmological system in Sect. 2 before introducing the NLS formulation in Sect. 3. Afterward, we consider each model of expansion separately. Exponential expansion is presented in Sect. 4 where we obtain exact solution from effective equation of state and later we solve for exact solution from Friedmann system. Afterward, in Sect. 5, we consider NLS formulation for the exponential expansion and solve for exact solutions in NLS framework. We analyze effective equation of state for the exponential expansion case in Sect. 6. When

[1] Considering Bianchi I scalar field cosmology, one can also construct a corresponding linear Schrödinger-type equation by redefining cosmological quantities [58].

considering power-law expansion, we work and organize the contents in the same spirit and order as in the previous sections. Beginning from Sects. 7, 8 and 9. Finally, conclusion is made in Sect. 10.

2 Cosmological system

To be realistic, two perfect fluids are considered in our system: barotropic fluid and scalar field fluid. The perfect barotropic fluid pressure p_γ and density ρ_γ obey an equation of state, $p_\gamma = (\gamma - 1)\rho_\gamma = w_\gamma \rho_\gamma$ while for scalar field, $p_\phi = w_\phi \rho_\phi$. Total density and total pressure are $\rho_{\text{tot}} = \rho_\gamma + \rho_\phi$ and $p_{\text{tot}} = p_\gamma + p_\phi$. The effective equation of state is weighed-value of these two components,

$$w_{\text{eff}} = \frac{\rho_\phi w_\phi + \rho_\gamma w_\gamma}{\rho_{\text{tot}}}. \tag{1}$$

For the barotropic fluid, its equation of state coefficient w_γ is written in term of n. We set $w_\gamma \equiv (n-3)/3$ so that $n = 3(1 + w_\gamma) = 3\gamma$, hence $w_\gamma = -1$ corresponds to $n = 0$, $w_\gamma = -1/3$ to $n = 2$, $w_\gamma = 0$ to $n = 3$, $w_\gamma = 1/3$ to $n = 4$, and $w_\gamma = 1$ to $n = 6$. The conservation equation is therefore

$$\dot{\rho}_\gamma = -nH\rho_\gamma \tag{2}$$

with solution obtained directly,

$$\rho_\gamma = \frac{D}{a^n}, \tag{3}$$

therefore $p_\gamma = [(n-3)/3](D/a^n)$, where a is scale factor, the dot denotes time derivative, $H = \dot{a}/a$ is Hubble parameter and $D \geq 0$ is a proportional constant. The scalar field considered here is minimally coupling to gravity with Lagrangian density, $\mathcal{L} = (1/2)\epsilon\dot{\phi}^2 - V(\phi)$, where $\epsilon = 1$ for non-phantom case and -1 for phantom case. Density and pressure of the field are given as

$$\rho_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi), \tag{4}$$

therefore

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\epsilon\dot{\phi}^2 - 2V(\phi)}{\epsilon\dot{\phi}^2 + 2V(\phi)}. \tag{5}$$

The field obeys conservation equation

$$\epsilon\left[\ddot{\phi} + 3H\dot{\phi}\right] + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0. \tag{6}$$

Considering FLRW universe, the Friedmann equation and acceleration equation are

$$H^2 = \frac{\kappa^2}{3}\rho_{\rm tot} - \frac{k}{a^2}, \tag{7}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}\rho_{\rm tot}(1+3w_{\rm eff}), \tag{8}$$

where $\kappa^2 \equiv 8\pi G = 1/M_{\rm P}^2$, G is Newton's gravitational constant, $M_{\rm P}$ is reduced Planck mass, k is spatial curvature. Using Eqs. (3), (4), (6) and (7), it is straightforward to show that

$$\epsilon\dot{\phi}(t)^2 = -\frac{2}{\kappa^2}\left[\dot{H} - \frac{k}{a^2}\right] - \frac{nD}{3a^n}, \tag{9}$$

$$V(\phi) = \frac{3}{\kappa^2}\left[H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right] + \left(\frac{n-6}{6}\right)\frac{D}{a^n}. \tag{10}$$

3 Non-linear Schrödinger-type formulation

Correspondence between NLS formulation for canonical scalar field cosmology with barotropic fluid was shown in [57] and was concluded recently in [59]. In the Schrödinger formulation, wave function $u(x)$ is related to scale factor in cosmology as

$$u(x) \equiv a(t)^{-n/2}, \tag{11}$$

and Schrödinger total energy E and Schrödinger potential $P(x)$ are linked to cosmology as

$$E \equiv -\frac{\kappa^2 n^2}{12}D, \tag{12}$$

$$P(x) \equiv \frac{\kappa^2 n}{4}a(t)^n\epsilon\dot{\phi}(t)^2. \tag{13}$$

These quantities satisfy NLS-type equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + [E - P(x)]u(x) = -\frac{nk}{2}u(x)^{(4-n)/n}. \tag{14}$$

The mapping from t to x is via $x = \sigma(t)$, such that

$$\dot{x}(t) = u(x), \tag{15}$$

$$\phi(t) = \psi(x). \tag{16}$$

We comment that the relation $\psi'(x)^2 = (4/n\kappa^2)P(x)$ in Ref. [57] does not include phantom field case. Modification is made in recent work [59] so that the solution includes the phantom field case, therefore

$$\psi(x) = \pm\frac{2}{\kappa\sqrt{n}}\int\sqrt{\frac{P(x)}{\epsilon}}\,\mathrm{d}x. \tag{17}$$

If $P(x) \neq 0$ and $n \neq 0$. There exists an inverse function of $\psi(x)$ as $\psi^{-1}(x)$. Therefore $x(t) = \psi^{-1}\circ\phi(t)$ and the scalar field potential, $V\circ\sigma^{-1}(x)$ can be expressed as function of time,

$$V(t) = \frac{12}{\kappa^2 n^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \frac{2u^2}{\kappa^2 n}P(x) + \frac{12u^2}{\kappa^2 n^2}E + \frac{3ku^{4/n}}{\kappa^2}. \tag{18}$$

4 Exponential expansion

4.1 Solution solved from effective equation of state for $k = 0$ case

Exponential expansion reads

$$a(t) = \exp(t/\tau), \tag{19}$$

where τ is a positive constant. Flat universe undergoes exponential expansion only when $w_{\mathrm{eff}} = -1$. The effective equation of state (Eq. 1), with Eqs. (4) and (5) can therefore be written as[2]

$$\epsilon\dot{\phi}^2 = -\frac{n}{3}\rho_\gamma, \tag{20}$$

which can be integrated directly, using Eq. (3), to

$$\phi(t) = \pm 2\tau\sqrt{\frac{D}{3n}}\,\mathrm{e}^{-nt/2\tau} + \phi_0. \tag{21}$$

The solution above is obtained when assuming phantom scalar field, i.e. $\epsilon = -1$. If the scalar field is not phantom, the solution is imaginary.

4.2 Solution solved from Friedmann formulation

Another way to find the exact solution is to use Eq. (19), in Eq. (9). Therefore

$$\epsilon\dot{\phi}(t)^2 = \frac{2k}{\kappa^2}\,\mathrm{e}^{-2t/\tau} - \frac{nD}{3}\,\mathrm{e}^{-nt/\tau}, \tag{22}$$

[2] We are not considering a cosmological constant but a dynamical scalar field and a barotropic fluid which together yield $w_{\mathrm{eff}} = -1$.

which gives an integration:

$$\phi(t) = \pm \int \sqrt{\frac{1}{\epsilon}\left(\frac{2k}{\kappa^2}\mathrm{e}^{-2t/\tau} - \frac{nD}{3}\mathrm{e}^{-nt/\tau}\right)}\mathrm{d}t. \tag{23}$$

4.2.1 Simplest case

In the case of $k = 0$ and $D = 0$, the integration yields a constant ϕ_0. Eq. (1) becomes $w_\phi = -1$. This is a cosmological constant as seen in simplest model of exponential expansion. When assuming only $k = 0$ and $\epsilon = -1$ but with $D \neq 0$, the solution of Eq. (23) is the same as the Eq. (21) previously. For a scalar field domination in a non-flat universe ($D = 0$, $k \neq 0$), the solution is

$$\phi(t) = \pm\frac{\tau}{\kappa}\sqrt{\frac{2k}{\epsilon}}\,\mathrm{e}^{-t/\tau} + \phi_0. \tag{24}$$

where k and ϵ must have the same sign, otherwise the solution is imaginary.

4.2.2 The case of non-zero k and non-zero D

When k and D are both not negligible. Performing integration to the Eq. (23) is more complicated and could be impossible unless assumption of barotropic fluid type. When assuming a particular type of fluid in the integration, i.e. $n = 0, 2, 3, 4$ and 6, analytical solution can be found for all n vales in complicated forms. For example, the simplest among these is dust case ($n = 3$) which has solution:

$$\phi(t) = \pm\frac{2\tau}{3D\sqrt{\epsilon}}\left(\frac{2k}{\kappa^2} - D\mathrm{e}^{-t/\tau}\right)^{3/2} + \phi_0, \tag{25}$$

with additional rule that $k \geq 0$ and $\epsilon = 1$ otherwise it is imaginary. In the next section, we will show how to obtain solution in NLS formulation for dust and radiation cases.

5 Exponential expansion: solutions solved with NLS formulation

For exponential expansion, following Eqs. (11) and (15), we get

$$u(x) = \dot{x}(t) = \exp\left(-nt/2\tau\right). \tag{26}$$

Integrating the above equation, hence parameters x and t scale as

$$x(t) = -\frac{2\tau}{n}\mathrm{e}^{-nt/2\tau} + x_0, \tag{27}$$

where x_0 is an integration constant. The reverse is

$$t(x) = -\frac{2\tau}{n} \ln\left[(-n/2\tau)(x - x_0)\right], \tag{28}$$

where the condition $x < x_0$ must be imposed. Now we can write wave function as

$$u(x) = -\frac{n}{2\tau}(x - x_0), \tag{29}$$

which is a linear function. Using Eq. (22), hence the Eq. (13) reads

$$P(t) = \frac{kn}{2}\, \mathrm{e}^{(n-2)t/\tau} - \frac{\kappa^2 n^2 D}{12}. \tag{30}$$

Here the Schrödinger kinetic energy term is

$$T(t) = -\frac{kn}{2}\, \mathrm{e}^{(n-2)t/\tau}. \tag{31}$$

Expressing in Schrödinger formulation, these functions are written in term of x,

$$P(x) = \frac{kn}{2}\left[-\frac{n}{2\tau}(x - x_0)\right]^{-2(n-2)/n} - \frac{\kappa^2 n^2 D}{12}, \tag{32}$$

$$T(x) = -\frac{kn}{2}\left[-\frac{n}{2\tau}(x - x_0)\right]^{-2(n-2)/n}. \tag{33}$$

In order to obtain the scalar field potential $V(t)$, we use Eqs. (11), (12), (13) in Eq. (18), we finally obtain

$$V(t) = \frac{3}{\kappa^2\tau^2} + \frac{2k}{\kappa^2}\, \mathrm{e}^{-2t/\tau} + \left(\frac{n-6}{6}\right) D\, \mathrm{e}^{-nt/\tau}. \tag{34}$$

which is checked by using Eq. (19) in standard formula (10). We use Eq. (32) in Eq. (17), then

$$\psi(x) = \frac{\pm 2}{\kappa\sqrt{n}} \times \int \sqrt{\frac{kn}{2\epsilon}\left[\frac{-n}{2\tau}(x - x_0)\right]^{-2(n-2)/n} - \frac{\kappa^2 n^2 D}{12\epsilon}}\, \mathrm{d}x. \tag{35}$$

We will integrate this equation in cases of $k = 0$ and $k \neq 0$.

5.1 The case $k = 0$

When $k = 0$ and $D \neq 0$ integrating Eq. (35) and transforming x to t with Eq. (27) yields same result as Eq. (21) obtained by solving effective equation of state equation or by integrating from the Friedmann formulation. Real solution exists only when the scalar field is phantom. With the solution (21), the scalar field potential in term of ϕ, reads

$$V(\phi) = \frac{3}{\kappa^2\tau^2} + \left(\frac{n-6}{6}\right)\frac{3n}{4\tau^2}(\phi - \phi_0)^2. \tag{36}$$

5.2 The case $k \neq 0$

When $k \neq 0$ and $D \neq 0$, the integral (35) can be integrated yielding complicated hypergeometric function even when n is not specified. The case $n = 0$ is excluded from our consideration by the reason mentioned in Sect. 3. For naturalness, we consider radiation ($n = 4$) and dust ($n = 3$).

5.2.1 Radiation case

Radiation fluid corresponds to $n = 4$, the integral (35) becomes

$$\psi(x) = \pm\frac{1}{\kappa}\int\sqrt{-\frac{k\tau}{\epsilon}\frac{1}{(x-x_0)} - \frac{4}{3}\frac{\kappa^2 D}{\epsilon}}\,\mathrm{d}x. \tag{37}$$

Here x could be negative, ϵ can possibly be either ± 1. The solution in radiation case is

$$\psi(x) = \pm\sqrt{\frac{1}{\epsilon}\left[-\frac{4}{3}D(x-x_0)^2 - \frac{k\tau}{\kappa^2}(x-x_0)\right]} \pm \frac{k\tau}{4\kappa^2}\sqrt{\frac{3}{D\epsilon}} \times \arctan\left\{\frac{[8\kappa^2 D(x-x_0)/3\epsilon] + k\tau/\epsilon}{[4\kappa\sqrt{D}/(\epsilon\sqrt{3})]\sqrt{-[4\kappa^2 D(x-x_0)^2/3] - k\tau(x-x_0)}}\right\} + \psi_0, \tag{38}$$

allowing only $\epsilon = 1$ case for the solution to be real. Transforming x scale to the t scale using Eq. (27), the solution therefore reads

$$\phi(t) = \pm\sqrt{\frac{1}{\epsilon}\left(-\frac{D\tau^2}{3}\,\mathrm{e}^{-4t/\tau} + \frac{k\tau^2}{2\kappa^2}\,\mathrm{e}^{-2t/\tau}\right)} \pm \frac{k\tau}{4\kappa^2}\sqrt{\frac{3}{D\epsilon}} \times \arctan\left\{\frac{-[4\kappa^2 D\tau/(3\epsilon)]\,\mathrm{e}^{-2t/\tau} + k\tau/\epsilon}{[4\kappa\sqrt{D}/(\epsilon\sqrt{3})]\sqrt{-(\kappa^2 D\tau^2/3)\,\mathrm{e}^{-4t/\tau} + (k\tau^2/2)\,\mathrm{e}^{-2t/\tau}}}\right\} + \phi_0. \tag{39}$$

The solution above, when assuming $k = 0$, reduces to the solution (21) when $n = 4$, confirming the correctness of the result obtained.

5.2.2 Dust case

The integral (35) in the dust case $n = 3$ reads

$$\psi(x) = \pm\frac{2}{\kappa\sqrt{3}}\int\sqrt{\left(\frac{3}{2}\tau^2\right)^{1/3}\frac{k}{\epsilon}\frac{1}{(x-x_0)^{2/3}} - \frac{3}{4}\frac{\kappa^2 D}{\epsilon}}\,\mathrm{d}x, \tag{40}$$

with solutions

$$\psi(x) = \pm\sqrt{\frac{D}{\epsilon}}\left[\left(\frac{3\tau^2}{2}\right)^{1/3}\frac{4k}{3\kappa^2 D} - (x-x_0)^{2/3}\right]^{3/2} + \psi_0. \tag{41}$$

With similar procedure to the radiation case, using (27), the solution is therefore,

$$\phi(t) = \pm\frac{2\tau}{3D\sqrt{\epsilon}}\left(\frac{2k}{\kappa^2} - D\mathrm{e}^{-t/\tau}\right)^{3/2} + \phi_0, \tag{42}$$

which is the same as Eq. (25) derived from Friedmann formulation. This solution when assuming $k = 0$ is exactly the same as the solution (21) when $n = 3$ (dust fluid). This also confirms that our results from NLS formulation are correct. The NLS solution can solve the case when k and D are non-zero together without knowing n value while the standard procedures in Sect. 4.2 cannot unless assuming a particular value $n = 0, 2, 3, 4, 6$. However, it must be noticed that one cannot reduce the NLS solutions (39) and (42) to the $D = 0$ case directly since there are mixed multiplication term of n and k in the solution and also the value of n has already been put in. Hence setting $D = 0$ in (39) and (42) cannot be considered as a pure scalar field dominant case.

6 Exponential expansion: analysis on effective equation of state coefficient

The exponential expansion in our scenario is caused from mixed effect of fluids and spatial curvature. We discuss mixed effect on equation of state here. Definition of effective equation of state coefficient, $w_{\mathrm{eff}} = (\rho_\phi w_\phi + \rho_\gamma w_\gamma)/\rho_{\mathrm{tot}}$ together with Eqs. (4), (22) and (34) in context of exponential expansion becomes

$$w_{\mathrm{eff}} = \frac{-1 - (k\tau^2/3)\mathrm{e}^{-2t/\tau}}{1 + k\tau^2\mathrm{e}^{-2t/\tau}}, \tag{43}$$

which is infinite when

$$t = \frac{\tau}{2}\ln(-k\tau^2). \tag{44}$$

Infinity can possibly happen only when $k = -1$ because logarithm function forbids negative domain. In order to acquire exponential expansion in flat universe, one needs to have $w_{\text{eff}} = -1$, but this is not true when k term is non-trivial. Therefore we can only express w_ϕ in term of w_{eff} as

$$w_\phi = \frac{[\,(3k/\kappa^2)\mathrm{e}^{-2t/\tau} + 3/(\kappa^2\tau^2)\,]w_{\text{eff}} \; - \; [(n-3)/3]\,D\mathrm{e}^{-nt/\tau}}{(3k/\kappa^2)\mathrm{e}^{-2t/\tau} \; + \; 3/(\kappa^2\tau^2) \; - \; D\mathrm{e}^{-nt/\tau}}, \tag{45}$$

for exponential expansion. The Eq. (43) does not depend on properties (n) or amount (D) of the barotropic fluid. It reduces to $w_{\text{eff}} = -1$ when $k = 0$ as expected. Considering Eq. (45), if $D = 0$ and $k = 0$, it yields $w_\phi = w_{\text{eff}}$ while setting $D = 0$ alone also gives the same result.

7 Power-law expansion

7.1 Bound value of $\phi(t)$ from effective equation of state for $k = 0$ case

In power-law expanding universe, scale factor evolves with time as

$$a(t) = t^q, \tag{46}$$

where $q > 0$ is a constant. In flat ($k = 0$) universe, it is known that the power-law expansion, is attained when $-1 < w_{\text{eff}} < -1/3$ where $q = 2/[3(1 + w_{\text{eff}})]$. The effective equation of state (Eq. 1), hence is a condition

$$-\frac{n}{3}\rho_\gamma < \epsilon\dot{\phi}^2 < \frac{2}{3}\left(\frac{\epsilon\dot{\phi}^2}{2} + V\right) + \left(\frac{2-n}{3}\right)\rho_\gamma, \tag{47}$$

i.e. $0 < \epsilon\dot{\phi}^2 + \rho_\gamma n/3 < (2/3)\rho_{\text{tot}}$. Both values of ϵ can be assigned and the power-law expansion is sustained as long as the condition is satisfied.

7.2 Solution solved from Friedmann formulation

If we directly consider Eq. (9), the solution for power-law expansion is an integration:

$$\phi(t) = \pm\int\sqrt{\frac{1}{\epsilon}\left(\frac{2q}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} - \frac{nD}{3t^{qn}}\right)}\mathrm{d}t. \tag{48}$$

7.2.1 Simplest case

Simplest integration case is when $k = 0$ and $D = 0$. The solution of Eq. (48) is well known [60],

$$\phi(t) = \pm\sqrt{\frac{2q}{\epsilon\kappa^2}}\ln t + \phi_0, \tag{49}$$

provided that q and ϵ have the same sign. Considering power-law inflation, the WMAP five-year combined analysis based on flat and scalar field domination assumption yields $q > 60$ at more than 99 % of confident level otherwise excluded while $q \sim 120$ is at boundary of 68% confident level [19]. These results base on single field model which we can applied the above solution to. When assuming only $k = 0$ with $D \neq 0$, the solution of Eq. (48) is

$$\phi(t) = \pm\frac{1}{qn-2}\sqrt{\frac{2q}{\epsilon\kappa^2}}\left\{\ln\left[\frac{t^{-qn+2}}{\left(1+\sqrt{1-(nD\kappa^2/6q)\,t^{-qn+2}}\right)^2}\right] + 2\sqrt{1-\left(\frac{nD\kappa^2}{6q}\right)t^{-qn+2}} + \ln\left(-\frac{nD\kappa^2}{6q}\right)\right\} + \phi_0, \tag{50}$$

where, when $q = 2/n$, the field has infinite value. The last logarithmic term in the bracket is an integrating constant which is valid only when $q < 0$. To attain power-law expansion, q must be positive. Hence, this term is not defined for power-law expansion. We will see later that the NLS result does not have this problem. For the reverse case, $D = 0$, $k \neq 0$, the solution is

$$\phi(t) = \pm\frac{1}{q-1}\sqrt{\frac{2q}{\epsilon\kappa^2}} \times\left\{\ln\left[\frac{t^{q-1}}{\sqrt{k/q}}\left(1+\sqrt{\left(\frac{k}{q}\right)t^{-2q+2}+1}\right)\right] - \sqrt{\left(\frac{k}{q}\right)t^{-2q+2}+1}\right\} + \phi_0, \tag{51}$$

which becomes infinite when $q = 1$. The values of q, k and ϵ must have the same sign in all terms of the solution otherwise becoming imaginary. Hence, for $q > 0$, the condition for the solution to be valid is $k = 1$ and $\epsilon = 1$.

7.2.2 The case of non-zero k and non-zero D

When considering non-negligible value of both k and D, the Eq. (48) cannot be integrated analytically except when setting $n = 2$ $(w_\gamma = -1/3)$ which is not natural fluid. Hence it is not considered.

8 Power-law expansion: solutions solved with NLS formulation

Power-law expansion cosmology in NLS-type formulation is presented and concluded in [59]. Important functions needed for evaluating the field exact solutions are

$$x = \sigma(t) = -\frac{t^{-\beta}}{\beta} + x_0, \tag{52}$$

$$t(x) = \frac{1}{[-\beta(x-x_0)]^{1/\beta}}, \tag{53}$$

$$\epsilon\dot{\phi}(t)^2 = \frac{2q}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} - \frac{nD}{3t^{qn}}, \tag{54}$$

$$P(x) = \frac{2qn}{(qn-2)^2}\frac{1}{(x-x_0)^2} + \frac{kn}{2}\left[\frac{-2}{(qn-2)(x-x_0)}\right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12}, \tag{55}$$

$$V(t) = \frac{q(3q-1)}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} + \left(\frac{n-6}{6}\right)\frac{D}{t^{qn}}. \tag{56}$$

We use Eq. (55) in Eq. (17), then

$$\psi(x) = \frac{\pm 2}{\kappa\sqrt{n}} \times \int\sqrt{\frac{2qn}{\epsilon(qn-2)^2}\frac{1}{(x-x_0)^2} + \frac{kn}{2\epsilon}\left[\frac{-2}{(qn-2)}\frac{1}{(x-x_0)}\right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12\epsilon}}\,\mathrm{d}x. \tag{57}$$

We consider the solution in cases of $k = 0$ and $k \neq 0$. Recall that setting $D = 0$ cannot be considered as an absence of barotropic fluid due to existence of n in the other terms.

8.1 The case $k = 0$

Solution to the integral (57) for $k = 0$ case is

$$\psi(x) = \pm\sqrt{\frac{8q}{\epsilon\kappa^2(qn-2)^2}}\left\{-\sqrt{1-\left[\frac{\kappa^2 Dn(qn-2)^2}{24q}(x-x_0)^2\right]} + \ln\left[\frac{1+\sqrt{1-\left[\kappa^2 Dn(qn-2)^2/24q\right](x-x_0)^2}}{(x-x_0)}\frac{4qn}{\epsilon(qn-2)^2}\right]\right\}. \tag{58}$$

Transforming to the t variable using Eq. (52), we obtain,

$$\phi(t) = \pm\frac{1}{qn-2}\sqrt{\frac{2q}{\epsilon\kappa^2}}\left\{\ln\left[\frac{t^{-qn+2}}{\left(1+\sqrt{1-(nD\kappa^2/6q)\,t^{-qn+2}}\right)^2}\right] + 2\sqrt{1-\left(\frac{nD\kappa^2}{6q}\right)t^{-qn+2}} + \ln\left(\frac{qn-2}{2qn}\right)^2\right\} + \phi_0. \tag{59}$$

This solution differs from the solution (50) only the last logarithmic term in the bracket which is only an integrating constant. When $q = 2/n$ or $n = 0$, the field has infinite value. The last logarithmic term does not restrict the sign of q. Only q and ϵ must have the same sign for the solution to be real.

8.2 The case $k \neq 0$

In case of non-zero k and non-zero D, the integral (57) cannot be integrated analytically even when assuming each n value except when $n = 2$ which is not natural fluid.

9 Power-law expansion: analysis on effective equation of state coefficient

Similar to the analysis in Sect. 6, mixed effect of the two fluids and spatial curvature results in power-law expansion. The coefficient $w_{\rm eff}$, with Eqs. (4), (54) and (56), reads

$$w_{\rm eff} = \frac{(-3q^2 + 2q)\, t^{2q-2} - k}{3q^2\, t^{2q-2} + 3k}, \tag{60}$$

which becomes infinity if

$$t = \left(\frac{-k}{q^2}\right)^{1/(2q-2)}. \tag{61}$$

We can also express w_ϕ in term of $w_{\rm eff}$ as

$$w_\phi = \frac{[(3q^2/\kappa^2)t^{-2} + (3k/\kappa^2)t^{-2q}\,]w_{\rm eff} \;-\; [(n-3)/3]\, Dt^{-qn}}{(3q^2/\kappa^2)t^{-2} + (3k/\kappa^2)t^{-2q} - Dt^{-qn}}, \tag{62}$$

for power-law expansion. The Eq. (62), when $D = 0$ and $k = 0$, yields $w_\phi = w_{\rm eff}$ as expected. Similar to the case of exponential expansion, setting $D = 0$ alone also yields $w_\phi = w_{\rm eff}$. In flat universe, power-law expansion happens when $w_{\rm eff}$ lies in an interval $(-1, -1/3)$. But in $k \neq 0$ universe, it is no longer true. Considering flat universe, setting $k = 0$ in Eq. (60) yields $q = 2/[3(1 + w_{\rm eff})]$. The condition $-1 < w_{\rm eff} < -1/3$ therefore corresponds to $q > 0$ as known. The condition also yields

$$-1 - (1 + w_\gamma)\frac{\rho_\gamma}{\rho_\phi} \;<\; w_\phi \;<\; -\frac{1}{3} - \left(\frac{1}{3} + w_\gamma\right)\frac{\rho_\gamma}{\rho_\phi}. \tag{63}$$

If there is more non-negligible radiation fluid (with $w_\gamma = 1/3$), it is noticed that the interval shifts to the more left. For example, setting $\rho_\gamma = 0.1\rho_\phi$, the interval shifts to about $-1.133 < w_\phi < -0.4$. If we assume more realistic situation when dust (dark matter and other matter elements) is presented. The dust density and dark energy is about 28% and 72% of total density, therefore $\rho_\gamma \simeq (28/72)\rho_\phi \simeq 0.389\rho_\phi$, the interval is $-1.389 < w_\phi < -0.463$ which covers valid range of recent observational data, assuming dynamical w with flat universe, $-1.38 < w_{\phi,0} < -0.86$ at 95% confident level [19].

10 Conclusions

This letter reports and demonstrates a method of solving for canonical scalar field exact solution in a non-flat FLRW universe with barotropic fluid using NLS-type formulation in comparison with the method in the standard Friedmann framework. We consider phantom and non-phantom scalar field cases with exponential and power-law accelerating expansion. We evaluate all NLS quantities needed to find the solution, e.g. non-normalizable wave function and Schrödinger potential. Our process is reverse to a problem solving in quantum mechanics that the wave function is expressed first by the expansion function, $a(t)$ before evaluating the Schrödinger potential based on a known expansion function. In NLS formulation the total energy E is negative. We do an analysis on effective equation of state to both cases of expansion. We expresses w_{eff} in term of q and k. In a flat universe, in order to have power-law expansion, the interval $(-1, -1/3)$ of the w_ϕ, is shifted leftward to more negative if more barotropic fluid density is presented.

Within framework of the standard Friedmann formulation, we obtained exact solution in various cases. Later we solved the problem using NLS formulation, in which the wave function is equivalent to the scalar field exact solution. NLS method is restricted by the fact that its scalar field solution is valid only when the barotropic fluid density is presented. Setting $D = 0$ does not imply the absence of barotropic fluid because the barotropic fluid parameter n still appears in the other terms of the Schrödinger potential. Therefore NLS formulation cannot be applied to situation when the scalar field is dominant and $D \sim 0$. Hence it is more suitable for a system of dark energy and dust dark matter fluid. This is a disadvantage point of the NLS formulation. Transforming from standard Friedmann formulation to NLS formulation makes n appear in all terms of the integrand and also changes fluid density term D from time-dependent term to a constant E. Hence the number of x(or equivalently t)-dependent terms is reduced by one. This is a good aspect of the NLS. In both Friedmann formulation and NLS formulation, the solutions when $k \neq 0$ and $D \neq 0$ are difficult or might be impossible to solve unless assuming values of q and n. Hence reduction number of x-dependent term helps simplifying the integration. There are also other good aspects of NLS formulation. Firstly, in the case of exponential expansion with NLS formulation, the solution when $k \neq 0$ and $D \neq 0$ can be obtained without assuming n value while $n = 0, 2, 3, 4, 6$ must be given if working within Friedmann formulation. Secondly, for power-law expansion with $k = 0$, the result (59) obtained from NLS formulation has integrating constant that does not restrict q value while (50) obtained from Friedmann formulation needs $q < 0$ which violates power-law expansion condition ($q > 0$). For power-law expansion, the most difficult case is when $k \neq 0$ with $D \neq 0$. In both formulations, the integral cannot be integrated unless assuming $n = 2$ (equivalent to $w_\gamma = -1/3$) which is not a physical fluid. We introduce here alternative method to obtain scalar field exact solution with advantage over and disadvantage to standard Friedmann formulation. The NLS formulation could render more interesting techniques for scalar field cosmology.

