สัญญาเลขที่ RSA/16/2543

รายงานวิจัยฉบับสมบูรณ์

โครงการ

คุณสมบัติเรขาคณิตและการแปลงเมทริกซ์ของปริภูมิลำดับ
Geometric Property and Matrix Transformations of Sequence Spaces

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

เกียรติคุณประกาศ

ผู้วิจัยขอขอบคุณสำนักงานสนับสนุนการวิจัย (สกว) ที่ให้ทุนสนับสนุนการวิจัยในโครงการนี้ โดยเฉพาะฝ่ายวิชาการ สกว. ที่อำนวยความสะดวกในการทำวิจัยจนทำให้โครงการวิจัยนี้ประสบความสำเร็จเป็นอย่างดียิ่ง และ ผู้วิจัยขอขอบคุณ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ ที่สนับสนุนให้ทำการวิจัย และ อำนวยความสะดวกในเรื่องของสถานที่วิจัย การใช้อุปกรณ์การวิจัย เช่น เครื่องคอมพิวเตอร์ และ เครื่องพิมพ์ ตลอดจนให้การสนับสนุนงบประมาณบางส่วนในการเดินทางไปเสนอผลงานต่างประเทศ

สารบัญ

Abstract

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The first purpose of this project is to study some geometric properties of some Banach sequence spaces and the second purpose is to give characterizations of infinite matrices mapping from certain sequence spaces into some certain sequence spaces. In this research we define three new sequence spaces, $ces(p)$, ces_M and $\ell(\Delta,p)$, where $p=(p_k)$ is a bounded positive sequence of real numbers and $M=(M_k)$ is a Musielak-Orlicz function. The spaces $ces(p)$ and ces_M are generalizations of the cesaro sequence spaces. We introduce the Luxemburg norrn on these spaces.
We show that $ces(p)$ and $\ell(\Delta,p)$ have property (H) if $p_k \geq 1$ for all $k \in N$ and $ces(p)$ is locally uniformly rotund if $p_k > 1$ for all $k \in N$. These results generalized the previous works of many mathematicians. We also show that the difference sequence spaces $\ell(\Delta,p)$ it is rotund if and only if $p_k > 1$ for all $k \in N$. In studying geometric properties on the spaces ces_M, we obtain that the space ces_M is (UKK) space when M satisfies δ_2- condition and the condition $(*)$, so it has property (H). We further study some geometric properties of Orlicz-sequence spaces of Bochner type.

For the second main purpose of this project, we give characterizations of infinite matrices mapping the Nakano vector -valued sequence spaces $\ell(X,p)$ and $F_r(X,p)$ into $E_r, \ell_\infty, \underline{\ell}_\infty(q), bs, cs$, when $p_k > 1$ for all $k \in N$, and we also we give characterizations of infinite matrices mapping from $\ell(X,p)$ and $M_0(X,p)$ into E_r when $p_k \leq 1$ for all $k \in N$. We can completely give characterizations of infinite matrices mapping from any FK- spaces into the spaces $c(q)$. From this result, we obtain many matrix transformations from certain FK-spaces into $c(q)$. Furthermore, we also we give characterizations of infinite matrices mapping from the sequence spaces of Maddox into Musielak-Orlicz sequence spaces.

Keywords : Geometric properties, property (H), rotund, locally uniformly rotund, uniform Kadec-Klee property, Cesaro sequence spaces, Musielak-Orlicz functions, Matrix transformations, Maddox sequence spaces, Vector-valued sequence spaces

บทคัดย่อ

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ชื่อโครงการ : คุณสมบัติเรขาคณิตและการแปลงเมทริกซ์ของปริภูมิลำดับ

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วัตถุประสงค์แรกของงานวิจัยนี้คือการศึกษาสมบัติเรขาคณิตของบางลำดับปริภูมิบานาค และ วัตถุประสงค์ที่สองคือการหาลักษณะเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิลำดับหนึ่งไปสู่อีกปริภูมิลำดับหนึ่ง ในงานวิจัยนี้เรานิยามปริภูมิลำดับใหม่ขึ้นมา 3 ปริภูมิลำดับคือ $ces(p)$, ces_M และ $\ell(\Delta,p)$ เมื่อ $p=(p_k)$ เป็นลำดับที่มีขอบเขตของจำนวนจริง และ $M=(M_k)$ เป็นฟังก์ชันมูสิลักซ์ ปริภูมิ $ces(p)$ และ ces_M เป็นปริภูมิที่วางนัยทั่วไปของปริภูมิลำดับเซซาโร เราศึกษาปริภูมิเหล่านี้ภายใต้นอร์มลักเซมเบิกร์ก เราได้แสดงว่าปริภูมิ $ces(p)$ และ $\ell(\Delta,p)$ มีสมบัติ (H) เมื่อ $p_k \geq 1$ สำหรับทุก $k \in N$ และปริภูมิ $ces(p)$ เป็นปริภูมิ LUR ถ้า $p_k > 1$ สำหรับทุก $k \in N$ ผลต่างๆเหล่านี้คลุมผลงานของนักคณิตศาสตร์หลายท่าน นอกจากนี้เรายังได้แสดงว่าปริภูมิ $\ell(\Delta,p)$ เป็น rotund ก็ต่อเมื่อ $p_k > 1$ สำหรับทุก $k \in N$ สำหรับการศึกษาสมบัติเรขาคณิตในปริภูมิ ces_M นั้นเราได้พบว่าปริภูมิ ces_M เป็นปริภูมิ (UKK) เมื่อ M สอดคล้องเงื่อนไข δ_2 และเงื่อนไข (*) นอกจากนี้เรายังได้ศึกษาสมบัติเรขาคณิตอีกหลายชนิดบนปริภูมิลำดับออร์ลิคซ์ชนิดบอคเนอร์ด้วย

สำหรับวัตถุประสงค์ที่สองนั้นเราได้ให้ลักษณะเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิลำดับค่าเวกเตอร์นาคาโน $\ell(\Delta,p)$ และ $F_r(X,p)$ ไปยังปริภูมิ $E_r, \ell_\infty, \underline{\ell}_\infty(q), bs, cs$ เมื่อ $p_k > 1$ สำหรับทุก $k \in N$ และเรายังได้ให้ให้ลักษณะเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิ $\ell(\Delta,p)$ และ $M_0(X,p)$ ไปยังปริภูมิ E_r เมื่อ $p_k \leq 1$ สำหรับทุก $k \in N$ เราสามารถให้ลักษณะเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิ FK ใดๆไปยังปริภูมิ $c(q)$ นอกเหนือจากนี้แล้วเราได้ให้ลักษณะเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิลำดับของแมดดอกซ์ไปยังปริภูมิลำดับมูสิลัก-ออร์ลิคซ์

คำหลัก : สมบัติเรขาคณิต สมบัติ (H) โรทุน โลคอลลียูนิฟอร์มโรทุน สมบัติยูนิฟอร์มคาเดค-คลี ปริภูมิลำดับเซซาโร ฟังก์ชันมูสิลัค-ออร์ลิคซ์ การแปลงเมทริกซ์ ปริภูมิลำดับแมดดอกซ์ ปริภูมิลำดับค่าเวกเตอร์

บทที่ 1
บทนำ

1.1ความสำคัญและที่มาของปัญหาที่ทำการวิจัย

ปริภูมิลำดับ (Sequence spaces) นับเป็นแขนงที่สำคัญแขนงหนึ่งในสาขาของการวิเคราะห์ฟังก์ชัน (Functional Analysis) ในปัจจุบันนักคณิตศาสตร์ได้ศึกษาและวิจัยกันอย่างต่อเนื่องในการคิดค้นทฤษฎีเพื่อหาองค์ความรู้ใหม่ ๆ ต่าง ๆมากมายในหลาย ๆหัวข้อของสาขานี้ เป็นที่ยอมรับว่าทฤษฎีและองค์ความรู้ใหม่ ๆที่เกิดจากการวิจัยนั้น นอกจากจะมีประโยชน์อย่างมากในการพัฒนาความรู้เชิงวิชาการในสาขาแล้ว บางครั้งสามารถนำไปประยุกต์ในสาขาอื่นๆ การพัฒนาทางวิชาการและการคิดค้นทฤษฎีและองค์ความรู้ใหม่ๆต่าง ๆในสาขานี้นับเป็นการพัฒนาที่สำคัญทางวิทยาศาสตร์พื้นฐาน (Basic science) อันถือเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

ปริภูมิลำดับถือได้ว่าเป็นตัวอย่างที่สำคัญของปริภูมิฟังก์ชัน (Function spaces) และในสาขาการวิเคราะห์ฟังก์ชันนั้นงานวิจัยโดยหลักแล้วก็เป็นการศึกษาคิดค้นทฤษฎีต่าง ๆเกี่ยวกับปริภูมิฟังก์ชันทั้งสิ้น ตัวอย่างที่สำคัญ ๆที่ทำให้เกิดแนวคิดที่กว้างขวางยิ่งขึ้นในสาขานี้ส่วนใหญ่เกิดจากตัวอย่างที่เกี่ยวกับปริภูมิลำดับ ดังนั้นจะเห็นได้ว่าเมื่อนักคณิตศาสตร์ได้คิดค้นและสร้างแนวคิดใหม่ ๆเกี่ยวกับปริภูมิฟังก์ชันก็มักจะหันมาศึกษาทฤษฎีและคุณสมบัติต่าง ๆที่เกี่ยวข้องกับปริภูมิลำดับเสมอ ในปัจจุบันมีนักคณิตศาสตร์จำนวนมากที่สนใจศึกษา วิจัยเกี่ยวกับคุณสมบัติทางเรขาคณิต (Geometric property) ของปริภูมิบานาค (Banach spaces) ซึ่งคุณสมบัติทางเรขาคณิตนั้นนับเป็นคุณสมบัติที่สำคัญที่ทำให้เราทราบถึงลักษณะและคุณสมบัติที่สำคัญต่าง ๆของปริภูมิบานาคมากขึ้น เช่น ถ้าปริภูมิบานาคมีคุณสมบัติบางประการทางเรขาคณิตแล้ว ปริภูมิบานาคนั้นจะเป็นปริภูมิสะท้อน (Reflexive space) ตัวอย่างเช่น ถ้าปริภูมิบานาคมีคุณสมบัติใดคุณสมบัติหนึ่งต่อไปนี้คือ

1. Uniform rotund
2. P-convexity
3. Q-convexity
4. Banach -saks property (BSP)

แล้วปริภูมินั้นจะเป็นปริภูมิสะท้อน (Reflexive space) ตัวอย่างอื่น ๆที่น่าสนใจคือทฤษฎีที่คิดค้นโดย Rolewicz [42] ที่พบว่าปริภูมิบานาค X มีคุณสมบัติ drop property ก็ต่อเมื่อ X เป็นปริภูมิสะท้อน และ X มีคุณสมบัติ H (H-property) Huff [30] ก็ได้แสดงว่า ทุก ๆปริภูมิ NUC (nearly uniformly convex) เป็นปริภูมิสะท้อนและมีคุณสมบัติ H

เราจะเห็นได้ว่าคุณสมบัติ H , drop property และ NUC ต่างก็เป็นคุณสมบัติเรขาคณิตที่สำคัญ ๆของปริภูมิบานาคทั้งสิ้น จากตัวอย่างข้างต้นและผลงานวิจัยอีกมากมายทำให้เกิดความตื่นตัวกันมากในวงการของการวิเคราะห์ฟังก์ชัน ทำให้นักคณิตศาสตร์ได้ศึกษาทฤษฎีและคุณสมบัติเรขาคณิตของปริภูมิบานาคมากขึ้น โดยเพียงเวลาไม่ถึง 20 ปี แต่มีการพัฒนาคิดค้นและวิจัยกันอย่างต่อเนื่องเกี่ยวกับคุณสมบัติเรขาคณิตของปริภูมิบานาค

ตามที่ได้กล่าวมาแล้วว่าปริภูมิลำดับเป็นตัวอย่างที่สำคัญของปริภูมิฟังก์ชันและปริภูมิบานาค ดังนั้นจึงมีนักคณิตศาสตร์จำนวนมาก (ดูได้จากเอกสารอ้างอิง) ได้ศึกษาคิดค้นและวิจัยเกี่ยวกับคุณสมบัติเรขาคณิตต่าง ๆในปริภูมิลำดับที่น่าสนใจ เช่น ในปริภูมิลำดับออร์ลิคซ์ (Orlicz sequence spaces) ปริภูมิลำดับเซซาโร (Cesaro sequence spaces) ปริภูมิลำดับมูซีลัก-ออร์ลิกซ์ (Musielak-Orlicz sequence spaces) ปริภูมิลำดับนาคาโน (Nakano sequence spaces)เป็นต้น

จากการศึกษาผลงานวิจัยต่าง ๆที่เกี่ยวข้องพบว่ายังมีปัญหาต่าง ๆมากมายเกี่ยวกับคุณสมบัติเรขาคณิตของปริภูมิลำดับข้างต้น ตลอดจนปริภูมิลำดับบานาค (Banach sequence spaces) อื่น ๆ อีกมากมาย

ปัญหาอีกปัญหาหนึ่งที่เป็นที่นิยมศึกษาวิจัยกันมากในสาขาของปริภูมิลำดับคือ ปัญหาการแปลงเมทริกซ์ของปริภูมิลำดับซึ่งเป็นปัญหาที่เกิดมาจากการศึกษา Summability Theory ของลำดับลู่ออก (divergent sequence) และอนุกรมลู่ออก (divergent series) ลักษณะของปัญหาดังกล่าวนี้คือ เป็นการศึกษาและให้ลักษณะของเมทริกซ์อนันต์ที่ส่ง (map) จากปริภูมิลำดับหนึ่งไปยังอีกปริภูมิลำดับหนึ่ง เช่น ทฤษฎีของโคจิมา-เชอร์ (Kojima-Shur Theorem) ได้ให้ลักษณะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิลำดับลู่เข้า (c) ไปยังปริภูมิลำดับลู่เข้า (c) ซึ่งถือเป็นการแปลงเมทริกซ์ที่รักษาคุณสมบัติเดิมของปริภูมิที่ถูกส่งไป ปัญหาการแปลงเมทริกซ์ในช่วงต้นนั้น ได้สนใจศึกษาเฉพาะเมทริกซ์อนันต์ของจำนวนจริงหรือจำนวนเชิงซ้อน และปริภูมิลำดับก็เป็นปริภูมิลำดับของจำนวนจริงหรือจำนวนเชิงซ้อน ต่อมานักคณิตศาสตร์จำนวนมากได้ขยายแนวคิดของปัญหาดังกล่าวไปสู่ปริภูมิลำดับค่าเวกเตอร์ (Vector-valued sequence spaces) และเมทริกซ์อนันต์ของตัวดำเนินการ เช่น ในเอกสาร [48 - 50], [73] ดังนั้นการศึกษาในปัญหาใหม่นี้นอกจากจะให้ทฤษฎีใหม่ ๆ แล้วผลงานที่ได้ยังสามารถตอบคำถามของปัญหาในแบบเดิมได้อีกด้วย

1.2 วัตถุประสงค์ของโครงการ

1 ศึกษาคุณสมบัติเรขาคณิตต่าง ๆของปริภูมิลำดับบานาค(Banach sequence spaces)

2 เพื่อหาลักษณะเฉพาะของปริภูมิลำดับบานาคต่าง ๆ ที่มีคุณสมบัติเรขาคณิตต่าง ๆ

3 เพื่อหาลักษณะของเมทริกซ์อนันต์ที่ส่งระหว่างปริภูมิลำดับ

4 เพื่อหาลักษณะของดวลต่าง ๆของปริภูมิลำดับ

5 เป็นการสร้างและพัฒนานักวิจัยรุ่นใหม่คือนักศึกษาระดับปริญญาโท และ เอก ในสาขาคณิตศาสตร์ของประเทศไทย

เนื้อหางานวิจัยนี้ได้แบ่งออกเป็นสองบทใหญ่ ๆ คือ การศึกษาสมบัติเรขาคณิตบางประการของปริภูมิลำดับบานาคในบทที่ 2 และ การศึกษาเกี่ยวกับการแปลงเมทริกซ์ของปริภูมิลำดับในบทที่ 3 สำหรับในภาคผนวกนั้นเป็นการรวบรวม reprints ของผลงานที่ได้รับในช่วงที่ทำการวิจัย

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บทที่ 2
สมบัติเรขาคณิตบางประการของปริภูมิลำดับบานาค
(Some Geometric Properties of Banach Sequence Spaces)

ในบทนี้เป็นการศึกษาสมบัติเรขาคณิตของบางลำดับปริภูมิบานาค ในงานวิจัยนี้เรานิยามปริฏมิลำดับใหม่ขึ้นมา 3 ปริภูมิลำดับคือ $ces(p)$, ces_M และ $\ell(\Delta, p)$ เมื่อ $p=(p_k)$ เป็นลำดับที่มีขอบเขตของจำนวนจริง และ $M=(M_k)$ เป็นฟังก์ชันมูสิลักซ์ ปริภูมิ $ces(p)$ และ ces_M เป็นปริภูมิที่วางนัยทั่วไปของปริภูมิลำดับเซซาโร เราศึกษาปริภูมิเหล่านี้ภายใต้นอร์มลักเซมเบิกร์ก และได้แสดงว่าปริภูมิ $ces(p)$ และ $\ell(\Delta, p)$ มีสมบัติ (H) เมื่อ $p_k \geq 1$ สำหรับทุก $k \in N$ และปริภูมิ $ces(p)$ เป็นปริภูมิ LUR ถ้า $p_k > 1$ สำหรับทุก $k \in N$ ผลต่าง ๆเหล่านี้คลุมผลงานของนักคณิตศาสตร์หลายท่าน นอกจากนี้เรายังได้แสดงว่าปริภูมิ $\ell(\Delta, p)$ เป็น rotund ก็ต่อเมื่อ $p_k > 1$ สำหรับทุก $k \in N$ สำหรับการศึกษาสมบัติเรขาคณิตในปริภูมิ ces_M นั้นเราได้พบว่าปริภูมิ ces_M เป็นปริภูมิ (UKK) เมื่อ M สอดคล้องเงื่อนไข δ_2 และเงื่อนไข $(*)$ นอกจากนี้เรายังได้ศึกษาสมบัติเรขาคณิตอีกหลายชนิดบนปริภูมิลำดับออร์ลิคซ์ด้วย

SOME GEOMETRIC PROPERTIES IN ORLICZ SEQUENCE SPACE OF BOCHNER TYPE

1. INTRODUCTION

The concept of Orlicz space was introduced by W.Orlicz early in 1932, however it was not until the last ten years that the theory of geometry of Orlicz space was developed extensively. A much richer field of examples is obtained by considering Orlicz sequence space. In 1986, S.Chen, C.Wui, T.Wang, and Y.Wang, published a book, *Theory of geometry of Orlicz spaces* (in Chinese), which collected the main results on geometry of Orlicz spaces as well as some applications obtained by that time, that was made great advances in the very short time and, at the same time, many geometric properties have been discuss more precisely, to the local behavior, to the pointwiseness. Moreover, since the book was published many open ploblems have been solved.

Chen and Wang[3] have studied **H**-*property* of Orlicz space. Cui, Hudzik and Meng[4] have studied some local geometry of Orlicz sequence space equipped with Luxemburg norm. Huff[7] has studied Banach space which are nearly uniform convex and proved that **(UKK)**⇒**(H)**, **(NUC)**⇔**(UKK)**+*reflexive*. Turett[15] has studied rotundity of Orlicz spaces.

The aim of this research is to generalize some geometric properties of Orlicz sequence space ℓ_M to the Orlicz sequence space of Bochner tpye $\ell_M(X)$, where X is a Banach space, and give characterizations of the Orlicz sequence space of Bocher tpye $\ell_M(X)$ to have the property **(R)**, **(H)**, **(UKK)**, **(LUR)**, **(CLUR)** and **(WCLUR)**. Furtheremore, we also give some relationship between those properties in this space.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2. PRELIMINARIES

In this section, we give some definitions, notation and some known results needed for the later section.

Throughout this research, we let $\mathbb{N}$ stand for the set of all positive integers. The field of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively. The symbol $\mathbb{F}$ stands for $\mathbb{R}$ or $\mathbb{C}$, and $\mathbb{R}_+$ is denoted by the set of all positive real numbers. The elements of $\mathbb{F}$ are called scalars.

2.1 Metric Space, Normed Space and Banach space.

Definition 2.1.1. A *metric space* is a pair (X, d), where X is a set and d a metric on X, that is $d : X \times X \to \mathbb{R}$ is a function such that the following three conditions are satisfied by all x, y and z in X:

(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x, z) \leq d(x, y) + d(y, z)$ (the triangle inequality).

Definition 2.1.2. Let (X, d) be a metric space. A sequence (x_n) of members of X *converge* to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$. When (x_n) converges to x, we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Definition 2.1.3. A sequence (x_n) in a metric space is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$.

Definition 2.1.4. A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges.

Definition 2.1.5. Let X be a linear space (or a vector space). A *norm* on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:

(1) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$.

(2) $||\alpha x|| = |\alpha|||x||$.

(3) $||x + y|| \leq ||x|| + ||y||$ (the triangle inequality).

The ordered pair $(X, ||\cdot||)$ is called a *normed space.* When there is no danger of confusion, it is customary to use the same symbol, such as X, to denote the normed space.

Definition 2.1.6. Let X be a normed space. The *metric induced by the norm* of X is the metric d on X defined by the formular $d(x, y) = ||x - y||$.

Definition 2.1.7. Let x be a normed space. The *closed unit ball* of X is $\{x : x \in X, ||x|| \leq 1\}$ and is denoted by $B(X)$. The *unit sphere* of X is $\{x : x \in X, ||x|| = 1\}$ and is denoted by $S(X)$.

Definition 2.1.8. A *Banach space* is a normed linear space which is complete under the metric induced by norm.

2.2 Linear Operator, Strong and Weak convergence

Definition 2.2.1. Let X and Y be normed spaces. Let $T : X \to Y$, if $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$ for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in X$, we say T is a *linear operator* or *linear transformation* from X to Y. When $Y = \mathbb{F}$, we say that T is a *linear functional* on X.

Definition 2.2.2. A linear operator $T : X \to Y$ is *bounded* if there exists $M \geq 0$ such that $||Tx|| \leq M||x||$ for all $x \in X$. The operator norm for a bounded linear operator T is defined as $||T|| = \sup\limits_{0 \neq x \in X} \frac{||Tx||}{||x||}$.

Theorem 2.2.3. A linear operator is bounded if and only if it is continuous. **Proof** See [11]. □

Definition 2.2.4. Let X be a normed space. Then the set of all bounded linear functional on X constitutes a normed linear space with normed defined by

$$||T|| = \sup_{0 \neq x \in X} \frac{||Tx||}{||x||}$$

which is called the *dual space of* X and it denoted by X'.

Definition 2.2.5. A sequence (x_n) in a normed space X is said to be *strongly convergent* (or *convergent in norm*) if there is an $x \in X$ such that $\lim_{n\to\infty} x_n = x$ (or $x_n \to x$).

Definition 2.2.6. A sequence (x_n) in a normed space X is said to be *weakly convergent* if there is an $x \in X$ such that for every $T \in X'$, $\lim_{n\to\infty} Tx_n = Tx$ (or $x_n \to x$, *weakly*).

2.3 Convex Functions and Orlicz Functions.

Definition 2.3.1. A continuous function $M : \mathbb{R} \to \mathbb{R}$ is called *convex* if

$$M\left(\frac{u+v}{2}\right) \le \frac{M(u)+M(v)}{2} \tag{2.1}$$

for all $u, v \in \mathbb{R}$. If, in addition, the two sides of (2.1) are not equal for all $u \neq v$, then we call M *strictly convex.*

Proposition 2.3.2. Let $M : \mathbb{R} \to \mathbb{R}$ be a continuous function. The following are equivalent:

(1) M is convex.

(2) There exists affine functions $L_n(u) = a_n u + b_n$ such that $M(u) = \sup_n L_n(u)$.

(3) For any $u, v \in \mathbb{R}$ and $\alpha \in [0, 1]$,

$$M(\alpha u + (1-\alpha)v) \le \alpha M(u) + (1-\alpha)M(v). \tag{2.2}$$

(4) For any $u_1, u_2, ..., u_n \in \mathbb{R}$ and $\alpha_i \ge 0$ with $\sum_{i=1}^{n} \alpha_i = 1$

$$M\left(\sum_{i=1}^{n} \alpha_i u_i\right) \le \sum_{i=1}^{n} \alpha_i M(u_i)$$

Proof. See [9]. □

Definition 2.3.3. A continuous function $M : \mathbb{R} \to \mathbb{R}$ is called an *Orlicz function* if it has the following properties:

(1) M is even, continuous, convex and $M(0) = 0$.

(2) $M(u) > 0$ for all $u \neq 0$.

(3) $\lim\limits_{u\to 0} \frac{M(u)}{u} = 0$ and $\lim\limits_{u\to\infty} \frac{M(u)}{u} = \infty$.

In addition, an Orlicz function N is called a *complementary function* of an Orlicz function M if

$$N(v) := \sup\{|v|u - M(u) \,:\, u \geq 0\}.$$

Definition 2.3.4. Let M be an Orlicz function. An interval $[a, b]$ is called a *structural affine interval* of M, or simply, SAI of M, is affine on $[a, b]$ and it is not affine on either $[a - \epsilon, b]$ or $[a, b + \epsilon]$ for any $\epsilon > 0$. Let $\{[a_i, b_i]\}_i$ be all the $SAIs$ of M. We call

$$S_M = \mathbb{R}\backslash[\bigcup_i (a_i, b_i)]$$

the set of *strictly convex points* of M.

It is well known that (see [2]) if $u, v \in \mathbb{R}, \alpha \in (0, 1)$ and $\alpha u + (1 - \alpha)v \in S_M$, then

$$M(\alpha u + (1 - \alpha)v) < \alpha M(u) + (1 - \alpha)M(v).$$

2.4 Convex Modular and Orlicz Vector-valued Sequence Spaces.

Definition 2.4.1 A sequence space is a vector space whose members are sequence under the usual addition and usual scalar multiplication.

Definition 2.4.2 For a real (or complex) vector space X, a function $f : X \to \mathbb{R}_+$ is a modular if the following properties are satisfied:

(1) $f(x) = 0$ if and only if $x = 0$.

(2) $f(\alpha x) = f(x)$ for all $\alpha \in \mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) with $|\alpha| = 1$.

(3) $f(\alpha x + \beta y) \leq f(x) + f(y)$ for all $\alpha, \beta \in \mathbb{R}_+$, such that $\alpha + \beta = 1$.

If the property (3) is replaced by:

(3′) $f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$ for all $\alpha, \beta \in \mathbb{R}_+$, such that $\alpha + \beta = 1$,

then we say that f is a *convex modular.*

Definition 2.4.3. Let X be a Banach space. Denote X^o the space of all sequence in X. For $x \in X^o$, we denote $x(i)$ the i^{th} term of x.

For a given Orlicz function M, we define $\varrho_M : X^o \to [0, \infty]$ by the formular

$$\varrho_M(x) = \sum_{i=1}^{\infty} M(\|x(i)\|).$$

We shall show in Chapter 3 that ϱ_M is a convex modular.
The Orlicz vector-valued sequence space $\ell_M(X)$ and its subspace $h_M(X)$ are defined as follows :

$$\ell_M(X) := \{x \in X^o : \varrho_M(cx) < \infty \quad \text{for some} \quad c > 0\}$$
$$h_M(X) := \{x \in \ell_M(X) : \varrho_M(cx) < \infty \quad \text{for all} \quad c > 0\}$$

We consider $\ell_M(X)$ equipped with the so-called *Luxemburg norm*

$$\|x\|_M = \inf\{\lambda > 0 : \varrho_M(x/\lambda) \le 1\}$$

under which we can show in Chapter 3 that $(\ell_M(X), \|\cdot\|_M)$ is a Banach space.
We call $(\ell_M(X), \|\cdot\|_M)$ the Orlicz vector-valued sequence space generated by the Orlicz function M and we will denote $\ell'_M(X)$ for the dual space of $\ell_M(X)$.

Recall that for an Orlicz function M, if $X = \mathbb{R}$ we denote $\ell_M(X)$ by ℓ_M it is known as an Orlicz sequence space and ϱ_M is a convex modular.

Definition 2.4.4. An Orlicz function M is said to be satisfies the $\delta_2 -$ *condition* ($M \in \delta_2$ for short) if ther exists constant $K \ge 2$, $u_0 > 0$ such that

$$M(2u) \le KM(u)$$

holds for every $u \in \mathbb{R}$ satisfying $|u| \le u_0$.

Theorem 2.4.5. The following are equivalent:

(1) $M \in \delta_2$.

(2) There exist $l > 1$, $u_0 > 0$, and $K > 1$ such that

$$M(lu) \le KM(u) \qquad (u < u_0)$$

(3) For any $l_1 > 1$ and $u_1 > 0$, there exists $K' > 0$ such that the inequality in (2) holds for $l = l_1$, $u_0 = u_1$ and $K = K'$.

(4) For any $l_2 > 1$ and $u_2 > 0$ there exists $\epsilon \in (0,1)$ such that

$$M((1+\epsilon)u) \leq l_2 M(u) \quad (u < u_2)$$

Proof. See [2]. □

For the dual space $\ell'_M(X)$ of $\ell_M(X)$, we say that $\varphi \in \ell'_M(X)$ is a singular functional if $\varphi(h_M(X)) = 0$.

Theorem 2.4.6. Any $f \in \ell'_M(X)$ has a unique decomposition

$$f = v + \varphi$$

where v and φ are respectively the regular and singular parts of f.
Proof. See [1]. □

Before ending this section, we introduce the relation between the distance from $x \in \ell_M(X)$ to $h_M(X)$:

$$\theta(x) = inf\{\lambda > 0 : \varrho_M(u/\lambda) < \infty\}$$

Note that $h_M(X) = \{x \in X^o | \theta(x) = 0\}$. For $x \in \ell_M(X)$ the next theorem show that $d(x, h_M(x)) = \theta(x)$.

Theorem 2.4.7. For any $x \in \ell_M(X)$, we have $d(x) = \theta(x)$, where

$$d(x) = inf\{\|x-u\|_M : u \in h_M(X)\}$$

Proof. See [2]. □

2.5 Some Geometric Properties of Banach Spaces.

In this section we introduce the definitions and notation concerning geometric properties of Banach spaces and give some known results that will be used in the later chapter.

Definition 2.5.1. For a Banach space X. A point $x \in B(X)$ is called an *extreme point* of $B(X)$ if $2x = y + z$ and $y, z \in B(X)$ imply $y = z$. The set of all extreme points of $B(X)$ is denoted by **Ext** $B(X)$. If **Ext** $B(X) = S(X)$, then X is called a *rotund* (**R**) space.

Definition 2.5.2. For a Banach space X, if for any $x_n, y_n \in B(X), \|x_n + y_n\| \to 2$ implies $\|x_n - y_n\| \to 0$, then X is called a *uniformly rotund* (**UR**) space.

Definition 2.5.3. For a Banach space X, if $x_n, y_n \in B(X)$ and $\|x_n + y_n\| \to 2$ imply $x_n - y_n \to 0$ *weakly*, then X is called a *weakly uniformly rotund* (**WUR**) space.

Definition 2.5.4. For a Banach space X, if for each $x \in S(X)$ and each sequence (x_n) in $S(X)$ such that $\lim\limits_{n\to\infty} \|x_n + x\| = 2$, there holds $\lim\limits_{n\to\infty} \|x_n - x\| = 0$, then X is called a *locally uniformly rotund* (**LUR**) space.

Definition 2.5.5. For a Banach space X, if for each $x \in S(X)$ and each sequence (x_n) in $S(X)$ such that $\lim\limits_{n\to\infty} \|x_n + x\| = 2$ imply $x_n \to x$ *weakly*, then X is called a *weakly locally uniformly rotund* (**WLUR**) space.

Theorem 2.5.6. Every uniformly rotund normed space is locally uniformly rotund, and every locally uniformly rotund is rotund. In symbols,

$$\mathbf{UR} \Rightarrow \mathbf{LUR} \Rightarrow \mathbf{R}.$$

Proof. See [11]. $\square$

Definition 2.5.7. For a Banach space X, if for each $x \in S(X)$ and each sequence (x_n) in $S(X)$ such that $\lim\limits_{n\to\infty} \|x_n + x\| \to 2$, there holds (x_n) is compact in $S(X)$, then X is called a *compactly locally uniformly rotund* (**CLUR**) space.

Definition 2.5.8. For a Banach space X, if for each $x \in S(X)$ and each sequence (x_n) in $S(X)$ with $\lim\limits_{n\to\infty} \|x_n + x\| \to 2$, there are $x' \in S(X)$ and a subsequence $\{x'_n\}$ of (x_n)

in such that $x'_n \to x'$ *weakly*, then X is called a *weakly compactly locally uniformly rotund* (**WCLUR**) space.

Theorem 2.5.9. Banach space X is **LUR** if and only if X is **CLUR** and **R**. **Proof.** See [13]. □

Definition 2.5.10. A Banach space X is said to have *property*(**H**)(or the *Kadec-Klee property)* if every weakly convergent sequence on the unit sphere $S(X)$ is convergent in norm.

Definition 2.5.11. A Banach space X is said to have *uniform Kadec-Klee property* (written **UKK**) if for any $\epsilon > 0$, there exists $\delta > 0$ such that $x_n \in S(X)$, $x_n \to x$ *weakly*, and $\|x_n - x_m\| \geq \epsilon$ $(n \neq m)$ imply $\|x\| \leq 1 - \delta$.

Theorem 2.5.12. Every **UKK** Banach space has *property*(**H**).
Proof. See [7]. □

3. MAIN RESULTS

This chapter is divided into two sections. The first section concerning rotundity of Orlicz vector-valued sequence space and the other section study about several geometric properties which are related to rotundity of Orlicz vector-valued sequence space, such as *property*(**H**), (**UKK**), (**CLUR**), (**WCLUR**).

3.1 Rotundity of Orlicz Vector - valued Sequence Space

In this section, characterizations of rotundity of Orlicz vector-valued sequence space equipped with the Luxemburg norm. At first, we begin with giving some important properties of the modular ϱ_M and the Luxemburg norm defined by ϱ_M. In the first proposition we show that ϱ_M is a convex modular on $\ell_M(X)$.

Proposition 3.1.1. ϱ_M is a convex modular.

Proof. Let $x, y \in X^o$ by definition of ϱ_M we have that

$$
\begin{aligned}
(i) \quad \varrho_M(x) = 0 &\Leftrightarrow \sum_{i=1}^{\infty} M(\|x(i)\|) = 0 \\
&\Leftrightarrow M(\|x(i)\|) = 0 \quad \text{for all} \quad i \\
&\Leftrightarrow \|x(i)\| = 0 \qquad \text{for all} \quad i \\
&\Leftrightarrow x = 0
\end{aligned}
$$

(ii) For $\alpha \in \mathbb{F}(\mathbb{F} = \mathbb{R}$ or $\mathbb{C})$ and $|\alpha| = 1$ we have

$$
\begin{aligned}
\varrho_M(\alpha x) &= \sum_{i=1}^{\infty} M(\|\alpha x(i)\|) \\
&= \sum_{i=1}^{\infty} M(|\alpha|\|x(i)\|) \\
&= \sum_{i=1}^{\infty} M(\|x(i)\|) \\
&= \varrho_M(x)
\end{aligned}
$$

(*iii*) For $\alpha, \beta \in \mathbb{R}_+$, $\alpha + \beta = 1$ we have

$$\begin{aligned}
\varrho_M(\alpha x + \beta y) &= \sum_{i=1}^{\infty} M(\|\alpha x(i) + \beta y(i)\|) \\
&\leq \sum_{i=1}^{\infty} M(\|\alpha x(i)\| + \|\beta y(i)\|) \\
&= \sum_{i=1}^{\infty} M(\alpha\|x(i)\| + \beta\|y(i)\|) \\
&\leq \sum_{i=1}^{\infty} \alpha M(\|x(i)\|) + \sum_{i=1}^{\infty} \beta M(\|y(i)\|) \\
&= \alpha \sum_{i=1}^{\infty} M(\|x(i)\|) + \beta \sum_{i=1}^{\infty} M(\|y(i)\|) \\
&= \alpha \varrho_M(x) + \beta \varrho_M(y)
\end{aligned}$$

from above we conclude that ϱ_M is a convex modular. □

The next proposition we give some important relationship between the modular ϱ_M and the Luxemburg norm.

Proposition 3.1.2. Let $x \in X^o$. Then

(1) For $\alpha \geq 1$, $\varrho_M(\alpha x) \geq \alpha \varrho_M(x)$.

(2) For $0 < \alpha < 1$, $\varrho_M(\alpha x) \leq \alpha \varrho_M(x)$.

(3) $\|x\|_M \leq 1 \Rightarrow \varrho_M(x) \leq \|x\|_M$.

(4) $\|x\|_M > 1 \Rightarrow \varrho_M(x) > \|x\|_M$.

Proof. (1) and (2) are immediately obtained from the following facts:

(1) $\alpha \geq 1 \Rightarrow M(\alpha x) \geq \alpha M(x)$.

(2) $0 < \alpha < 1 \Rightarrow M(\alpha x) \leq \alpha M(x)$.

(These two properties hold by the convexity of M)

(3) For any $x \in \ell_M(X)$, without loss of generality we may assume that $x \neq 0$. By definition of $\|\cdot\|_M$ there exists $\lambda_n \downarrow \|x\|_M$ such that $\varrho_M(x/\lambda_n) \leq 1$. Then $\sum_{i=1}^{\infty} M(\frac{\|x(i)\|}{\lambda_n}) \leq 1$ for all $n \in \mathbb{N}$. Thus, for $m \in \mathbb{N}$ we have $\sum_{i=1}^{m} M(\frac{\|x(i)\|}{\lambda_n}) \leq 1$. By taking $n \to \infty$ we get that $\sum_{i=1}^{m} M(\frac{\|x(i)\|}{\|x\|_M}) \leq 1$. By taking $m \to \infty$, we obtain that $\sum_{i=1}^{\infty} M(\frac{\|x(i)\|}{\|x\|_M}) \leq 1$, hence $\varrho_M(\frac{x}{\|x\|_M}) \leq 1$.

Thus by (1) we have

$$1 \geq \varrho_M\left(\frac{x}{\|x\|_M}\right) \geq \frac{1}{\|x\|_M}\varrho_M(x)$$

which yields $\|x\|_M \geq \varrho_M(x)$.

(4) If $\|x\|_M > 1$ then for all small $\epsilon > 0$ and by (2) we have

$$1 < \varrho_M\left(\frac{x}{(1-\epsilon)\|x\|_M}\right) \leq \frac{1}{(1-\epsilon)\|x\|_M}\varrho_M(x),$$

which implies that $(1-\epsilon)\|x\|_M < \varrho_M(x)$. Letting $\epsilon \to 0$ we have $\|x\|_M < \varrho_M(x)$. $\square$

Theorem 3.1.3. $(\ell_M(X), \|\cdot\|_M)$ is a Banach space.

Proof. Let $(x_n) = (x_n(i))$ be a Cauchy sequence in $\ell_M(X)$. Then for $\epsilon \in (0,1)$ we can choose $N \in \mathbb{N}$ for all $m, n > N$ such that $\|x_n - x_m\|_M < M(\epsilon)$ so we have by Proposition 3.1.2(3) that

$$\varrho_M(x_n - x_m) < M(\epsilon).$$

That is

$$\sum_{i=1}^{\infty} M(\|x_n(i) - x_m(i)\|) < M(\epsilon)$$

for all $m, n > N$.

So for each i, $M(\|x_n(i) - x_m(i)\|) < M(\epsilon)$ for all $m, n > N$.

Since M is convex, M is 1-1 on $[0, \infty]$, hence there exists the inverse function M^{-1} on $[0, \infty]$, so it follows that

$$\|x_n(i) - x_m(i)\| < \epsilon$$

for all $m, n \geq N$

Thus for each $i \in \mathbb{N}$ we have $\|x_n(i) - x_m(i)\| < \epsilon$ for all $m, n \geq N$. That is $(x_n(i))_{n=1}^{\infty}$ is a Cauchy sequence in X. Consequently, since X is a Banach space, there is $x(i) \in X$ such that

$$x_n(i) \to x(i) \text{ as } n \to \infty.$$

Since for all $m, n > N$ we have $\|x_n - x_m\|_M < \epsilon$ that is

$$\sum_{i=1}^{\infty} M\left(\frac{\|x_n(i) - x_m(i)\|}{\epsilon}\right) \leq 1$$

for all $m, n \geq N$.

Letting $m \to \infty$ we obtain

$$\varrho_{_M}(\frac{x_n - x}{\epsilon}) = \sum_{i=1}^{\infty} M\left(\frac{\|x_n(i) - x(i)\|}{\epsilon}\right) \leq 1 \qquad (*)$$

Putting $x = (x(1), x(2), ...)$
So from (*) we have that $x_n - x \in \ell_M(X)$ for all $n \geq N$. and since $\ell_M(X)$ is linear space we find that $x = (x - x_N) + x_N \in \ell_M(X)$.
And it follows from (*) that $\|x_n - x\|_{_M} \leq \epsilon$ for all $n \geq N$, that is, $x_n \to x$ as $n \to \infty$, this means that $(\ell_M(X), \|\cdot\|_{_M})$ is a Banach space. □

Theorem 3.1.4. An Orlicz function M satisfy $\delta_2-condition$ if and only if $\ell_M(X) = h_M(X)$.
Proof. Suppose that $M \in \delta_2$. Let $\lambda > 0$ and $x \in \ell_M(X)$. Then there exists a $\lambda_o > 0$ such that $\varrho_{_M}(\lambda_o x) < \infty$. If $\lambda \leq \lambda_o$ we have that $\varrho_{_M}(\lambda x) \leq \varrho_{_M}(\lambda_o x) < \infty$. If $\lambda > \lambda_o$, then $\frac{\lambda}{\lambda_o} > 1$. Since $M \in \delta_2$, it follows by Theorem 2.4.5(3) that there exist $u_o > 0$ and $K > 0$ such that

$$M(\frac{\lambda}{\lambda_o}u) \leq KM(u) \qquad (u \leq u_o) \qquad (*)$$

Since $\sum_{i=1}^{\infty} M(\lambda_o\|x(i)\|) < \infty$, then there is $i_o \in \mathbb{N}$ such that

$$\lambda_o\|x(i)\| < u_o \qquad \text{for all} \quad i \geq i_o \qquad (**)$$

By (*) and (**) we have

$$\begin{aligned}\varrho_{_M}(\lambda x) &= \sum_{i=1}^{i_o} M(\lambda\|x(i)\|) + \sum_{i=i_o+1}^{\infty} M(\lambda\|x(i)\|) \\ &= \sum_{i=1}^{i_o} M(\lambda\|x(i)\|) + \sum_{i=i_o+1}^{\infty} M(\frac{\lambda_o}{\lambda_o}(\lambda\|x(i)\|)) \\ &\leq \sum_{i=1}^{i_o} M(\lambda\|x(i)\|) + K\sum_{i=i_o+1}^{\infty} M(\lambda_o\|x(i)\|) < \infty\end{aligned}$$

Thus $\varrho_{_M}(\lambda x) < \infty$, so that $x \in h_M(X)$. Hence $\ell_{_M}(X) = h_{_M}(X)$.

If $M \notin \Delta_2$, then by Theorem 2.4.5(2), there exists $\alpha_k \downarrow 0$ such that $M(\alpha_k) < \frac{1}{2^{k+1}}$ and $M(v\alpha_k) > 2^{k+1}M(\alpha_k)$ where $v > 1$ and $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ choose an integer m_k such that

$$\frac{1}{2^{k+1}} \leq m_k M(\alpha_k) < \frac{1}{2^k}.$$

Put $x' \in S(X)$ and define

$$x = (\overbrace{\alpha_1 x', \alpha_1 x', ..., \alpha_1 x'}^{m_1 \ times}, \overbrace{\alpha_2 x', \alpha_2 x', ..., \alpha_2 x'}^{m_2 \ times}, ..., \overbrace{\alpha_k x', \alpha_k x', ..., \alpha_k x'}^{m_k \ times}, ...).$$

Observe that

$$\begin{aligned}
\varrho_M(x) &= \sum_{i=1}^{m_1} M(\|\alpha_1 x(i)\|) + \sum_{i=m_1+1}^{m_1+m_2} M(\|\alpha_2 x(i)\|) + \sum_{i=m_1+m_2+1}^{m_1+m_2+m_3} M(\|\alpha_3 x(i)\|) + \cdots \\
&+ \sum_{i=m_1+m_2+\cdots+m_{k-1}+1}^{m_1+m_2+\cdots+m_k} M(\|\alpha_k x(i)\|) + \cdots \\
&= m_1 M(\alpha_1) + m_2 M(\alpha_2) + \cdots + m_k M(\alpha_k) + \cdots \\
&= \sum_{k=1}^{\infty} m_k M(\alpha_k) \\
&< \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.
\end{aligned}$$

This show that $x \in \ell_M(X)$.

Now we consider for $v > 1$,

$$\varrho_M(vx) = \sum_{k=1}^{\infty} m_k M(v\alpha_k) > \sum_{k=1}^{\infty} m_k 2^{k+1} M(\alpha_k) \geq \sum_{k=1}^{\infty} 1 = \infty.$$

This mean that $x \notin h_M(X)$. Hence the theorem is proved. $\square$

Remark 3.1.5. For $x \in \ell_M(X)$, define $|x| = (\|x(1)\|, \|x(2)\|, ...)$ then $|x| \in \ell_M$ and we have $\||x|\|_M = \|x\|_M$ and $\varrho_M(x) = \varrho_M(|x|)$.

Proof. Obviously, $|x| \in \ell_M$. Now, observe that

$$\varrho_M(|x|) = \sum_{i=1}^{\infty} M(\|x(i)\|) = \varrho_M(x),$$

and

$$\begin{aligned}|||x|||_M &= \inf\{\lambda > 0 \ : \ \varrho_M(\frac{|x|}{\epsilon}) \leq 1\}\\ &= \inf\{\lambda > 0 \ : \ \sum_{i=1}^{\infty} M(\frac{\|x(i)\|}{\epsilon}) \leq 1\}\\ &= \|x\|_M.\end{aligned}$$

□

Theorem 3.1.6. The modular convergence and norm convergence are equivalent in $\ell_M(X)$ if and only if $M \in \delta_2$.

Proof. *Necessity.* If $M \notin \delta_2$, then as in the proof of Theorem 3.1.4, there exists x_n in $\ell_M(X)$ such that $\varrho_M(x_n) \to 0$ and $\varrho_M((1+\frac{1}{n})x_n) = \infty$. This implies that $\|x_n\|_M > \frac{1}{1+\frac{1}{n}}$ for all $n \in \mathbb{N}$, hence $\|x_n\|_M \not\to 0$.

Sufficiency. Let $M \in \delta_2$ and $\epsilon \in (0,1)$. Choose $u_0 = M^{-1}(\epsilon)$ and $L = 1/\epsilon$ then there exist $K \geq 1$ which

$$M(Lu) \leq KM(u) \qquad\qquad (u \leq u_0)$$

Since $\varrho_M(x_n) \to 0$ there exists $N \in \mathbb{N}$ such that $\varrho_M(x_n) < \frac{\epsilon}{K}$ for all $n \geq N$. Thus for $n \geq N$ we have

$$\varrho_M(\frac{x_n}{\epsilon}) = \sum_{i=1}^{\infty} M(\frac{\|x_n(i)\|}{\epsilon}) \leq K \sum_{i=1}^{\infty} M(\|x_n(i)\|) = K\varrho_M(x_n) < K\frac{\epsilon}{K} = \epsilon,$$

this implies that $\|x_n\|_M < \epsilon$ for all $n \geq N$, which we can deduce that $\|x_n\|_M \to 0$. □

Lemma 3.1.7. Let $x, x_n \in \ell_M(X)$ then

(1) $\|x_n\|_M \to \infty \Rightarrow \varrho_M(x_n) \to \infty$.

(2) $\varrho_M(x) = 1 \Rightarrow \|x\|_M = 1$.

If in addition, $M \in \delta_2$, then we have (3),(4) and the converse of (1),(2) are true ;

(3) For any $\epsilon > 0$, there exists $\beta > 0$ such that

$$\|x\|_M \geq \epsilon \Rightarrow \varrho_M(x) \geq \beta.$$

(4) For any $\epsilon \in (0,1)$, there exists $\beta \in (0,1)$ such that

$$\varrho_M(x) \leq 1-\epsilon \Rightarrow \|x\|_M \leq 1-\beta.$$

Proof. (1) For any $y \in \ell_M(X)$, we first note that if $\varrho_M(y) \leq K$, then $\|y\|_M \leq max\{1, K\}$. Indeed, suppose $\varrho_M(y) \leq K$ we consider in two case

Case 1 if $0 < K \leq 1$ it follows immediately that $\|y\|_M \leq 1$

Case 2 if $K > 1$ then by Proposition 3.1.2(2) we get $\varrho_M(y/K) \leq \frac{1}{K}\varrho_M(y) \leq 1$ this implies that $\|y\|_M \leq K$.

From above observation, we obtain (1). Conversely, suppose that $M \in \delta_2$ and $L > 0$. Since $M^{-1}(1) > 0$ we can choose $K \geq 1$ such that

$$M(Lu) \leq KM(u) \qquad (u \leq M^{-1}(1)) \tag{*}$$

If $y \in \ell_M(X)$ and $\|y\|_M \leq L$, then $\varrho_M(y/L) \leq 1$. It implies that $\frac{\|y(i)\|}{L} \leq M^{-1}(1)$ for all $i \in \mathbb{N}$. By (*), we have

$$\varrho_M(y) = \sum_{i=1}^{\infty} M(\|y(i)\|) = \sum_{i=1}^{\infty} M(L\frac{\|y(i)\|}{L}) \leq K\sum_{i=1}^{\infty} M(\frac{\|y(i)\|}{L}) \leq K.$$

So for any $L > 0$ we obtain that there exist $K \geq 1$ such that

$$\|y\|_M \leq L \Rightarrow \varrho_M(y) \leq K \tag{**}$$

The converse of (1) follows immediately from (**).

(2) Assume that $\varrho_M(x) = 1$. By definition of $\|\cdot\|_M$, we see that if $\|x\|_M < 1$ then Proposition 3.1.2(3) implies $\varrho_M(x) \leq \|x\|_M < 1$. This lead to a contradiction that $\varrho_M(x) = 1$, hence we conclude that $\|x\|_M = 1$.

Conversely, if $M \in \delta_2$ Theorem 3.1.4 ensures that $x \in \ell_M(X) = h_M(X)$. Since $\|x\|_M = 1$, we have $\varrho_M(x) \leq 1$. For $\epsilon > 0$, we have $\varrho_M(\frac{x}{1-\epsilon}) \geq 1$. Since $\varrho_M(\frac{x}{t})$ is a continuous function of t, we have

$$1 \leq \lim_{\epsilon \to 0} \varrho_M(\frac{x}{1-\epsilon}) = \varrho_M(x) \leq 1.$$

This implies that $\varrho_M(x) = 1$.

(3) is an immediate consequence of Proposition 3.1.2(4) and Theorem 3.1.6.

(4) If (4) does not hold, then there exists $\epsilon > 0$ and $x_n \in \ell_M(X)$ such that $\varrho_M(x_n) < 1-\epsilon$ and $\frac{1}{2} \leq \|x_n\|_M \uparrow 1$.
Since $2 \geq \frac{1}{\|x\|_M}$ implies that $1 \geq \frac{1}{\|x_n\|_M} - 1$. Put $a_n = \frac{1}{\|x_n\|_M} - 1$, so we have $a_n \downarrow 0$.
By (1), we can let $L = \sup_n\{\varrho_M(2x_n)\} < \infty$. And then

$$\begin{aligned}1 = \varrho_M\left(\frac{x_n}{\|x_n\|_M}\right) &= \varrho_M(2a_nx_n + (1-a_n)x_n)\\ &\leq a_n\varrho_M(2x_n) + (1-a_n)\varrho_M(x_n)\\ &\leq a_nL + (1-\epsilon).\end{aligned}$$

Since $a_n \to 0$, we obtain that $1 \leq 1-\epsilon$ which is a contradiction. Hence (4) holds. □

Theorem 3.1.8. $x \in S(\ell_M(X))$ is an extreme point of $B(\ell_M(X))$ if and only if

(i) $\varrho_M(x) = 1$

(ii) for all $i \in \mathbb{N}$ which $x(i) \neq 0$ then $\frac{x(i)}{\|x(i)\|} \in \mathbf{Ext}B(X)$

(iii) $\mu\{i : \|x(i)\| \in \mathbb{R}\backslash S_M\} \leq 1$.

Proof. *Necessity.* Suppose (i) does not hold, i.e, $\varrho_M(x) = c < 1$.
Since M is continuous we choose $\epsilon > 0$ so small such that

$$M(\|x(1)\| + \epsilon) \leq M(\|x(1)\|) + \frac{1-c}{2}.$$

By $\varrho_M(x) = \sum_{i=1}^{\infty} M(\|x(i)\|) < 1$, we have $\lim_{i\to\infty} M(\|x(i)\|) = 0$. This implies $\|x(i)\| \to 0$ as $i \to \infty$. So there exists $N \in \mathbb{N}$ for all $n \geq N$, $\|x(n)\| < \epsilon$.
Next, we select $x(n_0)$ for some $n_0 \geq N$ and defined $y = (y(i))_i$, $z = (z(i))_i$ by $y(1) = x(1) - x(n_0)$, $z(1) = x(1) + x(n_0)$ and $y(i) = x(i) = z(i)$ for all $i \geq 2$. Then

$2x = y + z$ and $y \neq z$. Then

$$\begin{aligned}
\varrho_M(y) &= M(\|x(1) - x(n_0)\|) + \sum_{i=2}^{\infty} M(\|x(i)\|) \\
&\le M(\|x(1)\| + \|x(n_0)\|) + \sum_{i=2}^{\infty} M(\|x(i)\|) \\
&\le M(\|x(1)\| + \epsilon) + \sum_{i=2}^{\infty} M(\|x(i)\|) \\
&\le M(\|x(1)\|) + \frac{1-c}{2} + \sum_{i=2}^{\infty} M(\|x(i)\|) \\
&= \varrho_M(x) + \frac{1-c}{2} \\
&= c + \frac{1-c}{2} \\
&= \frac{c+1}{2} < 1
\end{aligned}$$

and similary we can show that $\varrho_M(z) \le 1$. Thus $y, z \in B(\ell_M(X))$ which contradicts with the fact that $x \in \mathbf{Ext}B(\ell_M(X))$.

If (ii) is not true, then there exist $i_o \in \mathbb{N}$ and $u(i_o), v(i_o) \in B(X)$ with $u(i_o) \neq v(i_o)$. Define the sequence u' and v' by,

$$u'(i) = \begin{cases} \|x(i_o)\| u(i_o) & ; i = i_o \\ x(i) & ; i \neq i_o, \end{cases}$$

and

$$v'(i) = \begin{cases} \|x(i_o)\| v(i_o) & ; i = i_o \\ x(i) & ; i \neq i_o. \end{cases}$$

It is easy to see that $\varrho_M(u')$ and $\varrho_M(v') \le 1$. That is $u', v' \in B(\ell_M(X))$ and $2x = u' + v'$, $u' \neq v'$ which contradicts our hypothesis that x is an extreme point.

Next, assume (iii) does not holds, without loss of generality we may assume that $\|x(1)\|, \|x(2)\| \in \mathbb{R} \backslash S_M$ i.e, $\|x(1)\|, \|x(2)\|$ belong to some affine intervals $(a_1, b_1), (a_2, b_2)$ of M respectively.

Let $M(u) = k_i u + \beta_i$, $u \in (a_i, b_i)$ $(i = 1, 2)$. Select $\epsilon_1, \epsilon_2 > 0$ such that $k_1 \epsilon_1 \|x(1)\| = k_2 \epsilon_2 \|x(2)\|$ and $(1 \pm \epsilon_i)\|x(i)\| \in (a_i, b_i)$ $(i = 1, 2)$.

Define $y = (y(i))_i, \;\; z = (z(i))_i$ by

$$y(1) = (1+\epsilon_1)x(1) \qquad y(2) = (1-\epsilon_2)x(2)$$
$$z(1) = (1-\epsilon_1)x(1) \qquad z(2) = (1+\epsilon_2)x(2)$$

and $y(i) = z(i) = x(i)$ for all $i > 2$. So

$$\begin{aligned} y(1) + z(1) &= (1+\epsilon_1)x(1) + (1-\epsilon_1)x(1) \\ &= x(1) + \epsilon_1 x(1) + x(1) - \epsilon_1 x(1) \\ &= 2x(1) \\ y(2) + z(2) &= (1-\epsilon_2)x(2) + (1+\epsilon_2)x(2) \\ &= x(2) - \epsilon_2 x(2) + x(2) + \epsilon_2 x(2) \\ o &= 2x(2) \end{aligned}$$

and $y(i) + z(i) = 2x(i)$ for all $i > 2$.
$2x = y + z$ and $y \neq z$. Then

$$\begin{aligned} \varrho_M(y) &= M(\|y(1)\|) + M(\|y(2)\|) + \sum_{i=3}^{\infty} M(\|y(i)\|) \\ &= M(\|(1+\epsilon_1)x(1)\|) + M(\|(1-\epsilon_2)x(2)\|) + \sum_{i=3}^{\infty} M(\|x(i)\|) \\ &= k_1((1+\epsilon_1)\|x(1)\|) + \beta_1 + k_2((1-\epsilon_2)\|x(2)\|) + \beta_2 + \sum_{i=3}^{\infty} M(\|x(i)\|) \\ &= k_1\|x(1)\| + k_1\epsilon_1\|x(1)\| + \beta_1 + k_2\|x(2)\| - k_2\epsilon_2\|x(2)\| + \beta_2 + \sum_{i=3}^{\infty} M(\|x(i)\|) \\ &= k_1\|x(1)\| + \beta_1 + k_2\|x(2)\| + \beta_2 + \sum_{i=3}^{\infty} M(\|x(i)\|) \\ &= M(\|x(1)\|) + M(\|x(2)\|) + \sum_{i=3}^{\infty} M(\|x(i)\|) \\ &= \varrho_M(x) \leq 1. \end{aligned}$$

In the same way, we can get $\varrho_M(z) \leq 1$, and therefore $y, z \in B(\ell_M(X)))$ which contradicts the hypothesis $x \in \mathbf{Ext}B(\ell_M(X))$.

Sufficiency. Let $2x = y + z,\ \ y, z \in B(\ell_M(X))$. Since

$$1 = \varrho_M(x) = \varrho_M(\frac{y+z}{2}) \le \frac{1}{2}[\varrho_M(y) + \varrho_M(z)] \le 1,$$

we have

$$M(\frac{\|y(i)\| + \|z(i)\|}{2}) = \frac{1}{2}[M(\|y(i)\|) + M(\|z(i)\|)],$$

for all $i \in \mathbb{N}$. By (iii) there exists at most one $j \in \mathbb{N}$ such that $\|x(j)\| \in \mathbb{R}\backslash S_M$. This give $\|x(i)\| = \|y(i)\| = \|z(i)\|$ for all $i \neq j$. And since

$$1 = \sum_{i=1}^{\infty} M(\|x(i)\|) = \sum_{i=1}^{\infty} M(\|y(i)\|) = \sum_{i=1}^{\infty} M(\|z(i)\|),$$

we deduce $\|x(j)\| = \|y(j)\| = \|z(j)\|$. Since $2(x(i)) = y(i)+z(i)$ it implies that $\frac{2x(i)}{\|x(i)\|} = \frac{y(i)}{\|y(i)\|} + \frac{z(i)}{\|z(i)\|}$ then we can obtain by (ii) that $y = z$. Hence, $x \in$ **Ext** $B(\ell_M(X))$. □

Theorem 3.1.9. $\ell_M(X)$ is rotund if and only if

(1) $M \in \delta_2$

(2) X is rotund and

(3) M is strictly convex on $[0, M^{-1}(1/2)]$.

Proof. *Necessity.* If $M \notin \delta_2$, then as in the proof of Theorem 3.1.4 we can find $x \in \ell_M(X)$ such that $\|x\|_M = 1$ and $\varrho_M(x) < 1$, thus by (i) of Theorem 3.1.8 we have $x \notin$ **Ext** $B(\ell_M(X))$.

(2) If (2) is not true, then there exist $x, y, z \in S(X)$ with $2x = y + z$ and $y \neq z$. Pick $u \in S(\ell_M(X))$, by (1) and Lemma 3.1.7(2) we have $1 = \varrho_M(u) = \sum_{i=1}^{\infty} M(\|u(i)\|)$.

Define $x' = (x'(i))_i,\ \ y' = (y'(i))_i,\ \ z' = (z'(i))_i$ by

$$x'(i) = \|u(i)\|x,\ \ y'(i) = \|u(i)\|y \ \text{ and } \ z'(i) = \|u(i)\|z$$

Then

$$\varrho_M(x') = \sum_{i=1}^{\infty} M(\|\|u(i)\|x\|) = \sum_{i=1}^{\infty} M(\|u(i)\|\|x\|) = \sum_{i=1}^{\infty} M(\|u(i)\|) = 1 < \infty,$$

and in the same way we can get $\varrho_M(y'), \varrho_M(z') = 1$. That is $x', y', z' \in S(\ell_M(X))$. Moreover, we can see that $2x' = y' + z', y' \neq z'$ which implies that x' is not an extreme point of $B(\ell_M(X))$, contradicting the rotundity of $\ell_M(X)$.

If (3) does not hold, then M is affine on some interval $[a, b] \subseteq [0, M^{-1}(1/2)]$. Since $b \leq M^{-1}(1/2)$. So $2M(b) \leq 1$, thus we can find $c \in (a, b)$ and $d > 0$ such that $2M(c) + M(d) = 1$.
Choose $x' \in S(X)$ and defind

$$x = (cx', cx', dx', 0, 0, 0, ...) \in \ell_M(X).$$

Then

$$\begin{aligned}\varrho_M(x) &= \sum_{i=1}^{\infty} M(\|x(i)\|) \\ &= M(\|cx'\|) + M(\|cx'\|) + M(\|dx'\|) \\ &= M(c\|x'\|) + M(c\|x'\|) + M(d\|x'\|) \\ &= M(c) + M(c) + M(d) = 1\end{aligned}$$

But $x(1), x(2) \in \mathbb{R}\backslash S_M$. This contradicts whith Theorem 3.1.8 (iii). Hence $x \notin \mathbf{Ext}B(\ell_M(X))$.

Sufficiency. Let $x \in S(\ell_M(X))$. We have to verify (i)-(iii) of Theorem 3.1.8. We shall show that (i) and (iii) are true, which (ii) is obvious by (2). Since $M \in \delta_2$ we have by Lemma 3.1.7 (2) that $\varrho_M(x) = 1$.

Next, Let $I = \{i \in \mathbb{N} : \|x(i)\| \in \mathbb{R}\backslash S_M\}$. Then by (3), for any $i \in I$ we have $\|x(i)\| > M^{-1}(1/2)$ that is $M(\|x(i)\|) > 1/2$ and since

$$\varrho_M(x) = \sum_{i=1}^{\infty} M(\|x(i)\|) = 1,$$

it implies that I contain atmost a single point. Hence, we obtain by Theorem 3.1.8 that $x \in \mathbf{Ext}\ B(\ell_M(X))$. $\square$

Remark 3.1.10. Theorem 3.1.9 may be false if X is not $(\mathbf{R})$. For example, we shall we consider $X = \ell_1^2$ where the norm defined by

$$\|(\alpha_1, \alpha_2)\| = \sum_{j=1}^{2} |\alpha_j|$$

and $\alpha_j \in \mathbb{F}$ $(j = 1, 2)$. We have that ℓ_1^2 is a Banach space but not a rotund space (see [11]). Choose $\mathbb{F} = \mathbb{R}$, observe that $(\frac{1}{2}, 0), (0, \frac{1}{2}), (1, 0)$ and $(0, 1)$ are elements in ℓ_1^2. Let

$$\begin{aligned}
x &= ((\frac{1}{2}, 0), (0, \frac{1}{2}), (0, 0), (0, 0), ...),\\
y &= ((1, 0), (0, 0), (0, 0), (0, 0), ...),\\
z &= ((0, 0), (0, 1), (0, 0), (0, 0), ...).
\end{aligned}$$

Thus $x, y, z \in X^o$. Next, we define an Orlicz function $M : \mathbb{R}_+ \to \mathbb{R}_+$ by,

$$M(u) = \begin{cases} 2u^2 & ; \text{for } u \in [0, \frac{1}{2}] \\ u & ; \text{otherwise} \end{cases}$$

Note that $M^{-1}(\frac{1}{2}) = \frac{1}{2}$, and it easy to see that $M \in \delta_2$ and M is strictly convex on $[0, M^{-1}(\frac{1}{2})]$. Then

$$\begin{aligned}
\varrho_M(x) = \sum_{i=1}^{\infty} M(\|x(i)\|) &= M(\|x(1)\|) + M(\|x(2)\|) + M(\|x(3)\|) + \cdots \\
&= M(\frac{1}{2}) + M(\frac{1}{2}) + M(0) + \cdots \\
&= \frac{1}{2} + \frac{1}{2} + 0 + 0 + \cdots \\
&= 1,
\end{aligned}$$

and similary we can show that $\varrho_M(y) = \varrho_M(z) = 1$. Consequently, it follows by Lemma 3.1.7(2) that $x, y, z \in S(\ell_M(X))$. Moreover, we see that $2x = y + z$, and $y \neq z$. This implies that x is not extreme point of $B(\ell_M(X))$. Hence, $\ell_M(X)$ is not a rotund space.□

From Theorem 3.1.9, if $X = \mathbb{R}$ we have a corollary as in [2].

Corollary 3.1.11 (Chen[2], Theorem 2.7) ℓ_M is rotund iff

(1) $M \in \delta_2$ and

(2) M is strictly convex on $[0, M^{-1}(1/2)]$

Proof. By the fact that $\mathbb{R}$ is a rotund Banach space the corollary is obtained immediately by Theorem 3.1.9. $\square$

3.2 Some Geometric Properties Related to Rotundity

In this section show the relations between rotundity and some geometric properties in Orlicz vector-valued sequence spaces. We begin with giving some relation.

Theorem 3.2.1. For an Orlicz vector-valued sequence space $\ell_M(X)$. The following statements are equivalent

(1) $\ell_M(X)$ has the $uniform\ Kadec-Klee\ property$.

(2) $\ell_M(X)$ has the **H**-*property*.

(3) $M \in \delta_2$.

Proof. (1)$\Rightarrow$(2). Clearly by Theorem 2.5.12.

(2)$\Rightarrow$(3). Assume $M \notin \delta_2$. As in the proof of Theorem 3.1.4 we can find an element $x \in S(\ell_M(X))$ such that $\varrho_M(x) \le 1$ and $\varrho_M(\lambda x) = \infty$ for all $\lambda > 1$. This means that $\|\sum_{i=n}^{\infty} x(i)e_i\|_M = 1$ for all $n \in \mathbb{N}$, where $e_i = (\overbrace{0,0,0...,0}^{i-1\ times},1,0,0,0,...)$, consequently we can choose an increasing sequence (n_i) of natural numbers such that

$$\|\sum_{j=n_i+1}^{n_{i+1}} x(j)e_j\|_M \ge \frac{1}{2}$$

Define $x_i = (x(1),...,x(n_i),0,...,0,x(n_{i+1}+1),...)$ $,i = 1,2,...$

We will show that

(a) $\|x_i\|_M = 1 \quad i = 1,2,...$

(b) $x_i \to x$ weakly.

Since

$$\begin{aligned}\varrho_M(x) &= \sum_{j=1}^{\infty} M(\|x(j)\|) \\ &= \sum_{j=1}^{n_i} M(\|x(j)\|) + \sum_{j=n_i+1}^{n_{i+1}} M(\|x(j)\|) + \sum_{j=n_{i+1}+1}^{\infty} M(\|x(j)\|)\end{aligned}$$

and $\varrho_M(x) \le 1$, we have

$$\varrho_M(x_i) = \varrho_M(x) - \sum_{j=n_i+1}^{n_{i+1}} M(\|x(j)\|) \le 1 - \sum_{j=n_i+1}^{n_{i+1}} M(\|x(j)\|) < 1.$$

And from $\varrho_M(\lambda x) = \infty$ for all $\lambda > 1$. This give $\varrho_M(\lambda x_i) = \infty$ for all $\lambda > 1$. This show that $\|x_i\|_M = 1$.

Next, we will show that (b) holds. Let $f \in \ell'_M(X)$ so there is a unique decomposition $f = v + \varphi$ where v and φ are respectively the regular and singular parts of f, i.e, v is determined by a function $v_0 \in \ell_N(X)$, where N is the complementary function of M. Since $v_0 \in \ell_N(X)$, there exists $\lambda > 0$ such that

$$\sum_{i=1}^{\infty} N(\lambda\|v_0(i)\|) < \infty.$$

And since $x_i - x \in h_M(X)$. We have $\langle \varphi, x_i - x\rangle = 0$.
So we have

$$\begin{aligned}\langle f, x_i - x\rangle &= \langle v + \varphi, x_i - x\rangle = \langle v, x_i - x\rangle \\ &= \sum_{j=n_i+1}^{n_{i+1}} \|x(j)\|\|v_0(j)\| = \frac{1}{\lambda}\left(\sum_{j=n_i+1}^{n_{i+1}} \|x(j)\|\|\lambda v_0(j)\|\right)\end{aligned}$$

By Young inequality, we have

$$\langle f, x_i - x\rangle \le \frac{1}{\lambda}\sum_{j=n_i+1}^{n_{i+1}} (M(\|x(j)\|) + N(\lambda\|v_0(j)\|)) \to 0$$

as $i \to \infty$.
This show that $x_i \to x$ *weakly*, and therefore (2) holds. Moreover, we have that $\|x_i - x\|_M \ge \frac{1}{2}$ for all $i \in \mathbb{N}$. So we find that x is not an **H**-point of $B(\ell_M(X))$.

(3)$\Rightarrow$ (1) Since $M \in \delta_2$, for any given $\epsilon > 0$, by Lemma 3.1.7 (3) there exists $\beta > 0$ such that

$$\|x\|_M \ge \frac{\epsilon}{4} \Rightarrow \varrho_M(x) \ge \beta.$$

For this β, again by Lemma 3.1.7 (4) we can find $\eta \in (0, 1)$ such that

$$\varrho_M(x) \le 1 - \beta \Rightarrow \|x\|_M \le 1 - \eta.$$

Now, suppose $x_n \in B(\ell_M(X))$, $x_n \to x$ *weakly* and $\|x_n - x_m\|_M \ge \epsilon$ $(n \ne m)$. We shall show that $\|x\|_M \le 1 - \eta$. If not, that is $\|x\|_M > 1 - \eta$, then we can select a finite subset I of $\mathbb{N}$ such that $\|x_{|_I}\|_M > 1 - \eta$. Since the weak convergence of $\{x_n\}$ to x implies that $x_n \to x$ coordinatewise and since I is finite we deduce that $x_n \to x$ uniformly on I, consequently, there exists $k \in \mathbb{N}$ such that

$$\|x_{m_{|_I}}\|_M > 1 - \eta \ , \ \|(x_n - x_m)_{|_I}\|_M \leq \frac{\epsilon}{2}$$

for all $n, m > k$.
But the first inequality implies

$$\varrho_M(x_{m_{|_I}}) > 1 - \beta \quad (m > k)$$

and the second implies

$$\|(x_n - x_m)_{|_{\mathbb{N}\setminus I}}\|_M \geq \frac{\epsilon}{2} \quad (m, n > k, m \neq n).$$

(because $\|x_n - x_m\|_M \geq \epsilon$).
That is

$$\frac{\epsilon}{2} \leq \|(x_n - x_m)_{|_{\mathbb{N}\setminus I}}\|_M \leq \|x_{n_{|_{\mathbb{N}\setminus I}}}\|_M + \|x_{m_{|_{\mathbb{N}\setminus I}}}\|_M$$

which yield $\|x_{n_{|_{\mathbb{N}\setminus I}}}\|_M \geq \frac{\epsilon}{4}$ or $\|x_{m_{|_{\mathbb{N}\setminus I}}}\|_M \geq \frac{\epsilon}{4}$. Without loss of generality we may assume that $\|x_{m_{|_{\mathbb{N}\setminus I}}}\|_M \geq \frac{\epsilon}{4}$. Then $\varrho_M(x_{m_{|_{\mathbb{N}\setminus I}}}) \geq \beta$.
So $1 = (1 - \beta) + \beta < \varrho_M(x_{m_{|_I}}) + \varrho_M(x_{m_{|_I}}) = \varrho_M(x_m) \leq 1$ which is a contradiction.
Hence we obtain that $\|x\|_M \leq 1 - \eta$. That is $\ell_M(X)$ has the *Uniform Kadec-Klee property*
□

Corollary 3.2.2. The following statements are equivalent :

(1) $\ell_M(X)$ is rotund.

(2) $\ell_M(X)$ is **UKK**, X is rotund, and M is strictly convex on $[0, M^{-1}(1/2)]$.

(3) $\ell_M(X)$ has the *prqperty*(**H**), X is rotund, and M is strictly convex on $[0, M^{-1}(1/2)]$.

Proof. (1)⇒ (2). Since $\ell_M(X)$ is rotund, so by Theorem 3.1.9 we have that $M \in \delta_2$, X is rotund, and M is strictly convex on $[0, M^{-1}(1/2)]$. And by $M \in \delta_2$, it follows by Theorem 3.2.1 that $\ell_M(X)$ is **UKK**.

(2)⇒ (3). It follows by Theorem 2.5.12.

(3)⇒ (1). Since $\ell_M(X)$ has *property*(**H**), by Theorem 3.2.1 we have that $M \in \delta_2$. It follows by Theorem 3.1.9 that $\ell_M(X)$ is rotund. □

Corollary 3.2.3. $\ell_M(X)$ is **LUR** if and only if $M \in \delta_2$ and $\ell_M(X)$ is **WLUR**.
Proof. The proof is an immediate consequence of Theorem 3.2.1. □

Theorem 3.2.4. If $M \notin \delta_2$, then $\ell_M(X)$ is not a **WCLUR** space.
Proof. Since $M \notin \delta_2$, then as in the proof of Theorem 3.1.4 we can find an element $u \in S(\ell_M(X))$ such that $\varrho_M(u) \leq 1$ and $\varrho_M(\lambda u) = \infty$ for all $\lambda > 1$. Next, for $x \in S(\ell_M(X))$ by, convexity of M we can select a subsequence $z = (x(i_k))_{k=1}^{\infty}$ of $x = (x(i))_{i=1}^{\infty}$ such that $z \in h_M(X)$. Let $y = x - z$. Define $z_n = \sum_{i=1}^{n} z(i)e_i$, $u^{(n)} = \sum_{k=1}^{\infty} u(n+k)e_{n+k}$, where $e_i = (\overbrace{0,0,0...,0}^{i-1\ times},1,0,0,0,...)$ and $x_n = z_n + u^{(n)} + y$. Then we have

$$\|2x\|_M - \|\sum_{i=n+1}^{\infty} z(i)e_i\|_M \leq \|x + x_n\|_M \leq max\{2, 2\varrho_M(x) + \sum_{i=n+1}^{\infty} M(\|u(i)\|)\}$$

This implies that $\|x_n + x\|_M \to 2$. By the same method we can get $\|x_n\|_M \to 1$.

To complete the proof, we shall show that $\ell_M(X)$ is not a **WCLUR** space. If not, by $x_n \to x$ coordinatewise we may assume without loss of generality that $x_n \to x$, *weakly* (passing to a subsequence if necessary). Since

$$\begin{aligned} d(u, h_M(X)) &= inf\{\|u - x\|_M : x \in h_M(X)\} \\ &= inf\{\lambda > 0 : \varrho_M\left(\frac{u}{\lambda}\right) < \infty\} = \theta(u) = 1, \end{aligned}$$

thanks to the Hahn Banach theorem to obtained that there exist $f \in S(\ell'_M(X))$ such that $f(u) = 1$ and $f(x) = 0$ for all $x \in h_M(X)$. Consequently, we have $f(x_n - x) = f(z_n + u^{(n)} + y - x) = f(u - (u - u^{(n)})) = f(u) = 1$. This lead to a contradiction that $x_n \to x$ *weakly*, which complete the proof. □

Theorem 3.2.5 $\ell_M(X)$ is **CLUR** iff it is **WCLUR**
Proof. The proof follows immediately from Theorem 3.2.1, 3.2.4 and the general implications **CLUR⇒WCLUR**. □

Theorem 3.2.6. $\ell_M(X)$ is **LUR** if and only if $\ell_M(X)$ is **WCLUR** and **R**.
Proof. It follows immediately from Theorem 2.5.9. and Theorem 3.2.5. □

Corollary 3.2.7. The following statements are equivalent :

(1) $\ell_M(X)$ is **LUR**

(2) $\ell_M(X)$ is **CLUR**, X is rotund, and M is strictly convex on $[0, M^{-1}(1/2)]$

(3) $\ell_M(X)$ is **WCLUR**, X is rotund, and M is strictly convex on $[0, M^{-1}(1/2)]$

Proof. (1)$\Rightarrow$(2) and (2)$\Rightarrow$(3) are immediately obtained by Theorem 2.5.9. and 3.1.9. It suffices to show that (3)$\Rightarrow$(1). Since $\ell_M(X)$ is **WCLUR**, it follows from Theorem 3.2.4 that $M \in \delta_2$, it follows from Theorem 3.1.9. that $\ell_M(X)$ is **R** and then the proof is complete by Theorem 3.2.6. □

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LOCAL UNIFORM CONVEXITY OF CESARO MUSIELAK-ORLICZ SEQUENCE SPACES

1. INTRODUCTION

A sequence space is defined to be a linear space of scalar (real or complex) functions on $\mathbb{N}$. The study of sequence spaces is thus a special case of the more general study of function spaces, which in turn a branch of functional analysis. Geometric properties of Banach spaces play an important role in studying Banach spaces theories. Some geometric properties imply reflexivity of Banach spaces and some imply fixed point property or weak fixed point property. So the main problems of studying geometric properties of the given Banach spaces is to give necessary and sufficient conditions on the spaces which they have the given geometric properties. It is clear that the geometry of Banach spaces in the form of convexity would play a central role in the theory of Radon-Nikodym differentiation for vector-valued measures.

The theory of geometry of Orlicz sequence spaces was developed extensively by many authors. Y.A. Cui, H. Hudzik and C. Meng [3] have studied the $LUR, CLUR$ and $WCLUR$, and property (H) in Orlicz sequence spaces under the Luxemburg norm. Y. A. Cui and H. B. Thompson [6] have studied the LUR and property (β) in Musielak-Orlicz sequence spaces under the Luxemburg norm. It is obvious that both Orlicz sequence spaces and Musielak-Orlic sequence spaces are generalization of the l_p space.

In 1970, J. S. Shue introduced the Cesàro sequence spaces $ces_p (1 < p < \infty)$ and Y. Q. Liu, B. E. Wu and Y. Lee [9] studied many geometric properties in ces_p. It is know that ces_p is LUR and has property (H)(See [9]). Y. A. Cui and Hudzik [4] proved that ces_p has the Banach-Saks property of type p and Y. A. Cui and M. Chenghui [2] proved that ces_p has property (β) and has Banach-Saks property.

In this paper, we define the Cesàro-Musielak-Orlicz sequence spaces Ces_M, where $M = (M_k)$ is a Musielak-Orlicz function. This space is a generalization of the Cesàro sequence spaces ces_p. The main purpose of the thesis is to give sufficient conditions for M such that Ces_M is LUR and has property (H) under the Luxemburg norm defined by the convex modular introduced to Ces_M.

2. PRELIMINARIES

In this chapter, we give some definitions, notations and some useful results that will be used in the later chapter.

Throughout this thesis, we let $\mathbb{R}$ stand for the set of real numbers and $\mathbb{N}$ the set of natural numbers.

2.1 Norms and Normed spaces

Definition 2.1.1 Let X be a linear space. A norm on X is a nonnegative real valued function on X, written as $\|.\|$, such that the following conditions are satisfied by all $x, y \in X$ and each scalar α :

(1) $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\alpha x\| = |\alpha|\|x\|$,

(3) $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality).

Properly speaking, a *normed space* is an ordered pair $(X, \|.\|)$ consisting of a linear space X and a norm $\|.\|$ on X.

Definition 2.1.2 Let X be a normed space. The *metric induced by the norm of X* is the metric d on X defined by the formula $d(x, y) = \|x - y\|$. Moreover, a complete normed linear space is called a *Banach space.*

Definition 2.1.3 Let X be a normed space. The *closed unit ball* of X is the set $\{x \in X : \|x\| \leq 1\}$ and is denoted by $B(X)$. The *unit sphere* of X, denote by $S(X)$, is the set $\{x \in X : \|x\| = 1\}$.

Definition 2.1.4 A subset A of a normed space is *convex* if for each pair of its points, it contains the line segment joinning them. That is,

$$\{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\} \subseteq A,$$

for all $x, y \in A$.

2.2 Convex functions, Modular functions and Orlicz functions

Definition 2.2.1 A real valued continuous function $M : \mathbb{R} \to \mathbb{R}$ is called *convex* if

$$M(\frac{u+v}{2}) \leq \frac{M(u)+M(v)}{2} \tag{2.1}$$

for all $u, v \in \mathbb{R}$. If, in addition, the two sides of (2.1) are not equal for all $u \neq v$, then we call M a *strictly convex* function.

Proposition 2.2.2 Let $M : \mathbb{R} \to \mathbb{R}$ be a continuous function.Then the following statements are equivalent:

(1) M is convex.

(2) There exist affine functions $L_n(u) = a_n u + b_n$ such that $M(u) = \sup_n L_n(u)$.

(3) For any $u, v \in \mathbb{R}$ and $\alpha \in [0, 1]$,

$$M(\alpha u + (1-\alpha)v) \leq \alpha M(u) + (1-\alpha)M(v). \tag{2.2}$$

(4) For any $u_1, u_2, ..., u_n \in \mathbb{R}$ and $\alpha_i \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$,

$$M(\sum_{i=1}^{n} \alpha_i u_i) \leq \sum_{i=1}^{n} \alpha_i M(u_i).$$

In addition, M is strictly convex if and only if for any $u \neq v$ and $\alpha \in (0, 1)$, the inequality (2.2) is strict.

Proof. See [1, Proposition 1.3].

Theorem 2.2.3 Suppose that the Orlicz function M is strictly convex.

(1) For any $K > 1, \epsilon > 0$, there exists $\delta > 0$ such that

$$M(\frac{u+v}{2}) \leq (1-\delta)\frac{M(u)+M(v)}{2}$$

for all $u, v \in \mathbb{R}$ satisfying $|u|, |v| \leq K$ and $|u-v| \geq \epsilon$.

(2) For any $K > 0, \epsilon > 0$ and $[a, b] \subset (0, 1)$, there exists $\delta > 0$ such that $M(2u + (1-\alpha)v) \leq (1-\delta)[\alpha M(u) + (1-\alpha)M(u)]$ for all $\alpha \in [a, b]$ and $u, v \in \mathbb{R}$ satisfying $|u|, |v| \leq K$ and $|u-v| \geq \epsilon$.

Proof. See [1, Proposition 1.4].

Definition 2.2.4 A nonnegative function $M : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ is said to be an *Orlicz function* if M vanishes only at 0, M is even and convex.

Definition 2.2.5 Let X be a real vector space. A function $\rho : X \to [0, \infty]$ is called *modular* if it satisfies the following properties:

(1) $\rho(x) = 0$ if and only if $x = 0$;

(2) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$;

(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The modular ρ is called *convex* if

(4) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

If ρ is a modular on X, we define

$$X_\rho := \{x \in X : \lim_{\lambda \to 0^+} \rho(\lambda x) = 0\}$$

and

$$X_\rho^* := \{x \in X : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

It is clear that $X_\rho \subseteq X_\rho^*$. If ρ is a convex modular, for $x \in X_\rho$ we define

$$\|x\| = \inf\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\}. \tag{2.3}$$

Orlicz [13] proved that if ρ is a convex modular in X, then $X_\rho = X_\rho^*$ and $\|.\|$ is a norm on X_ρ for which it is a Banach space. The norm $\|.\|$ define as in (2.3) is called the *Luxemburg norm.*

A modular ρ on X is called

(a) *right-continuous* if $\lim_{\lambda \to 1^+} \rho(\lambda x) = \rho(x)$ for all $x \in X_\rho$

(b) *left-continuous* if $\lim_{\lambda \to 1^-} \rho(\lambda x) = \rho(x)$ for all $x \in X_\rho$

(c) *continuous* if it is both right-continuous and left-continuous.

The following known results gave some relationships between the modular ρ and the Luxemburg norm $\|.\|$ on X_ρ.

Theorem 2.2.6 Let ρ be a convex modular on X and let $x \in X_\rho$ and (x_n) a sequence in X_ρ. Then $\|x_n - x\| \to 0$ as $n \to \infty$ if and only if $\rho(\lambda(x_n - x)) \to 0$ as $n \to \infty$ for every $\lambda > 0$.

Proof. See [10, Theorem 1.3].

Theorem 2.2.7 Let ρ be a continuous convex modular on X and $x \in X_\rho$. Then

(1) $\|x\| < 1$ if and only if $\rho(x) < 1$.

(2) $\|x\| \leq 1$ if and only if $\rho(x) \leq 1$.

(3) $\|x\| = 1$ if and only if $\rho(x) = 1$.

Proof. See [10, Theorem 1.4].

2.3 Musielak-Orlicz Sequence Spaces and Cesàro-Musielak-Orlicz Sequence Spaces

Definition 2.3.1 A sequence $M = (M_k)$ of Orlicz functions is called a *Musielak- Orlicz function.* In addition, a Musielak-Orlicz function $N = (N_k)$ is called a *complementary function* of a Musielak-Orlicz function M if

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},$$

$k = 1, 2, ...$

Definition 2.3.2 Denote by l^0 the space of all real sequences. For a given Musielak-Orlicz function M, we define $I_M : l^0 \to [0, \infty]$ by the formula

$$I_M(x) = \sum_{k=1}^{\infty} M_k(x(k)), \quad x = (x(k)) \in l^0.$$

Then I_M is a convex modular. The *Musielak-Orlicz sequence space* l_M is the space

$$l_M := \{x \in l^0 : I_M(cx) < \infty \text{ for some } c > 0\}.$$

We consider l_M equipped with the *Luxemburg norm*

$$\|x\| = \inf\{k > 0 : I_M(\frac{x}{k}) \leq 1\}$$

or equipped with the *Orlicz norm*

$$\|x\|^0 = \inf\{\frac{1}{k}(1 + I_M(kx)) : k > 0\}.$$

To simplify notation, we put $l_M := (l_M, \|.\|)$ and $l_M^0 := (l_M, \|.\|^0)$. Both of them are Banach spaces (See [12]).

The subspace h_M of l_M, called the *finite (or order continuous) elements*, is defined by

$$h_M := \{x \in l^0 : I_M(\lambda x) < \infty \text{ for all } \lambda > 0\}.$$

Definition 2.3.3 For $1 \leq p < \infty$, the Cesàro sequence space ces_p is defined by

$$ces_p := \{x \in l^0 : \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^p < \infty\}.$$

We consider ces_p equipped with the norm

$$\|x\| = \left(\sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^p \right)^{\frac{1}{p}}.$$

This space was introduced by J. S. Shue [17]. It is sueful in the theory of matrix operator and others (See [9]). Some geometric properties of the Cesàro sequence space ces_p were studied by many mathematicians.

Definition 2.3.4 Let $p = (p_k)$ be a sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$. The Cesàro sequence space $ces(p)$ is defined by

$$ces(p) := \{x \in l^0 : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where $\rho(x) = \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k}$. We consider this space equipped with the Luxemburg norm

$$\|x\| = \inf\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\}$$

under which it is a Banach space. If $p = (p_k)$ is bounded, we have

$$ces(p) := \{x \in l^0 : \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} < \infty\}.$$

Definition 2.3.5 Let $M = (M_k)$ be the Musielak-Orlicz function. The *Cesàro-Musielak-Orlicz sequence space* is define by

$$Ces_M := \{x \in l^0 : \rho_M(cx) < \infty \text{ for some } c > 0\}$$

where $\rho_M(x) = \sum_{k=1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k} |x(i)|\right)$. We show in Theorem 3.1 that ρ_M is a convex modular on Ces_M. So in this space we can consider the Luxemburg norm induced by the convex modular ρ_M as follows:

$$\|x\| = \inf\{\lambda > 0 : \rho_M(\frac{x}{\lambda}) \leq 1\}.$$

To simplify notation, we put $Ces_M := (Ces_M, \|.\|)$, we have by [10] that Ces_M is a Banach space. We define the subspace $SCes_M$ of Ces_M by

$$SCes_M := \{x \in l^0 : \rho_M(cx) < \infty \text{ for all } c > 0\}.$$

Definition 2.3.6 We say a Musielak-Orlicz function M *satisfies the* δ_2*-condition* ($M \in \delta_2$ for short) if there exist constants $K \geq 2, u_0 > 0$ and a sequence (c_k) of positive numbers such that $\sum_{k=1}^{\infty} c_k < \infty$ and the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

hold for every $k \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $|u| \leq u_0$.

If $M \in \delta_2$ and $N \in \delta_2$, then we write $M \in \delta_2 \cap \delta_2^*$.

Moreover, we say that a Musielak-Orlicz function M *satisfies the* $(*)$*-condition* if for any $\epsilon \in (0,1)$ there exists a $\delta > 0$ such that $M_k((1+\delta)u) \leq 1$ whenever $M_k(u) \leq 1-\epsilon$ for all $k \in \mathbb{N}$ and $u \in \mathbb{R}$.

We shall show in Theorem 3.5 that if $M \in \delta_2$, then $SCes_M = Ces_M$.

2.4 Convergence

Definition 2.4.1 A sequence (x_n) in a Cesàro-Musielak-Orlicz sequence space Ces_M (Musielak-Orlicz sequence space l_M) is said to be

(1) *(norm) convergent,* if there exists an $x \in Ces_M (x \in l_M)$ such that $\|x_n - x\| \to 0$;

(2) *modular convergent,* if there exists an $x \in Ces_M (x \in l_M)$ such that $\rho_M(x_n - x) \to 0$ $(I_M(x_n - x) \to 0)$ as $n \to \infty$;

(3) *weakly convergent*, if there exists an $x \in Ces_M$ $(x \in l_M)$ such that $f(x_n - x) \to 0$ for all f in the dual space of Ces_M (l_M).

2.5 Rotund Spaces (R) and Locally Uniformly Rotund Spaces (LUR)

Definition 2.5.1 Let $(X, \|.\|)$ be a real Banach space. A point $x \in S(X)$ is said to be an *extreme point* if for every $y, z \in S(X)$ such that $x = \frac{y+z}{2}$, we have $z = y = x$. X is said to be *rotund* (R for short) if every $x \in S(X)$ is an extreme point.

Definition 2.5.2 (J. A. Clarkson, 1936). Let X be a normed space. Define a function $\delta_X : [0, 2] \to [0, 1]$ by the formula

$$\delta_X(\epsilon) = \inf\{1 - \|\frac{1}{2}(x + y)\| : x, y \in S(X), \|x - y\| \geq \epsilon\}$$

if $X \neq \{0\}$, and by the formula

$$\delta_X(\epsilon) = \begin{cases} 0 \text{ if } \epsilon = 0 \\ 1 \text{ if } 0 < \epsilon \leq 2 \end{cases}$$

if $X = \{0\}$. Then δ_X is the *modulus of rotundity* or *modulus of convexity* of X.. The space X is *uniformly rotund* or *uniformly convex* (UR for short) if $\delta_X(\epsilon) > 0$ whenever $0 < \epsilon \leq 2$.

Proposition 2.5.3 Every uniformly rotund normed space is rotund.

Proof. See [11, Proposition 5.2.6].

Proposition 2.5.4 Suppose that X is a normed space. Then the following are equivalent.

(a) The space X is UR.

(b) Whenever (x_n) and (y_n) are sequence in $S(X)$ and $\|\frac{1}{2}(x_n + y_n)\| \to 1$, it follows that $\|x_n - y_n\| \to 0$.

(c) Whenever (x_n) and (y_n) are sequence in $B(X)$ and $\|\frac{1}{2}(x_n + y_n)\| \to 1$, it follows that $\|x_n - y_n\| \to 0$.

(d) Whenever (x_n) and (y_n) are sequence in X and $\|x_n\|, \|y_n\|$ and $\|\frac{1}{2}(x_n + y_n)\|$ all tend to 1, it follows that $\|x_n - y_n\| \to 0$.

Proof. See [11, Proposition 5.2.8].

Definition 2.5.5 (A. R. Lovaglia, 1955). Suppose that X is a normed space. Define a function $\delta_X : [0,2] \times S(X) \to [0,1]$ by the formula

$$\delta_X(\epsilon, x) = \inf\{1 - \|\frac{1}{2}(x+y)\| : y \in S(X), \|x-y\| \geq \epsilon\}.$$

Then δ_X is the LUR modulus of X. The space X is *locally uniformly rotund* or *locally uniformly convex* (LUR for short) if $\delta_X(\epsilon, x) > 0$ whenever $0 < \epsilon \leq 2$ and $x \in S(X)$.

Proposition 2.5.6 Every uniformly rotund normed space is locally uniformly rotund, and every locally uniformly rotund normed space is rotund. In symbols, $UR \Rightarrow LUR \Rightarrow R$.

Proof. See [11, Proposition 5.3.3].

Proposition 2.5.7 Suppose that X is a normed space. Then the following are equivalent.

(a) The space X is LUR.

(b) When $x \in S(X)$ and (y_n) is a sequence in $S(X)$ such that $\|\frac{1}{2}(x+y_n)\| \to 1$, it follows that $\|x - y_n\| \to 0$.

(c) When $x \in S(X)$ and (y_n) is a sequence in $B(X)$ such that $\|\frac{1}{2}(x+y_n)\| \to 1$, it follows that $\|x - y_n\| \to 0$.

(d) When $x \in S(X)$ and (y_n) are sequence in X such that $\|y_n\|$ and $\|\frac{1}{2}(x+y_n)\|$ both tend to 1, it follows that $\|x - y_n\| \to 0$.

Proof. See [11, Proposition 5.3.5].

Definition 2.5.8 A Banach space X is said to have the *Kadee-Klee property* (X has the property (H) for short) if the weak convergence and the convergence in norm coincide in $S(X)$.

