รายงานวิจัยฉบับสมบูรณ์

โครงการ กราฟต่อเนื่อง-แกมมา
Gamma-Continuous Graphs

โดย ผู้ช่วยศาสตราจารย์ ดร. วราภรณ์ แสนพลพัฒน์
มหาวิทยาลัยศรีนครินทรวิโรฒ

มิถุนายน 2556

สัญญาเลขที่ RMU 53800031

รายงานวิจัยฉบับสมบูรณ์

โครงการ: กราฟต่อเนื่อง-แกมมา

ผู้ช่วยศาสตราจารย์ ดร. วราภรณ์ แสนพลพัฒน์ มหาวิทยาลัยศรีนครินทรวิโรฒ

สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา
สำนักงานกองทุนสนับสนุนการวิจัย และมหาวิทยาลัยศรีนครินทรวิโรฒ

กิตติกรรมประกาศ

ผู้วิจัยขอขอบระคุณ คณะกรรมการการอุดมศึกษา สำนักงานกองทุนสนับสนุน การวิจัย และมหาวิทยาลัยศรีนครินทรวิโรฒ ซึ่งได้ให้การสนับสนุนทุนวิจัย สำหรับการวิจัยครั้งนี้

ผู้ช่วยศาสตราจารย์ ดร. วราภรณ์ แสนพลพัฒน์
มหาวิทยาลัยศรีนครินทรวิโรฒ

บทคัดย่อ

รหัสโครงการ: RMU 53800031

ชื่อโครงการ: กราฟต่อเนื่อง-แกมมา

ชื่อนักวิจัย: ผู้ช่วยศาสตราจารย์ ดร. วราภรณ์ แสนพลพัฒน์
มหาวิทยาลัยศรีนครินทรวิโรฒ

E-mail Address: varaporn@swu.ac.th

ระยะเวลาโครงการ: วันที่ 15 มิถุนายน 2553 ถึงวันที่ 14 มิถุนายน 2556

วัตถุประสงค์:

1. ศึกษาแกมมาเลเบิลลิงของกราฟ
2. ศึกษาลักษณะและคุณสมบัติของกราฟต่อเนื่อง -แกมมา และกราฟต่อเนื่องสลับ -แกมมา
3. ศึกษาความสัมพันธ์ของกราฟต่อเนื่อง -แกมมา และกราฟต่อเนื่องสลับ -แกมมา
4. ศึกษาความหนาแน่น-แกมมาของกราฟ G ที่มีค่าน้อยที่สุด
5. สร้างองค์ความรู้ใหม่ซึ่งสามารถใช้เป็นพื้นฐานในการพัฒนาวิชาทฤษฎีกราฟ
6. เพื่อใช้เป็นฐานข้อมูลในการอ้างอิงต่อไป

วิธีวิจัย :

1. รวบรวมและศึกษาเอกสารที่เกี่ยวข้องจากแหล่งต่างๆ
2. คิดค้นเทคนิคที่เหมาะสมเพื่อใช้ในการแก้ปัญหา
3. อภิปรายแลกเปลี่ยนความคิดเห็นและขอคำแนะนำจากผู้เชี่ยวชาญในสาขานี้
4. สรุปผล นำเสนอผลงานในที่ประชุมวิชาการ
5. ปรับปรุงแก้ไขส่วนที่ยังขาดตกบกพร่อง (ถ้ามี) เพื่อทำต้นฉบับจริง และนำส่งเพื่อลงพิมพ์ในวารสารต่างประเทศ

สรุปงานวิจัย :

1. ค้นพบค่าสุดขีดของแกมมาเลเบิลของกราฟวัฏจักรที่มีสามเหลี่ยม
2. ค้นพบค่าสุดขีดของแกมมา-เดลต้าเลเบิลของกราฟวัฏจักร
3. ตอบคำถามเปิดของงานวิจัย Gamma-labelings of complete bipartite graphs โดย G. D. Bullington, L. L. Eroh, และ S. J. Winters ในวารสาร Discuss. Math. Graph Theory, 30 (2010) 45-54.
4. แสดงลักษณะเฉพาะของแกมมาเลเบิลสูงสุดของกราฟใดๆ
5. แสดงลักษณะเฉพาะของแกมมาเลเบิลสูงสุดของกราฟแบ่งกั้นสองส่วนบริบูรณ์และกราฟบริบูรณ์
6. สร้างการพิสูจน์ $val_{max}(K_{r,s})$ อีกแบบโดยใช้แกมมาเลเบิลต่ำสุดของกราฟบริบูรณ์

ข้อเสนอแนะสำหรับงานวิจัยในอนาคต :

เราควรทำการศึกษาลักษณะเฉพาะของแกมมาเลเบิลลิงต่ำสุดของกราฟใดๆ

คำหลัก: แกมมาเลเบิลลิง แกมมาเลเบิลต่ำสุด และ แกมมาเลเบิลสูงสุด

Abstract

Project Code: RMU 53800031

Project Title: Gamma-Continuous Graphs

Investigator: Asst. Prof. Dr. Varaporn Saenpholphat, Srinakharinwirot University

E-mail Address: varaporn@swu.ac.th

Project Period: June 15, 2010 – June 14, 2013

Purposes of this research:

1. To study gamma labeling of graphs.
2. To study properties of gamma-continuous graph and alternative gamma-continuous graph.
3. To study relationship between gamma-continuous graph and alternately gamma-continuous graph.
4. To study minimum gamma-density of graphs
5. To obtain original research results in graph theory emphasizing on gamma labeling.
6. To serve as gamma labeling research for future study and reference.

Research Methodology:

1. Studying and searching for literature review.
2. Developing tools and new technique or creating a new technique by combining existing known technique.
3. Discussing with graph theory experts.
4. Presenting our work to a conference in graph theory area
5. Submitting our final work to publish in leading journal.

Research Results:

1. We determine the extremal values of a gamma-labeling of a cycle with a triangle.
2. We determine the extremal values of a gamma-delta-labeling of some graphs.
3. We give answer to the open question posed by G. D. Bullington, L. L. Eroh, and S. J. Winters in Discuss. Math. Graph Theory, 30 (2010) 45-54.
4. We characterize gamma-min labelings of a graph.
5. We characterize gamma-max labelings of complete bipartite graph and complete graphs.
6. We provide an alternative and improved proof of $val_{max}(K_{r,s})$ given in the literature that employs gamma–min labelings of complete graphs.

Suggestion for Further research:

We should study and characterize gamma-max labeling of graphs.

Keyword: gamma labeling, gamma-min labeling and gamma-max labeling

Introduction

Let G be a graph of order n and size m. A γ-labeling of G is a one-to-one function $f\colon V(G) \to \{0,1,\ldots,m\}$ that induces an edge-labeling $f'\colon E(G) \to \{1,\ldots,m\}$ on G defined by

$$f'(uv) = |f(u) - f(v)|\,, \quad \text{for each edge } uv \text{ in } E(G)\,.$$

The value of f is defined by

$$\mathrm{val}(f) = \sum_{e \in E(G)} f'(e)\,.$$

This particular graph labeling was first introduced by Chartrand et al. [5], in 2005, and it was motivated by Rosa's influential paper [14], where a variety of types of "valuations" were introduced. (We remark that this notion should not be confused with the γ-labeling recently introduced by Blinco et al. in [1].)

When the labels $f'(e)$ are all distinct, the γ-labeling is called a graceful labeling. The origin of this concept is often attributed Gerhard Ringel [13]. The Ringel-Kotzig Conjecture, as it was coined in [12], and popularized by Golomb [10], states that all trees are graceful and it remains still unsolved. This conjecture is also known as Von Koch's Conjecture or Graceful Labeling Conjecture [14]. The study of graceful labelings and their applications has been intensively considered in the literature [2, 9, 11].

Obviously, since f is one-to-one, it follows that $f'(e) \geq 1$, for any edge e, and therefore, $\mathrm{val}(f) \geq m$. Moreover, G has a γ-labeling if and only if $m \geq n-1$ and every connected graph has a γ-labeling.

The maximum value and the minimum value of a γ-labeling of a graph G are defined in [5] as

$$\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f)\colon\ f \text{ is a } \gamma\text{-labeling of } G\}$$

and

$$\mathrm{val}_{\min}(G) = \min\{\mathrm{val}(f)\colon\ f \text{ is a } \gamma\text{-labeling of } G\}\,,$$

respectively.

A γ-labeling g of G is a γ-max *labeling* if $\mathrm{val}(g) = \mathrm{val}_{\max}(G)$ and a γ-labeling h is a γ-min *labeling* if $\mathrm{val}(h) = \mathrm{val}_{\min}(G)$. The *span* of a γ-labeling f of G is defined as

$$\mathrm{span}(f) = \max\{f(v) : v \in V(G)\} - \min\{f(v) : v \in V(G)\}.$$

The following results appear in [5].

Theorem A [[5]] *Let G be a connected graph of order n and let $f : V(G) \to \mathbf{Z}$ be a one-to-one function. Then there is a γ-labeling g on G with* $\mathrm{val}(g) \leq \mathrm{val}(f)$. *Furthermore, if* $\mathrm{span}(f) \geq n$, *then there is a γ-labeling g with* $\mathrm{val}(g) < \mathrm{val}(f)$.

For integers a and b with $a \leq b$, let

$$[a..b] = \{a, a+1, \ldots, b\}$$

be the set of integers between a and b. Such a set is referred to as a *consecutive set* of integers.

Theorem B [[5]] *If G is a connected graph of order n, then G has a γ-*min *labeling f such that $f(V(G)) = [0..n-1]$.*

The maximum and minimum values of γ-labelings of nontrivial complete graphs and complete bipartite graphs were determined in [5], as we stated next.

Theorem C [[5]] *For every integer $n \geq 2$,*

$$\begin{aligned}
\mathrm{val}_{\min}(K_n) &= \binom{n+1}{3} \\
\mathrm{val}_{\max}(K_n) &= \begin{cases} \frac{n(3n^3-5n^2+6n-4)}{24} & \textit{if } n \textit{ is even} \\ \frac{(n^2-1)(3n^2-5n+6)}{24} & \textit{if } n \textit{ is odd.} \end{cases}
\end{aligned}$$

Let K_n be a complete graph of order $n \geq 2$ with $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. If f is a γ-labeling of K_n such that $f(v_i) = a_i$ for $1 \leq i \leq n$ and

$$0 \leq a_1 < a_2 < \cdots < a_n \leq \binom{n}{2},$$

then the value of f was determined in [8], namely

$$\mathrm{val}(f) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (n-2i+1)(a_{n-i+1} - a_i). \tag{1}$$

Theorem D [[4]] *For any two positive integers $r \geq s$,*

$$\begin{aligned}
\mathrm{val}_{\min}(K_{r,s}) &= \frac{s(2s^2+1)}{3} + s^2(r-s) + s\left\lfloor \frac{(r-s)^2}{4} \right\rfloor \quad \textit{and} \\
\mathrm{val}_{\max}(K_{r,s}) &= rs(rs - \frac{1}{2}(r+s)+1).
\end{aligned}$$

The maximum and minimum values of a γ-labeling of path, star, cycle, and complete graph are determined. Next, we recall the extremal values of a γ-labeling of cycle of order n.

Theorem E [[5]] *If C_n is a cycle of order $n \geq 3$, then*

$$\mathrm{val}_{\min}(C_n) = 2(n-1)$$

and

$$\mathrm{val}_{\max}(C_n) = \begin{cases} \frac{(n-1)(n+3)}{2} & \textit{if } n \textit{ is odd} \\ \frac{n(n+2)}{2} & \textit{if } n \textit{ is even.} \end{cases}$$

The γ-spectrum of a graph G is also defined in [5] as

$$\operatorname{spec}(G) = \{\operatorname{val}(f) \,:\, f \text{ is a } \gamma\text{-labeling of } G\},$$

and the γ-spectrum of any star is determined as well. In [6], the same authors found the maximum and the minimum values of a γ-labeling of double star, and determine all graphs G of order n for which $\operatorname{val}_{\min}(G)$ is n or $n+1$. Some results about minimum value of a γ-labeling of a tree with maximum degree $\Delta = 3$ were established. Later, in [8], the γ-spectra of paths, cycles and complete graphs were determined.

In [5] a simple and useful connection between minimum and maximum values of a connected graph and that of a proper connected subgraph was found.

Proposition F [[5]] *If H is a proper connected subgraph of a connected graph G, then*

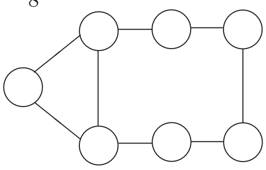
$$\operatorname{val}_{\min}(H) < \operatorname{val}_{\min}(G) \text{ and } \operatorname{val}_{\max}(H) < \operatorname{val}_{\max}(G).$$

The span of a γ-labeling f is defined as

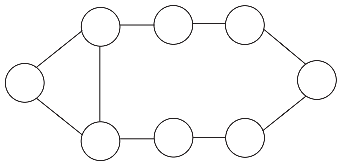
$$\operatorname{span}(f) = \max\{f(v) : v \in V(G)\} - \min\{f(v) : v \in V(G)\}.$$

Here we consider cycles of order $n \geq 5$ with a triangle, say $C_n^{\triangle}$, i.e., cycles with a chord joining two nonadjacent vertices but adjacent to some vertex in cycle. As below, we show $C_7^{\triangle}$ and $C_8^{\triangle}$:

$C_7^{\triangle}$:



$C_8^{\triangle}$:



In this paper, we determine $\operatorname{val}_{\min}(C_n^{\triangle})$ and $\operatorname{val}_{\max}(C_n^{\triangle})$. For the later case, we need to generalize the notion of γ-labeling of a connected graph. Several illustrative examples will be provided.

The reader is referred to Chartrand and Zhang [7] for basic definitions and terminology not mentioned here.

Main Results

1 The minimum value of $C_n^{\triangle}$

As in the case of the cycles, the minimum value of $C_n^{\triangle}$ is somewhat easier to determine.

Theorem 1.1 *For every integer $n \geq 5$,*

$$\mathrm{val}_{\min}(C_n^{\triangle}) = 2n - 1\,.$$

Proof. By Proposition F and Theorem E, we have

$$\mathrm{val}_{\min}(C_n^{\triangle}) \geq 2n - 1\,.$$

Hence it remains to show that $\mathrm{val}_{\min}(C_n^{\triangle}) \leq 2n - 1$.

Suppose that $C_n^{\triangle}$ is the cycle $C_n : v_1, v_2, \ldots, v_n, v_1$ with the triangle obtained by adding an edge $v_2 v_n$. Considering now the γ-labeling f of $C_n^{\triangle}$ defined by

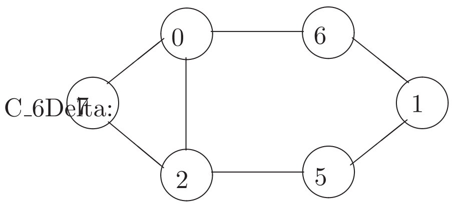
$$f(v_i) = \begin{cases} 0 & \text{if } i = 1 \\ i & \text{if } 2 \leq i \leq n-1 \\ 1 & \text{if } i = n, \end{cases}$$

we have

$$\begin{aligned} \mathrm{val}(f) &= \textstyle\sum_{i=2}^{n-2} |f(v_{i+1}) - f(v_i)| + |f(v_1) - f(v_2)| + \\ &\qquad + |f(v_2) - f(v_n)| + |f(v_n) - f(v_1)| + |f(v_{n-1}) - f(v_n)| \\ &= (n-3) \cdot 1 + 2 + 1 + 1 + (n-2) \;=\; 2n - 1\,. \end{aligned}$$

Therefore, $\mathrm{val}_{\min}(C_n^{\triangle}) \leq 2n - 1$. ■

To illustrate Theorem 1.1, we consider, for example, the following γ-min labeling f of $C_6^{\triangle}$:



Observe that $\mathrm{val}_{\min}(C_6^{\triangle}) = \mathrm{val}(f) = 11\,.$

2 The γ^δ-Labeling of a Connected Graph

We start this section with a slightly extension of the main notion of γ-labeling of a connected graph. For a nonnegative integer δ, a γ^δ-labeling of a connected graph G of order n and size m is a one-to-one function $f : V(G) \to \{0, 1, \ldots, m+\delta-1, m+\delta\}$ that induces an edge-labeling $f'\colon E(G) \to \{1, \ldots, m+\delta\}$ on G defined by

$$f'(uv) = |f(u) - f(v)|\,, \quad \text{for each edge } uv \text{ in } E(G)\,.$$

The value of a γ^δ-labeling f is defined by $\mathrm{val}(f) = \sum_{e\in E(G)} f'(e)$. The other definitions are similar to the γ-labeling case.

The following corollary is a consequence of Theorem A.

Corollary 2.1 *For any connected graph G of order n,*

$$\mathrm{val}^\delta_{\min}(G) = \mathrm{val}_{\min}(G)\,.$$

Next, we present the maximum value of a γ^δ-labeling of C_n when n is even.

Proposition 2.2 *For a nonnegative integer δ and an even integer n, with $n \geq 4$,*

$$\mathrm{val}^\delta_{\max}(C_n) = \mathrm{val}_{\max}(C_n) + n\delta = \frac{n(n+2\delta+2)}{2}\,.$$

Proof. Let $n = 2k$ be an even integer, with $n \geq 4$, and let $C_n : x_1, y_1, x_2, y_2, \ldots, x_k, y_k, x_1$. Define a γ^δ-labeling f of C_n by

$$f(x_i) = i - 1 \quad \text{and} \quad f(y_i) = 2k + \delta + 1 - i\,, \quad \text{for } 1 \leq i \leq k\,.$$

Then

$$\begin{aligned} \mathrm{val}(f) &= 2\left(\textstyle\sum_{i=1}^k f(y_i) - \sum_{i=1}^k f(x_i)\right) \\ &= 2\left(\textstyle\sum_{i=1}^k (2k+\delta+1-i) - \sum_{i=1}^k (i-1)\right) \\ &= 2k(k+\delta+1) \\ &= \tfrac{n(n+2\delta+2)}{2} = \mathrm{val}_{\max}(C_n) + n\delta\,. \end{aligned}$$

Hence

$$\mathrm{val}^\delta_{\max}(C_n) \geq \mathrm{val}_{\max}(C_n) + n\delta = \frac{n(n+2\delta+2)}{2}\,.$$

In order to show $\mathrm{val}^\delta_{\max}(C_n) \leq \frac{n(n+2\delta+2)}{2}$, let g be a γ^δ-max labeling of C_n, where $E(C_n) = \{e_1, e_2, \ldots, e_n\}$. For each integer i, with $1 \leq i \leq n$, let $e_i = u_i v_i$, where $g(u_i) < g(v_i)$. Then

$$\mathrm{val}(g) = \sum_{i=1}^n g(v_i) - \sum_{i=1}^n g(u_i)\,.$$

Since at most two vertices in $\{u_1, , \ldots, u_n\}$ can be assigned by each of the labels $0, 1, \ldots, k-1$, and at most two vertices in $\{v_1, \cdots, v_n\}$ can be assigned each of the labels $k+(\delta+1), k+(\delta+2), \ldots, k+(k+\delta-1), k+(k+\delta)$, it follows that

$$\sum_{i=1}^{n} g(u_i) \geq k^2 - k \quad \text{and} \quad \sum_{i=1}^{n} g(v_i) \leq 3k^2 + (2\delta+1)k\,.$$

Thus

$$\text{val}(g) \leq (3k^2 + (2\delta+1)k) - (k^2 - k) = 2k(k+\delta+1) = \frac{n(n+2\delta+2)}{2},$$

producing the desired result. ■

Proposition 2.3 *Let D be an oriented graph derived from a γ^δ-max labeling g of an even cycle C_n, with $n \geq 6$, by assigning to each edge uv the orientation (u, v), if $g(u) < g(v)$. Then for each vertex v either* $\text{id}(v) = 2$ *or* $\text{od}(v) = 2$.

Proof. For an even cycle C_n, we can have an oriented graph D from γ^δ-max labeling g of C_n by assigning to each edge uv the orientation (u, v), if $g(u) < g(v)$.

Let us assume that there exists a vertex such that indegree and outdegree are not two. Since C_n is an even cycle, it follows that there are at least two vertices, say u and v, such that

$$\text{id}(u) = \text{id}(v) = \text{od}(u) = \text{od}(v) = 1\,.$$

We consider two cases, according to the adjacency of the vertices u and v.

Case 1. *u is adjacent to v.* Then D contains a directed path x, u, v, y of order 4. If we delete the vertices u and v from C_n and join the vertices x and y, the resulting graph G is isomorphic to C_{n-2} and the restriction g' of g to $V(C_n) - \{u, v\}$ has the same value on G as g does on C_n. Therefore,

$$\text{val}_{\max}^{\delta+2}(C_{n-2}) \geq \text{val}(g) = \text{val}_{\max}^{\delta}(C_n),$$

which is a contradiction with $\text{val}(g) = \text{val}_{\max}^{\delta}(C_n) > \text{val}_{\max}^{\delta+2}(C_{n-2})$.

Case 2. *u and v are not adjacent.* The D contains two direct paths, s, u, t and x, v, y, of order 3. Therefore, we can construct a graph G that is isomorphic to C_{n-2} from C_n by deleting the vertices u and v, joining the vertices s and t, and then joining vertices x and y. The restriction g' of g to $V(C_n) - \{u, v\}$ has the same value on G as g does on C_n. Therefore,

$$\text{val}_{\max}^{\delta+2}(C_{n-2}) \geq \text{val}(g) = \text{val}_{\max}^{\delta}(C_n),$$

which is a contradiction with $\text{val}(g) = \text{val}_{\max}^{\delta}(C_n) > \text{val}_{\max}^{\delta+2}(C_{n-2})$. ■

We end this section with a relation between δ-max values of γ-labelings of cycles with consecutive order.

