THEOREM 2.8. *Let $w = f(z)$ be analytic and univalent in a nonempty connected domain R on the z-plane. Then f satisfies Property H iff f is a Möbius transformation of the form $uz + v$; $u \neq 0$.*

Proof of Theorem 2. Suppose that f is a Möbius transformation of the form $uz + v$, $u \neq 0$. Note that $f(\infty) = \infty$. Since $\frac{a-b}{c-b}$ is the cross-ratio of a, b, c, and $d = \infty$ and a Möbius transformation preserves the cross-ratio, we obtain

$$\frac{f(a) - f(b)}{f(c) - f(b)} = \frac{a-b}{c-b}$$

since $f(\infty) = \infty$. Therefore, $\arg(\frac{a-b}{c-b}) = \arg(\frac{f(a) - f(b)}{f(c) - f(b)})$ which implies that f satisfies Property H. Conversely, suppose that f satisfies Property H. Let T be a Möbius transformation of the form $\frac{s}{z+t}$ where $s \neq 0$. Let $g = T \circ f$. Let a, b, and c be three arbitrary distinct points in R such that $\arg(\frac{a-b}{c-b}) = \alpha$. Since f satisfies Property H, we obtain $\arg(\frac{f(a) - f(b)}{f(c) - f(b)}) = \alpha$. Since $\frac{a-b}{c-b}$ is the cross ratio of a, b, c, and $d = \infty$, it follows that $\frac{f(a) - f(b)}{f(c) - f(b)}$ is the cross ratio of $f(a)$, $f(b)$, $f(c)$, and $f(d) = \infty$ and since a Möbius transformation preserves the cross ratio, we obtain

$$\frac{g(a) - g(b)}{g(c) - g(b)} \cdot \frac{g(c)}{g(a)} = \frac{f(a) - f(b)}{f(c) - f(b)}$$

since $g(\infty) = 0$. From which it follows that

$$\arg\left(\frac{g(a) - g(b)}{g(c) - g(b)} \cdot \frac{g(c)}{g(a)}\right) = \arg\left(\frac{f(a) - f(b)}{f(c) - f(b)}\right) = \alpha.$$

By Theorem 1, g is a Möbius transformation of the form $\frac{u}{z+v}$, $u \neq 0$. Since $g = T \circ f$, we have $f = T^{-1} \circ g$ and it is easily seen that f is a Möbius transformation of the form $Uz + V$, where $U \neq 0$. ∎

Remark 2.2. In the results we obtained above, we may replace the use of argument by using modulus instead; for example, we can modify Property H by replacing $\arg(\frac{a-b}{c-b}) = \alpha$ and $\arg(\frac{f(a) - f(b)}{f(c) - f(b)}) = \alpha$ by $|\frac{a-b}{c-b}|$ and $|\frac{f(a) - f(b)}{f(c) - f(b)}|$, respectively. The proof will be almost the same except that we will need to use the Maximum Modulus Principle for analytic functions instead of using Lemma 1.4.

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A Note on the Characteristics of Möbius Transformations, II

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In this paper, we give some invariant characteristic properties of a certain class of Möbius transformations by means of their mapping properties. © 2001 Academic Press

Key Words: Möbius transformations; Schwarzian derivative; Newton derivative.

1. INTRODUCTION

Throughout the paper, unless otherwise stated, let $w = f(z)$ be a nonconstant meromorphic function on the complex plane C. It is well known that for $w = f(z)$ to be a Möbius transformation, it is necessary and sufficient that $w = f(z)$ satisfies Property A.

Property A. $w = f(z)$ maps circles in the z-plane onto circles in the w-plane, including straight lines among circles.

The following property of Möbius transformations is also well known:

THEOREM A. *$w = f(z)$ is a Möbius transformation iff $S_f(z) = 0$ for all $z \in \mathbb{C} - \{z : f'(z) = 0\}$, where $S_f(z) = (f''(z)/f'(z))' - \frac{1}{2}(f''(z)/f'(z))^2$, which is called the Schwarzian derivative of $f(z)$.*

Recently, in [1–4], Haruki and Rassias gave several new invariant characteristic properties of Möbius transformations by considering their mapping properties. In [5], we proved these results by using the fact that Möbius transformations preserve the cross ratio of four distinct points and gave several new invariant characteristic properties of Möbius transformations. For the sake of completeness, we shall give some definitions and results obtained in [5] which are related to the results in this paper.

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Definition 1.1 [5]. We define the Newton derivative of $f(z)$ as the first derivative of Newton's method of $f(z)$. In other words, we define the Newton derivative of $f(z)$ as

$$N_f(z) = \left(z - \frac{f(z)}{f'(z)}\right)' = \frac{f(z)f''(z)}{(f'(z))^2}.$$

Lemma 1.2 [5]. *$f(z)$ is a Möbius transformation of the form $\frac{u}{z+v}$, $u \neq 0$, iff $N_f(z) = 2$ for all $z \in \mathbb{C} - \{z : f'(z) = 0\}$.*

Property B [5]. Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R on the z-plane. Let α be an arbitrary fixed real number such that $\alpha \in (0, \pi)$. For three arbitrary distinct points a, b, and c in R satisfying

$$\arg\left(\frac{a-b}{c-b}\right) = \alpha,$$

we have

$$\arg\left(\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}\right) = \alpha.$$

Theorem 1.3 [5]. *Let $w = f(z)$ be analytic and univalent in a nonempty domain R on the z-plane. Then f satisfies Property B iff f is a Möbius transformation of the form $\frac{u}{z+v}$, $u \neq 0$.*

Remark 1. To prove the "sufficiency" part of Theorem 1.3, we show that $N_f(z) = 2$ for all $z \in \mathbb{C} - \{z : f'(z) = 0\}$, which implies that f is a Möbius transformation of the form $\frac{u}{z+v}$, $u \neq 0$, by Lemma 1.2.

In the next section, we will give some characterization of Möbius transformations of the form $\frac{uz+v}{z}$ where $v \neq 0$.

2. RESULTS

We first give the following result, which is similar to Theorem A and Lemma 1.2 in the previous section:

Lemma 2.1. *$f(z) = \frac{uz+v}{z}$, where $v \neq 0$ iff $f''(z)/f'(z) = -\frac{2}{z}$ for all $z \in \{z : f'(z) \neq 0\}$.*

Proof. Let $f(z) = \frac{uz+v}{z}$, where $v \neq 0$. It is straightforward to check that $f''(z)/f'(z) = -\frac{2}{z}$. Conversely, let $f''(z)/f'(z) = -\frac{2}{z}$ for all $z \in \{z : f'(z) \neq 0\}$. Then we obtain

$$\frac{1}{f'(z)} df'(z) = -\frac{2}{z} dz,$$

which implies that

$$\ln f'(z) = -\ln z^2 + \ln c$$

for some nonzero complex constant c. Thus

$$f'(z) = \frac{c}{z^2}$$

and we have

$$f(z) = -\frac{c}{z} + d$$

for some complex constant d. This completes the proof. ∎

In order to prove our results, we also need the following well-known result:

LEMMA 2.2. *If $f(z)$ and $g(z)$ are analytic functions in a nonempty domain R and $f(z)g(z) \neq 0$ in R and also $\arg(f(z)) = \arg(g(z))$ holds in R, then $f(z) = Kg(z)$ in R, where K is a positive real constant.*

We now consider the following.

Property C. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha \in (0, \pi)$. Let a, b, and c be three distinct points in R satisfying

$$\arg\left(\frac{a-b}{c-b} \cdot \frac{c}{a}\right) = \alpha.$$

Then

$$\arg\left(\frac{f(a)-f(b)}{f(c)-f(b)}\right) = \alpha.$$

THEOREM 2.3. *Let $w = f(z)$ be analytic and univalent on a nonempty domain R, where $0 \notin R$. Then $w = f(z)$ satisfies Property C iff $f(z) = \frac{uz+v}{z}$, where $v \neq 0$.*

Proof. Suppose that $f(z) = \frac{uz+v}{z}$, where $v \neq 0$. Let a, b, and c be three distinct points in R satisfying

$$\arg\left(\frac{a-b}{c-b} \cdot \frac{c}{a}\right) = \alpha.$$

Since $\frac{a-b}{c-b} \cdot \frac{c}{a}$ is the cross ratio of a, b, c, and 0 and the cross ratio is invariant under Möbius transformations, we obtain

$$\frac{a-b}{c-b} \cdot \frac{c}{a} = \frac{f(a)-f(b)}{f(c)-f(b)}$$

since $f(0) = \infty$. It follows that

$$\arg\left(\frac{f(a)-f(b)}{f(c)-f(b)}\right) = \arg\left(\frac{a-b}{c-b}\cdot\frac{c}{a}\right)$$
$$= \alpha.$$

In other words, f satisfies Property C.