2.6 Some Useful Results

Theorem 2.6.1 Let X be a real vector space and ρ a modular on X. For $x \in X$ let $f_x : \mathbb{R} \to \mathbb{R}$ be define by $f_x(c) = \rho(cx)$, $c \in \mathbb{R}$. Then f_x is a continuous (See [10]).

Theorem 2.6.2 [15] Let $x \in l_M$.

(1) If $0 < \alpha \leq 1$, then $\frac{1}{\alpha} I_M(\alpha x) \leq I_M(x) \leq \alpha I_M(\frac{x}{\alpha})$.

(2) If $\alpha > 1$, then $\alpha I_M(\frac{x}{\alpha}) \leq I_M(x) \leq \frac{1}{\alpha} I_M(\alpha x)$.

Theorem 2.6.3 [15] Let $x \in l_M$.

(1) If $\|x\| < 1$, then $I_M(x) \leq \|x\|$.

(2) If $\|x\| > 1$, then $I_M(x) \geq \|x\|$.

Theorem 2.6.4 [15] If a Musielak-Orlicz function $M = (M_k)$ satisfies the $(*)-$condition and $M \in \delta_2$, then, the norm convergence and modular convergence coincide.

Theorem 2.6.5 If the Musielak-Orlicz function $M = (M_k)$ satisfies the $(*)-$condition and $M \in \delta_2$, then

(1) [7] $\|x\| = 1 \Leftrightarrow I_M(x) = 1$,

(2) [8] for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x\| < 1 - \delta$ whenever $I_M(x) < 1 - \epsilon$,

(3) [8] for every $\epsilon > 0$ and $c > 0$ there exists a $\delta > 0$ such that for any $x, y \in l_M$, we have

$$|I_M(x + y) - I_M(x)| < \epsilon$$

whenever $I_M(x) \leq c$ and $I_M(y) \leq \delta$,

(4) [15] for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x\| > 1 + \delta$ whenever $I_M(x) > 1 + \epsilon$, and

(5) [15] for any sequence $(x_n) \subset l_M$, $\|x_n\| \to 1$ as $n \to \infty$ implies $I_M(x_n) \to 1$ as $n \to \infty$.

Theorem 2.6.6 [7] If the complementary of a Musielak-Orlicz function $M = (M_k)$ satisfies δ_2 i.e. $N \in \delta_2$, then there exists a $\theta \in (0, 1)$ and a sequence (h_k) of positive real numbers with $\sum_{k=1}^{\infty} M_k(h_k) < \infty$ such that

$$M_k(\frac{u}{2}) \leq \frac{1 - \theta}{2} M_k(u)$$

hold for every $k \in \mathbb{N}$ and u satisfying $M_k(h_k) \leq M_k(u) \leq 1$.

3. SOME CONVEXITY OF CESÀRO-MUSIELAK-ORLICZ SEQUENCE SPACES

In this section we show that the Cesàro-Musielak-Orlicz sequence spaces Ces_M equipped with the Luxemburg norm is locally uniformly rotund. So it is has property (H) and it is rotund. Before showing these results we give some useful facts and results concerning the modular ρ_M on Ces_M and relationships between the modular ρ_M and the Luxemburg norm on Ces_M. We start with showing that ρ_M is a convex modular on Ces_M.

Theorem 3.1 The functional ρ_M On the Cesàro-Musielak-Orlicz sequence space Ces_M given by

$$\rho_M(x) = \sum_{k=1}^{\infty} M_k \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right),$$

is a convex modular on Ces_M

Proof. Let $x, y \in Ces_M$. It is obvious that

(i) $\rho_M(x) = 0 \Leftrightarrow x = 0$

(ii) For $\alpha \in \mathbb{R}$, with $|\alpha| = 1$, we have

$$\begin{aligned}
\rho_M(\alpha x) &= \sum_{k=1}^{\infty} M_k \left(\frac{1}{k} \sum_{i=1}^{k} |\alpha x(i)| \right) \\
&= \sum_{k=1}^{\infty} M_k \left(|\alpha| \frac{1}{k} \sum_{i=1}^{k} |x(i)| \right) \\
&= \sum_{k=1}^{\infty} M_k \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right) \\
&= \rho_M(x)
\end{aligned}$$

(iii) For $\alpha, \beta \in \mathbb{R}$, with $\alpha, \beta \geq 0, \alpha + \beta = 1$, by convexity of M_k, we have

$$\begin{aligned}
\rho_M(\alpha x + \beta y) &= \sum_{k=1}^{\infty} M_k \left(\frac{1}{k} \sum_{i=1}^{k} |\alpha x(i) + \beta y(i)| \right) \\
&\leq \sum_{k=1}^{\infty} M_k \left(\frac{1}{k} \sum_{i=1}^{k} (\alpha |x(i)| + \beta |y(i)| \right) \\
&= \sum_{k=1}^{\infty} M_k \left(\frac{\alpha}{k} \sum_{i=1}^{k} |x(i)| + \frac{\beta}{k} \sum_{i=1}^{k} |y(i)| \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=1}^{\infty} \left(\alpha M_k \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right) + \beta M_k \left(\frac{1}{k} \sum_{i=1}^{k} |y(i)| \right) \right) \\
&= \alpha \sum_{k=1}^{\infty} M_k \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right) + \beta \sum_{k=1}^{\infty} M_k \left(\frac{1}{k} \sum_{i=1}^{k} |y(i)| \right) \\
&= \alpha \rho_M(x) + \beta \rho_M(y)
\end{aligned}$$

□

Proposition 3.2 Let $x \in Ces_M$.

(1) If $0 < \alpha \leq 1$, then $\frac{1}{\alpha}\rho_M(\alpha x) \leq \rho_M(x) \leq \alpha \rho_M(\frac{x}{\alpha})$

(2) If $\alpha > 1$, then $\alpha \rho_M(\frac{x}{\alpha}) \leq \rho_M(x) \leq \frac{1}{\alpha}\rho_M(\alpha x)$

Proof. Let $x = (x(i)) \in Ces_M$. For any $0 < \alpha \leq 1$, by convexity of each M_k, we have

$$M_k \left(\frac{1}{k} \sum_{i=1}^{k} |\alpha x(i)| \right) = M_k \left(\alpha \frac{1}{k} \sum_{i=1}^{k} |x(i)| \right) \leq \alpha M_k \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)$$

for all $k \in \mathbb{N}$. This implies $\rho_M(\alpha x) \leq \alpha \rho_M(x)$. Next, substituting x by $\frac{x}{\alpha}$, we obtain $\rho_M(x) \leq \alpha \rho_M(\frac{x}{\alpha})$. That is (1) holds. Next, let $\alpha > 1$. Then $0 < \frac{1}{\alpha} < 1$. By (1), we obtain that

$$\alpha \rho_M(\frac{x}{\alpha}) = \frac{1}{\frac{1}{\alpha}} \rho_M(\frac{x}{\alpha}) \leq \rho_M(x) \leq \frac{1}{\alpha} \rho_M(\frac{x}{\frac{1}{\alpha}}) = \frac{1}{\alpha} \rho_M(\alpha x)$$

, so (2) is satisfied. □

Proposition 3.3 For any $x \in Ces_M$, we have

(1) if $\|x\| \leq 1$, then $\rho_M(x) \leq \|x\|$

(2) if $\|x\| > 1$, then $\rho_M(x) \geq \|x\|$

Proof. (1) If $x = 0$, then the inequality holds. For $x \neq 0$, by the definition of $\|.\|$, there is a sequence $(\epsilon_n) \downarrow \|x\|$ such that $\rho_M(\frac{x}{\epsilon_n}) \leq 1$. This implies $\rho_M(\frac{x}{\|x\|}) \leq 1$, by Proposition 3.2(1), we have $\rho_M(x) \leq \|x\|\rho_M(\frac{x}{\|x\|}) \leq \|x\|$.

(2) Let $\|x\| > 1$. Then for $\epsilon \in (0, \frac{\|x\|-1}{\|x\|})$, we have $(1-\epsilon)\|x\| > 1$. By Proposition 3.2(1), we have

$$1 < \rho_M(\frac{x}{(1-\epsilon)\|x\|}) \leq \frac{\rho_M(x)}{(1-\epsilon)\|x\|}.$$

Letting $\epsilon \to 0$, we obtain (2). □

The following result is directly obtained from Proposition 3.3(1).

Corollary 3.4 If $x_n \to 0$ as $n \to \infty$ then $\rho_M(x_n) \to 0$ as $n \to \infty$

Theorem 3.5 If a Musielak-Orlicz function $M = (M_k) \in \delta_2$, then $SCes_M = Ces_M$.

Proof. Let $x \in Ces_M$. Thus $\rho_M(cx) < \infty$ for some $c > 0$. Since $M \in \delta_2$, there exists $K \geq 2$, $u_0 > 0$ and a positive sequence (c_k) such that $\sum_{k=1}^{\infty} c_k < \infty$ and

$$M_k(2u) \leq KM_k(u) + c_k$$

for all $k \in \mathbb{N}$ and $u \in \mathbb{R}$ with $|u| \leq u_0$. By $\rho_M(cx) < \infty$, we have that

$$\sum_{k=1}^{\infty} M_k\left(c\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) < \infty,$$

it follows that $M_k\left(c\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) \to 0$ as $k \to \infty$, and so $\frac{1}{k}\sum_{i=1}^{k}|x(i)| \to 0$ as $k \to \infty$. Let $\beta > 0$ and $t \in \mathbb{N}$ be such that $\frac{\beta}{c} \leq 2^t$. Then there exist $n_0 \in \mathbb{N}$ and a positive sequence (c'_k) with $\sum_{k=1}^{\infty} c'_k < \infty$ such that

$$c\frac{1}{k}\sum_{i=1}^{k}|x(i)| \leq \frac{u_0}{2^{t-1}} \text{ for all } k \geq n_0,$$

and

$$M_k(2^t u) \leq K^t M_k(u) + c'_k \text{ for all } k \in \mathbb{N} \text{ and } u \in \mathbb{R} \text{ with } |u| \leq \frac{u_0}{2^{t-1}}.$$

Thus

$$\begin{aligned}
\rho_M(\beta x) &= \sum_{k=1}^{\infty} M_k\left(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)\\
&= \sum_{k=1}^{n_0} M_k\left(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) + \sum_{k=n_0+1}^{\infty} M_k\left(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)\\
&= \sum_{k=1}^{n_0} M_k\left(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) + \sum_{k=n_0+1}^{\infty} M_k\left(\beta\frac{c}{c}\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)\\
&\leq \sum_{k=1}^{n_0} M_k\left(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) + \sum_{k=n_0+1}^{\infty} M_k\left(2^t c\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)\\
&\leq \sum_{k=1}^{n_0} M_k\left(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) + K^t\sum_{k=n_0+1}^{\infty} M_k\left(c\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) + \sum_{k=n_0+1}^{\infty} c_k' < \infty.
\end{aligned}$$

Therefore $Ces_M \subseteq SCes_M$. □

Lemma 3.6 On Cesàro-Musielak-Orlicz sequence space Ces_M, if the Musielak-Orlicz function $M = (M_k)$ satisfies the $(*)$−condition and $M \in \delta_2$, then

(1) $\|x\| = 1 \Leftrightarrow \rho_M(x) = 1$,

(2) for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x\| < 1-\delta$ whenever $\rho_M(x) < 1-\epsilon$,

(3) for every $\epsilon > 0$ and $c > 0$ there exists a $\delta > 0$ such that for any $x, y \in Ces_M$, we have

$$|\rho_M(x+y) - \rho_M(x)| < \epsilon$$

whenever $\rho_M(x) \leq c$ and $\rho_M(y) \leq \delta$,

(4) for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x\| > 1+\delta$ whenever $\rho_M(x) > 1+\epsilon$, and

(5) for any sequence $(x_n) \subset Ces_M$, $\|x_n\| \to 1$ implies $\rho_M(x_n) \to 1$.

Proof. (1) Assume that $\rho_M(x) = 1$. By definition of $\|.\|$, we have that $\|x\| \leq 1$. If $\|x\| < 1$, then we have by Proposition 3.3 (1) that $\rho_M(x) \leq \|x\| < 1$, which contradicts our assumption. Therefore $\|x\| = 1$.

Conversely, assume that $\|x\| = 1$. By Proposition 3.3(1), $\rho_M(x) \leq 1$. Suppose that $\rho_M(x) < 1$. By Theorem 3.5, we have $\rho_M(cx) < \infty$ for all $c > 1$. By Theorem 2.6.1 the function $c \mapsto \rho_M(cx)$ is continuous, so there exists an $c' > 1$ such that $\rho_M(c'x) = 1$.

By using the same proof as in the first path, we have that $\|c'x\| = 1$, so $c' = 1$ which is contradiction.

(2) Let $\epsilon > 0, x \in Ces_M$ such that $\rho_M(x) < 1-\epsilon$ and we put $a(k) = \frac{1}{k}\sum_{i=1}^{k}|x(i)|$, then $a = (a(k)) \in l_M$ and we have

$$I_M(a) = \rho_M(x), \|a\|_{l_M} = \|x\|_{Ces_M}.$$

By Theorem 2.6.5(2), there exists a $\delta > 0$ such that $\|a\| < 1 - \delta$ i.e. $\|x\| < 1 - \delta$.

(3) Let $x, y \in Ces_M, \epsilon > 0$ and $c > 0$, by Theorem 2.6.5(3), there exists a $\delta' > 0$ such that for any $a, b \in l_M$, we have

$$|I_M(a+b) - I_M(a)| < \epsilon \tag{3.1}$$

whenever $I_M(a) \leq c$ and $I_M(b) \leq \delta'$. For each $i \in \mathbb{N}$, let

$$s(i) = \begin{cases} sgn(x(i) + y(i)) \text{ if } x(i) + y(i) \neq 0, \\ 1 \text{ if } x(i) + y(i) = 0 \end{cases}$$

we note that

$$\begin{aligned} \rho_M(x+y) &= \sum_{k=1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)+y(i)|\right) \\ &= \sum_{k=1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}s(i)x(i) + \frac{1}{k}\sum_{i=1}^{k}s(i)y(i)\right). \end{aligned} \tag{3.2}$$

Let $a(k) = \frac{1}{k}\sum_{i=1}^{k}s(i)x(i)$ and $b(k) = \frac{1}{k}\sum_{i=1}^{k}s(i)y(i)$ for all $k \in \mathbb{N}$. Then $a = (a(k)) \in l_M$ and $b = (b(k)) \in l_M$, and from (3.2) we have

$$\rho_M(x+y) = I_M(a+b), I_M(a) \leq \rho_M(x) \text{ and } I_M(b) \leq \rho_M(y).$$

Choose $\delta = \delta'$. Let $\rho_M(x) \leq c$ and $\rho_M(y) \leq \delta$. Then $I_M(a) \leq c$ and $I_M(b) \leq \delta'$, by (3.1) we have

$$\rho_M(x+y) - \rho_M(x) \leq I_M(a+b) - I_M(a) < \epsilon$$

that is

$$\rho_M(x+y) < \rho_M(x) + \epsilon. \tag{3.3}$$

Proof. Suppose that $x_n(i) - y_n(i) \not\to 0$ as $n \to \infty$ for some $i \in \mathbb{N}$. Without loss of generality we may assume that $i = 1$, and then assume that , for some $\epsilon_0 > 0$,

$$|x_n(1) - y_n(1)| \geq \epsilon_0 \quad \text{for all } n \in \mathbb{N}.$$

Since M_1 is strictly convex by Theorem 2.2.3(1), there exists a $\delta \in (0,1)$ such that

$$M_1(\frac{x_n(1) + y_n(1)}{2}) < \frac{(1-\delta)}{2}(M_1(x_n(1)) + M_1(y_n(1))) \quad \text{for all } n \in \mathbb{N}.$$

Hence

$$\begin{aligned}
1 \leftarrow I_M(\frac{x_n + y_n}{2}) &= \sum_{k=1}^{\infty} M_k(\frac{x_n(k) + y_n(k)}{2}) \\
&= M_1(\frac{x_n(1) + y_n(1)}{2}) + \sum_{k=2}^{\infty} M_k(\frac{x_n(k) + y_n(k)}{2}) \\
&\leq \frac{1-\delta}{2}(M_1(x_n(1)) + M_1(y_n(1)) + \frac{1}{2}\sum_{k=2}^{\infty}(M_k(x_n(k)) + M_k(y_n(k))) \\
&= \frac{1}{2}\sum_{k=1}^{\infty}(M_k(x_n(k)) + M_k(y_n(k))) - \frac{\delta}{2}(M_1(x_n(1)) + M_1(y_n(1))) \\
&\leq \frac{1}{2}\sum_{k=1}^{\infty}(M_k(x_n(k)) + M_k(y_n(k))) - \delta M_1(|\frac{x_n(1) - y_n(1)}{2}|) \\
&= 1 - \delta M_1(2\epsilon_0),
\end{aligned}$$

which is a contradiction. □

Lemma 3.9 Suppose $(x_n) \subseteq B(Ces_M)$, $x \in S(Ces_M)$ and each M_k is strictly convex. If $\rho_M(\frac{x_n + x}{2}) \to 1$ as $n \to \infty$, then $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$ and $i \in \mathbb{N}$, let

$$s_n(i) = \begin{cases} sgn(x_n(i) + x(i)), & \text{if } x_n(i) + x(i) \neq 0, \\ 1, & \text{if } x_n(i) + x(i) = 0. \end{cases}$$

Thus

$$1 \leftarrow \rho_M(\frac{x_n + x}{2}) = \sum_{k=1}^{\infty} M_k \left(\frac{1}{k} \sum_{i=1}^{k} |\frac{x_n(i) + x(i)}{2}| \right)$$
$$= \sum_{k=1}^{\infty} M_k \left(\frac{1}{k} \sum_{i=1}^{k} s_n(i) \frac{x_n(i)}{2} + \frac{1}{k} \sum_{i=1}^{k} s_n(i) \frac{x(i)}{2} \right). \quad (3.7)$$

Let $a_n(k) = \frac{1}{k} \sum_{i=1}^{k} s_n(i) x_n(i)$ and $b_n(k) = \frac{1}{k} \sum_{i=1}^{k} s_n(i) x(i)$ for all $n, k \in \mathbb{N}$. Then $(a_n) \subseteq B(l_M)$ and $(b_n) \subseteq B(l_M)$. From (3.7) we have

$$I_M(\frac{a_n + b_n}{2}) \to 1 \quad \text{as } n \to \infty.$$

By Lemma 3.8, for each $i \in \mathbb{N}$, we have

$$a_n(i) - b_n(i) \to 0 \text{ as } n \to \infty. \quad (3.8)$$

Now, we shall show that $x_n(k) \to x(k)$ as $n \to \infty$ $\forall k \in \mathbb{N}$. By (3.8), we have that

$$s_n(1) x_n(1) - s_n(1) x(1) \to 0 \text{ as } n \to \infty.$$

This implies $x_n(1) \to x(1)$ as $n \to \infty$. If $x_n(i) \to x(i)$ as $n \to \infty$ $\forall i \leq k - 1$, then we have

$$s_n(i)(x_n(i) - x(i)) \to 0 \text{ as } n \to \infty \ \forall i \leq k - 1 \quad (3.9)$$

since

$$s_n(k)(x_n(k) - x(k)) = k(a_n(k) - b_n(k)) - \sum_{i=1}^{k-1} s_n(i)(x_n(i) - x(i)),$$

it follows from (3.8) and (3.9) that

$$s_n(k)(x_n(k) - x(k)) \to 0 \text{ as } n \to \infty,$$

hence $x_n(k) \to x(k)$ as $n \to \infty$. So we have by induction that $x_n(k) \to x(k)$ as $n \to \infty$ for all $k \in \mathbb{N}$. □

Theorem 3.10 If a Musielak-Orlicz function M satisfies the $(*)$-condition, $M \in \delta_2 \cap \delta_2^*$ and M is strictly convex , then the space Ces_M is LUR.

Proof. Let $(x_n) \subseteq B(Ces_M)$, $x \in S(Ces_M)$ be such that $\|x_n+x\| \to 2$ as $n \to \infty$. Then $\|\frac{x_n+x}{2}\| \to 1$ as $n \to \infty$. By Lemma 3.6(5), we have that $\rho_M(\frac{x_n+x}{2}) \to 1$ as $n \to \infty$. By Lemma 3.9, we have $x_n(i) \to x(i)$ as $n \to \infty$ $\forall i \in \mathbb{N}$. By Lemma 3.6(1), we have

$$\sum_{k=1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) = 1. \tag{3.10}$$

Now, let $\epsilon > 0$ be given. Since $M = (M_k)$ satisfies the $(*)$-condition and $M \in \delta_2$, there is a $\delta \in (0,1)$ such that

$$|\rho_M(u+v) - \rho_M(u)| < \frac{\epsilon}{3} \tag{3.11}$$

whenever $\rho_M(u) \le \frac{\epsilon}{3}$ and $\rho_M(v) \le \delta$. Since $N \in \delta_2$, by Theorem 2.6.6, there exists a $\theta \in (0,1)$ and a sequence (h_k) of positive numbers such that $\sum_{k=1}^{\infty} M_k(h_k) < \infty$ and

$$M_k(\frac{u}{2}) \le (1-\theta)\frac{M_k(u)}{2} \tag{3.12}$$

for all $k \in \mathbb{N}$ and $u \in \mathbb{R}$ with $M_k(h_k) \le M_k(u) \le 1$. Since $\rho_M(\frac{x_n+x}{2}) \to 1$ as $n \to \infty$, there exists $n' \in \mathbb{N}$ such that $1 - \frac{\theta\delta}{4} < \rho_M(\frac{x_n+x}{2})$ for all $n \ge n'$. First, we will show that there exists $j_0 \in \mathbb{N}$ such that for $j \ge j_0$,

$$\sup_{n \ge n'} \sum_{k=j+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|\right) \le \delta. \tag{3.13}$$

To show this, suppose that (3.13) dose not hold. Then there exists a sequence of positive integers $\{j_m\}$ with $j_m \to \infty$ as $m \to \infty$ and a sequence of positive integers (n_m) with $n_m \ge n'$ such that

$$\sum_{k=j_m+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x_{n_m}(i)|\right) > \delta. \tag{3.14}$$

for every $m \in \mathbb{N}$. Since $M = (M_k)$ satisfies the $(*)$-condition and $M \in \delta_2$, there exists a $\delta_1 > 0$ such that

$$|\rho_M(u+v) - \rho_M(u)| < \frac{\theta\delta}{8}, \tag{3.15}$$

whenever $\rho_M(u) \le 1$ and $\rho_M(v) \le \delta_1$. By $x \in Ces_M$, there exists a positive number k_1 such that

$$\sum_{k=k_1+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) \le \delta_1, \sum_{k=k_1+1}^{\infty} M_k(h_k) < \frac{\theta\delta}{4}. \tag{3.16}$$

Take m so large that $j_m > k_1$. Let

$$u_{n_m} = (\underbrace{0,0,...,0}_{k_1}, \sum_{i=1}^{k_1+1} |x_{n_m}(i)|, |x_{n_m}(k_1+2)|, |x_{n_m}(k_1+3), ...)$$

and

$$u = (\underbrace{0,0,...,0}_{k_1}, \sum_{i=1}^{k_1+1} |x(i)|, |x(k_1+2)|, |x(k_1+3)|, ...).$$

Then

$$\rho_M(\frac{u_{n_m}}{2}) = \sum_{k=k_1+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x_{n_m}(i)}{2}|\right) < 1$$

and

$$\rho_M(\frac{u}{2}) = \sum_{k=k_1+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x(i)}{2}|\right) < \delta_1.$$

By (3.15), $\rho_M(\frac{u_{n_m}+u}{2}) \leq \rho_M(\frac{u_{n_m}}{2}) + \frac{\theta\delta}{8}$. Thus

$$\sum_{k=k_1+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x_{n_m}(i)+x(i)}{2}|\right) \leq \sum_{k=k_1+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x_{n_m}(i)}{2}|\right) + \frac{\theta\delta}{8}. \tag{3.17}$$

By using convexity of each M_k, (3.10), (3.12), (3.14), (3.16) and (3.17), we have

$$\begin{aligned}
1 - \frac{\theta\delta}{4} < \rho_M(\frac{x_{n_m}+x}{2}) &= \sum_{k=1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x_{n_m}(i)+x(i)}{2}|\right) \\
&= \sum_{k=1}^{k_1} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x_{n_m}(i)+x(i)}{2}|\right) + \sum_{k=k_1+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x_{n_m}(i)+x(i)}{2}|\right) \\
&\leq \frac{1}{2}\sum_{k=1}^{k_1} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) + \frac{1}{2}\sum_{k=1}^{k_1} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x_{n_m}(i)|\right) + \\
&\quad \sum_{k=k_1+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x_{n_m}(i)}{2}|\right) + \frac{\theta\delta}{8}
\end{aligned}$$

$$\leq \frac{1}{2} + \frac{1}{2}\left[\sum_{k=1}^{\infty} M_k\left(\sum_{i=1}^{k}|x_{n_m}(i)|\right) - \sum_{k=k_1+1}^{\infty} M_k\left(\sum_{i=1}^{k}|x_{n_m}(i)|\right)\right] +$$

$$(\frac{1-\theta}{2})\sum_{k=k_1+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x_{n_m}(i)|\right) + \frac{1}{2}\sum_{k=k_1+1}^{\infty} M_k(h_k) + \frac{\theta\delta}{8}$$

$$= 1 + (\frac{1-\theta}{2} - \frac{1}{2})\sum_{k=k_1+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x_{n_m}(i)|\right) + \frac{\theta\delta}{8} + \frac{\theta\delta}{8}$$

$$< 1 - \frac{\theta\delta}{2} + \frac{\theta\delta}{4} = 1 - \frac{\theta\delta}{4},$$

this contradiction proves (3.13). From (3.13), there exists $k' \in \mathbb{N}$ such that

$$\sum_{k=k'+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|\right) \leq \delta,$$

for all $n \geq n'$. By $x \in Ces_M$, there exists $k'' \in \mathbb{N}$ such that

$$\sum_{k=k''+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) \leq \frac{\epsilon}{3}.$$

Choose $k_0 = \max\{k', k''\}$. Then, we have

$$\sum_{k=k_0+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) \leq \frac{\epsilon}{3}$$

$$\sum_{k=k_0+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|\right) \leq \delta$$

for all $n \geq n'$. Since $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$, then there exists $n'' \in \mathbb{N}$ such that

$$\sum_{k=1}^{k_0} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|\right) \leq \frac{\epsilon}{3}$$

for all $n \geq n''$. Choose $n_0 = \max\{n', n''\}$. Then we have

$$\sum_{k=k_0+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|\right) \leq \delta$$

$$\sum_{k=k_0+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) \le \frac{\epsilon}{3}$$

$$\sum_{k=1}^{k_0} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)-x(i)|\right) \le \frac{\epsilon}{3} \tag{3.18}$$

for all $n \geq n_0$. Put

$$u = (\underbrace{0,0,...,0}_{k_0}, \sum_{i=1}^{k_0+1}|x(i)|, |x(k_0+2)|, |x(k_0+3)|, ...)$$

$$u_n = (\underbrace{0,0,...,0}_{k_0}, \sum_{i=1}^{k_0+1}|x_n(i)|, |x_n(k_0+2)|, |x_n(k_0+3)|, ...)$$

for all $n \in \mathbb{N}$. Then

$$\rho_M(u) = \sum_{k=k_0+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) \le \frac{\epsilon}{3},$$

$$\rho_M(u_n) = \sum_{k=k_0+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|\right) \le \delta,$$

and

$$\rho_M(u_n+u) = \sum_{k=k_0+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}(|x_n(i)|+|x(i)|)\right).$$

Hence, we have by (3.11)

$$\sum_{k=k_0+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}(|x_n(i)|+|x(i)|)\right) < \frac{\epsilon}{3} + \rho_M(u). \tag{3.19}$$

By (3.18) and (3.19), we have for $n \geq n_0$,

$$\begin{aligned}
\rho_M(x_n - x) &= \sum_{k=1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|\right) \\
&= \sum_{k=1}^{k_0} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|\right) + \sum_{k=k_0+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|\right) \\
&\leq \frac{\epsilon}{3} + \sum_{k=k_0+1}^{\infty} M_k\left(\frac{1}{k}\sum_{i=1}^{k}(|x_n(i)| + |x(i)|)\right) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \rho_M(u) \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Hence $\rho_M(x_n - x) \to 0$ as $n \to \infty$. By Proposition 3.7, we have that $\|x_n - x\| \to 0$ as $n \to \infty$. Therefor Ces_M is LUR. □

It is well known that every LUR space has property (H), so the following result is obtained.

Corollary 3.11 If a Musielak-Orlicz function M satisfies the $(*)$-condition, $M \in \delta_2 \cap \delta_2^*$ and M is strictly convex, then the space Ces_M has the property (H).

Corollary 3.12 Suppose that $p = (p_k)$ is a bounded sequence of positive real numbers with $\inf_k p_k > 1$. Then $ces(p)$ is LUR.

Proof. We define the function $M_k : \mathbb{R} \to \mathbb{R}^+$ by $M_k(u) = |u|^{p_k}$ for all $k \in \mathbb{N}$. Thus we have the complementary function N_k of M_k is $N_k(v) = |v|^{q_k}$ where $\frac{1}{q_k} + \frac{1}{p_k} = 1$ for all $k \in \mathbb{N}$. Let $M = (M_k)$ and $N = (N_k)$. Clearly M_k is strictly convex for all $k \in \mathbb{N}$ and $M \in \delta_2$ since (p_k) is bounded. By the condition $\inf_k p_k > 1$, it implies that (q_k) is bounded, hence $N = (N_k) \in \delta_2$. To show that M satisfies the $(*)-$ condition. Let $\epsilon \in (0,1)$ and $M_k(u) < 1 - \epsilon$ for all $k \in \mathbb{N}$. Then $|u|^{p_k} < 1 - \epsilon$ for all $k \in \mathbb{N}$. We choose $\delta \in (0, (\frac{1}{1-\epsilon})^{\frac{1}{K}} - 1)$ where $K = \sup_k p_k$. Hence

$$\begin{aligned}
M_k((1+\delta)u) = |(1+\delta)u|^{p_k} = (1+\delta)^{p_k}|u|^{p_k} &< (1+\delta)^{p_k}(1-\epsilon) \\
&\leq \left(1 + (\frac{1}{1-\epsilon})^{\frac{1}{K}} - 1\right)^K (1-\epsilon) = 1.
\end{aligned}$$

Therefore M satisfies the $(*)-$condition. By Theorem 3.11, we conclude that $ces(p)$ is LUR. $\square$

Corollary 3.13 Suppose that $p = (p_k)$ is a bounded sequence of positive real numbers with $\inf\limits_k p_k > 1$. Then $ces(p)$ has the property (H).

When $p_k = p > 1$ for all $k \in \mathbb{N}$, we obtain immediatly the following result.

Corollary 3.14[9] For $1 < p < \infty$, the Cesàro sequence space ces_p is LUR and has property (H).

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SOME GEOMETRIC PROPERTIES OF CESARO SEQUENCE SPACE

WINATE SANHAN* AND SUTHEP SUANTAI

ABSTRACT. In this paper we define a modular on the Cesaro sequence space $ces(p)$ and consider it equipped with the Luxemburg norm. We give some relationships between the modular and the Luxemburg norm on this space and show that the space $ces(p)$ has property (H) but it is not rotund (R), where $p = (p_k)$ is a bounded sequence of positive real number with $p_k \geq 1$ for all $k \in \mathbb{N}$.

1. Introduction. Let $(X, \|.\|)$ be a real Banach space, and let $B(X)$ (resp. $S(X)$) be the closed unit ball (resp. the unit sphere) of X.

A point $x \in S(X)$ is an H-point of $B(X)$ if for any sequence (x_n) in X such that $\|x_n\| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x (write $x_n \xrightarrow{\omega} x$) implies that $\|x_n - x\| \to 0$ as $n \to \infty$. If every point in $S(X)$ is an H-point of $B(X)$, then X is said to have the property (H).

A point $x \in S(X)$ is an extreme point of $B(X)$, if for any $y, z \in S(X)$ the equality $2x = y + z$ implies $y = z$.

A point $x \in S(X)$ is an locally uniformly rotund point of $B(X)$ (LUR-point for short) if for any sequence (x_n) in $B(X)$ such that $\|x_n + x\| \to 2$ as $n \to \infty$ there holds $\|x_n - x\| \to 0$ as $n \to \infty$.

A Banach space X is said to be rotund (R), if every point of $S(X)$ is an extreme point of $B(X)$. If every point of $S(X)$ is a LUR-point of $B(X)$, then X is said to be locally uniformly rotund (LUR).

It is known that if X is LUR, then it is (R) and possesses property (H). For these geometric notions and their role in Mathematics we refer to the monographs [1], [2] , [6] and [13]. Some of them were studied for Orlicz spaces in [3],[7],[8],[9]and [14].

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Let l^0 be the space of all real sequences. For $1 \leq p < \infty$, the Cesaro sequence space (ces_p , for short) is defined by

$$ces_p = \{x \in l^0 : \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^p < \infty\}$$

equipped with the norm

$$\|x\| = (\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^p)^{\frac{1}{p}}$$

and

$$\|x\|_0 = (\sum_{r=0}^{\infty}(\frac{1}{2^r}\sum_r|x(i)|)^p)^{\frac{1}{p}}$$

where $\sum_r$ denotes a sum over the ranges $2^r \leq i < 2^{r+1}$

It is known that these two norms are equivalent and ces_p is Banach with respect to each of the two norms.

This space was introduced by J.S. Shue [15]. It is useful in the theory of matrix operator and others (see [10] and [12]). Some geometrio properties of the Cesáro sequence space $(ces_p, \|.\|)$ were studied by many mathematicians. It is known that $(ces_p, \|.\|)$ is LUR and possesses property (H) (see [12]). Y. A. Cui and H. Hudzik [4] proved that $(ces_p, \|.\|)$ has the Banach-Saks of type p if $p > 1$, and it was shown in [5] that $(ces_p, \|.\|)$ has property (β).

Now let $p = (p_k)$ be a bounded sequence of positive real number with $p_k \geq 1$ for all $k \in \mathbb{N}$. The Cesaro sequence space $ces(p)$ is defined by

$$ces(p) = \{x \in l^0 : \sum_{r=0}^{\infty}(\frac{1}{2^r}\sum_r|x(i)|)^{p_r}$$

where $\sum_r$ denotes a sum over the ranges $2^r \leq i < 2^{r+1}$.

For $x \in ces(p)$, let $\rho(x) = \sum_{r=0}^{\infty}(\frac{1}{2^r}\sum_r|x(i)|)^{p_r})$ and define the Luxemburg norm on $ces(p)$ by

$$\|x\| = \inf\{\varepsilon > 0 : \rho(\frac{x}{\varepsilon}) \leq 1\}, \quad x \in ces(p).$$

The main purpose of this paper is to show that the Cesaro sequence space $ces(p)$ equipped with the Luxemburg norm has property(H) but it is not rotund ,so it is not LUR.

Throughout this paper we let $M = \sup_r p_r$, and for $x \in l^0$ we put

$$x|_i = (x(1), x(2), ..., x(i), 0, 0, ...)$$

and

$$x|_{\mathbb{N}-i} = (0, 0, ..., 0, x(i+1), x(i+2), ...).$$

MAIN RESULTS

First, we show that ρ is a convex modular on $ces(p)$.

Proposition 2.1 *The functional ρ is a convex modular on $ces(p)$.*

Proof. It is obvious that $\rho(x) = 0 \Leftrightarrow x = 0$ and $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$.

Let $\alpha \geq 0, \beta \geq 0$ with $\alpha + \beta = 1$. By the convexity of the function $t \to |t|^{p_r}$ for every $r \in \mathbb{N}$, we have

$$\begin{aligned}
\rho(\alpha x + \beta y) &= \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |\alpha x(i) + \beta y(i)| \right)^{p_r} \\
&\leq \sum_{r=0}^{\infty} \left(\alpha \frac{1}{2^r} \sum_r |x(i)| + \beta \frac{1}{2^r} \sum_r |y(i)| \right)^{p_r} \\
&\leq \alpha \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |x(i)| \right)^{p_r} + \beta \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |y(i)| \right)^{p_r} \\
&= \alpha \rho(x) + \beta \rho(y).
\end{aligned}$$

Proposition 2.2 *For $x \in ces(p)$,the modular ρ on $ces(p)$ satisfies the following properties*

(i) if $0 < a < 1$, then $a^M \rho(\frac{x}{a}) \leq \rho(x)$ and $\rho(ax) \leq a\rho(x)$,

(ii) if $a > 1$, then $\rho(x) \leq a^M \rho(\frac{x}{a})$,

(iii) if $a \geq 1$, then $\rho(x) \leq a\rho(x) \leq \rho(ax)$.

Proof (i) Let $0 < a < 1$. Then we have

$$\begin{aligned}
\rho(x) &= \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |x_n| \right)^{p_r} \\
&= \sum_{r=0}^{\infty} \left(\frac{a}{2^r} \sum_r |\frac{x_n}{a}| \right)^{p_r} \\
&= \sum_{r=0}^{\infty} a^{p_r} \left(\frac{1}{2^r} \sum_r |\frac{x_n}{a}| \right)^{p_r}
\end{aligned}$$

$$\geq \sum_{r=0}^{\infty} a^M \left(\frac{1}{2^r} \sum_r |\frac{x_n}{a}| \right)^{p_r}$$
$$= a^M \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |\frac{x_n}{a}| \right)^{p_r}$$
$$= a^M \rho(\frac{x}{a}).$$

By convexity of ρ, we have $\rho(ax) \leq a\rho(x)$, so (i) is obtained
(ii) is an easy consequence of (i) when a is replaced by $\frac{1}{a}$.
(iii) follows from the convexity of ρ.

Proprosition 2.3 *For any $x \in ces(p)$, we have*

(i) if $\|x\| < 1$, then $\rho(x) \leq \|x\|$,
(ii) if $\|x\| > 1$, then $\rho(x) \geq \|x\|$,
(iii) $\|x\| = 1$ if and only if $\rho(x) = 1$,
(iv) $\|x\| < 1$ if and only if $\rho(x) < 1$ and
(v) $\|x\| > 1$ if and only if $\rho(x) > 1$.

Proof (i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - \|x\|$, so $\|x\| + \epsilon < 1$. By definition of $\|.\|$, there exists $\lambda > 0$ such that $\|x\| + \epsilon > \lambda$ and $\rho(\frac{x}{\lambda}) \leq 1$. By Proposition 2.2(i) and (iii), we have

$$\rho(x) \leq \rho\left(\frac{(\|x\| + \epsilon)}{\lambda} x\right)$$
$$= \rho\left((\|x\| + \epsilon)\frac{x}{\lambda}\right)$$
$$\leq (\|x\| + \epsilon)\rho(\frac{x}{\lambda})$$
$$\leq \|x\| + \epsilon,$$

which implies that $\rho(x) \leq \|x\|$. Hence (i) is satisfied.

(ii) Let $\epsilon > 0$ be such that $0 < \epsilon < \dfrac{\|x\| - 1}{\|x\|}$, then $1 < (1 - \epsilon)\|x\| < \|x\|$. By definition of $\|.\|$ and by Proposition 2.2(i), we have $1 < \rho\left(\frac{x}{(1-\epsilon)\|x\|}\right) \leq \frac{1}{(1-\epsilon)\|x\|}\rho(x)$, so $(1 - \epsilon)\|x\| < \rho(x)$ for all $\epsilon \in (0, \frac{\|x\|-1}{\|x\|})$, which implies that $\|x\| \leq \rho(x)$.

(iii) Assume that $\|x\| = 1$. Let $\epsilon > 0$, then there exists $\lambda > 0$ such that $1+\epsilon > \lambda > \|x\|$ and $\rho(\frac{x}{\lambda}) \leq 1$. By Proposition 2.2(ii), we have $\rho(x) \leq \lambda^M \rho(\frac{x}{\lambda}) \leq \lambda^M < (1+\epsilon)^M$, so $(\rho(x))^{\frac{1}{M}} < 1+\epsilon$ for all $\epsilon > 0$ which implies that $\rho(x) \leq 1$.

If $\rho(x) < 1$, let $a \in (0,1)$ such that $\rho(x) < a^M < 1$. From Proposition 2.2(i), we have $\rho(\frac{x}{a}) \leq \frac{1}{a^M}\rho(x) < 1$, hence $\|x\| \leq a < 1$, which is a contradiction. Thus, we have $\rho(x) = 1$.

Conversely, assume that $\rho(x) = 1$. By definition of $\|.\|$, we conclude that $\|x\| \leq 1$. If $\|x\| < 1$, then we have by (i) that $\rho(x) \leq \|x\| < 1$, which contradicts to our assumption, so we obtain that $\|x\| = 1$.

(iv) follows from (i) and (iii).

(v) follows from (iii) and (iv).

Proposition 2.4 *For $x \in ces(p)$ we have*

(i) if $0 < a < 1$ and $\|x\| > a$, then $\rho(x) > a^M$ and

(ii) if $a \geq 1$ and $\|x\| < a$, then $\rho(x) < a^M$.

Proof. (i) Suppose $0 < a < 1$ and $\|x\| > a$. Then $\|\frac{x}{a}\| > 1$. By Proposition 2.3(ii), we have $\rho(\frac{x}{a}) > 1$. Hence, by Proposition 2.2(i), we obtain that $\rho(x) \geq a^M \rho(\frac{x}{a}) > a^M$.

(ii) Suppose $a \geq 1$ and $\|x\| < a$. Then $\|\frac{x}{a}\| < 1$. By Proposition 2.3(i), we have $\rho(\frac{x}{a}) < 1$. If $a = 1$, we have $\rho(x) < 1 = a^M$. If $a > 1$, by Proposition 2.2(ii), we obtain that $\rho(x) < a^M \rho(\frac{x}{a}) < a^M$.

Proproposition 2.5 *Let (x_n) be a sequence in $ces(p)$.*

(i) If $\lim_{n\to\infty} \|x_n\| = 1$, then $\lim_{n\to\infty} \rho(x_n) = 1$.

(ii) If $\lim_{n\to\infty} \rho(x_n) = 0$ then $\lim_{n\to\infty} \|x_n\| = 0$.

Proof. (i) Suppose $\lim_{n\to\infty} \|x_n\| = 1$. Let $\epsilon \in (0,1)$. Then there exists $N \in \mathbb{N}$ such that $1-\epsilon < \|x_n\| < 1+\epsilon$ for all $n \geq \mathbb{N}$. By Proposition 2.4, $(1-\epsilon)^M < \rho(x_n) < (1+\epsilon)^M$ for all $n \geq \mathbb{N}$, which implies that $\lim_{n\to\infty} \rho(x_n) = 1$.

(ii) Suppose $\|x_n\| \not\to 0$. Then there is an $\epsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By Proproposition 2.4 (i), we obtain $\rho(x_{n_k}) > \epsilon^M$ for all $k \in \mathbb{N}$. This implies $\rho(x_n) \not\to 0$ as $n \to \infty$.

Lemma 2.6 *Let (x_n) be a sequence in $ces(p)$.If $\rho(x_n) \to \rho(x)$ and $x_n(k) \to x(k)$ $\forall k$, then $x_n \to x$ as $n \to \infty$.*

Proof Suppose that $x_n \not\to x$. By Proposition 2.5 (ii), we have $\rho(\frac{x_n - x}{2}) \not\to 0$.Without loss of generality we may assume that there exists $\epsilon \in (0,1)$ such that $\rho(\frac{x_n-x}{2}) > \epsilon$ for all $n \in \mathbb{N}$. Since $(\rho(\frac{x_n-x}{2}))_{n=1}^{\infty}$ is a bounded sequence, it must have a convergent subsequence. Passing through a subsequence, if necessary we can assume $\rho(\frac{x_n-x}{2}) \to \epsilon_0$ for some $\epsilon_0 \geq \epsilon$. Since $\rho(x) = \lim_{i\to\infty} \rho(x|_{2^i})$ and $(\rho(x|_{2^i}))_{i=0}^{\infty}$ is nondecreasing , we have $\rho(x) = \sup\{\rho(x|_{2^i}) : i \in \mathbb{N}\}$. So there exists $i \in \mathbb{N}$ such that $\rho(x|_{2^i}) > \rho(x) - \epsilon/2$. Thus

$$\rho(x|_{\mathbb{N}-2^i}) < \epsilon/2. \tag{2.1}$$

Since $x_n(k) \to x(k)$ for all $k \in \mathbb{N}$, we have

$$\rho(x_n|_{2^i}) \to \rho(x|_{2^i}) \text{ and } \rho(\frac{x_n - x}{2}|_{2^i}) \to 0 \text{ as } n \to \infty. \tag{2.2}$$

By the convexity of ρ together with (2.1) and (2.2), we have

$$\begin{aligned}
\varepsilon_0 &= \lim_{n\to\infty} \rho(\frac{x_n - x}{2}) \\
&= \lim_{n\to\infty} [\rho(\frac{x_n - x}{2}|_{2^i}) + \rho(\frac{x_n - x}{2}|_{\mathbb{N}-2^i}) \\
&= \lim_{n\to\infty} \rho(\frac{x_n - x}{2}|_{2^i}) + \lim_{n\to\infty} \rho(\frac{x_n - x}{2}|_{\mathbb{N}-2^i}) \\
&= 0 + \lim_{n\to\infty} \rho(\frac{x_n - x}{2}|_{\mathbb{N}-2^i}) \\
&\leq \frac{1}{2} \lim_{n\to\infty} \rho(x_n|_{\mathbb{N}-2^i}) + \frac{1}{2}\rho(x|_{\mathbb{N}-2^i}) \\
&= \frac{1}{2} \lim_{n\to\infty} (\rho(x_n) - \rho(x_n|_{2^i})) + \frac{1}{2}\rho(x|_{\mathbb{N}-2^i}) \\
&= \frac{1}{2}(\rho(x) - \rho(x|_{2^i})) + \frac{1}{2}\rho(x|_{\mathbb{N}-2^i}) \\
&= \frac{1}{2}\rho(x|_{\mathbb{N}-2^i}) + \frac{1}{2}\rho(x|_{\mathbb{N}-2^i}) \\
&= \rho(x|_{\mathbb{N}-2^i}) \\
&< \epsilon/2 \\
&< \epsilon_0,
\end{aligned}$$

which is a contradiction. Therefore $x_n \to x$ as $n \to \infty$.

Theorem 2.7 *The space $ces(p)$ has the property (H).*

Proof. Let $x \in S(ces(p))$, $x_n \in B(ces(p))$ for all $n \in \mathbb{N}$ such that $x_n \xrightarrow{\omega} x$ and $||x_n|| \to 1$ as $n \to \infty$. By Proposition 2.3(iii), we have $\rho(x) = 1$. By Proposition 2.5(i), we obtain that $\rho(x_n) \to 1$ as $n \to \infty$. So $\rho(x_n) \to \rho(x)$ as $n \to \infty$. Since $x_n \xrightarrow{\omega} x$ and the i^{th} coordinate mapping $p_i : ces(p) \to \mathbb{R}$, defined by $p_i(x) = x_i$, is continuous , it implies that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$. It follows from Lemma 2.6 that $x_n \to x$ as $n \to \infty$.

The following result is obtained directly from Theorem 2.7.

Corollary 2.8 *For $1 \leq p < \infty$, $(ces_p, \|.\|_0)$ has property (H)*

Remark 2.9 For a bounded sequence of positive real numbers $p = (p_k)$ with $p_k \geq 1$ for all $k \in \mathbb{N}$, the space $ces(p)$ equipped the Luxemburg norm is not rotund , so it is not LUR.

To see this we put

$$x = (0, 1, 1, 0, 0, ...) \text{ and } y = (0, 2, 0, 0, ...)$$

Then $x, y \in S(ces(p))$ because $\rho(x) = \rho(y) = 1$. Since $\rho(\frac{x+y}{2}) = 1$, we have by Proposition 2.3 (iii) that $||\frac{x+y}{2}|| = 1$. This shows that $ces(p)$ is not rotund, so it is not LUR.

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On the H-Property of Some Banach Sequence Spaces

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ABSTRACT. **In this paper, we define a generalized Cesáro sequence space $ces(p)$ and consider it equipped with the Luxemburg norm under which it is a Banach space , and we show that the space $ces(p)$ posses property (H) and property (G) , and it is rotund, where $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$.**

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1. Preliminaries.

For a Banach space X, we denote by $S(X)$ and $B(X)$ the unit sphere and unit ball of X, respectively. A point $x_0 \in S(X)$ is called

a) an *extreme point* if for every $x, y \in S(X)$ the equality $2x_0 = x + y$ implies $x = y$;

b) an *H-point* if for any sequence (x_n) in X such that $\|x_n\| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x_0 (write $x_n \xrightarrow{w} x_0$) implies that $\|x_n - x\| \to 0$ as $n \to \infty$;

c) a *denting point* if for every $\epsilon > 0$, $x_0 \notin \overline{conv}\{B(X)\backslash(x_0 + \epsilon B(X))\}$.

A Banach space X is said to be *rotund* (R), if every point of $S(X)$ is an extreme point.

A Banach space X is said to posses Property (H) (Property (G)) provided every ponit of $S(X)$ is H-point (denting point).

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For these geometric notions and their role in Mathematics we refer to the monographs [1], [2] , [6] and [13]. Some of them were studied for Orlicz spaces in [3],[7],[8],[9]and [14].

Let us denote by l^0 the space of all real sequences. For $1 \leq p < \infty$, the Cesáro sequence space (ces_p, for short) is defined by

$$ces_p = \{x \in l^0 : \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^p < \infty\}$$

equipped with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right)^p\right)^{\frac{1}{p}}.$$

This space was introduced by J.S. Shue [16]. It is useful in the theory of matrix operator and others (see [10] and [12]). Some geometric properties of the Cesáro sequence space ces_p were studied by many mathematicians. It is known that ces_p is LUR and posses property (H) (see [12]). Y. A. Cui and H. Hudzik [4] proved that ces_p has the Banach-Saks of type p if $p > 1$, and it was shown in [5] that ces_p has property (β).

Now, let $p = (p_k)$ be a sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$. The Nakano sequence space $l(p)$ is defined by

$$l(p) = \{x \in l^0 : \sigma(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where $\sigma(x) = \sum_{i=1}^{\infty}|x(i)|^{p_i}$. We consider the space $l(p)$ equipped with the norm

$$\|x\| = \inf\{\lambda > 0 : \sigma(\frac{x}{\lambda}) \leq 1\},$$

under which it is a Banach space. If $p = (p_k)$ is bounded, we have

$$l(p) = \{x \in l^0 : \sum_{i=1}^{\infty}|x(i)|^{p_i} < \infty\}.$$

Several geometric properties of $l(p)$ were studied in [1] and [4].

The Cesáro sequence space $ces(p)$ is defined by

$$ces(p) = \{x \in l^0 : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where $\varrho(x) = \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^{p_n}$. We consider the space $ces(p)$ equipped with the so-called Luxemburg norm

$$\|x\| = \inf\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\}$$

under which it is a Banach space. If $p = (p_k)$ is bounded, then we have

$$ces(p) = \{x = x(i) : \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^{p_n} < \infty\}.$$

W. Sanhan [15] proved that $ces(p)$ is nonsquare when $p_k > 1$ for all $k \in \mathbb{N}$. In this paper, we show that the Cesáro sequence space $ces(p)$ equipped with the Luxemburg norm is rotund (R) and posses property (H) and property (G) when $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$.

Throughout this paper we assume that $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$, and $M = \sup_k p_k$.

2. Main Results

We begin with giving some basic properties of modular on the space $ces(p)$.

Proposition 2.1 *The functional ϱ on the Cesaro sequence space $ces(p)$ is a convex modular.*

Proof. It is obvious that $\varrho(x) = 0 \Leftrightarrow x = 0$ and $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$. If $x, y \in ces(p)$ and $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta = 1$, by the convexity of the function $t \to |t|^{p_k}$ for every $k \in \mathbb{N}$, we have

$$\begin{aligned}
\varrho(\alpha x + \beta y) &= \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|\alpha x(i) + \beta y(i)|\right)^{p_k} \\
&\leq \sum_{k=1}^{\infty}\left(\alpha\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) + \beta\left(\frac{1}{k}\sum_{i=1}^{k}|y(i)|\right)\right)^{p_k} \\
&\leq \alpha\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} + \beta\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|y(i)|\right)^{p_k} \\
&= \alpha\varrho(x) + \beta\varrho(y).
\end{aligned}$$

Proposition 2.2 *For $x \in ces(p)$, the modular ϱ on $ces(p)$ satisfies the following properties :*

(i) if $0 < a < 1$, then $a^M \varrho(\frac{x}{a}) \leq \varrho(x)$ and $\varrho(ax) \leq a\varrho(x)$,

(ii) if $a \geq 1$, then $\varrho(x) \leq a^M \varrho(\frac{x}{a})$,

(iii) if $a \geq 1$, then $\varrho(x) \leq a\varrho(x) \leq \varrho(ax)$.

Proof. It is obvious that (iii) is satisfied by the convexity of ϱ . It remains to prove (i) and (ii) .

For $0 < a < 1$, we have

$$\begin{aligned}
\varrho(x) &= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \\
&= \sum_{k=1}^{\infty} \left(\frac{a}{k} \sum_{i=1}^{k} |\frac{x(i)}{a}| \right)^{p_k} \\
&= \sum_{k=1}^{\infty} a^{p_k} \left(\frac{1}{k} \sum_{i=1}^{k} |\frac{x(i)}{a}| \right)^{p_k} \\
&\geq \sum_{k=1}^{\infty} a^{M} \left(\frac{1}{k} \sum_{i=1}^{k} |\frac{x(i)}{a}| \right)^{p_k}
\end{aligned}$$

$$\begin{aligned}
&= a^M \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |\frac{x(i)}{a}| \right)^{p_k} \\
&= a^M \varrho(\frac{x}{a}),
\end{aligned}$$

and it implies by the convexity of ϱ that $\varrho(ax) \leq a\varrho(x)$, hence (i) is satisfied.

Now , suppose that $a \geq 1$. Then we have

$$\begin{aligned}
\varrho(x) &= \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} \\
&= \sum_{k=1}^{\infty} a^{p_k}\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x(i)}{a}|\right)^{p_k} \\
&\leq a^M \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x(i)}{a}|\right)^{p_k} \\
&= a^M \varrho(\frac{x}{a}).
\end{aligned}$$

So (ii) is obtained. □

Next, we give some relationships between the modular ϱ and the Luxemburg norm on $ces(p)$.

Proposition 2.3 *For any $x \in ces(p)$, we have*

(i) if $\|x\| < 1$, then $\varrho(x) \leq \|x\|$,

(ii) if $\|x\| > 1$, then $\varrho(x) \geq \|x\|$,

(iii) $\|x\| = 1$ if and only if $\varrho(x) = 1$,

(iv) $\|x\| < 1$ if and only if $\varrho(x) < 1$,

(v) $\|x\| > 1$ if and only if $\varrho(x) > 1$,

(vi) if $0 < a < 1$ and $\|x\| > a$, then $\varrho(x) > a^M$, and

(vii) if $a \geq 1$ and $\|x\| < a$, then $\varrho(x) < a^M$.

Proof. (i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - \|x\|$, so $\|x\| + \epsilon < 1$. By definition of $\|.\|$, there exists $\lambda > 0$ such that $\|x\| + \epsilon > \lambda$ *and* $\varrho(\frac{x}{\lambda}) \leq 1$. From Proposition 2.2(i) and (iii), we have

$$\begin{aligned}
\varrho(x) &\leq \varrho\left(\frac{(\|x\| + \epsilon)}{\lambda}x\right) \\
&= \varrho\left((\|x\| + \epsilon)\frac{x}{\lambda}\right) \\
&\leq (\|x\| + \epsilon)\varrho(\frac{x}{\lambda}) \\
&\leq \|x\| + \epsilon,
\end{aligned}$$

which implies that $\varrho(x) \leq \|x\|$, so (i) is satisfied.

(ii) Let $\epsilon > 0$ be such that $0 < \epsilon < \dfrac{\|x\|-1}{\|x\|}$, then $1 < (1-\epsilon)\|x\| < \|x\|$. By definition of $\|.\|$ and by Proposition 2.2 (i), we have

$$\begin{aligned} 1 &< \varrho\left(\frac{x}{(1-\epsilon)\|x\|}\right) \\ &\leq \frac{1}{(1-\epsilon)\|x\|}\varrho(x), \end{aligned}$$

so $(1-\epsilon)\|x\| < \varrho(x)$ for all $\epsilon \in (0, \dfrac{\|x\|-1}{\|x\|})$. This implies that $\|x\| \leq \varrho(x)$, hence (ii) is obtained.

(iii) Assume that $\|x\| = 1$. By definition of $\|x\|$, we have that for $\epsilon > 0$, there exists $\lambda > 0$ such that $1+\epsilon > \lambda > \|x\|$ and $\varrho(\frac{x}{\lambda}) \leq 1$. From Proposition 2.2(ii), we have $\varrho(x) \leq \lambda^M \varrho(\frac{x}{\lambda}) \leq \lambda^M < (1+\epsilon)^M$, so $(\varrho(x))^{\frac{1}{M}} < 1+\epsilon$ for all $\epsilon > 0$, which implies $\varrho(x) \leq 1$. If $\varrho(x) < 1$, then we can choose $a \in (0,1)$ such that $\varrho(x) < a^M < 1$. From Proposition 2.2(i), we have $\varrho(\frac{x}{a}) \leq \frac{1}{a^M}\varrho(x) < 1$, hence $\|x\| \leq a < 1$, which is a contradiction. Therefore $\varrho(x) = 1$.

On the other hand , assume that $\varrho(x) = 1$. Then $\|x\| \leq 1$. If $\|x\| < 1$, we have by (i) that $\varrho(x) \leq \|x\| < 1$, which contradicts our assumption. Therefore $\|x\| = 1$.

(iv) follows directly from (i) and (iii).

(v) follows from (iii) and (iv).

(vi) Suppose $0 < a < 1$ and $\|x\| > a$. Then $\left\|\dfrac{x}{a}\right\| > 1$. By (v), we have $\varrho(\dfrac{x}{a}) > 1$. Hence, by Proposition 2.2(i), we obtain that $\varrho(x) \geq a^M \varrho(\frac{x}{a}) > a^M$.

(vii) Suppose $a \geq 1$ and $\|x\| < a$. Then $\left\|\dfrac{x}{a}\right\| < 1$. By (iv), we have $\varrho(\dfrac{x}{a}) < 1$. If $a = 1$, it is obvious that $\varrho(x) < 1 = a^M$. If $a > 1$, then , by Proposition 2.2(ii), we obtain that $\varrho(x) \leq a^M \varrho(\dfrac{x}{a}) < a^M$.

Proposition 2.4 *Let (x_n) be a sequence in $ces(p)$.*

(i) If $\|x_n\| \to 1$ as $n \to \infty$, then $\varrho(x_n) \to 1$ as $n \to \infty$.

(ii) If $\varrho(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$ as $n \to \infty$.

Proof. (i) Suppose $\|x_n\| \to 1$ as $n \to \infty$. Let $\epsilon \in (0,1)$. Then there exists $N \in \mathbb{N}$ such that $1-\epsilon < \|x_n\| < 1+\epsilon$ for all $n \geq N$. By Proposition 2.3 (vi) and (vii), we have $(1-\epsilon)^M < \varrho(x_n) < (1+\epsilon)^M$ for all $n \geq N$, which implies that $\varrho(x_n) \to 1$ as $n \to \infty$.

(ii) Suppose $\|x_n\| \not\to 0$ as $n \to \infty$. Then there is an $\epsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By Proprosition 2.3 (vi), we have $\varrho(x_{n_k}) > \epsilon^M$ for all $k \in \mathbb{N}$. This implies $\varrho(x_n) \not\to 0$ as $n \to \infty$.

Next, we shall show that $ces(p)$ has the property (H). To do this, we need a lemma.

Lemma 2.5 *Let $x \in ces(p)$ and $(x_n) \subseteq ces(p)$. If $\varrho(x_n) \to \rho(x)$ as $n \to \infty$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$, then $x_n \to x$ as $n \to \infty$.*

Proof. Let $\epsilon > 0$ be given. Since $\rho(x) = \sum_{k=1}^{\infty}(\frac{1}{k}\sum_{i=1}^{k}|x(i)|)^{p_k} < \infty$, there is $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} < \frac{\epsilon}{3}\frac{1}{2^{M+1}}. \tag{2.1}$$

Since $\rho(x_n) - \sum_{k=1}^{k_0}(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|)^{p_k} \to \rho(x) - \sum_{k=1}^{k_0}(\frac{1}{k}\sum_{i=1}^{k}|x(i)|)^{p_k}$ as $n \to \infty$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that

$$\varrho(x_n) - \sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|\right)^{p_k} < \varrho(x) - \sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} + \frac{\epsilon}{3}\frac{1}{2^M} \tag{2.2}$$

for all $n \geq n_0$, and

$$\sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|\right)^{p_k} < \frac{\epsilon}{3}. \tag{2.3}$$

for all $n \geq n_0$.

It follows from (2.1), (2.2) and (2.3) that for $n \geq n_0$,

$$\begin{aligned}
\varrho(x_n - x) &= \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|\right)^{p_k} \\
&= \sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|\right)^{p_k} \\
&< \frac{\epsilon}{3} + 2^M\left(\sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k}\right) \\
&= \frac{\epsilon}{3} + 2^M\left(\varrho(x_n) - \sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k}\right) \\
&< \frac{\epsilon}{3} + 2^M\left(\varrho(x) - \sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} + \frac{\epsilon}{3}\frac{1}{2^M} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k}\right) \\
&= \frac{\epsilon}{3} + 2^M\left(\sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} + \frac{\epsilon}{3}\frac{1}{2^M} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k}\right) \\
&= \frac{\epsilon}{3} + 2^M\left(2\sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} + \frac{\epsilon}{3}\frac{1}{2^M}\right) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

This show that $\varrho(x_n - x) \to 0$ as $n \to \infty$. Hence, by Proposition 2.4 (ii), we have $\|x_n - x\| \to 0$ as $n \to \infty$.

Theorem 2.6 *The space $ces(p)$ has the property (H).*

Proof. Let $x \in S(ces(p))$ and $(x_n) \subseteq ces(p)$ such that $\|x_n\| \to 1$ and $x_n \xrightarrow{w} x$ as $n \to \infty$. From Proposition 2.3 (iii), we have $\varrho(x) = 1$, so it follows from Proposition 2.4 (i) that $\varrho(x_n) \to \varrho(x)$ as $n \to \infty$. Since the mapping $p_i : ces(p) \to \mathbb{R}$, defined by $p_i(y) = y(i)$, is a continuous linear functional on $ces(p)$, it follows that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$. Thus, we obtain by Lemma 2.5 that $x_n \to x$ as $n \to \infty$. $\square$

Theorem 2.7 *The space $ces(p)$ is rotund.*

Proof. Let $x \in S(ces(p))$ and $y, z \in B(ces(p))$ with $x = \frac{y+z}{2}$. By Proposition 2.3 and the convexity of ϱ we have

$$1 = \varrho(x) \leq \frac{1}{2}(\varrho(y) + \varrho(z)) \leq \frac{1}{2}(1+1) = 1\,,$$

so that $\varrho(x) = \frac{1}{2}(\varrho(y) + \varrho(z)) = 1$. This implies that

$$\left(\frac{1}{k}\sum_{i=1}^{k}\left|\frac{y(i)+z(i)}{2}\right|\right)^{p_k} = \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k}|y(i)|\right)^{p_k} + \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k}|z(i)|\right)^{p_k} \tag{2.4}$$

for all $k \in \mathbb{N}$.

We shall show that $y(i) = z(i)$ for all $i \in \mathbb{N}$.

From (2.4), we have

$$|x(1)|^{p_1} = \left|\frac{y(1)+z(1)}{2}\right|^{p_1} = \frac{1}{2}(|y(1)|^{p_1} + |z(1)|^{p_1}). \tag{2.5}$$

Since the mapping $t \to |t|^{p_1}$ is strictly convex, it implies by (2.5) that $y(1) = z(1)$.

Now assume that $y(i) = z(i)$ for all $i = 1, 2, 3, ..., k-1$. Then $y(i) = z(i) = x(i)$ for all $i = 1, 2, 3, ..., k-1$. From (2.4), we have

$$\begin{aligned}\left(\frac{1}{k}\sum_{i=1}^{k}\left|\frac{y(i)+z(i)}{2}\right|\right)^{p_k} &= \left(\frac{\frac{1}{k}\sum_{i=1}^{k}|y(i)| + \frac{1}{k}\sum_{i=1}^{k}|z(i)|}{2}\right)^{p_k} \\ &= \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k}|y(i)|\right)^{p_k} + \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k}|z(i)|\right)^{p_k} \end{aligned} \tag{2.6}$$

By convexity of the mapping $t \to |t|^{p_k}$, it implies that $\frac{1}{k}\sum_{i=1}^{k}|y(i)| = \frac{1}{k}\sum_{i=1}^{k}|z(i)|$. Since $y(i) = z(i)$ for all $i = 1, 2, 3, ..., k-1$, we get that

$$|y(k)| = |z(k)|. \tag{2.7}$$

If $y(k) = 0$, then we have $z(k) = y(k) = 0$. Suppose that $y(k) \neq 0$. Then $z(k) \neq 0$. If $y(k)z(k) < 0$, it follows from (2.7) that $y(k) + z(k) = 0$. This implies by

(2.6) and (2.7) that

$$\left(\frac{1}{k}\sum_{i=1}^{k-1}|x(i)|\right)^{p_k} = \left(\frac{1}{k}\left(\sum_{i=1}^{k-1}|x(i)|+|y(k)|\right)\right)^{p_k},$$

which is a contradiction. Thus, we have $y(k)z(k) > 0$. This implies by (2.5) that $y(k) = z(k)$. Thus, we have by induction that $y(i) = z(i)$ for all $i \in \mathbb{N}$, so $y = z$. □

Bor-Luh Lin , Pei-Kee Lin and S.L. Troyanski proved (cf. Theorem iii [11]) that element x in a bounded closed convex set K of a Banach space is a denting point of K iff x is an H-point of K and x is an extreme point of K. Combining this result with our results (Theorem 2.6 and Theorem 2.7), we obtain the following result.

Corollary 2.8 *The space $ces(p)$ has the property (G).*

For $1 < r < \infty$, let $p = (p_k)$ with $p_k = r$ for all $k \in \mathbb{N}$. We have that $ces_r = ces(p)$, so the following results are obtained directly from Theorem 2.6, Theorem 2.7 and Corollary 2.8, respectively.

Corollary 2.9 *For $1 < r < \infty$, the Cesáro sequence space ces_r has the property (H).*
Corollary 2.10 *For $1 < r < \infty$, the Cesáro sequence space ces_r is rotund.*
Corollary 2.11 *For $1 < r < \infty$, the Cesáro sequence space ces_r has the property (G).*

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On Some Convexity Properties of Generalized Cesáro Sequence Spaces

SUTHEP SUANTAI

ABSTRACT. **In this paper, we define a generalized Cesáro sequence space $ces(p)$ and consider it equipped with the Luxemburg norm under which it is a Banach space , and we show that the space $ces(p)$ is locally uniformly rotund (LUR), where $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$.**

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1. Preliminaries.

For a Banach space X, we denote by $S(X)$ and $B(X)$ the unit sphere and unit ball of X, respectively. A point $x_0 \in S(X)$ is called

a) an *extreme point* if for every $x, y \in S(X)$ the equality $2x_0 = x + y$ implies $x = y$;

b) a *locally uniformly rotund point* (LUR-point for short)if for any sequence (x_n) in $B(X)$ such that $\|x_n + x\| \to 2$ as $n \to \infty$ there holds $\|x_n - x\| \to 0$ as $n \to \infty$;

c) an *H-point* if for any sequence (x_n) in X such that $\|x_n\| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x_0 (write $x_n \xrightarrow{w} x_0$) implies that $\|x_n - x\| \to 0$ as $n \to \infty$;

d) a *denting point* if for every $\epsilon > 0$, $x_0 \notin \overline{conv}\{B(X)\backslash(x_0 + \epsilon B(X))\}$.

A Banach space X is said to be *rotund* (R), if every point of $S(X)$ is an extreme point.

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If every $x \in S(X)$ is a LUR-point, then X is said to be *locally uniformly rotund (LUR)*.

X is said to posses Property (H) (Property (G)) provided every ponit of $S(X)$ is H-point (denting point).

For these geometric notions and their role in Mathematics we refer to the monographs [1], [6] , [12] and [13]. Some of them were studied for Orlicz spaces in [1],[7],[8],[12] and [14].

Let X be a real vector space. A functional $\varrho : X \to [0, \infty]$ is called a *modular* if it satisfies the conditions

(i) $\varrho(x) = 0$ if and only if $x = 0$;

(ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$;

(iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$, for all $x, y \in X$ and all α, $\beta \geq 0$ with $\alpha + \beta = 1$.

The modular ϱ is called *convex* if

(iv) $\varrho(\alpha x + \beta y) \leq \alpha\varrho(x) + \beta\varrho(y)$, for all $x, y \in X$ and all α, $\beta \geq 0$ with $\alpha + \beta = 1$.

If ϱ is a modular in X, we define

$$X_\varrho = \{x \in X : \lim_{\lambda \to 0^+} \varrho(\lambda x) = 0\}$$

$$\text{and} \quad X_\varrho^* = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

It is clear that $X_\varrho \subseteq X_\varrho^*$. If ϱ is a convex modular, we define

$$\|x\| = \inf\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1\} \tag{1.1}$$

Orlicz [13] proved that if ϱ is a convex modular in X, then $X_\varrho = X_\varrho^*$ and $\|.\|$ is a norm on X_ϱ for which it is a Banach space. The norm $\|.\|$ defined as in (1.1) is called the Luxemburg norm.

A modular ϱ on X is called

(a) *right-continuous* if $\lim_{\lambda \to 1^+} \varrho(\lambda x) = \varrho(x)$ for all $x \in X_\varrho$

(b) *left-continuous* if $\lim_{\lambda \to 1^-} \varrho(\lambda x) = \varrho(x)$ for all $x \in X_\varrho$

(c) *cotinuous* if it is both right-continuous and left-continuous.

The following known results gave some relationships between the modular ϱ and the Luxemburg norm $\|.\|$ on X_ϱ.

Theorem 1.1 *Let ϱ be a convex modular on X and let $x \in X_\varrho$ and $(x - n)$ a sequence in X_ϱ. Then $\|x_n - x\| \to 0$ as $n \to \infty$ if and only if $\varrho(\lambda(x_n - x)) \to 0$ as $n \to \infty$ for every $\lambda > 0$.*
Proof. See [11, Theorem 1.3].

Theorem 1.2 *Let ϱ be a convex modular on X. Then*

(i) $\|x\| < 1$ *if and only if* $\varrho(x) < 1$.
(ii) $\|x\| \leq 1$ *if and only if* $\varrho(x) \leq 1$.
(iii) $\|x\| = 1$ *if and only if* $\varrho(x) = 1$.

Proof. See [11, Theorem 1.4].

Let us denote by l^0 the space of all real sequences. For $1 \leq p < \infty$, the Cesáro sequence space (ces_p, for short) is defined by

$$ces_p = \{x \in l^0 : \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^p < \infty\}$$

equipped with the norm

$$\|x\| = (\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^p)^{\frac{1}{p}}$$

This space was introduced by J.S. Shue [16]. It is useful in the theory of matrix operator and others (see [9] and [10]). Some geometric properties of the Cesáro sequence space ces_p were studied by many mathematicians. It is known that ces_p is LUR and posses property (H) (see [10]). Y. A. Cui and H. Hudzik [2] proved that ces_p has the Banach-Saks of type p if $p > 1$, and it was shown in [5] that ces_p has property (β).

Now, let $p = (p_k)$ be a sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$. The Nakano sequence space $l(p)$ is defined by

$$l(p) = \{x \in l^0 : \sigma(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where $\sigma(x) = \sum_{i=1}^{\infty}|x(i)|^{p_i}$. We consider the space $l(p)$ equipped with the norm

$$\|x\| = \inf\{\lambda > 0 : \sigma(\frac{x}{\lambda}) \leq 1\},$$

under which it is a Banach space. If $p = (p_k)$ is bounded, we have

$$l(p) = \{x \in l^0 : \sum_{i=1}^{\infty} |x(i)|^{p_i} < \infty\}.$$

Several geometric properties of $l(p)$ were studied in [1] and [4].

The Cesáro sequence space $ces(p)$ is defined by

$$ces(p) = \{x \in l^0 : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where $\varrho(x) = \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^{p_n}$. We consider this space equipped with the so-called Luxemburg norm

$$\|x\| = \inf\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\}$$

under which it is a Banach space. If $p = (p_k)$ is bounded, we have

$$ces(p) = \{x = x(i) : \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^{p_n} < \infty\}.$$

W. Sanhan [15] proved that $ces(p)$ is nonsquare when $p_k > 1$ for all $k \in \mathbb{N}$. In this paper, we show that the Cesáro sequence space $ces(p)$ equipped with the Luxemburg norm is LUR and posses property (H) and property (G) when $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$.

Throughout this paper we assume that $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$, and $M = \sup_k p_k$.

2. Main Results

We begin with giving some basic properties of modular on the space $ces(p)$.

Proposition 2.1 *The functional ϱ on the Cesaro sequence space $ces(p)$ is a convex modular.*

Proof. It is obvious that $\varrho(x) = 0 \Leftrightarrow x = 0$ and $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$. If $x, y \in ces(p)$ and $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta = 1$, by the convexity of the function $t \to |t|^{p_k}$ for every $k \in \mathbb{N}$, we have

$$
\begin{aligned}
\varrho(\alpha x + \beta y) &= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |\alpha x(i) + \beta y(i)| \right)^{p_k} \\
&\leq \sum_{k=1}^{\infty} \left(\alpha \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right) + \beta \left(\frac{1}{k} \sum_{i=1}^{k} |y(i)| \right) \right)^{p_k} \\
&\leq \alpha \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \beta \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |y(i)| \right)^{p_k} \\
&= \alpha \varrho(x) + \beta \varrho(y).
\end{aligned}
$$

Proposition 2.2 *For $x \in ces(p)$, the modular ϱ on $ces(p)$ satisfies the following properties :*

(i) if $0 < a < 1$, then $a^M \varrho(\frac{x}{a}) \leq \varrho(x)$ and $\varrho(ax) \leq a\varrho(x)$,

(ii) if $a \geq 1$, then $\varrho(x) \leq a^M \varrho(\frac{x}{a})$,

(iii) if $a \geq 1$, then $\varrho(x) \leq a\varrho(x) \leq \varrho(ax)$.

Proof. It is obvious that (iii) is satisfied by the convexity of ϱ . It remains to prove (i) and (ii) .

For $0 < a < 1$, we have

$$
\begin{aligned}
\varrho(x) &= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \\
&= \sum_{k=1}^{\infty} \left(\frac{a}{k} \sum_{i=1}^{k} |\frac{x(i)}{a}| \right)^{p_k} \\
&= \sum_{k=1}^{\infty} a^{p_k} \left(\frac{1}{k} \sum_{i=1}^{k} |\frac{x(i)}{a}| \right)^{p_k} \\
&\geq \sum_{k=1}^{\infty} a^{M} \left(\frac{1}{k} \sum_{i=1}^{k} |\frac{x(i)}{a}| \right)^{p_k}
\end{aligned}
$$

$$= a^M \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |\frac{x(i)}{a}| \right)^{p_k}$$
$$= a^M \varrho(\frac{x}{a}),$$

and it implies by the convexity of ϱ that $\varrho(ax) \leq a\varrho(x)$, hence (i) is satisfied.
Now , suppose that $a \geq 1$. Then we have

$$\begin{aligned}
\varrho(x) &= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \\
&= \sum_{k=1}^{\infty} a^{p_k} (\frac{1}{k} \sum_{i=1}^{k} |\frac{x(i)}{a}|)^{p_k} \\
&\leq a^M \sum_{k=1}^{\infty} (\frac{1}{k} \sum_{i=1}^{k} |\frac{x(i)}{a}|)^{p_k} \\
&= a^M \varrho(\frac{x}{a}).
\end{aligned}$$

So (ii) is obtained. □

Proposition 2.3 *The modular ϱ on $ces(p)$ is continuous.*
Proof. For $\lambda > 1$, by Proposition 2.2 (ii) and (iii), we have

$$\varrho(x) \leq \lambda\varrho(x) \leq \varrho(\lambda x) \leq \lambda^M \varrho(x) \tag{2.1}$$

By taking $\lambda \to 1^+$ in (2.1), we have $\lim_{\lambda \to 1^+} \varrho(\lambda x) = \varrho(x)$. Thus ϱ is right-continuous. If $0 < \lambda < 1$, by Proposition 2.2 (i), we have

$$\lambda^M \varrho(x) \leq \varrho(\lambda x) \leq \lambda\varrho(x) \tag{2.2}$$

By taking $\lambda \to 1^-$ in (2.2), we have that $\lim_{\lambda \to 1^-} \varrho(\lambda x) = \varrho(x)$, hence, ϱ is left-continuous. Thus ϱ is continuous.

Next, we give some relationships between the modular ϱ and the Luxemburg norm on $ces(p)$.

Proposition 2.4 *For any $x \in ces(p)$, we have*

(i) if $\|x\| < 1$, then $\varrho(x) \leq \|x\|$,

(ii) if $\|x\| > 1$, then $\varrho(x) \geq \|x\|$,

(iii) $\|x\| = 1$ if and only if $\varrho(x) = 1$,

(iv) $\|x\| < 1$ if and only if $\varrho(x) < 1$,

(v) $\|x\| > 1$ if and only if $\varrho(x) > 1$,

(vi) if $0 < a < 1$ and $\|x\| > a$, then $\varrho(x) > a^M$, and

(vii) if $a \geq 1$ and $\|x\| < a$, then $\varrho(x) < a^M$.

Proof. (i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - \|x\|$, so $\|x\| + \epsilon < 1$. By definition of $\|.\|$, there exists $\lambda > 0$ such that $\|x\| + \epsilon > \lambda$ *and* $\varrho(\frac{x}{\lambda}) \leq 1$. From Proposition 2.2(i) and (iii), we have

$$\begin{aligned}\varrho(x) &\leq \varrho\left(\frac{(\|x\| + \epsilon)}{\lambda}x\right)\\ &= \varrho\left((\|x\| + \epsilon)\frac{x}{\lambda}\right)\\ &\leq (\|x\| + \epsilon)\varrho(\frac{x}{\lambda})\\ &\leq \|x\| + \epsilon,\end{aligned}$$

which implies that $\varrho(x) \leq \|x\|$, so (i) is satisfied.

(ii) Let $\epsilon > 0$ be such that $0 < \epsilon < \dfrac{\|x\| - 1}{\|x\|}$, then $1 < (1-\epsilon)\|x\| < \|x\|$. By definition of $\|.\|$ and by Proposition 2.2 (i), we have

$$\begin{aligned}1 &< \varrho\left(\frac{x}{(1-\epsilon)\|x\|}\right)\\ &\leq \frac{1}{(1-\epsilon)\|x\|}\varrho(x),\end{aligned}$$

so $(1-\epsilon)\|x\| < \varrho(x)$ for all $\epsilon \in (0, \dfrac{\|x\| - 1}{\|x\|})$. This implies that $\|x\| \leq \varrho(x)$, hence (ii) is obtained.

(iii) and (iv) follow directly from Theorem 1.2.

(iv) follows directly from (i) and (iii).

(v) follows from (iii) and (iv).

(vi) Suppose $0 < a < 1$ and $\|x\| > a$. Then $\left\|\dfrac{x}{a}\right\| > 1$. By (v), we have $\varrho(\dfrac{x}{a}) > 1$. Hence, by Proposition 2.2(i), we obtain that $\varrho(x) \geq a^M \varrho(\frac{x}{a}) > a^M$.

(vii) Suppose $a \geq 1$ and $\|x\| < a$. Then $\left\|\frac{x}{a}\right\| < 1$. By (iv), we have $\varrho(\frac{x}{a}) < 1$. If $a = 1$, it is obvious that $\varrho(x) < 1 = a^M$. If $a > 1$, then , by Proposition 2.2(ii), we obtain that $\varrho(x) \leq a^M \varrho(\frac{x}{a}) < a^M$.

Proposition 2.5 *Let (x_n) be a sequence in $ces(p)$.*

(i) If $\|x_n\| \to 1$ as $n \to \infty$, then $\varrho(x_n) \to 1$ as $n \to \infty$.

(ii) $\|x_n\| \to 0$ as $n \to \infty$ if and only if $\varrho(x_n) \to 0$ as $n \to \infty$.

Proof. (i) Suppose $\|x_n\| \to 1$ as $n \to \infty$. Let $\epsilon \in (0,1)$. Then there exists $N \in \mathbb{N}$ such that $1 - \epsilon < \|x_n\| < 1 + \epsilon$ for all $n \geq N$. By Proposition 2.4 (vi) and (vii), we have $(1-\epsilon)^M < \varrho(x_n) < (1+\epsilon)^M$ for all $n \geq N$, which implies that $\varrho(x_n) \to 1$ as $n \to \infty$.
(ii) The only part of (ii) is true by Theorem 1.1, so we need to show only the if part of (ii). Suppose $\|x_n\| \not\to 0$ as $n \to \infty$. Then there is an $\epsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By Proprosition 2.4 (vi), we have $\varrho(x_{n_k}) > \epsilon^M$ for all $k \in \mathbb{N}$. This implies $\varrho(x_n) \not\to 0$ as $n \to \infty$.

Proposition 2.6 *Let $(x_n) \subseteq B(l(p))$ and $(y_n) \subseteq B(l(p))$. If $\sigma(\frac{x_n + y_n}{2}) \to 1$, then $x_n(i) - y_n(i) \to 0$ as $n \to \infty$ for all $i \in \mathbb{N}$.*

Proof . We first note that if $x \in B(\ell(p)$, then $\sigma(x) \leq 1$. Supose that $x_n(i) - y(i) \not\to 0$ as $n \to \infty$ for some $i \in \mathbb{N}$. Without loss of generality we may assume that $i = 1$, and then assume that , for some $\epsilon > 0$,

$$|x_n(1) - y_n(1)|^{p_1} \geq \epsilon \ \forall n \in \mathbb{N}$$

Thus

$$2^{p_1}(|x_n(1)|^{p_1} + |y_n(1)|^{p_1}) \geq \epsilon \ \forall n \in \mathbb{N}. \tag{2.3}$$

Since the function $t \to |t|^{p_1}$ is uniformly convex, there exists $\delta > 0$ such that

$$\left|\frac{x_n(1) + y_n(1)}{2}\right|^{p_1} \leq (1-\delta)\left(\frac{|x_n(1)|^{p_1} + |y_n(1)|^{p_1}}{2}\right) \ \forall n \in \mathbb{N}. \tag{2.4}$$

It follows from (2.3) and (2.4) that for each $n \in \mathbb{N}$,

$$\begin{aligned}
\sigma(\frac{x_n + y_n}{2}) &= \sum_{i=1}^{\infty} \left| \frac{x_n(i) + y_n(i)}{2} \right|^{p_i} \\
&= \left| \frac{x_n(1) + y_n(1)}{2} \right|^{p_1} + \sum_{i=2}^{\infty} \left| \frac{x_n(i) + y_n(i)}{2} \right|^{p_i} \\
&\leq (1-\delta) \left(\frac{|x_n(1)|^{p_1} + |y_n(1)|^{p_1}}{2} \right) + \frac{1}{2} \sum_{i=2}^{\infty} |x_n(i)|^{p_i} + \frac{1}{2} \sum_{i=2}^{\infty} |y_n(i)|^{p_i} \\
&= \frac{1}{2}\sigma(x_n) + \frac{1}{2}\sigma(y_n) - \delta \left(\frac{|x_n(1)|^{p_1} + |y_n(1)|^{p_1}}{2} \right) \\
&\leq \frac{1}{2} + \frac{1}{2} - \delta \frac{\epsilon}{2^{p_1+1}} \\
&= 1 - \delta \frac{\epsilon}{2^{p_1+1}}.
\end{aligned}$$

This implies that $\sigma(\frac{x_n + y_n}{2}) \not\to 1$ as $n \to \infty$.

Proposition 2.7 *Let $(x_n) \subseteq B(ces(p))$ and $x \in S(ces(p))$. If $\varrho(\frac{x_n + x}{2}) \to 1$ as $n \to \infty$, then $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$.*

Proof. For each $n \in \mathbb{N}$ and $i \in \mathbb{N}$, let

$$s_n(i) = \begin{cases} sgn(x_n(i) + x(i)) \text{ if } x_n(i) + x(i) \neq 0, \\ 1 \text{ if } x_n(i) + x(i) = 0. \end{cases}$$

Hence, we have

$$\begin{aligned}
1 \leftarrow \varrho(\frac{x_n + x}{2}) &= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} \left| \frac{x_n(i) + x(i)}{2} \right| \right)^{p_k} \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} s_n(i) \frac{x_n(i)}{2} + \frac{1}{k} \sum_{i=1}^{k} s_n(i) \frac{x(i)}{2} \right)^{p_k} \qquad (2.5)
\end{aligned}$$

Let $a_n(k) = \frac{1}{k} \sum_{i=1}^{k} s_n(i) x_n(i)$ and $b_n(k) = \frac{1}{k} \sum_{i=1}^{k} s_n(i) x(i)$ for all $n, k \in \mathbb{N}$. Then $(a_n) \in l(p)$ and $(b_n) \in l(p)$, and from (2.5) we have

$$\sigma(\frac{a_n + b_n}{2}) \to 1 \quad \text{as } n \to \infty.$$

Form Proposition 2.5, we have

$$a_n(i) - b_n(i) \to 0 \text{ as } n \to \infty \tag{2.6}$$

for all $i \in \mathbb{N}$. Now, we shall show that $x_n(k) \to x(k)$ as $n \to \infty$ for all $k \in \mathbb{N}$. From (2.6), we have

$$s_n(1)x_n(1) - s_n(1)x(1) \to 0 \text{ as } n \to \infty,$$

this implies $x_n(1) \to x(1)$ as $n \to \infty$. Assume that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \leq k-1$. Then we have

$$s_n(i)(x_n(i) - x(i)) \to 0 \text{ as } n \to \infty \tag{2.7}$$

for all $i \leq k-1$. Since $s_n(k)(x_n(k) - x(k)) = k(a_n(k) - b_n(k)) - \sum_{i=1}^{k-1} s_n(i)(x_n(i) - x(i))$, it follows from (2.6) and (2.7) that $s_n(k)(x_n(k) - x(k)) \to 0$ as $n \to \infty$. This implies $x_n(k) \to x(k)$ as $n \to \infty$. So we have by induction that $x_n(k) \to x(k)$ as $n \to \infty$ for all $k \in \mathbb{N}$.

Theorem 2.8 *The space $ces(p)$ is LUR.*

Proof. Let $(x_n) \subseteq B(ces(p))$ and $x \in S(ces(p))$ be such that $\|x_n + x\| \to 2$ as $n \to \infty$. Then $\|\frac{x_n + x}{2}\| \to 1$ as $n \to \infty$. By Proposition 2.5 (i), we have $\varrho(\frac{x_n + x}{2}) \to 1$ as $n \to \infty$. By Proposition 2.7, we have $x_n(i) \to x(i)$ as $n \to \infty$ $\forall i \in \mathbb{N}$.

Now, let $\epsilon > 0$ be given. Then there exists $k_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} < \frac{\epsilon}{3} \frac{1}{2^{M+1}}, \tag{2.8}$$

$$\sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^{p_k} < \frac{\epsilon}{3} \quad \text{for all } n \geq n_0, \tag{2.9}$$

$$\sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i)| \right)^{p_k} > \sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} - \frac{\epsilon}{3} \frac{1}{2^M}. \tag{2.10}$$

By Proposition 2.4 (i) and (iii), we have $\varrho(x_n) \leq 1$ for all $n \in \mathbb{N}$ and $\varrho(x) = 1$. From these together with (2.8), (2.9), (2.10) and the fact that $(a+b)^{p_k} \leq 2^{p_k}(a^{p_k} + b^{p_k})$ for

$a, b \geq 0$, we have that for all $n \geq n_0$,

$$\begin{aligned}
\varrho(x_n - x) &= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^{p_k} \\
&= \sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^{p_k} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^{p_k} \\
&< \frac{\epsilon}{3} + 2^M \left(\sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i)| \right)^{p_k} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \right) \\
&= \frac{\epsilon}{3} + 2^M \left(\varrho(x_n) - \sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i)| \right)^{p_k} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \right) \\
&\leq \frac{\epsilon}{3} + 2^M \left(1 - \sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i)| \right)^{p_k} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \right) \\
&< \frac{\epsilon}{3} + 2^M \left(1 - \sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \frac{\epsilon}{3} \frac{1}{2^M} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \right) \\
&= \frac{\epsilon}{3} + 2^M \left(\varrho(x) - \sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \frac{\epsilon}{3} \frac{1}{2^M} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \right) \\
&= \frac{\epsilon}{3} + 2^M \left(\sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \frac{\epsilon}{3} \frac{1}{2^M} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \right) \\
&= \frac{\epsilon}{3} + 2^M \left(2 \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \frac{\epsilon}{3} \frac{1}{2^M} \right) \\
&= \frac{\epsilon}{3} + 2^{M+1} \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \frac{\epsilon}{3} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

This shows that $\varrho(x_n - x) \to 0$ as $n \to \infty$. By Proposition 2.4(ii) , we have $\|x_n - x\| \to 0$ as $n \to \infty$. This completes the proof of the theorem. $\square$

It is known in general that $LUR \Rightarrow$ property (H). So we have the following result.

Corollary 2.9 The space $ces(p)$ posses property (H).

Bor-Luh Lin , Pei-Kee Lin and S.L. Troyanski proved (cf. Theorem iii [11]) that element x in a bounded closed convex set K of a Banach space is a denting point of K iff x is an H-point of K and x is an extreme point of K. Combining this result with our results (Theorem 2.8 and Corollary 2.9) and the general fact that $LUR \Rightarrow R$, we obtain the following result.

Corollary 2.10 *The space $ces(p)$ has the property (G).*

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On Property (H) and Rotundity of Difference Sequence Spaces

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ABSTRACT. **In this paper, we define a modular on difference sequence space $\ell(\Delta, p)$ and consider it equipped with the Luxemburg norm induced by the modular, where $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$. The main purpose of this paper is to show that $\ell(\Delta, p)$ has property (H) and we also show that $\ell(\Delta, p)$ is rotund if and only if $p_k > 1$ for all $k \in \mathbb{N}$.**

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1. Introduction.

Convexity properties in Banach space is an important topic in functional analysis and play an important role in infinite dimensional holomorphy. In order to study the geometric properties of Banach space, Clarkson [5] introduced the very important class of rotundity (strict convexity). Since Clarkson's paper many authors have defined and studied the classes of Bañach space lying between the uniform convexity and rotundity (see [2, 3, 5, 12, 14, 17].)

Among geometrical properties, property **(H)** in Banach spaces is important and it has been studied by various authors. Criteria of property **(H)** in Orlicz spaces and Orlicz sequence spaces were given by S. Chen and Y. Wang [4] and C. Wu, S. Chen and Y. Wang [20]. R. Pluciennik, T. Wang and Y. Zhang [19] obtained necessary and sufficient conditions for H- points and denting points in Orlicz sequence spaces. In [7], criteria for property **(H)** is given in Musielak-Orlicz sequence spaces.

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In this paper, we introduce the difference sequence space $\ell(\Delta, p)$, when $p = (p_k)$ is a bounded sequence of positive real number with $p_k \geq 1$ for all $k \in \mathbb{N}$, and consider it equipped the Luxemburg norm. We show that $\ell(\Delta, p)$ has property **(H)** and criteria for rotundity is given.

Now we introduced the basic notations and definitions. In the following, Let $\mathbb{R}$ be the real line and $\mathbb{N}$ the set of natural numbers.

For a Banach space X, we denote by $S(X)$ and $B(X)$ the unit sphere and unit ball of X, respectively. A point $x_0 \in S(X)$ is called

a) an *extreme point* if for every $x, y \in S(X)$ the equality $2x_0 = x + y$ implies $x = y$;

b) an *H-point* if for any sequence (x_n) in X such that $\|x_n\| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x_0 (write $x_n \xrightarrow{w} x_0$) implies that $\|x_n - x\| \to 0$ as $n \to \infty$;

A Banach space X is said to be *rotund* (R), if every point of $S(X)$ is an extreme point. X is said to posses property (H) provided every point of $S(X)$ is H-point .

For these geometric notions and their role in Mathematics we refer to the monographs [2], [8], and [17]. Some of them were studied for Orlicz spaces in [3], [6], [9], [10], [11], [19], and [20].

Let X be a real vector space. A functional $\varrho : X \to [0, \infty]$ is called a *modular* if it satisfies the conditions

(i) $\varrho(x) = 0$ if and only if $x = 0$;

(ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$;

(iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$, for all $x, y \in X$ and all α, $\beta \geq 0$ with $\alpha + \beta = 1$.

The modular ϱ is called *convex* if

(iv) $\varrho(\alpha x + \beta y) \leq \alpha\varrho(x) + \beta\varrho(y)$, for all $x, y \in X$ and all α, $\beta \geq 0$ with $\alpha + \beta = 1$.

If ϱ is a modular in X, we define

$$X_\varrho = \{x \in X : \lim_{\lambda \to 0^+} \varrho(\lambda x) = 0\},$$

$$\text{and} \quad X_\varrho^* = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

It is clear that $X_\varrho \subseteq X_\varrho^*$. If ϱ is a convex modular, we define

$$\|x\| = \inf\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1\}. \tag{1.1}$$

Orlicz [18] proved that if ϱ is a convex modular in X, then $X_\varrho = X_\varrho^*$ and $\|.\|$ is a norm on X_ϱ for which it is a Banach space. The norm $\|.\|$ defined as in (1.1) is called the Luxemburg norm.

A modular ϱ on X is called

(a) *right-continuous* if $\lim_{\lambda\to 1^+} \varrho(\lambda x) = \varrho(x)$ for all $x \in X_\varrho$,

(b) *left-continuous* if $\lim_{\lambda\to 1^-} \varrho(\lambda x) = \varrho(x)$ for all $x \in X_\varrho$,

(c) *cotinuous* if it is both right-continuous and left-continuous.

The following known results gave some relationships between the modular ϱ and the Luxemburg norm $\|.\|$ on X_ϱ.

