



รายงานวิจัยฉบับสมบูรณ์

**โครงการ : ทฤษฎีบทจุดคงที่สำหรับนัยทั่วไปของการส่งแบบ
นอนเอ็กซ์แพนซีฟ**

**Fixed point theorem for generalized of nonexpansive
mappings**

โดย

รองศาสตราจารย์ ดร.สมยศ พลับเที่ยง และคณะ

15 สิงหาคม 2548

รายงานวิจัยฉบับสมบูรณ์

โครงการ : ทฤษฎีบทจุดคงที่สำหรับนัยทั่วไปของการส่งแบบ
นอนเอกซ์แพนซีฟ

Fixed point theorem for generalized of nonexpansive
mappings

ผู้วิจัย : รองศาสตราจารย์ ดร.สมยศ พลับเที่ยง และคณะ
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์
มหาวิทยาลัยนเรศวร

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

งานวิจัยนี้ได้รับทุนสนับสนุนการวิจัยจากทุนพัฒนานักวิจัยประจำปี 2546 ของสำนักงาน
กองทุนสนับสนุนการวิจัย ผู้วิจัยขอขอบพระคุณเจ้าของทุนเป็นอย่างสูงมา ณ โอกาสนี้

ขอขอบพระคุณหัวหน้าภาควิชาคณิตศาสตร์ และคณบดีคณะวิทยาศาสตร์มหาวิทยาลัย
นเรศวร ที่ได้ให้การสนับสนุนและอำนวยความสะดวกในการใช้พัสดุ ครุภัณฑ์ในการวิจัยเป็นอย่างดี
ตลอดโครงการ และสุดท้ายขอขอบคุณคณะผู้ช่วยวิจัยซึ่งประกอบด้วย อาจารย์ระเบียบ วังศิริ
อาจารย์ภูมิ คำเอม อาจารย์อัสระ อินจันทร์ อาจารย์รัตนพร พันแพง อาจารย์อาทิตย์ แข็งธัญ
การ และ นายเกษมสุข อุงจิตตระกูล ซึ่งเป็นนิสิตระดับปริญญาโทและเอกที่ผู้วิจัยเป็นอาจารย์ที่
ปรึกษาวิทยานิพนธ์ ที่ได้ร่วมกันสร้างผลงานวิจัยที่ได้รวบรวมไว้ในรายงานวิจัยนี้

รองศาสตราจารย์ ดร.สมยศ พลับเที่ยง

Abstract

Project Code : RSA 4680022

Project Title : Fixed point theorems for generalized of nonexpansive mappings.

Investigator : Assoc. Dr. Somyot Plubtieng

Project Period : August 15, 2003-August 14, 2005

In this paper we present some results on fixed point theorems of mappings of nonexpansive, asymptotically nonexpansive, asymptotically quasi-nonexpansive, and asymptotically nonexpansive in the intermediate sense. Firstly, we proved weak and strong convergence theorems of three step (multi-step) iterative scheme with errors to a fixed point for generalized of nonexpansive mappings as above mention. Moreover, we also prove strong convergence theorem of implicit iteration process for self (nonself) mappings of nonexpansive and asymptotically nonexpansive, respectively. Finally, we proved the random fixed point theorems for nonexpansive random operators and multi-valued nonexpansive random operators.

Keywords : Fixed point theorem, Three-step iteration, Implicit iteration, Random operators, Random fixed point.

บทคัดย่อ

ชื่อโครงการ : ทฤษฎีบทจุดคงที่สำหรับนัยทั่วไปของการส่งแบบนอนเอกซ์แพนซีฟ
ชื่อนักวิจัย : รศ.ดร.สมยศ พลับเที่ยง
ระยะเวลาโครงการ : 15 สิงหาคม 2546 ถึง 14 สิงหาคม 2548

ในงานวิจัยนี้ ผู้วิจัยได้เสนอเนื้อหาบางอย่างของทฤษฎีบทจุดคงที่สำหรับการส่งแบบนอนเอกซ์แพนซีฟ แอสซิมโทติกอลีนอนเอกซ์แพนซีฟ แอสซิมโทติกวอไลนอนเอกซ์แพนซีฟ แอสซิมโทติกอลีนอนเอกซ์แพนซีฟในรูปแบบอินเทอร์มีเดียท

ประการแรกผู้วิจัยได้พิสูจน์ ทฤษฎีบทการลู่เข้าแบบแข็งและแบบอ่อนของกระบวนการทำซ้ำ 3 ขั้นตอน (หลายขั้นตอน) พร้อมด้วยค่าคาดเคลื่อนไปยังจุดคงที่สำหรับนัยทั่วไปของการส่งแบบนอนเอกซ์แพนซีฟซึ่งได้กล่าวไว้ข้างบนนี้ ยิ่งไปกว่านั้นผู้วิจัยได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบแข็งของกระบวนการกระทำซ้ำแบบอิมพลีชีท สำหรับการส่งแบบนอนเอกซ์แพนซีฟและแอสซิมโทติกอลีนอนเอกซ์แพนซีฟ สุดท้ายผู้วิจัยได้พิสูจน์ทฤษฎีบทจุดคงที่สำหรับตัวดำเนินการแบบสุมแบบนอนเอกซ์แพนซีฟตัว และตัวดำเนินการสุมค่าเซตแบบนอนเอกซ์แพนซีฟ.

คำหลัก : ทฤษฎีบทจุดคงที่ กระบวนการกระทำซ้ำสามขั้นตอน กระบวนการกระทำซ้ำแฝง
ตัวดำเนินการสุม จุดคงที่ของตัวดำเนินการสุม

บทนำ

ทฤษฎีบทจุดคงที่ (Fixed point theorem) ถือว่าเป็นทฤษฎีบทที่มีความสำคัญมากในการนำไปประยุกต์ทั้งในสาขาวิชาคณิตศาสตร์เองและในสาขาอื่นๆ การศึกษาวิจัยในเรื่องของทฤษฎีบทจุดคงที่บนปริภูมิบานาคเป็นการหาเงื่อนไขที่เพียงพอที่จะทำให้ฟังก์ชัน T ที่ส่งจากเซตย่อย K ของปริภูมิบานาค X ไปยัง K มีจุดคงที่ (นั่นคือจะมีจุด a ใน K ซึ่งทำให้ $T(a) = a$) ทฤษฎีบทจุดคงที่บนปริภูมิบานาคที่สำคัญเริ่มต้นจากในปี ค.ศ. 1922 Banach ได้พิสูจน์ว่าถ้า (X, d) เป็นปริภูมิเมตริกบริบูรณ์ และ $T : X \rightarrow X$ เป็น contraction (นั่นคือ จะมี $c \in (0, 1)$ ซึ่งทำให้ $\|T(x) - T(y)\| \leq c\|x - y\|$ ทุกๆ $x, y \in X$) แล้ว T จะมีจุดคงที่เพียงจุดเดียว ต่อมาในปี ค.ศ. 1930 Schauder ได้พิสูจน์ว่าถ้า K เป็นเซตย่อยที่ไม่เป็นเซตว่างซึ่งเป็นทั้งเซตคอมแพคต์และเซตคอนเวกซ์ของปริภูมิบานาค X และ $T : K \rightarrow K$ เป็นฟังก์ชันต่อเนื่องแล้ว T จะมีจุดคงที่ และต่อมาในปี ค.ศ. 1965 Browder ได้พิสูจน์ว่าถ้า K เป็นเซตย่อยที่ไม่เป็นเซตว่างซึ่งเป็น ทั้งเซตปิดที่มีขอบเขตและคอนเวกซ์ของ uniformly convex Banach space X และ $T : K \rightarrow K$ เป็น nonexpansive (นั่นคือ $\|T(x) - T(y)\| \leq \|x - y\|$ ทุกๆ $x, y \in K$) แล้ว T จะมีจุดคงที่ หลังจากนั้นเป็นต้นมาได้มีนักคณิตศาสตร์จำนวนมาก ที่ทำการศึกษาวิจัยเพื่อหาเงื่อนไขที่เป็นคุณสมบัติทางเรขาคณิตของปริภูมิบานาค เพื่อใช้พิสูจน์ทฤษฎีบทจุดคงที่สำหรับ nonexpansive mappings

จุดประสงค์ของการวิจัย

1. เพื่อศึกษาคุณสมบัติทางเรขาคณิตของปริภูมิบานาค ที่เพียงพอที่จะพิสูจน์ Fixed point theorems for generalized nonexpansive mappings.
2. เพื่อศึกษาคุณสมบัติทางเรขาคณิตของปริภูมิบานาค ที่เพียงพอที่จะพิสูจน์ Fixed point theorems for multivalued nonexpansive mappings
3. เพื่อศึกษาคุณสมบัติทางเรขาคณิตของปริภูมิบานาค ที่เพียงพอที่จะพิสูจน์ Random fixed point theorems for (multivalued) nonexpansive random operators
4. เพื่อศึกษาการลู่เข้าของลำดับที่เกิดจาก Iterative contraction of Mann iteration, Ishikawa iteration, Three-step iteration and multi-step iteration

ผลการวิจัย

1. Three-step and multi-step iteration

1.1 S. Plubtieng and R. Wangkeeree, Fixed point iteration for asymptotically quasi-nonexpansive mappings in Banach spaces,

Theorem 1 Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be uniformly L -Lipschitzian, completely continuous and asymptotically quasi-nonexpansive mapping with sequence $\{k_n\}_{n \geq 1}$ such

that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_0 \in C$ and for each $n \geq 0$,

$$\begin{aligned} z_n &= \alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n \\ y_n &= \alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned}$$

where $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are three bounded sequences in C and $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$ and $\{\gamma_n''\}$ are real sequences in $[0, 1]$ which satisfies the same assumptions as Lemma ?? and the additional assumption that $0 \leq \alpha < \alpha_n, \beta_n, \alpha_n', \beta_n' \leq \beta < 1$ for some α, β in $(0, 1)$. Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T .

1.2 S. Plubtieng and R. Wangkeeree, Noor Iterations with error for non-Lipschitzian mappings in Banach spaces,

Theorem 1 Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be a completely continuous asymptotically nonexpansive in the intermediate sense. Put

$$G_n = \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \forall n \geq 1.$$

Let $x_0 \in C$ and for each $n \geq 0$,

$$\begin{aligned} z_n &= \alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n \\ y_n &= \alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned}$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$ and $\{\gamma_n''\}$ are real sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are three bounded sequences in C such that

- (i) $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$.
- (ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$, $\sum_{n=1}^{\infty} \gamma_n'' < \infty$.
- (iii) $0 < \alpha \leq \alpha_n, \alpha_n' \leq \beta < 1$. Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converges strongly to a fixed point of T .

1.3 S. Plubtieng and R. Wangkeeree, Strong convergence theorems for multi-step Noor iterations with errors in Banach spaces,

Let C be a nonempty subset of normed space X and let $T : C \rightarrow C$ be a mapping. For a given $x_1 \in C$, and a fixed $m \in \mathbb{N}$ ($\mathbb{N}$ denote the set of all

positive integers), compute the iterative sequences $\{x_n^{(1)}\}, \dots, \{x_n^{(m)}\}$ defined by

$$\begin{aligned}
 x_n^{(1)} &= \alpha_n^{(1)} T^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)}, \\
 x_n^{(2)} &= \alpha_n^{(2)} T^n x_n^{(1)} + \beta_n^{(2)} x_n + \gamma_n^{(2)} u_n^{(2)}, \\
 x_n^{(3)} &= \alpha_n^{(3)} T^n x_n^{(2)} + \beta_n^{(3)} x_n + \gamma_n^{(3)} u_n^{(3)}, \\
 &\vdots \\
 x_n^{(m-1)} &= \alpha_n^{(m-1)} T^n x_n^{(m-2)} + \beta_n^{(m-1)} x_n + \gamma_n^{(m-1)} u_n^{(m-1)}, \\
 x_{n+1} = x_n^{(m)} &= \alpha_n^{(m)} T^n x_n^{(m-1)} + \beta_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)}, \quad n \geq 1
 \end{aligned}
 \tag{0.1}$$

where, $\{u_n^{(1)}\}, \dots, \{u_n^{(m)}\}$ are bounded sequences in C and $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ are appropriate real sequences in $[0, 1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $i \in \{1, 2, \dots, m\}$.

Theorem 1 Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X and $T : C \rightarrow C$ be a completely continuous asymptotically nonexpansive in the intermediate sense. Put

$$G_n = \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \quad \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let the sequence $\{x_n\}$ be defined by (0.2) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfies the following restrictions:

- (i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for all $i \in \{1, 2, \dots, m\}$ and for all $n \geq 1$;
- (ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i \in \{1, 2, \dots, m\}$. If $0 < \alpha \leq \alpha_n^{(m-1)}, \alpha_n^{(m)} \leq \beta < 1$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Then $\{x_n^{(k)}\}$ converges strongly to a fixed point of T for each $k = 1, 2, 3, \dots, m$.

Theorem 2 Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X and $T : C \rightarrow C$ be an uniformly L -Lipschitzian, completely continuous asymptotically quasi-nonexpansive with the sequence $\{r_n\}_{n \geq 1}$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (0.2) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfies the following restrictions:

- (i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for all $i \in \{1, 2, \dots, m\}$ and for all $n \geq 1$;
- (ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i \in \{1, 2, \dots, m\}$. If $0 < \alpha \leq \alpha_n^{(i)} \leq \beta < 1$ for all $i \in \{m-1, m\}$. Then $\{x_n^{(k)}\}$ converge strongly to a fixed point of T , for each $k = 1, 2, 3, \dots, m$.

1.4 S. Plubtieng and R. Wangkeeree, Strong convergence theorems for three-step iterations with errors for non-Lipschitzian nonself-mappings in Banach spaces, Algorithm 11(Three step iterative scheme for nonself maps with errors) Let C be a nonempty subset of normed space X . Let $P : X \rightarrow C$ be the nonexpansive retraction of X onto C and a mapping $T : C \rightarrow X$. For a given $x_0 \in C$,

compute the iteration sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ defined by

$$(0.2) \quad \begin{aligned} z_n &= P(\alpha_n'' T(PT)^{n-1} x_n + \beta_n'' x_n + \gamma_n'' u_n) \\ y_n &= P(\alpha_n' T(PT)^{n-1} z_n + \beta_n' x_n + \gamma_n' v_n) \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1} y_n + \beta_n x_n + \gamma_n w_n), \end{aligned}$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$ and $\{\gamma_n''\}$ are appropriate real sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are three bounded sequences in C .

Theorem 1 Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be an asymptotically nonexpansive in the intermediate sense nonself mapping with nonempty fixed point set $F(T)$. Put

$$G_n = \sup_{x, y \in C} (\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x - y\|) \vee 0, \forall n \geq 1.$$

Let the sequence $\{x_n\}$ be defined by (0.2) with the following restrictions

- (i) $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$.
- (ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$, $\sum_{n=1}^{\infty} \gamma_n'' < \infty$.
- (iii) $0 \leq \alpha < \alpha_n, \beta_n, \alpha_n', \beta_n' \leq \beta < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T .

1.5 S. Plubtieng and R. Wangkeeree Ishikawa Iteration Sequences for Asymptotically Quasi-Nonexpansive Nonself-Mappings with Error Members,

Let C be a nonempty closed convex subset of a real uniformly convex Banach space X . The following iteration process is studied:

$$(0.3) \quad \begin{aligned} x_1 \in C, x_{n+1} &= P(\alpha_n x_n + \beta_n T(PT)^{n-1} y_n + \gamma_n u_n), \\ y_n &= P(\alpha_n' x_n + \beta_n' T(PT)^{n-1} x_n + \gamma_n' v_n) \end{aligned}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$ and $\{\gamma_n'\}$ are sequences in $[0, 1]$ and P is a nonexpansive retraction of X onto C .

Theorem 1 Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let $T : C \rightarrow X$ be an asymptotically quasi-nonexpansive nonself-mapping with sequence $\{k_n\}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_1 \in C$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$ and $\{\gamma_n'\}$ be sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha_n' + \beta_n' + \gamma_n'$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n' < \infty$. Then the sequence $\{x_n\}$ defined by (0.3) strongly converges to a fixed point of T if and only if $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$, where $d(x, F(T))$ denote the distance of x to the set $F(T)$, i.e., $d(x, F(T)) = \inf_{y \in F(T)} d(x, y)$.

Theorem 2 Let X be a real uniformly convex Banach space, C a nonempty

closed convex subset of X . Let $T : C \rightarrow X$ be an uniformly L -Lipschitzian completely continuous and asymptotically quasi-nonexpansive nonself-mapping with sequence $\{k_n\}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_1 \in C$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ be sequences in $[0, 1]$ such that $0 < \alpha < \alpha_n, \beta_n, \alpha'_n, \beta'_n < \beta < 1, \alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n, \sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Then the sequence $\{x_n\}$ defined by (0.3) strongly converges to a fixed point of T .

1.6 I. Inchan and S. Plubtieng, Weak and strong convergence of scheme with errors for a finite family of nonexpansive mappings,

Let C be a nonempty subset of normed space X and let $T_1, T_2, \dots, T_N$ be nonexpansive mappings of C into itself. The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 = x \in C, \\ x_n^1 = \alpha_n^1 T_1 x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1, \\ x_n^2 = \alpha_n^2 T_2 x_n^1 + \beta_n^2 x_n + \gamma_n^2 u_n^2, \\ x_n^3 = \alpha_n^3 T_3 x_n^2 + \beta_n^3 x_n + \gamma_n^3 u_n^3, \\ x_n^4 = \alpha_n^4 T_4 x_n^3 + \beta_n^4 x_n + \gamma_n^4 u_n^4, \\ \vdots \\ x_{n+1} = x_n^N = \alpha_n^N T_N x_n^{N-1} + \beta_n^N x_n + \gamma_n^N u_n^N, n \geq 1, \end{cases} \quad (1.1)$$

where $\{\alpha_n^1\}, \dots, \{\alpha_n^N\}, \{\beta_n^1\}, \dots, \{\beta_n^N\}, \{\gamma_n^1\}, \dots, \{\gamma_n^N\}$ are sequences in $[0, 1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, 3, \dots, N$ and $\{u_n^1\}, \{u_n^2\}, \dots, \{u_n^N\}$ are bounded sequences in C .

Theorem 1 Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X . Let $T_1, T_2, \dots, T_N$ be a nonexpansive mappings of C into itself satisfying condition (A^N) and $\{x_n\}$ be a sequence as defined in (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2, \dots, N$. If $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point in F .

Theorem 2 Let X be a uniformly convex Banach space satisfying the Opial's condition, C its nonempty closed convex subset of X . Let $T_1, T_2, \dots, T_N$ be nonexpansive mappings of C into itself and $\{x_n\}$ be a sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2, \dots, N$. If $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point in F .

Theorem 3 Let C be a nonempty closed convex subset of uniformly convex Banach space X , and let $T_1, T_2, \dots, T_N$ nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. If P is a metric projection of C onto F and $\{x_n\}$

is a sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2, \dots, N$, then $\{Px_n\}$ converges strongly to a common fixed point in F .

1.7 S. Plubtieng, R. Punpeang and R. Wangkeeree, Weak and strong convergence of modified Noor iterations with errors for three asymptotically nonexpansive mappings

Let X be a normed space, C be a nonempty convex subset of X , and $T_1, T_2, T_3 : C \rightarrow C$ be three given mappings. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by

$$(0.4) \quad \begin{aligned} z_n &= \alpha_n'' T_3^n x_n + \beta_n'' x_n + \gamma_n'' u_n \\ y_n &= \alpha_n' T_2^n z_n + \beta_n' x_n + \gamma_n' v_n \\ x_{n+1} &= \alpha_n T_1^n y_n + \beta_n x_n + \gamma_n w_n \quad n \geq 1, \end{aligned}$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$ and $\{\gamma_n''\}$ are real sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in C .

Theorem 1 Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T_1, T_2 and T_3 be asymptotically nonexpansive self-maps of C with sequences $\{r_n^{(1)}\}$, $\{r_n^{(2)}\}$, $\{r_n^{(3)}\}$ respectively such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$ for all $i = 1, 2, 3$ and satisfying condition (A'') . Let $\{x_n\}$ be sequence as defined in (0.4) and some $\alpha, \beta \in (0, 1)$ with the following restrictions:

(i) $0 < \alpha \leq \alpha_n, \alpha_n', \alpha_n'' \leq \beta < 1, \forall n \geq n_0$ for some $n_0 \in \mathbb{N}$.

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma_n' < \infty, \sum_{n=1}^{\infty} \gamma_n'' < \infty$.

If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3 .

Theorem 2 Let X be a real uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed convex subset of X . Let T_1, T_2 and T_3 be asymptotically nonexpansive self-maps of C with sequence $\{r_n^{(1)}\}$, $\{r_n^{(2)}\}$, $\{r_n^{(3)}\}$ respectively such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$ for all $i = 1, 2, 3$. Let $\{x_n\}$ be sequence as defined in (0.4) and some $\alpha, \beta \in (0, 1)$ with the following restrictions:

(i) $0 < \alpha \leq \alpha_n, \alpha_n', \alpha_n'' \leq \beta < 1, \forall n \geq n_0$ for some $n_0 \in \mathbb{N}$.

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma_n' < \infty, \sum_{n=1}^{\infty} \gamma_n'' < \infty$.

If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3 .

2. Implicit iteration process

2.1 S. Plubtieng and R. Punpaeng, Implicit iteration process of nonexpansive nonself-mappings

In this paper, we extend Xu and Yin's results to study the contractions T_n, S_n and U_n define by

$$(0.5) \quad T_n x = (1 - \alpha_n)u + \alpha_n T[(1 - \beta_n)x + \beta_n T x]$$

$$(0.6) \quad S_n x = (1 - \alpha_n)u + \alpha_n P T[(1 - \beta_n)x + \beta_n P T x]$$

$$(0.7) \quad U_n x = P[(1 - \alpha_n)u + \alpha_n T P[(1 - \beta_n)x + \beta_n T x]],$$

where $\{\alpha_n\} \subseteq (0, 1)$, $0 \leq \beta_n \leq \beta < 1$, and P is the nearest point projection of H onto C .

Theorem 1 Let H be a real Hilbert space, C be a nonempty closed convex subset of H , and $T : C \rightarrow H$ be a nonexpansive nonself-mapping. Suppose that for some $u \in C$, $\{\alpha_n\} \subseteq (0, 1)$ and $0 \leq \beta_n \leq \beta < 1$, the mapping T_n defined by (0.5) has a (unique) fixed point $x_n \in C$ for all $n \geq 1$. Then T has a fixed point if and only if $\{x_n\}$ remains bounded as $\alpha_n \rightarrow 1$. In this case, $\{x_n\}$ converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T .

Theorem 2 Let H be a Hilbert space, C be a nonempty closed convex subset of H , $T : C \rightarrow H$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition, and $P : H \rightarrow C$ be the nearest point projection. Suppose that for some $u \in C$, each $\{\alpha_n\} \subseteq (0, 1)$ and $0 \leq \beta_n \leq \beta < 1$. Then, a mapping S_n defined by (0.20) has a unique fixed point $y_n \in C$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $\alpha_n \rightarrow 1$. In this case, $\{y_n\}$ converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T .

Theorem 3 Let $H, C, T, P, u, \{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem ???. Then a mapping U_n defined by (0.7) has a unique fixed point $z_n \in C$. Further, T has a fixed point if and only if $\{z_n\}$ remains bounded as $\alpha_n \rightarrow 1$ and $\beta_n \rightarrow 0$. In this case, $\{z_n\}$ converges strongly as $\alpha_n \rightarrow 1$ and $\beta_n \rightarrow 0$ to a fixed point of T .

2.2 S. Plubtieng and R. Wangkeeree, Strong convergence theorems of viscosity averaging iterations for asymptotically nonexpansive nonself-mappings

In this paper, we first show that, for an asymptotically nonexpansive nonself-mapping T with a sequence $\{k_n\} \subset [1, \infty)$, there exists two sequences $\{x_n\}$ and $\{y_n\}$ which defined by

$$(0.8) \quad x_n = a_n f(x_n) + (1 - a_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x_n, \quad \forall n \geq 1$$

and

$$(0.9) \quad y_n = \frac{1}{n} \sum_{j=1}^n P(a_n f(y_n) + (1 - a_n)(TP)^j y_n), \quad \forall n \geq 1$$

where

$$b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j}), \quad a_n = \frac{b_n - 1}{b_n - \beta}, \quad \forall n \geq 1,$$

$0 < \alpha < \beta < 1$, $f : C \rightarrow C$ is a contraction mapping with coefficient $\alpha \in (0, 1)$ and P is the metric projection from H onto C . **Theorem 1** Let C be a closed convex subset of a real Hilbert space H , P the metric projection from H onto C , T be an asymptotically nonexpansive nonself-mapping from C into H with Lipschitz constant k_n , and suppose that $F(T)$ is nonempty. Let $f : C \rightarrow C$ be a contraction mapping with coefficient $\alpha \in (0, 1)$,

$$b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j}) \text{ and } a_n = \frac{b_n - 1}{b_n - \beta},$$

where $0 < \alpha < \beta < 1$. If T satisfies (NNO) condition then the sequence $\{x_n\}$ defined by (0.8) converges strongly to z where, z is the unique solution in $F(T)$ to the variation inequality

$$(0.10) \quad \langle (I - f)z, x - z \rangle \geq 0, \quad x \in F(T)$$

or equivalently $z = G(f(z))$, where G is the metric projection from H onto $F(T)$.

Theorem 2 Let C be a closed convex subset of a real Hilbert space H , P the metric projection from H onto C , T be an asymptotically nonexpansive nonself-mapping from C into H with Lipschitz constant k_n , and suppose that $F(T)$ is nonempty. Let $f : C \rightarrow C$ be a contraction mapping with coefficient $\alpha \in (0, 1)$,

$$b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j}) \text{ and } a_n = \frac{b_n - 1}{b_n - \beta},$$

where $0 < \alpha < \beta < 1$. If T satisfies (NNO) condition then the sequence $\{y_n\}$ defined by (0.13) converges strongly to z where, z is the unique solution in $F(T)$ to the variation inequality

2.3 S. Plubtieng and R. Wangkeeree, Strong convergence theorems of viscosity averaging iterations for nonexpansive nonself-mappings in Hilbert spaces,

In this paper, we study the three type iterations process as follows: for $y_0, z_0 \in C$ and

$$(0.11) \quad x_n = t_n f(x_n) + (1 - t_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x_n$$

$$(0.12) \quad y_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n, n \geq 0$$

and

$$(0.13) \quad z_{n+1} = \frac{1}{n+1} \sum_{j=0}^n P(\alpha_n f(z_n) + (1 - \alpha_n)(TP)^j z_n), n \geq 0$$

where $\{t_n\} \subset (0, 1)$, $\{\alpha_n\}$ is a sequence such that $0 \leq \alpha_n \leq 1$, $f : C \rightarrow C$ is a contraction mapping and P is the metric projection of H onto C .

Theorem 1 Let H be a Hilbert space, C a nonempty closed convex subset of H , P the metric projection of H onto C and $T : C \rightarrow H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{t_n\}$ be sequence in $(0, 1)$ which satisfies $\lim_{n \rightarrow \infty} t_n = 0$. Then for a contraction mapping $f : C \rightarrow C$ with coefficient $\alpha \in (0, 1)$, the sequence $\{x_n\}$ defined by (0.11) converges strongly to z , where, z is the unique solution in $F(T)$ to the variation inequality

$$(0.14) \quad \langle (I - f)z, x - z \rangle \geq 0, x \in F(T)$$

or equivalently $z = G(f(z))$, where G is a metric projection mapping from H onto $F(T)$.

Theorem 2 Let C be a nonempty closed convex subset of a Hilbert space H , P be the metric projection of H onto C and $T : C \rightarrow H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ which satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then for a contraction mapping $f : C \rightarrow C$ with coefficient $\alpha \in (0, 1)$, the sequence $\{y_n\}$ defined by (0.12) converges strongly to z , where, z is the unique solution in $F(T)$ to the variation inequality

Theorem 3 Let C be a nonempty closed convex subset of a Hilbert space H , P the metric projection of H onto C and $T : C \rightarrow H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be sequence in $[0, 1]$ which satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then for a contraction mapping $f : C \rightarrow C$ with coefficient $\alpha \in (0, 1)$, the sequence $\{z_n\}$ defined by (0.13) converges strongly to z , where, z is the unique solution in $F(T)$ to the variation inequality

$$(0.15) \quad \langle (I - f)z, x - z \rangle \geq 0, x \in F(T)$$

or equivalently $z = G(f(z))$, where G is a metric projection mapping from H onto $F(T)$.

2.4 S. Plubtieng and R. Punpeang, Implicit iteration process of nonexpansive nonself-mappings in Banach spaces

In this paper, we extend Xu and Yin's results [?] to study the contractions T_n, S_n and U_n define by

$$(0.16) \quad T_n x = (1 - \alpha_n)u + \alpha_n T[(1 - \beta_n)x + \beta_n T x]$$

$$(0.17) \quad S_n x = (1 - \alpha_n)u + \alpha_n P T[(1 - \beta_n)x + \beta_n P T x]$$

$$(0.18) \quad U_n x = P[(1 - \alpha_n)u + \alpha_n T P[(1 - \beta_n)x + \beta_n T x]],$$

where $\{\alpha_n\} \subseteq (0, 1)$, $0 \leq \beta_n \leq \beta < 1$, and P is the nearest point projection of H onto C .

Theorem 1 Let E be a real reflexive Banach space with a uniformly *Gâteaux* differentiable norm. Let C be a nonempty closed convex subset of E which has normal structure, and $T : C \rightarrow C$ be a nonexpansive mapping. Suppose that for some $u \in C$, $\{\alpha_n\}_{n=1}^{\infty} \subseteq (0, 1)$ and $0 \leq \beta_n \leq \beta < 1$. Then, a mapping T_n defined by (0.19) has a unique fixed point $x_n \in C$. Further, T has a fixed point if and only if $\{x_n\}$ remains bounded as $\alpha_n \rightarrow 1$. In this case, $\{x_n\}$ converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T .

Theorem 2 Let E be a uniformly convex Banach space with a uniformly *Gâteaux* differentiable norm. Let C be a nonempty closed convex subset of E , and $T : C \rightarrow E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E and that for some $u \in C$, $\{\alpha_n\}_{n=1}^{\infty} \subseteq (0, 1)$ and $0 \leq \beta_n < \beta < 1$. Then, a mapping S_n defined by (0.20) has a unique fixed point $y_n \in C$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $\alpha_n \rightarrow 1$. In this case, $\{y_n\}$ converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T .

Theorem 3 Let E be a uniformly convex Banach space with a uniformly *Gâteaux* differentiable norm. Let C be a nonempty closed convex subset of E , and $T : C \rightarrow E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E , that for some $u \in C$, $\{\alpha_n\}_{n=1}^{\infty} \subseteq (0, 1)$, $0 \leq \beta_n \leq \beta < 1$. Then a mapping U_n defined by (0.21) has a unique fixed point $z_n \in C$. Further, then T has a fixed point if and only if $\{z_n\}$ remains bounded as $\alpha_n \rightarrow 1$ and $\beta_n \rightarrow 0$. In this case, $\{z_n\}$ converges strongly as $\alpha_n \rightarrow 1$ and $\beta_n \rightarrow 0$ to a fixed point of T .

2.5 A. Kangtunyakarn and S. Plubtieng, Strong convergence of an implicit iteration process for asymptotically nonexpansive mappings,

Theorem 1 Let C be a closed convex subset of Hilbert space H and T be asymptotically nonexpansive mapping on C into itself with Lipschitz condition

k_n and suppose that $F(T)$ is nonempty.

Let

$$b_n = \frac{1}{n} \sum_{j=1}^n \left(\frac{1}{2} + \left| \frac{1}{2} - k_j \right| + e^{-j} \right),$$

$$0 < a < \frac{1}{2}, 0 \leq a' < \frac{1}{2} \text{ and } x_0 \in C.$$

Then, a mapping T_n on C given by

$$T_n x = \alpha_n x_0 + (1 - \alpha_n) A_n [\beta_n x + (1 - \beta_n) A_n x] \quad \text{for all } x \in C$$

has a unique fixed point u_n in C , when

$$\alpha_n = \frac{b_n - \frac{1}{2}}{b_n - \frac{1}{2} + a}, \quad \beta_n = \frac{b_n - \frac{1}{2}}{b_n - \frac{1}{2} + a'}, \quad \text{and} \quad A_n = \frac{1}{n} \sum_{j=1}^n T^j.$$

Further $\{u_n\}$ converges strongly to the element of $F(T)$ which nearest to x_0 .

2.6 S. Plubtieng and R. Punpeang, Implicit iteration process with errors of nonexpansive nonself-mappings in Banach spaces,

In this paper, we extend Xu and Yin's results [?] to study the contractions T_n , S_n and U_n define by

$$(0.19) \quad T_n x = a_n u + b_n T[\bar{a}_n x + \bar{b}_n T x + \bar{c}_n u_n] + c_n v_n$$

$$(0.20) \quad S_n x = a_n u + b_n P T[\bar{a}_n x + \bar{b}_n P T x + \bar{c}_n u_n] + c_n v_n,$$

$$(0.21) \quad U_n x = P[a_n u + b_n T P[\bar{a}_n x + \bar{b}_n T x + \bar{c}_n u_n] + c_n v_n]$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a}_n\}$, $\{\bar{b}_n\}$, and $\{\bar{c}_n\}$ be real sequences on $[0,1]$ such that $a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1$, $0 \leq b_n \leq \beta < 1$, $\bar{b}_n \leq \beta < 1$, $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \bar{c}_n < \infty$, and P is the nearest point projection of H onto C .

Theorem 1 Let E be a real reflexive Banach space with a uniformly *Gâteaux* differentiable norm. Let C be a nonempty closed convex subset of E which has normal structure, and $T : C \rightarrow C$ be a nonexpansive mapping. Let $u \in C$, $\{u_n\}$ and $\{v_n\}$ be bounded sequences on C and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a}_n\}$, $\{\bar{b}_n\}$, and $\{\bar{c}_n\}$ be real sequences on $[0,1]$ satisfying the conditions:

- (i) $a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1$,
- (ii) $0 \leq b_n \leq \beta < 1$, $\bar{b}_n \leq \beta < 1$, $\forall n \geq 1$,
- (iii) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \bar{c}_n < \infty$.

Then the mapping T_n defined by (0.19) has a unique fixed point $x_n \in C$. Further, T has a fixed point if and only if $\{x_n\}$ remains bounded as $a_n \rightarrow 0$. In this case, $\{x_n\}$ converges strongly as $a_n \rightarrow 0$ to a fixed point of T .

Theorem 2 Let E be a uniformly convex Banach space with a uniformly *Gâteaux* differentiable norm. Let C be a nonempty closed convex subset of E , and $T : C \rightarrow E$ be a nonexpansive nonself-mapping satisfying the weak

inwardness condition. Suppose that C is a sunny nonexpansive retract of E and that for some $u \in C$, let $\{u_n\}$ and $\{v_n\}$ be bounded sequences on C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{\bar{a}_n\}, \{\bar{b}_n\}$, and $\{\bar{c}_n\}$ be real sequences on $[0,1]$ satisfying the conditions:

- (i) $a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1$,
- (ii) $0 \leq b_n \leq \beta < 1, \bar{b}_n \leq \beta < 1, \forall n \geq 1$,
- (iii) $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} \bar{c}_n < \infty$.

Then, a mapping S_n defined by (0.20) has a unique fixed point $y_n \in C$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $a_n \rightarrow 0$. In this case, $\{y_n\}$ converges strongly as $a_n \rightarrow 0$ to a fixed point of T .

Theorem 3 Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of E , and $T : C \rightarrow E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E and that for some $u \in C$, let $\{u_n\}$ and $\{v_n\}$ be bounded sequences on C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{\bar{a}_n\}, \{\bar{b}_n\}$, and $\{\bar{c}_n\}$ be real sequences on $[0,1]$ satisfying the conditions:

- (i) $a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1$,
- (ii) $0 \leq b_n \leq \beta < 1, \bar{b}_n \leq \beta < 1, \forall n \geq 1$,
- (iii) $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} \bar{c}_n < \infty$.

Then, a mapping U_n defined by (0.21) has a unique fixed point $z_n \in C$. Further, T has a fixed point if and only if $\{z_n\}$ remains bounded as $a_n \rightarrow 0$. In this case, $\{z_n\}$ converges strongly as $a_n \rightarrow 0$ to a fixed point of T .

2.7 A. Kangtunyakarn and S. Plubtieng, Strong convergence theorems of an implicit iteration process with errors for asymptotically nonexpansive mappings,

Theorem 1 Let C be a closed convex subset of Hilbert space H and T be asymptotically nonexpansive mapping on C into itself with Lipschitz condition k_n and suppose that $F(T)$ is nonempty.

Let

$$b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j}) \quad 0 < a < 1 \text{ and } x_0 \in C,$$

and let $T_n : C \rightarrow C$ be a mapping given by

$$T_n x = \alpha_n x_0 + \beta_n A_n x + \gamma_n v_n \quad ; \forall x \in C, v_n \in C,$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\alpha_n = \frac{b_n - 1}{b_n - 1 + a}$, and $\gamma_n < \frac{(b_n - 1)^2}{(b_n - 1)^2 + a}$ for all $n \geq 1$, $b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j})$, $A_n = \frac{1}{n} \sum_{j=1}^n T^j$, $0 < a' < 1$, $0 < a < 1$, and $\{v_n\}$ is a bounded sequence in C . Then T_n has a unique fixed point u_n in C . Further $\{u_n\}$ converges strongly to the element of $F(T)$ which nearest to x_0 .

3. Random fixed point theorems

3.1 P. Kumam and S. Plubtieng, The characteristic of noncompact convexity and random fixed point theorem for set-valued operators,

Theorem 1 Let C be a nonempty closed bounded convex subset of a Banach space X such that $\epsilon_\beta(X) < 1$, and $T : C \rightarrow KC(C)$ a nonexpansive mapping. Then T has a fixed point.

Theorem 2 Let C be a nonempty closed bounded convex separable subset of a Banach space X such that $\epsilon_\beta(X) < 1$, and $T : \Omega \times C \rightarrow KC(C)$ be a set-valued nonexpansive random operator. Then T has a random fixed point.

3.2 P. Kumam and S. Plubtieng, Random fixed point theorems for multivalued nonexpansive non-self random operators,

Theorem 1 Let C be a nonempty closed bounded convex separable subset of a Banach spaces X such that $\epsilon_\beta(X) < 1$, and $T : \Omega \times C \rightarrow KC(X)$ be a multivalued nonexpansive random operator and $1-\chi$ -contractive mapping, such that for each $\omega \in \Omega$, $T(\omega, C)$ is a bounded set, which satisfies the inwardness condition, i.e., for each $\omega \in \Omega$, $T(\omega, x) \subset I_C(x)$, $\forall x \in C$. Then T has a random fixed point.

Theorem 2 Let C be a nonempty closed bounded convex separable subset of a Banach spaces X such that $\epsilon_\alpha(X) < 1$, and $T : \Omega \times C \rightarrow KC(X)$ be a multivalued nonexpansive random operator and $1-\chi$ -contractive nonexpansive mapping, such that for each $\omega \in \Omega$, $T(\omega, C)$ is a bounded set, which satisfies the inwardness condition, i.e., for each $\omega \in \Omega$, $T(\omega, x) \subset I_C(x)$, $\forall x \in C$. Then T has a random fixed point.

3.3 P. Kumam and S. Plubtieng, Random fixed point theorems for asymptotically regular mappings,

Theorem 1 Let C be a nonempty weakly compact convex separable subset of a Banach space with $WCS(X) > 1$ and $T : \Omega \times C \rightarrow C$ be a random uniformly Lipschitzian mapping such that $\sigma(T(\omega, \cdot)) < \sqrt{WCS(X)}$ for all $\omega \in \Omega$. Suppose in addition that T is asymptotically regular on C . Then T has a random fixed point.

Theorem 2 Let X be a reflexive Banach space, C be a nonempty bounded convex separable subset of X and $T : \Omega \times C \rightarrow C$ be a random asymptotically

regular operator. If there exist a constant $c \in \mathbb{R}$ such that

$$\sigma(T(\omega, \cdot)) \leq c < \frac{1 + \sqrt{1 + 4WCS(X)(\kappa_\omega(X) - 1)}}{2}$$

for all $\omega \in \Omega$ then T has a random fixed point.

หนังสืออ้างอิง

1. R.E. Bruck, T. Kuczumow and S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform opial property*, Colloq. Math. 65 (1993) 169–179.
2. Y.J. Cho, H. Zhou and G. Guo, *Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings*, Comp. and Math. Appl. 47 (2004) 707–717.
3. R. Glowinski and P. Le Tallec, "Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics" Siam, Philadelphia, 1989.
4. K. Goebel and W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 35 (1972) 171–174.
5. S. Haubruge, V. H Nguyen, and J. J Surodiot, *Convergence analysis and applications of the Glowinski–Le Tallec splitting method for finding a zero of the sum of two maximal monotone operators*, J. Optim. Theory Appl. 97(1998), 645–673.
6. S. Ishikawa, *Fixed point by a new iterations*, Pro. Amer. Math. Soc. 44 (1974) 147–150.
7. J.U. Jeong, M. Aslam Noor and A. Rafiq, *Noor iterations for nonlinear Lipschitzian strongly accretive mappings*, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math., 11(4) (2004), 339–350.
8. T.H. Kim and J.W. Choi, *Asymptotic behavior of almost-orbits of non-Lipschitzian mappings in Banach spaces*, Math. Japonica 38(1993), 191–197.
9. G.E. Kim and T.H. Kim, *Mann and Ishikawa iterations with errors for non-Lipschitzian mappings in Banach spaces*, Comp. and Math. Appl. 42 (2001) 1565–1570.
10. Q. Liu, *Iterations sequence for asymptotically quasi-nonexpansive mapping with an error member*, J. Math. Anal. Appl. 259 (2001) 18–24.
11. W. R. Mann, *Mean value methods in iterations*, Pro. Amer. Math. Soc. 4 (1953) 506–510.
12. W. A. Kirk, *Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type*, Israel. J. Math. 17 (1974) 339–346.
13. M. Aslam Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl. 251, (2000), 217–229.
14. M. Aslam Noor, *Three-step iterative algorithms for multivalued quasi variational inclusions*, J. Math. Anal. Appl. 255 (2001).
15. M. Aslam Noor, T.M. Rassias and Z. Huang *Three-step iterations for nonlinear accretive operator equations*, J. Math. Anal. Appl. 274 (2002), 59–68.
16. B.E. Rhoades, *Fixed point iterations for certain nonlinear mappings*, J. Math. Anal. Appl. 183 (1994), 118–120.
17. J. Schu, *Iterative construction of fixed points of strictly quasicontractive mapping*, Appl. Anal. 40 (1991) 67–72.
18. S. Suantai, *Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings* J. Math. Anal. Appl. (in press).
19. K.K. Tan, H.K. Xu, *Approximating fixed points of nonexpansive mapping by the Ishikawa iterations process*, J. Math. Anal. Appl. 178 (1993) 301–308.

20. B.L. Xu and M. Aslam Noor, *Fixed-Point Iterations for Asymptotically Nonexpansive Mappings in Banach Spaces*, J. Math. Anal. Appl. 267, (2002) 444–453.

Output จากโครงการวิจัยที่ได้รับจาก สกว.

1. S. Plubtieng and R. Wangkeeree, Fixed point iteration for asymptotically quasi-nonexpansive mappings in Banach spaces, Inter. J. Math & Math. Sci. (2005), 1685-1692.
2. S. Plubtieng and R. Wangkeeree, Noor Iterations with error for non-Lipschitzian mappings in Banach spaces, Kyungpook Math. J. 45 (2005), In press.
3. S. Plubtieng and R. Punpaeng, Implicit iteration process of nonexpansive nonself-mappings, Inter. J. Math & Math. Sci. (2005) In press.
4. S. Plubtieng and R. Wangkeeree, Strong convergence theorems for multi-step Noor iterations with errors in Banach spaces, J. Math. Anal. Appl. (accepted).
5. P. Kumam and S. Plubtieng, The characteristic of noncompact convexity and random fixed point theorem for set-valued operators, Cze. Math. J. (accepted).
6. P. Kumam and S. Plubtieng, Random fixed point theorems for multivalued nonexpansive non-self random operators, J. Appl. Math. Stoc. Anal. (accepted).
7. S. Plubtieng and R. Wangkeeree, Strong convergence theorems of viscosity averaging iterations for asymptotically nonexpansive nonself-mappings, Proc. Amer. Math. Soc. (submitted).
8. S. Plubtieng and R. Wangkeeree, Strong convergence theorems of viscosity averaging iterations for nonexpansive nonself-mappings in Hilbert spaces, J. Korean. Math. Soc. (submitted).
9. S. Plubtieng and R. Punpaeng, Implicit iteration process of nonexpansive nonself-mappings in Banach spaces, Bull. Korean. Math. Soc. (submitted).
10. S. Plubtieng and R. Wangkeeree, Strong convergence theorems for three-step iterations with errors for non-Lipschitzian nonself-mappings in Banach spaces, Com. Math. Appl. (submitted).
11. A. Kangtanyakarn and S. Plubtieng, Strong convergence of an implicit iteration process for asymptotically nonexpansive mappings, Kyungpook Math. J. (submitted).
12. S. Plubtieng and R. Punpaeng, Implicit iteration process with errors of nonexpansive nonself-mappings in Banach spaces, Southeast Asian. Bull. Math. (submitted).
13. A. Kangtanyakarn and S. Plubtieng, Strong convergence theorems of an implicit iteration process with errors for asymptotically nonexpansive mappings, Acta. Scient. Math. (submitted).
14. S. Plubtieng and R. Wangkeeree, Ishikawa Iteration Sequences for Asymptotically Quasi-Nonexpansive Nonself-Mappings with Error Members, Cze. Math. J. (Submitted).

15. P. Kumam and S. Plubtieng, Random fixed point theorems for asymptotically regular mappings, *Stochastic Anal. Appl.* (submitted).
16. I. Inchan and S. Plubtieng, Weak and strong convergence of scheme with errors for a finite family of nonexpansive mappings, *Nonlinear Anal.* (submitted).
17. S. Plubtieng, R. Punpeang and R. Wangkeeree, Weak and strong convergence of modified Noor iterations with errors for three asymptotically nonexpansive mappings, *J. Math. Anal.* (submitted).

ภาคผนวก 1

**Fixed point iteration for asymptotically quasi-
nonexpansive mappings in Banach spaces**

S. Plubtieng and R. Wangkeeree

Inter. J. Math & Math. Sci. (2005), 1685-1692.

FIXED POINT ITERATION FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

SOMYOT PLUBTIENG AND RABIAN WANGKEEREE

Received 21 October 2004 and in revised form 20 April 2005

Suppose that C is a nonempty closed convex subset of a real uniformly convex Banach space X . Let $T : C \rightarrow C$ be an asymptotically quasi-nonexpansive mapping. In this paper, we introduce the three-step iterative scheme for such map with error members. Moreover, we prove that if T is uniformly L -Lipschitzian and completely continuous, then the iterative scheme converges strongly to some fixed point of T .

1. Introduction

Let C be a subset of normed space X , and let T be a self-mapping on C . T is said to be *nonexpansive* provided that $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 0$ such that $\|T^n x - T^n y\| \leq (1 + k_n)\|x - y\|$ for all $x, y \in C$ and $n \geq 1$. T is said to be an *asymptotically quasi-nonexpansive map* if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 0$ such that $\|T^n x - p\| \leq (1 + k_n)\|x - p\|$ for all $x \in C$ and $p \in F(T)$, and $n \geq 1$ ($F(T)$ denotes the set of fixed points of T , that is, $F(T) = \{x \in C : Tx = x\}$).

From the above definitions, if $F(T) \neq \emptyset$, then asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive mapping.

The concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk in 1972 [2]. In 2001, Noor [5, 6] introduced the three-step iterative scheme and he studied the approximate solutions of variational inclusions (inequalities) in Hilbert spaces. The three-step iterative approximation problems were studied extensively by Noor [5, 6], Glowinski and Le Tallec [1], and Haubruge et al. [3].

Recently, Xu and Noor [8] introduced the three-step iterative scheme for asymptotically nonexpansive mappings and they proved the following strong convergence theorem in Banach spaces.

THEOREM 1.1 (see [8, Theorem 2.1]). *Let X be a real uniformly convex Banach space, let C be a nonempty closed, bounded convex subset of X . Let T be a completely continuous and asymptotically nonexpansive self-mapping with sequence $\{k_n\}$ satisfying $k_n \geq 0$ and*

1686 Fixed points iteration

$\sum_{n=1}^{\infty} k_n < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying

(i) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$,

(ii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

For a given $x_0 \in D$, define

$$\begin{aligned} z_n &= \gamma_n T^n x_n + (1 - \gamma_n) x_n, \\ y_n &= \beta_n T^n z_n + (1 - \beta_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n. \end{aligned} \quad (1.1)$$

Then $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to a fixed point of T .

In this paper, we will extend the iterative scheme (1.1) to the iterative scheme of asymptotically quasi-nonexpansive mappings with error members. Moreover, we will prove the strong convergence of iterative scheme to a fixed point of T (C need not to be a bounded set), requiring T to be uniformly L -Lipschitzian and completely continuous. The results presented in this paper generalize and extend the corresponding main results of Xu and Noor [8].

2. Preliminaries

For the sake of convenience, we first recall some definitions and conclusions.

Definition 2.1 (see [2]). A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta_X(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \epsilon \right\} > 0 \quad (2.1)$$

for all $0 < \epsilon \leq 2$ (i.e., $\delta_X(\epsilon)$ is a function $(0, 2] \rightarrow (0, 1)$).

Definition 2.2. A mapping $T : C \rightarrow C$ is called *uniformly L -Lipschitzian* if there exists a constant $L > 0$ such that for all $x, y \in C$,

$$\|T^n x - T^n y\| \leq L \|x - y\|, \quad \forall n \geq 1. \quad (2.2)$$

In what follows, we will make use of the following lemmas.

LEMMA 2.3 (see [4]). Let the nonnegative number sequences $\{a_n\}$, $\{b_n\}$, and $\{d_n\}$ satisfy that

$$a_{n+1} \leq (1 + b_n) a_n + d_n, \quad \forall n = 1, 2, \dots, \sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} d_n < \infty. \quad (2.3)$$

Then,

- (1) $\lim_{n \rightarrow \infty} a_n$ exists;
- (2) if $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

LEMMA 2.4 ([7], J. Schu's Lemma). Let X be a real uniformly convex Banach space, $0 < \alpha \leq t_n \leq \beta < 1$, $x_n, y_n \in X$, $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$, and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = a$, $a \geq 0$. Then, $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

3. Main results

In this section, we prove our main theorem. First of all, we will need the following lemmas.

LEMMA 3.1. Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be an asymptotically quasi-nonexpansive mapping with sequence $\{k_n\}_{n \geq 1}$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_0 \in C$ and

$$\begin{aligned} z_n &= \alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n, \\ y_n &= \alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned} \quad (3.1)$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$, and $\{\gamma_n''\}$ are real sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are three bounded sequences in C such that

(i) $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$,

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$, $\sum_{n=1}^{\infty} \gamma_n'' < \infty$.

If $p \in F(T)$, then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists.

Proof. Let $p \in F(T)$. Since $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are bounded sequences in C , put

$$M = \sup_{n \geq 1} \|u_n - p\| \vee \sup_{n \geq 1} \|v_n - p\| \vee \sup_{n \geq 1} \|w_n - p\|. \quad (3.2)$$

Then M is a finite number. So for each $n \geq 1$, we note that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n - p\| \\ &\leq \alpha_n \|T^n y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\ &\leq \alpha_n (1 + k_n) \|y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\|, \end{aligned} \quad (3.3)$$

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n - p\| \\ &\leq \alpha_n' \|T^n z_n - p\| + \beta_n' \|x_n - p\| + \gamma_n' \|v_n - p\| \\ &\leq \alpha_n' (1 + k_n) \|z_n - p\| + \beta_n' \|x_n - p\| + \gamma_n' \|v_n - p\|, \end{aligned} \quad (3.4)$$

$$\|z_n - p\| \leq \alpha_n'' (1 + k_n) \|x_n - p\| + \beta_n'' \|x_n - p\| + \gamma_n'' \|u_n - p\|. \quad (3.5)$$

1688 Fixed points iteration

Substituting (3.5) into (3.4),

$$\begin{aligned}
 \|y_n - p\| &\leq \alpha'_n \alpha''_n (1 + k_n)^2 \|x_n - p\| \\
 &\quad + \alpha'_n \beta''_n (1 + k_n) \|x_n - p\| + \alpha'_n \gamma'_n (1 + k_n) \|u_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\
 &\leq (1 - \beta'_n - \gamma'_n) \alpha''_n (1 + k_n)^2 \|x_n - p\| + \beta'_n \|x_n - p\| \\
 &\quad + (1 - \beta'_n - \gamma'_n) \beta''_n \|x_n - p\| + m_n \\
 &\leq \beta'_n (1 + k_n)^2 \|x_n - p\| + (1 - \beta'_n) \alpha''_n (1 + k_n)^2 \|x_n - p\| \\
 &\quad + (1 - \beta'_n) \beta''_n (1 + k_n)^2 \|x_n - p\| + m_n \\
 &= \beta'_n (1 + k_n)^2 \|x_n - p\| + (1 - \beta'_n) (\alpha''_n + \beta''_n) (1 + k_n)^2 \|x_n - p\| + m_n \\
 &\leq \beta'_n (1 + k_n)^2 \|x_n - p\| + (1 - \beta'_n) (1 + k_n)^2 \|x_n - p\| + m_n \\
 &= (1 + k_n)^2 \|x_n - p\| + m_n,
 \end{aligned} \tag{3.6}$$

where $m_n = \gamma''_n (1 + k_n) M + \gamma'_n M$. Substituting (3.6) into (3.3) again, we have

$$\begin{aligned}
 \|x_{n+1} - p\| &\leq \alpha_n (1 + k_n) ((1 + k_n)^2 \|x_n - p\| + m_n) + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\
 &= \alpha_n (1 + k_n)^3 \|x_n - p\| + \alpha_n (1 + k_n) m_n + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\
 &\leq (\alpha_n + \beta_n) (1 + k_n)^3 \|x_n - p\| + (1 + k_n) m_n + \gamma_n \|w_n - p\| \\
 &\leq (1 + k_n)^3 \|x_n - p\| + (1 + k_n) m_n + \gamma_n \|w_n - p\| \\
 &\leq (1 + k_n)^3 \|x_n - p\| + (1 + k_n) m_n + \gamma_n M \\
 &= (1 + d_n) \|x_n - p\| + b_n,
 \end{aligned} \tag{3.7}$$

where $d_n = 3k_n + 3k_n^2 + k_n^3$ and $b_n = (1 + k_n) m_n + \gamma_n M$. Since $\sum_{n=1}^{\infty} d_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, by Lemma 2.3, we have that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. This completes the proof. $\square$

LEMMA 3.2. Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be an asymptotically quasi-nonexpansive mapping with sequence $\{k_n\}_{n \geq 1}$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_0 \in C$ and for each $n \geq 0$,

$$\begin{aligned}
 z_n &= \alpha''_n T^n x_n + \beta''_n x_n + \gamma''_n u_n, \\
 y_n &= \alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n, \\
 x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n,
 \end{aligned} \tag{3.8}$$

where $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are three bounded sequences in C and $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\beta'_n\}$, $\{\beta''_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\}$, and $\{\gamma''_n\}$ are real sequences in $[0, 1]$ which satisfy the same assumptions as Lemma 3.1 and the additional assumption that $0 \leq \alpha < \alpha_n$, $\beta_n, \alpha'_n, \beta'_n \leq \beta < 1$ for some α, β in $(0, 1)$. Then $\lim_{n \rightarrow \infty} \|T^n y_n - x_n\| = 0 = \lim_{n \rightarrow \infty} \|T^n z_n - x_n\|$.

Proof. For any $p \in F(T)$, it follows from Lemma 3.1, that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Let $\lim_{n \rightarrow \infty} \|x_n - p\| = a$ for some $a \geq 0$. From (3.6), we have

$$\|y_n - p\| \leq (1 + k_n)^2 \|x_n - p\| + m_n. \quad (3.9)$$

Taking $\limsup_{n \rightarrow \infty}$ in both sides, we obtain

$$\limsup_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = \lim_{n \rightarrow \infty} \|x_n - p\| = a. \quad (3.10)$$

Note that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|T^n y_n - p\| &\leq \limsup_{n \rightarrow \infty} (1 + k_n) \|y_n - p\| = \limsup_{n \rightarrow \infty} \|y_n - p\| \leq a, \\ a &= \lim_{n \rightarrow \infty} \|x_{n+1} - p\| = \lim_{n \rightarrow \infty} \|\alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n - p\| \\ &= \lim_{n \rightarrow \infty} \left\| \alpha_n \left[T^n y_n - p + \frac{\gamma_n}{2\alpha_n} (w_n - p) \right] + \beta_n \left[x_n - p + \frac{\gamma_n}{2\beta_n} (w_n - p) \right] \right\| \\ &= \lim_{n \rightarrow \infty} \left\| \alpha_n \left[T^n y_n - p + \frac{\gamma_n}{2\alpha_n} (w_n - p) \right] + (1 - \alpha_n) \left[x_n - p + \frac{\gamma_n}{2\beta_n} (w_n - p) \right] \right\|. \end{aligned} \quad (3.11)$$

By J. Schu's Lemma 2.4, we have

$$\lim_{n \rightarrow \infty} \left\| T^n y_n - x_n + \left(\frac{\gamma_n}{2\alpha_n} - \frac{\gamma_n}{2\beta_n} \right) (w_n - p) \right\| = 0. \quad (3.12)$$

Since $\lim_{n \rightarrow \infty} \|(\gamma_n/2\alpha_n - \gamma_n/2\beta_n)(w_n - p)\| = 0$, it follows that

$$\lim_{n \rightarrow \infty} \|T^n y_n - x_n\| = 0. \quad (3.13)$$

Finally, we will prove that $\lim_{n \rightarrow \infty} \|T^n z_n - x_n\| = 0$. To this end, we note that for each $n \geq 1$,

$$\|x_n - p\| \leq \|T^n y_n - x_n\| + \|T^n y_n - p\| \leq \|T^n y_n - x_n\| + (1 + k_n) \|y_n - p\|. \quad (3.14)$$

Since $\lim_{n \rightarrow \infty} \|T^n y_n - x_n\| = 0 = \lim_{n \rightarrow \infty} k_n$, we obtain that

$$a = \lim_{n \rightarrow \infty} \|x_n - p\| \leq \liminf_{n \rightarrow \infty} \|y_n - p\|. \quad (3.15)$$

It follows that

$$a \leq \liminf_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|y_n - p\| \leq a. \quad (3.16)$$

This implies that

$$\lim_{n \rightarrow \infty} \|y_n - p\| = a. \quad (3.17)$$

1690 Fixed points iteration

On the other hand, we note that

$$\begin{aligned} \|z_n - p\| &= \|\alpha'_n T^n x_n + \beta'_n x_n + \gamma'_n u_n - p\| \\ &\leq \alpha'_n (1 + k_n) \|x_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|u_n - p\| \\ &\leq \alpha'_n (1 + k_n) \|x_n - p\| + (1 - \alpha'_n) (1 + k_n) \|x_n - p\| + \gamma'_n \|u_n - p\| \\ &\leq (1 + k_n) \|x_n - p\| + \gamma'_n \|u_n - p\|. \end{aligned} \quad (3.18)$$

By boundedness of the sequence $\{u_n\}$ and $\lim_{n \rightarrow \infty} k_n = 0 = \lim_{n \rightarrow \infty} \gamma'_n$, we have

$$\limsup_{n \rightarrow \infty} \|z_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = a, \quad (3.19)$$

and so

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|T^n z_n - p\| &\leq \limsup_{n \rightarrow \infty} (1 + k_n) \|z_n - p\| \leq a, \\ a &= \lim_{n \rightarrow \infty} \|y_n - p\| = \lim_{n \rightarrow \infty} \|\alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n - p\| \\ &= \lim_{n \rightarrow \infty} \left\| \alpha'_n \left[T^n z_n - p + \frac{\gamma'_n}{2\alpha'_n} (v_n - p) \right] + \beta'_n \left[x_n - p + \frac{\gamma'_n}{2\beta'_n} (v_n - p) \right] \right\| \\ &= \lim_{n \rightarrow \infty} \left\| \alpha'_n \left[T^n z_n - p + \frac{\gamma'_n}{2\alpha'_n} (v_n - p) \right] + (1 - \alpha'_n) \left[x_n - p + \frac{\gamma'_n}{2\beta'_n} (v_n - p) \right] \right\|. \end{aligned} \quad (3.20)$$

By J. Schu's Lemma 2.4, we have

$$\lim_{n \rightarrow \infty} \left\| T^n z_n - x_n + \left(\frac{\gamma'_n}{2\alpha'_n} - \frac{\gamma'_n}{2\beta'_n} \right) (v_n - p) \right\| = 0. \quad (3.21)$$

Since $\lim_{n \rightarrow \infty} \|(\gamma'_n/2\alpha'_n - \gamma'_n/2\beta'_n)(v_n - p)\| = 0$, it follows that

$$\lim_{n \rightarrow \infty} \|T^n z_n - x_n\| = 0. \quad (3.22)$$

This completes the proof. $\square$

THEOREM 3.3. Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be uniformly L -Lipschitzian, completely continuous, and an asymptotically quasi-nonexpansive mapping with sequence $\{k_n\}_{n \geq 1}$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_0 \in C$ and for each $n \geq 0$,

$$\begin{aligned} z_n &= \alpha'_n T^n x_n + \beta'_n x_n + \gamma'_n u_n, \\ y_n &= \alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned} \quad (3.23)$$

where $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are three bounded sequences in C and $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\beta'_n\}$, $\{\beta''_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\}$, and $\{\gamma''_n\}$ are real sequences in $[0, 1]$ which satisfy the same assumptions as Lemma 3.1 and the additional assumption that $0 \leq \alpha < \alpha_n$, $\beta_n, \alpha'_n, \beta'_n \leq \beta < 1$ for some α, β in $(0, 1)$. Then $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to a fixed point of T .

Proof. It follows from Lemma 3.2 that

$$\lim_{n \rightarrow \infty} \|T^n y_n - x_n\| = 0 = \lim_{n \rightarrow \infty} \|T^n z_n - x_n\| \quad (3.24)$$

and this implies that

$$\|x_{n+1} - x_n\| \leq \alpha_n \|T^n y_n - x_n\| + \gamma_n \|w_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.25)$$

We note that

$$\begin{aligned} \|T^n x_n - x_n\| &\leq \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \leq L \|x_n - y_n\| + \|T^n y_n - x_n\| \\ &\leq \alpha'_n L \|x_n - T^n z_n\| + \gamma'_n L \|v_n - x_n\| + \|T^n y_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (3.26)$$

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1} x_{n+1}\| + \|T^{n+1} x_{n+1} - T^{n+1} x_n\| + \|T^{n+1} x_n - Tx_n\| \\ &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1} x_{n+1}\| + (1 + k_{n+1}) \|x_{n+1} - x_n\| + L \|T^n x_n - x_n\|. \end{aligned} \quad (3.27)$$

It follows from (3.25), (3.26), and the above inequality that

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0. \quad (3.28)$$

By Lemma 3.1, $\{x_n\}$ is bounded. It follows from our assumption that T is completely continuous and that there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \rightarrow p \in C$ as $k \rightarrow \infty$. Moreover, by (3.28), we have $\|Tx_{n_k} - x_{n_k}\| \rightarrow 0$ which implies that $x_{n_k} \rightarrow p$ as $k \rightarrow \infty$. By (3.28) again, we have

$$\|p - Tp\| = \lim_{k \rightarrow \infty} \|x_{n_k} - Tx_{n_k}\| = 0. \quad (3.29)$$

This shows that $p \in F(T)$. Furthermore, since $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, we have $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$, that is, $\{x_n\}$ converges to some fixed point of T . It follows that

$$\begin{aligned} \|y_n - x_n\| &\leq \alpha'_n \|T^n z_n - x_n\| + \gamma'_n \|v_n - x_n\| \rightarrow 0, \\ \|z_n - x_n\| &\leq \alpha''_n \|T^n x_n - x_n\| + \gamma''_n \|u_n - x_n\| \rightarrow 0. \end{aligned} \quad (3.30)$$

Therefore, $\lim_{n \rightarrow \infty} y_n = p = \lim_{n \rightarrow \infty} z_n$. This completes the proof. $\square$

References

- [1] R. Glowinski and P. Le Tallec, *Augmented Lagrangian and Operator-Splitting Methods in Non-linear Mechanics*, SIAM Studies in Applied Mathematics, vol. 9, SIAM, Pennsylvania, 1989.
- [2] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 35 (1972), 171–174.
- [3] S. Haubruge, V. H. Nguyen, and J. J. Strodiot, *Convergence analysis and applications of the Glowinski-Le Tallec splitting method for finding a zero of the sum of two maximal monotone operators*, J. Optim. Theory Appl. 97 (1998), no. 3, 645–673.

1692 Fixed points iteration

- [4] Q. Liu, *Iteration sequences for asymptotically quasi-nonexpansive mapping with an error member of uniform convex Banach space*, J. Math. Anal. Appl. 266 (2002), no. 2, 468–471.
- [5] M. A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl. 251 (2000), no. 1, 217–229.
- [6] ———, *Three-step iterative algorithms for multivalued quasi variational inclusions*, J. Math. Anal. Appl. 255 (2001), no. 2, 589–604.
- [7] J. Schu, *Iterative construction of fixed points of strictly pseudocontractive mappings*, Appl. Anal. 40 (1991), no. 2-3, 67–72.
- [8] B. Xu and M. A. Noor, *Fixed-point iterations for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. 267 (2002), no. 2, 444–453.

Somyot Plubtieng: Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

E-mail address: somyotp@nu.ac.th

Rabian Wangkeeree: Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

E-mail address: rabianw@nu.ac.th

International Journal of Mathematics and Mathematical Sciences

[Home](#) | [About this Journal](#) | [MS Tracking System](#) | [Author Index](#) | [Contents](#)

Volume 2005

Number 11

31 July 2005

CONTENTS

RESEARCH ARTICLES

- | | |
|--|------|
| A topological isomorphism invariant for certain AF algebras, Ryan J. Zerr | 1665 |
| On f-derivations of BCI-algebras, Jianming Zhan and Yong Lin Liu | 1675 |
| Fixed point iteration for asymptotically quasi-nonexpansive mappings in Banach spaces, Somyot Plubtieng and Rabian Wangkeeree | 1685 |
| Global and local optimality conditions in set-valued optimization problems, M. Durea | 1693 |
| Some generalized difference sequence spaces of invariant means defined by Orlicz functions, Ahmad H. A. Bataineh and Laith E. Azar | 1713 |
| Characterizations of fixed points of nonexpansive mappings, Tomonari Suzuki | 1723 |
| Transitive Courant algebroids, Izu Vaisman | 1737 |
| Brain-like functor control machine for general humanoid biodynamics, Vladimir Ivancevic and Nicholas Beagley | 1759 |
| The case of equality in Landau's problem, G. W. Hagerty and P. Nag | 1781 |
| Sensitivity analysis of solutions for a system of generalized parametric nonlinear quasivariational inequalities, Zeqing Liu, Beibei Zhu, Shin Min Kang, and Gwang Il Kim | 1795 |
| A necessary and sufficient condition for global existence for a quasilinear reaction-diffusion system, Alan V. Lair | 1809 |

Magneto-viscoelastic plane waves in rotating media in the generalized thermoelasticity II, S. K. Roy Choudhuri and Manidipa Banerjee (Chattopadhyay)

1819

Copyright © 2005 Hindawi Publishing Corporation. All rights reserved.

ภาคผนวก 2

**Noor Iterations with error for non-Lipschitzian
mappings in Banach spaces**

S. Plubtieng and R. Wangkeeree

Kyungpook Math. J. 45 (2005), In press.

KYUNGPOOK Math. J. 45(2005), 000-000

Noor Iterations with Error for Non-Lipschitzian Mappings in Banach Spaces

SOMYOT PLUBTIENG AND RABIAN WANGKEEREE

Department of Mathematics, Naresuan University, Pitsanulok 65000, Thailand

e-mail: Somyotp@nu.ac.th and Rabianw@nu.ac.th

ABSTRACT. Suppose C is a nonempty closed convex subset of a real uniformly convex Banach space X . Let $T : C \rightarrow C$ be an asymptotically nonexpansive in the intermediate sense mapping. In this paper we introduced the three-step iterative sequence for such map with error members. Moreover, we prove that, if T is completely continuous then the our iterative sequence converges strongly to a fixed point of T .

1. Introduction

Let C be a subset of real normed linear space X , and let T be a self-mapping on C . T is said to be *nonexpansive* provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $\lim_{n \rightarrow \infty} k_n = 1$ such that for each $x, y \in C$ and $n \geq 1$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|.$$

T is called *asymptotically nonexpansive in the intermediate sense* [1] provided T is uniformly continuous and

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically nonexpansive in the intermediate sense and asymptotically quasi-nonexpansive mapping. But the converges dose not holds as the following example:

Example 1.1 (see [6]). Let $X = \mathbb{R}$, $C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in C$, define

$$T(x) = \begin{cases} kx \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Received October 19, 2004, and, in revised form, December 16, 2004.

2000 Mathematics Subject Classification: 47H10, 47H09, 46B20.

Key words and phrases: asymptotically nonexpansive in the intermediate sense mappings, completely continuous, uniformly convex.

Supported by The Thailand Research Fund.

Then T is an asymptotically nonexpansive in the intermediate sense but it is not asymptotically nonexpansive mapping.

The concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk ([3]) in 1992. In 2001, Noor ([8], [9]) have introduced the three-step iterative sequences and he studied the approximate solutions of variational inclusions (inequalities) in Hilbert spaces. The three-step iterative approximation problems were studied extensively by Noor ([8], [9]), Glowinski and Le Tallec ([2]), Haubruge et al ([4]).

In 2002, Xu and Noor ([14]) introduced the three-step iterative for asymptotically nonexpansive mappings and they proved the following strong convergence theorem in Banach spaces;

Theorem XN ([14], Theorem 2.1). *Let X be a real uniformly convex Banach space, C be a nonempty closed, bounded convex subset of X . Let T be a completely continuous asymptotically nonexpansive self-mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying;*

- (i) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$, and
- (ii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

For a give $x_0 \in C$, define

$$(1.1) \quad \begin{aligned} z_n &= \gamma_n T^n x_n + (1 - \gamma_n) x_n \\ y_n &= \beta_n T^n z_n + (1 - \beta_n) x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n. \end{aligned}$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converges strongly to a fixed point of T .

