



รายงานวิจัยฉบับสมบูรณ์

โครงการ

**Modular Spectral Triples in
Non-commutative Geometry and
Physics**

**มอดูลาร์สเปกตรัลทริเปิ้ลใน เรขาคณิตแบบ
ไม่สลับที่และฟิสิกส์**

โดย

Paolo Bertozzini

26 พฤศจิกายน พ.ศ. 2548

สัญญาเลขที่ RSA4580030

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

Aknowledgments

The main investigator (P. B.) wish to thank first of all his collaborators and friends Dr. Roberto Conti and Dr. Wicharn Lewkeeratiyutkul, for the big effort and patience in order to carry on this research project.

Many thanks go also to the Thai Research Fund that has kindly supported this research and “waited some time” for the arrival of the this final report . . . and in particular to Khun Chonnapa for very kind cooperation in any matter related to the bureaucracy.

1 Abstract / บทคัดย่อ

Project Code: RSA4580030

(รหัสโครงการ)

Project Title: Modular Spectral Triples in Non-commutative Geometry and Physics
(ชื่อโครงการ) มอดูลาร์สเปกตรัลทริเปิลในเรขาคณิตแบบไม่สลับที่และฟิสิกส์

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Project Period: from 15 August 2002 to 14 August 2003 extended to 14 March 2004.
(ระยะเวลาโครงการ)

Inspired by the general goals of the proposed research project (to reinterpret Tomita-Takesaki modular theory in non-commutative geometry), we have worked out a definition of morphism and of category of A. Connes' spectral triples. As an application, we have constructed a model functor from the category of discrete groups equipped with a length function to our category of spectral triples.

A motivation for the introduction of these concepts comes from the consideration of non-commutative submanifolds and quotient manifolds in A. Connes spectral triples' context. This work opens the way to a systematic analysis of the notion of morphism of spectral triples, to be applied to different situations. Several variants of our notion of morphism are expected to be relevant for a complete categorical study of non-commutative geometry including a wide range of applications.

จากความพยายามในการอธิบายเรขาคณิตแบบไม่สลับที่ด้วยทฤษฎีของโทมิตะ-ทาเกซากิ ในงานนี้ เราได้ให้นิยามของสัจฐาน และประเภทของสเปกตรัลทริเปิลของคอนเนส ในการประยุกต์ เราได้สร้างตัวแบบฟังก์เตอร์จากประเภทของกรุ๊ปวิยัตต์ที่มีฟังก์ชันความยาว ไปยังประเภทของสเปกตรัลทริเปิล

แรงบันดาลใจสำหรับมโนคติเหล่านี้ได้มาจากการพิจารณาสับแมนิโฟลด์แบบไม่สลับที่ และแมนิโฟลด์ผลหารในบริบทของ สเปกตรัลทริเปิลของคอนเนสงานนี้ได้เปิดหนทางการวิเคราะห์อย่างเป็นระบบของแนวคิดเกี่ยวกับประเภทของสเปกตรัลทริเปิล เพื่อที่จะนำไปประยุกต์กับสถานการณ์ต่างๆ แนวคิดเกี่ยวกับสัจฐานในแบบต่างๆ คาดว่าจะเป็นสิ่งที่จำเป็นสำหรับการศึกษาที่สมบูรณ์สำหรับประเภทของเรขาคณิตแบบไม่สลับที่ ซึ่งรวมถึงการประยุกต์ในวงกว้าง

Keywords: Non-commutative Geometry, Spectral Triple, Morphism, Category.

(คำหลัก) เรขาคณิตแบบไม่สลับที่ สเปกตรัลทริเปิล สัจฐาน ประเภท

2 Executive Summary

2.1 Objectives

- Spectral construction of Space-Time from Tomita-Takesaki Modular theory;
- Discussing categorical structure of Connes' spectral triples;
- Construction and discussion of relevant model examples.

2.2 Results

In order to construct a “spectral space-time” from Tomita’s modular theory, we have been naturally led to the problem of “dimensional reduction” of the degrees of freedom of the theory i.e. to the possible definition of substructures in the context of A. Connes spectral triples. One way to implement such idea is by means of category theory.

In this direction, we have worked out the first definition of **morphism of spectral triples** in the framework of A. Connes’ non-commutative geometry and proposed a related notion of **category of spectral triples** including the “real” and “even” cases.

