



รายงานวิจัยฉบับสมบูรณ์

การแก้ปัญหาสนามแม่เหล็กไฟฟ้า
โดยใช้แบบจำลองทางคณิตศาสตร์ชนิดเวลาไม่ต่อเนื่อง

Finite-Difference Time-Domain Method
for Solving Electromagnetic Field Problems

รองศาสตราจารย์ ดร. ณัฏฐกา หอมทรัพย์

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Abstract

The research investigates numerical performances of various Finite-Difference Time-Domain(FDTD) schemes. Specifically, numerical solutions of Maxwell's equations in both two and three dimensions are obtained by these schemes. The problem of electromagnetic fields from mobiles telephones which interact with a human head are investigated in details. An important parameter such as the Specific Absorbtion Ratio (SAR) are calculated at various tissue types in the head model. The research also introduces new improved methods for an efficient solution of Maxwell's equations. Numerical schemes which are higher-order accurate and the one related to Spline interpolation are introduced and formulated for improving the performance of the standard FDTD scheme.

Keywords: FDTD, Maxwell's equations, Numerical methods, SAR, Spline

บทคัดย่อ

งานวิจัยนี้ได้ศึกษาถึงคุณลักษณะต่าง ๆ ของวิธีการใช้แบบจำลองทางคณิตศาสตร์ชนิดเวลาไม่ต่อเนื่อง โดยเฉพาะอย่างยิ่งการนำมาใช้แก้ปัญหасмการของแมกซ์เวลใน สอง และ สาม มิติ ตัวอย่างปัญหาเช่นผลกระทบอันเกิดจากคลื่นแม่เหล็กไฟฟ้าจากโทรศัพท์มือถือที่มีผลต่อศีรษะผู้ใช้ พารามิเตอร์ที่จะต้องคำนวณหาที่ส่วนต่างๆของศีรษะก็คือค่าอัตราส่วนการดูดกลืน (SAR) นอกจากนั้น ยังมีการวิจัยเพื่อหาวิธีการแก้ปัญหาใหม่ๆ ที่มีความละเอียดยิ่งขึ้น โดยอาศัย ฟังก์ชันสปราย เพื่อปรับปรุงวิธีการเดิมให้ดียิ่งขึ้น

1. Background

The finite-difference time-domain (FDTD) scheme is now widely used in various applications such as an analysis of antenna pattern in two and three dimensions, various fields pattern in wave guide, interaction of electromagnetic field with biological tissues, etc. The main advantages of FDTD-based techniques for solving the Maxwell's equations are simplicity and the ability to handle complex geometry. Derivation from Maxwell's equations is based on a method of approximating derivatives by finite difference and approximating line integrals, surface integrals, and volume integrals by summations. It uses basic arithmetic operations-addition, subtraction, multiplication, and division. Material properties are specified at each grid point. At the interface points of different materials, FDTD can model material through parameter averaging. The field at each grid point is calculated explicitly using only adjacent field's values at previous time. The disadvantages of the FDTD include the requirement of a high-speed computer and large amount of memory when the linear dimension of the object is large compared to the wavelength because of the dispersion introduced by the algorithm. Most of FDTD schemes are the second-order accurate. The scheme can be extended into a fourth order accurate-compact scheme. In fourth order scheme, the approximated spatial derivative can be represented by explicit or implicit method. In implicit method, the method requires the inversion of a tridiagonal matrix to find all spatial derivatives and all grid points must be uniform. Both methods are the extension of the standard Yee's scheme to higher-order accuracy. Thus, the scheme is at least a fourth order in space but second order in time. This is a reasonable option since the temporal accuracy can be improved by choosing a smaller time step. This increases the work only linearly and does not increase the storage. This higher order scheme in space enables one to choose a coarser mesh, This decreases the work in each space dimension and also decreases the storage.

