สกว

รายงานการวิจัยฉบับสมบูรณ์

โครงการ การวิจัยพลศาสตร์เชิงดิสครีตของฟังก์ชันตรรกยะบางฟังก์ชัน พลศาสตร์เชิงดิสครีตของฟังก์ชันในคลาส A และการวิจัยคุณสมบัติไม่แปรเปลี่ยนของการแปลงเชิงเส้นคู่

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Research on Discrete Dynamics of Certain Rational Functions, Discrete Dynamics of Functions of A_{∞} Class and Research on Invariant Properties of Mobius Transformations

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บทคัดย่อ

งานวิจัยนี้ได้ให้ความสำคัญในศึกษา 1. พลศาสตร์เชิงดิสครีตของฟังก์ชันตรรกยะบางฟังก์ชัน 2. ศึกษาคุณสมบัติไม่แปรเปลี่ยนของการแปลงเชิงเส้นคู่ และ 3. ผลเฉลยของสมการเชิงฟังก์ชันบางสมการ โดยความรู้พื้นฐานที่ใช้คือ การวิเคราะห์เชิงซ้อนและพลศาสตร์เชิงดิสครีต

Abstract

This research emphasizes on the study of 1. Discrete dynamics of certain rational functions 2. The invaraint properties of Mobius Transformations and 3. Solutions of certain functional equations in which the basic knowledge needed in the study are Complex analysis and Discrete dynamics.

Keywords: Discrete dynamics, Mobius Transformations, Rational functions

เนื้อหางานวิจัย

1. วัตถุประสงค์

1.1 เพื่อศึกษา วิจัย และอธิบาย พลศาสตร์เชิงดิสครีตของฟังก์ชันตรรกยะบางฟังก์ชัน

1.2 เพื่อศึกษา วิจัย และอธิบาย พลศาสตร์เชิงดิสครีตของฟังก์ชันในคลาส A_m

1.3 เพื่อศึกษา วิจัย และอธิบาย การมี Absolutely Continuous Invariant Measures ของฟังก์ชัน ใน 5.1 และ 5.2

1.4 เพื่อหาลักษณะเฉพาะของการแปลงเชิงเส้นคู่โดยอาศัยคุณสมบัติทางเรขาคณิต

1.5 เพื่อหาลักษณะเฉพาะของการแปลงเชิงเส้นคู่โดยอาศัยคุณสมบัติการไม่แปรเปลี่ยน

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2. ระเบียบวิธีวิจัย

2.1 ทำการรวบรวมเอกสารที่เกี่ยวข้องกับปัญหาที่จะทำวิจัยเพิ่มเติม

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2.3 ทำการวิจัยเพื่อหาผลการวิจัยใหม่ๆ

2.4 ปรึกษาและแลกเปลี่ยนความคิดเห็นเกี่ยวกับผลการวิจัยที่ได้รับกับผู้เชี่ยวชาญที่ได้ทำวิจัยร่วมกันในหัวข้อที่เกี่ยวข้อง

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3. ขอบเขตการวิจัย

3.1 ศึกษาพลศาสตร์เชิงดิสครีตของฟังก์ชันตรรกยะ (Rational Functions) บางฟังก์ชัน โดยเฉพาะฟังก์ชันตรรกยะที่ได้นำเสนอในหัวข้อที่ 7 (ผลงานวิจัยที่เกี่ยวข้อง)

3.2 หาโครงสร้างของเซตจูเลีย (Julia Sets) ของฟังก์ชันตรรกยะใน

3.3 หาขอบเขตล่าง (Lower Bound) และ ขอบเขตบน (Upper Bound) ของ Hausdorff dimension ของเซตจูเลียของฟังก์ชันตรรกยะใน

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3.8 หาลักษณะเฉพาะของการแปลงเชิงเส้นคู่โดยอาศัยคุณสมบัติทางเรขาคณิต

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สัญญาเลขที่ RSA/04/2544

รายงานการวิจัยฉบับสมบูรณ์

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มหาวิทยาลัยเชียงใหม่

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว.ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณสำนักงานกองทุนสนับสนุนการวิจัย (สกว.) ที่ได้ให้การสนับสนุนทุนวิจัยมาอย่างต่อเนื่อง ขอขอบพระคุณ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ ที่ได้ให้การสนับสนุนการทำวิจัยอย่างเต็มที่

ดร.ปิยะพงศ์ เนียมทรัพย์
หัวหน้าโครงการวิจัย

สารบัญ

Introduction

Part I: Julia sets and discrete dynamics of certain rational functions and functions of class A_∞.

Topic I. Julia sets and discrete dynamics of certain rational functions

Let $f(z)$ be a rational function (or an entire function) on the complex plane $\mathbb{C}$. We define the ith iterate as $f^0 = 1_{\mathbb{C}}$, $f^i = f^{i-1} \circ f$ and $f^{-i} = (f^i)^{-1}$ for $i \geq 1$. We call $z_0 \in \mathbb{C} \cup \{\infty\}$ a **periodic point** of f of period $p \geq 1$ if $f^p(z_0) = z_0$ and $f^i(z_0) \neq z_0$, $0 \leq i \leq p-1$. If $p = 1$ we say z_0 is a **fixed point of** f. Assume that z_0 is a periodic point of f of period p, we say that z_0 is an **attractive periodic point** of f if $|(f^p)'(z_0)| < 1$, we say z_0 is a **repulsive fixed point** if $|(f^p)'(z_0)| > 1$, and z_0 is an **indifferent fixed point** if $|(f^p)'(z_0)| = 1$. We are interested in the family of iterates of f, namely $\{f^i(z)\}_{i=1}^{\infty}$ where $z \in \mathbb{C} \cup \{\infty\}$ (or $z \in \mathbb{C}$), that is we are interested in the behavior of $f^i(z)$ for $i \geq 1$ and $z \in \mathbb{C} \cup \{\infty\}$. The studies of the modern theory of iteration of $f(z)$ has been traced back to around 1900 when G.Julia and P.Fatou had independently developed this branch of mathematics. The most important theory developed are involved with the following two sets, **Julia set** $J(f)$ and **Fatou set** $F(f)$ of function f where they are defined by

$$F(f) = \{z \in \mathbb{C} \cup \{\infty\} : \{f^i(z)\}_{i=1}^{\infty} \text{ is a normal family in a neighborhood of } z\}$$

$$J(f) = \{\mathbb{C} \cup \{\infty\}\} - F(f).$$

From the definition it follows that $J(f)$ is a perfect set and $F(f)$ is an open set. Results on the theory of iteration of rational can be found, for examples, in [8, 10, 15-16, 20, 44, 50, 52, 60, 72]. Results on the theory of iteration of transcendental meromorphic functions can be found in [1-6, 9, 16]. Results on the theory of iteration of entire functions can be found in [6, 50, 56, 63, 65]. In spite of extensive studies on the theory of iteration in recent years, there are still many interesting questions to be answered.

In this research project, we propose to do more research on theory of iteration of rational functions and entire functions. We will now give the literature reviews and related results. In this part, we focus on theory of iteration of rational functions. The topics we are interested is to describe dynamics of rational functions of certain forms. In [37], J.R.Kinney and T.S.Pitcher studied the Julia sets of rational functions $R(z)$ of the following forms:

$$R(z) = az - \sum_{i=1}^{n} \frac{b_i}{z - c_i}$$

where $a > 1$, $b_i \geq 0$, and the c_i are distinct real numbers arrangen in increasing order. They showed that $J(R) = \bigcap_{k=0}^{\infty} R^{-k}(I_0)$ where $I_0 = [x_0, x_n]$, x_0 and x_n are the smallest and largest real roots of $R(z) - z$. They also gave the upper and lower estimate of Hausdorff dimension $\dim(J(R))$ of $J(R)$ as follows:

$$\frac{\log(n+1)}{\log m} \leq \dim(J(R)) \leq \frac{\log(n+1)}{\log M}$$

where $m = \min_{x \in J(R)} \{R'(x)\}$ and $M = \max_{x \in J(R)} \{R'(x)\}$. From this results, we propose to extend the study of the rational functions of the following forms:

$$R(z) = az - \sum_{i=1}^{n} \frac{b_i z - c_i}{z - d_i} \tag{0.1}$$

where a, b_i, c_i, and d_i are real numbers satisfying $a \geq 1$, $d_1 < d_2 < \ldots < d_n$ and $b_i c_i - d_i > 0$ for all $1 \leq i \leq n$. Note that the rational functions we propose to study is in more general form that the ones studied by J.R.Kinney and T.S.Pitcher, in particular, we also study the case when $a = 1$ but J.R.Kinney and T.S.Pitcher didn't consider this case. We propose to study the structure of $J(R)$, find the upper and lower bounds for Hausdorff dimension of $J(R)$ (as of present, we have already obtained some results about $J(R)$ such as when $a > 1$, we obtain that $J(R) \subset [x_0, x_n]$ where x_0 and x_n are the smallest and largest real roots of $R(z) - z$ and the Lebesgue measure of $J(R)$ is qual to zero). We also propose to study symbolic dynamics (see [5, 7, 15] for details about symbolic dynamics) of function $R(z)$ on $J(R)$ in which we expect that when $a > 1$, $R(z)$ restricted to $J(R)$ is topological conjugate to the shift map σ on $n+1$ symbols. For the case $a = 1$, we have seen from several examples we have considered that $J(R)$ is unbounded subset of the real line, in this case the Lebesgure measure of $J(R)$ is not equal to zero and we could not obtain the upper and lower bounds for Hausdorff dimension of $J(R)$. However, the main point of interest in the case $a = 1$ is the symbolic dynamics of R restricted to $J(R)$ in which we have precise idea on how to deal with this question.

One of the most active branches of dynamical systems is the theory of absolutely continuous invariant measures (Absolutely continuous measure with respect to Lebesgue measure is a measure $\mu = f \cdot \lambda$ where λ is the Lebesgue measure. This means that for any measurable set A, $\mu(A) = \int_A$

$f\,d\lambda$. Measure μ is $\tau-$ *invariant* if for any measurable set A, we have $\mu(A)=\mu(\tau^{-1}(A)))$, especially for one-dimensional transformations. Let $\mathcal{M}$ be the set of all meromorphic functions which preserve the real line $\mathbb{R}$. As referred in [21], that Levin proved the following results:
Theorem A. A meromorphic function $g\in\mathcal{M}$ if and only if

$$g(z)=A+\varepsilon\left[Bz-\frac{C_0}{z}-\sum_{s}C_s\left(\frac{1}{z-p_s}+\frac{1}{p_s}\right)\right]$$

where $\varepsilon=\pm1$, A, B, C_s and p_s, $s=\pm1,\pm2,\ldots$ are real and $B\geq0$, $C_s\geq0$ and $\sum_{s}\frac{C_s}{p_s^2}<+\infty$. The poles p_s are numbered such that $p_s<p_{s+1}$ for $s=\pm1,\pm2,\ldots$ We assume that at least one C_s is different from 0. In [21], P.Gora and N.Obeid studied dynamics of functions $g(z)$ in class $\mathcal{M}$, where they considered four cases as follows:

Case I: $B<1$ and all fixed points of g in $\mathbb{R}$ are repelling.

Case II: $B<1$ and no fixed point of g in $\mathbb{R}$ is attracting and at least one is neutral.

Case III: $B=1$.

Case IV: $B>1$.

They have obtained results on absolutely continuous invariant measures for transformations for $\tau:\mathbb{R}\to\mathbb{R}$ where $\tau(x)=g(x)$, $x\in\mathbb{R}$ and $g\in\mathcal{M}$ for the above mentioned four cases where they used several previous known results and techniques, see [1-5, 29-31, 45, 66], to obtain their results.

Observe that the rational functions $R(z)$ in the form (1) are contained in $\mathcal{M}$ therefore we propose to study absolutely continuous invariant measure for $\tau:\mathbb{R}\to\mathbb{R}$ where $\tau(x)=R(x)$, $x\in\mathbb{R}$. Note that the techniques we need to obtain our results are not quite the same as in [21], since for the rational functions $R(z)$ we knows the precise structure of $J(R)$.

Topic II. Discrete dyamics of functions of class A_{∞}.

Suppose that M is the interval [-1,1] or the unit circle S^1 and f from M into itself is a C^1 map. We say f is an endomorphism of class $C^{1+\alpha}$ if the derivative f' of f is α-Hölder continuous and for every critical point c_i of f, there is a small neighborhood U_i of c_i such that $r(x)=\frac{f'(x)}{|x-c_i|^{\ell_i-1}}$ is α-Hölder continuous on $\{x<c_i\}\cap U_i$ and on $\{x>c_i\}\cap U_i$. Endomorphisms of class $C^{1+\alpha}$ are extensively studied for example in [32-35]. In all these papers the authors show distortation estimates for the family of mappings under consideration and they are crucial in the understanding of the dynamical properties of these mappings. These distortion estimates is closely related to the A_∞ condition we will define below. Again, let M be either a unit circle or an interval (or more generally a 1-dimensional manifold). A measure μ is said to be in A_∞ if there are constants $c > 0$ and $\delta \in [0, 1]$ so that

$$\frac{\mu(E)}{\mu(I)} \leq c\left(\frac{|E|}{|I|}\right)^{\delta} \tag{0.2}$$

for every measurable subset E of an interval $I \subset M$, where $|E|$ is the Lebesgue measurable of the set E. We abbreviate this with $\mu \in A_\infty(c, \delta)$.

Consider an endomorphism $f : M \to M$, the image $f_*\mu$ of a measure μ under f is defined by

$$f_*\mu(E) = \mu(f^{-1}(E)) \text{ for every measurable set } E \subset M.$$

Definition 1 *An Endomorphism $f : M \to M$ is called A_∞ mapping if the f^n – images of the Lebesgue measure dx are uniformly in A_∞, i.e., there is $c > 0$ and $\delta \in (0, 1]$ so that $(f^n)_*dx \in A_\infty(c, \delta)$ for every $n \in \mathbb{N}$.*

Observe that such a mapping is absolutely continuous with respect to the Lebesgue measure. If we write $(f^n)_*dx = \omega_n\, dx$, then (if f is differentiable) the density is

$$\omega_n(x) = \sum_{f^n(y)=x} \frac{1}{|(f^n)'(y)|}, \quad x \in M.$$

We call a point $c \in M$ *singular* if either f has no derivative at c or $f'(c) = 0$ and otherwise *regular*. The *singular set* is $C_f := \{c \in M : c \text{ singular}\}$. We propose to study first the following problem:

Problem 1 Establish elementary properties such as the periodic points of A_∞ mappings.

As we mentioned, $C^{1+\alpha}$ mappings are extensively studied in the literature where the authors have all established distortion results for the class of mappings they consider. For example, Shub and Sullivan considered expanding $C^{1+\alpha}$ circle endormorphisms. Lemma 1 of [62] says that there is a constant $c > 0$ such that if $I \subset S^1$ is an interval and f^n is one-to-one on I then

$$\frac{1}{c} < \left|\frac{(f^n)'(x)}{(f^n)'(y)}\right| < c \text{ for any } x,\ y \in I.$$

The estimated term is usually referred to be the distortion and this control of the distortion should be compared with the Koebe $\frac{1}{4}$ Theorem of analytic

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Problem 1 Establish elementary properties such as the periodic points of A_∞ mappings.

As we mentioned, $C^{1+\alpha}$ mappings are extensively studied in the literature where the authors have all established distortion results for the class of mappings they consider. For example, Shub and Sullivan considered expanding $C^{1+\alpha}$ circle endormorphisms. Lemma 1 of [62] says that there is a constant $c > 0$ such that if $I \subset S^1$ is an interval and f^n is one-to-one on I then

$$\frac{1}{c} < \left|\frac{(f^n)'(x)}{(f^n)'(y)}\right| < c \text{ for any } x,\ y \in I.$$

The estimated term is usually referred to be the distortion and this control of the distortion should be compared with the Koebe $\frac{1}{4}$ Theorem of analytic functionns. From this it is easy to see that such a mapping is in fact an A_∞ mapping. We thus are led to the following problem:

Problem 2 Give examples (if possible a description) of the smooth (for example $C^{1+\alpha}$) mappings that are A_∞. As we observe that the very good $C^{1+\alpha}$ mappings of [32-35] are A_∞. Note that these contain the mapping considered by Misiurewicz [45] and those of Jakobson [31]. What about the geometrically finite mappings in [31, 35]. Find more examples of A_∞ mappings.

We now focus on the absolutely continuous invariant measures. For the class of mappings he considered, Misiurewicz [45] gave a precise answer to the following question (see also [32-35]):

Problem 3 Show that an A_∞ mapping has always an absolutely continuous invariant measure. How many such absolutely continuous invariant measures do exist?

Part II Invariant properties of Möbius transformation

This part of our proposal is motivated by a serie of papers written by H.Haruki and T.M.Rassias on invariant characteristic property of Möbius transformations, [24-28]. Recall that a meromorphic functions on $\mathbb{C}$ is called a Möbius transformation if $f(z) = \frac{az+b}{cz+d}$ where a, b, c, and d are complex numbers satisfying $ad - bc \neq 0$. The set of all Möbius transformations under the usual composition of function is a group which is of particular interest when we consider some of its subgroups, for example, subgroup of all Möbius transformations which satisfy $ad - bc = 1$, see [13, 36, 47]. From now on we let $w = f(z)$ be a nonconstant meromorphic function on $\mathbb{C}$. Consider the following properties:

Property A. *$w = f(z)$ transforms circles in the $z-$plane onto circles in the $w-$plane, including straight lines among circles.*

Property B. *Suppose that $w = f(z)$ is analytic and univalent in a nonempty simply connected domain R on the $z-$plane. Let $ABCD$ be an arbitrary quadrilateral(not self-intersecting) contained in R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$ and if $A'B'C'D'$ is a quadrilateral on the $w-$plane which is not self-intersecting, then*

$$< A + < C = < A' + < C'$$

and

$$< B + < D = < B' + D'$$

hold.