In order to assess the validity of our proposed definitions, we have constructed a model functor from the category of discrete groups equipped with a length function to the category of spectral triples.

An output of this research appears in the following paper:

P. Bertozzini, R. Conti, W. Lewkeeratiyutkul, **A Category of Spectral Triples and Discrete Groups with Length Function**, Osaka Journal of Mathematics (to appear).

The need to reduce the “degrees of freedom” passing from the algebra of observables (phase space) to a suitable subalgebra (configuration space), has also led us to approach the delicate problem of a canonical choice of an appropriate “non-commutative polarization” inside the observables’ algebra. This is somehow related to the solution of an open problem mentioned by Fröhlich and collaborators [FGR4]: how to pass from operational spectral data on a phase space to spectral triples on a configuration space.

As a warming up towards a solution of the latter problem, we started examining some algebraic issues related to the mutual relationships between Clifford and Toeplitz algebras in the context of Hilbert modules equipped with a real structure.

Two of our Master's students in Thammasat university (Miss Areeya Sararak and Miss Supaporn Theesoongnern) have been directly involved in this part of the project that has led to the preparation of their respective Master's theses [Sa, Th]¹. In the specific:

- Using the language of category theory, we have performed (with Miss Areeya) a detailed and unified study of the several notions of **complexification and decomplexification** starting from the well-known case of vector spaces and introducing new appropriate definitions in the case of algebras, modules over algebras and module-algebras.
- We have reviewed (with Miss Supaporn) the study of the “multilinear algebra” of “second quantization” with particular emphasis on its extensions to the case of “inner product modules” over involutive algebras and we have defined (by universal factorization methods) **Hermitian-Clifford algebras of inner product modules**, i.e. analogs of the classical Clifford algebra of orthogonal spaces in the case of modules equipped with an Hermitian (or anti-Hermitian) product. These algebras are deemed to be relevant in the discussion of second quantization (field theory) on non-commutative space-times.

3 Content of the Research

Part of this section is a slightly modified version of the our original research proposal.

3.1 Motivation

• Generalities

Non-commutative Geometry is the name of a very young and fast developing mathematical theory that is making use of Operator Algebras (itself a branch of Functional Analysis created by von Neumann in 1929 [V]) to find algebraic generalizations of most of the structures currently available in mathematics: algebraic, measurable, topological, differential, metric etc. Even if, from the algebraic point of view, mathematicians have been dealing with “non-commutative” algebraic structures since a relatively long time ago (for example under the form of Groups, Matrices in Linear Algebra and Hamilton's Quaternions), the first seeds of Non-commutative Geometry (i.e. the idea to substitute

¹Abstracts of the results obtained so far are attached as pdf files.

a commutative algebra with a non-commutative one) is strictly linked with physics' developments and can be traced back to the beginning of Quantum Mechanics in the form of Heisenberg's Matrix Mechanics in 1925, where noncommuting matrices (operators) take the place of classical commuting observables of a physical system (hence the fashionable name of Quantum Mathematics often cited in the recent literature).

The fundamental idea, implicitly used in Non-commutative Geometry is that of trading geometric spaces X of points, with algebras of (say complex-valued) functions $f : X \rightarrow \mathbb{C}$ and to use these algebras to "reconstruct" the original geometric space X as a derived entity (the spectrum of the algebra), a technique that appeared for the first time in the work of Gel'fand on abelian C^* -algebras in 1939 [G] (similar ideas are well-known and used also in Algebraic Geometry). Anyway, it is only starting in 1980 (see [C1]), with the extraordinary work of Alain Connes, the real founder of Non-commutative Geometry, that a complete systematic theory capable of describing differential and metric structures becomes available.

In order to develop non-commutative geometries, we usually proceed as follows: first we find a suitable way to "codify" or translate the geometric properties of a space X (topology, measure, metric . . .) in algebraic terms, using a commutative algebra of functions over X , then we try to see if this codification "survives" generalizing to the case of non-commutative algebras. The generalized properties are taken as axioms defining what a "non-commutative" (topological, measurable, differential, metric, . . .) space is, without referring to any underlying point space. Of course the process of generalization of the properties from the commutative to the non-commutative algebra case is highly non trivial and, as a result, several alternative possible axiomatizations arise in the non-commutative case, corresponding to a unique "commutative limit".

