for all $k \in \mathbb{N}$ and u satisfies $|u| \leq u_0$. By $\rho_M(cx) < \infty$, we have $\sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \to 0$ as $k \to \infty$, it followes that $M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \to 0$ as $k \to \infty$, and so $\frac{1}{k}\sum_{i=1}^{k}|x(i)| \to 0$ as $k \to \infty$. Put any $\beta > 0$ and taking $t \in \mathbb{N}$ such that $\frac{\beta}{c} \leq 2^{t-1}$, there exists a positive sequence c_k' such that $\sum_{k=1}^{\infty} c_k' < \infty$ and

$$M_k(2^t u) \leq K^t M_k(u) + c_k'$$

for all $k \in \mathbb{N}$ and u satisfies $|u| \leq \frac{u_0}{2^{t-1}}$. By $\frac{1}{k}\sum_{i=1}^{k}|x(i)| \to 0$ as $k \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{k}\sum_{i=1}^{k}|x(i)| \leq \frac{\frac{u_0}{2^{t-1}}}{c}$ for all $k \geq n_0$. Hence

$$\begin{aligned}
\rho_M(\beta x) &= \sum_{k=1}^{\infty} M_k(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \\
&= \sum_{k=1}^{n_0} M_k(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|) + \sum_{k=n_0+1}^{\infty} M_k(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \\
&= \sum_{k=1}^{n_0} M_k(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|) + \sum_{k=n_0+1}^{\infty} M_k(2\beta\frac{c}{c}\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \\
&= \sum_{k=1}^{n_0} M_k(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|) + \sum_{k=n_0+1}^{\infty} M_k(2^t c\frac{1}{k}\sum_{i=1}^{k}|x(i)|) \\
&\leq \sum_{k=1}^{n_0} M_k(\beta\frac{1}{k}\sum_{i=1}^{k}|x(i)|) + K^t\sum_{k=n_0+1}^{\infty} M_k(c\frac{1}{k}\sum_{i=1}^{k}|x(i)|) + \sum_{k=n_0+1}^{\infty} c_k' < \infty.
\end{aligned}$$

Therefore $Ces_M \subseteq SCes_M$. □

Lemma 6 On Cesàro-Musielak-Orlicz sequence space, if the Musielak-Orlicz function $M = (M_k)$ satisfies condition $(*)$ and $M \in \delta_2$, then

(1) $\|x\| = 1 \Leftrightarrow \rho_M(x) = 1$,

(2) for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x\| < 1-\delta$ whenever $\rho_M(x) < 1-\epsilon$,

(3) for every $\epsilon > 0$ and $c > 0$ there exists a $\delta > 0$ such that for any $x, y \in Ces_M$, we have

$$|\rho_M(x+y) - \rho_M(x)| < \epsilon$$

whenever $\rho_M(x) \leq c$ and $\rho_M(y) \leq \delta$,

(4) for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x\| > 1+\delta$ whenever $\rho_M(x) > 1+\epsilon$, and

(5) for any sequence $(x_n) \subset Ces_M$, $\|x_n\| \to 1$ implies $\rho_M(x_n) \to 1$.

Proof (1) Assume that $\rho_M(x) = 1$. By definition of $\|.\|$, we have that $\|x\| \leq 1$. If $\|x\| < 1$, then we have by Proposition 3(1) that $\rho_M(x) \leq \|x\| < 1$, which contradicts our assumption. Therefore $\|x\| = 1$.

Conversely, assume that $\|x\| = 1$. By Proposition 3.3(1), $\rho_M(x) \leq 1$. Suppose that $\rho_M(x) < 1$. By Theorem 5, we have $\rho_M(cx) < \infty$ for all $c > 1$. By Theorem 2.6.1 the function $c \mapsto \rho_M(cx)$ is continuous, so there exists an $c' > 1$ such that $\rho_M(c'x) = 1$. By using the same proof as in the first path, we have that $\|c'x\| = 1$, so $c' = 1$ which is contradiction.

(2) Suppose (2)is not true. Then there exists a $\epsilon_0 > 0$ and $x_n \in Ces_M$ such that $\rho_M(x_n) < 1 - \epsilon_0$ and $\frac{1}{2} \leq \|x_n\|$ and $\|x_n\| \to 1$. Let $L = \sup_n\{\rho_M(2x_n)\}$ we have that $L < \infty$ since $M \in \delta_2$. Let $a_n = \frac{1}{\|x_n\|} - 1$ we have $a_n \leq 1$ and $a_n \to 0$. Then

$$\begin{aligned}
1 &= \rho_M(\frac{x_n}{\|x_n\|}) \\
&= \rho_M(2a_nx_n + (1-a_n)x_n) \\
&\leq a_n\rho_M(2x_n) + (1-a_n)\rho_M(x_n) \\
&\leq a_nL + (1-\epsilon).
\end{aligned}$$

Hence we have $1 \leq \lim_{n\to\infty}(a_nL + (1-\epsilon)) = 1-\epsilon$, which is a contradiction.

(3) Let $x, y \in Ces_M, \epsilon > 0$ and $c > 0$, by Theorem 2.6.5(3), there exists a $\delta' > 0$ such that for any $a, b \in l_M$, we have

$$|I_M(a+b) - I_M(a)| < \epsilon \tag{3.1}$$

whenever $I_M(a) \leq c$ and $I_M(b) \leq \delta'$. For each $i \in \mathbb{N}$, let

$$s(i) = \begin{cases} sgn(x(i)+y(i)) \text{ if } x(i)+y(i) \neq 0, \\ 1 \text{ if } x(i)+y(i) = 0 \end{cases}$$

we note that

$$\rho_M(x+y) = \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k} |x(i)+y(i)|)$$
$$= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k} s(i)x(i) + \frac{1}{k}\sum_{i=1}^{k} s(i)y(i)). \tag{3.2}$$

Let $a(k) = \frac{1}{k}\sum_{i=1}^{k} s(i)x(i)$ and $b(k) = \frac{1}{k}\sum_{i=1}^{k} s(i)y(i)$ for all $k \in \mathbb{N}$. Then $a = (a(k)) \in l_M$ and $b = (b(k)) \in l_M$, and from (3.2) we have

$$\rho_M(x+y) = I_M(a+b), I_M(a) \le \rho_M(x) \text{ and } I_M(b) \le \rho_M(y).$$

Choose $\delta = \delta'$. If $\rho_M(x) \le c$ and $\rho_M(y) \le \delta$ then $I_M(a) \le c$ and $I_M(b) \le \delta'$, by (1) we have

$$\rho_M(x+y) - \rho_M(x) \le I_M(a+b) - I_M(a) < \epsilon$$

that is

$$\rho_M(x+y) < \rho_M(x) + \epsilon. \tag{3.3}$$

Next, we shall show that

$$\rho_M(x) < \rho_M(x+y) + \epsilon. \tag{3.4}$$

For each $i \in \mathbb{N}$, let

$$s(i) = \begin{cases} sgn(x(i)) \text{ if } x(i) \neq 0, \\ 1 \text{ if } x(i) = 0 \end{cases}$$

we note that

$$\rho_M(x) = \rho_M((x+y)+(-y)) = \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k} |(x(i)+y(i)) + (-y(i))|)$$
$$= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k} s(i)(x(i)+y(i)) + \frac{1}{k}\sum_{i=1}^{k} s(i)(-y(i))). \tag{3.5}$$

Let $a(k) = \frac{1}{k}\sum_{i=1}^{k} s(i)(x(i)+y(i))$ and $b(k) = \frac{1}{k}\sum_{i=1}^{k} s(i)(-y(i))$ for all $k \in \mathbb{N}$. Then $a = (a(k)) \in l_M$ and $b = (b(k)) \in l_M$, and from (3.5) we have

$$\rho_M(x) = I_M(a+b), I_M(a) \le \rho_M(x+y) \text{ and } I_M(b) \le \rho_M(-y).$$

Choose $\delta = \delta'$. If $\rho_M(x) \leq c$ and $\rho_M(y) \leq \delta$ then $I_M(a+b) = \rho_M(x) \leq c$ and $I_M(-b) = I_M(b) \leq \rho_M(-y) = \rho_M(y) \leq \delta'$, by (3.1) we have

$$|I_M(a+b) - I_M(a)| = |I_M(a) - I_M(a+b)| = |I_M((a+b)+(-b)) - I_M(a+b)| < \epsilon$$

it follows that

$$\rho_M(x) - \rho_M(x+y) \leq I_M(a+b) - I_M(a) < \epsilon$$

that is

$$\rho_M(x) < \rho_M(x+y) + \epsilon$$

from (3.3) and (3.4), we have that

$$|\rho_M(x+y) - \rho_M(x)| < \epsilon$$

whenever $\rho_M(x) \leq c$ and $\rho_M(y) \leq \delta$.

(4) Given $\epsilon > 0$, by (3), there exists a $\delta \in (0,1)$ such that

$$\rho_M(u) \leq 1, \rho_M(v) \leq \delta \Rightarrow \rho_M(u+v) \leq \rho_M(u) + \epsilon.$$

Suppose that $\|x\| \leq 1+\delta$, then $\rho_M(\frac{x}{1+\delta}) \leq 1$ and $\rho_M(\frac{\delta x}{1+\delta}) \leq \delta\rho_M(\frac{x}{1+\delta}) \leq \delta$. This implies

$$\begin{aligned}
\rho_M(x) &= \rho_M(\frac{x}{1+\delta} + \frac{\delta x}{1+\delta}) \\
&\leq \rho_M(\frac{x}{1+\delta}) + \epsilon \\
&\leq 1 + \epsilon.
\end{aligned}$$

(5) Suppose that $\rho_M(x_n) \nrightarrow 1$ as $n \to \infty$, there exits a $\epsilon_0 > 0$ such that

$$|\rho_M(x_n) - 1| > \epsilon_0 \text{ for all } n \in \mathbb{N},$$

it follows that

$$\rho_M(x_n) - 1 > \epsilon_0 \text{ or } \rho_M(x_n) - 1 < -\epsilon_0 \text{ for all } n \in \mathbb{N}.$$

If $\rho_M(x_n) - 1 > \epsilon_0$, that is $\rho_M(x_n) > 1 + \epsilon_0$, by (4), there exists a $\delta > 0$ such that $\|x_n\| > 1+\delta$. If $\rho_M(x_n) - 1 < -\epsilon_0$, that is $\rho_M(x_n) < 1 - \epsilon_0$, by (2), there exists a $\delta' > 0$ such that $\|x_n\| < 1 - \delta'$, so that $\|x_n\| \nrightarrow 1$ as $n \to \infty$, which contradiction. □

Proposition 7 In Cesàro-Musielak-Orlicz sequence space. If a Musielak-Orlicz function $M = (M_k)$ satisfies condition $(*)$ and $M \in \delta_2$, then the norm convergence and modular convergence coincide.

Proof From Corollary 4, it suffices to prove that modular convergence implies norm convergence. For this let $\epsilon \in (0, \frac{1}{2})$, choose a positive integer K such that $\frac{1}{2^{K+1}} < \epsilon \leq \frac{1}{2^K}$. By Lemma 6(3), there exists a $\delta \in (0, \frac{1}{2^{K+1}})$ such that

$$\rho_M(u) \leq 1, \rho_M(v) \leq \delta \Rightarrow \rho_M(u+v) < \rho_M(u) + \epsilon.$$

Suppose that $\rho_M(x) < \delta$, we observe that

$$\rho_M(nx) < n\rho_M(x) + n\epsilon,$$

for $n = 1, ..., 2^{K-1}$. In particular, $\rho_M(\frac{x}{4\epsilon}) \leq \rho_M(2^{K-1}x) < 2^{K-1}\rho_M(x) + 2^{K-1}\epsilon < \frac{1}{2} + \frac{1}{2} = 1$. This implies $\|x\| < 4\epsilon$. □

Theorem 8 If $M \in \delta_2$ and M satisfies condition (*),then the space ces_M is **(UKK)**

Proof Assume that $M \in \delta_2$ and suppose that ces_M is not **(UKK)**.Then there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$ ther are a sequence (x_n) in $S(ces_M)$ and $x \in ces_M$ with $\text{sep}(x_n) \geq \varepsilon_0$,$x_n \overset{\omega}{\to} x$ and $\|x\| > 1 - \delta$. Since $\text{sep}(x_n) \geq \varepsilon_0$ passing subsequence we may assume that $\|x_n - x\| \geq \frac{\varepsilon_0}{2}$ for every $n \in \mathbb{N}$ Since $M \in \delta_2$ and M satisfies condition (*) and x can be assumed to have $\|x\|$ close to 1,ther exists $\eta > 0$ such that $\rho_M(x_n - x) \geq \eta$ and $\rho_M(x) > 1 - \frac{\eta}{5}$. Applying Lemma 6(3) there exists $\sigma \in (0, \frac{\eta}{5})$ such that

$$|\rho_M(x+y) - \rho_M(x)| < \frac{\eta}{5}$$

when ever $\rho_M(y) < \sigma$.

Since $(x_n) \subseteq S(ces_M)$ and $x_n \overset{\omega}{\to} x$,there exists $i_0 \in \mathbb{N}$ such that $\sum_{k=i_0+1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k} |x($ σ. By $x_n \overset{\omega}{\to} x$, which implies that $x_n \to x$ coordinatewise,hence there exists $n_0 \in \mathbb{N}$ such that

$$|\sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) - \sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|)| < \frac{\eta}{5} \text{ and } \sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|) < \frac{\eta}{5}$$

for $n \geq n_0$. So

$$1 = \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) = \sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) + \sum_{k=i_0+1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|)$$
$$\geq \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|) - \frac{\eta}{5} + \sum_{k=i_0+1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|)$$

Hence for every $n \geq n_0$ we have

$$\eta \leq \rho_M(x_n - x) = \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|)$$
$$= \sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|) + \sum_{k=i_0+1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|)$$
$$< \frac{\eta}{5} + \sum_{k=i_0+1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) + \frac{\eta}{5}$$
$$= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) - \sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) + \frac{2\eta}{5}$$
$$= 1 - \sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|) + \frac{2\eta}{5}$$
$$\leq 1 - \sum_{k=1}^{i_0} M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|) + \frac{\eta}{5} + \frac{2\eta}{5}$$
$$\leq 1 - (1-\sigma) + \frac{3\eta}{5}$$
$$\leq 1 - (1-\frac{\eta}{5}) + \frac{3\eta}{5} < \eta.$$

This is a contradiction.

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บทที่ 3
การแปลงเมทริกซ์ของปริภูมิลำดับ
(Matrix Transformations of Sequence Spaces)

ในบทนี้เราศึกษาเกี่ยวกับการแปลงเมทริกซ์ของปริภูมิลำดับ เราได้ให้ลักษณะเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิลำดับค่าเวกเตอร์นาคาโน $\ell(\Delta,p)$ และ $F_r(X,p)$ ไปยังปริภูมิ $E_r, \ell_\infty, \underline{\ell}_\infty(q), bs, cs$ เมื่อ $p_k > 1$ สำหรับทุก $k \in N$ และเรายังได้ให้ให้ลักษเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิ $\ell(\Delta,p)$ และ $M_0(X,p)$ ไปยังปริภูมิ E_r เมื่อ $p_k \le 1$ สำหรับทุก $k \in N$ เราสามารถให้ลักษณะเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิ FK ใดๆไปยังปริภูมิ $c(q)$ นอกเหนือจากนี้แล้วเราได้ให้ลักษณะเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิดับของแมดดอกซ์ไปยังปริภูมิลำดับมูสิลัก-ออร์ลิคซ์

Matrix Transformations of Nakano Vector-Valued Sequence Space

Suthep Suantai

Abstract. **In this paper, we give necessary and sufficient conditions for infinite matrices mapping Nakano vector-valued sequence space $\ell(X,p)$ into the sequence spaces $E_r(r \geq 0)$ and we also give the matrix characterlizations from $M_0(X,p)$ into the space E_r where $p = (p_k)$ is a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$.**

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1. INTRODUCTION

For $r \geq 0$, the normed sequence space E_r was first defined by Cooke [1] as follows:

$$E_r = \{ \; x = (x_k) \mid \sup_k \frac{|x_k|}{k^r} \; < \; \infty \; \}$$

equipped with the norm

$$\|x\| = \sup_k \frac{|x_k|}{k^r}.$$

Let $(X, \; \|.\|)$ be a Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. We write $x = (x_k)$ with x_k in X for all $k \in N$. The X-valued sequence spaces $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, and $M_0(X,p)$ are defined as

$$\begin{aligned}
c_0(X,p) &= \{x=(x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\},\\
c(X,p) &= \{x=(x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \ \text{ for some } a \in X\},\\
\ell_\infty(X,p) &= \{x=(x_k) : \sup_k \ \|x_k\|^{p_k} < \infty\},\\
\ell(X,p) &= \{x=(x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\},\\
M_0(X,p) &= \cup_{n=1}^{\infty} \ell(X)_{(n^{-1/p_k})}
\end{aligned}$$

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, $\ell(p)$, and $M_0(p)$, respectively. The spaces $c_0(p)$, $c(p)$, $\ell_\infty(p)$ are known as the sequence spaces of Maddox. These spaces were first introduced and studied by Simons [7], Maddox [4, 5]. The space $\ell(p)$ was first defined by Nakano [6] and it is known as the Nakano sequence space and the space $\ell(X,p)$ is known as the Nakano vector-valued sequence space. The spaces $M_0(p)$ was first introduced by Grosse-Erdmann [2] and he has investigated the structure of the spaces $c_0(p)$, $c(p)$ and $\ell_\infty(p)$. Grosse-Erdmann [3] gave the matrix characterizations between scalar-valued sequence spaces of Maddox. Wu and Liu [9] dealt with the problem of characterizations those infinite matrices mapping $c_0(X,p)$, $\ell_\infty(X,p)$ into $c_0(q)$ and $\ell_\infty(q)$ where $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers.