Algorithm 1.1 (Noor iterations with errors). Let C be a nonempty subset of normed space X and let $T : C \rightarrow C$ be a mapping. For a given $x_0 \in C$, find the sequence $\{x_{n+1}\}$ such that

$$(1.2) \quad \begin{aligned} z_n &= \alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n \\ y_n &= \alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned}$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$ and $\{\gamma_n''\}$ are real sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are three bounded sequences in C .

It is clear that the Mann and Ishikawa iterations processes are all special case of the Noor iterations with error.

In this paper, we will extend the process (1.1) to Noor iteration with error (1.2) for asymptotically nonexpansive in the intermediate sense and without boundedness conditions on C . The results presented in this paper generalize and extend the corresponding main results of Xu and Noor ([14]).

2. Preliminaries

For the sake of convenience, we first recall some definitions and conclusions.

Definition 2.1 (see [3]). A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta_X(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \epsilon \right\} > 0$$

for all $0 < \epsilon \leq 2$ (i.e., $\delta_X(\epsilon)$ is a function $(0, 2] \rightarrow (0, 1)$).

Lemma 2.2 (see [7]). Let the nonnegative number sequences $\{a_n\}$, $\{b_n\}$ and $\{d_n\}$ satisfy that

$$a_{n+1} \leq (1 + b_n)a_n + d_n, \forall n = 1, 2, \dots, \sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} d_n < \infty.$$

Then

- (1) $\lim_{n \rightarrow \infty} a_n$ exists;
- (2) If $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.3 ([13], J. Schu's Lemma). Let X be a real uniformly convex Banach space, $0 < \alpha \leq t_n \leq \beta < 1$, $x_n, y_n \in X$, $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$, and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = a$, $a \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

3. Main results

In this section, we prove our main theorem. First of all, we shall need the following lemmas.

Lemma 3.1. Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be an asymptotically nonexpansive in the intermediate sense. Put

$$G_n = \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let $x_0 \in C$ and

$$\begin{aligned} z_n &= \alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' v_n \\ y_n &= \alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned}$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$ and $\{\gamma_n''\}$ are real sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are three bounded sequences in C such that

- (i) $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$.

$$(ii) \sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty.$$

Then for each $p \in F(T)$, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists.

Proof. By the Schauder fixed-point theorem [12], we obtain that $F(T) \neq \emptyset$. Let $p \in F(T)$, since $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C , so we put

$$K = \sup_{n \geq 1} \|u_n - p\| \vee \sup_{n \geq 1} \|v_n - p\| \vee \sup_{n \geq 1} \|w_n - p\|.$$

For each $n \geq 1$, we note that

$$\begin{aligned} (3.1) \quad \|x_{n+1} - p\| &= \|\alpha_n x_n T^n y_n + \beta_n x_n + \gamma_n w_n - p\| \\ &\leq \alpha_n \|T^n y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\ &\leq \alpha_n \|y_n - p\| + G_n + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \end{aligned}$$

and

$$\begin{aligned} (3.2) \quad \|y_n - p\| &= \|\alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n - p\| \\ &\leq \alpha'_n \|T^n z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\ &\leq \alpha'_n \|z_n - p\| + G_n + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \end{aligned}$$

and

$$(3.3) \quad \|z_n - p\| \leq \alpha''_n \|x_n - p\| + G_n + \beta''_n \|x_n - p\| + \gamma''_n \|u_n - p\|.$$

Substituting (3.3) into (3.2),

$$\begin{aligned} (3.4) \quad \|y_n - p\| &\leq \alpha'_n \alpha''_n \|x_n - p\| + \alpha'_n G_n + \alpha'_n \beta''_n \|x_n - p\| + \alpha'_n \gamma''_n \|u_n - p\| \\ &\quad + G_n + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\ &\leq (1 - \beta'_n - \gamma'_n) \alpha''_n \|x_n - p\| + \beta'_n \|x_n - p\| + (1 - \beta'_n - \gamma'_n) \beta''_n \|x_n - p\| + m_n \\ &\leq \beta'_n \|x_n - p\| + (1 - \beta'_n) \alpha''_n \|x_n - p\| + (1 - \beta'_n) \beta''_n \|x_n - p\| + m_n \\ &= \beta'_n \|x_n - p\| + (1 - \beta'_n) (\alpha''_n + \beta''_n) \|x_n - p\| + m_n \\ &\leq \beta'_n \|x_n - p\| + (1 - \beta'_n) \|x_n - p\| + m_n \\ &= \|x_n - p\| + m_n, \end{aligned}$$

where $m_n = 2G_n + \gamma'_n \|v_n - p\| + \gamma''_n \|u_n - p\|$. Substituting (3.4) into (3.1) again, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n (\|x_n - p\| + m_n) + G_n + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\ &\leq (\alpha_n + \beta_n) \|x_n - p\| + \alpha_n m_n + G_n + \gamma_n \|w_n - p\| \\ &\leq \|x_n - p\| + m_n + G_n + \gamma_n \|w_n - p\| \\ &\leq \|x_n - p\| + 3G_n + (\gamma_n + \gamma'_n + \gamma''_n) M \\ &= \|x_n - p\| + b_n, \end{aligned}$$

where $b_n = 3G_n + (\gamma_n + \gamma'_n + \gamma''_n)M$. Since $\sum_{n=1}^{\infty} b_n < \infty$, by Lemma 2.2, we have $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. This completes the proof. $\square$

Lemma 3.2. Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be an asymptotically nonexpansive in the intermediate sense. Put

$$G_n = \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \forall n \geq 1.$$

Let $x_0 \in C$ and for each $n \geq 0$,

$$\begin{aligned} z_n &= \alpha''_n T^n x_n + \beta''_n x_n + \gamma''_n u_n \\ y_n &= \alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned}$$

where $\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}, \{\beta_n\}, \{\beta'_n\}, \{\beta''_n\}, \{\gamma_n\}, \{\gamma'_n\}$ and $\{\gamma''_n\}$ are real sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are three bounded sequences in C such that

- (i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$.
- (ii) $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty$.
- (iii) $0 \leq \alpha < \alpha_n, \alpha'_n \leq \beta < 1$. Then

- (a) $\lim_{n \rightarrow \infty} \|T^n y_n - x_n\| = 0$;
- (b) $\lim_{n \rightarrow \infty} \|T^n z_n - x_n\| = 0$.

Proof. (a). For any $p \in F(T)$, it follows from Lemma 3.1, we have $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Let $\lim_{n \rightarrow \infty} \|x_n - p\| = a$ for some $a \geq 0$. From (3.4), we have

$$\|y_n - p\| \leq \|x_n - p\| + m_n, \forall n \geq 1.$$

Taking $\limsup_{n \rightarrow \infty}$ in both sides, we obtain

$$\limsup_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = \lim_{n \rightarrow \infty} \|x_n - p\| = a.$$

Note that

$$\limsup_{n \rightarrow \infty} \|T^n y_n - p\| \leq \limsup_{n \rightarrow \infty} (\|y_n - p\| + G_n) = \limsup_{n \rightarrow \infty} \|y_n - p\| \leq a.$$

Next, consider

$$\|T^n y_n - p + \gamma_n(w_n - x_n)\| \leq \|T^n y_n - p\| + \gamma_n \|w_n - x_n\|.$$

Thus,

$$(3.5) \quad \limsup_{n \rightarrow \infty} \|T^n y_n - p + \gamma_n(w_n - x_n)\| \leq a.$$

Somyot Plubtieng

From: "KMJ" <kmj@knu.ac.kr>
 To: <Somyotp@nu.ac.th>; <Rabianw@nu.ac.th>
 Sent: Thursday, July 28, 2005 3:49 PM
 Attach: 04153.tex; 04153.dvi; 04153.pdf
 Subject: Galley proof of KMJ #04153

Dear Prof. Rabian Wangkeeree,

The proof of your article KMJ # 04153, entitled "Noor Iterations with Error for Non-Lipschitzian Mappings in Banach Spaces",
 (joint work with Somyot Plubtieng)
 to be published in Kyungpook Mathematical Journal,
 is attached to this e-mail both as the "TEX" , "DVI" and "PDF" file.

If, after reading this message, you would still prefer to receive your proofs by fax or mail
 then please inform us immediately by replying to this e-mail with full fax details.
 Please use this proof solely for checking the typesetting and editing,
 also the completeness and correctness of the text, tables and figures.
 Changes to the article as accepted for publication will not be considered at this stage.
 Please do not attempt to edit the PDF file (including adding <>type notes).

Please respond within 72 hours (even if you have no corrections) including the list of corrections,
 by using the "reply" button -
 note that Publishing committee may proceed with publication of your article if no response is received.

If, for any reason, this is not possible, mark the corrections
 and any other comments on a printout of the PDF file
 and send us the corrections via the fax number or the mail address below.

Finally, thank you for choosing this journal as your publishing medium.

Sincerely yours,
 Lee, Ji Hyun
 Editorial Assistant
 Kyungpook Mathematical Journal

Kyungpook Mathematical Journal Telephone : + 82-53-950-5306
 Department of Mathematics fax : + 82-53-950-6306
 College of Natural Sciences e-mail : kmj@knu.ac.kr
 Kyungpook National University URL : <http://kmj.knu.ac.kr/>
 Daegu 702-701, Korea

ภาคผนวก 3

**Implicit iteration process of nonexpansive
nonself-mappings**

S. Plubtieng and R. Punpeang

Inter. J. Math & Math. Sci, (2005) In press

IMPLICIT ITERATION PROCESS OF NONEXPANSIVE NONSELF-MAPPINGS

SOMYOT PLUBTIENG AND RATTANAPORN PUNPAENG

Department of Mathematics, Faculty of Science, Naresuan University, Pitsanulok 65000,
Thailand

ABSTRACT. Suppose C is a nonempty closed convex subset of real Hilbert space H . Let $T : C \rightarrow H$ be a nonexpansive nonself-mapping and P is the nearest point projection of H onto C . In this paper, we study the convergence of the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ satisfying

$$\begin{aligned}x_n &= (1 - \alpha_n)u + \alpha_n T[(1 - \beta_n)x_n + \beta_n T x_n] \\y_n &= (1 - \alpha_n)u + \alpha_n P T[(1 - \beta_n)y_n + \beta_n P T y_n], \text{ and} \\z_n &= P[(1 - \alpha_n)u + \alpha_n T P[(1 - \beta_n)z_n + \beta_n T z_n]]\end{aligned}$$

where $\{\alpha_n\} \subseteq (0, 1)$, $0 \leq \beta_n \leq \beta < 1$ and $\alpha_n \rightarrow 1$ as $n \rightarrow \infty$. The results obtained in this paper extend and improve the recent ones announced by Xu and Yin, and many others.

Keywords and phrases: Nonexpansive mapping, nearest point projection, fixed points, weak inwardness condition, strong convergence theorems.

2000 Mathematics Subject Classification: 47H10, 47H09, 46B20.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a Banach space E . Then a nonself-mapping T from C into E is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$

^{*}Corresponding author.

^{*}Email addresses: Somyotp@nu.ac.th(Somyot Plubtieng) and g46060088@nu.ac.th.
(Rattanaorn Punpaeng).

for all $x, y \in C$. Given $u \in C$ and $\{\alpha_n\}$ is a sequence such that $0 < \alpha_n < 1$. We can define a contraction $T_n : C \rightarrow E$ by

$$T_n x = (1 - \alpha_n)u + \alpha_n T x, \quad x \in C. \quad (1.1)$$

If T is a self-mapping (i.e. $T(C) \subset C$), then T_n maps C into itself, and hence, by Banach's contraction principle, T_n has a unique fixed point x_n in C , that is, we have

$$x_n = (1 - \alpha_n)u + \alpha_n T x_n, \quad \forall n \geq 1. \quad (1.2)$$

(Such a sequence $\{x_n\}$ is said to be an approximating fixed point of T since it possesses the property that if $\{x_n\}$ is bounded, then $\lim_{n \rightarrow \infty} \|T x_n - x_n\| = 0$ whenever $\lim_{n \rightarrow \infty} \alpha_n = 1$. The strong convergence of $\{x_n\}$ as $\alpha_n \rightarrow 1$ for a self-mapping T of a bounded C was proved in a Hilbert space independently by Browder [1] and Halpern [3] and in a uniformly smooth Banach space by Reich [7]. Thereafter, Singh and Watson [8] extended the result of Browder and Halpern to nonexpansive nonself-mapping T satisfying Rothe's boundary condition: $T(\partial C) \subset C$ (here ∂C denotes the boundary of C). Recently, Xu and Yin [11] proved that if C is a nonempty closed convex (not necessarily bounded) subset of Hilbert space H , if $T : C \rightarrow H$ is a nonexpansive nonself-mapping, and if $\{x_n\}$ is the sequence defined by (1.2) which is bounded, then $\{x_n\}$ converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T . Marino and Trombetta [5] defined contractions S_n and U_n from C into itself by

$$S_n x = (1 - \alpha_n)u + \alpha_n P T x \quad \text{for all } x \in C \quad (1.3)$$

and

$$U_n x = P[(1 - \alpha_n)u + \alpha_n T x] \quad \text{for all } x \in C, \quad (1.4)$$

where P is the nearest point projection of H onto C . Then by the Banach contraction principle, there exists a unique fixed point y_n (resp. z_n) of S_n (resp. U_n) in C i.e.

$$y_n = (1 - \alpha_n)u + \alpha_n P T y_n \quad (1.5)$$

and

$$z_n = P[(1 - \alpha_n)u + \alpha_n T z_n]. \quad (1.6)$$

Xu and Yin [11] also proved that if C is a nonempty closed convex subset of a Hilbert space H , if $T : C \rightarrow H$ is a nonexpansive nonself-mapping satisfying the weak inwardness condition, and $\{x_n\}$ is bounded, the $\{y_n\}$ (resp. $\{z_n\}$) defined by (1.5) (resp. (1.6)) converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T .

Let C be a nonempty convex subset of Banach space E . Then for $x \in C$ we define the inward set $I_c(x)$ as follows:

$$I_c(x) = \{y \in E : y = x + a(z - x) \text{ for some } z \in C \text{ and } a \geq 0\}.$$

A mapping $T : C \longrightarrow E$ is said to be *inward* if $Tx \in I_c(x)$ for all $x \in C$. T is also said to be *weakly inward* if for each $x \in C$, Tx belongs to the closure of $I_c(x)$.

In this paper, we extend Xu and Yin's results [11] to study the contractions T_n, S_n and U_n define by

$$T_n x = (1 - \alpha_n)u + \alpha_n T[(1 - \beta_n)x + \beta_n Tx] \quad (1.7)$$

$$S_n x = (1 - \alpha_n)u + \alpha_n PT[(1 - \beta_n)x + \beta_n PTx] \quad (1.8)$$

$$U_n x = P[(1 - \alpha_n)u + \alpha_n TP[(1 - \beta_n)x + \beta_n Tx]], \quad (1.9)$$

where $\{\alpha_n\} \subseteq (0, 1)$, $0 \leq \beta_n \leq \beta < 1$, and P is the nearest point projection of H onto C . We also prove the strong convergence of the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ satisfying

$$x_n = (1 - \alpha_n)u + \alpha_n T[(1 - \beta_n)x_n + \beta_n Tx_n], \quad (1.10)$$

$$y_n = (1 - \alpha_n)u + \alpha_n PT[(1 - \beta_n)y_n + \beta_n PTy_n], \quad (1.11)$$

$$z_n = P[(1 - \alpha_n)u + \alpha_n TP[(1 - \beta_n)z_n + \beta_n Tz_n]], \quad (1.12)$$

where $\alpha_n \longrightarrow 1$ as $n \longrightarrow \infty$.

We note that if $\beta_n \equiv 0$, then (1.10), (1.11), (1.12) reduces to (1.2), (1.5), and (1.6) respectively.

2. MAIN RESULTS

In this section, we prove the strong convergence theorems for nonexpansive nonself-mappings. To prove our results, we use the following Theorem.

Theorem 2.1. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H , and $T : C \longrightarrow H$ be a nonexpansive nonself-mapping. Suppose that for some $u \in C$, $\{\alpha_n\} \subseteq (0, 1)$ and $0 \leq \beta_n \leq \beta < 1$, the mapping T_n defined by (1.7) has a (unique) fixed point $x_n \in C$ for all $n \geq 1$. Then T has a fixed point if and only if $\{x_n\}$ remains bounded as $\alpha_n \longrightarrow 1$. In this case, $\{x_n\}$ converges strongly as $\alpha_n \longrightarrow 1$ to a fixed point of T .*

Proof. We denote by $F(T)$ the fixed point set of T . Suppose that $F(T)$ is nonempty. Let $w \in F(T)$. Then for each $n \geq 1$, we have

$$\begin{aligned} \|w - x_n\| &= \|w - (1 - \alpha_n)u - \alpha_n T[(1 - \beta_n)x_n + \beta_n T x_n]\| \\ &\leq (1 - \alpha_n)\|w - u\| + \alpha_n\|w - T[(1 - \beta_n)x_n + \beta_n T x_n]\| \\ &\leq (1 - \alpha_n)\|w - u\| + \alpha_n\|w - (1 - \beta_n)x_n - \beta_n T x_n\| \\ &\leq (1 - \alpha_n)\|w - u\| + \alpha_n(1 - \beta_n)\|w - x_n\| + \alpha_n\beta_n\|w - x_n\| \\ &= (1 - \alpha_n)\|w - u\| + \alpha_n\|w - x_n\| \end{aligned}$$

and hence $(1 - \alpha_n)\|w - x_n\| \leq (1 - \alpha_n)\|w - u\|$ for all $n \geq 1$. This implies $\|w - x_n\| \leq \|w - u\|$ for all $n \geq 1$. Then $\{x_n\}$ is a bounded sequence. Conversely, suppose that $\{x_n\}$ is bounded, z is a weak cluster point of $\{x_n\}$, and $\alpha_n \rightarrow 1$ as $n \rightarrow \infty$. Then we show that $F(T) \neq \emptyset$ and $\{x_n\}$ converges strongly to a fixed point of T . We choose a subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ with $\alpha_{n_i} \rightarrow 1$ such that $x_{n_i} \rightarrow z$ weakly, we can define a real valued function g on H given by

$$g(x) = \limsup_{i \rightarrow \infty} \|x_{n_i} - x\|^2 \text{ for every } x \in H.$$

Observeing $\|x_{n_i} - x\|^2 = \|x_{n_i} - z\|^2 + 2\langle x_{n_i} - z, z - x \rangle + \|z - x\|^2$. Since $x_{n_i} \rightarrow z$ weakly, we immediately get

$$g(x) = g(z) + \|x - z\|^2 \text{ for all } x \in H,$$

in particular,

$$g(Tz) = g(z) + \|Tz - z\|^2. \quad (2.1)$$

On the other hand, we have

$$\begin{aligned} \|x_{n_i} - T x_{n_i}\| &\leq (1 - \alpha_{n_i})\|u - T x_{n_i}\| + \alpha_{n_i}\|T[(1 - \beta_{n_i})x_{n_i} + \beta_{n_i} T x_{n_i}] - T x_{n_i}\| \\ &\leq (1 - \alpha_{n_i})\|u - T x_{n_i}\| + \alpha_{n_i}\|(1 - \beta_{n_i})x_{n_i} + \beta_{n_i} T x_{n_i} - x_{n_i}\| \\ &\leq (1 - \alpha_{n_i})\|u - T x_{n_i}\| + \beta_{n_i}\|T x_{n_i} - x_{n_i}\|, \end{aligned}$$

for all $i \geq 1$. This implies that $(1 - \beta_{n_i})\|x_{n_i} - T x_{n_i}\| \leq (1 - \alpha_{n_i})\|u - T x_{n_i}\|$ and hence

$$\begin{aligned} \|x_{n_i} - T x_{n_i}\| &= \frac{(1 - \alpha_{n_i})}{(1 - \beta_{n_i})}\|u - T x_{n_i}\| \\ &\leq \frac{(1 - \alpha_{n_i})}{(1 - \beta)}\|u - T x_{n_i}\| \rightarrow 0 \text{ as } i \rightarrow \infty. \end{aligned}$$

Note that,

$$\begin{aligned}\|x_{n_i} - Tz\|^2 &= \|x_{n_i} - Tx_{n_i} + Tx_{n_i} - Tz\|^2 \\ &\leq (\|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - Tz\|)^2 \\ &= \|x_{n_i} - Tx_{n_i}\|^2 + 2\|x_{n_i} - Tx_{n_i}\|\|Tx_{n_i} - Tz\| + \|Tx_{n_i} - Tz\|^2\end{aligned}$$

for all $n \in \mathbb{N}$. Hence

$$\begin{aligned}g(Tz) &= \limsup_{i \rightarrow \infty} \|x_{n_i} - Tz\|^2 \\ &\leq \limsup_{i \rightarrow \infty} \|Tx_{n_i} - Tz\|^2 \\ &\leq \limsup_{i \rightarrow \infty} \|x_{n_i} - z\|^2 = g(z).\end{aligned}$$

This, together with (2.1) implies that $Tz = z$ and z is a fixed point of T . Now since $F(T)$ is nonempty, closed and convex, there exists a unique $v \in F(T)$ that is closest to u ; namely, v is the nearest point projection of u onto $F(T)$. For any $y \in F(T)$, we have

$$\begin{aligned}\|(x_n - u) + \alpha_n(u - y)\|^2 &= \|((1 - \alpha_n)u + \alpha_n T[(1 - \beta_n)x_n + \beta_n Tx_n] - u) + \alpha_n(u - y)\|^2 \\ &= \alpha_n^2 \|T[(1 - \beta_n)x_n + \beta_n Tx_n] - y\|^2 \\ &\leq \alpha_n^2 \|(1 - \beta_n)x_n + \beta_n Tx_n - y\|^2 \\ &= \alpha_n^2 \|(1 - \beta_n)(x_n - y) + \beta_n(Tx_n - y)\|^2 \\ &\leq \alpha_n^2 ((1 - \beta_n)\|x_n - y\| + \beta_n\|Tx_n - y\|)^2 \\ &= \alpha_n^2 \|x_n - y\|^2 \\ &= \alpha_n^2 \|x_n - u + u - y\|^2,\end{aligned}$$

and so

$$\begin{aligned}\|x_n - u\|^2 + \alpha_n^2 \|u - y\|^2 + 2\alpha_n \langle x_n - u, u - y \rangle &\leq \alpha_n^2 (\|x_n - u\|^2 + \|u - y\|^2 \\ &\quad + 2\langle x_n - u, u - y \rangle) \\ &\leq \alpha_n \|x_n - u\|^2 + \alpha_n \|u - y\|^2 \\ &\quad + 2\alpha_n \langle x_n - u, u - y \rangle\end{aligned}$$

for all $n \geq 1$. It follows that

$$\|x_n - u\|^2 \leq \alpha_n \|y - u\|^2 \leq \|y - u\|^2 \text{ for all } y \in F(T) \text{ and } \{\alpha_n\} \subseteq (0, 1) \text{ for all } n \in \mathbb{N}.$$

Since the norm of H is weakly lower semicontinuous(w-l.s.c.), we get

$$\|z - u\| \leq \liminf_{i \rightarrow \infty} \|x_{n_i} - u\| \leq \|y - u\| \text{ for all } y \in F(T).$$

Therefore, we must have $z = v$ for v is the unique element in $F(T)$ that is closest to u . This shows that v is the only weak cluster point of $\{x_n\}$ with $\alpha_n \rightarrow 1$. It remains to verify that the convergence is strong. In fact, it follows

$$\begin{aligned}\|x_n - v\|^2 &= \|x_n - u\|^2 - \|u - v\|^2 - 2\langle x_n - v, v - u \rangle \\ &\leq -2\langle x_n - v, v - u \rangle \rightarrow 0 \text{ as } n \rightarrow \infty.\end{aligned}$$

This completes the proof. $\square$

Corollary 2.2. *Let H, C, T be as in Theorem 2.1. Suppose in addition that C is bounded and that the weak inwardness condition is satisfied. Then for each $u \in C$, the sequence $\{x_n\}$ satisfying (1.10) converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T .*

Theorem 2.3. *Let H be a Hilbert space, C be a nonempty closed convex subset of H , $T : C \rightarrow H$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition, and $P : H \rightarrow C$ be the nearest point projection. Suppose that for some $u \in C$, each $\{\alpha_n\} \subseteq (0, 1)$ and $0 \leq \beta_n \leq \beta < 1$. Then, a mapping S_n defined by (1.8) has a unique fixed point $y_n \in C$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $\alpha_n \rightarrow 1$. In this case, $\{y_n\}$ converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T .*

Proof. It is straightforward that $S_n : C \rightarrow C$ is a contraction for every $n \geq 1$. Therefore by the Banach contraction principle there exists a unique fixed point y_n of S_n in C satisfying (1.11). Let w be a fixed point of T . Then as in the proof of Theorem 2.1, $\{y_n\}$ is bounded. Conversely, suppose that $\{y_n\}$ is bounded. Apply Theorem 2.1, we obtain that $\{y_n\}$ converges strongly to a fixed point z of PT . Next, let us show that $z \in F(T)$. Since $z = PTz$ and P is the nearest point projection of H onto C , it follows by [9] that

$$\langle Tz - z, J(z - v) \rangle \geq 0 \text{ for all } v \in C.$$

On the other hand, Tz belongs to the closure of $I_C(z)$ by the weak inwardness conditions. Hence for each integer $n \geq 1$, there exists $z_n \in C$ and $a_n \geq 0$ such that the sequence

$$r_n := z + a_n(z_n - z) \rightarrow Tz.$$

Thus it follows that

$$\begin{aligned}0 &\leq a_n \langle Tz - z, z - z_n \rangle \\ &= \langle Tz - z, a_n(z - z_n) \rangle \\ &= \langle Tz - z, z - r_n \rangle \rightarrow \langle Tz - z, z - Tz \rangle \\ &= -\|Tz - z\|^2.\end{aligned}$$

Hence we have $Tz = z$. □

Corollary 2.4. ([11, Theorem 2]). Let H, C, T, P, u , and $\{\alpha_n\}$ be as in Theorem 2.3. Then, a mapping S_n given by (1.3) has a unique fixed point $y_n \in C$ such that $y_n = (1 - \alpha_n)u + \alpha_n PTy_n$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $\alpha_n \rightarrow 1$. In this case, $\{y_n\}$ converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T .

Theorem 2.5. Let $H, C, T, P, u, \{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem 2.3. Then a mapping U_n defined by (1.9) has a unique fixed point $z_n \in C$. Further, T has a fixed point if and only if $\{z_n\}$ remains bounded as $\alpha_n \rightarrow 1$ and $\beta_n \rightarrow 0$. In this case, $\{z_n\}$ converges strongly as $\alpha_n \rightarrow 1$ and $\beta_n \rightarrow 0$ to a fixed point of T .

Proof. It follows by the Banach contraction principle that there exists a unique fixed point z_n of U_n such that

$$z_n = P[(1 - \alpha_n)u + \alpha_n TP[(1 - \beta_n)z_n + \beta_n Tz_n]].$$

Let $w \in F(T)$. Then for each $n \geq 1$, we have

$$\begin{aligned} \|w - z_n\| &= \|Pw - P[(1 - \alpha_n)u + \alpha_n TP[(1 - \beta_n)z_n + \beta_n Tz_n]]\| \\ &\leq \|w - (1 - \alpha_n)u - \alpha_n TP[(1 - \beta_n)z_n + \beta_n Tz_n]\| \\ &\leq (1 - \alpha_n)\|w - u\| + \alpha_n\|w - TP[(1 - \beta_n)z_n + \beta_n Tz_n]\| \\ &\leq (1 - \alpha_n)\|w - u\| + \alpha_n(1 - \beta_n)\|w - z_n\| + \alpha_n\beta_n\|w - Tz_n\| \\ &\leq (1 - \alpha_n)\|w - u\| + \alpha_n(1 - \beta_n)\|w - z_n\| + \alpha_n\beta_n\|w - z_n\| \\ &= (1 - \alpha_n)\|w - u\| + \alpha_n\|w - z_n\| \end{aligned}$$

and hence $(1 - \alpha_n)\|w - z_n\| \leq (1 - \alpha_n)\|w - u\|$, $\forall n > 1$. This implies $\|w - z_n\| \leq \|w - u\|$, $\forall n > 1$. Then $\{z_n\}$ is bounded. Conversely, suppose that $\{z_n\}$ is bounded, $\alpha_n \rightarrow 1$ and $\beta_n \rightarrow 0$. To show that $F(T) \neq \emptyset$. For any subsequence $\{z_{n_i}\}$ of the sequence $\{z_n\}$ converging weakly to $\bar{z}$ such that $\alpha_{n_i} \rightarrow 1$, we can define a real valued function g on H given by

$$g(z) = \limsup_{i \rightarrow \infty} \|z_{n_i} - z\|^2 \text{ for every } z \in H. \quad (2.2)$$

Observing $\|z_{n_i} - z\|^2 = \|z_{n_i} - \bar{z}\|^2 + 2\langle z_{n_i} - \bar{z}, \bar{z} - z \rangle + \|\bar{z} - z\|^2$. Since $z_{n_i} \rightarrow \bar{z}$ weakly, we get

$$g(z) = g(\bar{z}) + \|\bar{z} - z\|^2 \text{ for all } z \in H,$$

in particular,

$$g(PT\bar{z}) = g(\bar{z}) + \|PT\bar{z} - \bar{z}\|^2. \quad (2.3)$$

For instance, that the straightforward verification gives

$$\begin{aligned}\|z_{n_i} - PTz_{n_i}\| &= \|P[(1 - \alpha_{n_i})u + \alpha_{n_i}TP((1 - \beta_{n_i})z_{n_i} + \beta_{n_i}Tz_{n_i})] - PTz_{n_i}\| \\ &\leq (1 - \alpha_{n_i})\|u - Tz_{n_i}\| + \alpha_{n_i}\beta_{n_i}\|Tz_{n_i} - z_{n_i}\|, \text{ for all } i \geq 1\end{aligned}$$

and this implies that $\|z_{n_i} - PTz_{n_i}\| \leq (1 - \alpha_{n_i})\|u - Tz_{n_i}\| + \alpha_{n_i}\beta_{n_i}\|Tz_{n_i} - z_{n_i}\| \rightarrow 0$ as $i \rightarrow \infty$. Moreover, we note that

$$\begin{aligned}\|z_{n_i} - PT\bar{z}\|^2 &= \|z_{n_i} - PTz_{n_i} + PTz_{n_i} - PT\bar{z}\|^2 \\ &\leq (\|z_{n_i} - PTz_{n_i}\| + \|PTz_{n_i} - PT\bar{z}\|)^2 \\ &= \|z_{n_i} - PTz_{n_i}\|^2 + 2\|z_{n_i} - PTz_{n_i}\|\|PTz_{n_i} - PT\bar{z}\| + \|PTz_{n_i} - PT\bar{z}\|^2\end{aligned}$$

for all $i \in \mathbb{N}$. It follows that

$$\begin{aligned}g(PT\bar{z}) &= \limsup_{i \rightarrow \infty} \|z_{n_i} - PT\bar{z}\|^2 \\ &\leq \limsup_{i \rightarrow \infty} \|PTz_{n_i} - PT\bar{z}\|^2 \\ &\leq \limsup_{i \rightarrow \infty} \|z_{n_i} - \bar{z}\|^2 = g(z)\end{aligned}$$

which in turn, together with (2.3), implies that $PT(\bar{z}) = \bar{z}$. Since T satisfies the weak inwardness condition, by the same argument as in the proof of Theorem 2.3, we see that $\bar{z}$ is a fixed point of T . For any $w \in F(T)$, we have

$$\begin{aligned}\alpha_n[TP((1 - \beta_n)w + \beta_n w) - u] + u &= \alpha_n(w - u) + u \\ &= \alpha_n w + (1 - \alpha_n)u \\ &= P(\alpha_n w + (1 - \alpha_n)u)\end{aligned}$$

for all $n \in \mathbb{N}$. By follows as in the proof of Theorem 2.1, we have

$$\|z_n - u\|^2 \leq \alpha_n \|w - u\|^2 \leq \|w - u\|^2 \text{ for all } w \in F(T) \text{ and } \{\alpha_n\} \subseteq (0, 1) \text{ for all } n \in \mathbb{N}. \quad (2.4)$$

From (2.4) and the w-l.s.c. of the norm of H , it follows that

$$\|\bar{z} - u\| \leq \liminf_{n \rightarrow \infty} \|z_n - u\| \leq \|w - u\|$$

for all $w \in F(T)$. Hence $\bar{z}$ is the nearest point projection z in $F(T)$ of u onto $F(T)$ which exists uniquely since $F(T)$ is nonempty, closed and convex. Moreover,

$$\begin{aligned}\|z_n - z\|^2 &= \|z_n - u\|^2 - \|u - z\|^2 - 2\langle z_n - z, z - u \rangle \\ &\leq -2\langle z_n - z, z - u \rangle \rightarrow 0 \text{ as } n \rightarrow \infty.\end{aligned}$$

This complete in the proof. □

Corollary 2.6. ([11, Theorem 3]). Let H, C, T, P, u , and $\{\alpha_n\}$ be as in Theorem 2.3. Then a mapping U_n defined by (1.4) has a unique fixed point $z_n \in C$. Further, T has a fixed point if and only if $\{z_n\}$ remains bounded as $\alpha_n \rightarrow 1$. In this case, $\{z_n\}$ converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T .

Acknowledgement. The authors would like to thanks The Thailand Research Fund for financial support.

REFERENCES

1. F. E. Browder, *Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Archs. Ratio. Mech. Anal., **24** (1967), 80-90.
2. K. Deimling, *Fixed points of condensing maps*, in Lecture Notes in Math., **737**. Springer, Berlin, (1979), 67-82.
3. B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc., **73** (1967), 957-961.
4. J.S. Jung and S.S. Kim, *Strong convergence theorems for nonexpansive nonself-mappings in Banach spaces*, Nonlinear Anal., **33**(3) (1998), 321-329.
5. G. Marino and G. Trombetta, *On approximating fixed points for nonexpansive maps*, Indian J. Math., **34** (1992), 91-98.
6. S. Reich, *Fixed points of condensing function*, J. Math. Anal. Appl., **41** (1973), 460-467.
7. S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach space*, J. Math. Anal. Appl., **75** (1980), 287-292.
8. S. P. Singh and B. Watson, *On approximating fixed points*, Proc. Symp. Pure Math., **45** (1986), 393-395.
9. W. Takahashi, *Nonlinear Functional Analysis*. Yokohama Publishers, Japan, (2000).
10. W. Takahashi and G. E. Kim, *Strong convergence of approximants to fixed points of nonexpansive nonself-mappings*, Nonlinear Anal., **32** (1998), 447-454.
11. H.K. Xu and X. M. Yin, *Strong convergence theorems for nonexpansive nonself-mappings*, Nonlinear Anal., **24** (1995), 223-228.

ภาคผนวก 3/10

Somyot Plubtieng

From: "IJMMS Administrative Office" <ijmms.admin@hindawi.com>
To: "Somyot Plubtieng" <somyotp@nu.ac.th>
Cc: <ijmms.admin@hindawi.com>
Sent: Thursday, August 04, 2005 3:41 PM
Attach: 50234.pdf
Subject: IJMMS/50234

Dear Prof. Plubtieng:

I am writing concerning the paper entitled "Implicit iteration process of nonexpansive nonself-mappings," with Prof. Rattanaporn Punpaeng that has been accepted for publication in the "International Journal of Mathematics and Mathematical Sciences." Please mention IJMMS/50234 in all correspondences about this paper. Attached is the copyright agreement of the paper. I would appreciate if you can sign this form and mail it to our address:

Hindawi Publishing Corporation
410 Park Avenue
15th Floor, #287 pmb
New York, NY 10022
USA

I would also appreciate if you can send a copy of the signed form to our fax number:

+1 215 893 4392 (Philadelphia, USA)

Being the corresponding author of the paper, you have the option to sign this form on behalf of your coauthor writing on the signed form after the signature: "on behalf of my coauthor."

Receiving the signed copyright form is essential to proceed with the publication process of your paper with no delay.

I will also appreciate if you send me the electronic files of your paper (TeX, LaTeX, or word files), and also the electronic file of the figures/graphs included. The best format for them is EPS, but other formats are possible as well.

In order to facilitate future communications regarding your paper, please send me your current address, phone and/or fax number, if you think we do not have them already.

A confirmation of this email would be highly appreciated.

Looking forward to hearing from you.

Best regards,

Sincerely yours,

--Hend El-Khawas

ภาคผนวก 4

**Strong convergence theorems for multi-step
Noor iterations with errors in Banach spaces**

S. Plubtieng and R. Wangkeeree

J. Math. Anal. Appl. (accepted).

Strong Convergence Theorems for Multi-Step Noor Iterations with Errors in Banach Spaces*

Somyot Plubtieng and Rabian Wangkeeree[†]

Department of Mathematics, Faculty of Science, Naresuan University,
Pitsanulok 65000, Thailand

Abstract

In this paper, we established two strong convergence theorems for a multi-step Noor iterative scheme with errors for mappings of asymptotically nonexpansive in the intermediate sense (asymptotically quasi-nonexpansive, respectively) in Banach spaces. Our results extend and improve the recent ones announced by Xu and Noor [20], Cho, Zhou and Guo [2], and many others.

keywords: Asymptotically nonexpansive in the intermediate sense; Asymptotically quasi-nonexpansive mappings; Completely continuous; Uniformly convex; Uniformly L -Lipschitzian.

1 Introduction

Let C be a subset of real normed linear space X . A mapping $T : C \rightarrow C$ is said to be *asymptotically nonexpansive* on C if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} r_n = 0$ such that for each $x, y \in C$,

$$\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|, \forall n \geq 1.$$

2000 Mathematics Subject Classification: 46B20, 47H09, 47H10.

*Corresponding author.

Email addresses: Somyotp@nu.ac.th (Somyot Plubtieng) and Rabianw@nu.ac.th (Rabian Wangkeeree)

[†]Supported by The Thailand Research Fund.

If $r_n \equiv 0$, then T is known as a *nonexpansive mapping*. T is called *asymptotically nonexpansive in the intermediate sense*[1] provided T is uniformly continuous and

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

T is said to be *asymptotically quasi-nonexpansive mapping*, if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} r_n = 0$ such that for all $x \in C$, $p \in F(T)$,

$$\|T^n x - p\| \leq (1 + r_n)\|x - p\|,$$

for all $n \geq 1$, where $F(T)$ denotes the set of fixed points of T i.e. $F(T) = \{x \in C : Tx = x\}$. T is said to be *uniformly L -Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for all $n \geq 1$ and $x, y \in C$.

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically nonexpansive in the intermediate sense, asymptotically quasi-nonexpansive mapping and L -Lipschitzian mapping. But the converges dose not holds such as the following example:

Example 1.1 (see [9]). Let $X = \mathbb{R}$, $C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in C$, define

$$T(x) = \begin{cases} kx \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then T is an asymptotically nonexpansive in the intermediate sense. It is well known in [8] that $T^n x \rightarrow 0$ uniformly, but is not a Lipschitzian mapping so that it is not asymptotically nonexpansive mapping.

Fixed-point iterations process for asymptotically nonexpansive mappings in Banach spaces including Mann and Ishikawa iterations process have been studied extensively by many authors to solve the nonlinear operator equations as well as variational inequations; see[6-14,16-18]. In 2000, Noor [13] introduced a three-step iterative scheme and studied the approximate solution of variational inclusion in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle. Glowinski and Le Tallec [3] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [3] that the three-step iterative scheme give better numerical results then the two-step and one step approximal iterations. In 1998, Haubruge, Nguyen and Strodiot[5] studied the convergence analysis

of three-step schemes of Glowinski and Le Tallec[3] and applied these schemes to obtain new splitting-type algorithms for solving variation inequalities. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions.

Recently, Xu and Noor [20] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach space. In 2004, Cho, Zhou and Guo[2] extended the work of Xu and Noor to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space. Moreover, Suantai [18] gave weak and strong convergence theorems for a new three-step iterative scheme of asymptotically nonexpansive mappings. Inspired and motivated by these facts, we introduce and study a multi-step scheme with errors for asymptotically nonexpansive mappings in the intermediate sense mapping and asymptotically quasi-nonexpansive mappings, respectively. Our results include the Ishikawa, Mann and Noor iterative schemes for solving variational inclusions (inequalities) and related problems as special case. The scheme is defined as follows.

Let C be a nonempty subset of normed space X and let $T : C \longrightarrow C$ be a mapping. For a given $x_1 \in C$, and a fixed $m \in \mathbb{N}$ ($\mathbb{N}$ denote the set of all positive integers), compute the iterative sequences $\{x_n^{(1)}\}, \dots, \{x_n^{(m)}\}$ defined by

$$\begin{aligned} x_n^{(1)} &= \alpha_n^{(1)} T^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)}, \\ x_n^{(2)} &= \alpha_n^{(2)} T^n x_n^{(1)} + \beta_n^{(2)} x_n + \gamma_n^{(2)} u_n^{(2)}, \\ x_n^{(3)} &= \alpha_n^{(3)} T^n x_n^{(2)} + \beta_n^{(3)} x_n + \gamma_n^{(3)} u_n^{(3)}, \\ &\vdots \\ x_n^{(m-1)} &= \alpha_n^{(m-1)} T^n x_n^{(m-2)} + \beta_n^{(m-1)} x_n + \gamma_n^{(m-1)} u_n^{(m-1)}, \\ x_{n+1} = x_n^{(m)} &= \alpha_n^{(m)} T^n x_n^{(m-1)} + \beta_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)}, \quad n \geq 1 \end{aligned} \tag{1.1}$$

where, $\{u_n^{(1)}\}, \dots, \{u_n^{(m)}\}$ are bounded sequences in C and $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ are appropriate real sequences in $[0, 1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $i \in \{1, 2, \dots, m\}$.

The iterative schemes (1.1) is called the multi-step Noor iterations with errors. This iterations include the Mann-Ishikawa-Noor iterations as special case. If $m = 3$ and $\beta_n^{(i)} = 1 - \alpha_n^{(i)} - \gamma_n^{(i)}$ for all $i = 1, 2, 3$ then (1.1) reduces to Noor iterations with

errors defined by Cho, Zhou and Guo [2]:

$$\begin{aligned} x_n^{(1)} &= \alpha_n^{(1)} T^n x_n + (1 - \alpha_n^{(1)} - \gamma_n^{(1)}) x_n + \gamma_n^{(1)} u_n^{(1)}, \\ x_n^{(2)} &= \alpha_n^{(2)} T^n x_n^{(1)} + (1 - \alpha_n^{(2)} - \gamma_n^{(2)}) x_n + \gamma_n^{(2)} u_n^{(2)}, \\ x_{n+1} = x_n^{(3)} &= \alpha_n^{(3)} T^n x_n^{(2)} + (1 - \alpha_n^{(3)} - \gamma_n^{(3)}) x_n + \gamma_n^{(3)} u_n^{(3)}, \end{aligned} \quad (1.2)$$

where $\{\alpha_n^{(i)}\}, \{\gamma_n^{(i)}\}$ are appropriate real sequences in $[0, 1]$ for all $i \in \{1, 2, 3\}$.

For $m = 3$ and $\gamma_n^{(1)} = \gamma_n^{(2)} = \gamma_n^{(3)} \equiv 0$, then (1.1) reduces to Noor iterations defined by Xu and Noor [20]:

$$\begin{aligned} x_n^{(1)} &= \alpha_n^{(1)} T^n x_n + (1 - \alpha_n^{(1)}) x_n, \\ x_n^{(2)} &= \alpha_n^{(2)} T^n x_n^{(1)} + (1 - \alpha_n^{(2)}) x_n, \\ x_{n+1} = x_n^{(3)} &= \alpha_n^{(3)} T^n x_n^{(2)} + (1 - \alpha_n^{(3)}) x_n, \quad n \geq 1, \end{aligned} \quad (1.3)$$

where $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \{\alpha_n^{(3)}\}$ are appropriate real sequences in $[0, 1]$.

The purpose of this paper is to establish several strong convergence theorems of the multi-step Noor iterative scheme with errors for mappings of asymptotically nonexpansive in the intermediate sense (asymptotically quasi-nonexpansive mappings, respectively) in a uniformly convex Banach space. This results presented in this paper extend and improve the corresponding ones announced by Xu and Noor [20], Cho, Zhou and Guo [2], and many others.

2 Preliminaries

In this section, we recall the well-known concepts and results.

Definition 2.1 (see [4]). A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta_X(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| = \epsilon \right\} > 0$$

for all $0 < \epsilon \leq 2$ (i.e., $\delta_X(\epsilon)$ is a function $(0, 2] \rightarrow (0, 1)$).

It is known [12] that if X is a uniformly convex Banach space and T is a self-mapping of bounded closed convex subset C of X which is an asymptotically nonexpansive in the intermediate sense, then $F(T) \neq \emptyset$.

Lemma 2.2 (see [10]). *Let $\{a_n\}, \{b_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \gamma_n)a_n + b_n, \forall n = 1, 2, \dots$$

If $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

- (i) $\lim_{n \rightarrow \infty} a_n$ exists;
- (ii) $\lim_{n \rightarrow \infty} a_n = 0$, whenever $\liminf_{n \rightarrow \infty} a_n = 0$.

Lemma 2.3 ([17], J. Schu's Lemma). *Let X be a uniformly convex Banach space, $0 < \alpha \leq t_n \leq \beta < 1$, $x_n, y_n \in X$, $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$, and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = a$, for some $a \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

3 Non-Lipschitzian mappings

Our first result is the strong convergence theorem for asymptotically nonexpansive in the intermediate sense mappings. Note the proof given below is different from that proof of Xu and Noor. In order to prove our main result, the following lemmas are needed.

Lemma 3.1. *Let X be a uniformly convex Banach space with $x_n, y_n \in X$, real numbers $a \geq 0$, $\alpha, \beta \in (0, 1)$ and $\{\alpha_n\}$ be a real sequence number which satisfying*

- (i) $0 < \alpha \leq \alpha_n \leq \beta < 1$, $\forall n \geq n_0$ and for some $n_0 \in \mathbb{N}$;
- (ii) $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$ and $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$;
- (iii) $\lim_{n \rightarrow \infty} \|\alpha_n x_n + (1 - \alpha_n)y_n\| = a$.

Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Proof. The proof is clear by Lemma 2.3. □

Lemma 3.2. *Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X and $T : C \rightarrow C$ be an asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let the sequence $\{x_n\}$ be defined by (1.1) with the following restrictions:

(i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for all $i \in \{1, 2, \dots, m\}$ and for all $n \geq 1$;

(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i \in \{1, 2, \dots, m\}$.

If $p \in F(T)$, then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists.

Proof. By [12], we have $F(T) \neq \emptyset$. Let $p \in F(T)$. For each $n \geq 1$, we note that

$$\begin{aligned} \|x_n^{(1)} - p\| &= \|\alpha_n^{(1)} T^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)} - p\| \\ &\leq \alpha_n^{(1)} \|T^n x_n - p\| + \beta_n^{(1)} \|x_n - p\| + \gamma_n^{(1)} \|u_n^{(1)} - p\| \\ &\leq \alpha_n^{(1)} \|x_n - p\| + \alpha_n^{(1)} G_n + \beta_n^{(1)} \|x_n - p\| + \gamma_n^{(1)} \|u_n^{(1)} - p\| \\ &= (\alpha_n^{(1)} + \beta_n^{(1)}) \|x_n - p\| + \alpha_n^{(1)} G_n + \gamma_n^{(1)} \|u_n^{(1)} - p\| \\ &\leq \|x_n - p\| + d_n^{(1)} \end{aligned} \quad (3.1)$$

where $d_n^{(1)} = \alpha_n^{(1)} G_n + \gamma_n^{(1)} \|u_n^{(1)} - p\|$. Since $\sum_{n=1}^{\infty} G_n < \infty$, we see that $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$. It follows from (3.1) that

$$\begin{aligned} \|x_n^{(2)} - p\| &\leq \alpha_n^{(2)} \|x_n^{(1)} - p\| + \alpha_n^{(2)} G_n + \beta_n^{(2)} \|x_n - p\| + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\ &\leq \alpha_n^{(2)} (\|x_n - p\| + d_n^{(1)}) + \alpha_n^{(2)} G_n + \beta_n^{(2)} \|x_n - p\| + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\ &= (\alpha_n^{(2)} + \beta_n^{(2)}) \|x_n - p\| + \alpha_n^{(2)} d_n^{(1)} + \alpha_n^{(2)} G_n + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\ &\leq \|x_n - p\| + d_n^{(2)} \end{aligned} \quad (3.2)$$

where $d_n^{(2)} = \alpha_n^{(2)} d_n^{(1)} + \alpha_n^{(2)} G_n + \gamma_n^{(2)} \|u_n^{(2)} - p\|$. Since $\sum_{n=1}^{\infty} G_n < \infty$ and $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$ it follows that $\sum_{n=1}^{\infty} d_n^{(2)} < \infty$. Moreover, we see that

$$\begin{aligned} \|x_n^{(3)} - p\| &\leq \alpha_n^{(3)} \|x_n^{(2)} - p\| + \alpha_n^{(3)} G_n + \beta_n^{(3)} \|x_n - p\| + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\ &\leq \alpha_n^{(3)} (\|x_n - p\| + d_n^{(2)}) + \alpha_n^{(3)} G_n + \beta_n^{(3)} \|x_n - p\| + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\ &= (\alpha_n^{(3)} + \beta_n^{(3)}) \|x_n - p\| + \alpha_n^{(3)} d_n^{(2)} + \alpha_n^{(3)} G_n + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\ &\leq \|x_n - p\| + d_n^{(3)} \end{aligned} \quad (3.3)$$

where $d_n^{(3)} = \alpha_n^{(3)} d_n^{(2)} + \alpha_n^{(3)} G_n + \gamma_n^{(3)} \|u_n^{(3)} - p\|$. So that $\sum_{n=1}^{\infty} d_n^{(3)} < \infty$. By continuing the above method, there are nonnegative real sequences $\{d_n^{(k)}\}$ such that $\sum_{n=1}^{\infty} d_n^{(k)} < \infty$ and

$$\|x_n^{(k)} - p\| \leq \|x_n - p\| + d_n^{(k)}, \text{ for all } k = 1, 2, \dots, m.$$

This together with Lemma 2.2, we have $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. This completes the proof. $\square$

Lemma 3.3. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X and $T : C \rightarrow C$ be an asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let the sequence $\{x_n\}$ be defined by (1.1) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfies the same assumptions as Lemma 3.2 for each $i \in \{1, 2, \dots, m\}$ and the additional assumption that $0 < \alpha \leq \alpha_n^{(m-1)}, \alpha_n^{(m)} \leq \beta < 1$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Then

$$(a). \lim_{n \rightarrow \infty} \|T^n x_n^{(m-1)} - x_n\| = 0;$$

$$(b). \lim_{n \rightarrow \infty} \|T^n x_n^{(m-2)} - x_n\| = 0.$$

Proof. (a). For any $p \in F(T)$, it follows from Lemma 3.2 that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Let $\lim_{n \rightarrow \infty} \|x_n - p\| = a$ for some $a \geq 0$. We note that

$$\|x_n^{(m-1)} - p\| \leq \|x_n - p\| + d_n^{(m-1)}, \forall n \geq 1$$

where $\{d_n^{(m-1)}\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} d_n^{(m-1)} < \infty$. It follows that

$$\limsup_{n \rightarrow \infty} \|x_n^{(m-1)} - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = \lim_{n \rightarrow \infty} \|x_n - p\| = a,$$

from which we have

$$\limsup_{n \rightarrow \infty} \|T^n x_n^{(m-1)} - p\| \leq \limsup_{n \rightarrow \infty} (\|x_n^{(m-1)} - p\| + G_n) = \limsup_{n \rightarrow \infty} \|x_n^{(m-1)} - p\| \leq a.$$

Next, we observe that

$$\|T^n x_n^{(m-1)} - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leq \|T^n x_n^{(m-1)} - p\| + \gamma_n^{(m)}\|u_n^{(m)} - x_n\|.$$

Thus we have

$$\limsup_{n \rightarrow \infty} \|T^n x_n^{(m-1)} - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leq a. \quad (3.4)$$

$$\text{Also, } \|x_n - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leq \|x_n - p\| + \gamma_n^{(m)}\|u_n^{(m)} - x_n\|,$$

gives that

$$\limsup_{n \rightarrow \infty} \|x_n - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leq a, \quad (3.5)$$

and note that

$$\begin{aligned}
 a &= \lim_{n \rightarrow \infty} \|x_n^{(m)} - p\| \\
 &= \lim_{n \rightarrow \infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} + \beta_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - p\| \\
 &= \lim_{n \rightarrow \infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} + (1 - \alpha_n^{(m)}) x_n - \gamma_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} \\
 &\quad - (1 - \alpha_n^{(m)}) p - \alpha_n^{(m)} p\| \\
 &= \lim_{n \rightarrow \infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} - \alpha_n^{(m)} p + \alpha_n^{(m)} \gamma_n^{(m)} u_n^{(m)} - \alpha_n^{(m)} \gamma_n^{(m)} x_n + (1 - \alpha_n^{(m)}) x_n \\
 &\quad - (1 - \alpha_n^{(m)}) p - \gamma_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - \alpha_n^{(m)} \gamma_n^{(m)} u_n^{(m)} + \alpha_n^{(m)} \gamma_n^{(m)} x_n\| \\
 &= \lim_{n \rightarrow \infty} \|\alpha_n^{(m)} (T^n x_n^{(m-1)} - p + \gamma_n^{(m)} (u_n^{(m)} - x_n)) \\
 &\quad + (1 - \alpha_n^{(m)}) (x_n - p + \gamma_n^{(m)} (u_n^{(m)} - x_n))\|.
 \end{aligned}$$

This together with (3.4), (3.5) and Lemma 3.1, we have

$$\lim_{n \rightarrow \infty} \|T^n x_n^{(m-1)} - x_n\| = 0. \quad (3.6)$$

This completes the proof of (a).

Proof of (b). For each $n \geq 1$,

$$\begin{aligned}
 \|x_n - p\| &\leq \|x_n - T^n x_n^{(m-1)}\| + \|T^n x_n^{(m-1)} - p\| \\
 &\leq \|x_n - T^n x_n^{(m-1)}\| + \|x_n^{(m-1)} - p\| + G_n.
 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|x_n - T^n x_n^{(m-1)}\| = 0 = \lim_{n \rightarrow \infty} G_n$, we obtain that

$$a = \lim_{n \rightarrow \infty} \|x_n - p\| \leq \liminf_{n \rightarrow \infty} \|x_n^{(m-1)} - p\|.$$

It follows that

$$a \leq \liminf_{n \rightarrow \infty} \|x_n^{(m-1)} - p\| \leq \limsup_{n \rightarrow \infty} \|x_n^{(m-1)} - p\| \leq a.$$

This implies that

$$\lim_{n \rightarrow \infty} \|x_n^{(m-1)} - p\| = a.$$

On the other hand, we note that

$$\|x_n^{(m-2)} - p\| \leq \|x_n - p\| + d_n^{(m-2)}, \forall n \geq 1$$

where $\{d_n^{(m-2)}\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} d_n^{(m-2)} < \infty$. So that

$$\limsup_{n \rightarrow \infty} \|x_n^{(m-2)} - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = a,$$

and hence

$$\limsup_{n \rightarrow \infty} \|T^n x_n^{(m-2)} - p\| \leq \limsup_{n \rightarrow \infty} (\|x_n^{(m-2)} - p\| + G_n) \leq a.$$

Next we observe that

$$\|T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \leq \|T^n x_n^{(m-2)} - p\| + \gamma_n^{(m-1)} \|u_n^{(m-1)} - x_n\|.$$

Thus,

$$\limsup_{n \rightarrow \infty} \|T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \leq a. \quad (3.7)$$

$$\text{Also, } \|x_n - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \leq \|x_n - p\| + \gamma_n^{(m-1)} \|u_n^{(m-1)} - x_n\|,$$

gives that

$$\limsup_{n \rightarrow \infty} \|x_n - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \leq a, \quad (3.8)$$

and note that

$$\begin{aligned} a &= \lim_{n \rightarrow \infty} \|x_n^{(m-1)} - p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n^{(m-1)} T^n x_n^{(m-2)} + \beta_n^{(m-1)} x_n + \gamma_n^{(m-1)} u_n^{(m-1)} - p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n^{(m-1)} (T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)) \\ &\quad + (1 - \alpha_n^{(m-1)})(x_n - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n))\|. \end{aligned} \quad (3.9)$$

It follows from (3.7), (3.8), (3.9) and Lemma 3.1 that

$$\lim_{n \rightarrow \infty} \|T^n x_n^{(m-2)} - x_n\| = 0.$$

This completes the proof of (b). $\square$

We now state and prove the first main result of this paper and this is the main motivation of our next result.

Theorem 3.4. *Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X and $T : C \rightarrow C$ be a completely continuous asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let the sequence $\{x_n\}$ be defined by (1.1) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfies the same assumptions as Lemma 3.2 for each $i \in \{1, 2, \dots, m\}$ and the additional assumption that $0 < \alpha \leq \alpha_n^{(m-1)}, \alpha_n^{(m)} \leq \beta < 1$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Then $\{x_n^{(k)}\}$ converges strongly to a fixed point of T for each $k = 1, 2, 3, \dots, m$.

Proof. It follows from Lemma 3.3 that

$$\lim_{n \rightarrow \infty} \|T^n x_n^{(m-1)} - x_n\| = 0 = \lim_{n \rightarrow \infty} \|T^n x_n^{(m-2)} - x_n\|$$

and this implies that,

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|x_n^{(m)} - x_n\| \leq \alpha_n^{(m)} \|T^n x_n^{(m-1)} - x_n\| + \gamma_n^{(m)} \|u_n^{(m)} - x_n\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (3.10)$$

It follows from (3.10) that

$$\begin{aligned} \|T^n x_n - x_n\| &\leq \|T^n x_n - T^n x_n^{(m-1)}\| + \|T^n x_n^{(m-1)} - x_n\| \\ &\leq \|x_n - x_n^{(m-1)}\| + G_n + \|T^n x_n^{(m-1)} - x_n\| \\ &\leq \alpha_n^{(m-1)} \|x_n - T^n x_n^{(m-2)}\| + G_n + \gamma_n^{(m-1)} \|u_n^{(m-1)} - x_n\| \\ &\quad + \|T^n x_n^{(m-1)} - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (3.11)$$

Since

$$\begin{aligned} \|x_n - T x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1} x_{n+1}\| \\ &\quad + \|T^{n+1} x_{n+1} - T^{n+1} x_n\| + \|T^{n+1} x_n - T x_n\|, \end{aligned}$$

it follows from (3.10), (3.11) and uniformly continuity of T that

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0. \quad (3.12)$$

Since $\{x_n\}$ is a bounded and T is completely continuous, there exists a subsequence $\{T x_{n_k}\}$ of $\{T x_n\}$ such that $T x_{n_k} \rightarrow p \in C$ as $k \rightarrow \infty$. Moreover, by (3.12), we have $\|T x_{n_k} - x_{n_k}\| \rightarrow 0$ which implies that $x_{n_k} \rightarrow p$ as $k \rightarrow \infty$. By (3.12) again, we have

$$\|p - T p\| = \lim_{k \rightarrow \infty} \|x_{n_k} - T x_{n_k}\| = 0.$$

It show that $p \in F(T)$. Since $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, we have $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$; that is $\lim_{n \rightarrow \infty} x_n^{(m)} = \lim_{n \rightarrow \infty} x_n = p$. Moreover, we observe that $\|x_n^{(k)} - p\| \leq \|x_n - p\| + d_n^{(k)}$ for all $k = 1, 2, 3, \dots, m-1$ and each $\lim_{n \rightarrow \infty} d_n^{(k)} = 0$. Therefore $\lim_{n \rightarrow \infty} x_n^{(k)} = p$ for all $k = 1, 2, 3, \dots, m-1$. The proof is completed. $\square$

4 Asymptotically quasi-nonexpansive mappings

In the next result, we prove strong convergence theorem for the multi-step Noor iterations (1.1) for asymptotically quasi-nonexpansive mapping in a uniformly convex Banach space. To do this, we need the following lemmas.

Lemma 4.1. Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X and T be an asymptotically quasi-nonexpansive with the sequence $\{r_n\}_{n \geq 1}$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (1.1) with the following restrictions:

- (i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for all $i \in \{1, 2, \dots, m\}$ and for all $n \geq 1$;
- (ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i \in \{1, 2, \dots, m\}$.

If $p \in F(T)$, then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists.

Proof. Let $p \in F(T)$. For each $n \geq 1$, we note that

$$\begin{aligned} \|x_n^{(1)} - p\| &= \|\alpha_n^{(1)} T^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)} - p\| \\ &\leq \alpha_n^{(1)} \|T^n x_n - p\| + \beta_n^{(1)} \|x_n - p\| + \gamma_n^{(1)} \|u_n^{(1)} - p\| \\ &\leq \alpha_n^{(1)} (1 + r_n) \|x_n - p\| + \beta_n^{(1)} \|x_n - p\| + \gamma_n^{(1)} \|u_n^{(1)} - p\| \\ &\leq (1 + r_n) \|x_n - p\| + d_n^{(1)} \end{aligned} \quad (4.1)$$

where $d_n^{(1)} = \gamma_n^{(1)} \|u_n^{(1)} - p\|$. Since $\{u_n^{(1)}\}$ is bounded and $\sum_{n=1}^{\infty} \gamma_n^{(1)} < \infty$ we see that $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$. It follows from (4.1) that

$$\begin{aligned} \|x_n^{(2)} - p\| &\leq \alpha_n^{(2)} (1 + r_n) \|x_n^{(1)} - p\| + \beta_n^{(2)} \|x_n - p\| + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\ &\leq \alpha_n^{(2)} (1 + r_n) ((1 + r_n) \|x_n - p\| + d_n^{(1)}) + \beta_n^{(2)} (1 + r_n)^2 \|x_n - p\| \\ &\quad + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\ &= (\alpha_n^{(2)} + \beta_n^{(2)}) (1 + r_n)^2 \|x_n - p\| + \alpha_n^{(2)} d_n^{(1)} (1 + r_n) + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\ &\leq (1 + r_n)^2 \|x_n - p\| + \alpha_n^{(2)} d_n^{(1)} (1 + r_n) + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\ &= (1 + r_n)^2 \|x_n - p\| + d_n^{(2)}, \end{aligned} \quad (4.2)$$

where $d_n^{(2)} = \alpha_n^{(2)} d_n^{(1)} (1 + r_n) + \gamma_n^{(2)} \|u_n^{(2)} - p\|$. Since $\{u_n^{(2)}\}$ is bounded and $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$, it follows that $\sum_{n=1}^{\infty} d_n^{(2)} < \infty$. Moreover, we see that

$$\begin{aligned} \|x_n^{(3)} - p\| &\leq \alpha_n^{(3)} (1 + r_n) \|x_n^{(2)} - p\| + \beta_n^{(3)} \|x_n - p\| + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\ &\leq \alpha_n^{(3)} (1 + r_n) ((1 + r_n)^2 \|x_n - p\| + d_n^{(2)}) + \beta_n^{(3)} (1 + r_n)^3 \|x_n - p\| \\ &\quad + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\ &\leq (\alpha_n^{(3)} + \beta_n^{(3)}) (1 + r_n)^3 \|x_n - p\| + \alpha_n^{(3)} d_n^{(2)} (1 + r_n) + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\ &\leq (1 + r_n)^3 \|x_n - p\| + \alpha_n^{(3)} d_n^{(2)} (1 + r_n) + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\ &= (1 + r_n)^3 \|x_n - p\| + d_n^{(3)}, \end{aligned} \quad (4.3)$$

where $d_n^{(3)} = \alpha_n^{(3)} d_n^{(2)} (1 + r_n) + \gamma_n^{(3)} \|u_n^{(3)} - p\|$. So that $\sum_{n=1}^{\infty} d_n^{(3)} < \infty$. By continuing the above method, there are nonnegative real sequence $\{d_n^{(k)}\}$ such that $\sum_{n=1}^{\infty} d_n^{(k)} < \infty$.

∞ and

$$\|x_n^{(k)} - p\| \leq (1 + r_n)^k \|x_n - p\| + d_n^{(k)}, \text{ for all } k = 1, 2, \dots, m.$$

By Lemma 2.2, we have $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. This completes the proof. $\square$

Lemma 4.2. *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X and $T : C \rightarrow C$ be an asymptotically quasi-nonexpansive with the sequence $\{r_n\}_{n \geq 1}$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (1.1) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfies the same assumptions as Lemma 4.1 for each $i \in \{1, 2, \dots, m\}$ and the additional assumption that $0 < \alpha \leq \alpha_n^{(m-1)}, \alpha_n^{(m)} \leq \beta < 1$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Then*

$$(a). \lim_{n \rightarrow \infty} \|T^n x_n^{(m-1)} - x_n\| = 0;$$

$$(b). \lim_{n \rightarrow \infty} \|T^n x_n^{(m-2)} - x_n\| = 0.$$

Proof. (a). For any $p \in F(T)$, it follows from Lemma 4.1 that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Let $\lim_{n \rightarrow \infty} \|x_n - p\| = a$ for some $a \geq 0$. We note that

$$\|x_n^{(m-1)} - p\| \leq (1 + r_n)^{m-1} \|x_n - p\| + d_n^{(m-1)}, \forall n \geq 1$$

where $\{d_n^{(m-1)}\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} d_n^{(m-1)} < \infty$. It follows that

$$\limsup_{n \rightarrow \infty} \|x_n^{(m-1)} - p\| \leq \limsup_{n \rightarrow \infty} ((1 + r_n)^{m-1} \|x_n - p\| + d_n^{(m-1)}) = \lim_{n \rightarrow \infty} \|x_n - p\| = a$$

and so

$$\limsup_{n \rightarrow \infty} \|T^n x_n^{(m-1)} - p\| \leq \limsup_{n \rightarrow \infty} (1 + r_n) \|x_n^{(m-1)} - p\| = \limsup_{n \rightarrow \infty} \|x_n^{(m-1)} - p\| \leq a.$$

Next, consider

$$\|T^n x_n^{(m-1)} - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leq \|T^n x_n^{(m-1)} - p\| + \gamma_n^{(m)} \|u_n^{(m)} - x_n\|.$$

Thus,

$$\limsup_{n \rightarrow \infty} \|T^n x_n^{(m-1)} - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leq a. \quad (4.4)$$

$$\text{Also, } \|x_n - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leq \|x_n - p\| + \gamma_n^{(m)} \|u_n^{(m)} - x_n\|,$$

gives that

$$\limsup_{n \rightarrow \infty} \|x_n - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leq a, \quad (4.5)$$

and we observe that

$$\begin{aligned}
 a &= \lim_{n \rightarrow \infty} \|x_n^{(m)} - p\| \\
 &= \lim_{n \rightarrow \infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} + \beta_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - p\| \\
 &= \lim_{n \rightarrow \infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} + (1 - \alpha_n^{(m)}) x_n - \gamma_n^{(m)} x_n \\
 &\quad + \gamma_n^{(m)} u_n^{(m)} - (1 - \alpha_n^{(m)}) p - \alpha_n^{(m)} p\| \\
 &= \lim_{n \rightarrow \infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} - \alpha_n^{(m)} p + \alpha_n^{(m)} \gamma_n^{(m)} u_n^{(m)} - \alpha_n^{(m)} \gamma_n^{(m)} x_n \\
 &\quad + (1 - \alpha_n^{(m)}) x_n - (1 - \alpha_n^{(m)}) p \\
 &\quad - \gamma_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - \alpha_n^{(m)} \gamma_n^{(m)} u_n^{(m)} + \alpha_n^{(m)} \gamma_n^{(m)} x_n\| \\
 &= \lim_{n \rightarrow \infty} \|\alpha_n^{(m)} (T^n x_n^{(m-1)} - p + \gamma_n^{(m)} (u_n^{(m)} - x_n)) + (1 - \alpha_n^{(m)}) (x_n - p + \gamma_n^{(m)} (u_n^{(m)} - x_n))\|.
 \end{aligned}$$

It follows from (4.4), (4.5) and Lemma 3.1 that

$$\lim_{n \rightarrow \infty} \|T^n x_n^{(m-1)} - x_n\| = 0.$$

This completes the proof of (a).

