



Final Report

Project: Projective Combinatorics and Invariant Theory

By: Dr. Matteo Mainetti

August 2003

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Sirindhorn International Institute of Technology

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
(ความคิดเห็นในรายงานนี้เป็นของผู้วิจัย สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

Abstract

Project code: RSA4580032

Project title: Projective Combinatorics and Invariant Theory

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(until April 2003)

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(current)

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Project period: 15 August 2002 – 15 August 2003

The objective of this research sponsored by the TRF was to study geometry from the point of view of linear lattices, a very versatile combinatorial tool rich with applications in other fields. As forecasted in the proposal, this approach could lead to results in related areas. Linear lattices are lattices of commuting equivalence relations. Studying them entails dealing closely with equivalence relations. Despite being so ubiquitous in mathematicians' everyday life, equivalence relations are still far from being fully understood. The algebra of equivalence relations (that is, the set of operations that can be defined on the family of all equivalence relations, and their properties), in particular, is still largely unknown. This research sprung from a previous work of the author [BMP01], where a class of operations on equivalence relations was introduced. In the current research project those operations are defined in a better way, and generalized in several directions. This has led to some interesting results that have been included partly in the Master degree thesis [Wan03], presented at an international conference, and accepted by one of the most renowned international journals in the field [Mai03].

Keywords: Equivalence relations, Jónsson's type, commuting equivalence relations, n-trusting organizations.

บทคัดย่อ

รหัสโครงการ: RSA4580032

ชื่อโครงการ: โพรเจกทีฟคอมบินาทอริกส์และทฤษฎีเอนิง

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จุดประสงค์ของงานวิจัยนี้ซึ่งได้รับการสนับสนุนจากสกว. คือการศึกษาเรขาคณิตในมุมมองของแลตติสเชิงเส้น ซึ่งเป็นระเบียบวิธีทางคอมบินาทอริกส์ที่มีประโยชน์หลากหลาย ทั้งยังมีการนำไปประยุกต์ใช้ในสาขาอื่นๆ ตามที่ได้คาดคะเนไว้ในเอกสารข้อเสนอโครงการ การใช้ระเบียบวิธีนี้ สามารถนำไปสู่ผลขยายในด้านอื่นๆที่มีความเกี่ยวข้องได้

แลตติสเชิงเส้นคือ กลุ่มแลตติสที่ประกอบด้วยความสัมพันธ์สมมูล ดังนั้นการศึกษากลุ่มแลตติสนี้จึงเกี่ยวเนื่องอย่างยิ่งกับความสัมพันธ์สมมูล ถึงแม้ว่าความสัมพันธ์สมมูลจะถูกนำไปใช้อย่างแพร่หลาย ในชีวิตประจำวันของนักคณิตศาสตร์ แต่ความสัมพันธ์สมมูลนั้นยังไม่เป็นที่เข้าใจอย่างถ่องแท้ โดยเฉพาะอย่างยิ่ง

ในด้านพีชคณิตและคุณสมบัติทางพีชคณิต

ซึ่งคือระเบียบวิธีการที่สามารถนิยามได้บนความสัมพันธ์สมมูล งานวิจัยนี้ซึ่งส่วนหนึ่งต่อเนื่อง มาจากงานวิจัยเดิมของผู้วิจัย [BMP01] ได้ขยายไปสู่ความเข้าใจเพิ่มเติมรวมทั้งการนิยามที่ครอบคลุมมากกว่าของระเบียบวิธีการเหล่านั้น

ผลงานบางส่วนได้ถูกนำเสนอ ในวิทยานิพนธ์ระดับปริญญาโท [Wan03]

ซึ่งได้มีนำเสนอในการประชุมวิชาการในต่างประเทศ และ

ผลงานที่สมบูรณ์ซึ่งจะถูกนำเสนอต่อไปในผลงานตีพิมพ์ ซึ่งจะลงในวารสารที่เป็นที่ยอมรับอย่างยิ่งในสาขานี้ [Mai03]

คำหลัก: ความสัมพันธ์สมมูล, Jónsson's type, commuting equivalence relations, n-trusting organizations.

