สกว

รายงานการวิจัยฉบับสมบูรณ์

โครงการ การวิจัยพลศาสตร์เชิงดิสครีตของฟังก์ชันตรรกยะบางฟังก์ชัน พลศาสตร์เชิงดิสครีตของฟังก์ชันในคลาส *A*, และการวิจัยคุณสมบัติไม่แปรเปลี่ยนของการแปลงเชิงเส้นคู่

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ธันวาคม 2546

Project Code: RSA/04/2544

Project Title: การวิจัยพลศาสตร์เชิงดิสครีตของฟังก์ชันตรรกยะบางฟังก์ชัน พลศาสตร์เชิงดิสครีตของฟังก์ชันในคลาส A_∞ และการวิจัยคุณสมบัติไม่แปรเปลี่ยนของการแปลงเชิงเส้นคู่

Research on Discrete Dynamics of Certain Rational Functions, Discrete Dynamics of Functions of A_∞ Class and Research on Invariant Properties of Mobius Transformations

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Project Period: 3 ปี

บทคัดย่อ

งานวิจัยนี้ได้ให้ความสำคัญในศึกษา 1. พลศาสตร์เชิงดิสครีตของฟังก์ชันตรรกยะบางฟังก์ชัน 2. ศึกษาคุณสมบัติไม่แปรเปลี่ยนของการแปลงเชิงเส้นคู่ และ 3. ผลเฉลยของสมการเชิงฟังก์ชันบางสมการ โดยความรู้พื้นฐานที่ใช้คือ การวิเคราะห์เชิงซ้อนและพลศาสตร์เชิงดิสครีต

Abstract

This research emphasizes on the study of 1. Discrete dynamics of certain rational functions 2. The invaraint properties of Mobius Transformations and 3. Solutions of certain functional equations in which the basic knowledge needed in the study are Complex analysis and Discrete dynamics.

Keywords: Discrete dynamics, Mobius Transformations, Rational functions

เนื้อหางานวิจัย

1. วัตถุประสงค์

1.1 เพื่อศึกษา วิจัย และอธิบาย พลศาสตร์เชิงดิสครีตของฟังก์ชันตรรกยะบางฟังก์ชัน

1.2 เพื่อศึกษา วิจัย และอธิบาย พลศาสตร์เชิงดิสครีตของฟังก์ชันในคลาส A_m

1.3 เพื่อศึกษา วิจัย และอธิบาย การมี Absolutely Continuous Invariant Measures ของฟังก์ชัน ใน 5.1 และ 5.2

1.4 เพื่อหาลักษณะเฉพาะของการแปลงเชิงเส้นคู่โดยอาศัยคุณสมบัติทางเรขาคณิต

1.5 เพื่อหาลักษณะเฉพาะของการแปลงเชิงเส้นคู่โดยอาศัยคุณสมบัติการไม่แปรเปลี่ยน

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2. ระเบียบวิธีวิจัย

2.1 ทำการรวบรวมเอกสารที่เกี่ยวข้องกับปัญหาที่จะทำวิจัยเพิ่มเติม

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2.5 เสนอผลงานเพื่อการตีพิมพ์ในวารสารระดับนานาชาติ

3. ขอบเขตการวิจัย

3.1 ศึกษาพลศาสตร์เชิงดิสครีตของฟังก์ชันตรรกยะ (Rational Functions) บางฟังก์ชัน โดยเฉพาะฟังก์ชันตรรกยะที่ได้นำเสนอในหัวข้อที่ 7 (ผลงานวิจัยที่เกี่ยวข้อง)

3.2 หาโครงสร้างของเซตจูเลีย (Julia Sets) ของฟังก์ชันตรรกยะใน

3.3 หาขอบเขตล่าง (Lower Bound) และ ขอบเขตบน (Upper Bound) ของ Hausdorff dimension ของเซตจูเลียของฟังก์ชันตรรกยะใน

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3.8 หาลักษณะเฉพาะของการแปลงเชิงเส้นคู่โดยอาศัยคุณสมบัติทางเรขาคณิต

3.9 หาลักษณะเฉพาะของการแปลงเชิงเส้นคู่โดยอาศัยคุณสมบัติที่ไม่แปรเปลี่ยน

สัญญาเลขที่ RSA/04/2544

รายงานการวิจัยฉบับสมบูรณ์

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว.ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณสำนักงานกองทุนสนับสนุนการวิจัย (สกว.) ที่ได้ให้การสนับสนุนทุนวิจัยมาอย่างต่อเนื่อง ขอขอบพระคุณ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ ที่ได้ให้การสนับสนุนการทำวิจัยอย่างเต็มที่

ดร.ปิยะพงศ์ เนียมทรัพย์
หัวหน้าโครงการวิจัย

สารบัญ

Introduction

Part I: Julia sets and discrete dynamics of certain rational functions and functions of class A_∞.

Topic I. Julia sets and discrete dynamics of certain rational functions

Let $f(z)$ be a rational function (or an entire function) on the complex plane $\mathbb{C}$. We define the ith iterate as $f^0 = 1_{\mathbb{C}}$, $f^i = f^{i-1} \circ f$ and $f^{-i} = (f^i)^{-1}$ for $i \geq 1$. We call $z_0 \in \mathbb{C} \cup \{\infty\}$ a **periodic point** of f of period $p \geq 1$ if $f^p(z_0) = z_0$ and $f^i(z_0) \neq z_0$, $0 \leq i \leq p-1$. If $p = 1$ we say z_0 is a **fixed point of** f. Assume that z_0 is a periodic point of f of period p, we say that z_0 is an **attractive periodic point** of f if $|(f^p)'(z_0)| < 1$, we say z_0 is a **repulsive fixed point** if $|(f^p)'(z_0)| > 1$, and z_0 is an **indifferent fixed point** if $|(f^p)'(z_0)| = 1$. We are interested in the family of iterates of f, namely $\{f^i(z)\}_{i=1}^{\infty}$ where $z \in \mathbb{C} \cup \{\infty\}$ (or $z \in \mathbb{C}$), that is we are interested in the behavior of $f^i(z)$ for $i \geq 1$ and $z \in \mathbb{C} \cup \{\infty\}$. The studies of the modern theory of iteration of $f(z)$ has been traced back to around 1900 when G.Julia and P.Fatou had independently developed this branch of mathematics. The most important theory developed are involved with the following two sets, **Julia set** $J(f)$ and **Fatou set** $F(f)$ of function f where they are defined by

$$F(f) = \{z \in \mathbb{C} \cup \{\infty\} : \{f^i(z)\}_{i=1}^{\infty} \text{ is a normal family in a neighborhood of } z\}$$
$$J(f) = \{\mathbb{C} \cup \{\infty\}\} - F(f).$$

From the definition it follows that $J(f)$ is a perfect set and $F(f)$ is an open set. Results on the theory of iteration of rational can be found, for examples, in [8, 10, 15-16, 20, 44, 50, 52, 60, 72]. Results on the theory of iteration of transcendental meromorphic functions can be found in [1-6, 9, 16]. Results on the theory of iteration of entire functions can be found in [6, 50, 56, 63, 65]. In spite of extensive studies on the theory of iteration in recent years, there are still many interesting questions to be answered.

In this research project, we propose to do more research on theory of iteration of rational functions and entire functions. We will now give the literature reviews and related results. In this part, we focus on theory of iteration of rational functions. The topics we are interested is to describe dynamics of rational functions of certain forms. In [37], J.R.Kinney and T.S.Pitcher studied the Julia sets of rational functions $R(z)$ of the following forms:

$$R(z) = az - \sum_{i=1}^{n} \frac{b_i}{z - c_i}$$

where $a > 1$, $b_i \geq 0$, and the c_i are distinct real numbers arrangen in increasing order. They showed that $J(R) = \bigcap_{k=0}^{\infty} R^{-k}(I_0)$ where $I_0 = [x_0, x_n]$, x_0 and x_n are the smallest and largest real roots of $R(z) - z$. They also gave the upper and lower estimate of Hausdorff dimension $\dim(J(R))$ of $J(R)$ as follows:

$$\frac{\log(n+1)}{\log m} \leq \dim(J(R)) \leq \frac{\log(n+1)}{\log M}$$

where $m = \min_{x \in J(R)} \{R'(x)\}$ and $M = \max_{x \in J(R)} \{R'(x)\}$. From this results, we propose to extend the study of the rational functions of the following forms:

$$R(z) = az - \sum_{i=1}^{n} \frac{b_i z - c_i}{z - d_i} \tag{0.1}$$

where a, b_i, c_i, and d_i are real numbers satisfying $a \geq 1$, $d_1 < d_2 < \ldots < d_n$ and $b_i c_i - d_i > 0$ for all $1 \leq i \leq n$. Note that the rational functions we propose to study is in more general form that the ones studied by J.R.Kinney and T.S.Pitcher, in particular, we also study the case when $a = 1$ but J.R.Kinney and T.S.Pitcher didn't consider this case. We propose to study the structure of $J(R)$, find the upper and lower bounds for Hausdorff dimension of $J(R)$ (as of present, we have already obtained some results about $J(R)$ such as when $a > 1$, we obtain that $J(R) \subset [x_0, x_n]$ where x_0 and x_n are the smallest and largest real roots of $R(z) - z$ and the Lebesgue measure of $J(R)$ is qual to zero). We also propose to study symbolic dynamics (see [5, 7, 15] for details about symbolic dynamics) of function $R(z)$ on $J(R)$ in which we expect that when $a > 1$, $R(z)$ restricted to $J(R)$ is topological conjugate to the shift map σ on $n+1$ symbols. For the case $a = 1$, we have seen from several examples we have considered that $J(R)$ is unbounded subset of the real line, in this case the Lebesgure measure of $J(R)$ is not equal to zero and we could not obtain the upper and lower bounds for Hausdorff dimension of $J(R)$. However, the main point of interest in the case $a = 1$ is the symbolic dynamics of R restricted to $J(R)$ in which we have precise idea on how to deal with this question.