Suantai [8] gave necessary and sufficient conditions for infinite matrices mapping $\ell(X,p)$ into ℓ_∞ and $\ell_\infty(q)$ where $p=(p_k)$ and $q=(q_k)$ are bounded sequence positive real numbers with $p_k \le 1$ for all $k \in N$.

In this paper we give characterizations of infinite matrices mapping $\ell(X,p)$ and $M_0(X,p)$ into the sequence space E_r when $p_k \le 1$ for all $k \in N$ and $r \ge 0$. Some results in [8] are obtained as special cases of this paper.

2. Notation and Definitions

Let $(X, \|.\|)$ be a Banach space. The space of all sequences and the space of all finite sequences in X are denoted by $W(X)$ and $\Phi(X)$, respectively. When X is K, the scalar field of X, the corresponding spaces are written as w and Φ.

A sequence space in X is a linear subspace of $W(X)$. Let E be an X-valued sequence space. For $x \in E$ and $k \in N$, we write x_k standing for the k^{th} term of x. For $x \in X$ and $k \in N$, let $e^k(x)$ be the sequence $(0, 0, ..., 0, x, 0, ...)$ with x in the k^{th} position and let $e(x)$ be the sequence $(x, x, x, ...)$. For a fixed scalar sequence $\mu = (\mu_k)$ the sequence space E_μ is defined as

$$E_\mu = \{x \in W(X) : (\mu_k x_k) \in E\} .$$

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map* E into F, written by $A : E \to F$ if for each $x = (x_k) \in E$, $A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$, and the sequence $Ax = (A_n(x)) \in F$. Let (E, F) denote for the set of all infinite matrices mapping from E into F.

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if for each $k \in N$ the k^{th} coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. If, in addition, (E, τ) is an Fréchet (Banach, LF-, LB-) space, then E is called an *FK- (BK-, LFK-, LBK-) space.* Now, suppose that E contains $\Phi(X)$. Then E is said to have *property AB* if the set $\{\sum_{k=1}^{n} e^k(x_k) : n \in N \}$ is bounded in E for every $x = (x_k) \in E$. It is said to have *property AK* if $\sum_{k=1}^{n} e^k(x_k) \to x$ in E as $n \to \infty$ for every $x = (x_k) \in E$. It has *property AD* if $\Phi(X)$ is dense in E.

It is known that E_r is a BK-sapce and $E_0 = \ell_\infty$. The space $\ell(X, p)$ is an FK-space with AK under the paranorm $g(x) = \left(\sum_{k=1}^{\infty} \|x_k\|^{p_k}\right)^{1/M}$, where $M = max\ \{1, \sup_k p_k\}$. In each of the space $\ell_\infty(X, p)$ and $c_0(X, p)$ we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = max\ \{1, \sup_k p_k\}$. It is known that $c_0(X, p)$ is an FK-space with AK under the paranorm g defined as above and $\ell_\infty(X, p)$ is a complete LBK-space with AB.

3. Main Results

We start with giving the matrix characterizations from $\ell(X, p)$ into E_r.

Theorem 3.1 *Let $r \geq 0$ and let $p = (p_k)$ be bounded sequences of positive real numbers with $p_k \leq 1$ and let $A = (f_k^n)$ be an infinite matrix. Then $A \in (\ell(X,p), E_r)$ if and only if there is $m_0 \in N$ such that $\sup_{n,\,k} m_0^{-1/p_k} n^{-r} \|f_k^n\| < \infty$.*

Proof. Assume that $A \in (\ell(X,p), E_r)$. In $\ell(X,p)$, we consider it as a paranormed space with the paranorm g defined as above and since $p_k \leq 1$ for all $k \in N$, we have $M = max\ \{1, \sup_k p_k\} = 1$. Now, we write $\|.\|$ standing for the paranorm g. By Zeller's theorem, $A : \ell(X,p) \to E_r$ is continuous. Then there is $m_0 \in N$ such that

$$\sup_n \Big|\sum_{k=1}^{\infty} f_k^n(x_k)\Big| \leq 1 \quad \text{for all } x \in \ell(X,p) \text{ with } \|x\| \leq \frac{1}{m_0} . \tag{3.1}$$

Let $n, k \in N$ be fixed and let $x_k \in X$ be such that $\|x_k\| \leq 1$. Then $e^{(k)}(m_0^{-1/p_k} x_k) \in \ell(X,p)$ and $\|e^{(k)}(m_0^{-1/p_k} x_k)\| \leq \dfrac{1}{m_0}$. By (3.1), we have

$$m_0^{-1/p_k} n^{-r} |f_k^n(x_k)| \leq \sup_{i \in N} i^{-r} |f_k^i(m_0^{-1/p_k} x_k)| = \|Ae^{(k)}(m_0^{-1/p_k} x_k)\| \leq 1 .$$

It implies that $\sup_{n,k} m_0^{-1/p_k} n^{-r} \|f_k^n\| < \infty$.

Conversely, assume that the condition holds. Let $x = (x_k) \in \ell(X,p)$. By assumption, there is a $C > 0$ such that

$$m_0^{-1/p_k} n^{-r} \|f_k^n\| < C \quad \text{for all } n, k \in N . \tag{3.2}$$

Since $\|m_0^{1/p_k} x_k\| \to 0$ as $k \to \infty$, there is a $k_0 \in N$ such that $\|m_0^{1/p_k} x_k\| < 1$ for all $k \geq k_0$. Since $0 < p_k \leq 1$ for all $k \in N$, we have

$$\|m_0^{1/p_k} x_k\| \leq \|m_0^{1/p_k} x_k\|^{p_k} \quad \text{for all } k \geq k_0. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned}\sum_{k=1}^{\infty}\|m_0^{1/p_k}x_k\| &= \sum_{k=1}^{k_0}\|m_0^{1/p_k}x_k\| + \sum_{k=k_0+1}^{\infty}\|m_0^{1/p_k}x_k\| \\ &\le \sum_{k=1}^{k_0}\|m_0^{1/p_k}x_k\| + \sum_{k=k_0+1}^{\infty}\|m_0^{1/p_k}x_k\|^{p_k} \\ &= K_1 + m_0\sum_{k=k_0+1}^{\infty}\|x_k\|^{p_k} \\ &\le K_1 + m_0\|x\|,\ K_1 = \sum_{k=1}^{k_0}\|m_0^{1/p_k}x_k\|. \end{aligned} \tag{3.4}$$

By (3.2) and (3.4) we have for $n \in N$,

$$\begin{aligned} n^{-r}|A_n x| &= n^{-r}\Big|\sum_{k=1}^{\infty} f_k^n\big(m_0^{-1/p_k}(m_0^{1/p_k}x_k)\big)\Big| \\ &\le \sum_{k=1}^{\infty} m_0^{-1/p_k} n^{-r}\|f_k^n\|.\|m_0^{1/p_k}x_k\| \\ &\le C\sum_{k=1}^{\infty}\|m_0^{1/p_k}x_k\| \\ &\le C(K_1 + m_0\|x\|). \end{aligned}$$

This implies that $\sup_n n^{-r}|A_n x| < \infty$, so that $Ax \in E_r$. This completes the proof. □

When $r = 0$, we see that $E_r = \ell_\infty$, so we obtain the following result directly from Theorem 3.1.

Corollary 3.2 *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k \le 1$ for all $k \in N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), \ell_\infty)$ if and only if there is $m_0 \in N$ such that $\sup_{n,k} m_0^{-1/p_k}\|f_k^n\| < \infty$.*

If $p_k = s \le 1$ for all $k \in N$, by Theorem 3.1 we obtain the following result:

Corollary 3.3 *Let $r \ge 0$ and $0 < s \le 1$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell_s(X),\ E_r)$ if and only if $\sup_{n,k} n^{-r}\|f_k^n\| < \infty$.*

When $p_k = 1$ for all $k \in N$ and $r = 0$, we obtain the following result by Corollary 3.3

Corollary 3.4 *For an infinite matrix $A = (f_k^n)$, $A \in (\ell(X),\ \ell_\infty)$ if and only if* $\sup\limits_{n,\,k} \|f_k^n\| < \infty$.

Theorem 3.5 *Let $r \geq 0$ and let $p = (p_k)$ be bounded sequences of positive real numbers and let $A = (f_k^n)$ be an infinite matrix. Then $A \in (M_0(X,p),\ E_r)$ if and only if for each $s \in N$, $\sup\limits_{n,\,k}\ n^{-r}s^{1/p_k}\|f_k^n\| \ < \ \infty$*

Proof. Since $M_0(X,p) = \cup_{n=1}^{\infty}\ell(X)_{(n^{-1/p_k})}$, we have

$$A \in (M_0(X,p),\ E_r) \iff A \in (\ell(X)_{(s^{-1/p_k})},\ E_r) \text{ for all } s \in N$$

For $s \in N$, we can easily show that

$$A \in (\ell(X)_{(s^{-1/p_k})},\ E_r) \iff \left(s^{1/p_k} f_k^n\right)_{n,k} \in (\ell(X),\ E_r)\ .$$

By Theorem 3.1, we obtain that for $s \in N$,

$$\left(s^{1/p_k} f_k^n\right)_{n,k} \in (\ell(X),\ E_r) \iff \sup_{n,\,k}\ n^{-r}s^{1/p_k}\|f_k^n\| \ < \ \infty\ .$$

Thus the theorem is proved. $\square$

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Suthep Suantai
Department of Mathematics, Faculty of Science,
Chiang Mai University, Chiang Mai 50200, Thailand.
Email : Scmti005@chiangmai.ac.th, malsutsu@science.cmu.ac.th

On Matrix Transformations Concerning the Nakano Vector-Valued Sequence Space

ABSTRACT. **In this paper, we give the matrix characterizations from Nakano vector-valued sequence space $\ell(X,p)$ and $F_r(X,p)$ into the sequence spaces E_r, ℓ_∞, $\underline{\ell}_\infty(q)$, bs and cs, where $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers such that $p_k>1$ for all $k\in I\!N$ and $r\geq 0$.**

Keywords : Matrix transformations, Nakano vector-valued sequence spaces

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1. INTRODUCTION

Let $(X,\|.\|)$ be a Banach space , $r\geq 0$ and $p=(p_k)$ a bounded sequence of positive real numbers. We write $x=(x_k)$ with x_k in X for all $k\in I\!N$. The X-valued sequence spaces $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, $E_r(X,p)$ and $F_r(X,p)$ are defined as

$$c_0(X,p)=\{x=(x_k):\lim_{k\to\infty}\|x_k\|^{p_k}=0\},$$
$$c(X,p)=\{x=(x_k):\lim_{k\to\infty}\|x_k-a\|^{p_k}=0 \text{ for some } a\in X\},$$
$$\ell_\infty(X,p)=\{x=(x_k):\sup_k\|x_k\|^{p_k}<\infty\},$$
$$\ell(X,p)=\{x=(x_k):\textstyle\sum_{k=1}^\infty\|x_k\|^{p_k}<\infty\},$$
$$E_r(X,p)=\{x=(x_k):\sup_k\|x_k\|^{p_k}/k^r<\infty\},$$
$$F_r(X,p)=\{x=(x_k):\textstyle\sum_{k=1}^\infty k^r\|x_k\|^{p_k}<\infty\},$$
$$\underline{\ell}_\infty(X,p)=\textstyle\bigcap_{n=1}^\infty\{x=(x_k)\ :\ \sup_k\|x_k\|n^{1/p_k}\}.$$

When $X=K$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$,$\ell(p)$, $E_r(p)$, $F_r(p)$ and $\underline{\ell}_\infty(p)$, respectively. The spaces $c_0(p)$, $c(p)$ and $\ell_\infty(p)$

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are known as the sequence spaces of Maddox. These spaces were first introduced and studied by Simons [7], Maddox [4, 5]. The space $\ell(p)$ was first defined by Nakano [6] and it is known as the Nakano sequence space and the space $\ell(X,p)$ is known as the Nakano vector-valued sequence space. When $p_k = 1$ for all $k \in I\!N$, the spaces $E_r(p)$ and $F_r(p)$ are written as E_r and F_r, respectively. These two sequence spaces were first introduced by Cooke [1]. The space $\underline{\ell}_\infty(p)$ was first defined by Grosse-Erdmann [2] and he has given in [3] characterizations of infinite matrices mapping between scalar-valued sequence spaces of Maddox. Wu and Liu [10] gave necessary and sufficient conditions for infinite matrices mapping from $c_0(X,p)$, $\ell_\infty(X,p)$ into $c_0(q)$ and $\ell_\infty(q)$. Suantai [8] has given characterizations of infinite matrices mapping $\ell(X,p)$ into ℓ_∞ and $\underline{\ell}_\infty(q)$ when $p_k \leq 1$ for all $k \in I\!N$ and he has also given in [9] characterizations of those infinite matrices mapping from $\ell(X,p)$ into the sequence space E_r when $p_k \leq 1$ for all $k \in I\!N$.

In this paper, we extend the results of [8] and [9] in the case that $p_k > 1$ for all $k \in I\!N$. Moreover, we also give the matrix characterizations from $\ell(X,p)$ and $F_r(X,p)$ into the sequence spaces bs and cs.

2. Notation and Definitions

Let $(X, \|.\|)$ be a Banach space, the space of all sequences in X is denoted by $W(X)$ and $\Phi(X)$ denotes for the space of all finite sequences in X. When X is K, the scalar field of X , the corresponding spaces are written as w and Φ.

A sequence space in X is a linear subspace of $W(X)$. Let E be an X-valued sequence space. For $x \in E$ and $k \in I\!N$, we write x_k stands for the k^{th} term of x. For $k \in I\!N$ denote by e_k the sequence $(0,0,...,0,1,0,...)$ with 1 in the k^{th} position and by e the sequence $(1,1,1,...)$. For $x \in X$ and $k \in I\!N$, let $e^k(x)$ be the sequence $(0,0,...,0,x,0,...)$ with x in the k^{th} position and let $e(x)$ be the sequence $(x,x,x,...)$. We call a sequence space E *normal* if $(t_k x_k) \in E$ for all $x = (x_k) \in E$ and $t_k \in K$ with $|t_k| = 1$ for all $t_k \in I\!N$. A normed sequence space $(E, \|.\|)$ is said to be *norm monotone* if $x = (x_k)$, $y = (y_k) \in E$ with $\|x_k\| \leq \|y_k\|$ for all $k \in I\!N$ implies $\|x\| \leq \|y\|$. For a fixed scalar sequence $\mu = (\mu_k)$ the sequence space E_μ is defined as

$$E_\mu = \{x \in W(X) : (\mu_k x_k) \in E\} .$$

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map* E into F, written by $A : E \to F$ if for each $x = (x_k) \in E$, $A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in I\!N$, and the sequence $Ax = (A_n(x)) \in F$. Let (E, F) denote for the set of all infinite matrices mapping from E into F.