Acknowledgement

I would like to thank the Thailand Research Fund for giving me the opportunity to pursue this research topic. From Chiang Mai, I would like to thank Srichan Arworn and Sorasak Leeratanavalee for useful hints and suggestions. Also, I would like to thank Thomas Britz and Catherine Yan, with whom I collaborated in the past, to take the time to review my manuscript. I am also grateful to Piyapong Niamsup for his help with Thai translation and advice.

Introduction

Throughout the history of Mathematics, the notion of equivalence relations has played a fundamental role. It dates back at least to when the natural numbers first were introduced: a non-negative integer may be thought of as a representative of the equivalence class of sets with the same cardinality. To express such a simple and "obvious" fact with equivalence relations may seem unnecessarily cumbersome. Nothing is further from the truth. Equivalence relations play a decisive role as building elements in every area of Mathematics. For instance, Algebra is firmly founded on equivalence relations: group theory, ring theory, modules, and fields would basically be impossible to define and use without equivalence relations.

Equivalence relations are so ubiquitous in everyday life that we often forget about their proactive existence. Much is still unknown about equivalence relations. Were this situation remedied, the theory of equivalence relations could initiate a chain process generating new insights and discoveries in many fields dependent upon it.

This research springs from a simple acknowledgement: the only operations on the family of equivalence relations fully studied, understood, and deployed are the binary join and meet operations. Very little is known about the possible classification of all operations on equivalence relations. Towards this aim, a new binary operation (and a class of n -ary operations) was introduced and studied in [BMP01], named "*" by the authors. The definition of * was quite hard to grasp, to work with, and to find an interpretation of. In other words, it was quite hard to answer the question "what does it really stand for?"

In the fall of 2002 I proposed to my Master degree student at Chiang Mai University, mister Apirut Wanichsombat, the problem of studying the work [BMP01] and see in what way we could improve it or generalize it. Our collaboration lead to some results, culminated with mister Apirut's master thesis and his presentation at the 18th Conference for Young algebraists, 21-23 March 2003, University of Potsdam, Potsdam, Germany. I finally improved and generalized the results in a paper to appear soon in Annals of Combinatorics [Mai03].

Results

We discover a new binary operation on the family of equivalence relations and prove it equivalent to the $*$ operation defined in [BMP01]. We get more than that: for any positive integer n , we build a binary operation $*_n$ that generalizes $*$ (the latter corresponding to the case $n=2$).

All these results are explained in details in the paper [Mai03] included at the end of this yearly report). The title of the paper, *n-Trusting Organizations*, is meant to catch the attention of the reader and focus it directly onto the interpretation of the operation $*_n$. We will explain this with a practical example: let S be an *organization*, where by organization we mean a set A and two equivalence relations R and T on A . For instance A could be a corporation, a random group of friends, an army, the guests at a wedding banquet, you name it. Let's take the example of the army, or better, an international army. The equivalence relations R and T could be "same nationality" and "graduated from the same cadet school," respectively. This equivalence relations form "bonds" between the elements of A . For instance, John feels close to Kevin because they are both American. Kevin, however, went to cadet school in Paris, together with Pierre, so Kevin also feels close to Pierre. John, in turns, considers Pierre "a friend of a friend": not as good as a friend, but better than a stranger. We can say that the element a *n-trusts* the element b if we can go from a to b in n (or less) steps as the ones described above.

The army will be divided into groups (which we call *departments*). Now suppose the army has to accomplish some very delicate mission, and suppose each department will be assigned a different task. To better achieve its mission, some conditions are needed. First of all, there must be cohesion and trust within each department. This means that we fix a trust level, n , and require that in any department E_i , every element n -trusts every other element. Secondly, in order to guarantee collaboration and friendship between the departments, the organization must be *envy-free*. By *envy-free* we roughly mean (the precise definition will be provided in the paper) that an element a of department E_i will not be jealous of department E_j for having kept his "buddy" b in its rank, or for having prevented a from joining E_j . Finally, the departments should be as big as possible, subject to the two constraints of n -trust and *envy-free*. It turns out that the only partition of A that satisfies the condition above is in fact $R *_n T$.