Definition 1. Let $\triangle ABC$ be an arbitrary triangle and L a point on the complex plane. We denote by $a = \overline{BC}$, $b = \overline{AC}$, $c = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$. If $ax = by = cz$ holds, then L is said to be an *Apollonius point* of $\triangle ABC$.

Definition 2. Let $ABCD$ be an arbitrary quadrilateral (not necessarily simple) on the complex plane. If $\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{AD}$ holds, then $ABCD$ is said to be an *Apollonius quadrilateral.*

Note that, from Definition 1, any triangle can have at most two Apollonius point.

Property C. *Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. Let $\triangle ABC$ be an arbitrary triangle contained in R and let its Apollonius point L be a point of R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $L' = f(L)$ and if the three different points A', B', C' form a triangle, (i.e. A', B', C' are not collinear), then the point L' is also an Apollonius point of $\triangle A'B'C'$.*

Property D. *Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. Let $ABCD$ be an arbitrary Apollonius quadrilateral contained in R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$, then $A'B'C'D'$ is also an Apollonius quadrilateral on the w-plane.*

The following result about invariant property of Möbius transformations is well-known:

Theorem A [36, 47] *The function $w = f(z)$ satisfies Property A if and only if $w = f(z)$ is a Möbius transformation.*

Later H.Haruki and T.M.Rassias proved the following results on invariant property of Möbius transformations:

Theorem B [25] *The function $w = f(z)$ satisfies Property B if and only if $w = f(z)$ is a Möbius transformation.*

Theorem C [26] *The function $w = f(z)$ satisfies Property C if and only if $w = f(z)$ is a Möbius transformation.*

Theorem D [27] *The function $w = f(z)$ satisfies Property D if and only if $w = f(z)$ is a Möbius transformation.*

Definition 3 The Schwarzian derivative of a function f, S_f is defined by

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The following result is well-known [47]:

Theorem 1 *Let f be a complex-valued function. Then $S_f(z) = 0$ for all z such that $f'(z) \neq 0$ if and only if f is a Möbius transformation.*

The key ingredients in the proof of the above Theorems are known, such as the Maximum Modulus Principle of analytic functions, the Reflection Principle of analytic functions, and some well-known lemmas. The strategies in all the results above are to show that if $w = f(z)$ satisfies either Property B, C, or D, then $w = f(z)$ has zero Schwarzian derivative in R and hence in $\mathbb{C}$ (by the Identity Theorem) and hence it must be a Möbius transformation. However, as we inspect the proof of these results carefully, we see that we can proof these results by using one of the most important properties of Möbius transformations, namely the invariance of cross ratio (recall that the cross ratio of four distinct points z_1, z_2, z_3, $z_4 \in \mathbb{C} \cup \{\infty\}$ is defined to be $\frac{z_1-z_3}{z_1-z_4} \cdot \frac{z_2-z_4}{z_2-z_3}$). Moreover, we are able to extend some definitions and results of H.Haruki and T.M.Rassias, for example, we have the following:

Definition 4. Let $k, l > 0$. Let $\triangle ABC$ be an arbitrary triangle and L a point on the complex plane. We denote by $a = \overline{BC}$, $b = \overline{AC}$, $c = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$. If $ax = k(by) = l(cz)$ holds, then L is said to be an *(k, l)-Apollonius point* of $\triangle ABC$.

Definition 5. Let $k > 0$. A quadrilateral $ABCD$ is called a *$k-$Apollonius quadrilateral* if $\overline{AB} \cdot \overline{CD} = k\,\overline{BC} \cdot \overline{AD}$.

By using the invariance of cross ratio under Möbius transformations, we have obtained the following results which generalize Theorem B, C, and D above:

Property C′. *Suppose that $w = f(z)$ is analytic and univalent in a non-empty domain R of the z-plane. Let $k, l > 0$. Let $\triangle ABC$ be an arbitrary triangle contained in R and let its (k, l)-Apollonius point L be a point of R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $L' = f(L)$ and if the three different points A', B', C' form a triangle, (i.e. A', B', C' are not collinear), then the point L' is also a (k, l)-Apollonius point of $\triangle A'B'C'$.*

Property D′. *Suppose that $w = f(z)$ is analytic and univalent in a non-empty domain R of the z-plane. Let $ABCD$ be an arbitrary k-Apollonius quadrilateral contained in R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$, then $A'B'C'D'$ is also a k-Apollonius quadrilateral on the w-plane.*

Theorem C′. The function $w = f(z)$ satisfies Property C′ if and only if $w = f(z)$ is a Möbius transformation.

Theorem D′. *The function $w = f(z)$ satisfies Property D′ if and only if $w = f(z)$ is a Möbius transformation.*

Moreover, we obtain a new result similar to Theorem 1 as follows:

Theorem 1′. *Let f be a meromorphic on the plane. Define the* **Newton derivative**, *N_f of a function f as $N_f(z) = z - \frac{f(z)}{f'(z)}$. Then $N_f(z) = 0$ for all z such that $f'(z) \neq 0$ if and only if f is a Möbius transformation of the form $f(z) = \frac{u}{z+u}$, $u \neq 0$.*

Motivated by the previously known results we have mentioned and new results we have obtained, we propose to study the following problems:

Problem I. From Theorem 1′, we propose to study (find) new invariance properties of Möbius transformations which have Newton derivative equal to zero. For examples, we propose to study Properties B, C, D, C′, and D′ for this class of Möbius transformations. (Currently, we have already submitted one paper related to Problem I to Journal of Mathematical Analysis and Applications which are now being refereed).

Problem II. Continue from Problem I, we will study some new invariant properties of general Möbius transformations. One possible direction is to consider hexagonal instead of quadrilateral in THeorem D and D′ above in which we have some precise idea of how to attack this problem.

Problem III. In [46, 48-49], it was shown that if an analytic function f in the unit disk $D(0,1) = \{z : |z| < 1\}$ satisfies the following inequality

$$|S_f(z)| \leq \frac{2}{(1-|z|^2)^2}$$

for all $z \in D(0,1)$ then f is univalent in $D(0,1)$ where the constant 2 is best possible. It is routine to check that if f is univalent in $D(0,1)$ then f satisfies the following inequality

$$|S_f(z)| \leq \frac{6}{(1-|z|^2)^2}$$

for all $z \in D(0,1)$ where again the constant 6 is best possible. The above result was generalized in [17] where it was shown that if an analytic function f in the unit disk $D(0,1)$ satisfies the following inequality

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{C}{1-|z|^2}$$

for all $z \in D(0,1)$ and for sufficiently small constant C then f is univalent in $D(0,1)$ and it was also shown that C can be taken to be $2(\sqrt{5}-2)$ or smaller.

Several authors have then tried to increase the value of C, see [17-18, 49]. From these results, we propose to find necessary and sufficient conditions for an analytic function in $D(0,1)$ to be univalent by considering the modulus of Newton derivative, $|N_f|$ instead of $|S_f|$ or $\left|\frac{f''(z)}{f'(z)}\right|$. Our results will be certainly new and can be applied to obtain some new criterion for univalence of analytic functions in $D(0,1)$. As of present, we have obtained a necessary condition already.

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ผลการวิจัยที่ได้รับ

1. สรุปรายละเอียดการวิจัย

1.1 การวิจัยในปีที่ 1 (เดือนที่ 1 – เดือนที่ 12) เน้นการวิจัยในหัวข้อการวิจัยคุณสมบัติไม่แปรเปลี่ยนบางประการของการแปลงเชิงเส้นคู่

1.2 การวิจัยในปีที่ 2 (เดือนที่ 13– เดือนที่ 24) เน้นการวิจัยในหัวข้อการวิจัยคุณสมบัติไม่แปรเปลี่ยนบางประการของการแปลงเชิงเส้นคู่ และพลศาสตร์เชิงดิสครีตของฟังก์ชันนิวตัน

1.3 การวิจัยในปีที่ 3 (เดือนที่ 25 – เดือนที่ 36) เน้นการศึกษาสมบัติบางประการของเซตจูเลียของฟังก์ชันตรรกยะ โดยเฉพาะการให้เงื่อนไขที่เพียงพอที่ทำให้เซตจูเลียของฟังก์ชันตรรกยะเป็น Lakes of Wada Continuum และการหาผลเฉลยของสมการเชิงฟังก์ชันนัยบางสมการ

2. ผลการวิจัยที่ได้รับ

2.1 ในการวิจัยหัวข้อการวิจัยคุณสมบัติไม่แปรเปลี่ยนบางประการของการแปลงเชิงเส้นคู่ ในช่วงเดือนที่ 1 - เดือนที่ 12 ได้งานวิจัยที่ได้รับการตีพิมพ์ในวารสารระดับนานาชาติจำนวน 2 ผลงานดังนี้

2.1.1 A Note on the Characteristics of Mobius Transformations, J. Math. Anal. Appls. 248 (2000), 203-215.

2.1.2 A Characterization of Mobius Transformations, Internat. J. Math. & Math. Sci., Vol.24, No.10 (2000), 663-666.

2.1.3 A Note on the Characteristics of Mobius Transformations II, J. Math. Anal. Appls. 261 (2001), 151-158.

2.2 การวิจัยในช่วงเดือนที่ 13 - เดือนที่ 24 ได้รับการตีพิมพ์จำนวน 1 เรื่อง คือ

2.2.1 Dynamics of Newton's Functions of Barna's Polynomials, Int. J. Math. & Math. Sc. 28:2 (2002) 79 – 84, 79 – 84.

2.3 การวิจัยในช่วงเดือนที่ 25 - เดือนที่ 36 ได้ผลงานวิจัย 2 เรื่องคือ

2.3.1 Julia Sets of Certain Rational Functions และอยู่ในระหว่างการเตรียมต้นฉบับเพื่อส่งตีพิมพ์

2.3.2 Meromorphic Solutions of Ceratin Functional Equation และอยู่ในระหว่างการเตรียมต้นฉบับเพื่อส่งตีพิมพ์

3. การนำผลงานวิจัยไปใช้ประโยชน์

เชิงสาธารณะ มีการประชาสัมพันธ์กิจกรรมการวิจัยและหาผู้วิจัยร่วมอย่างต่อเนื่องแต่ยังไม่ได้รับความสนใจนักเนื่องจากเป็นสาขาวิชาที่ค่อนข้างยากและใช้เวลาในการศึกษามาก

เชิงวิชาการ มีนักศึกษาปริญญาโทสำเร็จการศึกษาไปจำนวน 7 คนภายใต้โครงการนี้ และมีนักศึกษาปริญญาโท 3 คนและปริญญาเอกเอก 1 คนที่กำลังทำวิจัยในขณะนี้

4. นักศึกษาระดับปริญญาโทที่สำเร็จการศึกษา (โดยได้รับค่าจ้างวิจัยจากโครงการวิจัยนี้)

ชื่อนักศึกษา	ชื่อหัวข้อวิจัย/วิทยานิพนธ์ ระดับปริญญาโท	ค่าจ้าง (1,000 บาท/เดือน)
น.ส.ธนิกานต์ หอมแก่นจันทร์ (2544) (คณิตศาสตร์ประยุกต์)	Asymptotic Stability of Linear Difference Equations $x_{n+1} - a^2 x_{n-1} + b x_{n-k} = 0$	12 เดือน เป็นเงิน 12,000 บาท
นายสมเกียรติ ฤทธิ์ศิริ (คณิตศาสตร์ประยุกต์) (2544)	Asymptotic Stability of Linear Difference Equations with Delays	12 เดือน เป็นเงิน 12,000 บาท
นายสมบูรณ์ นิยม (คณิตศาสตร์ประยุกต์) (2544)	Asymptotic Stability of Nonlinear Systems Described by Difference Equations with Multiple Delays	12 เดือน เป็นเงิน 12,000 บาท
นายธีรพล สะอิ้วงศ์ (คณิตศาสตร์ประยุกต์) (2545)	Exponential Stability of Nonlinear Time-Varying Differential Equations	12 เดือน เป็นเงิน 12,000 บาท
นายจิตติ รักบุตร (คณิตศาสตร์) (2545)	Julia set of $az^n + b$	12 เดือน เป็นเงิน 12,000 บาท
น.ส.วิยะดา อมรรัตนไพจิตร (คณิตศาสตร์ประยุกต์) (2546)	Asymptotic Stability of Linear Difference Equations $\frac{x_n + a}{x_{n-1} + a x_n + b}$	12 เดือน เป็นเงิน 12,000 บาท
น.ส.พรทิพย์ ป้องชาลี (คณิตศาสตร์ประยุกต์) (2546)	Asymptotic Stability of Difference Equation $x_{n+1} - x_n = -a_n x_{n-k}$	12 เดือน เป็นเงิน 12,000 บาท

และยังมีนักศึกษาระดับปริญญาโทอีก 3 คนที่กำลังทำวิทยานิพนธ์และคาดว่าจะเสร็จภายในเดือน สิงหาคม 2547

5. ข้อเสนอแนะสำหรับงานวิจัยในอนาคต

เนื่องจากเนื้อหาการวิจัยที่เสนอมีค่อนข้างมากดังนั้นการวิจัยในส่วนของพลศาสตร์เชิงดิสครีตของฟังก์ชันในคลาส $A_\pm$ และอธิบาย การมี Absolutely Continuous Invariant Measures ของฟังก์ชันในคลาส $A_\pm$ ยังทำได้น้อย ดังนั้นในอนาคตจะได้ศึกษาวิจัยในส่วนนี้เพิ่มเติม นอกจากนั้นการวิจัยพลศาสตร์เชิงดิสครีตของฟังก์ชันตรรกยะบางฟังก์ชันก็ยังมีปัญหาในการวิจัยอีกมากเช่นกัน

ผลงานวิจัยที่ได้รับการตีพิมพ์

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A CHARACTERIZATION OF MÖBIUS TRANSFORMATIONS

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ABSTRACT. We give a new invariant characteristic property of Möbius transformations.

Keywords and phrases. Möbius transformations, Schwarzian derivative, Newton derivative.

2000 Mathematics Subject Classification. Primary 30C35.

1. Introduction. Throughout this paper, we let $w = f(z)$ be a nonconstant meromorphic function in $\mathbb{C}$ unless otherwise stated.

We consider the following properties.

PROPERTY 1.1. $w = f(z)$ transforms circles in the z-plane onto circles in the w-plane, including straight lines among circles.

PROPERTY 1.2. Suppose that $w = f(z)$ is analytic and univalent in a nonempty simply connected domain $\mathbb{R}$ on the z-plane. Let $ABCD$ be an arbitrary quadrilateral (not self-intersecting) contained in $\mathbb{R}$. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$ and if $A'B'C'D'$ is a quadrilateral on the w-plane which is not self-intersecting, then the following hold

$$\angle A + \angle C = \angle A' + \angle C', \qquad \angle B + \angle D = \angle B' + \angle D'. \tag{1.1}$$

The following is a well-known principle of circle transformation of Möbius transformations.

THEOREM 1.3. *$w = f(z)$ satisfies Property 1.1 if and only if $w = f(z)$ is a Möbius transformations.*

In [1], it is shown that Property 1.1 implies Property 1.2 and a new invariant characteristic property of Möbius transformations is given as follows.

THEOREM 1.4. *Let α be an arbitrary fixed real number such that $0 < \alpha < 2\pi$. Suppose that $w = f(z)$ is analytic and univalent in a nonempty simply connected domain $\mathbb{R}$ on the z-plane. Let $ABCD$ be an arbitrary quadrilateral (not self-intersecting) contained in $\mathbb{R}$ satisfying*

$$\angle A + \angle C = \alpha. \tag{1.2}$$

If $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$ is a quadrilateral on the w-plane which is not self-intersecting, then the only function which satisfies

$$\angle A' + \angle C' = \alpha \tag{1.3}$$

is a Möbius transformation.

Theorem 1.4 gives an alternative proof of "the only if part" of Theorem 1.3. Motivated by the above results, we consider the following property.

Property 1.5. Let k be an arbitrary positive real number. For three arbitrary distinct points a, b, and c in $\mathbb{R}$ satisfying

$$\left|\frac{a-b}{c-b}\right| = k, \tag{1.4}$$

we have

$$\left|\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}\right| = k. \tag{1.5}$$

In Section 3, we prove the following result concerning the mapping property of an analytic and univalent function on a connected domain.

Theorem 1.6. *Let k be an arbitrary positive real number. Let $w = f(z)$ be analytic and univalent in a nonempty connected domain $\mathbb{R}$ on the z-plane such that $f(z) \neq 0$ for all $z \in \mathbb{R}$. Then f satisfies Property 1.5 if and only if f is a Möbius transformation of the form $u/(z+v)$, $u \neq 0$.*

2. Lemmas

Definition 2.1. Let f be a complex-valued function. The Schwarzian derivative of f is defined as follows:

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{2.1}$$

Similar to Schwarzian derivative, we have the following.

Definition 2.2. Let f be a complex-valued function. We define the *Newton derivative* of f as follows:

$$N_f(z) = \left(z - \frac{f(z)}{f'(z)}\right)' = \frac{f(z)f''(z)}{(f'(z))^2}. \tag{2.2}$$

Remark 2.3. Note that $N_f(z)$ is the first derivative of Newton's method of f.

Remark 2.4. Let f be a complex-valued function. It is well known that $S_f(z) = 0$ if and only if f is a Möbius transformation.

