รายงานวิจัยฉบับสมบูรณ์

โครงการ: ปริภูมิโซโบเลฟโฮโลมอร์ฟิกและการแปลงซีกัล – บาร์กมันน์นัยทั่วไป

Holomorphic Sobolev Spaces and Generalized Segal-Bargmann Transform

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30 สิงหาคม 2551

สัญญาเลขที่ RSA4780022

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ผู้วิจัย

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
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ABSTRACT

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Project Title : Holomorphic Sobolev Spaces and Generalized Segal-Bargmann Transform

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The main objectives of this project are two folds. In one direction, we study a rotation-invariant form of the Segal-Bargmann transform. We consider the subspace of Segal-Bargmann space which is invariant under the action of the special orthogonal group. We establish a pointwise bound for a function in this space which is polynomially better than the pointwise bound for a function in the Segal-Bargmann space.

In the other direction, we continue to develop the categorical non-commutative geometry that was initiated in [BCL1]. More specifically:

1. we obtain a categorical version of Gel'fand duality theorem that generalizes the usual Gel'fand's duality theorem for the category of commutative unital C*-algebras and the category of compact Hausdorff spaces;

2. as a first step toward a bivariant Serre-Swan equivalence theory, we develop a spectral theorem for imprimitivity Hilbert C*-bimodules over commutative unital C*-algebras, in terms of Hermitian line bundles over the graph of a homeomorphism between the compact Hausdorff Gel'fand spectra of the two C*-algebras;

3. as the first effort in the direction of the construction of a full theory of morphisms of spectral geometries, we introduce a notion of metric morphism for A. Connes' spectral triples; we prove a duality between the category of isometries of compact Riemannian spin manifolds and the category of metric morphisms and we study the relationship between the metric category and the category of spectral triples already introduced in [BCL1].

Keywords : spectral triples, morphisms, categorification, C*-category, Segal-Bargmann transform

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ชื่อโครงการ : ปริภูมิโซโบเลฟโฮโลมอร์ฟิกและการแปลงซีกัล – บาร์กมันน์นัยทั่วไป

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จุดประสงค์หลักของงานวิจัยนี้มี 2 ประการ ในทิศทางหนึ่งเราศึกษารูปแบบที่ไม่แปรเปลี่ยนภายใต้การหมุนของการแปลงซีกัล – บาร์กมันน์ เราพิจารณาปริภูมิย่อยของปริภูมิซีกัล – บาร์กมันน์ที่ไม่แปรเปลี่ยนภายใต้การกระทำของกรุปเชิงตั้งฉากพิเศษ เราได้ขอบเขตรายจุดของฟังก์ชันในปริภูมินี้ซึ่งดีกว่าขอบเขตรายจุดของฟังก์ชันในปริภูมิซีกัล – บาร์กมันน์เป็นอันดับพหุนาม

ในอีกทิศทางหนึ่ง เราพัฒนาเรขาคณิตไม่สลับที่เชิงแคทีกอรีต่อเนื่องจากที่ได้เริ่มต้นไว้ในบทความวิจัย [BCL1] โดยเฉพาะอย่างยิ่งเราได้ผลต่อไปนี้

1. เราได้ทฤษฎีบทภาวะคู่กันของเกลฟานด์แบบแคทีกอรีซึ่งเป็นนัยทั่วไปของทฤษฎีบทภาวะคู่กันของเกลฟานด์แบบปกติสำหรับแคทีกอรีของพีชคณิตซีสตาร์สลับที่ที่มีเอกลักษณ์และแคทีกอรีของปริภูมิกระชับเฮาส์ดอฟฟ์

2. เพื่อเป็นขั้นแรกที่นำไปสู่ทฤษฎีสมมูลแซร์ – สวอนยืนยงคู่ เราพัฒนาทฤษฎีบทสเปกตรัมสำหรับฮิลเบิร์ตซีสตาร์ไบมอดูลไม่ปฐมฐานเหนือซีสตาร์สลับที่ที่มีเอกลักษณ์ในรูปของเฮอร์มิเชียนไลน์บันเดิลเหนือกราฟของสมานสัณฐานระหว่างเกลฟานด์สเปกตรัมกระชับเฮาส์ดอฟฟ์ของสองพีชคณิตซีสตาร์

3. เรานำเสนอแนวคิดของสัณฐานอิงระยะทางสำหรับสเปกตรัลทริเปิลของอลัน คอนน์ส เราพิสูจน์ภาวะคู่กันระหว่างแคทีกอรีของสมมิติของแมนิโฟลด์แบบสปินกระชับเชิงรีมันน์และแคทีกอรีของสัณฐานอิงระยะทาง และศึกษาความสัมพันธ์ระหว่างแคทีกอรีอิงระยะทางและแคทีกอรีของสเปกตรัลทริเปิลที่ได้นิยามไว้ใน [BCL1]

คำหลัก : สเปกตรัลทริเปิล, สัณฐาน, แคทีกอริฟิเคชัน, ซีสตาร์แคทีกอรี, ผลการแปลงซีกัล – บาร์กมันน์

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Holomorphic Sobolev Spaces and Generalized Segal-Bargmann Transform

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1. เนื้อหางานวิจัย : ปัญหา ที่มา และผลที่ได้

โครงการวิจัยนี้ได้ศึกษาปัญหาที่เกี่ยวข้องกันในสองทิศทาง จึงขออธิบายโดยแยกเป็นสองหัวข้อ ดังนี้

(1) Research work on the Segal-Bargmann transform

The Segal-Bargmann transform is the map $B_t : L^2(\mathbb{R}^n, \rho_t) \to \mathcal{H}(\mathbb{C}^n)$ defined by

$$B_t f(z) = (2\pi t)^{-n/2} \int_{R^n} f(x) e^{-(z-x)^2/2t} dx \qquad (z \in \mathbb{C}^n)$$

where $\mathcal{H}(\mathbb{C}^n)$ denotes the space of (entire) holomorphic functions on $\mathbb{C}^n$,and t is a positive real number. The density ρ_t is the Gaussian function given by

$$\rho_t(x)=(2\pi t)^{-n/2} e^{-x^2/2t}$$

This transform is also known as *coherent state transform* in physics literature. There are variants of the Segal-Bargmann transform, but the form above is the one we study in this work. It is easy to verify that the image of a function f in $L^2(\mathbb{R}^n, \rho_t)$ under B_t is a holomorphic function. In general, we have the following characterization of the image of B_t as follows:

Theorem 1 (Segal[Sg]-Bargmann[B]). B_t is an isometric isomorphism from $L^2(\mathbb{R}^n, \rho_t)$ onto the space $\mathcal{H}L^2(\mathbb{C}^n, \nu_t)$ consisting of the space of all holomorphic functions on $\mathbb{C}^n$ which is square-integrable with respect to the measure $d\mu_t = \mu_t dz$ where

$$\mu_t(z) = (\pi t)^{-n/2} e^{-|z|^2/t}$$

This isomorphism is central in quantum field theory because it was used for describing the wave-particle duality of light. For expository articles on the history and its relevance in physics, see [Gr], [H1], [H2].

The space $\mathcal{H}L^2(\mathbb{C}^n, \nu_t)$ is called the Segal-Bargmann space. Suppose that F is invariant under the action of $SO(d)$. By analytic continuation, it is also invariant under the action of $SO(d, \mathbb{C})$.

Theorem 2 [CL]. The Segal-Bargmann transform B_t preserves rotation. In other words, a function $f \in L^2(\mathbb{R}^n, \rho_t)$ is $SO(d)$-invariant if and only if $B_t(f)$ is $SO(d, \mathbb{C})$-invariant.

It is well-known that for any function $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, we have the pointwise bound

$$(1) \qquad \left|F(z)\right|^2 \le e^{|z|^2/t} \left\|F\right\|^2_{L^2(\mathbb{C}^d,\mu_t)} \quad (z \in \mathbb{C}^d).$$

By minimizing (1) on each orbit, for any $SO(d,\mathbb{C})$-invariant function F in the Segal-Bargmann space, we obtain the preliminary estimate

$$(2) \qquad \left|F(z)\right|^2 \le e^{|(z,z)|/t} \left\|F\right\|^2_{L^2(\mathbb{C}^d,\mu_t)} \quad (z \in \mathbb{C}^d),$$

where $(z,z) = z_1^2 + \cdots + z_d^2$. Since $\left|(z,z)\right| \le |z|^2$, this is already an improvement over the pointwise bound in (1).

The $SO(d)$ invariance means that F is determined by its values on $\{(z,0,\ldots,0)\} \simeq \mathbb{C}^1$. (By holomorphicity, F is determined by its values on $\mathbb{R}^d$, then any point in $\mathbb{R}^d$ can be rotated into $\mathbb{R}^1$.) Conversely, any even holomorphic function on $\mathbb{C}^1$ has an extension to an $SO(d)$-invariant function on $\mathbb{C}^d$. Then the space of $SO(d)$-invariant functions in the Segal-Bargmann space over $\mathbb{C}^d$ can be expressed as an L^2-space of holomorphic functions on $\mathbb{C}^1$, with some non-Gaussian measure. By estimating the reproducing kernel for this space, we obtain a sharp bound for an $SO(d)$-invariant function F in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, which will be polynomially better than (2). This bound is described in the following theorem.

Theorem 3 [KL]. There exists a constant C, depending only on d and t, such that for each $SO(d)$-invariant function F in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, we have

$$\left|F(z)\right|^2 \le \frac{Ce^{|(z,z)|/t}}{1 + \left|(z,z)\right|^{(d-1)/2}} \left\|F\right\|^2_{L^2(\mathbb{C}^d,\mu_t)} \quad (z \in \mathbb{C}^d).$$

(2) Research work on categorical non-commutative geometry

Non-commutative geometry, created by A. Connes, is a powerful extension of the ideas of R. Descartes' analytic geometry: to substitute "geometrical objects" with their Abelian algebras of functions; to "translate" the geometrical properties of spaces into algebraic properties of the associated algebras and to "reconstruct" the original geometric spaces as derived entities (the spectra of the algebras).

Whenever such "codifications" of geometry in algebraic terms still make sense if the Abelian condition is dropped, we can simply work with non-commutative algebras considered as "duals" of "non-commutative spaces."

The existence of dualities between categories of “geometrical spaces” and categories “constructed from Abelian algebras” is the starting point of any generalization of geometry to the non-commutative situation.

Typical examples of such (anti-) equivalences are:

- Gel’fand-Naimark duality between the category of continuous maps of compact Hausdorff spaces and the category of unital involutive homomorphisms of unital commutative C*-algebras ([G1], [G2]);

- Serre-Swan equivalence between the category of vector bundle maps of finite dimensional locally trivial vector bundles over a compact Hausdorff space and the category of homomorphisms of finite projective modules over a commutative unital C*-algebra ([Sr], [Sw]);

- Takahashi duality between the category of Hilbert bundles on (different) compact Hausdorff spaces and the category of Hilbert C*-modules over (different) commutative unital C*-algebras ([T1], [T2]).

Gel’fand-Naimark duality allows us to consider the non-commutative C*-algebras as non-commutative topological spaces, while Serre-Swan and Takahashi dualities allow us to consider Hilbert C*-modules as non-commutative Hilbert bundles.

In order to define “non-commutative manifolds”, we need categorical dualities between categories of manifolds and suitable categories constructed out of Abelian C*-algebras of functions over the manifolds.

A complete answer to the characterization of non-commutative manifolds is not yet known, but (at least in the case of compact finite-dimensional orientable Riemannian spin manifolds) the notion of Connes’ spectral triples and Connes-Rennie-Varilly reconstruction theorem ([C1], [C2], [R], [RV1], [RV2]) provide an appropriate starting point, suggesting to identify the objects of our non-commutative category with Connes’ spectral triples.

A (compact) **spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ is given by:

- a unital pre-C*-algebra $\mathcal{A}$;
- a (faithful) representation $[D, \pi(a)]_{-} \in \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$;
- a (generally unbounded) self-adjoint operator D on $\mathcal{H}$, called the Dirac operator, such that:
    a) the resolvent $(D-\lambda)^{-1}$ is a compact operator, $\forall \lambda \in \mathbb{C} \setminus \mathbb{R}$,

b) $[D,\pi(a)]_{-} \in \mathcal{B}(\mathcal{H})$, for every $a \in \mathcal{A}$, where $[x,y]_{-} := xy - yx$ denotes the commutator of $x,y \in \mathcal{B}(\mathcal{H})$.

Spectral triples with Abelian algebra $\mathcal{A}$ are naturally constructed from spinorial compact manifolds taking $\mathcal{A} = C^{\infty}(\Gamma)$, $\mathcal{H} = L^2(S(M))$, where $S(M)$ is a spinor bundle and $\not{D}$ the Atiyah-Singer Dirac operator. A theorem recently proved by A. Connes allows to reconstruct compact spin manifolds from commutative spectral triples that satisfies a number of additional technical requirements.

Since Connes' reconstruction theorem ([C1], [C2]) justified the fact that spectral triples are a possible definition for "non-commutative" compact finite-dimensional orientable Riemannian spin manifolds, it is our purpose to try to define suitable notions of morphisms and categories for these spectral triples in such a way that categorical dualities for noncommutative manifolds can be accomplished.

There are actually several possible notions of morphism, according to the amount of "background structure" of the manifold that we would like to see preserved:

- the metric, globally (isometries),

- the metric, locally (totally geodesic maps, in the differentiable case),

- the Riemannian structure,

- the differentiable structure.

In [BCL1] we proposed a notion of "totally-geodesic-spin" morphisms that (when applied to the case of spectral triples arising from the Atiyah-Singer Dirac operator) manifest a strong "spinorial rigidity", namely a morphism between two spectral triples

$(\mathcal{A}_j, \mathcal{H}_j, D_j)$, $j = 1,2$, is a pair (ϕ,Φ) with $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ and $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\Phi(ax) = \phi(a)\Phi(x)$ and $\Phi \circ D_1 = D_2 \circ \Phi$ for $a \in \mathcal{A}_1$ and $x \in \mathcal{H}_1$.

In [BCL5] we study a less rigid notion of metric morphisms preserving the Connes' distance formula on the set of pure states of the C*-algebras of the spectral triples

In practice, a "metric" morphism between two spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, $j = 1,2$, is a map $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ whose pull-back $\phi^{\bullet} : \mathcal{P}(\mathcal{A}_2) \to \mathcal{P}(\mathcal{A}_1)$, between the set of pure states of the C*-algebras $\mathcal{P}(\mathcal{A}_j)$, $j = 1,2$, is an isometry for the Connes distance defined by $d(\omega_1,\omega_2) = \sup\{|\,\omega_1(a) - \omega_2(a)\,| : \|[D,a]\| \leq 1\}$.

In [BCL5] we also examine the relation between the notions of "metric" and "totally geodesic-spin" morphisms.

These notions of morphism of spectral triples are only tentative: as pointed out by A. Rennie, it is likely that the "correct" definition of morphism will evolve, but it will surely reflect the basic structure suggested here.

Actually, the several notions of morphism of spectral triples described above are not as general as possible. In a wider perspective, a morphism of spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, where $j = 1,2$ might be formalized as a "suitable" functor $\mathfrak{F} : {}_{\mathcal{A}_1}\mathcal{M} \to {}_{\mathcal{A}_2}\mathcal{M}$ between the categories ${}_{\mathcal{A}_j}\mathcal{M}$ of $\mathcal{A}_j$-modules, having "appropriate intertwining" properties with the Dirac operators D_j.

Under some "mild" hypothesis, by Eilenberg-Gabriel-Watt theorem, any such functor is obtained by "tensorization" by a bimodule. These bimodules, suitably equipped with spectral data (as in the case of spectral triples), in our opinion, provide a natural setting for a general theory of morphisms of non-commutative spaces we are looking for.

In order to investigate more precisely the nature of bimodules as morphisms, we have turned our attention to a topological form of "categorification" where such bimodules appear naturally in a strict category (as non-diagonal blocks in a C*-category).

Categorification is the term, introduced by L. Crane-D. Yetter [CY], to denote the generic process to substitute ordinary algebraic structures with categorical counterparts. The term is now mostly used to denote a wide area of research (see J. Baez - J. Dolan [BD]) whose purpose is to use higher order categories to define categorial analogs of algebraic structures. This **vertical categorification** is usually done by promoting sets to categories, functions to functors, hence replacing a category with a 2-category and so on.

There are also more "trivial" forms of **horizontal categorification** in which ordinary algebraic associative structures are interpreted as categories with only one object and suitable analog categories with more than one object are defined.

In this case the passage is from endomorphisms of a single object to morphisms between different objects:

Monoids	Small Categories (Monoidoids)
Groups	Groupoids
Associative unital Rings	Ringoids
Associative unital Algebras	Algebroids
Unital C*-algebras	C*-categories (C*-algebroids)

It is an extremely interesting future topic of investigation to discuss the interplay between ideas of categorification and non-commutative geometry.

As a first step in the development of a "categorical non-commutative geometry", we have been looking at a possible "horizontal categorification" of Gel'fand duality.

In the setting of C*-categories, we [BCL3] provide a definition of "spectrum" of a commutative full C*-category (that we call spaceoid) as a one dimensional unital Fell-bundle over a suitable groupoid (equivalence relation) and we prove a categorical Gel'fand duality theorem generalizing the usual Gel'fand duality between the categories of Abelian C*-algebras and compact Hausdorff spaces.

On one side of the extended duality we have a "horizontal categorification" of the notion of commutative C*-algebra, namely that of "commutative full C*-category" whilst the corresponding replacement of spaces, the "spaceoids", are supposed to parametrize their spectra.

As a byproduct, we [BCL4] also obtain the following spectral theorem for imprimitivity bimodules over Abelian C*-algebras: every such bimodule is obtained by "twisting" (by the 2 projection homeomorphisms) the symmetric bimodule of sections of a unique Hermitian line bundle over the graph of a unique homeomorphism between the spectra of the two C*-algebras.

Rather surprisingly, as far as we know, our findings have not been discussed before, despite the fact that (mostly highly non-commutative) C*-categories have been somehow intensively exploited over the last 30 years in several areas of research, including Mackey induction, superselection structure in quantum field theory, abstract group duality, subfactors and the Baum-Connes conjecture.

Once we have a running definition of "spaceoid", it seems quite challenging in the next step to look for some natural occurrence of the notion of spaceoid in other contexts. For instance, we are not aware of any connection with the powerful concepts that have been introduced in algebraic topology to date. Also, the appearance of bundles in the structure of the spectrum suggests an intriguing connection to local gauge theory but we have not developed these ideas yet.

More structure is expected to emerge when our C*-categories are equipped with a differentiable/metric structure via Dirac operators. In particular one might be interested to define spectral triples on C*-categories and use them to provide a horizontal categorification of A. Connes non-commutative geometry as a first step before addressing a full (vertical) categorification of non-commutative geometry.

For more details on this account, a reader is recommended to read the survey paper [BCL2].

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2. ผลงานวิจัยที่ได้รับ แบ่งออกเป็นสองส่วนดังนี้

2.1 ผลงานวิจัยที่ได้รับการตีพิมพ์แล้ว

1. A pointwise bound for rotation-invariant holomorphic functions that are square-integrable with respect to a Gaussian measure, Taiwanese J. Math. 11 (2007), no.5, 1443-1455. (ISI Impact Factor =0.444)
2. Rotation-invariant Segal-Bargmann transform, East-West J. of Mathematics – a special volume 2007, 159-167.
3. Non-Commutative Geometry, Categories and Quantum Physics, East-West J. of Mathematics - a special volume 2007, 213-259.

2.2 ผลงานวิจัยที่ส่งไปยังวารสารเพื่อการตีพิมพ์และรอการตอบรับ

1. A horizontal categorification of Gel'fand duality, submitted to Advances in Mathematics (impact factor 2007 = 1.235).
2. A spectral theorem for imprimitivity C*-bimodule, submitted to Expositiones Mathematicae (impact factor 2007 = 0.311).
3. A remark on Gel'fand duality for spectral triples, submitted to Journal of Korean Mathematical Society (impact factor 2007 = 0.171).