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Slow-roll, acceleration, the big rip and the Wentzel–Kramers–Brillouin approximation in the non-linear Schrödinger-type formulation of scalar field cosmology

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Abstract. Aspects of the non-linear Schrödinger (NLS) type of formulation of scalar (phantom) field cosmology for slow-roll, acceleration, the Wentzel–Kramers–Brillouin (WKB) approximation and the big rip singularity are presented. Slow-roll parameters for the curvature and barotropic density terms are introduced. We re-express all slow-roll parameters, slow-roll conditions and the acceleration condition in NLS form. The WKB approximation in the NLS formulation is also discussed while simplifying to the linear case. Most of the Schrödinger potentials in the NLS formulation are very slowly varying; hence the WKB approximation is valid in some ranges. In the NLS form of the big rip singularity, two quantities are infinite instead of three. We also found that approaching the big rip, $w_{\rm eff} \to -1 + 2/3q$ ($q < 0$), which is the same as the effective phantom equation of state in the flat case.

Keywords: dark energy theory, inflation

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Contents

1. Introduction

Cosmology with a scalar field is one of today's main research aims. Although the scalar field has not yet been observed, it is motivated from many ideas in high energy physics and quantum gravities. Near future TeV scale experiments at LHC and Tevatron might discover its existence. It has been widely accepted in theoretical frameworks, especially in model building of contemporary cosmology, that the field sources acceleration expansion at early time, i.e. inflation, in order to solve horizon and flatness problems [1] and it also plays a similar role in explaining present acceleration observed and confirmed from the cosmic microwave background [2], large scale structure surveys [3] and supernovae of type Ia [4]–[6]. In the late acceleration, it plays the role of dark energy (see [7] for reviews). Both inflation and acceleration appear convincing from recent combined results [8], with the possibility that the scalar field could be a phantom, i.e. having an equation of state coefficient $w_\phi < -1$. The phantom equation of state is attained from negative kinetic energy terms in the Lagrangian density [9, 10]. Using the BBN constraint of the limit of the expansion rate [11, 12] with the most recent WMAP five-year result [13], $w_{\phi,0} = -1.09 \pm 0.12$ at 68% CL. The WMAP five-year result combined with baryon acoustic oscillation from the large scale structure survey (from SDSS and 2dFGRS) [14] and type Ia supernovae data (from HST [5], SNLS [6] and ESSENCE [15]), assuming dynamical w with a flat universe, yields $-1.38 < w_{\phi,0} < -0.86$ at 95% CL and $w_{\phi,0} = -1.12 \pm 0.13$ at 68% CL. Although the phantom field has room for observation, in a flat universe the idea suffers from an unwanted big rip singularity [16, 17]. However there have been many attempts to resolve the singularity from both phenomenological and fundamental inspiration [18].

Inflationary models in the presence of other fields behaving in a barotropic-like way apart from having only a single scalar field were considered, such as that in [19] where the scale invariant spectrum in the cosmic microwave background was claimed to be generated not from fluctuation of the scalar field alone but rather from both a scalar field and interaction between gravity and other gauge fields such as Dirac and gauge vector fields. This is similar to the situation in the late universe in which the acceleration happens in the presence of both dark matter fluid and scalar fluid (as dark energy). Proposals of mathematical alternatives to the standard Friedmann canonical scalar field cosmology with barotropic perfect fluid were advanced, such as the non-linear Ermakov–Pinney equation [20, 21]. There are also other applications of the Ermakov–Pinney equation; for example in [22], a link from standard cosmology with $k > 0$ in the Ermakov system to Bose–Einstein condensates was shown. Another example is a connection from the generalized Ermakov–Pinney equation with a perturbative scheme to the generalized WKB method of comparison equations [23]. Recently a link from standard canonical scalar field cosmology in the Friedmann–Lemaître–Robertson–Walker (FLRW) background with barotropic fluid to quantum mechanics was established. This was realized from the fact that solutions of the generalized Ermakov–Pinney equation correspond to solutions of the non-linear Schrödinger-type equation, hereafter the NLS equation [21, 24]. A connection from the NLS-type formulation to the Friedmann scalar field cosmology formulation is concluded in [25] where standard cosmological quantities are reinterpreted in the language of quantum mechanics assuming power-law expansion, $a \sim t^q$, and the phantom field case is included. The quantities in the new form satisfy a non-linear Schrödinger-type equation. In most circumstances, the scalar field exact solution $\phi(t)$ can be solved analytically only when assuming flat geometry ($k = 0$) and scalar field fluid domination. When $k \neq 0$ with more than one fluid component, it is not always possible to solve the system analytically in the standard Friedmann formulation. Transforming standard Friedmann cosmological quantities into NLS forms could help in obtaining the solution [26, 27]. In the NLS formulation, the independent variable t in the standard formulation is re-scaled to the variable x. However, pre-knowledge of the scale factor as a function of time, $a(t)$, must be assumed in order to express NLS quantities. It is interesting to see the other features of the field velocity, $\dot{\phi}$, e.g. the acceleration condition and slow-roll approximation, written in the NLS formulation. Mathematical tools such as the WKB approximation in quantum mechanics might also be interesting for application in standard scalar field cosmology. It is worthwhile to investigate this possibility. It is worth noting that a Schrödinger-type equation in scalar field cosmology was previously considered in a different procedure for studying inflation and phantom field problems [28].

We introduce the NLS formulation in section 2. The slow-roll conditions in both formulations are discussed in section 3 where we define slow-roll parameters for barotropic fluid and curvature terms. Then in section 4 we show acceleration conditions in NLS form. The WKB approximation is performed in section 5. The NLS form of the big rip singularity is in section 6 and finally conclusions are drawn in section 7.

2. Scalar field cosmology in the NLS formulation

Two perfect fluids are considered in an FLRW universe: barotropic fluid and the scalar field. The barotropic equation of state is $p_\gamma = w_\gamma \rho_\gamma$ with w_γ expressed as n where

$n = 3(1 + w_\gamma)$. The scalar field pressure obeys $p_\phi = w_\phi \rho_\phi$. The total density and pressure of the mixture are sums of the two components. The evolution of the barotropic density is governed by a conservation equation, $\dot{\rho}_\gamma = -nH\rho_\gamma$, with solution $\rho_\gamma = D/a^n$, where a is a scale factor, the dot denotes the time derivative, and $D \geq 0$ is a proportionality constant. Using the scalar field Lagrangian density, $\mathcal{L} = (1/2)\epsilon\dot{\phi}^2 - V(\phi)$, i.e. minimally coupling to gravity,

$$\rho_\phi = \tfrac{1}{2}\epsilon\dot{\phi}^2 + V(\phi), \qquad p_\phi = \tfrac{1}{2}\epsilon\dot{\phi}^2 - V(\phi). \tag{1}$$

The branch $\epsilon = 1$ is for the non-phantom case and $\epsilon = -1$ is for the phantom case [17]. Note that the phantom behaviour ($\rho_\phi < -p_\phi$) can also be obtained in the case of non-minimal coupling to gravity [29]. The dynamics of the field is controlled by the conservation equation

$$\epsilon\left(\ddot{\phi} + 3H\dot{\phi}\right) = -\frac{\mathrm{d}V}{\mathrm{d}\phi}. \tag{2}$$

The spatial expansion of the universe sources friction to dynamics of the field in equation (2) via the Hubble parameter H. The Hubble parameter is governed by the Friedmann equation,

$$H^2 = \frac{\kappa^2}{3}\rho_{\mathrm{tot}} - \frac{k}{a^2}, \tag{3}$$

where here $\rho_{\mathrm{tot}} = (1/2)\epsilon\dot{\phi}^2 + V + D/a^n$, and by the acceleration equation,

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}(\rho_{\mathrm{tot}} + 3p_{\mathrm{tot}}), \tag{4}$$

which does not depend on k. This gives the acceleration condition

$$p_{\mathrm{tot}} < -\frac{\rho_{\mathrm{tot}}}{3}. \tag{5}$$

Here $p_{\mathrm{tot}} = w_{\mathrm{eff}}\rho_{\mathrm{tot}}$, $\kappa^2 \equiv 8\pi G = 1/M_{\mathrm{P}}^2$, G is Newton's gravitational constant, M_{P} is the reduced Planck mass, k is the spatial curvature and $w_{\mathrm{eff}} = (\rho_\phi w_\phi + \rho_\gamma w_\gamma)/\rho_{\mathrm{tot}}$. Using these facts, it is straightforward to show that

$$\epsilon\dot{\phi}(t)^2 = -\frac{2}{\kappa^2}\left[\dot{H} - \frac{k}{a^2}\right] - \frac{nD}{3a^n}, \tag{6}$$

$$V(\phi) = \frac{3}{\kappa^2}\left[H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right] + \left(\frac{n-6}{6}\right)\frac{D}{a^n}. \tag{7}$$

The Friedmann formulation of scalar field cosmology above can be transformed to the NLS formulation as one defines NLS quantities [24]:

$$u(x) \equiv a(t)^{-n/2}, \tag{8}$$

$$E \equiv -\frac{\kappa^2 n^2}{12}D, \tag{9}$$

$$P(x) \equiv \frac{\kappa^2 n}{4}a(t)^n\epsilon\dot{\phi}(t)^2. \tag{10}$$

In the NLS formulation, there are no equations analogous to the Friedmann equation or the fluid equation since both of them are written together in the form of a non-linear Schrödinger-type equation[1]:

$$u''(x) + [E - P(x)]\, u(x) = -\frac{nk}{2} u(x)^{(4-n)/n}, \tag{11}$$

where $'$ denotes $\mathrm{d}/\mathrm{d}x$. The independent variable t is scaled to an NLS-independent variable x as $x = \sigma(t)$, such that

$$\dot{x}(t) = u(x), \tag{12}$$

$$\phi(t) = \psi(x). \tag{13}$$

Using equation (10) and $\epsilon\dot{\phi}(t)^2 = \epsilon\dot{x}^2\psi'(x)^2$, we get [25]

$$\epsilon\psi'(x)^2 = \frac{4}{\kappa^2 n} P(x), \tag{14}$$

and hence

$$\psi(x) = \pm\frac{2}{\kappa\sqrt{n}} \int \sqrt{\frac{P(x)}{\epsilon}}\, \mathrm{d}x. \tag{15}$$

The inverse function $\psi^{-1}(x)$ exists for $P(x) \neq 0$ and $n \neq 0$. In these circumstances, $x(t) = \psi^{-1} \circ \phi(t)$ and the scalar field potential, $V \circ \sigma^{-1}(x)$, and $\epsilon\dot{\phi}(t)^2$ can be expressed in the NLS formulation as

$$\epsilon\dot{\phi}(x)^2 = \frac{4}{\kappa^2 n} u u'' + \frac{2k}{\kappa^2} u^{4/n} + \frac{4E}{\kappa^2 n} u^2 = \frac{4P}{\kappa^2 n} u^2, \tag{16}$$

$$V(x) = \frac{12}{\kappa^2 n^2} (u')^2 - \frac{2P}{\kappa^2 n} u^2 + \frac{12E}{\kappa^2 n^2} u^2 + \frac{3k}{\kappa^2} u^{4/n}. \tag{17}$$

From equations (16) and (17), we can find

$$\rho_\phi = \frac{12}{\kappa^2 n^2} (u')^2 + \frac{12E}{\kappa^2 n^2} u^2 + \frac{3k}{\kappa^2} u^{4/n}, \tag{18}$$

$$p_\phi = -\frac{12}{\kappa^2 n^2} (u')^2 + \frac{4P}{\kappa^2 n} u^2 - \frac{12E}{\kappa^2 n^2} u^2 - \frac{3k}{\kappa^2} u^{4/n}. \tag{19}$$

We know that $\rho_\gamma = Du^2 = -12Eu^2/(\kappa^2 n^2)$ from equation (9) and the barotropic pressure is $p_\gamma = [(n-3)/3]\rho_\gamma$; therefore

$$\rho_{\mathrm{tot}} = \frac{12}{\kappa^2 n^2} (u')^2 + \frac{3k}{\kappa^2} u^{4/n}, \tag{20}$$

$$p_{\mathrm{tot}} = -\frac{12}{\kappa^2 n^2} (u')^2 + \frac{4u^2}{\kappa^2 n} [P - E] - \frac{3k}{\kappa^2} u^{4/n}. \tag{21}$$

Using the Schrödinger-type equation (11), then

$$p_{\mathrm{tot}} = -\frac{12}{\kappa^2 n^2} (u')^2 + \frac{4}{\kappa^2 n} u u'' - \frac{k}{\kappa^2} u^{4/n}. \tag{22}$$

[1] The NLS equation considered here is only dependent on x; hence it is not a partial differential equation with a localized soliton-like solution as in [30].

3. Slow-roll conditions

3.1. Slow-roll conditions: flat geometry and scalar field domination

In a flat universe with scalar field domination ($k = 0, \rho_\gamma = 0$), the Friedmann equation $H^2 = \kappa^2\rho_\phi/3$, together with equation (2), yields $\dot{H} = -\kappa^2\dot{\phi}^2\epsilon/2$. For $\epsilon = -1$, we get $\dot{H} > 0$ and

$$0 < aH^2 < \ddot{a}, \tag{23}$$

i.e. the acceleration is greater than the speed of expansion per Hubble radius, $\dot{a}/cH^{-1}$. On the other hand, for $\epsilon = 1$, we get $\dot{H} < 0$ and

$$0 < \ddot{a} < aH^2. \tag{24}$$

The slow-roll condition in [31, 32] assumes a negligible kinetic term; hence $|\epsilon\dot{\phi}^2/2| \ll V(\phi)$, and therefore $\rho_\phi \simeq V(\phi)$. Hence $H^2 \simeq \kappa^2 V/3$. With this approximation,

$$H^2 = -\frac{\dot{H}}{3} + \frac{\kappa^2}{3}V, \quad \Rightarrow \quad H^2 \simeq -\frac{\dot{H}}{3} + H^2. \tag{25}$$

This results in an approximation $|\dot{H}| \ll H^2$ from which the slow-roll parameter

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} \tag{26}$$

is defined. Then the condition $|\epsilon\dot{\phi}^2/2| \ll V(\phi)$ is equivalent to $|\varepsilon| \ll 1$, i.e. $-1 \ll \varepsilon < 0$ for the phantom field case and $0 < \varepsilon \ll 1$ for the non-phantom field case. For the non-phantom field, this condition is necessary for inflation to happen (though not sufficient) [32], but for the phantom field case, the slow-roll condition is not needed because the negative kinetic term results in acceleration with $w_\phi \leq -1$. The other slow-roll parameter is defined by balancing the magnitudes of the field friction and acceleration terms in equation (2). This is independent of k or ρ_γ. When friction dominates, $|\ddot{\phi}| \ll |3H\dot{\phi}|$, then

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{27}$$

is defined [32]. The condition is then $|\eta| \ll 1$ and the fluid equation is approximated to $\dot{\phi} \simeq -V_\phi/3\epsilon H$ which allows the field to roll up the hill when $\epsilon = -1$. Using two conditions, e.g. $|\epsilon\dot{\phi}^2/2| \ll V$ and $|\ddot{\phi}| \ll |3H\dot{\phi}|$, together, one can derive $\varepsilon = (1/2\kappa^2\epsilon)(V_\phi/V)^2$ and $\eta = (1/\kappa^2)(V_{\phi\phi}/V)$ as is well known.

3.2. Slow-roll conditions: non-flat geometry and non-negligible barotropic density

3.2.1. The Friedmann formulation. When considering the case of $k \neq 0$ and $\rho_\gamma \neq 0$, then

$$\dot{H} = -\frac{\kappa^2}{2}\dot{\phi}^2\epsilon + \frac{k}{a^2} - \frac{n\kappa^2}{6}\frac{D}{a^n}. \tag{28}$$

We can then write the slow-roll condition as $|\kappa^2\epsilon\dot{\phi}^2/6| \ll (\kappa^2 V/3) - (k/a^2) + (\kappa^2 D/3a^n)$ and hence $H^2 \simeq (\kappa^2 V/3) + (\kappa^2 D/3a^n) - (k/a^2)$. Using this approximation and equation (28) in (3),

$$H^2 \simeq -\frac{\dot{H}}{3} + \frac{k}{3a^2} - \frac{n\kappa^2}{18}\frac{D}{a^n} + H^2, \tag{29}$$

which implies $|-(\dot{H}/3) + (k/3a^2) - (n\kappa^2 D/18a^n)| \ll H^2$. We can re-express this slow-roll condition as

$$|\varepsilon + \varepsilon_k + \varepsilon_D| \ll 1, \tag{30}$$

where $\varepsilon_k \equiv k/a^2H^2$ and $\varepsilon_D \equiv -n\kappa^2 D/6a^n H^2$. Another slow-roll parameter, η, is defined as $\eta \equiv -\ddot{\phi}/H\dot{\phi}$, i.e. the same as for the flat scalar field dominated case since the condition $|\ddot{\phi}| \ll |3H\dot{\phi}|$ is derived from the fluid equation of the field which is independent of k and ρ_γ.

3.2.2. The NLS formulation. In the NLS formulation, the Hubble parameter takes the form

$$H = -\frac{2}{n}u', \tag{31}$$

with

$$\dot{H} = -\frac{2}{n}uu'' = \frac{2}{n}u^2\left[E - P(x)\right] + ku^{4/n}. \tag{32}$$

The slow-roll condition $|\epsilon\dot{\phi}^2/2| \ll V$ using equations (10) and (17) in NLS form is then

$$|P(x)| \ll \frac{3}{n}\left[\left(\frac{u'}{u}\right)^2 + E\right] + \frac{3}{4}knu^{(4-2n)/n}. \tag{33}$$

If the absolute sign is not used, the condition is then $\epsilon\dot{\phi}^2/2 \ll V$, allowing fast-roll negative kinetic energy. Then equation (33), when combined with the NLS equation (11), yields

$$u'' \ll \frac{3}{n}\frac{u'^2}{u} + \left(\frac{3}{n} - 1\right)Eu + \frac{kn}{4}u^{(4-n)/n}. \tag{34}$$

The Friedmann formulation analogue of this condition can be obtained simply by using equations (6) and (7) in the condition. Consider another aspect of slow-roll in the fluid equation; the field acceleration can be written in NLS form:

$$\ddot{\phi} = \frac{2Puu' + P'u^2}{\kappa\sqrt{P\epsilon n}}, \tag{35}$$

while the friction term in NLS form is

$$3H\dot{\phi} = -\frac{12u'u}{n\kappa}\sqrt{\frac{P}{\epsilon n}}. \tag{36}$$

The second slow-roll condition, $|\ddot{\phi}| \ll |3H\dot{\phi}|$, hence corresponds to

$$\left|\frac{P'}{P}\right| \ll \left|-2\left(\frac{6+n}{n}\right)\frac{u'}{u}\right|. \tag{37}$$

This condition yields the approximation $3H\epsilon\dot{\phi}^2 \simeq -\mathrm{d}V/\mathrm{d}\phi$. Using equations (16), (17), (31) and (32), one can express the approximation, $3H\epsilon\dot{\phi}^2 \simeq -\mathrm{d}V/\mathrm{d}\phi$, in NLS form as

$$\frac{P'}{P} \simeq -\frac{2u'}{u} = nHa^{n/2} \tag{38}$$

and finally the slow-roll parameters ε, ε_k and ε_D, introduced previously, become

$$\varepsilon = \frac{nuu''}{2u'^2}, \qquad \varepsilon_k = \frac{n^2ku^{4/n}}{4u'^2}, \qquad \varepsilon_D = \frac{nE}{2}\left(\frac{u}{u'}\right)^2, \tag{39}$$

in NLS form. With the help of NLS equation (11), the sum of the slow-roll parameters takes a simple form:

$$\varepsilon_{\mathrm{tot}} = \varepsilon + \varepsilon_k + \varepsilon_D = \frac{n}{2}\left(\frac{u}{u'}\right)^2 P(x). \tag{40}$$

Finally the slow-roll condition $|\varepsilon_{\mathrm{tot}}| \ll 1$ (equation (30)), in NLS form, is

$$\left|\left(\frac{u}{u'}\right)^2 P(x)\right| \ll 1. \tag{41}$$

Another slow-roll parameter, $\eta = -\ddot{\phi}/H\dot{\phi}$, can be found as follows. First considering $\psi(x) = \phi(t)$ (equation (13)), using the relation $\mathrm{d}/\mathrm{d}t = \dot{x}\,\mathrm{d}/\mathrm{d}x$ and equation (31), we obtain

$$\eta = \frac{n}{2}\left(\frac{u}{u'}\frac{\psi''}{\psi'} + 1\right). \tag{42}$$

Equation (15) yields

$$\psi' = \pm\frac{2}{\kappa}\sqrt{\frac{P}{n\epsilon}} \qquad \text{and} \qquad \psi'' = \pm\frac{P'}{\kappa\sqrt{nP\epsilon}}. \tag{43}$$

Hence

$$\eta = \frac{n}{2}\left(\frac{u}{u'}\frac{P'}{2P} + 1\right). \tag{44}$$

Finally, the slow-roll condition $|\eta| \ll 1$ then reads

$$\left|\frac{u}{u'}\frac{P'}{2P} + 1\right| \ll 1. \tag{45}$$

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4. The acceleration condition

The slow-roll condition is useful for a non-phantom field because it is a necessary condition for inflating acceleration. However, in the case of a phantom field, the kinetic term is always negative and could take any large negative value; hence the slow-roll condition is not necessary for the acceleration condition. More generally, to ensure acceleration, equation (4) must be positive. It is straightforward to show that, obeying the acceleration condition $\ddot{a} > 0$, equation (5) takes the form

$$\epsilon\dot{\phi}(x)^2 < -\left(\frac{n-2}{2}\right)\frac{D}{a^n} + V. \tag{46}$$

With equations (8)–(10) and (17), the acceleration condition (46) in the NLS-type formulation is

$$E - P > -\frac{2}{n}\left(\frac{u'}{u}\right)^2 - \frac{nk}{2}\left(\frac{u^{2/n}}{u}\right)^2. \tag{47}$$

With the help of the non-linear Schrödinger-type equation (11), it is simplified to

$$u'' < \frac{2}{n}\frac{u'^2}{u}. \tag{48}$$

Using equations (31) and (32), the acceleration condition is just $\varepsilon < 1$ without using any slow-roll assumptions.

5. The WKB approximation

The WKB approximation can be assumed when the coefficient of the highest order derivative term in the Schrödinger equation is small or when the potential is very slowly varying. Equation (11) when $k = 0$ is linear. It is then

$$-\frac{1}{n}u'' + \left[\tilde{P}(x) - \tilde{E}\right]u = 0, \tag{49}$$

where $\tilde{P}(x) \equiv P(x)/n$ and $\tilde{E} \equiv E/n$. For a slowly varying $P(x)$ with the assumption $n \gg 1$, the solution of equation (49) can be written as $u(x) \simeq A\exp[\pm inW_0(x)]$, where $W_0(x) = W(x_0)$ is the lowest order term in the Taylor expansion of the function $W(x)$ in $(1/n)$ about $x = x_0$,

$$W(x) = W(x_0) + W'(x_0)\frac{(x-x_0)}{n} + \cdots. \tag{50}$$

Then an approximation

$$W(x) = \pm\frac{1}{n}\int_{x_1}^{x_2} k(x)\,\mathrm{d}x \simeq W_0(x) \tag{51}$$

is made by analogy to the method in time-independent quantum mechanics. The Schrödinger wavenumber is hence

$$k(x) = \frac{2\pi}{\lambda(x)} = \sqrt{n\left[\tilde{E} - \tilde{P}(x)\right]}, \tag{52}$$

and the small variation in $\lambda(x)$ is

$$\frac{\delta\lambda}{\lambda(x)} = \left| \frac{\pi\tilde{P}'}{\sqrt{n}\left[\tilde{E}-\tilde{P}(x)\right]^{3/2}} \right| = \left| \frac{\pi P'}{[E-P(x)]^{3/2}} \right|. \tag{53}$$

For the WKB approximation, $\delta\lambda/\lambda(x) \ll 1$. In the real universe, we have $n=3$ (dust) or $n=4$ (radiation) which is not much greater than 1. However, if we are considering a range of very slowly varying potentials, $P' \simeq 0$ implying $\delta P/\delta x \sim 0$; hence $\delta k/\delta x \sim 0 \sim W'(x)$. Therefore $W(x) \simeq W_0(x)$ still holds in this range. Since $u(x) = a^{-n/2}$, using the WKB approximation, we get

$$a \sim A \exp\left[\pm(2/n)\mathrm{i}\int_{x_1}^{x_2} \sqrt{E-P(x)}\,\mathrm{d}x\right], \tag{54}$$

where A is a constant. Examples of Schrödinger potentials for exponential, power-law and phantom expansions are derived in [25]–[27]. These potentials are steep only in some small particular region but very slowly varying in most regions, especially at large values of $|x|$ for which the WKB approximation applies well.

6. The big rip singularity

When the field becomes a phantom, i.e. $\epsilon=-1$, in a flat FRLW universe it leads to a future big rip singularity [16, 17]. In a flat universe, when $w_{\mathrm{eff}} < -1$, i.e. for a phantom, the expansion obeys $a(t) \sim (t_{\mathrm{a}}-t)^q$, where $q = 2/3(1+w_{\mathrm{eff}}) < 0$ is a constant over time and t_{a} is a finite time[2]. The NLS phantom expansion was studied in [27] with inclusion of the non-zero k case. Therein, the same expansion function is assumed, with constant $q<0$, and x is related to the cosmic timescale, t, as $x(t) = (1/\beta)(t_{\mathrm{a}}-t)^{-\beta} + x_0$, so $u(x) = [\beta(x-x_0)]^{\alpha}$. Here $\alpha \equiv qn/(qn-2)$ and $\beta \equiv (qn-2)/2$, with the conditions $0<\alpha<1$ and $\beta<-1$ since $n>0$ always. The first and second x derivatives of u are[3]

$$u'(x) = \alpha\beta[\beta(x-x_0)]^{\alpha-1}, \tag{55}$$

$$u''(x) = \alpha(\alpha-1)\beta^2[\beta(x-x_0)]^{\alpha-2}, \tag{56}$$

where the exponents $\alpha-1$ and $\alpha-2$ are always negative. Using equations (20) and (22), then

$$\rho_{\mathrm{tot}} = \frac{12\alpha^2\beta^2}{\kappa^2 n^2}[\beta(x-x_0)]^{2(\alpha-1)} + \frac{3k}{\kappa^2}[\beta(x-x_0)]^{4\alpha/n}, \tag{57}$$

$$p_{\mathrm{tot}} = \frac{4\beta^2}{\kappa^2 n}[\beta(x-x_0)]^{2(\alpha-1)}\left[\left(1-\frac{3}{n}\right)\alpha^2-\alpha\right] - \frac{k}{\kappa^2}[\beta(x-x_0)]^{4\alpha/n} \tag{58}$$

$$= \frac{4u'^2}{\kappa^2 n}\left[\left(1-\frac{3}{n}\right)-\frac{1}{\alpha}\right] - \frac{k}{\kappa^2}[\beta(x-x_0)]^{4\alpha/n}. \tag{59}$$

[2] The relation $q = 2/3(1+w_{\mathrm{eff}}) < 0$ holds only when $k=0$.

[3] Note that $(x-x_0)$ and β are negative hence $(x-x_0)^{\alpha}$, β^{α}, $(x-x_0)^{\alpha-1}$ and $\beta^{\alpha-1}$ are imaginary.

The big rip, $(a, \rho_{\rm tot}, |p_{\rm tot}|) \to \infty$, happens when $t \to t_{\rm a}^{-}$. In the NLS formulation, if $a \to \infty$, then $u \to 0^{+}$ (equation (8)). From above, we see that the conditions for the big rip singularity are

$$\begin{aligned} & t \to t_{\rm a}^{-} \Leftrightarrow x \to x_0^{-}, \\ & a \to \infty \Leftrightarrow u(x) \to 0^{+}, \\ & \rho_{\rm tot} \to \infty \Leftrightarrow u'(x) \to \infty, \\ & |p_{\rm tot}| \to \infty \Leftrightarrow u'(x) \to \infty. \end{aligned} \tag{60}$$

The effective equation of state $w_{\rm eff} = p_{\rm tot}/\rho_{\rm tot}$ can also be stated in NLS language as a function of x. Approaching the big rip, $x \to x_0^{-}$, and the effective equation of state approaches a value

$$\lim_{x \to x_0^{-}} w_{\rm eff} = \frac{n}{3}\left(1 - \frac{1}{\alpha}\right) - 1 = -1 + \frac{2}{3q}, \tag{61}$$

which is similar to the equation of state in the flat case.