Lemma 2.4 *For every pair δ, k of nonnegative integers with $k \geq 2$,*

$$\mathrm{val}^{\delta}_{\max}(C_{2k+1}) = \mathrm{val}^{\delta+1}_{\max}(C_{2k})\,.$$

Proof. Let f be a $\gamma^{\delta+1}$-max labeling of C_{2k}. Suppose that a is a number in $\{0, 1, \cdots, 2k+\delta, 2k+\delta+1\}$ assigned to no vertex of C_{2k} by f. Consequently, we may define a γ^{δ}-labeling g of C_{2k+1} by

$$g(v_i) = \begin{cases} f(v_i) & \text{if } 1 \leq i \leq 2k \\ a & \text{if } i = 2k+1. \end{cases}$$

Thus $\mathrm{val}^{\delta}_{\max}(C_{2k+1}) \geq \mathrm{val}^{\delta+1}_{\max}(C_{2k})$. It remains to verify that $\mathrm{val}^{\delta+1}_{\max}(C_{2k}) \geq \mathrm{val}^{\delta}_{\max}(C_{2k+1})$.

Let h be a γ^{δ}-max labeling of C_{2k+1}. We construct an oriented graph D form C_{2k+1} by assigning to each edge uv the orientation (u, v), if $h(u) < h(v)$. Necessarily, D contains a directed path x, y, z of order 3. If we delete the vertex y from C_{2k+1} and join the vertices x and z, the resulting graph G is isomorphic to C_{2k} and the restriction $\tilde{h}$ of h to $V(C_{2k+1}) - \{y\}$ has the same value on G as h does on C_{2k+1}. The function $\tilde{h}$ is thus a $\gamma^{\delta+1}$-labeling of C_{2k}, and the result follows. ■

As a consequence of Lemma 2.4, we have the following result.

Corollary 2.5 *For a nonnegative integer δ and an odd integer n, with $n \geq 5$,*

$$\mathrm{val}^{\delta}_{\max}(C_n) = \mathrm{val}_{\max}(C_n) + (n-1)\delta = \frac{(n-1)(n+2\delta+3)}{2}\,.$$

3 The maximum value of $C_n^{\triangle}$ for an odd integer n

The simplest computation of the maximum value of a γ-labeling of $C_n^{\triangle}$ is when n an odd integer. The other case is more delicate as we will see in the next sections. Now, we consider a cycle with a triangle of order $n = 2k+1$, with $k \geq 2$, obtained from a cycle $C_{2k+1} : x_1, z, y_1, x_2, y_2, \cdots, x_k, y_k, x_1$ and a chord $x_1 y_1$.

Theorem 3.1 *For every odd integer $n \geq 5$,*

$$\mathrm{val}_{\max}(C_n^{\triangle}) = \mathrm{val}^{1}_{\max}(C_n^{\triangle}) + (n+1).$$

Proof. Setting $n = 2k+1$, with $k \geq 2$, and defining a γ-labeling f of $C_n^{\triangle}$ by

$$f(x_i) = i - 1\,, \quad f(y_i) = 2k + 3 - i\,, \ \text{ for } 1 \leq i \leq k\,, \ \text{ and } \ f(z) = k+1\,,$$

then

$$\begin{aligned}
\mathrm{val}(f) &= 2\left(\sum_{i=1}^{k} f(y_i) - \sum_{i=1}^{k} f(x_i)\right) + (f(y_1) - f(z)) + (f(z) - f(x_1)) \\
&= 2\left(\sum_{i=1}^{k}(2k+3-i) - \sum_{i=1}^{k}(i-1)\right) + (2k+2) \\
&= 2k(k+3) + (2k+2) \\
&= \tfrac{(n-1)(n-5)}{2} + (n+1) \\
&= \mathrm{val}^2_{\max}(C_{n-1}) + (n+1) \\
&= \mathrm{val}^1_{\max}(C_n) + (n+1)\,.
\end{aligned}$$

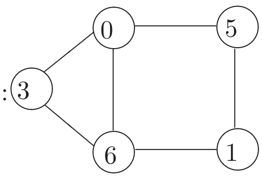
Thus $\mathrm{val}_{\max}(C_n^{\triangle}) \geq \mathrm{val}^1_{\max}(C_n) + (n+1)$.

In order to show that $\mathrm{val}_{\max}(C_n^{\triangle}) \leq \mathrm{val}^1_{\max}(C_n) + (n+1)$, let f be a γ-max labeling of $C_n^{\triangle}$, and let g be the restriction of f to C_n. Then

$$\mathrm{val}(f) = \mathrm{val}(g) + f'(e) \leq \mathrm{val}^1_{\max}(C_n) + (n+1)\,,$$

where e is a chord of $C_n^{\triangle}$. Therefore, $\mathrm{val}_{\max}(C_n^{\triangle}) \leq \mathrm{val}^1_{\max}(C_n) + (n+1)$. ∎

To illustrate Theorem 3.1, we consider, for example, the following γ-max labeling f of $C_5^{\triangle}$:

$C_5^{\triangle}$:

Observe that $\mathrm{val}_{\max}(C_5^{\triangle}) = \mathrm{val}(f) = 26\,.$

The next corollary is a consequence of Theorem 3.1.

Corollary 3.2 *For every pair of nonnegative integer δ and odd integer n with $n \geq 5$,*

$$\mathrm{val}^{\delta}_{\max}(C_n^{\triangle}) = \mathrm{val}^{\delta+1}_{\max}(C_n) + (\delta + n + 1).$$

Proof. Let $n = 2k+1$, with $k \geq 2$, and let f be a γ-labeling f of $C_n^{\triangle}$ defined by

$$f(x_i) = i-1\,, \quad f(y_i) = \delta + 2k + 3 - i, \text{ for } 1 \leq i \leq k\,, \text{ and } f(z) = k+1\,.$$

Then

$$\begin{aligned}
\mathrm{val}(f) &= 2\left(\sum_{i=1}^{k} f(y_i) - \sum_{i=1}^{k} f(x_i)\right) + (f(y_1) - f(z)) + (f(z) - f(x_1)) \\
&= 2\left(\sum_{i=1}^{k}(\delta + 2k + 3 - i) - \sum_{i=1}^{k}(i-1)\right) + (\delta + 2k + 2) \\
&= 2k(k+3) + 2\delta k + (\delta + 2k + 2) \\
&= k(2k + 2\delta + 6) + (\delta + 2k + 2) \\
&= \tfrac{(n-1)(n+2\delta+5)}{2} + (\delta + n + 1) \\
&= \mathrm{val}_{\max}^{\delta+2}(C_{n-1}) + (\delta + n + 1) \\
&= \mathrm{val}_{\max}^{\delta+1}(C_n) + (\delta + n + 1)\,.
\end{aligned}$$

Thus $\mathrm{val}_{\max}^{\delta}(C_n^{\triangle}) \geq \mathrm{val}_{\max}^{\delta+1}(C_n) + (\delta + n + 1)$.

To show that $\mathrm{val}_{\max}^{\delta}(C_n^{\triangle}) \leq \mathrm{val}_{\max}^{\delta+1}(C_n) + (\delta + n + 1)$, let f be a γ-max labeling of $C_n^{\triangle}$, and let g be the restriction of f to C_n. Then

$$\mathrm{val}(f) \;=\; \mathrm{val}(g) + f'(e) \;\leq\; \mathrm{val}_{\max}^{\delta+1}(C_n) + (\delta + n + 1),$$

where e is a chord of $C_n^{\triangle}$. Therefore, $\mathrm{val}_{\max}^{\delta}(C_n^{\triangle}) \leq \mathrm{val}_{\max}^{\delta+1}(C_n) + (\delta + n + 1)$. ■

4 The maximum value of $C_n^{\triangle}$ for an even integer n

In this section we consider a cycle with a triangle of order $n = 2k$, with $k \geq 3$, obtained from a cycle $C_{2k} : x_1, y_1, x_2, y_2, \cdots, x_k, y_k, x_1$ and a chord x_1x_k. We first present a lower bond for $\mathrm{val}_{\max}(C_n^{\triangle})$.

Lemma 4.1 *For every even integer $n \geq 6$,*

$$\mathrm{val}_{\max}(C_n^{\triangle}) \geq \begin{cases} \frac{n^2+5n-2}{2} & = 32 \quad \textit{if } n = 6 \\ \frac{n^2+5n-2}{2} = \frac{n^2+6n-10}{2} & = 51 \quad \textit{if } n = 8 \\ \frac{n^2+6n-10}{2} & \qquad\;\; \textit{if } n \geq 8. \end{cases}$$

Proof. If $k = 3$ or 4, we define a γ-labeling f of $C_{2k}^{\triangle}$ by

$$f(x_i) = i - 1 \quad \text{ and } \quad f(y_i) = n + 2 - i\,, \text{ for } 1 \leq i \leq k\,.$$

Then

$$\begin{aligned}
\mathrm{val}(f) &= 2\left(\sum_{i=1}^{k} f(y_i) - \sum_{i=1}^{k} f(x_i)\right) + f(x_k) - f(x_1) \\
&= 2\left(\sum_{i=1}^{k}(n + 2 - i) - \sum_{i=1}^{k}(i-1)\right) + (k-1) - 0 \\
&= \tfrac{n^2+5n-2}{2}\,.
\end{aligned}$$

Thus $\mathrm{val}_{\max}(C_n^{\triangle}) \geq \frac{n^2+5n-2}{2}$, for $n = 6, 8$.

Assume that $k \geq 4$, we define a γ-labeling f of $C_{2k}^{\triangle}$ by

$$f(x_i) = \begin{cases} n+2-i & \text{if } 1 \leq i \leq k-1 \\ 0 & \text{if } i = k \end{cases} \qquad \text{and} \qquad f(y_i) = i\,,\ \text{if } 1 \leq i \leq k.$$

Then

$$\begin{aligned} \mathrm{val}(f) &= 2\left(\textstyle\sum_{i=1}^{k-1} f(x_i) - \sum_{i=1}^{k-2} f(y_i)\right) + f(x_1) - 3f(x_k) \\ &= 2\left(\textstyle\sum_{i=1}^{k-1} (n+2-i) - \sum_{i=1}^{k-2} i\right) + (n+1) - 3(0) \\ &= \tfrac{n^2+6n-10}{2}\,. \end{aligned}$$

Hence $\mathrm{val}_{\max}(C_n^{\triangle}) \geq \frac{n^2+6n-10}{2}$, for $n \geq 8$. ∎

In order to present a formula for $\mathrm{val}_{\max}(C_n^{\triangle})$, for an even integer $n \geq 6$, we need some additional notation and new definitions. Let f be a γ-max labeling of $C_{2k}^{\triangle}$. For each integer i, with $1 \leq i \leq k$, we define the 3-term sequences

$$S_i(f) = (f(x_i), f(y_i), f(x_{i+1})) \text{ and } T_i(f) = (f(y_i), f(x_{i+1}), f(y_{i+1})),$$

where the addition is taken modulo k, and let

$$ST(f) = \{S_1(f), S_2(f), \ldots, S_k(f), T_1(f), T_2(f), \ldots, T_k(f)\}$$

be a set of 3-term sequences $S_i(f)$ and $T_i(f)$, for all i with $1 \leq i \leq k$. Furthermore, let $C_{2k}(f)$ be an oriented cycle obtained from $C_{2k}^{\triangle} - x_1x_k$ by assigning to the edge uv the orientation (u, v) if $f(u) < f(v)$.

We are ready to characterize any γ-max labeling f of $C_6^{\triangle}$.

Proposition 4.2 *Let f be a γ-max labeling of $C_6^{\triangle}$. Then $ST(f)$ contains no monotone elements.*

Proof. Assume, to the contrary, that some element of $ST(f)$ is monotone. Then an oriented cycle $C_6(f)$ contains a directed path a, b, c of order 3. If we delete the chord x_1x_3 and vertex b from $C_6^{\triangle}$ and then join the vertices a and c, the resulting graph G is isomorphic to C_5 and the restriction $\tilde{f}$ of f to $V(C_6^{\triangle} - x_1x_3) - \{b\}$ has the same value on G as f on $C_6^{\triangle} - x_1x_3$, that is

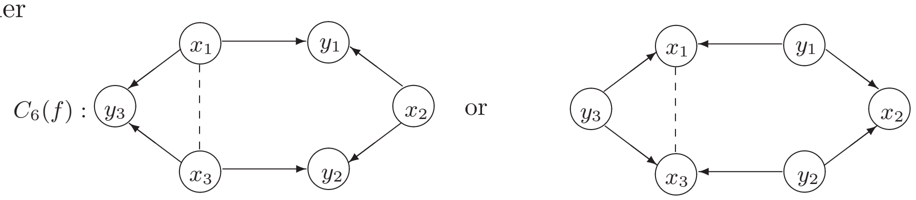
$$\mathrm{val}(\tilde{f}) = \mathrm{val}(f) - f'(x_1x_3) = \mathrm{val}_{\max}(C_6^{\triangle}) - f'(x_1x_3).$$

Then $\mathrm{val}_{\max}(C_6^{\triangle}) = \mathrm{val}(\tilde{f}) + f'(x_1x_3)$. Moreover, since $\tilde{f}$ is a γ^2-labeling of graph G that is isomorphic to C_5, it follows that $\mathrm{val}(\tilde{f}) \leq \mathrm{val}^2_{\max}(C_5) = \mathrm{val}^3_{\max}(C_4)$. Therefore

$$\mathrm{val}_{\max}(C_6^{\triangle}) \;\leq\; \mathrm{val}^3_{\max}(C_4) + f'(x_1x_3) \;\leq\; \frac{4(4+6+2)}{2} + 7 \;=\; 31,$$

which is a contradiction with Lemma 4.1. ■

The oriented cycle $C_6(f)$ defined by any γ-max labeling f of $C_6^{\triangle}$ in Proposition 4.2 is either



Observe that all 3-term sequences of $ST(f)$, which are

$S_1(f) = (f(x_1), f(y_1), f(x_2)), S_2(f) = (f(x_2), f(y_2), f(x_3)), S_3(f) = (f(x_3), f(y_3), f(x_1))$

$T_1(f) = (f(y_1), f(x_2), f(y_2)), T_2(f) = (f(y_2), f(x_3), f(y_3)), T_3(f) = (f(y_3), f(x_1), f(y_1))$,

are not monotone.

Combining Lemma 4.1 and Proposition 4.2, we are able to find $\mathrm{val}_{\max}(C_6^{\triangle})$.

Theorem 4.3

$$\mathrm{val}_{\max}(C_6^{\triangle}) = \mathrm{val}^1_{\max}(C_6) + 2 = 32.$$

Proof. By Lemma 4.1, we have $\mathrm{val}_{\max}(C_6^{\triangle}) \geq 32 = \mathrm{val}^1_{\max}(C_6) + 2$. Next, let f be a γ-max labeling of $C_6^{\triangle}$. According to Proposition 4.2, for each i with $1 \leq i \leq 3$, vertices x_i and y_i of oriented cycle $C_6(f)$ have

either $\mathrm{id}(x_i) = 0$, $\mathrm{id}(y_i) = 2$ **or** $\mathrm{id}(x_i) = 2$, $\mathrm{id}(y_i) = 0$.

We analyze the two cases separately.

Case 1. *For each* i *with* $1 \leq i \leq 3$, $\mathrm{id}(x_i) = 0$, $\mathrm{id}(y_i) = 2$ *of the oriented cycle* $C_6(f)$. So, $f(x_i) < f(y_i)$ and $f(x_{i+1}) < f(y_i)$ where addition is taken modulo 3. Then

$$\mathrm{val}_{\max}(C_6^{\triangle}) = \mathrm{val}(f) = 2\left(\sum_{i=1}^{3} f(y_i) - \sum_{i=1}^{3} f(x_i)\right) + |f(x_1) - f(x_3)|.$$

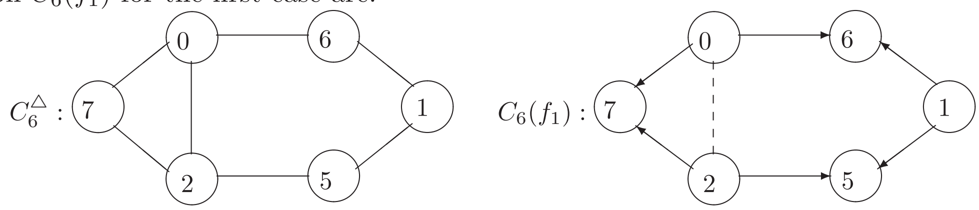
Since the vertices in $\{x_1, x_2, x_3\}$ can be labeled by each of the labels $0, 1, 2$ and the vertices in $\{y_1, y_2, y_3\}$ can be labeled by each of the labels $7, 6, 5$, it follows that

$$\sum_{i=1}^{3} f(x_i) \geq 0 + 1 + 2 = 3 \quad \text{and} \quad \sum_{i=1}^{3} f(y_i) \leq 7 + 6 + 5 = 18.$$

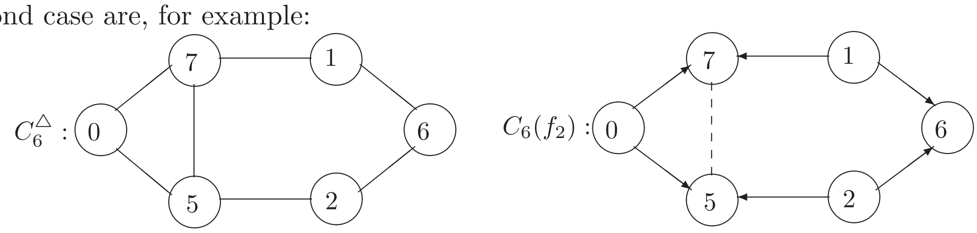
Then $\mathrm{val}_{\max}(C_6^{\triangle}) = \mathrm{val}(f) \leq 2(18 - 3) + 2 = \mathrm{val}^1_{\max}(C_6) + 2 = 32$.

Case 2. *For each* i *with* $1 \leq i \leq 3$, $\mathrm{id}(x_i) = 2$, $\mathrm{id}(y_i) = 0$ *of the oriented cycle* $C_6(f)$. So, $f(x_i) > f(y_i)$ and $f(x_{i+1}) > f(y_i)$ where addition is taken modulo 3. An argument similar to the one used in Case 1 shows that $\mathrm{val}_{\max}(C_6^{\triangle}) \leq \mathrm{val}^1_{\max}(C_6)+2 = 32$, producing the desired result. ■

To illustrate Theorem 4.3, an example of a γ-max labeling f_1 of $C_6^{\triangle}$ and an oriented graph $C_6(f_1)$ for the first case are:



On the another hand, a γ-max labeling f_2 of $C_6^{\triangle}$ and an oriented graph $C_6(f_2)$ in second case are, for example:



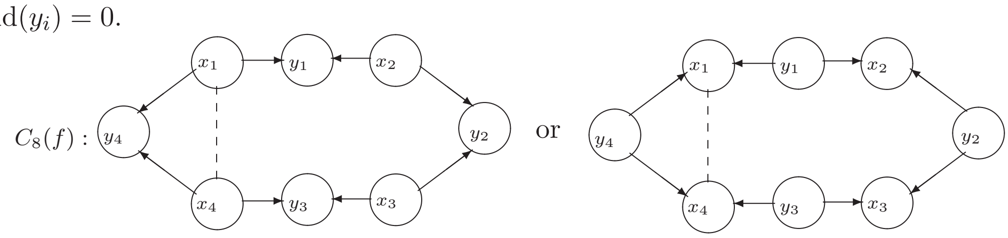
Observe that $\mathrm{val}_{\max}(C_6^{\triangle}) = \mathrm{val}(f_1) = \mathrm{val}(f_2) = 32$.

Theorem 4.4

$$\mathrm{val}_{\max}(C_8^{\triangle}) = \begin{cases} \mathrm{val}^1_{\max}(C_8) + 3 & = 51 \\ \mathrm{val}^2_{\max}(C_7) + 9 & = 51. \end{cases}$$

Proof. By Lemma 4.1, we have $\mathrm{val}_{\max}(C_8^{\triangle}) \geq 51 = \mathrm{val}^1_{\max}(C_8) + 3 = \mathrm{val}^2_{\max}(C_7) + 9$. On the other hand, let f be a γ-max labeling of $C_6^{\triangle}$. We consider the two cases according to the set $ST(f)$.

Case 1. *No element of $ST(f)$ is monotone.* Then, for each i with $1 \leq i \leq 4$, the vertices x_i and y_i of oriented cycle $C_8(f)$ have **either** $\mathrm{id}(x_i) = 0, \mathrm{id}(y_i) = 2$ **or** $\mathrm{id}(x_i) = 2, \mathrm{id}(y_i) = 0$.



An argument similar to the one used in Theorem 4.3 shows that $\mathrm{val}_{max}(C_8^{\triangle}) \leq \mathrm{val}^1_{max}(C_8) + 3 = 51$.

