รายงานการวิจัยเรื่องกราฟ k*-extendable

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กิตติกรรมประกาศ

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งานวิจัยฉบับนี้สำเร็จลุล่วงลงได้ด้วยดีด้วยทุนอุดหนุนการวิจัยจากสำนักงานกองทุนสนับสนุนการวิจัย ภายใต้โครงการทุนพัฒนานักวิจัย (1 กันยายน 2539 – 31 สิงหาคม 2542)

บทคัดย่อ

ชื่อโครงการวิจัย	กราฟ k*-extendable
ผู้วิจัย	รองศาสตราจารย์ ดร. นวรัตน์ อนันต์ชื่น ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยศิลปากร นครปฐม
แหล่งทุนอุดหนุน	สำนักงานกองทุนสนับสนุนการวิจัย
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ให้ G เป็นกราฟอย่างง่ายที่ไม่ขาดตอนซึ่งมี 2n จุดและมีการจับคู่สมบูรณ์ สำหรับจำนวนเต็มบวก k, $1 \le k \le n - 1$ เรากล่าวว่า G เป็นกราฟ k-extendable เมื่อทุก ๆ เซต M ที่เป็นเซตของเส้น k เส้นที่ไม่มีจุดปลายของเส้นคู่ใดๆ ร่วมกัน จะมีการจับคู่สมบูรณ์ใน G ที่ครอบคลุมเส้นทุกเส้นใน M สำหรับจำนวนเต็ม k, $0 \le k \le n - 2$ เรากล่าวว่า G เป็นกราฟ strongly k-extendable หรือเรียกสั้น ๆ ว่ากราฟ k*-extendable เมื่อ G - {u, v} เป็นกราฟ k-extendable สำหรับทุก ๆ จุด u และ v ใด ๆ ใน G ปัญหาที่เกิดขึ้นคือการศึกษาลักษณะเฉพาะเจาะจงของกราฟ k-extendable และ กราฟ k*-extendable ได้มีผู้ศึกษาลักษณะเฉพาะเจาะจงของกราฟ k-extendable แล้วมากมาย ในขณะที่ปัญหาการศึกษาลักษณะเฉพาะเจาะจงของกราฟ k*-extendable มีผู้ศึกษาเฉพาะในกรณีที่ k = 0 เท่านั้น ในงานวิจัยฉบับนี้เราได้ศึกษาลักษณะเฉพาะเจาะจงของกราฟ k*-extendable สำหรับกรณี k ใด ๆ ผลของการศึกษาเราได้คุณสมบัติของกราฟ k*-extendable มาจำนวนหนึ่งซึ่งรวมทั้งความสัมพันธ์ระหว่างกราฟ k-extendable และ กราฟ k*-extendable และเงื่อนไขจำเป็นและเงื่อนไขเพียงพอสำหรับการเป็นกราฟ k*-extendable เรายังได้พิจารณาค่าที่เป็นไปได้สำหรับดีกรีที่น้อยที่สุดของกราฟ k*-extendable พร้อมทั้งสร้างกราฟที่สอดคล้องกับเงื่อนไขของดีกรีที่น้อยที่สุดเหล่านี้ ผลของการศึกษาดังกล่าวนี้มีส่วนช่วยให้เราสามารถเสนอลักษณะเฉพาะอย่างสมบูรณ์ของกราฟ k*-extendable ที่มี 2n จุด เมื่อ k = n – 2 และ k = n – 3 นอกจากนั้นเรายังได้ศึกษา independence number ของ G[S] เมื่อ S เป็น minimum cutset ในกราฟ G ที่เป็นกราฟ k*-extendable พร้อมทั้งให้ขอบเขตบนของจำนวน component ใน G – S

Abstract

Research Title	On k*-extendable Graphs
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Let G be a simple connected graph on 2n vertices with a perfect matching. For a positive integer k, $1 \leq k \leq n - 1$, G is *k-extendable* if for every matching M of size k in G, there is a perfect matching in G containing all the edges of M. For an integer k, $0 \leq k \leq n - 2$, G is *strongly k-extendable* or simply *k*-extendable* if G - {u, v} is k-extendable for every pair of vertices u and v of G. The problem that arises is that of characterizing k-extendable graphs and k*-extendable graphs. The first of these problems has been considered by several authors while the latter has been investigated only for the case k = 0. In this paper, we focus on the problem of characterizing k*-extendable graphs for any k. We present a number of properties of k*-extendable graphs including a relationship between k-extendable and k*-extendable graphs and some necessary and sufficient conditions for k*-extendable graphs. We also determine the set of realizable values for minimum degree of k*-extendable graphs. A complete characterization of k*-extendable graphs on 2n vertices for $k = n - 2$ and $n - 3$ is also established. Further, we investigate the independence number of G[S] when S is a minimum cutset of a k*-extendable graph G. An upper bound on a number of components of G – S is also given.

Contents

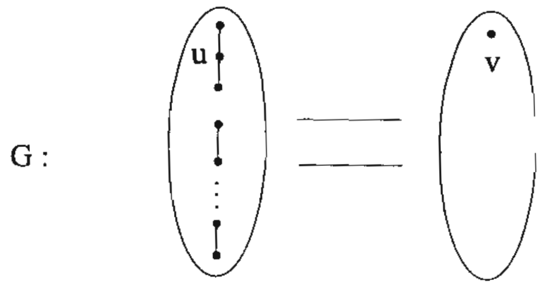
1. Introduction

All graphs considered in this paper are finite, connected, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [6]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices, $\varepsilon(G)$ edges, minimum degree $\delta(G)$, connectivity $\kappa(G)$ and independence number $\alpha(G)$. For $V' \subseteq V(G)$, $G[V']$ denotes the subgraph induced by V'. Similarly $G[E']$ denotes the subgraph induced by the edge set E' of G. $N_G(u)$ denotes the neighbour set of u in G and $\overline{N}_G(u)$ the non-neighbours of u. Note that $\overline{N}_G(u) = V(G) \setminus (N_G(u) \cup \{u\})$. The *join* $G \vee H$ of disjoint graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H.

A *matching* M in G is a subset of E(G) in which no two edges have a vertex in common. M is a *maximum matching* if $|M| \geq |M'|$ for any other matching M' in G. A vertex v is *saturated* by M if some edge of M is incident to v; otherwise, v is said to be *unsaturated*. A matching M is *perfect* if it saturates every vertex of the graph. For simplicity we let V(M) denote the vertex set of the subgraph G[M] induced by M.

Let G be a simple connected graph on 2n vertices with a perfect matching. For a given positive integer k, $1 \leq k \leq n - 1$, G is *k-extendable* if for every matching M of size k in G, there exists a perfect matching in G containing all the edges of M. For convenience, a graph with a perfect matching is said to be 0-extendable. For an integer k, $0 \leq k \leq n - 2$, we say that G is *strongly k-extendable* or simply *k*-extendable* if for every pair of vertices u and v of G, $G - \{u, v\}$ is k-extendable. A graph G is *bicritical* if $G - \{u, v\}$ has a perfect matching for every pair of vertices u and v. Clearly, 0*-extendable graphs are bicritical and a concept of k*-extendable graphs is a generalization of bicritical graphs.

Observe that the complete graph K_{2n} of order 2n is k*-extendable for all k, $0 \leq k \leq n - 2$ while the complete bipartite graph $K_{n,n}$ with bipartition (X, Y) is k-extendable, $0 \leq k \leq n - 2$, but not k*-extendable since a deletion of any two distinct vertices of X results in a graph $K_{n-2,n}$ which clearly has no perfect matching. In fact, k*-extendable graphs are not bipartite. Further, since a bipartite graph on 2n vertices with minimum degree at least $\frac{1}{2}(n + k)$ is k-extendable (see Ananchuen and Caccetta [5]), it follows that the classes of k*-extendable graphs and k-extendable graphs do not coincide. Moreover, there exists a k-extendable non-bipartite graph on 2n vertices, $0 \leq k \leq n - 2$, which is not k*-extendable. Such a graph is $G = G' \vee G''$, where $G' = P_3 \cup (n - k - 2)K_2$, P_3 is a path on 3 vertices, and $G'' = K_{2k+1}$ (see Figure 1.1). Note that in our diagrams a "double line" denotes the join. It is not difficult to



$G' = P_3 \cup (n - k - 2)K_2$ $G'' = K_{2k+1}$

Figure 1.1

show that G is k-extendable. Let u be the vertex of P_3 having degree 2 and v any vertex of G''. Consider $G_1 = G - \{u, v\}$. Clearly, $G'' - v$ contains a matching M of size k which cannot extend to a perfect matching in G_1, since $G_1 - V(M) = 2K_1 \cup (n - k - 2)K_2$.

A number of authors have studied k-extendable graphs. Excellent surveys are the papers of Plummer [13, 14]. Lovasz [7], Lovasz and Plummer [8, 9] and Plummer [10] have studied k*-extendable graphs for k = 0 (bicritical graphs) while k*-extendable graphs for $k \geq 1$ have not been previously investigated. In this paper, we focus on the problem of characterizing these graphs. We present a number of properties of k*-extendable graphs including a relationship between k-extendable and k*-extendable graphs and some necessary and sufficient conditions for k*-extendable graphs. We also determine the set of realizable values for minimum degree of k*-extendable graphs. A complete characterization of k*-extendable graphs on 2n vertices for $k = n - 2$ and $n - 3$ is also established. Further, we investigate the independence number of G[S] when S is a minimum cutset of a k*-extendable graph G. An upper bound on a number of components of $G - S$ is also given.

Section 2 contains some preliminary results that we make use of in establishing our results. In Section 3, we establish a number of results on properties of k*-extendable graphs. Some sufficient conditions for k*-extendable graphs are given in Section 4. In Section 5, we establish a necessary condition, in terms of minimum degree, for k*-extendable graphs. Further, we determine the set of realizable values for minimum degree of k*-extendable graphs. A complete characterization of k*-extendable graphs on 2n vertices for $k = n - 2$ and $n - 3$ is given in Section 6. In Section 7, we establish the independence number of G[S] when S is a minimum cutset of a k*-extendable graph G. Section 8 contains some results concerning an upper bound on a number of components of $G - S$.

2. Preliminaries

In this section we state a number of results which we make use of in our work. We begin with some fundamental results of k-extendable graphs proved by Plummer [10]:

Theorem 2.1: Let G be a k-extendable graph on 2n vertices, $1 \le k \le n - 1$. Then

(i) G is (k - 1)-extendable;

(ii) G is (k + 1)-connected. ❑

Theorem 2.2: Let G be a graph on 2n vertices and $1 \le k \le n - 1$. If $\delta(G) \ge n + k$, then G is k-extendable. ❑

Denoting the number of odd components in a graph H by o(H) we can now state Tutte's theorem which gives a necessary and sufficient condition of the existence of a perfect matching in a graph.

Theorem 2.3: Tutte's Theorem (see Bondy and Murty [6] p. 76)

A graph G has a perfect matching if and only if

$o(G - S) \le |S|$ for all $S \subset V(G)$. ❑

Our next result concerns a sufficient condition for a graph to be hamiltonian (see Bondy and Murty [6] p. 54).

Theorem 2.4: If G is a simple graph with $v(G) \ge 3$ and $\delta(G) \ge \frac{1}{2}v(G)$, then G is hamiltonian. ❑

Ananchuen and Caccetta [1, 2, 3] established the following three results, two of them are a characterization of k-extendable graphs on 2n vertices for $k = n - 1$ and $n - 2$.

Lemma 2.5: Let G be a connected graph on 2n vertices with $\delta(G) \ge n - 1$ having a maximum matching M of size n - 1. Then for M-unsaturated vertices u and v of G, $N_G(u) = N_G(v)$. Furthermore, no two vertices of $N_G(u)$ are joined by an edge of M, and the vertices of $V(G) \setminus N_G(u)$ form an independent set. ❑

Theorem 2.6: Let G be a graph on $2n \ge 4$ vertices. Then G is (n - 1)-extendable if and only if G is K_{2n} or $K_{n, n}$. ❑

Theorem 2.7: Let G be a graph on $2n \ge 10$ vertices with a perfect matching. Then G is (n - 2)-extendable if and only if G:

(i) is $K_{n, n}$ or K_{2n}, or

(ii) is a bipartite graph with minimum degree n - 1, or

(iii) has minimum degree 2n - 3 and $\alpha(G) \le 2$, or

(iv) has minimum degree 2n - 2. ❑

We conclude this section by stating a result proved by Plummer [11].

Theorem 2.8: Let G be k-connected, $k \geq 1$, let S be a minimum cutset in G, and let C be any component of G – S. Then given any subset $S' \subseteq S$, $S' \neq \emptyset$ and $|S'| \leq |V(C)|$, there exists a complete matching of S′ into V(C). ❑

3. Basic properties of k*-extendable graphs

Our first result concerns a necessary condition of k*-extendable graphs.

Lemma 3.1 : If G is a k*-extendable graph on 2n vertices; $1 \leq k \leq n - 2$, then G is (k - 1)*-extendable.

Proof: Let u, v be vertices of G and G* = G - {u, v}. Then G* is k-extendable, by Theorem 2.1, and so (k - 1)-extendable. Thus G is (k - 1)*-extendable as required. ❑

A consequence of Lemma 3.1 is the following corollary:

Corollary 3.2: If G is a k*-extendable graph on 2n vertices; $1 \leq k \leq n - 2$, then for $0 \leq t \leq k$, G is t*-extendable. ❑

The next result establishes a relationship between k*-extendable and k-extendable graphs.

Lemma 3.3 : If G is a k*-extendable graph on 2n vertices; $0 \leq k \leq n - 2$, then G is (k + 1)-extendable.

Proof: Let M be a matching of size k + 1 in G and uv an edge of M. Since G is k*-extendable, G - {u, v} has a perfect matching F containing M - {uv}. Thus $F \cup \{uv\}$ is a perfect matching containing M. This proves our result. ❑

Theorem 2.1 and Lemma 3.3 imply the following corollary.

Corollary 3.4: If G is a k*-extendable graph on 2n vertices; $0 \leq k \leq n - 2$, then G is t-extendable for $0 \leq t \leq k + 1$. ❑

Note that the converse of Lemma 3.3 is not true. The graphs G_1 and G_2 in Figure 3.1 are both (k + 1)-extendable (see Ananchuen and Caccetta [1]) but not k*-extendable since if we delete vertices u and v which are in diagonally opposite K_{k+1}' s (K_k and K_{k+2}) in the graph G_1 (G_2), then the resulting graph is not k-extendable.