Proof of (b). For each $n \geq 1$, we have

$$\begin{aligned}
 \|x_n - p\| &\leq \|x_n - T^n x_n^{(m-1)}\| + \|T^n x_n^{(m-1)} - p\| \\
 &\leq \|x_n - T^n x_n^{(m-1)}\| + (1 + r_n) \|x_n^{(m-1)} - p\|.
 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|x_n - T^n x_n^{(m-1)}\| = 0 = \lim_{n \rightarrow \infty} r_n$, we obtain that

$$a = \lim_{n \rightarrow \infty} \|x_n - p\| \leq \liminf_{n \rightarrow \infty} \|x_n^{(m-1)} - p\|.$$

It follows that

$$a \leq \liminf_{n \rightarrow \infty} \|x_n^{(m-1)} - p\| \leq \limsup_{n \rightarrow \infty} \|x_n^{(m-1)} - p\| \leq a,$$

which implies that

$$\lim_{n \rightarrow \infty} \|x_n^{(m-1)} - p\| = a.$$

On the other hand, we note that

$$\|x_n^{(m-2)} - p\| \leq (1 + r_n)^{m-2} \|x_n - p\| + d_n^{(m-2)}, \forall n \geq 1$$

where $\{d_n^{(m-2)}\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} d_n^{(m-2)} < \infty$. Thus

$$\limsup_{n \rightarrow \infty} \|x_n^{(m-2)} - p\| \leq \limsup_{n \rightarrow \infty} (1 + r_n)^{m-2} \|x_n - p\| = a,$$

and hence

$$\limsup_{n \rightarrow \infty} \|T^n x_n^{(m-2)} - p\| \leq \limsup_{n \rightarrow \infty} (1 + r_n) \|x_n^{(m-2)} - p\| \leq a.$$

Next, consider

$$\|T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \leq \|T^n x_n^{(m-2)} - p\| + \gamma_n^{(m-2)} \|u_n^{(m-1)} - x_n\|$$

Thus,

$$\limsup_{n \rightarrow \infty} \|T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \leq a. \quad (4.6)$$

$$\text{Also, } \|x_n - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \leq \|x_n - p\| + \gamma_n^{(m-1)} \|u_n^{(m-1)} - x_n\|,$$

gives that

$$\limsup_{n \rightarrow \infty} \|x_n - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \leq a, \quad (4.7)$$

and noth that

$$\begin{aligned} a &= \lim_{n \rightarrow \infty} \|x_n^{(m-1)} - p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n^{(m-1)} T^n x_n^{(m-2)} + \beta_n^{(m-1)} x_n + \gamma_n^{(m-1)} u_n^{(m-1)} - p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n^{(m-1)} (T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)} (u_n^{(m-1)} - x_n)) \\ &\quad + (1 - \alpha_n^{(m-1)}) (x_n - p + \gamma_n^{(m-1)} (u_n^{(m-1)} - x_n))\|. \end{aligned}$$

It follows from (4.6), (4.7) and Lemma 3.1 that

$$\lim_{n \rightarrow \infty} \|T^n x_n^{(m-2)} - x_n\| = 0.$$

This completes the proof of (b). $\square$

Theorem 4.3. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X and $T : C \rightarrow C$ be an uniformly L -Lipschitzian, completely continuous asymptotically quasi-nonexpansive with the sequence $\{r_n\}_{n \geq 1}$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (1.1) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfies the same assumptions as Lemma 4.1 for each $i \in \{1, 2, \dots, m\}$ and the additional assumption that $0 < \alpha \leq \alpha_n^{(i)} \leq \beta < 1$ for all $i \in \{m-1, m\}$. Then $\{x_n^{(k)}\}$ converge strongly to a fixed point of T , for each $k = 1, 2, 3, \dots, m$.*

Proof. It follows from Lemma 4.2 that

$$\lim_{n \rightarrow \infty} \|T^n x_n^{(m-1)} - x_n\| = 0 = \lim_{n \rightarrow \infty} \|T^n x_n^{(m-2)} - x_n\|.$$

This implies that,

$$\begin{aligned} \|x_{n+1} - x_n\| = \|x_n^{(m)} - x_n\| &\leq \alpha_n^{(m)} \|T^n x_n^{(m-1)} - x_n\| + \gamma_n^{(m)} \|u_n^{(m)} - x_n\| \\ &\longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \quad (4.8)$$

Thus, we have

$$\begin{aligned} \|T^n x_n - x_n\| &\leq \|T^n x_n - T^n x_n^{(m-1)}\| + \|T^n x_n^{(m-1)} - x_n\| \\ &\leq L \|x_n - x_n^{(m-1)}\| + \|T^n x_n^{(m-1)} - x_n\| \\ &\leq \alpha_n^{(m-1)} L \|x_n - T^n x_n^{(m-2)}\| + \gamma_n^{(m-1)} L \|u_n^{(m-1)} - x_n\| \\ &\quad + \|T^n x_n^{(m-1)} - x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty \end{aligned} \quad (4.9)$$

and we note that

$$\begin{aligned} \|x_n - T x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1} x_{n+1}\| \\ &\quad + \|T^{n+1} x_{n+1} - T^{n+1} x_n\| + \|T^{n+1} x_n - T x_n\| \\ &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1} x_{n+1}\| \\ &\quad + (1 + r_{n+1}) \|x_{n+1} - x_n\| + L \|T^n x_n - x_n\|. \end{aligned}$$

This together with (4.8) and (4.9) we obtain

$$\lim_{n \longrightarrow \infty} \|x_n - T x_n\| = 0. \quad (4.10)$$

By the boundedness of $\{x_n\}$ and our assumption that T is completely continuous, there exists a subsequence $\{T x_{n_k}\}$ of $\{T x_n\}$ such that $T x_{n_k} \longrightarrow p \in C$ as $k \longrightarrow \infty$. Moreover, by (4.10), we have $\|T x_{n_k} - x_{n_k}\| \longrightarrow 0$ which implies that $x_{n_k} \longrightarrow p$ as $k \longrightarrow \infty$. By (4.10) again, we have

$$\|p - T p\| = \lim_{k \longrightarrow \infty} \|x_{n_k} - T x_{n_k}\| = 0.$$

It show that $p \in F(T)$. Furthermore, since $\lim_{n \longrightarrow \infty} \|x_n - p\|$ exist we obtain $\lim_{n \longrightarrow \infty} \|x_n - p\| = 0$, that is $\lim_{n \longrightarrow \infty} x_n^{(m)} = \lim_{n \longrightarrow \infty} x_n = p$. Moreover we observe that $\|x_n^{(k)} - p\| \leq \|x_n - p\| + d_n^{(k)}$ for all $k = 1, 2, 3, \dots, m-1$ and each $\lim_{n \longrightarrow \infty} d_n^{(k)} = 0$. Therefore $\lim_{n \longrightarrow \infty} x_n^{(k)} = p$ for all $k = 1, 2, 3, \dots, m-1$. The proof is completed. $\square$

For $m = 3$ and $\beta_n^{(i)} = 1 - \alpha_n^{(i)} - \gamma_n^{(i)}$ for all $i = 1, 2, 3$ in Theorem 3.4 or Theorem 4.3, we obtain the following result.

Theorem 4.4. (see [2]) *Let X be uniformly convex Banach space and C be a nonempty closed convex subset of X . Let $T : C \rightarrow C$ be a completely continuous asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{r_n\}$ in $[0, \infty)$ and $\sum_{n=1}^{\infty} r_n < \infty$. Let a sequence $\{x_n\}$ be defined by (1.2) with the following restrictions:*

- (i) $0 < a \leq \alpha_n^{(3)} < b < 1$
- (ii) $\limsup_{n \rightarrow \infty} (1 + r_n) \alpha_n^{(2)} < 1$
- (iii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i = 1, 2, 3$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point p of T .

When $m = 3$ and $\gamma_n^{(1)} = \gamma_n^{(2)} = \gamma_n^{(3)} \equiv 0$ in Theorem 3.4 or Theorem 4.3, we obtain strong convergence theorem for Noor iteration as follows:

Theorem 4.5. [20, Theorem 2.1]. *Let X be a real uniformly convex Banach space, C be a nonempty closed, bounded convex subset of X . Let $T : C \rightarrow C$ be a completely continuous asymptotically nonexpansive self-mapping with sequence $\{r_n\}$ satisfying $r_n \geq 0$ and $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \{\alpha_n^{(3)}\}$ be real sequences in $[0, 1]$ satisfying;*

- (i) $0 < \liminf_{n \rightarrow \infty} \alpha_n^{(3)} \leq \limsup_{n \rightarrow \infty} \alpha_n^{(3)} < 1$, and
- (ii) $0 < \liminf_{n \rightarrow \infty} \alpha_n^{(2)} \leq \limsup_{n \rightarrow \infty} \alpha_n^{(2)} < 1$.

For a given $x_1 \in C$, the sequence $\{x_n\}, \{x_n^{(1)}\}, \{x_n^{(2)}\}$ defined by (1.3) converges strongly to a fixed point of T .

Proof. It follows from the condition (i) and (ii) that there are $\alpha, \beta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that

$$0 < \alpha \leq \alpha_n^{(2)}, \alpha_n^{(3)} \leq \beta < 1$$

for all $n \geq n_0$. So that the conclusion of Theorem follows from the Theorem 3.4 or Theorem 4.3. $\square$

Acknowledgment: The author thanks the Thailand Research Fund for their financial support.

References

- [1] R.E. Bruck, T. Kuczumow and S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform opial property*, Colloq. Math. **65** (1993) 169–179.
- [2] Y.J. Cho, H. Zhou and G. Guo, *Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings*, Comp. and Math. Appl. **47** (2004) 707–717.
- [3] R. Glowinski and P. Le Tallec, "Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics" Siam, Philadelphia, 1989.
- [4] K. Goebel and W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972) 171–174.
- [5] S. Haubruge, V. H Nguyen, and J. J Strodiot, *Convergence analysis and applications of the GlowinskiLe Tallec splitting method for finding a zero of the sum of two maximal monotone operators*, J. Optim. Theory Appl. **97**(1998), 645673.
- [6] S. Ishikawa, *Fixed point by a new iterations*, Pro. Amer. Math. Soc. **44** (1974) 147–150.
- [7] J.U. Jeong, M. Aslam Noor and A. Rafiq, *Noor iterations for nonlinear Lipschitzian strongly accretive mappings*, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math., **11**(4) (2004), 339-350.
- [8] T.H. Kim and J.W. Choi, *Asymptotic behavior of almost-orbits of non-Lipschitzian mappings in Banach spaces*, Math. Japonica **38**(1993), 191-197.
- [9] G.E. Kim and T.H. Kim, *Mann and Ishikawa iterations with errors for non-Lipschitzian mappings in Banach spaces*, Comp. and Math. Appl. **42** (2001) 1565–1570.
- [10] Q. Liu, *Iterations sequence for asymptotically quasi-nonexpansive mapping with an error member*, J. Math. Anal. Appl. **259** (2001) 18–24.
- [11] W. R. Mann, *Mean value methods in iterations*, Pro. Amer. Math. Soc. **4** (1953) 506–510.
- [12] W. A. Kirk, *Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type*, Israel. J. Math. **17** (1974) 339–346.

- [13] M. Aslam Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl. **251**, (2000), 217–229.
- [14] M. Aslam Noor, *Three-step iterative algorithms for multivalued quasi variational inclusions*, J. Math. Anal. Appl. **255** (2001).
- [15] M. Aslam Noor, T.M. Rassias and Z. Huang *Three-step iterations for nonlinear accretive operator equations*, J. Math. Anal. Appl. **274** (2002), 59–68.
- [16] B.E. Rhoades, *Fixed point iterations for certain nonlinear mappings*, J. Math. Anal. Appl. **183** (1994), 118–120.
- [17] J. Schu, *Iterative construction of fixed points of strictly quasicontractive mapping*, Appl. Anal. **40** (1991) 67–72.
- [18] S. Suantai, *Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings* J. Math. Anal. Appl. (in press).
- [19] K.K. Tan, H.K. Xu, *Approximating fixed points of nonexpansive mapping by the Ishikawa iterations process*, J. Math. Anal. Appl. **178** (1993) 301–308.
- [20] B.L. Xu and M. Aslam Noor, *Fixed-Point Iterations for Asymptotically Nonexpansive Mappings in Banach Spaces*, J. Math. Anal. Appl. **267**, (2002) 444–453.

ภาคผนวก 4/19

Somyot Plubtieng

From: "Rabian Wangkeeree" <rabianw@nu.ac.th>
To: "Somyot plubtieng" <somyotp@nu.ac.th>
Sent: Saturday, August 13, 2005 12:41 PM
Subject: FW: JMAA-05-862R3: Final Decision

From: JMAA (ELS) [mailto:jmaa@elsevier.com]
Sent: ศ. 12/8/2548 11:23
To: Rabian Wangkeeree
Subject: JMAA-05-862R3: Final Decision

Ms. No.: JMAA-05-862R3
Title: Strong Convergence Theorems for Multi-Step Noor Iterations
with Errors in Banach Spaces
Corresponding Author: Mr. Rabian Wangkeeree
Authors: Somyot Plubtieng

Dear Mr. Wangkeeree,

We are pleased to inform you that your manuscript referenced above has been accepted for publication in the Journal of Mathematical Analysis and Applications.

Many thanks for submitting your fine paper to the Journal of Mathematical Analysis and Applications. We look forward to receiving additional papers from you in the future.

With kind regards,

Muhammad Aslam Noor, Ph.D
Associate Editor

Editorial Board
Journal of Mathematical Analysis and Applications
Elsevier
525 B Street, Suite 1900
San Diego, CA 92101-4495
USA
jmaa@elsevier.com

8/16/2005

ภาคผนวก 5

**The characteristic of noncompact convexity and
random fixed point theorem for set-valued
operators**

P. Kumam and S. Plubtieng

Czechoslovak Math. J. (accepted).

THE CHARACTERISTIC OF NONCOMPACT CONVEXITY AND
RANDOM FIXED POINT THEOREM FOR SET-VALUED OPERATORS

POOM KUMAM AND SOMYOT PLUBTIENG

Abstract. Let (Ω, Σ) be a measurable space, X a Banach space whose characteristic of noncompact convexity is less than 1, C a bounded closed convex subset of X , $KC(C)$ the family of all compact convex subsets of C . We prove that a set-valued nonexpansive mapping $T : C \rightarrow KC(C)$ has a fixed point. Furthermore, if X is separable then we also prove that a set-valued nonexpansive operator $T : \Omega \times C \rightarrow KC(C)$ has a random fixed point.

Keywords : random fixed point, set-valued random operator, measure of noncompactness.

Mathematics Subject Classification 2000 : 47H10, 47H09, 47H04.

1. INTRODUCTION

The study of random fixed points has been a very active area of research in probabilistic operator theory in the last decade. In this direction, there have appeared various papers concerning random fixed point theorems for single-valued and set-valued random operators; see, for example, [6],[8],[10],[11],[12],[15],[21] and the references therein.

In 2002, P. Lorenzo Ramírez [10] proved the existence of a random fixed point theorems for a random nonexpansive operator in the framework of Banach spaces with the characteristic of noncompact convexity $\varepsilon_\alpha(X)$ less than 1. On the other hand, Domínguez Benavides and Ramírez [4] proved a fixed point theorem for a set-valued nonexpansive and $1-\chi$ -contractive mapping in the framework of Banach spaces whose characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ less than 1.

The purpose of the present paper is to prove a fixed point theorem for set-valued random nonexpansive operators in the framework of Banach spaces with characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ less than 1. Moreover, we also prove a fixed point theorem for set-valued nonexpansive mappings in Banach spaces with characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ less than 1. Our results can also be viewed as an extension of Theorem 6 in [10] and Theorem 4.2 in [4], respectively.

2. PRELIMINARIES

Through out this paper we will consider a measurable space (Ω, Σ) (where Σ is a σ -algebra of subsets of Ω) and (X, d) will be a metric space. We denote by $CL(X)$ (resp. $CB(X), KC(X)$) the family of all nonempty closed (resp. closed bounded, compact convex) subsets of X , and by H the Hausdorff metric on $CB(X)$ induced by d , i.e.,

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$

for $A, B \in CB(X)$, where $d(x, E) = \inf\{d(x, y) | y \in E\}$ is the distance from x to $E \subset X$.

Let C be a nonempty closed subset of a Banach space X . Recall now that a set-valued mapping $T : C \rightarrow 2^X$ is said to be upper semicontinuous on C if $\{x \in C : Tx \subset V\}$ is open in C whenever $V \subset X$ is open; T is said to be lower semicontinuous if $T^{-1}(V) := \{x \in C : Tx \cap V \neq \emptyset\}$ is open in C whenever $V \subset X$ is open; and T is said to be continuous if it is both upper and lower semicontinuous (cf. [2] and [3] for details). There is another different kind of continuity for multivalued operators: $T : C \rightarrow CB(X)$ is said to be continuous on C (with respect to the Hausdorff metric H) if $H(Tx_n, Tx) \rightarrow 0$ whenever $x_n \rightarrow x$. It is not hard to see (see Deimling [3]) that both definitions of continuity are equivalent if Tx is compact for every $x \in C$.

A set-valued operator $T : \Omega \rightarrow 2^X$ is called (Σ) -measurable if, for any open subset B of X ,

$$T^{-1}(B) := \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\}$$

belongs to Σ . A mapping $x : \Omega \rightarrow X$ is said to be a *measurable selector* of a measurable set-valued operator $T : \Omega \rightarrow 2^X$ if $x(\cdot)$ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. An operator $T : \Omega \times C \rightarrow 2^X$ is called a random operator if, for each fixed $x \in C$, the operator $T(\cdot, x) : \Omega \rightarrow 2^X$ is measurable. We will denote by $F(\omega)$ the fixed point set of $T(\omega, \cdot)$, i.e.,

$$F(\omega) := \{x \in C : x \in T(\omega, x)\}.$$

Note that if we do not assume the existence of a fixed point for the deterministic mapping $T(\omega, \cdot) : C \rightarrow 2^X$, $F(\omega)$ may be empty. A measurable operator $x : \Omega \rightarrow C$ is said to be a *random fixed point of an operator* $T : \Omega \times C \rightarrow 2^X$ if $x(\omega) \in T(\omega, x(\omega))$ for all $\omega \in \Omega$. Recall that $T : \Omega \times C \rightarrow 2^X$ is continuous if, for each fixed $\omega \in \Omega$, the operator $T : (\omega, \cdot) \rightarrow 2^X$ is continuous.

If C is a closed convex subset of a Banach space X , then a set-valued mapping $T : C \rightarrow CB(X)$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq k\|x - y\|, \quad x, y \in C,$$

and T is said to be *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in C.$$

A random operator $T : \Omega \times C \rightarrow 2^X$ is said to be *nonexpansive* if, for each fixed $\omega \in \Omega$, the map $T : (\omega, \cdot) \rightarrow C$ is nonexpansive.

For later convenience, we list the following results related to the concept of measurability.

Lemma 2.1. (Wagner cf.[14]) *Let (X, d) be a complete separable metric space and $F : \Omega \rightarrow CL(X)$ a measurable map. Then F has a measurable selector.*

Lemma 2.2. (Itoh 1977, cf.[8]) *Suppose $\{T_n\}$ is a sequence of measurable set-valued operator from Ω to $CB(X)$ and $T : \Omega \rightarrow CB(X)$ is an operators. If, for each $\omega \in \Omega$, $H(T_n(\omega), T(\omega)) \rightarrow 0$, then T is measurable.*

Lemma 2.3. (Tan and Yuan cf.[13]) *Let X be a separable metric space and Y a metric space. If $f : \Omega \times X \rightarrow Y$ is measurable in $\omega \in \Omega$ and continuous in $x \in X$, and if $x : \Omega \rightarrow X$ is measurable, then $f(\cdot, x(\cdot)) : \Omega \rightarrow Y$ is measurable.*

As an easy application of Proposition 3 of Itoh[8] we have the following result.

Lemma 2.4. *Let C be a closed separable subset of a Banach space X , $T : \Omega \times C \rightarrow C$ a random continuous operator and $F : \Omega \rightarrow 2^C$ a measurable closed-valued operator. Then for any $s > 0$, the operator $G : \Omega \rightarrow 2^C$ given by*

$$G(\omega) = \{x \in F(\omega) : \|x - T(\omega, x)\| < s\}, \quad \omega \in \Omega$$

is measurable and so is the operator $\text{cl}\{G(\omega)\}$ of the closure of $G(\omega)$.

Lemma 2.5. (Domínguez Benavidel, Lopez Acedo and Xu cf.[6]) *Suppose C is a weakly closed nonempty separable subset of a Banach space X , $F : \Omega \rightarrow 2^X$ a measurable map with weakly*

compact values and $f : \Omega \times C \rightarrow \mathbb{R}$ a measurable, continuous and weakly lower semicontinuous function. Then the marginal function $r : \Omega \rightarrow \mathbb{R}$ defined by

$$r(\omega) := \inf_{x \in F(x)} f(\omega, x)$$

and the marginal map $R : \Omega \rightarrow X$ defined by

$$R(\omega) := \{x \in F(x) : f(\omega, x) = r(\omega)\}$$

are measurable.

Recall that the Kuratowski and Hausdorff measures of noncompactness of a nonempty bounded subset B of X are respectively defined as the number

$$\alpha(B) = \inf \{r > 0 : B \text{ can be covered by finitely many sets of diameter } \leq r\},$$

$$\chi(B) = \inf \{r > 0 : B \text{ can be covered by finitely many balls of radius } \leq r\}.$$

The separation measure of noncompactness of a nonempty bounded subset B of X defined by

$$\beta(B) = \sup \{\varepsilon : \text{there exists a sequence } \{x_n\} \text{ in } B \text{ such that } \text{sep}(\{x_n\}) \geq \varepsilon\}.$$

Let X be a Banach space and $\phi = \alpha, \beta$ or χ . The modulus of noncompact convexity associated to ϕ is defined in the following way:

$$\Delta_{X,\phi}(\varepsilon) = \inf \{1 - d(0, A) : A \subset B_X \text{ is convex, } \phi(A) \geq \varepsilon\},$$

where B_X is the unit ball of X .

The characteristic of noncompact convexity of X associated with the measure of noncompactness ϕ is defined by

$$\varepsilon_\phi(X) = \sup \{\varepsilon \geq 0 : \Delta_{X,\phi}(\varepsilon) = 0\}.$$

The following relationships among the different moduli are easy to obtain

$$(2.1) \quad \Delta_{X,\alpha}(\varepsilon) \leq \Delta_{X,\beta}(\varepsilon) \leq \Delta_{X,\chi}(\varepsilon),$$

and consequently

$$(2.2) \quad \varepsilon_\alpha(X) \geq \varepsilon_\beta(X) \geq \varepsilon_\chi(X).$$

When X is a reflexive Banach space we have some alternative expressions for the moduli of noncompact convexity associated to β and χ .

$$\Delta_{X,\beta}(\varepsilon) = \inf \{1 - \|x\| : \{x_n\} \subset B_X, x = w - \lim_n x_n, \text{sep}(\{x_n\}) \geq \varepsilon\},$$

$$\Delta_{X,\chi}(\varepsilon) = \inf \{1 - \|x\| : \{x_n\} \subset B_X, x = w - \lim_n x_n, \chi(\{x_n\}) \geq \varepsilon\}.$$

Let C be a nonempty bounded closed subset of a Banach space X and $\{x_n\}$ a bounded sequence in X . We use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C , respectively, i.e.

$$r(C, \{x_n\}) = \inf \left\{ \limsup_n \|x_n - x\| : x \in C \right\},$$

$$A(C, \{x_n\}) = \left\{ x \in C : \limsup_n \|x_n - x\| = r(C, \{x_n\}) \right\}.$$

If D is a bounded subset of X , the *Chebyshev radius* of D relative to C is defined by

$$r_C(D) := \inf \{ \sup \{ \|x - y\| : y \in D \} : x \in C \}.$$

Let $\{x_n\}$ and C be nonempty bounded closed subsets of a Banach space X . Then $\{x_n\}$ is called *regular* with respect to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Moreover, we also need the following Lemmas.

Lemma 2.6. (Domínguez Benavides and Lorenzo Ramírez Theorem 4.3 cf. [4]) *Let C be a closed convex subset of a reflexive Banach space X , and let x_n be a bounded sequence in C which is regular with respect to C . Then*

$$(2.3) \quad r_C(A(C, x_n)) \leq (1 - \Delta_{X,\beta}(1^-))r(C, \{x_n\}).$$

Moreover, if X satisfies the nonstrict Opial condition then

$$(2.4) \quad r_C(A(C, x_n)) \leq (1 - \Delta_{X,X}(1^-))r(C, \{x_n\}).$$

The following result are now basic in the fixed point theorem for multivalued mappings.

Lemma 2.7. (Xu cf. Theorem 1.6 of [19]) *Let E be a nonempty bounded closed convex subset of a Banach space and $T : E \rightarrow KC(X)$ a contraction. Assume $Tx \cap \overline{I_E(x)} \neq \emptyset$ for all $x \in E$. Then T has a fixed point. (Here $I_E(x)$ is call the inward set at x defined by $I_E := \{x + \lambda(y - x) : \lambda \geq 0, y \in E\}$)*

Proposition 2.8. (Kirk-Massa Theorem cf.[16]) *Let C be a nonempty weakly compact separable subset of a Banach space X , $T : C \rightarrow K(C)$ a nonexpansive mapping, and $\{x_n\}$ a sequence in C such that $\lim_n d(x_n - Tx_n) = 0$. Then, there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that*

$$Tx \cap A \neq \emptyset, \forall x \in A := A(C, \{z_n\})$$

3. THE RESULTS

We begin this section by showing that in Benavides-Ramírez's result, the $1-\chi$ -contractive condition on T can be removed.

Theorem 3.1. *Let C be a nonempty closed bounded convex subset of a Banach space X such that $\varepsilon_\beta(X) < 1$, and $T : C \rightarrow KC(C)$ a nonexpansive mapping. Then T has a fixed point.*

Proof The condition $\varepsilon_\beta(X) < 1$ implies reflexivity [2], so C is weakly compact. Let $x_0 \in C$ be fixed and, for each $n \geq 1$, define $T_n : C \rightarrow KC(C)$ by

$$T_n x = \frac{1}{n} x_0 + (1 - \frac{1}{n}) T x, \quad \forall x \in C.$$

Then T_n is a set-valued contraction and hence has a fixed point x_n . It is easily seen that $\text{dist}(x_n, T x_n) \leq \frac{1}{n} \text{diam} C \rightarrow 0$ as $n \rightarrow \infty$. By Goebel and Kirk [7], we may assume that $\{x_n\}$ is regular with respect to C and using Proposition 2.8 we can also assume that

$$T x \cap A \neq \emptyset, \quad \forall x \in A := A(C, \{x_n\}).$$

We apply Lemma 2.6 to obtain

$$(3.1) \quad r_C(A) \leq \lambda r(C, \{x_n\}),$$

where $\lambda := (1 - \Delta_{X,\beta}(1^-)) < 1$.

It is clear that A is a weakly compact convex subset of C . Now fix $x_1 \in A$ and for each $n \geq 1$, define the contraction $T_n^1 : A \rightarrow KC(C)$ by

$$T_n^1(x) = \frac{1}{n} x_1 + (1 - \frac{1}{n}) T x, \quad \forall x \in A.$$

Since A is convex, each T_n^1 satisfies the same boundary condition as T does, that is, we have

$$T_n^1 x \cap \bar{I}_A(x) \neq \emptyset, \quad \forall x \in A,$$

Hence by Lemma 2.7, T_n^1 has a fixed point $z_n \in A$. Consequently, we can get a sequence $\{x_n^1\}$ in A satisfying $d(x_n^1, T(x_n^1)) \rightarrow 0$ as $n \rightarrow \infty$. Again, applying Lemma 2.6, we obtain

$$(3.2) \quad r_C(A^1) \leq \lambda r(C, \{x_n^1\}),$$

where $A^1 := A(C, \{x_n^1\})$. Since $\{x_n^1(\omega)\} \subset A$, we have

$$(3.3) \quad r(C, \{x_n^1\}) \leq r_C(A),$$

and then

$$(3.4) \quad r_C(A^1) \leq \lambda^2 r_C(A).$$

By induction, for each $m \geq 1$, we construct A^m , and $\{x_n^m\}_n$ where $A^m = A(C, \{x_n^m\})$, $x_n^m \subset A^{m-1}$ such that $d(x_n^m, Tx_n^m) \rightarrow 0$ as $n \rightarrow \infty$ and

$$(3.5) \quad r_C(A^m) \leq \lambda r_C(A) \leq \lambda^m r(C, \{x_n\}).$$

By assumption $\varepsilon_\beta(X) < 1$ and $\text{diam} A^m \leq 2r_C(A^m)$ leads to $\lim_{m \rightarrow \infty} \text{diam} A^m = 0$. Since $\{A^m\}$ is a descending sequence of weakly compact subsets of C , we have $\cap_m A^m = \{z\}$ for some $z \in C$. Finally, we will show that z is a fixed point of T . Indeed, for each $m \geq 1$, we have

$$\begin{aligned} d(z, Tz) &\leq \|z - x_n^m\| + d(x_n^m, Tx_n^m) + H(Tx_n^m, Tz) \\ &\leq 2\|z - x_n^m\| + d(x_n^m, Tx_n^m) \\ &\leq 2\text{diam} A^m + d(x_n^m, Tx_n^m). \end{aligned}$$

Taking the upper limit as $n \rightarrow \infty$,

$$d(z, Tz) \leq 2\text{diam} A^m.$$

Now taking the limit in m on both sides we obtain $z \in Tz$. $\square$

Corollary 3.2. (Domínguez Benavides and Lorenzo Ramírez. Theorem 4.2 in [4]) *Let C be a nonempty closed bounded convex subset of a Banach space X such that $\varepsilon_\beta(X) < 1$, and $T : C \rightarrow KC(C)$ a nonexpansive and 1 - χ -contractive mapping. Then T has a fixed point.*

Now we are ready to prove the main result of this paper.

Theorem 3.3. *Let C be a nonempty closed bounded convex separable subset of a Banach space X such that $\varepsilon_\beta(X) < 1$, and $T : \Omega \times C \rightarrow KC(C)$ be a set-valued nonexpansive random operator. Then T has a random fixed point.*

Proof For each $\omega \in \Omega$, and for every $n \geq 1$, we set

$$F(\omega) = \{x \in C : x \in T(\omega, x)\},$$

and

$$F_n(\omega) = \{x \in C : d(x, T(\omega, x)) \leq \frac{1}{n} \text{diam} C\}.$$

It follows from Theorem 3.1 that $F(\omega)$ is nonempty. Clearly $F(\omega) \subseteq F_n(\omega)$, and $F_n(\omega)$ is closed and convex. Furthermore, by [8, Proposition 3], each F_n is measurable. Then, by Lemma 2.1, each F_n admits a measurable selector $x_n(\omega)$ and

$$d(x_n(\omega), T(\omega, x_n(\omega))) \leq \frac{1}{n} \text{diam} C \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Define a function $f_1 : \Omega \times C \rightarrow \mathbb{R}^+$ by

$$f_1(\omega, x) = \limsup_n \|x_n(\omega) - x\|, \quad \forall \omega \in \Omega.$$

By Lemma 2.3, it is easily seen that for each $x \in C$, $f_1(\cdot, x) : \Omega \rightarrow \mathbb{R}^+$ is measurable and for each $\omega \in \Omega$, $f_1(\omega, \cdot) : C \rightarrow \mathbb{R}^+$ is continuous and convex (and hence weakly lower semicontinuous (w-l.s.c.)). Note that the condition $\varepsilon_\beta(X) < 1$ implies reflexivity (see [2]) and so C is weakly compact. Hence, by Lemma 2.5 the marginal functions

$$r_1(\omega) := \inf_{x \in C} f_1(\omega, x),$$

and

$$R_1(\omega) := \{x \in C : f_1(\omega, x) = r_1(\omega)\}$$

are measurable. By Goebel [7], for any $\omega \in \Omega$ we may assume that the sequence $\{x_n(\omega)\}$ is regular with respect to C . Observe that $R_1(\omega) = A(C, \{x_n(\omega)\})$ and $r_1(\omega) = r(C, \{x_n(\omega)\})$, thus we can apply Lemma 2.6 to obtain

$$(3.6) \quad r_C(R_1(\omega)) \leq \lambda r_1(\omega),$$

where $\lambda := 1 - \Delta_{X,\beta}(1^-) < 1$, since $\varepsilon_\beta(X) < 1$. It is clear that $R_1(\omega)$ is a weakly compact and convex subset of C . By Lemma 2.1 we can take $x_1(\omega)$ as a measurable selector of $R_1(\omega)$. For each $\omega \in \Omega$ and $n \geq 1$, we define the contraction $T_n^1(\omega, \cdot) : R_1(\omega) \rightarrow KC(C)$ by

$$T_n^1(\omega, x) = \frac{1}{n}x_1(\omega) + (1 - \frac{1}{n})T(\omega, x), \quad \forall x \in R_1(\omega).$$

Since $R_1(\omega)$ is convex, each T_n satisfies the same boundary condition as T does, that is, we have

$$T_n^1(\omega, x) \cap \bar{I}_{R_1}(\omega)(x) \neq \emptyset, \quad \forall x \in R_1(\omega).$$

Hence by Lemma 2.7, $T_n^1(\omega, \cdot)$ has a fixed point $z_n(\omega) \in R_1(\omega)$, i.e. $F(\omega) \cap R_1(\omega) \neq \emptyset$. Also it is easily seen that

$$\text{dist}(z_n(\omega), T(\omega, z_n(\omega))) \leq \frac{1}{n} \text{diam} C \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus $F_n^1(\omega) = \{x \in R_1(\omega) : d(x, T(\omega, x)) \leq \frac{1}{n} \text{diam} C\} \neq \emptyset$ for each $n \geq 1$, is closed and, by Lemma 2.4, measurable. Hence, by Lemma 2.1, we can choose x_n^1 a measurable selector of F_n^1 , and from its definition we have $x_n^1(\omega) \in R_1(\omega)$ and $d(x_n^1(\omega), T(\omega, x_n^1(\omega))) \rightarrow 0$ as $n \rightarrow \infty$. Consider the function $f_2 : \Omega \times C \rightarrow \mathbb{R}^+$ defined by

$$f_2(\omega, x) = \limsup_n \|x_n^1(\omega) - x\|, \quad \forall \omega \in \Omega.$$

As above, f_2 is a measurable function and weakly lower semicontinuous function. Thus the marginal functions

$$r_2(\omega) := \inf_{x \in R_1(\omega)} f_2(\omega, x)$$

and

$$R_2(\omega) := \{x \in R_1(\omega) : f_2(\omega, x) = r_2(\omega)\}$$

are measurable. Since $R_2(\omega) = A(R_1(\omega), \{x_n^1(\omega)\})$, it follows that $R_2(\omega)$ is weakly compact and convex. Also $r_2(\omega) = r(R_1(\omega), \{x_n^1(\omega)\})$. Again reasoning as above, for any $\omega \in \Omega$, we can

assume that the sequence $\{x_n^1(\omega)\}$ is regular with respect to $R_1(\omega)$. Again, applying Lemma 2.6, we obtain

$$(3.7) \quad r_C(R_2(\omega)) \leq \lambda r_2(\omega).$$

Furthermore, $\{x_n^1(\omega)\} \subset R_1(\omega)$. Hence

$$(3.8) \quad r_2(\omega) \leq r_C(R_1(\omega)),$$

and thus

$$(3.9) \quad r_C(R_2(\omega)) \leq \lambda^2 r_1(\omega).$$

By induction, for each $m \geq 1$, we construct $R_m(\omega), r_m(\omega)$ and $\{x_n^m(\omega)\}_n$ where $x_n^m(\omega) \in R_m(\omega)$ such that $d(x_n^m(\omega), T(\omega, x_n^m(\omega))) \rightarrow 0$ as $n \rightarrow \infty$ and

$$(3.10) \quad r_C(R_m(\omega)) \leq \lambda r_m(\omega) \leq \lambda^m r_1(\omega).$$

Since $\text{diam} R_m(\omega) \leq 2r_C(R_m(\omega))$ and $\lambda < 1$, it follows that $\lim_{m \rightarrow \infty} \text{diam} R_m(\omega) = 0$. Since $\{R_m(\omega)\}$ is a descending sequence of weakly compact subsets of C for each $\omega \in \Omega$, we have $\bigcap_m R_m(\omega) = \{z(\omega)\}$ for some $z(\omega) \in C$. Furthermore, we see that

$$H(R_m(\omega), \{z(\omega)\}) \leq \text{diam} R_m(\omega) \rightarrow 0 \text{ as } m \rightarrow +\infty.$$

Therefore, by Lemma 2.2, $z(\omega)$ is measurable. Finally, we will show that $z(\omega)$ is a fixed point of T . Indeed, for each $m \geq 1$, we have

$$\begin{aligned} d(z(\omega), T(\omega, z(\omega))) &\leq \|z(\omega) - x_n^m(\omega)\| + d(x_n^m(\omega), T(\omega, x_n^m(\omega))) \\ &\quad + H(T(\omega, x_n^m(\omega)), T(\omega, z(\omega))) \\ &\leq 2\|z(\omega) - x_n^m(\omega)\| + d(x_n^m(\omega), T(\omega, x_n^m(\omega))) \\ &\leq 2\text{diam} R_m(\omega) + d(x_n^m(\omega), T(\omega, x_n^m(\omega))). \end{aligned}$$

Taking the upper limit as $n \rightarrow \infty$,

$$d(z(\omega), T(\omega, z(\omega))) \leq 2\text{diam} R_m(\omega).$$

Finally, taking limit in m in both sides we obtain $z(\omega) \in T(\omega, z(\omega))$. □

Corollary 3.4. *Let C be a nonempty closed bounded convex separable subset of a Banach space X such that $\varepsilon_\beta(X) < 1$, and $T : \Omega \times C \rightarrow C$ a random nonexpansive operator. Then T has a random fixed point.*

Corollary 3.5. (Lorenzo Ramírez, Theorem 6 in [10]) *Let C be a nonempty closed bounded convex separable subset of a Banach space X such that $\varepsilon_\alpha(X) < 1$, and $T : \Omega \times C \rightarrow C$ a random nonexpansive operator. Then T has a random fixed point.*

Proof By (2.2) we see that $\varepsilon_\alpha(X) < 1$ implies $\varepsilon_\beta(X) < 1$.

Acknowledgement. The second author would like to thank The Thailand Research Fund for financial support.

REFERENCES

- [1] Aubin, J. P. and Frankowska H.: Set-valued Analysis. Birkhäuser. Boston. (1990).
- [2] Ayerbe Toledano, J. M., Domínguez Benavides, T. and López Acedo, G.: "Measures of noncompactness in metric fixed point theory": Advances and Applications Topics in metric fixed point theory. Birkhauser-Verlag. Basel. 99 (1997).
- [3] Deimling, K.: Nonlinear Functional Analysis. Springer-Verlag. Berlin. (1974).
- [4] Domínguez Benavides, T. and Lorenzo Ramírez P.: Fixed point theorem for multivalued nonexpansive mapping without uniform convexity. Abstr. Appl. Anal. 6 (2003), 375-386.
- [5] Domínguez Benavides, T. and Lorenzo Ramírez P. Ramírez: Fixed point theorem for multivalued non-expansive mapping satisfying inwardness conditions. J. Math. Anal. Appl. 291 (2004), 100-108.
- [6] Domínguez Benavides, T., Lopez Acedo, G. and Xu, H.-K.: Random fixed point of set-valued operator. Proc. Amer. Math. Soc. 124 (1996), 838-838.
- [7] Goebel, K. and Kirk, W. A.: Topics in metric fixed point theorem. Cambridge University Press, Cambridge (1990).
- [8] Itoh, S.: Random fixed point theorem for a multivalued contraction mapping. Pacific J. Math. 68 (1977), 85-90.
- [9] Kirk, W. A.: Nonexpansive mappings in product spaces, set-valued mappings, and k-uniform rotundity. Proceedings of the Symposium Pure Mathematics, Vol. 45, part 2, American Mathematical Society, Providence, RI, 1986, pp. 51-64.
- [10] Lorenzo Ramírez, P.: Some random fixed point theorems for nonlinear mappings. Nonlinear Anal. 50 (2002), 265-274.
- [11] Lorenzo Ramírez, P.: Random fixed point of uniformly Lipschitzian mappings. Nonlinear Anal. 57 (2004), 23-34.
- [12] Shahzad, N. and Latif, S.: Random fixed points for several classes of 1-ball-contractive and 1-set-contractive random maps. J. Math. Anal. Appl. 237 (1999), 83-92.
- [13] Tan, K.-K. and Yuan, X.Z.: Some random fixed point theorems. in: K.-K. Tan (Ed), Fixed Point Theory and Applications. World Scientific. Singapore. 1992, 334-345.
- [14] Wagner, D.-H.: Survey of measurable selection theorems. SIAM J. Control Optim. 15 (1977), 859-903.
- [15] Xu, H. K.: Some random fixed point for condensing and nonexpansive operators. Proc. Amer. Math. Soc. 110(1990), 395-400.
- [16] Xu, H. K.: Metric fixed point for multivalued mappings. Dissertationes Math. (Rozprawy Mat.) 389(2000), 39.
- [17] Xu, H. K.: A random theorem for multivalued nonexpansive operators in uniformly convex Banach spaces. Proc. Amer. Math. Soc. 117 (1993) No.4, 1089-1092.
- [18] Xu, H. K.: Random fixed point theorems for nonlinear uniform Lipschitzian mappings. Nonlinear Anal. 26 (1996) No.7, 1301-1311.
- [19] Xu, H. K.: Multivalued nonexpansive mappings in Banach spaces. Nonlinear Anal. 43 (2001), 693-706.
- [20] Reich, S.: Fixed points in locally convex spaces. Math. Z. 125 (1972), 17-31.
- [21] Yuan, X. and Yu, J.: Random fixed point theorems for nonself mappings. Nonlinear Anal. 26 (1996) No.6, 1097-1102.

ภาคผนวก 5/11

RANDOM FIXED POINT THEOREMS

11

POOM KUMAM, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KING MONGKUT'S UNIVERSITY OF TECHNOLOGY THONBURI, BANGKOK 10140. THAILAND. *e-mail:poom.kum@kmutt.ac.th*

AUTHORS ADDRESSES: SOMYOT PLUBTIENG, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NARESUAN UNIVERSITY, PITSANULOK 65000. THAILAND. *e-mail:somyotp@nu.ac.th*

Czechoslovak Mathematical Journal

Žitná 25, 115 67 Praha 1

Czech Republic

Dr. Somyot Plubtieng

Naresuan University

Dept. of Mathematics

Phitsanulok

65000

Thailand

Ref. No. 13/05

Praha 25.7.2005

Dear Professor:

we have the pleasure of informing you that your paper

The characteristic of noncompact convexity and random fixed point theorem
for set-valued operators

which had been submitted for publication in our journal, was accepted by the Editorial Board
and will appear as soon as possible.

We ask you kindly to supplement each item in References with the number under
which it is referred to in the database Zentralblatt MATH (<http://www.emis.de/ZMATH>),
hence e.g.

[3] R.Henstock: The General Theory of Integration. Clarendon Press, Oxford, U.K.,
1991. Zbl 0745.26006

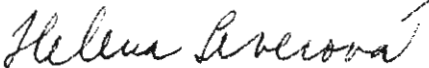
We should like to turn your attention to the comments of our referee and ask you to
consider them carefully.

We would like to ask you to send us a hardcopy and a T_EX file of corrected paper on
a diskette or by e-mail. If you do so, you will help us with the typesetting of our journal.

Enclosed you will find the Licence Agreement. If you find it acceptable, please sign it
and send it to our Editorial Office by return of mail. Note that signing the Licence Agreement
is a necessary condition for publication of your paper.

Thanking you in advance for your cooperation,

CZECHOSLOVAK MATHEMATICAL JOURNAL
REDAKCE:
PRAHA 1, ŽITNÁ ULICE 25
PSČ 115 67


Helena Severová
Executive Editor

ภาคผนวก 6

**Random fixed point theorems for multivalued
nonexpansive non-self random operators**

P. Kumam and S. Plubtieng

J. Appl. Math. Stoc. Anal. (accepted).

RANDOM FIXED POINT THEOREMS FOR MULTIVALUED NONEXPANSIVE NON-SELF RANDOM OPERATORS

S. PLUBTIENG AND P. KUMAM

Abstract. Let (Ω, Σ) be a measurable space, with Σ a sigma-algebra of subset of Ω , and let C be a nonempty bounded closed convex separable subset of a Banach space X , whose characteristic of noncompact convexity is less than 1, $KC(X)$ the family of all compact convex subsets of X . We prove that a multivalued nonexpansive non-self random operator $T : \Omega \times C \rightarrow KC(X)$, $1-\chi$ -contractive mapping, satisfying a inwardness condition has a random fixed point.

1. INTRODUCTION

In recent years there have appeared various random fixed point theorems for single-valued and set-valued random operator; see for example, Itoh [7], Ramírez [11], Tan and Yuan [12], Xu [14], and [15] Yuan and Yu [17] and references therein.

In 2002, P. L. Ramírez [11] proved the existence of random fixed point theorems for a random nonexpansive operator in the framework of a Banach spaces with a characteristic of noncompact convexity $\varepsilon_\alpha(X)$ is less than 1. On the other hand, Domínguez Benavides and Ramírez [3] proved a fixed point theorem for a set-valued nonexpansive self-mapping and $1-\chi$ -contractive mapping in the framework of a Banach spaces whose characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ is less than 1. In 2004, Domínguez Benavides and Ramírez [4] proved a fixed point theorem for a multivalued nonexpansive non-self mapping and $1-\chi$ -contractive mapping in the framework of a Banach spaces whose characteristic of noncompact convexity associated to the Kuratowski measure of noncompactness $\varepsilon_\alpha(X)$ is less than 1.

0

Key words and phrases: random fixed point, multivalued random operator, inwardness condition.
2000 Mathematics Subject Classification: 47H10, 47H09, 47H40.

The purpose of the present paper is to prove a random fixed point theorem for multivalued nonexpansive non-self random operators which is 1- χ -contractive mapping, in the framework of a Banach spaces with characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ less than 1. and satisfying a inwardness condition. Our result can also be seen as an extension of Theorem 3.4 in [4]

2. PRELIMINARIES AND NOTATIONS

We begin with establishing some preliminaries. By (Ω, Σ) we denote a measurable space with Σ a sigma-algebra of subset of Ω . Let (X, d) be a metric space. We denote by $CL(X)$ (resp $CB(X), KC(X)$) the family of all nonempty closed (resp. closed bounded, compact convex) subset of X , and by H the Hausdorff metric on $CB(X)$ induced by d , i.e.,

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$

for $A, B \in CB(X)$, where $d(x, E) = \inf \{d(x, y) | y \in E\}$ is the distance from x to $E \subset X$.

Let C be a nonempty closed subset of a Banach space X . Recall now that a Multivalued mapping $T : C \rightarrow 2^X$ is said to be upper semicontinuous on C if $\{x \in C : Tx \subset V\}$ is open in C whenever $V \subset X$ is open; T is said to be lower semicontinuous if $T^{-1}(V) := \{x \in C : Tx \cap V \neq \emptyset\}$ is open in C whenever $V \subset X$ is open; and T is said to be continuous if it is both upper and lower semicontinuous (cf.[1] and [2] for details). There is another different kind of continuity for multivalued operator: $T : C \rightarrow CB(X)$ is said to be continuous on C (with respect to the Hausdorff metric H) if $H(Tx_n, Tx) \rightarrow 0$ whenever $x_n \rightarrow x$. It is not hard to see (see Deimling [2]) that both definitions of continuity are equivalent if Tx is compact for every $x \in C$.

If C is a closed convex subset of a Banach spaces X , then a multivalued mapping $T : C \rightarrow CB(X)$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq k\|x - y\|, \quad x, y \in C,$$

and T is said to be *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in C,$$

A multivalued operator $T : \Omega \rightarrow 2^X$ is called (Σ) -measurable if, for any open subset B of X ,

$$T^{-1}(B) = \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\}$$

belongs to Σ . A mapping $x : \Omega \rightarrow X$ is said to be a *measurable selector* of a measurable Multivalued operator $T : \Omega \rightarrow 2^X$ if $x(\cdot)$ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. An operator $T : \Omega \times C \rightarrow 2^X$ is called a *random operator* if, for each fixed $x \in C$, the operator $T(\cdot, x) : \Omega \rightarrow 2^X$ is measurable. We will denote by $F(\omega)$ the fixed point set of $T(\omega, \cdot)$, i.e.,

$$F(\omega) := \{x \in C : x \in T(\omega, x)\}.$$

Note that if we do not assume the existence of fixed point for the deterministic mapping $T(\omega, \cdot) : C \rightarrow 2^X$, $F(\omega)$ may be empty. A measurable operator $x : \Omega \rightarrow C$ is said to be a *random fixed point of a operator* $T : \Omega \times C \rightarrow 2^X$ if $x(\omega) \in T(\omega, x(\omega))$ for all $\omega \in \Omega$. Recall that $T : \Omega \times C \rightarrow 2^X$ is continuous if, for each fixed $\omega \in \Omega$ the operator $T : (\omega, \cdot) \rightarrow 2^X$ is continuous.

A random operator $T : \Omega \times C \rightarrow 2^X$ is said to be *nonexpansive* if, for each fixed $\omega \in \Omega$ the map $T : (\omega, \cdot) \rightarrow C$ is nonexpansive.

For later convenience, we list the following results related to the concept of measurability.

Lemma 2.1. (Wagner cf.[13]). *Let (X, d) be a complete separable metric space and $F : \Omega \rightarrow CL(X)$ a measurable map. Then F has a measurable selector.*

Lemma 2.2. (Itoh 1977, cf.[7]). *Suppose $\{T_n\}$ is a sequence of measurable Multivalued operator from Ω to $CB(X)$ and $T : \Omega \rightarrow CB(X)$ is an operator. If, for each $\omega \in \Omega$, $H(T_n(\omega), T(\omega)) \rightarrow 0$, then T is measurable.*

Lemma 2.3. (Tan and Yuan cf.[12]). *Let X be a separable metric space and Y a metric space. If $f : \Omega \times X \rightarrow Y$ is a measurable in $\omega \in \Omega$ and continuous in $x \in X$, and if $x : \Omega \rightarrow X$ is measurable, then $f(\cdot, x(\cdot)) : \Omega \rightarrow Y$ is measurable.*

As an easy application of Proposition 3 of Itoh[7] we have the following result.

Lemma 2.4. *Let C be a closed separable subset of a Banach space X , $T : \Omega \times C \rightarrow C$ a random continuous operator and $F : \Omega \rightarrow 2^C$ a measurable closed-valued operator. Then for any $s > 0$, the operator $G : \Omega \rightarrow 2^C$ given by*

$$G(\omega) = \{x \in F(\omega) : \|x - T(\omega, x)\| < s\}, \quad \omega \in \Omega$$

is measurable and so is the operator $cl\{G(\omega)\}$ of the closure of $G(\omega)$.

Lemma 2.5. (Domínguez Benavidel and Lopez Acedo cf.[5]). *Suppose C is a weakly closed nonempty separable subset of a Banach space X , $F : \Omega \rightarrow 2^X$ a measurable with weakly compact values, $f : \Omega \times C \rightarrow \mathbb{R}$ is a measurable, continuous and weakly lower semicontinuous function. Then the marginal function $r : \Omega \rightarrow \mathbb{R}$ defined by*

$$r(\omega) := \inf_{x \in F(\omega)} f(\omega, x)$$

and the marginal map $R : \Omega \rightarrow X$ defined by

$$R(\omega) := \{x \in F(\omega) : f(\omega, x) = r(\omega)\}$$

are measurable.

Recall that the Kuratowski and Hausdorff measures of noncompactness of a nonempty bounded subset B of X are respectively defined as the number

$$\alpha(B) = \inf \{r > 0 : B \text{ can be covered by finitely many sets of diameter } \leq r\},$$

$$\chi(B) = \inf \{r > 0 : B \text{ can be covered by finitely many ball of radius } \leq r\}.$$

The separation measure of noncompactness of a nonempty bounded subset B of X defined by

$$\beta(B) = \sup \{\varepsilon : \text{there exists a sequence } \{x_n\} \text{ in } B \text{ such that } \text{sep}(\{x_n\}) \geq \varepsilon\}.$$

Then a multivalued mapping $T : C \rightarrow 2^X$ is called γ -condensing (resp., $1-\gamma$ -contractive) where $\gamma = \alpha(\cdot)$ or $\chi(\cdot)$ if, for each bounded subset B of C with $\gamma(B) > 0$, there holds the inequality

$$\gamma(T(B)) < \gamma(B) \quad (\text{resp. } \gamma(T(B)) \leq \gamma(B)).$$

Here $T(B) = \bigcup_{x \in B} Tx$. The random operator $T : \Omega \times C \rightarrow 2^X$ is said to be $1-\gamma$ -contractive if, for each $\omega \in \Omega$ the map $T : (\omega, \cdot) \rightarrow 2^X$ is $1-\gamma$ -contractive.

Definition 2.6. Let X be a Banach space and $\phi = \alpha, \beta$ or χ . The modulus of noncompact convexity associated to ϕ is defined in the following way:

$$\Delta_{X,\phi}(\varepsilon) = \inf \{1 - d(0, A) : A \subset B_X \text{ is convex, } \phi(A) \geq \varepsilon\},$$

where B_X is the unit ball of X .

The characteristic of noncompact convexity of X associated with the measure of noncompactness ϕ is defined by

$$\varepsilon_\phi(X) = \sup \{\varepsilon \geq 0 : \Delta_{X,\phi}(\varepsilon) = 0\}.$$

The following relationships among the different moduli are easy to obtain

$$(2.1) \quad \Delta_{X,\alpha}(\varepsilon) \leq \Delta_{X,\beta}(\varepsilon) \leq \Delta_{X,\chi}(\varepsilon),$$

and consequently

$$(2.2) \quad \varepsilon_\alpha(X) \geq \varepsilon_\beta(X) \geq \varepsilon_\chi(X).$$

When X is a reflexive Banach space we have some alternative expressions for the moduli of noncompact convexity associated β and χ .

$$\Delta_{X,\beta}(\varepsilon) = \inf \{1 - \|x\| : \{x_n\} \subset B_X, x = w - \lim x_n, \text{sep}(\{x_n\}) \geq \varepsilon\},$$

$$\Delta_{X,X}(\varepsilon) = \inf \{1 - \|x\| : \{x_n\} \subset B_X, x = w - \lim x_n, \chi(\{x_n\}) \geq \varepsilon\}.$$

In order to study the fixed point theory for non-self mappings we must introduce some terminology for boundary condition. The inward set of C at $x \in C$ defined by

$$I_C(x) := \{x + \lambda(y - x) : \lambda \geq 0, y \in C\}.$$

Clearly $C \subset I_C(x)$ and it is not hard to show that $I_C(x)$ is a convex set as C does. A multivalued mapping $T : C \rightarrow 2^X \setminus \{\emptyset\}$ is said to be *inward* on C if

$$Tx \subset I_C(x) \quad \forall x \in C.$$

Let $\bar{I}_C(x) := x + \{\lambda(z - x) : z \in C, \lambda \geq 1\}$. Note that for a convex C , we have $\bar{I}_C(x) = \overline{I_C(x)}$, and T is said to be *weakly inward* on C if

$$Tx \subset \bar{I}_C(x) \quad \forall x \in C.$$

Let C be a nonempty bounded closed subset of Banach spaces X and $\{x_n\}$ bounded sequence in X , we use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C , respectively, i.e.

$$\begin{aligned} r(C, \{x_n\}) &= \inf \left\{ \limsup_n \|x_n - x\| : x \in C \right\}, \\ A(C, \{x_n\}) &= \left\{ x \in C : \limsup_n \|x_n - x\| = r(C, \{x_n\}) \right\}. \end{aligned}$$

If D is a bounded subset of X , the *Chebyshev radius* of D relative to C is defined by

$$r_C(D) := \inf \{ \sup \{ \|x - y\| : y \in D \} : x \in C \}.$$

Obviously, the convexity of C implies that $A(C, \{x_n\})$ is convex. Notice that $A(C, \{x_n\})$ is a nonempty weakly compact set if C is weakly compact, or C is a closed convex subset of a reflexive Banach spaces X .

Let $\{x_n\}$ and C be a nonempty bounded closed subset of Banach spaces X . Then $\{x_n\}$ is called *regular* with respect to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$; while $\{x_n\}$ is called *asymptotically uniform* with respect to C if $A(C, \{x_n\}) = A(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Lemma 2.7. (Goebel[6] and Lim[10]). *Let $\{x_n\}$ and C be as above. Then we have*

- (i) There always exists a subsequence of $\{x_n\}$ which is regular with respect to C ;
- (ii) if C is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform with respect to C .

Moreover, we also need the following Lemma.

Lemma 2.8. (Domínguez Benavides and Ramírez. Theorem 3.4 cf. [3]). *Let C be a closed convex subset of a reflexive Banach spaces X , and let x_n be a bounded sequence in C which is regular with respect to C . Then*

$$(2.3) \quad r_C(A(C, x_n)) \leq (1 - \Delta_{X,\beta}(1^-))r(C, \{x_n\}).$$

Moreover, if X satisfies the nonstrict Opial condition then

$$(2.4) \quad r_C(A(C, x_n)) \leq (1 - \Delta_{X,X}(1^-))r(C, \{x_n\}).$$

Lemma 2.9. (Domínguez Benavides and Ramírez. Theorem 3.2 cf. [4]). *Let C be a closed convex subset of a reflexive Banach space X , and let $\{x_\beta : \beta \in D\}$ be a bounded ultranet. Then*

$$(2.5) \quad r_C(A(C, x_\beta)) \leq (1 - \Delta_{X,\alpha}(1^-))r(C, \{x_\beta\}).$$

The following result are now basic in the fixed point theorem for multivalued mappings.

Lemma 2.10. (Deimling 1992, cf. [2]). *Let X be a Banach space and $\emptyset \neq D \subset X$ be closed bounded convex. Let $F : D \rightarrow 2^X$ be upper semicontinuous γ -condensing with closed convex values, where $\gamma(\cdot) = \alpha(\cdot)$ or $\chi(\cdot)$. If $Fx \cap \overline{I_D(x)} \neq \emptyset$ for all $x \in C$, then F has a fixed point. (Here $I_D(x)$ is called the inward set at x defined by $I_D(x) := \{x + \lambda(y - x) : \lambda \geq 0, y \in D\}$)*

3. THE RESULT

In order to prove our first result, we need the following Lemma which is proved along the proof of Kirk-Massa theorem as it appear in [16].

Lemma 3.1. *Let C be a nonempty closed bounded convex separable subset of a Banach space X . $T : C \rightarrow KC(X)$ is a nonexpansive such that $T(C)$ is a bounded set and which satisfies $Tx \subset I_C(x)$, $\forall x \in C$, $\{x_n\}$ is a sequence in C such that $\lim_n d(x_n, Tx_n) = 0$. Then there exist a subsequence $\{z_n\}$ of $\{x_n\}$ such that $Tx \cap I_A(x) \neq \emptyset, \forall x \in A := A(C, \{z_n\})$.*

Lemma 3.1 is the part (more or less) of the proof of theorem 3.4 of [4].

The next result state the main result of this work.

Theorem 3.2. *Let C be a nonempty closed bounded convex separable subset of a Banach spaces X such that $\epsilon_\beta(X) < 1$, and $T : \Omega \times C \rightarrow KC(X)$ be a multivalued nonexpansive random operator and 1 - χ -contractive mapping, such that for each $\omega \in \Omega$, $T(\omega, C)$ is a bounded set, which satisfies the inwardness condition, i.e., for each $\omega \in \Omega$, $T(\omega, x) \subset I_C(x)$, $\forall x \in C$.*

Then T has a random fixed point.

Proof. Fix $x_0 \in C$, and consider the measurable function $x_0(\omega) \equiv x_0$. For each $n \geq 1$, define $T_n(\omega, \cdot) : C \rightarrow KC(X)$ by

$$T_n(\omega, x) = \frac{1}{n}x_0(\omega) + \left(\frac{n-1}{n}\right)T(\omega, x), \quad \forall x \in C.$$

Then $T_n(\omega, \cdot)$ is a multivalued contraction and $T_n(\omega, x) \subset I_C(x)$, $\forall x \in C$. Hence each T_n has a fixed point $z_n(\omega) \in C$. It is easily seen that $d(z_n(\omega), T(\omega, z_n(\omega))) \leq \frac{1}{n} \text{diam} C \rightarrow 0$ as $n \rightarrow \infty$. Thus the set

$$F_n(\omega) = \{x \in C : d(x, T(\omega, x)) \leq \frac{1}{n} \text{diam} C\}$$

is nonempty closed and convex. Furthermore, by Lemma 2.4, each F_n is measurable. Then, by Lemma 2.1, each F_n admits a measurable selector $x_n(\omega)$ such that

$$d(x_n(\omega), T(\omega, x_n(\omega))) \leq \frac{1}{n} \text{diam} C \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Define a function $f : \Omega \times C \rightarrow \mathbb{R}^+ := [0, \infty)$ by

$$f(\omega, x) = \limsup_n \|x_n(\omega) - x\|, \quad x \in C.$$

By Lemma 2.3, it is easily seen that $f(\cdot, x)$ is measurable and $f(\omega, \cdot)$ is continuous and convex, therefore it is a weakly lower semicontinuous function. Note that, condition $\epsilon_\beta(X) < 1$ implies reflexivity (see [1]) and so C is a weakly compact. Hence, by Lemma 2.5, the marginal functions

$$r(\omega) := \inf_{x \in C} f(\omega, x)$$

and

$$A(\omega) := \{x \in C : f(\omega, x) = r(\omega)\}$$

are measurable. It is clearly that $A(\omega)$ is a weakly compact convex subset of C . For any $\omega \in \Omega$, we may assume that the sequence $\{x_n(\omega)\}$ is regular with respect C . Note that $A(\omega) = A(C, \{x_n(\omega)\})$, and $r(\omega) = r(C, \{x_n(\omega)\})$. We can apply inequality (2.3) in Lemma 2.8 to obtain

$$(3.1) \quad r_C(A(\omega)) \leq \lambda r(C, \{x_n(\omega)\}),$$

where $\lambda = 1 - \Delta_{X, \beta}(1^-) < 1$, since $\epsilon_\beta(X) < 1$.

For each $\omega \in \Omega$ and $n \geq 1$, we define the multivalued contraction $T_n^1(\omega, \cdot) : A(\omega) \rightarrow KC(X)$ by

$$T_n^1(\omega, x) = \frac{1}{n}x_1(\omega) + \left(\frac{n-1}{n}\right)T(\omega, x),$$

for each $x \in C$. By Lemma 3.1 we note that $T(\omega, x) \cap I_{A(\omega)}(x) \neq \emptyset, \forall x \in A(\omega)$. Since $I_{A(\omega)}(x)$ is convex, it follow that $T_n^1(\omega, \cdot)$ satisfies the boundary condition i.e.,

$$(3.2) \quad T_n^1(\omega, x) \cap I_{A(\omega)}(x) \neq \emptyset, \forall x \in A(\omega).$$

Since $T_n^1(\omega, \cdot)$ is 1- χ -contractive mapping, it follows by [3, pp.382] that $T_n^1(\omega, \cdot)$ is χ -condensing. Hence, by Lemma 2.10, $T_n^1(\omega, \cdot)$ has a fixed point $z_n^1(\omega) \in A(\omega)$, i.e. $F(\omega) \cap A(\omega) \neq \emptyset$. Also it is easily seen that

$$\text{dist}(z_n^1(\omega), T(\omega, z_n^1(\omega))) \leq \frac{1}{n} \text{diam}C \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus $F_n^1(\omega) := \{x \in A(\omega) : d(x, T(\omega, x)) \leq \frac{1}{n} \text{diam}C\}$ is nonempty closed and convex for each $n \geq 1$. By Lemma 2.4, each F_n^1 are measurable. Hence, by Lemma 2.1, we can choose x_n^1 a measurable selector of F_n^1 . Thus we have $x_n^1(\omega) \in A(\omega)$ and $d(x_n^1(\omega), T(\omega, x_n^1(\omega))) \rightarrow 0$ as $n \rightarrow \infty$. Consider the function $f_2 : \Omega \times C \rightarrow \mathbb{R}^+$ defined by

$$f_2(\omega, x) = \limsup_n \|x_n^1(\omega) - x\|, \quad \forall \omega \in \Omega.$$

As above, f_2 is a measurable function and weakly lower semicontinuous function. Then the marginal function

$$r_2(\omega) := \inf_{x \in A(\omega)} f_2(\omega, x)$$

and

$$A^1(\omega) := \{x \in A(\omega) : f_2(\omega, x) = r_2(\omega)\}$$

are measurable. Since $A^1(\omega) = A(A(\omega), \{x_n^1(\omega)\})$, it follows that $A^1(\omega)$ is a weakly compact and convex. Moreover, we also note that $r_2(\omega) = r(A(\omega), \{x_n^1(\omega)\})$. Again reasoning as above, for any $\omega \in \Omega$, we can assume that the sequence $\{x_n^1(\omega)\}$ is regular with respect to $A^1(\omega)$. Moreover, we proceed as above using Lemma 3.1 and Lemma 2.8 to obtain that

$$T(\omega, x(\omega)) \cap I_{A^1}(x(\omega)) \neq \emptyset \quad \forall x(\omega) \in A^1 = A(A(\omega), \{x_n^1(\omega)\}),$$

and

$$(3.3) \quad r_C(A^1) \leq \lambda r(A(\omega), \{x_n^1(\omega)\}) \leq \lambda r_C(A(\omega)).$$

By induction, for each $m \geq 1$, we take a sequence $\{x_n^m(\omega)\}_n \subseteq A^{m-1}$ such that $r_C(A^m) \leq \lambda^m r_C(A(\omega))$ and $\lim_n d(x_n^m(\omega), T(\omega, x_n^m(\omega))) = 0$ for each fixed $\omega \in \Omega$, where $A^m := A(C, \{x_n^m(\omega)\})$. Since $\text{diam}R_m(\omega) \leq 2r_C(R_m(\omega))$ and $\lambda < 1$, it follows that $\lim_{m \rightarrow \infty} \text{diam}R_m(\omega) = 0$. Note that $\{R_m(\omega)\}$ is a descending sequence of weakly compact subset of C for each $\omega \in \Omega$. Thus we have $\cap_m R_m(\omega) = \{z(\omega)\}$ for some $z(\omega) \in C$. Furthermore, we see that

$$H(R_m(\omega), \{z(\omega)\}) \leq \text{diam}R_m(\omega) \rightarrow 0 \text{ as } m \rightarrow +\infty.$$

- [2] K. Deimling, "Nonlinear Functional Analysis", Springer-Verlag, Berlin (1974).
- [3] T. Domínguez Benavides and P. Lorenzo Ramírez, "Fixed point theorem for Multivalued nonexpansive mapping without uniform convexity", *Abstr. Appl. Anal.* 6(2003), 375-386.
- [4] T. Domínguez Benavides and P. Lorenzo Ramírez, "Fixed point theorem for multivalued nonexpansive mapping satisfying inwardness conditions", *J. Math. Anal. Appl.* 291 (2004), 100-108.
- [5] T. Domínguez Benavides G. Lopez Acedo and H.-K Xu, "Random fixed point of set-valued operator", *Proc. Amer. Math. Soc.* 124 (1996), 838-838.
- [6] K. Goebel "On the fixed point theorem for multivalued nonexpansive mappings," *Ann. Univ. M. Curie-Skłodowska* 29 (1975) 70-72.
- [7] S. Itoh, "Random fixed point theorem for a multivalued contraction mapping ", *Pacific J. Math.* 68 (1977), 85-90.
- [8] J.L. Kelly, *General Topology*, van Nostrand, Princeton, NJ, 1955.
- [9] W.A. Kirk, *Nonexpansive mappings in product spaces, set-valued mappings, and k-uniform rotundity*, *Proceedings of the Symposium Pure Mathematics*, Vol. 45, part 2, American Mathematical Society, Providence, RI, 1986, pp. 51-64.
- [10] T.C. Lim, "Remark on some fixed point theorems", *Proc. Amer. Math. Soc.* 60 (1976), 179-182.
- [11] P. Lorenzo Ramírez, "Some random fixed point theorems for nonlinear mappings", *Nonlinear Anal.* 50 (2002), 265-274.
- [12] K.-K. Tan and X.Z. Yuan, "Some Random fixed point theorem ", in: K.-K. Tan (Ed), *Fixed Point Theory and Applications*, World Scientific, Singapore, 1992, 334-345.
- [13] D.-H. Wagner, "Survey of measurable selection theorems", *SIAM J. Control Optim.* 15 (1977), 859-903.
- [14] H. K. Xu, "Some random fixed point for condensing and nonexpansive operators", *Proc. Amer. Math. Soc.* 110(1990), 395-400.
- [15] H. K. Xu, "Random fixed point theorems for nonlinear Uniform Lipschitzian mappings ", *Nonlinear Anal.* 26(1996)No.7, 1301-1311.
- [16] H. K. Xu, "Multivalued nonexpansive mappings in Banach spaces", *Nonlinear Anal.* 43(2001), 693-706.
- [17] X. Yuan and J. Yu, "Random fixed point theorems for nonself mappings ", *Nonlinear Anal.* 26(1996)No.6, 1097-1102.

SOMYOT PLUBTIENG, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NARESUAN UNIVERSITY, PITSANULOK 65000. THAILAND. *e-mail:somyotp@nu.ac.th*;

POOM KUMAM, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KING MONGKUT'S UNIVERSITY OF TECHNOLOGY THONBURI, BANGKOK 10140. THAILAND. *e-mail:poom.kum@kmutt.ac.th*

ภาคผนวก 6/11

Somyot Plubtieng

From: "JAMSA cfl" <jamsa@cfl.rr.com>
To: "Somyot Plubtieng" <somyotp@nu.ac.th>
Sent: Friday, June 10, 2005 8:59 AM
Subject: Re: revise (Somyot and Poom)

Thank you very much for your revision. The pdf file is fine.
I will forward all these files to the publisher and you will hear from them in due time.

Kind regards,
Eugene Dshalalow

----- Original Message -----

From: Somyot Plubtieng
To: revisions@jamsa.US
Sent: Thursday, June 09, 2005 4:09 AM
Subject: revise (Somyot and Poom)

Dear the Editor,

I was already revise the manuscript my paper (with Mr. Poom Kumam entitled "RANDOM FIXED POINT THEOREMS FOR MULTIVALUED NONEXPANSIVE NON-SELF RANDOM OPERATORS ") as the refferee's comment. I hope that you could open the pdf. file and tex.file which attached.

Thank you very much.

Somyot Plubtieng

8/9/2005

ภาคผนวก 6/12

Somyot Plubtieng

From: "Eugene Dshalalow" <edshalalow@cfl.rr.com>
To: "Poom KUMAM" <poom.kum@kmutt.ac.th>; <somyotp@nu.ac.th>
Cc: "JAMSA Editorial Office" <jamsa.ed@hindawi.com>
Sent: Tuesday, May 03, 2005 7:46 AM
Subject: Re: Submit manuscripts

Dear Authors:

I have heard from the referee regarding your MS submitted for publication to JAMSA. The referee has made suggestions right on the MS and I am forwarding you a hard copy of your article along with his comments, which you will receive by regular mail in due time.

Upon receipt of the report, please kindly acknowledge and then send your revision to

revisions@jamsa.US

Kind regards,

Eugene Dshalalow

eugene@jamsa.US

----- Original Message -----

From: "Poom KUMAM" <poom.kum@kmutt.ac.th>
To: <eugene@fit.edu>
Sent: Sunday, March 13, 2005 4:57 AM
Subject: Submit manuscripts

>

>

> The Editorial Board,

>

> Journal of Applied Mathematics and Stochastic Analysis (JAMSA)

> Department of Mathematical Sciences,

> College of Science and Liberal Arts,

> Florida Tech, 150 W. University

> Blvd., Melbourne, FL 32901-6988, U.S.A.

>

>

> 13 March 2005

>

> Dear Sir,

>

>

>

> Enclosed please find a files (the pdf.file) of my paper with Dr. Somyot

> Plubtieng entitled;

>

>

>

ภาคผนวก 7

**Strong convergence theorems of viscosity
averaging iterations for asymptotically
nonexpansive nonself-mappings**

S. Plubtieng and R. Wangkeeree

Proc. Amer. Math. Soc. (submitted).

STRONG CONVERGENCE THEOREMS OF VISCOSITY AVERAGING ITERATIONS FOR ASYMPTOTICALLY NONEXPANSIVE NONSELF-MAPPINGS †

SOMYOT PLUBTIENG AND RABIAN WANGKEEREE

ABSTRACT. Let C be a nonempty closed convex subset of a real Hilbert space H , P be the metric projection of H onto C , T be an asymptotically nonexpansive nonself-mapping from C into H with a sequence $\{k_n\} \subset [1, \infty)$ and $f : C \rightarrow C$ be a contraction mapping with coefficient $\alpha \in (0, 1)$. It proved that, for each $n \geq 1$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ which defined by

$$x_n = a_n f(x_n) + (1 - a_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x_n, \forall n \geq 1$$

and

$$y_n = \frac{1}{n} \sum_{j=1}^n P(a_n f(y_n) + (1 - \alpha_n)(TP)^j y_n), \forall n \geq 1,$$

where

$$b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j}), a_n = \frac{b_n - 1}{b_n - \beta}, \forall n \geq 1,$$

and $0 < \alpha < \beta < 1$. Then two sequences $\{x_n\}$ and $\{y_n\}$ converges strongly to a fixed point of T .