For an alternative to the army example, you can think of a graduation reunion example, where R and T are, respectively, year of graduation and common interests (football, gardening, etc.). The departments represent tables at the restaurant. You are organizing the banquet, what is the best way to divide the people in tables such that there will be n -trust within each table (once n is chosen) and *envy-free* across tables, so to assure a happy party?

In this setting, it comes natural to define the “leader”, or *director* of each department, to be the one who is the most popular in the department. Similarly, a *president* of the whole organization will be the one who is the most popular in the whole set A . These concepts are precisely defined and studied in my research.

Other results in this research include studying a correlation between the $*n$ operations and the classification given by Jönsson [Jon53] on the type of a pair of equivalence relations, classification that was already studied and expanded in [BMP01]. In addition, the $*n$ operations are generalized to become k -ary operations, and some questions are raised: are the $*n$ operations (or their generalizations) independent of each other? How do they relate to the k -ary operations introduced in [BMP01]? These questions pave the ground to further exploration and research directions.

Reference

- [Mai98] M. Mainetti, "Studies in Projective Combinatorics", Ph.D. Thesis, M.I.T., 1998.
- [MaYa98] M. Mainetti and C. H. Yan, *Graphical operations on projective spaces*, Annals of Combinatorics 2 No. 3 (1998), 245-291.
- [MaYa99] M. Mainetti and C. H. Yan, *Arguesian identities in linear lattices*, Advances in Mathematics 144 (1999), 50-93.
- [MaYa00] M. Mainetti and C. H. Yan, *Geometric identities in lattice theory*, Journal of Combinatorial Theory A, 91 No. 1/2 (2000), 411-450.
- [BMP01] T. Britz, M. Mainetti, and L. Pezzoli, *Some operations on the family of equivalence relations*. In "Algebraic Combinatorics and Computer Science," Springer-Verlag, Milan, 2001, 445-460.
- [Wan03] Apirut Wanichsombat, "Some Operations on the Family of Equivalence Relations", Master's thesis, Department of Mathematics, Chiang Mai University, 2003.
- [Mai03] M. Mainetti, *n-Trusting Organizations*, to appear in Annals of Combinatorics, (2003).

Output

Output of the project sponsored by TRF:

1. Paper *n-Trusting Organizations* by Matteo Mainetti, to appear in *Annals of Combinatorics*, 2003.
2. Master's thesis *Some Operations on the Family of Equivalence Relations* by Apirut Wanichsombat, Department of Mathematics, Chiang Mai University, March 2003.
3. Results presented at the 18th Conference for Young algebraists, 21-23 March 2003, University of Potsdam, Potsdam, Germany, by Mister Apirut Wanichsombat.

n -Trusting Organizations

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Abstract

In [BMP01] a new binary operation on the family of equivalence relations was introduced and studied. In this paper we give an equivalent, easier to grasp, definition of the same binary operation, and we prove it to be just an example of a bigger family of binary operations, which are shown to have interesting interpretation and meaning.

1 Introduction

Throughout the history of Mathematics, the notion of equivalence relations has played a fundamental role. It dates back at least to when the natural numbers first were introduced: a non-negative integer may be thought of as a representative of the equivalence class of sets with the same cardinality. To express such a simple and “obvious” fact with equivalence relations may seem unnecessarily cumbersome. Nothing is further from the truth. Equivalence relations play a decisive role as building elements in every area of Mathematics. For instance, Algebra is firmly founded on equivalence relations: group theory, ring theory, modules, and fields would basically be impossible to define and use without equivalence relations.