One of the most active branches of dynamical systems is the theory of absolutely continuous invariant measures (Absolutely continuous measure with respect to Lebesgue measure is a measure $\mu = f \cdot \lambda$ where λ is the Lebesgue measure. This means that for any measurable set A, $\mu(A) = \int_A$

$\int d\lambda$. Measure μ is τ − *invariant* if for any measurable set A, we have $\mu(A) = \mu(\tau^{-1}(A)))$, especially for one-dimensional transformations. Let $\mathcal{M}$ be the set of all meromorphic functions which preserve the real line $\mathbb{R}$. As referred in [21], that Levin proved the following results:
Theorem A. A meromorphic function $g \in \mathcal{M}$ if and only if

$$g(z) = A + \varepsilon\left[Bz - \frac{C_0}{z} - \sum_s C_s\left(\frac{1}{z-p_s} + \frac{1}{p_s}\right)\right]$$

where $\varepsilon = \pm 1$, A, B, C_s and p_s, $s = \pm 1, \pm 2, \ldots$ are real and $B \geq 0$, $C_s \geq 0$ and $\sum_s \frac{C_s}{p_s^2} < +\infty$. The poles p_s are numbered such that $p_s < p_{s+1}$ for $s = \pm 1, \pm 2, \ldots$ We assume that at least one C_s is different from 0. In [21], P.Gora and N.Obeid studied dynamics of functions $g(z)$ in class $\mathcal{M}$, where they considered four cases as follows:

Case I: $B < 1$ and all fixed points of g in $\mathbb{R}$ are repelling.

Case II: $B < 1$ and no fixed point of g in $\mathbb{R}$ is attracting and at least one is neutral.

Case III: $B = 1$.

Case IV: $B > 1$.

They have obtained results on absolutely continuous invariant measures for transformations for $\tau : \mathbb{R} \to \mathbb{R}$ where $\tau(x) = g(x)$, $x \in \mathbb{R}$ and $g \in \mathcal{M}$ for the above mentioned four cases where they used several previous known results and techniques, see [1-5, 29-31, 45, 66], to obtain their results.

Observe that the rational functions $R(z)$ in the form (1) are contained in $\mathcal{M}$ therefore we propose to study absolutely continuous invariant measure for $\tau : \mathbb{R} \to \mathbb{R}$ where $\tau(x) = R(x)$, $x \in \mathbb{R}$. Note that the techniques we need to obtain our results are not quite the same as in [21], since for the rational functions $R(z)$ we knows the precise structure of $J(R)$.

Topic II. Discrete dynamics of functions of class A_∞.

Suppose that M is the interval [-1,1] or the unit circle S^1 and f from M into itself is a C^1 map. We say f is an endomorphism of class $C^{1+\alpha}$ if the derivative f' of f is α-Hölder continuous and for every critical point c_i of f, there is a small neighborhood U_i of c_i such that $\tau(x) = \frac{f'(x)}{|x-c_i|^{\gamma_i - 1}}$ is α-Hölder continuous on $\{x < c_i\} \cap U_i$ and on $\{x > c_i\} \cap U_i$. Endomorphisms of class $C^{1+\alpha}$ are extensively studied for example in [32-35]. In all these papers the authors show distortation estimates for the family of mappings under consideration and they are crucial in the understanding of the dynamical properties of these mappings. These distortion estimates is closely related to the A_∞ condition we will define below. Again, let M be either a unit circle or an interval (or more generally a 1-dimensional manifold). A measure μ is said to be in A_∞ if there are constants $c > 0$ and $\delta \in [0,1]$ so that

$$\frac{\mu(E)}{\mu(I)} \le c\left(\frac{|E|}{|I|}\right)^{\delta} \tag{0.2}$$

for every measurable subset E of an interval $I \subset M$, where $|E|$ is the Lebesgue measurable of the set E. We abbreviate this with $\mu \in A_\infty(c,\delta)$.

Consider an endomorphism $f : M \to M$, the image $f_*\mu$ of a measure μ under f is defined by

$$f_*\mu(E) = \mu(f^{-1}(E)) \text{ for every measurable set } E \subset M.$$

Definition 1 *An Endomorphism $f : M \to M$ is called A_∞ mapping if the f^n – images of the Lebesgue measure dx are uniformly in A_∞, i.e., there is $c > 0$ and $\delta \in (0,1]$ so that $(f^n)_*dx \in A_\infty(c,\delta)$ for every $n \in \mathbb{N}$.*

Observe that such a mapping is absolutely continuous with respect to the Lebesgue measure. If we write $(f^n)_*dx = \omega_n\, dx$, then (if f is differentiable) the density is

$$\omega_n(x) = \sum_{f^n(y)=x} \frac{1}{|(f^n)'(y)|}, \quad x \in M.$$

We call a point $c \in M$ *singular* if either f has no derivative at c or $f'(c) = 0$ and otherwise *regular*. The *singular set* is $C_f := \{c \in M : c \text{ singular}\}$. We propose to study first the following problem:

Problem 1 Establish elementary properties such as the periodic points of A_∞ mappings.

As we mentioned, $C^{1+\alpha}$ mappings are extensively studied in the literature where the authors have all established distortion results for the class of mappings they consider. For example, Shub and Sullivan considered expanding $C^{1+\alpha}$ circle endormorphisms. Lemma 1 of [62] says that there is a constant $c > 0$ such that if $I \subset S^1$ is an interval and f^n is one-to-one on I then

$$\frac{1}{c} < \left|\frac{(f^n)'(x)}{(f^n)'(y)}\right| < c \text{ for any } x,\ y \in I.$$

The estimated term is usually referred to be the distortion and this control of the distortion should be compared with the Koebe $\frac{1}{4}$ Theorem of analytic

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Problem 1 Establish elementary properties such as the periodic points of A_∞ mappings.

As we mentioned, $C^{1+\alpha}$ mappings are extensively studied in the literature where the authors have all established distortion results for the class of mappings they consider. For example, Shub and Sullivan considered expanding $C^{1+\alpha}$ circle endormorphisms. Lemma 1 of [62] says that there is a constant $c > 0$ such that if $I \subset S^1$ is an interval and f^n is one-to-one on I then

$$\frac{1}{c} < \left| \frac{(f^n)'(x)}{(f^n)'(y)} \right| < c \text{ for any } x,\ y \in I.$$

The estimated term is usually referred to be the distortion and this control of the distortion should be compared with the Koebe $\frac{1}{4}$ Theorem of analytic functionns. From this it is easy to see that such a mapping is in fact an A_∞ mapping. We thus are led to the following problem:

Problem 2 Give examples (if possible a description) of the smooth (for example $C^{1+\alpha}$) mappings that are A_∞. As we observe that the very good $C^{1+\alpha}$ mappings of [32-35] are A_∞. Note that these contain the mapping considered by Misiurewicz [45] and those of Jakobson [31]. What about the geometrically finite mappings in [31, 35]. Find more examples of A_∞ mappings.

We now focus on the absolutely continuous invariant measures. For the class of mappings he considered, Misiurewicz [45] gave a precise answer to the following question (see also [32-35]):

Problem 3 Show that an A_∞ mapping has always an absolutely continuous invariant measure. How many such absolutely continuous invariant measures do exist?

Part II Invariant properties of Möbius transformation

This part of our proposal is motivated by a serie of papers written by H.Haruki and T.M.Rassias on invariant characteristic property of Möbius transformations, [24-28]. Recall that a meromorphic functions on $\mathbb{C}$ is called a Möbius transformation if $f(z) = \frac{az+b}{cz+d}$ where a, b, c, and d are complex numbers satisfying $ad - bc \neq 0$. The set of all Möbius transformations under the usual composition of function is a group which is of particular interest when we consider some of its subgroups, for example, subgroup of all Möbius transformations which satisfy $ad - bc = 1$, see [13, 36, 47]. From now on we let $w = f(z)$ be a nonconstant meromorphic function on $\mathbb{C}$. Consider the following properties:

Property A. *$w = f(z)$ transforms circles in the z − plane onto circles in the w − plane, including straight lines among circles.*

Property B. *Suppose that $w = f(z)$ is analytic and univalent in a nonempty simply connected domain R on the z − plane. Let $ABCD$ be an arbitrary quadrilateral(not self-intersecting) contained in R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$ and if $A'B'C'D'$ is a quadrilateral on the w − plane which is not self-intersecting, then*

$$< A + < C = < A' + < C'$$

and

$$< B + < D = < B' + D'$$

hold.

Definition 1. Let $\triangle ABC$ be an arbitrary triangle and L a point on the complex plane. We denote by $a = \overline{BC}$, $b = \overline{AC}$, $c = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$. If $ax = by = cz$ holds, then L is said to be an *Apollonius point* of $\triangle ABC$.

Definition 2. Let $ABCD$ be an arbitrary quadrilateral (not necessarily simple) on the complex plane. If $\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{AD}$ holds, then $ABCD$ is said to be an *Apollonius quadrilateral.*

Note that, from Definition 1, any triangle can have at most two Apollonius point.