1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Hilbert space H and let T be a mapping of C into itself. Then T is said to be nonexpansive provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is said to be asymptotically nonexpansive mapping if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$

Key words and phrases. Fixed point; Metric projection; Asymptotically Nonexpansive nonself-Mapping; Strong Convergence; Contraction mapping.

2000 *Mathematics Subject Classification:* 46C05, 47H09, 47H10.

† Supported by The Thailand Research Fund.

such that for each $x, y \in C$,

$$(1.1) \quad \|T^n x - T^n y\| \leq k_n \|x - y\|, \forall n = 1, 2, 3, \dots$$

Recall that a self-mapping $f : C \rightarrow C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\| \quad \forall x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T ; i.e. $F(T) = \{x \in C : Tx = x\}$. It is well known that if T is nonexpansive, then $F(T)$ is convex see [6]. In 1967, Browder[3] proved the following strong convergence theorem for nonexpansive mapping: let T be a nonexpansive mapping of a bounded closed convex subset C of H into itself. Let $u \in C$ and for each $t \in (0, 1)$, let $G_t x = tu + (1-t)Tx$. Then, G_t has a unique fixed point x_t in C , and $\{x_t\}$ converges strongly to a fixed point u_0 of T as $t \rightarrow 0$. The fixed point u_0 is uniquely specified as the fixed point of T in C closest to u . In 1975, Baillon [1], prove the first nonlinear ergodic theorem as follows: let C be a bounded closed convex subset of H and let T be a nonexpansive mapping of C into itself. Then for each $x \in C$

$$A_n x = \frac{1}{n} \sum_{k=1}^n T^k x$$

converges weakly to fixed point of T . In 1979, Hirano and Takahashi[5] extended Baillon's theorem to asymptotically nonexpansive mappings. By Using an idea of Browder[3], Shimizu and Takahashi[13] proved the following theorem for an asymptotically nonexpansive mapping in the framework of a Hilbert space:

Theorem 1.1. ([13]). *Let C be a closed convex subset of a real Hilbert space H , let T be an asymptotically nonexpansive mapping of C into itself with Lipschitz constants k_n and suppose that $F(T)$ is nonempty. Let*

$$b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j}), a_n = \frac{b_n - 1}{b_n - 1 + a},$$

where $0 < a < 1$. Let $x_0 \in C$. Then, a mapping T_n on C given by

$$T_n x = a_n x_0 + (1 - a_n) A_n x, \text{ for all } x \in C$$

has a unique fixed point u_n in C , when $A_n = \frac{1}{n} \sum_{j=1}^n T^j$. Further $\{u_n\}$ convergence strongly to the element of $F(T)$ which is nearest to x_0 .

On the other hand, Xu[16] extended Browder's result to studied two sequences $\{x_t\}$ and $\{x_n\}$ given by

$$x_t = tf(x_t) + (1-t)Tx_t$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, n = 1, 2, \dots,$$

where $t \in (0, 1)$, $\{\alpha_n\} \subset (0, 1)$ and f is a contraction mapping from C into itself. Xu[16] also proved the strong convergence of the sequences as $t \rightarrow 1$ and $\alpha_n \rightarrow 1$ to the unique solution z in $F(T)$ to the variational inequality $\langle (I - f)z = x - z \rangle \geq 0, x \in F(T)$ or equivalently to $z = P(f(z))$ where P is the metric projection from H onto $F(T)$.

In this paper, we first show that, for an asymptotically nonexpansive nonself-mapping T with a sequence $\{k_n\} \subset [1, \infty)$, there exists two sequences $\{x_n\}$ and $\{y_n\}$ which defined by

$$(1.2) \quad x_n = a_n f(x_n) + (1 - a_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x_n, \forall n \geq 1$$

and

$$(1.3) \quad y_n = \frac{1}{n} \sum_{j=1}^n P(a_n f(y_n) + (1 - a_n)(TP)^j y_n), \forall n \geq 1$$

where

$$b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j}), a_n = \frac{b_n - 1}{b_n - \beta}, \forall n \geq 1,$$

$0 < \alpha < \beta < 1$, $f : C \rightarrow C$ is a contraction mapping with coefficient $\alpha \in (0, 1)$ and P is the metric projection from H onto C . Finally we show that $\{x_n\}$ and $\{y_n\}$ converges strongly to a fixed point of T . Then the results presented in this paper generalized and extend the corresponding main results of Shimizu and Takahashi [13].

2. PRELIMINARIES

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ and let C be a closed convex subset of H . Recall the metric (nearest point)

projection P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: Given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

$P_C x$ is characterized as follows.

Lemma 2.1. *Let H be a real Hilbert space, C a closed convex subset of H . Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \forall z \in C.$$

Definition 2.2. A mapping $T : C \rightarrow H$ is said to satisfy nowhere normal outward condition ((NNO) for short) if and only if for each $x \in C$, $Tx \in S_x^C$, where $S_x = \{y \in H : y \neq x, Py = x\}$ and P is the metric projection from H onto C .

Lemma 2.3. ([9, Proposition 1]). *Let H be a Hilbert space, C a nonempty closed convex subset of H , P be the metric projection of H onto C and $T : C \rightarrow H$ be a nonself-mapping. Suppose that T satisfies (NNO) condition. Then $F(PT) = F(T)$.*

Lemma 2.4. ([13, Lemma 4]). *Let H be a Hilbert space, C a closed convex subset of H , and $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C and there exists a subsequence $\{x_{n_j}\}$ which converges weakly to $x \in C$ and $\{x_{n_j} - \frac{1}{n_j} \sum_{i=1}^{n_j} T^i x_{n_j}\}$ converges strongly to 0. Then x is a fixed point of T .*

Definition 2.5. ([4, Definition 3.1]). Let X be a real normed linear space, C a nonempty subset of X . Let $P : X \rightarrow C$ be the nonexpansive retraction of X onto C . A mapping $T : C \rightarrow X$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}_{n \geq 1} \subset [1, \infty)$, $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that for all $x, y \in C$, the following inequality holds:

$$(2.1) \quad \|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n \|x - y\|, \text{ for all } n \geq 1.$$

Remark 2.6. If X is a Hilbert space then we can replace the mapping P by the metric projection P .

Remark 2.7. If T is a self-map, then $PT = T$ so that (2.1) coincide with (1.1). Moreover, we note that $TP|_C = T$. So if a contraction mapping $f : C \rightarrow C$ defined by $f(x) = x_0 \in C, \forall x \in C$ and setting $\beta = 1 - a$ for some $0 < a < 1 - \alpha$ then, (1.2) and (1.3) reduce to the sequence $\{u_n\}$ in Theorem 1.1.

For a contraction mapping $f : C \rightarrow C$ with coefficient $\alpha \in (0, 1)$ and an asymptotically nonexpansive mapping T with a sequence $\{k_n\} \subset [1, \infty)$, we putting

$$b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j}) \text{ and } a_n = \frac{b_n - 1}{b_n - \beta} \text{ for } n = 1, 2, 3, \dots,$$

where $0 < \alpha < \beta < 1$. Then, we get the following facts:

- (i) $b_n > 1, \frac{1}{n} \sum_{j=1}^n k_j < b_n, 0 < a_n < 1, \forall n \geq 1$,
- (ii) $\lim_{n \rightarrow \infty} b_n = 1, \lim_{n \rightarrow \infty} a_n = 0$,
- (iii) $a_n > \frac{b_n - 1}{b_n - \alpha}$ or equivalently to $a_n(\alpha - b_n) + b_n < 1, \forall n \geq 1$.

Now, for each $n \geq 1$, we consider two mappings $S_n, U_n : C \rightarrow C$ given by

$$(2.2) \quad S_n x = a_n f(x) + (1 - a_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x, \text{ for all } x \in C$$

and

$$(2.3) \quad U_n y = \frac{1}{n} \sum_{j=1}^n P(a_n f(y) + (1 - a_n)(TP)^j y), \text{ for all } y \in C.$$

Then, we have the following three lemmas.

Lemma 2.8. *For each $n \geq 1$, S_n has a unique fixed point x_n in C .*

Proof. Let $x, y \in C$. Then for each $n \geq 1$, we have

$$\begin{aligned} \|S_n x - S_n y\| &= \|a_n(f(x) - f(y)) + (1 - a_n) \frac{1}{n} \sum_{j=1}^n ((PT)^j x - (PT)^j y)\| \\ &\leq a_n \alpha \|x - y\| + (1 - a_n) \frac{1}{n} \sum_{j=1}^n \|(PT)^j x - (PT)^j y\| \\ &\leq a_n \alpha \|x - y\| + (1 - a_n) \frac{1}{n} \sum_{j=1}^n \|T(PT)^{j-1} x - T(PT)^{j-1} y\| \\ &\leq a_n \alpha \|x - y\| + (1 - a_n) \frac{1}{n} \sum_{j=1}^n k_j \|x - y\| \end{aligned}$$

$$\begin{aligned} &\leq a_n \alpha \|x - y\| + (1 - a_n) b_n \|x - y\| \\ &= (a_n(\alpha - b_n) + b_n) \|x - y\|. \end{aligned}$$

Since $a_n(\alpha - b_n) + b_n < 1$, we get S_n is a contraction mapping on C . Therefore, by the Banach Contraction principle, S_n has a unique fixed point x_n in C . $\square$

Lemma 2.9. *For each $n \geq 1$, U_n has a unique fixed point y_n in C .*

Proof. Let $x, y \in C$. Since P is a nonexpansive mapping such that $Px = x$ and $Py = y$, it follows as in the proof of Lemma 2.8 that

$$\|U_n x - U_n y\| \leq (a_n(\alpha - b_n) + b_n) \|x - y\|.$$

Thus U_n is a contraction mapping and hence U_n has a unique fixed point y_n in C . $\square$

Lemma 2.10. *If $F(T)$ is a nonempty, then $\{x_n\}$ and $\{y_n\}$ are bounded sequences.*

Proof. Let $q \in F(T)$. Then, we have

$$\begin{aligned} \|x_n - q\| &= \|a_n(f(x_n) - q) + (1 - a_n) \frac{1}{n} \sum_{j=1}^n ((PT)^j x_n - q)\| \\ &\leq a_n \|f(x_n) - q\| + (1 - a_n) \frac{1}{n} \sum_{j=1}^n \|(PT)^j x_n - q\| \\ &\leq a_n \|f(x_n) - f(q)\| + a_n \|f(q) - q\| + (1 - a_n) \frac{1}{n} \sum_{j=1}^n k_j \|x_n - q\| \\ &\leq a_n \alpha \|x_n - q\| + a_n \|f(q) - q\| + (1 - a_n) b_n \|x_n - q\| \\ &= (a_n(\alpha - b_n) + b_n) \|x_n - q\| + a_n \|f(q) - q\|. \end{aligned}$$

We note that

$$\frac{a_n}{1 - [a_n(\alpha - b_n) + b_n]} = \frac{b_n - 1}{-\beta - b_n \alpha + \alpha + b_n \beta} = \frac{1}{\beta - \alpha}.$$

It follows that $\|x_n - q\| \leq \frac{a_n}{1 - [a_n(\alpha - b_n) + b_n]} \|f(q) - q\| = \frac{1}{\beta - \alpha} \|f(q) - q\|$. Hence $\{x_n\}$ is a bounded sequence. Then as in the proof above, $\{y_n\}$ is also bounded. This completely the proof. $\square$

3. MAIN RESULTS

In this section, we shall prove two strong convergence theorems for asymptotically nonexpansive nonself-mapping in a Hilbert space.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H , P the metric projection from H onto C , T be an asymptotically nonexpansive nonself-mapping from C into H with Lipschitz constant k_n , and suppose that $F(T)$ is nonempty. Let $f : C \rightarrow C$ be a contraction mapping with coefficient $\alpha \in (0, 1)$,*

$$b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j}) \text{ and } a_n = \frac{b_n - 1}{b_n - \beta},$$

where $0 < \alpha < \beta < 1$. If T satisfies (NNO) condition then the sequence $\{x_n\}$ defined by (1.2) converges strongly to z where, z is the unique solution in $F(T)$ to the variation inequality

$$(3.1) \quad \langle (I - f)z, x - z \rangle \geq 0, \quad x \in F(T)$$

or equivalently $z = G(f(z))$, where G is the metric projection from H onto $F(T)$.

Proof. By Lemma 2.10, we have $\{x_n\}$ is bounded so are $\{f(x_n)\}$ and $\{\frac{1}{n} \sum_{j=1}^n \|(TP)^j x_n\|\}$. Furthermore, we obtain

$$\begin{aligned} \|x_n - \frac{1}{n} \sum_{j=1}^n (PT)^j x_n\| &= \|a_n f(x_n) + (1 - a_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x_n - \frac{1}{n} \sum_{j=1}^n (PT)^j x_n\| \\ &= a_n \|f(x_n) - \frac{1}{n} \sum_{j=1}^n (PT)^j x_n\| \\ &\leq a_n \left[\|f(x_n)\| - \frac{1}{n} \sum_{j=1}^n \|(TP)^j x_n\| \right] \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This implies that $\{x_n - \frac{1}{n} \sum_{j=1}^n (PT)^j x_n\}$ converges strongly to 0. We next show that

$$(3.2) \quad \limsup_{n \rightarrow \infty} \langle z - x_n, z - f(z) \rangle \leq 0.$$

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{j \rightarrow \infty} \langle z - x_{n_j}, z - f(z) \rangle = \limsup_{n \rightarrow \infty} \langle z - x_n, z - f(z) \rangle,$$

and $x_{n_j} \rightarrow x \in C$. By Lemma 2.4 and Lemma 2.3, we get $x \in F(PT) = F(T)$. Hence, by (3.3) we obtain

$$\limsup_{n \rightarrow \infty} \langle z - x_n, z - f(z) \rangle = \langle z - x, z - f(z) \rangle \leq 0$$

as required. Finally we shall show that $x_n \rightarrow z$. For each $n \geq 1$, we note that

$$\begin{aligned} \|x_n - z\|^2 &= \|x_n - z + a_n(z - f(z)) - a_n(z - f(z))\|^2 \\ &\leq \|x_n - z + a_n(z - f(z))\|^2 + 2a_n \langle x_n - z, f(z) - z \rangle \\ &= \|a_n(f(x_n) - f(z)) + (1 - a_n) \frac{1}{n} \sum_{j=1}^n ((PT)^j x_n - z)\|^2 \\ &\quad + 2a_n \langle x_n - z, f(z) - z \rangle \\ &\leq \left\{ a_n \|f(x_n) - f(z)\| + (1 - a_n) \frac{1}{n} \sum_{j=1}^n \|((PT)^j x_n - z)\| \right\}^2 \\ &\quad + 2a_n \langle x_n - z, f(z) - z \rangle \\ &\leq \{a_n \alpha \|x_n - z\| + (1 - a_n) b_n \|x_n - z\|\}^2 \\ &\quad + 2a_n \langle x_n - z, f(z) - z \rangle \\ &\leq (a_n(\alpha - b_n) + b_n) \|x_n - z\|^2 + 2a_n \langle x_n - z, f(z) - z \rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_n - z\|^2 &\leq \frac{2a_n}{1 - [a_n(\alpha - b_n) + b_n]} \langle x_n - z, f(z) - z \rangle \\ &= \frac{2}{\beta - \alpha} \langle x_n - z, f(z) - z \rangle. \end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Then by the fact (3.2) there exists a natural number N such that

$$\langle x_n - z, f(z) - z \rangle \leq (\beta - \alpha) \frac{\epsilon}{2}, \forall n \geq N.$$

This implies that

$$\|x_n - z\|^2 \leq \epsilon, \forall n \geq N.$$

Hence the sequence $\{x_n\}$ converges strongly to a fixed point z of T . This completely the proof. $\square$

Theorem 3.2. *Let C be a closed convex subset of a real Hilbert space H , P the metric projection from H onto C , T be an asymptotically nonexpansive nonself-mapping from C into H with Lipschitz constant k_n , and suppose that $F(T)$ is nonempty. Let $f : C \rightarrow C$ be a contraction mapping with coefficient $\alpha \in (0, 1)$,*

$$b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j}) \text{ and } a_n = \frac{b_n - 1}{b_n - \beta},$$

where $0 < \alpha < \beta < 1$. If T satisfies (NNO) condition then the sequence $\{y_n\}$ defined by (1.3) converges strongly to z where, z is the unique solution in $F(T)$ to the variation inequality

$$(3.3) \quad \langle (I - f)z, x - z \rangle \geq 0, \quad x \in F(T)$$

or equivalently $z = G(f(z))$, where G is the metric projection from H onto $F(T)$.

Proof. By Lemma 2.10, we get $\{y_n\}$ is bounded so are $\{f(y_n)\}$ and $\{\frac{1}{n} \sum_{j=1}^n \|(TP)^j y_n\|\}$. Furthermore, we also have

$$\begin{aligned} \|y_n - \frac{1}{n} \sum_{j=1}^n (PT)^j y_n\| &= \left\| \frac{1}{n} \sum_{j=1}^n P(a_n f(y_n) + (1 - a_n)(TP)^j y_n) - \frac{1}{n} \sum_{j=1}^n (PT)^j y_n \right\| \\ &\leq \frac{1}{n} \sum_{j=1}^n \|a_n f(y_n) + (1 - a_n)(TP)^j y_n - T(PT)^{j-1} y_n\| \\ &= \frac{1}{n} \sum_{j=1}^n \|a_n f(y_n) + (1 - a_n)(TP)^j y_n - (TP)^j y_n\| \\ &= a_n \frac{1}{n} \sum_{j=1}^n \|f(y_n) - (TP)^j y_n\| \\ &\leq a_n \left[\|f(y_n)\| - \frac{1}{n} \sum_{j=1}^n \|(TP)^j y_n\| \right] \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This implies that $\{y_n - \frac{1}{n} \sum_{j=1}^n (PT)^j x_n\}$ converges strongly to 0. Then as in the proof of Theorem 3.1, we obtain

$$(3.4) \quad \limsup_{n \rightarrow \infty} \langle z - y_n, z - f(z) \rangle \leq 0.$$

Finally we shall show that $y_n \rightarrow z$. For each $n \geq 1$, we have

$$\begin{aligned}
 \|y_n - z\|^2 &\leq \|y_n - z + a_n(z - f(z))\|^2 + 2a_n\langle y_n - z, f(z) - z \rangle \\
 &\leq \left\{ \frac{1}{n} \sum_{j=1}^n \|P(a_n f(y_n) + (1 - a_n)(TP)^j y_n) - P(a_n f(z) + (1 - a_n)z)\| \right\}^2 \\
 &\quad + 2a_n\langle y_n - z, f(z) - z \rangle \\
 &\leq \left\{ \frac{1}{n} \sum_{j=1}^n (a_n \|f(y_n) - f(z)\| + (1 - a_n)\|(TP)^j y_n - z\|) \right\}^2 \\
 &\quad + 2a_n\langle y_n - z, f(z) - z \rangle \\
 &\leq \left\{ a_n \alpha \|y_n - z\| + (1 - a_n) \frac{1}{n} \sum_{j=1}^n \|(TP)^j y_n - z\| \right\}^2 \\
 &\quad + 2a_n\langle y_n - z, f(z) - z \rangle \\
 &\leq \left\{ a_n \alpha \|y_n - z\| + (1 - a_n) \frac{1}{n} \sum_{j=1}^n k_j \|y_n - z\| \right\}^2 \\
 &\quad + 2a_n\langle y_n - z, f(z) - z \rangle \\
 &\leq \{(a_n \alpha + (1 - a_n) b_n) \|y_n - z\|\}^2 \\
 &\quad + 2a_n\langle y_n - z, f(z) - z \rangle \\
 &\leq (a_n(\alpha - b_n) + b_n) \|y_n - z\|^2 + 2a_n\langle y_n - z, f(z) - z \rangle.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \|y_n - z\|^2 &\leq \frac{2a_n}{1 - [a_n(\alpha - b_n) + b_n]} \langle y_n - z, f(z) - z \rangle \\
 &= \frac{2}{\beta - \alpha} \langle y_n - z, f(z) - z \rangle.
 \end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Then by the fact (3.4) there exists a natural number N such that

$$\langle y_n - z, f(z) - z \rangle \leq (\beta - \alpha) \frac{\epsilon}{2}, \forall n \geq N.$$

This implies that

$$\|y_n - z\|^2 \leq \epsilon, \forall n \geq N.$$

Hence the sequence $\{y_n\}$ converges strongly to a fixed point z of T . This completely the proof. $\square$

REFERENCES

1. J. B. Baillon, *Un theoreme de type ergodique pour les contractions non linear dans un espace de Hilbert*, C.r. Acad. Sci. Paris, **280** (1975) 1511–1514.
2. H. Bauschke, *The approximation of Fixed points of compositions of nonexpansive mappings in Hilbert space*, J. Math. Anal. Appl. **202** (1996) 150–159.
3. F. E. Browder, *Convergence of approximates to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Archs ration. Mech. Analysis. **24** (1967) 82–90.
4. C. E. Chidume, E. U. Ofoedu, H. Zegeye, *Strong and weak convergence theorem for asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **280**, 364–374(2003).
5. N. Hirano and W. Takahashi, *Nonlinear ergodic theorems for nonexpansive mappings in Hilbert spaces*, Kodai math. J. **2**, 11–25(1979).
6. H. Ishihara and W. Takahashi, *A nonlinear ergodic theorem for a reversible semigroup of Lipschitzian mapping in a Hilbert space*, Proc. Amer. math. Soc. **104** (1988) 431–436.
7. A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl. **241** (2000) 46–55.
8. J. S. Jung, *Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **302** (2005) 509–520.
9. S. Matsushita and D. Kuroiwa, *Approximation of fixed points of nonexpansive nonself-mappings*, Sci. Math. Jpn. **57** (2003) 171–176.
10. S. Matsushita and D. Kuroiwa, *Strong convergence of averaging iterations of nonexpansive nonself-mappings*, J. Math. Anal. Appl. **294** (2004) 206–214.
11. G. O'Hara, P. Pillary and H. K. Xu, *Iterative approaches to finding nearest common fixed points of nonexpansive mappings in Hilbert spaces*, Nonlinear Anal. **54** (2003) 1417–1426.
12. T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl. **211** (1997) 71–83.
13. T. Shimizu and W. Takahashi, *Strong convergence for asymptotically nonexpansive mappings*, Nonlinear. Anal. **26** (1996) 265–272.
14. N. Shioji and W. Takahashi, *Strong convergence theorems for asymptotically nonexpansive semigroups in Hilbert spaces*, Nonlinear Anal. **34** (1998) 87–99.
15. W. Takahashi and G. E. Kim, *Strong convergence of approximants to fixed points of nonexpansive nonexpansive nonself-mappings in Banach spaces*, Nonlinear Anal. **32** (1998) 447–454.
16. H. K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298** (2004) 279–291.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NARESUAN UNIVERSITY,
PHITSANULOK 65000, THAILAND

E-mail address: Somyotp@nu.ac.th(Somyot Plubtieng) and Rabianw@nu.ac.th(Rabian Wangkeeree)

ภาคผนวก 7/12

Somyot Plubtieng

From: "Joseph Ball" <ball@calvin.math.vt.edu>
To: "Somyot Plubtieng" <somyotp@nu.ac.th>
Sent: Tuesday, February 08, 2005 9:29 PM
Subject: Re: submit paper

Dear Professor Plubtieng,

This is to acknowledge receipt of your paper (with Rabian Wangkeeree) "Strong convergence theorems of viscosity averaging iterations for asymptotically nonexpansive nonself-mappings" submitted for publication in Proceedings of the American Mathematical Society (reference number PAMS05_21). I will be in contact with you concerning this paper upon completion of the review process.

Sincerely,
J.A. Ball
Editor, PAMS

Professor Joseph A. Ball
Virginia Polytechnic

Institute and State University, Blacksburg,

VA 24061 USA
ball@math.vt.edu

February 8, 2005

Dear Sir,

Enclosed please find a files (the pdf. file) of my paper with Mr. Rabian Wangkeeree entitled;

Strong Convergence Theorems of Viscosity Averaging Iterations for Asymptotically Nonexpansive Nonself-Mappings

which I would like to submit for publication in the Proceeding of the American Mathematical Society.

I would like to thank you in advance for your consideration.

Your sincerely,

Somyot Plubtieng

ภาคผนวก 7/13

Associate Professor

Department of Mathematics

Naresuan University

Phitsanulok 65000

THAILAND

E-mail address : somyotp@nu.ac.th

Attachment converted: Macintosh HD:(9) Strong convergence of V.pdf (PDF /CARO)
(0007D7CF)

--
Prof. Joseph A. Ball
Department of Mathematics
Virginia Tech
Blacksburg, Virginia 24061-0123
USA

Office Phone: 540-231-7080
FAX: 540-231-5960
e-mail: ball@math.vt.edu
web page: <http://www.math.vt.edu/people>

ภาคผนวก 8

**Strong convergence theorems of vicosity
averaging iterations for nonexpansive nonself-
mappings in Hilbert spaces**

S. Plubtieng and R. Wangkeeree

J. Korean. Math. Soc. (submitted).

Strong Convergence Theorems of Viscosity Averaging Iterations for Nonexpansive Nonself-Mappings in Hilbert Spaces *

Somyot Plubtieng and Rabian Wangkeeree [†]

Department of Mathematics, Faculty of Science, Naresuan University, Pitsanulok 65000,
Thailand

Abstract

Let C be a nonempty closed convex subset of Hilbert space H , P a metric projection of H onto C and let T be a nonexpansive nonself-mapping from C into H . In this paper, we study the convergence of three sequences generated by

$$x_n = t_n f(x_n) + (1 - t_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x_n, n \geq 1$$

$$y_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n, n \geq 0,$$

and

$$z_{n+1} = \frac{1}{n+1} \sum_{j=0}^n P(\alpha_n f(z_n) + (1 - \alpha_n)(TP)^j z_n), n \geq 0,$$

where $y_0, z_0 \in C$, $\{t_n\} \subset (0, 1)$, $\{\alpha_n\}$ is a real sequence in an interval $[0, 1]$ and f is a contraction from C into itself.

Keywords: Fixed point; Metric projection; Nonexpansive Mapping; Strong Convergence.

2000 Mathematics Subject Classification: 46C05, 47H09, 47H10.

*Supported by The Thailand Research Fund.

[†]Corresponding author.

Email addresses: Somyotp@nu.ac.th. (Somyot Plubtieng) and Rabianw@nu.ac.th. (Rabian Wangkeeree)

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and let T be a mapping of C into itself. Then T is said to be nonexpansive provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Recall that a self-mapping $f : C \rightarrow C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\| \quad \forall x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T ; i.e. $F(T) = \{x \in C : Tx = x\}$. It is well known that if T is nonexpansive, then $F(T)$ is convex see [4]. In 1967, Browder[3] proved the following strong convergence theorem for nonexpansive mapping: let T be a nonexpansive mapping of a bounded closed convex subset C of H into itself. Let $u \in C$ and for each $t \in (0, 1)$, let $G_t x = tu + (1 - t)Tx$. Then, G_t has a unique fixed point x_t in C , and $\{x_t\}$ converges strongly to a fixed point u_0 of T as $t \rightarrow 0$. The fixed point u_0 is uniquely specified as the fixed point of T in C closest to u . In 1975, Baillon [1], prove the first nonlinear ergodic theorem as follows: let C be a bounded closed convex subset of H and let T be a nonexpansive mapping of C into itself. Then for each $x \in C$

$$A_n x = \frac{1}{n} \sum_{k=1}^n T^k x$$

converges weakly to fixed point of T . By Using an idea of Browder[3], Shimizu and Takahashi[11] studied the convergence of the following approximated sequence for an asymptotically nonexpansive mapping in the framework of a Hilbert space:

$$x_n = a_n x + (1 - a_n) \frac{1}{n} \sum_{j=1}^n T^j x_n, \quad n = 1, 2, \dots, \quad (1.1)$$

where $\{a_n\}$ is a real sequence satisfying $0 < a_n < 1$ and $a_n \rightarrow 0$.

In 1997, Shimizu and Takahashi [10] also studied the convergence of iteration process for a family of nonexpansive mappings in the framework of a Hilbert space as follows:

Theorem (Shimizu and Takahashi). Let C be a nonempty closed convex subset of a Hilbert space H , let T a nonexpansive self-mapping of C such that $F(T)$ is nonempty, and let P be the metric projection from C onto $F(T)$.

Let $\{a_n\}$ be a real sequence which satisfies $0 \leq a_n \leq 1$, $\lim_{n \rightarrow \infty} a_n = 0$ and $\sum_{n=0}^{\infty} a_n = \infty$. Let x and y_0 be element of C and let $\{y_n\}$ be the sequence defined by

$$y_{n+1} = a_n x + (1 - a_n) \frac{1}{n+1} \sum_{j=0}^n T^j y_n, n \geq 1. \quad (1.2)$$

Then $\{y_n\}$ converges strongly to Px .

Recently, Matsushita and Koroïwa[8] generalized the result of Shimizu and Takahashi [10] and prove the following theorems:

Theorem (Matsushita and Koroïwa). Let H be a Hilbert space, C a closed convex subset of H , P_1 the metric projection of H onto C and T be a non-expansive nonself-mapping from C into H such that $F(T)$ is nonempty, and $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that $\{x_n\}$ is given by $x_0, x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n (P_1 T)^j x_n, n \geq 0. \quad (1.3)$$

Then $\{x_n\}$ converges strongly to $P_2 x \in F(T)$, where P_2 is the metric projection from C onto $F(T)$.

Theorem (Matsushita and Koroïwa). Let H be a Hilbert space, C a closed convex subset of H , P_1 the metric projection of H onto C and T be a non-expansive nonself-mapping from C into H such that $F(T)$ is nonempty, and $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that $\{y_n\}$ is given by $y_0, y \in C$ and

$$y_{n+1} = \frac{1}{n+1} \sum_{j=0}^n P(\alpha_n y + (1 - \alpha_n)(TP_1)^j y_n, n \geq 0.) \quad (1.4)$$

Then $\{y_n\}$ converges strongly to $P_2 y \in F(T)$, where P_2 is the metric projection from C onto $F(T)$.

On the other hand, using the viscosity approximation method, Xu[14] studied the convergence of the following approximation for nonexpansive nonself-mapping in Hilbert space:

$$x_t = tf(x_t) + (1 - t)Tx_t \quad (1.5)$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, n = 1, 2, \dots, \quad (1.6)$$

where $t \in (0, 1)$, $\{\alpha_n\} \subset (0, 1)$ and f is a contraction mapping from C into itself. Xu[14] also proved the strong convergence of the sequences as $t \rightarrow 1$ and $\alpha_n \rightarrow 1$ to the unique solution z in $F(T)$ to the variational inequality $\langle (I - f)z = x - z \rangle \geq 0, x \in F(T)$ or equivalently to $z = P(f(z))$ where P is the metric projection from H onto $F(T)$.

In this paper, we study the three type iterations process which are mixed iteration process of (1.1) - (1.6) as follows: for $y_0, z_0 \in C$ and

$$x_n = t_n f(x_n) + (1 - t_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x_n \quad (1.7)$$

$$y_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n, n \geq 0 \quad (1.8)$$

and

$$z_{n+1} = \frac{1}{n+1} \sum_{j=0}^n P(\alpha_n f(z_n) + (1 - \alpha_n)(TP)^j z_n), n \geq 0 \quad (1.9)$$

where $\{t_n\} \subset (0, 1)$, $\{\alpha_n\}$ is a sequence such that $0 \leq \alpha_n \leq 1$, $f : C \rightarrow C$ is a contraction mapping and P is the metric projection of H onto C . We first establish the strong convergence of sequence $\{x_n\}$ defined by (1.7). In addition, we also prove the strong convergence of the approximation sequences $\{y_n\}$ and $\{z_n\}$ defined by (1.8) and (1.9) respectively. The results presented in this paper generalized and extend the corresponding main results of Baillon [1], Shimizu and Takahashi [10] and Matsushita and Koroiwa[8].

2 Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ and let C be a closed convex subset of H . Recall the metric (nearest point) projection P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: Given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

$P_C x$ is characterized as follows.

Lemma 2.1. *Let H be a real Hilbert space, C a closed convex subset of H . Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \forall z \in C.$$

Definition 2.2. A mapping $T : C \longrightarrow H$ is said to satisfy nowhere normal outward condition ((NNO) for short) if and only if for each $x \in C$, $Tx \in S_x^C$, where $S_x = \{y \in H : y \neq x, Py = x\}$ and P is the metric projection from H onto C .

The following results was proved by Matsushita and Koroiwa[7].

Lemma 2.3. ([7, Proposition 2, P. 208]). *Let H be a Hilbert space, C a nonempty closed convex subset of H , P the metric projection of H onto C and $T : C \longrightarrow H$ a nonexpansive nonself-mapping. If $F(T)$ is nonempty then T satisfies NNO condition.*

Lemma 2.4. ([7, Proposition 1, P. 208]). *Let H be a Hilbert space, C a nonempty closed convex subset of H , P the metric projection of H onto C and $T : C \longrightarrow H$ a nonself-mapping. Suppose that T satisfies (NNO) condition. Then $F(PT) = F(T)$.*

Further, we know the following lemmas actually hold for asymptotically nonexpansive[11]. But we only need its for nonexpansive version.

Lemma 2.5. ([11]). *Let H be a Hilbert space, C a closed convex subset of H , and $T : C \longrightarrow C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C and there exists a subsequence $\{x_{n_j}\}$ which converges weakly to $x \in C$ and $\{x_{n_j} - \frac{1}{n_j} \sum_{i=1}^{n_j} T^i x_{n_j}\}$ converges strongly to 0. Then x is a fixed point of T .*

Finally, the following two lemma are useful for the proof of our main theorems.

Lemma 2.6. ([14]). *Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ that satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{a_n\}$ be a sequence of nonnegative real numbers that satisfying:*

For all $\epsilon > 0$, there exists an integer $N \geq 1$ such that for all $n \geq N$,

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \epsilon.$$

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.7. [14] *Let H be a Hilbert space, C a nonempty closed convex subset of H , and $f : C \rightarrow C$ a contraction with coefficient $\alpha < 1$. Then*

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq (1 - \alpha)\|x - y\|^2, x, y \in C.$$

Remark 2.8. As in Lemma 2.7, if f is a nonexpansive mapping, then

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq 0, \forall x, y \in C.$$

3 Main results

In this section, we study the strong convergence properties of the three sequences (1.7), (1.8) and (1.9).

Theorem 3.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H , P the metric projection of H onto C and $T : C \rightarrow H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{t_n\}$ be sequence in $(0, 1)$ which satisfies $\lim_{n \rightarrow \infty} t_n = 0$. Then for a contraction mapping $f : C \rightarrow C$ with coefficient $\alpha \in (0, 1)$, the sequence $\{x_n\}$ defined by (1.7) converges strongly to z , where, z is the unique solution in $F(T)$ to the variation inequality*

$$\langle (I - f)z, x - z \rangle \geq 0, x \in F(T) \quad (3.1)$$

or equivalently $z = G(f(z))$, where G is a metric projection mapping from H onto $F(T)$.

Proof. Since $F(T)$ is nonempty, it follows that T satisfies (NNO) condition by Lemma 2.3. We first show that $\{x_n\}$ is bounded. Let $q \in F(T)$. We note that

$$\begin{aligned} \|x_n - q\| &= \|t_n f(x_n) + (1 - t_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x_n - q\| \\ &\leq \|t_n(f(x_n) - q) + (1 - t_n) \frac{1}{n} \sum_{j=1}^n ((PT)^j x_n - (PT)^j q)\| \\ &\leq t_n \|f(x_n) - q\| + (1 - t_n) \|x_n - q\|, \forall n \geq 1. \end{aligned}$$

So we get

$$\begin{aligned}\|x_n - q\| &\leq \|f(x_n) - q\| \\ &\leq \|f(x_n) - f(q)\| + \|f(q) - q\| \\ &\leq \alpha\|x_n - q\| + \|f(q) - q\|, \forall n \geq 1.\end{aligned}$$

Hence

$$\|x_n - q\| \leq \frac{1}{1-\alpha} \|f(q) - q\|, \forall n \geq 1.$$

This show that $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{\frac{1}{n} \sum_{j=1}^n (PT)^j x_n\}$. Further, we note that

$$\begin{aligned}\|x_n - \frac{1}{n} \sum_{j=1}^n (PT)^j x_n\| &= \|t_n f(x_n) + (1-t_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x_n - \frac{1}{n} \sum_{j=1}^n (PT)^j x_n\| \\ &= t_n \|f(x_n) - \frac{1}{n} \sum_{j=1}^n (PT)^j x_n\| \\ &\leq t_n (\|f(x_n)\| + \|\frac{1}{n} \sum_{j=1}^n (PT)^j x_n\|) \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned}$$

Thus $\{x_n - \frac{1}{n} \sum_{j=1}^n (PT)^j x_n\}$ converges strongly to 0. Since $\{x_n\}$ is a bounded sequence, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $z \in C$. By Lemma 2.5 and Lemma 2.4, we have $z \in F(T)$. For each $n \geq 1$, since

$$x_n - z = t_n(f(x_n) - z) + (1-t_n) \frac{1}{n} \sum_{j=1}^n ((PT)^j x_n - z),$$

so we get

$$\begin{aligned}\|x_n - z\|^2 &= (1-t_n) \langle \frac{1}{n} \sum_{j=1}^n ((PT)^j x_n - z), x_n - z \rangle + t_n \langle f(x_n) - z, x_n - z \rangle \\ &\leq (1-t_n) \|x_n - z\|^2 + t_n \langle f(x_n) - z, x_n - z \rangle.\end{aligned}$$

Hence

$$\begin{aligned}\|x_n - z\|^2 &\leq \langle f(x_n) - z, x_n - z \rangle \\ &= \langle f(x_n) - f(z), x_n - z \rangle + \langle f(z) - z, x_n - z \rangle \\ &\leq \alpha \|x_n - z\|^2 + \langle f(z) - z, x_n - z \rangle.\end{aligned}$$

This implies that

$$\|x_n - z\|^2 \leq \frac{1}{1-\alpha} \langle x_n - z, f(z) - z \rangle.$$

In particular, we have

$$\|x_{n_j} - z\|^2 \leq \frac{1}{1-\alpha} \langle x_{n_j} - z, f(z) - z \rangle.$$

Since $x_{n_j} \rightarrow z$, it follows that

$$x_{n_j} \rightarrow z \text{ as } j \rightarrow \infty.$$

Next we show that the inequality (3.1) is true. Indeed, from

$$x_n = t_n f(x_n) + (1-t_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x_n, \forall n \geq 1,$$

we have

$$(I-f)x_n = -\frac{1-t_n}{t_n} \left(x_n - \frac{1}{n} \sum_{j=1}^n (PT)^j x_n \right).$$

Thus for any $q \in F(T)$, we infer by Remark 2.8 that

$$\begin{aligned} \langle (I-f)x_n, x_n - q \rangle &= -\frac{1-t_n}{t_n} \langle (I - \frac{1}{n} \sum_{j=1}^n (PT)^j)x_n, x_n - q \rangle \\ &= -\frac{1-t_n}{t_n} \langle (I - \frac{1}{n} \sum_{j=1}^n (PT)^j)x_n - (I - \frac{1}{n} \sum_{j=1}^n (PT)^j)z, x_n - q \rangle \\ &\leq 0, \forall n \geq 1. \end{aligned}$$

In particular

$$\langle (I-f)x_{n_j}, x_{n_j} - q \rangle \leq 0, \forall j \geq 1.$$

Taking $j \rightarrow \infty$, so we obtain

$$\langle (I-f)z, z - q \rangle \leq 0, \forall q \in F(T), \quad (3.2)$$

or equivalent to $z = G(f(z))$. Finally, we shall show that $\{x_n\}$ convergence strongly to z . Let another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow z' \in C$ as $k \rightarrow \infty$. Then $z' \in F(T)$, it follows from the inequality (3.2) that

$$\langle (I-f)z, z - z' \rangle \leq 0. \quad (3.3)$$

Interchange z and z' to obtain

$$\langle (I-f)z', z' - z \rangle \leq 0. \quad (3.4)$$

Adding (3.3) and (3.4) and by Lemma 2.7 we get

$$(1-\alpha)\|z - z'\|^2 \leq \langle z - z', (I-f)z - (I-f)z' \rangle \leq 0.$$

This implies that $z = z'$. Hence $\{x_n\}$ converges strongly to z . This completely the proof. $\square$

Theorem 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H , P be the metric projection of H onto C and $T : C \rightarrow H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ which satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then for a contraction mapping $f : C \rightarrow C$ with coefficient $\alpha \in (0, 1)$, the sequence $\{y_n\}$ defined by (1.8) converges strongly to z , where, z is the unique solution in $F(T)$ to the variation inequality*

$$\langle (I - f)z, x - z \rangle \geq 0, \quad x \in F(T) \quad (3.5)$$

or equivalently $z = G(f(z))$, where G is a metric projection mapping from H onto $F(T)$.

Proof. Since $F(T)$ is nonempty, it follows that T satisfies (NNO) condition by Lemma 2.3. We first show that $\{y_n\}$ is bounded. Let $q \in F(T)$. We note that

$$\begin{aligned} \|y_{n+1} - q\| &= \|\alpha_n f(y_n) + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n - q\| \\ &\leq \alpha_n \|f(y_n) - q\| + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n \|(PT)^j y_n - q\| \\ &\leq \alpha_n \|f(y_n) - f(q)\| + \alpha_n \|f(q) - q\| + (1 - \alpha_n) \|y_n - q\| \\ &\leq \alpha_n \alpha \|y_n - q\| + \alpha_n \|f(q) - q\| + (1 - \alpha_n) \|y_n - q\| \\ &= (1 - \alpha_n(1 - \alpha)) \|y_n - q\| + \alpha_n \|f(q) - q\| \\ &\leq \max\{\|y_n - q\|, \frac{1}{1 - \alpha} \|f(q) - q\|\}, \quad \forall n \geq 1. \end{aligned}$$

So by induction, we get

$$\|y_n - q\| \leq \max\{\|y_0 - q\|, \frac{1}{1 - \alpha} \|f(q) - q\|\}, \quad n \geq 0.$$

This show that $\{y_n\}$ is bounded, so are $\{f(y_n)\}$ and $\{\frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n\}$. We observe that

$$\begin{aligned} \|y_{n+1} - \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n\| &= \|\alpha_n f(y_n) + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n - \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n\| \\ &= \alpha_n \|f(y_n) - \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n\| \\ &\leq \alpha_n (\|f(y_n)\| + \|\frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n\|). \end{aligned}$$

Hence $\{y_{n+1} - \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n\}$ converges strongly to 0. We next show that

$$\limsup_{n \rightarrow \infty} \langle z - y_n, z - f(z) \rangle \leq 0. \quad (3.6)$$

Let $\{y_{n_j}\}$ be a subsequence of $\{y_n\}$ such that

$$\lim_{j \rightarrow \infty} \langle z - y_{n_j}, z - f(z) \rangle = \limsup_{n \rightarrow \infty} \langle z - y_n, z - f(z) \rangle,$$

and $y_{n_j} \rightarrow q \in C$. It follows by Lemma 2.5 and Lemma 2.4 that $q \in F(PT) = F(T)$. By the inequality (3.5), we get

$$\limsup_{n \rightarrow \infty} \langle z - y_n, z - f(z) \rangle = \langle z - q, z - f(z) \rangle \leq 0.$$

Hence (3.6) is true. Finally we shall show that $y_n \rightarrow z$. For each $n \geq 0$, we have

$$\begin{aligned} \|y_{n+1} - z\|^2 &= \|y_{n+1} - z + \alpha_n(z - f(z)) - \alpha_n(z - f(z))\|^2 \\ &\leq \|y_{n+1} - z + \alpha_n(z - f(z))\|^2 + 2\alpha_n \langle y_{n+1} - z, f(z) - z \rangle \\ &= \|\alpha_n f(y_n) + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n - (\alpha_n f(z) + (1 - \alpha_n)z)\|^2 \\ &\quad + 2\alpha_n \langle y_{n+1} - z, f(z) - z \rangle \\ &= \|\alpha_n(f(y_n) - f(z)) + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n ((PT)^j y_n - z)\|^2 \\ &\quad + 2\alpha_n \langle y_{n+1} - z, f(z) - z \rangle \\ &\leq \left[\alpha_n \|f(y_n) - f(z)\| + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n \|(PT)^j y_n - z\| \right]^2 \\ &\quad + 2\alpha_n \langle y_{n+1} - z, f(z) - z \rangle \\ &\leq \left[\alpha_n \|y_n - z\| + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n \|y_n - z\| \right]^2 \\ &\quad + 2\alpha_n \langle y_{n+1} - z, f(z) - z \rangle \\ &= (1 - \alpha_n(1 - \alpha))^2 \|y_n - z\|^2 + 2\alpha_n \langle y_{n+1} - z, f(z) - z \rangle \\ &\leq (1 - \alpha_n(1 - \alpha)) \|y_n - z\|^2 + 2\alpha_n \langle y_{n+1} - z, f(z) - z \rangle. \end{aligned} \quad (3.7)$$

Now, let $\epsilon > 0$ be arbitrary. Then, by the fact (3.6), there exists a natural number N such that

$$\langle z - y_n, z - f(z) \rangle \leq \frac{\epsilon}{2}, \forall n \geq N.$$

From (3.7), we get

$$\|y_{n+1} - z\|^2 \leq (1 - \alpha_n(1 - \alpha))\|y_n - z\|^2 + \alpha_n \epsilon.$$

By Lemma 2.6, the sequence $\{y_n\}$ converges strongly to a fixed point z of T . This completely the proof. $\square$

Theorem 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H , P the metric projection of H onto C and $T : C \rightarrow H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be sequence in $[0, 1]$ which satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then for a contraction mapping $f : C \rightarrow C$ with coefficient $\alpha \in (0, 1)$, the sequence $\{z_n\}$ defined by (1.9) converges strongly to z , where, z is the unique solution in $F(T)$ to the variation inequality*

$$\langle (I - f)z, x - z \rangle \geq 0, \quad x \in F(T) \quad (3.8)$$

or equivalently $z = G(f(z))$, where G is a metric projection mapping from H onto $F(T)$.

Proof. Since $F(T)$ is nonempty, it follows that T satisfies (NNO) condition by Lemma 2.3. We first show that $\{z_n\}$ is bounded. Let $q \in F(T)$. We note that

$$\begin{aligned} \|z_{n+1} - q\| &= \left\| \frac{1}{n+1} \sum_{j=0}^n P(\alpha_n f(z_n) + (1 - \alpha_n)(TP)^j z_n) - q \right\| \\ &\leq \frac{1}{n+1} \sum_{j=0}^n \|P(\alpha_n f(z_n) + (1 - \alpha_n)(TP)^j z_n) - Pq\| \\ &\leq \frac{1}{n+1} \sum_{j=0}^n (\alpha_n \|f(z_n) - f(q)\| + \alpha_n \|f(q) - q\| + (1 - \alpha_n) \|z_n - q\|) \\ &\leq \alpha_n \alpha \|z_n - q\| + \alpha_n \|f(q) - q\| + (1 - \alpha_n) \|z_n - q\| \\ &= (1 - \alpha_n(1 - \alpha)) \|z_n - q\| + \alpha_n \|f(q) - q\| \\ &\leq \max\{\|x_n - q\|, \frac{1}{1 - \alpha} \|f(q) - q\|\}, \quad \forall n \geq 0. \end{aligned}$$

So by induction, we obtain

$$\|x_n - q\| \leq \max\{\|x_0 - q\|, \frac{1}{1 - \alpha} \|f(q) - q\|\}, \quad n \geq 0.$$

This show that $\{z_n\}$ is bounded, so are $\{f(x_n)\}$ and $\{\frac{1}{n+1} \sum_{j=0}^n \|(TP)^j z_n\|\}$. Furthermore, we also have

$$\begin{aligned}
 \|z_{n+1} - \frac{1}{n+1} \sum_{j=0}^n (PT)^j z_n\| &\leq \frac{1}{n+1} \sum_{j=0}^n \|P(\alpha_n f(z_n) + (1 - \alpha_n)(TP)^j z_n) - (PT)^j z_n\| \\
 &\leq \frac{1}{n+1} \sum_{j=0}^n \|\alpha_n f(z_n) + (1 - \alpha_n)(TP)^j z_n - T(PT)^{j-1} z_n\| \\
 &= \frac{1}{n+1} \sum_{j=0}^n \|\alpha_n f(z_n) + (1 - \alpha_n)(TP)^j z_n - (TP)^j z_n\| \\
 &= \alpha_n \frac{1}{n+1} \sum_{j=0}^n \|f(x_n) - (TP)^j z_n\| \\
 &\leq \alpha_n \left[\|f(x_n)\| - \frac{1}{n+1} \sum_{j=0}^n \|(TP)^j z_n\| \right] \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

This implies that $\{z_{n+1} - \frac{1}{n+1} \sum_{j=0}^n (PT)^j z_n\}$ converges strongly to 0. We next show that

$$\limsup_{n \rightarrow \infty} \langle z - z_n, z - f(z) \rangle \leq 0. \quad (3.9)$$

Let $\{z_{n_j}\}$ be a subsequence of $\{z_n\}$ such that

$$\lim_{j \rightarrow \infty} \langle z - z_{n_j}, z - f(z) \rangle = \limsup_{n \rightarrow \infty} \langle z - z_n, z - f(z) \rangle,$$

and $z_{n_j} \rightarrow q \in C$. By Lemma 2.5 and Lemma 2.4 we get $q \in F(PT) = F(T)$. From the inequality (3.8) we obtain

$$\limsup_{n \rightarrow \infty} \langle z - z_n, z - f(z) \rangle = \langle z - q, z - f(z) \rangle \leq 0.$$

This show that (3.9) is true. Finally we shall show that $z_n \rightarrow z$. For each

$n \geq 0$, we have

$$\begin{aligned}
 \|z_{n+1} - z\|^2 &= \|z_{n+1} - z + \alpha_n(z - f(z)) - \alpha_n(z - f(z))\|^2 \\
 &\leq \|z_{n+1} - z + \alpha_n(z - f(z))\|^2 + 2\alpha_n\langle z_{n+1} - z, f(z) - z \rangle \\
 &= \left\| \frac{1}{n+1} \sum_{j=0}^n P(\alpha_n f(z_n) + (1 - \alpha_n)(TP)^j z_n) - (\alpha_n f(z) + (1 - \alpha_n)z) \right\|^2 \\
 &\quad + 2\alpha_n\langle z_{n+1} - z, f(z) - z \rangle \\
 &\leq \left\{ \frac{1}{n+1} \sum_{j=0}^n \|P(\alpha_n f(z_n) + (1 - \alpha_n)(TP)^j z_n) - P(\alpha_n f(z) + (1 - \alpha_n)z)\| \right\}^2 \\
 &\quad + 2\alpha_n\langle z_{n+1} - z, f(z) - z \rangle \\
 &\leq \left\{ \frac{1}{n+1} \sum_{j=0}^n \|\alpha_n(f(z_n) - f(z)) + (1 - \alpha_n)((TP)^j z_n - z)\| \right\}^2 \\
 &\quad + 2\alpha_n\langle z_{n+1} - z, f(z) - z \rangle \\
 &\leq \{\alpha_n\|z_n - z\| + (1 - \alpha_n)\|z_n - z\|\}^2 \\
 &\quad + 2\alpha_n\langle z_{n+1} - z, f(z) - z \rangle \\
 &= (1 - \alpha_n(1 - \alpha))\|z_n - z\|^2 + 2\alpha_n\langle z_{n+1} - z, f(z) - z \rangle. \tag{3.10}
 \end{aligned}$$

Now, let $\epsilon > 0$ be arbitrary. Then, by the fact (3.9), there exists a natural number N such that

$$\langle z - z_n, z - f(z) \rangle \leq \frac{\epsilon}{2}, \forall n \geq N.$$

From (3.10), we have

$$\|z_{n+1} - z\|^2 \leq (1 - \alpha_n(1 - \alpha))\|z_n - z\|^2 + \alpha_n\epsilon.$$

By Lemma 2.6, the sequence $\{z_n\}$ converges strongly to a fixed point z of T . This completely the proof. $\square$

References

- [1] J. B. Baillon, *Un theoreme de type ergodique pour les contractions non linear dans un espace de Hilbert*, C.r. Acad. Sci. Paris, **280** (1975) 1511–1514.
- [2] H. Bauschke, *The approximation of Fixed points of compositions of non-expansive mappings in Hilbert space*, J. Math. Anal. Appl. **202** (1996) 150–159.

- [3] F. E. Browder, *Convergence of approximates to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Archs ration. Mech. Analysis. **24** (1967) 82–90.
- [4] H. Ishihara and W. Takahashi, *A nonlinear ergodic theorem for a reversible semigroup of Lipschitzian mapping in a Hilbert space*, Proc. Amer. math. Soc. **104** (1988) 431–436.
- [5] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl. **241** (2000) 46–55.
- [6] J. S. Jung, *Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **302** (2005) 509–520.
- [7] S. Matsushita and D. Kuroiwa, *Approximation of fixed points of nonexpansive nonself-mappings*, Sci. Math. Jpn. **57** (2003) 171–176.
- [8] S. Matsushita and D. Kuroiwa, *Strong convergence of averaging iterations of nonexpansive nonself-mappings*, J. Math. Anal. Appl. **294** (2004) 206–214.
- [9] G. O'Hara, P. Pillary and H. K. Xu, *Iterative approaches to finding nearest common fixed points of nonexpansive mappings in Hilbert spaces*, Nonlinear Anal. **54** (2003) 1417–1426.
- [10] T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl. **211** (1997) 71–83.
- [11] T. Shimizu and W. Takahashi, *Strong convergence for asymptotically nonexpansive mappings*, Nonlinear Anal. **26** (1996) 265–272.
- [12] N. Shioji and W. Takahashi, *Strong convergence theorems for asymptotically nonexpansive semigroups in Hilbert spaces*, Nonlinear Anal. **34** (1998) 87–99.
- [13] W. Takahashi and G. E. Kim, *Strong convergence of approximants to fixed points of nonexpansive nonexpansive nonself-mappings in Banach spaces*, Nonlinear Anal. **32** (1998) 447–454.
- [14] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298** (2004) 279–291.

ภาคผนวก 8/15

 You forwarded this message on 2/25/2005 2:10 PM.

Attachments can contain viruses that may harm your computer. Attachments may not display correctly.

Rabian Wangkeeree**From:** Rabian Wangkeeree**Sent:** Mon 2/21/2005 10:17 PM**To:** kms@kms.or.kr**Cc:****Subject:** Submission of Manuscripts**Attachments:**  (7) Strong convergence of Viscosity averaging iteration.pdf(169KB)

To the Editor,

Editorial Office,

Journal of the Korean Mathematical Society,

The Korean Mathematic Society

The Korea Science and Technology Center 214

635-4 Yeoksam-dong, Kangnam-ku

Seoul. 135-703, Korea

February 21, 2005

Dear Sir,

Enclosed please find the pdf.file of my paper with Dr. Somyot Plubtieng entitled;

"Strong Convergence Theorems of Viscosity Averaging Iterations for Nonexpansive Nonself-Mappings in Hilbert Spaces" which I would like to submit for publication in the Journal of the Korean Mathematical Society.

I would like to thank you in advance for your consideration.

Your sincerely,

Mr. Rabian Wangkeeree ,

Department of Mathematics ,

ภาคผนวก 9

**Implicit iteration process of nonexpansive
nonself-mappings in Banach spaces**

S. Plubtieng and R. Punpeang

Bull. Korean. Math. Soc. (submitted).

IMPLICIT ITERATION PROCESS OF NONEXPANSIVE
NONSELF-MAPPINGS IN BANACH SPACES

SOMYOT PLUBTIENG AND RATTANAPORN PUNPAENG

Department of Mathematics, Naresuan University Pitsanulok 65000,
Thailand

ABSTRACT. Suppose C is a nonempty bounded closed convex retract of a real uniformly convex Banach space E with a uniformly Gâteaux differentiable norm. Let $T : C \rightarrow E$ be a nonexpansive nonself-mapping and P is a sunny nonexpansive retraction of E onto C . In this paper, we study the convergence of the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ which defined by

$$\begin{aligned}x_n &= (1 - \alpha_n)u + \alpha_n T[(1 - \beta_n)x_n + \beta_n T x_n] \\y_n &= (1 - \alpha_n)u + \alpha_n P T[(1 - \beta_n)y_n + \beta_n P T y_n], \text{ and} \\z_n &= P[(1 - \alpha_n)u + \alpha_n T P[(1 - \beta_n)z_n + \beta_n T z_n]]\end{aligned}$$

where $\{\alpha_n\} \subseteq (0, 1)$, $0 \leq \beta_n \leq \beta < 1$ and $\alpha_n \rightarrow 1$ as $n \rightarrow \infty$.

Keywords: Nonexpansive mapping, sunny retraction, fixed point, weak inwardness condition, strong convergence theorem.

2000 Mathematics Subject Classification: 47H10, 47H09, 46B20.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a Banach space E . Then a nonself-mapping T from C into E is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Given $u \in C$ and $\{\alpha_n\}$ is a sequence such that $0 < \alpha_n < 1$. We can

Corresponding author.

Email addresses: Somyotp@nu.ac.th(Somyot Plubtieng) and g46060088@nu.ac.th.
(Rattanaorn Punpaeng).

ภาคผนวก 9/2

define a contraction $T_n : C \longrightarrow E$ by

$$(1.1) \quad T_n x = (1 - \alpha_n)u + \alpha_n T x, \quad x \in C.$$

If T is a self-mapping (i.e. $T(C) \subset C$), then T_n maps C into itself, and hence, by Banach's contraction principle, T_n has a unique fixed point x_n in C , that is, we have

$$(1.2) \quad x_n = (1 - \alpha_n)u + \alpha_n T x_n, \quad \forall n \geq 1.$$

(Such a sequence $\{x_n\}$ is said to be an approximating fixed point of T since it possesses the property that if $\{x_n\}$ is bounded, then $\lim_{n \rightarrow \infty} \|T x_n - x_n\| = 0$). The strong convergence of $\{x_n\}$ as $\alpha_n \rightarrow 1$ for a self-mapping T of a bounded C was proved in a Hilbert space independently by Browder [2] and Halpern [5] and in a uniformly smooth Banach space by Reich [11]. Thereafter, Singh and Watson [12] extended the result of Browder and Halpern to nonexpansive nonself-mapping T satisfying Rothe's boundary condition: $T(\partial C) \subset C$ (here ∂C denotes the boundary of C). Recently, Xu and Yin [16] proved that if C is a nonempty closed convex (not necessarily bounded) subset of Hilbert space H , if $T : C \longrightarrow H$ is a nonexpansive nonself-mapping, and if $\{x_n\}$ is the sequence define by (1.2) which is bounded, then $\{x_n\}$ converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T . Marino and Trombetta [9] defined contractions S_n and U_n from C into itself by

$$(1.3) \quad S_n x = (1 - \alpha_n)u + \alpha_n P T x \text{ for all } x \in C$$

and

$$(1.4) \quad U_n x = P[(1 - \alpha_n)u + \alpha_n T x] \text{ for all } x \in C,$$

where P is the nearest point projection of H onto C . Then by the Banach contraction principle, there exists a unique fixed point y_n (resp. z_n) of S_n (resp. U_n) in C i.e.

$$(1.5) \quad y_n = (1 - \alpha_n)u + \alpha_n P T y_n$$

and

$$(1.6) \quad z_n = P[(1 - \alpha_n)u + \alpha_n T z_n].$$

Xu and Yin [16] also proved that if C is a nonempty closed convex subset of a Hilbert space H , if $T : C \longrightarrow H$ is a nonexpansive nonself-mapping satisfying the weak inwardness condition, and $\{x_n\}$ is bounded, the $\{y_n\}$ (resp. $\{z_n\}$) defined by (1.5) (resp. (1.6)) converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T . Finally, Jung and Kim [6] extended Xu and Yin's result to a uniformly convex Banach space with a uniformly Gâteaux differentiable norm.

In this paper, we extend Xu and Yin's results [16] to study the contractions T_n, S_n and U_n define by

$$(1.7) \quad T_n x = (1 - \alpha_n)u + \alpha_n T[(1 - \beta_n)x + \beta_n T x]$$

$$(1.8) \quad S_n x = (1 - \alpha_n)u + \alpha_n P T[(1 - \beta_n)x + \beta_n P T x]$$

$$(1.9) \quad U_n x = P[(1 - \alpha_n)u + \alpha_n T P[(1 - \beta_n)x + \beta_n T x]],$$

where $\{\alpha_n\} \subseteq (0, 1)$, $0 \leq \beta_n \leq \beta < 1$, and P is the nearest point projection of H onto C . We also prove the strong convergence of the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ which defined by

$$(1.10) \quad x_n = (1 - \alpha_n)u + \alpha_n T[(1 - \beta_n)x_n + \beta_n T x_n],$$

$$(1.11) \quad y_n = (1 - \alpha_n)u + \alpha_n P T[(1 - \beta_n)y_n + \beta_n P T y_n],$$

$$(1.12) \quad z_n = P[(1 - \alpha_n)u + \alpha_n T P[(1 - \beta_n)z_n + \beta_n T z_n]],$$

where $\alpha_n \rightarrow 1$ as $n \rightarrow \infty$. The result presented in this paper generalized and extend the coresponding main results of Xu and Yin [16], and Jung and Kim [6].

2. PRELIMINARIES

Throughout this paper we denote by E and E^* a real Banach space and the dual space of E , respectively. The value of $x^* \in E^*$ at $x \in E$ will be denote by $\langle x, x^* \rangle$. We also denote by $\mathbb{R}$ and $\mathbb{R}^+$ the sets of all real numbers and all nonnegative real numbers, respectively. When $\{x_n\}$ is a sequence in E , then $x_n \rightarrow x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x . Let C be a nonempty closed convex subset of E and let T be a mapping of C into E . We denote to $F(T)$ the set of all fixed points of T , i.e. $F(T) = \{x \in C : Tx = x\}$. For every ϵ with $0 \leq \epsilon \leq 2$, the modulus $\delta(\epsilon)$ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon \right\}.$$

E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. If E is uniformly convex, then E is reflexive. Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$(2.1) \quad \lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in $S(E)$. It is also said to be *uniformly Gâteaux differentiable* if for each $y \in S(E)$, the limit (2.1) attained uniformly for x in $S(E)$. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

Then $J : E \longrightarrow E^*$ is said to be the *duality mapping*. It is well known if E is smooth, then the duality mapping J is single-valued and strong-weak* continuous. It is also known that if E has a uniformly Gâteaux differentiable norm, J is uniformly continuous on bounded sets when E has its strong topology while E^* has its weak star topology; see Day [3] or Diestel [4]. A closed convex subset C of a Banach space E is said to have normal structure, if for each bounded closed convex subset K of C , which contains at least two points, there exists an element of K which is not a diametral point of K . It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of Banach space has normal structure. Let D be a subset of C and let P be a mapping of C into D . Then P is said to be *sunny* if

$$P(Px + t(x - Px)) = Px,$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A subset K of E is said to be a retract of E if there exists a continuous map $P : E \longrightarrow K$ such that $Px = x$, for all $x \in K$. We note that every closed convex set of a uniformly convex Banach space is a retract. A mapping P of E into E is said to be a *retraction* if $P^2 = P$. If a mapping P of E is a retraction, then $Pz = z$ for every $z \in R(P)$, where $R(P)$ is the range of P . A subset D is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D ; for more details, see [8, 13].

Let C be a nonempty convex subset of Banach space E . Then for $x \in C$ we define the inward set $I_C(x)$ as follows:

$$I_C(x) = \{y \in E : y = x + a(z - x) \text{ for some } z \in C \text{ and } a \geq 0\}.$$

A mapping $T : C \longrightarrow E$ is said to be *inward* if $Tx \in I_C(x)$ for all $x \in C$. T is also said to be *weakly inward* if for each $x \in C$, Tx belongs to the closure of $I_C(x)$.

The following result was proved by Kirk [7].

Lemma 2.1. [7] *Let E be a reflexive Banach space and C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is nonempty.*

Let μ be a mean on positive integers N , i.e. a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on N if and only if

$$\inf\{a_n : n \in N\} \leq \mu(a) \leq \sup\{a_n : n \in N\}$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. According to time and circumstance, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on N is called a *Banach limit* if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. Using the Hahn-Banach theorem, or the Tychonoff fixed point theorem, we can prove the existence of a Banach limit. We know that if μ is a Banach limit, then

$$\liminf_{n \rightarrow \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \rightarrow \infty} a_n$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. So, if $a = (a_1, a_2, \dots) \in l^\infty$ and $a_n \rightarrow c$, as $n \rightarrow \infty$ we have $\mu_n(a_n) = \mu(a) = c$. Further, we will use the following Lemmas.

Lemma 2.2. [15] *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm, let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on N . Let $z \in C$. Then*

$$\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2$$

if and only if $\mu_n \langle y - z, J(x_n - z) \rangle \leq 0$ for all $y \in C$, where J is the duality mapping of E .

Lemma 2.3. [6] *Let C be a closed convex subset of a smooth Banach space E and let $T : E \rightarrow C$ be a retraction. Then the following are equivalent:*

- (a) $\langle x - Px, J(y - Px) \rangle \leq 0$ for all $x \in E$ and $y \in C$;
- (b) $\|Pz - Pw\|^2 \leq \langle z - w, J(Pz - Pw) \rangle$ for all z and w in E ;
- (c) P is both sunny and nonexpansive.

3. MAIN RESULTS

In this section, we give and prove our main theorems. Using Lemma 2.1 and Lemma 2.2, we have the following theorem.