Conversely, suppose that $w = f(z)$ satisfies Property C. Let α be an arbitrary real number such that $\alpha \in (0, \pi)$. Let z be an arbitrary point in R. Since $z \neq 0$ we can write $z = \frac{1}{a}$ for some $a \in \mathbb{C} - \{0\}$. Let $N_r(z)$ be an r circular neighborhood of z. Throughout the proof, let ABC be a triangle where $A = \frac{1}{a+\beta y}$, $B = \frac{1}{a+y}$, and $C = \frac{1}{a+\gamma y}$, where $\beta = -\frac{1}{2} - (\sqrt{3}/2)ki$, $\gamma = -\frac{1}{2} + (\sqrt{3}/2)ki$, $k > 0$, and y is some nonzero complex number. Without loss of generality, we let

$$k = \sqrt{\frac{3(1-\cos\alpha)}{1+\cos\alpha}}.$$

Since $\alpha \in (0, \pi)$, we have $k \in (0, +\infty)$. For example, if $\alpha = \frac{\pi}{3}$, then $k = 1$. It follows that $\arg(\frac{A-B}{C-B}\cdot\frac{C}{A}) = \alpha$. Since R is a nonempty connected domain, there exists a nonzero real number s such that $s < r$ and if $0 < |y| < s$ then ABC is contained in $N_r(z)$. Since $w = f(z)$ is univalent in R, $f(A) = f(\frac{1}{a+\beta y})$, $f(B) = f(\frac{1}{a+y})$, and $f(C) = f(\frac{1}{a+\gamma y})$ are distinct points. By assumption, we have

$$\arg\left(\frac{f(\frac{1}{a+\beta y}) - f(\frac{1}{a+y})}{f(\frac{1}{a+\gamma y}) - f(\frac{1}{a+y})}\right) = \alpha \tag{2.1}$$
$$= \arg(\exp(i\alpha))$$

for all y such that $0 < |y| < s$. Since $z \in R$ is arbitrarily fixed, we can set

$$h(y) = \frac{f(\frac{1}{a+\beta y}) - f(\frac{1}{a+y})}{f(\frac{1}{a+\gamma y}) - f(\frac{1}{a+y})}. \tag{2.2}$$

By (2.1) and (2.2), we obtain

$$\arg(h(y)) = \arg(\exp(i\alpha)) \tag{2.3}$$

for all y such that $0 < |y| < s$. By Riemann's Theorem, $h(y)$ will be analytic at $y = 0$ if we define

$$h(0) = \frac{3 - k^2 + (2\sqrt{3}k)i}{3 + k^2}. \tag{2.4}$$

Hence, $h(y)$ is analytic in $|y| < s$. Furthermore, it is routine to check that $\arg(h(0)) = \alpha$. By (2.2) and the fact that $w = f(z)$ is univalent in R, we obtain that $h(y) \neq 0$ in $|y| < s$. Hence, by (2.3) and Lemma 2.2, we have

$$h(y) = K \exp(i\alpha) \tag{2.5}$$

in $|y| < s$, where K is a positive real number. Setting $y = 0$ in (2.5) and using (2.4) we obtain

$$\frac{3 - k^2 + (2\sqrt{3}k)i}{3 + k^2} = K \exp(i\alpha). \tag{2.6}$$

By (2.5) and (2.6), we get

$$h(y) = \frac{3 - k^2 + (2\sqrt{3}k)i}{3 + k^2} \tag{2.7}$$

in $|y| < s$. By substituting (2.2) into (2.7) and removing the denominator in the resulting equality, it follows that

$$\begin{aligned} f\left(\frac{1}{a + \beta y}\right) - f\left(\frac{1}{a + y}\right) \\ = \left(\frac{3 - k^2 + (2\sqrt{3}k)i}{3 + k^2}\right)\left(f\left(\frac{1}{a + \gamma y}\right) - f\left(\frac{1}{a + y}\right)\right) \end{aligned} \tag{2.8}$$

in $|y| < s$. Differentiating twice both sides of (2.8) with respect to y and then setting $y = 0$ yields

$$\left(\frac{-3k^2 + (3\sqrt{3}k)i}{2a^4}\right)\left(2af'\left(\frac{1}{a}\right) + f''\left(\frac{1}{a}\right)\right) = 0. \tag{2.9}$$

It follows from (2.9) that

$$\frac{f''(\frac{1}{a})}{f'(\frac{1}{a})} = -\frac{2}{(\frac{1}{a})}$$

or $f''(x)/f'(x) = -\frac{2}{x}$. Since $x \in R$ is arbitrarily fixed, it follows that $f''(z)/f'(z) = -\frac{2}{z}$ for all $z \in R$. By the Identity Theorem, $f''(z)/f'(z) = -\frac{2}{z}$ holds for all z such that $f'(z) \neq 0$. Therefore, by Lemma 2.1, $w = f(z)$ is a Möbius transformation of the form $f(z) = \frac{uz+v}{z}$, $v \neq 0$. This completes the proof. ∎

Next, we consider the following.

Property D. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha \in (0, \pi)$. Let a, b, and c be three distinct points in R satisfying

$$\arg\left(\frac{a-b}{c-b} \cdot \frac{c}{a}\right) = \alpha.$$

Then

$$\arg\left(\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}\right) = \alpha.$$

For Möbius transformations satisfying Property D, we have the following result.

THEOREM 2.4. *Let $w = f(z)$ be analytic and univalent on a nonempty domain R, where $0 \in R$. Then $w = f(z)$ satisfies Property D if and only if $f(z) = \frac{uz}{z+v}$, where $u, v \neq 0$.*

Proof. Suppose first that f is a Möbius transformation of the form $\frac{uz}{z+v}$, $u, v \neq 0$. Note that $f(0) = 0$. Let a, b, and c be three distinct points in R satisfying

$$\arg\left(\frac{a-b}{c-b} \cdot \frac{c}{a}\right) = \alpha.$$

Since $\frac{a-b}{c-b} \cdot \frac{c}{a}$ is the cross ratio of a, b, c, and $d = 0$ and the cross ratio is preserved under the Möbius transformation, we obtain

$$\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)} = \frac{a-b}{c-b} \cdot \frac{c}{a}$$

since $f(0) = 0$. From this it follows that $\arg(\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}) = \arg(\frac{a-b}{c-b} \cdot \frac{c}{a}) = \alpha$, which implies that $w = f(z)$ satisfies Property D.

Conversely, suppose that $w = f(z)$ satisfies Property D. Let T be a Möbius transformation of the form $\frac{uz+v}{v}$, where $v \neq 0$. Let $g = T \circ f$. Let a, b, and c be three distinct points in R satisfying $\arg(\frac{a-b}{c-b} \cdot \frac{c}{a}) = \alpha$. Then $\arg(\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}) = \alpha$. Since $\frac{a-b}{c-b} \cdot \frac{c}{a}$ is the cross ratio of a, b, c, and $d = 0$, $\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}$ can be considered as the cross ratio of $f(a)$, $f(b)$, $f(c)$, and $f(d) = 0$. It follows that

$$\frac{g(a)-g(b)}{g(c)-g(b)} = \frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}$$

since $g(0) = \infty$. Thus we obtain

$$\arg\left(\frac{g(a)-g(b)}{g(c)-g(b)}\right) = \arg\left(\frac{f(a)-f(b)}{f(c)-f(b)}\cdot\frac{f(c)}{f(a)}\right) = \alpha.$$

By Theorem 2.3, g is a Möbius transformation of the form $\frac{sz+t}{z}$, where $t \neq 0$. Since $g = T \circ f$, we have $f = T^{-1} \circ g$ and we conclude that f is a Möbius transformation of the form $\frac{Uz}{z+V}$, $U, V \neq 0$. ∎

In what follows we shall denote $A = \arg(\frac{a-b}{c-b} + \frac{c-b}{a-b})$, $B = \arg(\frac{a-b}{c-b} - \frac{c-b}{a-b})$, $C = \arg(\frac{a-b}{c-b}\cdot\frac{c}{a} + \frac{c-b}{a-b}\cdot\frac{a}{c})$, $D = \arg(\frac{a-b}{c-b}\cdot\frac{c}{a} - \frac{c-b}{a-b}\cdot\frac{a}{c})$, $E = \arg(\frac{f(a)-f(b)}{f(c)-f(b)}\cdot\frac{f(c)}{f(a)} + \frac{f(c)-f(b)}{f(a)-f(b)}\cdot\frac{f(a)}{f(c)})$, $F = \arg(\frac{f(a)-f(b)}{f(c)-f(b)}\cdot\frac{f(c)}{f(a)} - \frac{f(c)-f(b)}{f(a)-f(b)}\cdot\frac{f(a)}{f(c)})$, $G = \arg(\frac{f(a)-f(b)}{f(c)-f(b)} + \frac{f(c)-f(b)}{f(a)-f(b)})$, and $H = \arg(\frac{f(a)-f(b)}{f(c)-f(b)} - \frac{f(c)-f(b)}{f(a)-f(b)})$. We now state the following.

Property E. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = 0$ or π. Let a, b, and c be three distinct points in R satisfying $A = \alpha$. Then $E = \alpha$.

Property F. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Let a, b, and c be three distinct points in R satisfying $B = \alpha$. Then $F = \alpha$.

Property G. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = 0$ or π. Let a, b, and c be three distinct points in R satisfying $A = \alpha$. Then $G = \alpha$.

Property H. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Let a, b, and c be three distinct points in R satisfying $B = \alpha$. Then $H = \alpha$.

Property I. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = 0$ or π. Let a, b, and c be three distinct points in R satisfying $C = \alpha$. Then $E = \alpha$.

Property J. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Let a, b, and c be three distinct points in R satisfying $D = \alpha$. Then $F = \alpha$.

Property K. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = 0$ or π. Let a, b, and c be three distinct points in R satisfying $C = \alpha$. Then $G = \alpha$.

Property L. Let $w = f(z)$ be analytic and univalent on a nonempty domain R. Let $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Let a, b, and c be three distinct points in R satisfying $D = \alpha$. Then $H = \alpha$.

The following results give some more characterizations of certain classes of Möbius transformations where the proof is a slight modification of the proof of Theorem 2.5 in [5] and will be omitted.

THEOREM 2.5. *Let $w = f(z)$ be analytic and univalent on a nonempty domain R where $0 \in R$. Then $w = f(z)$ satisfies Property E or F (or Property G or H) if and only if $f(z) = \frac{u}{z+v}$, where $u \neq 0$ (or $f(z) = uz + v$, $u \neq 0$).*

THEOREM 2.6. *Let $w = f(z)$ be analytic and univalent on a nonempty domain R where $0 \in R$. Then $w = f(z)$ satisfies Property I or J (or Property K or L)) if and only if $f(z) = \frac{uz}{z+v}$, where $u, v \neq 0$ (or $f(z) = \frac{uz+v}{z}$, $v \neq 0$).*

Remark 2. In the results we obtained above, we may replace the use of argument by using modulus instead; for example, we can modify Property D by replacing $\arg(\frac{a-b}{z-b} \cdot \frac{z}{a}) = \alpha$ and $\arg(\frac{f(a)-f(b)}{f(z)-f(b)} \cdot \frac{f(z)}{f(a)}) = \alpha$ with $|\frac{a-b}{z-b}|$ and $|\frac{f(a)-f(b)}{f(z)-f(b)}|$, respectively. The proof will be almost the same except that we will need to use the Maximum Modulus Principle for analytic functions instead of using Lemma 2.2.