1.1 Maxwell's equations

In a region of space that has no electric or magnetic current sources, but may have materials that absorb electromagnetic energy, the time-dependent Maxwell's equations are given in differential form by

$$\text{Faraday's Law:} \quad \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} - \vec{J}_m \quad (1)$$

$$\text{Ampere's Law:} \quad \frac{\partial \vec{D}}{\partial t} = \nabla \times \vec{H} - \vec{J}_e \quad (2)$$

$$\text{Gauss's Law:} \quad \nabla \cdot \vec{D} = 0 \quad (3)$$

$$\text{Gauss's Law} \quad \nabla \cdot \vec{B} = 0 \quad (4)$$

In linear, isotropic nondispersive materials, one can relate $\vec{B}$ to $\vec{H}$ and $\vec{D}$ to $\vec{E}$ using simple proportions:

$$\vec{B} = \mu \vec{H} \quad (5)$$

$$\vec{D} = \epsilon \vec{E} \quad (6)$$

$\vec{E}$ = The electric field vector in Volts per meter

$\vec{D}$ = The electric flux density vector in Coulombs per square meter

$\vec{H}$ = The magnetic field vector in Amperes per meter

$\vec{B}$ = The magnetic flux density vector in Weber per square meter

$\vec{J}_e$ = The electric conduction current density in Ampere per square meter

$\vec{J}_m$ = The equivalent magnetic conduction current density in Volts per square

meter

μ = The magnetic permeability in Henrys per meter

ϵ = The electric permittivity in Farad per meter

The possibility of electric and magnetic losses that can dissipate electromagnetic fields in materials via conversion to heat energy, $\vec{J}_m$ can be related to $\vec{H}$ using a linear relation:

$$\vec{J}_m = \rho' \vec{H} \quad (7)$$

And $\vec{J}_e$ to account for the electric loss mechanism:

$$\vec{J}_e = \sigma \vec{E} \quad (8)$$

Here ρ' is an equivalent magnetic resistivity in Ohms per meter and σ is the electric conductivity in Siemens per meter. Combining the assumptions of (5) through (8) and substituting into Maxwell's curl equations (1) and (2), one obtains

$$\frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \vec{E} - \frac{\rho'}{\mu} \vec{H} \quad (9)$$

$$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\epsilon} \nabla \times \vec{H} - \frac{\sigma}{\epsilon} \vec{E} \quad (10)$$

Writing out the vector components of the curl operator in (9) and (10) yields the system of six coupled scalar equations equivalent to Maxwell's curl equations in three dimensional rectangular coordinate system (x, y, z):

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu} \left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - \rho' H_x \right) \quad (11)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu} \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - \rho' H_y \right) \quad (12)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu} \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - \rho' H_z \right) \quad (13)$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon} \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \rho' E_x \right) \quad (14)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon} \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \rho' E_y \right) \quad (15)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon} \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \rho' E_z \right) \quad (16)$$

These six coupled partial differential equations of (11) through (16) forms the basis of FDTD numerical algorithm for electromagnetic wave interactions with general three-dimensional objects. The FDTD algorithm need not explicitly enforce the Gauss's law relations. This is because these relations are theoretically a direct consequence of the curl equations. However, the FDTD space grid and the numerical space derivative operations must be resulted in the enforcement of the Gauss's law implicitly .

1.2. Finite Difference Scheme

To simplify the notation, only the two-dimensional case with $\rho' = \sigma = 0$ is considered. The only sources for the problem are incident waves. The extension of the method to three space dimensions sources and variable coefficients is straightforward. The TM system is consider and the time is normalized with the speed of light, c. let $\tau = ct$, and $Z = \sqrt{\mu/\varepsilon}$, where z is a wave impedance. The TM wave equations then becomes:

$$\frac{\partial E_z}{\partial \tau} = Z \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \quad (17)$$

$$\frac{\partial H_x}{\partial \tau} = -\frac{1}{Z} \frac{\partial E_z}{\partial y} \quad (18)$$

$$\frac{\partial H_y}{\partial \tau} = \frac{1}{Z} \frac{\partial E_z}{\partial x} \quad (19)$$

Then, Yee's difference equations are presented in the form that will enable one to generalize them easily.

$$\delta_t H_x|_{i,j-\frac{1}{2}}^n = -\frac{1}{Z} \delta_y E_z|_{i,j-\frac{1}{2}}^n \quad (20)$$

$$\delta_t H_y|_{i-1/2,j}^n = \frac{1}{Z} \delta_x E_z|_{i-1/2,j}^n \quad (21)$$

$$\delta_t E_z|_{i,j}^{n+1/2} = Z \left[\delta_x H_y|_{i,j}^{n+1/2} - \delta_x H_x|_{i,j}^{n+1/2} \right] \quad (22)$$

Where

$$\delta_x U_{i,j}^n = (U_{i+1/2,j}^n - U_{i-1/2,j}^n) / \Delta x \quad (23)$$

$$\delta_t U_{i,j}^{n+1/2} = (U_{i,j}^{n+1} - U_{i,j}^n) / \Delta t \quad (24)$$

The second order accuracy of Yee algorithms can be extended to higher order as shown in Table 1.