7. Conclusions

We feature cosmological aspects of the NLS formulation of scalar field cosmology such as slow-roll conditions, the acceleration condition and the big rip. We conclude on these aspects in the standard Friedmann formulation before deriving them in the NLS formulation. We consider a non-flat FRLW universe filled with a scalar (phantom) field and barotropic fluid because, in the presence of a barotropic fluid density, the NLS-type formulation is consistent [26]. We obtain all NLS versions of the slow-roll parameters, the slow-roll conditions and the acceleration condition. This provides analytical tools in the NLS formulation. For a phantom field, due to its negative kinetic term, the slow-roll condition is not needed. When the NLS system is simplified to a linear equation (this happens when $k = 0$) as in time-independent quantum mechanics, we can apply the WKB approximation to the problem. When $n \gg 1$, the wavefunction is semi-classical, which is suitable for WKB approximation use. However, this does not work, since physically n cannot be much greater than unity, e.g. $n = 3$ for dust and $n = 4$ for radiation. However, the WKB approximation can still be clearly valid for a range of very slowly varying Schrödinger potentials $P(x)$ which were illustrated in [25]–[27]. Using the WKB approximation, we obtain an approximated scale factor function (equation (54)). For a flat universe with phantom expansion, the big rip singularity is its final fate. When the big rip happens, three quantities $(a(t), p(t)$ and $\rho(t))$ become infinite. Rewriting the singularity in NLS form (equation (60)), we can remove one infinity (see equation (60)). We found that near the big rip, $w_{\rm eff} \to -1 + 2/3q$ where $q < 0$ is a constant exponent of the expansion $a(t) \sim (t_{\rm a} - t)^{q}$. This limit is the same as the effective phantom equation of state in the case $k = 0$.

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RESEARCH ARTICLE

Phantom expansion with non-linear Schrödinger-type formulation of scalar field cosmology

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Abstract We describe non-flat standard Friedmann cosmology of canonical scalar field with barotropic fluid in form of non-linear Schrödinger-type (NLS) formulation in which all cosmological dynamical quantities are expressed in term of Schrödinger quantities as similar to those in time-independent quantum mechanics. We assume the expansion to be superfast, i.e. phantom expansion. We report all Schrödinger-analogous quantities to scalar field cosmology. Effective equation of state coefficient is analyzed and illustrated. We show that in a non-flat universe, there is no fixed $w_{\rm eff}$ value for the phantom divide. In a non-flat universe, even $w_{\rm eff} > -1$, the expansion can be phantom. Moreover, in open universe, phantom expansion can happen even with $w_{\rm eff} > 0$. We also report scalar field exact solutions within frameworks of the Friedmann formulation and the NLS formulation in non-flat universe cases.

Keywords Scalar field cosmology · Phantom expansion · Non-linear Schrödinger-type formulation of cosmology

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1 Introduction

Supernovae Type Ia data and cosmic microwave background observations show recently strong evidence of present accelerating phase of the universe [1–12] while nowadays inflationary paradigm in the early universe is one of the corner stones in cosmology [13–18]. Present acceleration and inflation of the universe are both believed to result from effect of either dynamical scalar field with time-dependent equation of state coefficient $w_\phi(t) < -1/3$ or a cosmological constant with $w = -1$. Alternative explanation for present acceleration to dark energy is modification of general relativity which includes braneworld models (for review, see [19] and references therein). Among these ideas, the scalar field catches most attention therefore many analysis in cosmological contexts and observations have been carried out [20–22]. Conventional formulation of canonical scalar field cosmology with barotropic perfect fluid, can also be expressed as non-linear Ermakov-Pinney equation as shown recently [23–28]. However, non-Ermakov-Pinney equation for such system can also be written in form of a non-linear Schrödinger-type equation (NLS). The solutions of the NLS-type equation correspond to solutions of the generalized Ermakov-Pinney equation of scalar field cosmology [28,29]. The NLS-type formulation was concluded and shown in case of power-law expansion in Ref. [30] where all Schrödinger-type quantities corresponding to scalar field cosmology are worked out. NLS-type formulation also provides an alternative way of solving for the scalar field exact solutions in various cases even with non-zero curvature [31].

Various observations allow scalar field equation of state coefficient, w_ϕ to be less than -1 [33–35]. Previous evidence from combined cosmic microwave background, large scale structure survey and supernovae type Ia without assuming flat universe yields $w_\phi = -1.06^{+0.13}_{-0.08}$ [36] while using supernovae data alone assuming flat universe yields $w_\phi = -1.07 \pm 0.09$ [37]. The most recent WMAP five-year result [38,39] combined with Baryon acoustic oscillation (BAO) of large scale structure survey: SDSS and 2dFGRS [40] and type Ia supernovae data from HST [10,11], SNLS [12] and ESSENCE [37] assuming dynamical w with flat universe yields $-1.33 < w_{\phi,0} < -0.79$ at 95% confident level [41]. Also this data with additional BBN constraint of limit of expansion rate [42,43] yields $-1.29 < w_{\phi,0} < -0.79$ at 95% confident level and $w_{\phi,0} = -1.04 \pm 0.13$ at 68% confident level [41]. This suggests that the scalar field could be phantom, i.e $w_\phi < -1$ [44–46]. For a canonical scalar field, phantom behavior can be attained by negative kinetic energy term of the scalar field Lagrangian density. In FLRW general relativistic cosmology, there is a Big Rip singularity with $a, \rho, |p| \to \infty$ at finite time [47–59], nevertheless singularity avoidance has been attempted in various ways [60–69]. Extension to include phantom field case in NLS-type formulation was made in [30]. In NLS-type formulation one can presume any law of expansion $a = a(t)$, e.g. power law $a \sim t^q$ or exponential expansion, $a \sim \exp(t/\tau)$ [30,31] and works out all NLS-quantities keeping open possibility for the field to be phantom or non-phantom and non-zero spatial curvature. Analogous studies to the slow-roll, WKB and the Big Rip in NLS formulation were done in [32].

To attain accelerating expansion, one needs to have effective equation of state coefficient, $w_{\rm eff} < -1/3$ where

$$w_{\text{eff}} = \frac{\rho_\phi w_\phi + \rho_\gamma w_\gamma}{\rho_{\text{tot}}}\,, \tag{1}$$

ρ_γ is density of barotropic fluid, ρ_ϕ is density of the scalar field and $\rho_{\text{tot}} = \rho_\phi + \rho_\gamma$. It has been known in standard cosmology that for flat universe ($k = 0$), if the expansion is $a \sim t^q$, then $-1 < w_{\text{eff}} < -1/3$; if $a \sim \exp(t/\tau)$, then $w_{\text{eff}} = -1$ and if $a \sim (t_{\text{a}} - t)^q$, then $w_{\text{eff}} < -1$. Here $q \equiv 2/[3(1+w_{\text{eff}})]$, τ, t_{a} are finite characteristic times. In the last case, $w_{\text{eff}} < -1$ corresponds to $q < 0$.

In this work, we consider phantom expansion $a \sim (t_{\text{a}} - t)^q$ in the NLS-type formulation with non-zero curvature k. We introduce cosmological system in Sect. 2, then NLS-type formulation in Sect. 3. The Schrödinger quantities for phantom expansion are presented in Sect. 4 where we analyze value of w_{eff} and show conditions of how much negative w_ϕ must be in order to keep the expansion phantom. We also illustrate parametric plots for w_{eff} with q and t. Scalar field exact solutions solved from both standard formulation and NLS-type formulation are given in Sect. 5 where we comment on both procedures of obtaining the solutions. Finally we conclude our work in Sect. 6.

2 Cosmological system

Barotropic fluid and scalar field fluid are major components in our scenario. The perfect barotropic fluid pressure p_γ and density ρ_γ obey an equation of state, $p_\gamma = (\gamma - 1)\rho_\gamma = w_\gamma \rho_\gamma$ while for scalar field, $p_\phi = w_\phi \rho_\phi$. Total density and total pressure are $\rho_{\text{tot}} = \rho_\gamma + \rho_\phi$ and $p_{\text{tot}} = p_\gamma + p_\phi$. For the barotropic fluid, w_γ is written in term of n. We set $w_\gamma \equiv (n-3)/3$ so that $n = 3(1 + w_\gamma) = 3\gamma$, hence $w_\gamma = -1$ corresponds to $n = 0$, $w_\gamma = -1/3$ to $n = 2$, $w_\gamma = 0$ to $n = 3$, $w_\gamma = 1/3$ to $n = 4$, and $w_\gamma = 1$ to $n = 6$. The conservation equation is hence

$$\dot{\rho}_\gamma = -nH\rho_\gamma\,, \tag{2}$$

with solution,

$$\rho_\gamma = \frac{D}{a^n}. \tag{3}$$

Therefore $p_\gamma = [(n-3)/3](D/a^n)$, where a is scale factor, the dot denotes time derivative, $H = \dot{a}/a$ is Hubble parameter and $D \geq 0$ is a proportional constant. We consider scalar field that is minimally coupling to gravity with Lagrangian density, $\mathcal{L} = (1/2)\epsilon\dot{\phi}^2 - V(\phi)$, of which $\epsilon = 1$ for non-phantom case and -1 for phantom case. Density and pressure of the field are given as

$$\rho_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi)\,, \tag{4}$$

therefore

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\epsilon\dot{\phi}^2 - 2V(\phi)}{\epsilon\dot{\phi}^2 + 2V(\phi)}. \tag{5}$$

The field obeys conservation equation

$$\epsilon\left[\ddot{\phi}+3H\dot{\phi}\right]+\frac{\mathrm{d}V}{\mathrm{d}\phi}=0. \tag{6}$$

Considering Friedmann-Lemaître-Robertson-Walker (FLRW) universe, the Friedmann equation and acceleration equation are

$$H^2=\frac{\kappa^2}{3}\rho_{\rm tot}-\frac{k}{a^2}\,, \tag{7}$$

$$\frac{\ddot{a}}{a}=-\frac{\kappa^2}{6}\rho_{\rm tot}(1+3w_{\rm eff})\,, \tag{8}$$

where $\kappa^2\equiv 8\pi G=1/M_{\rm P}^2$, G is Newton's gravitational constant, $M_{\rm P}$ is reduced Planck mass and k is spatial curvature. Using Eqs. (3), (4), (6) and (7), one can show that

$$\epsilon\dot{\phi}(t)^2=-\frac{2}{\kappa^2}\left[\dot{H}-\frac{k}{a^2}\right]-\frac{nD}{3a^n}\,, \tag{9}$$

$$V(\phi)=\frac{3}{\kappa^2}\left[H^2+\frac{\dot{H}}{3}+\frac{2k}{3a^2}\right]+\left(\frac{n-6}{6}\right)\frac{D}{a^n}. \tag{10}$$

3 NLS-type formulation

Non-linear Schrödinger-type formulation for canonical scalar field cosmology and barotropic fluid was proposed by J. D'Ambroise and F. L. Williams [29] and was also extended to include phantom field case [30].[1] In the Schrödinger formulation, wave function $u(x)$ is related to scale factor in cosmology as

$$u(x)\equiv a(t)^{-n/2}\,, \tag{11}$$

while Schrödinger total energy E and Schrödinger potential $P(x)$ are linked to cosmology as

$$E\equiv-\frac{\kappa^2n^2}{12}D\,, \tag{12}$$

$$P(x)\equiv\frac{\kappa^2 n}{4}a(t)^n\epsilon\dot{\phi}(t)^2. \tag{13}$$

These quantities satisfy a non-linear Schrödinger-type equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x)+[E-P(x)]\,u(x)=-\frac{nk}{2}u(x)^{(4-n)/n}\,, \tag{14}$$

[1] It is worth noting that Schrödinger-type equation in scalar field cosmology was previously considered in different procedure to study inflation and phantom field problems [70–74].

with a mapping from t to x is via

$$x = \sigma(t), \tag{15}$$

such that [30,31]

$$\dot{x}(t) = u(x)\,, \tag{16}$$

$$\phi(t) = \psi(x) = \frac{\pm 2}{\kappa\sqrt{n}} \int \sqrt{\frac{P(x)}{\epsilon}}\, \mathrm{d}x. \tag{17}$$

If $P(x) \neq 0$ and $n \neq 0$, inverse function of $\psi(x)$ exists as $\psi^{-1}(x)$. Therefore $x(t) = \psi^{-1} \circ \phi(t)$ and the scalar field potential, $V \circ \sigma^{-1}(x)$ can be expressed as,

$$V(t) = \frac{12}{\kappa^2 n^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \frac{2u^2}{\kappa^2 n} P(x) + \frac{12u^2}{\kappa^2 n^2} E + \frac{3ku^{4/n}}{\kappa^2}. \tag{18}$$

4 Phantom expansion

Expansion of the form $a \sim (t_a - t)^q$ is called phantom when $q < 0$ for a flat universe. Here in non-flat universe, q is considered to possess any value and the term phantom expansion also refers to expansion function of the form $a \sim (t_a - t)^q$ as in the flat case.

4.1 NLS-type formulation for phantom expansion

With the phantom expansion, $a \sim (t_a - t)^q$, we use Eqs. (11) and (16) to relate Schrödinger wave function to standard cosmological quantity as

$$u(x) = \dot{x}(t) = (t_a - t)^{-qn/2}. \tag{19}$$

Integrate the equation above so that the Schrödinger scale, x is related to cosmic time scale, t as

$$x(t) = \frac{1}{\beta}(t_a - t)^{-\beta} + x_0\,, \tag{20}$$

where $\beta \equiv (qn - 2)/2$ and x_0 is an integrating constant. Conversely,

$$t(x) = t_a - \frac{1}{[\beta(x - x_0)]^{1/\beta}}. \tag{21}$$

The Schrödinger wave function can be directly found from Eqs. (19) and (21) as

$$u(x) = [\beta(x - x_0)]^{qn/(qn-2)}\,. \tag{22}$$

For $a \sim (t_{\rm a}-t)^q$, we can find $\epsilon\dot{\phi}(t)^2$ from Eq. (9):

$$\epsilon\dot{\phi}(t)^2 = \frac{2q}{\kappa^2(t_{\rm a}-t)^2} + \frac{2k}{\kappa^2(t_{\rm a}-t)^{2q}} - \frac{nD}{3(t_{\rm a}-t)^{qn}}. \tag{23}$$

Using Eq. (23) with phantom expansion in Eq. (13), therefore

$$P(t) = \frac{qn}{2}(t_{\rm a}-t)^{qn-2} + \frac{kn}{2}(t_{\rm a}-t)^{q(n-2)} - \frac{\kappa^2 n^2 D}{12}, \tag{24}$$

which can be expressed in term of x using Eq. (21) as

$$P(x) = \frac{2qn}{(qn-2)^2}\frac{1}{(x-x_0)^2} + \frac{kn}{2}\left[\frac{2}{(qn-2)(x-x_0)}\right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12}. \tag{25}$$

One might have a thought that all functions in phantom expansion case can be changed to those in power-law expansion case by interchanging $(t_{\rm a}-t) \Leftrightarrow t$. However when $(t_{\rm a}-t)$ is differentiated, there is an extra minus sign. The Eq. (25) slightly defers from that of the power-law expansion case because in the power-law case, the numerator of the second term is -2 instead of 2. The Schrödinger kinetic energy T is negative value of the first two terms of the Schrödinger potential. At last, the scalar field potential obtained from Eq. (18) is

$$V(t) = \frac{q(3q-1)}{\kappa^2(t_{\rm a}-t)^2} + \frac{2k}{\kappa^2(t_{\rm a}-t)^{2q}} + \left(\frac{n-6}{6}\right)\frac{D}{(t_{\rm a}-t)^{qn}}. \tag{26}$$

which can be checked by using $a \sim (t_{\rm a}-t)^q$ in Eq. (10). Wave function of the NLS-formulation is found to be non-normalizable [30] as seen in Fig. 1 for case of phantom expansion with various types of barotropic fluid. Here q is chosen to -6.666. In flat universe $q = -6.666$ can be attained when $w_{\rm eff} = -1.1$. Figure 2 shows $P(x)$ plots for three cases of k with dust and radiation. In there $x_0 = 1$, therefore $P(x)$ goes to negative infinity at $x = 1$.

4.2 Analysis on effective equation of state coefficient

The definition of effective equation of state coefficient, $w_{\rm eff} = (\rho_\phi w_\phi + \rho_\gamma w_\gamma)/\rho_{\rm tot}$ together with Eq. (4) and the results in Eqs. (23) and (26) in context of phantom expansion $a \sim (t_{\rm a}-t)^q$, we can derive

$$w_{\rm eff} = \frac{(-3q^2+2q)(t_{\rm a}-t)^{2q-2} - k}{3q^2(t_{\rm a}-t)^{2q-2} + 3k}. \tag{27}$$

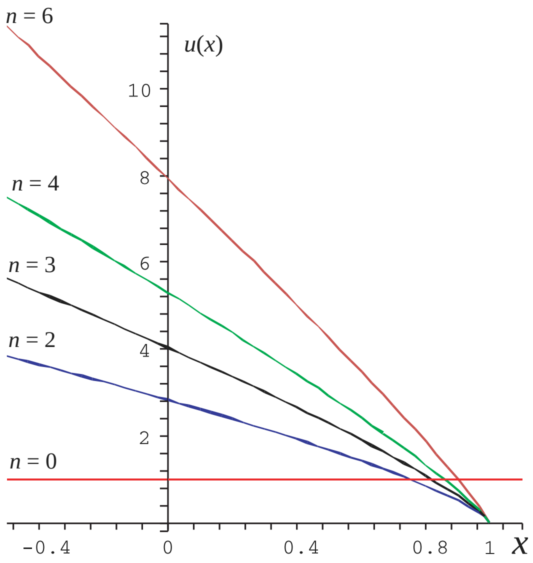


Fig. 1 Schrödinger wave function, $u(x)$ when assuming phantom expansion. $u(x)$ depends on only q, n and t_a but does not depend on k. Here we set $t_a = 1.0$ and $q = -6.666$. If $k = 0$, $q = -6.666$ corresponds to $w_{\text{eff}} = -1.1$

There is a locus,

$$t = t_a - \left(\frac{-k}{q^2}\right)^{1/(2q-2)}, \tag{28}$$

where w_{eff} becomes infinite along the locus. Hence for $k = -1$ the locus is $t = t_a - q^{-1/(q-1)}$ (in term of x, it is $x = [2/(qn-2)]q^{(qn-2)/2(q-1)} + x_0$). Hence for $k = 0$, the coefficient w_{eff} is infinite at $q = 0$ or $t = t_a$. It seems from the equation above that w_{eff} does not depend on properties, n or amount of the barotropic fluid, D. Indeed w_{eff} implicitly depends on D and n since time variable and q are related to ρ_γ in the Friedmann equation. If $k = 0$, it reduces to $q = 2/3(1 + w_{\text{eff}})$ and therefore the phantom condition $w_{\text{eff}} < -1$ implies $q < 0$ as it is known. This corresponds to a condition,

$$w_\phi < -1 - (1 + w_\gamma)\frac{\rho_\gamma}{\rho_\phi}. \tag{29}$$

Therefore for a fluid with $w_\gamma > -1$, w_ϕ is always less than -1 in a flat universe. In order to have the expansion $a \sim (t_a - t)^q$ in $k = 0$ universe, we must have $w_{\text{eff}} < -1$, i.e. in phantom region. We can rewrite w_ϕ in term of w_{eff} as

$$w_\phi = \frac{\left[\frac{3q^2}{\kappa^2}(t_a - t)^{-2} + \frac{3k}{\kappa^2}(t_a - t)^{-2q}\right] w_{\text{eff}} - \frac{n-3}{3} D(t_a - t)^{-qn}}{\frac{3q^2}{\kappa^2}(t_a - t)^{-2} + \frac{3k}{\kappa^2}(t_a - t)^{-2q} - D(t_a - t)^{-qn}}. \tag{30}$$

Equation (30), when $D = 0$ and $k = 0$, yields $w_\phi = w_{\text{eff}}$. Albeit we set only $D = 0$, it gives the same result since w_ϕ is independent of geometrical background. However,

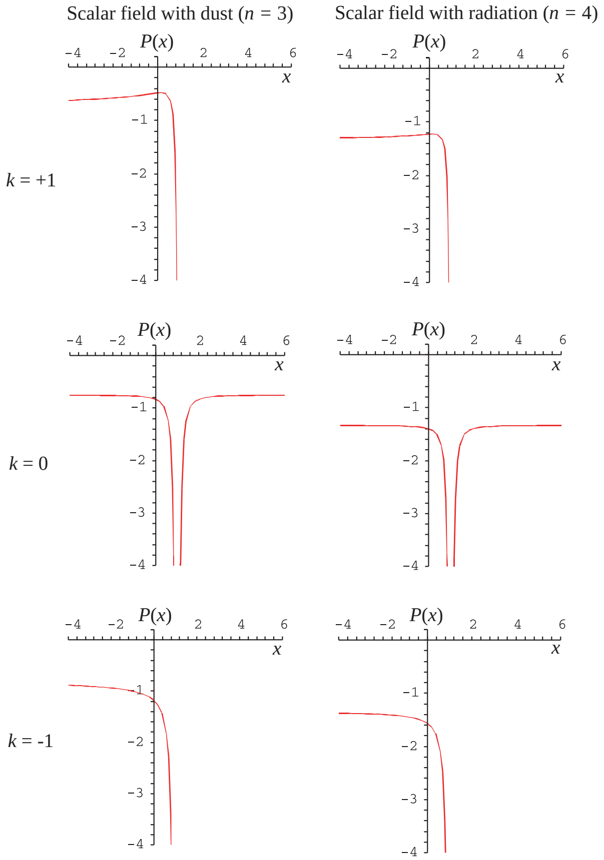


Fig. 2 Schrödinger potential in phantom expansion case for dust and radiation fluids with $k = 0, \pm 1$. Numerical parameters are as in the $u(x)$ plots (Fig. 1). x_0 is set to 1. For non-zero k, there is only one real branch of $P(x)$

since the expansion law is fixed, w_ϕ is tied up with D implicitly via Eq. (1). Note that w_ϕ has value in the range $(-\infty, -1]$ and $[1, \infty)$ so that the phantom crossing can not happen when the scalar field is dominant. However, presence of the dust barotropic fluid in the system gives a multiplication factor that is less than 1 to the equation of state, i.e.

$$w_{\text{eff}} = \left(\frac{\rho_\phi}{\rho_\phi + \rho_\gamma} \right) w_\phi . \tag{31}$$

We can see that the phantom crossing from $w_{\text{eff}} > -1$ to $w_{\text{eff}} < -1$ can happen in this situation. Figure 3 presents parametric plots of the w_{eff}, q, t diagram for various k values. From the figure, we see the locus in Eq. (28) where w_{eff} blows up. In the parametric plots, the value of w_{eff} at any instance can be obtained if we know the value of q. We need to know q from observation in order to know the realistic value of w_{eff} or the other way around. Figure 4 plotted from Eq. (27) setting $t_a = 1$ and $t = 0.7$

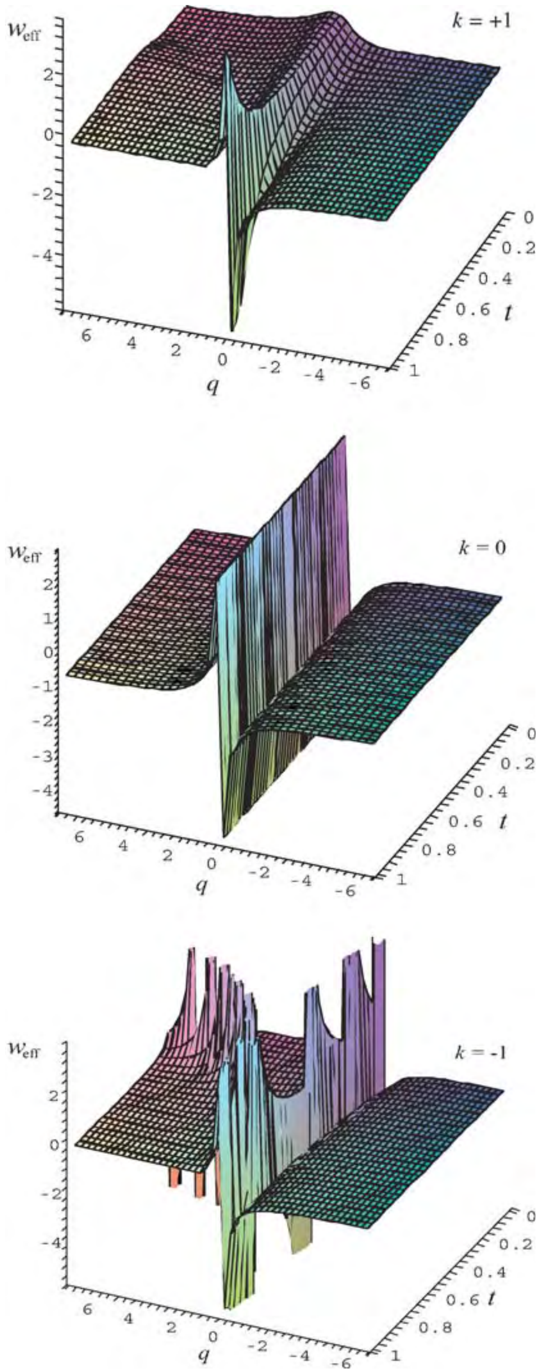
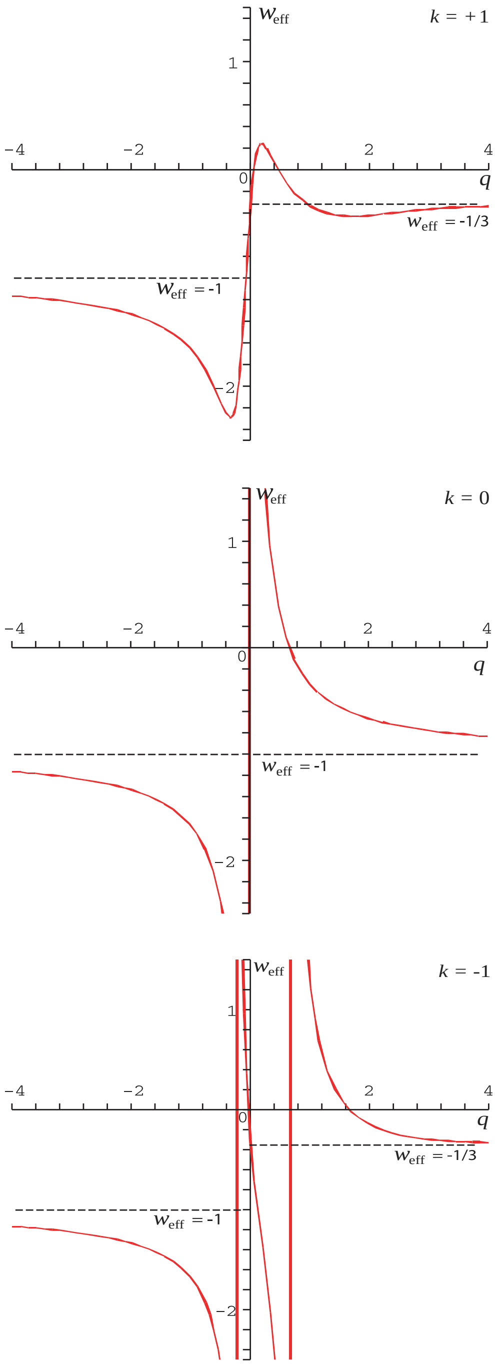


Fig. 3 Parametric plots of w_{eff} for the expansion $a \sim (t_{\text{a}} - t)^q$ in closed, flat and open universe. Here t_{a} is set to 1

Fig. 4 w_{eff} for the expansion $a \sim (t_a - t)^q$ in closed, flat and open universe. Here t_a is set to 1 and t is 0.7



shows that if $k = \pm 1$, q could be negative, i.e. phantom accelerated expansion, even when $w_{\rm eff} > -1$. Regardless of t_a and t,

$$\lim_{q \to -\infty} w_{\rm eff}(q) = -1 \quad \text{and} \quad \lim_{q \to +\infty} w_{\rm eff}(q) = -\frac{1}{3}, \tag{32}$$

for phantom expansion. In particular, for $k = -1$, $w_{\rm eff} > 0$ could give $q < 0$ and $w_{\rm eff}$ is infinite when $\ln q / \ln (t_{\rm a} - t) + q = 1$ (see Eq. (28)).

5 Scalar field exact solution

5.1 Bound value of $\phi(t)$ from effective equation of state for $k = 0$ case

In flat universe, the phantom expansion occurs when $w_{\rm eff} < -1$. Using Eqs. (4) and (5) in Eq. (1), we get a bound

$$\epsilon \dot{\phi}^2 < -\frac{n}{3} \rho_\gamma. \tag{33}$$

Assuming $a(t) = (t_{\rm a} - t)^q$ and phantom scalar field, i.e. $\epsilon = -1$ with using Eq. (3), the solution is found to be in the region,

$$\phi(t) > \frac{1}{\beta} \sqrt{\frac{Dn}{3}} (t_{\rm a} - t)^{-\beta} + \phi_0. \tag{34}$$

where $\beta \equiv (qn - 2)/2$.

5.2 Solution solved from Friedmann equation

5.2.1 Scalar field potential in flat and scalar field dominated case

A simplest case for analysis is when considering flat universe ($k = 0$) with negligible amount of barotropic fluid ($D = 0$). The Eq. (23) is hence simply integrated out. The solution is

$$\phi(t) = \pm \frac{1}{\kappa} \sqrt{\frac{2q}{\epsilon}} \ln(t_{\rm a} - t) + \phi_0 \tag{35}$$

Insert this result into Eq. (26), we obtain the scalar field potential,

$$V(\phi) = \frac{q(3q - 1)}{\kappa^2} \exp \left\{ \pm \kappa \sqrt{\frac{2\epsilon}{q}} [\phi(t) - \phi_0] \right\}. \tag{36}$$

The solutions above are real only when q and ϵ have the same sign, i.e. when $\epsilon = 1, q > 0$ and $\epsilon = -1, q < 0$. This looks similar to potential that gives power-law expansion as well-known [75]. It is not surprised since in our case ($q < 0$) it has been known that phantom field, when rolling up the hill of slope-varying exponential potential (varying q), results in phantom expansion $a \sim (t_a - t)^q$ [47–59].