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DYNAMICS OF NEWTON'S FUNCTIONS OF BARNA'S POLYNOMIALS

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We define Barna's polynomials as real polynomials with all real roots of which at least four are distinct. In this paper, we study the dynamics of Newton's functions of such polynomials. We also give the upper and lower bounds of the Hausdorff dimension of exceptional sets of these Newton's functions.

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1. Introduction. Newton's method is a well-known iterative method used to locate the roots of functions. Barna, [1, 2, 3, 4], proved that for a real polynomial $P(x)$ with only simple real roots of which at least four are distinct, the exceptional set of initial points of its Newton's function $N(x)$ (the set of $x \in \mathbb{R}$ such that $N^j(x)$ does not converge to any root of P, where $N^j(x)$ denotes the jth iterate of N) is homeomorphic to a Cantor subset of $\mathbb{R}$ which has the Lebesgue measure zero. Wong [7], generalized this result to real polynomials having all real roots (not necessarily simple) of which at least four are distinct (which will be called Barna's polynomials) by using a symbolic dynamics approach. In this paper, we will investigate the symbolic dynamics of Newton's functions of Barna's polynomials. Furthermore, we give the upper and lower bounds of the Hausdorff dimension of the exceptional sets.

2. Symbolic dynamics of Newton's functions

Definition 2.1. A real polynomial with all real roots of which at least four are distinct is called a *Barna's polynomial*. Thus $P(x)$ is a Barna's polynomial if and only if

$$P(x) = c\prod_{i=1}^{n}(x-r_i)^{m_i}, \tag{2.1}$$

where c is a nonzero real constant, $r_1 < r_2 < \cdots < r_n$, $n \geq 4$, and $m_i \geq 1$ for all $1 \leq i \leq n$.

Definition 2.2. The Newton's function $N_f(x)$ of a function $f(x)$ is defined as

$$N_f(x) = x - \frac{f(x)}{f'(x)}, \tag{2.2}$$

where $f'(x)$ is the derivative of $f(x)$.

Let $P(x) = c\prod_{i=1}^{n}(x - r_i)^{m_i}$ be a Barna's polynomial and $N_P(x)$ be the Newton's function of P. The following are well-known properties of N_P (see [1, 2, 3, 4, 7]).

(a) For each $i = 1, 2, \ldots, n-1$, there exists $c_i \in (r_i, r_{i+1})$ such that c_i is a zero of $P'(x)$, the derivative of $P(x)$, and $\{c_1, c_2, \ldots, c_{n-1}\}$ is exactly the set of all zeros of P' which are not zeros of P.

(b) In each of the intervals $(-\infty, c_1)$ and $(c_{n-1}, +\infty)$, N_P has exactly one critical point. If they are denoted by a_1 and a_n, respectively, then $a_1 \in [r_1, c_1)$ and $a_n \in (c_{n-1}, r_n]$. Moreover, N_P is monotone increasing on $(-\infty, a_1]$ and $[a_n, +\infty)$ and monotone decreasing on $[a_1, c_1)$ and $(c_{n-1}, a_n]$.

(c) In each (c_{i-1}, c_i), $2 \le i \le n-1$, N_P has two critical points s_1^i and s_2^i, where $s_1^i < s_2^i$. If r_i is a multiple root of N_P, then $s_1^i < r_i < s_2^i$. If r_i is a simple root, then either $r_i = s_1^i$, or $r_i = s_2^i$. The function N_P is monotone increasing on (s_1^i, s_2^i) and monotone decreasing on (c_{i-1}, s_1^i) and (s_2^i, c_i). Moreover, N_P is monotone increasing on (c_{i-1}, s_1^i) and monotone decreasing on (s_2^i, c_i).

(d) $\lim_{x \to c_i^-} N_P(x) = -\infty$ and $\lim_{x \to c_i^+} N_P(x) = +\infty$, for all $1 \le i \le n-1$.

(e) $\lim_{k \to \infty} N_P^k(x) = r_1$ for all $x \in (-\infty, c_1)$ and $\lim_{k \to \infty} N_P^k(x) = r_n$ for all $x \in (c_n, +\infty)$.

(f) For each $i = 2, 3, \ldots, n-1$, the interval (c_{i-1}, c_i) contains exactly one period-two cycle of N_P, say, at $\{\alpha_i, \beta_i\}$ where $\alpha_i < s_1^i < s_2^i < \beta_i$. Also $N_P'(a) < -1$, $N_P'(b) < -1$, and $\lim_{k \to \infty} N_P^k(x) = r_i$ for all $x \in (\alpha_i, \beta_i)$.

Definition 2.3. Let $P(x)$ and $N_P(x)$ be as above. The exceptional set Λ of N_P is defined as the complement of the set of real numbers x such that $N_P^j(x) = \infty$ for some $j \ge 0$ or $\lim_{k \to \infty} N_P^k(x) = r_i$ for some $1 \le i \le n$.

Remark 2.4. Note that Λ consists of points where N_P^k is well defined for each $k \in \mathbb{N}$ and never converge to any r_i.

Since our main interest is on the set Λ, those points which are not in Λ together with their preimages will be removed from $\mathbb{R}$. From this we have the following result on period-two cycle of Newton's function.

Proposition 2.5. *The function N_P has a period-two cycle at $\{\alpha, \beta\}$ such that $c_1 < \alpha < \alpha_2$ and $\beta_{n-1} < \beta < c_{n-1}$.*

Proof. Since $(-\infty, c_1)$ and (c_{n-1}, ∞) are not in Λ, we remove these sets together with their preimages. Let $y_0 = N_P^{-1}(c_1)$ such that $y_0 \in (\beta_{n-1}, c_{n-1})$. Then $(y_0, +\infty) \subseteq \Lambda$ and we remove this interval. Next let $z_1 = N_P^{-1}(y_0)$ such that $z_1 \in (c_1, \alpha_2)$. Then $(-\infty, z_1) \subseteq \Lambda$ and we remove this interval. Applying this procedure repeatedly we get two sequences of points $\{y_i\}_{i=1}^{\infty}$ and $\{z_j\}_{j=1}^{\infty}$ where

$$\begin{gathered} y_i = N_P^{-1}(z_i) \in (\beta_{n-1}, c_{n-1}), \\ \beta_{n-1} < \cdots < y_i < \cdots < y_2 < y_1 < y_0 < c_{n-1}, \\ z_j = N_P^{-1}(y_{j-1}) \in (c_1, \alpha_2), \\ c_1 < z_1 < z_2 < \cdots < z_j < \cdots < \alpha_2. \end{gathered} \tag{2.3}$$

Thus $\lim_{i \to \infty} y_i = \beta$ and $\lim_{j \to \infty} z_j = \alpha$ exist. As a result,

$$N_P(\beta) = N_P(\lim_{i \to \infty} y_i) = \lim_{i \to \infty} N_P(y_i) = \lim_{i \to \infty} N_P(N_P^{-1}(z_i)) = \lim_{i \to \infty} z_i = \alpha. \tag{2.4}$$

Similarly, we have $N_P(\alpha) = \beta$. Since $c_1 < \alpha \le \alpha_2$ and $\beta_{n-1} \le \beta < c_{n-1}$, we get $\alpha \ne \beta$. Finally, $\alpha \ne \alpha_2$ because $N_P(\alpha) = \beta \ge \beta_{n-1} > \beta_2$. This completes the proof of the proposition. □

For each $i = 2,3,\dots,n-1$, we have $N_P((c_{i-1},\alpha_i]) = [\beta_i,+\infty)$, and $N_P([\beta_i,c_i)) = (-\infty,\alpha_i]$. Thus there exist $t_i \in (c_{i-1},\alpha_i]$ and $u_i \in [\beta_i,c_i)$ such that $N_P(t_i) = \beta$ and $N_P(u_i) = \alpha$. Then $N_P((c_{i-1},t_i]) = [\beta,+\infty)$ and $N_P([u_i,c_i)) = (-\infty,\alpha]$. Denote the $2n-4$ intervals

$$[t_2,\alpha_2],[\beta_2,u_2],[t_3,\alpha_3],[\beta_3,u_3],\dots,[t_{n-1},\alpha_{n-1}],[\beta_{n-1},u_{n-1}] \tag{2.5}$$

by $I_1,I_2,\dots,I_{2n-4}$, respectively, and let $I = \bigcup_{i=1}^{2n-4} I_i$. With a similar approach used by Wong [7], we shall define the transition matrix V associated to N_P. This matrix V will determine the symbolic dynamics of N_P. Let $V = (v_{ij})$ be a $(2n-4)\times(2n-4)$ matrix of zeros and ones defined by

$$v_{ij} = \begin{cases} 1 & \text{if } I_i \cap N_P^{-1}(I_j) \ne \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{2.6}$$

for $i,j \in \{1,2,\dots,2n-4\}$. From this definition and properties of N_P, it is easily seen that V is a $(2n-4)\times(2n-4)$ matrix built from the following 2×2 matrices:

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}. \tag{2.7}$$

In fact V can be interpreted as an $(n-2)\times(n-2)$ matrix of matrices as follows:
(1) $V_{ii} = J$ for $1 \le i \le n-2$,
(2) $V_{ij} = M$ for $1 \le i \le n-3$, for $j > i$,
(3) $V_{ij} = N$ for $2 \le i \le n-2$, for $j < i$.
For example, if $n = 6$, then the matrix V has the form

$$\begin{bmatrix} J & M & M & M \\ N & J & M & M \\ N & N & J & M \\ N & N & N & J \end{bmatrix}. \tag{2.8}$$

With the same technique in [7], V is irreducible and we can show that N_P restricted to Λ is conjugate to the one-sided shift map σ on the set Σ_{2n-4}^V where

$$\Sigma_{2n-4}^V = \{s = s_0 s_1 \cdots s_n \cdots \in \Sigma_{2n-4} \mid v_{s_i s_{i+1}} = 1 \ \forall i \ge 0\} \tag{2.9}$$

is the symbolic sequences space consisting of $2n-4$ symbols (cf. [6]).