Equivalence relations are so ubiquitous in everyday life that we often forget about their proactive existence. Much is still unknown about equivalence relations. Were this situation remedied, the theory of equivalence relations could initiate a chain process generating new insights and discoveries in many fields dependent upon it.

*Supported by the Thailand Research Fund, grant RSA4580032

This paper springs from a simple acknowledgement: the only operations on the family of equivalence relations fully studied, understood, and deployed are the binary join $\vee$ and meet $\wedge$ operations. Very little is known about the possible classification of all operations on equivalence relations. Towards this aim, a new binary operation (and a class of n -ary operations) was introduced and studied in [BMP01], named “ $*$ ” by the authors. The definition of $*$ was quite hard to grasp, to work with, and to find an interpretation of. In other words, it was quite hard to answer the question “what does it really stand for?”

In this paper, we discover a new binary operation on the family of equivalence relations and prove it equivalent to the $*$ operation defined in [BMP01]. We get more than that: firstly, for any positive integer n , we build a new binary operation $*_n$ that generalizes $*$ (the latter corresponding to the case $n = 2$), and secondly, we find a modified version of $*_n$, which we denote by $*_n^w$, which is independent of the former.

The title we chose, *n-Trusting Organizations*, is meant to catch the attention of the reader and focus it directly onto the interpretation of the operation $*_n$. We will explain this with a practical example: let $\mathcal{O}$ be an *organization*, where by organization we mean a set $\mathcal{A}$ and two equivalence relations R and T on $\mathcal{A}$. For instance $\mathcal{A}$ could be a corporation, a random group of friends, an army, the guests at a wedding banquet, you name it. Let's take the example of the army, or better, an international army. The equivalence relations R and T could be “*same nationality*” and “*graduated from the same cadet school*,” respectively. This equivalence relations form “*bonds*” between the elements of $\mathcal{A}$. For instance, John feels close to Kevin because they are both American. Kevin, however, went to cadet school in Paris, together with Pierre, so Kevin also feels close to Pierre. John, in turns, considers Pierre “*a friend of a friend*,” not as good as a friend, but better than a stranger. We can say that the element a *n-trusts* the element b if we can go from a to b in n (or less) steps as the ones described above.

The army will be divided into groups (which we call *departments*). Now suppose the army has to accomplish some very delicate mission, and suppose each department will be assigned a different task. To better achieve its mission, some conditions are needed. First of all, there must be cohesion and trust within each department. This means that we fix a trust level, n , and require that in any department E_i , every element n -trusts every other element. Secondly, in order to guarantee collaboration and friendship between the departments, the organization must be *envy-free*. By *envy-free* we roughly mean (the precise definition will be provided in the paper) that an element a of department E_i will not be jealous of department E_j for



having kept its “buddy” b in its rank, or for having prevented a from joining E_j . Finally, the departments should be as big as possible, subject to the two constraints of n -trust and envy-freeness. It turns out that the only partition of $\mathcal{A}$ that satisfies the condition above is in fact $R *_{\mathbf{n}} T$.

In case you don't like the army example, you can think of a graduation reunion example, where R and T are, respectively, year of graduation and common interests (football, gardening, etc.). The departments represent tables at the restaurant. You are organizing the banquet, what is the best way to divide the people in tables such that there will be n -trust within each table (once n is chosen) and envy-freeness across tables, so to assure a happy party?

In this setting, it comes natural to define the “leader,” or *director* of each department, to be the one who is the most popular in the department. Similarly, a *president* of the whole organization will be the one who is the most popular in the whole set $\mathcal{A}$. These concepts are precisely defined and studied in the paper.

Other results in this paper include studying a correlation between the $*_{\mathbf{n}}$ operations and the classification given by Jónsson [Jon53] on the type of a pair of equivalence relations, classification that was already studied and expanded in [BMP01].