Theorem 1.1 *Let ϱ be a convex modular on X and let $x \in X_\varrho$ and (x_n) a sequence in X_ϱ. Then $\|x_n - x\| \to 0$ as $n \to \infty$ if and only if $\varrho(\lambda(x_n - x)) \to 0$ as $n \to \infty$ for every $\lambda > 0$.*

Proof. See [16, Theorem 1.3].

Theorem 1.2 *Let ϱ be a continuous convex modular on X. Then*

(i) $\|x\| < 1$ if and only if $\varrho(x) < 1$.

(ii) $\|x\| \le 1$ if and only if $\varrho(x) \le 1$.

(iii) $\|x\| = 1$ if and only if $\varrho(x) = 1$.

Proof. See [16, Theorem 1.4].

Let us denoted by ℓ^0 the space of all real sequences and let $p = (p_k)$ be a sequence of positive real numbers. In [13], Kizmaz introduced the sequence spaces $\ell_\infty(\Delta)$, $c_0(\Delta)$ and $c(\Delta)$ by considering the difference sequence $\Delta x = (x_k - x_{k+1})_{k=1}^\infty$ for any sequence $x \in l^0$, where ℓ_∞, c_0 an c are Banach spaces of bounded, null and convergent sequences, respectively. In [1], these sequence spaces were extended to $\ell_\infty(\Delta, p)$, $c_0(\Delta, p)$ and $c(\Delta, p)$, e.g.

$$\ell_\infty(\Delta, p) = \{x \in l^0 \ : \ \Delta x \in \ell_\infty(p) \ \}$$

where

$$\ell_\infty(p) = \{x \in l^0 \ : \ \sup_k |x_k|^{p_k} < \infty \ \}.$$

In [1] and [13] the authers determined the Köthe-Töeplitz and generalized Köthe-Töeplitz duals of these spaces and consider various matrix transformations.

In this paper we introduced the space $\ell(\Delta, p)$ analogously as follows and study some of its geometric properties.

$$\ell(\Delta, p) = \{x \in l^0 \ : \ \Delta x \in \ell(p) \ \},$$

where

$$\ell(p) = \{x \in l^0 \ : \ \sum_{k=1}^{\infty} |x(k)|^{p_k} < \infty \ \}.$$

For the detail of spaces $\ell_\infty(p)$ and $\ell(p)$, we refer to [15].

For $x \in \ell(\Delta, p)$, we define

$$\varrho_p(x) = |x(1)| + \sum_{k=1}^{\infty} |x(k) - x(k+1)|^{p_k}$$

If $p_k \geq 1$ for all $k \in \mathbb{N}$, by convexity of the functions $t \mapsto |t|^{p_k}$ for each $k \in \mathbb{N}$, we have that ϱ_p is a convex modular on $\ell(\Delta, p)$. We consider $\ell(\Delta, p)$ equipped with the Luxemburg norm given by

$$\|x\| = inf\{\varepsilon > 0 : \varrho_p(\frac{x}{\varepsilon}) \leq 1\}.$$

A normed sequence space S is said to be a *K-space* if each coordinate mapping P_k, defined by $P_k(x) = x_k$, is continuous. If S is both Banach and K-space, it is called a BK-space.

Throughout this paper we let $M = sup_k \, p_k$ and assume that $p_k \geq 1$ for all $k \in \mathbb{N}$.

2. Main Results

We begin with giving some basic properties of modular on the space $\ell(\Delta, p)$.

Proposition 2.1 *For $x \in \ell(\Delta,p)$, the modular ϱ_p on $\ell(\Delta,p)$ satisfies the following properties :*

(i) if $0 < a \leq 1$, then $a^M \varrho_p(\frac{x}{a}) \leq \varrho_p(x)$ and $\varrho_p(ax) \leq a\varrho_p(x)$,

(ii) if $a \geq 1$, then $\varrho_p(x) \leq a^M \varrho_p(\frac{x}{a})$,

(iii) if $a \geq 1$, then $\varrho_p(x) \leq a\varrho_p(x) \leq \varrho_p(ax)$.

Proof. It is obvious that (iii) is satisfied by the convexity of ϱ_p. It remains to prove (i) and (ii).

For $0 < a \leq 1$, we have

$$\begin{aligned}
\varrho_p(x) &= |x(1)| + \sum_{k=1}^{\infty} |x(k) - x(k+1)|^{p_k} \\
&= a\left|\frac{x(1)}{a}\right| + \sum_{k=1}^{\infty}\left|\frac{a(x(k) - x(k+1))}{a}\right|^{p_k} \\
&\geq a^M\left|\frac{x(1)}{a}\right| + a^M\sum_{k=1}^{\infty}\left|\frac{x(k) - x(k+1)}{a}\right|^{p_k} \\
&= a^M \varrho_p(\frac{x}{a}).
\end{aligned}$$

It follows by the convexity of ϱ that $\varrho_p(ax) \leq a\varrho_p(x)$, hence (i) is satisfied.
Now, suppose that $a \geq 1$. Then $\frac{1}{a} \leq 1$. It follows from (i) that

$$\left(\frac{1}{a}\right)^M \varrho_p(x) = \left(\frac{1}{a}\right)^M \varrho_p\left(\frac{x/a}{1/a}\right) \leq \varrho_p\left(\frac{x}{a}\right),$$

so that $\varrho_p(x) \leq a^M \varrho_p\left(\frac{x}{a}\right)$, hence (ii) is obtained. □

Proposition 2.2 *The modular ϱ_p on $\ell(\Delta,p)$ is continuous.*

Proof. For $\lambda > 1$, by Proposition 2.1 (ii) and (iii), we have

$$\varrho_p(x) \leq \lambda\varrho_p(x) \leq \varrho_p(\lambda x) \leq \lambda^M \varrho_p(x) \tag{2.1}$$

By taking $\lambda \to 1^+$ in (2.1), we have $\lim_{\lambda\to 1^+} \varrho_p(\lambda x) = \varrho_p(x)$. Thus ϱ_p is right-continuous. If $0 < \lambda < 1$, by Proposition 2.1 (i), we have

$$\lambda^M \varrho_p(x) \leq \varrho_p(\lambda x) \leq \lambda \varrho_p(x) \tag{2.2}$$

By taking $\lambda \to 1^-$ in (2.2), we have that $\lim_{\lambda\to 1^-} \varrho_p(\lambda x) = \varrho_p(x)$, hence, ϱ_p is left-continuous. Thus ϱ_p is continuous. $\square$

Next, we give some relationships between the modular ϱ_p and the Luxemburg norm on $\ell(\Delta, p)$.

Proposition 2.3 *For any $x \in \ell(\Delta, p)$, we have*

(i) if $\|x\| < 1$, then $\varrho_p(x) \leq \|x\|$,

(ii) if $\|x\| > 1$, then $\varrho_p(x) \geq \|x\|$,

(iii) $\|x\| = 1$ if and only if $\varrho_p(x) = 1$,

(iv) $\|x\| < 1$ if and only if $\varrho_p(x) < 1$,

(v) $\|x\| > 1$ if and only if $\varrho_p(x) > 1$,

(vi) if $0 < a < 1$ and $\|x\| > a$, then $\varrho_p(x) > a^M$, and

(vii) if $a \geq 1$ and $\|x\| < a$, then $\varrho_p(x) < a^M$.

Proof. (i) Let $\epsilon > 0$ be such that $0 < \epsilon < 1 - \|x\|$, so $\|x\| + \epsilon < 1$. By definition of $\|\cdot\|$, there exists $\lambda > 0$ such that $\|x\| + \epsilon > \lambda$ *and* $\varrho(\frac{x}{\lambda}) \leq 1$. From Proposition 2.1(i) and (iii), we have

$$\begin{aligned}\varrho_p(x) &\leq \varrho_p\left(\frac{(\|x\|+\epsilon)}{\lambda}x\right)\\ &= \varrho_p\left((\|x\|+\epsilon)\frac{x}{\lambda}\right)\\ &\leq (\|x\|+\epsilon)\varrho(\frac{x}{\lambda})\\ &\leq \|x\|+\epsilon,\end{aligned}$$

which implies that $\varrho_p(x) \leq \|x\|$, so (i) is satisfied.

(ii) Let $\epsilon > 0$ be such that $0 < \epsilon < \dfrac{\|x\| - 1}{\|x\|}$, then $1 < (1-\epsilon)\|x\| < \|x\|$. By definition of $\|\cdot\|$ and by Proposition 2.1 (i), we have

$$\begin{aligned} 1 &< \varrho_p\left(\frac{x}{(1-\epsilon)\|x\|}\right) \\ &\leq \frac{1}{(1-\epsilon)\|x\|}\varrho_p(x), \end{aligned}$$

so $(1-\epsilon)\|x\| < \varrho_p(x)$ for all $\epsilon \in (0, \dfrac{\|x\|-1}{\|x\|})$. This implies that $\|x\| \leq \varrho_p(x)$, hence (ii) is obtained.

Since ϱ_p is continuous (Proposition 2.2) , (iii) and (iv) follow directly from Theorem 1.2.

(iv) follows directly from (i) and (iii).

(v) follows from (iii) and (iv).

(vi) Suppose $0 < a < 1$ and $\|x\| > a$. Then $\left\|\dfrac{x}{a}\right\| > 1$. By (v), we have $\varrho_p(\dfrac{x}{a}) > 1$. Hence, by Proposition 2.1(i), we obtain that $\varrho_p(x) \geq a^M \varrho_p(\frac{x}{a}) > a^M$.

(vii) Suppose $a \geq 1$ and $\|x\| < a$. Then $\left\|\dfrac{x}{a}\right\| < 1$. By (iv), we have $\varrho_p(\dfrac{x}{a}) < 1$. If $a = 1$, it is obvious that $\varrho_p(x) < 1 = a^M$. If $a > 1$, by Proposition 2.1(ii), we obtain that $\varrho_p(x) \leq a^M \varrho_p(\dfrac{x}{a}) < a^M$. □

Proposition 2.4 *Let (x_n) be a sequence in $\ell(\Delta, p)$.*

(i) If $\|x_n\| \to 1$ as $n \to \infty$, then $\varrho_p(x_n) \to 1$ as $n \to \infty$.

(ii) $\|x_n\| \to 0$ as $n \to \infty$ if and only if $\varrho_p(x_n) \to 0$ as $n \to \infty$.

Proof. (i) Suppose $\|x_n\| \to 1$ as $n \to \infty$. Let $\epsilon \in (0,1)$. Then there exists $N \in \mathbb{N}$ such that $1 - \epsilon < \|x_n\| < 1 + \epsilon$ for all $n \geq N$. By Proposition 2.3 (vi) and (vii), we have $(1-\epsilon)^M < \varrho_p(x_n) < (1+\epsilon)^M$ for all $n \geq N$, which implies that $\varrho_p(x_n) \to 1$ as $n \to \infty$.
(ii) The only part of (ii) is true by Theorem 1.1, so we need to show only the if part of (ii). Suppose $\|x_n\| \not\to 0$ as $n \to \infty$. Then there is an $\epsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By Proprosition 2.3 (vi), we have $\varrho_p(x_{n_k}) > \epsilon^M$ for all $k \in \mathbb{N}$. This implies $\varrho_p(x_n) \not\to 0$ as $n \to \infty$. □

Next, we shall show that $\ell(\Delta, p)$ has the property (H). To do this, we need two lemmas.

Lemma 2.5 *The space $\ell(\Delta, p)$ is a BK-space.*

Proof. Since $\ell(\Delta, p)$ equipped with the Luxemburg norm is Banach, we need to show only that $\ell(\Delta, p)$ is a $K-$space. Suppose $(x_n) \subset \ell(\Delta, p)$ suth that $x_n \to 0$ as $n \to \infty$. It follows by Proposition 2.4(ii) that $\varrho_p(x_n) \to 0$ as $n \to \infty$. This implies that

$$|x_n(1)| \to 0 \quad \text{as} \quad n \to \infty \quad \text{and}$$

$$|x_n(k) - x_n(k+1)| \to 0 \ \text{ as } n \to \infty \ \text{ for all } \ k \in \mathbb{N}.$$

By induction, we have $x_n(k) \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}$. Hence $P_k(x_n) \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}$. This implies that P_k is continuous for all $k \in \mathbb{N}$.

Lemma 2.6 *Let $x \in \ell(\Delta, p)$ and $(x_n) \subseteq \ell(\Delta, p)$. If $\varrho_p(x_n) \to \varrho_p(x)$ as $n \to \infty$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$, then $x_n \to x$ as $n \to \infty$.*

Proof. Let $\epsilon > 0$ be given. Since $\varrho_p(x) = |x(1)| + \sum_{k=1}^{\infty} |x(k) - x(k+1)|^{p_k} < \infty$, there is $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty} |x(k) - x(k+1)|^{p_k} < \frac{\epsilon}{3} \cdot \frac{1}{2^{M+1}}. \tag{2.3}$$

Since $\varrho_p(x_n) \to \varrho_p(x)$ as $n \to \infty$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\varrho_p(x_n) - \left(|x_n(1)| + \sum_{k=1}^{k_0} |x_n(k) - x_n(k+1)|^{p_k} \right) < \varrho_p(x) - \left(|x(1)| + \sum_{i=1}^{k_0} |x(k) - x(k+1)|^{p_k} \right) + \frac{\varepsilon}{3 \cdot 2^M} \tag{2.4}$$

and

$$|x_n(1) - x(1)| + \sum_{k=1}^{k_0} |(x_n(k) + x(k)) - (x_n(k+1) - x(k+1))|^{p_k} < \frac{\varepsilon}{3}. \tag{2.5}$$

It follows from (2.3), (2.4) and (2.5) that for $n \geq n_0$,

$$
\begin{aligned}
\varrho_p(x_n - x) &= |x_n(1) - x(1)| + \sum_{k=1}^{\infty} |(x_n(k) - x(k)) - (x_n(k+1) - x(k+1))|^{p_k} \\
&= |x_n(1) - x(1)| + \sum_{k=1}^{k_0} |(x_n(k) - x(k)) - (x_n(k+1) - x(k+1))|^{p_k} \\
&+ \sum_{k=k_0+1}^{\infty} |(x_n(k) - x(k)) - (x_n(k+1) - x(k+1))|^{p_k} \\
&< \frac{\varepsilon}{3} + 2^M \Big(\sum_{k=k_0+1}^{\infty} |x_n(k) - x_n(k+1)|^{p_k} + \sum_{k=k_0+1}^{\infty} |x(k) - x(k+1)|^{p_k} \Big) \\
&= \frac{\varepsilon}{3} + 2^M \Big(\varrho_p(x_n) - \big(|x_n(1)| + \sum_{k=1}^{k_0} |x_n(k) - x_n(k+1)|^{p_k}\big) + \sum_{k=k_0+1}^{\infty} |x(k) - x(k+1)|^{p_k} \\
&< \frac{\varepsilon}{3} + 2^M \Big(\varrho_p(x) - \big(|x(1)| + \sum_{k=1}^{k_0} |x(k) - x(k+1)|^{p_k}\big) + \frac{\varepsilon}{3 \cdot 2^M} + \sum_{k_0+1}^{\infty} |x(k) - x(k+1)| \\
&= \frac{\varepsilon}{3} + 2^M \Big(2 \sum_{k=k_0+1}^{\infty} |x(k) - x(k+1)|^{p_k} + \frac{\varepsilon}{3 \cdot 2^M} \Big) \\
&< \frac{\varepsilon}{3} + 2^{M+1} \frac{\varepsilon}{3.2^{M+1}} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}
$$

This show that $\varrho_p(x_n - x) \to 0$ as $n \to \infty$. Hence, by Proposition 2.4 (ii), we have $\|x_n - x\| \to 0$ as $n \to \infty$.

Theorem 2.7 *The space $\ell(\Delta, p)$ has the property (H).*

Proof. Let $x \in S(\ell(\Delta, p))$ and $(x_n) \subset \ell(\Delta, p)$ such that $\|x_n\| \to 1$ and $x_n \xrightarrow{w} x$ as $n \to \infty$. From Proposition 2.3 (iii), we have $\varrho_p(x) = 1$, so it follows from Proposition 2.4 (i) that $\varrho_p(x_n) \to \varrho_p(x)$ as $n \to \infty$. By Lemma 2.5, we have that the coordinate mapping $P_i : \ell(\Delta, p) \to \mathbb{R}$ is continuous, so it follows that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$. Thus, we have by Lemma 2.6 that $x_n \to x$ as $n \to \infty$. $\square$

Theorem 2.8 *The space $\ell(\Delta, p)$ is rotund if and only if $p_k > 1$ for all $k \in \mathbb{N}$.*

Proof. Necessity. Suppose that there is $k_0 \in \mathbb{N}$ such that $p_0 = 1$. Let $x = (1,1,1,...)$ and $y = (\underbrace{0,0,0...,0}_{k_0},1,1,1,...)$. Then $x \neq y$ and it is easy to see that

$$\varrho_p(x) = \varrho_p(y) = \varrho_p\left(\frac{x+y}{2}\right) = 1.$$

By Proposition 2.3(iii) , we have x, y and $\frac{x+y}{2} \in S(\ell(\Delta,p))$, so that $\ell(\Delta,p)$ is not rotund.

Sufficiency. Suppose that $p_k > 1$ for all $k \in \mathbb{N}$. Let $x \in S(\ell(\Delta,p))$ and $y, z \in B(\ell(\Delta,p))$ with $x = \frac{y+z}{2}$. By convexity of ϱ_p and Proposition 2.3(iii), we have

$$1 = \varrho_p(x) \leq \frac{1}{2}(\varrho_p(y) + \varrho_p(z)) \leq \frac{1}{2} + \frac{1}{2} = 1.$$

This implies that

$$\varrho_p(y) = \varrho_p(z) = 1 \tag{2.6}$$

$$\varrho_p(x) = \frac{1}{2}(\varrho_p(y) + \varrho_p(z)). \tag{2.7}$$

By (2.7), we have

$$\begin{aligned}
&\left|\frac{y(1)+z(1)}{2}\right| + \sum_{k=1}^{\infty}\left|\frac{(y(k)-y(k+1)) + (z(k)-z(k+1))}{2}\right|^{p_k} \\
&= \frac{1}{2}\left(|y(1)| + \sum_{k=1}^{\infty}|y(k)-y(k+1)|^{p_k}\right) + \frac{1}{2}\left(|z(1)| + \sum_{k=1}^{\infty}|z(k)-z(k+1)|^{p_k}\right) \\
&= \frac{1}{2}(|y(1)| + |z(1)|) + \frac{1}{2}\left(\sum_{k=1}^{\infty}|y(k)-y(k+1)|^{p_k} + \sum_{k=1}^{\infty}|z(k)-z(k+1)|^{p_k}\right),
\end{aligned}$$

which implies that

$$|y(1)+z(1)| = |y(1)| + |z(1)|, \tag{2.8}$$

$$\left|\frac{(y(k)-y(k+1)) + (z(k)-z(k+1))}{2}\right|^{p_k} = \frac{1}{2}\left(|y(k)-y(k+1)|^{p_k} + |z(k)-z(k+1)|^{p_k}\right) \tag{2.9}$$

for all $k \in \mathbb{N}$.

Since the function $t \mapsto |t|^{p_k}$ is strickly convex for every $k \in \mathbb{N}$, it implies by (2.9) that

$$y(k) - y(k+1) = z(k) - z(k+1) \quad \text{for all k } \in \mathbb{N}. \tag{2.10}$$

It follows from (2.6) and (2.10) that $|y(1)| = |z(1)|$. This implies by (2.8) that $y(1) = z(1)$. This together with (2.10), by using induction, we obtain that $y(k) = z(k)$ for all $k \in \mathbb{N}$. Hence $y = z$.

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ON PROPERTY (UKK) IN CESARO MUSIELAK-ORICZ SEQUENCE SPACES.

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ABSTRACT. In this paper we define a generalized Cesaro sequence space ces_M , where M is a Musielak-Orlicz function , and consider it equipped with the Luxemburg norm . The main purpose of the paper is to show that ces_M is a (UKK) spaace, when $M \in \delta_2$ and satisfies condition (*).

1. Introduction. Geometric of Banach spaces is an important topic in functional analysis and plays an important role in the theory of approximation and optimization. The property of uniform rotundity ensures, for example, the existence and unicity of nearest pionts in best approximation problems. Among geometrical properties, H-property and (UKK) are also improtant. Both of them follow from the uniform convexity (UC) and that (UKK) implies H-property, and nearly uniform convexity (NUC) implies (UKK).

Summarizing the above discussion we have

$$(UC) \Rightarrow (NUC) \Rightarrow (UKK) \Rightarrow H-property$$

The criteria of the H-property in Orlicz function space and Orlicz sequence space were given by S.Chen and Y.Wang[2] and C.Wu, S.Chen and Y. Wang [18]. R.Pluciennik, T.Wang and Y.Zhang[13] considered the problem more precisely and obtained all the criteria for H-points and denting points in both Orlicz function spaces and Orlicz sequence spaces. In [4] criteria for (NUC),(UKK) and H-property are given for Musielak-Orlicz sequence spaces.

The Cesaro sequence space ces_p $(1 < p < \infty)$ were introduced by J.S.Shue[17].They are useful for theory of matrix operators. Y.Q.Liu, B.E.Wu and P.Y.Lee[9] showed that ces_p has H-property. S.Suantai[16] defined a generalized Cesaro sequence space $ces(p)$, where $p = (p_k)$ is a bounded sequence of positive real number with $p_k > 1$ for all $k \in \mathbb{N}$,

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and consider it equipped the Luxemburg norm. He has also showed that $ces(p)$ has H-property.

In this paper we define a new sequence space, ces_M, which is a generalization of the space $ces(p)$ by using a Musielak-Orlicz function. We call the space ces_M,Cesaro Musielak-Orlicz sequence spaces. We show in this paper that if $M \in \delta_2$ and M satisfies the condition (*), then ces_M is (UKK), so it has H-property. These results generalize those in [].

Now we introduce the basic notations and defitions. In the following, let $\mathbb{R}$ be the real line and $\mathbb{N}$ the set of natural numbers

Let $(X, \|.\|)$ be a real Banach space, and let $B(X)$ (resp. $S(X)$) be the closed unit ball (resp. the unit sphere) of X. For any subset A of X, we denote by conv (A) (resp. $\overline{\text{conv}}(A)$) the convex hull (resp. the closed convex hull) of A. Clarkson [1] introduced the concept of uniform convexity.

The norm $\|.\|$ is called *uniformly convex* (write **(UC)**) if for each $\epsilon > 0$ there is $\delta > 0$ such that for $x, y \in S(X)$ inequality $\|x - y\| > \epsilon$ implies

$$\|\frac{1}{2}(x+y)\| < 1 - \delta$$

A Banach space X is said to have the *Kadac-Klee property* (or *property* **(H)**) if every weakly convergent sequence on the unit sphere is convergent in norm.

Recall that a sequence $\{x_n\} \subset X$ is said to be $\epsilon-$*separated sequence* for some $\epsilon > 0$ if

$$sep(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} > \epsilon.$$

A Banach space is said to have the *unifrom Kadec-Klee property* (written **UKK**) if for every $\varepsilon > 0$ there exists $\delta > 0$ for every sequence (x_n) in $S(X)$ with $sep(x_n) > \varepsilon$ and $x_n \xrightarrow{\omega} x$,we have $\|x\| < 1 - \delta$. Every **(UKK)** Banach space has property **(H)**(see [5])

A Banach space is said to be *nearly uniformly convex* (write **(NUC)**) if for every $\epsilon > 0$ there exists $\delta \in (0,1)$ such that for every sequence $(x_n) \subseteq B(X)$ with $sep(x_n) > \epsilon$, we have

$$conv(x_n) \cap ((1-\delta)B(X)) \neq \emptyset$$

Huff [5] proved that every (**NUC**) Banach space is reflexive and it has a property (**H**) and he proved that X is (**NUC**) if and only if X is reflexive and (**UKK**).

Let X be a real vector space. A functional $\varrho : X \to [0, \infty]$ is called a *modular* if it satisfies the conditions

(i) $\varrho(x) = 0$ if and only if $x = 0$;

(ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$;

(iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$, for all $x, y \in X$ and all α, $\beta \geq 0$ with $\alpha + \beta = 1$.

The modular ϱ is called *convex* if

(iv) $\varrho(\alpha x + \beta y) \leq \alpha\varrho(x) + \beta\varrho(y)$, for all $x, y \in X$ and all α, $\beta \geq 0$ with $\alpha + \beta = 1$.

If ϱ is a modular in X, we define

$$X_\varrho = \{x \in X : \lim_{\lambda \to 0^+} \varrho(\lambda x) = 0 \}$$

$$\text{and} \quad X_\varrho^* = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0 \}.$$

It is clear that $X_\varrho \subseteq X_\varrho^*$. If ϱ is a convex modular, for $x \in X_\varrho$ we define

$$\|x\| = \inf\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1 \} \tag{1.1}$$

Orlicz [13] proved that if ϱ is a convex modular on X, then $X_\varrho = X_\varrho^*$ and $\|.\|$ is a norm on X_ϱ for which it is a Banach space. The norm $\|.\|$ defined as in (1.1) is called the Luxemburg norm.

A modular ϱ on X is called

(a) *right-continuous* if $\lim_{\lambda \to 1^+} \varrho(\lambda x) = \varrho(x)$ for all $x \in X_\varrho$

(b) *left-continuous* if $\lim_{\lambda \to 1^-} \varrho(\lambda x) = \varrho(x)$ for all $x \in X_\varrho$

(c) *continuous* if it is both left-continuous and right-continuous .

The following known results gave some relationships between the modular ϱ and the Luxemburg norm $\|.\|$ on X_ϱ.

Theorem 1.1 *Let ϱ be a convex modular on X and let $x \in X_\varrho$ and (x_n) a sequence in X_ϱ. Then $\|x_n - x\| \to 0$ as $n \to \infty$ if and only if $\varrho(\lambda(x_n - x)) \to 0$ as $n \to \infty$ for every $\lambda > 0$.*

Proof. See [11, Theorem 1.3].

Theorem 1.2 *Let ϱ be a convex modular on X and $x \in X_\varrho$.*

(i) If ϱ is right-continuous, then $\|x\| < 1$ if and only if $\varrho(x) < 1$.

(ii) If ϱ is left-continuous, then $\|x\| \le 1$ if and only if $\varrho(x) \le 1$.

(iii) If ϱ is continuous, then $\|x\| = 1$ if and only if $\varrho(x) = 1$.

Proof. See [11, Theorem 1.4].

Let l^0 be the space of all real sequences. For $1 \le p < \infty$, the Cesaro sequence space (ces_p , for short) is defined by

$$ces_p = \{x \in l^0 : \textstyle\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^p < \infty\}$$

equipped with the norm

$$\|x\| = (\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^p)^{\frac{1}{p}}$$

This space was introduced by Shue [16]. It is useful in the theory of Matrix operator and others (see [7] and [8]). Some geometric properties of the Cesaro sequence space ces_p were studied by many authors. It is known that $(ces_p, \|.\|)$ is LUR and has property (H)(see [8]). Cui and Meng [2] prove that $(ces_p, \|.\|)$ has property (β).

A map $\phi : \mathbb{R} \to [0, \infty$ is said to be an *Orlicz function* if ϕ vanishes only at 0, and ϕ is even and convex.

A sequence $M = (M_k)$ of Orlicz functions is called a *Musielak-Orlicz function.* In addition, a Musielak-Orlicz function $N = (N_k)$ is called a *complementary function* of a Musielak-Orlicz function M if

$$N_k(v) = \sup\{|v|u - M_k(u) : u \ge 0\}\ , \quad k = 1, 2, ...$$

For a given Musielak-Orlicz function M, the *Musielak-Orlicz sequence space* l_M is defined by

$$l_M := \{x \in l^0 : I_M(cx) < \infty \text{ for some } c > 0\}.$$

where I_M is a convex modular defined by

$$I_M(x) = \sum_{k=1}^{\infty} M_k(x(k)), \quad x = (x(k)) \in l_M.$$

This space equipped with the *Luxemburg norm*

$$\|x\| = \inf\{k > 0 : I_M(\frac{x}{k}) \leq 1\}$$

equivalent one

$$\|x\|^0 = \inf\{\frac{1}{k}(1 + I_M(kx)) : k > 0\},$$

called the Orlicz (or Amemiya) norm is a Banach space. To simplify notation, we put $l_M := (l_M, \|.\|)$ and $l_M^0 := (l_M, \|.\|^0)$. Both of them are Banach spaces .

Let $M = (M_k)$ be the Musielak-Orlicz function. The *Cesàro-Musielak-Orlicz sequence space* is define by

$$Ces_M := \{x \in l^0 : \rho_M(cx) < \infty \text{ for some } c > 0\},$$

where $\rho_M(x) = \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|)$. We show in Theorem 3.1 that ρ_M is a convex modular on Ces_M. In this space, we consider the Luxemburg norm induced by the modular ρ_M as follows:

$$\|x\| = \inf\{\lambda > 0 : \rho_M(\frac{x}{\lambda}) \leq 1\}.$$

We have by [10] that Ces_M is a Banach space. We define the subspace $SCes_M$ of Ces_M by

$$SCes_M := \{x \in l^0 : \rho_M(cx) < \infty \text{ for all } c > 0\}.$$

We say that a Musielak-Orlicz function M *satisfies the δ_2-condition* (we will write $M \in \delta_2$ for short) if there exist constants $K \geq 2, u_0 > 0$ and a sequence (c_k) of positive numbers such that $\sum_{k=1}^{\infty} c_k < \infty$ and the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

hold for every $k \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $|u| \leq u_0$.

If $M \in \delta_2$ and $N \in \delta_2$, then we write $M \in \delta_2 \cap \delta_2^*$.

Moreover, we say that a Musielak-Orlicz function M *satisfies the $(*)$-condition* if for any $\epsilon \in (0,1)$ there exists a $\delta > 0$ such that $M_k((1+\delta)u) \leq 1$ whenever $M_k(u) \leq 1 - \epsilon$ for all $k \in \mathbb{N}$ and $u \in \mathbb{R}$.

We shall show in Theorem 5 that if $M \in \delta_2$, then $SCes_M = Ces_M$.

MAIN RESULT

We start with showing that ρ_M is a convex modular on Ces_M.

Theorem 1 The functional ρ_M on the Cesàro-Musielak-Orlicz sequence space Ces_M given by

$$\rho_M(x) = \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|),$$

is a modular on Ces_M

Proof Let $x, y \in Ces_M$. It is obvious that

(i) $\rho_M(x) = 0 \Leftrightarrow x = 0$

(ii) For $\alpha \in \mathbb{R}$, with $|\alpha| = 1$, we have

$$\begin{aligned}\rho_M(\alpha x) &= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|\alpha x(i)|) \\ &= \sum_{k=1}^{\infty} M_k(|\alpha|\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \\ &= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \\ &= \rho_M(x)\end{aligned}$$

(iii) For $\alpha, \beta \in \mathbb{R}$, with $\alpha, \beta \geq 0, \alpha + \beta = 1$, by convexity of M_k, we have

$$\begin{aligned}\rho_M(\alpha x + \beta y) &= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|\alpha x(i) + \beta y(i)|) \\ &\leq \sum_{k=1}^{\infty} M_k(\frac{\alpha}{k}\sum_{i=1}^{k}|x(i)| + \frac{\beta}{k}\sum_{i=1}^{k}|y(i)|) \\ &\leq \sum_{k=1}^{\infty}(\alpha M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|) + \beta M_k(\frac{1}{k}\sum_{i=1}^{k}|y(i)|)) \\ &= \alpha\sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|) + \beta\sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|y(i)|) \\ &= \alpha\rho_M(x) + \beta\rho_M(y) \leq \rho_M(x) + \rho_M(y)\end{aligned}$$

□

Proposition 2. Let $x \in Ces_M$.

(1) If $0 < \alpha \leq 1$, then $\frac{1}{\alpha}\rho_M(\alpha x) \leq \rho_M(x) \leq \alpha\rho_M(\frac{x}{\alpha})$

(2) If $\alpha > 1$, then $\alpha\rho_M(\frac{x}{\alpha}) \leq \rho_M(x) \leq \frac{1}{\alpha}\rho_M(\alpha x)$

Proof. Let $x = (x(i)) \in Ces_M$. For any $0 < \alpha \leq 1$, by convexity of each M_k, we have

$$M_k(\frac{1}{k}\sum_{i=1}^{k}|\alpha x(i)|) = M_k(\alpha\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \leq \alpha M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|)$$

for all $k \in \mathbb{N}$. This implies $\rho_M(\alpha x) \leq \alpha\rho_M(x)$. By substituting x by $\frac{x}{\alpha}$, we obtain $\rho_M(x) \leq \alpha\rho_M(\frac{x}{\alpha})$, so that (1) holds. Next, let $\alpha > 1$, then $0 < \frac{1}{\alpha} < 1$. By (1), we obtain that

$$\alpha\rho_M(\frac{x}{\alpha}) = \frac{1}{\frac{1}{\alpha}}\rho_M(\frac{x}{\alpha}) \leq \rho_M(x) \leq \frac{1}{\alpha}\rho_M(\frac{x}{\frac{1}{\alpha}}) = \frac{1}{\alpha}\rho_M(\alpha x)$$

, hence (2) is satisfied. □

Proposition 3. For any $x \in Ces_M$, we have

(1) if $\|x\| \leq 1$, then $\rho_M(x) \leq \|x\|$

(2) if $\|x\| > 1$, then $\rho_M(x) \geq \|x\|$

Proof (1) If $x = 0$, then the inequality holds. Let $x \neq 0$. By the definition of $\|.\|$, there is a sequence (ϵ_n) such that $\epsilon_n \downarrow \|x\|$ such that $\rho_M(\frac{x}{\epsilon_n}) \leq 1$. This implies $\rho_M(\frac{x}{\|x\|}) \leq 1$, by Proposition 2(1), we have $\rho_M(x) \leq \|x\|\rho_M(\frac{x}{\|x\|}) \leq \|x\|$.

(2) Let $\|x\| > 1$. Then for $\epsilon \in (0, \frac{\|x\|-1}{\|x\|})$, we have $(1-\epsilon)\|x\| > 1$. By Proposition 2(1), we have

$$1 < \rho_M(\frac{x}{(1-\epsilon)\|x\|}) \leq \frac{\rho_M(x)}{(1-\epsilon)\|x\|}.$$

Letting $\epsilon \to 0$, we obtain (2).

The following result is directly obtained from Proposition 3(1).

Corollary 4 If $x_n \to 0$ as $n \to \infty$ then $\rho_M(x_n) \to 0$ as $n \to \infty$ □

Theorem 5 If a Musielak-Orlicz function $M = (M_k) \in \delta_2$, then $SCes_M = Ces_M$.

Proof Let $x \in Ces_M$. Thus $\rho_M(cx) < \infty$ for some $c > 0$. Since $M \in \delta_2$, there exists $K \geq 2$, $u_0 > 0$ and a positive sequence c_k such that $\sum_{k=1}^{\infty} c_k < \infty$ and

$$M_k(2u) \leq KM_k(u) + c_k$$

for all $k \in \mathbb{N}$ and u satisfies $|u| \leq u_0$. By $\rho_M(cx) < \infty$, we have $\sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \to 0$ as $k \to \infty$, it followes that $M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \to 0$ as $k \to \infty$, and so $\frac{1}{k}\sum_{i=1}^{k}|x(i)| \to 0$ as $k \to \infty$. Put any $\beta > 0$ and taking $t \in \mathbb{N}$ such that $\frac{\beta}{c} \leq 2^{t-1}$, there exists a positive sequence c_k' such that $\sum_{k=1}^{\infty} c_k' < \infty$ and

$$M_k(2^t u) \leq K^t M_k(u) + c_k'$$

for all $k \in \mathbb{N}$ and u satisfies $|u| \leq \frac{u_0}{2^{t-1}}$. By $\frac{1}{k}\sum_{i=1}^{k}|x(i)| \to 0$ as $k \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{k}\sum_{i=1}^{k}|x(i)| \leq \frac{\frac{u_0}{2^{t-1}}}{c}$ for all $k \geq n_0$. Hence

$$\begin{aligned}
\rho_M(\beta x) &= \sum_{k=1}^{\infty} M_k(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \\
&= \sum_{k=1}^{n_0} M_k(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|) + \sum_{k=n_0+1}^{\infty} M_k(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \\
&= \sum_{k=1}^{n_0} M_k(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|) + \sum_{k=n_0+1}^{\infty} M_k(2\beta\frac{c}{c}\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \\
&= \sum_{k=1}^{n_0} M_k(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|) + \sum_{k=n_0+1}^{\infty} M_k(2^t c\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \\
&\leq \sum_{k=1}^{n_0} M_k(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|) + K^t \sum_{k=n_0+1}^{\infty} M_k(c\frac{1}{k}\sum_{i=1}^{k}|x(i)|) + \sum_{k=n_0+1}^{\infty} c_k' < \infty.
\end{aligned}$$

Therefore $Ces_M \subseteq SCes_M$. □

Lemma 6 On Cesàro-Musielak-Orlicz sequence space, if the Musielak-Orlicz function $M = (M_k)$ satisfies condition $(*)$ and $M \in \delta_2$, then

(1) $\|x\| = 1 \Leftrightarrow \rho_M(x) = 1$,

(2) for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x\| < 1-\delta$ whenever $\rho_M(x) < 1-\epsilon$,

(3) for every $\epsilon > 0$ and $c > 0$ there exists a $\delta > 0$ such that for any $x, y \in Ces_M$, we have

$$|\rho_M(x+y) - \rho_M(x)| < \epsilon$$

whenever $\rho_M(x) \leq c$ and $\rho_M(y) \leq \delta$,

(4) for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x\| > 1+\delta$ whenever $\rho_M(x) > 1+\epsilon$, and

(5) for any sequence $(x_n) \subset Ces_M$, $\|x_n\| \to 1$ implies $\rho_M(x_n) \to 1$.

Proof (1) Assume that $\rho_M(x) = 1$. By definition of $\|.\|$, we have that $\|x\| \leq 1$. If $\|x\| < 1$, then we have by Proposition 3(1) that $\rho_M(x) \leq \|x\| < 1$, which contradicts our assumption. Therefore $\|x\| = 1$.

Conversely, assume that $\|x\| = 1$. By Proposition 3.3(1), $\rho_M(x) \leq 1$. Suppose that $\rho_M(x) < 1$. By Theorem 5, we have $\rho_M(cx) < \infty$ for all $c > 1$. By Theorem 2.6.1 the function $c \mapsto \rho_M(cx)$ is continuous, so there exists an $c' > 1$ such that $\rho_M(c'x) = 1$. By using the same proof as in the first path, we have that $\|c'x\| = 1$, so $c' = 1$ which is contradiction.

(2) Suppose (2)is not true. Then there exists a $\epsilon_0 > 0$ and $x_n \in Ces_M$ such that $\rho_M(x_n) < 1 - \epsilon_0$ and $\frac{1}{2} \leq \|x_n\|$ and $\|x_n\| \to 1$. Let $L = \sup_n\{\rho_M(2x_n)\}$ we have that $L < \infty$ since $M \in \delta_2$. Let $a_n = \frac{1}{\|x_n\|} - 1$ we have $a_n \leq 1$ and $a_n \to 0$. Then

$$\begin{aligned}
1 &= \rho_M(\frac{x_n}{\|x_n\|}) \\
&= \rho_M(2a_n x_n + (1 - a_n)x_n) \\
&\leq a_n \rho_M(2x_n) + (1 - a_n)\rho_M(x_n) \\
&\leq a_n L + (1 - \epsilon).
\end{aligned}$$

Hence we have $1 \leq \lim_{n\to\infty}(a_n L + (1 - \epsilon)) = 1 - \epsilon$, which is a contradiction.

(3) Let $x, y \in Ces_M, \epsilon > 0$ and $c > 0$, by Theorem 2.6.5(3), there exists a $\delta' > 0$ such that for any $a, b \in l_M$, we have

$$|I_M(a + b) - I_M(a)| < \epsilon \tag{3.1}$$

whenever $I_M(a) \leq c$ and $I_M(b) \leq \delta'$. For each $i \in \mathbb{N}$, let

$$s(i) = \begin{cases} sgn(x(i) + y(i)) \text{ if } x(i) + y(i) \neq 0, \\ 1 \text{ if } x(i) + y(i) = 0 \end{cases}$$

we note that

$$\begin{aligned}\rho_M(x+y) &= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)+y(i)|) \\ &= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}s(i)x(i)+\frac{1}{k}\sum_{i=1}^{k}s(i)y(i)).\end{aligned} \tag{3.2}$$

Let $a(k)=\frac{1}{k}\sum_{i=1}^{k}s(i)x(i)$ and $b(k)=\frac{1}{k}\sum_{i=1}^{k}s(i)y(i)$ for all $k\in\mathbb{N}$. Then $a=(a(k))\in l_M$ and $b=(b(k))\in l_M$, and from (3.2) we have

$$\rho_M(x+y)=I_M(a+b), I_M(a)\le\rho_M(x) \text{ and } I_M(b)\le\rho_M(y).$$

Choose $\delta=\delta'$. If $\rho_M(x)\le c$ and $\rho_M(y)\le\delta$ then $I_M(a)\le c$ and $I_M(b)\le\delta'$, by (1) we have

$$\rho_M(x+y)-\rho_M(x)\le I_M(a+b)-I_M(a)<\epsilon$$

that is

$$\rho_M(x+y)<\rho_M(x)+\epsilon. \tag{3.3}$$

Next, we shall show that

$$\rho_M(x)<\rho_M(x+y)+\epsilon. \tag{3.4}$$

For each $i\in\mathbb{N}$, let

$$s(i)=\begin{cases} sgn(x(i)) \text{ if } x(i)\neq 0, \\ 1 \text{ if } x(i)=0\end{cases}$$

we note that

$$\begin{aligned}\rho_M(x)=\rho_M((x+y)+(-y)) &= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|(x(i)+y(i))+(-y(i))|) \\ &= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}s(i)(x(i)+y(i))+\frac{1}{k}\sum_{i=1}^{k}s(i)(-y(i))).\end{aligned} \tag{3.5}$$

Let $a(k)=\frac{1}{k}\sum_{i=1}^{k}s(i)(x(i)+y(i))$ and $b(k)=\frac{1}{k}\sum_{i=1}^{k}s(i)(-y(i))$ for all $k\in\mathbb{N}$. Then $a=(a(k))\in l_M$ and $b=(b(k))\in l_M$, and from (3.5) we have

$$\rho_M(x)=I_M(a+b), I_M(a)\le\rho_M(x+y) \text{ and } I_M(b)\le\rho_M(-y).$$

Choose $\delta = \delta'$. If $\rho_M(x) \leq c$ and $\rho_M(y) \leq \delta$ then $I_M(a+b) = \rho_M(x) \leq c$ and $I_M(-b) = I_M(b) \leq \rho_M(-y) = \rho_M(y) \leq \delta'$, by (3.1) we have

$$|I_M(a+b) - I_M(a)| = |I_M(a) - I_M(a+b)| = |I_M((a+b)+(-b)) - I_M(a+b)| < \epsilon$$

it follows that

$$\rho_M(x) - \rho_M(x+y) \leq I_M(a+b) - I_M(a) < \epsilon$$

that is

$$\rho_M(x) < \rho_M(x+y) + \epsilon$$

from (3.3) and (3.4), we have that

$$|\rho_M(x+y) - \rho_M(x)| < \epsilon$$

whenever $\rho_M(x) \leq c$ and $\rho_M(y) \leq \delta$.

(4) Given $\epsilon > 0$, by (3), there exists a $\delta \in (0,1)$ such that

$$\rho_M(u) \leq 1, \rho_M(v) \leq \delta \Rightarrow \rho_M(u+v) \leq \rho_M(u) + \epsilon.$$

Suppose that $\|x\| \leq 1+\delta$, then $\rho_M(\frac{x}{1+\delta}) \leq 1$ and $\rho_M(\frac{\delta x}{1+\delta}) \leq \delta\rho_M(\frac{x}{1+\delta}) \leq \delta$. This implies

$$\begin{aligned}\rho_M(x) &= \rho_M(\frac{x}{1+\delta} + \frac{\delta x}{1+\delta}) \\ &\leq \rho_M(\frac{x}{1+\delta}) + \epsilon \\ &\leq 1+\epsilon.\end{aligned}$$

(5) Suppose that $\rho_M(x_n) \nrightarrow 1$ as $n \to \infty$, there exits a $\epsilon_0 > 0$ such that

$$|\rho_M(x_n) - 1| > \epsilon_0 \text{ for all } n \in \mathbb{N},$$

it follows that

$$\rho_M(x_n) - 1 > \epsilon_0 \text{ or } \rho_M(x_n) - 1 < -\epsilon_0 \text{ for all } n \in \mathbb{N}.$$

If $\rho_M(x_n) - 1 > \epsilon_0$, that is $\rho_M(x_n) > 1+\epsilon_0$, by (4), there exists a $\delta > 0$ such that $\|x_n\| > 1+\delta$. If $\rho_M(x_n) - 1 < -\epsilon_0$, that is $\rho_M(x_n) < 1-\epsilon_0$, by (2), there exists a $\delta' > 0$ such that $\|x_n\| < 1-\delta'$, so that $\|x_n\| \nrightarrow 1$ as $n \to \infty$, which contradiction. $\square$

Proposition 7 In Cesàro-Musielak-Orlicz sequence space. If a Musielak-Orlicz function $M = (M_k)$ satisfies condition $(*)$ and $M \in \delta_2$, then the norm convergence and modular convergence coincide.

Proof From Corollary 4, it suffices to prove that modular convergence implies norm convergence. For this let $\epsilon \in (0, \frac{1}{2})$, choose a positive integer K such that $\frac{1}{2^{K+1}} < \epsilon \leq \frac{1}{2^K}$. By Lemma 6(3), there exists a $\delta \in (0, \frac{1}{2^{K+1}})$ such that

$$\rho_M(u) \leq 1, \rho_M(v) \leq \delta \Rightarrow \rho_M(u+v) < \rho_M(u) + \epsilon.$$

Suppose that $\rho_M(x) < \delta$, we observe that

$$\rho_M(nx) < n\rho_M(x) + n\epsilon,$$

for $n = 1, ..., 2^{K-1}$. In particular, $\rho_M(\frac{x}{4\epsilon}) \leq \rho_M(2^{K-1}x) < 2^{K-1}\rho_M(x) + 2^{K-1}\epsilon < \frac{1}{2} + \frac{1}{2} = 1$. This implies $\|x\| < 4\epsilon$. □

Theorem 8 If $M \in \delta_2$ and M satisfies condition (*),then the space ces_M is **(UKK)**

Proof Assume that $M \in \delta_2$ and suppose that ces_M is not **(UKK)**.Then there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$ ther are a sequence (x_n) in $S(ces_M)$ and $x \in ces_M$ with $\text{sep}(x_n) \geq \varepsilon_0$,$x_n \overset{\omega}{\to} x$ and $\|x\| > 1 - \delta$. Since $\text{sep}(x_n) \geq \varepsilon_0$ passing subsequence we may assume that $\|x_n - x\| \geq \frac{\varepsilon_0}{2}$ for every $n \in \mathbb{N}$ Since $M \in \delta_2$ and M satisfies condition (*) and x can be assumed to have $\|x\|$ close to 1,ther exists $\eta > 0$ such that $\rho_M(x_n - x) \geq \eta$ and $\rho_M(x) > 1 - \frac{\eta}{5}$. Applying Lemma 6(3) there exists $\sigma \in (0, \frac{\eta}{5})$ such that

$$|\rho_M(x+y) - \rho_M(x)| < \frac{\eta}{5}$$

when ever $\rho_M(y) < \sigma$.

Since $(x_n) \subseteq S(ces_M)$ and $x_n \overset{\omega}{\to} x$,there exists $i_0 \in \mathbb{N}$ such that $\sum_{k=i_0+1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k} |x($ σ. By $x_n \overset{\omega}{\to} x$, which implies that $x_n \to x$ coordinatewise,hence there exists $n_0 \in \mathbb{N}$ such that

$$|\sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k} |x_n(i)|) - \sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k} |x(i)|)| < \frac{\eta}{5} \text{ and } \sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k} |x_n(i) - x(i)|) < \frac{\eta}{5}$$

for $n \geq n_0$. So

$$\begin{aligned} 1 &= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) = \sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) + \sum_{k=i_0+1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) \\ &\geq \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|) - \frac{\eta}{5} + \sum_{k=i_0+1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) \end{aligned}$$

Hence for every $n \geq n_0$ we have

$$\begin{aligned} \eta \leq \rho_M(x_n - x) &= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|) \\ &= \sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|) + \sum_{k=i_0+1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|) \\ &< \frac{\eta}{5} + \sum_{k=i_0+1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) + \frac{\eta}{5} \\ &= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) - \sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) + \frac{2\eta}{5} \\ &= 1 - \sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) + \frac{2\eta}{5} \\ &\leq 1 - \sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|) + \frac{\eta}{5} + \frac{2\eta}{5} \\ &\leq 1 - (1-\sigma) + \frac{3\eta}{5} \\ &\leq 1 - (1 - \frac{\eta}{5}) + \frac{3\eta}{5} < \eta. \end{aligned}$$

This is a contradiction.

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บทที่ 3

การแปลงเมทริกซ์ของปริภูมิลำดับ

(Matrix Transformations of Sequence Spaces)

ในบทนี้เราศึกษาเกี่ยวกับการแปลงเมทริกซ์ของปริภูมิลำดับ เราได้ให้ลักษณะเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิลำดับค่าเวกเตอร์นาคาโน $\ell(\Delta,p)$ และ $F_r(X,p)$ ไปยังปริภูมิ $E_r, \ell_\infty, \underline{\ell}_\infty(q), bs, cs$ เมื่อ $p_k > 1$ สำหรับทุก $k \in N$ และเรายังได้ให้ให้ลักษเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิ $\ell(\Delta,p)$ และ $M_0(X,p)$ ไปยังปริภูมิ E_r เมื่อ $p_k \le 1$ สำหรับทุก $k \in N$ เราสามารถให้ลักษณะเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิ FK ใดๆไปยังปริภูมิ $c(q)$ นอกเหนือจากนี้แล้วเราได้ให้ลักษณะเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิดับของแมดดอกซ์ไปยังปริภูมิลำดับมูสิลัก-ออร์ลิคซ์

Matrix Transformations of Nakano Vector-Valued Sequence Space

SUTHEP SUANTAI

ABSTRACT. **In this paper, we give necessary and sufficient conditions for infinite matrices mapping Nakano vector-valued sequence space $\ell(X,p)$ into the sequence spaces $E_r(r \geq 0)$ and we also give the matrix characterlizations from $M_0(X,p)$ into the space E_r where $p = (p_k)$ is a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$.**

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1. INTRODUCTION

For $r \geq 0$, the normed sequence space E_r was first defined by Cooke [1] as follows:

$$E_r = \{ \, x = (x_k) \mid \sup_k \frac{|x_k|}{k^r} < \infty \, \}$$

equipped with the norm

$$\|x\| = \sup_k \frac{|x_k|}{k^r}.$$

Let $(X, \ \|.\|)$ be a Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. We write $x = (x_k)$ with x_k in X for all $k \in N$. The X-valued sequence

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spaces $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, and $M_0(X,p)$ are defined as

$$c_0(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\},$$
$$c(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X\},$$
$$\ell_\infty(X,p) = \{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\},$$
$$\ell(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\},$$
$$M_0(X,p) = \cup_{n=1}^{\infty} \ell(X)_{(n^{-1/p_k})}$$

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, $\ell(p)$, and $M_0(p)$, respectively. The spaces $c_0(p)$, $c(p)$, $\ell_\infty(p)$ are known as the sequence spaces of Maddox. These spaces were first introduced and studied by Simons [7], Maddox [4, 5]. The space $\ell(p)$ was first defined by Nakano [6] and it is known as the Nakano sequence space and the space $\ell(X,p)$ is known as the Nakano vector-valued sequence space. The spaces $M_0(p)$ was first introduced by Grosse-Erdmann [2] and he has investigated the structure of the spaces $c_0(p)$, $c(p)$ and $\ell_\infty(p)$. Grosse-Erdmann [3] gave the matrix characterizations between scalar-valued sequence spaces of Maddox. Wu and Liu [9] dealt with the problem of characterizations those infinite matrices mapping $c_0(X,p)$, $\ell_\infty(X,p)$ into $c_0(q)$ and $\ell_\infty(q)$ where $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers.

Suantai [8] gave necessary and sufficient conditions for infinite matrices mapping $\ell(X,p)$ into ℓ_∞ and $\ell_\infty(q)$ where $p = (p_k)$ and $q = (q_k)$ are bounded sequence positive real numbers with $p_k \leq 1$ for all $k \in N$.

In this paper we give characterizations of infinite matrices mapping $\ell(X,p)$ and $M_0(X,p)$ into the sequence space E_r when $p_k \leq 1$ for all $k \in N$ and $r \geq 0$. Some results in [8] are obtained as special cases of this paper.

2. Notation and Definitions

Let $(X, \|.\|)$ be a Banach space. The space of all sequences and the space of all finite sequences in X are denoted by $W(X)$ and $\Phi(X)$, respectively. When X is K, the scalar field of X, the corresponding spaces are written as w and Φ.

A sequence space in X is a linear subspace of $W(X)$. Let E be an X-valued sequence space. For $x \in E$ and $k \in N$, we write x_k standing for the k^{th} term of x. For $x \in X$ and $k \in N$, let $e^k(x)$ be the sequence $(0, 0, ..., 0, x, 0, ...)$ with x in the k^{th} position and let $e(x)$ be the sequence $(x, x, x, ...)$. For a fixed scalar sequence $\mu = (\mu_k)$ the sequence space E_μ is defined as

$$E_\mu = \{x \in W(X) : (\mu_k x_k) \in E\} .$$

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map* E into F, written by $A : E \to F$ if for each $x = (x_k) \in E$, $A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$, and the sequence $Ax = (A_n(x)) \in F$. Let (E, F) denote for the set of all infinite matrices mapping from E into F.

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if for each $k \in N$ the k^{th} coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. If, in addition, (E, τ) is an Fréchet (Banach, LF-, LB-) space, then E is called an *FK- (BK-, LFK-, LBK-) space.* Now, suppose that E contains $\Phi(X)$. Then E is said to have *property AB* if the set $\{\sum_{k=1}^{n} e^k(x_k) : n \in N \}$ is bounded in E for every $x = (x_k) \in E$. It is said to have *property AK* if $\sum_{k=1}^{n} e^k(x_k) \to x$ in E as $n \to \infty$ for every $x = (x_k) \in E$. It has *property AD* if $\Phi(X)$ is dense in E.

It is known that E_r is a BK-sapce and $E_0 = \ell_\infty$. The space $\ell(X, p)$ is an FK-space with AK under the paranorm $g(x) = \left(\sum_{k=1}^{\infty} \|x_k\|^{p_k}\right)^{1/M}$, where $M = max\ \{1, \sup_k p_k\}$. In each of the space $\ell_\infty(X, p)$ and $c_0(X, p)$ we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = max\ \{1, \sup_k p_k\}$. It is known that $c_0(X, p)$ is an FK-space with AK under the paranorm g defined as above and $\ell_\infty(X, p)$ is a complete LBK-space with AB.

3. Main Results

We start with giving the matrix characterizations from $\ell(X, p)$ into E_r.

Theorem 3.1 *Let $r \geq 0$ and let $p = (p_k)$ be bounded sequences of positive real numbers with $p_k \leq 1$ and let $A = (f_k^n)$ be an infinite matrix. Then $A \in (\ell(X,p), E_r)$ if and only if there is $m_0 \in N$ such that $\sup\limits_{n,\,k} m_0^{-1/p_k} n^{-r} \|f_k^n\| < \infty$.*

Proof. Assume that $A \in (\ell(X,p), E_r)$. In $\ell(X,p)$, we consider it as a paranormed space with the paranorm g defined as above and since $p_k \leq 1$ for all $k \in N$, we have $M = max\ \{1, \sup\limits_k p_k\} = 1$. Now, we write $\|.\|$ standing for the paranorm g. By Zeller's theorem, $A : \ell(X,p) \to E_r$ is continuous. Then there is $m_0 \in N$ such that

$$\sup_n \Big|\sum_{k=1}^{\infty} f_k^n(x_k)\Big| \leq 1 \quad \text{for all } x \in \ell(X,p) \ \text{ with } \|x\| \leq \frac{1}{m_0} . \tag{3.1}$$

Let $n, k \in N$ be fixed and let $x_k \in X$ be such that $\|x_k\| \leq 1$. Then $e^{(k)}(m_0^{-1/p_k} x_k) \in \ell(X,p)$ and $\|e^{(k)}(m_0^{-1/p_k} x_k)\| \leq \dfrac{1}{m_0}$. By (3.1), we have

$$m_0^{-1/p_k} n^{-r} |f_k^n(x_k)| \leq \sup_{i \in N} i^{-r} |f_k^i(m_0^{-1/p_k} x_k)| = \|Ae^{(k)}(m_0^{-1/p_k} x_k)\| \leq 1 .$$

It implies that $\sup\limits_{n,k} m_0^{-1/p_k} n^{-r} \|f_k^n\| < \infty$.

Conversely, assume that the condition holds. Let $x = (x_k) \in \ell(X,p)$. By assumption, there is a $C > 0$ such that

$$m_0^{-1/p_k} n^{-r} \|f_k^n\| < C \quad \text{for all } n, k \in N . \tag{3.2}$$

Since $\|m_0^{1/p_k} x_k\| \to 0$ as $k \to \infty$, there is a $k_0 \in N$ such that $\|m_0^{1/p_k} x_k\| < 1$ for all $k \geq k_0$. Since $0 < p_k \leq 1$ for all $k \in N$, we have

$$\|m_0^{1/p_k} x_k\| \leq \|m_0^{1/p_k} x_k\|^{p_k} \quad \text{for all } k \geq k_0. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned}\sum_{k=1}^{\infty}\|m_0^{1/p_k}x_k\| &= \sum_{k=1}^{k_0}\|m_0^{1/p_k}x_k\| + \sum_{k=k_0+1}^{\infty}\|m_0^{1/p_k}x_k\| \\ &\leq \sum_{k=1}^{k_0}\|m_0^{1/p_k}x_k\| + \sum_{k=k_0+1}^{\infty}\|m_0^{1/p_k}x_k\|^{p_k} \\ &= K_1 + m_0\sum_{k=k_0+1}^{\infty}\|x_k\|^{p_k} \\ &\leq K_1 + m_0\|x\|,\ K_1 = \sum_{k=1}^{k_0}\|m_0^{1/p_k}x_k\|. \end{aligned} \tag{3.4}$$

By (3.2) and (3.4) we have for $n \in N$,

$$\begin{aligned} n^{-r}|A_n x| &= n^{-r}\Big|\sum_{k=1}^{\infty} f_k^n\big(m_0^{-1/p_k}(m_0^{1/p_k}x_k)\big)\Big| \\ &\leq \sum_{k=1}^{\infty} m_0^{-1/p_k} n^{-r}\|f_k^n\|.\|m_0^{1/p_k}x_k\| \\ &\leq C\sum_{k=1}^{\infty}\|m_0^{1/p_k}x_k\| \\ &\leq C(K_1 + m_0\|x\|). \end{aligned}$$

This implies that $\sup_n n^{-r}|A_n x| < \infty$, so that $Ax \in E_r$. This completes the proof. □

When $r = 0$, we see that $E_r = \ell_\infty$, so we obtain the following result directly from Theorem 3.1.

Corollary 3.2 *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), \ell_\infty)$ if and only if there is $m_0 \in N$ such that $\sup_{n,k} m_0^{-1/p_k}\|f_k^n\| < \infty$.*

If $p_k = s \leq 1$ for all $k \in N$, by Theorem 3.1 we obtain the following result:

Corollary 3.3 *Let $r \geq 0$ and $0 < s \leq 1$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell_s(X),\ E_r)$ if and only if $\sup_{n,k} n^{-r}\|f_k^n\| < \infty$.*

When $p_k = 1$ for all $k \in N$ and $r = 0$, we obtain the following result by Corollary 3.3

Corollary 3.4 *For an infinite matrix $A = (f_k^n)$, $A \in (\ell(X),\ \ell_\infty)$ if and only if* $\sup_{n,\,k} \|f_k^n\| < \infty$.

Theorem 3.5 *Let $r \geq 0$ and let $p = (p_k)$ be bounded sequences of positive real numbers and let $A = (f_k^n)$ be an infinite matrix. Then $A \in (M_0(X,p),\ E_r)$ if and only if for each $s \in N$, $\sup_{n,\,k}\ n^{-r} s^{1/p_k} \|f_k^n\| \ < \ \infty$*

Proof. Since $M_0(X,p) = \cup_{n=1}^{\infty} \ell(X)_{(n^{-1/p_k})}$, we have

$$A \in (M_0(X,p),\ E_r) \iff A \in (\ell(X)_{(s^{-1/p_k})},\ E_r) \text{ for all } s \in N$$

For $s \in N$, we can easily show that

$$A \in (\ell(X)_{(s^{-1/p_k})},\ E_r) \iff \left(s^{1/p_k} f_k^n\right)_{n,k} \in (\ell(X),\ E_r)\ .$$

By Theorem 3.1, we obtain that for $s \in N$,

$$\left(s^{1/p_k} f_k^n\right)_{n,k} \in (\ell(X),\ E_r) \iff \sup_{n,\,k}\ n^{-r} s^{1/p_k} \|f_k^n\| \ < \ \infty\ .$$

Thus the theorem is proved. □

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On Matrix Transformations Concerning the Nakano Vector-Valued Sequence Space

ABSTRACT. **In this paper, we give the matrix characterizations from Nakano vector-valued sequence space $\ell(X,p)$ and $F_r(X,p)$ into the sequence spaces E_r, ℓ_∞, $\underline{\ell}_\infty(q)$, bs and cs, where $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers such that $p_k>1$ for all $k\in I\!N$ and $r\geq 0$.**

Keywords : Matrix transformations, Nakano vector-valued sequence spaces

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1. INTRODUCTION

Let $(X,\|.\|)$ be a Banach space , $r\geq 0$ and $p=(p_k)$ a bounded sequence of positive real numbers. We write $x=(x_k)$ with x_k in X for all $k\in I\!N$. The X-valued sequence spaces $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, $E_r(X,p)$ and $F_r(X,p)$ are defined as

$$
\begin{aligned}
&c_0(X,p)=\{x=(x_k):\lim_{k\to\infty}\|x_k\|^{p_k}=0\},\\
&c(X,p)=\{x=(x_k):\lim_{k\to\infty}\|x_k-a\|^{p_k}=0 \text{ for some } a\in X\},\\
&\ell_\infty(X,p)=\{x=(x_k):\sup_k\|x_k\|^{p_k}<\infty\},\\
&\ell(X,p)=\{x=(x_k):\textstyle\sum_{k=1}^\infty\|x_k\|^{p_k}<\infty\},\\
&E_r(X,p)=\{x=(x_k):\sup_k\|x_k\|^{p_k}/k^r<\infty\},\\
&F_r(X,p)=\{x=(x_k):\textstyle\sum_{k=1}^\infty k^r\|x_k\|^{p_k}<\infty\},\\
&\underline{\ell}_\infty(X,p)=\textstyle\bigcap_{n=1}^\infty\{x=(x_k)\ :\ \sup_k\|x_k\|n^{1/p_k}\}.
\end{aligned}
$$

When $X=K$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$,$\ell(p)$, $E_r(p)$, $F_r(p)$ and $\underline{\ell}_\infty(p)$, respectively. The spaces $c_0(p)$, $c(p)$ and $\ell_\infty(p)$

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are known as the sequence spaces of Maddox. These spaces were first introduced and studied by Simons [7], Maddox [4, 5]. The space $\ell(p)$ was first defined by Nakano [6] and it is known as the Nakano sequence space and the space $\ell(X,p)$ is known as the Nakano vector-valued sequence space. When $p_k = 1$ for all $k \in I\!N$, the spaces $E_r(p)$ and $F_r(p)$ are written as E_r and F_r, respectively. These two sequence spaces were first introduced by Cooke [1]. The space $\underline{\ell}_\infty(p)$ was first defined by Grosse-Erdmann [2] and he has given in [3] characterizations of infinite matrices mapping between scalar-valued sequence spaces of Maddox. Wu and Liu [10] gave necessary and sufficient conditions for infinite matrices mapping from $c_0(X,p)$, $\ell_\infty(X,p)$ into $c_0(q)$ and $\ell_\infty(q)$. Suantai [8] has given characterizations of infinite matrices mapping $\ell(X,p)$ into ℓ_∞ and $\underline{\ell}_\infty(q)$ when $p_k \leq 1$ for all $k \in I\!N$ and he has also given in [9] characterizations of those infinite matrices mapping from $\ell(X,p)$ into the sequence space E_r when $p_k \leq 1$ for all $k \in I\!N$.

In this paper, we extend the results of [8] and [9] in the case that $p_k > 1$ for all $k \in I\!N$. Moreover, we also give the matrix characterizations from $\ell(X,p)$ and $F_r(X,p)$ into the sequence spaces bs and cs.

2. Notation and Definitions

Let $(X, \|.\|)$ be a Banach space, the space of all sequences in X is denoted by $W(X)$ and $\Phi(X)$ denotes for the space of all finite sequences in X. When X is K, the scalar field of X , the corresponding spaces are written as w and Φ.

A sequence space in X is a linear subspace of $W(X)$. Let E be an X-valued sequence space. For $x \in E$ and $k \in I\!N$, we write x_k stands for the k^{th} term of x. For $k \in I\!N$ denote by e_k the sequence $(0,0,...,0,1,0,...)$ with 1 in the k^{th} position and by e the sequence $(1,1,1,...)$. For $x \in X$ and $k \in I\!N$, let $e^k(x)$ be the sequence $(0,0,...,0,x,0,...)$ with x in the k^{th} position and let $e(x)$ be the sequence $(x,x,x,...)$. We call a sequence space E *normal* if $(t_k x_k) \in E$ for all $x = (x_k) \in E$ and $t_k \in K$ with $|t_k| = 1$ for all $t_k \in I\!N$. A normed sequence space $(E, \|.\|)$ is said to be *norm monotone* if $x = (x_k)$, $y = (y_k) \in E$ with $\|x_k\| \leq \|y_k\|$ for all $k \in I\!N$ implies $\|x\| \leq \|y\|$. For a fixed scalar sequence $\mu = (\mu_k)$ the sequence space E_μ is defined as

$$E_\mu = \{x \in W(X) : (\mu_k x_k) \in E\} .$$

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map* E into F, written by $A : E \to F$ if for each $x = (x_k) \in E$, $A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in \mathbb{N}$, and the sequence $Ax = (A_n(x)) \in F$. Let (E, F) denote for the set of all infinite matrices mapping from E into F.

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if for each $k \in \mathbb{N}$, the k^{th} coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. If, in addition, (E, τ) is an Fréchet (Banach, LF-, LB-) space, then E is called an *FK- (BK-, LFK-, LBK-) space.* Now, suppose that E contains $\Phi(X)$, then E is said to have *property AB* if the set $\{\sum_{k=1}^{n} e^k(x_k) : n \in \mathbb{N} \}$ is bounded in E for every $x = (x_k) \in E$. It is said to have *property AK* if $\sum_{k=1}^{n} e^k(x_k) \to x$ in E as $n \to \infty$ for every $x = (x_k) \in E$. It has *property AD* if $\Phi(X)$ is dense in E.

It is known that the Nakano sequence space $\ell(X, p)$ is an FK-space with property AK under the paranorm $g(x) = \left(\sum_{k=1}^{\infty} \|x_k\|^{p_k}\right)^{1/M}$, where $M = max\ \{1, \sup_k p_k\}$. If $p_k > 1$ for all $k \in \mathbb{N}$, then $\ell(X,p)$ is a BK-space with the Luxemburg norm defined by

$$\|(x_k)\| = \inf \{\varepsilon : \sum_{k=1}^{\infty} \|x_k/\varepsilon\|^{p_k} \leq 1 \}$$

3. Main Results

We first give a characterization of an infinite matrix mapping from $\ell(X,p)$ into E_r when $p_k > 1$ for all $k \in \mathbb{N}$. To do this, we need a lemma.

Lemma 3.1 *Let E be an X-valued BK-space which is normal and norm monotone and $A = (f_k^n)$ an infinite matrix. Then $A : E \to E_r$ if and only if $\sup_n \sum_{k=1}^{\infty} |f_k^n(x_k)|/n^r <$ ∞ for every $x = (x_k) \in E$.*

Proof If the condition holds true, it follows that $\sup_n |\sum_{k=1}^{\infty} f_k^n(x_k)|/n^r \leq$ $\sup_n \sum_{k=1}^{\infty} |f_k^n(x_k)|/n^r < \infty$ for every $x = (x_k) \in E$, hence $A : E \to E_r$.

Conversely, assume that $A : E \to E_r$. Since E and E_r are BK-spaces, by Zeller's Theorem, $A : E \to E_r$ is bounded, so there exists $M > 0$ such that

$$\sup_{\substack{n\in I\!N \\ \|(x_k)\|\leq 1}} \Big|\sum_{k=1}^{\infty} f_k^n(x_k)\Big|/n^r \leq M. \tag{3.1}$$

Let $x = (x_k) \in E$ be such that $\|x\| = 1$. For each $n \in I\!N$, we can choose a scalar sequence (t_k) with $|t_k| = 1$ and $f_k^n(t_k x_k) = |f_k^n(x_k)|$ for all $k \in I\!N$. Since E is normal and norm monotone, we have $(t_k x_k) \in E$ and $\|(t_k x_k)\| \leq 1$. It follows from (3.1) that $\sum_{k=1}^{\infty} |f_k^n(x_k)|/n^r = \big|\sum_{k=1}^{\infty} f_k^n(t_k x_k)\big|/n^r \leq M$, which implies

$$\sup_{n\in I\!N} \sum_{k=1}^{\infty} |f_k^n(x_k)|/n^r \leq M. \tag{3.2}$$

It follows from (3.2) that for every $x = (x_k) \in E$,

$$\sup_{n\in I\!N} \sum_{k=1}^{\infty} |f_k^n(x_k)|/n^r \leq M\|x\|.$$

This complete the proof. □

Theorem 3.2 *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in I\!N$ and $1/p_k + 1/q_k = 1$ for all $k \in I\!N$, and let $r \geq 0$. For an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), E_r)$ if and only if there is $m_0 \in I\!N$ such that*

$$\sup_n \sum_{k=1}^{\infty} \|f_k^n\|^{q_k} n^{-rq_k} m_0^{-q_k} < \infty\,. \tag{3.3}$$

Proof. Let $x = (x_k) \in \ell(X,p)$. By the condition, there are $m_0 \in I\!N$ and $K > 1$ such that

$$\sum_{k=1}^{\infty} \|f_k^n\|^{q_k} n^{-rq_k} m_0^{-q_k} < K \text{ for all } n \in I\!N. \tag{3.4}$$

Note that for a, $b \geq 0$, we have

$$ab \leq a^{p_k} + b^{q_k}\,. \tag{3.5}$$

It follows by (3.4) and (3.5) that for $n \in \mathbb{N}$,

$$\begin{aligned} n^{-r}\Big|\sum_{k=1}^{\infty} f_k^n(x_k)\Big| &= n^{-r}\Big|\sum_{k=1}^{\infty} f_k^n(m_0^{-1}.m_0 x_k)\Big| \\ &\leq \sum_{k=1}^{\infty}(n^{-r}m_0^{-1}\|f_k^n\|)(\|m_0 x_k\|) \\ &\leq \sum_{k=1}^{\infty} n^{-rq_k} m_0^{-q_k}\|f_k^n\|^{q_k} \ + \ m_0^{\alpha}\sum_{k=1}^{\infty}\|x_k\|^{p_k} \\ &\leq K + m_0^{\alpha}\sum_{k=1}^{\infty}\|x_k\|^{p_k} \ , \text{ where } \ \alpha = \sup_k p_k. \end{aligned}$$

Hence $\sup_n \ n^{-r}\big|\sum_{k=1}^{\infty} f_k^n(x_k)\big| < \infty$, so that $Ax \in E_r$.

For necessity, assume that $A \in (\ell(X,p), \ E_r)$. For each $k \in \mathbb{N}$, we have $\sup_n n^{-r}|f_k^n(x)| < \infty$ for all $x \in X$ since $e^{(k)}(x) \in \ell(X,p)$. It follows by the uniform bounded principle that for each $k \in \mathbb{N}$ there is $C_k > 1$ such that

$$\sup_n n^{-r}\|f_k^n\| \leq C_k \ . \tag{3.6}$$

Suppose that (3.3) is not true. Then

$$\sup_n \sum_{k=1}^{\infty}\|f_k^n\|^{q_k} n^{-rq_k} m^{-q_k} = \infty \quad \text{for every } m \in \mathbb{N} \ . \tag{3.7}$$

For $n \in \mathbb{N}$, we have by (3.6) that for $k, m \in \mathbb{N}$,

$$\begin{aligned} \sum_{j=1}^{\infty}\|f_j^n\|^{q_j} n^{-rq_j} m^{-q_j} &= \sum_{j=1}^{k}\|f_j^n\|^{q_j} n^{-rq_j} m^{-q_j} + \sum_{j>k}\|f_j^n\|^{q_j} n^{-rq_j} m^{-q_j} \\ &\leq \sum_{j=1}^{k} C_j^{q_j} m^{-q_j} + \sum_{j>k}\|f_j^n\|^{q_j} n^{-rq_j} m^{-q_j}. \end{aligned}$$

This together with (3.7), we have

$$\sup_n \sum_{j>k}\|f_j^n\|^{q_j} n^{-rq_j} m^{-q_j} = \infty \quad \text{for all } k, m \in \mathbb{N}. \tag{3.8}$$

By (3.8) we can choose $0 = k_0 < k_1 < k_2 < \dots$, $m_1 < m_2 < \dots$, $m_i > 4^i$ and a subsequence (n_i) of positive intergers such that for all $i \geq 1$

$$\sum_{k_{i-1}<j\leq k_i} \|f_j^{n_i}\|^{q_j} n_i^{-rq_j} m_i^{-q_j} > 2^i.$$

For each $i \in I\!N$, we can choose $x_j \in X$ with $\|x_j\| = 1$, for $k_{i-1} < j \leq k_i$ such that

$$\sum_{k_{i-1}<j\leq k_i} |f_j^{n_i}(x_j)|^{q_j} n_i^{-rq_j} m_i^{-q_j} > 2^i$$

For each $i \in I\!N$, let $F_i : (0,\infty) \to (0,\infty)$ be defined by

$$F_i(M) = \sum_{k_{i-1}<j\leq k_i} |f_j^{n_i}(x_j)|^{q_j} n_i^{-rq_j} M^{-q_j}.$$

Then F_i is continuous and nonincreasing such that $F(M) \to 0$ as $M \to \infty$. Thus there exists $M_i > 0$ such that $M_i > m_i$ and

$$F(M_i) = \sum_{k_{i-1}<j\leq k_i} |f_j^{n_i}(x_j)|^{q_j} n_i^{-rq_j} M_i^{-q_j} = 2^i \tag{3.9}$$

Put $y = (y_j)$, $y_j = 4^{-i} M_i^{-(q_j-1)} n_i^{-rq_j/p_j} |f_j^{n_i}(x_j)|^{q_j-1} x_j$ for $k_{i-1} < j \leq k_i$. Thus

$$\begin{aligned}
\sum_{j=1}^{\infty} \|y_j\|^{p_j} &= \sum_{i=1}^{\infty} \sum_{k_{i-1}<j\leq k_i} 4^{-ip_j} M_i^{-p_j(q_j-1)} n_i^{-rq_j} \||f_j^{n_i}(x_j)|^{p_j(q_j-1)} \\
&\leq \sum_{i=1}^{\infty} 4^{-i} \sum_{k_{i-1}<j\leq k_i} M_i^{-q_j} n_i^{-rq_j} |f_j^{n_i}(x_j)|^{q_j} \\
&= \sum_{i=1}^{\infty} 4^{-i}.2^i \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} = 1.
\end{aligned}$$

Thus $y = (y_j) \in \ell(X,p)$. Since $\ell(X,p)$ is a BK-space which is normal and norm monotone under the Luxemburg norm, by Lemma 3.1, we obtain that

$$\sup_n \sum_{k=1}^{\infty} \frac{|f_k^n(y_k)|}{n^r} < \infty. \tag{3.10}$$

But we have

$$\begin{aligned}
\sup_n \sum_{j=1}^{\infty} |f_j^n(y_j)|/n^r &\geq \sup_i \sum_{j=1}^{\infty} |f_j^{n_i}(y_j)|/n_i^r \\
&\geq \sup_i \sum_{k_{i-1}<j\leq k_i} |f_j^{n_i}(y_j)|/n_i^r \\
&= \sup_i \sum_{k_{i-1}<j\leq k_i} 4^{-i} M_i^{-(q_j-1)} n_i^{-r(q_j/p_j+1)} |f_j^{n_i}(x_j)|^{q_j} \\
&= \sup_i \sum_{k_{i-1}<j\leq k_i} 4^{-i} M_i^{-(q_j-1)} n_i^{-rq_j} |f_j^{n_i}(x_j)|^{q_j} \\
&= \sup_i \sum_{k_{i-1}<j\leq k_i} \left(|f_j^{n_i}(x_j)|^{q_j} n_i^{-rq_j} M_i^{-q_j}\right) 4^{-i} M_i \\
&\geq \sup_i 2^i = \infty, \text{ because } M_i > 4^i
\end{aligned}$$

This is contradictory with (3.10). Therefore (3.3) is satisfied. □

Theorem 3.3 *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k > 1$ for all $k \in \mathbb{N}$, $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$, $r \geq 0$ and $s \geq 0$. Then for an infinite matrix $A = (f_k^n)$, $A \in (F_r(X,p),\ E_s)$ if and only if there is $m_0 \in \mathbb{N}$ such that*

$$\sup_n \sum_{k=1}^{\infty} \left(k^{-rq_k/p_k} \|f_k^n\|^{q_k} n^{-sq_k} m_0^{-q_k}\right) < \infty.$$

Proof. Since $F_r(X,p) = \ell(X,p)_{(k^{r/p_k})}$, it is easy to see that

$$A \in (F_r(X,p),\ E_s) \iff \left(k^{-r/p_k} f_k^n\right)_{n,k} \in (\ell(X,p)\ E_s)$$

By Theorem 3.2, we have $\left(k^{-r/p_k} f_k^n\right)_{n,k} \in (\ell(X,p)\ E_s)$ if and only if there is $m_0 \in \mathbb{N}$ such that

$\sup_n \sum_{k=1}^{\infty} \left(k^{-rq_k/p_k} \|f_k^n\|^{q_k} n^{-sq_k} m_0^{-q_k}\right. < \infty$. Thus the theorem is proved. □

Since $E_0 = \ell_\infty$, the following two results are obtained directly from Theorem 3.2 and Theorem 3.3, respectively.

Corollary 3.4 *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in I\!N$ and let $1/p_k + 1/q_k = 1$ for all $k \in I\!N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p),\ \ell_\infty)$ if and only if there is $m_0 \in I\!N$ such that* $\sup_n \sum_{k=1}^{\infty} \|f_k^n\|^{q_k} m_0^{-q_k} < \infty$.

Corollary 3.5 *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in I\!N$ and let $1/p_k + 1/q_k = 1$ for all $k \in I\!N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (F_r(X,p),\ \ell_\infty)$ if and only if there is $m_0 \in I\!N$ such that* $\sup_n \sum_{k=1}^{\infty} \left(k^{-rq_k/p_k} \|f_k^n\|^{q_k} m_0^{-q_k}\right) < \infty$.

Theorem 3.6 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in I\!N$ and let $1/p_k + 1/t_k = 1$ for all $k \in I\!N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), \underline{\ell}_\infty(q))$ if and only if for each $r \in I\!N$, there is $m_r \in I\!N$ such that $\sup_{n,\,k}\ r^{t_k/q_n} \|f_k^n\|^{t_k} m_r^{-t_k} < \infty$.*

Proof. Since $\underline{\ell}_\infty(q) = \cap_{r=1}^{\infty} \ell_{\infty(r^{1/q_k})}$, it follows that

$$A \in (\ell(X,p), \underline{\ell}_\infty(q)) \iff A \in (\ell(X,p),\ \ell_{\infty(r^{1/q_k})}) \text{ for all } r \in I\!N$$

It is easy to show that for $r \in I\!N$,

$$A \in (\ell(X,p),\ \ell_{\infty(r^{1/q_k})}) \iff \left(r^{1/q_n} f_k^n\right)_{n,k} \in (\ell(X,p),\ \ell_\infty)\ .$$

We obtain by Corollary 3.4 that for $r \in I\!N$,

$$\left(r^{1/q_n} f_k^n\right)_{n,k} \in (\ell(X,p),\ \ell_\infty) \iff \text{there is } m_r \in N \text{ such that } \sup_n \sum_{k=1}^{\infty} r^{t_k/q_n} \|f_k^n\|^{t_k} m_r^{-t_k} < \infty$$

Thus the theorem is proved. □

Theorem 3.7 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in I\!N$ and let $1/p_k + 1/t_k = 1$ for all $k \in I\!N$. For an* *infinite matrix* $A = (f_k^n)$, $A \in \big(F_r(X,p),\ \ell_\infty(q)\big)$ *if and only if for each* $i \in I\!N$, *there is* $m_i \in I\!N$ *such that*

$$sup_n \sum_{k=1}^{\infty} i^{t_k/q_n} k^{-rt_k/p_k} \|f_k^n\|^{t_k} m_i^{-t_k} < \infty.$$

Proof. Since $F_r(X,p) = \ell(X,p)_{(k^{r/p_k})}$, it implies that

$$A \in (F_r(X,p), \ell_\infty(q)) \iff \big(k^{-r/p_k} f_k^n\big)_{n,k} \in (\ell(X,p), \ell_\infty(q)).$$

It follows from Thoerem 3.6 that $A \in (F_r(X,p), \ell_\infty(q))$ if and only if for each $i \in I\!N$, there is $m_i \in I\!N$ such that $sup_n \sum_{k=1}^{\infty} i^{t_k/q_n} k^{-rt_k/p_k} \|f_k^n\|^{t_k} m_i^{-t_k} < \infty$.

Theorem 3.8 *Let* $p = (p_k)$ *be bounded sequence of positive real numbers with* $p_k > 1$ *for all* $n \in I\!N$ *and let* $1/p_k + 1/q_k = 1$ *for all* $k \in I\!N$. *Then for an infinite matrix* $A = (f_k^n)$, $A \in (\ell(X,p),\ bs)$ *if and only if there is* $m_0 \in I\!N$ *such that* $sup_n \sum_{k=1}^{\infty} \| \sum_{i=1}^{n} f_k^i \|^{q_k} m_0^{-q_k} < \infty$.

Proof. For an infinite matrix $A = (f_k^n)$, we can easily show that

$$A \in (\ell(X,p),\ bs) \iff \big(\sum_{i=1}^{n} f_k^i\big)_{n,k} \in (\ell(X,p),\ \ell_\infty) .$$

This implies by Corollary 3.4 that $A \in (\ell(X,p),\ bs)$ if and only if there is $m_0 \in I\!N$ such that

$$sup_n \sum_{k=1}^{\infty} \| \sum_{i=1}^{n} f_k^i \|^{q_k} m_0^{-q_k} < \infty .$$

□

Theorem 3.9 *Let* $p = (p_k)$ *be a bounded sequence of positive real numbers with* $p_k > 1$ *for all* $k \in I\!N$ *and let* $1/p_k + 1/q_k = 1$ *for all* $k \in I\!N$. *Then for an infinite matrix* $A = (f_k^n)$, $A \in (\ell(X,p),\ cs)$ *if and only if*

(1) there is $m_0 \in I\!N$ *such that* $sup_n \sum_{k=1}^{\infty} \| \sum_{i=1}^{n} f_k^i \|^{q_k} m_0^{-q_k} < \infty$ *and*

(2) for each $k \in I\!N$ *and* $x \in X$, $\sum_{n=1}^{\infty} f_k^n(x)$ *converges.*

Proof. The necessity is obtained by Theorem 3.8 and by the fact that $e^{(k)}(x) \in \ell(X,p)$ for every $k \in I\!N$ and $x \in X$.

Now, suppose that (1) and (2) hold. By Theorem 3.8, we have $A : \ell(X,p) \to bs$. Let $x = (x_k) \in \ell(X,p)$. Since $\ell(X,p)$ has the AK property, we have $x = \lim\limits_{n\to\infty} \sum_{k=1}^{n} e^{(k)}(x_k)$. By Zeller's theorem, $A : \ell(X,p) \to bs$ is continuous. It implies that $Ax = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} Ae^{(k)}(x_k)$. By (2), $Ae^{(k)}(x_k) \in cs$ for all $k \in I\!N$. Since cs is a closed subspace of bs, it implies that $Ax \in cs$, that is $A : \ell(X,p) \to cs$. $\square$

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On β-Dual of Vector-Valued Sequence Spaces of Maddox

Abstract. In this paper, the β-dual of a vector-valued sequence space is defined and studied. We show that if an X-valued sequence space $S(X)$ is a BK-space having AK property, then the dual space of $S(X)$ and its β-dual are isometrically isomorphic. We also give characterizations of β-daul of vector-valued sequence spaces of Maddox $\ell(X,p)$, $\ell_\infty(X,p)$, $c_0(X,p)$, and $c(X,p)$.

Keywords: β-dual ; vector-valued sequence spaces of Maddox

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1. Introduction

Let $(X, \|.\|)$ be a Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. Let N be the set of all natural numbers, we write $x = (x_k)$ with x_k in X for all $k \in N$. The $X-$valued sequence spaces of Maddox are defined as

$c_0(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\}$;

$c(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X\}$;

$\ell_\infty(X,p) = \{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\}$;

$\ell(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\}$.

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p), c(p), \ell_\infty(p)$, and $\ell(p)$, respectively. All of these spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons [7] and Maddox [3 - 5]. The space $\ell(p)$ was first defined by Nakano [6] and is known as the Nakano sequence space. Grosse-Erdmann [1] has investigated the structure of the spaces $c_0(p), c(p), \ell(p)$, and $\ell_\infty(p)$ and has given characterizations of β-dual of scalar-valued sequence spaces of Maddox.

In [8], Wu and Bu gave characterizations of Köthe dual of the vector-valued sequence space $\ell_p[X]$, where $\ell_p[X]$ $(1 < p < \infty)$ is defined by

$$\ell_p[X] = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |f(x_k)|^p < \infty \text{ for each } f \in X' \right\}.$$

In this paper, the β-dual of a vector-valued sequence spaces is defined and studied, and we give characterizations of β-dual of vector-valued sequence spaces of Maddox $\ell(X,p)$, $\ell_\infty(X,p)$, $c_0(X,p)$, and $c(X,p)$. Some results, obtained in this paper, are generalizations of some in [3].

2. Notation and Definitions

Let $(X, \|.\|)$ be a Banach space. Let $W(X)$ and $\Phi(X)$ denote the space of all sequences in X and the space of all finite sequences in X, respectively. A sequence space in X is a linear subspace of $W(X)$. Let E be an X- valued sequence space. For $x \in E$ and $k \in N$ we write that x_k stand for the kth term of x. For $x \in X$ and $k \in N$, we let $e^{(k)}(x)$ be the sequence $(0,0,0,...,0,x,0,...)$ with x in the kth position and let $e(x)$ be the sequence $(x,x,x,...)$. For a fixed scalar sequence $u = (u_k)$ the sequence space E_u is defined as

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}.$$

An X-valued sequence space E is said to be *normal* if $(x_k) \in E$ and $(y_k) \in W(X)$ with $\|y_k\| \leq \|x_k\|$ for all $k \in N$ implies that $(y_k) \in E$. Suppose the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if, for each $k \in N$ the kth coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. In addition, if (E, τ) is a $Fre'chet(Banach)$ space, then E is called an $FK-(BK-)$ space . Now, suppose that E contains $\Phi(X)$. Then E is said to have *property AK* if $\sum_{k=1}^{n} e^{(k)}(x_k) \to x$ in E as $n \to \infty$ for every $x = (x_k) \in E$

The spaces $c_0(p)$ and $c(p)$ are FK-spaces. In $c_0(X,p)$, we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = \max\,\{1, \sup_k p_k\}$, as a paranorm on $c_0(X,p)$, and it is known that $c_0(X,p)$ is an FK-space having property AK under the paranorm g defined as above. In $\ell(X,p)$, we consider it as a paranormed sequence space with the paranorm given by $\|(x_k)\| = \left(\sum_{k=1}^{\infty} \|x_k\|^{p_k}\right)^{1/M}$, where $M = \max\,\{1, \sup_k p_k\}$. It is known that $\ell(X,p)$ is an FK-space under the paranorm defined as above.

For an X-valued sequence space $S(X)$, define its Köthe dual with respect to the dual pair (X, X')

(see [2]) as follow :

$$S(X)^{\times}|_{(X,X')} = \left\{ (f_k) \subset X' : \sum_{k=1}^{\infty} |f_k(x_k)| < \infty \text{ for all } x = (x_k) \in S(X) \right\}.$$

Sometime we denote $S(X)^{\times}|_{(X,X')}$ by $S(X)^{\alpha}$ and it is called the α-dual of $S(X)$.

For a sequence space $S(X)$, the β-dual of $S(X)$ is defined by

$$S(X)^{\beta} = \left\{ (f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x_k) \text{ converges for all } (x_k) \in S(X) \right\}.$$

It is easy to see that $S(X)^{\alpha} \subseteq S(X)^{\beta}$.

For the sake of completeness we introduce some further sequence spaces that will be considered as β-dual of the vector-valued sequence spaces of Maddox :

$M_0(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\| M^{-1/p_k} < \infty \text{ for some } M \in N \}$;

$M_{\infty}(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\| n^{1/p_k} < \infty \text{ for all } n \in N \}$;

$\ell_0(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} M^{-(p_k-1)} < \infty \text{ for some } M \in N \}$; where $p_k > 1$ for all $k \in N$,

$cs[X'] = \{(f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x) \text{ converges for all } x \in X \}$.

When $X = K$, the scalar field of X, the corresponding first two sequence spaces are written as $M_0(p)$ and $M_{\infty}(p)$, respectively. These spaces were first introduced by Grosse-Erdmann [3].

3. Main Results

We begin with giving some general properties of β-dual of vector-valued sequence spaces.

Proposition 3.1. *Let X be a Banach space and let $S(X)$, $S_1(X)$, and $S_2(X)$ be X-valued sequence spaces. Then*

(i) $S(X)^{\alpha} \subseteq S(X)^{\beta}$.

(ii) If $S_1(X) \subseteq S_2(X)$, *then* $S_2(X)^{\beta} \subseteq S_1(X)^{\beta}$.

(iii) If $S(X) = S_1(X) + S_2(X)$, *then* $S(X)^{\beta} = S_1(X)^{\beta} \cap S_2(X)^{\beta}$.

(iv) If $S(X)$ *is normal, then* $S(X)^{\alpha} = S(X)^{\beta}$.

Proof Assertions (i) - (iii) are immediately obtained by the definitions. To prove (iv), by (i), it suffices to show only that $S(X)^{\beta} \subseteq S(X)^{\alpha}$. Let $(f_k) \in S(X)^{\beta}$ and $x = (x_k) \in S(X)$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges. Choose a scalar sequence (t_k) such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Since $S(X)$ is normal, $(t_k x_k) \in S(X)$. Thus $\sum_{k=1}^{\infty} |f_k(x_k)| = \sum_{k=1}^{\infty} f_k(t_k x_k)$ and the series $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges. This implies that $(f_k) \in S(X)^{\alpha}$. □

If $S(X)$ is an BK-space, we define a norm on $S(X)^{\beta}$ by the formular

$$\|(f_k)\|_{S(X)^{\beta}} = \sup_{\|(x_k)\| \leq 1} \left| \sum_{k=1}^{\infty} f_k(x_k) \right|.$$

It is easy to show that $\|.\|_{S(X)^{\beta}}$ is a norm on $S(X)^{\beta}$

Next, we give some relations between β-dual of a sequence space and its dual. Indeed, we need a lemma.

Lemma 3.2. *Let $S(X)$ be an X-valued sequence space which is an FK-space and contains $\Phi(X)$. Then for each $k \in N$, the mapping $T_k : X \to S(X)$, defined by $T_k x = e^k(x)$, is continuous.*

Proof. Let $V = \{e^k(x) : x \in X\}$. Then V is a closed subspace of $S(X)$, so it is an FK-space because $S(X)$ is an FK-space. Since $S(X)$ is a K-space, the coordinate mapping $p_k : V \to X$ is continuous and bijective. It follows from the open mapping theorem that p_k is open, which implies that $p_k^{-1} : X \to V$ is continuous. But since $T_k = p_k^{-1}$, we thus obtain that T_k is continuous. □

Theorem 3.3. *If $S(X)$ is a BK-space having property AK, then $S(X)^{\beta}$ and $S(X)'$ are isometrically isomorphic.*

Proof. We first show that for $x = (x_k) \in S(X)$ and $f \in S(X)'$,

$$f(x) = \sum_{k=1}^{\infty} f(e^k(x_k)) \tag{3.1}$$

To show this, let $x = (x_k) \in S(X)$ and $f \in S(X)'$. Since $S(X)$ has property AK, $x = \lim_{n\to\infty} \sum_{k=1}^{n} e^{(k)}(x_k)$. By the continuity of f, it follows that $f(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f(e^{(k)}(x_k)) =$

$\sum_{k=1}^{\infty} f(e^{(k)}(x_k))$, so (3.1) is obtained. For each $k \in N$, let $T_k : X \to S(X)$ be defined as in Lemma 3.2. Since $S(X)$ is a BK-space, by Lemma 3.2, T_k is continuous. Hence $f \circ T_k \in X'$ for all $k \in N$. It follows from (3.1) that

$$f(x) = \sum_{k=1}^{\infty}(f \circ T_k)(x_k) \text{ for all } x = (x_k) \in S(X). \tag{3.2}$$

We have by (3.2) that $(f \circ T_k)_{k=1}^{\infty} \in S(X)^{\beta}$. Define $\varphi : S(X)' \to S(X)^{\beta}$ by

$$\varphi(f) = (f \circ T_k)_{k=1}^{\infty} \text{ for all } f \in S(X)'.$$

It is easy to see that φ is linear. Now, we shall show that φ is onto. Let $(f_k) \in S(X)^{\beta}$. Define $f : S(X) \to K$, where K is the scalar field of X, by

$$f(x) = \sum_{k=1}^{\infty} f_k(x_k) \text{ for all } x = (x_k) \in S(X) \tag{3.3}$$

For each $k \in N$,let p_k be the kth coordinate mapping on $S(X)$. Then we have

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty}(f_k \circ p_k)(x) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n}(f \circ p_k)(x). \end{aligned}$$

Since f_k and p_k are continuous linear, so is $f \circ p_k$. It follows by Banach-Steinhaus theorem that $f \in S(X)'$, and we have by (3.3) that for each $k \in N$ and each $z \in X$, $(f \circ T_k)(z) = f(e^{(k)}(z)) = f_k(z)$. Thus $f \circ T_k = f_k$ for all $k \in N$, which implies $\varphi(f) = (f_k)$, hence φ is onto.