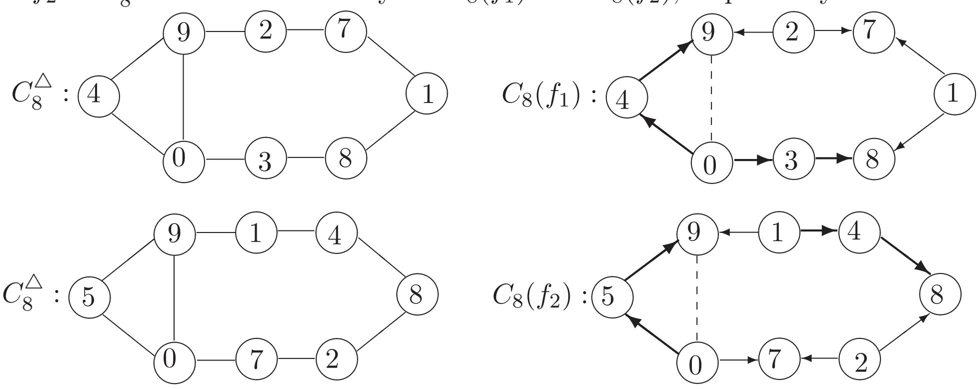
Case 2. *Some element of $ST(f)$ is monotone.* Then the oriented cycle $C_8(f)$ contains a directed path a, b, c of order 3. If we delete the chord x_1x_4 and vertex b from

$C_8^{\triangle}$ and join the vertices a and c, the resulting graph G is isomorphic to C_7 and the restriction $\widetilde{f}$ of f to $V\left(C_8^{\triangle} - x_1x_4\right) - \{b\}$ has the same value on G as f on $C_8^{\triangle} - x_1x_4$, that is

$$\mathrm{val}(\widetilde{f}) = \mathrm{val}(f) - f'(x_1x_4) \geq \mathrm{val}_{\max}(C_8^{\triangle}) - 9.$$

Moreover, since $\widetilde{f}$ is a γ^2-labeling of a graph G isomorphic to C_7, it follows that $\mathrm{val}(\widetilde{f}) \leq \mathrm{val}^2_{\max}(C_7)$. Therefore, $\mathrm{val}_{\max}(C_8^{\triangle}) \leq \mathrm{val}^2_{\max}(C_7) + 9 = 51$. ■

To illustrate Case 2 of Theorem 4.4, we consider, for example, the γ-max labelings f_1 and f_2 of $C_8^{\triangle}$ and the oriented cycles $C_8(f_1)$ and $C_8(f_2)$, respectively:



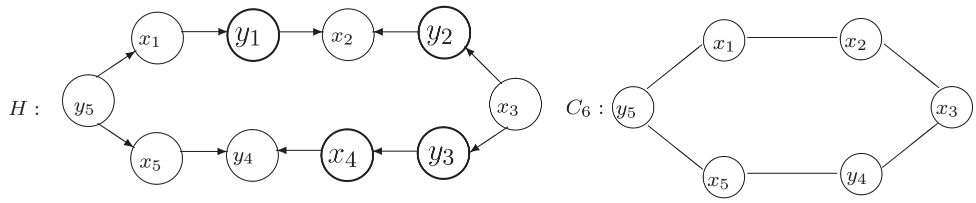
Observe that $\mathrm{val}_{\max}(C_8^{\triangle}) = \mathrm{val}(f_1) = \mathrm{val}(f_2) = 51$.

We remark that, since for any integer k with $k \geq 4$, the size of the oriented cycle $C_{2k}(f)$ is even, it follows that there are ℓ directed paths of order 3 in the oriented cycle $C_{2k}(f)$, for some even integer ℓ with $0 \leq \ell \leq 2k$. This observation leads us to the following lemma.

Lemma 4.5 *Let $n = 2k$ be an even integer, with $n \geq 8$, and let f be a γ-max labeling of $C_{2k}^{\triangle}$. If some element of $ST(f)$ is monotone, then there are exactly two monotone elements in $ST(f)$.*

Proof. Suppose that some element of $ST(f)$ is monotone. Then the oriented cycle $C_{2k}(f)$ contains a directed path of order 3. Since the size of the oriented cycle $C_{2k}(f)$ is even, it follows that there are ℓ directed paths of order 3 in $C_{2k}(f)$, for some even integer ℓ with $2 \leq \ell \leq 2k$. Next, we show that there are no more than two directed paths of order 3 in oriented cycle $C_{2k}(f)$. Assume, to the contrary, that there are at least 4 directed paths of order 3 in the oriented graph $C_{2k}(f)$. For each i with $1 \leq i \leq 4$, let P_i be a directed path of order 3 having internal vertex u_i in the oriented graph $C_{2k}(f)$. We can see that the cycle $C_{2k}^{\triangle} - x_1x_k$ is a subdivision of the cycle C_{n-4} by every two vertices of C_{n-4} which are connected by either an edge or a path, all of whose interior vertices are internal vertices u_1, u_2, u_3, u_4.

To illustrate this concept of subdivision, let us define:



For example, considering the oriented cycle $H : x_1, y_1, x_2, y_2, \cdots, x_5, y_5, x_1$ of order 10 having 6 directed paths of order 3. Let $P_1 : x_1, y_1, x_2, P_2 : x_2, y_2, x_3, P_3 : x_3, y_3, x_4$ and $P_4 : y_3, x_4, y_4$ be directed paths of order 3 in H. Therefore the underlying cycle H is a subdivision of the cycle $C_6 : x_1, x_2, x_3, y_4, x_5, y_5, x_1$, where the internal vertices y_1, y_2, y_3, x_4 of the directed paths P_1, P_2, P_4, P_4, respectively, are inserted into one or more edges of C_6.

Since the cycle $C_{2k}^{\triangle} - x_1x_k$ is a subdivision of the cycle C_{n-4}, we can construct a graph G which is isomorphic to C_{n-4} from $C_{2k}^{\triangle}$ by deleting the chord x_1x_k and the internal vertices u_1, u_2, u_3, u_4 and then adding one or more edges to G. Then the restriction of f to $V\left(C_{2k}^{\triangle} - x_1x_k\right) - \{u_1, u_2, u_3, u_4\}$, $\tilde{f}$, has the same value on G as f does on $C_{2k}^{\triangle} - x_1x_k$, that is

$$\mathrm{val}(\widetilde{f}) = \mathrm{val}(f) - f'(x_1x_k) \geq \mathrm{val}_{\max}(C_n^{\triangle}) - (n+1).$$

Moreover, since $\tilde{f}$ is a γ^5-labeling of a graph G which is isomorphic to C_{n-4}, it follows that $\mathrm{val}(\widetilde{f}) \leq \mathrm{val}_{\max}^5(C_{n-4})$. Therefore

$$\mathrm{val}_{\max}(C_n^{\triangle}) \leq \mathrm{val}_{\max}^5(C_{n-4}) + (n+1) = \frac{n^2 + 6n - 30}{2},$$

a contradiction with Lemma 4.1. Thus, for $k \geq 4$, the oriented cycle $C_{2k}(f)$ contains exactly two directed paths of order 3 and there are also exactly two monotone elements in $ST(f)$. ■

By Theorem 4.4 and Lemma 4.5, we are able to characterize any γ-max labeling f of $C_8^{\triangle}$, in terms of set $ST(f)$ and the oriented cycle $C_8(f)$.

Proposition 4.6 *Let f be a γ-max labeling of $C_8^{\triangle}$. Then either no element of $ST(f)$ is monotone or there are exactly two monotone elements in $ST(f)$.*

For any integers p and q, with $p \leq q$, let

$$[p, q] = \{p, p+1, \ldots, q\}$$

be the set of all the integers between p and q. Next we characterize any γ-max labeling f of $C_n^{\triangle}$, for $n \geq 10$.

Proposition 4.7 *For any integer k, with $k \geq 5$, let f be a γ-max labeling of $C_{2k}^{\triangle}$. Then $ST(f)$ contains exactly two monotone of $ST(f)$.*

Proof. Assume, to the contrary, that the property does not verify. Then, by Lemma 4.5, $ST(f)$ contains no monotone element. Hence, for the vertices x_i and y_i of the oriented cycle $C_{2k}(f)$,

$$\textbf{either}\ \ \mathrm{id}(x_i)=0,\ \mathrm{id}(y_i)=2\ \ \textbf{or}\ \ \mathrm{id}(x_i)=2,\ \mathrm{id}(y_i)=0 \text{ for each } i \text{ with } 1\le i\le k\ .$$

We consider the two cases separately.

Case 1. *For each i with $1 \le i \le k$, id$(x_i) = 0$, id$(y_i) = 2$ of the oriented cycle $C_{2k}(f)$.* Thus $f(x_i) < f(y_i)$ and $f(x_{i+1}) < f(y_i)$ where addition is taken modulo k. Then

$$\mathrm{val}_{\max}(C_{2k}^{\triangle}) = \mathrm{val}(f) = 2\sum_{i=1}^{k} f(y_i) - 2\sum_{i=1}^{k} f(x_i) + |f(x_1) - f(x_k)|.$$

Before we proceed, we prove two claims.

Claim 1. $\{f(y_1), f(y_2), \ldots, f(y_k)\} = [k+2, n+1]$.

Proof of Claim 1. Assume, to the contrary, that

$$\{f(y_1), f(y_2), \ldots, f(y_k)\} \neq [k+2, n+1]\,.$$

Then there are two numbers a and b, with $a \in [0, k+1]$ and $b \in [k+2, n+1]$, such that a is assigned by f to some vertex of $\{y_1, y_2, \ldots, y_k\}$, namely y_r and b is assigned by f to either no vertex of $C_{2k}^{\triangle}$ or some vertex of $\{x_1, x_2, \ldots, x_k\}$.

If b is assigned to no vertex of $C_{2k}^{\triangle}$, then we define a new γ-labeling g of $C_{2k}^{\triangle}$ by

$$g(u) = \begin{cases} f(u) & \text{if } u \neq y_r \\ b & \text{if } u = y_r. \end{cases}$$

Then

$$\begin{aligned}
\mathrm{val}(g) &= 2\textstyle\sum_{i=1}^{k} g(y_i) - 2\sum_{i=1}^{k} g(x_i) + |g(x_1) - g(x_k)| \\
&= \Big(2\textstyle\sum_{i=1}^{k} f(y_i) - 2f(y_r) + 2g(y_r)\Big) - 2\sum_{i=1}^{k} f(x_i) + |f(x_1) - f(x_k)| \\
&= \mathrm{val}(f) - 2a + 2b \ = \ \mathrm{val}(f) + 2(b-a) \ > \ \mathrm{val}(f)\,,
\end{aligned}$$

which is a contradiction. Then b must be assigned to some vertex of $\{x_1, x_2, \ldots, x_k\}$, namely x_s. Then we define the γ-labeling g of $C_{2k}^{\triangle}$ by

$$g(u) = \begin{cases} f(u) & \text{if } u \neq x_s, y_r \\ f(y_r) = a & \text{if } u = x_s \\ f(x_s) = b & \text{if } u = y_r \end{cases}$$

and, therefore,

$$\begin{aligned}
\mathrm{val}(g) &= 2\sum_{i=1}^{k} g(y_i) - 2\sum_{i=1}^{k} g(x_i) + |g(x_1) - g(x_k)| \\
&= \Big(2\sum_{i=1}^{k} f(y_i) - 2f(y_r) + 2g(y_r)\Big) - \Big(2\sum_{i=1}^{k} f(x_i) - 2f(x_s) + 2g(x_s)\Big) \\
&\quad + |g(x_1) - g(x_k)| + (|f(x_1) - f(x_k)| - |f(x_1) - f(x_k)|) \\
&= \mathrm{val}(f) + 4(b-a) + |g(x_1) - g(x_k)| - |f(x_1) - f(x_k)|\,.
\end{aligned}$$

If $s \neq 1, k$, then

$$\begin{aligned}
\mathrm{val}(g) &= \mathrm{val}(f) + 4(b-a) + |f(x_1) - f(x_k)| - |f(x_1) - f(x_k)| \\
&= \mathrm{val}(f) + 4(b-a) \;>\; \mathrm{val}(f)\,.
\end{aligned}$$

Otherwise, assuming without loss of generality that $s = 1$, we get

$$\begin{aligned}
\mathrm{val}(g) &= \mathrm{val}(f) + 4(b-a) + |a - f(x_k)| - |b - f(x_k)| \\
&= \mathrm{val}(f) + (|b-a| + |a - f(x_k)|) - |b - f(x_k)| + 3(b-a) \\
&\geq \mathrm{val}(f) + |b - f(x_k)| - |b - f(x_k)| + 3(b-a) \\
&= \mathrm{val}(f) + 3(b-a) \;>\; \mathrm{val}(f)\,,
\end{aligned}$$

which is again a contradiction. Hence $\{f(y_1), f(y_2), \ldots, f(y_k)\} = [k+2, n+1]$. Consequently, $\{f(x_1), f(x_k), \ldots, f(x_k)\} \subset [0, k+1]$.

This is complete the proof of Claim 1.

Claim 2. $\{f(x_1), f(x_k), \ldots, f(x_k)\} = [0, k-1]$.

Proof of Claim 2. Assume, to the contrary, that

$$\{f(x_1), f(x_k), \ldots, f(x_k)\} \neq [0, k-1]\,.$$

Then there are two integer numbers a and b, with $a \in [k, k+1]$ and $b \in [0, k-1]$, such that a is assigned by f to some vertex of $\{x_1, x_2, \ldots, x_k\}$, namely x_r, and b is assigned by f to no vertex of $C_{2k}^{\triangle}$. Then we define a new γ-labeling g of $C_{2k}^{\triangle}$ by

$$g(u) = \begin{cases} f(u) & \text{if } u \neq x_r \\ b & \text{if } u = x_r. \end{cases}$$

Then

$$\begin{aligned}
\mathrm{val}(g) &= 2\sum_{i=1}^{k} g(y_i) - 2\sum_{i=1}^{k} g(x_i) + |g(x_1) - g(x_k)| \\
&= 2\sum_{i=1}^{k} f(y_i) - \Big(2\sum_{i=1}^{k} f(x_i) - 2f(x_r) + 2g(x_r)\Big) \\
&\quad + |g(x_1) - g(x_k)| + (|f(x_1) - f(x_k)| - |f(x_1) - f(x_k)|) \\
&= \mathrm{val}(f) + 2(a-b) + |g(x_1) - g(x_k)| - |f(x_1) - f(x_k)|\,.
\end{aligned}$$

If $r \neq 1, k$, then

$$\begin{aligned} \mathrm{val}(g) &= \mathrm{val}(f) + 2(a-b) + |f(x_1) - f(x_k)| - |f(x_1) - f(x_k)| \\ &= \mathrm{val}(f) + 2(a-b) \; > \; \mathrm{val}(f)\,. \end{aligned}$$

Otherwise assuming, without loss of generality, that $r = 1$, then

$$\begin{aligned} \mathrm{val}(g) &= \mathrm{val}(f) + 2(a-b) + |b - f(x_k)| - |a - f(x_k)| \\ &= \mathrm{val}(f) + (|a-b| + |b - f(x_k)|) - |a - f(x_k)| + (a-b) \\ &\geq \mathrm{val}(f) + |a - f(x_k)| - |a - f(x_k)| + (a-b) \\ &= \mathrm{val}(f) + (a-b) \; > \; \mathrm{val}(f)\,. \end{aligned}$$

Again we reach a contradiction. Therefore, $\{f(x_1), f(x_k), \ldots, f(x_k)\} = [0, k-1]$. This completes the proof of Claim 2.

Therefore, the function f is defined by

$$\{f(y_1), f(y_2), \ldots, f(y_k)\} = [k+2, n+1] \text{ and } \{f(x_1), f(x_k), \ldots, f(x_k)\} = [0, k-1].$$

So

$$\begin{aligned} \mathrm{val}_{\max}(C_{2k}^{\triangle}) = \mathrm{val}(f) &= 2\textstyle\sum_{i=1}^{k} f(y_i) - 2\sum_{i=1}^{k} f(x_i) + |f(x_1) - f(x_k)| \\ &= 3k(k-1) - k(k-1) + (k-1) \\ &= 2k^2 + 5k - 1\,. \end{aligned}$$

However, by Lemma 4.1, we have $\mathrm{val}_{\max}(C_{2k}^{\triangle}) \geq \frac{(2k)^2+6(2k)-10}{2} = 2k^2 + 6k - 5$, which contradicts our assumption.

Case 2. *For each i with $1 \leq i \leq k$, $id(x_i) = 2$, $id(y_i) = 0$ of the oriented cycle $C_{2k}(f)$.* So, $f(x_i) > f(y_i)$ and $f(x_{i+1}) > f(y_i)$, where the addition is taken modulo k. A similar argument to the one used in Case 1 shows that $\mathrm{val}_{\max}(C_{2k}^{\triangle}) = 2k^2 + 5k - 1$, which is a contradiction. ■

We are now prepared to establish a general formula for $\mathrm{val}_{\max}(C_n^{\triangle})$, when $n \geq 10$.

Theorem 4.8 *For every even integer $n \geq 10$,*

$$\mathrm{val}_{\max}(C_n^{\triangle}) = \mathrm{val}_{\max}^2(C_{n-1}) + (n+1) = \frac{n^2 + 6n - 10}{2}.$$

Proof. By Lemma 4.1, we have

$$\mathrm{val}_{\max}(C_n^{\triangle}) \geq \frac{n^2 + 6n - 10}{2} = \mathrm{val}_{\max}^2(C_{n-1}) + (n+1).$$

Next, we show that $\text{val}_{\max}(C_n^{\triangle}) \leq \text{val}^2_{\max}(C_{n-1}) + (n+1)$. Let f be a γ-max labeling of $C_n^{\triangle}$. By Proposition 4.7, the oriented cycle $C_n(f)$ contains a directed path a, b, c of order 3. If we delete the chord x_1x_k and vertex b from $C_n^{\triangle}$ and then join the vertices a and c, the resulting graph G is isomorphic to C_{n-1} and the restriction $\tilde{f}$ of f to $V(C_n^{\triangle} - x_1x_k) - \{b\}$, verifies

$$\text{val}(\widetilde{f}) = \text{val}(f) - f'(x_1x_k) \geq \text{val}_{max}(C_n^{\triangle}) - (n+1).$$

Moreover, since $\widetilde{f}$ is a γ^2-labeling of a graph G that is isomorphic to C_{n-1}, it follows that $\text{val}(\widetilde{f}) \leq \text{val}^2_{max}(C_{n-1})$. Therefore,

$$\text{val}_{max}(C_n^{\triangle}) \leq \text{val}^2_{max}(C_{n-1}) + (n+1) = \frac{n^2+6n-10}{2}$$

producing the desired result. ■

5 On Spans of γ-Min and γ-Max Labelings of Graphs

In order to show a formula for spans of γ-min and γ-max labelings of graphs, we first present a useful lemma.

Lemma 5.1 *Let f be a γ-max labeling of a nontrivial graph G of order n and size m and let $u, w \in V(G)$ with $f(u) = \min\{f(v) : v \in V(G)\}$ and $f(w) = \max\{f(v) : v \in V(G)\}$. Then neighborhoods of u and v are not empty.*

Proof. For any nontrivial connected graph G, it is obvious that $|N(u)|$ and $|N(w)|$ are not empty. Let G be a disconnected graph. Assume, to the contrary, that $|N(u)| = 0$. Therefore, u is an isolated vertex of G. Since G has γ-labeling, it follows that $m \geq n-1$. Therefore, there is a component G_1 of G with $V(G_1) \geq 2$. Let $x \in V(G_1)$ with $f(x) = \min\{f(v) : v \in V(G_1)\}$. Let g be a γ-labeling of G defined by

$$g(v) = \begin{cases} f(x) & \text{if } v = u \\ f(u) & \text{if } v = x \\ f(v) & \text{if } v \neq u, x. \end{cases}$$

Since

$$\begin{aligned} \text{val}(g) &= \text{val}(f) - \sum_{v \in N(x)} (f(v) - f(x)) + \sum_{v \in N(x)} (g(v) - g(x)) \\ &= \text{val}(f) - \sum_{v \in N(x)} (f(v) - f(x)) + \sum_{v \in N(x)} (f(v) - f(u)) \\ &= \text{val}(f) + |N(x)|(f(x) - f(u)), \end{aligned}$$

and $f(x) > f(u)$, it follows that $\text{val}(g) > \text{val}(f)$, which is a contradiction.

It is a similar argument to verify that neighborhood of v is also not empty. ■

Theorem 5.2 *Let G be a nontrivial graph of order n and size m and let f be a γ-labeling of G.*

(a) *If f is a γ-*min *labeling of G, then* $\mathrm{span}(f) = n - 1$.

(b) *If f is a γ-*max *labeling of G, then* $\mathrm{span}(f) = m$.

Proof. We first verify (a). Let $\mathrm{val}_{\min}(G) = \mathrm{val}(f)$ and assume, to the contrary, that $\mathrm{span}(f) \geq n$. It then follows by Theorem A that there is a γ-labeling g with $\mathrm{val}(g) < \mathrm{val}(f)$, which is a contradiction.

Next we verify (b). Let f be a γ-max labeling of G and let $u, w \in V(G)$ with $f(u) = \min\{f(v) : v \in V(G)\}$ and $f(w) = \max\{f(v) : v \in V(G)\}$. Then $f(u) \geq 0$ and $f(w) \leq m$. Assume, to the contrary, that $\mathrm{span}(f) < m$. Then $f(w) - f(u) < m$. Therefore,

$$f(u) > 0 \text{ or } m - f(w) > 0. \tag{2}$$

Let g be a γ-labeling of G defined by

$$g(v) = \begin{cases} 0 & \text{if } v = u \\ f(v) & \text{if } v \neq u, w \\ m & \text{if } v = w. \end{cases}$$

Since

$$\begin{aligned} \mathrm{val}(g) &= \mathrm{val}(f) - \sum_{v \in N(u)} (f(v) - f(u)) + \sum_{v \in N(u)} (g(v) - g(u)) \\ &\quad - \sum_{v \in N(w)} (f(w) - f(v)) + \sum_{v \in N(w)} (g(w) - g(v)) \\ &= \mathrm{val}(f) - \sum_{v \in N(u)} (f(v) - f(u)) + \sum_{v \in N(u)} (f(v) - 0) \\ &\quad - \sum_{v \in N(w)} (f(w) - f(v)) + \sum_{v \in N(w)} (m - f(v)) \\ &= \mathrm{val}(f) + |N(u)|(f(u) - 0) + |N(w)|(m - f(w)), \end{aligned}$$

it follows by Lemma 5.1 and (2) that $\mathrm{val}(g) > \mathrm{val}(f)$, which is a contradiction. ∎

The following results are consequences of Theorem 5.2.