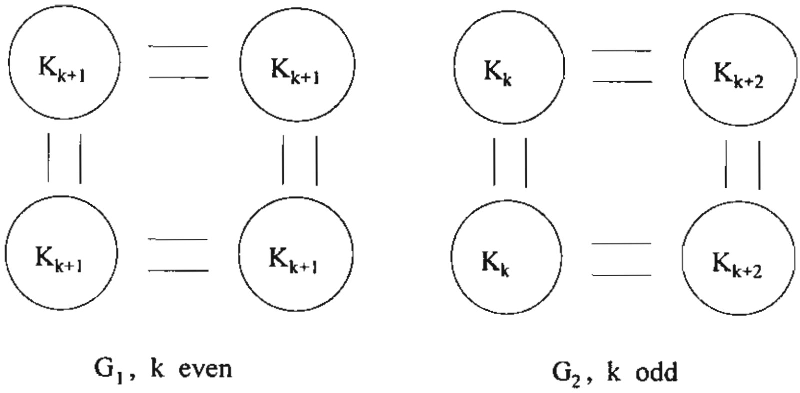


Figure 3.1

We have observed that if G is k*-extendable, then G is not bipartite. The following lemma establishes that G - V(M) is also a non-bipartite graph for every matching M in G of size at most k.

Lemma 3.5: Let G be a k*-extendable graph on 2n vertices, $0 \le k \le n - 2$. If M is a matching of size $t \le k$ in G, then G - V(M) is not a bipartite graph.

Proof: Suppose $G' = G - V(M)$ is a bipartite graph for some matching M of size $t \le k$ in G. Let (V_1, V_2) be bipartition of G'. Since G is k*-extendable, by Corollary 3.4, G' has a perfect matching. Thus $|V_1| = |V_2| = n - t \ge n - k \ge 2$. Let x and y be vertices of V_1 and $G'' = G - \{x, y\}$. Since $G'' - V(M)$ is a bipartite graph with bipartitioning sets of order $|V_1| - 2$ and $|V_2| (= |V_1|)$, $G'' - V(M)$ has no perfect matching. Hence, G is not t*-extendable. This contradicts Corollary 3.2 and completes the proof of our lemma.

❑

Our next two theorems yield a necessary and sufficient condition for k*-extendable graphs.

Theorem 3.6: Let G be a graph on 2n vertices. For $0 \le k \le n - 2$, G is k*-extendable if and only if for every matching M in G of size t, $0 \le t \le k$, G - V(M) is (k - t)*-extendable.

Proof: Suppose G is k*-extendable. For a matching M, in G, of size t, $0 \le t \le k$, let $G' = G - V(M)$. Further, let $a, b \in V(G')$ and consider $G'' = G' - \{a, b\}$. For a matching M'', in G'', of size k - t, $M \cup M''$ is a matching, in G - {a, b}, of size t + (k - t) = k. Since G is k*-extendable, there exists a perfect matching F in G - {a, b} containing $M \cup M''$. Thus $F \setminus M$ is a perfect matching, in G'', containing M''. Hence, G - V(M) is (k - t)*-extendable.

Conversely, let x, y be a pair of vertices of G and M_1 a matching of size k in G - {x, y}. By our hypothesis, $G - V(M_1)$ is 0*-extendable. Then $G - (V(M_1) \cup \{x, y\})$ contains a perfect matching F_1. Consequently, $F_1 \cup M_1$ is a perfect matching in G - {x, y} containing M_1. Hence, G is k*-extendable. This completes the proof of our theorem. ❑

Denoting a maximum matching in G[S] by M(S) for any $S \subseteq V(G)$ we can now establish another theorem giving a necessary and sufficient condition for k*-extendable graphs.

Theorem 3.7: Let G be a graph on 2n vertices. For $0 \le k \le n - 2$, G is k*-extendable if and only if for all $S \subseteq V(G)$

$$o(G - S) \le \begin{cases} |S| - 2t, & \text{for } |S| \le 2k+1 \\ |S| - 2t - 2, & \text{for } |S| \ge 2k+2 \end{cases}$$

where $t = \min\{|M(S)|, k\}$.

Proof: Suppose G is k*-extendable. Let $S \subseteq V(G)$ and $t = \min\{|M(S)|, k\}$. If $|S| \le 2k + 1$, $|M(S)| \le k$. Thus $t = |M(S)|$. Since G is k*-extendable, by Corollary 3.4, G - V(M(S)) has a perfect matching. By Theorem 2.3,

$$o(G - S) = o((G - V(M(S))) - (S \setminus V(M(S)))) \le |S \setminus V(M(S))| = |S| - 2t,$$

as required.

Next we consider the case $|S| \ge 2k + 2$. For this case we distinguish two subcases according to $|M(S)|$.

Case 1: $|M(S)| \le k$. Then $t = |M(S)|$. Let $x, y \in S \setminus V(M(S))$ and put

$$G' = G - (V(M(S)) \cup \{x, y\})$$

and

$$S' = S \setminus (V(M(S)) \cup \{x, y\}).$$

Since G is k*-extendable, Corollary 3.2 implies that G′ has a perfect matching. By Theorem 2.3,

$$o(G' - S') \le |S'|.$$

Thus $$o(G - S) = o(G' - S') \le |S'| = |S| - 2t - 2.$$

Case 2: $|M(S)| \ge k + 1$. Then $t = k$. Let M′ be a subset of M(S) with $|M'| = k$ and $x, y \in S \setminus V(M')$. Put

$$G'' = G - (V(M') \cup \{x, y\})$$

and

$$S'' = S \setminus (V(M') \cup \{x, y\}).$$

By the same argument as in the proof of Case 1, we have

$$o(G - S) = o(G'' - S'') \le |S''| = |S| - 2k - 2 = |S| - 2t - 2.$$

This proves sufficiency.

Conversely, suppose that for all $S \subseteq V(G)$

$$o(G - S) \le \begin{cases} |S| - 2t, & \text{for } |S| \le 2k+1 \\ |S| - 2t - 2, & \text{for } |S| \ge 2k+2 \end{cases}$$

where $t = \min\{|M(S)|, k\}$. Let x, y be vertices of G and M a matching of size k in $G - \{x, y\}$. Put

$$G' = G - (V(M) \cup \{x, y\}).$$

Let $S' \subseteq V(G')$ and $S = S' \cup (V(M) \cup \{x, y\})$. Clearly,

$$|S| = |S'| + 2k + 2 \geq 2k + 2$$

and

$$o(G' - S') = o(G - S).$$

By our hypothesis, $o(G - S) \leq |S| - 2k - 2 = |S'|$. Thus $o(G' - S') \leq |S'|$. By Theorem 2.3, G′ has a perfect matching. This proves that G is k*-extendable and completes the proof of our theorem. ❑

Theorem 3.7 implies a following corollary which was also proved by Lovasz [7].

Corollary 3.8: Let G be a graph on 2n vertices. Then G is bicritical if and only if for every $S \subseteq V(G)$, $|S| \geq 2$, $G - S$ has at most $|S| - 2$ odd components. ❑

4. Some sufficient conditions for k*-extendable graphs

In this section we establish a number of sufficient conditions for a graph to be k*-extendable. We start with a following result:

Lemma 4.1: Let G be a graph on 2n vertices and $0 \leq k \leq n - 2$. If $\delta(G) \geq n + k + 1$, then G is k*-extendable. Further, the bound is sharp.

Proof: Let u and v be vertices of G and $G' = G - \{u, v\}$. Since $\delta(G) \geq n + k + 1$, $\delta(G') \geq (n + k + 1) - 2 = (n - 1) + k$. By Theorem 2.2, G′ is k-extendable. Hence, G is k*-extendable as required.

To see that the bound is sharp, let $G_1 = K_{n+k}$, $G_2 = \overline{K}_{n-k}$ and $G = G_1 \vee G_2$. Clearly, $\delta(G) = n + k$. Let x and y be vertices of G_1 and M a matching of size k in $G_1 - \{x, y\}$. But then M does not extend to a perfect matching in $G - \{x, y\}$ since $G - (V(M) \cup \{x, y\}) = K_{n-k-2} \vee \overline{K}_{n-k}$. Thus G is not k*-extendable. ❑

Remark 4.1: There exists a graph on 2n vertices with minimum degree $n + k + 1$, $0 \leq k \leq n - 2$. Such a graph is $K_1 \vee K_{n+k+1} \vee K_{n-k-2}$ which is k*-extendable by Lemma 4.1.

As a corollary we have:

Corollary 4.2: Let G be a graph on $2n \geq 4$ vertices. If $\delta(G) \geq n + 1$, then G is bicritical. ❑

Theorem 4.3: Let G be a (k + 1)-extendable non-bipartite graph on 2n vertices; $0 \leq k \leq n - 2$, with $\delta(G) = n + k$. If $n - k - 1$ is even or $\kappa(G) \geq 2k + 3$, then G is k*-extendable.

Proof: The case $k = n - 2$ follows directly from Theorem 2.6. So we only need to prove the remaining case $0 \le k \le n - 3$.

Let u, v be vertices of G and M a matching of size k in $G - \{u, v\}$. Put $G' = G - (\{u, v\} \cup V(M))$. We need to show that G′ contains a perfect matching. First we assume that $\kappa(G) \ge 2k + 3$. Then G′ is connected. Suppose G′ has no perfect matching. Clearly $uv \notin E(G)$. Further, since $\nu(G') = 2n - 2k - 2$ it follows from Theorem 2.4 that $\delta(G') = n - k - 2$.

Let M′ be a maximum matching in G′. Then $|M'| \le n - k - 2$. If $|M'| \le n - k - 3$, then M cannot extend to a perfect matching in G since $G - V(M)$ contains at least 2 independent vertices, a contradiction. Thus $|M'| = n - k - 2$. Let x and y be the M′-unsaturated vertices of G′. Since $\nu(G') = 2n - 2k - 2$ and $\delta(G') = n - k - 2$, it follows from Lemma 2.5 that $N_{G'}(x) = N_{G'}(y)$. Further, no two vertices of $N_{G'}(x)$ are joined by an edge of M′ and $A = V(G') \setminus N_{G'}(x)$ is an independent set. Consequently, $|N_{G'}(x)| = n - k - 2$ and $|A| = n - k$.

Let $x' \in N_{G'}(x)$. If $ux' \in E(G)$, then $M_1 = M \cup \{ux'\}$ is a matching of size $k + 1$ in G which does not extend to a perfect matching since $G - V(M_1)$ contains A as an independent set of order $n - k$ and $\nu(G - V(M_1)) = 2n - 2k - 2$. Hence, $ux' \notin E(G)$ for all $x' \in N_{G'}(x)$. Similarly, $vx' \notin E(G)$ for all $x' \in N_{G'}(x)$.

Suppose $1 \le k \le n - 3$. Since $\delta(G) = n + k$, , there exists an edge ab of M such that $ua, vb \in E(G)$. But then $M_2 = (M \setminus \{ab\}) \cup \{ua, vb\}$ is a matching of size $k + 1$ which does not extend to a perfect matching in G since $G - V(M_2) = G'$, a contradiction. Hence, $k = 0$. If $N_{G'}(x)$ is an independent set , then G is a bipartite graph with bipartitioning sets A and $N_{G'}(x) \cup \{u, v\}$, contradicting the hypothesis of our theorem. Thus there exists an edge x_1x_2 of G with $x_1, x_2 \in N_{G'}(x)$. But then $\{x_1x_2\}$ does not extend to a perfect matching in $G - \{x_1, x_2\}$ since $G - \{x_1, x_2\}$ contains A as an independent set of order $n - k = n$ and $\nu(G - \{x_1, x_2\}) = 2n - 2$, contradicting the extendability of G. This proves that G′ has a perfect matching.

Next we suppose that $n - k - 1$ is even. If G′ is connected, then by applying a similar argument as above, G′ has a perfect matching. Hence we may assume that G′ is disconnected. Since $\nu(G') = 2n - 2k - 2$ and $\delta(G') \ge n - k - 2$, G′ contains exactly 2 components, H_1 and H_2 say. Further, $\nu((H_1) = \nu((H_2) = n - k - 1$. Then H_1 and H_2 are complete. Consequently, G′ has a perfect matching since $n - k - 1$ is even. This completes the proof of our theorem. ❑

Theorem 4.3 is best possible in the sense that there exists a $(k + 1)$-extendable non-bipartite graph G on 2n vertices with $\delta(G) = n + k$ and $\kappa(G) = 2k + 2$ but G is not k^*-extendable when $n - k - 1$ is odd. Let $G = (\overline{K}_k \vee \overline{K}_{k+2}) \vee 2K_{n-k-1}$ (see Figure

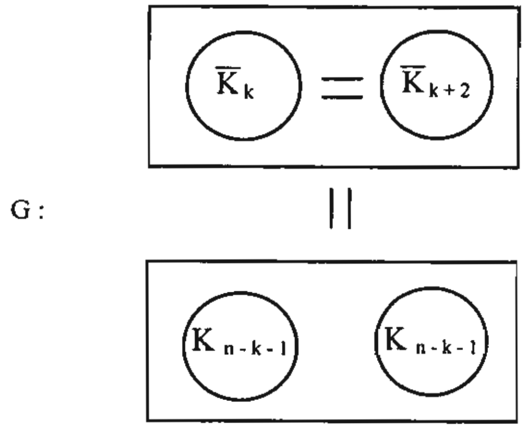


Figure 4.1

4.1.) For $n \geq 2k + 3$ it is not difficult to verify that G is $(k + 1)$-extendable with $\delta(G) = n + k$ and $\kappa(G) = 2k + 2$. But G is not k*-extendable when $n - k - 1$ is odd, since $G - (V(\overline{K}_k) \cup V(\overline{K}_{k+2})) = 2K_{n-k-1}$ has no perfect matching where $G[V(\overline{K}_k) \cup V(\overline{K}_{k+2})]$ contains a pair of vertices u and v and a matching M of size k for which $V(M) \cup \{u, v\} = (V(\overline{K}_k) \cup V(\overline{K}_{k+2}))$.

Theorem 4.4: Let G be a graph on 2n vertices with $\delta(G) = n + k$; $0 \leq k \leq n - 2$. If $n - k - 1$ is even and $\alpha(G) \leq n - k - 1$, then G is k*-extendable.

