RSA4680022 รศ.ดร.สมยศ พลับเที่ยง

รายงานวิจัยฉบับสมบูรณ์

โครงการ : ทฤษฎีบทจุดคงที่สำหรับนัยทั่วไปของการส่งแบบนอนเอกซ์แพนซีฟ

Fixed point theorem for generalized of nonexpansive mappings

โดย

รองศาสตราจารย์ ดร.สมยศ พลับเที่ยง และคณะ

15 สิงหาคม 2548

สัญญาเลขที่ RSA4680022

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

งานวิจัยนี้ได้รับทุนสนับสนุนการวิจัยจากทุนพัฒนานักวิจัยประจำปี 2546 ของสำนักงานกองทุนสนับสนุนการวิจัย ผู้วิจัยขอขอบพระคุณเจ้าของทุนเป็นอย่างสูงมา ณ โอกาสนี้

ขอขอบพระคุณหัวหน้าภาควิชาคณิตศาสตร์ และคณบดีคณะวิทยาศาสตร์มหาวิทยาลัยนเรศวร ที่ได้ให้การสนับสนุนและอำนวยความสะดวกในการใช้พัสดุ ครุภัณฑ์ในการวิจัยเป็นอย่างดีตลอดโครงการ และสุดท้ายขอขอบคุณคณะผู้ช่วยวิจัยซึ่งประกอบด้วย อาจารย์ระเบียน วังคีรี อาจารย์ภูมิ คำเอม อาจารย์อิสระ อินจันทร์ อาจารย์รัตนาพร พันแพง อาจารย์อาทิตย์ แข็งธัญการ และ นายเกษมสุข อุงจิตตระกูล ซึ่งเป็นนิสิตระดับปริญญาโทและเอกที่ผู้วิจัยเป็นอาจารย์ที่ปรึกษาวิทยานิพนธ์ ที่ได้ร่วมกันสร้างผลงานวิจัยที่ได้รวบรวมไว้ในรายงานวิจัยนี้

รองศาสตราจารย์ ดร.สมยศ พลับเที่ยง

Abstract

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Investigator : Assoc. Dr. Somyot Plubtieng

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In this paper we present some results on fixed point theorems of mappings of nonexpansive, asymptotically nonexpansive, asymptotically quasi-nonexpansive, and asymptotically nonexpansive in the intermediate sense. Firstly, we proved weak and strong convergence theorems of three step (multi-step) iterative scheme with errors to a fixed point for generalized of nonexpansive mappings as above mension. Moreover, we also prove strong convergence theorem of implicit iteration process for self (nonself) mappings of nonexpansive and asymptotically nonexpansive, respectively. Finally, we proved the random fixed point theorems for nonexpansive random operators and multi-valued nonexpansive random operators.

Keywords : Fixed point theorem, Three-step iteration, Implicit iteration, Random operators, Random fixed point.

บทคัดย่อ

ชื่อโครงการ : ทฤษฎีบทจุดคงที่สำหรับนัยทั่วไปของการส่งแบบนอนเอกซ์แพนซีฟ

ชื่อนักวิจัย : รศ.ดร.สมยศ พลับเที่ยง

ระยะเวลาโครงการ : 15 สิงหาคม 2546 ถึง 14 สิงหาคม 2548

ในงานวิจัยนี้ ผู้วิจัยได้เสนอเนื้อหาบางอย่างของทฤษฎีบทจุดคงที่สำหรับการส่งแบบนอนเอกแพนซีฟ แอสซิมโททิคอลีนอนเอกแพนซีฟ แอสซิมโททิควอไซนอนเอกแพนซิฟ แอสซิมโททิคอลีนอนเอกแพนซีฟในรูปแบบอินเทอร์มิเดท

ประการแรกผู้วิจัยได้พิสูจน์ ทฤษฎีบทการลู่เข้าแบบแข็งและแบบอ่อนของกระบวนการทำซ้ำ 3 ขั้นตอน (หลายขั้นตอน) พร้อมด้วยค่าคาดเคลื่อนไปยังจุดคงที่สำหรับนัยทั่วไปของการส่งแบบนอนเอกแพนซีฟซึ่งได้กล่าวไว้ข้างบนนี้ ยิ่งไปกว่านั้นผู้วิจัยได้พิสูจน์ทฤษฎีการลู่เข้าแบบแข็งของกระบวนการกระทำซ้ำแบบอิมพลิซิท สำหรับการส่งแบบนอนเอกแพนซีฟและแอสซิมโททิคอลีนอนเอกแพนซีฟ สุดท้ายผู้วิจัยได้พิสูจน์ทฤษฎีบทจุดคงที่สำหรับตัวดำเนินการแบบสุ่มแบบนอนเอกแพนซีฟตัว และตัวดำเนินการสุ่มค่าเซตแบบนอนเอกแพนซีฟ.

คำหลัก : ทฤษฎีบทจุดคงที่ กระบวนการกระทำซ้ำสามขั้นตอน กระบวนการกระทำซ้ำแฝง ตัวดำเนินการสุ่ม จุดคงที่ของตัวดำเนินการสุ่ม

บทนำ

ทฤษฎีบทจุดคงที่ (Fixed point theorem) ถือว่าเป็นทฤษฎีบทที่มีความสำคัญมากในการนำไป ประยุกต์ทั้งในสาขาวิชาคณิตศาสตร์เองและในสาขาอื่นๆ การศึกษาวิจัยในเรื่องของทฤษฎีบทจุดคงที่บนปริภูมิบานาคเป็น การหาเงื่อนไขที่เพียงพอที่จะทำให้ฟังก์ชัน T ที่ส่งจากเซตย่อย K ของปริภูมิบานาค X ไปยัง K มีจุดคงที่ (นั่นคือจะมีจุด a ใน K ซึ่งทำให้ $T(a) = a$) ทฤษฎีบทจุดคงที่บนปริภูมิบานาคที่สำคัญเริ่มต้นจากในปี ค.ศ. 1922 Banach ได้พิสูจน์ว่าถ้า (X, d) เป็นปริภูมิเมตริกบริบูรณ์ และ $T : X \to X$ เป็น contraction (นั่นคือ จะมี $c \in (0,1)$ ซึ่งทำให้ $\|T(x)-T(y)\| \leqslant c\|x-y\|$ ทุกๆ $x, y \in X$) แล้ว T จะมีจุดคงที่เพียงจุดเดียว ต่อมาในปี ค.ศ. 1930 Schauder ได้พิสูจน์ว่าถ้า K เป็นเซตย่อยที่ไม่เป็นเซตว่างซึ่ง เป็นทั้งเซตคอมแพกต์และเซตคอนเวกซ์ของปริภูมิบานาค X และ $T : K \to K$ เป็นฟังก์ชันต่อเนื่องแล้ว T จะมีจุดคงที่ และต่อมาในปี ค.ศ. 1965 Browder ได้พิสูจน์ว่าถ้า K เป็นเซตย่อยที่ไม่เป็นเซตว่างซึ่งเป็น ทั้งเซตปิดที่มีขอบเขตและคอนเวกซ์ของ uniformly convex Banach space X และ $T : K \to K$ เป็น nonexpansive (นั่นคือ $\|T(x)-T(y)\| \leqslant \|x-y\|$ ทุกๆ $x, y \in K$) แล้ว T จะมีจุดคงที่ หลังจากนั้นเป็นต้นมาได้มีนักคณิตศาสตร์จำนวนมาก ที่ทำการศึกษาวิจัยเพื่อหาเงื่อนไขที่เป็นคุณสมบัติทางเรขาคณิตของปริภูมบานาค เพื่อใช้พิสูจน์ทฤษฎีบทจุดคงที่สำหรับ nonexpansive mappings

จุดประสงค์ของการวิจัย

1. เพื่อศึกษาคุณสมบัติทางเรขาคณิตของปริภูมิบานาค ที่เพียงพอที่จะพิสูจน์ Fixed point theorems for generalized nonexpansive mappings.
2. เพื่อศึกษาคุณสมบัติทางเรขาคณิตของปริภูมิบานาค ที่เพียงพอที่จะพิสูจน์ Fixed point theorems for multivalued nonexpansive mappings
3. เพื่อศึกษาคุณสมบัติทางเรขาคณิตของปริภูมิบานาค ที่เพียงพอที่จะพิสูจน์ Random fixed point theorems for (multivalued) nonexpansive random operators
4. เพื่อศึกษาการลู่เข้าของลำดับที่เกิดจาก Iterative contraction of Mann iteration, Ishikawa iteration, Three-step iteration and multi-step iteration

ผลการวิจัย

1. Three-step and multi-step iteration
   1.1 **S. Plubtieng and R. Wangkeeree, Fixed point iteration for asymptotically quasi-nonexpansive mappings in Banach spaces,**
   Theorem 1 Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let T be uniformly L-Lipschitzian, completely continuous and asymptotically quasi-nonexpansive mapping with sequence $\{k_n\}_{n\geq 1}$ such

that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_0 \in C$ and for each $n \geq 0$,

$$\begin{aligned} z_n &= \alpha''_n T^n x_n + \beta''_n x_n + \gamma''_n u_n \\ y_n &= \alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned}$$

where $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are three bounded sequences in C and $\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}$, $\{\beta_n\}, \{\beta'_n\}, \{\beta''_n\}, \{\gamma_n\}, \{\gamma'_n\}$ and $\{\gamma''_n\}$ are real sequences in $[0, 1]$ which satisfies the same assumptions as Lemma ?? and the additional assumption that $0 \leq \alpha < \alpha_n, \beta_n, \alpha'_n, \beta'_n \leq \beta < 1$ for some α, β in $(0, 1)$. Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

1.2 **S. Plubtieng and R. Wangkeeree, Noor Iterations with error for non-Lipschitzian mappings in Banach spaces,**

Theorem 1 Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let T be a completely continuous asymptotically nonexpansive in the intermediate sense. Put

$$G_n = \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \forall n \geq 1.$$

Let $x_0 \in C$ and for each $n \geq 0$,

$$\begin{aligned} z_n &= \alpha''_n T^n x_n + \beta''_n x_n + \gamma''_n u_n \\ y_n &= \alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned}$$

where $\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}, \{\beta_n\}, \{\beta'_n\}, \{\beta''_n\}, \{\gamma_n\}, \{\gamma'_n\}$ and $\{\gamma''_n\}$ are real sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are three bounded sequences in C such that

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$.

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty$.

(iii) $0 < \alpha \leq \alpha_n, \alpha'_n \leq \beta < 1$. Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converges strongly to a fixed point of T.

1.3 **S. Plubtieng and R. Wangkeeree, Strong convergence theorems for multi-step Noor iterations with errors in Banach spaces,**

Let C be a nonempty subset of normed space X and let $T : C \to C$ be a mapping. For a given $x_1 \in C$, and a fixed $m \in \mathbb{N}$ ($\mathbb{N}$ denote the set of all positive integers), compute the iterative sequences $\{x_n^{(1)}\}, ..., \{x_n^{(m)}\}$ defined by

$$
\begin{aligned}
x_n^{(1)} &= \alpha_n^{(1)} T^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)}, \\
x_n^{(2)} &= \alpha_n^{(2)} T^n x_n^{(1)} + \beta_n^{(2)} x_n + \gamma_n^{(2)} u_n^{(2)}, \\
x_n^{(3)} &= \alpha_n^{(3)} T^n x_n^{(2)} + \beta_n^{(3)} x_n + \gamma_n^{(3)} u_n^{(3)}, \\
&\vdots \\
x_n^{(m-1)} &= \alpha_n^{(m-1)} T^n x_n^{(m-2)} + \beta_n^{(m-1)} x_n + \gamma_n^{(m-1)} u_n^{(m-1)}, \\
x_{n+1} = x_n^{(m)} &= \alpha_n^{(m)} T^n x_n^{(m-1)} + \beta_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)}, \quad n \geq 1
\end{aligned} \tag{0.1}
$$

where, $\{u_n^{(1)}\}, ..., \{u_n^{(m)}\}$ are bounded sequences in C and $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ are appropriate real sequences in $[0, 1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $i \in \{1, 2, ..., m\}$.

Theorem 1 Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X and $T : C \to C$ be a completely continuous asymptotically nonexpansive in the intermediate sense. Put

$$G_n = \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let the sequence $\{x_n\}$ be defined by (0.2) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfies the following restrictions:
(i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for all $i \in \{1, 2, ..., m\}$ and for all $n \geq 1$;
(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i \in \{1, 2, ..., m\}$. If $0 < \alpha \leq \alpha_n^{(m-1)}, \alpha_n^{(m)} \leq \beta < 1$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Then $\{x_n^{(k)}\}$ converges strongly to a fixed point of T for each $k = 1, 2, 3, ..., m$.

Theorem 2 Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X and $T : C \to C$ be an uniformly L-Lipschitzian, completely continuous asymptotically quasi-nonexpansive with the sequence $\{r_n\}_{n \geq 1}$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (0.2) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfies the following restrictions:
(i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for all $i \in \{1, 2, ..., m\}$ and for all $n \geq 1$;
(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i \in \{1, 2, ..., m\}$. If $0 < \alpha \leq \alpha_n^{(i)} \leq \beta < 1$ for all $i \in \{m - 1, m\}$. Then $\{x_n^{(k)}\}$ converge strongly to a fixed point of T, for each $k = 1, 2, 3, ..., m$.

1.4 **S. Plubtieng and R. Wangkeeree, Strong convergence theorems for three-step iterations with errors for non-Lipschitzian nonself-mappings in Banach spaces,**
Algorithm 1.1(Three step iterative scheme for nonself maps with errors) Let C be a nonempty subset of normed space X. Let $P : X \to C$ be the nonexpansive retraction of X onto C and a mapping $T : C \to X$. For a given $x_0 \in C$,

compute the iteration sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by

(0.2)
$$\begin{aligned} z_n &= P\left(\alpha_n'' T(PT)^{n-1} x_n + \beta_n'' x_n + \gamma_n'' u_n\right) \\ y_n &= P\left(\alpha_n' T(PT)^{n-1} z_n + \beta_n' x_n + \gamma_n' v_n\right) \\ x_{n+1} &= P\left(\alpha_n T(PT)^{n-1} y_n + \beta_n x_n + \gamma_n w_n\right), \end{aligned}$$

where $\{\alpha_n\}, \{\alpha_n'\}, \{\alpha_n''\}, \{\beta_n\}, \{\beta_n'\}, \{\beta_n''\}, \{\gamma_n\}, \{\gamma_n'\}$ and $\{\gamma_n''\}$ are appropriate real sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are three bounded sequences in C.

Theorem 1 Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let T be an asymptotically nonexpansive in the intermediate sense nonself mapping with nonempty fixed point set $F(T)$. Put

$$G_n = \sup_{x,y\in C} \left(\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x - y\|\right) \vee 0, \forall n \geq 1.$$

Let the sequence $\{x_n\}$ be defined by (0.2) with the following restrictions

(i) $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$.

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma_n' < \infty, \sum_{n=1}^{\infty} \gamma_n'' < \infty$.

(iii) $0 \leq \alpha < \alpha_n, \beta_n, \alpha_n', \beta_n' \leq \beta < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.

1.5 S. Plubtieng and R. Wangkeeree Ishikawa Iteration Sequences for Asymptotically Quasi-Nonexpansive Nonself-Mappings with Error Members,

Let C be a nonempty closed convex subset of a real uniformly convex Banach space X. The following iteration process is studied:

(0.3)
$$\begin{aligned} x_1 \in C, x_{n+1} &= P\left(\alpha_n x_n + \beta_n T(PT)^{n-1} y_n + \gamma_n u_n\right), \\ y_n &= P\left(\alpha_n' x_n + \beta_n' T(PT)^{n-1} x_n + \gamma_n' v_n\right) \end{aligned}$$

where $\{u_n\}, \{v_n\}$ are bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n'\}\{\beta_n'\}$ and $\{\gamma_n'\}$ are sequences in $[0, 1]$ and P is a nonexpansive retraction of X onto C.

Theorem 1 Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let $T : C \to X$ be an asymptotically quasi-nonexpansive nonself-mapping with sequence $\{k_n\}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_1 \in C, \{\alpha_n\}, \{\beta_n\}$ $\{\gamma_n\}, \{\alpha_n'\}, \{\beta_n'\}$ and $\{\gamma_n'\}$ be sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha_n' + \beta_n' + \gamma_n', \sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n' < \infty$. Then the sequence $\{x_n\}$ defined by (0.3) strongly converges to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x, F(T))$ denote the distance of x to the set $F(T)$, i.e., $d(x, F(T)) = \inf_{y\in F(T)} d(x, y)$.

Theorem 2 Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let $T : C \to X$ be an uniformly L-Lipschitzian completely continuous and asymptotically quasi-nonexpansive nonself-mapping with sequence $\{k_n\}$ in $[0,\infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_1 \in C, \{\alpha_n\}, \{\beta_n\}$ $\{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ be sequences in $[0,1]$ such that $0 < \alpha < \alpha_n, \beta_n, \alpha'_n, \beta'_n < \beta < 1, \alpha_n+\beta_n+\gamma_n = 1 = \alpha'_n+\beta'_n+\gamma'_n, \sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$ Then the sequence $\{x_n\}$ defined by (0.3) strongly converges to a fixed point of T.

1.6 **I. Inchan and S. Plubtieng, Weak and strong convergence of scheme with errors for a finite family of nonexpansive mappings,**

Let C be a nonempty subset of normed space X and let $T_1, T_2, ..., T_N$ be nonexpansive mappings of C into itself. The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 = x \in C, \\ x_n^1 = \alpha_n^1 T_1 x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1, \\ x_n^2 = \alpha_n^2 T_2 x_n^1 + \beta_n^2 x_n + \gamma_n^2 u_n^2, \\ x_n^3 = \alpha_n^3 T_3 x_n^2 + \beta_n^3 x_n + \gamma_n^3 u_n^3, \\ x_n^4 = \alpha_n^4 T_4 x_n^3 + \beta_n^4 x_n + \gamma_n^4 u_n^4, \\ \vdots \\ x_{n+1} = x_n^N = \alpha_n^N T_N x_n^{N-1} + \beta_n^N x_n + \gamma_n^N u_n^N, n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n^1\}, ..., \{\alpha_n^N\}, \{\beta_n^1\}, ..., \{\beta_n^N\}, \{\gamma_n^1\}, ..., \{\gamma_n^N\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1,2,3,...,N$ and $\{u_n^1\}, \{u_n^2\}, ..., \{u_n^N\}$ are bounded sequences in C.

Theorem 1 Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T_1, T_2, \ldots, T_N$ be a nonexpansive mappings of C into itself satisfying condition (A^N) and $\{x_n\}$ be a sequence as defined in (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2, \ldots, N$. If $F = \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$, then $\{x_n\}$ converges strongly to a common fixed point in F.

Theorem 2 Let X be a uniformly convex Banach space satisfying the Opial's condition, C its nonempty closed convex subset of X. Let $T_1, T_2, ..., T_N$ be nonexpansive mappings of C into itself and $\{x_n\}$ be a sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2, \ldots, N$. If $F = \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$, then $\{x_n\}$ converges weakly to a common fixed point in F.

Theorem 3 Let C be a nonempty closed convex subset of uniformly convex Banach space X, and let $T_1, T_2, \ldots, T_N$ nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$. If P is a metric projection of C onto F and $\{x_n\}$

is a sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $0 < \alpha < \alpha_n^i < \beta < 1$ for all $n \in \mathbb{N}$ and for all $i = 1, 2, \ldots, N$, then $\{Px_n\}$ converges strongly to a common fixed point in F.

1.7 **S. Plubtieng, R. Punpeang and R. Wangkeeree, Weak and strong convergence of modified Noor iterations with errors for three asymptotically nonexpansive mappings**

Let X be a normed space, C be a nonempty convex subset of X, and $T_1, T_2, T_3 : C \to C$ be three given mappings. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by

$$
\begin{aligned}
z_n &= \alpha_n'' T_3^n x_n + \beta_n'' x_n + \gamma_n'' u_n \\
y_n &= \alpha_n' T_2^n z_n + \beta_n' x_n + \gamma_n' v_n \\
x_{n+1} &= \alpha_n T_1^n y_n + \beta_n x_n + \gamma_n w_n \quad n \geq 1,
\end{aligned}
\tag{0.4}
$$

where $\{\alpha_n\}, \{\alpha_n'\}, \{\alpha_n''\}, \{\beta_n\}, \{\beta_n'\}, \{\beta_n''\}, \{\gamma_n\}, \{\gamma_n'\}$ and $\{\gamma_n''\}$ are real sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$ and $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C.

Theorem 1 Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let T_1, T_2 and T_3 be asymptotically nonexpansive self-maps of C with sequences $\{r_n^{(1)}\}, \{r_n^{(2)}\}, \{r_n^{(3)}\}$ respectively such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$ for all $i = 1, 2, 3$ and satisfying condition (A''). Let $\{x_n\}$ be sequence as defined in (0.4) and some $\alpha, \beta \in (0,1)$ with the following restrictions:
(i) $0 < \alpha \leq \alpha_n, \alpha_n', \alpha_n'' \leq \beta < 1$, $\forall n \geq n_0$ for some $n_0 \in \mathbb{N}$.
(ii)$\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma_n' < \infty, \sum_{n=1}^{\infty} \gamma_n'' < \infty$.
If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then $\{x_n\}, \{y_n\}, \{z_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.

Theorem 2 Let X be a real uniformly convex Banach space which satisfies Opial' s condition, and C a nonempty closed convex subset of X. Let T_1, T_2 and T_3 be asymptotically nonexpansive self-maps of C with sequence $\{r_n^{(1)}\}, \{r_n^{(2)}\}, \{r_n^{(3)}\}$ respectively such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$ for all $i = 1, 2, 3$. Let $\{x_n\}$ be sequence as defined in (0.4) and some $\alpha, \beta \in (0,1)$ with the following restrictions:
(i) $0 < \alpha \leq \alpha_n, \alpha_n', \alpha_n'' \leq \beta < 1$, $\forall n \geq n_0$ for some $n_0 \in \mathbb{N}$.
(ii)$\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma_n' < \infty, \sum_{n=1}^{\infty} \gamma_n'' < \infty$.
If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then $\{x_n\}, \{y_n\}, \{z_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3.

2. Implicit iteration process

2.1 **S. Plubtieng and R. Punpaeng, Implicit iteration process of nonexpansive nonself-mappings**

In this paper, we extend Xu and Yin's results to study the contractions T_n, S_n and U_n define by

$$\begin{aligned} T_n x &= (1-\alpha_n)u + \alpha_n T[(1-\beta_n)x + \beta_n Tx] && (0.5) \\ S_n x &= (1-\alpha_n)u + \alpha_n PT[(1-\beta_n)x + \beta_n PTx] && (0.6) \\ U_n x &= P[(1-\alpha_n)u + \alpha_n TP[(1-\beta_n)x + \beta_n Tx]], && (0.7) \end{aligned}$$

where $\{\alpha_n\} \subseteq (0,1), 0 \leq \beta_n \leq \beta < 1$, and P is the nearest point projection of H onto C.

Theorem 1 Let H be a real Hilbert space, C be a nonempty closed convex subset of H, and $T : C \to H$ be a nonexpansive nonself-mapping. Suppose that for some $u \in C$, $\{\alpha_n\} \subseteq (0,1)$ and $0 \leq \beta_n \leq \beta < 1$, the mapping T_n defined by (0.5) has a (unique) fixed point $x_n \in C$ for all $n \geq 1$. Then T has a fixed point if and only if $\{x_n\}$ remains bounded as $\alpha_n \to 1$. In this case, $\{x_n\}$ converges strongly as $\alpha_n \to 1$ to a fixed point of T.

Theorem 2 Let H be a Hilbert space, C be a nonempty closed convex subset of H, $T : C \to H$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition, and $P : H \to C$ be the nearest point projection. Suppose that for some $u \in C$, each $\{\alpha_n\} \subseteq (0,1)$ and $0 \leq \beta_n \leq \beta < 1$. Then, a mapping S_n defined by (0.20) has a unique fixed point $y_n \in C$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $\alpha_n \to 1$. In this case, $\{y_n\}$ converges strongly as $\alpha_n \to 1$ to a fixed point of T.

Theorem 3 Let $H, C, T, P, u, \{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem ??. Then a mapping U_n defined by (0.7) has a unique fixed point $z_n \in C$. Further, T has a fixed point if and only if $\{z_n\}$ remains bounded as $\alpha_n \to 1$ and $\beta_n \to 0$. In this case, $\{z_n\}$ converges strongly as $\alpha_n \to 1$ and $\beta_n \to 0$ to a fixed point of T.

2.2 **S. Plubtieng and R. Wangkeeree, Strong convergence theorems of viscosity averaging iterations for asymptotically nonexpansive nonself-mappings**

In this paper, we first show that, for an asymptotically nonexpansive nonself-mapping T with a sequence $\{k_n\} \subset [1,\infty)$, there exists two sequences $\{x_n\}$ and $\{y_n\}$ which defined by

$$x_n = a_n f(x_n) + (1-a_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n, \ \forall n \geq 1 \qquad (0.8)$$

and

(0.9) $$y_n = \frac{1}{n}\sum_{j=1}^{n} P(a_n f(y_n) + (1-a_n)(TP)^j y_n),\ \forall n \geq 1$$

where

$$b_n = \frac{1}{n}\sum_{j=1}^{n}(1 + |1-k_j| + e^{-j}), a_n = \frac{b_n - 1}{b_n - \beta}, \forall n \geq 1,$$

$0 < \alpha < \beta < 1$, $f : C \to C$ is a contraction mapping with coefficient $\alpha \in (0,1)$ and P is the metric projection from H onto C. **Theorem 1** Let C be a closed convex subset of a real Hilbert space H, P the metric projection from H onto C, T be an asymptotically nonexpansive nonself-mapping from C into H with Lipschitz constant k_n, and suppose that $F(T)$ is nonempty. Let $f : C \to C$ be a contraction mapping with coefficient $\alpha \in (0,1)$,

$$b_n = \frac{1}{n}\sum_{j=1}^{n}(1 + |1-k_j| + e^{-j}) \text{ and } a_n = \frac{b_n - 1}{b_n - \beta},$$

where $0 < \alpha < \beta < 1$. If T satisfies (NNO) condition then the sequence $\{x_n\}$ defined by (0.8) converges strongly to z where, z is the unique solution in $F(T)$ to the variation inequality

(0.10) $$\langle (I-f)z, x-z\rangle \geq 0,\ x \in F(T)$$

or equivalently $z = G(f(z))$, where G is the metric projection from H onto $F(T)$.

Theorem 2 Let C be a closed convex subset of a real Hilbert space H, P the metric projection from H onto C, T be an asymptotically nonexpansive nonself-mapping from C into H with Lipschitz constant k_n, and suppose that $F(T)$ is nonempty. Let $f : C \to C$ be a contraction mapping with coefficient $\alpha \in (0,1)$,

$$b_n = \frac{1}{n}\sum_{j=1}^{n}(1 + |1-k_j| + e^{-j}) \text{ and } a_n = \frac{b_n - 1}{b_n - \beta},$$

where $0 < \alpha < \beta < 1$. If T satisfies (NNO) condition then the sequence $\{y_n\}$ defined by (0.13) converges strongly to z where, z is the unique solution in $F(T)$ to the variation inequality

2.3 S. Plubtieng and R. Wangkeeree, Strong convergence theorems of vicosity averaging iterations for nonexpansive nonself-mappings in Hilbert spaces,

In this paper, we study the three type iterations process as follows: for $y_0, z_0 \in C$ and

(0.11) $$x_n = t_n f(x_n) + (1 - t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n$$

(0.12) $$y_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n, n \geq 0$$

and

(0.13) $$z_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n f(z_n) + (1 - \alpha_n)(TP)^j z_n), n \geq 0$$

where $\{t_n\} \subset (0,1)$, $\{\alpha_n\}$ is a sequence such that $0 \leq \alpha_n \leq 1$, $f : C \to C$ is a contraction mapping and P is the metric projection of H onto C.

Theorem 1 Let H be a Hilbert space, C a nonempty closed convex subset of H, P the metric projection of H onto C and $T : C \to H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{t_n\}$ be sequence in $(0,1)$ which satisfies $\lim_{n\to\infty} t_n = 0$. Then for a contraction mapping $f : C \to C$ with coefficient $\alpha \in (0,1)$, the sequence $\{x_n\}$ defined by (0.11)converges strongly to z, where, z is the unique solution in $F(T)$ to the variation inequality

(0.14) $$\langle (I - f)z, x - z\rangle \geq 0, \ x \in F(T)$$

or equivalently $z = G(f(z))$, where G is a metric projection mapping from H onto $F(T)$.