3. กิจกรรมอื่น ๆที่เกี่ยวข้อง

3.1 การนำเสนอผลงานในที่ประชุมวิชาการ

รองศาสตราจารย์ ดร.วิชาญ ลิ่วกีรติยุตกุล อ.ดร.อารีรักษ์ ชัยวร และ Dr.Paolo Bertozzini ได้เข้าร่วมประชุมวิชาการทั้งในประเทศและต่างประเทศและได้นำเสนอผลงานในที่ประชุมต่าง ๆ ดังนี้

- "Rotation-invariant functions in the Segal-Bargmann space", AMS 2006 Spring Central Sectional Meeting, University of Notre Dame, April 2006.
- "Non-commutative Geometry and Categories", ICAA 2006, International Conference on Mathematical Analysis and Its Applications, Chulalongkorn University, Montien Hotel, Bangkok, Thailand, 24 May 2006.
- "Categories in Non-commutative Geometry", Center for Mathematical Physics, University of Queensland, Brisbane, Australia, 8 June 2006.
- A talk at the TRF annual meeting, Petchaburi, October 2006.
- "Categorical Non-commutative Geometry", Colloquium, Mathematics Science Institute, Australian National University, Canberra, Australia, 19 October 2006.

- "Horizontal Categorification of Gel'fand Theory and Categorical Non-commutative Geometry", Australian Category Seminar, Mathematics Department, Macquarie University, Sydney, Australia, 25 October 2006.
- "C*-categories, Spaceoids and Categorical Non-commutative Geometry", Center for Mathematics Physics, University of Queensland, Brisbane, Australia, 1 November 2006.
- "A Panorama of Non-commutative Geometry", Department of Mathematics, Universita' di Bologna, Italy, 21 March 2007.
- "Categorical Non-commutative Geometry, Gel'fand Duality and Physics", Mathematical Physics Sector, SISSA, International School for Advanced Studies, Trieste, Italy, 22 March 2007.
- "Categorification of Gel'fand Duality and Non-commutative Geometry", Department of Mathematics, Universita' di Roma II "Tor Vergata", Italy, 28 March 2007.
- "An Overview of Categorical Non-commutative Geometry", Department of Mathematics, Politecnico di Milano, Italy, 23 April 2007.
- "A Pointwise Bound for Rotation-Invariant Holomorphic Functions that are Square Integrable with Respect to a Gaussian Measure", ICMA-MU 2007, International Conference on Mathematics and Applications, Mahidol University, Century Park Hotel, Bangkok, Thailand, 15 August 2007.
- "Non-commutative Geometry, Categories and Quantum Physics", ICMA-MU 2007, International Conference on Mathematics and Applications, Mahidol University, Century Park Hotel, Bangkok, Thailand, 15 August 2007.
- "C*-categories, Non-commutative Geometry and Quantum Physics", Australian Mathematical Society Meeting 2007, La Trobe University, Melbourne, 27 September 2007.
- "Categorical Non-commutative Geometry and Quantum Theory", Paul Baum Fest, School of Mathematical and Physical Sciences, University of Newcastle, Australia, 2 October 2007.
- "Functional Analysis with Respect to Heat Kernel Measure", National Workshop and Conference on Mathematics, Silpakorn University, 5 October 2007.
- "Spectral Theorem for Commutative Full C*-categories", School of Mathematical and Physical Sciences, University of Newcastle, Australia, 11 October 2007.

- "Categories of Spectral Geometries", Second Workshop on Categories Logic and Physics, Imperial College, London, UK, 14 May 2008.
- "Spectral Theory for C*-categories and Categorical Non-commutative Geometry", University of Cardiff, UK, 15 May 2008.
- "Categories in Non-commutative Geometry: an Overview", University of Swansea, UK, 16 May 2008.
- "Categorical Spectral Geometries and Modular Quantum Gravity", CPT, Marseille, France, 23 May 2008.
- "Spectral Spaceoids and Categories of Spectral Geometries", Department of Mathematics, Universita' di Roma II "Tor Vergata", Italy, 28 May 2008.
- "Modular Spectral Geometries for Algebraic Quantum Gravity", QG2-2008 Quantum Geometry and Quantum Gravity Conference, University of Nottingham, UK, 1 July 2008.

3.2 การผลิตบัณฑิตระดับปริญญาตรี โท เอก

งานวิจัยนี้ได้ผลิตบัณฑิตทั้งในระดับปริญญาตรี โท และ เอก รวม 12 คน ดังนี้

1. นางสาวอารีรักษ์ แก้วเทพ
   วิทยาศาสตร์ดุษฎีบัณฑิต จุฬาลงกรณ์มหาวิทยาลัย 2549
   วิทยานิพนธ์เรื่อง "A pointwise bound for rotation-invariant holomorphic functions that are square-integrable with respect to a Gaussian measure"
2. นายบุญยงค์ ศรีพลแผ้ว
   วิทยาศาสตร์มหาบัณฑิต จุฬาลงกรณ์มหาวิทยาลัย 2548
   วิทยานิพนธ์เรื่อง "Segal-Bargmann transform on spheres"
3. นางสาวอารียา สาระรักษ์
   วิทยาศาสตร์มหาบัณฑิต มหาวิทยาลัยธรรมศาสตร์ 2548
   วิทยานิพนธ์เรื่อง "Real and Complex Structures in Modules"
4. นางสาวสุภาพร ถีสูงเนิน
   วิทยาศาสตร์มหาบัณฑิต มหาวิทยาลัยธรรมศาสตร์ 2548
   วิทยานิพนธ์เรื่อง "Algebraic aspects of second quantization in bimodules"
5. นายสารัตน์ ศิลปวงษา
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   วิทยานิพนธ์เรื่อง "Heat equation on a compact Lie group"

6. นายประพันธ์พงศ์ พงศ์ศรีเอี่ยม
   วิทยาศาสตร์มหาบัณฑิต จุฬาลงกรณ์มหาวิทยาลัย 2550
   วิทยานิพนธ์เรื่อง “Action by automorphisms on the dual of a group”
7. นายสุทธิพงษ์ แก้วอำไพ
   วิทยาศาสตร์มหาบัณฑิต มหาวิทยาลัยธรรมศาสตร์ 2550
   วิทยานิพนธ์เรื่อง “Krein C*-modules”
8. นายพิชญ์กิตติ บรรณางกูร
   วิทยาศาสตร์มหาบัณฑิต จุฬาลงกรณ์มหาวิทยาลัย 2552
   วิทยานิพนธ์เรื่อง “Spectral theory for a commutative Krein C*-algebra”
9. นายเกษมสันต์ รุทธิ์อมร
   วิทยาศาสตร์มหาบัณฑิต มหาวิทยาลัยธรรมศาสตร์ 2552
   วิทยานิพนธ์เรื่อง “Krein C*-categories”
10. นายนพคุณ สุทธิจิตรานนท์
    วิทยาศาสตร์บัณฑิต มหาวิทยาลัยมหิดล 2550
    ปริญญานิพนธ์เรื่อง “Higher C*-categories”
11. นายเรวัต ถนัดกิจหิรัญ
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    ปริญญานิพนธ์เรื่อง “Tensor product of Krein spaces”
12. นายอมรสิทธิ์ อัจฉริยบดี
    วิทยาศาสตร์บัณฑิต จุฬาลงกรณ์มหาวิทยาลัย 2552
    ปริญญานิพนธ์เรื่อง “States for Krein C*-algebras”

A pointwise bound for rotation-invariant holomorphic functions that are square-integrable with respect to a Gaussian measure

Taiwanese Journal of Mathematics 11 (2007), no.5, 1443-1455

MR2368662 (2008k:46082) 46E20 (43A32 81S30)
Kaewthep, Areerak (THA-CHULS); **Lewkeeratiyutkul, Wicharn** (THA-CHULS)
A pointwise bound for rotation-invariant holomorphic functions that are square integrable with respect to a Gaussian measure. (English summary)
Taiwanese J. Math. **11** (2007), *no. 5,* 1443–1455.

The objects of interest in this article are arbitrary functions $F \in \mathcal{H}L^2(\mathbf{C}^d, \mu_t)$ (the Segal-Bargmann space of holomorphic functions that are square integrable with respect to the Gaussian measure μ_t on $\mathbf{C}^d$ with density $(\pi t)^{-d} e^{-|z|^2/t}$ with respect to Lebesgue measure where $t > 0$ and $z \in \mathbf{C}^d$) which are invariant under the action of the special orthogonal group $\mathrm{SO}(d, \mathbf{C})$, namely $F(Az) = F(z)$ for all $z \in \mathbf{C}^d$ and all $A \in \mathrm{SO}(d, \mathbf{C})$. The authors establish two pointwise bounds for such functions. For their first bound they take the well-known pointwise bound for *any* $F \in \mathcal{H}L^2(\mathbf{C}^d, \mu_t)$,

(1) $$|F(z)|^2 \leq e^{|z|^2/t} \|F\|^2,$$

where $\|F\|$ denotes the norm in $L^2(\mathbf{C}^d, \mu_t)$, and then they minimize this bound over the orbits of the $\mathrm{SO}(d, \mathbf{C})$ action. While this gives the improved bound

(2) $$|F(z)|^2 \leq e^{|(z,z)|/t} \|F\|^2$$

for any $\mathrm{SO}(d, \mathbf{C})$ invariant F, it is not optimal. Here $(z, z) = z_1^2 + \cdots + z_d^2$ for $z = (z_1, \ldots, z_d)$. The straightforward inequality $|(z, z)| \leq |z|^2$ shows that (2) is a better bound than (1). In fact, the authors show that (2) is not optimal by proving the major result of the paper, which is the following pointwise bound for any $\mathrm{SO}(d, \mathbf{C})$ invariant F in the Segal-Bargmann space:

(3) $$|F(z)|^2 \leq \frac{C e^{|(z,z)|/t}}{1 + |(z, z)|^{(d-1)/2}} \|F\|^2,$$

where the constant C depends only on the dimension d and the parameter t. This is done by estimating the reproducing kernel function of the subspace of $\mathrm{SO}(d, \mathbf{C})$ invariant functions of the Segal-Bargmann space. In a remark at the end of the paper, the authors indicate why the inequality (3) is sharp. However, they do not identify the optimal constant C. They do sketch a proof for showing that the functional form of the right side of (3) is sharp.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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Kaewthep, Areerak; Lewkeeratiyutkul, Wicharn

A pointwise bound for rotation-invariant holomorphic functions that are square integrable with respect to a Gaussian measure. (English)

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The Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$ is the space of holomorphic functions on $\mathbb{C}^d$ that are square-integrable with respect to the Gaussian measure $\mu_t(z)dz = (\pi t)^{-d} e^{-|z|^2/t}\, dz$. Here t is a fixed positive real number. In this paper, the authors consider the subspace of the standard Segal-Bargmann space that is invariant under the special orthogonal group $\mathrm{SO}(d)$. The goal of the paper is to compare two bounds for functions in this space, a simple bound obtained by minimizing the standard bounds in the full Segal-Bargmann space over the orbits of the group, and a sharp bound obtained by directly estimating the reproducing kernel for the subspace. The authors show that the sharp bounds are polynomially better than the simple bounds, with the difference between the two growing larger and larger as the dimension d goes to infinity. It is well known that for any function $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, we have the pointwise bound

$$|F(z)|^2 \leq e^{|z|^2/t} \|F\|^2_{L^2(\mathbb{C}^d,\mu_t)} \quad (z \in \mathbb{C}^d). \tag{1}$$

Now suppose that F is invariant under the action of $\mathrm{SO}(d)$, and therefore, by analytic continuation, under the action of $\mathrm{SO}(d, \mathbb{C})$. By minimizing (1) on each orbit, for any $\mathrm{SO}(d)$-invariant function F in the Segal-Bargmann space, we obtain the preliminary estimate

$$|F(z)|^2 \leq e^{|(z,z)|^2/t} \|F\|^2_{L^2(\mathbb{C}^d,\mu_t)} \quad (z \in \mathbb{C}^d). \tag{2}$$

Since $|(z, z)| \leq |z|^2$, this is already an improvement over the pointwise bound in (1). The $\mathrm{SO}(d)$ invariance means that F is determined by its values on $\{(z, 0, \dots, 0)\} \simeq \mathbb{C}^1$. (By holomorphy, F is determined by its values on $\mathbb{R}^d$, then any point in $\mathbb{R}^d$ can be rotated into $\mathbb{R}^1$.) Conversely, any even holomorphic function on $\mathbb{C}^1$ has an extension to an $\mathrm{SO}(d)$-invariant function on $\mathbb{C}^d$. Then the space of $\mathrm{SO}(d)$-invariant functions in the Segal-Bargmann space over $\mathbb{C}^d$ can be expressed as an L^2-space of holomorphic functions on $\mathbb{C}^1$, with some non-Gaussian measure. By estimating the reproducing kernel for this space, the authors obtain a sharp bound for an $\mathrm{SO}(d)$-invariant function F in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, which is polynomially better than (2). This bound is described in the following theorem: There exists a constant C, depending only on d and t, such that for each $\mathrm{SO}(d)$-invariant function F in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$,

$$|F(z)|^2 \leq \frac{Ce^{|(z,z)|/t}}{1 + |(z, z)|^{(d-1)/2}} \|F\|^2_{L(\mathbb{C}^d,\mu_t)} \quad (z \in \mathbb{C}^d).$$

This bound is sharp.

Vasily A. Chernecky (Odessa)

Keywords : Segal-Bargmann space; rotation-invariant function; Gaussian measure; Bargmann's pointwise bound
Classification :
∗46E20 Hilbert spaces of functions defined by smoothness properties
43A32 Other transforms and operators of Fourier type

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A POINTWISE BOUND FOR ROTATION-INVARIANT HOLOMORPHIC FUNCTIONS THAT ARE SQUARE INTEGRABLE WITH RESPECT TO A GAUSSIAN MEASURE

Areerak Kaewthep and Wicharn Lewkeeratiyutkul

Abstract. We consider the subspace of Segal-Bargmann space which is invariant under the action of the special orthogonal group. We establish a pointwise bound for a function in this space which is polynomially better than the pointwise bound for a function in the Segal-Bargmann space.

1. Introduction

The Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$ is the space of holomorphic functions on $\mathbb{C}^d$ that are square-integrable with respect to the Gaussian measure $\mu_t(z)\, dz = (\pi t)^{-d} e^{-|z|^2/t}\, dz$, where $|z|^2 = |z_1|^2 + \cdots + |z_d|^2$. Here t is a fixed positive real number. See [1, 5, 7, 8, 10, 11, 15], for details about the importance of this space.

Various generalizations of the Segal-Bargmann space have been considered. An important part of the study of such generalizations is to obtain sharp pointwise bounds on the functions. (See, for example, [2, 4, 9, 12, 14]). Such bounds amount to estimates for the reproducing kernel on the diagonal.

In this paper, we consider the subspace of the standard Segal-Bargmann space that is invariant under the special orthogonal group. The goal of the paper is to compare two bounds for functions in this space, a simple bound obtained by minimizing the standard bounds in the full Segal-Bargmann space over the orbits of the group, and a sharp bound obtained by directly estimating the reproducing kernel for the subspace. We show that the sharp bounds are polynomially better than the simple bounds, with the difference between the two growing larger and larger as the dimension d goes to infinity.

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Key words and phrases: Segal-Bargmann, Rotation-Invariant, Gaussian measure, Pointwise bound.
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This analysis is motivated in part by a comparison of [3] and [9]. In [3], Driver obtains (among other things) bounds for a *generalized* Segal-Bargmann space by representing it as the subspace of a certain infinite-dimensional *standard* Segal-Bargmann space that is invariant under a certain group action. (See also [7, 16, 13]). Meanwhile, in [9], Hall obtains sharp bounds for the relevant generalized Segal-Bargmann space by directly estimating the reproducing kernel. The difference between the two bounds is significant; the sharper bounds of [9] are essential, for example, in the analysis in [14].

It is well-known that for any function $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, we have the pointwise bound

$$|F(z)|^2 \leq e^{|z|^2/t} \|F\|^2_{L^2(\mathbb{C}^d,\mu_t)} \qquad (z \in \mathbb{C}^d). \tag{1}$$

Now suppose that F is invariant under the action of $SO(d)$, and therefore, by analytic continuation, under the action of $SO(d, \mathbb{C})$. By minimizing (1) on each orbit, for any $SO(d)$-invariant function F in the Segal-Bargmann space, we obtain the preliminary estimate

$$|F(z)|^2 \leq e^{|(z,z)|/t} \|F\|^2_{L^2(\mathbb{C}^d,\mu_t)} \qquad (z \in \mathbb{C}^d), \tag{2}$$

where $(z, z) = z_1^2 + \cdots + z_d^2$. Since $|(z, z)| \leq |z|^2$, this is already an improvement over the pointwise bound in (1).

The $SO(d)$ invariance means that F is determined by its values on $\{(z, 0, ..., 0)\} \simeq \mathbb{C}^1$. (By holomorphicity, F is determined by its values on $\mathbb{R}^d$, then any point in $\mathbb{R}^d$ can be rotated into $\mathbb{R}^1$.) Conversely, any even holomorphic function on $\mathbb{C}^1$ has an extension to an $SO(d)$-invariant function on $\mathbb{C}^d$. Then the space of $SO(d)$-invariant functions in the Segal-Bargmann space over $\mathbb{C}^d$ can be expressed as an L^2-space of holomorphic functions on $\mathbb{C}^1$, with some non-Gaussian measure. By estimating the reproducing kernel for this space, we obtain a sharp bound for an $SO(d)$-invariant function F in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, which will be polynomially better than (2). This bound is described in the following theorem.

Theorem 1. *There exists a constant C, depending only on d and t, such that for each $SO(d)$-invariant function F in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, we have*

$$|F(z)|^2 \leq \frac{C\, e^{|(z,z)|/t}}{1 + |(z, z)|^{(d-1)/2}} \|F\|^2_{L(\mathbb{C}^d,\mu_t)} \qquad (z \in \mathbb{C}^d).$$

2. $SO(d, \mathbb{C})$-Invariant Measure on a Complex Sphere

Denote by $SO(d, \mathbb{C})$ the set of $d \times d$ complex orthogonal matrices with determinant one. Elements of $SO(d, \mathbb{C})$ preserve the bilinear form $(\cdot, \cdot)$ on $\mathbb{C}^d$ defined by

$$(z, \xi) = z_1\xi_1 + z_2\xi_2 + \cdots + z_d\xi_d$$

for any $z, \xi \in \mathbb{C}^d$. For each $w \in \mathbb{C}$, we define

$$S_w = \{z \in \mathbb{C}^d \mid (z, z) = w^2\}.$$

In particular, $S_0 = \{z \in \mathbb{C}^d \mid (z, z) = 0\}$. Using the nondegeneracy of the form $(\cdot, \cdot)$, it is not hard to show that $SO(d, \mathbb{C})$ acts transitively on S_w for all $w \in \mathbb{C} - \{0\}$. Moreover, let

$$S = \{z \in \mathbb{C}^d \mid (z, z) \in (-\infty, 0]\}.$$

By the Implicit Function Theorem, $S_0 - \{0\}$ and $S - S_0$ are submanifolds of $\mathbb{C}^d$ with dimensions less than the dimension of $\mathbb{C}^d$. This implies that S has Lebesgue measure zero.

Denote by $\mathbb{H}^+ = \{z \in \mathbb{C} \mid \Re(z) > 0\}$ the open right-half plane of $\mathbb{C}$. Define $\Psi\colon \mathbb{C}^d - S \to \mathbb{H}^+ \times S_1$ by

$$\Psi(z) = (w, z')$$

where $w = |(z, z)|^{1/2} e^{i\frac{\theta}{2}}$, θ is the principal value of $\arg(z, z)$, $\theta \in (-\pi, \pi)$, and $z' = \frac{z}{w}$. It is easy to verify that Ψ is a continuous bijective map whose inverse is $\Psi^{-1}(w, z') = wz'$. We can think of this map as a "complex polar form" of an element in $\mathbb{C}^d$ that is not in S. Let m be Lebesgue measure on $\mathbb{C}^d$ and m_* the Borel measure on $\mathbb{H}^+ \times S_1$ such that $m_*(E) = m(\Psi^{-1}(E))$. The next theorem shows that the pushed-forward measure m_* on $\mathbb{H}^+ \times S_1$ can be written as a product measure $m_* = \rho \times \alpha$, where ρ is a measure on $\mathbb{H}^+$ defined by

$$\rho(A) = \int_A |w|^{2d-2} dw$$

and α is an $SO(d, \mathbb{C})$-invariant Borel measure on S_1.