Theorem 3.1. *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of E which has normal structure, and $T : C \rightarrow C$ be a nonexpansive mapping. Suppose that for some $u \in C$, $\{\alpha_n\}_{n=1}^\infty \subseteq (0, 1)$ and $0 \leq \beta_n \leq \beta < 1$. Then, a mapping T_n defined by (1.7) has a unique fixed point $x_n \in C$. Further, T has a fixed point if and only if $\{x_n\}$ remains bounded as $\alpha_n \rightarrow 1$. In this case, $\{x_n\}$ converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T .*

Proof. For every $n \geq 1$ and $x, y \in C$, we have

$$\begin{aligned} \|T_n x - T_n y\| &\leq \alpha_n \|T[(1 - \beta_n)x + \beta_n T x] - T[(1 - \beta_n)y + \beta_n T y]\| \\ &\leq \alpha_n (1 - \beta_n) \|x - y\| + \alpha_n \beta_n \|T x - T y\| \\ &= \alpha_n \|x - y\|. \end{aligned}$$

Since $0 < \alpha_n < 1$, T_n is a contraction. Therefore by the Banach contraction principle, T_n has a unique fixed point $x_n \in C$ such that

$$x_n = (1 - \alpha_n)u + \alpha_n T[(1 - \beta_n)x_n + \beta_n T x_n].$$

Let $w \in F(T)$. Then for each $n \geq 1$, we have

$$\begin{aligned} \|w - x_n\| &= \|w - (1 - \alpha_n)u - \alpha_n T[(1 - \beta_n)x_n + \beta_n T x_n]\| \\ &\leq (1 - \alpha_n)\|w - u\| + \alpha_n \|w - T[(1 - \beta_n)x_n + \beta_n T x_n]\| \\ &\leq (1 - \alpha_n)\|w - u\| + \alpha_n \|w - (1 - \beta_n)x_n - \beta_n T x_n\| \\ &\leq (1 - \alpha_n)\|w - u\| + \alpha_n (1 - \beta_n)\|w - x_n\| + \alpha_n \beta_n \|w - T x_n\| \\ &\leq (1 - \alpha_n)\|w - u\| + \alpha_n (1 - \beta_n)\|w - x_n\| + \alpha_n \beta_n \|w - x_n\| \\ &= (1 - \alpha_n)\|w - u\| + \alpha_n \|w - x_n\| \end{aligned}$$

and hence $(1 - \alpha_n)\|w - x_n\| \leq (1 - \alpha_n)\|w - u\| \forall n \geq 1$. This implies $\|w - x_n\| \leq \|w - u\| \forall n \geq 1$. Then $\{x_n\}$ is a bounded sequence. Suppose that $\{x_n\}$ is bounded and $\alpha_n \rightarrow 1$ as $n \rightarrow \infty$. Then we show that $F(T) \neq \emptyset$ and $\{x_n\}$ converges strongly to a fixed point of T . For any subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ such that $\alpha_{n_i} \rightarrow 1$, we can define a real valued function g on C given by

$$(3.1) \quad g(z) = \mu_i \|x_{n_i} - z\| \text{ for every } z \in C,$$

where μ is a Banach limit. Define the set

$$(3.2) \quad M = \{v \in C : g(v) = \inf_{z \in C} g(z)\}.$$

Since E is reflexive it follows by Barbu ([1], P.79) that M is nonempty. Moreover, M is closed and convex. Further, we note that

$$\begin{aligned} \|x_{n_i} - T x_{n_i}\| &\leq (1 - \alpha_{n_i})\|u - T x_{n_i}\| + \alpha_{n_i} \|T[(1 - \beta_{n_i})x_{n_i} + \beta_{n_i} T x_{n_i}] - T x_{n_i}\| \\ &\leq (1 - \alpha_{n_i})\|u - T x_{n_i}\| + \alpha_{n_i} \|(1 - \beta_{n_i})x_{n_i} + \beta_{n_i} T x_{n_i} - x_{n_i}\| \\ &\leq (1 - \alpha_{n_i})\|u - T x_{n_i}\| + \beta_{n_i} \|T x_{n_i} - x_{n_i}\|, \quad \forall i \geq 1. \end{aligned}$$

This implies that $(1 - \beta_{n_i})\|x_{n_i} - T x_{n_i}\| \leq (1 - \alpha_{n_i})\|u - T x_{n_i}\|$ and hence

$$\begin{aligned} \|x_{n_i} - T x_{n_i}\| &= \frac{(1 - \alpha_{n_i})}{(1 - \beta_{n_i})} \|u - T x_{n_i}\| \\ &\leq \frac{(1 - \alpha_{n_i})}{(1 - \beta)} \|u - T x_{n_i}\| \rightarrow 0 \text{ as } i \rightarrow \infty. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \mu_i \|x_{n_i} - T v\| &\leq \mu_i \|x_{n_i} - T x_{n_i}\| + \mu_i \|T x_{n_i} - T v\| \\ &\leq \mu_i \|x_{n_i} - v\|, \quad \forall i \geq 1. \end{aligned}$$

This implies that M is T -invariant. By Lemma 2.1, we have a fixed point z_0 of T in M . On the other hand, let us show that $\{x_n\}$ converges strongly as $n \rightarrow \infty$ to a fixed point of T . For any $w \in F(T)$, we have

$$\begin{aligned} \langle x_n - Tx_n, J(x_n - w) \rangle &= \langle x_n - Tw + Tw - Tx_n, J(x_n - w) \rangle \\ &= \|x_n - Tw\|^2 - \langle Tx_n - Tw, J(x_n - w) \rangle \\ &\geq \|x_n - Tw\|^2 - \|Tx_n - Tw\| \|x_n - w\| \\ &\geq \|x_n - w\|^2 - \|x_n - w\|^2 = 0 \end{aligned}$$

for all n . Since $x_n = (1 - \alpha_n)u + \alpha_n T[(1 - \beta_n)x_n + \beta_n Tx_n]$, we get

$$\begin{aligned} 0 &\leq \langle (1 - \alpha_n)u + \alpha_n T[(1 - \beta_n)x_n + \beta_n Tx_n] - Tx_n, J(x_n - w) \rangle \\ &= \langle (1 - \alpha_n)(u - Tx_n) + \alpha_n [T((1 - \beta_n)x_n + \beta_n Tx_n) - Tx_n], J(x_n - w) \rangle \\ &= (1 - \alpha_n) \langle u - Tx_n, J(x_n - w) \rangle + \alpha_n \langle T((1 - \beta_n)x_n + \beta_n Tx_n) - Tx_n, J(x_n - w) \rangle \\ &\leq \langle u - Tx_n, J(x_n - w) \rangle + \alpha_n \|T((1 - \beta_n)x_n + \beta_n Tx_n) - Tx_n\| \|J(x_n - w)\| \\ &\leq \langle u - Tx_n, J(x_n - w) \rangle + \alpha_n \|(1 - \beta_n)x_n + \beta_n Tx_n - x_n\| \|x_n - w\| \\ &= \langle u - Tx_n, J(x_n - w) \rangle + \alpha_n \beta_n \|Tx_n - x_n\| \|x_n - w\|, \text{ for all } n. \end{aligned}$$

Since $\{x_n\}$ is bounded and μ is a Banach limit, we have

$$\mu_i \langle u - Tx_{n_i}, J(x_{n_i} - w) \rangle + \mu_i (\alpha_{n_i} \beta_{n_i} \|Tx_{n_i} - x_{n_i}\| \|x_{n_i} - w\|) \geq 0,$$

and hence

$$\mu_i \langle Tx_{n_i} - u, J(x_{n_i} - w) \rangle \leq 0, \quad \forall i \geq 1.$$

This implies that

$$\begin{aligned} \mu_i \langle x_{n_i} - u, J(x_{n_i} - w) \rangle &= \mu_i \langle x_{n_i} - Tx_{n_i}, J(x_{n_i} - w) \rangle + \mu_i \langle Tx_{n_i} - u, J(x_{n_i} - w) \rangle \\ (3.3) \quad &= \mu_i \langle Tx_{n_i} - u, J(x_{n_i} - w) \rangle \leq 0. \end{aligned}$$

Further, since z_0 is the minimizer of the function g on C , by Lemma 2.2 we have

$$(3.4) \quad \mu_i \langle z - z_0, J(x_{n_i} - z_0) \rangle \leq 0, \text{ for all } z \in C.$$

So, putting $w = z_0$ and $z = u$, from (3.3) and (3.4), we have

$$\mu_i \langle x_{n_i} - u, J(x_{n_i} - z_0) \rangle \leq 0$$

and

$$\mu_i \langle u - z_0, J(x_{n_i} - z_0) \rangle \leq 0.$$

Then we get

$$\mu_i \langle x_{n_i} - z_0, J(x_{n_i} - z_0) \rangle = \mu_i \|x_{n_i} - z_0\|^2 \leq 0.$$

Therefore, there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges strongly to z_0 . To show that $\{x_n\}$ converges strongly as $n \rightarrow \infty$ to a fixed point of T , let $x_{s_k} \rightarrow z$ and $x_{t_k} \rightarrow z'$. Since

$$\begin{aligned} \|z - Tz\| &\leq \|z - x_{s_k}\| + \|x_{s_k} - Tx_{s_k}\| + \|Tx_{s_k} - Tz\| \\ &\leq 2\|z - x_{s_k}\| + \|x_{s_k} - Tx_{s_k}\| \rightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned}$$

We obtain $z = Tz$. Similarly, we have $z' = Tz'$. From (3.3), we have that

$$\mu_i \langle x_{s_{k_i}} - u, J(x_{s_{k_i}} - z') \rangle \leq 0$$

and

$$\mu_i \langle x_{t_{k_i}} - u, J(x_{t_{k_i}} - z) \rangle \leq 0.$$

Then, we have

$$\langle z - u, J(z - z') \rangle \leq 0$$

and

$$\langle u - z', J(z - z') \rangle = \langle z' - u, J(z' - z) \rangle \leq 0.$$

Adding these two inequalities yields

$$0 \geq \langle z - z', J(z - z') \rangle = \|z - z'\|^2$$

and thus $z = z'$. Therefore $\{x_n\}$ converges strongly to a fixed point of T . $\square$

Theorem 3.2. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of E , and $T : C \rightarrow E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E and that for some $u \in C$, $\{\alpha_n\}_{n=1}^\infty \subseteq (0, 1)$ and $0 \leq \beta_n < \beta < 1$. Then, a mapping S_n defined by (1.8) has a unique fixed point $y_n \in C$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $\alpha_n \rightarrow 1$. In this case, $\{y_n\}$ converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T .*

Proof. By the Banach contraction principle there exists a unique fixed point y_n of S_n in C such that

$$y_n = (1 - \alpha_n)u + \alpha_n PT[(1 - \beta_n)y_n + \beta_n PTy_n].$$

Let w be a fixed point of T . Then as in the proof of Theorem 3.1, $\{y_n\}$ is bounded. Conversely, suppose that $\{y_n\}$ is bounded. Apply Theorem 3.1, we obtain that $\{y_n\}$ converges strongly to a fixed point z of PT . Next, let us show that $z \in F(T)$. Since $z = PTz$ and P is a sunny nonexpansive retraction of E onto C , it follows by [13] that

$$\langle Tz - z, J(z - v) \rangle \geq 0 \text{ for all } v \in C.$$

On the other hand, Tz belongs to the closure of $I_c(z)$ by the weak inwardness conditions. Hence for each integer $n \geq 1$, there exists $z_n \in C$ and $a_n \geq 0$ such that the sequence

$$r_n := z + a_n(z_n - z) \longrightarrow Tz.$$

Thus it follows that

$$\begin{aligned} 0 &\leq a_n \langle Tz - z, J(z - z_n) \rangle \\ &= \langle Tz - z, J\{a_n(z - z_n)\} \rangle \\ &= \langle Tz - z, J(z - r_n) \rangle \longrightarrow \langle Tz - z, J(z - Tz) \rangle \\ &= -\|Tz - z\|^2. \end{aligned}$$

Hence we have $Tz = z$. □

From Theorem 3.2, we are able to derive the following corollaries.

Corollary 3.3. *Let H be a Hilbert space, C a nonempty closed convex subset of H , $T : C \longrightarrow H$ a nonexpansive nonself-mapping satisfying the weak inwardness condition, $P : H \longrightarrow C$ the nearest point projection. If $\{\alpha_n\}_{n=1}^\infty \subseteq (0, 1)$ and $0 \leq \beta_n \leq \beta < 1$. Then a mapping S_n defined by (1.8) has unique fixed point $y_n \in C$ such that $y_n = (1 - \alpha_n)u + \alpha_n PT[(1 - \beta_n)y_n + \beta_n PTy_n]$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $\alpha_n \longrightarrow 1$. In this case, $\{y_n\}$ converges strong as $\alpha_n \longrightarrow 1$ to a fixed point of T .*

Proof. Note that the nearest point projection P of Hilbert space H onto a closed convex subset C is a sunny and nonexpansive retraction. Thus the result follows from Theorem 3.2. □

Corollary 3.4. ([6, Theorem 2]). *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E , and $T : C \longrightarrow E$ a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E and that for some $u \in C$, $\{\alpha_n\}_{n=1}^\infty \subseteq (0, 1)$. Then, a mapping S_n given by $S_n x = (1 - \alpha_n)u + PTx$ for all $x \in C$ has a unique fixed point $y_n \in C$ such that $y_n = (1 - \alpha_n)u + \alpha_n PTy_n$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $\alpha_n \longrightarrow 1$. In this case, $\{y_n\}$ converges strongly as $\alpha_n \longrightarrow 1$ to a fixed point of T .*

Theorem 3.5. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of E , and $T : C \longrightarrow E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E , that for some $u \in C$,*

$\{\alpha_n\}_{n=1}^\infty \subseteq (0, 1), 0 \leq \beta_n \leq \beta < 1$. Then a mapping U_n defined by (1.9) has a unique fixed point $z_n \in C$. Further, then T has a fixed point if and only if $\{z_n\}$ remains bounded as $\alpha_n \rightarrow 1$ and $\beta_n \rightarrow 0$. In this case, $\{z_n\}$ converges strongly as $\alpha_n \rightarrow 1$ and $\beta_n \rightarrow 0$ to a fixed point of T .

Proof. It follows by the Banach contraction principle that there exists a unique fixed point z_n of R_n such that

$$z_n = P[(1 - \alpha_n)u + \alpha_n TP[(1 - \beta_n)z_n + \beta_n Tz_n]].$$

Let $w \in F(T)$. Then for each $n \geq 1$, we have

$$\begin{aligned} \|w - z_n\| &= \|Pw - P[(1 - \alpha_n)u + \alpha_n TP[(1 - \beta_n)z_n + \beta_n Tz_n]]\| \\ &\leq \|w - (1 - \alpha_n)u - \alpha_n TP[(1 - \beta_n)z_n + \beta_n Tz_n]\| \\ &\leq (1 - \alpha_n)\|w - u\| + \alpha_n\|w - TP[(1 - \beta_n)z_n + \beta_n Tz_n]\| \\ &\leq (1 - \alpha_n)\|w - u\| + \alpha_n(1 - \beta_n)\|w - z_n\| + \alpha_n\beta_n\|w - Tz_n\| \\ &\leq (1 - \alpha_n)\|w - u\| + \alpha_n(1 - \beta_n)\|w - z_n\| + \alpha_n\beta_n\|w - z_n\| \\ &= (1 - \alpha_n)\|w - u\| + \alpha_n\|w - z_n\| \end{aligned}$$

and hence $(1 - \alpha_n)\|w - z_n\| \leq (1 - \alpha_n)\|w - u\|$, $\forall n > 1$. This implies $\|w - z_n\| \leq \|w - u\|$, $\forall n > 1$. Then $\{z_n\}$ is bounded. Conversely, suppose that $\{z_n\}$ is bounded as $\alpha_n \rightarrow 1$ and $\beta_n \rightarrow 0$. To show that $F(T) \neq \emptyset$. For any subsequence $\{z_{n_i}\}$ of the sequence $\{z_n\}$ such that $\alpha_{n_i} \rightarrow 1$, we can define a real valued function g on C given by

$$(3.5) \quad g(z) = \mu_i \|z_{n_i} - z\| \text{ for every } z \in C,$$

where μ is a Banach limit. Define the set

$$(3.6) \quad M = \{v \in C : g(v) = \inf_{z \in C} g(z)\}.$$

Then M is nonempty, bounded and convex. As in the proof of Theorem 3.1, M is PT -invariant. Hence, by Lemma 2.1, we have a fixed point y of PT in M . Thus $y = PTy$. So from [13],

$$\langle Ty - y, J(y - v) \rangle \geq 0 \text{ for all } v \in C.$$

Since T is weak inwardness condition, we get Ty belong to the closure of $I_c(y)$. Then for each integer $n \geq 1$, there exists $w_n \in C$ and $a_n \geq 0$ such that the sequence

$$r_n := y + a_n(w_n - y) \rightarrow Ty.$$

As in the proof of theorem 3.2, we have $Ty = y$. For any $w \in F(T)$, we have

$$\begin{aligned}\alpha_n[TP((1 - \beta_n)w + \beta_n w) - u] + u &= \alpha_n(w - u) + u \\ &= \alpha_n w + (1 - \alpha_n)u \\ &= P(\alpha_n w + (1 - \alpha_n)u),\end{aligned}$$

and hence

$$\begin{aligned}\|z_n - u - \alpha_n(w - u)\|^2 &= \|P[(1 - \alpha_n)u + \alpha_n TP((1 - \beta_n)z_n + \beta_n Tz_n)] \\ &\quad - P(\alpha_n w + (1 - \alpha_n)u)\|^2 \\ &\leq \|(1 - \alpha_n)u + \alpha_n TP((1 - \beta_n)z_n + \beta_n Tz_n) \\ &\quad - \alpha_n w - (1 - \alpha_n)u\|^2 \\ &= \alpha_n^2 \|TP((1 - \beta_n)z_n + \beta_n Tz_n) - w\|^2 \\ &\leq \alpha_n^2 [(1 - \beta_n)\|z_n - w\| + \beta_n \|Tz_n - w\|]^2 \\ &\leq \alpha_n^2 \|z_n - w\|^2 \\ &= \|\alpha_n(z_n - u) - \alpha_n(w - u)\|^2, \quad \forall n \geq 1.\end{aligned}$$

So, we have

$$\begin{aligned}0 &\geq \|z_n - u - \alpha_n(w - u)\|^2 - \|\alpha_n(z_n - u) - \alpha_n(w - u)\|^2 \\ &\geq 2\langle z_n - u - \alpha_n(w - u) - \alpha_n(z_n - u) + \alpha_n(w - u), J(\alpha_n(z_n - w)) \rangle \\ &\geq 2\langle (1 - \alpha_n)(z_n - u), J(\alpha_n(z_n - w)) \rangle \\ &\geq 2(1 - \alpha_n)\alpha_n \langle z_n - u, J(z_n - w) \rangle\end{aligned}$$

and hence

$$(3.7) \quad \langle z_n - u, J(z_n - w) \rangle \leq 0.$$

Then, putting $w=y$, we have

$$(3.8) \quad \langle z_n - u, J(z_n - y) \rangle \leq 0$$

and hence

$$(3.9) \quad \mu_i \langle z_{n_i} - u, J(z_{n_i} - y) \rangle \leq 0.$$

On the other hand, since

$$g(y) = \min_{z \in C} g(z),$$

it follows by Lemma 2.2, that

$$\mu_i \langle z - y, J(z_{n_i} - y) \rangle \leq 0 \quad \text{for all } z \in C.$$

Putting $z=u$, we have

$$(3.10) \quad \mu_i \langle u - y, J(z_{n_i} - y) \rangle \leq 0.$$

As in the proof of Theorem 3.1, from (3.7), (3.8), (3.9) and (3.10), we have that $\{z_n\}$ converges strongly as $\alpha_n \rightarrow 1$ and $\beta_n \rightarrow 0$ to a fixed point of T . $\square$

Corollary 3.6. ([16, Theorem 3]). Let H be a Hilbert space, C a nonempty closed convex subset of H , $T : C \rightarrow H$ a nonexpansive nonself-mapping satisfying the weak inwardness condition, $P : H \rightarrow C$ the nearest projection. If $\{\alpha_n\} \subseteq (0, 1)$, then a mapping U_n defined by (1.4) has unique fixed point $z_n \in C$. Further, T has a fixed point if and only if $\{z_n\}$ is bounded as $\alpha_n \rightarrow 1$. In this case, $\{z_n\}$ converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T .

Acknowledgement. The authors would like to thank The Thailand Research Fund for financial support.

REFERENCES

1. V. Barbu and T. Precupanu, *Convexity and Optimization in Banach Space*, Editura. Academiiei. R.S.R., Bucharest, (1978).
2. F. E. Browder, *Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Archs. Ratio. Mech. Anal., **24** (1967), 80-90.
3. M. M. Day, *Normed Linear Spaces*, 3rd ed. Springer-Verlag, Berlin, (1973).
4. J. Diestel, *Geometry of Banach Spaces-Selected Topics*, Lecture Notes in Math., Vol. 485. Springer-Verlag, Berlin, (1975).
5. B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc., **73** (1967), 957-961.
6. J.S. Jung and S.S. Kim, *Strong convergence theorems for nonexpansive nonself-mappings in Banach spaces*, Nonlinear Analysis, **33**(3) (1998), 321-329.
7. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004-1006.
8. S. Kitahara and W. Takahashi, *Image recovery by convex combinations of sunny nonexpansive retractions*, Nonlinear Analysis, **2** (1993), 333-342.
9. G. Marino and G. Trombetta, *On approximating fixed points for nonexpansive maps*, Indian J. Math., **34** (1992), 91-98.
10. S. Reich, *Product formula, nonlinear semigroups, and accretive operators*, J. Functional Anal., **36** (1980), 147-168.
11. S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach space*, J. Math. Anal. Appl., **75** (1980), 287-292.
12. S. P. Singh and B. Watson, *On approximating fixed points*, Proc. Symp. Pure Math., **45** (1986), 393-395.
13. W. Takahashi, *Nonlinear Functional Analysis*. Yokohama Publishers, Japan, (2000).
14. W. Takahashi and G. E. Kim, *Strong convergence of approximants to fixed points of nonexpansive nonself-mappings*, Nonlinear Analysis, **32** (1998), 447-454.
15. W. Takahashi and Y. Ueda, *On Reich's strong convergence theorems for resolvents of accretive operators*, J. Math. Anal. Appl., **104** (1984), 546-553.
16. H.K. Xu and X. M. Yin, *Strong convergence theorems for nonexpansive nonself-mappings*, Nonlinear Analysis, **24** (1995), 223-228.

ภาคผนวก 9/13

Somyot Plubtieng

From: "Jin GyoTaek" <JinGyoTaek@kaist.ac.kr>
To: <somyotp@nu.ac.th>
Cc: "KMS" <kms@kms.or.kr>; "สุพิณวธ" <kye@math.snu.ac.kr>
Sent: Monday, January 24, 2005 1:50 PM
Subject: [Bull of KMS] Acknowledgement of Receipt (B05-0122)



Dear Professor Somyot Plubtieng,

Manuscript No. : B05-0122

Authors : Somyot Plubtieng and Rattanaorn Punpaeng

Paper: Implicit iteration process of nonexpansive nonself-mappings in Banach spaces

Received Date : January 21, 2005

This message acknowledges receipt of the above manuscript, which you recently submitted for possible publication in the Bulletin of the Korean Mathematical Society. Your manuscript will be sent to a referee, and you will be notified as soon as a report is received. Thanks for submitting your paper to the Bulletin of the Korean Mathematical Society.

Sincerely,

Gyo Taek Jin,

Editor-in-Chief
Bulletin of the Korean Mathematical Society

Department of Mathematics
Korea Advanced Institute of Science and Technology
Daejeon 305-701, Korea

E-mail: trefoil@kaist.ac.kr

ภาคผนวก 10

**Strong convergence theorems for three-step
iterations with errors for non-Lipschitzian
nonself-mappings in Banach spaces**

S. Plubtieng and R. Wangkeeree

Com. and Math. With Appl. (submitted).

Strong Convergence Theorems for Three step Iterations with Errors for Non-Lipschitzian Nonsself-Mappings in Banach Spaces*

Somyot Plubtieng and Rabian Wangkeeree[†]

Department of Mathematics, Faculty of Science, Naresuan University,
Pitsanulok 65000, Thailand

Abstract

Suppose C is a nonempty closed convex subset of a real uniformly convex Banach space X with P is a nonexpansive retraction of X onto C . Let $T : C \rightarrow X$ be an asymptotically nonexpansive in the intermediate sense nonsself-mapping. In this paper we introduced the three step iterative sequence for such map with errors. Moreover, we prove that, if T is completely continuous, then the three step iterative sequences converges strongly to a fixed point of T .

Keywords and phrases. Asymptotically nonexpansive in the intermediate sense mappings; Asymptotically nonexpansive in the intermediate sense nonsself-mappings; completely continuous; uniformly convex.

1 Introduction

Let C be a subset of real normed linear space X , and let T be a self-mapping on C . T is said to be *nonexpansive* provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is

2000 *Mathematics Subject Classification*: 47H10, 47H09, 46B20.

*Corresponding author.

Email addresses: Somyotp@nu.ac.th. (Somyot Plubtieng) and Rabianw@nu.ac.th. (Rabian Wangkeeree)

[†]Supported by The Thailand Research Fund.

called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $\lim_{n \rightarrow \infty} k_n = 1$ such that for each $x, y \in C$ and $n \geq 1$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|.$$

T is called *asymptotically nonexpansive in the intermediate sense*[1] provided T is uniformly continuous and

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \quad (1.1)$$

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically nonexpansive mapping in the intermediate sense.

The concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk[4] in 1992. In 2001, Noor[8, 9] have introduced the three-step iterative sequences and he studied the approximate solutions of variational inclusions(inequalities) in Hilbert spaces. The three-step iterative approximation problems were studied extensively by Noor[8, 9], Glowinski and Le Tallec[3], Haubruge et al[5].

In 2002, Xu and Noor[14] introduced the three-step iterative for asymptotically nonexpansive mappings and they proved the following strong convergence theorem in Banach spaces;

Theorem XN([14], Theorem 2.1). *Let X be a real uniformly convex Banach space, C be a nonempty closed, bounded convex subset of X . Let T be a completely continuous asymptotically nonexpansive self-mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying;*

$$(i) \ 0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1, \text{ and}$$

$$(ii) \ 0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1.$$

For a given $x_0 \in C$, define

$$\begin{aligned} z_n &= \gamma_n T^n x_n + (1 - \gamma_n) x_n \\ y_n &= \beta_n T^n z_n + (1 - \beta_n) x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n. \end{aligned} \quad (1.2)$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converges strongly to a fixed point of T .

Recently, Y. J. Cho, H. Zhou and G. Guo[2] introduced and studied a three step scheme to approximate fixed points of asymptotically nonexpansive mappings as follows:

Theorem CZG([2]). Let X be a real uniformly convex Banach space, C be a nonempty closed convex subset of X . Let $T : C \rightarrow C$ be a completely continuous and asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{k_n\}$ of positive real numbers be such that $k_n \geq 1$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $x_1 \in C$, the sequence $\{x_n\}$ defined by

$$\begin{aligned} z_n &= (1 - \gamma_n - \nu_n)x_n + \gamma_n T^n x_n + \nu_n u_n, n \geq 1 \\ y_n &= (1 - \beta_n - \mu_n)x_n + \beta_n T^n z_n + \mu_n v_n, n \geq 1 \\ x_{n+1} &= (1 - \alpha_n - \lambda_n)x_n + \alpha_n T^n y_n + \lambda_n w_n, n \geq 1, \end{aligned} \quad (1.3)$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}, \{\mu_n\}$ and $\{\nu_n\}$ are real sequences in $[0, 1]$ with the following restrictions:

$$(i) 0 < a \leq \alpha_n \leq b < 1,$$

$$(ii) \limsup_{n \rightarrow \infty} k_n \beta_n < 1,$$

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty, \sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \nu_n < \infty$. Then the sequence $\{x_n\}$ converges strongly to a fixed point p of T .

A subset C of X is called *retract* of X if there exists a continuous mapping $P : X \rightarrow C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \rightarrow C$ is called *retraction* if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P .

Algorithm 1.1(Three step iterative scheme for nonself maps with errors) Let C be a nonempty subset of normed space X . Let $P : X \rightarrow C$ be the nonexpansive retraction of X onto C and a mapping $T : C \rightarrow X$. For a given $x_0 \in C$, compute the iteration sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by

$$\begin{aligned} z_n &= P(\alpha_n'' T(PT)^{n-1} x_n + \beta_n'' x_n + \gamma_n'' u_n) \\ y_n &= P(\alpha_n' T(PT)^{n-1} z_n + \beta_n' x_n + \gamma_n' v_n) \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1} y_n + \beta_n x_n + \gamma_n w_n), \end{aligned} \quad (1.4)$$

where $\{\alpha_n\}, \{\alpha_n'\}, \{\alpha_n''\}, \{\beta_n\}, \{\beta_n'\}, \{\beta_n''\}, \{\gamma_n\}, \{\gamma_n'\}$ and $\{\gamma_n''\}$ are appropriate real sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are three bounded sequences in C .

Algorithm 1.2 If $\alpha_n'' = \gamma_n'' \equiv 0$ and $\beta_n'' \equiv 1$, then iteration(1.4) reduces to the modified Ishikawa iterative scheme for nonself maps with errors

$$\begin{aligned} y_n &= P(\alpha_n' T(PT)^{n-1} x_n + \beta_n' x_n + \gamma_n' v_n) \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1} y_n + \beta_n x_n + \gamma_n w_n), \end{aligned} \quad (1.5)$$

where $\{\alpha_n\}, \{\alpha'_n\}, \{\beta_n\}, \{\beta'_n\}, \{\gamma_n\}$ and $\{\gamma'_n\}$ are appropriate real sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are three bounded sequences in C .

Algorithm 1.3 If $\alpha''_n = \gamma''_n = \alpha'_n = \gamma'_n \equiv 0$ and $\beta''_n = \beta'_n \equiv 1$, then iteration(1.4) reduces to the modified Mann iterative scheme for nonself maps with errors

$$x_{n+1} = P(\alpha_n T(PT)^{n-1} x_n + \beta_n x_n + \gamma_n w_n), \quad (1.6)$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are appropriate real sequences in $[0, 1]$ and $\{w_n\}$ is a bounded sequence in C .

It is clear that the Modified Mann and Ishikawa iterations processes are all special case of the three step iterative scheme for nonself maps with errors(1.4).

In this paper, we will extend the iterations process(1.3) to the three step iterative scheme for nonself maps with errors(1.4) for asymptotically nonexpansive in the intermediate sense nonself maps and without the condition $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. The results presented in this paper generalize and extend the corresponding main results of Cho, Zhou and Guo[2] and Xu and Noor[14].

2 Preliminaries

For the sake of convenience, we first recall some definitions and conclusions.

Definition 2.1 (see [4]). A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta_X(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \epsilon \right\} > 0$$

for all $0 < \epsilon \leq 2$ (i.e., $\delta_X(\epsilon)$ is a function $(0, 2] \rightarrow (0, 1)$).

Lemma 2.2 (see [7]). Let the nonnegative number sequences $\{a_n\}, \{b_n\}$ and $\{d_n\}$ satisfy that

$$a_{n+1} \leq (1 + b_n)a_n + d_n, \forall n = 1, 2, \dots, \sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} d_n < \infty.$$

Then

(1) $\lim_{n \rightarrow \infty} a_n$ exists;

(2) If $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.3 ([13], J. Schu's Lemma). *Let X be a real uniformly convex Banach space, $0 < \alpha \leq t_n \leq \beta < 1$, $x_n, y_n \in X$, $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$, and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = a$, $a \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

3 Main results

In this section, we give new definition and prove our main theorems.

Definition 3.1. Let C be a nonempty subset of a Banach space X . A mapping $T : C \rightarrow X$ is said to be *asymptotically nonexpansive in the intermediate sense nonself-mapping* provided T is uniformly continuous and

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x - y\|) \leq 0, \quad (3.1)$$

where P is a nonexpansive retraction of X onto C .

Remark 3.2. If T is a self-map, then $PT = T$, so that (3.1) coincide with (1.1) and the three step iterative scheme for nonself-maps with errors(1.4) coincide with the three step iterative with errors(1.3).

The following lemma is crucial in proving the main Theorem.

Lemma 3.3. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be an asymptotically nonexpansive in the intermediate sense nonself mapping with the nonempty fixed points set $F(T)$. Put*

$$G_n = \sup_{x, y \in C} (\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x - y\|) \vee 0, \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let the sequence $\{x_n\}$ be defined by (1.4) with the following restrictions

$$(i) \alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1.$$

$$(ii) \sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty.$$

Then for each $x^* \in F(T)$, $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists.

Proof. Let $x^* \in F(T)$, since $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C , so we put

$$K = \sup_{n \geq 1} \|u_n - p\| \vee \sup_{n \geq 1} \|v_n - p\| \vee \sup_{n \geq 1} \|w_n - p\|.$$

For each $n \geq 1$, we note that

$$\begin{aligned}
 \|x_{n+1} - x^*\| &= \|P(\alpha_n T(PT)^{n-1} y_n + \beta_n x_n + \gamma_n w_n) - P(x^*)\| \\
 &\leq \alpha_n \|T(PT)^{n-1} y_n - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|w_n - x^*\| \\
 &= \alpha_n \|T(PT)^{n-1} y_n - T(PT)^{n-1} x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|w_n - x^*\| \\
 &\leq \alpha_n \|y_n - x^*\| + G_n + \beta_n \|x_n - x^*\| + \gamma_n \|w_n - x^*\| \quad (3.2)
 \end{aligned}$$

and

$$\begin{aligned}
 \|y_n - x^*\| &= \|P(\alpha'_n T(PT)^{n-1} z_n + \beta'_n x_n + \gamma'_n v_n) - x^*\| \\
 &\leq \alpha'_n \|z_n - x^*\| + G_n + \beta'_n \|x_n - x^*\| + \gamma'_n \|v_n - x^*\| \quad (3.3)
 \end{aligned}$$

and

$$\|z_n - x^*\| = \| \leq \alpha''_n \|x_n - x^*\| + G_n + \beta''_n \|x_n - x^*\| + \gamma''_n \|u_n - x^*\|. \quad (3.4)$$

Substituting (3.4) into (3.3), we have

$$\begin{aligned}
 \|y_n - x^*\| &\leq \alpha'_n \alpha''_n \|x_n - x^*\| + \alpha'_n G_n + \alpha'_n \beta''_n \|x_n - x^*\| + \alpha'_n \gamma''_n \|u_n - x^*\| \\
 &\quad + G_n + \beta'_n \|x_n - x^*\| + \gamma'_n \|v_n - x^*\| \\
 &\leq (1 - \beta'_n - \gamma'_n) \alpha''_n \|x_n - x^*\| + \beta'_n \|x_n - x^*\| + (1 - \beta'_n - \gamma'_n) \beta''_n \|x_n - x^*\| + m_n \\
 &\leq \beta'_n \|x_n - x^*\| + (1 - \beta'_n) \alpha''_n \|x_n - x^*\| + (1 - \beta'_n) \beta''_n \|x_n - x^*\| + m_n \\
 &= \beta'_n \|x_n - x^*\| + (1 - \beta'_n) (\alpha''_n + \beta''_n) \|x_n - x^*\| + m_n \\
 &\leq \beta'_n \|x_n - x^*\| + (1 - \beta'_n) \|x_n - x^*\| + m_n \\
 &= \|x_n - x^*\| + m_n, \quad (3.5)
 \end{aligned}$$

where $m_n = 2G_n + \gamma'_n \|v_n - x^*\| + \gamma''_n \|u_n - x^*\|$. Substituting (3.5) into (3.2) again, we have

$$\begin{aligned}
 \|x_{n+1} - x^*\| &\leq \alpha_n (\|x_n - x^*\| + m_n) + G_n + \beta_n \|x_n - x^*\| + \gamma_n \|w_n - x^*\| \\
 &\leq (\alpha_n + \beta_n) \|x_n - x^*\| + \alpha_n m_n + G_n + \gamma_n \|w_n - x^*\| \\
 &\leq \|x_n - x^*\| + m_n + G_n + \gamma_n \|w_n - x^*\| \\
 &\leq \|x_n - x^*\| + 3G_n + (\gamma_n + \gamma'_n + \gamma''_n) M \\
 &= \|x_n - x^*\| + b_n,
 \end{aligned}$$

where $b_n = 3G_n + (\gamma_n + \gamma'_n + \gamma''_n) M$. Since $\sum_{n=1}^{\infty} b_n < \infty$, by Lemma 2.2, we have $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists. This completes the proof. $\square$

Lemma 3.4. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be an asymptotically nonexpansive in the intermediate sense nonself mapping with the nonempty fixed points set $F(T)$. Put*

$$G_n = \sup_{x,y \in C} (\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x - y\|) \vee 0, \forall n \geq 1.$$

Let the sequence $\{x_n\}$ be defined by (1.4) with the following restrictions

$$(i) \alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1.$$

$$(ii) \sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty.$$

$$(iii) 0 \leq \alpha < \alpha_n, \beta_n, \alpha'_n, \beta'_n \leq \beta < 1. \text{ Then}$$

$$(a) \lim_{n \rightarrow \infty} \|T(PT)^{n-1}y_n - x_n\| = 0;$$

$$(b) \lim_{n \rightarrow \infty} \|T(PT)^{n-1}z_n - x_n\| = 0.$$

Proof. (a). For any $x^* \in F(T)$, it follows from Lemma 3.4, we have $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists. Let $\lim_{n \rightarrow \infty} \|x_n - x^*\| = a$ for some $a \geq 0$. From (3.5), we have

$$\|y_n - x^*\| \leq \|x_n - x^*\| + m_n, \forall n \geq 1.$$

Taking $\limsup_{n \rightarrow \infty}$ in both sides, we obtain

$$\limsup_{n \rightarrow \infty} \|y_n - x^*\| \leq \limsup_{n \rightarrow \infty} \|x_n - x^*\| = \lim_{n \rightarrow \infty} \|x_n - x^*\| = a. \quad (3.6)$$

It follows that

$$\limsup_{n \rightarrow \infty} \|T(PT)^{n-1}y_n - x^*\| \leq \limsup_{n \rightarrow \infty} (\|y_n - x^*\| + G_n) = \limsup_{n \rightarrow \infty} \|y_n - x^*\| \leq a,$$

and

$$\begin{aligned} a = \lim_{n \rightarrow \infty} \|x_{n+1} - x^*\| &\leq \lim_{n \rightarrow \infty} \|\alpha_n T(PT)^{n-1}y_n + \beta_n x_n + \gamma_n w_n - x^*\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n [(T(PT)^{n-1}y_n - x^*) + \frac{\gamma_n}{2\alpha_n}(w_n - x^*)] \\ &\quad + \beta_n [(x_n - x^*) + \frac{\gamma_n}{2\beta_n}(w_n - x^*)]\| \\ &\leq \liminf_{n \rightarrow \infty} \alpha_n \|T(PT)^{n-1}y_n - x^*\| + \liminf_{n \rightarrow \infty} \beta_n \|x_n - x^*\| \\ &\leq \liminf_{n \rightarrow \infty} \alpha_n \|y_n - x^*\| + \liminf_{n \rightarrow \infty} \alpha_n G_n + \liminf_{n \rightarrow \infty} \beta_n \|x_n - x^*\| \\ &= \liminf_{n \rightarrow \infty} \alpha_n \|y_n - x^*\| + \liminf_{n \rightarrow \infty} \beta_n \|x_n - x^*\| \\ &\leq \liminf_{n \rightarrow \infty} (\alpha_n \|x_n - x^*\| + \alpha_n m_n + \beta_n \|x_n - x^*\|) \\ &= \liminf_{n \rightarrow \infty} (\alpha_n + \beta_n) \|x_n - x^*\| \leq \liminf_{n \rightarrow \infty} \|x_n - x^*\| = a. \end{aligned}$$

Hence

$$a = \lim_{n \rightarrow \infty} \left\| \alpha_n [(T(PT)^{n-1}y_n - x^*) + \frac{\gamma_n}{2\alpha_n}(w_n - x^*)] + \beta_n [(x_n - x^*) + \frac{\gamma_n}{2\beta_n}(w_n - x^*)] \right\|.$$

By J. Schu's Lemma 2.3, we have

$$\lim_{n \rightarrow \infty} \|T(PT)^{n-1}y_n - x_n + (\frac{\gamma_n}{2\alpha_n} - \frac{\gamma_n}{2\beta_n})(w_n - x^*)\| = 0. \quad (3.7)$$

Since $\lim_{n \rightarrow \infty} \|(\frac{\gamma_n}{2\alpha_n} - \frac{\gamma_n}{2\beta_n})(w_n - x^*)\| = 0$, we obtain that

$$\lim_{n \rightarrow \infty} \|T(PT)^{n-1}y_n - x_n\| = 0.$$

This completes the proof of (a).

Proof of (b). For each $n \geq 1$,

$$\begin{aligned} \|x_n - x^*\| &\leq \|x_n - T(PT)^{n-1}y_n\| + \|T(PT)^{n-1}y_n - x^*\| \\ &\leq \|x_n - T(PT)^{n-1}y_n\| + \|y_n - x^*\| + G_n. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|x_n - T(PT)^{n-1}y_n\| = 0 = \lim_{n \rightarrow \infty} G_n$, we obtain that

$$a = \lim_{n \rightarrow \infty} \|x_n - x^*\| \leq \liminf_{n \rightarrow \infty} \|y_n - x^*\|.$$

It follows by (3.6),

$$a \leq \liminf_{n \rightarrow \infty} \|y_n - x^*\| \leq \limsup_{n \rightarrow \infty} \|y_n - x^*\| \leq a.$$

This implies that

$$\lim_{n \rightarrow \infty} \|y_n - x^*\| = a.$$

On the other hand, we note that

$$\begin{aligned} \|z_n - x^*\| &\leq \|\alpha_n'' T(PT)^{n-1}x_n + \beta_n'' x_n + \gamma_n'' u_n - x^*\| \\ &\leq \alpha_n'' \|x_n - x^*\| + G_n + \beta_n'' \|x_n - x^*\| + \gamma_n'' \|u_n - x^*\| \\ &\leq \alpha_n'' \|x_n - x^*\| + G_n + (1 - \alpha_n'') \|x_n - x^*\| + \gamma_n'' \|u_n - x^*\| \\ &\leq \|x_n - x^*\| + G_n + \gamma_n'' \|u_n - x^*\|. \end{aligned}$$

By boundedness of $\{u_n\}$ and $\lim_{n \rightarrow \infty} G_n = 0 = \lim_{n \rightarrow \infty} \gamma_n''$, we have

$$\limsup_{n \rightarrow \infty} \|z_n - x^*\| \leq \limsup_{n \rightarrow \infty} \|x_n - x^*\| = a,$$

so

$$\limsup_{n \rightarrow \infty} \|T(PT)^{n-1}z_n - x^*\| \leq \limsup_{n \rightarrow \infty} (\|z_n - x^*\| + G_n) \leq a$$

and

$$\begin{aligned}
 a = \lim_{n \rightarrow \infty} \|y_n - p\| &\leq \lim_{n \rightarrow \infty} \|\alpha'_n T(PT)^{n-1} z_n + \beta'_n x_n + \gamma'_n w_n - x^*\| \\
 &= \lim_{n \rightarrow \infty} \|\alpha'_n [T(PT)^{n-1} z_n - x^* + \frac{\gamma'_n}{2\alpha'_n} (v_n - x^*)] \\
 &\quad + \beta'_n [(x_n - x^*) + \frac{\gamma'_n}{2\beta'_n} (v_n - x^*)]\| \\
 &\leq \liminf_{n \rightarrow \infty} \alpha'_n \|T(PT)^{n-1} z_n - x^*\| + \liminf_{n \rightarrow \infty} \beta'_n \|x_n - x^*\| \\
 &\leq \liminf_{n \rightarrow \infty} \alpha'_n \|z_n - x^*\| + \liminf_{n \rightarrow \infty} \alpha'_n G_n + \liminf_{n \rightarrow \infty} \beta'_n \|x_n - x^*\| \\
 &= \liminf_{n \rightarrow \infty} (\alpha'_n \|z_n - x^*\| + \beta'_n \|x_n - x^*\|) \\
 &= \liminf_{n \rightarrow \infty} (\alpha'_n \|x_n - x^*\| + \alpha'_n G_n + \alpha'_n \gamma'_n \|u_n - x^*\| + \beta'_n \|x_n - x^*\|) \\
 &= \liminf_{n \rightarrow \infty} (\alpha'_n + \beta'_n) \|x_n - x^*\| \leq \liminf_{n \rightarrow \infty} \|x_n - x^*\| = a.
 \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} \|\alpha'_n [T(PT)^{n-1} z_n - x^* + \frac{\gamma'_n}{2\alpha'_n} (v_n - x^*)] + \beta'_n [(x_n - x^*) + \frac{\gamma'_n}{2\beta'_n} (v_n - x^*)]\| = a.$$

By J. Schu's Lemma 2.3, we have

$$\lim_{n \rightarrow \infty} \|T(PT)^{n-1} z_n - x_n + (\frac{\gamma'_n}{2\alpha'_n} - \frac{\gamma'_n}{2\beta'_n})(v_n - x^*)\| = 0. \quad (3.8)$$

Since $\lim_{n \rightarrow \infty} \|(\frac{\gamma'_n}{2\alpha'_n} - \frac{\gamma'_n}{2\beta'_n})(v_n - x^*)\| = 0$, it follows that

$$\lim_{n \rightarrow \infty} \|T(PT)^{n-1} z_n - x_n\| = 0.$$

This completes the proof of (b). $\square$

Theorem 3.5. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be an asymptotically nonexpansive in the intermediate sense nonself mapping with nonempty fixed point set $F(T)$. Put*

$$G_n = \sup_{x, y \in C} (\|T(PT)^{n-1} x - T(PT)^{n-1} y\| - \|x - y\|) \vee 0, \forall n \geq 1.$$

Let the sequence $\{x_n\}$ be defined by (1.4) with the following restrictions

$$(i) \alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1.$$

$$(ii) \sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty.$$

(iii) $0 \leq \alpha < \alpha_n, \beta_n, \alpha'_n, \beta'_n \leq \beta < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T .

Proof. It follows from Lemma 3.4, that

$$\lim_{n \rightarrow \infty} \|T(PT)^{n-1}y_n - x_n\| = 0 = \lim_{n \rightarrow \infty} \|T(PT)^{n-1}z_n - x_n\|$$

and this implies that,

$$\|x_{n+1} - x_n\| \leq \alpha_n \|T(PT)^{n-1}y_n - x_n\| + \gamma_n \|w_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus

$$\begin{aligned} \|T(PT)^{n-1}x_n - x_n\| &\leq \|T(PT)^{n-1}x_n - T(PT)^{n-1}y_n\| + \|T(PT)^{n-1}y_n - x_n\| \\ &\leq \|x_n - y_n\| + G_n + \|T(PT)^{n-1}y_n - x_n\| \\ &\leq \alpha'_n \|x_n - T(PT)^{n-1}z_n\| + G_n + \gamma'_n \|v_n - x_n\| \\ &\quad + \|T(PT)^{n-1}y_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (3.9)$$

Since

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T(PT)^n x_{n+1}\| \\ &\quad + \|T(PT)^n x_{n+1} - T(PT)^n x_n\| + \|T(PT)^n x_n - Tx_n\|, \end{aligned}$$

and by uniform continuity of T and (3.9), we have

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0. \quad (3.10)$$

By Lemma 3.3, $\{x_n\}$ is a bounded. It follows by our assumption that T is completely continuous, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \rightarrow x^* \in C$ as $k \rightarrow \infty$. Moreover, by (3.10), we have $\|Tx_{n_k} - x_{n_k}\| \rightarrow 0$ which implies that $x_{n_k} \rightarrow x^*$ as $k \rightarrow \infty$. By (3.10) again, we have

$$\|x^* - Tx^*\| = \lim_{k \rightarrow \infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

It show that $x^* \in F(T)$. Furthermore, since $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists. Therefore $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$, that is $\{x_n\}$ converges to a fixed point of T . This completes the proof. $\square$

Corollary 3.6. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be an asymptotically nonexpansive in the intermediate sense nonself mapping with nonempty fixed point set $F(T)$. Put*

$$G_n = \sup_{x, y \in C} (\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x - y\|) \vee 0, \forall n \geq 1.$$

Let the sequence $\{y_n\}$ and $\{z_n\}$ be defined by (1.4) with the following restrictions

$$(i) \alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1.$$

$$(ii) \sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty.$$

(iii) $0 \leq \alpha < \alpha_n, \beta_n, \alpha'_n, \beta'_n \leq \beta < 1$. Then $\{y_n\}$ and $\{z_n\}$ converges strongly to a fixed point of T .

Proof. By Theorem 3.5, we have the sequence $\{x_n\}$ converges to $x^* \in F(T)$, so that it is also bounded sequence. Thus by Lemma 3.4(b) and 3.9, we obtain that

$$\|y_n - x_n\| \leq \alpha'_n \|T(PT)^{n-1} z_n - x_n\| + \gamma'_n \|u_n - x_n\| \rightarrow 0,$$

and

$$\|z_n - x_n\| \leq \alpha''_n \|T(PT)^{n-1} x_n - x_n\| + \gamma''_n \|u_n - x_n\| \rightarrow 0.$$

Therefore $\lim_{n \rightarrow \infty} y_n = x^* = \lim_{n \rightarrow \infty} z_n$. This completes the proof. $\square$

References

- [1] R. E. Bruck, T. Kuczumow and S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform opial property*, Colloq. Math. 65 (1993) 169-179.
- [2] Y. J. Cho, H. Zhou and G. Guo, *Weak and Strong Convergence Theorems for Three-step Iterations with Errors for asymptotically nonexpansive mappings*, Comp. & Math. with Appl. 47 (2004) 707-717.
- [3] R. Glowinski and P. Le Tallec, "Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics" Siam, Philadelphia, 1989.
- [4] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 35 (1972) 171-174.
- [5] S. Haubruge, V. H Nguyen, and J. J Strodiot, *Convergence analysis and applications of the GlowinskiLe Tallec splitting method for finding a zero of the sum of two maximal monotone operators*, J. Optim. Theory Appl. 97, (1998), 645-673.
- [6] J. U. Jeong, M. Aslam Noor and A. Rafiq, *Noor iterations for nonlinear Lipschitzian strongly accretive mappings*, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math., 11(4) (2004), 339-350.

- [7] Q. Liu, *Iteration sequence for asymptotically quasi-nonexpansive mapping with an error member*, J. Math. Anal. Appl. **259** (2001) 18–24.
- [8] M Aslam Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl. **251**, (2000), 217–229.
- [9] M Aslam Noor, *Three-step iterative algorithms for multivalued quasi variational inclusions*, J. Math. Anal. Appl. **255** (2001).
- [10] M Aslam Noor, T. M. Rassias and Z. Huang *Three-step iterations for nonlinear accretive operator equations*, J. Math. Anal. Appl. **274** (2002), 59–68.
- [11] R. E. Rhoades and S. M. Soltuz, *The equivalence between Mann- Ishikawa Iterations and multistep iteration*, Nonlinear Analysis, **58** (2004), 219–228.
- [12] J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, Studia. Math. **2** (1930) 171–180.
- [13] J. Schu, *Iterative construction of fixed points of strictly quasicontractive mapping*, Appl. Anal. **40** (1991) 67–72.
- [14] B. Xu and M Aslam Noor, *Fixed-Point Iterations for Asymptotically Nonexpansive Mappings in Banach Spaces*, J. Math. Anal. Appl. **267**, (2002) 444–453.

ภาคผนวก 10/13

Rabian Wangkeeree

From: Rabian Wangkeeree **Sent:** Sat 4/23/2005 12:20 PM
To: Rabian Wangkeeree
Cc:
Subject: Fw: Acknowledgement of CAM6222, "Strong Convergence Theorems for Three Step..."
Attachments:

----- Original Message -----

From: Ervin Y. Rodin
To: rabianw@nu.ac.th
Sent: Wednesday, January 19, 2005 9:03 PM
Subject: Acknowledgement of CAM6222, "Strong Convergence Theorems for Three Step..."

January 19, 2005

Dear Dr. Wangkeeree.

I am pleased to acknowledge receipt of your manuscript, "Strong Convergence Theorems for Three Step Iterations with Errors for Non-Lipschitzian Nonself-Mappings in Banach Spaces" which you have submitted to **Computers and Mathematics with Applications**. Your interest in the Journal is greatly appreciated.

Please note, however, that the requirements for authors submitting papers to us are:

1.) A list of 6-8 possible reviewers from the North American Region (US & CANADA (full names, complete current address, and email address).

******If you have submitted to our office previously, please note that you must have a different list of reviewers for each manuscript submitted.**

2.) Reviewers MUST BE knowledgeable in the subject area of the paper.

3.) Reviewers MUST NOT be members of the Journal's Editorial Board.

http://www.elsevier.com/wps/find/journaleditorialboard.cws_home/301/editorialboard#editorialboard

Thus, we shall be waiting to receive your list of potential referees, so that we can start the reviewing procedure. Feel free to send this list via email in order to expedite the process.

Due to overwhelming submissions of reviewers that are not knowledgeable in the author's area of research and the lack of reviewer information requested (full name, complete address, AND email) we will be rejecting manuscripts without the proper requirements!!

Michelle Davis
 Prod. Assn't to EY Rodin

Computers and Mathematics with Applications
 Applied Mathematics Letters
 Mathematical and Computer Modelling

Ervin Y. Rodin, Editor in Chief Tel. 314-935-6007
 C. Zimmermann, Editorial Assn't. Tel. 314-935-5806
 M. Davis, Production Assn't. Tel. 314-935-5806
 FAX Tel. 314-935-6121
 Washington University, Campus Box 1040
 One Brookings Dr., St. Louis, MO 63130-4899
 U.S.A.

Computers and Mathematics with Applications
 Applied Mathematics Letters
 Mathematical and Computer Modelling

Ervin Y. Rodin, Editor in Chief Tel. 314-935-6007
 C. Zimmermann, Editorial Assn't. Tel. 314-935-5806
 M. Davis, Production Assn't. Tel. 314-935-5806
 FAX Tel. 314-935-6121
 Washington University, Campus Box 1040

ภาคผนวก 11

**Strong convergence of an implicit iteration
process for asymptotically nonexpansive
mappings**

A. Kangtunyakarn and S. Plubtieng

Kyungpook Math. J. (submitted).

**STRONG CONVERGENCE OF AN IMPLICIT ITERATION
PROCESS FOR ASYMPTOTICALLY NONEXPANSIVE
MAPPINGS**

SOMYOT PLUBTIENG AND ATID KANGTUNYAKARN

*Department of Mathematics, Naresuan University Pitsanulok 65000,
Thailand*

ABSTRACT. Let C a nonempty closed convex subset of a real Hilbert space H , T be an asymptotically nonexpansive self-mapping of C into itself. In this paper, we study the convergence of a sequence $\{x_n\}$ generated by

$$x_n = \alpha_n x_0 + (1 - \alpha_n) A_n [\beta_n x_n + (1 - \beta_n) A_n x_n],$$

where

$$x_0 \in C, A_n = \frac{1}{n} \sum_{j=1}^n T^j, \alpha_n = \frac{b_n - \frac{1}{2}}{b_n - \frac{1}{2} + a}, \beta_n = \frac{b_n - \frac{1}{2}}{b_n - \frac{1}{2} + a'},$$
$$b_n = \frac{1}{n} \sum_{j=1}^n \left(\frac{1}{2} + \left| \frac{1}{2} - k_j \right| + e^{-j} \right), 0 < a < \frac{1}{2} \quad \text{and} \quad 0 \leq a' < \frac{1}{2}.$$

Keywords: Fixed point, asymptotically nonexpansive mapping, strong convergence.

2000 Mathematics Subject Classification: 47H10, 47H09, 46C05.

1. INTRODUCTION

Let C be a closed convex subset of Hilbert space H and let T be a mapping of C into itself. Then, T is said to be a *Lipschitzian mapping* if for each $n \geq 1$ there exists a positive real number k_n such that

Corresponding author.

Email addresses: Somyotp@nu.ac.th (Somyot Plubtieng) and g46060090@nu.ac.th.
(Atid kangtunyakarn).

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$. A Lipschitzian mapping is said to be *nonexpensive* if $k_n = 1$ for all $n \geq 1$ and *asymptotically nonexpensive* [3] if $\lim_{n \rightarrow \infty} k_n = 1$, respectively. We denote by $F(T)$ the set of fixed points of T . It is well known that if T is asymptotically nonexpensive, then $F(T)$ is convex: see [5]. In 1967, Browder [2] proved that the following strong convergence theorem for nonexpensive mapping of a bounded closed convex subset of H into itself. Let $x_0 \in C$, and for each k with $0 < k < 1$, let $T_k x = kx_0 + (1 - k)Tx$. Then, T_k has a unique fixed point u_k in C , and u_k converges as $k \rightarrow 0$ strongly to a fixed point u_0 of T in C . The fixed point u_0 is uniquely specified as the fixed point of T in C closest to x_0 . After Browder's result, such a problem has been investigated by several authors; see Singh and Watson [8], Marino and Trombetta [6], and other

On the other hand, Baillon [1] proved the first nonlinear ergodic theorem: let C be a bounded closed convex subset of H and let T be a nonexpansive mapping of C into itself. Then for each x in C

$$A_n x = \frac{1}{n} \sum_{k=1}^n T^k x$$

converges weakly to a fixed point of T . Recently, Hirano and Takahashi [4] extended Baillon's theorem to asymptotically nonexpensive mapping. Finally, Shimizu and Takahashi [7] studied a sequence $\{x_n\}$ defined by

$$x_n = \alpha_n x_0 + (1 - \alpha_n) A_n x_n$$

where

$$b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j}), \quad \alpha_n = \frac{b_n - 1}{b_n - 1 + a} \quad \text{for } n \geq 1$$

and $0 < a < 1$. They also proved that the sequence $\{x_n\}$ converges strongly to element of $F(T)$ which is nearest to x_0 .

In this paper, we extend Shimizu and Takahashi's result [7] to study a sequence $\{x_n\}$ defined by

$$x_n = \alpha_n x_0 + (1 - \alpha_n) A_n [\beta_n x_n + (1 - \beta_n) A_n x_n], \quad (1.1)$$

where

$$x_0 \in C, \quad b_n = \frac{1}{n} \sum_{j=1}^n \left(\frac{1}{2} + \left| \frac{1}{2} - k_j \right| + e^{-j} \right), \quad \alpha_n = \frac{b_n - \frac{1}{2}}{b_n - \frac{1}{2} + a}, \quad \beta_n = \frac{b_n - \frac{1}{2}}{b_n - \frac{1}{2} + a'},$$

$0 < a < \frac{1}{2}$, and $0 \leq a' < \frac{1}{2}$. Moreover, we prove the strong convergence of a sequence $\{x_n\}$ given by (1.1).

2. LEMMAS

To prove the main theorem we need the following lemmas.

Lemma 2.1 For each $n \geq 1$, T_n has unique fixed point u_n in C .

Proof. Let $x, y \in C$, we have

$$\begin{aligned}
 \|T_n x - T_n y\| &= (1 - \alpha_n) \|A_n[\beta_n x + (1 - \beta_n)A_n x] - A_n[\beta_n y + (1 - \beta_n)A_n y]\| \\
 &\leq (1 - \alpha_n) \frac{1}{n} \sum_{j=1}^n \|T^j[\beta_n x + (1 - \beta_n)A_n x] - T^j[\beta_n y + (1 - \beta_n)A_n y]\| \\
 &\leq (1 - \alpha_n) \frac{1}{n} \sum_{j=1}^n k_j \|\beta_n x + (1 - \beta_n)A_n x - \beta_n y - (1 - \beta_n)A_n y\| \\
 &\leq (1 - \alpha_n) b_n \|\beta_n x + (1 - \beta_n)A_n x - \beta_n y - (1 - \beta_n)A_n y\| \\
 &\leq (1 - \alpha_n) b_n \|\beta_n x + (1 - \beta_n)A_n x - \beta_n y - (1 - \beta_n)A_n y\| \\
 &\leq (1 - \alpha_n) b_n \|\beta_n x + (1 - \beta_n)A_n x - \beta_n y - (1 - \beta_n)A_n y\| \\
 &\leq (1 - \alpha_n) b_n \|\beta_n x + (1 - \beta_n)A_n x - \beta_n y - (1 - \beta_n)A_n y\|.
 \end{aligned}$$

Since $(1 - \beta_n)b_n < \frac{1}{2}$ and $(1 - \alpha_n)b_n < \frac{1}{2}$, it follows that T_n is a contraction mappings. Hence, by the Banach contraction principle, T_n has a unique fixed point u_n in C . ■

Lemma 2.2 If $F(T)$ is nonempty, u_n is bounded.

Proof. Let $v \in F(T)$ and $\{u_n\}$ is fixed point of T_n . Thus we have

$$\begin{aligned}
 \|u_n - v\| &= \|T_n u_n - v\| \\
 &= \|\alpha_n(x_0 - v) + (1 - \alpha_n) \frac{1}{n} \sum_{j=1}^n T^j[\beta_n u_n + (1 - \beta_n)A_n u_n] - (1 - \alpha_n) \frac{1}{n} \sum_{j=1}^n T^j v\| \\
 &\leq \alpha_n \|x_0 - v\| + (1 - \alpha_n) \frac{1}{n} \sum_{j=1}^n \|T^j[\beta_n u_n + (1 - \beta_n)A_n u_n] - T^j v\| \\
 &\leq \alpha_n \|x_0 - v\| + (1 - \alpha_n) \frac{1}{n} \sum_{j=1}^n k_j \|\beta_n u_n + (1 - \beta_n)A_n u_n - v\| \\
 &\leq \alpha_n \|x_0 - v\| + (1 - \alpha_n) b_n \|\beta_n u_n + (1 - \beta_n)A_n u_n - v\| \\
 &= \alpha_n \|x_0 - v\| + (1 - \alpha_n) b_n \|\beta_n u_n + (1 - \beta_n)A_n u_n - v\| \\
 &\leq \alpha_n \|x_0 - v\| + (1 - \alpha_n) b_n \|\beta_n (u_n - v)\| + (1 - \beta_n) \frac{1}{n} \sum_{j=1}^n \|T^j u_n - T^j v\| \\
 &\leq \alpha_n \|x_0 - v\| + (1 - \alpha_n) b_n \|\beta_n (u_n - v)\| + (1 - \beta_n) \frac{1}{n} \sum_{j=1}^n k_j \|u_n - v\|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \alpha_n \|x_0 - v\| + (1 - \alpha_n) b_n [\|\beta_n(u_n - v)\| + (1 - \beta_n) \frac{1}{n} \sum_{j=1}^n k_j \|u_n - v\|] \\
 &\leq \alpha_n \|x_0 - v\| + (1 - \alpha_n) b_n [\|\beta_n(u_n - v)\| + (1 - \beta_n) b_n \|u_n - v\|] \\
 &\leq \alpha_n \|x_0 - v\| + (1 - \alpha_n) b_n \|u_n - v\| + (1 - \alpha_n) b_n \|u_n - v\| \\
 &= \alpha_n \|x_0 - v\| + 2(1 - \alpha_n) b_n \|u_n - v\|.
 \end{aligned}$$

This implies that

$$\|u_n - v\| \leq \frac{\alpha_n \|x_0 - v\|}{[1 - (1 - \alpha_n) 2b_n]} = \frac{\|x_0 - v\|}{(1 - 2a)}.$$

Therefore, $\{u_n\}$ is bounded. ■

Lemma 2.3 [7] Let B be a nonempty bounded subset of C and let $F(T)$ be nonempty. Then, for a positive number ϵ , There exist a positive integer l_ϵ such that for any integer $l \geq l_\epsilon$, there is a positive integer n_l satisfying

$$\|A_n x - T^l A_n x\| < \epsilon \text{ for all } x \in B \text{ and } n \geq n_l.$$

Lemma 2.4 Let $\{x_n\}$ be a sequence in C and let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $x \in C$ and $\{x_{n_i} - (\frac{1}{n_i}) \sum_{j=1}^{n_i} T^j A_{n_i} x_{n_i}\}$ converges strong to 0. Then, x is a fixed point of T .

Proof. We first show that $T^l x$ converges strongly to x . Assume that $T^l x$ does not converges strongly to x . Thus, there exist a positive number ϵ and a subsequence $\{T^{l_k} x\}$ of $\{T^l x\}$ such that $\|T^{l_k} x - x\| \geq \epsilon$ for every k . Since $\{x_{n_i}\}$ converges weakly to x , for each $y \in C$ with $y \neq x$, we have

$$\liminf_{i \rightarrow \infty} \|x_{n_i} - x\| < \liminf_{i \rightarrow \infty} \|x_{n_i} - y\|. \quad (2.1)$$

Let $r = \liminf_{i \rightarrow \infty} \|x_{n_i} - x\|$ and choose $\delta > 0$ such that

$$\delta < \sqrt{r^2 + \frac{\epsilon^2}{4}} - r.$$

Thus, there exist a subsequence $\{x_{m_i}\}$ of $\{x_{n_i}\}$ such that $\|x_{m_i} - x\| < r + \frac{\delta}{4}$ for every i . On the other hand, we note that

$$\begin{aligned} \|x_{m_i} - T^l x\| &\leq \|x_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j[A_{m_i} x_{m_i}]\| + \|\frac{1}{m_i} \sum_{j=1}^{m_i} T^j[A_{m_i} x_{m_i}] - T^l[\frac{1}{m_i} \sum_{j=1}^{m_i} T^j[A_{m_i} x_{m_i}]]\| \\ &\quad + \|T^l[\frac{1}{m_i} \sum_{j=1}^{m_i} T^j[A_{m_i} x_{m_i}]] - T^l x\| \\ &\leq \|x_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j[A_{m_i} x_{m_i}]\| + \|\frac{1}{m_i} \sum_{j=1}^{m_i} T^j[A_{m_i} x_{m_i}] - T^l[\frac{1}{m_i} \sum_{j=1}^{m_i} T^j[A_{m_i} x_{m_i}]]\| \\ &\quad + k_l \|\frac{1}{m_i} \sum_{j=1}^{m_i} T^j[A_{m_i} x_{m_i}] - x\| \\ &\leq \|x_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j[A_{m_i} x_{m_i}]\| + \|\frac{1}{m_i} \sum_{j=1}^{m_i} T^j[A_{m_i} x_{m_i}] - T^l[\frac{1}{m_i} \sum_{j=1}^{m_i} T^j[A_{m_i} x_{m_i}]]\| \\ &\quad + k_l \|\frac{1}{m_i} \sum_{j=1}^{m_i} T^j[A_{m_i} x_{m_i}] - x_{m_i}\| + k_l \|x_{m_i} - x\|. \end{aligned}$$

Since x_{m_i} is bounded, it follow that $A_{m_i} x_{m_i} = \frac{1}{m_i} \sum_{j=1}^{m_i} T^j x_{m_i}$ is bounded. Hence, by Lemma 2.3, there exist a positive integer L_0 such that for every $l \geq L_0$ there is a positive integer i_l satisfying

$$\|\frac{1}{m_i} \sum_{j=1}^{m_i} T^j(A_{m_i} x_{m_i}) - T^l[\frac{1}{m_i} \sum_{j=1}^{m_i} T^j(A_{m_i} x_{m_i})]\| \leq \frac{\delta}{4} \quad \text{for all } i \geq i_l.$$

Since $\lim_{l \rightarrow \infty} k_l = 1$, there exist a positive integer L_1 such that

$$k_l \|x_{m_i} - x\| < r + \frac{\delta}{4} \quad \text{for all } l \geq L_1.$$

Moreover, since

$$\lim_{i \rightarrow \infty} \|x_{n_i} - (\frac{1}{n_i}) \sum_{j=1}^{n_i} T^j A_{n_i} x_{n_i}\| = 0$$

and $\{k_i\}$ is bounded, it follow that there exist a positive integer i_0 such that

$$k_l \|x_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j[A_{m_i} x_{m_i}]\| < \frac{\delta}{4}$$

and

$$\|x_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j[A_{m_i} x_{m_i}]\| < \frac{\delta}{4}$$

for all l and $i \geq i_0$. So, for any $l \geq \max\{L_0, L_1\}$ and $i \geq \max\{i_l, i_0\}$, we have

$$\|x_{m_i} - T^l x\| < \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{4} + r + \frac{\delta}{4} = r + \delta.$$

Choose $l_k \geq \max\{L_0, L_1\}$ and $i \geq \max\{i_{l_k}, i_0\}$, we have

$$\begin{aligned} \|x_{m_i} - \frac{T^{l_k}x + x}{2}\|^2 &= 2\|\frac{x_{m_i} - T^{l_k}x}{2}\|^2 + 2\|\frac{x_{m_i} - x}{2}\|^2 - \|\frac{T^{l_k}x - x}{2}\|^2 \\ &< \frac{(r+\delta)^2}{2} + \frac{(r+\frac{\epsilon}{4})^2}{2} - \frac{\epsilon^2}{4} \\ &< (r+\delta)^2 - \frac{\epsilon^2}{4} \\ &< r^2. \end{aligned}$$

This contradicts (2.1). Therefore, $\{T^{l_k}x\}$ converges strongly to x and hence x is a fixed point of T . ■

3. MAIN RESULT

Now, we have a strong convergence theorem for asymptotically nonexpansive mapping in Hilbert space.

Theorem 3.1. Let C be a closed convex subset of Hilbert space H and T be asymptotically nonexpansive mapping on C into itself with Lipschitz condition k_n and suppose that $F(T)$ is nonempty.

Let

$$b_n = \frac{1}{n} \sum_{j=1}^n (\frac{1}{2} + |\frac{1}{2} - k_j| + e^{-j}), \quad 0 < a < \frac{1}{2}, \quad 0 \leq a' < \frac{1}{2} \text{ and } x_0 \in C.$$

Then, a mapping T_n on C given by

$$T_n x = \alpha_n x_0 + (1 - \alpha_n) A_n [\beta_n x + (1 - \beta_n) A_n x] \quad \text{for all } x \in C$$

has a unique fixed point u_n in C , when

$$\alpha_n = \frac{b_n - \frac{1}{2}}{b_n - \frac{1}{2} + a}, \quad \beta_n = \frac{b_n - \frac{1}{2}}{b_n - \frac{1}{2} + a'}, \quad \text{and} \quad A_n = \frac{1}{n} \sum_{j=1}^n T^j.$$

Further $\{u_n\}$ converges strongly to the element of $F(T)$ which nearest to x_0 .

Proof. By Lemma 2.1, T_n has a unique fixed point u_n in C so, we show that $\{u_n\}$ converges strongly to element of $F(T)$ which nearest to x_0 . Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$. It is sufficient to find a subsequence $\{u_{m_i}\}$ of $\{u_{n_i}\}$ converges strongly to element of $F(T)$ which nearest to x_0 .

Suppose a subsequence $\{u_{m_i}\}$ of $\{u_{n_i}\}$ converges weakly to v in C . Since $F(T)$ is nonempty closed convex, there exists the element $u_0 \in F(T)$ which nearest to x_0 . By Lemma 2.2, $\{u_{m_i}\}$ is bounded so is $\frac{1}{m_i} \sum_{j=1}^{m_i} T^j(A_{m_i} u_{m_i})$.

We now show that

$$u_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j(A_{m_i} u_{m_i}) \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty.$$

We note that

$$\begin{aligned}
 \|u_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j(A_{m_i} u_{m_i})\| &= \|T_{m_i} u_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j(A_{m_i} u_{m_i})\| \\
 &= \|\alpha_{m_i} x_0 + (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} T^j[\beta_{m_i} u_{m_i} + (1 - \beta_{m_i}) A_{m_i} u_{m_i}] \\
 &\quad - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j(A_{m_i} u_{m_i})\| \\
 &= \|\alpha_{m_i} (x_0 - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j(A_{m_i} u_{m_i})) \\
 &\quad + (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} T^j[\beta_{m_i} u_{m_i} + (1 - \beta_{m_i}) A_{m_i} u_{m_i}] \\
 &\quad - (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} T^j(A_{m_i} u_{m_i})\| \\
 &\leq \alpha_{m_i} \|x_0 - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j(A_{m_i} u_{m_i})\| \\
 &\quad + (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} \|T^j[\beta_{m_i} u_{m_i} + (1 - \beta_{m_i}) A_{m_i} u_{m_i}] - T^j(A_{m_i} u_{m_i})\| \\
 &\leq \alpha_{m_i} \|x_0 - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j(A_{m_i} u_{m_i})\| \\
 &\quad + (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} k_j \|\beta_{m_i} u_{m_i} + (1 - \beta_{m_i}) A_{m_i} u_{m_i} - A_{m_i} u_{m_i}\| \\
 &\leq \alpha_{m_i} \|x_0 - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j(A_{m_i} u_{m_i})\| + \beta_{m_i} \|u_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}\|.
 \end{aligned}$$

Since $\alpha_{m_i} \rightarrow 0$, $\beta_{m_i} \rightarrow 0$ as $i \rightarrow \infty$, $\frac{1}{m_i} \sum_{j=1}^{m_i} T^j(A_{m_i} u_{m_i})$, and $\frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}$ are bounded, we get

$$\|u_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j(A_{m_i} u_{m_i})\| \rightarrow 0 \text{ as } i \rightarrow \infty.$$

Hence, by Lemma 2.4, v is a fixed point in T . On the other hand, since $T_{m_i} u_{m_i} = u_{m_i}$ and $Tu_0 = u_0$, we have

$$u_{m_i} = \alpha_{m_i} x_0 + (1 - \alpha_{m_i}) \sum_{j=1}^{m_i} T^j[\beta_{m_i} u_{m_i} + (1 - \beta_{m_i}) A_{m_i} u_{m_i}],$$

and

$$\alpha_{m_i} x_0 = u_{m_i} - (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} T^j[\beta_{m_i} u_{m_i} + (1 - \beta_{m_i}) A_{m_i} u_{m_i}].$$

Thus, we have

$$\begin{aligned}
 \alpha_{m_i} \langle x_o - u_o, u_{m_i} - u_o \rangle &= \langle \alpha_{m_i} x_o - \alpha_{m_i} u_o, u_{m_i} - u_o \rangle \\
 &= \langle u_{m_i} - (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} T^j [\beta_{m_i} u_{m_i} + (1 - \beta_{m_i}) A_{m_i} u_{m_i}] - u_o \\
 &\quad + (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_o, u_{m_i} - u_o \rangle \\
 &= \langle u_{m_i} - u_o, u_{m_i} - u_o \rangle - (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} \langle T^j [\beta_{m_i} u_{m_i} + (1 - \beta_{m_i}) A_{m_i} u_{m_i}] \\
 &\quad - T^j u_o, u_{m_i} - u_o \rangle \\
 &\geq \|u_{m_i} - u_o\|^2 - (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} \|T^j [\beta_{m_i} u_{m_i} + (1 - \beta_{m_i}) A_{m_i} u_{m_i}] - T^j u_o\| \\
 &\quad \|u_{m_i} - u_o\| \\
 &\geq \|u_{m_i} - u_o\|^2 - (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} k_j \|\beta_{m_i} u_{m_i} + (1 - \beta_{m_i}) A_{m_i} u_{m_i} - u_o\| \\
 &\quad \|u_{m_i} - u_o\| \\
 &\geq \|u_{m_i} - u_o\|^2 - (1 - \alpha_{m_i}) b_{m_i} \|\beta_{m_i} u_{m_i} + (1 - \beta_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} - u_o\| \\
 &\quad \|u_{m_i} - u_o\| \\
 &= \|u_{m_i} - u_o\|^2 - (1 - \alpha_{m_i}) b_{m_i} \|\beta_{m_i} (u_{m_i} - u_o) + (1 - \beta_{m_i}) [\frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} - u_o]\| \\
 &\quad \|u_{m_i} - u_o\| \\
 &\geq \|u_{m_i} - u_o\|^2 - (1 - \alpha_{m_i}) b_{m_i} [\|\beta_{m_i} (u_{m_i} - u_o)\| \\
 &\quad + (1 - \beta_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} \|T^j u_{m_i} - T^j u_o\|] \|u_{m_i} - u_o\| \\
 &\geq \|u_{m_i} - u_o\|^2 - (1 - \alpha_{m_i}) b_{m_i} \beta_{m_i} \|u_{m_i} - u_o\|^2 - (1 - \alpha_{m_i}) b_{m_i} \|u_{m_i} - u_o\| \\
 &\quad (1 - \beta_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} k_j \|u_{m_i} - u_o\| \\
 &\geq \|u_{m_i} - u_o\|^2 - (1 - \alpha_{m_i}) b_{m_i} \beta_{m_i} \|u_{m_i} - u_o\|^2 - (1 - \alpha_{m_i}) b_{m_i} (1 - \beta_{m_i}) b_{m_i} \\
 &\quad \|u_{m_i} - u_o\|^2 \\
 &\geq \|u_{m_i} - u_o\|^2 - (1 - \alpha_{m_i}) b_{m_i} \|u_{m_i} - u_o\|^2 - (1 - \alpha_{m_i}) b_{m_i} \|u_{m_i} - u_o\|^2 \\
 &= \|u_{m_i} - u_o\|^2 - (1 - \alpha_{m_i}) 2b_{m_i} \|u_{m_i} - u_o\|^2 \\
 &= [1 - (1 - \alpha_{m_i}) 2b_{m_i}] \|u_{m_i} - u_o\|^2.
 \end{aligned}$$

It follow that

$$\begin{aligned}\|u_{m_i} - u_0\|^2 &\leq \frac{\alpha_{m_i} \langle x_0 - u_0, u_{m_i} - u_0 \rangle}{[1 - (1 - \alpha_{m_i})2b_{m_i}]} \\ &= \frac{\alpha_{m_i} \langle x_0 - u_0, u_{m_i} - u_0 \rangle}{\alpha_{m_i}(1 - 2a)} \\ &= \frac{\langle x_0 - u_0, u_{m_i} - u_0 \rangle}{(1 - 2a)} \quad \text{for every } i \geq 1.\end{aligned}\tag{3.1}$$

Since u_0 is the element of $F(T)$ with is nearest to x_0 , it follow by [9, Lemma 3.1.3] that

$$\begin{aligned}\langle x_0 - u_0, u_{m_i} - u_0 \rangle &= \langle x_0 - u_0, v - u_0 \rangle + \langle x_0 - u_0, u_{m_i} - v \rangle \\ &\leq \langle x_0 - u_0, u_{m_i} - v \rangle.\end{aligned}$$

So, we have $\|u_{m_i} - u_0\|^2 \leq \frac{\langle x_0 - u_0, u_{m_i} - v \rangle}{(1 - 2a)}$ for every i . Since u_{m_i} converges weakly to v , we get $\langle x_0 - u_0, u_{m_i} - v \rangle \rightarrow 0$ as $i \rightarrow \infty$. Hence $\{u_{m_i}\}$ is converges strongly to u_0 . This complete the prove. ■

The following corollary follows from theorem 3.1 .