Order of Derivative	Order of accuracy	Approximation at x=0 x-coordinates at nodes:							
		-7/2	-5/2	-3/2	-1/2	1/2	3/2	5/2	7/2
1	2				-1	1			
	4			$\frac{1}{24}$	$-\frac{9}{8}$	$\frac{9}{8}$	$-\frac{1}{24}$		
	6		$-\frac{3}{640}$	$\frac{25}{384}$	$-\frac{75}{64}$	$\frac{75}{64}$	$-\frac{25}{384}$	$\frac{3}{640}$	
	8	$\frac{5}{7168}$	$-\frac{49}{5120}$	$\frac{245}{3072}$	$-\frac{1225}{1024}$	$\frac{1225}{1024}$	$-\frac{245}{3072}$	$\frac{49}{5120}$	$-\frac{5}{7168}$

Table 1. Coefficients for centered finite difference approximations at a "half-way" point [6].

Any function U of space and time are evaluated at a discrete point in the grid and at a discrete point in time as

$$U_{i,j}^n = U(i\Delta x, j\Delta y, n\Delta t) \quad (25)$$

Where Δt is the time increment, and $\Delta x, \Delta y$ are space increment in x and y direction respectively. The disadvantage of a high-order scheme is that it is not compact scheme. In particular, field components over a wider stencil are required. This is very troublesome when dealing with material discontinuities. A compromise to use a high-order difference scheme at location space cells away from any discontinuities, and to apply the basic second-order Yee algorithm at field component location near the discontinuities.

1.3. Absorbing Boundary

Recently, a technique for truncation of the computational domains in finite difference methods was proposed . In this method, which is called by the author's transparent absorbing boundary (TAB), a physical problem to be solved is transformed into a problem for auxiliary fields. These fields are equal to zero at the closed boundary. Since the relationship between the physical fields and their auxiliary counterparts is explicitly known and the former can be found from the latter within the computational domain.

In the TAB boundary formulation, the auxiliary vectors $\tilde{\vec{E}}$ and $\tilde{\vec{H}}$ are introduced, which are related to the physical fields $\vec{E}_o$ and $\vec{H}_o$ by

$$\tilde{\vec{E}}(t, r) = F(r)\tilde{\vec{E}}_o(t, r) \quad (26)$$

$$\tilde{\vec{H}}(t, r) = F(r)\tilde{\vec{H}}_o(t, r) \quad (27)$$

For $r \leq r_0$ when r_0 defines the closed truncation boundary.

$F(r)$ is a scalar function and $F(r_0) = 0$ while the physical fields $(\vec{E}_o, \vec{H}_o)$ are described by Maxwell's equations, governing equations for the auxiliary field $(\tilde{\vec{E}}, \tilde{\vec{H}})$ are

obtained by substituting $\vec{E}_o$ and $\vec{H}_o$ of (26) and (27) into Maxwell's equations. These equations are expressed in the form of

$$\frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu} \left(\nabla \times \vec{E} - \frac{1}{F} \nabla F \times \vec{E} \right) - \frac{\rho'}{\mu} \vec{H} \quad (28)$$

$$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon} \left(\nabla \times \vec{H} - \frac{1}{F} \nabla F \times \vec{H} \right) - \frac{\rho'}{\varepsilon} \vec{E} \quad (29)$$

$$\nabla \cdot \vec{E} = \frac{1}{F} \nabla F \cdot \vec{E} \quad (30)$$

$$\nabla \cdot \vec{H} = \frac{1}{F} \nabla F \cdot \vec{H} \quad (31)$$

With the boundary conditions of

$$\vec{E} = F(r_o)E_o = 0 \quad (32)$$

$$\vec{H} = F(r_o)H_o = 0 \quad (33)$$

Consider TM wave in (17) through (19), the field attenuation in the auxiliary system is prescribed by the function