The general reference source for Non-commutative Geometry is Alain Connes' book [C3]. For a detailed study we suggest the recent textbook by Gracia-Bondia, Figueroa and Varilly [FGV]. Shorter introductions are contained in Landi's book [La], in the earlier papers by Coquereaux [Co1, Co2], Gracia-Bondia and Varilly [GV, Va] or, for the real beginner, in our seminar's notes [B].

Spectral Triples

A. Connes has developed a spectral notion of non-commutative spin manifold, that has been very effective in generalizing Atiyah-Singer index theory in several directions. Accordingly, he has started a deep theory of such non-commutative spaces that is customarily referred to as Non-commutative Geometry.

In this research work, we consider only some aspects of “metric” Non-commutative Geometry i.e. those parts of Non-commutative Geometry dealing with the non-commutative generalizations of notions such as Riemannian and Semi-Riemannian Manifolds. Furthermore, for the moment, we focus only on Connes’ Spectral Triple definition: there are in fact also several other alternative and interesting definitions of non-commutative geometries making use for example of Sauvageot’s Modules [S], Lipschitz Algebras (see N. Weaver [W]), Lipschitz seminorms and compact Lie groups ergodic actions (see the works by Rieffel [Ri1, Ri2, Ri3]) or modules of derivations (see Bratteli [Br], Madore [M] and references there).

Following A. Connes, “non-commutative manifolds”² are axiomatized by the following structure, called a (compact) Spectral Triple³:

A (compact) **Spectral Triple** $(\mathcal{A}, \mathcal{H}, D)$ is given by a unital C^* -algebra⁴ $\mathcal{A}$; a Hilbert space $\mathcal{H}$ on which the algebra $\mathcal{A}$ is represented by bounded operators $\pi(a) \in \mathcal{B}(\mathcal{H})$ for every $a \in \mathcal{A}$; and a (non-necessarily bounded) self-adjoint operator D , called the Dirac Operator, whose resolvent $(D - \lambda)^{-1}$ is compact for every $\lambda \in \mathbb{C} - \mathbb{R}$ and $[D, \pi(a)]_- \in \mathcal{B}(\mathcal{H})$, for every $a \in \mathcal{A}$, where $[x, y]_- := xy - yx$ denotes the commutator of x, y . The spectral triple is called **even** if there exists a Grading Operator, i.e. a bounded self-adjoint operator $\Gamma \in \mathcal{B}(\mathcal{H})$ such that:

$$\Gamma^2 = \text{Id}_{\mathcal{H}}; \quad [\Gamma, \pi(a)]_- = 0, \forall a \in \mathcal{A}; \quad [\Gamma, D]_+ = 0,$$

where $[x, y]_+ := xy + yx$ is the anticommutator of x, y . A spectral triple that is not even is called **odd**.

A spectral triple is **regular** if $\pi(a), [D, \pi(a)]_- \in \cap_{m=1}^{\infty} \text{Dom} \delta^m$, where $\delta(x) := [|D|, x]_-$.

The spectral triple is **n -dimensional** iff there exists an integer n such that the Dixmier trace of $|D|^{-n}$ is finite nonzero.

A spectral triple is **θ -summable** if $\exp(-tD^2)$ is a trace class operator for every $t > 0$.

²at least in the case of compact, finite dimensional, Riemannian, orientable, spin^c manifolds

³or K-cycle, in the earlier literature

⁴actually a unital pre- C^* -algebra is sufficient

A spectral triple is **real** if there exists an antiunitary operator $J : \mathcal{H} \rightarrow \mathcal{H}$ such that:

$$\begin{aligned} [\pi(a), J\pi(b^*)J^{-1}]_- &= 0, \forall a, b \in \mathcal{A}; \\ [[D, \pi(a)]_-, J\pi(b^*)J^{-1}]_- &= 0, \forall a, b \in \mathcal{A} \quad \textbf{first order condition}; \\ J^2 &= \pm \text{Id}_{\mathcal{H}}; \quad [J, D]_{\pm} = 0; \quad \text{and, only in the even case,} \quad [J, \Gamma]_{\pm} = 0, \end{aligned}$$

where the choice of $\pm$ in the last three formulas depends on the “dimension” n of the spectral triple modulo 8.