Corollary 3.2 Let C be a closed convex subset of Hilbert space H and T be asymptotically nonexpensive mapping on C into itself with Lipschitz condition k_n and suppose that $F(T)$ is nonempty.

Let

$$b_n = \frac{1}{n} \sum_{j=1}^n \left(\frac{1}{2} + \left| \frac{1}{2} - k_j \right| + e^{-j} \right), \quad 0 < a < \frac{1}{2} \text{ and } x_0 \in C.$$

Then, a mapping T_n on C given by

$$T_n x = \alpha_n x_0 + (1 - \alpha_n) A_n x \quad ; \forall x \in C$$

has unique fixed point u_n in C , when

$$\alpha_n = \frac{b_n - \frac{1}{2}}{b_n - \frac{1}{2} + a} \quad \text{and} \quad A_n = \frac{1}{n} \sum_{j=1}^n T^j.$$

Further $\{u_n\}$ converges strongly to the element of $F(T)$ which nearest to x_0 .

Acknowledgement, The authors would like to thanks the Thailand Research fund for financial support.

REFERENCES

1. BAILLON J. B., Un theoreme de type ergodique pour les contraction non lineares dans un espace de Hibert, *C. r. Acad. Sci. Paris.* **280**, 1511-1514 (1975).
2. BROWDER F. E., Convergen of approximates to fixed points of nonexpansive nonlinear mapping in Banach space, *Archs ration. Math. Analysis.* **24**, 82-90 (1967).

3. GOEBEL K. & KIRK W. A., A fixed point theorem for asymptotically nonexpansive mapping, *Proc. Am. math. Soc.* **35**, 171-174 (1972).
4. HIRANO N. & TAKAHASHI W., Nonlinear ergodic theorems for nonexpansive mappings in Hilbert space, *Kodai math. J.* **2**, 11-25 (1979).
5. ISHIHARA H. & TAKAHASHI W., A nonlinear erodic theorem for a reversible semigroup of Lipschitzian mapping in Hilbert space, *Proc. Am. math. Soc.* **104**, 431-436 (1988).
6. MARINO G. & TROMBETTA G., On approximating fixed point for nonexpansive maps. *Indian J. Math.* **34**, 91-98 (1992).
7. SHIMIZU T. & TAKAHASHI W., Strong converge theorem for asymptotically nonexpansive mappings. *Nonlinear Analysis.* **26**, 265-272 (1996).
8. SINGH S.P. & WATSON B. On approximating fixed points, *Proc. Symp. Pure Math.* **45** (part 2) 393-395 (1986).
9. TAKAHASHI, W. Nonlinear function Functional analysis. Japan: Yokohama (2000).
10. TAN K. K. & XU H. K., The nonlinear theorem for asymptotically nonexpansive mappings in Banach space, *Proc. Am. math. soc.* **114**, 399-404 (1992).

ภาคผนวก 11/11

Somyot Plubtieng

From: "KMJ" <kmj@knu.ac.kr>
To: <somyotp@nu.ac.th>
Sent: Monday, February 28, 2005 12:33 PM
Subject: Kyungpook Mathematical Journal

Dear Dr.Prof. Somyot Plubtieng,

I am writing to acknowledge the receipt of your paper, entitled
"STRONG CONVERGENCE OF AN IMPLICIT ITERATION PROCESS FOR
ASYMPTOTICALLY NONEXPANSIVE MAPPINGS(# 0524)"

(joint work with Atid Kangtunyakarn)

submitted to Kyungpook Mathematical Journal (KMJ) for publication.

Your paper is assigned the KMJ number # 0524.

Please refer to this number for the future correspondence.

We will send the paper soon to referees and the process should take approximately
10 weeks.

If you have any questions on your paper, please tell us
via the e-mail address (kmj@knu.ac.kr) or the mail address or the fax number
below.

Thank you for your interest in publishing with us.

Sincerely yours,

Lee, Ji Hyun

Editorial Assistant

Kyungpook Mathematical Journal

Kyungpook Mathematical Journal Telephone : + 82-53-950-5306

Department of Mathematics

fax : + 82-53-950-6306

College of Natural Sciences

e-mail : kmj@knu.ac.kr

Kyungpook National University

URL : <http://kmj.knu.ac.kr/>

Daegu 702-701, Korea

ภาคผนวก 12

**Implicit iteration process with errors of
nonexpansive nonself-mappings in Banach
spaces**

S. Plubtieng and R. Punpeang

Southeast Asian. Bull. Math. (submitted).

IMPLICIT ITERATION PROCESS WITH ERROR OF NONEXPANSIVE NONSELF-MAPPINGS IN BANACH SPACES

SOMYOT PLUBTIENG AND RATTANAPORN PUNPAENG

Department of Mathematics, Faculty of Science, Naresuan University Pitsanulok 65000,
Thailand

ABSTRACT. Suppose C is a nonempty bounded closed convex retract of a real uniformly convex Banach space E with a uniformly Gâteaux differentiable norm. Let $T : C \rightarrow E$ be a nonexpansive nonself-mapping and P is a sunny nonexpansive retraction of E onto C . Let $u \in C$, $\{u_n\}$ and $\{v_n\}$ be bounded sequence on C , and let $\{a_n\}, \{b_n\}, \{c_n\}, \{\bar{a}_n\}, \{\bar{b}_n\}$, and $\{\bar{c}_n\}$ be real sequences on $[0,1]$. In this paper, we study the convergence of the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ which defined by

$$\begin{aligned}x_n &= a_n u + b_n T[\bar{a}_n x_n + \bar{b}_n T x_n + \bar{c}_n u_n] + c_n v_n \\y_n &= a_n u + b_n P T[\bar{a}_n x_n + \bar{b}_n P T x_n + \bar{c}_n u_n] + c_n v_n, \\z_n &= P[a_n u + b_n T P[\bar{a}_n x_n + \bar{b}_n T x_n + \bar{c}_n u_n] + c_n v_n]\end{aligned}$$

where $a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1$, $0 \leq b_n \leq \beta < 1$, $\bar{b}_n \leq \beta < 1$, $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \bar{c}_n < \infty$, and $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Keywords: Nonexpansive mapping, sunny retraction, fixed point, weak inwardness condition, strong convergence theorem.

2000 Mathematics Subject Classification: 47H10, 47H09, 46B20.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a Banach space E . Then a nonself-mapping T from C into E is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$

Corresponding author.

Email addresses: Somyotp@nu.ac.th (Somyot Plubtieng) and g46060088@nu.ac.th. (Rattanaorn Punpaeng).

for all $x, y \in C$. Given $u \in C$ and $\{\alpha_n\}$ is a sequence such that $0 < \alpha_n < 1$. We can define a contraction $T_n : C \rightarrow E$ by

$$T_n x = (1 - \alpha_n)u + \alpha_n T x, \quad x \in C. \quad (1.1)$$

If T is a self-mapping (i.e. $T(C) \subset C$), then T_n maps C into itself, and hence, by Banach's contraction principle, T_n has a unique fixed point x_n in C , that is, we have

$$x_n = (1 - \alpha_n)u + \alpha_n T x_n, \quad \forall n \geq 1. \quad (1.2)$$

(Such a sequence $\{x_n\}$ is said to be an approximating fixed point of T since it possesses the property that if $\{x_n\}$ is bounded, then $\lim_{n \rightarrow \infty} \|T x_n - x_n\| = 0$). The strong convergence of $\{x_n\}$ as $\alpha_n \rightarrow 1$ for a self-mapping T of a bounded C was proved in a Hilbert space independently by Browder [2] and Halpern [5] and in a uniformly smooth Banach space by Reich [11]. Thereafter, Singh and Watson [12] extended the result of Browder and Halpern to nonexpansive nonself-mapping T satisfying Rothe's boundary condition: $T(\partial C) \subset C$ (here ∂C denotes the boundary of C). Recently, Xu and Yin [16] proved that if C is a nonempty closed convex (not necessarily bounded) subset of Hilbert space H , if $T : C \rightarrow H$ is a nonexpansive nonself-mapping, and if $\{x_n\}$ is the sequence define by (1.2) which is bounded, then $\{x_n\}$ converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T . Marino and Trombetta [9] defined contractions S_n and U_n from C into itself by

$$S_n x = (1 - \alpha_n)u + \alpha_n P T x \text{ for all } x \in C \quad (1.3)$$

and

$$U_n x = P[(1 - \alpha_n)u + \alpha_n T x] \text{ for all } x \in C, \quad (1.4)$$

where P is the nearest point projection of H onto C . Then by the Banach contraction principle, there exists a unique fixed point y_n (resp. z_n) of S_n (resp. U_n) in C i.e.

$$y_n = (1 - \alpha_n)u + \alpha_n P T y_n \quad (1.5)$$

and

$$z_n = P[(1 - \alpha_n)u + \alpha_n T z_n]. \quad (1.6)$$

Xu and Yin [16] also proved that if C is a nonempty closed convex subset of a Hilbert space H , if $T : C \rightarrow H$ is a nonexpansive nonself-mapping satisfying the weak inwardness condition, and $\{x_n\}$ is bounded, the $\{y_n\}$ (resp. $\{z_n\}$) defined by (1.5) (resp. (1.6)) converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T . Finally, Jung and Kim [6] extended Xu and Yin's result to a uniformly convex Banach space with a uniformly Gâteaux differentiable norm.

In this paper, we extend Xu and Yin's results [16] to study the contractions T_n, S_n and U_n define by

$$T_n x = a_n u + b_n T[\bar{a}_n x + \bar{b}_n T x + \bar{c}_n u_n] + c_n v_n \quad (1.7)$$

$$S_n x = a_n u + b_n PT[\bar{a}_n x + \bar{b}_n PT x + \bar{c}_n u_n] + c_n v_n, \quad (1.8)$$

$$U_n x = P[a_n u + b_n TP[\bar{a}_n x + \bar{b}_n T x + \bar{c}_n u_n] + c_n v_n] \quad (1.9)$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\bar{a}_n\}, \{\bar{b}_n\}$, and $\{\bar{c}_n\}$ be real sequences on $[0,1]$ such that $a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1$, $0 \leq b_n \leq \beta < 1, \bar{b}_n \leq \beta < 1, \sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} \bar{c}_n < \infty$, and P is the nearest point projection of H onto C . We also prove the strong convergence of the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ which defined by

$$x_n = a_n u + b_n T[\bar{a}_n x_n + \bar{b}_n T x_n + \bar{c}_n u_n] + c_n v_n \quad (1.10)$$

$$y_n = a_n u + b_n PT[\bar{a}_n x_n + \bar{b}_n PT x_n + \bar{c}_n u_n] + c_n v_n, \quad (1.11)$$

$$z_n = P[a_n u + b_n TP[\bar{a}_n x_n + \bar{b}_n T x_n + \bar{c}_n u_n] + c_n v_n] \quad (1.12)$$

where $a_n \rightarrow 0$ as $n \rightarrow \infty$. To results presented in this paper generalized and extend the corresponding main results of Xu and Yin [16], and Jung and Kim [6].

2. PRELIMINARIES

Throughout this paper we denote by E and E^* a real Banach space and the dual space of E , respectively. The value of $x^* \in E^*$ at $x \in E$ will be denote by $\langle x, x^* \rangle$. We also denote by $\mathbb{R}$ and $\mathbb{R}^+$ the sets of all real numbers and all nonnegative real numbers, respectively. When $\{x_n\}$ is a sequence in E , then $x_n \rightarrow x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x . Let C be a nonempty closed convex subset of E and let T be a mapping of C into E . We denote to $F(T)$ the set of all fixed points of T , i.e. $F(T) = \{x \in C : Tx = x\}$. For every ϵ with $0 \leq \epsilon \leq 2$, the modulus $\delta(\epsilon)$ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon \right\}.$$

E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. If E is uniformly convex, then E is reflexive. Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.1)$$

exists for each x and y in $S(E)$. It is also said to be *uniformly Gâteaux differentiable* if for each $y \in S(E)$, the limit (2.1) attained uniformly for x in $S(E)$. With each

$x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

Then $J : E \rightarrow E^*$ is said to be the *duality mapping*. It is well known if E is smooth, then the duality mapping J is single-valued and strong-weak* continuous. It is also known that if E has a uniformly Gâteaux differentiable norm, J is uniformly continuous on bounded sets when E has its strong topology while E^* has its weak star topology; see Day [3] or Diestel [4]. A closed convex subset C of a Banach space E is said to have normal structure, if for each bounded closed convex subset K of C , which contains at least two points, there exists an element of K which is not a diametral point of K . It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of Banach space has normal structure. Let D be a subset of C and let P be a mapping of C into D . Then P is said to be *sunny* if

$$P(Px + t(x - Px)) = Px,$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A subset K of E is said to be a retract of E if there exists a continuous map $P : E \rightarrow K$ such that $Px = x$, for all $x \in K$. We note that every closed convex set of a uniformly convex Banach space is a retract. A mapping P of E into E is said to be a *retraction* if $P^2 = P$. If a mapping P of E is a retraction, then $Pz = z$ for every $z \in R(P)$, where $R(P)$ is the range of P . A subset D is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D ; for more details, see [8, 13]. Let C be a nonempty convex subset of Banach space E . Then for $x \in C$ we define the inward set $I_C(x)$ as follows:

$$I_C(x) = \{y \in E : y = x + a(z - x) \text{ for some } z \in C \text{ and } a \geq 0\}.$$

A mapping $T : C \rightarrow E$ is said to be *inward* if $Tx \in I_C(x)$ for all $x \in C$. T is also said to be *weakly inward* if for each $x \in C$, Tx belongs to the closure of $I_C(x)$.

Let μ be a mean on positive integers N , i.e. a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on N if and only if

$$\inf\{a_n : n \in N\} \leq \mu(a) \leq \sup\{a_n : n \in N\}$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. According to time and circumstance, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on N is called a *Banach limit* if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. Using the Hahn-Banach theorem, or the Tychonoff fixed point theorem, we can prove the existence of a Banach limit. We know that if

μ is a Banach limit, then

$$\liminf_{n \rightarrow \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \rightarrow \infty} a_n$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. So, if $a = (a_1, a_2, \dots) \in l^\infty$ and $a_n \rightarrow c$, as $n \rightarrow \infty$ we have $\mu_n(a_n) = \mu(a) = c$. Further, we will use the following Lemmas.

Lemma 2.1. [15] *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm, let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on N . Let $z \in C$. Then*

$$\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2$$

if and only if $\mu_n \langle y - z, J(x_n - z) \rangle \leq 0$ for all $y \in C$, where J is the duality mapping of E .

Lemma 2.2. [6] *Let C be a closed convex subset of a smooth Banach space E and let $T : E \rightarrow C$ be a retraction. Then the following are equivalent:*

- (a) $\langle x - Px, J(y - Px) \rangle \leq 0$ for all $x \in E$ and $y \in C$;
- (b) $\|Px - Pw\|^2 \leq \langle z - w, J(Pz - Pw) \rangle$ for all z and w in E ;
- (c) P is both sunny and nonexpansive.

3. MAIN RESULTS

In this section, we prove our main theorems. Using Lemma 2.1, we have the following theorem.

Theorem 3.1. *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of E which has normal structure, and $T : C \rightarrow C$ be a nonexpansive mapping. Let $u \in C$, $\{u_n\}$ and $\{v_n\}$ be bounded sequences on C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{\bar{a}_n\}, \{\bar{b}_n\}$, and $\{\bar{c}_n\}$ be real sequences on $[0,1]$ satisfying the conditions:*

- (i) $a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1$,
- (ii) $0 \leq b_n \leq \beta < 1, \bar{b}_n \leq \beta < 1, \forall n \geq 1$,
- (iii) $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} \bar{c}_n < \infty$.

Then the mapping T_n defined by (1.7) has a unique fixed point $x_n \in C$. Further, T has a fixed point if and only if $\{x_n\}$ remains bounded as $a_n \rightarrow 0$. In this case, $\{x_n\}$ converges strongly as $a_n \rightarrow 0$ to a fixed point of T .

Proof. For every $n \geq 1$ and $x, y \in C$, we have

$$\begin{aligned} \|T_n x - T_n y\| &\leq b_n \|T[\bar{a}_n x + \bar{b}_n T x + \bar{c}_n u_n] - T[\bar{a}_n y + \bar{b}_n T y + \bar{c}_n u_n]\| \\ &\leq b_n \bar{a}_n \|x - y\| + b_n \bar{b}_n \|T x - T y\| \\ &\leq b_n (\bar{a}_n + \bar{b}_n) \|x - y\| \\ &\leq (\bar{a}_n + \bar{b}_n + \frac{\bar{c}_n}{2}) \|x - y\|. \end{aligned}$$

Since $0 < \bar{a}_n + \bar{b}_n + \frac{\bar{c}_n}{2} < 1$, T_n is a contraction. Therefore by the Banach contraction principle, T_n has a unique fixed point $x_n \in C$ such that

$$x_n = a_n u + b_n T[\bar{a}_n x_n + \bar{b}_n T x_n + \bar{c}_n u_n] + c_n v_n.$$

Let $w \in F(T)$ and $M = \max\{\|w - u\|, \sup \|w - u_n\|, \sup \|w - v_n\|\}$. Then for each $n \geq 1$, we have

$$\begin{aligned} \|w - x_n\| &= \|w - a_n u - b_n T[\bar{a}_n x_n + \bar{b}_n T x_n + \bar{c}_n u_n] - c_n v_n\| \\ &\leq a_n \|w - u\| + b_n \|w - T[\bar{a}_n x_n + \bar{b}_n T x_n + \bar{c}_n u_n]\| + c_n \|w - v_n\| \\ &\leq a_n \|w - u\| + b_n \bar{a}_n \|w - x_n\| + b_n \bar{b}_n \|w - T x_n\| \\ &\quad + b_n \bar{c}_n \|w - u_n\| + c_n \|w - v_n\| \\ &\leq a_n \|w - u\| + b_n (\bar{a}_n + \bar{b}_n) \|w - x_n\| + b_n \bar{c}_n \|w - u_n\| + c_n \|w - v_n\| \\ &\leq a_n \|w - u\| + b_n \|w - x_n\| + b_n \bar{c}_n \|w - u_n\| + c_n \|w - v_n\| \\ &\leq a_n M + b_n \|w - x_n\| + b_n M + c_n M. \end{aligned}$$

This implies $\|w - x_n\| \leq \frac{1}{1-\beta} M$ for all $n \geq 1$. Then $\{x_n\}$ is a bounded sequence. Conversely, we suppose that $\{x_n\}$ is bounded as $a_n \rightarrow 0$. Then we show that $F(T) \neq \emptyset$ and $\{x_n\}$ converges strongly as $a_n \rightarrow 0$ to a fixed point of T . For any subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ such that $a_{n_i} \rightarrow 0$, we can define a real valued function g on C given by

$$g(z) = \mu_i \|x_{n_i} - z\| \text{ for every } z \in C, \quad (3.1)$$

where μ is a Banach limit. Define the set

$$N = \{v \in C : g(v) = \inf_{z \in C} g(z)\}. \quad (3.2)$$

Since E is reflexive, it follows by Barbu ([1], P.79) that N is nonempty. We note that N is closed and convex. Further, we note that

$$\begin{aligned}
 \|x_{n_i} - Tx_{n_i}\| &\leq \|a_{n_i}u + b_{n_i}T[a_{n_i}x_{n_i} + b_{n_i}Tx_{n_i} + c_{n_i}u_{n_i}] + c_{n_i}v_{n_i} - Tx_{n_i}\| \\
 &\leq a_{n_i}\|u - Tx_{n_i}\| + b_{n_i}\|T[a_{n_i}x_{n_i} + b_{n_i}Tx_{n_i} + c_{n_i}u_{n_i}] - Tx_{n_i}\| \\
 &\quad + c_{n_i}\|v_{n_i} - Tx_{n_i}\| \\
 &\leq a_{n_i}\|u - Tx_{n_i}\| + b_{n_i}\|a_{n_i}x_{n_i} + b_{n_i}Tx_{n_i} + c_{n_i}u_{n_i} - x_{n_i}\| \\
 &\quad + c_{n_i}\|v_{n_i} - Tx_{n_i}\| \\
 &\leq a_{n_i}\|u - Tx_{n_i}\| + b_{n_i}b_{n_i}\|Tx_{n_i} - x_{n_i}\| + b_{n_i}c_{n_i}\|u_{n_i} - x_{n_i}\| \\
 &\quad + c_{n_i}\|v_{n_i} - Tx_{n_i}\| \\
 &\leq a_{n_i}\|u - Tx_{n_i}\| + \beta^2\|Tx_{n_i} - x_{n_i}\| + b_{n_i}c_{n_i}\|u_{n_i} - x_{n_i}\| \\
 &\quad + c_{n_i}\|v_{n_i} - Tx_{n_i}\|.
 \end{aligned}$$

This implies that $(1 - \beta^2)\|x_{n_i} - Tx_{n_i}\| \leq a_{n_i}\|u - Tx_{n_i}\| + b_{n_i}c_{n_i}\|u_{n_i} - x_{n_i}\| + c_{n_i}\|v_{n_i} - Tx_{n_i}\|$ and hence

$$\begin{aligned}
 \|x_{n_i} - Tx_{n_i}\| &\leq \frac{1}{(1 - \beta^2)}[a_{n_i}\|u - Tx_{n_i}\| + b_{n_i}c_{n_i}\|u_{n_i} - x_{n_i}\| \\
 &\quad + c_{n_i}\|v_{n_i} - Tx_{n_i}\|] \longrightarrow 0 \text{ as } i \longrightarrow \infty.
 \end{aligned}$$

Thus, we obtain

$$\begin{aligned}
 \mu_i\|x_{n_i} - Tv\| &\leq \mu_i\|x_{n_i} - Tx_{n_i}\| + \mu_i\|Tx_{n_i} - Tv\| \\
 &\leq \mu_i\|x_{n_i} - v\|, \forall i \geq 1.
 \end{aligned}$$

This implies that N is T -invariant. It follows by Kirk's theorem [7] that T has a fixed point z_0 in N . On the other hand, let us show that $\{x_n\}$ converges strongly as $a_n \longrightarrow 0$ to a fixed point of T . For any $w \in F(T)$ and for all $n \geq 1$, we have

$$\begin{aligned}
 \langle x_n - Tx_n, J(x_n - w) \rangle &= \langle x_n - Tw + Tw - Tx_n, J(x_n - w) \rangle \\
 &= \|x_n - Tw\|^2 - \langle Tx_n - Tw, J(x_n - w) \rangle \\
 &\geq \|x_n - Tw\|^2 - \|Tx_n - Tw\|\|x_n - w\| \\
 &\geq \|x_n - w\|^2 - \|x_n - w\|^2 = 0.
 \end{aligned}$$

From (1.10), we get

$$\begin{aligned}
 0 &\leq \langle a_n u + b_n T[\bar{a}_n x_n + \bar{b}_n T x_n + \bar{c}_n u_n] + c_n v_n - T x_n, J(x_n - w) \rangle \\
 &= \langle a_n(u - T x_n) + b_n[T(\bar{a}_n x_n + \bar{b}_n T x_n + \bar{c}_n u_n) - T x_n] \\
 &\quad + c_n(v_n - T x_n), J(x_n - w) \rangle \\
 &= a_n \langle u - T x_n, J(x_n - w) \rangle + b_n \langle T(\bar{a}_n x_n + \bar{b}_n T x_n + \bar{c}_n u_n) \\
 &\quad - T x_n, J(x_n - w) \rangle + c_n \langle (v_n - T x_n), J(x_n - w) \rangle \\
 &\leq \langle u - T x_n, J(x_n - w) \rangle + b_n \|T(\bar{a}_n x_n + \bar{b}_n T x_n + \bar{c}_n u_n) - T x_n\| \\
 &\quad \|J(x_n - w)\| + c_n \|v_n - T x_n\| \|J(x_n - w)\| \\
 &\leq \langle u - T x_n, J(x_n - w) \rangle + b_n \|\bar{a}_n x_n + \bar{b}_n T x_n + \bar{c}_n u_n - x_n\| \|x_n - w\| \\
 &\quad + c_n \|v_n - T x_n\| \|x_n - w\| \\
 &= \langle u - T x_n, J(x_n - w) \rangle + b_n \bar{b}_n \|T x_n - x_n\| \|x_n - w\| + b_n \bar{c}_n \|u_n - x_n\| \|x_n - w\| \\
 &\quad + c_n \|v_n - T x_n\| \|x_n - w\|, \quad \forall n \geq 1.
 \end{aligned}$$

Since $\{x_n\}$ is bounded and μ is a Banach limit, we have

$$\begin{aligned}
 &\mu_i \langle u - T x_{n_i}, J(x_{n_i} - w) \rangle + \mu_i (b_{n_i} \bar{b}_{n_i} \|T x_{n_i} - x_{n_i}\| \|x_{n_i} - w\|) \\
 &+ \mu_i (b_{n_i} \bar{c}_{n_i} \|u_{n_i} - x_{n_i}\| \|x_{n_i} - w\|) + \mu_i (c_{n_i} \|v_{n_i} - T x_{n_i}\| \|x_{n_i} - w\|) \geq 0,
 \end{aligned}$$

and hence

$$\mu_i \langle T x_{n_i} - u, J(x_{n_i} - w) \rangle \leq 0, \quad \forall i \geq 1.$$

This implies that

$$\begin{aligned}
 \mu_i \langle x_{n_i} - u, J(x_{n_i} - w) \rangle &= \mu_i \langle x_{n_i} - T x_{n_i}, J(x_{n_i} - w) \rangle + \mu_i \langle T x_{n_i} - u, J(x_{n_i} - w) \rangle \\
 &= \mu_i \langle T x_{n_i} - u, J(x_{n_i} - w) \rangle \leq 0.
 \end{aligned} \tag{3.3}$$

Further, since z_0 is the minimizer of the function g on C , by Lemma 2.1 we have

$$\mu_i \langle z - z_0, J(x_{n_i} - z_0) \rangle \leq 0, \quad \text{for all } z \in C. \tag{3.4}$$

So, putting $w = z_0$ and $z = u$, from (3.3) and (3.4), we have

$$\mu_i \langle x_{n_i} - u, J(x_{n_i} - z_0) \rangle \leq 0$$

and

$$\mu_i \langle u - z_0, J(x_{n_i} - z_0) \rangle \leq 0.$$

Then we get

$$\mu_i \langle x_{n_i} - z_0, J(x_{n_i} - z_0) \rangle = \mu_i \|x_{n_i} - z_0\|^2 \leq 0.$$

Therefore, there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges strongly to z_0 . To show that $\{x_n\}$ converges strongly as $a_n \rightarrow 0$ to a fixed point of T , let $x_{s_k} \rightarrow z$ and $x_{t_k} \rightarrow z'$. Since

$$\begin{aligned} \|z - Tz\| &\leq \|z - x_{s_k}\| + \|x_{s_k} - Tx_{s_k}\| + \|Tx_{s_k} - Tz\| \\ &\leq 2\|z - x_{s_k}\| + \|x_{s_k} - Tx_{s_k}\| \rightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned}$$

We obtain $z = Tz$. Similarly, we have $z' = Tz'$. From (3.3), we get

$$\mu_i \langle x_{s_{k_i}} - u, J(x_{s_{k_i}} - z') \rangle \leq 0$$

and

$$\mu_i \langle x_{t_{k_i}} - u, J(x_{t_{k_i}} - z) \rangle \leq 0.$$

Thus, we obtain

$$\langle z - u, J(z - z') \rangle \leq 0$$

and

$$\langle u - z', J(z - z') \rangle = \langle z' - u, J(z' - z) \rangle \leq 0.$$

Adding these two inequalities yields

$$0 \geq \langle z - z', J(z - z') \rangle = \|z - z'\|^2$$

and thus $z = z'$. Therefore $\{x_n\}$ converges strongly as $a_n \rightarrow 0$ to a fixed point of T . $\square$

Theorem 3.2. Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of E , and $T : C \rightarrow E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E and that for some $u \in C$, let $\{u_n\}$ and $\{v_n\}$ be bounded sequences on C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{\bar{a}_n\}, \{\bar{b}_n\}$, and $\{\bar{c}_n\}$ be real sequences on $[0,1]$ satisfying the conditions:

- (i) $a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1$,
- (ii) $0 \leq b_n \leq \beta < 1, \bar{b}_n \leq \beta < 1, \forall n \geq 1$,
- (iii) $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} \bar{c}_n < \infty$.

Then, a mapping S_n defined by (1.8) has a unique fixed point $y_n \in C$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $a_n \rightarrow 0$. In this case, $\{y_n\}$ converges strongly as $a_n \rightarrow 0$ to a fixed point of T .

Proof. By the Banach contraction principle there exists a unique fixed point y_n of S_n in C such that

$$y_n = a_n u + b_n PT[\bar{a}_n y_n + \bar{b}_n PT y_n + \bar{c}_n u_n] + c_n v_n.$$

Let w be a fixed point of T . Then as in the proof of Theorem 3.1, $\{y_n\}$ is bounded. Conversely, suppose that $\{y_n\}$ is bounded. Apply Theorem 3.1, we obtain that $\{y_n\}$ converges strongly to a fixed point z of PT . Next, let us show that $z \in F(T)$. Since $z = PTz$ and P is a sunny nonexpansive retraction of E onto C , it follows by [13] that

$$\langle Tz - z, J(z - v) \rangle \geq 0 \text{ for all } v \in C.$$

On the other hand, Tz belongs to the closure of $I_c(z)$ by the weak inwardness conditions. Hence for each integer $n \geq 1$, there exists $z_n \in C$ and $a_n \geq 0$ such that the sequence

$$r_n := z + a_n(z_n - z) \longrightarrow Tz.$$

Thus it follows that

$$\begin{aligned} 0 &\leq a_n \langle Tz - z, J(z - z_n) \rangle \\ &= \langle Tz - z, J\{a_n(z - z_n)\} \rangle \\ &= \langle Tz - z, J(z - r_n) \rangle \longrightarrow \langle Tz - z, J(z - Tz) \rangle \\ &= -\|Tz - z\|^2. \end{aligned}$$

Hence we have $Tz = z$. □

From Theorem 3.2, we are able to derive the following corollaries.

Corollary 3.3. [6, Theorem 2] Let E be a uniformly convex Banach space with a uniformly *Gâteaux* differentiable norm, C a nonempty closed convex subset of E , and $T : C \longrightarrow E$ a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E and that for some $u \in C$, $\{\alpha_n\}_{n=1}^\infty \subseteq (0, 1)$. Then, a mapping S_n given by $S_n x = (1 - \alpha_n)u + PTx$ for all $x \in C$ has a unique fixed point $y_n \in C$ such that $y_n = (1 - \alpha_n)u + \alpha_n PTy_n$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $\alpha_n \longrightarrow 1$. In this case, $\{y_n\}$ converges strongly as $\alpha_n \longrightarrow 1$ to a fixed point of T .

Theorem 3.4. Let E be a uniformly convex Banach space with a uniformly *Gâteaux* differentiable norm. Let C be a nonempty closed convex subset of E , and $T : C \longrightarrow E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E and that for some $u \in C$, let $\{u_n\}$ and $\{v_n\}$ be bounded sequences on C and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a}_n\}$, $\{\bar{b}_n\}$, and $\{\bar{c}_n\}$ be real sequences on $[0, 1]$ satisfying the conditions:

- (i) $a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1$,
- (ii) $0 \leq b_n \leq \beta < 1, \bar{b}_n \leq \beta < 1, \forall n \geq 1$,
- (iii) $\sum_{n=1}^\infty c_n < \infty, \sum_{n=1}^\infty \bar{c}_n < \infty$.

Then, a mapping U_n defined by (1.9) has a unique fixed point $z_n \in C$. Further, T has a fixed point if and only if $\{z_n\}$ remains bounded as $a_n \rightarrow 0$. In this case, $\{z_n\}$ converges strongly as $a_n \rightarrow 0$ to a fixed point of T .

Proof. It follows by the Banach contraction principle that there exists a unique fixed point z_n of U_n such that

$$z_n = P[a_n u + b_n T P[\bar{a}_n z_n + \bar{b}_n T z_n + \bar{c}_n u_n] + c_n v_n].$$

Let $w \in F(T)$, $M = \max\{\sup \|w - u_n\|, \sup \|w - v_n\| + \|w - u\|\}$. Then for each $n \geq 1$, we have

$$\begin{aligned} \|w - z_n\| &= \|Pw - P[a_n u + b_n T P[\bar{a}_n z_n + \bar{b}_n T z_n + \bar{c}_n u_n] + c_n v_n]\| \\ &\leq \|w - a_n u - b_n T P[\bar{a}_n z_n + \bar{b}_n T z_n + \bar{c}_n u_n] - c_n v_n\| \\ &\leq a_n \|w - u\| + b_n \|w - T P[\bar{a}_n z_n + \bar{b}_n T z_n + \bar{c}_n u_n]\| + c_n \|w - v_n\| \\ &\leq a_n \|w - u\| + b_n \bar{a}_n \|w - z_n\| + b_n \bar{b}_n \|w - T z_n\| + b_n \bar{c}_n \|w - u_n\| + c_n \|w - v_n\| \\ &\leq a_n M + b_n (\bar{a}_n + \bar{b}_n) \|w - z_n\| + b_n M + c_n M \\ &\leq M + b_n \|w - z_n\| \\ &\leq M + \beta \|w - z_n\|. \end{aligned}$$

This implies $\|w - z_n\| \leq \frac{M}{1-\beta}$, $\forall n \geq 1$. Then $\{z_n\}$ is bounded. Conversely, suppose that $\{z_n\}$ is bounded as $a_n \rightarrow 0$. To show that $F(T) \neq \emptyset$. For any subsequence $\{z_{n_i}\}$ of the sequence $\{z_n\}$ such that $a_{n_i} \rightarrow 0$, we can define a real valued function g on C given by

$$g(z) = \mu_i \|z_{n_i} - z\| \text{ for every } z \in C, \quad (3.5)$$

where μ is a Banach limit. Define the set

$$N = \{v \in C : g(v) = \min_{z \in C} g(z)\}. \quad (3.6)$$

Then N is nonempty, closed and convex. As in the proof of Theorem 3.1, N is PT -invariant. Hence, by Kirk's theorem [7], we have a fixed point y of PT in N . Thus $y = PTy$. So from [13],

$$\langle Ty - y, J(y - v) \rangle \geq 0 \text{ for all } v \in C.$$

Since T is weak inwardness condition, we get Ty belong to the closure of $I_c(y)$. Then for each integer $n \geq 1$, there exists $w_n \in C$ and $d_n \geq 0$ such that the sequence

$$r_n := y + d_n(w_n - y) \rightarrow Ty.$$

As in the proof of Theorem 3.2, we have $Ty = y$. For any $w \in F(T)$ and for all $n \geq 1$, we have

$$\begin{aligned}(b_n + c_n)(w - u) + u &= a_n u + (b_n + c_n)w \\ &= P(a_n u + (b_n + c_n)w).\end{aligned}$$

This implies that

$$\begin{aligned}\|z_n - u - (b_n + c_n)(w - u)\|^2 &= \|P[a_n u + b_n TP(\bar{a}_n z_n + \bar{b}_n Tz_n + \bar{c}_n u_n) + c_n v_n] \\ &\quad - P(a_n u + (b_n + c_n)w)\|^2 \\ &\leq \|(a_n u + b_n TP(\bar{a}_n z_n + \bar{b}_n Tz_n + \bar{c}_n u_n) \\ &\quad + c_n v_n - a_n u - (b_n + c_n)w)\|^2 \\ &\leq (b_n \|TP(\bar{a}_n z_n + \bar{b}_n Tz_n + \bar{c}_n u_n) - w\| \\ &\quad + c_n \|v_n - w\|)^2 \\ &\leq (b_n \|\bar{a}_n z_n + \bar{b}_n Tz_n + \bar{c}_n u_n - w\| + c_n \|v_n - w\|)^2 \\ &\leq (b_n \bar{a}_n \|z_n - w\| + b_n \bar{b}_n \|Tz_n - w\| \\ &\quad + b_n \bar{c}_n \|u_n - w\| + c_n \|v_n - w\|)^2 \\ &\leq (b_n (\bar{a}_n + \bar{b}_n) \|z_n - w\| + b_n \bar{c}_n \|u_n - w\| \\ &\quad + c_n \|v_n - w\|)^2 \\ &\leq (b_n \|z_n - w\| + b_n \bar{c}_n \|u_n - w\| + c_n \|v_n - w\|)^2 \\ &= b_n^2 \|z_n - w\|^2 + 2b_n^2 \bar{c}_n \|z_n - w\| \|u_n - w\| \\ &\quad + 2b_n c_n \|v_n - w\| \|z_n - w\| + b_n^2 \bar{c}_n^2 \|u_n - w\|^2 \\ &\quad + 2b_n c_n \bar{c}_n \|u_n - w\| \|v_n - w\| + c_n^2 \|v_n - w\|^2.\end{aligned}$$

Hence for all $n \geq 1$, we have

$$\begin{aligned}&2\langle (1 - b_n)(z_n - u) - c_n(w - u), J(b_n(z_n - w)) \rangle \\ &\leq \|z_n - u - (b_n + c_n)(w - u)\|^2 - \|b_n(z_n - u) - b_n(w - u)\|^2 \\ &= \|z_n - u - (b_n + c_n)(w - u)\|^2 - b_n^2 \|z_n - w\|^2 \\ &\leq 2b_n^2 \bar{c}_n \|z_n - w\| \|u_n - w\| + 2b_n c_n \|v_n - w\| \|z_n - w\| + 2b_n c_n \bar{c}_n \|u_n - w\| \|v_n - w\| \\ &\quad + b_n^2 \bar{c}_n^2 \|u_n - w\|^2 + c_n^2 \|v_n - w\|^2\end{aligned}$$

Since

$$\begin{aligned}2\langle (1 - b_n)(z_n - u) - c_n(w - u), J(b_n(z_n - w)) \rangle &= 2(1 - b_n)b_n \langle z_n - u, J(z_n - w) \rangle \\ &\quad - 2c_n b_n \langle w - u, J(z_n - w) \rangle\end{aligned}$$

it follow that

$$\begin{aligned} \langle z_n - u, J(z_n - w) \rangle &\leq \frac{b_n}{1 - b_n} \bar{c}_n \|z_n - w\| \|u_n - w\| + \frac{c_n}{1 - b_n} \|v_n - w\| \|z_n - w\| \\ &\quad + \frac{\bar{c}_n \bar{c}_n}{1 - b_n} \|u_n - w\| \|v_n - w\| + \frac{b_n \bar{c}_n^2}{2(1 - b_n)} \|u_n - w\|^2 \\ &\quad + \frac{c_n^2}{2(1 - b_n)b_n} \|v_n - w\|^2 + \frac{c_n}{1 - b_n} \|w - u\| \|z_n - w\| \end{aligned}$$

and hence

$$\mu_i \langle z_{n_i} - u, J(z_{n_i} - w) \rangle \leq 0. \quad (3.7)$$

Then, putting $w=y$, we have

$$\mu_i \langle z_{n_i} - u, J(z_{n_i} - y) \rangle \leq 0 \quad (3.8)$$

On the other hand, since

$$g(y) = \min_{z \in C} g(z),$$

it follows by Lemma 2.1, that

$$\mu_i \langle z - y, J(z_{n_i} - y) \rangle \leq 0 \quad \text{for all } z \in C.$$

Putting $z=u$, we have

$$\mu_i \langle u - y, J(z_{n_i} - y) \rangle \leq 0. \quad (3.9)$$

As in the proof of Theorem 3.1, from (3.7), (3.8), and (3.9), we have that $\{z_n\}$ converges strongly as $a_n \rightarrow 0$ to a fixed point of T . $\square$

Corollary 3.5. [16, Theorem 2] Let H be a Hilbert space, C a nonempty closed convex subset of H , $T : C \rightarrow H$ a nonexpansive nonself-mapping satisfying the weak inwardness condition, $P : H \rightarrow C$ the nearest projection. If $\{\alpha_n\} \subseteq (0, 1)$, then a mapping U_n defined by (1.4) has unique fixed point $z_n \in C$. Further, T has a fixed point if and only if $\{z_n\}$ is bounded as $\alpha_n \rightarrow 1$. In this case, $\{z_n\}$ converges strongly as $\alpha_n \rightarrow 1$ to a fixed point of T .

Acknowledgement. The authors would like to thanks The Thailand Research Fund for financial support.

REFERENCES

1. V. Barbu and T. Precupanu, Convexity and Optimization in Banach Space, Editura. Acadimiei. R.S.R., Bucharest, (1978).
2. F. E. Browder, Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces, Archs. Ratio. Mech. Anal., 24 (1967), 80-90.
3. M. M. Day, Normed Linear Spaces, 3rd ed. Springer-Verlag, Berlin, (1973).

4. J. Diestel, *Geometry of Banach Spaces-Selected Topics*, Lecture Notes in Math., Vol. 485. Springer-Verlag, Berlin, (1975).
5. B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc., **73** (1967), 957-961.
6. J.S. Jung and S.S. Kim, *Strong convergence theorems for nonexpansive nonself-mappings in Banach spaces*, Nonlinear Anal., **33**(3) (1998), 321-329.
7. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004-1006.
8. S. Kitahara and W. Takahashi, *Image recovery by convex combinations of sunny nonexpansive retractions*, Nonlinear Anal., **2** (1993), 333-342.
9. G. Marino and G. Trombetta, *On approximating fixed points for nonexpansive maps*, Indian J. Math., **34** (1992), 91-98.
10. S. Reich, *Product formula, nonlinear semigroups, and accretive operators*, J. Funct. Anal., **36** (1980), 147-168.
11. S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach space*, J. Math. Anal. Appl., **75** (1980), 287-292.
12. S. P. Singh and B. Watson, *On approximating fixed points*, Proc. Symp. Pure Math., **45** (1986), 393-395.
13. W. Takahashi, *Nonlinear Functional Analysis*. Yokohama Publishers, Japan, (2000).
14. W. Takahashi and G. E. Kim, *Strong convergence of approximants to fixed points of nonexpansive nonself-mappings*, Nonlinear Anal., **32** (1998), 447-454.
15. W. Takahashi and Y. Ueda, *On Reich's strong convergence theorems for resolvents of accretive operators*, J. Math. Anal. Appl., **104** (1984), 546-553.
16. H.K. Xu and X. M. Yin, *Strong convergence theorems for nonexpansive nonself-mappings*, Nonlinear Anal., **24** (1995), 223-228.

ภาคผนวก 12/15

Somyot Plubtieng

From: "Prof Yuqun Chen" <yqchen@scnu.edu.cn>
To: "Somyot Plubtieng" <somyotp@nu.ac.th>
Cc: "kpshum" <kpshum@scnu.edu.cn>
Sent: Monday, January 31, 2005 5:57 PM
Subject: Re: submit paper(change to generalized results)

That is ok.

----- Original Message -----
From: Somyot Plubtieng
To: yqchen@scnu.edu.cn
Sent: Monday, January 31, 2005 2:42 PM
Subject: Fw: submit paper(change to generalized results)

----- Original Message -----
From: Somyot Plubtieng
To: seams
Sent: Monday, January 31, 2005 10:31 AM
Subject: Re: submit paper(change to generalized results)

Dear Professor Yuqun Chen,

According my paper entitled "Implicit iteration process of nonexpansive nonself-mappings in Banach spaces" which submitted for publication in the Southeast Asian Bulletin of Mathematics. Now, I get the generalized result of that paper and I think that it is a good way if change the manuscript. So, I was send the pdf.file of our paper entitled "Implicit iteration process with error of nonexpansive nonself-mappings in Banach spaces". I hope that you will be consider it.
 Thank you very much for your kind arrangement.

----- Original Message -----
From: seams
To: Somyot Plubtieng
Cc: Yuqun Chen
Sent: Tuesday, January 25, 2005 3:57 PM
Subject: Re: submit paper

Dear Professor Somyot Plubtieng,

This is to acknowledge that I have received your paper "Implicit iteration process of nonexpansive nonself-mappings in Banach spaces" for possible publication in the Southeast Asian Bulletin of Mathematics. I have forwarded your paper to referees. I will inform you the outcome when I receive the referee report. Usually, it will take 12 weeks to get an answer from the referees.

Thanks for your attention to our journal.

With My Best wishes,

ภาคผนวก 12/16

Prof Yuqun Chen (陈裕群)
Managing Editor of SEA Bull Math
Department of Mathematics
South China Normal University
Guangzhou 510631
China
Tel: 86-20-85216957
Fax: 86-20-85216705
Email: yqchen@scnu.edu.cn

Website of SEA Bull Math: <http://www.scnu.edu.cn/seam-bulletin/>

----- Original Message -----

From: Somyot Plubtieng
To: seams@scnu.edu.cn
Sent: Monday, January 10, 2005 9:44 AM
Subject: Fw: submit paper

----- Original Message -----

From: Somyot Plubtieng
To: seams@scnu.edu.cn
Sent: Thursday, January 06, 2005 8:52 AM
Subject: submit paper

The Editor,
Southeast Asian Bulletin of Mathematics,
Department of Mathematics,
South China Normal University,
Guangzhou 510631,
CHINA

January 6, 2004

Dear the editor;

Enclosed please find the files (pdf.file) of my paper with Miss. Rattanaorn Punpaeng entitled;

Implicit iteration process of nonexpansive nonself-mappings in Banach spaces

which I would like to submit for publication in the Southeast Asian Bulletin of Mathematics.

I would like to thank you in advance for your consideration.

ภาคผนวก 13

**Strong convergence theorems of an implicit
iteration process with errors for asymptotically
nonexpansive mappings**

A. Kangtunyakarn and S. Plubtieng

Acta. Scient. Math. (submitted).

STRONG CONVERGE THEOREM FOR ASYMPTOTICALLY NONEXPANSIVE MAPPING WITH ERROR TERM

SOMYOT PLUBTIENG AND ATID KANGTUNYAKARN

*Department of Mathematics, Naresuan University Pitsanulok 65000,
Thailand*

ABSTRACT. Let C a nonempty closed convex subset of a Hilbert space H , T be an asymptotically nonexpansive self-mapping of C into itself. In this paper, we study the convergence of a sequence $\{x_n\}$ generated by

$$x_n = \alpha_n x_0 + \beta_n A_n x_n + \gamma_n v_n$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $[0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\alpha_n = \frac{b_n-1}{b_n-1+a}$ and $\gamma_n < \frac{(b_n-1)^2}{(b_n-1)^2+a}$ for all $n \geq 1$, v_n is bounded sequence in C , and $A_n = \frac{1}{n} \sum_{j=1}^n T^j$, $0 < a' < 1$, and $0 < a < 1$.

Keywords: Fixed point, asymptotically nonexpansive mapping, strong convergence.

2000 Mathematics Subject Classification: 47H10, 47H09, 46C05.

1. INTRODUCTION

Let C be a closed convex subset of Hilbert space H and let T be a mapping of C into itself. Then, T is said to be a *Lipchitzian mapping* if for each $n \geq 1$ there exist a positive real number k_n such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$. A Lipchitzian mapping is said to be *nonexpansive* if $k_n = 1$ for all $n \geq 1$ and *asymptotically nonexpansive* [3] if $\lim_{n \rightarrow \infty} k_n = 1$, respectively. We denote by $F(T)$ the set of fixed points of T . It is well know that if T is asymptotically nonexpansive, then $F(T)$ is convex: see [5]. In 1967, Browder [2] proved that

Corresponding author.

Email addresses: Somyotp@nu.ac.th(Somyot Plubtieng) and g46060090@nu.ac.th.
(Atid kangtunyakarn).

the following strong convergence theorem for nonexpensive mapping of a bounded a closed convex subset of H into itself. Let $x_0 \in C$, and for each k with $0 < k < 1$, let $T_k x = kx_0 + (1 - k)Tx$. Then, T_k has a unique fixed point u_k in C , and u_k converges as $k \rightarrow 0$ strongly to a fixed point u_0 of T in C . The fixed point u_0 is uniquely specified as the fixed point of T in C closet to x_0 . After Browder's result, such a problem has been investigated by several authors; see Singh and Watson [8], Marino and Trombetta [6] and other

On the other hand, Baillon [1] proved the first nonlinear ergodic theorem: let C be a bounded closed convex subset of H and let T be a nonexpansive mapping of C into itself. Then for each x in C

$$A_n x = \frac{1}{n} \sum_{k=1}^n T^k x$$

converges weakly to a fixed point of T . Recently, Hirano and Takahashi [4] extended Baillon's theorem to asymptotically nonexpensive mapping. Finally, Shimizu and Takahashi [7] studied a sequence $\{x_n\}$ define by

$$x_n = \alpha_n x_0 + (1 - \alpha_n) A_n x_n$$

where

$$b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j}), \quad \alpha_n = \frac{b_n - 1}{b_n - 1 + a} \quad \text{for } n \geq 1$$

and $0 < a < 1$. They also proved that the sequence $\{x_n\}$ converges strongly to element of $F(T)$ which is nearest to x_0 .

In this paper, we extend Shimizu and Takahashi's result [7] to study the sequence $\{x_n\}$ defined by

$$x_n = \alpha_n x_0 + \beta_n A_n x_n + \gamma_n v_n \quad (1.1)$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\alpha_n = \frac{b_n - 1}{b_n - 1 + a}$ and $\gamma_n < \frac{(b_n - 1)^2}{(b_n - 1)^2 + a'}$ for all $n \geq 1$, v_n is bounded sequence in C , and $A_n = \frac{1}{n} \sum_{j=1}^n T^j$, $0 < a' < 1$, and $0 < a < 1$. Moreover, we prove the strong converges theorem of a sequence $\{x_n\}$ given by (1.1).

2. LEMMAS

To prove the main theorem we need the following lemmas.

Lemma 2.1 For each $n \geq 1$, T_n has unique fixed point u_n in C .

Proof. Let $x, y \in C$, we have

$$\begin{aligned}
 \|T_n x - T_n y\| &= \|\alpha_n x_0 + \beta_n (\frac{1}{n} \sum_{j=1}^n T^j x) + \gamma_n v_n - \alpha_n x_0 - \beta_n (\frac{1}{n} \sum_{j=1}^n T^j y) + \gamma_n v_n\| \\
 &\leq \beta_n \frac{1}{n} \sum_{j=1}^n \|T^j x - T^j y\| \\
 &\leq \beta_n \frac{1}{n} \sum_{j=1}^n k_j \|x - y\| \\
 &\leq \beta_n b_n \|x - y\| \\
 &= (1 - \alpha_n - \gamma_n) b_n \|x - y\| \\
 &\leq (1 - \alpha_n) b_n \|x - y\|, \quad \forall n \geq 1.
 \end{aligned}$$

Since $0 < (1 - \alpha_n) b_n < 1$, it follows that T_n is contraction mapping on C . Hence by Banach contraction principle, T_n has unique fixed point u_n . $\square$

Lemma 2.2 If $F(T)$ is nonempty, u_n is bounded.

Proof. Let $v \in F(T)$ and $\{u_n\}$ is fixed point of T_n . Thus, we have

$$\begin{aligned}
 \|u_n - v\| &= \|T_n u_n - v\| \\
 &= \|\alpha_n x_0 + \beta_n (\frac{1}{n} \sum_{j=1}^n T^j u_n) + \gamma_n v_n - v\| \\
 &= \|\alpha_n x_0 + \beta_n (\frac{1}{n} \sum_{j=1}^n T^j u_n) + \gamma_n v_n - (\alpha_n + \beta_n + \gamma_n) v\| \\
 &= \|\alpha_n (x_0 - v) + \beta_n (\frac{1}{n} \sum_{j=1}^n T^j u_n) - v + \gamma_n (v_n - v)\| \\
 &= \|\alpha_n (x_0 - v) + \beta_n [\frac{1}{n} \sum_{j=1}^n T^j u_n - \frac{1}{n} \sum_{j=1}^n T^j v] + \gamma_n (v_n - v)\| \\
 &= \|\alpha_n (x_0 - v) + \beta_n \frac{1}{n} \sum_{j=1}^n (T^j u_n - T^j v) + \gamma_n (v_n - v)\| \\
 &\leq \alpha_n \|x_0 - v\| + \beta_n \frac{1}{n} \sum_{j=1}^n \|T^j u_n - T^j v\| + \gamma_n \|v_n - v\| \\
 &\leq \alpha_n \|x_0 - v\| + \beta_n \frac{1}{n} \sum_{j=1}^n k_j \|u_n - v\| + \gamma_n \|v_n - v\| \\
 &\leq \alpha_n \|x_0 - v\| + \beta_n b_n \|u_n - v\| + \gamma_n \|v_n - v\|.
 \end{aligned}$$

This implies that

$$(1 - \beta_n b_n) \|u_n - v\| \leq \alpha_n \|x_0 - v\| + \gamma_n \|v_n - v\|.$$

ภาคผนวก 13/4

4

S. PLUBTIENG AND A. KANGTUNYAKARN

Since $(1 - \beta_n b_n) = [1 - (1 - \alpha_n - \gamma_n)b_n] \geq [1 - (1 - \alpha_n)b_n] > 0$.

Thus

$$\|u_n - v\| \leq \frac{\alpha_n \|x_0 - v\|}{(1 - (1 - \alpha_n)b_n)} + \frac{\gamma_n \|v_n - v\|}{(1 - (1 - \alpha_n)b_n)}.$$

Since $[1 - (1 - \alpha_n)b_n] = \alpha_n(1 - a)$, and $(1 - \alpha_n)b_n = \frac{ab_n}{b_n - 1 + a}$.

It follow that

$$\begin{aligned} \|u_n - v\| &\leq \frac{\|x_0 - v\|}{(1 - a)} + \frac{\gamma_n \|v_n - v\|}{[1 - \frac{ab_n}{b_n - 1 + a}]} \\ &\leq \frac{\|x_0 - v\|}{(1 - a)} + \frac{\gamma_n \|v_n - v\| b_n}{(b_n - 1)(1 - a)}. \end{aligned}$$

Since

$$\gamma_n < \frac{(b_n - 1)^2}{(b_n - 1)^2 + a'}$$

This implies that

$$\begin{aligned} \|u_n - v\| &\leq \frac{\|x_0 - v\|}{(1 - a)} + \frac{(b_n - 1)^2 \|v_n - v\| b_n}{[(b_n - 1)^2 + a'](b_n - 1)(1 - a)} \\ &= \frac{\|x_0 - v\|}{(1 - a)} + \frac{(b_n - 1) \|v_n - v\| b_n}{[(b_n - 1)^2 + a'](1 - a)}. \end{aligned}$$

Since $b_n \rightarrow 1$ as $n \rightarrow \infty$, $\{u_n\}$ is bounded. □

Lemma 2.3. [7] Let B be a nonempty bounded subset of C and let $F(T)$ be nonempty. Then, for a positive number ϵ , There exist a positive integer l_ϵ such that for any integer $l \geq l_\epsilon$, there is a positive integer n_l satisfying

$$\|A_n x - T^l A_n x\| < \epsilon \text{ for all } x \in B \text{ and } n \geq n_l.$$

Lemma 2.4. [7] Let $\{x_n\}$ be a sequence in C and let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $x \in C$ and $\{x_{n_i} - (\frac{1}{n_i}) \sum_{j=1}^{n_i} T^j x_{n_i}\}$ converges strong to 0. Then, x is a fixed point of T .

3. MAIN RESULTS

Now, we have a strong convergence theorem for asymptotically nonexpansive mappings in Hilbert space.

Theorem 3.1 Let C be a closed convex subset of Hilbert space H and T be asymptotically nonexpensive mapping on C into itself with Lipschitz condition k_n and suppose that $F(T)$ is nonempty.

Let

$$b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j})$$

$$0 < a < 1 \text{ and } x_0 \in C,$$

and let $T_n : C \rightarrow C$ be a mapping given by

$$T_n x = \alpha_n x_0 + \beta_n A_n x + \gamma_n v_n \quad ; \forall x \in C, v_n \in C,$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\alpha_n = \frac{b_n-1}{b_n-1+a}$ and $\gamma_n < \frac{(b_n-1)^2}{(b_n-1)^2+a}$ for all $n \geq 1$, v_n is bounded sequence in C , and $A_n = \frac{1}{n} \sum_{j=1}^n T^j$, $0 < a' < 1$, and $0 < a < 1$. Further $\{u_n\}$ converges strongly to the element of $F(T)$ which nearest to x_0 .

Proof. By Lemma 2.1, T_n has a unique fixed point u_n in C so we show that $\{u_n\}$ converges strongly to element of $F(T)$ which nearest to x_0 . Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$. it is sufficient to find a subsequence $\{u_{m_i}\}$ of $\{u_{n_i}\}$ converges strongly to element of $F(T)$ which nearest to x_0 .

Suppose a subsequence $\{u_{m_i}\}$ of $\{u_{n_i}\}$ converges weakly to v in C . Since $F(T)$ is nonempty closed convex, there exists the element $u_0 \in F(T)$ which nearest to x_0 . By Lemma 2.2, $\{u_{m_i}\}$ is bounded so is $\frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}$.

We now show that

$$u_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty.$$

We note that

$$\begin{aligned} u_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} &= T_{m_i} u_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} \\ &= \alpha_{m_i} x_0 + \beta_{m_i} \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} + \gamma_{m_i} v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} \\ &= \alpha_{m_i} x_0 + (1 - \alpha_{m_i} - \gamma_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} + \gamma_{m_i} v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} \\ &= \alpha_{m_i} x_0 + \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} - \alpha_{m_i} \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} \\ &\quad - \gamma_{m_i} \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} + \gamma_{m_i} v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} \\ &= \alpha_{m_i} (x_0 - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}) + \gamma_{m_i} (v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}) \rightarrow 0 \text{ as } i \rightarrow \infty \end{aligned}$$

since $\alpha_{m_i} \rightarrow 0$ and $\gamma_{m_i} \rightarrow 0$ as $i \rightarrow \infty$. We get

$$u_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} \rightarrow 0.$$

Hence, by Lemma 2.4, v is a fixed point of T . On the other hand, since $T_{m_i} u_{m_i} = u_{m_i}$ and $T u_0 = u_0$, we have

$$u_{m_i} = \alpha_{m_i} x_0 + \beta_{m_i} \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} + \gamma_{m_i} v_{m_i},$$

and so

$$\alpha_{m_i} x_0 = u_{m_i} - \beta_{m_i} \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} - \gamma_{m_i} v_{m_i}.$$

Thus, we have

$$\begin{aligned}
 \alpha_{m_i} \langle x_0 - u_0, u_{m_i} - u_0 \rangle &= \langle \alpha_{m_i} x_0 - \alpha_{m_i} u_0, u_{m_i} - u_0 \rangle \\
 &= \langle u_{m_i} - \beta_{m_i} \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} - \gamma_{m_i} v_{m_i} - u_0 + (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_0, u_{m_i} - u_0 \rangle \\
 &= \langle u_{m_i} - (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} + \gamma_{m_i} \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i} - \gamma_{m_i} v_{m_i} - u_0 \\
 &\quad + (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_0, u_{m_i} - u_0 \rangle \\
 &= \langle u_{m_i} - u_0, u_{m_i} - u_0 \rangle - (1 - \alpha_{m_i}) \langle \frac{1}{m_i} \sum_{j=1}^{m_i} (T^j u_{m_i} - T^j u_0), u_{m_i} - u_0 \rangle \\
 &\quad - \gamma_{m_i} \langle v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}, u_{m_i} - u_0 \rangle \\
 &\geq \|u_{m_i} - u_0\|^2 - (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} \|T^j u_{m_i} - T^j u_0\| \|u_{m_i} - u_0\| \\
 &\quad - \gamma_{m_i} \|v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}\| \|u_{m_i} - u_0\| \\
 &\geq \|u_{m_i} - u_0\|^2 - (1 - \alpha_{m_i}) \frac{1}{m_i} \sum_{j=1}^{m_i} k_j \|u_{m_i} - u_0\|^2 - \gamma_{m_i} \|v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}\| \\
 &\quad \|u_{m_i} - u_0\| \\
 &\geq \|u_{m_i} - u_0\|^2 - (1 - \alpha_{m_i}) b_{m_i} \|u_{m_i} - u_0\|^2 - \gamma_{m_i} \|v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}\| \\
 &\quad \|u_{m_i} - u_0\| \\
 &= [1 - (1 - \alpha_{m_i}) b_{m_i}] \|u_{m_i} - u_0\|^2 - \gamma_{m_i} \|v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}\| \|u_{m_i} - u_0\|.
 \end{aligned}$$

Since $[1 - (1 - \alpha_{m_i}) b_{m_i}] = \alpha_{m_i} (1 - a)$. It follow that

$$\begin{aligned}
 \|u_{m_i} - u_0\|^2 &\leq \frac{\alpha_{m_i} \langle x_0 - u_0, u_{m_i} - u_0 \rangle}{\alpha_{m_i} (1 - a)} + \frac{\gamma_{m_i} \|v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}\| \|u_{m_i} - u_0\|}{\alpha_{m_i} (1 - a)} \\
 &= \frac{\langle x_0 - u_0, u_{m_i} - u_0 \rangle}{(1 - a)} + \frac{\gamma_{m_i} \|v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}\| \|u_{m_i} - u_0\|}{[1 - (1 - \alpha_{m_i}) b_{m_i}]}.
 \end{aligned}$$

Since $(1 - \alpha_{m_i}) b_{m_i} = \frac{ab_{m_i}}{b_{m_i} - 1 + a}$. It implies that

$$\begin{aligned}
 \|u_{m_i} - u_0\|^2 &\leq \frac{\langle x_0 - u_0, u_{m_i} - u_0 \rangle}{(1 - a)} + \frac{\gamma_{m_i} \|v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}\| \|u_{m_i} - u_0\|}{1 - \frac{ab_{m_i}}{b_{m_i} - 1 + a}} \\
 &\leq \frac{\langle x_0 - u_0, u_{m_i} - u_0 \rangle}{(1 - a)} + \frac{\gamma_{m_i} \|v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}\| \|u_{m_i} - u_0\| b_{m_i}}{b_{m_i} - 1 + a - ab_{m_i}}.
 \end{aligned}$$

Since u_0 is the element of $F(T)$ with is nearest to x_0 , it follow by [9, Lemma 3.1.3] that

$$\begin{aligned}\langle x_0 - u_0, u_{m_i} - u_0 \rangle &= \langle x_0 - u_0, v - u_0 \rangle + \langle x_0 - u_0, u_{m_i} - v \rangle \\ &\leq \langle x_0 - u_0, u_{m_i} - v \rangle.\end{aligned}$$

It implies that

$$\begin{aligned}\|u_{m_i} - u_0\|^2 &\leq \frac{\langle x_0 - u_0, u_{m_i} - v \rangle}{(1-a)} + \frac{\gamma_{m_i} \|v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}\| \|u_{m_i} - u_0\| b_{m_i}}{(b_{m_i} - 1)(1-a)} \\ &\leq \frac{\langle x_0 - u_0, u_{m_i} - v \rangle}{(1-a)} + \frac{(b_{m_i} - 1)^2 \|v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}\| \|u_{m_i} - u_0\| b_{m_i}}{[(b_{m_i} - 1)^2 + a'](b_{m_i} - 1)(1-a)} \\ &= \frac{\langle x_0 - u_0, u_{m_i} - v \rangle}{(1-a)} + \frac{(b_{m_i} - 1) \|v_{m_i} - \frac{1}{m_i} \sum_{j=1}^{m_i} T^j u_{m_i}\| \|u_{m_i} - u_0\| b_{m_i}}{[(b_{m_i} - 1)^2 + a'](1-a)}.\end{aligned}$$

Since u_{m_i} converges weakly to v , we get $\langle x_0 - u_0, u_{m_i} - v \rangle \rightarrow 0$ and $b_{m_i} \rightarrow 1$ as $i \rightarrow \infty$. Hence $\{u_{m_i}\}$ is converges strongly to u_0 . This complete the proof.

Corollary 3.2 [7] Let C be a closed convex subset of Hilbert space H and T be asymptotically nonexpensive mapping on C into itself with Lipschitz condition k_n and suppose that $F(T)$ is nonempty.

Let

$$b_n = \frac{1}{n} \sum_{j=1}^n (1 + |1 - k_j| + e^{-j}) \quad 0 < a < 1 \text{ and } x_0 \in C.$$

Then a mapping T_n on C given by

$$T_n x = \alpha_n x_0 + (1 - \alpha_n) A_n x \quad ; \forall x \in C$$

has unique fixed point u_n in C , when

$$\alpha_n = \frac{b_n - 1}{b_n - 1 + a} \text{ and } A_n = \frac{1}{n} \sum_{j=1}^n T^j.$$

Further $\{u_n\}$ converges strongly to the element of $F(T)$ which nearest to x_0 .

Acknowledgement, The authors would like to thanks the Thailand Research fund for financial support.

REFERENCES

1. BAILLON J. B., Un theoreme de type ergodique pour les contraction non lineares dans un espace de Hilbert, *C. r. Acad. Sci. Paris.* **280**, 1511-1514 (1975).
2. BROWDER F. E., Convergen of approximates to fixed points of nonexpansive nonlinear mapping in Banach space, *Archs ration. Math. Analysis.* **24**, 82-90 (1967).
3. GOEBEL K. & KIRK W. A., A fixed point theorem for asymptotically nonexpansive mapping, *Proc. Am. math. Soc.* **35**, 171-174 (1972).
4. HIRANO N. & TAKAHASHI W., Nonlinear ergodic theorems for nonexpansive mappings in Hilbert space, *Kodai math. J.* **2**, 11-25 (1979).
5. ISHIHARA H. & TAKAHASHI W., A nonlinear erodic theorem for a reversible semigroup of Lipschitzian mapping in Hilbert space, *Proc. Am. math. Soc.* **104**, 431-436 (1988).

ภาคผนวก 13/8

8

S. PLUBTIENG AND A. KANGTUNYAKARN

6. MARINO G. & TROMBETTA G., On approximating fixed point for nonexpansive maps. *Indian J. Math.* **34**, 91-98 (1992).
7. SHIMIZU T. & TAKAHASHI W., Strong converge theorem for asymptotically nonexpansive mappings. *Nonlinear Analysis.* **26**, 265-272 (1996).
8. SINGH S.P. & WATSON B. On approximating fixed points, *Proc. Symp. Pure Math.* **45** (part 2) 393-395 (1986).
9. TAKAHASHI, W. *Nonlinear function Functional analysis.* Japan: Yokohama (2000).
10. TAN K. K. & XU H. K., The nonlinear theorem for asymptotically nonexpansive mappings in Banach space, *Proc. Am. math. soc.* **114**, 399-404 (1992).

From: "Laszlo Kerchy" <kerchy@math.tamu.edu>
To: <sornyotp@nu.ac.th>; "Laszlo Imre Szabo" <lszabo@math.u-szeged.hu>
Sent: Thursday, March 31, 2005 2:21 AM
Subject: acknowledgement of receipt

Thanks for your paper submitted to our journal.

L. Kerchy
Bolyai Institute
University of Szeged
Aradi vertanuk tere 1
6720 Szeged, Hungary.

Yours sincerely,
Laszlo Kerchy
Editor-in-Chief

> March 30, 2005

 $\geq *$

- > STRONG CONVERGENCE OF AN IMPLICIT ITERATION PROCESS WITH ERROR FOR
- >
- > ASYMPTOTICALLY NONEXPANSIVE MAPPINGS
- >

> which I would like to submit for publication in the Acta Scientiarum
> Mathematicarum.

>

> I would like to thank you in advance for your consideration.

>

>

>

> Your sincerely,

>

>

>

>

>

> Somyot Plubtieng

>

> Associate Professor

>

> Department of Mathematics

>

> Naresuan University

>

> Phitsanulok 65000

>

> THAILAND

>

> E-mail address : somyotp@nu.ac.th <<mailto:somyotp@nu.ac.th>>

>

>

>

> *

ภาคผนวก 14

**Ishikawa Iteration Sequences for Asymptotically
Quasi-Nonexpansive Nonself-Mappings with
Error Members**

S. Plubtieng and R. Wangkeeree

Czechoslovak Math. J. (Submitted).