Proof: Let u and v be vertices of G and M a matching of size k in $G - \{u, v\}$. Put $G' = G - (\{u, v\} \cup V(M))$. Suppose G' is disconnected. Since $\delta(G') \geq n + k - (2k + 2) = n - k - 2$ and $\nu(G') = 2n - 2k - 2$, $G' = 2K_{n-k-1}$. Clearly, G' contains a perfect matching since $n - k - 1$ is even. Next we suppose that G' is connected and has no perfect matching. Let M' be a maximum matching in G'. By a similar argument as that in the proof of Theorem 4.3, there are exactly two M' - unsaturated vertices of G', x and y say. Further, $V(G') \setminus N_{G'}(x)$ is an independent set of order $n - k$. This contradicts the hypothesis that $\alpha(G) \leq n - k - 1$. Thus G' has a perfect matching. This proves that G is k*-extendable and completes the proof of our theorem. ❑

The condition in Theorem 4.4 is best possible in the sense that there exists a graph G on 2n vertices with minimum degree $n + k$; $0 \leq k \leq n - 2$, which is not k*-extendable when $n - k - 1$ is odd or $\alpha(G) \geq n - k$. Such graphs are $K_{2k+2} \vee 2K_{n-k-1}$ and $K_{2k} \vee (\overline{K}_{n-k} \vee \overline{K}_{n-k})$. Clearly, $K_{2k+2} \vee 2K_{n-k-1}$ is not k*-extendable if $n - k - 1$ is odd since deleting vertices u and v of K_{2k+2} and a matching of size k in $K_{2k+2} - \{u, v\}$ results in the graph $2K_{n-k-1}$. Further, the graph $K_{2k} \vee (\overline{K}_{n-k} \vee \overline{K}_{n-k})$ which contains an independent set of vertices of order $n - k$ is not k*-extendable since deleting vertices x and y of one of $\overline{K}_{n-k}$'s and a matching of size k in K_{2k} results in a graph $\overline{K}_{n-k-2} \vee \overline{K}_{n-k}$.

We need the following lemmas in establishing our main result in this section.

Lemma 4.5: Suppose G is a $(k + 1)$-extendable graph on 2n vertices; $0 \leq k \leq n - 2$ and M is a matching of size $t \leq k$ in G. For every non-empty even set $A \subseteq V(G) \setminus V(M)$ with $|A| < 2(n - k)$ there exists an edge e joining a vertex of A to a vertex of $V(G) \setminus (V(M) \cup A)$.

Proof: Suppose to the contrary that there exists a non-empty even set $A \subseteq V(G) \setminus V(M)$ with $|A| < 2(n - k)$ which vertices of A and $B = V(G) \setminus (V(M) \cup A)$ are not adjacent. Since G is $(k + 1)$-extendable, by Theorem 2.1, G is $(k + 2)$-connected. So there are at least $k + 2$ vertices of V(M) which are adjacent to vertices of A. Similarly, there are at least $k + 2$ vertices of V(M) which are adjacent to vertices of B. Since $|V(M)| = 2t \leq 2k$, there must be an edge of M, x_1y_1 say, such that $xx_1, yy_1 \in E(G)$ with $x \in A$ and $y \in B$. Then $(M \setminus \{x_1y_1\}) \cup \{xx_1, yy_1\}$ is a matching of size $t + 1 \leq k + 1$ in G which does not extend to a perfect matching in G since $A \setminus \{x\}$ becomes an isolated odd component in $G - (V(M) \cup \{x, y\})$. This contradicts the $(k + 1)$-extendability of G and completes the proof of our lemma. ❑

Lemma 4.6: Suppose G is a $(k + 1)$-extendable graph on 2n vertices; $0 \leq k \leq n - 2$. Let u and v be vertices of G and M a matching of size k in $G - \{u, v\}$. If $S \subseteq V(G_1)$ where $G_1 = G - (V(M) \cup \{u, v\})$ with $o(G_1 - S) \geq |S| + 2$, then $G[V(M) \cup S \cup \{u, v\}]$ contains a maximum matching of size exactly k. Further, $S \cup \{u, v\}$ is an independent set.

Proof: Clearly, $G_2 = G[V(M) \cup S \cup \{u, v\}]$ contains M as a matching of size k. Suppose M_1 is a matching of size $k + 1$ in G_2. Let

$$S_1 = V(G_2) \setminus V(M_1).$$

Then
$$\begin{aligned} |S_1| &= |V(G_2)| - |V(M_1)| \\ &= (2k + |S| + 2) - (2k + 2) \\ &= |S|. \end{aligned}$$

Since $o((G - V(M_1)) - S_1) = o(G_1 - S) \geq |S| + 2 > |S_1|$, M_1 does not extend to a perfect matching in G, contradicting the $(k + 1)$ - extendablity of G. Thus $G[V(M) \cup S \cup \{u, v\}]$ contains a maximum matching of size exactly k and hence $S \cup \{u, v\}$ is an independent set, completing the proof of our lemma. ❑

Lemma 4.7: Let G be a $(k + 2)$-extendable graph on 2n vertices; $0 \leq k \leq n - 3$. Suppose $G_1 = G - (V(M) \cup \{u, v\})$ has no perfect matching for some vertices u and v of G and a matching M of size k in $G - \{u, v\}$. Then there exists a set $S \subseteq V(G_1)$ such that

(i) $o(G_1 - S) = |S| + 2$ and $G_1 - S$ has no even components, and
(ii) each odd component of $G_1 - S$ is a singleton set.

Proof: Since G_1 has no perfect matching, there exists, by Theorem 2.3, a set $S \subseteq V(G_1)$ such that $o(G_1 - S) > |S|$. Because $\nu(G_1)$ is even, $o(G_1 - S)$ and $|S|$ have the same parity. So $o(G_1 - S) \geq |S| + 2$. Since $o((G - V(M)) - (S \cup \{u, v\})) = o(G_1 - S)$, if $o(G_1 - S) > |S| + 2 = |S \cup \{u, v\}|$, then $G - V(M)$ has no perfect matching. This implies that M does not extend to a perfect matching in G, contradicting the (k + 2) - extendability of G. Hence, $o(G_1 - S) = |S| + 2$.

Next we will show that $G_1 - S$ has no even components. Suppose to the contrary that H is an even component of $G_1 - S$. Further, let $S' = V(G) \setminus (V(M) \cup V(H))$. By Lemma 4.5, there exists an edge e = xy of G joining a vertex x of H to a vertex y of S′. Then $y \in S \cup \{u, v\}$. But then $M \cup \{e\}$ does not extend to a perfect matching in G since the odd components of $G_1 - S$ together with H - x form at least $|S| + 3$ odd components of $(G - (V(M) \cup \{x, y\})) - ((S \cup \{u, v\}) \setminus \{y\})$ and $|(S \cup \{u, v\}) \setminus \{y\}| = |S| + 1$. This contradicts the fact that G is (k + 2) - extendable. Hence, $G_1 - S$ has no even components. This proves (i).

Now we establish (ii). Suppose to the contrary that $G_1 - S$ contains H_0 as an odd component with $v(H_0) \geq 3$. Consider $E_1 = \{ab \in E(G) \mid a \in S \cup \{u, v\}; b \in V(H_0)\}$.

Suppose e_1 and e_2 are independent edges of E_1. Then $M_2 = M \cup \{e_1, e_2\}$ is a matching of size k + 2. But then M_2 does not extend to a perfect matching in G since $v(H_0) \geq 3$ and

$$\begin{aligned} |S| + 2 = o(G_1 - S) &= o((G - V(M_2)) - ((S \cup \{u, v\}) \setminus V(M_2))) \\ &> |S| \\ &= |(S \cup \{u, v\}) \setminus V(M_2)|. \end{aligned}$$

This contradicts the fact that G is (k + 2) - extendable. Hence, $G_3 = G[E_1] \cong K_{1,s}$ for some integer $s \geq 1$.

Let (V_1, V_2) be bipartition of $K_{1,s}$ where $V_1 = \{w\}$. Then $w \in V(H_0)$ or $w \in S \cup \{u, v\}$.

Suppose $w \in V(H_0)$. Figure 4.2 illustrates the situation with the edges of M drawn in solid lines.

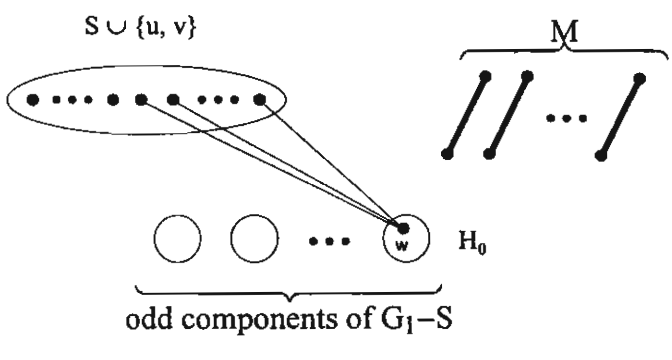


Figure 4.2

Since $v(H_0) \geq 3$, there exists a vertex w′ of H_0 such that $ww' \in E(G)$. Let $M_3 = M \cup \{ww'\}$. Clearly, $|M_3| = k + 1$ and $H_0 - V(M_3)$ becomes an isolated odd component in $G - V(M_3)$. Thus M_3 does not extend to a perfect matching in G, a contradiction to the (k + 2) - extendability of G. Hence, $w \notin V(H_0)$. Consequently, $w \in S \cup \{u, v\}$. Figure 4.3 illustrates the situation.

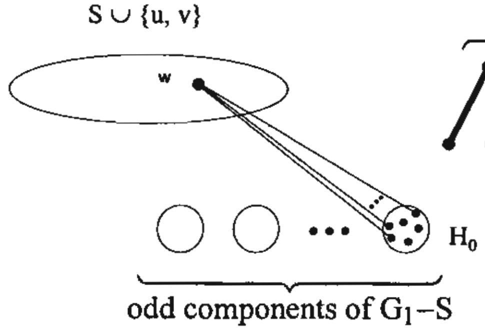


Figure 4.3

We will show that w is not adjacent to any vertex of $V(G) \setminus (V(M) \cup V(H_0))$. Suppose there exists a vertex $w_1 \in V(G) \setminus (V(M) \cup V(H_0))$ such that $ww_1 \in E(G)$. Let $M_4 = M \cup \{ww_1\}$. Clearly, $|M_4| = k + 1$. Since there is no edge joining a vertex of $(S \cup \{u, v\}) \setminus \{w\}$ to a vertex of $V(H_0)$ and $v(H_0)$ is odd, M_4 does not extend to a perfect matching in G, a contradiction. Hence, w is not adjacent to any vertex of $V(G) \setminus (V(M) \cup V(H_0))$. Let

$$A = V(H_0) \cup \{w\}$$

and

$$B = V(G) \setminus (V(M) \cup A).$$

Figure 4.4 depicts the situation.

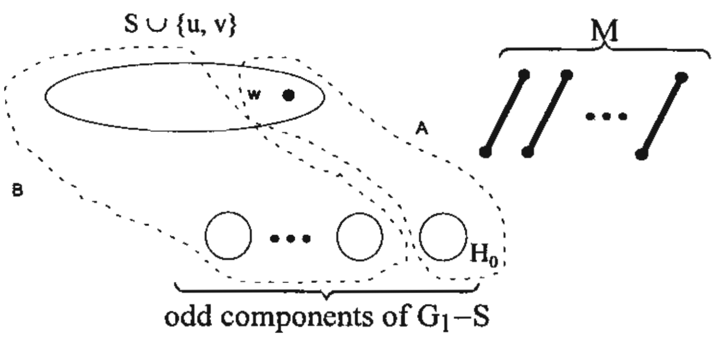


Figure 4.4

Clearly, $A \subseteq V(G) \setminus V(M)$ which $|A|$ is even and there is no edge joining a vertex of A to a vertex of B, contradicting Lemma 4.5. This proves (ii) and completes the proof of our lemma. ❑

Now we are ready to prove our main result.

Theorem 4.8: If G is a (k + 2)-extendable non-bipartite graph on 2n vertices; $0 \leq k \leq n - 3$, then G is k^*-extendable.

Proof: Suppose to the contrary that there exist vertices u and v of G and a matching M of size k in $G - \{u, v\}$ which does not extend to a perfect matching in $G - \{u, v\}$. Let

$G_1 = G - (V(M) \cup \{u, v\})$. Since G_1 has no perfect matching, by Lemma 4.7, there exists a set $S \subseteq V(G_1)$ such that $G_1 - S$ contains exactly $|S| + 2$ odd components, all of them are singletons. Let C be a set of vertices of these components. Clearly, C is an independent set and $|C| = |S| + 2$. Further, $V(G) = V(M) \cup \{u, v\} \cup S \cup C$. Note that, by Lemma 4.6, $G[V(M) \cup S \cup \{u, v\}]$ contains a maximum matching of size exactly k and $S \cup \{u, v\}$ is independent. This implies:

Claim 1: For every vertex w of $S \cup \{u, v\}$, if $wx \in E(G)$ where xy is an edge of M, then $zy \notin E(G)$ for every $z \in (S \cup \{u, v\}) \setminus \{w\}$.

We now establish a number of further claims.

Claim 2: $G[V(M) \cup C]$ contains a maximum matching of size exactly k. This claim follows immediately from the fact that $V(G) = V(M) \cup \{u, v\} \cup S \cup C$, $S \cup \{u, v\}$ and C are independent and $|C| = |S| + 2$.

Claim 3 : Every vertex w of $S \cup \{u, v\}$ is adjacent to at most one end vertex of an edge e of M.

Suppose to the contrary that there exist a vertex x' of $S \cup \{u, v\}$ and an edge $e = xy$ of M such that $x'x, x'y \in E(G)$. By Claim 1, $xy', yy' \notin E(G)$ for all $y' \in (S \cup \{u, v\}) \setminus \{x'\}$. Let $M_1 = (M \setminus \{xy\}) \cup \{x'x\}$. Since G is $(k + 2)$ - extendable, there is a perfect matching F containing the edges of M_1. Let $yz \in F$. Clearly, z is a vertex of C. Similarly, there exists a perfect matching F_1 containing the edges of $(M \setminus \{xy\}) \cup \{x'y\}$ and $xz_1 \in F_1$ where $z_1 \in C$. Then $z = z_1$; otherwise, $(M \setminus \{xy\}) \cup \{xz_1, yz\}$ becomes a matching of size $k + 1$ in $G[V(M) \cup C]$, a contradiction to Claim 2. By Claim 2, $xc, yc \notin E(G)$ for all $c \in C \setminus \{z\}$. Further, by similar argument to the one used in the proof of Lemma 4.6, $G[V(M_1) \cup C]$ contains a maximum matching of size exactly k. Thus $x'c \notin E(G)$ for all $c \in C \setminus \{z\}$.

Let $A_1 = \{x, y, z, x'\}$

and $B_1 = V(G) \setminus (V(M \setminus \{xy\}) \cup A_1)$.

By Lemma 4.5, there is an edge $e = wb$ joining a vertex w of A_1 to a vertex b of B_1. This implies that $w = z$. Then y becomes an isolated vertex of $G - V((M \setminus \{xy\}) \cup \{zb, xx'\})$ since $yy' \notin E(G)$ for all $y' \in (S \cup \{u, v\}) \setminus \{x'\}$ and $yc \notin E(G)$ for all $c \in C \setminus \{z\}$. This implies that $(M \setminus \{xy\}) \cup \{zb, xx'\}$ does not extend to a perfect matching in G, contradicting the $(k + 2)$ - extendability of G. This proves Claim 3.

The above argument can be used to prove:

Claim 4: Every vertex c of C is adjacent to at most one end vertex of an edge e of M.