Theorem 2 Let C be a nonempty closed convex subset of a Hilbert space H, P be the metric projection of H onto C and $T : C \to H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ which satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then for a contraction mapping $f : C \to C$ with coefficient $\alpha \in (0,1)$, the sequence $\{y_n\}$ defined by (0.12)converges strongly to z, where, z is the unique solution in $F(T)$ to the variation inequality

Theorem 3 Let C be a nonempty closed convex subset of a Hilbert space H, P the metric projection of H onto C and $T : C \to H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be sequence in $[0,1]$ which satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then for a contraction mapping $f : C \to C$ with coefficient $\alpha \in (0,1)$, the sequence $\{z_n\}$ defined by (0.13)converges strongly to z, where, z is the unique solution in $F(T)$ to the variation inequality

(0.15) $$\langle (I - f)z, x - z\rangle \geq 0, \ x \in F(T)$$

or equivalently $z = G(f(z))$, where G is a metric projection mapping from H onto $F(T)$.

2.4 **S. Plubtieng and R. Punpeang, Implicit iteration process of nonexpansive nonself-mappings in Banach spaces**

In this paper, we extend Xu and Yin's results [?] to study the contractions T_n, S_n and U_n define by

$$T_n x = (1-\alpha_n)u + \alpha_n T[(1-\beta_n)x + \beta_n Tx] \tag{0.16}$$

$$S_n x = (1-\alpha_n)u + \alpha_n PT[(1-\beta_n)x + \beta_n PTx] \tag{0.17}$$

$$U_n x = P[(1-\alpha_n)u + \alpha_n TP[(1-\beta_n)x + \beta_n Tx]], \tag{0.18}$$

where $\{\alpha_n\} \subseteq (0,1), 0 \leq \beta_n \leq \beta < 1$, and P is the nearest point projection of H onto C.

Theorem 1 Let E be a real reflexive Banach space with a uniformly *Gâteaux* differentiable norm. Let C be a nonempty closed convex subset of E which has normal structure, and $T : C \to C$ be a nonexpansive mapping. Suppose that for some $u \in C$, $\{\alpha_n\}_{n=1}^{\infty} \subseteq (0,1)$ and $0 \leq \beta_n \leq \beta < 1$. Then, a mapping T_n defined by (0.19) has a unique fixed point $x_n \in C$. Futher, T has a fixed point if and only if $\{x_n\}$ remains bounded as $\alpha_n \to 1$. In this case, $\{x_n\}$ converges strongly as $\alpha_n \to 1$ to a fixed point of T.

Theorem 2 Let E be a uniformly convex Banach space with a uniformly *Gâteaux* differentiable norm. Let C be a nonempty closed convex subset of E, and $T : C \to E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E and that for some $u \in C$, $\{\alpha_n\}_{n=1}^{\infty} \subseteq (0,1)$ and $0 \leq \beta_n < \beta < 1$. Then, a mapping S_n defined by (0.20) has a unique fixed point $y_n \in C$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $\alpha_n \to 1$. In this case, $\{y_n\}$ converges strongly as $\alpha_n \to 1$ to a fixed point of T.

Theorem 3 Let E be a uniformly convex Banach space with a uniformly *Gâteaux* differentiable norm. Let C be a nonempty closed convex subset of E, and $T : C \to E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E, that for some $u \in C$, $\{\alpha_n\}_{n=1}^{\infty} \subseteq (0,1), 0 \leq \beta_n \leq \beta < 1$. Then a mapping U_n defined by (0.21) has a unique fixed point $z_n \in C$. Further, then T has a fixed point if and only if $\{z_n\}$ remains bounded as $\alpha_n \to 1$ and $\beta_n \to 0$. In this case, $\{z_n\}$ converges strongly as $\alpha_n \to 1$ and $\beta_n \to 0$ to a fixed point of T.

2.5 **A. Kangtunyakarn and S. Plubtieng, Strong convergence of an implicit iteration process for asymptotically nonexpansive mappings,**

Theorem 1 Let C be a closed convex subset of Hilbert space H and T be asymptotically nonexpensive mapping on C into itself with Lipschitz condition

k_n and suppose that $F(T)$ is nonempty.
Let
$b_n = \frac{1}{n}\sum_{j=1}^{n}(\frac{1}{2} + |\frac{1}{2} - k_j| + e^{-j})$, $0 < a < \frac{1}{2}$, $0 \leq a' < \frac{1}{2}$ and $x_0 \in C$.
Then, a mapping T_n on C given by

$$T_nx = \alpha_n x_0 + (1 - \alpha_n)A_n[\beta_n x + (1 - \beta_n)A_n x] \qquad \text{for all } x \in C$$

has a unique fixed point u_n in C, when

$$\alpha_n = \frac{b_n - \frac{1}{2}}{b_n - \frac{1}{2} + a}, \ \beta_n = \frac{b_n - \frac{1}{2}}{b_n - \frac{1}{2} + a'} \qquad \text{and} \qquad A_n = \frac{1}{n}\sum_{j=1}^{n} T^j.$$

Further $\{u_n\}$ converges strongly to the element of $F(T)$ which nearest to x_0.

2.6 S. Plubtieng and R. Punpeang, Implicit iteration process with errors of nonexpansive nonself-mappings in Banach spaces,

In this paper, we extend Xu and Yin's results [?] to study the contractions T_n, S_n and U_n define by

$$T_nx = a_nu + b_nT[\bar{a_n}x + \bar{b_n}Tx + \bar{c_n}u_n] + c_nv_n \tag{0.19}$$
$$S_nx = a_nu + b_nPT[\bar{a_n}x + \bar{b_n}PTx + \bar{c_n}u_n] + c_nv_n, \tag{0.20}$$
$$U_nx = P[a_nu + b_nTP[\bar{a_n}x + \bar{b_n}Tx + \bar{c_n}u_n] + c_nv_n] \tag{0.21}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\bar{a_n}\}, \{\bar{b_n}\}$, and $\{\bar{c_n}\}$ be real sequences on [0,1] such that $a_n + b_n + c_n = \bar{a_n} + \bar{b_n} + \bar{c_n} = 1$, $0 \leq b_n \leq \beta < 1, \bar{b_n} \leq \beta < 1$, $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \bar{c_n} < \infty$, and P is the nearest point projection of H onto C.

Theorem 1 Let E be a real reflexive Banach space with a uniformly *Gâteaux* differentiable norm. Let C be a nonempty closed convex subset of E which has normal structure, and $T : C \to C$ be a nonexpansive mapping. Let $u \in C$, $\{u_n\}$ and $\{v_n\}$ be bounded sequences on C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{\bar{a_n}\}, \{\bar{b_n}\}, and\{\bar{c_n}\}$ be real sequences on [0,1] satisfying the conditions:

(i) $a_n + b_n + c_n = \bar{a_n} + \bar{b_n} + \bar{c_n} = 1$,
(ii) $0 \leq b_n \leq \beta < 1, \bar{b_n} \leq \beta < 1, \ \forall n \geq 1$,
(iii) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \bar{c_n} < \infty$.

Then the mapping T_n defined by (0.19) has a unique fixed point $x_n \in C$. Futher, T has a fixed point if and only if $\{x_n\}$ remains bounded as $a_n \to 0$. In this case, $\{x_n\}$ converges strongly as $a_n \to 0$ to a fixed point of T.

Theorem 2 Let E be a uniformly convex Banach space with a uniformly *Gâteaux* differentiable norm. Let C be a nonempty closed convex subset of E, and $T : C \to E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E and that for some $u \in C$, let $\{u_n\}$ and $\{v_n\}$ be bounded sequences on C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{\bar{a_n}\}, \{\bar{b_n}\}, and\{\bar{c_n}\}$ be real sequences on [0,1] satisfying the conditions:

(i) $a_n + b_n + c_n = \bar{a_n} + \bar{b_n} + \bar{c_n} = 1$,

(ii) $0 \leq b_n \leq \beta < 1, \bar{b_n} \leq \beta < 1, \ \forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} \bar{c_n} < \infty$.

Then, a mapping S_n defined by (0.20) has a unique fixed point $y_n \in C$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $a_n \to 0$. In this case, $\{y_n\}$ converges strongly as $a_n \to 0$ to a fixed point of T.

Theorem 3 Let E be a uniformly convex Banach space with a uniformly *Gâteaux* differentiable norm. Let C be a nonempty closed convex subset of E, and $T : C \to E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E and that for some $u \in C$, let $\{u_n\}$ and $\{v_n\}$ be bounded sequences on C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{\bar{a_n}\}, \{\bar{b_n}\}, and\{\bar{c_n}\}$ be real sequences on [0,1] satisfying the conditions:

(i) $a_n + b_n + c_n = \bar{a_n} + \bar{b_n} + \bar{c_n} = 1$,

(ii) $0 \leq b_n \leq \beta < 1, \bar{b_n} \leq \beta < 1, \ \forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} c_n < \infty, \ \sum_{n=1}^{\infty} \bar{c_n} < \infty$.

Then, a mapping U_n defined by (0.21) has a unique fixed point $z_n \in C$. Further, T has a fixed point if and only if $\{z_n\}$ remains bounded as $a_n \to 0$. In this case, $\{z_n\}$ converges strongly as $a_n \to 0$ to a fixed point of T.

2.7 A. Kangtunyakarn and S. Plubtieng, Strong convergence theorems of an implicit iteration process with errors for asymptotically nonexpansive mappings,

Theorem 1 Let C be a closed convex subset of Hilbert space H and T be asymptotically nonexpensive mapping on C into itself with Lipschitz condition k_n and suppose that $F(T)$ is nonempty.

Let

$$b_n = \frac{1}{n}\sum_{j=1}^{n}(1 + |1 - k_j| + e^{-j}) \qquad 0 < a < 1 \ \text{ and } \ x_0 \in C,$$

and let $T_n : C \to C$ be a mapping given by

$$T_n x = \alpha_n x_0 + \beta_n A_n x + \gamma_n v_n \qquad ;\forall x \in C, v_n \in C,$$

where $\{\alpha_n\}$ $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in [0,1) such that $\alpha_n + \beta_n + \gamma_n = 1$, $\alpha_n = \frac{b_n - 1}{b_n - 1 + a}$, and $\gamma_n < \frac{(b_n-1)^2}{(b_n-1)^2 + a'}$ for all $n \geq 1$, $b_n = \frac{1}{n}\sum_{j=1}^{n}(1 + |1 - k_j| + e^{-j})$, $A_n = \frac{1}{n}\sum_{j=1}^{n} T^j, 0 < a' < 1, 0 < a < 1$, and $\{v_n\}$ is a bounded sequence in C. Then T_n has a unique fixed point u_n in C. Further $\{u_n\}$ converges strongly to the element of $F(T)$ which nearest to x_0.

3. Random fixed point theorems

3.1 **P. Kumam and S. Plubtieng, The characteristic of noncompact convexity and random fixed point theorem for set-valued operators,**

Theorem 1 Let C be a nonempty closed bounded convex subset of a Banach space X such that $\varepsilon_\beta(X) < 1$, and $T : C \to KC(C)$ a nonexpansive mapping. Then T has a fixed point.

Theorem 2 Let C be a nonempty closed bounded convex separable subset of a Banach space X such that $\varepsilon_\beta(X) < 1$, and $T : \Omega \times C \to KC(C)$ be a set-valued nonexpansive random operator. Then T has a random fixed point.

3.2 **P. Kumam and S. Plubtieng, Random fixed point theorems for multivalued nonexpansive non-self random operators,**

Theorem 1 Let C be a nonempty closed bounded convex separable subset of a Banach spaces X such that $\epsilon_\beta(X) < 1$, and $T : \Omega \times C \to KC(X)$ be a multivalued nonexpansive random operator and 1-χ-contractive mapping, such that for each $\omega \in \Omega$, $T(\omega, C)$ is a bounded set, which satisfies the inwardness condition, i.e., for each $\omega \in \Omega$, $T(\omega, x) \subset I_C(x)$, $\forall x \in C$.
Then T has a random fixed point.

Theorem 2 Let C be a nonempty closed bounded convex separable subset of a Banach spaces X such that $\epsilon_\alpha(X) < 1$, and $T : \Omega \times C \to KC(X)$ be a multivalued nonexpansive random operator and 1-χ-contractive nonexpansive mapping, such that for each $\omega \in \Omega$, $T(\omega, C)$ is a bounded set, which satisfies the inwardness condition, i.e., for each $\omega \in \Omega$, $T(\omega, x) \subset I_C(x)$, $\forall x \in C$.
Then T has a random fixed point.

3.3 **P. Kumam and S. Plubtieng, Random fixed point theorems for asymptotically regular mappings,**

Theorem 1 Let C be a nonempty weakly compact convex separable subset of a Banach space with $WCS(X) > 1$ and $T : \Omega \times C \to C$ be a random uniformly Lipschitzian mapping such that $\sigma(T(\omega, \cdot)) < \sqrt{WCS(X)}$ for all $\omega \in \Omega$. Suppose in addition that T is asymptotically regular on C. Then T has a random fixed point.

Theorem 2 Let X be a reflexive Banach space, C be a nonempty bounded convex separable subset of X and $T : \Omega \times C \to C$ be a random asymptotically regular operator. If there exist a constant $c \in \mathbb{R}$ such that

$$\sigma(T(\omega,\cdot)) \leq c < \frac{1+\sqrt{1+4WCS(X)(\kappa_\omega(X)-1}}{2}$$

for all $\omega \in \Omega$ then T has a random fixed point.

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ภาคผนวก 1

Fixed point iteration for asymptotically quasi-nonexpansive mappings in Banach spaces

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FIXED POINT ITERATION FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Suppose that C is a nonempty closed convex subset of a real uniformly convex Banach space X. Let $T : C \to C$ be an asymptotically quasi-nonexpansive mapping. In this paper, we introduce the three-step iterative scheme for such map with error members. Moreover, we prove that if T is uniformly L-Lipschitzian and completely continuous, then the iterative scheme converges strongly to some fixed point of T.

1. Introduction

Let C be a subset of normed space X, and let T be a self-mapping on C. T is said to be *nonexpansive* provided that $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$; T is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} k_n = 0$ such that $\|T^n x - T^n y\| \le (1+k_n)\|x - y\|$ for all $x, y \in C$ and $n \ge 1$. T is said to be an *asymptotically quasi-nonexpansive map* if there exists a sequence $\{k_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} k_n = 0$ such that $\|T^n x - p\| \le (1+k_n)\|x - p\|$ for all $x \in C$ and $p \in F(T)$, and $n \ge 1$ ($F(T)$ denotes the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$).

From the above definitions, if $F(T) \ne \emptyset$, then asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive mapping.

The concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk in 1972 [2]. In 2001, Noor [5, 6] introduced the three-step iterative scheme and he studied the approximate solutions of variational inclusions (inequalities) in Hilbert spaces. The three-step iterative approximation problems were studied extensively by Noor [5, 6], Glowinski and Le Tallec [1], and Haubruge et al. [3].

Recently, Xu and Noor [8] introduced the three-step iterative scheme for asymptotically nonexpansive mappings and they proved the following strong convergence theorem in Banach spaces.

Theorem 1.1 (see [8, Theorem 2.1]). *Let X be a real uniformly convex Banach space, let C be a nonempty closed, bounded convex subset of X. Let T be a completely continuous and asymptotically nonexpansive self-mapping with sequence $\{k_n\}$ satisfying $k_n \ge 0$ and*

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$\sum_{n=1}^{\infty} k_n < \infty$. *Let* $\{\alpha_n\}, \{\beta_n\}$, *and* $\{\gamma_n\}$ *be real sequences in* $[0,1]$ *satisfying*

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

For a given $x_0 \in D$, *define*

$$\begin{aligned} z_n &= \gamma_n T^n x_n + (1-\gamma_n)x_n, \\ y_n &= \beta_n T^n z_n + (1-\beta_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n. \end{aligned} \tag{1.1}$$

Then $\{x_n\}, \{y_n\}$, *and* $\{z_n\}$ *converge strongly to a fixed point of* T.

In this paper, we will extend the iterative scheme (1.1) to the iterative scheme of asymptotically quasi-nonexpansive mappings with error members. Moreover, we will prove the strong convergence of iterative scheme to a fixed point of T (C need not to be a bounded set), requiring T to be uniformly L-Lipschitzian and completely continuous. The results presented in this paper generalize and extend the corresponding main results of Xu and Noor [8].

2. Preliminaries

For the sake of convenience, we first recall some definitions and conclusions.

Definition 2.1 (see [2]). A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1,\ \|x-y\| = \epsilon\right\} > 0 \tag{2.1}$$

for all $0 < \epsilon \le 2$ (i.e., $\delta_X(\epsilon)$ is a function $(0,2] \to (0,1)$).

Definition 2.2. A mapping $T : C \to C$ is called *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that for all $x, y \in C$,

$$\|T^n x - T^n y\| \le L\|x-y\|, \quad \forall n \ge 1. \tag{2.2}$$

In what follows, we will make use of the following lemmas.

Lemma 2.3 (see [4]). *Let the nonnegative number sequences* $\{a_n\}, \{b_n\}$, *and* $\{d_n\}$ *satisfy that*

$$a_{n+1} \le (1+b_n)a_n + d_n, \quad \forall n = 1,2,\dots, \sum_{n=1}^{\infty} b_n < \infty,\ \sum_{n=1}^{\infty} d_n < \infty. \tag{2.3}$$

Then,

(1) $\lim_{n\to\infty} a_n$ *exists;*

(2) *if* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

LEMMA 2.4 ([7], J. Schu's Lemma). *Let X be a real uniformly convex Banach space, $0 < \alpha \leq t_n \leq \beta < 1$, $x_n, y_n \in X$, $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$, and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$, $a \geq 0$. Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Main results

In this section, we prove our main theorem. First of all, we will need the following lemmas.

LEMMA 3.1. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let T be an asymptotically quasi-nonexpansive mapping with sequence $\{k_n\}_{n\geq 1}$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_0 \in C$ and*

$$\begin{aligned} z_n &= \alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n, \\ y_n &= \alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$, and $\{\gamma_n''\}$ are real sequences in $[0,1]$ and $\{u_n\}, \{v_n\}$, and $\{w_n\}$ are three bounded sequences in C such that

(i) $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$,

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$, $\sum_{n=1}^{\infty} \gamma_n'' < \infty$.

If $p \in F(T)$, then $\lim_{n\to\infty} \|x_n - p\|$ exists.

Proof. Let $p \in F(T)$. Since $\{u_n\}, \{v_n\}$, and $\{w_n\}$ are bounded sequences in C, put

$$M = \sup_{n\geq 1} \|u_n - p\| \vee \sup_{n\geq 1} \|v_n - p\| \vee \sup_{n\geq 1} \|w_n - p\|. \tag{3.2}$$

Then M is a finite number. So for each $n \geq 1$, we note that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n - p\| \\ &\leq \alpha_n \|T^n y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\ &\leq \alpha_n (1+k_n) \|y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\|, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n - p\| \\ &\leq \alpha_n' \|T^n z_n - p\| + \beta_n' \|x_n - p\| + \gamma_n' \|v_n - p\| \\ &\leq \alpha_n' (1+k_n) \|z_n - p\| + \beta_n' \|x_n - p\| + \gamma_n' \|v_n - p\|, \end{aligned} \tag{3.4}$$

$$\|z_n - p\| \leq \alpha_n'' (1+k_n) \|x_n - p\| + \beta_n'' \|x_n - p\| + \gamma_n'' \|u_n - p\|. \tag{3.5}$$

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Substituting (3.5) into (3.4),

$$
\begin{aligned}
\|y_n - p\| &\le \alpha_n'\alpha_n''(1+k_n)^2\|x_n - p\| \\
&\quad + \alpha_n'\beta_n''(1+k_n)\|x_n - p\| + \alpha_n'\gamma_n''(1+k_n)\|u_n - p\| + \beta_n'\|x_n - p\| + \gamma_n'\|v_n - p\| \\
&\le (1-\beta_n'-\gamma_n')\alpha_n''(1+k_n)^2\|x_n - p\| + \beta_n'\|x_n - p\| \\
&\quad + (1-\beta_n'-\gamma_n')\beta_n''\|x_n - p\| + m_n \\
&\le \beta_n'(1+k_n)^2\|x_n - p\| + (1-\beta_n')\alpha_n''(1+k_n)^2\|x_n - p\| \\
&\quad + (1-\beta_n')\beta_n''(1+k_n)^2\|x_n - p\| + m_n \\
&= \beta_n'(1+k_n)^2\|x_n - p\| + (1-\beta_n')(\alpha_n''+\beta_n'')(1+k_n)^2\|x_n - p\| + m_n \\
&\le \beta_n'(1+k_n)^2\|x_n - p\| + (1-\beta_n')(1+k_n)^2\|x_n - p\| + m_n \\
&= (1+k_n)^2\|x_n - p\| + m_n,
\end{aligned}
\tag{3.6}
$$

where $m_n = \gamma_n''(1+k_n)M + \gamma_n'M$. Substituting (3.6) into (3.3) again, we have

$$
\begin{aligned}
\|x_{n+1} - p\| &\le \alpha_n(1+k_n)\left((1+k_n)^2\|x_n - p\| + m_n\right) + \beta_n\|x_n - p\| + \gamma_n\|w_n - p\| \\
&= \alpha_n(1+k_n)^3\|x_n - p\| + \alpha_n(1+k_n)m_n + \beta_n\|x_n - p\| + \gamma_n\|w_n - p\| \\
&\le (\alpha_n+\beta_n)(1+k_n)^3\|x_n - p\| + (1+k_n)m_n + \gamma_n\|w_n - p\| \\
&\le (1+k_n)^3\|x_n - p\| + (1+k_n)m_n + \gamma_n\|w_n - p\| \\
&\le (1+k_n)^3\|x_n - p\| + (1+k_n)m_n + \gamma_nM \\
&= (1+d_n)\|x_n - p\| + b_n,
\end{aligned}
\tag{3.7}
$$

where $d_n = 3k_n + 3k_n^2 + k_n^3$ and $b_n = (1+k_n)m_n + \gamma_nM$. Since $\sum_{n=1}^{\infty} d_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, by Lemma 2.3, we have that $\lim_{n\to\infty}\|x_n - p\|$ exists. This completes the proof. □

Lemma 3.2. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let T be an asymptotically quasi-nonexpansive mapping with sequence $\{k_n\}_{n\ge1}$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \ne \emptyset$. Let $x_0 \in C$ and for each $n \ge 0$,*

$$
\begin{aligned}
z_n &= \alpha_n''T^nx_n + \beta_n''x_n + \gamma_n''u_n, \\
y_n &= \alpha_n'T^nz_n + \beta_n'x_n + \gamma_n'v_n, \\
x_{n+1} &= \alpha_nT^ny_n + \beta_nx_n + \gamma_nw_n,
\end{aligned}
\tag{3.8}
$$

where $\{u_n\}, \{v_n\}$, and $\{w_n\}$ are three bounded sequences in C and $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$, and $\{\gamma_n''\}$ are real sequences in $[0,1]$ which satisfy the same assumptions as Lemma 3.1 and the additional assumption that $0 \le \alpha < \alpha_n, \beta_n, \alpha_n', \beta_n' \le \beta < 1$ for some α, β in $(0,1)$. Then $\lim_{n\to\infty}\|T^ny_n - x_n\| = 0 = \lim_{n\to\infty}\|T^nz_n - x_n\|$.

Proof. For any $p \in F(T)$, it follows from Lemma 3.1, that $\lim_{n\to\infty}\|x_n - p\|$ exists. Let $\lim_{n\to\infty}\|x_n - p\| = a$ for some $a \geq 0$. From (3.6), we have

$$\|y_n - p\| \leq (1+k_n)^2\|x_n - p\| + m_n. \tag{3.9}$$

Taking $\limsup_{n\to\infty}$ in both sides, we obtain

$$\limsup_{n\to\infty}\|y_n - p\| \leq \limsup_{n\to\infty}\|x_n - p\| = \lim_{n\to\infty}\|x_n - p\| = a. \tag{3.10}$$

Note that

$$\begin{aligned}
&\limsup_{n\to\infty}\|T^n y_n - p\| \leq \limsup_{n\to\infty}(1+k_n)\|y_n - p\| = \limsup_{n\to\infty}\|y_n - p\| \leq a,\\
a &= \lim_{n\to\infty}\|x_{n+1} - p\| = \lim_{n\to\infty}\|\alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n - p\|\\
&= \lim_{n\to\infty}\left\|\alpha_n\left[T^n y_n - p + \frac{\gamma_n}{2\alpha_n}(w_n - p)\right] + \beta_n\left[x_n - p + \frac{\gamma_n}{2\beta_n}(w_n - p)\right]\right\|\\
&= \lim_{n\to\infty}\left\|\alpha_n\left[T^n y_n - p + \frac{\gamma_n}{2\alpha_n}(w_n - p)\right] + (1-\alpha_n)\left[x_n - p + \frac{\gamma_n}{2\beta_n}(w_n - p)\right]\right\|.
\end{aligned} \tag{3.11}$$

By J. Schu's Lemma 2.4, we have

$$\lim_{n\to\infty}\left\|T^n y_n - x_n + \left(\frac{\gamma_n}{2\alpha_n} - \frac{\gamma_n}{2\beta_n}\right)(w_n - p)\right\| = 0. \tag{3.12}$$

Since $\lim_{n\to\infty}\|(\gamma_n/2\alpha_n - \gamma_n/2\beta_n)(w_n - p)\| = 0$, it follows that

$$\lim_{n\to\infty}\|T^n y_n - x_n\| = 0. \tag{3.13}$$

Finally, we will prove that $\lim_{n\to\infty}\|T^n z_n - x_n\| = 0$. To this end, we note that for each $n \geq 1$,

$$\|x_n - p\| \leq \|T^n y_n - x_n\| + \|T^n y_n - p\| \leq \|T^n y_n - x_n\| + (1+k_n)\|y_n - p\|. \tag{3.14}$$

Since $\lim_{n\to\infty}\|T^n y_n - x_n\| = 0 = \lim_{n\to\infty}k_n$, we obtain that

$$a = \lim_{n\to\infty}\|x_n - p\| \leq \liminf_{n\to\infty}\|y_n - p\|. \tag{3.15}$$

It follows that

$$a \leq \liminf_{n\to\infty}\|y_n - p\| \leq \limsup_{n\to\infty}\|y_n - p\| \leq a. \tag{3.16}$$

This implies that

$$\lim_{n\to\infty}\|y_n - p\| = a. \tag{3.17}$$

On the other hand, we note that

$$\begin{aligned}\|z_n - p\| &= \|\alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n - p\| \\ &\le \alpha_n''(1+k_n)\|x_n - p\| + \beta_n''\|x_n - p\| + \gamma_n''\|u_n - p\| \\ &\le \alpha_n''(1+k_n)\|x_n - p\| + (1-\alpha_n'')(1+k_n)\|x_n - p\| + \gamma_n''\|u_n - p\| \\ &\le (1+k_n)\|x_n - p\| + \gamma_n''\|u_n - p\|.\end{aligned} \tag{3.18}$$

By boundedness of the sequence $\{u_n\}$ and $\lim_{n\to\infty} k_n = 0 = \lim_{n\to\infty} \gamma_n''$, we have

$$\limsup_{n\to\infty}\|z_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| = a, \tag{3.19}$$

and so

$$\begin{aligned}\limsup_{n\to\infty}\|T^n z_n - p\| &\le \limsup_{n\to\infty}(1+k_n)\|z_n - p\| \le a, \\ a = \lim_{n\to\infty}\|y_n - p\| &= \lim_{n\to\infty}\|\alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n - p\| \\ &= \lim_{n\to\infty}\left\|\alpha_n'\left[T^n z_n - p + \frac{\gamma_n'}{2\alpha_n'}(v_n - p)\right] + \beta_n'\left[x_n - p + \frac{\gamma_n'}{2\beta_n'}(v_n - p)\right]\right\| \\ &= \lim_{n\to\infty}\left\|\alpha_n'\left[T^n z_n - p + \frac{\gamma_n'}{2\alpha_n'}(v_n - p)\right] + (1-\alpha_n')\left[x_n - p + \frac{\gamma_n'}{2\beta_n'}(v_n - p)\right]\right\|.\end{aligned} \tag{3.20}$$

By J. Schu's Lemma 2.4, we have

$$\lim_{n\to\infty}\left\|T^n z_n - x_n + \left(\frac{\gamma_n'}{2\alpha_n'} - \frac{\gamma_n'}{2\beta_n'}\right)(v_n - p)\right\| = 0. \tag{3.21}$$

Since $\lim_{n\to\infty}\|(\gamma_n'/2\alpha_n' - \gamma_n'/2\beta_n')(v_n - p)\| = 0$, it follows that

$$\lim_{n\to\infty}\|T^n z_n - x_n\| = 0. \tag{3.22}$$

This completes the proof. □

Theorem 3.3. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let T be uniformly L-Lipschitzian, completely continuous, and an asymptotically quasi-nonexpansive mapping with sequence $\{k_n\}_{n\ge1}$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \ne \varnothing$. Let $x_0 \in C$ and for each $n \ge 0$,*

$$\begin{aligned} z_n &= \alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n, \\ y_n &= \alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n,\end{aligned} \tag{3.23}$$

where $\{u_n\}, \{v_n\}$, and $\{w_n\}$ are three bounded sequences in C and $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$, and $\{\gamma_n''\}$ are real sequences in $[0,1]$ which satisfy the same assumptions as Lemma 3.1 and the additional assumption that $0 \le \alpha < \alpha_n, \beta_n, \alpha_n', \beta_n' \le \beta < 1$ for some α, β in $(0,1)$. Then $\{x_n\}, \{y_n\}$, and $\{z_n\}$ converge strongly to a fixed point of T.