Three additional axioms: Finiteness, Orientability and Poincaré Duality must be added in order to recover spin^c manifolds in the commutative case. The first complete proof of such a theorem (Connes’ theorem), i.e. the fact that, in the case of a commutative C^* -algebra $\mathcal{A}$, a finite dimensional, regular, real spectral triple that satisfies the axioms of finiteness, orientability and Poincaré duality allows us to recover a compact, finite dimensional, orientable, spin^c manifold, has been given by A. Rennie in [R1].

Axioms for non-compact manifolds have been proposed by Connes and are under investigation by Rennie [R2] and also by Gracia-Bondia, Lizzi, Marmo and Vitale [GLMV]. A proposal for an axiomatic definition of spectral triples describing “non-commutative” compact, finite dimensional, Riemann manifold that are not necessarily spin^c has been recently presented by S. Lord in [L]. Furthermore several papers (see for example the works of Parfionov-Zapatrin [PZ], Kopf-Pasche [Ko, KP1, KP2] and especially the recent paper by Strohmaier [St]) have been dealing with the problem to find an axiomatization of “non-commutative” semi-Riemannian manifolds (whose importance should be clear from the point of view of physics, since space-time is described in general relativity by a Lorentzian manifold).

Modular Theory

The Modular Theory of von Neumann algebras has been created by M. Tomita in 1967 [T] and perfected by M. Takesaki around 1970 [Ta1]. It is a very deep theory that, to every von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ acting on a Hilbert space $\mathcal{H}$, and equipped with a generating (i.e. $\mathcal{M}u^\perp = \mathcal{H}$) and separating (i.e. $Au = 0 \Rightarrow A = 0$ when $A \in \mathcal{M}$) vector $u \in \mathcal{H}$, associates a one parameter unitary group $t \mapsto \Delta^{it}$ and an antilinear isometry J such that:

$$\begin{aligned} \Delta^{it}\mathcal{M}\Delta^{-it} &= \mathcal{M}, \quad \forall t \in \mathbb{R}; \\ J\mathcal{M}J &= \mathcal{M}' \quad \text{where } \mathcal{M}' \text{ is the commutant of } \mathcal{M}. \end{aligned}$$

More generally, given a von Neumann algebra $\mathcal{M}$ and a faithful, normal, semifinite weight (in particular a faithful, normal state) ω on the algebra $\mathcal{M}$, the modular theory allows to create a one parameter group of automorphism $t \mapsto \sigma_t^\omega$ of the algebra $\mathcal{M}$ such that, in the Gel'fand-Naimark-Segal representation π_ω induced by the weight ω , on the Hilbert space $\mathcal{H}_\omega$, the automorphism group σ^ω is implemented by a unitary one parameter group $t \mapsto \Delta_\omega^{it}$ and there is an antilinear isometry J_ω whose adjoint action maps $\pi_\omega(\mathcal{M})$ in its commutant i.e.:

$$\begin{aligned}\pi_\omega(\sigma_t^\omega(x)) &= \Delta_\omega^{it} \pi_\omega(x) \Delta_\omega^{-it}, \quad \forall x \in \mathcal{M}, \quad \forall t \in \mathbb{R}; \\ J_\omega \pi_\omega(x) J_\omega &\in \pi_\omega(\mathcal{M})', \quad \forall x \in \mathcal{M}.\end{aligned}$$

J_ω and Δ_ω are called respectively the modular conjugation and the modular operator induced by ω . We will call “modular generator” the operator $K_\omega := \log \Delta_\omega$ that is the generator of the unitary one parameter group $t \mapsto \Delta_\omega^{it}$. The modular automorphism group associated with ω is the only one parameter automorphism group that satisfies the Kubo-Martin-Schwinger (KMS) condition (see Kadison Ringrose vol. 2 [KR] for details) with respect to ω .

Tomita-Takesaki theory has been central in most of the rigorous formulations of quantum field theory and quantum statistical mechanics [H] [BR, vol. 2].

The most elementary background in functional analysis is covered for example in Sunder's book [Su]; general references in Operator Algebras, include Kadison-Ringrose [KR], Bing Ren [Bi], Bratteli-Robinson [BR, vol. 1] and Takesaki [Ta2]. For modular theory, the most complete book source is Strătilă [Str].