Remark 2.6. From [6], we have $\operatorname{card}(\operatorname{Per}_k \sigma) = \operatorname{Tr}(V^k)$, where $\operatorname{card}(\operatorname{Per}_k \sigma)$ denotes the number of points of period k of the shift map σ and $\operatorname{Tr}(V^k)$ is the trace of V^k.

Remark 2.7. By MATHEMATICA, we compute that $\operatorname{Tr}(V^k) = (n-2)^k + (-1)^k(n-2)$ where V is the transition matrix associated to Newton's function of a Barna's polynomial with n distinct real roots.

We summarize this section as follows.

Theorem 2.8. *Let $P(x) = c\prod_{i=1}^{n}(x-r_i)^{m_i}$ be a Barna's function and N_P the Newton's function of P. Let Λ be the exceptional set of N_P. Then Λ is a Cantor subset of $\mathbb{R}$ and N_P restricted to Λ is conjugate to the one-sided shift map on Σ^V_{2n-4}.*

Remark 2.9. There is some difference between our proof of Theorem 2.8 and the proof of a similar result by Wong in [7]. In our proof we use the fact that the exceptional set Λ lies between the period-two cycle $\{\alpha,\beta\}$ as stated in Proposition 2.5 and hence we can explicitly define the transition matrix V.

3. Hausdorff dimension of exceptional sets. Let Λ be the exceptional set of Newton's function of a Barna's polynomial with n distinct real roots. In this section, we give the upper and lower bounds of the Hausdorff dimension of Λ. The technique we will use here is similar to the one used in [5]. We first note that N_P^{-1} has $n-2$ branches $N_{P,i}^{-1}$ where $N_{P,i}((c_i,c_{i+1})) = \mathbb{R}$ for all $1 \le i \le n-2$. We will write $N^{-k}_{s_0s_1\cdots s_{k-1}}$ for the inverse N_P^{-k} using specific branches $N^{-1}_{P,s_0}, N^{-1}_{P,s_1},\dots,N^{-1}_{P,s_{k-1}}$. Let the interval I be the same as in the previous section. Then I has $n-2$ preimages under N_P each in the interval (c_{i-1},c_i), $2 \le i \le n-1$. For each $k \ge 1$, we have

$$N_P^{-k}(I) = \bigcup_{s_0,s_1,\dots,s_{k-1}=1}^{n-2} I_{s_0s_1\cdots s_{k-1}}, \tag{3.1}$$

where $I_{s_0s_1\cdots s_{k-1}} = N^{-k}_{s_0s_1\cdots s_{k-1}}(I)$. Let $\Lambda_k = \{x \mid N_P^k(x) \in I\}$. Then $\Lambda_k = N_P^{-k}(I)$ and $\Lambda = \bigcap_{k\ge 0}\Lambda_k$. Define

$$\begin{aligned} m_k &= \min\{|N_P'(x)| \mid x \in \Lambda_k\},\\ m &= \min\{|N_P'(x)| \mid x \in \Lambda\},\\ M_k &= \max\{|N_P'(x)| \mid x \in \Lambda_k\},\\ M &= \max\{|N_P'(x)| \mid x \in \Lambda\}. \end{aligned} \tag{3.2}$$

Remark 3.1. For each $k \ge 1$, $M_k \ge M_{k+1}$, and $m_k \le m_{k+1}$ since $\Lambda_k \supset \Lambda_{k+1}$.

We now state and prove the result on the estimation of the Hausdorff dimension of Λ.

Theorem 3.2. $\ln(n-2)/\ln M \le \dim\Lambda \le \ln(n-2)/\ln m$.

Proof. For each $k \ge 1$, let S_k and L_k be the lengths of the smallest and largest intervals in Λ_k, respectively. For each $k \ge 0$, we get

$$-M_k\left|I_{s_0s_1\cdots s_k}\right| \le -M_{k+1}\left|I_{s_0s_1\cdots s_k}\right| \le \int_{I_{s_0s_1\cdots s_k}} N'(x)\,dx = -\left|I_{s_1\cdots s_k}\right| \le -S_k. \tag{3.3}$$

Hence, $|I_{s_0s_1\cdots s_k}| \ge S_k/M_k$. By iterating, we get $|I_{s_0s_1\cdots s_{k+p-1}}| \ge S_k/(M_k)^p$. Similarly, we have $|I_{s_0s_1\cdots s_{k+p-1}}| \le L_k/(m_k)^p$. Since Λ is compact, any covering $\{U_i\}$ of Λ can be refined to a finite cover, where each element of this cover contains exactly one $I_{s_0s_1\cdots s_{k+p-1}}$

for some sufficiently large p. Then we get

$$\begin{aligned}\sum |U_i|^\alpha &\ge \sum_{n_0,\dots,n_{k+p-1}=1}^{n-2} \left|I_{n_0n_1\cdots n_{k+p-1}}\right|^\alpha \ge \sum_{n_0,\dots,n_{k+p-1}=1}^{n-2} \left(\frac{S_k}{(M_k)^p}\right)^\alpha \\ &= (S_k)^\alpha \frac{(n-2)^{k+p}}{(M_k)^{\alpha p}} = (S_k)^\alpha (n-2)^k \left(\frac{n-2}{(M_k)^\alpha}\right)^p.\end{aligned} \tag{3.4}$$

If $\alpha < \ln(n-2)/\ln M_k$, then this diverges as $p \to \infty$, that is, as the covering gets smaller. Thus $\dim\Lambda \ge \ln(n-2)/\ln M_k$. By letting $k \to \infty$, we have $\dim\Lambda \ge \ln(n-2)/\ln M$. Similarly, for a given $\epsilon > 0$ and for some sufficiently large p, there exists a covering $\{U_i\}_{i=1}^{(n-2)^{k+p}}$ of Λ such that each element of the cover contains exactly one interval $I_{n_0n_1\cdots n_{k+p-1}}$ and $|U_i|^{1+\epsilon} \le |I_{n_0n_1\cdots n_{k+p-1}}|$. Thus

$$\begin{aligned}\sum |U_i|^{(1+\epsilon)\alpha} &\le \sum_{n_0,\dots,n_{k+p-1}=1}^{n-2} \left|I_{n_0n_1\cdots n_{k+p-1}}\right|^\alpha \\ &\le \sum_{n_0,\dots,n_{k+p-1}=1}^{n-2} \left(\frac{L_k}{(m_k)^p}\right)^\alpha \\ &= (L_k)^\alpha (n-2)^k \left(\frac{n-2}{(m_k)^\alpha}\right)^p\end{aligned} \tag{3.5}$$

and this goes to zero as $p \to \infty$ if $\alpha > \ln(n-2)/\ln m_k$. Consequently, $\dim\Lambda \le (1+\epsilon)\ln(n-2)/\ln m_k$. By letting $k \to \infty$ and $\epsilon \to 0$ we have $\dim\Lambda \le \ln(n-2)/\ln m$. □

Example 3.3. Let $P(x) = (x+1)(x+2)(x-1)(x-2)$ be a Barna's polynomial and $N_P(x) = (3x^4 - 5x^2 - 4)/(4x^3 - 10x)$ be the Newton's function of P. Then N_P has three period-two cycles approximately at

$$\begin{aligned}\{x_0, x_1\} &= \{-1.5435941, 1.5435941\}, \\ \{x_2, x_3\} &= \{-1.4790145, -0.3142616\}, \\ \{x_4, x_5\} &= \{0.3142616, 1.4790145\}.\end{aligned} \tag{3.6}$$

These are the only period-two cycles by Remark 2.4. From Proposition 2.5, we obtain $\{x_0, x_1\}$ by removing the sequence of points which are the successive preimages of -2 and 2. Since $(N_P^2)'(x_0) = (N_P^2)'(x_1) > 1$, $\{x_0, x_1\} \in \Lambda$. Hence, in order to find the maximum and minimum values of $|N'|$ we must also consider the values of $|N_P'|$ at the preimages of x_0 and x_1. By computation, $x_6 = N_P^{-1}(x_0) = -0.2965502$ and $x_7 = N_P^{-1}(x_1) = 0.2965502$. Since $N_P(x)$ is an odd function, we get, by computation,

$$\begin{aligned}|N_P'(x_0)| &= |N_P'(x_1)| = |N_P'(x_6)| = |N_P'(x_7)| = 3.8985101, \\ |N_P'(x_2)| &= |N_P'(x_5)| = 10.2443746, \\ |N_P'(x_3)| &= |N_P'(x_4)| = 3.4016188.\end{aligned} \tag{3.7}$$

It follows that

$$\begin{aligned} m &= \min\{|N_P'(x)| \mid x \in \Lambda\} \approx 3.4016188, \\ M &= \max\{|N_P'(x)| \mid x \in \Lambda\} \approx 10.2443746. \end{aligned} \tag{3.8}$$

Consequently, we have

$$0.2979063 \approx \frac{\ln 2}{\ln M} \le \dim\Lambda \le \frac{\ln 2}{\ln m} \approx 0.5661804. \tag{3.9}$$

Remark 3.4. Let $P(x) = c\prod_{i=1}^{n}(x-r_i)^{m_i}$ be a Barna's polynomial and let $N_P(x)$ be its Newton's function. Let $M(x) = kN_P(x/k)$, k is a nonzero real constant. Then M is the Newton's function of Barna's polynomial of the form $Q(x) = c_0(x-1)^{m_n}\prod_{i=1}^{n-1}(x-r_i/r_n)^{m_i}$ and M is conjugate to N_P via the map $h(x) = kx$. As a result, α is a periodic point of N_P if and only if $k\alpha$ is a periodic point of M and $N_P'(\alpha)$ is equal to $M'(k\alpha)$. Consequently, dynamics of M and N_P on their exceptional sets are the same and the Hausdorff dimensions of their exceptional sets are equal. As a result, it suffices to consider the dynamics of Newton's functions of Barna's polynomials which have 1 as the largest root.