Claim 5 : If $wx \in E(G)$ for some $w \in S \cup \{u, v\}$ and $xy \in M$, then $xc \notin E(G)$ for all $c \in C$.

Suppose to the contrary that there exist vertices $w_1 \in S \cup \{u, v\}$, $c_1 \in C$ and edge $x_1y_1 \in M$ such that $w_1x_1, x_1c_1 \in E(G)$. Let F_2 be a perfect matching containing the edges of $(M \setminus \{x_1y_1\}) \cup \{w_1x_1\}$. Then $y_1z \in F_2$. Since $G[V(M) \cup S \cup \{u, v\}]$ contains a maximum matching of size exactly k, $z \notin (S \cup \{u, v\} \setminus \{w_1\}$. Then $z \in C$. Since $x_1c_1 \in E(G)$ and c_1 is adjacent to at most one end vertex of an edge of M, $z \neq c_1$. Consequently, $(M \setminus \{x_1y_1\}) \cup \{x_1c_1, y_1z\}$ is a matching of size $k + 1$ in $G[V(M) \cup C]$, contradicting Claim 2. This proves Claim 5.

Claim 6 : For every edge $xy \in M$, if $xw \notin E(G)$ for all $w \in S \cup \{u, v\}$, then $yc \notin E(G)$ for all $c \in C$.
Suppose to the contrary that there exist edge $x_2y_2 \in M$ and a vertex $c_2 \in C$ such that $x_2w \notin E(G)$ for all $w \in S \cup \{u, v\}$ but $y_2c_2 \in E(G)$. Consider $M_2 = (M \setminus \{x_2y_2\}) \cup \{y_2c_2\}$. Clearly, $|M_2| = k$. Since $x_2w \notin E(G)$ for all $w \in S \cup \{u, v\}$, the set $S \cup \{u, v, x_2\}$ is independent. Because $G - V(M_2)$ contains $S \cup \{u, v, x_2\}$ and $C - \{c_2\}$ as independent sets of order $|S| + 3$ and $|S| + 1$ respectively, $G - V(M_2)$ does not have a perfect matching. Thus M_2 does not extend to a perfect matching in G. This contradicts the $(k + 2)$-extendability of G and completes the proof of Claim 6.

Now let $M = \{x_1y_1, x_2y_2, \dots, x_ky_k\}$. Consider x_1y_1. If $x_1w \notin E(G)$ for all $w \in S \cup \{u, v\}$, then, by Claim 6, $y_1c \notin E(G)$ for all $c \in C$. Put

$$X_1 = S \cup \{u, v\} \cup \{x_1\}$$

and

$$Y_1 = C \cup \{y_1\}.$$

If $x_1w_1 \in E(G)$ for some $w_1 \in S \cup \{u, v\}$, then, by Claim 5, $x_1c \notin E(G)$ for all $c \in C$. Further, by Lemma 4.6 and Claim 3, $y_1w \notin E(G)$ for all $w \in S \cup \{u, v\}$. Put

$$X_1 = S \cup \{u, v\} \cup \{y_1\}$$

and

$$Y_1 = C \cup \{x_1\}.$$

For each edge $x_iy_i \in M$; $2 \leq i \leq k$, we can construct sets X_i and Y_i in a similar fashion as we do with the edge x_1y_1. Until the step k, we have

$$X_k = S \cup \{u, v\} \cup \{a_1, a_2, \dots, a_k\}$$

and

$$Y_k = C \cup \{b_1, b_2, \dots, b_k\}$$

where a_i and b_i $(1 \leq i \leq k)$ are end vertices of edge $a_ib_i = x_iy_i$ of M. Clearly $|X_k| = |Y_k| = |S| + k + 2$. Further, by our construction, there is no edge joining a vertex of $S \cup \{u, v\}$ to a vertex of $\{a_1, a_2, \dots, a_k\}$ and a vertex of C to a vertex of $\{b_1, b_2, \dots, b_k\}$.

Since $S \cup \{u, v\}$ and C are independent sets, to show that (X_k, Y_k) is a bipartition of G it is sufficient to prove that $\{a_1, a_2, \dots, a_k\}$ and $\{b_1, b_2, \dots, b_k\}$ are independent sets. Suppose to the contrary that $\{a_1, a_2, \dots, a_k\}$ is not independent. Without any loss of generality, we may assume that $a_1a_2 \in E(G)$. If $b_1w_1 \in E(G)$ for some $w_1 \in S \cup \{u, v\}$, then $M_3 = \{a_1a_2, b_1w_1\} \cup \{a_ib_i \mid 3 \leq i \leq k\}$ is a matching of size $(k - 2) + 2 = k$ in G which does not extend to a perfect matching in G since $G - V(M_3)$ contains $(S \cup \{u, v\}) \setminus \{w_1\}$ and $C \cup \{b_2\}$ as independent sets of order $|S| + 1$ and $|S| + 3$ respectively. Thus $b_1w \notin E(G)$ for all $w \in S \cup \{u, v\}$. Similarly, $b_2w \notin E(G)$ for all $w \in S \cup \{u, v\}$.

Let $A_2 = \{b_1, b_2\}$

and

$$B_2 = C \cup S \cup \{u, v\}.$$

Figure 4.5 depicts the situation with the edges of M drawn in solid lines.

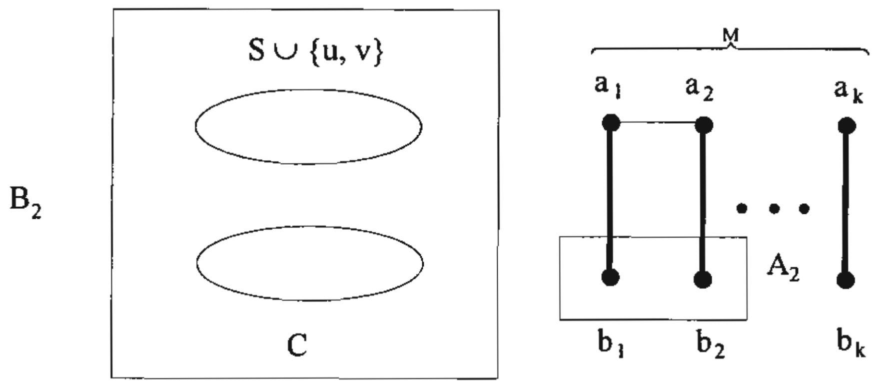


Figure 4.5

Let $M_4 = (M \setminus \{a_1b_1, a_2b_2\}) \cup \{a_1a_2\}$. Clearly, $|M_4| = k - 1$. Notice that $A_2 \subseteq V(G) \setminus V(M_4)$ and $B_2 = V(G) \setminus (V(M_4) \cup A_2)$. By lemma 4.5, there is an edge e joining a vertex of A_2 to a vertex of B_2 which is impossible since b_1 and b_2 are not adjacent to any vertex of $C \cup S \cup \{u, v\}$. This contradiction proves that $\{a_1, a_2, \dots, a_k\}$ is an independent set. Similarly, $\{b_1, b_2, \dots, b_k\}$ is an independent set. Hence, G is a bipartite graph with bipartition (X_k, Y_k). This contradicts the hypothesis of our theorem and completes the proof. ❑

An immediate consequence of Theorem 4.8 is the following result of Plummer [10].

Corollary 4.9: If G is a 2-extendable non-bipartite graph on $2n \geq 6$ vertices, then G is bicritical. ❑

A converse of Theorem 4.8 is not true. For integers n, k; $0 \leq k \leq n - 3$, let $G_1 = K_{n+k+1}$, $G_2 = \overline{K}_{n-k-1}$. Clearly, $G = G_1 \vee G_2$ is a graph on 2n vertices with minimum degree $n + k + 1$. By Lemma 4.1, G is k*-extendable. Let M be a matching of size $k + 2$ in G_1. Then $G - V(M) = K_{n-k-3} \vee \overline{K}_{n-k-1}$ has no perfect matching. Thus G is not $(k + 2)$-extendable.

For $1 \leq k \leq n - 1$, let $\mathcal{G}(2n, k)$ denote the class of k-extendable non-bipartite graphs on 2n vertices. Further, for $0 \leq k \leq n - 2$, let $\mathcal{G}^*(2n, k^*)$ denote the class of k*-extendable graphs on 2n vertices. Then Lemma 3.3, Theorems 2.1 and 4.8 imply that these classes are "nested" as follows :

$\mathcal{G}(2n, 1) \supset \mathcal{G}^*(2n, 0^*) \supset \mathcal{G}(2n, 2) \supset \mathcal{G}^*(2n, 1^*) \supset \dots \supset \mathcal{G}(2n, n - 2) \supset \mathcal{G}^*(2n, (n - 3)^*) \supset \mathcal{G}(2n, n - 1)$.

5. Minimum Degree of k*-Extendable Graphs

In this section we establish a necessary condition, in terms of the minimum degree, for k*-extendable graphs. We start with a following lemma:

Lemma 5.1: Let G be a k*-extendable graph on 2n vertices; $1 \le k \le n - 2$. Then G is $(k + 3)$-connected.

Proof: Since $G - \{u, v\}$ is k-extendable for every pair of vertices u and v of G, $G - \{u, v\}$ is $(k + 1)$-connected by Theorem 2.1. Thus G is $(k + 3)$-connected. ❑

Remark 5.1: Note that for any positive integer r, a graph $K_2 \vee 2K_{2r}$ is 0*-extendable which is 2-connected. Thus the bound on k in Lemma 5.1 is sharp. However, if G is 0*-extendable, then $\delta(G) \ge 3$ by the definition of 0*-extendable graphs. This fact together with Lemma 5.1 assures that if G is a k*-extendable graph on 2n vertices, $0 \le k \le n - 2$, then $\delta(G) \ge k + 3$.

Our next result concerns the size of a maximum matching in an induced subgraph of a neighbour set of a vertex in a k*-extendable graph.

Lemma 5.2: Let G be a k*-extendable graph on 2n vertices; $0 \le k \le n - 2$, and u a vertex of degree $k + t$; $3 \le t \le k + 2$, of G. Then $G[N_G(u)]$ has a matching of size at most $t - 3$.

Proof: Suppose not. Then there exists a vertex u of G of degree $k + t$; $3 \le t \le k + 2$ such that $G[N_G(u)]$ has a maximum matching of size at least $t - 2$.

Let M be a maximum matching in $G[N_G(u)]$ of size $s \ge t - 2$. Since G is k*-extendable and $d_G(u) \le 2k + 2$, $s < k$. Further, $|N_G(u) \setminus V(M)| \ge 3$. Suppose $|\overline{N}_G(u)| = 1$. Then $d_G(u) = |N_G(u)| = 2n - 2$. Since $d_G(u) \le 2k + 2$ and the assumption on k, $k = n - 2$. Let $u' \in \overline{N}_G(u)$. Because G is k*-extendable, $G - \{u, u'\} = G[N_G(u)]$ contains a perfect matching. Thus $n - 1 = |M| = s < k = n - 2$, a contradiction. Hence, $|\overline{N}_G(u)| \ge 2$. Let x and y be vertices of $\overline{N}_G(u)$ and $G^* = G - \{x, y\}$. Clearly G^* is k-extendable. Further, $N_G(u) = N_{G^*}(u)$ and $G[N_G(u)] = G^*[N_{G^*}(u)]$.

Let F be a perfect matching in G^* containing M. Then there exists a vertex v of $N_G(u) \setminus V(M)$ such that $uv \in F$. Put

$$F_1 = \{ab \in F \mid a \in N_G(u) \setminus (V(M) \cup \{v\}),\ b \in \overline{N}_G(u)\}.$$

Since $|N_G(u) \setminus V(M)| \ge 3$, $|F_1| = k + t - 2s - 1 \ge 2$. Let $wz \in F_1$ where $w \in N_G(u) \setminus (V(M) \cup \{v\})$ and $z \in \overline{N}_G(u)$. Consider $F_2 = M \cup (F_1 \setminus \{wz\})$. Since $s \ge t - 2$,

$$|F_2| = s + (k + t - 2s - 1) - 1 = k + t - s - 2 \le k.$$

But then F_2 does not extend to a perfect matching in $G - \{v, w\}$ as u becomes an isolated vertex in $G - (\{v, w\} \cup V(F_2))$. This contradicts k*-extendability of G and completes the proof of our lemma. ❑

We now prove the main result in this section.

Theorem 5.3: If G is a k*-extendable graph on 2n vertices; $0 \le k \le n - 2$, then $k + 3 \le \delta(G) \le n - 2$ or $\delta(G) \ge 2k + 3$.

Proof: The assertion is true for k = 0 by Remark 5.1. Suppose to the contrary that G is a k*-extendable graph on 2n vertices, $1 \le k \le n - 2$, with $n - 1 \le \delta(G) \le 2k + 2$. Let u be a vertex of G with $d_G(u) = \delta(G) = r$ and M a maximum matching in $G[N_G(u)]$. By Lemma 5.2, $|M| \le r - k - 3 \le k - 1$ and $|N_G(u) \setminus V(M)| \ge r - 2(r - k - 3) = 2k - r + 6 \ge 4$.

By applying similar argument as in the proof of Lemma 5.2, $|\overline{N}_G(u)| \ge 2$. Let $x, y \in \overline{N}_G(u)$ and $G_1^* = G - \{x, y\}$. Since $|N_G(u) \setminus V(M)| \ge 4$, there is a vertex $v \in N_G(u) \setminus V(M)$. Because G is k*-extendable and $M \cup \{uv\}$ is a matching in G of size at most k, there is a perfect matching F in G_1^* containing $M \cup \{uv\}$. Let

$$F_1 = \{ab \in F \mid a \in N_G(u) \setminus (V(M) \cup \{v\}), b \in \overline{N}_G(u) \setminus \{x, y\}\}$$

and

$$F_2 = \{ab \in F \mid a, b \in \overline{N}_G(u) \setminus \{x, y\}\}.$$

Clearly, $|F_1| = r - 2|M| - 1$
and

$$\begin{aligned} |F_2| &= \frac{1}{2}[(2n - r - 3) - (r - 2|M| - 1)] \\ &= n - r + |M| - 1. \end{aligned}$$

Suppose $G[\overline{N}_G(u)]$ contains M′ as a matching of size $n - r + |M| \le n - k - 3 \le k$.
Since

$$\begin{aligned} |\overline{N}_G(u) \setminus V(M')| &= (2n - r - 1) - 2(n - r + |M|) \\ &= r - 2|M| - 1 \\ &\ge r - 2(r - k - 3) - 1 \\ &= 2k - r + 5 \\ &\ge 3, \end{aligned}$$

there exist vertices $x', y' \in \overline{N}_G(u) \setminus V(M')$. But then M′ does not extend to a perfect matching in $G_2^* = G - \{x', y'\}$ since $G_2^*[N_G(u) \setminus V(M)] = G[N_G(u) \setminus V(M)]$ is an independent set of order $r - 2|M|$ and

$$\begin{aligned} |\overline{N}_{G_2^*}(u) \setminus (V(M') \cup \{x', y'\})| &= (2n - r - 1) - [2(n - r + |M|) + 2] \\ &= r - 2|M| - 3. \end{aligned}$$

Hence, a size of maximum matching in $G[\overline{N}_G(u)]$ is $n - r + |M| - 1$.