$$F(x, y) = f(x)g(y) \quad (34)$$

The governing equations for the auxiliary field (E,H) are expressed in the form of

$$\frac{\partial E_z}{\partial \tau} = Z \left[\frac{\partial H_y}{\partial x} - \frac{f'}{f} H_y + \frac{g'}{g} H_x - \frac{\partial H_x}{\partial y} \right] \quad (35)$$

$$\frac{\partial H_x}{\partial \tau} = -\frac{1}{Z} \left[\frac{\partial E_z}{\partial y} - \frac{g'}{g} E_z \right] \quad (36)$$

$$\frac{\partial H_y}{\partial \tau} = \frac{1}{Z} \left[\frac{\partial E_z}{\partial y} - \frac{f'}{f} E_z \right] \quad (37)$$

Where

$$f' = \frac{\partial f}{\partial x} \quad \text{and} \quad g' = \frac{\partial g}{\partial y}$$

2. Objective

The research objective is to investigate a numerical performance of various FDTD schemes. Specifically, numerical solutions of Maxwell's equations in both two and three dimensions will be obtained by these schemes. The problem of electromagnetic field from mobile telephone interaction with a human head will be investigated in details. An important parameter such as the Specific Absorbtion Ratio(SAR) will be calculated at various tissue types in the head model. The research will introduced a new improved method for an efficient solution of Maxwell's equations. A few schemes which are related a high-order accurate and the Spline interpolation will be introduced and formulated for improving the performance of the standard FDTD scheme.

3. Research Methodology

Begin with Maxwell's equations in two-dimensions, the FDTD scheme is used to obtained the EM field distribution in the bounded domain. The absorbing boundary condition is implemented at area near the boundary of the computational domain. Next step is to implement the FDTD scheme for the three-dimension problem. The formulation of the Maxwell's equations in the total field and the scattered field will be implemented in the numerical simulation. The chosen problem is the interaction of the

EM field from a mobile telephone with the human head. The human head is modeled as three-dimensional grid points which each grid point is represented by a dielectric constant value of various tissue types, such as bone, brain, blood, skin, etc. The applied electromagnetic field will be modelled by both a plane wave and a dipole. All calculations in this research is implemented using MATLAB.

4. Conclusion

Research results during the period of three years can be summarized as the extension of standard FDTD schemes and having the improvement in two areas. First, we investigated the application of the higher-order accuracy scheme with the transparency boundary condition to the EM field interaction. Second, the Spline and the B-Spline method is introduced as an alternative scheme for solving the Maxwell's equations.

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FDTD: A Numerical Method for Electromagnetic Problems

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Abstract

Personal computers (PC) are playing an increasingly important role in a wide range of engineering computations. Students from all disciplines benefit from using PCs in solving engineering problems. Within various schools of engineering, the electrical engineering department currently teaches electromagnetic courses. The basic equations for electromagnetic theory are Maxwell's equations. It is rare for the electromagnetic problems to fall into a class that can be solved by analytical methods. There are many situations when the analytic methods are fail: the solution domain is complex; the medium is inhomogeneous and/or anisotropy; the equations have discontinuous coefficients and/or time varying; the boundary conditions are complicated. Whenever a problem with such complexity arises, numerical solutions must be employed. This paper presents a numerical method called the finite different time domain (FDTD) as a tool for analyzing the electromagnetic problems encountered in various electromagnetic courses. This FDTD scheme can be implemented using a low-cost PC.

INTRODUCTION

Maxwell's partial differential equations represent a fundamental unification of electric and magnetic fields predicting electromagnetic wave phenomena. The solution of Maxwell's equations has been found to be remarkably robust, providing accurate modeling predictions for a wide variety of free-space and guided wave electromagnetic interaction problems. There is frequency-domain and time-domain technique for solving Maxwell's equations. Frequency-domain techniques are based on the Fourier analysis while the time-domain technique employs finite element or finite difference methods. The disadvantage of the frequency-domain technique is its inability to handle the nonlinear problems arising from the inhomogeneity of the computational domain. The decomposition of the computational domain into finite element has been introduced and used to solve different electromagnetic problems. The disadvantage is its computational cost and its complexity of computer program. Also it can not handle the unbounded domain.