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if for each $k \in I\!N$, the k^{th} coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. If, in addition, (E, τ) is an Fréchet (Banach, LF-, LB-) space, then E is called an *FK- (BK-, LFK-, LBK-) space.* Now, suppose that E contains $\Phi(X)$, then E is said to have *property AB* if the set $\{\sum_{k=1}^{n} e^k(x_k) : n \in I\!N \}$ is bounded in E for every $x = (x_k) \in E$. It is said to have *property AK* if $\sum_{k=1}^{n} e^k(x_k) \to x$ in E as $n \to \infty$ for every $x = (x_k) \in E$. It has *property AD* if $\Phi(X)$ is dense in E.

It is known that the Nakano sequence space $\ell(X, p)$ is an FK-space with property AK under the paranorm $g(x) = \left(\sum_{k=1}^{\infty} \|x_k\|^{p_k}\right)^{1/M}$, where $M = max\ \{1, \sup_k p_k\}$. If $p_k > 1$ for all $k \in I\!N$, then $\ell(X, p)$ is a BK-space with the Luxemburg norm defined by

$$\|(x_k)\| = \inf \{\varepsilon : \sum_{k=1}^{\infty} \|x_k/\varepsilon\|^{p_k} \leq 1 \}$$

3. Main Results

We first give a characterization of an infinite matrix mapping from $\ell(X, p)$ into E_r when $p_k > 1$ for all $k \in I\!N$. To do this, we need a lemma.

Lemma 3.1 *Let E be an $X-$valued $BK-$space which is normal and norm monotone and $A = (f_k^n)$ an infinite matrix. Then $A : E \to E_r$ if and only if $sup_n \sum_{k=1}^{\infty} |f_k^n(x_k)|/n^r <$ ∞ for every $x = (x_k) \in E$.*

Proof If the condition holds true, it follows that $sup_n |\sum_{k=1}^{\infty} f_k^n(x_k)|/n^r \leq$ $\sup_n \sum_{k=1}^{\infty} |f_k^n(x_k)|/n^r < \infty$ for every $x = (x_k) \in E$, hence $A : E \to E_r$.

Conversely, assume that $A : E \to E_r$. Since E and E_r are BK-spaces, by Zeller's Theorem, $A : E \to E_r$ is bounded, so there exists $M > 0$ such that

$$\sup_{\substack{n\in I\!N \\ \|(x_k)\|\le 1}} \Big|\sum_{k=1}^{\infty} f_k^n(x_k)\Big|/n^r \le M. \tag{3.1}$$

Let $x = (x_k) \in E$ be such that $\|x\| = 1$. For each $n \in I\!N$, we can choose a scalar sequence (t_k) with $|t_k| = 1$ and $f_k^n(t_k x_k) = |f_k^n(x_k)|$ for all $k \in I\!N$. Since E is normal and norm monotone, we have $(t_k x_k) \in E$ and $\|(t_k x_k)\| \le 1$. It follows from (3.1) that $\sum_{k=1}^{\infty} |f_k^n(x_k)|/n^r = \big|\sum_{k=1}^{\infty} f_k^n(t_k x_k)\big|/n^r \le M$, which implies

$$\sup_{n\in I\!N} \sum_{k=1}^{\infty} |f_k^n(x_k)|/n^r \le M. \tag{3.2}$$

It follows from (3.2) that for every $x = (x_k) \in E$,

$$\sup_{n\in I\!N} \sum_{k=1}^{\infty} |f_k^n(x_k)|/n^r \le M\|x\|.$$

This complete the proof. □

Theorem 3.2 *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in I\!N$ and $1/p_k + 1/q_k = 1$ for all $k \in I\!N$, and let $r \ge 0$. For an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), E_r)$ if and only if there is $m_0 \in I\!N$ such that*

$$\sup_n \sum_{k=1}^{\infty} \|f_k^n\|^{q_k} n^{-rq_k} m_0^{-q_k} < \infty\,. \tag{3.3}$$

Proof. Let $x = (x_k) \in \ell(X,p)$. By the condition, there are $m_0 \in I\!N$ and $K > 1$ such that

$$\sum_{k=1}^{\infty} \|f_k^n\|^{q_k} n^{-rq_k} m_0^{-q_k} < K \text{ for all } n \in I\!N. \tag{3.4}$$

Note that for a, $b \ge 0$, we have

$$ab \le a^{p_k} + b^{q_k}\,. \tag{3.5}$$

It follows by (3.4) and (3.5) that for $n \in I\!N$,

$$\begin{aligned} n^{-r}\Big|\sum_{k=1}^{\infty} f_k^n(x_k)\Big| &= n^{-r}\Big|\sum_{k=1}^{\infty} f_k^n(m_0^{-1}.m_0x_k)\Big| \\ &\leq \sum_{k=1}^{\infty}(n^{-r}m_0^{-1}\|f_k^n\|)(\|m_0x_k\|) \\ &\leq \sum_{k=1}^{\infty} n^{-rq_k}m_0^{-q_k}\|f_k^n\|^{q_k} + m_0^{\alpha}\sum_{k=1}^{\infty}\|x_k\|^{p_k} \\ &\leq K + m_0^{\alpha}\sum_{k=1}^{\infty}\|x_k\|^{p_k}\ , \text{ where } \alpha = \sup_k p_k. \end{aligned}$$

Hence $\sup_n n^{-r}\big|\sum_{k=1}^{\infty} f_k^n(x_k)\big| < \infty$, so that $Ax \in E_r$.

For necessity, assume that $A \in (\ell(X,p),\ E_r)$. For each $k \in I\!N$, we have $\sup_n n^{-r}|f_k^n(x)| < \infty$ for all $x \in X$ since $e^{(k)}(x) \in \ell(X,p)$. It follows by the uniform bounded principle that for each $k \in I\!N$ there is $C_k > 1$ such that

$$\sup_n n^{-r}\|f_k^n\| \leq C_k\ . \tag{3.6}$$

Suppose that (3.3) is not true. Then

$$\sup_n \sum_{k=1}^{\infty}\|f_k^n\|^{q_k}n^{-rq_k}m^{-q_k} = \infty \quad \text{for every } m \in I\!N\ . \tag{3.7}$$

For $n \in I\!N$, we have by (3.6) that for $k, m \in I\!N$,

$$\begin{aligned} \sum_{j=1}^{\infty}\|f_j^n\|^{q_j}n^{-rq_j}m^{-q_j} &= \sum_{j=1}^{k}\|f_j^n\|^{q_j}n^{-rq_j}m^{-q_j} + \sum_{j>k}\|f_j^n\|^{q_j}n^{-rq_j}m^{-q_j} \\ &\leq \sum_{j=1}^{k}C_j^{q_j}m^{-q_j} + \sum_{j>k}\|f_j^n\|^{q_j}n^{-rq_j}m^{-q_j}. \end{aligned}$$

This together with (3.7), we have

$$\sup_n \sum_{j>k}\|f_j^n\|^{q_j}n^{-rq_j}m^{-q_j} = \infty \quad \text{for all } k, m \in I\!N. \tag{3.8}$$

By (3.8) we can choose $0 = k_0 < k_1 < k_2 < \dots$, $m_1 < m_2 < \dots$, $m_i > 4^i$ and a subsequence (n_i) of positive intergers such that for all $i \geq 1$

$$\sum_{k_{i-1}<j\leq k_i} \|f_j^{n_i}\|^{q_j} n_i^{-rq_j} m_i^{-q_j} > 2^i.$$

For each $i \in I\!N$, we can choose $x_j \in X$ with $\|x_j\| = 1$, for $k_{i-1} < j \leq k_i$ such that

$$\sum_{k_{i-1}<j\leq k_i} |f_j^{n_i}(x_j)|^{q_j} n_i^{-rq_j} m_i^{-q_j} > 2^i$$

For each $i \in I\!N$, let $F_i : (0,\infty) \to (0,\infty)$ be defined by

$$F_i(M) = \sum_{k_{i-1}<j\leq k_i} |f_j^{n_i}(x_j)|^{q_j} n_i^{-rq_j} M^{-q_j}.$$

Then F_i is continuous and nonincreasing such that $F(M) \to 0$ as $M \to \infty$. Thus there exists $M_i > 0$ such that $M_i > m_i$ and

$$F(M_i) = \sum_{k_{i-1}<j\leq k_i} |f_j^{n_i}(x_j)|^{q_j} n_i^{-rq_j} M_i^{-q_j} = 2^i \tag{3.9}$$

Put $y = (y_j)$, $y_j = 4^{-i} M_i^{-(q_j-1)} n_i^{-rq_j/p_j} |f_j^{n_i}(x_j)|^{q_j-1} x_j$ for $k_{i-1} < j \leq k_i$. Thus

$$\begin{aligned}
\sum_{j=1}^{\infty} \|y_j\|^{p_j} &= \sum_{i=1}^{\infty} \sum_{k_{i-1}<j\leq k_i} 4^{-ip_j} M_i^{-p_j(q_j-1)} n_i^{-rq_j} \|f_j^{n_i}(x_j)|^{p_j(q_j-1)} \\
&\leq \sum_{i=1}^{\infty} 4^{-i} \sum_{k_{i-1}<j\leq k_i} M_i^{-q_j} n_i^{-rq_j} |f_j^{n_i}(x_j)|^{q_j} \\
&= \sum_{i=1}^{\infty} 4^{-i}.2^i \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} = 1.
\end{aligned}$$

Thus $y = (y_j) \in \ell(X,p)$. Since $\ell(X,p)$ is a BK-space which is normal and norm monotone under the Luxemburg norm, by Lemma 3.1, we obtain that

$$\sup_n \sum_{k=1}^{\infty} \frac{|f_k^n(y_k)|}{n^r} < \infty. \tag{3.10}$$

But we have

$$\begin{aligned}
\sup_n \sum_{j=1}^{\infty} |f_j^n(y_j)|/n^r &\geq \sup_i \sum_{j=1}^{\infty} |f_j^{n_i}(y_j)|/n_i^r \\
&\geq \sup_i \sum_{k_{i-1}<j\leq k_i} |f_j^{n_i}(y_j)|/n_i^r \\
&= \sup_i \sum_{k_{i-1}<j\leq k_i} 4^{-i} M_i^{-(q_j-1)} n_i^{-r(q_j/p_j+1)} |f_j^{n_i}(x_j)|^{q_j} \\
&= \sup_i \sum_{k_{i-1}<j\leq k_i} 4^{-i} M_i^{-(q_j-1)} n_i^{-rq_j} |f_j^{n_i}(x_j)|^{q_j} \\
&= \sup_i \sum_{k_{i-1}<j\leq k_i} \left(|f_j^{n_i}(x_j)|^{q_j} n_i^{-rq_j} M_i^{-q_j}\right) 4^{-i} M_i \\
&\geq \sup_i 2^i = \infty, \text{ because } M_i > 4^i
\end{aligned}$$

This is contradictory with (3.10). Therefore (3.3) is satisfied. □

Theorem 3.3 *Let $p = (p_k)$ be a bounded sequence of positive real numbers such that $p_k > 1$ for all $k \in \mathbb{N}$, $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$, $r \geq 0$ and $s \geq 0$. Then for an infinite matrix $A = (f_k^n)$, $A \in (F_r(X,p),\ E_s)$ if and only if there is $m_0 \in \mathbb{N}$ such that*

$$\sup_n \sum_{k=1}^{\infty} \left(k^{-rq_k/p_k} \|f_k^n\|^{q_k} n^{-sq_k} m_0^{-q_k}\right) < \infty.$$

Proof. Since $F_r(X,p) = \ell(X,p)_{(k^{r/p_k})}$, it is easy to see that

$$A \in (F_r(X,p),\ E_s) \iff \left(k^{-r/p_k} f_k^n\right)_{n,k} \in (\ell(X,p)\ E_s)$$

By Theorem 3.2, we have $\left(k^{-r/p_k} f_k^n\right)_{n,k} \in (\ell(X,p)\ E_s)$ if and only if there is $m_0 \in \mathbb{N}$ such that

$\sup_n \sum_{k=1}^{\infty} \left(k^{-rq_k/p_k} \|f_k^n\|^{q_k} n^{-sq_k} m_0^{-q_k} < \infty\right.$. Thus the theorem is proved. □

Since $E_0 = \ell_\infty$, the following two results are obtained directly from Theorem 3.2 and Theorem 3.3, respectively.

Corollary 3.4 *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in I\!N$ and let $1/p_k + 1/q_k = 1$ for all $k \in I\!N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p),\ \ell_\infty)$ if and only if there is $m_0 \in I\!N$ such that* $\sup_n \sum_{k=1}^{\infty} \|f_k^n\|^{q_k} m_0^{-q_k} < \infty$.

Corollary 3.5 *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in I\!N$ and let $1/p_k + 1/q_k = 1$ for all $k \in I\!N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (F_r(X,p),\ \ell_\infty)$ if and only if there is $m_0 \in I\!N$ such that* $\sup_n \sum_{k=1}^{\infty} \left(k^{-rq_k/p_k} \|f_k^n\|^{q_k} m_0^{-q_k}\right) < \infty$.

Theorem 3.6 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in I\!N$ and let $1/p_k + 1/t_k = 1$ for all $k \in I\!N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), \underline{\ell}_\infty(q))$ if and only if for each $r \in I\!N$, there is $m_r \in I\!N$ such that $\sup_{n,\, k}\ r^{t_k/q_n} \|f_k^n\|^{t_k} m_r^{-t_k} < \infty$.*

Proof. Since $\underline{\ell}_\infty(q) = \cap_{r=1}^{\infty} \ell_{\infty(r^{1/q_k})}$, it follows that

$$A \in (\ell(X,p), \underline{\ell}_\infty(q)) \iff A \in (\ell(X,p),\ \ell_{\infty(r^{1/q_k})}) \text{ for all } r \in I\!N$$

It is easy to show that for $r \in I\!N$,

$$A \in (\ell(X,p),\ \ell_{\infty(r^{1/q_k})}) \iff \left(r^{1/q_n} f_k^n\right)_{n,k} \in (\ell(X,p),\ \ell_\infty) .$$

We obtain by Corollary 3.4 that for $r \in I\!N$,

$$\left(r^{1/q_n} f_k^n\right)_{n,k} \in (\ell(X,p),\ \ell_\infty) \iff \text{there is } m_r \in N \text{ such that } \sup_n \sum_{k=1}^{\infty} r^{t_k/q_n} \|f_k^n\|^{t_k} m_r^{-t_k} < \infty$$

Thus the theorem is proved. ☐

Theorem 3.7 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in I\!N$ and let $1/p_k + 1/t_k = 1$ for all $k \in I\!N$. For an* *infinite matrix* $A = (f_k^n)$, $A \in \big(F_r(X,p),\ \underline{\ell}_\infty(q)\big)$ *if and only if for each* $i \in I\!N$, *there is* $m_i \in I\!N$ *such that*

$$\sup_n \sum_{k=1}^{\infty} i^{t_k/q_n} k^{-rt_k/p_k} \|f_k^n\|^{t_k} m_i^{-t_k} < \infty.$$

Proof. Since $F_r(X,p) = \ell(X,p)_{(k^{r/p_k})}$, it implies that

$$A \in (F_r(X,p), \underline{\ell}_\infty(q)) \iff \big(k^{-r/p_k} f_k^n\big)_{n,k} \in (\ell(X,p), \underline{\ell}_\infty(q)).$$

It follows from Thoerem 3.6 that $A \in (F_r(X,p), \underline{\ell}_\infty(q))$ if and only if for each $i \in I\!N$, there is $m_i \in I\!N$ such that $\sup\limits_n \sum_{k=1}^{\infty} i^{t_k/q_n} k^{-rt_k/p_k} \|f_k^n\|^{t_k} m_i^{-t_k} < \infty$.