5.2.2 *Solution for* $k = 0,\ D \neq 0$ *case*

For the case $k = 0$ with $D \neq 0$, the solution of Eq. (23) is

$$\begin{aligned}\phi(t) = &\pm \frac{1}{qn-2}\sqrt{\frac{2q}{\epsilon\kappa^2}} \\ &\times \left\{ \ln\left[\frac{(t_a - t)^{-qn+2}}{\left(1 + \sqrt{1 - (nD\kappa^2/6q)(t_a - t)^{-qn+2}}\right)^2} \right] \right. \\ &\left. + 2\sqrt{1 - \frac{nD\kappa^2 (t_a - t)^{-qn+2}}{6q}} + \ln\left(\frac{-nD\kappa^2}{6q}\right) \right\} + \phi_0 \,, \qquad (37)\end{aligned}$$

which is infinite when $q = 2/n$. The last logarithmic term in the bracket is an integrating constant. Logarithmic function is valid only when $q < 0$.

5.2.3 *Solution for* $k \neq 0,\ D = 0$ *case*

For the reverse case, $k \neq 0,\ D = 0$, the solution is

$$\begin{aligned}\phi(t) = &\pm \frac{1}{q-1}\sqrt{\frac{2q}{\epsilon\kappa^2}} \\ &\times \left\{ \ln\left[\frac{(t_a - t)^{q-1}}{\sqrt{k/q}} \left(1 + \sqrt{\left(\frac{k}{q}\right)(t_a - t)^{-2q+2} + 1}\right)\right] \right. \\ &\left. - \sqrt{\left(\frac{k}{q}\right)(t_a - t)^{-2q+2} + 1} \right\} + \phi_0 \,, \qquad (38)\end{aligned}$$

which becomes infinite when $q = 1$. The values q and ϵ must have the same sign for it to be real-value function. The case $k \neq 0$ with $D \neq 0$ can not be found analytically except when setting $n = 2\ (w_\gamma = -1/3)$ which is not natural fluid.

5.3 Solution solved with NLS-type formulation

One can obtain exact solution of Eq. (23) indirectly via NLS-type formulation. Consider Eq. (25), we notice that setting $D = 0$ does not make sense in NLS-formulation since even D vanishes, n (barotropic fluid parameter) still appears in other terms.

Therefore we can only consider non-zero D case. Assuming $k = 0$ with $D \neq 0$ and using Eq. (25) in Eq. (17), the solution is

$$\psi(x) = \pm\sqrt{\frac{8q}{\epsilon\kappa^2(qn-2)^2}} \times \left\{ -\sqrt{1-\left[\frac{\kappa^2 Dn(qn-2)^2}{24q}(x-x_0)^2\right]} + \ln\left[\frac{1+\sqrt{1-\left[\kappa^2 Dn(qn-2)^2/24q\right](x-x_0)^2}}{(x-x_0)}\frac{4qn}{\epsilon(qn-2)^2}\right]\right\}. \tag{39}$$

Transforming to t variable using Eq. (20),

$$\phi(t) = \pm\frac{1}{qn-2}\sqrt{\frac{2q}{\epsilon\kappa^2}} \times \left\{ \ln\left[\frac{(t_a-t)^{-qn+2}}{\left(1+\sqrt{1-(nD\kappa^2/6q)(t_a-t)^{-qn+2}}\right)^2}\right] + 2\sqrt{1-\frac{nD\kappa^2(t_a-t)^{-qn+2}}{6q}} + \ln\left(\frac{qn-2}{2qn}\right)^2\right\} + \phi_0. \tag{40}$$

The only difference from the solution (37) obtained from standard method is the logarithmic integrating constant term in the bracket. In case of $k \neq 0$ with $D \neq 0$, the integral (17) can not be integrated analytically even when assuming n value except for $n = 2$ which is integrable. However $n = 2$ is not natural fluid. This is similar to using standard method in Sect. 5.2.3.

6 Conclusions

We consider a system of FLRW cosmology of scalar field and barotropic fluid assuming phantom acceleration. We have worked out cosmological quantities in the NLS-formulation of the system for flat and non-flat curvature. The Schrödinger wave functions are illustrated in Fig. 1 for various types of barotropic fluid. These wave functions are non-normalizable. We show Schrödinger potential plots for dust and radiation cases in closed, flat and open universe. The procedure considered here is reverse to a problem solving in quantum mechanics in which the Schrödinger potential must be known before solving for wave function. In NLS formulation, the Schrödinger equation is non-linear (reducible to linear in some cases) and the wave function is expressed first by the expansion function, $a(t)$. Afterward the Schrödinger potential is worked out based on expansion function assumed. Moreover, the NLS total energy E is negative (see Eq. (12)). We also perform analysis on effective equation of state. We expresses $w_{\rm eff}$ in term of q and k. In a non-flat universe, there is no fixed $w_{\rm eff}$ value for a phantom divide. We show this by analyzing Eq. (27) and by presenting illustrations in Figs. 3 and 4. In these plots, even $w_{\rm eff} > -1$, the expansion can still be phantom, i.e. q can be negative. Especially, in $k = -1$ case, positive $w_{\rm eff}$ could also give $q < 0$. The value of $w_{\rm eff}$ approaches -1 when $q \to -\infty$ and $-1/3$ when $q \to +\infty$. In open universe, $w_{\rm eff}$ blows up when $\ln q / \ln (t_{\rm a} - t) + q = 1$.

The last part of this work is to solve for scalar field exact solution for phantom expansion. Within framework of the standard Friedmann formulation, we obtained exact solution in simplest case where scalar field is dominated in flat universe. Apart from that we also obtained exact solutions in the cases of non-flat universe with scalar field domination and flat universe with mixture of barotropic fluid and scalar field. Afterward, we use NLS formulation, in which the wave function is equivalent to the scalar field exact solution, to solve for the exact solutions. We can apply the NLS method to solve for the solution only when the barotropic fluid density is non-negligible. Setting $D = 0$ in NLS framework is not sensible because even if D term vanishes, the barotropic fluid parameter n still appears in other terms of the wave function. This is a disadvantage point of the NLS formulation.

Transforming standard Friedmann formulation to NLS formulation renders a few effects to the integration. In standard form (Eq. (23)), n appears in only D-term and all terms are t-dependent. In NLS-form (Eq. (25) when inserted in Eq. (17)), D-term becomes a constant (E), hence the number of x(or equivalently t)-dependent terms is reduced by one. This is a good aspect of the NLS. In both Friedmann-form and NLS-form, the solutions when $k \neq 0$ and $D \neq 0$ are difficult or might be impossible to integrate unless assuming values of q and n. Therefore reduction number of x-dependent term helps simplifying the integration.

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Generalized DBI quintessence

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We investigate the phase space of a quintessence theory governed by a generalized version of the DBI action, using a combination of numeric and analytic methods. The additional degrees of freedom lead to a vastly richer phase-space structure, where the field covers the full equation of state parameter space: $-1 \leq \omega \leq 1$. We find many nontrivial solution curves to the equations of motion which indicate that DBI quintessence is an interesting candidate for a viable k-essence model.

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I. INTRODUCTION

The dark energy problem continues to be a sticking point for theoretical physicists. The simplest solution to this problem is to postulate the existence of a vacuum energy or cosmological constant which agrees with all the current observational bounds [1–4]. However, we are then left with a secondary problem, namely, explaining why the vacuum energy is tuned to such a small value without some obvious symmetry to protect it. For many years we have hoped that UV complete theories of gravity would shed light on this issue, which is in effect an extremely embarrassing IR problem from this perspective. However, despite much effort, neither string theory nor loop quantum gravity has shed any compelling light on this issue—although there have been many interesting proposals.

An alternative approach is to assume that the cosmological constant is exactly zero, since supersymmetry can then be invoked as the regulating symmetry in this case. However, one then has to account for the fact that low energy supersymmetry must be broken and an alternative explanation for the current expansion, and for the vanishing of the cosmological constant, must be sought. One way to deal with the latter problem is to assume that the dark energy phase is driven by a dynamical field, implying that the equation of state is an explicit function of time [5,6]. Currently this cannot be ruled out by our best observations and therefore remains a possible solution to the dark energy problem. However, one cannot just consider *ad hoc* scalar fields coupling to gravity, since the low energy theory will still be sensitive to high energy physics. In particular, we must ensure that any additional scalars are neutral under all the standard model symmetries, and that they do not introduce additional fifth forces. Therefore, one must search for viable models of dynamical dark energy within UV sensitive theories.

Phenomenological models of our Universe have proven difficult to construct within string theory, due to technical difficulties arising from moduli stabilization, whereby we assume that the extra dimensions of the theory are compactified on manifolds with $SU(3) \times SU(3)$ structure (in the type IIB case) [7], and orientifolded to preserve the minimal amount of supersymmetry in four dimensions. Most of the work in this area assumes that the compact space is a Calabi-Yau threefold, which is a special limit of the $SU(3)$ structure manifold class.

As a result, embedding realistic cosmology into string theory has proven difficult. One area which has been well explored in recent years is inflation driven by the open string sector through dynamical Dp-branes. This is the so-called Dirac-Born-Infeld action (DBI) inflation [8,9]—which lies in a special class of K-inflation models. It was originally thought that such models yielded large levels of non-Gaussian perturbations which could be used as a falsifiable signature of string theory [10]. However, subsequent work has shown that this may not be the case, and that the simplest DBI models are essentially indistinguishable from standard field theoretic slow roll models [11–13].[1] The problem is that the WMAP 5 yr data set [2] imposes very tight constraints on the allowed tuning of the free parameters in the theory. We are then left with the choice of either having large non-Gaussianities but with vanishing tensors, or assume that the tensor spectrum will be visible—in which case there is no non-Gaussian signature. The models are only falsifiable once these conditions are relaxed. One can get around these conditions by considering more complicated models such as multiple fields [15,16], multiple branes [17–19], wrapped branes [20], or

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[1]Note, however, that the models proposed in [13,14] evade such problems.

monodromies [21]—but even here there are still problems with fine-tuning, backreaction, and the apparent breakdown of perturbation theory in the inflationary regime [22].

In models of dynamical dark energy, on the other hand [5,6,23], the WMAP constraints can be relaxed and therefore DBI models may still have some use as an explanation for a dynamical equation of state. Moreover, this fits in nicely with several intuitive ideas from string theory. Namely, that inflation can still occur, albeit only through the closed string sector—where one (or more) of the geometric pseudomoduli are actually responsible for the initial inflationary epoch (see [24] for the phenomenologically most viable proposals). After inflation the universe lives on branes that wrap various cycles within the compact space and are extended along the large Minkowski directions. In this sense we see that a grand unified theory or electroweak (EW) phase transition can manifest through a geometric fashion—namely, the Higgsing of branes in the compact bulk space. This suggests that dark energy may well be a dynamical process, and moreover, in the light of these open string constructions, it retains a sense of being geometric in nature.

With this in mind, various authors have begun to explore the phase space of DBI-driven dark energy [25,26]. The initial works have dealt with the dynamics of a solitary $D3$-brane moving through a particular warped compactification of type IIB. In this paper we wish to generalize this further to a more phenomenological class of models that include multiple and partially wrapped branes. We believe that this may be a more generic situation to consider, since typically one should expect branes of varying degrees to be wrapped on nontrivial cycles of the compact space. Our work is a first step into considerations of a more general setup for quintessence in IIB (open) string theory, and we hope it will be a valuable starting point for further endeavours.

II. DYNAMICS OF THE EFFECTIVE THEORY

To begin let us assume that the universe at such late times can be adequately described by a flat Friedmann-Robertson-Walker (FRW) metric and that the matter sector consists of a dynamical scalar field and a perfect fluid, which are both separately conserved. The usual cosmological equations of motion are therefore independent of any particular model and can be written as

$$H^2 = \frac{(\rho + \rho_\phi)}{3M_p^2}, \qquad \dot{\rho}_i = -3H(P_i + \rho_i), \tag{2.1}$$

where i runs over the contributing components. The equation of state is given by $\omega_i = P_i/\rho_i$; however, if ω of the fluid component is assumed to be constant, then we can integrate the appropriate conservation equation exactly to obtain

$$\rho \propto a^{-3(1+\omega)}, \tag{2.2}$$

where the scale factor varies as a function of time such that $a(t) \sim t^{2/(3[1+\omega])}$.

The model dependence arises in the parametrization of the scalar field sector. In our case we are assuming that the dark energy is driven by open string modes, which at low energies are described by fluctuations of a Dp-brane whose dynamics are governed by the DBI—which is a generalization of nonlinear electrodynamics [8,9]. Typically one assumes that the standard model is localized on an intersecting brane stack, in one of the many warped throats that are attached to the internal space. For consistency reasons, in the simplest cases, these are taken to be either $D3$- or $D7$-branes. In this paper we will consider a bottom-up approach; therefore, we shall not worry too much about the geometric deformations of the compact space, nor about any constraints imposed by orientifolded Op-planes—aside from those that ensure that all tadpoles are consistently canceled so that we can trust the low energy supergravity theory.

The action we consider is a generalized form of the DBI one coupled to Einstein-Hilbert gravity, which can be embedded into this background and takes the following generalized form[2]:

$$S = -\int d^4x a^3(t)\left(T(\phi)W(\phi)\sqrt{1 - \frac{\dot{\phi}^2}{T(\phi)}} - T(\phi) + \tilde{V}(\phi)\right) + S_M, \tag{2.3}$$

where $T(\phi)$ is the warped tension of the brane and S_M is the action for matter localized in the standard model (SM) sector. Thus our assumption here is that our dynamical open string sector is coupled only gravitationally to the SM sector, and so we do not have to worry about additional forces or particle production. There are two potential terms for the scalar field, which are denoted by $W(\phi)$ and $\tilde{V}(\phi)$. The first of these terms can arise in different places within the theory. First, if the brane is actually a non-Bogomol'nyi-Prasad-Sommerfield (BPS) one [28], then the scalar field mode is actually tachyonic and the potential is therefore of the usual runaway form. If there are N multiple coincident branes, then the world-volume field theory is a $U(N)$ non-Abelian gauge theory and the potential term is simply a reflection of the additional degrees of freedom [29]. Through the dielectric effect, one can also see that this configuration is related to a $D5$-brane wrapping a two-cycle within the compact space and carrying a nonzero magnetic flux along this cycle. Both of these configurations lead to an additional potential multiplying the usual DBI kinetic term.

[2]We refer the more interested readers to [19] for more details on the precise structure and origin of this action. The important thing to note is that ϕ is a matrix valued field. For recent work in a related direction see [27].

The origin of the $\tilde{V}(\phi)$ term is less explicit—but is a sum of terms. One expects open or closed string interactions to generate a scalar potential $V(\phi)$; however, the precise form of such an interaction depends upon many factors such as the number of additional branes and geometric moduli, the number of nontrivial cycles in the compact space, and the choice of embedding for branes on these cycles. Typically one can only compute this in special cases in the full string theory. There are also additional terms coming from coupling of the brane to any background Ramond-Ramond form fields. The action above is assumed to be that of a $D3$-brane(s) filling the space-time directions, which naturally couples to the field $C^{(4)}$ through the Chern-Simons part of the action. However, for wrapped $D5$-branes there is also the possibility of a coupling $C^{(4)} \wedge F$, where F is the magnetic field through the two-cycle. For example, in the warped deformed conifold, one can see that $dC^{(6)} = \star dC^{(2)}$, and therefore there is an additional term in the DBI action,

$$S \sim \int d^4 x a^3(t) g_s^{-1} M \alpha' T(\phi), \qquad (2.4)$$

up to a normalization factor of order 1. Terms such as this have been added to the interaction potential to define the *full* scalar potential $\tilde{V}(\phi)$. Recent extensions to standard DBI inflation have included the contribution from higher dimensional bulk forms, with the remarkable result that they cancel one another up to third order in the action and therefore do not affect the leading order perturbations [30]. Extending this work to higher orders is therefore extremely interesting.

The corresponding equations for the energy density and pressure of the DBI can then be written succinctly as

$$P_\phi = \frac{T(\phi)}{\gamma}[\gamma - W(\phi)] - \tilde{V}(\phi),$$
$$\rho_\phi = T(\phi)[W(\phi)\gamma - 1] + \tilde{V}(\phi), \qquad (2.5)$$

where $\gamma = [1 - \dot{\phi}^2/T(\phi)]^{-1/2}$ is the usual generalization of the relativistic factor. The subscript ϕ denotes the scalar field component here. We can also immediately define the equation of state parameter for the quintessence field to be

$$\omega_\phi = \frac{T(\phi)[\gamma - W(\phi)] - \tilde{V}(\phi)\gamma}{T(\phi)\gamma[W(\phi)\gamma - 1] + \tilde{V}(\phi)\gamma}, \qquad (2.6)$$

from which one clearly sees that it is dynamically sensitive and can take a wide range of values. For instance, we only recover $\omega_\phi \sim -1$ in the limit that the field is nonrelativistic and the entire solution is dominated by the $\tilde{V}(\phi)$ terms—which will clearly require large amounts of fine-tuning to accomplish. There are clearly several regions of parameter space that are of interest. First let us assume that the potential term is zero, either because it is suppressed or there is an unlikely cancellation between the contributing terms. The more general case with nonzero $\tilde{V}$ leads to a wide variety of complex behavior. We can therefore identify several limits of interest—focusing on the behavior of W:

(i) $W(\phi) = 1$—which reduces the action back to the usual DBI case which has $\omega_\phi = 1/\gamma$ as discussed in [25].

(ii) $W(\phi) = \alpha\gamma$—which leads to constant γ if α is constant, since the two are related via $\gamma^2 \omega_\phi \alpha = 1 - \alpha + \omega_\phi$. Moreover, this again means that $\dot{\phi} \propto t^{-(1+\omega_\phi)/(1+\omega)}$, as in the case where $W = 1$.

(iii) $W(\phi) \to 0$—as could occur in the case of a tachyonic theory, which mimics a dark energy dominated phase with $\omega_\phi = -1$. However, one must be careful if this is to be representative of non-BPS D-brane actions, since the coupling to the form field is noncanonical in this instance. In fact, the coupling term will typically be of the form $d\phi \wedge C$. This means that there is no solitary $T(\phi)$ term in the action, and therefore the equation of state in this instance will vary like $-1/\gamma^2$.

(iv) $W(\phi) \gg \gamma$—which can occur in the multibrane/wrapped brane case and yields $\omega_\phi \sim -1/\gamma^2$.

Note that in all cases the equation of state parameter remains bounded in the range $-1 \le \omega_\phi \le 1$.

One can combine the expressions for the energy-momentum tensor components, and together with the continuity equation, we obtain the following equation of motion (assuming that the scalar field follows a monotonic path):

$$\ddot{\phi} + \frac{3H\dot{\phi}}{\gamma^2} + \frac{3T_\phi}{2\gamma^2} + \frac{1}{W\gamma^3}(\tilde{V}_\phi - T_\phi) - \frac{T_\phi}{2} + \frac{TW_\phi}{W\gamma^2} = 0, \qquad (2.7)$$

which is a generalization of the Klein-Gordon equation for the DBI Lagrangian. The subscript ϕ of T, W, and $\tilde{V}$ denotes a derivative with respect to the field value. The other dynamical equation of motion for the Hubble parameter can be written as

$$\dot{H} = -\frac{1}{2M_p^2}[\rho(1+\omega) + \gamma W(\phi)\dot{\phi}^2], \qquad (2.8)$$

where we have defined the pressure of the noninteracting barotropic fluid to be$P = \omega\rho$. We leave the interesting case of interacting pressure for future endeavours.

Let us consider, as an example solution, the case where there is a scaling solution with $W = 1$, which has been reviewed elsewhere [26]. We will find it convenient to define the quantity

$$X = \frac{1 + \omega_\phi}{1 + \omega}, \qquad (2.9)$$

in which case we see that $\dot{\phi} \sim t^{-X}$. This allows us to reconstruct the tension of the brane as follows:

$$T(\phi) = \mathcal{M}^4 e^{-\lambda\phi}, \qquad X = 1,$$
$$T(\phi) = \mathcal{M}^{4+\alpha}\phi^{-\alpha}, \qquad X \neq 1, \tag{2.10}$$

where $\mathcal{M}$ is a dimensionful mass scale, λ is a constant, and $\alpha = 2X/(1-X)$. Using the fact that $\omega_\phi = 1/\gamma$, we can then see that for $X \neq 1$ the solution is physically valid only when $\omega > 2/\alpha$ since we define γ to be the positive root. Let us now consider the phase-space dynamics of the theory in more detail, following along the lines of [5]. It is initially convenient to define the following new variables:

$$x = \sqrt{\frac{T(\phi)W(\phi)\gamma}{3}}\frac{1}{HM_p}, \qquad \mu_1 = \frac{\sqrt{T}M_p\tilde{V}_\phi}{\tilde{V}^{3/2}},$$
$$y = \sqrt{W(\phi)\gamma}\frac{\dot{\phi}}{HM_p}, \qquad \mu_2 = -\frac{\sqrt{T}M_pT_\phi}{\tilde{V}^{3/2}}, \tag{2.11}$$
$$z = \sqrt{\frac{\tilde{V}}{3}}\frac{1}{HM_p}, \qquad \mu_3 = \frac{W_\phi M_p}{W^{3/2}\gamma^{5/2}},$$

in terms of which we can see that $\gamma = [1 - y^2/(3x^2)]^{-1/2}$, and the fluid density parameter can be written as

$$\Omega = 1 - \Omega_\phi = 1 - \left(z^2 + x^2\left[1 - \frac{1}{W(\phi)\gamma}\right]\right), \tag{2.12}$$

while the equation of state in dimensionless variables will become

$$\omega_\phi = \frac{1}{\gamma}\left(\frac{x^2[\gamma - W(\phi)] - z^2W(\phi)\gamma^2}{x^2[W(\phi)\gamma - 1] + z^2W(\phi)\gamma}\right). \tag{2.13}$$

As is customary we will now switch to dimensionless derivatives, denoted by a prime, replacing time derivatives by derivatives with respect to the e-folding number $\mathcal{N}$. Therefore we can easily determine

$$\frac{H'}{H} = -\frac{y^2}{2} - \frac{3(1+\omega)}{2}\left(1 - z^2 - x^2\left[1 - \frac{1}{W(\phi)\gamma}\right]\right). \tag{2.14}$$

A useful quantity to calculate is the variation of the kinetic function, which we can write in the following manner using the equation of motion,

$$\frac{\dot{\gamma}}{\gamma} = -\frac{3H\dot{\phi}^2}{T} - \frac{W_\phi\dot{\phi}}{W} - \frac{T_\phi\dot{\phi}}{T} - \frac{\dot{\phi}}{\gamma WT}(\tilde{V}_\phi - T_\phi). \tag{2.15}$$

We can then determine the dynamical equations for the dimensionless fields as derivatives with respect to $\mathcal{N}$,

$$x' = -\frac{1}{2}(\mu_1 + \mu_2)\frac{yz^3}{x^2} - \frac{y^2}{2x} - x\frac{H'}{H},$$
$$y' = -3y\left(1 - \frac{y^2}{6x^2}\right)\left(1 + \frac{z^3}{xy}[\mu_1 + \mu_2]\right) + \frac{3\mu_2z^3W}{\gamma x} - 3x^2\mu_3 - y\frac{H'}{H}, \tag{2.16}$$
$$z' = \frac{z^2y\mu_1}{2x} - z\frac{H'}{H},$$

and the remaining parametric solutions are

$$\mu_1' = \frac{\mu_1^2yz}{x}\left(-\frac{3}{2} + \frac{\tilde{V}_{\phi\phi}\tilde{V}}{\tilde{V}_\phi^2} + \frac{T_\phi\tilde{V}}{\tilde{V}\tilde{V}_\phi}\right),$$
$$\mu_2' = \frac{\mu_1\mu_2yz}{x}\left(-\frac{3}{2} + \frac{T_\phi\tilde{V}}{2T\tilde{V}_\phi} + \frac{T_{\phi\phi}\tilde{V}}{T_\phi\tilde{V}_\phi}\right),$$
$$\mu_3' = y\mu_3^2\gamma^{3/2}\left(1 + \frac{W_{\phi\phi}W}{W_\phi^2} + \frac{5T_\phi W}{2TW_\phi} + \frac{5}{2T\gamma W_\phi}[\tilde{V}_\phi - T_\phi]\right) + \frac{5\mu_3y^2}{2x^2}. \tag{2.17}$$

Note that if the μ_i are constants, then the previous three equations form an autonomous set and should uniquely specify the dynamics of the quintessence field. We will consider this case as the simplest (canonical) example. If we wish to appeal to string theoretic constructions, then we restrict the parameter space of solutions. It is more interesting to consider the above equations in the context of a phenomenological model and see what kind of functions yield the correct behavior. Explicit constructions of string backgrounds are typically difficult, and there are only a few well-known examples that are ritually invoked; however, if we take string theory seriously, then there are undoubtedly other nontrivial backgrounds that are cosmologically interesting but not yet constructed. Since an analytic analysis of this generalized system is highly complicated, it is convenient to use a combination of analytic and numerical methods to understand the dynamics of the system. For a numeric analysis it is necessary to rewrite the fluid equation in terms of more useful variables. It turns out that the simplest variables to use are the following:

$$\phi' = \Phi, \tag{2.18}$$

$$\Phi' = -\frac{3\Phi}{\gamma^2} + \frac{3M_pz^3}{x}\left(\frac{\sqrt{W\gamma}\mu_2}{2}\left[\frac{3}{\gamma^2} - 1\right] - \frac{(\mu_1 + \mu_2)}{\sqrt{W}\gamma^{5/2}}\right) - \frac{3M_px^2\mu_3}{\sqrt{W\gamma}} - \Phi\frac{H'}{H}, \tag{2.19}$$

which are easily derivable from the terms written above. Equations (2.14), (2.16), (2.17), (2.18), and (2.19), together with the barotropic fluid equation $\rho' = -3\rho(\mathcal{N})(1+w)$, hence form a closed ten-dimensional autonomous system if

T, W, or $\tilde{V}$ is given as an explicit function of ϕ or as a constant.

A. Case I

Let us take the canonical string theoretic example arising when the local geometry can be approximated by an anti-de Sitter space. This geometry typically arises in the near horizon limit of coincident $D3$-branes (or flux). In this case we see that (at leading order)

$$T(\phi) = \frac{\phi^4}{\lambda^4}, \qquad \tilde{V}(\phi) = \frac{m^2\phi^2}{2}, \qquad W(\phi) = W, \tag{2.20}$$

where we have also included an effective ϕ^2 potential for the system. This means that $\mu_3 = 0$, and we also have a constant μ_1 which allows us to write the remaining μ terms as

$$\mu_1 = \frac{2\sqrt{2}M_p}{m\lambda^2}, \qquad \mu_2 = -\frac{2x^2\mu_1}{W\gamma z^2}, \tag{2.21}$$

and therefore the dynamical equations reduce to

$$\begin{aligned} x' &= -\frac{\mu_1 y z^3}{2x^2}\left(1 - \frac{2x^2}{W\gamma z^2}\right) - \frac{y^2}{2x} - \frac{xH'}{H}, \\ y' &= -3y\left(1 - \frac{y^2}{6x^2}\right)\left(1 + \frac{z^3\mu_1}{xy}\left[1 - \frac{2x^2}{W\gamma z^2}\right]\right) \\ &\quad - \frac{6\mu_1 z x}{\gamma^2} - \frac{yH'}{H}, \\ z' &= \frac{z^2 y\mu_1}{2x} - \frac{zH'}{H}. \end{aligned} \tag{2.22}$$

The simplest way to proceed with the analysis is to consider the final equation above, since this splits the solution space neatly into two components. Thus we search for solutions where either $z = 0$ or $z = (2x/y\mu_1)H'/H$ as initial conditions.

The first subset of solutions admits (0, 0, 0) as a (trivial) fixed point, which is a fluid dominated solution since $\Omega = 1$ in this instance. Let us remark here that this fixed point solution will occur for *all* the cases we consider; however, since this implies a vanishing of the action, causality implies that this fixed point must be unstable—i.e. phase-space trajectories will flow away from it. By making this field a phantom scalar, one can evade this causal bound and the point can become a stable fixed point. This behavior arises in many places in the literature, so we will not discuss it further here.

There is also a critical point at $(1, \sqrt{3}, 0)$ which is a kinetic dominated solution. This solution actually exists as solutions to the quadratic expression $y^2 = 3x^2$ which corresponds to the limit $\gamma \to \infty$. In terms of the density parameter, a quick calculation shows that along the general curve (parametrized by y_0 and x_0), we find $\Omega = 1 - x_0^2$. Thus at the trivial fixed point we see $\Omega \to 1$; however, for $x_0 \to 1$ we see that $\Omega \to 0$, corresponding to nonrelativistic matter, i.e. dust. In this instance we also find $a(t) \sim t^{2/3}$ as expected from the cosmological evolution equations. Again due to the special algebraic properties of the DBI action, we anticipate that this solution will also be found for the other cases of interest.

The second subset of solutions is more interesting, as initially one can solve the system by slicing the phase space at $y = 0$.[3] One can use the condition on H' to fix z through $z^2 = 1 - x^2(W - 1)/W$. Combining this with the equations of motion gives us the following fixed point (taking positive signs of all roots for simplicity):

$$x = \sqrt{\frac{W}{1 - W}}, \qquad y = 0, \qquad z = 1, \tag{2.23}$$

which is valid for all $W < 1$ in order for these points to be real, and at finite distance in phase space. If we then compute the density of the fluid, we find $\Omega = 0$ since $\Omega_\phi = 1$, which corresponds to a purely dustlike solution. Note that this class of solutions does not exist for the simple $D3$-brane analysis as in [26], since it arises from additional degrees of freedom which are neglected in these models. The remaining solutions in this subset are difficult to find analytically.