Finally, we shall show that φ is linear isometry. For $f \in S(X)'$, we have

$$\begin{aligned}
\|f\| &= \sup_{\|(x_k)\|\leq 1} |f((x_k))| \\
&= \sup_{\|(x_k)\|\leq 1} \left|\sum_{k=1}^{\infty} f(e^{(k)}(x_k))\right| \quad \text{(by (3.1))} \\
&= \sup_{\|(x_k)\|\leq 1} \left|\sum_{k=1}^{\infty} (f \circ T_k)(x_k)\right| \\
&= \|(f \circ T_k)_{k=1}^{\infty}\|_{S(X)^\beta} \\
&= \|\varphi(f)\|_{S(X)^\beta}.
\end{aligned}$$

Hence φ is isometry. Therefore $\varphi : S(X)' \to S(X)^\beta$ is an isometrically isomorphism form $S(X)'$ onto $S(X)^\beta$, so the theorem is proved. □

We next give characterizations of β-dual of the sequence space $\ell(X,p)$ when $p_k > 1$ for all $k \in N$.

Theorem 3.4. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$. Then $\ell(X,p)^\beta = \ell_0(X',q)$, where $q = (q_k)$ is a sequence of positive real numbers such that $1/p_k + 1/q_k = 1$ for all $k \in N$.*

Proof. Suppose that $(f_k) \in \ell_0(X',q)$. Then $\sum_{k=1}^{\infty} \|f_k\|^{q_k} M^{-(q_k-1)} < \infty$ for some $M \in N$.

Then for each $x = (x_k) \in \ell(X,p)$, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} |f_k(x_k)| &\leq \sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} M^{1/p_k} \|x_k\| \\
&\leq \sum_{k=1}^{\infty} \left(\|f_k\|^{q_k} M^{-q_k/p_k} + M\|x_k\|^{p_k}\right) \\
&= \sum_{k=1}^{\infty} \|f_k\|^{q_k} M^{-(q_k-1)} + M\sum_{k=1}^{\infty} \|x_k\|^{p_k} \quad < \infty ,
\end{aligned}$$

which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so $(f_k) \in \ell(X,p)^\beta$.

On the other hand, assume that $(f_k) \in \ell(X,p)^\beta$, then $\sum_{k=1}^\infty f_k(x_k)$ converges for all $x = (x_k) \in \ell(X,p)$. For each $x = (x_k) \in \ell(X,p)$, choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_kx_k) = |f_k(x_k)|$ for all $k \in N$. Since $(t_kx_k) \in \ell(X,p)$, by our assumption, we have $\sum_{k=1}^\infty f_k(t_kx_k)$ converges, so that

$$\sum_{k=1}^{\infty}|f_k(x_k)| < \infty \quad \text{for all } x \in \ell(X,p). \tag{3.4}$$

We want to show that $(f_k) \in \ell_0(X',q)$, that is $\sum_{k=1}^\infty \|f_k\|^{q_k} M^{-(q_k-1)} < \infty$ for some $M \in N$. If it is not true, then

$$\sum_{k=1}^{\infty}\|f_k\|^{q_k} m^{-(q_k-1)} = \infty, \text{ for all } m \in N. \tag{3.5}$$

It implies by (3.5) that for each $k \in N$,

$$\sum_{i>k}\|f_i\|^{q_i} m^{-(q_i-1)} = \infty, \text{ for all } m \in N. \tag{3.6}$$

By (3.5), let $m_1 = 1$, then there is a $k_1 \in N$ such that

$$\sum_{k\le k_1} \|f_k\|^{q_k} m_1^{-(q_k-1)} > 1.$$

By (3.6), we can choose $m_2 > m_1$ and $k_2 > k_1$ with $m_2 > 2^2$ such that

$$\sum_{k_1<k\le k_2} \|f_k\|^{q_k} m_2^{-(q_k-1)} > 1. \tag{3.7}$$

Proceeding in this way, we can choose sequences of positive integers (k_i) and (m_i) with $1 = k_0 < k_1 < k_2 < \ldots$ and $m_1 < m_2 < \ldots$, such that $m_i > 2^i$ and

$$\sum_{k_{i-1}<k\le k_i} \|f_k\|^{q_k} m_i^{-(q_k-1)} > 1\,.$$

For each $i \in N$, choose x_k in X with $\|x_k\| = 1$ for all $k \in N$, $k_{i-1} < k \le k_i$ such that

$$\sum_{k_{i-1}<k\le k_i} |f_k(x_k)|^{q_k} m_i^{-(q_k-1)} > 1 \text{ for all } i \in N.$$

Let $a_i = \sum_{k_{i-1}<k\leq k_i} |f_k(x_k)|^{q_k} m_i^{-(q_k-1)}$. Put $y = (y_k)$, $y_k = a_i^{-1} m_i^{-(q_k-1)} |f_k(x_k)|^{q_k-1} x_k$ for all k $k_{i-1} < k \leq k_i$. For each $i \in N$, we have

$$\begin{aligned}
\sum_{k_{i-1}<k\leq k_i} \|y_k\|^{p_k} &= \sum_{k_{i-1}<k\leq k_i} \left\| a_i^{-1} m_i^{-(q_k-1)} |f_k(x_k)|^{q_k-1} x_k \right\|^{p_k} \\
&= \sum_{k_{i-1}<k\leq k_i} a_i^{-p_k} m_i^{-q_k} |f_k(x_k)|^{q_k} \\
&\leq \sum_{k_{i-1}<k\leq k_i} a_i^{-1} m_i^{-1} m_i^{-(q_k-1)} |f_k(x_k)|^{q_k} \\
&= a_i^{-1} m_i^{-1} a_i \\
&= m_i^{-1} \\
&< 1/2^i.
\end{aligned}$$

So we have that $\sum_{k=1}^{\infty} \|y_k\|^{p_k} \leq \sum_{i=1}^{\infty} 1/2^i < \infty$. Hence, $y = (y_k) \in \ell(X,p)$. For each $i \in N$, we have

$$\begin{aligned}
\sum_{k_{i-1}<k\leq k_i} |f_k(y_k)| &= \sum_{k_{i-1}<k\leq k_i} \left| f_k(a_i^{-1} m_i^{-(q_k-1)} |f_k(x_k)|^{q_k-1} x_k) \right| \\
&= \sum_{k_{i-1}<k\leq k_i} a_i^{-1} m_i^{-(q_k-1)} |f_k(x_k)|^{q_k} \\
&= a_i^{-1} \sum_{k_{i-1}<k\leq k_i} m_i^{-(q_k-1)} |f_k(x_k)|^{q_k} \\
&= 1.
\end{aligned}$$

So that $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts to (3.4). Hence $(f_k) \in \ell_0(X', q)$. The proof is now complete. $\square$

The following theorem give a characterization of β-dual of $\ell(X,p)$ when $p_k \leq 1$ for all $k \in N$. To do this, the following lemma is needed.

Lemma 3.5. *Let $p = (p_k)$ be a bounded sequences of positive real numbers. Then $\ell_\infty(X,p) = \bigcup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$.*

Proof. Let $x \in \ell_\infty(X,p)$, then there is some $n \in N$ with $\|x_k\|^{p_k} \leq n$ for all $k \in N$. Hence $\|x_k\| n^{-1/p_k} \leq 1$ for all $k \in N$, so that $x \in \ell_\infty(X)_{(n^{-1/p_k})}$. On the other hand, if $x \in \cup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$, then there are some $n \in N$ and $M > 1$ such that $\|x_k\| n^{-1/p_k} \le M$ for every $k \in N$. Then we have $\|x_k\|^{p_k} \le nM^{p_k} \le nM^\alpha$ for all $k \in N$, where $\alpha = \sup_k p_k$. Hence $x \in \ell_\infty(X,p)$. □

Theorem 3.6. *Let $p = (p_k)$ be a bounded sequence of positve real numbers with $p_k \le 1$ for all $k \in N$. Then $\ell(X,p)^\beta = \ell_\infty(X',p)$.*

Proof. If $(f_k) \in \ell(X,p)^\beta$, then $\sum_{k=1}^\infty f_k(x_k)$ converges for every $x = (x_k) \in \ell(X,p)$, using the same proof as in Theorem 3.4, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \text{for all} \quad x = (x_k) \in \ell(X,p) \tag{3.8}$$

If $(f_k) \notin \ell_\infty(X',p)$, it follows by Lemma 3.5 that $\sup_k \|f_k\| m^{-1/p_k} = \infty$ for all $m \in N$. For each $i \in N$, choose sequences (m_i) and (k_i) of positive integers with $m_1 < m_2 < \ldots$ and $k_1 < k_2 < \ldots$ such that $m_i > 2^i$ and $\|f_{k_i}\| m_i^{-1/p_{k_i}} > 1$. Choose $x_{k_i} \in X$ with $\|x_{k_i}\| = 1$ such that

$$|f_{k_i}(x_{k_i})| m_i^{-1/p_{k_i}} > 1\ . \tag{3.9}$$

Let $y = (y_k)$, $y_k = m_i^{-1/p_{k_i}} x_{k_i}$ if $k = k_i$ for some i, and 0 otherwise. Then $\sum_{k=1}^\infty \|y_k\|^{p_k} = \sum_{i=1}^\infty 1/m_i < \sum_{i=1}^\infty 1/2^i = 1$, so that $(y_k) \in \ell(X,p)$ and

$$\begin{aligned}\sum_{k=1}^{\infty} |f_k(y_k)| &= \sum_{i=1}^{\infty} |f_{k_i}(m_i^{-1/p_{k_i}} x_{k_i})| \\ &= \sum_{i=1}^{\infty} m_i^{-1/p_{k_i}} |f_{k_i}(x_{k_i})| = \infty \ \text{ by (3.9)},\end{aligned}$$

and this is contradictory to (3.8), hence $(f_k) \in \ell_\infty(X',p)$.

Conversely, assume that $(f_k) \in \ell_\infty(X',p)$. By Lemma 3.5, there exists $M \in N$ such that $\sup_k \|f_k\| M^{-1/p_k} < \infty$. Let $x = (x_k) \in \ell(X,p)$, then there is a $K > 0$ such that

$$\|f_k\| \le K M^{1/p_k} \quad \text{for all} \quad k \in N \tag{3.10}$$

and there is a $k_0 \in N$ such that $M^{1/p_k}\|x_k\| \le 1$ for all $k \ge k_0$. By $p_k \le 1$ for all $k \in N$, we have that for all $k \ge k_0$,

$$M^{1/p_k}\|x_k\| \le (M^{1/p_k}\|x_k\|)^{p_k} = M\|x_k\|^{p_k}. \tag{3.11}$$

Then

$$\begin{aligned}
\sum_{k=1}^{\infty} |f_k(x_k)| &\leq \sum_{k=1}^{k_o} \|f_k\|\|x_k\| + \sum_{k=k_0+1}^{\infty} \|f_k\|\|x_k\| \\
&\leq \sum_{k=1}^{k_o} \|f_k\|\|x_k\| + K \sum_{k=k_0+1}^{\infty} M^{1/p_k}\|x_k\| \quad \text{(by (3.10))} \\
&\leq \sum_{k=1}^{k_o} \|f_k\|\|x_k\| + KM \sum_{k=k_0+1}^{\infty} \|x_k\|^{p_k} \quad \text{(by (3.11))} \\
&< \infty.
\end{aligned}$$

This implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, hence $(f_k) \in \ell(X,p)^{\beta}$. □

Theorem 3.7. *Let $p = (p_k)$ be a bounded sequence of positve real numbers. Then $\ell_{\infty}(X,p)^{\beta} = M_{\infty}(X',p)$.*

Proof. If $(f_k) \in M_{\infty}(X',p)$, then $\sum_{k=1}^{\infty} \|f_k\| m^{1/p_k} < \infty$ for all $m \in N$, we have that for each $x = (x_k) \in \ell_{\infty}(X,p)$, there is $m_0 \in N$ such that $\|x_k\| \leq m_0^{1/p_k}$ for all $k \in N$, hence $\sum_{k=1}^{\infty} |f_k(x_k)| \leq \sum_{k=1}^{\infty} \|f_k\|\|x_k\| \leq \sum_{k=1}^{\infty} \|f_k\| m_0^{1/p_k} < \infty$, which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so that $(f_k) \in \ell(X,p)^{\beta}$.

Conversely, assume that $(f_k) \in \ell_{\infty}(X,p)^{\beta}$, then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_{\infty}(X,p)$, by using the same proof as in Theorem 3.4, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \text{ for all } x = (x_k) \in \ell_{\infty}(X,p). \tag{3.12}$$

If $(f_k) \notin M_{\infty}(X',p)$, then $\sum_{k=1}^{\infty} \|f_k\| M^{1/p_k} = \infty$ for some $M \in N$. Then we can choose a sequence (k_i) of positive integers with $0 = k_0 < k_1 < k_2 < \ldots$ such that

$$\sum_{k_{i-1} < k \leq k_i} \|f_k\| M^{1/p_k} > i \quad \text{for all } i \in N.$$

And we choose x_k in X with $\|x_k\| = 1$ such that for all $i \in N$,

$$\sum_{k_{i-1} < k \leq k_i} |f_k(x_k)| M^{1/p_k} > i.$$

Put $y = (y_k)$, $y_k = M^{1/p_k} x_k$. Clearly, $y \in \ell_\infty(X,p)$ and

$$\sum_{k=1}^{\infty} |f_k(y_k)| \geq \sum_{k_{i-1}<k\leq k_i}^{\infty} |f_k(x_k)| M^{1/p_k} > i \quad \text{for all } i \in N.$$

Hence $\sum\limits_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts to (3.12). Hence $(f_k) \in M_\infty(X',p)$. The proof is now complete. □

Theorem 3.8. *Let $p = (p_k)$ be a bounded sequence of positve real numbers. Then $c_0(X,p)^\beta = M_0(X',p)$.*

Proof. Suppose $(f_k) \in M_0(X',p)$, then $\sum_{k=1}^\infty \|f_k\| M^{-1/p_k} < \infty$ for some $M \in N$. Let $x = (x_k) \in c_0(X,p)$. Then there is a positive integer K_0 such that $\|x_k\|^{p_k} < 1/M$ for all $k \geq K_0$, hence $\|x_k\| < M^{-1/p_k}$ for all $k \geq K_0$. Then we have

$$\sum_{k=K_0}^{\infty} |f_k(x_k)| \leq \sum_{k=K_0}^{\infty} \|f_k\| \|x_k\| \leq \sum_{k=K_0}^{\infty} \|f_k\| M^{-1/p_k} < \infty.$$

It follows that $\sum_{k=1}^\infty f_k(x_k)$ converges, so that $(f_k) \in c_0(X,p)^\beta$.

On the other hand, assume that $(f_k) \in c_0(X,p)^\beta$, then $\sum_{k=1}^\infty f_k(x_k)$ converges for all $x = (x_k) \in c_0(X,p)$. For each $x = (x_k) \in c_0(X,p)$, choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Since $(t_k x_k) \in c_0(X,p)$, by our assumption, we have $\sum_{k=1}^\infty f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \text{for all } x \in c_0(X,p). \tag{3.13}$$

Now, suppose that $(f_k) \notin M_0(X',p)$. Then $\sum_{k=1}^\infty \|f_k\| m^{-1/p_k} = \infty$ for all $m \in N$. Choose $m_1, k_1 \in N$ such that

$$\sum_{k\leq k_1} \|f_k\| m_1^{-1/p_k} > 1$$

and choose $m_2 > m_1$ and $k_2 > k_1$ such that

$$\sum_{k_1<k\leq k_2} \|f_k\| m_2^{-1/p_k} > 2.$$

Proceeding in this way, we can choose $m_1 < m_2 < ...$, and $0 = k_1 < k_2 < ...$ such that

$$\sum_{k_{i-1}<k\leq k_i} \|f_k\| m_i^{-1/p_k} > i.$$

Take x_k in X with $\|x_k\| = 1$ for all $k, k_{i-1} < k \leq k_i$ such that

$$\sum_{k_{i-1}<k\leq k_i} |f_k(x_k)| m_i^{-1/p_k} > i \quad \text{for all } i \in N.$$

Put $y = (y_k)$, $y_k = m_i^{-1/p_k} x_k$ for $k_{i-1} < k \leq k_i$, then $y \in c_0(X, p)$ and

$$\sum_{k=1}^{\infty} |f_k(y_k)| \geq \sum_{k_{i-1}<k\leq k_i} |f_k(x_k)| m_i^{-1/p_k} > i \quad \text{for all } i \in N.$$

Hence we have $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$ which contradicts to (3.13), therefore $(f_k) \in M_0(X', p)$. This completes the proof. □

Theorem 3.9. *Let $p = (p_k)$ be a bounded sequence of positve real numbers. Then* $c(X,p)^{\beta} = M_0(X', p) \cap cs[X']$.

Proof. Since $c(X,p) = c_0(X,p) + E$, where $E = \{e(x) : x \in X\}$, it follows by Proposition 3.1(iii) and Theorem 3.8 that $c(X,p)^{\beta} = M_0(X',p) \cap E^{\beta}$. It is obvious by the definition that $E^{\beta} = \{(f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x) \text{ converges for all } x \in X\} = cs[X']$. Hence we have the theorem. □

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MATRIX TRANSFORMATIONS OF SOME VECTOR-VALUED SEQUENCE SPACES

Necessary and sufficient conditions have been established for an infinite matrix $A = (f_n^k)$ of continuous linear functionals on a Banach space X to transform the vector-valued sequence spaces of Maddox $\ell_\infty(X,p)$, $\ell(X,p)$, $c_0(X,p)$, and $c(X,p)$ into the scalar-valued sequence space $c(q)$, where $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers.

Keywords: Matrix transformations, Maddox vector-valued sequence spaces

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1. Introduction

The study of matrix transformations of scalar- valued sequence spaces is known since the turn of the century. In seventies, Maddox[12], Gupta[4] studied matrix transformations of continuous linear mappings on vector-valued sequence spaces. Das and Choudhury[1] gave conditions on the matrix $A = (f_k^n)$ of continuous linear mappings from a normed linear space X into a normed linear space Y under which A maps $c_0(X)$ into $c_0(Y)$, $\ell_1(X)$ into $\ell_\infty(Y)$, and $\ell_1(X)$ into $\ell_p(Y)$. Liu and Wu[22] gave the matrix characterizations from vector-valued sequence spaces $c_0(X,p)$, $\ell(X,p)$, and $\ell_\infty(X,p)$ into scalar-valued sequence spaces $c_0(q)$ and $\ell_\infty(q)$. Suantai[20] gave the matrix characterizations from the Nakano vector-valued sequence space $\ell(X,p)$ into the vector-valued sequence spaces $c_0(Y,q)$, $c(Y)$, and $\ell_r(Y)$. In this paper, we continue the study of matrix transformations of continuous linear mappings on vector-valued sequence spaces.

The main purpose of this paper is to give the matrix characterizations from $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, and $\ell(X,p)$ into $c(q)$, where $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, and $\ell(X,p)$ are the vector-valued sequence spaces of Maddox as defined in Section 2. When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, and $\ell(p)$, respectively. Several papers deal with the problem of characterizing those matrices that map a scalar-valued sequence space of Maddox into anoher such

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spaces, see [6, 7, 11, 13, 15, 17, 18, 19, 21]. Some of these results become particular cases of our theorems. Also some more interesting results are derived.

Section 2 deals with necessary preliminaries and some known results quoted as lemmas which are needed to characterize an infinite matrix $A = (f_k^n)$ such that A maps the vector- valued sequence spaces of Maddox into $c(q)$, and we also give some auxiliary results in Section 3. The main results of the paper is in Section 4.

2 Preliminaries and Lemmas

Let $(X, \|.\|)$ be a Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. Let N be the set of all natural numbers, we write $x = (x_k)$ with x_k in X for all $k \in N$. Let $W(X)$ and $\Phi(X)$ denote the space of all sequences and the space of all finite sequences in X, respectively. When $X = K$, the scalar field of X, the corresponding spaces are written as w and ϕ, respectively. An $X-$valued sequence space is a linear subspace of $W(X)$. The sequence spaces of Maddox are defined as

$c_0(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\}$,

$c(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X\}$,

$\ell_\infty(X,p) = \{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\}$,

$\ell(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\}$.

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p), c(p), \ell_\infty(p)$, and $\ell(p)$, respectively. All of these spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons[16] and Maddox[8,9]. The space $\ell(p)$ was first defined by Nakano[14] and it is known as the Nakano sequence space. Also, we need to define the following sequence space :

$M_0(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\| n^{-1/p_k} < \infty \text{ for some } n \in N\}$.

When $X = K$, the scalar field of X, the corresponding space is written as $M_0(p)$. This space was first introduced by Maddox[10]. Grosse-Erdmann[2] has investigated the structure of the spaces $c_0(p), c(p), \ell(p)$, and $\ell_\infty(p)$ and he also gave the matrix characterizations between scalar-valued sequence spaces of Maddox in [3]. Let E be an X-valued sequence space. For $x \in E$ and $k \in N$ we write that x_k stand for the kth term of x and for $x \in X$ and $k \in N$, let $e^{(k)}(x)$ be the sequence $(0, 0, 0, ..., 0, x, 0, ...)$ with x in the kth position and let $e(x)$ be the sequence $(x, x, x, ...)$, and we denote by e the the sequence $(1,1,1,...)$. An $X-$valued sequence space E is said to be *normal* if $(x_k) \in E$ and $(y_k) \in W(X)$ with $\|y_k\| \leq \|x_k\|$ for all $k \in N$ implies that $(y_k) \in E$. For a fixed scalar sequence $u = (u_k)$ the sequence space E_u is defined as

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}.$$

The $\alpha-$, $\beta-$ and $\gamma-$ duals of a scalar-valued sequence space F are defined as

$$F^{\varsigma} = \{x \in w : (x_k y_k) \in X_{\varsigma} \text{ for every } \ y \in F\}$$

for $\zeta = \alpha,\ \beta,\ \gamma$ and $X_\alpha = \ell_1$, $X_\beta = cs$, and $X_\gamma = bs$, where ℓ_1, cs and bs are defined as

$\ell_1 = \{x = (x_k) \in w : \sum_{k=1}^{\infty} |x_k| < \infty \}$,
$cs = \{x = (x_k) \in w : \sum_{k=1}^{\infty} x_k \text{ converges } \}$,
$bs = \{x = (x_k) \in w : \sup_n | \sum_{k=1}^{n} x_k| < \infty \}$.

In the same manner, for an $X-$valued sequence space E, the $\alpha-$, $\beta-$ and $\gamma-$ duals of E are defined as

$$E^{\varsigma} = \{(f_k) \subset X' : (f_k(x_k)) \in X_{\varsigma} \text{ for every } \ x = (x_k) \in E\}$$

for $\zeta = \alpha,\ \beta,\ \gamma$, where $X_\alpha = \ell_1$, $X_\beta = cs$ and $X_\gamma = bs$.

It is obvious from the definition that $E^\alpha \subseteq E^\beta \subseteq E^\gamma$ and it is easy to see that if E is normal, then $E^\alpha = E^\beta = E^\gamma$.

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is an X-valued sequence space and F a scalar-valued sequence space. Then A is said to *map E into F*, written by $A : E \to F$ if, for each $x = (x_k) \in E, A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$ and the sequence $Ax = (A_n(x)) \in F$. We denote by (E, F) the class of all infinite matrices mapping E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$${}_u(E, F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E, F)\}.$$

If $u_k \neq 0$ for all $k \in N$, we put $u^{-1} = (1/u_k)$. Suppose the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if, for each $k \in N$ the kth coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. A K-space that is a Fréchet(Banach) space is called an $FK-(BK-)$ space.

The spaces $c_0(p)$ and $c(p)$ are FK-spaces. In $c_0(X,p)$, we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = \max\{1, \sup_k p_k\}$, as a paranorm on $c_0(X,p)$, and it is known that $c_0(X,p)$ is an FK-space under the paranorm g defined as above. In $\ell(X,p)$, we consider it as a paranormed sequence space with the paranorm given by $\|(x_k)\| = \left(\sum_{k=1}^{\infty} \|x_k\|^{p_k}\right)^{1/M}$. It is known that $\ell(X,p)$ is an FK-space under the paranorm defined as above.

Now let us quote some known results as the following.

Lemma 2.1[10] *If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$, then*

$$\ell(p)^{\beta} = \{x \in w : \sum_{k=1}^{\infty} |x_k|^{t_k} M^{-t_k} < \infty \text{ for some } M \in N\}$$

where $1/p_k + 1/t_k = 1$ for all $k \in N$.

Lemma 2.2[16] *If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in N$, then $\ell(p)^{\beta} = \ell_{\infty}(p)$.*

Lemma 2.3[6] *If $p = (p_k)$ is a bounded sequence of positive real numbers , then*

$$\ell_{\infty}(p)^{\beta} = \{x \in w : \sum_{k=1}^{\infty} |x_k| n^{1/p_k} < \infty \text{ for all } n \in N\}.$$

Lemma 2.4[10] *If $p = (p_k)$ is a bounded sequence of positive real numbers, then $c_0(p)^{\beta} = M_0(p)$.*

Lemma 2.5[22] *Let $p = (p_k)$ be a bounded sequence of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c_0(X,p) \to c_0$ if and only if*

(1) $f_k^n \overset{w^*}{\to} 0$ *as $n \to \infty$ for each $k \in N$ and*

(2) $\lim_{m\to\infty} \sup_n \sum_{k=1}^{\infty} \|f_k^n\| m^{-1/p_k} = 0.$

Lemma 2.6[22] *Let $p = (p_k)$ be a bounded sequence of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : \ell_\infty(X, p) \to c_0$ if and only if*

(1) $f_k^n \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for each* $k \in N$ *and*

(2) *for each* $M \in N$, $\sum_{j>k} \|f_j^n\| M^{1/p_j} \to 0$ *as* $k \to \infty$ *uniformly on* $n \in N$.

Lemma 2.7[22] *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ and $1/p_k + 1/t_k = 1$ for all $k \in N$ and let $A = (f_k^n)$ be an infinite matrix. Then $A : \ell(X, p) \to c_0$ if and only if*

(1) $f_k^n \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for each* $k \in N$ *and*

(2) $\sum_{k=1}^{\infty} \|f_k^n\|^{t_k} m^{-t_k} \to 0$ *as* $m \to \infty$ *uniformly on* $n \in N$.

Lemma 2.8[22] *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in N$ and let $A = (f_k^n)$ be an infinite matrix. Then $A : \ell(X, p) \to c_0$ if and only if*

(1) $f_k^n \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for each* $k \in N$ *and*

(2) $\sup_{n,k} \|f_k^n\|^{p_k} < \infty$.

3. Some Auxiliary Results

Suppose that E and F are sequence spaces and that we want to characterize the matrix space (E, F). If E and/or F can be derived from simpler sequence spaces in some fashion, then, in many cases, the problem reduces to the characterization of the corresponding simpler matrix spaces. We begin with giving various useful results in this direction.

Proposition 3.1. *Let E and $E_n (n \in N)$ be X-valued sequence spaces, and F and $F_n (n \in N)$ scalar-valued sequence spaces, and let u and v be scalar sequences with $u_k \neq 0, v_k \neq 0$ for all $k \in N$. Then*

(i) $(\cup_{n=1}^{\infty} E_n, F) = \cap_{n=1}^{\infty} (E_n, F)$,

(ii) $(E, \cap_{n=1}^{\infty} F_n) = \cap_{n=1}^{\infty} (E, F_n)$,

(iii) $(E_1 + E_2, F) = (E_1, F) \cap (E_2, F)$,

(iv) $(E_u, F_v) = {}_v(E, F)_{u^{-1}}$.

Proof. All of them are obtained directly from the definitions.

Propostion 3.2. *Let $p = (p_k)$ be a bounded sequences of positive real numbers . Then*

(i) $c(X,p) = c_0(X,p) + \{e(x) : x \in X\}$,

(ii) $M_0(X,p) = \cup_{n=1}^{\infty} \ell(X)_{(n^{-1/p_k})}$,

(iii) $\ell_\infty(X,p) = \cup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$.

Proof. Assertions (i) and (ii) are immediately obtained from the definitions. To show (iii), let $x \in \ell_\infty(X,p)$, then there is some $n \in N$ with $\|x_k\|^{p_k} \leq n$ for all $k \in N$. Hence $\|x_k\| n^{-1/p_k} \leq 1$ for all $k \in N$, so that $x \in \ell_\infty(X)_{(n^{-1/p_k})}$. On the other hand, if $x \in \cup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$, then there are some $n \in N$ and $M > 1$ such that $\|x_k\| n^{-1/p_k} \leq M$ for every $k \in N$. Then we have $\|x_k\|^{p_k} \leq nM^{p_k} \leq nM^{\alpha}$ for all $k \in N$, where $\alpha = \sup_k p_k$. Hence $x \in \ell_\infty(X,p)$. □

The next proposition give a relationship between the $\beta-$ dual of vector-valued and scalar-valued sequence spaces.

Proposition 3.3 *Let X be a Banach space and F a normal scalar-valued sequence space and define $F(X) = \{(x_k) \in W(X) : (\|x_k\|) \in F \}$. then for $(f_k) \subset X'$, the topological dual of X, $(f_k) \in F(X)^\beta$ if and only if $(\|f_k\|) \in F^\beta$.*

Proof. If $(\|f_k\|) \in F^\beta$, then for $x = (x_k) \in F(X)$ we have $\sum_{k=1}^{\infty} |f_k(x_k)| \leq \sum_{k=1}^{\infty} \|f_k\| \|x_k\| < \infty$, so that $x \in F(X)^\beta$.

Conversely, suppose that $(f_k) \in F(X)^\beta$ and $a = (a_k) \in F$. Since F is normal, $(|a_k|) \in F$. For each $k \in N$, we can choose $x_k \in X$ such that $\|x_k\| = 1$ and $|f_k(x_k)| \geq \dfrac{\|f_k\|}{2}$. Let $y = (a_k x_k)$, then $y \in F(X)$. Choose a sequence (t_k) of scalars such that $|t_k| \leq 1$ and $f_k(t_k a_k x_k) = |f_k(x_k)||a_k|$ for all $k \in N$. Since F is normal, $(t_k y_k) \in F(X)$, so we obtain that $\sum_{k=1}^{\infty} f_k(t_k y_k)$ converges. This implies $\sum_{k=1}^{\infty} \|f_k\| |a_k| \leq 2 \sum_{k=1}^{\infty} |f_k(x_k)||a_k| < \infty$. It follows that $(\|f_k\|) \in F^\beta$.

By using Proposition 3.3, the following results are obtained immediately from Lemma 2.1 - 2.4, respectively.

Proposition 3.4 *If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$, then*

$$\ell(X,p)^\beta = \{(f_k) \subset X' : \sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-t_k} < \infty \text{ for some } M \in N\}$$

where $1/p_k + 1/t_k = 1$ for all $k \in N$.

Proposition 3.5 *If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in N$, then $\ell(X,p)^\beta = \ell_\infty(X',p)$.*

Proposition 3.6 *If $p = (p_k)$ is a bounded sequence of positive real numbers, then*

$$\ell_\infty(X,p)^\beta = \{(f_k) \subset X' : \sum_{k=1}^{\infty} \|f_k\| n^{1/p_k} < \infty \text{ for all } n \in N\}.$$

Proposition 3.7 *If $p = (p_k)$ is a bounded sequence of positive real numbers, then $c_0(X,p)^\beta = M_0(X',p)$.*

4. Main Results

We begin with the following useful result.

Theorem 4.1. *Let $q = (q_k)$ be a bounded sequence of positive real numbers and let E be a normal $X-$ valued sequence space which is an FK-space containing $\Phi(X)$. Then*

$$(E, c(q)) = (E, c_0(q)) \oplus (E, < e >).$$

To prove this theorem, we need the following two lemmas.

Lemma 4.1. *Let E be an X-valued sequence space which is an FK-space containing $\Phi(X)$. Then for each $k \in N$, the mapping $T_k : X \to E$, defined by $T_k x = e^k(x)$, is continuous.*

Proof. For each $k \in N$, we have that $V = \{e^k(x) : x \in X\}$ is a closed subspace of E, so it is an FK-space. Since E is a K-space, the coordinate mapping $p_k : V \to X$ is continuous and bijective. It follows from the open mapping theorem that p_k is open, hence, $p_k^{-1} : X \to V$ is continuous. It follows that T_k is continuous because $T_k = p_k^{-1}$.□

Lemma 4.2. *If E and F are scalar-valued sequence spaces such that E is normal containing ϕ, F is an FK-space and there is a subsequence (n_k) with $x_{n_k} \to 0$ as $k \to \infty$ for all $x = (x_n) \in F$, then $(E,\ F \oplus < e >) = (E,\ F) \oplus (E,\ < e >)$.*

Proof. See [3, Proposition 3.1(vi)]. □

Proof of Theorem 4.1 Since $c(q) = c_0(q) \oplus < e >$, it is clear that $(E,\ c_0(q)) + (E, < e >) \subseteq (E, c_0(q) \oplus < e >) = (E, c(q))$. Moreover, if $A \in (E, c_0(q)) \bigcap (E, < e >)$, then $A \in (E,\ c_0(q) \bigcap < e >)$, so that $A \in (E, 0)$, which implies that $A = 0$ because E contain $\Phi(X)$. Hence $(E, c_0(q)) + (E, < e >)$ is a direct sum. Now, we will show that $(E, c(q)) \subseteq (E, c_0(q)) \oplus (E, < e >)$. Let $A = (f_k^n) \in (E, c(q)) = (E, c_0(q) \oplus < e >)$. For $x \in X$ and $k \in N$, we have $(f_k^n(x))_{n=1}^\infty = Ae^k(x) \in c_0(q) \oplus < e >$, so that there exist unique $(b_k^n(x))_{n=1}^\infty \in c_0(q)$ and $(c_k^n(x))_{n=1}^\infty \in < e >$ with

$$(f_k^n(x))_{n=1}^\infty = (b_k^n(x))_{n=1}^\infty + (c_k^n(x))_{n=1}^\infty. \tag{4.1}$$

For each $n, k \in N$, let g_k^n and h_k^n be the functionals on X defined by

$$g_k^n(x) = b_k^n(x) \quad \text{and} \quad h_k^n(x) = c_k^n(x) \quad \text{for all } x \in X.$$

Clearly, g_k^n and h_k^n are linear, and by (4.1)

$$f_k^n = g_k^n + h_k^n \quad \text{for all } n, k \in N. \tag{4.2}$$

Note that $c_0(q) \oplus < e >$ is an FK-space in its direct sum topology. By Zeller's theorem, $A : E \to c_0(q) \oplus < e >$ is continuous. For each $k \in N$, let $T_k : X \to E$ be defined by $T_k(x) = e^k(x)$. By Lemma 4.1, we have that T_k is continuous for all $k \in N$. Since the projection P_1 of $c_0(q) \oplus < e >$ onto $c_0(q)$ and the projection P_2 of $c_0(q) \oplus < e >$ onto $< e >$ are continuous and $g_k^n = p_n \circ P_1 \circ A \circ T_k$ and $h_k^n = p_n \circ P_2 \circ A \circ T_k$ for all $n, k \in N$, we obtain that g_k^n and h_k^n are continuous, so $g_k^n, h_k^n \in X'$ for all $n, k \in N$. Let

$B = (g_k^n)$ and $C = (h_k^n)$. By (4.1) and (4.2), we have $A = B + C$, $B = (g_k^n) \in (\Phi(X), c_0(q))$ and $C = (h_k^n) \in (\Phi(X), < e >)$. We will show that $B \in (E, c_0(q))$ and $C \in (E, < e >)$. To do this, let $x = (x_k) \in E$. Then for $\alpha = (\alpha_k) \in \ell_\infty$, we have $\|\alpha_k x_k\| = |\alpha_k| \|x_k\| \le \|M x_k\|$, where $M = \sup_k |\alpha_k|$. Then the normality of E implies that $(\alpha_k x_k) \in E$. Hence $(f_k^n(x_k))_{n,k} \in (\ell_\infty,\ c_0(q) \oplus < e >)$, moreover, we have $(g_k^n(x_k))_{n,k} \in (\Phi,\ c_0(q))$, $(h_k^n(x_k))_{n,k} \in (\Phi,\ < e >)$, and $(f_k^n(x_k))_{n,k} = (g_k^n(x_k))_{n,k} + (h_k^n(x_k))_{n,k}$. Since ℓ_∞ is normal containing ϕ and $c_0(q) \subseteq c_0$, it follows from Lemma 4.2 that $(g_k^n(x_k))_{n,k} \in (\ell_\infty,\ c_0(q))$ and $(h_k^n(x_k))_{n,k} \in (\ell_\infty,\ < e >)$. This implies that $Bx \in c_0(q)$ and $Cx \in < e >$, so we have $B \in (E, c_0(q))$ and $C \in (E, < e >)$, hence $A \in (E,\ c_0(q)) \oplus (E,\ < e >)$. This completes the proof. □

Theorem 4.2. *Let $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : \ell_\infty(X) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| < \infty$,

(2) $m^{1/q_n} (f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for every $k, m \in N$ and

(3) for each $m \in N$, $\sum_{j>k} m^{1/q_n} \|f_j^n - f_j\| \to 0$ as $k \to \infty$ uniformly on $n \in N$.

Proof. Necessity. Let $A \in (\ell_\infty(X), c(q))$. It follows from Theorem 4.1 that $A = B + C$, where $B \in (\ell_\infty(X),\ c_0(q))$ and $C \in (\ell_\infty(X),\ < e >)$. Then there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that $C = (f_k)_{n,k}$ and $B = (f_k^n - f_k)_{n,k} \in (\ell_\infty(X),\ c_0(q))$, which implies that $(f_k) \in \ell_\infty(X)^\beta$, so (1) is obtained by Proposition 3.6. Since $c_0(q) = \cap_{m=1}^{\infty} c_{0(m^{1/q_k})}$ (by [2, Theorem 0 (i)]), we have by Proposition 3.1 (ii) and (iv) that for each $m \in N$, $\left(m^{1/q_n}(f_k^n - f_k)_{n,k}\right) : \ell_\infty(X) \to c_0$. Hence, (2) and (3) are obtained by Lemma 2.6.

Sufficiency. Suppose that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that conditions (1), (2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. It is obvious that $A = B + C$. By condition (2) and (3), we obtain by Lemma 2.6 and Proposition 3.1(ii) and (iv) that $B \in (\ell_\infty(X), c_0(q))$. By Proposition 3.6 , condition (1) implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X)$, which implies that $C \in (\ell_\infty(X), < e >)$. Hence, we obtain by Theorem 4.1 that $A \in (\ell_\infty(X), c(q))$. This completes the proof. □

Theorem 4.3. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : \ell_\infty(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that (1), (2) and (3) are satisfied, where*

(1) for each $m \in N$, $\sum_{k=1}^{\infty} \|f_k\| m^{1/p_k} < \infty$,

(2) $r^{1/q_n}(f_k^n - f_k) \overset{w^*}{\to} 0$ as $n \to \infty$ for every $k, r \in N$ and

(3) for each $m, r \in N$, $r^{1/q_n} \sum_{j>k} m^{1/p_j} \|f_j^n - f_j\| \to 0$ as $k \to \infty$ uniformly on $n \in N$.

Moreover, (3) is equivalent to (3′), where

(3′) for each $m \in N$, $\lim_{k\to\infty} \sup_n \left(\sum_{j>k} m^{1/p_j} \|f_j^n - f_j\|\right)^{q_n} = 0$.

Proof. Necessity. Suppose that $A : \ell_\infty(X,p) \to c(q)$. By Theorem 4.1, $A = B + C$, where $B \in (\ell_\infty(X,p), c_0(q))$ and $C \in (\ell_\infty(X,p), < e >)$. Then there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that $C = (f_k)_{n,k}$ and $B = (f_k^n - f_k) \in (\ell_\infty(X,p), c_0(q))$. Since $C = (f_k)_{n,k} : \ell_\infty(X,p) \to < e >$, it implies by Proposition 3.6 that (1) holds. Since $c_0(q) = \cap_{m=1}^{\infty} c_{0(m^{1/q_k})}$, we have by Proposition 3.1 (ii) that for each $r \in N$, $\left(r^{1/q_n}(f_k^n - f_k)\right)_{n,k} : \ell_\infty(X,p) \to c_0$. Hence , (2) and (3) holds by an application of Lemma 2.6.

Sufficiency. Suppose that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that condition (1), (2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. It is obvious that $A = B + C$. By condition (2) and (3), we obtain by Lemma 2.6 and Proposition 3.1(ii) and (iv) that $B \in (\ell_\infty(X,p), c_0(q))$. By Proposition 3.6 , condition (1) implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X,p)$, which implies that $C \in (\ell_\infty(X,p), < e >)$. Hence, we obtain by Theorem 4.1 that $A \in (\ell_\infty(X,p), c(q))$.

Now we shall show that (3) and (3′) are equivalent. Suppose (3) holds and let $\varepsilon > 0$. Choose $r \in N$ such that $1/r < \varepsilon$. By (3), there exists $k_0 \in N$ such that

$$r^{1/q_n} \sum_{j>k} m^{1/p_j} \|f_j^n - f_j\| < 1 \text{ for all } k \geq k_0 \text{ and all } n \in N,$$

which implies that

$$\sup_n \left(\sum_{j>k} m^{1/p_j} \|f_j^n - f_j\|\right)^{q_n} \leq 1/r < \varepsilon \text{ for } k \geq k_0,$$

hence, (3′) holds.

Conversely, assume that (3′) holds. Let $m, r \in N$ and $0 < \varepsilon < 1$. Then there exists $k_0 \in N$ such that

$$\sup_n \left(\sum_{j>k} m^{1/p_j} \|f_j^n - f_j\| \right)^{q_n} < \varepsilon^H / r \quad \text{for all } k \geq k_0$$

where $H = sup_n q_n$. This implies that

$$r^{1/q_n} \sum_{j>k} m^{1/p_j} \|f_j^n - f_j\| < \varepsilon^{H/q_n} < \varepsilon \quad \text{for all } k \geq k_0 \text{ and all } \; n \in N$$

hence, (3) holds. □

Theorem 4.4. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c_0(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that (1), (2), and (3) are satisfied, where*

(1) $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in N$,

(2) $m^{\frac{1}{q_n}} (f_k^n - f_k) \overset{w^*}{\to} 0$ as $n \to \infty$ for every $m, k \in N$ and

(3) for each $m \in N$, $\sup_n \left(m^{1/q_n} \sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} \right) \to 0$ as $r \to \infty$.

Moreover, (3) is equivalent to (3′) where

(3′) $\lim_{r\to\infty} \sup_n \left(\sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} \right)^{q_n} = 0$.

Proof. Necessity. Suppose that $A : c_0(X,p) \to c(q)$. By Theorem 4.1, we have $A = B + C$, where $B \in (c_0(X,p), c_0(q))$ and $C \in (c_0(X,p), < e >)$. It follows that there is a sequence $(f_k) \subset X'$ such that $C = (f_k)_{n,k}$ and $B = (f_k^n - f_k)_{n,k}$. Since $c_0(q) = \cap_{r=1}^{\infty} c_{0(r^{1/q_k})}$, it follows from Proposition 3.1 (ii) and (iv) that for each $m \in N$, $(m^{1/q_n}(f_k^n - f_k))_{n,k} \in (c_0(X,p),\ c_0)$, hence, conditions (2) and (3) hold by using the result from Lemma 2.5. Since $C = (f_k)_{n,k} \in (c_0(X,p),\ < e >)$, we have that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = x_k \in c_0(X,p)$, so that $(f_k) \in c_0(X,p)^{\beta}$, hence, by Proposition 3.7, we obtain that there exists $M \in N$ such that $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$. Hence, (1) is obtained.

Sufficiency. Assume that there is a sequence $(f_k) \subset X'$ such that conditions (1),(2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. Then $A = B + C$. By conditions (2)and (3), we obtain from Proposition 3.1(ii) and (iv) and Lemma 2.5 that $B \in \left(c_0(X,p), c_0(q)\right)$. The condition (1) implies by Proposition 3.7 that $\sum_{k=1}^{\infty} f_k(x_k)$

converges for all $x = (x_k) \in c_0(X,p)$, so that $C \in (c_0(X,p), \; < e >)$. Hence, by Theorem 4.1, we obtain that $A \in (c_0(X,p), c(q))$.

Now, we shall show that conditions (3) and (3′) are equivalent. To do this, suppose that (3) holds and let $\varepsilon > 0$. Choose $m \in N$, $1/m < \varepsilon$. From (3), there is $r_0 \in N$ such that

$$\sup_n \; m^{1/q_n} \sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} \leq 1 \text{ for all } r \geq r_0.$$

This implies that $\sup_n \left(\sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k}\right)^{q_n} \leq 1/m < \varepsilon$ for all $r \geq r_0$. Hence, (3′) holds.

Conversely, suppose that (3′) holds. Let $m \in N$ and $0 < \varepsilon < 1$. Then there exists $r_0 \in N$ such that $\sup_n \left(\sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k}\right)^{q_n} < \varepsilon^H/m$ for all $r \geq r_0$, where $H = sup_n q_n$. Hence, we have

$$m^{1/q_n} \sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} < \varepsilon^{H/q_n} \leq \varepsilon \;\text{ for all }\; r \geq r_0 \text{ and } \; n \in N,$$

so that (3) holds. This completes the proof. □

Theorem 4.5. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that (1), (2), (3) and (4) are satisfied, where*

(1) $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in N$,

(2) for each $m, k \in N$, $m^{1/q_n}(f_k^n - f_k) \overset{w^*}{\to} 0$ as $n \to \infty$,

(3) for each $m \in N$, $\sup_n \; m^{1/q_n} \sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} \to 0$ as $r \to \infty$ and

(4) $\left(\sum_{k=1}^{\infty} f_k^n(x)\right)_{n=1}^{\infty} \in c(q)$ for all $x \in X$.

Moreover, (3) is equivalent to (3′) where

(3′) $\lim_{r\to\infty} \sup_n \left(\sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k}\right)^{q_n} = 0$.

Proof. Since $c(X,p) = c_0(X,p) + \{e(x) : x \in X\}$ (Proposition 3.2 (i)), it follows from Proposition 3.1(iii) that $A \in (c(X,p), c(q))$ if and only if $A \in (c_0(X,p), c(q))$ and $A \in (\{e(x) : x \in X\}, c(q))$. By Theorem 4.4, we have $A \in (c_0(X,p), c(q))$ if and only if conditions (1)-(3) hold and it is clear that $A \in \left(\{e(x) : x \in X\}, c(q)\right)$ if and only if (4) holds. We have by Theorem 4.4 that (3) and (3′) are equivalent. Hence, the theorem is proved. □

Wu and Liu (Lemma 2.7) have given a characterization of an infinite matrix A such that $A \in (\ell(X,p), c_0)$ when $p_k > 1$ for all $k \in N$. By applications of Proposition 3.1(ii) and (iv), Proposition 3.4, and Theorem 4.1, and using the fact that $c_0(q) = \bigcap_{m=1}^{\infty} c_{0(m^{1/q_k})}$, we obtain the following result.

Theorem 4.6. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in N$ and $1/p_k + 1/t_k = 1$ for all $k \in N$, and let $A = (f_k^n)$ be an infinite matrix. Then $A : \ell(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-t_k} < \infty$ *for some $M \in N$,*

(2) $m^{1/q_n}(f_k^n - f_k) \overset{w^*}{\to} 0$ *as $n \to \infty$ for all $m, k \in N$ and*

(3) for each $m \in N$, $\sum_{k=1}^{\infty} m^{t_k/q_n} \|f_k^n - f_k\|^{t_k} r^{-t_k} \to 0$ as $r \to \infty$ uniformly on $n \in N$.

By using Lemma 2.8, Proposition 3.1(ii) and (iv), Proposition 3.5 and Theorem 4.1, we also obtain the following result.

Theorem 4.7. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k \leq 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sup_k \|f_k\|^{p_k} < \infty$,

(2) $m^{1/q_n}(f_k^n - f_k) \overset{w^*}{\to} 0$ *as $n \to \infty$ for all $m, k \in N$ and*

(3) $\sup_{n,k} m^{p_k/q_n} \|f_k^n - f_k\|^{p_k} < \infty$ *for all $m \in N$.*

When $p_k = 1$ for all $k \in N$, we obtain the following.

Corollary 4.8. *Let $q = (q_k)$ be a bounded sequence of positive real numbers and let $A = (f_k^n)$ be an infinite matrix. Then $A : \ell_1(X) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sup_k \|f_k\| < \infty$,

(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for all* $m, k \in N$ *and*

(3) $\sup_{n,k} m^{1/q_n}\|f_k^n - f_k\| < \infty$ *for every* $m \in N$.

Theorem 4.9. *Let* $p = (p_k)$ *be a bounded sequence of positive real numbers and* $A = (f_k^n)$ *an infinite matrix. Then* $A : M_0(X,p) \to c(q)$ *if and only if there is a sequence* (f_k) *of bounded linear functionals on* X *such that*

(1) $\sup_k m^{1/p_k}\|f_k\| < \infty$ *for all* $m \in N$,

(2) for each $m, r \in N$, $r^{1/q_n} m^{1/p_k}(f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for all* $k \in N$ *and*

(3) for each $m, r \in N$, $\sup_{n,k} r^{1/q_n} m^{1/p_k}\|f_k^n - f_k\| < \infty$.

Proof. It follows from Theorem 4.1 that $A \in (M_0(X,p), c_0(q) \oplus < e >)$ if and only if there is a sequence (f_k) of bounded linear functionals on X such that $A = B + (f_k)_{n,k}$ where $B : M_0(X,p) \to c_0(q)$ and $(f_k)_{n,k} : M_0(X,p) \to < e >$. Since $B = (f_k^n - f_k)_{n,k}$ and $M_0(X,p) = \cup_{m=1}^{\infty} \ell_1(X)_{(m^{-1/p_k})}$ (by Proposition 3.2 (ii)), we have by Proposition 3.1 (i) and (iv) that $B : M_0(X,p) \to c_0(q)$ if and only if $\left(m^{1/p_k}(f_k^n - f_k)\right)_{n,k} : \ell_1(X) \to c_0(q)$ for all $m \in N$. Since $c_0(q) = \cap_{r=1}^{\infty} c_{0(r^{1/q_k})}$, by Proposition 3.1 (ii) and (iv), we have $\left(m^{1/p_k}(f_k^n - f_k)\right)_{n,k} : \ell_1(X) \to c_0(q)$ if and only if $\left(r^{1/q_n} m^{1/p_k}(f_k^n - f_k)\right)_{n,k} : \ell_1(X) \to c_0$ for all $r \in N$. By Lemma 2.8 , we have

$\left(r^{1/q_n} m^{1/p_k}(f_k^n - f_k)\right)_{n,k} : \ell_1(X) \to c_0$ if and only if

(a) $r^{1/q_n} m^{1/p_k}(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for all $k \in N$ and

(b) $\sup_{n,k} r^{1/q_n} m^{1/p_k}\|f_k^n - f_k\| < \infty$.

By Proposition 3.1 (i) and (iv), we have $(f_k)_{n,k} : M_0(X,p) \to < e >$ if and only if $(m^{1/p_k} f_k)_{n,k} : \ell_1(X) \to < e >$ for all $m \in N$. By Proposition 3.5, we obtain that $(m^{1/p_k} f_k)_{n,k} : \ell_1(X) \to < e >$ if and only if $\sup_k m^{1/p_k}\|f_k\| < \infty$. Hence, the theorem is proved. □.

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$e^{(k)}(z)$ be the sequence $(0,0,0,...,0,z,0,...)$ with z in the k^{th} position. For a fixed scalar sequence $u = (u_k)$ the sequence space E_u is defined by

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}.$$

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if for each $n \in N$ the n^{th} coordinate mapping $p_n : E \to X$, defined by $p_n(x) = x_n$, is continuous on E. If, in addition, (E,τ) is an $Fre'chet(Banach)$ space, then E is called an $FK-(BK-)$ space . Now, suppose that E contains $\Phi(X)$. Then E is said to have *property AB* if the set $\{\sum_{k=1}^{n} e^k(x_k) : n \in N\}$ is bounded in E for every $x = (x_k) \in E$. It is said to have *property AK* if $\sum_{k=1}^{n} e^k(x_k) \to x \in E$ as $n \to \infty$ for every $x = (x_k) \in E$. It has *property AD* if $\Phi(X)$ is dense in E.

If $p_k > 1$ for all $k \in N$, the space $\ell(p)$ is an BK-space with AK under the Luxemburg norn defined by

$$\|x\| = inf\{\varepsilon > 0 : \sum_{k=1}^{\infty} |\frac{x_k}{\varepsilon}|^{p_k} \leq 1\} .$$

For more detail about the space $\ell(p)$ see [3]. The space $c_0(p)$ is an FK-space with AK, $c(p)$ is an FK-space and $\ell_\infty(p)$ is a complete LBK-space with AB (see [3]). In each of the space $\ell_\infty(X,p)$ and $c_0(X,p)$ we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = max\ \{1, \sup_k p_k\}$, as a paranorm on $\ell_\infty(X,p)$ and $c_0(X,p)$ and it is known that $c_0(X,p)$ is an FK-space with AK under the paranorm g defined as above and $\ell_\infty(X,p)$ is a complete LBK-space with AB.

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map E into F*, written $A : E \to F$ if for each $x = (x_k) \in E, A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$ and the sequence $Ax = (A_n(x)) \in F$. We denote by (E,F) the set of all infinite matrices mapping E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$${}_u(E,F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E,F)\}.$$

If $u_k \neq 0$ for all $k \in N$,we write $u^{-1} = (\frac{1}{u_k})$.

Let E be an X-valued sequence space. The $\beta-$ daul of E is defined to be

$$E^{\beta} = \{(f_k) \subset X' : \sum_{k=1}^{\infty} f(x_k) \text{ converges for all } x = (x_k) \in E.\}$$

By the defintion, we see that if $A = (f_k^n)$ maps the sequence space E into a scalar sequence space, then each row of A belongs to E^{β}, i.e., $(f_k^n)_{k=1}^{\infty} \in E^{\beta}$, so this is a necessary condition for an infinite matrix A mapping from one sequence space into the other. We shall give characterizations of the $\beta-$ dual of some vector-valued sequence spaces in Section 3.

3. The $\beta-$ Dual of some Vector-Valued Sequence Spaces

We start with characterizations of the $\beta-$ dual of the space $c_0(X,p)$

Proposition 3.1 *Let $p = (p_k)$ be a bounded sequences of positive real numbers. Then*

$$c_0(X,p)^{\beta} = \{(f_k) \subset X' : \sum_{k=1}^{\infty} \|f_k\| M^{-\frac{1}{p_k}} < \infty \text{ for some } M \in N.\}$$

Proof. Suppose that $\sum_{k=1}^{\infty} \|f_k\| M^{-\frac{1}{p_k}} < \infty$ for some $M \in N$. Let $x = (x_k) \in c_0(X,p)$. Then there is a positive integer K_0 such that $\|x_k\|^{p_k} < \frac{1}{M}$ for all $k \geq K_0$, hence

$$\|x_k\| < M^{-\frac{1}{p_k}} \text{ for all } k \geq K_0.$$

Then we have

$$\sum_{k=K_0}^{\infty} |f_k(x_k)| \leq \sum_{k=K_0}^{\infty} \|f_k\| \|x_k\| \leq \sum_{k=K_0}^{\infty} \|f_k\| M^{-\frac{1}{p_k}} < \infty.$$

It follows that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so $(f_k) \in c_0(X,p)^{\beta}$.

On the other hand, assume that $(f_k) \in c_0(X,p)^{\beta}$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X,p)$. For each $x = (x_k) \in c_0(X,p)$, choose scalar sequence (t_k) with

$|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Since $(t_k x_k) \in c_0(X,p)$, by our assumption, we have $\sum\limits_{k=1}^{\infty} f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \text{ for all } x \in c_0(X,p). \tag{3.1}$$

Now, suppose that $\sum\limits_{k=1}^{\infty} \|f_k\| m^{-\frac{1}{p_k}} = \infty$ for all $m \in N$. Choose $m_1, k_1 \in N$ such that

$$\sum_{k \leq k_1} \|f_k\| m_1^{-\frac{1}{p_k}} > 1$$

and choose $m_2 > m_1$ and $k_2 > k_1$ such that

$$\sum_{k_1 < k \leq k_2} \|f_k\| m_2^{-\frac{1}{p_k}} > 2.$$

Proceeding in this way, we can choose $m_1 < m_2 < ...$, and $0 = k_1 < k_2 < ...$ such that

$$\sum_{k_{i-1} < k \leq k_i} \|f_k\| m_i^{-\frac{1}{p_k}} > i.$$

Take x_k in X with $\|x_k\| = 1$ for all $k, k_{i-1} < k \leq k_i$ such that

$$\sum_{k_{i-1} < k \leq k_i} |f_k(x_k)| m_i^{-\frac{1}{p_k}} > i \text{ for all } i \in N.$$

Put $y = (y_k), (y_k) = m_i^{-\frac{1}{p_k}} x_k$ for $k_{i-1} < k \leq k_i$, then $y \in c_0(X,p)$ and we have

$$\sum_{k=1}^{\infty} |f_k(y_k)| \geq \sum_{k_{i-1} < k \leq k_i} |f_k(x_k)| m_i^{-\frac{1}{p_k}} > i \text{ for all } i \in N.$$

Hence we have $\sum\limits_{k=1}^{\infty} |f_k(y_k)| = \infty$ which contradicts with (3.1). Hence $(f_k) \in \{(g_k) \subset X' : \sum\limits_{k=1}^{\infty} \|g_k\| M^{-\frac{1}{p_k}} < \infty$ for some $M \in N.\}$. Thus the proposition is proved. □

Proposition 3.2 *Let $p = (p_k)$ be a bounded sequences of positive real numbers. Then*

$$\ell_\infty(X,p)^\beta = \{(f_k) \subset X' : \sum_{k=1}^{\infty} \|f_k\| m^{\frac{1}{p_k}} < \infty \textit{ for all } m \in N.\}$$

Proof. If $\sum_{k=1}^{\infty}\|f_k\|m^{\frac{1}{p_k}} < \infty$ for all $m \in N$, then we have that for each $x = (x_k) \in \ell_\infty(X,p)$, there is $m_0 \in N$ such that $\|x_k\| \le m_0^{\frac{1}{p_k}}$ for all $k \in N$, hence $\sum_{k=1}^{\infty}|f_k(x_k)| \le \sum_{k=1}^{\infty}\|f_k\|\|x_k\| \le \sum_{k=1}^{\infty}\|f_k\|m_0^{\frac{1}{p_k}} < \infty$, which implies that $\sum_{k=1}^{\infty}f_k(x_k)$ converges, so $(f_k) \in \ell_\infty(X,p)^\beta$.

Conversely, assume that $(f_k) \in \ell_\infty(X,p)^\beta$. Then $\sum_{k=1}^{\infty}f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X,p)$. We first note that, by using the same proof as in Proposition 3.3, we have

$$\sum_{k=1}^{\infty}|f_k(x_k)| < \infty \text{ for all } x = (x_k) \in \ell_\infty(X,p). \tag{3.2}$$

Now, suppose that $\sum_{k=1}^{\infty}\|f_k\|M^{\frac{1}{p_k}} = \infty$ for some $M \in N$. Then we can choose a sequence (k_i) of positive integers with $0 = k_0 < k_1 < k_2 < ...$ such that

$$\sum_{k_{i-1}<k\le k_i}\|f_k\|M^{\frac{1}{p_k}} > i \text{ for all } i \in N.$$

Taking x_k in X with $\|x_k\| = 1$ such that for all $i \in N$,

$$\sum_{k_{i-1}<k\le k_i}\|f_k(x_k)\|M^{\frac{1}{p_k}} > i.$$

Put $y = (y_k) = (M^{\frac{1}{p_k}}x_k)_{k=1}^{\infty}$. Clearly, $y \in \ell_\infty(X,p)$ and

$$\sum_{k=1}^{\infty}|f_k(y_k)| \ \ge \sum_{k_{i-1}<k\le k_i}\|f_k(x_k)\|M^{\frac{1}{p_k}} > i \text{ for all } i \in N.$$

Hence $\sum_{k=1}^{\infty}|f_k(y_k)| = \infty$, which contradicts with (3.2). Thus $(f_k) \in \{(g_k) \subset X' : \sum_{k=1}^{\infty}\|g_k\|m^{\frac{1}{p_k}} < \infty \text{ for all } m \in N.\}$. Hence $\ell_\infty(X,p)^\beta = \{(g_k) \subset X' : \sum_{k=1}^{\infty}\|g_k\|m^{\frac{1}{p_k}} < \infty \text{ for all } m \in N.\}$. □

Proposition 3.3 *Let $p = (p_k)$ be a bounded sequences of positive real numbers. Then*

$$\ell(X,p)^\beta = \{(f_k) \subset X' : \sum_{k=1}^{\infty}\|f_k\|^{t_k}M^{-(t_k-1)} < \infty \textit{ for some } M \in N\}$$

where $\frac{1}{p_k} + \frac{1}{t_k} = 1$ *for all* $k \in N$.

Proof. Suppose that $\sum\limits_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty$ for some $M \in N$.
Then we have that for each $x = (x_k) \in \ell(X,p)$,

$$\begin{aligned}\sum_{k=1}^{\infty}|f_k(x_k)| &\leq \sum_{k=1}^{\infty}\|f_k\|M^{-\frac{1}{p_k}}M^{\frac{1}{p_k}}\|x_k\| \\ &\leq \sum_{k=1}^{\infty}\left(\|f_k\|^{t_k}M^{-\frac{t_k}{p_k}} + M\|x_k\|^{p_k}\right) \\ &= \sum_{k=1}^{\infty}\|f_k\|^{t_k}M^{-(t_k-1)} + M\sum_{k=1}^{\infty}\|x_k\|^{p_k} \quad < \infty\end{aligned}$$

which implies that $\sum\limits_{k=1}^{\infty} f_k(x_k)$ converges, so $(f_k) \in \ell(X,p)^{\beta}$.

On the other hand, assume that $(f_k) \in \ell(X,p)^{\beta}$. Then $\sum\limits_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell(X,p)$. We first note that, by using the same proof as in Proposition 3.1, we have

$$\sum_{k=1}^{\infty}|f_k(x_k)| < \infty \text{ for all } x = (x_k) \in \ell(X,p). \tag{3.3}$$

We want to show that there exists $M \in N$ such that

$$\sum_{k=1}^{\infty}\|f_k\|^{t_k}M^{-(t_k-1)} < \infty$$

If it is not true, then

$$\sum_{k=1}^{\infty}\|f_k\|^{t_k}m^{-(t_k-1)} = \infty, \text{ for all } m \in N. \tag{3.4}$$

And (3.4) implies that for each $k_0 \in N$.

$$\sum_{k>k_0}\|f_k\|^{t_k}m^{-(t_k-1)} = \infty, \text{ for all } m \in N. \tag{3.5}$$

By (3.4), let $m_1 = 1$, then there is a $k_1 \in N$ such that

$$\sum_{k\leq k_1}\|f_k\|^{t_k}m_1^{-(t_k-1)} > 1.$$

By (3.5), we can choose $m_2 > m_1$ and $m_2 > 2^2$ and $k_2 > k_1$ such that

$$\sum_{k_1 < k \leq k_2} \|f_k\|^{t_k} m_2^{-(t_k-1)} > 1. \tag{3.6}$$

By continueing in this way, we obtain sequences (k_i) and (m_i) of positive integers with $1 = k_0 < k_1 < k_2 < \ldots, m_1 < m_2 < \ldots, m_i > 2^i$ and

$$\sum_{k_{i-1} < k \leq k_i} \|f_k\|^{t_k} m_i^{-(t_k-1)} > 1.$$

Choose x_k in X with $\|x_k\| = 1$ such that for all $i \in N$,

$$\sum_{k_{i-1} < k \leq k_i} |f_k(x_k)|^{t_k} m_i^{-(t_k-1)} > 1 \text{ for all } i \in N.$$

Let $a_i = \sum_{k_{i-1} < k \leq k_i} |f_k(x_k)|^{t_k} m_i^{-(t_k-1)}$. Put $y = (y_k)$, $y_k = a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k$ for all k, $k \leq k_i$. For each $i \in N$, we have

$$\begin{aligned}
\sum_{k_{i-1} < k \leq k_i} \|y_k\|^{p_k} &= \sum_{k_{i-1} < k \leq k_i} \left\| a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k \right\|^{p_k} \\
&= \sum_{k_{i-1} < k \leq k_i} a_i^{-p_k} m_i^{-t_k} |f_k(x_k)|^{t_k} \\
&\leq \sum_{k_{i-1} < k \leq k_i} a_i^{-1} m_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k} \\
&= a_i^{-1} m_i^{-1} a_i \\
&= m_i^{-1} \\
&< \frac{1}{2^i}.
\end{aligned}$$

So we have that

$$\sum_{k=1}^{\infty} \|y_k\|^{p_k} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty.$$

Hence,

$$y = (y_k) \in \ell(X, p). \tag{3.7}$$

ON MATRIX TRANSFORMATIONS OF VECTOR-VALUED SEQUENCE SPACES OF MADDOX

Abstract. In this paper, characterizations of infinite matrices mapping the vector-valued sequence spaces of Maddox into Musielak-Orlicz sequence space are given.

1. Introduction. Let $(X, \|.\|)$ be a real Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. Let N be the set of all natural numbers, we write $x = (x_k)$ with x_k in X for all $k \in N$. The $X-$valued sequence spaces $c_0(X,p), c(X,p), \ell_\infty(X,p)$, and $\ell(X,p)$ are defined by

$$c_0(X,p) = \left\{ x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0 \right\},$$

$$c(X,p) = \left\{ x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X \right\},$$

$$\ell_\infty(X,p) = \left\{ x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty \right\}, and$$

$$\ell(X,p) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty \right\}.$$

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p), c(p), \ell(p)$, and $\ell_\infty(p)$,
respectively. The first three spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons[9] and Maddox[5, 6]. The space $\ell(p)$ was first defined by Nakano[8] and is known as the Nakano sequence space. Grosse-Erdmann [3] investigated the structure of the spaces $c_0(p), c(p), \ell(p)$ and $\ell_\infty(p)$.

A function $f : R \to [0,\infty)$ is called an *Orlicz function* if it has the following properties:

(1) f is even, continuous and convex,

(2) $f(x) = 0 \Longleftrightarrow x = 0$,

(3) $\displaystyle lim_{x\to 0} \frac{f(x)}{x} = 0$ and $\displaystyle lim_{x\to\infty} \frac{f(x)}{x} = \infty$.

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Let $M = (M_n)$ be a sequence of Orlicz functions, for a given real sequence $x = (x_n)$ define

$$\varrho_M(x) = \sum_{n=1}^{\infty} M_n(x_n) \quad .$$

Let $\ell_M = \{x = (x_n) : \varrho_M(\lambda x) < \infty \text{ for some } \lambda > 0.\}$ and for $x = (x_n) \in \ell_M$, define the Luxemburg norm of x by

$$\|x\| = inf\{\lambda > 0 : \varrho_M(\frac{x}{\lambda}) \leq 1.\}$$

The sequence space $(\ell_M, \|.\|)$ was defined by Musielak [7] and it is called the Musielak-Orlicz sequence space with the Luxemburg norm. If $M_n = M_{n+1}$ for all $n \in N$, the space ℓ_M is known as the Orlicz sequence space. For more details about the Orlicz sequence space and Musielak-Orlicz sequence space see [1] and [7].

In [4], Grosse-Erdmann gave characterizations of infinite matrices mapping between the scalar sequence spaces of Maddox. Wu [12] gave characterizations of matrix transformations from the space $\ell(X,p)$, $c_0(X,p)$ and $\ell_\infty(X,p)$ into the space $c_0(q)$ and $\ell_\infty(q)$. These results generalized some of those in [4]. In [11], Suantai gave characterizations of infinite matrices of bounded linear functuionals on X mapping the Nakano sequence space $\ell(X,p)$ into $\ell_\infty(q)$ and $\underline{\ell}_\infty(q)$ and $M_0(X,p)$ into $\underline{\ell}_\infty(q)$. Choudhur [1] gave necessary and sufficient conditions for an infinite matrix of continuous linear operators which maps the vector-valued sequence space $c_0(X)$ into $c_0(Y), \ell_1(X)$ into $\ell_\infty(Y)$ and $\ell_1(X)$ into $\ell_p(Y)$ where Y is a Banach space. Suantai [10] gave characterizations of infinte matrices of bounded linear operators mapping from the Nakano vector-valued sequence space $\ell(X,p)$ into any BK-space. In this paper, we use some technics in [10] and another new technics to give the matrix characterizations from the sequence spaces of Maddox $\ell_\infty(X,p)$, $\ell(X,p)$ and $c_0(X,p)$ into the Musielak-Orlicz sequence space ℓ_M..

2. Notation and Definitions. Let $(X, \|.\|)$ be a Banach space. The space of all sequences in X is denoted by $W(X)$ and $\Phi(X)$ denotes the space of all finite sequences in X.

A sequence space in X is a linear subspace of $W(X)$. Let E be an X- valued sequence space. For $x \in E$ we write $x = (x_k), k \in N$. For $z \in X$ and $k \in N$,we let

For each $i \in N$, we have

$$\begin{aligned}\sum_{k_{i-1}<k\le k_i} |f_k(y_k)| &= \sum_{k_{i-1}<k\le k_i} \left|f_k(a_i^{-1}m_i^{-(t_k-1)}|f_k(x_k)|^{t_k-1}x_k)\right| \\ &= \sum_{k_{i-1}<k\le k_i} a_i^{-1}m_i^{-(t_k-1)}|f_k(x_k)|^{t_k} \\ &= a_i^{-1}\sum_{k_{i-1}<k\le k_i} m_i^{-(t_k-1)}|f_k(x_k)|^{t_k} \\ &= 1.\end{aligned}$$

So that $\sum_{k=1}^{\infty}|f_k(y_k)| = \infty$, which contradicts with (3.3). Thus $(f_k) \in \{(g_k) \subset X' : \sum_{k=1}^{\infty}\|g_k\|^{t_k}M^{-(t_k-1)} < \infty$ for some $M \in N\}$. Hence the proposition is proved. □

Proposition 3.4 *Let $p = (p_k)$ be a bounded sequences of positive real numbers. Then $\ell_\infty(X,p) = \cup_{n=1}^{\infty}\ell_\infty(X)_{(n^{-1/p_k})}$.*

Proof. If $x \in \ell_\infty(X,p)$, then there is some $n \in N$ with $\|x_k\|^{p_k} \le n$ for all $k \in N$. Hence $\|x_k\|n^{-1/p_k} \le 1$ for all $k \in N$, so that $x \in \ell_\infty(X)_{(n^{-1/p_k})}$. On the other hand, if $x \in \cup_{n=1}^{\infty}\ell_\infty(X)_{(n^{-1/p_k})}$, then there are some $n \in N$ and $M > 1$ such that $\|x_k\|n^{-1/p_k} \le M$ for every $k \in N$. Then we have $\|x_k\|^{p_k} \le nM^{p_k} \le nM^{\alpha}$ for all $k \in N$, where $\alpha = \sup_k p_k$. Hence $x \in \ell_\infty(X,p)$ □.

4. Main Results. Now, we turn to our objective. We begin with giving general characterizations of matrix transformations from an FK-space of vector sequences with AK property into an FK-space of scalar sequences.

Theorem 4.1 *Let $E \subseteq W(X)$ be an FK-space with AK property and F an FK-space of scalar sequences. Then, for an infinite matrix $A = (f_k^n), A : E \to F$ if and only if*

(1) for each $n \in N$, $\sum_{k=1}^{\infty} f_k^n(x_k)$ converges for all $x = (x_k) \in E$,

(2) for each $k \in N$, $(f_k^n(z))_{n=1}^{\infty} \in F$ for all $z \in X$, and

(3) $A : \Phi(X) \to F$ is continuous when $\Phi(X)$ is considered as a subspace of E.

Proof. Assume that $A : E \to F$. Then we have that for any $x = (x_k) \in E$, $\sum_{k=1}^{\infty} f_k^n(x_k)$ converges for all $n \in N$, so (1) holds. Since $e^k(z) \in E$ for all $k \in N$ and all $z \in X$, we obtain that for each $k \in N$,

$$(f_k^n(z))_{n=1}^{\infty} = Ae^k(z) \in F,$$

hence (2) holds. Since E and F are FK-spaces, by Zeller's theorem, $A : E \to F$ is continuous, so (3) is obtained.

Conversely, assume that the conditions hold. By (1), we have

$$Ax = \left(\sum_{k=1}^{\infty} f_k^n(x_k)\right)_{n=1}^{\infty} \in W, \text{ for all } x = (x_k) \in E.$$

It follows from (2) that $Ae^k(z) \in F$ for all $k \in N$ and $z \in X$, which implies that $A : \Phi(X) \to F$. By (3), we have $A : \Phi(X) \to F$ is continuous. Let $x = (x_k) \in E$. Since E has the AK property, we have

$$x = \lim_{n \to \infty} \sum_{k=1}^{n} e^k(x_k).$$

Then $\left(\sum_{k=1}^{n} e^k(x_k)\right)_{n=1}^{\infty}$ is a Cuachy sequence in E. Since $A : \Phi(X) \to F$ is continuous and linear, it implies that $\left(\sum_{k=1}^{n} Ae^k(x_k)\right)_{n=1}^{\infty}$ is a Cauchy sequence in F. Since F is complete, we have $\left(\sum_{k=1}^{n} Ae^k(x_k)\right)_{n=1}^{\infty}$ converges in F. Since F is a K-space, it implies that $\left(\sum_{k=1}^{\infty} f_k^n(x_k)\right)_{n=1}^{\infty} \in F$, so that $Ax \in F$. This shows that $A : E \to F$. $\square$

If $p_k > 1$ for all $k \in N$, it is the same as the space $\ell(p)$ (see [ ]), we have that the space $\ell(X,p)$ is an BK-space with AK property under the Luxemburg norm

$$\|x\| = inf\{\varepsilon > 0 : \sum_{k=1}^{\infty} \|\frac{x_k}{\varepsilon}\|^{p_k} \leq 1\}$$

The following proposition gives some useful properties conerning about the Luxemburg norm.

Proposition 4.2 *Let $p = (p_k)$ be a bounded sequences of positive real numbers such that $p_k > 1$ for all $k \in N$ and let $x = (x_k) \in \ell(X, p)$. Then*

(1) $\|x\| \leq 1$ *if and only if* $\sum_{k=1}^{\infty} \|x_k\|^{p_k} \leq 1$, *and*

(2) If $\|x\| = 1$, *then* $\sum_{k=1}^{\infty} \|x_k\|^{p_k} = 1$.

Proof. If $\sum_{k=1}^{\infty} \|x_k\|^{p_k} \leq 1$, we have by the definition of the Luxemburg norm that $\|x\| \leq 1$.

If $\|x\| \leq 1$, then $\|x\| < 1 + \frac{1}{n}$ for all $n \in N$, it implies that $\sum_{k=1}^{\infty} \|\frac{x_k}{1+\frac{1}{n}}\|^{p_k} \leq 1$. Since $\frac{1}{(1+\frac{1}{n})^\alpha} \sum_{k=1}^{\infty} \|x_k\|^{p_k} \leq \sum_{k=1}^{\infty} \|\frac{x_k}{1+\frac{1}{n}}\|^{p_k}$, where $\alpha = \sup_k p_k$, it follows that

$$\sum_{k=1}^{\infty} \|x_k\|^{p_k} \leq (1 + \frac{1}{n})^\alpha \quad \text{for all } n \in N \tag{4.1}$$

By taking $n \to \infty$ in 4.1, we obtain $\sum_{k=1}^{\infty} \|x_k\|^{p_k} \leq 1$.

(2) Assume that $\|x\| = 1$. By (1) we have $\sum_{k=1}^{\infty} \|x_k\|^{p_k} \leq 1$. If $\sum_{k=1}^{\infty} \|x_k\|^{p_k} < 1$, then for each $\varepsilon > 0$ such that $\sum_{k=1}^{\infty} \|x_k\|^{p_k} < \varepsilon < 1$, we have that $\sum_{k=1}^{\infty} \|\frac{x_k}{\varepsilon}\|^{p_k} > 1$. Since $(\frac{1}{\varepsilon})^\alpha \geq (\frac{1}{\varepsilon})^{p_k}$ for all $k \in N$, where $\alpha = \sup_k p_k$, we have $(\frac{1}{\varepsilon})^\alpha \sum_{k=1}^{\infty} \|x_k\|^{p_k} \geq \sum_{k=1}^{\infty} \|\frac{x_k}{\varepsilon}\|^{p_k} > 1$, hence

$$\sum_{k=1}^{\infty} \|x_k\|^{p_k} > \varepsilon^\alpha . \tag{4.2}$$

By taking $\varepsilon \to 1^-$ in (4.2) we obtain that $\sum_{k=1}^{\infty} \|x_k\|^{p_k} \geq 1$ which is a contradiction. Hence $\sum_{k=1}^{\infty} \|x_k\|^{p_k} = 1$. $\square$

Theorem 4.3 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers such that $p_k > 1$ for all $k \in N$, and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X,p) \to \ell_M$ if and only if*

(1) for each $n \in N$ there exists $M_n \in N$ such that

$$\sum_{k=1}^{\infty} \|f_k^n\|^{t_k} M_n^{-(t_k-1)} < \infty, \text{ where } \frac{1}{p_k} + \frac{1}{t_k} = 1 \text{ for all } k \in N,$$

(2) for each $k \in N$ and $x \in X$, $(f_k^n(x))_{n=1}^{\infty} \in \ell_M$, and

(3) there exists $\lambda > 0$ such that

$$sup \{\sum_{n=1}^{\infty} M_n(\frac{1}{\lambda}\sum_{k\in K} f_k^n(x_k)) : K \subset N \text{ is finite, } x_k \in X \text{ for all } k \in K \text{ and } \sum_{k\in K} \|x_k\|^{p_k} \leq 1\}$$

Proof Assume that $A : \ell(X,p) \to \ell_M$. By Proposition 3.3 and Theorem 4.1, the conditions (1) and (2) are satisfied. Since $\ell(X,p)$ and ℓ_M are BK-spaces with the Luxemburg norm, by Zeller's Theorem, we have that $A : \ell(X,p) \to \ell_M$ is continuous, so $A : \Phi(X) \to \ell_M$ is continuous when $\Phi(X)$ is considered as a subspace of $\ell(X,p)$. It implies that A is bounded, hence there exists $\lambda > 0$ such that $\|Ax\| \leq 1$ for all $x \in \Phi(X)$ such that $\|x\| \leq 1$. By Proposition 4.2(1), we have

$$\varrho_M(\frac{1}{\lambda}Ax) = \sum_{n=1}^{\infty} M_n(\frac{1}{\lambda}\sum_{k=1}^{\infty} f_k^n(x_k)) \leq 1 \tag{4.3}$$

for all $x = (x_k) \in \Phi(X)$ such that $\|x\| \leq 1$.

Let $K \subset N$ be finite and $x_k \in X$ for all $k \in N$ such that $\sum_{k\in K} \|x_k\|^{p_k} \leq 1$. Let $z = (z_k)$ where $z_k = x_k$ if $k \in K$ and $z_k = 0$ otherwise. Then $\sum_{k\in K} \|z_k\|^{p_k} \leq 1$. It implies by Proposition 4.2 (1) that $\|z\| \leq 1$. By (4.3), we have $\sum_{n=1}^{\infty} M_n(\frac{1}{\lambda}\sum_{k\in K} f_k^n(x_k)) = \sum_{n=1}^{\infty} M_n(\frac{1}{\lambda}\sum_{k=1}^{\infty} f_k^n(z_k)) \leq 1$. This implies that the condition (3) is satisfied.

Conversely, assume that the conditions (1), (2) and (3) hold. We will show that the conditions (1), (2) and (3) of Theorem 4.1 are satisfied. The condition (1) implies by Proposition 3.3 that $(f_k^n)_{k=1}^{\infty} \in \ell(X,p)^{\beta}$ for all $n \in N$, so $\sum_{k+1}^{\infty} f_k^n(x_k)$ converges for

Proof. Assume that $\left(\sum_{k=1}^{\infty}\|f_k^n\|\right)_{n=1}^{\infty} \in \ell_M$. Then there exists $\lambda > 0$ and $\alpha > 1$ such that $\sum_{n=1}^{\infty} M(\lambda \sum_{k=1}^{\infty}\|f_k^n\|) \le \alpha$. Let $x = (x_k) \in \ell_\infty(X)$ and $\|x\| \le 1$. Then $\|x_k\| \le 1$ for all $k \in N$, so $|f_k^n(x_k)| \le \|f_k^n\|$ for all $n, k \in N$. Putting $K = \dfrac{\alpha}{\lambda}$. Since M_n is convex, even, and increasing on $[0,\infty)$, it follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} M_n\Big(\frac{1}{K}\sum_{k=1}^{\infty} f_k^n(x_k)\Big) &= \sum_{n=1}^{\infty} M_n\left(\frac{\lambda}{\alpha}\left|\sum_{k=1}^{\infty} f_k^n(x_k)\right|\right) \\
&\le \frac{1}{\alpha}\sum_{n=1}^{\infty} M_n\left(\lambda\left|\sum_{k=1}^{\infty} f_k^n(x_k)\right|\right) \\
&\le \frac{1}{\alpha}\sum_{n=1}^{\infty} M_n\Big(\lambda\sum_{k=1}^{\infty}|f_k^n(x_k)|\Big) \\
&\le \frac{1}{\alpha}\sum_{n=1}^{\infty} M_n\Big(\lambda\sum_{k=1}^{\infty}\|f_k^n\|\Big) \\
&\le 1.
\end{aligned}$$

It follows by Theorem 4.5 that $A \in (\ell_\infty(X),\ \ell_M)$. □

Theorem 4.7 *Let $A = (f_k^n)$ be an infinite matrix and let $p = (p_k)$ be a bounded sequence of positive real numbers. Then $A \in (\ell_\infty(X,p),\ \ell_M)$ if and only if*

(1) $\sum_{k=1}^{\infty} m^{1/p_k}\|f_k^n\| < \infty$ for all $m, n \in N$, and

(2) for each $m \in N$, there exists $K_m > 0$ such that

$$\sum_{n=1}^{\infty} M\Big(\frac{1}{K_m}\sum_{k=1}^{\infty} m^{1/p_k} f_k^n(x_k)\Big) \le 1$$

for every sequence (x_k) with $\|x_k\| \le 1$ for all $k \in N$.

Proof. By Proposition 3.4, we have $\ell_\infty(X,p) = \bigcup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$. It implies by [10 , Proposition 3.1(i)] that

$$A \in (\ell_\infty(X,p),\ \ell_M) \Longleftrightarrow A \in (\ell_\infty(X)_{(m^{-1/p_k})},\ \ell_M) \text{ for all } m \in N$$

By [10, Proposition 3.1(iii)], we have

$$A \in (\ell_\infty(X)_{(m^{-1/p_k})},\ \ell_M) \Longleftrightarrow (m^{1/p_k} f_k^n)_{n,k} \in (\ell_\infty(X),\ \ell_M)$$

We have by Theorem 4.5 that

$$(m^{1/p_k} f_k^n)_{n,k} \in (\ell_\infty(X),\ \ell_M) \Longleftrightarrow \text{ the conditions (1) and (2) hold.}$$

□

Theorem 4.8 *Let $A = (f_k^n)$ be an infinite matrix and let $p = (p_k)$ be a bounded sequence of positive real numbers. Then $A \in (c_0(X,p),\ \ell_M)$ if and only if*

(1) for each $n \in N$, there exists $K_n \in N$ such that $\sum_{k=1}^{\infty} K_n^{-1/p_k}\|f_k^n\| < \infty$ for all $m, n \in N$, and

(2) for each $r \in N$, there exists $m_r \in N$ such that

$$x = (x_k) \in \Phi(X)\ , \sum_{k=1}^{\infty} \|x_k\|^{p_k} \leq 1 \Longrightarrow \sum_{n=1}^{\infty} M_n(r\sum_{k=1}^{\infty} f_k^n(x_k)) \leq 1.$$

Proof. By Proposition 3.1, the condition (1) is equivalent to (1) of Theorem 4.1 and it is easy to see that the condition (3) is equivalent to the condition (3) of Theorem 4.1. Hence the theorem is obtained by Theorem 4.1. □

5. Matrix Transformations Between Some Scalar Sequence Spaces

In this section, we apply some results of Section 4 for givng characterizations of matrix transformation from the spaces $\ell(p)$ and $\ell_\infty(p)$ into ℓ_M.

Remark 5.1 Let $A = (a_{nk})$ be an infinite matrix of real numbers. For each $n, k \in N$, let $f_k^n : R \to R$ be defined by

$$f_k^n(x) = a_{nk}x \quad , x \in R\ .$$

all $n \in N$ and all $x = (x_k) \in \ell(X,p)$. Thus the condition (1) of Theorem 4.1 holds. It is clear that the condition (2) of Theorem 4.1 hold. By (3), there exists $\lambda > 0$ such that

$$sup\,\{\sum_{k=1}^{\infty} M_n(\frac{1}{\lambda}\sum_{k\in K} f_k^n(x_k)) : K \subset N \text{ is finite, } x_k \in X \text{ for all } k \in K \text{ and } \sum_{k\in K} \|x_k\|^{p_k} \leq 1\} \leq 1 \tag{4.4}$$

Let $x = (x_k) \in \Phi(X)$ be such that $\|x\| \leq 1$. By Proposition 4.2 (1) we have that $\sum_{k\in K} \|x_k\|^{p_k} \leq 1$ for some finite subset K of N. It follows by (4.4) that

$$\sum_{n=1}^{\infty} M_n(\frac{1}{\lambda}\sum_{k\in K} f_k^n(x_k)) \leq 1\,,$$

which implies $\|Ax\| \leq \lambda$, hence A is bounded, so $A : \Phi(X) \to \ell_M$ is continuous. Thus thye condition (3) of Theorem 4.1 is satisfied, so we have by Theorem 4.1 that $A : \ell(X,p) \to \ell_M$. The proof is now complete. □

Theorem 4.4 *Let $p = (p_k)$ be a bounded sequences of positive real numbers such that $p_k \leq 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X,p) \to \ell_M$ if and only if*

(1) for each $k \in N$ and $x \in X$, $(f_k^n(x))_{n=1}^{\infty} \in \ell_M$, and

(2) there exists $m_0 \in N$ such that $\sup\limits_{\substack{k\in N\\ \|x\|\leq 1}} \sum_{n=1}^{\infty} M_n(m_0^{-\frac{1}{p_k}} f_k^n(x)) \leq 1$.

Proof. By [10, Theorem 4.1], we have that

$A : \ell(X,p) \to \ell_M$ if and only if

(i) for each $k \in N$ and $x \in X$, $(f_k^n(x))_{n=1}^{\infty} \in \ell_M$, and

(ii) there exists $m_0' \in N$ such that

$$\sup_{\substack{k\in N\\ \|x\|\leq 1}} \|A(m_0^{-\frac{1}{p_k}} e^k(x)\| \leq 1\,.$$

Proof. By Proposition 4.2 (1), we have that the condition (ii) above is equivalent to (2). Hence $A : \ell(X,p) \to \ell_M$ if and only if the conditions (1) and (2) are satisfied. □

Theorem 4.5 *Let $A = (f_k^n)$ be an infinite matrix and $E \in \{\ell_\infty(X), c_0(X)\}$. Then $A \in (E,\ \ell_M)$ if and only if*

(1) $\sum_{k=1}^{\infty} \|f_k^n\| < \infty$ for every $n \in N$, and

(2) there exists $K > 0$ such that $\sum_{n=1}^{\infty} M_n\big(\frac{1}{K}\sum_{k=1}^{\infty} f_k^n(x_k)\big) \leq 1$ for every $(x_k) \in E$ with $\|x_k\| \leq 1$ for all $k \in N$.

Proof. Assume that $A \in (E, \ell_\infty)$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in E$. Hence (1) holds by Proposition 3.1 and 3.2. Since E and ℓ_M are BK-spaces, by Zeller's theorem, A is continuous. It follows that there exists $K > 0$ such that

$$\|Ax\| \leq K \tag{4.5}$$

for every $x = (x_k) \in E$ with $\|x_k\| \leq 1$ for all $k \in N$.
Then we have $\|A(\frac{1}{K}x)\| \leq 1$ for all $x = (x_k) \in E$ with $\|x_k\| \leq 1$ for all $k \in N$. By Proposition 4.2(1), we have

$$\sum_{n=1}^{\infty} M_n\Big(\frac{1}{K}\sum_{k=1}^{\infty} f_k(x_k)\Big) \leq 1$$

for every $x = (x_k) \in E$ with $\|x_k\| \leq 1$ for all $k \in N$. Hence (2) holds.

Conversely, assume that (1) and (2) hold. By Proposition 3.1 and 3.2, we have $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in E$. Let $K > 0$ be such that $\sum_{n=1}^{\infty} M\big(\frac{1}{K}\sum_{k=1}^{\infty} f_k^n(x_k)\big) \leq 1$ for every $x = (x_k) \in E$ with $\|x_k\| \leq 1$ for all $k \in N$. Then for $x = (x_k) \in E$ and $x \neq 0$, we have

$$\sum_{n=1}^{\infty} M\Big(\frac{1}{K\|x\|}\sum_{k=1}^{\infty} f_k^n(x_k)\Big) = \sum_{n=1}^{\infty} M\Big(\frac{1}{K}\sum_{k=1}^{\infty} f_k^n\big(\frac{x_k}{\|x\|}\big)\Big) \leq 1$$

which implies that $Ax \in \ell_M$, hence we have $A \in (E,\ \ell_M)$. $\square$

Corollary 4.6 *Let $A = (f_k^n)$ be an infinite matrix. If $\big(\sum_{k=1}^{\infty} \|f_k^n\|\big)_{n=1}^{\infty} \in \ell_M$, then $A \in (\ell_\infty(X),\ \ell_M)$.*

Then f_k^n is a continuous linear functional on R and $\|f_k^n\| = |a_{nk}|$. Let $B = (f_k^n)$, for a real sequence $x = (x_k)$, we see that $Ax = Bx$, hence if E and F are scalar sequence spaces, then

$$A : E \to F \Longleftrightarrow B : E \to F$$

Theorem 5.1 *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$, and let $A = (a_{nk})$ be an infinite matrix of real numbers. Then $A : \ell(p) \to \ell_M$ if and only if*

(1) for each $k \in N$, $(a_{nk})_{n=1}^{\infty} \in \ell_M$, and

(2) there exists $m_0 \in N$ such that $\sup_k \sum_{n=1}^{\infty} M_n(m_0^{-1/p_k} a_{nk}) \leq 1$.

Proof. Let $B = (f_k^n)$ be the matrix defined as in Remark 5.1. Then

$$A : \ell(p) \to \ell_M \Longleftrightarrow B : \ell(p) \to \ell_M$$

It implies by Theorem 4.4 that

$$B : \ell(p) \to \ell_M \Longleftrightarrow \text{the conditions (1) and (2) hold.}$$

Hence $A : \ell(p) \to \ell_M \Longleftrightarrow$ (1) and (2)hold. □

If $q = (q_k)$ is a bounded sequence of positive real numbers such that $q_k > 1$ for all $k \in N$, then the function $M_k(x) = \|x|^{p_k}$ is an Orlicz function for all $k \in N$ and we see that the Nakano sequence spaces $\ell(q) = \ell_M$. Hence the following result is directly obtained by Theorem 5.1.

Corollary 5.2 *Let $p = (p_k)$ and $q = (q_k)$ be a bounded sequence of positive real numbers such that $p_k \leq 1$ and $q_k > 1$ for all $k \in N$, and $A = (a_{nk})$ an infinite matrix of real numbers. Then $A : \ell(p) \to \ell(q)$ if and only if*

(1) for each $k \in N$, $(a_{nk})_{n=1}^{\infty} \in \ell(q)$, and

(2) there exists $m_0 \in N$ such that $\sup_k \sum_{n=1}^{\infty} m_0^{-\frac{q_k}{p_k}} |a_{nk}|^{q_k} \leq 1$.

Theorem 5.3 *Let $p = (p_k)$ be a bounded sequence of positive real numbers and $A = (a_{nk})$ an infinite matrix of nonnegative real numbers. Then $A : \ell_\infty(p) \to \ell_M$ if and only if*

(1) $\sum_{k=1}^{\infty} m^{1/p_k}|a_{nk}| < \infty$ *for all $m, n \in N$, and*

(2) for each $m \in N$, there exists $K_m > 0$ such that

$$\sum_{n=1}^{\infty} M_n\Big(\frac{1}{K_m}\sum_{k=1}^{\infty} m^{1/p_k} a_{nk}\Big) \leq 1 .$$

Proof. Let $B = (f_k^n)$ be the matrix defined as in Remark 5.1. Then

$$A : \ell_\infty(p) \to \ell_M \Longleftrightarrow B : \ell_\infty(p) \to \ell_M .$$

It follows by Theorem 4.6 that $B : \ell_\infty(p) \to \ell_M$ if and only if the conditions (1) and (2) hold. □

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Matrix Transformations of Some Vector-Valued Sequence Spaces

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ABSTRACT. **In this paper, we give the matrix characterizations from vector-valued sequence spaces $\ell_\infty(X,p)$, and $\underset{\sim}{c}_0(X,p)$ into the Orlicz sequence space ℓ_M where $p=(p_k)$ is a bounded sequences of positive real numbers.**

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1. INTRODUCTION

Let $(X, \|.\|)$ be a real Banach space and $p=(p_k)$ a bounded sequence of positive real numbers. We write $x=(x_k)$ with x_k in X for all $k \in N$. The X-valued sequence spaces $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, and $\underset{\sim}{c}_0(X,p)$ are defined as

$$c_0(X,p)=\{x=(x_k): \lim_{k\to\infty} \|x_k\|^{p_k}=0\},$$
$$c(X,p)=\{x=(x_k): \lim_{k\to\infty} \|x_k-a\|^{p_k}=0 \text{ for some } a \in X\},$$
$$\ell_\infty(X,p)=\{x=(x_k): \sup_k \|x_k\|^{p_k}<\infty\},$$
$$\ell(X,p)=\{x=(x_k): \sum_{k=1}^{\infty} \|x_k\|^{p_k}<\infty\}$$
$$\underset{\sim}{c}_0(X,p)=\{x=(x_k): \sup_k \|\frac{x_k}{\delta_k}\|^{p_k}<\infty \text{ for some } (\delta_k)\in c_0 \text{ with } \delta_k \neq 0 \text{ for all } k \in N\}$$

When $X=R$, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$,$\ell(p)$, and $\underset{\sim}{c}_o(p)$ respectively. Each of the first four spaces are known as the sequence spaces of

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A sequence space in X is a linear subspace of $W(X)$. Let E be any X-valued sequence space. For $x \in E$ and $k \in N$, we write x_k stands for the k^{th} term of x. For $k \in N$ denote by e_k the sequence $(0,0,...,0,1,0,...)$ with 1 in the k^{th} position and by e the sequence $(1,1,1,...)$. For $x \in X$ and $k \in N$, let $e^k(x)$ be the sequence $(0,0,...,0,x,0,...)$ with x in the k^{th} position and let $e(x)$ be the sequence $(x,x,x,...)$. For a fixed scalar sequence $\mu = (\mu_k)$ the sequence space E_μ is defined as

$$E_\mu = \{x \in W(X) : (\mu_k x_k) \in E\} .$$

The sequence space E is called *normal* if $x \in E$ and $y \in W(X)$ with $\|y_k\| \leq \|x_k\|$ for all $k \in N$ implies that $y \in E$.