Property C. *Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. Let $\triangle ABC$ be an arbitrary triangle contained in R and let its Apollonius point L be a point of R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $L' = f(L)$ and if the three different points A', B', C' form a triangle, (i.e. A', B', C' are not collinear), then the point L' is also an Apollonius point of $\triangle A'B'C'$.*

Property D. *Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. Let $ABCD$ be an arbitrary Apollonius quadrilateral contained in R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$, then $A'B'C'D'$ is also an Apollonius quadrilateral on the w-plane.*

The following result about invariant property of Möbius transformations is well-known:

Theorem A [36, 47] *The function $w = f(z)$ satisfies Property A if and only if $w = f(z)$ is a Möbius transformation.*

Later H.Haruki and T.M.Rassias proved the following results on invariant property of Möbius transformations:

Theorem B [25] *The function $w = f(z)$ satisfies Property B if and only if $w = f(z)$ is a Möbius transformation.*

Theorem C [26] *The function $w = f(z)$ satisfies Property C if and only if $w = f(z)$ is a Möbius transformation.*

Theorem D [27] *The function $w = f(z)$ satisfies Property D if and only if $w = f(z)$ is a Möbius transformation.*

Definition 3 The Schwarzian derivative of a function f, S_f is defined by

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The following result is well-known [47]:

Theorem 1 *Let f be a complex-valued function. Then $S_f(z) = 0$ for all z such that $f'(z) \neq 0$ if and only if f is a Möbius transformation.*

The key ingredients in the proof of the above Theorems are known, such as the Maximum Modulus Principle of analytic functions, the Reflection Principle of analytic functions, and some well-known lemmas. The strategies in all the results above are to show that if $w = f(z)$ satisfies either Property B, C, or D, then $w = f(z)$ has zero Schwarzian derivative in R and hence in $\mathbb{C}$ (by the Identity Theorem) and hence it must be a Möbius transformation. However, as we inspect the proof of these results carefully, we see that we can proof these results by using one of the most important properties of Möbius transformations, namely the invariance of cross ratio (recall that the cross ratio of four distinct points z_1, z_2, z_3, $z_4 \in \mathbb{C} \cup \{\infty\}$ is defined to be $\frac{z_1-z_2}{z_1-z_4} \cdot \frac{z_3-z_4}{z_3-z_2}$). Moreover, we are able to extend some definitions and results of H.Haruki and T.M.Rassias, for example, we have the following:

Definition 4. Let $k, l > 0$. Let $\triangle ABC$ be an arbitrary triangle and L a point on the complex plane. We denote by $a = \overline{BC}$, $b = \overline{AC}$, $c = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$. If $ax = k(by) = l(cz)$ holds, then L is said to be an *(k, l)-Apollonius point* of $\triangle ABC$.

Definition 5. Let $k > 0$. A quadrilateral $ABCD$ is called a *$k-$Apollonius quadrilateral* if $\overline{AB} \cdot \overline{CD} = k\,\overline{BC} \cdot \overline{AD}$.

By using the invariance of cross ratio under Möbius transformations, we have obtained the following results which generalize Theorem B, C, and D above:

Property C$'$. *Suppose that $w = f(z)$ is analytic and univalent in a non-empty domain R of the z-plane. Let $k, l > 0$. Let $\triangle ABC$ be an arbitrary triangle contained in R and let its (k, l)-Apollonius point L be a point of R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $L' = f(L)$ and if the three different points A', B', C' form a triangle, (i.e. A', B', C' are not collinear), then the point L' is also a (k, l)-Apollonius point of $\triangle A'B'C'$.*

Property D$'$. *Suppose that $w = f(z)$ is analytic and univalent in a non-empty domain R of the z-plane. Let $ABCD$ be an arbitrary k-Apollonius quadrilateral contained in R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$, then $A'B'C'D'$ is also a k-Apollonius quadrilateral on the w-plane.*

Theorem C$'$. The function $w = f(z)$ satisfies Property C$'$ if and only if $w = f(z)$ is a Möbius transformation.

Theorem D$'$. *The function $w = f(z)$ satisfies Property D$'$ if and only if $w = f(z)$ is a Möbius transformation.*

Moreover, we obtain a new result similar to Theorem 1 as follows:

Theorem 1′. *Let f be a meromorphic on the plane. Define the* ***Newton derivative****, N_f of a function f as $N_f(z) = z - \frac{f(z)}{f'(z)}$. Then $N_f(z) = 0$ for all z such that $f'(z) \neq 0$ if and only if f is a Möbius transformation of the form $f(z) = \frac{u}{z+u}$, $u \neq 0$.*

Motivated by the previously known results we have mentioned and new results we have obtained, we propose to study the following problems:

Problem I. From Theorem 1′, we propose to study (find) new invariance properties of Möbius transformations which have Newton derivative equal to zero. For examples, we propose to study Properties B, C, D, C′, and D′ for this class of Möbius transformations. (Currently, we have already submitted one paper related to Problem I to Journal of Mathematical Analysis and Applications which are now being refereed).

Problem II. Continue from Problem I, we will study some new invariant properties of general Möbius transformations. One possible direction is to consider hexagonal instead of quadrilateral in THeorem D and D′ above in which we have some precise idea of how to attack this problem.

Problem III. In [46, 48-49], it was shown that if an analytic function f in the unit disk $D(0,1) = \{z : |z| < 1\}$ satisfies the following inequality

$$|S_f(z)| \leq \frac{2}{\left(1-|z|^2\right)^2}$$

for all $z \in D(0,1)$ then f is univalent in $D(0,1)$ where the constant 2 is best possible. It is routine to check that if f is univalent in $D(0,1)$ then f satisfies the following inequality

$$|S_f(z)| \leq \frac{6}{\left(1-|z|^2\right)^2}$$

for all $z \in D(0,1)$ where again the constant 6 is best possible. The above result was generalized in [17] where it was shown that if an analytic function f in the unit disk $D(0,1)$ satisfies the following inequality

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{C}{1-|z|^2}$$

for all $z \in D(0,1)$ and for sufficiently small constant C then f is univalent in $D(0,1)$ and it was also shown that C can be taken to be $2(\sqrt{5}-2)$ or smaller.

Several authors have then tried to increase the value of C, see [17-18, 49]. From these results, we propose to find necessary and sufficient conditions for an analytic function in $D(0,1)$ to be univalent by considering the modulus of Newton derivative, $|N_f|$ instead of $|S_f|$ or $\left|\frac{f''(z)}{f'(z)}\right|$. Our results will be certainly new and can be applied to obtain some new criterion for univalence of analytic functions in $D(0,1)$. As of present, we have obtained a necessary condition already.

เอกสารอ้างอิง

[1] I.N.Baker, J.Kotus, and L.Yinian, Dynamics of Meromorphic Functions II, *J. London Math. Soc.*, 42(1990), 267-278

[2] I.N.Baker, J.Kotus, and L.Yinian, Dynamics of Meromorphic Functions I, *Ergod. Th. & Dynam. Sys.*, 11(1991), 241-248

[3] I.N.Baker, J.Kotus, and L.Yinian, Dynamics of Meromorphic Functions III, , *Ergod. Th. & Dynam. Sys.*, 11(1991), 603-618

[4] I.N.Baker, J.Kotus, and L.Yinian, Dynamics of Meromorphic Functions IV, *Results in Mathematics*, 22(1992), 651-656

[5] K.Baranski, Hausdorff Dimension and Measure on Julia Sets of Some Meromorphic Maps, *Fundamenta Mathematicae*, 147(1995), 239-260

[6] I.N.Baker and A.P.Singh, Wandering Domains in the Iteration of Compositions of Entire Functions, *Ann. Acad. Sci. Fenn., Series A I, Mathematica*, 20(1995), 149-153

[7] A.F.Beardon, On the Hausdorff Dimension of General Cantor Sets, *Proc. Camb. Phil. Soc.*, 61(1965), 679-694

[8] A.F.Beardon, "*Iteration of Rational Functions*", Graduate Texts in Math., 132(1991), Springer-Verlag

[9] W.Bergweiler, Iteration of Meromorphic Fuunctions, *Bull. Aer. Math. Soc.*, 29:2 (1993), 151-188

[10] P.Blanchard, Complex Analytic Dynamics on the Riemann Sphere, *Bull. Amer. Math. Soc.*, 1(1984), 85-141

[11] A.Boyarsky, "*Laws of Chaos*", Birkhauser, Boston, 1997.

[12] H.Brolin, Invariant Sets Under Iteration of Rational Functions, *Arkiv for Matematik*, 6 (1965), 103-144

[13] J.B.Conway, "*Functions of One Complex Variable*", Springer-Verlag, 1978

[14] J.W.Dettman, "*Applied Complex Variables*", Dover Publications Inc., New York, 1984.

[15] R.Devaney, "*An Introduction to Chaotic Dynamical Systems*", Addison-Wesley, 1989

[16] P.Dominguez, Dynamics of Transcendental Meromorphic Functions, *Ann. Acad. Sci. Fenn., Series A I. Mathematica*, 23(1998), 225-250

[17] P.Duren, H.Shapiro, and A.Shields, Singular Measures and Domains Not of Smirnov Type, *Duke Math. J.*, 33(1966), 247-254

[18] P.Duren and O.Lehto, Schwarzian Derivatives and Homeomorphic Extensions, *Ann. Acad. Sci. Fenn., Series A I Mathematica*, 477(1970), 11 pages.