This paper is organized as follows: Section 2 serves as background. In Section 3 the operation $*$ is introduced and proven equivalent to the one introduced in [BMP01]. In Section 4 the operation $*$ is generalized, and in Section 5 the operation $*^w$ is introduced. Section 6 correlates these operations with the concept of “type” of pairs of equivalence relations. Section 7 gives some applications, and finally, in Section 8 some open questions and working problems are hinted. Pictures and examples throughout the paper are provided.

2 Types of Pairs of Equivalence Relations

For the rest of this work $\mathcal{A}$ will denote a fixed set, that we will assume finite, although most of the results will remain valid even in the case of an arbitrary set. If R is an equivalence relation on $\mathcal{A}$, and $a, b \in \mathcal{A}$, we will denote by aRb whenever a and b are R -equivalent. If $b \in \mathcal{A}$, the R -equivalence class of b will be denoted by $R(b)$. That is,

$$R(b) = \{a : aRb\}.$$

Saying it in another way, $R(b)$ is $\cdot Rb$, that is, the set of all elements that can be placed instead of the dot “.” to make the relation hold.

In a similar fashion, if $a \in \mathcal{A}$, one can define $(a)R$ to be

$$(a)R = \{b : aRb\},$$

and the fact that R is an equivalence relation implies that, for every a in $\mathcal{A}$, $(a)R = R(a)$. The sets $R(a)$'s are called *blocks* of the partition associated to the equivalence relation R . This apparent triviality, and unusual notation, will be helpful quite soon.

Definition 2.1 *If R and T are equivalence relations, their composition is defined as*

$$RT = \{(a, c) \in \mathcal{A} \times \mathcal{A} : aRb, bTc \text{ for some } b \in \mathcal{A}\}. \quad (1)$$

Following the same notation introduced for equivalence relations, if (a, c) belongs to the set (1), we will write $aRTc$. Similarly, we define

$$RT(c) = \{a \in \mathcal{A} : aRTc\} \text{ that is, } \{a \in \mathcal{A} : aRb, bTc \text{ for some } b \in \mathcal{A}\},$$

and

$$(a)RT = \{c \in \mathcal{A} : aRTc\}.$$

Notice that it is no longer valid in general that $(a)RT = RT(a)$, nor is it valid that the set (of sets)

$$\{RT(x) : x \in \mathcal{A}\}$$

forms a partition of $\mathcal{A}$. It does give a covering of $\mathcal{A}$, but the sets $RT(x)$ could overlap. The composition of equivalence relations can be defined for an arbitrary (finite) sequence of them, not just two, and the notation would be similar to the one introduced above. We decided to give in detail only the binary case here for clarity purposes.

Denote by $\Pi[\mathcal{A}]$ the set of all equivalence relations on $\mathcal{A}$. Endowed with the partial order of refinement, that is, $R \leq T$ if and only if every block of R is contained in a block of T , $\Pi[\mathcal{A}]$ is seen to be a lattice with unique minimal and maximal elements, namely $\hat{0}$, the partition composed of one-element blocks, and $\hat{1}$, the partition with the one block $\{\mathcal{A}\}$. Meet and join of equivalence relations can be explicitly expressed in terms of intersection and union of compositions, as in the following proposition, whose proof is straightforward, and can be found in most papers in our reference list.

Proposition 2.2

$$R \wedge T = R \cap T; \quad (2)$$

$$R \vee T = R \cup RT \cup RTR \cup \dots \cup T \cup TR \cup TRT \cup \dots \quad (3)$$

In what follows we will make extensive use of the following notation:

$$\frac{RT}{n} \doteq \overbrace{RTR \dots}^n$$

Remark 2.3 The relation $\frac{RT}{n}$ is reflexive and, for n odd, symmetric.

Notice that the infinite union in (3) consists of increasing terms (in each line). The pair of equivalence relations (R, T) is said to be of *finite type* if (3) can be expressed with a finite union. It is immediate to see that if

$$\frac{RT}{n} = \frac{TR}{n},$$

then (R, T) is of finite type, since in this case we would have, for every $m \geq n$,

$$\frac{RT}{m} = \frac{TR}{m} = \frac{RT}{n} = \frac{TR}{n}.$$

Jónsson [Jon53] called such pairs (R, T) pairs of *type* $n - 1$. His definition was further generalized in [BMP01], as follows.