2.2 Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map* E into F, written by $A : E \to F$ if for each $x = (x_k) \in E$, $A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$, and the sequence $Ax = (A_n(x)) \in F$. Let (E,F) denote for the set of all infinite matrices mapping from E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$$_u(E,F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E,F)\}$$

If $u_k \neq 0$ for all $k \in N$, we write $u^{-1} = (\dfrac{1}{u_k})$.

2.3 Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if for each $n \in N$ the n^{th} coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. If, in addition, (E,τ) is an Fréchet (Banach, LF-, LB-) space, then E is called an *FK- (BK-, LFK-, LBK-) space.* Now, suppose that E contains $\Phi(X)$. Then E is said to have *property AB* if the set $\{\sum_{k=1}^{n} e^k(x_k) : n \in N\}$ is bounded in E for every $x = (x_k) \in E$. It is said to have *property AK* if $\sum_{k=1}^{n} e^k(x_k) \to x$ in E as $n \to \infty$ for every $x = (x_k) \in E$. It has *property AD* if $\Phi(X)$ is dense in E.

3. Some Auxiliary Results

Maddox. These spaces were first introduced and studied by Simons [7], Maddox [4, 5], and Nakano [6]. In [2] the structure of the spaces $c_0(p)$, $c(p)$, and $\ell_\infty(p)$ have been investigated.

Let $M : R \to [0, \infty)$ be convex, even, continuous and $M(u) = 0 \Longleftrightarrow u = 0$. For a given real sequence $x = (x_n)$, define

$$\varrho_M(x) = \sum_{n=1}^{\infty} M(x_n) ,$$

$$\ell_M = \{x = (x_k) : \varrho_M(\lambda x) < \infty \text{ for some} \lambda > 0\}, \text{ and}$$

$$||x|| = inf\{\lambda > 0 : \varrho_M(\frac{x}{\lambda}) \leq 1\} \text{ for } x \in \ell_M.$$

The sequence space $(\ell_M, ||.||)$ is known as the Orlicz sequence space and it is a BK-space.

In this paper we consider the problem of characterizing those matrices that map an X-valued sequence spaces $\ell_\infty(X,p)$ and $\underline{c}_0(X,p)$ into the Orlicz sequence spaces. Wu and Liu [8] deal with the problem of characterization those infinite matrices mapping from $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$ and $\ell(X,p)$ into the scalar-sequence spaces of Maddox with some conditions on the sequences (p_k) and (q_k). Grosse-Erdmann [3] has given characterizations of matrix transformations between the scalar-valued sequence spaces of Maddox. Their characterizations are derived from functional analytic principles. Our approach here is different. We use a method of reduction introduced by Grosse-Erdmann [3]. In [2] it is pointed out that $c_0(p)$ is an echelon space of order 0 and that $\ell_\infty(p)$ is a co-echelon space of order ∞. In this paper we also show that $\underline{c}_0(X,p)$ and $\ell_\infty(X,p)$ is a co-echelon space of order ∞. Therefore these spaces are made up of simpler spaces. We will use certain auxiliary results(Section 3) to reduce our problem to the characterisations of matrix mapping between much simpler spaces.

2. Notation and Definitions

2.1 Let $(X, ||.||)$ be a real Banach space, the space of all sequences in X is denoted by $W(X)$ and $\Phi(X)$ is denoted for the space of all finite sequences in X. When $X = R$, the corresponding spaces are written as w and Φ.

In this section we give various useful results that can be used to reduce our problems into some simpler forms.

Proposition 3.1 *Let E and $E_n (n \in N)$ be X-valued sequence spaces, and F and $F_n (n \in N)$ scalar sequence spaces, and let u and v be sequences of real numbers with $u_k \neq 0$, $v_k \neq 0$ for all $k \in N$. Then we have*

(i) $\left(\cup_{n=1}^{\infty} E_n, F\right) = \cap_{n=1}^{\infty} (E_n, F)$

(ii) $(E, \cap_{n=1}^{\infty} F_n) = \cap_{n=1}^{\infty} (E, F_n)$

(iii) $(E_1 + E_2, F) = (E_1, F) \cap (E_2, F)$

(iv) $(E, F_1) = (E, F_2) \cap (\Phi(X), F_1)$ *if E is an FK-space with AD, F_2 is an FK-space and F_1 is a closed subspace of F_2.*

(v) $(E_u, F_v) = {}_v(E, F)_{u^{-1}}$.

Proof. Assertions (i), (ii), (iii), and (v) are immediate. To show (iv), assume that E is an FK-space with AD, F_2 is an FK-space and F_1 is a closed subspace of F_2. Clearly, $(E, F_1) \subseteq (E, F_2) \cap (\Phi(X), F_1)$ is always the case. Now, assume that $A = (f_k^n) \in (E, F_2) \cap (\phi(X), F_1)$ and $x \in E$. By Zeller's theorem, $A : E \to F_2$ is continuous. Since E has AD, there is a sequence $(y^{(n)})$ with $y^{(n)} \in \Phi(X)$ for all $n \in N$ such that $y^{(n)} \to x$ in E as $n \to \infty$. By the continuity of A, we have $Ay^{(n)} \to Ax$ in F_2 as $n \to \infty$. Since $Ay^{(n)} \in F_1$ for all $n \in N$ and F_1 is a closed subspace of F_2, we obtain that $Ax \in F_1$. Hence $A \in (E, F_1)$, so that $(E, F_2) \cap (\Phi(X), F_1) \subseteq (E, F_1)$. This complete the proof. $\square$

Proposition 3.2 *Let $p = (p_k)$ be a bounded sequences of positive real numbers. Then*

(i) $\underline{c}_0(X, p) = \cup_{n=1}^{\infty} c_0(X)_{(n^{-1/p_k})}$. *Hence $c_0(X, p)$ is an echelon space of order 0.*

(ii) $\ell_\infty(X, p) = \cup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$. *Hence $\ell_\infty(X, p)$ is a co-echelon space of order ∞.*

Proof. (i) Let $x = (x_k) \in \underline{c}_0(X, p)$. Then there is a sequence $(\delta_k) \in c_0$ with $\delta_k \neq 0$ for all $k \in N$ such that $\sup_k \|\frac{x_k}{\delta_k}\|^{p_k} < \infty$. Hence there exists $\alpha > 0$ such that $\|x_k\| \leq \alpha^{1/p_k} |\delta_k|$ for all $k \in N$. Choose $n_0 \in N$ with $n_0 > \alpha$. Then

$\|x_k\|n_0^{-1/p_k} \leq (\frac{\alpha}{n_0})^{1/p_k}|\delta_k| < |\delta_k|$ which implies that $\lim_{k\to\infty} \|x_k\|n_0^{-1/p_k} = 0$, hence $x = (x_k) \in c_0(X)_{(n^{-1/p_k})} \subseteq \cup_{n=1}^{\infty} c_0(X)_{(n^{-1/p_k})}$. On the other hand, suppose that $x = (x_k) \in \cup_{n=1}^{\infty} c_0(X)_{(n^{-1/p_k})}$. Then $\lim_{k\to\infty} \|x_k\|n^{-1/p_k} = 0$ for some $n \in N$. Let $\delta = (\delta_k)$ be the sequence definded by

$$\delta_k = \begin{cases} \|x_k\|n^{-1/p_k} & \text{if } \|x_k\| \neq 0 \\ \dfrac{1}{k} & \text{otherwise.} \end{cases}$$

Clearly $(\delta_k) \in c_0$ and $\|\frac{x_k}{\delta_k}\|^{p_k} \leq n$ for all $k \in N$, hence $\sup_k \|\frac{x_k}{\delta_k}\|^{p_k} \leq n$, so $x = (x_k) \in \underset{\sim}{c}_0(X, p)$.

Now we show (ii). If $x \in \ell_\infty(X, p)$, then there is some $n \in N$ with $\|x_k\|^{p_k} \leq n$ for all $k \in N$. Hence $\|x_k\|.n^{-1/p_k} \leq 1$ for all $k \in N$, so that $x \in \ell_\infty(X)_{(n^{-1/p_k})}$. On the other hand, if $x \in \cup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$, then there are some $n \in N$ and $M > 1$ such that $\|x_k\|.n^{-1/p_k} \leq M$ for every $k \in N$. Then we have $\|x_k\|^{p_k} \leq n.M^{p_k} \leq n.M^\alpha$ for all $k \in N$, where $\alpha = \sup_k p_k$. Hence $x \in \ell_\infty(X, p)$ □.

3. Main Results

We now turn to our main objective. We begin with giving characterisations of matrix transformations from $\ell_\infty(X)$ and $c_0(X)$ into ℓ_M. To do this we need a lemma.

Lemma 4.1 Let $E \in \{\ell_\infty(X), c_0(X)\}$ and (f_k) a sequence of continuous linear functionals on X. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in E$ if and only if $\sum_{k=1}^{\infty} \|f_k\| < \infty$

Proof. If $\sum_{k=1}^{\infty} \|f_k\| < \infty$, then for each $x = (x_k) \in E$, $\sum_{k=1}^{\infty} |f_k(x_k)| \leq \sum_{k=1}^{\infty} \|f_k\|\|x_k\| \leq \|x\| \sum_{k=1}^{\infty} \|f_k\| < \infty$, so that $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

Conversley, assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in E$. Define $T : E \to R$ by $Tx = \sum_{k=1}^{\infty} f_k(x_k)$. Clearly, T is linear. For each $n \in N$, let $s_n = \sum_{k=1}^{n} f_k o p_k$. Then $s_n \in E'$ since E is a K-space. It is clear that $s_n(x) \to Tx$ as $n \to \infty$ for all $x \in E$. It follows by Banach-Steinhaus theorem that $T \in E'$. Hence there is a positive real number α such that

$$\left|\sum_{k=1}^{\infty} f_k(x_k)\right| \leq \alpha \tag{4.1}$$

for all $x = (x_k) \in E$ with $\|x\| \leq 1$.

For each $x = (x_k) \in E$ with $\|x\| \leq 1$, we can choose a real sequence (t_k) with $|t_k| = 1$ for all $k \in N$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Clearly, $(t_k x_k) \in E$ and $\|(t_k x_k)\| \leq 1$. It follows by (4.1)

$$\sum_{k=1}^{\infty} |f_k(x_k)| = \left|\sum_{k=1}^{\infty} f_k(t_k x_k)\right| \leq \alpha \tag{4.2}$$

for all $x = (x_k) \in E$ with $\|x\| \leq 1$.
It implies by (4.2) that

$$\sum_{k=1}^{n} |f_k(x_k)| \leq \alpha \tag{4.3}$$

for all $n \in N$ and all $x_k \in X$ with $\|x_k\| \leq 1$.
It follows from (4.3) that $\sum_{k=1}^{n} \|f_k\| \leq \alpha$ for all $n \in N$, hence $\sum_{k=1}^{\infty} \|f_k\| \leq \alpha$. This complete the proof. □

Theorem 4.2 *Let $A = (f_k^n)$ be an infinite matrix and $E \in \{\ell_\infty(X), c_0(X)\}$. Then $A \in (E,\ \ell_M)$ if and only if*

(1) $\sum_{k=1}^{\infty} \|f_k^n\| < \infty$ for every $n \in N$, and

(2) There exists $K > 0$ such that $\sum_{n=1}^{\infty} M\left(\frac{1}{K}\sum_{k=1}^{\infty} f_k^n(x_k)\right) \leq 1$ for every $(x_k) \in E$ with $\|x_k\| \leq 1$ for all $k \in N$.

Proof. Assume that $A \in (E, \ell_\infty)$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in E$. Hence (1) holds by Lemma 4.1. Since E and ℓ_∞ are BK-spaces, by Zeller's theorem, A is continuous. It follows that there exists $K > 0$ such that

$$\|Ax\| \le K \tag{4.4}$$

for every $x = (x_k) \in E$ with $\|x_k\| \le 1$ for all $k \in N$.
Then we have $\|A(\frac{1}{K}x) \le 1$ for all $x = (x_k) \in E$ with $\|x_k\| \le 1$ for all $k \in N$. By [1, Theorem 1.38(1)], we have

$$\sum_{n=1}^{\infty} M\big(\frac{1}{K}\sum_{k=1}^{\infty} f_k(x_k)\big) \le 1$$

for every $x = (x_k) \in E$ with $\|x_k\| \le 1$ for all $k \in N$. Hence (2) holds.

Conversely, assume that (1) and (2) hold. By Lemma 4.1 , we have $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in E$. Let $K > 0$ be such that $\sum_{n=1}^{\infty} M\big(\frac{1}{K}\sum_{k=1}^{\infty} f_k^n(x_k)\big) \le 1$ for every $x = (x_k) \in E$ with $\|x_k\| \le 1$ for all $k \in N$. Then for $x = (x_k) \in E$ and $x \ne 0$, we have

$$\sum_{n=1}^{\infty} M\big(\frac{1}{K\|x\|}\sum_{k=1}^{\infty} f_k^n(x_k)\big) = \sum_{n=1}^{\infty} M\big(\frac{1}{K}\sum_{k=1}^{\infty} f_k^n(\frac{x_k}{\|x\|})\big) \le 1$$

which implies that $Ax \in \ell_M$, hence we have $A \in (E,\ \ell_M)$. □

Corollary 4.3 *Let $A = (f_k^n)$ be an infinite matrix. If $\big(\sum_{k=1}^{\infty} \|f_k^n\|\big)_{n=1}^{\infty} \in \ell_M$, then $A \in (\ell_\infty(X),\ \ell_M)$.*

Proof. Assume that $\big(\sum_{k=1}^{\infty} \|f_k^n\|\big)_{n=1}^{\infty} \in \ell_M$. Then there exists $\lambda > 0$ and $\alpha > 1$ such that $\sum_{n=1}^{\infty} M\big(\lambda\sum_{k=1}^{\infty} \|f_k^n\|\big) \le \alpha$. Let $x = (x_k) \in \ell_\infty(X)$ and $\|x\| \le 1$. Then $\|x_k\| \le 1$ for all $k \in N$, so $|f_k^n(x_k)| \le \|f_k^n\|$ for all $n, k \in N$. Putting $K = \frac{\alpha}{\lambda}$. Since M is convex, even, and increasing on $[0, \infty)$, it follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} M\Big(\frac{1}{K}\sum_{k=1}^{\infty} f_k^n(x_k)\Big) &= \sum_{n=1}^{\infty} M\left(\frac{\lambda}{\alpha}\left|\sum_{k=1}^{\infty} f_k^n(x_k)\right|\right) \\
&\leq \frac{1}{\alpha}\sum_{n=1}^{\infty} M\left(\lambda\left|\sum_{k=1}^{\infty} f_k^n(x_k)\right|\right) \\
&\leq \frac{1}{\alpha}\sum_{n=1}^{\infty} M\Big(\lambda\sum_{k=1}^{\infty} |f_k^n(x_k)|\Big) \\
&\leq \frac{1}{\alpha}\sum_{n=1}^{\infty} M\Big(\lambda\sum_{k=1}^{\infty} \|f_k^n\|\Big) \\
&\leq 1.
\end{aligned}$$

It follows by Theorem 4.2 that $A \in (\ell_\infty(X),\ \ell_M)$. $\square$

Theorem 4.4 *Let $A = (f_k^n)$ be an infinite matrix and let $p = (p_k)$ be a bounded sequence of positive real numbers. Then $A \in (\ell_\infty(X,p),\ \ell_M)$ if and only if*

(1) $\sum_{k=1}^{\infty} m^{1/p_k}\|f_k^n\| < \infty$ *for all $m, n \in N$, and*

(2) There exists $K > 0$ such that

$$\sum_{n=1}^{\infty} M\Big(\frac{1}{K}\sum_{k=1}^{\infty} m^{1/p_k} f_k^n(x_k)\Big) \leq 1$$

for every sequence (x_k) with $\|x_k\| \leq 1$ for all $k \in N$.

Proof. By Proposition 3.2(ii), we have $\ell_\infty(X,p) = \bigcup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$. It implies by Proposition 3.1(i) that

$$A \in (\ell_\infty(X,p),\ \ell_M) \iff A \in (\ell_\infty(X)_{(m^{-1/p_k})},\ \ell_M) \text{ for all } m \in N$$

By Proposition 3.1(v), we have

$$A \in (\ell_\infty(X)_{(m^{-1/p_k})},\ \ell_M) \iff (m^{1/p_k} f_k^n)_{n,k} \in (\ell_\infty(X),\ \ell_M)$$

We have by Theorem 4.2 that

$$(m^{1/p_k} f_k^n)_{n,k} \in (\ell_\infty(X),\ \ell_M) \iff \text{ the conditions (1) and (2) hold.}$$

Hence the theorem is proved. $\square$

Theorem 4.5 *Let $A = (f_k^n)$ be an infinite matrix and let $p = (p_k)$ be a bounded sequence of positive real numbers. Then $A \in (\underline{c}_0(X,p),\ \ell_M)$ if and only if*

(1) $\sum_{k=1}^{\infty} m^{1/p_k} \|f_k^n\| < \infty$ *for all $m, n \in N$, and*

(2) There exists $K > 0$ such that

$$\sum_{n=1}^{\infty} M\Big(\frac{1}{K}\sum_{k=1}^{\infty} m^{1/p_k} f_k^n(x_k)\Big) \leq 1$$

for every sequence $(x_k) \in c_0(X)$ with $\|x_k\| \leq 1$ for all $k \in N$.

Proof. Since $\underline{c}_0(X,p) = \cup_{n=1}^{\infty} c_0(X)_{(n^{-1/p_k})}$, we have by Proposition 3.1(i) that

$$A \in (\underline{c}_0(X,p),\ \ell_M) \Longleftrightarrow A \in (c_0(X)_{(m^{-1/p_k})},\ \ell_M) \text{ for all } m \in N$$

By Proposition 3.1(v), we have

$$A \in (c_0(X)_{(m^{-1/p_k})},\ \ell_M) \Longleftrightarrow (m^{1/p_k} f_k^n)_{n,k} \in (c_0(X),\ \ell_M)$$

It follows by Theorem 4.2 that

$$(m^{1/p_k} f_k^n)_{n,k} \in (\ell_\infty(X),\ \ell_M) \Longleftrightarrow \text{ the conditions (1) and (2) hold.}$$

Hence we have the theorem. $\square$

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MATRIX TRANSFORMATIONS ON THE MADDOX VECTOR-VALUED SEQUENCE SPACES

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Abstract. In this paper, we give the matrix characterizations from any normal vector-valued FK-space containing $\phi(X)$ into scalar-valued sequence space $c(q)$ and by applying this result, we also obtain necessary and sufficient conditions for infinite matrices mapping the sequence spaces $c_0(X,p), c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, $\underline{c_0}(X,p)$,$E_r(X,p)$, and $F_r(X,p)$ into the space $c(q)$, where $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers and $r \geq 0$.

Keywords: Matrix transformations, Maddox vector-valued sequence spaces

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1. Introduction

Let $(X, \|.\|)$ be a Banach space and $p=(p_k)$ a bounded sequence of positive real numbers. Let N be the set of all natural numbers, we write $x=(x_k)$ with x_k in X for all $k \in N$. The $X-$valued sequence spaces $c_0(X,p), c(X,p), \ell_\infty(X,p), \ell(X,p), \underline{c_0}(X,p), E_r(X,p)$, and $F_r(X,p)$ are defined as

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(a) $c_0(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\}$;
(b) $c(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0$ for some $a \in X\}$;
(c) $\ell_\infty(X,p) = \{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\}$;
(d) $\ell(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\}$;
(e) $\underline{c_0}(X,p) = \{x = (x_k) : \sup_k \|x_k/\delta_k\|^{p_k} < \infty$ for some $(\delta_k) \in c_0$ with $\delta_k \neq 0$ for all $k \in N\}$;
(f) $E_r(X,p) = \{x = (x_k) : \sup_k k^{-r}\|x_k\|^{p_k} < \infty\}$; .
(g) $F_r(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} k^r\|x_k\|^{p_k} < \infty\}$.

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p), c(p), \ell_\infty(p), \ell(p)$, $\underline{c_0}(p), E_r(p)$ and $F_r(p)$, respectively. The first three spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons [9] and Maddox [5 - 7]. The space $\ell(p)$ was first defined by Nakano [8] and is known as the Nakano sequence space. The spaces $\underline{c_0}(p)$ was first introduced by Grosse-Erdmann [3] and he investigated in [3] the structure of the spaces $c_0(p), c(p), \ell(p)$, and $\ell_\infty(p)$. Grosse-Erdmann [4] gave the matrix characterizations between scalar-valued sequence spaces of Maddox. When $p_k = 1$, for all $k \in N$, the spaces $E_r(p)$ and $F_r(p)$ are written as E_r and F_r, respectively. These two spaces were first introduced by Cooke [2]. Now the problem of matrix transformations becomes more general, we consider infinite matrices of bounded linear operators instead of matrices of real or complex numbers and we consider on vector-valued sequence spaces instead of scalar-valued sequence spaces. Choudhury [1] gave the matrix characterizations mapping $c_0(X)$ into $c_0(Y)$, $\ell_1(X)$ into $\ell_\infty(Y)$, and $\ell_1(X)$ into $\ell_p(Y)$. Wu and Liu [12] deal with the problem of characterizing infinite matrices mapping $c_0(X,p)$ and $\ell_\infty(X,p)$ into $c_0(q)$ and $\ell_\infty(q)$, where $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers. Suantai [10] has given matrix characterizations from $\ell(X,p)$ into the vector-valued sequence spaces $c_0(Y,q), c(Y)$ and $\ell_s(Y)$, where $q = (q_k)$ is a sequence of positive real numbers, Y is a Banach space and $s \geq 1$. He has also given in [11] necessary and sufficient conditions for infinite matrices mapping $\ell(X,p)$ into ℓ_∞ and $\underline{\ell_\infty}(q)$.

In this paper, we extend some results in [10] and [11] and generalize some results in [4]. We also obtain some related results as mentioned in the abstract.

2. Notation and Definitions

Let $(X, \|.\|)$ be a Banach space. Let $W(X)$ and $\Phi(X)$ denote the space of all sequences in X and the space of all finite sequences in X. When $X = K$, the scalar field of X, the corresponding spaces are written as w and Φ, respectively. A sequence space in X is a linear subspace of $W(X)$. Let E be an X- valued sequence space. For $x \in E$ and $k \in N$ we write that x_k stand for the kth term of x. For $x \in X$ and $k \in N$, we let $e^{(k)}(x)$ be the sequence $(0, 0, 0, ..., 0, x, 0, ...)$ with x in the kth position and let $e(x)$ be the sequence $(x, x, x, ...)$, and we denote by e the the sequence $(1, 1, 1, ...)$. For a fixed scalar sequence $u = (u_k)$ the sequence space E_u is defined as

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}.$$

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose E is an X-valued sequence space and F a scalar-valued sequence space. Then A is said to *map E into F*, written by $A : E \to F$ if, for each $x = (x_k) \in E, A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$, and the sequence $Ax = (A_n(x)) \in F$. We denote by (E, F) the set of all infinite matrices mapping E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$$_u(E, F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E, F)\}.$$

If $u_k \neq 0$ for all $k \in N$, we put $u^{-1} = (1/u_k)$. An $X-$valued sequence space E is said to be *normal* if $(x_k) \in E$ and $(y_k) \in W(X)$ with $\|y_k\| \leq \|x_k\|$ for all $k \in N$ implies that $(y_k) \in E$.

Suppose the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if, for each $k \in N$ the kth coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. In addition, if (E, τ) is a $Fre'chet(Banach)$ space, then E is called an $FK-(BK-)$ space .

The spaces $c_0(p)$ and $c(p)$ are FK-spaces. In $c_0(X, p)$, we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = \max \{1, \sup_k p_k\}$, as a paranorm on $c_0(X, p)$, and it is known that $c_0(X, p)$ is an FK-space under the paranorm g defined as above. In $\ell(X, p)$, we consider it as a paranormed sequence space with the paranorm given by $\|(x_k)\| = (\sum_{k=1}^{\infty} \|x_k\|^{p_k})^{1/M}$, where $M = \max \{1, \sup_k p_k\}$. It is known that $\ell(X, p)$ is an FK-space under the paranorm defined as above.

3. Some Auxiliary Results

We start with the following useful results that will reduce our problems into some simpler forms.

Proposition 3.1. *Let E and $E_n (n \in N)$ be X-valued sequence spaces, and F and $F_n (n \in N)$ scalar-valued sequence spaces, and let μ and ν be scalar sequences with $\mu_k \neq 0, \nu_k \neq 0$ for all $k \in N$. Then*

(i) $(\cup_{n=1}^{\infty} E_n, F) = \cap_{n=1}^{\infty} (E_n, F)$;

(ii) $(E, \cap_{n=1}^{\infty} F_n) = \cap_{n=1}^{\infty} (E, F_n)$;

(iii) $(E_1 + E_2, F) = (E_1, F) \cap (E_2, F)$;

(iv) $(E_u, F_v) = {}_v(E, F)_{u^{-1}}$.

Proof. All assertions are immediately obtained directly by the definition.

Propostion 3.2. *Let $p = (p_k)$ be a bounded sequences of positive real numbers and $r \geq 0$. Then*

(i) $c(X, p) = c_0(X, p) + \{e(x) : x \in X\}$;

(ii) $\underline{c_0}(X, p) = \cup_{n=1}^{\infty} c_0(X)_{(n^{-\frac{1}{p_k}})}$;

(iii) $E_r(X, p) = \ell_\infty(X, p)_{(k^{-\frac{r}{p_k}})}$;

(iv) $F_r(X, p) = \ell(X, p)_{(k^{\frac{r}{p_k}})}$;

(v) $\ell_\infty(X, p) = \cup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$.

Proof. Assertions (i), (iii) and (iv) are immediately obtained by the definition. To show (ii), let $x = (x_k) \in \underline{c_0}(X, p)$. Then there is a sequence $(\delta_k) \in c_0$ with $\delta_k \neq 0$ for all $k \in N$ such that $\sup_k \|x_k/\delta_k\|^{p_k} < \infty$. Hence there exists $\alpha > 0$ such that $\|x_k\| \leq \alpha^{1/p_k} |\delta_k|$ for all $k \in N$. Choose $n_0 \in N$ so that $n_0 > \alpha$. Then

$$\|x_k\| n_0^{-1/p_k} \leq (\alpha/n_0)^{1/p_k} |\delta_k| < |\delta_k| ,$$

which implies that $\lim_{k\to\infty} \|x_k\| n_0^{-1/p_k} = 0$, hence $x = (x_k) \in c_0(X)_{(n_0^{-1/p_k})} \subseteq \cup_{n=1}^{\infty} c_0(X)_{(n^{-1/}}$ On the other hand, suppose $x = (x_k) \in \cup_{n=1}^{\infty} c_0(X)_{(n^{-1/p_k})}$. Then $\lim_{k\to\infty} \|x_k\| n^{-1/p_k} = 0$ for some $n \in N$. Let $\delta = (\delta_k)$ be the sequence defined by

$$\delta_k = \begin{cases} \|x_k\| n^{-1/p_k}, & \text{if } x_k \neq 0 \\ 1/k, & \text{otherwise.} \end{cases}$$

Clearly, $(\delta_k) \in c_0$ and $\|x_k/\delta_k\|^{p_k} \leq n$ for all $k \in N$, hence $\sup_k \|x_k/\delta_k\|^{p_k} \leq n$, so $x = (x_k) \in \underline{c_0}(X,p)$.

It remains to show (v). If $x \in \ell_\infty(X,p)$, then there is some $n \in N$ with $\|x_k\|^{p_k} \leq n$ for all $k \in N$. Hence $\|x_k\| n^{-1/p_k} \leq 1$ for all $k \in N$, so that $x \in \ell_\infty(X)_{(n^{-1/p_k})}$. On the other hand, if $x \in \cup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$, then there are some $n \in N$ and $M > 1$ such that $\|x_k\| n^{-1/p_k} \leq M$ for every $k \in N$. Then we have $\|x_k\|^{p_k} \leq nM^{p_k} \leq nM^\alpha$ for all $k \in N$, where $\alpha = \sup_k p_k$. Hence $x \in \ell_\infty(X,p)$. □

Proposition 3.3. *Let (f_k) be a sequence of continuous linear functionals on X. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X,p)$ if and only if $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in N$.*

Proof. Suppose $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in N$. Let $x = (x_k) \in c_0(X,p)$. Then there is a positive integer K_0 such that $\|x_k\|^{p_k} < 1/M$ for all $k \geq K_0$, hence $\|x_k\| < M^{-1/p_k}$ for all $k \geq K_0$. Then we have

$$\sum_{k=K_0}^{\infty} |f_k(x_k)| \leq \sum_{k=K_0}^{\infty} \|f_k\| \|x_k\| \leq \sum_{k=K_0}^{\infty} \|f_k\| M^{-1/p_k} < \infty.$$

It follows that $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

On the other hand, assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X,p)$. For each $x = (x_k) \in c_0(X,p)$, choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Since $(t_k x_k) \in c_0(X,p)$, by our assumption, we have $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \text{for all } x \in c_0(X,p). \tag{3.1}$$

Now, suppose that $\sum_{k=1}^{\infty} \|f_k\| m^{-1/p_k} = \infty$ for all $m \in N$. Choose $m_1, k_1 \in N$ such that

$$\sum_{k \leq k_1} \|f_k\| m_1^{-1/p_k} > 1$$

and choose $m_2 > m_1$ and $k_2 > k_1$ such that

$$\sum_{k_1<k\le k_2} \|f_k\| m_2^{-1/p_k} > 2.$$

Proceeding in this way, we can choose $m_1 < m_2 < ...$, and $0 = k_1 < k_2 < ...$ such that

$$\sum_{k_{i-1}<k\le k_i} \|f_k\| m_i^{-1/p_k} > i.$$

Take x_k in X with $\|x_k\| = 1$ for all $k, k_{i-1} < k \le k_i$ such that

$$\sum_{k_{i-1}<k\le k_i} |f_k(x_k)| m_i^{-1/p_k} > i \quad \text{for all } i \in N.$$

Put $y = (y_k)$, $y_k = m_i^{-1/p_k} x_k$ for $k_{i-1} < k \le k_i$, then $y \in c_0(X,p)$ and

$$\sum_{k=1}^{\infty} |f_k(y_k)| \ge \sum_{k_{i-1}<k\le k_i} |f_k(x_k)| m_i^{-1/p_k} > i \quad \text{for all } i \in N.$$

Hence we have $\text{sum}_{k=1}^{\infty} |f_k(y_k)| = \infty$ which contradicts with (3.1). This completes the proof. □

Proposition 3.4. *Let (f_k) be a sequence of continuous linear functionals on X. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X,p)$ if and only if $\sum_{k=1}^{\infty} \|f_k\| m^{1/p_k} < \infty$ for all $m \in N$.*

Proof. If $\sum_{k=1}^{\infty} \|f_k\| m^{1/p_k} < \infty$ for all $m \in N$, we have that for each $x = (x_k) \in \ell_\infty(X,p)$, there is $m_0 \in N$ such that $\|x_k\| \le m_0^{1/p_k}$ for all $k \in N$, hence $\sum_{k=1}^{\infty} |f_k(x_k)| \le \sum_{k=1}^{\infty} \|f_k\| \|x_k\| \le \sum_{k=1}^{\infty} \|f_k\| m_0^{1/p_k} < \infty$, which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

Conversely, if $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X,p)$, by using the same proof as in Proposition 3.3, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \text{ for all } x = (x_k) \in \ell_\infty(X,p). \tag{3.2}$$

Now, suppose that $\sum_{k=1}^{\infty} \|f_k\| M^{1/p_k} = \infty$, for some $M \in N$. Then we can choose a sequence (k_i) of positive integers with $0 = k_0 < k_1 < k_2 < \ldots$ such that

$$\sum_{k_{i-1}<k\leq k_i} \|f_k\| M^{1/p_k} > i \quad \text{for all } i \in N.$$

And we choose x_k in X with $\|x_k\| = 1$ such that for all $i \in N$,

$$\sum_{k_{i-1}<k\leq k_i} |f_k(x_k)| M^{1/p_k} > i.$$

Put $y = (y_k)$, $y_k = M^{1/p_k} x_k$. Clearly, $y \in \ell_\infty(X,p)$ and

$$\sum_{k=1}^{\infty} |f_k(y_k)| \;\geq \sum_{k_{i-1}<k\leq k_i}^{\infty} |f_k(x_k)| M^{1/p_k} > i \quad \text{for all } i \in N.$$

Hence $\sum\limits_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts with (3.2). The proof is now complete. $\square$

Proposition 3.5. *Let (f_k) be a sequence of continuous linear functionals on X and $p = (p_k)$ a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell(X,p)$ if and only if $\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty$ for some $M \in N$, where $1/p_k + 1/t_k = 1$ for all $k \in N$.*

Proof. Suppose $\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty$ for some $M \in N$.
Then we have that for each $x = (x_k) \in \ell(X,p)$,

$$\begin{aligned}
\sum_{k=1}^{\infty} |f_k(x_k)| \;&\leq \sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} M^{1/p_k} \|x_k\| \\
&\leq \sum_{k=1}^{\infty} \left(\|f_k\|^{t_k} M^{-t_k/p_k} + M \|x_k\|^{p_k} \right) \\
&= \sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} + M \sum_{k=1}^{\infty} \|x_k\|^{p_k} \quad < \infty ,
\end{aligned}$$

which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

On the other hand, assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell(X,p)$. By using the same proof as in Proposition 3.3, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \text{ for all } x = (x_k) \in \ell(X,p). \tag{3.3}$$

We want to show that there exists $M \in N$ such that $\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty$. If it is not true, then

$$\sum_{k=1}^{\infty} \|f_k\|^{t_k} m^{-(t_k-1)} = \infty, \text{ for all } m \in N. \tag{3.4}$$

It implies by (3.4) that for each $k \in N$,

$$\sum_{i>k} \|f_i\|^{t_i} m^{-(t_i-1)} = \infty, \text{ for all } m \in N. \tag{3.5}$$

By (3.4), let $m_1 = 1$, then there is a $k_1 \in N$ such that

$$\sum_{k \le k_1} \|f_k\|^{t_k} m_1^{-(t_k-1)} > 1.$$

By (3.5), we can choose $m_2 > m_1$ and $k_2 > k_1$ with $m_2 > 2^2$ such that

$$\sum_{k_1 < k \le k_2} \|f_k\|^{t_k} m_2^{-(t_k-1)} > 1. \tag{3.6}$$

Proceeding in this way, we can choose sequences of positive integers (k_i) and (m_i) with $1 = k_0 < k_1 < k_2 < \dots$ and $m_1 < m_2 < \dots$, such that $m_i > 2^i$ and

$$\sum_{k_{i-1} < k \le k_i} \|f_k\|^{t_k} m_i^{-(t_k-1)} > 1 \; .$$

For each $i \in N$, choose x_k in X with $\|x_k\| = 1$ for all $k \in N$, $k_{i-1} < k \le k_i$ such that

$$\sum_{k_{i-1} < k \le k_i} |f_k(x_k)|^{t_k} m_i^{-(t_k-1)} > 1 \text{ for all } i \in N.$$

Let $a_i = \sum_{k_{i-1} < k \le k_i} |f_k(x_k)|^{t_k} m_i^{-(t_k-1)}$. Put $y = (y_k)$, $y_k = a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k$

for all k $k_{i-1} < k \leq k_i$. For each $i \in N$, we have

$$\begin{aligned}
\sum_{k_{i-1}<k\leq k_i} \|y_k\|^{p_k} &= \sum_{k_{i-1}<k\leq k_i} \left\| a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k \right\|^{p_k} \\
&= \sum_{k_{i-1}<k\leq k_i} a_i^{-p_k} m_i^{-t_k} |f_k(x_k)|^{t_k} \\
&\leq \sum_{k_{i-1}<k\leq k_i} a_i^{-1} m_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k} \\
&= a_i^{-1} m_i^{-1} a_i \\
&= m_i^{-1} \\
&< 1/2^i.
\end{aligned}$$

So we have that $\sum_{k=1}^{\infty} \|y_k\|^{p_k} \leq \sum_{i=1}^{\infty} 1/2^i < \infty$. Hence, $y = (y_k) \in \ell(X,p)$. For each $i \in N$, we have

$$\begin{aligned}
\sum_{k_{i-1}<k\leq k_i} |f_k(y_k)| &= \sum_{k_{i-1}<k\leq k_i} \left| f_k(a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k) \right| \\
&= \sum_{k_{i-1}<k\leq k_i} a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k} \\
&= a_i^{-1} \sum_{k_{i-1}<k\leq k_i} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k} \\
&= 1.
\end{aligned}$$

So that $\sum\limits_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts with (3.3). The proof is now complete. □

Proposition 3.6. *Let $p = (p_k)$ be a bounded sequence of positve real numbers with $p_k \leq 1$ for all $k \in N$ and $(f_k) \subset X'$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in \ell(X,p)$ if and only if there exists $M \in N$ such that $\sup_k \|f_k\| M^{-1/p_k} < \infty$.*

Proof. If $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in \ell(X,p)$, using the same proof as in Proposition 3.3, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \text{for all} \quad x = (x_k) \in \ell(X,p) \tag{3.7}$$

Suppose that $\sup_k \|f_k\| m^{-1/p_k} = \infty$ for all $m \in N$. For each $i \in N$, choose sequences (m_i) and (k_i) of positive integers with $m_1 < m_2 < \ldots$ and $k_1 < k_2 < \ldots$ such that $m_i > 2^i$ and $\|f_{k_i}\| m_i^{-1/p_{k_i}} > 1$. Choose $x_{k_i} \in X$ with $\|x_{k_i}\| = 1$ such that

$$|f_{k_i}(x_{k_i})| m_i^{-1/p_{k_i}} > 1 . \tag{3.8}$$

Let $y = (y_k)$, $y_k = m_i^{-1/p_{k_i}} x_{k_i}$ if $k = k_i$ for some i, and 0 otherwise. Then $\sum_{k=1}^{\infty} \|y_k\|^{p_k} = \sum_{i=1}^{\infty} 1/m_i < \sum_{i=1}^{\infty} 1/2^i = 1$, so that $(y_k) \in \ell(X,p)$ and

$$\begin{aligned}\sum_{k=1}^{\infty} |f_k(y_k)| &= \sum_{i=1}^{\infty} |f_{k_i}(m_i^{-1/p_{k_i}} x_{k_i})| \\ &= \sum_{i=1}^{\infty} m_i^{-1/p_{k_i}} |f_{k_i}(x_{k_i})| = \infty \text{ by (3.8)},\end{aligned}$$

and this is contradictory with (3.7). Therefore, there exists $M \in N$ such that $\sup_k \|f_k\| M^{-1/p_k} < \infty$.

Conversely, assume that there exists $M \in N$ such that $\sup_k \|f_k\| M^{-1/p_k} < \infty$. Let $x = (x_k) \in \ell(X,p)$, then there is a $K > 0$ such that

$$\|f_k\| \leq K M^{1/p_k} \quad \text{for all} \quad k \in N \tag{3.9}$$

and there is a $k_0 \in N$ such that $M^{1/p_k} \|x_k\| \leq 1$ for all $k \geq k_0$. By $p_k \leq 1$ for all $k \in N$, we have that for all $k \geq k_0$,

$$M^{1/p_k} \|x_k\| \leq (M^{1/p_k} \|x_k\|)^{p_k} = M \|x_k\|^{p_k}. \tag{3.10}$$

Then

$$\begin{aligned}\sum_{k=1}^{\infty} |f_k(x_k)| &\leq \sum_{k=1}^{k_o} \|f_k\| \|x_k\| + \sum_{k=k_0+1}^{\infty} \|f_k\| \|x_k\| \\ &\leq \sum_{k=1}^{k_o} \|f_k\| \|x_k\| + K \sum_{k=k_0+1}^{\infty} M^{1/p_k} \|x_k\| \quad \text{(by (3.9))} \\ &\leq \sum_{k=1}^{k_o} \|f_k\| \|x_k\| + KM \sum_{k=k_0+1}^{\infty} \|x_k\|^{p_k} \quad \text{(by (3.10))} \\ &< \infty.\end{aligned}$$

This implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges. □

4. Main Results

We begin with the following useful result.

Theorem 4.1. *Let $q = (q_k)$ be a bounded sequence of positive real numbers and let E be a normal $X-$ valued sequence space which is an FK-space and contains $\Phi(X)$. Then*

$$(E, c(q)) \;=\; (E, c_0(q)) \;\oplus\; (E, < e >).$$

To prove this theorem, we need the following two lemmas.

Lemma 4.1. *Let E be an X-valued sequence space which is an FK-space and contains $\Phi(X)$. Then for each $k \in N$, the mapping $T_k : X \to E$, defined by $T_k x = e^k(x)$, is continuous.*

Proof. Let $V = \{e^k(x) : x \in X\}$. Then V is a closed subspace of E, so it is an FK-space because E is an FK-space. Since E is a K-space, the coordinate mapping $p_k : V \to X$ is continuous and bijective. It follows from the open mapping theorem that p_k is open, which implies that $p_k^{-1} : X \to V$ is continuous. But since $T_k = p_k^{-1}$, we thus obtain that T_k is continuous. □

Lemma 4.2. *Let $q = (q_k)$ be a bounded sequence of positive real numbers. If E and F are scalar-valued sequence spaces such that E is normal containing Φ and F is an FK-space with the property that for each $x = (x_k) \in F$, there is a subsequence (x_{n_k}) of (x_k) with $x_{n_k} \to 0$ as $k \to \infty$, then $(E,\ F \oplus\ < e >) = (E,\ F)\ \oplus\ (E,\ < e >)$.*

Proof. See [2, Proposition 3.1(vi)]. □

Proof of Theorem 4.1 Since $c(q) = c_0(q) \oplus < e >$, it is clear that $(E,\ c_0(q)) + (E, < e >) \ \subseteq\ (E, c_0(q) \oplus\ < e >) = (E, c(q))$. Moreover, if $A \in (E, c_0(q)) \bigcap (E, < e >)$, then $A \in (E,\ c_0(q) \bigcap < e >)$, so that $A \in (E, 0)$, which implies that $A = 0$ because E contain $\Phi(X)$. Hence $(E, c_0(q)) + (E, < e >)$ is a direct sum. Now, we will show that $(E, c(q)) \subseteq (E, c_0(q)) \oplus (E, < e >)$. Let $A = (f_k^n) \in (E, c(q)) = (E, c_0(q) \oplus < e >)$. For $x \in X$ and $k \in N$, we have $(f_k^n(x))_{n=1}^{\infty} = Ae^k(x) \in c_0(q) \oplus < e >$, so that there exist unique $(b_k^n(x))_{n=1}^{\infty} \in c_0(q)$ and $(c_k^n(x))_{n=1}^{\infty} \in < e >$ with

$$(f_k^n(z))_{n=1}^{\infty} = (b_k^n(z))_{n=1}^{\infty} + (c_k^n(z))_{n=1}^{\infty}. \tag{4.1}$$

For each $n, k \in N$, let g_k^n and h_k^n be the functionals on X defined by

$$g_k^n(x) = b_k^n(x) \quad \text{and} \quad h_k^n(x) = c_k^n(x) \quad \text{for all } x \in X.$$

Clearly, g_k^n and h_k^n are linear, and by (4.1)

$$f_k^n = g_k^n + h_k^n \quad \text{for all } n, k \in N. \tag{4.2}$$

Note that $c_0(q) \oplus < e >$ is an FK-space in its direct sum topology. By Zeller's theorem, $A : E \to c_0(q) \oplus < e >$ is continuous. For each $k \in N$, let $T_k : X \to E$ be defined by $T_k(x) = e^k(x)$. By Lemma 4.1, we have that T_k is continuous for all $k \in N$. Since the projection P_1 of $c_0(q) \oplus < e >$ onto $c_0(q)$ and the projection P_2 of $c_0(q) \oplus < e >$ onto $< e >$ are continuous and $g_k^n = p_n \circ P_1 \circ A \circ T_k$ and $h_k^n = p_n \circ P_2 \circ A \circ T_k$ for all $n, k \in N$, we obtain that g_k^n and h_k^n are continuous, so $g_k^n, h_k^n \in X'$ for all $n, k \in N$. Let $B = (g_k^n)$ and $C = (h_k^n)$. By (4.1) and (4.2) we have $A = B + C$, $B = (g_k^n) \in (\Phi(X), c_0(q))$ and $C = (h_k^n) \in (\Phi(X), < e >)$. We will show that $B \in (E, c_0(q))$ and $C \in (E, < e >)$. To do this, let $x = (x_k) \in E$. Then for $\alpha = (\alpha_k) \in \ell_\infty$, we have $\|\alpha_k x_k\| = |\alpha_k| \|x_k\| \leq \|M x_k\|$,where $M = \sup_k |\alpha_k|$. Then the normality of E implies that $(\alpha_k x_k) \in E$. Hence $(f_k^n(x_k))_{n,k} \in (\ell_\infty,\ c_0(q) \oplus < e >)$, moreover, we have $(g_k^n(x_k))_{n,k} \in (\Phi,\ c_0(q))$, $(h_k^n(x_k))_{n,k} \in (\Phi,\ < e >)$, and $(f_k^n(x_k))_{n,k} = (g_k^n(x_k))_{n,k} + (h_k^n(x_k))_{n,k}$. Since ℓ_∞ is normal containing Φ and $c_0(q) \subseteq c_0$, it follows from Lemma 4.2 that $(g_k^n(x_k))_{n,k} \in (\ell_\infty,\ c_0(q))$ and $(h_k^n(x_k))_{n,k} \in (\ell_\infty,\ < e >)$. This implies that $Bx \in c_0(q)$ and $Cx \in < e >$, so we have $B \in (E, c_0(q))$ and $C \in (E, < e >)$, hence $A \in (E,\ c_0(q)) \oplus (E,\ < e >)$. This completes the proof. $\square$

Theorem 4.2. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c_0(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in N$,

(2) $m^{\frac{1}{q_n}} (f_k^n - f_k) \overset{w^*}{\to} 0$ as $n \to \infty$ for every $m, k \in N$ and

(3) $\sum_{k=1}^{\infty} m^{1/q_n} \|f_k^n - f_k\| r^{-1/p_k} \to 0$ as $n, r \to \infty$ for each $m \in N$.

Proof. If $A \in (c_0(X,p), c(q))$ we have $A \in (c_0(X,p), c_0(q) \oplus < e >)$ since $c(q) = c_0(q) \oplus < e >$. It follows from Theorem 4.1 that $A = B + C$, where $B \in (c_0(X,p), c_0(q))$ and $C \in (c_0(X,p), < e >)$. Let $C = (g_k^n)$. Since $\Phi(X) \subseteq c_0(X,p)$, we have $(g_k^n(x))_{n=1}^{\infty} \in < e >$ for all $x \in X$ and $k \in N$, which implies that $g_k^n = g_k^{n+1}$ for all $n, k \in N$, let $f_k = g_k^1$. Then we have $B = (f_k^n - f_k)_{n,k} \in (c_0(X,p), c_0(q))$. By [3, Theorem 0 (i)), we have $c_0(q) = \cap_{m=1}^{\infty} c_{0(m^{1/p_k})}$. It follows from Proposition 3.1(ii) and (iv) that $\left(m^{1/q_n}(f_k^n - f_k)\right)_{n,k} \in (c_0(X,p),\ c_0)$ for all $m \in N$. By Wu [11, Theorem 2.4], we have that the conditions (2) and (3) hold. Since $C = (f_k)_{n,k} \in (c_0(X,p),\ < e >)$, we have $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = x_k \in c_0(X,p)$, hence (1) is obtained by Proposition 3.3.

Conversely, assume that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that conditions (1),(2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. It is obvious that $A = B + C$. By conditions (2) and (3), we obtain by Proposition 3.1(ii) and (iv), and Wu [11, Theorem 2.4] that $B \in \left(c_0(X,p), c_0(q)\right)$. The condition (1) implies by Proposition 3.3 that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X,p)$, which implies that $C \in \left(c_0(X,p), < e >\right)$. Hence we have by Theorem 4.1 that $A \in \left(c_0(X,p), c(q)\right)$. This completes the proof. □

Theorem 4.3. *Let $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : \ell_\infty(X) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| < \infty$,

(2) $m^{1/q_n} (f_k^n - f_k) \overset{w^*}{\to}$ as $n \to \infty$ for every $k, m \in N$ and

(3) for each $m, r \in N$, $\sum_{j>k} m^{1/q_n} \|f_j^n - f_j\| r^{1/p_j} \to 0$ as $k \to \infty$ uniformly on $n \in N$.

Proof. If $A \in (\ell_\infty(X), c(q))$, then the condition (1) holds by Proposition 3.4. It follows from Theorem 4.1 that $A = B+C$, where $B \in (\ell_\infty(X),\ c_0(q))$ and $C \in (\ell_\infty(X),\ < e >)$. Using the same proof as in Theorem 4.2, there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that $C = (f_k)_{n,k}$ and $B = (f_k^n - f_k)_{n,k} \in (\ell_\infty(X),\ c_0(q))$. Since $c_0(q) = \cap_{k=1}^{\infty} c_{0_{(m^{\frac{1}{p_k}})}}$, we thus obtain (2) and (3) by Proposition 3.1(ii) and (iv), and Wu [11, Theorem 2.9].

Conversely, assume that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that condition (1), (2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. It is obvious that $A = B + C$. By conditions (2) and (3), we obtain by Proposition 3.1(ii) and (iv), and Wu [11, Theorem 2.9] that $B \in (\ell_\infty(X), c_0(q))$. The condition (1) implies by Proposition 3.4 that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X)$, which implies that $C \in (\ell_\infty(X), < e >)$. Hence, we have by Theorem 4.1 that $A \in (\ell_\infty(X), c(q))$. This completes the proof. □

Theorem 4.4. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : \ell_\infty(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| m^{1/p_k} < \infty$ for all $m \in N$,

(2) $r^{1/q_n} \left(m^{1/p_k} f_k^n - f_k\right) \xrightarrow{w^*} 0$ as $n \to \infty$ for every $m, k, r \in N$ and

(3) for each $m, r, s \in N$, $\sum_{j>k} r^{1/q_n} \|m^{1/p_j} f_j^n - f_j\| s^{1/p_j} \to 0$ as $k \to \infty$ uniformly on $n \in N$.

Proof. By Proposition 3.2 (v), $\ell_\infty(X,p) = \cup_{m=1}^{\infty} \ell_\infty(X)_{(m^{-1/p_k})}$. It follows from Proposition 3.1(i) and (iv), Proposition 3.4 and Theorem 4.3 that

$$\begin{aligned} A : \ell_\infty(X,p) \to c(q) &\iff \left(m^{1/p_k} f_k^n\right)_{n,k} : \ell_\infty(X) \to c(q) \quad \text{for all} \quad m \in N \\ &\iff \text{the conditions (1), (2), and (3) hold.} \end{aligned}$$

Theorem 4.5. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in N$,

(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for every* $m, k \in N$,

(3) $\sum_{k=1}^{\infty} m^{1/q_n}\|f_k^n - f_k\| r^{-1/p_k} \to 0$ *as* $n, r \to \infty$ *for every* $m \in N$ *and*

(4) $(\sum_{k=1}^{\infty} f_k^n(x))_{n=1}^{\infty} \in c(q)$ *for all* $x \in X$.

Proof. Since $c(X,p) = c_0(X,p) + \{e(x) : x \in X\}$ (Proposition 3.2 (i)), it follows from Proposition 3.1(iii) that $A \in (c(X,p), c(q))$ if and only if $A \in (c_0(X,p), c(q))$ and $A \in (\{e(x) : x \in X\}, c(q))$. By Theorem 4.2, we have $A \in (c_0(X,p), c(q))$ if and only if conditions (1)-(3) hold and it is clear that $A \in \big(\{e(x) : x \in X\}, c(q)\big)$ if and only if (4) holds. Hence, the theorem is proved. $\square$

Wu [12, Theorem 2.7] has given a characterization of an infinite matrix A such that $A \in (\ell(X,p), c_0)$ when $p_k > 1$ for all $k \in N$. By applying of Proposition 3.1(ii) and (iv), Proposition 3.5 and Theorem 4.1, and using the fact that $\cap_{m=1}^{\infty} c_{0(m^{1/p_k})}$, we obtain the following result.

Theorem 4.6. *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequences of positive real numbers with* $p_k > 1$ *for all* $k \in N$ *and* $1/p_k + 1/t_k = 1$ *for all* $k \in N$, *and let* $A = (f_k^n)$ *be an infinite matrix. Then* $A : \ell(X,p) \to c(q)$ *if and only if there is a sequence* (f_k) *with* $f_k \in X'$ *for all* $k \in N$ *such that*

(1) $\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty$ *for some* $M \in N$,

(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for all* $m, k \in N$ *and*

(3) *for each* $m \in N$, $\big(\sum_{k=1}^{\infty} m^{t_k/q_n}\|f_k^n - f_k\|^{t_k} r^{-(t_k-1)}\big) \to 0$ *as* $r \to \infty$ *uniformly on* $n \in N$.

By using [12, Theorem 2.6], Proposition 3.1(ii) and (iv), Proposition 3.6 and Theorem 4.1, we also obtain the following result.

Theorem 4.7. *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequences of positive real numbers with* $p_k \leq 1$ *for all* $k \in N$ *and* $A = (f_k^n)$ *an infinite matrix. Then* $A : \ell(X,p) \to c(q)$ *if and only if there is a sequence* (f_k) *with* $f_k \in X'$ *for all* $k \in N$ *such that*

(1) $\sup_k \|f_k\| M^{-1/p_k} < \infty$ *for some* $M \in N$,

(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for all* $m, k \in N$ *and*

(3) $\sup_{n,k} m^{p_k/q_n}\|f_k^n - f_k\|^{p_k} < \infty$ *for all* $m \in N$.

Theorem 4.8. *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequences of positive real numbers and* $A = (f_k^n)$ *an infinite matrix. Then* $A : \underline{c_0}(X,p) \to c(q)$ *if and only if there is a sequence* (f_k) *with* $f_k \in X'$ *for all* $k \in N$ *such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| < \infty$,

(2) $m^{1/q_n}(s^{1/p_k} f_k^n - f_k) \overset{w^*}{\to} 0$ *as* $n \to \infty$ *for every* $m, k, s \in N$ *and*

(3) $\frac{1}{r}\sum_{k=1}^{\infty} m^{1/q_n}\|s^{1/p_k} f_k^n - f_k\| \to 0$ *as* $n, r \to \infty$ *for each* $m, s \in N$.

Proof. By Proposition 3.2(ii), we have $\underline{c_0}(X,p) = \cup_{s=1}^{\infty} c_0(X)_{(s^{-1/p_k})}$. By Proposition 3.2(i) and (iv) and Theorem 4.2, we have

$$\begin{aligned} A : \underline{c_0}(X,p) \to c(q) &\iff A : \cup_{s=1}^{\infty} c_0(X)_{(s^{-1/p_k})} \to c(q) \\ &\iff A : c_0(X)_{(s^{-1/p_k})} \to c(q), \text{for all } s \in N \\ &\iff \left(s^{1/p_k} f_k^n\right)_{n,k} : c_0(X) \to c(q), \text{for all } s \in N \\ &\iff \text{the conditions (1), (2) and (3) hold.} \end{aligned}$$ □

Theorem 4.9. *Let* $p = (p_k)$ *and* $q = (q_k)$ *be a bounded sequences of positive real numbers and* $r \geq 0$, *and let* $A = (f_k^n)$ *be an infinite matrix. Then* $A : E_r(X,p) \to c(q)$ *if and only if there is a sequence* (f_k) *with* $f_k \in X'$ *for all* $k \in N$ *such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| m^{1/p_k} < \infty$ *for all* $m \in N$,

(2) $r^{1/q_n}\left(m^{1/p_k} k^{r/p_k} f_k^n - f_k\right) \overset{w^*}{\to} 0$ *as* $n \to \infty$ *for every* $m, k, r \in N$ *and*

(3) *for each* $m, r, s \in N$, $\sum_{j>k} r^{1/q_n}\|m^{1/p_j} j^{r/p_j} f_j^n - f_j\| s^{1/p_j} \to 0$ *as* $k \to \infty$ *uniformly on* $n \in N$.

Proof. By Proposition 3.2(iii), we have $E_r(X,p) = \ell_\infty(X,p)_{(k^{-r/p_k})}$. By Proposition 3.1 (iv) and Theorem 4.4, we have

$$\begin{aligned} A : E_r(X,p) \to c(q) &\iff A : \ell_\infty(X,p)_{(k^{-r/p_k})} \to c(q) \\ &\iff \left(k^{r/p_k} f_k^n\right)_{n,k} : \ell_\infty(X,p) \to c(q) \\ &\iff \text{the conditions (1), (2) and (3) hold.} \end{aligned}$$ □

In the last theorem, we give a characterization of a matrix transformation from the space $F_r(X,p)$ into $c(q)$. It is known by Proposition 3.2 (iv) that $F_r(X,p) = \ell(X,p)_{(k^{r/p_k})}$. By Proposition 3.1(iv), for a scalar sequence space E and an infinite matrix $A = (f_k^n)$, we have

$$A : F_r(X,p) \to E \Longleftrightarrow \left(k^{-r/p_k} f_k^n\right)_{n,k} : \ell(X,p) \to E.$$

So we the following theorem is obtained by applying Theorem 4.6.

Theorem 4.10. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in N$, $1/p_k + 1/t_k = 1$ for all $k \in N$ and $r \geq 0$, and let $A = (f_k^n)$ be an infinite matrix. Then $A : F_r(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\|^{t_k} k^{-rt_k/p_k} M^{-(t_k-1)} < \infty$ for some $M \in N$,

(2) $m^{1/q_n} \left(k^{-r/p_k} f_k^n - f_k\right) \xrightarrow{w^*} 0$ as $n \to \infty$ for all $m, k \in N$ and

(3) for each $m \in N$, $\sum_{k=1}^{\infty} m^{t_k/q_n} \|k^{-r/p_k} f_k^n - f_k\|^{t_k} r^{-(t_k-1)} \to 0$ as $r \to \infty$ uniformly on r N.

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MATRIX TRANSFORMATIONS ON THE NAKANO VECTOR-VALUED SEQUENCE SPACE

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Abstract: In this paper, we give the matrix characterizations from any FK-space of vector sequences with AK property into any FK-space of scalar sequences, and by applying this result we also obtain necessary and sufficient conditions for infinite matrices mapping the spaces $\ell(X,p)$ into Maddox sequence spaces $c_0(q)$ and $\ell(q)$ where $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers such that $p_k \geq 1$ for all $k \in N$.

1. Introduction: Let $(X, \|.\|)$ be a Banach space and $p=(p_k)$ a bounded sequence of positive real numbers. Let N be the set of all natural numbers, we write $x=(x_k)$ with x_k in X for all $k \in N$. The $X-$valued sequence spaces $c_0(X,p), c(X,p), \ell_\infty(X,p)$, and $\ell(X,p)$ are defined by

$$c_0(X,p)=\left\{x=(x_k): \lim_{k\to\infty}\|x_k\|^{p_k}=0\right\},$$

$$c(X,p)=\left\{x=(x_k): \lim_{k\to\infty}\|x_k-a\|^{p_k}=0 \text{ for some } a\in X\right\},$$

$$\ell_\infty(X,p)=\left\{x=(x_k): \sup_k\|x_k\|^{p_k}<\infty\right\}, and$$

$$\ell(X,p)=\left\{x=(x_k): \sum_{k=1}^{\infty}\|x_k\|^{p_k}<\infty\right\}.$$

When $X=K$,the scalar field of X,the corresponding spaces are written as $c_0(p), c(p), \ell(p)$, and $\ell_\infty(p)$,

respectively. The first three spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons[7] and Maddox[4, 5]. The space $\ell(p)$ was first defined by Nakano[6] and is known as the Nakano sequence space, and the space $\ell(X,p)$ is known as the Nakano vector-valued sequence space. Choudhur[1] gave

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necessary and sufficient conditions for an infinite matrix of continuous linear operators which maps the vector-valued sequence space $c_0(X)$ into $c_0(Y), \ell_1(X)$ into $\ell_\infty(Y)$ and $\ell_1(X)$ into $\ell_p(Y)$ where Y is a Banach space. Grosse-Erdmann[2] investigated the structure of the spaces $c_0(p), c(p), \ell(p)$ and $\ell_\infty(p)$ and the problem of characterizing a matrix that maps a sequence space of Maddox into another such space is studied by them in[3]. Suantai[9, 10, 11] gave the matrix characterizations from $\ell(X,p)$ into the space $c_0(Y,p), \underline{\ell_\infty}(q)$ and F_s in the case $p_k \leq 1$ for all $k \in N$ and $s \geq 0$, where Y is a Banach space. Wu[12] gave characterizations of matrix transformations from the space $\ell(X,p)$ into the space c_0 and $\ell_\infty(q)$. The characterizations of matrix transformations from the space $\ell(X,p)$ into $\ell(q)$ and $c_0(q)$ can not be expected to be characterized completely in term of Toeplitz conditions, but however we can give characterizations of these matrix transformations in term of other conditions. Even the classical pair (ℓ_p, ℓ_q) is an open problem when $1 < p, q < \infty$, and $(p,q) \neq (2,2)$. Also, in the case $(\ell(p), \ell(q))$ is an open problem if $q_k < 1$ for all $k \in N$.

2. Notation and Definitions: Let $(X, \|.\|)$ be a Banach space. The space of all sequences in X is denoted by $W(X)$ and $\Phi(X)$ denote for the space of all finite sequences in X.

A sequence space in X is a linear subspace of $W(X)$. Let E be an X- valued sequence space. For $x \in E$ we write $x = (x_k), k \in N$. For $z \in X$ and $k \in N$,we let $e^{(k)}(z)$ be the sequence $(0,0,0,...,0,z,0,...)$ with z in the k^{th} position. For a fixed scalar sequence $u = (u_k)$ the sequence space E_u is defined by

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}.$$

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a **K-space** if for each $n \in N$ the n^{th} coordinate mapping $p_n : E \to X$, defined by $p_n(x) = x_n$, is continuous on E. If, in addition, (E, τ) is an $Fre'chet(Banach, LF-, LB-)$ space, then E is called an $FK-(BK-, LFK-, LBK-)$ space. Now, suppose that E contains $\Phi(X)$. Then E is said to have **property AB** if the set $\{\sum_{k=1}^{n} e^k(x_k) : n \in N\}$ is bounded in E for every $x = (x_k) \in E$. It is said to have **property AK** if $\sum_{k=1}^{n} e^k(x_k) \to x \in E$ as $n \to \infty$ for every $x = (x_k) \in E$. It has **property AD** if $\Phi(X)$ is dense in E. Let $A = (f_k^n)$ with f_k^n in X',the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to **map E into F**, written $A : E \to F$ if for each $x = (x_k) \in E, A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$ and if the sequence $Ax = (A_n(x)) \in F$. We denote by (E, F) the set of all infinite matrices mapping E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$$_u(E,F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E,F)\}.$$

If $u_k \neq 0$ for all $k \in N$,we write $u^{-1} = (\frac{1}{u_k})$.

3. Some Auxiliary Results: We start with the following useful results that will reduce our problems into some simpler forms.

Proposition 3.1 Let E and $E_n (n \in N)$ be X-valued sequence spaces, and F and $F_n (n \in N)$ scalar sequence spaces, and let μ and ν be scalar sequences with $\mu_k \neq 0, \nu_k \neq 0$ for all $k \in N$. Then

(i) $(E, \bigcap_{n=1}^{\infty} F_n) = \bigcap_{n=1}^{\infty} (E, F_n)$ and

(ii) $(E_u, F_v) = {}_v(E,F)_{u^{-1}}$.

Proof (i) and (ii) are immediately obtained by the definition. □

Proposition 3.2 *Let (f_k) be a sequence of continuous linear functional on X and $p = (p_k)$ a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell(X,p)$ if and only if*

$$\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty \text{ for some } M \in N,$$

where $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$.

Proof. Suppose that $\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty$ for some $M \in N$.

Then we have that for each $x = (x_k) \in \ell(X,p)$,

$$\sum_{k=1}^{\infty} |f_k(x_k)| \leq \sum_{k=1}^{\infty} \|f_k\| M^{-\frac{1}{p_k}} M^{\frac{1}{p_k}} \|x_k\|$$

$$\leq \sum_{k=1}^{\infty}\left(\|f_k\|^{t_k}M^{-\frac{t_k}{p_k}}+M\|x_k\|^{p_k}\right)=\sum_{k=1}^{\infty}\|f_k\|^{t_k}M^{-(t_k-1)}+M\sum_{k=1}^{\infty}\|x_k\|^{p_k}<\infty$$

which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

On the other hand, assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x=(x_k)\in \ell(X,p)$. For each $x=(x_k)\in \ell(X,p)$, choose scalar sequence (t_k) with $|t_k|=1$ such that $f_k(t_kx_k)=|f_k(x_k)|$ for all $k\in N$. Since $(t_kx_k)\in \ell(X,p)$, by our assumption, we have $\sum_{k=1}^{\infty} f_k(t_kx_k)$ converges, so that

$$\sum_{k=1}^{\infty}|f_k(x_k)|<\infty \text{ for all } x=(x_k)\in \ell(X,p). \tag{3.1}$$

We want to show that there exists $M\in N$ such that

$$\sum_{k=1}^{\infty}\|f_k\|^{t_k}M^{-(t_k-1)}<\infty$$

If it is not true, then

$$\sum_{k=1}^{\infty}\|f_k\|^{t_k}m^{-(t_k-1)}=\infty,\ \text{ for all } m\in N. \tag{3.2}$$

And (3.2) implies that for each $k_0\in N$.

$$\sum_{k>k_0}\|f_k\|^{t_k}m^{-(t_k-1)}=\infty,\ \text{ for all } m\in N. \tag{3.3}$$

By (3.2). let $m_1=1$, then there is a $k_1\in N$ such that

$$\sum_{k\leq k_1}\|f_k\|^{t_k}m_1^{-(t_k-1)}>1.$$

By (3.3), we can choose $m_2>m_1$ and $m_2>2^2$ and $k_2>k_1$, such that

$$\sum_{k_1<k\leq k_2}\|f_k\|^{t_k}m_2^{-(t_k-1)}>1.$$

By continueing in this way, we obtain sequences (k_i) and (m_i) of positive integers such that $1 = k_0 < k_1 < k_2 < \ldots$ and $m_1 < m_2 < \ldots, m_i > 2^i$ and

$$\sum_{k_{i-1}<k\leq k_i} \|f_k\|^{t_k} m_i^{-(t_k-1)} > 1.$$

Choose x_k in X with $\|x_k\| = 1$ such that for all $i \in N$,

$$\sum_{k_{i-1}<k\leq k_i} |f_k(x_k)|^{t_k} m_i^{-(t_k-1)} > 1 \text{ for all } i \in N.$$

Let $a_i = \sum\limits_{k_{i-1}<k\leq k_i} |f_k(x_k)|^{t_k} m_i^{-(t_k-1)}$.
Put $y = (y_k)$, $y_k = a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k$ for all k, $k_{i-1} < k \leq k_i$.
For each $i \in N$, we have

$$\sum_{k_{i-1}<k\leq k_i} \|y_k\|^{p_k} = \sum_{k_{i-1}<k\leq k_i} \left\|a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k\right\|^{p_k} = \sum_{k_{i-1}<k\leq k_i} a_i^{-p_k} m_i^{-t_k} |f_k$$
$$\leq \sum_{k_{i-1}<k\leq k_i} a_i^{-1} m_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k} = a_i^{-1} m_i^{-1} a_i = m_i^{-1} < \tfrac{1}{2^i}.$$

So we have that

$$\sum_{k=1}^{\infty} \|y_k\|^{p_k} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty.$$

Hence $y = (y_k) \in \ell(X,p)$.
For each $i \in N$, we have

$$\begin{aligned}\sum_{k_{i-1}<k\leq k_i} |f_k(y_k)| &= \sum_{k_{i-1}<k\leq k_i} \left| f_k(a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k)\right| \\ &= \sum_{k_{i-1}<k\leq k_i} a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k} \\ &= a_i^{-1} \sum_{k_{i-1}<k\leq k_i} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k} \\ &= 1.\end{aligned}$$

So that $\sum\limits_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts with (3.4). The proof is now complete. □

4. Main Results: Now, we turn to our objective. We begin with giving characterizations of matrix transformations from an FK-space of vector sequences with AK property into an FK-space of scalar sequences.

Theorem 4.1.1 *Let $E \subseteq W(X)$ be an FK-space with AK property and F an FK-space of scalar sequences. Then, for an infinite matrix $A = (f_k^n), A : E \to F$ if and only if*

(1) for each $n \in N$, $\sum\limits_{k=1}^{\infty} f_k^n(x_k)$ converges for all $x = (x_k) \in E$,

(2) for each $k \in N$, $(f_k^n(z))_{n=1}^{\infty} \in F$ for all $z \in X$, and

(3) $A : \Phi(X) \to F$ is continuous when $\Phi(X)$ is considered as a subspace of E.

Proof. Assume that $A : E \to F$. Then we have that for any $x = (x_k) \in E$, $\sum\limits_{k=1}^{\infty} f_k^n(x_k)$ converges for all $n \in N$, so (1) holds. Since $e^k(z) \in E$ for all $k \in N$ and all $z \in X$, we obtain that for each $k \in N$,

$$(f_k^n(z))_{n=1}^{\infty} = Ae^k(z) \in F,$$

hence (2) holds. Since E and F are FK-spaces, by Zeller's theorem, $A : E \to F$ is continuous, so (3) is obtained.

Conversely, assume that the conditions hold. By (1), we have

$$Ax = \left(\sum_{k=1}^{\infty} f_k^n(x_k)\right)_{n=1}^{\infty} \in W, \text{ for all } x = (x_k) \in E.$$

It follows from (2) that $Ae^k(z) \in F$, for all $k \in N$ and $z \in X$, which implies that $A : \Phi(X) \to F$. By (3), we have $A : \Phi(X) \to F$ is continuous. Let $x = (x_k) \in E$. Since E has the AK property, we have

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} e^k(x_k).$$

Then $\left(\sum\limits_{k=1}^{n} e^k(x_k)\right)_{n=1}^{\infty}$ is a Cuachy sequence in E. Since $A : \Phi(X) \to F$ is continuous and linear, it implies that $\left(\sum\limits_{k=1}^{n} Ae^k(x_k)\right)_{n=1}^{\infty}$ is a Cauchy sequence in F. Since F is complete, we have $\left(\sum\limits_{k=1}^{n} Ae^k(x_k)\right)_{n=1}^{\infty}$ converges in F. Since F is a K-space, it implies that $\left(\sum\limits_{k=1}^{\infty} f_k^n(x_k)\right)_{n=1}^{\infty} \in F$, so that $Ax \in F$. This shows that $A : E \to F$. □

It is known that the space $\ell(X,p)$ is an FK-space with AK property under the paranorm

$$g(x) = \left(\sum_{k=1}^{\infty} \|x_k\|^{p_k}\right)^{\frac{1}{M}}, \text{ when } M = \max\{1, \sup_k p_k\}.$$

By Proposition 3.2 and Theorem 4.1.1, we have the following theorem.

Theorem 4.1.2 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers such that $p_k > 1$ for all $k \in N$, and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X,p) \to \ell(q)$ if and only if*

(1) for each $n \in N$ there exists $M_n \in N$ such that

$$\sum_{k=1}^{\infty} \|f_k^n\|^{t_k} M_n^{-(t_k-1)} < \infty, \text{ where } \frac{1}{p_k} + \frac{1}{t_k} = 1 \text{ for all } k \in N,$$

(2) for each $k \in N$ and $z \in X$, $\sum_{n=1}^{\infty} |f_k^n(z)|^{q_n} < \infty$, and

(3) for each $r \in N$ there exists $M_r \in N$ such that

$$\sum_{k\in K} \|x_k\|^{p_k} < \frac{1}{M_r} \Rightarrow \sum_{n=1}^{\infty} |\sum_{k\in K} f_k^n(x_k)|^{q_n} < \frac{1}{r},$$

for all $x = (x_k) \in \Phi(X)$ and all finite subsets K of N.

Now, we have the sufficient conditions for an infinite matrix $A = (f_k^n)$ that maps $\ell(X,p)$ into $\ell(q)$.

Theorem 4.1.3 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers such that $p_k > 1$ and $q_k > 1$ for all $k \in N$, and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X,p) \to \ell(q)$ if the following two conditions hold;*

(1) for each $n \in N$ there exists $M_n \in N$ such that

$$\sum_{k=1}^{\infty} \|f_k^n\|^{t_k} M_n^{-(t_k-1)} < \infty \text{ where } \frac{1}{p_k} + \frac{1}{t_k} = 1 \text{ for all } k \in N, \text{and}$$

(2) there exists $M_0 \in N$ such that

$$\sup_K \sum_{n=1}^{\infty} \left(\sum_{k\in K} \|f_k^n\| M_0^{-\frac{1}{p_k}} \right)^{q_n} < \infty,$$

where supremum is taken over all finite subsets K of N.

Proof. Suppose that the two conditions hold. Then by Proposition 3.2 the condition (1) implies the condition (1) of Theorem 4.1.1. By the condition (2), we have that there exists $M_0, L \in N$ such that

$$\sum_{n=1}^{\infty} \left(\sum_{k\in K} \|f_k^n\| M_0^{-\frac{1}{p_k}} \right)^{q_n} < L, \tag{4.1}$$

for all the finite subsets K of N. Then, for each $z \in X - \{0\}$ we can choose $M_1 > M_0$ such that $M_1\|z\| > 1$. Then for each $k \in N$, we have by (4.1) that

$$\begin{aligned}
\sum_{n=1}^{\infty} |f_k^n(z)|^{q_n} &\leq \sum_{n=1}^{\infty} (\|f_k^n\|\|z\|)^{q_n} \\
&= \sum_{n=1}^{\infty} \left(\|f_k^n\| M_1^{-\frac{1}{p_k}} M_1^{\frac{1}{p_k}} \|z\| \right)^{q_n} \\
&\leq \sum_{n=1}^{\infty} \left(\|f_k^n\| M_1^{-\frac{1}{p_k}} M_1 \|z\| \right)^{q_n} \\
&\leq (M_1\|z\|)^{\beta} \sum_{n=1}^{\infty} \left(\|f_k^n\| M_1^{-\frac{1}{p_k}} \right)^{q_n} \quad ; \ \beta = \sup_n q_n \\
&\leq (M_1\|z\|)^{\beta} L \\
&< \infty.
\end{aligned}$$

So, we have that $(f_k^n(z))_{n=1}^{\infty} \in \ell(q)$ for all $z \in N$ and $k \in N$. Hence the condition (2) of Theorem 4.1.1 holds. We shall now show that the condition (3) of Theorem 4.1.1 is satisfied. To show this, let $\varepsilon > 0$ and $x = (x_k) \in \Phi(X)$. Recall that $\|x\| = \left(\sum_{k=1}^{\infty} \|x_k\|^{p_k} \right)^{\frac{1}{M}}$ where $M = \sup_n q_n$. If $\|x\| \leq 1$, then for all $k \in N$ we have

$$\|x_k\| \leq \|x\|^{\frac{M}{p_k}} \leq \|x\|. \tag{4.2}$$

Since $x = (x_k) \in \Phi(X)$, there is a finite subset K_0 of N such that

$$\sum_{k=1}^{\infty} f_k^n(x_k) = \sum_{k \in K_0} f_k^n(x_k) \text{ for all } n \in N. \tag{4.3}$$

So, we have by (4.1), (4.2), and (4.3) that

$$\begin{aligned}
\|Ax\| &= \left(\sum_{n=1}^{\infty} |\sum_{k=1}^{\infty} f_k^n(x_k)|^{q_n}\right)^{\frac{1}{G}} \\
&= \left(\sum_{n=1}^{\infty} |\sum_{k \in K_0} f_k^n(x_k)|^{q_n}\right)^{\frac{1}{G}} \\
&\leq \left(\sum_{n=1}^{\infty} (\sum_{k \in K_0} \|f_k^n\| \|x_k\|)^{q_n}\right)^{\frac{1}{G}} \\
&\leq \left(\sum_{n=1}^{\infty} (\sum_{k \in K_0} \|f_k^n\| M_0^{-\frac{1}{p_k}} M_0^{\frac{1}{p_k}} \|x\|)^{q_n}\right)^{\frac{1}{G}} \\
&\leq \left(M_0^G \|x\| \sum_{n=1}^{\infty} (\sum_{k \in K_0} \|f_k^n\| M_0^{-\frac{1}{p_k}})^{q_n}\right)^{\frac{1}{G}} \\
&\leq M_0(\|x\| L)^{\frac{1}{G}}, \quad ,G = \sup_n q_n.
\end{aligned} \tag{4.4}$$

It implies by (4.4) that $A : \Phi(X) \to \ell(q)$. Now choose $\delta = \min\{1, \frac{1}{L}(\frac{\varepsilon}{M_0})^G\}$. It follows by (4.4) that

$$\|x\| < \delta \Rightarrow \|Ax\| < \varepsilon.$$

It follows that $A : \Phi(X) \to \ell(q)$ is continuous. Hence, by Theorem 4.1.1, we have that $A : \ell(X, p) \to \ell(q)$. □

By using the previous auxiliary results and Theorem 1.6 in [12], we obtain necessary and sufficient conditions for infinite matrices mapping the space $\ell(X, p)$ into $c_0(q)$.

Theorem 4.1.4 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers such that $p_k > 1$ for all $k \in N$, and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X, p) \to c_0(q)$ if and only if*

(1) for all $m, k \in N$, $m^{\frac{1}{q_n}} f_k^n \to 0$ weakly as $n \to \infty$, and

(2) for each $m \in N$,

$$\left(\sum_{k=1}^{\infty} m^{\frac{t_k}{q_n}} \|f_k^n\|^{t_k} r^{-(t_k-1)}\right) \to 0 \text{ uniformly for } n \geq 1 \text{ as } r \to \infty,$$

where $\frac{1}{p_k} + \frac{1}{t_k} = 1$ *for all* $k \in N$.

Proof. By Theorem 0 in [2], we have $c_0(q) = \bigcap_{s=1}^{\infty} c_{0(s^{\frac{1}{q_n}})}$. By Proposition 2.1(i) and (ii) and Theorem 1.6 in [12], we have

$$\begin{aligned} A : \ell(X,p) \to c_0(q) &\iff A : \ell(X,p) \to \bigcap_{m=1}^{\infty} c_{0(m^{\frac{1}{q_n}})} \\ &\iff A : \ell(X,p) \to c_{0(m^{\frac{1}{q_n}})}, \text{ for all } m \in N \\ &\iff \left(m^{\frac{1}{q_n}} f_k^n\right)_{n,k} : \ell(X,p) \to c_0, \text{ for all } m \in N \\ &\iff \text{the conditions (1) and (2) hold.} \end{aligned}$$

□

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Output จากโครงการวิจัยที่ได้รับทุนจาก สกว.

ผลงานที่ได้จากโครงการวิจัยนี้เป็นผลงานวิจัยที่ได้รับการตอบรับให้ตีพิมพ์ในวารสารระดับนานาชาติต่างๆ รวมทั้งที่รอผลการตอบรับจากวารสาร จำนวน 14 เรื่อง (papers) และเป็นผลงานที่เสนอในที่ประชุมนานาชาติอีก 3 เรื่อง ตลอดจนการผลิตนักวิจัยรุ่นใหม่อีกจำนวน 6 คน

1. ผลงานวิจัยที่เป็น paper ตีพิมพ์ในวารสารต่างๆ จำนวน 14 เรื่อง ดังนี้

1 ได้คิดค้นทฤษฎีใหม่เกี่ยวกับการแปลงเมทริกซ์ของปริภูมิลำดับค่าเวกเตอร์นาคาโน $\ell(X,p)$ เมื่อ $p=(p_k)$ เป็นลำดับของจำนวนจริงซึ่ง $p_k \le 1$ สำหรับทุกค่าของจำนวนนับ k ไปยังปริภูมิลำดับ $E_r (r \ge 0)$ และ ได้ให้เงื่อนไขที่จำเป็นและเพียงพอสำหรับเมทริกซ์อนันต์ที่ส่งจากปริภูมิ $M_0(X,p)$ ไปยังปริภูมิ $E_r (r \ge 0)$

ผลงานนี้ชื่อ "Matrix transformations of Nakano vector-valued sequence spaces" ตีพิมพ์ในวารสาร Kyungpook Mathematical Journal, Vol. 40, No. 1 (2000) 93 - 97.

2. ได้รับเชิญจากสมาคมคณิตศาสตร์ CALCUTTA ประเทศอินเดีย ในฐานะวิทยากรรับเชิญให้ไปเสนอผลงานวิจัยในที่ประชุมนานาชาติที่ CALCUTTA MATHEMATICAL SOCIETY ระหว่างวันที่ 23 - 26 มกราคม 2543 และผู้วิจัยได้นำเสนอผลงานเรื่อง " On Matrix Transformations of vector-valued sequence spaces of Maddox"

ผลงานนี้ตีพิมพ์ในวารสาร Bulletin of Culcutta Mathematical society, vol 9, no. 1&2 (2001), 99 - 110.

ในงานวิจัยนี้ได้ให้เงื่อนไขที่จำเป็นและเพียงพอสำหรับเมทริกซ์อนันต์ที่ส่งจากปริภูมิลำดับ Maddox ไปยังปริภูมิ Musielak-Orlicz sequence space

3. ได้ผลงานตีพิมพ์ชื่อ "On matrix transformations concerning the Nakano vector-valued sequence space" **ตีพิมพ์ในวารสาร International Journal of Mathematics and Mathematical Science (IJMMS) vol. 26, no. 11 (2001), 671 - 678.**

งานวิจัยนี้ได้ให้เงื่อนไขที่จะเป็นและเพียงพอสำหรับเมทริกซ์อนันต์ที่ส่งจากปริภูมิลำดับนาคาโน $\ell(X,p)$ และ $F_r(X,p)$ ไปยังปริภูมิลำดับ $E_r, \ell_\infty, \underline{\ell}_\infty(q), bs, cs$

4. ได้ผลงานตีพิมพ์ชื่อ "On $\beta-$dual of vector-valued sequence spaces of Maddox" **ตีพิมพ์ในวารสารชื่อ สาร International Journal of Mathematics and Mathematical Science (IJMMS) vol. 30, no. 7 (2002), 383 - 392.**
ผลงานดังกล่าวเป็นการศึกษาหาลักษณะเฉพาะของ $\beta-$dual ของปริภูมิลำดับค่าเวกเตอร์ของแมดดอกซ์ $\ell(X,p), \ell_\infty(X,p), c(X,p), c_0(X,p)$

5. ได้ผลงานตีพิมพ์ชื่อ "Matrix Transformations of Some Vector-Valued Sequence Spaces" **ผลงานนี้ได้รับการตอบรับให้ลงตีพิมพ์ ในวารสารระดับนานาชาติ ชื่อ Indian Journal of Pure and Applied Mathematics**

งานวิจัยนี้ได้ทฤษฎีองค์ความรู้ใหม่เกี่ยวกับการแปลงเมทริกซ์ที่ส่งจากปริภูมิลำดับที่เป็นนอร์มัล FK- space ไปยังปริภูมิ c(q) และโดยการประยุกต์ผลดังกล่าวนี้ให้สามารถให้ลักษณะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิลำดับของแมดดอกซ์ไปยังปริภูมิ c(q)

6. ได้ผลงานตีพิมพ์ชื่อ "Matrix Transformations on the Nakano vector-valued sequence spaces" ซึ่งได้รับการตอบรับให้ตีพิมพ์ในวารสาร **Kyungpook Mathematical Journal**

ผลงานดังกล่าวนี้ได้ให้เงื่อนไขที่จำเป็นและเพียงพอสำหรับเมตริซ์อนันต์ที่ส่งจากปริภูมิลำดับนาคาโน $\ell(X,p)$ ไปยังปริภูมิลำดับ E_r เมื่อ $p_k \le 1$ ทุกๆ $k \in N$

7. ได้ผลงานตีพิมพ์ชื่อ "Matrix Transformations of Orlicz Sequence Spaces" ผลงานดังกล่าวนี้ ได้ส่งไปเพื่อตีพิมพ์ในวารสาร Journal of Mathematical Analysis and Applications ผลงานนี้ได้ให้ลักษณะเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิลับนาคาโน $\ell(X,p)$ ไปยังปริภูมิลำดับออร์ลิคซ์ ℓ_M และส่งจากปริภูมิลำดับออร์ลิคซ์ไปยังปริภูมิลำดับแมดดอกซ์

8. ได้ผลงานตีพิมพ์ชื่อ "Some Geometric Properties of Cesaro Sequence spaces" **ผลงานนี้ได้รับการตอบรับให้ตีพิมพ์ลงในวารสาร Kyungpook Mathematical Journal**

ผลงานนี้ได้พบว่าปริภูมิลำดับเซซาโร $ces(p)$ ภายใต้ Luxemburg norm เป็นปริภูมิ ที่มี Property (H) แต่ไม่ rotund (R)

9. ได้ผลงานตีพิมพ์ชื่อ "On the H-property of Some Banach sequence Spaces" ผลงานนี้ได้รับการตอบรับให้ตีพิมพ์ในวารสาร Archivum Mathematicum

ผลงานดังกล่าวได้แสดงว่าปริภูมิลำดับเซซาโร $ces(p)$ ภายใต้ Luxemburg norm (ที่แตกต่างจาก นอร์มในผลงานวิจัย 6) มีสมบัติ (H) และเป็นปริภูมิ rotund

10. ได้ผลงานตีพิมพ์ชื่อ "On some convexity property of generalized Cesaro sequence spaces" ผลงานดังกล่าวนี้ ได้รับการยอมรับให้ตีพิมพ์ในวารสาร Georgian Mathematical Journal
ผลงานนี้ได้แสดงว่า ปริภูมิลำดับเซซาโร $ces(p)$ ภายใต้ Luxemburg norm เป็นปริภูมิ locally uniformly rotund เมื่อ $p_k > 1$ ทุก ๆ $k \in N$

11. ได้ผลงานตีพิมพ์ชื่อ "On the property (H) and Rotundity of Difference Sequence Spaces" ผลงานดังกล่าวนี้ ได้รับการตอบรับให้เพื่อตีพิมพ์ในวารสาร Journal of Nonlinear Analysis and Convex Analysis ผลงานนี้ได้แสดงว่า ปริภูมิ $\ell(\Delta, p)$ เป็นปริภูมิ rotund และมีสมบัติ (H) ภายใต้ Luxemburg norm เมื่อ $p_k > 1$ ทุก ๆ $k \in N$

12. ได้ผลงานตีพิมพ์ชื่อ "On property (UKK) in Cesaro-Musielak-Orlicz Sequence Spaces" ผลงานดังกล่าวนี้ ได้ส่งไปเพื่อตีพิมพ์ในวารสาร Czechoslovak Mathematical Journal ผลงานนี้ได้แสดงว่า ปริภูมิ Cesaro-Musielak-Orlicz Ces_M มีสมบัติ (UKK) และ สมบัติ (H) ภายใต้ Luxemburg norm เมื่อ M เป็นฟังก์ชัน Musielak-Orlicz

13. ได้ผลงานตีพิมพ์ชื่อ "Some Geometric Properties in Orlicz Sequence Spaces of Bocher Type" ผลงานดังกล่าวนี้ ได้ส่งไปเพื่อตีพิมพ์ในวาร nonlinear Analysis and Applications ผลงานนี้ได้ให้ลักษณะของปริภูมิลำดับออร์ลิคซ์ ชนิดบอชเนอร์ $\ell_M(X)$ ที่เป็นปริภูมิ rotund (R), locally unifromly (LUR) , CLUR, WCLUR, มีสมบัติ (H) , (UKK) ภายใต้ Luxemburg norm เมื่อ M เป็นฟังก์ชันออร์ลิคซ์

14. ได้ผลงานตีพิมพ์ชื่อ Local Uniform Convexity of Cesaro Musielak-Orlicz Sequence Spaces ผลงานนี้ได้ส่งไปเพื่อลงตีพิมพ์ในวารสาร Bulletin of Australian

Mathematics. ผลงานนี้ได้แสดงว่า ปริภูมิ Ces_M ภายใต้นอร์มลักเซมเบิร์กเป็นปริภูมิ LUR เมื่อ M สอดคล้องเงื่อนไข บางประการ

2. ผลงานวิจัยที่นำเสนอในที่ประชุมนานาชาติ

1. เรื่อง ON MATRIX TRANSFORMATIONS OF VECTOR-VALUED SEQUENCESPACES OF MADDOX

นำเสนอในที่ประชุมนานาชาติ International Conference in Mathematics ที่ Calcutta Mathematical Society ประเทศอินเดีย ระหว่างวันที่ 23 - 26 มกราคม 2543

2. เรื่อง Some Geometric Properties in Cesaro Sequence Space with the Luxemburg norm เสนอในที่ประชุมนานาชาติ The Third Asian Mathematical Conference (AMC 2000) ซึ่งจัดที่ University of the Philippines, Diliman, Quezon City, Philippines ระหว่างวันที่ 23 –27 ตุลาคม 2543

3. เรื่อง On the H-Property of Some Banach Sequence Spaces

ซึ่งเสนอในที่ประชุมนานาชาติ Function Spaces VI ซึ่งจัดที่ Institute of Mathematics Wroclaw University of Technology, Wroclaw, Poland

ระหว่างวันที่ 3 – 8 กันยายน 2544

3. การผลิตนักวิจัยรุ่นใหม่

การผลิตบัณฑิตระดับปริญญาโท และ เอก

ได้ผลิตบัณฑิตระดับปริญญาโท สาขาคณิตศาสตร์ จำนวน 5 คน ดังนี้

1. นาย วิเนตร แสนหาญ ปีที่จบ เมษายน 2543

   หัวข้อวิทยานิพนธ์ Some Geometric Properties of Banach Sequence Spaces

   ปัจจุบันกำลังศึกษาปริญญาเอกอยู่ที่ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่

2. นางสาว อรวรรณ ตรีพักตร์ ปีที่จบ เมษายน 2543

หัวข้อวิทยานิพนธ์ Matrix Transformations Between Vector-Valued Sequence Spaces

ปัจจุบันทำงานที่ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยสงขลานครินทร์

3. นาย ภักดี เจริญสวรรค์ ปีที่จบ มีนาคม 2544

หัวข้อวิทยานิพนธ์ Some Geometric Properties of Banach Sequence Spaces

ปัจจุบันทำงานที่ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัย เชียงใหม่

4 นาย นรินทร์ เพชร์โรจน์ ปีที่จบ มีนาคม 2544

หัวข้อวิทยานิพนธ์ Rotundity in Orlicz Vector-valued Sequence Spaces

ปัจจุบันทำงานที่ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยนเรศวร

5. นายระเบียน วังคีรี

หัวข้อวิทยานิพนธ์ Local Uniform Rotundity in Some Banach Sequence Spaces

ปัจจุบันทำงานที่ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยนเรศวร

6. ได้ผลิตบัณฑิตระดับปริญญาเอก สาขาคณิตศาสตร์ จำนวน 1 คน ดังนี้

นาย ชานันท์ สุดสุข ปีที่จบการศึกษา พฤษภาคม 2543

หัวข้อวิทยานิพนธ์ Matrix Transformations of Some Vector-Valued Sequence Spaces

ปัจจุบันทำงานที่ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเกษตรศาสตร์ วิทยาเขตกำแพงแสน

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ON MATRIX TRANSFORMATIONS CONCERNING THE NAKANO VECTOR-VALUED SEQUENCE SPACE

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ABSTRACT. We give the matrix characterizations from Nakano vector-valued sequence space $\ell(X,p)$ and $F_r(X,p)$ into the sequence spaces E_r, ℓ_∞, $\underline{\ell}_\infty(q)$, bs, and cs, where $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers such that $p_k>1$ for all $k\in\mathbb{N}$ and $r\geq 0$.