ISHIKAWA ITERATION SEQUENCE FOR ASYMPTOTICALLY
QUASI-NONEXPANSIVE NONSELF-MAPPINGS
WITH ERROR MEMBER

SOMYOT PLUBTIENG AND RABIAN WANGKEEREE

Department of Mathematics, Naresuan University Pitsanulok 65000,
Thailand.

ABSTRACT. Suppose C is a nonempty closed convex retract of a real uniformly convex Banach space X with P as a nonexpansive retraction of X onto C . Let $T : C \rightarrow X$ be an asymptotically quasi-nonexpansive nonself-mapping with sequence $\{k_n\}_{n \geq 1} \subset [0, \infty)$, $\lim k_n = 0$, $F(T) = \{x \in C : Tx = x\} \neq \emptyset$. Suppose $\{x_n\}_{n \geq 1}$ is generated iteratively by

$$\begin{aligned}x_1 \in C, x_{n+1} &= P((\alpha_n x_n + \beta_n T(PT)^{n-1} y_n + \gamma_n u_n), \\y_n &= P(\alpha'_n x_n + \beta'_n T(PT)^{n-1} x_n + \gamma'_n v_n), n \geq 1\end{aligned}$$

where $\{u_n\}, \{v_n\}$ are bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ are sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ and $0 < \alpha < \alpha_n, \beta_n, \alpha'_n, \beta'_n < \beta < 1$. It is prove that if $\sum_{n=1}^{\infty} k_n < \infty$ and T is completely continuous and uniformly L -Lipschitzian, $\{x_n\}$ strongly converges to some fixed point $x^* \in F(T)$.

keywords: Asymptotically quasi-nonexpansive nonself-maps; Completely continuous; nonexpansive retraction; uniformly convex

1. INTRODUCTION

Let C be a subset of normed space X , and let T be a self-mapping on C . T is said to be *nonexpansive* provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 0$ such that for each $x, y \in C$ and $n \geq 1$, $\|T^n x - T^n y\| \leq$

Corresponding author.

Email addresses: Somyotp@nu.ac.th(Somyot Plubtieng) and Rabian@nu.ac.th.

(Rabian Wangkeeree).

$(1 + k_n)\|x - y\|$. T is said to be an *asymptotically quasi-nonexpansive*, if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 0$ such that

$$(1.1) \quad \|T^n x - x^*\| \leq (1 + k_n)\|x - x^*\|, \forall x \in C, x^* \in F(T),$$

for all $n \geq 1$, ($F(T)$ denotes the set of fixed points of T i.e. $F(T) = \{x \in C : Tx = x\}$). T is said to be an *uniformly L -Lipschitzian*, if there exists a constant $L > 0$ such that for each $x, y \in C$, $\|T^n x - T^n y\| \leq L\|x - y\|$, $\forall n \geq 1$.

From the above definitions, it follows that if $F(T)$ is nonempty then, nonexpansive mapping must be quasi-nonexpansive and an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive. But the converse does not hold.

The concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk [6] in 1972. The iterative approximation problems for nonexpansive mapping asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping were studied extensively by Browder [1, 2], Goebel and Kirk [6], Ghosh and Debnath[7] and Liu[9, 10, 11].

In 1991 J. Schu introduced a modified Mann iteration process to approximate fixed point of asymptotically nonexpansive self-mappings defined on nonempty closed convex and bounded subsets of Hilbert space H . More precisely, he proved the following theorem:

Theorem JS[12]. *Let H be a Hilbert spaces, C closed convex bounded nonempty subset of H . Let $T : C \rightarrow C$ be completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_n^2 + 2k_n) < \infty$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the condition $\epsilon \leq \alpha_n \leq 1 - \epsilon, \forall n \geq 1$ and for some $\epsilon > 0$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$, by*

$$(1.2) \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, n \geq 1,$$

converges strongly to some fixed point of T .

Recently, Chidume, Ofoedu and Zegeye[4] have introduced the class of asymptotically nonexpansive nonself-maps and proved demiclosed principle for such maps. Moreover, they proved the strong and weak convergence theorems of a Mann iteration process for asymptotically nonexpansive nonself-mappings.

It is our purpose in this paper first to introduce the class of asymptotically quasi-nonexpansive nonself-mappings. Moreover, we prove the strong convergence theorem of an Ishikawa iteration sequence with error members for such maps. Our

theorem improve and generalized important related results of Chidume, Ofoedu, and Zegeye[4], and Liu[9].

2. PRELIMINARIES

For the sake of convenience, we first recall some definitions and conclusions.

Definition 2.1 (see [6]). A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta_X(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \epsilon \right\} > 0$$

for all $0 < \epsilon \leq 2$ (i.e., $\delta_X(\epsilon)$ is a function $(0, 2] \rightarrow (0, 1)$).

A subset C of X is called *retract* of X if there exists a continuous mapping $P : X \rightarrow C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \rightarrow C$ is called *retraction* if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P .

Definition 2.2 (see [4]). Let X be a real normed linear space, C a nonempty subset of X . Let $P : X \rightarrow C$ be the nonexpansive retraction of X onto C . A map $T : C \rightarrow X$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 0$ such that the following inequality holds:

$$(2.1) \quad \|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq (1 + k_n)\|x - y\|; \forall x, y \in C, n \geq 1.$$

T is called *uniformly L -Lipschitzian* if there exists a constant $L > 0$ such that:

$$(2.2) \quad \|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|; \forall x, y \in C, n \geq 1.$$

Theorem 2.3 ([4, Theorem 3.7]). Let X be a real uniformly convex Banach space, C closed convex nonempty subset of X . Let $T : C \rightarrow X$ be completely continuous and asymptotically nonexpansive map with sequence $\{k_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_n^2 + 2k_n) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$ be the such that $\epsilon \leq 1 - \alpha_n \leq 1 - \epsilon, \forall n \geq 1$ and some $\epsilon > 0$. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by

$$(2.3) \quad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), n \geq 1.$$

Then $\{x_n\}$ converges strongly to some fixed point of T .

We shall make use of the following lemmas.

Lemma 2.4 ([9, Lemma 2]). *Let the nonnegative real number sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ satisfy that*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \forall n = 1, 2, \dots, \sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} c_n < \infty.$$

Then

(1) $\lim_{n \rightarrow \infty} a_n$ exists;

(2) If $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.5 ([12], J. Schu's Lemma). *Let X be a real uniformly convex Banach space, $0 < \alpha \leq t_n \leq \beta < 1$, $x_n, y_n \in X$, $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$, and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = a$, $a \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

3. MAIN RESULTS

In this section, we give new definitions and prove our main theorems.

Definition 3.1. Let C be a nonempty subset of a Banach space X . A mapping $T : C \rightarrow X$ is said to be *asymptotically quasi-nonexpansive* nonself-map if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 0$ such that:

$$(3.1) \quad \|T(PT)^{n-1}x - x^*\| \leq (1 + k_n)\|x - x^*\|; \forall x \in C, x^* \in F(T), n \geq 1,$$

where P is a nonexpansive retraction of X onto C .

Remark 3.2. If T is a self-map, then $PT = T$, so that (3.1) coincide with (1.1).

Let C be a nonempty closed convex subset of a real uniformly convex Banach space X . The following iteration process is studied:

$$(3.2) \quad \begin{aligned} x_1 \in C, x_{n+1} &= P(\alpha_n x_n + \beta_n T(PT)^{n-1}y_n + \gamma_n u_n), \\ y_n &= P(\alpha'_n x_n + \beta'_n T(PT)^{n-1}x_n + \gamma'_n v_n) \end{aligned}$$

where $\{u_n\}, \{v_n\}$ are bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ are sequences in $[0, 1]$ and P is a nonexpansive retraction of X onto C .

The following lemma is crucial in proving the main Theorem.

Lemma 3.3. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let $T : C \rightarrow X$ be an asymptotically quasi-nonexpansive nonself-mapping of C with sequence $\{k_n\}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and*

$F(T) \neq \emptyset$. Let $x_1 \in C$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ are sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Then the sequence $\{x_n\}$ defined by (3.2) satisfies the following:

- (1) For each $x^* \in F(T)$ and for each $n \geq 1$, we have $\|x_{n+1} - x^*\| \leq (1+k_n)^2 \|x_n - x^*\| + d_n$, where $\{d_n\}$ is a nonnegative sequence with $\sum_{n=1}^{\infty} d_n < \infty$.
- (2) For each $m \geq 1$, there exists a constant $K > 0$ such that $\|x_{n+m} - x^*\| \leq K \|x_n - x^*\| + K \sum_{j=1}^m d_j; \forall x^* \in F(T), n \geq 1$.

Proof. Let $x^* \in F(T)$ and $M = \sup_{n \geq 1} \{\|u_n - x^*\| \vee \|v_n - x^*\|\}$. Then, for each $n \geq 1$, we have

$$\begin{aligned}
 \|x_{n+1} - x^*\| &= \|P(\alpha_n x_n + \beta_n T(PT)^{n-1} y_n + \gamma_n u_n) - Px^*\| \\
 &\leq \|\alpha_n x_n + \beta_n T(PT)^{n-1} y_n + \gamma_n u_n - x^*\| \\
 &\leq \alpha_n \|x_n - x^*\| + \beta_n \|T(PT)^{n-1} y_n - x^*\| + \gamma_n \|u_n - x^*\| \\
 (3.3) \quad &\leq \alpha_n \|x_n - x^*\| + \beta_n (1+k_n) \|y_n - x^*\| + \gamma_n M
 \end{aligned}$$

and

$$\begin{aligned}
 \|y_n - x^*\| &= \|P(\alpha'_n x_n + \beta'_n T(PT)^{n-1} x_n + \gamma'_n v_n) - Px^*\| \\
 &\leq \|\alpha'_n x_n + \beta'_n T(PT)^{n-1} x_n + \gamma'_n v_n - x^*\| \\
 &\leq \alpha'_n \|x_n - x^*\| + \beta'_n \|T(PT)^{n-1} x_n - x^*\| + \gamma'_n \|v_n - x^*\| \\
 (3.4) \quad &\leq \alpha'_n \|x_n - x^*\| + \beta'_n (1+k_n) \|x_n - x^*\| + \gamma'_n M.
 \end{aligned}$$

Substituting (3.4) into (3.3), it can be obtained that

$$\begin{aligned}
 \|x_{n+1} - x^*\| &\leq \alpha_n \|x_n - x^*\| + \beta_n \alpha'_n (1+k_n) \|x_n - x^*\| \\
 &\quad + \beta_n \beta'_n (1+k_n)^2 \|x_n - x^*\| + \beta_n (1+k_n) \gamma'_n M + M \gamma_n \\
 &\leq \alpha_n \|x_n - x^*\| + (1-\alpha_n) \alpha'_n (1+k_n)^2 \|x_n - x^*\| \\
 &\quad + (1-\alpha_n) \beta'_n (1+k_n)^2 \|x_n - x^*\| + d_n \\
 &= \alpha_n \|x_n - x^*\| + (1-\alpha_n) (1+k_n)^2 (\alpha'_n + \beta'_n) \|x_n - x^*\| + d_n \\
 &\leq \alpha_n \|x_n - x^*\| + (1-\alpha_n) (1+k_n)^2 \|x_n - x^*\| + d_n \\
 &\leq \alpha_n \|x_n - x^*\| + (1-\alpha_n + 2k_n + k_n^2) \|x_n - x^*\| + d_n \\
 &= (1+2k_n + k_n^2) \|x_n - x^*\| + d_n \\
 &= (1+k_n)^2 \|x_n - x^*\| + d_n
 \end{aligned}$$

where $d_n = (1+k_n) \gamma'_n M + M \gamma_n$. Since $\sum_{n=1}^{\infty} k_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, we have $\sum_{n=1}^{\infty} d_n < \infty$.

We now to prove (2). Notice that when $x \geq 0$, $1 + x \leq e^x$. For any $x^* \in F(T)$, it follows from (1) that

$$\begin{aligned}
 \|x_{n+m} - x^*\| &\leq (1 + k_{n+m-1})^2 \|x_{n+m-1} - x^*\| + d_{n+m-1} \\
 &= e^{2k_{n+m-1}} \|x_{n+m-1} - x^*\| + d_{n+m-1} \\
 &\leq e^{2(k_{n+m-1} + k_{n+m-2})} \|x_{n+m-2} - x^*\| + e^{2k_{n+m-1}} d_{n+m-2} + d_{n+m-1} \\
 &\vdots \\
 &\leq e^{2\sum_{j=n}^{n+m-1} k_j} \|x_n - x^*\| + e^{\sum_{j=n}^{n+m-1} k_j} \sum_{j=n}^{n+m-1} d_j \\
 &\leq e^{2\sum_{j=1}^{\infty} k_j} \|x_n - x^*\| + e^{2\sum_{j=1}^{\infty} k_j} \sum_{j=1}^{\infty} d_j \\
 &\leq K \|x_n - x^*\| + K \sum_{j=1}^{\infty} d_j, \forall n \geq 1,
 \end{aligned}$$

where $K = e^{2\sum_{j=1}^{\infty} k_j} > 0$. Thus,

$$\|x_{n+m} - x^*\| \leq K \|x_n - x^*\| + K \sum_{j=1}^{\infty} d_j, \forall n, m \geq 1, x^* \in F(T).$$

This completes the proof of (2). □

We now to prove the following theorems.

Theorem 3.4. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let $T : C \rightarrow X$ be an asymptotically quasi-nonexpansive nonself-mapping of C with sequence $\{k_n\}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_1 \in C$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ be sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Then the sequence $\{x_n\}$ defined by (3.2) strongly converges to a fixed point of T if and only if $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$, where $d(x, F(T))$ denote the distance of x to the set $F(T)$, i.e., $d(x, F(T)) = \inf_{y \in F(T)} d(x, y)$.*

Proof. The necessity of the conditions is obvious. Thus we will only prove the sufficiency. For any $x^* \in F(T)$, from (3.2), it follows by Lemma 3.3 that

$$\|x_{n+1} - x^*\| \leq (1 + k_n)^2 \|x_n - x^*\| + d_n, \forall n \geq 1.$$

This implies that $d(x_{n+1}, F(T)) \leq (1 + k_n)^2 d(x_n, F(T)) + d_n$. From Lemma 2.4, we have $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$. Hereafter, we will prove that $\{x_n\}$ is a Cauchy

sequence. From Lemma 3.3, there exists a constant $K > 0$ such that

$$(3.5) \quad \|x_{n+m} - x^*\| \leq K\|x_n - x^*\| + K \sum_{j=1}^{\infty} d_j, \forall n, m \geq 1.$$

Let $\epsilon > 0$. Since $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$, there exists a natural number N_1 such that for each $n \geq N_1$

$$d(x_n, F(T)) < \frac{\epsilon}{3K} \text{ and } \sum_{k=n}^{\infty} d_k < \frac{\epsilon}{6K}.$$

In particular, we have $d(x_{N_1}, F) < \frac{\epsilon}{3K}$. This implies that there exists a point $y' \in F(T)$ such that

$$\|x_{N_1} - y'\| < \frac{\epsilon}{3K}.$$

It follows, from (3.5), that when $n, m \geq N_1$,

$$\|x_{n+m} - x_n\| \leq \|x_{m+n} - y'\| + \|x_n - y'\| \leq K\|x_{N_1} - y'\| + K\|x_{N_1} - y'\| + 2K \sum_{j=1}^{\infty} d_j < \epsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Because the space is complete, the sequence $\{x_n\}$ is convergent. Let $\lim_{n \rightarrow \infty} x_n = y$. Moreover, we note that

$$d(y, F(T)) \leq d(x_n, F(T)) + \|x_n - y\|, \forall n \geq 1.$$

Since $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$ and the set $F(T)$ is closed, we have $y \in F(T)$, i.e. y is a fixed point of T . This complete the proof. $\square$

Corollary 3.5. *Suppose that condition are as same as in Theorem 3.4. Then the sequence $\{x_n\}$ generated by (3.2) converges to a fixed point of T if and only if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to y .*

Theorem 3.6. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let $T : C \rightarrow X$ be an uniformly L -Lipschitzian completely continuous and asymptotically quasi-nonexpansive nonself-mapping of C with sequence $\{k_n\}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_1 \in C, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ be sequences in $[0, 1]$ such that $0 < \alpha < \alpha_n, \beta_n, \alpha'_n, \beta'_n < \beta < 1, \alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n, \sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Then the sequence $\{x_n\}$ defined by (3.2) strongly converges to a fixed point in of T .*

Proof. Let $x^* \in F(T)$. From Lemma 3.3, we have

$$\|x_{n+1} - x^*\| \leq (1 + k_n)^2 \|x_n - x^*\| + d_n, \forall n \geq 1.$$

Since $\sum_{n=1}^{\infty} k_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$, it follows by Lemma 2.4 that $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists. Let $\lim_{n \rightarrow \infty} \|x_n - p\| = c$ for some $c \geq 0$. From the proof of Lemma 3.3, we have that

$$\|y_n - x^*\| \leq (1 + k_n)\|x_n - x^*\| + \gamma'_n\|v_n - x^*\|.$$

Taking $\limsup_{n \rightarrow \infty}$ in both sides, we obtain

$$\limsup_{n \rightarrow \infty} \|y_n - x^*\| \leq \limsup_{n \rightarrow \infty} \|x_n - x^*\| = \lim_{n \rightarrow \infty} \|x_n - x^*\| = c.$$

Note that

$$\limsup_{n \rightarrow \infty} \|T(PT)^{n-1}y_n - x^*\| \leq \limsup_{n \rightarrow \infty} \|y_n - x^*\| \leq \lim_{n \rightarrow \infty} \|x_n - x^*\| = c.$$

and, by (3.4), we have

$$\begin{aligned} c = \lim_{n \rightarrow \infty} \|x_{n+1} - x^*\| &\leq \lim_{n \rightarrow \infty} \|\alpha_n x_n + \beta_n T(PT)^{n-1}y_n + \gamma_n u_n - x^*\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n [(x_n - x^*) + \frac{\gamma_n}{2\alpha_n}(u_n - x^*)] \\ &\quad + \beta_n [(T(PT)^{n+1}y_n - x^*) + \frac{\gamma_n}{2\beta_n}(u_n - x^*)]\| \\ &\leq \lim_{n \rightarrow \infty} \alpha_n \|x_n - x^*\| + \lim_{n \rightarrow \infty} \beta_n \|T(PT)^{n+1}y_n - x^*\| \\ &\leq \lim_{n \rightarrow \infty} \alpha_n \|x_n - x^*\| + \lim_{n \rightarrow \infty} \beta_n (1 + k_n) \|y_n - x^*\| \\ &= \lim_{n \rightarrow \infty} (\alpha_n \|x_n - x^*\| + \beta_n (1 + k_n) \|y_n - x^*\|) \\ &\leq \lim_{n \rightarrow \infty} (\alpha_n \|x_n - x^*\| + \beta_n (1 + k_n)^2 \|x_n - x^*\| \\ &\quad + \gamma'_n \beta_n (1 + k_n) \|v_n - x^*\|) \\ &\leq \lim_{n \rightarrow \infty} (\alpha_n (1 + k_n)^2 \|x_n - x^*\| + (1 - \alpha_n) (1 + k_n)^2 \|x_n - x^*\|) \\ &\leq \lim_{n \rightarrow \infty} (1 + k_n)^2 \|x_n - x^*\| = c. \end{aligned}$$

Hence

$$c = \lim_{n \rightarrow \infty} \|\alpha_n [x_n - x^* + \frac{\gamma_n}{2\alpha_n}(u_n - x^*)] + \beta_n [T(PT)^{n-1}y_n - x^* + \frac{\gamma_n}{2\beta_n}(u_n - x^*)]\|$$

By J. Schu's Lemma, we have

$$\lim_{n \rightarrow \infty} \|x_n - T(PT)^{n-1}y_n + (\frac{\gamma_n}{2\alpha_n} - \frac{\gamma_n}{2\beta_n})(u_n - x^*)\| = 0.$$

Since $\lim_{n \rightarrow \infty} \|(\frac{\gamma_n}{2\alpha_n} - \frac{\gamma_n}{2\beta_n})(u_n - x^*)\| = 0$. Then

$$(3.6) \quad \lim_{n \rightarrow \infty} \|x_n - T(PT)^{n-1}y_n\| = 0.$$

It follows that

$$(3.7) \quad \|x_{n+1} - x_n\| \leq \alpha_n \|x_n - T(PT)^{n-1}y_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

On the other hand, we obtain that

$$\begin{aligned}
 \|x_{n+1} - x^*\| &= \|\alpha_n(x_n - x^*) + \beta_n(T(PT)^{n-1}y_n - x^*) + \gamma_n(u_n - x^*)\| \\
 &\leq \alpha_n\|x_n - x^*\| + \beta_n(1 + k_n)\|y_n - x^*\| + \gamma_n\|u_n - x^*\| \\
 &\leq \alpha_n\|x_n - T(PT)^{n-1}y_n\| + \alpha_n\|T(PT)^{n-1}y_n - x^*\| \\
 &\quad + \beta_n(1 + k_n)\|y_n - x^*\| + \gamma_n\|u_n - x^*\| \\
 &\leq \alpha_n\|x_n - T(PT)^{n-1}y_n\| + \alpha_n(1 + k_n)\|y_n - x^*\| \\
 &\quad + \beta_n(1 + k_n)\|y_n - x^*\| + \gamma_n\|u_n - x^*\| \\
 &\leq \alpha_n\|x_n - T(PT)^{n-1}y_n\| + (1 - \beta_n)(1 + k_n)\|y_n - x^*\| \\
 &\quad + \beta_n(1 + k_n)\|y_n - x^*\| + \gamma_n\|u_n - x^*\| \\
 &\leq \alpha_n\|x_n - T(PT)^{n-1}y_n\| + (1 + k_n)\|y_n - x^*\| + \gamma_n\|u_n - x^*\|.
 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|x_n - T(PT)^{n-1}y_n\| = 0 = \lim_{n \rightarrow \infty} \gamma_n$, it follows that

$$c = \lim_{n \rightarrow \infty} \|x_n - x^*\| \leq \liminf_{n \rightarrow \infty} \|y_n - x^*\|.$$

Hence

$$c \leq \liminf_{n \rightarrow \infty} \|y_n - x^*\| \leq \limsup_{n \rightarrow \infty} \|y_n - x^*\| \leq c,$$

and so

$$\lim_{n \rightarrow \infty} \|y_n - x^*\| = c.$$

This implies that

$$\begin{aligned}
 c = \lim_{n \rightarrow \infty} \|y_n - x^*\| &\leq \lim_{n \rightarrow \infty} \|\alpha'_n x_n + \beta'_n T(PT)^{n-1} x_n + \gamma'_n v_n - x^*\| \\
 &= \lim_{n \rightarrow \infty} \|\alpha'_n [x_n - x^* + \frac{\gamma'_n}{2\alpha'_n} (v_n - x^*)] \\
 &\quad + \beta'_n [T(PT)^{n-1} x_n - x^* + \frac{\gamma'_n}{2\beta'_n} (v_n - x^*)]\| \\
 &\leq \lim_{n \rightarrow \infty} \alpha'_n \|x_n - x^*\| + \lim_{n \rightarrow \infty} \beta'_n \|T(PT)^{n-1} x_n - x^*\| \\
 &\leq \lim_{n \rightarrow \infty} \alpha'_n \|x_n - x^*\| + \lim_{n \rightarrow \infty} \beta'_n (1 + k_n) \|x_n - x^*\| \\
 &\leq \lim_{n \rightarrow \infty} (\alpha'_n \|x_n - x^*\| + (1 - \alpha'_n)(1 + k_n) \|x_n - x^*\|) \\
 &\leq \lim_{n \rightarrow \infty} (1 + k_n) \|x_n - x^*\| = c.
 \end{aligned}$$

Then

$$c = \lim_{n \rightarrow \infty} \|\alpha'_n [x_n - x^* + \frac{\gamma'_n}{2\alpha'_n} (v_n - x^*)] + \beta'_n [T(PT)^{n-1} x_n - x^* + \frac{\gamma'_n}{2\beta'_n} (v_n - x^*)]\|.$$

By J. Schu's Lemma, we have

$$\lim_{n \rightarrow \infty} \|x_n - T(PT)^{n-1} x_n + (\frac{\gamma'_n}{2\alpha'_n} - \frac{\gamma'_n}{2\beta'_n}) (v_n - x^*)\| = 0.$$

Since $\lim_{n \rightarrow \infty} \|(\frac{\gamma'_n}{2\alpha'_n} - \frac{\gamma'_n}{2\beta'_n})(v_n - x^*)\| = 0$, it follows that

$$(3.8) \quad \lim_{n \rightarrow \infty} \|x_n - T(PT)^{n-1}x_n\| = 0.$$

We now to show that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. First, we note that

$$\begin{aligned} \|x_n - T(PT)^{n-2}x_n\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - T(PT)^{n-2}x_{n-1}\| \\ &\quad + \|T(PT)^{n-2}x_{n-1} - T(PT)^{n-2}x_n\| \\ &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - T(PT)^{n-2}x_{n-1}\| + L\|x_{n-1} - x_n\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus from above inequality, we have

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - T(PT)^{n-1}x_n\| + \|T(PT)^{n-1}x_n - Tx_n\| \\ &= \|x_n - T(PT)^{n-1}x_n\| + \|T(PT)^{1-1}(PT)^{n-1}x_n - T(PT)^{1-1}x_n\| \\ &\leq \|x_n - T(PT)^{n-1}x_n\| + L\|PT(PT)^{n-2}x_n - x_n\| \\ &\leq \|x_n - T(PT)^{n-1}x_n\| + L\|T(PT)^{n-2}x_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

It implies that

$$(3.9) \quad \lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0.$$

We note from Lemma 3.3 that $\{x_n\}$ is bounded. Since T is completely continuous, it follows that there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \rightarrow x^*$ as $k \rightarrow \infty$. Moreover, by (3.9), we have $\|Tx_{n_k} - x_{n_k}\| \rightarrow 0$ which implies that $x_{n_k} \rightarrow x^*$ as $k \rightarrow \infty$. By (3.9) again, we have

$$\|x^* - Tx^*\| = \lim_{k \rightarrow \infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

It show that $x^* \in F(T)$. Furthermore, since $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists. Therefore $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$. This complete the proof. $\square$

Acknowledgement. The authors would like to thanks The Thailand Research Fund for financial support.

REFERENCES

1. F. E. Browder, *Nonexpansive nonlinear operators in Banach spaces*, Proc. Natl. Acad. Sci. USA **54** (1965) 1041-1044.
2. F. E. Browder, *Convergence of approximates to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Arch. Rational Mech. Anal. **24** (1967) 82-90.
3. S. S. Chang, Y. J. Cho, J. K. Kim and K. H. Kim, *Iterative approximation of fixed points for asymptotically nonexpansive type mappings in Banach spaces*, Panamer. Math. J. **11** (2001) 53-63.

ภาคผนวก 15

**Random fixed point theorems for asymptotically
regular mappings**

P. Kumam and S. Plubtieng

Stochastic Anal. Appl. (submitted).

ภาคผนวก 15/1

Random fixed point theorems for asymptotically regular mappings

poom kumam^{†,*} and Somyot Plubtieng[‡]

[†]Department of Mathematics King Mongkut's University of Technology Thonburi,
Bangkok 10140. THAILAND.

[‡]Department of Mathematics, Naresuan University, Pitsanulok 65000. THAILAND.

May 7, 2005

Abstract. Let (Ω, Σ) be a measurable space, X a Banach space, C a weakly compact convex subset of X and $T : \Omega \times C \rightarrow C$ a random operator. We prove the random version of a deterministic fixed point theorem when T is uniformly Lipschitzian mapping such that $\sigma(T(\omega, \cdot)) < \sqrt{WCS(X)}$ for all $\omega \in \Omega$ and T is asymptotically regular on C . Let $WCS(X)$ be the weakly convergent sequence coefficient of X and $\kappa_\omega(X)$ its Lifschitz characteristic. If T is asymptotically regular and there exists a constant c such that

$$\sigma(T(\omega, \cdot)) \leq c < \frac{1 + \sqrt{1 + 4WCS(X)(\kappa_\omega(X) - 1)}}{2},$$

we prove that T has a random fixed point.

2000 Mathematics Subject Classification: 47H10, 47H09, 47H04.

Key words and phrases: random fixed point, uniformly Lipschitzian mapping, Lifschitz characteristic, random asymptotically regular.

*Corresponding author.

Email addresses: poom.kum@kmutt.ac.th (P. Kumam) and somyot@nu.ac.th (S. Plubtieng)

1 Introduction

The study of random fixed point theorems was initiated by the Prague school of probability in the 1950s. Random operator theory is needed for the study of various classes of random equations (see [8] and references therein). Random fixed point theory has received much attention for the last two decades because of its importance in probabilistic functional analysis; the reader is referred to Beg and Shahzad [2], Shahzad and Latif [10] and Tan and Yaun [11]. Generalizations of the random fixed point theorems for continuous selfmaps to the case of non-selfmaps have been considered by many authors (see e.g. Beg et al. [2], and Shahzad and Latif [10]). On the other hand, the first fixed point theorem for uniformly Lipschitzian mapping in Banach spaces was given by Goebel and Kirk [7] who state a relationship between the existence of fixed point for uniformly Lipschitzian mappings and clarkson modulus of convexity. Recently, Benavides and Xu [6] link the coefficients $\kappa_w(X)$ and $WCS(X)$ to fixed points of uniformly Lipschitzian mappings. Letter, Benavides [3] was improved a result in [6] and given a class of spaces X whose $\kappa_w(X) < WCS(X)$.

The main goal of this paper is to establish some random fixed point theorems for Uniformly Lipschitzian and asymptotically regular operator. Firstly, we will prove the random fixed point theorems for nonlinear uniformly Lipschitzian mappings in Banach spaces. Moreover, we also stat the random version of a fixed point of a fixed point result based on the Lifshitz's constant of a Banach space due to Domínguez Benavides [3].

2 Preliminaries and notations

Through this paper we will consider a measurable spaces (Ω, Σ) (where Σ is a σ -algebra of subset of Ω) and (X, d) will be a metric spaces. We denote by $CL(X)$ (resp. $CB(X)$, $KC(X)$) the family of all nonempty closed (resp. closed bounded, compact) subset of X , and by H the Hausdorff metric on $CB(X)$ induced by d , i.e.,

$$H(A, B) = \max \{d(a, B)_{a \in A}, d(b, A)_{b \in B}\}$$

for $A, B \in CB(X)$, where $d(x, E) = \inf\{d(x, y) | y \in E\}$ is the distance from x to $E \subset X$.

A set-valued operator $T : \Omega \rightarrow 2^X$ is call (Σ) - measurable if, for any open subset B of X ,

$$T^{-1}(B) := \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\}$$

belongs to Σ . A mapping $x : \Omega \rightarrow X$ is said to be a *measurable selector* of a measurable set-valued

operator $T : \Omega \rightarrow 2^X$ if $x(\cdot)$ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. Let M be a nonempty closed subset of X . An operator $T : \Omega \times M \rightarrow 2^X$ is called a random operator if, for each fixed $x \in M$, the operator $T(\cdot, x) : \Omega \rightarrow 2^X$ is measurable. We will denote by $F(\omega)$ the fixed point set of $T(\omega, \cdot)$, i.e.,

$$F(\omega) := \{x \in M : x \in T(\omega, x)\}.$$

Note that if we do not assume the existence of fixed point for the deterministic mapping $T(\omega, \cdot) : M \rightarrow 2^X$, $F(\omega)$ may be empty. A measurable operator $x : \Omega \rightarrow M$ is said to be a *random fixed point of a operator* $T : \Omega \times M \rightarrow 2^X$ if $x(\omega) \in T(\omega, x(\omega))$ for all $\omega \in \Omega$. Recall that $T : \Omega \times M \rightarrow 2^X$ is continuous if, for each fixed $\omega \in \Omega$, the operator $T : (\Omega, \cdot) \rightarrow 2^X$ is continuous.

Let C be a closed bounded convex subset of a Banach spaces X . A random operator $T : \Omega \times C \rightarrow C$ is said to be *nonexpansive* if, for fixed $\omega \in \Omega$ the map $T : (\omega, \cdot) \rightarrow C$ is nonexpansive. We will say that T is *uniformly Lipschitzian* if there exists a function $k : \Omega \rightarrow [1, +\infty)$ such that

$$\|T^n(\omega, x) - T^n(\omega, y)\| \leq k(\omega)\|x - y\|$$

for all $x, y \in C$ and for each integer $n \geq 1$. Here $T^n(\omega, x)$ is the value at x of the n th iterate of the map $T(\omega, \cdot)$. We will say that T is *asymptotically nonexpansive* if there exists a sequence of function $k_n : \Omega \rightarrow [1, +\infty)$ such that for each fixed $\omega \in \Omega$, $\lim_{n \rightarrow \infty} k_n(\omega) = 1$ and

$$\|T^n(\omega, x) - T^n(\omega, y)\| \leq k_n(\omega)\|x - y\|$$

for all $x, y \in C$ and integer $n \geq 1$. The nonexpansive random map T is called *asymptotically regular* if for each $x \in C$,

$$\lim_{n \rightarrow \infty} \|T^{n+1}(\omega, x) - T^n(\omega, x)\| = 0$$

for each $\omega \in \Omega$.

Now recall the weakly convergent sequence coefficient $WCS(X)$ [6] of X is defined by

$$WCS(X) = \inf \left\{ \frac{A(\{x_n\})}{\inf_{y \in co\{x_n\}} \limsup_{n \rightarrow \infty} \|x_n - y\|} : \{x_n\} \text{ is a weakly convergent sequence which is not norm-convergent} \right\},$$

where $A(\{x_n\}) = \limsup_{n \rightarrow \infty} \{\|x_i - x_j\| : i, j \geq n\}$ is the asymptotic diameter of $\{x_n\}$. We will use next relationship between the asymptotically center of a sequence and the characteristic of convexity of the space. Let C be a nonempty bounded closed subset of Banach spaces X and $\{x_n\}$ bounded sequence in X , we use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C , respectively, i.e.

$$\begin{aligned} r(C, \{x_n\}) &= \inf \{r(x, \{x_n\}) : x \in C\}, \text{ where } r(x, \{x_n\}) = \limsup_n \|x_n - x\|, \\ A(C, \{x_n\}) &= \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}. \end{aligned}$$

If D is a bounded subset of X , the *Chebyshev radius* of D relative to C is defined by

$$r_C(D) := \inf \{\sup \{\|x - y\| : y \in D\} : x \in C\}.$$

Definition 2.1. Let $\{x_n\}$ and C be a nonempty bounded closed subset of Banach spaces X . Then $\{x_n\}$ is called regular with respect to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

We are going to list several result related to the concept of measurability which will be used repeatedly in next section.

Theorem 2.2. (cf. Wagner [12]). Let (X, d) be a complete separable metric spaces and $F : \Omega \rightarrow CL(X)$ a measurable map. Then F has a measurable selector.

Theorem 2.3. (cf. Tan and Yuan [11]). Let X be a separable metric spaces and Y a metric spaces. If $f : \Omega \times X \rightarrow Y$ is a measurable in $\omega \in \Omega$ and continuous in $x \in X$, and if $x : \Omega \rightarrow X$ is measurable, then $f(\cdot, x(\cdot)) : \Omega \rightarrow Y$ is measurable.

Theorem 2.4. (Benavidel, Lopez and Xu cf.[4]). Suppose C is a weakly closed nonempty separable subset of a Banach space X , $F : \Omega \rightarrow 2^X$ a measurable with weakly compact values, $f : \Omega \times C \rightarrow \mathbb{R}$ is a measurable, continuous and weakly lower semicontinuous function. Then the marginal function $r : \Omega \rightarrow \mathbb{R}$ defined by

$$r(\omega) := \inf_{x \in F(\omega)} f(\omega, x)$$

and the marginal map. $R : \Omega \rightarrow X$ defined by

$$R(\omega) := \{x \in F(\omega) : f(\omega, x) = r(\omega)\}$$

are measurable.

Proposition 2.5. (Xu cf.[13]) Let M be a separable metric space and $f : \Omega \times C \rightarrow \mathbb{R}$ be a Carathéodory map, i.e., for every $x \in M$, then the map $f(\cdot, x) : \Omega \rightarrow \mathbb{R}$ is measurable and for every $\omega \in \Omega$, the map $f(\omega, \cdot) : M \rightarrow \mathbb{R}$ is continuous. Then for any $s \in \mathbb{R}$, the map $F_s : \Omega \rightarrow M$ defined by

$$F_s(\omega) = \{x \in M : f(\omega, x) < s, \quad \omega \in \Omega\}$$

is measurable.

Let M be a bounded convex subset of a Banach space X . We recall that the Lifschitz characteristic for asymptotically regular mappings, is defined;

- (i) A number $b \geq 0$ is said to have property (P_ω) with respect to M if there exists some $a > 1$ such that for all $x, y \in M$ and $r > 0$ with $\|x - y\| > r$ and each weakly convergent sequence $\{\xi_n\} \subset M$ for which $\limsup \|\xi_n - x\| \leq ar$ and $\limsup \|\xi_n - y\| \leq br$, there exists some $z \in M$ such that $\liminf \|\xi_n - z\| \leq r$;
- (ii) $\kappa_\omega(M) = \sup\{b > 0 : b \text{ has property } (P_\omega) \text{ w.r.t. } M\}$;
- (iii) $\kappa_\omega(X) = \inf\{\kappa_\omega(m) : M \text{ as above}\}$.

If S is a mapping from a set C into itself, then we use the symbol $|S|$ to denote the Lipschitz constant of S , i.e.

$$|S| = \sup \left\{ \frac{\|Sx - Sy\|}{\|x - y\|} : x, y \in C, x \neq y \right\}.$$

For a mapping T on C , we set

$$\sigma(T) = \liminf_{n \rightarrow \infty} |T^n|.$$

Theorem 2.6. (Domínguez Benavides and Xu [6]) *Let X be a Banach spaces. Suppose also C is a closed convex bounded subset of X and $T : C \rightarrow C$ is an asymptotically regular and uniformly Lipschitzian mapping. Then, if $\sigma(T) < \kappa_\omega(C)$, T has a fixed point.*

3 The results

The following is the random version of theorem 3.2 of Domínguez Benavides and Xu [6].

Theorem 3.1. *Let C be a nonempty weakly compact convex separable subset of a Banach space with $WCS(X) > 1$ and $T : \Omega \times C \rightarrow C$ be a random uniformly Lipschitzian mapping such that $\sigma(T(\omega, \cdot)) < \sqrt{WCS(X)}$ for all $\omega \in \Omega$. Suppose in addition that T is asymptotically regular on C . Then T has a random fixed point.*

Proof. It is easy to see that (cf. [6]),

$$WCS(X) = \sup \left\{ M > 0 : M \cdot \limsup_{n \rightarrow \infty} \|x_n - x_\infty\| \leq D\{x_n\} \right\},$$

where the supremum is taken over all weakly (not strongly) convergent sequence $\{x_n\}$ in X and x_∞ is the weak limit of $\{x_n\}$ and $D\{x_n\} = \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \|x_n - x_m\|$. The separability of C makes it possible to select a subsequence $\{n_j\}$ of positive integer such that

$$\sigma(T(\omega, \cdot)) < \sqrt{WCS(X)},$$

and $\{T^{n_j}(\omega, x)\}$ converges weakly for every $x \in C$. Choose an arbitrary $x_0 \in C$ and set $x_0(\omega) = x_0$, this x_0 is obvious measurable. Now we can construct a sequence $\{x_m\}$ of measurable function $x_m : \Omega \rightarrow C$ such that for each $\omega \in \Omega$ and integer $m \geq 1$,

$$x_m(\omega) = w - \lim_{j \rightarrow \infty} T^{n_j}(\omega, x_{m-1}(\omega)).$$

Note that the asymptotic regular of $T(\omega, \cdot)$ we have

$$x_m(\omega) = w - \lim_{j \rightarrow \infty} T^{n_j+p}(\omega, x_{m-1}(\omega)), \quad \forall p \geq 0.$$

We now show that $\{x_m(\omega)\}$ converges strongly to fixed point of T . Set for each integer $m \geq 1$,

$$B_m(\omega) = \limsup_j \|T^{n_j}(\omega, x_m(\omega)) - x_{m+1}(\omega)\|,$$

and

$$\alpha = \frac{[\sigma(T)]^2}{WCS(X)}.$$

Then $\alpha < 1$ and by the above definition of $WCS(X)$, for each $\omega \in \Omega$, we have

$$B_m(\omega) \leq \frac{1}{WCS(X)} D(\{T^{n_j}(\omega, x_m(\omega))\})$$

However, from the w-lower semicontinuous of the norm of X , it follows that

$$\begin{aligned} D(\{T^{n_j}(\omega, x_m(\omega))\}) &= \limsup_j \limsup_i \|T^{n_i}(\omega, x_m(\omega)) - T^{n_j}(\omega, x_m(\omega))\| \\ &= \limsup_j \limsup_i \|T^{n_i+n_j}(\omega, x_m(\omega)) - T^{n_j}(\omega, x_m(\omega))\| \\ &\leq \limsup_j |T^{n_j}| \limsup_i \|T^{n_i}(\omega, x_m(\omega)) - x_m(\omega)\| \\ &\leq \sigma(T(\omega, \cdot)) \limsup_i \|T^{n_i}(\omega, x_m(\omega)) - x_m(\omega)\| \\ &\leq \sigma(T(\omega, \cdot)) \limsup_i (\liminf_j \|T^{n_i}(\omega, x_m(\omega)) - T^{n_j}(\omega, x_{m-1}(\omega))\|) \\ &\leq \sigma(T(\omega, \cdot)) \limsup_i |T^{n_i}| \limsup_j \|x_m(\omega) - T^{n_j}(\omega, x_{m-1}(\omega))\| \\ &= [\sigma(T(\omega, \cdot))]^2 B_{m-1}(\omega). \end{aligned}$$

We, therefore conclude that

$$B_m(\omega) \leq \frac{[\sigma(T(\omega, \cdot))]^2}{WCS(X)} B_{m-1}(\omega) \leq \alpha B_{m-1}(\omega).$$

Now using the w-lower semicontinuous of the norm of X again, we deduce that

$$\begin{aligned} \|x_m(\omega) - x_{m+1}(\omega)\| &\leq \limsup_i \|x_m(\omega) - T^{n_i}(\omega, x_m(\omega))\| + \limsup_i \|T^{n_i}(\omega, x_m(\omega)) - x_{m+1}(\omega)\| \\ &\leq \limsup_i \limsup_j \|T^{n_j}(\omega, x_{m-1}(\omega)) - T^{n_i}(\omega, x_m(\omega))\| + B_m(\omega) \\ &\leq \limsup_i |T^{n_i}| \limsup_j \|T^{n_j}(\omega, x_{m-1}(\omega)) - x_m(\omega)\| + B_m(\omega) \\ &= \sigma(T(\omega, \cdot)) B_{m-1}(\omega) + B_m(\omega). \end{aligned}$$

This implies that $\{x_m(\omega)\}$ is Cauchy sequence for each $\omega \in \Omega$. For any $\omega \in \Omega$, let $x(\omega) = \lim x_m(\omega)$.

We will that $x(\omega)$ is a random fixed point of T . Indeed, for each $j \geq 1$ we have

$$\begin{aligned} \|x(\omega) - T^{n_j}(\omega, x(\omega))\| &\leq \|x(\omega) - x_{m+1}(\omega)\| + \|x_{m+1}(\omega) - T^{n_j}(\omega, x_m(\omega))\| \\ &\quad + \|T^{n_j}(\omega, x_m(\omega)) - T^{n_j}(\omega, x(\omega))\| \\ &\leq \|x(\omega) - x_{m+1}(\omega)\| + \|x_{m+1}(\omega) - T^{n_j}(\omega, x_m(\omega))\| + |T^{n_j}| \|x_m(\omega) - x(\omega)\| \end{aligned}$$

Taking the upper limit as $j \rightarrow \infty$ yields

$$\limsup_j \|x(\omega) - T^{n_j}(\omega, x(\omega))\| \leq \|x(\omega) - x_{m+1}(\omega)\| + B_m(\omega) + \sigma(T(\omega, \cdot)) \|x_m(\omega) - x(\omega)\|$$

which implies $T^{n_j}(\omega, x(\omega)) - x(\omega) \rightarrow 0$ as $m \rightarrow \infty$. Since $T(\omega, \cdot)$ is continuous and asymptotic regular, it follows that $x(\omega) = T(\omega, x(\omega))$. Observe that $x(\omega)$ is the limit of measurable mappings, so it is measurable. Hence $x(\omega)$ is a random fixed point of T . This completes the proof. $\square$

Theorem 3.2. *Let X be a reflexive Banach space, C be a nonempty bounded convex separable subset of X and $T : \Omega \times C \rightarrow C$ be a random asymptotically regular operator. If there exist a constant $c \in \mathbb{R}$ such that*

$$\sigma(T(\omega, \cdot)) \leq c < \frac{1 + \sqrt{1 + 4WCS(X)(\kappa_\omega(X) - 1)}}{2}$$

for all $\omega \in \Omega$ then T has a random fixed point.

Proof. Denote $W = WCS(X)$ and $\kappa_\omega = \kappa_\omega(X)$. According to the stochastic version of Banach's contraction principle [1], we only need to prove the result if $(1 + \sqrt{1 + 4WCS(X)(\kappa_\omega(X) - 1)})/2 > 1$. Then we can assume that $c > 1$. Furthermore, since $\kappa_\omega \leq W$ (see Lemma 2 [3]) we have

$$\frac{1 + \sqrt{1 + 4W(\kappa_\omega - 1)}}{2} \leq W.$$

Hence $c < W$. On the other hand, the condition $c < 1 + \sqrt{1 + 4W(\kappa_\omega - 1)}/2$ is equivalent to $c(c-1) < W(\kappa_\omega - 1)$. choose $b < \kappa_\omega$ such that $c(c-1) < W(b-1)$. We shall consider a fixed element $x_0 \in C$, and for every $\omega \in \Omega$ define

$$R(\omega, x_0) = \inf \{ \liminf_n \|T^n(\omega, y) - x_0\| : y \in C \}.$$

We start by proving that $R(\cdot, x_0)$ is a measurable function. Set for each $\omega \in \Omega$, $f(\omega, y) = \liminf_n \|T^n(\omega, y) - x_0\|$. We can apply Theorem 2.3 to deduce that $f(\cdot, y)$ is measurable for each $y \in C$. Since C is a separable subset of X , it follows that there exists a countable dens subset $\{y_n\}$ of C . Therefore for each $\omega \in \Omega$ we have

$$R(\omega, x_0) = \inf_{n \geq 1} f(\omega, y_n),$$

which implies that $R(\cdot, y)$ is measurable. Take $\varepsilon > 0$ such that $(1 + \varepsilon)/a := \alpha < 1$. Set $G(\omega) = \{y \in C : f(\omega, y) \leq R(\omega, x_0)(1 + \varepsilon)\}$. It is clear that $G(\omega)$ is a nonempty subset C and since $f(\omega, \cdot)$ is continuous in C , it follows from Proposition 2.5 that $G(\cdot)$ is measurable. Hence, by Theorem 2.2, we can find a measurable selector $y(\omega)$ of $G(\omega)$, which verifies

$$\liminf_n \|T^n(\omega, y(\omega)) - x_0\| < R(\omega, x_0)(1 + \varepsilon).$$

Choose a sequence $\{n_k\}$ of positive integer such that $\sigma(T(\omega, \cdot)) = \lim_{k \rightarrow \infty} |T^{n_k}(\omega, \cdot)|$ and set $L_k(\omega) = |T^{n_k}(\omega, \cdot)|$ for all $\omega \in \Omega$. Consider in Ω the partition given by the set:

$$\Omega_1 := \left\{ \omega \in \Omega : \sup_k \|x_0 - T^{n_k}(\omega, x_0)\| \leq \frac{WR(\omega, x_0)(1 + \varepsilon)}{ca} \right\}$$

and

$$\Omega_2 := \left\{ \omega \in \Omega : \sup_k \|x_0 - T^{n_k}(\omega, x_0)\| > \frac{WR(\omega, x_0)(1 + \varepsilon)}{ca} \right\}.$$

It is easy to prove that both set are measurable. Assume that $\Omega_1 \neq \emptyset$, and fix $\omega \in \Omega_1$. We can choose a subsequence $\{n_{k'}\}$ of n_k such that

$$\lim_{k'} \|x_0 - T^{n_{k'}}(\omega, x_0)\| = \limsup_k \|x_0 - T^{n_k}(\omega, x_0)\|$$

and $T^{n_{k'}}(\omega, x(\omega))$ converges weakly to a point, say $z(\omega)$. Since T is asymptotically regular, it follows that

$$\lim_{k'} \|T^{n_{k'}+m}(\omega, x_0) - T^{n_{k'}}(\omega, x_0)\| = 0$$

for any fixed m . Thus, we have

$$\begin{aligned} \limsup_{p'} \limsup_{q'} \|T^{n_{p'}}(\omega, x_0) - T^{n_{q'}}(\omega, x_0)\| &\leq \limsup_{p'} \limsup_{q'} L_{p'}(\omega) \|T^{n_{q'}}(\omega, x_0) - x_0\| \\ &\leq \limsup_{p'} L_{p'}(\omega) \frac{WR(\omega, x_0)(1+\varepsilon)}{ca} \\ &\leq \frac{WR(\omega, x_0)(1+\varepsilon)}{a} \end{aligned}$$

Since C is weakly compact set. Define a function $E: \Omega_1 \rightarrow 2^C$ by

$$E(\omega) := \left\{ z \in C : \limsup_{k'} \|T^{n_{k'}}(\omega, x_0) - z\| \leq D(\{T^{n_{k'}}\})/W \right\}$$

is measurable in Ω_1 , where $D(\{T^n\}) = \limsup_n \limsup_m \|T^n(\omega, x_0) - T^m(\omega, x_0)\|$. It follows from Proposition 2.5 that $E(\omega)$ is measurable in Ω_1 . It follows from Theorem 2.2 that there exists a measurable selector. Take $z_1: \Omega_1 \rightarrow C$ a measurable selector of $E(\cdot)$. By Theorem 1 in [5] we know that

$$W = \inf \left\{ \frac{\limsup_n \limsup_m \|x_n - x_m\|}{\limsup_n \|x_n - x_\infty\|} \right\},$$

where the infimum is taken over all weakly (not strong) converging sequences $\{x_n\}$ and $x_\infty = w - \lim x_n$. Then, the above form of W gives

$$\limsup_{k'} \|z_1(\omega) - T^{n_{k'}}(\omega, x_0)\| \leq \frac{R(\omega, x_0)(1+\varepsilon)}{a}$$

which implies

$$\liminf_k \|z_1(\omega) - T^{n_k}(\omega, x_0)\| \leq \frac{R(\omega, x_0)(1+\varepsilon)}{a}$$

and so $R(\omega, z_1(\omega)) \leq \frac{(1+\varepsilon)R(\omega, x_0)}{a} = \alpha R(\omega, x_0)$. Furthermore from the weakly-lower semicontinuous of the norm we have $\|z_1(\omega) - x_0\| \leq \|z_1(\omega) - z(\omega)\| + \|z(\omega) - x_0\|$. This implies that

$$\begin{aligned} \|z_1(\omega) - x_0\| &\leq \liminf_{k'} \|z_1(\omega) - T^{n_{k'}}(\omega, x(\omega))\| + \liminf_{k'} \|T^{n_{k'}}(\omega, x(\omega)) - x_0\| \\ &\leq R(\omega, z_1(\omega)) + \frac{WR(\omega, x_0)(1+\varepsilon)}{ca} \\ &\leq \alpha R(\omega, x_0) + \alpha \frac{WR(\omega, x_0)}{c} = \alpha \left(1 + \frac{W}{c}\right) R(\omega, x_0) \end{aligned}$$

Now, assume that $\Omega_2 \neq \emptyset$. In this case, for $\omega \in \Omega$, there exists $i \in \mathbb{N}$ such that $\|x_0 - T^{n_i}(\omega, x_0)\| > WR(\omega, x_0)(1+\varepsilon)/ca$ and $c(L_i(\omega)-1) < W(\frac{b}{a}-1)$. If $\sigma(T(\omega, \cdot)) > 1$, then we can assume that $L_i(\omega) > 1$. Choose $y \in C$ such that $\liminf_k \|x_0 - T^{n_k}(\omega, y)\| < R(\omega, x_0)(1+\varepsilon)$ and a subsequence $\{T^{n_{k'}}(\omega, y)\}$ of $\{T^{n_k}(\omega, y)\}$ such that $\liminf_k \|x_0 - T^{n_k}(\omega, y)\| = \lim_{k'} \|x_0 - T^{n_{k'}}(\omega, y)\|$ and $\{T^{n_{k'}}(\omega, y)\}$ converges weakly to a point, say $v(\omega)$. Using again the asymptotic regularity of T we obtain

$$\begin{aligned} \limsup_{k'} \|T^{n_{k'}}(\omega, x_0) - T^{n_{k'}}(\omega, y)\| &\leq L_i(\omega) \limsup_{k'} \|T^{n_{k'}}(\omega, y) - x_0\| \\ &= L_i(\omega) \liminf_k \|T^{n_k}(\omega, y) - x_0\| \\ &\leq L_i(\omega) R(\omega, x_0)(1+\varepsilon) \end{aligned}$$

Choose $\lambda \in [0, 1]$ such that $\frac{c}{W} < \lambda < (\frac{b}{a} - 1)/(L_i(\omega) - 1)$ if $L_i(\omega) > 1$ or $\lambda = 1$ otherwise. Then

$$\begin{aligned} \limsup_{k'} \|T^{n_{k'}}(\omega, y) - \lambda T^{n_i}(\omega, x_0) - (1 - \lambda)x_0\| &\leq \limsup_{k'} \lambda \|T^{n_{k'}}(\omega, y) - T^{n_i}(\omega, x_0)\| \\ &\quad + (1 - \lambda) \limsup_{k'} \|T^{n_{k'}}(\omega, y) - x_0\| \\ &\leq \lambda L_i(\omega) R(\omega, x_0)(1 + \varepsilon) + (1 - \lambda) R(\omega, x_0)(1 + \varepsilon) \\ &\leq \frac{bR(\omega, x_0)(1 + \varepsilon)}{a}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \|x_0 - \lambda T^{n_i}(\omega, x_0) - (1 - \lambda)x_0\| &= \lambda \|T^{n_i}(\omega, x_0) - x_0\| \\ &\geq \frac{\lambda W R(\omega, x_0)(1 + \varepsilon)}{a} \\ &> \frac{R(\omega, x_0)(1 + \varepsilon)}{a}. \end{aligned}$$

By the condition which b satisfies there exist $z(\omega) \in C$ such that

$$\liminf_{k'} \|T^{n_{k'}}(\omega, y) - z(\omega)\| \leq \frac{R(\omega, x_0)(1 + \varepsilon)}{a} = \alpha R(\omega, x_0).$$

Since

$$\liminf_k \|T^{n_k}(\omega, y) - z(\omega)\| \leq \liminf_{k'} \|T^{n_{k'}}(\omega, y) - z(\omega)\|$$

we obtain $R(\omega, z) \leq \alpha R(\omega, x_0)$. Therefore, from Proposition 2.5 the mapping $F : \Omega_2 \rightarrow 2^C$ given by

$$F(\omega) = \left\{ z \in C : \limsup_k \|T^{n_k}(\omega, y(\omega)) - z(\omega)\| \leq \alpha R(\omega, x_0) \right\}$$

is measurable. Thus it admit a measurable selector $z_2 : \Omega_2 \rightarrow C$ which satisfies $R(\omega, z_2(\omega)) < \alpha R(\omega, x_0)$. Hence by the weakly lower semicontinuous of the function norm, we have $\|x_0 - z_2(\omega)\| \leq \|x_0 - v(\omega)\| + \|v(\omega) - z_2(\omega)\|$. This implies that

$$\begin{aligned} \|x_0 - z_2(\omega)\| &\leq \liminf_{k'} \|x_0 - T^{n_{k'}}(\omega, y(\omega))\| + \liminf_{k'} \|T^{n_{k'}}(\omega, y(\omega)) - z_2(\omega)\| \\ &\leq \liminf_k \|x_0 - T^{n_k}(\omega, y(\omega))\| + \limsup_k \|T^{n_k}(\omega, y(\omega)) - z_2(\omega)\| \\ &\leq (1 + \varepsilon)R(\omega, x_0) + \alpha R(\omega, x_0) = (1 + \varepsilon + \alpha)R(\omega, x_0). \end{aligned}$$

consider $z : \Omega \rightarrow C$ given by $z(\omega) = z_1(\omega)$ if $\omega \in \Omega_1$ and $z(\omega) = z_2(\omega)$ if $\omega \in \Omega_2$. Clearly $z(\cdot)$ is a measurable function. Choose an arbitrary $x_0 \in C$ and set $x_0(\omega) = x_0$. We defined $x_1(\omega) = z(\omega)$ as above and then we can inductively construct a sequence $\{x_m(\omega)\}$ of measurable functions $x_m : \Omega \rightarrow C$ such that for each $\omega \in \Omega$ we have

$$R(\omega, x_m(\omega)) \leq \alpha R(\omega, x_{m-1}(\omega)) \leq \dots \leq \alpha^m R(\omega, x_0(\omega)).$$

We shall prove that $\{x_m(\omega)\}$ is a Cauchy sequence. Indeed, if $M = \max\{1 + \varepsilon + \alpha, \alpha(1 + \frac{W}{c})\}$ we have

$$\|x_m(\omega) - x_{m+1}(\omega)\| \leq MR(\omega, x_m(\omega)) \leq M\alpha^m R(\omega, x_0(\omega)).$$

Thus $\{x_m(\omega)\}$ converges to some $x(\omega) \in C$. It is readily see that $R(\omega, x(\omega)) = 0$ for every $\omega \in \Omega$ which implies that $x(\omega)$ is a random fixed point of T . Indeed, if $R(\omega, x(\omega)) = 0$ for any $\varepsilon > 0, \omega \in \Omega$ there exist $y(\omega) \in C$ such that

$$\liminf_k \|x(\omega) - T^{n_k}(\omega, y(\omega))\| \leq \varepsilon.$$

Take a subsequence $\{T^{n_{k''}}(\omega, y(\omega))\}$ of $\{T^{n_k}(\omega, y(\omega))\}$ such that

$$\liminf_k \|x(\omega) - T^{n_k}(\omega, y(\omega))\| = \lim_{k''} \|x(\omega) - T^{n_{k''}}(\omega, y(\omega))\|.$$

By assumption there exists a positive integer m such that $|T^m(\omega, \cdot)| < \infty$. Then

$$\limsup_{k''} \|x(\omega) - T^{n_{k''}+m}(\omega, y(\omega))\| = \limsup_{k''} \|x(\omega) - T^{n_{k''}}(\omega, y(\omega))\| \leq \varepsilon.$$

Thus

$$\begin{aligned} \|x(\omega) - T^m(\omega, x(\omega))\| &\leq \limsup_{k''} (\|x(\omega) - T^{n_{k''}+m}(\omega, y(\omega))\| + \|T^{n_{k''}+m}(\omega, y(\omega)) - T^m(\omega, x(\omega))\|) \\ &\leq \limsup_{k''} \|x(\omega) - T^{n_{k''}+m}(\omega, y(\omega))\| + |T^m(\omega, \cdot)| \lim_{k''} \|T^{n_{k''}}(\omega, y(\omega)) - x(\omega)\| \\ &= (1 + L_m(\omega)) \liminf_k \|x(\omega) - T^{n_k}(\omega, y(\omega))\| \\ &\leq (1 + L_m(\omega))\varepsilon \rightarrow 0, \text{ as } \varepsilon \rightarrow 0, \end{aligned}$$

yielding for each $\omega \in \Omega$, $T^m(\omega, x(\omega)) \rightarrow x(\omega)$ and thus $x(\omega) = T(\omega, x(\omega))$ by the continuity and asymptotic regularity of $T(\omega, \cdot)$. This completes the proof. $\square$

Acknowledgement. The second author would like to thanks The Thailand Research Fund for financial support.

References

- [1] A.T. Bharucha-Reid, "Fixed point theorem in proobabilistic analysis," Bull. Amer. Math. Soc. 82(1976) 641-645.
- [2] I. Beg and N. Shahzad. Random approximations and random fixed point theorems, *J. Appl. Math. Stoch. Anal.* 7, 2, 145-150 (1994).
- [3] T. Domínguez Benavides, "Fixed point theorems for uniformly Lipschitziane mappings and asymptotically regular mappings", *Nonlinear Anal.* 32 No. 1 (1998), 15-27.
- [4] T. Domínguez Benavides G. Lopez Acedo and H.-K Xu, "Random fixed point of set-valued operator", *Proc. Amer. Math. Soc.* 124 (1996), 838-838.
- [5] T. Domínguez Benavides G. Lopez Acedo and H.-K Xu "Weak uniform normal structure and iterative fixed points of nonexpansive mappings", *Coll. Math.*, (1995), LXVIII(I), 17-23.
- [6] T. Domínguez Benavides and H.-K Xu "A new geometrical coefficient for Banach spaces and its applications in fixed point theory", *Nonlinear Anal.* 25 No. 3 (1995), 311-325.
- [7] K. Goebel and W. A. Kirk, "Topic in metric fixed point theorem," Cambridge University Press, Cambridge (1990).

ภาคผนวก 15/11

- [8] S. Itoh, "Random fixed point theorem for a multivalued contraction mapping", *Pacific J. Math.* 68 (1977), 85-90.
- [9] P. Lorenzo Ramírez, "Random fixed point of uniformly Lipschitzian mappings", *Nonlinear Anal.* 57 (2004), 23-34.
- [10] N. Shahzad and S. Latif, "Random fixed points for several classes of 1-ball-contractive and 1-set-contractive random maps", *J. Math. Anal. Appl.* 237 (1999), 83-92.
- [11] K.-K. Tan and X.Z. Yuan, "Some random fixed point theorem", in: K.-K. Tan (Ed), *Fixed Point Theory and Applications*, World Scientific, Singapore, 1992, 334-345.
- [12] D.-H. Wagner, "Survey of measurable selection theorems", *SIAM J. Control Optim.* 15 (1977), 859-903.
- [13] H. K. Xu, "Random fixed point theorems for nonlinear uniform Lipschitzian mappings", *Nonlinear Anal.* 26(1996)No.7, 1301-1311.
- [14] S. Reich, "Fixed point in locally convex spaces", *Math. Z.* 125(1972), 17-31.
- [15] X. Yuan and J. Yu, "Random fixed point theorems for nonself mappings", *Nonlinear Anal.* 26(1996)No.6, 1097-1102.

ภาคผนวก 15/12

Somyot Plubtieng

From: "Poom KUMAM" <poom.kum@kmutt.ac.th>
To: <somyotp@nu.ac.th>
Sent: Friday, August 05, 2005 1:15 PM
Subject: [Fwd: Stochastic Analysis and Applications]

----- Original Message -----

Subject: Stochastic Analysis and Applications
From: "Poom KUMAM" <poom.kum@kmutt.ac.th>
Date: Thu, August 4, 2005 6:54 pm
To: somyotp@nu.ac.th

----- Original Message -----

Subject: RE: Submit manuscripts new version
From: "Sally Ellingson" <sellings@fit.edu>
Date: Sat, May 14, 2005 12:14 am
To: "Poom KUMAM" <poom.kum@kmutt.ac.th>

I have received your files and forwarded them to the appropriate people.
They will inform you of any decisions on your paper.

Sincerely,

Sally R. Ellingson
Editorial/Technical Assistant
Florida Institute of Technology
Department of Mathematical Sciences
sellings@fit.edu
321-674-7412

-----Original Message-----

From: Poom KUMAM [<mailto:poom.kum@kmutt.ac.th>]
Sent: Friday, May 13, 2005 5:28 AM
To: lakshmik@fit.edu
Subject: Submit manuscripts new version

The Editorial Board, Stochastic Analysis and Applications

13 May 2005

8/9/2005

ภาคผนวก 15/13

Dear Prof. V. Lakshmikantham ,

Enclosed please find a files (the TEX and pdf.file new version) of my paper with Dr. Somyot

Plubtieng entitled;

"Random fixed point theorems for asymptotically regular Mappings"

which I would like to submit for publication in the Journal of Stochastic Analysis and Applications.

I would like to thank you in advance for your consideration.

Your sincerely,

Poom Kumam

-
- >
 - > Title: Random fixed point theorems for asymptotically regular
 - > mappings
 - > Nonlinear Analysis Series A: Theory, Methods & Applications
 - >
 - > Dear Dr. Poom Kumam,
 - >
 - > Unfortunately, your submission is not suitable for publication in
 - Nonlinear Analysis
 - > Series A: Theory, Methods & Applications, as it is not within the scope
 - of the
 - > journal. We think that it may be appropriate for the Journal Stochastic
 - Analysis and
 - > Applications. If you would like for your paper to be considered for this
 - journal.
 - > please email your source file and a pdf to lakshmik@fit.edu or
 - > sellings@fit.edu.
 - >
 - > Thank you for giving us the opportunity to consider your work.
 - >
 - > Yours sincerely,
 - >
 - > V. Lakshmikantham
 - > Editor-in-Chief
 - > Nonlinear Analysis Series A: Theory, Methods & Applications
 - >

8/9/2005

ภาคผนวก 16

**Weak and strong convergence of scheme with
errors for a finite family of nonexpansive
mappings**

I. Inchan and S. Plubtieng,

Nonlinear Anal. (submitted).

Weak and strong convergence of a scheme with errors for a finite family of nonexpansive mappings

Somyot Plubtieng * and Issara Inchan

Department of Mathematics, Naresuan University, Phitsanulok 65000, THAILAND

e-mail : somyotp@nu.ac.th

Abstract

In this paper, we are concerned with the study of an iterative scheme with error for a finite family of nonexpansive mappings. Weak and strong convergence theorems are established for a multi-step iterative scheme for a finite family of nonexpansive mappings in Banach spaces. The results obtained in this paper extend and improve the recent ones announced by Khan and Fukhar-ud-din, Takahashi and Tamura, and many others.

1. Introduction

Let C be a subset of real normed linear space X , and let T be a self-mapping on C . T is said to be *nonexpansive* provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

Fixed-point iteration processes for nonexpansive mapping in Banach spaces including Mann and Ishikawa iteration processes have been studied

2000 Mathematics Subject Classification : 47H10, 47H09, 46B20

Keywords : Fixed point ; nonexpansive mappings; Uniformly convex; Opial condition.

*Supported by Thailand Research Fund

2 Weak and strong convergence of a scheme for a finite family of mappings

extensively by many others; see [5, 8, 10, 11, 15, 16]. In 1998, Xu [18] introduced and studied the Mann iterative scheme with errors and Ishikawa iterative scheme with errors. On the other hand, Das and Debata [4] and Takahashi and Tamura [14] introduced and studied a generalization of Ishikawa iterative schemes for a pair of nonexpansive mappings. Recently, Khan and Fukhar-ud-din [6] extended their scheme to a modified Ishikawa iterative schemes with errors for two mappings and gave weak and strong convergence theorems. Inspired and motivated by these facts, a new class of multi-step iterative scheme for a finite family of mappings is introduced and studied in this paper. This scheme can be viewed as an extension for a scheme with errors for two mappings of Khan and Fukhar-ud-din [6]. The scheme is defined as follows.

Let C be a nonempty subset of normed space X and let $T_1, T_2, \dots, T_N$ be nonexpansive mappings of C into itself. The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 = x \in C, \\ x_n^1 = \alpha_n^1 T_1 x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1, \\ x_n^2 = \alpha_n^2 T_2 x_n^1 + \beta_n^2 x_n + \gamma_n^2 u_n^2, \\ x_n^3 = \alpha_n^3 T_3 x_n^2 + \beta_n^3 x_n + \gamma_n^3 u_n^3, \\ x_n^4 = \alpha_n^4 T_4 x_n^3 + \beta_n^4 x_n + \gamma_n^4 u_n^4, \\ \vdots \\ x_{n+1} = x_n^N = \alpha_n^N T_N x_n^{N-1} + \beta_n^N x_n + \gamma_n^N u_n^N, n \geq 1, \end{cases} \quad (1.1)$$

where $\{\alpha_n^1\}, \dots, \{\alpha_n^N\}, \{\beta_n^1\}, \dots, \{\beta_n^N\}, \{\gamma_n^1\}, \dots, \{\gamma_n^N\}$ are sequences in $[0, 1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, 3, \dots, N$ and $\{u_n^1\}, \{u_n^2\}, \dots, \{u_n^N\}$ are bounded sequences in C .

If $T_1 = T_2 = T_3 = \dots = T_N := T$, then (1.1) reduces to the modified multi-step iterative scheme with errors defined by:

$$\begin{cases} x_1 = x \in C, \\ x_n^1 = \alpha_n^1 T x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1, \\ x_n^2 = \alpha_n^2 T x_n^1 + \beta_n^2 x_n + \gamma_n^2 u_n^2, \\ x_n^3 = \alpha_n^3 T x_n^2 + \beta_n^3 x_n + \gamma_n^3 u_n^3, \\ x_n^4 = \alpha_n^4 T x_n^3 + \beta_n^4 x_n + \gamma_n^4 u_n^4, \\ \vdots \\ x_{n+1} = x_n^N = \alpha_n^N T x_n^{N-1} + \beta_n^N x_n + \gamma_n^N u_n^N, n \geq 1, \end{cases} \quad (1.2)$$

where $\{\alpha_n^1\}, \dots, \{\alpha_n^N\}, \{\beta_n^1\}, \dots, \{\beta_n^N\}, \{\gamma_n^1\}, \dots, \{\gamma_n^N\}$ are sequences in $[0, 1]$ with

$\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, 3, \dots, N$ and $\{u_n^1\}, \{u_n^2\}, \dots, \{u_n^N\}$ are bounded sequences in C .

For $N = 2$, then (1.1) reduces to the scheme with errors for two mappings defined by Khan and Fukhar-ud-din [6]:

$$\begin{cases} x_1 = x \in C, \\ y_n := x_n^1 = \alpha_n^1 T_1 x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1, \\ x_{n+1} = \alpha_n^2 T_2 x_n^1 + \beta_n^2 x_n + \gamma_n^2 u_n^2, n \geq 1, \end{cases} \quad (1.3)$$

where $\{\alpha_n^1\}, \{\alpha_n^2\}, \{\beta_n^1\}, \{\beta_n^2\}, \{\gamma_n^1\}, \{\gamma_n^2\}$ are sequences in $[0, 1]$ with $\alpha_n^1 + \beta_n^1 + \gamma_n^1 = 1 = \alpha_n^2 + \beta_n^2 + \gamma_n^2$ and $\{u_n^1\}, \{u_n^2\}$ are bounded sequences in C .