Now let w, z be vertices of $N_G(u) \setminus (V(M) \cup \{v\})$ and $G_3^* = G - \{w, z\}$. Clearly, $M \cup \{uv\}$ is a matching of size at most k in G_3^*. Since $N_G(u) \setminus (V(M) \cup \{v, w, z\})$ is an independent set of order $r - 2|M| - 3$ and $\overline{N}_G(u)$ has a maximum matching of size $n - r + |M| - 1$, if $M \cup \{uv\}$ extended to a perfect matching F′ in G_3^*, then F′ would have to map the $r - 2|M| - 3$ vertices of $N_G(u) \setminus (V(M) \cup \{v, w, z\})$ onto $(2n - r - 1) - 2(n - r + |M| - 1) = r - 2|M| + 1$ vertices of $\overline{N}_{G_3^*}(u) \setminus V(M'')$ where M″ is a maximum matching in $G[\overline{N}_G(u)]$, which is impossible. This contradicts the k*-extendability of G and completes the proof of our theorem. ❑

Corollary 5.4: Let G be a k*-extendable graph on 2n vertices; $0 \leq k \leq n - 2$. Then G is complete or $n \geq k + 3$.

Proof: Suppose $n \leq k + 2$. By Theorem 5.3, $\delta(G) \geq 2k + 3$. This implies that $\nu(G) = 2k + 4$. Consequently, G is complete. ❑

Corollary 5.5: Let G be a k*-extendable graph on 2n vertices with $\delta(G) \leq 2k + 2$. Then $2n \geq 4k + 8$.

Proof: By Theorem 5.3, it follows that $2k + 2 \leq n - 2$. Thus $2n \geq 4k + 8$ as required. ❑

Next we consider the realizability problem associated with Theorem 5.3. We start with the following lemma.

Lemma 5.6: For any non-negative integers n, k, r with $2k + 3 \leq r \leq 2n - 1$, there exists a k*-extendable graph on 2n vertices with minimum degree r.

Proof: Let $G_1 = K_1$, $G_2 = K_r$ and $G_3 = K_{2n-r-1}$. Then $G = G_1 \vee G_2 \vee G_3$ is a graph on 2n vertices with minimum degree r. Figure 5.1 depicts the graph G. Note that in our diagrams a "double line" denotes the join.

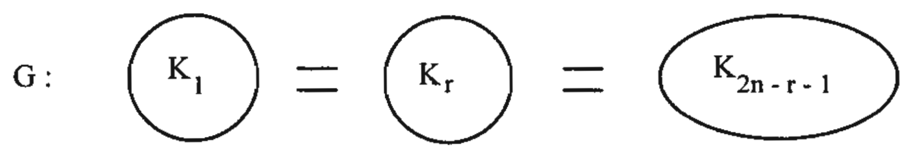


Figure 5.1

Let u and v be any pair of vertices of G and M a matching of size k in G - {u, v}. Put

$$A = \{u, v\} \cup V(M).$$

If $V(G_1) \subseteq A$, then $G - A = K_{2n-2k-2}$ has a perfect matching. Next we suppose that $V(G_1) \cap A = \varnothing$. Let $s = |A \cap V(G_2)|$. Then $G - A = K_1 \vee K_{r-s} \vee K_{2n+s-r-2k-3}$. Clearly, $0 \leq s \leq 2k + 2 < r$ and $2k + 2 - s \leq 2n - r - 1$. Thus $r - s \geq 1$ and $2n + s - r - 2k - 3 \geq 0$. Consequently, G - A contains a perfect matching. Hence, G is k*-extendable as required. ❑

Lemma 5.7: For any positive integers n, k, r with $k + 3 \leq r \leq 2k + 2$ and $2n = 4k + 2s + 8$ for some integer $s \geq 0$, there exists a k*-extendable graph on 2n vertices with minimum degree r.

Proof: For integers s and t with $3 \leq t \leq k + 2$ and $s \geq 0$, let $G_1 = K_1$, $G_2 = \overline{K}_{t-3}$, $G_3 = \overline{K}_{k+3}$ and $G_4 = K_{3k-t+2s+7}$. Then $G = G_1 \vee (G_2 \vee G_3) \vee G_4$ is a graph of order $4k + 2s + 8$ with minimum degree $k + t$. Figure 5.2 illustrates our notation.

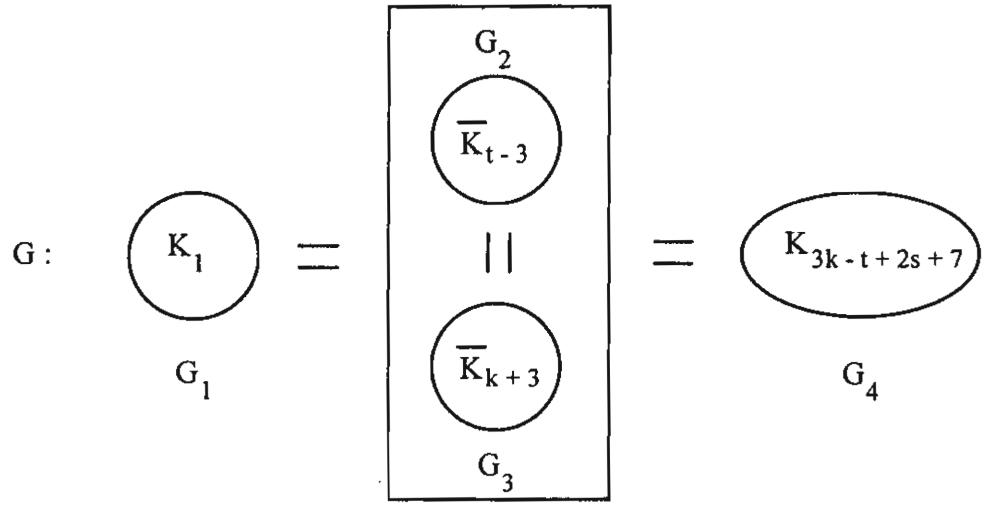


Figure 5.2

Observe that $G_2 \vee G_3$ contains a maximum matching of size $t - 3 \le k - 1$. We will show that G is k*-extendable. Let $u, v \in V(G)$ and M a matching of size k in $G - \{u, v\}$. To complete the proof of our lemma we need to show that $G' = G - (V(M) \cup \{u, v\})$ contains a perfect matching. Let

$$A = V(M) \cup \{u, v\}$$
$$a_1 = |V(G_1) \cap A|$$
$$a_2 = |V(G_2) \cap A|$$
$$a_3 = |V(G_3) \cap A|$$

and

$$a_4 = |V(G_4) \cap A|.$$

Notice that $a_1 + a_2 + a_3 + a_4 = |A| = 2k + 2$ and $0 \le a_1 \le 1$. We distinguish two cases according to a_1.

Case 1: $a_1 = 1$.
Then $G' = (\overline{K}_{t-3-a_2} \vee \overline{K}_{k+3-a_3}) \vee K_{k+2s+6-t+a_2+a_3}$. Let M_1 be a maximum matching in $G'[(V(G_2) \cup V(G_3)) \setminus A]$. Then

$$|M_1| = \min\{t - 3 - a_2, k + 3 - a_3\}.$$

Consider $B = (V(G_2) \cup V(G_3)) \setminus (A \cup V(M_1))$. Clearly,

$$|B| = (k + t) - (a_2 + a_3 + 2|M_1|)$$
$$= \begin{cases} k - t + a_2 - a_3 + 6, & \text{for } |M_1| = t - 3 - a_2 \\ t - k - a_2 + a_3 - 6, & \text{for } |M_1| = k + 3 - a_3. \end{cases}$$

Since $t \le k + 2$,

$$|V(G_4) \setminus A| - |B| = k + 2s + 6 - t + a_2 + a_3 - |B|$$
$$= \begin{cases} 2s + 2a_3 \ge 0, & \text{for } |M_1| = t - 3 - a_2 \\ 2k - 2t + 2s + 2a_2 + 12 \ge 8, & \text{for } |M_1| = k + 3 - a_3 \end{cases}$$

and then there is a matching M_2 which maps each vertex of B to a vertex of $V(G_4) \setminus A$. Clearly,

$$G'[V(G_4) \setminus (A \cup V(M_2))] = \begin{cases} K_{2s+2a_3}, & \text{for} \quad |M_1| = t-3-a_2 \\ K_{2k-2t+2s+2a_2+12}, & \text{for} \quad |M_1| = k+3-a_3 \end{cases}$$

contains a perfect matching M_3. Hence, $M_1 \cup M_2 \cup M_3$ forms a perfect matching in G'.

Case 2: $a_1 = 0$.
If $V(G_3) \setminus A = \emptyset$, then $|M| \geq (k+3) - 2 = k+1$, a contradiction. Thus $V(G_3) \setminus A \neq \emptyset$. Let $xy \in E(G)$ where $x \in V(G_1)$ and $y \in V(G_3) \setminus A$. Clearly, $G' - \{x, y\} = (\overline{K}_{t-3-a_2} \vee \overline{K}_{k+2-a_3}) \vee K_{k+2s+5-t+a_2+a_3}$. By similar argument as in the proof of Case 1, $G' - \{x, y\}$ contains F as a perfect matching in $G' - \{x, y\}$. Hence, $F \cup \{xy\}$ forms a perfect matching in G'. This completes the proof of our lemma. ❑

Let G be a k*-extendable graph on 2n vertices, $0 \leq k \leq n-2$, with minimum degree r. By Theorem 5.3 and Corollary 5.5, notice that

$$r \in \begin{cases} [k+3,\ 2n-1], & \text{for } n \geq 2k+4 \\ [2k+3,\ 2n-1], & \text{for } n \leq 2k+3. \end{cases} \qquad (5.1)$$

Corollary 5.5 and Lemmas 5.6 and 5.7 yield the following theorem:

Theorem 5.8: For any integers n, k and r with $0 \leq k \leq n-2$, there exists a k*-extendable graph on 2n vertices with minimum degree r if r satisfies (5.1). ❑

6. A Characterization of (n - 2)*-Extendable and (n - 3)*-Extendable Graphs

We now turn our attention to a characterization of k*-extendable graphs on 2n vertices for k = n - 2 and n - 3. We begin with (n - 2)*-extendable graphs.

Theorem 6.1: G is an (n - 2)*-extendable graph on $2n \geq 4$ vertices if and only if G is K_{2n}.

Proof: It follows directly from Corollary 5.4 and the fact that K_{2n} is k*-extendable for $0 \leq k \leq n-2$. ❑

Our next result concerns an independence number of k*-extendable graphs which is useful for establishing a characterization of (n - 3)*-extendable graphs.

Lemma 6.2: Let G be a k*-extendable graph on 2n vertices; $0 \leq k \leq n-2$. Then $\alpha(G) \leq n-k-1$.

Proof: The case k = n - 2 is obvious since the only (n - 2)*-extendable graph is K_{2n}. So we need to consider the case $0 \leq k \leq n-3$. Suppose to the contrary that $\alpha(G) \geq$

$n - k$. Let S be an independent set of vertices of G of order $n - k$. Further let $u \in S$ and $v \in N_G(u)$. Since G is k*-extendable, there is a perfect matching F containing the edge uv. Let $F_1 = \{xy \in F \mid x \in S\}$. Then $|F \setminus F_1| = k$. Next let z, w be vertices of G such that $zz', ww' \in F_1$ and $z', w' \in S$. Then $G' = G - (V(F \setminus F_1) \cup \{z, w\})$ contains S as an independent set of order $n - k$. Since $\nu(G') = 2n - 2k - 2$, G′ has no perfect matching. This implies that $F \setminus F_1$ cannot extend to a perfect matching in $G - \{z, w\}$, a contradiction to the extendability of G. Hence, $\alpha(G) \le n - k - 1$, completing the proof of our lemma. ❑

Lemma 6.2 is best possible since there exists a k*-extendable graph G with $\alpha(G) = n - k - 1$. Such a graph is $K_{2k+2} \vee (\overline{K}_{n-k-1} \vee \overline{K}_{n-k-1})$.

We now characterize $(n - 3)$*-extendable graphs on 2n vertices.

Theorem 6.3: Let G be a graph on $2n \ge 6$ vertices. Then G is $(n - 3)$*-extendable if and only if G:

(i) is K_{2n}, or
(ii) has minimum degree $2n - 2$, or
(iii) has minimum degree $2n - 3$ and $\alpha(G) \le 2$.

Proof: The necessity follows directly from Theorem 5.3 and Lemma 6.2. Now we prove the sufficiency. Clearly, K_{2n} is $(n - 3)$*-extendable. If $\delta(G) = 2n - 2$, then, by Lemma 4.1, G is $(n - 3)$*-extendable. The last case follows directly from Theorem 4.4. This completes the proof of our theorem. ❑

Remark 6.1: There exist $(n - 3)$*-extendable graphs for each type specified in Theorem 6.3. Clearly, $2K_1 \vee K_{2n-2}$ satisfies type (ii) and $2K_2 \vee K_{2n-4}$ is of type (iii).

A consequence of Theorems 2.7 and 6.3 is the following theorem:

Theorem 6.4: Let G be a graph on $2n \ge 10$ vertices. Then G is $(n - 3)$*-extendable if and only if G is $(n - 2)$-extendable non-bipartite . ❑

Let $\mathcal{G}(2n, k)$ and $\mathcal{G}^*(2n, k^*)$ denote the classes of k-extendable non-bipartite graphs and k*-extendable graphs on 2n vertices, respectively. Theorem 6.4 assures that for $2n \ge 10$

$$\mathcal{G}(2n, n - 2) = \mathcal{G}^*(2n, (n - 3)^*).$$

The bound on the number of vertices of a graph in Theorem 6.4 is best possible since there exist $(n - 2)$-extendable non-bipartite graphs on $2n = 6$ and $2n = 8$ vertices which are not $(n - 3)$*-extendable. Such graphs are displayed in Figure 6.1

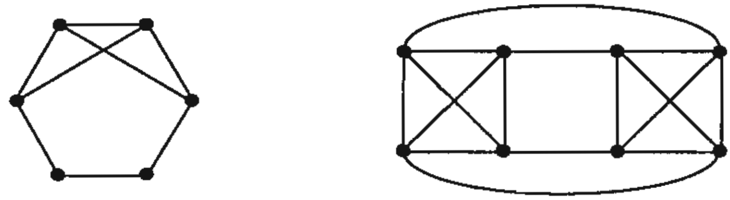


Figure 6.1

For $2n \geq 6$, Lemma 3.3 implies that $\mathcal{G}^*(2n, (n-3)^*) \subseteq \mathcal{G}(2n, n-2)$. The graphs in Figure 6.1 ensure that $\mathcal{G}^*(2n, (n-3)^*)$ is the proper subclass of $\mathcal{G}(2n, n-2)$. By take advantage of a characterization of (n - 2)-extendable graphs on $2n \geq 6$ vertices, proved by Ananchuen and Caccetta [3, 4], we can now state the following corollary.