Theorem 3.8 *Let* $p = (p_k)$ *be bounded sequence of positive real numbers with* $p_k > 1$ *for all* $n \in I\!N$ *and let* $1/p_k + 1/q_k = 1$ *for all* $k \in I\!N$. *Then for an infinite matrix* $A = (f_k^n)$, $A \in (\ell(X,p),\ bs)$ *if and only if there is* $m_0 \in I\!N$ *such that* $\sup\limits_n \sum_{k=1}^{\infty} \| \sum_{i=1}^{n} f_k^i\|^{q_k} m_0^{-q_k} < \infty$.

Proof. For an infinite matrix $A = (f_k^n)$, we can easily show that

$$A \in (\ell(X,p),\ bs) \iff \Big(\sum_{i=1}^{n} f_k^i\Big)_{n,k} \in (\ell(X,p),\ \ell_\infty) .$$

This implies by Corollary 3.4 that $A \in (\ell(X,p),\ bs)$ if and only if there is $m_0 \in I\!N$ such that

$$\sup_n \sum_{k=1}^{\infty} \| \sum_{i=1}^{n} f_k^i\|^{q_k} m_0^{-q_k} < \infty .$$

□

Theorem 3.9 *Let* $p = (p_k)$ *be a bounded sequence of positive real numbers with* $p_k > 1$ *for all* $k \in I\!N$ *and let* $1/p_k + 1/q_k = 1$ *for all* $k \in I\!N$. *Then for an infinite matrix* $A = (f_k^n)$, $A \in (\ell(X,p),\ cs)$ *if and only if*

(1) there is $m_0 \in I\!N$ *such that* $\sup\limits_n \sum\limits_{k=1}^{\infty} \| \sum\limits_{i=1}^{n} f_k^i\|^{q_k} m_0^{-q_k} < \infty$ *and*

(2) for each $k \in I\!N$ *and* $x \in X$, $\sum\limits_{n=1}^{\infty} f_k^n(x)$ *converges.*

Proof. The necessity is obtained by Theorem 3.8 and by the fact that $e^{(k)}(x) \in \ell(X,p)$ for every $k \in I\!N$ and $x \in X$.

Now, suppose that (1) and (2) hold. By Theorem 3.8, we have $A : \ell(X,p) \to bs$. Let $x = (x_k) \in \ell(X,p)$. Since $\ell(X,p)$ has the AK property, we have $x = \lim\limits_{n\to\infty} \sum_{k=1}^{n} e^{(k)}(x_k)$. By Zeller's theorem, $A : \ell(X,p) \to bs$ is continuous. It implies that $Ax = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} Ae^{(k)}(x_k)$. By (2), $Ae^{(k)}(x_k) \in cs$ for all $k \in I\!N$. Since cs is a closed subspace of bs, it implies that $Ax \in cs$, that is $A : \ell(X,p) \to cs$. □

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Department of Mathematics, Faculty of Science,
Chiang Mai University, Chiang Mai 50200, Thailand.
Email : malsuthe@science.cmu.ac.th scmti005@chiangmai.ac.th

On β-Dual of Vector-Valued Sequence Spaces of Maddox

Abstract. In this paper, the β-dual of a vector-valued sequence space is defined and studied. We show that if an X-valued sequence space $S(X)$ is a BK-space having AK property, then the dual space of $S(X)$ and its β-dual are isometrically isomorphic. We also give characterizations of β-daul of vector-valued sequence spaces of Maddox $\ell(X,p)$, $\ell_\infty(X,p)$, $c_0(X,p)$, and $c(X,p)$.

Keywords: β-dual ; vector-valued sequence spaces of Maddox

AMS Mathematics Subject Classification (2000): 46A45.

1. Introduction

Let $(X, \|.\|)$ be a Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. Let N be the set of all natural numbers, we write $x = (x_k)$ with x_k in X for all $k \in N$. The $X-$valued sequence spaces of Maddox are defined as

$c_0(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\}$;

$c(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X\}$;

$\ell_\infty(X,p) = \{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\}$;

$\ell(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\}$.

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p), c(p), \ell_\infty(p)$, and $\ell(p)$, respectively. All of these spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons [7] and Maddox [3 - 5]. The space $\ell(p)$ was first defined by Nakano [6] and is known as the Nakano sequence space. Grosse-Erdmann [1] has investigated the structure of the spaces $c_0(p), c(p), \ell(p)$, and $\ell_\infty(p)$ and has given characterizations of β-dual of scalar-valued sequence spaces of Maddox.

In [8], Wu and Bu gave characterizations of Köthe dual of the vector-valued sequence space $\ell_p[X]$, where $\ell_p[X]$ $(1 < p < \infty)$ is defined by

$$\ell_p[X] = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |f(x_k)|^p < \infty \text{ for each } f \in X' \right\}.$$

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In this paper, the β-dual of a vector-valued sequence spaces is defined and studied, and we give characterizations of β-dual of vector-valued sequence spaces of Maddox $\ell(X,p)$, $\ell_\infty(X,p)$, $c_0(X,p)$, and $c(X,p)$. Some results, obtained in this paper, are generalizations of some in [3].

2. Notation and Definitions

Let $(X, \|.\|)$ be a Banach space. Let $W(X)$ and $\Phi(X)$ denote the space of all sequences in X and the space of all finite sequences in X, respectively. A sequence space in X is a linear subspace of $W(X)$. Let E be an X- valued sequence space. For $x \in E$ and $k \in N$ we write that x_k stand for the kth term of x. For $x \in X$ and $k \in N$, we let $e^{(k)}(x)$ be the sequence $(0,0,0,...,0,x,0,...)$ with x in the kth position and let $e(x)$ be the sequence $(x,x,x,...)$. For a fixed scalar sequence $u = (u_k)$ the sequence space E_u is defined as

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}.$$

An X-valued sequence space E is said to be *normal* if $(x_k) \in E$ and $(y_k) \in W(X)$ with $\|y_k\| \leq \|x_k\|$ for all $k \in N$ implies that $(y_k) \in E$. Suppose the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if, for each $k \in N$ the kth coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. In addition, if (E, τ) is a *Fre'chet*(*Banach*) space, then E is called an $FK-(BK-)$ space . Now, suppose that E contains $\Phi(X)$. Then E is said to have *property AK* if $\sum_{k=1}^{n} e^{(k)}(x_k) \to x$ in E as $n \to \infty$ for every $x = (x_k) \in E$

The spaces $c_0(p)$ and $c(p)$ are FK-spaces. In $c_0(X,p)$, we consider the function $g(x) = \sup\limits_k \|x_k\|^{p_k/M}$, where $M = \max\ \{1, \sup_k p_k\}$, as a paranorm on $c_0(X,p)$, and it is known that $c_0(X,p)$ is an FK-space having property AK under the paranorm g defined as above. In $\ell(X,p)$, we consider it as a paranormed sequence space with the paranorm given by $\|(x_k)\| = \left(\sum_{k=1}^{\infty} \|x_k\|^{p_k}\right)^{1/M}$, where $M = \max\ \{1, \sup_k p_k\}$. It is known that $\ell(X,p)$ is an FK-space under the paranorm defined as above.

For an X-valued sequence space $S(X)$, define its Köthe dual with respect to the dual pair (X, X')

(see [2]) as follow :

$$S(X)^{\times}|_{(X,X')} = \left\{ (f_k) \subset X' : \sum_{k=1}^{\infty} |f_k(x_k)| < \infty \text{ for all } x = (x_k) \in S(X) \right\}.$$

Sometime we denote $S(X)^{\times}|_{(X,X')}$ by $S(X)^{\alpha}$ and it is called the α-dual of $S(X)$.

For a sequence space $S(X)$, the β-dual of $S(X)$ is defined by

$$S(X)^{\beta} = \left\{ (f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x_k) \text{ converges for all } (x_k) \in S(X) \right\}.$$

It is easy to see that $S(X)^{\alpha} \subseteq S(X)^{\beta}$.

For the sake of completeness we introduce some further sequence spaces that will be considered as β-dual of the vector-valued sequence spaces of Maddox :

$M_0(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\| M^{-1/p_k} < \infty \text{ for some } M \in N \}$;

$M_{\infty}(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\| n^{1/p_k} < \infty \text{ for all } n \in N \}$;

$\ell_0(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} M^{-(p_k-1)} < \infty \text{ for some } M \in N \}$; where $p_k > 1$ for all $k \in N$,

$cs[X'] = \{(f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x) \text{ converges for all } x \in X \}$.

When $X = K$, the scalar field of X, the corresponding first two sequence spaces are written as $M_0(p)$ and $M_{\infty}(p)$, respectively. These spaces were first introduced by Grosse-Erdmann [3].

3. Main Results

We begin with giving some general properties of β-dual of vector-valued sequence spaces.

Proposition 3.1. *Let X be a Banach space and let $S(X)$, $S_1(X)$, and $S_2(X)$ be X-valued sequence spaces. Then*

(i) $S(X)^{\alpha} \subseteq S(X)^{\beta}$.

(ii) If $S_1(X) \subseteq S_2(X)$, *then* $S_2(X)^{\beta} \subseteq S_1(X)^{\beta}$.

(iii) If $S(X) = S_1(X) + S_2(X)$, *then* $S(X)^{\beta} = S_1(X)^{\beta} \cap S_2(X)^{\beta}$.

(iv) If $S(X)$ *is normal, then* $S(X)^{\alpha} = S(X)^{\beta}$.

Proof Assertions (i) - (iii) are immediately obtained by the definitions. To prove (iv), by (i), it suffices to show only that $S(X)^\beta \subseteq S(X)^\alpha$. Let $(f_k) \in S(X)^\beta$ and $x = (x_k) \in S(X)$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges. Choose a scalar sequence (t_k) such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Since $S(X)$ is normal, $(t_k x_k) \in S(X)$. Thus $\sum_{k=1}^{\infty} |f_k(x_k)| = \sum_{k=1}^{\infty} f_k(t_k x_k)$ and the series $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges. This implies that $(f_k) \in S(X)^\alpha$. $\square$

If $S(X)$ is an BK-space, we define a norm on $S(X)^\beta$ by the formular

$$\|(f_k)\|_{S(X)^\beta} = \sup_{\|(x_k)\| \leq 1} \left| \sum_{k=1}^{\infty} f_k(x_k) \right| .$$

It is easy to show that $\|.\|_{S(X)^\beta}$ is a norm on $S(X)^\beta$

Next, we give some relations between β-dual of a sequence space and its dual. Indeed, we need a lemma.

Lemma 3.2. *Let $S(X)$ be an X-valued sequence space which is an FK-space and contains $\Phi(X)$. Then for each $k \in N$, the mapping $T_k : X \to S(X)$, defined by $T_k x = e^k(x)$, is continuous.*

Proof. Let $V = \{e^k(x) : x \in X\}$. Then V is a closed subspace of $S(X)$, so it is an FK-space because $S(X)$ is an FK-space. Since $S(X)$ is a K-space, the coordinate mapping $p_k : V \to X$ is continuous and bijective. It follows from the open mapping theorem that p_k is open, which implies that $p_k^{-1} : X \to V$ is continuous. But since $T_k = p_k^{-1}$, we thus obtain that T_k is continuous. $\square$

Theorem 3.3. *If $S(X)$ is a BK-space having property AK, then $S(X)^\beta$ and $S(X)'$ are isometrically isomorphic.*

Proof. We first show that for $x = (x_k) \in S(X)$ and $f \in S(X)'$,

$$f(x) = \sum_{k=1}^{\infty} f(e^k(x_k)) \tag{3.1}$$

To show this, let $x = (x_k) \in S(X)$ and $f \in S(X)'$. Since $S(X)$ has property AK, $x = \lim_{n\to\infty} \sum_{k=1}^{n} e^{(k)}(x_k)$. By the continuity of f, it follows that $f(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f(e^{(k)}(x_k)) =$

$\sum_{k=1}^{\infty} f(e^{(k)}(x_k))$, so (3.1) is obtained. For each $k \in N$, let $T_k : X \to S(X)$ be defined as in Lemma 3.2. Since $S(X)$ is a BK-space, by Lemma 3.2, T_k is continuous. Hence $f \circ T_k \in X'$ for all $k \in N$. It follows from (3.1) that

$$f(x) = \sum_{k=1}^{\infty} (f \circ T_k)(x_k) \text{ for all } x = (x_k) \in S(X). \tag{3.2}$$

We have by (3.2) that $(f \circ T_k)_{k=1}^{\infty} \in S(X)^{\beta}$. Define $\varphi : S(X)' \to S(X)^{\beta}$ by

$$\varphi(f) = (f \circ T_k)_{k=1}^{\infty} \text{ for all } f \in S(X)'.$$

It is easy to see that φ is linear. Now, we shall show that φ is onto. Let $(f_k) \in S(X)^{\beta}$. Define $f : S(X) \to K$, where K is the scalar field of X, by

$$f(x) = \sum_{k=1}^{\infty} f_k(x_k) \text{ for all } x = (x_k) \in S(X) \tag{3.3}$$

For each $k \in N$,let p_k be the kth coordinate mapping on $S(X)$. Then we have

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} (f_k \circ p_k)(x) \\ &= \lim_{n \to \infty} \sum_{k=1}^{n} (f \circ p_k)(x). \end{aligned}$$

Since f_k and p_k are continuous linear, so is $f \circ p_k$. It follows by Banach-Steinhaus theorem that $f \in S(X)'$, and we have by (3.3) that for each $k \in N$ and each $z \in X$, $(f \circ T_k)(z) = f(e^{(k)}(z)) = f_k(z)$. Thus $f \circ T_k = f_k$ for all $k \in N$, which implies $\varphi(f) = (f_k)$, hence φ is onto.

Finally, we shall show that φ is linear isometry. For $f \in S(X)'$, we have

$$\begin{aligned}
\|f\| &= \sup_{\|(x_k)\|\leq 1} |f((x_k))| \\
&= \sup_{\|(x_k)\|\leq 1} \left|\sum_{k=1}^{\infty} f(e^{(k)}(x_k))\right| \quad \text{(by (3.1))} \\
&= \sup_{\|(x_k)\|\leq 1} \left|\sum_{k=1}^{\infty} (f \circ T_k)(x_k)\right| \\
&= \|(f \circ T_k)_{k=1}^{\infty}\|_{S(X)^\beta} \\
&= \|\varphi(f)\|_{S(X)^\beta}.
\end{aligned}$$

Hence φ is isometry. Therefore $\varphi : S(X)' \to S(X)^\beta$ is an isometrically isomorphism form $S(X)'$ onto $S(X)^\beta$, so the theorem is proved. □

We next give characterizations of β-dual of the sequence space $\ell(X,p)$ when $p_k > 1$ for all $k \in N$.

Theorem 3.4. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$. Then $\ell(X,p)^\beta = \ell_0(X',q)$, where $q = (q_k)$ is a sequence of positive real numbers such that $1/p_k + 1/q_k = 1$ for all $k \in N$.*

Proof. Suppose that $(f_k) \in \ell_0(X',q)$. Then $\sum_{k=1}^{\infty} \|f_k\|^{q_k} M^{-(q_k-1)} < \infty$ for some $M \in N$.

Then for each $x = (x_k) \in \ell(X,p)$, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} |f_k(x_k)| &\leq \sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} M^{1/p_k} \|x_k\| \\
&\leq \sum_{k=1}^{\infty} \left(\|f_k\|^{q_k} M^{-q_k/p_k} + M\|x_k\|^{p_k} \right) \\
&= \sum_{k=1}^{\infty} \|f_k\|^{q_k} M^{-(q_k-1)} + M \sum_{k=1}^{\infty} \|x_k\|^{p_k} \quad < \infty\,,
\end{aligned}$$

which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so $(f_k) \in \ell(X,p)^\beta$.