[19] H.M. Emily, "*The Dynamics of Newton's Method on the Exponential Function in the Complex Plane*", PhD thesis, Boston University, 1992

[20] V.Garber, On the Iteration of Rational Functions, *Proc. Camb. Phil. Soc.*, 84(1978), 497-504

[21] P.Gora and N.Obeid, Absolutely Continuous Invariant Measures for a Class of Meromorphic Transformations, *Proc. Amer. Math. Soc.*, 12(1997), 1-13

[22] L.S.Hahn and B.Epstein, "*Classical Complex Analysis*", Jones and Bartlett Publishers, 1996.

[23] W.K.Hayman, "*Meromorphic Functions*", Clarendon Press, Oxford, 1964.

[24] H.Haruki, A Proof of the Principle of Circle Transformation by Use of a Theorem on Univalent Functions, *Enseign. Math.*, 18(1972), 145-146

[25] H.Haruki an T.M.Rassias, A New Invariant Characteristic Property of Mobius Transformations From the Standpoint of Conformal Mapping, *J. Math. Anal. Appls.*, 181 (1994), 320-327

[26] H.Haruki an T.M.Rassias, A New Characteristic of Mobius Transformations by Use of Apollonius Points of Triangles, *J. Math. Anal. Appls*, 197(1996), 14-22

[27] H.Haruki an T.M.Rassias, A New Characteristic of Mobius Transformations by Use of Apollonius Quadrilaterals, *Proc. Amer. Math. Soc.*, 126(1998), 2857-2861

[28] H.Haruki an T.M.Rassias, A New Characteristic of Mobius Transformations by Use of Apollonius Quadrilaterals, *Topol. Anal.* To appear

[29] M. Jablonski and A.Lasota, Absolutely Continuous Invariant Measures for Transformations on the Real Line, *Zeszyty Nauk Univ. Jagiellon Prace Mat. Zeszyt*, 22 (1981), 7-13

[30] M.Jablonski, P.Gora, and A.Boyarsky, A General Existence Theorem for Absolutely Continuous Invariant Measures on Bounded and Unbounded Intervals, *Nonlinear World*, 3(1996), 183-200

[31] M.Jakobson, Quasisymmetric Conjugacy for Some One-Dimensional Maps Including Expansion, Preprint, 1989.

[32] Y.P.Jiang, Dynamics of Certain Smooth One-Dimensional Mappings, II. Geometrically finite one-dimensional mappings, SUNY, Preprint, 1990

[33] Y.P.Jiang, Dynamics of Certain Smooth One-Dimensional Mappings, I. The $C^{1+\alpha}$-Denjoy-Koebe distortion lemma, SUNY, Preprint, 1991

[34] Y.P.Jiang , Dynamics of Certain Smooth One-Dimensional Mappings, III Scaling function geometry, SUNY, Preprint, 1992.

[35] Y.P.Jiang, Geometry of Geometrically Finite One-Dimensional Maps, *Commun. Math. Phys*, 1993, 125-134

[36] G.A.Jones and D.Singerman, "*Complex Functions*", Cambridge University Press, 1987.

[37] J.R.Kinney and T.S.Pitcher, Julia Sets for Certain Rational Functions, *Adv. In Appl. Math.*, 9(1988), 51-55

[38] V.Mayer, Cyclic Parabolic Quasiconformal Groups That Are Not Quasiconformal Conjugates of Mobius Groups, *Ann. Acad. Sci. Fenn., Series A I. Mathematica*, 18 (1993), 147-154

[39] V.Mayer, Trajectories of one-parameter group of quasi-isometries, *Revista Mathematica Iberoamericana*, 11(1995), 143-164

[40] V.Mayer, Uniformly Quasiregular Mappings of Lattes Type, *Conform. Geom. Dynam.*, 1(1997), 104-111

[41] V.Mayer, Bilipschitz Group Actions and Homogeneous Jordan Curves, Pub. Irma, Lille, 46(1998), 1-25

[42] V.Mayer, Quasiregular Analogues of Critically Finite Rational Functions With Parabolic Orbifold, *J. Analyse Math.*, 75(1998), 105-119

[43] V.Mayer, Behavior of Quasiregular Semigroups Near Attracting Fixed Points, *Ann. Acad. Sci. Fenn., Series A I. Mathematica*, 24(1999), 1-9

[44] J.Milnors, "*Dynamics in One Complex Variables: Introductory Lectures*", SUNY Stony Brook, 1990

[45] M.Misiurewicz, Absolutely Continuous Measures for Certain Maps of an Interval, *Inst. Hautes. Etudes. Sci. Publ. Math.*, 53(1978), 17-51

[46] Z.Nehari, The Schwarzian Derivative and Schlicht Functions, Bul. Amer. Math. Soc., 55(1949), 545-551

[47] Z.Nehari, "*Conformal Mappings*", Mcgraw-Hill, New York, 1952

[48] Z.Nehari, Some Criteria of Univalence, *Proc. Amer. Math. Soc.*, 5(1954), 700-704

[49] Z.Nehari, Univalence Criteria Depending on the Schwarzian Derivative, *Illinois J. Math.*, 23(1979), 345-351

[50] P.Niamsup, "*Julia Sets and Symbolic Dynamics of Certain Rational and Entire Functions*", PhD Thesis, University of Illinois, 1997.

[51] P.Niamsup, Julia Sets of Certain Exponential Functions, to appear in Journal of Mathematical Analysis and Applications (see the letter of acceptance for publication).

[52] P.Niamsup, Dynamics of Certain Rational Functions, (Submitted for Publication)

[53] P.Niamsup, J.Palmore, and Y.Lenbury, The Composition of Halley's Functions and Newton's Functions and Its Schwarzian Derivatives, submitted for publication to Complex Variables.

[54] P.Niamsup, J.Palmore, and Y.Lenbury, Some Relations Among Halley's Functions, Newton's Functions and Successive Approximations, submitted for publications to SEAM Bulletin.

[55] P.Niamsup and V.N.Phat, Asymptotic Stability of Nonlinear Control Systems Described by Difference Equations with Multiple Delays, *J. Diff. Eqns.* (electronic version), 2000(2000), no.11, 1-17

[56] P.Petek and M.S.Rugelj, The Dynamics of $\lambda + z + \exp(z)$, *J. Math. Anal. Appls.*, 222(1998), 38-63

[57] C.Robinson, "*Dynamical Systems*", CRC Press, 1994.

[58] W.Schwick, Normality Criteria for Families of Meromorphic Functions, *J. Anal. Math.*, 52(1989), 241-289.

[59] W.Schwick, Exceptional Functions and Normality, *Bull. London Math. Soc.*, 29 (1997), 425-432.

[60] M.Shishikura, On the Complex Dynamical Systems on the Riemann Sphere, *Sugaku Expositions*, 6(1993), 165-184

[61] M.Shub, Endomorphisms of Compact Differential Manifolds, *Amer. J. Math.* XCI (1969), 175-199

[62] M.Shub and D.Sullivan, Expanding Endomorphisms of Circle Revisited, *Ergod. Th. & Dynam. Sys.*, 5(1985), 285-289

[63] G.M.Stallard, The Hausdorff Dimensions of Julia Sets of Entire Functions, *Ergod. Th. & Dynam. Sys.*, 11(1991), 769-777

[64] G.M.Stallard, The Hausdorff Dimensions of Julia Sets of Meromorphic Functions, *J. London Math. Soc.*, 49(1994), 281-295

[65] G.M.Stallard, The Hausdorff Dimensions of Julia Sets of Entire Functions II, *Proc. Camb. Phil. Soc.*, 119(1996), 513-536

[66] P.Walters, Invariant Measures and Equilibrium States for Some Mappings Which Expand Distances, *Trans. Amer. Math. Soc.*, 236(1978), 121-153

[67] S. Wong, Newton's Method and Symbolic Dynamics, *Proc. Amer. Math. Soc*, 91 (1984), 245-253

[68] P.YaKochina and O.I.Shishorina, Linear Fractional Transformations and Equations of Empirical Curves, *Prikl. Mat. Mekh.*, 57(1993), 215-219

[69] P.YaKochina and N.N.Kochina, Hydromechanics of Subterranean Water and Problems of Irrigation, *Fizmatlit*, Moscow, 1994

[70] P.YaKochina and N.N.Kochina, Problems of Motion with a Free Surface in Subterranean Hydrodynamics, *Redaktsiya*, Moscow, 1996

[71] P.YaKochina and N.N.Kochina, Some Properties of a Linear Fractional Transformation, *J. Appl. Maths. Mechs.*, 63(1999), 161-163

[72] Y.Yongcheng, On the Julia Swts of Quadratic Rational Maps, *Complex Variables*, 18(1992), 141-147

ผลการวิจัยที่ได้รับ

1. สรุปรายละเอียดการวิจัย

1.1 การวิจัยในปีที่ 1 (เดือนที่ 1 – เดือนที่ 12) เน้นการวิจัยในหัวข้อการวิจัยคุณสมบัติไม่แปรเปลี่ยนบางประการของการแปลงเชิงเส้นคู่

1.2 การวิจัยในปีที่ 2 (เดือนที่ 13– เดือนที่ 24) เน้นการวิจัยในหัวข้อการวิจัยคุณสมบัติไม่แปรเปลี่ยนบางประการของการแปลงเชิงเส้นคู่ และพลศาสตร์เชิงดิสครีตของฟังก์ชันนิวตัน