From Remark 2.4, we have observed that a similar result holds true when we replace Schwarzian derivative by the Newton derivative.

Lemma 2.5. *Let f be a complex-valued function. Then $N_f(z) = 2$ if and only if f is a Möbius transformation of the form $u/(z+v)$, $u \neq 0$.*

Proof. Let f be a Möbius transformation of the form $u/(z+v)$, $u \neq 0$, then it is easily checked that $N_f(z) = 2$. Let f be a complex-valued function such that $N_f(z) = 2$. It follows that

$$\left(z - \frac{f(z)}{f'(z)}\right)' = 2 \tag{2.3}$$

which implies that

$$z - \frac{f(z)}{f'(z)} = 2z - c_1, \tag{2.4}$$

where c_1 is a complex constant, thus

$$\frac{f(z)}{f'(z)} = -z + c_1 \tag{2.5}$$

or

$$\frac{1}{f(z)} \frac{df(z)}{dz} = \frac{1}{-z + c_1}. \tag{2.6}$$

From which it follows by a simple calculation that f is a Möbius transformation of the form $u/(z+v)$, $u \neq 0$. □

3. Main result. In this section, we assume that $w = f(z)$ is analytic and univalent on a nonempty connected domain $\mathbb{R}$ on the z-plane such that $f(z) \neq 0$ for all $z \in \mathbb{R}$.

Proof of Theorem 1.6. Let $f(z)$ be a Möbius transformation of the form $u/(z+v)$, $u \neq 0$. Let a, b, and c be arbitrary three distinct points in $\mathbb{R}$ such that

$$\left| \frac{a-b}{c-b} \right| = k. \tag{3.1}$$

We observe that

$$\frac{a-b}{c-b} \tag{3.2}$$

is the cross-ratio of a, b, c, and d, where d is the point at infinity. Since $f(z) = u/(z+v)$, $u \neq 0$, we have $f(d) = 0$. Since Möbius transformations preserve the cross-ratio, we obtain

$$\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)} = \frac{a-b}{c-b} \tag{3.3}$$

which implies that

$$\left| \frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)} \right| = \left| \frac{a-b}{c-b} \right| = k. \tag{3.4}$$

Therefore, any Möbius transformation of the form $u/(z+v)$, $u \neq 0$ satisfies Property 1.5.

Conversely, let x be an arbitrary fixed point in $\mathbb{R}$. Then there exists a positive real number r such that the r circular neighborhood $N_r(x)$ of x is contained in $\mathbb{R}$.

Throughout the proof let $A = x + ky$, $B = x$, $C = x - y$. Since $\mathbb{R}$ is a nonempty connected domain on the z-plane, there exists a positive real number s such that if

$$0 < |y| < s, \tag{3.5}$$

then A, B, and C are contained in $N_r(x)$.

Since $w = f(z)$ is univalent in $\mathbb{R}$, $f(A) = f(x+ky)$, $f(B) = f(x)$, and $f(C) = f(x-y)$ are distinct points. By assumption, we have

$$\left|\frac{f(x+ky)-f(x)}{f(x-y)-f(x)}\cdot\frac{f(x-y)}{f(x+ky)}\right| = k \tag{3.6}$$

for all y such that $0 < |y| < s$.

Let

$$h(y) = \frac{f(x+ky)-f(x)}{f(x-y)-f(x)}\cdot\frac{f(x-y)}{f(x+ky)}. \tag{3.7}$$

Then

$$|h(y)| = k \tag{3.8}$$

for all y such that $0 < |y| < s$. The function $h(y)$ extends analytically at zero by $h(0) = -k$. Hence, by the maximum modulus principle, we have $h(y) = -k$ for all y with $|y| < s$. In other words, we have

$$\frac{f(x+ky)-f(x)}{f(x-y)-f(x)}\cdot\frac{f(x-y)}{f(x+ky)} = -k \tag{3.9}$$

in $|y| < s$. This equality implies that

$$(f(x+ky)-f(x))f(x-y) = -k(f(x-y)-f(x))f(x+ky). \tag{3.10}$$

Differentiate this equality twice with respect to y and then set $y = 0$, we obtain

$$-k(k+1)(2(f'(x))^2 - f(x)f''(x)) = 0 \tag{3.11}$$

which implies that

$$2(f'(x))^2 - f(x)f''(x) = 0 \tag{3.12}$$

or

$$\frac{f(x)f''(x)}{(f'(x))^2} = 2. \tag{3.13}$$

By the identity theorem and Lemma 2.5, we conclude that f is a Möbius transformation of the form $u/(z+v)$, $u \neq 0$. □

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A Note on the Characteristics of Möbius Transformations

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We give some new invariant characteristic properties of Möbius transformations by means of their mapping properties. © 2000 Academic Press

Key Words: Möbius transformations; Schwarzian derivative.

1. INTRODUCTION

Throughout the paper, unless otherwise stated, let $w = f(z)$ be a nonconstant meromorphic function on the complex plane $\mathbb{C}$. It is well known that for $w = f(z)$ to be a Möbius transformation, it is necessary and sufficient that $w = f(z)$ satisfies the following Property A:

Property A. $w = f(z)$ maps circles in the z-plane onto circles in the w-plane, including straight lines among circles.

The following are some definitions and mapping properties which were introduced in [5–7].

DEFINITION 1.1. Let $\triangle ABC$ be an arbitrary triangle and L a point on $\mathbb{C}$. We denote $a = \overline{BC}$, $b = \overline{AC}$, $c = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$. If $ax = by = cz$ holds, then L is said to be an Apollonius point of $\triangle ABC$.

DEFINITION 1.2. Let $ABCD$ be an arbitrary quadrilateral (not necessarily simple) on $\mathbb{C}$. If $\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{DA}$ holds, then $ABCD$ is said to be an Apollonius quadrilateral.

Property B. Suppose that $w = f(z)$ is analytic and univalent in a nonempty simply connected domain R on the z-plane. Let $ABCD$ be an arbitrary quadrilateral (not self-intersecting) contained in R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$ and if $A'B'C'D'$ is a quadri-

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lateral on the w-plane which is not self-intersecting, then

$$\angle A + \angle C = \angle A' + \angle C'$$

and

$$\angle B + \angle D = \angle B' + \angle D'$$

hold.

Property C. Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. Let $\triangle ABC$ be an arbitrary triangle contained in R and let its Apollonius point L be a point of R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $L' = f(L)$, and if the three different points A', B', C' form a triangle (i.e., A', B', C' are not collinear), then the point L' is also an Apollonius point of $\triangle A'B'C'$.

Property D. Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. Let $ABCD$ be an arbitrary Apollonius quadrilateral contained in R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$, then $A'B'C'D'$ is also an Apollonius quadrilateral.

Recently, in [5–7] Haruki and Rassias gave several new characteristics of Möbius transformations from the standpoint of conformal mapping and elementary geometry. The following are the results they obtained:

THEOREM A [5]. *The function $w = f(z)$ satisfies Property B iff $w = f(z)$ is a Möbius transformation.*

THEOREM B [6]. *The function $w = f(z)$ satisfies Property C iff $w = f(z)$ is a Möbius transformation.*

THEOREM C [7]. *The function $w = f(z)$ satisfies Property D iff $w = f(z)$ is a Möbius transformation.*

These results are interesting in the sense that they illustrated some connections between geometric properties and analytic properties of analytic univalent mappings. The proof of the "only if" part of these results requires some known results from geometry together with the following key lemmas which are also well known:

LEMMA 1.3. *If the function $w = f(z)$ is analytic and univalent in a nonempty domain R, then $f'(z) \neq 0$ in R.*

LEMMA 1.4. *If $f(z)$ and $g(z)$ are analytic functions in a nonempty domain R and $f(z)g(z) \neq 0$ in R and also $\arg(f(z)) = \arg(g(z))$ holds in R, then $f(z) = Kg(z)$ in R where K is a positive real constant.*

LEMMA 1.5. *Let $w = f(z)$ be meromorphic on $\mathbb{C}$. Then $w = f(z)$ is a Möbius transformation iff $S_f(z) = 0$ for all $z \in \mathbb{C} - \{z : f'(z) = 0\}$, where*

$S_f(z) = (f''(z)/f'(z))' - (1/2)(f''(z)/f'(z))^2$ *which is called the Schwarzian derivative of* $f(z)$.

We now introduce the *Newton derivative* of a function $f(z)$ as follows:

DEFINITION 1.6. Let $f(z)$ be a function on $\mathbf{C}$. We define the Newton derivative of $f(z)$ as the first derivative of the Newton's method of $f(z)$. In other words, we define the Newton's derivative of $f(z)$ as

$$N_f(z) = \left(z - \frac{f(z)}{f'(z)}\right)' = \frac{f(z)f''(z)}{(f'(z))^2}.$$

The main purpose of this paper is to generalize Theorems A, B, C and to prove the obtained results by means of the invariance of cross-ratio of four distinct points on $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ under a Möbius transformation. We will also give some new invariant characteristic properties of Möbius transformations. In particular, we will characterize Möbius transformations which have Newton derivative equal to 2 instead of having zero Schwarzian derivative.

2. MAIN RESULTS

First, we give another proof of Theorem A by means of the invariance of cross-ratio of four distinct points on $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ under a Möbius transformation.

Proof of Theorem A. Suppose that $w = f(z)$ is a Möbius transformation and let $ABCD$ be an arbitrary quadrilateral in R. Then we obtain

$$\angle A = \arg\left(\frac{A-D}{A-B}\right)$$

and

$$\angle C = \arg\left(\frac{C-B}{C-D}\right)$$

which implies that

$$\angle A + \angle C = \arg\left(\frac{A-D}{A-B}\right) + \arg\left(\frac{C-B}{C-D}\right) = \arg\left(\frac{A-D}{A-B}\cdot\frac{C-B}{C-D}\right).$$

Since $\frac{A-D}{A-B}\cdot\frac{C-B}{C-D}$ is the cross-ratio of four distinct points A, D, C, and B, we obtain

$$\frac{f(A)-f(D)}{f(A)-f(B)}\cdot\frac{f(C)-f(B)}{f(C)-f(D)} = \frac{A-D}{A-B}\cdot\frac{C-B}{C-D}$$

since the cross-ratio is invariant under mapping by a Möbius transformation. It follows that

$$\begin{aligned}\angle A' + \angle C' &= \arg\left(\frac{f(A)-f(D)}{f(A)-f(B)} \cdot \frac{f(C)-f(B)}{f(C)-f(D)}\right)\\ &= \arg\left(\frac{A-D}{A-B} \cdot \frac{C-B}{C-D}\right)\\ &= \angle A + \angle C\end{aligned}$$

which implies that $w = f(z)$ satisfies Property B. The other direction of the proof is the same as in [5].

We now generalize Definitions 1.1 and 1.2 as follows:

DEFINITION 2.1. Let $\triangle ABC$ be an arbitrary triangle and L a point on $\mathbb{C}$. We denote $a = \overline{BC}$, $b = \overline{AC}$, $c = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$. If $ax = k(by) = l(cz)$ holds where $k, l > 0$, then L is said to be a (k, l)-Apollonius point of $\triangle ABC$.

DEFINITION 2.2. Let $ABCD$ be an arbitrary quadrilateral (not necessarily simple) on $\mathbb{C}$. If $\overline{AB} \cdot \overline{CD} = k(\overline{BC} \cdot \overline{DA})$ holds, then $ABCD$ is said to be a k-Apollonius quadrilateral.

Remark 2.1. If L is a (k, l)-Apollonius point of $\triangle ABC$, then the quadrilateral $BCAL$ is a k-Apollonius quadrilateral, and the quadrilateral $BCLA$ is an l-Apollonius quadrilateral, where the sense of any four points is counterclockwise.

Consider the following properties:

Property C'. Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. Let $\triangle ABC$ be an arbitrary triangle contained in R and let its (k, l)-Apollonius point L be a point of R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $L' = f(L)$, and if the three different points A', B', C' form a triangle, (i.e., A', B', C' are not collinear), then the point L' is also a (k, l)-Apollonius point of $\triangle A'B'C'$.

Property D'. Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. Let $ABCD$ be an arbitrary k-Apollonius quadrilateral contained in R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$, then $A'B'C'D'$ is also a (k, l)-Apollonius quadrilateral.

The "if" parts of the following results can be proved using the same technique as in the proof of Theorem A above. The "only if" parts can be proved similarly as in [6, 7].

THEOREM 2.3. *The function $w = f(z)$ satisfies Property C' iff $w = f(z)$ is a Möbius transformation.*

THEOREM 2.4. *The function $w = f(z)$ satisfies Property D' iff $w = f(z)$ is a Möbius transformation.*

We now state

Property E. Let $w = f(z)$ be analytic and univalent in a nonempty domain R on the z-plane. Let $\alpha = 0$ or π. Let a, b, c, and d be four distinct points in R such that

$$\arg\left(\frac{a-b}{a-d}\cdot\frac{c-d}{c-b}+\frac{a-d}{a-b}\cdot\frac{c-b}{c-d}\right)=\alpha.$$

Then we have

$$\arg\left(\frac{f(a)-f(b)}{f(a)-f(d)}\cdot\frac{f(c)-f(d)}{f(c)-f(b)}+\frac{f(a)-f(d)}{f(a)-f(b)}\cdot\frac{f(c)-f(b)}{f(c)-f(d)}\right)=\alpha.$$

Property F. Let $w = f(z)$ be analytic and univalent in a nonempty domain R on the z-plane. Let $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Let a, b, c, and d be four distinct points in R such that

$$\arg\left(\frac{a-b}{a-d}\cdot\frac{c-d}{c-b}-\frac{a-d}{a-b}\cdot\frac{c-b}{c-d}\right)=\alpha.$$

Then we have

$$\arg\left(\frac{f(a)-f(b)}{f(a)-f(d)}\cdot\frac{f(c)-f(d)}{f(c)-f(b)}-\frac{f(a)-f(d)}{f(a)-f(b)}\cdot\frac{f(c)-f(b)}{f(c)-f(d)}\right)=\alpha.$$

We now give some more invariant characteristics of Möbius transformations as follows.

THEOREM 2.5. *Let $w = f(z)$ be analytic and univalent in a nonempty domain R on the z-plane. Then $w = f(z)$ satisfies Property E (or F) iff $w = f(z)$ is a Möbius transformation.*

Proof. The "if" part follows as in the proof of the "if" part of Theorem A. We now proceed to prove the "only if" part. Since $w = f(z)$ is analytic and univalent in the domain R, we have $f'(z) \neq 0$ in R. If x is an arbitrary fixed point in R, then we obtain $f'(x) \neq 0$. Let E be the point represented by x. Since $E \in R$, there exists a positive real number r such that the r circular neighborhood of E is contained in R. Throughout the proof let $ABCD$ denote an arbitrary rhombus in R with center at E

where A, B, C, and D are distinct points. Here the sense of A, B, C, and D is counterclockwise. Since $ABCD$ is a rhombus contained in R, we can represent A, B, C, and D by complex numbers

$$z+y, \quad z+iky, \quad z-y, \quad z-iky,$$

respectively for some positive real number k. Without loss of generality, we may assume that $k > 1+\sqrt{2}$. Since R is a nonempty domain R on the z-plane, there exists a nonzero real number s such that $s < r$ and if $0 < |y| < s$ then $ABCD$ is contained in R. Since $w = f(z)$ is univalent in R, $f(A) = f(z+y)$, $f(B) = f(z+iky)$, $f(C) = f(z-y)$, $f(D) = f(z-iky)$ are distinct points. By assumption, we have

$$\arg\left(\begin{array}{l}\dfrac{f(z+y)-f(z+iky)}{f(z+y)-f(z-iky)}\cdot\dfrac{f(z-y)-f(z-iky)}{f(z-y)-f(z+iky)}\\[2ex] +\dfrac{f(z+y)-f(z-iky)}{f(z+y)-f(z+iky)}\cdot\dfrac{f(z-y)-f(z+iky)}{f(z-y)-f(z-iky)}\end{array}\right) = 0$$

$$= \arg(1) \tag{2.1}$$

for all y such that $0 < |y| < s$.

Since $z \in R$ is arbitrarily fixed, we can set

$$h(y) = \frac{f(z+y)-f(z+iky)}{f(z+y)-f(z-iky)}\cdot\frac{f(z-y)-f(z-iky)}{f(z-y)-f(z+iky)} + \frac{f(z+y)-f(z-iky)}{f(z+y)-f(z+iky)}\cdot\frac{f(z-y)-f(z+iky)}{f(z-y)-f(z-iky)}. \tag{2.2}$$

By (2.1) and (2.2) we obtain

$$\arg(h(y)) = \arg(1) \tag{2.3}$$

for all y such that $0 < |y| < s$. Now we prove that $h(y)$ is analytic at $y = 0$ and that (2.3) still holds at $y = 0$. To this end we apply Riemann's Theorem on removable singularities. As $y \to 0$, by L'Hopital's Rule, we obtain that

$$h(y) \to \left(\frac{1+ik}{1-ik}\right)^2 + \left(\frac{-1+ik}{1+ik}\right)^2 = \frac{2(1-6k^2+k^4)}{(1+k^2)^2}. \tag{2.4}$$

If we define

$$h(0) = \frac{2(1-6k^2+k^4)}{(1+k^2)^2} \tag{2.5}$$

by (2.4), by Riemann's Theorem on removable singularities, the function $h(y)$ is analytic at $y = 0$. Furthermore, (2.3) still holds at $y = 0$. The function $h(y)$ is analytic in $|y| < s$. By (2.2) and the fact that $w = f(z)$ is univalent in R, we obtain that $h(y) \neq 0$ in $|y| < s$. Hence by Lemma 1.4 we have

$$h(y) = K \tag{2.6}$$

in $|y| < s$, where K is a positive real constant. Setting $y = 0$ in (2.6) and using (2.5), it yields

$$\frac{2(1 + 6k^2 + k^4)}{(1 + k^2)^2} = K. \tag{2.7}$$

By (2.7) and (2.6) we obtain

$$h(y) = \frac{2(1 - 6k^2 + k^4)}{(1 + k^2)^2} \tag{2.8}$$

in $|y| < s$.