The finite difference techniques are based upon approximations, which permit replacing differential equations by finite difference equations. These finite differential equations relate the value of the dependent variable at a point in the computational domain to the value at some neighboring points.

The unique solution of finite difference equations is obtained using the prescribed boundary condition and/or initial conditions. There are many finite difference schemes for partial differential equation. The choice of the finite difference schemes is dictated by the nature of the problem being solved, the computational domains, and the boundary conditions.

The well-known finite-difference-time-domain (FDTD) for the electromagnetic fields problems was first introduced by Yee[1] in 1966 and later developed by Lubber, Taflove and others [2]–[8]. This FDTD scheme is a direct solution of Maxwell's curl equations.

FINITE DIFFERENCE TIME DOMAIN (FDTD) ALGORITHM

In an isotropic medium, Maxwell's equation can be written as

$$\nabla \times E = -\mu \frac{\partial H}{\partial t} \quad (1a)$$

$$\nabla \times H = \sigma E + \varepsilon \frac{\partial E}{\partial t} \quad (1b)$$

The vector Eq.(1) represents a system of six scalar equations, which can be expressed in rectangular coordinate system (x,y,z) as:

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu} \left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right) \quad (2a)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu} \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \quad (2b)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu} \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) \quad (2c)$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon} \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \sigma E_x \right) \quad (2d)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon} \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \sigma E_y \right) \quad (2e)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon} \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma E_z \right) \quad (2f)$$

Following Yee's notation, a grid point is defined as

$$(i,j,k) = (i \Delta x, j \Delta y, i \Delta z) \quad (3)$$

And any function of space and time as

$$F^n(i, j, k) = F(i \Delta x, j \Delta y, k \Delta z, n \Delta t)$$

Where $\Delta x, \Delta y, \Delta z$ are the space increments and Δt is the time increment, while i, j, k and n are integers.

Using the central finite difference approximations with an second-order accurate, the space and time derivative can be approximated

$$\frac{\partial F^n(i, j, k)}{\partial x} = \frac{F^n(i + \frac{1}{2}, j, k) - F^n(i - \frac{1}{2}, j, k)}{\delta} + O(\Delta x^2) \quad (4)$$

$$\frac{\partial F^n(i, j, k)}{\partial t} = \frac{F^{n+\frac{1}{2}}(i, j, k) - F^{n-\frac{1}{2}}(i, j, k)}{\Delta t} + O(\Delta t^2) \quad (5)$$

In order to apply Eq.(4)&(5) for this algorithm, Yee used an electric field (E) grid which was offset both spatially and temporally from a magnetic field (H) grid to obtain update equations that yield the present fields throughout the computational domain interms of the past fields. A Yee's unit cell is shown in Fig.1. Thus the explicit finite difference approximation of Eq.(2c) can be obtained as:

$$H_x^{n+\frac{1}{2}}(i, j + \frac{1}{2}, k + \frac{1}{2}) = H_x^{n-\frac{1}{2}}(i, j + \frac{1}{2}, k + \frac{1}{2}) + \frac{\Delta t}{\mu(i, j + \frac{1}{2}, k + \frac{1}{2})\delta} \left[E_y^n(i, j + \frac{1}{2}, k + 1) - E_y^n(i, j + \frac{1}{2}, k) - E_z^n(i, j, k + \frac{1}{2}) - E_z^n(i, j + 1, k + \frac{1}{2}) \right] \quad (6)$$

The update expression for other field components is obtained in similar fashions.

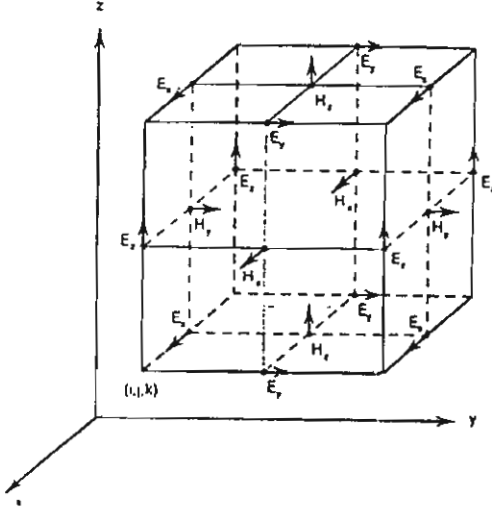


Figure 1. A unit cell of Yee's lattice.