Corollary 6.5: For $2n \geq 6$, $|\mathcal{G}(2n, n-2) \setminus \mathcal{G}^*(2n, (n-3)^*)| = 11$. Such graphs are displayed in Figure 6.2.

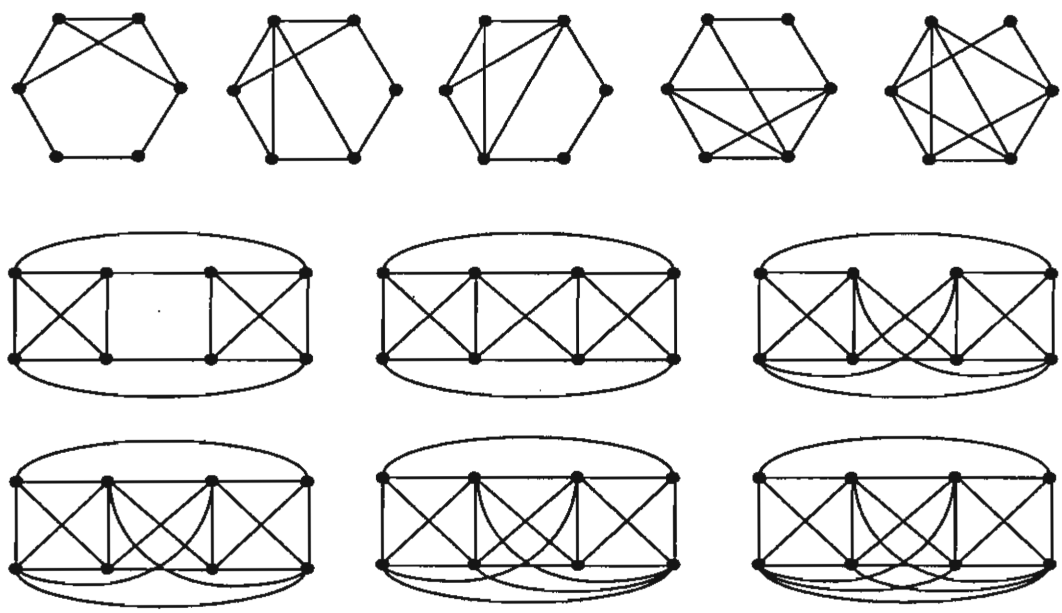


Figure 6.2 ❑

7. The independence number of a minimum cutset

In this section, we investigate the independence number of a minimum cutset of strongly k-extendable graphs. By Theorem 6.1, the only $(n-2)^*$-extendable graph on 2n vertices is K_{2n} which is clearly $(2n-1)$-connected. Hence, in the rest of this paper, we will restrict our attention to k^*-extendable graphs on 2n vertices for $0 \leq k \leq n-3$. It follows directly from the definition of bicritical graphs (0^*-extendable) that such graphs are 2-connected. A graph $K_2 \vee 2K_{2r}$ for any positive integer r is an example of a bicritical graph which is 2-connected. For $1 \leq k \leq n-3$, it follows from Lemma 5.1 that k^*-extendable graphs on 2n vertices are $(k+3)$-connected. Our first result establishes the independence number of a minimum cutset of k^*-extendable graphs.

Theorem 7.1: Let G be a k^*-extendable graph on 2n vertices with $2 \leq k \leq n-3$ and suppose $S \subseteq V(G)$ is a minimum cutset of G with $|S| = k + t$ for $t \geq 3$, then $\alpha(G[S]) \geq k + 6 - t$ or $\alpha(G[S]) \leq 2$.

Proof: Suppose to the contrary that there is a minimum cutset S of G with $|S| = k + t$, $t \geq 3$ and $3 \leq \alpha(G[S]) \leq k + 5 - t$. Then $k \geq t - 2$ and $2|M(S)| \geq (k + t) - (k + 5 - t) = 2t - 5$. Thus $|S| \geq 2t - 2$ and $|M(S)| \geq t - 2$. Let M be a matching of size $t - 2$ in G[S] and let u and v be vertices of $S \setminus V(M)$. Such vertices exist since $|S| \geq 2t - 2$. Put

$$S_1 = S \setminus (V(M) \cup \{u, v\}).$$

Then $|S_1| = (k + t) - 2(t - 2) - 2 = k - t + 2 \geq 0$. Let $S_1 = \{x_1, x_2, \dots, x_{k-t+2}\}$. Further, let $C_1, C_2, \dots, C_r$ be components of $G - S$. We claim that $|V(C_i)| \leq k - t + 1$ for all i, $1 \leq i \leq r$. Suppose to the contrary that there exists a component C_j with $|V(C_j)| \geq k - t + 2$. By Theorem 2.8, there is a matching M_1 which matches vertices of S_1 into $V(C_j)$. Let $M_1 = \{x_1y_1, x_2y_2, \dots, x_{k-t+2}y_{k-t+2}\}$. Clearly, $M \cup M_1$ is a matching of size $(t - 2) + (k - t + 2) = k$. Since $G - \{u, v\}$ has a perfect matching containing all the edges of $M \cup M_1$, $C_j \setminus (V(M_1))$ is an even component of $G - (S \cup V(M_1))$.

Now x_1 must be adjacent to some vertex $w_1 \in V(C_i)$ for some $i \neq j$ since S is a minimum cutset. Then $M_2 = (M \cup M_1 \cup \{x_1w_1\}) \setminus \{x_1y_1\}$ is a matching of size k which does not extend to a perfect matching in $G - \{u, v\}$ since M_2 covers $S \setminus \{u, v\}$ and $G - (S \cup V(M_2))$ contains $C_j \setminus V(M_2)$ as an isolated odd component, a contradiction. Hence, $|V(C_i)| \leq k - t + 1$ for all i, $1 \leq i \leq r$.

Next we let $V(C_1) = \{w_1, w_2, \dots, w_m\}$ where $m = |V(C_1)|$. By Theorem 2.8, there is a matching M_3 which matches vertices of $V(C_1)$ into S_1. Let this matching be $\{x_1w_1, x_2w_2, \dots, x_mw_m\}$. Clearly, $|S_1 \setminus V(M_3)| = k - t + 2 - m \geq 1$. Suppose $\left|\bigcup_{i=2}^{r} V(C_i)\right| > k - t + 2 - m$. Then, in view of Theorem 2.8, there is a matching M_4 of size $k - t + 3 - m$ which matches vertices of $\{x_m, x_{m+1}, \dots, x_{k-t+2}\}$ into $\bigcup_{i=2}^{r} V(C_i)$. Let $x_mz \in M_4$ where $z \in \bigcup_{i=2}^{r} V(C_i)$. Now $M_5 = (M \cup (M_3 \setminus \{x_mw_m\}) \cup M_4)$ is a matching of size $(t - 2) + (m - 1) + (k - t + 3 - m) = k$ in $G - \{u, v\}$ which does not extend to a perfect matching in $G - \{u, v\}$ since M_5 covers $S \setminus \{u, v\}$ and $G - (S \cup V(M_5))$ contains w_m as an isolated vertex. Thus $\left|\bigcup_{i=2}^{r} V(C_i)\right| \leq k - t + 2 - m$. But then

$$2n = \nu(G) = |S| + \left|\bigcup_{i=1}^{r} V(C_i)\right| \leq k + t + m + (k - t + 2 - m) = 2k + 2 \leq 2n - 4,$$

a contradiction. This completes the proof of our theorem. ❑

Corollary 7.2: Let G be a k*-extendable graph on 2n vertices with $2 \le k \le n-3$ and suppose $S \subseteq V(G)$ is a minimum cutset of G with $|S| = k + t$ for $3 \le t \le k + 2$, then $|M(S)| \le t - 3$.

Proof: It follows directly from the proof of Theorem 7.1. ❑

As a consequence of Lemma 5.1 and Corollary 7.2, we have the following corollary:

Corollary 7.3: Let G be a k*-extendable graph on 2n vertices; $2 \le k \le n - 3$. If $S \subseteq V(G)$ is a cutset of G with $|S| = k + 3$, then S is independent. ❑

Remark 7.1: For $n \ge 3$, a graph $K_{1,3} \vee 2K_{2n}$ is 1*-extendable which contains $K_{1,3}$ as a cutset of order 4. Clearly, $\alpha(K_{1,3}) = 3$. Hence, the lower bound on k in Theorem 7.1 and Corollaries 7.2 and 7.3 is best possible.

Theorem 4.8 together with Theorem 7.1 yields the following corollary:

Corollary 7.4: Let G be a k-extendable graph on 2n vertices with $4 \le k \le n - 1$ and suppose $S \subseteq V(G)$ is a minimum cutset of G with $|S| = k + t - 2$ for $t \ge 3$, then $\alpha(G[S]) \ge k + 4 - t$ or $\alpha(G[S]) \le 2$. ❑

We conclude this section by establishing a necessary condition, in terms of connectivity, for k*-extendable graphs which are locally connected. A graph G is said to be *locally connected* if for every vertex u of G, the induced subgraph $G[N_G(u)]$ is connected.

Theorem 7.5: Let G be a k*-extendable graph on 2n vertices with $2 \le k \le n - 3$. If G is locally connected, then G is (k+4)-connected.

Proof: Suppose to the contrary that G is not (k + 4)- connected. By Lemma 5.1, $\kappa(G) = k + 3$. Let S be a cutset of order k + 3 of G. Then S is independent by Corollary 7.3. But then $G[N_G(u)]$ is disconnected for any vertex $u \in S$, contradicting the locally connected of G. Hence, G is (k + 4)-connected as required. ❑

Remark 7.2: (1) For an odd integer $n \ge 5$, $G_1 = K_2 \vee 2K_{n-1}$ and $G_2 = K_4 \vee (K_{n-1} \cup K_{n-3})$ are k*-extendable for k = 0 and 1, respectively. Clearly, G_1 and G_2 are locally connected but $\kappa(G_1) = 2 < 4$ and $\kappa(G_2) = 4 < 5$. Hence, the lower bound on k in Theorem 7.5 is best possible.

(2) Theorem 7.5 is best possible in the sense that there exists a graph G on 2n vertices which is k*-extendable, locally connected and $\kappa(G) = k + 4$. Such graph is $(K_1 \vee \overline{K}_{k+3}) \vee (K_{2k} \cup K_{k+4})$.

8. Results on a number of components

In this section, we establish some results concerning an upper bound on a number of components of $G - S$ when S is a minimum cutset of a k^*-extendable graph G. We begin with a minimum cutset of order at most $2k + 1$.

Theorem 8.1: Let G be a k^*-extendable graph on $2n$ vertices with $2 \le k \le n - 3$ and let S be a minimum cutset of G and $M(S)$ a maximum matching in $G[S]$. If $|S| \le 2k + 1$, then $2 \le \omega(G - S) \le |S| - |M(S)| - k - 1$.

Proof: Clearly, since S is a cutset, $\omega(G - S) \ge 2$. Now we suppose to the contrary that $\omega(G - S) \ge |S| - |M(S)| - k$. Since G is k^*-extendable and S is a minimum cutset, by Corollary 7.2, $|M(S)| \le |S| - k - 3 \le (2k + 1) - k - 3 = k - 2$. Thus, $|S| - 2|M(S)| \ge k - |M(S)| + 3$. Let $x, y \in V(G) \setminus S$. Since $S \setminus V(M(S))$ is independent and G is k^*-extendable, $\nu(G - (S \cup \{x, y\}) \ge |S \setminus V(M(S))| = |S| - 2|M(S)|$. Thus $\nu(G - S) \ge |S| - 2|M(S)| + 2$. Now let $C_1, C_2, \dots, C_r$ be components of $G - S$. Clearly, $r \ge |S| - |M(S)| - k \ge 3$. We claim that there is a subset of $\bigcup_{i=1}^{r} V(C_i)$ of cardinality $k - |M(S)| \ge 2$ with deleting this set from $G - S$ results in a graph with at least $|S| - |M(S)| - k - 1 \ge 2$ odd components. Suppose there is no such subset. Among subsets of $\bigcup_{i=1}^{r} V(C_i)$ with cardinality $k - |M(S)|$, let A be a subset of $\bigcup_{i=1}^{r} V(C_i)$ with $|A| = k - |M(S)|$ and $o(G - (S \cup A))$ is as large as possible. Notice that $\nu(G - (S \cup A)) \ge |S| - 2|M(S)| + 2 - (k - |M(S)|) = |S| - |M(S)| - k + 2$. Suppose $\omega(G - (S \cup A)) = 1$. This implies that $G - (S \cup A)$ is connected and then there exists a component of $G - S$, C_1 say, which $V(C_1) \setminus A \ne \varnothing$ and $V(C_i) \cap A = V(C_i)$; $2 \le i \le r$. Since $\nu(G - (S \cup A)) \ge |S| - |M(S)| - k + 2$, $|V(C_1) \setminus A| \ge |S| - |M(S)| - k + 2$. Let $x_1, x_2, \dots, x_{|S| - |M(S)| - k - 1} \in V(C_1) \setminus A$ and $y_i \in V(C_i) \cap A$, $2 \le i \le |S| - |M(S)| - k$. Put

$$A_1 = (A \cup \{x_1, x_2, \dots, x_{|S| - |M(S)| - k - 1}\}) \setminus \{y_2, y_3, \dots, y_{|S| - |M(S)| - k}\}.$$

Clearly, $|A_1| = |A|$ and $G - (S \cup A_1)$ contains at least $|S| - |M(S)| - k - 1 \ge 2$ odd components. This contradicts the choice of A. Hence, $\omega(G - (S \cup A)) \ge 2$. Now we suppose that $G - (S \cup A)$ contains only odd components. Since $o(G - (S \cup A)) \le |S| - |M(S)| - k - 2$, there are at least 2 components of $G - S$, C_j and $C_{j'}$ say, which $V(C_i) \cap A = V(C_i)$ for $i = j, j'$. Further, there exists an odd component of $G - (S \cup A)$, H_1 say, which $\nu(H_1) \ge 3$. Let $a_1, a_2 \in V(H_1)$, $b_1 \in V(C_j)$ and $b_2 \in V(C_{j'})$. Put $A_2 = (A \cup \{a_1, a_2\}) \setminus \{b_1, b_2\}$. Clearly, $|A_2| = |A|$ and $o(G - (S \cup A_2)) = o(G - (S \cup A)) + 2$, a contradiction. Thus $G - (S \cup A)$ contains at least one even component. Suppose there is a component of $G - S$, $C_{j''}$ say, which $V(C_{j''}) \cap A = V(C_{j''})$. Let $w \in V(C_{j''})$ and $z \in V(H_j)$ for some an even component H_j of $G - (S \cup A)$. Then $A_3 = (A \cup \{z\}) \setminus \{w\}$ has the same cardinality with A and $o(G - (S \cup A_3)) = o(G - (S \cup A)) + 2$, a contradiction. Hence, $V(C_j) \setminus A \neq \emptyset$ for all j, $1 \leq j \leq r$. Consequently, $\omega(G-(S \cup A)) = \omega(G-S) = r$ and $G-(S \cup A)$ contains at least 2 even components.