On the other hand, assume that $(f_k) \in \ell(X,p)^\beta$, then $\sum_{k=1}^\infty f_k(x_k)$ converges for all $x = (x_k) \in \ell(X,p)$. For each $x = (x_k) \in \ell(X,p)$, choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Since $(t_k x_k) \in \ell(X,p)$, by our assumption, we have $\sum_{k=1}^\infty f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^{\infty}|f_k(x_k)| < \infty \quad \text{for all } x \in \ell(X,p). \tag{3.4}$$

We want to show that $(f_k) \in \ell_0(X',q)$, that is $\sum_{k=1}^\infty \|f_k\|^{q_k} M^{-(q_k-1)} < \infty$ for some $M \in N$. If it is not true, then

$$\sum_{k=1}^{\infty}\|f_k\|^{q_k} m^{-(q_k-1)} = \infty, \text{ for all } m \in N. \tag{3.5}$$

It implies by (3.5) that for each $k \in N$,

$$\sum_{i>k}\|f_i\|^{q_i} m^{-(q_i-1)} = \infty, \text{ for all } m \in N. \tag{3.6}$$

By (3.5), let $m_1 = 1$, then there is a $k_1 \in N$ such that

$$\sum_{k \le k_1} \|f_k\|^{q_k} m_1^{-(q_k-1)} > 1.$$

By (3.6), we can choose $m_2 > m_1$ and $k_2 > k_1$ with $m_2 > 2^2$ such that

$$\sum_{k_1 < k \le k_2} \|f_k\|^{q_k} m_2^{-(q_k-1)} > 1. \tag{3.7}$$

Proceeding in this way, we can choose sequences of positive integers (k_i) and (m_i) with $1 = k_0 < k_1 < k_2 < \ldots$ and $m_1 < m_2 < \ldots$, such that $m_i > 2^i$ and

$$\sum_{k_{i-1} < k \le k_i} \|f_k\|^{q_k} m_i^{-(q_k-1)} > 1 .$$

For each $i \in N$, choose x_k in X with $\|x_k\| = 1$ for all $k \in N$, $k_{i-1} < k \le k_i$ such that

$$\sum_{k_{i-1} < k \le k_i} |f_k(x_k)|^{q_k} m_i^{-(q_k-1)} > 1 \text{ for all } i \in N.$$

Let $a_i = \sum\limits_{k_{i-1}<k\leq k_i} |f_k(x_k)|^{q_k} m_i^{-(q_k-1)}$. Put $y=(y_k)$, $y_k = a_i^{-1}m_i^{-(q_k-1)}|f_k(x_k)|^{q_k-1}x_k$ for all k $k_{i-1}<k\leq k_i$. For each $i\in N$, we have

$$\begin{aligned}\sum_{k_{i-1}<k\leq k_i}\|y_k\|^{p_k} &= \sum_{k_{i-1}<k\leq k_i}\left\|a_i^{-1}m_i^{-(q_k-1)}|f_k(x_k)|^{q_k-1}x_k\right\|^{p_k}\\ &= \sum_{k_{i-1}<k\leq k_i} a_i^{-p_k}m_i^{-q_k}|f_k(x_k)|^{q_k}\\ &\leq \sum_{k_{i-1}<k\leq k_i} a_i^{-1}m_i^{-1}m_i^{-(q_k-1)}|f_k(x_k)|^{q_k}\\ &= a_i^{-1}m_i^{-1}a_i\\ &= m_i^{-1}\\ &< 1/2^i.\end{aligned}$$

So we have that $\sum_{k=1}^{\infty}\|y_k\|^{p_k}\leq\sum_{i=1}^{\infty}1/2^i<\infty$. Hence, $y=(y_k)\in\ell(X,p)$. For each $i\in N$, we have

$$\begin{aligned}\sum_{k_{i-1}<k\leq k_i}|f_k(y_k)| &= \sum_{k_{i-1}<k\leq k_i}\left|f_k(a_i^{-1}m_i^{-(q_k-1)}|f_k(x_k)|^{q_k-1}x_k)\right|\\ &= \sum_{k_{i-1}<k\leq k_i} a_i^{-1}m_i^{-(q_k-1)}|f_k(x_k)|^{q_k}\\ &= a_i^{-1}\sum_{k_{i-1}<k\leq k_i} m_i^{-(q_k-1)}|f_k(x_k)|^{q_k}\\ &= 1.\end{aligned}$$

So that $\sum\limits_{k=1}^{\infty}|f_k(y_k)|=\infty$, which contradicts to (3.4). Hence $(f_k)\in\ell_0(X',q)$. The proof is now complete. $\square$

The following theorem give a characterization of β-dual of $\ell(X,p)$ when $p_k\leq 1$ for all $k\in N$. To do this, the following lemma is needed.

Lemma 3.5. *Let $p=(p_k)$ be a bounded sequences of positive real numbers. Then $\ell_\infty(X,p)=\bigcup_{n=1}^{\infty}\ell_\infty(X)_{(n^{-1/p_k})}$.*

Proof. Let $x\in\ell_\infty(X,p)$, then there is some $n\in N$ with $\|x_k\|^{p_k}\leq n$ for all $k\in N$. Hence $\|x_k\|n^{-1/p_k}\leq 1$ for all $k\in N$, so that $x\in\ell_\infty(X)_{(n^{-1/p_k})}$. On the other hand, if $x \in \cup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$, then there are some $n \in N$ and $M > 1$ such that $\|x_k\| n^{-1/p_k} \leq M$ for every $k \in N$. Then we have $\|x_k\|^{p_k} \leq n M^{p_k} \leq n M^{\alpha}$ for all $k \in N$, where $\alpha = \sup_k p_k$. Hence $x \in \ell_\infty(X, p)$. □

Theorem 3.6. *Let $p = (p_k)$ be a bounded sequence of positve real numbers with $p_k \leq 1$ for all $k \in N$. Then $\ell(X,p)^\beta = \ell_\infty(X', p)$.*

Proof. If $(f_k) \in \ell(X,p)^\beta$, then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in \ell(X,p)$, using the same proof as in Theorem 3.4, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \text{for all} \quad x = (x_k) \in \ell(X,p) \tag{3.8}$$

If $(f_k) \notin \ell_\infty(X', p)$, it follows by Lemma 3.5 that $\sup_k \|f_k\| m^{-1/p_k} = \infty$ for all $m \in N$. For each $i \in N$, choose sequences (m_i) and (k_i) of positive integers with $m_1 < m_2 < \ldots$ and $k_1 < k_2 < \ldots$ such that $m_i > 2^i$ and $\|f_{k_i}\| m_i^{-1/p_{k_i}} > 1$. Choose $x_{k_i} \in X$ with $\|x_{k_i}\| = 1$ such that

$$|f_{k_i}(x_{k_i})| m_i^{-1/p_{k_i}} > 1\ . \tag{3.9}$$

Let $y = (y_k)$, $y_k = m_i^{-1/p_{k_i}} x_{k_i}$ if $k = k_i$ for some i, and 0 otherwise. Then $\sum_{k=1}^{\infty} \|y_k\|^{p_k} = \sum_{i=1}^{\infty} 1/m_i < \sum_{i=1}^{\infty} 1/2^i = 1$, so that $(y_k) \in \ell(X,p)$ and

$$\begin{aligned}\sum_{k=1}^{\infty} |f_k(y_k)| &= \sum_{i=1}^{\infty} |f_{k_i}(m_i^{-1/p_{k_i}} x_{k_i})| \\ &= \sum_{i=1}^{\infty} m_i^{-1/p_{k_i}} |f_{k_i}(x_{k_i})| = \infty \text{ by (3.9)},\end{aligned}$$

and this is contradictory to (3.8), hence $(f_k) \in \ell_\infty(X', p)$.

Conversely, assume that $(f_k) \in \ell_\infty(X', p)$. By Lemma 3.5, there exists $M \in N$ such that $\sup_k \|f_k\| M^{-1/p_k} < \infty$. Let $x = (x_k) \in \ell(X,p)$, then there is a $K > 0$ such that

$$\|f_k\| \leq K M^{1/p_k} \quad \text{for all} \quad k \in N \tag{3.10}$$

and there is a $k_0 \in N$ such that $M^{1/p_k} \|x_k\| \leq 1$ for all $k \geq k_0$. By $p_k \leq 1$ for all $k \in N$, we have that for all $k \geq k_0$,

$$M^{1/p_k} \|x_k\| \leq (M^{1/p_k} \|x_k\|)^{p_k} = M \|x_k\|^{p_k}. \tag{3.11}$$

Then

$$\begin{aligned}
\sum_{k=1}^{\infty}|f_k(x_k)| &\leq \sum_{k=1}^{k_o}\|f_k\|\|x_k\| + \sum_{k=k_0+1}^{\infty}\|f_k\|\|x_k\| \\
&\leq \sum_{k=1}^{k_o}\|f_k\|\|x_k\| + K\sum_{k=k_0+1}^{\infty} M^{1/p_k}\|x_k\| \quad \text{(by (3.10))} \\
&\leq \sum_{k=1}^{k_o}\|f_k\|\|x_k\| + KM\sum_{k=k_0+1}^{\infty}\|x_k\|^{p_k} \quad \text{(by (3.11))} \\
&< \infty.
\end{aligned}$$

This implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, hence $(f_k) \in \ell(X,p)^{\beta}$. □

Theorem 3.7. *Let $p = (p_k)$ be a bounded sequence of positve real numbers. Then $\ell_\infty(X,p)^\beta = M_\infty(X',p)$.*

Proof. If $(f_k) \in M_\infty(X',p)$, then $\sum_{k=1}^{\infty}\|f_k\|m^{1/p_k} < \infty$ for all $m \in N$, we have that for each $x = (x_k) \in \ell_\infty(X,p)$, there is $m_0 \in N$ such that $\|x_k\| \leq m_0^{1/p_k}$ for all $k \in N$, hence $\sum_{k=1}^{\infty}|f_k(x_k)| \leq \sum_{k=1}^{\infty}\|f_k\|\|x_k\| \leq \sum_{k=1}^{\infty}\|f_k\|m_0^{1/p_k} < \infty$, which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so that $(f_k) \in \ell(X,p)^\beta$.

Conversely, assume that $(f_k) \in \ell_\infty(X,p)^\beta$, then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X,p)$, by using the same proof as in Theorem 3.4, we have

$$\sum_{k=1}^{\infty}|f_k(x_k)| < \infty \text{ for all } x = (x_k) \in \ell_\infty(X,p). \tag{3.12}$$

If $(f_k) \notin M_\infty(X',p)$, then $\sum_{k=1}^{\infty}\|f_k\|M^{1/p_k} = \infty$ for some $M \in N$. Then we can choose a sequence (k_i) of positive integers with $0 = k_0 < k_1 < k_2 < \ldots$ such that

$$\sum_{k_{i-1}<k\leq k_i}\|f_k\|M^{1/p_k} > i \quad \text{for all } i \in N.$$

And we choose x_k in X with $\|x_k\| = 1$ such that for all $i \in N$,

$$\sum_{k_{i-1}<k\leq k_i}|f_k(x_k)|M^{1/p_k} > i.$$

Put $y = (y_k)$, $y_k = M^{1/p_k} x_k$. Clearly, $y \in \ell_\infty(X,p)$ and

$$\sum_{k=1}^{\infty} |f_k(y_k)| \ \geq \sum_{k_{i-1}<k\leq k_i}^{\infty} |f_k(x_k)| M^{1/p_k} > i \quad \text{for all } i \in N.$$

Hence $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts to (3.12). Hence $(f_k) \in M_\infty(X', p)$. The proof is now complete. □

Theorem 3.8. *Let $p = (p_k)$ be a bounded sequence of positve real numbers. Then $c_0(X,p)^\beta = M_0(X',p)$.*

Proof. Suppose $(f_k) \in M_0(X',p)$, then $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in N$. Let $x = (x_k) \in c_0(X,p)$. Then there is a positive integer K_0 such that $\|x_k\|^{p_k} < 1/M$ for all $k \geq K_0$, hence $\|x_k\| < M^{-1/p_k}$ for all $k \geq K_0$. Then we have

$$\sum_{k=K_0}^{\infty} |f_k(x_k)| \leq \sum_{k=K_0}^{\infty} \|f_k\| \|x_k\| \leq \sum_{k=K_0}^{\infty} \|f_k\| M^{-1/p_k} < \infty.$$

It follows that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so that $(f_k) \in c_0(X,p)^\beta$.

On the other hand, assume that $(f_k) \in c_0(X,p)^\beta$, then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X,p)$. For each $x = (x_k) \in c_0(X,p)$, choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Since $(t_k x_k) \in c_0(X,p)$, by our assumption, we have $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \text{for all } x \in c_0(X,p). \tag{3.13}$$

Now, suppose that $(f_k) \notin M_0(X',p)$. Then $\sum_{k=1}^{\infty} \|f_k\| m^{-1/p_k} = \infty$ for all $m \in N$. Choose $m_1, k_1 \in N$ such that

$$\sum_{k\leq k_1} \|f_k\| m_1^{-1/p_k} > 1$$

and choose $m_2 > m_1$ and $k_2 > k_1$ such that

$$\sum_{k_1<k\leq k_2} \|f_k\| m_2^{-1/p_k} > 2.$$

Proceeding in this way, we can choose $m_1 < m_2 < ...$, and $0 = k_1 < k_2 < ...$ such that

$$\sum_{k_{i-1}<k\leq k_i} \|f_k\| m_i^{-1/p_k} > i.$$

Take x_k in X with $\|x_k\| = 1$ for all $k, k_{i-1} < k \leq k_i$ such that

$$\sum_{k_{i-1}<k\leq k_i} |f_k(x_k)| m_i^{-1/p_k} > i \quad \text{for all } i \in N.$$

Put $y = (y_k),\ y_k = m_i^{-1/p_k} x_k$ for $k_{i-1} < k \leq k_i$, then $y \in c_0(X, p)$ and

$$\sum_{k=1}^{\infty} |f_k(y_k)| \geq \sum_{k_{i-1}<k\leq k_i} |f_k(x_k)| m_i^{-1/p_k} > i \quad \text{for all } i \in N.$$

Hence we have $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$ which contradicts to (3.13), therefore $(f_k) \in M_0(X', p)$. This completes the proof. □

Theorem 3.9. *Let $p = (p_k)$ be a bounded sequence of positve real numbers. Then $c(X,p)^{\beta} = M_0(X',p) \cap cs[X']$.*

Proof. Since $c(X,p) = c_0(X,p) + E$, where $E = \{e(x) : x \in X\}$, it follows by Proposition 3.1(iii) and Theorem 3.8 that $c(X,p)^{\beta} = M_0(X',p) \cap E^{\beta}$. It is obvious by the definition that $E^{\beta} = \{(f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x) \text{ converges for all } x \in X\} = cs[X']$. Hence we have the theorem. □

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MATRIX TRANSFORMATIONS OF SOME VECTOR-VALUED SEQUENCE SPACES

Necessary and sufficient conditions have been established for an infinite matrix $A = (f_n^k)$ of continuous linear functionals on a Banach space X to transform the vector-valued sequence spaces of Maddox $\ell_\infty(X,p)$, $\ell(X,p)$, $c_0(X,p)$, and $c(X,p)$ into the scalar-valued sequence space $c(q)$, where $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers.

Keywords: Matrix transformations, Maddox vector-valued sequence spaces

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1. Introduction

The study of matrix transformations of scalar- valued sequence spaces is known since the turn of the century. In seventies, Maddox[12], Gupta[4] studied matrix transformations of continuous linear mappings on vector-valued sequence spaces. Das and Choudhury[1] gave conditions on the matrix $A = (f_k^n)$ of continuous linear mappings from a normed linear space X into a normed linear space Y under which A maps $c_0(X)$ into $c_0(Y)$, $\ell_1(X)$ into $\ell_\infty(Y)$, and $\ell_1(X)$ into $\ell_p(Y)$. Liu and Wu[22] gave the matrix characterizations from vector-valued sequence spaces $c_0(X,p)$, $\ell(X,p)$, and $\ell_\infty(X,p)$ into scalar-valued sequence spaces $c_0(q)$ and $\ell_\infty(q)$. Suantai[20] gave the matrix characterizations from the Nakano vector-valued sequence space $\ell(X,p)$ into the vector-valued sequence spaces $c_0(Y,q)$, $c(Y)$, and $\ell_r(Y)$. In this paper, we continue the study of matrix transformations of continuous linear mappings on vector-valued sequence spaces.

The main purpose of this paper is to give the matrix characterizations from $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, and $\ell(X,p)$ into $c(q)$, where $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, and $\ell(X,p)$ are the vector-valued sequence spaces of Maddox as defined in Section 2. When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, and $\ell(p)$, respectively. Several papers deal with the problem of characterizing those matrices that map a scalar-valued sequence space of Maddox into anoher such

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spaces, see [6, 7, 11, 13, 15, 17, 18, 19, 21]. Some of these results become particular cases of our theorems. Also some more interesting results are derived.