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1. Introduction. Let $(X,\|\cdot\|)$ be a Banach space, $r\geq 0$ and $p=(p_k)$ a bounded sequence of positive real numbers. We write $x=(x_k)$ with x_k in X for all $k\in\mathbb{N}$. The X-valued sequence spaces $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, $E_r(X,p)$, $F_r(X,p)$, and $\underline{\ell}_\infty(X,p)$ are defined as

$$\begin{aligned}
c_0(X,p)&=\Big\{x=(x_k):\lim_{k\to\infty}\|x_k\|^{p_k}=0\Big\},\\
c(X,p)&=\Big\{x=(x_k):\lim_{k\to\infty}\|x_k-a\|^{p_k}=0,\ \text{for some } a\in X\Big\},\\
\ell_\infty(X,p)&=\Big\{x=(x_k):\sup_k\|x_k\|^{p_k}<\infty\Big\},\\
\ell(X,p)&=\Big\{x=(x_k):\sum_{k=1}^{\infty}\|x_k\|^{p_k}<\infty\Big\},\\
E_r(X,p)&=\Big\{x=(x_k):\sup_k\frac{\|x_k\|^{p_k}}{k^r}<\infty\Big\},\\
F_r(X,p)&=\Big\{x=(x_k):\sum_{k=1}^{\infty}k^r\|x_k\|^{p_k}<\infty\Big\},\\
\underline{\ell}_\infty(X,p)&=\bigcap_{n=1}^{\infty}\Big\{x=(x_k):\sup_k\|x_k\|n^{1/p_k}\Big\}.
\end{aligned}\tag{1.1}$$

When $X=K$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, $\ell(p)$, $E_r(p)$, $F_r(p)$, and $\underline{\ell}_\infty(p)$, respectively. The spaces $c_0(p)$, $c(p)$, and $\ell_\infty(p)$ are known as the sequence spaces of Maddox. These spaces were first introduced and studied by Simons [7] and Maddox [4, 5]. The space $\ell(p)$ was first defined by Nakano [6] and it is known as the Nakano sequence space and the space $\ell(X,p)$ is known as the Nakano vector-valued sequence space. When $p_k=1$ for all $k\in\mathbb{N}$, the spaces $E_r(p)$ and $F_r(p)$ are written as E_r and F_r, respectively. These two sequence spaces were first introduced by Cooke [1]. The space $\underline{\ell}_\infty(p)$ was first defined by Grosse-Erdmann [2] and he has given in [3] characterizations of infinite matrices mapping between scalar-valued sequence spaces of Maddox. Wu and Liu [10] gave necessary and sufficient conditions for infinite matrices mapping from $c_0(X,p)$ and $\ell_\infty(X,p)$ into $c_0(q)$ and $\ell_\infty(q)$. Suantai [8] has given characterizations of infinite matrices mapping $\ell(X,p)$ into ℓ_∞ and $\underline{\ell}_\infty(q)$ when $p_k \le 1$ for all $k \in \mathbb{N}$ and he has also given in [9] characterizations of those infinite matrices mapping from $\ell(X,p)$ into the sequence space E_r when $p_k \le 1$ for all $k \in \mathbb{N}$.

In this paper, we extend the results of [8, 9] in case $p_k > 1$ for all $k \in \mathbb{N}$. Moreover, we also give the matrix characterizations from $\ell(X,p)$ and $F_r(X,p)$ into the sequence spaces bs and cs.

2. Notations and definitions. Let $(X, \|\cdot\|)$ be a Banach space, the space of all sequences in X is denoted by $W(X)$, and $\Phi(X)$ denotes the space of all finite sequences in X. When $X = K$, the scalar field of X, the corresponding spaces are written as w and Φ.

A sequence space in X is a linear subspace of $W(X)$. Let E be an X-valued sequence space. For $x \in E$ and $k \in \mathbb{N}$, x_k stands for the kth term of x. For $k \in \mathbb{N}$, we denote by e_k the sequence $(0,0,\dots,0,1,0,\dots)$ with 1 in the kth position and by e the sequence $(1,1,1,\dots)$. For $x \in X$ and $k \in \mathbb{N}$, let $e^k(x)$ be the sequence $(0,0,\dots,0,x,0,\dots)$ with x in the kth position and let $e(x)$ be the sequence $(x,x,x,\dots)$. We call a sequence space E normal if $(t_k x_k) \in E$ for all $x = (x_k) \in E$ and $t_k \in K$ with $|t_k| = 1$ for all $t_k \in \mathbb{N}$. A normed sequence space $(E, \|\cdot\|)$ is said to be *norm monotone* if $x = (x_k)$, $y = (y_k) \in E$ with $\|x_k\| \le \|y_k\|$ for all $k \in \mathbb{N}$ we have $\|x\| \le \|y\|$. For a fixed scalar sequence $\mu = (\mu_k)$, the sequence space E_μ is defined as

$$E_\mu = \{x \in W(X) : (\mu_k x_k) \in E\}. \tag{2.1}$$

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map* E into F, written by $A : E \to F$, if for each $x = (x_k) \in E$, $A_n(x) = \sum_{k=1}^\infty f_k^n(x_k)$ converges for each $n \in \mathbb{N}$, and the sequence $Ax = (A_n(x)) \in F$. Let (E,F) denote the set of all infinite matrices mapping from E into F.

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a K-space if for each $k \in \mathbb{N}$, the kth coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. If, in addition, (E,τ) is a Fréchet (Banach) space, then E is called an FK- (BK-) space. Now, suppose that E contains $\Phi(X)$. Then E is said to have *property* AB if the set $\{\sum_{k=1}^n e^k(x_k) : n \in \mathbb{N}\}$ is bounded in E for every $x = (x_k) \in E$. It is said to have *property* AK if $\sum_{k=1}^n e^k(x_k) \to x$ in E as $n \to \infty$ for every $x = (x_k) \in E$. It has *property* AD if $\Phi(X)$ is dense in E.

It is known that the Nakano sequence space $\ell(X,p)$ is an FK-space with property AK under the paranorm $g(x) = (\sum_{k=1}^\infty \|x_k\|^{p_k})^{1/M}$, where $M = \max\{1, \sup_k p_k\}$. If $p_k > 1$ for all $k \in \mathbb{N}$, then $\ell(X,p)$ is a BK-space with the Luxemburg norm defined by

$$\|(x_k)\| = \inf\left\{\varepsilon > 0 : \sum_{k=1}^\infty \left\|\frac{x_k}{\varepsilon}\right\|^{p_k} \le 1\right\}. \tag{2.2}$$

3. Main results. We first give a characterization of an infinite matrix mapping from $\ell(X,p)$ into E_r when $p_k > 1$ for all $k \in \mathbb{N}$. To do this, we need the following lemma.

Lemma 3.1. *Let E be an X-valued BK-space which is normal and norm monotone and let $A = (f_k^n)$ be an infinite matrix. Then $A : E \to E_r$ if and only if $\sup_n \sum_{k=1}^{\infty} |f_k^n(x_k)|/n^r < \infty$ for every $x = (x_k) \in E$.*

Proof. If the condition holds true, it follows that

$$\sup_n \frac{|\sum_{k=1}^{\infty} f_k^n(x_k)|}{n^r} \le \sup_n \sum_{k=1}^{\infty} \frac{|f_k^n(x_k)|}{n^r} < \infty \tag{3.1}$$

for every $x = (x_k) \in E$, hence $A : E \to E_r$.

Conversely, assume that $A : E \to E_r$. Since E and E_r are BK-spaces, by Zeller's theorem, $A : E \to E_r$ is bounded, so there exists $M > 0$ such that

$$\sup_{\substack{n \in \mathbb{N} \\ \|(x_k)\| \le 1}} \frac{|\sum_{k=1}^{\infty} f_k^n(x_k)|}{n^r} \le M. \tag{3.2}$$

Let $x = (x_k) \in E$ be such that $\|x\| = 1$. For each $n \in \mathbb{N}$, we can choose a scalar sequence (t_k) with $|t_k| = 1$ and $f_k^n(t_k x_k) = |f_k^n(x_k)|$ for all $k \in \mathbb{N}$. Since E is normal and norm monotone, we have $(t_k x_k) \in E$ and $\|(t_k x_k)\| \le 1$. It follows from (3.2) that

$$\sum_{k=1}^{\infty} \frac{|f_k^n(x_k)|}{n^r} = \frac{|\sum_{k=1}^{\infty} f_k^n(t_k x_k)|}{n^r} \le M, \tag{3.3}$$

which implies

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} \frac{|f_k^n(x_k)|}{n^r} \le M. \tag{3.4}$$

It follows from (3.4) that for every $x = (x_k) \in E$,

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} \frac{|f_k^n(x_k)|}{n^r} \le M\|x\|. \tag{3.5}$$

This completes the proof. □

Theorem 3.2. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$ and $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$, and let $r \ge 0$. For an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), E_r)$ if and only if there is $m_0 \in \mathbb{N}$ such that*

$$\sup_n \sum_{k=1}^{\infty} \|f_k^n\|^{q_k} n^{-rq_k} m_0^{-q_k} < \infty. \tag{3.6}$$

Proof. Let $x = (x_k) \in \ell(X,p)$. By (3.6), there are $m_0 \in \mathbb{N}$ and $K > 1$ such that

$$\sum_{k=1}^{\infty} \|f_k^n\|^{q_k} n^{-rq_k} m_0^{-q_k} < K, \quad \forall n \in \mathbb{N}. \tag{3.7}$$

Note that for $a, b \ge 0$, we have

$$ab \le a^{p_k} + b^{q_k}. \tag{3.8}$$

It follows by (3.7) and (3.8) that for $n \in \mathbb{N}$,

$$\begin{aligned} n^{-r}\left|\sum_{k=1}^{\infty} f_k^n(x_k)\right| &= n^{-r}\left|\sum_{k=1}^{\infty} f_k^n(m_0^{-1}\cdot m_0 x_k)\right| \\ &\le \sum_{k=1}^{\infty}(n^{-r}m_0^{-1}\|f_k^n\|)(\|m_0 x_k\|) \\ &\le \sum_{k=1}^{\infty} n^{-rq_k} m_0^{-q_k}\|f_k^n\|^{q_k} + m_0^{\alpha}\sum_{k=1}^{\infty}\|x_k\|^{p_k} \\ &\le K + m_0^{\alpha}\sum_{k=1}^{\infty}\|x_k\|^{p_k}, \quad \text{where } \alpha = \sup_k p_k. \end{aligned} \tag{3.9}$$

Hence $\sup n^{-r}|\sum_{k=1}^{\infty} f_k^n(x_k)| < \infty$, so that $Ax \in E_r$.

For necessity, assume that $A \in (\ell(X,p), E_r)$. For each $k \in \mathbb{N}$, we have $\sup_n n^{-r}|f_k^n(x)| < \infty$ for all $x \in X$ since $e^{(k)}(x) \in \ell(X,p)$. It follows by the uniform bounded principle that for each $k \in \mathbb{N}$ there is $C_k > 1$ such that

$$\sup_n n^{-r}\|f_k^n\| \le C_k. \tag{3.10}$$

Suppose that (3.6) is not true. Then

$$\sup_n \sum_{k=1}^{\infty}\|f_k^n\|^{q_k} n^{-rq_k} m^{-q_k} = \infty, \quad \forall m \in \mathbb{N}. \tag{3.11}$$

For $n \in \mathbb{N}$, we have by (3.10) that for $k, m \in \mathbb{N}$,

$$\begin{aligned} \sum_{j=1}^{\infty}\left\|f_j^n\right\|^{q_j} n^{-rq_j} m^{-q_j} &= \sum_{j=1}^{k}\left\|f_j^n\right\|^{q_j} n^{-rq_j} m^{-q_j} + \sum_{j>k}\left\|f_j^n\right\|^{q_j} n^{-rq_j} m^{-q_j} \\ &\le \sum_{j=1}^{k} C_j^{q_j} m^{-q_j} + \sum_{j>k}\left\|f_j^n\right\|^{q_j} n^{-rq_j} m^{-q_j}. \end{aligned} \tag{3.12}$$

This together with (3.11) give

$$\sup_n \sum_{j>k}\left\|f_j^n\right\|^{q_j} n^{-rq_j} m^{-q_j} = \infty, \quad \forall k, m \in \mathbb{N}. \tag{3.13}$$

By (3.13) we can choose $0 = k_0 < k_1 < k_2 < \cdots$, $m_1 < m_2 < \cdots$, $m_i > 4^i$ and a subsequence (n_i) of positive integers such that for all $i \ge 1$,

$$\sum_{k_{i-1} < j \le k_i}\left\|f_j^{n_i}\right\|^{q_j} n_i^{-rq_j} m_i^{-q_j} > 2^i. \tag{3.14}$$

For each $i \in \mathbb{N}$, we can choose $x_j \in X$ with $\|x_j\| = 1$, for $k_{i-1} < j \le k_i$ such that

$$\sum_{k_{i-1} < j \le k_i}\left|f_j^{n_i}(x_j)\right|^{q_j} n_i^{-rq_j} m_i^{-q_j} > 2^i. \tag{3.15}$$

For each $i \in \mathbb{N}$, let $F_i : (0, \infty) \to (0, \infty)$ be defined by

$$F_i(M) = \sum_{k_{i-1} < j \le k_i} \left| f_j^{n_i}(x_j) \right|^{q_j} n_i^{-rq_j} M^{-q_j}. \tag{3.16}$$

Then F_i is continuous and non-increasing such that $F(M) \to 0$ as $M \to \infty$. Thus there exists $M_i > 0$ such that $M_i > m_i$ and

$$F(M_i) = \sum_{k_{i-1} < j \le k_i} \left| f_j^{n_i}(x_j) \right|^{q_j} n_i^{-rq_j} M_i^{-q_j} = 2^i. \tag{3.17}$$

Put

$$y = (y_j), \quad y_j = 4^{-i} M_i^{-(q_j-1)} n_i^{-rq_j/p_j} \left| f_j^{n_i}(x_j) \right|^{q_j-1} x_j \text{ for } k_{i-1} < j \le k_i. \tag{3.18}$$

Thus

$$\begin{aligned}
\sum_{j=1}^{\infty} \|y_j\|^{p_j} &= \sum_{i=1}^{\infty} \sum_{k_{i-1} < j \le k_i} 4^{-ip_j} M_i^{-p_j(q_j-1)} n_i^{-rq_j} \left| f_j^{n_i}(x_j) \right|^{p_j(q_j-1)} \\
&\le \sum_{i=1}^{\infty} 4^{-i} \sum_{k_{i-1} < j \le k_i} M_i^{-q_j} n_i^{-rq_j} \left| f_j^{n_i}(x_j) \right|^{q_j} \\
&= \sum_{i=1}^{\infty} 4^{-i} \cdot 2^i \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} = 1.
\end{aligned} \tag{3.19}$$

Thus $y = (y_j) \in \ell(X, p)$. Since $\ell(X, p)$ is a BK-space which is normal and norm monotone under the Luxemburg norm, by Lemma 3.1, we obtain that

$$\sup_n \sum_{k=1}^{\infty} \frac{|f_k^n(y_k)|}{n^r} < \infty. \tag{3.20}$$

But we have

$$\begin{aligned}
\sup_n \sum_{j=1}^{\infty} \frac{\left| f_j^n(y_j) \right|}{n^r} &\ge \sup_i \sum_{j=1}^{\infty} \frac{\left| f_j^{n_i}(y_j) \right|}{n_i^r} \ge \sup_i \sum_{k_{i-1} < j \le k_i} \frac{\left| f_j^{n_i}(y_j) \right|}{n_i^r} \\
&= \sup_i \sum_{k_{i-1} < j \le k_i} 4^{-i} M_i^{-(q_j-1)} n_i^{-r(q_j/p_j+1)} \left| f_j^{n_i}(x_j) \right|^{q_j} \\
&= \sup_i \sum_{k_{i-1} < j \le k_i} 4^{-i} M_i^{-(q_j-1)} n_i^{-rq_j} \left| f_j^{n_i}(x_j) \right|^{q_j} \\
&= \sup_i \sum_{k_{i-1} < j \le k_i} \left(\left| f_j^{n_i}(x_j) \right|^{q_j} n_i^{-rq_j} M_i^{-q_j} \right) 4^{-i} M_i \\
&\ge \sup_i 2^i = \infty, \quad \text{because } M_i > 4^i.
\end{aligned} \tag{3.21}$$

This is contradictory with (3.20). Therefore (3.6) is satisfied. □

THEOREM 3.3. *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k > 1$ for all $k \in \mathbb{N}$, $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$, $r \geq 0$ and $s \geq 0$. Then for an infinite matrix $A = (f_k^n)$, $A \in (F_r(X,p), E_s)$ if and only if there is $m_0 \in \mathbb{N}$ such that*

$$\sup_n \sum_{k=1}^{\infty} \left(k^{-rq_k/p_k} \|f_k^n\|^{q_k} n^{-sq_k} m_0^{-q_k} \right) < \infty. \tag{3.22}$$

PROOF. Since $F_r(X,p) = \ell(X,p)_{(k^{r/p_k})}$, it is easy to see that

$$A \in (F_r(X,p), E_s) \Longleftrightarrow (k^{-r/p_k} f_k^n)_{n,k} \in (\ell(X,p)E_s). \tag{3.23}$$

By Theorem 3.2, we have $(k^{-r/p_k} f_k^n)_{n,k} \in (\ell(X,p)E_s)$ if and only if there is $m_0 \in \mathbb{N}$ such that

$$\sup_n \sum_{k=1}^{\infty} \left(k^{-rq_k/p_k} \|f_k^n\|^{q_k} n^{-sq_k} m_0^{-q_k} \right) < \infty. \tag{3.24}$$

Thus the theorem is proved. □

Since $E_0 = \ell_\infty$, the following two results are obtained directly from Theorems 3.2 and 3.3, respectively.

COROLLARY 3.4. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$ and let $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), \ell_\infty)$ if and only if there is $m_0 \in \mathbb{N}$ such that*

$$\sup_n \sum_{k=1}^{\infty} \|f_k^n\|^{q_k} m_0^{-q_k} < \infty. \tag{3.25}$$

COROLLARY 3.5. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$ and let $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$. Then for an infinite matrix $A = (f_k^n)$, $A \in (F_r(X,p), \ell_\infty)$ if and only if there is $m_0 \in \mathbb{N}$ such that*

$$\sup_n \sum_{k=1}^{\infty} \left(k^{-rq_k/p_k} \|f_k^n\|^{q_k} m_0^{-q_k} \right) < \infty. \tag{3.26}$$

THEOREM 3.6. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$ and let $1/p_k + 1/t_k = 1$ for all $k \in \mathbb{N}$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), \underline{\ell}_\infty(q))$ if and only if for each $r \in \mathbb{N}$, there is $m_r \in \mathbb{N}$ such that*

$$\sup_{n,k} \sum_{k=1}^{\infty} r^{t_k/q_n} \|f_k^n\|^{t_k} m_r^{-t_k} < \infty. \tag{3.27}$$

PROOF. Since $\underline{\ell}_\infty(q) = \cap_{r=1}^{\infty} \ell_{\infty(r^{1/q_k})}$, it follows that

$$A \in (\ell(X,p), \underline{\ell}_\infty(q)) \Longleftrightarrow A \in \left(\ell(X,p), \ell_{\infty(r^{1/q_k})}\right), \quad \forall r \in \mathbb{N}. \tag{3.28}$$

It is easy to show that for $r \in \mathbb{N}$,

$$A \in \left(\ell(X,p), \ell_{\infty(r^{1/q_k})}\right) \Longleftrightarrow (r^{1/q_n} f_k^n)_{n,k} \in (\ell(X,p), \ell_\infty). \tag{3.29}$$

We obtain by Corollary 3.4 that for $r \in \mathbb{N}$, $(r^{1/q_n} f_k^n)_{n,k} \in (\ell(X,p), \ell_\infty)$ if and only if there is $m_r \in \mathbb{N}$ such that

$$\sup_n \sum_{k=1}^{\infty} r^{t_k/q_n} \|f_k^n\|^{t_k} m_r^{-t_k} < \infty. \tag{3.30}$$

Thus the theorem is proved. □

Theorem 3.7. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$ and let $1/p_k + 1/t_k = 1$ for all $k \in \mathbb{N}$. For an infinite matrix $A = (f_k^n)$, $A \in (F_r(X,p), \underline{\ell}_\infty(q))$ if and only if for each $i \in \mathbb{N}$, there is $m_i \in \mathbb{N}$ such that*

$$\sup_n \sum_{k=1}^{\infty} i^{t_k/q_n} k^{-rt_k/p_k} \|f_k^n\|^{t_k} m_i^{-t_k} < \infty. \tag{3.31}$$

Proof. Since $F_r(X,p) = \ell(X,p)_{(k^{r/p_k})}$, it implies that

$$A \in (F_r(X,p), \underline{\ell}_\infty(q)) \Longleftrightarrow (k^{-r/p_k} f_k^n)_{n,k} \in (\ell(X,p), \underline{\ell}_\infty(q)). \tag{3.32}$$

It follows from Theorem 3.6 that $A \in (F_r(X,p), \underline{\ell}_\infty(q))$ if and only if for each $i \in \mathbb{N}$, there is $m_i \in \mathbb{N}$ such that

$$\sup_n \sum_{k=1}^{\infty} i^{t_k/q_n} k^{-rt_k/p_k} \|f_k^n\|^{t_k} m_i^{-t_k} < \infty. \tag{3.33}$$

□

Theorem 3.8. *Let $p = (p_k)$ be bounded sequence of positive real numbers with $p_k > 1$ for all $n \in \mathbb{N}$ and let $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), bs)$ if and only if there is $m_0 \in \mathbb{N}$ such that*

$$\sup_n \sum_{k=1}^{\infty} \left\| \sum_{i=1}^{n} f_k^i \right\|^{q_k} m_0^{-q_k} < \infty. \tag{3.34}$$

Proof. For an infinite matrix $A = (f_k^n)$, we can easily show that

$$A \in (\ell(X,p), bs) \Longleftrightarrow \left(\sum_{i=1}^{n} f_k^i \right)_{n,k} \in (\ell(X,p), \ell_\infty). \tag{3.35}$$

This implies by Corollary 3.4 that $A \in (\ell(X,p), bs)$ if and only if there is $m_0 \in \mathbb{N}$ such that

$$\sup_n \sum_{k=1}^{\infty} \left\| \sum_{i=1}^{n} f_k^i \right\|^{q_k} m_0^{-q_k} < \infty. \tag{3.36}$$

□

Theorem 3.9. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$ and let $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), cs)$ if and only if*

(1) *there is $m_0 \in \mathbb{N}$ such that $\sup_n \sum_{k=1}^{\infty} \|\sum_{i=1}^{n} f_k^i\|^{q_k} m_0^{-q_k} < \infty$ and*
(2) *for each $k \in \mathbb{N}$ and $x \in X$, $\sum_{n=1}^{\infty} f_k^n(x)$ converges.*

PROOF. The necessity is obtained by Theorem 3.8 and by the fact that $e^{(k)}(x) \in \ell(X,p)$ for every $k \in \mathbb{N}$ and $x \in X$.

Now, suppose that (1) and (2) hold. By Theorem 3.8, we have $A : \ell(X,p) \to bs$. Let $x = (x_k) \in \ell(X,p)$. Since $\ell(X,p)$ has the AK property, we have $x = \lim_{n\to\infty} \sum_{k=1}^{n} e^{(k)}(x_k)$. By Zeller's theorem, $A : \ell(X,p) \to bs$ is continuous. It implies that

$$Ax = \lim_{n\to\infty} \sum_{k=1}^{n} Ae^{(k)}(x_k). \tag{3.37}$$

By (2), $Ae^{(k)}(x_k) \in cs$ for all $k \in \mathbb{N}$. Since cs is a closed subspace of bs, it implies that $Ax \in cs$, that is, $A : \ell(X,p) \to cs$. □

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ON β-DUAL OF VECTOR-VALUED SEQUENCE SPACES OF MADDOX

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The β-dual of a vector-valued sequence space is defined and studied. We show that if an X-valued sequence space E is a BK-space having AK property, then the dual space of E and its β-dual are isometrically isomorphic. We also give characterizations of β-dual of vector-valued sequence spaces of Maddox $\ell(X,p)$, $\ell_\infty(X,p)$, $c_0(X,p)$, and $c(X,p)$.

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1. Introduction. Let $(X,\|\cdot\|)$ be a Banach space and $p=(p_k)$ a bounded sequence of positive real numbers. Let $\mathbb{N}$ be the set of all natural numbers, we write $x=(x_k)$ with x_k in X for all $k\in\mathbb{N}$. The X-valued sequence spaces of Maddox are defined as

$$\begin{aligned}
c_0(X,p)&=\left\{x=(x_k):\lim_{k\to\infty}\|x_k\|^{p_k}=0\right\};\\
c(X,p)&=\left\{x=(x_k):\lim_{k\to\infty}\|x_k-a\|^{p_k}=0\text{ for some }a\in X\right\};\\
\ell_\infty(X,p)&=\left\{x=(x_k):\sup_k\|x_k\|^{p_k}<\infty\right\};\\
\ell(X,p)&=\left\{x=(x_k):\sum_{k=1}^{\infty}\|x_k\|^{p_k}<\infty\right\}.
\end{aligned}\tag{1.1}$$

When $X=\mathbb{K}$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, and $\ell(p)$, respectively. All of these spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons [7] and Maddox [3, 4, 5]. The space $\ell(p)$ was first defined by Nakano [6] and is known as the Nakano sequence space. Grosse-Erdmann [1] has investigated the structure of the spaces $c_0(p)$, $c(p)$, $\ell(p)$, and $\ell_\infty(p)$ and has given characterizations of β-dual of scalar-valued sequence spaces of Maddox.

In [8], Wu and Bu gave characterizations of Köthe dual of the vector-valued sequence space $\ell_p[X]$, where $\ell_p[X]$, $1<p<\infty$, is defined by

$$\ell_p[X]=\left\{x=(x_k):\sum_{k=1}^{\infty}|f(x_k)|^p<\infty\text{ for each }f\in X'\right\}.\tag{1.2}$$

In this paper, the β-dual of a vector-valued sequence space is defined and studied and we give characterizations of β-dual of vector-valued sequence spaces of Maddox

$\ell(X,p)$, $\ell_\infty(X,p)$, $c_0(X,p)$, and $c(X,p)$. Some results, obtained in this paper, are generalizations of some in [1, 3].

2. Notation and definitions. Let $(X,\|\cdot\|)$ be a Banach space. Let $W(X)$ and $\Phi(X)$ denote the space of all sequences in X and the space of all finite sequences in X, respectively. A sequence space in X is a linear subspace of $W(X)$. Let E be an X-valued sequence space. For $x \in E$ and $k \in \mathbb{N}$ we write that x_k stand for the kth term of x. For $x \in X$ and $k \in \mathbb{N}$, we let $e^{(k)}(x)$ be the sequence $(0,0,0,\dots,0,x,0,\dots)$ with x in the kth position and let $e(x)$ be the sequence $(x,x,x,\dots)$. For a fixed scalar sequence $u = (u_k)$, the sequence space E_u is defined as

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}. \tag{2.1}$$

An X-valued sequence space E is said to be *normal* if $(y_k) \in E$ whenever $\|y_k\| \le \|x_k\|$ for all $k \in \mathbb{N}$ and $(x_k) \in E$. Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if, for each $k \in \mathbb{N}$, the kth coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. In addition, if (E,τ) is a *Fréchet (Banach) space*, then E is called an FK-(BK)-space. Now, suppose that E contains $\Phi(X)$, then E is said to have *property AK* if $\sum_{k=1}^{n} e^{(k)}(x_k) \to x$ in E as $n \to \infty$ for every $x = (x_k) \in E$.

The spaces $c_0(p)$ and $c(p)$ are FK-spaces. In $c_0(X,p)$, we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = \max\{1, \sup_k p_k\}$, as a paranorm on $c_0(X,p)$, and it is known that $c_0(X,p)$ is an FK-space having property AK under the paranorm g defined as above. In $\ell(X,p)$, we consider it as a paranormed sequence space with the paranorm given by $\|(x_k)\| = (\sum_{k=1}^{\infty} \|x_k\|^{p_k})^{1/M}$. It is known that $\ell(X,p)$ is an FK-space under the paranorm defined as above.

For an X-valued sequence space E, define its Köthe dual with respect to the dual pair (X,X') (see [2]) as follows:

$$E^{\times}|_{(X,X')} = \left\{(f_k) \subset X' : \sum_{k=1}^{\infty} |f_k(x_k)| < \infty \ \forall x = (x_k) \in E\right\}. \tag{2.2}$$

In this paper, we denote $E^{\times}|_{(X,X')}$ by E^{α} and it is called the α-dual of E.

For a sequence space E, the β-dual of E is defined by

$$E^{\beta} = \left\{(f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x_k) \text{ converges } \forall \ (x_k) \in E\right\}. \tag{2.3}$$

It is easy to see that $E^{\alpha} \subseteq E^{\beta}$.

For the sake of completeness we introduce some further sequence spaces that will be considered as β-dual of the vector-valued sequence spaces of Maddox:

$$M_0(X,p) = \left\{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\| M^{-1/p_k} < \infty \text{ for some } M \in \mathbb{N}\right\};$$

$$M_\infty(X,p) = \left\{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\| n^{1/p_k} < \infty \ \forall n \in \mathbb{N}\right\};$$

$$\ell_0(X,p)=\left\{x=(x_k):\sum_{k=1}^{\infty}\|x_k\|^{p_k}M^{-p_k}<\infty \text{ for some } M\in\mathbb{N}\right\},\quad p_k>1\ \forall k\in N;$$

$$cs[X']=\left\{(f_k)\subset X':\sum_{k=1}^{\infty}f_k(x) \text{ converges } \forall\, x\in X\right\}. \tag{2.4}$$

When $X=\mathbb{K}$, the scalar field of X, the corresponding first two sequence spaces are written as $M_0(p)$ and $M_\infty(p)$, respectively. These two spaces were first introduced by Grosse-Erdmann [1].

3. Main results. We begin by giving some general properties of β-dual of vector-valued sequence spaces.

PROPOSITION 3.1. *Let X be a Banach space and let E, E_1, and E_2 be X-valued sequence spaces. Then*

(i) $E^\alpha\subseteq E^\beta$.

(ii) *If $E_1\subseteq E_2$, then $E_2^\beta\subseteq E_1^\beta$.*

(iii) *If $E=E_1+E_2$, then $E^\beta=E_1^\beta\cap E_2^\beta$.*

(iv) *If E is normal, then $E^\alpha=E^\beta$.*

PROOF. Assertions (i), (ii), and (iii) are immediately obtained by the definitions. To prove (iv), by (i), it suffices to show only that $E^\beta\subseteq E^\alpha$. Let $(f_k)\in E^\beta$ and $x=(x_k)\in E$. Then $\sum_{k=1}^{\infty}f_k(x_k)$ converges. Choose a scalar sequence (t_k) with $|t_k|=1$ and $f_k(t_kx_k)=|f_k(x_k)|$ for all $k\in\mathbb{N}$. Since E is normal, $(t_kx_k)\in E$. It follows that $\sum_{k=1}^{\infty}|f_k(x_k)|$ converges, hence $(f_k)\in E^\alpha$. □

If E is a BK-space, we define a norm on E^β by the formula

$$\|(f_k)\|_{E^\beta}=\sup_{\|(x_k)\|\le 1}\left|\sum_{k=1}^{\infty}f_k(x_k)\right|. \tag{3.1}$$

It is easy to show that $\|\cdot\|_{E^\beta}$ is a norm on E^β.

Next, we give a relationship between β-dual of a sequence space and its continuous dual. Indeed, we need a lemma.

LEMMA 3.2. *Let E be an X-valued sequence space which is an FK-space containing $\Phi(X)$. Then for each $k\in\mathbb{N}$, the mapping $T_k:X\to E$, defined by $T_kx=e^k(x)$, is continuous.*

PROOF. Let $V=\{e^k(x):x\in X\}$. Then V is a closed subspace of E, so it is an FK-space because E is an FK-space. Since E is a K-space, the coordinate mapping $p_k:V\to X$ is continuous and bijective. It follows from the open mapping theorem that p_k is open, which implies that $p_k^{-1}:X\to V$ is continuous. But since $T_k=p_k^{-1}$, we thus obtain that T_k is continuous. □

THEOREM 3.3. *If E is a BK-space having property AK, then E^β and E' are isometrically isomorphic.*

PROOF. We first show that for $x = (x_k) \in E$ and $f \in E'$,

$$f(x) = \sum_{k=1}^{\infty} f(e^k(x_k)). \tag{3.2}$$

To show this, let $x = (x_k) \in E$ and $f \in E'$. Since E has property AK,

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} e^{(k)}(x_k). \tag{3.3}$$

By the continuity of f, it follows that

$$f(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f(e^{(k)}(x_k)) = \sum_{k=1}^{\infty} f(e^{(k)}(x_k)), \tag{3.4}$$

so (3.2) is obtained. For each $k \in \mathbb{N}$, let $T_k : X \to E$ be defined as in Lemma 3.2. Since E is a BK-space, by Lemma 3.2, T_k is continuous. Hence $f \circ T_k \in X'$ for all $k \in \mathbb{N}$. It follows from (3.2) that

$$f(x) = \sum_{k=1}^{\infty} (f \circ T_k)(x_k) \quad \forall x = (x_k) \in E. \tag{3.5}$$

It implies, by (3.5), that $(f \circ T_k)_{k=1}^{\infty} \in E^{\beta}$. Define $\varphi : E' \to E^{\beta}$ by

$$\varphi(f) = (f \circ T_k)_{k=1}^{\infty} \quad \forall f \in E'. \tag{3.6}$$

It is easy to see that φ is linear. Now, we show that φ is onto. Let $(f_k) \in E^{\beta}$. Define $f : E \to K$, where K is the scalar field of X, by

$$f(x) = \sum_{k=1}^{\infty} f_k(x_k) \quad \forall x = (x_k) \in E. \tag{3.7}$$

For each $k \in \mathbb{N}$, let p_k be the kth coordinate mapping on E. Then we have

$$f(x) = \sum_{k=1}^{\infty} (f_k \circ p_k)(x) = \lim_{n\to\infty} \sum_{k=1}^{n} (f \circ p_k)(x). \tag{3.8}$$

Since f_k and p_k are continuous linear, so is also continuous $f \circ p_k$. It follows by Banach-Steinhaus theorem that $f \in E'$ and we have by (3.7) that; for each $k \in \mathbb{N}$ and each $z \in X$, $(f \circ T_k)(z) = f(e^{(k)}(z)) = f_k(z)$. Thus $f \circ T_k = f_k$ for all $k \in \mathbb{N}$, which implies that $\varphi(f) = (f_k)$, hence φ is onto.

Finally, we show that φ is linear isometry. For $f \in E'$, we have

$$\begin{aligned} \|f\| &= \sup_{\|(x_k)\|\le 1} |f((x_k))| \\ &= \sup_{\|(x_k)\|\le 1} \left| \sum_{k=1}^{\infty} f(e^{(k)}(x_k)) \right| \quad \text{(by (3.2))} \\ &= \sup_{\|(x_k)\|\le 1} \left| \sum_{k=1}^{\infty} (f \circ T_k)(x_k) \right| \\ &= \|(f \circ T_k)_{k=1}^{\infty}\|_{E^{\beta}} \\ &= \|\varphi(f)\|_{E^{\beta}}. \end{aligned} \tag{3.9}$$

Hence φ is isometry. Therefore, $\varphi : E' \to E^\beta$ is an isometrically isomorphism from E' onto E^β. This completes the proof. □

We next give characterizations of β-dual of the sequence space $\ell(X,p)$ when $p_k > 1$ for all $k \in \mathbb{N}$.

Theorem 3.4. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$. Then $\ell(X,p)^\beta = \ell_0(X',q)$, where $q = (q_k)$ is a sequence of positive real numbers such that $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$.*

Proof. Suppose that $(f_k) \in \ell_0(X',q)$. Then $\sum_{k=1}^\infty \|f_k\|^{q_k} M^{-q_k} < \infty$ for some $M \in \mathbb{N}$. Then for each $x = (x_k) \in \ell(X,p)$, we have

$$\begin{aligned}
\sum_{k=1}^\infty |f_k(x_k)| &\le \sum_{k=1}^\infty \|f_k\| M^{-1/p_k} M^{1/p_k} \|x_k\| \\
&\le \sum_{k=1}^\infty \left(\|f_k\|^{q_k} M^{-q_k/p_k} + M\|x_k\|^{p_k}\right) \\
&= \sum_{k=1}^\infty \|f_k\|^{q_k} M^{-(q_k-1)} + M\sum_{k=1}^\infty \|x_k\|^{p_k} \\
&= M\sum_{k=1}^\infty \|f_k\|^{q_k} M^{-q_k} + M\sum_{k=1}^\infty \|x_k\|^{p_k} \\
&< \infty,
\end{aligned} \tag{3.10}$$

which implies that $\sum_{k=1}^\infty f_k(x_k)$ converges, so $(f_k) \in \ell(X,p)^\beta$.

On the other hand, assume that $(f_k) \in \ell(X,p)^\beta$, then $\sum_{k=1}^\infty f_k(x_k)$ converges for all $x = (x_k) \in \ell(X,p)$. For each $x = (x_k) \in \ell(X,p)$, choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in \mathbb{N}$. Since $(t_k x_k) \in \ell(X,p)$, by our assumption, we have $\sum_{k=1}^\infty f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^\infty |f_k(x_k)| < \infty \quad \forall x \in \ell(X,p). \tag{3.11}$$

We want to show that $(f_k) \in \ell_0(X',q)$, that is, $\sum_{k=1}^\infty \|f_k\|^{q_k} M^{-q_k} < \infty$ for some $M \in \mathbb{N}$. If it is not true, then

$$\sum_{k=1}^\infty \|f_k\|^{q_k} m^{-q_k} = \infty \quad \forall m \in \mathbb{N}. \tag{3.12}$$

It implies by (3.12) that for each $k \in \mathbb{N}$,

$$\sum_{i>k} \|f_i\|^{q_i} m^{-q_i} = \infty \quad \forall m \in \mathbb{N}. \tag{3.13}$$

By (3.12), let $m_1 = 1$, then there is a $k_1 \in \mathbb{N}$ such that

$$\sum_{k\le k_1} \|f_k\|^{q_k} m_1^{-q_k} > 1. \tag{3.14}$$

By (3.13), we can choose $m_2 > m_1$ and $k_2 > k_1$ with $m_2 > 2^2$ such that

$$\sum_{k_1<k\le k_2} \|f_k\|^{q_k} m_2^{-q_k} > 1. \tag{3.15}$$

Proceeding in this way, we can choose sequences of positive integers (k_i) and (m_i) with $1 = k_0 < k_1 < k_2 < \cdots$ and $m_1 < m_2 < \cdots$, such that $m_i > 2^i$ and

$$\sum_{k_{i-1}<k\le k_i} \|f_k\|^{q_k} m_i^{-q_k} > 1. \tag{3.16}$$

For each $i \in \mathbb{N}$, choose x_k in X with $\|x_k\| = 1$ for all $k \in \mathbb{N}$, $k_{i-1} < k \le k_i$ such that

$$\sum_{k_{i-1}<k\le k_i} |f_k(x_k)|^{q_k} m_i^{-q_k} > 1 \quad \forall i \in \mathbb{N}. \tag{3.17}$$

Let $a_i = \sum_{k_{i-1}<k\le k_i} |f_k(x_k)|^{q_k} m_i^{-q_k}$. Put $y = (y_k)$, $y_k = a_i^{-1} m_i^{-q_k} |f_k(x_k)|^{q_k-1} x_k$ for all $k \in \mathbb{N}$ with $k_{i-1} < k \le k_i$. By using the fact that $p_k q_k = p_k + q_k$ and $p_k(q_k - 1) = q_k$ for all $k \in \mathbb{N}$, we have that for each $i \in \mathbb{N}$,

$$\begin{aligned}
\sum_{k_{i-1}<k\le k_i} \|y_k\|^{p_k} &= \sum_{k_{i-1}<k\le k_i} \left\|a_i^{-1} m_i^{-q_k} |f_k(x_k)|^{q_k-1} x_k\right\|^{p_k} \\
&= \sum_{k_{i-1}<k\le k_i} a_i^{-p_k} m_i^{-p_k q_k} |f_k(x_k)|^{q_k} \\
&= \sum_{k_{i-1}<k\le k_i} a_i^{-p_k} m_i^{-p_k} m_i^{-q_k} |f_k(x_k)|^{q_k} \\
&\le a_i^{-1} m_i^{-1} \sum_{k_{i-1}<k\le k_i} m_i^{-q_k} |f_k(x_k)|^{q_k} \\
&\le a_i^{-1} m_i^{-1} a_i \\
&= m_i^{-1} \\
&< \frac{1}{2^i},
\end{aligned} \tag{3.18}$$

so we have that $\sum_{k=1}^{\infty} \|y_k\|^{p_k} \le \sum_{i=1}^{\infty} 1/2^i < \infty$. Hence, $y = (y_k) \in \ell(X, p)$. For each $i \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{k_{i-1}<k\le k_i} |f_k(y_k)| &= \sum_{k_{i-1}<k\le k_i} \left|f_k\left(a_i^{-1} m_i^{-q_k} |f_k(x_k)|^{q_k-1} x_k\right)\right| \\
&= \sum_{k_{i-1}<k\le k_i} a_i^{-1} m_i^{-q_k} |f_k(x_k)|^{q_k} \\
&= a_i^{-1} \sum_{k_{i-1}<k\le k_i} m_i^{-q_k} |f_k(x_k)|^{q_k} \\
&= 1,
\end{aligned} \tag{3.19}$$

so that $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts (3.11). Hence $(f_k) \in \ell_0(X', q)$. The proof is now complete. □

The following theorem gives a characterization of β-dual of $\ell(X,p)$ when $p_k \le 1$ for all $k \in \mathbb{N}$. To do this, the following lemma is needed.

Lemma 3.5. *Let $p = (p_k)$ be a bounded sequence of positive real numbers. Then $\ell_\infty(X,p) = \bigcup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$.*

Proof. Let $x \in \ell_\infty(X,p)$, then there is some $n \in \mathbb{N}$ with $\|x_k\|^{p_k} \le n$ for all $k \in \mathbb{N}$. Hence $\|x_k\| n^{-1/p_k} \le 1$ for all $k \in \mathbb{N}$, so that $x \in \ell_\infty(X)_{(n^{-1/p_k})}$. On the other hand, if $x \in \bigcup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$, then there are some $n \in \mathbb{N}$ and $M > 1$ such that $\|x_k\| n^{-1/p_k} \le M$ for every $k \in \mathbb{N}$. Then we have $\|x_k\|^{p_k} \le nM^{p_k} \le nM^{\alpha}$ for all $k \in \mathbb{N}$, where $\alpha = \sup_k p_k$. Hence $x \in \ell_\infty(X,p)$. □

Theorem 3.6. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k \le 1$ for all $k \in \mathbb{N}$. Then $\ell(X,p)^{\beta} = \ell_\infty(X',p)$.*

Proof. If $(f_k) \in \ell(X,p)^{\beta}$, then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in \ell(X,p)$, using the same proof as in Theorem 3.4, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \forall x = (x_k) \in \ell(X,p). \tag{3.20}$$

If $(f_k) \notin \ell_\infty(X',p)$, it follows by Lemma 3.5 that $\sup_k \|f_k\| m^{-1/p_k} = \infty$ for all $m \in \mathbb{N}$. For each $i \in \mathbb{N}$, choose sequences (m_i) and (k_i) of positive integers with $m_1 < m_2 < \cdots$ and $k_1 < k_2 < \cdots$ such that $m_i > 2^i$ and $\|f_{k_i}\| m_i^{-1/p_{k_i}} > 1$. Choose $x_{k_i} \in X$ with $\|x_{k_i}\| = 1$ such that

$$|f_{k_i}(x_{k_i})| m_i^{-1/p_{k_i}} > 1. \tag{3.21}$$

Let $y = (y_k)$, $y_k = m_i^{-1/p_{k_i}} x_{k_i}$ if $k = k_i$ for some i, and 0 otherwise. Then $\sum_{k=1}^{\infty} \|y_k\|^{p_k} = \sum_{i=1}^{\infty} 1/m_i < \sum_{i=1}^{\infty} 1/2^i = 1$, so that $(y_k) \in \ell(X,p)$ and

$$\begin{aligned}\sum_{k=1}^{\infty} |f_k(y_k)| &= \sum_{i=1}^{\infty} \left| f_{k_i}\left(m_i^{-1/p_{k_i}} x_{k_i}\right)\right| \\ &= \sum_{i=1}^{\infty} m_i^{-1/p_{k_i}} |f_{k_i}(x_{k_i})| \\ &= \infty \quad (\text{by } (3.21)),\end{aligned} \tag{3.22}$$

and this is contradictory to (3.20), hence $(f_k) \in \ell_\infty(X',p)$.

Conversely, assume that $(f_k) \in \ell_\infty(X',p)$. By Lemma 3.5, there exists $M \in \mathbb{N}$ such that $\sup_k \|f_k\| M^{-1/p_k} < \infty$, so there is a $K > 0$ such that

$$\|f_k\| \le KM^{1/p_k} \quad \forall k \in \mathbb{N}. \tag{3.23}$$

Let $x = (x_k) \in \ell(X,p)$. Then there is a $k_0 \in \mathbb{N}$ such that $M^{1/p_k}\|x_k\| \le 1$ for all $k \ge k_0$. By $p_k \le 1$ for all $k \in \mathbb{N}$, we have that, for all $k \ge k_0$,

$$M^{1/p_k}\|x_k\| \le \left(M^{1/p_k}\|x_k\|\right)^{p_k} = M\|x_k\|^{p_k}. \tag{3.24}$$

Then

$$\begin{aligned}\sum_{k=1}^{\infty} |f_k(x_k)| &\le \sum_{k=1}^{k_0} \|f_k\|\|x_k\| + \sum_{k=k_0+1}^{\infty} \|f_k\|\|x_k\| \\ &\le \sum_{k=1}^{k_0} \|f_k\|\|x_k\| + K \sum_{k=k_0+1}^{\infty} M^{1/p_k}\|x_k\| \quad \text{(by (3.23))} \\ &\le \sum_{k=1}^{k_0} \|f_k\|\|x_k\| + KM \sum_{k=k_0+1}^{\infty} \|x_k\|^{p_k} \quad \text{(by (3.24))} \\ &< \infty.\end{aligned} \tag{3.25}$$

This implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, hence $(f_k) \in \ell(X,p)^{\beta}$. □

Theorem 3.7. *Let $p = (p_k)$ be a bounded sequence of positive real numbers. Then $\ell_\infty(X,p)^{\beta} = M_\infty(X',p)$.*

Proof. If $(f_k) \in M_\infty(X',p)$, then $\sum_{k=1}^{\infty} \|f_k\| m^{1/p_k} < \infty$ for all $m \in \mathbb{N}$, we have that for each $x = (x_k) \in \ell_\infty(X,p)$, there is $m_0 \in \mathbb{N}$ such that $\|x_k\| \le m_0^{1/p_k}$ for all $k \in \mathbb{N}$, hence $\sum_{k=1}^{\infty} |f_k(x_k)| \le \sum_{k=1}^{\infty} \|f_k\|\|x_k\| \le \sum_{k=1}^{\infty} \|f_k\| m_0^{1/p_k} < \infty$, which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so that $(f_k) \in \ell_\infty(X,p)^{\beta}$.

Conversely, assume that $(f_k) \in \ell_\infty(X,p)^{\beta}$, then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X,p)$, by using the same proof as in Theorem 3.4, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \forall x = (x_k) \in \ell_\infty(X,p). \tag{3.26}$$

If $(f_k) \notin M_\infty(X',p)$, then $\sum_{k=1}^{\infty} \|f_k\| M^{1/p_k} = \infty$ for some $M \in \mathbb{N}$. Then we can choose a sequence (k_i) of positive integers with $0 = k_0 < k_1 < k_2 < \cdots$ such that

$$\sum_{k_{i-1}<k\le k_i} \|f_k\| M^{1/p_k} > i \quad \forall i \in \mathbb{N}. \tag{3.27}$$

And we choose x_k in X with $\|x_k\| = 1$ such that for all $i \in \mathbb{N}$,

$$\sum_{k_{i-1}<k\le k_i} |f_k(x_k)| M^{1/p_k} > i. \tag{3.28}$$

Put $y = (y_k)$, $y_k = M^{1/p_k} x_k$. Clearly, $y \in \ell_\infty(X,p)$ and

$$\sum_{k=1}^{\infty} |f_k(y_k)| \ge \sum_{k_{i-1}<k\le k_i} |f_k(x_k)| M^{1/p_k} > i \quad \forall i \in \mathbb{N}. \tag{3.29}$$

Hence $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts (3.26). Hence $(f_k) \in M_\infty(X',p)$. The proof is now complete. □

Theorem 3.8. *Let $p = (p_k)$ be a bounded sequence of positive real numbers. Then $c_0(X,p)^{\beta} = M_0(X',p)$.*

Proof. Suppose $(f_k) \in M_0(X',p)$, then $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in \mathbb{N}$. Let $x = (x_k) \in c_0(X,p)$. Then there is a positive integer K_0 such that $\|x_k\|^{p_k} < 1/M$ for all $k \geq K_0$, hence $\|x_k\| < M^{-1/p_k}$ for all $k \geq K_0$. Then we have

$$\sum_{k=K_0}^{\infty} |f_k(x_k)| \leq \sum_{k=K_0}^{\infty} \|f_k\| \|x_k\| \leq \sum_{k=K_0}^{\infty} \|f_k\| M^{-1/p_k} < \infty. \tag{3.30}$$

It follows that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so that $(f_k) \in c_0(X,p)^{\beta}$.

On the other hand, assume that $(f_k) \in c_0(X,p)^{\beta}$, then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X,p)$. For each $x = (x_k) \in c_0(X,p)$, choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in \mathbb{N}$. Since $(t_k x_k) \in c_0(X,p)$, by our assumption, we have $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \forall x \in c_0(X,p). \tag{3.31}$$

Now, suppose that $(f_k) \notin M_0(X',p)$. Then $\sum_{k=1}^{\infty} \|f_k\| m^{-1/p_k} = \infty$ for all $m \in \mathbb{N}$. Choose $m_1, k_1 \in \mathbb{N}$ such that

$$\sum_{k \leq k_1} \|f_k\| m_1^{-1/p_k} > 1 \tag{3.32}$$

and choose $m_2 > m_1$ and $k_2 > k_1$ such that

$$\sum_{k_1 < k \leq k_2} \|f_k\| m_2^{-1/p_k} > 2. \tag{3.33}$$

Proceeding in this way, we can choose $m_1 < m_2 < \cdots$, and $0 = k_1 < k_2 < \cdots$ such that

$$\sum_{k_{i-1} < k \leq k_i} \|f_k\| m_i^{-1/p_k} > i. \tag{3.34}$$

Take x_k in X with $\|x_k\| = 1$ for all $k, k_{i-1} < k \leq k_i$ such that

$$\sum_{k_{i-1} < k \leq k_i} |f_k(x_k)| m_i^{-1/p_k} > i \quad \forall i \in \mathbb{N}. \tag{3.35}$$

Put $y = (y_k)$, $y_k = m_i^{-1/p_k} x_k$ for $k_{i-1} < k \leq k_i$, then $y \in c_0(X,p)$ and

$$\sum_{k=1}^{\infty} |f_k(y_k)| \geq \sum_{k_{i-1} < k \leq k_i} |f_k(x_k)| m_i^{-1/p_k} > i \quad \forall i \in \mathbb{N}. \tag{3.36}$$

Hence we have $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts (3.31), therefore $(f_k) \in M_0(X',p)$. This completes the proof. □

Theorem 3.9. *Let $p = (p_k)$ be a bounded sequence of positive real numbers. Then $c(X,p)^{\beta} = M_0(X',p) \cap cs[X']$.*

Proof. Since $c(X,p) = c_0(X,p) + E$, where $E = \{e(x) : x \in X\}$, it follows by Proposition 3.1(iii) and Theorem 3.8 that $c(X,p)^{\beta} = M_0(X',p) \cap E^{\beta}$. It is obvious by definition that $E^{\beta} = \{(f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x)$ converges for all $x \in X\} = cs[X']$. Hence we have the theorem. □

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Matrix Transformations Between Some Vector-Valued Sequence Spaces

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Abstract. In this paper, we give necessary and sufficient conditions for infinite matrices mapping from the Nakano vector-valued sequence space $\ell(X, p)$ into any BK-space, and by using this result, we obtain the matrix characterizations from $\ell(X, p)$ into the sequence spaces $\ell_\infty(Y)$, $c_0(Y, q)$, $c(Y)$, $\ell_s(Y)$, $E_r(Y)$, and $F_r(Y)$, where $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers such that $p_k \leq 1$ for all $k \in N$, $r \geq 0$, and $s \geq 1$.

Keywords: matrix transformations, vector-valued sequences spaces

1. Introduction

Let $(X, \|.\|)$ be a Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. We write $x = (x_k)$ with x_k in X for all $k \in N$. The X-valued sequence spaces $c_0(X, p)$, $c(X, p)$, $\ell_\infty(X, p)$, $\ell(X, p)$, $E_r(X, p)$, and $F_r(X, p)$ are defined as

(a) $c_0(X, p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\}$;
(b) $c(X, p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0$ for some $a \in X\}$;
(c) $\ell_\infty(X, p) = \{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\}$;
(d) $\ell(X, p) = \{x = (x_k) : \Sigma_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\}$;
(e) $E_r(X, p) = \{x = (x_k) : \sup_k \|x_k\|^{p_k}/n^r < \infty\}$;
(f) $F_r(X, p) = \{x = (x_k) : \Sigma_{k=1}^{\infty} k^r \|x_k\|^{p_k} < \infty\}$.

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, $\ell(p)$, $E_r(p)$, and $F_r(p)$, respectively and the first three spaces are known as *the sequence spaces of Maddox*. These spaces were first introduced and studied by Simons [8] and Maddox [4–6]. The space $\ell(p)$ was first introduced by Nakano [7]. When $p_k = 1$, for all $k \in N$, the space $E_r(p)$ and $F_r(p)$ are written as E_r and F_r, respectively. These two spaces were first defined by Cooke [1]. The structure of sequence spaces $c_0(p)$, $c(p)$, and $\ell_\infty(p)$ have been investigated by Grosse-Erdmann [2] and he has given in [3] characterizations of matrix transformations between the scalar-valued sequence spaces of Maddox. Wu and Liu [9] deal with the problem of characterizations of infinite matrices mapping from $c_0(X, p)$ and $\ell_\infty(X, p)$ into $c_0(q)$ and $\ell_\infty(q)$.

2. Notation and Definitions

Let $(X, \|.\|)$ and $(Y, \|.\|)$ be Banach spaces, and the space of all continuous linear operators from X into Y is denoted by $\mathcal{L}(X, Y)$. Let $W(X)$ and $\Phi(X)$ denote for the space of all sequences in X and the space of all finite sequences in X. When $X = K$, the scalar filed of X, the corresponding spaces are written as W and Φ, respectively.

A sequence spaces in X is a linear subspace of $W(X)$. Let E be any X-valued sequence space. For $x \in E$ and $k \in N$, we write that x_k stands for the kth term of x. For $x \in X$ and $k \in N$, let $e^k(x)$ be the sequence $(0, 0, ..., 0, x, 0, ...)$ with x in the kth position and let $e(x)$ be the sequence $(x, x, x, ...)$. For a fixed scalar sequence $\mu = (\mu_k)$, the sequence space E_μ is defined as

$$E_\mu = \{x \in W(X) : (\mu_k x_k) \in E\} .$$

Let $A = (T_k^n)$ with T_k^n in $\mathcal{L}(X, Y)$. Suppose E is an X-valued sequence space and F a Y-valued sequence space. Then A is said to *map* E into F, written by $A : E \to F$ if, for each $x = (x_k) \in E$, $A_n(x) = \Sigma_{k=1}^\infty T_k^n(x_k)$ converges for each $n \in N$, and the sequence $Ax = \big(A_n(x)\big) \in F$. Let (E, F) denote the set of all infinite matrices mapping from E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$$_u(E, F)_v = \{A = (T_k^n) : (u_n v_k T_k^n)_{n,k} \in (E, F)\}$$

If $u_k \neq 0$ for all $k \in N$, we write $u^{-1} = (1/u_k)$.

Suppose the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if, for each $k \in N$, the kth coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. In addition, if (E, τ) is a Fréchet (Banach, LF-, LB-) space, then E is called an *FK- (BK-, LFK-, LBK-) space*. Now, suppose that E contains $\phi(X)$. Then E is said to have *property AB* if the set $\{\Sigma_{k=1}^n e^k(x_k) : n \in N\}$ is bounded in E for every $x = (x_k) \in E$. It is said to have *property AK* if $\Sigma_{k=1}^n e^k(x_k) \to x$ in E as $n \to \infty$ for every $x = (x_k) \in E$. It has *property AD* if $\Phi(X)$ is dense in E.

The space $\ell(p)$ is an FK-space with AK under the paranorm $g(x) = (\Sigma_{k=1}^\infty |x_k|^{p_k})^{1/M}$, where $M = \max\{1, \sup_k p_k\}$ (see [6]). The space $c_0(p)$ is an FK-space with AK, $c(p)$ is an FK-space, and $\ell_\infty(p)$ is a complete LBK-space with AB (see [2, 6]). It is known that the space $\ell(X, p)$ is an FK-space with AK under the paranorm $g(x) = (\Sigma_{k=1}^\infty \|x_k\|_k^p)^{1/M}$, where $M = \max\{1, \sup_k p_k\}$. In each of the space $\ell_\infty(X, p)$ and $c_0(X, p)$, we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = \max\{1, \sup_k p_k\}$. It is known that $c_0(X, p)$ is an FK-space with AK under the paranorm g defined as above and $\ell_\infty(X, p)$ is a complete LBK-space with AB.

3. Some Auxiliary Results

In this section we give some useful results that can be used to reduce our problems into some simpler forms.

Proposition 3.1. *Let E and $E_n (n \in N)$ be X-valued sequence spaces, and F and $F_n (n \in N)$ Y-valued sequence spaces, and let u and v be sequences of real numbers with $u_k \neq 0$, $v_k \neq 0$ for all $k \in N$. Then we have*

(i) $(\cup_{n=1}^{\infty} E_n, F) = \cap_{n=1}^{\infty}(E_n, F)$;
(ii) $(E, \cap_{n=1}^{\infty} F_n) = \cap_{n=1}^{\infty}(E, F_n)$;
(iii) $(E_u, F_v) = {}_v(E, F)_{u^{-1}}$.

Proof. Assertion (i)–(iii) are immediately obtained directly by the definitions. □

Proposition 3.2. *Let $p = (p_k)$ be a bounded sequence of positive real numbers and $r \geq 0$. Then*

(i) $F_r(X, p) = \ell(X, p)_{(n^r)}$;
(ii) $c_0(X, p) = \cap_{n=1}^{\infty} c_0(X)_{(n^{1/p_k})}$.

Proof. Assertion (i) is obviously obtained by the definition.

To show (ii), let $x \in c_0(X, p)$. Then $\|x_k\|^{p_k} \to 0$ as $k \to \infty$. For each $n \in N$, let $\delta_k = \|x_k\|^{p_k} n$ for all $k \in N$. We have that $\delta_k \to 0$ as $k \to \infty$; hence, $\|x_k\| n^{1/p_k} = \delta_k^{1/p_k} \to 0$ as $k \to \infty$ (because $p \in \ell_\infty$), so we have $x \in c_0(X)_{(n^{1/p_k})}$. Conversely, assume $x \in \cap_{n=1}^{\infty} c_0(X)_{(n^{1/p_k})}$. Then, $\lim_{k\to\infty} \|x_k\| n^{1/p_k} = 0$ for every $n \in N$. Then, for $n \in N$, we have $\|x_k\|^{p_k} \leq 1/n$ for large k, hence, $x \in c_0(X, p)$. □

4. Main Results

We begin by giving the matrix characterizations mapping from $\ell(X, p)$ into a BK-space, where $p_k \leq 1$ for all $k \in N$.

Theorem 4.1. *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$ and let E be a Y-valued sequence space which is a BK-space. Then for an infinite matrix $A = (T_k^n)$, $A \in \big(\ell(X, p), E\big)$ if and only if*

(1) *for each $k \in N$, $\big(T_k^n(x)\big)_{n=1}^{\infty} \in E$ for all $x \in X$ and*
(2) *there exists $m_0 \in N$ such that*

$$\sup_k \sup_{\|x\|\leq 1} \|A\big(m_0^{-1/p_k} e^k(x)\big)\| \leq 1.$$

Proof. Suppose $A \in \big(\ell(X, p), E\big)$. Since $e^k(x) \in \ell(X, p)$ for all $x \in X$ and all $k \in N$, we have $Ae^k(x) \in E$, so (1) is obtained. Now, we shall show that condition (2) is satisfied. By Zeller's theorem, we have that $A : \ell(X, p) \to E$ is continuous. Then there exists $m_0 \in N$ such that

$$x = (x_k) \in \ell(X, p), \quad \|x\| \leq \frac{1}{m_0} \quad \Rightarrow \quad \|Ax\| \leq 1\,. \tag{4.1}$$

Let $x \in X$ with $\|x\| \leq 1$ and $k \in N$. We have $m_0^{-1/p_k} e^k(x) \in \ell(X, p)$ and $\|m_0^{-1/p_k} e^k(x)\| \leq 1/m_0$. By (4.1), we have $\|A\big(m_0^{-1/p_k} e^k(x)\big)\| \leq 1$. This implies that

$$\sup_k \sup_{\|x\|\leq 1} \|A\big(m_0^{-1/p_k} e^k(x)\big)\| \leq 1.$$

Hence, (2) holds.

Conversely, assume that conditions (1) and (2) hold. By (2), there is $m_0 \in N$ such that

$$\sup_k \|A(m_0^{-1/p_k} e^k(x))\| \leq 1 \tag{4.2}$$

for all $x \in X$ with $\|x\| \leq 1$.

This implies by (4.2) that

$$\sup_k \|A(m_0^{-1/p_k} e^k(x))\| \leq \|x\| \tag{4.3}$$

for all $x \in X$.

Let $x = (x_k) \in \ell(X, p)$. For each $k \in N$, by (4.3), we have that

$$\begin{aligned} \|Ae^k(x_k)\| &= \|A(m_0^{1/p_k}(m_0^{-1/p_k} e^k(x_k)))\| \\ &= m_0^{1/p_k} \|A(m_0^{-1/p_k} e^k(x_k))\| \\ &\leq m_0^{1/p_k} \|x_k\|. \end{aligned} \tag{4.4}$$

Since $(m_0^{1/p_k} x_k) \in \ell(X, p)$, so $(m_0^{1/p_k} x_k) \in c_0(X, p) \subseteq c_0(X)$. Hence, there is a $k_0 \in N$ such that $m_0^{1/p_k} \|x_k\| < 1$ for all $k > k_0$. Since $0 < p_k \leq 1$ for all $k \in N$, we have

$$m_0^{1/p_k} \|x_k\| \leq (m_0^{1/p_k} \|x_k\|)^{p_k} = m_0 \|x_k\|^{p_k} \tag{4.5}$$

for all $k > k_0$.

It follows by (4.4) and (4.5) that

$$\begin{aligned} \sum_{k=1}^{\infty} \|Ae^k(x_k)\| &\leq \sum_{k=1}^{\infty} m_0^{1/p_k} \|x_k\| \\ &= \sum_{k=1}^{k_0} m_0^{1/p_k} \|x_k\| + \sum_{k=k_0+1}^{\infty} m_0^{1/p_k} \|x_k\| \\ &\leq \sum_{k=1}^{k_0} m_0^{1/p_k} \|x_k\| + m_0 \sum_{k=k_0+1}^{\infty} \|x_k\|^{p_k} \\ &< \infty . \end{aligned}$$

Hence, $\Sigma_{k=1}^{\infty} Ae^k(x_k)$ converges absolutely in E. Since E is Banach, $\Sigma_{k=1}^{\infty} Ae^k(x_k)$ converges in E. Let $y = (y_k) \in E$ be the sum of $\Sigma_{k=1}^{\infty} Ae^k(x_k)$. Since E is a K-space, we have that, for each $m \in N$, p_m is continuous, so that

$$y_m = p_m(y) = \lim_{n\to\infty} \sum_{k=1}^{n} p_m(Ae^k(x_k)) = \lim_{n\to\infty} \sum_{k=1}^{n} T_k^m(x_k).$$

This implies that Ax exists and $(Ax)_m = \Sigma_{k=1}^{\infty} T_k^n(x_k) = y_m$, hence, $Ax \in E$.

This completes the proof. □

When $p_k = 1$ for all $k \in N$, the following results are obtained directly from Theorem 4.1.

Theorem 4.2. *Let E be a Y-valued sequence space which is a BK-space and $A = (T_k^n)$ an infinite matrix. Then $A \in (\ell(X), E)$ if and only if*

(1) *for each $k \in N$, $(T_k^n(x))_{n=1}^{\infty} \in E$ for all $x \in X$ and*
(2) $\sup_k \sup_{\|x\| \leq 1} \|Ae^k(x)\| < \infty$. □

Theorem 4.3. *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$ and let $A = (T_k^n)$ be an infinite matrix. Then $A \in (\ell(X, p), \ell_\infty(Y))$ if and only if*

(1) *for each $k \in N$, $\sup_n \|T_k^n\| < \infty$ and*
(2) *there exists $m_0 \in N$ such that*

$$\sup_{n,k} m_0^{-1/p_k} \|T_k^n\| \leq 1.$$

Proof. By Theorem 4.1, to prove this theorem we only show that conditions (1) and (2) are equivalent to conditions (1′) and (2′), respectively, where

(1′) for each $k \in N$, $Ae^k(x) \in \ell_\infty(Y)$ for all $x \in X$ and
(2′) there exists $m_0 \in N$ such that

$$\sup_k \sup_{\|x\| \leq 1} \|A(m_0^{-1/p_k} e^k(x))\| \leq 1 .$$

Conditions (1) and (1′) are equivalent by the uniform bounded principle.

If (2) holds, for $k, n \in N$ and $x \in X$ with $\|x\| \leq 1$, we have $m_0^{-1/p_k} \|T_k^n x\| \leq m_0^{-1/p_k} \|T_k^n\| \|x\| \leq m_0^{-1/p_k} \|T_k^n\| \leq 1$, which implies

$$\sup_k \sup_{\|x\| \leq 1} \|A(m_0^{-1/p_k} e^k(x))\| = \sup_k \sup_{\|x\| \leq 1} \sup_n m_0^{-1/p_k} \|T_k^n x\| \leq 1.$$

so (2′) is obtained.

Now, suppose that (2′) holds. Then there exists $m_0 \in N$ such that

$$\sup_n m_0^{-1/p_k} \|T_k^n x\| = \|A(m_0^{-1/p_k} e^k(x))\| \leq 1 \tag{4.6}$$

for all $k \in N$ and all $x \in N$ with $\|x\| \leq 1$.

It follows by (4.6) that, for each $n, k \in N$, $m_0^{-1/p_k} \|T_k^n\| \leq 1$, so (2) is obtained. □

By using the same proof as in Theorem 4.3, we obtain

Theorem 4.4. *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$ and let $A = (T_k^n)$ be an infinite matrix. Then $A \in (\ell(X, p), c_0(Y))$ if and only if*

(1) *for each $k \in N$, $T_k^n(x) \to 0$ as $n \to \infty$ for all $x \in X$ and*
(2) *there exists $m_0 \in N$ such that*

$$\sup_{n,k} m_0^{-1/p_k} \|T_k^n\| \leq 1.$$

Theorem 4.5. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers such that $p_k \leq 1$ for all $k \in N$ and let $A = (T_k^n)$ be an infinite matrix. Then $A \in \big(\ell(X, p),\ c_0(Y, q)\big)$ if and only if*

(1) *for each $k \in N$ and $m \in N$, $m^{1/p_n} T_k^n(x) \to 0$ as $n \to \infty$ for all $x \in X$ and*
(2) *for each $m \in N$, there exists $r_m \in N$ such that*

$$\sup_{n,k} r_m^{-1/p_k} m^{1/p_n} \|T_k^n\| \leq 1.$$

Proof. By Proposition 3.1(ii) and 3.2(ii), we have

$$A \in \big(\ell(X, p),\ c_0(Y, q)\big) \iff A \in \big(\ell(X, p),\ c_0(Y)_{(m^{1/q_k})}\big) \text{ for all } m \in N.$$

It follows by Proposition 3.1(iii) that, for each $m \in N$,

$$A \in \big(\ell(X, p),\ c_0(Y)_{(m^{1/q_k})}\big) \iff \big(m^{1/q_n} T_k^n\big)_{n,k} \in \big(\ell(X, p),\ c_0(Y)\big).$$

By Theorem 4.4, we obtain that

$$\big(m^{1/q_n} T_k^n\big)_{n,k} \in \big(\ell(X, p),\ c_0(Y)\big) \iff \text{(1) and (2) are satisfied.}$$

□

By applying Theorem 4.1, and using the same proof as in Theorem 4.3, we obtain

Theorem 4.6. *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$ and let $A = (T_k^n)$ be an infinite matrix. Then $A \in \big(\ell(X, p),\ c(Y)\big)$ if and only if*

(1) *for each $k \in N$, $\lim_{n\to\infty} T_k^n(x)$ exists for all $x \in X$ and*
(2) *there exists $m_0 \in N$ such that*

$$\sup_{n,k}\ m_0^{-1/p_k} \|T_k^n\| \leq 1.$$

By applying Theorem 4.1, we also have the following result.

Theorem 4.7. *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$, $s \geq 1$ and let $A = (T_k^n)$ be an infinite matrix. Then $A \in \big(\ell(X, p),\ \ell_s(Y)\big)$ if and only if*

(1) *for each $k \in N$, $\big(T_k^n(x)\big)_{n=1}^{\infty} \in \ell_s(Y)$ for all $x \in X$ and*
(2) *there exists $m_0 \in N$ such that*

$$\sup_k \sup_{\|x\|\leq 1} \sum_{n=1}^{\infty} m_0^{s/p_k} \|T_k^n x\|^s \leq 1.$$

Since $E_r(Y) = \ell_\infty(Y)_{(k^{r})}$, the following result is obtained by Proposition 3.1(iii) and Theorem 4.3.

Theorem 4.8. *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$, $r \geq 0$ and let $A = (T_k^n)$ be an infinite matrix. Then $A \in \big(\ell(X, p),\ E_r(Y)\big)$ if and only if*

(1) *for each $k \in N$, $\sup_n \|n^{-r} T_k^n\| < \infty$ and*
(2) *there exists $m_0 \in N$ such that*

$$\sup_{n,k} m_0^{-1/p_k} n^{-r} \|T_k^n\| \leq 1.$$

Since $F_r(Y) = \ell(Y)_{(k^r)}$, by applying Proposition 3.1(iii) and Theorem 4.1, we obtain

Theorem 4.9. *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$, $r \geq 0$ and let $A = (T_k^n)$ be an infinite matrix. Then $A \in \big(\ell(X, p),\ F_r(Y)\big)$ if and only if*

(1) *for each $k \in N$, $\big(n^r T_k^n(x)\big)_{n=1}^{\infty} \in \ell(Y)$ for all $x \in X$ and*
(2) *there exists $m_0 \in N$ such that*

$$\sup_k \sup_{\|x\| \leq 1} \sum_{n=1}^{\infty} m_0^{-1/p_k} n^r \|T_k^n x\| \leq 1\ .$$

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Subject: Paper entitled Matrix Transformations of Some Vector-Valued Sequence Spaces

Dear Dr. Suantai,

I am pleased to inform you that the above paper has been accepted for publication in the INDIAN JOURNAL OF PURE AND APPLIED MATHEMATICS.

Your paper is scheduled to appear in____________________

Yours sincerely

(O.P. Bhutani)

Accepted to be published in Indian Journal of Pure and Applied Mathematics

MATRIX TRANSFORMATIONS OF SOME VECTOR-VALUED SEQUENCE SPACES

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Necessary and sufficient conditions are established for an infinite matrix $A = (f_n^k)$ of continuous linear functionals on a Banach space X to transform the vector-valued sequence spaces of Maddox $\ell_\infty(X,p)$, $\ell(X,p)$, $c_0(X,p)$, and $c(X,p)$ into the scalar-valued sequence space $c(q)$, where $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers.

Keywords: Matrix transformations, Maddox vector-valued sequence spaces

AMS Mathematics Subject Classification (2000): 46A45.

1. Introduction

The study of matrix transformations of scalar- valued sequence spaces is known since the turn of the century. In seventies, Maddox[12], Gupta[4] studied matrix transformations of continuous linear mappings on vector-valued sequence spaces. Das and Choudhury[1] gave conditions on the matrix $A = (f_k^n)$ of continuous linear mappings from a normed linear space X into a normed linear space Y under which A maps $c_0(X)$ into $c_0(Y)$, $\ell_1(X)$ into $\ell_\infty(Y)$, and $\ell_1(X)$ into $\ell_p(Y)$. Liu and Wu[22] gave the matrix characterizations from vector-valued sequence spaces $c_0(X,p)$, $\ell(X,p)$, and $\ell_\infty(X,p)$ into scalar-valued sequence spaces $c_0(q)$ and $\ell_\infty(q)$. Suantai[20] gave the matrix characterizations from the Nakano vector-valued sequence space $\ell(X,p)$ into the vector-valued sequence spaces $c_0(Y,q)$, $c(Y)$, and $\ell_r(Y)$. In this paper, we continue the study of matrix transformations of continuous linear mappings on vector-valued sequence spaces.

The main purpose of this paper is to give the matrix characterizations from $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, and $\ell(X,p)$ into $c(q)$, where $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, and $\ell(X,p)$ are the vector-valued sequence spaces of Maddox as defined in Section 2. When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, and $\ell(p)$, respectively. Several papers deal with the problem of characterizing those matrices that map a scalar-valued sequence space of Maddox into anoher such spaces, see [6, 7, 11, 13, 15, 17, 18, 19, 21]. Some of these results become particular cases of our theorems. Also some more interesting results are derived.

Section 2 deals with necessary preliminaries and some known results quoted as lemmas which are needed to characterize an infinite matrix $A = (f_k^n)$ such that A maps the vector- valued sequence spaces of Maddox into $c(q)$, and we also give some auxiliary results in Section 3. The main results of the paper is in Section 4.

2 Preliminaries and Lemmas

Let $(X, \|.\|)$ be a Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. Let N be the set of all natural numbers, we write $x = (x_k)$ with x_k in X for all $k \in N$. Let $W(X)$ and $\Phi(X)$ denote the space of all sequences and the space of all finite sequences in X, respectively. When $X = K$, the scalar field of X, the corresponding spaces are written as w and ϕ, respectively. An $X-$valued sequence space is a linear subspace of $W(X)$. The sequence spaces of Maddox are defined as

$c_0(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\}$,

$c(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X\}$,

$\ell_\infty(X,p) = \{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\}$,

$\ell(X,p) = \{x = (x_k) : \sum_{k=1}^\infty \|x_k\|^{p_k} < \infty\}$.

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p), c(p), \ell_\infty(p)$, and $\ell(p)$, respectively. All of these spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons[16] and Maddox[8,9]. The space $\ell(p)$ was first defined by Nakano[14] and it is known as the Nakano sequence space. Also, we need to define the following sequence space :

$M_0(X,p) = \{x = (x_k) : \sum_{k=1}^\infty \|x_k\| n^{-1/p_k} < \infty \text{ for some } n \in N\}$.

When $X = K$, the scalar field of X, the corresponding space is written as $M_0(p)$. This space was first introduced by Maddox[10]. Grosse-Erdmann[2] has investigated the structure of the spaces $c_0(p), c(p), \ell(p)$, and $\ell_\infty(p)$ and he also gave the matrix characterizations between scalar-valued sequence spaces of Maddox in [3]. Let E be an X- valued sequence space. For $x \in E$ and $k \in N$ we write that x_k stand for the kth term of x and for $x \in X$ and $k \in N$, let $e^{(k)}(x)$ be the sequence $(0, 0, 0, ..., 0, x, 0, ...)$ with x in the kth position and let $e(x)$ be the sequence $(x, x, x, ...)$, and we denote by e the the sequence $(1, 1, 1, ...)$. An $X-$valued sequence space E is said to be *normal* if $(x_k) \in E$ and $(y_k) \in W(X)$ with $\|y_k\| \le \|x_k\|$ for all $k \in N$ implies that $(y_k) \in E$. For a fixed scalar sequence $u = (u_k)$ the sequence space E_u is defined as

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}.$$

The $\alpha-$, $\beta-$ and $\gamma-$ duals of a scalar-valued sequence space F are defined as

$$F^\zeta = \{x \in w : (x_k y_k) \in X_\zeta \text{ for every } y \in F\}$$

for $\zeta = \alpha, \beta, \gamma$ and $X_\alpha = \ell_1$, $X_\beta = cs$, and $X_\gamma = bs$, where ℓ_1, cs and bs are defined as

$\ell_1 = \{x = (x_k) \in w : \sum_{k=1}^\infty |x_k| < \infty\}$,

$cs = \{x = (x_k) \in w : \sum_{k=1}^\infty x_k \text{ converges}\}$,

$bs = \{x = (x_k) \in w : \sup_n |\sum_{k=1}^n x_k| < \infty\}$.

In the same manner, for an $X-$valued sequence space E, the $\alpha-$, $\beta-$ and $\gamma-$ duals of E are defined as

$$E^\zeta = \{(f_k) \subset X' : (f_k(x_k)) \in X_\zeta \text{ for every } x = (x_k) \in E\}$$

for $\zeta = \alpha, \beta, \gamma$, where $X_\alpha = \ell_1$, $X_\beta = cs$ and $X_\gamma = bs$.

It is obvious from the definition that $E^\alpha \subseteq E^\beta \subseteq E^\gamma$ and it is easy to see that if E is normal, then $E^\alpha = E^\beta = E^\gamma$.

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is an X-valued sequence space and F a scalar-valued sequence space. Then A is said to *map E into F*, written by $A : E \to F$ if, for each $x = (x_k) \in E$, $A_n(x) = \sum_{k=1}^\infty f_k^n(x_k)$ converges for each $n \in N$ and the sequence $Ax = (A_n(x)) \in F$. We denote by (E, F) the class of all infinite matrices mapping E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$${}_u(E,F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E,F)\}.$$

If $u_k \neq 0$ for all $k \in N$, we put $u^{-1} = (1/u_k)$. Suppose the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if, for each $k \in N$ the kth coordinate mapping

$p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. A K-space that is a Fréchet(Banach) space is called an $FK-(BK-)$ space.

The spaces $c_0(p)$ and $c(p)$ are FK-spaces. In $c_0(X,p)$, we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = \max\{1, \sup_k p_k\}$, as a paranorm on $c_0(X,p)$, and it is known that $c_0(X,p)$ is an FK-space under the paranorm g defined as above. In $\ell(X,p)$, we consider it as a paranormed sequence space with the paranorm given by $\|(x_k)\| = \left(\sum_{k=1}^{\infty} \|x_k\|^{p_k}\right)^{1/M}$. It is known that $\ell(X,p)$ is an FK-space under the paranorm defined as above.

Now let us quote some known results as the following.

Lemma 2.1[10] *If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$, then*

$$\ell(p)^{\beta} = \{x \in w : \sum_{k=1}^{\infty} |x_k|^{t_k} M^{-t_k} < \infty \ \text{ for some } \ M \in N\}$$

where $1/p_k + 1/t_k = 1$ for all $k \in N$.

Lemma 2.2[16] *If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in N$, then $\ell(p)^{\beta} = \ell_{\infty}(p)$.*

Lemma 2.3[6] *If $p = (p_k)$ is a bounded sequence of positive real numbers , then*

$$\ell_{\infty}(p)^{\beta} = \{x \in w : \sum_{k=1}^{\infty} |x_k| n^{1/p_k} < \infty \text{ for all } \ n \in N\}.$$

Lemma 2.4[10] *If $p = (p_k)$ is a bounded sequence of positive real numbers, then $c_0(p)^{\beta} = M_0(p)$.*

Lemma 2.5[22] *Let $p = (p_k)$ be a bounded sequence of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c_0(X,p) \to c_0$ if and only if*

(1) $f_k^n \xrightarrow{w^*} 0$ as $n \to \infty$ for each $k \in N$ and

(2) $\lim_{m \to \infty} \sup_n \sum_{k=1}^{\infty} \|f_k^n\| m^{-1/p_k} = 0$.

Lemma 2.6[22] *Let $p = (p_k)$ be a bounded sequence of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : \ell_{\infty}(X,p) \to c_0$ if and only if*

(1) $f_k^n \xrightarrow{w^*} 0$ as $n \to \infty$ for each $k \in N$ and

(2) for each $M \in N$, $\sum_{j>k} \|f_j^n\| M^{1/p_j} \to 0$ as $k \to \infty$ uniformly on $n \in N$.

Lemma 2.7[22] *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ and $1/p_k + 1/t_k = 1$ for all $k \in N$ and let $A = (f_k^n)$ be an infinite matrix. Then $A : \ell(X,p) \to c_0$ if and only if*

(1) $f_k^n \xrightarrow{w^*} 0$ as $n \to \infty$ for each $k \in N$ and

(2) $\sum_{k=1}^{\infty} \|f_k^n\|^{t_k} m^{-t_k} \to 0$ as $m \to \infty$ uniformly on $n \in N$.

Lemma 2.8[22] *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in N$ and let $A = (f_k^n)$ be an infinite matrix. Then $A : \ell(X,p) \to c_0$ if and only if*

(1) $f_k^n \xrightarrow{w^*} 0$ as $n \to \infty$ for each $k \in N$ and

(2) $\sup_{n,k} \|f_k^n\|^{p_k} < \infty$.

3. Some Auxiliary Results

Suppose that E and F are sequence spaces and that we want to characterize the matrix space (E, F). If E and/or F can be derived from simpler sequence spaces in some fashion, then, in many cases, the problem reduces to the characterization of the corresponding simpler matrix spaces. We begin with giving various useful results in this direction.

Proposition 3.1. *Let E and $E_n (n \in N)$ be X-valued sequence spaces, and F and $F_n (n \in N)$ scalar-valued sequence spaces, and let u and v be scalar sequences with $u_k \neq 0, v_k \neq 0$ for all $k \in N$. Then*

(i) $(\cup_{n=1}^{\infty} E_n, F) = \cap_{n=1}^{\infty} (E_n, F)$,

(ii) $(E, \cap_{n=1}^{\infty} F_n) = \cap_{n=1}^{\infty} (E, F_n)$,

(iii) $(E_1 + E_2, F) = (E_1, F) \cap (E_2, F)$,

(iv) $(E_u, F_v) = {}_v(E, F)_{u^{-1}}$.

Proof. All of them are obtained directly from the definitions.

Propostion 3.2. *Let $p = (p_k)$ be a bounded sequences of positive real numbers . Then*

(i) $c(X, p) = c_0(X, p) + \{e(x) : x \in X\}$,

(ii) $M_0(X, p) = \cup_{n=1}^{\infty} \ell(X)_{(n^{-1/p_k})}$,

(iii) $\ell_\infty(X, p) = \cup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$.

Proof. Assertions (i) and (ii) are immediately obtained from the definitions. To show (iii), let $x \in \ell_\infty(X, p)$, then there is some $n \in N$ with $\|x_k\|^{p_k} \leq n$ for all $k \in N$. Hence $\|x_k\| n^{-1/p_k} \leq 1$ for all $k \in N$, so that $x \in \ell_\infty(X)_{(n^{-1/p_k})}$. On the other hand, if $x \in \cup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$, then there are some $n \in N$ and $M > 1$ such that $\|x_k\| n^{-1/p_k} \leq M$ for every $k \in N$. Then we have $\|x_k\|^{p_k} \leq nM^{p_k} \leq nM^\alpha$ for all $k \in N$, where $\alpha = \sup_k p_k$. Hence $x \in \ell_\infty(X, p)$. □

The next proposition give a relationship between the $\beta-$ dual of vector-valued and scalar-valued sequence spaces.

Proposition 3.3 *Let X be a Banach space and F a normal scalar-valued sequence space and define $F(X) = \{(x_k) \in W(X) : (\|x_k\|) \in F\}$. then for $(f_k) \subset X'$, the topological dual of X, $(f_k) \in F(X)^\beta$ if and only if $(\|f_k\|) \in F^\beta$.*

Proof. If $(\|f_k\|) \in F^\beta$, then for $x = (x_k) \in F(X)$ we have $\sum_{k=1}^{\infty} |f_k(x_k)| \leq \sum_{k=1}^{\infty} \|f_k\| \|x_k\| < \infty$, so that $x \in F(X)^\beta$.

Conversely, suppose that $(f_k) \in F(X)^\beta$ and $a = (a_k) \in F$. Since F is normal, $(|a_k|) \in F$. For each $k \in N$, we can choose $x_k \in X$ such that $\|x_k\| = 1$ and $|f_k(x_k)| \geq \dfrac{\|f_k\|}{2}$. Let $y = (a_k x_k)$, then $y \in F(X)$. Choose a sequence (t_k) of scalars such that $|t_k| \leq 1$ and $f_k(t_k a_k x_k) = |f_k(x_k)||a_k|$ for all $k \in N$. Since F is normal, $(t_k y_k) \in F(X)$, so we obtain that $\sum_{k=1}^{\infty} f_k(t_k y_k)$ converges. This implies $\sum_{k=1}^{\infty} \|f_k\| |a_k| \leq 2 \sum_{k=1}^{\infty} |f_k(x_k)||a_k| < \infty$. It follows that $(\|f_k\|) \in F^\beta$.

By using Proposition 3.3, the following results are obtained immediately from Lemma 2.1 - 2.4, respectively.

Proposition 3.4 *If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$, then*

$$\ell(X,p)^\beta = \{(f_k) \subset X' : \sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-t_k} < \infty \text{ for some } M \in N\}$$

where $1/p_k + 1/t_k = 1$ for all $k \in N$.

Proposition 3.5 *If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in N$, then $\ell(X,p)^\beta = \ell_\infty(X', p)$.*

Proposition 3.6 *If $p = (p_k)$ is a bounded sequence of positive real numbers, then*

$$\ell_\infty(X,p)^\beta = \{(f_k) \subset X' : \sum_{k=1}^{\infty} \|f_k\| n^{1/p_k} < \infty \text{ for all } n \in N\}.$$

Proposition 3.7 *If $p = (p_k)$ is a bounded sequence of positive real numbers, then $c_0(X,p)^\beta = M_0(X', p)$.*

4. Main Results

We begin with the following useful result.

Theorem 4.1. *Let $q = (q_k)$ be a bounded sequence of positive real numbers and let E be a normal $X-$valued sequence space which is an FK-space containing $\Phi(X)$. Then*

$$(E, c(q)) = (E, c_0(q)) \oplus (E, < e >).$$

To prove this theorem, we need the following two lemmas.

Lemma 4.1. *Let E be an X-valued sequence space which is an FK-space containing $\Phi(X)$. Then for each $k \in N$, the mapping $T_k : X \to E$, defined by $T_k x = e^k(x)$, is continuous.*

Proof. For each $k \in N$, we have that $V = \{e^k(x) : x \in X\}$ is a closed subspace of E, so it is an FK-space. Since E is a K-space, the coordinate mapping $p_k : V \to X$ is continuous and bijective. It follows from the open mapping theorem that p_k is open, hence, $p_k^{-1} : X \to V$ is continuous. It follows that T_k is continuous because $T_k = p_k^{-1}$. □

Lemma 4.2. *If E and F are scalar-valued sequence spaces such that E is normal containing ϕ, F is an FK-space and there is a subsequence (n_k) with $x_{n_k} \to 0$ as $k \to \infty$ for all $x = (x_n) \in F$, then $(E, F \oplus < e >) = (E, F) \oplus (E, < e >)$.*

Proof. See [3, Proposition 3.1(vi)]. □

Proof of Theorem 4.1 Since $c(q) = c_0(q) \oplus < e >$, it is clear that $(E, c_0(q)) + (E, < e >) \subseteq (E, c_0(q) \oplus < e >) = (E, c(q))$. Moreover, if $A \in (E, c_0(q)) \bigcap (E, < e >)$, then $A \in (E, c_0(q) \bigcap < e >)$, so that $A \in (E, 0)$, which implies that $A = 0$ because E contain $\Phi(X)$. Hence $(E, c_0(q)) + (E, < e >)$ is a direct sum. Now, we will show that $(E, c(q)) \subseteq (E, c_0(q)) \oplus (E, < e >)$. Let $A = (f_k^n) \in (E, c(q)) =$

$(E, c_0(q) \oplus < e >)$. For $x \in X$ and $k \in N$, we have $(f_k^n(x))_{n=1}^{\infty} = Ae^k(x) \in c_0(q) \oplus < e >$, so that there exist unique $(b_k^n(x))_{n=1}^{\infty} \in c_0(q)$ and $(c_k^n(x))_{n=1}^{\infty} \in < e >$ with

$$(f_k^n(x))_{n=1}^{\infty} = (b_k^n(x))_{n=1}^{\infty} + (c_k^n(x))_{n=1}^{\infty}. \tag{4.1}$$

For each $n, k \in N$, let g_k^n and h_k^n be the functionals on X defined by

$$g_k^n(x) = b_k^n(x) \text{ and } h_k^n(x) = c_k^n(x) \text{ for all } x \in X.$$

Clearly, g_k^n and h_k^n are linear, and by (4.1)

$$f_k^n = g_k^n + h_k^n \quad \text{for all } n, k \in N. \tag{4.2}$$

Note that $c_0(q) \oplus < e >$ is an FK-space in its direct sum topology. By Zeller's theorem, $A : E \to c_0(q) \oplus < e >$ is continuous. For each $k \in N$, let $T_k : X \to E$ be defined by $T_k(x) = e^k(x)$. By Lemma 4.1, we have that T_k is continuous for all $k \in N$. Since the projection P_1 of $c_0(q) \oplus < e >$ onto $c_0(q)$ and the projection P_2 of $c_0(q) \oplus < e >$ onto $< e >$ are continuous and $g_k^n = p_n \circ P_1 \circ A \circ T_k$ and $h_k^n = p_n \circ P_2 \circ A \circ T_k$ for all $n, k \in N$, we obtain that g_k^n and h_k^n are continuous, so $g_k^n, h_k^n \in X'$ for all $n, k \in N$. Let $B = (g_k^n)$ and $C = (h_k^n)$. By (4.1) and (4.2), we have $A = B + C$, $B = (g_k^n) \in (\Phi(X), c_0(q))$ and $C = (h_k^n) \in (\Phi(X), < e >)$. We will show that $B \in (E, c_0(q))$ and $C \in (E, < e >)$. To do this, let $x = (x_k) \in E$. Then for $\alpha = (\alpha_k) \in \ell_\infty$, we have $\|\alpha_k x_k\| = |\alpha_k|\|x_k\| \leq \|Mx_k\|$, where $M = \sup_k |\alpha_k|$. Then the normality of E implies that $(\alpha_k x_k) \in E$. Hence $(f_k^n(x_k))_{n,k} \in (\ell_\infty,\ c_0(q) \oplus < e >)$, moreover, we have $(g_k^n(x_k))_{n,k} \in (\Phi,\ c_0(q))$, $(h_k^n(x_k))_{n,k} \in (\Phi,\ < e >)$, and $(f_k^n(x_k))_{n,k} = (g_k^n(x_k))_{n,k} + (h_k^n(x_k))_{n,k}$. Since ℓ_∞ is normal containing ϕ and $c_0(q) \subseteq c_0$, it follows from Lemma 4.2 that $(g_k^n(x_k))_{n,k} \in (\ell_\infty,\ c_0(q))$ and $(h_k^n(x_k))_{n,k} \in (\ell_\infty,\ < e >)$. This implies that $Bx \in c_0(q)$ and $Cx \in < e >$, so we have $B \in (E, c_0(q))$ and $C \in (E, < e >)$, hence $A \in (E,\ c_0(q)) \oplus (E,\ < e >)$. This completes the proof. □

Theorem 4.2. *Let $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : \ell_\infty(X) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| < \infty$,

(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for every $k, m \in N$ and

(3) for each $m \in N$, $\sum_{j>k} m^{1/q_n}\|f_j^n - f_j\| \to 0$ as $k \to \infty$ uniformly on $n \in N$.

Proof. Necessity. Let $A \in (\ell_\infty(X), c(q))$. It follows from Theorem 4.1 that $A = B + C$, where $B \in (\ell_\infty(X),\ c_0(q))$ and $C \in (\ell_\infty(X),\ < e >)$. Then there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that $C = (f_k)_{n,k}$ and $B = (f_k^n - f_k)_{n,k} \in (\ell_\infty(X),\ c_0(q))$, which implies that $(f_k) \in \ell_\infty(X)^\beta$, so (1) is obtained by Proposition 3.6. Since $c_0(q) = \cap_{m=1}^{\infty} c_{0(m^{1/q_k})}$ (by [2, Theorem 0 (i)]), we have by Proposition 3.1 (ii) and (iv) that for each $m \in N$, $\left(m^{1/q_n}(f_k^n - f_k)_{n,k}\right) : \ell_\infty(X) \to c_0$. Hence, (2) and (3) are obtained by Lemma 2.6.

Sufficiency. Suppose that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that conditions (1), (2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. It is obvious that $A = B + C$. By condition (2) and (3), we obtain by Lemma 2.6 and Proposition 3.1(ii) and (iv) that $B \in (\ell_\infty(X), c_0(q))$. By Proposition 3.6 , condition (1) implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X)$, which implies that $C \in (\ell_\infty(X), < e >)$. Hence, we obtain by Theorem 4.1 that $A \in (\ell_\infty(X), c(q))$. This completes the proof. □

Theorem 4.3. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : \ell_\infty(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that (1), (2) and (3) are satisfied, where*

(1) for each $m \in N$, $\sum_{k=1}^{\infty} \|f_k\| m^{1/p_k} < \infty$,

(2) $r^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for every $k, r \in N$ and

(3) for each $m, r \in N$, $r^{1/q_n} \sum_{j>k} m^{1/p_j} \|f_j^n - f_j\| \to 0$ as $k \to \infty$ uniformly on $n \in N$.

Moreover, (3) is equivalent to (3′), where

(3′) for each $m \in N$, $\lim_{k\to\infty} \sup_n \left(\sum_{j>k} m^{1/p_j} \|f_j^n - f_j\|\right)^{q_n} = 0$.

Proof. Necessity. Suppose that $A : \ell_\infty(X,p) \to c(q)$. By Theorem 4.1, $A = B + C$, where $B \in (\ell_\infty(X,p), c_0(q))$ and $C \in (\ell_\infty(X,p), < e >)$. Then there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that $C = (f_k)_{n,k}$ and $B = (f_k^n - f_k) \in (\ell_\infty(X,p), c_0(q))$. Since $C = (f_k)_{n,k} : \ell_\infty(X,p) \to < e >$, it implies by Proposition 3.6 that (1) holds. Since $c_0(q) = \cap_{m=1}^{\infty} c_{0(m^{1/q_k})}$, we have by Proposition 3.1 (ii) that for each $r \in N$, $\left(r^{1/q_n}(f_k^n - f_k)\right)_{n,k} : \ell_\infty(X,p) \to c_0$. Hence , (2) and (3) holds by an application of Lemma 2.6.

Sufficiency. Suppose that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that condition (1), (2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. It is obvious that $A = B + C$. By condition (2) and (3), we obtain by Lemma 2.6 and Proposition 3.1(ii) and (iv) that $B \in (\ell_\infty(X,p), c_0(q))$. By Proposition 3.6 , condition (1) implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X,p)$, which implies that $C \in (\ell_\infty(X,p), < e >)$. Hence, we obtain by Theorem 4.1 that $A \in (\ell_\infty(X,p), c(q))$.