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Proof. It follows from Lemma 3.2 that

$$\lim_{n\to\infty} \|T^n y_n - x_n\| = 0 = \lim_{n\to\infty} \|T^n z_n - x_n\| \tag{3.24}$$

and this implies that

$$\|x_{n+1} - x_n\| \le \alpha_n \|T^n y_n - x_n\| + \gamma_n \|w_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.25}$$

We note that

$$\begin{aligned}\|T^n x_n - x_n\| &\le \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \le L\|x_n - y_n\| + \|T^n y_n - x_n\| \\ &\le \alpha'_n L\|x_n - T^n z_n\| + \gamma'_n L\|v_n - x_n\| + \|T^n y_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{3.26}$$

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\ &\le \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + (1 + k_{n+1})\|x_{n+1} - x_n\| + L\|T^n x_n - x_n\|.\end{aligned} \tag{3.27}$$

It follows from (3.25), (3.26), and the above inequality that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.28}$$

By Lemma 3.1, $\{x_n\}$ is bounded. It follows from our assumption that T is completely continuous and that there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \to p \in C$ as $k \to \infty$. Moreover, by (3.28), we have $\|Tx_{n_k} - x_{n_k}\| \to 0$ which implies that $x_{n_k} \to p$ as $k \to \infty$. By (3.28) again, we have

$$\|p - Tp\| = \lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0. \tag{3.29}$$

This shows that $p \in F(T)$. Furthermore, since $\lim_{n\to\infty} \|x_n - p\|$ exists, we have $\lim_{n\to\infty} \|x_n - p\| = 0$, that is, $\{x_n\}$ converges to some fixed point of T. It follows that

$$\begin{aligned}\|y_n - x_n\| &\le \alpha'_n \|T^n z_n - x_n\| + \gamma'_n \|v_n - x_n\| \longrightarrow 0, \\ \|z_n - x_n\| &\le \alpha''_n \|T^n x_n - x_n\| + \gamma''_n \|u_n - x_n\| \longrightarrow 0.\end{aligned} \tag{3.30}$$

Therefore, $\lim_{n\to\infty} y_n = p = \lim_{n\to\infty} z_n$. This completes the proof. □

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Noor Iterations with error for non-Lipschitzian mappings in Banach spaces

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Noor Iterations with Error for Non-Lipschitzian Mappings in Banach Spaces

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ABSTRACT. Suppose C is a nonempty closed convex subset of a real uniformly convex Banach space X. Let $T : C \to C$ be an asymptotically nonexpansive in the intermediate sense mapping. In this paper we introduced the three-step iterative sequence for such map with error members. Moreover, we prove that, if T is completely continuous then the our iterative sequence converges strongly to a fixed point of T.

1. Introduction

Let C be a subset of real normed linear space X, and let T be a self-mapping on C. T is said to be *nonexpansive* provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $\lim_{n\to\infty} k_n = 1$ such that for each $x, y \in C$ and $n \geq 1$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|.$$

T is called *asymptotically nonexpansive in the intermediate sense* [1] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically nonexpansive in the intermediate sense and asymptotically quasi-nonexpansive mapping. But the converges dose not holds as the following example:

Example 1.1 (see [6]). Let $X = \mathbb{R}$, $C = [\frac{-1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in C$, define

$$T(x) = \begin{cases} kx \sin\frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

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Then T is an asymptotically nonexpansive in the intermediate sense but it is not asymptotically nonexpansive mapping.

The concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk ([3]) in 1992. In 2001, Noor ([8], [9]) have introduced the three-step iterative sequences and he studied the approximate solutions of variational inclusions(inequalities) in Hilbert spaces. The three-step iterative approximation problems were studied extensively by Noor ([8], [9]), Glowinski and Le Tallec ([2]), Haubruge et al ([4]).

In 2002, Xu and Noor ([14]) introduced the three-step iterative for asymptotically nonexpansive mappings and they proved the following strong convergence theorem in Banach spaces;

Theorem XN ([14], Theorem 2.1). *Let X be a real uniformly convex Banach space, C be a nonempty closed, bounded convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0,1]$ satisfying;*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, *and*

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

For a give $x_0 \in C$, define

$$\begin{aligned} z_n &= \gamma_n T^n x_n + (1-\gamma_n)x_n \\ y_n &= \beta_n T^n z_n + (1-\beta_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n. \end{aligned} \tag{1.1}$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converges strongly to a fixed point of T.

Algorithm 1.1(Noor iterations with errors). Let C be a nonempty subset of normed space X and let $T : C \to C$ be a mapping. For a given $x_0 \in C$, find the sequence $\{x_{n+1}\}$ such that

$$\begin{aligned} z_n &= \alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n \\ y_n &= \alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}, \{\alpha_n'\}, \{\alpha_n''\}, \{\beta_n\}, \{\beta_n'\}, \{\beta_n''\}, \{\gamma_n\}, \{\gamma_n'\}$ and $\{\gamma_n''\}$ are real sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are three bounded sequences in C.

It is clear that the Mann and Ishikawa iterations processes are all special case of the Noor iterations with error.

In this paper, we will extend the process (1.1) to Noor iteration with error (1.2) for asymptotically nonexpansive in the intermediate sense and without boundedness conditions on C. The results presented in this paper generalize and extend the corresponding main results of Xu and Noor ([14]).

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2. Preliminaries

For the sake of convenience, we first recall some definitions and conclusions.

Definition 2.1 (see [3]). A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta_X(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \epsilon\} > 0$$

for all $0 < \epsilon \leq 2$ (i.e., $\delta_X(\epsilon)$ is a function $(0,2] \to (0,1)$).

Lemma 2.2 (see [7]). *Let the nonnegative number sequences $\{a_n\}, \{b_n\}$ and $\{d_n\}$ satisfy that*

$$a_{n+1} \leq (1+b_n)a_n + d_n, \forall n = 1, 2, \cdots, \sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} d_n < \infty.$$

Then

(1) $\lim_{n\to\infty} a_n$ *exists;*

(2) *If* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([13], J. Schu's Lemma). *Let X be a real uniformly convex Banach space, $0 < \alpha \leq t_n \leq \beta < 1$, x_n, $y_n \in X$, $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$, and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$, $a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Main results

In this section, we prove our main theorem. First of all, we shall need the following lammas.

Lemma 3.1. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let T be an asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \vee 0, \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let $x_0 \in C$ and

$$\begin{aligned} z_n &= \alpha''_n T^n x_n + \beta''_n x_n + \gamma''_n u_n \\ y_n &= \alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned}$$

where $\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}, \{\beta_n\}, \{\beta'_n\}, \{\beta''_n\}, \{\gamma_n\}, \{\gamma'_n\}$ and $\{\gamma''_n\}$ are real sequences in $[0,1]$ and $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are three bounded sequences in C such that

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1.$

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(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty, \quad \sum_{n=1}^{\infty} \gamma'_n < \infty, \quad \sum_{n=1}^{\infty} \gamma''_n < \infty.$

Then for each $p \in F(T), \lim_{n\to\infty} \|x_n - p\|$ exists.

Proof. By the Schauder fixed-point theorem [12], we obtain that $F(T) \neq \emptyset$. Let $p \in F(T)$, since $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are bounded sequences in C, so we put

$$K = \sup_{n\geq 1} \|u_n - p\| \vee \sup_{n\geq 1} \|v_n - p\| \vee \sup_{n\geq 1} \|w_n - p\|.$$

For each $n \geq 1$, we note that

$$\begin{aligned} (3.1) \qquad \|x_{n+1} - p\| &= \|\alpha_n x_n T^n y_n + \beta_n x_n + \gamma_n w_n - p\| \\ &\leq \alpha_n \|T^n y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\ &\leq \alpha_n \|y_n - p\| + G_n + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \end{aligned}$$

and

$$\begin{aligned} (3.2) \qquad \|y_n - p\| &= \|\alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n - p\| \\ &\leq \alpha'_n \|T^n z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\ &\leq \alpha'_n \|z_n - p\| + G_n + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \end{aligned}$$

and

$$(3.3) \qquad \|z_n - p\| = \| \leq \alpha''_n \|x_n - p\| + G_n + \beta''_n \|x_n - p\| + \gamma''_n \|u_n - p\|.$$

Substituting (3.3) into (3.2),

$$\begin{aligned} (3.4) \quad & \|y_n - p\| \\ \leq\ & \alpha'_n \alpha''_n \|x_n - p\| + \alpha'_n G_n + \alpha'_n \beta''_n \|x_n - p\| + \alpha'_n \gamma''_n \|u_n - p\| \\ & + G_n + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\ \leq\ & (1 - \beta'_n - \gamma'_n)\alpha''_n \|x_n - p\| + \beta'_n \|x_n - p\| + (1 - \beta'_n - \gamma'_n)\beta''_n \|x_n - p\| + m_n \\ \leq\ & \beta'_n \|x_n - p\| + (1 - \beta'_n)\alpha''_n \|x_n - p\| + (1 - \beta'_n)\beta''_n \|x_n - p\| + m_n \\ =\ & \beta'_n \|x_n - p\| + (1 - \beta'_n)(\alpha''_n + \beta''_n)\|x_n - p\| + m_n \\ \leq\ & \beta'_n \|x_n - p\| + (1 - \beta'_n)\|x_n - p\| + m_n \\ =\ & \|x_n - p\| + m_n, \end{aligned}$$

where $m_n = 2G_n + \gamma'_n \|v_n - p\| + \gamma''_n \|u_n - p\|$. Substituting (3.4) into (3.1) again, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n(\|x_n - p\| + m_n) + G_n + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\ &\leq (\alpha_n + \beta_n)\|x_n - p\| + \alpha_n m_n + G_n + \gamma_n \|w_n - p\| \\ &\leq \|x_n - p\| + m_n + G_n + \gamma_n \|w_n - p\| \\ &\leq \|x_n - p\| + 3G_n + (\gamma_n + \gamma'_n + \gamma''_n)M \\ &= \|x_n - p\| + b_n, \end{aligned}$$

where $b_n = 3G_n + (\gamma_n + \gamma'_n + \gamma''_n)M$. Since $\sum_{n=1}^{\infty} b_n < \infty$, by Lemma 2.2, we have $\lim_{n\to\infty} \|x_n - p\|$ exists. This completes the proof. □

Lemma 3.2. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let T be an asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \forall n \geq 1.$$

Let $x_0 \in C$ and for each $n \geq 0$,

$$\begin{aligned} z_n &= \alpha''_n T^n x_n + \beta''_n x_n + \gamma''_n u_n \\ y_n &= \alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned}$$

where $\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}, \{\beta_n\}, \{\beta'_n\}, \{\beta''_n\}, \{\gamma_n\}, \{\gamma''_n\}$ and $\{\gamma''_n\}$ are real sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are three bounded sequences in C such that

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$.

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty, \sum_{n=1}^{\infty} \gamma''_n < \infty$.

(iii) $0 \leq \alpha < \alpha_n, \alpha'_n, \leq \beta < 1$. *Then*

(a) $\lim_{n\to\infty} \|T^n y_n - x_n\| = 0$;

(b) $\lim_{n\to\infty} \|T^n z_n - x_n\| = 0$.

Proof. (a). For any $p \in F(T)$, it follows from Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = a$ for some $a \geq 0$. From (3.4), we have

$$\|y_n - p\| \leq \|x_n - p\| + m_n, \forall n \geq 1.$$

Taking $\limsup_{n\to\infty}$ in both sides, we obtain

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|x_n - p\| = a.$$

Note that

$$\limsup_{n\to\infty} \|T^n y_n - p\| \leq \limsup_{n\to\infty} (\|y_n - p\| + G_n) = \limsup_{n\to\infty} \|y_n - p\| \leq a.$$

Next, consider

$$\|T^n y_n - p + \gamma_n(w_n - x_n)\| \leq \|T^n y_n - p\| + \gamma_n \|w_n - x_n\|.$$

Thus,

$$\limsup_{n\to\infty} \|T^n y_n - p + \gamma_n(w_n - x_n)\| \leq a. \quad (3.5)$$

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Dear Prof. Rabian Wangkeeree,

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ภาคผนวก 3

Implicit iteration process of nonexpansive nonself-mappings

S. Plubtieng and R. Punpeang

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IMPLICIT ITERATION PROCESS OF NONEXPANSIVE NONSELF-MAPPINGS

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ABSTRACT. Suppose C is a nonempty closed convex subset of real Hilbert space H. Let $T : C \longrightarrow H$ be a nonexpansive nonself-mapping and P is the nearest point projection of H onto C. In this paper, we study the convergence of the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ satisfying

$$\begin{aligned} x_n &= (1-\alpha_n)u + \alpha_n T[(1-\beta_n)x_n + \beta_n Tx_n] \\ y_n &= (1-\alpha_n)u + \alpha_n PT[(1-\beta_n)y_n + \beta_n PTy_n], \text{ and} \\ z_n &= P[(1-\alpha_n)u + \alpha_n TP[(1-\beta_n)z_n + \beta_n Tz_n]] \end{aligned}$$

where $\{\alpha_n\} \subseteq (0,1)$, $0 \leq \beta_n \leq \beta < 1$ and $\alpha_n \longrightarrow 1$ as $n \longrightarrow \infty$. The results obtained in this paper extend and improve the recent ones announced by Xu and Yin, and many others.

Keywords and phrases: Nonexpansive mapping, nearest point projection, fixed points, weak inwardness condition, strong convergence theorems.

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1. INTRODUCTION

Let C be a nonempty closed convex subset of a Banach space E. Then a nonself-mapping T from C into E is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$

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for all $x, y \in C$. Given $u \in C$ and $\{\alpha_n\}$ is a sequence such that $0 < \alpha_n < 1$. We can define a contraction $T_n : C \longrightarrow E$ by

$$T_n x = (1-\alpha_n)u + \alpha_n Tx, \ x \in C. \tag{1.1}$$

If T is a self-mapping(i.e. $T(C) \subset C$), then T_n maps C into itself, and hence, by Banach's contraction principle, T_n has a unique fixed point x_n in C, that is, we have

$$x_n = (1-\alpha_n)u + \alpha_n Tx_n, \forall n \geq 1. \tag{1.2}$$

(Such a sequence $\{x_n\}$ is said to be an approximating fixed point of T since it possesses the property that if $\{x_n\}$ is bounded, then $\lim_{n\longrightarrow\infty} \|Tx_n - x_n\| = 0$) whenever $\lim_{n\longrightarrow\infty} \alpha_n = 1$. The strong convergence of $\{x_n\}$ as $\alpha_n \longrightarrow 1$ for a self-mapping T of a bounded C was proved in a Hilbert space independently by Browder [1] and Halpern [3] and in a uniformly smooth Banach space by Reich [7]. Thereafter, Singh and Watson [8] extended the result of Browder and Halpern to nonexpansive nonself-mapping T satisfying Rothe's boundary condition: $T(\partial C) \subset C$ (here ∂C denotes the boundary of C). Recently, Xu and Yin [11] proved that if C is a nonempty closed convex(not necessarily bounded) subset of Hilbert space H, if $T : C \longrightarrow H$ is a nonexpansive nonself-mapping, and if $\{x_n\}$ is the sequence defined by (1.2) which is bounded, then $\{x_n\}$ converges strongly as $\alpha_n \longrightarrow 1$ to a fixed point of T. Marino and Trombetta [5] defined contractions S_n and U_n from C into itself by

$$S_n x = (1-\alpha_n)u + \alpha_n PTx \text{ for all } x \in C \tag{1.3}$$

and

$$U_n x = P[(1-\alpha_n)u + \alpha_n Tx] \text{ for all } x \in C, \tag{1.4}$$

where P is the nearest point projection of H onto C. Then by the Banach contraction principle, there exists a unique fixed point y_n(resp. z_n) of S_n(resp. U_n) in C i.e.

$$y_n = (1-\alpha_n)u + \alpha_n PTy_n \tag{1.5}$$

and

$$z_n = P[(1-\alpha_n)u + \alpha_n Tz_n]. \tag{1.6}$$

Xu and Yin [11] also proved that if C is a nonempty closed convex subset of a Hilbert space H, if $T : C \longrightarrow H$ is a nonexpansive nonself-mapping satisfying the weak inwardness condition, and $\{x_n\}$ is bounded, the $\{y_n\}$ (resp. $\{z_n\}$) defined by (1.5) (resp.(1.6)) converges strongly as $\alpha_n \longrightarrow 1$ to a fixed point of T.

Let C be a nonempty convex subset of Banach space E. Then for $x \in C$ we define the inward set $I_c(x)$ as follows:

$$I_c(x) = \{y \in E : y = x + a(z-x) \text{ for some } z \in C \text{ and } a \geq 0\}.$$

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A mapping $T : C \longrightarrow E$ is said to be *inward* if $Tx \in I_c(x)$ for all $x \in C$. T is also said to be *weakly inward* if for each $x \in C, Tx$ belongs to the closure of $I_c(x)$.

In this paper, we extend Xu and Yin's results [11] to study the contractions T_n, S_n and U_n define by

$$\begin{aligned} T_n x &= (1-\alpha_n)u + \alpha_n T[(1-\beta_n)x + \beta_n Tx] & (1.7)\\ S_n x &= (1-\alpha_n)u + \alpha_n PT[(1-\beta_n)x + \beta_n PTx] & (1.8)\\ U_n x &= P[(1-\alpha_n)u + \alpha_n TP[(1-\beta_n)x + \beta_n Tx]], & (1.9) \end{aligned}$$

where $\{\alpha_n\} \subseteq (0,1), 0 \le \beta_n \le \beta < 1$, and P is the nearest point projection of H onto C. We also prove the strong convergence of the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ satisfying

$$\begin{aligned} x_n &= (1-\alpha_n)u + \alpha_n T[(1-\beta_n)x_n + \beta_n Tx_n], & (1.10)\\ y_n &= (1-\alpha_n)u + \alpha_n PT[(1-\beta_n)y_n + \beta_n PTy_n], & (1.11)\\ z_n &= P[(1-\alpha_n)u + \alpha_n TP[(1-\beta_n)z_n + \beta_n Tz_n]], & (1.12) \end{aligned}$$

where $\alpha_n \longrightarrow 1$ as $n \longrightarrow \infty$.
We note that if $\beta_n \equiv 0$, then (1.10), (1.11), (1.12) reduces to (1.2), (1.5), and (1.6) respectively.

2. Main results

In this section, we prove the strong convergence theorems for nonexpansive nonself-mappings. To prove our results, we use the following Theorem.

Theorem 2.1. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, and $T : C \longrightarrow H$ be a nonexpansive nonself-mapping. Suppose that for some $u \in C$, $\{\alpha_n\} \subseteq (0,1)$ and $0 \le \beta_n \le \beta < 1$, the mapping T_n defined by (1.7) has a (unique) fixed point $x_n \in C$ for all $n \ge 1$. Then T has a fixed point if and only if $\{x_n\}$ remains bounded as $\alpha_n \longrightarrow 1$. In this case, $\{x_n\}$ converges strongly as $\alpha_n \longrightarrow 1$ to a fixed point of T.*

Proof. We denote by $F(T)$ the fixed point set of T. Suppose that $F(T)$ is nonempty. Let $w \in F(T)$. Then for each $n \geq 1$, we have

$$\begin{aligned}
\|w - x_n\| &= \|w - (1-\alpha_n)u - \alpha_n T[(1-\beta_n)x_n + \beta_n Tx_n]\| \\
&\leq (1-\alpha_n)\|w - u\| + \alpha_n\|w - T[(1-\beta_n)x_n + \beta_n Tx_n]\| \\
&\leq (1-\alpha_n)\|w - u\| + \alpha_n\|w - (1-\beta_n)x_n - \beta_n Tx_n\| \\
&\leq (1-\alpha_n)\|w - u\| + \alpha_n(1-\beta_n)\|w - x_n\| + \alpha_n\beta_n\|w - x_n\| \\
&= (1-\alpha_n)\|w - u\| + \alpha_n\|w - x_n\|
\end{aligned}$$

and hence $(1-\alpha_n)\|w - x_n\| \leq (1-\alpha_n)\|w - u\|$ for all $n \geq 1$. This implies $\|w - x_n\| \leq \|w - u\|$ for all $n \geq 1$. Then $\{x_n\}$ is a bounded sequence. Conversely, suppose that $\{x_n\}$ is bounded, z is a weak cluster point of $\{x_n\}$, and $\alpha_n \longrightarrow 1$ as $n \longrightarrow \infty$. Then we show that $F(T) \neq \emptyset$ and $\{x_n\}$ converges strongly to a fixed point of T. We choose a subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ with $\alpha_{n_i} \longrightarrow 1$ such that $x_{n_i} \longrightarrow z$ weakly, we can define a real valued function g on H given by

$$g(x) = \limsup_{i \longrightarrow \infty} \|x_{n_i} - x\|^2 \text{ for every } x \in H.$$

Observeing $\|x_{n_i} - x\|^2 = \|x_{n_i} - z\|^2 + 2\langle x_{n_i} - z, z - x\rangle + \|z - x\|^2$. Since $x_{n_i} \longrightarrow z$ weakly, we immediately get

$$g(x) = g(z) + \|x - z\|^2 \text{ for all } x \in H,$$

in particular,

$$g(Tz) = g(z) + \|Tz - z\|^2. \tag{2.1}$$

On the other hand, we have

$$\begin{aligned}
\|x_{n_i} - Tx_{n_i}\| &\leq (1-\alpha_{n_i})\|u - Tx_{n_i}\| + \alpha_{n_i}\|T[(1-\beta_{n_i})x_{n_i} + \beta_{n_i}Tx_{n_i}] - Tx_{n_i}\| \\
&\leq (1-\alpha_{n_i})\|u - Tx_{n_i}\| + \alpha_{n_i}\|(1-\beta_{n_i})x_{n_i} + \beta_{n_i}Tx_{n_i} - x_{n_i}\| \\
&\leq (1-\alpha_{n_i})\|u - Tx_{n_i}\| + \beta_{n_i}\|Tx_{n_i} - x_{n_i}\|,
\end{aligned}$$

for all $i \geq 1$. This implies that $(1-\beta_{n_i})\|x_{n_i} - Tx_{n_i}\| \leq (1-\alpha_{n_i})\|u - Tx_{n_i}\|$ and hence

$$\begin{aligned}
\|x_{n_i} - Tx_{n_i}\| &= \frac{(1-\alpha_{n_i})}{(1-\beta_{n_i})}\|u - Tx_{n_i}\| \\
&\leq \frac{(1-\alpha_{n_i})}{(1-\beta)}\|u - Tx_{n_i}\| \longrightarrow 0 \text{ as } i \longrightarrow \infty.
\end{aligned}$$

Note that,

$$
\begin{aligned}
\|x_{n_i} - Tz\|^2 &= \|x_{n_i} - Tx_{n_i} + Tx_{n_i} - Tz\|^2 \\
&\le (\|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - Tz\|)^2 \\
&= \|x_{n_i} - Tx_{n_i}\|^2 + 2\|x_{n_i} - Tx_{n_i}\|\|Tx_{n_i} - Tz\| + \|Tx_{n_i} - Tz\|^2
\end{aligned}
$$

for all $n \in \mathbb{N}$. Hence

$$
\begin{aligned}
g(Tz) &= \limsup_{i\longrightarrow\infty} \|x_{n_i} - Tz\|^2 \\
&\le \limsup_{i\longrightarrow\infty} \|Tx_{n_i} - Tz\|^2 \\
&\le \limsup_{i\longrightarrow\infty} \|x_{n_i} - z\|^2 = g(z).
\end{aligned}
$$

This, together with (2.1) implies that $Tz = z$ and z is a fixed point of T. Now since $F(T)$ is nonempty, closed and convex, there exists a unique $v \in F(T)$ that is closest to u; namely, v is the nearest point projection of u onto $F(T)$. For any $y \in F(T)$, we have

$$
\begin{aligned}
\|(x_n - u) + \alpha_n(u - y)\|^2 &= \|((1-\alpha_n)u + \alpha_n T[(1-\beta_n)x_n + \beta_n Tx_n] - u) + \alpha_n(u - y)\|^2 \\
&= \alpha_n^2\|T[(1-\beta_n)x_n + \beta_n Tx_n] - y\|^2 \\
&\le \alpha_n^2\|(1-\beta_n)x_n + \beta_n Tx_n - y\|^2 \\
&= \alpha_n^2\|(1-\beta_n)(x_n - y) + \beta_n(Tx_n - y)\|^2 \\
&\le \alpha_n^2((1-\beta_n)\|x_n - y\| + \beta_n\|x_n - y\|)^2 \\
&= \alpha_n^2\|x_n - y\|^2 \\
&= \alpha_n^2\|x_n - u + u - y\|^2,
\end{aligned}
$$

and so

$$
\begin{aligned}
\|x_n - u\|^2 + \alpha_n^2\|u - y\|^2 + 2\alpha_n\langle x_n - u, u - y\rangle &\le \alpha_n^2(\|x_n - u\|^2 + \|u - y\|^2 \\
&\quad + 2\langle x_n - u, u - y\rangle) \\
&\le \alpha_n\|x_n - u\|^2 + \alpha_n\|u - y\|^2 \\
&\quad + 2\alpha_n\langle x_n - u, u - y\rangle
\end{aligned}
$$

for all $n \ge 1$. It follows that

$\|x_n - u\|^2 \le \alpha_n\|y - u\|^2 \le \|y - u\|^2$ for all $y \in F(T)$ and $\{\alpha_n\} \subseteq (0,1)$ for all $n \in \mathbb{N}$.

Since the norm of H is weakly lower semicontinuous(w-l.s.c.), we get

$$\|z - u\| \le \liminf_{i\longrightarrow\infty} \|x_{n_i} - u\| \le \|y - u\| \text{ for all } y \in F(T).$$

Therefore, we must have $z = v$ for v is the unique element in $F(T)$ that is closest to u. This shows that v is the only weak cluster point of $\{x_n\}$ with $\alpha_n \longrightarrow 1$. It remains to verify that the convergence is strong. In fact, it follows

$$\begin{aligned}\|x_n - v\|^2 &= \|x_n - u\|^2 - \|u - v\|^2 - 2\langle x_n - v, v - u\rangle \\ &\leq -2\langle x_n - v, v - u\rangle \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned}$$

This completes the proof. □

Corollary 2.2. *Let H, C, T be as in Theorem 2.1. Suppose in addition that C is bounded and that the weak inwardness condition is satisfied. Then for each $u \in C$, the sequence $\{x_n\}$ satisfying (1.10) converges strongly as $\alpha_n \longrightarrow 1$ to a fixed point of T.*

Theorem 2.3. *Let H be a Hilbert space, C be a nonempty closed convex subset of H, $T : C \longrightarrow H$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition, and $P : H \longrightarrow C$ be the nearest point projection. Suppose that for some $u \in C$, each $\{\alpha_n\} \subseteq (0,1)$ and $0 \leq \beta_n \leq \beta < 1$. Then, a mapping S_n defined by (1.8) has a unique fixed point $y_n \in C$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $\alpha_n \longrightarrow 1$. In this case, $\{y_n\}$ converges strongly as $\alpha_n \longrightarrow 1$ to a fixed point of T.*

Proof. It is straightforward that $S_n : C \longrightarrow C$ is a contraction for every $n \geq 1$. Therefore by the Banach contraction principle there exists a unique fixed point y_n of S_n in C satisfying (1.11). Let w be a fixed point of T. Then as in the proof of Theorem 2.1, $\{y_n\}$ is bounded. Conversely, suppose that $\{y_n\}$ is bounded. Apply Theorem 2.1, we obtain that $\{y_n\}$ converges strongly to a fixed point z of PT. Next, let us show that $z \in F(T)$. Since $z = PTz$ and P is the nearest point projection of H onto C, it follows by [9] that

$$\langle Tz - z, J(z - v)\rangle \geq 0 \text{ for all } v \in C.$$

On the other hand, Tz belongs to the closure of $I_c(z)$ by the weak inwardness conditions. Hence for each integer $n \geq 1$, there exists $z_n \in C$ and $a_n \geq 0$ such that the sequence

$$r_n := z + a_n(z_n - z) \longrightarrow Tz.$$

Thus it follows that

$$\begin{aligned}0 &\leq a_n\langle Tz - z, z - z_n\rangle \\ &= \langle Tz - z, a_n(z - z_n)\rangle \\ &= \langle Tz - z, z - r_n\rangle \longrightarrow \langle Tz - z, z - Tz\rangle \\ &= -\|Tz - z\|^2.\end{aligned}$$

Hence we have Tz = z. □

Corollary 2.4. ([11, Theorem 2]). Let H, C, T, P, u, and $\{\alpha_n\}$ be as in Theorem 2.3. Then, a mapping S_n given by (1.3) has a unique fixed point $y_n \in C$ such that $y_n = (1-\alpha_n)u + \alpha_n PTy_n$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $\alpha_n \longrightarrow 1$. In this case, $\{y_n\}$ converges strongly as $\alpha_n \longrightarrow 1$ to a fixed point of T.