Section 2 deals with necessary preliminaries and some known results quoted as lemmas which are needed to characterize an infinite matrix $A = (f_k^n)$ such that A maps the vector- valued sequence spaces of Maddox into $c(q)$, and we also give some auxiliary results in Section 3. The main results of the paper is in Section 4.

2 Preliminaries and Lemmas

Let $(X, \|.\|)$ be a Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. Let N be the set of all natural numbers, we write $x = (x_k)$ with x_k in X for all $k \in N$. Let $W(X)$ and $\Phi(X)$ denote the space of all sequences and the space of all finite sequences in X, respectively. When $X = K$, the scalar field of X, the corresponding spaces are written as w and ϕ, respectively. An $X-$valued sequence space is a linear subspace of $W(X)$. The sequence spaces of Maddox are defined as

$c_0(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\}$,

$c(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X\}$,

$\ell_\infty(X,p) = \{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\}$,

$\ell(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\}$.

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p), c(p), \ell_\infty(p)$, and $\ell(p)$, respectively. All of these spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons[16] and Maddox[8,9]. The space $\ell(p)$ was first defined by Nakano[14] and it is known as the Nakano sequence space. Also, we need to define the following sequence space :

$M_0(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\| n^{-1/p_k} < \infty \text{ for some } n \in N \}$.

When $X = K$, the scalar field of X, the corresponding space is written as $M_0(p)$. This space was first introduced by Maddox[10]. Grosse-Erdmann[2] has investigated the structure of the spaces $c_0(p), c(p), \ell(p)$, and $\ell_\infty(p)$ and he also gave the matrix characterizations between scalar-valued sequence spaces of Maddox in [3]. Let E be an X-valued sequence space. For $x \in E$ and $k \in N$ we write that x_k stand for the kth term of x and for $x \in X$ and $k \in N$, let $e^{(k)}(x)$ be the sequence $(0, 0, 0, ..., 0, x, 0, ...)$ with x in the kth position and let $e(x)$ be the sequence $(x, x, x, ...)$, and we denote by e the the sequence $(1,1,1,...)$. An X−valued sequence space E is said to be *normal* if $(x_k) \in E$ and $(y_k) \in W(X)$ with $\|y_k\| \leq \|x_k\|$ for all $k \in N$ implies that $(y_k) \in E$. For a fixed scalar sequence $u = (u_k)$ the sequence space E_u is defined as

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}.$$

The $\alpha-$, $\beta-$ and $\gamma-$ duals of a scalar-valued sequence space F are defined as

$$F^{\varsigma} = \{x \in w : (x_k y_k) \in X_\zeta \text{ for every } y \in F\}$$

for $\zeta = \alpha,\ \beta,\ \gamma$ and $X_\alpha = \ell_1$, $X_\beta = cs$, and $X_\gamma = bs$, where ℓ_1, cs and bs are defined as

$\ell_1 = \{x = (x_k) \in w : \sum_{k=1}^{\infty} |x_k| < \infty\ \}$,

$cs = \{x = (x_k) \in w : \sum_{k=1}^{\infty} x_k \text{ converges }\}$,

$bs = \{x = (x_k) \in w : \sup_n |\sum_{k=1}^{n} x_k| < \infty\ \}$.

In the same manner, for an X−valued sequence space E, the $\alpha-$, $\beta-$ and $\gamma-$ duals of E are defined as

$$E^{\varsigma} = \{(f_k) \subset X' : (f_k(x_k)) \in X_\zeta \text{ for every } x = (x_k) \in E\}$$

for $\zeta = \alpha,\ \beta,\ \gamma$, where $X_\alpha = \ell_1$, $X_\beta = cs$ and $X_\gamma = bs$.

It is obvious from the definition that $E^\alpha \subseteq E^\beta \subseteq E^\gamma$ and it is easy to see that if E is normal, then $E^\alpha = E^\beta = E^\gamma$.

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is an X-valued sequence space and F a scalar-valued sequence space. Then A is said to *map E into F*, written by $A : E \to F$ if, for each $x = (x_k) \in E, A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$ and the sequence $Ax = (A_n(x)) \in F$. We denote by (E,F) the class of all infinite matrices mapping E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$${}_u(E,F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E,F)\}.$$

If $u_k \neq 0$ for all $k \in N$, we put $u^{-1} = (1/u_k)$. Suppose the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if, for each $k \in N$ the kth coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. A K-space that is a Fréchet(Banach) space is called an $FK-(BK-)$ space.

The spaces $c_0(p)$ and $c(p)$ are FK-spaces. In $c_0(X,p)$, we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = \max\{1, \sup_k p_k\}$, as a paranorm on $c_0(X,p)$, and it is known that $c_0(X,p)$ is an FK-space under the paranorm g defined as above. In $\ell(X,p)$, we consider it as a paranormed sequence space with the paranorm given by $\|(x_k)\| = (\sum_{k=1}^{\infty} \|x_k\|^{p_k})^{1/M}$. It is known that $\ell(X,p)$ is an FK-space under the paranorm defined as above.

Now let us quote some known results as the following.

Lemma 2.1[10] *If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$, then*

$$\ell(p)^\beta = \{x \in w : \sum_{k=1}^{\infty} |x_k|^{t_k} M^{-t_k} < \infty \ \ \textit{for some} \ \ M \in N\}$$

where $1/p_k + 1/t_k = 1$ for all $k \in N$.

Lemma 2.2[16] *If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in N$, then $\ell(p)^\beta = \ell_\infty(p)$.*

Lemma 2.3[6] *If $p = (p_k)$ is a bounded sequence of positive real numbers , then*

$$\ell_\infty(p)^\beta = \{x \in w : \sum_{k=1}^{\infty} |x_k| n^{1/p_k} < \infty \ \textit{for all} \ \ n \in N\}.$$

Lemma 2.4[10] *If $p = (p_k)$ is a bounded sequence of positive real numbers, then $c_0(p)^\beta = M_0(p)$.*

Lemma 2.5[22] *Let $p = (p_k)$ be a bounded sequence of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c_0(X,p) \to c_0$ if and only if*

(1) $f_k^n \xrightarrow{w^*} 0$ as $n \to \infty$ for each $k \in N$ and

(2) $\lim_{m\to\infty} \sup_n \sum_{k=1}^{\infty} \|f_k^n\| m^{-1/p_k} = 0$.

Lemma 2.6[22] *Let $p = (p_k)$ be a bounded sequence of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : \ell_\infty(X,p) \to c_0$ if and only if*

(1) $f_k^n \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for each* $k \in N$ *and*

(2) *for each* $M \in N$, $\sum_{j>k} \|f_j^n\| M^{1/p_j} \to 0$ *as* $k \to \infty$ *uniformly on* $n \in N$.

Lemma 2.7[22] *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ and $1/p_k + 1/t_k = 1$ for all $k \in N$ and let $A = (f_k^n)$ be an infinite matrix. Then $A : \ell(X,p) \to c_0$ if and only if*

(1) $f_k^n \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for each* $k \in N$ *and*

(2) $\sum_{k=1}^{\infty} \|f_k^n\|^{t_k} m^{-t_k} \to 0$ *as* $m \to \infty$ *uniformly on* $n \in N$.

Lemma 2.8[22] *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in N$ and let $A = (f_k^n)$ be an infinite matrix. Then $A : \ell(X,p) \to c_0$ if and only if*

(1) $f_k^n \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for each* $k \in N$ *and*

(2) $\sup_{n,k} \|f_k^n\|^{p_k} < \infty$.

3. Some Auxiliary Results

Suppose that E and F are sequence spaces and that we want to characterize the matrix space (E, F). If E and/or F can be derived from simpler sequence spaces in some fashion, then, in many cases, the problem reduces to the characterization of the corresponding simpler matrix spaces. We begin with giving various useful results in this direction.

Proposition 3.1. *Let E and $E_n (n \in N)$ be X-valued sequence spaces, and F and $F_n (n \in N)$ scalar-valued sequence spaces, and let u and v be scalar sequences with $u_k \neq 0, v_k \neq 0$ for all $k \in N$. Then*

(i) $(\cup_{n=1}^{\infty} E_n, F) = \cap_{n=1}^{\infty} (E_n, F)$,

(ii) $(E, \cap_{n=1}^{\infty} F_n) = \cap_{n=1}^{\infty} (E, F_n)$,

(iii) $(E_1 + E_2, F) = (E_1, F) \cap (E_2, F)$,

(iv) $(E_u, F_v) = {}_v(E, F)_{u^{-1}}$.

Proof. All of them are obtained directly from the definitions.

Propostion 3.2. *Let $p = (p_k)$ be a bounded sequences of positive real numbers . Then*

(i) $c(X, p) = c_0(X, p) + \{e(x) : x \in X\}$,

(ii) $M_0(X, p) = \cup_{n=1}^{\infty} \ell(X)_{(n^{-1/p_k})}$,

(iii) $\ell_\infty(X, p) = \cup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$.

Proof. Assertions (i) and (ii) are immediately obtained from the definitions. To show (iii), let $x \in \ell_\infty(X, p)$, then there is some $n \in N$ with $\|x_k\|^{p_k} \le n$ for all $k \in N$. Hence $\|x_k\| n^{-1/p_k} \le 1$ for all $k \in N$, so that $x \in \ell_\infty(X)_{(n^{-1/p_k})}$. On the other hand, if $x \in \cup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$, then there are some $n \in N$ and $M > 1$ such that $\|x_k\| n^{-1/p_k} \le M$ for every $k \in N$. Then we have $\|x_k\|^{p_k} \le nM^{p_k} \le nM^{\alpha}$ for all $k \in N$, where $\alpha = \sup_k p_k$. Hence $x \in \ell_\infty(X, p)$. □

The next proposition give a relationship between the $\beta-$ dual of vector-valued and scalar-valued sequence spaces.

Proposition 3.3 *Let X be a Banach space and F a normal scalar-valued sequence space and define $F(X) = \{(x_k) \in W(X) : (\|x_k\|) \in F\}$. then for $(f_k) \subset X'$, the topological dual of X, $(f_k) \in F(X)^\beta$ if and only if $(\|f_k\|) \in F^\beta$.*

Proof. If $(\|f_k\|) \in F^\beta$, then for $x = (x_k) \in F(X)$ we have $\sum_{k=1}^{\infty} |f_k(x_k)| \le \sum_{k=1}^{\infty} \|f_k\| \|x_k\| < \infty$, so that $x \in F(X)^\beta$.

Conversely, suppose that $(f_k) \in F(X)^\beta$ and $a = (a_k) \in F$. Since F is normal, $(|a_k|) \in F$. For each $k \in N$, we can choose $x_k \in X$ such that $\|x_k\| = 1$ and $|f_k(x_k)| \ge \dfrac{\|f_k\|}{2}$. Let $y = (a_k x_k)$, then $y \in F(X)$. Choose a sequence (t_k) of scalars such that $|t_k| \le 1$ and $f_k(t_k a_k x_k) = |f_k(x_k)||a_k|$ for all $k \in N$. Since F is normal, $(t_k y_k) \in F(X)$, so we obtain that $\sum_{k=1}^{\infty} f_k(t_k y_k)$ converges. This implies $\sum_{k=1}^{\infty} \|f_k\| |a_k| \le 2 \sum_{k=1}^{\infty} |f_k(x_k)||a_k| < \infty$. It follows that $(\|f_k\|) \in F^\beta$.

By using Proposition 3.3, the following results are obtained immediately from Lemma 2.1 - 2.4, respectively.

Proposition 3.4 *If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$, then*

$$\ell(X,p)^\beta = \{(f_k) \subset X' : \sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-t_k} < \infty \text{ for some } M \in N\}$$

where $1/p_k + 1/t_k = 1$ for all $k \in N$.

Proposition 3.5 *If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in N$, then $\ell(X,p)^\beta = \ell_\infty(X',p)$.*

Proposition 3.6 *If $p = (p_k)$ is a bounded sequence of positive real numbers, then*

$$\ell_\infty(X,p)^\beta = \{(f_k) \subset X' : \sum_{k=1}^{\infty} \|f_k\| n^{1/p_k} < \infty \text{ for all } n \in N\}.$$

Proposition 3.7 *If $p = (p_k)$ is a bounded sequence of positive real numbers, then $c_0(X,p)^\beta = M_0(X',p)$.*

4. Main Results

We begin with the following useful result.

Theorem 4.1. *Let $q = (q_k)$ be a bounded sequence of positive real numbers and let E be a normal $X-$ valued sequence space which is an FK-space containing $\Phi(X)$. Then*

$$(E, c(q)) = (E, c_0(q)) \oplus (E, <e>).$$

To prove this theorem, we need the following two lemmas.

Lemma 4.1. *Let E be an X-valued sequence space which is an FK-space containing $\Phi(X)$. Then for each $k \in N$, the mapping $T_k : X \to E$, defined by $T_k x = e^k(x)$, is continuous.*

Proof. For each $k \in N$, we have that $V = \{e^k(x) : x \in X\}$ is a closed subspace of E, so it is an FK-space. Since E is a K-space, the coordinate mapping $p_k : V \to X$ is continuous and bijective. It follows from the open mapping theorem that p_k is open, hence, $p_k^{-1} : X \to V$ is continuous. It follows that T_k is continuous because $T_k = p_k^{-1}$.□

Lemma 4.2. *If E and F are scalar-valued sequence spaces such that E is normal containing ϕ, F is an FK-space and there is a subsequence (n_k) with $x_{n_k} \to 0$ as $k \to \infty$ for all $x = (x_n) \in F$, then $(E,\ F \oplus < e >) = (E,\ F) \oplus (E,\ < e >)$.*

Proof. See [3, Proposition 3.1(vi)]. □

Proof of Theorem 4.1 Since $c(q) = c_0(q) \oplus < e >$, it is clear that $(E,\ c_0(q)) + (E, < e >) \subseteq (E, c_0(q) \oplus < e >) = (E, c(q))$. Moreover, if $A \in (E, c_0(q)) \bigcap (E, < e >)$, then $A \in (E,\ c_0(q) \bigcap < e >)$, so that $A \in (E, 0)$, which implies that $A = 0$ because E contain $\Phi(X)$. Hence $(E, c_0(q)) + (E, < e >)$ is a direct sum. Now, we will show that $(E, c(q)) \subseteq (E, c_0(q)) \oplus (E, < e >)$. Let $A = (f_k^n) \in (E, c(q)) = (E, c_0(q) \oplus < e >)$. For $x \in X$ and $k \in N$, we have $(f_k^n(x))_{n=1}^{\infty} = Ae^k(x) \in c_0(q) \oplus < e >$, so that there exist unique $(b_k^n(x))_{n=1}^{\infty} \in c_0(q)$ and $(c_k^n(x))_{n=1}^{\infty} \in < e >$ with

$$(f_k^n(x))_{n=1}^{\infty} = (b_k^n(x))_{n=1}^{\infty} + (c_k^n(x))_{n=1}^{\infty}. \tag{4.1}$$

For each $n, k \in N$, let g_k^n and h_k^n be the functionals on X defined by

$$g_k^n(x) = b_k^n(x) \quad \text{and} \quad h_k^n(x) = c_k^n(x) \quad \text{for all } x \in X.$$

Clearly, g_k^n and h_k^n are linear, and by (4.1)

$$f_k^n = g_k^n + h_k^n \quad \text{for all } n, k \in N. \tag{4.2}$$

Note that $c_0(q) \oplus < e >$ is an FK-space in its direct sum topology. By Zeller's theorem, $A : E \to c_0(q) \oplus < e >$ is continuous. For each $k \in N$, let $T_k : X \to E$ be defined by $T_k(x) = e^k(x)$. By Lemma 4.1, we have that T_k is continuous for all $k \in N$. Since the projection P_1 of $c_0(q) \oplus < e >$ onto $c_0(q)$ and the projection P_2 of $c_0(q) \oplus < e >$ onto $< e >$ are continuous and $g_k^n = p_n \circ P_1 \circ A \circ T_k$ and $h_k^n = p_n \circ P_2 \circ A \circ T_k$ for all $n, k \in N$, we obtain that g_k^n and h_k^n are continuous, so $g_k^n, h_k^n \in X'$ for all $n, k \in N$. Let

$B = (g_k^n)$ and $C = (h_k^n)$. By (4.1) and (4.2), we have $A = B + C$, $B = (g_k^n) \in (\Phi(X), c_0(q))$ and $C = (h_k^n) \in (\Phi(X), < e >)$. We will show that $B \in (E, c_0(q))$ and $C \in (E, < e >)$. To do this, let $x = (x_k) \in E$. Then for $\alpha = (\alpha_k) \in \ell_\infty$, we have $\|\alpha_k x_k\| = |\alpha_k| \|x_k\| \leq \|M x_k\|$, where $M = \sup_k |\alpha_k|$. Then the normality of E implies that $(\alpha_k x_k) \in E$. Hence $(f_k^n(x_k))_{n,k} \in (\ell_\infty, c_0(q) \oplus < e >)$, moreover, we have $(g_k^n(x_k))_{n,k} \in (\Phi, c_0(q))$, $(h_k^n(x_k))_{n,k} \in (\Phi, < e >)$, and $(f_k^n(x_k))_{n,k} = (g_k^n(x_k))_{n,k} + (h_k^n(x_k))_{n,k}$. Since ℓ_∞ is normal containing ϕ and $c_0(q) \subseteq c_0$, it follows from Lemma 4.2 that $(g_k^n(x_k))_{n,k} \in (\ell_\infty, c_0(q))$ and $(h_k^n(x_k))_{n,k} \in (\ell_\infty, < e >)$. This implies that $Bx \in c_0(q)$ and $Cx \in < e >$, so we have $B \in (E, c_0(q))$ and $C \in (E, < e >)$, hence $A \in (E, c_0(q)) \oplus (E, < e >)$. This completes the proof. $\square$

Theorem 4.2. *Let $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : \ell_\infty(X) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| < \infty$,

(2) $m^{1/q_n} (f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for every $k, m \in N$ and

(3) for each $m \in N$, $\sum_{j>k} m^{1/q_n} \|f_j^n - f_j\| \to 0$ as $k \to \infty$ uniformly on $n \in N$.