Now we shall show that (3) and (3′) are equivalent. Suppose (3) holds and let $\varepsilon > 0$. Choose $r \in N$ such that $1/r < \varepsilon$. By (3), there exists $k_0 \in N$ such that

$$r^{1/q_n} \sum_{j>k} m^{1/p_j} \|f_j^n - f_j\| < 1 \text{ for all } k \geq k_0 \text{ and all } n \in N,$$

which implies that

$$\sup_n \left(\sum_{j>k} m^{1/p_j} \|f_j^n - f_j\|\right)^{q_n} \leq 1/r < \varepsilon \quad \text{for } k \geq k_0,$$

hence, (3′) holds.

Conversely, assume that (3′) holds. Let $m, r \in N$ and $0 < \varepsilon < 1$. Then there exists $k_0 \in N$ such that

$$\sup_n \left(\sum_{j>k} m^{1/p_j} \|f_j^n - f_j\|\right)^{q_n} < \varepsilon^H / r \quad \text{for all } k \geq k_0$$

where $H = sup_n q_n$. This implies that

$$r^{1/q_n} \sum_{j>k} m^{1/p_j} \|f_j^n - f_j\| < \varepsilon^{H/q_n} < \varepsilon \quad \text{for all } k \geq k_0 \text{ and all } n \in N$$

hence, (3) holds. □

Theorem 4.4. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c_0(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that (1), (2), and (3) are satisfied, where*

(1) $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in N$,

(2) $m^{\frac{1}{q_n}}(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for every $m, k \in N$ and

(3) for each $m \in N$, $\sup_n \left(m^{1/q_n} \sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k}\right) \rightarrow 0$ as $r \rightarrow \infty$.
Moreover, (3) is equivalent to (3′) where
(3′) $\lim_{r\to\infty} \sup_n \left(\sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k}\right)^{q_n} = 0$.

Proof. Necessity. Suppose that $A : c_0(X,p) \to c(q)$. By Theorem 4.1, we have $A = B + C$, where $B \in (c_0(X,p), c_0(q))$ and $C \in (c_0(X,p), < e >)$. It follows that there is a sequence $(f_k) \subset X'$ such that $C = (f_k)_{n,k}$ and $B = (f_k^n - f_k)_{n,k}$. Since $c_0(q) = \cap_{r=1}^{\infty} c_{0(r^{1/q_k})}$, it follows from Proposition 3.1 (ii) and (iv) that for each $m \in N$, $(m^{1/q_n}(f_k^n - f_k))_{n,k} \in (c_0(X,p),\ c_0)$, hence, conditions (2) and (3) hold by using the result from Lemma 2.5. Since $C = (f_k)_{n,k} \in (c_0(X,p), < e >)$, we have that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = x_k \in c_0(X,p)$, so that $(f_k) \in c_0(X,p)^{\beta}$, hence, by Proposition 3.7, we obtain that there exists $M \in N$ such that $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$. Hence, (1) is obtained.

Sufficiency. Assume that there is a sequence $(f_k) \subset X'$ such that conditions (1),(2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. Then $A = B + C$. By conditions (2)and (3), we obtain from Proposition 3.1(ii) and (iv) and Lemma 2.5 that $B \in \left(c_0(X,p), c_0(q)\right)$. The condition (1) implies by Proposition 3.7 that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X,p)$, so that $C \in (c_0(X,p), < e >)$. Hence, by Theorem 4.1, we obtain that $A \in (c_0(X,p), c(q))$.

Now, we shall show that conditions (3) and (3′) are equivalent. To do this, suppose that (3) holds and let $\varepsilon > 0$. Choose $m \in N$, $1/m < \varepsilon$. From (3), there is $r_0 \in N$ such that

$$\sup_n m^{1/q_n} \sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} \leq 1 \text{ for all } r \geq r_0.$$

This implies that $\sup_n \left(\sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k}\right)^{q_n} \leq 1/m < \varepsilon$ for all $r \geq r_0$. Hence, (3′) holds.

Conversely, suppose that (3′) holds. Let $m \in N$ and $0 < \varepsilon < 1$. Then there exists $r_0 \in N$ such that $\sup_n \left(\sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k}\right)^{q_n} < \varepsilon^H/m$ for all $r \geq r_0$, where $H = sup_n q_n$. Hence, we have

$$m^{1/q_n} \sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} < \varepsilon^{H/q_n} \leq \varepsilon \text{ for all } r \geq r_0 \text{ and } n \in N,$$

so that (3) holds. This completes the proof. □

Theorem 4.5. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that (1), (2), (3) and (4) are satisfied, where*
(1) $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in N$,
(2) for each $m, k \in N$, $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$,
(3) for each $m \in N$, $\sup_n\ m^{1/q_n} \sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} \to 0$ as $r \to \infty$ and
(4) $(\sum_{k=1}^{\infty} f_k^n(x))_{n=1}^{\infty} \in c(q)$ for all $x \in X$.
Moreover, (3) is equivalent to (3′) where
(3′) $\lim_{r\to\infty} \sup_n \left(\sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k}\right)^{q_n} = 0$.

Proof. Since $c(X,p) = c_0(X,p) + \{e(x) : x \in X\}$ (Proposition 3.2 (i)), it follows from Proposition 3.1(iii) that $A \in (c(X,p), c(q))$ if and only if $A \in (c_0(X,p), c(q))$ and $A \in (\{e(x) : x \in X\}, c(q))$. By Theorem 4.4, we have $A \in (c_0(X,p), c(q))$ if and only if conditions (1)-(3) hold and it is clear that $A \in \left(\{e(x) : x \in X\}, c(q)\right)$ if and only if (4) holds. We have by Theorem 4.4 that (3) and (3′) are equivalent. Hence, the theorem is proved. □

Wu and Liu (Lemma 2.7) have given a characterization of an infinite matrix A such that $A \in (\ell(X,p), c_0)$ when $p_k > 1$ for all $k \in N$. By applications of Proposition 3.1(ii) and (iv), Proposition 3.4, and Theorem 4.1, and using the fact that $c_0(q) = \cap_{m=1}^{\infty} c_{0(m^{1/q_k})}$, we obtain the following result.

Theorem 4.6. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in N$ and $1/p_k + 1/t_k = 1$ for all $k \in N$, and let $A = (f_k^n)$ be an infinite matrix. Then $A : \ell(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-t_k} < \infty$ for some $M \in N$,

(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for all $m, k \in N$ and

(3) for each $m \in N$, $\sum_{k=1}^{\infty} m^{t_k/q_n} \|f_k^n - f_k\|^{t_k} r^{-t_k} \to 0$ as $r \to \infty$ uniformly on $n \in N$.

By using Lemma 2.8, Proposition 3.1(ii) and (iv), Proposition 3.5 and Theorem 4.1, we also obtain the following result.

Theorem 4.7. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k \le 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sup_k \|f_k\|^{p_k} < \infty$,

(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for all $m, k \in N$ and

(3) $\sup_{n,k} m^{p_k/q_n} \|f_k^n - f_k\|^{p_k} < \infty$ for all $m \in N$.

When $p_k = 1$ for all $k \in N$, we obtain the following.

Corollary 4.8. *Let $q = (q_k)$ be a bounded sequence of positive real numbers and let $A = (f_k^n)$ be an infinite matrix. Then $A : \ell_1(X) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sup_k \|f_k\| < \infty$,

(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for all $m, k \in N$ and

(3) $\sup_{n,k} m^{1/q_n} \|f_k^n - f_k\| < \infty$ for every $m \in N$.

Theorem 4.9. *Let $p = (p_k)$ be a bounded sequence of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : M_0(X,p) \to c(q)$ if and only if there is a sequence (f_k) of bounded linear functionals on X such that*

(1) $\sup_k m^{1/p_k} \|f_k\| < \infty$ for all $m \in N$,

(2) for each $m, r \in N$, $r^{1/q_n} m^{1/p_k}(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for all $k \in N$ and

(3) for each $m, r \in N$, $\sup_{n,k} r^{1/q_n} m^{1/p_k} \|f_k^n - f_k\| < \infty$.

Proof. It follows from Theorem 4.1 that $A \in (M_0(X,p), c_0(q) \oplus \langle e \rangle)$ if and only if there is a sequence (f_k) of bounded linear functionals on X such that $A = B + (f_k)_{n,k}$ where $B : M_0(X,p) \to c_0(q)$ and $(f_k)_{n,k} : M_0(X,p) \to \langle e \rangle$. Since $B = (f_k^n - f_k)_{n,k}$ and $M_0(X,p) = \cup_{m=1}^{\infty} \ell_1(X)_{(m^{-1/p_k})}$ (by Proposition 3.2 (ii)), we have by Proposition 3.1 (i) and (iv) that $B : M_0(X,p) \to c_0(q)$ if and only if $\left(m^{1/p_k}(f_k^n - f_k)\right)_{n,k} : \ell_1(X) \to c_0(q)$ for all $m \in N$. Since $c_0(q) = \cap_{r=1}^{\infty} c_{0(r^{1/q_k})}$, by Proposition 3.1 (ii) and (iv), we have $\left(m^{1/p_k}(f_k^n - f_k)\right)_{n,k} : \ell_1(X) \to c_0(q)$ if and only if $\left(r^{1/q_n}m^{1/p_k}(f_k^n - f_k)\right)_{n,k} : \ell_1(X) \to c_0$ for all $r \in N$. By Lemma 2.8 , we have

$\left(r^{1/q_n}m^{1/p_k}(f_k^n - f_k)\right)_{n,k} : \ell_1(X) \to c_0$ if and only if

(a) $r^{1/q_n}m^{1/p_k}(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for all $k \in N$ and

(b) $\sup_{n,k} r^{1/q_n}m^{1/p_k}\|f_k^n - f_k\| < \infty$.

By Proposition 3.1 (i) and (iv), we have $(f_k)_{n,k} : M_0(X,p) \to < e >$ if and only if $(m^{1/p_k}f_k)_{n,k} : \ell_1(X) \to < e >$ for all $m \in N$. By Proposition 3.5, we obtain that $(m^{1/p_k}f_k)_{n,k} : \ell_1(X) \to < e >$ if and only if $\sup_k m^{1/p_k}\|f_k\| < \infty$. Hence, the theorem is proved. □.

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ON PROPERTY (H) AND ROTUNDITY OF DIFFERENCE SEQUENCE SPACES.

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ABSTRACT. In this paper, we define a modular on difference sequence space $\ell(\Delta, p)$ and consider it equipped with the Luxemburg norm induced by the modular, where $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$. The main purpose of this paper is to show that $\ell(\Delta, p)$ has property (H) and we also show that $\ell(\Delta, p)$ is rotund if and only if $p_k > 1$ for all $k \in \mathbb{N}$.

1. INTRODUCTION.

Convexity properties of Banach spaces is an important topic in functional analysis and plays an important role in infinite dimensional holomorphy. In order to study the geometric properties of Banach spaces, Clarkson [5] introduced the very important class of rotund (strictly convex) spaces. Since Clarkson's paper, many authors have defined and studied various convexity properties lying between uniform convexity and rotundity (see [2, 3, 5, 12, 14, 17].)

Among the geometrical properties of Banach spaces, property **(H)** has proved to be particularly important and has been studied by various authors. Criteria for property (H) in Orlicz spaces and Orlicz sequence spaces were given by S. Chen and Y. Wang [4] and C. Wu, S. Chen and Y. Wang [20]. R. Pluciennik, T. Wang and Y. Zhang [19] obtained necessary and sufficient conditions for H- points and denting points in Orlicz sequence spaces.

In [7], criteria are given for Musielak-Orlicz sequence spaces to have property (H).

In this paper, we introduce the difference sequence space $\ell(\Delta, p)$, when $p = (p_k)$ is a bounded sequence of positive real number with $p_k \geq 1$ for all $k \in \mathbb{N}$, and consider it equipped the Luxemburg norm. We show that $\ell(\Delta, p)$ has property **(H)** and establish criteria for rotundity.

We begin by introducing the basic notations and definitions. In the following, Let $\mathbb{R}$ be the real line and $\mathbb{N}$ the set of natural numbers.

For a Banach space X, we denote by $S(X)$ and $B(X)$ the unit sphere and unit ball of X, respectively. A point $x_0 \in S(X)$ is called:

a) an *extreme point* if for every $x, y \in S(X)$ the equality $2x_0 = x + y$ implies $x = y$;

b) an *H-point* if for any sequence (x_n) in X such that $\|x_n\| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x_0 (written $x_n \xrightarrow{w} x_0$) implies that $\|x_n - x\| \to 0$ as $n \to \infty$;

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A Banach space X is said to be *rotund* (R), if every point of $S(X)$ is an extreme point. X is said to posses property (H) provided every point of $S(X)$ is an H-point.

For these geometric notions and their role in Mathematics we refer to the monographs [2], [8], and [17]. Some of them were studied for Orlicz spaces in [3], [6], [9], [10], [11], [19], and [20].

Let X be a real vector space. A functional $\varrho : X \to [0, \infty]$ is called a *modular* if it satisfies the conditions

(i) $\varrho(x) = 0$ if and only if $x = 0$;

(ii) $\varrho(\alpha x) = \varrho(x)$ for all scalars α with $|\alpha| = 1$;

(iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$, for all $x, y \in X$ and all α, $\beta \geq 0$ with $\alpha + \beta = 1$.

The modular ϱ is called *convex* if

(iv) $\varrho(\alpha x+\beta y) \leq \alpha\varrho(x)+\beta\varrho(y)$, for all $x, y \in X$ and all α, $\beta \geq 0$ with $\alpha+\beta = 1$.

If ϱ is a modular in X, we define

$$X_\varrho = \{x \in X : \lim_{\lambda\to 0^+} \varrho(\lambda x) = 0 \},$$

$$\text{and} \quad X_\varrho^* = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0 \}.$$

It is clear that $X_\varrho \subseteq X_\varrho^*$. If ϱ is a convex modular, we define

$$\|x\| = \inf\{\lambda > 0 : \ \varrho\left(\frac{x}{\lambda}\right) \leq 1 \}. \tag{1.1}$$

Orlicz [18] proved that if ϱ is a convex modular in X, then $X_\varrho = X_\varrho^*$ and $\|.\|$ is a norm on X_ϱ for which it is a Banach space. The norm $\|.\|$ defined as in (1.1) is called the Luxemburg norm.

A modular ϱ on X is called

(a) *right-continuous* if $\lim_{\lambda\to 1^+} \varrho(\lambda x) = \varrho(x)$ for all $x \in X_\varrho$,

(b) *left-continuous* if $\lim_{\lambda\to 1^-} \varrho(\lambda x) = \varrho(x)$ for all $x \in X_\varrho$,

(c) *cotinuous* if it is both right-continuous and left-continuous.

The following known results gave some relationships between the modular ϱ and the Luxemburg norm $\|.\|$ on X_ϱ.

Theorem 1.1 *Let ϱ be a convex modular on X and let $x \in X_\varrho$ and (x_n) a sequence in X_ϱ. Then $\|x_n - x\| \to 0$ as $n \to \infty$ if and only if $\varrho(\lambda(x_n - x)) \to 0$ as $n \to \infty$ for every $\lambda > 0$.*

Proof. See [16, Theorem 1.3].

Theorem 1.2 *Let ϱ be a continuous convex modular on X. Then*

(i) *$\|x\| < 1$ if and only if $\varrho(x) < 1$.*

(ii) *$\|x\| \leq 1$ if and only if $\varrho(x) \leq 1$.*

(iii) *$\|x\| = 1$ if and only if $\varrho(x) = 1$.*

Proof. See [16, Theorem 1.4].

Let us denoted by ℓ^0 the space of all real sequences and let $p = (p_k)$ be a bounded sequence of positive real numbers. In [13], Kizmaz introduced the sequence spaces $\ell_\infty(\Delta)$, $c_0(\Delta)$ and $c(\Delta)$ by considering the difference sequence $\Delta x = (x_k - x_{k+1})_{k=1}^\infty$ for any sequence $x \in l^0$, where ℓ_∞, c_0 an c are Banach spaces of bounded, null and convergent sequences, respectively. In [1], these sequence spaces were extended to $\ell_\infty(\Delta,p)$, $c_0(\Delta,p)$ and $c(\Delta,p)$, where, for example

$$\ell_\infty(\Delta,p)=\{x\in l^0\ :\ \Delta x\in \ell_\infty(p)\ \}$$

with

$$\ell_\infty(p)=\{x\in l^0\ :\ \sup_k |x_k|^{p_k}<\infty\ \}.$$

In [1] and [13] the authers determined the Köthe-Töeplitz and generalized Köthe-Töeplitz duals of these spaces and consider various matrix transformations.

In this paper we introduce the space $\ell(\Delta,p)$ defined analogously as follows,

$$\ell(\Delta,p)=\{x\in l^0\ :\ \Delta x\in \ell(p)\ \},$$

where

$$\ell(p)=\{x\in l^0\ :\ \sum_{k=1}^{\infty}|x(k)|^{p_k}<\infty\ \}.$$

and study some of its geometric properties.

For details of the spaces $\ell_\infty(p)$ and $\ell(p)$, we refer to [15].

For $x\in \ell(\Delta,p)$, we define

$$\varrho_p(x)=|x(1)|+\sum_{k=1}^{\infty}|x(k)-x(k+1)|^{p_k}$$

If $p_k\geq 1$ for all $k\in\mathbb{N}$, we have, by the convexity of the functions $t\mapsto |t|^{p_k}$ for each $k\in\mathbb{N}$, that ϱ_p is a convex modular on $\ell(\Delta,p)$. We consider $\ell(\Delta,p)$ equipped with the Luxemburg norm given by

$$\|x\|=inf\,\{\varepsilon>0:\varrho_p(\frac{x}{\varepsilon})\leq 1\}.$$

A normed sequence space S is said to be a *K-space* if each coordinate mapping P_k, defined by $P_k(x)=x_k$, is continuous. If S is both a Banach and a K-space, it is called a BK-space.

Throughout this paper we let $M=\sup_k\ p_k$ and assume that $p_k\geq 1$ for all $k\in\mathbb{N}$.

2. Main Results

We begin by giving some basic properties of the modular on the space $\ell(\Delta,p)$.

Proposition 2.1 *For $x\in\ell(\Delta,p)$, the modular ϱ_p on $\ell(\Delta,p)$ satisfies the following:*

(i) if $0<a\leq 1$, then $a^M\varrho_p(\frac{x}{a})\leq\varrho_p(x)$ and $\varrho_p(ax)\leq a\varrho_p(x)$,

(ii) if $a\geq 1$, then $\varrho_p(x)\leq a^M\varrho_p(\frac{x}{a})$,

(iii) if $a\geq 1$, then $\varrho_p(x)\leq a\varrho_p(x)\leq\varrho_p(ax)$.

Proof. It is obvious that (iii) is satisfied by the convexity of ϱ_p. It remains to prove (i) and (ii).

For $0<a\leq 1$, we have

$$\varrho_p(x)=|x(1)|+\sum_{k=1}^{\infty}|x(k)-x(k+1)|^{p_k}$$

$$
\begin{aligned}
&= a\left|\frac{x(1)}{a}\right| + \sum_{k=1}^{\infty}\left|\frac{a(x(k)-x(k+1))}{a}\right|^{p_k} \\
&\geq a^M\left|\frac{x(1)}{a}\right| + a^M\sum_{k=1}^{\infty}\left|\frac{x(k)-x(k+1)}{a}\right|^{p_k} \\
&= a^M\varrho_p(\frac{x}{a}).
\end{aligned}
$$

It follows by the convexity of ϱ that $\varrho_p(ax) \leq a\varrho_p(x)$, hence (i) is satisfied. Now, suppose that $a \geq 1$. Then $\frac{1}{a} \leq 1$. It follows from (i) that

$$\left(\frac{1}{a}\right)^M \varrho_p(x) = \left(\frac{1}{a}\right)^M \varrho_p\left(\frac{x/a}{1/a}\right) \leq \varrho_p\left(\frac{x}{a}\right),$$

so that $\varrho_p(x) \leq a^M\varrho_p\left(\frac{x}{a}\right)$, hence (ii) is obtained. □

Proposition 2.2 *The modular ϱ_p on $\ell(\Delta,p)$ is continuous.*
Proof. For $\lambda > 1$, by Proposition 2.1 (ii) and (iii), we have

$$\varrho_p(x) \leq \lambda\varrho_p(x) \leq \varrho_p(\lambda x) \leq \lambda^M\varrho_p(x) \tag{2.1}$$

By taking $\lambda \to 1^+$ in (2.1), we have $\lim_{\lambda\to 1^+}\varrho_p(\lambda x) = \varrho_p(x)$. Thus ϱ_p is right-continuous. If $0 < \lambda < 1$, by Proposition 2.1 (i), we have

$$\lambda^M\varrho_p(x) \leq \varrho_p(\lambda x) \leq \lambda\varrho_p(x) \tag{2.2}$$

By taking $\lambda \to 1^-$ in (2.2), we have that $\lim_{\lambda\to 1^-}\varrho_p(\lambda x) = \varrho_p(x)$, hence, ϱ_p is left-continuous. Thus ϱ_p is continuous. □

Next, we give some relationships between the modular ϱ_p and the Luxemburg norm on $\ell(\Delta,p)$.

Proposition 2.3 *For any $x \in \ell(\Delta,p)$, we have*
(i) if $\|x\| < 1$, then $\varrho_p(x) \leq \|x\|$,
(ii) if $\|x\| > 1$, then $\varrho_p(x) \geq \|x\|$,
(iii) $\|x\| = 1$ if and only if $\varrho_p(x) = 1$,
(iv) $\|x\| < 1$ if and only if $\varrho_p(x) < 1$,
(v) $\|x\| > 1$ if and only if $\varrho_p(x) > 1$,
(vi) if $0 < a < 1$ and $\|x\| > a$, then $\varrho_p(x) > a^M$, and
(vii) if $a \geq 1$ and $\|x\| < a$, then $\varrho_p(x) < a^M$.
Proof. (i) Let $\epsilon > 0$ be such that $0 < \epsilon < 1 - \|x\|$, so $\|x\| + \epsilon < 1$. By definition of $\|\cdot\|$, there exists $\lambda > 0$ such that $\|x\| + \epsilon > \lambda$ *and* $\varrho(\frac{x}{\lambda}) \leq 1$. From Proposition 2.1(i) and (iii), we have

$$
\begin{aligned}
\varrho_p(x) &\leq \varrho_p\left(\frac{(\|x\|+\epsilon)}{\lambda}x\right) \\
&= \varrho_p\left((\|x\|+\epsilon)\frac{x}{\lambda}\right)
\end{aligned}
$$

$$\leq (\|x\| + \epsilon)\varrho(\frac{x}{\lambda})$$
$$\leq \|x\| + \epsilon,$$

which implies that $\varrho_p(x) \leq \|x\|$, so (i) is satisfied.

(ii) Let $\epsilon > 0$ be such that $0 < \epsilon < \dfrac{\|x\| - 1}{\|x\|}$, then $1 < (1-\epsilon)\|x\| < \|x\|$. By definition of $\|\cdot\|$ and by Proposition 2.1 (i), we have

$$1 < \varrho_p\left(\frac{x}{(1-\epsilon)\|x\|}\right)$$
$$\leq \frac{1}{(1-\epsilon)\|x\|}\varrho_p(x),$$

so $(1-\epsilon)\|x\| < \varrho_p(x)$ for all $\epsilon \in (0, \dfrac{\|x\|-1}{\|x\|})$. This implies that $\|x\| \leq \varrho_p(x)$, hence (ii) is obtained.

Since ϱ_p is continuous (Proposition 2.2) , (iii) and (iv) follow directly from Theorem 1.2.

(iv) follows directly from (i) and (iii).

(v) follows from (iii) and (iv).

(vi) Suppose $0 < a < 1$ and $\|x\| > a$. Then $\left\|\dfrac{x}{a}\right\| > 1$. By (v), we have $\varrho_p(\dfrac{x}{a}) > 1$. Hence, by Proposition 2.1(i), we obtain that $\varrho_p(x) \geq a^M \varrho_p(\frac{x}{a}) > a^M$.

(vii) Suppose $a \geq 1$ and $\|x\| < a$. Then $\left\|\dfrac{x}{a}\right\| < 1$. By (iv), we have $\varrho_p(\dfrac{x}{a}) < 1$. If $a = 1$, it is obvious that $\varrho_p(x) < 1 = a^M$. If $a > 1$, by Proposition 2.1(ii), we obtain that $\varrho_p(x) \leq a^M \varrho_p(\dfrac{x}{a}) < a^M$. □

Proposition 2.4 *Let (x_n) be a sequence of elements of $\ell(\Delta, p)$.*

(i) If $\|x_n\| \to 1$ as $n \to \infty$, then $\varrho_p(x_n) \to 1$ as $n \to \infty$.

(ii) $\|x_n\| \to 0$ as $n \to \infty$ if and only if $\varrho_p(x_n) \to 0$ as $n \to \infty$.

Proof. (i) Suppose $\|x_n\| \to 1$ as $n \to \infty$. Let $\epsilon \in (0,1)$. Then there exists $N \in \mathbb{N}$ such that $1 - \epsilon < \|x_n\| < 1 + \epsilon$ for all $n \geq N$. By Proposition 2.3 (vi) and (vii), we have $(1-\epsilon)^M < \varrho_p(x_n) < (1+\epsilon)^M$ for all $n \geq N$, which implies that $\varrho_p(x_n) \to 1$ as $n \to \infty$.

(ii) The only part of (ii) is true by Theorem 1.1, so we need to show only the if part. Suppose $\|x_n\| \not\to 0$ as $n \to \infty$. Then there is an $\epsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By Proprosition 2.3 (vi), we have $\varrho_p(x_{n_k}) > \epsilon^M$ for all $k \in \mathbb{N}$. This implies $\varrho_p(x_n) \not\to 0$ as $n \to \infty$. □

Next, we shall show that $\ell(\Delta, p)$ has property (H). To do this, we need two lemmas.

Lemma 2.5 *The space $\ell(\Delta, p)$ is a BK-space.*

Proof. Since $\ell(\Delta, p)$ equipped with the Luxemburg norm is Banach, we need only show that $\ell(\Delta, p)$ is a $K-$space. Suppose $(x_n) \subset \ell(\Delta, p)$ suth that $x_n \to 0$ as $n \to \infty$. It follows by Proposition 2.4(ii) that $\varrho_p(x_n) \to 0$ as $n \to \infty$. This implies that

$$|x_n(1)| \to 0 \quad \text{as} \quad n \to \infty \quad \text{and}$$
$$|x_n(k) - x_n(k+1)| \to 0 \ \text{ as } n \to \infty \ \text{ for all } \ k \in \mathbb{N}.$$

By induction, we have $x_n(k) \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}$. Hence $P_k(x_n) \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}$. This implies that P_k is continuous for all $k \in \mathbb{N}$.

Lemma 2.6 *Let $x \in \ell(\Delta, p)$ and $(x_n) \subseteq \ell(\Delta, p)$. If $\varrho_p(x_n) \to \varrho_p(x)$ as $n \to \infty$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$, then $x_n \to x$ as $n \to \infty$.*

Proof. Let $\epsilon > 0$ be given. Since $\varrho_p(x) = |x(1)| + \sum_{k=1}^{\infty} |x(k) - x(k+1)|^{p_k} < \infty$, there is $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty} |x(k) - x(k+1)|^{p_k} < \frac{\epsilon}{3} \cdot \frac{1}{2^{M+1}}. \tag{2.3}$$

Since $\varrho_p(x_n) \to \varrho_p(x)$ as $n \to \infty$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\begin{aligned} \varrho_p(x_n) - \left(|x_n(1)| + \sum_{k=1}^{k_0} |x_n(k) - x_n(k+1)|^{p_k} \right) \\ < \varrho_p(x) - \left(|x(1)| + \sum_{i=1}^{k_0} |x(k) - x(k+1)|^{p_k} \right) + \frac{\varepsilon}{3 \cdot 2^M} \end{aligned} \tag{2.4}$$

and

$$|x_n(1) - x(1)| + \sum_{k=1}^{k_0} |(x_n(k) + x(k)) - (x_n(k+1) - x(k+1))|^{p_k} < \frac{\varepsilon}{3}. \tag{2.5}$$

It follows from (2.3), (2.4) and (2.5) that for $n \geq n_0$,

$$\begin{aligned} \varrho_p(x_n - x) &= |x_n(1) - x(1)| + \sum_{k=1}^{\infty} |(x_n(k) - x(k)) - (x_n(k+1) - x(k+1))|^{p_k} \\ &= |x_n(1) - x(1)| + \sum_{k=1}^{k_0} |(x_n(k) - x(k)) - (x_n(k+1) - x(k+1))|^{p_k} \\ &+ \sum_{k=k_0+1}^{\infty} |(x_n(k) - x(k)) - (x_n(k+1) - x(k+1))|^{p_k} \\ &< \frac{\varepsilon}{3} + 2^M (\sum_{k=k_0+1}^{\infty} |x_n(k) - x_n(k+1)|^{p_k} + \sum_{k=k_0+1}^{\infty} |x(k) - x(k+1)|^{p_k}) \\ &= \frac{\varepsilon}{3} + 2^M (\varrho_p(x_n) - (|x_n(1)| + \sum_{k=1}^{k_0} |x_n(k) - x_n(k+1)|^{p_k}) \\ &+ \sum_{k=k_0+1}^{\infty} |x(k) - x(k+1)|^{p_k}) \end{aligned}$$

$$< \frac{\varepsilon}{3} + 2^M(\varrho_p(x) - (|x(1)| + \sum_{k=1}^{k_0} |x(k) - x(k+1)|^{p_k}) + \frac{\varepsilon}{3 \cdot 2^M}$$
$$+ \sum_{k_0+1}^{\infty} |x(k) - x(k+1)|^{p_k})$$
$$= \frac{\varepsilon}{3} + 2^M(2 \sum_{k=k_0+1}^{\infty} |x(k) - x(k+1)|^{p_k} + \frac{\varepsilon}{3 \cdot 2^M})$$
$$< \frac{\varepsilon}{3} + 2^{M+1}\frac{\varepsilon}{3.2^{M+1}} + \frac{\varepsilon}{3}$$
$$= \varepsilon.$$

This show that $\varrho_p(x_n - x) \to 0$ as $n \to \infty$. Hence, by Proposition 2.4 (ii), we have $\|x_n - x\| \to 0$ as $n \to \infty$.

Theorem 2.7 *The space $\ell(\Delta, p)$ has property (H).*

Proof. Let $x \in S(\ell(\Delta, p))$ and $(x_n) \subset \ell(\Delta, p)$ such that $\|x_n\| \to 1$ and $x_n \xrightarrow{w} x$ as $n \to \infty$. From Proposition 2.3 (iii), we have $\varrho_p(x) = 1$, so it follows from Proposition 2.4 (i) that $\varrho_p(x_n) \to \varrho_p(x)$ as $n \to \infty$. By Lemma 2.5, we have that the coordinate mapping $P_i : \ell(\Delta, p) \to \mathbb{R}$ is continuous, so it follows that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$. Thus, we have by Lemma 2.6 that $x_n \to x$ as $n \to \infty$. □

Theorem 2.8 *The space $\ell(\Delta, p)$ is rotund if and only if $p_k > 1$ for all $k \in \mathbb{N}$.*

Proof. Necessity. Suppose that there is $k_0 \in \mathbb{N}$ such that $p_{k_0} = 1$. Let $x = (1, 1, 1, ...)$ and $y = (\underbrace{0, 0, 0..., 0}_{k_0}, 1, 1, 1, ...)$. Then $x \neq y$ and it is easy to see that

$$\varrho_p(x) = \varrho_p(y) = \varrho_p\left(\frac{x+y}{2}\right) = 1.$$

By Proposition 2.3(iii) , we have x, y and $\frac{x+y}{2} \in S(\ell(\Delta, p))$, so that $\ell(\Delta, p)$ is not rotund.

Sufficiency. Suppose that $p_k > 1$ for all $k \in \mathbb{N}$. Let $x \in S(\ell(\Delta, p))$ and $y, z \in B(\ell(\Delta, p))$ with $x = \frac{y+z}{2}$. By convexity of ϱ_p and Proposition 2.3(iii), we have

$$1 = \varrho_p(x) \leq \frac{1}{2}(\varrho_p(y) + \varrho_p(z)) \leq \frac{1}{2} + \frac{1}{2} = 1.$$

This implies that

$$\varrho_p(y) = \varrho_p(z) = 1 \tag{2.6}$$

$$\varrho_p(x) = \frac{1}{2}(\varrho_p(y) + \varrho_p(z)). \tag{2.7}$$

By (2.7), we have

$$\left|\frac{y(1)+z(1)}{2}\right| + \sum_{k=1}^{\infty}\left|\frac{(y(k)-y(k+1))+(z(k)-z(k+1))}{2}\right|^{p_k}$$

$$= \frac{1}{2}\left(|y(1)| + \sum_{k=1}^{\infty}|y(k)-y(k+1)|^{p_k}\right) + \frac{1}{2}\left(|z(1)| + \sum_{k=1}^{\infty}|z(k)-z(k+1)|^{p_k}\right)$$

$$= \frac{1}{2}(|y(1)|+|z(1)|) + \frac{1}{2}\left(\sum_{k=1}^{\infty}|y(k)-y(k+1)|^{p_k} + \sum_{k=1}^{\infty}|z(k)-z(k+1)|^{p_k}\right),$$

which implies that

$$|y(1)+z(1)| = |y(1)|+|z(1)| \quad \text{and} \tag{2.8}$$

$$\left|\frac{(y(k)-y(k+1))+(z(k)-z(k+1))}{2}\right|^{p_k} = \frac{1}{2}\left(|y(k)-y(k+1)|^{p_k} + |z(k)-z(k+1)|^{p_k}\right) \tag{2.9}$$

for all $k \in \mathbb{N}$.
Since the function $t \mapsto |t|^{p_k}$ is strictly convex for every $k \in \mathbb{N}$, we see that(2.9) implies,

$$y(k) - y(k+1) = z(k) - z(k+1) \quad \text{for all k } \in \mathbb{N}. \tag{2.10}$$

It follows from (2.6) and (2.10) that $|y(1)| = |z(1)|$. This implies by (2.8) that $y(1) = z(1)$. This, together with (2.10), yields by an inductive argument that $y(k) = z(k)$ for all $k \in \mathbb{N}$. Hence $y = z$.

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Matrix Transformations of the Maddox Vector-Valued Sequence Spaces

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Abstract. In this paper, we give the matrix characterizations from any normal vector-valued FK-space containing $\phi(X)$ into scalar-valued sequence space $c(q)$ and by applying this result, we also obtain necessary and sufficient conditions for infinite matrices mapping the sequence spaces $c_0(X,p), c(X,p), \ell_\infty(X,p), \ell(X,p), \underline{c_0}(X,p), E_r(X,p)$, and $F_r(X,p)$ into the space $c(q)$, where $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers and $r \geq 0$.

Keywords: Matrix transformations, Maddox vector-valued sequence spaces

1. Introduction

Let $(X, \|.\|)$ be a Banach space and $p=(p_k)$ a bounded sequence of positive real numbers. Let N be the set of all natural numbers, we write $x=(x_k)$ with x_k in X for all $k \in N$. The X-valued sequence spaces $c_0(X,p), c(X,p), \ell_\infty(X,p), \ell(X,p)$, $\underline{c_0}(X,p), E_r(X,p)$, and $F_r(X,p)$ are defined as

(a) $c_0(X,p)=\{x=(x_k) : \lim_{k\to\infty}\|x_k\|^{p_k}=0\}$;
(b) $c(X,p)=\{x=(x_k) : \lim_{k\to\infty}\|x_k-a\|^{p_k}=0$ for some $a \in X\}$;
(c) $\ell_\infty(X,p)=\{x=(x_k) : \sup_k\|x_k\|^{p_k}<\infty\}$;
(d) $\ell(X,p)=\{x=(x_k) : \sum_{k=1}^{\infty}\|x_k\|^{p_k}<\infty\}$;
(e) $\underline{c_0}(X,p)=\{x=(x_k) : \sup_k\|x_k/\delta_k\|^{p_k}<\infty$ for some $(\delta_k) \in c_0$ with $\delta_k \neq 0$ for all $k \in N\}$;

(f) $E_r(X,p) = \{x = (x_k) : \sup_k k^{-r}\|x_k\|^{p_k} < \infty\}$;
(g) $F_r(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} k^r\|x_k\|^{p_k} < \infty\}$.

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p), c(p), \ell_\infty(p), \ell(p), \underline{c_0}(p), E_r(p)$ and $F_r(p)$, respectively. The first three spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons [9] and Maddox [5–7]. The space $\ell(p)$ was first defined by Nakano [8] and is known as the Nakano sequence space. The spaces $\underline{c_0}(p)$ was first introduced by Grosse-Erdmann [3] and he investigated in [3] the structure of the spaces $c_0(p), c(p), \ell(p)$, and $\ell_\infty(p)$. Grosse-Erdmann [4] gave the matrix characterizations between scalar-valued sequence spaces of Maddox. When $p_k = 1$, for all $k \in N$, the spaces $E_r(p)$ and $F_r(p)$ are written as E_r and F_r, respectively. These two spaces were first introduced by Cooke [2]. Now the problem of matrix transformations becomes more general, we consider infinite matrices of bounded linear operators instead of matrices of real or complex numbers and we consider on vector-valued sequence spaces instead of scalar-valued sequence spaces. Choudhury [1] gave the matrix characterizations mapping $c_0(X)$ into $c_0(Y), \ell_1(X)$ into $\ell_\infty(Y)$, and $\ell_1(X)$ into $\ell_p(Y)$. Wu and Liu [12] deal with the problem of characterizing infinite matrices mapping $c_0(X,p)$ and $\ell_\infty(X,p)$ into $c_0(q)$ and $\ell_\infty(q)$, where $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers. Suantai [10] has given matrix characterizations from $\ell(X,p)$ into the vector-valued sequence spaces $c_0(Y,q), c(Y)$ and $\ell_s(Y)$, where $q = (q_k)$ is a sequence of positive real numbers, Y is a Banach space and $s \geq 1$. He has also given in [11] necessary and sufficient conditions for infinite matrices mapping $\ell(X,p)$ into ℓ_∞ and $\underline{\ell_\infty}(q)$.

In this paper, we extend some results in [10] and [11] and generalize some results in [4]. We also obtain some related results as mentioned in the abstract.

2. Notation and Definitions

Let $(X, \|.\|)$ be a Banach space. Let $W(X)$ and $\Phi(X)$ denote the space of all sequences in X and the space of all finite sequences in X. When $X = K$, the scalar field of X, the corresponding spaces are written as w and Φ, respectively. A sequence space in X is a linear subspace of $W(X)$. Let E be an X-valued sequence space. For $x \in E$ and $k \in N$ we write that x_k stand for the kth term of x. For $x \in X$ and $k \in N$, we let $e^{(k)}(x)$ be the sequence $(0,0,0,\ldots,0,x,0,\ldots)$ with x in the kth position and let $e(x)$ be the sequence $(x,x,x,\ldots)$, and we denote by e the the sequence $(1,1,1,\ldots)$. For a fixed scalar sequence $u = (u_k)$ the sequence space E_u is defined as

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}.$$

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose E is an X-valued sequence space and F a scalar-valued sequence space. Then A is said to *map E into F*, written by $A : E \to F$ if, for each $x = (x_k) \in E$, $A_n(x) =$

$\sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$, and the sequence $Ax = \big(A_n(x)\big) \in F$. We denote by (E, F) the set of all infinite matrices mapping E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$$_u(E,F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E,F)\}.$$

If $u_k \neq 0$ for all $k \in N$, we put $u^{-1} = (1/u_k)$. An X-valued sequence space E is said to be *normal* if $(x_k) \in E$ and $(y_k) \in W(X)$ with $\|y_k\| \leq \|x_k\|$ for all $k \in N$ implies that $(y_k) \in E$.

Suppose the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if, for each $k \in N$ the *kth* coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. In addition, if (E, τ) is a *Fre'chet* (*Banach*) space, then E is called an *FK-*(*BK-*)space.

The spaces $c_0(p)$ and $c(p)$ are FK-spaces. In $c_0(X,p)$, we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = \max\{1, \sup_k p_k\}$, as a paranorm on $c_0(X,p)$, and it is known that $c_0(X,p)$ is an FK-space under the paranorm g defined as above. In $\ell(X,p)$, we consider it as a paranormed sequence space with the paranorm given by $\|(x_k)\| = (\sum_{k=1}^{\infty} \|x_k\|^{p_k})^{1/M}$, where $M = \max\{1, \sup_k p_k\}$. It is known that $\ell(X,p)$ is an FK-space under the paranorm defined as above.

3. Some Auxiliary Results

We start with the following useful results that will reduce our problems into some simpler forms.

Proposition 3.1. *Let E and E_n ($n \in N$) be X-valued sequence spaces, and F and F_n ($n \in N$) scalar-valued sequence spaces, and let μ and v be scalar sequences with $\mu_k \neq 0$, $v_k \neq 0$ for all $k \in N$. Then*

(i) $(\bigcup_{n=1}^{\infty} E_n, F) = \bigcap_{n=1}^{\infty} (E_n, F)$;
(ii) $(E, \bigcap_{n=1}^{\infty} F_n) = \bigcap_{n=1}^{\infty} (E, F_n)$;
(iii) $(E_1 + E_2, F) = (E_1, F) \cap (E_2, F)$;
(iv) $(E_u, F_v) = {}_v(E,F)_{u^{-1}}$.

Proof. All assertions are immediately obtained directly by the definition.

Propostion 3.2. *Let $p = (p_k)$ be a bounded sequences of positive real numbers and $r \geq 0$. Then*

(i) $c(X,p) = c_0(X,p) + \{e(x) : x \in X\}$;
(ii) $\underline{c_0}(X,p) = \bigcup_{n=1}^{\infty} c_0(X)_{(n^{-1/p_k})}$;
(iii) $\overline{E}_r(X,p) = \ell_\infty(X,p)_{(k^{-r/p_k})}$;
(iv) $F_r(X,p) = \ell(X,p)_{(k^{r/p_k})}$;
(v) $\ell_\infty(X,p) = \bigcup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$.

Proof. Assertions (i), (iii) and (iv) are immediately obtained by the definition. To show (ii), let $x = (x_k) \in \underline{c_0}(X, p)$. Then there is a sequence $(\delta_k) \in c_0$ with $\delta_k \neq 0$ for all $k \in N$ such that $\sup_k \|x_k/\delta_k\|^{p_k} < \infty$. Hence there exists $\alpha > 0$ such that $\|x_k\| \leq \alpha^{1/p_k}|\delta_k|$ for all $k \in N$. Choose $n_0 \in N$ so that $n_0 > \alpha$. Then

$$\|x_k\| n_0^{-1/p_k} \leq (\alpha/n_0)^{1/p_k}|\delta_k| < |\delta_k|,$$

which implies that $\lim_{k\to\infty}\|x_k\|n_0^{-1/p_k} = 0$, hence $x = (x_k) \in c_0(X)_{(n_0^{-1/p_k})} \subseteq \bigcup_{n=1}^{\infty} c_0(X)_{(n^{-1/p_k})}$. On the other hand, suppose $x = (x_k) \in \bigcup_{n=1}^{\infty} c_0(X)_{(n^{-1/p_k})}$. Then $\lim_{k\to\infty}\|x_k\|n^{-1/p_k} = 0$ for some $n \in N$. Let $\delta = (\delta_k)$ be the sequence defined by

$$\delta_k = \begin{cases} \|x_k\|n^{-1/p_k}, & \text{if } x_k \neq 0 \\ 1/k, & \text{otherwise.} \end{cases}$$

Clearly, $(\delta_k) \in c_0$ and $\|x_k/\delta_k\|^{p_k} \leq n$ for all $k \in N$, hence $\sup_k\|x_k/\delta_k\|^{p_k} \leq n$, so $x = (x_k) \in \underline{c_0}(X, p)$.

It remains to show (v). If $x \in \ell_\infty(X, p)$, then there is some $n \in N$ with $\|x_k\|^{p_k} \leq n$ for all $k \in N$. Hence $\|x_k\|n^{-1/p_k} \leq 1$ for all $k \in N$, so that $x \in \ell_\infty(X)_{(n^{-1/p_k})}$. On the other hand, if $x \in \bigcup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$, then there are some $n \in N$ and $M > 1$ such that $\|x_k\|n^{-1/p_k} \leq M$ for every $k \in N$. Then we have $\|x_k\|^{p_k} \leq nM^{p_k} \leq nM^{\alpha}$ for all $k \in N$, where $\alpha = \sup_k p_k$. Hence $x \in \ell_\infty(X, p)$. ∎

Proposition 3.3. *Let (f_k) be a sequence of continuous linear functionals on X. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X, p)$ if and only if $\sum_{k=1}^{\infty} \|f_k\|M^{-1/p_k} < \infty$ for some $M \in N$.*

Proof. Suppose $\sum_{k=1}^{\infty} \|f_k\|M^{-1/p_k} < \infty$ for some $M \in N$. Let $x = (x_k) \in c_0(X, p)$. Then there is a positive integer K_0 such that $\|x_k\|^{p_k} < 1/M$ for all $k \geq K_0$, hence $\|x_k\| < M^{-1/p_k}$ for all $k \geq K_0$. Then we have

$$\sum_{k=K_0}^{\infty} |f_k(x_k)| \leq \sum_{k=K_0}^{\infty} \|f_k\|\,\|x_k\| \leq \sum_{k=K_0}^{\infty} \|f_k\|M^{-1/p_k} < \infty.$$

It follows that $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

On the other hand, assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X, p)$. For each $x = (x_k) \in c_0(X, p)$, choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_kx_k) = |f_k(x_k)|$ for all $k \in N$. Since $(t_kx_k) \in c_0(X, p)$, by our assumption, we have $\sum_{k=1}^{\infty} f_k(t_kx_k)$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \text{for all } x \in c_0(X, p). \tag{3.1}$$

Now, suppose that $\sum_{k=1}^{\infty} \|f_k\|m^{-1/p_k} = \infty$ for all $m \in N$. Choose $m_1, k_1 \in N$ such that

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$$\sum_{k \le k_1} \|f_k\| m_1^{-1/p_k} > 1$$

and choose $m_2 > m_1$ and $k_2 > k_1$ such that

$$\sum_{k_1 < k \le k_2} \|f_k\| m_2^{-1/p_k} > 2.$$

Proceeding in this way, we can choose $m_1 < m_2 < \cdots$, and $0 = k_1 < k_2 < \cdots$ such that

$$\sum_{k_{i-1} < k \le k_i} \|f_k\| m_i^{-1/p_k} > i.$$

Take x_k in X with $\|x_k\| = 1$ for all k, $k_{i-1} < k \le k_i$ such that

$$\sum_{k_{i-1} < k \le k_i} |f_k(x_k)| m_i^{-1/p_k} > i \quad \text{for all } i \in N.$$

Put $y = (y_k)$, $y_k = m_i^{-1/p_k} x_k$ for $k_{i-1} < k \le k_i$, then $y \in c_0(X, p)$ and

$$\sum_{k=1}^{\infty} |f_k(y_k)| \ge \sum_{k_{i-1} < k \le k_i} |f_k(x_k)| m_i^{-1/p_k} > i \quad \text{for all } i \in N.$$

Hence we have sum$_{k=1}^{\infty} |f_k(y_k)| = \infty$ which contradicts with (3.1). This completes the proof. ■

Proposition 3.4. *Let (f_k) be a sequence of continuous linear functionals on X. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X, p)$ if and only if $\sum_{k=1}^{\infty} \|f_k\| m^{1/p_k} < \infty$ for all $m \in N$.*

Proof. If $\sum_{k=1}^{\infty} \|f_k\| m^{1/p_k} < \infty$ for all $m \in N$, we have that for each $x = (x_k) \in \ell_\infty(X, p)$, there is $m_0 \in N$ such that $\|x_k\| \le m_0^{1/p_k}$ for all $k \in N$, hence $\sum_{k=1}^{\infty} |f_k(x_k)| \le \sum_{k=1}^{\infty} \|f_k\| \, \|x_k\| \le \sum_{k=1}^{\infty} \|f_k\| m_0^{1/p_k} < \infty$, which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

Conversely, if $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X, p)$, by using the same proof as in Proposition 3.3, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \text{for all } x = (x_k) \in \ell_\infty(X, p). \tag{3.2}$$

Now, suppose that $\sum_{k=1}^{\infty} \|f_k\| M^{1/p_k} = \infty$, for some $M \in N$. Then we can choose a sequence (k_i) of positive integers with $0 = k_0 < k_1 < k_2 < \cdots$ such that

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$$\sum_{k_{i-1}<k\le k_i} \|f_k\|M^{1/p_k} > i \quad \text{for all } i \in N.$$

And we choose x_k in X with $\|x_k\| = 1$ such that for all $i \in N$,

$$\sum_{k_{i-1}<k\le k_i} |f_k(x_k)|M^{1/p_k} > i.$$

Put $y = (y_k)$, $y_k = M^{1/p_k}x_k$. Clearly, $y \in \ell_\infty(X, p)$ and

$$\sum_{k=1}^{\infty} |f_k(y_k)| \ge \sum_{k_{i-1}<k\le k_i}^{\infty} |f_k(x_k)|M^{1/p_k} > i \quad \text{for all } i \in N.$$

Hence $\sum\limits_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts with (3.2). The proof is now completed. ∎

Proposition 3.5. *Let (f_k) be a sequence of continuous linear functionals on X and $p = (p_k)$ a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell(X, p)$ if and only if $\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty$ for some $M \in N$, where $1/p_k + 1/t_k = 1$ for all $k \in N$.*

Proof. Suppose $\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty$ for some $M \in N$.
Then we have that for each $x = (x_k) \in \ell(X, p)$,

$$\begin{aligned}\sum_{k=1}^{\infty} |f_k(x_k)| &\le \sum_{k=1}^{\infty} \|f_k\|M^{-1/p_k}M^{1/p_k}\|x_k\| \\ &\le \sum_{k=1}^{\infty} (\|f_k\|^{t_k} M^{-t_k/p_k} + M\|x_k\|^{p_k}) \\ &= \sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} + M\sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty,\end{aligned}$$

which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

On the other hand, assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell(X, p)$. By using the same proof as in Proposition 3.3, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \text{for all } x = (x_k) \in \ell(X, p). \tag{3.3}$$

We want to show that there exists $M \in N$ such that $\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty$. If it is not true, then

$$\sum_{k=1}^{\infty} \|f_k\|^{t_k} m^{-(t_k-1)} = \infty, \quad \text{for all } m \in N. \tag{3.4}$$

It implies by (3.4) that for each $k \in N$,

$$\sum_{i>k} \|f_i\|^{t_i} m^{-(t_i-1)} = \infty, \quad \text{for all } m \in N. \tag{3.5}$$

By (3.4), let $m_1 = 1$, then there is a $k_1 \in N$ such that

$$\sum_{k \le k_1} \|f_k\|^{t_k} m_1^{-(t_k-1)} > 1.$$

By (3.5), we can choose $m_2 > m_1$ and $k_2 > k_1$ with $m_2 > 2^2$ such that

$$\sum_{k_1 < k \le k_2} \|f_k\|^{t_k} m_2^{-(t_k-1)} > 1. \tag{3.6}$$

Proceeding in this way, we can choose sequences of positive integers (k_i) and (m_i) with $1 = k_0 < k_1 < k_2 < \cdots$ and $m_1 < m_2 < \cdots$, such that $m_i > 2^i$ and

$$\sum_{k_{i-1} < k \le k_i} \|f_k\|^{t_k} m_i^{-(t_k-1)} > 1.$$

For each $i \in N$, choose x_k in X with $\|x_k\| = 1$ for all $k \in N$, $k_{i-1} < k \le k_i$ such that

$$\sum_{k_{i-1} < k \le k_i} |f_k(x_k)|^{t_k} m_i^{-(t_k-1)} > 1 \quad \text{for all } i \in N.$$

Let $a_i = \sum_{k_{i-1} < k \le k_i} |f_k(x_k)|^{t_k} m_i^{-(t_k-1)}$. Put $y = (y_k)$, $y_k = a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k$ for all k $k_{i-1} < k \le k_i$. For each $i \in N$, we have

$$\begin{aligned}
\sum_{k_{i-1} < k \le k_i} \|y_k\|^{p_k} &= \sum_{k_{i-1} < k \le k_i} \|a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k\|^{p_k} \\
&= \sum_{k_{i-1} < k \le k_i} a_i^{-p_k} m_i^{-t_k} |f_k(x_k)|^{t_k} \\
&\le \sum_{k_{i-1} < k \le k_i} a_i^{-1} m_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k} \\
&= a_i^{-1} m_i^{-1} a_i \\
&= m_i^{-1} \\
&< 1/2^i.
\end{aligned}$$

So we have that $\sum_{k=1}^{\infty}\|y_k\|^{p_k} \leq \sum_{i=1}^{\infty} 1/2^i < \infty$. Hence, $y = (y_k) \in \ell(X, p)$. For each $i \in N$, we have

$$\begin{aligned}\sum_{k_{i-1}<k\le k_i} |f_k(y_k)| &= \sum_{k_{i-1}<k\le k_i} |f_k(a_i^{-1}m_i^{-(t_k-1)}|f_k(x_k)|^{t_k-1}x_k)| \\ &= \sum_{k_{i-1}<k\le k_i} a_i^{-1}m_i^{-(t_k-1)}|f_k(x_k)|^{t_k} \\ &= a_i^{-1}\sum_{k_{i-1}<k\le k_i} m_i^{-(t_k-1)}|f_k(x_k)|^{t_k} \\ &= 1.\end{aligned}$$

So that $\sum_{k=1}^{\infty}|f_k(y_k)| = \infty$, which contradicts with (3.3). The proof is now complete. ■

Proposition 3.6. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k \le 1$ for all $k \in N$ and $(f_k) \subset X'$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in \ell(X, p)$ if and only if there exists $M \in N$ such that $\sup_k \|f_k\| M^{-1/p_k} < \infty$.*

Proof. If $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in \ell(X, p)$, using the same proof as in Proposition 3.3, we have

$$\sum_{k=1}^{\infty}|f_k(x_k)| < \infty \quad \text{for all } x = (x_k) \in \ell(X, p) \tag{3.7}$$

Suppose that $\sup_k \|f_k\| m^{-1/p_k} = \infty$ for all $m \in N$. For each $i \in N$, choose sequences (m_i) and (k_i) of positive integers with $m_1 < m_2 < \cdots$ and $k_1 < k_2 < \cdots$ such that $m_i > 2^i$ and $\|f_{k_i}\| m_i^{-1/p_{k_i}} > 1$. Choose $x_{k_i} \in X$ with $\|x_{k_i}\| = 1$ such that

$$|f_{k_i}(x_{k_i})| m_i^{-1/p_{k_i}} > 1. \tag{3.8}$$

Let $y = (y_k)$, $y_k = m_i^{-1/p_{k_i}} x_{k_i}$ if $k = k_i$ for some i, and 0 otherwise. Then $\sum_{k=1}^{\infty}\|y_k\|^{p_k} = \sum_{i=1}^{\infty} 1/m_i < \sum_{i=1}^{\infty} 1/2^i = 1$, so that $(y_k) \in \ell(X, p)$ and

$$\begin{aligned}\sum_{k=1}^{\infty}|f_k(y_k)| &= \sum_{i=1}^{\infty}|f_{k_i}(m_i^{-1/p_{k_i}} x_{k_i})| \\ &= \sum_{i=1}^{\infty} m_i^{-1/p_{k_i}}|f_{k_i}(x_{k_i})| = \infty \quad \text{by (3.8)},\end{aligned}$$

and this is contradictory with (3.7). Therefore, there exists $M \in N$ such that $\sup_k \|f_k\| M^{-1/p_k} < \infty$.

Conversely, assume that there exists $M \in N$ such that $\sup_k \|f_k\| M^{-1/p_k} < \infty$. Let $x = (x_k) \in \ell(X, p)$, then there is a $K > 0$ such that

$$\|f_k\| \leq KM^{1/p_k} \quad \text{for all } k \in N \tag{3.9}$$

and there is a $k_0 \in N$ such that $M^{1/p_k}\|x_k\| \leq 1$ for all $k \geq k_0$. By $p_k \leq 1$ for all $k \in N$, we have that for all $k \geq k_0$,

$$M^{1/p_k}\|x_k\| \leq (M^{1/p_k}\|x_k\|)^{p_k} = M\|x_k\|^{p_k}. \tag{3.10}$$

Then

$$\begin{aligned}
\sum_{k=1}^{\infty} |f_k(x_k)| &\leq \sum_{k=1}^{k_o} \|f_k\|\,\|x_k\| + \sum_{k=k_0+1}^{\infty} \|f_k\|\,\|x_k\| \\
&\leq \sum_{k=1}^{k_o} \|f_k\|\,\|x_k\| + K\sum_{k=k_0+1}^{\infty} M^{1/p_k}\|x_k\| \quad \text{(by (3.9))} \\
&\leq \sum_{k=1}^{k_o} \|f_k\|\,\|x_k\| + KM\sum_{k=k_0+1}^{\infty} \|x_k\|^{p_k} \quad \text{(by (3.10))} \\
&< \infty.
\end{aligned}$$

This implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges. ∎

4. Main Results

We begin with the following useful result.

Theorem 4.1. *Let $q = (q_k)$ be a bounded sequence of positive real numbers and let E be a normal X-valued sequence space which is an FK-space and contains $\Phi(X)$. Then*

$$(E, c(q)) = (E, c_0(q)) \oplus (E, \langle e \rangle).$$

To prove this theorem, we need the following two lemmas.

Lemma 4.1. *Let E be an X-valued sequence space which is an FK-space and contains $\Phi(X)$. Then for each $k \in N$, the mapping $T_k : X \to E$, defined by $T_k x = e^k(x)$, is continuous.*

Proof. Let $V = \{e^k(x) : x \in X\}$. Then V is a closed subspace of E, so it is an FK-space because E is an FK-space. Since E is a K-space, the coordinate mapping $p_k : V \to X$ is continuous and bijective. It follows from the open mapping theorem that p_k is open, which implies that $p_k^{-1} : X \to V$ is continuous. But since $T_k = p_k^{-1}$, we thus obtain that T_k is continuous. ∎

Lemma 4.2. *Let $q = (q_k)$ be a bounded sequence of positive real numbers. If E and F are scalar-valued sequence spaces such that E is normal containing Φ and F is an FK-space with the property that for each $x = (x_k) \in F$, there is a subsequence (x_{n_k}) of (x_k) with $x_{n_k} \to 0$ as $k \to \infty$, then $(E, F \oplus \langle e \rangle) = (E, F) \oplus (E, \langle e \rangle)$.*

Proof. See [2, Proposition 3.1(vi)]. ∎

Proof of Theorem 4.1. Since $c(q) = c_0(q) \oplus \langle e \rangle$, it is clear that $(E, c_0(q)) + (E, \langle e \rangle) \subseteq (E, c_0(q) \oplus \langle e \rangle) = (E, c(q))$. Moreover, if $A \in (E, c_0(q)) \cap (E, \langle e \rangle)$, then $A \in (E, c_0(q) \cap \langle e \rangle)$, so that $A \in (E, 0)$, which implies that $A = 0$ because E contain $\Phi(X)$. Hence $(E, c_0(q)) + (E, \langle e \rangle)$ is a direct sum. Now, we will show that $(E, c(q)) \subseteq (E, c_0(q)) \oplus (E, \langle e \rangle)$. Let $A = (f_k^n) \in (E, c(q)) = (E, c_0(q) \oplus \langle e \rangle)$. For $x \in X$ and $k \in N$, we have $(f_k^n(x))_{n=1}^{\infty} = Ae^k(x) \in c_0(q) \oplus \langle e \rangle$, so that there exists unique $(b_k^n(x))_{n=1}^{\infty} \in c_0(q)$ and $(c_k^n(x))_{n=1}^{\infty} \in \langle e \rangle$ with

$$(f_k^n(z))_{n=1}^{\infty} = (b_k^n(z))_{n=1}^{\infty} + (c_k^n(z))_{n=1}^{\infty}. \tag{4.1}$$

For each $n, k \in N$, let g_k^n and h_k^n be the functionals on X defined by

$$g_k^n(x) = b_k^n(x) \quad \text{and} \quad h_k^n(x) = c_k^n(x) \quad \text{for all } x \in X.$$

Clearly, g_k^n and h_k^n are linear, and by (4.1)

$$f_k^n = g_k^n + h_k^n \quad \text{for all } n, k \in N. \tag{4.2}$$

Note that $c_0(q) \oplus \langle e \rangle$ is an *FK*-space in its direct sum topology. By Zeller's theorem, $A : E \to c_0(q) \oplus \langle e \rangle$ is continuous. For each $k \in N$, let $T_k : X \to E$ be defined by $T_k(x) = e^k(x)$. By Lemma 4.1, we have that T_k is continuous for all $k \in N$. Since the projection P_1 of $c_0(q) \oplus \langle e \rangle$ onto $c_0(q)$ and the projection P_2 of $c_0(q) \oplus \langle e \rangle$ onto $\langle e \rangle$ are continuous and $g_k^n = p_n \circ P_1 \circ A \circ T_k$ and $h_k^n = p_n \circ P_2 \circ A \circ T_k$ for all $n, k \in N$, we obtain that g_k^n and h_k^n are continuous, so $g_k^n, h_k^n \in X'$ for all $n, k \in N$. Let $B = (g_k^n)$ and $C = (h_k^n)$. By (4.1) and (4.2) we have $A = B + C$, $B = (g_k^n) \in (\Phi(X), c_0(q))$ and $C = (h_k^n) \in (\Phi(X), \langle e \rangle)$. We will show that $B \in (E, c_0(q))$ and $C \in (E, \langle e \rangle)$. To do this, let $x = (x_k) \in E$. Then for $\alpha = (\alpha_k) \in \ell_\infty$, we have $\|\alpha_k x_k\| = |\alpha_k| \, \|x_k\| \le \|M x_k\|$, where $M = \sup_k |\alpha_k|$. Then the normality of E implies that $(\alpha_k x_k) \in E$. Hence $(f_k^n(x_k))_{n,k} \in (\ell_\infty, c_0(q) \oplus \langle e \rangle)$, moreover, we have $(g_k^n(x_k))_{n,k} \in (\Phi, c_0(q))$, $(h_k^n(x_k))_{n,k} \in (\Phi, \langle e \rangle)$, and $(f_k^n(x_k))_{n,k} = (g_k^n(x_k))_{n,k} + (h_k^n(x_k))_{n,k}$. Since ℓ_∞ is normal containing Φ and $c_0(q) \subseteq c_0$, it follows from Lemma 4.2 that $(g_k^n(x_k))_{n,k} \in (\ell_\infty, c_0(q))$ and $(h_k^n(x_k))_{n,k} \in (\ell_\infty, \langle e \rangle)$. This implies that $Bx \in c_0(q)$ and $Cx \in \langle e \rangle$, so we have $B \in (E, c_0(q))$ and $C \in (E, \langle e \rangle)$, hence $A \in (E, c_0(q)) \oplus (E, \langle e \rangle)$. This completes the proof. ∎

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Theorem 4.2. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c_0(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ *for some* $M \in N$,
(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for every* $m, k \in N$ *and*
(3) $\sum_{k=1}^{\infty} m^{1/q_n} \|f_k^n - f_k\| r^{-1/p_k} \to 0$ *as* $n, r \to \infty$ *for each* $m \in N$.

Proof. If $A \in (c_0(X,p), c(q))$ we have $A \in (c_0(X,p), c_0(q) \oplus \langle e \rangle)$ since $c(q) = c_0(q) \oplus \langle e \rangle$. It follows from Theorem 4.1 that $A = B + C$, where $B \in (c_0(X,p), c_0(q))$ and $C \in (c_0(X,p), \langle e \rangle)$. Let $C = (g_k^n)$. Since $\Phi(X) \subseteq c_0(X,p)$, we have $(g_k^n(x))_{n=1}^{\infty} \in \langle e \rangle$ for all $x \in X$ and $k \in N$, which implies that $g_k^n = g_k^{n+1}$ for all $n, k \in N$, let $f_k = g_k^1$. Then we have $B = (f_k^n - f_k)_{n,k} \in (c_0(X,p), c_0(q))$. By [3, Theorem 0 (i)], we have $c_0(q) = \bigcap_{m=1}^{\infty} c_0(m^{1/p_k})$. It follows from Proposition 3.1(ii) and (iv) that $(m^{1/q_n}(f_k^n - f_k))_{n,k} \in (c_0(X,p), c_0)$ for all $m \in N$. By Wu [11, Theorem 2.4], we have that the conditions (2) and (3) hold. Since $C = (f_k)_{n,k} \in (c_0(X,p), \langle e \rangle)$, we have $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = x_k \in c_0(X,p)$, hence (1) is obtained by Proposition 3.3.

Conversely, assume that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that conditions (1), (2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. It is obvious that $A = B + C$. By conditions (2) and (3), we obtain by Proposition 3.1(ii) and (iv), and Wu [11, Theorem 2.4] that $B \in (c_0(X,p), c_0(q))$. The condition (1) implies by Proposition 3.3 that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X,p)$, which implies that $C \in (c_0(X,p), \langle e \rangle)$. Hence we have by Theorem 4.1 that $A \in (c_0(X,p), c(q))$. This completes the proof. ∎

Theorem 4.3. *Let $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : \ell_\infty(X) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| < \infty$,
(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*}$ *as* $n \to \infty$ *for every* $k, m \in N$ *and*
(3) *for each* $m, r \in N$, $\sum_{j>k} m^{1/q_n} \|f_j^n - f_j\| r^{1/p_j} \to 0$ *as* $k \to \infty$ *uniformly on* $n \in N$.

Proof. If $A \in (\ell_\infty(X), c(q))$, then the condition (1) holds by Proposition 3.4. It follows from Theorem 4.1 that $A = B + C$, where $B \in (\ell_\infty(X), c_0(q))$ and $C \in (\ell_\infty(X), \langle e \rangle)$. Using the same proof as in Theorem 4.2, there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that $C = (f_k)_{n,k}$ and $B = (f_k^n - f_k)_{n,k} \in (\ell_\infty(X), c_0(q))$. Since $c_0(q) = \bigcap_{k=1}^{\infty} c_{0_{(m^{1/p_k})}}$, we thus obtain (2) and (3) by Proposition 3.1(ii) and (iv), and Wu [11, Theorem 2.9].

Conversely, assume that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that condition (1), (2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. It is obvious that $A = B + C$. By conditions (2) and (3), we obtain by Proposition 3.1(ii) and (iv), and Wu [11, Theorem 2.9] that $B \in (\ell_\infty(X), c_0(q))$. The condition (1) implies by Proposition 3.4 that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X)$, which implies that $C \in (\ell_\infty(X), \langle e \rangle)$. Hence, we have by Theorem 4.1 that $A \in (\ell_\infty(X), c(q))$. This completes the proof. ∎

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Theorem 4.4. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : \ell_\infty(X, p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^\infty \|f_k\| m^{1/p_k} < \infty$ *for all* $m \in N$,
(2) $r^{1/q_n}(m^{1/p_k} f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for every* $m, k, r \in N$ *and*
(3) *for each* $m, r, s \in N$, $\sum_{j>k} r^{1/q_n} \|m^{1/p_j} f_j^n - f_j\| s^{1/p_j} \to 0$ *as* $k \to \infty$ *uniformly on* $n \in N$.

Proof. By Proposition 3.2 (v), $\ell_\infty(X, p) = \bigcup_{m=1}^\infty \ell_\infty(X)_{(m^{-1/p_k})}$. It follows from Proposition 3.1(i) and (iv), Proposition 3.4 and Theorem 4.3 that

$$A : \ell_\infty(X, p) \to c(q) \Leftrightarrow (m^{1/p_k} f_k^n)_{n,k} : \ell_\infty(X) \to c(q) \quad \text{for all } m \in N$$

$$\Leftrightarrow \text{the conditions (1), (2), and (3) hold.}$$

Theorem 4.5. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c(X, p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^\infty \|f_k\| M^{-1/p_k} < \infty$ *for some* $M \in N$,
(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for every* $m, k \in N$,
(3) $\sum_{k=1}^\infty m^{1/q_n} \|f_k^n - f_k\| r^{-1/p_k} \to 0$ *as* $n, r \to \infty$ *for every* $m \in N$ *and*
(4) $\left(\sum_{k=1}^\infty f_k^n(x)\right)_{n=1}^\infty \in c(q)$ *for all* $x \in X$.

Proof. Since $c(X, p) = c_0(X, p) + \{e(x) : x \in X\}$ (Proposition 3.2 (i)), it follows from Proposition 3.1(iii) that $A \in (c(X, p), c(q))$ if and only if $A \in (c_0(X, p), c(q))$ and $A \in (\{e(x) : x \in X\}, c(q))$. By Theorem 4.2, we have $A \in (c_0(X, p), c(q))$ if and only if conditions (1)–(3) hold and it is clear that $A \in (\{e(x) : x \in X\}, c(q))$ if and only if (4) holds. Hence, the theorem is proved. ∎

Wu [12, Theorem 2.7] has given a characterization of an infinite matrix A such that $A \in (\ell(X, p), c_0)$ when $p_k > 1$ for all $k \in N$. By applying of Proposition 3.1(ii) and (iv), Proposition 3.5 and Theorem 4.1, and using the fact that $\bigcap_{m=1}^\infty c_{0_{(m^{1/p_k})}}$, we obtain the following result.

Theorem 4.6. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in N$ and $1/p_k + 1/t_k = 1$ for all $k \in N$, and let $A = (f_k^n)$ be an infinite matrix. Then $A : \ell(X, p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^\infty \|f_k\|^{t_k} M^{-(t_k-1)} < \infty$ *for some* $M \in N$,
(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for all* $m, k \in N$ *and*
(3) *for each* $m \in N$, $\left(\sum_{k=1}^\infty m^{t_k/q_n} \|f_k^n - f_k\|^{t_k} r^{-(t_k-1)}\right) \to 0$ *as* $r \to \infty$ *uniformly on* $n \in N$.

By using [12, Theorem 2.6], Proposition 3.1(ii) and (iv), Proposition 3.6 and Theorem 4.1, we also obtain the following result.

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Theorem 4.7. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k \leq 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X, p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $sup_k \|f_k\| M^{-1/p_k} < \infty$ *for some* $M \in N$,
(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for all* $m, k \in N$ *and*
(3) $\sup\limits_{n,k} m^{p_k/q_n} \|f_k^n - f_k\|^{p_k} < \infty$ *for all* $m \in N$.

Theorem 4.8. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : \underline{c_0}(X, p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| < \infty$,
(2) $m^{1/q_n}(s^{1/p_k} f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for every* $m, k, s \in N$ *and*
(3) $\dfrac{1}{r} \sum\limits_{k=1}^{\infty} m^{1/q_n} \|s^{1/p_k} f_k^n - f_k\| \to 0$ *as* $n, r \to \infty$ *for each* $m, s \in N$.

Proof. By Proposition 3.2(ii), we have $\underline{c_0}(X, p) = \bigcup_{s=1}^{\infty} c_0(X)_{(s^{-1/p_k})}$. By Proposition 3.2(i) and (iv) and Theorem 4.2, we have

$$\begin{aligned} A : \underline{c_0}(X, p) \to c(q) &\Leftrightarrow A : \bigcup_{s=1}^{\infty} c_0(X)_{(s^{-1/p_k})} \to c(q) \\ &\Leftrightarrow A : c_0(X)_{(s^{-1/p_k})} \to c(q), \quad \text{for all } s \in N \\ &\Leftrightarrow (s^{1/p_k} f_k^n)_{n,k} : c_0(X) \to c(q), \quad \text{for all } s \in N \\ &\Leftrightarrow \text{the conditions (1), (2) and (3) hold.} \end{aligned}$$ ■

Theorem 4.9. *Let $p = (p_k)$ and $q = (q_k)$ be a bounded sequences of positive real numbers and $r \geq 0$, and let $A = (f_k^n)$ be an infinite matrix. Then $A : E_r(X, p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| m^{1/p_k} < \infty$ *for all* $m \in N$,
(2) $r^{1/q_n}(m^{1/p_k} k^{r/p_k} f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for every* $m, k, r \in N$ *and*
(3) *for each* $m, r, s \in N$, $\sum\limits_{j>k} r^{1/q_n} \|m^{1/p_j} j^{r/p_j} f_j^n - f_j\| s^{1/p_j} \to 0$ *as* $k \to \infty$ *uniformly on* $n \in N$.

Proof. By Proposition 3.2(iii), we have $E_r(X, p) = \ell_\infty(X, p)_{(k^{-r/p_k})}$. By Proposition 3.1 (iv) and Theorem 4.4, we have

$$\begin{aligned} A : E_r(X, p) \to c(q) &\Leftrightarrow A : \ell_\infty(X, p)_{(k^{-r/p_k})} \to c(q) \\ &\Leftrightarrow (k^{r/p_k} f_k^n)_{n,k} : \ell_\infty(X, p) \to c(q) \\ &\Leftrightarrow \text{the conditions (1), (2) and (3) hold.} \end{aligned}$$ ■

In the last theorem, we give a characterization of a matrix transformation from

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the space $F_r(X,p)$ into $c(q)$. It is known by Proposition 3.2 (iv) that $F_r(X,p) = \ell(X,p)_{(k^{r/p_k})}$. By Proposition 3.1(iv), for a scalar sequence space E and an infinite matrix $A = (f_k^n)$, we have

$$A : F_r(X,p) \to E \Leftrightarrow (k^{-r/p_k} f_k^n)_{n,k} : \ell(X,p) \to E.$$

So we the following theorem is obtained by applying Theorem 4.6.

Theorem 4.10. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in N$, $1/p_k + 1/t_k = 1$ for all $k \in N$ and $r \geq 0$, and let $A = (f_k^n)$ be an infinite matrix. Then $A : F_r(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\|^{t_k} k^{-rt_k/p_k} M^{-(t_k-1)} < \infty$ *for some* $M \in N$,
(2) $m^{1/q_n}(k^{-r/p_k} f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for all* $m, k \in N$ *and*
(3) *for each* $m \in N$, $\sum_{k=1}^{\infty} m^{t_k/q_n} \|k^{-r/p_k} f_k^n - f_k\|^{t_k} r^{-(t_k-1)} \to 0$ *as* $r \to \infty$ *uniformly on* $n \in N$.

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Matrix transformations of Nakano vector-valued sequence space

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Matrix Transformations of Nakano Vector-valued Sequence Space

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In this paper, we give necessary and sufficient conditions for infinite matrices mapping Nakano vector-valued sequence space $\ell(X,p)$ into the sequence spaces $E_r (r \geq 0)$ and we also give the matrix characterlizations from $M_0(X,p)$ into the space E_r where $p = (p_k)$ is a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$.

1. Introduction

For $r \geq 0$, the normed sequence space E_r was first defined by Cooke [1] as follows :

$$E_r = \{ x = (x_k) \mid \sup_k \frac{|x_k|}{k^r} < \infty \}$$

equipped with the norm

$$\|x\| = \sup_k \frac{|x_k|}{k^r}.$$

Let $(X, \|.\|)$ be a Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. We write $x = (x_k)$ with x_k in X for all $k \in N$. The X-valued sequence spaces $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, and $M_0(X,p)$ are defined as

$$\begin{aligned}
c_0(X,p) &= \{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\}, \\
c(X,p) &= \{x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X\}, \\
\ell_\infty(X,p) &= \{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\}, \\
\ell(X,p) &= \{x = (x_k) : \textstyle\sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\}, \\
M_0(X,p) &= \textstyle\bigcup_{n=1}^{\infty} \ell(X)_{(n^{-1/p_k})}
\end{aligned}$$

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, $\ell(p)$, and $M_0(p)$, respectively. The spaces $c_0(p)$, $c(p)$, $\ell_\infty(p)$ are known as the sequence spaces of Maddox. These spaces were first introduced and studied

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by Simons [7], Maddox [4, 5]. The space $\ell(p)$ was first defined by Nakano [6] and it is known as the Nakano sequence space and the space $\ell(X,p)$ is known as the Nakano vector-valued sequence space. The spaces $M_0(p)$ was first introduced by Grosse-Erdmann [2] and he has investigated the structure of the spaces $c_0(p)$, $c(p)$ and $\ell_\infty(p)$. Grosse-Erdmann [3] gave the matrix characterizations between scalar-valued sequence spaces of Maddox. Wu and Liu [9] dealt with the problem of characterizations those infinite matrices mapping $c_0(X,p)$, $\ell_\infty(X,p)$ into $c_0(q)$ and $\ell_\infty(q)$ where $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers. Suantai [8] gave necessary and sufficient conditions for infinite matrices mapping $\ell(X,p)$ into ℓ_∞ and $\ell_\infty(q)$ where $p=(p_k)$ and $q=(q_k)$ are bounded sequence positive real numbers with $p_k \leq 1$ for all $k \in N$.

In this paper we give characterizations of infinite matrices mapping $\ell(X,p)$ and $M_0(X,p)$ into the sequence space E_r when $p_k \leq 1$ for all $k \in N$ and $r \geq 0$. Some results in [8] are obtained as special cases of this paper.

2. Notation and Definitions

Let $(X, \|.\|)$ be a Banach space. The space of all sequences and the space of all finite sequences in X are denoted by $W(X)$ and $\Phi(X)$, respectively. When X is K, the scalar field of X, the corresponding spaces are written as w and Φ.

A sequence space in X is a linear subspace of $W(X)$. Let E be an X-valued sequence space. For $x \in E$ and $k \in N$, we write x_k standing for the k^{th} term of x. For $x \in X$ and $k \in N$, let $e^k(x)$ be the sequence $(0,0,...,0,x,0,...)$ with x in the k^{th} position and let $e(x)$ be the sequence $(x,x,x,...)$. For a fixed scalar sequence $\mu=(\mu_k)$ the sequence space E_μ is defined as

$$E_\mu = \{x \in W(X) : (\mu_k x_k) \in E\}\ .$$

Let $A=(f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map* E into F, written by $A: E \to F$ if for each $x=(x_k) \in E$, $A_n(x) = \sum_{k=1}^\infty f_k^n(x_k)$ converges for each $n \in N$, and the sequence $Ax = (A_n(x)) \in F$. Let (E,F) denote for the set of all infinite matrices mapping from E into F.

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if for each $k \in N$ the k^{th} coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. If, in addition, (E,τ) is an Fréchet (Banach, LF-, LB-) space, then E is called an *FK- (BK-, LFK-, LBK-) space*. Now, suppose that E contains $\Phi(X)$. Then E is said to have *property AB* if the set $\{\sum_{k=1}^n e^k(x_k) : n \in N\}$ is bounded in E for every $x=(x_k) \in E$. It is said to have *property AK* if $\sum_{k=1}^n e^k(x_k) \to x$ in E as $n \to \infty$ for every $x=(x_k) \in E$. It has *property AD* if $\Phi(X)$ is dense in E.

It is known that E_r is a BK-sapce and $E_0 = \ell_\infty$. The space $\ell(X,p)$ is an FK-space with AK under the paranorm $g(x) = \left(\sum_{k=1}^\infty \|x_k\|^{p_k}\right)^{1/M}$, where $M =$

$max\ \{1, \sup_k p_k\}$. In each of the space $\ell_\infty(X,p)$ and $c_0(X,p)$ we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = max\ \{1, \sup_k p_k\}$. It is known that $c_0(X,p)$ is an FK-space with AK under the paranorm g defined as above and $\ell_\infty(X,p)$ is a complete LBK-space with AB.

3. Main Results

We start with giving the matrix characterizations from $\ell(X,p)$ into E_r.

Theorem 3.1. *Let $r \geq 0$ and let $p = (p_k)$ be bounded sequences of positive real numbers with $p_k \leq 1$ and let $A = (f_k^n)$ be an infinite matrix. Then $A \in (\ell(X,p), E_r)$ if and only if there is $m_0 \in N$ such that $\sup_{n,k} m_0^{-1/p_k} n^{-r} \|f_k^n\| < \infty$.*

Proof. Assume that $A \in (\ell(X,p), E_r)$. In $\ell(X,p)$, we consider it as a paranormed space with the paranorm g defined as above and since $p_k \leq 1$ for all $k \in N$, we have $M = max\ \{1, \sup_k p_k\} = 1$. Now, we write $\|.\|$ standing for the paranorm g. By Zeller's theorem, $A : \ell(X,p) \to E_r$ is continuous. Then there is $m_0 \in N$ such that

$$\sup_n n^{-r} \Big|\sum_{k=1}^{\infty} f_k^n(x_k)\Big| \leq 1 \quad \text{for all } x \in \ell(X,p) \ \text{ with } \|x\| \leq \frac{1}{m_0} . \tag{3.1}$$

Let $n, k \in N$ be fixed and let $x_k \in X$ be such that $\|x_k\| \leq 1$. Then $e^{(k)}(m_0^{-1/p_k} x_k) \in \ell(X,p)$ and $\|e^{(k)}(m_0^{-1/p_k} x_k)\| \leq \frac{1}{m_0}$. By (3.1), we have

$$m_0^{-1/p_k} n^{-r} |f_k^n(x_k)| \leq \sup_{i \in N} i^{-r} |f_k^i(m_0^{-1/p_k} x_k)| = \|Ae^{(k)}(m_0^{-1/p_k} x_k)\| \leq 1 .$$

It implies that $\sup_{n,k} m_0^{-1/p_k} n^{-r} \|f_k^n\| < \infty$.

Conversely, assume that the condition holds. Let $x = (x_k) \in \ell(X,p)$. By assumption, there is a $C > 0$ such that

$$m_0^{-1/p_k} n^{-r} \|f_k^n\| < C \quad \text{for all } n, k \in N . \tag{3.2}$$

Since $\|m_0^{1/p_k} x_k\| \to 0$ as $k \to \infty$, there is a $k_0 \in N$ such that $\|m_0^{1/p_k} x_k\| < 1$ for all $k \geq k_0$. Since $0 < p_k \leq 1$ for all $k \in N$, we have

$$\|m_0^{1/p_k} x_k\| \leq \|m_0^{1/p_k} x_k\|^{p_k} \quad \text{for all } k \geq k_0. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned} \textstyle\sum_{k=1}^{\infty} \|m_0^{1/p_k} x_k\| &= \textstyle\sum_{k=1}^{k_0} \|m_0^{1/p_k} x_k\| + \sum_{k=k_0+1}^{\infty} \|m_0^{1/p_k} x_k\| \\ &\leq \textstyle\sum_{k=1}^{k_0} \|m_0^{1/p_k} x_k\| + \sum_{k=k_0+1}^{\infty} \|m_0^{1/p_k} x_k\|^{p_k} \\ &= K_1 + m_0 \textstyle\sum_{k=k_0+1}^{\infty} \|x_k\|^{p_k} \\ &\leq K_1 + m_0\|x\|, \ K_1 = \textstyle\sum_{k=1}^{k_0} \|m_0^{1/p_k} x_k\|. \end{aligned} \tag{3.4}$$

By (3.2) and (3.4) we have for $n \in N$,

$$\begin{aligned} n^{-r}|A_n x| &= n^{-r}\left|\sum_{k=1}^{\infty} f_k^n\left(m_0^{-1/p_k}(m_0^{1/p_k} x_k)\right)\right| \\ &\leq \sum_{k=1}^{\infty} m_0^{-1/p_k} n^{-r}\|f_k^n\|.\|m_0^{1/p_k} x_k\| \\ &\leq C\sum_{k=1}^{\infty}\|m_0^{1/p_k} x_k\| \\ &\leq C(K_1 + m_0\|x\|). \end{aligned}$$

This implies that $\sup_n n^{-r}|A_n x| < \infty$, so that $Ax \in E_r$. This completes the proof. □

When $r = 0$, we see that $E_r = \ell_\infty$, so we obtain the following result directly from Theorem 3.1.

Corollary 3.2. *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), \ell_\infty)$ if and only if there is $m_0 \in N$ such that $\sup_{n,k} m_0^{-1/p_k}\|f_k^n\| < \infty$.*

If $p_k = s \leq 1$ for all $k \in N$, by Theorem 3.1 we obtain the following result:

Corollary 3.3. *Let $r \geq 0$ and $0 < s \leq 1$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell_s(X),\ E_r)$ if and only if $\sup_{n,k} n^{-r}\|f_k^n\| < \infty$.*

When $p_k = 1$ for all $k \in N$ and $r = 0$, we obtain the following result by Corollary 3.3.

Corollary 3.4. *For an infinite matrix $A = (f_k^n)$, $A \in (\ell(X),\ \ell_\infty)$ if and only if $\sup_{n,\,k} \|f_k^n\| < \infty$.*

Theorem 3.5. *Let $r \geq 0$ and let $p = (p_k)$ be bounded sequences of positive real numbers and let $A = (f_k^n)$ be an infinite matrix. Then $A \in (M_0(X,p),\ E_r)$ if and only if for each $s \in N$, $\sup_{n,\,k} n^{-r}s^{1/p_k}\|f_k^n\| < \infty$.*

Proof. Since $M_0(X,p) = \bigcup_{n=1}^{\infty} \ell(X)_{(n^{-1/p_k})}$, we have

$$A \in (M_0(X,p),\ E_r) \iff A \in (\ell(X)_{(s^{-1/p_k})},\ E_r) \text{ for all } s \in N$$

For $s \in N$, we can easily show that

$$A \in (\ell(X)_{(s^{-1/p_k})},\ E_r) \iff \left(s^{1/p_k} f_k^n\right)_{n,k} \in (\ell(X),\ E_r)\ .$$

By Theorem 3.1, we obtain that for $s \in N$,

$$\left(s^{1/p_k} f_k^n\right)_{n,k} \in (\ell(X),\ E_r) \iff \sup_{n,\,k}\ n^{-r}s^{1/p_k}\|f_k^n\| < \infty\ .$$

Thus the theorem is proved. □

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Matrix Transformations on the Nakano Vector-valued Sequence Space

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Matrix Transformations on the Nakano Vector-valued Sequence Space

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In this paper, we give the matrix characterizations from any FK-space of vector sequences with AK property into any FK-space of scalar sequences, and by applying this result we also obtain necessary and sufficient conditions for infinite matrices mapping the spaces $\ell(X,p)$ into Maddox sequence spaces $c_0(q)$ and $\ell(q)$ where $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers such that $p_k \geq 1$ for all $k \in N$.

1. Introduction

Let $(X, \|.\|)$ be a Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. Let N be the set of all natural numbers, we write $x = (x_k)$ with x_k in X for all $k \in N$. The $X-$valued sequence spaces $c_0(X,p), c(X,p), \ell_\infty(X,p)$, and $\ell(X,p)$ are defined by

$$\begin{aligned}
c_0(X,p) &= \left\{ x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0 \right\}, \\
c(X,p) &= \left\{ x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X \right\}, \\
\ell_\infty(X,p) &= \left\{ x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty \right\}, \text{and} \\
\ell(X,p) &= \left\{ x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty \right\}.
\end{aligned}$$

When $X = K$,the scalar field of X,the corresponding spaces are written as

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$c_0(p), c(p), \ell(p)$, and $\ell_\infty(p)$, respectively. The first three spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons[7] and Maddox[4, 5]. The space $\ell(p)$ was first defined by Nakano[6] and is known as the Nakano sequence space, and the space $\ell(X,p)$ is known as the Nakano vector-valued sequence space. Choudhur[1] gave necessary and sufficient conditions for an infinite matrix of continuous linear operators which maps the vector-valued sequence space $c_0(X)$ into $c_0(Y), \ell_1(X)$ into $\ell_\infty(Y)$ and $\ell_1(X)$ into $\ell_p(Y)$ where Y is a Banach space. Grosse-Erdmann[2] investigated the structure of the spaces $c_0(p), c(p), \ell(p)$ and $\ell_\infty(p)$ and the problem of characterizing a matrix that maps a sequence space of Maddox into another such space is studied by them in[3]. Suantai[8, 9, 10, 11] gave the matrix characterizations from $\ell(X,p)$ into the space $c_0(Y,p), \ell_\infty(q)$ and F_s in the case $p_k \leq 1$ for all $k \in N$ and $s \geq 0$, where Y is a Banach space. Wu[12] gave characterizations of matrix transformations from the space $\ell(X,p)$ into the space c_0 and $\ell_\infty(q)$. The characterizations of matrix transformations from the space $\ell(X,p)$ into $\ell(q)$ and $c_0(q)$ can not be expected to be characterized completely in term of Toeplitz conditions, but however we can give characterizations of these matrix transformations in term of other conditions. Even the classical pair (ℓ_p, ℓ_q) is an open problem when $1 < p, q < \infty$, and $(p,q) \neq (2,2)$. Also, in the case $(\ell(p), \ell(q))$ is an open problem if $q_k < 1$ for all $k \in N$.

2. Notation and Definitions

Let $(X, \|.\|)$ be a Banach space. The space of all sequences in X is denoted by $W(X)$ and $\Phi(X)$ denote for the space of all finite sequences in X.
A sequence space in X is a linear subspace of $W(X)$. Let E be an X- valued sequence space. For $x \in E$ we write $x = (x_k), k \in N$. For $z \in X$ and $k \in N$,we let $e^{(k)}(z)$ be the sequence $(0,0,0,...,0,z,0,...)$ with z in the k^{th} position. For a fixed scalar sequence $u = (u_k)$ the sequence space E_u is defined by

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}.$$

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a **K-space** if for each $n \in N$ the n^{th} coordinate mapping $p_n : E \to X$, defined by $p_n(x) = x_n$, is continuous on E. If, in addition, (E, τ) is an $Fre'chet(Banach, LF-, LB-)$ space, then E is called an $FK-(BK-, LFK-, LBK-)$ space. Now, suppose that E contains $\Phi(X)$. Then E is said to have **property AB** if the set $\{\sum_{k=1}^{n} e^k(x_k) : n \in N\}$ is bounded in E for every $x = (x_k) \in E$. It is said to have **property AK** if $\sum_{k=1}^{n} e^k(x_k) \to x \in E$ as $n \to \infty$ for every $x = (x_k) \in E$. It has **property AD** if $\Phi(X)$ is dense in E.
Let $A = (f_k^n)$ with f_k^n in X',the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to **map** E **into** F, written $A : E \to F$ if for each $x = (x_k) \in E, A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$ and if the sequence $Ax = (A_n(x)) \in F$. We denote by (E, F) the set of all infinite matrices mapping E into F. If $u = (u_k)$ and

$v = (v_k)$ are scalar sequences, let

$$_u(E,F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E,F)\}.$$

If $u_k \neq 0$ for all $k \in N$,we write $u^{-1} = (\frac{1}{u_k})$.

3. Some Auxiliary Results

We start with the following useful results that will reduce our problems into some simpler forms.

Proposition 3.1. *Let E and $E_n (n \in N)$ be X-valued sequence spaces, and F and $F_n (n \in N)$ scalar sequence spaces, and let μ and ν be scalar sequences with $\mu_k \neq 0, \nu_k \neq 0$ for all $k \in N$. Then*

(i) $(E, \bigcap_{n=1}^{\infty} F_n) = \bigcap_{n=1}^{\infty} (E, F_n)$ *and*

(ii) $(E_u, F_v) = {}_v(E,F)_{u^{-1}}$.

Proof. (i) and (ii) are immediately obtained by the definition. □

Proposition 3.2. *Let (f_k) be a sequence of continuous linear functional on X and $p = (p_k)$ a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell(X,p)$ if and only if*

$$\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty \text{ for some } M \in N,$$

where $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$.

Proof. Suppose that $\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty$ for some $M \in N$.
Then we have that for each $x = (x_k) \in \ell(X,p)$,

$$\begin{aligned}
&\sum_{k=1}^{\infty} |f_k(x_k)| \leq \sum_{k=1}^{\infty} \|f_k\| M^{-\frac{1}{p_k}} M^{\frac{1}{p_k}} \|x_k\| \\
&\leq \sum_{k=1}^{\infty} \left(\|f_k\|^{t_k} M^{-\frac{t_k}{p_k}} + M \|x_k\|^{p_k} \right) = \sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} + M \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty
\end{aligned}$$

which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

On the other hand, assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell(X,p)$. For each $x = (x_k) \in \ell(X,p)$, choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Since $(t_k x_k) \in \ell(X,p)$, by our assumption, we have $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \text{ for all } x = (x_k) \in \ell(X, p). \tag{3.1}$$

We want to show that there exists $M \in N$ such that

$$\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty$$

If it is not true, then

$$\sum_{k=1}^{\infty} \|f_k\|^{t_k} m^{-(t_k-1)} = \infty, \text{ for all } m \in N. \tag{3.2}$$

And (3.2) implies that for each $k_0 \in N$.

$$\sum_{k>k_0} \|f_k\|^{t_k} m^{-(t_k-1)} = \infty, \text{ for all } m \in N. \tag{3.3}$$

By (3.2), let $m_1 = 1$, then there is a $k_1 \in N$ such that

$$\sum_{k \leq k_1} \|f_k\|^{t_k} m_1^{-(t_k-1)} > 1.$$

By (3.3), we can choose $m_2 > m_1$ and $m_2 > 2^2$ and $k_2 > k_1$, such that

$$\sum_{k_1 < k \leq k_2} \|f_k\|^{t_k} m_2^{-(t_k-1)} > 1.$$

By continueing in this way, we obtain sequences (k_i) and (m_i) of positive integers such that $1 = k_0 < k_1 < k_2 < \ldots$ and $m_1 < m_2 < \ldots, m_i > 2^i$ and

$$\sum_{k_{i-1} < k \leq k_i} \|f_k\|^{t_k} m_i^{-(t_k-1)} > 1.$$

Choose x_k in X with $\|x_k\| = 1$ such that for all $i \in N$,

$$\sum_{k_{i-1} < k \leq k_i} |f_k(x_k)|^{t_k} m_i^{-(t_k-1)} > 1 \text{ for all } i \in N.$$

Let $a_i = \sum_{k_{i-1} < k \leq k_i} |f_k(x_k)|^{t_k} m_i^{-(t_k-1)}$. Put $y = (y_k)$, $y_k = a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k$ for all k $k_{i-1} < k \leq k_i$.