Theorem 2.5. *Let $H, C, T, P, u, \{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem 2.3. Then a mapping U_n defined by (1.9) has a unique fixed point $z_n \in C$. Further, T has a fixed point if and only if $\{z_n\}$ remains bounded as $\alpha_n \longrightarrow 1$ and $\beta_n \longrightarrow 0$. In this case, $\{z_n\}$ converges strongly as $\alpha_n \longrightarrow 1$ and $\beta_n \longrightarrow 0$ to a fixed point of T.*

Proof. It follows by the Banach contraction principle that there exists a unique fixed point z_n of U_n such that

$$z_n = P[(1-\alpha_n)u + \alpha_n TP[(1-\beta_n)z_n + \beta_n Tz_n]].$$

Let $w \in F(T)$. Then for each $n \geq 1$, we have

$$\begin{aligned}
\|w - z_n\| &= \|Pw - P[(1-\alpha_n)u + \alpha_n TP((1-\beta_n)z_n + \beta_n Tz_n)]\| \\
&\leq \|w - (1-\alpha_n)u - \alpha_n TP[(1-\beta_n)z_n + \beta_n Tz_n]\| \\
&\leq (1-\alpha_n)\|w-u\| + \alpha_n\|w - TP[(1-\beta_n)z_n + \beta_n Tz_n]\| \\
&\leq (1-\alpha_n)\|w-u\| + \alpha_n(1-\beta_n)\|w-z_n\| + \alpha_n\beta_n\|w - Tz_n\| \\
&\leq (1-\alpha_n)\|w-u\| + \alpha_n(1-\beta_n)\|w-z_n\| + \alpha_n\beta_n\|w - z_n\| \\
&= (1-\alpha_n)\|w-u\| + \alpha_n\|w-z_n\|
\end{aligned}$$

and hence $(1-\alpha_n)\|w-z_n\| \leq (1-\alpha_n)\|w-u\|$, $\forall n > 1$. This implies $\|w-z_n\| \leq \|w-u\|$, $\forall n > 1$. Then $\{z_n\}$ is bounded. Conversely, suppose that $\{z_n\}$ is bounded, $\alpha_n \longrightarrow 1$ and $\beta_n \longrightarrow 0$. To show that $F(T) \neq \emptyset$. For any subsequence $\{z_{n_i}\}$ of the sequence $\{z_n\}$ converging weakly to $\bar{z}$ such that $\alpha_{n_i} \longrightarrow 1$, we can define a real valued function g on H given by

$$g(z) = \limsup_{i \longrightarrow \infty} \|z_{n_i} - z\|^2 \text{ for every } z \in H. \tag{2.2}$$

Observing $\|z_{n_i} - z\|^2 = \|z_{n_i} - \bar{z}\|^2 + 2\langle z_{n_i} - \bar{z}, \bar{z} - z\rangle + \|\bar{z} - z\|^2$. Since $z_{n_i} \longrightarrow \bar{z}$ weakly, we get

$$g(z) = g(\bar{z}) + \|\bar{z} - z\|^2 \text{ for all } z \in H,$$

in particular,

$$g(PT\bar{z}) = g(\bar{z}) + \|PT\bar{z} - \bar{z}\|^2. \tag{2.3}$$

For instance, that the straightforward verification gives

$$\begin{aligned}\|z_{n_i} - PTz_{n_i}\| &= \|P[(1-\alpha_{n_i})u + \alpha_{n_i}TP((1-\beta_{n_i})z_{n_i} + \beta_{n_i}Tz_{n_i})] - PTz_{n_i}\| \\ &\leq (1-\alpha_{n_i})\|u - Tz_{n_i}\| + \alpha_{n_i}\beta_{n_i}\|Tz_{n_i} - z_{n_i}\|, \text{ for all } i \geq 1\end{aligned}$$

and this implies that $\|z_{n_i} - PTz_{n_i}\| \leq (1-\alpha_{n_i})\|u - Tz_{n_i}\| + \alpha_{n_i}\beta_{n_i}\|Tz_{n_i} - z_{n_i}\| \longrightarrow 0$ as $i \longrightarrow \infty$. Moreover, we note that

$$\begin{aligned}\|z_{n_i} - PT\bar{z}\|^2 &= \|z_{n_i} - PTz_{n_i} + PTz_{n_i} - PT\bar{z}\|^2 \\ &\leq (\|z_{n_i} - PTz_{n_i}\| + \|PTz_{n_i} - PT\bar{z}\|)^2 \\ &= \|z_{n_i} - PTz_{n_i}\|^2 + 2\|z_{n_i} - PTz_{n_i}\|\|PTz_{n_i} - PT\bar{z}\| + \|PTz_{n_i} - PT\bar{z}\|^2\end{aligned}$$

for all $i \in \mathbb{N}$. It follows that

$$\begin{aligned}g(PT\bar{z}) &= \limsup_{i \longrightarrow \infty} \|z_{n_i} - PT\bar{z}\|^2 \\ &\leq \limsup_{i \longrightarrow \infty} \|PTz_{n_i} - PT\bar{z}\|^2 \\ &\leq \limsup_{i \longrightarrow \infty} \|z_{n_i} - \bar{z}\|^2 = g(z)\end{aligned}$$

which in turn, together with (2.3), implies that $PT(\bar{z}) = \bar{z}$. Since T satisfies the weak inwardness condition, by the same argument as in the proof of Theorem 2.3, we see that $\bar{z}$ is a fixed point of T. For any $w \in F(T)$, we have

$$\begin{aligned}\alpha_n[TP((1-\beta_n)w + \beta_n w) - u] + u &= \alpha_n(w - u) + u \\ &= \alpha_n w + (1-\alpha_n)u \\ &= P(\alpha_n w + (1-\alpha_n)u)\end{aligned}$$

for all $n \in \mathbb{N}$. By follows as in the proof of Theorem 2.1, we have

$$\|z_n - u\|^2 \leq \alpha_n\|w - u\|^2 \leq \|w - u\|^2 \text{ for all } w \in F(T) \text{ and } \{\alpha_n\} \subseteq (0,1) \text{ for all } n \in \mathbb{N}. \tag{2.4}$$

From (2.4) and the w-l.s.c. of the norm of H, it follows that

$$\|\bar{z} - u\| \leq \liminf_{n \longrightarrow \infty} \|z_n - u\| \leq \|w - u\|$$

for all $w \in F(T)$. Hence $\bar{z}$ is the nearest point projection z in $F(T)$ of u onto $F(T)$ which exists uniquely since $F(T)$ is nonempty, closed and convex. Moreover,

$$\begin{aligned}\|z_n - z\|^2 &= \|z_n - u\|^2 - \|u - z\|^2 - 2\langle z_n - z, z - u\rangle \\ &\leq -2\langle z_n - z, z - u\rangle \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned}$$

This complete in the proof. □

Corollary 2.6. ([11,Theorem 3]). Let H, C, T, P, u, and $\{\alpha_n\}$ be as in Theorem 2.3. Then a mapping U_n defined by (1.4) has a unique fixed point $z_n \in C$. Further, T has a fixed point if and only if $\{z_n\}$ remains bounded as $\alpha_n \longrightarrow 1$. In this case, $\{z_n\}$ converges strongly as $\alpha_n \longrightarrow 1$ to a fixed point of T.

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ภาคผนวก 4

Strong convergence theorems for multi-step Noor iterations with errors in Banach spaces

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Strong Convergence Theorems for Multi-Step Noor Iterations with Errors in Banach Spaces*

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Abstract

In this paper, we established two strong convergence theorems for a multi-step Noor iterative scheme with errors for mappings of asymptotically nonexpansive in the intermediate sense(asymptotically quasi-nonexpansive, respectively) in Banach spaces. Our results extend and improve the recent ones announced by Xu and Noor [20], Cho, Zhou and Guo [2], and many others.

keywords: Asymptotically nonexpansive in the intermediate sense; Asymptotically quasi-nonexpansive mappings; Completely continuous; Uniformly convex; Uniformly L-Lipschitzian.

1 Introduction

Let C be a subset of real normed linear space X. A mapping $T : C \longrightarrow C$ is said to be *asymptotically nonexpansive* on C if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n\longrightarrow\infty} r_n = 0$ such that for each $x, y \in C$,

$$\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|, \forall n \geq 1.$$

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If $r_n \equiv 0$, then T is known as a *nonexpansive mapping.* T is called *asymptotically nonexpansive in the intermediate sense*[1] provided T is uniformly continuous and

$$\limsup_{n\longrightarrow\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \leq 0.$$

T is said to be *asymptotically quasi-nonexpansive mapping*, if there exists a sequence $\{r_n\}$ in $[0,\infty)$ with $\lim_{n\longrightarrow\infty} r_n = 0$ such that for all $x \in C$, $p \in F(T)$,

$$\|T^n x - p\| \leq (1+r_n)\|x-p\|,$$

for all $n \geq 1$, where $F(T)$ denotes the set of fixed points of T i.e. $F(T) = \{x \in C : Tx = x\}$. T is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x-y\|,$$

for all $n \geq 1$ and $x, y \in C$.

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically nonexpansive in the intermediate sense, asymptotically quasi-nonexpansive mapping and L-Lipschitzian mapping. But the converges dose not holds such as the following example:

Example 1.1 (see [9]). *Let $X = \mathbb{R}$, $C = [\frac{-1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in C$, define*

$$T(x) = \begin{cases} kx\sin\frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then T is an asymptotically nonexpansive in the intermediate sense. It is well known in [8] that $T^n x \longrightarrow 0$ uniformly, but is not a Lipschitzian mapping so that it is not asymptotically nonexpansive mapping.

Fixed-point iterations process for asymptotically nonexpansive mappings in Banach spaces including Mann and Ishikawa iterations process have been studied extensively by many authors to solve the nonlinear operator equations as well as variational inequations; see[6-14,16-18]. In 2000, Noor [13] introduced a three-step iterative scheme and studied the approximate solution of variational inclusion in Hilbert spaces by using the techniques of updating the solution and the auxiliary priciple. Glowinski and Le Tallec [3] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [3] that the three-step iterative scheme give better numerical results then the two-step and one step approximal iterations. In 1998, Haubruge, Nguyen and Strodiot[5] studied the convergence analysis

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of three-step schemes of Glowinski and Le Tallec[3] and applied these schemes to obtain new splitting-type algorithms for solving variation inequalities. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions.

Recently, Xu and Noor [20] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach space. In 2004, Cho, Zhou and Guo[2] extended the work of Xu and Noor to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space. Moreover, Suantai [18] gave weak and strong convergence theorems for a new three-step iterative scheme of asymptotically nonexpansive mappings. Inspired and motivated by these facts, we introduce and study a multi-step scheme with errors for asymptotically nonexpansive mappings in the intermediate sense mapping and asymptotically quasi-nonexpansive mappings, respectively. Our results include the Ishikawa, Mann and Noor iterative schemes for solving variational inclusions (inequalities) and related problems as special case. The scheme is defined as follows.

Let C be a nonempty subset of normed space X and let $T : C \longrightarrow C$ be a mapping. For a given $x_1 \in C$, and a fixed $m \in \mathbb{N}$ ($\mathbb{N}$ denote the set of all positive integers), compute the iterative sequences $\{x_n^{(1)}\}, ..., \{x_n^{(m)}\}$ defined by

$$\begin{aligned}
x_n^{(1)} &= \alpha_n^{(1)} T^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)}, \\
x_n^{(2)} &= \alpha_n^{(2)} T^n x_n^{(1)} + \beta_n^{(2)} x_n + \gamma_n^{(2)} u_n^{(2)}, \\
x_n^{(3)} &= \alpha_n^{(3)} T^n x_n^{(2)} + \beta_n^{(3)} x_n + \gamma_n^{(3)} u_n^{(3)}, \\
&\ \vdots \\
x_n^{(m-1)} &= \alpha_n^{(m-1)} T^n x_n^{(m-2)} + \beta_n^{(m-1)} x_n + \gamma_n^{(m-1)} u_n^{(m-1)}, \\
x_{n+1} = x_n^{(m)} &= \alpha_n^{(m)} T^n x_n^{(m-1)} + \beta_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)}, \ n \geq 1
\end{aligned} \tag{1.1}$$

where, $\{u_n^{(1)}\}, ..., \{u_n^{(m)}\}$ are bounded sequences in C and $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ are appropriate real sequences in $[0,1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $i \in \{1, 2, ..., m\}$.

The iterative schemes (1.1) is called the multi-step Noor iterations with errors. This iterations include the Mann-Ishikawa-Noor iterations as special case. If $m = 3$ and $\beta_n^{(i)} = 1 - \alpha_n^{(i)} - \gamma_n^{(i)}$ for all $i = 1, 2, 3$ then (1.1) reduces to Noor iterations with errors defined by Cho, Zhou and Guo [2]:

$$\begin{aligned} x_n^{(1)} &= \alpha_n^{(1)}T^n x_n + (1-\alpha_n^{(1)}-\gamma_n^{(1)})x_n + \gamma_n^{(1)}u_n^{(1)}, \\ x_n^{(2)} &= \alpha_n^{(2)}T^n x_n^{(1)} + (1-\alpha_n^{(2)}-\gamma_n^{(2)})x_n + \gamma_n^{(2)}u_n^{(2)}, \\ x_{n+1} = x_n^{(3)} &= \alpha_n^{(3)}T^n x_n^{(2)} + (1-\alpha_n^{(3)}-\gamma_n^{(3)})x_n + \gamma_n^{(3)}u_n^{(3)}, \end{aligned} \tag{1.2}$$

where $\{\alpha_n^{(i)}\}, \{\gamma_n^{(i)}\}$ are appropriate real sequences in $[0,1]$ for all $i \in \{1,2,3\}$.

For $m=3$ and $\gamma_n^{(1)} = \gamma_n^{(2)} = \gamma_n^{(3)} \equiv 0$, then (1.1) reduces to Noor iterations defined by Xu and Noor [20]:

$$\begin{aligned} x_n^{(1)} &= \alpha_n^{(1)}T^n x_n + (1-\alpha_n^{(1)})x_n, \\ x_n^{(2)} &= \alpha_n^{(2)}T^n x_n^{(1)} + (1-\alpha_n^{(2)})x_n, \\ x_{n+1} = x_n^{(3)} &= \alpha_n^{(3)}T^n x_n^{(2)} + (1-\alpha_n^{(3)})x_n, \ \ n \geq 1, \end{aligned} \tag{1.3}$$

where $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \{\alpha_n^{(3)}\}$ are appropriate real sequences in $[0,1]$.

The purpose of this paper is to establish several strong convergence theorems of the multi-step Noor iterative scheme with errors for mappings of asymptotically nonexpansive in the intermediate sense (asymptotically quasi-nonexpansive mappings, respectively) in a uniformly convex Banach space. This results presented in this paper extend and improve the corresponding ones announced by Xu and Noor [20], Cho, Zhou and Guo [2], and many others.

2 Preliminaries

In this section, we recall the well-known concepts and results.

Definition 2.1 (see [4]). A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta_X(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \epsilon\} > 0$$

for all $0 < \epsilon \leq 2$(i.e., $\delta_X(\epsilon)$ is a function $(0,2] \longrightarrow (0,1)$).

It is known [12] that if X is a uniformly convex Banach space and T is a self-mapping of bounded closed convex subset C of X which is an asymptotically nonexpansive in the intermediate sense, then $F(T) \neq \emptyset$.

Lemma 2.2 (see [10]). *Let* $\{a_n\}, \{b_n\}$ *and* $\{\gamma_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\gamma_n)a_n + b_n, \forall n = 1, 2, \ldots .$$

If $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, *then*

(i) $\lim_{n\longrightarrow\infty} a_n$ *exists;*

(ii) $\lim_{n\longrightarrow\infty} a_n = 0$, *whenever* $\liminf_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.3 ([17], J. Schu's Lemma). *Let* X *be a uniformly convex Banach space,* $0 < \alpha \leq t_n \leq \beta < 1, x_n, y_n \in X, \limsup_{n\longrightarrow\infty} \|x_n\| \leq a, \limsup_{n\longrightarrow\infty} \|y_n\| \leq a$, *and* $\lim_{n\longrightarrow\infty} \|t_n x_n + (1-t_n)y_n\| = a$, *for some* $a \geq 0$. *Then* $\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0$.

3 Non-Lipschitzian mappings

Our first result is the strong convergence theorem for asymptotically nonexpansive in the intermediate sense mappings. Note the proof given below is different from that proof of Xu and Noor. In order to prove our main result, the following lemmas are needed.

Lemma 3.1. *Let* X *be a uniformly convex Banach space with* $x_n, y_n \in X$, *real numbers* $a \geq 0, \alpha, \beta \in (0,1)$ *and* $\{\alpha_n\}$ *be a real sequence number which satisfying*

(i) $0 < \alpha \leq \alpha_n \leq \beta < 1$, $\forall n \geq n_0$ *and for some* $n_0 \in \mathbb{N}$;

(ii) $\limsup_{n\longrightarrow\infty} \|x_n\| \leq a$ *and* $\limsup_{n\longrightarrow\infty} \|y_n\| \leq a$;

(iii) $\lim_{n\longrightarrow\infty} \|\alpha_n x_n + (1-\alpha_n)y_n\| = a$.

Then $\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0$.

Proof. The proof is clear by Lemma 2.3. □

Lemma 3.2. *Let* X *be a uniformly convex Banach space,* C *a nonempty closed bounded convex subset of* X *and* $T : C \longrightarrow C$ *be an asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. *Let the sequence* $\{x_n\}$ *be defined by (1.1) with the following restrictions:*

(i) $\alpha_n^{(i)}+\beta_n^{(i)}+\gamma_n^{(i)}=1$ *for all* $i\in\{1,2,...,m\}$ *and for all* $n\geq 1$;

(ii) $\sum_{n=1}^{\infty}\gamma_n^{(i)}<\infty$ *for all* $i\in\{1,2,...,m\}$.

If $p\in F(T)$, *then* $\lim_{n\longrightarrow\infty}\|x_n-p\|$ *exists.*

Proof. By [12], we have $F(T)\neq\emptyset$. Let $p\in F(T)$. For each $n\geq 1$, we note that

$$\begin{aligned}
\|x_n^{(1)}-p\| &= \|\alpha_n^{(1)}T^nx_n+\beta_n^{(1)}x_n+\gamma_n^{(1)}u_n^{(1)}-p\|\\
&\leq \alpha_n^{(1)}\|T^nx_n-p\|+\beta_n^{(1)}\|x_n-p\|+\gamma_n^{(1)}\|u_n^{(1)}-p\|\\
&\leq \alpha_n^{(1)}\|x_n-p\|+\alpha_n^{(1)}G_n+\beta_n^{(1)}\|x_n-p\|+\gamma_n^{(1)}\|u_n^{(1)}-p\|\\
&= (\alpha_n^{(1)}+\beta_n^{(1)})\|x_n-p\|+\alpha_n^{(1)}G_n+\gamma_n^{(1)}\|u_n^{(1)}-p\|\\
&\leq \|x_n-p\|+d_n^{(1)} \qquad (3.1)
\end{aligned}$$

where $d_n^{(1)}=\alpha_n^{(1)}G_n+\gamma_n^{(1)}\|u_n^{(1)}-p\|$. Since $\sum_{n=1}^{\infty}G_n<\infty$, we see that $\sum_{n=1}^{\infty}d_n^{(1)}<\infty$. It follows from (3.1) that

$$\begin{aligned}
\|x_n^{(2)}-p\| &\leq \alpha_n^{(2)}\|x_n^{(1)}-p\|+\alpha_n^{(2)}G_n+\beta_n^{(2)}\|x_n-p\|+\gamma_n^{(2)}\|u_n^{(2)}-p\|\\
&\leq \alpha_n^{(2)}(\|x_n-p\|+d_n^{(1)})+\alpha_n^{(2)}G_n+\beta_n^{(2)}\|x_n-p\|+\gamma_n^{(2)}\|u_n^{(2)}-p\|\\
&= (\alpha_n^{(2)}+\beta_n^{(2)})\|x_n-p\|+\alpha_n^{(2)}d_n^{(1)}+\alpha_n^{(2)}G_n+\gamma_n^{(2)}\|u_n^{(2)}-p\|\\
&\leq \|x_n-p\|+d_n^{(2)} \qquad (3.2)
\end{aligned}$$

where $d_n^{(2)}=\alpha_n^{(2)}d_n^{(1)}+\alpha_n^{(2)}G_n+\gamma_n^{(2)}\|u_n^{(2)}-p\|$. Since $\sum_{n=1}^{\infty}G_n<\infty$ and $\sum_{n=1}^{\infty}d_n^{(1)}<\infty$ it follows that $\sum_{n=1}^{\infty}d_n^{(2)}<\infty$. Moreover, we see that

$$\begin{aligned}
\|x_n^{(3)}-p\| &\leq \alpha_n^{(3)}\|x_n^{(2)}-p\|+\alpha_n^{(3)}G_n+\beta_n^{(3)}\|x_n-p\|+\gamma_n^{(3)}\|u_n^{(3)}-p\|\\
&\leq \alpha_n^{(3)}(\|x_n-p\|+d_n^{(2)})+\alpha_n^{(3)}G_n+\beta_n^{(3)}\|x_n-p\|+\gamma_n^{(3)}\|u_n^{(3)}-p\|\\
&= (\alpha_n^{(3)}+\beta_n^{(3)})\|x_n-p\|+\alpha_n^{(3)}d_n^{(2)}+\alpha_n^{(3)}G_n+\gamma_n^{(3)}\|u_n^{(3)}-p\|\\
&\leq \|x_n-p\|+d_n^{(3)} \qquad (3.3)
\end{aligned}$$

where $d_n^{(3)}=\alpha_n^{(3)}d_n^{(2)}+\alpha_n^{(3)}G_n+\gamma_n^{(3)}\|u_n^{(3)}-p\|$. So that $\sum_{n=1}^{\infty}d_n^{(3)}<\infty$. By continuiting the above method, there are nonnegative real sequences $\{d_n^{(k)}\}$ such that $\sum_{n=1}^{\infty}d_n^{(k)}<\infty$ and

$$\|x_n^{(k)}-p\|\leq\|x_n-p\|+d_n^{(k)},\ \text{for all } k=1,2,...,m.$$

This together with Lemma 2.2, we have $\lim_{n\longrightarrow\infty}\|x_n-p\|$ exists. This completes the proof. □

Lemma 3.3. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X and $T : C \longrightarrow C$ be an asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \vee 0, \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let the sequence $\{x_n\}$ be defined by (1.1) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfies the same assumptions as Lemma 3.2 for each $i \in \{1, 2, ..., m\}$ and the additional assumption that $0 < \alpha \leq \alpha_n^{(m-1)}, \alpha_n^{(m)} \leq \beta < 1$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Then

(a). $\lim_{n\longrightarrow\infty} \|T^n x_n^{(m-1)} - x_n\| = 0$;

(b). $\lim_{n\longrightarrow\infty} \|T^n x_n^{(m-2)} - x_n\| = 0$.

Proof. (a). For any $p \in F(T)$, it follows from Lemma 3.2 that $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists. Let $\lim_{n\longrightarrow\infty} \|x_n - p\| = a$ for some $a \geq 0$. We note that

$$\|x_n^{(m-1)} - p\| \leq \|x_n - p\| + d_n^{(m-1)}, \forall n \geq 1$$

where $\{d_n^{(m-1)}\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} d_n^{(m-1)} < \infty$. It follows that

$$\limsup_{n\longrightarrow\infty} \|x_n^{(m-1)} - p\| \leq \limsup_{n\longrightarrow\infty} \|x_n - p\| = \lim_{n\longrightarrow\infty} \|x_n - p\| = a,$$

from which we have

$$\limsup_{n\longrightarrow\infty} \|T^n x_n^{(m-1)} - p\| \leq \limsup_{n\longrightarrow\infty}(\|x_n^{(m-1)} - p\| + G_n) = \limsup_{n\longrightarrow\infty} \|x_n^{(m-1)} - p\| \leq a.$$

Next, we observe that

$$\|T^n x_n^{(m-1)} - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leq \|T^n x_n^{(m-1)} - p\| + \gamma_n^{(m)}\|u_n^{(m)} - x_n\|.$$

Thus we have

$$\limsup_{n\longrightarrow\infty} \|T^n x_n^{(m-1)} - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leq a. \tag{3.4}$$

Also, $\|x_n - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leq \|x_n - p\| + \gamma_n^{(m)}\|u_n^{(m)} - x_n\|$,

gives that

$$\limsup_{n\longrightarrow\infty} \|x_n - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leq a, \tag{3.5}$$

and note that

$$\begin{aligned}
a &= \lim_{n\longrightarrow\infty} \|x_n^{(m)} - p\| \\
&= \lim_{n\longrightarrow\infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} + \beta_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - p\| \\
&= \lim_{n\longrightarrow\infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} + (1-\alpha_n^{(m)}) x_n - \gamma_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} \\
&\quad - (1-\alpha_n^{(m)}) p - \alpha_n^{(m)} p\| \\
&= \lim_{n\longrightarrow\infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} - \alpha_n^{(m)} p + \alpha_n^{(m)} \gamma_n^{(m)} u_n^{(m)} - \alpha_n^{(m)} \gamma_n^{(m)} x_n + (1-\alpha_n^{(m)}) x_n \\
&\quad - (1-\alpha_n^{(m)}) p - \gamma_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - \alpha_n^{(m)} \gamma_n^{(m)} u_n^{(m)} + \alpha_n^{(m)} \gamma_n^{(m)} x_n\| \\
&= \lim_{n\longrightarrow\infty} \|\alpha_n^{(m)} (T^n x_n^{(m-1)} - p + \gamma_n^{(m)} (u_n^{(m)} - x_n)) \\
&\quad + (1-\alpha_n^{(m)})(x_n - p + \gamma_n^{(m)} (u_n^{(m)} - x_n))\|.
\end{aligned}$$

This together with (3.4), (3.5) and Lemma 3.1, we have

$$\lim_{n\longrightarrow\infty} \|T^n x_n^{(m-1)} - x_n\| = 0. \tag{3.6}$$

This completes the proof of (a).

Proof of (b). For each $n \geq 1$,

$$\begin{aligned}
\|x_n - p\| &\leq \|x_n - T^n x_n^{(m-1)}\| + \|T^n x_n^{(m-1)} - p\| \\
&\leq \|x_n - T^n x_n^{(m-1)}\| + \|x_n^{(m-1)} - p\| + G_n.
\end{aligned}$$

Since $\lim_{n\longrightarrow\infty} \|x_n - T^n x_n^{(m-1)}\| = 0 = \lim_{n\longrightarrow\infty} G_n$, we obtain that

$$a = \lim_{n\longrightarrow\infty} \|x_n - p\| \leq \liminf_{n\longrightarrow\infty} \|x_n^{(m-1)} - p\|.$$

It follows that

$$a \leq \liminf_{n\longrightarrow\infty} \|x_n^{(m-1)} - p\| \leq \limsup_{n\longrightarrow\infty} \|x_n^{(m-1)} - p\| \leq a.$$

This implies that

$$\lim_{n\longrightarrow\infty} \|x_n^{(m-1)} - p\| = a.$$

On the other hand, we note that

$$\|x_n^{(m-2)} - p\| \leq \|x_n - p\| + d_n^{(m-2)}, \forall n \geq 1$$

where $\{d_n^{(m-2)}\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} d_n^{(m-2)} < \infty$. So that

$$\limsup_{n\longrightarrow\infty} \|x_n^{(m-2)} - p\| \leq \limsup_{n\longrightarrow\infty} \|x_n - p\| = a,$$

and hence

$$\limsup_{n\longrightarrow\infty}\|T^n x_n^{(m-2)} - p\| \le \limsup_{n\longrightarrow\infty}(\|x_n^{(m-2)} - p\| + G_n) \le a.$$

Next we observe that

$$\|T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \le \|T^n x_n^{(m-2)} - p\| + \gamma_n^{(m-1)}\|u_n^{(m-1)} - x_n\|.$$

Thus,

$$\limsup_{n\longrightarrow\infty}\|T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \le a. \tag{3.7}$$

Also, $\|x_n - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \le \|x_n - p\| + \gamma_n^{(m-1)}\|u_n^{(m-1)} - x_n\|$,

gives that

$$\limsup_{n\longrightarrow\infty}\|x_n - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \le a, \tag{3.8}$$

and note that

$$\begin{aligned} a &= \lim_{n\longrightarrow\infty}\|x_n^{(m-1)} - p\| \\ &= \lim_{n\longrightarrow\infty}\|\alpha_n^{(m-1)}T^n x_n^{(m-2)} + \beta_n^{(m-1)}x_n + \gamma_n^{(m-1)}u_n^{(m-1)} - p\| \\ &= \lim_{n\longrightarrow\infty}\|\alpha_n^{(m-1)}(T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)) \\ &\quad + (1 - \alpha_n^{(m-1)})(x_n - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n))\|. \end{aligned} \tag{3.9}$$

It follows from (3.7), (3.8), (3.9) and Lemma 3.1 that

$$\lim_{n\longrightarrow\infty}\|T^n x_n^{(m-2)} - x_n\| = 0.$$

This completes the proof of (b). □

We now state and prove the first main result of this paper and this is the main motivation of our next result.