Proof. Necessity. Let $A \in (\ell_\infty(X), c(q))$. It follows from Theorem 4.1 that $A = B + C$, where $B \in (\ell_\infty(X), c_0(q))$ and $C \in (\ell_\infty(X), < e >)$. Then there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that $C = (f_k)_{n,k}$ and $B = (f_k^n - f_k)_{n,k} \in (\ell_\infty(X), c_0(q))$, which implies that $(f_k) \in \ell_\infty(X)^\beta$, so (1) is obtained by Proposition 3.6. Since $c_0(q) = \cap_{m=1}^{\infty} c_{0(m^{1/q_k})}$ (by [2, Theorem 0 (i)]), we have by Proposition 3.1 (ii) and (iv) that for each $m \in N$, $\left(m^{1/q_n}(f_k^n - f_k)_{n,k}\right) : \ell_\infty(X) \to c_0$. Hence, (2) and (3) are obtained by Lemma 2.6.

Sufficiency. Suppose that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that conditions (1), (2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. It is obvious that $A = B + C$. By condition (2) and (3), we obtain by Lemma 2.6 and Proposition 3.1(ii) and (iv) that $B \in (\ell_\infty(X), c_0(q))$. By Proposition 3.6 , condition (1) implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X)$, which implies that $C \in (\ell_\infty(X), < e >)$. Hence, we obtain by Theorem 4.1 that $A \in (\ell_\infty(X), c(q))$. This completes the proof. $\square$

Theorem 4.3. *Let $p=(p_k)$ and $q=(q_k)$ be bounded sequences of positive real numbers and $A=(f_k^n)$ an infinite matrix. Then $A:\ell_\infty(X,p)\to c(q)$ if and only if there is a sequence (f_k) with $f_k\in X'$ for all $k\in N$ such that (1), (2) and (3) are satisfied, where*

(1) for each $m\in N$, $\sum_{k=1}^\infty \|f_k\|m^{1/p_k}<\infty$,

(2) $r^{1/q_n}(f_k^n-f_k)\overset{w^*}{\to} 0$ as $n\to\infty$ for every $k,r\in N$ and

(3) for each $m,r\in N$, $r^{1/q_n}\sum_{j>k} m^{1/p_j}\|f_j^n-f_j\|\to 0$ as $k\to\infty$ uniformly on $n\in N$.

Moreover, (3) is equivalent to (3′), where

(3′) for each $m\in N$, $\lim_{k\to\infty}\sup_n\left(\sum_{j>k} m^{1/p_j}\|f_j^n-f_j\|\right)^{q_n}=0$.

Proof. Necessity. Suppose that $A:\ell_\infty(X,p)\to c(q)$. By Theorem 4.1, $A=B+C$, where $B\in(\ell_\infty(X,p),c_0(q))$ and $C\in(\ell_\infty(X,p),<e>)$. Then there is a sequence (f_k) with $f_k\in X'$ for all $k\in N$ such that $C=(f_k)_{n,k}$ and $B=(f_k^n-f_k)\in(\ell_\infty(X,p),c_0(q))$. Since $C=(f_k)_{n,k}:\ell_\infty(X,p)\to<e>$, it implies by Proposition 3.6 that (1) holds. Since $c_0(q)=\cap_{m=1}^\infty c_{0(m^{1/q_k})}$, we have by Proposition 3.1 (ii) that for each $r\in N$, $\left(r^{1/q_n}(f_k^n-f_k)\right)_{n,k}:\ell_\infty(X,p)\to c_0$. Hence , (2) and (3) holds by an application of Lemma 2.6.

Sufficiency. Suppose that there is a sequence (f_k) with $f_k\in X'$ for all $k\in N$ such that condition (1), (2) and (3) hold. Let $B=(f_k^n-f_k)_{n,k}$ and $C=(f_k)_{n,k}$. It is obvious that $A=B+C$. By condition (2) and (3), we obtain by Lemma 2.6 and Proposition 3.1(ii) and (iv) that $B\in(\ell_\infty(X,p),c_0(q))$. By Proposition 3.6 , condition (1) implies that $\sum_{k=1}^\infty f_k(x_k)$ converges for all $x=(x_k)\in\ell_\infty(X,p)$, which implies that $C\in(\ell_\infty(X,p),<e>)$. Hence, we obtain by Theorem 4.1 that $A\in(\ell_\infty(X,p),c(q))$.

Now we shall show that (3) and (3′) are equivalent. Suppose (3) holds and let $\varepsilon>0$. Choose $r\in N$ such that $1/r<\varepsilon$. By (3), there exists $k_0\in N$ such that

$$r^{1/q_n}\sum_{j>k} m^{1/p_j}\|f_j^n-f_j\|<1 \text{ for all } k\ge k_0 \text{ and all } n\in N,$$

which implies that

$$\sup_n\left(\sum_{j>k} m^{1/p_j}\|f_j^n-f_j\|\right)^{q_n}\le 1/r<\varepsilon \text{ for } k\ge k_0,$$

hence, (3′) holds.

Conversely, assume that (3′) holds. Let $m, r \in N$ and $0 < \varepsilon < 1$. Then there exists $k_0 \in N$ such that

$$\sup_n \left(\sum_{j>k} m^{1/p_j} \|f_j^n - f_j\| \right)^{q_n} < \varepsilon^H / r \quad \text{for all } k \geq k_0$$

where $H = sup_n q_n$. This implies that

$$r^{1/q_n} \sum_{j>k} m^{1/p_j} \|f_j^n - f_j\| < \varepsilon^{H/q_n} < \varepsilon \quad \text{for all } k \geq k_0 \text{ and all } \; n \in N$$

hence, (3) holds. □

Theorem 4.4. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c_0(X, p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that (1), (2), and (3) are satisfied, where*

(1) $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in N$,

(2) $m^{\frac{1}{q_n}} (f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for every $m, k \in N$ and

(3) for each $m \in N$, $\sup_n \left(m^{1/q_n} \sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} \right) \to 0$ as $r \to \infty$.

Moreover, (3) is equivalent to (3′) where

(3′) $\lim_{r\to\infty} \sup_n \left(\sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} \right)^{q_n} = 0$.

Proof. Necessity. Suppose that $A : c_0(X, p) \to c(q)$. By Theorem 4.1, we have $A = B + C$, where $B \in (c_0(X, p), c_0(q))$ and $C \in (c_0(X, p), < e >)$. It follows that there is a sequence $(f_k) \subset X'$ such that $C = (f_k)_{n,k}$ and $B = (f_k^n - f_k)_{n,k}$. Since $c_0(q) = \cap_{r=1}^{\infty} c_{0(r^{1/q_k})}$, it follows from Proposition 3.1 (ii) and (iv) that for each $m \in N$, $(m^{1/q_n}(f_k^n - f_k))_{n,k} \in (c_0(X, p),\ c_0)$, hence, conditions (2) and (3) hold by using the result from Lemma 2.5. Since $C = (f_k)_{n,k} \in (c_0(X, p),\ < e >)$, we have that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = x_k \in c_0(X, p)$, so that $(f_k) \in c_0(X, p)^{\beta}$, hence, by Proposition 3.7, we obtain that there exists $M \in N$ such that $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$. Hence, (1) is obtained.

Sufficiency. Assume that there is a sequence $(f_k) \subset X'$ such that conditions (1),(2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. Then $A = B + C$. By conditions (2)and (3), we obtain from Proposition 3.1(ii) and (iv) and Lemma 2.5 that $B \in \left(c_0(X, p), c_0(q)\right)$. The condition (1) implies by Proposition 3.7 that $\sum_{k=1}^{\infty} f_k(x_k)$

converges for all $x = (x_k) \in c_0(X,p)$, so that $C \in (c_0(X,p), \; < e >)$. Hence, by Theorem 4.1, we obtain that $A \in (c_0(X,p), c(q))$.

Now, we shall show that conditions (3) and (3′) are equivalent. To do this, suppose that (3) holds and let $\varepsilon > 0$. Choose $m \in N$, $1/m < \varepsilon$. From (3), there is $r_0 \in N$ such that

$$\sup_n \; m^{1/q_n} \sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} \leq 1 \text{ for all } r \geq r_0.$$

This implies that $\sup_n \left(\sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k}\right)^{q_n} \leq 1/m < \varepsilon$ for all $r \geq r_0$. Hence, (3′) holds.

Conversely, suppose that (3′) holds. Let $m \in N$ and $0 < \varepsilon < 1$. Then there exists $r_0 \in N$ such that $\sup_n \left(\sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k}\right)^{q_n} < \varepsilon^H/m$ for all $r \geq r_0$, where $H = sup_n q_n$. Hence, we have

$$m^{1/q_n} \sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} < \varepsilon^{H/q_n} \leq \varepsilon \text{ for all } r \geq r_0 \text{ and } n \in N,$$

so that (3) holds. This completes the proof. □

Theorem 4.5. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that (1), (2), (3) and (4) are satisfied, where*

(1) $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ *for some* $M \in N$,

(2) for each $m, k \in N$, $m^{1/q_n}(f_k^n - f_k) \overset{w^*}{\to} 0$ *as* $n \to \infty$,

(3) for each $m \in N$, $\sup_n \; m^{1/q_n} \sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} \to 0$ *as* $r \to \infty$ *and*

(4) $\left(\sum_{k=1}^{\infty} f_k^n(x)\right)_{n=1}^{\infty} \in c(q)$ *for all* $x \in X$.

Moreover, (3) is equivalent to (3′) where

(3′) $\lim_{r \to \infty} \sup_n \left(\sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k}\right)^{q_n} = 0.$

Proof. Since $c(X,p) = c_0(X,p) + \{e(x) : x \in X\}$ (Proposition 3.2 (i)), it follows from Proposition 3.1(iii) that $A \in (c(X,p), c(q))$ if and only if $A \in (c_0(X,p), c(q))$ and $A \in (\{e(x) : x \in X\}, c(q))$. By Theorem 4.4, we have $A \in (c_0(X,p), c(q))$ if and only if conditions (1)-(3) hold and it is clear that $A \in \left(\{e(x) : x \in X\}, c(q)\right)$ if and only if (4) holds. We have by Theorem 4.4 that (3) and (3′) are equivalent. Hence, the theorem is proved. □

Wu and Liu (Lemma 2.7) have given a characterization of an infinite matrix A such that $A \in (\ell(X,p), c_0)$ when $p_k > 1$ for all $k \in N$. By applications of Proposition 3.1(ii) and (iv), Proposition 3.4, and Theorem 4.1, and using the fact that $c_0(q) = \bigcap_{m=1}^{\infty} c_{0(m^{1/q_k})}$, we obtain the following result.

Theorem 4.6. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in N$ and $1/p_k + 1/t_k = 1$ for all $k \in N$, and let $A = (f_k^n)$ be an infinite matrix. Then $A : \ell(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-t_k} < \infty$ *for some $M \in N$,*

(2) $m^{1/q_n}(f_k^n - f_k) \overset{w^*}{\to} 0$ *as $n \to \infty$ for all $m, k \in N$ and*

(3) for each $m \in N$, $\sum_{k=1}^{\infty} m^{t_k/q_n} \|f_k^n - f_k\|^{t_k} r^{-t_k} \to 0$ *as $r \to \infty$ uniformly on $n \in N$.*

By using Lemma 2.8, Proposition 3.1(ii) and (iv), Proposition 3.5 and Theorem 4.1, we also obtain the following result.

Theorem 4.7. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k \le 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sup_k \|f_k\|^{p_k} < \infty$,

(2) $m^{1/q_n}(f_k^n - f_k) \overset{w^*}{\to} 0$ *as $n \to \infty$ for all $m, k \in N$ and*

(3) $\sup_{n,k} m^{p_k/q_n} \|f_k^n - f_k\|^{p_k} < \infty$ *for all $m \in N$.*

When $p_k = 1$ for all $k \in N$, we obtain the following.

Corollary 4.8. *Let $q = (q_k)$ be a bounded sequence of positive real numbers and let $A = (f_k^n)$ be an infinite matrix. Then $A : \ell_1(X) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sup_k \|f_k\| < \infty$,

(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for all* $m, k \in N$ *and*

(3) $\sup_{n,k} m^{1/q_n}\|f_k^n - f_k\| < \infty$ *for every* $m \in N$.

Theorem 4.9. *Let* $p = (p_k)$ *be a bounded sequence of positive real numbers and* $A = (f_k^n)$ *an infinite matrix. Then* $A : M_0(X,p) \to c(q)$ *if and only if there is a sequence* (f_k) *of bounded linear functionals on* X *such that*

(1) $\sup_k m^{1/p_k}\|f_k\| < \infty$ *for all* $m \in N$,

(2) for each $m, r \in N$, $r^{1/q_n} m^{1/p_k}(f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for all* $k \in N$ *and*

(3) for each $m, r \in N$, $\sup_{n,k} r^{1/q_n} m^{1/p_k}\|f_k^n - f_k\| < \infty$.

Proof. It follows from Theorem 4.1 that $A \in (M_0(X,p), c_0(q) \oplus < e >)$ if and only if there is a sequence (f_k) of bounded linear functionals on X such that $A = B + (f_k)_{n,k}$ where $B : M_0(X,p) \to c_0(q)$ and $(f_k)_{n,k} : M_0(X,p) \to < e >$. Since $B = (f_k^n - f_k)_{n,k}$ and $M_0(X,p) = \cup_{m=1}^{\infty} \ell_1(X)_{(m^{-1/p_k})}$ (by Proposition 3.2 (ii)), we have by Proposition 3.1 (i) and (iv) that $B : M_0(X,p) \to c_0(q)$ if and only if $\left(m^{1/p_k}(f_k^n - f_k)\right)_{n,k} : \ell_1(X) \to c_0(q)$ for all $m \in N$. Since $c_0(q) = \cap_{r=1}^{\infty} c_{0(r^{1/q_k})}$, by Proposition 3.1 (ii) and (iv), we have $\left(m^{1/p_k}(f_k^n - f_k)\right)_{n,k} : \ell_1(X) \to c_0(q)$ if and only if $\left(r^{1/q_n} m^{1/p_k}(f_k^n - f_k)\right)_{n,k} : \ell_1(X) \to c_0$ for all $r \in N$. By Lemma 2.8 , we have

$\left(r^{1/q_n} m^{1/p_k}(f_k^n - f_k)\right)_{n,k} : \ell_1(X) \to c_0$ if and only if

(a) $r^{1/q_n} m^{1/p_k}(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for all $k \in N$ and

(b) $\sup_{n,k} r^{1/q_n} m^{1/p_k}\|f_k^n - f_k\| < \infty$.