• Motivations and origins of the problem

The main goal of this project is to construct “spectral triple like objects” canonically induced by the modular theory of a von Neumann algebra with respect to a faithful normal state (or more generally a faithful, normal, semifinite weight). There are several different motivations to look for such a structure:

- a) First of all there is a purely mathematical interest in trying to define geometrical objects associated with von Neumann algebras and in particular factors and to classify them;
- b) Then there is an interesting possibility of using this construction to create a huge family of examples of non-commutative geometries: it is well-known that one of the major problems in non-commutative geometry is the small number of natural

examples available. On the other side, normal faithful states over von Neumann algebras are abundant!

- c) The final and main interest in developing such a project is directly related to physics and it is the hope that such a construction might cast some light on how to formulate a purely algebraic theory of quantum relativity. The solution to the problem of finding a coherent theory (often called quantum gravity) embracing both quantum mechanics and general relativity has eluded mathematical physicists since 1926. Some of the problems of the standard formulation of quantum field theory are due to the rigidity of the notion of the underlying space-time.

Non-commutative geometry provides us with the a “quantum” version of geometry that should play a role completely analogous to that of classical Lorentzian geometry in the formulation of General Relativity (as an example see the proposal of quantum space-time by Doplicher-Fredenhagen-Roberts [DFR]).

The tentative usage of Tomita-Takesaki modular theory in physics has a long history (see [H] and [CR]). The first to conjecture a deep relation between modular theory, supersymmetry and non-commutative geometry has been A. Connes himself as reported in Daniel Kastler’s paper in 1988 [K]. The topic received immediately considerable attention in the works of Jaffe, Lesniewski and Osterwalder [JLO1, JLO2, J] on supersymmetric KMS functionals defining K-cycles in the context of supersymmetric quantum field theories.

We initially got involved in this kind of research long time ago, around 1990, when we realized that non-commutative geometry was offering the possibility to define “quantized” space-time in a spectral way and that the only missing ingredient for a physical theory of quantum space-time, was a method to obtain spectral triples from the operationally defined physical concepts of states and observables. Motivated by the fact that in quantum physics we always deal with C^* -algebras $\mathcal{A}$ of observables and that states ω over $\mathcal{A}$ naturally provide representations $\pi_\omega : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\omega)$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$ by GNS construction, we started in 1994 to ask if also the information contained in the specification of the Dirac operator of a spectral triple could have been obtained from the knowledge of suitable state on the algebra of observables and in April 1995 we conjectured that the Tomita-Takesaki modular automorphisms group could be used for this purpose. When Connes’ definition of real spectral triples (making use of the modular conjugation operator!) appeared in print [C4], we became

completely convinced that the original intuition was on the right track. Between 1995 and 1997 we tried to work on the problem using the generator of the modular automorphism group as a “first order” operator (a sort of “non-supersymmetric square root” of the D’Alembertian of a manifold), but at that time no viable axiomatizations for “non-supersymmetric” non-commutative geometries (read manifolds that are not spin^c) were known, nor useful definitions of semi-Riemannian non-commutative manifolds. After 1997, we have been working on several other aspects partly related to the original problem: in particular its possible relations with Isham’s History Projection Operator Theory⁵ and Algebraic Quantum Field Theory⁶ (but also questions related to the foundational aspects of physics such as covariance and equivalence principle). In the meantime, other people have started to publish results that can partially help to overcome some of the technical difficult points in the original project, opening the way for a positive solution:

- Several possible variants of axiomatizations for spectral triples and their relations with supersymmetry have been the object of the investigations of J. Fröhlich, O. Grandjean and A. Recknagel in a series of papers [FGR1, FGR2, FGR3, FGR4] from 1996 to 1999;
- A clear detailed proof of the Connes’ theorem (eliminating unnecessary hypothesis on the algebra $\mathcal{A}$) has been available through the work of A. Rennie [R1] in 1999;
- An important axiomatization of Riemannian (non- spin^c manifolds) has been given by S. Lord [L] in 2000;
- A first approach to the definition of semi-Riemannian, non-commutative manifold, using Krein spaces⁷, has been developed by A. Strohmaier [St] in 2001.

It is in this context, that after discussing some of the ideas with Dr. Roberto Conti (during his visit to Thammasat University in August 2001), we proposed to Dr. Roberto Conti and to Dr. Wicharn Lewkeeratiyutkul to join us in the study of this topic under this project.

⁵See Saviddu [Sv] and Isham, Linden [I, IL] for a review.

⁶See the Araki’s [A] and Hagg’s [H] books for details.