Corollary 5.3 *Let G be a nontrivial graph of order n and size m and let f be a γ-labeling of G.*

(a) *If f is a γ-*min *labeling of G, then $f(V(G))$ is a consecutive subset of $[0..m]$, that is, $f(V(G)) = [k..k + (n - 1)]$ for some integer k with $0 \leq k \leq m - (n - 1)$.*

(b) *If f is a γ-*max *labeling of G, then $\{0, m\} \subset f(V(G))$.*

Proof. To verify (a), let f be a γ-min labeling of G with $f(V(G)) = \{a_1, a_2, \dots, a_n\}$ where

$$0 \le a_1 < a_2 < \cdots < a_n \le m.$$

Then $\operatorname{span}(f) = a_n - a_1$. By Theorem 5.2 (a), $\operatorname{span}(f) = n-1$. Therefore, $a_n - a_1 = n-1$. This implies that $a_1, a_2, \dots, a_n$ are consecutive integers in the set $[0..m]$ and so $f(V(G))$ is a consecutive subset of $[0..m]$. That is $f(V(G)) = [k..k+(n-1)]$ for some integer k with $0 \le k \le m - (n-1)$. Statement (b) is an immediate consequence of Theorem 5.2 (b). ■

Corollary 5.4 *Let G be a nontrivial connected graph of order n and size m. Suppose that f and g are γ-min and γ-max labelings of G, respectively. Then $\operatorname{span}(f) = \operatorname{span}(g)$ if and only if G is a tree.*

Proof. First, assume that $\operatorname{span}(f) = \operatorname{span}(g)$. By Theorem 5.2, $\operatorname{span}(f) = n-1$ and $\operatorname{span}(g) = m$. Thus $n-1 = m$, which implies that G is a tree. For the converse, assume that G is a tree. Then $m = n-1$. It then follows by Theorem 5.2 that $\operatorname{span}(f) = n-1 = m = \operatorname{span}(g)$, as desired. ■

Now, we are able to answer the following open question in [4] : For any connected graph G, does exist a γ-max labeling of G with a vertex label set that is the union of no more than two sets of consecutive numbers?

Theorem 5.5 *Let G be a connected graph of order n and size m. Then vertex label set of γ-max labeling of G is a set of consecutive numbers if and only if G is a tree.*

Proof. It is straightforward to verify that vertex label set of γ-max labeling of tree is a set of consecutive numbers. For the converse, let f be a γ-max labeling of a connected graph G with a vertex label set that is a set of consecutive numbers. Then $\operatorname{span}(f) = n-1$. Moreover, by Theorem 5.2 (b), we have $\operatorname{span}(f) = m$ Then $m = n-1$. Therefore, G is a tree. ■

6 On γ-Max Labelings of Complete Bipartite and Complete Graphs

In this section, we begin characterize γ-max labelings of complete bipartite which also appeared in [4]. We provide alternative and improved proof of formula for $\operatorname{val}(K_{r,s})$ that uses γ-min labelings of complete graphs.

Theorem 6.1 *Let f be a γ-labeling of complete bipartite graph $K_{r,s}$ with partite sets V_r and V_s of cardinality r and s, respectively. Then f is a γ-max labeling of $K_{r,s}$ if and only if*

(*a*) $f(V_r) = [0..r-1]$ *and* $f(V_s) = [rs-(s-1)..rs]$, *or*

(*b*) $f(V_r) = [rs-(r-1)..rs]$ *and* $f(V_s) = [0..s-1]$.

Proof. Let $K_{r,s}$ be a complete bipartite graph with partite sets V_r and V_s and let f be a γ-max labeling of $K_{r,s}$ with $f(V(K_{r,s})) = \{a_1, a_2, \ldots, a_{r+s}\}$ where

$$1 \leq a_1 < a_2 < \cdots < a_{r+s} \leq rs.$$

Now, consider complete graph K_{r+s} with $V(K_{r+s}) = V_r \cup V_s$. Then K_{r+s} contains three edge-disjoint graphs, namely

$$K_r \text{ with } V(K_r) = V_r,$$
$$K_s \text{ with } V(K_s) = V_s \text{ and}$$
$$K_{r,s} \text{ with partite sets } V_r \text{ and } V_s.$$

That is,

$$E(K_{r+s}) = E(K_r) \cup E(K_s) \cup E(K_{r,s}).$$

For each $i \in \{r, s, r+s\}$, let f_i be the restriction of f to K_i and then

$$f_i : V(K_i) \to f(V(K_{r,s})) \text{ is a one to one function}$$

and

$$\text{val}(f_{r+s}) = \text{val}(f_r) + \text{val}(f_s) + \text{val}(f)$$

and so

$$\text{val}(f) = \text{val}(f_{r+s}) - \text{val}(f_r) - \text{val}(f_s).$$

Since $f_{r+s}(V(K_{r+s})) = f(V(K_{r,s})) = \{a_1, a_2, \ldots, a_{r+s}\}$, it follows by (1) that $\text{val}(f_{r+s})$ gives us same value even though we switch some labels of $\{a_1, a_2, \ldots, a_{r+s}\}$ to $V(K_{r+s})$. Furthermore, since $\text{val}(f) = \text{val}_{\max}(K_{r,s})$, it follows that

$$\text{val}(f_r) = \min \{\text{val}(g) \mid g : V(K_r) \to \{a_1, a_2, \ldots, a_{r+s}\}\} \tag{3}$$

and

$$\text{val}(f_s) = \min \{\text{val}(g) \mid g : V(K_s) \to \{a_1, a_2, \ldots, a_{r+s}\}\}.$$

We claim that $f_r(V_r)$ and $f_s(V_s)$ are consecutive subsets of $\{a_1, a_2, \ldots, a_{r+s}\}$. Let

$$f_r(V_r) = f_r(V(K_r)) = \{a_{\ell_1}, a_{\ell_2}, \ldots, a_{\ell_j}, a_{\ell_{j+1}}, \ldots, a_{\ell_r}\},$$

where $1 \leq \ell_1 < \ell_2 < \cdots < \ell_j < \ell_{j+1} < \cdots < \ell_r \leq r+s$. By (1), we then have

$$\text{val}(f_r) = \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} (r - 2i + 1)(a_{\ell_{r-i+1}} - a_{\ell_i}).$$

Assume, to the contrary, that there is an integer j with $2 \leq j \leq r$ such that $\ell_j - \ell_{j-1} > 1$.

We first consider the case when $2 \le j \le \lfloor \frac{r}{2} \rfloor + 1$. In this case, we define a one to one function $h : f_r(V_r) \to \Big[f_r(V_r) \cup \{a_{(\ell_j)-1}\}\Big] - \{a_{\ell_{j-1}}\}$ by

$$h(a_{\ell_i}) = \begin{cases} a_{\ell_i} & \text{if } i \neq j-1 \\ a_{(\ell_j)-1} & \text{if } i = j-1. \end{cases}$$

Certainly, $g = h \circ f_r$ is a one to one function from V_r to $\{a_1, a_2, \ldots, a_{r+s}\}$. Furthermore,

$$\begin{aligned} \mathrm{val}(g) &= \mathrm{val}(f_r) - (r - 2(j-1) + 1)(a_{\ell_{r-(j-1)+1}} - a_{\ell_{j-1}}) \\ &\quad +(r - 2(j-1) + 1)(a_{\ell_{r-(j-1)+1}} - a_{(\ell_j)-1}) \\ &= \mathrm{val}(f_r) + (r - 2j + 3)(a_{\ell_{j-1}} - a_{(\ell_j)-1}). \end{aligned}$$

Since

$$\ell_j - \ell_{j-1} > 1,$$

it follows that

$$\ell_j - 1 > \ell_{j-1}$$

and so that

$$a_{(\ell_j)-1} > a_{\ell_{j-1}}.$$

Thus $\mathrm{val}(g) < \mathrm{val}(f_r)$, which is a contradiction with (3).

Next, we consider the case when $\lfloor \frac{r}{2} \rfloor + 2 \le j \le r = \lfloor \frac{r}{2} \rfloor + \lceil \frac{r}{2} \rceil$. In this case,we define a one to one function $h : f_r(V_r) \to \Big[f_r(V_r) \cup \{a_{(\ell_j)-1}\}\Big] - \{a_{\ell_j}\}$ by

$$h(a_{\ell_i}) = \begin{cases} a_{\ell_i} & \text{if } i \neq j \\ a_{(\ell_j)-1} & \text{if } i = j. \end{cases}$$

Certainly, $g = h \circ f_r$ is a one to one function from V_r to $\{a_1, a_2, \ldots, a_{r+s}\}$. Furthermore, let δ be an integer such that $j = r - \delta + 1$. Consequently, $1 \le \delta \le \lceil \frac{r}{2} \rceil - 1$. Therefore,

$$\begin{aligned} \mathrm{val}(g) &= \mathrm{val}(f_r) - (r - 2\delta + 1)(a_{\ell_j} - a_{\ell_\delta}) + (r - 2\delta + 1)(a_{(\ell_j)-1} - a_{\ell_\delta}) \\ &= \mathrm{val}(f_r) + (r - 2\delta + 1)(a_{(\ell_j)-1} - a_{\ell_j}). \end{aligned}$$

Since

$$\ell_j - \ell_{j-1} > 1,$$

it follows that

$$\ell_{j-1} < \ell_j - 1 < \ell_j$$

and so that

$$a_{\ell_{j-1}} < a_{(\ell_j)-1} < a_{\ell_j}.$$

Thus $\mathrm{val}(g) < \mathrm{val}(f_r)$, which is a contradiction with (3).

With a similar argument as above, we can conclude that $f_s(V_s)$ is a consecutive subset of $\{a_1, a_2, \ldots, a_{r+s}\}$. Therefore, either

$$f(V_r) = \{a_1, a_2, \ldots, a_r\} \text{ and } f(V_s) = \{a_{r+1}, a_{r+2}, \ldots, a_{r+s-1}, a_{r+s}\}$$

or

$$f(V_r) = \{a_{s+1}, a_{s+2}, \ldots, a_{s+r-1}, a_{s+r}\} \text{ and } f(V_s) = \{a_1, a_2, \ldots, a_s\}.$$

Now, we consider two cases according to $f(V_r)$ and $f(V_s)$.

Case 1. $f(V_r) = \{a_1, a_2, \ldots, a_r\}$ *and* $f(V_s) = \{a_{r+1}, a_{r+2}, \ldots, a_{r+s-1}, a_{r+s}\}$. Since

$$\begin{aligned}
\mathrm{val}_{\max}(K_{r,s}) &= \mathrm{val}(f) \\
&= (a_{r+1} - a_1) + (a_{r+1} - a_2) + \cdots + (a_{r+1} - a_r) \\
&\quad +(a_{r+2} - a_1) + (a_{r+2} - a_2) + \cdots + (a_{r+2} - a_r) \\
&\quad \vdots \\
&\quad +(a_{r+s} - a_1) + (a_{r+s} - a_2) + \cdots + (a_{r+s} - a_r) \\
&= (ra_{r+1} - \sum_{i=1}^{r} a_i) + (ra_{r+2} - \sum_{i=1}^{r} a_i) + \cdots + (ra_{r+s} - \sum_{i=1}^{r} a_i) \\
&= r\sum_{i=1}^{s} a_{r+i} - s\sum_{i=1}^{r} a_i,
\end{aligned}$$

it follows that $a_{r+i} = rs - i + 1$ for $i = 1, 2, \ldots, s$ and $a_i = i - 1$, for $i = 1, 2, \ldots, r$, that is, $f(V_r) = [0..r-1]$ and $f(V_s) = [rs - (s-1)..rs]$.

Case 2. $f(V_r) = \{a_{s+1}, a_{s+2}, \ldots, a_{s+r-1}, a_{s+r}\}$ *and* $f(V_s) = \{a_1, a_2, \ldots, a_s\}$. Since

$$\begin{aligned}
\mathrm{val}_{\max}(K_{r,s}) &= \mathrm{val}(f) \\
&= (a_{s+1} - a_1) + (a_{s+1} - a_2) + \cdots + (a_{s+1} - a_s) \\
&\quad +(a_{s+2} - a_1) + (a_{s+2} - a_2) + \cdots + (a_{s+2} - a_s) \\
&\quad \vdots \\
&\quad +(a_{s+r} - a_1) + (a_{s+r} - a_2) + \cdots + (a_{s+r} - a_s) \\
&= (sa_{s+1} - \sum_{i=1}^{s} a_i) + (sa_{s+2} - \sum_{i=1}^{s} a_i) + \cdots + (sa_{s+r} - \sum_{i=1}^{s} a_i) \\
&= s\sum_{i=1}^{r} a_{s+i} - r\sum_{i=1}^{s} a_i,
\end{aligned}$$

it follows that $a_{s+i} = rs - i + 1$ for $i = 1, 2, \ldots, r$ and $a_i = i - 1$, for $i = 1, 2, \ldots, s$, that is, $f(V_r) = [rs - (r-1)..rs]$ and $f(V_s) = [0..s-1]$. ∎

Next, we characterize γ-max labelings of complete graphs.

Theorem 6.2 *Let f be a γ-labeling of complete graph K_n. Then f is a γ-max labeling of K_n if and only if*

$$f(V(K_n)) = \begin{cases} \left[0..\left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1..\binom{n}{2}\right] & \text{if } n \text{ is even} \\ \left[0..\left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1..\binom{n}{2}\right] \cup \{k\} & \text{if } n \text{ is odd,} \end{cases}$$

where $k \in \left[\left\lfloor \frac{n}{2} \right\rfloor..\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor\right]$.

Proof. Let f be a γ-max labeling of K_n such that $f(v_i) = a_i$ for $1 \le i \le n$ and

$$0 \le a_1 < a_2 < \cdots < a_n \le \binom{n}{2}.$$

Therefore by (1),

$$\mathrm{val}_{\max}(K_n) = \mathrm{val}(f) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (n - 2i + 1)a_{n-i+1} - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (n - 2i + 1)a_i.$$

Then for $i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$, $a_{n-i+1} = \binom{n}{2} - i + 1$ and $a_i = i - 1$. That is,

$$f(V(K_n)) = \begin{cases} \left[0..\left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1..\binom{n}{2}\right] & \text{if } n \text{ is even} \\ \left[0..\left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1..\binom{n}{2}\right] \cup \{k\} & \text{if } n \text{ is odd,} \end{cases}$$

where $k \in \left[\left\lfloor \frac{n}{2} \right\rfloor..\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor\right]$. ■

7 Final Remarks

In [5] and [6] the maximum value of γ-labeling of path P_n, cycle C_n, double star $S_{p,q}$ and cycle with a triangle $C_n^{\triangle}$ was determined. It would be interesting to characterize γ-max labeling of those graphs.

In [8] the γ-spectra of paths, cycles, and complete graphs are determined. It would be interesting to determine the γ-spectra of $C_n^{\triangle}$ and find whether it is continuous or not. Observe that the value of any γ-labeling of a cycle is always even.

References

[1] A. Blinco, S. El-Zanati, C. Vanden Eynden, On the cyclic decomposition of complete graphs into almost-bipartite graphs, Discrete Math. 284 (2004), no. 1-3, 71-81.

[2] G.S. Bloom, S.W. Golomb, Numbered complete graphs, unusual rulers, and assorted applications, in *Theory and Applications of Graphs*, Lecture Notes in Math. 642, Springer-Verlag (1978), 53-65.

[3] G. D. Bullington, L. L. Eroh, and S. J. Winters, *Gamma-labelings of complete bipartite graphs*, Discuss. Math. Graph Theory, **30** (2010) 45-54.

[4] G. D. Bullington, L. L. Eroh, and S. J. Winters, *Gamma-labelings of complete bipartite graphs*, Discuss. Math. Graph Theory, **30** (2010) 45-54.

[5] G. Chartrand, D. Erwin, D.W. VanderJagt, P. Zhang, γ-labelings of graphs, Bull. Inst. Combin. Appl. 44 (2005), 51-68.

[6] G. Chartrand, D. Erwin, D.W. VanderJagt, P. Zhang, On γ-labelings of trees, Discuss. Math. Graph Theory 25 (2005), no. 3, 363-383.

[7] G. Chartrand, P. Zhang, *Introduction to Graph Theory*, The Walter Rudin student series in advanced mathematics, McGraw-Hill Higher Education, Boston, 2005.

[8] C.M. da Fonseca, V. Saenpholphat, P. Zhang, The γ-spectrum of a graph, Ars Combin., to appear.

[9] J.A. Gallian, A dynamic survey of graph labeling, Electron. J. Combin. 5 (1998), Dynamic Survey 6, 43 pp.

[10] S.W. Golomb, How to number a graph, in *Graph Theory and Computing*, pp. 23-37, Academic Press, New York, 1972.

[11] R. Häggkvist, Decompositions of complete bipartite graphs, in *Surveys in Combinatorics*, 1989 (Norwich, 1989), 115-147, London Math. Soc. Lecture Note Ser., 141, Cambridge Univ. Press, Cambridge, 1989.

[12] C. Huang, A. Kotzig, A. Rosa, Further results on tree labellings, Utilitas Math. 21 (1982), 31-48.

[13] G. Ringel, Problem 25, in *Theory of Graphs and its Applications*, Proceedings of the Symposium in Smolenice 1963, 1964, p. 162.

[14] A. Rosa, On certain valuations of the vertices of a graph, in *Theory of Graphs* (Internat. Sympos., Rome, 1966) pp. 349-355, Gordon and Breach, New York; Dunod, Paris.

Output จากโครงการวิจัย

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ

1.1 งานวิจัย **Extremal values for a gamma-labeling of a cycle with a triangle.** (C. M. da Fonseca, V. Saenpholphat and P. Zhang)
ได้ตอบรับให้ตีพิมพ์ในวารสาร Utilitas Mathematica

1.2 งานวิจัย **On gamma-labeling of graphs.** (C. M. da Fonseca, V. Saenpholphat and P. Zhang ได้ตอบรับให้ตีพิมพ์ในวารสาร Utilitas Mathematica

2. การนำผลงานวิจัยไปใช้ประโยชน์

2.1 ได้ผลงานวิจัยที่เป็นทางด้านคณิตศาสตร์บริสุทธิ์ซึ่งเป็นประโยชน์โดยตรงในการเผยแพร่ ชื่อเสียงของประเทศชาติไปสู่นานาอารยะประเทศทางด้านวิชาการ
2.2 ได้นำผลงานวิจัยนี้ในการสอนนิสิตที่ศึกษาวิชาทฤษฎีกราฟในระดับปริญญาตรีและระดับบัณฑิตศึกษา

3. การเสนอผลงานในที่ประชุมวิชาการ

3.1 **Oral Presentation: On gamma labelings of graphs**. การประชุมนักวิจัยรุ่นใหม่พบเมธีวิจัยอาวุโส สกว. ครั้งที่ 12, โรงแรมฮอลิเดย์ อินน์ รีสอร์ท รีเจ้นท์ บีช ชะอำ จังหวัดเพชรบุรี, 10 - 12 ตุลาคม 2555

3.2 **Oral Presentation: Extremal values for a gamma-labeling of graph**. Thailand-Japan Joint Conference on Computational Geometry and Graphs (TJJCCGG2012), Srinakharinwirot University, Bangkok, Thailand, December 6 - 8, 2012

3.3 **Oral Presentation: On extremal values for a gamma-labeling of graph**. การประชุมนักวิจัยรุ่นใหม่พบเมธีวิจัยอาวุโส สกว. ครั้งที่ 13, โรงแรมเดอะรีเจนท์ รีเจ้นท์ บีช รีสอร์ท หัวหิน ชะอำ จังหวัดเพชรบุรี, 16 - 18 ตุลาคม 2556

ภาคผนวก

Subject: Re: new pdf file: Utilitas Math paper
From: "Math Journals" <MathJournals@attglobal.net>
Date: Mon, January 17, 2011 10:28 pm
To: varaporn@swu.ac.th
Priority: Normal
Options: View Full Header | View Printable Version | Download this as a file

```
Professor Saenpholphat

Thank you for your file, it printed out
correctly. I have passed it on to production. You
will be receiving an invoice for reprints in due
course. If you do not want reprints, please let
me know.

Currently, the backlog for accepted papers is now over 3 years.

Yours sincerely,
J.L. Allston
Managing Editor
Utilitas Mathematica

>Dear Professor Allston
>
>Thank you so much for your letter of January 7, 2011, in which you informed
>us that the paper
>
>"Extremal values for a $\gamma$-labeling of a cycle with a triangle"
>
>(by C.M. da Fonseca, P. Zhang and myself) was accepted for publication in
>Utilitas Mathematica.
>
>Enclosed is the pdf file of this paper.
>
>Sincerely,
>
>Varaporn Saenpholphat
>
>---------------------------- Original Message ----------------------------
>Subject: Utilitas Math paper
>From:    "Math Journals" <MathJournals@attglobal.net>
>Date:    Fri, January 7, 2011 4:56 pm
>To:      cmf@mat.uc.pt
>--------------------------------------------------------------------------
>
>Professor da Fonseca,
>
>I have now received the referees report on your paper with Varaporn
>Saenpholphat and Ping Zhang entitled, "Extremal values for a
>$\gamma$-labeling of a cycle with a triangle".".
>
>The referee's report is attached as a .pdf file. As you can see, the
>referee recommends publication with revisions.
>
>Therefore, I am happy to accept it for publication in Utilitas
>Mathematica once the corrections are done.
```

Extremal values for a γ-labeling of a cycle with a triangle

C.M. da Fonseca [1]

Department of Mathematics
University of Coimbra
3001-454 Coimbra, Portugal

Varaporn Saenpholphat [2]

Department of Mathematics
Srinakharinwirot University,
Sukhumvit 23, Bangkok 10110, Thailand

Ping Zhang

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA

ABSTRACT

For any graph G of order n and size m, a γ-labeling of G is defined as a one-to-one function $f\colon V(G) \to \{0,1,\ldots,m\}$ that induces an edge-labeling $f'\colon E(G) \to \{1,\ldots,m\}$ on G defined by

$$f'(uv) = |f(u) - f(v)|\,, \quad \text{for each edge } uv \text{ in } E(G)\,.$$

The value of f is defined by

$$\mathrm{val}(f) = \sum_{e\in E(G)} f'(e)\,.$$

In this paper, we determine the extremal values of a γ-labeling of a cycle with a triangle. Several examples are considered. More generally, for a nonnegative integer δ, we define a γ^δ-labeling of G as a one-to-one function $f : V(G) \to \{0,1,\ldots,m+\delta-1, m+\delta\}$ that induces an edge-labeling $f'\colon E(G) \to \{1,\ldots,m+\delta\}$ on G defined by

$$f'(uv) = |f(u) - f(v)|\,, \quad \text{for each edge } uv \text{ in } E(G)\,.$$

The value of a γ^δ-labeling f is defined by $\mathrm{val}(f) = \sum_{e\in E(G)} f'(e)$. We determine the extremal values of a γ^δ-labeling of some graphs.