Theorem 3.4. *Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X and $T : C \longrightarrow C$ be a completely continuous asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0, \forall n \ge 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let the sequence $\{x_n\}$ be defined by (1.1) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfies the same assumptions as Lemma 3.2 for each $i \in \{1, 2, ..., m\}$ and the additional assumption that $0 < \alpha \le \alpha_n^{(m-1)}, \alpha_n^{(m)} \le \beta < 1$ for all $n \ge n_0$, for some $n_0 \in \mathbb{N}$. Then $\{x_n^{(k)}\}$ converges strongly to a fixed point of T for each $k = 1, 2, 3, ..., m$.

Proof. It follows from Lemma 3.3 that

$$\lim_{n\longrightarrow\infty}\|T^n x_n^{(m-1)} - x_n\| = 0 = \lim_{n\longrightarrow\infty}\|T^n x_n^{(m-2)} - x_n\|$$

and this implies that,

$$\begin{aligned}\|x_{n+1} - x_n\| = \|x_n^{(m)} - x_n\| &\leq \alpha_n^{(m)}\|T^n x_n^{(m-1)} - x_n\| + \gamma_n^{(m)}\|u_n^{(m)} - x_n\| \\ &\longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}\tag{3.10}$$

It follows from (3.10) that

$$\begin{aligned}\|T^n x_n - x_n\| &\leq \|T^n x_n - T^n x_n^{(m-1)}\| + \|T^n x_n^{(m-1)} - x_n\| \\ &\leq \|x_n - x_n^{(m-1)}\| + G_n + \|T^n x_n^{(m-1)} - x_n\| \\ &\leq \alpha_n^{(m-1)}\|x_n - T^n x_n^{(m-2)}\| + G_n + \gamma_n^{(m-1)}\|u_n^{(m-1)} - x_n\| \\ &\quad + \|T^n x_n^{(m-1)} - x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned}\tag{3.11}$$

Since

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\ &+ \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\|,\end{aligned}$$

it follows from (3.10), (3.11) and uniformly continuity of T that

$$\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0. \tag{3.12}$$

Since $\{x_n\}$ is a bounded and T is completely continuous, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \longrightarrow p \in C$ as $k \longrightarrow \infty$. Moreover, by (3.12), we have $\|Tx_{n_k} - x_{n_k}\| \longrightarrow 0$ which implies that $x_{n_k} \longrightarrow p$ as $k \longrightarrow \infty$. By (3.12) again, we have

$$\|p - Tp\| = \lim_{k\longrightarrow\infty}\|x_{n_k} - Tx_{n_k}\| = 0.$$

It show that $p \in F(T)$. Since $\lim_{n\longrightarrow\infty}\|x_n - p\|$ exists, we have $\lim_{n\longrightarrow\infty}\|x_n - p\| = 0$; that is $\lim_{n\longrightarrow\infty} x_n^{(m)} = \lim_{n\longrightarrow\infty} x_n = p$. Moreover, we observe that $\|x_n^{(k)} - p\| \leq \|x_n - p\| + d_n^{(k)}$ for all $k = 1, 2, 3, ..., m-1$ and each $\lim_{n\longrightarrow\infty} d_n^{(k)} = 0$. Therefore $\lim_{n\longrightarrow\infty} x_n^{(k)} = p$ for all $k = 1, 2, 3, ..., m-1$. The proof is completed. □

4 Asymptotically quasi-nonexpansive mappings

In the next result, we prove strong convergence theorem for the multi-step Noor iterations (1.1) for asymptotically quasi-nonexpansive mapping in a uniformly convex Banach space. To do this, we need the following lemmas.

Lemma 4.1. *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X and T be an asymptotically quasi-nonexpansive with the sequence $\{r_n\}_{n\geq 1}$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (1.1) with the following restrictions:*

(i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ *for all* $i \in \{1, 2, ..., m\}$ *and for all* $n \geq 1$;

(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ *for all* $i \in \{1, 2, ..., m\}$.

If $p \in F(T)$, *then* $\lim_{n\longrightarrow\infty} \|x_n - p\|$ *exists.*

Proof. Let $p \in F(T)$. For each $n \geq 1$, we note that

$$\begin{aligned}
\|x_n^{(1)} - p\| &= \|\alpha_n^{(1)} T^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)} - p\| \\
&\leq \alpha_n^{(1)} \|T^n x_n - p\| + \beta_n^{(1)} \|x_n - p\| + \gamma_n^{(1)} \|u_n^{(1)} - p\| \\
&\leq \alpha_n^{(1)} (1 + r_n) \|x_n - p\| + \beta_n^{(1)} \|x_n - p\| + \gamma_n^{(1)} \|u_n^{(1)} - p\| \\
&\leq (1 + r_n) \|x_n - p\| + d_n^{(1)} \qquad (4.1)
\end{aligned}$$

where $d_n^{(1)} = \gamma_n^{(1)} \|u_n^{(1)} - p\|$. Since $\{u_n^{(1)}\}$ is bounded and $\sum_{n=1}^{\infty} \gamma_n^{(1)} < \infty$ we see that $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$. It follows from (4.1) that

$$\begin{aligned}
\|x_n^{(2)} - p\| &\leq \alpha_n^{(2)} (1 + r_n) \|x_n^{(1)} - p\| + \beta_n^{(2)} \|x_n - p\| + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\
&\leq \alpha_n^{(2)} (1 + r_n)((1 + r_n)\|x_n - p\| + d_n^{(1)}) + \beta_n^{(2)} (1 + r_n)^2 \|x_n - p\| \\
&\quad + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\
&= (\alpha_n^{(2)} + \beta_n^{(2)})(1 + r_n)^2 \|x_n - p\| + \alpha_n^{(2)} d_n^{(1)} (1 + r_n) + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\
&\leq (1 + r_n)^2 \|x_n - p\| + \alpha_n^{(2)} d_n^{(1)} (1 + r_n) + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\
&= (1 + r_n)^2 \|x_n - p\| + d_n^{(2)}, \qquad (4.2)
\end{aligned}$$

where $d_n^{(2)} = \alpha_n^{(2)} d_n^{(1)} (1 + r_n) + \gamma_n^{(2)} \|u_n^{(2)} - p\|$. Since $\{u_n^{(2)}\}$ is bounded and $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$, it follows that $\sum_{n=1}^{\infty} d_n^{(2)} < \infty$. Moreover, we see that

$$\begin{aligned}
\|x_n^{(3)} - p\| &\leq \alpha_n^{(3)} (1 + r_n) \|x_n^{(2)} - p\| + \beta_n^{(3)} \|x_n - p\| + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\
&\leq \alpha_n^{(3)} (1 + r_n)((1 + r_n)^2 \|x_n - p\| + d_n^{(2)}) + \beta_n^{(3)} (1 + r_n)^3 \|x_n - p\| \\
&\quad + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\
&\leq (\alpha_n^{(3)} + \beta_n^{(3)})(1 + r_n)^3 \|x_n - p\| + \alpha_n^{(3)} d_n^{(2)} (1 + r_n) + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\
&\leq (1 + r_n)^3 \|x_n - p\| + \alpha_n^{(3)} d_n^{(2)} (1 + r_n) + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\
&= (1 + r_n)^3 \|x_n - p\| + d_n^{(3)}, \qquad (4.3)
\end{aligned}$$

where $d_n^{(3)} = \alpha_n^{(3)} d_n^{(2)} (1 + r_n) + \gamma_n^{(3)} \|u_n^{(3)} - p\|$. So that $\sum_{n=1}^{\infty} d_n^{(3)} < \infty$. By continuiting the above method, there are nonnegative real sequence $\{d_n^{(k)}\}$ such that $\sum_{n=1}^{\infty} d_n^{(k)} <$

∞ and

$$\|x_n^{(k)} - p\| \le (1+r_n)^k \|x_n - p\| + d_n^{(k)}, \text{ for all } k = 1, 2, ..., m.$$

By Lemma 2.2 , we have $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists. This completes the proof. □

Lemma 4.2. *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X and $T: C \longrightarrow C$ be an asymptotically quasi-nonexpansive with the sequence $\{r_n\}_{n\ge1}$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $F(T) \ne \emptyset$. Let the sequence $\{x_n\}$ be defined by (1.1) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfies the same assumptions as Lemma 4.1 for each $i \in \{1, 2, ..., m\}$ and the additional assumption that $0 < \alpha \le \alpha_n^{(m-1)}, \alpha_n^{(m)} \le \beta < 1$ for all $n \ge n_0$, for some $n_0 \in \mathbb{N}$. Then*

(a). $\lim_{n\longrightarrow\infty} \|T^n x_n^{(m-1)} - x_n\| = 0$;

(b). $\lim_{n\longrightarrow\infty} \|T^n x_n^{(m-2)} - x_n\| = 0$.

Proof. (a). For any $p \in F(T)$, it follows from Lemma 4.1 that $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists. Let $\lim_{n\longrightarrow\infty} \|x_n - p\| = a$ for some $a \ge 0$. We note that

$$\|x_n^{(m-1)} - p\| \le (1+r_n)^{m-1} \|x_n - p\| + d_n^{(m-1)}, \forall n \ge 1$$

where $\{d_n^{(m-1)}\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} d_n^{(m-1)} < \infty$. It follows that

$$\limsup_{n\longrightarrow\infty} \|x_n^{(m-1)} - p\| \le \limsup_{n\longrightarrow\infty} ((1+r_n)^{m-1}\|x_n - p\| + d_n^{(m-1)}) = \lim_{n\longrightarrow\infty} \|x_n - p\| = a$$

and so

$$\limsup_{n\longrightarrow\infty} \|T^n x_n^{(m-1)} - p\| \le \limsup_{n\longrightarrow\infty} (1+r_n)\|x_n^{(m-1)} - p\| = \limsup_{n\longrightarrow\infty} \|x_n^{(m-1)} - p\| \le a.$$

Next, consider

$$\|T^n x_n^{(m-1)} - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \le \|T^n x_n^{(m-1)} - p\| + \gamma_n^{(m)} \|u_n^{(m)} - x_n\|.$$

Thus,

$$\limsup_{n\longrightarrow\infty} \|T^n x_n^{(m-1)} - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \le a. \tag{4.4}$$

Also, $\|x_n - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \le \|x_n - p\| + \gamma_n^{(m)}\|u_n^{(m)} - x_n\|$,

gives that

$$\limsup_{n\longrightarrow\infty} \|x_n - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \le a, \tag{4.5}$$

and we observe that

$$
\begin{aligned}
a &= \lim_{n\longrightarrow\infty} \|x_n^{(m)} - p\| \\
&= \lim_{n\longrightarrow\infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} + \beta_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - p\| \\
&= \lim_{n\longrightarrow\infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} + (1-\alpha_n^{(m)}) x_n - \gamma_n^{(m)} x_n \\
&\quad + \gamma_n^{(m)} u_n^{(m)} - (1-\alpha_n^{(m)})p - \alpha_n^{(m)} p\| \\
&= \lim_{n\longrightarrow\infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} - \alpha_n^{(m)} p + \alpha_n^{(m)}\gamma_n^{(m)} u_n^{(m)} - \alpha_n^{(m)}\gamma_n^{(m)} x_n \\
&\quad + (1-\alpha_n^{(m)}) x_n - (1-\alpha_n^{(m)}) p \\
&\quad - \gamma_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - \alpha_n^{(m)}\gamma_n^{(m)} u_n^{(m)} + \alpha_n^{(m)}\gamma_n^{(m)} x_n\| \\
&= \lim_{n\longrightarrow\infty} \|\alpha_n^{(m)}(T^n x_n^{(m-1)} - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)) + (1-\alpha_n^{(m)})(x_n - p + \gamma_n^{(m)}(u_n^{(m)} - x_n))\|.
\end{aligned}
$$

It follows from (4.4), (4.5) and Lemma 3.1 that

$$\lim_{n\longrightarrow\infty} \|T^n x_n^{(m-1)} - x_n\| = 0.$$

This completes the proof of (a).

Proof of (b). For each $n \geq 1$, we have

$$
\begin{aligned}
\|x_n - p\| &\leq \|x_n - T^n x_n^{(m-1)}\| + \|T^n x_n^{(m-1)} - p\| \\
&\leq \|x_n - T^n x_n^{(m-1)}\| + (1+r_n)\|x_n^{(m-1)} - p\|.
\end{aligned}
$$

Since $\lim_{n\longrightarrow\infty} \|x_n - T^n x_n^{(m-1)}\| = 0 = \lim_{n\longrightarrow\infty} r_n$, we obtain that

$$a = \lim_{n\longrightarrow\infty} \|x_n - p\| \leq \liminf_{n\longrightarrow\infty} \|x_n^{(m-1)} - p\|.$$

It follows that

$$a \leq \liminf_{n\longrightarrow\infty} \|x_n^{(m-1)} - p\| \leq \limsup_{n\longrightarrow\infty} \|x_n^{(m-1)} - p\| \leq a,$$

which implies that

$$\lim_{n\longrightarrow\infty} \|x_n^{(m-1)} - p\| = a.$$

On the other hand, we note that

$$\|x_n^{(m-2)} - p\| \leq (1+r_n)^{m-2}\|x_n - p\| + d_n^{(m-2)}, \forall n \geq 1$$

where $\{d_n^{(m-2)}\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} d_n^{(m-2)} < \infty$. Thus

$$\limsup_{n\longrightarrow\infty} \|x_n^{(m-2)} - p\| \leq \limsup_{n\longrightarrow\infty} (1+r_n)^{m-2}\|x_n - p\| = a,$$

and hence

$$\limsup_{n\longrightarrow\infty} \|T^n x_n^{(m-2)} - p\| \le \limsup_{n\longrightarrow\infty}(1+r_n)\|x_n^{(m-2)} - p\| \le a.$$

Next, consider

$$\|T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \le \|T^n x_n^{(m-2)} - p\| + \gamma_n^{(m-2)}\|u_n^{(m-1)} - x_n\|$$

Thus,

$$\limsup_{n\longrightarrow\infty} \|T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \le a. \tag{4.6}$$

Also, $\|x_n - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \le \|x_n - p\| + \gamma_n^{(m-1)}\|u_n^{(m-1)} - x_n\|$,

gives that

$$\limsup_{n\longrightarrow\infty} \|x_n - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \le a, \tag{4.7}$$

and noth that

$$\begin{aligned} a &= \lim_{n\longrightarrow\infty} \|x_n^{(m-1)} - p\| \\ &= \lim_{n\longrightarrow\infty} \|\alpha_n^{(m-1)} T^n x_n^{(m-2)} + \beta_n^{(m-1)} x_n + \gamma_n^{(m-1)} u_n^{(m-1)} - p\| \\ &= \lim_{n\longrightarrow\infty} \|\alpha_n^{(m-1)}(T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)) \\ &\quad + (1-\alpha_n^{(m-1)})(x_n - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n))\|. \end{aligned}$$

It follows from (4.6), (4.7) and Lemma 3.1 that

$$\lim_{n\longrightarrow\infty} \|T^n x_n^{(m-2)} - x_n\| = 0.$$

This completes the proof of (b). □

Theorem 4.3. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X and $T : C \longrightarrow C$ be an uniformly L-Lipschitzian, completely continuous asymptotically quasi-nonexpansive with the sequence $\{r_n\}_{n\ge1}$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (1.1) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfies the same assumptions as Lemma 4.1 for each $i \in \{1, 2, ..., m\}$ and the additional assumption that $0 < \alpha \le \alpha_n^{(i)} \le \beta < 1$ for all $i \in \{m-1, m\}$. Then $\{x_n^{(k)}\}$ converge strongly to a fixed point of T, for each $k = 1, 2, 3, ..., m$.*

Proof. It follows from Lemma 4.2 that

$$\lim_{n\longrightarrow\infty} \|T^n x_n^{(m-1)} - x_n\| = 0 = \lim_{n\longrightarrow\infty} \|T^n x_n^{(m-2)} - x_n\|.$$

This implies that,

$$\begin{aligned}\|x_{n+1} - x_n\| = \|x_n^{(m)} - x_n\| &\leq \alpha_n^{(m)}\|T^n x_n^{(m-1)} - x_n\| + \gamma_n^{(m)}\|u_n^{(m)} - x_n\| \\ &\longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{4.8}$$

Thus, we have

$$\begin{aligned}\|T^n x_n - x_n\| &\leq \|T^n x_n - T^n x_n^{(m-1)}\| + \|T^n x_n^{(m-1)} - x_n\| \\ &\leq L\|x_n - x_n^{(m-1)}\| + \|T^n x_n^{(m-1)} - x_n\| \\ &\leq \alpha_n^{(m-1)} L\|x_n - T^n x_n^{(m-2)}\| + \gamma_n^{(m-1)} L\|u_n^{(m-1)} - x_n\| \\ &\quad + \|T^n x_n^{(m-1)} - x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty \end{aligned} \tag{4.9}$$

and we note that

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\ &+ \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\ &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\ &+ (1 + r_{n+1})\|x_{n+1} - x_n\| + L\|T^n x_n - x_n\|. \end{aligned}$$

This together with (4.8) and (4.9) we obtain

$$\lim_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0. \tag{4.10}$$

By the boundedness of $\{x_n\}$ and our assumption that T is completely continuous, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \longrightarrow p \in C$ as $k \longrightarrow \infty$. Moreover, by (4.10), we have $\|Tx_{n_k} - x_{n_k}\| \longrightarrow 0$ which implies that $x_{n_k} \longrightarrow p$ as $k \longrightarrow \infty$. By (4.10) again, we have

$$\|p - Tp\| = \lim_{k \longrightarrow \infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

It show that $p \in F(T)$. Furthermore, since $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exist we obtain $\lim_{n\longrightarrow\infty} \|x_n - p\| = 0$, that is $\lim_{n\longrightarrow\infty} x_n^{(m)} = \lim_{n\longrightarrow\infty} x_n = p$. Moreover we observe that $\|x_n^{(k)} - p\| \leq \|x_n - p\| + d_n^{(k)}$ for all $k = 1, 2, 3, ..., m-1$ and each $\lim_{n\longrightarrow\infty} d_n^{(k)} = 0$. Therefore $\lim_{n\longrightarrow\infty} x_n^{(k)} = p$ for all $k = 1, 2, 3, ..., m-1$. The proof is completed. □

For $m = 3$ and $\beta_n^{(i)} = 1 - \alpha_n^{(i)} - \gamma_n^{(i)}$ for all $i = 1, 2, 3$ in Theorem 3.4 or Theorem 4.3, we obtain the following result.

Theorem 4.4. (see [2]) *Let X be uniformly convex Banach space and C be a non-empty closed convex subset of X. Let $T : C \longrightarrow C$ be an completely continuous asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{r_n\}$ in $[0, \infty)$ and $\sum_{n=1}^{\infty} r_n < \infty$. Let a sequence $\{x_n\}$ be defined by (1.2) with the following restrictions:*

(i) $0 < a \leq \alpha_n^{(3)} < b < 1$

(ii) $\limsup_{n \longrightarrow \infty}(1 + r_n)\alpha_n^{(2)} < 1$

(iii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ *for all* $i = 1, 2, 3$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point p of T.

When $m = 3$ and $\gamma_n^{(1)} = \gamma_n^{(2)} = \gamma_n^{(3)} \equiv 0$ in Theorem 3.4 or Theorem 4.3, we obtain strong convergence theorem for Noor iteration as follows:

Theorem 4.5. [20, Theorem 2.1]. *Let X be a real uniformly convex Banach space, C be a nonempty closed, bounded convex subset of X. Let $T : C \longrightarrow C$ be a completely continuous asymptotically nonexpansive self-mapping with sequence $\{r_n\}$ satisfying $r_n \geq 0$ and $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \{\alpha_n^{(3)}\}$ be real sequences in $[0, 1]$ satisfying;*

(i) $0 < \liminf_{n \longrightarrow \infty} \alpha_n^{(3)} \leq \limsup_{n \longrightarrow \infty} \alpha_n^{(3)} < 1$, *and*

(ii) $0 < \liminf_{n \longrightarrow \infty} \alpha_n^{(2)} \leq \limsup_{n \longrightarrow \infty} \alpha_n^{(2)} < 1$.

For a given $x_1 \in C$, the sequence $\{x_n\}, \{x_n^{(1)}\}, \{x_n^{(2)}\}$ defined by (1.3) converges strongly to a fixed point of T.

Proof. It follows from the condition (i) and (ii) that there are $\alpha, \beta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that

$$0 < \alpha \leq \alpha_n^{(2)}, \alpha_n^{(3)} \leq \beta < 1$$

for all $n \geq n_0$. So that the conclusion of Theorem follows from the Theorem 3.4 or Theorem 4.3. □

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ภาคผนวก 5

The characteristic of noncompact convexity and random fixed point theorem for set-valued operators

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ภาคผนวก 5/1

THE CHARACTERISTIC OF NONCOMPACT CONVEXITY AND RANDOM FIXED POINT THEOREM FOR SET-VALUED OPERATORS

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Abstract. Let (Ω, Σ) be a measurable space, X a Banach space whose characteristic of noncompact convexity is less than 1, C a bounded closed convex subset of X, $KC(C)$ the family of all compact convex subsets of C. We prove that a set-valued nonexpansive mapping $T : C \to KC(C)$ has a fixed point. Furthermore, if X is separable then we also prove that a set-valued nonexpansive operator $T : \Omega \times C \to KC(C)$ has a random fixed point.

Keywords : random fixed point, set-valued random operator, measure of noncompacness.

Mathematics Subject Classification 2000 : 47H10, 47H09, 47H04.

1. Introduction

The study of random fixed points has been a very active area of research in probabilistic operator theory in the last decade. In this direction, there have appeared various papers concerning random fixed point theorems for single-valued and set-valued random operators; see, for example, [6],[8],[10],[11],[12][15],[21] and the references therein.

In 2002, P. Lorenzo Ramírez [10] proved the existence of a random fixed point theorems for a random nonexpansive operator in the framework of Banach spaces with the characteristic of noncompact convexity $\varepsilon_\alpha(X)$ less than 1. On the other hand, Domínguez Benavides and Ramírez [4] proved a fixed point theorem for a set-valued nonexpansive and 1-χ-contractive mapping in the framework of Banach spaces whose characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ less than 1.

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The purpose of the present paper is to prove a fixed point theorem for set-valued random nonexpansive operators in the framework of Banach spaces with characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ less than 1. Moreover, we also prove a fixed point theorem for set-valued nonexpansive mappings in Banach spaces with characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ less than 1. Our results can also be viewed as an extension of Theorem 6 in [10] and Theorem 4.2 in [4], respectively.

2. Preliminaries

Through out this paper we will consider a measurable space (Ω, Σ) (where Σ is a $\sigma-$algebra of subsets of Ω) and (X, d) will be a metric space. We denote by $CL(X)$(resp. $CB(X), KC(X)$) the family of all nonempty closed (resp. closed bounded, compact convex) subsets of X, and by H the Hausdorff metric on $CB(X)$ induced by d, i.e.,

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

for $A, B \in CB(X)$, where $d(x,E) = \inf\{d(x,y)|y \in E\}$ is the distance from x to $E \subset X$.

Let C be a nonempty closed subset of a Banach space X. Recall now that a set-valued mapping $T : C \to 2^X$ is said to be upper semicontinuous on C if $\{x \in C : Tx \subset V\}$ is open in C whenever $V \subset X$ is open; T is said to be lower semicontinuous if $T^{-1}(V) := \{x \in C : Tx \cap V \neq \emptyset\}$ is open in C whenever $V \subset X$ is open; and T is said to be continuous if it is both upper and lower semicontinuous (cf.[2] and [3] for details). There is another different kind of continuity for multivalued operators: $T : C \to CB(X)$ is said to be continuous on C (with respect to the Hausdorff metric H) if $H(Tx_n, Tx) \to 0$ whenever $x_n \to x$. It is not hard to see (see Deimling [3]) that both definitions of continuity are equivalent if Tx is compact for every $x \in C$.

A set-valued operator $T : \Omega \to 2^X$ is called $(\Sigma)-$ measurable if, for any open subset B of X,

$$T^{-1}(B) := \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\}$$

belongs to Σ. A mapping $x : \Omega \to X$ is said to be a *measurable selector* of a measurable set-valued operator $T : \Omega \to 2^X$ if $x(\cdot)$ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. An operator $T : \Omega \times C \to 2^X$ is called a random operator if, for each fixed $x \in C$, the operator $T(\cdot, x) : \Omega \to 2^X$ is measurable. We will denote by $F(\omega)$ the fixed point set of $T(\omega, \cdot)$, i.e.,

$$F(\omega) := \{x \in C : x \in T(\omega, x)\}.$$

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Note that if we do not assume the existence of a fixed point for the deterministic mapping $T(\omega,\cdot) : C \to 2^X, F(\omega)$ may be empty. A measurable operator $x : \Omega \to C$ is said to be a *random fixed point of an operator* $T : \Omega \times C \to 2^X$ if $x(\omega) \in T(\omega, x(\omega))$ for all $\omega \in \Omega$. Recall that $T : \Omega \times C \to 2^X$ is continuous if, for each fixed $\omega \in \Omega$, the operator $T : (\omega,\cdot) \to 2^X$ is continuous.

If C is a closed convex subset of a Banach space X, then a set-valued mapping $T : C \to CB(X)$ is said to be a *contraction* if there exists a constant $k \in [0,1)$ such that

$$H(Tx, Ty) \leq k\|x - y\|, \quad x, y \in C,$$

and T is said to be *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in C.$$

A random operator $T : \Omega \times C \to 2^X$ is said to be *nonexpansive* if, for each fixed $\omega \in \Omega$, the map $T : (\omega,\cdot) \to C$ is nonexpansive.

For later convenience, we list the following results related to the concept of measurability.

Lemma 2.1. (Wagner cf.[14]) *Let (X, d) be a complete separable metric space and $F : \Omega \to CL(X)$ a measurable map. Then F has a measurable selector.*

Lemma 2.2. (Itoh 1977, cf.[8]) *Suppose $\{T_n\}$ is a sequence of measurable set-valued operator from Ω to $CB(X)$ and $T : \Omega \to CB(X)$ is an operators. If, for each $\omega \in \Omega$, $H(T_n(\omega), T(\omega)) \to 0$, then T is measurable.*

Lemma 2.3. (Tan and Yuan cf.[13]) *Let X be a separable metric space and Y a metric space. If $f : \Omega \times X \to Y$ is measurable in $\omega \in \Omega$ and continuous in $x \in X$, and if $x : \Omega \to X$ is measurable, then $f(\cdot, x(\cdot)) : \Omega \to Y$ is measurable.*

As an easy application of Proposition 3 of Itoh[8] we have the following result.

Lemma 2.4. *Let C be a closed separable subset of a Banach space X, $T : \Omega \times C \to C$ a random continuous operator and $F : \Omega \to 2^C$ a measurable closed-valued operator. Then for any $s > 0$, the operator $G : \Omega \to 2^C$ given by*

$$G(\omega) = \{x \in F(\omega) : \|x - T(\omega, x)\| < s\}, \quad \omega \in \Omega$$

is measurable and so is the operator $\mathrm{cl}\{G(\omega)\}$ of the closure of $G(\omega)$.

Lemma 2.5. (Domínguez Benavidel, Lopez Acedo and Xu cf.[6]) *Suppose C is a weakly closed nonempty separable subset of a Banach space X, $F : \Omega \to 2^X$ a measurable map with weakly*

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compact values and $f : \Omega \times C \to \mathbb{R}$ *a measurable, continuous and weakly lower semicontinuous function. Then the marginal function* $r : \Omega \to \mathbb{R}$ *defined by*

$$r(\omega) := \inf_{x \in F(x)} f(\omega, x)$$

and the marginal map $R : \Omega \to X$ *defined by*

$$R(\omega) := \{x \in F(x) : f(\omega, x) = r(\omega)\}$$

are measurable.

Recall that the Kuratowski and Hausdorff measures of noncompactness of a nonempty bounded subset B of X are respectively defined as the number

$$\alpha(B) = \inf\{r > 0 : B \text{ can be covered by finitely many sets of diameter } \leq r\},$$

$$\chi(B) = \inf\{r > 0 : B \text{ can be covered by finitely many balls of radius } \leq r\}.$$

The separation measure of noncompacness of a nonempty bounded subset B of X defined by

$$\beta(B) = \sup\{\varepsilon : \text{there exists a sequence } \{x_n\} \text{ in } B \text{ such that } \operatorname{sep}(\{x_n\}) \geq \varepsilon\}.$$

Let X be a Banach space and $\phi = \alpha, \beta$ or χ. The modulus of noncompact convexity associated to ϕ is defined in the following way:

$$\Delta_{X,\phi}(\varepsilon) = \inf\{1 - d(0, A) : A \subset B_X \text{ is convex, } \phi(A) \geq \varepsilon\},$$

where B_X is the unit ball of X.