⁷See J. Bognar [Bo] for the definitions.

3.2 Obtained Results

Motivated by physically meaningful examples (such as quantum mechanics of a particle on a manifold, see [FGR1, FGR2]) and further elaborating on it, we were led to conjecture the existence of a close relationship between modular theory and geometry of phase and configuration spaces respectively: the generator of the Tomita-Takesaki modular automorphism group of a suitable KMS-state, might define a first order operator on the algebra of observables of the physical system (phase space) and, when considered on an appropriate subalgebra (configuration space), the same modular generator should actually provide the information necessary to recover a second order operator (the Laplacian) whose square root is exactly the Dirac operator necessary in order to define the Connes' spectral triple associated to the configuration space of the system.

As an immediate consequence of this insight, in order to continue and complete our initial research project, it appeared necessary to consider in more precise terms the exact definitions of submanifold and quotient manifold in the non-commutative case. A bit surprisingly, we found that this problem, at least in the case of Connes' spectral triples approach to non-commutative geometry, had not been considered before.

We decided to fill the gap and chose to approach these issues in the context of category theory that is a powerful tool for expressing physical concepts related to covariance, such as “coordinate transformation” and “group of transformations”, in a mathematically sound form (see e.g. J. Baez [Ba]).

We have thus shifted our attention to uncover the categorical structure of spectral triples providing tentative definitions of “morphism”, “category of spectral triples”, and we have investigated some of their properties. In detail, given two spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, with $j = 1, 2$, a **morphism of spectral triples** is a pair

$$(\phi, \Phi) \in \text{Mor}_{\mathcal{S}}[(\mathcal{A}_1, \mathcal{H}_1, D_1), (\mathcal{A}_2, \mathcal{H}_2, D_2)],$$

$$(\mathcal{A}_1, \mathcal{H}_1, D_1) \xrightarrow{(\phi, \Phi)} (\mathcal{A}_2, \mathcal{H}_2, D_2),$$

where $\phi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is a $*$ -morphism between the pre-C*-algebras $\mathcal{A}_1, \mathcal{A}_2$ and $\Phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a bounded linear map in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ that “intertwines” the representations $\pi_1, \pi_2 \circ \phi$ and the Dirac operators D_1, D_2 :

$$\pi_2(\phi(x)) \circ \Phi = \Phi \circ \pi_1(x), \quad \forall x \in \mathcal{A}_1,$$

$$D_2 \circ \Phi = \Phi \circ D_1,$$

i.e. such that the following diagrams commute for every $x \in \mathcal{A}_1$:

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \\ D_1 \downarrow & \circlearrowleft & \downarrow D_2 \\ \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \end{array} \quad \begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \\ \pi_1(x) \downarrow & \circlearrowleft & \downarrow \pi_2 \circ \phi(x) \\ \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \end{array}$$

Such definition carries over without substantial changes to the case of real and even spectral triples.

It should be mentioned that in the process of developing formal definitions of several new concepts (such as Morphism of Spectral Triples, Immersions of Spectral Triples, Submersions of Spectral Triples, Sub-Spectral Triples, Quotient Spectral Triples) in the general setting that reduce to the usual notions in the commutative case, we run into a series of technical obstacles:

- there is neither an available notion of “pull-back” for spinor fields between Riemannian spin manifolds of different dimension nor formal definitions of “spin-preserving” maps in such case;
- some results circulating in the literature were affected by serious flaws and needed major revision.

In order to show the validity of the proposed definition of morphisms, we considered a construction by A. Connes. Given a length function ℓ on a discrete group G , there is a spectral triple

$$(\mathbb{C}G, \ell^2(G), M_\ell)$$

where M_ℓ is the multiplication operator by ℓ . We have constructed a natural covariant functor from the category of discrete groups equipped with a length function to our category of spectral triples. We have also outlined the main differences between the functor in the case of inclusion of a subgroup and a relator in the case of a projection onto a quotient group, that point to general features of the construction.

In a wider perspective a morphism of the spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, with $j = 1, 2$, might be formalized as a “suitable” functor $\mathcal{F} : \mathcal{A}_2\mathcal{M} \rightarrow \mathcal{A}_1\mathcal{M}$, between the categories $\mathcal{A}_j\mathcal{M}$ of $\mathcal{A}_j$ -modules, having “appropriate intertwining” properties with the Dirac operators D_j .