More generally, we can see that the above solution is a special case of the more general case I behavior, which we parametrize by

$$T(\phi) = \frac{\phi^\alpha}{\lambda^\alpha}, \qquad \tilde{V}(\phi) = \frac{m^\beta\phi^\beta}{\beta}, \qquad W(\phi) = W, \tag{2.24}$$

where we can then explicitly write

$$\mu_1 = A\left(\frac{x}{z\gamma^{1/2}}\right)^{(\alpha-\beta-2)/(\alpha-\beta)}, \qquad \mu_2 = -\frac{\alpha}{\beta}\frac{\mu_1}{W\gamma}\frac{x^2}{z^2}, \qquad \mu_3 = 0, \tag{2.25}$$

where A is a (real, positive) constant provided that $\beta > 0$,

$$A = \frac{M_p\beta^{3/2}}{\lambda^{\alpha/2}m^{\beta/2}}\left(\frac{\lambda^\alpha m^\beta}{\beta W}\right)^{(\alpha-\beta-2)/2(\alpha-\beta)}, \tag{2.26}$$

but which simplifies in the limit $\alpha = \beta + 2$. As before, the solution space splits into two disconnected subsets; therefore, in the first instance where we take slices through $z = 0$, we find the following bound:

$$\frac{2}{(\alpha - \beta)} > 0, \tag{2.27}$$

which implies that $\alpha > \beta$ and so the brane tension should dominate the dynamics (in the large field regime). Let us therefore assume that α, β are chosen such that this

[3]Note that one cannot do this for $x = 0$ since the action becomes singular and ill defined.

condition is satisfied—then we find the solution branch is governed again by the relation $y^2 = 3x^2$ as expected—which contains the solution (0, 0, 0) as a special case. Moreover, this is valid for all values of α, β satisfying the above constraint. The secondary solution branch occurs when we find solutions to

$$\frac{zy\mu_1}{2x} = \frac{H'}{H}, \tag{2.28}$$

which is generally very complicated. A simple set of solutions does arise when we consider slices at $y = 0$, since the fixed points are localized along the curve

$$x = \pm\sqrt{\frac{\beta W}{(\alpha-\beta)(1-W)}}, \qquad y = 0, \qquad z = \pm\sqrt{\frac{\alpha}{(\alpha-\beta)}}, \tag{2.29}$$

which corresponds to a dustlike solution: $\Omega = 0\ \forall\ \alpha, \beta$. The reality constraint here demands that $\alpha > \beta$, which in turn fixes $W < 1$. However there are also additional solutions where $\beta < 0$ and α is positive—provided that $W > 1$. Explicit realizations of this scenario within a string theory context can arise through potentials arising from brane/antibrane interactions and it is therefore a nontrivial and interesting solution.

Figures 1 and 2 show the numerical solutions in phase space. For the $W = 1$ case, the numerical constants are given as $M_p = 1$, $m = 1$, $\lambda = 1$, and $w = 0$ (dust case). Other parameters are $\alpha = 4$, $\beta = 2$, and $A = 2\sqrt{2}$. As expected, the (five) fixed points all lie along the curve $y^2 = 3x^2$. We also plot the evolution of each parameter (x, y, z)

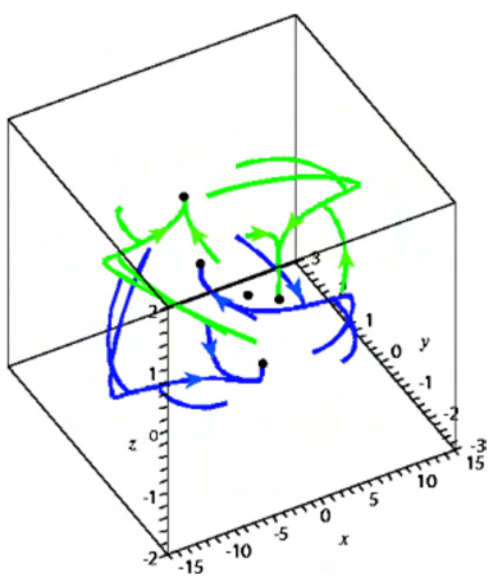

FIG. 1 (color online). Case I: 3D xyz phase-space trajectories for $T(\phi) = \phi^4/\lambda^4$, $\tilde{V}(\phi) = m^2\phi^2/2$, and $W(\phi) = W$. Here we have set $M_p = 1$, $W = 1$, $m = 1$, $\lambda = 1$, and $w = 0$ (dust case).

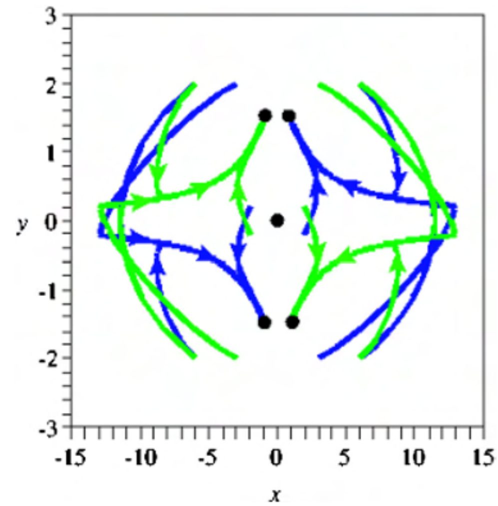

FIG. 2 (color online). Case I: Phase-space trajectories in the xy plane. Four attractors $(\pm1, \pm\sqrt{3}, 0)$ and one unstable node (0, 0, 0) can be seen here. z is bounded within the $(-1, 1)$ range.

as a function of the e-folding number in Fig. 3, where each of the coordinates tends to its critical value. As expected the phase-space dynamics are $\mathbb{Z}_2$ symmetric about the origin. Note that in the case of $y(N)$ one can keep y suppressed for a few e-foldings with enough tuning, before it eventually evolves towards the points $\pm\sqrt{3}$ at late times. The full numerical solution of the case $W < 1$ is illustrated in Fig. 4 where $W = 0.95$, which uniquely fixes the critical points to be $x = \pm\sqrt{20}$, $y = 0$, $z = 1$. As one can see from the resulting plot, this is an unstable node because the general behavior is divergent. Note that $x \rightarrow \infty$ in this regime effectively solves all the dynamical equations trivially.

B. Case II

Analogous to the first case, let us now consider another branch of solutions where this time the tension of the brane is taken to be constant. This dramatically alters the relativistic rolling of the scalar field since the γ factor is no longer warped. Initially, let us consider the ansatz

$$\tilde{V}(\phi) = \frac{m^2\phi^2}{2}, \qquad T(\phi) = T, \qquad W(\phi) = \frac{\phi^4}{\lambda^4}, \tag{2.30}$$

which implies that

$$\mu_1 = \left(\frac{4\sqrt{2T^3}M_p}{\lambda^4 m^3}\right)\frac{z^2\gamma}{x^2}, \qquad \mu_2 = 0, \qquad \mu_3 = \frac{2z\mu_1}{\gamma^2 x}, \tag{2.31}$$

and the corresponding field equations become

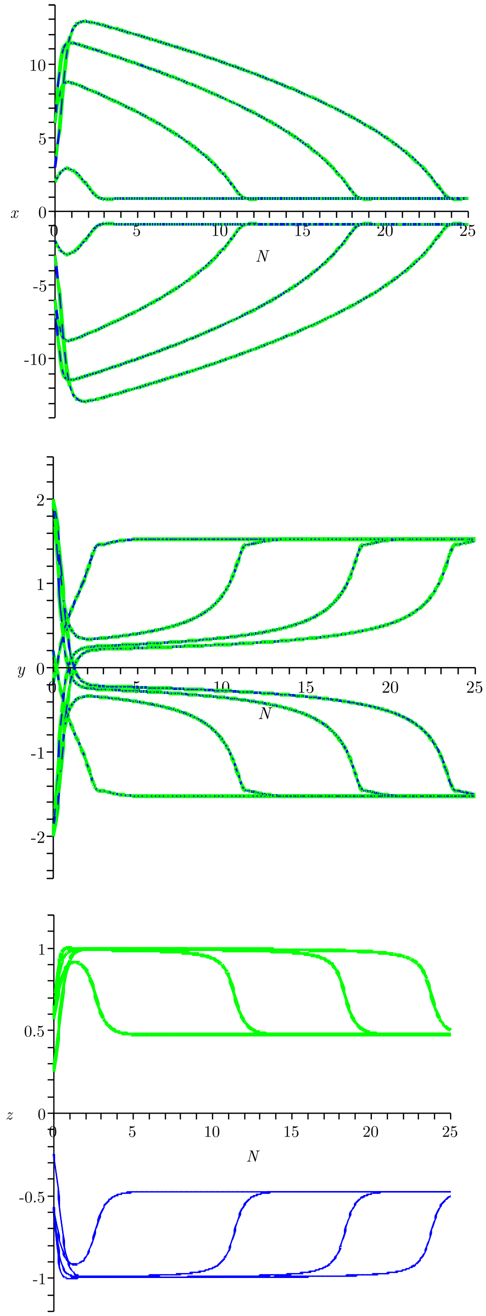

FIG. 3 (color online). Case I: Evolution of x, y, z versus the e-folding number, setting $W = 1$.

$$x' = -\frac{\alpha\gamma y z^5}{x^4} - \frac{y^2}{2x} - \frac{xH'}{H},$$
$$y' = -3y\left(1 - \frac{y^2}{6x^2}\right)\left(1 + \frac{\alpha\gamma z^5}{2x^3 y}\right) - \frac{6\alpha z^3}{\gamma x} - \frac{yH'}{H}, \quad (2.32)$$
$$z' = \frac{\alpha\gamma z^4}{2x^3} - \frac{zH'}{H},$$

where we have defined α as the constant prefactor in the definition of μ_1.

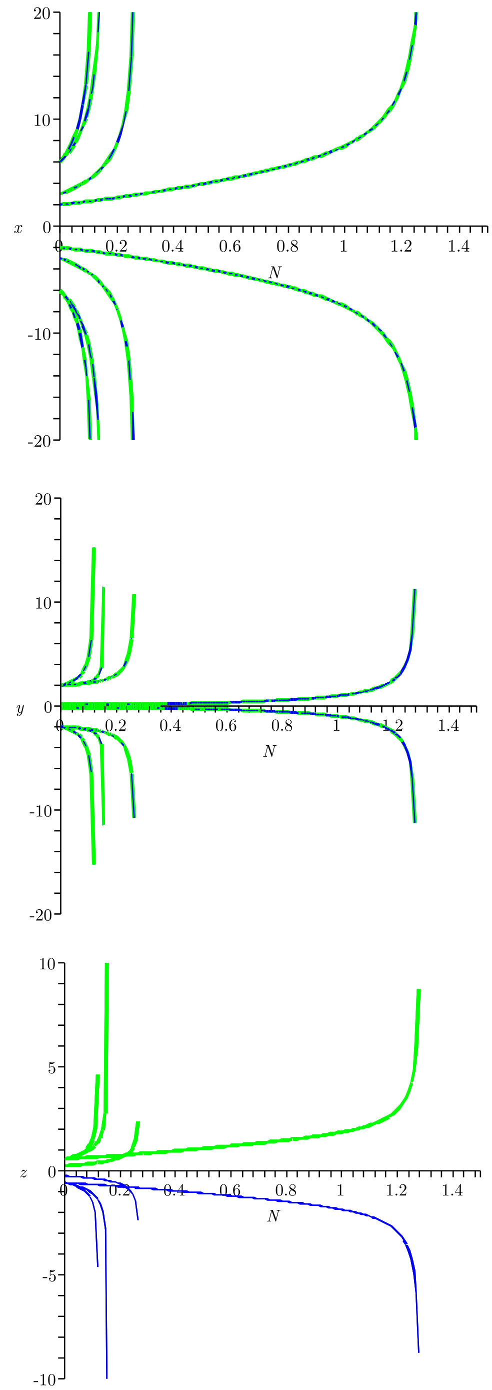

FIG. 4 (color online). Case I: Evolution of x, y, z versus the e-folding number, setting $W = 0.95$. All solutions diverge from the origin.

As before we separate the solution space into two—first finding solutions to $z = 0$ and then solutions to $H'/H = \alpha\gamma z^3/(2x^3)$. In the first case it is straightforward to see that there are the usual fixed point solutions at (0, 0, 0) and $(1, \sqrt{3}, 0)$ (with their respective partner solutions) respectively coming from the usual condition that $y^2 = 3x^2$. The secondary branch of solutions also admits fixed points when $y = 0$; however, the condition on z is that $z = 0, -4x^2$. Since we want real solutions we are forced to set $z = 0$ as a secondary constraint. This forces W to diverge, and therefore in the limit that $z \to 0$, we find that $x^2 \to \pm 1$ which is a unique solution. Again the density parameter vanishes identically in this limit as one would expect. The remaining solutions are actually extremely difficult to solve analytically as they correspond to high order polynomials. As a result we are forced to sketch their behavior numerically.

Phenomenologically, we see that the ansatz presented above is a special class of the more general solution

$$T(\phi) = T, \qquad \tilde{V}(\phi) = \frac{m^\beta \phi^\beta}{\beta}, \qquad W(\phi) = \frac{\phi^\alpha}{\lambda^\alpha}, \tag{2.33}$$

which has the parametrization constraints

$$\mu_1 = A\left(\frac{z\gamma^{1/2}}{x}\right)^{(2+\beta)/(\alpha-\beta)}, \qquad \mu_2 = 0,$$
$$\mu_3 = B\gamma^{(2-4\alpha+5\beta)/2(\alpha-\beta)}\left(\frac{z}{x}\right)^{(2+\alpha)/(\alpha-\beta)}, \tag{2.34}$$

where A, B are both constants. One can see from the dynamical equations that fixed points with $z = 0$ can only occur when the following condition is met:

$$\frac{2(1+\alpha) - \beta}{\alpha - \beta} > 0, \tag{2.35}$$

which is trivially satisfied for cases where $\alpha > \beta$ (which we assume as an initial constraint).

More generically we see that, provided $\alpha > -2$, we recover the usual fixed point equation $y^2 = 3x^2$. However, we need to be careful here because if this condition is satisfied, then W becomes undefined. Since this is the overall prefactor multiplying the DBI action, the action is undefined in this limit and it should therefore correspond to a point of instability in the phase space. In the limit where $\alpha = -2$, which implies that $\beta > -2$, the fixed point solution now lives on the zeros of the polynomial

$$3x^4 B\gamma^{-5(2+\beta)/2(1+\beta)} + 3x^2 y - y^3 = 0, \tag{2.36}$$

which can be used to fix $x = x(y)$ or vice versa. This solution is actually indicative of a more general branch of physical solutions where we take $\beta > 2(1+\alpha)$. The resulting fixed point equation (provided $\alpha \neq 2$) is trivially calculated to be $y^2 = 3x^2$ as before, but now we see that W vanishes identically. In turn this means that the kinetic terms also vanish and the solution is dominated solely by the potential interaction. One could imagine a situation such as this occurring in the condensation of an open string tachyon mode on a non-BPS brane, where the vanishing of W indicates that we are living in the closed string vacuum. For dynamic solutions it seems reasonable to consider this particular case as the late-time attractor for the solution $z \to 0$.

The second subset of solutions is again complicated, but again we can analytically understand the plane at $y = 0$, which gives us the fixed point solutions

$$x = \pm\sqrt{-\frac{\beta z^2}{\alpha}}, \qquad y = 0,$$
$$z = \pm\left(1 - \frac{\beta}{\alpha}\left(1 - \left[-\frac{\alpha T}{\lambda^\alpha m^\beta}\right]^{\alpha/(\alpha-\beta)}\right)\right)^{-1/2}. \tag{2.37}$$

Clearly, for the solution to be real we require that α, β have opposite signs. This satisfies our primary constraint, and therefore is a physical possibility. Moreover, in the limit where we set $\beta = -\alpha$, we find that $\Omega = 0$, which is again the dust solution. Illustrations of numerical solutions for case II are shown in Figs. 5–7. Constants are set as $M_p = 1$, $T = 1$, $m = 1$, $\lambda = 1$, and $w = 0$ (dust case). Other parameters are $\alpha = 4$, $\beta = 2$. From the numerical analysis one sees that there are six saddle nodes, only two attractors, and one repulsive point which is the origin (0, 0, 0) as expected. The dynamical trajectories are particularly interesting due to their apparent lack of monotonicity as a function of e-fold number. The z term, in particular, appears to have a large variation in trajectory, diverging in some instances while rapidly reaching zero in other instances. Conversely, the y variable displays very uniform (physical) trajectory behavior, with several curves almost on top of one another at $y = 0$ and the remainder smoothly driven to the (unstable) critical point $y_c \sim 1.8$ in the example given.

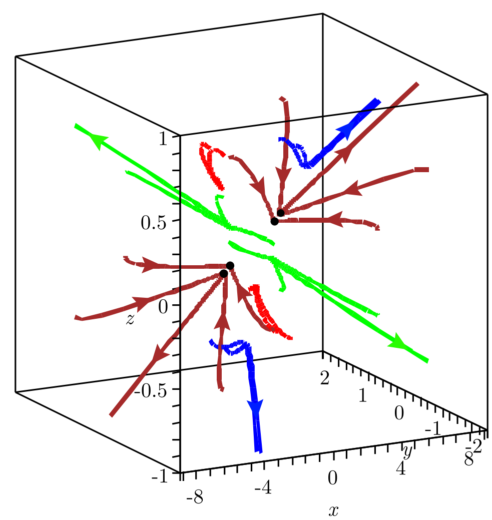

FIG. 5 (color online). Case II: 3D xyz phase-space trajectories for $T(\phi) = T$, $\tilde{V}(\phi) = m^2\phi^2/2$, and $W(\phi) = \phi^4/\lambda^4$. Here we have set $M_p = 1$, $T = 1$, $m = 1$, $\lambda = 1$, and $w = 0$ (dust case).

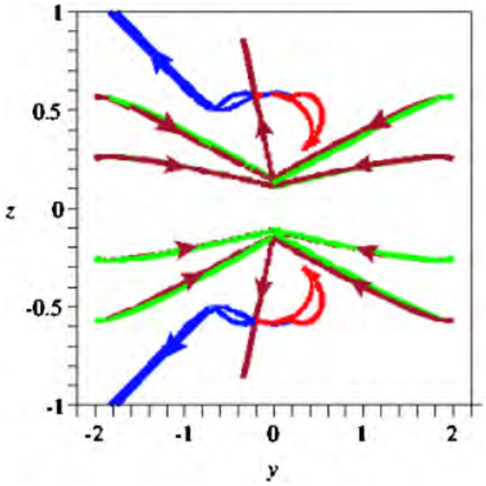

FIG. 6 (color online). Case II: Trajectory slice through the yz plane.

C. Case III

Let us now consider a new case where only $W = W(\phi)$, with all the other terms being constant. We will take $W = \phi^\alpha/\lambda^\alpha$ for generality—which in turn should impose a constraint on the allowed values of α. In this case we see that

$$\mu_1 = 0, \qquad \mu_2 = 0, \qquad \mu_3 = \alpha A\left(\frac{z}{x}\right)^{(\alpha+2)/\alpha}\frac{1}{\gamma^{(2\alpha-1)/\alpha}}, \tag{2.38}$$

where A is a function of the constant parameters $A = M_p/\lambda(T/\tilde{V})^{(\alpha+2)/2\alpha}$. Because only μ_3 is nonzero, the resulting dynamical expressions are considerably easy to work with,

$$x' = -\frac{y^2}{2x} - \frac{xH'}{H},$$
$$y' = -3y\left(1 - \frac{y^2}{6x^2}\right) - 3\alpha A z^{(\alpha+2)/\alpha}x^{(\alpha-2)/\alpha} - \frac{yH'}{H},$$
$$z' = -\frac{zH'}{H}. \tag{2.39}$$

Considering the slice again through $z = 0$, we see that the solutions split into two types depending upon the integer α. We recover the usual $y^2 = 3x^2$ curve only when $\alpha > 0$ or when $\alpha < -2$. If $\alpha = -2$ then the corresponding polynomial equation becomes

$$y\gamma^{9/2} = 2Ax^2 \tag{2.40}$$

which is difficult to solve analytically due to the dependence of γ on both x, y. This expression does not admit anything but the trivial solution if we set y to zero.[4] Again we see that there is a potential problem here since the potential W goes like $1/z^2$, and is therefore divergent in this limit. Solutions to this expression are possible, but complicated. Interestingly there does exist a solution curve given by

$$y^2 = ax_c^2, \qquad x_c = \frac{81}{2A}\frac{\sqrt{3a}}{(9-3a)^{9/4}} \tag{2.41}$$

where the parameter a factor must satisfy $0 \le a < 3$ for this solution to be physical. Since a need not be integer,

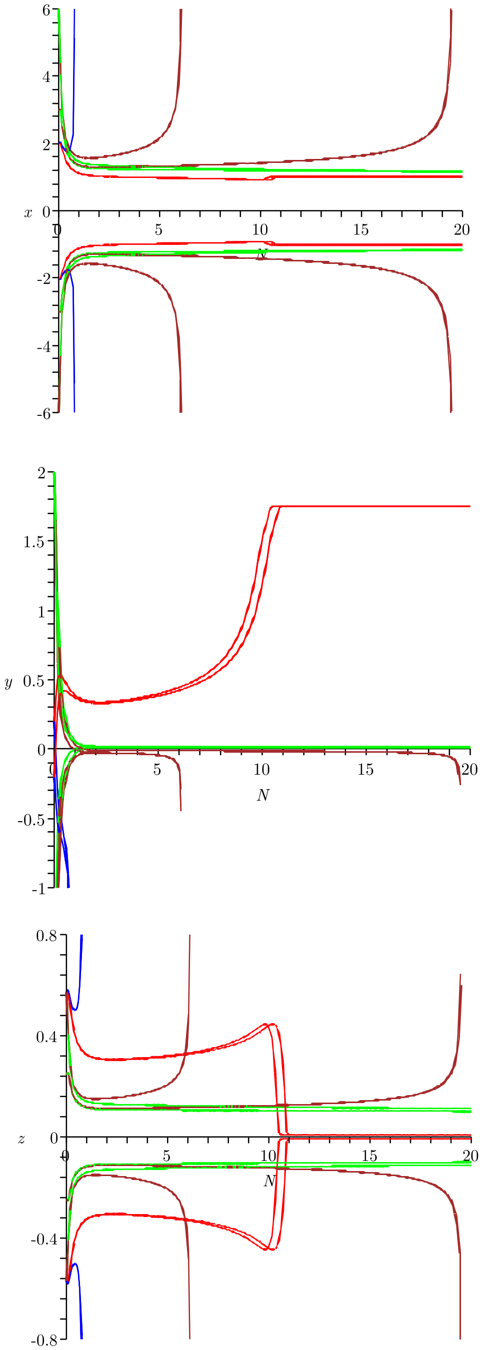

FIG. 7 (color online). Case II: Evolution of x, y, z versus the e-folding number.

[4]By trivial we mean the point (0, 0, 0).

there is essentially a continuum of curves giving rise to fixed points in this theory.

The secondary branch of solutions again admits fixed point behavior for $y = 0$; however, things are more complicated since the fixed points are now obtained by solving more nonlinear expressions. There are two cases of immediate interest though. First, if we have $\alpha = 2$ then we see that $z^2 = -1/(2A)$ which is only real when A is negative. Since we have chosen our parametrization such that this quantity is positive, this particular branch of solutions is ruled out. Interestingly, when $\alpha = -2$ there is a unique fixed point located at

$$x = \pm\frac{1}{2A}, \qquad y = 0, \qquad z = \pm\frac{1}{\sqrt{T/\tilde{V} - 1}}\sqrt{\frac{1}{2A} - 1}, \tag{2.42}$$

which corresponds to a positive definite equation of state parameter

$$\Omega = \frac{2T^2 A(A-1) + \tilde{V}^2(T/\tilde{V} - 1)}{AT\tilde{V}(T/\tilde{V} - 1)(2A - 1)}. \tag{2.43}$$

Note that we must require $T > \tilde{V}$ for this solution to be nonsingular, which means (again) that the tension term dominates the energetics of the theory. What is also obvious is that demanding $A = 1/2$ leads to a novel fixed point at $(\pm 1, 0, 0)$ regardless of the ratio $T/\tilde{V}$. Using the definition of A, this fixes $\lambda = 2M_p$, and therefore W is vanishingly small unless the scalar is trans-Planckian. This is manifest in a divergence in the equation of state parameter and is therefore unphysical. Therefore we must ensure that $A < 1/2$, implying that $\lambda > 2M_p$. Since this is the largest scale in our theory, one again expects this to be unphysical.

The more general solutions can be found numerically and correspond to $x_0^2 = 1 + z_0^2(T/\tilde{V} - 1)$, where z_0^2 are the characteristic solutions to the nonlinear equation

$$1 + \alpha A z^{(2+\alpha)/2}(1 + z^2(T/\tilde{V} - 1))^{(\alpha-2)/2\alpha} = 0. \tag{2.44}$$

In this more general case we can set $T = \tilde{V}$ without the solution diverging, and we therefore find the corresponding fixed point solution is thus given by

$$x = \pm 1, \qquad y = 0, \qquad z = \left(-\frac{1}{A\alpha}\right)^{2/(2+\alpha)} \tag{2.45}$$

which implies that α is negative. Moreover we see that Ω is again zero here for all physical values of α, although there is no additional constraint upon the magnitude of A. Now, we see numerical solutions in Figs. 8–10. Constants are set as $M_p = 1$, $T = 1$, $\tilde{V} = 1$, $\lambda = 1$, and $w = 0$ (dust case). Other parameters are $\alpha = 1$ and $A = 1$.

D. Case IV

Following on from the previous class of models, we can find solutions where the scalar potential is now constant,

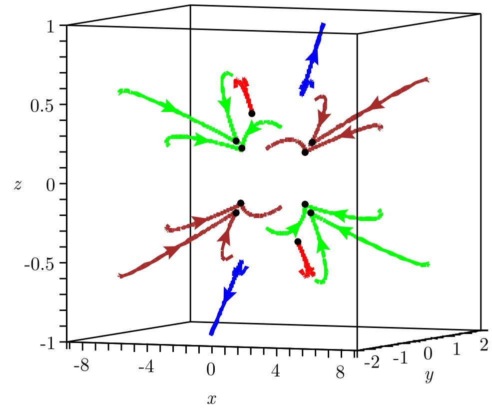

FIG. 8 (color online). Case III: 3D xyz phase-space trajectories for $T(\phi) = T$, $\tilde{V}(\phi) = V$, and $W(\phi) = \phi^\alpha/\lambda^\alpha$. Here we set $M_p = 1$, $T = V = 1$, $m = 1$, $\lambda = 1$, and $w = 0$ (dust case), $\alpha = 1$.

using the ansatz

$$\tilde{V} = V, \qquad T(\phi) = \left(\frac{\phi}{\lambda}\right)^\alpha, \qquad W(\phi) = \left(\frac{\phi}{\delta}\right)^\beta \tag{2.46}$$

where λ, δ are terms of the requisite dimensionality. From this expression we see that μ_1 is identically zero. It will be convenient to define the following function, $Q = V\lambda^\alpha\delta^\beta$, which in turn can be used in the definitions of the remaining μ_i functions

$$\mu_2 = -\frac{\alpha M_p}{\lambda^{\alpha/2}V}\left(\frac{Qx^2}{\gamma z^2}\right)^{n_1},$$
$$\mu_3 = \frac{\beta M_p \delta^{\beta/2}}{\gamma^{4/2}}\left(-\frac{\mu_2\lambda^{\alpha/2}V^{3/2}}{\alpha M_p}\right)^{-n_2}, \tag{2.47}$$
$$n_1 = \frac{3\alpha - 2}{2(\alpha + \beta)}, \qquad n_2 = \frac{1 + \beta}{3\alpha - 2},$$

and now the dynamical equations simplify to become

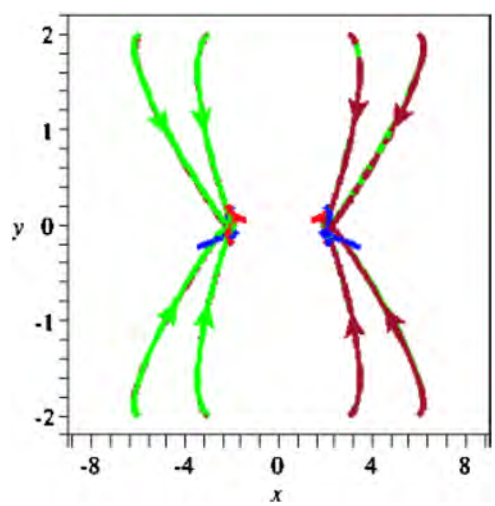

FIG. 9 (color online). Case III: Phase-space trajectories in the xy plane.