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ผลงานวิจัยที่เสนอเพื่อการตีพิมพ์

JULIA SET OF BICRITICAL RATIONAL FUNCTION [†]

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ABSTRACT. We study the Julia sets of bicritical rational functions R of degree at least two with the completely invariant attracting Fatou component and give some necessary and sufficient conditions which imply that the Julia sets are Lakes of Wada continuums.

1. Introduction

Let $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational function of degree at least two and let $R^n : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be its nth iterate . *The Fatou set* $F(R)$ is a subset of $\bar{\mathbb{C}}$ consists of points at which the sequence of iterates $\{R^n\}$ is normal. The complement of $F(R)$ in $\bar{\mathbb{C}}$, denoted $J(R)$, is called *Julia set*.

By a *continuum*, we mean a non-empty connected compact metric space. A *subcontinuum* is a continuum as a subset of a metric space.

The Julia set $J(R)$ is a non-empty, perfect, and completely invariant closed set, namely $R^{-1}(J(R)) = J(R) = R(J(R))$. In this paper we further assume that $J(R)$ is connected, hence, a continuum. On the other hand, the Fatou set $F(R)$ is also a completely invariant open set, but possibly empty. In this paper,however, we are concerned with nonempty Fatou set. Therefore, the Julia set $J(R)$ is a proper subcontimuum of $\bar{\mathbb{C}}$. Each component U of $F(R)$ is called a Fatou component.

In [3], Morosawa shows that every Fatou component U is eventually periodic and all periodic Fatou component can be classified into five types, namely, component of immediate super-attractive basin, component of immediate attractive basin, component of immediate parabolic basin, Siegel disc or Herman ring. In [1], Beardon presents a topological picture of Fatou set.

We say that a continuum of the complex sphere $\bar{\mathbb{C}}$, is a *Lakes of Wada continuum* if it forms a common boundary of three or more open connected mutually disjoint sets.

Let R be a rational function of degree at least two. The *residual Julia set* $J_0(R)$ of R is the sets of points in $J(R)$ that do not lie on the boundary of any Fatou

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component. For example of rational functions which has residual Julia set, see [1] page 266. If there is a completely invariant Fatou component D under some iterate R^n, then $\partial D = J(R)$, hence $J_0(R)$ is empty.This always occurs for polynomials since any polynomial has a completely invariant Fatou component, namely the unbounded Fatou component.

In [5], Yeshun Sun and Chung-Chun Yang proved that if $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational function of degree $d \geq 3$ which has exactly two critical points and satisfies the conditions: $J(R)$ is a proper subcontinuum of $\bar{\mathbb{C}}$, $J_0(R)$ is empty, and there is no completely invariant Fatou component under the second iterate R^2, then the $J(R)$ is a Lakes of Wada continuum.

In [4], some necessary and sufficient conditions which imply that the Julia set of polynomials $P(z) = az^n + b, n \geq 2$ to be Lakes of Wada continuums are given.

In this paper, we extend this result by given some necessary and sufficient conditions which imply that the Julia set of rational functions with two critical points to be Lakes of Wada continuums.

We call a rational function with two critical points *bicritical rational function*. By conjugation, we may assume that two critical points are at 0 and ∞, in which bicritical rational function will be the form $\frac{az^n+b}{cz^n+d}$, see in [2]. In what follows, bicritical rational function will be rational function with two critical points at 0 and ∞.

2. Main results

Lemma 2.1. *Let R be a rational function with a completely invariant attracting Fatou component F_0. Then the following statements are equivalent:*

(a) There exists a component D of $F(R) \neq F_0$ with $\partial D = J(R)$.

(b) There exists a periodic component $D_0 \neq F_0$ of $F(R)$ such that $\partial D_0 = J(R)$.

(c) $J(R) = \bigcup_{i=1}^{M} \partial D_i$, where $\{D_i\}_{i=1}^{M}$ is the set of the immediate basins of all (super-) attracting cycles of R, the immediate basins of all rationally indifferent cycles of R, cycles of all Siegel discs of R and cycles of all Herman rings of R, except F_0.

Proof. (a) $\Rightarrow$ (b). Suppose that $J(R) = \partial D$ for some component $D \neq F_0$ of $F(R)$. Then by the no wandering domains theorem and the complete invariance of F_0, there exists a non-negative integer N such that $R^N(D) = D_0$ for some periodic component $D_0 \neq F_0$ of $F(R)$. Hence $\partial D_0 = \partial R^N(D) = R^N(\partial D) = J(R)$.

(b) $\Rightarrow$ (c). This is trivial.

(c) $\Rightarrow$ (a). See the first part of the proof of Theorem 3 of [3]. □

Lemma 2.2. *Let R be a bicritical rational function with a completely invariant attracting Fatou component F_0 containing 0. Assume that $\infty \in J(R)$. Then the following statements are equivalent:*
(a) There exists a bounded component $D \neq F_0$ of $F(R)$ with $\partial D = J(R)$.
(b) There exists a bounded periodic component $D_0 \neq F_0$ of $F(R)$ such that $\partial D_0 = J(R)$.
(c) $J(R) = \bigcup_{i=1}^{M} \partial D_i$, where $\{D_i\}_{i=1}^{M}$ is the set of the immediate basins of all (super-) attracting cycles of R, the immediate basins of all rationally indifferent cycles of R, cycles of all Siegel discs of R and cycles of all Herman rings of R, except F_0.

Proof. (a) $\Rightarrow$ (b). Suppose that $J(R) = \partial D$ for some bounded component $D \neq F_0$ of $F(R)$. Then by the no wandering domains theorem and the complete invariance of F_0, there exists a non-negative integer N such that $R^N(D) = D_0$ for some bounded periodic component $D_0 \neq F_0$ of $F(R)$. Hence $\partial D_0 = \partial R^N(D) = R^N(\partial D) = J(R)$.
(b) $\Rightarrow$ (c). This is trivial.
(c) $\Rightarrow$ (a). See the first part of the proof of Theorem 3 of [3]. □

We say that a rational function is *hyperbolic* if $\overline{\bigcup_{n=0}^{\infty} R^n(C(R))} \cap J(R) = \emptyset$, where $C(R)$ is the set of all critical points of R. This occurs if and only if each critical point of R has the forward orbit that accumulates at a (super-)attracting cycle of R (see [2] page 90).

Remark 2.3. Let R be a rational function of deg $R = d \geq 2$. If R is hyperbolic, then for each completely invariant Fatou component containing critical point is a (super-) attracting basin.

Remark 2.4. Let R be a bicritical rational function of deg $R = d \geq 2$ with the completely invariant attracting Fatou component F_0 containing 0.
Then R is hyperbolic if and only if $\infty \in F(R)$ and F_∞ is a (super-) attracting basin.

Remark 2.5. Let R be a bicritical rational function of deg $R = d \geq 2$ with the completely invariant attracting Fatou component F_0 containing 0.
If $\infty \in F_0$, then $J(R)$ is a Cantor set, that is $J(R)$ is disconnected.

Lemma 2.6. *Let R be a rational function.*
If R is hyperbolic, then R^q is hyperbolic for any positive integer q.

Proof. See Lemma 3.2 of [4]. □

Lemma 2.7. *Let R be a hyperbolic rational function with degree at least two. Assume that $J(R)$ is connected. If there exists a forward invariant component D_0 of $F(R)$ with $\partial D_0 = J(R)$, then D_0 is completely invariant.*

Proof. See Lemma 2 of [3]. □

Proposition 2.8. *Let R be a rational function with deg $R \geq 2$. Assume that $J(R)$ is connected and $F(R)$ has infinitely many components. If R is hyperbolic, then $F(R)$ has at most one component such that its boundary coincides with the Julia set, and such a component is periodic.*

Proof. See Proposition 3.4 of [4]. □

Corollary 2.9. *Let R be a bicritical rational function, deg $R = d \geq 2$ with the completely invariant Fatou component F_0 containing 0. Assume that $F(R)$ has infinitely many components and $J(R)$ is connected. If R is hyperbolic, then F_0 is the only component of $F(R)$ such that $\partial F_0 = J(R)$.*

Proof. This follows from Proposition 2.8. □

Theorem 2.10. *Let R be a bicritical rational function, deg $R = d \geq 2$ with the completely invariant Fatou component F_0 containing 0. Assume that $F(R)$ has infinitely many components and $J(R)$ is a subcontinuum of $\hat{\mathbb{C}}$.*
Then the following statements are equivalent:
(a) $J(R)$ is a Lakes of Wada continuum.
(b) There exists a Fatou component $D \neq F_0$ with $\partial D = J(R)$.
(c) $F(R)$ has no immediate basin of (super-) attracting cycles except F_0, and $J(R) = \partial D_0$ for some periodic Fatou component $D_0 \neq F_0$ which lies in an immediate basin of rationally indifferent cycle or a cycle of Siegel discs of R.
(d) $J(R) = \bigcup_{i=1}^{M} \partial D_i$, where $\{D_i\}_{i=1}^{M}$ is the set of all periodic Fatou components, except F_0.