The morphisms described in the paper are only a special case of a more general picture. However for the present purposes that level of generality was not necessary, and so we stuck to the more restrictive definition provided by homomorphisms $\phi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ of algebras with an intertwining operators $\Phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ between the Dirac operators. The main technical result contained in our paper states that Connes' construction of spectral triples from group algebras [C2] is functorial in nature.

3.3 Potential Future Work

As we said before, the long term goal of our research is to identify new possible axiomatizations of spectral geometry that might be more efficient than spectral triples in order to deal with non-commutative geometric objects (useful in Quantum Physics), with particular emphasis on those originating from Tomita-Takesaki modular theory. The quest for a suitable formalization of non-commutative geometries arising from Tomita's modular theory (and its applications to Quantum Physics) is still at its beginning and the present work has been mainly dealing with some of the important preliminary conceptual and technical problems.

During our previous study we have recognized that all the relevant information necessary to prove theorems about spectral triples is conveniently codified in certain “algebra modules” equipped with additional structures (inner products, real structures . . .) and that such “module-algebras” might exist also in cases when the underlying spectral triple cannot be constructed. The categorical properties of non-commutative spaces are probably better dealt with inside the categories of these “algebra modules” and (extremely important for our purposes) a KMS-state ω of an observable algebra $\mathcal{A}$ seems to give rise directly to such an object.

We plan to explore the relationships between modular pairs $(\mathcal{A}, \omega)$ and the algebras with real structure and the possibility to perform an intrinsic “polarization” for every such modular pair leading to a spectral definition of configuration space, which is one of our ultimate goals.

As more immediate short term goals:

- a) Now that some notion of morphism (and category) of spectral triple is available, we can proceed to the study of “substructures” i.e. “non-commutative submanifolds” and “dimensional reduction”.

- b) After the completion of the preliminary study initiated with our Master's students on “real/complex structures for modules” [Sa] and “Hermitian-Clifford algebras for Hermitian modules” [Th], we expect a couple of publications (together with Miss Areeya Sararak and Miss Supaporn Theesoongnern, respectively) that might lead to the definition of “non-commutative polarizations” and “second quantization on non-commutative spaces”.

As an unexpected direct fallout, this project (specifically the definition of morphism of spectral triples) has opened the way towards a full investigation of the **categorical properties of spectral triples** and more generally non-commutative spaces that look extremely promising in their relations with other branches such as algebraic geometry and that we deem necessary to consider in detail in the near future.

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4 Output

One research paper⁸ with the following title:

P. Bertozzini, R. Conti, W. Lewkeeratiyutkul,

“A Category of Spectral Triples and Discrete Groups with Length Function”

has been accepted for publication in the **Osaka Journal of Mathematics** on 20 June 2005 and is currently in press.

The preprint is also available on-line on the ArXive at the following URL:

<http://xxx.lanl.gov/math.OA/0502583>

Partial reports of the present research project have been presented in the form of seminars⁹:

- Paolo Bertozzini,
“Non-commutative Geometry and Spectral Space-Time”
Physics Department, Mahidol University, Bangkok, Thailand
August 19, 2003;
- Paolo Bertozzini,
“Non-commutative Submanifolds and Quotient Manifolds”
Mathematics Department, Thammasat University, Bangkok, Thailand
December 22, 2003;
- Roberto Conti,
“Observables and Space-Time”
Department of Mathematics, University of Oslo, Norway
20 April 2005.

⁸See attachment a) in the appendix.

⁹See the pdf files on the CDs.

5 Appendix

The following documents are attached to this final report:

- a) Budget summary.
- b) A copy of the preprint of the paper “A Category of Spectral Triples and Discrete Groups with Length Function” submitted to the Osaka Journal of Mathematics on February 28, 2005.
- c) A copy of the submission letter to the Osaka Journal of Mathematics.
- d) A copy of the acceptance e-mail from the editor of the Osaka Journal of Mathematics on June 20, 2005.
- e) An electronic copy (only on the attached CDs) of:
 - two seminars’ slides containing material produced during this research project, but not yet published (NOT TO BE DISTRIBUTED!),
 - two seminars’ slides with a summary of the theses’ results produced by the two Master’s students in Thammasat University involved in the research.
- f) Two CDs containing an electronic copy of all of the abovementioned documents a)-e).