Theorem 2. *There is an $SO(d, \mathbb{C})$-invariant Borel measure α on S_1 such that $m_* = \rho \times \alpha$. If f is a Borel function on $\mathbb{C}^d$ such that $f \geq 0$ or $f \in L^1(\mathbb{C}^d, m)$, then*

$$\int_{\mathbb{C}^d} f(z)\, dz = \int_{\mathbb{C}} \int_{S_1} f(wz')\, d\alpha(z')\, |w|^{2d-2}\, dw, \tag{3}$$

where dw denotes the two-dimensional Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$.

Proof. Since S has Lebesgue measure zero, (3) is equivalent to

$$\int_{\mathbb{C}^d - S} f(z)\, dz = \int_{\mathbb{C}} \int_{S_1} f(wz')\, d\alpha(z')\, |w|^{2d-2}\, dw. \tag{4}$$

First, we need to construct α. If E is a Borel set in S_1, let E_1 be the set in $\mathbb{C}^d$ given by

$$E_1 = \{wz' \mid w \in \mathbb{H}^+, |w| < 1, z' \in E\}.$$

If (4) is to hold when $f = \chi_{E_1}$, we must have

$$m(E_1) = \frac{1}{2}\int_{D_1}\int_E d\alpha(z')\,|w|^{2d-2}\,dw = \frac{\pi}{2d}\,\alpha(E).$$

Hence, for any Borel set E in S_1, we define

$$\alpha(E) = \frac{2d}{\pi}\,m(E_1).$$

Since the map $E \mapsto E_1$ takes Borel sets to Borel sets and commutes with unions, intersections and complements, α is a Borel measure on S_1. If E is a Borel set in S_1 and $A \in SO(d,\mathbb{C})$ then

$$\alpha(AE) = \frac{2d}{\pi}\,m((AE)_1) = \frac{2d}{\pi}\,m(A(E)_1) = \frac{2d}{\pi}\,\det(A)m(E_1) = \alpha(E),$$

where $\det(A)$ is the determinant of A over $\mathbb{R}$, which is 1. Hence α is $SO(d,\mathbb{C})$-invariant. Following a similar argument to the real polar coordinates formula (see, e.g., [6, Theorem 2.49]) we can show that $m_* = \rho \times \alpha$ on all Borel sets. Hence equation (4) holds when f is a characteristic function of a Borel set and it follows for general f by the usual linearity and approximation argument. ■

The measure α in Theorem 2 is uniquely determined and can be given explicitly. There is a diffeomorphism between the tangent bundle $T(S^{d-1})$ of the real unit sphere S^{d-1} and the complex unit sphere S_1 given by

$$\mathbf{a}(\mathbf{x},\mathbf{p}) = \cosh(p)\,\mathbf{x} + \frac{i}{p}\sinh(p)\,\mathbf{p} \quad \text{for any } \mathbf{x} \in S^{d-1} \text{ and } \mathbf{x}\cdot\mathbf{p} = 0$$

where $p = |\mathbf{p}|$. See [15] for more details. Using these coordinates, we can write the measure α explicitly as follows:

Lemma 3. *The measure α is given by*

$$\alpha(z) = a_0\Big(\frac{\sinh 2p}{2p}\Big)^{d-2} 2^{d-1}\,d\mathbf{p}\,d\mathbf{x}.$$

Here $z = \mathbf{a}(\mathbf{x},\mathbf{p})$, a_0 is a constant, $d\mathbf{x}$ is the surface area measure on S^{d-1} and $d\mathbf{p}$ is Lebesgue measure on $\mathbb{R}^d$.

Proof. The measure α and the measure $\big(\frac{\sinh 2p}{2p}\big)^{d-2} 2^{d-1}\,d\mathbf{p}\,d\mathbf{x}$ are both $SO(d,\mathbb{C})$-invariant(Lemma 3 of [15]) and finite on compact sets. Thus, by Theorem 8.36 of [17], these two measures must agree up to a constant. ■

3. Pointwise Bound for a Function in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}}$

Denote by $\mathcal{H}(\mathbb{C}^d)^{\mathcal{O}}$ the space of $SO(d,\mathbb{C})$-invariant holomorphic functions on $\mathbb{C}^d$, i.e., the space of holomorphic functions f for which $f(Az) = f(z)$ for all $z \in \mathbb{C}^d$ and $A \in SO(d,\mathbb{C})$. In this section, we will establish a pointwise bound for a function in the space $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}} := \mathcal{H}(\mathbb{C}^d)^{\mathcal{O}} \cap L^2(\mathbb{C}^d, \mu_t)$.

By minimizing over each orbit, we obtain the following pointwise bound:

Proposition 4. *For any $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}}$ and for any $z \in \mathbb{C}^d$*

$$|F(z)|^2 \leq e^{|(z,z)|/t} \|F\|^2_{L^2(\mathbb{C}^d,\mu_t)}. \tag{5}$$

Proof. Note that $|(z,z)| = |(Az,Az)| \leq |Az|^2$ for any $z \in \mathbb{C}^d$ and $A \in SO(d,\mathbb{C})$. If $z \notin S_0$, we have that $(\sqrt{(z,z)}, 0, \ldots, 0) \in \{Az \mid A \in SO(d,\mathbb{C})\}$, because $SO(d,\mathbb{C})$ acts transitively on S_w where $w = \sqrt{(z,z)}$, and thus

$$|(z,z)| = \inf\left\{|Az|^2 : A \in SO(d,\mathbb{C})\right\}.$$

But S_0^c is dense in $\mathbb{C}^d$, so this equation is also true for all $z \in \mathbb{C}^d$. This immediately gives (5). ∎

This simple technique yields an improvement from the Bargmann's pointwise bound (1). However, we will establish a polynomially-better bound than the bound in (5). Our strategy is to construct a non-Gaussian measure λ on $\mathbb{C}$ so that we can express $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}}$ in terms of the space $\mathcal{H}L^2(\mathbb{C}, \lambda)^e$ of holomorphic even functions on $\mathbb{C}$ that are square-integrable with respect to λ and then estimate the reproducing kernel of the latter space.

Proposition 5. *Let $\mathcal{H}(\mathbb{C})^e$ be the set of all holomorphic even functions on $\mathbb{C}$. Then for any $d \geq 2$, the map $\phi\colon \mathcal{H}(\mathbb{C}^d)^{\mathcal{O}} \to \mathcal{H}(\mathbb{C})^e$ defined by*

$$\phi(f)(x) = f(x, 0, \ldots, 0),$$

for all $f \in \mathcal{H}(\mathbb{C}^d)^{\mathcal{O}}$ and all $x \in \mathbb{C}$, is a linear isomorphism whose inverse is given by

$$\psi(g)(z) = g\left(\sqrt{(z,z)}\right)$$

for all $g \in \mathcal{H}(\mathbb{C})^e$ and all $z \in \mathbb{C}^d$.

Note that since g is even, the value of $\psi(g)(z)$ is independent of the choice of square root of (z,z). Again because g is even, $\psi(g)$ will be given by a convergent power series in integer powers of $(z,z) = z_1^2 + \cdots + z_d^2$, and therefore $\psi(g)$ will be holomorphic on $\mathbb{C}^d$.

Proof. It is clear that ϕ is a linear map and $\phi(f)$ is a holomorphic function on $\mathbb{C}$ for any $f \in \mathcal{H}(\mathbb{C}^d)^{\mathcal{O}}$. Moreover, $\phi(f)$ is even since $A(-w,0,\ldots,0) = (w,0,\ldots,0)$ for any $w \in \mathbb{C}$, where $A = \mathrm{diag}(-1,-1,1,1,\ldots,1)$.

On the other hand, ψ is a linear map and $\psi(g)$ is holomorphic on $\mathbb{C}^d$ for each $g \in \mathcal{H}(\mathbb{C})^e$. Since the bilinear form is preserved under the action of the orthogonal group, $\psi(g)$ is $SO(d,\mathbb{C})$-invariant. It is straightforward to verify that $\phi\circ\psi = \mathrm{I}_{\mathcal{H}(\mathbb{C})^e}$ and $\psi\circ\phi = \mathrm{I}_{\mathcal{H}(\mathbb{C}^d)^{\mathcal{O}}}$, so the theorem is proved. ■

Henceforth, we will choose an argument of $w \in \mathbb{C}$ so that $-\pi < \arg(w) \leq \pi$. Denote by $\mathcal{B}_d$ the Borel σ-algebra in $\mathbb{C}^d$ and by $\mathcal{B}$ the Borel σ-algebra in $\mathbb{C}$. Define $\Phi_i\colon (\mathbb{C}^d, \mathcal{B}_d, \mu_t) \to (\mathbb{C}, \mathcal{B})$, $i = 1,2$ to be the branch of $\sqrt{(z,z)}$ with smaller and larger argument, respectively, and for each $E \in \mathcal{B}$ define

$$\lambda_i(E) = \mu_t(\Phi_i^{-1}(E)).$$

Then define $\lambda = (\lambda_1 + \lambda_2)/2$. It is easy to check that λ is a Borel measure on $\mathbb{C}$ and for any measurable function g and any $E \in \mathcal{B}$

$$\int_E g\, d\lambda \;=\; \frac{1}{2}\int_{\Phi_1^{-1}(E)} g\circ\Phi_1\, d\mu_t + \frac{1}{2}\int_{\Phi_2^{-1}(E)} g\circ\Phi_2\, d\mu_t.$$

It is now straightforward to verify that the restriction of ϕ to $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}}$ is a unitary map onto $\mathcal{H}L^2(\mathbb{C}, \lambda)^e$.

Proposition 6. *The measure λ is absolutely continuous with respect to Lebesgue measure on $\mathbb{C}$ with density given by*

$$\Lambda(w) = \frac{|w|^{2d-2}}{(\pi t)^d}\int_{S_1} e^{-|wz|^2/t} d\alpha(z). \tag{6}$$

Proof. If E is a Borel set in $\mathbb{C}$, then by Theorem 2

$$\begin{aligned}\lambda(E) &= \frac{1}{2}\int_{\Phi_1^{-1}(E)} \frac{e^{-|z|^2/t}}{(\pi t)^d}\, dz + \frac{1}{2}\int_{\Phi_2^{-1}(E)} \frac{e^{-|z|^2/t}}{(\pi t)^d}\, dz \\ &= \int_{\mathbb{C}}\int_{S_1} \chi_E(w)\frac{|w|^{2d-2}}{(\pi t)^d}\, e^{-|wz'|^2/t}\, d\alpha(z')\, dw \\ &= \int_E \Lambda(w)\, dw\end{aligned}$$

where Λ is given by (6). ■

Next, we will approximate the density Λ of λ and show that on holomorphic functions, the L^2-norm with respect to λ is equivalent to the L^2-norm with respect to the measure $\beta(w)dw$, where

$$\beta(w) = \frac{e^{-|w|^2/t}}{t\pi}\,|w|^{d-1} \qquad (w \in \mathbb{C}). \tag{7}$$

Proposition 7. *There exist constants* $m, M > 0$*, depending on* d *and* t*, such that the density function* Λ *of* λ *satisfies*

$$m\,\beta(w) \le \Lambda(w) \le M\,\beta(w)$$

for all $w \in \mathbb{C}$ *with* $|w| \ge 1$.

Proof. From Lemma 3, for any $w \in \mathbb{C}$

$$\begin{aligned}
\textstyle\int_{S_1} e^{-|wz|^2/t}\,d\alpha(z) &= a_0 \int_{S^{d-1}} \int_{\mathbf{x}\cdot\mathbf{p}=\mathbf{0}} e^{-|w\,\mathbf{a}(\mathbf{x},\mathbf{p})|^2/t} \Big(\frac{\sinh 2p}{2p}\Big)^{d-2} 2^{d-1}\,d\mathbf{p}\,d\mathbf{x} \\
&= a_d \int_0^\infty e^{-(\cosh 2p)|w|^2/t} \Big(\frac{\sinh 2p}{2p}\Big)^{d-2} 2^{d-1}\,p^{d-2}\,dp \\
&= a_d\, e^{-|w|^2/t} \int_0^\infty e^{-x|w|^2/t}\,(x^2+2x)^{(d-3)/2}\,dx,
\end{aligned}$$

with $a_d = a_0\sigma(S^{d-1})\sigma(S^{d-2})$, where σ is the surface measure. The last equality follows from the change of variables $\cosh 2p = x+1$.

Now, let us consider the case $d \ge 3$. To approximate the above integral, we expand $(x^2+2x)^{d-3}$ using the binomial theorem, apply the inequalities

$$\frac{1}{\sqrt{n}}(\sqrt{a_1}+\cdots+\sqrt{a_n}\,) \le \sqrt{a_1+\cdots+a_n} \le \sqrt{a_1}+\cdots+\sqrt{a_n}$$

to $(x^2+2x)^{(d-3)/2}$ and then use the formula for the Gamma function in order to obtain

$$\frac{1}{\sqrt{d-2}} P\left(\frac{\sqrt{t}}{|w|}\right) \le \int_0^\infty e^{-x|w|^2/t}\,(x^2+2x)^{(d-3)/2}\,dx \le P\left(\frac{\sqrt{t}}{|w|}\right),$$

where

$$P(x) = \sum_{k=0}^{d-3} a_k^{1/2}\Gamma\left(\frac{d-1+k}{2}\right) x^{d-1+k} \quad \text{and} \quad a_k = \binom{d-3}{k} 2^{d-3-k}.$$

This shows that

$$\frac{a_d}{\sqrt{d-2}} P\left(\frac{\sqrt{t}}{|w|}\right) e^{-|w|^2/t} \;\le\; \int_{S_1} e^{-|wz|^2/t}\,d\alpha(z) \;\le\; a_d P\left(\frac{\sqrt{t}}{|w|}\right) e^{-|w|^2/t}.$$

It follows from (6) that

$$\frac{e^{-|w|^2/t}}{t\pi\sqrt{d-2}}Q\left(\frac{|w|}{\sqrt{t}}\right) \;\le\; \Lambda(w) \;\le\; \frac{e^{-|w|^2/t}}{t\pi}Q\left(\frac{|w|}{\sqrt{t}}\right) \tag{8}$$

where

$$Q(x) \;=\; \frac{a_d}{\pi^{d-1}}\sum_{k=0}^{d-3} a_k^{1/2}\Gamma\left(\frac{d-1+k}{2}\right)x^{d-1-k} \;=\; \sum_{k=2}^{d-1} b_k x^k.$$

From this (7) easily follows for the case $d \ge 3$.

Meanwhile in the $d = 2$ case we have

$$\begin{aligned}\int_{S_1} e^{-|wz|^2/t}\,d\alpha(z) &= a_2\, e^{-|w|^2/t}\int_0^\infty \frac{e^{-x|w|^2/t}}{\sqrt{x^2+2x}}\,dx \\ &= \frac{a_2}{\sqrt{2}}\, e^{-|w|^2/t}\int_0^\infty e^{-u}\sqrt{\left(\frac{|w|^2}{tu}-\frac{|w|^2}{2|w|^2+tu}\right)}\,\frac{t}{|w|^2}\,du \\ &\ge \frac{a_2\sqrt{t}}{\sqrt{2}|w|}\, e^{-|w|^2/t}\left(\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\,du - \int_0^\infty \frac{e^{-u}}{\sqrt{2|w|^2/t+u}}\,du\right).\end{aligned}$$

The function

$$\phi(r) = \int_0^\infty \frac{e^{-u}}{\sqrt{r+u}}\,du \qquad (r \ge 0)$$

is a strictly decreasing function. Hence, if we let $\delta = 2/t$ and $\varepsilon = \phi(0) - \phi(\delta)$, then $\phi(0) - \phi\left(2|w|^2/t\right) \ge \phi(0) - \phi(\delta) = \varepsilon$ for any w with $2|w|^2/t \ge \delta$. It follows that

$$\begin{aligned}\Lambda(w) &= \frac{|w|^2}{(\pi t)^2}\int_{S_1} e^{-|wz|^2/t}d\alpha(z) \\ &\ge \frac{\varepsilon a_2}{\pi\sqrt{2t}}\,\frac{e^{-|w|^2/t}}{\pi t}|w|\end{aligned}$$

for any $w \in \mathbb{C}$ with $|w| \ge 1$.

On the other hand,

$$\begin{aligned}\int_{S_1} e^{-|wz|^2/t}\,d\alpha(z) &\le a_2\, e^{-|w|^2/t}\int_0^\infty \frac{e^{-x|w|^2/t}}{\sqrt{2x}}\,dx \\ &= \frac{a_2\sqrt{t\pi}}{\sqrt{2}|w|}\, e^{-|w|^2/t}.\end{aligned}$$

Hence

$$\Lambda(w) \leq \frac{a_2}{\sqrt{2\pi t}} \frac{e^{-|w|^2/t}}{\pi t}|w|. \qquad \blacksquare$$

Corollary 8. *The norms $\|\cdot\|_{L^2(\mathbb{C},\beta)}$ and $\|\cdot\|_{L^2(\mathbb{C},\lambda)}$ are equivalent, i.e., there are constants $k, K > 0$, depending on d and t, such that*

$$k\|f\|_{L^2(\mathbb{C},\beta)} \leq \|f\|_{L^2(\mathbb{C},\lambda)} \leq K\|f\|_{L^2(\mathbb{C},\beta)}, \tag{9}$$

for all $f \in \mathcal{H}L^2(\mathbb{C}, \lambda)$.

Proof. First, we will show that there is a constant $D > 0$, depending on d and t, such that

$$\|f\|^2_{L^2(\mathbb{C},\beta)} \leq D\,\|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)}$$

for any $f \in \mathcal{H}L^2(\mathbb{C}, \lambda)$, where $\mathbb{D} = \{w \in \mathbb{C} : |w| \leq 1\}$.

Let $w \in \mathbb{D}$. Denote by $A(w)$ the annulus $\{z \in \mathbb{C} : 2 \leq |z - w| \leq 3\}$. If f is in $\mathcal{H}L^2(\mathbb{C}, \lambda)$ then a simple power series argument shows that

$$\int_{A(w)} f(v)\,dv = (9\pi - 4\pi)f(w) = 5\pi f(w).$$

This implies that

$$\begin{aligned} |f(w)| &= \frac{1}{5\pi}\Big|\int_{A(w)} f(v)\,dv\Big| \\ &= \frac{1}{5\pi}\Big|\Big\langle \chi_{A(w)}\frac{1}{\Lambda}, f\Big\rangle_{L^2(\mathbb{C}-\mathbb{D},\lambda)}\Big| \\ &\leq \frac{1}{5\pi}\Big\|\chi_{A^*}\frac{1}{\Lambda}\Big\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)}\|f\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)}, \end{aligned}$$

where $A^* = \{z \in \mathbb{C} : 1 < |z| < 4\}$, which contains each $A(w)$, $w \in \mathbb{D}$. It follows that there exists a constant c such that for any $w \in \mathbb{D}$

$$|f(w)| \;\leq\; c\|f\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)}.$$

It now follows from Proposition 7 that

$$\begin{aligned} \int_{\mathbb{C}} |f(w)|^2\beta(w)\,dw &= \int_{\mathbb{D}} |f(w)|^2\beta(w)\,dw + \int_{\mathbb{C}-\mathbb{D}} |f(w)|^2\beta(w)\,dw \\ &\leq c^2\|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)}\int_{\mathbb{D}}\beta(w)\,dw + \frac{1}{m}\|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)} \\ &\leq D\,\|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)} \end{aligned}$$

for some constant $D > 0$ depending on d and t. This gives the first inequality in (9). The second inequality in (9) can be proved in the same way. ■

Having established Corollary 8, it remains only to obtain pointwise bounds for elements in $\mathcal{H}L^2(\mathbb{C},\lambda)$. We do this by reducing to the standard Segal-Bargmann space (if d is odd) or to the space $\mathcal{H}L^2\big(\mathbb{C},(t\pi)^{-1}|w|e^{-|w|^2/t}\,dw\big)$ (if d is even). We now establish pointwise bound in the latter space.