1.3 การวิจัยในปีที่ 3 (เดือนที่ 25 – เดือนที่ 36) เน้นการศึกษาสมบัติบางประการของเซตจูเลียของฟังก์ชันตรรกยะ โดยเฉพาะการให้เงื่อนไขที่เพียงพอที่ทำให้เซตจูเลียของฟังก์ชันตรรกยะเป็น Lakes of Wada Continuum และการหาผลเฉลยของสมการเชิงฟังก์ชันนัยบางสมการ

2. ผลการวิจัยที่ได้รับ

2.1 ในการวิจัยหัวข้อการวิจัยคุณสมบัติไม่แปรเปลี่ยนบางประการของการแปลงเชิงเส้นคู่ ในช่วงเดือนที่ 1 - เดือนที่ 12 ได้งานวิจัยที่ได้รับการตีพิมพ์ในวารสารระดับนานาชาติจำนวน 2 ผลงานดังนี้

2.1.1 A Note on the Characteristics of Mobius Transformations, J. Math. Anal. Appls. 248 (2000), 203-215.

2.1.2 A Characterization of Mobius Transformations, Internat. J. Math. & Math. Sci., Vol.24, No.10 (2000), 663-666.

2.1.3 A Note on the Characteristics of Mobius Transformations II, J. Math. Anal. Appls. 261 (2001), 151-158.

2.2 การวิจัยในช่วงเดือนที่ 13 - เดือนที่ 24 ได้รับการตีพิมพ์จำนวน 1 เรื่อง คือ

2.2.1 Dynamics of Newton's Functions of Barna's Polynomials, Int. J. Math. & Math. Sc. 28:2 (2002) 79 – 84, 79 – 84.

2.3 การวิจัยในช่วงเดือนที่ 25 - เดือนที่ 36 ได้ผลงานวิจัย 2 เรื่องคือ

2.3.1 Julia Sets of Certain Rational Functions และอยู่ในระหว่างการเตรียมต้นฉบับเพื่อส่งตีพิมพ์

2.3.2 Meromorphic Solutions of Ceratin Functional Equation และอยู่ในระหว่างการเตรียมต้นฉบับเพื่อส่งตีพิมพ์

3. การนำผลงานวิจัยไปใช้ประโยชน์

เชิงสาธารณะ มีการประชาสัมพันธ์กิจกรรมการวิจัยและหาผู้วิจัยร่วมอย่างต่อเนื่องแต่ยังไม่ได้รับความสนใจนักเนื่องจากเป็นสาขาวิชาที่ค่อนข้างยากและใช้เวลาในการศึกษามาก

เชิงวิชาการ มีนักศึกษาปริญญาโทสำเร็จการศึกษาไปจำนวน 7 คนภายใต้โครงการนี้ และมีนักศึกษาปริญญาโท 3 คนและปริญญาเอกเอก 1 คนที่กำลังทำวิจัยในขณะนี้

4. นักศึกษาระดับปริญญาโทที่สำเร็จการศึกษา (โดยได้รับค่าจ้างวิจัยจากโครงการวิจัยนี้)

ชื่อนักศึกษา	ชื่อหัวข้อวิจัย/วิทยานิพนธ์ ระดับปริญญาโท	ค่าจ้าง (1,000 บาท/เดือน)
น.ส.ธนิกานต์ หอมแก่นจันทร์ (2544) (คณิตศาสตร์ประยุกต์)	Asymptotic Stability of Linear Difference Equations $x_{n+1} - a^2 x_{n-1} + b x_{n-k} = 0$	12 เดือน เป็นเงิน 12,000 บาท
นายสมเกียรติ ฤทธิ์ศิริ (คณิตศาสตร์ประยุกต์) (2544)	Asymptotic Stability of Linear Difference Equations with Delays	12 เดือน เป็นเงิน 12,000 บาท
นายสมบูรณ์ นิยม (คณิตศาสตร์ประยุกต์) (2544)	Asymptotic Stability of Nonlinear Systems Described by Difference Equations with Multiple Delays	12 เดือน เป็นเงิน 12,000 บาท
นายธีรพล สะอึ้วงศ์ (คณิตศาสตร์ประยุกต์) (2545)	Exponential Stability of Nonlinear Time-Varying Differential Equations	12 เดือน เป็นเงิน 12,000 บาท
นายจิตติ รักบุตร (คณิตศาสตร์) (2545)	Julia set of $az^n + b$	12 เดือน เป็นเงิน 12,000 บาท
น.ส.วิยะดา อมรรัตนไพจิตร (คณิตศาสตร์ประยุกต์) (2546)	Asymptotic Stability of Linear Difference Equations $\frac{x_n + a}{x_{n-1} + a x_n + b}$	12 เดือน เป็นเงิน 12,000 บาท
น.ส.พรทิพย์ ป้องชาลี (คณิตศาสตร์ประยุกต์) (2546)	Asymptotic Stability of Difference Equation $x_{n+1} - x_n = -p_n x_{n-k}$	12 เดือน เป็นเงิน 12,000 บาท

และยังมีนักศึกษาระดับปริญญาโทอีก 3 คนที่กำลังทำวิทยานิพนธ์และคาดว่าจะเสร็จภายในเดือน สิงหาคม 2547

5. ข้อเสนอแนะสำหรับงานวิจัยในอนาคต

เนื่องจากเนื้อหาการวิจัยที่เสนอมีค่อนข้างมากดังนั้นการวิจัยในส่วนของพลศาสตร์เชิงดิสครีตของฟังก์ชันในคลาส $A_\pm$ และอธิบาย การมี Absolutely Continuous Invariant Measures ของฟังก์ชันในคลาส $A_\pm$ ยังทำได้น้อย ดังนั้นในอนาคตจะได้ศึกษาวิจัยในส่วนนี้เพิ่มเติม นอกจากนั้นการวิจัยพลศาสตร์เชิงดิสครีตของฟังก์ชันตรรกยะบางฟังก์ชันก็ยังมีปัญหาในการวิจัยอีกมากเช่นกัน

ผลงานวิจัยที่ได้รับการตีพิมพ์

Internat. J. Math. & Math. Sci.
Vol. 24, No. 10 (2000) 683–686
S0161171200010255

A CHARACTERIZATION OF MÖBIUS TRANSFORMATIONS

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(Received 9 March 2000 and in revised form 5 June 2000)

ABSTRACT. We give a new invariant characteristic property of Möbius transformations.

Keywords and phrases. Möbius transformations, Schwarzian derivative, Newton derivative.

2000 Mathematics Subject Classification. Primary 30C35.

1. Introduction. Throughout this paper, we let $w = f(z)$ be a nonconstant meromorphic function in $\mathbb{C}$ unless otherwise stated.

We consider the following properties.

PROPERTY 1.1. $w = f(z)$ transforms circles in the z-plane onto circles in the w-plane, including straight lines among circles.

PROPERTY 1.2. Suppose that $w = f(z)$ is analytic and univalent in a nonempty simply connected domain $\mathbb{R}$ on the z-plane. Let $ABCD$ be an arbitrary quadrilateral (not self-intersecting) contained in $\mathbb{R}$. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$ and if $A'B'C'D'$ is a quadrilateral on the w-plane which is not self-intersecting, then the following hold

$$\angle A + \angle C = \angle A' + \angle C', \qquad \angle B + \angle D = \angle B' + \angle D'. \tag{1.1}$$

The following is a well-known principle of circle transformation of Möbius transformations.

THEOREM 1.3. *$w = f(z)$ satisfies Property 1.1 if and only if $w = f(z)$ is a Möbius transformations.*

In [1], it is shown that Property 1.1 implies Property 1.2 and a new invariant characteristic property of Möbius transformations is given as follows.

THEOREM 1.4. *Let α be an arbitrary fixed real number such that $0 < \alpha < 2\pi$. Suppose that $w = f(z)$ is analytic and univalent in a nonempty simply connected domain $\mathbb{R}$ on the z-plane. Let $ABCD$ be an arbitrary quadrilateral (not self-intersecting) contained in $\mathbb{R}$ satisfying*

$$\angle A + \angle C = \alpha. \tag{1.2}$$

If $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$ is a quadrilateral on the w-plane which is not self-intersecting, then the only function which satisfies

$$\angle A' + \angle C' = \alpha \tag{1.3}$$

is a Möbius transformation.

Theorem 1.4 gives an alternative proof of "the only if part" of Theorem 1.3. Motivated by the above results, we consider the following property.

Property 1.5. Let k be an arbitrary positive real number. For three arbitrary distinct points a, b, and c in $\mathbb{R}$ satisfying

$$\left|\frac{a-b}{c-b}\right| = k, \tag{1.4}$$

we have

$$\left|\frac{f(a)-f(b)}{f(c)-f(b)}\cdot\frac{f(c)}{f(a)}\right| = k. \tag{1.5}$$

In Section 3, we prove the following result concerning the mapping property of an analytic and univalent function on a connected domain.

Theorem 1.6. *Let k be an arbitrary positive real number. Let $w = f(z)$ be analytic and univalent in a nonempty connected domain $\mathbb{R}$ on the z-plane such that $f(z) \neq 0$ for all $z \in \mathbb{R}$. Then f satisfies Property 1.5 if and only if f is a Möbius transformation of the form $u/(z+v)$, $u \neq 0$.*

2. Lemmas

Definition 2.1. Let f be a complex-valued function. The Schwarzian derivative of f is defined as follows:

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{2.1}$$

Similar to Schwarzian derivative, we have the following.

Definition 2.2. Let f be a complex-valued function. We define the *Newton derivative* of f as follows:

$$N_f(z) = \left(z - \frac{f(z)}{f'(z)}\right)' = \frac{f(z)f''(z)}{(f'(z))^2}. \tag{2.2}$$

Remark 2.3. Note that $N_f(z)$ is the first derivative of Newton's method of f.