The characteristic of noncompact convexity of X associated with the measure of noncompactness ϕ is defined by

$$\varepsilon_\phi(X) = \sup\{\varepsilon \geq 0 : \Delta_{X,\phi}(\varepsilon) = 0\}.$$

The following relationships among the different moduli are easy to obtain

$$\Delta_{X,\alpha}(\varepsilon) \leq \Delta_{X,\beta}(\varepsilon) \leq \Delta_{X,\chi}(\varepsilon), \tag{2.1}$$

and consequently

$$\varepsilon_\alpha(X) \geq \varepsilon_\beta(X) \geq \varepsilon_\chi(X). \tag{2.2}$$

When X is a reflexive Banach space we have some alternative expressions for the moduli of noncompact convexity associated to β and χ.

$$\Delta_{X,\beta}(\varepsilon) = \inf\{1 - \|x\| : \{x_n\} \subset B_X, x = w - lim_n\, x_n, \operatorname{sep}(\{x_n\}) \geq \varepsilon\},$$

$$\Delta_{X,\chi}(\varepsilon) = \inf\{1 - \|x\| : \{x_n\} \subset B_X, x = w - lim_n\, x_n, \chi(\{x_n\}) \geq \varepsilon\}.$$

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Let C be a nonempty bounded closed subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. We use $r(C,\{x_n\})$ and $A(C,\{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C, respectively, i.e.

$$\begin{aligned} r(C,\{x_n\}) &= \inf\left\{\limsup_n \|x_n - x\| : x \in C\right\}, \\ A(C,\{x_n\}) &= \left\{x \in C : \limsup_n \|x_n - x\| = r(C,\{x_n\})\right\}. \end{aligned}$$

If D is a bounded subset of X, the *Chebyshev radius* of D relative to C is defined by

$$r_C(D) := \inf\{\sup\{\|x-y\| : y \in D\} : x \in C\}.$$

Let $\{x_n\}$ and C be nonempty bounded closed subsets of a Banach space X. Then $\{x_n\}$ is called *regular* with respect to C if $r(C,\{x_n\}) = r(C,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Moreover, we also need the following Lemmas.

Lemma 2.6. (Domínguez Benavides and Lorenzo Ramírez Theorem 4.3 cf. [4]) *Let C be a closed convex subset of a reflexive Banach space X, and let x_n be a bounded sequence in C which is regular with respect to C. Then*

$$r_C(A(C,x_n)) \le (1 - \Delta_{X,\beta}(1^-))r(C,\{x_n\}). \tag{2.3}$$

Moreover, if X satisfies the nonstrict Opial condition then

$$r_C(A(C,x_n)) \le (1 - \Delta_{X,\chi}(1^-))r(C,\{x_n\}). \tag{2.4}$$

The following result are now basic in the fixed point theorem for multivalued mappings.

Lemma 2.7. (Xu cf. Theorem 1.6 of [19]) *Let E be a nonempty bounded closed closed convex subset of a Banach space and $T : E \to KC(X)$ a contraction. Assume $Tx \cap \overline{I_E(x)} \neq \emptyset$ for all $x \in E$. Then T has a fixed point. (Here $I_E(x)$ is call the inward set at x defined by $I_E := \{x + \lambda(y - x) : \lambda \ge 0, y \in E\}$)*

Proposition 2.8. (Kirk-Massa Theorem cf.[16]) *Let C be a nonempty weakly compact separable subset of a Banach space X, $T : C \to K(C)$ a nonexpansive mapping, and $\{x_n\}$ a sequence in C such that $\lim_n d(x_n - Tx_n) = 0$. Then, there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that*

$$Tx \cap A \neq \emptyset, \forall x \in A := A(C,\{z_n\})$$

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3. The Results

We begin this section by showing that in Benavides-Ramírez's result, the 1-χ-contractive condition on T can be removed.

Theorem 3.1. *Let C be a nonempty closed bounded convex subset of a Banach space X such that $\varepsilon_\beta(X) < 1$, and $T : C \to KC(C)$ a nonexpansive mapping. Then T has a fixed point.*

Proof The condition $\varepsilon_\beta(X) < 1$ implies reflexivity [2], so C is weakly compact. Let $x_0 \in C$ be fixed and, for each $n \geq 1$, define $T_n : C \to KC(C)$ by

$$T_n x = \frac{1}{n}x_0 + (1 - \frac{1}{n})Tx, \ \ \forall x \in C.$$

Then T_n is a set-valued contraction and hence has a fixed point x_n. It is easily seen that $\operatorname{dist}(x_n, Tx_n) \leq \frac{1}{n}\operatorname{diam}C \to 0$ as $n \to \infty$. By Goebel and Kirk [7], we may assume that $\{x_n\}$ is regular with respect to C and using Proposition 2.8 we can also assume that

$$Tx \cap A \neq \emptyset, \ \ \forall x \in A := A(C, \{x_n\}).$$

We apply Lemma 2.6 to obtain

$$r_C(A) \leq \lambda r(C, \{x_n\}), \tag{3.1}$$

where $\lambda := (1 - \Delta_{X,\beta}(1^-)) < 1$.

It is clear that A is a weakly compact convex subset of C. Now fix $x_1 \in A$ and for each $n \geq 1$, define the contraction $T_n^1 : A \to KC(C)$ by

$$T_n^1(x) = \frac{1}{n}x_1 + (1 - \frac{1}{n})Tx, \ \ \forall x \in A.$$

Since A is convex, each T_n^1 satisfies the same boundary condition as T does, that is, we have

$$T_n^1 x \cap \overline{I}_A(x) \neq \emptyset, \ \ \forall x \in A,$$

Hence by Lemma 2.7, T_n^1 has a fixed point $z_n \in A$. Consequently, we can get a sequence $\{x_n^1\}$ in A satisfying $d(x_n^1, T(x_n^1)) \to 0$ as $n \to \infty$. Again, applying Lemma 2.6, we obtain

$$r_C(A^1) \leq \lambda r(C, \{x_n^1\}), \tag{3.2}$$

where $A^1 := A(C, \{x_n^1\})$. Since $\{x_n^1(\omega)\} \subset A$, we have

$$r(C, \{x_n^1\}) \leq r_C(A), \tag{3.3}$$

and then

$$r_C(A^1) \leq \lambda^2 r_C(A). \tag{3.4}$$

By induction, for each $m \geq 1$, we construct A^m, and $\{x_n^m\}_n$ where $A^m = A(C, \{x_n^m\}), x_n^m \subset A^{m-1}$ such that $d(x_n^m, Tx_n^m) \to 0$ as $n \to \infty$ and

$$r_C(A^m) \leq \lambda r_C(A) \leq \lambda^m r(C, \{x_n\}). \tag{3.5}$$

By assumption $\varepsilon_\beta(X) < 1$ and $\operatorname{diam} A^m \leq 2r_C(A^m)$ leads to $\lim_{m\to\infty} \operatorname{diam} A^m = 0$. Since $\{A^m\}$ is a descending sequence of weakly compact subsets of C, we have $\cap_m A^m = \{z\}$ for some $z \in C$. Finally, we will show that z is a fixed point of T. Indeed, for each $m \geq 1$, we have

$$\begin{aligned} d(z, Tz) &\leq \|z - x_n^m\| + d(x_n^m, Tx_n^m) + H(Tx_n^m, Tz) \\ &\leq 2\|z - x_n^m\| + d(x_n^m, Tx_n^m) \\ &\leq 2\operatorname{diam} A^m + d(x_n^m, Tx_n^m). \end{aligned}$$

Taking the upper limit as $n \to \infty$,

$$d(z, Tz) \leq 2\operatorname{diam} A^m.$$

Now taking the limit in m on both sides we obtain $z \in Tz$. □

Corollary 3.2. (Domínguez Benavides and Lorenzo Ramírez. Theorem 4.2 in [4]) *Let C be a nonempty closed bounded convex subset of a Banach space X such that $\varepsilon_\beta(X) < 1$, and $T : C \to KC(C)$ a nonexpansive and 1-χ-contractive mapping. Then T has a fixed point.*

Now we are ready to prove the main result of this paper.

Theorem 3.3. *Let C be a nonempty closed bounded convex separable subset of a Banach space X such that $\varepsilon_\beta(X) < 1$, and $T : \Omega \times C \to KC(C)$ be a set-valued nonexpansive random operator. Then T has a random fixed point.*

Proof For each $\omega \in \Omega$, and for every $n \geq 1$, we set

$$F(\omega) = \{x \in C : x \in T(\omega, x)\},$$

and

$$F_n(\omega) = \{x \in C : d(x, T(\omega, x)) \leq \frac{1}{n}\operatorname{diam} C.\}$$

It follows from Theorem 3.1 that $F(\omega)$ is nonempty. Clearly $F(\omega) \subseteq F_n(\omega)$, and $F_n(\omega)$ is closed and convex. Furthermore, by [8, Proposition 3], each F_n is measurable. Then, by Lemma 2.1, each F_n admits a measurable selector $x_n(\omega)$ and

$$d(x_n(\omega), T(\omega, x_n(\omega))) \leq \frac{1}{n}\operatorname{diam} C \to 0 \text{ as } n \to \infty.$$

Define a function $f_1 : \Omega \times C \to \mathbb{R}^+$ by

$$f_1(\omega, x) = \limsup_n \|x_n(\omega) - x\|, \ \forall \omega \in \Omega.$$

By Lemma 2.3, it is easily seen that for each $x \in C$, $f_1(\cdot, x) : \Omega \to \mathbb{R}^+$ is measurable and for each $\omega \in \Omega$, $f_1(\omega, \cdot) : C \to \mathbb{R}^+$ is continuous and convex (and hence weakly lower semicontinuous (w-l.s.c.)). Note that the condition $\varepsilon_\beta(X) < 1$ implies reflexivity (see [2]) and so C is weakly compact. Hence, by Lemma 2.5 the marginal functions

$$r_1(\omega) := \inf_{x \in C} f_1(\omega, x),$$

and

$$R_1(\omega) := \{x \in C : f_1(\omega, x) = r_1(\omega)\}$$

are measurable. By Goebel [7], for any $\omega \in \Omega$ we may assume that the sequence $\{x_n(\omega)\}$ is regular with respect to C. Observe that $R_1(\omega) = A(C, \{x_n(\omega)\})$ and $r_1(\omega) = r(C, \{x_n(\omega)\})$, thus we can apply Lemma 2.6 to obtain

$$r_C(R_1(\omega)) \leq \lambda r_1(\omega), \tag{3.6}$$

where $\lambda := 1 - \Delta_{X,\beta}(1^-) < 1$, since $\varepsilon_\beta(X) < 1$. It is clear that $R_1(\omega)$ is a weakly compact and convex subset of C. By Lemma 2.1 we can take $x_1(\omega)$ as a measurable selector of $R_1(\omega)$. For each $\omega \in \Omega$ and $n \geq 1$, we define the contraction $T_n^1(\omega, \cdot) : R_1(\omega) \to KC(C)$ by

$$T_n^1(\omega, x) = \frac{1}{n} x_1(\omega) + (1 - \frac{1}{n}) T(\omega, x), \ \ \forall x \in R_1(\omega).$$

Since $R_1(\omega)$ is convex, each T_n satisfies the same boundary condition as T does, that is, we have

$$T_n^1(\omega, x) \cap \overline{I}_{R_1}(\omega)(x) \neq \emptyset, \ \ \forall x \in R_1(\omega).$$

Hence by Lemma 2.7, $T_n^1(\omega, \cdot)$ has a fixed point $z_n(\omega) \in R_1(\omega)$, *i.e.* $F(\omega) \cap R_1(\omega) \neq \emptyset$. Also it is easily seen that

$$\mathrm{dist}(z_n(\omega), T(\omega, z_n(\omega))) \leq \frac{1}{n} \mathrm{diamC} \to 0 \text{ as n} \to \infty.$$

Thus $F_n^1(\omega) = \{x \in R_1(\omega) : d(x, T(\omega, x)) \leq \frac{1}{n}\mathrm{diam}C\} \neq \emptyset$ for each $n \geq 1$, is closed and, by Lemma 2.4, measurable. Hence, by Lemma 2.1, we can choose x_n^1 a measurable selector of F_n^1, and from its definition we have $x_n^1(\omega) \in R_1(\omega)$ and $d(x_n^1(\omega), T(\omega, x_n^1(\omega))) \to 0$ as $n \to \infty$. Consider the function $f_2 : \Omega \times C \to \mathbb{R}^+$ defined by

$$f_2(\omega, x) = \limsup_n \|x_n^1(\omega) - x\|, \ \ \forall \omega \in \Omega.$$

As above, f_2 is a measurable function and weakly lower semicontinuous function. Thus the marginal functions

$$r_2(\omega) := \inf_{x \in R_1(\omega)} f_2(\omega, x)$$

and

$$R_2(\omega) := \{x \in R_1(\omega) : f_2(\omega, x) = r_2(\omega)\}$$

are measurable. Since $R_2(\omega) = A(R_1(\omega), \{x_n^1(\omega)\})$, it follows that $R_2(\omega)$ is weakly compact and convex. Also $r_2(\omega) = r(R_1(\omega), \{x_n^1(\omega)\})$. Again reasoning as above, for any $\omega \in \Omega$, we can assume that the sequence $\{x_n^1(\omega)\}$ is regular with respect to $R_1(\omega)$. Again, applying Lemma 2.6, we obtain

$$r_C(R_2(\omega)) \leq \lambda r_2(\omega). \tag{3.7}$$

Furthermore, $\{x_n^1(\omega)\} \subset R_1(\omega)$. Hence

$$r_2(\omega) \leq r_C(R_1(\omega)), \tag{3.8}$$

and thus

$$r_C(R_2(\omega)) \leq \lambda^2 r_1(\omega). \tag{3.9}$$

By induction, for each $m \geq 1$, we construct $R_m(\omega), r_m(\omega)$ and $\{x_n^m(\omega)\}_n$ where $x_n^m(\omega) \in R_m(\omega)$ such that $d(x_n^m(\omega), T(\omega, x_n^m(\omega))) \to 0$ as $n \to \infty$ and

$$r_C(R_m(\omega)) \leq \lambda r_m(\omega) \leq \lambda^m r_1(\omega). \tag{3.10}$$

Since $\operatorname{diam} R_m(\omega) \leq 2r_C(R_m(\omega))$ and $\lambda < 1$, it follows that $\lim_{m\to\infty} \operatorname{diam} R_m(\omega) = 0$. Since $\{R_m(\omega)\}$ is a descending sequence of weakly compact subsets of C for each $\omega \in \Omega$, we have $\cap_m R_m(\omega) = \{z(\omega)\}$ for some $z(\omega) \in C$. Furthermore, we see that

$$H(R_m(\omega), \{z(\omega)\}) \leq \operatorname{diam} R_m(\omega) \to 0 \quad \text{as } n \to +\infty.$$

Therefore, by Lemma 2.2, $z(\omega)$ is measurable. Finally, we will show that $z(\omega)$ is a fixed point of T. Indeed, for each $m \geq 1$, we have

$$\begin{aligned} d(z(\omega), T(\omega, z(\omega)) &\leq \|z(\omega) - x_n^m(\omega)\| + d(x_n^m(\omega), T(\omega, x_n^m(\omega))) \\ &+ H(T(\omega, x_n^m(\omega)), T(\omega, z(\omega))) \\ &\leq 2\|z(\omega) - x_n^m(\omega)\| + d(x_n^m(\omega), T(\omega, x_n^m(\omega))) \\ &\leq 2\operatorname{diam} R_m(\omega) + d(x_n^m(\omega), T(\omega, x_n^m(\omega))). \end{aligned}$$

Taking the upper limit as $n \to \infty$,

$$d(z(\omega), T(\omega, z(\omega))) \leq 2\operatorname{diam} R_m(\omega).$$

Finally, taking limit in m in both sides we obtain $z(\omega) \in T(\omega, z(\omega))$. □

Corollary 3.4. *Let C be a nonempty closed bounded convex separable subset of a Banach space X such that $\varepsilon_\beta(X) < 1$, and $T : \Omega \times C \to C$ a random nonexpansive operator. Then T has a random fixed point.*

Corollary 3.5. (Lorenzo Ramírez, Theorem 6 in [10]) *Let C be a nonempty closed bounded convex separable subset of a Banach space X such that $\varepsilon_\alpha(X) < 1$, and $T : \Omega \times C \to C$ a random nonexpansive operator. Then T has a random fixed point.*

Proof By (2.2) we see that $\varepsilon_\alpha(X) < 1$ implies $\varepsilon_\beta(X) < 1$.

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Random fixed point theorems for multivalued nonexpansive non-self random operators

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RANDOM FIXED POINT THEOREMS FOR MULTIVALUED NONEXPANSIVE NON-SELF RANDOM OPERATORS

S. PLUBTIENG AND P. KUMAM

Abstract. Let (Ω, Σ) be a measurable space, with $\sum$ a sigma-algebra of subset of Ω, and let C be a nonempty bounded closed convex separable subset of a Banach space X, whose characteristic of noncompact convexity is less than 1, KC(X) the family of all compact convex subsets of X. We prove that a multivalued nonexpansive non-self random operator $T : \Omega \times C \to KC(X)$, 1-$\chi$-contractive mapping, satisfying a inwardness condition has a random fixed point.

1. Introduction

In recent years there have appeared various random fixed point theorems for single-valued and set-valued random operator; see for example, Itoh [7], Ramírez [11], Tan and Yuan [12], Xu [14], and [15] Yuan and Yu [17] and references therein.

In 2002, P. L. Ramírez [11] proved the existence of random fixed point theorems for a random nonexpansive operator in the framework of a Banach spaces with a characteristic of noncompact convexity $\varepsilon_\alpha(X)$ is less than 1. On the other hand, Domínguez Benavides and Ramírez [3] proved a fixed point theorem for a set-valued nonexpansive self-mapping and 1-χ-contractive mapping in the framework of a Banach spaces whose characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ is less than 1. In 2004, Domínguez Benavides and Ramírez [4] proved a fixed point theorem for a multivalued nonexpansive non-self mapping and 1-χ-contractive mapping in the framework of a Banach spaces whose characteristic of noncompact convexity associated to the Kuratowski measure of noncompactness $\varepsilon_\alpha(X)$ is less than 1.

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The purpose of the present paper is to prove a random fixed point theorem for multivalued nonexpansive non-self random operators which is 1-χ-contractive mapping, in the framework of a Banach spaces with characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ less than 1. and satisfying a inwardness condition. Our result can also be seen as an extension of Theorem 3.4 in [4]

2. Preliminaries and notations

We begin with establishing some preliminaries. By (Ω, Σ) we denote a measurable space with Σ a sigma-algebra of subset of Ω. Let (X, d) be a metric space. We denote by $CL(X)$(resp $CB(X), KC(X)$) the family of all nonempty closed (resp. closed bounded, compact convex) subset of X, and by H the Hausdorff metric on $CB(X)$ induced by d, i.e.,

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

for $A, B \in CB(X)$, where $d(x, E) = \inf\{d(x, y)|y \in E\}$ is the distance from x to $E \subset X$.

Let C be a nonempty closed subset of a Banach space X. Recall now that a Multivalued mapping $T : C \to 2^X$ is said to be upper semicontinuous on C if $\{x \in C : Tx \subset V\}$ is open in C whenever $V \subset X$ is open; T is said to be lower semicontinuous if $T^{-1}(V) := \{x \in C : Tx \cap V \neq \emptyset\}$is open in C whenever $V \subset X$ is open; and T is said to be continuous if it is both upper and lower semicontinuous (cf.[1] and [2] for details). There is another different kind of continuity for multivalued operator: $T : C \to CB(X)$ is said to be continuous on C (with respect to the Hausdorff metric H) if $H(Tx_n, Tx) \to 0$ whenever $x_n \to x$. It is not hard to see (see Deimling [2])that both definitions of continuity are equivalent if Tx is compact for every $x \in C$.

If C is a closed convex subset of a Banach spaces X, then a multivalued mapping $T : C \to CB(X)$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq k\|x - y\|, \quad x, y \in C,$$

and T is said to be *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in C,$$

A multivalued operator $T : \Omega \to 2^X$ is called $(\Sigma)-$ measurable if, for any open subset B of X,

$$T^{-1}(B) = \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\}$$

belongs to Σ. A mapping $x : \Omega \to X$ is said to be a *measurable selector* of a measurable Multivalued operator $T : \Omega \to 2^X$ if $x(\cdot)$ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. An operator $T : \Omega \times C \to 2^X$ is called a random operator if, for each fixed $x \in C$, the operator $T(\cdot, x) : \Omega \to 2^X$ is measurable. We will denote by $F(\omega)$ the fixed point set of $T(\omega, \cdot)$, i.e.,

$$F(\omega) := \{x \in C : x \in T(\omega, x)\}.$$

Note that if we do not assume the existence of fixed point for the deterministic mapping $T(\omega, \cdot) : C \to 2^X, F(\omega)$ may be empty. A measurable operator $x : \Omega \to C$ is said to be a *random fixed point of a operator* $T : \Omega \times C \to 2^X$ if $x(\omega) \in T(\omega, x(\omega))$ for all $\omega \in \Omega$. Recall that $T : \Omega \times C \to 2^X$ is continuous if, for each fixed $\omega \in \Omega$ the operator $T : (\omega, \cdot) \to 2^X$ is continuous.

A random operator $T : \Omega \times C \to 2^X$ is said to be *nonexpansive* if, for each fixed $\omega \in \Omega$ the map $T : (\omega, \cdot) \to C$ is nonexpansive.

For later convenience, we list the following results related to the concept of measurability.

Lemma 2.1. (Wagner cf.[13]). *Let (X, d) be a complete separable metric space and $F : \Omega \to CL(X)$ a measurable map. Then F has a measurable selector.*

Lemma 2.2. (Itoh 1977, cf.[7]). *Suppose $\{T_n\}$ is a sequence of measurable Multivalued operator from Ω to $CB(X)$ and $T : \Omega \to CB(X)$ is an operator. If, for each $\omega \in \Omega$, $H(T_n(\omega), T(\omega)) \to 0$, then T is measurable.*

Lemma 2.3. (Tan and Yuan cf.[12]). *Let X be a separable metric space and Y a metric space. If $f : \Omega \times X \to Y$ is a measurable in $\omega \in \Omega$ and continuous in $x \in X$, and if $x : \Omega \to X$ is measurable, then $f(\cdot, x(\cdot)) : \Omega \to Y$ is measurable.*

As an easy application of Proposition 3 of Itoh[7] we have the following result.

Lemma 2.4. *Let C be a closed separable subset of a Banach space $X, T : \Omega \times C \to C$ a random continuous operator and $F : \Omega \to 2^C$ a measurable closed-valued operator. Then for any $s > 0$, the operator $G : \Omega \to 2^C$ given by*

$$G(\omega) = \{x \in F(\omega) : \|x - T(\omega, x)\| < s\}, \quad \omega \in \Omega$$

is measureble and so is the operator $cl\{G(\omega)\}$ of the closure of $G(\omega)$.

Lemma 2.5. (Domínguez Benavidel and Lopez Acedo cf.[5]). *Suppose C is a weakly closed nonempty separable subset of a Banach space $X, F : \Omega \to 2^X$ a measurable with weakly compact values, $f : \Omega \times C \to \mathbb{R}$ is a measurable, continuous and weakly lower semicontinuous function. Then the marginal function $r : \Omega \to \mathbb{R}$ defined by*

$$r(\omega) := \inf_{x \in F(\omega)} f(\omega, x)$$

and the marginal map $R : \Omega \to X$ *defined by*

$$R(\omega) := \{x \in F(\omega) : f(\omega, x) = r(\omega)\}$$

are measurable.

Recall that the Kuratowski and Hausdorff measures of noncompactness of a nonempty bounded subset B of X are respectively defined as the number

$$\alpha(B) = \inf\{r > 0 : B \text{ can be covered by finitely many sets of diameter} \le r\},$$

$$\chi(B) = \inf\{r > 0 : B \text{ can be covered by finitely many ball of radius} \le r\}.$$

The separation measure of noncompacness of a nonempty bounded subset B of X defined by

$$\beta(B) = \sup\{\varepsilon : \text{there exists a sequence } \{x_n\} \text{ in } B \text{ such that } sep(\{x_n\}) \ge \varepsilon\}.$$

Then a multivalued mapping $T : C \to 2^X$ is called *γ-condensing (resp., 1-γ-contractive)* where $\gamma = \alpha(\cdot)$ or $\chi(\cdot)$ if, for each bounded subset B of C with $\gamma(B) > 0$, there holds the inequality

$$\gamma(T(B)) < \gamma(B) \quad (\text{resp.}\gamma(T(B)) \le \gamma(B)).$$

Here $T(B) = \bigcup_{x\in B} Tx$. The random operator $T : \Omega \times C \to 2^X$ is said to be *1-γ-contractive* if, for each $\omega \in \Omega$ the map $T : (\omega, \cdot) \to 2^X$ is 1-γ-contractive.

Definition 2.6. *Let X be a Banach space and $\phi = \alpha, \beta$ or χ. The modulus of noncompact convexity associated to ϕ is defined in the following way:*

$$\Delta_{X,\phi}(\varepsilon) = inf\{1 - d(0, A) : A \subset B_X \text{ is convex}, \phi(A) \ge \varepsilon\},$$

where B_X is the unit ball of X.

The characteristic of noncompact convexity of X associated with the measure of noncompactness ϕ is defined by

$$\varepsilon_\phi(X) = \sup\{\varepsilon \ge 0 : \Delta_{X,\phi}(\varepsilon) = 0\}.$$

The following relationshops among the different moduli are easy to obtain

$$\Delta_{X,\alpha}(\varepsilon) \le \Delta_{X,\beta}(\varepsilon) \le \Delta_{X,\chi}(\varepsilon), \tag{2.1}$$

and consequently

$$\varepsilon_\alpha(X) \ge \varepsilon_\beta(X) \ge \varepsilon_\chi(X). \tag{2.2}$$

When X is a reflexive Banach space we have some alternative expressions for the moduli of noncompact convexity associated β and χ.

$$\Delta_{X,\beta}(\varepsilon) = \inf\{1 - \|x\| : \{x_n\} \subset B_X, x = w - \lim x_n, sep(\{x_n\}) \ge \varepsilon\},$$

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$$\Delta_{X,\chi}(\varepsilon) = \inf \{1 - \|x\| : \{x_n\} \subset B_X, x = w - \lim x_n, \chi(\{x_n\}) \geq \varepsilon\}.$$

In order to study the fixed point theory for non-self mappings we must introduce some terminology for boundary condition. The inward set of C at $x \in C$ defined by

$$I_C(x) := \{x + \lambda(y - x) : \lambda \geq 0, y \in C\}.$$

Clearly $C \subset I_C(x)$ and it is not hard to show that $I_C(x)$ is a convex set as C does. A multivalued mapping $T : C \to 2^X\{\emptyset\}$ is said to be *inward* on C if

$$Tx \subset I_C(x) \ \forall x \in C.$$

Let $\bar{I}_C(x) := x + \{\lambda(z - x) : z \in C, \lambda \geq 1\}$. Note that for a convex C, we have $\bar{I}_C(x) = \overline{I_C(x)}$, and T is said to be *weakly inward* on C if

$$Tx \subset \bar{I}_C(x) \ \forall x \in C.$$

Let C be a nonempty bounded closed subset of Banach spaces X and $\{x_n\}$ bounded sequence in X, we use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C, respectively, i.e.

$$\begin{aligned} r(C, \{x_n\}) &= \inf \left\{ \limsup_n \|x_n - x\| : x \in C \right\}, \\ A(C, \{x_n\}) &= \left\{ x \in C : \limsup_n \|x_n - x\| = r(C, \{x_n\}) \right\}. \end{aligned}$$

If D is a bounded subset of X, the *Chebyshev radius* of D relative to C is defined by

$$r_C(D) := \inf \{\sup\{\|x - y\| : y \in D\} : x \in C\}.$$

Obviously, the convexity of C implies that $A(C, \{x_n\})$ is convex. Notice that $A(C, \{x_n\})$ is a nonempty weakly compact set if C is weakly compact, or C is a closed convex subset of a reflexive Banach spaces X.

Let $\{x_n\}$ and C be a nonempty bounded closed subset of Banach spaces X. Then $\{x_n\}$ is called *regular* with respect to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$; while $\{x_n\}$ is called *asymptotically uniform* with respect to C if $A(C, \{x_n\}) = A(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Lemma 2.7. (Goebel[6] and Lim[10]). *Let $\{x_n\}$ and C be as above. Then we have*

(i) *There always exists a subsequence of* $\{x_n\}$ *which is regular with respect to* C;
(ii) *if* C *is separable, then* $\{x_n\}$ *contains a subsequence which is asymptotically uniform with respect to* C.

Moreover, we also need the following Lemma.

