THE THAILAND RESEARCH FUND

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SURFACE-DISTURBANCE OF A MOVING STREAM

by

Jack Asavanant
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Chulalongkorn University

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ABSTRACT

This project concerns the two-dimensional flows of an inviscid and incompressible fluid in domain bounded below by a rigid bottom and above partially by a free surface. Different types of forcing are considered. We study the effect of an applied pressure distribution on the free surface. For supercritical solutions, the free surface profiles are found to be symmetric whereas in the case of subcritical flows the solutions are characterized by a train of nonlinear waves behind the pressure distribution. However the numerical scheme does not converge as the Froude number $\uparrow 1$. This is due to the occurence of unexpected periodic disturbance on the upstream free surface. Free-surface flows past an object with smooth attachments without gravity is also considered. Our results confirm the previous conjecture about the existence of solutions as the Froude number $\to \infty$. Furthermore, we consider flows of two immiscible fluids over an obstruction. The forcing is assumed to be of compact support. Existence of symmetric solutions are proved and computed numerically. Free-surface flows past a flat-bottomed object with stagnation points are studied. It is found that splashless solutions exist for certain values of the Froude number. These two-parameter family of solutions are subfamily of the more general three-parameter family of solution which might include breaking solutions.

Chapter 1
Introduction

Efforts to analyze the hydrodynamical characteristics of free-surface flow with surface-disturbance have been divided primarily between theoretical and experimental considerations. There are various types of surface-disturbance occurred in nature and some are due to man-made structures. Most of the theoretical studies lie mainly in the two-dimensional framework and were based on global analysis. Results from the laboratory experiments provided, on the other hand, small scale analysis for both two- and three-dimensional problems.

We devote this report .to the investigations of steady two-dimensional potential flow of an inviscid and incompressible fluid. These 2-D models allow us to utilize various mathematical tools to solve the problems, for example, the use of complex analysis particularly conformal transformation. This simplification will not only provide qualitative behaviors but also give some insights to the real flow situations to gain more understanding. Though the assumption of steadiness may seem unreal but we can always choose the appropriate moving frame of reference in such a way that the flow becomes steady.

In this study, we consider fully nonlinear free-surface flow problems with three different types of surface-disturbance. The fluid domains are of finite depth with no vertical boundaries in the far fields. One problem is associated with pressure distribution applied to a portion of the free surface. Next problem concerns flows over a bottom obstruction. Lastly, we investigate the the existence of particular types of flow due to the motion of ship hull.

Generally, problems in free-surface hydrodynamics under the influence of gravity are too difficult to solve exactly. Thus appropriate techniques of mathematical approximations are usually sought. These can be classified as analytic approximations and direct numerical calculations. Asymptotic analysis is one of the classic approaches of analytic approximations and is used here in this study. We also introduce the integral equation formulation and the method based on analytic function theory which may prove useful in treating a variety of fluid flow problems with free boundaries. Numerical results from these two methods are obtained after a few Newtonian iterations.

Free-surface flows past an applied pressure distribution in water of infinite depth was studied long ago. Literatures and discussions on the linear theory of two-dimensional pressure distribution can be found in [1]. Lamb [1] also provided the analysis of three-dimensional problem without any calculations. Schwartz [2] formulated the problem into a system of nonlinear integral equations and solved numerically. Results were found to be qualitatively similar. That is, in the case of subcritical flows, both linear and nonlinear theoy predicted the existence of wave train propagating downstream. When the flow is supercritical, the free surface profile was always symmetric indicating the drag-free situation. Vanden-Broeck and Tuck [3] used perturbation technique to demonstrate the existence of some families of solutions that do not generate waves. All of these results are for the fluid domain of infinite depth. In the case of finite depth, Von-Kerczek and Salvesen [4] performed direct numerical calculations by placing a network of mesh points over the entire domain and used the Successive Over Relaxation scheme. They found that the numerical results were restricted to certain values of the pressure-distribution-length to the depth ratio. Here we propose the boundary integral formulation which requires the uses of Cauchy inegral formula and Schwarz reflection principle. Integral equation is derived in terms of unknown variables on the free surface. After discretization, we obtain the system of nonlinear algebraic equations to be solved by Newton's method. Results are calculated for both supercritical and subcritical flow regimes. Due to the existence of a nonlinear wave train behind the disturbed free surface, there are periodic disturbance of small amplitude on the upstream free surface. This numerical disturbance grows with a given parameter. Discussion of these errors can be found in chapter 2.

Another type of surface-disturbance in a canal is due to a bottom obstruction. This part is the continuation of Choi *et al* [5]. Flow domain of interest consists of two immiscible fluids of constant but different denstities. The bottom of the canal is described by a function of compact support. Derivation of the governing equations is carried out by using a unified asymptotic approach. Existence of the symmetric solutions is proved and numerical calculations are also given by using the shooting method.

Next we study the special case of flow past a surface-piercing object with smooth attachments at the separation points when gravity is excluded. This is known as the free-streamline problem and is the special case of the problem considered by Asavanant and Vanden-Broeck [6]. Here we construct an analytic function representing the complex velocity in the flow domain of interest by expressing in the form of power series. We determine the unknown coefficients of the series by requiring the complex velocity to satisfy the dynamic and kinematic boundary conditions. This approach proves to be more efficient than solving the Laplace equation directly.

Lastly we investigate free-surface flows past an object for which the separations occur at the stagnation points (i.e. points at which the fluid velocity vanishes). The object is of flat-bottomed body with two vertical faces. Some of the previous results can be summarized as follows. Dagan and Tulin [7] solved the steady flow past a semi-infinite two dimensional flat-bottomed body of draft H in deep water by perturbation procedure. The expansion in power series of the Froude number $F_H = U/\sqrt{gH}$ was of second order. However the algebra involved in the extension of their approach to higher order in F_H is formidable, even for computer use. Vanden-Broeck, Schwartz and Tuck [8] derived a nonlinear singular integro-differential equation which allowed them to continue the series indefinitely. They showed analytically and numerically that there are no continuous solutions for bow flows for which stagnation point occurs at the separation point and the stream approaches a uniform velocity in the far field. In the case of stern flows, they found continuous solutions with waves in the far field. A new family of stern flows for which the flow separates at the corner with smooth attachment was described analytically and numerically by Vanden-Broeck [9]. Madurasinghe and Tuck [10] constructed solutions for splashless bow flows past a semi-infinite object with smooth attachment. A model of bow flows with splash (or spray) at the leading contact point of the object was proposed by Dias and Vanden-Broeck [11]. The splash (or spray) was assumed to take on the form of a jet rising along the bow to a stagnation point and falling down onto the oncoming stream which was considered to be another Riemann sheet. In the case of flow past a semi- infinite bow with a flat bottom in water of finite depth, there are continuous solutions for which the flow rises up along the vertical front of the body and separates at a stagnation point (Vanden-Broeck [12]). Asavanant and Vanden-Broeck [6] considered the complete nonlinear flow past a curved object of finite length. Their results confirmed those obtained analytically by Craig and Sternberg [13]. In this report, we show that there are splashless solutions of flows past a flat-bottomed with vertical faces. These particular solutions represent flows for which the separations occur at the stagnation points. We also generalize the problem to the case of inclined faces.

Discussions of flows with an applied pressure distribution are given in chapter 2. Flows of two immiscible fluids over a bottom obstruction is considered in chapter 3. Free-streamline solutions of flow past a surface-piercing object are summarized in chapter 4. Chapter 5 concludes the results obtained for the problem of flows past a two-dimensional ship with stagnation points on the hull.

Finally the author wishes to acknowledge the Thailand Research Fund (TR-F). This work could have never been completed without the financial support from the TRF. Special thanks should be given to the Center for the Mathematical Sciences at the University of Wisconsin-Madison for allowing the calculations on the computer. During the past two years, the author collaborated with Professor J.-M. Vanden-Broeck at the University of Wisconsin-Madison, USA, and Associate Professor J.W. Choi at the Korea University, Korea.

Chapter 2
Model equation of flows with free surface pressure distribution

2.1 Introduction

Steady two-dimensional free-surface flow past an applied pressure distribution on the free surface is considered (see Figure 1). The stream is of finite depth. The fluid is assumed to be inviscid and incompressible and flows irrotationally with constant horizontal velocity U at infinity. The pressure function is also assumed to be of compact support. This flow configuration can serve as a model of moving vehicles such as hovercraft in a long canal. We restrict our attention to flows which approach a uniform depth H as $x \to -\infty$. The flow is characterized by a nondimensional parameter, the Froude number

$$F = \frac{U}{\sqrt{gH}} \tag{1}$$

where g denotes the acceleration of gravity.

The problem of free surface pressure distributions has been studied quite extensively in the case of infinite depth for over 150 years. The classical linearized version of the two-dimensional problem was solved long ago and was discussed in detail by Lamb [1]. It was shown that for some pressure distributions the motion is drag-free. That is, the free surface is symmetric with respect to the applied pressure-distribution without a train of sinusoidal waves in the far field. Schwartz [2] reformulated the problem into a boundary integral equation and solved numerically. He showed that for some values of the Froude number (defined by using the span length of the pressure distribution as the length scale) nonlinear theory gave drag-free solution while the linearized theory did not. He also found nonlinear wave train in the form of narrow crests and broad troughs which are essentially periodic and propagate downstream. Vanden-Broeck and Tuck [3] demonstrated, by perturbation procedure, the existence of some families of free-surface pressure distributions that do not generate waves. Their analysis includes asymptotic solutions up to terms of second order. In the case of finite depth, Von-Kerczek and Salvesen [4] placed a network of mesh points over the entire flow domain and performed finite-difference calculations to obtain nonlinear solutions. Their numerical procedure was restricted to certain values of the ratio of pressure-distribution-length to the depth of the flow domain.

In this chapter we consider the fluid domain of finite depth. We solve the problem numerically by the boundary integral equation method for arbitrary values of the Froude number, magnitude and span length of the pressure distribution. Our results show that the free surface profile is always symmetric (drag free) when the flow is supercritical. For subcritical flow, the solutions are characterized by a train of nonlinear waves downstream while the flow satisfies the radiation condition on the upstream side. Difficulty in the numerical process occurs when the Froude number is increasing to unity from below because of the unpredicted numerical disturbances in periodic forms on the upstream free surface. This has sabotaged the convergence of the numerical scheme. However we conjecture that, for subcritical solutions, the wavelength would approach infinity as $F \uparrow 1$ and would extend to the supercritical regime. Also, as F decreases, the downstream free surface would approach its limiting configuration in the form of Stokes' wave (sharp crest and broad trough).

2.2 Formulation

Let us consider the steady two-dimensional, irrotational flow of an inviscid incompressible fluid in the region shown in Figure 1. We choose Cartesian coordinates with the x-axis along the free surface at $x = -\infty$ and the y-axis directed vertically upwards through the center of the pressure distribution. Gravity is acting in the negative y-direction. The components of the velocity in the x- and the y-directions are denoted by u and v respectively. As $x \to -\infty$, the flow is assumed to approach a uniform stream with constant velocity U and uniform depth H.

We introduce the complex potential $f = \phi + i\psi$, in which ϕ and ψ represent the potential function and the streamfunction respectively. Without loss of generality, we choose $\psi = 0$ on the free surface ABEF. The bottom AF defines another streamline on which $\phi = -UH$. The nonlinear free surface condition for this problem can be expressed by

$$\frac{1}{2}q^2 + gy + \frac{p}{\rho} = \text{constant on the free surface} \tag{2}$$

Here q, p and ρ denote the magnitude of the velocity, the pressure function and fluid density respectively.

Let us choose U as the unit velocity and H as the unit length. From this choice of dimensionless variables, (2) becomes

$$q^2 + \frac{2}{F^2}y + \tilde{p} = 1 \tag{3}$$

where F is the Froude number defined by (1), $\tilde{p} = \frac{p - p_o}{\frac{1}{2}\rho U^2 H}$ for which p_o represents the atmospheric pressure. The kinematic condition on the bottom AF is

$$v(\phi, \psi) = 0 \quad on \quad \psi = -1 \tag{4}$$

Next we define the complex function $\xi = u - iv - 1$ and the flow region in the f-plane $D = \{(\phi, \psi) \mid -\infty < \phi < \infty, -1 < \psi < 0\}$. The function ξ is analytic in the domain D and is real on the boundary $\psi = -1$. We satisfy the kinematic condition (4) on AF by reflecting the flow field in the physical z-plane about the line $y = -H$. Let Ω and $\overline{\Omega}$ denote the fluid domain in the z-plane and its reflection. By Schwarz reflection principle, the function ξ can be extended to a function Ξ which is analytic in $\Omega \cup \overline{\Omega}$ and is defined as

$$\Xi(z) = \begin{cases} \xi(z) & , \text{for } z \in \Omega \\ \overline{\xi(\bar{z})} & , \text{for } z \in \overline{\Omega}. \end{cases} \tag{5}$$

Applying the Cauchy integral formula to the function Ξ in the extended region in the f-plane which is the strip $-2 \leq \psi \leq 0$, we obtain

$$\Xi(f) = u - iv - 1 = -\frac{1}{2\pi i} \oint_{\Gamma} \frac{u(f') - iv(f') - 1}{f' - f} \, df'. \tag{6}$$

Here Γ is the negatively oriented contour given by

$$\Gamma : \begin{cases} f = \phi & ; -R \leq \phi \leq R \\ f = R + i\psi & ; 0 \geq \psi \geq -2 \\ f = \phi - 2i & ; R \geq \phi \geq -R \\ f = -R + i\psi & ; -2 \leq \psi \leq 0. \end{cases}$$

Letting f approaches the boundary $\psi = 0$, we obtain

$$\Xi(\phi, 0) = u(\phi, 0) - iv(\phi, 0) - 1 = -\frac{1}{\pi i} \oint_{\Gamma} \frac{u(f') - iv(f') - 1}{f' - \phi} \, df'. \tag{7}$$

We denote by $u(\phi)$ and $v(\phi)$ the velocity components in the x- and y- directions on the free surface $\psi = 0$. Consequently, (7) becomes

$$u(\phi) - iv(\phi) - 1 = -\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{u(\phi') - iv(\phi') - 1}{\phi' - \phi} \, d\phi' + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\bar{u}(\phi') - i\bar{v}(\phi') - 1}{\phi' - 2i - \phi} \, d\phi' \tag{8}$$

where $\bar{u}(\phi)$ and $\bar{v}(\phi)$ represent the horizontal and vertical components of the velocity on the image $\psi = -2$ of the free surface. Using (5), we can relate the integral over the reflection of the free surface to the integral over the free surface itself. Finally, after taking the real part, we can rewrite (8) as

$$u(\phi) - 1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(\phi')}{\phi' - \phi} \, d\phi' + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(\phi')(\phi' - \phi) + 2(u(\phi') - 1)}{(\phi' - \phi)^2 + 4} \, d\phi'. \tag{9}$$

Using the identity

$$\frac{\partial x}{\partial \phi} + i \frac{\partial y}{\partial \phi} = \frac{1}{u - iv}, \tag{10}$$

the Bernoulli equation (3) can be written in terms of $u(\phi)$ and $v(\phi)$ as

$$u^2(\phi) + v^2(\phi) + \frac{2}{F^2}\int_{-\infty}^{\phi} \frac{v(\phi')}{u^2(\phi') + v^2(\phi')}\, d\phi' + \tilde{p} = 1\ ;\ -\infty < \phi < \infty. \quad (11)$$

We consider now that the distribution of pressure be described by function with compact support and be defined by

$$\tilde{p} = \begin{cases} 0 & ,\ \text{for } |\phi| > 1 \\ \epsilon\, e^{\frac{1}{|\phi|^2 - 1}} & ,\ \text{for } |\phi| < 1. \end{cases}$$

The problem becomes that of finding $u(\phi)$ and $v(\phi)$ satisfying (9) and (11). The shape of the unknown free surface can be determined by integrating numerically the identity (10).

2.3 Numerical Procedure and Discussion

To obtain nonlinear solutions to (9) and (11), it is necessary to resort to a numerical method. We solve this system of integral equations by discretizing the free surface in the f-plane. Thus we introduce the M mesh points

$$\phi_i = (i-1)E, \quad i = 1, 2, ..., M \quad (12)$$

where E is the discretization interval. The values of $u(\phi)$ and $v(\phi)$ are computed at the mid points

$$\phi_{i-\frac{1}{2}} = \frac{\phi_i + \phi_{i+1}}{2}, \quad i = 1, 2, ..., M-1.$$

Equations (9) and (11) are to be satisfied at these mid points. We denote the values of u and v at the mesh points by u_i and v_i. The integrals in (9) are truncated downstream at the point ϕ_M subject to the requirement that the pressure distribution is applied on the free surface sufficiently far from the end points. The error due to this truncation can be estimated by comparing the solutions for different values of M and E.

We approximate the integral in (9) by using the trapezoidal rule with summation over ϕ_i. Since the spacing points are symmetric with respect to the pole. the singularity is subtracted from the Cauchy principal-value integral leaving nonsingular integrals. The values of u and v at the mid points are related to the values at the mesh points by linear interpolation. Next we replace (9) by

$$u_{i-\frac{1}{2}} - 1 = \frac{1}{\pi}\int_0^{(M-1)E} v_i\left[\frac{1}{\phi' - \phi_{i-\frac{1}{2}}}\right] d\phi' + \frac{1}{\pi}\int_0^{(M-1)E} \frac{v_i(\phi' - \phi_{i-\frac{1}{2}}) + 2(u_i - 1)}{(\phi' - \phi_{i-\frac{1}{2}})^2 + 4}\, d\phi'. \quad (13)$$

The radiation condition $v \to 0$ as $\phi \to -\infty$ is now applied at the first mesh point, ie.,

$$v_1 = 0. \tag{14}$$

The Bernoulli equation (11) is satisfied at the mid points

$$u^2_{i-\frac{1}{2}} + v^2_{i-\frac{1}{2}} + \frac{2}{F^2}\, y_{i-\frac{1}{2}} + \tilde{p}_{i-\frac{1}{2}} = 1.\ ;\ i = 2, 3, .., M. \tag{15}$$

We obtain $2M$ equations from (13) – (15) for the $2M$ unknowns u_i and v_i. It is convenient to write this system of equations in the form

$$f_i(\eta_1, \eta_2, ..., \eta_{2M}) = 0\ ;\ i = 1, 2, ..., 2M \tag{16}$$

where $\{\,\eta_j\,\}_{j=1}^{M} = \{\,u_j\,\}_{j=1}^{M}$ and $\{\,\eta_j\,\}_{j=M+1}^{2M} = \{\,v_j\,\}_{j=1}^{M}$. We solve (16) by Newton's method. Thus if $\eta_j^{(k)}$ is an approximation to the solution, the next approximation $\eta_j^{(k+1)}$ is obtained by

$$\eta_j^{(k+1)} = \eta_j^{(k)} - \triangle_j^{(k)}\ ;\ i = 1, 2, ..., 2M \tag{17}$$

where the correction $\triangle_j^{(k)}$ are calculated from

$$\sum_{j=1}^{2M} \left[\frac{\partial f_i}{\partial \eta_j}\right]^{(k)} \triangle_j^{(k)} = f_i^{(k)}\ ;\ i = 1, 2, ..., 2M \tag{18}$$

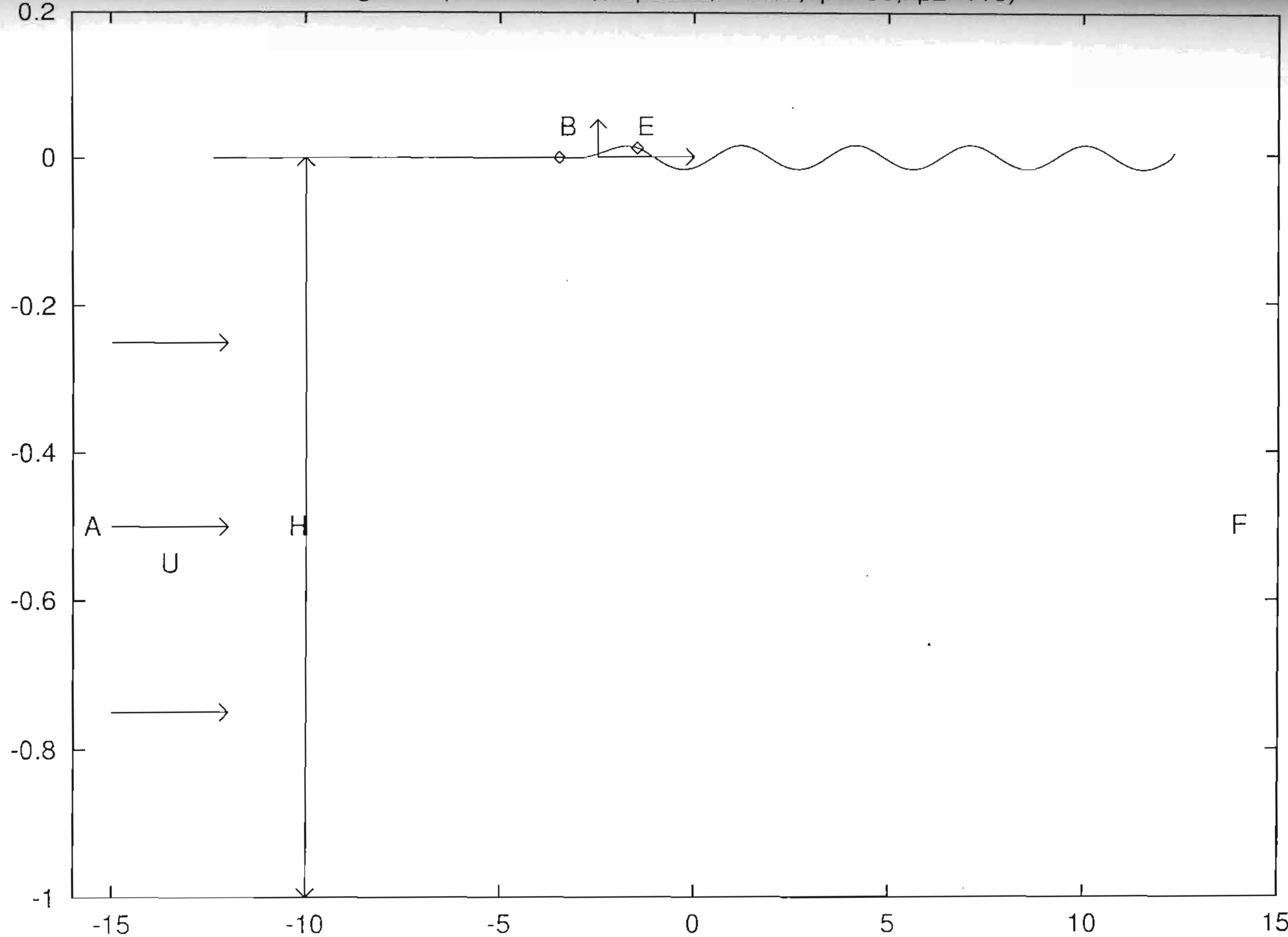
The elements $\frac{\partial f_i}{\partial \eta_j}$ are determined by exact differentiation of (16).

We use the numerical scheme described earlier to compute solutions for various values of F^2, ϵ and ϕ_e. We found that the behavior of the solutions for different values of $\phi_e > 0$ is qualitatively similar. We can thus fix value of ϕ_e and calculate solutions for various values of F^2 and ϵ. The numerical accuracy is checked by increasing M while keeping E fixed and vice versa.

When $F > 1$, the flow is supercritical throughtout the fluid domain. It is found that the free surface is symmetric with repect to the pressure distribution in both cases,i.e. $\epsilon > 0$ and $\epsilon < 0$. The pressure is applied onto the free surface when $\epsilon > 0$ and the negative pressure relative to the atmospheric pressure is the case when $\epsilon < 0$. The stream is undisturbed when $\epsilon = 0$.

When $F < 1$, the flow is subcritical which characterizes by the train of waves on the free surface behind the pressure distribution. Our numerical scheme can only compute solutions for certain values of the Froude number. This is due to numerical disturbances in periodic form which may cause from the stability of the numerical sheme. This type of numerical distubances occurs also in the case of applied pressure distribution on the stream of infinite depth under the influence of surface tension (Vanden-Broeck, private communication). At the present stage, there is no remedy to eliminate such disturbances. Other mathematical approaches should be used to further investigate this problem.