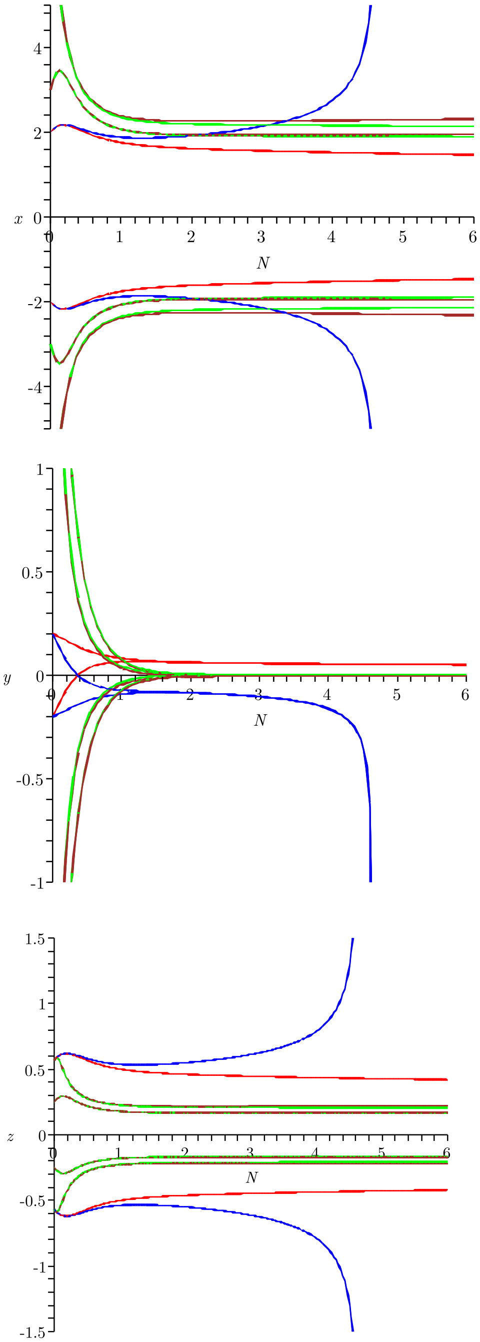

FIG. 10 (color online). Case III: Evolution of x, y, z versus the e-folding number.

$$
\begin{aligned}
x' &= -\frac{\mu_2 y z^3}{2x^2} - \frac{y^2}{2x} - \frac{xH'}{H},\\
y' &= -3y\left(1 - \frac{y^2}{6x^2}\right)\left(1 + \frac{z^3\mu_2}{xy}\right)\\
&\quad + 3\mu_2\left(\frac{z^{3\alpha+\beta}}{x^{\beta-\alpha}y^{2\beta+\alpha}}\left[\frac{Q}{\delta}\right]^{\beta}\right)^{1/(\alpha+\beta)} - 3x^2\mu_3 - \frac{yH'}{H},\\
z' &= -\frac{zH'}{H}.
\end{aligned} \tag{2.48}
$$

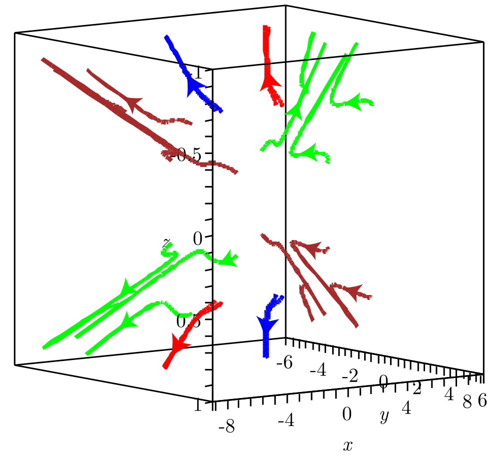

FIG. 11 (color online). Case IV: 3D xyz phase-space trajectories for $T(\phi) = (\phi/\lambda)^{\alpha}$, $\tilde{V}(\phi) = V$, and $W(\phi) = (\phi/\delta)^{\beta}$. Here, $M_p = 1$, $V = 1$, $m = 1$, $\lambda = 1$, $\alpha = 1$, $\beta = 1$, $\delta = 1$, and $w = 0$ (dust case).

The resulting analysis is far more complicated than in the previous cases. Let us again start with the simplest solution slices at $z = 0$. The expressions for x' and z' readily simplify in this instance; however, the equation for y' requires us to be more careful. We see that in order for the $z^3\mu_2$ term to vanish in this limit, we require $(2 + 3\beta)/(\alpha + \beta) > 0$. The remaining μ_2 term only vanishes if this condition is tightened to $(2 + \beta)/(\alpha + \beta) > 0$, and the term coming from μ_3 only vanishes if $(1 + \beta)/(\alpha + \beta) > 0$. If these inequalities are reversed, for example, then these terms diverge in the $z \to 0$ limit. If we restrict ourselves to well-behaved solutions such that α, β satisfy the above bounds (either by both α, $\beta \geq 0$ or by $\alpha \geq 0$, $\beta \leq 0$ with $|\beta| > |\alpha|$), then we obtain the solution curve $y^2 = 3x^2$ as usual. If the parameters α, β do not satisfy at

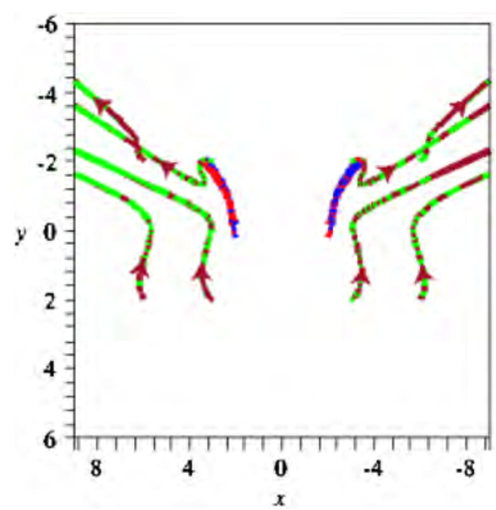

FIG. 12 (color online). Case IV: Phase-space trajectories in the xy plane.

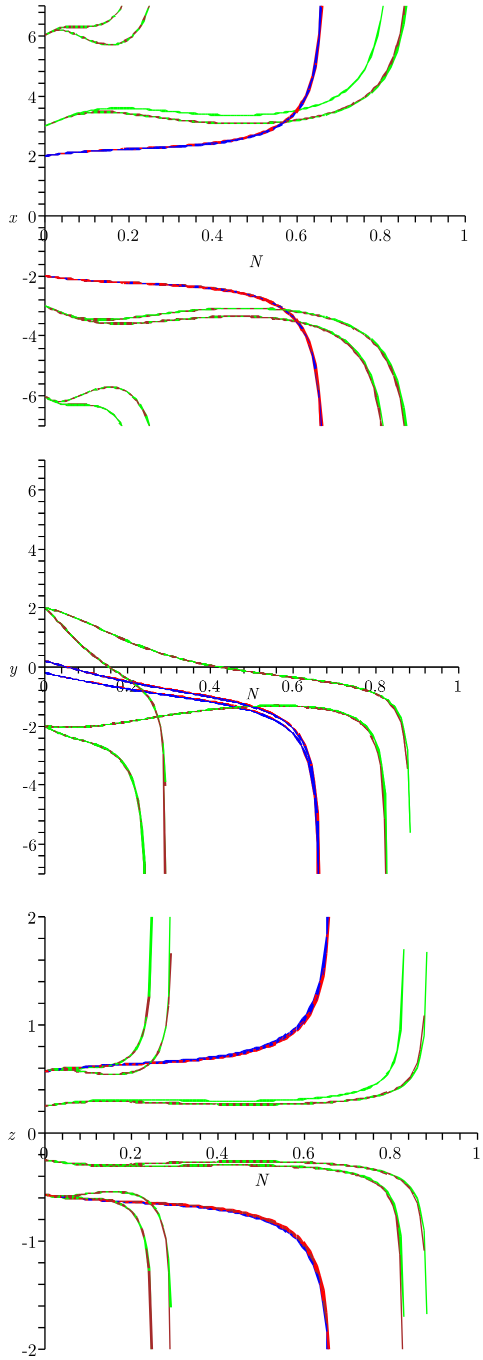

FIG. 13 (color online). Case IV: Evolution of x, y, z versus the e-folding number.

least the minimal bound, then one can only solve these expressions numerically.

The only other solution branch occurs when $H'/H = 0$. This is again a complicated solution; however, things simplify somewhat when we slice through $y = 0$, but also tune the solution such that $\alpha = \beta$, which gives us

$$z = \frac{x\delta^{\alpha}}{2\sqrt{Q}}\left(1 \pm \sqrt{1 - \frac{4Q(x^2 - 1)}{x^2\delta^{2\alpha}}}\right), \tag{2.49}$$

and therefore the fixed point solution in this instance is given by solutions of the polynomial

$$x\sqrt{\frac{Q}{\delta}} + \left(\frac{\sqrt{Q}}{z}\right)^{1/(2\alpha)} \frac{x^{(1+8\alpha)/(4\alpha)}}{\lambda^{\alpha/2}\delta^{\alpha/2}} = 1. \tag{2.50}$$

This can actually be solved exactly when $\alpha = -1$, but numerically for more general α. The exact case gives us the following solution:

$$x_0 = \frac{Q\delta - 2\sqrt{Q} + \delta^2 \pm \delta\sqrt{F(\lambda, \delta)}}{2\lambda\delta^4},$$
$$F(\lambda, \delta) = Q^2 + \delta^2 - 4\sqrt{Q^3\delta} + 6Q\delta - 4\sqrt{Q\delta^3} + 16\sqrt{Q^3\delta^7} - 4Q^3\delta^2 + 16\sqrt{Q^{5/2}\delta^{5/2}} - 24Q^2\delta^3 - 4Q\delta^4 + 4\delta^6\lambda, \tag{2.51}$$

where z_0 is given by the term written above. This is a highly complicated solution, but one sees that, in principle, there are many fixed points along the plane $(x_0, 0, z_0)$ depending on the constants λ, δ. One also sees that there is a simple solution when $x = 1$, since this implies that $z_0 = \delta^{\alpha}/\sqrt{Q}$ or $z_0 = 0$, the latter again giving rise to the point (1, 0, 0) which corresponds to the nonpropagating end point of the brane dynamics as shown in Figs. 11–13.

E. Case V

Finally let us comment on perhaps the most general form of the solution one could obtain from this model, namely, that corresponding to turning on all the relevant degrees of freedom. One can therefore see that cases I–IV are actually slices through the full phase space described in this section. We will take the following parametrization for simplicity:

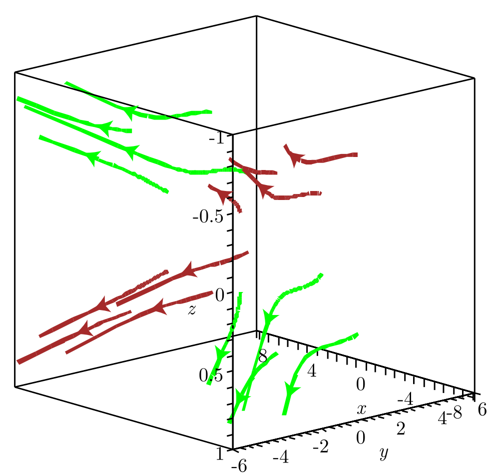

FIG. 14 (color online). Case V: 3D xyz phase-space trajectories for $T(\phi) = (\phi/\lambda)^{\alpha}$, $\tilde{V}(\phi) = (m\phi)^{\xi}/\xi$, and $W(\phi) = (\phi/\delta)^{\beta}$. Here, $M_p = 1$, $V = 1$, $m = 1$, $\lambda = 1$, $\alpha = 1$, $\beta = 1$, $\delta = 1$, $\xi = 2$, and $w = 0$ (dust case).

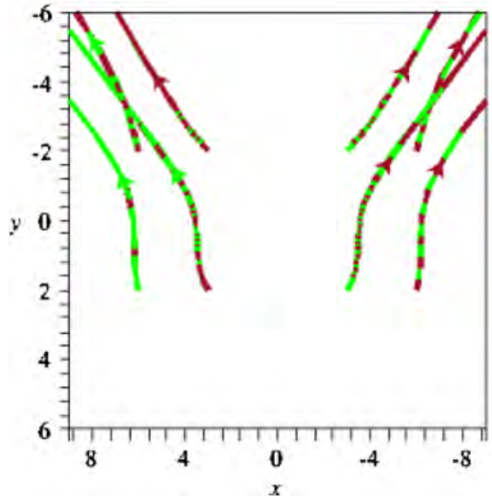

FIG. 15 (color online). Case V: Phase-space trajectories in the xy plane.

$$T = \left(\frac{\phi}{\lambda}\right)^{\alpha}, \qquad W = \left(\frac{\phi}{\delta}\right)^{\beta}, \qquad \tilde{V} = \frac{m^{\xi}\phi^{\xi}}{\xi}. \tag{2.52}$$

In this case all three μ_i will be nonzero, which complicates the analysis somewhat, and reality again imposes the condition that $\xi > 0$. Let us initially search for the fixed points around $z = 0$. The primary constraint equation for this becomes

$$\frac{\alpha - \xi + 2(1-\beta)}{(\alpha + \beta - \xi)} > 0. \tag{2.53}$$

Let us initially assume that the denominator is positive definite. Going through the same analysis as before yields the usual solution curve $y^2 = 3x^2$, provided that we tune $\beta > 0$ and $\alpha + \beta > \xi$. However, with reference to the action, we see that this situation leads to both W, T diverging, and therefore we should be wary of this part of the solution. Returning to the constraint equation, let us therefore assume that $\xi > \alpha + \beta$ and redo the analysis. We then find that the $y^2 = 3x^2$ is perfectly valid, and moreover the parameters W, T are not divergent, provided that the parameters satisfy $\alpha + \beta - \xi < -(2 + \beta)$. Furthermore, we also see that β is bounded from above such that $\beta < -2/3$—thus severely restricting the form of the variable phase space.

If we search for solutions along the $y = 0$ slicing, things are again complicated. However, we can simplify things by identifying $\alpha = \xi$, since we can then solve explicitly for x via

$$x^2 = 1 + z^2\left(\frac{\xi^2}{\lambda^{\xi} m^{\xi}} - 1\right). \tag{2.54}$$

The remaining equation coming from $y' = 0$ has several solutions. The simplest ones are $z^2 = 0$, $(\lambda^{-\xi} m^{-\xi}\xi^2 - 1)^{-1}$, which give rise to the points

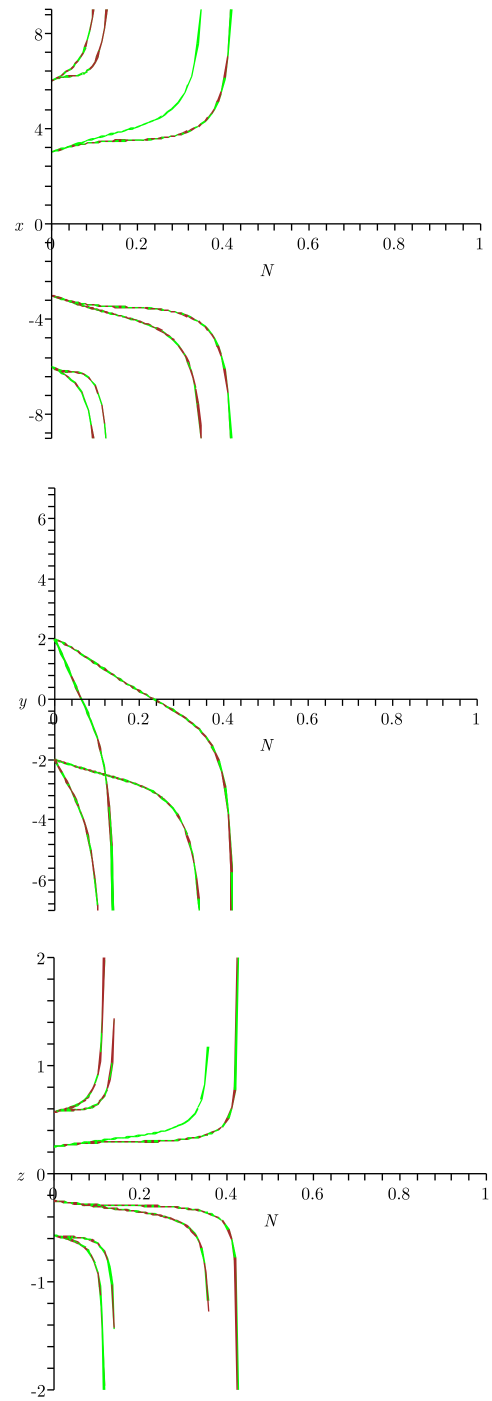

FIG. 16 (color online). Case V: Evolution of x, y, z versus the e-folding number.

$$x_0 = \pm\sqrt{2}, \qquad y_0 = 0, \qquad z_0 = \frac{1}{\sqrt{\lambda^{-\xi} m^{-\xi}\xi^2 - 1}},$$
$$x_0 = \pm 1, \qquad y_0 = 0, \qquad z_0 = 0. \tag{2.55}$$

However, the first of these conditions also requires that $\xi^{2/\xi} > \lambda m$ for the solution to be real. The maximal value of $\xi^{2/\xi}$ is actually given by $\xi = e^1$, which imposes a tight constraint on the background parameters which can only be satisfied through substantial fine-tuning. Again, more general solutions are only available through numeric methods as shown in Figs. 14–16.

III. PERTURBATIONS AND FIXED POINT STABILITY

We now need to evaluate the stability of these fixed point solutions. Clearly one may anticipate that solutions such as (0, 0, 0) may well be unstable. We must perturb the field equations about small values; therefore we need

$$x \to x_0 + \delta x, \qquad y \to y_0 + \delta y, \qquad z \to z_0 + \delta z. \tag{3.1}$$

Now the analysis is more complicated than in standard models due to the complexity of the DBI action and the general (unknown) phase-space dependence of the variables T, W, $\tilde{V}$. Since γ is independent of any particular parametrization, we can calculate the general result.

$$\gamma \to \gamma\left(1 + \frac{\gamma^2 y_0 \delta y}{3x_0^2} - \frac{\gamma^2 y_0^2 \delta x}{3x_0^3} + \ldots\right). \tag{3.2}$$

Using this we can write the perturbation in H'/H. In general, we can Taylor expand the function W such that we have $W(x^i + \epsilon^i) \sim W(x_0^i) + \partial_i W \epsilon^i$, and therefore the general result is true:

$$\delta\left(\frac{H'}{H}\right) = -y_0\delta y - \frac{3(1+\omega)}{2} \times \left(-2z_0\delta z - 2x_0\delta x\left[1 - \frac{1}{W\gamma}\right] - \frac{x_0^2}{\gamma W}\left\{-\frac{\gamma^2 y_0 \delta y}{3x_0^2} + \frac{\gamma^2 y_0^2 \delta x}{3x_0^3} - \frac{\partial_i W \epsilon^i}{W}\right\}\right),$$

where all terms such as γ, W are evaluated on the classical solution and there is a summation over Latin indices.

The general equations, even for the linear perturbation, are shown below for case V—which encompasses all the other solutions in the relevant limit:

$$\begin{aligned}
\delta x' &= -\frac{yz^3}{2x^2}(\mu_1+\mu_2)\left(\frac{\delta y}{y} + 3\frac{\delta z}{z} - 2\frac{\delta x}{x}\right) - \frac{yz^3}{2x}(\mu_1\delta\mu_1 + \mu_2\delta\mu_2) - \frac{y^2}{2x}\left(2\frac{\delta y}{y} - \frac{\delta x}{x}\right) - \delta x\frac{H_0'}{H_0} - x\delta\left(\frac{H'}{H}\right),\\
\delta y' &= -\frac{3z^3}{x}\left\{\mu_1\delta\mu_1 + \mu_2\delta\mu_2 + [\mu_1+\mu_2]\left(3\frac{\delta z}{z} - \frac{\delta x}{x} - \frac{\delta y}{y}\right)\right\} - 3x^2\mu_3\left(2\frac{\delta x}{x} + \delta\mu_3\right) + \left(1 + \frac{z^3}{xy}[\mu_1+\mu_2]\right)\\
&\quad \times\left(\frac{y^3}{x^2}\left[\frac{\delta y}{y} - \frac{\delta x}{x}\right] - 3\delta y\right) - \delta y\frac{H_0'}{H_0} - y\delta\left(\frac{H'}{H}\right)\\
&= \frac{3z^3\mu_2 W}{\gamma x}\left(\delta\mu_2 + 3\frac{\delta z}{z}\left(1 - \frac{2\beta}{3n}\right) - \frac{\gamma^2 y\delta y}{3x^2}\left(1 + \frac{\beta}{n} + \frac{\delta x}{x}\left\{\frac{2\beta}{n} - 1 + \frac{\gamma^2 y^2}{3x^2}\left(1+\frac{\beta}{n}\right)\right\}\right)\right),\\
\delta z' &= \frac{z^2 y\mu_1}{2x}\left(2\frac{\delta z}{z} + \frac{\delta y}{y} - \frac{\delta x}{x} + \delta\mu_1\right) - \delta z\frac{H_0'}{H_0} - z\delta\left(\frac{H'}{H}\right),
\end{aligned} \tag{3.3}$$

where we have defined $n = \alpha + \beta - \rho$ for simplicity and also the following terms:

$$\begin{aligned}
\delta\mu_1 &= -\frac{2(\alpha-2\rho)}{n}\frac{\delta z}{z} - \frac{(\alpha-2-\rho)}{2n}\frac{\gamma^2 y\delta y}{3x^2} + \frac{2(\alpha-2-\rho)}{n}\frac{\delta x}{x}\left(1 + \frac{\gamma^2 y^2}{12x^2}\right),\\
\delta\mu_2 &= -\frac{4(\alpha-1-\rho)}{n}\frac{\delta z}{z} - \frac{(3\alpha-3\rho-2)}{n}\frac{\gamma^2 y\delta y}{6x^2} + \frac{\delta x}{nx}\left(4(\alpha-1-\rho) + \frac{(3\alpha-3\rho-2)\gamma^2 y^2}{6x^2}\right),\\
\delta\mu_3 &= \frac{2(\alpha+2+3\rho+2\beta)}{n}\frac{\delta z}{z} + \frac{(2\alpha-2-10\rho+\beta)\gamma^2 y\delta y}{6nx^2} + \frac{\delta x}{nx}\frac{(-2\alpha+2+10\rho-\beta)\gamma^2 y^2}{6x^2}\\
&\quad - \frac{\delta x}{x}\frac{(2\alpha+4+6\rho+4\beta)}{n}.
\end{aligned} \tag{3.4}$$

We will work through an explicit example to illustrate the formalism, namely, the case I solutions. First, we can calculate the following expression,

$$\delta\left(\frac{H'}{H}\right) \sim -y_0\delta y + \frac{3(1+\omega)}{2}\left(2z_0\delta z + 2x_0\delta x\left[1 - \frac{1}{W\gamma}\right] + \frac{x_0^2\gamma y_0}{3Wx_0^2}\left(\delta y - \frac{y\delta x}{x_0}\right)\right), \tag{3.5}$$

which will allow us to calculate the perturbed phase-space variables. The perturbed dynamic expressions then take the following form:

$$\delta x' = \frac{yz^3\mu_1}{2x^2}\left(\frac{\alpha x^2}{\beta W\gamma^2}\left[\frac{2\delta x}{x}\left(1+\frac{\gamma^2 y^2}{6x^3}\right) - 2\frac{\delta z}{z} - \frac{\gamma^2 y\delta y}{3x^2}\right]\right) - \frac{y^2}{2x}\left(\frac{2\delta y}{y} - \frac{\delta x}{x}\right) - \frac{yz^3\mu_1}{2x^2}\left(1 - \frac{\alpha x^2}{\beta W\gamma^2}\right)$$
$$\times\left(\frac{\delta y}{y}\left(1 - \frac{n\gamma^2 y}{6x^2}\right) + (3-n)\frac{\delta z}{z} + \frac{\delta x}{x}\left(n - 2 + \frac{n\gamma^2 y^2}{6x^2}\right)\right) - \delta x\frac{H_0'}{H_0} - x\delta\left(\frac{H'}{H}\right),$$
$$\delta y' = 3z^3\mu_1\left(1 - \frac{y^2}{6x^2}\right)\frac{\alpha x}{\beta W\gamma^2}\left[2\frac{\delta x}{x}\left(1 + \frac{\gamma^2 y^2}{6x^2}\right) - 2\frac{\delta z}{z} - \frac{\gamma^2 y\delta y}{3x^2}\right] - \frac{3z^3\mu_1}{2x}\left(1 - \frac{\alpha x^2}{\beta W\gamma^2}\right)\left(1 - \frac{y^2}{6x^2}\right)$$
$$\times\left((3-n)\frac{\delta z}{z} + \frac{\delta x}{x}\left(n - 1 + \frac{n\gamma^2 y^2}{6x^2}\right) - \frac{\delta y}{y}\left(1 + \frac{n\gamma^2 y^2}{6x^2}\right)\right) + \left(1 + \frac{z^3\mu_1}{xy}\left[1 - \frac{\alpha x^2}{\beta W\gamma^2}\right]\right)$$
$$\times\left\{\frac{y^3}{x^2}\left[\frac{\delta y}{y} - \frac{\delta x}{x}\right] - 3\delta y\left(1 - \frac{y^2}{6x^2}\right)\right\} - \frac{3xz\alpha\mu_1}{\beta\gamma^2}\left(\frac{\delta x}{x}\left[1 + n + \frac{2\gamma^2 y^2}{3x^2}\left(1 + \frac{n}{4}\right)\right] + (1-n)\frac{\delta z}{z} - \frac{\delta y}{y}\frac{2\gamma^2 y^2}{3x^2}\left(1 + \frac{n}{4}\right)\right)$$
$$- y\delta\left(\frac{H'}{H}\right) - \delta y\frac{H_0'}{H_0},$$
$$\delta z' = \frac{z^2 y\mu_1}{2x}\left((2-n)\frac{\delta z}{z} + \frac{\delta x}{x}\left[n - 1 + \frac{n\gamma^2 y^2}{6x^2}\right] + \frac{\delta y}{y}\left[1 - \frac{n\gamma^2 y^2}{6x^2}\right]\right) - \delta z\frac{H_0'}{H_0} - z\delta\left(\frac{H'}{H}\right), \quad (3.6)$$

where the notation H_0'/H_0 implies that we take this function evaluated at the critical points, and we have defined $n = (\alpha - \beta - 2)/(\alpha - \beta)$ for simplicity. Note that these are the leading order solutions only, and that all terms proportional to δ^2 have been neglected.

The stability of the fixed point solutions is therefore determined by the eigenvalues of the resulting perturbation matrix. A lengthy calculation which we will omit here shows that the point (0, 0, 0) leads to the eigenvalues

$$\lambda_1 = \frac{3(\omega - 1)}{2}, \qquad \lambda_2 = \frac{3(1+\omega)}{2}, \qquad \lambda_3 = \frac{3(1+\omega)}{2}, \quad (3.7)$$

which indicates that this is never a point of stability for the theory unless the equation of state is phantom, i.e. $\omega < -1$. In fact, this statement will be true for all the various cases we have considered in the physical limit, since the dynamical equations of motion all reduce to the exact same form in this instance.

Another relatively simple case to consider is that in case III. For slices through the (x, y) plane at $z = 0$, we find the eigenvalues

$$\lambda = \frac{1}{2}(x^2 + y^2) + \frac{3}{2}(\omega(x^2 - 1) - 1), \qquad \lambda_\pm = \frac{1}{4x^2}(-6x^4(1+\omega) - 6x^2 + 2y^2x^2 + 5y^2 \pm F(x, y)),$$
$$F(x, y) = \sqrt{12y^2x^2\omega + 96y^2x^4\omega - 48y^2x^6\omega - 8y^4x^2 + 48y^2x^4 + 16y^4x^4 - 48y^2x^6 + 36\omega^2x^4 + 17y^4}. \quad (3.8)$$

If one now slices this through $y = 0$, we see that we are left with the same situation discussed above (as expected), indicative of a phantom equation of state.

On the other hand, through the $y = 0$ plane we see that the eigenvalues become

$$\lambda = \frac{3}{2}(1+\omega)\left(1 - z^2 - x^2\left(1 - \frac{Qz^2}{x^2}\right)\right),$$
$$\lambda_\pm = -\frac{3}{2x}(-Qz^2 - x + 2xz^2 + x^3 - xz^2Q + x^2 \pm F(x, y)), \quad (3.9)$$

where F is another polynomial in x, z, and we have defined $Q = T/\tilde{V}$ for simplicity. In the limit that $z \to 0$, we find that these simplify to yield

$$\lambda \to \tfrac{3}{2}(1+\omega)(1 - x^2),$$
$$\lambda_\pm \to \tfrac{3}{2}(1+\omega)(x^2 - 1 - x^2(1 \pm 1)). \quad (3.10)$$

Note that two of the eigenvalues are therefore degenerate as before, requiring a phantom equation of state; however, the final eigenvalue has the opposite sign, and therefore this fixed point is always unstable.

The remaining fixed points can be analyzed in precisely the same manner, although the analysis is somewhat awkward. We will postpone the relevant discussion and return to it in a follow-up publication.

IV. DISCUSSION

We have initiated an alternate approach to the problem of k-essence, or DBI quintessence [25], using a more generalized form of the DBI action. Since this has more degrees of freedom, the resulting analysis is typically complicated, but the phase-space structure is far richer. We have attempted to make some headway by restricting the phase-space volume to various two-dimensional slices, and by attempting to identify the relevant solution curves upon which the fixed points may lie. Our ansatz for each of the unknown functions is also potentially restrictive; however, we are confident that it represents the leading semiclassical contributions which may (or may not) be derivable from a full string theory embedding of our model.

What is clear is that the ratio of the (warped) brane tension to the potential is an important factor in the dynamics of the theory, where we found $T \geq \tilde{V}$ in several cases. Moreover, the additional multiplicative factor $W(\phi)$ plays a crucial role, even when it is a constant, since it comes into the field equations nontrivially in the expression for H'/H. In the usual DBI analysis, $W = 1$ and the tension is the sole term responsible for the interesting quintessence behavior. In some string compactifications, where the warp factor has no cutoff at small distances, we typically find W is constant and greater than unity. However, there may be entire classes of solutions where $W \leq 1$, which can lead to novel phase-space trajectories. Since our approach has been phenomenological, and since there may be additional string backgrounds of interest that have yet to be fully explored, we cannot rule out $W < 1$—which is vital for obtaining fixed point solutions in case I, for example.

Our numerical results have shown that there is indeed a rich phase-space structure present due to the increased number of degrees of freedom. We expect many of these to yield highly nontrivial stable fixed points in the full analysis, which is beyond the scope of the current paper. We have classified the nature of as many of the fixed points as is feasible within the current analysis. Ultimately we hope that this will lead to a renewed interest in dynamical dark energy models driven by a more generalized approach to $D3$-brane dynamics.

In light of the recent developments in holographic dark energy [31,32] and the apparent relation to agegraphic [33,34] dark energy, we hope that it may be possible to reconstruct the various potentials in our generalized model along the lines of [35].