This Yee scheme is second order accurate both in time and space. In order to improve the accuracy of this scheme, the difference operator in Eq. (4) is replaced by the following fourth-order accurate scheme:

$$\frac{\partial F^n(i, j, k)}{\partial x} = \frac{1}{24\Delta x} \left[F^n(i - \frac{3}{2}, j, k) - 27F^n(i - \frac{1}{2}, j, k) + 27F^n(i + \frac{1}{2}, j, k) - F^n(i + \frac{3}{2}, j, k) \right] \quad (7)$$

Thereafter Eq.(4) is called the Yee scheme and Eq. (7) the explicit (2,4) scheme. At the first and last points of bounded spatial domain, the fourth-order accurate one side approximation to the derivative is used. This is used only in order to globally approximate the derivative. These one-sided approximations are as follows:

$$\frac{\partial F(\frac{1}{2}, j, k)}{\partial x} = \frac{1}{24\Delta x} \left[-22F(0, j, k) + 17F(1, j, k) + 9F(2, j, k) - 5F(3, j, k) + F(4, j, k) \right]$$

$$\frac{\partial F(N - \frac{1}{2}, j, k)}{\partial x} = \frac{1}{24\Delta x} [22F(N, j, k) - 17F(N-1, j, k) - 9F(N-2, j, k) + 5F(N-3, j, k) - F(N-4, j, k)] \quad (8)$$

The only difference between the Yee scheme and the explicit (2,4) is the replacement of Eq. (4) by Eq. (7). The fourth order scheme in space enables one to choose a coarser mesh. This decreases the work in each space dimension and also decreases the storage.

COMPUTATIONAL RESULTS

In this section the standard Yee and the explicit (2,4) are compared. All schemes are advanced in time by the leapfrog method. To simplify the notation, the two dimensional transverse Magnetic (TM) case is considered. Letting, $\tau = \frac{t}{\sqrt{\mu\epsilon}}$ and $Z = \sqrt{\frac{\mu}{\epsilon}}$,

$$\begin{aligned} \frac{\partial E_z}{\partial \tau} &= Z \left(\frac{\partial H_y}{\partial y} - \frac{\partial H_x}{\partial x} \right) \\ \frac{\partial H_x}{\partial \tau} &= -\frac{1}{Z} \frac{\partial E_z}{\partial y} \\ \frac{\partial H_y}{\partial \tau} &= \frac{1}{Z} \frac{\partial E_z}{\partial x} \end{aligned} \quad (9)$$

For the rest of this paper, let $\epsilon = \mu = 1$. The exact solution as a basis for comparison in the box $[0,1] \times [0,1] \times [0,1]$ is used in this simulation. The walls are perfect conductors. The following boundary and initial conditions are obtained:

$$E_z(x, y, 0) = 0$$

$$H_x(x, y, \frac{\Delta t}{2}) = \frac{1}{\sqrt{2}} \cos(\omega_0 \frac{\Delta t}{2}) \sin \pi x \cos \pi y$$

$$H_y(x, y, \frac{\Delta t}{2}) = -\frac{1}{\sqrt{2}} \cos(\omega_0 \frac{\Delta t}{2}) \cos \pi x \sin \pi y$$

$$E_z(0, y, t) = E_z(1, y, t) = 0$$

$$E_z(x,0,t) = E_z(x,1,t) = 0$$

A test case is a TM_{110} mode with the following exact solution:

$$E_z = \sin \omega_0 t \sin \pi x \sin \pi y$$

$$H_x = \frac{1}{\sqrt{2}} \cos \omega_0 t \sin \pi x \cos \pi y$$

$$H_y = -\frac{1}{\sqrt{2}} \cos \omega_0 t \cos \pi x \sin \pi y$$

where $\omega_0 = \sqrt{2}\pi$. So that $f = \frac{1}{\sqrt{2}}$ and $\lambda = \sqrt{2}$. Assume $\hat{E}_z(i, j)$ is a solution obtained from the FDTD scheme. Define the error as:

$$\text{error} = \frac{\sqrt{\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} |\hat{E}_z(i, j) - E_z(i, j)|^2}}{N_x N_y}$$

where N_x and N_y are the number of grid points in the x and y directions respectively. Figure 2-5 shows the logarithmic errors as a function of time. Tables 1-2 compare the performance of both schemes over the time interval $[0, 5]$. The maximum error for each mesh size is shown in Table 1.