Proof. (a) $\Rightarrow$ (b). This is obvious.
(b) $\Rightarrow$ (a). Suppose that there exists a Fatou component D_0 such that $\partial D_0 = J(R)$. If $\infty \notin D_0$, then $R(D_0)$ does not contain critical values of R. From this, we obtain by the Riemann-Hurwitz formula that there exist Fatou components $D_1, D_2, ..., D_{d-1}$ such that $R|_{D_k} : D_k \to R(D_0)$ is a homeomorphism for all $k = 0, ..., d-1$. For each k, let $S_k : R(D_0) \to D_k$ be the inverse of $R|_{D_k}$, these are all distinct branches of $R^{-1}|_{R(D_0)}$. Then we get that $S_k R|_{D_0}(z) = \omega^k z$ for all $z \in D_0, k = 0, ..., d-1$ where ω is an nth

primitive root of the unity. For each k, let $\omega^k : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a Möbious transformation which is defined by $\omega^k(z) = \omega^k z$. Then $S_k R|_{D_0} = \omega^k|_{D_0}$ for all k, and so we obtain that $\omega^k(D_0) = D_k$ for all k. We note that for each k, $R\omega^k = R$ from which it is easy to see that $\omega^k(J(R)) = J(R)$ for all k. Thus $\partial D_0 = \partial D_1 = ... = \partial D_{d-1} = \partial F_0 = J(R)$ and it follows that $J(R)$ is a Lakes of Wada continuum. Assume $\infty \in D_0$. Since $\partial D = J(R)$, $\partial R(D) = J(R)$. If $D_0 = R(D_0)$, then by the Riemann-Hurwitz formula, we get that $R|_{D_0} : D_0 \to D_0$ is an d-fold map. Hence D_0 is completely invariant, and so $F(R)$ has only two components. This is a contradiction. Thus $D_0 \neq P(D_0)$. From $\partial F_0 = J(R)$, $\partial D_0 = J(R)$ and $\partial R(D_0) = J(R)$, it follows that $J(R)$ is a Lakes of Wada continuum.

(b) $\Rightarrow$ (c). Assume that there exists a Fatou component D such that $\partial D = J(R)$. Then by Lemma 2.1, $J(R) = \partial D_0$ for some a periodic component $D_0 \neq F_0$ of $F(R)$. So by Proposition 2.9, R is not hyperbolic. Suppose that $F(R)$ has an immediate basin of (super-) attracting cycle of R, say F_1, which $F_1 \neq F_0$. By Theorem 9.3.1 in [3], F_1 must contains a critical point of R, so F_1 contains ∞. Thus F_0 and F_1 are immediate basins of (super-) attracting cycle of R and 0 accumulate to F_0, ∞ accumulate to F_1. This implies that R is hyperbolic, which is a contradiction.So $F(R)$ has no immediate basin of (super-) attracting cycle, except F_0. Thus D_0 either lies in an immediate basin of rationally indifferent cycle or cycle of Siegel discs of R.

(c) $\Rightarrow$ (d). This is trivial.

(d) $\Rightarrow$ (b). This follows immediately from Lemma 2.1.

The proof is complete. □

Theorem 2.11. *Let R be a bicritical rational function, deg $R = d \geq 2$ with the completely invariant Fatou component F_0 containing 0. Assume that $\infty \in J(R)$, $F(R)$ has infinitely many components and $J(R)$is a subcontinuum of $\bar{\mathbb{C}}$.*

Then the following statements are equivalent:

(a) $J(R)$ is a Lakes of Wada continuum.

(b) There exists a bounded Fatou component $D \neq F_0$ with $\partial D = J(R)$.

(c) $F(R)$ has no bounded immediate basin of (super-) attracting cycles except F_0, and $J(R) = \partial D_0$ for some periodic Fatou component $D_0 \neq F_0$ which lies in an immediate basin of rationally indifferent cycle or a cycle of Siegel discs of R.

(d) $J(R) = \bigcup_{i=1}^{M} \partial D_i$, where $\{D_i\}_{i=1}^{M}$ is the set of all periodic Fatou components, except F_0.

Proof. (a) $\Rightarrow$ (b). This is obvious.

(b) ⇒ (a). Suppose that there exists a bounded Fatou component D_0 such that $\partial D_0 = J(R)$. Since $0 \notin D_0$, then $R(D_0)$ does not contain critical values of R. From this, we obtain by the Riemann-Hurwitz formula that there exist bounded Fatou components $D_1, D_2, ..., D_{d-1}$ such that $R|_{D_k} : D_k \to R(D_0)$ is a homeomorphism for all $k = 0, ..., d-1$. For each k, let $S_k : R(D_0) \to D_k$ be the inverse of $R|_{D_k}$, these are all distinct branches of $R^{-1}|_{D_0}$. Then we get that $S_k R|_{D_0}(z) = \omega^k z$ for all $z \in D_0, k = 0, ..., d-1$ where ω be an nth primitive root of the unity. For each k, let $\omega^k : \mathbb{C} \to \mathbb{C}$ is a Möbious transformation which is defined by $\omega^k(z) = \omega^k z$. Then $S_k R|_{D_0} = \omega^k|_{D_0}$ for all k, and so we obtain that $\omega^k(D_0) = D_k$ for all k. We note that for each k, $R\omega^k = R$, from this, it is easy to say that $\omega^k(J(R)) = J(R)$ for all k. Thus $\partial D_0 = \partial D_1 = ... = \partial D_{d-1} = \partial F_0 = J(R)$. It follows that $J(R)$ is a Lakes of Wada continuum.

(b) ⇒ (c). Assume that there exists a bounded Fatou component $D \neq F_0$ such that $\partial D = J(R)$.Then by Lemma 2.2, $J(R) = \partial D_0$ for some a bounded periodic component $D_0 \neq F_0$ of $F(R)$. So by Proposition 2.9, R is not hyperbolic. Suppose that $F(R)$ has an bounded immediate basin of (super-) attracting cycle of R, say F_1 which $F_1 \neq F_0$. By Theorem 9.3.1 in [3], F_1 must contains a critical point of R, so F_1 contains ∞. This implies that $\infty \in F(R)$, which is a contradiction. So $F(R)$ has no bounded immediate basin of (supper-)attracting cycle, except F_0. Thus D_0 either lies in an immediate basin of rationally indifferent cycle or cycle of Siegel discs of R.

(c) ⇒ (d). This is trivial.

(d) ⇒ (b). This follows immediately from Lemma 2.2.

The proof is complete. □

Theorem 2.12. *Let R be a bicritical rational function, deg $R = d \geq 2$ with the completely invariant Fatou component F_0 containing 0. Assume that $\infty \in J(R)$ and $J(R)$is a subcontinuum of $\bar{\mathbb{C}}$. Then the following statements are equivalent:*

(a) $J(R)$ is a Lakes of Wada continuum.

(b) There exists a bounded Fatou component $D \neq F_0$ with $\partial D = J(R)$.

(c) $J(R)$ coincides with the union of boundaries of cycles of all Siegel discs of R.

(d) The infinite critical point, ∞, lies in the union of boundaries of cycles of all Siegel discs of R and the forward obit $\{R^n(\infty) : n \geq 0\}$ of ∞ is dense in $J(R)$.

Proof. Let $\{D_i\}_{i=1}^M$ be the set of all Siegel discs of R. Then $\bigcup_{i=1}^{M} \partial D_i \subseteq \overline{\{R^n(\infty) : n \geq 0\}}$. As $\infty \in J(R)$, $\{D_i\}_{i=1}^M$ and the set of all bounded Fatou components, except F_0, coincide.

(a) ⇒ (b). This is trivial.

(b) ⇒ (a). Suppose that (*b*) holds. We will show that $F(R)$ has infinitely many components. Assume that $F(R)$ has only two components. Then D is completely invariant under R. From this by the Riemann-Herwitz formula that D contains ∞. This is a contradiction. Hence $F(R)$ has infinitely many components. So, by Theorem 2.11, $J(R)$ is a Lakes of Wada continuum.

(b) ⇔ (c). This is follows from Lemma 2.2.

(c) ⇒ (d). Assume that $J(R) = \bigcup_{i=1}^{M} \partial D_i$. Then $J(R) \supseteq \overline{\{R^n(\infty) : n \geq 0\}} \supseteq \bigcup_{i=1}^{M} \partial D_i = J(R)$, so $J(R) = \overline{\{R^n(\infty) : n \geq 0\}}$.

(d) ⇒ (c). Suppose that (*d*) holds. By the assumption of the set $\{D_i\}_{i=1}^{M}$, we get that $R\left(\bigcup_{i=1}^{M} \partial D_i\right) = \bigcup_{i=1}^{M} \partial D_i$. Hence $J(R) = \overline{\{R^n(\infty) : n \geq 0\}} = \bigcup_{i=1}^{M} \partial D_i$. □

Remark 2.13. Under the condition that $\infty \in J(R)$, the rational function in Theorem 2.12 is not hyperbolic.

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SOLUTIONS OF FUNCTIONAL EQUATIONS $f \circ S = S^k \circ f$

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1. Abstract

Let S be a Möbius transformation which has two fixed points, say a and b in $\mathbb{C}$. Without loss of generality we may assume that a is an attracting fixed point and b is a repelling fixed point of S. We are interested in finding solutions f of the following functional equation

$$f \circ S = S^k \circ f \tag{1}$$

where $k \geq 2$. We will show that for a given complex number α distinct from a and b, there exists a unique solution of (1) which fixes α, a, and b. We also show that the Julia sets of rational solutions of (1) are circles on the sphere.