Lemma 9. *The set* $\left\{\frac{w^n}{\left(t^{(2n+1)/2}\Gamma(n+\frac{3}{2})\right)^{1/2}}\right\}_{n=0}^{\infty}$ *is an orthonormal basis for the Hilbert space* $\mathcal{H}L^2\big(\mathbb{C},|w|\frac{e^{-|w|^2/t}}{t\pi}dw\big)$. *Hence for any* $g\in\mathcal{H}L^2\big(\mathbb{C},|w|\frac{e^{-|w|^2/t}}{t\pi}dw\big)$,

$$|g(w)|^2 \le \frac{e^{|w|^2/t}}{|w|}\operatorname{erf}\Big(\frac{|w|}{\sqrt{t}}\Big)\,\|g\|^2 \qquad (w\in\mathbb{C}),$$

where the error function erf *is defined by*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy = e^{-x^2}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{\Gamma(n+\frac{3}{2})}.$$

Proof. The proof of the orthonormal basis part uses the same technique as in [1, 7] and [10], which we will omit. Then the pointwise bound for a function g in this space is

$$|g(w)|^2 \le \sum_{n=0}^{\infty}\frac{|w|^{2n}}{t^{(2n+1)/2}\Gamma(n+\frac{3}{2})}\|g\|^2 = \frac{e^{|w|^2/t}}{|w|}\operatorname{erf}\Big(\frac{|w|}{\sqrt{t}}\Big)\,\|g\|^2$$

for any $w\in\mathbb{C}$. ■

Theorem 10. *There is a constant B, depending on d and t, such that for any $f\in\mathcal{H}L^2(\mathbb{C},\lambda)$ and any $w\in\mathbb{C}-\{0\}$,*

$$|f(w)|^2 \le \frac{B}{|w|^{d-1}}\,e^{|w|^2/t}\,\|f\|^2_{L^2(\mathbb{C},\lambda)}. \tag{10}$$

Proof.
Let $f\in\mathcal{H}L^2(\mathbb{C},\lambda)$. Then $f\in\mathcal{H}L^2(\mathbb{C},\beta)$, and thus

$$\int_{\mathbb{C}}|w|^{d-1}|f(w)|^2\,\frac{e^{-|w|^2/t}}{\pi t}\,dw<\infty.$$

If $d-1$ is an even number, then

$$w^{(d-1)/2}f(w) \in \mathcal{H}L^2(\mathbb{C}, \frac{e^{-|w|^2/t}}{t\pi}\, dw).$$

This is the one-dimensional Segal-Bargmann space. Using Bargmann's pointwise bound (1) for this space, we obtain

$$|w|^{d-1}|f(w)|^2 \ \le\ \|f\|^2_{L^2(\mathbb{C},\beta)}\, e^{|w|^2/t} \ \le\ \tfrac{1}{k^2}\, \|f\|^2_{L^2(\mathbb{C},\lambda)}\, e^{|w|^2/t}$$

for all $w \in \mathbb{C}$, where k is the constant in Corollary 8.

On the other hand, if $d-1$ is an odd number, then

$$w^{(d-2)/2}f(w) \in \mathcal{H}L^2(\mathbb{C}, |w|\frac{e^{-|w|^2/t}}{t\pi} dw).$$

Following Lemma 9, we have

$$\begin{aligned}|w|^{d-2}|f(w)|^2 &\le \|f\|^2_{L^2(\mathbb{C},\beta)}\frac{e^{|w|^2/t}}{|w|}\operatorname{erf}\Big(\frac{|w|}{\sqrt{t}}\Big)\\ &\le \frac{1}{k^2}\|f\|^2_{L^2(\mathbb{C},\lambda)}\frac{e^{|w|^2/t}}{|w|}\end{aligned}$$

for all $w \in \mathbb{C} - \{0\}$.

In either case we obtain the pointwise (10) with $B = 1/k^2$. ■

Proof of Theorem 1. We will transform the pointwise bound (10) to a function in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}}$. Let $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}}$. Then $F(w, 0, \ldots, 0) \in \mathcal{H}L^2(\mathbb{C}, \lambda)^e$, which implies

$$|F(z)|^2 = |F(w,0,\ldots,0)|^2 \le B\,\frac{e^{|w|^2/t}}{|w|^{d-1}}\,\|F\|^2_{L^2(\mathbb{C}^d,\mu_t)}$$

where $w = \sqrt{(z,z)}$ for any $z \in \mathbb{C}^d$ with $(z,z) \neq 0$. In particular,

$$|F(z)|^2 \le \frac{B\, e^{|(z,z)|/t}}{|(z,z)|^{(d-1)/2}}\|F\|^2_{L^2(\mathbb{C}^d,\mu_t)}.$$

On the other hand, from Proposition 4,

$$|F(z)|^2 \le e^{|(z,z)|/t}\|F\|^2_{L^2(\mathbb{C}^d,\mu_t)} \quad \text{for any } z \in \mathbb{C}^d.$$

Applying the inequality

$$\min\left\{1, \frac{1}{x}\right\} \le \frac{2}{x+1} \qquad \text{for each } x > 0,$$

we have

$$|F(z)|^2 \leq \frac{C\, e^{|(z,z)|/t}}{|(z,z)|^{(d-1)/2}+1}\|F\|^2_{L^2(\mathbb{C}^d,\mu_t)}$$

for each $z \in \mathbb{C}^d$, where C is a constant depending on d and t. This completes the proof of Theorem 1. ■

Remark on the sharpness. The bound in Theorem 1 is indeed sharp. We only outline the proof here since the argument relies heavily on properties of special functions. We can show that the reproducing kernel of the Hilbert space $\mathcal{H}L^2(\mathbb{C},\lambda)^e$ is given by

$$K(w,w) = \frac{\Gamma(d/2)}{a_0 2^{d/2+1}}\text{BesselI}\Big(\frac{d-2}{2}, \frac{|w|^2}{t}\Big)\Big(\frac{t}{|w|^2}\Big)^{d/2-1}$$

where BesselI is the modified Bessel function of the first kind ([18, 19]). Asymptotically, BesselI$(\alpha, x) \sim \frac{e^x}{\sqrt{x}}$ if x is large enough when $\alpha > 0$ is fixed. Hence,

$$K(w,w) \sim C\frac{e^{|w|^2/t}}{|w|^{d-1}}$$

for any w such that $|w|$ is large enough, where C is a constant depending on d and t. The result follows by transforming this estimate back to the space $\mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{\mathcal{O}}$.

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Rotation-invariant Segal-Bargmann transform

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In the present study the authors deal with the Segal-Bargmann transform applied to rotation-invariant functions. They show that it preserves rotation and hence the closed subspaces of rotation-invariant functions become Hilbert spaces. The authors show that the values of rotation-invariant functions are determined by a one-dimensional parameter and hence they establish these Hilbert spaces as L^2-spaces on $\mathbb{R}^1$ and $\mathbb{C}^1$ with respect to some non-Gaussian measures. These non-Gaussian measures are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and $\mathbb{C}$, respectively. The authors obtain formulas for these densities.

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ROTATION-INVARIANT SEGAL-BARGMANN TRANSFORM

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Abstract

The Segal-Bargmann transform on $\mathbb{R}^d$ maps a square-integrable function on $\mathbb{R}^d$ with respect to a Gaussian measure into a holomorphic function on $\mathbb{C}^d$ which is squre-integrable function with respect to a complex Gaussian measure. In this paper we show that it preserves rotation and hence the closed subspaces of rotation-invariant functions become Hilbert spaces. The values of rotation-invariant functions will be determined by a one-dimensional parameter and hence we can established these Hilbert spaces as L^2-spaces on $\mathbb{R}^1$ and $\mathbb{C}^1$ with respect to some non-Gaussian measures. We find the densities of these measures with respect to Lebesgue measure and establish unitarity among relevant Hilbert spaces.

1 Introduction

The Segal-Bargmann transform is an integral transform B_t which maps $L^2(\mathbb{R}^d, \rho_t)$, the set of all functions on $\mathbb{R}^d$ that are square integrable with respect to the real Gaussian measure $\rho_t(x)dx = (2\pi t)^{-d/2}e^{-x^2/2t}dx$, onto $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, the set of all holomorphic functions on $\mathbb{C}^d$ that are square integrable with

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respect to the complex Gaussian measure $\mu_t(z)dz = (\pi t)^{-d}e^{-z^2/t}dz$, for all positive real numbers t. The transform B_t is given by this formula

$$(B_t f)(z) = \int_{\mathbb{R}^d} f(x)\frac{e^{-(z-x)^2/2t}}{(2\pi t)^{d/2}}dx$$

for all $f \in L^2(\mathbb{R}^d, \rho_t)$ and $z \in \mathbb{C}^d$. Here we use notation $x^2 = x_1^2 + x_2^2 + \cdots + x_d^2$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The space $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$ is also called the Segal-Bargmann space. See [1], [4], [6], [7], [8], [9], [11] for details about the importance of this space.

In this paper, we consider the Segal-Bargmann transform applied to rotation-invariant functions. It turns out that if f is a rotation-invariant function in $L^2(\mathbb{R}^d, \rho_t)$, then $F = B_t(f)$ is also invariant under complex rotation. The rotation invariance of a real-valued function f means that it is determined by its values on $\{(x, 0, ..., 0)\} \simeq \mathbb{R}^1$ and the resulting function on $\mathbb{R}^1$ will be an even function. Similarly, a complex rotation-invariant function F is determined by its values on $\{(z, 0, ..., 0)\} \simeq \mathbb{C}^1$ and it is a complex even function. Conversely, any even function on $\mathbb{R}^1$ has an extension to a rotation-invariant on $\mathbb{R}^d$ and any even holomorphic function on $\mathbb{C}^1$ has an extension to a complex rotation-invariant holomorphic function on $\mathbb{C}^d$. Thus the space of rotation-invariant functions in $L^2(\mathbb{R}^d, \rho_t)$ can be expressed as an L^2-space of functions on $\mathbb{R}^1$ with respect to some non-Gaussian measure and also the space of complex rotation-invariant functions in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$ can be expressed as an L^2-space of holomorphic functions on $\mathbb{C}^1$ with respect to some non-Gaussian measure. These non-Gaussian measures are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ and $\mathbb{C}$ respectively. We obtain the formulas for these densities. Finally we establish unitarity among these Hilbert spaces

2 Main Results

Denote by $SO(d)$ the set of $d \times d$ real orthogonal matrices with determinant one and by $SO(d, \mathbb{C})$ the set of $d \times d$ complex orthogonal matrices with determinant one. Define a bilinear form $(\cdot, \cdot)$ on $\mathbb{F}^d$ by

$$(x, y) = x_1y_1 + x_2y_2 + \cdots + x_dy_d$$

for all $x, y \in \mathbb{F}^d$. Then the elements of $SO(d)$ and $SO(d, \mathbb{C})$ preserve the bilinear form on $\mathbb{R}^d$ and $\mathbb{C}^d$ respectively.

Definition 1. *Let F be a function on $\mathbb{F}^d$ where $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{R}$ and let G be a group of $d \times d$ matrices. We say that F is **G-invariant** if*

$$F(Ax) = F(x) \quad \text{for all } A \in G \text{ and all } x \in \mathbb{F}^d.$$

Notice that if F is an $SO(d)$-invariant holomorphic function on $\mathbb{C}^d$, then by analytic continuation it is $SO(d,\mathbb{C})$-invariant.

Denote by $\mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ the set of all $SO(d,\mathbb{C})$-invariant holomorphic functions on $\mathbb{C}^d$, and $\mathcal{F}(\mathbb{R}^d)^{SO(d)}$ the set of all $SO(d)$-invariant functions on $\mathbb{R}^d$.

Denote by $\mathcal{H}(\mathbb{C})^e$ the set of all holomorphic even functions on $\mathbb{C}$, and $\mathcal{F}(\mathbb{R})^e$ the set of all even functions on $\mathbb{R}$.

Theorem 1. *For any $d \geq 2$, the map $\Phi\colon \mathcal{F}(\mathbb{R}^d)^{SO(d)} \to \mathcal{F}(\mathbb{R})^e$ defined by*

$$\Phi(G)(s) = G(s, 0, \ldots, 0)$$

for all $G \in \mathcal{F}(\mathbb{R}^d)^{SO(d)}$ and all $s \in \mathbb{R}$, is a linear isomorphism whose inverse is given by

$$\Psi(g)(x) = g\left(\sqrt{(x,x)}\right) = g(|x|)$$

for all $g \in \mathcal{F}(\mathbb{R})^e$ and all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$.

Proof. It is clear that Φ is a linear map. Moreover, $\Phi(G)$ is even since

$$G(-s, 0, \ldots, 0) = G(A(s, 0, \ldots, 0)) = G(s, 0, \ldots, 0)$$

where $A = \operatorname{diag}(-1, -1, 1, 1, \ldots, 1) \in SO(d)$. On the other hand, Ψ is a linear map and $\Psi(g)$ is $SO(d)$-invariant because $(Ax, Ax) = (x, x)$ for all $x \in \mathbb{R}^d$ and all $A \in SO(d)$. It is easy to see that $\Phi \circ \Psi = \mathrm{id}_{\mathcal{F}(\mathbb{R})^e}$ and $\Psi \circ \Phi = \mathrm{id}_{\mathcal{F}(\mathbb{R}^d)^{SO(d)}}$, so the theorem is proved. □

Similarly, we have the following theorem for complex case.

Theorem 2. *For any $d \geq 2$, the map $\phi\colon \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})} \to \mathcal{H}(\mathbb{C})^e$ defined by*

$$\phi(f)(\xi) = f(\xi, 0, \ldots, 0)$$

for any $f \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ and any $\xi \in \mathbb{C}$, is a linear isomorphism whose inverse is given by

$$\psi(g)(z) = g\left(\sqrt{(z,z)}\right)$$

for any $g \in \mathcal{H}(\mathbb{C})^e$ and any $z \in \mathbb{C}^d$.

Note that since g is even, the value of $\phi(g)(z)$ is independent of the choice of square root of (z, z).

Denote by $\mathcal{B}_d$ the Borel σ-algebra in $\mathbb{R}^d$ and by $\mathcal{B}$ the Borel σ-algebra in $\mathbb{R}$. Define the maps $\Psi_i\colon (\mathbb{R}^d, \mathcal{B}_d, \rho_t) \to (\mathbb{R}, \mathcal{B})$, $i = 1, 2$ by

$$\Psi_1(x) = |x| \quad \text{and} \quad \Psi_2(x) = -|x|$$

for all $x \in \mathbb{R}^d$. For each $E \in \mathcal{B}$ let

$$\gamma_i(E) = \rho_t(\Psi_i^{-1}(E)),$$

and let $\gamma = (\gamma_1 + \gamma_2)/2$. It is easy to check that γ is a Borel measure on $\mathbb{R}$ and for any measurable function g and any $E \in \mathcal{B}$

$$\int_E g\, d\gamma \;=\; \frac{1}{2}\int_{\Psi_1^{-1}(E)} g \circ \Psi_1 \, d\rho_t + \frac{1}{2}\int_{\Psi_2^{-1}(E)} g \circ \Psi_2 \, d\rho_t.$$

We write

$$\begin{aligned} L^2(\mathbb{R}^d, \rho_t)^{SO(d)} &= \mathcal{F}(\mathbb{R}^d)^{SO(d)} \cap L^2(\mathbb{R}^d, \rho_t) \\ L^2(\mathbb{R}, \gamma)^e &= \mathcal{F}(\mathbb{R})^e \cap L^2(\mathbb{R}, \gamma). \end{aligned}$$

It is easy to see that $L^2(\mathbb{R}^d, \rho_t)^{SO(d)}$ and $L^2(\mathbb{R}, \gamma)^e$ are closed subspaces of $L^2(\mathbb{R}^d, \rho_t)$ and $L^2(\mathbb{R}, \gamma)$ respectively and hence are Hilbert spaces.

Theorem 3. *The Hilbert spaces $L^2(\mathbb{R}^d, \rho_t)^{SO(d)}$ and $L^2(\mathbb{R}, \gamma)^e$ are unitarily equivalent.*

Proof. From Theorem 1 we have that the function

$$\Psi \colon \mathcal{F}(\mathbb{R})^e \to \mathcal{F}(\mathbb{R}^d)^{SO(d)}$$

is a linear isomorphism. We consider the restriction of Ψ to the space $L^2(\mathbb{R}, \gamma)^e$. Let $g \in \mathcal{F}(\mathbb{R})^e$ and $G \in \mathcal{F}(\mathbb{R}^d)^{SO(d)}$ be such that $G = \Psi(g)$. Thus

$$\begin{aligned} \int_{\mathbb{R}} |g|^2 d\gamma &= \frac{1}{2}\int_{\Psi_1^{-1}(\mathbb{R})} |g \circ \Psi_1(x)|^2 \, \rho_t(x)\, dx + \frac{1}{2}\int_{\Psi_2^{-1}(\mathbb{R})} |g \circ \Psi_2(x)|^2 \, \rho_t(x)\, dx \\ &= \int_{\mathbb{R}^d} |g(|x|)|^2 \, \rho_t(x)\, dx \\ &= \int_{\mathbb{R}^d} |\Psi(g)\,(x)|^2 \, \rho_t(x)\, dx \\ &= \int_{\mathbb{R}^d} |G(x)|^2 \, \rho_t(x)\, dx. \end{aligned}$$

So $\|g\|_{L^2(\mathbb{R},\gamma)} = \|G\|_{L^2(\mathbb{R}^d,\rho_t)}$. Hence, $G \in L^2(\mathbb{R}^d, \rho_t)^{SO(d)}$ if and only if $g \in L^2(\mathbb{R}, \gamma)^e$. This shows that Ψ is a unitary map from $L^2(\mathbb{R}, \gamma)^e$ onto $L^2(\mathbb{R}^d, \rho_t)^{SO(d)}$. □

We next show that the measure γ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$.

Theorem 4. *The measure γ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ with density given by*

$$\Delta(s) = \frac{\sigma(S^{d-1})}{(2\pi t)^{d/2}}\, |s|^{d-1} e^{-s^2/2t}. \tag{1}$$

where S^{d-1} is a unit sphere on $\mathbb{R}^d$ and σ is the surface measure on S^{d-1}.

Proof. Let E be a Borel set in $\mathbb{R}$. Then

$$\begin{aligned}\gamma(E) &= \frac{1}{2}\int_{\Psi_1^{-1}(E)} \frac{e^{-|x|^2/2t}}{(2\pi t)^{d/2}}\,dx + \frac{1}{2}\int_{\Psi_2^{-1}(E)} \frac{e^{-|x|^2/2t}}{(2\pi t)^{d/2}}\,dx \\ &= \frac{1}{2}\int_0^\infty \int_{S^{d-1}} \chi_{\Psi_1^{-1}(E)}(rx')\frac{r^{d-1}}{(2\pi t)^{d/2}}\,e^{-|rx'|^2/2t}\,d\sigma(x')\,dr \\ &+ \frac{1}{2}\int_0^\infty \int_{S^{d-1}} \chi_{\Psi_2^{-1}(E)}(rx')\frac{r^{d-1}}{(2\pi t)^{d/2}}\,e^{-|rx'|^2/2t}\,d\sigma(x')\,dr \\ &= \frac{1}{2}\int_0^\infty \chi_E(r)\frac{r^{d-1}}{(2\pi t)^{d/2}}\,e^{-r^2/2t}\,\sigma(S^{d-1})\,dr \\ &+ \frac{1}{2}\int_0^\infty \chi_E(-r)\frac{r^{d-1}}{(2\pi t)^{d/2}}\,e^{-r^2/2t}\,\sigma(S^{d-1})\,dr \\ &= \int_{\mathbb{R}} \chi_E(s)\frac{|s|^{d-1}}{(2\pi t)^{d/2}}\,e^{-s^2/2t}\sigma(S^{d-1})\,ds \\ &= \int_E \Delta(s)\,ds\end{aligned}$$

where Δ is given by (1). □

In the same way denote by $\mathcal{B}(\mathbb{C}^d)$ the Borel σ-algebra in $\mathbb{C}^d$ and by $\mathcal{B}(\mathbb{C})$ the Borel σ-algebra in $\mathbb{C}$ and define $\Phi_i\colon (\mathbb{C}^d, \mathcal{B}(\mathbb{C}^d), \mu_t) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$, $i = 1, 2$ to be the branch of $\sqrt{(z,z)}$ with a smaller and larger argument respectively and for each $E \in \mathcal{B}(\mathbb{C})$ define

$$\lambda_i(E) = \mu_t(\Phi_i^{-1}(E)),$$

and let $\lambda = (\lambda_1 + \lambda_2)/2$.