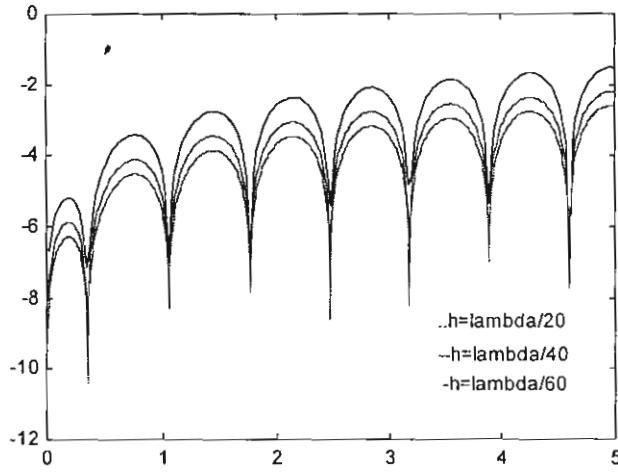


Figure 2. $\log_{10}(\text{error})$ for the Yee scheme

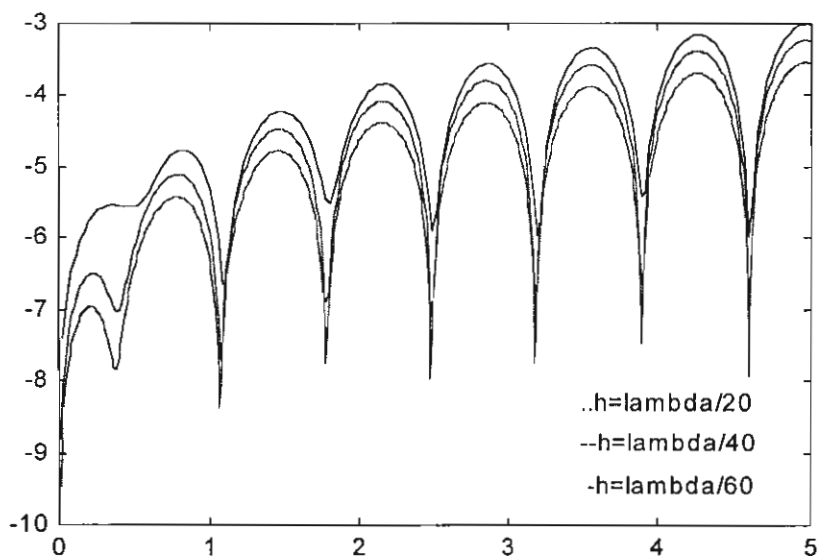


Figure 3. $\log_{10}(\text{error})$ for the explicit (2,4)

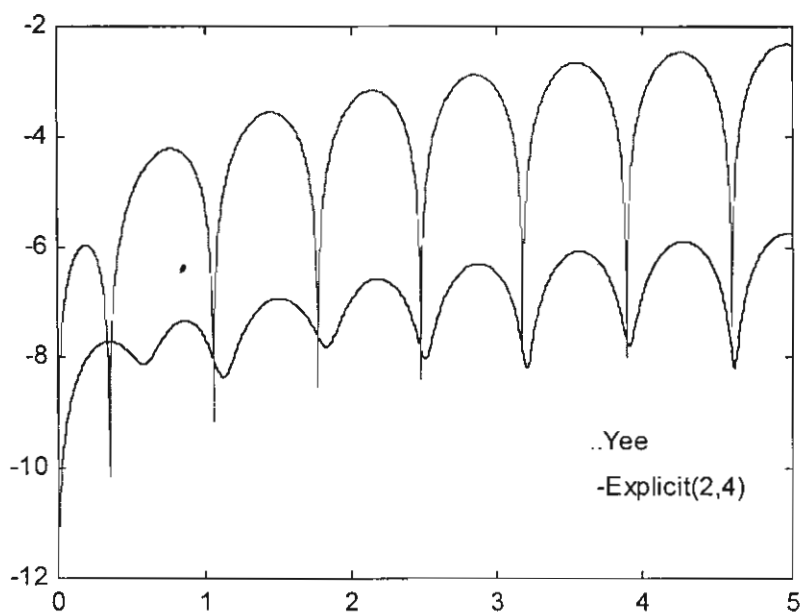


Figure 4. $\log_{10}(\text{error})$ for the Yee and the Explicit (2,4) with $h = \frac{\lambda}{60}$, $\frac{c\Delta t}{\Delta x} = 0.2$