Figure 1 (F^2=0.45,ϵ=-0.05,np1=90,np2=110)



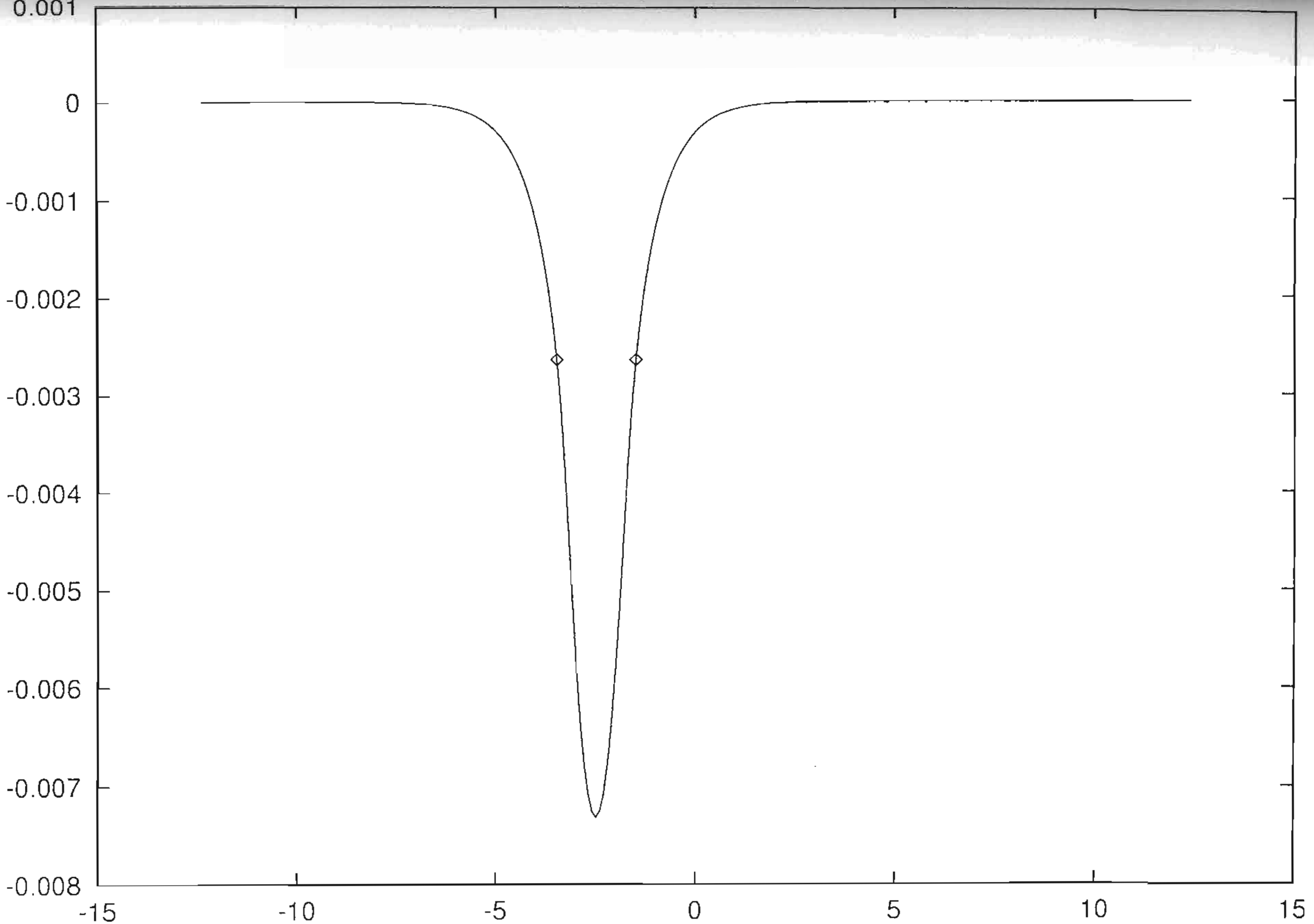


Figure 2 (\$F^2\$=4,\$\epsilon\$=0.05,np1=90,np2=110)

Figure 3 (F^2=0.155,ϵ=0.5,np1=90,np2=110)

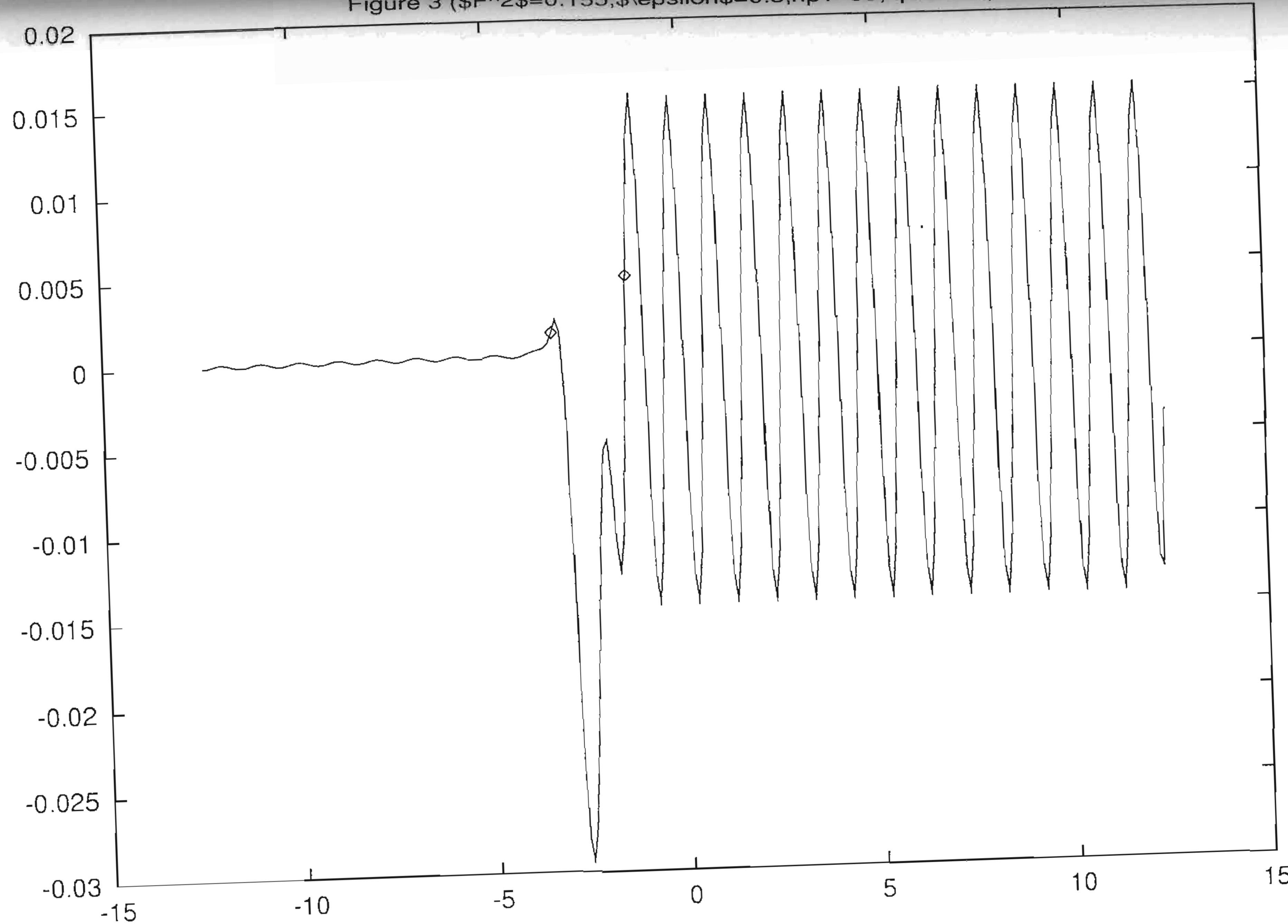
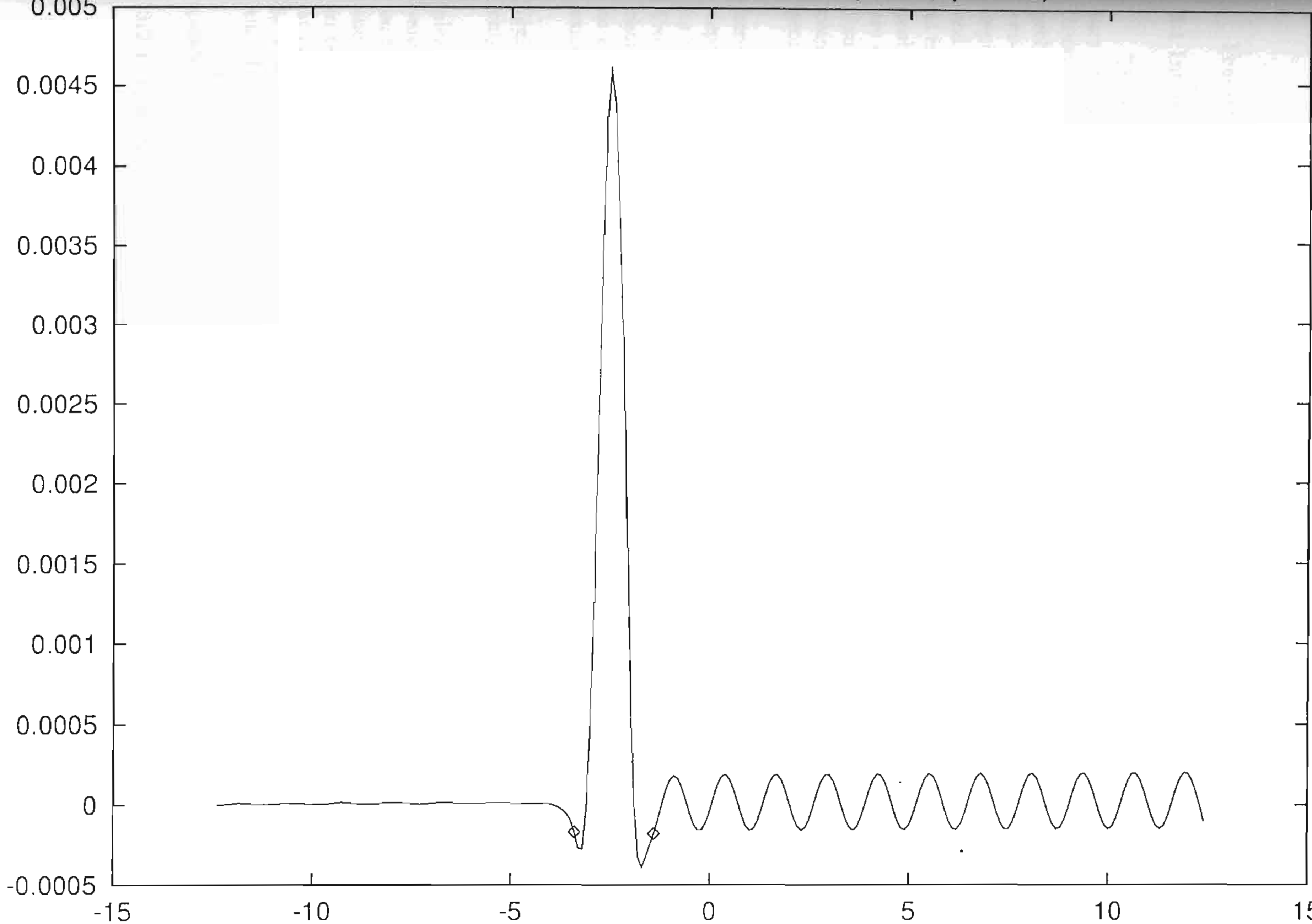


Figure 4 (F^2=0.25,ϵ=-0.05,np1=90,np2=110)



Chapter 3

Free-streamline solutions of flows past a surface-piercing object

3.1 Introduction

The classical free-streamline theory introduced in the mid nineteenth century by Helmholtz and Kirchhoff based on the use of hodograph plane and of Schwarz- Christoeffel transformation cannot be used to find exact solutions for problems with curved boundaries. In this chapter we consider a problem of steady two-dimensional flows past a parabolic obstacle of finite length in water of finite depth without gravity. This is the extension of the case considered by Asavanant and Vanden-Broeck [6] when the Froude number $F \to \infty$. The fluid is treated as inviscid and incompressible and the flow is assumed to be irrotational. The problem is solved by series truncation technique. Accurate numerical solutions are obtained by collocation procedure. As we shall see, there is a family of continuous solutions for which the free surfaces attach tangentially at the separation points. The solutions are found to depend on two parameters, that is the object geometry and the location of one of the separation points

Figure 1 (see Appendix A) shows the flow configuration to be considered here. Far upstream the flow approaches a uniform stream with velocity U and depth H. Asavanant and Vanden-Broeck [6] solved this problem numerically by including the effect of gravity. In that case, the nondimensional parameter Froude number F defined by the ratio of the inertial force to the gravity force characterizes the flows. They conjectured that there exist continuous solutions as $F \to \infty$. We shall use an efficient numerical technique to show that such flows exist when $F = \infty$.

Birkhoff and Zarantonello [14], Gurevich [15] gave systematic reports on high- speed. incompressible hydrodynamics problems. However they considered only flows around a flat plae in the presence of the wall.

The idea behind the series truncation technique is to seek an analytic complex velocity function satisfying the required boundary conditions and does not vanish anywhere in the flow domain. The construction of this function is based on the analog of Levi-Civita's function $\Omega(t)$ which is bounded and continuous on the unit semi-circle $|t| \leq 1$ and analytic in the interior. Thus we represent $\Omega(t)$ in terms of power series. The local behavior of this complex velocity function at each singularities must be prescribed appropriately. Once its representation is obtained, the velocity components on the free surfaces and on the object can then be determined by collocation method.

The main results were published in Advances in Fluid Mechanics (see Appendix A).

3.2 Formulation

Let us consider the steady two-dimensional irrotational flow of an inviscid incompressible fluid past a parabolic object lying on a free surface in water of finite depth. We introduce Cartesian coordinates with the x-axis on the bottom and the y-axis directed vertically upwards through the vertex of the object. Gravity is excluded and the object is described by

$$y = \frac{1}{2}\epsilon(x - x_o)^2 + y_o. \tag{1}$$

We define dimensionless variables by choosing U as the unit velocity and H as the unit length. We denote the potential function by ϕ and the streamfunction by ψ. In addition, we introduce the complex potential $f = \phi + i\psi$ and the complex velocity $\zeta = u + iv = \frac{df}{dz}$. Here u and v are the velocity components in the x- and y- directions respectively, and $z = x + iy$. Both f and ζ are analytic functions of z. The function ζ does not vanish anywhere in the flow domain.

Without loss of generality, we choose $\psi = 0$ on the bottom AF and $\phi = 0$ at the vertex C. It follows from the choice of dimensionless variables that $\psi = 1$ on the free surfaces AB, EF and the object BE. The flow region in the complex potential plane is an infinite strip $\{(\phi, \psi) \mid -\infty < \phi < \infty,\ 0 \le \psi \le 1\}$.

On the free surface, we impose the dynamic boundary condition stating that the pressure is constant along the free surface by using the Bernoulli equation

$$u^2 + v^2 = \text{ constant on the free surfaces.} \tag{2}$$

It is easy to see that the constant can be chosen to be U^2. The kinematic boundary conditions on AF and BE can be described by

$$\frac{\partial \phi}{\partial y} = 0, \psi = 0; \quad -\infty < \phi < \infty. \tag{3}$$

$$\frac{\partial \phi}{\partial n} = 0, \psi = 1; \quad -b < \phi < b. \tag{4}$$

Here $\frac{\partial \phi}{\partial n}$ is the normal derivative of ϕ and $\pm b$ are the values of ϕ at the separation points E and B respectively.

This completes the formulation of the problem. We shall seek ζ as an analytic function of f in the strip $0 < \psi < 1$ satisfying the conditions (2) - (4).

3.3 Numerical procedure and discussion

We map the flow domain in the complex f-plane conformally and symmetrically onto the unit semicircle Γ : $|t| \le 1$, $\mathrm{Im}(t) \ge 0$ in the auxiliary t-plane by

$$f = \frac{2}{\pi}\log\left(\frac{1+t}{1-t}\right). \tag{5}$$

This transformation maps the bottom AF onto the real diameter, and the free surfaces AB, EF and the object BE onto the circumference. At B and E, where the separations occur, the behavior of the flow is similar to those of other free-streamline problems. Therefore the free-streamline theory suggests that ζ behave at these separation points like

$$\zeta \sim G + H\left[f - (\pm b + i)\right]^{\frac{1}{2}} \text{ as } f \to \pm b + i. \tag{6}$$

Here $b + i$ and $-b + i$ denote the values of the complex potential f at E and B respectively. The constants G and H will be determined as part of the solutions.

We use the notation $t = r\, e^{i\sigma}$ so that the free surfaces and the object are described by $r = 1$ and $0 \leq \sigma \leq \pi$. The point $t = e^{i\beta}$ and $t = e^{-i\beta}$ correspond to the separation points E and B respectively. The appropriate behaviors of the flow near E and B can now be described by

$$\zeta \sim G + H\left[t^2 - 2t\cos\beta + 1\right]^{\frac{1}{2}} \text{ as } t \to e^{i\beta}, \tag{7}$$

$$\zeta \sim G + H\left[t^2 + 2t\cos\beta + 1\right]^{\frac{1}{2}} \text{ as } t \to e^{-i\beta}. \tag{8}$$

We now define the analytic function $\zeta(t)$ by

$$\zeta(t) = e^{\Omega(t)}. \tag{9}$$

From (7), (8) and the symmetry of the problem, we find that $\Omega(t)$ has the expansion

$$\Omega(t) = B_1\left[(t^2+1)^2 - (2t\cos\beta)^2\right]^{\frac{1}{2}} - B_1\left[4 - 4\cos^2\beta\right]^{\frac{1}{2}} + \sum_{n=1}^{\infty} a_n(t^{2n} - 1). \tag{10}$$

The kinematic condition (3) on the bottom AF is satisfied by requiring the coefficients a_n in the infinite series expansion (10) and B_1 to be real. It can be easily shown that (9) satisfies (7) and (8). Therefore we expect the series in (10) to converge for $|t| < 1$. The coefficients a_n and the constant B_1 must be determined so that (9) satisfies the free surface condition (2) and the kinematic condition (4).

It is convenient to eliminate y from (2) and (4). These give

$$u(\sigma)\, u_\sigma(\sigma) + v(\sigma)\, v_\sigma(\sigma) = 0\,, \tag{11}$$

$$v(\sigma) = \epsilon u(\sigma) \int_{\pi-\beta_1}^{\beta} \frac{u}{u^2+v^2}\frac{df}{d\sigma}\, d\sigma, \beta \leq \beta_1 \leq \frac{\pi}{2}. \tag{12}$$

In the numerical procedure we take advantage of the symmetry with respect to the y-axis of the problem by restricting the collocation points to the circular arc $t = e^{i\sigma}$, $0 \le \sigma \le \frac{\pi}{2}$. The infinite series in (10) is truncated after $N+1$ terms. We introduce the $N+2$ collocation points

$$\sigma_i = \frac{\pi}{2(N+2)} \left(i - \frac{1}{2} \right), \; i = 1, ..., N+2. \tag{13}$$

For simplicity, we assume values of β of the form

$$\beta = \frac{\pi}{2(N+2)} M, \tag{14}$$

where M is an integer smaller than $N+2$. Using (9) we obtain u, v, u_σ, v_σ at the mesh points σ_i in terms of the coefficients a_n and B_1. Substituting these expressions into (11) and (12), we obtain $N+2$ nonlinear algebraic equations for the unknowns $\{ a_n \}_{n=1}^{N+1}$ and B_1. We solve this system of equations by using Newton's method.

The numerical scheme described above was used to compute solutions for various values of β and ϵ. It was found that the coefficient a_n of the infinite series decrease rapidly as n increases. Most of the calculations were performed with $N = 180$.

As $\epsilon \to 0$, the object approaches a flat plate and the flow reduces to a uniform stream for all values of β. For $\beta = \frac{\pi}{2}$, the solutions correspond to a uniform stream with no object.

We define the amplitude parameter by

$$\alpha = \frac{W}{H}. \tag{15}$$

Here W is the distance from the bottom AF to the vertex of the object. We found relation of $\alpha - 1$ and ϵ for different values of β. These suggest that our solutions depend on two parameters β and ϵ. The computed free surface profiles were found to be similar to those obtained by Asavanant and Vanden-Broeck [6] for large values of the Froude number.

Details of the discussion of our numerical results can be found in Appendix A.

Chapter 4
Flows of a two-layer fluid over an obstruction

4.1 Introduction

In this chapter we study steady two dimensional waves in a two-layer fluid bounded above by a free surface and below by a horizontal rigid boundary with a small obstruction. Two critical speeds for the waves are obtained , near either one of which an FKdV for steady flow can be derived and has been studied extensively in [16] and [17]. Forbes [18], Belward and Forbes [19], Sha and Vanden-Broeck [20], and Moni and King [21] studied steady flow of a two layer fluid over a bump or a step bounded by a free of rigid boundary numerically. An asymptotic approach for the case of a rigid upper boundary was developed without surface tension by Shen [22] on the basis of FKdV theory, and with surface tension by Choi *et.al.* [5]. The case of free upper boundary was studied with surface tension by Choi *et.al.* [23] asymptotically on the basis of EKdV theory. Near the smaller critical speed, the derivation of the usual forced KdV equation (FKdV) fails when the coefficient of the nonlinear term in the FKdV vanishes. To overcome this difficulty, a new equation called a Steady Modified KdV equation with forcing term (SFMKdV) governing interfacial wave forms is derived by a refined asymptotic method. By using SMKdV we find the traveling soliton-like solutions and symmetric wave solutions for different choices of parameters. Existence theorems are proved and numerical results of this equation are presented.

In section 4.2. we formulate the problem and develop the asymptotic scheme to derive the SFMKdV. In section 4.3. existence theorems are proved and numerical solutions of soliton-like solutions and symmetric wave solutions are presented for different values of parameters. The parameters are determined along the density ratios of the two fluids, depth ratio of the two, and the perturbation of horizontal velocity at far upstream.

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4.2. Formulation and Successive Approximate Equations

We consider steady internal gravity waves between two immiscible, inviscid and incompressible fluids of constant but different densities bounded above by a free surface and below by a horizontal rigid boundary with a small obstruction of compact support. The domains of the upper fluid with a constant density ρ^{*+} and the lower fluid with a constant density ρ^{*-} are denoted by Ω^{*+} and Ω^{*-} respectively (Figure 1 in Appendix B). Assume that the small obstruction is moving with a constant speed C. In reference to a coordinate system moving with the obstruction, the flow is steady and moving with the speed C far upstream. The governing equations and boundary conditions are given by the following Euler equations:

In $\Omega^{*\pm}$,

$$
\begin{aligned}
&u^{*\pm}_{x^*} + v^{*\pm}_{y^*} = 0\,,\\
&u^{*\pm}u^{*\pm}_{x^*} + v^{*\pm}u^{*\pm}_{y^*} = -p^{*\pm}_{x^*}/\rho^{*\pm}\,,\\
&u^{*\pm}v^{*\pm}_{x^*} + v^{*\pm}v^{*\pm}_{y^*} = -p^{*\pm}_{y^*}/\rho^{*\pm} - g\,;
\end{aligned}
$$

at the free surface, $y^* = h^{*+} + \eta_1^*$,

$$
\begin{aligned}
&u^{*+}\eta^*_{1x^*} - v^{*+} = 0\,,\\
&p^{*+} = 0\,;
\end{aligned}
$$

at the interface, $y^* = \eta_2^*$,

$$
\begin{aligned}
&p^{*+} - p^{*-} = 0\,,\\
&u^{*\pm}\eta^*_{2x^*} - v^{*\pm} = 0\,;
\end{aligned}
$$

at the rigid bottom, $y^* = -h^{*-} + b^*(x^*)$,

$$
v^{*-} - b^*_{x^*}u^{*-} = 0\,,
$$

where $u^{*\pm}$ and $v^{*\pm}$ are horizontal and vertical velocities, $p^{*\pm}$ are pressures, g is the gravitational acceleration constant. We define the following nondimensional variables:

$$
\begin{aligned}
&\epsilon = H/L << 1,\ \ \eta_1 = \epsilon^{-1}\eta_1^*/h^{*-},\ \ \eta_2 = \epsilon^{-1}\eta_2^*/h^{*-},\ \ p^{\pm} = p^{*\pm}/gh^{*-}\rho^{*-}\,,\\
&(x,y) = (\epsilon x^*, y^*)/h^{*-},\ \ (u^{\pm}, v^{\pm}) = (gh^{*-})^{-1/2}(u^{*\pm}, \epsilon^{-1}v^{*\pm})\,,\\
&\rho^+ = \rho^{*+}/\rho^{*-} < 1,\ \ \rho^- = \rho^{*-}/\rho^{*-} = 1,\ \ U = C/(gh^{*-})^{1/2}\,,\\
&h = h^{*+}/h^{*-}, b(x) = b^*(x^*)(h^{*-}\epsilon^3)^{-1}\,,
\end{aligned}
$$

where L is the horizontal length scale, H is the vertical length scale, $b(x) = b^*(x)(h^{*-}\epsilon^3)^{-1}$, h^{*+} and h^{*-} are the equilibrium depths of the upper and lower fluids at $x^* = -\infty$ respectively, and $y^* = -h^{*-} + b^*(x)$ is the equation of the obstruction. In terms of the nondimensional quantities, the above equations become in $\Omega^{\pm}$,

$$
u^{\pm}_x + v^{\pm}_y = 0\,, \tag{1}
$$

$$
u^{\pm}u^{\pm}_x + v^{\pm}u^{\pm}_y = -p^{\pm}_x/\rho^{\pm}\,, \tag{2}
$$

$$
\epsilon^2u^{\pm}v^{\pm}_x + \epsilon^2v^{\pm}v^{\pm}_y = -p^{\pm}_y/\rho^{\pm} - 1\,; \tag{3}
$$

at $y = h + \epsilon\eta_1$,

$$p^+ = 0, \tag{4}$$
$$\epsilon u^+ \eta_{1x} - v^+ = 0; \tag{5}$$

at $y = \epsilon\eta_2$,

$$\epsilon u^- \eta_{2x} - v^- = 0, \tag{6}$$
$$\epsilon u^+ \eta_{2x} - v^+ = 0, \tag{7}$$
$$p^+ - p^- = 0; \tag{8}$$

at $y = -1 + \epsilon^3 b(x)$,

$$v^- = \epsilon^3 u b_x, \tag{9}$$

where $b(x)$ has a compact support.