Substituting (2.2) into (2.6) and removing the denominator in the resulting equality it follows that

$$\begin{aligned}
&(f(x+y) - f(x+iky))^2(f(x-y) - f(x-iky))^2\\
&\quad + (f(x+y) - f(x-iky))^2(f(x-y) - f(x+iky))^2\\
&= \frac{2(1 - 6k^2 + k^4)}{(1 + k^2)^2}(f(x+y) - f(x+iky))\\
&\quad \times (f(x-y) - f(x-iky))(f(x+y) - f(x-iky))\\
&\quad \times (f(x-y) - f(x+iky))
\end{aligned} \tag{2.9}$$

in $|y| < s$.

Using Leibnitz's Rule for differentiation, differentiate six times both sides of (2.7) with respect to y; setting $y = 0$ yields

$$-1920k^2(-1 + k^2)(f'(x))^2\left(-3(f''(x))^2 + 2f'(x)f'''(x)\right) = 0. \tag{2.10}$$

Since k is a positive real number which is greater than $1 + \sqrt{2}$, we have $k^2(-1 + k^2) \neq 0$. Hence by (2.8) we obtain $f'''(x)f'(x) - \frac{3}{2}(f''(x))^2 = 0$. Since $x \in R$ was arbitrarily fixed, we can replace x by a variable z, and by (2) we have $f'''(z)f'(z) - \frac{3}{2}(f''(z))^2 = 0$ in R. By the Identity Theorem the above equality holds in C. Hence,

$$\frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = 0$$

holds for all z satisfying $f'(z) \neq 0$. Thus, the Schwarzian derivative of f vanishes for all z satisfying $f'(z) \neq 0$. Therefore, by Lemma 1.5, $f(z)$ is a Möbius transformation of z. ∎

We now give some invariant properties for Möbius transformations which have Newton derivative equal to 2. First, we state the following

Property G. Suppose that $w = f(z)$ is analytic and univalent in a nonempty simply connected domain R on the z-plane. Let α be an arbitrarily fixed real number such that $\alpha \in (0, \pi)$. For three arbitrary distinct points a, b, and c in R satisfying

$$\arg\left(\frac{a-b}{c-b}\right) = \alpha,$$

we have

$$\arg\left(\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}\right) = \alpha$$

For Möbius transformations which have Newton derivative equal to 2, we have the following result:

LEMMA 2.6. *Let f be a complex-valued function. Then $N_f(z) = 2$ for all $z \in \mathbb{C} - \{z : f'(z) = 0\}$ iff f is a Mobius transformation of the form $\frac{u}{z+v}$, $u \neq 0$.*

Proof. Let f be a Möbius transformation of the form $\frac{u}{z+v}$, $u \neq 0$; then it is easily checked that $N_f(z) = 2$. Let f be a complex-valued function such that $N_f(z) = 2$. It follows that

$$\left(z - \frac{f(z)}{f'(z)}\right)' = 2$$

which implies that

$$z - \frac{f(z)}{f'(z)} = 2z - c_1,$$

where c_1 is a complex constant,

$$\frac{f(z)}{f'(z)} = -z + c_1$$

or

$$\frac{1}{f(z)} \frac{df(z)}{dz} = \frac{1}{-z + c_1}.$$

It follows by a simple calculation that f is a Möbius transformation of the form $\frac{u}{z+c}$, $u \neq 0$. ∎

We are now ready to prove the following

THEOREM 2.7. *Let $w = f(z)$ be analytic and univalent in a nonempty connected domain R on the z-plane. Then f satisfies Property G iff f is a Möbius transformation of the form $\frac{u}{z+c}$, $u \neq 0$.*

Proof. Let $f(z)$ be a Möbius transformation of the form $\frac{u}{z+c}$, $u \neq 0$. Let a, b, and c be arbitrary three distinct points in R such that

$$\arg\left(\frac{a-b}{c-b}\right) = \alpha.$$

We observe that

$$\frac{a-b}{c-b}$$

is the cross-ratio of a, b, c, and d where d is the point at infinity. Since $f(z) = \frac{u}{z+c}$, $u \neq 0$, we have $f(d) = 0$. Since Möbius transformations preserve the cross-ratio, we obtain

$$\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)} = \frac{a-b}{c-b}$$

which implies that

$$\arg\left(\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}\right) = \arg\left(\frac{a-b}{c-b}\right) = \alpha.$$

Therefore, any Möbius transformation of the form $\frac{u}{z+c}$, $u \neq 0$ satisfies Property G. Conversely, let α be an arbitrary real number such that $\alpha \in (0, \pi) - \{\frac{\pi}{2}\}$. Let x be an arbitrary fixed point in R; then we obtain $f'(x) \neq 0$. Since $x \in R$, there exists a positive real number r such that the r circular neighborhood of x is contained in R. Throughout the proof let ABC denote an arbitrary isosceles triangle in R with center at x where A, B, and C are distinct points. Here the sense of A, B, and C are counterclockwise. Since ABC is an isosceles triangle contained in R, we can represent A, B, and C by complex numbers

$$x + ay, \qquad x + y, \qquad x + by,$$

respectively where $a = (-1/2 - (\sqrt{3}/2)ki)$, $b = (-1/2 + (\sqrt{3}/2)ki)$, $k > 0$, and y is some nonzero complex number. Without loss of generality, we let

$$k = \sqrt{\frac{3(1-\cos\alpha)}{1+\cos\alpha}}.$$

Since $\alpha \in (0, \pi) - \{\frac{\pi}{3}\}$, we have $k \in (0, +\infty) - \{\sqrt{3}\}$. For example, if $\alpha = \frac{\pi}{4}$, then $k = 1$. Since R is a nonempty connected domain R on the z-plane, there exists a nonzero real number s such that $s < r$ and if $0 < |y| < s$ then ABC is contained in R. Since $w = f(z)$ is univalent in R, $f(A) = f(x + ay)$, $f(B) = f(x + y)$, and $f(C) = f(x + by)$ are distinct points. By assumption, we have

$$\arg\left(\frac{f(x+ay) - f(x+y)}{f(x+by) - f(x+y)} \cdot \frac{f(x+by)}{f(x+ay)}\right) = \alpha$$

$$= \arg(\exp(i\alpha)) \quad (2.11)$$

for all y such that $0 < |y| < s$. Since $x \in R$ is arbitrarily fixed, we can set

$$h(y) = \frac{f(x+ay) - f(x+y)}{f(x+by) - f(x+y)} \cdot \frac{f(x+by)}{f(x+ay)}. \quad (2.12)$$

By (2.11) and (2.12) we obtain

$$\arg(h(y)) = \arg(\exp(i\alpha)) \quad (2.13)$$

for all y such that $0 < |y| < s$. Similar to the proof of Theorem 2.5, $h(y)$ is analytic at $y = 0$ if we define

$$h(0) = 1 + \frac{2\sqrt{3}k}{3 - k^2}i. \quad (2.14)$$

Hence, $h(y)$ is analytic in $|y| < s$. Furthermore, it is routine to check that $\arg(h(0)) = \alpha$. By (2.12) and the fact that $w = f(z)$ is univalent in R, we obtain that $h(y) \neq 0$ in $|y| < s$. Hence by Lemma 1.4 we have

$$h(y) = K\exp(i\alpha) \quad (2.15)$$

in $|y| < s$, where K is a positive real constant. Setting $y = 0$ in (2.14) and using (2.13), it yields

$$1 + \frac{2\sqrt{3}k}{3 - k^2}i = K\exp(i\alpha). \quad (2.16)$$

By (2.14) and (2.15) we obtain $K = 1$ and

$$h(y) = 1 + \frac{2\sqrt{3}k}{3 - k^2}i \quad (2.17)$$

in $|y| < s$.

Substituting (2.12) into (2.17) and removing the denominator in the resulting equality it follows that

$$(f(x+ay) - f(x+y)) \cdot f(x+by)$$
$$-\left(1 + \frac{2\sqrt{3}\,k}{3-k^2}i\right)(f(x+by) - f(x+y)) \cdot f(x+ay) = 0 \quad (2.18)$$

in $|y| < s$. Differentiate twice both sides of (2.18) with respect to y and setting $y = 0$ yields

$$\frac{k(\sqrt{3}k^2 + 6ki + 9\sqrt{3})}{k^2 - 3}\left(2(f'(x))^2 - f(x)f''(x)\right) = 0.$$

Since $k \in (0, +\infty) - \{\sqrt{3}\}$, we obtain

$$2(f'(x))^2 - f(x)f''(x) = 0$$

which implies that

$$N_f(x) = 2.$$

Since $x \in R$ was arbitrarily fixed, we can replace x by a variable z, and we get

$$N_f(z) = 2$$

in R. By the Identity Theorem, the above equality holds in C. Hence, f is a Möbius transformation of the form $\frac{u}{z+v}$, $u \neq 0$. The case $\alpha = \frac{\pi}{2}$ can be proved similarly by choosing suitable triangle which contains x inside and will be omitted. The proof is complete. ∎

Finally, we consider the following

Property H. Suppose that $w = f(z)$ is analytic and univalent in a nonempty simply connected domain R on the z-plane. Let α be an arbitrary fixed real number such that $\alpha \in (0, \pi)$. For three arbitrary distinct points a, b, and c in R satisfying

$$\arg\left(\frac{a-b}{c-b}\right) = \alpha,$$

we have

$$\arg\left(\frac{f(a)-f(b)}{f(c)-f(b)}\right) = \alpha.$$

THEOREM 2.8. *Let $w = f(z)$ be analytic and univalent in a nonempty connected domain R on the z-plane. Then f satisfies Property H iff f is a Möbius transformation of the form $uz + v$; $u \neq 0$.*

Proof of Theorem 2. Suppose that f is a Möbius transformation of the form $uz + v$, $u \neq 0$. Note that $f(\infty) = \infty$. Since $\frac{a-b}{c-b}$ is the cross-ratio of a, b, c, and $d = \infty$ and a Möbius transformation preserves the cross-ratio, we obtain

$$\frac{f(a) - f(b)}{f(c) - f(b)} = \frac{a - b}{c - b}$$

since $f(\infty) = \infty$. Therefore, $\arg(\frac{a-b}{c-b}) = \arg(\frac{f(a) - f(b)}{f(c) - f(b)})$ which implies that f satisfies Property H. Conversely, suppose that f satisfies Property H. Let T be a Möbius transformation of the form $\frac{1}{z+s}$ where $s \neq 0$. Let $g = T \circ f$. Let a, b, and c be three arbitrary distinct points in R such that $\arg(\frac{a-b}{c-b}) = \alpha$. Since f satisfies Property H, we obtain $\arg(\frac{f(a) - f(b)}{f(c) - f(b)}) = \alpha$. Since $\frac{a-b}{c-b}$ is the cross ratio of a, b, c, and $d = \infty$, it follows that $\frac{f(a) - f(b)}{f(c) - f(b)}$ is the cross ratio of $f(a)$, $f(b)$, $f(c)$, and $f(d) = \infty$ and since a Möbius transformation preserves the cross ratio, we obtain

$$\frac{g(a) - g(b)}{g(c) - g(b)} \cdot \frac{g(c)}{g(a)} = \frac{f(a) - f(b)}{f(c) - f(b)}$$

since $g(\infty) = 0$. From which it follows that

$$\arg\left(\frac{g(a) - g(b)}{g(c) - g(b)} \cdot \frac{g(c)}{g(a)}\right) = \arg\left(\frac{f(a) - f(b)}{f(c) - f(b)}\right) = \alpha.$$

By Theorem 1, g is a Möbius transformation of the form $\frac{u}{z+v}$, $u \neq 0$. Since $g = T \circ f$, we have $f = T^{-1} \circ g$ and it is easily seen that f is a Möbius transformation of the form $Uz + V$, where $U \neq 0$. ∎

Remark 2.2. In the results we obtained above, we may replace the use of argument by using modulus instead; for example, we can modify Property H by replacing $\arg(\frac{a-b}{c-b}) = \alpha$ and $\arg(\frac{f(a) - f(b)}{f(c) - f(b)}) = \alpha$ by $|\frac{a-b}{c-b}|$ and $|\frac{f(a) - f(b)}{f(c) - f(b)}|$, respectively. The proof will be almost the same except that we will need to use the Maximum Modulus Principle for analytic functions instead of using Lemma 1.4.

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A Note on the Characteristics of Möbius Transformations, II

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In this paper, we give some invariant characteristic properties of a certain class of Möbius transformations by means of their mapping properties.

Key Words: Möbius transformations; Schwarzian derivative; Newton derivative.

1. INTRODUCTION

Throughout the paper, unless otherwise stated, let $w = f(z)$ be a nonconstant meromorphic function on the complex plane C. It is well known that for $w = f(z)$ to be a Möbius transformation, it is necessary and sufficient that $w = f(z)$ satisfies Property A.

Property A. $w = f(z)$ maps circles in the z-plane onto circles in the w-plane, including straight lines among circles.

The following property of Möbius transformations is also well known:

THEOREM A. *$w = f(z)$ is a Möbius transformation iff $S_f(z) = 0$ for all $z \in \mathrm{C} - \{z : f'(z) = 0\}$, where $S_f(z) = (f''(z)/f'(z))' - \frac{1}{2}(f''(z)/f'(z))^2$, which is called the Schwarzian derivative of $f(z)$.*

Recently, in [1–4], Haruki and Rassias gave several new invariant characteristic properties of Möbius transformations by considering their mapping properties. In [5], we proved these results by using the fact that Möbius transformations preserve the cross ratio of four distinct points and gave several new invariant characteristic properties of Möbius transformations. For the sake of completeness, we shall give some definitions and results obtained in [5] which are related to the results in this paper.

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Definition 1.1 [5]. We define the Newton derivative of $f(z)$ as the first derivative of Newton's method of $f(z)$. In other words, we define the Newton derivative of $f(z)$ as

$$N_f(z) = \left(z - \frac{f(z)}{f'(z)}\right)' = \frac{f(z)f''(z)}{(f'(z))^2}.$$

Lemma 1.2 [5]. *$f(z)$ is a Möbius transformation of the form $\frac{u}{z+v}$, $u \neq 0$, iff $N_f(z) = 2$ for all $z \in \mathbb{C} - \{z : f'(z) = 0\}$.*

Property B [5]. Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R on the z-plane. Let α be an arbitrary fixed real number such that $\alpha \in (0, \pi)$. For three arbitrary distinct points a, b, and c in R satisfying

$$\arg\left(\frac{a-b}{c-b}\right) = \alpha,$$

we have

$$\arg\left(\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}\right) = \alpha.$$

Theorem 1.3 [5]. *Let $w = f(z)$ be analytic and univalent in a nonempty domain R on the z-plane. Then f satisfies Property B iff f is a Möbius transformation of the form $\frac{u}{z+v}$, $u \neq 0$.*

Remark 1. To prove the "sufficiency" part of Theorem 1.3, we show that $N_f(z) = 2$ for all $z \in \mathbb{C} - \{z : f'(z) = 0\}$, which implies that f is a Möbius transformation of the form $\frac{u}{z+v}$, $u \neq 0$, by Lemma 1.2.

In the next section, we will give some characterization of Möbius transformations of the form $\frac{uz+v}{z}$ where $v \neq 0$.

2. RESULTS

We first give the following result, which is similar to Theorem A and Lemma 1.2 in the previous section:

Lemma 2.1. *$f(z) = \frac{uz+v}{z}$, where $v \neq 0$ iff $f''(z)/f'(z) = -\frac{2}{z}$ for all $z \in \{z : f'(z) \neq 0\}$.*

Proof. Let $f(z) = \frac{uz+v}{z}$, where $v \neq 0$. It is straightforward to check that $f''(z)/f'(z) = -\frac{2}{z}$. Conversely, let $f''(z)/f'(z) = -\frac{2}{z}$ for all $z \in \{z : f'(z) \neq 0\}$. Then we obtain

$$\frac{1}{f'(z)} df'(z) = -\frac{2}{z} dz,$$

which implies that

$$\ln f'(z) = -\ln z^2 + \ln c$$

for some nonzero complex constant c. Thus

$$f'(z) = \frac{c}{z^2}$$

and we have

$$f(z) = -\frac{c}{z} + d$$

for some complex constant d. This completes the proof. ∎

In order to prove our results, we also need the following well-known result:

LEMMA 2.2. *If $f(z)$ and $g(z)$ are analytic functions in a nonempty domain R and $f(z)g(z) \neq 0$ in R and also $\arg(f(z)) = \arg(g(z))$ holds in R, then $f(z) = Kg(z)$ in R, where K is a positive real constant.*

We now consider the following.