Definition 2.4 A pair (R, T) of equivalence relations on the set A is said to be of *type*

$$\begin{array}{ll} n \text{ strong} & \text{if } \frac{RT}{n+1} = \frac{TR}{n+1} \text{ (original Jónsson type } n) \\ n \text{ weak} & \text{if } \frac{RT}{n+1} \subseteq \frac{TR}{n+1} \text{ or } \frac{TR}{n+1} \subseteq \frac{RT}{n+1} \\ n.5 & \text{if } R \vee T = \frac{RT}{n+1} \cup \frac{TR}{n+1} \end{array}$$

Notice that in all three cases above (n strong, n weak, $n.5$), the remark following Remark 2.3 implies that

$$R \vee T = \frac{RT}{n+1} \cup \frac{TR}{n+1}.$$

For a pair (R, T) we clearly have the sequence of implications:

$$n \text{ strong} \implies n \text{ weak} \implies n.5 \implies n+1 \text{ strong}$$

More specifically, if we denote the families of pairs (R, T) of type $n.5$, type n weak, and type n strong by $\mathcal{H}_n$, $\mathcal{W}_n$ and $\mathcal{S}_n$, respectively, then the following holds [BMP01].

Proposition 2.5 *The families $\mathcal{H}_n$, $\mathcal{W}_n$, and $\mathcal{S}_n$ form the sequence*

$$\emptyset \neq \mathcal{S}_0 \subsetneq \mathcal{W}_0 \subsetneq \mathcal{H}_0 \subsetneq \mathcal{S}_1 = \mathcal{W}_1 \subsetneq \mathcal{H}_1 \subsetneq \mathcal{S}_2 \subsetneq \mathcal{W}_2 \subsetneq \mathcal{H}_2 \subsetneq \mathcal{S}_3 = \mathcal{W}_3 \subsetneq \dots,$$

that is, all types are distinct except for n odd, when n strong and n weak coincide.

M.-L. Dubreil and her husband P. Dubreil investigated pairs of types 1 and 2 [Dub50, DJD39]. The originator of the term “type”, B. Jónsson, investigated pairs of type 1, 2, and 3 in relation to embeddings of lattices [Jon53]. Specifically, he showed that any lattice admits a representation of type 3 (that is, for any lattice L there is a set $\mathcal{A}$ and a sublattice of $\Pi[\mathcal{A}]$ isomorphic to L such that any two equivalence relations in $\Pi[\mathcal{A}]$ are of type 3); for type 2 he showed that L is modular if and only if L admits a representation of type 2. Lattices that admit representation of type 1 are sometimes called “lattices of commuting equivalence relations”, or “linear lattices”, by some authors [Hai85, Hai91, FMR96]. They are “special” in the sense that the composition of any two equivalence relations is already an equivalence relation. G. Hutchinson [Hut81] – among many other authors – studied pairs of type n . In view of the importance of the classification of pairs of equivalence relations according to their type, it is reasonable to believe that refining such definition can only lead to new important discoveries and understandings.

In [BMP01] the author, T. Britz and L. Pezzoli studied these “new” generalized types, finding structural characterizations of type 1.5, 2 strong and 2 weak. Among other results, they provided minimal representation of pairs of each type.

If we represent the elements of $\mathcal{A}$ as little black squares and arrange them into a grid in such a way that R equivalence classes correspond to rows and T equivalence classes to columns, then Figure 1 shows some examples of pairs (R, T) of all type $n \geq 1$. Notice that those example are what we refer to as “proper” type, in the sense that they belong to the specified type class and not to stronger (that is, smaller) type classes.

3 The $\square$ and $*$ Operations

Britz *et al.* [BMP01] introduced and studied a new binary operation on the family of equivalence relations. We will begin this section with reviewing some basic definitions and results from that paper.