Remark 2.4. Let f be a complex-valued function. It is well known that $S_f(z) = 0$ if and only if f is a Möbius transformation.

From Remark 2.4, we have observed that a similar result holds true when we replace Schwarzian derivative by the Newton derivative.

Lemma 2.5. *Let f be a complex-valued function. Then $N_f(z) = 2$ if and only if f is a Möbius transformation of the form $u/(z+v)$, $u \neq 0$.*

Proof. Let f be a Möbius transformation of the form $u/(z+v)$, $u \neq 0$, then it is easily checked that $N_f(z) = 2$. Let f be a complex-valued function such that $N_f(z) = 2$. It follows that

$$\left(z - \frac{f(z)}{f'(z)}\right)' = 2 \tag{2.3}$$

which implies that

$$z - \frac{f(z)}{f'(z)} = 2z - c_1, \tag{2.4}$$

where c_1 is a complex constant, thus

$$\frac{f(z)}{f'(z)} = -z + c_1 \tag{2.5}$$

or

$$\frac{1}{f(z)}\frac{df(z)}{dz} = \frac{1}{-z + c_1}. \tag{2.6}$$

From which it follows by a simple calculation that f is a Möbius transformation of the form $u/(z+v)$, $u \neq 0$. □

3. Main result. In this section, we assume that $w = f(z)$ is analytic and univalent on a nonempty connected domain $\mathbb{R}$ on the z-plane such that $f(z) \neq 0$ for all $z \in \mathbb{R}$.

Proof of Theorem 1.6. Let $f(z)$ be a Möbius transformation of the form $u/(z+v)$, $u \neq 0$. Let a, b, and c be arbitrary three distinct points in $\mathbb{R}$ such that

$$\left|\frac{a-b}{c-b}\right| = k. \tag{3.1}$$

We observe that

$$\frac{a-b}{c-b} \tag{3.2}$$

is the cross-ratio of a, b, c, and d, where d is the point at infinity. Since $f(z) = u/(z+v)$, $u \neq 0$, we have $f(d) = 0$. Since Möbius transformations preserve the cross-ratio, we obtain

$$\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)} = \frac{a-b}{c-b} \tag{3.3}$$

which implies that

$$\left|\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}\right| = \left|\frac{a-b}{c-b}\right| = k. \tag{3.4}$$

Therefore, any Möbius transformation of the form $u/(z+v)$, $u \neq 0$ satisfies Property 1.5.

Conversely, let x be an arbitrary fixed point in $\mathbb{R}$. Then there exists a positive real number r such that the r circular neighborhood $N_r(x)$ of x is contained in $\mathbb{R}$.

Throughout the proof let $A = x + ky$, $B = x$, $C = x - y$. Since $\mathbb{R}$ is a nonempty connected domain on the z-plane, there exists a positive real number s such that if

$$0 < |y| < s, \tag{3.5}$$

then A, B, and C are contained in $N_r(x)$.

Since $w = f(z)$ is univalent in $\mathbb{R}$, $f(A) = f(x+ky)$, $f(B) = f(x)$, and $f(C) = f(x-y)$ are distinct points. By assumption, we have

$$\left|\frac{f(x+ky)-f(x)}{f(x-y)-f(x)}\cdot\frac{f(x-y)}{f(x+ky)}\right| = k \tag{3.6}$$

for all y such that $0 < |y| < s$.

Let

$$h(y) = \frac{f(x+ky)-f(x)}{f(x-y)-f(x)}\cdot\frac{f(x-y)}{f(x+ky)}. \tag{3.7}$$

Then

$$|h(y)| = k \tag{3.8}$$

for all y such that $0 < |y| < s$. The function $h(y)$ extends analytically at zero by $h(0) = -k$. Hence, by the maximum modulus principle, we have $h(y) = -k$ for all y with $|y| < s$. In other words, we have

$$\frac{f(x+ky)-f(x)}{f(x-y)-f(x)}\cdot\frac{f(x-y)}{f(x+ky)} = -k \tag{3.9}$$

in $|y| < s$. This equality implies that

$$(f(x+ky)-f(x))f(x-y) = -k(f(x-y)-f(x))f(x+ky). \tag{3.10}$$

Differentiate this equality twice with respect to y and then set $y = 0$, we obtain

$$-k(k+1)(2(f'(x))^2 - f(x)f''(x)) = 0 \tag{3.11}$$

which implies that

$$2(f'(x))^2 - f(x)f''(x) = 0 \tag{3.12}$$

or

$$\frac{f(x)f''(x)}{(f'(x))^2} = 2. \tag{3.13}$$

By the identity theorem and Lemma 2.5, we conclude that f is a Möbius transformation of the form $u/(z+v)$, $u \neq 0$. □

Acknowledgement. I would like to thank the referees for valuable comments and suggestions. This work is supported by the Thailand Research Fund.

References

[1] H. Haruki and T. M. Rassias, *A new invariant characteristic property of Möbius transformations from the standpoint of conformal mapping*, J. Math. Anal. Appl. 181 (1994), no. 2, 320–327. MR 94m:30018. Zbl 796.30008.

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Journal of Mathematical Analysis and Applications 248, 203–215 (2000)
doi:10.1006/jmaa.2000.6888, available online at http://www.idealibrary.com on IDEAL®

A Note on the Characteristics of Möbius Transformations

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Submitted by T. M. Rassias

Received February 8, 2000

We give some new invariant characteristic properties of Möbius transformations by means of their mapping properties.

Key Words: Möbius transformations; Schwarzian derivative.

1. INTRODUCTION

Throughout the paper, unless otherwise stated, let $w = f(z)$ be a nonconstant meromorphic function on the complex plane C. It is well known that for $w = f(z)$ to be a Möbius transformation, it is necessary and sufficient that $w = f(z)$ satisfies the following Property A:

Property A. $w = f(z)$ maps circles in the z-plane onto circles in the w-plane, including straight lines among circles.

The following are some definitions and mapping properties which were introduced in [5–7].

Definition 1.1. Let $\triangle ABC$ be an arbitrary triangle and L a point on C. We denote $a = \overline{BC}$, $b = \overline{AC}$, $c = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$. If $ax = by = cz$ holds, then L is said to be an Apollonius point of $\triangle ABC$.

Definition 1.2. Let $ABCD$ be an arbitrary quadrilateral (not necessarily simple) on C. If $\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{DA}$ holds, then $ABCD$ is said to be an Apollonius quadrilateral.

Property B. Suppose that $w = f(z)$ is analytic and univalent in a nonempty simply connected domain R on the z-plane. Let $ABCD$ be an arbitrary quadrilateral (not self-intersecting) contained in R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$ and if $A'B'C'D'$ is a quadri-

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lateral on the w-plane which is not self-intersecting, then

$$\angle A + \angle C = \angle A' + \angle C'$$

and

$$\angle B + \angle D = \angle B' + \angle D'$$

hold.

Property C. Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. Let $\triangle ABC$ be an arbitrary triangle contained in R and let its Apollonius point L be a point of R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $L' = f(L)$, and if the three different points A', B', C' form a triangle (i.e., A', B', C' are not collinear), then the point L' is also an Apollonius point of $\triangle A'B'C'$.

Property D. Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. Let $ABCD$ be an arbitrary Apollonius quadrilateral contained in R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$, then $A'B'C'D'$ is also an Apollonius quadrilateral.

Recently, in [5–7] Haruki and Rassias gave several new characteristics of Möbius transformations from the standpoint of conformal mapping and elementary geometry. The following are the results they obtained:

THEOREM A [5]. *The function $w = f(z)$ satisfies Property B iff $w = f(z)$ is a Möbius transformation.*

THEOREM B [6]. *The function $w = f(z)$ satisfies Property C iff $w = f(z)$ is a Möbius transformation.*

THEOREM C [7]. *The function $w = f(z)$ satisfies Property D iff $w = f(z)$ is a Möbius transformation.*

These results are interesting in the sense that they illustrated some connections between geometric properties and analytic properties of analytic univalent mappings. The proof of the "only if" part of these results requires some known results from geometry together with the following key lemmas which are also well known:

LEMMA 1.3. *If the function $w = f(z)$ is analytic and univalent in a nonempty domain R, then $f'(z) \neq 0$ in R.*

LEMMA 1.4. *If $f(z)$ and $g(z)$ are analytic functions in a nonempty domain R and $f(z)g(z) \neq 0$ in R and also $\arg(f(z)) = \arg(g(z))$ holds in R, then $f(z) = Kg(z)$ in R where K is a positive real constant.*

LEMMA 1.5. *Let $w = f(z)$ be meromorphic on $\mathbb{C}$. Then $w = f(z)$ is a Möbius transformation iff $S_f(z) = 0$ for all $z \in \mathbb{C} - \{z : f'(z) = 0\}$, where*

$S_f(z) = (f''(z)/f'(z))' - (1/2)(f''(z)/f'(z))^2$ *which is called the Schwarzian derivative of* $f(z)$.

We now introduce the *Newton derivative* of a function $f(z)$ as follows:

DEFINITION 1.6. Let $f(z)$ be a function on C. We define the Newton derivative of $f(z)$ as the first derivative of the Newton's method of $f(z)$. In other words, we define the Newton's derivative of $f(z)$ as

$$N_f(z) = \left(z - \frac{f(z)}{f'(z)}\right)' = \frac{f(z)f''(z)}{(f'(z))^2}.$$

The main purpose of this paper is to generalize Theorems A, B, C and to prove the obtained results by means of the invariance of cross-ratio of four distinct points on $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ under a Möbius transformation. We will also give some new invariant characteristic properties of Möbius transformations. In particular, we will characterize Möbius transformations which have Newton derivative equal to 2 instead of having zero Schwarzian derivative.