Define

$$\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})} = \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})} \cap L^2(\mathbb{C}^d, \mu_t)$$

and

$$\mathcal{H}L^2(\mathbb{C}, \lambda)^e = \mathcal{H}(\mathbb{C})^e \cap L^2(\mathbb{C}, \lambda).$$

Then they are also Hilbert spaces. We now have the following theorem whose proof is similar to that of Theorem 3.

Theorem 5. *The Hilbert spaces $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ and $\mathcal{H}L^2(\mathbb{C}, \lambda)^e$ are unitarily equivalent.*

Denote by $\mathbb{H}^+ = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. Consider the map $\mathbb{C}^d - S \to \mathbb{H}^+ \times S_1$, $z \mapsto (w, z')$, where $w = |(z,z)|^{1/2} e^{i\frac{\theta}{2}}$, θ is the principal value of $\arg(z,z)$, $\theta \in (-\pi, \pi)$, and $z' = \frac{z}{w}$. This map is a continuous bijection whose inverse is given by $(w, z') \mapsto wz'$. We can think of it as a "complex polar form" of an element in $\mathbb{C}^d$ whose bilinear form is nonzero.

Let m be Lebesgue measure on $\mathbb{C}^d$ and m_* the "push-forward" Borel measure of m on $\mathbb{H}^+ \times S_1$. Then m_* can be written as a product measure $m_* = \rho \times \alpha$, where ρ is a measure on $\mathbb{H}^+$ defined by

$$\rho(A) = \int_A |w|^{2d-2} dw$$

and α is an $SO(d,\mathbb{C})$-invariant Borel measure on S_1. There is an $SO(d,\mathbb{C})$-invariant Borel measure α on S_1 such that $m_* = \rho \times \alpha$. If f is a Borel measurable function on $\mathbb{C}^d$ such that $f \geq 0$ or $f \in L^1(\mathbb{C}^d, m)$, then

$$\int_{\mathbb{C}^d} f(z)\, dz = \int_{\mathbb{C}} \int_{S_1} f(wz') d\alpha(z')\, |w|^{2d-2} dw, \tag{2}$$

where dw denotes the two-dimensional Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$. Details can be found in [13].

Theorem 6. *[13] The measure λ is absolutely continuous with respect to Lebesgue measure on $\mathbb{C}$ with density given by*

$$\Lambda(w) = \frac{|w|^{2d-2}}{(\pi t)^d} \int_{S_1} e^{-(wz)^2/t} d\alpha(z). \tag{3}$$

Proof. See Proposition 6 in [13]. □

Unlike the real case, we do not have an explicit form of the formula for the density of λ with respect to Lebesgue measure. However, in [13], we established that the density Λ is equivalent to the function $|w|^{d-1}\, e^{-|w|^2/t}$ for all $w \in \mathbb{C}$ bounded away from zero.

Theorem 7. *Let $B_t \colon L^2(\mathbb{R}^d, \rho_t) \to \mathcal{H}L^2(\mathbb{C}^d, \mu_t)$ be the Segal-Bargmann transform given by the formula:*

$$(B_t f)(z) = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-(z-x)^2/2t}\, dx.$$

Then B_t preserves the rotation action. In other words, a function $f \in L^2(\mathbb{R}^d, \rho_t)$ is $SO(d)$-invariant if and only if $B_t(f)$ is $SO(d,\mathbb{C})$-invariant. Hence, we can consider the Segal-Bargmannn transform B_t as a unitary map from $L^2(\mathbb{R}^d, \rho_t)^{SO(d)}$ onto $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$.

Proof. Let $f \in L^2(\mathbb{R}^d, \rho_t)$ and $F = B_t(f)$. First assume that f is $SO(d)$-invariant. Recall that the bilinear form $(\cdot,\cdot)$ preserves the action of $SO(d)$ and

$SO(d,\mathbb{C})$. If $A \in SO(d)$ and $z \in \mathbb{C}^d$, then

$$\begin{aligned} F(Az) &= \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-(Az-x)^2/2t}\, dx \\ &= \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-(z-A^{-1}x)^2/2t}\, dx \\ &= \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} f(Ax) e^{-(z-x)^2/2t}\, d(Ax) \\ &= \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-(z-x)^2/2t}\, dx \\ &= F(z). \end{aligned}$$

Notice that we use the fact that $\det(A) = 1$ in the change of variables above. Hence, F is $SO(d)$-invariant. By analytic continuation, it is $SO(d,\mathbb{C})$-invariant. Conversely, assume that F is $SO(d,\mathbb{C})$-invariant. Fix $A \in SO(d)$ and let

$$g(x) = f(Ax) \quad \text{for any } x \in \mathbb{R}^d.$$

Then $g \in L^2(\mathbb{R}^d, \rho_t)$. Moreover, for any $z \in \mathbb{C}^d$

$$\begin{aligned} B_t g(z) &= \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} g(x) e^{-(z-x)^2/2t}\, dx \\ &= \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} f(Ax) e^{-(z-x)^2/2t}\, dx \\ &= \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-(z-A^{-1}x)^2/2t}\, d(A^{-1}x) \\ &= \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-(Az-x)^2/2t}\, dx \\ &= F(Az) = F(z). \end{aligned}$$

Hence, $B_t g = B_t f$. Since B_t is 1-1, we must have $g = f$, i.e. f is $SO(d)$-invariant. □

Theorem 8. *$L^2(\mathbb{R},\gamma)^e$ and $\mathcal{H}L^2(\mathbb{C},\lambda)^e$ are unitarily equivalent. Moreover the following diagram is commutative*

$$\begin{array}{ccc} L^2(\mathbb{R},\gamma)^e & \xrightarrow{\Psi} & L^2(\mathbb{R}^d,\rho_t)^{SO(d)} \\ \beta\downarrow & & \downarrow B_t \\ \mathcal{H}L^2(\mathbb{C},\lambda)^e & \xrightarrow{\psi} & \mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})} \end{array}$$

where $\beta\colon L^2(\mathbb{R},\gamma)^e \to \mathcal{H}L^2(\mathbb{C},\lambda)^e$ *is given by*

$$\beta(f)(w) = \frac{1}{(2\pi t)^{d/2}} \int_0^\infty f(r) \int_{S^{d-1}} e^{-((w,0,\dots,0)-rx')^2/2t}\, d\sigma(x')\, r^{d-1}\, dr$$

for all $f \in L^2(\mathbb{R},\gamma)^e$ *and* $w \in \mathbb{C}$.

Proof. We know that ψ^{-1}, B_t and Ψ are unitary, so $\psi^{-1}\circ B_t \circ \Psi\colon L^2(\mathbb{R},\gamma)^e \to \mathcal{H}L^2(\mathbb{C},\lambda)^e$ is also unitary. For any $f \in L^2(\mathbb{C},\gamma)^e$ and any $w \in \mathbb{C}$, we have

$$\begin{aligned}(\psi^{-1}\circ B_t\circ\Psi)(f)(w) &= (B_t\circ\Psi)(f)(w,0,\dots,0)\\ &= \frac{1}{(2\pi t)^{d/2}}\int_{\mathbb{R}^d}\Psi(f)(x)e^{-((w,0,\dots,0)-x)^2/2t}\,dx\\ &= \frac{1}{(2\pi t)^{d/2}}\int_{\mathbb{R}^d} f(|x|)e^{-((w,0,\dots,0)-x)^2/2t}\,dx\\ &= \frac{1}{(2\pi t)^{d/2}}\int_0^\infty\int_{S^{d-1}} f(r)e^{-((w,0,\dots,0)-rx')^2/2t}\,r^{d-1}d\sigma(x')\,dr.\end{aligned}$$

Hence $\beta = \psi^{-1}\circ B_t\circ\Psi$, so β is a unitary map. The theorem is proved. □

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The article gives a quite nice and up-to-date overview of noncommutative geometry, covering different approaches to the subject, including an outline of the authors' own categorical approach. The last section has some discussion on the possible connections with and applications to physics, such as quantum gravity. The bibliograpghy at the end of the paper is quite exhaustive and will be valuable to any researcher in the field.

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NON-COMMUTATIVE GEOMETRY, CATEGORIES AND QUANTUM PHYSICS

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Abstract

After an introduction to some basic issues in non-commutative geometry (Gel'fand duality, spectral triples), we present a "panoramic view" of the status of our current research program on the use of categorical methods in the setting of A. Connes' non-commutative geometry: morphisms/categories of spectral triples, categorification of Gel'fand duality. We conclude with a summary of the expected applications of "categorical non-commutative geometry" to structural questions in relativistic quantum physics: (hyper)covariance, quantum space-time, (algebraic) quantum gravity.

Key words: Non-commutative Geometry, Spectral Triple, Category, Morphisms, Quantum Physics, Space-Time.
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1 Introduction.

The purpose of this review paper is to present the status of our research work on categorical non-commutative geometry and to contextualize it providing appropriate references.

The paper is organized as follows. In section 2, we first provide a review of the basic dualities (Gelf'and, Serre-Swan and Takahashi) that constitute the main categorical motivation for non-commutative geometry and then we pass to introduce the definition of A. Connes spectral triple.

In the first part of section 3, we give an overview of our proposed definitions of morphisms between spectral triples and categories of spectral triples. In the second part of section 3 we show how to generalize Gel'fand duality to the setting of commutative full C*-categories and we suggest how to apply this insight to the purpose of defining "bivariant" spectral triples as a correct notion of metric morphism.

The last section 4, is mainly intended for an audience of mathematicians and tries to explain how categorical and non-commutative notions enter the context of quantum mathematical physics and how we hope to see such notions emerge in a non-perturbative treatment of quantum gravity.

The last part (section 4.4) is more speculative and contains a short overview of our present research program in quantum gravity based on Tomita-Takesaki modular theory and categorical non-commutative geometry.

We have tried to provide an extensive bibliography in order to help to place our research in a broader landscape and to suggest as much as possible future links with interesting ideas already developed. Of course missing references are sole responsability of the ignorance of the authors, that are still trying to learn their way through the material. We will be grateful for any suggestion to improve the on-line version of the document.

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2 Non-commutative Geometry (Objects).

For an introduction to the subject we refer to the books by A. Connes [58], G. Landi [157], H. Figueroa-J. Gracia-Bondia-J. Varilly [108] (see also [223]) and M. Khalkhali [142]. For the basic definitions on category theory we refer to S. McLane [171] and M. Barr-C. Wells [9].

Non-commutative geometry, created by A. Connes, is a powerful extension of the ideas of R. Decartes' analytic geometry: to substitute "geometrical objects" with their Abelian algebras of functions; to "translate" the geometrical properties of spaces into algebraic properties of the associated algebras[1] and to "reconstruct" the original geometric spaces as a derived entities (the spectra of the algebras), a technique that appeared for the first time in the work of I. Gel'fand on Abelian C*-algebras in 1939.[2]

Whenever such "codifications" of geometry in algebraic terms still make sense if the Abelian condition is dropped, we can simply work with non-commutative algebras considered as "duals" of "non-commutative spaces".

The existence of dualities between categories of "geometrical spaces" and categories "constructed from Abelian algebras" is the starting point of any generalization of geometry to the non-commutative situation. Here are some examples.

2.1 Non-commutative Topology.

2.1.1 Gel'fand Theorem.

For the details on operator algebras, the reader may refer to R. Kadison-J. Ringrose [137], M. Takesaki [219] and B. Blackadar [20]. A complex unital **algebra** $\mathcal{A}$ is a vector space over $\mathbb{C}$ with an associative unital bilinear multiplication. $\mathcal{A}$ is **Abelian** (commutative) if $ab = ba$, for all $a, b \in \mathcal{A}$. An **involution** on $\mathcal{A}$ is a conjugate linear map $* : \mathcal{A} \to \mathcal{A}$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$, for all $a, b \in \mathcal{A}$. An involutive complex unital algebra is $\mathcal{A}$ called a **C*-algebra** if $\mathcal{A}$ is a Banach space with a norm $a \mapsto \|a\|$ such that $\|ab\| \leq \|a\| \cdot \|b\|$ and $\|a^*a\| = \|a\|^2$, for all $a, b \in \mathcal{A}$. Notable examples are the algebras of continuous complex valued functions $C(X; \mathbb{C})$ on a compact topological space with the "sup norm" and the algebras of linear bounded operators $\mathcal{B}(H)$ on the Hilbert space H.

Theorem 2.1 (Gel'fand; see e.g. [165]). *There exists a duality* $(\Gamma^{(1)}, \Sigma^{(1)})$ *between the category* $\mathscr{T}^{(1)}$*, of continuous maps between compact Hausdorff topological spaces, and the category* $\mathscr{A}^{(1)}$*, of unital homomorphisms of commutative unital C*-algebras.*

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[1] A line of thought already present in J.L. Koszul algebraization of differential geometry.

[2] Although similar ideas, previously developed by D. Hilbert, are well known and used also in P. Cartier-A. Grothendieck's definition of schemes in algebraic geometry.

$\Gamma^{(1)}$ is the functor that associates to every compact Hausdorff topological space $X \in \mathrm{Ob}_{\mathscr{T}^{(1)}}$ the unital commutative C*-algebra $\mathrm{C}(X;\mathbb{C})$ of complex valued continuous functions on X (with pointwise multiplication and conjugation and supremum norm) and that to every continuous map $f : X \to Y$ associates the unital $*$-homomorphism $f^\bullet : C(Y;\mathbb{C}) \to C(X;\mathbb{C})$ given by the pull-back of continuous functions by f.

$\Sigma^{(1)}$ is the functor that associates to every unital commutative C*-algebra $\mathcal{A}$ its spectrum $\mathrm{Sp}(\mathcal{A}) := \{\omega \mid \omega : \mathcal{A} \to \mathbb{C}$ is a unital $*$-homomorphism$\}$ (as a topological space with the weak topology induced by the evaluation maps $\omega \mapsto \omega(x)$, for all $x \in \mathcal{A}$) and that to every unital $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ of algebras associates the continuous map $\phi^\bullet : \mathrm{Sp}(\mathcal{B}) \to \mathrm{Sp}(\mathcal{A})$ given by the pull-back under ϕ.

The natural isomorphism $\mathfrak{G} : \mathcal{I}_{\mathscr{A}^{(1)}} \to \Gamma^{(1)} \circ \Sigma^{(1)}$ is given by the **Gel'fand transforms** $\mathfrak{G}_{\mathcal{A}} : \mathcal{A} \to C(\mathrm{Sp}(\mathcal{A}))$ defined by $\mathfrak{G}_{\mathcal{A}} : a \mapsto \hat{a}$, where $\hat{a} : \mathrm{Sp}(\mathcal{A}) \to \mathbb{C}$ is the Gel'fand transform of a i.e. $\hat{a} : \omega \mapsto \omega(a)$.

The natural isomorphism $\mathfrak{E} : \mathcal{I}_{\mathscr{T}^{(1)}} \to \Sigma^{(1)} \circ \Gamma^{(1)}$ is given by the **evaluation** homeomorphisms $\mathfrak{E}_X : X \to \mathrm{Sp}(C(X))$ defined by $\mathfrak{E}_X : p \mapsto \mathrm{ev}_p$, where $\mathrm{ev}_p : C(X) \to \mathbb{C}$ is the p-evaluation i.e. $\mathrm{ev}_p : f \mapsto f(p)$.

In view of this result, compact Hausdorff spaces and Abelian unital C*-algebras are essentially the same thing and we can freely translate properties of the geometrical space in algebraic properties of its Abelian algebra of functions.

In the spirit of non-commutative geometry, we can simply consider non-Abelian unital C*-algebras as "duals" of "non-commutative compact Hausdorff topological spaces".

2.1.2 Serre-Swan and Takahashi Theorems.

A **left pre-Hilbert-C*-module** ${}_{\mathcal{A}}M$ over the unital C*-algebra $\mathcal{A}$ (whose positive part is denoted by $\mathcal{A}_+ := \{x^*x \mid x \in \mathcal{A}\}$) is a unital left module M over the unital ring $\mathcal{A}$ that is equipped with an $\mathcal{A}$-valued inner product $M \times M \to \mathcal{A}$ denoted by $(x, y) \mapsto {}_{\mathcal{A}}\langle x \mid y\rangle$ such that, for all $x, y, z \in M$ and $a \in \mathcal{A}$, $\langle x + y \mid z\rangle = \langle x \mid z\rangle + \langle y \mid z\rangle$, $\langle a \cdot x \mid z\rangle = a\langle x \mid z\rangle$, $\langle y \mid x\rangle = \langle x \mid y\rangle^*$, $\langle x \mid x\rangle \in \mathcal{A}_+$, $\langle x \mid x\rangle = 0_{\mathcal{A}} \Rightarrow x = 0_M$. A similar definition of a right pre-Hilbert-C*-module is given with multiplication by elements of the algebra on the right.

A left Hilbert C*-module ${}_{\mathcal{A}}M$ is a left pre-Hilbert C*-module that is complete in the norm defined by $x \mapsto \sqrt{\|{}_{\mathcal{A}}\langle x \mid x\rangle\|}$.[3] We say that a left pre-Hilbert C*-module ${}_{\mathcal{A}}M$ is **full** if $\overline{\mathrm{span}\{\langle x \mid y\rangle \mid x, y \in M\}} = \mathcal{A}$, where the closure is in the norm topology of the C*-algebra $\mathcal{A}$. A **pre-Hilbert-C*-bimodule** ${}_{\mathcal{A}}M_{\mathcal{B}}$ over the unital C*-algebras $\mathcal{A}, \mathcal{B}$, is a left pre-Hilbert module over $\mathcal{A}$ and a

[3] A similar definition applies for right modules.

right pre-Hilbert C*-module over $\mathcal{B}$ such that:

$$(a \cdot x) \cdot b = a \cdot (x \cdot b), \quad \forall a \in \mathcal{A},\ x \in M,\ b \in \mathcal{B}.$$

A full Hilbert C*-bimodule is said to be an **imprimitivity bimodule** or an **equivalence bimodule** if:

$$_{\mathcal{A}}\langle x \mid y\rangle \cdot z = x \cdot \langle y \mid z\rangle_{\mathcal{B}}, \quad \forall x, y, z \in M.$$

A bimodule $_{\mathcal{A}}M_{\mathcal{A}}$ is called **symmetric** if $ax = xa$ for all $x \in M$ and $a \in \mathcal{A}$.[4] A module $_{\mathcal{A}}M$ is **free** if it is isomorphic to a module of the form $\oplus_J \mathcal{A}$ for some index set J. A module $_{\mathcal{A}}M$ is **projective** if there exists another module $_{\mathcal{A}}N$ such that $M \oplus N$ is a free module.

An "equivalence result" strictly related to Gel'fand theorem, is the following "Hermitian" version of Serre-Swan theorem (see for example M. Frank [109, Theorem 7.1], N. Weaver [226, Theorem 9.1.6] and also H. Figueroa-J. Gracia-Bondia-J. Varilly [108, Theorem 2.10 and page 68]) that provides a "spectral interpretation" of symmetric finite projective bimodules over a commutative unital C*-algebra as Hermitian vector bundles over the spectrum of the algebra.[5]

Theorem 2.2 (Serre-Swan; see e.g. [226, 108]). *Let X be a compact Hausdorff topological space. Let $\mathscr{M}_{C(X)}$ be the category of symmetric projective finite Hilbert C*-bimodules over the commutative C*-algebra $C(X;\mathbb{C})$ with $C(X;\mathbb{C})$-bimodule morphisms. Let $\mathscr{E}_X$ be the category of Hermitian vector bundles over X with bundle morphisms*[6].

The functor $\Gamma : \mathscr{E}_X \to \mathscr{M}_{C(X)}$, that to every Hermitian vector bundle associates its symmetric $C(X)$-bimodule of sections, is an equivalence of categories.

In practice, to every Hermitian vector bundle $\pi : H \to X$ over the compact Hausdorff space X, we associate the symmetric Hilbert C*-bimodule $\Gamma(X; H)$, the continuous sections of H, over the C*-algebra $C(X;\mathbb{C})$.

Since, in the light of Gel'fand theorem, non-Abelian unital C*-algebras are to be interpreted as "non-commutative compact Hausdorff topological spaces", Serre-Swan theorem suggests that also finite projective Hilbert C*-bimodules over unital C*-algebras should be considered as "Hermitian bundles over non-commutative Hausdorff compact spaces".

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[4] Of course this definition make sense only for bimodules over a commutative algebra $\mathcal{A}$.