Next we use a unified asymptotic method to derive the equations for $\eta_1(x)$ and $\eta_2(x)$. We assume that $u^\pm, v^\pm$, and $p^\pm$ are functions of x, y near the equilibrium state $u^\pm = u_0$, $v^\pm = 0$, $p^+ = -\rho^+ y + \rho^+ h$ and $p^- = -\rho^- y + \rho^+ h$, where u_0 is a constant, and possess asymptotic expansions:

$$\begin{aligned}(u^\pm, v^\pm, p^\pm) &= (u_0, 0, -\rho^\pm y + \rho^+ h) + \epsilon(u_1^\pm, v_1^\pm, p_1^\pm) \\ &\quad + \epsilon^2(u_2^\pm, v_2^\pm, p_2^\pm) + \epsilon^3(u_3^\pm, v_3^\pm, p_3^\pm) + O(\epsilon^4).\end{aligned} \tag{10}$$

By inserting (10) into (1) to (4) and (7) to (9) and arranging the resulting equations according to the powers of ϵ, it follows that $(u_0, 0, -\rho^\pm y + \rho^+ h)$ are the solutions of the zeroth order system of equations and the equations of the order ϵ are as follows:

$$u_{1x}^\pm + v_{1y}^\pm = 0, \tag{11}$$
$$u_0 u_{1x}^\pm = -p_{1x}^\pm / \rho^\pm, \tag{12}$$
$$p_{1y}^\pm = 0; \tag{13}$$

at $y = h$,

$$p_1^+ + \eta_1 p_{0y}^+ = 0, \tag{14}$$

at $y = 0$,

$$p_1^+ - p_1^- + \eta_2(p_{0y}^+ - p_{0y}^-) = 0, \tag{15}$$
$$u_0 \eta_{2x} - v_1^+ = 0; \tag{16}$$

at $y = -1$

$$v_1^- = 0\,. \tag{17}$$

Hereafter for the sake of convenience we shall use ρ to denote ρ^+ and set ρ^- equal to 1. From (13), $p_1^\pm$ are functions of x only. $p_1^+ = \rho\eta_1$ by (14) and $p_1^- = \rho\eta_1 + \eta_2(1-\rho)$ by (15). We can find $v_1^\pm$ by using (11), (12), (15), and (17) so that

$$\begin{aligned} v_1^+ &= y(\eta_{1x}/u_0) + u_0\eta_{2x}\,, \\ v_1^- &= (y+1)(\rho\eta_{1x} + (1-\rho)\eta_{2x})/u_0\,. \end{aligned} \tag{18}$$

$u_1^\pm$ are also derived from (11)

$$\begin{aligned} u_1^+ &= -\eta_1/u_0\,, \\ u_1^- &= (-\rho\eta_1 - (1-\rho)\eta_2)/u_0\,, \end{aligned} \tag{19}$$

where we assume $\eta_1(x=-\infty) = \eta_2(x=-\infty) = 0$, $u_1^\pm(x=-\infty) = 0$.

Similarly, we can find $p_2^\pm, v_2^\pm, u_2^\pm, p_3^\pm, v_3^\pm, u_3^\pm$ in terms of η_1 and η_2 without using the kinematic conditions (5) and (6). From (5) and(6), and the asymptotic expansion of u^- and v^-, we have
at $y = h$,

$$\begin{aligned} &u_0\eta_{1x} - v_1^+ + \epsilon(u_1^+\eta_{1x} - \eta_1 v_{1y}^+ - v_2^+) \\ &+ \epsilon^2(u_2^+\eta_{1x} + \eta_1\eta_{1x}u_{1y}^+ - v_{1yy}^+\eta_1^2 - \eta_1 v_{2y}^+ - v_3^+) + O(\epsilon^3) = 0, \end{aligned} \tag{20}$$

and at $y = 0$,

$$\begin{aligned} &u_0\eta_{2x} - v_1^- + \epsilon(u_1^-\eta_{2x} - \eta_2 v_{1y}^- - v_2^-) \\ &+ \epsilon^2(u_2^-\eta_{2x} + \eta_2\eta_{2x}u_{1y}^- - v_{1yy}^-\eta_2^2 - \eta_2 v_{2y}^- - v_3^-) + O(\epsilon^3) = 0. \end{aligned} \tag{21}$$

Then we make use of these equations to find the equations of the free surface $\eta_1(x)$ and the interface $\eta_2(x)$. By substituting $u_0, u_1^\pm, v_1^\pm, u_2^\pm, v_2^\pm, v_3^\pm$ into (20) and (21) and eliminating η_1, we obtain

$$\begin{aligned} &(u_0 - \rho c_1/u_0 - (1-\rho)/u_0)\eta_{2x} + \epsilon(E\eta_2\eta_{2x} + E_2\eta_{2x}) \\ &\quad + \epsilon^2(F_1\eta_2^2\eta_{2x} + F_2\eta_{2x} + F_3\eta_{2xxx} + F_4 b_x) \\ &\quad + O(\epsilon^3) = 0\,, \end{aligned} \tag{22}$$

where if we let $c_1 = (2u_0^2-(1-\rho))/(\rho+u_0^2-h)$, $D_1 = u_0/(\rho+u_0^2-h)$, $\lambda = u_2^{\pm}(-\infty)$, and $R = \rho c_1 + 1 - \rho$, then

$$E = -(R^2 + 2Ru_0^2)u_0^{-3} - \rho D_1((hc_1^2 - R^2)u_0^{-4} + (2c_1^2 - 2R - 2c_1)u_0^{-2}),$$

$$\begin{aligned} F_1 = &-\rho D_1 u_0^{-1}((3c_1^3 - 3c_1^2 + R^2/2)u_0^{-3} + (3hc_1^3/2 - 3R^3/2)u_0^{-5} \\ &+ 3D_1(\rho u_0^{-1} + \rho R u_0^{-3})((3R/2 + c_1 - c_1^2)u_0^{-1} + (R^2/2 - hc_1^2/2)u_0^{-3}) \\ &- 3R^2 u_0^{-3}/2 - 3R^3 u_0^{-5}/2, \end{aligned}$$

$$F_2 = \lambda((-\rho D_1 u_0^{-1})(2 + Ru_0^{-2} - c_1 - hc_1 u_0^{-2}) + (1 + Ru_0^{-2})),$$

$$\begin{aligned} F_3 = &(-\rho D_1 u_0^{-1})(-c_1(\rho h^2/2 + \rho/3)u_0^{-1} - (u_0^2 \rho h + (1-\rho)/3)u_0^{-1} \\ &+ (c_1(\rho h^3/3)/u_0\rho) + u_0 h^2/2) \\ &- c_1(\rho h^2/2 + \rho/3)u_0^{-1} - (u_0^2 \rho h + (1-\rho)/3)u_0^{-1}, \end{aligned}$$

$$F_4 = \rho D_1 - u_0.$$

4.3. Steady Forced Modified KdV Equation (SFMKdV)

From the zeroth order term of (22), we obtain

$$u_0 - (\rho c_1/u_0) - (1-\rho)/u_0 = 0,$$

and by the expression for c_1 in (22), it follows that

$$u_0^4 - (1+h)u_0^2 + h(1-\rho) = 0, \tag{23}$$

and

$$u_0^2 = (1 + h \pm ((1-h)^2 + 4\rho h)^{1/2})/2.$$

We denote the two values of u_0^2 by u_{01}^2 and u_{02}^2 respectively corresponding to the plus and minus signs. Without loss of generality we assume u_{01} and u_{02} are both positive and call them critical speeds, near each of which a nonlinear theory for the motion of the interface has to be developed.

Next we consider the coefficients of $\eta_2\eta_{2x}$ in the first order terms of the equation (22). If E in (22) is not zero, an FKdV can be derived if we assume $b(x) = b^*(x^*)(h^{*-}\epsilon^2)^{-1}$ and $x = \epsilon^{1/2}x^*/h^{*-}$ in nondimensional varibles and similar results as in [16] can be obtained. However, E may vanish. First, let us simplify the expression of E,

$$
\begin{aligned}
E &= -((\rho c_1 + 1 - \rho)^2/u_0^3) - 2((\rho c_1 + 1 - \rho)/u_0) \\
&\quad - \rho D_1[-2((\rho c_1 + 1 - \rho)/u_0) - ((\rho c_1 - \rho + 1)^2/u_0^3) + 2(c_1^2/u_0) + h(c_1^2/u_0^3) \\
&\qquad - 2(c_1/u_0)]/u_0 \\
&= 3(u_0\rho)^{-1}(u_0^2 + \rho - h)(\rho(u_0^2 h - u_0^4 - u_0^2 + 1) - u_0^4 + 2u_0^2 - 1), \\
&= 3u_0(1 - u_0^2)(\rho h(u_0^2 + \rho - h))^{-1}(u_0^4 + (1 - 2h)u_0^2 + h^2 - 1).
\end{aligned}
$$

where (23) has been used. When u_0 satisfies the equation (23), it is seen that u_0^2 is neither 1 nor $h - \rho$. Hence $E = 0$ implies $u_0^4 + (1 - 2h)u_0^2 + h^2 - 1 = 0$. Let $u_0 = u_{01}$ or u_{02}. Then

$$
u_{01}^4 + (1 - 2h)u_{01}^2 + h^2 - 1 = 1 + h\rho + (2 - h)((1 - h)^2 + 4\rho h)^{1/2}, \tag{24}
$$

$$
u_{02}^4 + (1 - 2h)u_{02}^2 + h^2 - 1 = 1 + h\rho - (2 - h)((1 - h)^2 + 4\rho h)^{1/2}. \tag{25}
$$

Equation (24) tells us that E does not vanish if we take u_{01} as a critical speed. Suppose both sides of (24) vanish. Then real u_{01}^2 implies $h < 5/4$ and the right hand side of (24) is greater than zero. This is a contradiction. Thus the only possible case for $E = 0$ is that the critical speed u_0^2 is equal to u_{02}^2, and it is easy to show that $E = 0$ if $u_0^2 = u_{02}^2$, and

$$
1 + h\rho = (2 - h)((1 - h)^2 + 4\rho h)^{1/2}. \tag{26}
$$

With the conditions (21) and (25), we obtain a Steady FMKdV,

$$
F_1 \eta_2^2 \eta_{2x} + F_2 \eta_{2x} + F_3 \eta_{2xxx} + F_4 b_x = 0, \tag{27}
$$

where

$$
\begin{aligned}
F_1 &= 3u_0(4\rho + 3h - u_0^2), \\
F_2 &= \lambda(2(1 + h)u_0^2 - 4h(1 - \rho))u_0^{-2}, \\
F_3 &= u_0^{-1}(h(1 + h) - u_0^2(h^2 + 1 + 3\rho h), \\
F_4 &= u_0(h - u_0^2).
\end{aligned}
$$

The coefficients F_1 to F_4 here are the simplified forms of F_1 to F_4 in the previous section by using (23). The sign of $F_3 F_1$ determines the existence of solutions of (27). In the following sections, we assume $F_3 F_1 > 0$ and the case for $F_3 F_1 < 0$ is considered in subsequent study [24].

4.3.1. Symmetric soliton-like waves

We assume $U = u_0 + \lambda\epsilon^2 + O(\epsilon^3)$ and consider (27) for $F_1/F_3 > 0$ and $F_2/F_3 < 0$. (27) can be rewritten as

$$\eta_{2xxx} = -A_1\eta_2^2\eta_{2x} + A_2\eta_x + A_3 b_x\,, \tag{28}$$

where $A_1 = F_1/F_3 > 0$, $A_2 = -F_2/F_3 > 0$, $A_3 = -F_4/F_3$. When $b_x \equiv 0$, (28) has soliton solutions whose value is 0 at $x = \pm\infty$ for $A_2 \geq 0$:

$$\eta_2(x) = \pm(6A_2/A_1)^{1/2}\operatorname{sech}((A_2)^{1/2}x)\,, \tag{29}$$

For $A_2 \leq 0$, there is no soliton solution. The solutions in (29) are obtained as in the classical case by taking the limit of elliptic functions in the periodic solutions of (28) for $b_x = 0$ when the wave length tends to infinity. Next we consider (28) when $b_x \neq 0$ but of compact support.

We look for a solution $\eta_2(x)$ such that $A_2 > 0$ and

$$\lim_{|x|\to\infty}(d/dx)^j\eta_2(x) = 0 \qquad j = 0, 1, 2.$$

Integrating (28) from $-\infty$ to x, it follows that

$$A_2\eta_2 - \eta_{2xx} = A_1\eta_2^3 - A_3 b(x), \quad -\infty < x < \infty\,. \tag{30}$$

(30) can be converted to the following in tegral equation:

$$\eta_2(x) = \int_{-\infty}^{\infty} K(x,\xi)(A_1\eta_2^3(\xi)/3 - A_3 b(\xi))d\xi\,,$$

where $K(x,\xi) = \exp(-\sqrt{A_2}|x-\xi|)/(2\sqrt{A_2})$ is a Green function of $A_2K(x,\xi) - K_{xx}(x,\xi) = \delta(x,\xi), -\infty < x < \infty$.

Define

$$\begin{aligned}
T(\eta_2) &= \int_{-\infty}^{\infty} K(x,\xi)(A_1\eta_2^3(\xi)/3 - A_3 b(\xi))d\xi\,,\\
\|u\| &= \|u\|_\infty = \sup_{x\in\Re}|\eta_2(x)|\,,\\
H &= \{u \mid u \in C(\Re), \|\exp(\sqrt{A_2}|x|)u\| < \infty\}\,,\\
B_M &= \{u \mid u \in H, \|u\| \leq M, 0 < M < \infty\}\,.
\end{aligned}$$

Then clearly H is a complete metric space and B_M is a closed ball in H, and the following theorem can be proved by Contraction Mapping Theorem.

Theorem 1. (30) has a solution in $C^3(\Re)$ which decays exponentially at $|x| = \infty$ if A_2 is sufficiently large.

We have shown that (28) has an exponentially decaying solution as x tends to $\pm\infty$. In the following we use numerical computation to find symmetric soliton-like solutions of (28) when the obstruction $b(x)$ is given by $b(x) = R(1-x^2)^{1/2}$ for $|x| \leq 1$ and $b(x) = 0$ for $|x| > 1$, where R is a given constant.

Let

$$\eta_2(x) = \pm(6A_2/A_1)^{1/2}\mathrm{sech}((A_2)^{1/2}(x-x_0)), \tag{31}$$

where x_0 is a phase shift. To find a solution in $|x| \leq 1$, we need only consider (31) in $-1 \leq x \leq 0$ subject to $(\eta_2'(x))^2 = -A_1\eta_2^4/6 + A_2\eta_2^2$ at $x = -1$ and $\eta_2(x) = 0$ at $x = 0$. This problem can be solved numerically by a shooting method and the phase shift x_0 is determined by (31) for $x = -1$. The numerical results are given in Appendix B. Four typical soliton-like solutions are also given we show the dependence of soliton-like solution at $x = 0$ and λ. In both numerical results, we assume $R = 1$.

4.3.2. Symmetric waves with zero behind and ahead of the obstruction

Similar to the section 3.1, we consider the equation

$$\eta_{2xxx} = -A_1\eta_2^2\eta_{2x} + A_2\eta_x + A_3 b_x\,, \tag{32}$$

where $A_1 = F_1/F_3 > 0$, $A_2 = -F_2/F_3$, $A_3 = -F_4/F_3$. Integrating (32) from $-\infty$ to x, we obtain

$$\eta_{2xx} = -A_1\eta_2^3/3 + A_2\eta_2 + A_3 b(x)\,, \tag{33}$$

where $b(x)$ is assumed to have compact support and $\eta_2(-\infty) = 0$.. We assume $\eta_2 \equiv 0$ in $(-\infty, x_-)$ where $[x_-, x_+]$ is the support of the obstruction. We can show that the solution of (33) exists and is bounded with initial values $\eta_2(x_-) = \eta_{2x}(x_-) = 0$. In the following, we use numerical computation to find symmetric wave solution of (33) which is zero behind and ahead of the elliptic obstruction. Similar methods as in section 3.1 is used to find the solutions of this problem. To find a solution in $|x| \leq 1$, we need only consider (33) in $-1 \leq x \leq 0$ subject to $\eta_2'(x) = \eta_2(x) = 0$ at $x = -1$ and $\eta_{2x}(x) = 0$ at $x = 0$. Same assumption as in section 4.3.1 has been given for obstruction and the numerical results can be found in Appendix B. Relationships between symmetric solutions and positive values of λ are also presented. The relations between R, which represents the hight of the obstruction, and $\eta_2(0)$ are also given. We note that, for a given R, symmetric solution is embedded in periodic solutions.

Chapter 5

Free-surface flow past an object with stagnation points

5.1 Introduction

We consider the steady two-dimensional irrotational flow of an inviscid incompressible fluid past a rectangular-shaped object lying on the free surface in water of finite depth (see Figure 1 in Appendix C). The problem models a barge-like vessel moving at a constant velocity in a canal without breaking or splash at the front of the vessel. Generally the breaking of the incoming stream at the contact point of the vessel is likely to occur in the real flow situation. It causes several problems due to the force generated at the breaking. Therefore it is of interest to determine whether the solutions without breaking exist. We restrict out attention to flows which approach a uniform stream with velocity U and depth H at far upstream and downstream. As we shall see, the flow can be characterized by the Froude number

$$F = U/\sqrt{gH} \tag{1}$$

where g is the acceleration due to gravity.

The problem of free-surface flows past an object has been considered by many investigators for the past two decades. Analytical and numerical results have been proposed for different flow configurations. In water of infinite depth, Vanden-Broeck and Tuck [3], Vanden-Broeck, Schwartz and Tuck [8], and Vanden-Broeck [9] showed that there are no continuous solutions for flows past a semi-infinite object such that the flows separate at a stagnation point and approach a uniform stream in the far field. In addition, they found that there are solutions with a train of nonlinear waves at infinity. This is the near-stern flow model. Madurasinghe and Tuck [10] constructed a near-bow flow model with splashless and continuous free surface profile at which the free surface attaches tangentially at the separation point. Diaz and Vanden-Broeck [11] considered solution of flows past a semi-infinite body by allowing discontinuity on the free surface in the form of breaking jet at the contact point. In water of finite depth, Vanden-Broeck [12] provided numerical evidence that there are continuous solutions for which the free surface rises up along the vertical side of the semi-infinite flat-bottomed object to the stagnation point at which separation occurs. The corresponding flows approach a uniform stream at infinity. This can be used as a model for near-bow flows in shallow water. Such flows exist for the values of the Froude number between $1.22 < F < \sqrt{2}$. Analytical results of Craig and Sternberg [13] provided the existence of flows past a ship hull for which the free surfaces make contact with the object with continuous tangent. Asavanant and Vanden-Broeck [6] calculated numerically solutions for flows past a curved object of finite length.

These flows have smooth tangents at both ends of the object at which separation occurs. Their numerical calculations showed that there are up to three supercritical solutions if the object is concave upwards and up to two solutions if the object is concave downwards, and there are subcritical solutions with waves behind the object.

In this chapter, we compute accurate numerical solutions for the fully nonlinear problem of flows past an object for which the free surfaces separate at the stagnation points. The problem is solved by series truncation method for arbitrary values of the width and the height of the object. The numerical procedure is similar to the one used by Vanden-Broeck and Keller [25], Vanden-Broeck [12] and Asavanant and Vanden-Broeck [6]. Our results include those obtained by Vanden-Broeck [12] as a particular case. We show that there exists a splashless solution for only some values of the Froude number.

In section 5.2 we formulate the problem for the flow configuration shown in Figure 1. The numerical procedure and discussion of the results are given in section 5.3. Generalization of this problem is presented in section 5.4.

Manuscript of this work are submitted to the European Journal of Mechanics B/Fluids and can be found in Appendix C.

5.2 Formulation of the problem

The steady two-dimensional irrotational flow of an inviscid incompressible fluid past a ship hull in water of finite depth is considered (see Figure 1 in Appendix C). The hull is assumed to have flat-bottomed with two vertical fronts for which the free surfaces separate at the stagnation points K and N. Here we choose Cartesian coordinates with the $x-$ axis along the bottom and the $y-$ axis directed vertically upwards through the middle of the hull. Gravity is acting in the negative $y-$ direction. The coordinate system moving with the hull is chosen so that the hull is stationary. As $|x| \to \infty$, the flow approaches a uniform stream with constant velocity U and uniform depth H.

It is convenient to define dimensionless variables by taking U as the unit velocity and H as the unit length. We denote the velocity potential by $\phi(x,y)$ and the streamfunction by $\psi(x,y)$. Let the complex potential be $f = \phi + i\psi$ and the complex velocity be defined by $\zeta = u - iv = df/dz$. Here u and v are the velocity components in the $x-$ and $y-$ directions respectively, and $z = x + iy$. Without loss of generality, we choose $\phi = 0$ at the middle of the bottom LM of the object and $\psi = 1$ on the free surfaces IK, NJ and the object KLMN. The flow domain in the $f-$ plane is an infinite strip. In it we let $\pm a$ denote the values of the potential functions at the two separation points K and N, and $\pm b$ denote the values of the potential functions at the corner points L and M of the object on $\psi = 1$. On the free surfaces, the constant pressure condition corresponding to the Bernoulli equation in dimensionless coordinates takes on the form

$$|\zeta|^2 + 2(y-1)/F^2 = 1, \text{ on IK and NJ.} \tag{2}$$

Here $|\zeta|$ is the magnitude of the velocity and F is the Froude number defined by (1).

The kinematic condition on the bottom IJ, and on the object KL, LM, MN can be expressed as

$$\mathrm{Im}\zeta = 0, \psi = 0 \text{ on } -\infty < \phi < \infty. \tag{3}$$

$$\mathrm{Re}\zeta = 0, \psi = 1 \text{ on } -a < \phi < -b \text{ and } b < \phi < a. \tag{4}$$

$$\mathrm{Im}\zeta = 0, \psi = 1 \text{ on } -b < \phi < b. \tag{5}$$

As $|\phi| \to \infty$, the flow approaches a uniform stream with constant unit velocity. For $F > 1$, we expect the approach to be described by exponentially decaying terms. So the complex velocity ζ can be expressed by

$$\zeta \sim 1 + De^{\mp\pi\lambda f} \text{ as } \phi \to \pm\infty. \tag{6}$$

Here D is a constant to be determined as part of the solution and λ is the smallest positive root of

$$\pi\lambda F^2 - \tan\pi\lambda = 0. \tag{7}$$

In addition, there are singularities at the corner points K, L, M and N. The appropriate behaviors of ζ near these singularities are

$$\zeta \sim H(f \pm b - i)^{1/2} \text{ as } f \to \mp b + i \tag{8}$$

$$\zeta \sim S(f \pm a - i)^{-1/2} \text{ as } f \to \mp a + i, \tag{9}$$

where H and S are constants to be determined as part of the solution. The problem now becomes that of finding ζ as an analytic function of f in the strip $0 < \psi < 1$ satisfying equations (2) - (6), (8) and (9).

5.3 Numerical procedure and discussion of the results

Following the method which was used successfully by Vanden-Broeck [12], Asavanant and Vanden-Broeck [6] and others, we map the flow domain in the complex $f-$ plane onto the upper half of the unit circle. The appropriate expression for the complex velocity ζ is sought as an analytic function inside the unit circle based upon the above formulation. Numerical calculation is then performed at each collocation points on the circumference of the upper half unit circle using Newton's method.

The transformation from the $f-$ plane onto the upper half of the unit circle in the $t-$ plane is given by

$$f = (2/\pi)\ \log[(1+t)/(1-t)]. \tag{10}$$

This maps the bottom IJ onto the real diameter, and the free surfaces IK, NJ and the object KL, LM, MN onto the circumference. We use the notation $t = re^{i\sigma}$ so that the free surfaces and the object are described by $r = 1$ and $0 < \sigma < \pi$. The points $t = e^{i\gamma_1}$ and $t = -e^{-i\gamma_1}$ are the images of the stagnation points N and K. The points $t = e^{i\gamma_2}$ and $t = -e^{-i\gamma_2}$ are the images of the corner points M and L of the object. By using (10), we find that γ_1 and b, γ_2 and a are related by

$$\gamma_1 = 2\ \arctan[exp(-\pi b/2)] \tag{11}$$

$$\gamma_2 = 2\ \arctan[exp(-\pi a/2)]. \tag{12}$$

We now seek the complex velocity ζ as a series representation in terms of t. Taking into account the local behaviors of the flow in (6), (8), (9) and its symmetry, we have the expression for the complex velocity

$$\zeta = [((t^2+1)^2-4t^2\cos^2\gamma_1)/(4-4\cos^2\gamma_1)]^{1/2}[((t^2+1)^2-4t^2\cos^2\gamma_2)/(4-4\cos^2\gamma_2)]^{-1/2}e^{\Omega(t)} \tag{13}$$

where $\Omega(t)$ has the expansion

$$\Omega(t) = A(1-t^2)^{2\lambda} + \sum_{n=1}^{\infty} a_n(t^{2n}-1). \tag{14}$$

The kinematic condition (3) on the bottom IJ implies that the expression $\Omega(t)$ has real coefficients. The representation (13) factors out the singular behaviors of the velocity at the corner points and the stagnation points. It can easily be verified that (13) satisfied (6), (8), and (9). The unknown constants A and the coefficients a_n of the power series must be determined so that the dynamic boundary condition (2) on the free surface, and the kinematic conditions (4) and (5) on the object are satisfied. It is now convenient to eliminate y from (2) by differentiating this equation with respect to σ. By using the identity

$$\partial x/\partial\phi + i\partial y/\partial\phi = 1/\zeta, \tag{15}$$

we obtain

$$F^2[u(\sigma)u_\sigma(\sigma) + v(\sigma)v_\sigma(\sigma)] - (2/\pi\sin\sigma)[v(\sigma)/(u^2(\sigma)+v^2(\sigma))] = 0. \tag{16}$$

We now solve the problem numerically by truncating the infinite series in (13) after N terms. There are $N+3$ unknowns λ, A, F and the coefficients a_n to be determined by collocation. Thus we introduce the $N+2$ mesh points

$$\sigma_I = (\pi/[2(N+2)])(I-1/2),\ I = 1, ..., N+2. \tag{17}$$

Here we take advantage of the symmetry of the problem. For simplicity, we consider values of γ_1 and γ_2 in the form of

$$\begin{aligned}\gamma_1 &= \pi M_1/[2(N+2)]\\ \gamma_2 &= \pi M_2/[2(N+2)].\end{aligned} \tag{18}$$

where $M_1 \leq M_2$ and both are integers smaller than $N+2$. We obtain $N+2$ equations by satisfying (16) at the mesh points $I = 1, ..., M_1$, (4) at the mesh points $I = M_1+1, ..., M_2$, and (5) at the mesh points $I = M_2+1, ..., N+2$. The last equation is provided by imposing the relation (7). For given values of M_1 and M_2, we solve this system of nonlinear algebraic equations by Newton's method. Once it is solved we obtain the shape of the free surface and the object by integrating numerically the relations

$$dx/d\sigma = (-2/\pi \sin\sigma)[u(\sigma)/(u^2(\sigma)+v^2(\sigma))] \tag{19}$$

$$\text{and } dy/d\sigma = (-2/\pi \sin\sigma)[v(\sigma)/(u^2(\sigma)+v^2(\sigma))]. \tag{20}$$

Numerical scheme described earlier was employed to compute solutions of the flow configuration in Figure 1 for various values of γ_1 and γ_2 corresponding to the values of MM_1 and MM_2 respectively. Here MM_1 and MM_2 are given by

$$MM_1 = M_1/(N+2) \text{ and } MM_2 = M_2/(N+2).$$

The coefficients a_n were found to decrease rapidly. For example, $|a_{10}/a_1| \approx 0.29\times10^{-1}$,$|a_{40}/a_1| \approx 0.36\times10^{-2}$, $|a_{200}/a_1| \approx 0.24\times10^{-3}$,$|a_{370}/a_1| \approx 0.15\times10^{-6}$ for $MM_1 = 1/2$ and $MM_2 = 7/10$. Most of the calculations were performed with 400 coefficients.