Property C. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha \in (0, \pi)$. Let a, b, and c be three distinct points in R satisfying

$$\arg\left(\frac{a-b}{c-b} \cdot \frac{c}{a}\right) = \alpha.$$

Then

$$\arg\left(\frac{f(a)-f(b)}{f(c)-f(b)}\right) = \alpha.$$

THEOREM 2.3. *Let $w = f(z)$ be analytic and univalent on a nonempty domain R, where $0 \notin R$. Then $w = f(z)$ satisfies Property C iff $f(z) = \frac{uz+v}{z}$, where $v \neq 0$.*

Proof. Suppose that $f(z) = \frac{uz+v}{z}$, where $v \neq 0$. Let a, b, and c be three distinct points in R satisfying

$$\arg\left(\frac{a-b}{c-b} \cdot \frac{c}{a}\right) = \alpha.$$

Since $\frac{a-b}{c-b} \cdot \frac{c}{a}$ is the cross ratio of a, b, c, and 0 and the cross ratio is invariant under Möbius transformations, we obtain

$$\frac{a-b}{c-b} \cdot \frac{c}{a} = \frac{f(a)-f(b)}{f(c)-f(b)}$$

since $f(0) = \infty$. It follows that

$$\arg\left(\frac{f(a)-f(b)}{f(c)-f(b)}\right) = \arg\left(\frac{a-b}{c-b}\cdot\frac{c}{a}\right)$$
$$= \alpha.$$

In other words, f satisfies Property C.

Conversely, suppose that $w = f(z)$ satisfies Property C. Let α be an arbitrary real number such that $\alpha \in (0, \pi)$. Let z be an arbitrary point in R. Since $z \neq 0$ we can write $z = \frac{1}{a}$ for some $a \in \mathbb{C} - \{0\}$. Let $N_r(z)$ be an r circular neighborhood of z. Throughout the proof, let ABC be a triangle where $A = \frac{1}{a+\beta y}$, $B = \frac{1}{a+y}$, and $C = \frac{1}{a+\gamma y}$, where $\beta = -\frac{1}{2} - (\sqrt{3}/2)ki$, $\gamma = -\frac{1}{2} + (\sqrt{3}/2)ki$, $k > 0$, and y is some nonzero complex number. Without loss of generality, we let

$$k = \sqrt{\frac{3(1-\cos\alpha)}{1+\cos\alpha}}.$$

Since $\alpha \in (0, \pi)$, we have $k \in (0, +\infty)$. For example, if $\alpha = \frac{\pi}{3}$, then $k = 1$. It follows that $\arg(\frac{A-B}{C-B}\cdot\frac{C}{A}) = \alpha$. Since R is a nonempty connected domain, there exists a nonzero real number s such that $s < r$ and if $0 < |y| < s$ then ABC is contained in $N_r(z)$. Since $w = f(z)$ is univalent in R, $f(A) = f(\frac{1}{a+\beta y})$, $f(B) = f(\frac{1}{a+y})$, and $f(C) = f(\frac{1}{a+\gamma y})$ are distinct points. By assumption, we have

$$\arg\left(\frac{f\left(\frac{1}{a+\beta y}\right) - f\left(\frac{1}{a+y}\right)}{f\left(\frac{1}{a+\gamma y}\right) - f\left(\frac{1}{a+y}\right)}\right) = \alpha \tag{2.1}$$
$$= \arg(\exp(i\alpha))$$

for all y such that $0 < |y| < s$. Since $z \in R$ is arbitrarily fixed, we can set

$$h(y) = \frac{f\left(\frac{1}{a+\beta y}\right) - f\left(\frac{1}{a+y}\right)}{f\left(\frac{1}{a+\gamma y}\right) - f\left(\frac{1}{a+y}\right)}. \tag{2.2}$$

By (2.1) and (2.2), we obtain

$$\arg(h(y)) = \arg(\exp(i\alpha)) \tag{2.3}$$

for all y such that $0 < |y| < s$. By Riemann's Theorem, $h(y)$ will be analytic at $y = 0$ if we define

$$h(0) = \frac{3 - k^2 + (2\sqrt{3}k)i}{3 + k^2}. \tag{2.4}$$

Hence, $h(y)$ is analytic in $|y| < s$. Furthermore, it is routine to check that $\arg(h(0)) = \alpha$. By (2.2) and the fact that $w = f(z)$ is univalent in R, we obtain that $h(y) \neq 0$ in $|y| < s$. Hence, by (2.3) and Lemma 2.2, we have

$$h(y) = K\exp(i\alpha) \tag{2.5}$$

in $|y| < s$, where K is a positive real number. Setting $y = 0$ in (2.5) and using (2.4) we obtain

$$\frac{3 - k^2 + (2\sqrt{3}k)i}{3 + k^2} = K\exp(i\alpha). \tag{2.6}$$

By (2.5) and (2.6), we get

$$h(y) = \frac{3 - k^2 + (2\sqrt{3}k)i}{3 + k^2} \tag{2.7}$$

in $|y| < s$. By substituting (2.2) into (2.7) and removing the denominator in the resulting equality, it follows that

$$\begin{aligned} f\left(\frac{1}{a + \beta y}\right) - f\left(\frac{1}{a + y}\right) \\ = \left(\frac{3 - k^2 + (2\sqrt{3}k)i}{3 + k^2}\right)\left(f\left(\frac{1}{a + \gamma y}\right) - f\left(\frac{1}{a + y}\right)\right) \end{aligned} \tag{2.8}$$

in $|y| < s$. Differentiating twice both sides of (2.8) with respect to y and then setting $y = 0$ yields

$$\left(\frac{-3k^2 + (3\sqrt{3}k)i}{2a^4}\right)\left(2af'\left(\frac{1}{a}\right) + f''\left(\frac{1}{a}\right)\right) = 0. \tag{2.9}$$

It follows from (2.9) that

$$\frac{f''(\frac{1}{a})}{f'(\frac{1}{a})} = -\frac{2}{(\frac{1}{a})}$$

or $f''(z)/f'(z) = -\frac{2}{z}$. Since $z \in R$ is arbitrarily fixed, it follows that $f''(z)/f'(z) = -\frac{2}{z}$ for all $z \in R$. By the Identity Theorem, $f''(z)/f'(z) = -\frac{2}{z}$ holds for all z such that $f'(z) \neq 0$. Therefore, by Lemma 2.1, $w = f(z)$ is a Möbius transformation of the form $f(z) = \frac{uz + v}{z}$, $v \neq 0$. This completes the proof. ∎

Next, we consider the following.

Property D. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha \in (0, \pi)$. Let a, b, and c be three distinct points in R satisfying

$$\arg\left(\frac{a-b}{c-b} \cdot \frac{c}{a}\right) = \alpha.$$

Then

$$\arg\left(\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}\right) = \alpha.$$

For Möbius transformations satisfying Property D, we have the following result.

THEOREM 2.4. *Let $w = f(z)$ be analytic and univalent on a nonempty domain R, where $0 \in R$. Then $w = f(z)$ satisfies Property D if and only if $f(z) = \frac{uz}{z+v}$, where $u, v \neq 0$.*

Proof. Suppose first that f is a Möbius transformation of the form $\frac{uz}{z+v}$, $u, v \neq 0$. Note that $f(0) = 0$. Let a, b, and c be three distinct points in R satisfying

$$\arg\left(\frac{a-b}{c-b} \cdot \frac{c}{a}\right) = \alpha.$$

Since $\frac{a-b}{c-b} \cdot \frac{c}{a}$ is the cross ratio of a, b, c, and $d = 0$ and the cross ratio is preserved under the Möbius transformation, we obtain

$$\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)} = \frac{a-b}{c-b} \cdot \frac{c}{a}$$

since $f(0) = 0$. From this it follows that $\arg(\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}) = \arg(\frac{a-b}{c-b} \cdot \frac{c}{a}) = \alpha$, which implies that $w = f(z)$ satisfies Property D.

Conversely, suppose that $w = f(z)$ satisfies Property D. Let T be a Möbius transformation of the form $\frac{uz+v}{z}$, where $v \neq 0$. Let $g = T \circ f$. Let a, b, and c be three distinct points in R satisfying $\arg(\frac{a-b}{c-b} \cdot \frac{c}{a}) = \alpha$. Then $\arg(\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}) = \alpha$. Since $\frac{a-b}{c-b} \cdot \frac{c}{a}$ is the cross ratio of a, b, c, and $d = 0$, $\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}$ can be considered as the cross ratio of $f(a)$, $f(b)$, $f(c)$, and $f(d) = 0$. It follows that

$$\frac{g(a)-g(b)}{g(c)-g(b)} = \frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}$$

since $g(0) = \infty$. Thus we obtain

$$\arg\left(\frac{g(a)-g(b)}{g(c)-g(b)}\right) = \arg\left(\frac{f(a)-f(b)}{f(c)-f(b)}\cdot\frac{f(c)}{f(a)}\right) = \alpha.$$

By Theorem 2.3, g is a Möbius transformation of the form $\frac{az+t}{z}$, where $t \neq 0$. Since $g = T \circ f$, we have $f = T^{-1} \circ g$ and we conclude that f is a Möbius transformation of the form $\frac{Uz}{z+V}$, $U, V \neq 0$. ∎

In what follows we shall denote $A = \arg(\frac{a-b}{c-b} + \frac{c-b}{a-b})$, $B = \arg(\frac{a-b}{c-b} - \frac{c-b}{a-b})$, $C = \arg(\frac{a-b}{c-b}\cdot\frac{c}{a} + \frac{c-b}{a-b}\cdot\frac{a}{c})$, $D = \arg(\frac{a-b}{c-b}\cdot\frac{c}{a} - \frac{c-b}{a-b}\cdot\frac{a}{c})$, $E = \arg(\frac{f(a)-f(b)}{f(c)-f(b)}\cdot\frac{f(c)}{f(a)} + \frac{f(c)-f(b)}{f(a)-f(b)}\cdot\frac{f(a)}{f(c)})$, $F = \arg(\frac{f(a)-f(b)}{f(c)-f(b)}\cdot\frac{f(c)}{f(a)} - \frac{f(c)-f(b)}{f(a)-f(b)}\cdot\frac{f(a)}{f(c)})$, $G = \arg(\frac{f(a)-f(b)}{f(c)-f(b)} + \frac{f(c)-f(b)}{f(a)-f(b)})$, and $H = \arg(\frac{f(a)-f(b)}{f(c)-f(b)} - \frac{f(c)-f(b)}{f(a)-f(b)})$. We now state the following.

Property E. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = 0$ or π. Let a, b, and c be three distinct points in R satisfying $A = \alpha$. Then $E = \alpha$.

Property F. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Let a, b, and c be three distinct points in R satisfying $B = \alpha$. Then $F = \alpha$.

Property G. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = 0$ or π. Let a, b, and c be three distinct points in R satisfying $A = \alpha$. Then $G = \alpha$.

Property H. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Let a, b, and c be three distinct points in R satisfying $B = \alpha$. Then $H = \alpha$.

Property I. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = 0$ or π. Let a, b, and c be three distinct points in R satisfying $C = \alpha$. Then $E = \alpha$.

Property J. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Let a, b, and c be three distinct points in R satisfying $D = \alpha$. Then $F = \alpha$.

Property K. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = 0$ or π. Let a, b, and c be three distinct points in R satisfying $C = \alpha$. Then $G = \alpha$.

Property L. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Let a, b, and c be three distinct points in R satisfying $D = \alpha$. Then $H = \alpha$.

The following results give some more characterizations of certain classes of Möbius transformations where the proof is a slight modification of the proof of Theorem 2.5 in [5] and will be omitted.

THEOREM 2.5. *Let $w = f(z)$ be analytic and univalent on a nonempty domain R where $0 \in R$. Then $w = f(z)$ satisfies Property E or F (or Property G or H) if and only if $f(z) = \frac{u}{z+v}$, where $u \neq 0$ (or $f(z) = uz + v$, $u \neq 0$).*

THEOREM 2.6. *Let $w = f(z)$ be analytic and univalent on a nonempty domain R where $0 \in R$. Then $w = f(z)$ satisfies Property I or J (or Property K or L)) if and only if $f(z) = \frac{uz}{z+v}$, where $u, v \neq 0$ (or $f(z) = \frac{uz+v}{z}$, $v \neq 0$).*

Remark 2. In the results we obtained above, we may replace the use of argument by using modulus instead; for example, we can modify Property D by replacing $\arg(\frac{a-b}{z-b} \cdot \frac{z}{a}) = \alpha$ and $\arg(\frac{f(a)-f(b)}{f(z)-f(b)} \cdot \frac{f(z)}{f(a)}) = \alpha$ with $|\frac{a-b}{z-b}|$ and $|\frac{f(a)-f(b)}{f(z)-f(b)}|$, respectively. The proof will be almost the same except that we will need to use the Maximum Modulus Principle for analytic functions instead of using Lemma 2.2.

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DYNAMICS OF NEWTON'S FUNCTIONS OF BARNA'S POLYNOMIALS

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We define Barna's polynomials as real polynomials with all real roots of which at least four are distinct. In this paper, we study the dynamics of Newton's functions of such polynomials. We also give the upper and lower bounds of the Hausdorff dimension of exceptional sets of these Newton's functions.

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1. Introduction. Newton's method is a well-known iterative method used to locate the roots of functions. Barna, [1, 2, 3, 4], proved that for a real polynomial $P(x)$ with only simple real roots of which at least four are distinct, the exceptional set of initial points of its Newton's function $N(x)$ (the set of $x \in \mathbb{R}$ such that $N^j(x)$ does not converge to any root of P, where $N^j(x)$ denotes the jth iterate of N) is homeomorphic to a Cantor subset of $\mathbb{R}$ which has the Lebesgue measure zero. Wong [7], generalized this result to real polynomials having all real roots (not necessarily simple) of which at least four are distinct (which will be called Barna's polynomials) by using a symbolic dynamics approach. In this paper, we will investigate the symbolic dynamics of Newton's functions of Barna's polynomials. Furthermore, we give the upper and lower bounds of the Hausdorff dimension of the exceptional sets.

2. Symbolic dynamics of Newton's functions

Definition 2.1. A real polynomial with all real roots of which at least four are distinct is called a *Barna's polynomial*. Thus $P(x)$ is a Barna's polynomial if and only if

$$P(x) = c\prod_{i=1}^{n}(x-r_i)^{m_i}, \tag{2.1}$$

where c is a nonzero real constant, $r_1 < r_2 < \cdots < r_n$, $n \geq 4$, and $m_i \geq 1$ for all $1 \leq i \leq n$.

Definition 2.2. The Newton's function $N_f(x)$ of a function $f(x)$ is defined as

$$N_f(x) = x - \frac{f(x)}{f'(x)}, \tag{2.2}$$

where $f'(x)$ is the derivative of $f(x)$.

Let $P(x) = c\prod_{i=1}^{n}(x - r_i)^{m_i}$ be a Barna's polynomial and $N_P(x)$ be the Newton's function of P. The following are well-known properties of N_P (see [1, 2, 3, 4, 7]).

(a) For each $i = 1, 2, \ldots, n-1$, there exists $c_i \in (r_i, r_{i+1})$ such that c_i is a zero of $P'(x)$, the derivative of $P(x)$, and $\{c_1, c_2, \ldots, c_{n-1}\}$ is exactly the set of all zeros of P' which are not zeros of P.

(b) In each of the intervals $(-\infty, c_1)$ and $(c_{n-1}, +\infty)$, N_P has exactly one critical point. If they are denoted by a_1 and a_n, respectively, then $a_1 \in [r_1, c_1)$ and $a_n \in (c_{n-1}, r_n]$. Moreover, N_P is monotone increasing on $(-\infty, a_1]$ and $[a_n, +\infty)$ and monotone decreasing on $[a_1, c_1)$ and $(c_{n-1}, a_n]$.

(c) In each (c_{i-1}, c_i), $2 \le i \le n-1$, N_P has two critical points s_1^i and s_2^i, where $s_1^i < s_2^i$. If r_i is a multiple root of N_P, then $s_1^i < r_i < s_2^i$. If r_i is a simple root, then either $r_i = s_1^i$, or $r_i = s_2^i$. The function N_P is monotone increasing on (s_1^i, s_2^i) and monotone decreasing on (c_{i-1}, s_1^i) and (s_2^i, c_i). Moreover, N_P^2 is monotone increasing on (c_{i-1}, s_1^i) and monotone decreasing on (s_2^i, c_i).

(d) $\lim_{x \to c_i^-} N_P(x) = -\infty$ and $\lim_{x \to c_i^+} N_P(x) = +\infty$, for all $1 \le i \le n-1$.

(e) $\lim_{k \to \infty} N_P^k(x) = r_1$ for all $x \in (-\infty, c_1)$ and $\lim_{k \to \infty} N_P^k(x) = r_n$ for all $x \in (c_n, +\infty)$.

(f) For each $i = 2, 3, \ldots, n-1$, the interval (c_{i-1}, c_i) contains exactly one period-two cycle of N_P, say, at $\{\alpha_i, \beta_i\}$ where $\alpha_i < s_1^i < s_2^i < \beta_i$. Also $N_P'(a) < -1$, $N_P'(b) < -1$, and $\lim_{k \to \infty} N_P^k(x) = r_i$ for all $x \in (\alpha_i, \beta_i)$.

Definition 2.3. Let $P(x)$ and $N_P(x)$ be as above. The exceptional set Λ of N_P is defined as the complement of the set of real numbers x such that $N_P^j(x) = \infty$ for some $j \ge 0$ or $\lim_{k \to \infty} N_P^k(x) = r_i$ for some $1 \le i \le n$.

Remark 2.4. Note that Λ consists of points where N_P^k is well defined for each $k \in \mathbb{N}$ and never converge to any r_i.

Since our main interest is on the set Λ, those points which are not in Λ together with their preimages will be removed from $\mathbb{R}$. From this we have the following result on period-two cycle of Newton's function.