Let $W_1, W_2, \ldots, W_t$ be odd components of $G-(S \cup A)$ and $W_{t+1}, W_{t+2}, \ldots, W_r$ be even components of $G-(S \cup A)$ where $t \leq |S| - |M(S)| - k - 2$. Without any loss of generality, we may assume that $V(W_i) = V(C_i) \setminus A$; $1 \leq i \leq r$. Suppose $V(C_{t+1}) \cap A \neq \emptyset$. Let $w' \in V(C_{t+1}) \cap A$ and $z' \in V(W_{t+2})$. Put $A_4 = (A \cup \{z'\}) \setminus \{w'\}$. Then $|A_4| = |A|$ and $o(G-(S \cup A_4)) = o(G-(S \cup A)) + 2$, contradicting the choice of A. Thus, $V(C_{t+1}) \cap A = \emptyset$. Similarly, $V(C_i) \cap A = \emptyset$, $t+2 \leq i \leq r$. This implies that $V(W_i) = V(C_i)$; $t+1 \leq i \leq r$. Now we will show that $|V(C_i) \cap A| \leq 1$, $1 \leq i \leq t$. Suppose there is an odd component W_j, $1 \leq j \leq t$, which $|V(C_j) \cap A| \geq 2$. Let $w_1, w_2 \in V(C_j) \cap A$, $z_1 \in V(W_{t+1})$, $z_2 \in V(W_{t+2})$. Then $A_5 = (A \cup \{z_1, z_2\}) \setminus \{w_1, w_2\}$ has the same cardinality with A and $o(G-(S \cup A_5)) = o(G-(S \cup A)) + 2$, a contradiction. Hence, $|V(C_i) \cap A| \leq 1$, $1 \leq i \leq t$. Now $k - |M(S)| = |A| = \sum_{i=1}^{t} |V(C_i) \cap A| \leq t \leq |S| - |M(S)| - k - 2$. Thus $|S| \geq 2k+2$. This contradicts our assumption on $|S|$ and proves our claim.

Now let B be a subset of $\bigcup_{i=1}^{r} V(C_i)$ with $|B| = k - |M(S)|$ and $o(G-(S \cup B)) \geq |S| - |M(S)| - k - 1$. Since $|S| - 2|M(S)| \geq k - |M(S)| + 3$, in view of Theorem 2.8, there is a complete matching F of size $k - |M(S)|$ joining vertices of B to vertices of $S' \subseteq S \setminus V(M(S))$. Clearly, $|S| - (2|M(S)| + |S'|) \geq 3$. Let $c_1, c_2 \in S \setminus (V(M(S)) \cup S')$. Then $F \cup M(S)$ is a matching of size $k - |M(S)| + |M(S)| = k$ which does not extend to a perfect matching in $G - \{c_1, c_2\}$ since $S'' = S \setminus (V(M(S)) \cup S' \cup \{c_1, c_2\}) \subseteq V(G-(V(M(S) \cup F) \cup \{c_1, c_2\})$ of order $|S| - (2|M(S)| + k - |M(S)| + 2) = |S| - |M(S)| - k - 2$ and $G - (V(M(S) \cup F) \cup \{c_1, c_2\} \cup S'') = G-(S \cup B)$ contains at least $|S| - |M(S)| - k - 1$ odd components. This contradicts the k*-extendablity of G and completes the proof of our theorem. ❑

Corollary 8.2: Let G be a k*-extendable graph on 2n vertices with $2 \leq k \leq n-3$. Let S be a minimum cutset of order at most $2k+1$ which S is independent. Then

$$o(G-S) \leq \begin{cases} |S| - k - 2, & \text{for k is even} \\ |S| - k - 1, & \text{for k is odd.} \end{cases}$$

Proof: By Theorem 8.1, $o(G-S) \leq \omega(G-S) \leq |S| - k - 1$. Thus we only need to prove the case k is even. Suppose k is even and

$$o(G-S) = |S| - k - 1.$$

Since $\nu(G)$ is even, $|S|$ and $|S| - k - 1$ must have the same parity. This implies that $k+1$ is even and hence k is odd, a contradiction. This completes the proof of our corollary. ❑

Remark 8.1: Let s and k be positive integers with $k+3 \leq s \leq 2k+1$. Let $G_1 = \overline{K}_s \vee (s-k-1)K_{2s+1}$ for an odd $k \geq 3$ and $G_2 = \overline{K}_s \vee (K_{2s} \cup (s-k-2)K_{2s+1})$ for an even $k \geq 2$. It is not difficult to show that G_1 and G_2 are both k*-extendable. Clearly, $V(\overline{K}_s)$ is a cutset of G_i, $i = 1, 2$ and $G_1 - S$ and $G_2 - S$ contain exactly $s - k - 1$ and $s - k - 2$ odd components, respectively. Thus Corollary 8.2 is best possible.

The next corollary follows immediately from Theorem 8.1, Corollaries 7.3 and 8.2.

Corollary 8.3: Let G be a k*-extendable graph on 2n vertices with $2 \leq k \leq n - 3$. Suppose S is a cutset of G with $|S| = k + 3$. Then $G - S$ contains exactly 2 components. Further,

(i) If k is odd, then both components of $G - S$ are odd or even.

(ii) If k is even, then one of components of $G - S$ is odd and the other is even. ❑

We make an observation here that $k + 3$ is the smallest order of a cutset of k*-extendable graphs for $1 \leq k \leq n - 3$. Corollary 8.3 presents the number of components of $G - S$ when S is a cutset of order $k + 3$ of a k*-extendable graph G for $2 \leq k \leq n - 3$. Our next lemma concerns a similar result for $k = 0$ and 1. Note that 0*-extendable graphs are 2 connected and 1*-extendable graphs are 4-connected.

Lemma 8.4: Let G be a 0*-extendable graph on $2n \geq 4$ vertices. Suppose S is a cutset of G with $|S| = 2$. Then $G - S$ contains at least 2 even components and no odd components.

Proof: It follows directly from the definition of 0*-extendable graphs and the fact that $|S|$ is even. ❑

Lemma 8.5: Let G be a 1*-extendable graph on $2n \geq 6$ vertices. Suppose S is a cutset of G with $|S| = 4$.

(i) If G[S] contains an edge, then $G - S$ contains at least 2 even components but no odd components.

(ii) If S is an independent set, then $G - S$ contains exactly 2 odd components and no even components or at least 2 even components but no odd components.

Proof: Let $S = \{a, b, c, d\}$ be a cutset of G. Without any loss of generality, we may assume that $ab \in E(G)$. If $G - S$ contains an odd component, then the edge ab does not extend to a perfect matching in $G - \{c, d\}$. This contradicts 1*-extendability of G. Hence, $G - S$ has no odd components. Since S is a cutset of G, $G - S$ contains at least 2 even components but no odd components. This proves (i).

Now we suppose that S is independent and $G - S$ contains an odd component (and hence, by parity, at least 2 odd components). Further, we suppose that $G - S$ contains H_0 as an even component. Since $|S| = 4$, by Lemma 5.1, S is a minimum cutset. Thus there exists an edge $e = xy$ joining a vertex x of S to a vertex y of H_0. Without any loss of generality, we may assume that $x = a$. Then the edge ay does not extend to a perfect matching in $G - \{b, c\}$ since the odd components of $G - S$ together with $H_0 \setminus y$ form at least 3 odd components of $(G - (S \cup \{y\})$ and $|S \setminus \{a, b, c\}| = |\{d\}| = 1$, a contradiction. Hence, $G - S$ contains only odd components. It follows from Theorem 3.7 that $G - S$ contains exactly 2 odd components and no even components. If $G - S$ has no odd components, then $G - S$ contains at least 2 even components as S is a cutset. This completes the proof of our lemma. ❑

Remark 8.2: (1) For $n \geq 3$, a graph $K_2 \vee (n - 1) K_2$ is 0*-extendable which satisfies Lemma 8.4.

(2) For $n \geq 4$ a graph $K_4 \vee (n - 2)K_2$ is 1*-extendable which satisfies Lemma 8.5 (i) and for $2n \geq 12$ graphs $\overline{K}_4 \vee (K_1, \cup K_{2n-5})$ and $\overline{K}_4 \vee (n - 2)K_2$ are both 1*-extendable which satisfy Lemma 8.5 (ii).

Theorem 4.8 together with Theorem 8.1 yields the following corollary:

Corollary 8.6: Let G be a k-extendable graph on 2n vertices with $4 \leq k \leq n - 1$ and let S be a minimum cutset of G and M(S) a maximum matching in G[S]. If $|S| \leq 2k - 3$, then $2 \leq \omega(G - S) \leq |S| - |M(S)| - k + 1$. ❑

Theorem 8.1 gives an upper bound on a number of components of $G - S$ when S is a minimum cutset of order at most $2k + 1$ of a k*-extendable graph G. One might expect a similar result for $|S| \geq 2k + 2$ but this is not the case. For non-negative integers s and t, a graph $G_1 = (K_{2k} \cup \overline{K}_{t+2}) \vee (s + t + 2)K_{2k+4}$ for t is even and a graph $G_2 = (K_{2k} \cup \overline{K}_{t+2}) \vee [(s + t + 1)K_{2k+4} \cup K_{2k+3}]$ for t is odd are k*-extendable with a minimum cutset $S = V(K_{2k} \cup \overline{K}_{t+2})$. Clearly, $\omega(G_i - S) = s + t + 2 \geq 2$ for $i = 1, 2$. However, if a number of odd components of $G - S$ is sufficiently large, then an upper bound on a number of even components of $G - S$ can be given with some restriction on the size of M(S). Our next result establishes this.

Theorem 8.7: Let G be a k*-extendable graph on 2n vertices with $1 \leq k \leq n - 3$ and let S be a minimum cutset of G with $|S| \geq 2k + 2$ and M(S) a maximum matching in G[S]. Suppose $o(G - S) = |S| - 2|M(S)| - 2 - r$ for some non-negative integer r. If $2|M(S)| + r \leq 2k - 2$, then the number of even components of $G - S$ is at most $|M(S)| + \left\lfloor \frac{r}{2} \right\rfloor$.

Proof: Let $\eta(G - S)$ be a number of even components of $G - S$. Suppose to the contrary that $\eta(G - S) \geq |M(S)| + \left\lfloor \frac{r}{2} \right\rfloor + 1 = t$. Let $H_1, H_2, \ldots, H_t$ be even components of $G - S$. Choose $x_i \in V(H_i)$, $1 \leq i \leq t$. Since $2|M(S)| + r \leq 2k - 2$, $t = |M(S)| + \left\lfloor \frac{r}{2} \right\rfloor + 1 \leq k$ and $|S| \geq 2k + 2 \geq t + 2$. Let $y_1, y_2, \ldots, y_t, y_{t+1}, y_{t+2} \in S$. In view of Theorem 2.8, there is a matching M′ of size t joining vertices of $\{x_1, x_2, \ldots, x_t\}$ to vertices of $\{y_1, y_2, \dots, y_t\}$. Clearly, $G - (V(M') \cup S)$ contains $|S| - 2|M(S)| - 2 - r + t = |S| - |M(S)| - \left\lceil \frac{r}{2} \right\rceil - 1$ odd components. Further $|S \setminus (V(M') \cup \{y_{t+1}, y_{t+2}\})| = |S| - (t + 2) = |S| - |M(S)| - \left\lfloor \frac{r}{2} \right\rfloor - 3$. If M' extended to a perfect matching in $G - \{y_{t+1}, y_{t+2}\}$, then each odd component of $G - (V(M') \cup S)$ would be joined to at least one vertex of $S \setminus (V(M') \cup \{y_{t+1}, y_{t+2}\})$. But this is impossible since $o(G - (V(M') \cup S)) = |S| - |M(S)| - \left\lceil \frac{r}{2} \right\rceil - 1$ while $|S \setminus (V(M') \cup \{y_{t+1}, y_{t+2}\})| = |S| - |M(S)| - \left\lfloor \frac{r}{2} \right\rfloor - 3$. Hence, $\eta(G - S) \le |M(S)| + \left\lfloor \frac{r}{2} \right\rfloor$ as required. ❑

Our next result concerns an upper bound on a number of odd components of $G - S$ when S is an independent cutset of a k^*-extendable graph G with $|S| \ge 2k + 2$.

Corollary 8.8: Let G be a k^*-extendable graph on $2n$ vertices with $2 \le k \le n - 3$ and let S be a minimum cutset of G with $|S| \ge 2k + 2$. If S is independent, then $o(G - S) \le |S| - 4$. Further, if $k \ge 3$ and $|S| - 5 \le o(G - S)$, then $G - S$ has no even components.

Proof: Suppose to the contrary that $o(G - S) \ge |S| - 3$. It follows from Theorem 8.7 that $G - S$ has no even components. Let $C_1, C_2, \dots, C_t$ be odd components of $G - S$. If $|V(C_i)| = 1$; $1 \le i \le t$, then G is bipartite which is impossible since G is k^*-extendable. Hence, there is a component of $G - S$, C_1 say, with $|V(C_1)| \ge 3$. Let $x, y \in V(C_1)$ and $a, b, c, d \in S$. In view of Theorem 2.8, there is a matching M of size 2 joining vertices of $\{x, y\}$ to vertices of $\{a, b\}$. But then M does not extend to a perfect matching in $G - \{c, d\}$ since $G - (S \cup \{x, y\})$ contains at least $|S| - 3$ odd components while $|S \setminus \{a, b, c, d\}| = |S| - 4$. This contradicts the k^*-extendability of G and proves that $o(G - S) \le |S| - 4$.

Further, we assume that $k \ge 3$ and $|S| - 5 \le o(G - S)$. Since $\nu(G)$ is even, $|S|$ and $o(G - S)$ have the same parity. This implies that $o(G - S) = |S| - 4$. By Theorem 8.7, $G - S$ has at most one even component.