Typical profiles are given in Appendix C. As $\gamma_1 \to \pi/2$ and $\gamma_2 \to \pi/2$, the height MN, KL and the bottom width LM of the object reduce to zero and we recover the case of the steepest solitary wave. It is found that the limiting configuration with sharp crest and a 120^0 angle is obtained at $F = 1.29$. The value of this critical Froude number is found to be is good agreement with the one obtained by Asavanant and Vanden-Broeck [6], Hunter and Vanden-Broeck [26], and Lenau [27].

As $MM_1 \to MM_2$, the height MN, KL of the object reduce to zero and the problem becomes that of flows past a flat plate at which the free surfaces separate at the stagnation point with 120^0 angle corner. Since the corner point coincides with the stagnation point, the representation (13) of the complex velocity fails to converge in the numerical calculations. Thus we replace the singularities at these separation points to a more appropriate one. The local behavior of this stagnation points can be expressed by

$$\zeta \sim K(f \pm b - i)^{1/3} \text{ as } f \to \mp b + i. \tag{21}$$

The complex velocity ζ can be expanded as

$$\zeta = [((t^2+1)^2 - 4t^2\cos^2\gamma)/(4-4\cos^2\gamma)]^{1/3} e^{A(1-t^2)^{2\lambda}+\sum_{n=1}^{\infty} a_n(t^{2n}-1)}. \qquad (22)$$

The limiting configuration of flow past a flat plate with 120^0 angle corner is also calculated. The results for $\gamma = 0$ corresponds to flow past a semi-infinite flat plate with $F = \sqrt{2}$.

We now define the amplitude parameter as

$$\alpha = W/H, \qquad (23)$$

where W is the distance from the bottom IJ to the bottom LM of the object. Thus the bottom of the object lies above the undisturbed free surface level when $\alpha - 1 > 0$ and below when $\alpha - 1 < 0$. Numerical values of $\alpha - 1$ versus MM_1 are presented in Appendix C for various values of MM_2.

5.4 Flows past an object with inclined fronts

We now consider a more realistic situation of flows past a ship hull problem .i.e the two fronts of the object are inclined at an angle β. The solution described in the previous section are the special case of the present problem for $\beta = \pi/2$.

It is not obvious that there are solutions without splash jet for $0 < \beta < \pi/2$ (Dagan and Tulin [7], Vanden-Broeck and Tuck [25], Diaz and Vanden-Broeck [11]). The formulation can be generalized from the one described in section 5.2. In the complex potential plane, the images $-b+i$, $b+i$,$-a+i$ and $a+i$ again denote the values of the potential function at the corner points L, M and the separation points K, N respectively. As before, the points $t = e^{i\gamma_1}$, $-e^{-i\gamma_1}$, $e^{i\gamma_2}$ and $-e^{-i\gamma_2}$ are the images of N, K, M and L in the $t-$ plane respectively.

At the stagnation points K and N, the velocity vanishes which implies that the complex function ζ is singular at these stagnation points. According to Dagan and Tulin [7], the local behavior of the flow near these points is described by

$$\zeta \sim (f \pm b - i)^{\theta/\pi} \text{ as } f \to \mp b + i, \qquad (24)$$

where

$$\theta = \beta \text{ if } \pi/3 \leq \beta \leq \pi/2,$$
$$\theta = \frac{\pi}{3} \text{ if } 0 \leq \beta \leq \pi/3. \qquad (25)$$

At the corner point, the singularity is given by

$$\zeta \sim (f \pm a - i)^{-\beta/\pi} \text{ as } f \to \mp a + i. \qquad (26)$$

By using the transformation (10), we seek ζ as a function of t based upon the prescribed local behavior at infinity (6) and appropriate singularities (24) and (26). We represent the complex velocity ζ by

$$\zeta = [((t^2+1)^2-4t^2\cos^2\gamma_1)/(4-4\cos^2\gamma_1)]^{\theta/\pi}[((t^2+1)^2-4t^2\cos^2\gamma_2)/(4-4\cos^2\gamma_2)]^{-\beta/\pi}e^{\Omega(t)} \tag{26}$$

where $\Omega(t)$ has the expansion

$$\Omega(t) = A(1-t^2)^{2\lambda} + \sum_{n=1}^{\infty} a_n(t^{2n}-1). \tag{27}$$

The coefficients must be real in order to satisfy the kinematic boundary condition (3) on IJ. Furthermore $\zeta(\pm 1) = 1$.

The unknown constants A, a_n, λ and the Froude number F must be found for given values of γ_1, γ_2 and β so that the Bernoulli equation (16), the kinematic conditions (4) and

$$v = u \tan\beta \quad \text{on the inclined fronts of the object,} \tag{28}$$

and the relation (7). To determine this we truncate the infinite series in (27) after a finite number of terms and use the numerical scheme described in section 3.

It was found that the series converges rapidly. Typical profiles for two values of front inclinations can be found in Appendix C.

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Appendix A

Free-streamline solutions

Main results published in *Advances in Fluid Mechanics*

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Computation of free-streamline solutions of flows past a surface-piercing object

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ABSTRACT

The classical free-streamline theory introduced in the middle of the nineteenth century by Helmholtz and Krichhoff, i.e., the use of the hodograph plane and of the Schwarz-Christoeffel transformation, cannot be used to find exact solutions when some part of the rigid boundary is curved. We consider a problem of steady two-dimensional flows past a parabolic obstacle of finite length in water of finite depth in the absence of gravity. The fluid is treated as inviscid and incompressible and the flow is assumed to be irrotational. The problem is solved by using a series truncation technique. Accurate numerical solutions are obtained by collocation method. It is shown that there is a family of continuous solutions for which the free surfaces attach tangentially at the separation points. Furthermore, these solutions are found to depend on two parameters: the object geometry and the location of one of the separation points.

I INTRODUCTION

We consider the free-surface flow past a curved object lying on the free surface in water of finite depth (see Figure 1). Far upstream the flow approaches a uniform stream with velocity U and depth H. The original problem, which was considered by Asavanant and Vanden-Broeck,[1] took into account the effect of gravity. When gravity is included, the nondimensional parameter Froude number F defined by the ratio of the inertial force to the gravity force characterizes the flows. It was conjectured that there exist continuous solutions as $F \to \infty$. It is the purpose of this paper to use an efficient numerical tool to show that such flows exist when $F = \infty$.

The conventional free-streamline theory introduced by Helmholtz and Kirchhoff is based on the use of velocity hodograph and of the Schwarz-Christoffel transformation in which the rigid boundary must be straight-line segments. Birkhoff and Zarantonello,[2] Gurevich (1961)[3] gave systematic reports on high-speed, incompressible hydrodynamics problems. They considered only flows around a flat plate in the presence of a wall.

In this paper we solve the problem by the series truncation technique used by Vanden-Broeck and Keller.[4] The idea behind this approach is to seek an analytic complex velocity function satisfying the boundary conditions and does not vanish anywhere in the flow domain. The construction of this function is based on the analog of Levi-Civita's function $\Omega(t)$, which is bounded and continuous on the unit circle $|t| \le 1$ and analytic in the interior. Thus we represent $\Omega(t)$ in terms of power series. The local behavior of this complex velocity function at each singularities need to be prescribed appropriately, Once its representation is obtained, the velocity components on the free surfaces and on the object can then be determined by collocation method.

The problem is formulated in Sec. II. The numerical procedure is described in Sec. III and the results are presented in Sec. IV.

II MATHEMATICAL FORMULATION

Let us consider the steady two-dimensional irrotational flow of an inviscid incompressible fluid past a parabolic object lying on a free surface in water of finite depth (see Figure 1). We introduce Cartesian coordinates with th x-axis on the bottom and the y-axis directed vertically upwards through the vertex of the object. Gravity in neglected and the object is described by

$$y = \frac{1}{2}\varepsilon\left(x - x_0\right)^2 + y_0 \qquad (1)$$

We define dimensionless variables by taking U as the unit velocity and H as the unit length. We denote the potential function by ϕ and the streamfunction by ψ. In addition, we introduce the complex potential $f = \phi + i\psi$ and the complex velocity $\zeta = u - iv = df/dz$. Here u and v are the components of the velocity in the x- and y- directions respectively, and $z = x + iy$. Both f and ζ are analytic function of z. The function ζ does not vanish anywhere in the flow domain.

Without loss of generality we choose $\psi = 0$ on the bottom AF and ϕ = 0 at the vertex C. It follows from the choice of dimensionless variables that $\psi = 1$ on the free surfaces AB, EF and the object BE. The flow region in the complex potential plane is an infinite strip $\{(\phi, \psi): -\infty < \phi < \infty, 0 < \psi < 1\}$ as illustrated in Figure 2.

On the free surface, where the pressure is constant, the Bernoulli equation reduces to

$$u^2 + v^2 = \text{constant on the free surfaces} \qquad (2)$$

It is easy to see that the constant can be chosen to be U^2. The kinematic boundary conditions on AF and BE are

$$\frac{\partial\phi}{\partial y} = 0\,,\ \psi = 0 \quad ; -\infty < \phi < \infty \qquad (3)$$

$$\frac{\partial\phi}{\partial n} = 0\,,\ \psi = 1 \quad ; -b < \phi < b. \qquad (4)$$

Here $\partial\phi/\partial n$ is the normal derivative of ϕ and $\pm b$ are the values of ϕ at the separation points E and B respectively.

This completes the formulation of the problem. We seek ζ as an analytic function of f in the strip $0 < \psi < 1$ satisfying the conditions (2) - (4).

III NUMERICAL PROCEDURE

We map the flow domain in the f - plane conformally and symmetrically onto the unit semicircle $\Gamma: |t| \le 1, \mathrm{Im}(t) \ge 0$ in the auxiliary t - plane (see Figure 3) by

$$f = \frac{2}{\pi}\log\left(\frac{1+t}{1-t}\right) \qquad (5)$$

This transformation maps the bottom AF onto the real diameter, and the free surfaces AB, EF and the object BE onto the circumference. At B and E where the separations occur, the behavior of the flow is similar to those of free-streamline problems, e.g. flow from an orifice (see Batchelor).[5] Therefore the free-streamline theory suggests that ζ behave at these separation points like

$$\zeta \sim \mathrm{G} + \mathrm{H}\left[f - (\pm b + i)\right]^{\frac{1}{2}} \quad \text{as } f \to \pm b + i \qquad (6)$$

Here $b + i$ and $-b + i$ denote the values of the complex potential function f at E and B respectively. The constants G and H will be determined as part of the solution.

We use the notation $t = r\,e^{i\sigma}$ so that the free surfaces and the object are described by $r = 1$ and $0 \le \sigma \le \pi$. The points $t = e^{i\beta}$ and $t = e^{-i\beta}$ correspond to the separation points E and B respectively. The appropriate behaviors of the flow near E and B can now be expressed, in terms of t, as

$$\zeta \sim \mathrm{G} + \mathrm{H}\left[t^2 - 2t\cos\beta + 1\right]^{\frac{1}{2}} \quad \text{as } t \to e^{i\beta} \qquad (7)$$

$$\zeta \sim \mathrm{G} + \mathrm{H}\left[t^2 + 2t\cos\beta + 1\right]^{\frac{1}{2}} \quad \text{as } t \to e^{-i\beta} \qquad (8)$$

We now define the analytic function $\Omega(t)$ by the relation

$$\zeta(t) = e^{\Omega(t)} \tag{9}$$

From (7), (8), and the symmetry of the problem, we find that $\Omega(t)$ has the expansion

$$\Omega(t) = B_1\left[(t^2+1)^2 - (2t\cos\beta)^2\right]^{\frac{1}{2}} - B_1\left[4 - 4\cos^2\beta\right]^{\frac{1}{2}} + \sum_{n=1}^{\infty} a_n\left(t^{2n}-1\right) \tag{10}$$

The kinematic condition (3) on the bottom AF is satisfied by requiring the coefficients a_n in the infinite series expansion (10) and B_1 to be real. It can be checked that (9) satisfies (7) and (8). Therefore we expect the series in (10) to converge for $|t|<1$. The coefficients a_n and the constant B_1 must be determined so that (9) satisfies the free surface condition (2) and the kinematic condition (4).

It is convenient for calculation purposes to rewrite (2) and (4) in the form

$$u(\sigma)u_\sigma(\sigma) + v(\sigma)v_\sigma(\sigma) = 0 \tag{11}$$

and

$$v(\sigma) = \varepsilon u(\sigma) \int_{\pi-\beta_1}^{\beta_1} \frac{u}{u^2+v^2}\frac{df}{d\sigma}\,d\sigma \quad , \beta \le \beta_1 \le \pi/2. \tag{12}$$

In the numerical procedure we take advantage of the symmetry of the problem with respect to the y-axis by restricting the collocation points to the circular arc $t = e^{i\sigma}$, $0 \le \sigma \le \pi/2$. The infinite series in (10) is truncated after $N + 1$ terms. We introduce the $N + 2$ collocation points

$$\sigma_i = \frac{\pi}{2(N+2)}\left(i - \frac{1}{2}\right) \quad , i = 1, \ldots, N+2 \tag{13}$$

For simplicity, we assume values of β of the form

$$\beta = \frac{\pi}{2(N+2)} M \tag{14}$$

where M is an integer smaller than $N + 2$. Using (9) we obtain u, v, u_σ, v_σ at the mesh points σ_i in terms of the coefficients a_n and B_1. Substituting these expressions into (11) and (12), we obtain $N + 2$ nonlinear algebraic equations for the unknowns $\{a_n\}_{n=1}^{N+1}$ and B_1. We solve this system of equations by Newton's method. Once it is solved, we obtain the shape of the free surface by integrating numerically the identity

$$\frac{\partial x}{\partial \phi} + i\frac{\partial y}{\partial \phi} = \frac{1}{u - iv} = \frac{1}{\zeta} . \tag{15}$$

IV DISCUSSION OF THE RESULTS

The numerical scheme described in Sec. III was used to compute solutions for various values of β and ε. As n increases, the coefficients a_n decrease rapidly. Table 1 shows some of these coefficients for $\beta = 2\pi/3$ and for various values of ε. Most of the calculations were performed with $N = 180$.

ε	a_1	a_{10}	a_{100}	a_{180}
-1.0	0.19	0.23×10^{-2}	-0.31×10^{-4}	0.17×10^{-5}
-0.5	-0.61×10^{-1}	0.13×10^{-2}	-0.56×10^{-4}	0.50×10^{-5}
0.5	-0.37×10^{-1}	-0.10×10^{-2}	0.16×10^{-4}	-0.31×10^{-5}
1.0	-0.79×10^{-1}	-0.21×10^{-2}	0.23×10^{-4}	-0.53×10^{-5}

Table 1. Numerical values of the coefficients a_n coefficients for $\beta = 2\pi/3$.

Typical profiles are presented in Figures 4 (a) and (b). As $\varepsilon \to 0$, the object approaches a flat plate and the flow reduces to a uniform stream for all values of β. For $\beta = \pi/2$, the solutions correspond to a uniform stream with no object.

Following Asavanant and Vanden-Broeck (1994), we define the amplitude parameter

$$\alpha = \frac{W}{H} . \tag{16}$$

Here W is the distance from the bottom AF to the vertex of the object. Values of $\alpha - 1$ versus ε, for various values of β, are shown in Figure 5. As we can see, the solutions depend on two parameters β and ε. The computed free-surface profiles are found to be similar to those obtained for large values of the Froude number by Asavanant and Vanden-Broeck[1] for large values of Froude number.

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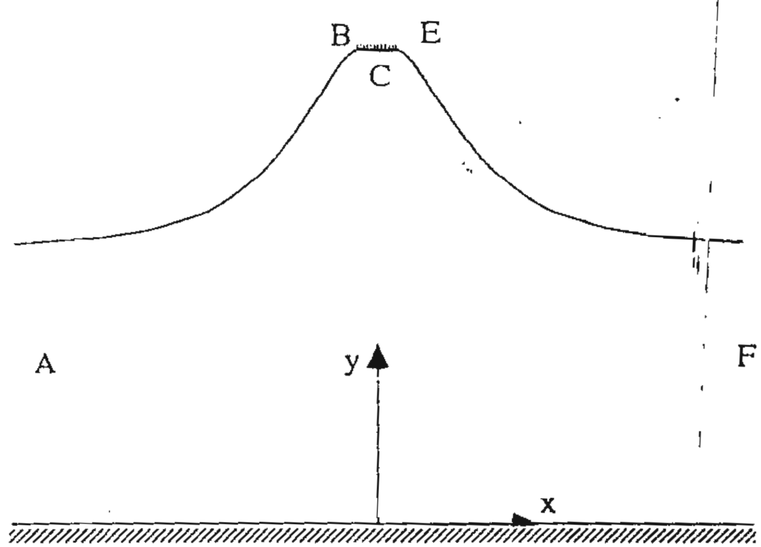

Figure 1 Sketch of the flow domain and of the coordinate system. Here C denotes the vertex of the object.

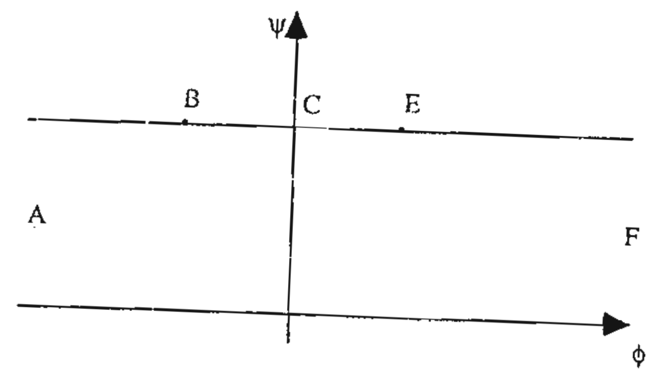

Figure 2 Flow in the f – plane.

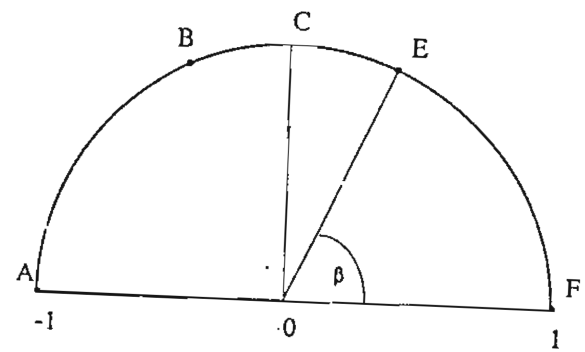

Figure 3 Flow in the t – plane.

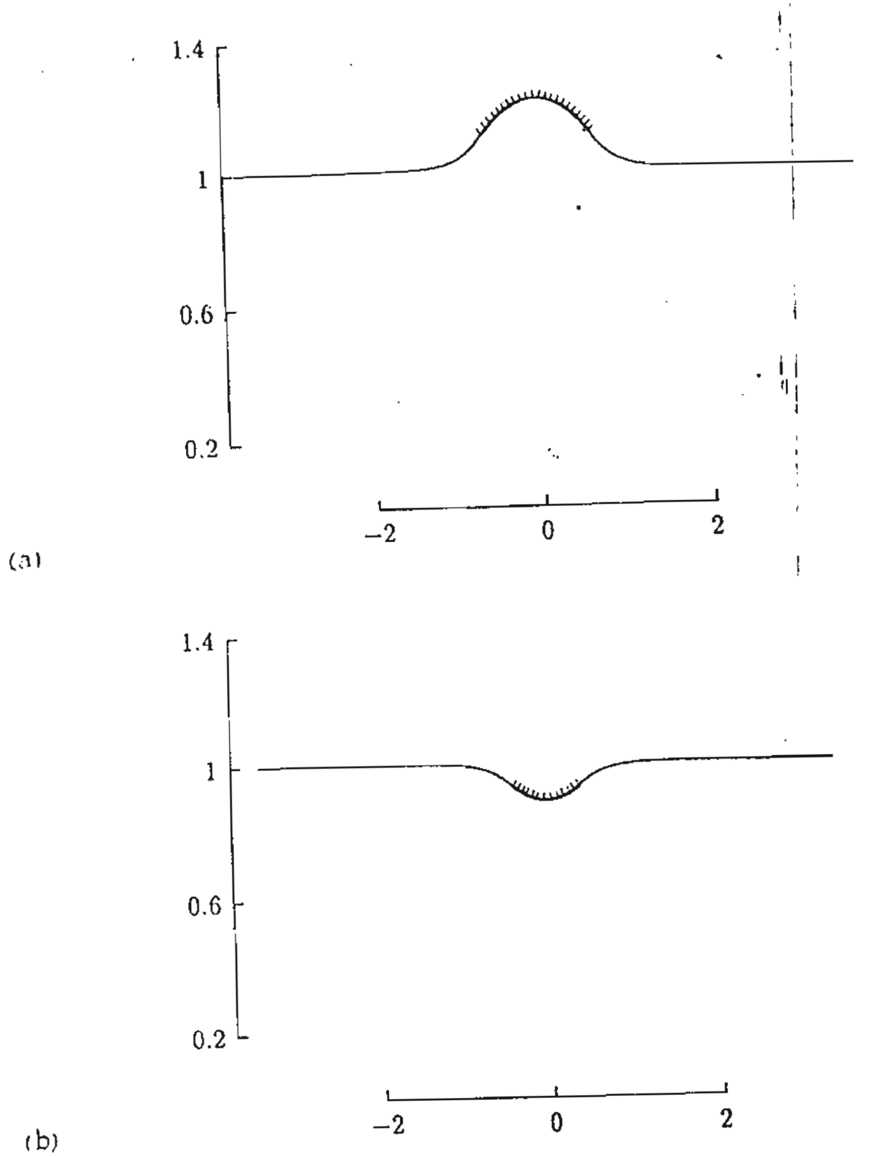

Figure 4 Computed free surface profiles for flow in the absence of gravity

(a) $\varepsilon = -0.5$ and $\beta = \dfrac{5\pi}{9}$

(b) $\varepsilon = 0.5$ and $\beta = \dfrac{5\pi}{9}$.

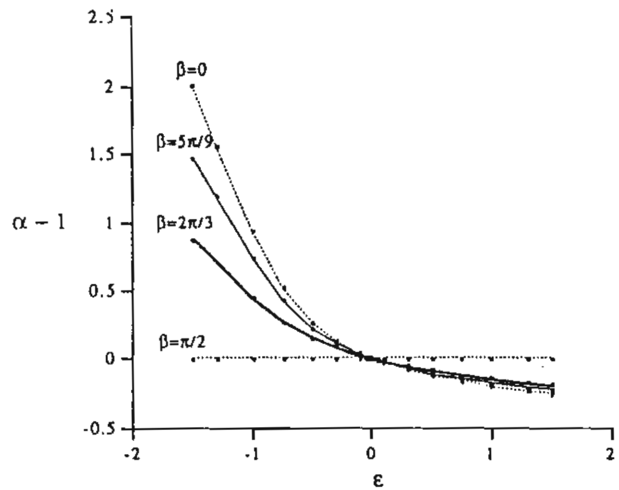

Figure 5 Values of $\alpha - 1$ versus ε for four values of β. The horizontal line for $\beta = \pi/2$ corresponds to a uniform stream ($\alpha - 1 = 0$) with no object, the middle curves for $\beta = 2\pi/3$ and $5\pi/9$ correspond to objects of finite length, and the curve for $\beta=0$ corresponds to a semi-infinite object.

Appendix B

Two-layer flows over an obstruction

Main results published in *Journal of Science Society of Thailand*

RESEARCH ARTICLES

SYMMETRIC WAVES OF A TWO-LAYER FLUID WITH FREE SURFACE OVER AN UNEVEN BOTTOM

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ABSTRACT

In this paper we study steady two dimensional waves in a two-layer fluid bounded above by a free surface and below by a horizontal rigid boundary with a small obstruction. Two critical speeds for the waves are obtained. Near the smaller critical speed, the derivation of the usual forced KdV equation (FKdV) fails when the coefficient of the nonlinear term in the FKdV vanishes. To overcome this difficulty, a new equation called a Steady Modified KdV equation with forcing term (SFMKdV) governing interfacial wave forms is obtained by a refined asymptotic method. By using SFMKdV we find the traveling solition-like solutions and symmetric wave solutions for different choices of parameters. Existence theorems are proved and numerical results of this equation are presented.