2. MAIN RESULTS

First, we give another proof of Theorem A by means of the invariance of cross-ratio of four distinct points on $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ under a Möbius transformation.

Proof of Theorem A. Suppose that $w = f(z)$ is a Möbius transformation and let $ABCD$ be an arbitrary quadrilateral in R. Then we obtain

$$\angle A = \arg\left(\frac{A-D}{A-B}\right)$$

and

$$\angle C = \arg\left(\frac{C-B}{C-D}\right)$$

which implies that

$$\angle A + \angle C = \arg\left(\frac{A-D}{A-B}\right) + \arg\left(\frac{C-B}{C-D}\right) = \arg\left(\frac{A-D}{A-B}\cdot\frac{C-B}{C-D}\right).$$

Since $\frac{A-D}{A-B}\cdot\frac{C-B}{C-D}$ is the cross-ratio of four distinct points A, D, C, and B, we obtain

$$\frac{f(A)-f(D)}{f(A)-f(B)}\cdot\frac{f(C)-f(B)}{f(C)-f(D)} = \frac{A-D}{A-B}\cdot\frac{C-B}{C-D}$$

since the cross-ratio is invariant under mapping by a Möbius transformation. It follows that

$$\angle A' + \angle C' = \arg\left(\frac{f(A)-f(D)}{f(A)-f(B)}\cdot\frac{f(C)-f(B)}{f(C)-f(D)}\right)$$
$$= \arg\left(\frac{A-D}{A-B}\cdot\frac{C-B}{C-D}\right)$$
$$= \angle A + \angle C$$

which implies that $w = f(z)$ satisfies Property B. The other direction of the proof is the same as in [5].

We now generalize Definitions 1.1 and 1.2 as follows:

DEFINITION 2.1. Let $\triangle ABC$ be an arbitrary triangle and L a point on $\mathbb{C}$. We denote $a = \overline{BC}$, $b = \overline{AC}$, $c = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$. If $ax = k(by) = l(cz)$ holds where $k, l > 0$, then L is said to be a (k,l)-Apollonius point of $\triangle ABC$.

DEFINITION 2.2. Let $ABCD$ be an arbitrary quadrilateral (not necessarily simple) on $\mathbb{C}$. If $\overline{AB}\cdot\overline{CD} = k(\overline{BC}\cdot\overline{DA})$ holds, then $ABCD$ is said to be a k-Apollonius quadrilateral.

Remark 2.1. If L is a (k,l)-Apollonius point of $\triangle ABC$, then the quadrilateral $BCAL$ is a k-Apollonius quadrilateral, and the quadrilateral $BCLA$ is an l-Apollonius quadrilateral, where the sense of any four points is counterclockwise.

Consider the following properties:

Property C'. Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. Let $\triangle ABC$ be an arbitrary triangle contained in R and let its (k,l)-Apollonius point L be a point of R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $L' = f(L)$, and if the three different points A', B', C' form a triangle, (i.e., A', B', C' are not collinear), then the point L' is also a (k,l)-Apollonius point of $\triangle A'B'C'$.

Property D'. Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. Let $ABCD$ be an arbitrary k-Apollonius quadrilateral contained in R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$, then $A'B'C'D'$ is also a (k,l)-Apollonius quadrilateral.

The "if" parts of the following results can be proved using the same technique as in the proof of Theorem A above. The "only if" parts can be proved similarly as in [6, 7].

THEOREM 2.3. *The function $w = f(z)$ satisfies Property C' iff $w = f(z)$ is a Möbius transformation.*

THEOREM 2.4. *The function $w = f(z)$ satisfies Property D' iff $w = f(z)$ is a Möbius transformation.*

We now state

Property E. Let $w = f(z)$ be analytic and univalent in a nonempty domain R on the z-plane. Let $\alpha = 0$ or π. Let a, b, c, and d be four distinct points in R such that

$$\arg\left(\frac{a-b}{a-d}\cdot\frac{c-d}{c-b}+\frac{a-d}{a-b}\cdot\frac{c-b}{c-d}\right)=\alpha.$$

Then we have

$$\arg\left(\frac{f(a)-f(b)}{f(a)-f(d)}\cdot\frac{f(c)-f(d)}{f(c)-f(b)}+\frac{f(a)-f(d)}{f(a)-f(b)}\cdot\frac{f(c)-f(b)}{f(c)-f(d)}\right)=\alpha.$$

Property F. Let $w = f(z)$ be analytic and univalent in a nonempty domain R on the z-plane. Let $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Let a, b, c, and d be four distinct points in R such that

$$\arg\left(\frac{a-b}{a-d}\cdot\frac{c-d}{c-b}-\frac{a-d}{a-b}\cdot\frac{c-b}{c-d}\right)=\alpha.$$

Then we have

$$\arg\left(\frac{f(a)-f(b)}{f(a)-f(d)}\cdot\frac{f(c)-f(d)}{f(c)-f(b)}-\frac{f(a)-f(d)}{f(a)-f(b)}\cdot\frac{f(c)-f(b)}{f(c)-f(d)}\right)=\alpha.$$

We now give some more invariant characteristics of Möbius transformations as follows.

THEOREM 2.5. *Let $w = f(z)$ be analytic and univalent in a nonempty domain R on the z-plane. Then $w = f(z)$ satisfies Property E (or F) iff $w = f(z)$ is a Möbius transformation.*

Proof. The "if" part follows as in the proof of the "if" part of Theorem A. We now proceed to prove the "only if" part. Since $w = f(z)$ is analytic and univalent in the domain R, we have $f'(z) \neq 0$ in R. If x is an arbitrary fixed point in R, then we obtain $f'(x) \neq 0$. Let E be the point represented by x. Since $E \in R$, there exists a positive real number r such that the r circular neighborhood of E is contained in R. Throughout the proof let $ABCD$ denote an arbitrary rhombus in R with center at E

where A, B, C, and D are distinct points. Here the sense of A, B, C, and D is counterclockwise. Since $ABCD$ is a rhombus contained in R, we can represent A, B, C, and D by complex numbers

$$x + y, \quad x + iky, \quad x - y, \quad x - iky,$$

respectively for some positive real number k. Without loss of generality, we may assume that $k > 1 + \sqrt{2}$. Since R is a nonempty domain R on the z-plane, there exists a nonzero real number s such that $s < r$ and if $0 < |y| < s$ then $ABCD$ is contained in R. Since $w = f(z)$ is univalent in R, $f(A) = f(x+y)$, $f(B) = f(x+iky)$, $f(C) = f(x-y)$, $f(D) = f(x - iky)$ are distinct points. By assumption, we have

$$\arg\left(\begin{array}{l}\dfrac{f(x+y)-f(x+iky)}{f(x+y)-f(x-iky)}\cdot\dfrac{f(x-y)-f(x-iky)}{f(x-y)-f(x+iky)}\\ +\dfrac{f(x+y)-f(x-iky)}{f(x+y)-f(x+iky)}\cdot\dfrac{f(x-y)-f(x+iky)}{f(x-y)-f(x-iky)}\end{array}\right) = 0$$
$$= \arg(1) \tag{2.1}$$

for all y such that $0 < |y| < s$.

Since $x \in R$ is arbitrarily fixed, we can set

$$h(y) = \frac{f(x+y)-f(x+iky)}{f(x+y)-f(x-iky)}\cdot\frac{f(x-y)-f(x-iky)}{f(x-y)-f(x+iky)} + \frac{f(x+y)-f(x-iky)}{f(x+y)-f(x+iky)}\cdot\frac{f(x-y)-f(x+iky)}{f(x-y)-f(x-iky)}. \tag{2.2}$$

By (2.1) and (2.2) we obtain

$$\arg(h(y)) = \arg(1) \tag{2.3}$$

for all y such that $0 < |y| < s$. Now we prove that $h(y)$ is analytic at $y = 0$ and that (2.3) still holds at $y = 0$. To this end we apply Riemann's Theorem on removable singularities. As $y \to 0$, by L'Hopital's Rule, we obtain that

$$h(y) \to \left(\frac{1+ik}{1-ik}\right)^2 + \left(\frac{-1+ik}{1+ik}\right)^2 = \frac{2(1-6k^2+k^4)}{(1+k^2)^2}. \tag{2.4}$$

If we define

$$h(0) = \frac{2(1-6k^2+k^4)}{(1+k^2)^2} \tag{2.5}$$

by (2.4), by Riemann's Theorem on removable singularities, the function $h(y)$ is analytic at $y = 0$. Furthermore, (2.3) still holds at $y = 0$. The function $h(y)$ is analytic in $|y| < s$. By (2.2) and the fact that $w = f(z)$ is univalent in R, we obtain that $h(y) \neq 0$ in $|y| < s$. Hence by Lemma 1.4 we have

$$h(y) = K \tag{2.6}$$

in $|y| < s$, where K is a positive real constant. Setting $y = 0$ in (2.6) and using (2.5), it yields

$$\frac{2(1 + 6k^2 + k^4)}{(1 + k^2)^2} = K. \tag{2.7}$$

By (2.7) and (2.6) we obtain

$$h(y) = \frac{2(1 - 6k^2 + k^4)}{(1 + k^2)^2} \tag{2.8}$$

in $|y| < s$.