[5] The result, as it is stated in the previously given references [109, 226] and [108, page 68], is actually formulated without the finiteness and projectivity conditions on the modules and with Hilbert bundles (see J. Fell-R. Doran [107, Section 13] or [108, Definition 2.9] for a detailed definition) in place of Hermitian bundles. Note that Hilbert bundles are not necessarily locally trivial, but they become so if they have finite constant rank (see for example J. Fell-R. Doran [107, Remark 13.9]) and hence the more general equivalence between the category of Hilbert bundles with the category of Hilbert C*-modules actually entails the Hermitian version of Serre-Swan theorem presented here.

[6] Continuous, fiberwise linear maps, preserving the base points.

↬ Problem: Serre-Swan theorem deals only with categories of bundles over a fixed topological space (categories of modules over a fixed algebra, respectively). In order to extend the theorem to categories of bundles over different spaces, it is necessary to define generalized notions of morphism between modules over different algebras. The easiest solution is to define a morphism from the $\mathcal{A}$-module ${}_{\mathcal{A}}\mathcal{M}$ to the $\mathcal{B}$-module ${}_{\mathcal{B}}\mathcal{N}$ as a pair (ϕ, Φ), where $\phi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of algebras and $\Phi : \mathcal{M} \to \mathcal{N}$ is a $\mathbb{C}$-linear map of the bimodules such that $\Phi(am) = \phi(a)\Phi(m)$, for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$. This is the notion that we have used in [15], and that appeared also in [217, 218, 108, 129]. A more appropriate solution would be to consider "congruences" of bimodules and reformulate Serre-Swan theorem in terms of relators (as defined in [15]). Work on this topic is in progress.

↬ Problem: note that Serre-Swan theorem gives an equivalence of categories (and not a duality), this will create problems of "covariance" for any generalization of the well-known covariant functors between categories of manifolds and categories of their associated vector (tensor, Clifford) bundles, to the case of non-commutative spaces and their "bundles". Again a more appropriate approach using relators should deal with this issue.

A first immediate solution to both the above problems is provided by Takahashi duality theorem below. Serre-Swan equivalence is actually a particular case of the following general (and surprisingly almost unnoticed) Gel'fand duality result that was obtained in 1971 by A. Takahashi [217, 218].[7] In this formulation, one actually considers much more general C*-modules and Hilbert bundles at the price of losing contact with K-theory; anyway (as described in the footonote 5 at page 217) the Hermitian version of Serre-Swan theorem can be recovered considering bundles with constant finite rank (over a fixed compact Hausdorff topological space).

Theorem 2.3 (Takahashi [217, 218]). *There is a (weak $*$-monoidal) category* ${}_{\bullet}\mathscr{M}$ *of left Hilbert C*-modules* ${}_{\mathcal{A}}M, {}_{\mathcal{B}}N$ *over unital commutative C*-algebras, whose morphisms are given by pairs* (ϕ, Φ) *where* $\phi : \mathcal{A} \to \mathcal{B}$ *is a unital $*$-homomorphism of C*-algebras and* $\Phi : M \to N$ *is a continuous map such that* $\Phi(ax) = \phi(a)\Phi(x)$*, for all* $a \in \mathcal{A}$ *and* $x \in M$*.*

There is a (weak $*$-monoidal) category $\mathscr{E}$ *of Hilbert bundles* $(\mathcal{E}, \pi, \mathcal{X})$*, over compact Hausdorff topological spaces with morphisms given by pairs* $(f, \mathcal{F})$ *with* $f : \mathcal{X} \to \mathcal{Y}$ *a continuous map and* $\mathcal{F} : f^\bullet(\mathcal{F}) \to \mathcal{E}$ *satisfying* $\pi \circ \mathcal{F} = \rho^f$*, where* $(f^\bullet(\mathcal{F}), \rho^f, \mathcal{X})$ *denotes the pull-back of the bundle* $(\mathcal{F}, \rho, \mathcal{Y})$ *under* f*.*

There is an equivalence (of weak $*$-monoidal) categories given by the functor Γ *that associates to every Hilbert bundle* $(\mathcal{E}, \pi, \mathcal{X})$ *the set of sections* $\Gamma(\mathcal{X}; \mathcal{E})$

[7]Note that our Gel'fand duality result for commutative full C*-categories (that we will present later in section 3.2.1) can be seen as "strict"-$*$-monoidal version of Takahashi duality.

and that to every section $\sigma \in \Gamma(\mathcal{Y}; \mathcal{F})$ *associates the section* $\mathcal{F} \circ f^\bullet(\sigma) \in \Gamma(\mathcal{X}; \mathcal{E})$.

2.2 Non-commutative (Spin) Differential Geometry.

What are "non-commutative manifolds"?

In order to define "non-commutative manifolds", we have to find a categorical duality between a category of manifolds and a suitable category constructed out of Abelian C*-algebras of functions over the manifolds. The complete answer to the question is not yet known, but (at least in the case of compact finite-dimensional orientable Riemannian spin manifolds) the notion of Connes' spectral triples and Connes-Rennie-Varilly [60, 66], [198] reconstruction theorem provide an appropriate starting point, specifying the objects of our non-commutative category[8].

2.2.1 Connes Spectral Triples.

A. Connes (see [58, 108]) has proposed a set of axioms for "non-commutative manifolds" (at least in the case of a compact finite-dimensional orientable Riemannian spin manifolds), called a (compact) spectral triple or an (unbounded) K-cycle.

- A (compact) **spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ is given by:
  - a unital pre-C*-algebra[9] $\mathcal{A}$;
  - a (faithful) representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$;
  - a (generally unbounded) self-adjoint operator D on $\mathcal{H}$, called the Dirac operator, such that:
    a) the resolvent $(D - \lambda)^{-1}$ is a compact operator, $\forall \lambda \in \mathbb{C} \setminus \mathbb{R}$,[10]
    b) $[D, \pi(a)]_- \in \mathcal{B}(\mathcal{H})$, for every $a \in \mathcal{A}$,
    where $[x, y]_- := xy - yx$ denotes the commutator of $x, y \in \mathcal{B}(\mathcal{H})$.
- A spectral triple is called **even** if there exists a grading operator, i.e. a bounded self-adjoint operator $\Gamma \in \mathcal{B}(\mathcal{H})$ such that:

  $$\Gamma^2 = \mathrm{Id}_\mathcal{H}; \quad [\Gamma, \pi(a)]_- = 0, \forall a \in \mathcal{A}; \quad [\Gamma, D]_+ = 0,$$

  where $[x, y]_+ := xy + yx$ is the anticommutator of x, y.

  A spectral triple that is not even is called **odd**.

[8] We will of course deal later with the morphisms in section 3.1.

[9] Sometimes $\mathcal{A}$ is required to be closed under holomorphic functional calculus.

[10] As already noticed by Connes, this condition has to be weakened in the case of non-compact manifolds, cf. [122, 114, 195, 196].

- A spectral triple is **regular** if the function

$$\Xi_x : t \mapsto \exp(it|D|)x \exp(-it|D|)$$

is regular, i.e. $\Xi_x \in C^\infty(\mathbb{R}, \mathcal{B}(\mathcal{H}))$,[11] for every $x \in \Omega_D(\mathcal{A})$, where [12]

$$\Omega_D(\mathcal{A}) := \mathrm{span}\{\pi(a_0)[D,\pi(a_1)]_- \cdots [D,\pi(a_n)]_- \mid n \in \mathbb{N},\ a_0,\dots,a_n \in \mathcal{A}\}$$

- A spectral triple is n**-dimensional** iff there exists an integer n such that the Dixmier trace of $|D|^{-n}$ is finite nonzero.

- A spectral triple is θ**-summable** if $\exp(-tD^2)$ is a trace-class operator for all $t > 0$.

- A spectral triple is **real** if there exists an antiunitary operator $J : \mathcal{H} \to \mathcal{H}$ such that:

$$[\pi(a), J\pi(b^*)J^{-1}]_- = 0, \quad \forall a, b \in \mathcal{A};$$
$$[\,[D,\pi(a)]_-, J\pi(b^*)J^{-1}]_- = 0, \quad \forall a, b \in \mathcal{A}, \quad \textbf{first order condition};$$
$$J^2 = \pm \mathrm{Id}_{\mathcal{H}}; \quad [J, D]_\pm = 0; \quad \text{and, only in the even case,} \quad [J, \Gamma]_\pm = 0,$$

where the choice of $\pm$ in the last three formulas depends on the "dimension" n of the spectral triple modulo 8 in accordance to the following table:

n	0	1	2	3	4	5	6	7
$J^2 = \pm \mathrm{Id}_{\mathcal{H}}$	+	+	−	−	−	−	+	+
$[J, D]_\pm = 0$	−	+	−	−	−	+	−	−
$[J, \Gamma]_\pm = 0$	−		+		−		+	

- A spectral triple is **finite** if $\mathcal{H}_\infty := \cap_{k=1}^\infty \mathrm{Dom}\, D^k$ is a finite projective $\mathcal{A}$-bimodule and **absolutely continuous** if, there exists an Hermitian form $(\xi, \eta) \mapsto (\xi \mid \eta)$ on $\mathcal{H}_\infty$ such that, for all $a \in \mathcal{A}$, $\langle \xi \mid \pi(a)\eta\rangle$ is the Dixmier trace of $\pi(a)(\xi \mid \eta)|D|^{-n}$.

- An n-dimensional spectral triple is said to be **orientable** if there is a Hochschild cycle $c = \sum_{j=1}^m a_0^{(j)} \otimes a_1^{(j)} \otimes \cdots \otimes a_n^{(j)}$ such that its "representation" on the Hilbert space $\mathcal{H}$,

$$\pi(c) = \sum_{j=1}^m \pi(a_0^{(j)})[D, \pi(a_1^{(j)})]_- \cdots [D, \pi(a_n^{(j)})]_-$$

[11] This condition is equivalent to $\pi(a), [D,\pi(a)]_- \in \cap_{m=1}^\infty \mathrm{Dom}\,\delta^m$, for all $a \in \mathcal{A}$, where δ is the derivation given by $\delta(x) := [|D|, x]_-$.

[12] We assume that for $n = 0 \in \mathbb{N}$ the term in the formula simply reduces to $\pi(a_0)$.

is the grading operator in the even case or the identity operator in the odd case[13].

- A real spectral triple is said to satisfy **Poincaré duality** if its fundamental class in the KR-homology of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ induces (via Kasparov intersection product) an isomorphism between the K-theory $\mathrm{K}_\bullet(\mathcal{A})$ and the K-homology $\mathrm{K}^\bullet(\mathcal{A})$ of $\mathcal{A}$.[14]

- A spectral triple will be called **Abelian** or commutative whenever $\mathcal{A}$ is Abelian.

- Finally a spectral triple is **irreducible** if there is no non-trivial closed subspace in $\mathcal{H}$ that is invariant for $\pi(\mathcal{A}), D, J, \Gamma$.

To every spectral triple $(\mathcal{A}, \mathcal{H}, D)$ there is a naturally associated quasi-metric[15] on the set of pure states $\mathcal{P}(\mathcal{A})$, called Connes' distance and given for all pure states ω_1, ω_2 by:

$$d_D(\omega_1, \omega_2) := \sup\{|\omega_1(x) - \omega_2(x)| \mid x \in \mathcal{A}, \ \|[D, \pi(x)]\| \leq 1\}.$$

Theorem 2.4 (Connes; see e.g. [58, 108]). *Given an orientable compact Riemannian spin m-dimensional differentiable manifold M, with a given complex spinor bundle $S(M)$, a given spinorial charge conjugation C_M and a given volume form μ_M,[16] define:*

> $\mathcal{A}_M := C^\infty(M; \mathbb{C})$ *the algebra of complex valued regular functions on the differentiable manifold M,*
>
> $\mathcal{H}_M := L^2(M; S(M))$ *the Hilbert space of "square integrable" sections of the given spinor bundle $S(M)$ of the manifold M i.e. the completion of the space $\Gamma^\infty(M; S(M))$ of smooth sections of the spinor bundle $S(M)$ equipped with the inner product given by $\langle \sigma \mid \tau \rangle := \int_M \langle \sigma(p) \mid \tau(p) \rangle_p \, d\mu_M$, where $\langle \, | \, \rangle_p$, with $p \in M$, is the unique inner product on $S_p(M)$ compatible with the Clifford action and the Clifford product.*

[13] In the following, in order to simplify the discussion, we will always refer to a "grading operator" Γ that actually coincides with the grading operator in the even case and that is by definition the identity operator in the odd case.

[14] In [198] some of the axioms are reformulated in a different form, in particular this condition is replaced by the requirement that the C*-module completion of $\mathcal{H}_\infty$ is a Morita equivalence bimodule between (the norm completions of) $\mathcal{A}$ and $\Omega_D(\mathcal{A})$.

[15] In general d_D can take the value $+\infty$ unless the spectral triple is irreducible.

[16] Remember that an orientable manifolds admits two different orientations and that, on a Riemannian manifold, the choice of an orientation canonically determines a volume form μ_M. Recall also [210] that a spin manifold M admits several inequivalent spinor bundles and for every choice of a complex spinor bundle $S(M)$ (whose isomorphism class define the spin^c structure of M) there are inequivalent choices of spinorial charge conjugations C_M that define, up to bundle isomorphisms, the spin structure of M.

D_M the Atiyah-Singer Dirac operator i.e. the closure of the operator that is obtained by "contracting" the unique spinorial covariant derivative $\nabla^{S(M)}$ (induced on $\Gamma^\infty(M;S(M))$ by the Levi-Civita covariant derivative of M, see [108, Theorem 9.8]) with the Clifford multiplication;

J_M the unique antilinear unitary extension $J_M : \mathcal{H}_M \to \mathcal{H}_M$ of the operator determined by the spinorial charge conjugation C_M, that is defined as $(J_M\sigma)(p) := C_M(\sigma(p))$ for $\sigma \in \Gamma^\infty(M;S(M))$ and $p \in M$;

Γ_M the unique unitary extension on $\mathcal{H}_M$ of the operator given by fiberwise grading on $S_p(M)$, with $p \in M$.[17]

The data $(\mathcal{A}_M, \mathcal{H}_M, D_M)$ define a spectral triple that is Abelian regular finite absolutely continuous m-dimensional real, with real structure J_M, orientable, with grading Γ_M, and that satisfies Poincaré duality.

Theorem 2.5 (Connes [60, 66]). *Let $(\mathcal{A}, \mathcal{H}, D)$ be an irreducible commutative real (with real structure J and grading Γ) strongly regular*[18] *m-dimensional finite absolutely continuous orientable spectral triple, with totally antisymmetric Hochschild cycle in the last m entries, and satisfying Poincaré duality. The spectrum of (the norm closure of) $\mathcal{A}$ can be endowed, in a unique way, with the structure of an m-dimensional connected compact spin Riemannian manifold M with an irreducible complex spinor bundle $S(M)$, a charge conjugation J_M and a grading Γ_M such that: $\mathcal{A} \simeq C^\infty(M;\mathbb{C})$, $\mathcal{H} \simeq L^2(M, S(M))$, $D \simeq D_M$, $J \simeq J_M$, $\Gamma \simeq \Gamma_M$.*

↬ A. Connes first proved the previous theorem under the additional condition that $\mathcal{A}$ is already given as the algebra of smooth complex-valued functions over a differentiable manifold M, namely $\mathcal{A} = C^\infty(M;\mathbb{C})$, and conjectured [61, Theorem 6, Remark (a)] [60] the result for general commutative pre-C*-algebras $\mathcal{A}$.

A tentative proof of this last fact has been published by A. Rennie [194]; some gaps were pointed out in the original argument, a different revised, but still incorrect, proof appears in [198] (see also [199]) under some additional technical conditions. Recently A. Connes [66] finally provided the missing steps in the proof of the result.

As a consequence, there exists a one-to-one correspondence between unitary equivalence classes of spectral triples and connected compact oriented Riemannian spin manifolds up to spin-preserving isometric diffeomorphisms.

Similar results should also be available for spin^c manifolds [61, Theorem 6, Remark (e)].

[17] The grading is actually the identity in odd dimension.
[18] In the sense of [66, Definition 6.1].

2.3 Examples.

Of course, the most inspiring examples of spectral triples (starting from those arising from Riemannian spin-manifolds) are contained in A. Connes' book [58] and an updated account of most of the available constructions is contained in A. Connes-M. Marcolli's lecture notes [74]. Here below we provide a short guide to some of the relevant literature:

- Abelian spectral triples arising from the Atiyah-Singer Dirac Operator on Riemannian spin manifolds, A. Connes [58], and classical compact homogeneous spaces, M. Rieffel [203].

- Spectral triples for the non-commutative tori, A. Connes [58].

- Discrete spectral triples, T. Krajewski [152], M. Paschke-A. Sitarz [188].

- Spectral triples from Moyal planes (these are examples of "non-compact" triples), V. Gayral-J.M. Gracia-Bondia-B. Iochum-T. Schüker-J. Varilly [114].

- Examples of Non-commutative Lorentzian Spectral Triples (following the definition given by A. Strohmaier [213]), W. D. Suijlekom [214].

- Spectral Triples related to the Kronecker foliation (following the general construction by A. Connes-H. Moscovici [76] of spectral triples associated to crossed product algebras related to foliations), R. Matthes-O. Richter-G. Rudolph [180].

- Dirac operators as multiplication by length functions on finitely generated discrete (amenable) groups, A. Connes [57], M. Rieffel [201].

- K-cycles and (twisted) spectral triples arising from supersymmetric quantum field theory, A. Jaffe-A. Lesniewski-K. Osterwalder [133, 134], D. Kastler [138], A. Connes [58], D. Goswami [120].

- Spectral triples associated to quantum groups (in some case it is necessary to modify the first order condition involving the Dirac operator, requiring it to hold only up to compact operators), P. Chakraborty-A. Pal [39, 40, 41, 42, 43, 44, 45, 46, 47], D. Goswami [119], A. Connes [63], L.Dabrowski-G.Landi-A.Sitarz-W.van Suijlekom-J. Varilly [92, 93], J. Kustermans-G. Murphy-L. Tuset [156], S. Neshveyev-L. Tuset [184]; and also spectral triples associated to homogeneus spaces of quantum groups, see e.g. L. Dabrowski [88], L. Dabrowski-G. Landi-M. Paschke- A. Sitarz [91], F. D'Andrea-L. Dabrowski [96], F. D'Andrea-L. Dabrowski-G. Landi [97], [95] (the latter is "twisted" according to A. Connes-H. Moscovici [78]).

- Non-commutative manifolds and instantons, A. Connes-G. Landi [72], L. Dabrowski G. Landi-T. Masuda [90], L. Dabrowski-G. Landi [89], G. Landi [159, 160], G. Landi-W. van Suijlekom [163, 164].
- Non-commutative spherical manifolds A. Connes-M. Dubois-Violette [68, 69, 70].
- Spectral triples for some classes of fractal spaces, A. Connes [58], D. Guido-T. Isola [124, 125, 126], C. Antonescu-E. Christensen [53], E. Christensen C. Ivan-M. Lapidus [54].
- Spectral Triples for AF C*-algebras, C. Antonescu-E. Christensen [53].
- Spectral triples in number theory: A. Connes [58], A. Connes-M. Marcolli [74], R. Meyer [181]; spectral triples from Arakelov Geometry, from Mumford curves and hyperbolic Riemann surfaces, C. Consani-M. Marcolli [79, 80, 81, 82], G. Cornelissen-M. Marcolli-K. Reihani-A. Vdovina [84], G. Cornelissen-M. Marcolli [83].
- Spectral triples of the standard model in particle physics, for instance A. Connes-J. Lott [73], J. Gracia-Bondia-J.Varilly [123], D. Kastler [140, 141], A. Connes [59, 60, 65], J. Barrett [10], A. Chamseddine-A. Connes [48, 49, 50], A. Chamseddine-A. Connes-M. Marcolli [51], A. Connes-M. Marcolli [74, 75].