Suppose H is an even component of $G - S$. We will show that $\nu(H) = 2$. Suppose to the contrary that $\nu(H) \ge 4$. Let $z_1, z_2, z_3 \in V(H)$ and $w_1, w_2, w_3, w_4, w_5 \in S$. By Theorem 2.8, there is a matching M_1 of size 3 joining vertices of $\{z_1, z_2, z_3\}$ to vertices of $\{w_1, w_2, w_3\}$. By applying a similar argument used as above, M_1 does not extend to a perfect matching in $G - \{w_4, w_5\}$, a contradiction. Hence, $\nu(H) = 2$. Since G has a perfect matching and S is independent, $\nu(G - S) \ge |S|$. Because $\nu(H) = 2$ and $o(G - S) = |S| - 4$, there is an odd component of $G - S$, C say, with $\nu(C) \ge 3$. Now let $a_1, a_2 \in V(C)$ and $b \in V(H)$. Then, in view of Theorem 2.8, there is a matching M_2 of size 3 joining vertices of $\{a_1, a_2, b\}$ to vertices of $\{w_1, w_2, w_3\}$. Again, M_2 does not extend to a perfect matching in $G - \{w_4, w_5\}$, a contradiction. This proves that $G - S$ has no even components and completes the proof of our corollary. ❑

Remark 8.3: For a positive integer $s \geq 4$, a graph $G_1 = \overline{K_s} \vee (s-2)K_{2s+1}$ is 1*-extendable containing $V(\overline{K_s})$ as a minimum cutset. Clearly, $G_1 - V(\overline{K_s})$ contains $s-2$ odd components. Further, for a positive integer $s \geq 5$, a graph $G_2 = \overline{K_s} \vee [(s-4)K_{2s+1} \cup K_{2s}]$ is 2*-extendable which $V(\overline{K_s})$ is a minimum cutset and $G_2 - V(\overline{K_s})$ contains $s-4$ odd components and an even component. Thus the bound on k in Corollary 8.8 is best possible.

Our next result concerns a minimum cutset of a k*-extendable graph which its induced subgraph has a small independence number. We begin with the following lemma.

Lemma 8.9: Let G be a simple graph with $\alpha(G) \leq 2$ and M a maximum matching in G. Then $|M| = \frac{\nu(G)-1}{2}$ for $\nu(G)$ is odd and $|M| \geq \frac{\nu(G)}{2} - 1$ for $\nu(G)$ is even.

Proof: Let $\nu(G)$ be odd. Suppose $|M| < \frac{\nu(G)-1}{2}$. Clearly, $|M| \leq \frac{\nu(G)-3}{2}$ and $G - V(M)$ is independent since M is a maximum matching. Then $G - V(M)$ contains at least $\nu(G) - 2|M| \geq 3$ independent vertices, contradicting the fact that $\alpha(G) \leq 2$. Hence, $|M| = \frac{\nu(G)-1}{2}$. By applying a similar argument, $|M| \geq \frac{\nu(G)}{2} - 1$ for $\nu(G)$ is even. ❑

Theorem 8.10: Let G be a k*-extendable graph on 2n vertices with $0 \leq k \leq n-3$ and let $S \subseteq V(G)$ be a minimum cutset of G. Suppose $\alpha(G[S]) \leq 2$. Then $|S| \geq 2k+2$ and $o(G-S) \leq |S| - 2k - 2$.

Proof: By Theorem 3.7 and the fact that 0*-extendable graphs are 2-connected, our theorem follows immediately for $k = 0$. So we only need to consider the case $k \geq 1$. Since G is $(k+3)$-connected, $|S| \geq k+3 \geq 4$. Suppose $|S| \leq 2k+1$. Let M be a maximum matching in G[S]. We will show that $G - S$ contains only even components. Suppose to the contrary that $G - S$ contains an odd component. Assume that $G - S$ contains exactly one odd component. Then |S| is odd by the fact that $\nu(G)$ is even. Further, since S is a cutset, $G - S$ contains an even component, H say. By Lemma 8.9, $|M| = \frac{|S|-1}{2} \leq k$. Let $x \in S \setminus V(M)$ and $y \in V(H)$. Then M does not extend to a perfect matching in $G - \{x, y\}$ since $G - (V(M) \cup \{x, y\})$ contains $o(G-S) + 1 = 2$ isolated odd components, a contradiction. Hence, $G - S$ contains at least 2 odd components. Clearly, |S| is odd otherwise G is not k*-extendable since $\frac{|S|}{2} - 1 \leq |M| \leq k$ and $|S \setminus V(M)| = 0$ or 2. Consequently, $G - S$ contains at least 3 odd components. Let C_1 be an odd component of $G - S$ and let $z \in V(C_1)$. By Lemma

8.9, $|M| = \frac{|S|-1}{2} \leq k$ and there is a vertex $x \in S \setminus V(M)$. Now M does not extend to a perfect matching in $G - \{x, z\}$ since $G - (V(M) \cup \{x, z\})$ contains $o(G - S) - 1 \geq 2$ isolated odd components, again a contradiction. This proves that $G - S$ contains only even components. Consequently, $|S|$ is even and $|S| \leq 2k$. Further, $G - S$ contains at least two even components, H_1 and H_2 say. By Lemma 8.9, $\frac{|S|}{2} - 1 \leq |M| \leq k$. Let $a \in V(H_1)$ and $b \in V(H_2)$. If $|M| = \frac{|S|}{2} \leq k$, then M does not extend to a perfect matching in $G - \{a, b\}$ since $G - (V(M) \cup \{a, b\})$ contains $H_1 - a$ and $H_2 - b$ as isolated odd components. This contradicts the fact that G is k*-extendable. Thus $|M| = \frac{|S|}{2} - 1 \geq 1$ since $|S| \geq 4$. Let $a_1b_1 \in M$, a_2 and b_2 belong to $S \setminus V(M)$. Since S is a minimum cutset, in view of Theorem 2.8, there is a matching $M_1 = \{a_1x_1, b_1x_2 \mid x_1 \in V(H_1)$ and $x_2 \in V(H_2)\}$. Then $M_2 = (M \cup M_1) \setminus \{a_1b_1\}$ is a matching of size $(\frac{|S|}{2} - 1) + 2 - 1 = \frac{|S|}{2} \leq k$. Clearly M_2 does not extend to a perfect matching in $G - \{a_2, b_2\}$ since $G - (V(M_2) \cup \{a_2, b_2\})$ contains $H_1 - x_1$ and $H_2 - x_2$ as isolated odd components. This contradiction proves that $|S| \geq 2k + 2$. It follows immediately from Theorem 3.7 that $o(G - S) \leq |S| - 2k - 2$. This completes the proof of our theorem. ❑

Remark 8.4: Theorem 8.10 is best possible in the sense that there is a k*-extendable graph G with a cutset S satisfies conditions of the theorem and $G - S$ contains a number of odd components up to $|S| - 2k - 2$.

Let $G_1 = K_{2k+2+r} - \{\text{an edge}\}$, $G_2 = \bigcup_{i=1}^{q} K_{2a_i+1}$ and $G_3 = \bigcup_{j=1}^{m} K_{2b_j}$ where r, q, m, a_i, b_j are non-negative integers, $q + m \geq 2$, $q \leq r$ and $q \equiv r \pmod 2$. Put $G = G_1 \vee (G_2 \cup G_3)$. Figure 8.1 depicts the graph G. It is not too difficult to show that G is

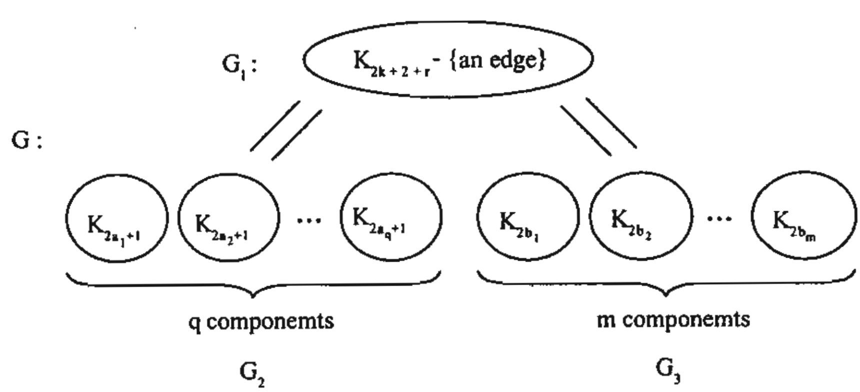


Figure 8.1

k*-extendable containing $V(G_1)$ as a cutset of order $2k + 2 + r$. Notice that the number of components of $G - V(G_1)$ can be any integer which is at least 2.

Theorem 4.8 together with Theorem 8.10 yields the following corollary:

Corollary 8.11: Let G be a k-extendable graph on 2n vertices with $2 \le k \le n - 1$ and let S be a minimum cutset of G. Suppose $\alpha(G[S]) \le 2$. Then $|S| \ge 2k - 2$ and $o(G - S) \le |S| - 2k + 2$. ❑

We conclude our paper by establishing a lower bound on an order of k*-extendable graphs in terms of an order of a minimum cutset.

Theorem 8.12: Let G be a k*-extendable graph on 2n vertices with $0 \le k \le n - 3$ and let S be a minimum cutset of G and M(S) a maximum matching in G[S]. If

(i) $|S| \le 2k + 2$, or

(ii) $|S| \ge 2k + 3$ and $|M(S)| \le k$

then $2n \ge 2|S| + 2k - 2|M(S)| + 2$.

Proof: Clearly, by the assumption on |S| and Corollary 7.2, $|S| - 2|M(S)| \ge 3$. Let x and y be vertices of $S \setminus V(M(S)$. Since G is k*-extendable, there is a perfect matching F in $G - \{x, y\}$ containing all the edges of M(S). Put

$$F_1 = \{ab \in F \mid a \in S \setminus (V(M(S)) \cup \{x, y\}), b \notin S\}$$

and

$$F_2 = \{ab \in F \mid a, b \notin S\}.$$

Then

$$|F_1| = |S| - 2|M(S)| - 2 \ge 1$$

and

$$\begin{aligned}|F_2| &= \frac{1}{2}[2n - |S| - |F_1|] \\ &= \frac{1}{2}[2n - |S| - (|S| - 2|M(S)| - 2)] \\ &= n - |S| + |M(S)| + 1.\end{aligned}$$

If $|F_2| = 0$, then M(S) does not extend to a perfect matching in G since $G - V(M(S))$ contains $S \setminus V(M(S))$ as an independent set of order $|S| - 2|M(S)|$ and $\nu(G - V(M(S))) = |S| - 2|M(S)| + (|S| - 2|M(S)| - 2) = 2|S| - 4|M(S)| - 2$, contradicting the k*-extendability of G. Thus $|F_2| \ge 1$. Let $zw \in F_2$. Suppose $|F_2| \le k + 1$. Then $F_2 \setminus \{zw\}$ does not extend to a perfect matching in $G - \{z, w\}$ since $G - V(F_2)$ contains $S \setminus V(M(S))$ as an independent set of order $|S| - 2|M(S)|$ and $\nu(G - (S \cup V(F_2)) = |F_1| = |S| - 2|M(S)| - 2$, again a contradiction. Hence, $n - |S| + |M(S)| + 1 = |F_2| \ge k + 2$. Thus $2n \ge 2|S| + 2k - 2|M(S)| + 2$ as required. This completes the proof of our theorem. ❑

As a corollary we have:

Corollary 8.13: Let G be a k-extendable graph on 2n vertices with $2 \le k \le n - 1$ and let S be a minimum cutset of G and M(S) a maximum matching in G[S]. If

(i) $|S| \le 2k - 2$, or

(ii) $|S| \ge 2k - 1$ and $|M(S)| \le k - 2$

then $2n \ge 2|S| + 2k - 2|M(S)| - 2$. ❑

Remark 8.5: Theorems 8.1 and 8.12 are best possible in the sense that for $k \ge 2$ there is a k*-extendable graph G on $2n \ge 2|S| + 2k - 2|M(S)| + 2$ vertices containing a minimum cutset S of order at most $2k + 1$ with $2 \le \omega(G - S) \le |S| - |M(S)| - k - 1$. For non-negative integers k, s, t, q, r, m with

(i) $k + 3 \le s \le 2k + 1$

(ii) $0 \le t \le s - k - 3$

(iii) $0 \le 2q + r \le s - t - k - 3$,

let $G = (K_{2t} \cup \overline{K}_{s-2t}) \vee [K_{s-2q} \cup K_{2k+2-2r-2t+2m} \cup (2q)K_1 \cup rK_2]$. Figure 8.2 illustrates the graph G. It is not too difficult to show that G is k*-extendable. Clearly, $S = V(K_{2t} \cup \overline{K}_{s-2t})$ is a cutset of order s, $\nu(G) = 2s + 2k - 2t + 2 + 2m$ and $2 \le \omega(G - S) = 2q + r + 2 \le s - t - k - 1$.

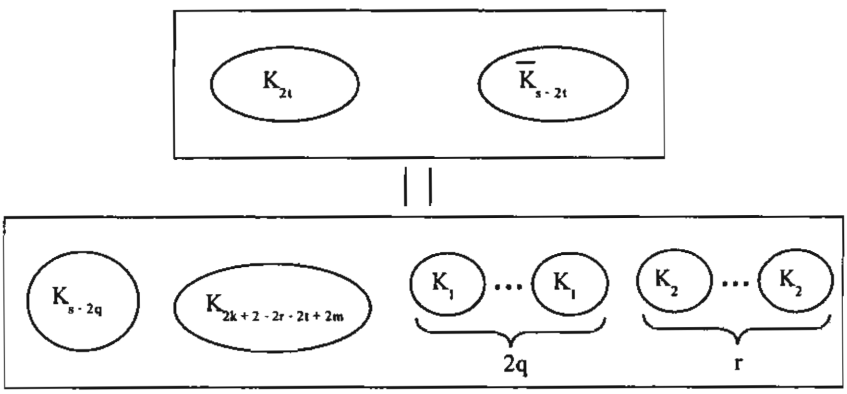


Figure 8.2

9. References

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10. Output

จากผลงานวิจัยเรื่องกราฟ k*-extendable ที่ศึกษามาเราสามารถนำมาเขียนเรียบเรียงเป็นบทความทางวิชาการเพื่อส่งตีพิมพ์ในวารสารทางวิชาการได้ 3 บทความดังนี้

1. N. Ananchuen, *On Strongly k-extendable Graphs,* Journal of Combinatorial Mathematics and Combinatorial Computing (in press).
2. N. Ananchuen, *On Minimum Degree of Strongly k-extendable Graphs* (submitted).
3. N. Ananchuen, *On a Minimum Cutset of Strongly k-extendable Graphs* (submitted).

ซึ่งจำนวนบทความข้างต้นได้บรรลุเป้าหมายที่วางไว้ว่าผลงานวิจัยดังกล่าวสามารถเขียนเรียบเรียงเป็นบทความทางวิชาการได้อย่างน้อย 2 บทความ