Substituting (2.2) into (2.6) and removing the denominator in the resulting equality it follows that

$$\begin{aligned}(f(x + y) - f(x + iky))^2(f(x - y) - f(x - iky))^2 \\ + (f(x + y) - f(x - iky))^2(f(x - y) - f(x + iky))^2 \\ = \frac{2(1 - 6k^2 + k^4)}{(1 + k^2)^2}(f(x + y) - f(x + iky)) \\ \times (f(x - y) - f(x - iky))(f(x + y) - f(x - iky)) \\ \times (f(x - y) - f(x + iky))\end{aligned} \tag{2.9}$$

in $|y| < s$.

Using Leibnitz's Rule for differentiation, differentiate six times both sides of (2.7) with respect to y; setting $y = 0$ yields

$$-1920k^2(-1 + k^2)(f'(x))^2\left(-3(f''(x))^2 + 2f'(x)f'''(x)\right) = 0. \tag{2.10}$$

Since k is a positive real number which is greater than $1 + \sqrt{2}$, we have $k^2(-1 + k^2) \neq 0$. Hence by (2.8) we obtain $f'''(x)f'(x) - \frac{3}{2}(f''(x))^2 = 0$. Since $x \in R$ was arbitrarily fixed, we can replace x by a variable z, and by (2) we have $f'''(z)f'(z) - \frac{3}{2}(f''(z))^2 = 0$ in R. By the Identity Theorem the above equality holds in C. Hence,

$$\frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = 0$$

holds for all z satisfying $f'(z) \neq 0$. Thus, the Schwarzian derivative of f vanishes for all z satisfying $f'(z) \neq 0$. Therefore, by Lemma 1.5, $f(z)$ is a Möbius transformation of z. ∎

We now give some invariant properties for Möbius transformations which have Newton derivative equal to 2. First, we state the following

Property G. Suppose that $w = f(z)$ is analytic and univalent in a nonempty simply connected domain R on the z-plane. Let α be an arbitrarily fixed real number such that $\alpha \in (0, \pi)$. For three arbitrary distinct points a, b, and c in R satisfying

$$\arg\left(\frac{a-b}{c-b}\right) = \alpha,$$

we have

$$\arg\left(\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}\right) = \alpha$$

For Möbius transformations which have Newton derivative equal to 2, we have the following result:

LEMMA 2.6. *Let f be a complex-valued function. Then $N_f(z) = 2$ for all $z \in \mathbb{C} - \{z : f'(z) = 0\}$ iff f is a Mobius transformation of the form $\frac{u}{z+v}$, $u \neq 0$.*

Proof. Let f be a Möbius transformation of the form $\frac{u}{z+v}$, $u \neq 0$; then it is easily checked that $N_f(z) = 2$. Let f be a complex-valued function such that $N_f(z) = 2$. It follows that

$$\left(z - \frac{f(z)}{f'(z)}\right)' = 2$$

which implies that

$$z - \frac{f(z)}{f'(z)} = 2z - c_1,$$

where c_1 is a complex constant,

$$\frac{f(z)}{f'(z)} = -z + c_1$$

or

$$\frac{1}{f(z)}\frac{df(z)}{dz} = \frac{1}{-z+c_1}.$$

It follows by a simple calculation that f is a Möbius transformation of the form $\frac{u}{z+c}$, $u \neq 0$. ∎

We are now ready to prove the following

THEOREM 2.7. *Let $w = f(z)$ be analytic and univalent in a nonempty connected domain R on the z-plane. Then f satisfies Property G iff f is a Möbius transformation of the form $\frac{u}{z+c}$, $u \neq 0$.*

Proof. Let $f(z)$ be a Möbius transformation of the form $\frac{u}{z+c}$, $u \neq 0$. Let a, b, and c be arbitrary three distinct points in R such that

$$\arg\left(\frac{a-b}{c-b}\right) = \alpha.$$

We observe that

$$\frac{a-b}{c-b}$$

is the cross-ratio of a, b, c, and d where d is the point at infinity. Since $f(z) = \frac{u}{z+c}$, $u \neq 0$, we have $f(d) = 0$. Since Möbius transformations preserve the cross-ratio, we obtain

$$\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)} = \frac{a-b}{c-b}$$

which implies that

$$\arg\left(\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}\right) = \arg\left(\frac{a-b}{c-b}\right) = \alpha.$$

Therefore, any Möbius transformation of the form $\frac{u}{z+c}$, $u \neq 0$ satisfies Property G. Conversely, let α be an arbitrary real number such that $\alpha \in (0, \pi) - \{\frac{\pi}{2}\}$. Let z be an arbitrary fixed point in R; then we obtain $f'(z) \neq 0$. Since $z \in R$, there exists a positive real number r such that the r circular neighborhood of z is contained in R. Throughout the proof let ABC denote an arbitrary isosceles triangle in R with center at z where A, B, and C are distinct points. Here the sense of A, B, and C are counterclockwise. Since ABC is an isosceles triangle contained in R, we can represent A, B, and C by complex numbers

$$z + ay, \qquad z + y, \qquad z + by,$$

respectively where $a = (-1/2 - (\sqrt{3}/2)ki)$, $b = (-1/2 + (\sqrt{3}/2)ki)$, $k > 0$, and y is some nonzero complex number. Without loss of generality, we let

$$k = \sqrt{\frac{3(1-\cos\alpha)}{1+\cos\alpha}}.$$

Since $\alpha \in (0, \pi) - \{\frac{\pi}{3}\}$, we have $k \in (0, +\infty) - \{\sqrt{3}\}$. For example, if $\alpha = \frac{\pi}{4}$, then $k = 1$. Since R is a nonempty connected domain R on the z-plane, there exists a nonzero real number s such that $s < r$ and if $0 < |y| < s$ then ABC is contained in R. Since $w = f(z)$ is univalent in R, $f(A) = f(x + ay)$, $f(B) = f(x + y)$, and $f(C) = f(x + by)$ are distinct points. By assumption, we have

$$\arg\left(\frac{f(x+ay) - f(x+y)}{f(x+by) - f(x+y)} \cdot \frac{f(x+by)}{f(x+ay)}\right) = \alpha$$

$$= \arg(\exp(i\alpha)) \quad (2.11)$$

for all y such that $0 < |y| < s$. Since $x \in R$ is arbitrarily fixed, we can set

$$h(y) = \frac{f(x+ay) - f(x+y)}{f(x+by) - f(x+y)} \cdot \frac{f(x+by)}{f(x+ay)}. \quad (2.12)$$

By (2.11) and (2.12) we obtain

$$\arg(h(y)) = \arg(\exp(i\alpha)) \quad (2.13)$$

for all y such that $0 < |y| < s$. Similar to the proof of Theorem 2.5, $h(y)$ is analytic at $y = 0$ if we define

$$h(0) = 1 + \frac{2\sqrt{3}\,k}{3 - k^2} i. \quad (2.14)$$

Hence, $h(y)$ is analytic in $|y| < s$. Furthermore, it is routine to check that $\arg(h(0)) = \alpha$. By (2.12) and the fact that $w = f(z)$ is univalent in R, we obtain that $h(y) \neq 0$ in $|y| < s$. Hence by Lemma 1.4 we have

$$h(y) = K\exp(i\alpha) \quad (2.15)$$

in $|y| < s$, where K is a positive real constant. Setting $y = 0$ in (2.14) and using (2.13), it yields

$$1 + \frac{2\sqrt{3}\,k}{3 - k^2} i = K\exp(i\alpha). \quad (2.16)$$

By (2.14) and (2.15) we obtain $K = 1$ and

$$h(y) = 1 + \frac{2\sqrt{3}\,k}{3 - k^2} i \quad (2.17)$$

in $|y| < s$.

Substituting (2.12) into (2.17) and removing the denominator in the resulting equality it follows that

$$(f(x+ay)-f(x+y))\cdot f(x+by)$$
$$-\left(1+\frac{2\sqrt{3}\,k}{3-k^2}i\right)(f(x+by)-f(x+y))\cdot f(x+ay)=0 \quad (2.18)$$

in $|y|<s$. Differentiate twice both sides of (2.18) with respect to y and setting $y=0$ yields

$$\frac{k(\sqrt{3}k^2+6ki+9\sqrt{3})}{k^2-3}\left(2(f'(x))^2-f(x)f''(x)\right)=0.$$

Since $k\in(0,+\infty)-\{\sqrt{3}\}$, we obtain

$$2(f'(x))^2-f(x)f''(x)=0$$

which implies that

$$N_f(x)=2.$$

Since $x\in R$ was arbitrarily fixed, we can replace x by a variable z, and we get

$$N_f(z)=2$$

in R. By the Identity Theorem, the above equality holds in C. Hence, f is a Möbius transformation of the form $\frac{u}{z+v}$, $u\neq 0$. The case $\alpha=\frac{\pi}{2}$ can be proved similarly by choosing suitable triangle which contains x inside and will be omitted. The proof is complete. ∎

Finally, we consider the following

Property H. Suppose that $w=f(z)$ is analytic and univalent in a nonempty simply connected domain R on the z-plane. Let α be an arbitrary fixed real number such that $\alpha\in(0,\pi)$. For three arbitrary distinct points a, b, and c in R satisfying

$$\arg\left(\frac{a-b}{c-b}\right)=\alpha,$$

we have

$$\arg\left(\frac{f(a)-f(b)}{f(c)-f(b)}\right)=\alpha.$$