Key Words: Graph; cycle; γ-labeling; γ^δ-labeling.
AMS Subject Classification: 05C78.

[1] Research supported by CMUC - Centro de Matemática da Universidade de Coimbra.
[2] Corresponding author, Research supported in part by Srinakharinwirot University, Commission on Higher Education and the Thailand Research Fund-RMU5380031.

1 Introduction

Let G be a graph of order n and size m. A γ-labeling of G is a one-to-one function $f\colon V(G) \to \{0,1,\ldots,m\}$ that induces an edge-labeling $f'\colon E(G) \to \{1,\ldots,m\}$ on G defined by

$$f'(uv) = |f(u) - f(v)|\,, \quad \text{for each edge } uv \text{ in } E(G)\,.$$

The value of f is defined by

$$\mathrm{val}(f) = \sum_{e\in E(G)} f'(e)\,.$$

This particular graph labeling was first introduced by Chartrand et al. [3], in 2005, and it was motivated by Rosa's influential paper [12], where a variety of types of "valuations" were introduced. (We remark that this notion should not be confused with the γ-labeling recently introduced by Blinco et al. in [1].)

When the labels $f'(e)$ are all distinct, the γ-labeling is called a graceful labeling. The origin of this concept is often attributed Gerhard Ringel [11]. The Ringel-Kotzig Conjecture, as it was coined in [10], and popularized by Golomb [8], states that all trees are graceful and it remains still unsolved. This conjecture is also known as Von Koch's Conjecture or Graceful Labeling Conjecture [12]. The study of graceful labelings and their applications has been intensively considered in the literature [2, 7, 9].

Obviously, since f is one-to-one, it follows that $f'(e) \geq 1$, for any edge e, and therefore, $\mathrm{val}(f) \geq m$. Moreover, G has a γ-labeling if and only if $m \geq n-1$ and every connected graph has a γ-labeling.

The maximum value and the minimum value of a γ-labeling of a graph G are defined in [3] as

$$\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f)\colon\ f \text{ is a } \gamma\text{-labeling of } G\}$$

and

$$\mathrm{val}_{\min}(G) = \min\{\mathrm{val}(f)\colon\ f \text{ is a } \gamma\text{-labeling of } G\}\,,$$

respectively. In that paper, the maximum and minimum values of a γ-labeling of path, star, cycle, and complete graph are determined. Next, we recall the extremal values of a γ-labeling of cycle of order n.

Theorem 1.1 ([3]) *If C_n is a cycle of order $n \geq 3$, then*

$$\mathrm{val}_{\min}(C_n) = 2(n-1)$$

and

$$\mathrm{val}_{\max}(C_n) = \begin{cases} \frac{(n-1)(n+3)}{2} & \text{if } n \text{ is odd} \\ \frac{n(n+2)}{2} & \text{if } n \text{ is even.} \end{cases}$$

The γ-spectrum of a graph G is also defined in [3] as

$$\mathrm{spec}(G) = \{\mathrm{val}(f)\ :\ f \text{ is a } \gamma\text{-labeling of } G\}\,,$$

and the γ-spectrum of any star is determined as well. In [4], the same authors found the maximum and the minimum values of a γ-labeling of double star, and determine all graphs G of order n for which $\mathrm{val}_{\min}(G)$ is n or $n+1$. Some results about minimum value of a γ-labeling of a tree with maximum degree $\Delta = 3$ were established. Later, in [6], the γ-spectra of paths, cycles and complete graphs were determined.

In [3] a simple and useful connection between minimum and maximum values of a connected graph and that of a proper connected subgraph was found.

Proposition 1.2 ([3]) *If H is a proper connected subgraph of a connected graph G, then*

$$\mathrm{val}_{\min}(H) < \mathrm{val}_{\min}(G) \text{ and } \mathrm{val}_{\max}(H) < \mathrm{val}_{\max}(G).$$

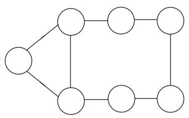
The span of a γ-labeling f is defined as

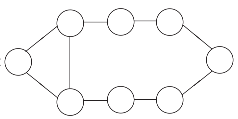
$$\mathrm{span}(f) = \max\{f(v) : v \in V(G)\} - \min\{f(v) : v \in V(G)\}.$$

The following lemma that appeared in [3] will be useful to us.

Lemma 1.3 ([3]) *Let G be a connected graph of order n and $f : V(G) \to \mathbf{Z}$ a one-to-one function. Then there is a γ-labeling g on G with $\mathrm{val}(g) \le \mathrm{val}(f)$. Furthermore, if $\mathrm{span}(f) \ge n$, then there is a γ-labeling g with $\mathrm{val}(g) < \mathrm{val}(f)$.*

Here we consider cycles of order $n \ge 5$ with a triangle, say $C_n^\triangle$, i.e., cycles with a chord joining two nonadjacent vertices but adjacent to some vertex in cycle. As below, we show $C_7^\triangle$ and $C_8^\triangle$:

$C_7^\triangle$:


$C_8^\triangle$:


In this paper, we determine $\mathrm{val}_{\min}(C_n^\triangle)$ and $\mathrm{val}_{\max}(C_n^\triangle)$. For the later case, we need to generalize the notion of γ-labeling of a connected graph. Several illustrative examples will be provided.

The reader is referred to Chartrand and Zhang [5] for basic definitions and terminology not mentioned here.

2 The minimum value of $C_n^\triangle$

As in the case of the cycles, the minimum value of $C_n^\triangle$ is somewhat easier to determine.

Theorem 2.1 *For every integer $n \ge 5$,*

$$\mathrm{val}_{\min}(C_n^\triangle) = 2n - 1\,.$$

Proof. By Proposition 1.2 and Theorem 1.1, we have

$$\mathrm{val}_{\min}(C_n^\triangle) \ge 2n - 1\,.$$

Hence it remains to show that $\mathrm{val}_{\min}(C_n^\triangle) \le 2n-1$.

Suppose that $C_n^\triangle$ is the cycle $C_n : v_1, v_2, \ldots, v_n, v_1$ with the triangle obtained by adding an edge v_2v_n. Considering now the γ-labeling f of $C_n^\triangle$ defined by

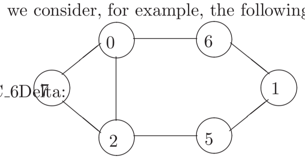
$$f(v_i) = \begin{cases} 0 & \text{if } i = 1 \\ i & \text{if } 2 \le i \le n-1 \\ 1 & \text{if } i = n, \end{cases}$$

we have

$$\begin{aligned}\mathrm{val}(f) &= \textstyle\sum_{i=2}^{n-2} |f(v_{i+1}) - f(v_i)| + |f(v_1) - f(v_2)| + \\ &\quad + |f(v_2) - f(v_n)| + |f(v_n) - f(v_1)| + |f(v_{n-1}) - f(v_n)| \\ &= (n-3)\cdot 1 + 2 + 1 + 1 + (n-2) \;=\; 2n-1\,.\end{aligned}$$

Therefore, $\mathrm{val}_{\min}(C_n^\triangle) \le 2n - 1$. ■

To illustrate Theorem 2.1, we consider, for example, the following γ-min labeling f of $C_6^\triangle$:

C_6Delta:


Observe that $\mathrm{val}_{\min}(C_6^\triangle) = \mathrm{val}(f) = 11\,.$

3 The γ^δ-Labeling of a Connected Graph

We start this section with a slightly extension of the main notion of γ-labeling of a connected graph. For a nonnegative integer δ, a γ^δ-labeling of a connected graph G of order n and size m is a one-to-one function $f : V(G) \to \{0, 1, \ldots, m+\delta-1, m+\delta\}$ that induces an edge-labeling $f' \colon E(G) \to \{1, \ldots, m+\delta\}$ on G defined by

$$f'(uv) = |f(u) - f(v)|\,, \quad \text{for each edge } uv \text{ in } E(G)\,.$$

The value of a γ^δ-labeling f is defined by $\mathrm{val}(f) = \sum_{e \in E(G)} f'(e)$. The other definitions are similar to the γ-labeling case.

The following corollary is a consequence of Lemma 1.3.

Corollary 3.1 *For any connected graph G of order n,*

$$\mathrm{val}^\delta_{\min}(G) = \mathrm{val}_{\min}(G)\,.$$

Next, we present the maximum value of a γ^δ-labeling of C_n when n is even.

Proposition 3.2 *For a nonnegative integer δ and an even integer n, with $n \ge 4$,*

$$\mathrm{val}^\delta_{\max}(C_n) = \mathrm{val}_{\max}(C_n) + n\delta = \frac{n(n+2\delta+2)}{2}\,.$$

Proof. Let $n = 2k$ be an even integer, with $n \ge 4$, and let $C_n : x_1, y_1, x_2, y_2, \ldots, x_k, y_k, x_1$. Define a γ^δ-labeling f of C_n by

$$f(x_i) = i - 1 \quad \text{and} \quad f(y_i) = 2k + \delta + 1 - i\,, \quad \text{for } 1 \le i \le k\,.$$

Then

$$\begin{aligned}\mathrm{val}(f) &= 2\left(\sum_{i=1}^{k} f(y_i) - \sum_{i=1}^{k} f(x_i)\right)\\ &= 2\left(\sum_{i=1}^{k}(2k+\delta+1-i) - \sum_{i=1}^{k}(i-1)\right)\\ &= 2k(k+\delta+1)\\ &= \tfrac{n(n+2\delta+2)}{2} = \mathrm{val}_{\max}(C_n) + n\delta\,.\end{aligned}$$

Hence

$$\mathrm{val}^{\delta}_{\max}(C_n) \geq \mathrm{val}_{\max}(C_n) + n\delta = \frac{n(n+2\delta+2)}{2}\,.$$

In order to show $\mathrm{val}^{\delta}_{\max}(C_n) \leq \frac{n(n+2\delta+2)}{2}$, let g be a γ^{δ}-max labeling of C_n, where $E(C_n) = \{e_1, e_2, \ldots, e_n\}$. For each integer i, with $1 \leq i \leq n$, let $e_i = u_iv_i$, where $g(u_i) < g(v_i)$. Then

$$\mathrm{val}(g) = \sum_{i=1}^{n} g(v_i) - \sum_{i=1}^{n} g(u_i)\,.$$

Since at most two vertices in $\{u_1, \ldots, u_n\}$ can be assigned by each of the labels $0, 1, \ldots, k-1$, and at most two vertices in $\{v_1, \cdots, v_n\}$ can be assigned each of the labels $k+(\delta+1), k+(\delta+2), \ldots, k+(k+\delta-1), k+(k+\delta)$, it follows that

$$\sum_{i=1}^{n} g(u_i) \geq k^2 - k \quad \text{and} \quad \sum_{i=1}^{n} g(v_i) \leq 3k^2 + (2\delta+1)k\,.$$

Thus

$$\mathrm{val}(g) \leq (3k^2 + (2\delta+1)k) - (k^2 - k) = 2k(k+\delta+1) = \frac{n(n+2\delta+2)}{2},$$

producing the desired result. ■

Proposition 3.3 *Let D be an oriented graph derived from a γ^{δ}-max labeling g of an even cycle C_n, with $n \geq 6$, by assigning to each edge uv the orientation (u, v), if $g(u) < g(v)$. Then for each vertex v either $\mathrm{id}(v) = 2$ or $\mathrm{od}(v) = 2$.*

Proof. For an even cycle C_n, we can have an oriented graph D from γ^{δ}-max labeling g of C_n by assigning to each edge uv the orientation (u, v), if $g(u) < g(v)$.

Let us assume that there exists a vertex such that indegree and outdegree are not two. Since C_n is an even cycle, it follows that there are at least two vertices, say u and v, such that

$$\mathrm{id}(u) = \mathrm{id}(v) = \mathrm{od}(u) = \mathrm{od}(v) = 1\,.$$

We consider two cases, according to the adjacency of the vertices u and v.

Case 1. *u is adjacent to v.* Then D contains a directed path x, u, v, y of order 4. If we delete the vertices u and v from C_n and join the vertices x and y, the resulting graph G is isomorphic to C_{n-2} and the restriction g' of g to $V(C_n) - \{u, v\}$ has the same value on G as g does on C_n. Therefore,

$$\mathrm{val}^{\delta+2}_{\max}(C_{n-2}) \geq \mathrm{val}(g) = \mathrm{val}^{\delta}_{\max}(C_n),$$

which is a contradiction with $\mathrm{val}(g) = \mathrm{val}^{\delta}_{\max}(C_n) > \mathrm{val}^{\delta+2}_{\max}(C_{n-2})$.

Case 2. *u and v are not adjacent.* The D contains two direct paths, s, u, t and x, v, y, of order 3. Therefore, we can construct a graph G that is isomorphic to C_{n-2} from C_n by deleting the vertices u and v, joining the vertices s and t, and then joining vertices x and y. The restriction g' of g to $V(C_n) - \{u, v\}$ has the same value on G as g does on C_n. Therefore,

$$\mathrm{val}^{\delta+2}_{\max}(C_{n-2}) \geq \mathrm{val}(g) = \mathrm{val}^{\delta}_{\max}(C_n),$$

which is a contradiction with $\mathrm{val}(g) = \mathrm{val}^{\delta}_{\max}(C_n) > \mathrm{val}^{\delta+2}_{\max}(C_{n-2})$. ■

We end this section with a relation between δ-max values of γ-labelings of cycles with consecutive order.

Lemma 3.4 *For every pair δ, k of nonnegative integers with $k \geq 2$,*

$$\mathrm{val}^{\delta}_{\max}(C_{2k+1}) = \mathrm{val}^{\delta+1}_{\max}(C_{2k})\,.$$

Proof. Let f be a $\gamma^{\delta+1}$-max labeling of C_{2k}. Suppose that a is a number in $\{0, 1, \cdots, 2k+\delta, 2k+\delta+1\}$ assigned to no vertex of C_{2k} by f. Consequently, we may define a γ^{δ}-labeling g of C_{2k+1} by

$$g(v_i) = \begin{cases} f(v_i) & \text{if } 1 \leq i \leq 2k \\ a & \text{if } i = 2k+1. \end{cases}$$

Thus $\mathrm{val}^{\delta}_{\max}(C_{2k+1}) \geq \mathrm{val}^{\delta+1}_{\max}(C_{2k})$. It remains to verify that $\mathrm{val}^{\delta+1}_{\max}(C_{2k}) \geq \mathrm{val}^{\delta}_{\max}(C_{2k+1})$.

Let h be a γ^{δ}-max labeling of C_{2k+1}. We construct an oriented graph D form C_{2k+1} by assigning to each edge uv the orientation (u, v), if $h(u) < h(v)$. Necessarily, D contains a directed path x, y, z of order 3. If we delete the vertex y from C_{2k+1} and join the vertices x and z, the resulting graph G is isomorphic to C_{2k} and the restriction $\tilde{h}$ of h to $V(C_{2k+1}) - \{y\}$ has the same value on G as h does on C_{2k+1}. The function $\tilde{h}$ is thus a $\gamma^{\delta+1}$-labeling of C_{2k}, and the result follows. ■

As a consequence of Lemma 3.4, we have the following result.

Corollary 3.5 *For a nonnegative integer δ and an odd integer n, with $n \geq 5$,*

$$\mathrm{val}^{\delta}_{\max}(C_n) = \mathrm{val}_{\max}(C_n) + (n-1)\delta = \frac{(n-1)(n+2\delta+3)}{2}\,.$$

4 The maximum value of $C_n^{\triangle}$ for an odd integer n

The simplest computation of the maximum value of a γ-labeling of $C_n^{\triangle}$ is when n an odd integer. The other case is more delicate as we will see in the next sections. Now, we consider a cycle with a triangle of order $n = 2k+1$, with $k \geq 2$, obtained from a cycle C_{2k+1} : $x_1, z, y_1, x_2, y_2, \cdots, x_k, y_k, x_1$ and a chord x_1y_1.

Theorem 4.1 *For every odd integer $n \geq 5$,*

$$\mathrm{val}_{\max}(C_n^{\triangle}) = \mathrm{val}^{1}_{\max}(C_n^{\triangle}) + (n+1).$$

Proof. Setting $n = 2k+1$, with $k \geq 2$, and defining a γ-labeling f of $C_n^{\triangle}$ by

$$f(x_i) = i-1\,, \quad f(y_i) = 2k+3-i\,, \quad \text{for } 1 \leq i \leq k\,, \text{ and } f(z) = k+1\,,$$

then

$$\begin{aligned}
\mathrm{val}(f) &= 2\left(\textstyle\sum_{i=1}^k f(y_i) - \sum_{i=1}^k f(x_i)\right) + (f(y_1) - f(z)) + (f(z) - f(x_1)) \\
&= 2\left(\textstyle\sum_{i=1}^k (2k+3-i) - \sum_{i=1}^k (i-1)\right) + (2k+2) \\
&= 2k(k+3) + (2k+2) \\
&= \tfrac{(n-1)(n-5)}{2} + (n+1) \\
&= \mathrm{val}_{\max}^2(C_{n-1}) + (n+1) \\
&= \mathrm{val}_{\max}^1(C_n) + (n+1)\,.
\end{aligned}$$

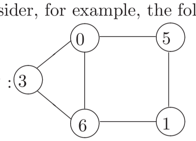
Thus $\mathrm{val}_{\max}(C_n^{\triangle}) \geq \mathrm{val}_{\max}^1(C_n) + (n+1)$.

In order to show that $\mathrm{val}_{\max}(C_n^{\triangle}) \leq \mathrm{val}_{\max}^1(C_n) + (n+1)$, let f be a γ-max labeling of $C_n^{\triangle}$, and let g be the restriction of f to C_n. Then

$$\mathrm{val}(f) = \mathrm{val}(g) + f'(e) \leq \mathrm{val}_{\max}^1(C_n) + (n+1)\,,$$

where e is a chord of $C_n^{\triangle}$. Therefore, $\mathrm{val}_{\max}(C_n^{\triangle}) \leq \mathrm{val}_{\max}^1(C_n) + (n+1)$. ■

To illustrate Theorem 4.1, we consider, for example, the following γ-max labeling f of $C_5^{\triangle}$:

$C_5^{\triangle}$:


Observe that $\mathrm{val}_{\max}(C_5^{\triangle}) = \mathrm{val}(f) = 26$.

The next corollary is a consequence of Theorem 4.1.

Corollary 4.2 *For every pair of nonnegative integer δ and odd integer n with $n \geq 5$,*

$$\mathrm{val}_{\max}^{\delta}(C_n^{\triangle}) = \mathrm{val}_{\max}^{\delta+1}(C_n) + (\delta+n+1).$$

Proof. Let $n = 2k+1$, with $k \geq 2$, and let f be a γ-labeling f of $C_n^{\triangle}$ defined by

$$f(x_i) = i-1\,, \quad f(y_i) = \delta + 2k + 3 - i, \text{ for } 1 \leq i \leq k\,, \text{ and } f(z) = k+1\,.$$

Then

$$\begin{aligned}
\mathrm{val}(f) &= 2\left(\textstyle\sum_{i=1}^k f(y_i) - \sum_{i=1}^k f(x_i)\right) + (f(y_1) - f(z)) + (f(z) - f(x_1)) \\
&= 2\left(\textstyle\sum_{i=1}^k (\delta + 2k+3-i) - \sum_{i=1}^k (i-1)\right) + (\delta + 2k+2) \\
&= 2k(k+3) + 2\delta k + (\delta + 2k+2) \\
&= k(2k+2\delta+6) + (\delta+2k+2) \\
&= \tfrac{(n-1)(n+2\delta+5)}{2} + (\delta+n+1) \\
&= \mathrm{val}_{\max}^{\delta+2}(C_{n-1}) + (\delta+n+1) \\
&= \mathrm{val}_{\max}^{\delta+1}(C_n) + (\delta+n+1)\,.
\end{aligned}$$

Thus $\mathrm{val}_{\max}^{\delta}(C_n^{\triangle}) \geq \mathrm{val}_{\max}^{\delta+1}(C_n) + (\delta+n+1)$.