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4 Scalar field cosmology: Its non-linear Schrödinger-type formulation

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Scalar field cosmology is a model for dark energy and inflation. It has been recently found that the standard Friedmann formulation of the scalar field cosmology can be expressed in a nonlinear Schrödinger-type equation. The new mathematical formulation is hence called non-linear Schrödinger (NLS) formulation which is suitable for a FRLW cosmological system with non-negligible barotropic fluid density. Its major features are reviewed here.

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1. Introduction

The present universe is under accelerating expansion. This is convinced by many present observational data from cosmic microwave background [1], large scale structure surveys [2] and supernovae type Ia [3-5]. There are many ideas to explain such an expanding state, mainly it can be classified into three types: braneworlds and modification of gravitational theory (e.g. [6]), backreaction effect from inhomogeneity [7] and dark energy (for review, see [8]). Dark energy is a type of cosmological fluid appearing in the matter term of the Einstein equation with equation of state $w_{\rm D.E.} < -1/3$ so that it can generate repulsive gravity and therefore accelerating the universe. The simplest dark energy model is just a cosmological constant with $w_\Lambda = -1$. However the cosmological constant suffers from fine-tuning problem. Observational data suggests that the present value of $w_{\rm D.E.}$ is very close to -1 and it also allows possibility that dark energy could be dynamical. Therefore scalar field model of dark energy became interesting topic in cosmology since time-evolving behavior of the scalar field gives hope for resolving the fine-tuning problem. Although the scalar field has not yet been observed, it is motivated from many ideas in high energy physics and quantum gravities. Theoretical predictions of its existence at TeV scale could be tested at LHC and Tevatron in very near future. Phenomenologically the scalar field is also motivated in model building of inflation where super-fast acceleration happens in the early universe [9]. Cosmic microwave background data combined with other results allows possibility that the scalar field could be phantom, i.e. having equation of state coefficient $w_\phi < -1$ [10]. The phantom equation of state is attained from negative kinetic energy term in its Lagrangian density [11,12]. The most recent five-year WMAP result [13] combined with Baryon Acoustic Oscillation of large scale structure survey from SDSS and 2dFGRS [14] and type Ia supernovae data from HST [4], SNLS [5] and ESSENCE [15] assuming dynamical w with flat universe yields $-1.38 < w_{\phi,0} < -0.86$ at 95% CL and $w_{\phi,0} = -1.12 \pm 0.13$ at 68% CL. With additional BBN constraint of limit of expansion rate [16,17], $w_{\phi,0} = -1.09 \pm 0.12$ at 68% CL. The phantom field will finally dominate the universe in future, leading to Big Rip singularity [18]. There have been many attempts to resolve the singularity from both phenomenological and fundamental inspirations [19]. However fundamental physics of the phantom field is still incomplete due to severe UV instability of the field's quantum vacuum state [20].

This review interests in non-linear Schrödinger-type formulation of scalar field cosmology. We shall call the formulation, NLS formulation. In our NLS system, cosmological ingredients are scalar field and a barotropic fluid with constant equation of state, $p_\gamma = w_\gamma \rho_\gamma$. We also have non-zero spatial curvature. This is a system resembling of our present universe filled with scalar field dark energy and barotropic cold dark matter or of the early inflationary universe in presence of inflaton and other fields behaving barotropic-like considered in e.g. [32]. In such a model, the scale invariant spectrum in the cosmic microwave background was claimed to be generated not only from fluctuation of scalar field alone but rather from both scalar field and interaction between gravity to other gauge fields such as Dirac and gauge vector fields.

Not long ago, mathematical alternatives to the standard Friedmann canonical scalar field cosmology with barotropic perfect fluid, was proposed e.g. non-linear Ermakov-Pinney equation [21,22]. Expressing standard cosmology with $k > 0$ in Ermakov equation system yields a system similar to Bose-Einstein condensates [23]. Another example is a connection from a generalized Ermakov-Pinney equation with perturbative scheme to a generalized WKB method of comparison equation [24]. It was then realized that solutions of the generalized Ermakov-Pinney equation are correspondent to solutions of a non-linear Schrödinger-type equation, and then the NLS version of the Friedmannıtre-Robertson-Walker (FLRW) cosmology was formulated [22]. In the NLS framework, the system of FLRW cosmological equations: Friedmann equation, acceleration equation and fluid equation are written in a single nonlinear Schrödinger-type equation. We will not prove it here but instead, referring to Ref. [25]. Few recent applications [26-29] of the NLS formulation have been made and this review intends to conclude its major aspects.

2. Scalar field cosmology

A. Friedman formulation

We set up major concepts in this section before considering its application later. In the Friedmann system, barotropic fluid has pressure p_γ and density ρ_γ with an equation of state, $p_\gamma = w_\gamma \rho_\gamma = [(n-3)/3]\rho_\gamma$ where $n = 3(1 + w_\gamma)$. Scalar field pressure obeys $p_\phi = w_\phi \rho_\phi$. To sum up, $\rho_{\rm tot} = \rho_\gamma + \rho_\varphi$ and $p_{\rm tot} = p_\gamma + p_\phi$. Therefore $n = 0$ means $w_\gamma = -1$. The others are: $n = 2$ for $w_\gamma = -1/3$; $n = 3$ for $w_\gamma = 0$; $n = 4$ for $w_\gamma = 1/3$; $n = 6$ for $w_\gamma = 1$. Barotropic fluid and scalar fluid are conserved separately. Dynamics of the barotropic is governed by fluid equation, $\dot{\rho}_\gamma = -nH\rho_\gamma$ with solution, $\rho_\gamma = D/a^n$, where a is scale factor. The dot denotes time derivative. $H = \dot{a}/a$ is Hubble parameter and

$D \geq 0$ is a proportional constant. Scalar field is minimally-coupled to gravity with Lagrangian density, $\mathcal{L} = (1/2)\varepsilon \dot{\phi}^2 - V(\varphi)$ and is homogenously spread all over the universe. The scalar field density and pressure are

$$\rho_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi), \qquad p_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi). \tag{1}$$

The branch $\varepsilon = 1$ is for non-phantom field case and $\in = -1$ is for phantom field case. Dynamics of the scalar field is controlled by conservation equation, $\varepsilon(\ddot{\phi} + 3H\dot{\phi}) = -\mathrm{d}V/\mathrm{d}\phi$, in which the spatial expansion H of the universe sources friction to dynamics of the field. The Hubble parameter is governed by Friedmann equation, $H^2 = (\kappa^2/3)\rho_{\text{tot}} - k/a^2$, and by acceleration equation, $\ddot{a}/a = -(\kappa^2/6)(\rho_{\text{tot}} + 3p_{\text{tot}})$, which gives acceleration condition $p_{\text{tot}} < -\rho_{\text{tot}}/3$. Here $p_{\text{tot}} = w_{\text{eff}}\,\rho_{\text{tot}}$, $\kappa^2 \equiv 8\pi G = 1/M_{\text{P}}^2$. G is Newton's gravitational constant. M_{P} is reduced Planck mass. k is spatial curvature and

$$w_{\text{eff}} = \frac{\rho_\phi w_\phi + \rho_\gamma w_\gamma}{\rho_{\text{tot}}}. \tag{2}$$

If we express the field speed and the field potential in term of $a(t)$ and time derivative of $a(t)$, then

$$\epsilon\dot{\phi}(t)^2 = -\frac{2}{\kappa^2}\left[\dot{H} - \frac{k}{a^2}\right] - \frac{nD}{3a^n} \quad \text{and} \quad V(\phi) = \frac{3}{\kappa^2}\left[H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right] + \left(\frac{n-6}{6}\right)\frac{D}{a^n}. \tag{3}$$

B. NLS formulation

NLS formulation is a mathematical alternative to the standard Friedmann formulation with hope that the new formulation might suggest some new mathematical tackling to problems in scalar field cosmology. In the NLS formulation, there is no such an analogous equation to Friedmann equation or fluid equation. Instead both of them combine in single non-linear Schrödinger-type equation,

$$u''(x) + [E - P(x)]\,u(x) = -\frac{nk}{2}u(x)^{(4-n)/n}. \tag{4}$$

The links to cosmology are valid as one defines NLS quantities [25],

$$u(x) \equiv a(t)^{-n/2}, \quad E \equiv -\frac{\kappa^2 n^2}{12} D, \quad P(x) \equiv \frac{\kappa^2 n}{4} a(t)^n \epsilon \dot{\phi}(t)^2 . \tag{5}$$

where ' denotes d/dx. Independent variable t is scaled to NLS independent variable x as $x = \sigma(t)$, such that

$$\dot{x}(t) = u(x) \quad \text{and} \quad \phi(t) = \psi(x) , \tag{6}$$

which gives $\epsilon\dot{\phi}(t)^2 = \epsilon\dot{x}^2\,\psi'(x)^2$. Hence $\epsilon\,\psi'(x)^2 = (4/\kappa^2 n)\,P(x)$, and

$$\psi(x) = \pm\frac{2}{\kappa\sqrt{n}} \int \sqrt{\frac{P(x)}{\epsilon}}\, \mathrm{d}x . \tag{7}$$

Inverse function $\psi^{-1}(x)$ exists for $P(x) \neq 0$ and n $\neq$ 0. In this circumstance, $x(t) = \psi^{-1} \circ \phi(t)$ and the scalar field potential, $V \circ \sigma^{-1}(x)$ and $\epsilon\dot{\phi}(t)^2$ can be expressed in NLS formulation as

$$\epsilon\dot{\phi}(x)^2 = \frac{4}{\kappa^2 n} u u'' + \frac{2k}{\kappa^2} u^{4/n} + \frac{4E}{\kappa^2 n} u^2 , \tag{8}$$

and

$$V(x) = \frac{12}{\kappa^2 n^2}(u')^2 - \frac{2P}{\kappa^2 n} u^2 + \frac{12E}{\kappa^2 n^2} u^2 + \frac{3k}{\kappa^2} u^{4/n} . \tag{9}$$

The other equations are

$$\rho_\phi \;=\; \frac{12}{\kappa^2 n^2}(u')^2 + \frac{12E}{\kappa^2 n^2} u^2 + \frac{3k}{\kappa^2} u^{4/n} ; \tag{10}$$

$$p_\phi \;=\; -\frac{12}{\kappa^2 n^2}(u')^2 + \frac{4P}{\kappa^2 n} u^2 - \frac{12E}{\kappa^2 n^2} u^2 - \frac{3k}{\kappa^2} u^{4/n} , \tag{11}$$

$$\rho_{\text{tot}} \;=\; \frac{12}{\kappa^2 n^2}(u')^2 + \frac{3k}{\kappa^2} u^{4/n} , \tag{12}$$

$$p_{\text{tot}} \;=\; -\frac{12}{\kappa^2 n^2}(u')^2 + \frac{4}{\kappa^2 n} u u'' - \frac{k}{\kappa^2} u^{4/n} . \tag{13}$$

$$H = -\frac{2}{n}u', \qquad \dot{H} = -\frac{2}{n}uu'', \tag{14}$$

$$\ddot{\phi} = \frac{2Puu' + P'u^2}{\kappa\sqrt{P\epsilon n}}, \quad 3H\dot{\phi} = -\frac{12u'u}{n\kappa}\sqrt{\frac{P}{\epsilon n}}. \tag{15}$$

We shall see later examples that the program of NLS formulation must start from presuming the "wave function", $u(x) \equiv a^{-n/2} = \dot{x}(t)$, before proceeding to calculate the other quantities. We know that normalization condition for a wave function is $\int_{-\infty}^{\infty} |u(x)|^2 \mathrm{d}x = 1$. If applying this to our NLS wave function, then $\int_{-\infty}^{\infty} \dot{x}^2 \mathrm{d}x = 1$. In order to satisfy the condition, x must be constant (hence so is t) with an integrating constant = 1. In connecting Friedmann formulation to NLS formulation, we are forced to have $u(x) = \dot{x}(t)$. Therefore $u(x)$ is, in general, non-normalizable.

3. Slow-roll conditions

A. Slow-roll conditions: Flat geometry and scalar field domination

In flat universe with scalar field domination, $\dot{H} = -(\kappa^2/2)\dot{\phi}^2\epsilon$. Hencefor $\varepsilon = -1$(phantom field),

$$0 < aH^2 < \ddot{a}, \tag{16}$$

i.e. the acceleration is greater than speed of expansion per Hubble radius, $\dot{a}/cH^{-1}$ and for $\in = 1$ (non-phantom field),

$$0 < \ddot{a} < aH^2. \tag{17}$$

Slow-roll condition [30,31] assumes negligible kinetic term, i.e. $|\epsilon\dot{\phi}^2/2| \ll V(\phi)$ which makes an approximation $H^2 \simeq \kappa^2 V/3$. This results in a condition $\dot{H}| \ll H^2$. Slow-roll parameter, $\varepsilon \equiv -\dot{H}/H^2$ is hence defined from this relation. The condition $|\epsilon\dot{\phi}^2/2| \ll V(\phi)$ is then equivalent to $|\varepsilon| << 1$, i.e. $-1 << \varepsilon < 0$ for phantom field case and $0 < \varepsilon << 1$ for non-phantom field case. Considering $\dot{H} \simeq 0$ implying approximative constancy in H during the slow-rolling regime. For non-phantom field, this condition is necessary for inflation to happen (though not sufficient) [31] however, for phantom field case, the negative kinetic term always results in acceleration with $w_\phi \le -1$ then it does not need the slow-roll approximation. Another slow-roll parameter can be defined when the friction term dominates $|\ddot{\phi}| \ll |3H\dot{\phi}|$. This gives the second parameter, $\eta \equiv -\ddot{\phi}/H\dot{\phi}$ and the approximation is made to $|\eta| << 1$ [31]. The field fluid equation is then $\dot{\phi} \simeq -V_\phi/3\epsilon H$ which implies that if $\varepsilon = -1$, the field can roll up the hill. With all assumptions imposed here, i.e. $k = 0$, $\rho_\gamma = 0$, $|\epsilon\dot{\phi}^2/2| \ll V$ and $|\ddot{\phi}| \ll |3H\dot{\phi}|$, one can derive $\varepsilon = (1/2\kappa^2\varepsilon)(V_\phi/V)^2$ and $\eta = (1/\kappa^2)(V_{\phi\phi}/V)$ as known where the subscript ϕ denotes $d/d\phi$.

B. Slow-roll conditions: Non-flat geometry and non-negligible barotropic density

There are also inflationary models in presence of other field behaving barotropic-like apart from having only single scalar fluid [32]. The scale invariant spectrum in the cosmic microwave background was claimed to be generated not only from fluctuation of scalar field alone but rather from both scalar field and interaction between gravity to other gauge fields. Assuming this scenario with $k \neq 0$ and $\rho_\gamma = 0$, then

$$\dot{H} = -\frac{\kappa^2}{2}\dot{\phi}^2\epsilon + \frac{k}{a^2} - \frac{n\kappa^2}{6}\frac{D}{a^n}\,. \tag{18}$$

The slow-roll condition becomes $\kappa^2\epsilon\dot{\phi}^2/6| \ll (\kappa^2 V/3) - (k/a^2) + (\kappa^2 D/3a^n)$ hence

$$H^2 \simeq -\frac{\dot{H}}{3} + \frac{k}{3a^2} - \frac{n\kappa^2}{18}\frac{D}{a^n} + H^2\,, \tag{19}$$

implying $|-(\dot{H}/3) + (k/3a^2) - (n\kappa^2 D/18a^n)| \ll H^2$. We can reexpress this slow-roll condition as

$$|\,\varepsilon + \varepsilon_k + \varepsilon_D\,| << 1, \tag{20}$$

where $\varepsilon_k \equiv k/\,a^2H^2$ and $\varepsilon_D \equiv -n\kappa^2\, D/6a^nH^2$. Another slow-roll parameter η is defined as $\eta \equiv -\ddot{\phi}/H\dot{\phi}$, i.e. the same as the flat scalar field dominated case since the condition $|\ddot{\phi}| \ll |3H\dot{\phi}|$ is independent of k and ρ_γ.

Writing the condition $\epsilon\dot{\phi}^2/2| \ll V$ in NLS form using Eqs. (5) and (9),

$$|P(x)| \ll \frac{3}{n}\left[\left(\frac{u'}{u}\right)^2 + E\right] + \frac{3}{4}k\,n\,u^{(4-2n)/n}\,. \tag{21}$$

If the absolute sign is not used, the condition is then $\epsilon\dot{\phi}^2/2 \ll V$, allowing fast-roll negative kinetic energy. Then Eq. (21), when combined with the NLS equation (4), yields

$$u'' \ll \frac{3}{n}\frac{u'^2}{u} + \left(\frac{3}{n} - 1\right)Eu + \frac{kn}{4}u^{(4-n)/n}\,. \tag{22}$$

Friedman analog of this condition can be obtained simply by using Eq. (3) in the condition. Using Eq. (15), the second slow-roll condition, $|\ddot{\phi}| \ll |3H\dot{\phi}|$ in the NLS form is written as,

$$\left|\frac{P'}{P}\right| \ll \left|-2\left(\frac{6+n}{n}\right)\frac{u'}{u}\right|\,. \tag{23}$$

This condition yields the approximation $3H\epsilon\dot{\phi}^2 \simeq -\mathrm{d}V/\mathrm{d}\phi$ which, in NLS form, is

$$\frac{P'}{P} \simeq -\frac{2u'}{u} = nHa^{n/2}\,. \tag{24}$$

The slow-roll parameters ε, ε_k and ε_D, in NLS form, are

$$\varepsilon = \frac{nuu''}{2u'^2}\,, \quad \varepsilon_k = \frac{n^2ku^{4/n}}{4u'^2}\,, \quad \varepsilon_D = \frac{nE}{2}\left(\frac{u}{u'}\right)^2\,, \tag{25}$$

therefore

$$\varepsilon_{\text{tot}} = \varepsilon + \varepsilon_k + \varepsilon_D = \frac{n}{2}\left(\frac{u}{u'}\right)^2 P(x)\,. \tag{26}$$

Hence the slow-rollcondition, $|\varepsilon_{\text{tot}}| << 1$, is just

$$\left|\left(\frac{u}{u'}\right)^2 P(x)\right| \ll 1\,. \tag{27}$$

Another slow-roll parameter $\eta = -\ddot{\phi}/H\dot{\phi}$ can be found as follow. First considering $\psi(x) = \phi(t)$ and Eq. (7), using relation $\mathrm{d}/\mathrm{d}t = \dot{x}\,\mathrm{d}/\mathrm{d}x$, one can obtain

$$\eta = \frac{n}{2}\left(\frac{u}{u'}\frac{\psi''}{\psi'}+1\right) = \frac{n}{2}\left(\frac{u}{u'}\frac{P'}{2P}+1\right). \tag{28}$$

The slow-roll condition $|\eta| << 1$ in NLS form is just

$$\frac{u}{u'}\frac{P'}{2P}+1\Big| \ll 1\,. \tag{29}$$

4. Acceleration condition

For the phantom field, since its kinetic term is always negative and could take any large negative values, the slow-roll condition is not needed. The acceleration equation is taken as acceleration condition straightforwardly, i.e. $\ddot{a} > 0$ hence

$$\epsilon\dot{\phi}(x)^2 \; < \; -\left(\frac{n-2}{2}\right)\frac{D}{a^n}+V\,. \tag{30}$$

This, in NLS-type form, is equivalent to

$$E-P \; > \; -\frac{2}{n}\left(\frac{u'}{u}\right)^2 - \frac{nk}{2}\left(\frac{u^{2/n}}{u}\right)^2, \tag{31}$$

which is reduced to

$$u'' < \frac{2}{n}\frac{u'^2}{u}\,. \tag{32}$$

with help of the Eq. (4). Using Eqs. (14), the acceleration condition is just $\varepsilon < 1$.

5. Power-law cosmology

The power-law expansion $a(t) = t^q$, with $q > 1$ is assumed here as the first step of calculation. In some high-energy physics models, during inflation, flat geometry and scalar field domination are assumed. The universe was driven by an exponential potential $V(\phi) = [q(3q - 1)/(\kappa^2\, t_0^{\,2})]$ exp $\left\{-\kappa\sqrt{2/q}\,[\phi(t)-\phi(t_0)]\right\}$ [33]. Also, at late time with dark matter component, the expansion could be power-law. Recent results from X-Ray gas of galaxy clusters put a constraint of $q \sim 2.3$ for $k = 0$, $q \sim 1.14$ for $k = -1$ and $q \sim 0.95$ for $k = 1$ [34]. For a flat universe, the power law expansion,

$a = t^q$, is attained when $-1 < w_{\rm eff} < -1/3$ where $q = 2/[3(1 + w_{\rm eff})]$. If using $q = 2.3$ as above, it gives $w_{\rm eff} = -0.71$ (only flat case). Latest combined WMAP5 results with SNI and BAO yield $-0.0175 < \Omega_k < 0.0085$ at 95% maximum likelihood [13]. The mean is Ω_k = -0.0045 corresponding to closed universe with $q = 0.986$ [35]. Assuming power-law expansion, the Schrödinger wave function is [26]

$$u(x) = \dot{x}(t) = t^{-qn/2}\,. \tag{33}$$

Integrating the equation above so that the Schrödinger scale, x is related to cosmic time scale, t as

$$x = x(t) = -\frac{t^{-\beta}}{\beta} + x_0, \quad \text{and} \quad t(x) = \frac{1}{[-\beta(x-x_0)]^{1/\beta}}\,, \tag{34}$$

where $\beta \equiv (qn - 2)/2$ and x_0 is an integrating constant. The parameters x and t have the same dimension since β is only a number. Then the wave function is

$$u(x) = \left[\left(-\frac{1}{2}qn + 1\right)(x - x_0)\right]^{qn/(qn-2)}\,, \tag{35}$$

which depends on only q and n. Wave functions for a range of barotropic fluids are presented in Fig. 1. The result is confirmed by substituting Eq. (35) into Eq. (4). The field speed and scalar potential are:

$$\epsilon\dot{\phi}(t)^2 = \frac{2q}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} - \frac{nD}{3t^{qn}} \quad \text{and} \quad V(t) = \frac{q(3q-1)}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} + \left(\frac{n-6}{6}\right)\frac{D}{t^{qn}}\,. \tag{36}$$

From Eq. (5), therefore the Schrödinger potential is found to be

$$P(x) \;=\; \frac{2qn}{(qn-2)^2}\frac{1}{(x-x_0)^2} + \frac{kn}{2}\left[\frac{-2}{(qn-2)(x-x_0)}\right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12}\,. \tag{37}$$

With $E = -\kappa^2 n^2 D/12$, the Schrödinger kinetic energy is

$$T(x) \;=\; -\frac{2qn}{(qn-2)^2}\frac{1}{(x-x_0)^2} - \frac{kn}{2}\left[\frac{-2}{(qn-2)(x-x_0)}\right]^{2q(n-2)/(qn-2)}\,. \tag{38}$$

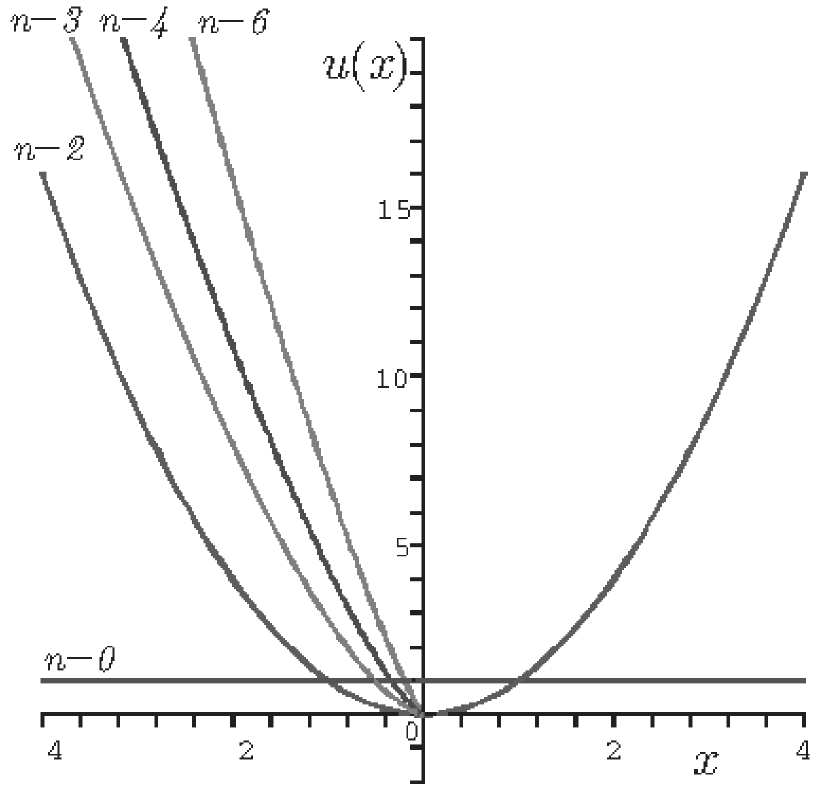

Figure 1. $u(x)$ versus x for power-law cosmology with $q = 2$. We set $x_0 = 0$. There is no real-value wave function for $n = 3$, $n = 4$ and $n = 6$ unless $x < 0$.

A disadvantage of Eq. (37) is that we can not use it in the case of scalar field domination. Dropping D term in Eq. (37) can not be considered as scalar field domination case since the barotropic fluid coefficient n still appears in the other terms. The non-linear Schrödinger-type formulation is therefore suitable when there are both scalar field and a barotropic fluid together such as the situation when dark matter and scalar field dark energy live together in the late universe or in the inflationary models in presence of other fields behaving barotropic-like and single scalar fluid [32]. $P(x)$ is plotted versus x for power-law expansion with $q = 2$ in closed, flat and open universe in Fig. 2. One can check that the acceleration condition (32) for the power-law case is just $q > 1$.

There is application of the NLS scalar field function ψ in Eq. (7) to solve for scalar field exact solutions in power-law, phantom expansion ($a \sim (t_a - t)^q$, $q < 0$) and exponential (de Sitter) expansion $a \sim \exp(t/\tau)$ [27,28]. For example in power-law case:

$$\psi(x) = \frac{\pm 2}{\kappa\sqrt{n}} \times \int \sqrt{\frac{2qn}{\epsilon(qn-2)^2}\frac{1}{(x-x_0)^2} + \frac{kn}{2\epsilon}\left[\frac{-2}{(qn-2)}\frac{1}{(x-x_0)}\right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12\epsilon}}\, dx \,. \quad (39)$$

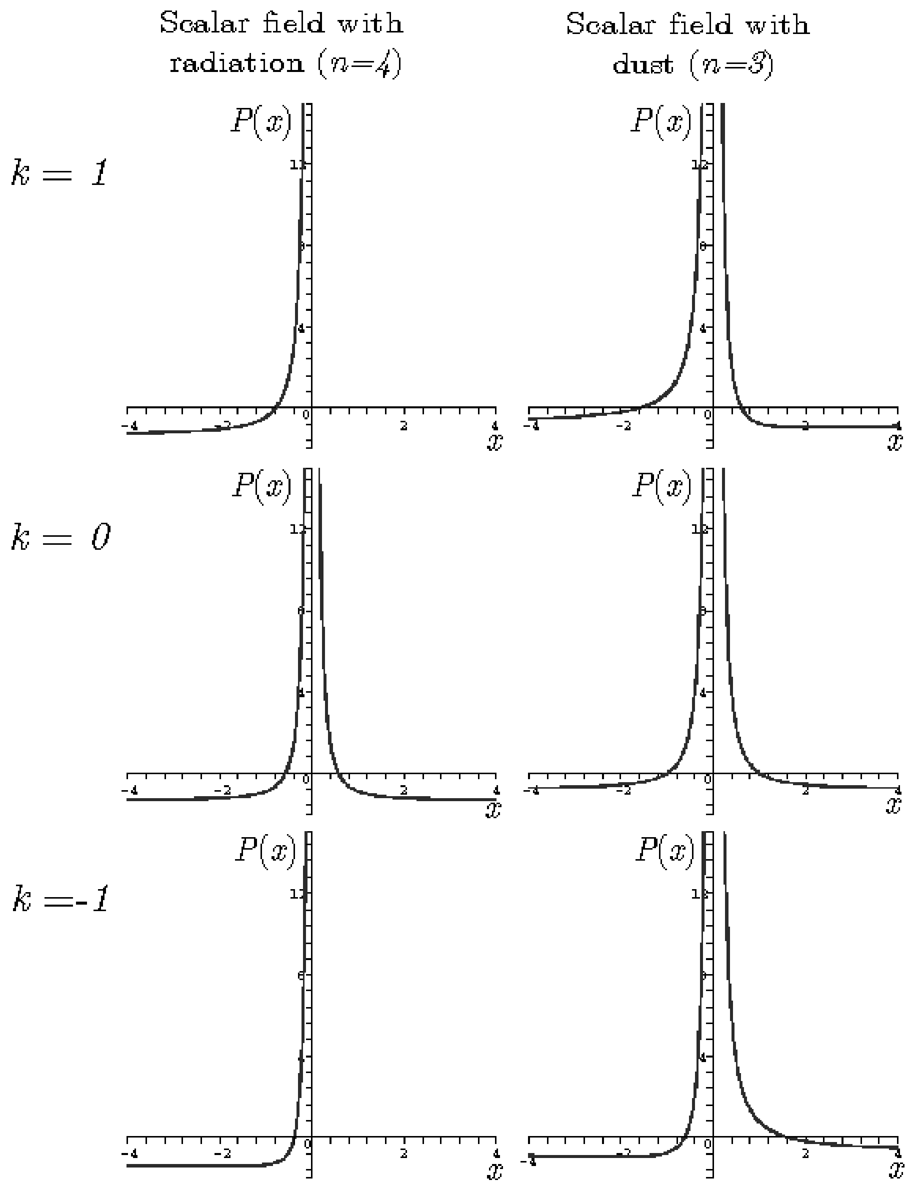

Figure 2. $P(x)$ plotted versus x for power-law expansion. We set $q = 2$, $\kappa = 1$, $D = 1$ and x_0. There is only a real-value $P(x)$ for the cases $k = \pm 1$ with $n = 4$ because, when $x > 0$, $P(x)$ becomes imaginary in these cases. The physical value is when $x < 0$ since t has a reverse sign of x.