2. Introduction

Halley's method, Newton's method of a given function $P(z)$ are defined respectively as follows

$$H(z) = z - \frac{P(z)}{P'(z) - \frac{P(z)P''(z)}{2P'(z)}},$$

$$N(z) = z - \frac{P(z)}{P'(z)}.$$

A successive approximation $S(z)$, of $P(z)$ can be obtained by setting $P(z) = 0$ and then write this equation as $z = S(z)$. In our case $P(z)$ is a quadratic polynomial with roots a and b such that $0 < |a| < |b|$, then $S(z) = \frac{-ab}{z-(a+b)}$ is a successive approximation of $P(z)$ that $z = a$ as a global attractor.

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In [3], the functional equation (1) where f is a rational function of degree k of the form

$$f(z) = \frac{a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0}{b_k z^k + b_{k-1} z^{k-1} + \cdots + b_1 z + b_0}, \tag{2}$$

where $a_i, b_j \in \mathbb{C}$, $(a_0, b_0) \neq (0, 0)$ was studied. The main results in [3] are as follows:

Theorem 2.1. *Let f_k be a rational solution of (1) of the form (2).*

(a) If $a_k \neq 0$, then

$$f_k = T_k \circ f_{0,k}$$

where $f_{0,k}(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}$, $T_k(z) = \frac{z - abb_k}{b_k z + (1-(a+b)b_k)}$ *and* $b_k \in \mathbb{C}$.

(b) If $a_k = 0$, and $a_{k-1} \neq 0$ then there is only one rational solution in this form for (1) and we can explicitly find such the solution.

(c) If $a_k = a_{k-1} = 0$, then there are no nonzero rational solutions for (1) of this form.

Conversely, if T is any mapping such that $T \circ S = S \circ T$, then $f_0 \circ T$ and $T \circ f_0$ are solutions of (1).

Remark 2.1. *When $k = 2$, $f_{0,2}(z)$ is the Newton's method for P and when $k = 3$, $f_{0,3}(z)$ is the Halley's method for P, where $P(z) = (z - a)(z - b)$.*

In [3], the rational solutions f of (1) are solved directly from a linear system of equations. In this paper, we study the functional equation (1) more analytically and we also describe the Julia sets of rational solutions of (1).

3. Main Results

Let S and f be as in the previous section. We have

Theorem 3.1. *For any $i, j \in \mathbb{N}$,*

$$f^i \circ S^j = S^{jk^i} \circ f^i. \tag{3}$$

Proof. For fix $i = 1$, let $P(j) = f \circ S^j = S^{jk} \circ f$. Then for $j = 2$,

$$\begin{aligned}(f \circ S) \circ S &= (S^k \circ f) \circ S \\ &= S^k \circ (f \circ S) \\ &= S^k \circ (S^k \circ f) \\ &= S^{2k} \circ f.\end{aligned}$$

This implies $P(2)$ holds. Assume that $P(N)$ holds. Then

$$\begin{aligned} f \circ S^{N+1} &= (f \circ S^N) \circ S \\ &= (S^{Nk} \circ f) \circ S \\ &= S^{Nk} \circ (f \circ S) \\ &= S^{Nk} \circ (S^k \circ f) \\ &= S^{N+1} \circ f \end{aligned}$$

which implies that $P(N+1)$ holds. Therefore $f \circ S^j = S^{jk} \circ f$ hold for all $j \in \mathbb{N}$. Similarly for a fixed $j \in \mathbb{N}$, we may show that $f^i \circ S^j = S^{jk^i} \circ f^i$ holds for all $i \in \mathbb{N}$. We conclude that $f^i \circ S^j = S^{jk^i} \circ f^i$ for all $i, j \in \mathbb{N}$. This completes the proof. ∎

Theorem 3.2. *Let f be a solution of (1). If $f(b) \neq a$ then a and b are fixed points of f.*

Proof. Firstly, we show that a, b are not poles of f. For if a was a pole of f, then $f(a) = \infty$. From (3) and for $i = 1$ we have

$$\infty = f(a) = f \circ S^j(a) = S^{jk}(f(a)) = s^{jk}(\infty).$$

this implies that $a = \infty$ or $b = \infty$ which is a contradiction. Thus a and b are not poles of f. From (3) if we take $i = 1$, then for $z \notin f^{-1}(b) \cup \{a, b\}$ we have, by continuity of f,

$$f\left(S^j(z)\right) = S^{jk}\left(f(z)\right).$$

Thus

$$\begin{aligned} f(a) &= f\left(\lim_{j \to +\infty} S^j(z)\right) = \lim_{j \to +\infty} f\left(S^j(z)\right) \\ &= \lim_{j \to +\infty} S^{jk}\left(f(z)\right) = a \end{aligned}$$

which implies that a is a fixed point of f. From (3) if we take $z = b$, then

$$f(b) = S^{jk}(f(b)).$$

As we assume that $f(b) \neq a$ we conclude that $f(b) = b$. This completes the proof. ∎

Remark 3.1. *Let f be a solution of (1) such that $f(b) \neq a$. Then a, b are super-attracting fixed points of f.*

Proof. Consider

$$f \circ S(z) = S^k \circ f(z)$$

by differentiate both sides we obtain

$$f'(a)S'(b) = S'(S^{k-1} \circ f(z)) \cdot S'(S^{k-2} \circ f(z)) \cdot \ldots \cdot S'(f(z)) \cdot f'(z).$$

For $z = a$,

$$f'(a) \cdot S'(a) = [S'(a)]^k \cdot f'(a)$$

and since $S'(a) \neq 0$, we conclude that $f'(a) = 0$. That is, a is a super-attracting fixed point of f. Similarly, we can show that $f'(b) = 0$. This completes the proof. ∎

Theorem 3.3. *For a given complex number $\alpha \neq a, b$. There exists a unique solution of (1) which fixes α, a and b.*

Proof. Let f and g be solutions of (1) which fix a, b and α. From (3), take $i = 1$ we have

$$f \circ S^j(\alpha) = S^{jk} \circ f(\alpha) = S^{jk}(\alpha)$$

and

$$g \circ S^j(\alpha) = S^{jk} \circ g(\alpha) = S^{jk}(\alpha).$$

Since $\alpha \neq a, b$ and a is a global attractor of S, $S^j(\alpha) \to a$ as $j \to \infty$. This implies that a is a limit point of $\{S^j(\alpha) : j \in \mathbb{N}\}$. As

$$\{S^j(\alpha) : j \in \mathbb{N}\} \subseteq \{z \in \overline{\mathbb{C}} : f(z) = g(z)\},$$

we have, by the Identity Theorem, $f \equiv g$ on $\overline{\mathbb{C}}$. Therefore, there is a unique solution of (1) which fixes α, a and b where $\alpha \neq a, b$. This completes the proof. ∎

Remark 3.2. *Let f be a solution of (1) which fixes a, b and $\alpha(\neq a, b)$. For all Möbius transformation which fixes a, b we may show that if $T(f(\alpha)) = \alpha$, then $T \circ S = S \circ T$.*

Theorem 3.4. *Let f be a solution of (1). Then $f \circ T$ and $T \circ f$ are solutions of (1) where T is any transformation which satisfies $S \circ T = T \circ S$.*

Proof. Put $g = f \circ T$ and $h = T \circ f$. Then

$$\begin{aligned} g \circ S &= (f \circ T) \circ S \\ &= f \circ (T \circ S) \\ &= f \circ (S \circ T) \\ &= (f \circ S) \circ T \\ &= (S^k \circ f) \circ T \\ &= S^k \circ (f \circ T) \\ &= S^k \circ g. \end{aligned}$$

That is, g is a solution of (1). And

$$\begin{aligned} h \circ S &= (T \circ f) \circ S \\ &= T \circ (f \circ S) \\ &= T \circ (S^k \circ f) \\ &= (T \circ S) \circ (S^{k-1} \circ f) \\ &= S \circ (T \circ S) \circ (S^{k-2} \circ f) \\ &\vdots \\ &= S^k \circ (T \circ f) \\ &= S^k \circ h. \end{aligned}$$

That is, h is a solution of (1). This completes the proof. ∎

Theorem 3.5. *Let f and g be solutions of (1) which f fixes a, b and α ($\alpha \neq a, b$) and g fixes a, b and β ($\beta \neq a, b$). Then g can be expressed in the form*

$$g = T \circ f$$

where T is a Möbius transformation which fixes a, b and $T(f(\beta)) = \beta$.

Proof. It is easy to see that $S \circ T = T \circ S$. By Theorem 3.4, $T \circ f$ is a solution of (1). Since

$$T \circ f(a) = T(a) = a$$

$$T \circ f(b) = T(b) = b$$

$$T \circ f(\beta) = T(f(\beta)) = \beta,$$

this implies $T \circ f$ is a solution of (1) which fixes a, b and β. By Theorem 3.3, we obtain $g = T \circ f$. ∎

Theorem 3.6. *Let f be a solution of (1) which fixes a, b. Then f is a rational function.*

Proof. First, we consider $S(z) = \lambda z, \lambda \neq 0$. Let g be a solution of (1) which fixes $0, \infty$ and S is defined as above. So

$$(*) \qquad g(\lambda z) = \lambda^k g(z).$$

Set

$$g(z) = \Sigma_{n=1}^{\infty} a_n z^n$$

where $a_n \in \mathbb{C}, \forall n$. We have

$$g(\lambda z) = \Sigma_{n=1}^{\infty} a_n \lambda^n z^n$$

and

$$\lambda^k g(z) = \Sigma_{n=1}^{\infty} a_n \lambda^k z^n.$$

From (*), we obtain

$$\Sigma_{n=1}^{\infty} a_n \lambda^n z^n = \Sigma_{n=1}^{\infty} a_n \lambda^k z^n$$
$$\Sigma_{n=1}^{\infty} a_n (\lambda^n - \lambda^k) z^n = 0$$
$$a_n(\lambda^n - \lambda^k) = 0, \forall n$$

For $n \neq k, a_n = 0$, so that $g(z) = a_k z^k$. This implies that g is a rational function.