Proposition 2.5. *The function N_P has a period-two cycle at $\{\alpha, \beta\}$ such that $c_1 < \alpha < \alpha_2$ and $\beta_{n-1} < \beta < c_{n-1}$.*

Proof. Since $(-\infty, c_1)$ and (c_{n-1}, ∞) are not in Λ, we remove these sets together with their preimages. Let $y_0 = N_P^{-1}(c_1)$ such that $y_0 \in (\beta_{n-1}, c_{n-1})$. Then $(y_0, +\infty) \subseteq \Lambda$ and we remove this interval. Next let $z_1 = N_P^{-1}(y_0)$ such that $z_1 \in (c_1, \alpha_2)$. Then $(-\infty, z_1) \subseteq \Lambda$ and we remove this interval. Applying this procedure repeatedly we get two sequences of points $\{y_i\}_{i=1}^{\infty}$ and $\{z_j\}_{j=1}^{\infty}$ where

$$
\begin{gathered}
y_i = N_P^{-1}(z_i) \in (\beta_{n-1}, c_{n-1}), \\
\beta_{n-1} < \cdots < y_i < \cdots < y_2 < y_1 < y_0 < c_{n-1}, \\
z_j = N_P^{-1}(y_{j-1}) \in (c_1, \alpha_2), \\
c_1 < z_1 < z_2 < \cdots < z_j < \cdots < \alpha_2.
\end{gathered}
\tag{2.3}
$$

Thus $\lim_{i \to \infty} y_i = \beta$ and $\lim_{j \to \infty} z_j = \alpha$ exist. As a result,

$$
N_P(\beta) = N_P(\lim_{i \to \infty} y_i) = \lim_{i \to \infty} N_P(y_i) = \lim_{i \to \infty} N_P(N_P^{-1}(z_i)) = \lim_{i \to \infty} z_i = \alpha. \tag{2.4}
$$

Similarly, we have $N_P(\alpha) = \beta$. Since $c_1 < \alpha \le \alpha_2$ and $\beta_{n-1} \le \beta < c_{n-1}$, we get $\alpha \ne \beta$. Finally, $\alpha \ne \alpha_2$ because $N_P(\alpha) = \beta \ge \beta_{n-1} > \beta_2$. This completes the proof of the proposition. □

For each $i = 2,3,\dots,n-1$, we have $N_P((c_{i-1},\alpha_i]) = [\beta_i,+\infty)$, and $N_P([\beta_i,c_i)) = (-\infty,\alpha_i]$. Thus there exist $t_i \in (c_{i-1},\alpha_i]$ and $u_i \in [\beta_i,c_i)$ such that $N_P(t_i) = \beta$ and $N_P(u_i) = \alpha$. Then $N_P((c_{i-1},t_i]) = [\beta,+\infty)$ and $N_P([u_i,c_i)) = (-\infty,\alpha]$. Denote the $2n-4$ intervals

$$[t_2,\alpha_2],[\beta_2,u_2],[t_3,\alpha_3],[\beta_3,u_3],\dots,[t_{n-1},\alpha_{n-1}],[\beta_{n-1},u_{n-1}] \tag{2.5}$$

by $I_1,I_2,\dots,I_{2n-4}$, respectively, and let $I = \bigcup_{i=1}^{2n-4} I_i$. With a similar approach used by Wong [7], we shall define the transition matrix V associated to N_P. This matrix V will determine the symbolic dynamics of N_P. Let $V = (v_{ij})$ be a $(2n-4)\times(2n-4)$ matrix of zeros and ones defined by

$$v_{ij} = \begin{cases} 1 & \text{if } I_i \cap N_P^{-1}(I_j) \ne \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{2.6}$$

for $i,j \in \{1,2,\dots,2n-4\}$. From this definition and properties of N_P, it is easily seen that V is a $(2n-4)\times(2n-4)$ matrix built from the following 2×2 matrices:

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}. \tag{2.7}$$

In fact V can be interpreted as an $(n-2)\times(n-2)$ matrix of matrices as follows:

(1) $V_{ii} = J$ for $1 \le i \le n-2$,
(2) $V_{ij} = M$ for $1 \le i \le n-3$, for $j > i$,
(3) $V_{ij} = N$ for $2 \le i \le n-2$, for $j < i$.

For example, if $n = 6$, then the matrix V has the form

$$\begin{bmatrix} J & M & M & M \\ N & J & M & M \\ N & N & J & M \\ N & N & N & J \end{bmatrix}. \tag{2.8}$$

With the same technique in [7], V is irreducible and we can show that N_P restricted to Λ is conjugate to the one-sided shift map σ on the set Σ_{2n-4}^V where

$$\Sigma_{2n-4}^V = \{s = s_0 s_1 \cdots s_n \cdots \in \Sigma_{2n-4} \mid v_{s_i s_{i+1}} = 1 \ \forall i \ge 0\} \tag{2.9}$$

is the symbolic sequences space consisting of $2n-4$ symbols (cf. [6]).

Remark 2.6. From [6], we have $\operatorname{card}(\operatorname{Per}_k \sigma) = \operatorname{Tr}(V^k)$, where $\operatorname{card}(\operatorname{Per}_k \sigma)$ denotes the number of points of period k of the shift map σ and $\operatorname{Tr}(V^k)$ is the trace of V^k.

Remark 2.7. By MATHEMATICA, we compute that $\operatorname{Tr}(V^k) = (n-2)^k + (-1)^k(n-2)$ where V is the transition matrix associated to Newton's function of a Barna's polynomial with n distinct real roots.

We summarize this section as follows.

Theorem 2.8. *Let $P(x) = c\prod_{i=1}^{n}(x-r_i)^{m_i}$ be a Barna's function and N_P the Newton's function of P. Let Λ be the exceptional set of N_P. Then Λ is a Cantor subset of $\mathbb{R}$ and N_P restricted to Λ is conjugate to the one-sided shift map on Σ_{2n-4}^{V}.*

Remark 2.9. There is some difference between our proof of Theorem 2.8 and the proof of a similar result by Wong in [7]. In our proof we use the fact that the exceptional set Λ lies between the period-two cycle $\{\alpha,\beta\}$ as stated in Proposition 2.5 and hence we can explicitly define the transition matrix V.

3. Hausdorff dimension of exceptional sets. Let Λ be the exceptional set of Newton's function of a Barna's polynomial with n distinct real roots. In this section, we give the upper and lower bounds of the Hausdorff dimension of Λ. The technique we will use here is similar to the one used in [5]. We first note that N_P^{-1} has $n-2$ branches $N_{P,i}^{-1}$ where $N_{P,i}((c_i,c_{i+1})) = \mathbb{R}$ for all $1 \le i \le n-2$. We will write $N_{s_0s_1\cdots s_{k-1}}^{-k}$ for the inverse N_P^{-k} using specific branches $N_{P,s_0}^{-1}, N_{P,s_1}^{-1},\dots,N_{P,s_{k-1}}^{-1}$. Let the interval I be the same as in the previous section. Then I has $n-2$ preimages under N_P each in the interval (c_{i-1},c_i), $2 \le i \le n-1$. For each $k \ge 1$, we have

$$N_P^{-k}(I) = \bigcup_{s_0,s_1,\dots,s_{k-1}=1}^{n-2} I_{s_0s_1\cdots s_{k-1}}, \tag{3.1}$$

where $I_{s_0s_1\cdots s_{k-1}} = N_{s_0s_1\cdots s_{k-1}}^{-k}(I)$. Let $\Lambda_k = \{x \mid N_P^k(x) \in I\}$. Then $\Lambda_k = N_P^{-k}(I)$ and $\Lambda = \bigcap_{k\ge 0}\Lambda_k$. Define

$$\begin{aligned} m_k &= \min\{|N_P'(x)| \mid x \in \Lambda_k\}, \\ m &= \min\{|N_P'(x)| \mid x \in \Lambda\}, \\ M_k &= \max\{|N_P'(x)| \mid x \in \Lambda_k\}, \\ M &= \max\{|N_P'(x)| \mid x \in \Lambda\}. \end{aligned} \tag{3.2}$$

Remark 3.1. For each $k \ge 1$, $M_k \ge M_{k+1}$, and $m_k \le m_{k+1}$ since $\Lambda_k \supset \Lambda_{k+1}$.

We now state and prove the result on the estimation of the Hausdorff dimension of Λ.

Theorem 3.2. $\ln(n-2)/\ln M \le \dim\Lambda \le \ln(n-2)/\ln m$.

Proof. For each $k \ge 1$, let S_k and L_k be the lengths of the smallest and largest intervals in Λ_k, respectively. For each $k \ge 0$, we get

$$-M_k\left|I_{s_0s_1\cdots s_k}\right| \le -M_{k+1}\left|I_{s_0s_1\cdots s_k}\right| \le \int_{I_{s_0s_1\cdots s_k}} N'(x)\,dx = -\left|I_{s_1\cdots s_k}\right| \le -S_k. \tag{3.3}$$

Hence, $|I_{s_0s_1\cdots s_k}| \ge S_k/M_k$. By iterating, we get $|I_{s_0s_1\cdots s_{k+p-1}}| \ge S_k/(M_k)^p$. Similarly, we have $|I_{s_0s_1\cdots s_{k+p-1}}| \le L_k/(m_k)^p$. Since Λ is compact, any covering $\{U_i\}$ of Λ can be refined to a finite cover, where each element of this cover contains exactly one $I_{s_0s_1\cdots s_{k+p-1}}$

for some sufficiently large p. Then we get

$$\begin{aligned}\sum |U_i|^{\alpha} &\geq \sum_{\eta_0,\ldots,\eta_{k+p-1}=1}^{n-2} \left|I_{\eta_0\eta_1\cdots\eta_{k+p-1}}\right|^{\alpha} \geq \sum_{\eta_0,\ldots,\eta_{k+p-1}=1}^{n-2} \left(\frac{S_k}{(M_k)^{p}}\right)^{\alpha} \\ &= (S_k)^{\alpha}\frac{(n-2)^{k+p}}{(M_k)^{\alpha p}} = (S_k)^{\alpha}(n-2)^{k}\left(\frac{n-2}{(M_k)^{\alpha}}\right)^{p}.\end{aligned} \tag{3.4}$$

If $\alpha < \ln(n-2)/\ln M_k$, then this diverges as $p \to \infty$, that is, as the covering gets smaller. Thus $\dim\Lambda \geq \ln(n-2)/\ln M_k$. By letting $k \to \infty$, we have $\dim\Lambda \geq \ln(n-2)/\ln M$. Similarly, for a given $\epsilon > 0$ and for some sufficiently large p, there exists a covering $\{U_i\}_{i=1}^{(n-2)^{k+p}}$ of Λ such that each element of the cover contains exactly one interval $I_{\eta_0\eta_1\cdots\eta_{k+p-1}}$ and $|U_i|^{1+\epsilon} \leq |I_{\eta_0\eta_1\cdots\eta_{k+p-1}}|$. Thus

$$\begin{aligned}\sum |U_i|^{(1+\epsilon)\alpha} &\leq \sum_{\eta_0,\ldots,\eta_{k+p-1}=1}^{n-2} \left|I_{\eta_0\eta_1\cdots\eta_{k+p-1}}\right|^{\alpha} \\ &\leq \sum_{\eta_0,\ldots,\eta_{k+p-1}=1}^{n-2} \left(\frac{L_k}{(m_k)^{p}}\right)^{\alpha} \\ &= (L_k)^{\alpha}(n-2)^{k}\left(\frac{n-2}{(m_k)^{\alpha}}\right)^{p}\end{aligned} \tag{3.5}$$

and this goes to zero as $p \to \infty$ if $\alpha > \ln(n-2)/\ln m_k$. Consequently, $\dim\Lambda \leq (1+\epsilon)\ln(n-2)/\ln m_k$. By letting $k \to \infty$ and $\epsilon \to 0$ we have $\dim\Lambda \leq \ln(n-2)/\ln m$. □

Example 3.3. Let $P(x) = (x+1)(x+2)(x-1)(x-2)$ be a Barna's polynomial and $N_P(x) = (3x^4 - 5x^2 - 4)/(4x^3 - 10x)$ be the Newton's function of P. Then N_P has three period-two cycles approximately at

$$\begin{aligned}\{x_0, x_1\} &= \{-1.5435941, 1.5435941\}, \\ \{x_2, x_3\} &= \{-1.4790145, -0.3142616\}, \\ \{x_4, x_5\} &= \{0.3142616, 1.4790145\}.\end{aligned} \tag{3.6}$$

These are the only period-two cycles by Remark 2.4. From Proposition 2.5, we obtain $\{x_0, x_1\}$ by removing the sequence of points which are the successive preimages of -2 and 2. Since $(N_P^2)'(x_0) = (N_P^2)'(x_1) > 1$, $\{x_0, x_1\} \in \Lambda$. Hence, in order to find the maximum and minimum values of $|N'|$ we must also consider the values of $|N_P'|$ at the preimages of x_0 and x_1. By computation, $x_6 = N_P^{-1}(x_0) = -0.2965502$ and $x_7 = N_P^{-1}(x_1) = 0.2965502$. Since $N_P(x)$ is an odd function, we get, by computation,

$$\begin{aligned}|N_P'(x_0)| = |N_P'(x_1)| = |N_P'(x_6)| = |N_P'(x_7)| &= 3.8985101, \\ |N_P'(x_2)| = |N_P'(x_5)| &= 10.2443746, \\ |N_P'(x_3)| = |N_P'(x_4)| &= 3.4016188.\end{aligned} \tag{3.7}$$

It follows that

$$\begin{aligned} m &= \min\{|N_P'(x)| \mid x \in \Lambda\} \approx 3.4016188, \\ M &= \max\{|N_P'(x)| \mid x \in \Lambda\} \approx 10.2443746. \end{aligned} \tag{3.8}$$

Consequently, we have

$$0.2979063 \approx \frac{\ln 2}{\ln M} \le \dim \Lambda \le \frac{\ln 2}{\ln m} \approx 0.5661804. \tag{3.9}$$

Remark 3.4. Let $P(x) = c\prod_{i=1}^{n}(x - r_i)^{m_i}$ be a Barna's polynomial and let $N_P(x)$ be its Newton's function. Let $M(x) = kN_P(x/k)$, k is a nonzero real constant. Then M is the Newton's function of Barna's polynomial of the form $Q(x) = c_0(x-1)^{m_1}\prod_{i=1}^{n-1}(x - r_i/r_n)^{m_i}$ and M is conjugate to N_P via the map $h(x) = kx$. As a result, α is a periodic point of N_P if and only if $k\alpha$ is a periodic point of M and $N_P'(\alpha)$ is equal to $M'(k\alpha)$. Consequently, dynamics of M and N_P on their exceptional sets are the same and the Hausdorff dimensions of their exceptional sets are equal. As a result, it suffices to consider the dynamics of Newton's functions of Barna's polynomials which have 1 as the largest root.

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ผลงานวิจัยที่เสนอเพื่อการตีพิมพ์

JULIA SET OF BICRITICAL RATIONAL FUNCTION [†]

PIYAPONG NIAMSUP AND KEAITSUDA MANEERUK

ABSTRACT. We study the Julia sets of bicritical rational functions R of degree at least two with the completely invariant attracting Fatou component and give some necessary and sufficient conditions which imply that the Julia sets are Lakes of Wada continuums.

1. INTRODUCTION

Let $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational function of degree at least two and let $R^n : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be its nth iterate . *The Fatou set* $F(R)$ is a subset of $\bar{\mathbb{C}}$ consists of points at which the sequence of iterates $\{R^n\}$ is normal. The complement of $F(R)$ in $\bar{\mathbb{C}}$, denoted $J(R)$, is called *Julia set*.

By a *continuum*, we mean a non-empty connected compact metric space. A *subcontinuum* is a continuum as a subset of a metric space.

The Julia set $J(R)$ is a non-empty, perfect, and completely invariant closed set, namely $R^{-1}(J(R)) = J(R) = R(J(R))$. In this paper we further assume that $J(R)$ is connected, hence, a continuum. On the other hand, the Fatou set $F(R)$ is also a completely invariant open set, but possibly empty. In this paper,however, we are concerned with nonempty Fatou set. Therefore, the Julia set $J(R)$ is a proper subcontimuum of $\bar{\mathbb{C}}$. Each component U of $F(R)$ is called a Fatou component.

In [3], Morosawa shows that every Fatou component U is eventually periodic and all periodic Fatou component can be classified into five types, namely, component of immediate super-attractive basin, component of immediate attractive basin, component of immediate parabolic basin, Siegel disc or Herman ring. In [1], Beardon presents a topological picture of Fatou set.

We say that a continuum of the complex sphere $\bar{\mathbb{C}}$, is a *Lakes of Wada continuum* if it forms a common boundary of three or more open connected mutually disjoint sets.

Let R be a rational function of degree at least two. The *residual Julia set* $J_0(R)$ of R is the sets of points in $J(R)$ that do not lie on the boundary of any Fatou

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component. For example of rational functions which has residual Julia set, see [1] page 266. If there is a completely invariant Fatou component D under some iterate R^n, then $\partial D = J(R)$, hence $J_0(R)$ is empty. This always occurs for polynomials since any polynomial has a completely invariant Fatou component, namely the unbounded Fatou component.

In [5], Yeshun Sun and Chung-Chun Yang proved that if $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational function of degree $d \geq 3$ which has exactly two critical points and satisfies the conditions: $J(R)$ is a proper subcontinuum of $\bar{\mathbb{C}}$, $J_0(R)$ is empty, and there is no completely invariant Fatou component under the second iterate R^2, then the $J(R)$ is a Lakes of Wada continuum.

In [4], some necessary and sufficient conditions which imply that the Julia set of polynomials $P(z) = az^n + b, n \geq 2$ to be Lakes of Wada continuums are given.

In this paper, we extend this result by given some necessary and sufficient conditions which imply that the Julia set of rational functions with two critical points to be Lakes of Wada continuums.