2.4 Other Spectral Geometries.

In the last few years several others variants and extensions of "spectral geometries" have been considered or proposed:

- Lorentzian spectral triples (A. Strohmaier [213], M.Paschke-R.Verch [191] and also M. Paschke-A. Sitarz [189]),
- Riemannian non-spin (S. Lord [169]),
- Laplacian, Kähler (J. Fröhlich-O. Grandjean-A. Recknagel [110, 111, 112, 113]),
- Following works by M. Breuer [23, 24] on Fredholm modules on von Neumann algebras, M-T. Benameur-T. Fack [12] and more recently in a series of papers [29, 35, 36, 31, 32, 33, 34, 37, 11, 192, 30], M-T. Benameur-A. Carey-D. Pask-J. Phillips-A. Rennie-F. Sukochev-K. Wojciechowski (see also J. Kaad-R. Nest-A. Rennie [135]), have been trying to generalize the formalism of Connes' spectral triples when the algebra of bounded operators on the Hilbert space of the triple is replaced by a more general semifinite von Neumann algebra.

- Although non-commutative differential geometry, following A. Connes, has been mainly developed in the axiomatic framework of spectral triples, that essentially generalize the structures available for the Atiyah-Singer theory of first order differential elliptic operators of the Dirac type, it is very likely that suitable "spectral geometries" might be developed using operators of higher order (the Laplacian type being the first notable example). Since "topological obstructions" (such us non-orientability, non-spinoriality) are expected to survive essentially unaltered in the transition from the commutative to the non commutative world, these "higher-order non-commutative geometries" will deal with more general situations compared to usual spectral triples.

- Apart from the "spectral approaches" to non-commutative geometry, more or less directly inspired by A. Connes' spectral triples, there are other lines of development that are worth investigating and whose "relation" with spectral triples is not yet clear:

  - J.-L. Sauvageot [209] and F. Cipriani [55] are developing a version of non-commutative geometry described by Hilbert C*-bimodules associated to a semigroup of completely positive contractions, an approach that is directly related to the analysis of the properties of the heat-kernel of the Laplacian on Riemannian manifolds (see N. Berline-E. Getzler-M. Vergne [13]);
  - M. Rieffel [202], and along similar lines N. Weaver [225, 226], have developed a theory of non-commutative compact metric spaces based on Lipschitz algebras.
  - Following an idea of G. Parfionov-R. Zapatrin [186], V. Moretti [183] has generalized Connes' distance formula (using the D'Alembert operator) to the case of Lorentzian globally hyperbolic manifolds and has developed an approach to Lorentzian non-commutative geometry based on C*-algebras whose relations with Strohmaier's spectral triples is intriguing.
  - In algebraic quantum field theory (see section 4.2), S. Doplicher-K. Fredenhagen J. Roberts [104, 105] (and also S. Doplicher [101, 102, 103]) have developed a model of Poincaré covariant quantum spacetime.
  - O. Bratteli and collaborators [21, 22] and later M. Madore [170] have been approaching the definition of non-commutative differential geometries through modules of derivations over the algebra of "smooth functions".
  - Strictly related to the previous approach there is a formidable literature (see for example S. Majid [174, 175]) on non-commutative geometry based on "quantum groups" structures (Hopf algebras).

– Most of the physics literature use the term non-commutative geometry to indicate non-commutative spaces obtained by a quantum "deformation" of a classical commutative space.

3 Categories in Non-Commutative Geometry.

After the discussion of "objects" in non-commutative geometry, we now shift our attention to some tentative definitions of morphism of non-commutative spaces and of categories of non-commutative spaces.

In the first subsection we present morphisms of "spectral geometries". We limit our discussion essentially to the case of morphisms of A. Connes' spectral triples, although we expect that similar notions might be developed also for other spectral geometries.

In the second subsection we describe some other extremely important categories of "non-commutative spaces" that arise, at the "topological level", from "variations on the theme" of Morita equivalence.

Finally we indicate some direction of future research.

3.1 Morphisms of Spectral Triples.

Having described A. Connes spectral triples and somehow justified the fact that spectral triples are a possible definition for "non-commutative" compact finite-dimensional orientable Riemannian spin manifolds, our next goal here is to discuss definitions of "morphisms" between spectral triples and to construct categories of spectral triples.

Even for spectral triples, there are actually several possible notions of morphism, according to the amount of "background structure" of the manifold that we would like to see preserved:[19]

- the metric, globally (isometries),
- the metric, locally (totally geodesic maps, in the differentiable case),
- the Riemannian structure,
- the differentiable structure,

3.1.1 Totally-Geodesic-Spin Morphisms.

This is the notion of morphism of spectral triples that we proposed in [15].

[19] And also depending on the kind of topological properties that we would like to "attach" to our morphisms: orientation, spinoriality, ...

Given two spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, with $j = 1, 2$, a **morphism of spectral triples** is a pair

$$(\mathcal{A}_1, \mathcal{H}_1, D_1) \xrightarrow{(\phi, \Phi)} (\mathcal{A}_2, \mathcal{H}_2, D_2),$$

where $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a $*$-morphism between the pre-C*-algebras $\mathcal{A}_1, \mathcal{A}_2$ and $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear map in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ that "intertwines" the representations $\pi_1, \pi_2 \circ \phi$ and the Dirac operators D_1, D_2 :

$$\begin{aligned} \pi_2(\phi(x)) \circ \Phi = \Phi \circ \pi_1(x), \quad \forall x \in \mathcal{A}_1, \\ D_2 \circ \Phi = \Phi \circ D_1, \end{aligned} \tag{3.1}$$

i.e. such that the following diagrams commute for every $x \in \mathcal{A}_1$:

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \\ {\scriptstyle D_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle D_2} \\ \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \end{array} \qquad \begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \\ {\scriptstyle \pi_1(x)}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \pi_2\circ\phi(x)} \\ \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \end{array}$$

Of course, the intertwining relation between the Dirac operators makes sense only on the domain of D_1.

It is possible (in the case of even and/or real spectral triples) to require also commutations between Φ and the grading operators and/or the real structures. More specifically:

> a **morphism of real spectral triples** $(\mathcal{A}_j, \mathcal{H}_j, D_j, J_j)$, is a morphism of spectral triples, as above, such that Φ also "intertwines" the real structure operators J_1, J_2: $J_2 \circ \Phi = \Phi \circ J_1$;
>
> a **morphism of even spectral triples** $(\mathcal{A}_j, \mathcal{H}_j, D_j, \Gamma_j)$, with $j = 1, 2$, is a morphism of spectral triples, as above, such that Φ also "intertwines" with the grading operators Γ_1, Γ_2: $\Gamma_2 \circ \Phi = \Phi \circ \Gamma_1$.

Clearly this definition of morphism contains as a special case the notion of (unitary) equivalence of spectral triples [108, pp. 485-486] and implies quite a strong relationship between the spectra of the Dirac operators of the two spectral triples.

Loosely speaking, for ϕ epi and Φ coisometric (respectively mono and isometric), in the commutative case one expects such definition to become relevant only for maps that "preserve the geodesic structures" (totally geodesic immersions and respectively totally geodesic submersions). Note that (already in the commutative case) these maps might not necessarily be metric isometries: totally geodesic maps are local isometries but not always global isometries (but we do not have a counterexample yet).

Furthermore these morphisms depend, at least in some sense, on the spin structures:[20] this "spinorial rigidity" (at least in the case of morphisms of real even spectral triples) requires that such morphisms between spectral triples of different dimensions might be possible only when the difference in dimension is a multiple of 8.

It might be interesting to examine alternative sets of conditions on the pairs (ϕ, Φ) that allow for example to formalize the notion of "immersion" of a non-commutative manifold into another with arbitrary higher dimension, avoiding the requirements coming from the spinorial structures. Some preliminary considerations along similar lines have been independently proposed by A. Sitarz [211] in his habilitation thesis. There it was suggested that the appropriate morphisms satisfy some "graded intertwining relations" with the relevant operators, indicating the possibility to formalize suitable sign rules depending on the involved dimensions (modulo 8). We plan to elaborate on this topic elsewhere.

3.1.2 Metric Morphisms.

In [16] we introduce the following notion of metric morphisms. Given two spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, with $j = 1, 2$, denote by $\mathcal{P}(\mathcal{A}_j)$ the sets of pure states over (the norm closure of) $\mathcal{A}_j$. A **metric morphism** of spectral triples

$$(\mathcal{A}_1, \mathcal{H}_1, D_1) \xrightarrow{\phi} (\mathcal{A}_2, \mathcal{H}_2, D_2)$$

is by definition a unital epimorphism[21] $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ of pre-C*-algebras whose pull-back $\phi^\bullet : \mathcal{P}(\mathcal{A}_2) \to \mathcal{P}(\mathcal{A}_1)$ is an isometry, i.e.

$$d_{D_1}(\phi^\bullet(\omega_1), \phi^\bullet(\omega_2)) = d_{D_2}(\omega_1, \omega_2), \quad \forall \omega_1, \omega_2 \in \mathcal{P}(\mathcal{A}_2).$$

This notion of morphism is "essentially blind" to the spin structures of the non-commutative manifolds (that in this case appears only as a necessary complication[22]).

3.1.3 Riemannian Morphisms.

A less rigid notion of morphism of spectral triples (a definition that, for unitary maps, was introduced by R. Verch and M. Paschke [190]) consists of relaxing the "intertwining" condition (3.1) between Φ and the Dirac operators, imposing

[20] In the case of morphisms of even real spectral triples, the map should preserve in the strongest possible sense the spin and orientation structures of the manifolds (whatever this might mean).

[21] Note that if ϕ is an epimorphism, its pull-back $\phi^\bullet$ maps pure states into pure states.

[22] Since it is possible to define functional distances using also Laplacian operators, we expect this notion to continue to make sense once a suitable notion of "Laplacian non-commutative manifold" is developed.

only "intertwining relations" with the commutators of Dirac operators with elements of the algebras. In more detail: given two spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, with $j = 1, 2$, a **Riemannian morphism of spectral triples** is a pair

$$(\mathcal{A}_1, \mathcal{H}_1, D_1) \xrightarrow{(\phi,\Phi)} (\mathcal{A}_2, \mathcal{H}_2, D_2),$$

where $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a $*$-morphism between the pre-C*-algebras $\mathcal{A}_1, \mathcal{A}_2$ and $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear map in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ that "intertwines" the representations $\pi_1, \pi_2 \circ \phi$ and the commutators of the Dirac operators D_1, D_2 with the elements $x \in \mathcal{A}_1, \phi(x) \in \mathcal{A}_2$:

$$\pi_2(\phi(x)) \circ \Phi = \Phi \circ \pi_1(x), \quad \forall x \in \mathcal{A}_1,$$
$$[D_2, \phi(x)] \circ \Phi = \Phi \circ [D_1, x], \quad \forall x \in \mathcal{A}_1,$$

i.e. such that the following diagrams commute for every $x \in \mathcal{A}_1$:

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \\ {\scriptstyle [D_1,x]}\downarrow & \circlearrowleft & \downarrow{\scriptstyle [D_2,\phi(x)]} \\ \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \end{array} \qquad \begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \\ {\scriptstyle \pi_1(x)}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \pi_2\circ\phi(x)} \\ \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \end{array}$$

Again the intertwining relation containing the Dirac operators makes sense only on the relevant domain.

In the commutative case, when ϕ is epi and Φ is coisometric (respectively mono and isometric), this definition is expected to correspond to the Riemannian isometries (respectively coisometries) of compact finite-dimensional orientable Riemannian spin manifolds.

- These notions of morphism of spectral triples are only tentative and more examples need to be tested. As pointed out by A. Rennie, it is likely that the "correct" definition of morphism will evolve, but it will surely reflect the basic structure suggested here. At the "topological level" pair of maps (ϕ, Φ) that intertwine the actions of the algebras on the respective Hilbert spaces (but not the Dirac operators or their commutators), have recently been used by P. Ivankov-N. Ivankov [131] for the definition of finite covering (and fundamental group) of a spectral triple.

- The several notions of morphism of spectral triples described above are not as general as possible. In a wider perspective, a morphism of spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, where $j = 1, 2$, might be formalized as a "suitable" functor $\mathcal{F} : {}_{\mathcal{A}_2}\mathscr{M} \to {}_{\mathcal{A}_1}\mathscr{M}$, between the categories ${}_{\mathcal{A}_j}\mathscr{M}$ of $\mathcal{A}_j$-modules, having "appropriate intertwining" properties with the Dirac operators D_j. Now, under some "mild" hypothesis, by Eilenberg-Gabriel-Watt theorem, any such functor is given by "tensorization" by a bimodule. These bimodules, suitably equipped with spectral data (as in the case of spectral triples), provide the natural setting for a general theory of morphisms of non-commutative spaces.

3.1.4 Morita Morphisms.

In the previous subsections we described in some detail some proposed notions of morphism of "non-commutative spaces" (described as spectral triples) at the "metric" level. A few other discussions of non-commutative geometry in a suitable categorical framework, have already appeared in the literature in a more or less explicit form. Most of them deal essentially with morphisms at the "topological level" and are making use of the notion of Morita equivalence that we are going to introduce.

Definition 3.1. *Two unital C*-algebras $\mathcal{A}, \mathcal{B}$ are said to be* ***strongly Morita equivalent*** *if there exists an imprimitivity bimodule ${}_{\mathcal{A}}X_{\mathcal{B}}$.*

It is a standard procedure in algebraic geometry, to define "spaces" dually by their "spectra" i.e. by the categories of (equivalence classes of) representations of their algebras. Hence, for a given unital C*-algebra $\mathcal{A}$, we consider its category ${}_{\mathcal{A}}\mathscr{M}$ of (isomorphism classes of) left C*-Hilbert $\mathcal{A}$-modules with morphisms given by (equivalence classes of) $\mathcal{A}$-linear module maps.

Morphisms between these "non-commutative spectra" are given by covariant functors between the categories of modules.[23]

The Eilenberg-Gabriel-Watt theorem assures that under suitable conditions every functor $\mathfrak{F} : {}_{\mathcal{A}}\mathscr{M} \to {}_{\mathcal{B}}\mathscr{M}$ coincides "up to a natural equivalence" with the functor given by left tensorization with a C*-Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule ${}_{\mathcal{B}}X_{\mathcal{A}}$ (with X unique up to isomorphism of bimodules) i.e.:

$$\mathfrak{F}({}_{\mathcal{A}}E) \simeq {}_{\mathcal{B}}X_{\mathcal{A}} \otimes {}_{\mathcal{A}}E.$$

Y. Manin [176] has been advocating the use of such "Morita morphisms" (tensorizations with Hilbert C*-bimodules) as the natural notion of morphism of non-commutative spaces. In [59, 60, 62] A. Connes already discussed how to transfer a given Dirac operator using Morita equivalence bimodules and compatible connections on them, thus leading to the concept of "inner deformations" of a spectral geometry underlying the "transformation rule" $\widetilde{D} = D + A + JAJ^{-1}$ (where A denotes the "connection"). It is possible to define a strictly related category of spectral triples, based on the notions of connection on a Morita morphism, that contains "inner deformations" as isomorphisms.

More specifically, given two spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, with $j = 1, 2$, by a **Morita-Connes** morphism of spectral triples, we mean a pair (X, ∇) where

[23] This kind of "ideology" about categories of "non-commutative spectra" is very fashionable in "non-commutative algebraic geometry" (see for example M. Kontsevich and A. Rosenberg [145, 146, 205]).

X is Morita morphism from $\mathcal{A}_1$ to $\mathcal{A}_2$ i.e. an $\mathcal{A}_2$-$\mathcal{A}_1$-bimodule that is a Hilbert C*-module over $\mathcal{A}_2$ and ∇ is a Riemannian connection on the bimodule X (the Dirac operators are related to the connection ∇ by the "inner deformation" formula). The composition of two Morita-Connes morphisms (X^1, ∇^1) and (X^2, ∇^2) is defined by taking the tensor product $X^3 := X^1 \otimes_{\mathcal{A}_2} X^2$ of the bimodules and taking the connection ∇^3 on X^3 given by:

$$\nabla^3(\xi_1 \otimes \xi_2)(h_1) := \xi_1 \otimes (\nabla^2 \xi_2)(h_1) + (\nabla^1 \xi_1)(\xi_2 \otimes h_1), \quad h_1 \in \mathcal{H}_1,\ \xi_j \in X^j.$$

In a remarkable recent paper, A. Connes-C. Consani-M. Marcolli [67] have been pushing even further the notion of "Morita morphism" defining morphisms between two algebras $\mathcal{A}, \mathcal{B}$ as "homotopy classes" of bimodules in G. Kasparov KK-theory $KK(\mathcal{A}, \mathcal{B})$. In this way, every morphism is determined by a bimodule that is further equipped with additional structure (Fredholm module).

In the same paper [67], A. Connes and collaborators provide ground for considering "cyclic cohomology" as an "absolute cohomology of non-commutative motives" and the category of modules over the "cyclic category" (already defined by A. Connes-H. Moscovici [77]) as a "non-commutative motivic cohomology" .

- ↬ All the notions of categories of non-commutative spaces developed from the notion of Morita morphism, seem to be confined to the topological setting. Morita equivalence in itself is a non-commutative "topological" notion. It is widely believed that Morita equivalent algebras should be considered as describing the "same" space. This comes from the fact that most of the "geometric functors" for commutative spaces when suitably extended to the non-commutative case are invariant under Morita equivalences (because Morita equivalence reduces to isomorphism for commutative algebras). Anyway, most of the success of Connes' non-commutative geometry actually comes from the fact that some commutative algebras are replaced with some other Morita equivalent non-commutative algebras that are able to describe in a much better way the geometry of the "singular space". In a more direct way, it seems that the correct way to associate a C*-algebra to a space, requires the direct input of the natural symmetries of the space (hence Morita equivalence is broken).

  Although the formalization of the notion of morphism as a bimodule is probably here to stay, additional structures on the bimodule will be required to account for different level of "rigidity" (metric, Riemannian, differential, ...) and some of these, are probably going to break Morita equivariance as long as non-topological properties are concerned.

- ↬ Finally we note that we have not been discussing here the role of quantum groups as possible symmetries of spectral triples (see for example the recent paper by D. Goswami [121] discussing quantum isometries of spectral triples).

3.2 Categorification (Topological Level).

Categorification is the term, introduced by L. Crane-D. Yetter [87], to denote the generic process to substitute ordinary algebraic structures with categorical counterparts. The term is now mostly used to denote a wide area of research (see J. Baez - J. Dolan [7]) whose purpose is to use higher order categories to define categorial analogs of algebraic structures. This **vertical categorification**[24] is usually done by promoting sets to categories, functions to functors, ... hence replacing a category with a bi-category and so on. In non-commutative geometry, where usually spaces are defined "dually" by "spectra" i.e. categories of representations of their algebras of functions, this is a kind of compulsory step: morphisms of non-commutative spaces are actually particular functors between "spectra". In this sense, non-commutative geometry (and also ordinary commutative algebraic geometry of schemes) is already a kind of vertical categorification.

There are also more "trivial" forms of **horizontal categorification** in which ordinary algebraic associative structures are interpreted as categories with only one object and suitable analog categories with more than one object are defined. In this case the passage is from endomorphisms of a single object to morphisms between different objects[25]:

Monoids	Small Categories (Monoidoids)
Groups	Groupoids
Associative Unital Rings	Ringoids
Associative Unital Algebras	Algebroids
Unital C*-algebras	C*-categories (C*-algebroids)

It is an extremely interesting future topic of investigation to discuss the interplay between ideas of categorification and non-commutative geometry ... Here we are really only at the beginning of a long journey and we can present only a few ideas.[26]

3.2.1 Horizontal Categorification of Gel'fand Duality.

As a first step in the development of a "categorical non-commutative geometry", we have been looking at a possible "horizontal categorification" of Gel'fand duality (theorem 2.1). In practice, the purpose is:

[24] In general a n-category get replaced with a $n+1$-category, increasing the "depth" of the available morphisms, hence the terminology "vertical" adopted here.

[25] Hence the name "horizontal", adopted here, that implies that no jump in the "depth" of morphisms is required. J. Baez [21] prefers to use the term **oidization** for this case.

[26] Other approaches to the abstract concept of "categorification" have turned out to be useful in the theory of knots and links, see [143, 144].