For each $i \in N$, we have

$$\begin{aligned}
&\sum_{k_{i-1}<k\le k_i} \|y_k\|^{p_k} \\
&= \sum_{k_{i-1}<k\le k_i} \left\| a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k \right\|^{p_k} \\
&= \sum_{k_{i-1}<k\le k_i} a_i^{-p_k} m_i^{-t_k} |f_k(x_k)|^{t_k} \\
&\le \sum_{k_{i-1}<k\le k_i} a_i^{-1} m_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k} \\
&= a_i^{-1} m_i^{-1} a_i \\
&= m_i^{-1} < \tfrac{1}{2^i}.
\end{aligned}$$

So we have that

$$\sum_{k=1}^{\infty} \|y_k\|^{p_k} \le \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty.$$

Hence $y = (y_k) \in \ell(X,p)$. For each $i \in N$, we have

$$\begin{aligned}
\sum_{k_{i-1}<k\le k_i} |f_k(y_k)| &= \sum_{k_{i-1}<k\le k_i} \left| f_k(a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k) \right| \\
&= \sum_{k_{i-1}<k\le k_i} a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k} \\
&= a_i^{-1} \sum_{k_{i-1}<k\le k_i} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k} \\
&= 1.
\end{aligned}$$

So that $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts with (3. 4). The proof is now complete. □

4. Main Results

Now, we turn to our objective. We begin with giving characterizations of matrix transformations from an FK-space of vector sequences with AK property into an FK-space of scalar sequences.

Theorem 4.1.1. *Let $E \subseteq W(X)$ be an FK-space with AK property and F an FK-space of scalar sequences. Then, for an infinite matrix $A = (f_k^n), A : E \to F$ if and only if*

(1) *for each $n \in N$, $\sum_{k=1}^{\infty} f_k^n(x_k)$ converges for all $x = (x_k) \in E$,*
(2) *for each $k \in N$, $(f_k^n(z))_{n=1}^{\infty} \in F$ for all $z \in X$, and*
(3) *$A : \Phi(X) \to F$ is continuous when $\Phi(X)$ is considered as a subspace of E.*

Proof. Assume that $A : E \to F$. Then we have that for any $x = (x_k) \in E$, $\sum_{k=1}^{\infty} f_k^n(x_k)$ converges for all $n \in N$, so (1) holds. Since $e^k(z) \in E$ for all $k \in N$ and all $z \in X$, we obtain that for each $k \in N$,

$$(f_k^n(z))_{n=1}^{\infty} = Ae^k(z) \in F,$$

hence (2) holds. Since E and F are FK-spaces, by Zeller's theorem, $A : E \to F$ is continuous, so (3) is obtained.

Conversely, assume that the conditions hold. By (1), we have

$$Ax = \left(\sum_{k=1}^{\infty} f_k^n(x_k)\right)_{n=1}^{\infty} \in W, \text{ for all } x = (x_k) \in {}^{\prime}E.$$

It follows from (2) that $Ae^k(z) \in F$, for all $k \in N$ and $z \in X$, which implies that $A : \Phi(X) \to F$. By (3), we have $A : \Phi(X) \to F$ is continuous. Let $x = (x_k) \in E$. Since E has the AK property, we have

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} e^k(x_k).$$

Then $\left(\sum\limits_{k=1}^{n} e^k(x_k)\right)_{n=1}^{\infty}$ is a Cuachy sequence in E. Since $A : \Phi(X) \to F$ is continuous

and linear, it implies that $\left(\sum\limits_{k=1}^{n} Ae^k(x_k)\right)_{n=1}^{\infty}$ is a Cauchy sequence in F. Since F is complete, we have $\left(\sum\limits_{k=1}^{n} Ae^k(x_k)\right)_{n=1}^{\infty}$ converges in F. Since F is a K-space, it implies that $\left(\sum\limits_{k=1}^{\infty} f_k^n(x_k)\right)_{n=1}^{\infty} \in F$, so that $Ax \in F$. This shows that $A : E \to F$.
□

It is known that the space $\ell(X, p)$ is an FK-space with AK property under the paranorm

$$g(x) = \left(\sum_{k=1}^{\infty} \|x_k\|^{p_k}\right)^{\frac{1}{M}}, \text{ when } M = \max\{1, \underset{k}{\to} \sup\, p_k\}.$$

By Proposition 3.2 and Theorem 4.1.1, we have the following theorem.

Theorem 4.1.2. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers such that $p_k > 1$ for all $k \in N$, and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X, p) \to \ell(q)$ if and only if*
(1) *for each $n \in N$ there exists $M_n \in N$ such that*

$$\sum_{k=1}^{\infty} \|f_k^n\|^{t_k} M_n^{-(t_k-1)} < \infty, \text{ where } \frac{1}{p_k} + \frac{1}{t_k} = 1 \text{ for all } k \in N,$$

(2) *for each* $k \in N$ *and* $z \in X$, $\sum_{n=1}^{\infty} |f_k^n(z)|^{q_n} < \infty$, *and*
(3) *for each* $r \in N$ *there exists* $M_r \in N$ *such that*

$$\sum_{k\in K} \|x_k\|^{p_k} < \frac{1}{M_r} \Rightarrow \sum_{n=1}^{\infty} |\sum_{k\in K} f_k^n(x_k)|^{q_n} < \frac{1}{r},$$

for all $x = (x_k) \in \Phi(X)$ *and all finite subsets* K *of* N.

Now, we have the sufficient conditions for an infinite matrix $A = (f_k^n)$ that maps $\ell(X,p)$ into $\ell(q)$.

Theorem 4.1.3. *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequences of positive real numbers such that* $p_k > 1$ *and* $q_k > 1$ *for all* $k \in N$, *and* $A = (f_k^n)$ *an infinite matrix. Then* $A : \ell(X,p) \to \ell(q)$ *if the following two conditions hold* ;
(1) *for each* $n \in N$ *there exists* $M_n \in N$ *such that*

$$\sum_{k=1}^{\infty} \|f_k^n\|^{t_k} M_n^{-(t_k-1)} < \infty \text{ where } \frac{1}{p_k} + \frac{1}{t_k} = 1 \text{ for all } k \in N, \text{ and}$$

(2) *there exists* $M_0 \in N$ *such that*

$$\sup_K \sum_{n=1}^{\infty} \left(\sum_{k\in K} \|f_k^n\| M_0^{-\frac{1}{p_k}} \right)^{q_n} < \infty,$$

where supremum is taken over all finite subsets K *of* N.

Proof. Suppose that the two conditions hold. Then by Proposition 3.2 the condition (1) implies the condition (1) of Theorem 4.1.1. By the condition (2), we have that there exists $M_0, L \in N$ such that

$$\sum_{n=1}^{\infty} \left(\sum_{k\in K} \|f_k^n\| M_0^{-\frac{1}{p_k}} \right)^{q_n} < L, \tag{4.1}$$

for all the finite subsets K of N. Then, for each $z \in X - \{0\}$ we can choose $M_1 > M_0$ such that $M_1\|z\| > 1$. Then for each $k \in N$, we have by (4.1) that

$$\begin{aligned}
\sum_{n=1}^{\infty} |f_k^n(z)|^{q_n} &\le \sum_{n=1}^{\infty} (\|f_k^n\|\|z\|)^{q_n} \\
&= \sum_{n=1}^{\infty} \left(\|f_k^n\| M_1^{-\frac{1}{p_k}} M_1^{\frac{1}{p_k}} \|z\| \right)^{q_n} \\
&\le \sum_{n=1}^{\infty} \left(\|f_k^n\| M_1^{-\frac{1}{p_k}} M_1 \|z\| \right)^{q_n} \\
&\le (M_1\|z\|)^{\beta} \sum_{n=1}^{\infty} \left(\|f_k^n\| M_1^{-\frac{1}{p_k}} \right)^{q_n} \ ; \ \beta = \sup_n q_n \\
&\le (M_1\|z\|)^{\beta} L \\
&< \infty.
\end{aligned}$$

So, we have that $(f_k^n(z))_{n=1}^{\infty} \in \ell(q)$ for all $z \in N$ and $k \in N$. Hence the condition (2) of Theorem 4.1.1 holds. We shall now show that the condition (3) of Theorem 4.1.1 is satisfied. To show this, let $\varepsilon > 0$ and $x = (x_k) \in \Phi(X)$. Recall that $\|x\| = \left(\sum_{k=1}^{\infty} \|x_k\|^{p_k}\right)^{\frac{1}{M}}$ where $M = \sup_n q_n$. If $\|x\| \leq 1$, then for all $k \in N$ we have

$$\|x_k\| \leq \|x\|^{\frac{M}{p_k}} \leq \|x\|. \tag{4.2}$$

Since $x = (x_k) \in \Phi(X)$, there is a finite subset K_0 of N such that

$$\sum_{k=1}^{\infty} f_k^n(x_k) = \sum_{k \in K_0} f_k^n(x_k) \text{ for all } n \in N. \tag{4.3}$$

So, we have by (4.1), (4.2), and (4.3) that

$$\begin{aligned}
\|Ax\| &= \left(\sum_{n=1}^{\infty} |\sum_{k=1}^{\infty} f_k^n(x_k)|^{q_n}\right)^{\frac{1}{G}} \\
&= \left(\sum_{n=1}^{\infty} |\sum_{k \in K_0} f_k^n(x_k)|^{q_n}\right)^{\frac{1}{G}} \\
&\leq \left(\sum_{n=1}^{\infty} (\sum_{k \in K_0} \|f_k^n\| \|x_k\|)^{q_n}\right)^{\frac{1}{G}} \\
&\leq \left(\sum_{n=1}^{\infty} (\sum_{k \in K_0} \|f_k^n\| M_0^{-\frac{1}{p_k}} M_0^{\frac{1}{p_k}} \|x\|)^{q_n}\right)^{\frac{1}{G}} \\
&\leq \left(M_0^G \|x\| \sum_{n=1}^{\infty} (\sum_{k \in K_0} \|f_k^n\| M_0^{-\frac{1}{p_k}})^{q_n}\right)^{\frac{1}{G}} \\
&\leq M_0(\|x\|L)^{\frac{1}{G}}, \ ,G = \sup_n q_n.
\end{aligned} \tag{4.4}$$

It implies by (4.4) that $A : \Phi(X) \to \ell(q)$. Now choose $\delta = \min\{1, \frac{1}{L}(\frac{\varepsilon}{M_0})^G\}$. It follows by (4.4) that

$$\|x\| < \delta \Rightarrow \|Ax\| < \varepsilon.$$

It follows that $A : \Phi(X) \to \ell(q)$ is continuous. Hence, by Theorem 4.1.1, we have that $A : \ell(X,p) \to \ell(q)$. $\square$

By using the previous auxiliary results and Theorem 1.6 in [12], we obtain necessary and sufficient conditions for infinite matrices mapping the space $\ell(X,p)$ into $c_0(q)$.

Theorem 4.1.4. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers such that $p_k > 1$ for all $k \in N$, and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X,p) \to c_0(q)$ if and only if*

(1) *for all* $m, k \in N$, $m^{\frac{1}{q_n}} f_k^n \to 0$ *weakly as* $n \to \infty$, *and*
(2) *for each* $m \in N$,

$$\left(\sum_{k=1}^{\infty} m^{\frac{t_k}{q_n}} \|f_k^n\|^{t_k} r^{-(t_k-1)}\right) \to 0 \text{ uniformly for } n \geq 1 \text{ as } r \to \infty,$$

where $\frac{1}{p_k} + \frac{1}{t_k} = 1$ *for all* $k \in N$.

Proof. By Theorem 0 in [2], we have $c_0(q) = \bigcap_{s=1}^{\infty} c_{0(s^{\frac{1}{q_n}})}$. By Proposition 2.1(i) and (ii) and Theorem 1.6 in [12], we have

$$\begin{aligned} A : \ell(X,p) \to c_0(q) &\iff A : \ell(X,p) \to \bigcap_{m=1}^{\infty} c_{0(m^{\frac{1}{q_n}})} \\ &\iff A : \ell(X,p) \to c_{0(m^{\frac{1}{q_n}})}, \text{ for all } m \in N \\ &\iff \left(m^{\frac{1}{q_n}} f_k^n\right)_{n,k} : \ell(X,p) \to c_0, \text{ for all } m \in N \\ &\iff \text{the conditions (1) and (2) hold.} \end{aligned}$$ □

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ON MATRIX TRANSFORMATIONS OF SOME VECTOR-VALUED SEQUENCE SPACES

SUTHEP SUANTAI

Abstract. In this paper, chracterizations of infinite matrices mapping theNakano vector-valued sequence space into Musielak-Orilicz sequence space are given, and we also give characterizations of the β-dual of the Nakano-vector valued sequence space.

1. Introduction. Let $(X, \|\cdot\|)$ be a Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. Let N be the set of all natural, we write $x = (x_k)$ with x_k in X for all $k \in N$. The X-valued sequence space $c_0(X,p), c(X,p), \ell_\infty(X,p)$, and $\ell(X,p)$ are defined by

$$c_0(X,p) = \left\{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\right\},$$

$$c(X,p) = \left\{x = (x_k) : \lim_{k\in\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X\right\},$$

$$\ell_\infty(X,p) = \left\{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\right\},$$

$$\text{and} \quad \ell(X,p) = \left\{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\right\}.$$

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p), c(p), \ell_\infty(p)$, and $\ell(p)$, respectively. The first three spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons (1965) and Maddox (1967, 1968). The space $\ell(p)$ was first defined by Nakano (1951) and is known as the Nakano sequence space. Grosse-Erdmann (1992) investigated the structure of the spaces $c_0(p), c(p), \ell_\infty(p)$, and $\ell(p)$.

A function $f : R \to [0, \infty)$ is called an *Orlicz function* if it is even, continuous, convex and vanishing at 0 and $f(x) \to \infty$ as $x \to \infty$. Let $M = (M_n)$ be a sequence of Orlicz functions, for a given real sequence $x = (x_n)$, define

$$\varrho_M(x) = \sum_{n=1}^{\infty} M_n(x_n).$$

Let $\ell_M = \{x = (x_n) : \varrho_M(\lambda x) < \infty \text{ for some } \lambda > 0\}$ and for $x = (x_n) \in \ell_M$, the Luxemburg norm of x is defined by the formula

$$\|x\| = \inf\left\{\lambda > 0 : \varrho_M\left(\frac{x}{\lambda}\right) \le 1\right\}$$

The sequence space $(\ell_M, \|\cdot\|)$ was first defined by Musielak (1983) and it is called the Musielak-Orlicz sequence space with the Luxemburg norm. If $M_n = M_{n+1}$ for $n \in N$, the space ℓ_M is known as the Orlicz sequence space. We also see that if $p = (p_k)$ is a bounded sequnce of positive real numbers with $p_k \geq 1$ for all $k \in N$, then the Musielak-Orlicz sequence space ℓ_M, where $M_k(t) = |t|^{p_k}$ is the Nakano sequence space $\ell(p)$. So the Musielak-Orlicz sequence space is a generalization of both Orlicz and Nakano sequence spaces. For more details about the orlicz sequence space and Musielak-Orlicz sequence space see Chen (1996) and Musielak (1983).

$|t|^{p_k}$

Grosse-Erdmann (1993) gave characterizations of infinite matrices mapping between scalar-valued sequence spaces of Maddox. Wu and Liu (1993) gave characterizations of matrix transformations from the spaces $\ell(X,p)$, $c_0(X,p)$ and $\ell_\infty(X,p)$ into the spaces $c_0(q)$ and $\ell_\infty(q)$. These results generalized some of Grosse-Erdmann (1993). Suantai (1999) gave characterizations of infinite matrices of bounded linear functionals on X mapping the Nakano sequence space $\ell(X,p)$ into $\ell_\infty(q)$ and $\ell_{-\infty}(q)$ and $M_0(X,p)$ into $\ell_{-\infty}(q)$. Choudhur (1992) gave necessary and sufficient conditions for an infinite matrix of continuous linear operators which maps the vector-valued sequence space $c_0(X)$ into $c_0(Y)$, $\ell_1(X)$ into $\ell_\infty(Y)$ and $\ell_1(X)$ into $\ell_o(Y)$, where Y is a Banach space. Suantai (2000) gave characterizations of infinite matrices of bounded linear operators mapping from the Nakano vector-valued sequence space $\ell(X,p)$ into any BK-space.

In this paper, we use some technics of Suantai (2000) and other technics to give the matrix characterizations from the Nakano vector-valued sequence space $\ell(X,p)$ into the Musielak-Orlicz sequence space ℓ_M.

2. Notation and definitions. Let $(X, \|\cdot\|)$ be a Banach space. The space of all sequences in X is denoted by $W(X)$ and $\Phi(X)$ denotes the space of all finite sequences in X.

A sequence space in X is a linear subspace of $W(X)$. Let E be an X-valued sequence space. For $x \in E$ we write $x = (x_k)$, $k \in N$. For $z \in X$ and $k \in N$, we let $e^{(k)}(z)$ be the sequence $(0,0,0,\cdots,0,z,0,\cdots)$ with z in the kth position. Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a K-space if for each $n \in N$ the nth coordinate mapping $p_n : E \to X$, defined by $p_n(x) = x_n$, is continuous on E. If, in addition, (E,τ) is an *Fréchet(Banach)* space, then E is called an $FK-(BK-)$ space. Now, suppose that E contains $\Phi(X)$. Then E is said to have property AK if $\sum_{k=1}^{n} e^k(x_k) \to x$ as $n \to \infty$ for all $x = (x_k) \in E$.

If $p_k \geq 1$ for all $k \in N$, the space $\ell(p)$ is a BK-space with AK property under the Luxemburg norm defined by

$$\|x\| = \inf\left\{\varepsilon > 0 : \sum_{k=1}^{\infty} \left|\frac{x_k}{\varepsilon}\right|^{p_k} \leq 1\right\}.$$

For more detail about the space ~~the space~~ $\ell(p)$ see Gross-Erdmann (1992). By using the same argument as in the case of real sequence space $\ell(q)$, we can also obtain that $\ell(X,p)$ is a BK-space with AK property under the Luxemburg norm defined by the formula

$$\|x\| = \inf\left\{\varepsilon > 0 : \sum_{k=1}^{\infty}\left\|\frac{x_k}{\varepsilon}\right\|^{p_k} \le 1\right\}.$$

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map* E *into* F, written by $A : E \to F$ if for each $x = (x_k) \in E$, $A_n(x) = \sum\limits_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$ and the sequnce $Ax = (A_n(x)) \in F$.

Let E be an X-valued sequence space. The β-dual of E is defined to be

$$E^\beta = \left\{(f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x_k) \text{ converges for all } x = (x_k) \in E\right\},$$

We see that if $A = (f_k^n)$ maps a seqeunce space E into a scalar sequence space, then each row of A must belong to E^β, i.e., $(f_k^n)_{k=1}^{\infty} \in E^\beta$, so this is a necessary condition for an infinite matrix A mapping from one sequence space into the other. In this paper, we also give characterizations of the β-dual of Nakano vector-valued sequence space. of the

3. Main results. We begin with giving characterizations of the β-dual ofthe Nakano vector-valued sequence space.

PROPOSITION 3.1. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$. Then*

$$\ell(X,p)^\beta = \left\{(f_k) \subset X' : \sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k - 1)} < \infty \text{ for some } M \in N\right\}$$

where $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$.

Proof: Suppose that $\sum\limits_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty$ for some $M \in N$. By using the fact that $ab \le a^{t_k} + b^{p_k}$ for all $a, b \ge 0$ and all $k \in N$, we then obtain that for each $x = (x_k) \in \ell(X,p)$,

$$\begin{aligned}
\sum_{k=1}^{\infty} |f_k(x_k)| &\le \sum_{k=1}^{\infty} \|f_k\| M^{-\frac{1}{p_k}} M^{\frac{1}{p_k}} \|x_k\| \\
&\le \sum_{k=1}^{\infty} \left(\|f_k\|^{t_k} M^{-\frac{t_k}{p_k}} + M\|x_k\|^{p_k}\right) \\
&= \sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} + M \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty
\end{aligned}$$

which implies that $\sum\limits_{k=1}^{\infty} f_k(x_k)$ converges, so $(f_k) \in \ell(X,p)^\beta$.

On the other hand, assume that $(f_k) \in \ell(X,p)^\beta$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell(X,p)$. For each $x = (x_k) \in \ell(X,p)$, choose a scalar seqeunce (t_k) with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Since $(t_k x_k) \in \ell(X,p)$, by our assumption, we have $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \text{ for all } x = (x_k) \in \ell(X,p). \tag{3.1}$$

We want to show that there exists $M \in N$ such that

$$\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{(-t_k-1)} < \infty.$$

If it is not true, then

$$\sum_{k=1}^{\infty} \|f_k\|^{t_k} m^{-(t_k-1)} = \infty, \text{ for all } m \in N. \tag{3.2}$$

And (3.2) implies that for each $k_0 \in N$,

$$\sum_{k>k_0} \|f_k\|^{t_k} m^{-(t_k-1)} = \infty, \text{ for all } m \in N. \tag{3.3}$$

By (3.2), let $m_1 = 1$, then there is a $k_1 \in N$ such that

$$\sum_{k \le k_1} \|f_k\|^{t_k} m_1^{-(t_k-1)} > 1.$$

By (3.3), we can choose $m_2 > m_1$ and $m_2 > 2^2$ and $k_2 > k_1$ such that

$$\sum_{k_1 < k \le k_2} \|f_k\|^{t_k} m_2^{-(t_k-1)} > 1. \tag{3.4}$$

By continuing in this way, we obtain sequences (k_i) and (m_i) of positive integers with $1 = k_0 < k_1 < k_2 < \cdots, m_1 < m_2 < \cdots, m_i > 2^i$ and

$$\sum_{k_{i-1} < k \le k_i} \|f_k\|^{t_k} m_i^{-(t_k-1)} > 1.$$

Choose x_k in X with $\|x_k\| = 1$ such that for all $i \in N$,

$$\sum_{k_{i-1} < k \le k_i} |f_k(x_k)|^{t_k} m_i^{-(t_k-1)} > 1 \text{ for all } i \in N.$$

Let

$$a_i = \sum_{k_{i-1}<k\leq k_i} |f_k(x_k)|^{t_k} m_i^{-(t_k-1)}.$$

Put $y = (y_k)$, $y_k = a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k$ for all $k, k_{i-1} < k \leq k_i$. For each $i \in N$, we have

$$\begin{aligned}
\sum_{k_{i-1}<k\leq k_i} \|y_k\|^{p_k} &= \sum_{k_{i-1}<k\leq k_i} \left\| a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k \right\|^{p_k} \\
&= \sum_{k_{i-1}<k\leq k_i} a_i^{-p_k} m_i^{-t_k} |f_k(x_k)|^{t_k} \\
&\leq \sum_{k_{i-1}<k\leq k_i} a_i^{-1} m_i^{-1} m_i^{(-t_k-1)} |f_k(x_k)|^{t_k} \\
&= a_i^{-1} m_i^{-1} a_i = m_i^{-1} < \frac{1}{2^i}.
\end{aligned}$$

So we have that

$$\sum_{k=1}^{\infty} \|y_k\|^{p_k} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty.$$

Hence,

$$y = (y_k) \in \ell(X, p).$$

For each $i \in N$, we have

$$\begin{aligned}
\sum_{k_{i-1}<k\leq k_i} |f_k(y_k)| &= \sum_{k_{i-1}<k\leq k_i} \left| f_k\left(a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k \right) \right| \\
&= \sum_{k_{i-1}<k\leq k_i} a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k} \\
&= a_i^{-1} \sum_{k_{i-1}<k\leq k_i} m^{-(t_k-1)} |f_k(x_k)|^{t_k} \\
&= 1.
\end{aligned}$$

So that $\sum\limits_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts (3.1). Thus

$$(f_k) \in \left\{ (g_k) \subset X' : \sum_{k=1}^{\infty} \|g_k\|^{t_k} M^{-(t_k-1)} < \infty \text{ for some } M \in N \right\}.$$

Hence the proposition is proved.

PROPOSITION 3.2. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k \le 1$ for all $k \in N$ and $(f_k) \subset X'$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in \ell(X,p)$ if and only if there exists $M \in N$ such that $\sup_k \|f_k\| M^{-1/p_k} < \infty$.*

Proof: if $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in \ell(X,p)$, using the same proof as in Proposition 3.1, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \text{ for all } x = (x_k) \in \ell(X,p). \tag{3.5}$$

Suppose that $\sup_k \|f_k\| m^{-1/p_k} = \infty$ for all $m \in N$. For each $i \in N$, choose sequences (m_i) and (k_i) of positive integers with $m_1 < m_2 < \cdots$ and $k_1 < k_2 < \cdots$ such that $m_i > 2^i$ and $\|f_{k_i}\| m_i^{-1/p_{k_i}} > 1$. Choose $x_{k_i} \in X$ with $\|x_{k_i}\| = 1$ such that

$$|f_{k_i}(x_{k_i})| m_i^{-1/p_{k_i}} > 1. \tag{3.6}$$

Let $y = (y_k)$, $y_k = m_i^{-1/p_{k_i}} x_{k_i}$ if $k = k_i$ for some i, and 0 otherwise. Then

$$\sum_{k=1}^{\infty} \|y_k\|^{p_k} = \sum_{i=1}^{\infty} 1/m_i < \sum_{i=1}^{\infty} 1/2^i = 1,$$

so that $(y_k) \in \ell(X,p)$ and

$$\begin{aligned} \sum_{k=1}^{\infty} |f_k(y_k)| &= \sum_{i=1}^{\infty} \left| f_{k_i}(m_i^{-1/p_{k_i}} x_{k_i}) \right| \\ &= \sum_{i=1}^{\infty} m_i^{-1/p_{k_i}} |f_{k_i}(x_{k_i})| = \infty \text{ by (3.6)}, \end{aligned}$$

and this is contradictory to (3.5). Therefore, there exists $M \in N$ such that $\sup_k \|f_k\| M^{-1/p_k} < \infty$.

Conversely, assume that there exists $M \in N$ such that $\sup_k \|f_k\| M^{-1/p_k} < \infty$. Let $x = (x_k) \in \ell(X,p)$, then there is a $K > 0$ such that

$$\|f_k\| \le K M^{1/p_k} \text{ for all } k \in N \tag{3.7}$$

and there is a $k_0 \in N$ such that $M^{1/p_k} \|x_k\| \le 1$ for all $k \ge k_0$. By $p_k \le 1$ for all $k \in N$, we have that for all $k \ge k_0$,

$$M^{1/p_k} \|x_k\| \le (M^{1/p_k} \|x_k\|)^{p_k} = M \|x_k\|^{p_k}. \tag{3.8}$$

Then

$$\begin{aligned}\sum_{k=1}^{\infty}|f_k(x_k)| &\leq \sum_{k-1}^{k_0}\|f_k\|\,\|x_k\| + \sum_{k=k_0+1}^{\infty}\|f_k\|\,\|x_k\| \\ &\leq \sum_{k=1}^{k_0}\|f_k\|\,\|x_k\| + K\sum_{k=k_0+1}^{\infty}M^{1/p_k}\|x_k\| \quad \text{by (3.7)} \\ &\leq \sum_{k=1}^{k_0}\|f_k\|\,\|x_k\| + KM\sum_{k=k_0+1}^{\infty}\|x_k\|^{p_k} \quad \text{by (3.8)} \\ &< \infty.\end{aligned}$$

This implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

The following proposition gives some useful properties concerning the Luxemburg norm on $\ell(X;p)$.

PROPOSITION 3.3. *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k > 1$ for all $k \in N$ and let $x = (x_k) \in \ell(X,p)$. Then*

(1) $\|x\| \leq 1$ *if and only if* $\sum_{k=1}^{\infty}\|x_k\|^{p_k} \leq 1$, *and*

(2) *If* $\|x\| = 1$, *then* $\sum_{k=1}^{\infty}\|x_k\|^{p_k} = 1$.

Proof: If $\sum_{k=1}^{\infty}\|x_k\|^{p_k} \leq 1$, we have by the definition of the Luxemburg norm that $\|x\| \leq 1$.

If $\|x\| \leq 1$, then $\|x\| < 1 + \frac{1}{n}$ for all $n \in N$, which implies that

$$\sum_{k=1}^{\infty}\|x_k/(1+1/n)\|^{p_k} \leq 1.$$

Since

$$(1/(1+1/n))^{\alpha}\sum_{k=1}^{\infty}\|x_k\|^{p_k} \leq \sum_{k=1}^{\infty}\|x_k/(1+1/n)\|^{p_k},$$

where $\alpha = \sup_k p_k$, it follows that

$$\sum_{k=1}^{\infty}\|x_k\|^{p_k} \leq (1+1/n)^{\alpha} \text{ for all } n \in N. \tag{3.9}$$

By taking $n \to \infty$ in (3.9), we obtain $\sum_{k=1}^{\infty}\|x_k\|^{p_k} \leq 1$.

(2) Assume that $\|x\| = 1$. By (1) we have

$$\sum_{k=1}^{\infty} \|x_k\|^{p_k} \leq 1.$$

If $\sum_{k=1}^{\infty} \|x_k\|^{p_k} < 1$, then for each $\varepsilon > 0$ such that

$$\sum_{k=1}^{\infty} \|x_k\|^{p_k} < \varepsilon < 1,$$

we have that

$$\sum_{k=1}^{\infty} \|x_k/\varepsilon\|^{p_k} > 1.$$

Since $(1/\varepsilon)^{\alpha} \geq (1/\varepsilon)^{p_k}$ for all $k \in N$, where $\alpha = \sup_k p_k$, we have

$$(1/\varepsilon)^{\alpha} \sum_{k=1}^{\infty} \|x_k\|^{p_k} > 1,$$

hence

$$\sum_{k=1}^{\infty} \|x_k\|^{p_k} > \varepsilon^{\alpha}. \tag{3.10}$$

By taking $\varepsilon \to 1^-$ in (3.10) we obtain that $\sum_{k=1}^{\infty} \|x_k\|^{p_k} \geq 1$ which is a contradiction. Hence $\sum_{k=1}^{\infty} \|x_k\|^{p_k} = 1$.

In the next theorem, we give necessary and sufficient conditions for infinite matrices mapping the Nakano sequence space $\ell(X,p)$, when $p_k \leq 1$ for all $k \in N$, into the Musielak-Orlicz sequence space ℓ_M

THEOREM 3.4. *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X,p) \to \ell_M$ if and only if*

(1) *for each $k \in N$ and $x \in X$, $(f_k^n(x))_{n=1}^{\infty} \in \ell_M$, and*

(2) *there exists $m_0 \in N$ such that*

$$\sup_{\substack{k \in N \\ \|x\| \leq 1}} \cancel{\left\| A\left(m_0^{-\frac{1}{p_k}} e^k(x)\right)\right\|} \leq 1. \quad \sup_{\substack{k \in N \\ \|x\| \leq 1}} \sum_{n=1}^{\infty} M_n\left(m_0^{-1/p_k} f_k^n(x)\right) \leq 1.$$

~~By Proposition 3.3 (1), we have that the condition (ii) above is equivalent to (2). Hence $A : \ell(X,p) \to \ell_M$ if and only if the conditions (1) and (2) are satisfied.~~

Proof: By Suantai (2000) (Theorem 4.1), we obtain that $A : \ell(X,p) \to \ell_M$ iff (1) holds and

(2)′ there exists $m_0 \in \mathbf{N}$ such that $\sup_{\substack{k \in N \\ \|x\| \leq 1}} \left\| A\left(m_0^{-\frac{1}{p_k}} e^k(x)\right)\right\| \leq 1.$

By general results in Musielak (1983), it implies that (2) and (2)′ are equivalent. Hence the theorem is ok

We next give the matrix characterizations from $\ell(X,p)$ into ℓ_M when $p_k > 1$ for all $k \in N$. To obtain this, we first give a general result concerning matrix transformations between FK-spaces.

THEOREM 3.5. *Let $E \subseteq W(X)$ be an FK-space with AK property and F an FK-space of scalar sequences. Then, for an infinite matrix $A = (f_k^n)$, $A : E \to F$ if and only if*

(1) *for each $n \in N$, $\sum_{k=1}^{\infty} f_k^n(x_k)$ converges for all $x = (x_k) \in E$,*

(2) *for each $k \in N$, $(f_k^n(x))_{n=1}^{\infty} \in F$ for all $x \in X$, and*

(3) *$A : \Phi(X) \to F$ is continuous when $\Phi(X)$ is considered as a subspace of E.*

Proof: Assume that $A : E \to F$. Then we have that for any $x = (x_k) \in E$, $\sum_{k=1}^{\infty} f_k^n(x_k)$ converges for all $n \in N$, so (1) holds. Since $e^k(z) \in E$ for all $k \in N$ and all $z \in X$, we obtain that for each $k \in N$,

$$(f_k^n(z))_{n=1}^{\infty} = Ae^k(z) \in F,$$

hence (2) holds. Since E and F are FK-spaces, by Zeller's theorem, $A : E \to F$ is continuous, so (3) is obtained.

Conversely, assume that the conditions hold. By (1), we have

$$Ax = \left(\sum_{k=1}^{\infty} f_k^n(x_k) \right)_{n=1}^{\infty} \in \overset{\omega}{\cancel{W}}. \text{ for all } x = (x_k) \in E.$$

It follows from (2) that $Ae^k(z) \in F$ for all $k \in N$ and $z \in X$, which implies that $A : \Phi(X) \to F$. By (3), we have $A : \Phi(X) \to F$ is continuous. Let $x = (x_k) \in E$. Since E has the AK property, we have

$$x = \lim_{n \to \infty} \sum_{k=1}^{n} e^k(x_k).$$

Then $\left(\sum_{k=1}^{n} e^k(x_k) \right)_{n=1}^{\infty}$ is a Cauchy sequence in E. Since $A : \Phi(X) \to F$ is continuous and linear, it implies that $\left(\sum_{k=1}^{n} Ae^k(x_k) \right)_{n=1}^{\infty}$ is a Cauchy sequence in F. Since F is complete, we have $\left(\sum_{k=1}^{n} Ae^k(x_k) \right)_{n=1}^{\infty}$ converges in F. Since F is a K-spcae, it implies that

$$\left(\sum_{k=1}^{n} f_k^n(x_k) \right)_{n=1}^{\infty} \in F,$$

so that $Ax \in F$. This shows that $A : E \to F$.

m_n

THEOREM 3.6. *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k > 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X,p) \to \ell_M$ if and only if*

(1) *for each $n \in N$ there exists $m_n \in N$ ~~there exists $M_n \in N$~~ such that*

$$\sum_{k=1}^{\infty} \|f_k^n\|^{t_k} m_n^{-(t_k-1)} < \infty, \quad \text{where } \frac{1}{p_k} + \frac{1}{t_k} = 1 \text{ for all } k \in N,$$

(2) *for each $k \in N$ and $x \in X$, $(f_k^n(x))_{n=1}^{\infty} \in \ell_M$, and*

(3) *there exists $\lambda > 0$ such that*

$$\sup\left\{\sum_{n=1}^{\infty} M_n\left(\frac{1}{\lambda}\sum_{k\in K} f_k^n(x_k)\right) : K \subset N \text{ is finite, } x_k \in X \text{ for all } k \in K \text{ and } \sum_{k\in K}\|x_k\|^{p_k} \le 1\right\} \le 1.$$

Proof: Assume that $A : \ell(X,p) \to \ell_M$. By Proposition 3.1 and Theorem 3.5, conditions (1) and (2) are satisfied. Since $\ell(X,p)$ and ℓ_M are BK-spaces with the Luxemburg norm, by Zeller's Theorem, we have that $A : \ell(X,p) \to \ell_M$ is continuous, so $A : \Phi(X) \to \ell_M$ is continuous when $\Phi(X)$ is considered as a subspace of $\ell(X,p)$. It follows that A is a bounded, hence there exists $\lambda > 0$ such that $\|Ax\| \le \lambda$ for all $x \in \Phi(X)$ such that $\|x\| \le 1$. By ~~Proposition 3.2(i), we have~~ using general results in Musielak (1983), we have

$$\varrho_M\left(\frac{1}{\lambda}Ax\right) = \sum_{n=1}^{\infty} M_n\left(\frac{1}{\lambda}\sum_{k=1}^{\infty} f_k^n(x_k)\right) \le 1 \tag{3.11}$$

for all $x = (x_k) \in \Phi(X)$ such that $\|x\| \le 1$.

Let $K \subset N$ be finite and $x_k \in X$ for all $k \in N$ such that

$$\sum_{k\in K}\|x_k\|^{p_k} \le 1.$$

Let $z = (z_k)$ where $z_k = x_k$ if $k \in K$ and $z_k = 0$ otherwise. Then

$$\sum_{k\in K}\|z_k\|^{p_k} \le 1.$$

This implies by Proposition 3.3(1) that $\|z\| \le 1$. By (3.11), we have

$$\sum_{n=1}^{\infty} M_n\left(\frac{1}{\lambda}\sum_{k\in K} f_k^n(x_k)\right) = \sum_{n=1}^{\infty} M_n\left(\frac{1}{\lambda}\sum_{k=1}^{\infty} f_k^n(z_k)\right) \le 1.$$

This implies that condition (3) is satisfied.

$k \in K$

Conversely, assume that conditions (1), (2) and (3) hold. We will show that the conditions (1), (2) and (3) of Theorem 3.5 are satisfied. The condition (1) implies by Proposition 3.1 that $(f_k^n)_{k=1}^{\infty} \in \ell(X,p)^{\beta}$ for all $n \in N$, so $\sum_{k=1}^{\infty} f_k^n(x_k)$ converges for all $n \in N$ and all $x = (x_k) \in \ell(X,p)$. Thus condition (1) of Theorem 3.5 holds. It is clear that condition (2) of Theorem 3.5 hold. By (3), there exists $\lambda > 0$ such that

$$\sup\left\{\sum_{k=1}^{\infty} M_n\left(\frac{1}{\lambda}\sum_{k\in K} f_k^n(x_k)\right) : K \subset N \text{ is finite,}\right.$$

$$\left. x_k \in X \text{ for all } k \in K \text{ and } \sum_{k\in K} \|x_k\|^{p_k} \leq 1\right\} \leq 1. \tag{3.12}$$

Let $x = (x_k) \in \Phi(X)$ be such that $\|x\| \leq 1$. By Proposition 3.3(1) we have that

$$\sum_{k\in K} \|x_k\|^{p_k} \leq 1$$

for some finite subset K of N. It follows by (3.12) that

$$\sum_{n=1}^{\infty} M_n\left(\frac{1}{\lambda}\sum_{k\in K} f_k^n(x_k)\right) \leq 1,$$

which implies $\|Ax\| \leq \lambda$, hence A is bounded, so $A : \Phi(X) \to \ell_M$ is continuous. Thus condition (3) of Theorem 3.5 is satisfied, so we have by Theorem 3.5 that $A : \ell(X,p) \to \ell_M$. The proof is now complete.

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SOME GEOMETRIC PROPERTIES OF CESARO SEQUENCE SPACE

WINATE SANHAN* AND SUTHEP SUANTAI

ABSTRACT. In this paper we define a modular on the Cesaro sequence space $ces(p)$ and consider it equipped with the Luxemburg norm. We give some relationships between the modular and the Luxemburg norm on this space and show that the space $ces(p)$ has property (H) but it is not rotund (R), where $p = (p_k)$ is a bounded sequence of positive real number with $p_k \geq 1$ for all $k \in \mathbb{N}$.

1. Introduction. Let $(X, \|.\|)$ be a real Banach space, and let $B(X)$ (resp. $S(X)$) be the closed unit ball (resp. the unit sphere) of X.

A point $x \in S(X)$ is an H-point of $B(X)$ if for any sequence (x_n) in X such that $\|x_n\| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x (write $x_n \xrightarrow{\omega} x$) implies that $\|x_n - x\| \to 0$ as $n \to \infty$. If every point in $S(X)$ is an H-point of $B(X)$, then X is said to have the property (H)..

A point $x \in S(X)$ is an extreme point of $B(X)$, if for any $y, z \in S(X)$ the equality $2x = y + z$ implies $y = z$.

A point $x \in S(X)$ is an locally uniformly rotund point of $B(X)$ (LUR-point for short) if for any sequence (x_n) in $B(X)$ such that $\|x_n + x\| \to 2$ as $n \to \infty$ there holds $\|x_n - x\| \to 0$ as $n \to \infty$.

A Banach space X is said to be rotund (R), if every point of $S(X)$ is an extreme point of $B(X)$. If every point of $S(X)$ is a LUR-point of $B(X)$, then X is said to be locally uniformly rotund (LUR).

It is known that if X is LUR, then it is (R) and possesses property (H). For these geometric notions and their role in Mathematics we refer to the monographs [1], [2] , [6] and [13]. Some of them were studied for Orlicz spaces in [3],[7],[8],[9]and [14].

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Let l^0 be the space of all real sequences. For $1 \leq p < \infty$, the Cesaro sequence space (ces_p , for short) is defined by

$$ces_p = \{x \in l^0 : \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^p < \infty\}$$

equipped with the norm

$$\|x\| = (\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^{p_r})^{\frac{1}{p}}$$

and

$$\|x\|_0 = (\sum_{r=0}^{\infty}(\frac{1}{2^r}\sum_r|x(i)|)^p)^{\frac{1}{p}}$$

where $\sum_r$ denotes a sum over the ranges $2^r \leq i < 2^{r+1}$

It is known that these two norms are equivalent and ces_p is Banach with respect to each of the two norms.

This space was introduced by J.S. Shue [15]. It is useful in the theory of matrix operator and others (see [10] and [12]). Some geometric properties of the Cesáro sequence space $(ces_p, \|.\|)$ were studied by many mathematicians. It is known that $(ces_p, \|.\|)$ is LUR and posses property (H) (see [12]). Y. A. Cui and H. Hudzik [4] proved that $(ces_p, \|.\|)$ has the Banach-Saks of type p if $p > 1$, and it was shown in [5] that $(ces_p, \|.\|)$ has property (β).

Now let $p = (p_k)$ be a bounded sequence of positive real number with $p_k \geq 1$ for all $k \in \mathbb{N}$. The Cesaro sequence space $ces(p)$ is defined by

$$ces(p) = \{x \in l^0 : \sum_{r=0}^{\infty}(\frac{1}{2^r}\sum_r|x(i)|)^{p_r}$$

where $\sum_r$ denotes a sum over the ranges $2^r \leq i < 2^{r+1}$.

For $x \in ces(p)$, let $\rho(x) = \sum_{r=0}^{\infty}(\frac{1}{2^r}\sum_r|x(i)|)^{p_r})$ and define the Luxemburg norm on $ces(p)$ by

$$\|x\| = \inf\{\varepsilon > 0 : \rho(\frac{x}{\varepsilon}) \leq 1\}, \quad x \in ces(p).$$

The main purpose of this paper is to show that the Cesaro sequence space $ces(p)$ equipped with the Luxemburg norm has property(H) but it is not rotund ,so it is not LUR.

Throughout this paper we let $M = \sup_r p_r$, and for $x \in l^0$ we put

$$x|_i = (x(1), x(2), ..., x(i), 0, 0, ...)$$

and

$$x|_{\mathbb{N}-i} = (0, 0, ..., 0, x(i+1), x(i+2), ...).$$

we have

$$\begin{aligned}\rho(x) &\leq \rho\left(\frac{(\|x\|+\epsilon)}{\lambda}x\right)\\ &= \rho\left((\|x\|+\epsilon)\frac{x}{\lambda}\right)\\ &\leq (\|x\|+\epsilon)\rho(\frac{x}{\lambda})\\ &\leq \|x\|+\epsilon,\end{aligned}$$

which implies that $\rho(x) \leq \|x\|$. Hence (i) is satisfied.

(ii) Let $\epsilon > 0$ be such that $0 < \epsilon < \dfrac{\|x\|-1}{\|x\|}$, then $1 < (1-\epsilon)\|x\| < \|x\|$. By definition of $\|.\|$ and by Proposition 2.2(i), we have $1 < \rho\left(\frac{x}{(1-\epsilon)\|x\|}\right) \leq \frac{1}{(1-\epsilon)\|x\|}\rho(x)$, so $(1-\epsilon)\|x\| < \rho(x)$ for all $\epsilon \in (0, \frac{\|x\|-1}{\|x\|})$, which implies that $\|x\| \leq \rho(x)$.

(iii) Assume that $\|x\| = 1$. Let $\epsilon > 0$, then there exists $\lambda > 0$ such that $1+\epsilon > \lambda > \|x\|$ and $\rho(\frac{x}{\lambda}) \leq 1$. By Proposition 2.2(ii), we have $\rho(x) \leq \lambda^M \rho(\frac{x}{\lambda}) \leq \lambda^M < (1+\epsilon)^M$, so $(\rho(x))^{\frac{1}{M}} < 1+\epsilon$ for all $\epsilon > 0$ which implies that $\rho(x) \leq 1$.

If $\rho(x) < 1$, let $a \in (0,1)$ such that $\rho(x) < a^M < 1$. From Proposition 2.2(i), we have $\rho(\frac{x}{a}) \leq \frac{1}{a^M}\rho(x) < 1$, hence $\|x\| \leq a < 1$, which is a contradiction. Thus, we have $\rho(x) = 1$.

Conversely, assume that $\rho(x) = 1$. By definition of $\|.\|$, we conclude that $\|x\| \leq 1$. If $\|x\| < 1$, then we have by (i) that $\rho(x) \leq \|x\| < 1$, which contradicts to our assumption, so we obtain that $\|x\| = 1$.

(iv) follows from (i) and (iii).

(v) follows from (iii) and (iv).

Proposition 2.4 *For $x \in ces(p)$ we have*

(i) if $0 < a < 1$ and $\|x\| > a$, then $\rho(x) > a^M$ and

(ii) if $a \geq 1$ and $\|x\| < a$, then $\rho(x) < a^M$.

Proof. (i) Suppose $0 < a < 1$ and $\|x\| > a$. Then $\|\frac{x}{a}\| > 1$. By Proposition 2.3(ii), we have $\rho(\frac{x}{a}) > 1$. Hence, by Proposition 2.2(i), we obtain that $\rho(x) \geq a^M\rho(\frac{x}{a}) > a^M$.

(ii) Suppose $a \geq 1$ and $\|x\| < a$. Then $\|\frac{x}{a}\| < 1$. By Proposition 2.3(i), we have $\rho(\frac{x}{a}) < 1$. If $a = 1$, we have $\rho(x) < 1 = a^M$. If $a > 1$, by Proposition 2.2(ii), we obtain that $\rho(x) < a^M \rho(\frac{x}{a}) < a^M$.

Proprosition 2.5 *Let* (x_n) *be a sequence in* $ces(p)$.

(i) If $\lim_{n\to\infty} \|x_n\| = 1$, *then* $\lim_{n\to\infty} \rho(x_n) = 1$.

(ii) If $\lim_{n\to\infty} \rho(x_n) = 0$ *then* $\lim_{n\to\infty} \|x_n\| = 0$.

Proof. (i) Suppose $\lim_{n\to\infty} \|x_n\| = 1$. Let $\epsilon \in (0,1)$. Then there exists $N \in \mathbb{N}$ such that $1-\epsilon < \|x_n\| < 1+\epsilon$ for all $n \geq \mathbb{N}$. By Proposition 2.4, $(1-\epsilon)^M < \rho(x_n) < (1+\epsilon)^M$ for all $n \geq \mathbb{N}$, which implies that $\lim_{n\to\infty} \rho(x_n) = 1$.

(ii) Suppose $\|x_n\| \not\to 0$. Then there is an $\epsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By Proprosition 2.4 (i), we obtain $\rho(x_{n_k}) > \epsilon^M$ for all $k \in \mathbb{N}$. This implies $\rho(x_n) \not\to 0$ as $n \to \infty$.

Lemma 2.6 *Let* (x_n) *be a sequence in* $ces(p)$.*If* $\rho(x_n) \to \rho(x)$ *and* $x_n(k) \to x(k)$ $\forall k$, *then* $x_n \to x$ *as* $n \to \infty$.

Proof Suppose that $x_n \not\to x$. By Proposition 2.5 (ii), we have $\rho(\frac{x_n-x}{2}) \not\to 0$.Without loss of generality we may assume that there exists $\epsilon \in (0,1)$ such that $\rho(\frac{x_n-x}{2}) > \epsilon$ for all $n \in \mathbb{N}$. Since $(\rho(\frac{x_n-x}{2}))_{n=1}^{\infty}$ is a bounded sequence, it must have a convergent subsequence. Passing through a subsequence, if necessary we can assume $\rho(\frac{x_n-x}{2}) \to \epsilon_0$ for some $\epsilon_0 \geq \epsilon$. Since $\rho(x) = \lim_{i\to\infty} \rho(x|_{2^i})$ and $(\rho(x|_{2^i}))_{i=0}^{\infty}$ is nondecreasing , we have $\rho(x) = \sup\{\rho(x|_{2^i}) : i \in \mathbb{N}\}$. So there exists $i \in \mathbb{N}$ such that $\rho(x|_{2^i}) > \rho(x) - \epsilon/2$. Thus

$$\rho(x|_{\mathbb{N}-2^i}) < \epsilon/2. \tag{2.1}$$

Since $x_n(k) \to x(k)$ for all $k \in \mathbb{N}$, we have

$$\rho(x_n|_{2^i}) \to \rho(x|_{2^i}) \text{ and } \rho(\frac{x_n-x}{2}|_{2^i}) \to 0 \text{ as } n \to \infty. \tag{2.2}$$

MAIN RESULTS

First, we show that ρ is a convex modular on $ces(p)$.

Proposition 2.1 *The functional ρ is a convex modular on $ces(p)$.*

Proof. It is obvious that $\rho(x) = 0 \Leftrightarrow x = 0$ and $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$.

Let $\alpha \geq 0, \beta \geq 0$ with $\alpha + \beta = 1$. By the convexity of the function $t \to |t|^{p_r}$ for every $r \in \mathbb{N}$, we have

$$\begin{aligned}
\rho(\alpha x + \beta y) &= \sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_r |\alpha x(i) + \beta y(i)|\right)^{p_r} \\
&\leq \sum_{r=0}^{\infty}\left(\alpha\frac{1}{2^r}\sum_r |x(i)| + \beta\frac{1}{2^r}\sum_r |y(i)|\right)^{p_r} \\
&\leq \alpha\sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_r |x(i)|\right)^{p_r} + \beta\sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_r |y(i)|\right)^{p_r} \\
&= \alpha\rho(x) + \beta\rho(y).
\end{aligned}$$

Proposition 2.2 *For $x \in ces(p)$, the modular ρ on $ces(p)$ satisfies the following property*
(i) if $0 < a < 1$, then $a^M\rho(\frac{x}{a}) \leq \rho(x)$ and $\rho(ax) \leq a\rho(x)$,
(ii) if $a > 1$, then $\rho(x) \leq a^M\rho(\frac{x}{a})$,
(iii) if $a \geq 1$, then $\rho(x) \leq a\rho(x) \leq \rho(ax)$.

Proof (i) Let $0 < a < 1$. Then we have

$$\begin{aligned}
\rho(x) &= \sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_r |x_n|\right)^{p_r} \\
&= \sum_{r=0}^{\infty}\left(\frac{a}{2^r}\sum_r |\frac{x_n}{a}|\right)^{p_r} \\
&= \sum_{r=0}^{\infty} a^{p_r}\left(\frac{1}{2^r}\sum_r |\frac{x_n}{a}|\right)^{p_r}
\end{aligned}$$

$$\geq \sum_{r=0}^{\infty} a^M \left(\frac{1}{2^r}\sum_r |\frac{x_n}{a}|\right)^{p_r}$$
$$= a^M \sum_{r=0}^{\infty} \left(\frac{1}{2^r}\sum_r |\frac{x_n}{a}|\right)^{p_r}$$
$$= a^M \rho(\frac{x}{a}).$$

By convexity of ρ, we have $\rho(ax) \leq a\rho(x)$, so (i) is obtained

(ii) Let $a > 1$. Then

$$\rho(x) = \sum_{r=0}^{\infty} \left(\frac{1}{2^r}\sum_r |x_n|\right)^{p_r}$$
$$= \sum_{r=0}^{\infty} a^{p_r} \left(\frac{1}{2^r}\sum_r |\frac{x_n}{a}|\right)^{p_r}$$
$$\leq a^M \sum_{r=0}^{\infty} \left(\frac{1}{2^r}\sum_r |\frac{x_n}{a}|\right)^{p_r}$$
$$= a^M \rho(\frac{x}{a}).$$

Hence (ii) is satisfied. (iii) follows from the convexity of ρ.

Proprosition 2.3 *For any $x \in ces(p)$, we have*

(i) if $\|x\| < 1$, then $\rho(x) \leq \|x\|$,

(ii) if $\|x\| > 1$, then $\rho(x) \geq \|x\|$,

(iii) $\|x\| = 1$ if and only if $\rho(x) = 1$,

(iv) $\|x\| < 1$ if and only if $\rho(x) < 1$ and

(v) $\|x\| > 1$ if and only if $\rho(x) > 1$.

Proof (i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - \|x\|$, so $\|x\| + \epsilon < 1$. By definition of $\|.\|$, there exists $\lambda > 0$ such that $\|x\| + \epsilon > \lambda$ and $\rho(\frac{x}{\lambda}) \leq 1$. By Proposition 2.2(i) and (iii), By the convexity of ρ together with (2.1) and (2.2), we have

$$\begin{aligned}
\varepsilon_0 &= \lim_{n\to\infty} \rho(\frac{x_n - x}{2}) \\
&= \lim_{n\to\infty} [\rho(\frac{x_n - x}{2}|_{2^i}) + \rho(\frac{x_n - x}{2}|_{\mathbb{N}-2^i}) \\
&= \lim_{n\to\infty} \rho(\frac{x_n - x}{2}|_{2^i}) + \lim_{n\to\infty} \rho(\frac{x_n - x}{2}|_{\mathbb{N}-2^i}) \\
&= 0 + \lim_{n\to\infty} \rho(\frac{x_n - x}{2}|_{\mathbb{N}-2^i}) \\
&\leq \frac{1}{2} \lim_{n\to\infty} \rho(x_n|_{\mathbb{N}-2^i}) + \frac{1}{2}\rho(x|_{\mathbb{N}-2^i}) \\
&= \frac{1}{2} \lim_{n\to\infty} (\rho(x_n) - \rho(x_n|_{2^i})) + \frac{1}{2}\rho(x|_{\mathbb{N}-2^i}) \\
&= \frac{1}{2}(\rho(x) - \rho(x|_{2^i})) + \frac{1}{2}\rho(x|_{\mathbb{N}-2^i}) \\
&= \frac{1}{2}\rho(x|_{\mathbb{N}-2^i}) + \frac{1}{2}\rho(x|_{\mathbb{N}-2^i}) \\
&= \rho(x|_{\mathbb{N}-2^i}) \\
&< \epsilon/2 \\
&< \epsilon_0,
\end{aligned}$$

which is a contradiction. Therefore $x_n \to x$ as $n \to \infty$.

Theorem 2.7 *The space ces(p) has the property (H).*

Proof. Let $x \in S(ces(p))$, $x_n \in B(ces(p))$ for all $n \in \mathbb{N}$ such that $x_n \xrightarrow{\omega} x$ and $\|x_n\| \to 1$ as $n \to \infty$. By Proposition 2.3(iii), we have $\rho(x) = 1$. By Proposition 2.5(i), we obtain that $\rho(x_n) \to 1$ as $n \to \infty$. So $\rho(x_n) \to \rho(x)$ as $n \to \infty$. Since $x_n \xrightarrow{\omega} x$, it implies that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$. It follows from Lemma 2.6 that $x_n \to x$ as $n \to \infty$.

The following result is obtained directly from Theorem 2.7.

Corollary 2.8 *For $1 \leq p < \infty$, $(ces_p, \|.\|_0)$ has property (H)*

Remark 2.9 For a bounded sequence of positive real numbers $p = (p_k)$ with $p_k \geq 1$ for all $k \in \mathbb{N}$, the space $ces(p)$ equipped the Luxemburg norm is not rotund , so it is not LUR.

To see this we put

$$x = (0, 1, 1, 0, 0, ...) \text{ and } y = (0, 2, 0, 0, ...)$$

Then $x, y \in S(ces(p))$ because $\rho(x) = \rho(y) = 1$. Since $\rho(\frac{x+y}{2}) = 1$, we have by Proposition 2.3 (iii) that $\|\frac{x+y}{2}\| = 1$. This shows that $ces(p)$ is not rotund, so it is not LUR.

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Brno, April 16, 2002

Dear Dr. Suantai,

We are pleased to inform you that your paper entitled *On the H-property of some Banach sequence spaces* has been accepted for a publication.

Yours sincerely,

Jiří Rosický
Editor-in-Chief

Accepted for publication in Archivum Mathematicum

On the H-Property of Some Banach Sequence Spaces

SUTHEP SUANTAI

ABSTRACT. **In this paper, we define a generalized Cesáro sequence space $ces(p)$ and consider it equipped with the Luxemburg norm under which it is a Banach space , and we show that the space $ces(p)$ posses property (H) and property (G) , and it is rotund, where $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$.**

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1. Preliminaries.

For a Banach space X, we denote by $S(X)$ and $B(X)$ the unit sphere and unit ball of X, respectively. A point $x_0 \in S(X)$ is called

a) an *extreme point* if for every $x, y \in S(X)$ the equality $2x_0 = x + y$ implies $x = y$;

b) an *H-point* if for any sequence (x_n) in X such that $\|x_n\| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x_0 (write $x_n \xrightarrow{w} x_0$) implies that $\|x_n - x\| \to 0$ as $n \to \infty$;

c) a *denting point* if for every $\epsilon > 0$, $x_0 \notin \overline{conv}\{B(X) \backslash (x_0 + \epsilon B(X))\}$.

A Banach space X is said to be *rotund* (R), if every point of $S(X)$ is an extreme point.

A Banach space X is said to posses Property (H) (Property (G)) provided every ponit of $S(X)$ is H-point (denting point).

For these geometric notions and their role in Mathematics we refer to the monographs [1], [2] , [6] and [13]. Some of them were studied for Orlicz spaces in [3],[7],[8],[9]and [14].

Let us denote by l^0 the space of all real sequences. For $1 \leq p < \infty$, the Cesáro sequence space (ces_p, for short) is defined by

$$ces_p = \{x \in l^0 : \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^p < \infty\}$$

equipped with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right)^p\right)^{\frac{1}{p}} .$$

This space was introduced by J.S. Shue [16]. It is useful in the theory of matrix operator and others (see [10] and [12]). Some geometric properties of the Cesáro sequence space ces_p were studied by many mathematicians. It is known that ces_p is LUR and posses property (H) (see [12]). Y. A. Cui and H. Hudzik [4] proved that ces_p has the Banach-Saks of type p if $p > 1$, and it was shown in [5] that ces_p has property (β).

Now, let $p = (p_k)$ be a sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$. The Nakano sequence space $l(p)$ is defined by

$$l(p) = \{x \in l^0 : \sigma(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where $\sigma(x) = \sum_{i=1}^{\infty}|x(i)|^{p_i}$. We consider the space $l(p)$ equipped with the norm

$$\|x\| = \inf\{\lambda > 0 : \sigma(\frac{x}{\lambda}) \leq 1\},$$

under which it is a Banach space. If $p = (p_k)$ is bounded, we have

$$l(p) = \{x \in l^0 : \sum_{i=1}^{\infty}|x(i)|^{p_i} < \infty\}.$$

Several geometric properties of $l(p)$ were studied in [1] and [4].

The Cesáro sequence space $ces(p)$ is defined by

$$ces(p) = \{x \in l^0 : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where $\varrho(x) = \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^{p_n}$. We consider the space $ces(p)$ equipped with the so-called Luxemburg norm

$$\|x\| = \inf\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\}$$

under which it is a Banach space. If $p = (p_k)$ is bounded, then we have

$$ces(p) = \{x = x(i) : \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^{p_n} < \infty\}.$$

W. Sanhan [15] proved that $ces(p)$ is nonsquare when $p_k > 1$ for all $k \in \mathbb{N}$. In this paper, we show that the Cesáro sequence space $ces(p)$ equipped with the Luxemburg norm is rotund (R) and posses property (H) and property (G) when $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$.

Throughout this paper we assume that $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$, and $M = \sup_k p_k$.

2. Main Results

We begin with giving some basic properties of modular on the space $ces(p)$.

Proposition 2.1 *The functional ϱ on the Cesaro sequence space $ces(p)$ is a convex modular.*

Proof. It is obvious that $\varrho(x) = 0 \Leftrightarrow x = 0$ and $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$. If $x, y \in ces(p)$ and $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta = 1$, by the convexity of the function $t \longrightarrow |t|^{p_k}$ for every $k \in \mathbb{N}$, we have

$$\begin{aligned}
\varrho(\alpha x + \beta y) &= \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|\alpha x(i) + \beta y(i)|\right)^{p_k} \\
&\leq \sum_{k=1}^{\infty}\left(\alpha\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right) + \beta\left(\frac{1}{k}\sum_{i=1}^{k}|y(i)|\right)\right)^{p_k} \\
&\leq \alpha\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} + \beta\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|y(i)|\right)^{p_k} \\
&= \alpha\varrho(x) + \beta\varrho(y).
\end{aligned}$$

Proposition 2.2 *For $x \in ces(p)$, the modular ϱ on $ces(p)$ satisfies the following properties :*

(i) if $0 < a < 1$, then $a^M \varrho(\frac{x}{a}) \leq \varrho(x)$ and $\varrho(ax) \leq a\varrho(x)$,

(ii) if $a \geq 1$, then $\varrho(x) \leq a^M \varrho(\frac{x}{a})$,

(iii) if $a \geq 1$, then $\varrho(x) \leq a\varrho(x) \leq \varrho(ax)$.

Proof. It is obvious that (iii) is satisfied by the convexity of ϱ . It remains to prove (i) and (ii) .

For $0 < a < 1$, we have

$$\begin{aligned}
\varrho(x) &= \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} \\
&= \sum_{k=1}^{\infty}\left(\frac{a}{k}\sum_{i=1}^{k}|\frac{x(i)}{a}|\right)^{p_k} \\
&= \sum_{k=1}^{\infty}a^{p_k}\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x(i)}{a}|\right)^{p_k} \\
&\geq \sum_{k=1}^{\infty}a^{M}\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x(i)}{a}|\right)^{p_k} \\
&= a^M\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x(i)}{a}|\right)^{p_k} \\
&= a^M \varrho(\frac{x}{a}),
\end{aligned}$$

and it implies by the convexity of ϱ that $\varrho(ax) \leq a\varrho(x)$, hence (i) is satisfied.

Now , suppose that $a \geq 1$. Then we have

$$\begin{aligned}
\varrho(x) &= \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} \\
&= \sum_{k=1}^{\infty} a^{p_k}\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x(i)}{a}|\right)^{p_k} \\
&\leq a^M \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|\frac{x(i)}{a}|\right)^{p_k} \\
&= a^M \varrho(\frac{x}{a}).
\end{aligned}$$

So (ii) is obtained. □

Next, we give some relationships between the modular ϱ and the Luxemburg norm on $ces(p)$.

Proposition 2.3 *For any $x \in ces(p)$, we have*

(i) if $\|x\| < 1$, then $\varrho(x) \leq \|x\|$,

(ii) if $\|x\| > 1$, then $\varrho(x) \geq \|x\|$,

(iii) $\|x\| = 1$ if and only if $\varrho(x) = 1$,

(iv) $\|x\| < 1$ if and only if $\varrho(x) < 1$,

(v) $\|x\| > 1$ if and only if $\varrho(x) > 1$,

(vi) if $0 < a < 1$ and $\|x\| > a$, then $\varrho(x) > a^M$, and

(vii) if $a \geq 1$ and $\|x\| < a$, then $\varrho(x) < a^M$.

Proof. (i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - \|x\|$, so $\|x\| + \epsilon < 1$. By definition of $\|.\|$, there exists $\lambda > 0$ such that $\|x\| + \epsilon > \lambda$ *and* $\varrho(\frac{x}{\lambda}) \leq 1$. From Proposition 2.2(i) and (iii), we have

$$\begin{aligned}
\varrho(x) &\leq \varrho\left(\frac{(\|x\| + \epsilon)}{\lambda}x\right) \\
&= \varrho\left((\|x\| + \epsilon)\frac{x}{\lambda}\right) \\
&\leq (\|x\| + \epsilon)\varrho(\frac{x}{\lambda}) \\
&\leq \|x\| + \epsilon,
\end{aligned}$$

which implies that $\varrho(x) \leq \|x\|$, so (i) is satisfied.
(ii) Let $\epsilon > 0$ be such that $0 < \epsilon < \dfrac{\|x\| - 1}{\|x\|}$, then $1 < (1-\epsilon)\|x\| < \|x\|$. By definition of $\|.\|$ and by Proposition 2.2 (i), we have

$$\begin{aligned} 1 &< \varrho\left(\frac{x}{(1-\epsilon)\|x\|}\right) \\ &\leq \frac{1}{(1-\epsilon)\|x\|}\varrho(x); \end{aligned}$$

so $(1-\epsilon)\|x\| < \varrho(x)$ for all $\epsilon \in (0, \dfrac{\|x\|-1}{\|x\|})$. This implies that $\|x\| \leq \varrho(x)$, hence (ii) is obtained.
(iii) Assume that $\|x\| = 1$. By definition of $\|x\|$, we have that for $\epsilon > 0$, there exists $\lambda > 0$ such that $1 + \epsilon > \lambda > \|x\|$ and $\varrho(\frac{x}{\lambda}) \leq 1$. From Proposition 2.2(ii), we have $\varrho(x) \leq \lambda^M \varrho(\frac{x}{\lambda}) \leq \lambda^M < (1+\epsilon)^M$, so $(\varrho(x))^{\frac{1}{M}} < 1 + \epsilon$ for all $\epsilon > 0$, which implies $\varrho(x) \leq 1$. If $\varrho(x) < 1$, then we can choose $a \in (0,1)$ such that $\varrho(x) < a^M < 1$. From Proposition 2.2(i), we have $\varrho(\frac{x}{a}) \leq \frac{1}{a^M}\varrho(x) < 1$, hence $\|x\| \leq a < 1$, which is a contradiction. Therefore $\varrho(x) = 1$.

On the other hand , assume that $\varrho(x) = 1$. Then $\|x\| \leq 1$. If $\|x\| < 1$, we have by (i) that $\varrho(x) \leq \|x\| < 1$, which contradicts our assumption. Therefore $\|x\| = 1$.
(iv) follows directly from (i) and (iii).
(v) follows from (iii) and (iv).
(vi) Suppose $0 < a < 1$ and $\|x\| > a$. Then $\left\|\dfrac{x}{a}\right\| > 1$. By (v), we have $\varrho(\dfrac{x}{a}) > 1$. Hence, by Proposition 2.2(i), we obtain that $\varrho(x) \geq a^M \varrho(\frac{x}{a}) > a^M$.
(vii) Suppose $a \geq 1$ and $\|x\| < a$. Then $\left\|\dfrac{x}{a}\right\| < 1$. By (iv), we have $\varrho(\dfrac{x}{a}) < 1$. If $a = 1$, it is obvious that $\varrho(x) < 1 = a^M$. If $a > 1$, then , by Proposition 2.2(ii), we obtain that $\varrho(x) \leq a^M \varrho(\dfrac{x}{a}) < a^M$.

Proposition 2.4 *Let (x_n) be a sequence in $ces(p)$.*

(i) If $\|x_n\| \to 1$ as $n \to \infty$, then $\varrho(x_n) \to 1$ as $n \to \infty$.

(ii) If $\varrho(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$ as $n \to \infty$.

Proof. (i) Suppose $\|x_n\| \to 1$ as $n \to \infty$. Let $\epsilon \in (0,1)$. Then there exists $N \in \mathbb{N}$ such that $1 - \epsilon < \|x_n\| < 1 + \epsilon$ for all $n \geq N$. By Proposition 2.3 (vi) and (vii), we have $(1-\epsilon)^M < \varrho(x_n) < (1+\epsilon)^M$ for all $n \geq N$, which implies that $\varrho(x_n) \to 1$ as $n \to \infty$.

(ii) Suppose $\|x_n\| \not\to 0$ as $n \to \infty$. Then there is an $\epsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By Proprosition 2.3 (vi), we have $\varrho(x_{n_k}) > \epsilon^M$ for all $k \in \mathbb{N}$. This implies $\varrho(x_n) \not\to 0$ as $n \to \infty$.

Next, we shall show that $ces(p)$ has the property (H). To do this, we need a lemma.

Lemma 2.5 *Let $x \in ces(p)$ and $(x_n) \subseteq ces(p)$. If $\varrho(x_n) \to \rho(x)$ as $n \to \infty$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$, then $x_n \to x$ as $n \to \infty$.*

Proof. Let $\epsilon > 0$ be given. Since $\rho(x) = \sum_{k=1}^{\infty}(\frac{1}{k}\sum_{i=1}^{k}|x(i)|)^{p_k} < \infty$, there is $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} < \frac{\epsilon}{3}\frac{1}{2^{M+1}}. \tag{2.1}$$

Since $\rho(x_n) - \sum_{k=1}^{k_0}(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|)^{p_k} \to \rho(x) - \sum_{k=1}^{k_0}(\frac{1}{k}\sum_{i=1}^{k}|x(i)|)^{p_k}$ as $n \to \infty$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that

$$\varrho(x_n) - \sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|\right)^{p_k} < \varrho(x) - \sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} + \frac{\epsilon}{3}\frac{1}{2^M} \tag{2.2}$$

for all $n \geq n_0$, and

$$\sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|\right)^{p_k} < \frac{\epsilon}{3}. \tag{2.3}$$

for all $n \geq n_0$.

It follows from (2.1), (2.2) and (2.3) that for $n \geq n_0$,

$$
\begin{aligned}
\varrho(x_n - x) &= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^{p_k} \\
&= \sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^{p_k} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^{p_k} \\
&< \frac{\epsilon}{3} + 2^M \left(\sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i)| \right)^{p_k} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \right) \\
&= \frac{\epsilon}{3} + 2^M \left(\varrho(x_n) - \sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i)| \right)^{p_k} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \right) \\
&< \frac{\epsilon}{3} + 2^M \left(\varrho(x) - \sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \frac{\epsilon}{3} \frac{1}{2^M} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \right) \\
&= \frac{\epsilon}{3} + 2^M \left(\sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \frac{\epsilon}{3} \frac{1}{2^M} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \right) \\
&= \frac{\epsilon}{3} + 2^M \left(2 \sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \frac{\epsilon}{3} \frac{1}{2^M} \right) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}
$$

This show that $\varrho(x_n - x) \to 0$ as $n \to \infty$. Hence, by Proposition 2.4 (ii), we have $\|x_n - x\| \to 0$ as $n \to \infty$.

Theorem 2.6 *The space $ces(p)$ has the property (H).*

Proof. Let $x \in S(ces(p))$ and $(x_n) \subseteq ces(p)$ such that $\|x_n\| \to 1$ and $x_n \xrightarrow{w} x$ as $n \to \infty$. From Proposition 2.3 (iii), we have $\varrho(x) = 1$, so it follows from Proposition 2.4 (i) that $\varrho(x_n) \to \varrho(x)$ as $n \to \infty$. Since the mapping $p_i : ces(p) \to \mathbb{R}$, defined by $p_i(y) = y(i)$, is a continuous linear functional on $ces(p)$, it follows that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$. Thus, we obtain by Lemma 2.5 that $x_n \to x$ as $n \to \infty$. $\square$

Theorem 2.7 *The space $ces(p)$ is rotund.*

Proof. Let $x \in S(ces(p))$ and $y, z \in B(ces(p))$ with $x = \dfrac{y+z}{2}$. By Proposition 2.3 and the convexity of ϱ we have

$$1 = \varrho(x) \leq \frac{1}{2}(\varrho(y) + \varrho(z)) \leq \frac{1}{2}(1+1) = 1\,,$$

so that $\varrho(x) = \dfrac{1}{2}(\varrho(y) + \varrho(z)) = 1$. This implies that

$$\left(\frac{1}{k}\sum_{i=1}^{k} |\frac{y(i)+z(i)}{2}|\right)^{p_k} = \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k} |y(i)|\right)^{p_k} + \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k} |z(i)|\right)^{p_k} \tag{2.4}$$

for all $k \in \mathbb{N}$.

We shall show that $y(i) = z(i)$ for all $i \in \mathbb{N}$.

From (2.4), we have

$$|x(1)|^{p_1} = \left|\frac{y(1)+z(1)}{2}\right|^{p_1} = \frac{1}{2}(|y(1)|^{p_1} + |z(1)|^{p_1}). \tag{2.5}$$

Since the mapping $t \to |t|^{p_1}$ is strictly convex, it implies by (2.5) that $y(1) = z(1)$.

Now assume that $y(i) = z(i)$ for all $i = 1, 2, 3, ..., k-1$. Then $y(i) = z(i) = x(i)$ for all $i = 1, 2, 3, ..., k-1$. From (2.4), we have

$$\begin{aligned}\left(\frac{1}{k}\sum_{i=1}^{k} |\frac{y(i)+z(i)}{2}|\right)^{p_k} &= \left(\frac{\frac{1}{k}\sum_{i=1}^{k}|y(i)| + \frac{1}{k}\sum_{i=1}^{k}|z(i)|}{2}\right)^{p_k} \\ &= \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k} |y(i)|\right)^{p_k} + \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k} |z(i)|\right)^{p_k} \end{aligned} \tag{2.6}$$

By convexity of the mapping $t \to |t|^{p_k}$, it implies that $\frac{1}{k}\sum_{i=1}^{k}|y(i)| = \frac{1}{k}\sum_{i=1}^{k}|z(i)|$. Since $y(i) = z(i)$ for all $i = 1, 2, 3, ..., k-1$, we get that

$$|y(k)| = |z(k)|. \tag{2.7}$$

If $y(k) = 0$, then we have $z(k) = y(k) = 0$. Suppose that $y(k) \neq 0$. Then $z(k) \neq 0$. If $y(k)z(k) < 0$, it follows from (2.7) that $y(k) + z(k) = 0$. This implies by

(2.6) and (2.7) that

$$\left(\frac{1}{k}\sum_{i=1}^{k-1}|x(i)|\right)^{p_k} = \left(\frac{1}{k}\left(\sum_{i=1}^{k-1}|x(i)|+|y(k)|\right)\right)^{p_k},$$

which is a contradiction. Thus, we have $y(k)z(k) > 0$. This implies by (2.5) that $y(k) = z(k)$. Thus, we have by induction that $y(i) = z(i)$ for all $i \in \mathbb{N}$, so $y = z$. $\square$

Bor-Luh Lin , Pei-Kee Lin and S.L. Troyanski proved (cf. Theorem iii [11]) that element x in a bounded closed convex set K of a Banach space is a denting point of K iff x is an H-point of K and x is an extreme point of K. Combining this result with our results (Theorem 2.6 and Theorem 2.7), we obtain the following result.

Corollary 2.8 *The space $ces(p)$ has the property (G).*

For $1 < r < \infty$, let $p = (p_k)$ with $p_k = r$ for all $k \in \mathbb{N}$. We have that $ces_r = ces(p)$, so the following results are obtained directly from Theorem 2.6, Theorem 2.7 and Corollary 2.8, respectively.

Corollary 2.9 *For $1 < r < \infty$, the Cesáro sequence space ces_r has the property (H).*

Corollary 2.10 *For $1 < r < \infty$, the Cesáro sequence space ces_r is rotund.*

Corollary 2.11 *For $1 < r < \infty$, the Cesáro sequence space ces_r has the property (G).*

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On Some Convexity Properties of Generalized Cesáro Sequence Spaces

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ABSTRACT. In this paper, we define a generalized Cesáro sequence space $ces(p)$ and consider it equipped with the Luxemburg norm under which it is a Banach space , and we show that the space $ces(p)$ is locally uniformly rotund (LUR), where $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$.

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1. Preliminaries.

For a Banach space X, we denote by $S(X)$ and $B(X)$ the unit sphere and unit ball of X, respectively. A point $x_0 \in S(X)$ is called

a) an *extreme point* if for every $x, y \in S(X)$ the equality $2x_0 = x + y$ implies $x = y$;

b) a *locally uniformly rotund point* (LUR-point for short)if for any sequence (x_n) in $B(X)$ such that $\|x_n + x\| \to 2$ as $n \to \infty$ there holds $\|x_n - x\| \to 0$ as $n \to \infty$;

c) an *H-point* if for any sequence (x_n) in X such that $\|x_n\| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x_0 (write $x_n \xrightarrow{w} x_0$) implies that $\|x_n - x\| \to 0$ as $n \to \infty$;

A Banach space X is said to be *rotund* (R), if every point of $S(X)$ is an extreme point.

If every $x \in S(X)$ is a LUR-point, then X is said to be *locally uniformly rotund (LUR)*.

X is said to possess property (H) provided every ponit of $S(X)$ is H-point .

For these geometric notions and their role in Mathematics we refer to the monographs [1], [6] , [12] and [13]. Some of them were studied for Orlicz spaces in [1],[7],[8],[12] and [14].

Let X be a real vector space. A functional $\varrho : X \to [0, \infty]$ is called a *modular* if it satisfies the conditions

(i) $\varrho(x) = 0$ if and only if $x = 0$;

(ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$;

(iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$, for all $x, y \in X$ and all α, $\beta \geq 0$ with $\alpha + \beta = 1$.

The modular ϱ is called *convex* if

(iv) $\varrho(\alpha x + \beta y) \leq \alpha\varrho(x) + \beta\varrho(y)$, for all $x, y \in X$ and all α, $\beta \geq 0$ with $\alpha + \beta = 1$.

If ϱ is a modular in X, we define

$$X_\varrho = \{x \in X : \lim_{\lambda \to 0^+} \varrho(\lambda x) = 0 \}$$

$$\text{and} \quad X^*_\varrho = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0 \}.$$

It is clear that $X_\varrho \subseteq X^*_\varrho$. If ϱ is a convex modular, for $x \in X_\varrho$ we define

$$\|x\| = \inf\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1 \} \tag{1.1}$$

Orlicz [13] proved that if ϱ is a convex modular in X, then $X_\varrho = X^*_\varrho$ and $\|.\|$ is a norm on X_ϱ for which it is a Banach space. The norm $\|.\|$ defined as in (1.1) is called the Luxemburg norm.

A modular ϱ on X is called

(a) *right-continuous* if $\lim_{\lambda \to 1^+} \varrho(\lambda x) = \varrho(x)$ for all $x \in X_\varrho$

(b) *left-continuous* if $\lim_{\lambda \to 1^-} \varrho(\lambda x) = \varrho(x)$ for all $x \in X_\varrho$

(c) *continuous* if it is both left-continuous and right-continuous .

The following known results gave some relationships between the modular ϱ and the Luxemburg norm $\|.\|$ on X_ϱ.

Theorem 1.1 *Let ϱ be a convex modular on X and let $x \in X_\varrho$ and (x_n) a sequence in X_ϱ. Then $\|x_n - x\| \to 0$ as $n \to \infty$ if and only if $\varrho(\lambda(x_n - x)) \to 0$ as $n \to \infty$ for every $\lambda > 0$.*

Proof. See [11, Theorem 1.3].

Theorem 1.2 *Let ϱ be a convex modular on X .and $x \in X_\varrho$.*

(i) If ϱ is right-continuous, then $\|x\| < 1$ if and only if $\varrho(x) < 1$.

(ii) If ϱ is left-continuous, then $\|x\| \leq 1$ if and only if $\varrho(x) \leq 1$.

(iii) If ϱ is continuous, then $\|x\| = 1$ if and only if $\varrho(x) = 1$.

Proof. See [11, Theorem 1.4].

Let us denote by l^0 the space of all real sequences. For $1 \leq p < \infty$, the Cesáro sequence space (ces_p, for short) is defined by

$$ces_p = \{x \in l^0 : \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^p < \infty\}$$

equipped with the norm

$$\|x\| = (\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^p)^{\frac{1}{p}}$$

This space was introduced by J.S. Shue [16]. It is useful in the theory of matrix operators and others (see [9] and [10]). Some geometric properties of the Cesáro sequence space ces_p were studied by many mathematicians. It is known that ces_p is LUR and possesses property (H) (see [10]). Y. A. Cui and H. Hudzik [2] proved that ces_p has the Banach-Saks property , and it was shown in [5] that ces_p has property (β).

Now, let $p = (p_k)$ be a sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$. The Nakano sequence space $l(p)$ is defined by

$$l(p) = \{x \in l^0 : \sigma(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where $\sigma(x) = \sum_{i=1}^{\infty}|x(i)|^{p_i}$. We consider the space $l(p)$ equipped with the norm

$$\|x\| = \inf\{\lambda > 0 : \sigma(\frac{x}{\lambda}) \leq 1\},$$

under which it is a Banach space. If $p = (p_k)$ is bounded, we have

$$l(p) = \{x \in l^0 : \sum_{i=1}^{\infty} |x(i)|^{p_i} < \infty\}.$$

Several geometric properties of $l(p)$ were studied in [1] and [4].

The generalized Cesáro sequence space $ces(p)$ is defined by

$$ces(p) = \{x \in l^0 : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where $\varrho(x) = \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^{p_n}$. We consider this space equipped with the so-called Luxemburg norm

$$\|x\| = \inf\{\lambda > 0 : \varrho(\frac{x}{\lambda}) \leq 1\}$$

under which it is a Banach space. If $p = (p_k)$ is bounded, we have

$$ces(p) = \{x = x(i) : \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^{p_n} < \infty\}.$$

W. Sanhan [15] proved that $ces(p)$ is nonsquare when $p_k > 1$ for all $k \in \mathbb{N}$. In this paper, we show that the Cesáro sequence space $ces(p)$ equipped with the Luxemburg norm is LUR and has property (H) when $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$.

Throughout this paper we assume that $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$, and $M = \sup_k p_k$.

2. Main Results

We begin with giving some basic properties of the modular ϱ on the space $ces(p)$. By convexity of the function $t \to |t|^{p_k}$ for every $k \in \mathbb{N}$, we have that ϱ is a convex modular. So we have the following proposition.

Proposition 2.1 *The functional ϱ on the Cesaro sequence space $ces(p)$ is a convex modular.*

Proposition 2.2 *For $x \in ces(p)$, the modular ϱ on $ces(p)$ satisfies the following properties :*

(i) if $0 < a < 1$, *then* $a^M \varrho(\frac{x}{a}) \leq \varrho(x)$ *and* $\varrho(ax) \leq a\varrho(x)$,

(ii) if $a \geq 1$, *then* $\varrho(x) \leq a^M \varrho(\frac{x}{a})$,

(iii) if $a \geq 1$, *then* $\varrho(x) \leq a\varrho(x) \leq \varrho(ax)$.

Proof. All assertions are clearly obtained by definition of ϱ. □

Proposition 2.3 *The modular* ϱ *on* $ces(p)$ *is continuous.*

Proof. For $\lambda > 1$, by Proposition 2.2 (ii) and (iii), we have

$$\varrho(x) \leq \lambda\varrho(x) \leq \varrho(\lambda x) \leq \lambda^M \varrho(x) \tag{2.1}$$

By taking $\lambda \to 1^+$ in (2.1), we have $\lim_{\lambda\to 1^+} \varrho(\lambda x) = \varrho(x)$. Thus ϱ is right-continuous. If $0 < \lambda < 1$, by Proposition 2.2 (i), we have

$$\lambda^M \varrho(x) \leq \varrho(\lambda x) \leq \lambda\varrho(x) \tag{2.2}$$

By taking $\lambda \to 1^-$ in (2.2), we have that $\lim_{\lambda\to 1^-} \varrho(\lambda x) = \varrho(x)$, hence, ϱ is left-continuous. Thus ϱ is continuous.

Next, we give some relationships between the modular ϱ and the Luxemburg norm on $ces(p)$.

Proposition 2.4 *For any* $x \in ces(p)$, *we have*

(i) if $\|x\| < 1$, *then* $\varrho(x) \leq \|x\|$,

(ii) if $\|x\| > 1$, *then* $\varrho(x) \geq \|x\|$,

(iii) $\|x\| = 1$ *if and only if* $\varrho(x) = 1$,

(iv) $\|x\| < 1$ *if and only if* $\varrho(x) < 1$,

(v) $\|x\| > 1$ *if and only if* $\varrho(x) > 1$,

(vi) if $0 < a < 1$ *and* $\|x\| > a$, *then* $\varrho(x) > a^M$, *and*

(vii) if $a \geq 1$ *and* $\|x\| < a$, *then* $\varrho(x) < a^M$.

Proof. If $\|x\| \leq 1$, it follows by convexity and continuity of ϱ that $\varrho(x) = \varrho\left(\|x\|\frac{x}{\|x\|}\right) \leq \|x\|\varrho\left(\frac{x}{\|x\|}\right) \leq \|x\|$. So (i) is obtained. If $\|x\| > 1$, then there is $\varepsilon_0 > 0$ such that $\|x\| - \varepsilon > 1$ for all $\varepsilon \in (0, \varepsilon_0)$. Consequently, $\varrho(x) = \varrho\left((\|x\| - \varepsilon)\frac{x}{\|x\| - \varepsilon}\right) \geq$

$(\|x\|-\varepsilon)\varrho\left(\frac{x}{\|x\|-\varepsilon}\right) > \|x\|-\varepsilon$, so (ii) is satisfied. It is clear that (iii), (iv) and (v) follow by Theorem 1.2, and properties (vi) and (vii) follow by Proposition 2.2. □

Proposition 2.5 *Let (x_n) be a sequence in ces(p).*

(i) If $\|x_n\| \to 1$ as $n \to \infty$, then $\varrho(x_n) \to 1$ as $n \to \infty$.

(ii) $\|x_n\| \to 0$ as $n \to \infty$ if and only if $\varrho(x_n) \to 0$ as $n \to \infty$.

Proof. (i) Suppose $\|x_n\| \to 1$ as $n \to \infty$. Let $\epsilon \in (0,1)$. Then there exists $N \in \mathbb{N}$ such that $1-\epsilon < \|x_n\| < 1+\epsilon$ for all $n \geq N$. By Proposition 2.4 (vi) and (vii), we have $(1-\epsilon)^M < \varrho(x_n) < (1+\epsilon)^M$ for all $n \geq N$, which implies that $\varrho(x_n) \to 1$ as $n \to \infty$. (ii) It follows from Theorem 1.1 that if $\|x_n\| \to 0$ as $n \to \infty$, then $\varrho(x_n) \to 0$ as $n \to \infty$. For the converse, suppose $\|x_n\| \not\to 0$ as $n \to \infty$. Then there is an $\epsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By Proprosition 2.4 (vi), we have $\varrho(x_{n_k}) > \epsilon^M$ for all $k \in \mathbb{N}$. This implies $\varrho(x_n) \not\to 0$ as $n \to \infty$. □

Proposition 2.6 *Let $(x_n) \subseteq B(l(p))$ and $(y_n) \subseteq B(l(p))$. If $\sigma(\frac{x_n+y_n}{2}) \to 1$, then $x_n(i) - y_n(i) \to 0$ as $n \to \infty$ for all $i \in \mathbb{N}$.*

Proof . We first note that if $x \in B(\ell(p)$, then $\sigma(x) \leq 1$. Supose that $x_n(i) - y(i) \not\to 0$ as $n \to \infty$ for some $i \in \mathbb{N}$. Without loss of generality we may assume that $i = 1$, and then assume without loss of generality (passing to a subsequence if necessary) that , for some $\epsilon > 0$,

$$|x_n(1) - y_n(1)|^{p_1} \geq \epsilon \ \forall n \in \mathbb{N}$$

Thus

$$2^{p_1}(|x_n(1)|^{p_1} + |y_n(1)|^{p_1}) \geq \epsilon \ \forall n \in \mathbb{N}. \tag{2.3}$$

Since the function $t \to |t|^{p_1}$ is uniformly convex, there exists $\delta > 0$ such that

$$\left|\frac{x_n(1)+y_n(1)}{2}\right|^{p_1} \leq (1-\delta)\left(\frac{|x_n(1)|^{p_1} + |y_n(1)|^{p_1}}{2}\right) \ \forall n \in \mathbb{N}. \tag{2.4}$$

It follows from (2.3) and (2.4) that for each $n \in \mathbb{N}$,

$$\begin{aligned}
\sigma(\frac{x_n + y_n}{2}) &= \sum_{i=1}^{\infty} \left| \frac{x_n(i) + y_n(i)}{2} \right|^{p_i} \\
&= \left| \frac{x_n(1) + y_n(1)}{2} \right|^{p_1} + \sum_{i=2}^{\infty} \left| \frac{x_n(i) + y_n(i)}{2} \right|^{p_i} \\
&\leq (1-\delta) \left(\frac{|x_n(1)|^{p_1} + |y_n(1)|^{p_1}}{2} \right) + \frac{1}{2} \sum_{i=2}^{\infty} |x_n(i)|^{p_i} + \frac{1}{2} \sum_{i=2}^{\infty} |y_n(i)|^{p_i} \\
&= \frac{1}{2}\sigma(x_n) + \frac{1}{2}\sigma(y_n) - \delta \left(\frac{|x_n(1)|^{p_1} + |y_n(1)|^{p_1}}{2} \right) \\
&\leq \frac{1}{2} + \frac{1}{2} - \delta \frac{\epsilon}{2^{p_1+1}} \\
&= 1 - \delta \frac{\epsilon}{2^{p_1+1}}.
\end{aligned}$$

This implies that $\sigma(\frac{x_n + y_n}{2}) \not\to 1$ as $n \to \infty$, a contradiction, which finishes the proof.
□

Proposition 2.7 *Let* $(x_n) \subseteq B(ces(p))$ *and* $x \in S(ces(p))$. *If* $\varrho(\frac{x_n + x}{2}) \to 1$ *as* $n \to \infty$, *then* $x_n(i) \to x(i)$ *as* $n \to \infty$ *for all* $i \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$ and $i \in \mathbb{N}$, let

$$s_n(i) = \begin{cases} sgn(x_n(i) + x(i)) \text{ if } x_n(i) + x(i) \neq 0, \\ 1 \text{ if } x_n(i) + x(i) = 0. \end{cases}$$

Hence, we have

$$\begin{aligned}
1 \leftarrow \varrho(\frac{x_n + x}{2}) &= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} \left| \frac{x_n(i) + x(i)}{2} \right| \right)^{p_k} \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} s_n(i) \frac{x_n(i)}{2} + \frac{1}{k} \sum_{i=1}^{k} s_n(i) \frac{x(i)}{2} \right)^{p_k} \qquad (2.5)
\end{aligned}$$

Let $a_n(k) = \frac{1}{k} \sum_{i=1}^{k} s_n(i) x_n(i)$ and $b_n(k) = \frac{1}{k} \sum_{i=1}^{k} s_n(i) x(i)$ for all $n, k \in \mathbb{N}$. Then $(a_n) \in l(p)$ and $(b_n) \in l(p)$, and from (2.5) we have

$$\sigma(\frac{a_n + b_n}{2}) \to 1 \quad \text{as } n \to \infty.$$

Form Proposition 2.6, we have

$$a_n(i) - b_n(i) \to 0 \text{ as } n \to \infty \tag{2.6}$$

for all $i \in \mathbb{N}$. Now, we shall show that $x_n(k) \to x(k)$ as $n \to \infty$ for all $k \in \mathbb{N}$. From (2.6), we have

$$s_n(1)x_n(1) - s_n(1)x(1) \to 0 \text{ as } n \to \infty,$$

this implies $x_n(1) \to x(1)$ as $n \to \infty$. Assume that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \leq k-1$. Then we have

$$s_n(i)(x_n(i) - x(i)) \to 0 \text{ as } n \to \infty \tag{2.7}$$

for all $i \leq k-1$. Since $s_n(k)(x_n(k)-x(k)) = k(a_n(k)-b_n(k)) - \sum_{i=1}^{k-1} s_n(i)(x_n(i)-x(i))$, it follows from (2.6) and (2.7) that $s_n(k)(x_n(k) - x(k)) \to 0$ as $n \to \infty$. This implies $x_n(k) \to x(k)$ as $n \to \infty$. So we have by induction that $x_n(k) \to x(k)$ as $n \to \infty$ for all $k \in \mathbb{N}$.

Theorem 2.8 *The space $ces(p)$ is LUR.*

Proof. Let $(x_n) \subseteq B(ces(p))$ and $x \in S(ces(p))$ be such that $\|x_n + x\| \to 2$ as $n \to \infty$. Then $\|\frac{x_n + x}{2}\| \to 1$ as $n \to \infty$. By Proposition 2.5 (i), we have $\varrho(\frac{x_n + x}{2}) \to 1$ as $n \to \infty$. By Proposition 2.7, we have $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$.

Now, let $\epsilon > 0$ be given. Then there exists $k_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} < \frac{\epsilon}{3} \frac{1}{2^{M+1}}, \tag{2.8}$$

$$\sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^{p_k} < \frac{\epsilon}{3} \quad \text{for all } n \geq n_0, \tag{2.9}$$

$$\sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x_n(i)| \right)^{p_k} > \sum_{k=1}^{k_0} \left(\frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} - \frac{\epsilon}{3} \frac{1}{2^M} \text{ for all } n \geq n_0. \tag{2.10}$$

By Proposition 2.4 (i) and (iii), we have $\varrho(x_n) \leq 1$ for all $n \in \mathbb{N}$ and $\varrho(x) = 1$. From these together with (2.8), (2.9), (2.10) and the fact that $(a+b)^{p_k} \leq 2^{p_k}(a^{p_k} + b^{p_k})$ for

$a, b \geq 0$, we have that for all $n \geq n_0$,

$$\begin{aligned}
\varrho(x_n - x) &= \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|\right)^{p_k} \\
&= \sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|\right)^{p_k} \\
&< \frac{\epsilon}{3} + 2^M\left(\sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k}\right) \\
&= \frac{\epsilon}{3} + 2^M\left(\varrho(x_n) - \sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k}\right) \\
&\leq \frac{\epsilon}{3} + 2^M\left(1 - \sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k}\right) \\
&< \frac{\epsilon}{3} + 2^M\left(1 - \sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} + \frac{\epsilon}{3}\frac{1}{2^M} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k}\right) \\
&= \frac{\epsilon}{3} + 2^M\left(\varrho(x) - \sum_{k=1}^{k_0}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} + \frac{\epsilon}{3}\frac{1}{2^M} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k}\right) \\
&= \frac{\epsilon}{3} + 2^M\left(\sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} + \frac{\epsilon}{3}\frac{1}{2^M} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k}\right) \\
&= \frac{\epsilon}{3} + 2^M\left(2\sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} + \frac{\epsilon}{3}\frac{1}{2^M}\right) \\
&= \frac{\epsilon}{3} + 2^{M+1}\sum_{k=k_0+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^{p_k} + \frac{\epsilon}{3} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

This shows that $\varrho(x_n - x) \to 0$ as $n \to \infty$. By Proposition 2.5(ii) , we have $\|x_n - x\| \to 0$ as $n \to \infty$. This completes the proof of the theorem. □

It is known in general that a locally uniformly rotund space has property (H). So we have the following result.

Corollary 2.9 The space $ces(p)$ possesses property (H).

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