1. INTRODUCTION

This paper concerns the symmetric wave solutions between two immiscible, inviscid, and imcompressible fluids of different but constant densities in the presence of small elliptic obstruction of compact support at the rigid bottom when the effect of gravity is considered (Fig. 1). We assume that the upper boundary is a free surface and the two dimensional obstruction is moving along the lower rigid boundary at a constant speed. By choosing a coordinate system moving with the object, the fluid motion becomes steady. Two critical speeds are obtained, near either one of which an FKdV for steady flow can be derived and has been studied extensively in [1] and [2]. Forbes [3], Belward and Forbes [4], Sha and Vanden-Broeck [5], and Mohi and King [6] studied numerically steady flow of a two layer fluid over a bump or a step bounded by a free surface and a rigid boundary. An asymptotic approach for the case of a rigid upper boundary was developed without surface tension by Shen [7] on the basis of FKdV theory, and with surface tension by Choi *et al.* [8]. The case of free upper boundary was studied with surface tension by Choi *et al.* [9] asymptotically on the basis of EKdV theory. In the case considered here, when the wave speed is near the smaller critical speed for internal wave, the nonlinear term in the FKdV may vanish and the derivation of FKdV fails. To overcome this difficulty. a refined asymptotic method is used to derive the Steady Modified KdV equation with forcing term (SFMKdV) in the following form:

$$(A\eta_2^2 + B)\eta_{2x} + C\eta_{2xxx} + Db_x = 0,$$

where A to D are constants depending on several parameters and $b(x)$ is a function with compact support due to the obstruction on the rigid lower boundary. We investigate solutions of the SFMKdV, which represent possible interfacial wave forms.

In section 2, we formulate the problem and develop the asymptotic scheme to derive the SFMKdV. In section 3, existence theorems are proved and numerical solutions of soliton-like solutions and symmetric wave solutions are presented for different values of parameters. The parameters are determined along density ratios of the two fluids, depth ratio of the two fluids, and perturbation of the horizontal velocity at far upstream.

2. FORMULATION AND SUCCESSIVE APPROXIMATE EQUATIONS

We consider steady internal gravity waves between two immiscible, inviscid and incompressible fluids of constant but different densities bounded above by a free surface and below by a horizontal rigid boundary with a small obstruction of compact support. The domains of the upper fluid with a constant density ρ^{*+} and the lower fluid with a constant density ρ^{*-} are denoted by Ω^{*+} and Ω^{*-} respectively (Fig. 1). Assume that the small obstruction is moving with a constant speed C. In reference to a coordinate system moving with the obstruction, the flow is steady and moving with the speed C far upstream. The governing equations and boundary conditions are given by the following Euler equations:

In $\Omega^{*\pm}$,

$$u^{*\pm}_{x^*} + v^{*\pm}_{y^*} = 0,$$

$$u^{*\pm}u^{*\pm}_{x^*} + v^{*\pm}u^{*\pm}_{y^*} = -p^{*\pm}_{x^*}/\rho^{*\pm},$$

$$u^{*\pm}v^{*\pm}_{x^*} + v^{*\pm}v^{*\pm}_{y^*} = -p^{*\pm}_{y^*}/\rho^{*\pm} - g\,;$$

at the free surface, $y^* = h^{*+} + \eta^*_1$,

$$u^{*+}\eta^*_{1x^*} - v^{*+} = 0,$$

$$p^{*+} = 0;$$

at the interface, $y^* = \eta^*_2$,

$$p^{*+} - p^{*-} = 0,$$

$$u^{*\pm}\eta^*_{2x^*} - v^{*\pm} = 0;$$

at the rigid bottom, $y^* = -h^{*-} + b^*(x^*)$

$$v^{*-} - b^*_x u^{*-} = 0$$

where $u^{*\pm}$ and $v^{*\pm}$ are horizontal and vertical velocities, $p^{*\pm}$ are pressures, g is the gravitational acceleration constant. We define the following nondimensional variables:

$$\varepsilon = H/L << 1,\ \eta_1 = \varepsilon^{-1}\eta^*_1/h^{*-},\ \eta_2 = \varepsilon^{-1}\eta^*_2/h^{*-},\ p^{\pm} = p^{*\pm}/gh^{*-}\rho^{*-},$$

$$(x,y) = (\varepsilon x^*, y^*)/h^{*-},\ (u^{\pm}, v^{\pm}) = (gh^{*-})^{-1/2}(u^{*\pm}, \varepsilon^{-1}v^{*\pm}),$$

$$\rho^{+} = \rho^{*+}/\rho^{*-} < 1,\ \rho^{-} = \rho^{*-}/\rho^{*-} = 1,\ \mathrm{U} = \mathrm{C}/(gh^{*-})^{1/2},$$

$$h = h^{*+}/h^{*-},\ b(x) = b^*(x^*)(h^{*-}\varepsilon^3)^{-1},$$

where L is the horizontal scale, H is the vertical scale, $b(x) = b^*(x^*)(h^*\varepsilon^3)^{-1}$, h^{*-} and h^{*+} are the equilibrium depths of the upper and lower fluids at $x^* = -\infty$ respectively, and $y^* = -h^{*-} + b^*(x^*)$ is the equation of the obstruction. In terms of the nondimensional quantities, the above

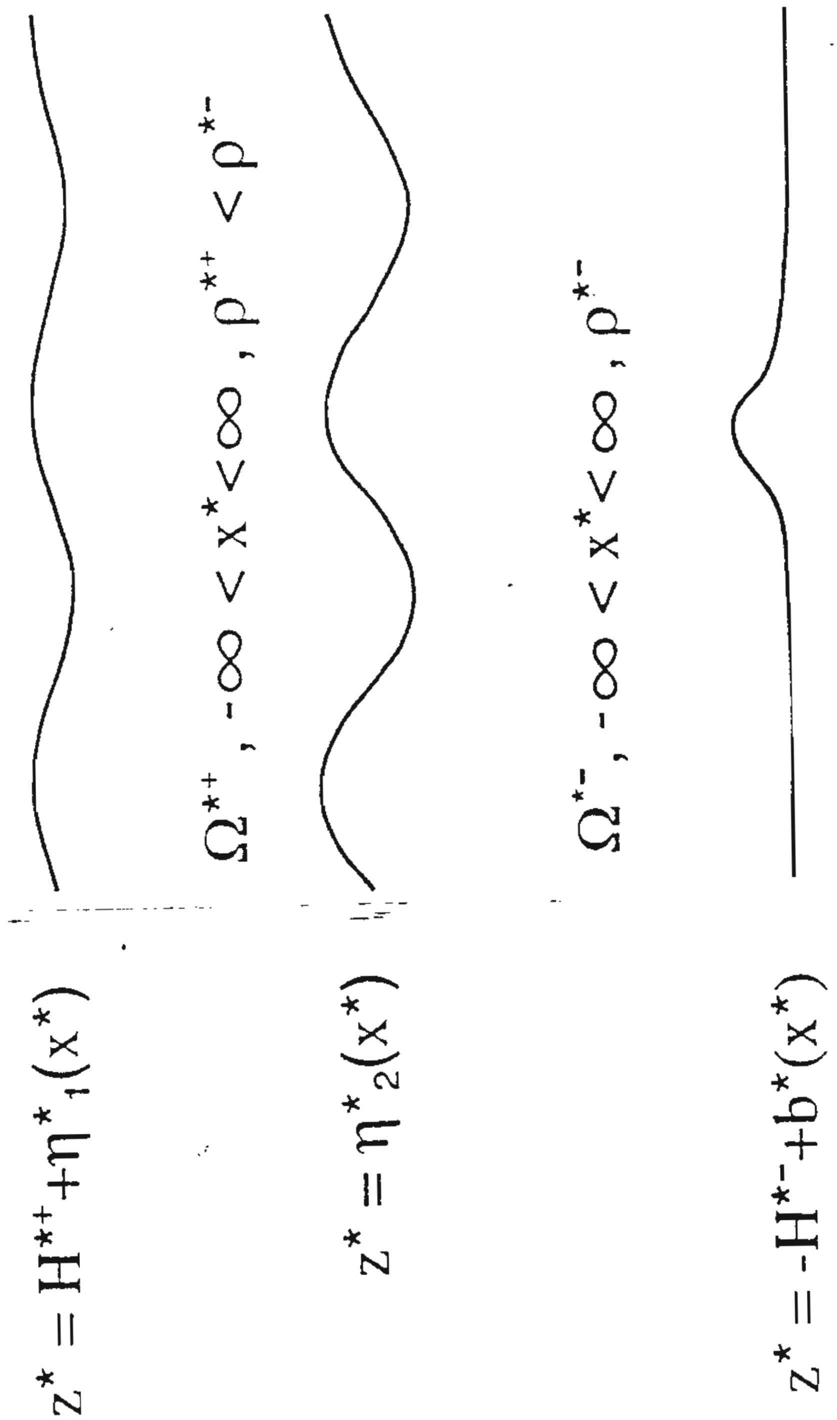

Fig. 1. Fluid Domain

equations become in $\Omega^{\pm}$,

$$u_x^{\pm} + v_y^{\pm} = 0, \tag{1}$$

$$u^{\pm}u_x^{\pm} + v^{\pm}u_y^{\pm} = -p_x^{\pm}/\rho^{\pm}, \tag{2}$$

$$\varepsilon^2 u^{\pm}v_x^{\pm} + \varepsilon^2 v^{\pm}v_y^{\pm} = -p_y^{\pm}/\rho^{\pm} - 1; \tag{3}$$

at $y = h + \varepsilon\eta_1$,

$$p^{+} = 0 \tag{4}$$

$$\varepsilon u^{+}\eta_{1x} - v^{+} = 0; \tag{5}$$

at $y = \varepsilon\eta_2$,

$$\varepsilon u^{-}\eta_{2x} - v^{-} = 0; \tag{6}$$

$$\varepsilon u^{+}\eta_{2x} - v^{+} = 0; \tag{7}$$

$$p^{+} - p^{-} = 0; \tag{8}$$

at $y = -1 + \varepsilon^3 b(x)$,

$$v^{-} = \varepsilon^3 u b_x\,, \tag{9}$$

where $b(x)$ has a compact support.

In the following, we use a unified asymptotic method to derive the equations for $\eta_1(x)$ and $\eta_2(x)$. We assume that $u^{\pm}$, $v^{\pm}$, and $p^{\pm}$ are functions of x, y near the equilibrium state $u^{\pm} = u_0$, $v^{\pm} = 0$, $p^{+} = -\rho^{+}y + \rho^{+}h$ and $p^{-} = -\rho^{-}y + \rho^{+}h$, where u_0 is a constant, and possess asymptotic expansions:

$$\begin{aligned}(u^{\pm}, v^{\pm}, p^{\pm}) &= (u_0, 0, -\rho^{\pm}y + \rho^{+}h) + \varepsilon(u_1^{\pm}, v_1^{\pm}, p_1^{\pm}) \\ &\quad + \varepsilon^2(u_2^{\pm}, v_2^{\pm}, p_2^{\pm}) + \varepsilon^3(u_3^{\pm}, v_3^{\pm}, p_3^{\pm}) + O(\varepsilon^4).\end{aligned} \tag{10}$$

By inserting (10) into (1) to (4) and (7) to (9) and arranging the resulting equations according to the powers of ε, it follows that $(u_0, 0, -\rho^{\pm}y + \rho^{+}h)$ are the solutions of the zeroth order system of equations and the equations of the order ε are as follows:

$$u_{1x}^{\pm} + v_{1y}^{\pm} = 0, \tag{11}$$

$$u_0 u_{1x}^{\pm} = -p_{1x}^{\pm}/\rho^{\pm}, \tag{12}$$

$$p_{1y}^{\pm} = 0; \tag{13}$$

at $y = h$,

$$p_1^{+} + \eta_1 p_{0y}^{+} = 0; \tag{14}$$

at $y = 0$,

$$(p_1^{+} - p_1^{-} + \eta_2(p_{0y}^{+} - p_{0y}^{-}) = 0; \tag{15}$$

$$u_0\eta_{2x} - v_1^{+} = 0; \tag{16}$$

at $y = -1$,

$$v_1^{-} = 0. \tag{17}$$

Hereafter for the sake of convenience we shall use ρ to denote ρ^{-} and set ρ^{-} equal to 1. From (13), $p_1^{\pm}$ are functions of x only. $p_1^{+} = \rho\eta_1$ by (14) and $p_1^{-} = \rho\eta_1 + \eta_2(1 - \rho)$ by (15). We can find $v_1^{\pm}$ by using (11), (12), (15), and (17) so that

$$v_1^+ = y(\eta_{1x}/u_0) + u_0\eta_{2x}, \qquad (18)$$
$$v_1^- = (y+1)(\rho\eta_{1x} + (1-\rho)\eta_{2x})/u_0 .$$

$u^\pm_1$ are also derived from (11)

$$u_1^+ = -\eta_1/u_0, \qquad (19)$$
$$u_1^- = (-\rho\eta_1 - (1-\rho)\eta_2)/u_0,$$

where we assume $\eta_1(x=-\infty) = \eta_2(x=-\infty) = 0$, $u_1^\pm(x=-\infty) = 0$.

Similarly, we can find $p^\pm_2$, $v^\pm_2$, $u^\pm_2$, $p^\pm_3$, $v^\pm_3$, $u^\pm_3$ in terms of η_1 and η_2 without using the kinematic conditions (5) and (6). From (5) and (6), and the asymptotic expansions of u^- and v^-, we have

at $y = h$,

$$\begin{aligned} &u_0\eta_{1x} - v_1^+ + \varepsilon(u_1^-\eta_{1x} - \eta_1 v_{1y}^+ - v_2^+) \\ &+\varepsilon^2(u_2^+\eta_{1x} + \eta_1\eta_{1x}u_{1y}^+ - v_{1yy}^+\eta_1^2 - \eta_1 v_{2y}^+ - v_3^+) + O(\varepsilon^3) = 0, \end{aligned} \qquad (20)$$

and at $y = 0$,

$$\begin{aligned} &u_0\eta_{2x} - v_1^- + \varepsilon(u_1^-\eta_{2x} - \eta_2 v_{1y}^- - v_2^-) \\ &+\varepsilon^2(u_2^-\eta_{2x} + \eta_2\eta_{2x}u_{1y}^- - v_{1yy}^-\eta_2^2 - \eta_2 v_{2y}^- - v_3^-) + O(\varepsilon^3) = 0. \end{aligned} \qquad (21)$$

Then we make use of these equations to find the equations of the free surface $\eta_1(x)$ and the interface $\eta_2(x)$. By substituting u_0, $u^\pm_1$, $v^\pm_1$, $u^\pm_2$, $v^\pm_2$, $v^\pm_3$ into (20) and (21) and eliminating η_1, we obtain

$$\begin{aligned} &(u_0 - \rho c_1/u_0 - (1-\rho)/u_0)\eta_{2x} + \varepsilon(E\eta_2\eta_{2x} + E_2\eta_{2x}) \\ &+ \varepsilon^2(F_1\eta_2^2\eta_{2x} + F_2\eta_{2x} + F_3\eta_{2xxx} + F_4 b_x) \\ &+ O(\varepsilon^3) = 0, \end{aligned} \qquad (22)$$

where if we let $c_1 = (2u_0^2 - (1-\rho))/(\rho + u_0^2 - h)$, $D_1 = u_0/(\rho + u_0^2 - h)$, $\lambda = u_2^\pm(-\infty)$, and $R = \rho c_1 + 1 - \rho$, then

$$E = -(R^2 + 2Ru_0^2)u_0^{-3} - \rho D_1((hc_1^2 - R^2)u_0^{-4} + (2c_1^2 - 2R - 2c_1)u_0^{-2}),$$

$$\begin{aligned} F_1 = \; & -\rho D_1 u_0^{-1}((3c_1^3 - 3c_1^2 + R^2/2)u_0^{-3} + (3hc_1^3/2 - 3R^3/2)u_0^{-5} \\ & + 3D_1(\rho u_0^{-1} + \rho R u_0^{-3})((3R/2 + c_1 - c_1^2)u_0^{-1} + (R^2/2 - hc_1^2/2)u_0^{-3}) \\ & - 3R^2u_0^{-3}/2 - 3R^3u_0^{-5}/2, \end{aligned}$$

$$F_2 = \lambda((-\rho D_1 u_0^{-1})(2 + Ru_0^{-2} - c_1 - hc_1u_0^{-2}) + (1 + Ru_0^{-2})),$$

$$\begin{aligned} F_3 = \; & (-\rho D_1 u_0^{-1})(-c_1(\rho h^2/2 + \rho/3)u_0^{-1} - (u_0^2\rho h + (1-\rho)/3)u_0^{-1} \\ & + (c_1(\rho h^3/3)/u_0\rho) + u_0h^2/2) \\ & - c_1(\rho h^2/2) + \rho/3)u_0^{-1} - (u_0^2\rho h + (1-\rho)/3)u_0^{-1}, \end{aligned}$$

$$F_4 = \rho D_1 - u_0 .$$

3. STEADY MODIFIED KdV EQUATION WITH FORCING (SFMKdV)

From the zeroth order term of (22), we obtain

$$u_0 - (\rho c_1/u_0) - (1-\rho)/u_0 = 0 ,$$

and by the expression for c_1 in (22), it follows that

$$u_0^4 - (1+h)u_0^2 + h(1-\rho) = 0 , \tag{23}$$

and

$$u_0^2 = (1 + h \pm ((1-h)^2 + 4\rho h)^{1/2})/2.$$

We denote the two values of u_0^2 by u_{01}^2 and u_{02}^2 respectively corresponding to the plus and minus signs. Without loss of generality we assume u_{01} and u_{02} are both positive and call them critical speeds, near each of which a nonlinear theory for the motion of the interface has to be developed.

Next we consider the coefficients of $\eta_2\eta_{2x}$ in the first order terms of the equation (22). If E in (22) is not zero, an FKdV can be derived if we assume $b(x) = b^*(x^*)(h^{*-}\varepsilon^2)^{-1}$ and $x = \varepsilon^{1/2}x^*/h^{*-}$ in nondimensional variables and similar results as in [1] can be obtained. However, E may vanish. First, let us simplify the expression of E,

$$\begin{aligned} E &= -((\rho c_1 + 1 - \rho)^2/u_0^3) - 2((\rho c_1 + 1 - \rho)/u_0) \\ &\quad - \rho D_1[-2((\rho c_1 + 1 - \rho)/u_0) - ((\rho c_1 - \rho + 1)^2/u_0^3) + 2(c_1^2/u_0) \\ &\quad + h(c_1^2/u_0^3) - 2(c_1/u_0)]/u_0 \\ &= 3(u_0\rho)^{-1}(u_0^2 + \rho - h)(\rho(u_0^2 h - u_0^4 - u_0^2 + 1) - u_0^4 + 2u_0^2 - 1), \\ &= 3u_0(1 - u_0^2)(\rho h(u_0^2 + \rho - h))^{-1}(u_0^4 + (1-2h)u_0^2 + h^2 - 1). \end{aligned}$$

where (23) has been used. When u_0 satisfies the equation (23), it is seen that u_0^2 is neither 1 nor $h - \rho$. Hence $E = 0$ implies $u_0^4 + (1-2h)u_0^2 + h^2 - 1 = 0$. Let $u_0 = u_{01}$ or u_{02}. Then

$$u_{01}^4 + (1-2h)u_{01}^2 + h^2 - 1 = 1 + h\rho + (2-h)((1-h)^2 + 4h\rho)^{1/2} , \tag{24}$$

$$u_{02}^4 + (1-2h)u_{02}^2 + h^2 - 1 = 1 + h\rho - (2-h)((1-h)^2 + 4h\rho)^{1/2} , \tag{25}$$

Equation (24) tells us that E does not vanish if we take u_{01} as a critical speed. Suppose both sides of (24) vanish. Then real u_{01}^2 imples $h < 5/4$ and the right hand side of (24) is greater than zero. This is a contradiction. Thus the only possible case for $E = 0$ is that the critical speed u_0^2 is equal to u_{02}^2, and it is easy to show that $E = 0$ if $u_0^2 = u_{02}^2$, and

$$1 + h\rho = (2-h)((1-h)^2 + 4h\rho)^{1/2}. \tag{26}$$

With the conditions (21) and (25), we obtain a Steady FMKdV,

$$F_1\eta_2^2\eta_{2x} + F_2\eta_{2x} + F_3\eta_{2xxx} + F_4 b_x = 0 \tag{27}$$

where

$$F_1 = 3u_0(4\rho + 3h - u_0^2),$$
$$F_2 = \lambda(2(1+h)u_0^2 - 4h(1-\rho))u_0^{-2},$$
$$F_3 = u_0^{-1}(h(1+h) - u_0^2(h^2 + 1 + 3\rho h),$$
$$F_4 = u_0(h - u_0^2).$$

The coefficients F_1 to F_4 here are the simplified forms of F_1 to F_4 in the previous section by using (23). The sign of F_3F_1 determines the existence of solutions of (27). In the following sections, we assume $F_3F_1 > 0$ and the case for $F_3F_1 < 0$ is considered in subsequent study [10].

3.1 Symmetric solition-like waves

We assume the speed $U^{\pm}$ of the fluid at $x = \pm\infty$ are the same and given by $U = u_0 + \lambda\varepsilon^2 + O(\varepsilon^3)$ and consider (27) $F_1/F_3 > 0$ and $F_2/F_3 < 0$. (27) can be rewritten as

$$\eta_{2xxx} = -A_1\eta_2^2\eta_{2x} + A_2\eta_{2x} + A_3b_x, \tag{28}$$

where $A_1 = F_1/F_3 > 0$, $A_2 = -F_2/F_3 > 0$, $A_3 = -F_4/F_3$. Here λ is a parameter determining the flow regime, eg. $\lambda > 0$ and $\lambda < 0$ represent the supercritical and subcritical cases respectively. Where $b_x \equiv 0$, (28) has soliton solutions whose value is 0 at $x = \pm\infty$ for $A_2 \geq 0$:

$$\eta_2(x) = \pm(6A_2/A_1)^{1/2}\text{sech}((A_2)^{1/2}x), \tag{29}$$

For $A_2 \leq 0$, there is no soliton solution. The solutions in (29) are obtained as in the classical case by taking the limit of elliptic functions in the periodic solutions of (28) for $b_x = 0$ when the wave length tends to infinity. Next we consider (28) when $b_x \neq 0$ but of compact support.

We look for a solution $\eta_2(x)$ such that $A_2 > 0$ and

$$\lim_{|x|\to\infty}(d/dx)^j\eta_2(x) = 0 \qquad j = 0, 1, 2.$$

Integrating (28) from $-\infty$ to x, it follows that

$$A_2\eta_2 - \eta_{2xx} = A_1\eta_2^3/3 - A_3b(x), \quad \infty < x < \infty. \tag{30}$$

(30) can be converted to the following integral equation:

$$\eta_2(x) = \int_{-\infty}^{\infty}K(x,\xi)(A_1\eta_2^3(\xi)/3 - A_3b(\xi))\,d\xi,$$

where $K(x,\xi) = \exp(-\sqrt{A_2}\,|\,x-\xi\,|)/(2\sqrt{A_2})$ is a Green function of $A_2K(x,\xi) - K_{xx}(x,\xi) = \delta(x,\xi)$, $-\infty < x < \infty$.

Define

$$T(\eta_2) = \int_{-\infty}^{\infty}K(x,\xi)(A_1\eta_2^3(\xi)/3 - A_3b(\xi))\,d\xi,$$
$$\|u\| = \|u\|_\infty = \sup_{x\in\Re}|\eta_2(x)|,$$
$$H = \{u\,|\,u \in C(\Re), \|\exp(\sqrt{A_2}\,|x|)u\| < \infty\},$$

$$B_M = \{u \mid u \in H, \|u\| \le M, 0 < M < \infty\}.$$

Then clearly H is a complete metric space and B_M is a closed ball in H, and the following theorem can be proved by Contraction Mapping Theorem[8].

Theorem 1. (30) has a solution in $C^3(\Re)$ which decays exponentially at $|x| = \infty$ if A_2 is sufficiently large.

We have shown that (28) has an exponentially decaying solution as x tends to $\pm\infty$. In the following we use numerical computation to find symmetric soliton-like solutions of (28) when the obstruction $b(x)$ is given by $b(x) = R(1 - x^2)^{1/2}$ for $|x| \le 1$ and $b(x) = 0$ for $|x| > 1$, where R is a given constant.

Let

$$\eta_2(x) = \pm(6A_2/A_1)^{1/2}\text{sech}((A_1)^{1/2}(x - x_0)), \qquad \textbf{(31)}$$

where x_0 is a phase shift. To find a solution in $|x| \le 1$, we need only consider (30) in $-1 \le x \le 0$ subject to $(\eta_2'(x))^2 = -A_1\eta_2^4/6 + A_2\eta_2^2$ at $x = -1$ and $\eta_2'(x) = 0$ at $x = 0$. This problem can be solved numerically by a shooting method and the phase shift x_0 is determined by (31) for $x = -1$. There are three parameters involved in this analysis: the depth ratio h, the perturbation of the horizontal velocity at far upstream λ and the density ratio ρ. The numerical results are given in Fig. 2 and Fig. 3. Since solutions for different values of h and ρ are qualitatively similar, we choose $h = 0.98$ and $r = 0.25$ in all calculations. Four typical soliton-like solutions are shown in Fig. 2. Fig. 3 shows the relation between the value of soliton-like solution at $x = 0$ as a function of λ. In both numerical results, we assume $R = 1$.

We remark that the shooting method for two-point boundary value problem is simple. The differential equation is solved as an initial value problem in some form over the given domain for a succession of trial values of η which are adjusted till the boundary conditions at both ends can be satisfied at once. The simplest way to do is to shoot from one end to the other, that is to say we choose η such that the left-end boundary is satisfied. The second trial for shooting is done with the corrected value of η which is adjusted according to the miss-distance from the first shooting. We repeat the process until η satisfies the right-end boundary condition.