We call a rational function with two critical points *bicritical rational function*. By conjugation, we may assume that two critical points are at 0 and ∞, in which bicritical rational function will be the form $\frac{az^n+b}{cz^n+d}$, see in [2]. In what follows, bicritical rational function will be rational function with two critical points at 0 and ∞.

2. Main results

Lemma 2.1. *Let R be a rational function with a completely invariant attracting Fatou component F_0. Then the following statements are equivalent:*

(a) There exists a component D of $F(R) \neq F_0$ with $\partial D = J(R)$.

(b) There exists a periodic component $D_0 \neq F_0$ of $F(R)$ such that $\partial D_0 = J(R)$.

(c) $J(R) = \bigcup_{i=1}^{M} \partial D_i$, where $\{D_i\}_{i=1}^{M}$ is the set of the immediate basins of all (super-) attracting cycles of R, the immediate basins of all rationally indifferent cycles of R, cycles of all Siegel discs of R and cycles of all Herman rings of R, except F_0.

Proof. (a) $\Rightarrow$ (b). Suppose that $J(R) = \partial D$ for some component $D \neq F_0$ of $F(R)$. Then by the no wandering domains theorem and the complete invariance of F_0, there exists a non-negative integer N such that $R^N(D) = D_0$ for some periodic component $D_0 \neq F_0$ of $F(R)$. Hence $\partial D_0 = \partial R^N(D) = R^N(\partial D) = J(R)$.

(b) $\Rightarrow$ (c). This is trivial.

(c) $\Rightarrow$ (a). See the first part of the proof of Theorem 3 of [3]. ◻

Lemma 2.2. *Let R be a bicritical rational function with a completely invariant attracting Fatou component F_0 containing 0. Assume that $\infty \in J(R)$. Then the following statements are equivalent:*
(a) There exists a bounded component $D \neq F_0$ of $F(R)$ with $\partial D = J(R)$.
(b) There exists a bounded periodic component $D_0 \neq F_0$ of $F(R)$ such that $\partial D_0 = J(R)$.
(c) $J(R) = \bigcup_{i=1}^{M} \partial D_i$, where $\{D_i\}_{i=1}^{M}$ is the set of the immediate basins of all (super-) attracting cycles of R, the immediate basins of all rationally indifferent cycles of R, cycles of all Siegel discs of R and cycles of all Herman rings of R, except F_0.

Proof. (a) $\Rightarrow$ (b). Suppose that $J(R) = \partial D$ for some bounded component $D \neq F_0$ of $F(R)$. Then by the no wandering domains theorem and the complete invariance of F_0, there exists a non-negative integer N such that $R^N(D) = D_0$ for some bounded periodic component $D_0 \neq F_0$ of $F(R)$. Hence $\partial D_0 = \partial R^N(D) = R^N(\partial D) = J(R)$.
(b) $\Rightarrow$ (c). This is trivial.
(c) $\Rightarrow$ (a). See the first part of the proof of Theorem 3 of [3]. □

We say that a rational function is *hyperbolic* if $\overline{\bigcup_{n=0}^{\infty} R^n(C(R))} \cap J(R) = \emptyset$, where $C(R)$ is the set of all critical points of R. This occurs if and only if each critical point of R has the forward orbit that accumulates at a (super-)attracting cycle of R (see [2] page 90).

Remark 2.3. Let R be a rational function of deg $R = d \geq 2$. If R is hyperbolic, then for each completely invariant Fatou component containing critical point is a (super-) attracting basin.

Remark 2.4. Let R be a bicritical rational function of deg $R = d \geq 2$ with the completely invariant attracting Fatou component F_0 containing 0.
Then R is hyperbolic if and only if $\infty \in F(R)$ and F_∞ is a (super-) attracting basin.

Remark 2.5. Let R be a bicritical rational function of deg $R = d \geq 2$ with the completely invariant attracting Fatou component F_0 containing 0.
If $\infty \in F_0$, then $J(R)$ is a Cantor set, that is $J(R)$ is disconnected.

Lemma 2.6. *Let R be a rational function.*
If R is hyperbolic, then R^q is hyperbolic for any positive integer q.

Proof. See Lemma 3.2 of [4]. □

Lemma 2.7. *Let R be a hyperbolic rational function with degree at least two. Assume that $J(R)$ is connected. If there exists a forward invariant component D_0 of $F(R)$ with $\partial D_0 = J(R)$, then D_0 is completely invariant.*

Proof. See Lemma 2 of [3]. □

Proposition 2.8. *Let R be a rational function with $\deg R \geq 2$. Assume that $J(R)$ is connected and $F(R)$ has infinitely many components. If R is hyperbolic, then $F(R)$ has at most one component such that its boundary coincides with the Julia set, and such a component is periodic.*

Proof. See Proposition 3.4 of [4]. □

Corollary 2.9. *Let R be a bicritical rational function, $\deg R = d \geq 2$ with the completely invariant Fatou component F_0 containing 0. Assume that $F(R)$ has infinitely many components and $J(R)$ is connected. If R is hyperbolic, then F_0 is the only component of $F(R)$ such that $\partial F_0 = J(R)$.*

Proof. This follows from Proposition 2.8. □

Theorem 2.10. *Let R be a bicritical rational function, $\deg R = d \geq 2$ with the completely invariant Fatou component F_0 containing 0. Assume that $F(R)$ has infinitely many components and $J(R)$ is a subcontinuum of $\hat{\mathbb{C}}$.*
Then the following statements are equivalent:
(a) $J(R)$ is a Lakes of Wada continuum.
(b) There exists a Fatou component $D \neq F_0$ with $\partial D = J(R)$.
(c) $F(R)$ has no immediate basin of (super-) attracting cycles except F_0, and $J(R) = \partial D_0$ for some periodic Fatou component $D_0 \neq F_0$ which lies in an immediate basin of rationally indifferent cycle or a cycle of Siegel discs of R.
(d) $J(R) = \bigcup_{i=1}^{M} \partial D_i$, where $\{D_i\}_{i=1}^{M}$ is the set of all periodic Fatou components, except F_0.

Proof. (a) $\Rightarrow$ (b). This is obvious.
(b) $\Rightarrow$ (a). Suppose that there exists a Fatou component D_0 such that $\partial D_0 = J(R)$. If $\infty \notin D_0$, then $R(D_0)$ does not contain critical values of R. From this, we obtain by the Riemann-Hurwitz formula that there exist Fatou components $D_1, D_2, ..., D_{d-1}$ such that $R|_{D_k} : D_k \to R(D_0)$ is a homeomorphism for all $k = 0, ..., d-1$. For each k, let $S_k : R(D_0) \to D_k$ be the inverse of $R|_{D_k}$, these are all distinct branches of $R^{-1}|_{R(D_0)}$. Then we get that $S_k R|_{D_0}(z) = \omega^k z$ for all $z \in D_0, k = 0, ..., d-1$ where ω is an nth

primitive root of the unity. For each k, let $\omega^k : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a Möbious transformation which is defined by $\omega^k(z) = \omega^k z$. Then $S_k R|_{D_0} = \omega^k|_{D_0}$ for all k, and so we obtain that $\omega^k(D_0) = D_k$ for all k. We note that for each k, $R\omega^k = R$ from which it is easy to see that $\omega^k(J(R)) = J(R)$ for all k. Thus $\partial D_0 = \partial D_1 = ... = \partial D_{d-1} = \partial F_0 = J(R)$ and it follows that $J(R)$ is a Lakes of Wada continuum. Assume $\infty \in D_0$. Since $\partial D = J(R)$, $\partial R(D) = J(R)$. If $D_0 = R(D_0)$, then by the Riemann-Hurwitz formula, we get that $R|_{D_0} : D_0 \to D_0$ is an d-fold map. Hence D_0 is completely invariant, and so $F(R)$ has only two components. This is a contradiction. Thus $D_0 \neq P(D_0)$. From $\partial F_0 = J(R)$, $\partial D_0 = J(R)$ and $\partial R(D_0) = J(R)$, it follows that $J(R)$ is a Lakes of Wada continuum.

(b) $\Rightarrow$ (c). Assume that there exists a Fatou component D such that $\partial D = J(R)$. Then by Lemma 2.1, $J(R) = \partial D_0$ for some a periodic component $D_0 \neq F_0$ of $F(R)$. So by Proposition 2.9, R is not hyperbolic. Suppose that $F(R)$ has an immediate basin of (super-) attracting cycle of R, say F_1, which $F_1 \neq F_0$. By Theorem 9.3.1 in [3], F_1 must contains a critical point of R, so F_1 contains ∞. Thus F_0 and F_1 are immediate basins of (super-) attracting cycle of R and 0 accumulate to F_0, ∞ accumulate to F_1. This implies that R is hyperbolic, which is a contradiction.So $F(R)$ has no immediate basin of (super-) attracting cycle, except F_0. Thus D_0 either lies in an immediate basin of rationally indifferent cycle or cycle of Siegel discs of R.

(c) $\Rightarrow$ (d). This is trivial.

(d) $\Rightarrow$ (b). This follows immediately from Lemma 2.1.

The proof is complete. □

Theorem 2.11. *Let R be a bicritical rational function, $\deg R = d \geq 2$ with the completely invariant Fatou component F_0 containing 0. Assume that $\infty \in J(R)$, $F(R)$ has infinitely many components and $J(R)$is a subcontinuum of $\bar{\mathbb{C}}$.*

Then the following statements are equivalent:

(a) $J(R)$ is a Lakes of Wada continuum.

(b) There exists a bounded Fatou component $D \neq F_0$ with $\partial D = J(R)$.

(c) $F(R)$ has no bounded immediate basin of (super-) attracting cycles except F_0, and $J(R) = \partial D_0$ for some periodic Fatou component $D_0 \neq F_0$ which lies in an immediate basin of rationally indifferent cycle or a cycle of Siegel discs of R.

(d) $J(R) = \bigcup_{i=1}^{M} \partial D_i$, where $\{D_i\}_{i=1}^{M}$ is the set of all periodic Fatou components, except F_0.

Proof. (a) $\Rightarrow$ (b). This is obvious.

(b) $\Rightarrow$ (a). Suppose that there exists a bounded Fatou component D_0 such that $\partial D_0 = J(R)$. Since $0 \notin D_0$, then $R(D_0)$ does not contain critical values of R. From this, we obtain by the Riemann-Hurwitz formula that there exist bounded Fatou components $D_1, D_2, \dots, D_{d-1}$ such that $R|_{D_k} : D_k \to R(D_0)$ is a homeomorphism for all $k = 0, \dots, d-1$. For each k, let $S_k : R(D_0) \to D_k$ be the inverse of $R|_{D_k}$, these are all distinct branches of $R^{-1}|_{D_0}$. Then we get that $S_k R|_{D_0}(z) = \omega^k z$ for all $z \in D_0, k = 0, \dots, d-1$ where ω be an nth primitive root of the unity. For each k, let $\omega^k : \mathbb{C} \to \mathbb{C}$ is a Möbious transformation which is defined by $\omega^k(z) = \omega^k z$. Then $S_k R|_{D_0} = \omega^k|_{D_0}$ for all k, and so we obtain that $\omega^k(D_0) = D_k$ for all k. We note that for each k, $R\omega^k = R$, from this, it is easy to say that $\omega^k(J(R)) = J(R)$ for all k. Thus $\partial D_0 = \partial D_1 = \dots = \partial D_{d-1} = \partial F_0 = J(R)$. It follows that $J(R)$ is a Lakes of Wada continuum.

(b) $\Rightarrow$ (c). Assume that there exists a bounded Fatou component $D \neq F_0$ such that $\partial D = J(R)$.Then by Lemma 2.2, $J(R) = \partial D_0$ for some a bounded periodic component $D_0 \neq F_0$ of $F(R)$. So by Proposition 2.9, R is not hyperbolic. Suppose that $F(R)$ has an bounded immediate basin of (super-) attracting cycle of R, say F_1 which $F_1 \neq F_0$. By Theorem 9.3.1 in [3], F_1 must contains a critical point of R, so F_1 contains ∞. This implies that $\infty \in F(R)$, which is a contradiction. So $F(R)$ has no bounded immediate basin of (supper-)attracting cycle, except F_0. Thus D_0 either lies in an immediate basin of rationally indifferent cycle or cycle of Siegel discs of R.

(c) $\Rightarrow$ (d). This is trivial.

(d) $\Rightarrow$ (b). This follows immediately from Lemma 2.2.

The proof is complete. □

Theorem 2.12. *Let R be a bicritical rational function, deg $R = d \geq 2$ with the completely invariant Fatou component F_0 containing 0. Assume that $\infty \in J(R)$ and $J(R)$is a subcontinuum of $\hat{\mathbb{C}}$. Then the following statements are equivalent:*

(a) $J(R)$ is a Lakes of Wada continuum.

(b) There exists a bounded Fatou component $D \neq F_0$ with $\partial D = J(R)$.

(c) $J(R)$ coincides with the union of boundaries of cycles of all Siegel discs of R.

(d) The infinite critical point, ∞, lies in the union of boundaries of cycles of all Siegel discs of R and the forward obit $\{R^n(\infty) : n \geq 0\}$ of ∞ is dense in $J(R)$.

Proof. Let $\{D_i\}_{i=1}^M$ be the set of all Siegel discs of R. Then $\bigcup_{i=1}^{M} \partial D_i \subseteq \overline{\{R^n(\infty) : n \geq 0\}}$. As $\infty \in J(R)$, $\{D_i\}_{i=1}^M$ and the set of all bounded Fatou components, except F_0, coincide.

(a) ⇒ (b). This is trivial.

(b) ⇒ (a). Suppose that (*b*) holds. We will show that $F(R)$ has infinitely many components. Assume that $F(R)$ has only two components. Then D is completely invariant under R. From this by the Riemann-Herwitz formula that D contains ∞. This is a contradiction. Hence $F(R)$ has infinitely many components. So, by Theorem 2.11, $J(R)$ is a Lakes of Wada continuum.

(b) ⇔ (c). This is follows from Lemma 2.2.

(c) ⇒ (d). Assume that $J(R) = \bigcup_{i=1}^{M} \partial D_i$. Then $J(R) \supseteq \overline{\{R^n(\infty) : n \geq 0\}} \supseteq \bigcup_{i=1}^{M} \partial D_i = J(R)$, so $J(R) = \overline{\{R^n(\infty) : n \geq 0\}}$.

(d) ⇒ (c). Suppose that (*d*) holds. By the assumption of the set $\{D_i\}_{i=1}^{M}$, we get that $R\left(\bigcup_{i=1}^{M} \partial D_i\right) = \bigcup_{i=1}^{M} \partial D_i$. Hence $J(R) = \overline{\{R^n(\infty) : n \geq 0\}} = \bigcup_{i=1}^{M} \partial D_i$. □

Remark 2.13. Under the condition that $\infty \in J(R)$, the rational function in Theorem 2.12 is not hyperbolic.

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SOLUTIONS OF FUNCTIONAL EQUATIONS $f \circ S = S^k \circ f$

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1. Abstract

Let S be a Möbius transformation which has two fixed points, say a and b in $\mathbb{C}$. Without loss of generality we may assume that a is an attracting fixed point and b is a repelling fixed point of S. We are interested in finding solutions f of the following functional equation

$$f \circ S = S^k \circ f \tag{1}$$

where $k \geq 2$. We will show that for a given complex number α distinct from a and b, there exists a unique solution of (1) which fixes α, a, and b. We also show that the Julia sets of rational solutions of (1) are circles on the sphere.

2. Introduction

Halley's method, Newton's method of a given function $P(z)$ are defined respectively as follows

$$\begin{aligned} H(z) &= z - \frac{P(z)}{P'(z) - \frac{P(z)P''(z)}{2P'(z)}}, \\ N(z) &= z - \frac{P(z)}{P'(z)}. \end{aligned}$$

A successive approximation $S(z)$, of $P(z)$ can be obtained by setting $P(z) = 0$ and then write this equation as $z = S(z)$. In our case $P(z)$ is a quadratic polynomial with roots a and b such that $0 < |a| < |b|$, then $S(z) = \frac{-ab}{z-(a+b)}$ is a successive approximation of $P(z)$ that $z = a$ as a global attractor.

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In [3], the functional equation (1) where f is a rational function of degree k of the form

$$f(z) = \frac{a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0}{b_k z^k + b_{k-1} z^{k-1} + \cdots + b_1 z + b_0}, \tag{2}$$

where a_i, $b_j \in \mathbb{C}$, $(a_0, b_0) \neq (0, 0)$ was studied. The main results in [3] are as follows:

Theorem 2.1. *Let f_k be a rational solution of (1) of the form (2).*

(a) If $a_k \neq 0$, then

$$f_k = T_k \circ f_{0,k}$$

where $f_{0,k}(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}$, $T_k(z) = \frac{z - abb_k}{b_k z + (1-(a+b)b_k)}$ *and* $b_k \in \mathbb{C}$.

(b) If $a_k = 0$, and $a_{k-1} \neq 0$ then there is only one rational solution in this form for (1) and we can explicitly find such the solution.

(c) If $a_k = a_{k-1} = 0$, then there are no nonzero rational solutions for (1) of this form.

Conversely, if T is any mapping such that $T \circ S = S \circ T$, then $f_0 \circ T$ and $T \circ f_0$ are solutions of (1).

Remark 2.1. *When $k = 2$, $f_{0,2}(z)$ is the Newton's method for P and when $k = 3$, $f_{0,3}(z)$ is the Halley's method for P, where $P(z) = (z - a)(z - b)$.*

In [3], the rational solutions f of (1) are solved directly from a linear system of equations. In this paper, we study the functional equation (1) more analytically and we also describe the Julia sets of rational solutions of (1).