By Proposition 3.1 (i) and (iv), we have $(f_k)_{n,k} : M_0(X,p) \to < e >$ if and only if $(m^{1/p_k} f_k)_{n,k} : \ell_1(X) \to < e >$ for all $m \in N$. By Proposition 3.5, we obtain that $(m^{1/p_k} f_k)_{n,k} : \ell_1(X) \to < e >$ if and only if $\sup_k m^{1/p_k}\|f_k\| < \infty$. Hence, the theorem is proved. □.

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$e^{(k)}(z)$ be the sequence $(0,0,0,...,0,z,0,...)$ with z in the k^{th} position. For a fixed scalar sequence $u=(u_k)$ the sequence space E_u is defined by

$$E_u = \{x=(x_k) \in W(X) : (u_k x_k) \in E\}.$$

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if for each $n \in N$ the n^{th} coordinate mapping $p_n : E \to X$, defined by $p_n(x) = x_n$, is continuous on E. If, in addition, (E,τ) is an $Fre'chet(Banach)$ space, then E is called an $FK-(BK-)$ space . Now, suppose that E contains $\Phi(X)$. Then E is said to have *property AB* if the set $\{\sum_{k=1}^{n} e^k(x_k) : n \in N\}$ is bounded in E for every $x=(x_k) \in E$. It is said to have *property AK* if $\sum_{k=1}^{n} e^k(x_k) \to x \in E$ as $n \to \infty$ for every $x=(x_k) \in E$. It has *property AD* if $\Phi(X)$ is dense in E.

If $p_k > 1$ for all $k \in N$, the space $\ell(p)$ is an BK-space with AK under the Luxemburg norn defined by

$$\|x\| = inf\{\varepsilon > 0 : \sum_{k=1}^{\infty} |\frac{x_k}{\varepsilon}|^{p_k} \leq 1\} .$$

For more detail about the space $\ell(p)$ see [3]. The space $c_0(p)$ is an FK-space with AK, $c(p)$ is an FK-space and $\ell_\infty(p)$ is a complete LBK-space with AB (see [3]). In each of the space $\ell_\infty(X,p)$ and $c_0(X,p)$ we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = max\ \{1, \sup_k p_k\}$, as a paranorm on $\ell_\infty(X,p)$ and $c_0(X,p)$ and it is known that $c_0(X,p)$ is an FK-space with AK under the paranorm g defined as above and $\ell_\infty(X,p)$ is a complete LBK-space with AB.

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map E into F*, written $A : E \to F$ if for each $x=(x_k) \in E, A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$ and the sequence $Ax = (A_n(x)) \in F$. We denote by (E,F) the set of all infinite matrices mapping E into F. If $u=(u_k)$ and $v=(v_k)$ are scalar sequences, let

$$_u(E,F)_v = \{A=(f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E,F)\}.$$

If $u_k \neq 0$ for all $k \in N$,we write $u^{-1} = (\frac{1}{u_k})$.

Let E be an X-valued sequence space. The $\beta-$ daul of E is defined to be

$$E^{\beta} = \{(f_k) \subset X' : \sum_{k=1}^{\infty} f(x_k) \text{ converges for all } x = (x_k) \in E.\}$$

By the defintion, we see that if $A = (f_k^n)$ maps the sequence space E into a scalar sequence space, then each row of A belongs to E^{β}, i.e., $(f_k^n)_{k=1}^{\infty} \in E^{\beta}$, so this is a necessary condition for an infinite matrix A mapping from one sequence space into the other. We shall give characterizations of the $\beta-$ dual of some vector-valued sequence spaces in Section 3.

3. The $\beta-$ Dual of some Vector-Valued Sequence Spaces

We start with characterizations of the $\beta-$ dual of the space $c_0(X,p)$

Proposition 3.1 *Let $p = (p_k)$ be a bounded sequences of positive real numbers. Then*

$$c_0(X,p)^{\beta} = \{(f_k) \subset X' : \sum_{k=1}^{\infty} \|f_k\| M^{-\frac{1}{p_k}} < \infty \textit{ for some } M \in N.\}$$

Proof. Suppose that $\sum_{k=1}^{\infty} \|f_k\| M^{-\frac{1}{p_k}} < \infty$ for some $M \in N$. Let $x = (x_k) \in c_0(X,p)$. Then there is a positive integer K_0 such that $\|x_k\|^{p_k} < \frac{1}{M}$ for all $k \geq K_0$, hence

$$\|x_k\| < M^{-\frac{1}{p_k}} \text{ for all } k \geq K_0.$$

Then we have

$$\sum_{k=K_0}^{\infty} |f_k(x_k)| \leq \sum_{k=K_0}^{\infty} \|f_k\|\|x_k\| \leq \sum_{k=K_0}^{\infty} \|f_k\| M^{-\frac{1}{p_k}} < \infty.$$

It follows that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so $(f_k) \in c_0(X,p)^{\beta}$.

On the other hand, assume that $(f_k) \in c_0(X,p)^{\beta}$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X,p)$. For each $x = (x_k) \in c_0(X,p)$, choose scalar sequence (t_k) with

$|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Since $(t_k x_k) \in c_0(X,p)$, by our assumption, we have $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \text{ for all } x \in c_0(X,p). \tag{3.1}$$

Now, suppose that $\sum_{k=1}^{\infty} \|f_k\| m^{-\frac{1}{p_k}} = \infty$ for all $m \in N$. Choose $m_1, k_1 \in N$ such that

$$\sum_{k \leq k_1} \|f_k\| m_1^{-\frac{1}{p_k}} > 1$$

and choose $m_2 > m_1$ and $k_2 > k_1$ such that

$$\sum_{k_1 < k \leq k_2} \|f_k\| m_2^{-\frac{1}{p_k}} > 2.$$

Proceeding in this way, we can choose $m_1 < m_2 < \ldots$, and $0 = k_1 < k_2 < \ldots$ such that

$$\sum_{k_{i-1} < k \leq k_i} \|f_k\| m_i^{-\frac{1}{p_k}} > i.$$

Take x_k in X with $\|x_k\| = 1$ for all $k, k_{i-1} < k \leq k_i$ such that

$$\sum_{k_{i-1} < k \leq k_i} |f_k(x_k)| m_i^{-\frac{1}{p_k}} > i \text{ for all } i \in N.$$

Put $y = (y_k), (y_k) = m_i^{-\frac{1}{p_k}} x_k$ for $k_{i-1} < k \leq k_i$, then $y \in c_0(X,p)$ and we have

$$\sum_{k=1}^{\infty} |f_k(y_k)| \geq \sum_{k_{i-1} < k \leq k_i} |f_k(x_k)| m_i^{-\frac{1}{p_k}} > i \text{ for all } i \in N.$$

Hence we have $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$ which contradicts with (3.1). Hence $(f_k) \in \{(g_k) \subset X' : \sum_{k=1}^{\infty} \|g_k\| M^{-\frac{1}{p_k}} < \infty$ for some $M \in N.\}$. Thus the proposition is proved. □

Proposition 3.2 *Let $p = (p_k)$ be a bounded sequences of positive real numbers. Then*

$$\ell_\infty(X,p)^\beta = \{(f_k) \subset X' : \sum_{k=1}^{\infty} \|f_k\| m^{\frac{1}{p_k}} < \infty \textit{ for all } m \in N.\}$$

Proof. If $\sum_{k=1}^{\infty}\|f_k\|m^{\frac{1}{p_k}} < \infty$ for all $m \in N$, then we have that for each $x = (x_k) \in \ell_\infty(X,p)$, there is $m_0 \in N$ such that $\|x_k\| \leq m_0^{\frac{1}{p_k}}$ for all $k \in N$, hence $\sum_{k=1}^{\infty}|f_k(x_k)| \leq \sum_{k=1}^{\infty}\|f_k\|\|x_k\| \leq \sum_{k=1}^{\infty}\|f_k\|m_0^{\frac{1}{p_k}} < \infty$, which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so $(f_k) \in \ell_\infty(X,p)^\beta$.

Conversely, assume that $(f_k) \in \ell_\infty(X,p)^\beta$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X,p)$. We first note that, by using the same proof as in Proposition 3.3, we have

$$\sum_{k=1}^{\infty}|f_k(x_k)| < \infty \text{ for all } x = (x_k) \in \ell_\infty(X,p). \tag{3.2}$$

Now, suppose that $\sum_{k=1}^{\infty}\|f_k\|M^{\frac{1}{p_k}} = \infty$ for some $M \in N$. Then we can choose a sequence (k_i) of positive integers with $0 = k_0 < k_1 < k_2 < \ldots$ such that

$$\sum_{k_{i-1}<k\leq k_i}\|f_k\|M^{\frac{1}{p_k}} > i \text{ for all } i \in N.$$

Taking x_k in X with $\|x_k\| = 1$ such that for all $i \in N$,

$$\sum_{k_{i-1}<k\leq k_i}\|f_k(x_k)\|M^{\frac{1}{p_k}} > i.$$

Put $y = (y_k) = (M^{\frac{1}{p_k}}x_k)_{k=1}^{\infty}$. Clearly, $y \in \ell_\infty(X,p)$ and

$$\sum_{k=1}^{\infty}|f_k(y_k)| \;\geq \sum_{k_{i-1}<k\leq k_i}\|f_k(x_k)\|M^{\frac{1}{p_k}} > i \text{ for all } i \in N.$$

Hence $\sum_{k=1}^{\infty}|f_k(y_k)| = \infty$, which contradicts with (3.2). Thus $(f_k) \in \{(g_k) \subset X' : \sum_{k=1}^{\infty}\|g_k\|m^{\frac{1}{p_k}} < \infty \text{ for all } m \in N.\}$. Hence $\ell_\infty(X,p)^\beta = \{(g_k) \subset X' : \sum_{k=1}^{\infty}\|g_k\|m^{\frac{1}{p_k}} < \infty \text{ for all } m \in N.\}$. $\square$

Proposition 3.3 *Let $p = (p_k)$ be a bounded sequences of positive real numbers. Then*

$$\ell(X,p)^\beta = \{(f_k) \subset X' : \sum_{k=1}^{\infty}\|f_k\|^{t_k}M^{-(t_k-1)} < \infty \text{ for some } M \in N\}$$

where $\frac{1}{p_k} + \frac{1}{t_k} = 1$ *for all* $k \in N$.

Proof. Suppose that $\sum_{k=1}^{\infty} \|f_k\|^{t_k} M^{-(t_k-1)} < \infty$ for some $M \in N$.
Then we have that for each $x = (x_k) \in \ell(X,p)$,

$$\begin{aligned}\sum_{k=1}^{\infty}|f_k(x_k)| &\leq \sum_{k=1}^{\infty}\|f_k\|M^{-\frac{1}{p_k}}M^{\frac{1}{p_k}}\|x_k\| \\ &\leq \sum_{k=1}^{\infty}\left(\|f_k\|^{t_k}M^{-\frac{t_k}{p_k}} + M\|x_k\|^{p_k}\right) \\ &= \sum_{k=1}^{\infty}\|f_k\|^{t_k}M^{-(t_k-1)} + M\sum_{k=1}^{\infty}\|x_k\|^{p_k} \quad < \infty\end{aligned}$$

which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so $(f_k) \in \ell(X,p)^{\beta}$.

On the other hand, assume that $(f_k) \in \ell(X,p)^{\beta}$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell(X,p)$. We first note that, by using the same proof as in Proposition 3.1, we have

$$\sum_{k=1}^{\infty}|f_k(x_k)| < \infty \text{ for all } x = (x_k) \in \ell(X,p). \tag{3.3}$$

We want to show that there exists $M \in N$ such that

$$\sum_{k=1}^{\infty}\|f_k\|^{t_k}M^{-(t_k-1)} < \infty$$

If it is not true, then

$$\sum_{k=1}^{\infty}\|f_k\|^{t_k}m^{-(t_k-1)} = \infty, \text{ for all } m \in N. \tag{3.4}$$

And (3.4) implies that for each $k_0 \in N$.

$$\sum_{k>k_0}\|f_k\|^{t_k}m^{-(t_k-1)} = \infty, \text{ for all } m \in N. \tag{3.5}$$

By (3.4), let $m_1 = 1$, then there is a $k_1 \in N$ such that

$$\sum_{k\leq k_1}\|f_k\|^{t_k}m_1^{-(t_k-1)} > 1.$$

By (3.5), we can choose $m_2 > m_1$ and $m_2 > 2^2$ and $k_2 > k_1$ such that

$$\sum_{k_1<k\leq k_2} \|f_k\|^{t_k} m_2^{-(t_k-1)} > 1. \tag{3.6}$$

By continueing in this way, we obtain sequences (k_i) and (m_i) of positive integers with $1 = k_0 < k_1 < k_2 < ..., m_1 < m_2 < ..., m_i > 2^i$ and

$$\sum_{k_{i-1}<k\leq k_i} \|f_k\|^{t_k} m_i^{-(t_k-1)} > 1.$$

Choose x_k in X with $\|x_k\| = 1$ such that for all $i \in N$,

$$\sum_{k_{i-1}<k\leq k_i} |f_k(x_k)|^{t_k} m_i^{-(t_k-1)} > 1 \text{ for all } i \in N.$$

Let $a_i = \sum_{k_{i-1}<k\leq k_i} |f_k(x_k)|^{t_k} m_i^{-(t_k-1)}$. Put $y = (y_k),\ y_k = a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k$ for all k, $k \leq k_i$. For each $i \in N$, we have

$$\begin{aligned}
\sum_{k_{i-1}<k\leq k_i} \|y_k\|^{p_k} &= \sum_{k_{i-1}<k\leq k_i} \left\| a_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k-1} x_k \right\|^{p_k} \\
&= \sum_{k_{i-1}<k\leq k_i} a_i^{-p_k} m_i^{-t_k} |f_k(x_k)|^{t_k} \\
&\leq \sum_{k_{i-1}<k\leq k_i} a_i^{-1} m_i^{-1} m_i^{-(t_k-1)} |f_k(x_k)|^{t_k} \\
&= a_i^{-1} m_i^{-1} a_i \\
&= m_i^{-1} \\
&< \frac{1}{2^i}.
\end{aligned}$$

So we have that

$$\sum_{k=1}^{\infty} \|y_k\|^{p_k} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty.$$

Hence,

$$y = (y_k) \in \ell(X, p). \tag{3.7}$$

ON MATRIX TRANSFORMATIONS OF VECTOR-VALUED SEQUENCE SPACES OF MADDOX

Abstract. In this paper, characterizations of infinite matrices mapping the vector-valued sequence spaces of Maddox into Musielak-Orlicz sequence space are given.

1. Introduction. Let $(X, \|.\|)$ be a real Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. Let N be the set of all natural numbers, we write $x = (x_k)$ with x_k in X for all $k \in N$. The $X-$valued sequence spaces $c_0(X,p), c(X,p), \ell_\infty(X,p)$, and $\ell(X,p)$ are defined by

$$c_0(X,p) = \left\{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\right\},$$

$$c(X,p) = \left\{x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X\right\},$$

$$\ell_\infty(X,p) = \left\{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\right\}, and$$

$$\ell(X,p) = \left\{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\right\}.$$

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p), c(p), \ell(p)$, and $\ell_\infty(p)$,
respectively. The first three spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons[9] and Maddox[5, 6]. The space $\ell(p)$ was first defined by Nakano[8] and is known as the Nakano sequence space. Grosse-Erdmann [3] investigated the structure of the spaces $c_0(p), c(p), \ell(p)$ and $\ell_\infty(p)$.

A function $f : R \to [0,\infty)$ is called an *Orlicz function* if it has the following properties:

(1) f is even, continuous and convex,

(2) $f(x) = 0 \Longleftrightarrow x = 0$,

(3) $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = 0$ and $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \infty$.

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