To show that $\mathrm{val}_{\max}^{\delta}(C_n^{\triangle}) \leq \mathrm{val}_{\max}^{\delta+1}(C_n) + (\delta+n+1)$, let f be a γ-max labeling of $C_n^{\triangle}$, and let g be the restriction of f to C_n. Then

$$\mathrm{val}(f) \;=\; \mathrm{val}(g) + f'(e) \;\leq\; \mathrm{val}_{\max}^{\delta+1}(C_n) + (\delta+n+1),$$

where e is a chord of $C_n^{\triangle}$. Therefore, $\mathrm{val}_{\max}^{\delta}(C_n^{\triangle}) \leq \mathrm{val}_{\max}^{\delta+1}(C_n) + (\delta+n+1)$. ■

5 The maximum value of $C_n^{\triangle}$ for an even integer n

In this section we consider a cycle with a triangle of order $n = 2k$, with $k \geq 3$, obtained from a cycle $C_{2k} : x_1, y_1, x_2, y_2, \cdots, x_k, y_k, x_1$ and a chord x_1x_k. We first present a lower bond for $\mathrm{val}_{\max}(C_n^{\triangle})$.

Lemma 5.1 *For every even integer $n \geq 6$,*

$$\mathrm{val}_{\max}(C_n^{\triangle}) \geq \begin{cases} \frac{n^2+5n-2}{2} & = 32 \quad \text{if } n = 6 \\ \frac{n^2+5n-2}{2} = \frac{n^2+6n-10}{2} & = 51 \quad \text{if } n = 8 \\ \frac{n^2+6n-10}{2} & \quad\quad\;\; \text{if } n \geq 8. \end{cases}$$

Proof. If $k = 3$ or 4, we define a γ-labeling f of $C_{2k}^{\triangle}$ by

$$f(x_i) = i-1 \quad \text{and} \quad f(y_i) = n+2-i\,, \text{ for } 1 \leq i \leq k\,.$$

Then

$$\begin{aligned}
\mathrm{val}(f) &= 2\left(\textstyle\sum_{i=1}^k f(y_i) - \sum_{i=1}^k f(x_i)\right) + f(x_k) - f(x_1) \\
&= 2\left(\textstyle\sum_{i=1}^k (n+2-i) - \sum_{i=1}^k (i-1)\right) + (k-1) - 0 \\
&= \tfrac{n^2+5n-2}{2}\,.
\end{aligned}$$

Thus $\mathrm{val}_{\max}(C_n^{\triangle}) \geq \frac{n^2+5n-2}{2}$, for $n = 6, 8$.

Assume that $k \geq 4$, we define a γ-labeling f of $C_{2k}^{\triangle}$ by

$$f(x_i) = \begin{cases} n+2-i & \text{if } 1 \leq i \leq k-1 \\ 0 & \text{if } i = k \end{cases} \quad \text{and} \quad f(y_i) = i\,, \text{ if } 1 \leq i \leq k.$$

Then

$$\begin{aligned}
\mathrm{val}(f) &= 2\left(\textstyle\sum_{i=1}^{k-1} f(x_i) - \sum_{i=1}^{k-2} f(y_i)\right) + f(x_1) - 3f(x_k) \\
&= 2\left(\textstyle\sum_{i=1}^{k-1} (n+2-i) - \sum_{i=1}^{k-2} i\right) + (n+1) - 3(0) \\
&= \tfrac{n^2+6n-10}{2}\,.
\end{aligned}$$

Hence $\mathrm{val}_{\max}(C_n^{\triangle}) \geq \frac{n^2+6n-10}{2}$, for $n \geq 8$. ■

In order to present a formula for $\mathrm{val}_{\max}(C_n^{\triangle})$, for an even integer $n \geq 6$, we need some additional notation and new definitions. Let f be a γ-max labeling of $C_{2k}^{\triangle}$. For each integer i, with $1 \leq i \leq k$, we define the 3-term sequences

$$S_i(f) = (f(x_i), f(y_i), f(x_{i+1})) \text{ and } T_i(f) = (f(y_i), f(x_{i+1}), f(y_{i+1})),$$

where the addition is taken modulo k, and let

$$ST(f) = \{S_1(f), S_2(f), \ldots, S_k(f), T_1(f), T_2(f), \ldots, T_k(f)\}$$

be a set of 3-term sequences $S_i(f)$ and $T_i(f)$, for all i with $1 \leq i \leq k$. Furthermore, let $C_{2k}(f)$ be an oriented cycle obtained from $C_{2k}^{\triangle} - x_1x_k$ by assigning to the edge uv the orientation (u, v) if $f(u) < f(v)$.

We are ready to characterize any γ-max labeling f of $C_6^{\triangle}$.

Proposition 5.2 *Let f be a γ-max labeling of $C_6^{\triangle}$. Then $ST(f)$ contains no monotone elements.*

Proof. Assume, to the contrary, that some element of $ST(f)$ is monotone. Then an oriented cycle $C_6(f)$ contains a directed path a, b, c of order 3. If we delete the chord x_1x_3 and vertex b from $C_6^{\triangle}$ and then join the vertices a and c, the resulting graph G is isomorphic to C_5 and the restriction $\tilde{f}$ of f to $V(C_6^{\triangle} - x_1x_3) - \{b\}$ has the same value on G as f on $C_6^{\triangle} - x_1x_3$, that is

$$\text{val}(\tilde{f}) = \text{val}(f) - f'(x_1x_3) = \text{val}_{\max}(C_6^{\triangle}) - f'(x_1x_3).$$

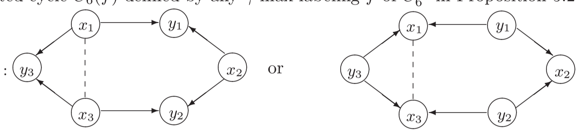
Then $\text{val}_{\max}(C_6^{\triangle}) = \text{val}(\tilde{f}) + f'(x_1x_3)$. Moreover, since $\tilde{f}$ is a γ^2-labeling of graph G that is isomorphic to C_5, it follows that $\text{val}(\tilde{f}) \leq \text{val}^2_{\max}(C_5) = \text{val}^3_{\max}(C_4)$. Therefore

$$\text{val}_{\max}(C_6^{\triangle}) \;\leq\; \text{val}^3_{\max}(C_4) + f'(x_1x_3) \;\leq\; \frac{4(4+6+2)}{2} + 7 \;=\; 31,$$

which is a contradiction with Lemma 5.1. ∎

The oriented cycle $C_6(f)$ defined by any γ-max labeling f of $C_6^{\triangle}$ in Proposition 5.2 is either

$C_6(f)$:



Observe that all 3-term sequences of $ST(f)$, which are

$$S_1(f) = (f(x_1), f(y_1), f(x_2)), S_2(f) = (f(x_2), f(y_2), f(x_3)), S_3(f) = (f(x_3), f(y_3), f(x_1))$$

$$T_1(f) = (f(y_1), f(x_2), f(y_2)), T_2(f) = (f(y_2), f(x_3), f(y_3)), T_3(f) = (f(y_3), f(x_1), f(y_1)),$$

are not monotone.

Combining Lemma 5.1 and Proposition 5.2, we are able to find $\text{val}_{\max}(C_6^{\triangle})$.

Theorem 5.3

$$\text{val}_{\max}(C_6^{\triangle}) = \text{val}^1_{\max}(C_6) + 2 = 32.$$

Proof. By Lemma 5.1, we have $\text{val}_{\max}(C_6^{\triangle}) \geq 32 = \text{val}^1_{\max}(C_6) + 2$. Next, let f be a γ-max labeling of $C_6^{\triangle}$. According to Proposition 5.2, for each i with $1 \leq i \leq 3$, vertices x_i and y_i of oriented cycle $C_6(f)$ have

$$\textbf{either} \;\; \text{id}(x_i) = 0, \text{id}(y_i) = 2 \;\; \textbf{or} \;\; \text{id}(x_i) = 2, \text{id}(y_i) = 0.$$

We analyze the two cases separately.

Case 1. *For each i with $1 \leq i \leq 3$, $\text{id}(x_i) = 0$, $\text{id}(y_i) = 2$ of the oriented cycle $C_6(f)$.* So, $f(x_i) < f(y_i)$ and $f(x_{i+1}) < f(y_i)$ where addition is taken modulo 3. Then

$$\text{val}_{\max}(C_6^{\triangle}) = \text{val}(f) = 2\left(\sum_{i=1}^{3} f(y_i) - \sum_{i=1}^{3} f(x_i)\right) + |f(x_1) - f(x_3)|.$$

Since the vertices in $\{x_1, x_2, x_3\}$ can be labeled by each of the labels $0, 1, 2$ and the vertices in $\{y_1, y_2, y_3\}$ can be labeled by each of the labels $7, 6, 5$, it follows that

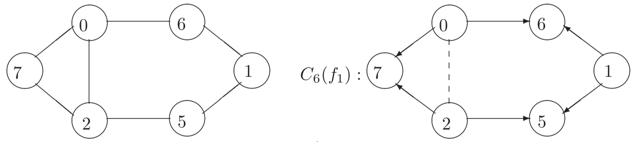
$$\sum_{i=1}^{3} f(x_i) \geq 0 + 1 + 2 = 3 \;\text{ and }\; \sum_{i=1}^{3} f(y_i) \leq 7 + 6 + 5 = 18.$$

Then $\text{val}_{\max}(C_6^{\triangle}) = \text{val}(f) \leq 2(18 - 3) + 2 = \text{val}^1_{\max}(C_6) + 2 = 32$.

Case 2. *For each i with $1 \leq i \leq 3$, $\text{id}(x_i) = 2$, $\text{id}(y_i) = 0$ of the oriented cycle $C_6(f)$.* So, $f(x_i) > f(y_i)$ and $f(x_{i+1}) > f(y_i)$ where addition is taken modulo 3. An argument similar to the one used in Case 1 shows that $\text{val}_{\max}(C_6^{\triangle}) \leq \text{val}^1_{\max}(C_6) + 2 = 32$, producing the desired result. ∎

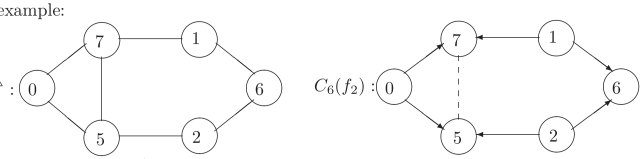
To illustrate Theorem 5.3, an example of a γ-max labeling f_1 of $C_6^{\triangle}$ and an oriented graph $C_6(f_1)$ for the first case are:

$C_6^{\triangle}$:



On the another hand, a γ-max labeling f_2 of $C_6^{\triangle}$ and an oriented graph $C_6(f_2)$ in second case are, for example:

$C_6^{\triangle}$:



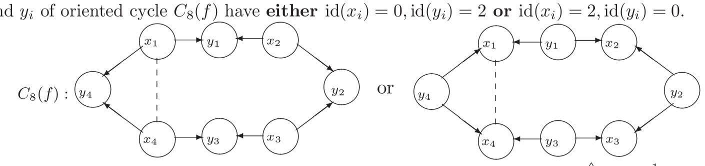
Observe that $\text{val}_{\max}(C_6^{\triangle}) = \text{val}(f_1) = \text{val}(f_2) = 32$.

Theorem 5.4

$$\text{val}_{\max}(C_8^{\triangle}) = \begin{cases} \text{val}^1_{\max}(C_8) + 3 & = 51 \\ \text{val}^2_{\max}(C_7) + 9 & = 51. \end{cases}$$

Proof. By Lemma 5.1, we have $\text{val}_{\max}(C_8^{\triangle}) \geq 51 = \text{val}^1_{\max}(C_8) + 3 = \text{val}^2_{\max}(C_7) + 9$. On the other hand, let f be a γ-max labeling of $C_6^{\triangle}$. We consider the two cases according to the set $ST(f)$.

Case 1. *No element of $ST(f)$ is monotone.* Then, for each i with $1 \le i \le 4$, the vertices x_i and y_i of oriented cycle $C_8(f)$ have **either** $\mathrm{id}(x_i) = 0, \mathrm{id}(y_i) = 2$ **or** $\mathrm{id}(x_i) = 2, \mathrm{id}(y_i) = 0$.



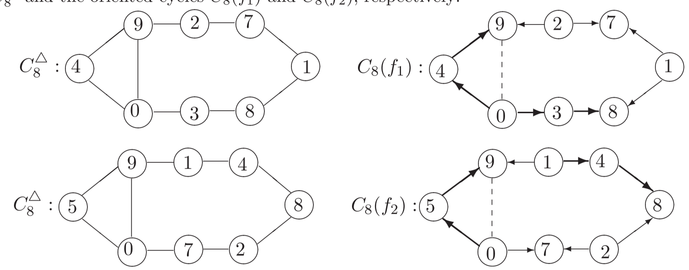
An argument similar to the one used in Theorem 5.3 shows that $\mathrm{val}_{max}(C_8^{\triangle}) \le \mathrm{val}^1_{max}(C_8) + 3 = 51$.

Case 2. *Some element of $ST(f)$ is monotone.* Then the oriented cycle $C_8(f)$ contains a directed path a, b, c of order 3. If we delete the chord x_1x_4 and vertex b from $C_8^{\triangle}$ and join the vertices a and c, the resulting graph G is isomorphic to C_7 and the restriction $\widetilde{f}$ of f to $V\left(C_8^{\triangle} - x_1x_4\right) - \{b\}$ has the same value on G as f on $C_8^{\triangle} - x_1x_4$, that is

$$\mathrm{val}(\widetilde{f}) = \mathrm{val}(f) - f'(x_1x_4) \ge \mathrm{val}_{\max}(C_8^{\triangle}) - 9.$$

Moreover, since $\widetilde{f}$ is a γ^2-labeling of a graph G isomorphic to C_7, it follows that $\mathrm{val}(\widetilde{f}) \le \mathrm{val}^2_{\max}(C_7)$. Therefore, $\mathrm{val}_{\max}(C_8^{\triangle}) \le \mathrm{val}^2_{\max}(C_7) + 9 = 51$. ■

To illustrate Case 2 of Theorem 5.4, we consider, for example, the γ-max labelings f_1 and f_2 of $C_8^{\triangle}$ and the oriented cycles $C_8(f_1)$ and $C_8(f_2)$, respectively:



Observe that $\mathrm{val}_{\max}(C_8^{\triangle}) = \mathrm{val}(f_1) = \mathrm{val}(f_2) = 51$.

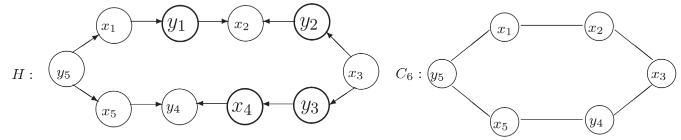
We remark that, since for any integer k with $k \ge 4$, the size of the oriented cycle $C_{2k}(f)$ is even, it follows that there are ℓ directed paths of order 3 in the oriented cycle $C_{2k}(f)$, for some even integer ℓ with $0 \le \ell \le 2k$. This observation leads us to the following lemma.

Lemma 5.5 *Let $n = 2k$ be an even integer, with $n \ge 8$, and let f be a γ-max labeling of $C_{2k}^{\triangle}$. If some element of $ST(f)$ is monotone, then there are exactly two monotone elements in $ST(f)$.*

Proof. Suppose that some element of $ST(f)$ is monotone. Then the oriented cycle $C_{2k}(f)$ contains a directed path of order 3. Since the size of the oriented cycle $C_{2k}(f)$ is even, it follows that there are ℓ directed paths of order 3 in $C_{2k}(f)$, for some even integer ℓ with $2 \le \ell \le 2k$. Next, we show that there are no more than two directed paths of order 3 in oriented cycle $C_{2k}(f)$.

Assume, to the contrary, that there are at least 4 directed paths of order 3 in the oriented graph $C_{2k}(f)$. For each i with $1 \le i \le 4$, let P_i be a directed path of order 3 having internal vertex u_i in the oriented graph $C_{2k}(f)$. We can see that the cycle $C_{2k}^{\triangle} - x_1x_k$ is a subdivision of the cycle C_{n-4} by every two vertices of C_{n-4} which are connected by either an edge or a path, all of whose interior vertices are internal vertices u_1, u_2, u_3, u_4.

To illustrate this concept of subdivision, let us define:



For example, considering the oriented cycle $H : x_1, y_1, x_2, y_2, \cdots, x_5, y_5, x_1$ of order 10 having 6 directed paths of order 3. Let $P_1 : x_1, y_1, x_2$, $P_2 : x_2, y_2, x_3$, $P_3 : x_3, y_3, x_4$ and $P_4 : y_3, x_4, y_4$ be directed paths of order 3 in H. Therefore the underlying cycle H is a subdivision of the cycle $C_6 : x_1, x_2, x_3, y_4, x_5, y_5, x_1$, where the internal vertices y_1, y_2, y_3, x_4 of the directed paths P_1, P_2, P_4, P_4, respectively, are inserted into one or more edges of C_6.

Since the cycle $C_{2k}^{\triangle} - x_1x_k$ is a subdivision of the cycle C_{n-4}, we can construct a graph G which is isomorphic to C_{n-4} from $C_{2k}^{\triangle}$ by deleting the chord x_1x_k and the internal vertices u_1, u_2, u_3, u_4 and then adding one or more edges to G. Then the restriction of f to $V\left(C_{2k}^{\triangle} - x_1x_k\right) - \{u_1, u_2, u_3, u_4\}$, $\tilde{f}$, has the same value on G as f does on $C_{2k}^{\triangle} - x_1x_k$, that is

$$\mathrm{val}(\widetilde{f}) = \mathrm{val}(f) - f'(x_1x_k) \ge \mathrm{val}_{\max}(C_n^{\triangle}) - (n+1).$$

Moreover, since $\tilde{f}$ is a γ^5-labeling of a graph G which is isomorphic to C_{n-4}, it follows that $\mathrm{val}(\widetilde{f}) \le \mathrm{val}^5_{\max}(C_{n-4})$. Therefore

$$\mathrm{val}_{\max}(C_n^{\triangle}) \le \mathrm{val}^5_{\max}(C_{n-4}) + (n+1) = \frac{n^2 + 6n - 30}{2},$$

a contradiction with Lemma 5.1. Thus, for $k \ge 4$, the oriented cycle $C_{2k}(f)$ contains exactly two directed paths of order 3 and there are also exactly two monotone elements in $ST(f)$. ■

By Theorem 5.4 and Lemma 5.5, we are able to characterize any γ-max labeling f of $C_8^{\triangle}$, in terms of set $ST(f)$ and the oriented cycle $C_8(f)$.

Proposition 5.6 *Let f be a γ-max labeling of $C_8^{\triangle}$. Then either no element of $ST(f)$ is monotone or there are exactly two monotone elements in $ST(f)$.*

For any integers p and q, with $p \le q$, let

$$[p, q] = \{p, p+1, \ldots, q\}$$

be the set of all the integers between p and q. Next we characterize any γ-max labeling f of $C_n^{\triangle}$, for $n \ge 10$.

Proposition 5.7 *For any integer k, with $k \ge 5$, let f be a γ-max labeling of $C_{2k}^{\triangle}$. Then $ST(f)$ contains exactly two monotone of $ST(f)$.*

Proof. Assume, to the contrary, that the property does not verify. Then, by Lemma 5.5, $ST(f)$ contains no monotone element. Hence, for the vertices x_i and y_i of the oriented cycle $C_{2k}(f)$,

$$\textbf{either}\ \ \mathrm{id}(x_i)=0,\ \mathrm{id}(y_i)=2\ \ \textbf{or}\ \ \mathrm{id}(x_i)=2,\ \mathrm{id}(y_i)=0 \text{ for each } i \text{ with } 1\le i\le k\,.$$

We consider the two cases separately.

Case 1. *For each i with $1 \le i \le k$, $id(x_i) = 0$, $id(y_i) = 2$ of the oriented cycle $C_{2k}(f)$.* Thus $f(x_i) < f(y_i)$ and $f(x_{i+1}) < f(y_i)$ where addition is taken modulo k. Then

$$\mathrm{val}_{\max}(C_{2k}^{\triangle}) = \mathrm{val}(f) = 2\sum_{i=1}^{k} f(y_i) - 2\sum_{i=1}^{k} f(x_i) + |f(x_1) - f(x_k)|.$$

Before we proceed, we prove two claims.

Claim 1. $\{f(y_1), f(y_2), \ldots, f(y_k)\} = [k+2, n+1]$.

Proof of Claim 1. Assume, to the contrary, that

$$\{f(y_1), f(y_2), \ldots, f(y_k)\} \neq [k+2, n+1]\,.$$

Then there are two numbers a and b, with $a \in [0, k+1]$ and $b \in [k+2, n+1]$, such that a is assigned by f to some vertex of $\{y_1, y_2, \ldots, y_k\}$, namely y_r and b is assigned by f to either no vertex of $C_{2k}^{\triangle}$ or some vertex of $\{x_1, x_2, \ldots, x_k\}$.