Lemma 2.8. (Domínguez Benavides and Ramírez. Theorem 3.4 cf. [3]). *Let* C *be a closed convex subset of a reflexive Banach spaces* X, *and let* x_n *be a bounded sequence in* C *which is regular with respect to* C. *Then*

$$r_C(A(C,x_n)) \leq (1-\Delta_{X,\beta}(1^-))r(C,\{x_n\}). \tag{2.3}$$

Moreover, if X *satisfies the nonstrict Opial condition then*

$$r_C(A(C,x_n)) \leq (1-\Delta_{X,\chi}(1^-))r(C,\{x_n\}). \tag{2.4}$$

Lemma 2.9. (Domínguez Benavides and Ramírez. Theorem 3.2 cf. [4]). *Let* C *be a closed convex subset of a reflexive Banach space* X, *and let* $\{x_\beta : \beta \in D\}$ *be a bounded ultranet. Then*

$$r_C(A(C,x_\beta)) \leq (1-\Delta_{X,\alpha}(1^-))r(C,\{x_\beta\}). \tag{2.5}$$

The following result are now basic in the fixed point theorem for multivalued mappings.

Lemma 2.10. (Deimling 1992, cf. [2]). *Let* X *be a Banach space and* $\emptyset \neq D \subset X$ *be closed bounded convex. Let* $F : D \to 2^X$ *be upper semicontinuous* $\gamma-$*condensing with closed convex values, where* $\gamma(\cdot) = \alpha(\cdot)$ *or* $\chi(\cdot)$. *If* $Fx \cap \overline{I_D(x)} \neq \emptyset$ *for all* $x \in C$, *then* F *has a fixed point. (Here* $I_D(x)$ *is called the inward set at* x *defined by* $I_D(x) := \{x + \lambda(y-x) : \lambda \geq 0, y \in D\}$*)*

3. The Result

In order to prove our first result, we need the following Lemma which is proved along the proof of Kirk-Massa theorem as it appear in [16].

Lemma 3.1. *Let* C *be a nonempty closed bounded convex separable subset of a Banach space* X. $T : C \to KC(X)$ *is a nonexpansive such that* $T(C)$ *is a bounded set and which satisfies* $Tx \subset I_C(x)$, $\forall x \in C$, $\{x_n\}$ *is a sequence in* C *such that* $\lim_n d(x_n, Tx_n) = 0$. *Then there exist a subsequence* $\{z_n\}$ *of* $\{x_n\}$ *such that* $Tx \cap I_A(x) \neq \emptyset, \forall x \in A := A(C,\{z_n\})$.

Lemma 3.1 is the part (more or less) of the proof of theorem 3.4 of [4].

The next result state the main result of this work.

Theorem 3.2. *Let C be a nonempty closed bounded convex separable subset of a Banach spaces X such that $\epsilon_\beta(X) < 1$, and $T : \Omega \times C \to KC(X)$ be a multivalued nonexpansive random operator and 1-χ-contractive mapping, such that for each $\omega \in \Omega$, $T(\omega, C)$ is a bounded set, which satisfies the inwardness condition, i.e., for each $\omega \in \Omega$, $T(\omega, x) \subset I_C(x)$, $\forall x \in C$.*
Then T has a random fixed point.

Proof. Fix $x_0 \in C$, and consider the measurable function $x_0(\omega) \equiv x_0$. For each $n \geq 1$, define $T_n(\omega, \cdot) : C \to KC(X)$ by

$$T_n(\omega, x) = \frac{1}{n}x_0(\omega) + (\frac{n-1}{n})T(\omega, x), \ \ \forall x \in C.$$

Then $T_n(\omega, \cdot)$ is a multivalued contraction and $T_n(\omega, x) \subset I_C(x)$, $\forall x \in C$. Hence each T_n has a fixed point $z_n(\omega) \in C$. It is easily seen that $d(z_n(\omega), T(\omega, z_n(\omega))) \leq \frac{1}{n} diamC \to 0$ as $n \to \infty$. Thus the set

$$F_n(\omega) = \{x \in C : d(x, T(\omega, x)) \leq \frac{1}{n} diamC\}$$

is nonempty closed and convex. Furthermore, by Lemma 2.4, each F_n is measurable. Then, by Lemma 2.1, each F_n admits a measurable selector $x_n(\omega)$ such that

$$d(x_n(\omega), T(\omega, x_n(\omega))) \leq \frac{1}{n} diamC \to 0 \text{ as } n \to \infty.$$

Defin a function $f : \Omega \times C \to \mathbb{R}^+ := [0, \infty)$ by

$$f(\omega, x) = \limsup_n \|x_n(\omega) - x\|, \ \ x \in C.$$

By Lemma 2.3, it is easily seen that $f(\cdot, x)$ is measurable and $f(\omega, \cdot)$ is continuous and convex, therefore it is a weakly lower semicontinuous function. Note that, condition $\varepsilon_\beta(X) < 1$ implies reflexivity (see [1]) and so C is a weakly compact. Hence, by Lemma 2.5, the marginal functions

$$r(\omega) := \inf_{x \in C} f(\omega, x)$$

and

$$A(\omega) := \{x \in C : f(\omega, x) = r(\omega)\}$$

are measurable. It is clearly that $A(\omega)$ is a weakly compact convex subset of C. For any $\omega \in \Omega$, we may assume that the sequence $\{x_n(\omega)\}$ is regular with respect C. Note that $A(\omega) = A(C, \{x_n(\omega)\})$, and $r(\omega) = r(C, \{x_n(\omega)\})$. We can apply inequality (2.3) in Lemma 2.8 to obtain

$$r_C(A(\omega)) \leq \lambda r(C, \{x_n(\omega)\}), \tag{3.1}$$

where $\lambda = 1 - \Delta_{X,\beta}(1^-) < 1$, since $\varepsilon_\beta(X) < 1$.

For each $\omega \in \Omega$ and $n \geq 1$, we define the multivalued contraction $T_n^1(\omega, \cdot) : A(\omega) \to KC(X)$ by

$$T_n^1(\omega, x) = \frac{1}{n}x_1(\omega) + (\frac{n-1}{n})T(\omega, x),$$

for each $x \in C$. By Lemma 3.1 we note that $T(\omega, x) \cap I_{A(\omega)}(x) \neq \emptyset, \forall x \in A(\omega)$. Since $I_{A(\omega)}(x)$ is convex, it follow that $T_n^1(\omega, \cdot)$ satisfies the boundary condition i.e.,

$$T_n^1(\omega, x) \cap I_{A(\omega)}(x) \neq \emptyset, \forall x \in A(\omega). \tag{3.2}$$

Since $T_n^1(\omega, \cdot)$ is 1-χ-contractive mapping, it follows by [3, pp.382] that $T_n^1(\omega, \cdot)$ is χ-condensing. Hence, by Lemma 2.10, $T_n^1(\omega, \cdot)$ has a fixed point $z_n^1(\omega) \in A(\omega), i.e.F(\omega) \cap A(\omega) \neq \emptyset$. Also it is easily seen that

$$dist(z_n^1(\omega), T(\omega, z_n^1(\omega))) \leq \frac{1}{n}diamC \to 0 \text{ as } n \to \infty.$$

Thus $F_n^1(\omega) := \{x \in A(\omega) : d(x, T(\omega, x)) \leq \frac{1}{n}diamC\}$ is nonempty closed and convex for each $n \geq 1$. By Lemma 2.4, each F_n^1 are measurable. Hence, by Lemma 2.1, we can choose x_n^1 a measurable selector of F_n^1. Thus we have $x_n^1(\omega) \in A(\omega)$ and $d(x_n^1(\omega), T(\omega, x_n^1(\omega))) \to 0$ as $n \to \infty$. Consider the function $f_2 : \Omega \times C \to \mathbb{R}^+$ defined by

$$f_2(\omega, x) = \limsup_n \|x_n^1(\omega) - x\|, \ \forall \omega \in \Omega.$$

As above, f_2 is a measurable function and weakly lower semicontunuous function. Then the marginal function

$$r_2(\omega) := \inf_{x \in A(\omega)} f_2(\omega, x)$$

and

$$A^1(\omega) := \{x \in A(\omega) : f_2(\omega, x) = r_2(\omega)\}$$

are measurable. Since $A^1(\omega) = A(A(\omega), \{x_n^1(\omega)\})$, it follows that $A^1(\omega)$ is a weakly compact and convex. Moreover, we also note that $r_2(\omega) = r(A(\omega), \{x_n^1(\omega)\})$. Again reasoning as above, for any $\omega \in \Omega$, we can assume that the sequence $\{x_n^1(\omega)\}$ is regular with respect to $A^1(\omega)$. Moreover, we proceed as above using Lemma 3.1 and Lemma 2.8 to obtain that

$$T(\omega, x(\omega)) \cap I_{A^1}(x(\omega)) \neq \emptyset \ \forall x(\omega) \in A^1 = A(A(\omega), \{x_n^1(\omega)\}),$$

and

$$r_C(A^1) \leq \lambda r(A(\omega), \{x_n^1(\omega)\}) \leq \lambda r_C(A(\omega)). \tag{3.3}$$

By induction, for each $m \geq 1$, we take a sequence $\{x_n^m(\omega)\}_n \subseteq A^{m-1}$ such that $r_C(A^m) \leq \lambda^m r_C(A(\omega))$ and $\lim_n d(x_n^m(\omega), T(\omega, x_n^m(\omega))) = 0$ for each fixed $\omega \in \Omega$, where $A^m := A(C, \{x_n^m(\omega)\})$. Since $diamR_m(\omega) \leq 2r_C(R_m(\omega))$ and $\lambda < 1$, it follows that $\lim_{m\to\infty} diamR_m(\omega) = 0$. Note that $\{R_m(\omega)\}$ is a descending sequence of weakly compact subset of C for each $\omega \in \Omega$. Thus we have $\cap_m R_m(\omega) = \{z(\omega)\}$ for some $z(\omega) \in C$. Furthermore, we see that

$$H(R_m(\omega), \{z(\omega)\}) \leq diamR_m(\omega) \to 0 \quad \text{as } n \to +\infty.$$

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ภาคผนวก 7

Strong convergence theorems of viscosity averaging iterations for asymptotically nonexpansive nonself-mappings

S. Plubtieng and R. Wangkeeree

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ภาคผนวก 7/1

STRONG CONVERGENCE THEOREMS OF VISCOSITY AVERAGING ITERATIONS FOR ASYMPTOTICALLY NONEXPANSIVE NONSELF-MAPPINGS †

SOMYOT PLUBTIENG AND RABIAN WANGKEEREE

ABSTRACT. Let C be a nonempty closed convex subset of a real Hilbert space H, P be the metric projection of H onto C, T be an asymptotically nonexpansive nonself-mapping from C into H with a sequence $\{k_n\} \subset [1,\infty)$ and $f : C \longrightarrow C$ be a contraction mapping with coefficient $\alpha \in (0,1)$. It proved that, for each $n \geq 1$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ which defined by

$$x_n = a_n f(x_n) + (1 - a_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n, \ \forall n \geq 1$$

and

$$y_n = \frac{1}{n}\sum_{j=1}^{n} P(a_n f(y_n) + (1 - \alpha_n)(TP)^j y_n), \ \forall n \geq 1,$$

where

$$b_n = \frac{1}{n}\sum_{j=1}^{n}(1 + |1 - k_j| + e^{-j}), a_n = \frac{b_n - 1}{b_n - \beta}, \forall n \geq 1,$$

and $0 < \alpha < \beta < 1$. Then two sequences $\{x_n\}$ and $\{y_n\}$ converges strongly to a fixed point of T.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Hilbert space H and let T be a mapping of C into itself. Then T is said to be nonexpansive provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is said to be asymptotically nonexpansive mapping if there exits a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\longrightarrow\infty} k_n = 1$

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such that for each $x, y \in C$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \forall n = 1, 2, 3, \ldots \tag{1.1}$$

Recall that a self-mapping $f : C \longrightarrow C$ is a contraction on C if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\| \ \forall x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T; i.e. $F(T) = \{x \in C : Tx = x\}$. It is well know that if T is nonexpansive, then $F(T)$ is convex see [6]. In 1967, Browder[3] proved the following strong convergence theorem for nonexpansive mapping: let T be a nonexpansive mapping of a bounded closed convex subset C of H into itself. Let $u \in C$ and for each $t \in (0,1)$, let $G_t x = tu + (1-t)Tx$. Then, G_t has a unique fixed point x_t in C, and $\{x_t\}$ converges strongly to a fixed point u_0 of T as $t \longrightarrow 0$. The fixed point u_0 is uniquely specified as the fixed point of T in C closest to u. In 1975, Baillon [1], prove the first nonlinear egodic theorem as follows: let C be a bounded closed convex subset of H and let T be a nonexpansive mapping of C into itself. Then for each $x \in C$

$$A_n x = \frac{1}{n} \sum_{k=1}^{n} T^k x$$

converges weakly to fixed point of T. In 1979, Hirano and Takahashi[5] extended Baillon's theorem to asymptotically nonexpansive mappings. By Using an idea of Browder[3], Shimizu and Takahashi[13] proved the following theorem for an asymptotically nonexpansive mapping in the framework of a Hilbert space:

Theorem 1.1. ([13]). *Let C be a closed convex subset of a real Hilbert space H, let T be an asymptotically nonexpansive mapping of C into itself with Lipschitz constants k_n and suppose that $F(T)$ is nonempty. Let*

$$b_n = \frac{1}{n} \sum_{j=1}^{n} (1 + |1 - k_j| + e^{-j}), a_n = \frac{b_n - 1}{b_n - 1 + a},$$

where $0 < a < 1$. Let $x_0 \in C$. Then, a mapping T_n on C given by

$$T_n x = a_n x_0 + (1 - a_n) A_n x, \ \textit{for all } x \in C$$

has a unique fixed point u_n in C, when $A_n = \frac{1}{n} \sum_{j=1}^{n} T^j$. Further $\{u_n\}$ convergence strongly to the element of $F(T)$ which is nearest to x_0.

On the other hand, Xu[16] extended Browder's result to studied two sequences $\{x_t\}$ and $\{x_n\}$ given by

$$x_t = tf(x_t) + (1-t)Tx_t$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, n = 1, 2, ...,$$

where $t \in (0,1), \{\alpha_n\} \subset (0,1)$ and f is a contraction mapping from C into itself. Xu[16] also proved the strong convergence of the sequences as $t \longrightarrow 1$ and $\alpha_n \longrightarrow 1$ to the unique solution z in $F(T)$ to the variational inequality $\langle (I-f)z = x - z\rangle \geq 0, x \in F(T)$ or equivalently to $z = P(f(z))$ where P is the metric projection from H onto $F(T)$.

In this paper, we first show that, for an asymptotically nonexpansive nonself-mapping T with a sequence $\{k_n\} \subset [1, \infty)$, there exists two sequences $\{x_n\}$ and $\{y_n\}$ which defined by

$$x_n = a_n f(x_n) + (1-a_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n, \ \forall n \geq 1 \tag{1.2}$$

and

$$y_n = \frac{1}{n}\sum_{j=1}^{n} P(a_n f(y_n) + (1-a_n)(TP)^j y_n), \ \forall n \geq 1 \tag{1.3}$$

where

$$b_n = \frac{1}{n}\sum_{j=1}^{n}(1 + |1-k_j| + e^{-j}), a_n = \frac{b_n - 1}{b_n - \beta}, \forall n \geq 1,$$

$0 < \alpha < \beta < 1$, $f : C \longrightarrow C$ is a contraction mapping with coefficient $\alpha \in (0,1)$ and P is the metric projection from H onto C. Finally we show that $\{x_n\}$ and $\{y_n\}$ converges strongly to a fixed point of T. Then the results presented in this paper generalized and extend the corresponding main results of Shimizu and Takahashi [13].

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. Recall the metric (nearest point) projection P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: Given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

$P_C x$ is characterized as follows.

Lemma 2.1. *Let H be a real Hilbert space, C a closed convex subset of H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \forall z \in C.$$

Definition 2.2. A mapping $T : C \longrightarrow H$ is said to satisfy nowhere normal outward condition ((NNO) for short) if and only if for each $x \in C$, $Tx \in S_x^C$, where $S_x = \{y \in H : y \neq x, Py = x\}$ and P is the metric projection from H onto C.

Lemma 2.3. ([9, Proposition 1]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, P be the metric projection of H onto C and $T : C \longrightarrow H$ be a nonself-mapping. Suppose that T satisfies (NNO) condition. Then $F(PT) = F(T)$.*

Lemma 2.4. ([13, Lemma 4]). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \longrightarrow C$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C and there exists a subsequence $\{x_{n_j}\}$ which converges weakly to $x \in C$ and $\{x_{n_j} - \frac{1}{n_j}\sum_{i=1}^{n_j} T^i x_{n_j}\}$ converges strongly to 0. Then x is a fixed point of T.*

Definition 2.5. ([4, Definition 3.1]). Let X be a real normed linear space, C a nonempty subset of X. Let $P : X \longrightarrow C$ be the nonexpansive retraction of X onto C. A mapping $T : C \longrightarrow X$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}_{n\geq 1} \subset [1, \infty)$, $k_n \longrightarrow 1$ as $n \longrightarrow \infty$ such that for all $x, y \in C$, the following inequality holds:

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\|, \text{for all } n \geq 1. \tag{2.1}$$

Remark 2.6. If X is a Hilbert space then we can replace the mapping P by the metric projection P.

Remark 2.7. If T is a self-map, then $PT = T$ so that (2.1) coincide with (1.1). Moreover, we note that $TP |_C = T$. So if a contraction mapping $f : C \longrightarrow C$ defined by $f(x) = x_0 \in C, \forall x \in C$ and setting $\beta = 1 - a$ for some $0 < a < 1 - \alpha$ then, (1.2) and (1.3) reduce to the sequence $\{u_n\}$ in Theorem 1.1.

For a contraction mapping $f : C \longrightarrow C$ with coefficient $\alpha \in (0,1)$ and an asymptotically nonexpansive mapping T with a sequence $\{k_n\} \subset [1,\infty)$, we putting

$$b_n = \frac{1}{n}\sum_{j=1}^{n}(1+|1-k_j|+e^{-j}) \text{ and } a_n = \frac{b_n-1}{b_n-\beta} \text{ for } n = 1,2,3,...,$$

where $0 < \alpha < \beta < 1$. Then, we get the following facts:

(i) $b_n > 1, \frac{1}{n}\sum_{j=1}^{n} k_j < b_n, 0 < a_n < 1, \forall n \geq 1,$

(ii) $\lim_{n\longrightarrow\infty} b_n = 1, \lim_{n\longrightarrow\infty} a_n = 0,$

(iii) $a_n > \frac{b_n-1}{b_n-\alpha}$ or equivalently to $a_n(\alpha - b_n) + b_n < 1, \forall n \geq 1.$

Now, for each $n \geq 1$, we consider two mappings $S_n, U_n : C \longrightarrow C$ given by

$$S_n x = a_n f(x) + (1-a_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x, \text{ for all } x \in C \tag{2.2}$$

and

$$U_n y = \frac{1}{n}\sum_{j=1}^{n} P(a_n f(y) + (1-a_n)(TP)^j y), \text{ for all } y \in C. \tag{2.3}$$

Then, we have the following three lemmas.

Lemma 2.8. *For each $n \geq 1$, S_n has a unique fixed point x_n in C.*

Proof. Let $x, y \in C$. Then for each $n \geq 1$, we have

$$\begin{aligned}
\|S_n x - S_n y\| &= \|a_n(f(x)-f(y)) + (1-a_n)\frac{1}{n}\sum_{j=1}^{n}((PT)^j x - (PT)^j y)\| \\
&\leq a_n\alpha\|x-y\| + (1-a_n)\frac{1}{n}\sum_{j=1}^{n}\|(PT)^j x - (PT)^j y\| \\
&\leq a_n\alpha\|x-y\| + (1-a_n)\frac{1}{n}\sum_{j=1}^{n}\|T(PT)^{j-1}x - T(PT)^{j-1}y\| \\
&\leq a_n\alpha\|x-y\| + (1-a_n)\frac{1}{n}\sum_{j=1}^{n}k_j\|x-y\|
\end{aligned}$$

$$
\begin{aligned}
&\le a_n\alpha\|x-y\| + (1-a_n)b_n\|x-y\| \\
&= (a_n(\alpha-b_n)+b_n)\|x-y\|.
\end{aligned}
$$

Since $a_n(\alpha-b_n)+b_n < 1$, we get S_n is a contraction mapping on C. Therefore, by the Banach Contraction principle, S_n has a unique fixed point x_n in C. □

Lemma 2.9. *For each $n \ge 1$, U_n has a unique fixed point y_n in C.*

Proof. Let $x, y \in C$. Since P is a nonexpansive mapping such that $Px = x$ and $Py = y$, it follows as in the proof of Lemma 2.8 that

$$\|U_n x - U_n y\| \le (a_n(\alpha-b_n)+b_n)\|x-y\|.$$

Thus U_n is a contraction mapping and hence U_n has a unique fixed point y_n in C. □

Lemma 2.10. *If $F(T)$ is a nonempty, then $\{x_n\}$ and $\{y_n\}$ are bounded sequences.*

Proof. Let $q \in F(T)$. Then, we have

$$
\begin{aligned}
\|x_n - q\| &= \|a_n(f(x_n)-q) + (1-a_n)\frac{1}{n}\sum_{j=1}^{n}((PT)^j x_n - q)\| \\
&\le a_n\|f(x_n)-q\| + (1-a_n)\frac{1}{n}\sum_{j=1}^{n}\|(PT)^j x_n - q\| \\
&\le a_n\|f(x_n)-f(q)\| + a_n\|f(q)-q\| + (1-a_n)\frac{1}{n}\sum_{j=1}^{n}k_j\|x_n-q\| \\
&\le a_n\alpha\|x_n-q\| + a_n\|f(q)-q\| + (1-a_n)b_n\|x_n-q\| \\
&= (a_n(\alpha-b_n)+b_n)\|x_n-q\| + a_n\|f(q)-q\|.
\end{aligned}
$$

We note that

$$\frac{a_n}{1-[a_n(\alpha-b_n)+b_n]} = \frac{b_n-1}{-\beta-b_n\alpha+\alpha+b_n\beta} = \frac{1}{\beta-\alpha}.$$

It follows that $\|x_n-q\| \le \frac{a_n}{1-[a_n(\alpha-b_n)+b_n]}\|f(q)-q\| = \frac{1}{\beta-\alpha}\|f(q)-q\|$. Hence $\{x_n\}$ is a bounded sequence. Then as in the proof above, $\{y_n\}$ is also bounded. This completely the proof. □

3. Main results

In this section, we shall prove two strong convergence theorems for asymptotically nonexpansive nonself-mapping in a Hilbert space.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H, P the metric projection from H onto C, T be an asymptotically nonexpansive nonself-mapping from C into H with Lipschitz constant k_n, and suppose that $F(T)$ is nonempty. Let $f : C \longrightarrow C$ be a contraction mapping with coefficient $\alpha \in (0,1)$,*

$$b_n = \frac{1}{n}\sum_{j=1}^{n}(1 + |1 - k_j| + e^{-j}) \text{ and } a_n = \frac{b_n - 1}{b_n - \beta},$$

where $0 < \alpha < \beta < 1$. If T satisfies (NNO) condition then the sequence $\{x_n\}$ defined by (1.2) converges strongly to z where, z is the unique solution in $F(T)$ to the variation inequality

$$\langle (I - f)z, x - z\rangle \geq 0, \; x \in F(T) \tag{3.1}$$

or equivalently $z = G(f(z))$, where G is the metric projection from H onto $F(T)$.

Proof. By Lemma 2.10, we have $\{x_n\}$ is bounded so are $\{f(x_n)\}$ and $\{\frac{1}{n}\sum_{j=1}^{n}\|(TP)^j x_n\|\}$. Furthermore, we obtain

$$\begin{aligned}
\|x_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\| &= \|a_n f(x_n) + (1 - a_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\| \\
&= a_n\|f(x_n) - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\| \\
&\leq a_n\left[\|f(x_n)\| - \frac{1}{n}\sum_{j=1}^{n}\|(TP)^j x_n\|\right] \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

This implies that $\{x_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\}$ converges strongly to 0. We next show that

$$\limsup_{n\longrightarrow\infty}\langle z - x_n, z - f(z)\rangle \leq 0. \tag{3.2}$$

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{j\longrightarrow\infty}\langle z - x_{n_j}, z - f(z)\rangle = \limsup_{n\longrightarrow\infty}\langle z - x_n, z - f(z)\rangle,$$

and $x_{n_j} \rightharpoonup x \in C$. By Lemma 2.4 and Lemma 2.3, we get $x \in F(PT) = F(T)$. Hence, by (3.3) we obtain

$$\limsup_{n\longrightarrow\infty}\langle z - x_n, z - f(z)\rangle = \langle z - x, z - f(z)\rangle \leq 0$$

as required. Finally we shall show that $x_n \longrightarrow z$. For each $n \geq 1$, we note that

$$\begin{aligned}
\|x_n - z\|^2 &= \|x_n - z + a_n(z - f(z)) - a_n(z - f(z))\|^2 \\
&\leq \|x_n - z + a_n(z - f(z))\|^2 + 2a_n\langle x_n - z, f(z) - z\rangle \\
&= \|a_n(f(x_n) - f(z)) + (1 - a_n)\frac{1}{n}\sum_{j=1}^{n}((PT)^j x_n - z)\|^2 \\
&\quad +2a_n\langle x_n - z, f(z) - z\rangle \\
&\leq \left\{a_n\|f(x_n) - f(z)\| + (1 - a_n)\frac{1}{n}\sum_{j=1}^{n}\|((PT)^j x_n - z)\|\right\}^2 \\
&\quad +2a_n\langle x_n - z, f(z) - z\rangle \\
&\leq \{a_n\alpha\|x_n - z\| + (1 - a_n)b_n\|x_n - z\|\}^2 \\
&\quad +2a_n\langle x_n - z, f(z) - z\rangle \\
&\leq (a_n(\alpha - b_n) + b_n)\|x_n - z\|^2 + 2a_n\langle x_n - z, f(z) - z\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_n - z\|^2 &\leq \frac{2a_n}{1 - [a_n(\alpha - b_n) + b_n]}\langle x_n - z, f(z) - z\rangle \\
&= \frac{2}{\beta - \alpha}\langle x_n - z, f(z) - z\rangle.
\end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Then by the fact (3.2) there exists a natural number N such that

$$\langle x_n - z, f(z) - z\rangle \leq (\beta - \alpha)\frac{\epsilon}{2}, \forall n \geq N.$$

This implies that

$$\|x_n - z\|^2 \leq \epsilon, \forall n \geq N.$$

Hence the sequence $\{x_n\}$ converges strongly to a fixed point z of T. This completely the proof. □

Theorem 3.2. *Let C be a closed convex subset of a real Hilbert space H, P the metric projection from H onto C, T be an asymptotically nonexpansive nonself-mapping from C into H with Lipschitz constant k_n, and suppose that $F(T)$ is nonempty. Let $f : C \longrightarrow C$ be a contraction mapping with coefficient $\alpha \in (0,1)$,*

$$b_n = \frac{1}{n}\sum_{j=1}^{n}(1 + |1-k_j| + e^{-j}) \text{ and } a_n = \frac{b_n - 1}{b_n - \beta},$$

where $0 < \alpha < \beta < 1$. If T satisfies (NNO) condition then the sequence $\{y_n\}$ defined by (1.3) converges strongly to z where, z is the unique solution in $F(T)$ to the variation inequality

$$\langle (I-f)z, x-z\rangle \geq 0, \ x \in F(T) \tag{3.3}$$

or equivalently $z = G(f(z))$, where G is the metric projection from H onto $F(T)$.