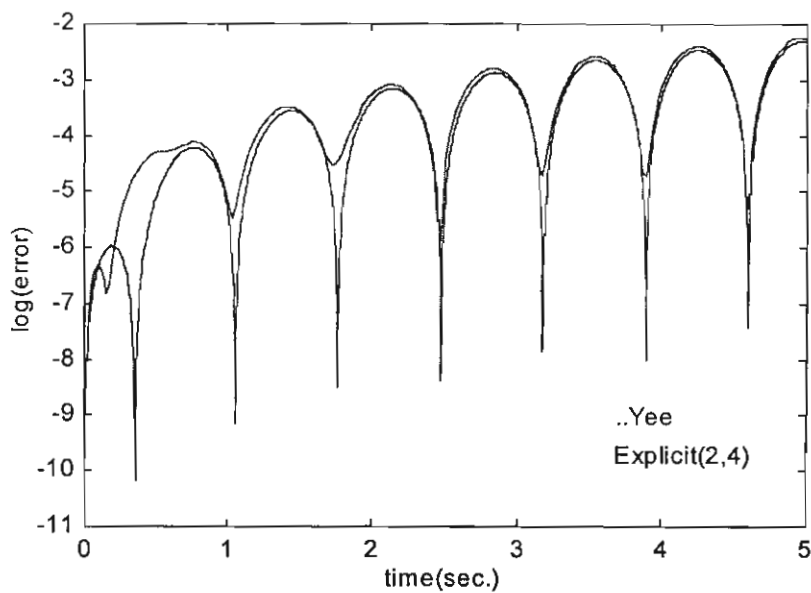


Figure 5. $\log_{10}(\text{error})$ for the Yee with $h = \lambda/60$ and the Explicit (2,4) with $h = \lambda/13$

Table 1.The maximum errors for each mesh size

Scheme	H_0	$c\Delta t/\Delta x$	Max($\log(\text{error})$) $0 \leq t \leq 5$
Yee	$\lambda/20$	0.4	-1.51
Yee	$\lambda/40$	0.4	-2.30
Yee	$\lambda/60$	0.4	-2.61
Explicit(2,4)	$\lambda/20$	0.4	-3.00
Explicit (2,4)	$\lambda/40$	0.4	-3.23
Explicit (2,4)	$\lambda/60$	0.4	-3.50
Yee	$\lambda/40$	0.2	-1.90
Yee	$\lambda/40$	0.6	-3.09
Explicit(2,4)	$\lambda/40$	0.2	-6.80
Explicit(2,4)	$\lambda/40$	0.6	-2.26

As shown in Fig.5, these mesh sizes are chosen in order to get the same error. The comparison is shown in Table 2. The programs were written in MATLAB and run on Pentium II Personal Computer. The CPU time needed to achieve the same accuracy in Yee's case is more than 27 times larger than is required for the explicit(2,4) scheme.

Table 2. CPU time using explicit (2,4) and the Yee scheme

Scheme	H	$c\Delta t/\Delta x$	Max (log(error))	CPU-time
Yee	1/60	0.2	-2.31	531.86 sec
Explicit(2,4)	1/13	0.2	-2.23	19.75 sec

CONCLUSION

Two FDTD techniques presented here offer a simple mathematical model for the analysis of the electromagnetic problems. The results demonstrate that a coarser mesh can be used with the fourth order scheme and still get the same accuracy as with the Yee scheme. The explicit (2,4) is easier to modify an existing code based on the Yee schemes and make it a fourth order accurate. From numerical examples, the CPU time needed to achieve the same accuracy in Yee's case is more than 27 times larger than is required for the explicit (2,4) scheme.

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