If b is assigned to no vertex of $C_{2k}^{\triangle}$, then we define a new γ-labeling g of $C_{2k}^{\triangle}$ by

$$g(u) = \begin{cases} f(u) & \text{if } u \neq y_r \\ b & \text{if } u = y_r. \end{cases}$$

Then

$$\begin{aligned}
\mathrm{val}(g) &= 2\textstyle\sum_{i=1}^{k} g(y_i) - 2\sum_{i=1}^{k} g(x_i) + |g(x_1) - g(x_k)| \\
&= \Big(2\textstyle\sum_{i=1}^{k} f(y_i) - 2f(y_r) + 2g(y_r)\Big) - 2\sum_{i=1}^{k} f(x_i) + |f(x_1) - f(x_k)| \\
&= \mathrm{val}(f) - 2a + 2b \ = \ \mathrm{val}(f) + 2(b-a) \ > \ \mathrm{val}(f)\,,
\end{aligned}$$

which is a contradiction. Then b must be assigned to some vertex of $\{x_1, x_2, \ldots, x_k\}$, namely x_s. Then we define the γ-labeling g of $C_{2k}^{\triangle}$ by

$$g(u) = \begin{cases} f(u) & \text{if } u \neq x_s, y_r \\ f(y_r) = a & \text{if } u = x_s \\ f(x_s) = b & \text{if } u = y_r \end{cases}$$

and, therefore,

$$\begin{aligned}
\mathrm{val}(g) &= 2\textstyle\sum_{i=1}^{k} g(y_i) - 2\sum_{i=1}^{k} g(x_i) + |g(x_1) - g(x_k)| \\
&= \Big(2\textstyle\sum_{i=1}^{k} f(y_i) - 2f(y_r) + 2g(y_r)\Big) - \Big(2\sum_{i=1}^{k} f(x_i) - 2f(x_s) + 2g(x_s)\Big) \\
&\quad + |g(x_1) - g(x_k)| + (|f(x_1) - f(x_k)| - |f(x_1) - f(x_k)|) \\
&= \mathrm{val}(f) + 4(b-a) + |g(x_1) - g(x_k)| - |f(x_1) - f(x_k)|\,.
\end{aligned}$$

If $s \neq 1, k$, then

$$\begin{aligned}
\mathrm{val}(g) &= \mathrm{val}(f) + 4(b-a) + |f(x_1) - f(x_k)| - |f(x_1) - f(x_k)| \\
&= \mathrm{val}(f) + 4(b-a) \ > \ \mathrm{val}(f)\,.
\end{aligned}$$

Otherwise, assuming without loss of generality that $s = 1$, we get

$$\begin{aligned}
\mathrm{val}(g) &= \mathrm{val}(f) + 4(b-a) + |a - f(x_k)| - |b - f(x_k)| \\
&= \mathrm{val}(f) + (|b-a| + |a - f(x_k)|) - |b - f(x_k)| + 3(b-a) \\
&\geq \mathrm{val}(f) + |b - f(x_k)| - |b - f(x_k)| + 3(b-a) \\
&= \mathrm{val}(f) + 3(b-a) \ > \ \mathrm{val}(f)\,,
\end{aligned}$$

which is again a contradiction. Hence $\{f(y_1), f(y_2), \ldots, f(y_k)\} = [k+2, n+1]$. Consequently, $\{f(x_1), f(x_k), \ldots, f(x_k)\} \subset [0, k+1]$.

This is complete the proof of Claim 1.

Claim 2. $\{f(x_1), f(x_k), \ldots, f(x_k)\} = [0, k-1]$.

Proof of Claim 2. Assume, to the contrary, that

$$\{f(x_1), f(x_k), \ldots, f(x_k)\} \neq [0, k-1]\,.$$

Then there are two integer numbers a and b, with $a \in [k, k+1]$ and $b \in [0, k-1]$, such that a is assigned by f to some vertex of $\{x_1, x_2, \ldots, x_k\}$, namely x_r, and b is assigned by f to no vertex of $C_{2k}^{\triangle}$. Then we define a new γ-labeling g of $C_{2k}^{\triangle}$ by

$$g(u) = \begin{cases} f(u) & \text{if } u \neq x_r \\ b & \text{if } u = x_r. \end{cases}$$

Then

$$\begin{aligned}
\mathrm{val}(g) &= 2\textstyle\sum_{i=1}^{k} g(y_i) - 2\sum_{i=1}^{k} g(x_i) + |g(x_1) - g(x_k)| \\
&= 2\textstyle\sum_{i=1}^{k} f(y_i) - \Big(2\sum_{i=1}^{k} f(x_i) - 2f(x_r) + 2g(x_r)\Big) \\
&\quad + |g(x_1) - g(x_k)| + (|f(x_1) - f(x_k)| - |f(x_1) - f(x_k)|) \\
&= \mathrm{val}(f) + 2(a-b) + |g(x_1) - g(x_k)| - |f(x_1) - f(x_k)|\,.
\end{aligned}$$

If $r \neq 1, k$, then

$$\begin{aligned}
\mathrm{val}(g) &= \mathrm{val}(f) + 2(a-b) + |f(x_1) - f(x_k)| - |f(x_1) - f(x_k)| \\
&= \mathrm{val}(f) + 2(a-b) \ > \ \mathrm{val}(f)\,.
\end{aligned}$$

Otherwise assuming, without loss of generality, that $r = 1$, then

$$\begin{aligned}
\mathrm{val}(g) &= \mathrm{val}(f) + 2(a-b) + |b - f(x_k)| - |a - f(x_k)| \\
&= \mathrm{val}(f) + (|a-b| + |b - f(x_k)|) - |a - f(x_k)| + (a-b) \\
&\geq \mathrm{val}(f) + |a - f(x_k)| - |a - f(x_k)| + (a-b) \\
&= \mathrm{val}(f) + (a-b) \ > \ \mathrm{val}(f)\,.
\end{aligned}$$

Again we reach a contradiction. Therefore, $\{f(x_1), f(x_k), \ldots, f(x_k)\} = [0, k-1]$. This completes the proof of Claim 2.

Therefore, the function f is defined by

$$\{f(y_1), f(y_2), \ldots, f(y_k)\} = [k+2, n+1] \text{ and } \{f(x_1), f(x_k), \ldots, f(x_k)\} = [0, k-1].$$

So

$$\begin{aligned}\mathrm{val}_{\max}(C_{2k}^{\triangle}) = \mathrm{val}(f) &= 2\textstyle\sum_{i=1}^{k} f(y_i) - 2\sum_{i=1}^{k} f(x_i) + |f(x_1) - f(x_k)| \\ &= 3k(k-1) - k(k-1) + (k-1) \\ &= 2k^2 + 5k - 1\,.\end{aligned}$$

However, by Lemma 5.1, we have $\mathrm{val}_{\max}(C_{2k}^{\triangle}) \geq \frac{(2k)^2+6(2k)-10}{2} = 2k^2 + 6k - 5$, which contradicts our assumption.

Case 2. *For each i with $1 \leq i \leq k$, $id(x_i) = 2$, $id(y_i) = 0$ of the oriented cycle $C_{2k}(f)$.* So, $f(x_i) > f(y_i)$ and $f(x_{i+1}) > f(y_i)$, where the addition is taken modulo k. A similar argument to the one used in Case 1 shows that $\mathrm{val}_{\max}(C_{2k}^{\triangle}) = 2k^2 + 5k - 1$, which is a contradiction. ∎

We are now prepared to establish a general formula for $\mathrm{val}_{\max}(C_n^{\triangle})$, when $n \geq 10$.

Theorem 5.8 *For every even integer $n \geq 10$,*

$$\mathrm{val}_{\max}(C_n^{\triangle}) = \mathrm{val}^2_{\max}(C_{n-1}) + (n+1) = \frac{n^2 + 6n - 10}{2}.$$

Proof. By Lemma 5.1, we have

$$\mathrm{val}_{\max}(C_n^{\triangle}) \geq \frac{n^2 + 6n - 10}{2} = \mathrm{val}^2_{\max}(C_{n-1}) + (n+1).$$

Next, we show that $\mathrm{val}_{\max}(C_n^{\triangle}) \leq \mathrm{val}^2_{\max}(C_{n-1}) + (n+1)$. Let f be a γ-max labeling of $C_n^{\triangle}$. By Proposition 5.7, the oriented cycle $C_n(f)$ contains a directed path a, b, c of order 3. If we delete the chord x_1x_k and vertex b from $C_n^{\triangle}$ and then join the vertices a and c, the resulting graph G is isomorphic to C_{n-1} and the restriction $\tilde{f}$ of f to $V(C_n^{\triangle} - x_1x_k) - \{b\}$, verifies

$$\mathrm{val}(\widetilde{f}) = \mathrm{val}(f) - f'(x_1x_k) \geq \mathrm{val}_{max}(C_n^{\triangle}) - (n+1).$$

Moreover, since $\widetilde{f}$ is a γ^2-labeling of a graph G that is isomorphic to C_{n-1}, it follows that $\mathrm{val}(\widetilde{f}) \leq \mathrm{val}^2_{max}(C_{n-1})$. Therefore,

$$\mathrm{val}_{max}(C_n^{\triangle}) \leq \mathrm{val}^2_{max}(C_{n-1}) + (n+1) = \frac{n^2 + 6n - 10}{2}$$

producing the desired result. ∎

6 Final remarks

In [6] the γ-spectra of paths, cycles, and complete graphs are determined. It would be interesting to determine the γ-spectra of $C_n^{\triangle}$ and find whether it is continuous or not. Observe that the value of any γ-labeling of a cycle is always even.

References

[1] A. Blinco, S. El-Zanati, C. Vanden Eynden, On the cyclic decomposition of complete graphs into almost-bipartite graphs, Discrete Math. 284 (2004), no. 1-3, 71-81.

[2] G.S. Bloom, S.W. Golomb, Numbered complete graphs, unusual rulers, and assorted applications, in *Theory and Applications of Graphs*, Lecture Notes in Math. 642, Springer-Verlag (1978), 53-65.

[3] G. Chartrand, D. Erwin, D.W. VanderJagt, P. Zhang, γ-labelings of graphs, Bull. Inst. Combin. Appl. 44 (2005), 51-68.

[4] G. Chartrand, D. Erwin, D.W. VanderJagt, P. Zhang, On γ-labelings of trees, Discuss. Math. Graph Theory 25 (2005), no. 3, 363-383.

[5] G. Chartrand, P. Zhang, *Introduction to Graph Theory*, The Walter Rudin student series in advanced mathematics, McGraw-Hill Higher Education, Boston, 2005.

[6] C.M. da Fonseca, V. Saenpholphat, P. Zhang, The γ-spectrum of a graph, Ars Combin., to appear.

[7] J.A. Gallian, A dynamic survey of graph labeling, Electron. J. Combin. 5 (1998), Dynamic Survey 6, 43 pp.

[8] S.W. Golomb, How to number a graph, in *Graph Theory and Computing*, pp. 23-37, Academic Press, New York, 1972.

[9] R. Häggkvist, Decompositions of complete bipartite graphs, in *Surveys in Combinatorics*, 1989 (Norwich, 1989), 115-147, London Math. Soc. Lecture Note Ser., 141, Cambridge Univ. Press, Cambridge, 1989.

[10] C. Huang, A. Kotzig, A. Rosa, Further results on tree labellings, Utilitas Math. 21 (1982), 31-48.

[11] G. Ringel, Problem 25, in *Theory of Graphs and its Applications*, Proceedings of the Symposium in Smolenice 1963, 1964, p. 162.

[12] A. Rosa, On certain valuations of the vertices of a graph, in *Theory of Graphs* (Internat. Sympos., Rome, 1966) pp. 349-355, Gordon and Breach, New York; Dunod, Paris.

Current Folder: INBOX | Sign Out
Compose Addresses Folders Options Search Help Calendar Bookmarks | SquirrelMail

Search Results | Delete | Forward | Forward as Attachment | Reply | Reply All

Subject: [Fwd: Utilitas Math paper]
From: cmf@mat.uc.pt
Date: Wed, April 11, 2012 6:19 pm
To: varaporn@swu.ac.th (more)
Priority: Normal
Mailer: SquirrelMail/1.4.19
Options: View Full Header | View Printable Version | Download this as a file

```
---------------------------- Original Message ----------------------------
Subject: Utilitas Math paper
From:    "Math Journals" <MathJournals@attglobal.net>
Date:    Tue, April 10, 2012 6:28 pm
To:      cmf@mat.uc.pt
--------------------------------------------------------------------------
```

Professor da Fonseca,

I have now received the referees second report on your paper with Varaporn Saenpholphat and Ping Zhang entitled, "On γ-labelings of graphs".

The report is attached below. The referee now recommends publication with revisions. Therefore, I am happy to accept it for publication in Utilitas Mathematica.

Would you please send me a new copy: single spaced, in 10 point Times or Computer Modern font, 4.5 x 7.125 inches (11.4 x 18.1 cm) in text size. I use your copy as camera-ready for publication. A .pdf file by email will suffice.

Yours sincerely,
J.L. Allston
Managing Editor
Utilitas Mathematica

Prof. Allston,

Thank you for sending the revision of the paper by Fonseca, Saenpholphat and Zhang. I've read through it briefly and they have addressed all of my concerns. I'm happy to recommend the paper for publication.
--

Attachments:

untitled-[2]	**1.4 k**	[text/html]	Download | View

On γ-Labelings of Graphs

C. M. da Fonseca [1]
Department of Mathematics
University of Coimbra
3001-454 Coimbra, Portugal

Varaporn Saenpholphat [2]
Department of Mathematics
Srinakharinwirot University,
Sukhumvit Soi 23, Bangkok, 10110, Thailand
varaporn@swu.ac.th

Ping Zhang
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA

March 13, 2011

ABSTRACT

Let G be a graph of order n and size m. A γ-labeling of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ of the edges of G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. The value of a γ-labeling f is defined as

$$\mathrm{val}(f) = \sum_{e \in E(G)} f'(e).$$

The maximum value of a γ-labeling of G is defined as

$$\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\};$$

while the minimum value of a γ-labeling of G is

$$\mathrm{val}_{\min}(G) = \min\{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

A γ-labeling of G whose value is $\mathrm{val}_{\min}(G)$ (resp. $\mathrm{val}_{\max}(G)$) is called a γ-min (resp. γ-max) labeling of G. In this paper, we give answer to the open question posed by G. D. Bullington, L. L. Eroh, and S. J. Winters [1]. We also characterize γ-min labelings of a graph and γ-max labelings of complete bipartite graph and complete graphs. Furthermore, we provide an alternative and improved proof of $\mathrm{val}_{\max}(K_{r,s})$ given in the literature that employs γ-min labelings of complete graphs.

Key Words: γ-labeling, γ-min and γ-max labelings.
AMS Subject Classification: 05C78.

[1] Research supported by CMUC - Centro de Matemática da Universidade de Coimbra.

[2] Corresponding author, Research supported in part by Srinakharinwirot University, Commission on Higher Education and the Thailand Research Fund-RMU5380031.

1 Introduction

For a graph G of order n and size m, a *γ-labeling* of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ of the edges of G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. Therefore,

$$\text{graph } G \text{ has a } \gamma\text{-labeling if and only if } m \geq n - 1. \tag{1}$$

In particular, every connected graph has a γ-labeling. Each γ-labeling f of a graph G of order n and size m is assigned a *value* denoted by $\mathrm{val}(f)$ and defined by

$$\mathrm{val}(f) = \sum_{e \in E(G)} f'(e).$$

Since f is a one-to-one function from $V(G)$ to $\{0, 1, 2, \ldots, m\}$, it follows that $f'(e) \geq 1$ for each edge e in G and so $\mathrm{val}(f) \geq m$. For a graph G of order n and size m, the *maximum value* of a γ-labeling of a graph G is defined as

$$\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\};$$

while the *minimum value* of a γ-labeling of G is

$$\mathrm{val}_{\min}(G) = \min\{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

A γ-labeling g of G is a γ-max *labeling* if $\mathrm{val}(g) = \mathrm{val}_{\max}(G)$ and a γ-labeling h is a γ-min *labeling* if $\mathrm{val}(h) = \mathrm{val}_{\min}(G)$. The *span* of a γ-labeling f of G is defined as

$$\mathrm{span}(f) = \max\{f(v) : v \in V(G)\} - \min\{f(v) : v \in V(G)\}.$$

The following results appear in [2].

Theorem 1.1 ([2]) *Let G be a connected graph of order n and let $f : V(G) \to \mathbf{Z}$ be a one-to-one function. Then there is a γ-labeling g on G with $\mathrm{val}(g) \leq \mathrm{val}(f)$. Furthermore, if $\mathrm{span}(f) \geq n$, then there is a γ-labeling g with $\mathrm{val}(g) < \mathrm{val}(f)$.*

For integers a and b with $a \leq b$, let

$$[a..b] = \{a, a+1, \ldots, b\}$$

be the set of integers between a and b. Such a set is referred to as a *consecutive set* of integers.

Theorem 1.2 ([2]) *If G is a connected graph of order n, then G has a γ-min labeling f such that $f(V(G)) = [0..n-1]$.*

The maximum and minimum values of γ-labelings of nontrivial complete graphs and complete bipartite graphs were determined in [2], as we stated next.

Theorem 1.3 ([2]) *For every integer $n \geq 2$,*

$$\begin{aligned}
\mathrm{val}_{\min}(K_n) &= \binom{n+1}{3} \\
\mathrm{val}_{\max}(K_n) &= \begin{cases} \frac{n(3n^3-5n^2+6n-4)}{24} & \text{if } n \text{ is even} \\ \frac{(n^2-1)(3n^2-5n+6)}{24} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Let K_n be a complete graph of order $n \geq 2$ with $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. If f is a γ-labeling of K_n such that $f(v_i) = a_i$ for $1 \leq i \leq n$ and

$$0 \leq a_1 < a_2 < \cdots < a_n \leq \binom{n}{2},$$

then the value of f was determined in [6], namely

$$\mathrm{val}(f) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (n-2i+1)(a_{n-i+1} - a_i). \tag{2}$$

Theorem 1.4 ([1]) *For any two positive integers $r \geq s$,*

$$\begin{aligned}
\mathrm{val}_{\min}(K_{r,s}) &= \frac{s(2s^2+1)}{3} + s^2(r-s) + s\left\lfloor \frac{(r-s)^2}{4} \right\rfloor \text{ and} \\
\mathrm{val}_{\max}(K_{r,s}) &= rs(rs - \frac{1}{2}(r+s) + 1).
\end{aligned}$$

The reader is referred to Chartrand and Zhang [4] for basic definitions and terminology not mentioned here.

2 On Spans of γ-Min and γ-Max Labelings of Graphs

In order to show a formula for spans of γ-min and γ-max labelings of graphs, we first present a useful lemma.

Lemma 2.1 *Let f be a γ-max labeling of a nontrivial graph G of order n and size m and let $u, w \in V(G)$ with $f(u) = \min\{f(v) : v \in V(G)\}$ and $f(w) = \max\{f(v) : v \in V(G)\}$. Then neighborhoods of u and v are not empty.*

Proof. For any nontrivial connected graph G, it is obvious that $|N(u)|$ and $|N(w)|$ are not empty. Let G be a disconnected graph. Assume, to the contrary, that $|N(u)| = 0$. Therefore, u is an isolated vertex of G. By (1), we then have $m \geq n-1$. Therefore, there is a component G_1 of G with $V(G_1) \geq 2$. Let $x \in V(G_1)$ with $f(x) = \min\{f(v) : v \in V(G_1)\}$. Let g be a γ-labeling of G defined by

$$g(v) = \begin{cases} f(x) & \text{if } v = u \\ f(u) & \text{if } v = x \\ f(v) & \text{if } v \neq u, x. \end{cases}$$

Since

$$
\begin{aligned}
\mathrm{val}(g) &= \mathrm{val}(f) - \sum_{v\in N(x)} (f(v)-f(x)) + \sum_{v\in N(x)} (g(v)-g(x)) \\
&= \mathrm{val}(f) - \sum_{v\in N(x)} (f(v)-f(x)) + \sum_{v\in N(x)} (f(v)-f(u)) \\
&= \mathrm{val}(f) + |N(x)|(f(x)-f(u)),
\end{aligned}
$$

and $f(x) > f(u)$, it follows that $\mathrm{val}(g) > \mathrm{val}(f)$, which is a contradiction.

It is a similar argument to verify that neighborhood of v is also not empty. ■

Theorem 2.2 *Let G be a nontrivial graph of order n and size m and let f be a γ-labeling of G.*

(*a*) *If f is a γ-*min *labeling of G, then* $\mathrm{span}(f) = n-1$.

(*b*) *If f is a γ-*max *labeling of G, then* $\mathrm{span}(f) = m$.

Proof. We first verify (a). Let $\mathrm{val}_{\min}(G) = \mathrm{val}(f)$ and assume, to the contrary, that $\mathrm{span}(f) \geq n$. It then follows by Theorem 1.1 that there is a γ-labeling g with $\mathrm{val}(g) < \mathrm{val}(f)$, which is a contradiction.

Next we verify (b). Let f be a γ-max labeling of G and let $u, w \in V(G)$ with $f(u) = \min\{f(v) : v \in V(G)\}$ and $f(w) = \max\{f(v) : v \in V(G)\}$. Then $f(u) \geq 0$ and $f(w) \leq m$. Assume, to the contrary, that $\mathrm{span}(f) < m$. Then $f(w) - f(u) < m$. Therefore,

$$
f(u) > 0 \text{ or } m - f(w) > 0. \tag{3}
$$

Let g be a γ-labeling of G defined by

$$
g(v) = \begin{cases} 0 & \text{if } v = u \\ f(v) & \text{if } v \neq u, w \\ m & \text{if } v = w. \end{cases}
$$

Since

$$
\begin{aligned}
\mathrm{val}(g) &= \mathrm{val}(f) - \sum_{v\in N(u)} (f(v)-f(u)) + \sum_{v\in N(u)} (g(v)-g(u)) \\
&\quad - \sum_{v\in N(w)} (f(w)-f(v)) + \sum_{v\in N(w)} (g(w)-g(v)) \\
&= \mathrm{val}(f) - \sum_{v\in N(u)} (f(v)-f(u)) + \sum_{v\in N(u)} (f(v)-0) \\
&\quad - \sum_{v\in N(w)} (f(w)-f(v)) + \sum_{v\in N(w)} (m-f(v)) \\
&= \mathrm{val}(f) + |N(u)|(f(u)-0) + |N(w)|(m-f(w)),
\end{aligned}
$$

it follows by Lemma 2.1 and (3) that $\mathrm{val}(g) > \mathrm{val}(f)$, which is a contradiction. ■

The following results are consequences of Theorem 2.2.