3. Main Results

Let S and f be as in the previous section. We have

Theorem 3.1. *For any $i, j \in \mathbb{N}$,*

$$f^i \circ S^j = S^{jk^i} \circ f^i. \tag{3}$$

Proof. For fix $i = 1$, let $P(j) = f \circ S^j = S^{jk} \circ f$. Then for $j = 2$,

$$\begin{aligned}(f \circ S) \circ S &= (S^k \circ f) \circ S \\ &= S^k \circ (f \circ S) \\ &= S^k \circ (S^k \circ f) \\ &= S^{2k} \circ f.\end{aligned}$$

This implies $P(2)$ holds. Assume that $P(N)$ holds. Then

$$\begin{aligned} f \circ S^{N+1} &= (f \circ S^N) \circ S \\ &= (S^{Nk} \circ f) \circ S \\ &= S^{Nk} \circ (f \circ S) \\ &= S^{Nk} \circ (S^k \circ f) \\ &= S^{N+1} \circ f \end{aligned}$$

which implies that $P(N+1)$ holds. Therefore $f \circ S^j = S^{jk} \circ f$ hold for all $j \in \mathbb{N}$. Similarly for a fixed $j \in \mathbb{N}$, we may show that $f^i \circ S^j = S^{jk^i} \circ f^i$ holds for all $i \in \mathbb{N}$. We conclude that $f^i \circ S^j = S^{jk^i} \circ f^i$ for all $i, j \in \mathbb{N}$. This completes the proof. ∎

Theorem 3.2. *Let f be a solution of (1). If $f(b) \neq a$ then a and b are fixed points of f.*

Proof. Firstly, we show that a, b are not poles of f. For if a was a pole of f, then $f(a) = \infty$. From (3) and for $i = 1$ we have

$$\infty = f(a) = f \circ S^j(a) = S^{jk}(f(a)) = s^{jk}(\infty).$$

this implies that $a = \infty$ or $b = \infty$ which is a contradiction. Thus a and b are not poles of f. From (3) if we take $i = 1$, then for $z \notin f^{-1}(b) \cup \{a, b\}$ we have, by continuity of f,

$$f\left(S^j(z)\right) = S^{jk}\left(f(z)\right).$$

Thus

$$\begin{aligned} f(a) &= f\left(\lim_{j \to +\infty} S^j(z)\right) = \lim_{j \to +\infty} f\left(S^j(z)\right) \\ &= \lim_{j \to +\infty} S^{jk}\left(f(z)\right) = a \end{aligned}$$

which implies that a is a fixed point of f. From (3) if we take $z = b$, then

$$f(b) = S^{jk}(f(b)).$$

As we assume that $f(b) \neq a$ we conclude that $f(b) = b$. This completes the proof. ∎

Remark 3.1. *Let f be a solution of (1) such that $f(b) \neq a$. Then a, b are super-attracting fixed points of f.*

Proof. Consider

$$f \circ S(z) = S^k \circ f(z)$$

by differentiate both sides we obtain

$$f'(a)S'(b) = S'(S^{k-1} \circ f(z)) \cdot S'(S^{k-2} \circ f(z)) \cdot \ldots \cdot S'(f(z)) \cdot f'(z).$$

For $z = a$,

$$f'(a) \cdot S'(a) = [S'(a)]^k \cdot f'(a)$$

and since $S'(a) \neq 0$, we conclude that $f'(a) = 0$. That is, a is a super-attracting fixed point of f. Similarly, we can show that $f'(b) = 0$. This completes the proof. ∎

Theorem 3.3. *For a given complex number $\alpha \neq a, b$. There exists a unique solution of (1) which fixes α, a and b.*

Proof. Let f and g be solutions of (1) which fix a, b and α. From (3), take $i = 1$ we have

$$f \circ S^j(\alpha) = S^{jk} \circ f(\alpha) = S^{jk}(\alpha)$$

and

$$g \circ S^j(\alpha) = S^{jk} \circ g(\alpha) = S^{jk}(\alpha).$$

Since $\alpha \neq a, b$ and a is a global attractor of S, $S^j(\alpha) \to a$ as $j \to \infty$. This implies that a is a limit point of $\{S^j(\alpha) : j \in \mathbb{N}\}$. As

$$\{S^j(\alpha) : j \in \mathbb{N}\} \subseteq \{z \in \overline{\mathbb{C}} : f(z) = g(z)\},$$

we have, by the Identity Theorem, $f \equiv g$ on $\overline{\mathbb{C}}$. Therefore, there is a unique solution of (1) which fixes α, a and b where $\alpha \neq a, b$. This completes the proof. ∎

Remark 3.2. *Let f be a solution of (1) which fixes a, b and $\alpha(\neq a, b)$. For all Möbius transformation which fixes a, b we may show that if $T(f(\alpha)) = \alpha$, then $T \circ S = S \circ T$.*

Theorem 3.4. *Let f be a solution of (1). Then $f \circ T$ and $T \circ f$ are solutions of (1) where T is any transformation which satisfies $S \circ T = T \circ S$.*

Proof. Put $g = f \circ T$ and $h = T \circ f$. Then

$$\begin{aligned} g \circ S &= (f \circ T) \circ S \\ &= f \circ (T \circ S) \\ &= f \circ (S \circ T) \\ &= (f \circ S) \circ T \\ &= (S^k \circ f) \circ T \\ &= S^k \circ (f \circ T) \\ &= S^k \circ g. \end{aligned}$$

That is, g is a solution of (1). And

$$\begin{aligned} h \circ S &= (T \circ f) \circ S \\ &= T \circ (f \circ S) \\ &= T \circ (S^k \circ f) \\ &= (T \circ S) \circ (S^{k-1} \circ f) \\ &= S \circ (T \circ S) \circ (S^{k-2} \circ f) \\ &\;\vdots \\ &= S^k \circ (T \circ f) \\ &= S^k \circ h. \end{aligned}$$

That is, h is a solution of (1). This completes the proof. ∎

Theorem 3.5. *Let f and g be solutions of (1) which f fixes a, b and α ($\alpha \neq a, b$) and g fixes a, b and β ($\beta \neq a, b$). Then g can be expressed in the form*

$$g = T \circ f$$

where T is a Möbius transformation which fixes a, b and $T(f(\beta)) = \beta$.

Proof. It is easy to see that $S \circ T = T \circ S$. By Theorem 3.4, $T \circ f$ is a solution of (1). Since

$$T \circ f(a) = T(a) = a$$

$$T \circ f(b) = T(b) = b$$

$$T \circ f(\beta) = T(f(\beta)) = \beta,$$

this implies $T \circ f$ is a solution of (1) which fixes a, b and β. By Theorem 3.3, we obtain $g = T \circ f$. ∎

Theorem 3.6. *Let f be a solution of (1) which fixes a, b. Then f is a rational function.*

Proof. First, we consider $S(z) = \lambda z, \lambda \neq 0$. Let g be a solution of (1) which fixes $0, \infty$ and S is defined as above. So

$$(*) \qquad g(\lambda z) = \lambda^k g(z).$$

Set

$$g(z) = \Sigma_{n=1}^{\infty} a_n z^n$$

where $a_n \in \mathbb{C}, \forall n$. We have

$$g(\lambda z) = \Sigma_{n=1}^{\infty} a_n \lambda^n z^n$$

and

$$\lambda^k g(z) = \Sigma_{n=1}^{\infty} a_n \lambda^k z^n.$$

From (*), we obtain

$$\Sigma_{n=1}^{\infty} a_n \lambda^n z^n = \Sigma_{n=1}^{\infty} a_n \lambda^k z^n$$
$$\Sigma_{n=1}^{\infty} a_n (\lambda^n - \lambda^k) z^n = 0$$
$$a_n(\lambda^n - \lambda^k) = 0, \forall n$$

For $n \neq k, a_n = 0$, so that $g(z) = a_k z^k$. This implies that g is a rational function.

Now, we consider S which fixes a, b. Then S is conjugate to a map $z \mapsto \lambda z, \lambda \neq 0$ by the Möbius transformation that send $z = a$ to 0 and $z = b$ to ∞, namely

$$M(z) = \frac{-z + a}{-z + b}.$$

Let f be a solution of (1) which fixes a, b. Then f is conjugate to g with the same Möbius transformation. Therefore f is a rational function. This completes the proof. ∎

Proposition 3.1. *Let f be a solution of (1) which fixes a and b. For $\alpha \neq a, b$, if $f(\alpha) = \alpha$, then f is conjugate to a map $\left(\frac{-\alpha+b}{-\alpha+a}\right)^{k-1} z^k$.*

Proof. Assume that f is a solution of (1) which fixes a and b. Given $\alpha \neq a, b$. From Theorem 3.6, f is conjugate to a map g where $g(z) = Kz^k, \exists K \neq 0$ by the Möbius transformation that send $z = a$ to 0 and $z = b$ to ∞, namely

$$M(z) = \frac{-z + a}{-z + b}.$$

Since $f(\alpha) = \alpha$, so

$$M^{-1}gM(\alpha) = \alpha.$$
$$g(M(\alpha)) = M(\alpha).$$

That is $M(\alpha)$ is a fixed point of g. But g has fixed points at $0, \infty$ and $(k-1)^{th}$ roots of $\frac{1}{K}$. Then

$$M(\alpha) = \left(\frac{1}{K}\right)^{\frac{1}{k-1}}$$
$$\frac{-\alpha + a}{-\alpha + b} = \left(\frac{1}{K}\right)^{\frac{1}{k-1}}$$
$$\left(\frac{-\alpha + a}{-\alpha + b}\right)^{k-1} = \frac{1}{K}$$
$$K = \left(\frac{-\alpha + b}{-\alpha + a}\right)^{k-1}.$$

This implies that f is conjugate to a map $\left(\frac{-\alpha+b}{-\alpha+a}\right)^{k-1} z^k$. This completes a proof. ∎

Example 3.1. *For $k = 2$, N is the rational solution of (1) which fixes a, b and ∞.*
Let f be a solution of (1) which f fixes a, b and α $(\alpha \neq a, b)$.
Then $f = T \circ N$ where T is a Möbius transformation which fixes a, b and $T(f(\alpha)) = \alpha$.

Definition 3.1. *Let f be a rational function and let f^i denote the ith iterate of f.* ***The Julia set*** *$J(f)$ of f is defined as follows:*

$$J(f) = \overline{\mathbb{C}} - F(f),$$

where $F(f) = \{z \in \overline{\mathbb{C}} : \{f^i\}_{i=0}^{\infty}$ is a normal family in a neighborhood of $z\}$. The set $F(f)$ is called ***the Fatou set*** *of f.*

Theorem 3.7. *The Julia set of the rational solutions of (1) are circles on the sphere.*

Proof. In [3], we know that

$$f_k(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}$$

is the rational solution of (1). Let f be a rational solution of (1) which fixes a, b and α $(\alpha \neq a, b)$. Theorem 3.5 shows that $f = T_k \circ f_k$ where T_k is a Möbius transformation which fixes a, b and $T_k(f_k(\alpha)) = \alpha$. For $k \geq 2$, the function f_k is conjugate to a map $w \mapsto w^k$ and T_k is conjugate to a map $w \mapsto Kw^k$ where $|K| < 1$ by the Möbius transformation that send $w = 0$ to a and $w = \infty$ to b, namely

$$M(w) = \frac{bw - a}{w - 1}.$$

The inverse M^{-1}, of M is given by

$$M^{-1}(z) = \frac{-z + a}{-z + b}.$$

This implies f is conjugate to the map Kz^{k^2} where $|K| < 1$. So we obtain that $J(f)$ is a circle on the sphere. ∎

We now consider the case when the Möbius transformation has exactly one fixed point in the complex plane. Let R be a Möbius transformation which has only one fixed point, say $a \in \mathbf{C}$ (so a is the global attractor of R). We are interested in finding solutions f of the following functional equation

$$f \circ R = R^k \circ f \tag{4}$$

where $k \geq 2$ and R is defined as above. We have

Remark 3.3. *Let f be a solution of (4). Then a is a fixed point of f.*

Remark 3.4. *For a given complex number $\alpha \neq a$. There exists a unique solution of (4) which fixes α and a.*

Remark 3.5. *Let f be a solution of (4). If T is any Möbius transformation such that $T \circ R = R \circ T$, then $f \circ T$ and $T \circ f$ are solutions of (4).*

Theorem 3.8. *Let f be a solution of (4). Then f is conjugate to a map*

$$kz + P(e^{\frac{-2\pi i}{c}z}) + Q(e^{\frac{2\pi i}{c}z})$$

where P, Q are meromorphic functions.

Proof. Without loss of generality we may assume that $R(z) = z + c, c \neq 0$. Assume that g is a solution of (4). Then $g(z + c) = g(z) + kz$. Note that $g(z + c) = g(z) + kz$ if and only if $g(z) = kz + H(z)$ where $H(z) = g(z) - kz$. So $H(z) = H(z + c)$, that is, H is periodic. Since rational functions cannot have a period, this implies that $H(z) = P(e^{\frac{-2\pi i}{c}z}) + Q(e^{\frac{2\pi i}{c}z})$ where P, Q are meromorphic functions.

Now, we consider S which fixes a. Then S is conjugate to a map $z \mapsto z + c, c \neq 0$ by the Möbius transformation that send $z = a$ to 0, namely

$$M(z) = \frac{1}{-z + a}.$$

Let f be a solution of (4). Then f is conjugate to g with the same Möbius transformation. This completes the proof. ∎

Example 3.2. *$f(z) = 2z + e^{-z} + e^{z}$ is a solution of the functional equation $f \circ R = R^2 \circ f$ where $R(z) = z + 2\pi i$.*

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สัญญาเลขที่ RSA/04/2544

โครงการ การวิจัยพลศาสตร์เชิงดิสครีตของฟังก์ชันตรรกยะบางฟังก์ชัน พลศาสตร์เชิงดิสครีตของฟังก์ชันในคลาส A_∞ และการวิจัยคุณสมบัติไม่แปรเปลี่ยนของการแปลงเชิงเส้นคู่

รายงานสรุปการเงินในรอบ 36 เดือน

==

ชื่อหัวหน้าโครงการ ดร.ปิยะพงศ์ เนียมทรัพย์

รายงานในช่วงตั้งแต่วันที่ 1 ธันวาคม 2543 ถึง 30 พฤศจิกายน 2546

<u>รายจ่าย</u>

หมวด (ตามสัญญา)	รายจ่ายสะสมจากรายงานครั้งก่อน	ค่าใช้จ่ายงวดปัจจุบัน	รวมรายจ่ายสะสมจนถึงงวดปัจจุบัน	งบประมาณที่ตั้งไว้ (รวมสะสมจนถึงปัจจุบัน)	คงเหลือ(หรือเกิน)
1. ค่าตอบแทน	450,000	90,000	540,000	540,000	-
2. ค่าจ้าง	90,000	18,000	108,000	108,000	-
3. ค่าใช้สอย	135,860.45	4,520	140,380.45	232,000	91,619.55
4. ค่าวัสดุ	195,831.75	-	195,831.75	120,000	เกิน 75,831.75
5. ค่าครุภัณฑ์	30,973	-	30,973	30,000	เกิน 973
6. ค่าเดินทางไปต่างประเทศ	30,000**	-	30,000**	50,000	20,000
รวม	932,665.20	112,520	1,045,185.20	1,080,000	34,814.80

<u>หมายเหตุ **</u>

** 1. เดินทางไปเสนอผลงาน The 10th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications, Busan, Korea, July 29 – August, 2002 เรื่อง Julia set of $az^n + b$ โดยได้รับงบประมาณบางส่วนจากคณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่

** 2. เดินทางไปร่วมประชุม International Congress of Mathematic 2002, Beijing, China, August 19 – 29, 2002 โดยได้รับงบประมาณบางส่วนจากผู้จัดการประชุม

<u>จำนวนเงินที่ได้รับและจำนวนคงเหลือ</u>

<u>จำนวนเงินที่ได้รับ</u>

งวดที่ 1	360,000	บาท	เมื่อ 1 ธันวาคม 2543
งวดที่ 2	360,000	บาท	เมื่อ 31 ธันวาคม 2544
งวดที่ 3	270,000	บาท	เมื่อ 31 ธันวาคม 2545
ดอกเบี้ยครั้งที่ 1	-	บาท	
ดอกเบี้ยครั้งที่ 2	-	บาท	
ดอกเบี้ยครั้งที่ 3	-	บาท	
ฯลฯ			
รวม	**990,000**	**บาท**	(1)

<u>ค่าใช้จ่าย</u>

งวดที่ 1 (เดือนที่ 1 – 6) เป็นเงิน	149,799.50	บาท	
งวดที่ 2 (เดือนที่ 7 – 12) เป็นเงิน	155,063.95	บาท	
งวดที่ 3 (เดือนที่ 13 – 18) เป็นเงิน	186,100.50	บาท	
งวดที่ 4 (เดือนที่ 19 – 24) เป็นเงิน	245,870.50	บาท	
งวดที่ 5 (เดือนที่ 25 – 30) เป็นเงิน	195,830.75	บาท	
งวดที่ 6 (เดือนที่ 31 – 36) เป็นเงิน	112,520.00	บาท	
รวม	**1,045,185.20**	**บาท**	(2)
<u>จำนวนเงินคงเหลือ</u> (1) – (2)	- **46,185.20**	**บาท**	

ลงนามหัวหน้าโครงการ..............................

(ดร.ปิยะพงศ์ เนียมทรัพย์)

วันที่ 10 มิ.ค. [illegible]