The solution can be found only when assuming $k = 0$,

$$\phi(t) = \pm\frac{1}{qn-2}\sqrt{\frac{2q}{\epsilon\kappa^2}}\left\{\ln\left[\frac{t^{-qn+2}}{\left(1+\sqrt{1-(nD\kappa^2/6q)\,t^{-qn+2}}\right)^2}\right]+2\sqrt{1-\left(\frac{nD\kappa^2}{6q}\right)t^{-qn+2}}+\ln\left(\frac{qn-2}{2qn}\right)^2\right\}+\phi_0 \tag{40}$$

When $q = 2/n$ or $n = 0$, the field has infinite value. q and ε must have the same sign for the solution to be real. The last logarithmic term does not restrict sign of q. This is unlike the solution obtained from Friedmann formulation which requires $q < 0$ which violates power-law expansion condition ($q > 1$). Working in neither of them can obtain exact solution with $k \neq 0$. In NLS formulation, we can not set D to zero while n is multiplied to the other terms then it can not be reduced to the scalar dominant case. This is a weak aspect. Obviously, the most difficult case is when $k \neq 0$ with $D \neq 0$. This case can not be integrated out in both frameworks unless assuming $n = 2$ (equivalent to $w_\gamma = -1/3$) which is not physical.

There are other good aspects of the NLS formulation. Since transforming standard Friedmann formulation (t as independent variable) to NLS formulation (x as independent variable) makes n appear in all terms of the integrand and also changes fluid density term D from time-dependent term to a constant E, therefore the number of x (or equivalently t)-dependent terms is reduced by one and hence simplifying the integral (7). In the case of exponential (de Sitter) expansion using NLS formulation, the solution when $k \neq 0$ and $D \neq 0$ can be obtained without assuming n value but n = 0, 2, 3, 4, 6 must be given if working within Friedmann formulation. The phantom expansion case is very similar to the power-law case but only with different sign (see Ref. [28]).

6. Phantom cosmology and big rip singularity

If we assume the expansion to a form, $a(t) \sim (t_a - t)^q$ with a finite time t_a, one can see that $q = 2/3(1 + w_{\text{eff}}) < 0$ (for a flat universe). This corresponds to $w_{\text{eff}} < -1$. Such equation of state is called phantom. The Schrödinger scale, x is related to cosmic time scale, t as

$$x(t) = \frac{1}{\beta}\left[(t_{\rm a} - t)^{-\beta}\right] + x_0\,, \tag{41}$$

and the wave function is

$$u(x) = [\beta(x-x_0)]^{qn/(qn-2)} \tag{42}$$

which is plotted in Fig. 3 with various types of barotropic fluid [28].
Therefore

$$P(x) \;=\; \frac{2qn}{(qn-2)^2}\frac{1}{(x-x_0)^2} + \frac{kn}{2}\left[\frac{2}{(qn-2)(x-x_0)}\right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12}. \tag{43}$$

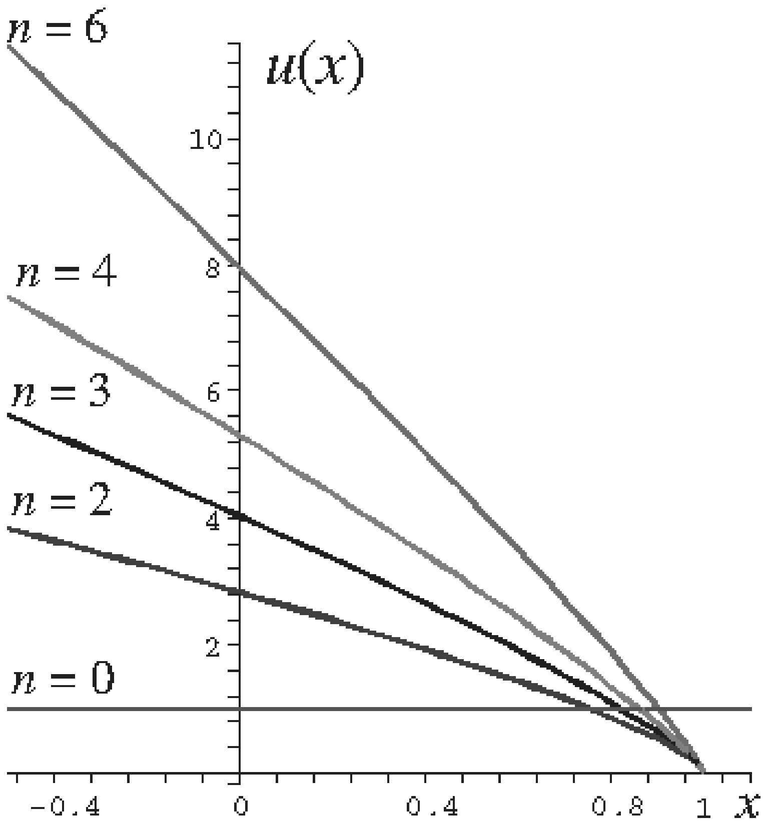

Figure 3. Schrödinger wave function, $u(x)$ when assuming phantom expansion. $u(x)$ depends on only q, n and t_a. Here we set $t_a = 1.0$ and $q = -6.666$. If $k = 0$, $q = -6.666$ corresponds to $w_{\rm eff} = -1.1$.

Fig. 4 shows $P(x)$ plots for three cases of k with dust and radiation. $P(x)$ goes to negative infinity at $x = x_0 = 1$. Expansion of the form, $a(t) \sim (t_a - t)^q$ leads to unwanted future Big Rip singularity [11]. The Big Rip conditions are that $(a, \rho_{\rm tot}, |p_{\rm tot}|) \rightarrow \infty$ which happen when $t \rightarrow t_a^{-n}$ in finite future time. Written in NLS language, if $a \rightarrow \infty$, $u \rightarrow 0^+$ and then $u \rightarrow 0$ (see Fig. 3). Considering also Eqs. (12) and (13), hence conditions of the Big Rip singularity are [29]

$$\begin{aligned} t \rightarrow t_{\rm a}^- \quad &\Leftrightarrow \quad x \rightarrow x_0^- \\ a \rightarrow \infty \quad &\Leftrightarrow \quad u(x) \rightarrow 0^+ \\ \rho_{\rm tot} \rightarrow \infty \quad &\Leftrightarrow \quad u'(x) \rightarrow \infty \\ |p_{\rm tot}| \rightarrow \infty \quad &\Leftrightarrow \quad u'(x) \rightarrow \infty \,. \end{aligned} \tag{44}$$

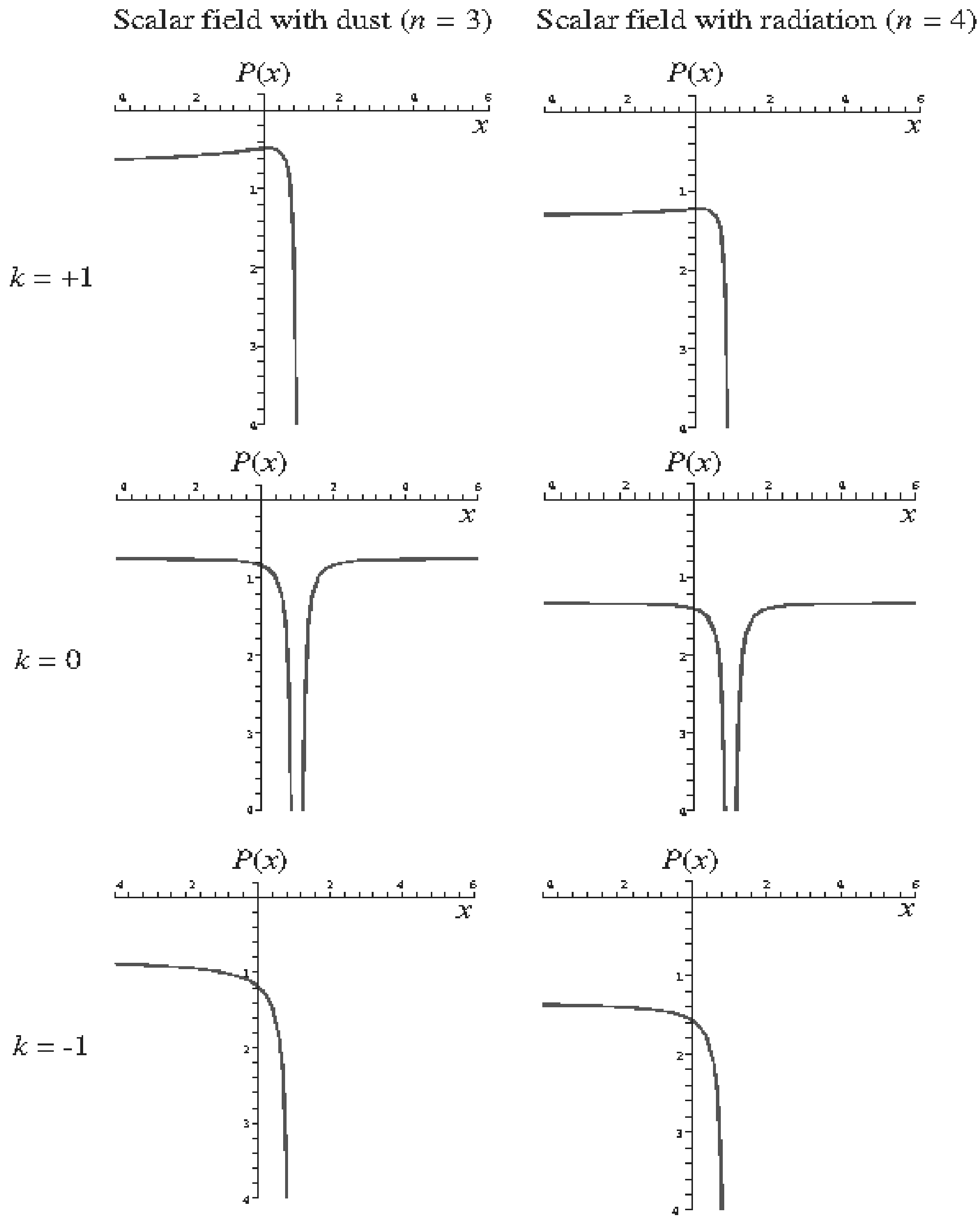

Figure 4. Schrödinger potential in phantom expansion case for dust and radiation fluids with $k = 0, \pm 1$. Numerical parameters are as in the $u(x)$ plots (Fig. 3). x_0 is set to 1. For non-zero k, there is only one real branch of $P(x)$.

We have one less infinite value in NLS Big Rip condition, i.e. $u(x)$ goes to zero. The NLS effective equation of state $w_{\text{eff}} = p_{\text{tot}} / \rho_{\text{tot}}$ can be expressed using Eqs. (12) and (13). Approaching the Big Rip, $x \to x_0^-$, $u \to 0^+$, then $w_{\text{eff}} \to -1 + 2/3q$, where $q < 0$ is a constant. This limit is the same as the effective phantom equation of state in the case $k = 0$. It is important to note that scalar field potential here is built phenomenologically based on expansion function, not on fundamental physics.

7. WKB approximation

WKB approximation in quantum mechanics is a tool to obtain wave function. However, in NLS formulation of scalar field cosmology, the wave function is first presumed before working out the shape of $P(x)$. Procedure is opposite to that of quantum mechanics. Hence the WKB approximation might not be needed at all for the NLS. Anyway, if one wants to test the WKB approximation in the NLS formulation, these below are some results. The WKB are valid when the coefficient of highest-order derivative term in the Schrödinger equation is small or when the potential is very slowly-varying. Consider linear case of Eq. (4), ($k = 0$),

$$-u'' + [P(x) - E]\, u = 0. \tag{45}$$

In Figs 2 and 4, the left-hand side of $P(x)$ is physical since it corresponds to positive time. In most regions, there are ranges of slowly varying $P(x)$ at large value of $|x|$, in which the WKB is valid. The approximation gives

$$a \sim A \exp\left[\pm(2/n) i \int_{x_1}^{x_2} \sqrt{E - P(x)}\, \mathrm{d}x\right], \tag{46}$$

where A is a constant.

8. Conclusions

Here we conclude aspects of NLS-type formulation of scalar field cosmology. The NLS-type formulation is well-applicable in presence of barotropic fluid and a canonical scalar field. There are few advantages of the NLS formulation as well as disadvantages to the conventional Friedmann formulation. With hope that some more interesting and useful features could be revealed in future.

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Determining scalar field potential in power-law cosmology with observational data

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In power-law cosmology, we determine potential function of a canonical scalar field in FLRW universe in presence of barotropic perfect fluid. The combined WMAP5+BAO+SN dataset and WMAP5 dataset are used here to determine the value of the potential. The datasets suggest slightly closed universe. If the universe is closed, the exponents of the power-law cosmology are $q = 1.01$ (WMAP5 dataset) and $q = 0.985$ (combined dataset). The lower limits of a_0 (closed geometry) are 5.1×10^{26} for WMAP5 dataset and 9.85×10^{26} for the combined dataset. The domination of the power-law term over the curvature and barotropic density terms is characterised by the inflection of the potential curve. This happens when the universe is 5.3 Gyr old for both datasets.

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I. INTRODUCTION

The presence of a scalar field is motivated by many ideas in high energy physics and quantum gravities, although it has not been discovered experimentally. TeV-scale experiments at LHC and Tevatron may be able to confirm its existence. It is nevertheless widely accepted in several theoretical modeling frameworks, especially in contemporary cosmology, in which an early-time accelerated expansion, i.e., inflation, is proposed to be driven by a scalar field in order to solve horizon and flatness problems [1]. After inflation, components of barotropic fluids such as radiation and other non-relativistic matter were produced during reheating and cooling-down processes. A scalar field was also believed to be responsible for the present acceleration in various models of dark energy [2]. The present acceleration is strongly backed up by various observations, e.g. the cosmic microwave background [3], large-scale structure surveys [4] and SNe type Ia observations [5–7].

Power-law cosmology, where $a \propto t^q$, describes an acceleration phase if $q > 1$. Modelling the present expansion with a power-law function where $q \sim 1$ was found to be consistent with nucleosynthesis [8, 9], the age of high-redshift objects such as globular clusters [8, 10–12], SNe Ia data [11], SNe Ia with $H(z)$ data [13], and X-ray gas mass fraction measurement of galaxy clusters [14, 15]. In the context of the power-law model, other aspects such as gravitational lensing statistics [12], angular size-redshift data of compact radio sources [16], and SN Ia magnitude-redshift relation [11, 17] have also been studied. Originally, the power-law expansion has its motivation from the simplest inflationary model that can remove the flatness and horizon problems with simple spectrum [18]. For the present universe, the idea of linear coasting cosmology ($a \propto t$) [19] can resolve the age problem of the CDM model [10] while as well agreeing with the nucleosynthesis constraint. The coasting model arises from non-minimally coupled scalar-tensor theory in which the scalar field couples to the curvature to contribute to the energy density that cancels out the vacuum energy [9, 20]. The model could also be a result of the domination of an SU(2) cosmological instanton [21].

Here our assumption is that the universe is expanding in the form of the power law function. Two major ingredients are scalar field dark energy evolving under the scalar field potential $V(\phi)$, and barotropic fluid consisting of cold dark matter and baryons. We derive the potential, and use the combined WMAP5 data [22] as well as the WMAP5 data alone to determine the values of q and other relevant parameters of the potential. The numerical results are subsequently compared and discussed.

II. COSMOLOGICAL SYSTEM WITH POWER-LAW EXPANSION

Two perfect fluids, the cold dark matter and scalar field $\phi \equiv \phi(t)$, in the late FLRW universe of the simplest CDM model with zero cosmological constant are considered. The time evolution of the barotropic fluid is governed by the fluid equation $\dot{\rho}_\gamma = -3H\rho_\gamma$, with a solution

$$\rho_\gamma = \frac{D}{a^n}, \tag{1}$$

where $n \equiv 3(1 + w_\gamma)$ and $D \geq 0$ is a proportional constant. For the scalar field, supposed that it is minimally coupled to gravity, its Lagrangian density is $\mathcal{L} = \dot{\phi}^2/2 - V(\phi)$. The energy density and pressure are

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{2}$$

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The fluid equation of the field describing its energy conservation as the universe expands is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}}{\mathrm{d}\phi}V = 0. \tag{3}$$

Total energy density $\rho_{\rm tot}$ and total pressure $p_{\rm tot}$ of the mixture are simply the sums of those contributed by each fluid, for which the Friedmann equation is

$$H^2 = \frac{8\pi G}{3}\rho_{\rm tot} - \frac{kc^2}{a^2}. \tag{4}$$

It is straightforward to show that

$$V(\phi) = \frac{3}{8\pi G}\left(H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right) + \left(\frac{n-6}{6}\right)\frac{D}{a^n}, \tag{5}$$

where $8\pi G$ is related to the reduced Planck mass $M_{\rm P}$ by $8\pi G = M_{\rm P}^{-2}$. The power-law scale factor is

$$a(t) = a_0\left(\frac{t}{t_0}\right)^q, \tag{6}$$

without fixing $a_0 = 1$ at the present time because we have implicitly rescaled it to allow for k taking only either one of the three discrete values $0, \pm 1$. The Hubble parameter is

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{q}{t}. \tag{7}$$

Our goal is to construct $V(t)$ using recent observational data, as far as the simplest CDM model is concerned.

III. SCALAR FIELD POTENTIAL

We will work with observational data in SI units. Restoring the physical constants in place, we obtain

$$V(\phi) = \frac{3M_{\rm P}^2 c}{\hbar}\left(H^2 + \frac{\dot{H}}{3} + \frac{2kc^2}{3a^2}\right) - \frac{Dc^2}{2a^3}, \tag{8}$$

where $M_{\rm P}^2 = \hbar c/8\pi G$ and we have set $n = 3$ ($w_\gamma = 0$ for dust). Incorporating (6) and (7) into the above equation, we obtain

$$V(t) = \frac{M_{\rm P}^2 c}{\hbar}\left(\frac{3q^2 - q}{t^2} + \frac{2kc^2 t_0^{2q}}{a_0^2 t^{2q}}\right) - \frac{Dc^2}{2}\frac{t_0^{3q}}{a_0^3 t^{3q}}. \tag{9}$$

We shall consider contribution of the first term alone in comparison to total contribution when including the second (the curvature) and the third (density) terms. It is worth noting that reconstruction of scalar field potential $V(\phi)$ was considered previously in context of flat universe with non-specified expansion law and using luminosity function of redshift z from SNe Ia observation [**?**].

A. Cosmological Parameters

Using the equation for the Hubble parameter (7) at the present time, we have

$$q = H_0 t_0. \tag{10}$$

The sign of k depends on the sign of the density parameter $\Omega_k \equiv -kc^2/a^2H^2$. In our convention here, $k = 1$ ($\Omega_k < 0$) for a closed universe, $k = 0$ for a flat one, and $k = -1$ ($\Omega_k > 0$) for an open one. The present value of the scale factor can be found from the definition of $\Omega_{k,0}$, that is,

$$a_0 = \frac{c}{H_0}\sqrt{\frac{-k}{\Omega_{k,0}}}. \tag{11}$$

The density constant D can be found from (1),

$$D = \rho_{\gamma,0}a_0^3 = \Omega_{\gamma,0}\rho_{c,0}a_0^3, \tag{12}$$

where $\Omega_{\gamma,0} = \Omega_{\rm CDM,0} + \Omega_{b,0}$, i.e. the sum of the present density parameters of the barotropic fluid components. $\rho_{c,0}$ is the present value of the critical density. The neutrino contribution is assumed to be negligible. The values of H_0, t_0, $\Omega_{k,0}, \Omega_{\rm CDM,0}$, and $\Omega_{b,0}$ are taken from observational data.

B. Observational Data

We work on two sets of data provided by [22]. One comes solely from the WMAP5 data and the other is the WMAP5 data combined with distance measurements from Type Ia supernovae (SN) and the Baryon Acoustic Oscillations (BAO) in the distribution of galaxies. For t_0, H_0, $\Omega_{b,0}$, and $\Omega_{\rm CDM,0}$, we take their maximum likelihood values. The curvature density parameter $\Omega_{k,0}$ comes as a range with 95% confidence level on deviation from the simplest ΛCDM model. The data are shown in Table I.

IV. RESULTS AND DISCUSSIONS

Using combined WMAP5+BAO+SN dataset, the potential is

$$V(t) = \frac{1.03\times 10^{26}}{t^2} + \frac{1.5\times 10^{23}}{t^{1.97}} - \frac{1.5\times 10^{42}}{t^{2.96}}, \tag{13}$$

whereas, for WMAP5 dataset alone,

$$V(t) = \frac{1.11\times 10^{26}}{t^2} + \frac{7.6\times 10^{24}}{t^{2.03}} - \frac{4.6\times 10^{43}}{t^{3.04}}. \tag{14}$$

in SI units. We use the mean of each $\Omega_{k,0}$ interval to represent $\Omega_{k,0}$ in each of the above equations. Their plots are shown in Fig. 1. In both cases, $\bar{\Omega}_{k,0}$ is negative (a closed universe). The points at which the potential, its derivative, and its second-order derivative, are zero ($t_{\rm intercept}$,

Parameter	WMAP5+BAO+SN	WMAP5
t_0	13.72 Gyr	13.69 Gyr
H_0	70.2 km/s/Mpc	72.4 km/s/Mpc
$\Omega_{b,0}$	0.0459	0.0432
$\Omega_{\mathrm{CDM},0}$	0.231	0.206
$\Omega_{k,0}$	$-0.0179 < \Omega_{k,0} < 0.0081$	$-0.063 < \Omega_{k,0} < 0.017$

TABLE I: Observational data used in the construction of our scalar-field potentials [22]

	WMAP5+BAO+SN		WMAP5	
	$\bar{\Omega}_{k,0} = -0.0045$	$-0.0175 < \Omega_{k,0} < 0.0085$	$\bar{\Omega}_{k,0} = -0.023$	$-0.063 < \Omega_{k,0} < 0.017$
q	0.985	0.985	1.01	1.01
a_0	1.9×10^{27}	$a_0 > 9.85 \times 10^{26}$ (closed) $a_0 > 1.5 \times 10^{27}$ (open)	8.4×10^{26}	$a_0 > 5.1 \times 10^{26}$ (closed) $a_0 > 9.8 \times 10^{26}$ (open)
$t_{\mathrm{intercept}}$	2.7 Gyr	2.62 Gyr $< t <$ 2.7 Gyr	2.7 Gyr	2.6 Gyr $< t <$ 2.8 Gyr
$t_{\max}$	4.0 Gyr	3.94 Gyr $< t <$ 4.0 Gyr	4.0 Gyr	3.8 Gyr $< t <$ 4.1 Gyr
$t_{\mathrm{inflection}}$	5.3 Gyr	5.26 Gyr $< t <$ 5.4 Gyr	5.3 Gyr	5.1 Gyr $< t <$ 5.5 Gyr

TABLE II: A summary of numerical results. Times are shown in Gyr for comprehensibility. Positive and negative Ω_k's correspond to open and closed universes, respectively.

$t_{\max}$, and $t_{\mathrm{inflection}}$, respectively) are also determined, for both $\bar{\Omega}_{k,0}$ and each end of the $\Omega_{k,0}$ interval. The results are summarised in Table II.

The values of the exponent q from the two sets of data are only slightly different, but only the latter is an accelerated expansion as $q > 1$. The determination of q from X-Ray gas mass fractions in galaxy clusters favours open universe with $q > 1$ ($q = 1.14 \pm 0.05$) [15] and combined analysis from SNLS and $H(z)$ data (from Germini Deep Deep Survey) assuming open geometry yields $q = 1.31$ [13]. Note that, in the power-law regime, q only depends on the observed values of the Hubble constant and t_0. This may give an impression that the maximum likelihood values from the combined data has yet to be relied upon, but the power-law expansion has not been proven to be the case nonetheless.

After $t_{\mathrm{inflection}}$, the potential from each data behaves like its first term, i.e. decreasing in its value while increasing in its slope (being less and less negative). The other terms quickly become weaker. This can be seen in Fig. 1. Since the first term is contributed only by $H(t)$ (and its time derivative), it is dominant in the post-inflection phase. In fact, the convergence to zero of the potential is slower than its first term alone (see (13) and (14)), because the sum of the last two terms consequently becomes positive before converging to zero. This means that the plots of each potential and its first term in Fig. 1 eventually crosses, but it occurs much, much later at $t = 8.8 \times 10^{67}$ Gyr. Along with the potential function in (5), we also obtain the solution

$$\phi(t) = \int \sqrt{-\frac{2M_{\mathrm{P}}^2 c}{\hbar}\left(\dot{H} - \frac{kc^2}{a^2}\right) - \frac{Dc^2}{a^3}}\, \mathrm{d}t \qquad (15)$$

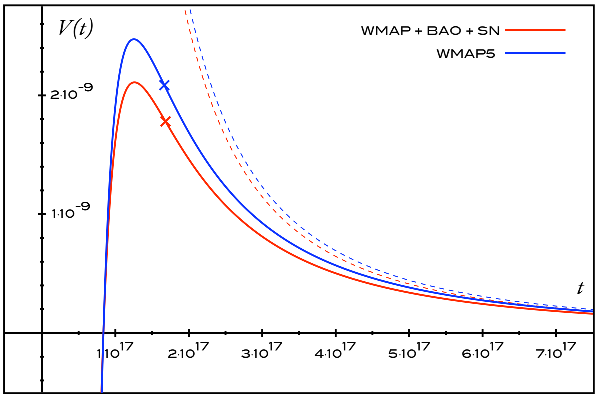

FIG. 1: The potentials in (13) and (14). The units of the abscissa and ordinate axes are sec and J/m^3, respectively. The crosses mark their inflection points. Also plotted in dash lines are their first terms. Each potential does not actually converge to its first term, but later intersect with and deviate from it, though still very close together. However, this occurs much later (at $t = 2.8 \times 10^{84}$ sec $= 8.8 \times 10^{67}$ Gyr in both cases).

in SI units. Using WMAP5+BAO+SN dataset,

$$\phi(t) = \int \sqrt{\frac{1.06 \times 10^{26}}{t^2} + \frac{1.5 \times 10^{23}}{t^{1.97}} - \frac{3.0 \times 10^{42}}{t^{2.96}}}\, \mathrm{d}t, \qquad (16)$$

where, for WMAP5 dataset alone,

$$\phi(t) = \int \sqrt{\frac{1.09 \times 10^{26}}{t^2} + \frac{7.6 \times 10^{24}}{t^{2.03}} - \frac{9.3 \times 10^{43}}{t^{3.04}}}\, \mathrm{d}t. \qquad (17)$$

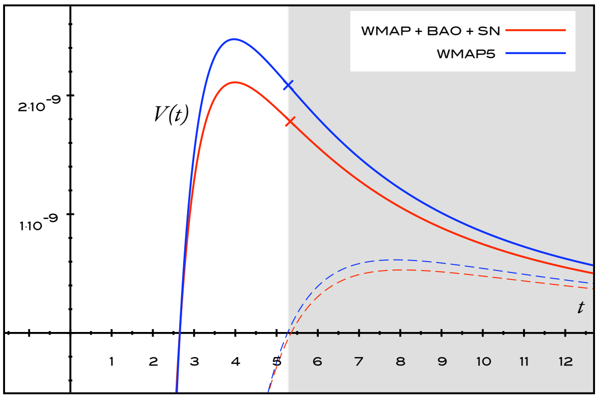

FIG. 2: The potentials in (13) and (14) along with the radicands of the integrands in (16) and (17) (dash line). The shaded region is the post-inflection phase. The unit of the abscissa axis is sec. After $t_{\rm inflection}$, $\phi(t)$ is real.

In the late post-inflection phase, the first term is dominant over the k and D terms then the last two terms of the radicands are negligible (Fig. 2). The above two equations are approximated as

$$\phi(t) \approx 1.04 \times 10^{13} \ln t, \qquad (18)$$

whereas, for WMAP5 dataset alone,

$$\phi(t) \approx 1.03 \times 10^{13} \ln t. \qquad (19)$$

The radicand in (17) of the WMAP5 dataset is zero at approximately $t_{\rm inflection} = 5.3$ Gyr (see Fig. 2), therefore so does $\phi(t)$. While the combined dataset has the zero radicand (then zero $\phi(t)$) in (16) later at approximately $t = 5.4$ Gyr. Scalar field exact solutions for the power-law cosmology with non-zero curvature and non-zero matter density are reported in [23]. It is also worth noting that the general exact form of the potential, that renders scaling solution, is some negative powers of a hyperbolic sine [24].

V. CONCLUSION

We consider a potential function of a homogeneous scalar field in late-time FLRW universe of the simplest CDM model with zero cosmological constant, assuming power-law expansion. The scalar field is minimally coupled to gravity and the other fluid is non-relativistic barotropic perfect fluid. We use two sets of observational data, combined WMAP5+BAO+SN dataset and WMAP5 dataset, as the inputs. Potential functions are obtained using numerical values from the observations. Mean values of both sets suggest slightly closed geometry. The WMAP5 dataset implies accelerated expansion ($q = 1.01$) while the combined dataset gives $q = 0.985$. This is slightly lower than the value obtained from SNLS and $H(z)$ data ($q = 1.31$) [13] and X-Ray gas mass fraction ($q = 1.14 \pm 0.05$) [15]. Our result is independent of the geometry unlike q obtained from [13] which assumes open geometry. For closed universe, the WMAP5 dataset puts the lower limit of 5.1×10^{26} for a_0 while the combined dataset puts the lower limit of 9.85×10^{26}. We characterise the domination of the first term of (9) by using the inflection of the potential plots from which the first term is found to be dominant to the potential 5.3 Gyr after the Big Bang in both datasets.

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