Corollary 2.3 *Let G be a nontrivial graph of order n and size m and let f be a γ-labeling of G.*

(*a*) *If f is a γ-*min *labeling of G, then $f(V(G))$ is a consecutive subset of $[0..m]$, that is, $f(V(G)) = [k..k+(n-1)]$ for some integer k with $0 \leq k \leq m-(n-1)$.*

(*b*) *If f is a γ-*max *labeling of G, then $\{0, m\} \subset f(V(G))$.*

Proof. To verify (a), let f be a γ-min labeling of G with $f(V(G)) = \{a_1, a_2, \ldots, a_n\}$ where

$$
0 \leq a_1 < a_2 < \cdots < a_n \leq m.
$$

Then $\mathrm{span}(f) = a_n - a_1$. By Theorem 2.2 (a), $\mathrm{span}(f) = n-1$. Therefore, $a_n - a_1 = n-1$. This implies that $a_1, a_2, \ldots, a_n$ are consecutive integers in the set $[0..m]$ and so $f(V(G))$ is a consecutive subset of $[0..m]$. That is $f(V(G)) = [k..k+(n-1)]$ for some integer k with $0 \leq k \leq m-(n-1)$. Statement (b) is an immediate consequence of Theorem 2.2 (b). ■

Corollary 2.4 *Let G be a nontrivial connected graph of order n and size m. Suppose that f and g are γ-*min *and γ-*max *labelings of G, respectively. Then* $\mathrm{span}(f) = \mathrm{span}(g)$ *if and only if G is a tree.*

Proof. First, assume that $\mathrm{span}(f) = \mathrm{span}(g)$. By Theorem 2.2, $\mathrm{span}(f) = n-1$ and $\mathrm{span}(g) = m$. Thus $n-1 = m$, which implies that G is a tree. For the converse, assume that G is a tree. Then $m = n-1$. It then follows by Theorem 2.2 that $\mathrm{span}(f) = n-1 = m = \mathrm{span}(g)$, as desired. ∎

Now, we are able to answer the following open question in [1] : For any connected graph G, does exist a γ-max labeling of G with a vertex label set that is the union of no more than two sets of consecutive numbers?

Theorem 2.5 *Let G be a connected graph of order n and size m. Then vertex label set of γ-max labeling of G is a set of consecutive numbers if and only if G is a tree.*

Proof. It is straightforward to verify that vertex label set of γ-max labeling of tree is a set of consecutive numbers. For the converse, let f be a γ-max labeling of a connected graph G with a vertex label set that is a set of consecutive numbers. Then $\mathrm{span}(f) = n-1$. Moreover, by Theorem 2.2 (b), we have $\mathrm{span}(f) = m$ Then $m = n-1$. Therefore, G is a tree. ∎

3 On γ-Max Labelings of Complete Bipartite and Complete Graphs

In this section, we begin characterize γ-max labelings of complete bipartite which also appeared in [1]. We provide alternative and improved proof of formula for $\mathrm{val}(K_{r,s})$ that uses γ-min labelings of complete graphs.

Theorem 3.1 *Let f be a γ-labeling of complete bipartite graph $K_{r,s}$ with partite sets V_r and V_s of cardinality r and s, respectively. Then f is a γ-max labeling of $K_{r,s}$ if and only if*

(*a*) $f(V_r) = [0..r-1]$ *and* $f(V_s) = [rs-(s-1)..rs]$, *or*

(*b*) $f(V_r) = [rs-(r-1)..rs]$ *and* $f(V_s) = [0..s-1]$.

Proof. Let $K_{r,s}$ be a complete bipartite graph with partite sets V_r and V_s and let f be a γ-max labeling of $K_{r,s}$ with $f(V(K_{r,s})) = \{a_1, a_2, \ldots, a_{r+s}\}$ where

$$1 \le a_1 < a_2 < \cdots < a_{r+s} \le rs.$$

Now, consider complete graph K_{r+s} with $V(K_{r+s}) = V_r \cup V_s$. Then K_{r+s} contains three edge-disjoint graphs, namely

$$K_r \text{ with } V(K_r) = V_r,$$
$$K_s \text{ with } V(K_s) = V_s \text{ and}$$
$$K_{r,s} \text{ with partite sets } V_r \text{ and } V_s.$$

That is,

$$E(K_{r+s}) = E(K_r) \cup E(K_s) \cup E(K_{r,s}).$$

For each $i \in \{r, s, r+s\}$, let f_i be the restriction of f to K_i and then

$$f_i : V(K_i) \to f(V(K_{r,s})) \text{ is a one to one function}$$

and

$$\mathrm{val}(f_{r+s}) = \mathrm{val}(f_r) + \mathrm{val}(f_s) + \mathrm{val}(f)$$

and so

$$\mathrm{val}(f) = \mathrm{val}(f_{r+s}) - \mathrm{val}(f_r) - \mathrm{val}(f_s).$$

Since $f_{r+s}(V(K_{r+s})) = f(V(K_{r,s})) = \{a_1, a_2, \ldots, a_{r+s}\}$, it follows by (2) that $\mathrm{val}(f_{r+s})$ gives us same value even though we switch some labels of $\{a_1, a_2, \ldots, a_{r+s}\}$ to $V(K_{r+s})$. Furthermore, since $\mathrm{val}(f) = \mathrm{val}_{\max}(K_{r,s})$, it follows that

$$\mathrm{val}(f_r) = \min\{\mathrm{val}(g) \mid g : V(K_r) \to \{a_1, a_2, \ldots, a_{r+s}\}\} \tag{4}$$

and

$$\mathrm{val}(f_s) = \min\{\mathrm{val}(g) \mid g : V(K_s) \to \{a_1, a_2, \ldots, a_{r+s}\}\}.$$

We claim that $f_r(V_r)$ and $f_s(V_s)$ are consecutive subsets of $\{a_1, a_2, \ldots, a_{r+s}\}$. Let

$$f_r(V_r) = f_r(V(K_r)) = \{a_{\ell_1}, a_{\ell_2}, \ldots, a_{\ell_j}, a_{\ell_{j+1}}, \ldots, a_{\ell_r}\},$$

where $1 \le \ell_1 < \ell_2 < \cdots < \ell_j < \ell_{j+1} < \cdots < \ell_r \le r+s$. By (2), we then have

$$\mathrm{val}(f_r) = \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} (r-2i+1)(a_{\ell_{r-i+1}} - a_{\ell_i}).$$

Assume, to the contrary, that there is an integer j with $2 \le j \le r$ such that $\ell_j - \ell_{j-1} > 1$.

We first consider the case when $2 \le j \le \lfloor \frac{r}{2} \rfloor + 1$. In this case, we define a one to one function $h : f_r(V_r) \to \left[f_r(V_r) \cup \{a_{(\ell_j)-1}\}\right] - \{a_{\ell_{j-1}}\}$ by

$$h(a_{\ell_i}) = \begin{cases} a_{\ell_i} & \text{if } i \neq j-1 \\ a_{(\ell_j)-1} & \text{if } i = j-1. \end{cases}$$

Certainly, $g = h \circ f_r$ is a one to one function from V_r to $\{a_1, a_2, \ldots, a_{r+s}\}$. Furthermore,

$$\begin{aligned} \mathrm{val}(g) &= \mathrm{val}(f_r) - (r - 2(j-1)+1)(a_{\ell_{r-(j-1)+1}} - a_{\ell_{j-1}}) \\ &\quad +(r - 2(j-1)+1)(a_{\ell_{r-(j-1)+1}} - a_{(\ell_j)-1}) \\ &= \mathrm{val}(f_r) + (r - 2j + 3)(a_{\ell_{j-1}} - a_{(\ell_j)-1}). \end{aligned}$$

Since

$$\ell_j - \ell_{j-1} > 1,$$

it follows that

$$\ell_j - 1 > \ell_{j-1}$$

and so that

$$a_{(\ell_j)-1} > a_{\ell_{j-1}}.$$

Thus $\mathrm{val}(g) < \mathrm{val}(f_r)$, which is a contradiction with (4).

Next, we consider the case when $\lfloor \frac{r}{2} \rfloor + 2 \le j \le r = \lfloor \frac{r}{2} \rfloor + \lceil \frac{r}{2} \rceil$. In this case,we define a one to one function $h : f_r(V_r) \to \left[f_r(V_r) \cup \{a_{(\ell_j)-1}\}\right] - \{a_{\ell_j}\}$ by

$$h(a_{\ell_i}) = \begin{cases} a_{\ell_i} & \text{if } i \neq j \\ a_{(\ell_j)-1} & \text{if } i = j. \end{cases}$$

Certainly, $g = h \circ f_r$ is a one to one function from V_r to $\{a_1, a_2, \ldots, a_{r+s}\}$. Furthermore, let δ be an integer such that $j = r - \delta + 1$. Consequently, $1 \le \delta \le \lceil \frac{r}{2} \rceil - 1$. Therefore,

$$\begin{aligned} \mathrm{val}(g) &= \mathrm{val}(f_r) - (r - 2\delta + 1)(a_{\ell_j} - a_{\ell_\delta}) + (r - 2\delta + 1)(a_{(\ell_j)-1} - a_{\ell_\delta}) \\ &= \mathrm{val}(f_r) + (r - 2\delta + 1)(a_{(\ell_j)-1} - a_{\ell_j}). \end{aligned}$$

Since

$$\ell_j - \ell_{j-1} > 1,$$

it follows that

$$\ell_{j-1} < \ell_j - 1 < \ell_j$$

and so that

$$a_{\ell_{j-1}} < a_{(\ell_j)-1} < a_{\ell_j}.$$

Thus $\mathrm{val}(g) < \mathrm{val}(f_r)$, which is a contradiction with (4).

With a similar argument as above, we can conclude that $f_s(V_s)$ is a consecutive subset of $\{a_1, a_2, \ldots, a_{r+s}\}$. Therefore, either

$$f(V_r) = \{a_1, a_2, \ldots, a_r\} \text{ and } f(V_s) = \{a_{r+1}, a_{r+2}, \ldots, a_{r+s-1}, a_{r+s}\}$$

or

$$f(V_r) = \{a_{s+1}, a_{s+2}, \ldots, a_{s+r-1}, a_{s+r}\} \text{ and } f(V_s) = \{a_1, a_2, \ldots, a_s\}.$$

Now, we consider two cases according to $f(V_r)$ and $f(V_s)$.

Case 1. $f(V_r) = \{a_1, a_2, \ldots, a_r\}$ *and* $f(V_s) = \{a_{r+1}, a_{r+2}, \ldots, a_{r+s-1}, a_{r+s}\}$. Since

$$\begin{aligned} \mathrm{val}_{\max}(K_{r,s}) &= \mathrm{val}(f) \\ &= (a_{r+1} - a_1) + (a_{r+1} - a_2) + \cdots + (a_{r+1} - a_r) \\ &\quad +(a_{r+2} - a_1) + (a_{r+2} - a_2) + \cdots + (a_{r+2} - a_r) \\ &\quad \vdots \\ &\quad +(a_{r+s} - a_1) + (a_{r+s} - a_2) + \cdots + (a_{r+s} - a_r) \\ &= (ra_{r+1} - \sum_{i=1}^{r} a_i) + (ra_{r+2} - \sum_{i=1}^{r} a_i) + \cdots + (ra_{r+s} - \sum_{i=1}^{r} a_i) \\ &= r\sum_{i=1}^{s} a_{r+i} - s\sum_{i=1}^{r} a_i, \end{aligned}$$

it follows that $a_{r+i} = rs - i + 1$ for $i = 1, 2, \ldots, s$ and $a_i = i - 1$, for $i = 1, 2, \ldots, r$, that is, $f(V_r) = [0..r-1]$ and $f(V_s) = [rs - (s-1)..rs]$.

Case 2. $f(V_r) = \{a_{s+1}, a_{s+2}, \ldots, a_{s+r-1}, a_{s+r}\}$ *and* $f(V_s) = \{a_1, a_2, \ldots, a_s\}$. Since

$$\begin{aligned}
\mathrm{val}_{\max}(K_{r,s}) &= \mathrm{val}(f) \\
&= (a_{s+1} - a_1) + (a_{s+1} - a_2) + \cdots + (a_{s+1} - a_s) \\
&\quad +(a_{s+2} - a_1) + (a_{s+2} - a_2) + \cdots + (a_{s+2} - a_s) \\
&\quad \vdots \\
&\quad +(a_{s+r} - a_1) + (a_{s+r} - a_2) + \cdots + (a_{s+r} - a_s) \\
&= (sa_{s+1} - \sum_{i=1}^{s} a_i) + (sa_{s+2} - \sum_{i=1}^{s} a_i) + \cdots + (sa_{s+r} - \sum_{i=1}^{s} a_i) \\
&= s\sum_{i=1}^{r} a_{s+i} - r\sum_{i=1}^{s} a_i,
\end{aligned}$$

it follows that $a_{s+i} = rs - i + 1$ for $i = 1, 2, \ldots, r$ and $a_i = i - 1$, for $i = 1, 2, \ldots, s$, that is, $f(V_r) = [rs - (r-1)..rs]$ and $f(V_s) = [0..s-1]$. ■

Next, we characterize γ-max labelings of complete graphs.

Theorem 3.2 *Let f be a γ-labeling of complete graph K_n. Then f is a γ-max labeling of K_n if and only if*

$$f(V(K_n)) = \begin{cases} \left[0..\left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1..\binom{n}{2}\right] & \text{if } n \text{ is even} \\ \left[0..\left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1..\binom{n}{2}\right] \cup \{k\} & \text{if } n \text{ is odd,} \end{cases}$$

where $k \in \left[\left\lfloor \frac{n}{2} \right\rfloor..\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor\right]$.

Proof. Let f be a γ-max labeling of K_n such that $f(v_i) = a_i$ for $1 \le i \le n$ and

$$0 \le a_1 < a_2 < \cdots < a_n \le \binom{n}{2}.$$

Therefore by (2),

$$\mathrm{val}_{\max}(K_n) = \mathrm{val}(f) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (n - 2i + 1)a_{n-i+1} - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (n - 2i + 1)a_i.$$

Then for $i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$, $a_{n-i+1} = \binom{n}{2} - i + 1$ and $a_i = i - 1$. That is,

$$f(V(K_n)) = \begin{cases} \left[0..\left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1..\binom{n}{2}\right] & \text{if } n \text{ is even} \\ \left[0..\left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1..\binom{n}{2}\right] \cup \{k\} & \text{if } n \text{ is odd,} \end{cases}$$

where $k \in \left[\left\lfloor \frac{n}{2} \right\rfloor..\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor\right]$. ■

4 Final Remarks

In [2], [3] and [5] the maximum value of γ-labeling of path P_n, cycle C_n, double star $S_{p,q}$ and cycle with a triangle $C_n^{\triangle}$ was determined. It would be interesting to characterize γ-max labeling of those graphs.

References

[1] G. D. Bullington, L. L. Eroh, and S. J. Winters, *Gamma-labelings of complete bipartite graphs*, Discuss. Math. Graph Theory, **30** (2010) 45-54.

[2] G. Chartrand, D. Erwin, D. W. VanderJagt, and P. Zhang, *γ-Labelings of graphs*, Bull. Inst. Combin. Appl., **44** (2005) 51-68.

[3] G. Chartrand, D. Erwin, D. W. VanderJagt, and P. Zhang, *On γ-labelings of trees*, Discuss. Math. Graph Theory, **25** (2005) 363-383.

[4] G. Chartrand, P. Zhang, Introduction to Graph Theory (The Walter Rudin student series in advanced mathematics, McGraw-Hill Higher Education, Boston, 2005).

[5] C. Fonseca, V. Saenpholphat, and P. Zhang, *Extremal values for a γ-labeling of a cycle with a triangle*, Utilitas Math., to appear.

[6] C. Fonseca, V. Saenpholphat, and P. Zhang, *The γ-spectrum of a graph*, Ars Combin., to appear.

On γ-Labelings of Graphs

Saenpholphat, V.[1*], Fonseca, C. M.[2], Zhang, P.[3]

[1]*Department of Mathematics, Srinakharinwirot University, Bangkok 10110, Thailand*
[2]*Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal*
[3]*Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA*

Abstract

Let G be a graph of order n and size m. A γ-labeling of G is a one-to-one function $f: V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f': E(G) \to \{1, 2, \ldots, m\}$ of the edges of G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. The value of a γ-labeling f is defined as

$$\text{val}(f) = \sum_{e \in E(G)} f'(e).$$

The maximum value of a γ-labeling of G is defined as

$$\text{val}_{\max}(G) = \max\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\};$$

while the minimum value of a γ-labeling of G is

$$\text{val}_{\min}(G) = \min\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

A γ-labeling of G whose value is $\text{val}_{\min}(G)$ (resp. $\text{val}_{\max}(G)$) is called a γ-min (resp. γ-max) labeling of G.

We determine the extremal values of a γ-labeling of a cycle with a triangle. We also show a formula for spans of γ-min and γ-max labelings of graphs. Furthermore, we characterize γ-max labelings of complete bipartite and complete graphs.

Keywords: γ-labeling, γ-min labeling, γ-max labeling

Outputs

1. Saenpholphat V, Fonseca C, Zhang P. *Extremal values for a gamma-labeling of a cycle with a triangle*. Utilitas Mathematica **2011** (accepted).
2. Saenpholphat V, Fonseca C, Zhang P. *On gamma-labeling of graphs*. Utilitas Mathematica **2012** (accepted).

*Corresponding author.
Tel.: 0-2664-1000 ext. 8607; Fax: 0-2649-5622
E-mail: varaporn@swu.ac.th

Extremal Values for a γ-Labeling of Graph

Varaporn Saenpholphat [1]
Department of Mathematics
Srinakharinwirot University,
Sukhumvit 23, Bangkok, 10110, Thailand
varaporn@swu.ac.th

ABSTRACT

For any graph G of order n and size m, a γ-labeling of G is defined as a one-to-one function $f: V(G) \to \{0, 1, \ldots, m\}$ that induces an edge-labeling $f': E(G) \to \{1, \ldots, m\}$ on G defined by

$$f'(uv) = |f(u) - f(v)|, \quad \text{for each edge } uv \text{ in } E(G).$$

The value of f is defined by

$$\mathrm{val}(f) = \sum_{e \in E(G)} f'(e).$$

In this paper, we determine the extremal values of a γ-labeling of a cycle with a triangle. Several examples are considered. More generally, for a nonnegative integer δ, we define a γ^δ-labeling of G as a one-to-one function $f: V(G) \to \{0, 1, \ldots, m+\delta-1, m+\delta\}$ that induces an edge-labeling $f': E(G) \to \{1, \ldots, m+\delta\}$ on G defined by

$$f'(uv) = |f(u) - f(v)|, \quad \text{for each edge } uv \text{ in } E(G).$$

The value of a γ^δ-labeling f is defined by $\mathrm{val}(f) = \sum_{e \in E(G)} f'(e)$. We determine the extremal values of a γ^δ-labeling of some graphs.

Key Words: Graph; cycle; γ-labeling; γ^δ-labeling.
AMS Subject Classification: *05C78*.

References

[1] G. Chartrand, D. Erwin, D.W. VanderJagt, P. Zhang, γ-labelings of graphs, Bull. Inst. Combin. Appl. 44 (2005), 51-68.

[1] Research supported in part by Srinakharinwirot University, Commission on Higher Education and the Thailand Research Fund-RMU5380031.

On Extremal Values for γ-Labeling of Graph

Saenpholphat, V.[1*], Fonseca C. M.[2], Zhang P.[3]

[1]*Department of Mathematics, Srinakharinwirot University, Bangkok, 10110, Thailand*
[2]*Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal*
[3]*Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA*

Abstract

Let G be a graph of order n and size m. A γ-labeling of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ of the edges of G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. The value of a γ-labeling f is defined as

$$\mathrm{val}(f) = \sum_{e \in E(G)} f'(e).$$

The maximum value of a γ-labeling of G is defined as

$$\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\};$$

while the minimum value of a γ-labeling of G is

$$\mathrm{val}_{\min}(G) = \min\{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

A γ-labeling of G whose value is $\mathrm{val}_{\min}(G)$ (resp. $\mathrm{val}_{\max}(G)$) is called a γ-min (resp. γ-max) labeling of G.

We show formula for spans of γ-max and γ-min labelings of graph. We also characterize γ-min labelings of graph and γ-max labelings of complete bipartite and complete graphs. Furthermore, we determine the extremal values of a γ-labeling of a cycle with a triangle.

Keywords: γ-labeling, γ-min labeling, γ-max labeling

Output

1. Saenpholphat V., Fonseca C., Zhang P. *Extremal values for a gamma-labeling of a cycle with a triangle*. Utilitas Mathematica; accepted, **2011**.
2. Saenpholphat V., Fonseca C., Zhang P. *On gamma-labeling of graphs*. Utilitas Mathematica; accepted, **2012**.

---

*Corresponding author.
Tel.: 0-2649-5000 ext. 8607; Fax: 0-2649-5622
E-mail: varaporn@swu.ac.th