3.2 SYMMETRIC WAVES WITH ZERO BEHIND AND AHEAD OF THE OBSTRUCTION

Similar to section 3.1, we consider the equation

$$\eta_{2xxx} = -A_1\eta_2^2\eta_{2x} + A_2\eta_{2x} + A_3 b_x, \qquad \textbf{(32)}$$

where $A_1 = F_1/F_3 > 0$, $A_2 = -F_2/F_3 > 0$, $A_3 = -F_4/F_3$. Integrating (32) from $-\infty$ to x, we obtain

$$\eta_{2xx} = -A_1\eta_2^3/3 + A_2\eta_2 + A_3 b(x), \qquad \textbf{(33)}$$

where $b(x)$ is assumed to have compact support and $\eta_2(-\infty) = 0$. We assume $\eta_2 \equiv 0$ in $(-\infty, x_-)$ where $[x_-, x_-]$ is the support of the obstruction. We can show that the solution of (33) exists and is bounded with initial values $\eta_2(x_-) = \eta_{2x}(x_-) = 0$. [8] In the following, we use numerical computation to find symmetric wave solution of (33) which is zero behind and

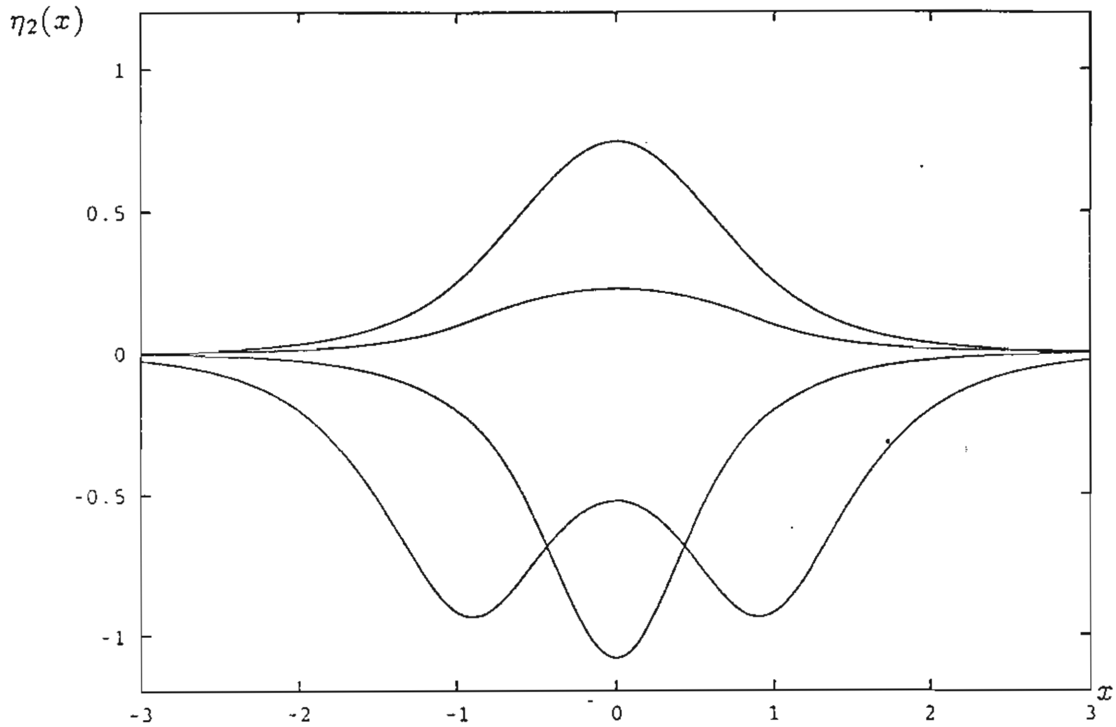

Fig. 2. Four typical soliton-like solution.
h = 0.98, R = 1, λ = -4

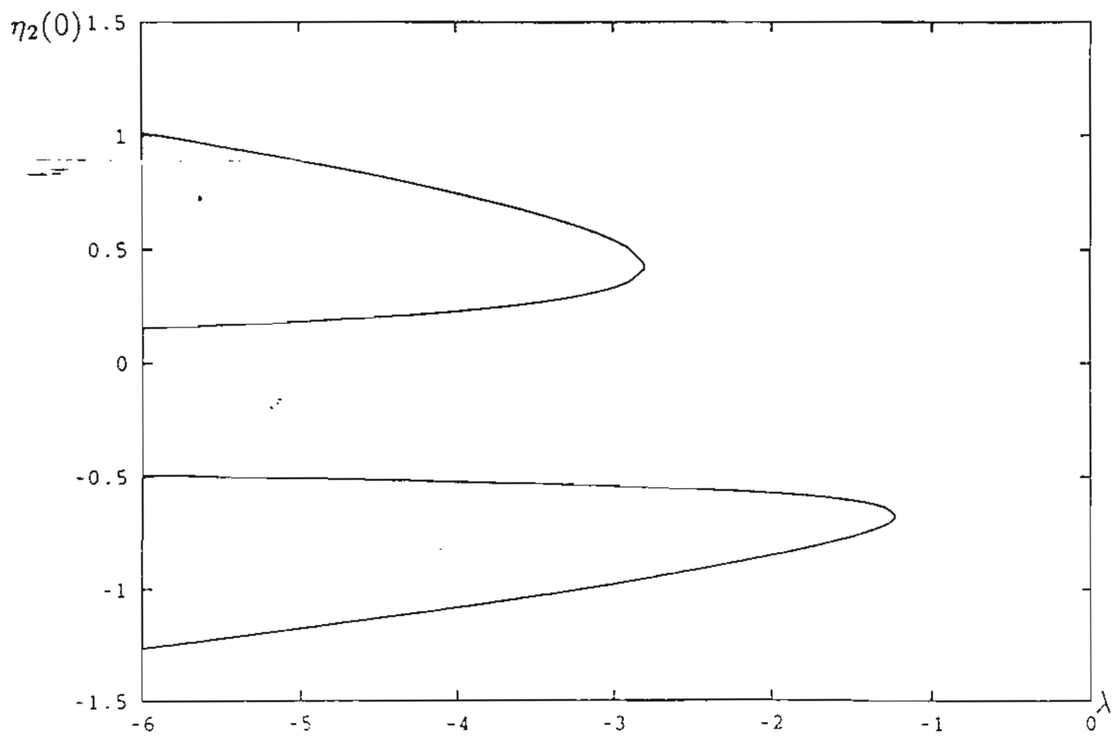

Fig. 3. Relationship between $\eta_2(0)$ and λ
h = 0.98, R = 1

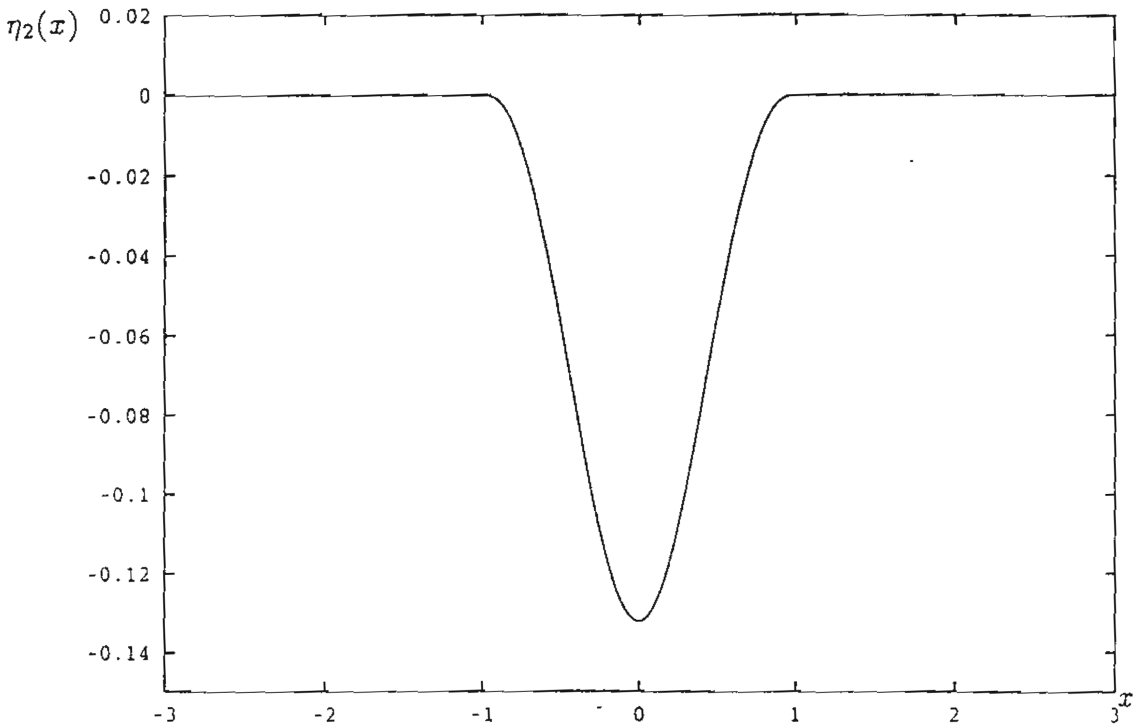

Fig. 4. Symmetric solution with one hump.
$h = 0.98$, $R = 1$, $\lambda = 14.000283$

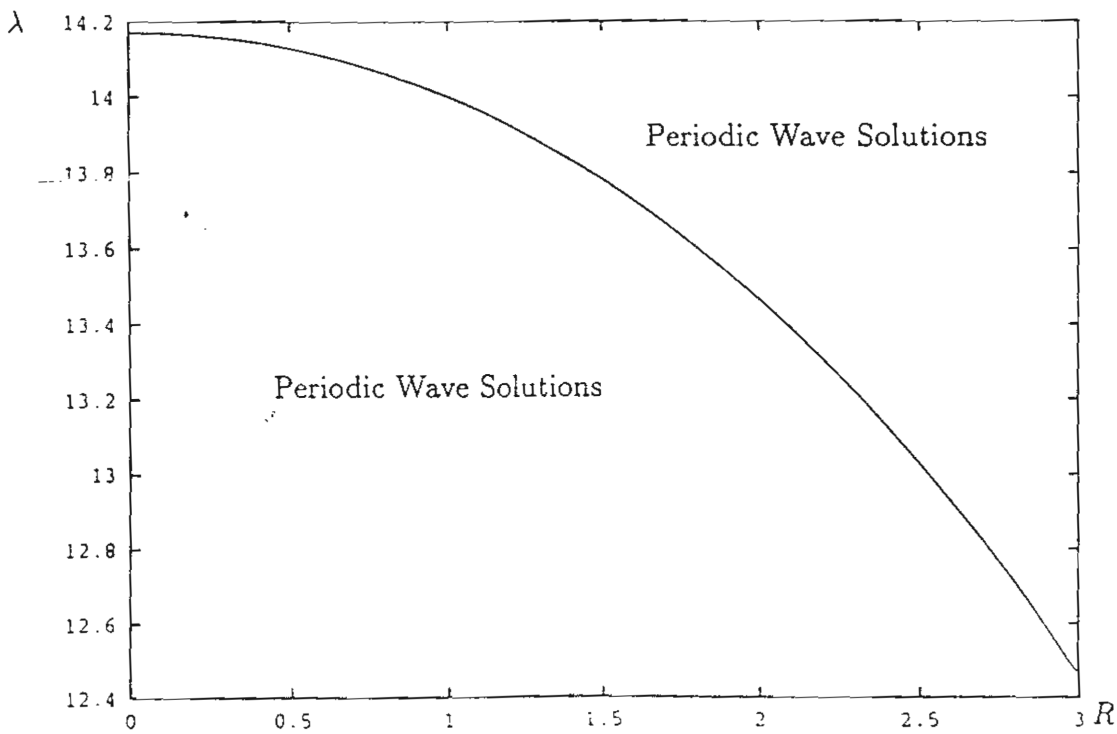

Fig. 5. Solution curve of symmetric solutions with one hump. $h = 0.98$

ahead of the elliptic obstruction. Similar methods as in section 3.1 is used to find the solutions of this problem. To find a solution in $|x| \leq 1$, we need only consider (33) in $-1 \leq x \leq 0$ subject to $\eta_2'(x) = \eta_2(x) = 0$ at $x = -1$ and $\eta_{2x}(x) = 0$ at $x = 0$. The same assumptions as in section 3.1 have been made for the obstruction and the numerical results are shown in Fig. 4 and 5. Fig. 4 shows the symmetric solutions for positive values of λ. The relations between R, which represents the height of the obstruction, and λ are given in Fig. 5. We note that, for a given R, symmetric solution is embedded in periodic solutions.

We have shown that there exist two types of symmetric solutions of the interfacial wave forms both analytically and numerically. First type of solutions depends strongly on the values of λ as shown in section 3.1 and corresponds to symmetric soliton-like solutions. The other type of solutions is the limiting case of the solutions with waves behind the bump and zero ahead. These correspond to symmetric wave solution with one hump.

ACKNOWLEDGEMENT

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Appendix C

Flows past a ship hull

Main results submitted to *European Journal of Mechanics B/Fluids*

FREE-SURFACE FLOW PAST A TWO-DIMENSIONAL SHIP WITH STAGNATION POINTS ON THE HULL

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Abstract

Two-dimensional free-surface potential flow past a ship in water of finite depth is considered. The ship is modelled as a flat-bottomed object with vertical faces. Gravity is included in the dynamic boundary condition. It is assumed that the intersections of the free surface with the hull are stagnation points. Numerical solutions are obtained by a series truncation procedure when the flow is supercritical in the far field. It is found that there is a two-parameter family of splashless solutions. A generalization of the problem to the case of inclined faces is also presented.

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1. Introduction

We consider the steady two-dimensional irrotational flow of an inviscid incompressible fluid past a rectangular-shaped object lying on the free surface in water of finite depth (see Figure 1). When the level of the bottom of the object is below the level of the free surface at infinity, the problem models a barge-like vessel movinng at a constant velocity in a canal. When it is above (like in Figure 1) we refer to it as a surfing flow. Observation of real ships and the theoritical considerations described in the next paragraphs suggest that there is in general a splash at the bow of the vessel. In this paper we show that there are particular solutions without splashes. We restrict our attention to supercritical flows, i.e. flows for which the Froude number

$$F = U/\sqrt{gH} \tag{1}$$

is greater than one. Here U and H are the velocity and depth in the far field and g is the acceleration of gravity.

The problem of free-surface flows past two-dimensional objects has been considered by many previous investigators. Analytical and numerical results have been proposed for different flow configurations. In water of infinite depth, Vanden-Broeck and Tuck [1], Vânden-Broeck, Schwartz and Tuck [2], and Vanden-Broeck [3] showed that there are no continuous solutions for flows past a semi-infinite two-dimensional object with a flat bottom and a vertical front such that the flow separate at a stagnation point and approach a uniform stream in the far field. The implication is that there is either a train of waves on the free surface or a splash. These two situations correspond respectively to near stern flows and near bow flows. The near stern flows were calculated in [1]–[3] and the near bow flow was constructed in [4]. Furthermore Madurasinghe and Tuck [5] constructed a near-bow flow without splash by replacing the assumption of a stagnation point with that of a smooth detachment.

Vanden-Broeck [6] generalized the flow configuration in [1]–[3] (i.e. the flow past a semi-infinite two-dimensional object with a flat bottom, a vertical front and a stagnation point) to water of finite depth. He showed that there are splashless supercritical flows. The corresponding problem with smooth detachment was considered by Hocking [7]. All the above results are for semi-infinite objects. Results for objects of finite length were obtained analytically by Craig and Sternberg [8] and numerically by Asavanant and Vanden-Broeck [9]. These authors assume that the free surfaces separate smoothly from the object.

In the present paper we examine the flow past an object of finite length when there are stagnation points at the intersection of the free surface with the hull (see Figure 1). The problem is solved by a series truncation method for arbitrary values of the width and the height of the object. The numerical procedure is similar to the one used by Vanden-Broeck and Keller [10], Vanden-Broeck [6] and Asavanant and Vanden-Broeck [11]. Our results include those obtained by Vanden-Broeck [6] as a particular case. We show that there is a two-parameter family of splashless solutions.

In section 2 we formulate the problem for the flow configuration shown in Figure 1. The numerical procedure is described in section 3. In section 4 we discuss the numerical results. The problem is generalized to a geometry with inclined faces in section 5 and concluding remarks are presented in section 6.

2. Formulation of the problem

We consider the flow configuration shown in Figure 1. The hull is assumed to have a flat bottom and two vertical faces from which the free surface separates at the stagnation points K and N. We introduce cartesian coordinates with the x-axis along the bottom, the y-axis directed vertically upwards and the origin in the middle of the bottom of the hull. Gravity is acting in the negative y-direction. The coordinate system is moving with the hull, so that the flow is steady. We assume that the flow is supercritical in the far field (i.e. $F \geq 1$). Therefore there are no waves on the free surface and the flow approaches a uniform stream with constant velocity U and uniform depth H as $|x| \to \infty$.

It is convenient to define dimensionless variables by taking U as the unit velocity and H as the unit length. We introduce the velocity potential $\phi(x, y)$ and the streamfunction $\psi(x, y)$. Next we define the complex potential $f = \phi + i\psi$ and the complex velocity by $\zeta = u - iv = df/dz$. Here u and v are the velocity components in the x and y directions, and $z = x + iy$. Without loss of generality, we choose $\phi = 0$ at the point in the middle of the bottom LM of the object and $\psi = 1$ on the free surfaces IK, NJ and on the object KLMN. It follows that $\psi = 0$ on the bottom. The flow domain in the $f-$ plane is an infinite strip (see Figure 2). We denote by $\pm a$ and $\pm b$ the values of the potential function at the corner points L and M and at the two separation points K and N respectively (see Figure 2). On the free surface, the pressure is constant and the Bernoulli equation in dimensionless form yields

$$|\zeta|^2 + 2(y - 1)/F^2 = 1, \text{ on IK and NJ.} \tag{2}$$

Here F is the Froude number defined by (1).

The kinematic condition on the bottom IJ, and on the object KL, LM, MN can be expressed as

$$\text{Im}\zeta = 0, \psi = 0 \text{ on } -\infty < \phi < \infty, \tag{3}$$
$$\text{Re}\zeta = 0, \psi = 1 \text{ on } -b < \phi < -a \text{ and } a < \phi < b, \tag{4}$$
$$\text{Im}\zeta = 0, \psi = 1 \text{ on } -a < \phi < a. \tag{5}$$

As $|\phi| \to \infty$, the flow approaches a uniform stream with constant unit velocity. It can easily be shown by linearizing around a uniform stream that the approach is described by exponentially decaying terms, i.e.

$$\zeta \sim 1 + De^{\mp\pi\lambda f} \text{ as } \phi \to \pm\infty. \tag{6}$$

Here D is a constant to be determined as part of the solution and λ is the smallest positive root of

$$\pi\lambda F^2 - \tan\pi\lambda = 0. \tag{7}$$

We note that there are singularities at the corner points K, L, M and N corresponding to flows inside and around corners.. The appropriate behaviors of ζ near these singularities are

$$\zeta \sim H(f \pm b - i)^{1/2} \text{ as } f \to \mp b + i \tag{8}$$
$$\zeta \sim S(f \pm a - i)^{-1/2} \text{ as } f \to \mp a + i, \tag{9}$$

where H and S are constants to be determined as part of the solution. The problem now becomes that of finding ζ as an analytic function of f in the strip $0 < \psi < 1$ satisfying the equations (2) - (6), (8) and (9).

3. Numerical procedure

Following Vanden-Broeck [6], Asavanant and Vanden-Broeck [11] and others, we map the flow domain from the complex f- plane onto the upper half of the unit circle in the complex t-plane. The transformation is given by

$$f = (2/\pi) \log[(1 + t)/(1 - t)]. \tag{10}$$

Its maps the bottom IJ onto the real diameter, and the free surfaces IK, NJ and the object KL, LM, MN onto the circumference (see Figure 3). We use the notation $t = re^{i\sigma}$ so that the free surface and the object are described by $r = 1$ and $0 < \sigma < \pi$. The points $t = e^{i\gamma_1}$ and $t = -e^{-i\gamma_1}$ are the images of the stagnation points N and K. The points $t = e^{i\gamma_2}$ and $t = -e^{-i\gamma_2}$ are the images of the corner points M and L of the object. By using (10), we find that γ_1 and γ_2 are related to b and a by

$$\gamma_1 = 2 \arctan[\exp(-\pi b/2)] \tag{11}$$

$$\gamma_2 = 2 \arctan[\exp(-\pi a/2)]. \tag{12}$$

We now seek the complex velocity ζ as a function of t. Taking into account the local behaviors of the flow in (6), (8), (9) and the symmetry of the flow about $y = 0$, we write the complex velocity as

$$\zeta = [((t^2+1)^2 - 4t^2\cos^2\gamma_1)/(4-4\cos^2\gamma_1)]^{1/2}[((t^2+1)^2 - 4t^2\cos^2\gamma_2)/(4-4\cos^2\gamma_2)]^{-1/2}e^{\Omega(t)} \tag{13}$$

where $\Omega(t)$ has the expansion

$$\Omega(t) = A(1-t^2)^{2\lambda} + \sum_{n=1}^{\infty} a_n(t^{2n} - 1). \tag{14}$$

The kinematic condition (3) on the bottom IJ implies the coefficients A and a_n are real. The representation (13) factors out the singular behaviors of the velocity at the corner points and the stagnation points. It can easily be verified that (13) satisfied (6), (8), and (9). Therefore we can expect the expansion in (14) to converge for $|t| \leq 1$. The unknown constants A and the coefficients a_n of the power series must be determined so that the dynamic boundary condition (2) on the free surface. and the kinematic conditions (4) and (5) on the object are satisfied. We first eliminate y from (2) by differentiating this equation with respect to σ. By using the identity

$$\partial x/\partial\phi + i\partial y/\partial\phi = 1/\zeta, \tag{15}$$

we obtain

$$F^2[u(\sigma)u_\sigma(\sigma) + v(\sigma)v_\sigma(\sigma)] - 2/(\pi\sin\sigma)[v(\sigma)/(u^2(\sigma) + v^2(\sigma))] = 0. \tag{16}$$

We now solve the problem numerically by truncating the infinite series in (14) after N terms. There are $N+3$ unknowns λ, A, F and the coefficients a_n to be determined by collocation. Thus we introduce the $N+2$ mesh points

$$\sigma_I = (\pi/[2(N+2)])(I - 1/2),\ I = 1, ..., N+2. \tag{17}$$

Here we take advantage of the symmetry of the problem by using mesh points only for $0 \leq \sigma \leq \pi/2$. For simplicity, we consider values of γ_1 and γ_2 in the form of

$$\begin{aligned} \gamma_1 &= \pi M_1/[2(N+2)] \\ \gamma_2 &= \pi M_2/[2(N+2)]. \end{aligned} \tag{18}$$

where $M_1 \leq M_2$ and both are integers smaller than $N+2$. We obtain $N+2$ equations by satisfying (16) at the mesh points $I = 1, ..., M_1$, (4) at the mesh points $I = M_1 + 1, ..., M_2$, and (5) at the mesh points $I = M_2 + 1, ..., N+2$. The last equation is provided by imposing the relation (7). For given values of M_1 and M_2, we solve this system of $N+3$ nonlinear algebraic equations with

$N+3$ unknowns by Newton's method. Once it is solved we obtain the shape of the free surface and the object by integrating numerically the relations

$$dx/d\sigma = -2/(\pi \sin\sigma)[u(\sigma)/(u^2(\sigma)+v^2(\sigma))] \tag{19}$$

$$dy/d\sigma = -2/(\pi \sin\sigma)[v(\sigma)/(u^2(\sigma)+v^2(\sigma))]. \tag{20}$$

4. Discussion of the results

The numerical scheme described in the previous section was used to compute solutions for various values of γ_1 and γ_2. Using (18), we specify γ_1 and γ_2 by fixing

$$\delta_1 = M_1/(N+2) \text{ and } \delta_2 = M_2/(N+2).$$

The coefficients a_n were found to decrease rapidly. For example, $|a_{10}/a_1| \approx 0.29\times 10^{-1}$,$|a_{40}/a_1| \approx 0.36\times 10^{-2}$, $|a_{200}/a_1| \approx 0.24\times 10^{-3}$,$|a_{370}/a_1| \approx 0.15\times 10^{-6}$ for $\delta_1 = 1/2$ and $\delta_2 = 7/10$. Most of the calculations were performed with 400 coefficients. In all the calculations presented we checked that the results are independent of N within graphical accuracy.

Typical profiles are shown in Figures 1, 4 and 5. In Figure 5, the bottom of the object is below the level of the free surface at infinty. However the bottom of the object is above this level in Figures 1 and 4. Therefore Figures 1 and 4 do not model a ship. Following Vanden-Broeck and Keller (10), we refer to these flows as "surfing flows". As $\gamma_1 \to \pi/2$ and $\gamma_2 \to \pi/2$, the height MN, KL and the bottom width LM of the object reduce to zero and we recover the case of the steepest solitary wave. It is found that the limiting configuration with sharp crest and a 120° angle is obtained at $F = 1.29$ (see Figure 6). The value of this critical Froude number is found to be in good agreement with the one obtained by Asavanant and Vanden-Broeck [9] and Hunter and Vanden-Broeck [12].

In Figure 7, we present the values of the Froude number versus δ_1 for various values of δ_2. These results show that there is a two-parameter family of splasless solutions.

As $\delta_1 \to \delta_2$, the height MN, KL of the object approaches zero and the problem reduces to a flow past a flat plate considered by Vanden-Broeck and Keller [10]. It is a configuration with two stagnation points and 120° angles at the end of the plate. To compute accurately these flows, we note that (8) should be replaced by a singular behavior corresponding to a flow inside a 120°, i.e.

$$\zeta \sim K(f \pm b - i)^{1/3} \text{ as } f \to \mp b + i. \tag{21}$$

The complex velocity ζ is then expanded as

$$\zeta = [((t^2+1)^2 - 4t^2\cos^2\gamma)/(4-4\cos^2\gamma)]^{1/3} e^{A(1-t^2)^{2\lambda}+\sum_{n=1}^{\infty} a_n(t^{2n}-1)}. \tag{22}$$

Here $\gamma = \gamma_1 = \gamma_2$. A typical computed profile is shown in Figure 8 and the values of F versus δ_1 are shown in Figure 7 (broken line). The particular value $\gamma = 0$ corresponds to a semi-infinite flat plate. It can then be shown analytically that $F = \sqrt{2}$ (see [10]). Our numerical value of F for $\delta_1 = 0$ agrees with this value.