Proof. By Lemma 2.10, we get $\{y_n\}$ is bounded so are $\{f(y_n)\}$ and $\{\frac{1}{n}\sum_{j=1}^{n}\|(TP)^j y_n\|\}$. Furthermore, we also have

$$\begin{aligned}
\|y_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j y_n\| &= \|\frac{1}{n}\sum_{j=1}^{n}P(a_n f(y_n) + (1-a_n)(TP)^j y_n) - \frac{1}{n}\sum_{j=1}^{n}(PT)^j y_n\| \\
&\leq \frac{1}{n}\sum_{j=1}^{n}\|a_n f(y_n) + (1-a_n)(TP)^j y_n - T(PT)^{j-1}y_n\| \\
&= \frac{1}{n}\sum_{j=1}^{n}\|a_n f(y_n) + (1-a_n)(TP)^j y_n - (TP)^j y_n\| \\
&= a_n\frac{1}{n}\sum_{j=1}^{n}\|f(y_n) - (TP)^j y_n\| \\
&\leq a_n\left[\|f(y_n)\| - \frac{1}{n}\sum_{j=1}^{n}\|(TP)^j y_n\|\right] \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

This implies that $\{y_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\}$ converges strongly to 0. Then as in the proof of Theorem 3.1, we obtain

$$\limsup_{n\longrightarrow\infty}\langle z - y_n, z - f(z)\rangle \leq 0. \tag{3.4}$$

Finally we shall show that $y_n \longrightarrow z$. For each $n \geq 1$, we have

$$\begin{aligned}
\|y_n - z\|^2 &\leq \|y_n - z + a_n(z - f(z))\|^2 + 2a_n\langle y_n - z, f(z) - z\rangle \\
&\leq \left\{\frac{1}{n}\sum_{j=1}^{n}\|P(a_n f(y_n) + (1-a_n)(TP)^j y_n) - P(a_n f(z) + (1-a_n)z)\|\right\}^2 \\
&\quad + 2a_n\langle y_n - z, f(z) - z\rangle \\
&\leq \left\{\frac{1}{n}\sum_{j=1}^{n}(a_n\|f(y_n) - f(z)\| + (1-a_n)((TP)^j y_n - z)\|)\right\}^2 \\
&\quad + 2a_n\langle y_n - z, f(z) - z\rangle \\
&\leq \left\{a_n\alpha\|y_n - z\| + (1-a_n)\frac{1}{n}\sum_{j=1}^{n}\|(TP)^j y_n - z\|\right\}^2 \\
&\quad + 2a_n\langle y_n - z, f(z) - z\rangle \\
&\leq \left\{a_n\alpha\|y_n - z\| + (1-a_n)\frac{1}{n}\sum_{j=1}^{n}k_j\|y_n - z\|\right\}^2 \\
&\quad + 2a_n\langle y_n - z, f(z) - z\rangle \\
&\leq \{(a_n\alpha + (1-a_n)b_n)\|y_n - z\|\}^2 \\
&\quad + 2a_n\langle y_n - z, f(z) - z\rangle \\
&\leq (a_n(\alpha - b_n) + b_n)\|y_n - z\|^2 + 2a_n\langle y_n - z, f(z) - z\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_n - z\|^2 &\leq \frac{2a_n}{1 - [a_n(\alpha - b_n) + b_n]}\langle y_n - z, f(z) - z\rangle \\
&= \frac{2}{\beta - \alpha}\langle y_n - z, f(z) - z\rangle.
\end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Then by the fact (3.4) there exists a natural number N such that

$$\langle y_n - z, f(z) - z\rangle \leq (\beta - \alpha)\frac{\epsilon}{2}, \forall n \geq N.$$

This implies that

$$\|y_n - z\|^2 \leq \epsilon, \forall n \geq N.$$

Hence the sequence $\{y_n\}$ converges strongly to a fixed point z of T. This completely the proof. $\square$

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Somyot Plubtieng

From: "Joseph Ball" <ball@calvin.math.vt.edu>
To: "Somyot Plubtieng" <somyotp@nu.ac.th>
Sent: Tuesday, February 08, 2005 9:29 PM
Subject: Re: submit paper

Dear Professor Plubtieng,

This is to acknowledge receipt of your paper (with Rabian Wangkeeree) "Strong convergence theorems of viscosity averaging iterations for asymptotically nonexpansive nonself-mappings" submitted for publication in Proceedings of the American Mathematical Society (reference number PAMS05_21). I will be in contact with you concerning this paper upon completion of the review process.

Sincerely,
J.A. Ball
Editor, PAMS

Professor Joseph A. Ball
Virginia Polytechnic

Institute and State University, Blacksburg,

VA 24061 USA
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February 8, 2005

Dear Sir,

Enclosed please find a files (the pdf. file) of my paper with Mr. Rabian Wangkeeree entitled;

Strong Convergence Theorems of Viscosity Averaging Iterations for Asymptotically Nonexpansive Nonself-Mappings

which I would like to submit for publication in the Proceeding of the American Mathematical Society.

I would like to thank you in advance for your consideration.

Your sincerely,

Somyot Plubtieng

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Attachment converted: Macintosh HD:(9) Strong convergence of V.pdf (PDF /CARO) (0007D7CF)

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ภาคผนวก 8

Strong convergence theorems of vicosity averaging iterations for nonexpansive nonself-mappings in Hilbert spaces

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Strong Convergence Theorems of Viscosity Averaging Iterations for Nonexpansive Nonself-Mappings in Hilbert Spaces *

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Abstract

Let C be a nonempty closed convex subset of Hilbert space H, P a metric projection of H onto C and let T be a nonexpansive nonself-mapping from C into H. In this paper, we study the convergence of three sequences generated by

$$x_n = t_n f(x_n) + (1-t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n \ , n \geq 1$$

,

$$y_{n+1} = \alpha_n f(y_n) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n \ , n \geq 0,$$

and

$$z_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n f(z_n) + (1-\alpha_n)(TP)^j z_n), \ n \geq 0,$$

where $y_0, z_0 \in C$, $\{t_n\} \subset (0,1)$, $\{\alpha_n\}$ is a real sequence in an interval $[0,1]$ and f is a contraction from C into itself.

Keywords: Fixed point; Metric projection; Nonexpansive Mapping; Strong Convergence.

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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and let T be a mapping of C into itself. Then T is said to be nonexpansive provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Recall that a self-mapping $f : C \longrightarrow C$ is a contraction on C if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\| \ \forall x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T; i.e. $F(T) = \{x \in C : Tx = x\}$. It is well know that if T is nonexpansive, then $F(T)$ is convex see [4]. In 1967, Browder[3] proved the following strong convergence theorem for nonexpansive mapping: let T be a nonexpansive mapping of a bounded closed convex subset C of H into itself. Let $u \in C$ and for each $t \in (0,1)$, let $G_t x = tu + (1-t)Tx$. Then, G_t has a unique fixed point x_t in C, and $\{x_t\}$ converges strongly to a fixed point u_0 of T as $t \longrightarrow 0$. The fixed point u_0 is uniquely specified as the fixed point of T in C closest to u. In 1975, Baillon [1], prove the first nonlinear egodic theorem as follows: let C be a bounded closed convex subset of H and let T be a nonexpansive mapping of C into itself. Then for each $x \in C$

$$A_n x = \frac{1}{n}\sum_{k=1}^{n} T^k x$$

converges weakly to fixed point of T. By Using an idea of Browder[3], Shimizu and Takahashi[11] studied the convergence of the following approximated sequence for an asymptotically nonexpansive mapping in the framework of a Hilbert space:

$$x_n = a_n x + (1 - a_n)\frac{1}{n}\sum_{j=1}^{n} T^j x_n \,, n = 1, 2, ..., \tag{1.1}$$

where $\{a_n\}$ is a real sequence satisfying $0 < a_n < 1$ and $a_n \longrightarrow 0$.

In 1997, Shimizu and Takahashi [10] also studied the convergence of iteration process for a family of nonexpansive mappings in the framework of a Hilbert space as follows:

Theorem (Shimizu and Takahashi). Let C be a nonempty closed convex subset of a Hilbert space H, let T a nonexpansive self-mapping of C such that$F(T)$ is nonempty, and let P be the metric projection from C onto $F(T)$.

Let $\{a_n\}$ be a real sequence which satisfies $0 \leq a_n \leq 1$, $\lim_{n\longrightarrow\infty} a_n = 0$ and $\sum_{n=0}^{\infty} a_n = \infty$. Let x and y_0 be element of C and let $\{y_n\}$ be the sequence defined by

$$y_{n+1} = a_n x + (1 - a_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j y_n\,, n \geq 1. \tag{1.2}$$

Then $\{y_n\}$ converges strongly to Px.

Recently, Matsushita and Koroiwa[8] generalized the result of Shimizu and Takahashi [10] and prove the following theorems:

Theorem (Matsushita and Koroiwa). Let H be a Hilbert space, C a closed convex subset of H, P_1 the metric projection of H onto C and T be a nonexpansive nonself-mapping from C into H such that $F(T)$ is nonempty, and $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that $\{x_n\}$ is given by $x_0, x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(P_1 T)^j x_n,\ n \geq 0. \tag{1.3}$$

Then $\{x_n\}$ converges strongly to $P_2 x \in F(T)$, where P_2 is the metric projection from C onto $F(T)$.

Theorem (Matsushita and Koroiwa). Let H be a Hilbert space, C a closed convex subset of H, P_1 the metric projection of H onto C and T be a nonexpansive nonself-mapping from C into H such that $F(T)$ is nonempty, and $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that $\{y_n\}$ is given by $y_0, y \in C$ and

$$y_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n y + (1 - \alpha_n)(TP_1)^j y_n,\ n \geq 0.) \tag{1.4}$$

Then $\{y_n\}$ converges strongly to $P_2 y \in F(T)$, where P_2 is the metric projection from C onto $F(T)$.

On the other hand, using the viscosity approximation method, Xu[14] studied the convergence of the following approximation for nonexpansive nonself-mapping in Hilbert space:

$$x_t = tf(x_t) + (1 - t)Tx_t \tag{1.5}$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, n = 1, 2, ..., \tag{1.6}$$

where $t \in (0,1)$, $\{\alpha_n\} \subset (0,1)$ and f is a contraction mapping from C into itself. Xu[14] also proved the strong convergence of the sequences as $t \longrightarrow 1$ and $\alpha_n \longrightarrow 1$ to the unique solution z in $F(T)$ to the variational inequality $\langle (I-f)z = x - z\rangle \geq 0, x \in F(T)$ or equivalently to $z = P(f(z))$ where P is the metric projection from H onto $F(T)$.

In this paper, we study the three type iterations process which are mixed iteration process of (1.1) - (1.6) as follows: for $y_0, z_0 \in C$ and

$$x_n = t_n f(x_n) + (1-t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n \tag{1.7}$$

$$y_{n+1} = \alpha_n f(y_n) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n, n \geq 0 \tag{1.8}$$

and

$$z_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n f(z_n) + (1-\alpha_n)(TP)^j z_n), n \geq 0 \tag{1.9}$$

where $\{t_n\} \subset (0,1)$, $\{\alpha_n\}$ is a sequence such that $0 \leq \alpha_n \leq 1$, $f : C \longrightarrow C$ is a contraction mapping and P is the metric projection of H onto C. We first estabish the strong convergence of sequence $\{x_n\}$ defined by (1.7). In addition, we also prove the strong convergence of the approximation sequences $\{y_n\}$ and $\{z_n\}$ defined by (1.8) and (1.9) respectively. The results presented in this paper generalized and extend the corresponding main results of Baillon [1], Shimizu and Takahashi [10] and Matsushita and Koroiwa[8].

2 Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. Recall the metric (nearest point) projection P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: Given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x-y\| : y \in C\}.$$

$P_C x$ is characterized as follows.

Lemma 2.1. *Let H be a real Hilbert space, C a closed convex subset of H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \forall z \in C.$$

Definition 2.2. A mapping $T : C \longrightarrow H$ is said to satisfy nowhere normal outward condition ((NNO) for short) if and only if for each $x \in C$, $Tx \in S_x^C$, where $S_x = \{y \in H : y \neq x, Py = x\}$ and P is the metric projection from H onto C.

The following results was proved by Matsushita and Koroiwa[7].

Lemma 2.3. ([7, Proposition 2, P. 208]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, P the metric projection of H onto C and $T : C \longrightarrow H$ a nonexpansive nonself-mapping. If $F(T)$ is nonempty then T satisfies NNO condition.*

Lemma 2.4. ([7, Proposition 1, P. 208]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, P the metric projection of H onto C and $T : C \longrightarrow H$ a nonself-mapping. Suppose that T satisfies (NNO) condition. Then $F(PT) = F(T)$.*

Further, we know the following lemmas actually hold for asymptotically nonexpansive[11]. But we only need its for nonexpansive version.

Lemma 2.5. ([11]). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \longrightarrow C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C and there exists a subsequence $\{x_{n_j}\}$ which converges weakly to $x \in C$ and $\{x_{n_j} - \frac{1}{n_j}\sum_{i=1}^{n_j} T^i x_{n_j}\}$ converges strongly to 0. Then x is a fixed point of T.*

Finally, the following two lemma are useful for the proof of our main theorems.

Lemma 2.6. ([14]). *Let $\{\alpha_n\}$ be a sequence in $[0,1]$ that satisfies $\lim_{n \longrightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{a_n\}$ be a sequence of nonnegative real numbers that satisfying:*

For all $\epsilon > 0$, there exists an integer $N \geq 1$ such that for all $n \geq N$,

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \epsilon.$$

Then $\lim_{n\longrightarrow\infty} a_n = 0.$

Lemma 2.7. [14] *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $f : C \longrightarrow C$ a contraction with coefficient $\alpha < 1$. Then*

$$\langle x-y, (I-f)x-(I-f)y\rangle \geq (1-\alpha)\|x-y\|^2, x, y \in C.$$

Remark 2.8. As in Lemma 2.7, if f is a nonexpansive mapping, then

$$\langle x-y, (I-f)x-(I-f)y\rangle \geq 0, \forall x, y \in C.$$

3 Main results

In this section, we study the strong convergence properties of the three sequences (1.7), (1.8) and (1.9).

Theorem 3.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, P the metric projection of H onto C and $T : C \longrightarrow H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{t_n\}$ be sequence in $(0,1)$ which satisfies* $\lim_{n\longrightarrow\infty} t_n = 0$. *Then for a contraction mapping $f : C \longrightarrow C$ with coefficient $\alpha \in (0,1)$, the sequence $\{x_n\}$ defined by (1.7)converges strongly to z, where, z is the unique solution in $F(T)$ to the variation inequality*

$$\langle (I-f)z, x-z\rangle \geq 0, \; x \in F(T) \tag{3.1}$$

or equivalently $z = G(f(z))$, where G is a metric projection mapping from H onto $F(T)$.

Proof. Since $F(T)$ is nonempty, it follows that T satisfies (NNO) condition by Lemma 2.3. We first show that $\{x_n\}$ is bounded. Let $q \in F(T)$. We note that

$$\begin{aligned}\|x_n - q\| &= \|t_n f(x_n) + (1-t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n - q\| \\ &\leq \|t_n(f(x_n)-q) + (1-t_n)\frac{1}{n}\sum_{j=1}^{n}((PT)^j x_n - (PT)^j q)\| \\ &\leq t_n\|f(x_n)-q\| + (1-t_n)\|x_n - q\|, \forall n \geq 1.\end{aligned}$$

So we get

$$\begin{aligned}\|x_n - q\| &\leq \|f(x_n) - q\| \\ &\leq \|f(x_n) - f(q)\| + \|f(q) - q\| \\ &\leq \alpha\|x_n - q\| + \|f(q) - q\|, \forall n \geq 1.\end{aligned}$$

Hence

$$\|x_n - q\| \leq \frac{1}{1-\alpha}\|f(q) - q\|, \forall n \geq 1.$$

This show that $\{x_n\}$ is bounded, so are $\{f(x_n)\}, \{\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\}$. Further, we note that

$$\begin{aligned}\|x_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\| &= \|t_n f(x_n) + (1-t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\| \\ &= t_n\|f(x_n) - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\| \\ &\leq t_n(\|f(x_n)\| + \|\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\|) \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned}$$

Thus $\{x_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\}$ converges strongly to 0. Since $\{x_n\}$ is a bounded sequence, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $z \in C$. By Lemma 2.5 and Lemma 2.4, we have $z \in F(T)$. For each $n \geq 1$, since

$$x_n - z = t_n(f(x_n) - z) + (1-t_n)\frac{1}{n}\sum_{j=1}^{n}((PT)^j x_n - z),$$

so we get

$$\begin{aligned}\|x_n - z\|^2 &= (1-t_n)\langle\frac{1}{n}\sum_{j=1}^{n}((PT)^j x_n - z), x_n - z\rangle + t_n\langle f(x_n) - z, x_n - z\rangle \\ &\leq (1-t_n)\|x_n - z\|^2 + t_n\langle f(x_n) - z, x_n - z\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\|x_n - z\|^2 &\leq \langle f(x_n) - z, x_n - z\rangle \\ &= \langle f(x_n) - f(z), x_n - z\rangle + \langle f(z) - z, x_n - z\rangle \\ &\leq \alpha\|x_n - z\|^2 + \langle f(z) - z, x_n - z\rangle.\end{aligned}$$

This implies that

$$\|x_n - z\|^2 \leq \frac{1}{1-\alpha}\langle x_n - z, f(z) - z\rangle.$$

In particular, we have

$$\|x_{n_j} - z\|^2 \le \frac{1}{1-\alpha}\langle x_{n_j} - z, f(z) - z\rangle.$$

Since $x_{n_j} \rightharpoonup z$, it follows that

$$x_{n_j} \longrightarrow z \text{ as } j \longrightarrow \infty.$$

Next we show that the inequality (3.1) is true. Indeed, from

$$x_n = t_n f(x_n) + (1 - t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n, \forall n \ge 1,$$

we have

$$(I-f)x_n = -\frac{1-t_n}{t_n}(x_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n).$$

Thus for any $q \in F(T)$, we infer by Remark 2.8 that

$$\begin{aligned}
\langle (I-f)x_n, x_n - q\rangle &= -\frac{1-t_n}{t_n}\langle (I - \frac{1}{n}\sum_{j=1}^{n}(PT)^j)x_n, x_n - q\rangle \\
&= -\frac{1-t_n}{t_n}\langle (I - \frac{1}{n}\sum_{j=1}^{n}(PT)^j)x_n - (I - \frac{1}{n}\sum_{j=1}^{n}(PT)^j)z, x_n - q\rangle \\
&\le 0, \ \forall n \ge 1.
\end{aligned}$$

In particular

$$\langle (I-f)x_{n_j}, x_{n_j} - q\rangle \le 0, \forall j \ge 1.$$

Taking $j \longrightarrow \infty$, so we obtain

$$\langle (I-f)z, z-q\rangle \le 0, \forall q \in F(T), \tag{3.2}$$

or equivalent to $z = G(f(z))$. Finally, we shall show that $\{x_n\}$ convergence strongly to z. Let another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow z' \in C$ as $k \longrightarrow \infty$. Then $z' \in F(T)$, it follows from the inequality (3.2) that

$$\langle (I-f)z, z-z'\rangle \le 0. \tag{3.3}$$

Interchange z and z' to obtain

$$\langle (I-f)z', z'-z\rangle \le 0. \tag{3.4}$$

Adding (3.3) and (3.4) and by Lemma 2.7 we get

$$(1-\alpha)\|z-z'\|^2 \le \langle z-z', (I-f)z - (I-f)z'\rangle \le 0.$$

This implies that $z = z'$. Hence $\{x_n\}$ converges strongly to z. This completely the proof. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H, P be the metric projection of H onto C and $T : C \longrightarrow H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ which satisfies $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then for a contraction mapping $f : C \longrightarrow C$ with coefficient $\alpha \in (0,1)$, the sequence $\{y_n\}$ defined by (1.8)converges strongly to z, where, z is the unique solution in $F(T)$ to the variation inequality*

$$\langle (I-f)z, x-z\rangle \geq 0,\ x \in F(T) \tag{3.5}$$

or equivalently $z = G(f(z))$, where G is a metric projection mapping from H onto $F(T)$.

Proof. Since $F(T)$ is nonempty, it follows that T satisfies (NNO) condition by Lemma 2.3. We first show that $\{y_n\}$ is bounded. Let $q \in F(T)$. We note that

$$\begin{aligned}
\|y_{n+1} - q\| &= \|\alpha_n f(y_n) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n - q\| \\
&\leq \alpha_n\|f(y_n) - q\| + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}\|(PT)^j y_n - q\| \\
&\leq \alpha_n\|f(y_n) - f(q)\| + \alpha_n\|f(q) - q\| + (1-\alpha_n)\|y_n - q\| \\
&\leq \alpha_n\alpha\|y_n - q\| + \alpha_n\|f(q) - q\| + (1-\alpha_n)\|y_n - q\| \\
&= (1-\alpha_n(1-\alpha))\|y_n - q\| + \alpha_n\|f(q) - q\| \\
&\leq \max\{\|y_n - q\|, \frac{1}{1-\alpha}\|f(q) - q\|\}, \forall n \geq 1.
\end{aligned}$$

So by induction, we get

$$\|y_n - q\| \leq \max\{\|y_0 - q\|, \frac{1}{1-\alpha}\|f(q) - q\|\}, n \geq 0.$$

This show that $\{y_n\}$ is bounded, so are $\{f(y_n)\}$ and $\{\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\}$. We observe that

$$\begin{aligned}
\|y_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\| &= \|\alpha_n f(y_n) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n \\
&= \alpha_n\|f(y_n) - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\| \\
&\leq \alpha_n(\|f(y_n)\| + \|\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\|).
\end{aligned}$$

Hence $\{y_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\}$ converges strongly to 0. We next show that

$$\limsup_{n\longrightarrow\infty}\langle z - y_n, z - f(z)\rangle \leq 0. \tag{3.6}$$

Let $\{y_{n_j}\}$ be a subsequence of $\{y_n\}$ such that

$$\lim_{j\longrightarrow\infty}\langle z - y_{n_j}, z - f(z)\rangle = \limsup_{n\longrightarrow\infty}\langle z - y_n, z - f(z)\rangle,$$

and $y_{n_j} \rightharpoonup q \in C$. It follows by Lemma 2.5 and Lemma 2.4 that $q \in F(PT) = F(T)$. By the inequality (3.5), we get

$$\limsup_{n\longrightarrow\infty}\langle z - y_n, z - f(z)\rangle = \langle z - q, z - f(z)\rangle \leq 0.$$

Hence (3.6) is true. Finally we shall show that $y_n \longrightarrow z$. For each $n \geq 0$, we have

$$\begin{aligned}
\|y_{n+1} - z\|^2 &= \|y_{n+1} - z + \alpha_n(z - f(z)) - \alpha_n(z - f(z))\|^2 \\
&\leq \|y_{n+1} - z + \alpha_n(z - f(z))\|^2 + 2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle \\
&= \|\alpha_n f(y_n) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n - (\alpha_n f(z) + (1-\alpha_n)z)\|^2 \\
&\quad +2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle \\
&= \|\alpha_n(f(y_n) - f(z)) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}((PT)^j y_n - z)\|^2 \\
&\quad +2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle \\
&\leq \left[\alpha_n\|f(y_n) - f(z)\| + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}\|(PT)^j y_n - z\|\right]^2 \\
&\quad +2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle \\
&\leq \left[\alpha_n\alpha\|y_n - z\| + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}\|y_n - z\|\right]^2 \\
&\quad +2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle \\
&= (1 - \alpha_n(1-\alpha))^2\|y_n - z\|^2 + 2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle \\
&\leq (1 - \alpha_n(1-\alpha))\|y_n - z\|^2 + 2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle. \qquad (3.7)
\end{aligned}$$

Now, let $\epsilon > 0$ be arbitrary. Then, by the fact (3.6), there exists a natural number N such that

$$\langle z - y_n, z - f(z)\rangle \leq \frac{\epsilon}{2}, \forall n \geq N.$$

From (3.7), we get

$$\|y_{n+1} - z\|^2 \leq (1 - \alpha_n(1-\alpha))\|y_n - z\|^2 + \alpha_n \epsilon.$$

By Lemma 2.6, the sequence $\{y_n\}$ converges strongly to a fixed point z of T. This completely the proof. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H, P the metric projection of H onto C and $T : C \longrightarrow H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be sequence in $[0,1]$ which satisfies $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then for a contraction mapping $f : C \longrightarrow C$ with coefficient $\alpha \in (0,1)$, the sequence $\{z_n\}$ defined by (1.9)converges strongly to z, where, z is the unique solution in $F(T)$ to the variation inequality*

$$\langle (I-f)z, x-z\rangle \geq 0, \ x \in F(T) \tag{3.8}$$

or equivalently $z = G(f(z))$, where G is a metric projection mapping from H onto $F(T)$.

Proof. Since $F(T)$ is nonempty, it follows that T satisfies (NNO) condition by Lemma 2.3. We first show that $\{z_n\}$ is bounded. Let $q \in F(T)$. We note that

$$\begin{aligned}
\|z_{n+1} - q\| &= \|\frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n f(z_n) + (1-\alpha_n)(TP)^j z_n) - q\| \\
&\leq \frac{1}{n+1}\sum_{j=0}^{n} \|P(\alpha_n f(z_n) + (1-\alpha_n)(TP)^j z_n) - Pq\| \\
&\leq \frac{1}{n+1}\sum_{j=0}^{n} (\alpha_n\|f(z_n) - f(q)\| + \alpha_n\|f(q) - q\| + (1-\alpha_n)\|z_n - q\|) \\
&\leq \alpha_n \alpha\|z_n - q\| + \alpha_n\|f(q) - q\| + (1-\alpha_n)\|z_n - q\|) \\
&= (1 - \alpha_n(1-\alpha))\|z_n - q\| + \alpha_n\|f(q) - q\| \\
&\leq \max\{\|x_n - q\|, \frac{1}{1-\alpha}\|f(q) - q\|\}, \forall n \geq 0.
\end{aligned}$$

So by induction, we obtain

$$\|x_n - q\| \leq \max\{\|x_0 - q\|, \frac{1}{1-\alpha}\|f(q) - q\|\}, n \geq 0.$$

This show that $\{z_n\}$ is bounded, so are $\{f(x_n)\}$ and $\{\frac{1}{n+1}\sum_{j=0}^{n}\|(TP)^j z_n\|\}$. Furthermore, we also have

$$\begin{aligned}
\|z_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j z_n\| &\leq \frac{1}{n+1}\sum_{j=0}^{n}\|P(\alpha_n f(z_n) + (1-\alpha_n)(TP)^j z_n) - (PT)^j z_n\| \\
&\leq \frac{1}{n+1}\sum_{j=0}^{n}\|\alpha_n f(z_n) + (1-\alpha_n)(TP)^j z_n - T(PT)^{j-1} z_n\| \\
&= \frac{1}{n+1}\sum_{j=0}^{n}\|\alpha_n f(z_n) + (1-\alpha_n)(TP)^j z_n - (TP)^j z_n\| \\
&= \alpha_n \frac{1}{n+1}\sum_{j=0}^{n}\|f(x_n) - (TP)^j z_n\| \\
&\leq \alpha_n \left[\|f(x_n)\| - \frac{1}{n+1}\sum_{j=0}^{n}\|(TP)^j z_n\|\right] \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

This implies that $\{z_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j z_n\}$ converges strongly to 0. We next show that

$$\limsup_{n\longrightarrow\infty}\langle z - z_n, z - f(z)\rangle \leq 0. \tag{3.9}$$

Let $\{z_{n_j}\}$ be a subsequence of $\{z_n\}$ such that

$$\lim_{j\longrightarrow\infty}\langle z - z_{n_j}, z - f(z)\rangle = \limsup_{n\longrightarrow\infty}\langle z - z_n, z - f(z)\rangle,$$

and $z_{n_j} \rightharpoonup q \in C$. By Lemma 2.5 and Lemma 2.4 we get $q \in F(PT) = F(T)$. From the inequality (3.8) we obtain

$$\limsup_{n\longrightarrow\infty}\langle z - z_n, z - f(z)\rangle = \langle z - q, z - f(z)\rangle \leq 0.$$

This show that (3.9) is true. Finally we shall show that $z_n \longrightarrow z$. For each

$n \geq 0$, we have

$$\begin{aligned}
\|z_{n+1} - z\|^2 &= \|z_{n+1} - z + \alpha_n(z - f(z)) - \alpha_n(z - f(z))\|^2 \\
&\leq \|z_{n+1} - z + \alpha_n(z - f(z))\|^2 + 2\alpha_n\langle z_{n+1} - z, f(z) - z\rangle \\
&= \|\frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n f(z_n) + (1-\alpha_n)(TP)^j z_n) - (\alpha_n f(z) + (1-\alpha_n)z)\|^2 \\
&\quad +2\alpha_n\langle x_{n+1} - z, f(z) - z\rangle \\
&\leq \left\{\frac{1}{n+1}\sum_{j=0}^{n} \|P(\alpha_n f(z_n) + (1-\alpha_n)(TP)^j z_n) - P(\alpha_n f(z) + (1-\alpha_n)z)\|\right\}^2 \\
&\quad +2\alpha_n\langle x_{n+1} - z, f(z) - z\rangle \\
&\leq \left\{\frac{1}{n+1}\sum_{j=0}^{n} \|\alpha_n(f(z_n) - f(z)) + (1-\alpha_n)((TP)^j z_n - z)\|\right\}^2 \\
&\quad +2\alpha_n\langle x_{n+1} - z, f(z) - z\rangle \\
&\leq \{\alpha_n\alpha\|z_n - z\| + (1-\alpha_n)\|z_n - z\|\}^2 \\
&\quad +2\alpha_n\langle x_{n+1} - z, f(z) - z\rangle \\
&= (1 - \alpha_n(1-\alpha))\|z_n - z\|^2 + 2\alpha_n\langle z_{n+1} - z, f(z) - z\rangle. \qquad (3.10)
\end{aligned}$$

Now, let $\epsilon > 0$ be arbitrary. Then, by the fact (3.9), there exists a natural number N such that

$$\langle z - z_n, z - f(z)\rangle \leq \frac{\epsilon}{2}, \forall n \geq N.$$

From (3.10), we have

$$\|z_{n+1} - z\|^2 \leq (1 - \alpha_n(1-\alpha))\|z_n - z\|^2 + \alpha_n\epsilon.$$

By Lemma 2.6, the sequence $\{z_n\}$ converges strongly to a fixed point z of T. This completely the proof. □

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