For $N = 2$ and $\gamma_n^1 = \gamma_n^2 \equiv 0$, then (1.1) reduces to the modified Ishikawa iterative scheme for two nonexpansive mappings defined by Das and Debata [4] and Takahashi and Tamura [14]:

$$\begin{cases} x_1 = x \in C, \\ y_n := x_n^1 = \alpha_n^1 T_1 x_n + (1 - \alpha_n^1) x_n, \\ x_{n+1} = \alpha_n^2 T_2 x_n^1 + (1 - \alpha_n^2) x_n, n \geq 1, \end{cases} \quad (1.4)$$

where $\{\alpha_n^1\}, \{\alpha_n^2\}$ are sequences in $[0, 1]$ and $\{u_n^1\}, \{u_n^2\}$ are bounded sequences in C .

For $N = 2$ and $T_1 = T_2 \equiv T$, then (1.1) reduces to the modified Ishikawa iterative scheme with errors defined by

$$\begin{cases} x_1 = x \in C \\ y_n := x_n^1 = \alpha_n^1 T x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1, \\ x_{n+1} = \alpha_n^2 T x_n^1 + \beta_n^2 x_n + \gamma_n^2 u_n^2, n \geq 1, \end{cases} \quad (1.5)$$

where $\{\alpha_n^1\}, \{\alpha_n^2\}, \{\beta_n^1\}, \{\beta_n^2\}, \{\gamma_n^1\}, \{\gamma_n^2\}$ are sequences in $[0, 1]$ with $\alpha_n^1 + \beta_n^1 + \gamma_n^1 = 1 = \alpha_n^2 + \beta_n^2 + \gamma_n^2$ and $\{u_n^1\}, \{u_n^2\}$ are bounded sequences in C .

The purpose of this paper is to establish several weak and strong convergence results of the multi-step iterative scheme given in (1.1) for a finite family of nonexpansive mappings. Our results extend and improve the corresponding ones announced by Khan and Fukhar-ud-din [6], and others.

Now, we recall the well-known concepts and results.

A Banach space X is said to satisfy Opial's condition [9] if for any sequence $\{x_n\}$ in X , $x_n \rightarrow x$ implies that $\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$ for all $y \in X$ with $y \neq x$.

4 Weak and strong convergence of a scheme for a finite family of mappings

A mapping $T : C \longrightarrow X$ is called demiclosed with respect to $y \in X$ if for each sequence $\{x_n\}$ in C and each $x \in X$, $x_n \rightharpoonup x$ and $Tx_n \rightarrow y$ implies that $x \in X$ and $Tx = y$.

Lemma 1.1 (Schu [12]). Suppose that X is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n . Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequence of X such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$ and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 1.2 ([15, Lemma1]). Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be nonnegative sequences of real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n \text{ for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(1). $\lim_{n \rightarrow \infty} a_n$ exists.

(2). $\lim_{n \rightarrow \infty} a_n = 0$ whenever $\liminf_{n \rightarrow \infty} a_n = 0$.

Lemma 1.3 (Browder [1]). Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X . Let T be a nonexpansive mapping of C into itself. Then $I - T$ is demiclosed with respect to zero.

The mapping $T : C \longrightarrow C$ with $F(T) \neq \phi$, is said to satisfy condition (A) if there exists a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\|x - Tx\| \geq f(d(x, F(T)))$$

for all $x \in C$ where $d(x, F(T)) = \inf_{p \in F(T)} \|x - p\|$. Recently, Khan and Fukkar-ud-din [6] modified the condition (A') for two mappings as follows: Two mappings $S, T : C \longrightarrow X$ where C a subset of X , are said to satisfy condition (A') if there exists a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\frac{1}{2}(\|x - Tx\| + \|x - Sx\|) \geq f(d(x, F))$$

for all $x \in C$ where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(S) \cap F(T)\}$. Note that condition (A') reduces to condition (A) when $T = S$. We now modify condition for N mappings $T_1, T_2, \dots, T_N : C \longrightarrow C$ as follows.

Mappings $T_1, T_2, \dots, T_N : C \longrightarrow C$ where C a subset of X , is said to satisfy condition (A^N) if there exists a nondecreasing function $f : [0, \infty) \longrightarrow$

$[0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that $\frac{1}{N} \sum_{j=1}^N \|x - T_j x\| \geq f(d(x, F))$ for all $x \in C$ where $d(x, F) = \inf\{\|x - x^*\| : x^* \in F = \bigcap_{j=1}^N F(T_j)\}$. Note that condition (A^N) reduces to condition (A') when $N = 2$, and condition (A^N) reduces to condition (A) when $T_1 = T_2 = T_3 = \dots = T_N := T$.

2. Main Theorems.

In this section, we prove and strong convergence theorems of the iterative scheme given in (1.1) for a finite family of nonexpansive mappings in a Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 2.1. Let X be a normed space and C its nonempty bounded convex subset of X . Let $T_1, T_2, \dots, T_N$ be nonexpansive mappings of C into itself. Let $\{x_n\}$ be the sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ for all $i \in \{1, 2, 3, \dots, N\}$. If $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$.

Proof. Let $p \in F$ for each $n \geq 1$, we note that

$$\begin{aligned} \|x_n^1 - p\| &= \|\alpha_n^1 T_1 x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1 - p\| \\ &\leq \alpha_n^1 \|T_1 x_n - p\| + \beta_n^1 \|x_n - p\| + \gamma_n^1 \|u_n^1 - p\| \\ &\leq \alpha_n^1 \|x_n - p\| + \beta_n^1 \|x_n - p\| + \gamma_n^1 \|u_n^1 - p\| \end{aligned}$$

and

$$\|x_n^2 - p\| \leq \alpha_n^2 \|x_n^1 - p\| + \beta_n^2 \|x_n - p\| + \gamma_n^2 \|u_n^2 - p\|. \quad (2)$$

Put $d_n^0 = \gamma_n^1 \|u_n^1 - p\|$ for all $n \in \mathbb{N}$. Thus $\|x_n^1 - p\| \leq \|x_n - p\| + d_n^0$ and $\sum_{n=1}^{\infty} d_n^0 < \infty$.

Substituting (1) and (2), we have

$$\begin{aligned} \|x_n^2 - p\| &\leq \alpha_n^2 [\alpha_n^1 \|x_n - p\| + \beta_n^1 \|x_n - p\| + \gamma_n^1 \|u_n^1 - p\|] + \beta_n^2 \|x_n - p\| \\ &\quad + \gamma_n^2 \|u_n^2 - p\| \\ &= \alpha_n^2 \alpha_n^1 \|x_n - p\| + \alpha_n^2 \beta_n^1 \|x_n - p\| + \alpha_n^2 \gamma_n^1 \|u_n^1 - p\| + \beta_n^2 \|x_n - p\| \\ &\quad + \gamma_n^2 \|u_n^2 - p\| \\ &= (1 - \beta_n^2 - \gamma_n^2) \alpha_n^1 \|x_n - p\| + (1 - \beta_n^2 - \gamma_n^2) \beta_n^1 \|x_n - p\| \\ &\quad + \beta_n^2 \|x_n - p\| + d_n^1 \end{aligned}$$

6 Weak and strong convergence of a scheme for a finite family of mappings

$$\begin{aligned}
 &\leq \beta_n^2 \|x_n - p\| + (1 - \beta_n^2) \alpha_n^1 \|x_n - p\| + (1 - \beta_n^2) \beta_n^1 \|x_n - p\| + d_n^1 \\
 &= \beta_n^2 \|x_n - p\| + (1 - \beta_n^2) (\alpha_n^1 + \beta_n^1) \|x_n - p\| + d_n^1 \\
 &\leq \beta_n^2 \|x_n - p\| + (1 - \beta_n^2) \|x_n - p\| + d_n^1 \\
 &= \|x_n - p\| + d_n^1
 \end{aligned}$$

where $d_n^1 = \alpha_n^2 \gamma_n^1 \|u_n^1 - p\| + \gamma_n^2 \|u_n^2 - p\|$. Since $\sum_{n=1}^{\infty} \gamma_n^i < \infty$, we get $\sum_{n=1}^{\infty} d_n^1 < \infty$. Similarly, we have

$$\begin{aligned}
 \|x_n^3 - p\| &\leq \|x_n - p\| + \alpha_n^3 d_n^1 + \gamma_n^3 \|u_n^3 - p\| \\
 &= \|x_n - p\| + d_n^2,
 \end{aligned}$$

where $d_n^2 = \alpha_n^3 d_n^1 + \gamma_n^3 \|u_n^3 - p\|$. So that $\sum_{n=1}^{\infty} d_n^2 < \infty$.

By continuing the above method, for any $k = 1, 2, \dots, N$ there exists a nonnegative real sequence $\{d_n^{k-1}\}$ such that $\sum_{n=1}^{\infty} d_n^{k-1} < \infty$ and

$$\|x_n^k - p\| \leq \|x_n - p\| + d_n^{k-1}.$$

Thus we have $\|x_{n+1} - p\| = \|x_n^N - p\| \leq \|x_n - p\| + d_n^{N-1}$ for all $n \in \mathbb{N}$. Hence, by Lemma 1.2, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. This complete the proof. $\square$

Lemma 2.2. Let C be a subset of uniformly convex Banach space X . Let $T_1, T_2, \dots, T_N$ be nonexpansive mappings of C into itself and $\{x_n\}$ be a sequence as in (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2, \dots, N$. Then $\lim_{n \rightarrow \infty} \|T_i x_n - x_n\| = 0$ for all $i = 1, 2, \dots, N$.

Proof. By Lemma 2.1, we have $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and

$$\|x_n^{N-1} - p\| \leq \|x_n - p\| + d_n^{N-2}$$

where $\sum_{n=1}^{\infty} d_n^{N-2} < \infty$. Let $\lim_{n \rightarrow \infty} \|x_n - p\| = c \geq 0$. This implies that

$$\limsup_{n \rightarrow \infty} \|x_n - p + \gamma_n^N (u_n^N - x_n)\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| + \limsup_{n \rightarrow \infty} \|\gamma_n^N (u_n^N - x_n)\| \leq c$$

and so

$$\limsup_{n \rightarrow \infty} \|T_N x_n^{N-1} - p + \gamma_n^N (u_n^N - x_n)\| \leq \limsup_{n \rightarrow \infty} \|x_n^{N-1} - p\| + \limsup_{n \rightarrow \infty} \|\gamma_n^N (u_n^N - x_n)\| \leq c.$$

We note that

$$\begin{aligned}
 c = \lim_{n \rightarrow \infty} \|x_{n+1} - p\| &= \lim_{n \rightarrow \infty} \|\alpha_n^N T_N x_n^{N-1} + \beta_n^N x_n + \gamma_n^N u_n^N - p\| \\
 &= \lim_{n \rightarrow \infty} \|\alpha_n^N T_N x_n^{N-1} + (1 - \alpha_n^N)x_n - \gamma_n^N x_n \\
 &\quad + \gamma_n^N u_n^N - (1 - \alpha_n^N)p - \alpha_n^N p\| \\
 &= \lim_{n \rightarrow \infty} \|\alpha_n^N T_N x_n^{N-1} - \alpha_n^N p - \alpha_n^N \gamma_n^N x_n + \alpha_n^N \gamma_n^N u_n^N \\
 &\quad + (1 - \alpha_n^N)x_n - (1 - \alpha_n^N)p - \gamma_n^N x_n + \gamma_n^N u_n^N - \alpha_n^N \gamma_n^N u_n^N \\
 &\quad + \alpha_n^N \gamma_n^N x_n\|. \\
 &= \lim_{n \rightarrow \infty} \|\alpha_n^N (T_N x_n^{N-1} - p + \gamma_n^N (u_n^N - x_n)) \\
 &\quad + (1 - \alpha_n^N)(x_n - p + \gamma_n^N (u_n^N - x_n))\|
 \end{aligned}$$

By Lemma 1.1, we have

$$\lim_{n \rightarrow \infty} \|T_N x_n^{N-1} - x_n\| = 0.$$

Note that

$$\begin{aligned}
 \|x_n - p\| &\leq \|x_n - T_N x_n^{N-1}\| + \|T_N x_n^{N-1} - p\| \\
 &\leq \|x_n - T_N x_n^{N-1}\| + \|x_n^{N-1} - p\|, \text{ for all } n \in \mathbb{N}.
 \end{aligned}$$

Then

$$c = \lim_{n \rightarrow \infty} \|x_n - p\| \leq \liminf_{n \rightarrow \infty} \|x_n^{N-1} - p\|,$$

and hence

$$c \leq \liminf_{n \rightarrow \infty} \|x_n^{N-1} - p\| \leq \limsup_{n \rightarrow \infty} \|x_n^{N-1} - p\| \leq c.$$

This implies that $\lim_{n \rightarrow \infty} \|x_n^{N-1} - p\| = c$.

Again, by the proof of Lemma 2.1, we have

$$\|x_n^{N-2} - p\| \leq \|x_n - p\| + d_n^{N-3} \text{ for all } n \geq 1.$$

Hence $\limsup_{n \rightarrow \infty} \|x_n^{N-2} - p\| \leq c$, and so

$$\limsup_{n \rightarrow \infty} \|T_{N-1} x_n^{N-2} - p\| \leq \limsup_{n \rightarrow \infty} \|x_n^{N-2} - p\| \leq c.$$

Thus, we have $\lim_{n \rightarrow \infty} \|x_n - p + r_n^{N-1}(u_n^{N-1} - x_n)\| \leq c$ and

$$\limsup_{n \rightarrow \infty} \|T_{N-1} x_n^{N-2} - p + r_n^{N-1}(u_n^{N-1} - x_n)\| \leq c.$$

8 Weak and strong convergence of a scheme for a finite family of mappings

Moreover, we note that

$$\begin{aligned}
 c &= \lim_{n \rightarrow \infty} \|x_n^{N-1} - p\| = \lim_{n \rightarrow \infty} \|\alpha_n^N T_{N-1} x_n^{N-2} + \beta_n^{N-1} x_n + \gamma_n^{N-1} u_n^{N-1} - p\| \\
 &= \lim_{n \rightarrow \infty} \|\alpha_n^{N-1} T_N x_n^{N-2} + (1 - \alpha_n^{N-1}) x_n - \gamma_n^{N-1} x_n \\
 &\quad + \gamma_n^{N-1} u_n^{N-1} - (1 - \alpha_n^{N-1}) p - \alpha_n^{N-1} p\| \\
 &= \lim_{n \rightarrow \infty} \|\alpha_n^{N-1} T_N x_n^{N-2} - \alpha_n^{N-1} p - \alpha_n^{N-1} \gamma_n^{N-1} x_n \\
 &\quad + \alpha_n^{N-1} \gamma_n^{N-1} u_n^{N-1} + (1 - \alpha_n^{N-1}) x_n - (1 - \alpha_n^{N-1}) p - \gamma_n^{N-1} x_n \\
 &\quad + \gamma_n^{N-1} u_n^{N-1} - \alpha_n^{N-1} \gamma_n^{N-1} u_n^{N-1} + \alpha_n^{N-1} \gamma_n^{N-1} x_n\|. \\
 &= \lim_{n \rightarrow \infty} \|\alpha_n^{N-1} (T_{N-1} x_n^{N-2} - p + \gamma_n^{N-1} (u_n^{N-1} - x_n)) \\
 &\quad + (1 - \alpha_n^{N-1}) (x_n - p + \gamma_n^{N-1} (u_n^{N-1} - x_n))\|
 \end{aligned}$$

Hence, by Lemma 1.1, we have

$$\lim_{n \rightarrow \infty} \|T_{N-1} x_n^{N-2} - x_n\| = 0.$$

Thus, we have

$$\begin{aligned}
 \|x_n - T_N x_n\| &\leq \|x_n - T_N x_n^{N-1}\| + \|T_N x_n^{N-1} - T_N x_n\| \\
 &\leq \|x_n^{N-1} - x_n\| + \|x_n - T_N x_n^{N-1}\| \\
 &\leq \|(\alpha_n^{N-1} + \beta_n^{N-1} + \gamma_n^{N-1}) x_n - (\alpha_n^{N-1} T_{N-1} x_n^{N-2} + \beta_n^{N-1} x_n \\
 &\quad + \gamma_n^{N-1} u_n^{N-1})\| + \|x_n - T_N x_n^{N-1}\| \\
 &\leq \alpha_n^{N-1} \|x_n - T_{N-1} x_n^{N-2}\| + \gamma_n^{N-1} \|u_n^{N-1} - x_n\| + \|x_n - T_N x_n^{N-1}\|.
 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|T_N x_n^{N-1} - x_n\| = 0$, $\lim_{n \rightarrow \infty} \|T_{N-1} x_n^{N-2} - x_n\| = 0$ and $\sum_{n=1}^{\infty} \gamma_n^{N-1} < \infty$, it follows that

$$\lim_{n \rightarrow \infty} \|x_n - T_N x_n\| = 0.$$

Similarly, by using as argument in the proof above, we obtain $\lim_{n \rightarrow \infty} \|T_k x_n^{k-1} - x_n\| = 0$ for all $k = 2, 3, \dots, N$, and hence $\lim_{n \rightarrow \infty} \|T_k x_n - x_n\| = 0$ for all $k = 2, 3, \dots, N$. We now show that $\lim_{n \rightarrow \infty} \|T_1 x_n - x_n\| = 0$.

$$\begin{aligned}
 \|x_n^1 - p\| &\leq \alpha_n^1 \|x_n - p\| + \beta_n^1 \|x_n - p\| + \gamma_n^1 \|u_n^1 - p\| \\
 &\leq \|x_n - p\| + \gamma_n^1 \|u_n^1 - p\| \quad \text{for all } n \in \mathbb{N},
 \end{aligned}$$

it follows that

$$\limsup_{n \rightarrow \infty} \|x_n^1 - p\| \leq c.$$

Note that

$$\begin{aligned}\|x_n - p\| &\leq \|x_n - T_2 x_n^1\| + \|T_2 x_n^1 - p\| \\ &\leq \|x_n - T_2 x_n^1\| + \|x_n^1 - p\|, \text{ for all } n \in \mathbb{N}.\end{aligned}$$

Thus, we have

$$c = \lim_{n \rightarrow \infty} \|x_n - p\| \leq \liminf_{n \rightarrow \infty} \|x_n^1 - p\| \leq \limsup_{n \rightarrow \infty} \|x_n^1 - p\| \leq c.$$

Hence, $\lim_{n \rightarrow \infty} \|x_n - p + r_n^1(u_n^1 - x_n)\| \leq c$ and

$$\limsup_{n \rightarrow \infty} \|T_{N-1} x_n^{N-2} - p + \gamma_n^{N-1}(u_n^{N-1} - x_n)\| \leq \limsup_{n \rightarrow \infty} \|x_n^{N-2} - p\| \leq c.$$

Moreover, we observe that

$$\begin{aligned}c = \lim_{n \rightarrow \infty} \|x_n^1 - p\| &= \lim_{n \rightarrow \infty} \|\alpha_n^1 T_1 x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1 - p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n^1 T_N x_n + (1 - \alpha_n^1) x_n - \gamma_n^1 x_n \\ &\quad + \gamma_n^1 u_n^1 - (1 - \alpha_n^1) p - \alpha_n^1 p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n^1 T_N x_n - \alpha_n^1 p - \alpha_n^1 \gamma_n^1 x_n + \alpha_n^1 \gamma_n^1 u_n^1 \\ &\quad + (1 - \alpha_n^1) x_n - (1 - \alpha_n^1) p - \gamma_n^1 x_n + \gamma_n^1 u_n^1 - \alpha_n^1 \gamma_n^1 u_n^1 + \alpha_n^1 \gamma_n^1 x_n\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n^1 (T_1 x_n - p + \gamma_n^1 (u_n^1 - x_n)) \\ &\quad + (1 - \alpha_n^1) (x_n - p + \gamma_n^1 (u_n^1 - x_n))\|\end{aligned}$$

By Lemma 1.1, we have

$$\lim_{n \rightarrow \infty} \|T_1 x_n - x_n\| = 0.$$

Therefore $\lim_{n \rightarrow \infty} \|T x_n - x_n\| = 0$. □

Theorem 2.3. Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X . Let $T_1, T_2, \dots, T_N$ be a nonexpansive mappings of C into itself satisfying condition (A^N) and $\{x_n\}$ be a sequence as defined in (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2, \dots, N$. If $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point in F .

Proof. By Lemma 2.1, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$. Let $\lim_{n \rightarrow \infty} \|x_n - p\| = c$ for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$.

10 Weak and strong convergence of a scheme for a finite family of mappings

By Lemma 2.2, we obtain

$$\lim_{n \rightarrow \infty} \|T_N x_n - x_n\| = 0 = \lim_{n \rightarrow \infty} \|T_{N-1} x_n - x_n\|.$$

Again by the proof of Lemma 2.1 we have

$$\|x_{n+1} - p\| = \|x_n^N - p\| \leq \|x_n - p\| + d_n^{N-1},$$

where $\sum_{n=1}^{\infty} d_n^{N-1} < \infty$. It implies that

$$\inf_{p \in F} \|x_{n+1} - p\| \leq \inf_{p \in F} \|x_n - p\| + d_n^{N-1},$$

and hence $d(x_{n+1}, F) \leq d(x_n, F) + d_n^{N-1}$ for all $n \in \mathbb{N}$.

Hence, by Lemma 1.2, we note that $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. Next by condition (A^N) , $f(d(x_n, F)) \leq \frac{1}{N}(\sum_{j=1}^N \|T_j x_n - x_n\|)$ for all $n \in \mathbb{N}$. Taking $n \rightarrow \infty$, we obtain $\lim_{n \rightarrow \infty} f(d(x_n, F)) = 0$. Since f is a decreasing function and $f(0) = 0$, it follows that $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. Now we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{y_j\} \subset F$ such that $\|x_{n_j} - y_j\| < 2^{-j}$. By the following method of proof of Tan and Xu [15], we get that $\{y_j\}$ is a Cauchy sequence in F and so it converges. Let $y_j \rightarrow y$. Since F is closed, therefore $y \in F$ and then $x_{n_j} \rightarrow y$. As $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, $x_n \rightarrow y \in F = \bigcap_{i=1}^N F(T_i)$. $\square$

For $T_1 = T_2 = \dots = T_N := T$ in Theorem 2.3, we can obtain the following result.

Corollary 2.4 Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X . Let $T : C \rightarrow C$ be a nonexpansive mapping satisfying condition (A) , and $\{x_n\}$ be a sequence defined by (1.2) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2, \dots, N$. If $F = F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to a fixed point of T .

When $N = 2$ in Theorem 2.3, we can obtain Ishikawa-type convergence result for two mappings.

Corollary 2.5 [6, Theorem 2] Let X be a uniformly convex Banach space and C its nonempty closed convex subset of X . Let T_1, T_2 be nonexpansive mappings of C into itself satisfying condition (A') , and $\{x_n\}$ be a sequence defined by (1.3) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2$. If $F = F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2 .

The following is a weak convergence theorem for a finite family of non-expansive mappings in a Banach space.

Theorem 2.6 Let X be a uniformly convex Banach space satisfying the Opial's condition, C its nonempty closed convex subset of X . Let $T_1, T_2, \dots, T_N$ be nonexpansive mappings of C into itself and $\{x_n\}$ be a sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2, \dots, N$. If $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point in F .

Proof. Let $p \in F$. Then, by Lemma 2.1, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in F . To prove this, let z_1, z_2 be weak limits of the subsequence $\{x_{n_i}\}, \{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 2.2, $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \dots, N$. Moreover, we note by Lemma 2.2 that $I - T_i$ are demiclosed with respect zero for all $i = 1, 2, \dots, N$. Therefore we obtain $T_i z_1 = z_1$ and $T_i z_2 = z_2$ for all $i = 1, 2, \dots, N$. Then $z_1, z_2 \in F = \bigcap_{i=1}^N F(T_i)$. Next, we prove the uniqueness. Suppose that $z_1 \neq z_2$, then by the Opial's condition

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - z_1\| &= \lim_{i \rightarrow \infty} \|x_{n_i} - z_1\| \\ &< \lim_{i \rightarrow \infty} \|x_{n_i} - z_2\| \\ &= \lim_{n \rightarrow \infty} \|x_n - z_2\| \\ &= \lim_{j \rightarrow \infty} \|x_{n_j} - z_2\| \\ &< \lim_{j \rightarrow \infty} \|x_{n_j} - z_1\| \\ &= \lim_{n \rightarrow \infty} \|x_n - z_1\|. \end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to a point in F . $\square$

For $T_1 = T_2 = \dots = T_N := T$ in Theorem 2.6, we can obtain the following result.

Corollary 2.7 Let X be a uniformly convex Banach space satisfying the Opial's condition and C be a nonempty closed convex subset of X . Let $T : C \rightarrow C$ be nonexpansive mapping and $\{x_n\}$ be a sequence as in (1.2) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2, \dots, N$. If $F = F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to a fixed point of T .

12 Weak and strong convergence of a scheme for a finite family of mappings

When $N = 2$ in Theorem 2.6, we can obtain Ishikawa-type convergence result for two mappings.

Corollary 2.8 [6, Theorem 1] Let X be a uniformly convex Banach space satisfying the Opial's condition and C be a nonempty closed convex subset of X . Let T_1 and T_2 be nonexpansive mappings of C into itself, and $\{x_n\}$ be a sequence defined by (1.3) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2, \dots, N$. If $F = F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2 .

Finally, we prove a strong convergence theorem which is connected with result of [14].

Theorem 2.9 Let C be a nonempty closed convex subset of uniformly convex Banach space X , and let $T_1, T_2, \dots, T_N$ nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. If P is a metric projection of C onto F and $\{x_n\}$ is a sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2, \dots, N$, then $\{Px_n\}$ converges strongly to a common fixed point in F .

Proof. By the proof of Lemma 2.1, we have

$\|x_{n+1} - p\| = \|x_n^N - p\| \leq \|x_n - p\| + d_n^{N-1}$ for every $p \in F$ and for all $n \in \mathbb{N}$. Since $Px_n \in F$, we get

$$\|Px_{n+1} - x_{n+1}\| \leq \|Px_n - x_{n+1}\| \leq \|x_n - Px_n\| + d_n^{N-1}$$

where $\sum_{n=1}^{\infty} d_n^{N-1} < \infty$. It follows by Lemma 1.2 that $\lim_{n \rightarrow \infty} \|Px_n - x_n\|$ exists. We denote the limit of $\{\|Px_n - x_n\|\}$ by r . Next, we show that for each $n, k \in \mathbb{N}$,

$$\|Px_n - x_{n+k}\| \leq \|Px_n - x_n\| + \sum_{j=n}^{n+k-1} d_j^{N-1},$$

by mathematical induction. For $k = 1$, we have

$$\|Px_n - x_{n+1}\| \leq \|Px_n - x_n\| + d_n^{N-1}.$$

We assume that for $k = l$,

$$\|Px_n - x_{n+l}\| \leq \|Px_n - x_n\| + \sum_{j=n}^{n+l-1} d_j^{N-1}$$

Then, we get

$$\begin{aligned}\|Px_n - x_{n+l}^N\| &= \|Px_n - x_{n+l+1}\| \leq \|Px_n - x_{n+l}\| + d_{n+l}^{N-1} \\ &\leq \|Px_n - x_n\| + \sum_{j=1}^{n+l} d_j\end{aligned}$$

Hence $\|Px_n - x_{n+k}\| \leq \|Px_n - x_n\| + \sum_{j=n}^{n+k-1} d_j^{N-1} \leq \|Px_n - x_n\| + \sum_{j=n}^{\infty} d_j$ for all $n, k \in \mathbb{N}$. Now, we show that $\{Px_n\}$ is a Cauchy sequence. Put $b_n = \sum_{j=n}^{\infty} d_j$. If $r = \lim_{n \rightarrow \infty} \|Px_n - x_n\| = 0$, for any arbitrary positive number ϵ , there exists integer n_0 such that $\|Px_n - x_n\| < \frac{\epsilon}{6}$ and $b_n < \frac{\epsilon}{6}$ for all $n \geq n_0$. Then, we have for $m, n \in \mathbb{N}$ with $m > n \geq n_0$,

$$\begin{aligned}\|Px_n - Px_m\| &\leq \|Px_n - Px_{n_0}\| + \|Px_{n_0} - Px_m\| \\ &\leq \|Px_n - x_n\| + \|x_n - Px_{n_0}\| + \|Px_{n_0} - x_m\| + \|x_m - Px_m\| \\ &\leq \|Px_n - x_n\| + \|x_{n_0} - Px_{n_0}\| + b_{n_0} + \|Px_{n_0} - x_{n_0}\| + b_{n_0} \\ &\quad + \|x_m - Px_m\| \\ &< \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} \\ &= \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\{Px_n\}$ is a Cauchy sequence. Next we assume that $r > 0$ and $\{Px_n\}$ is not a Cauchy sequence. Then there exists a positive number ϵ and two subsequence $\{Px_{n_i}\}, \{Px_{m_i}\}$ of $\{Px_n\}$ such that $\|Px_{n_i} - Px_{m_i}\| \geq \epsilon$ for all $i \in \mathbb{N}$. Also, there exists a positive integer d such that $(r + d)(1 - \delta(\frac{\epsilon}{r+d})) < r$. By the definition of r and b_n , there exists a positive integer n_0 such that $r \leq \|Px_n - x_n\| < r + \frac{d}{2}$ and $b_n < \frac{d}{2}$ for all $n \geq n_0$. Let $n_i, m_i \geq n_0$ and $l \geq n_i, m_i$. Thus, we have

$$\|Px_{n_i} - x_l\| \leq \|Px_{n_i} - x_{n_i}\| + b_{n_i} < r + \frac{d}{2} + \frac{d}{2} = r + d$$

and

$$\|Px_{m_i} - x_l\| \leq \|Px_{m_i} - x_{m_i}\| + b_{m_i} < r + \frac{d}{2} + \frac{d}{2} = r + d.$$

By uniformly convex of X , we get

$$r < \|Px_l - x_l\| \leq \left\| \frac{Px_{n_i} + Px_{m_i}}{2} - x_l \right\| \leq (r + d)(1 - \delta(\frac{\epsilon}{r+d})) < r.$$

It is a contradiction. This complete the proof. $\square$

For $N = 2$ and $\gamma_n^1 = \gamma_n^2 \equiv 0$ in Theorems 2.9, we can obtain Ishikawa iteration convergence result for two mappings.

14 Weak and strong convergence of a scheme for a finite family of mappings

Corollary 2.10 [14, Theorem 4.4] Let C be a nonempty closed convex subset of a uniformly convex Banach space X , and let T_1, T_2 be nonexpansive mappings of C into itself such that $F := F(T_1) \cap F(T_2) \neq \emptyset$. If P is a metric projection of C onto $F(T_1) \cap F(T_2)$ and $\{x_n\}$ is a sequence defined by (1.3) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2, \dots, N$. Then $\{Px_n\}$ converges strongly to an element of $F(T_1) \cap F(T_2)$.

References

- [1] F. E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, Proceedings of the Symposium on Pure Mathematics, vol. 18, Proc. Amer. Math. Soc., Providence, RI, 1976.
- [2] C. E. Chidume, Chika Moore, Fixed points iteration for pseudocontractive maps, Proc. Amer. Math. Soc. 127(4)(1999) 1163-1170.
- [3] Y. J. Cho, H. Y. Zhou, G. Guo, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings, Comput. Math. Appl. 47(2004) 707-717.
- [4] G. Das, J. P. Debata, Fixed points of quasi-nonexpansive mappings, Indian J. Pure Appl. Math. 17(1986) 1263-1269.
- [5] S. Ishikawa, Fixed point by a new iteration, Proc. Amer. Math. Soc. 4(1974)147-150.
- [6] S. H. Khan, H. Fukhar-ud-din, Weak and strong convergence of a scheme with errors for two nonexpansive mappings, Nonlinear Anal. 61 (2005) 1295-1301.
- [7] M. Maiti, M. K. Ghosh, Approximating fixed points by Ishikawa iterates, Bull. Austral. Math. Soc. 40(1989) 113-117.
- [8] W. R. Mann, Mean value method in iteration, Proc. Amer. Math. Soc. 4(1953) 506-510.
- [9] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73(1967) 591-597.
- [10] B. E. Rhoades, Fixed point iterations for certain nonlinear mappings, J. Math. Anal. Appl. 183(1994) 118-120.

- [11] J. Schu, Iterative construction of fixed point of asymptotically nonexpansive mappings, *J. Math. Anal. Appl.* 158(1991) 407-413.
- [12] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, *Bull. Austral. Math. Soc.* 43(1991) 153-159.
- [13] H. F. Senter, W. G. Dotson, Approximating fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc.* 44(2)(1974) 375-380.
- [14] W. Takahashi, T. Tamura, Convergence theorems for a pair of nonexpansive mappings, *J. Convex Analysis* 5(1)(1998) 45-58.
- [15] K.K. Tan, H. K. Xu, Approximating fixed point of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.* 178 (1993) 301-308.
- [16] K. K. Tan, H. K. Xu, Fixed point iteration processes for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* 122(1994) 733-739.
- [17] B. L. Xu, M. Aslam Noor, Fixed point iterations for asymptotically nonexpansive mappings in Banach space, *J. Math. Anal. Appl.* 267(2002) 444-453.
- [18] Y. Xu, Ishikawa and Mann Iteration process with errors for nonlinear strongly accretive equation, *J. Math. Anal.* 224 (1998) 91-101.

ภาคผนวก 16/16

Somyot Plubtieng

From: "Journal of Nonlinear Analysis - A: Theory and Me" <lakshmik@fit.edu>
To: <somyotp@nu.ac.th>
Sent: Wednesday, June 01, 2005 12:33 AM
Subject: A manuscript number has been assigned: NA-D-05-00253

Ms. Ref. No.: NA-D-05-00253

Title: Weak and strong convergence of a scheme with errors for a finite family of nonexpansive mappings
Nonlinear Analysis Series A: Theory, Methods & Applications

Dear Yos,

Your submission entitled "Weak and strong convergence of a scheme with errors for a finite family of nonexpansive mappings" has been assigned the following manuscript number: NA-D-05-00253.

You may check on the progress of your paper by logging on to the Elsevier Editorial System as an author. The URL is <http://ees.elsevier.com/na/>. Your username is: somyotp and Your password is:

[REDACTED]

Thank you for submitting your work to this journal.

Kind regards,

V. Lakshmikantham

Editor-in-Chief

Nonlinear Analysis Series A: Theory, Methods & Applications

8/16/2005

ภาคผนวก 17

**Weak and strong convergence of modified Noor
iterations with errors for three asymptotically
nonexpansive mappings**

S. Plubtieng, R. Punpeang and R. Wangkeeree,

J. Math. Anal. (submitted).

**WEAK AND STRONG CONVERGENCE OF MODIFIED NOOR
ITERATIONS WITH ERRORS FOR THREE ASYMPTOTICALLY
NONEXPANSIVE MAPPINGS**

SOMYOT PLUBTIENG, RABIAN WANGKEEREE AND RATTANAPORN PUNPAENG

*Department of Mathematics, Faculty of Science, Naresuan University ,
Pitsanulok 65000 Thailand*

ABSTRACT. In this paper, several weak and strong convergence theorems are established for a Modified Noor iterative scheme with errors for three asymptotically nonexpansive mappings in Banach spaces. Mann-type, Ishikawa-type, and Noor-type iterations are included by the new iteration scheme. Our results extend and improve the recent ones announced by Xu and Noor [16], and many others.

1. INTRODUCTION

In recent years, one-step and two-step iterative schemes (including Mann and Ishikawa iteration processes) have been studied extensively by many authors: see [2, 4, 11, 13]. In [16], Xu and Noor introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings. Recently, Cho, Zhou and Guo [3] extended their schemes to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mappings.

2000 Mathematics Subject Classification: 46B20, 47H09, 47H10.

Supported by The Thailand Research Fund.

Corresponding author.

Email addresses: Somyotp@nu.ac.th (Somyot Plubtieng), Rabianw@nu.ac.th (Rabian Wangkeeree) and g46060088@nu.ac.th (Rattanaorn Punpaeng) .

On the other hand, Xu [15] introduced and studied the Mann iterative scheme with errors. A generalization of Ishikawa iterative schemes for two mappings was given by Das and Debata [4] and Takahashi and Tamura [13]. Finally, Khan and Fukhar-un-din [5] extended their scheme to a modified Ishikawa iterative scheme with errors for two mappings and gave weak and strong convergence theorems.

Inspired and motivated by these facts, a new class of three-step iterative scheme, for three nonexpansive mappings, is introduced and studied in this paper. This scheme can be viewed as an extension for three-step iterative schemes of Xu and Noor [16], and Cho, Zhou and Guo [3]. This scheme defined as follows.

Let X be a normed space, C be a nonempty convex subset of X , and $T_1, T_2, T_3 : C \rightarrow C$ be three given mappings. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by

$$\begin{aligned} z_n &= \alpha_n'' T_3^n x_n + \beta_n'' x_n + \gamma_n'' u_n \\ y_n &= \alpha_n' T_2^n z_n + \beta_n' x_n + \gamma_n' v_n \\ x_{n+1} &= \alpha_n T_1^n y_n + \beta_n x_n + \gamma_n w_n \quad n \geq 1, \end{aligned} \quad (1.1)$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$ and $\{\gamma_n''\}$ are real sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in C .

The iteration schemes (1.1) are called the modified Noor iterations with errors for three mappings. Noor iteration include the Mann-Ishikawa iteration as special case. If $T_1 = T_2 = T_3 := T$, then (1.1) reduces to three-step iterative scheme define by Cho, Zhou and Guo [3]:

$$\begin{aligned} z_n &= \alpha_n'' T^n x_n + (1 - \alpha_n'' - \gamma_n'') x_n + \gamma_n'' u_n \\ y_n &= \alpha_n' T^n z_n + (1 - \alpha_n' - \gamma_n') x_n + \gamma_n' v_n \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n - \gamma_n) x_n + \gamma_n w_n \quad n \geq 1, \end{aligned} \quad (1.2)$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$ and $\{\gamma_n''\}$ are appropriate real sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in C .

For $T_1 = T_2 = T_3 := T$ and $\gamma_n = \gamma_n' = \gamma_n'' \equiv 0$, then (1.1) reduced to the Noor iterations defined by Xu and Noor[16]

$$\begin{aligned} z_n &= \alpha_n'' T^n x_n + (1 - \alpha_n'') x_n \\ y_n &= \alpha_n' T^n z_n + (1 - \alpha_n') x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geq 1, \end{aligned} \quad (1.3)$$

where $\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}$ are appropriate sequences in $[0, 1]$.

Asymptotically nonexpansive mappings since their introduction have been extensively studied by many authors in different frames of work. One is the convergence of iteration schemes constructed through asymptotically nonexpansive mappings. It is our purpose in this paper to establish several weak and strong convergence results presented in this paper extend and improve the corresponding ones announced by Xu and Noor [16], Cho, Zhou and Guo [3], and others. Now, we recall the well-known concepts and results.

Let C be a nonempty subset of normed space X . A mapping $T : C \rightarrow C$ is said to be *asymptotically nonexpansive* on C if there exists a sequence $\{r_n\}$ in $[0, \infty)$, with $\lim_{n \rightarrow \infty} r_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|,$$

for all $x, y \in C$ and each $n \geq 1$.

If $r_n \equiv 0$, then T is well known as a nonexpansive mapping. A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *demiclosed* at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p , then $Tx^* = p$.

A Banach space X is said to satisfy *Opial's condition* [5] if any sequence $\{x_n\}$ in C , $x_n \rightarrow x$ weakly as $n \rightarrow \infty$ implies that $\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$ for all $y \in C$ with $y \neq x$.

The mapping $T : C \rightarrow X$ with $F(T) \neq \emptyset$ is said to satisfy *condition (A)* [11] if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\|x - Tx\| \geq f(d(x, F(T)))$$

for all $x \in C$ where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$. Recently, Khan and Un-din [3] modified the condition (A) for two mappings as follows: Two mappings $T_1, T_2 : C \rightarrow X$ where C a subset of X , are said to satisfy *condition (A')* if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\frac{1}{2}(\|x - Tx\| + \|x - Sx\|) \geq f(d(x, F))$$

for all $x \in C$ where $F := F(T_1) \cap F(T_2)$ and $d(x, F) = \inf\{\|x - x^*\| : x^* \in F\}$. Note that condition (A') reduces to condition (A) when $T = S$. We modify this condition for three mappings $T_1, T_2, T_3 : C \rightarrow C$ as follows:

Three mappings $T_1, T_2, T_3 : C \rightarrow C$ where C a subset of X , are said to satisfy condition (A'') if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\frac{1}{3}(\|x - T_1x\| + \|x - T_2x\| + \|x - T_3x\|) \geq f(d(x, F))$$

for all $x \in C$ where $F := F(T_1) \cap F(T_2) \cap F(T_3)$. Note that condition (A'') reduces to condition (A) when $T_1 = T_2 = T_3$. It is well known that every continuous and demicompact mapping must satisfy condition (A) (see [11]). Since every completely continuous $T : C \rightarrow C$ is continuous and demicompact so that it satisfies Condition (A) . Thus we shall use condition (A) instead the complete continuity of the mapping T to study the strong convergence of $\{x_n\}$ defined in (1.1).

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.1 (see [14]). *Let the nonnegative number sequences $\{a_n\}, \{b_n\}$ and $\{d_n\}$ satisfy that*

$$a_{n+1} \leq (1 + b_n)a_n + d_n, \forall n = 1, 2, \dots$$

If $\sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} d_n < \infty$, then

- (i) $\lim_{n \rightarrow \infty} a_n$ exists.
- (ii) $\lim_{n \rightarrow \infty} a_n = 0$ whenever $\liminf_{n \rightarrow \infty} a_n = 0$.

Lemma 1.2 (see [10]). *Let X be a uniformly convex Banach space, $0 < \alpha \leq t_n \leq \beta < 1$ for all positive integers n . Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq a, \limsup_{n \rightarrow \infty} \|y_n\| \leq a$, and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = a, a \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Lemma 1.3 (see [3]). *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X , and $T : C \rightarrow C$ be an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at 0.*

2. MAIN RESULTS

In this section, we prove weak and strong convergence theorem of the modified Noor iteration with errors for three mappings in a Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 2.1. *Let X be a uniformly convex Banach space with $x_n, y_n \in X$, real number $a \geq 0, \alpha, \beta \in (0, 1)$ and $\{\alpha_n\}$ be a real sequence number which satisfying*

- (i) $0 < \alpha \leq \alpha_n \leq \beta < 1, \forall n \geq n_0$ and for some $n_0 \in \mathbb{N}$.

$$(ii) \limsup_{n \rightarrow \infty} \|x_n\| \leq a \text{ and } \limsup_{n \rightarrow \infty} \|y_n\| \leq a,$$

$$(iii) \lim_{n \rightarrow \infty} \|\alpha_n x_n + (1 - \alpha_n)y_n\| = a.$$

Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Proof. This is clear by Lemma 1.2. □

Lemma 2.2. Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T_1, T_2 and T_3 be asymptotically nonexpansive self-maps of C with sequences $\{r_n^{(1)}\}, \{r_n^{(2)}\}, \{r_n^{(3)}\}$ respectively such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$ for all $i = 1, 2, 3$ and $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty$. If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F(T)$.

Proof. Let $p \in F(T_1) \cap F(T_2) \cap F(T_3)$. Since $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are bounded sequences in C , we can put

$$M = \sup_{n \geq 1} \|u_n - p\| \vee \sup_{n \geq 1} \|v_n - p\| \vee \sup_{n \geq 1} \|w_n - p\|.$$

Then M is a finite number. For each $n \geq 1$, let $r_n = \max\{r_n^{(1)}, r_n^{(2)}, r_n^{(3)}\}$. Then $r_n \geq 0$, and $\lim_{n \rightarrow \infty} r_n = 0$. Moreover, we note that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n T_1^n y_n + \beta_n x_n + \gamma_n w_n - p\| \\ &\leq \alpha_n \|T_1^n y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\ &\leq \alpha_n (1 + r_n) \|y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\|, \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} \|y_n - p\| &= \|\alpha'_n T_2^n z_n + \beta'_n x_n + \gamma'_n v_n - p\| \\ &\leq \alpha'_n \|T_2^n z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\ &\leq \alpha'_n (1 + r_n) \|z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\|, \end{aligned} \quad (2.2)$$

and

$$\|z_n - p\| \leq \alpha''_n (1 + r_n) \|x_n - p\| + \beta''_n \|x_n - p\| + \gamma''_n \|u_n - p\|. \quad (2.3)$$

Substituting (2.3) into (2.2), we obtain

$$\begin{aligned}
 \|y_n - p\| &\leq \alpha'_n \alpha''_n (1 + r_n)^2 \|x_n - p\| + \alpha'_n \beta''_n (1 + r_n) \|x_n - p\| + \alpha'_n \gamma''_n (1 + r_n) \|u_n - p\| \\
 &\quad + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\
 &\leq (1 - \beta'_n - \gamma'_n) \alpha''_n (1 + r_n)^2 \|x_n - p\| + \beta'_n \|x_n - p\| \\
 &\quad + (1 - \beta'_n - \gamma'_n) \beta''_n (1 + r_n) \|x_n - p\| + m_n \\
 &\leq \beta'_n (1 + r_n)^2 \|x_n - p\| + (1 - \beta'_n) \alpha''_n (1 + r_n)^2 \|x_n - p\| \\
 &\quad + (1 - \beta'_n) \beta''_n (1 + r_n)^2 \|x_n - p\| + m_n \\
 &= \beta'_n (1 + r_n)^2 \|x_n - p\| + (1 - \beta'_n) (\alpha''_n + \beta''_n) (1 + r_n)^2 \|x_n - p\| + m_n \\
 &\leq \beta'_n (1 + r_n)^2 \|x_n - p\| + (1 - \beta'_n) (1 + r_n)^2 \|x_n - p\| + m_n \\
 &= (1 + r_n)^2 \|x_n - p\| + m_n.
 \end{aligned} \tag{2.4}$$

where $m_n = \gamma''_n (1 + r_n) M + \gamma'_n M$. So that $\sum_{n=1}^{\infty} m_n < \infty$. Substituting (2.4) into (2.1), we have

$$\begin{aligned}
 \|x_{n+1} - p\| &\leq \alpha_n (1 + r_n) ((1 + r_n)^2 \|x_n - p\| + m_n) + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\
 &= \alpha_n (1 + r_n)^3 \|x_n - p\| + \alpha_n (1 + r_n) m_n + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\
 &\leq (\alpha_n + \beta_n) (1 + r_n)^3 \|x_n - p\| + (1 + r_n) m_n + \gamma_n \|w_n - p\| \\
 &\leq (1 + r_n)^3 \|x_n - p\| + (1 + r_n) m_n + \gamma_n M \\
 &\leq (1 + r_n)^3 \|x_n - p\| + (1 + r_n) m_n + \gamma_n M \\
 &= (1 + d_n) \|x_n - p\| + b_n
 \end{aligned} \tag{2.5}$$

where $d_n = 3r_n + 3r_n^2 + r_n^3$ and $b_n = (1 + r_n) m_n + \gamma_n M$. Since $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} m_n < \infty$, we get $\sum_{n=1}^{\infty} d_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. It follows from Lemma 1.1 that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. This completes the proof. $\square$

Lemma 2.3. Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T_1, T_2 and T_3 be asymptotically nonexpansive self-maps of C with sequence $\{r_n^{(1)}\}, \{r_n^{(2)}\}, \{r_n^{(3)}\}$ respectively such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$ for all $i = 1, 2, 3$ and $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined by (1.1) and some α, β in $(0, 1)$ with the following restrictions:

(i) $0 < \alpha \leq \alpha_n, \alpha'_n, \alpha''_n \leq \beta < 1, \forall n \geq n_0$ for some $n_0 \in \mathbb{N}$.

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty$.

Then $\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = \lim_{n \rightarrow \infty} \|x_n - T_2 x_n\| = \lim_{n \rightarrow \infty} \|x_n - T_3 x_n\| = 0$.

Proof. For any $p \in F(T_1) \cap F(T_2) \cap F(T_3)$, it follows from Lemma 2.2, we have $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Let $\lim_{n \rightarrow \infty} \|x_n - p\| = a$ for some $a \geq 0$. For each $n \geq 1$

let $r_n = \max\{r_n^{(1)}, r_n^{(2)}, r_n^{(3)}\}$. Then $r_n \geq 0$ and $\lim_{n \rightarrow \infty} r_n = 0$. It follows from (2.4) that

$$\|y_n - p\| \leq (1 + r_n)^2 \|x_n - p\| + m_n.$$

Taking $\limsup_{n \rightarrow \infty}$ in both sides, we obtain

$$\limsup_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = \lim_{n \rightarrow \infty} \|x_n - p\| = a.$$

So that

$$\limsup_{n \rightarrow \infty} \|T_1^n y_n - p\| \leq \limsup_{n \rightarrow \infty} (1 + r_n) \|y_n - p\| = \limsup_{n \rightarrow \infty} \|y_n - p\| \leq a.$$

Next, consider

$$\|T_1^n y_n - p + \gamma_n(w_n - x_n)\| \leq \|T_1^n y_n - p\| + \gamma_n \|w_n - x_n\|.$$

Thus, we have

$$\limsup_{n \rightarrow \infty} \|T_1^n y_n - p + \gamma_n(w_n - x_n)\| \leq a, \quad (2.6)$$

and $\|x_n - p + \gamma_n(w_n - x_n)\| \leq \|x_n - p\| + \gamma_n \|w_n - x_n\|$.

This implies that

$$\limsup_{n \rightarrow \infty} \|x_n - p + \gamma_n(w_n - x_n)\| \leq a, \quad (2.7)$$

and

$$\begin{aligned} a &= \lim_{n \rightarrow \infty} \|x_{n+1} - p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n T_1^n y_n + \beta_n x_n + \gamma_n w_n - p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n T_1^n y_n + (1 - \alpha_n)x_n - \gamma_n x_n + \gamma_n w_n - (1 - \alpha_n)p - \alpha_n p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n T_1^n y_n - \alpha_n p + \alpha_n \gamma_n w_n - \alpha_n \gamma_n x_n + (1 - \alpha_n)x_n - (1 - \alpha_n)p \\ &\quad - \gamma_n x_n + \gamma_n w_n - \alpha_n \gamma_n w_n + \alpha_n \gamma_n x_n\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n (T_1^n y_n - p + \gamma_n(w_n - x_n)) + (1 - \alpha_n)(x_n - p + \gamma_n(w_n - x_n))\|. \end{aligned}$$

By J. Schu's Lemma 2.1, we have

$$\lim_{n \rightarrow \infty} \|T_1^n y_n - x_n\| = 0. \quad (2.8)$$

Next, we shall prove that $\lim_{n \rightarrow \infty} \|T_2^n z_n - x_n\| = 0$. For each $n \geq 1$,

$$\begin{aligned} \|x_n - p\| &\leq \|T_1^n y_n - x_n\| + \|T_1^n y_n - p\| \\ &\leq \|T_1^n y_n - x_n\| + (1 + r_n) \|y_n - p\|. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|T_1^n y_n - x_n\| = 0 = \lim_{n \rightarrow \infty} r_n$, we obtain

$$a = \lim_{n \rightarrow \infty} \|x_n - p\| \leq \liminf_{n \rightarrow \infty} \|y_n - p\|.$$

It follows that

$$a \leq \liminf_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|y_n - p\| \leq a.$$

This implies that

$$\lim_{n \rightarrow \infty} \|y_n - p\| = a.$$

On the other hand from (2.3), we have

$$\|z_n - p\| \leq (1 + r_n)\|x_n - p\| + \gamma_n''\|u_n - p\|.$$

By boundedness of the sequence $\{u_n\}$ and $\lim_{n \rightarrow \infty} r_n = 0 = \lim_{n \rightarrow \infty} \gamma_n''$, we have

$$\limsup_{n \rightarrow \infty} \|z_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = a,$$

and so

$$\limsup_{n \rightarrow \infty} \|T_2^n z_n - p\| \leq \limsup_{n \rightarrow \infty} (1 + r_n)\|z_n - p\| \leq a.$$

Next, consider

$$\|T_2^n z_n - p + \gamma_n'(v_n - x_n)\| \leq \|T_2^n z_n - p\| + \gamma_n'\|v_n - x_n\|.$$

Thus we have

$$\limsup_{n \rightarrow \infty} \|T_2^n z_n - p + \gamma_n'(v_n - x_n)\| \leq a,$$

and

$$\|x_n - p + \gamma_n'(v_n - x_n)\| \leq \|x_n - p\| + \gamma_n'\|v_n - x_n\|.$$

It follows that

$$\limsup_{n \rightarrow \infty} \|x_n - p + \gamma_n'(v_n - x_n)\| \leq a$$

and

$$\begin{aligned} a = \lim_{n \rightarrow \infty} \|y_n - p\| &= \lim_{n \rightarrow \infty} \|\alpha_n' T_2^n z_n + \beta_n' x_n + \gamma_n' v_n - p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n' [T_2^n z_n - x_n + \gamma_n'(v_n - x_n)] \\ &\quad + (1 - \alpha_n') [x_n - p + \gamma_n'(v_n - x_n)]\|. \end{aligned}$$

By J. Schu's Lemma 2.1, we have

$$\lim_{n \rightarrow \infty} \|T_2^n z_n - x_n\| = 0. \quad (2.9)$$

Similarly, by using the same argument as in the proof above, we have

$$\lim_{n \rightarrow \infty} \|T_3^n x_n - x_n\| = 0.$$

Hence

$$\lim_{n \rightarrow \infty} \|T_1^n y_n - x_n\| = 0, \lim_{n \rightarrow \infty} \|T_2^n z_n - x_n\| = 0, \lim_{n \rightarrow \infty} \|T_3^n x_n - x_n\| = 0$$

and this implies that

$$\|x_{n+1} - x_n\| \leq \alpha_n \|T_1^n y_n - x_n\| + \gamma_n \|w_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.10)$$

We note that

$$\begin{aligned}
 \|T_1^n x_n - x_n\| &\leq \|T_1^n x_n - T_1^n y_n\| + \|T_1^n y_n - x_n\| \\
 &\leq (1 + r_n)\|x_n - y_n\| + \|T_1^n y_n - x_n\| \\
 &\leq \alpha'_n(1 + r_n)\|x_n - T_2^n z_n\| + \gamma'_n(1 + r_n)\|v_n - x_n\| \\
 &\quad + \|T_1^n y_n - x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty,
 \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
 \|x_n - T_1 x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T_1^{n+1} x_{n+1}\| \\
 &\quad + \|T_1^{n+1} x_{n+1} - T_1^{n+1} x_n\| + \|T_1^{n+1} x_n - T_1 x_n\| \\
 &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T_1^{n+1} x_{n+1}\| \\
 &\quad + (1 + k_{n+1})\|x_{n+1} - x_n\| + (1 + k_1)\|T_1^n x_n - x_n\|.
 \end{aligned} \tag{2.12}$$

It follows from (2.10), (2.11) and the above inequality that

$$\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = 0. \tag{2.13}$$

Next, consider

$$\begin{aligned}
 \|T_2^n x_n - x_n\| &\leq \|T_2^n x_n - T_2^n z_n\| + \|T_2^n z_n - x_n\| \\
 &\leq (1 + r_n)\|x_n - z_n\| + \|T_2^n z_n - x_n\| \\
 &\leq \alpha''_n(1 + r_n)\|x_n - T_2^n z_n\| + \gamma''_n(1 + r_n)\|u_n - x_n\| \\
 &\quad + \|T_2^n z_n - x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty,
 \end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
 \|x_n - T_2 x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T_2^{n+1} x_{n+1}\| \\
 &\quad + \|T_2^{n+1} x_{n+1} - T_2^{n+1} x_n\| + \|T_2^{n+1} x_n - T_2 x_n\| \\
 &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T_2^{n+1} x_{n+1}\| \\
 &\quad + (1 + k_{n+1})\|x_{n+1} - x_n\| + (1 + k_1)\|T_2^n x_n - x_n\|.
 \end{aligned}$$

It follows from (2.10), (2.14) and the above inequality that

$$\lim_{n \rightarrow \infty} \|x_n - T_2 x_n\| = 0.$$

We have

$$\begin{aligned}
 \|x_n - T_3 x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T_3^{n+1} x_{n+1}\| \\
 &\quad + \|T_3^{n+1} x_{n+1} - T_3^{n+1} x_n\| + \|T_3^{n+1} x_n - T_3 x_n\| \\
 &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T_3^{n+1} x_{n+1}\| \\
 &\quad + (1 + k_{n+1})\|x_{n+1} - x_n\| + (1 + k_1)\|T_3^n x_n - x_n\|.
 \end{aligned}$$

It follows from (2.10), $\lim_{n \rightarrow \infty} \|T_3^n x_n - x_n\| = 0$, and the above inequality that

$$\lim_{n \rightarrow \infty} \|x_n - T_3 x_n\| = 0.$$

Therefore

$$\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = \lim_{n \rightarrow \infty} \|x_n - T_2 x_n\| = \lim_{n \rightarrow \infty} \|x_n - T_3 x_n\| = 0.$$

This completes the proof. $\square$

Theorem 2.4. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T_1, T_2 and T_3 be asymptotically nonexpansive self-maps of C with sequences $\{r_n^{(1)}\}, \{r_n^{(2)}\}, \{r_n^{(3)}\}$ respectively such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$ for all $i = 1, 2, 3$ and satisfying condition (A'') . Let $\{x_n\}$ be sequence as defined in (1.1) and some $\alpha, \beta \in (0, 1)$ with the following restrictions:*

$$(i) \ 0 < \alpha \leq \alpha_n, \alpha'_n, \alpha''_n \leq \beta < 1, \forall n \geq n_0 \text{ for some } n_0 \in \mathbb{N}.$$

$$(ii) \sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty.$$

If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then $\{x_n\}, \{y_n\}, \{z_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3 .

Proof. By Lemma 2.2, we readily see that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F(T_1) \cap F(T_2) \cap F(T_3)$. Let $\lim_{n \rightarrow \infty} \|x_n - p\| = a$ for some $a \geq 0$. Without loss of generality, we may assume $a > 0$. As proved in Lemma 2.2, we have

$$\|x_{n+1} - p\| \leq (1 + d_n)\|x_n - p\| + b_n$$

which gives that

$$d(x_{n+1}, F) \leq (1 + d_n)d(x_n, F) + b_n$$

where $F := F(T_1) \cap F(T_2) \cap F(T_3)$. Applying Lemma 1.1 to the above inequality, we obtain that $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. Now by the combined effect condition (A'') and Lemma 2.3, we get that

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) \leq \lim_{n \rightarrow \infty} \left(\frac{1}{3} (\|x_n - T_1 x_n\| + \|x_n - T_2 x_n\| + \|x_n - T_3 x_n\|) \right) = 0.$$

It show that $\lim_{n \rightarrow \infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\}$ in F such that $\|x_{n_k} - y_k\| < \frac{1}{2^k}$. It follows from the proof of Tan and Xu[14], we get that $\{y_k\}$ is Cauchy sequence in F , and so $y_k \rightarrow y$ for some $y \in F$. It follows that $x_{n_k} \rightarrow y$. Since $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, $x_n \rightarrow y$.

Since

$$\begin{aligned} \|y_n - x_n\| &\leq \alpha'_n \|T^n z_n - x_n\| + \gamma'_n \|v_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty \\ , \text{ and } \|z_n - x_n\| &\leq \alpha''_n \|T^n z_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

it follows that $\lim_{n \rightarrow \infty} y_n = y$ and $\lim_{n \rightarrow \infty} z_n = y$. This completely the proof. $\square$

If $T_1 = T_2 = T_3 := T$, then (1.1) reduces to modified Noor iteration scheme with errors and so we obtain the following result:

Corollary 2.5. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping the nonempty fixed point set $F(T)$ and a sequence $\{r_n\}$ in $[0, \infty)$ with $\sum r_n < \infty$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{aligned} z_n &= \alpha''_n T^n x_n + (1 - \alpha''_n - \gamma''_n) x_n + \gamma''_n u_n \\ y_n &= \alpha'_n T^n z_n + (1 - \alpha'_n - \gamma'_n) x_n + \gamma'_n v_n \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n - \gamma_n) x_n + \gamma_n w_n \quad n \geq 1, \end{aligned}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C and $\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}, \{\gamma_n\}, \{\gamma'_n\}, \{\gamma''_n\}$ are real sequences in $[0, 1]$ satisfying

- (i) $0 < \alpha \leq \alpha_n, \alpha'_n \leq \beta < 1$ (α''_n need not in $[\alpha, \beta]$), $\forall n \geq n_0$ for some $n_0 \in \mathbb{N}$.
- (ii) $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T .

Proof. Setting $T_1 = T_2 = T_3 = T$ in Lemma 2.3. It follows from (2.8), (2.9) and (2.12) that $\lim_{n \rightarrow \infty} \|T x_n - x_n\| = 0$. Since T satisfies condition (A) with respect to the sequence $\{x_n\}$, and so the conclusion of the corollary follows from Theorem 2.4. This completes the proof. $\square$

Corollary 2.6. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let $T : C \rightarrow C$ be completely continuous asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and a sequence $\{r_n\}$ in $[0, \infty)$ with $\sum r_n < \infty$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{aligned} z_n &= \alpha''_n T^n x_n + \beta''_n x_n + \gamma''_n u_n \\ y_n &= \alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \quad n \geq 1 \end{aligned}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequence in C and $\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}, \{\beta_n\}, \{\beta'_n\}, \{\beta''_n\}, \{\gamma_n\}, \{\gamma'_n\}, \{\gamma''_n\}$ are real sequences in $[0, 1]$ satisfying

- (i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$.
- (ii) $0 < \alpha \leq \alpha_n, \alpha'_n \leq \beta < 1$ (α''_n need not in $[\alpha, \beta]$), $\forall n \geq n_0$ for some $n_0 \in \mathbb{N}$.
- (iii) $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty$.

Then $\{x_n\}, \{y_n\}, \{z_n\}$ converges strongly to a fixed point of T .

Proof. Setting $T_1 = T_2 = T_3 = T$ in Lemma 2.3. It follows from (2.8), (2.9) and (2.12) that $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$. Since T is completely continuous, so it satisfies condition (A) on C , and so the conclusion of the corollary follows from Theorem 2.4. This completes the proof. $\square$

For $\gamma_n = \gamma'_n = \gamma''_n \equiv 0$, in Corollary (2.6), we can obtain the following result.

Corollary 2.7. [16, Theorem 2.1]. *Let X be a real uniformly convex Banach space, C be a nonempty closed, bounded convex subset of X . Let T be a completely continuous and asymptotically nonexpansive self-mapping with sequence $\{r_n\}$ satisfying $r_n \geq 0$ and $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}$ be real sequences in $[0, 1]$ satisfying;*

- (i) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$, and
- (ii) $0 < \liminf_{n \rightarrow \infty} \alpha'_n \leq \limsup_{n \rightarrow \infty} \alpha'_n < 1$.

For a give $x_1 \in C$, the sequence $\{x_n\}, \{y_n\}, \{z_n\}$ defined by

$$\begin{aligned} z_n &= \alpha''_n T_n x_n + (1 - \alpha''_n) x_n \\ y_n &= \alpha'_n T_n z_n + (1 - \alpha'_n) x_n \\ x_{n+1} &= \alpha_n T_n y_n + (1 - \alpha_n) x_n, \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}, \{y_n\}, \{z_n\}$ converges strongly to a fixed point of T .

Proof. It follows from the condition (i) and (ii) that there exists $\alpha, \beta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that

$$0 < \alpha \leq \alpha_n, \alpha'_n \leq \beta < 1$$

for all $n \geq n_0$. So that the conclusion of the Corollary follows from Corollary 2.6. $\square$

In the next result, we prove weak convergence for the modified Noor iterations with errors for three asymptotically nonexpansive mappings in a Banach space satisfying Opial's condition.

Lemma 2.8. *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X . Let $u, v \in X$ be such that $\lim_{n \rightarrow \infty} \|x_n - u\|$ and $\lim_{n \rightarrow \infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequence of x_n which converge weakly to u and v , respectively, then $u = v$.*

Proof. See [12] □

Theorem 2.9. *Let X be a real uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed convex subset of X . Let T_1, T_2 and T_3 be asymptotically nonexpansive self-maps of C with sequence $\{r_n^{(1)}\}, \{r_n^{(2)}\}, \{r_n^{(3)}\}$ respectively such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$ for all $i = 1, 2, 3$. Let $\{x_n\}$ be sequence as defined in (1.1) and some $\alpha, \beta \in (0, 1)$ with the following restrictions:*

$$(i) \ 0 < \alpha \leq \alpha_n, \alpha'_n, \alpha''_n \leq \beta < 1, \forall n \geq n_0 \text{ for some } n_0 \in \mathbb{N},$$

$$(ii) \sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty.$$

If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then $\{x_n\}, \{y_n\}, \{z_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3 .

Proof. It follows from Lemma 2.3 that

$$\lim_{n \rightarrow \infty} \|T_1 x_n - x_n\| = 0, \lim_{n \rightarrow \infty} \|T_2 x_n - x_n\| = 0, \lim_{n \rightarrow \infty} \|T_3 x_n - x_n\| = 0.$$

Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \rightarrow q$ weakly as $n \rightarrow \infty$, without loss of generality. By Lemma 1.3, we have $q \in F(T_1) \cap F(T_2) \cap F(T_3)$. Suppose that subsequence $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v respectively. From Lemma 1.3, $u, v \in F(T_1) \cap F(T_2) \cap F(T_3)$. By Lemma 2.2, $\lim_{n \rightarrow \infty} \|x_n - u\|$ and $\lim_{n \rightarrow \infty} \|x_n - v\|$ exist. It follows from Lemma 2.8 that $u = v$. Therefore $\{x_n\}$ converges weakly to a point in $F(T_1) \cap F(T_2) \cap F(T_3)$. □

For $T_1 = T_2 = T_3 := T$ then we obtain the following results:

Corollary 2.10. *Let X be a real uniformly convex Banach space which satisfies Opial's condition, C a nonempty closed convex subset of X . Let $T : C \rightarrow C$ be asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and a sequence $\{r_n\}$ in $[0, \infty)$ with $\sum r_n < \infty$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{aligned} z_n &= \alpha''_n T^n x_n + \beta''_n x_n + \gamma''_n u_n \\ y_n &= \alpha'_n T^n z_n + \beta'_n z_n + \gamma'_n v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n \quad n \geq 1 \end{aligned}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequence in C and $\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}, \{\beta_n\}, \{\beta'_n\}, \{\beta''_n\}, \{\gamma_n\}, \{\gamma'_n\}, \{\gamma''_n\}$ are real sequences in $[0, 1]$ satisfying

- (i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$.
- (ii) $0 < \alpha \leq \alpha_n, \alpha'_n \leq \beta < 1$ (α''_n need not in $[\alpha, \beta]$), $\forall n \geq n_0$ for some $n_0 \in \mathbb{N}$,
- (iii) $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty$.

Then $\{x_n\}, \{y_n\}, \{z_n\}$ converges weakly to a fixed point of T .

For $\gamma_n = \gamma'_n = \gamma''_n \equiv 0$, in the Corollary (2.10), we can obtain the following result.

Corollary 2.11. *Let X be a real uniformly convex Banach space which satisfies Opial's condition, C be a nonempty closed, bounded convex subset of X . Let T be an asymptotically nonexpansive self-mapping with sequence $\{r_n\}$ satisfying $r_n \geq 0$ and $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}$ be real sequences in $[0, 1]$ satisfying;*

- (i) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$, and
- (ii) $0 < \liminf_{n \rightarrow \infty} \alpha'_n \leq \limsup_{n \rightarrow \infty} \alpha'_n < 1$.

For a give $x_1 \in C$, the sequence $\{x_n\}, \{y_n\}, \{z_n\}$ defined by

$$\begin{aligned} z_n &= \alpha''_n T_n x_n + (1 - \alpha''_n) x_n \\ y_n &= \alpha'_n T_n z_n + (1 - \alpha'_n) x_n \\ x_{n+1} &= \alpha_n T_n y_n + (1 - \alpha_n) x_n, \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}, \{y_n\}, \{z_n\}$ converges weakly to a fixed point of T .

Acknowledgment: The author thanks the Thailand Research Fund for their financial support.

REFERENCES

1. F.E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, Proceedings of the Symposium on Pure Mathematics, vol. 18, Proc. Amer. Soc., Providence, RI, 1976.
2. C.E. Chidume, Chika Moore, Fixed points iteration for pseudocontractive maps, Proc. Amer. Math. Soc. 127(4)(1999) 1163-1170.
3. Y.J. Cho, H. Zhou and G. Guo, weak and strong convergence theorems for three-step iterations with errors for Asymptotically nonexpansive mappings, Com. and Math. with Appl. 47(2004) 707-717.

4. G. Das, J.P. Debata, Fixed points of quasi-nonexpansive mappings, Indian J. Pure Appl. Math. 17(1986) 1263-1269.
5. S. H. Khan, H.F.-ud-din, weak and strong convergence of a scheme with errors for two nonexpansive mappings. 61(2005) 1295-1301.
6. M. Maiti, M.K. Gosh, Approximating fixed points by Ishikawa iterates, Bull. Austral. Math. Soc. 40(1989) 113-117.
7. Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 733(1967) 591-597.
8. B.E. Rhoades, Fixed point iteration for certain nonlinear mappings, J. Math. Anal. Appl. 183(1994) 118-120.
9. J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. 158(1991) 407-413.
10. J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. austral. Math. Soc.
11. H. F. Senter, W. G. Dotson, Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc. 44(2) (1974) 375-380.
12. S. Suantai, Weak and strong convergence criteria of Noor iteration for asymptotically nonexpansive mappings, J. Math. Anal. Appl. (to appear)
13. W. Takahashi, T. Tamura, Convergence theorems for a pair of nonexpansive mappings, J. Convex Analysis 5 (1)(1998) 45-58.
14. K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178(1993) 301-308.
15. Y. Xu, Ishikawa and Mann Iteration process with errors for nonlinear strongly accretive operator equations, J. Math. Anal. Appl. 224(1998) 91-101.
16. B. L. Xu and M. Aslam Noor, Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 267(2002) 444-453.

ภาคผนวก 17/16

Somyot Plubtieng

From: "JMAA (ELS)" <jmaa@elsevier.com>
To: <somyotp@nu.ac.th>
Sent: Tuesday, June 14, 2005 11:16 PM
Subject: JMAA Submission - Manuscript Number Assigned

Ms. No.: JMAA-05-1168

Title: Weak and strong convergence of modified noor iterations with errors for three asymptotically nonexpansive mappings

Corresponding Author: Mr Somyot - Plubtieng

Authors: RABIAN WANGKEEREE; RATTANAPORN PUNPAENG

Dear Mr Plubtieng,

Your submission, referenced above, has been assigned the following manuscript number: JMAA-05-1168

You will be able to check on the progress of your paper by logging on to the Elsevier Editorial System as an author:

<http://ees.elsevier.com/jmaa/>

Your username is: somyotp@nu.ac.th

Your password is: XXXXXXXXXX

Thank you for submitting your work to the Journal of Mathematical Analysis and Applications.

Kind regards,

Allyn Molina & Kristin Stair

Journal Managers

Journal of Mathematical Analysis and Applications

E-mail: jmaa@elsevier.com

8/16/2005