Now, we consider S which fixes a, b. Then S is conjugate to a map $z \mapsto \lambda z, \lambda \neq 0$ by the Möbius transformation that send $z = a$ to 0 and $z = b$ to ∞, namely

$$M(z) = \frac{-z + a}{-z + b}.$$

Let f be a solution of (1) which fixes a, b. Then f is conjugate to g with the same Möbius transformation. Therefore f is a rational function. This completes the proof. ∎

Proposition 3.1. *Let f be a solution of (1) which fixes a and b. For $\alpha \neq a, b$, if $f(\alpha) = \alpha$, then f is conjugate to a map $\left(\frac{-\alpha+b}{-\alpha+a}\right)^{k-1} z^k$.*

Proof. Assume that f is a solution of (1) which fixes a and b. Given $\alpha \neq a, b$. From Theorem 3.6, f is conjugate to a map g where $g(z) = Kz^k, \exists K \neq 0$ by the Möbius transformation that send $z = a$ to 0 and $z = b$ to ∞, namely

$$M(z) = \frac{-z + a}{-z + b}.$$

Since $f(\alpha) = \alpha$, so

$$M^{-1}gM(\alpha) = \alpha.$$
$$g(M(\alpha)) = M(\alpha).$$

That is $M(\alpha)$ is a fixed point of g. But g has fixed points at $0, \infty$ and $(k-1)^{th}$ roots of $\frac{1}{K}$. Then

$$M(\alpha) = \left(\frac{1}{K}\right)^{\frac{1}{k-1}}$$
$$\frac{-\alpha + a}{-\alpha + b} = \left(\frac{1}{K}\right)^{\frac{1}{k-1}}$$
$$\left(\frac{-\alpha + a}{-\alpha + b}\right)^{k-1} = \frac{1}{K}$$
$$K = \left(\frac{-\alpha + b}{-\alpha + a}\right)^{k-1}.$$

This implies that f is conjugate to a map $\left(\frac{-\alpha+b}{-\alpha+a}\right)^{k-1} z^k$. This completes a proof. ∎

Example 3.1. *For $k = 2$, N is the rational solution of (1) which fixes a, b and ∞.*
Let f be a solution of (1) which f fixes a, b and α ($\alpha \neq a, b$).
Then $f = T \circ N$ where T is a Möbius transformation which fixes a, b and $T(f(\alpha)) = \alpha$.

Definition 3.1. *Let f be a rational function and let f^i denote the ith iterate of f.* ***The Julia set*** *$J(f)$ of f is defined as follows:*

$$J(f) = \overline{\mathbb{C}} - F(f),$$

where $F(f) = \{z \in \overline{\mathbb{C}} : \{f^i\}_{i=0}^{\infty}$ is a normal family in a neighborhood of $z\}$. The set $F(f)$ is called ***the Fatou set*** *of f.*

Theorem 3.7. *The Julia set of the rational solutions of (1) are circles on the sphere.*

Proof. In [3], we know that

$$f_k(z) = \frac{a(z-b)^k - b(z-a)^k}{(z-b)^k - (z-a)^k}$$

is the rational solution of (1). Let f be a rational solution of (1) which fixes a, b and α ($\alpha \neq a, b$). Theorem 3.5 shows that $f = T_k \circ f_k$ where T_k is a Möbius transformation which fixes a, b and $T_k(f_k(\alpha)) = \alpha$. For $k \geq 2$, the function f_k is conjugate to a map $w \mapsto w^k$ and T_k is conjugate to a map $w \mapsto Kw^k$ where $|K| < 1$ by the Möbius transformation that send $w = 0$ to a and $w = \infty$ to b, namely

$$M(w) = \frac{bw - a}{w - 1}.$$

The inverse M^{-1}, of M is given by

$$M^{-1}(z) = \frac{-z + a}{-z + b}.$$

This implies f is conjugate to the map Kz^{k^2} where $|K| < 1$. So we obtain that $J(f)$ is a circle on the sphere. ∎

We now consider the case when the Möbius transformation has exactly one fixed point in the complex plane. Let R be a Möbius transformation which has only one fixed point, say $a \in \mathbf{C}$ (so a is the global attractor of R). We are interested in finding solutions f of the following functional equation

$$f \circ R = R^k \circ f \tag{4}$$

where $k \geq 2$ and R is defined as above. We have

Remark 3.3. *Let f be a solution of (4). Then a is a fixed point of f.*

Remark 3.4. *For a given complex number $\alpha \neq a$. There exists a unique solution of (4) which fixes α and a.*

Remark 3.5. *Let f be a solution of (4). If T is any Möbius transformation such that $T \circ R = R \circ T$, then $f \circ T$ and $T \circ f$ are solutions of (4).*

Theorem 3.8. *Let f be a solution of (4). Then f is conjugate to a map*

$$kz + P(e^{\frac{-2\pi i}{c}z}) + Q(e^{\frac{2\pi i}{c}z})$$

where P, Q are meromorphic functions.

Proof. Without loss of generality we may assume that $R(z) = z+c, c \neq 0$. Assume that g is a solution of (4). Then $g(z+c) = g(z) + kz$. Note that $g(z+c) = g(z) + kz$ if and only if $g(z) = kz + H(z)$ where $H(z) = g(z) - kz$. So $H(z) = H(z+c)$, that is, H is periodic. Since rational functions cannot have a period, this implies that $H(z) = P(e^{\frac{-2\pi i}{c}z}) + Q(e^{\frac{2\pi i}{c}z})$ where P, Q are meromorphic functions.

Now, we consider S which fixes a. Then S is conjugate to a map $z \mapsto z + c, c \neq 0$ by the Möbius transformation that send $z = a$ to 0, namely

$$M(z) = \frac{1}{-z + a}.$$

Let f be a solution of (4). Then f is conjugate to g with the same Möbius transformation. This completes the proof. ∎

Example 3.2. *$f(z) = 2z + e^{-z} + e^{z}$ is a solution of the functional equation $f \circ R = R^2 \circ f$ where $R(z) = z + 2\pi i$.*

REFERENCES

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สัญญาเลขที่ RSA/04/2544

โครงการ การวิจัยพลศาสตร์เชิงดิสครีตของฟังก์ชันตรรกยะบางฟังก์ชัน พลศาสตร์เชิงดิสครีตของฟังก์ชันในคลาส A_e และการวิจัยคุณสมบัติไม่แปรเปลี่ยนของการแปลงเชิงเส้นคู่

รายงานสรุปการเงินในรอบ 36 เดือน

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ชื่อหัวหน้าโครงการ ดร.ปิยะพงศ์ เนียมทรัพย์

รายงานในช่วงตั้งแต่วันที่ 1 ธันวาคม 2543 ถึง 30 พฤศจิกายน 2546

รายจ่าย

หมวด (ตามสัญญา)	รายจ่ายสะสมจากรายงานครั้งก่อน	ค่าใช้จ่ายงวดปัจจุบัน	รวมรายจ่ายสะสมจนถึงงวดปัจจุบัน	งบประมาณที่ตั้งไว้ (รวมสะสมจนถึงปัจจุบัน)	คงเหลือ(หรือเกิน)
1. ค่าตอบแทน	450,000	90,000	540,000	540,000	-
2. ค่าจ้าง	90,000	18,000	108,000	108,000	-
3. ค่าใช้สอย	135,860.45	4,520	140,380.45	232,000	91,619.55
4. ค่าวัสดุ	195,831.75	-	195,831.75	120,000	เกิน 75,831.75
5. ค่าครุภัณฑ์	30,973	-	30,973	30,000	เกิน 973
6. ค่าเดินทางไปต่างประเทศ	30,000**	-	30,000**	50,000	20,000
รวม	932,665.20	112,520	1,045,185.20	1,080,000	34,814.80

หมายเหตุ **

** 1. เดินทางไปเสนอผลงาน The 10th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications, Busan, Korea, July 29 – August, 2002 เรื่อง Julia set of $az^n + b$ โดยได้รับงบประมาณบางส่วนจากคณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่

** 2. เดินทางไปร่วมประชุม International Congress of Mathematic 2002, Beijing, China, August 19 – 29, 2002 โดยได้รับงบประมาณบางส่วนจากผู้จัดการประชุม

จำนวนเงินที่ได้รับและจำนวนคงเหลือ

จำนวนเงินที่ได้รับ

งวดที่ 1	360,000	บาท เมื่อ 1 ธันวาคม 2543
งวดที่ 2	360,000	บาท เมื่อ 31 ธันวาคม 2544
งวดที่ 3	270,000	บาท เมื่อ 31 ธันวาคม 2545
ดอกเบี้ยครั้งที่ 1	-	บาท
ดอกเบี้ยครั้งที่ 2	-	บาท
ดอกเบี้ยครั้งที่ 3	-	บาท
ฯลฯ		
รวม	990,000	บาท (1)

ค่าใช้จ่าย

งวดที่ 1 (เดือนที่ 1 – 6) เป็นเงิน	149,799.50	บาท
งวดที่ 2 (เดือนที่ 7 – 12) เป็นเงิน	155,063.95	บาท
งวดที่ 3 (เดือนที่ 13 – 18) เป็นเงิน	186,100.50	บาท
งวดที่ 4 (เดือนที่ 19 – 24) เป็นเงิน	245,870.50	บาท
งวดที่ 5 (เดือนที่ 25 – 30) เป็นเงิน	195,830.75	บาท
งวดที่ 6 (เดือนที่ 31 – 36) เป็นเงิน	112,520.00	บาท
รวม	1,045,185.20	บาท (2)
จำนวนเงินคงเหลือ (1) – (2)	- 46,185.20	บาท

ลงนามหัวหน้าโครงการ..

(ดร.ปิยะพงศ์ เนียมทรัพย์)

วันที่.......10 มี.ค. ๔๘๖..........................