We now define the amplitude parameter as

$$\alpha = W/H. \tag{23}$$

where W is the distance from the bottom IJ to the bottom LM of the object. Thus the bottom of the object lies above the undisturbed free surface level when $\alpha - 1 > 0$ and below when $\alpha - 1 < 0$. Numerical values of $\alpha - 1$ versus δ_1 are presented in Figure 9 for various values of δ_2.

The length of the bottom LM and the depth of the hull KL and MN of the object are shown in Figures 10 and 11 as functions of δ_1 and δ_2.

5. Flows past an object with inclined faces

We now consider a more realistic shape for the hull , i.e the two faces of the object are inclined at an angle β (see Figure 12). The solution described in section 4 is a special case of the present problem for $\beta = \pi/2$.

We show in this section that there are solutions without splashes for $0 < \beta < \pi/2$. The formulation is very similar to the one described in section 2. Therefore we need only to mention the differences.

At the stagnation points K and N, the flow is locally a flow inside a 120° angle when $0 \leq \beta \leq \pi/3$. When $\pi/3 \leq \beta \leq \pi/2$, the free surface is horizontal at the stagnation points (see [14] for details). Therefore

$$\zeta \sim (f \pm b - i)^{\theta/\pi} \text{ as } f \to \mp b + i \tag{24}$$

where

$$\theta = \beta \text{ if } \pi/3 \leq \beta \leq \pi/2,,$$
$$\theta = \pi/3 \text{ if } 0 \leq \beta \leq \pi/3,. \tag{25}$$

At the corner point, the singularity is given by

$$\zeta \sim (f \pm a - i)^{-\beta/\pi} \text{ as } f \to \mp a + i \tag{26}$$

Following the formulation of section 2 and using (24) and (25) instead of (8) and (9), we represent the complex velocity ζ by

$$\zeta = [((t^2+1)^2 - 4t^2\cos^2\gamma_1)/(4-4\cos^2\gamma_1)]^{\theta/\pi}[((t^2+1)^2 - 4t^2\cos^2\gamma_2)/(4-4\cos^2\gamma_2)]^{-\beta/\pi}e^{\Omega(t)} \tag{26}$$

where $\Omega(t)$ has the expansion

$$\Omega(t) = A(1-t^2)^{2\lambda} + \sum_{n=1}^{\infty} a_n(t^{2n} - 1). \tag{27}$$

The coefficients must be real in order to satisfy the kinematic boundary condition (3) on IJ. Furthermore $\zeta(\pm 1) = 1$.

For given values of γ_1, γ_2 and β, the unknown A, a_n, λ and F must be found such that the Bernoulli equation (16). the kinematic conditions (4) and

$$v = u \tan\beta \text{ on the inclined faces of the object,.} \tag{28}$$

and the relation (7) are satisfied. This is achieved by truncating the infinite series in (27) after a finite number of terms and using the numerical scheme described in section 3.

It was found that the series converges rapidly. Typical profiles for two values of face inclinations are shown in Figure 12 and 13.

6. Conclusions

We have presented numerical solutions for the free surface flow past a two-dimensional ship in water of finite depth. We have assumed that the free surface attaches to the hull at stagnation points. The results supplement previous studies in which smooth attachment is assumed.

We have shown that there is a two-parameter family of splashless solutions. Physically we expect that the flow configuration of Figure 1 depends on three parameters: the width of the ship, the draft (i.e. the distance between the bottom of the hull and the level of the free surface at infinity) and the Froude number. Furthermore we expect the members of this three-parameter family to have a splash at the bow. The findings of the present paper identifies among these members a sub-family (depending on two parameters) of splashless solutions. Splashless solutions are of particular interest, since an important concern in ship hydrodynamics is the reduction of the splash at the bow of a ship.

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List of Figures

Figure 12. Computed profile for $\delta_1 = \frac{1}{4}$, $\delta_2 = \frac{1}{2}$ $\beta = \frac{2\pi}{5}$ and $F = 1.25$. The inclination angle β is measured counterclockwise from the horizontal plane to the right front of the object.

Figure 13. Computed profile for $\delta_1 = \frac{1}{4}$, $\delta_2 = \frac{1}{2}$ $\beta = \frac{\pi}{6}$ and $F = 1.28$.

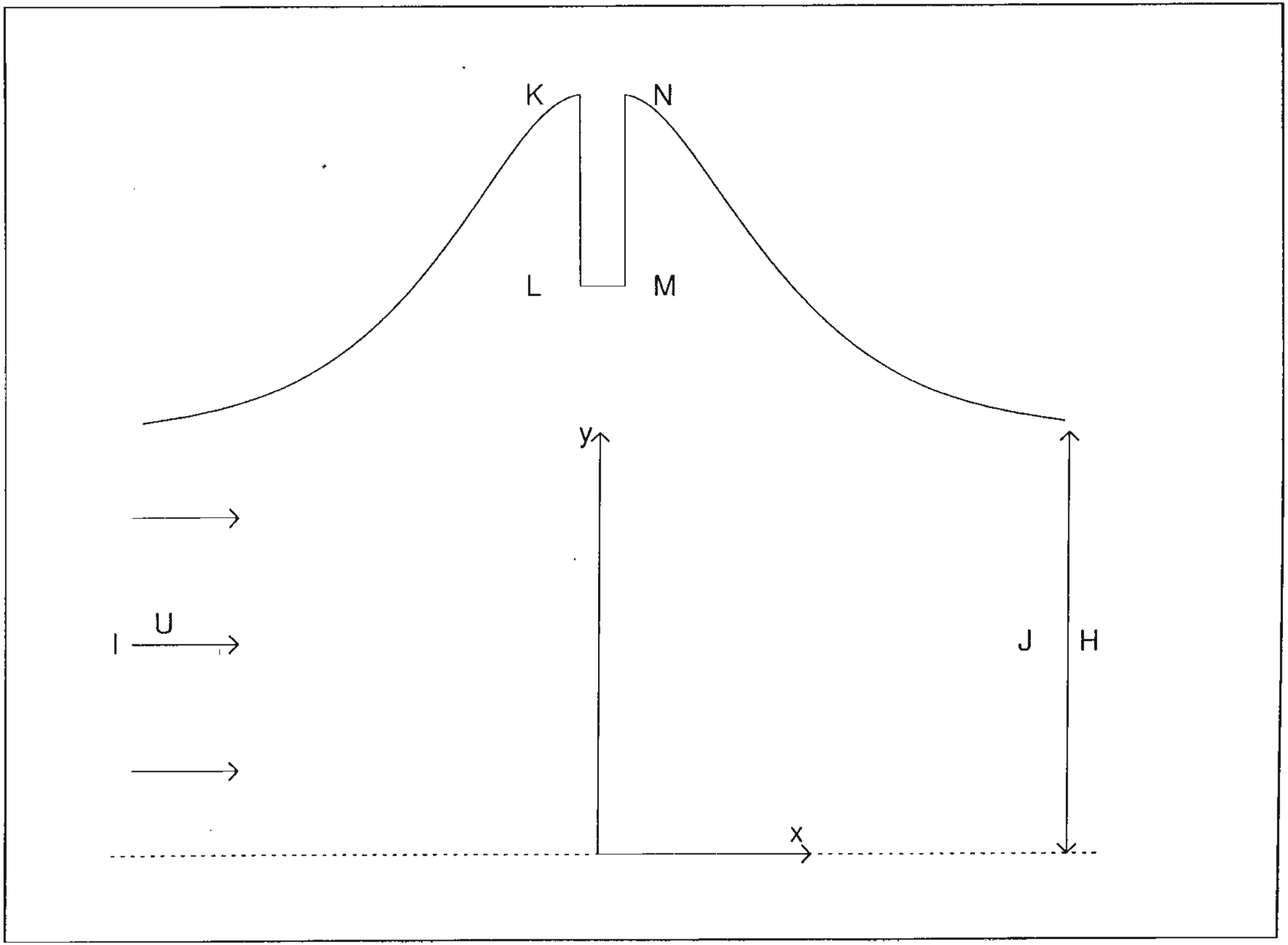


Figure 1

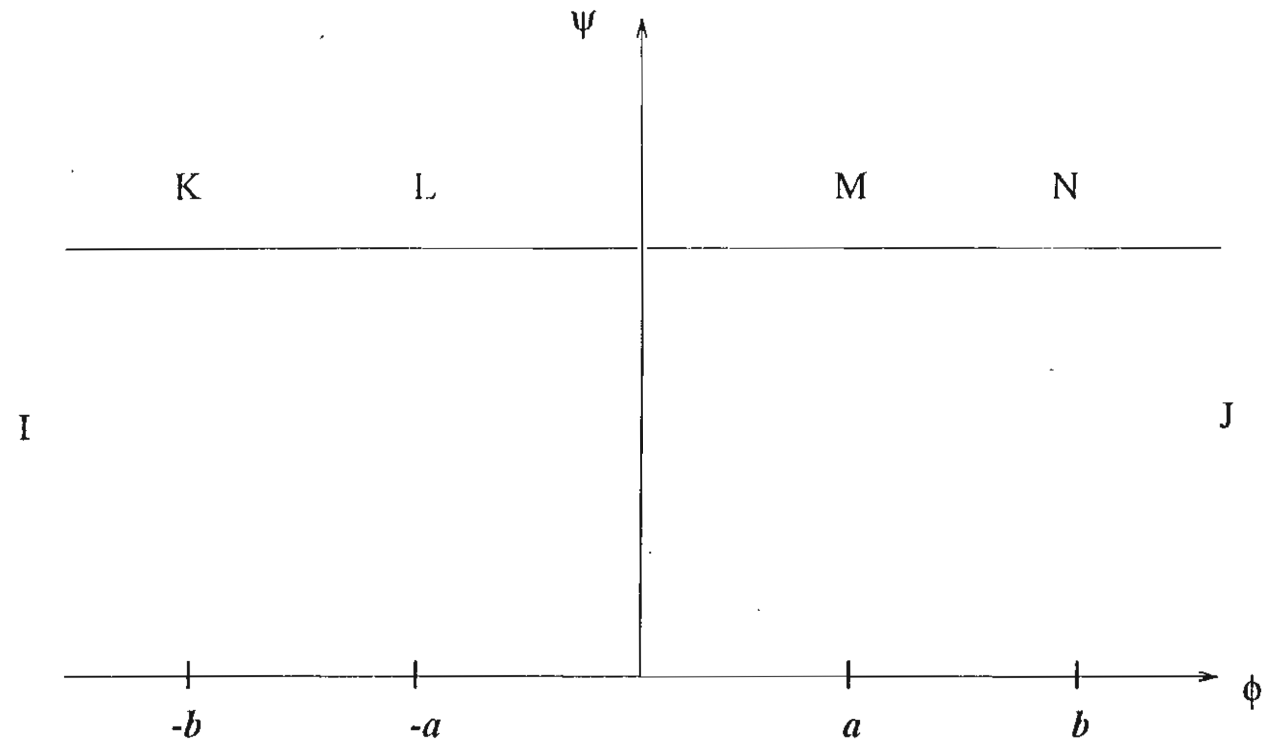


Figure 2

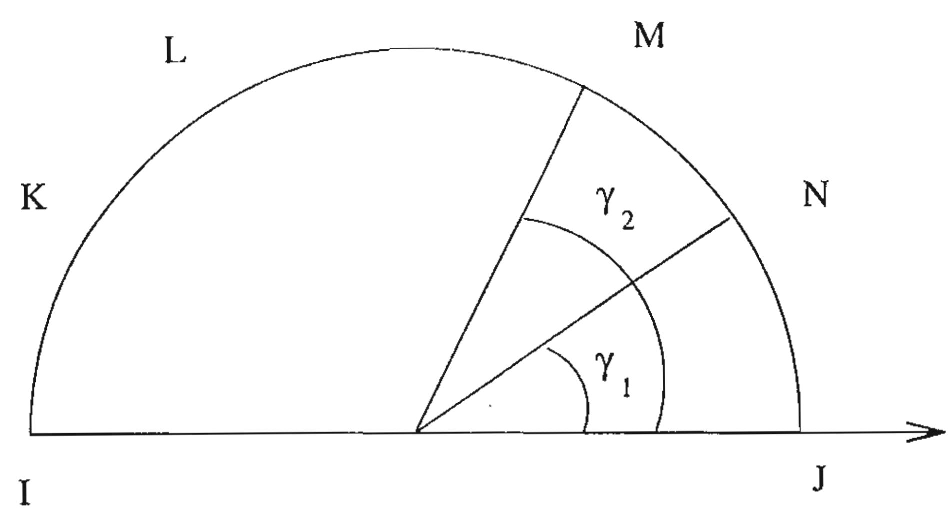


Figure 3

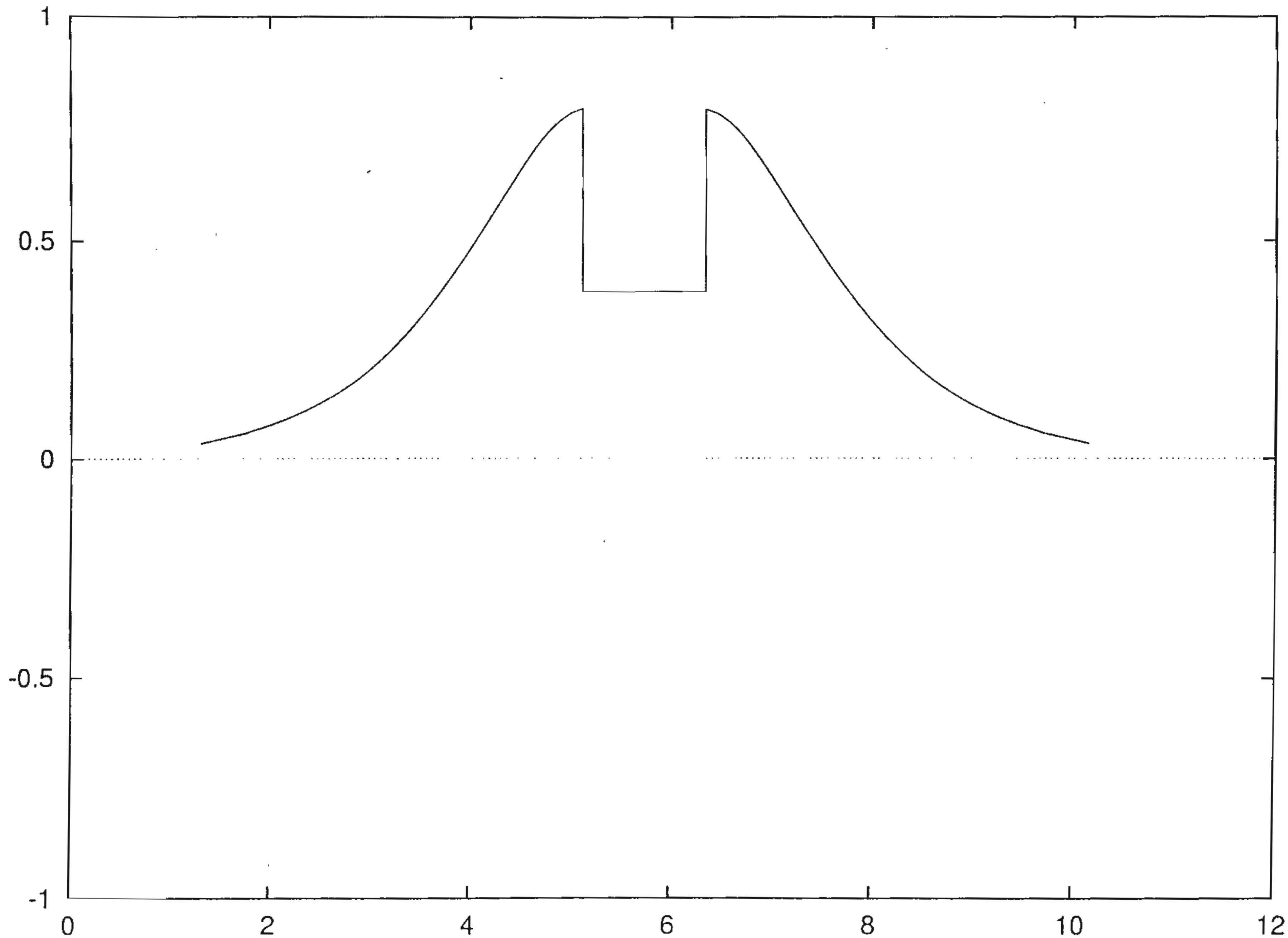


Figure 4

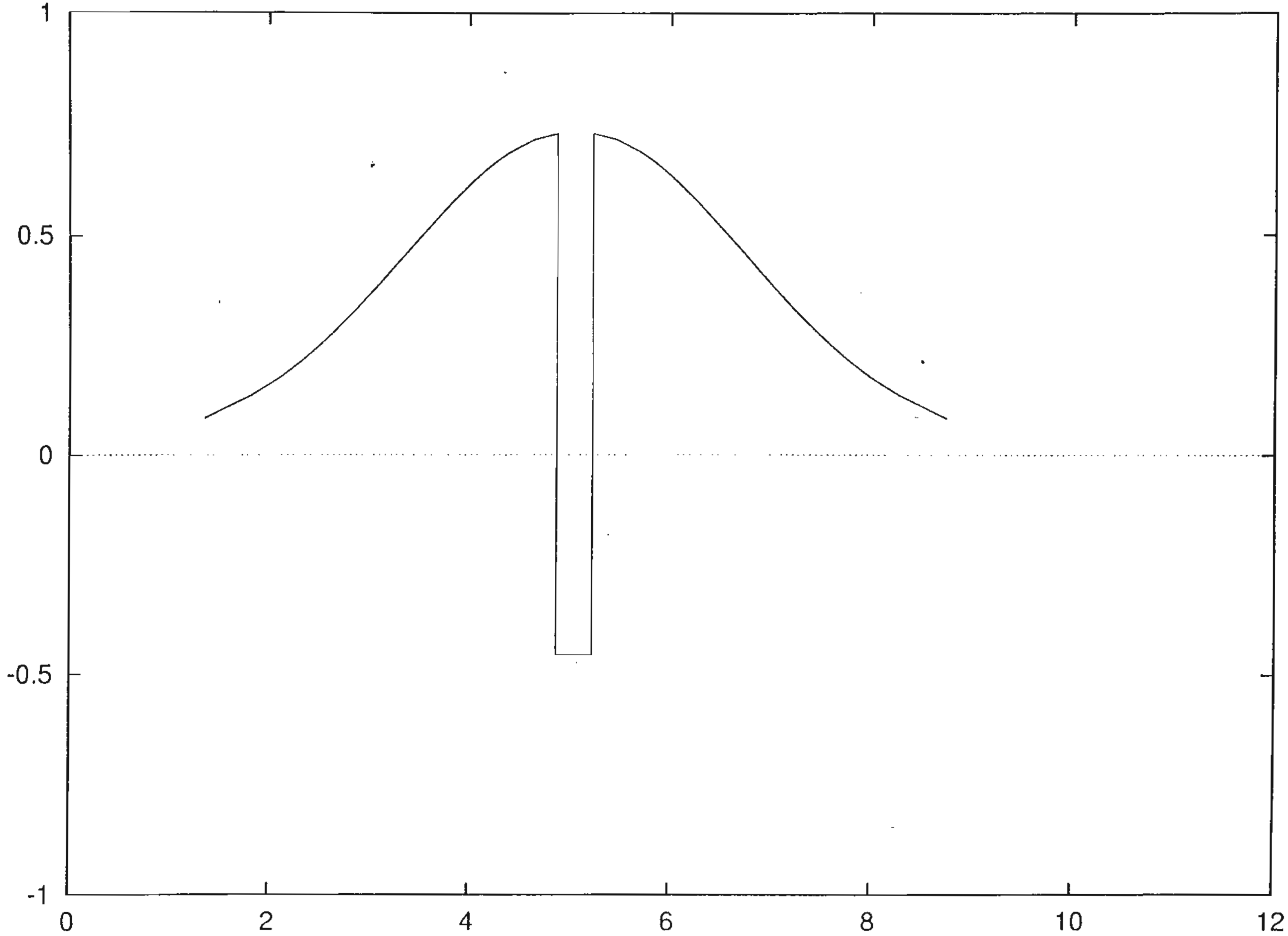


Figure 5

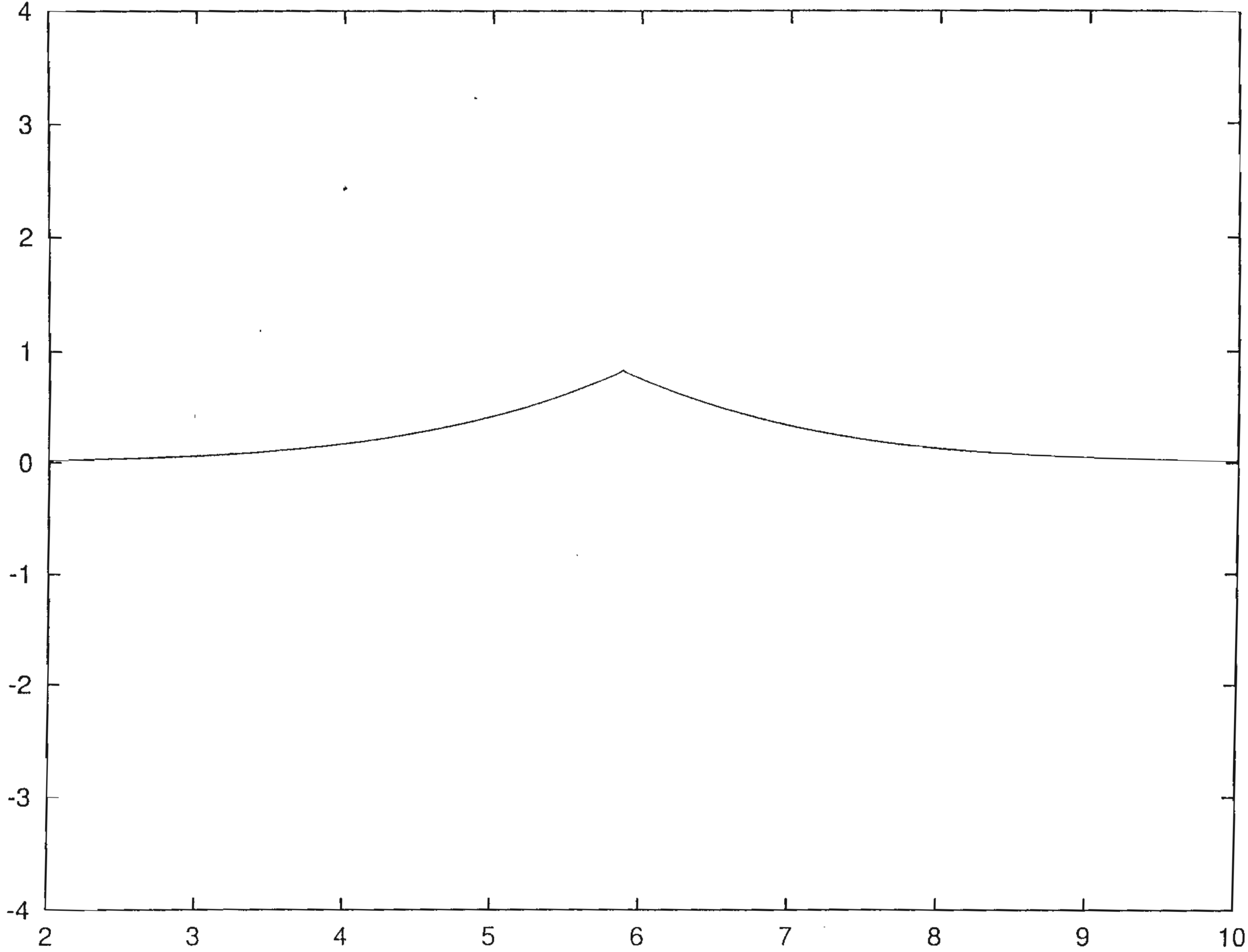
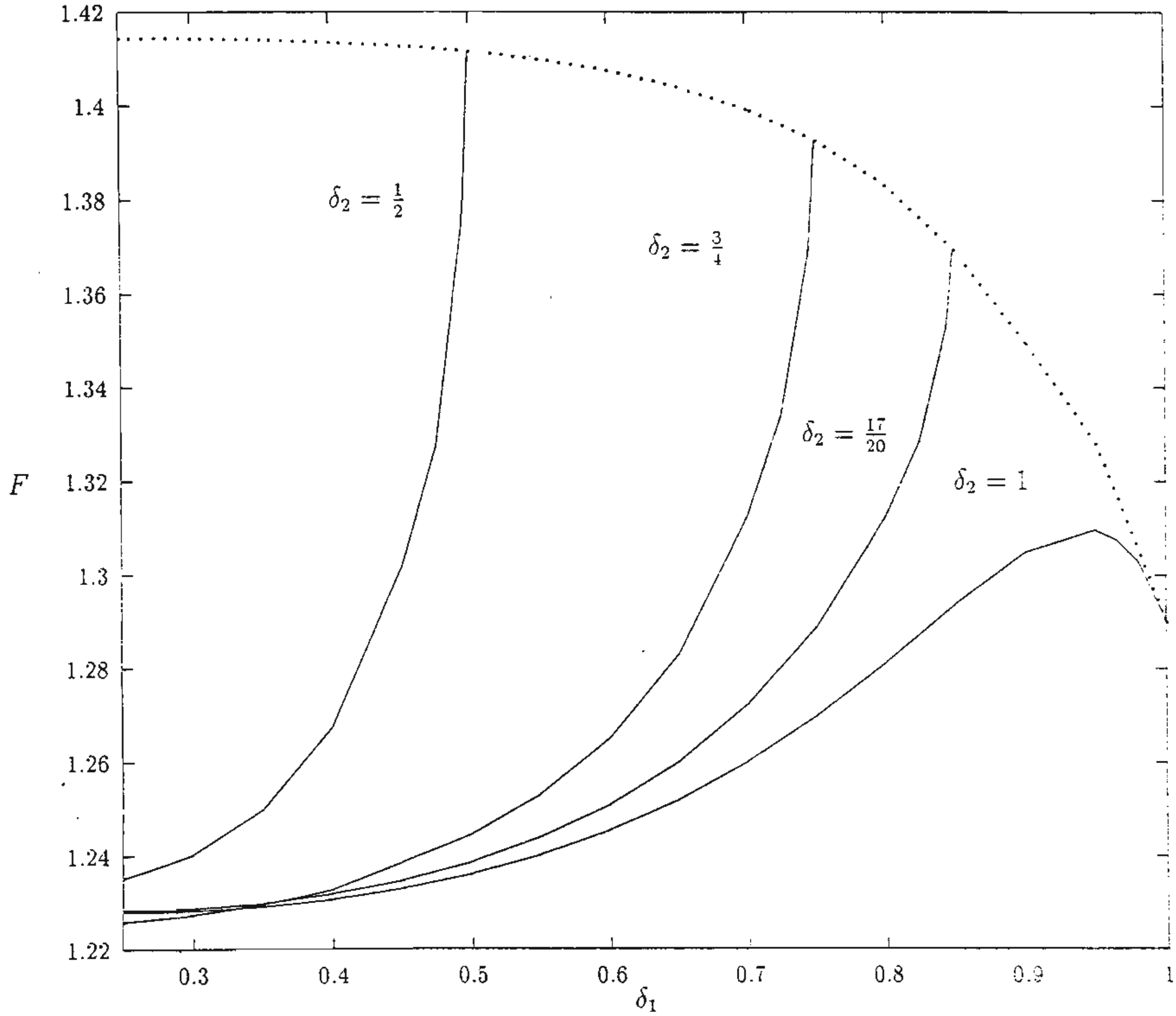


Figure 6



1.42
1.4
1.38
1.36
1.34
1.32
1.3
1.28
1.26
1.24
1.22
F
$\delta_2 = \frac{1}{2}$
$\delta_2 = \frac{3}{4}$
$\delta_2 = \frac{17}{20}$
$\delta_2 = 1$
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
δ_1

Figure 7

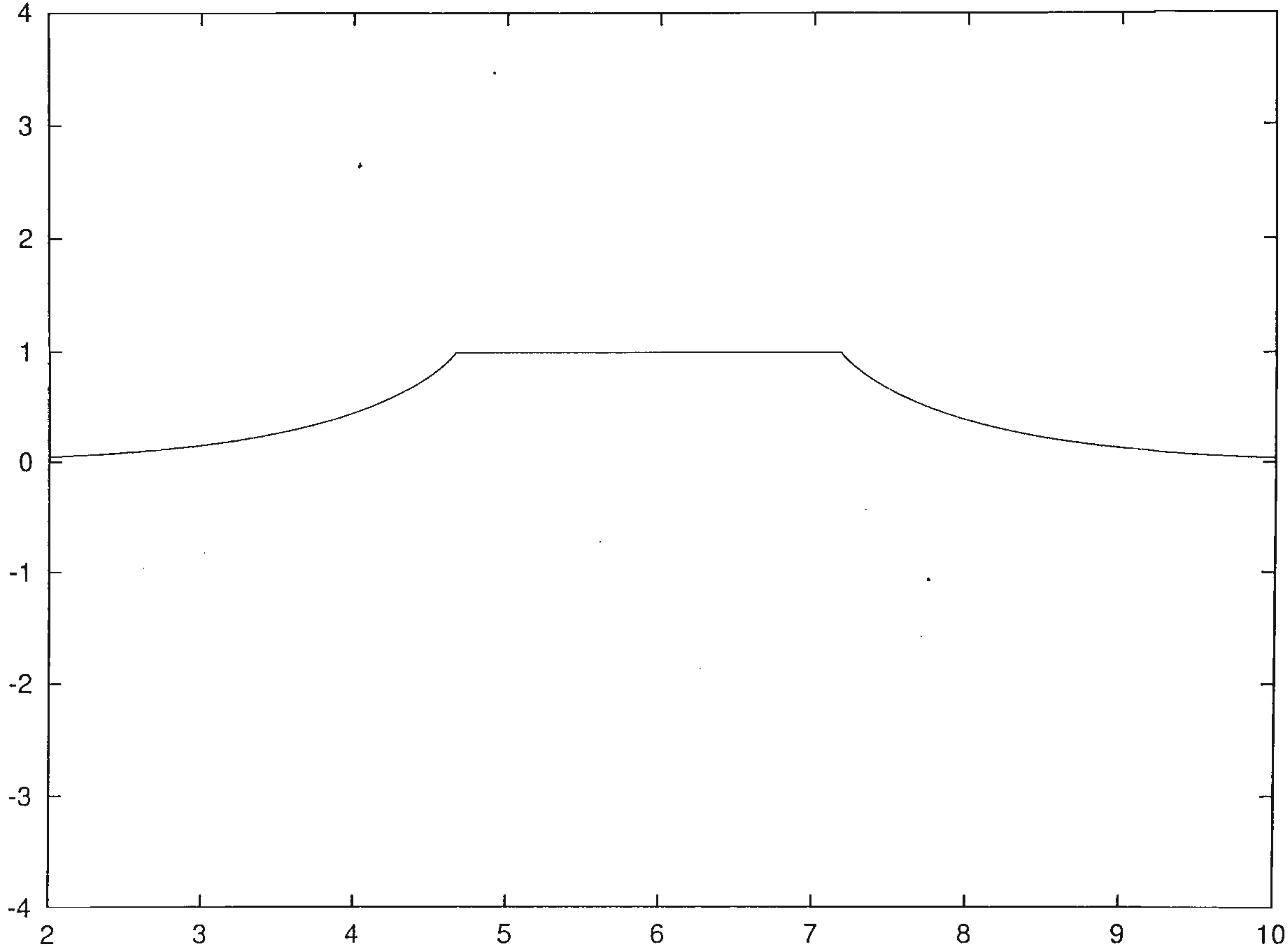


Figure 8

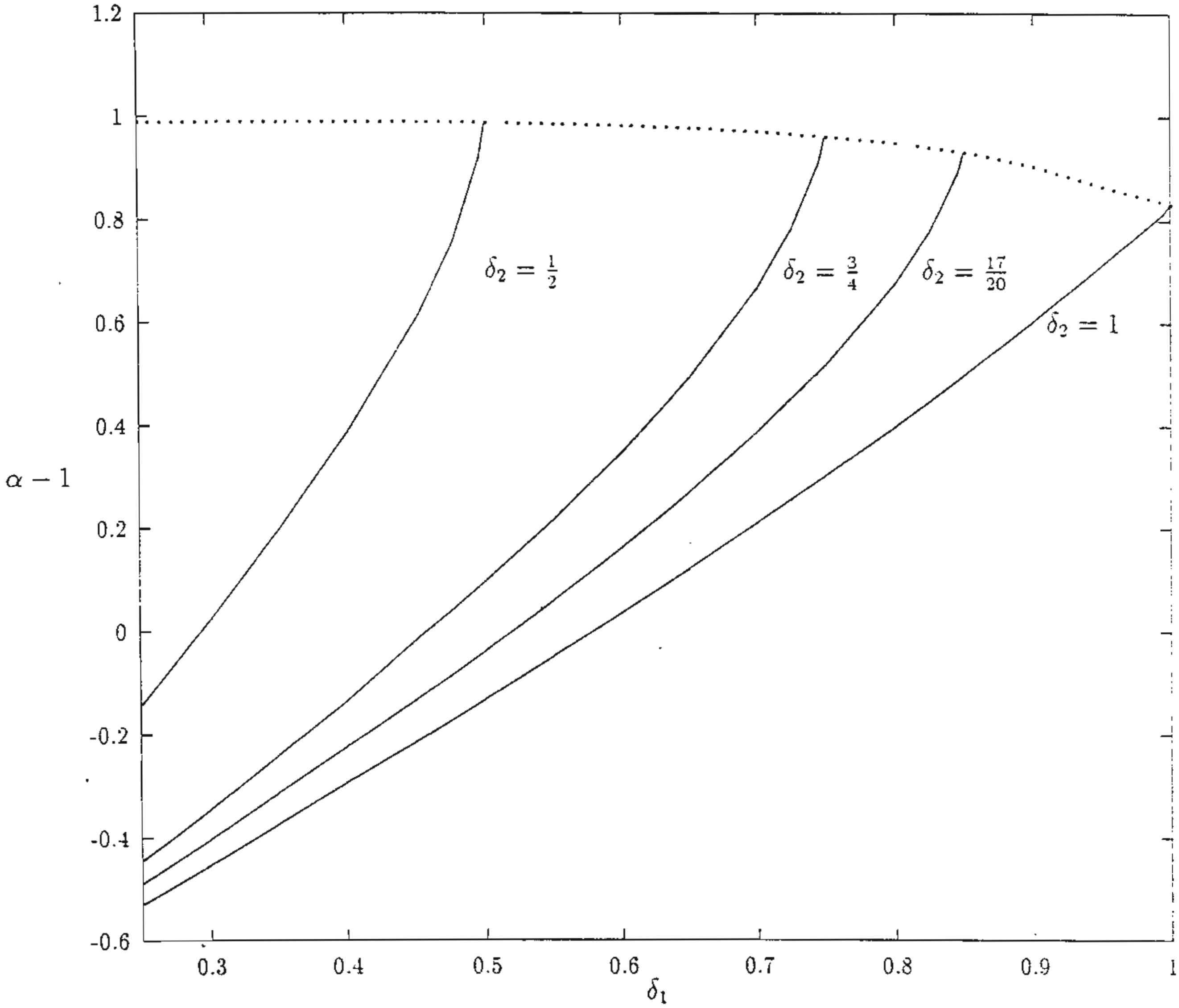
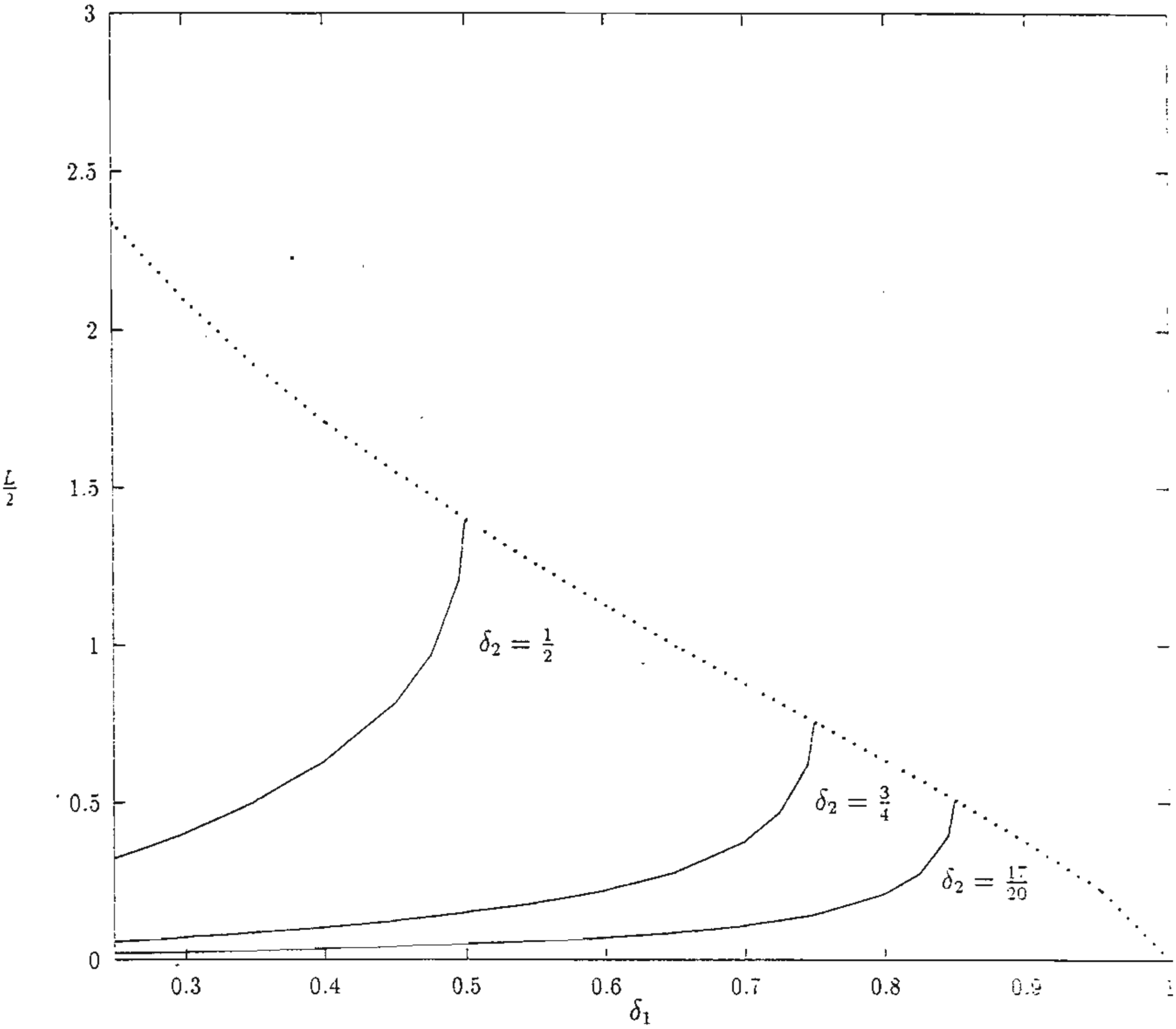

1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
$\alpha - 1$
$\delta_2 = \frac{1}{2}$
$\delta_2 = \frac{3}{4}$
$\delta_2 = \frac{17}{20}$
$\delta_2 = 1$
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
δ_1

Figure 9



3
2.5
2
1.5
1
0.5
0
$\frac{L}{2}$
$\delta_2 = \frac{1}{2}$
$\delta_2 = \frac{3}{4}$
$\delta_2 = \frac{17}{20}$
0.3
0.4
0.5
0.6
0.7
0.8
0.9
δ_1

Figure 10

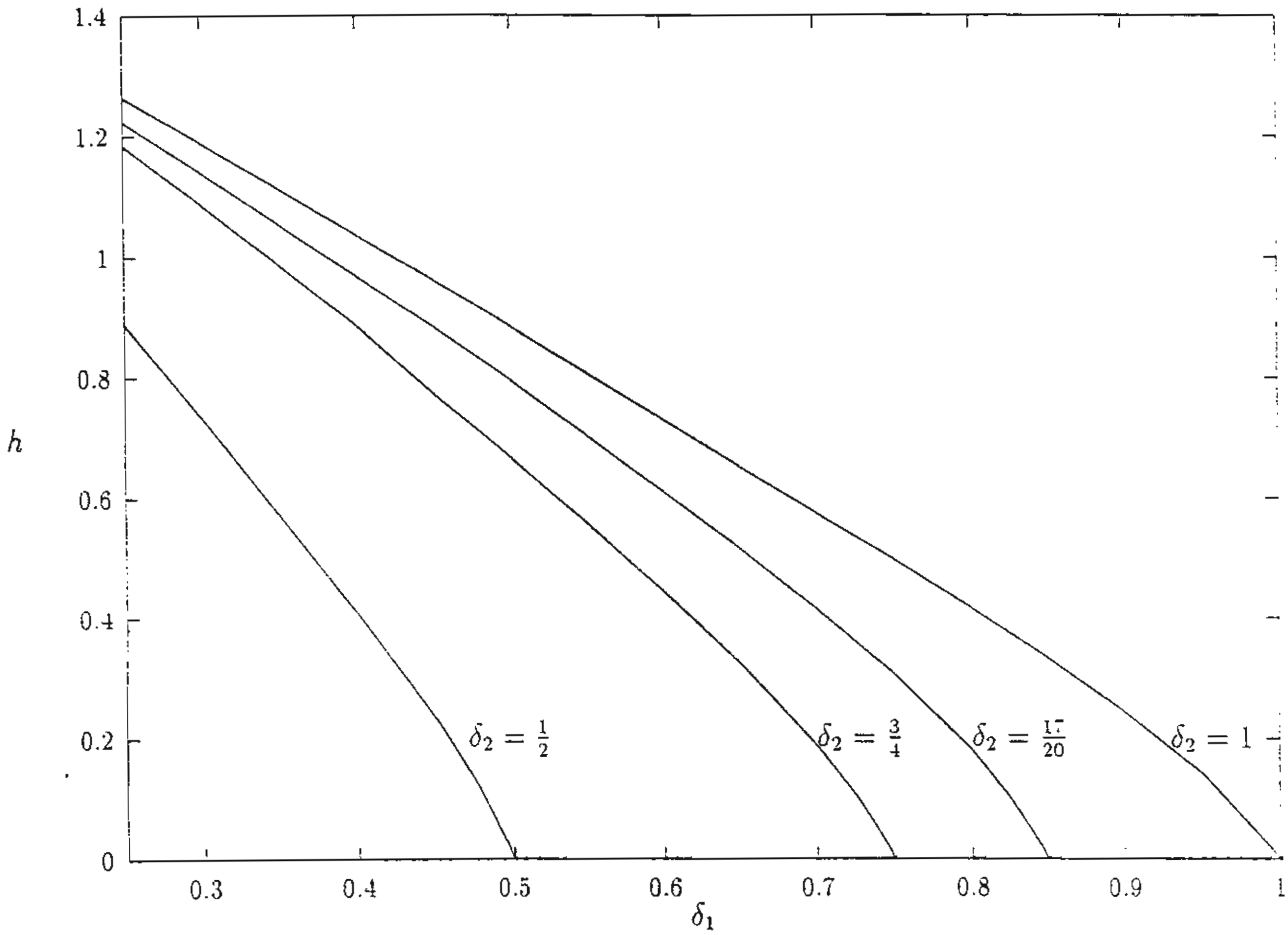

1.4
1.2
1
0.8
h
0.6
0.4
0.2
0
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
δ_1
$\delta_2 = \frac{1}{2}$
$\delta_2 = \frac{3}{4}$
$\delta_2 = \frac{17}{20}$
$\delta_2 = 1$

Figure 11

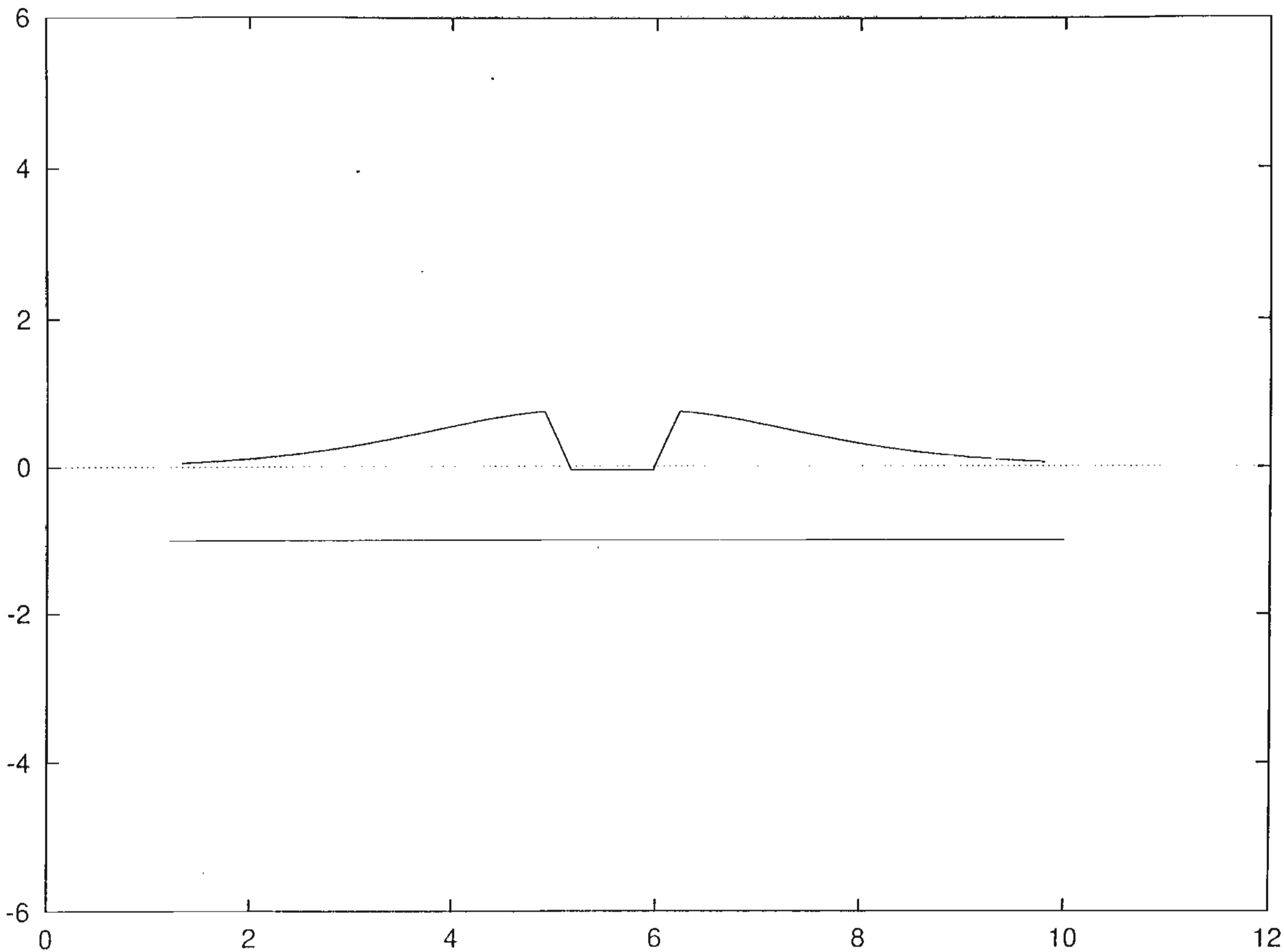


Figure 12

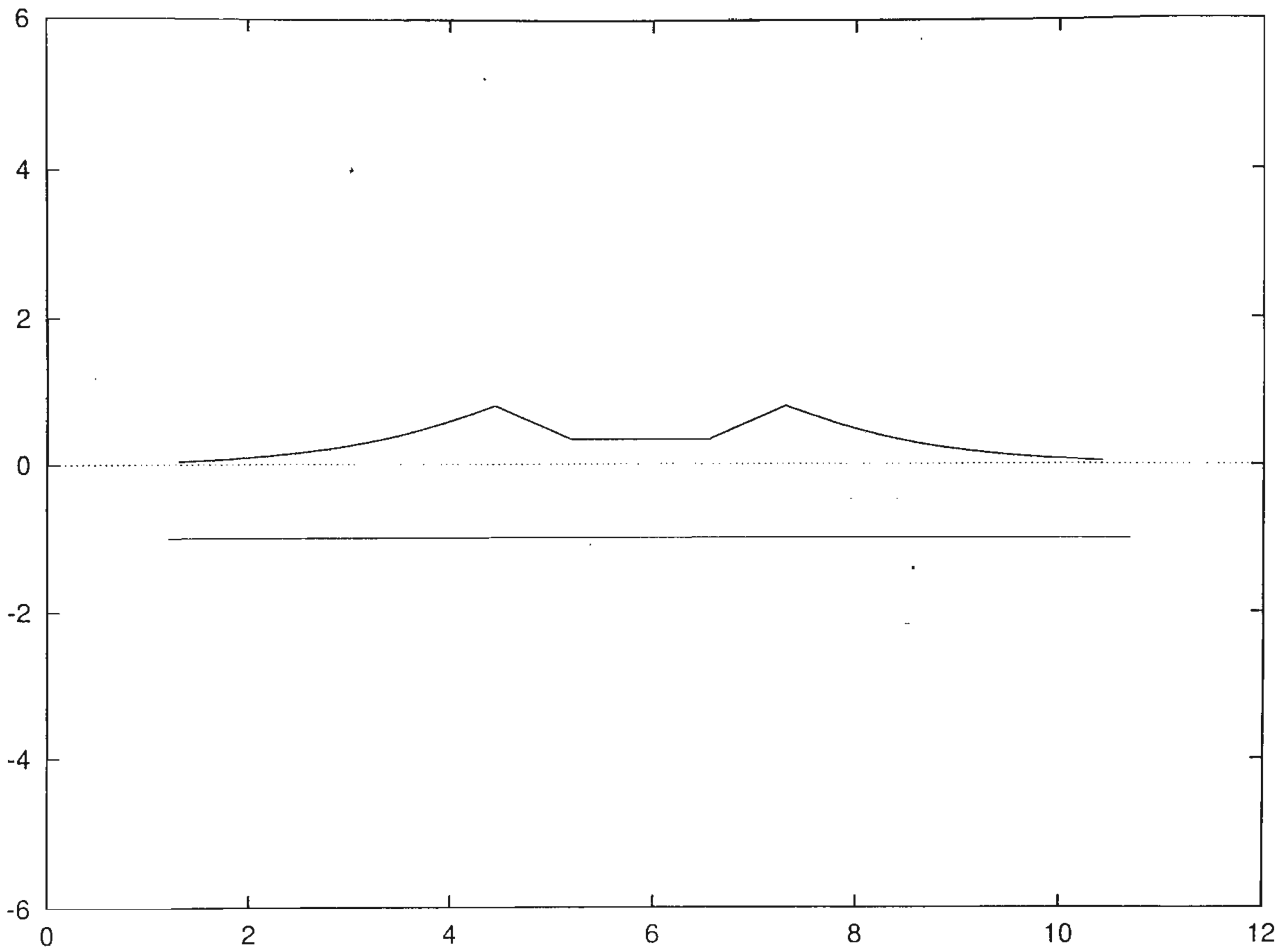


Figure 13