- to find "suitable embedding functors" $F : \mathscr{T}^{(1)} \to \mathscr{T}$ and $G : \mathscr{A}^{(1)} \to \mathscr{A}$ of the categories $\mathscr{T}^{(1)}$ (of compact Hausdorff topological spaces) and $\mathscr{A}^{(1)}$ (of unital commutative C*-algebras) into two categories $\mathscr{T}$ and $\mathscr{A}$;

- to extend the categorical duality $(\Gamma^{(1)}, \Sigma^{(1)})$ between $\mathscr{T}^{(1)}$ and $\mathscr{A}^{(1)}$ provided by Gel'fand theorem, to a categorical duality between $\mathscr{T}$ and $\mathscr{A}$ in such a way that the following diagrams are commutative up to natural isomorphisms η, ξ:

$$\begin{array}{ccc} \mathscr{T}^{(1)} & \underset{\Sigma^{(1)}}{\overset{\Gamma^{(1)}}{\longleftrightarrow}} & \mathscr{A}^{(1)} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \mathscr{T} & \underset{\Sigma}{\overset{\Gamma}{\longleftrightarrow}} & \mathscr{A}, \end{array} \qquad \begin{array}{c} F \circ \Sigma^{(1)} \xrightarrow{\eta} \Sigma \circ G, \\ \\ G \circ \Gamma^{(1)} \xrightarrow[\xi]{} \Gamma \circ F. \end{array}$$

Since $\mathscr{A}^{(1)}$ is a full subcategory of the category of C*-algebras, we identify the horizontal categorification of $\mathscr{A}^{(1)}$ as a subcategory of the category of small C*-categories.

In [17], in the setting of C*-categories, we provide a definition of "spectrum" of a commutative full C*-category as a one dimensional (saturated) unital Fell-bundle over a suitable groupoid (equivalence relation) and we prove a categorical Gel'fand duality theorem generalizing the usual Gel'fand duality between the categories of Abelian C*-algebras and compact Hausdorff spaces.

As a byproduct, we also obtain the following spectral theorem for imprimitivity bimodules over Abelian C*-algebras: every such bimodule is obtained by "twisting" (by the 2 projection homeomorphisms) the symmetric bimodule of sections of a unique Hermitian line bundle over the graph of a unique homeomorphism between the spectra of the two C*-algebras.

Theorem 3.2 (P. Bertozzini-R. Conti-W. Lewkeeratiyutkul [18]). *Given an imprimitivity Hilbert C*-bimodule* ${}_{\mathcal{A}}M_{\mathcal{B}}$ *over the Abelian unital C*-algebras* $\mathcal{A}, \mathcal{B}$, *there exists a canonical homeomorphism*[27] $R_{BA} : \mathrm{Sp}(\mathcal{A}) \to \mathrm{Sp}(\mathcal{B})$ *and a Hermitian line bundle* E *over* R_{BA} *such that* ${}_{\mathcal{A}}M_{\mathcal{B}}$ *is isomorphic to the (left/right) "twisting"*[28] *of the symmetric bimodule* $\Gamma(R_{BA}; E)_{C(R_{BA};\mathbb{C})}$ *of sections of the bundle* E *by the "pull-back" isomorphisms* $\pi_A^\bullet : \mathcal{A} \to C(R_{BA}; \mathbb{C})$, $\pi_B^\bullet : \mathcal{B} \to C(R_{BA}; \mathbb{C})$.

↬ This reconstruction theorem for imprimitivity bimodules is actually only the starting point for the development of a complete "bivariant" version of Serre-Swan and Takahashi's dualities. In this case we will generalize

[27] R_{BA} is a compact Hausdorff subspace of $\mathrm{Sp}(\mathcal{A}) \times \mathrm{Sp}(\mathcal{B})$ homeomorphic to $\mathrm{Sp}(\mathcal{A})$ (resp. $\mathrm{Sp}(\mathcal{B})$) via the projections $\pi_A : R_{BA} \to \mathrm{Sp}(\mathcal{A})$ (resp. $\pi_B : R_{BA} \to \mathrm{Sp}(\mathcal{B})$).

[28] If M is a left module over $\mathcal{C}$ and $\phi : \mathcal{A} \to \mathcal{C}$ is an isomorphism, the left twisting of M by ϕ is the module over $\mathcal{A}$ defined by $a \cdot x := \phi(a)x$ for $a \in \mathcal{A}$ and $x \in M$.

the previous spectral theorem to (classes of) bimodules over commutative unital C*-algebras that are more general than imprimitivity bimodules; furthermore the appropriate notion of morphism will be introduced in order to get a categorical duality. We plan to return to this subject elsewhere.

A **C*-category** [118, 182] is a category $\mathcal{C}$ such that the sets $\mathcal{C}_{AB} := \mathrm{Hom}_{\mathcal{C}}(B, A)$ are complex Banach spaces and the compositions are bilinear maps, there is an involutive antilinear contravariant functor $* : \mathrm{Hom}_{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{C}}$ acting identically on the objects such that x^*x is a positive element in the $*$-algebra $\mathcal{C}_{AA}$ for every $x \in \mathcal{C}_{BA}$ (that is, $x^*x = y^*y$ for some $y \in \mathcal{C}_{AA}$), $\|xy\| \leq \|x\| \cdot \|y\|$, $\forall x \in \mathcal{C}_{AB}$, $y \in \mathcal{C}_{BC}$, $\|x^*x\| = \|x\|^2$, $\forall x \in \mathcal{C}_{BA}$.

In a C*-category $\mathcal{C}$, the sets $\mathcal{C}_{AA} := \mathrm{Hom}_{\mathcal{C}}(A, A)$ are unital C*-algebras for all $A \in \mathrm{Ob}_{\mathcal{C}}$. The sets $\mathcal{C}_{AB} := \mathrm{Hom}_{\mathcal{C}}(B, A)$ have a natural structure of unital Hilbert C*-bimodule on the C*-algebras $\mathcal{C}_{AA}$ on the right and $\mathcal{C}_{BB}$ on the left.

A C*-category is **commutative** if the C*-algebras $\mathcal{C}_{AA}$ are Abelian for all $A \in \mathrm{Ob}_{\mathcal{C}}$. The C*-category $\mathcal{C}$ is **full** if all the bimodules $\mathcal{C}_{AB}$ are full[29]. A basic example is the C*-category of linear bounded maps between Hilbert spaces.

A **Banach bundle** [107, Section I.13] (E, p, X) is given by a continuous open surjection $p : E \to X$ of Hausdorff topological spaces, whose total space E is equipped with a continuous partial operation of addition $+ : \{(e_1, e_2) \mid p(e_1) = p(e_2)\} \to E$, a continuous operation of multiplication by scalars $\cdot : \mathbb{C} \times E \to E$ and a continuous norm $\| \cdot \| : E \to \mathbb{R}$, making all the fibers $E_x := p^{-1}(x)$ Banach spaces and such that, for all $x \in X$, the sets of the form $B_{U,\epsilon} := \{e \in E \mid p(e) \in U, \|e\| < \epsilon\}$, where $\epsilon > 0$ and U is a neighbourhood of $x \in X$, constitute a base of neighbourhoods of $0_x \in E_x$ in the topology of E.

If the topological space X is equipped with the algebraic structure of category (let X^o be the set of its units, $r, s : X \to X^o$ its range and source maps and $X^n := \{(x_1, \ldots, x_n) \in \times_{j=1}^n X \mid s(x_j) = r(x_{j+1})\}$ its set of n-composable morphisms), we further require that the composition $\circ : X^2 \to X$ is a continuous map.

If X is an involutive category i.e. there is a map $* : X \to X$ with the properties $(x^*)^* = x$ and $(x \circ y)^* = y^* \circ x^*$, for all $(x, y) \in X^2$, we also require $*$ to be continuous.

A **Fell bundle** [107, 155, 17] over the involutive category X is a Banach bundle (E, p, X) whose total space E is equipped with a continuous multiplication defined on the set $E^2 := \{(e, f) \mid (p(e), p(f)) \in X^2\}$, denoted by

[29] In this case $\mathcal{C}_{AB}$ are imprimitivity bimodules.

$(e, f) \mapsto ef$, and a continuous involution $* : E \to E$, $* : e \mapsto e^*$ such that

$$
\begin{gathered}
e(fg) = (ef)g, \quad \forall (p(e), p(f), p(g)) \in X^3, \\
p(ef) = p(e) \circ p(f), \quad \forall e, f \in E^2, \\
\forall x, y \in X^2, \text{ the restriction of } (e, f) \mapsto ef \text{ to } E_x \times E_y \text{ is bilinear}, \\
\|ef\| \leq \|e\| \cdot \|f\|, \quad \forall e, f \in E^2, \\
(e^*)^* = e, \quad \forall e \in E, \\
p(e^*) = p(e)^*, \quad \forall e \in E, \\
\forall x \in X, \text{ the restriction of } e \mapsto e^* \text{ to } E_x \text{ is conjugate linear}, \\
(ef)^* = f^* e^*, \quad \forall e, f \in E^2, \\
\|e^* e\| = \|e\|^2, \quad \forall e \in E \text{ such that } p(e^* e) \in X^o, \\
e^* e \geq 0, \quad \forall e \in E, \text{ such that } p(e^* e) \in X^o,
\end{gathered}
$$

where in the last line we mean that e^*e is a positive element in the C*-algebra $E_{p(e^*e)}$. It is in fact easy to see that for every $x \in X^o$, E_x is a C*-algebra.
A Fell bundle (E, p, X) is said to be **unital** if the C*-algebras E_x, for $x \in X^o$, are unital. Note that the fiber E_x has a natural structure of Hilbert C*-bimodule over the C*-algebras $E_{r(x)}$ on the left and $E_{s(x)}$ on the right. A Fell bundle is said to be **saturated** if the above Hilbert C*-bimodules E_x are full. Note also that in a saturated Fell bundle, the Hilbert C*-bimodules E_x are imprimitivity bimodules.

Let $\mathcal{O}$ be a set and X a compact Hausdorff topological space. We denote by $\mathcal{R}_{\mathcal{O}} := \{(A, B) \mid A, B \in \mathcal{O}\}$ the "total" equivalence relation in $\mathcal{O}$ and by $\Delta_X := \{(p, p) \mid p \in X\}$ the "diagonal" equivalence relation in X.

Definition 3.3. *A* ***topological spaceoid*** $(\mathcal{E}, \pi, \mathcal{X})$ *is a saturated unital rank-one Fell bundle over the product involutive topological category* $\mathcal{X} := \Delta_X \times \mathcal{R}_{\mathcal{O}}$.

Definition 3.4. *Let* $(\mathcal{E}_j, \pi_j, \mathcal{X}_j)$*, for* $j = 1, 2$*, be two spaceoids.*[30] *A morphism of spaceoids* $(\mathcal{E}_1, \pi_1, \mathcal{X}_1) \xrightarrow{(f,\mathcal{F})} (\mathcal{E}_2, \pi_2, \mathcal{X}_2)$ *is a pair* $(f, \mathcal{F})$ *where*

- $f := (f_\Delta, f_\mathcal{R})$ *with* $f_\Delta : \Delta_1 \to \Delta_2$ *a continuous map of topological spaces and* $f_\mathcal{R} : \mathcal{R}_1 \to \mathcal{R}_2$ *an isomorphism of equivalence relations;*

- $\mathcal{F} : f^\bullet(\mathcal{E}_2) \to \mathcal{E}_1$ *is a continuous fiberwise linear* $*$*-functor such that* $\pi_1 \circ \mathcal{F} = (\pi_2)^f$*, where* $(f^\bullet(\mathcal{E}_2), \pi_2^f, \mathcal{X}_1)$ *denotes a given choice of an* f*-pull-back of* $(\mathcal{E}_2, \pi_2, \mathcal{X}_2)$.

[30] Where $\mathcal{X}_j = \Delta_{X_j} \times \mathcal{R}_{\mathcal{O}_j}$, with $\mathcal{O}_j$ sets and X_j compact Hausdorff topological spaces for $j = 1, 2$.

Topological spaceoids constitute a category with composition defined by

$$(g, \mathcal{G}) \circ (f, \mathcal{F}) := (g \circ f, \mathcal{F} \circ f^\bullet(\mathcal{G}) \circ \Theta),$$

where Θ is the natural isomorphism from $f^\bullet(g^\bullet(\mathcal{E}_3))$ to $(g \circ f)^\bullet(\mathcal{E}_3)$, and with identities

$$\iota_{(\mathcal{E},\pi,\mathcal{X})} := (\iota_{\mathcal{X}}, \iota_{\mathcal{E}}).$$

Note that we have chosen $(\mathcal{E}, \pi, \mathcal{X})$ to be the $\iota_{\mathcal{X}}$-pull-back of itself.

The category $\mathscr{T}^{(1)}$ of continuous maps between compact Hausdorff spaces can be naturally identified with the full subcategory of the category $\mathscr{T}$ of spaceoids with index set $\mathcal{O}$ containing a single element.

To every object $X \in \mathrm{Ob}_{\mathscr{T}^{(1)}}$ we associate the trivial $\mathbb{C}$-line bundle $\mathcal{X}_X \times \mathbb{C}$ over the involutive category $\mathcal{X}_X := \Delta_X \times \mathcal{R}_{\mathcal{O}_X}$ with $\mathcal{O}_X := \{X\}$ the one point set.

To every continuous map $f : X \to Y$ in $\mathscr{T}^{(1)}$ we associate the morphism $(g, \mathcal{G})$ with $g_\Delta(p, p) := (f(p), f(p))$, $g_\mathcal{R} : (X, X) \mapsto (Y, Y)$ and $\mathcal{G} := \iota_{\mathcal{X}_X \times \mathbb{C}}$.

Note that the trivial bundle over $\mathcal{X}_X$ is naturally a f-bull-back of the trivial bundle over $\mathcal{X}_Y$ hence $\mathcal{G}$ can be taken as the identity map.

Let $\mathcal{C}$ and $\mathcal{D}$ be two full commutative small C*-categories (with the same cardinality of the set of objects). Denote by $\mathcal{C}_o$ and $\mathcal{D}_o$ their sets of identities.

A morphism $\Phi : \mathcal{C} \to \mathcal{D}$ is an object bijective $*$-functor, i.e.

$$\Phi(x + y) = \Phi(x) + \Phi(y), \ \forall x, y \in \mathcal{C}_{AB},$$
$$\Phi(a \cdot x) = a \cdot \Phi(x), \ \forall x \in \mathcal{C}, \ \forall a \in \mathbb{C},$$
$$\Phi(x \circ y) = \Phi(x) \circ \Phi(y), \ \forall x \in \mathcal{C}_{CB}, \ y \in \mathcal{C}_{BA}$$
$$\Phi(x^*) = \Phi(x)^*, \ \forall x \in \mathcal{C}_{AB},$$
$$\Phi(\iota) \in \mathcal{D}_o, \ \forall \iota \in \mathcal{C}_o,$$
$$\Phi_o := \Phi|_{\mathcal{C}_o} : \mathcal{C}_o \to \mathcal{D}_o \quad \text{is bijective.}$$

To every spaceoid $(\mathcal{E}, \pi, \mathcal{X})$, with $\mathcal{X} := \Delta_X \times \mathcal{R}_\mathcal{O}$, we can associate a full commutative C*-category $\Gamma(\mathcal{E})$ as follows:

- $\mathrm{Ob}_{\Gamma(\mathcal{E})} := \mathcal{O}$;
- For all $A, B \in \mathrm{Ob}_{\Gamma(\mathcal{E})}$, $\mathrm{Hom}_{\Gamma(\mathcal{E})}(B, A) := \Gamma(\Delta_X \times \{(A, B)\}; \mathcal{E})$, where we denote with $\Gamma(\Delta_X \times \{(A, B)\}; \mathcal{E})$ the set of continuous sections $\sigma : \Delta_X \times \{(A, B)\} \to \mathcal{E}$, $\sigma : p_{AB} \mapsto \sigma_p^{AB} \in \mathcal{E}_{p_{AB}}$ of the restriction of $\mathcal{E}$ to the base space $\Delta_X \times \{(A, B)\} \subset \mathcal{X}$;

- for all $\sigma \in \mathrm{Hom}_{\Gamma(\mathcal{E})}(A,B)$ and $\rho \in \mathrm{Hom}_{\Gamma(\mathcal{E})}(B,C)$:
$$\rho \circ \sigma : p_{AC} \mapsto (\rho\circ\sigma)_p^{AC} := \rho_p^{AB} \circ \sigma_p^{BC},$$
$$\sigma^* : p_{BA} \mapsto (\sigma^*)_p^{BA} := (\sigma_p^{AB})^*,$$
$$\|\sigma\| := \sup_{p\in\Delta_X} \|\sigma_p^{AB}\|_{\mathcal{E}},$$
with operations taken in the total space $\mathcal{E}$ of the Fell bundle.

We extend now the definition of Γ to the morphism of $\mathscr{T}$ in order to obtain a contravariant functor.

Let $(f,\mathcal{F})$ be a morphism in $\mathscr{T}$ from $(\mathcal{E}_1,\pi_1,\mathcal{X}_1)$ to $(\mathcal{E}_2,\pi_2,\mathcal{X}_2)$.

Given $\sigma \in \Gamma(\mathcal{E}_2)$, we consider the unique section $f^\bullet(\sigma) : \mathcal{X}_1 \to f^\bullet(\mathcal{E}_2)$ such that $f^{\pi_2} \circ f^\bullet(\sigma) = \sigma \circ f$ and the composition $\mathcal{F} \circ f^\bullet(\sigma)$.

In this way we get a map
$$\Gamma_{(f,\mathcal{F})} : \Gamma(\mathcal{E}_2) \to \Gamma(\mathcal{E}_1), \quad \Gamma_{(f,\mathcal{F})} : \sigma \mapsto \mathcal{F}\circ f^\bullet(\sigma), \quad \forall \sigma \in \Gamma(\mathcal{E}_2).$$

Proposition 3.5 ([17]). *For any morphism* $(\mathcal{E}_1,\pi_1,\mathcal{X}_1) \xrightarrow{(f,\mathcal{F})} (\mathcal{E}_2,\pi_2,\mathcal{X}_2)$ *in* $\mathscr{T}$*, the map* $\Gamma_{(f,\mathcal{F})} : \Gamma(\mathcal{E}_2) \to \Gamma(\mathcal{E}_1)$ *is a morphism in* $\mathscr{A}$*.*

The pair of maps $\Gamma : (\mathcal{E},\pi,\mathcal{X}) \mapsto \Gamma(\mathcal{E})$ *and* $\Gamma : (f,\mathcal{F}) \mapsto \Gamma_{(f,\mathcal{F})}$ *gives a contravariant functor from the category* $\mathscr{T}$ *of spaceoids to the category* $\mathscr{A}$ *of small full commutative C*-categories.*

We proceed to associate to every commutative full C*-category $\mathcal{C}$ its spectral spaceoid $\Sigma(\mathcal{C}) := (\mathcal{E}^{\mathcal{C}}, \pi^{\mathcal{C}}, \mathcal{X}^{\mathcal{C}})$, see [17] for details.

- The set $[\mathcal{C};\mathbb{C}]$ of $\mathbb{C}$-valued $*$-functors $\omega : \mathcal{C} \to \mathbb{C}$, with the weakest topology making all evaluations continuous, is a compact Hausdorff topological space.
- By definition two $*$-functors $\omega_1,\omega_2 \in [\mathcal{C};\mathbb{C}]$ are **unitarily equivalent** if there exists a "unitary" natural trasformation $A \mapsto \nu_A \in \mathbb{T}$ between them. This is true iff $\omega_1|_{\mathcal{C}_{AA}} = \omega_2|_{\mathcal{C}_{AA}}$ for all $A \in \mathrm{Ob}_{\mathcal{C}}$.
- Let $\mathrm{Sp}_b(\mathcal{C}) := \{[\omega] \mid \omega \in [\mathcal{C};\mathbb{C}]\}$ denote the **base spectrum** of $\mathcal{C}$, defined as the set of unitary equivalence classes of $*$-functors in $[\mathcal{C};\mathbb{C}]$. It is a compact Hausdorff space with the quotient topology induced by the map $\omega \mapsto [\omega]$.
- Let $\mathcal{X}^{\mathcal{C}} := \Delta^{\mathcal{C}} \times \mathcal{R}^{\mathcal{C}}$ be the direct product topological $*$-category of the compact Hausdorff $*$-category $\Delta^{\mathcal{C}} := \Delta_{\mathrm{Sp}_b(\mathcal{C})}$ and the topologically discrete $*$-category $\mathcal{R}^{\mathcal{C}} := \mathcal{C}/\mathcal{C} \simeq \mathcal{R}_{\mathrm{Ob}_{\mathcal{C}}}$.
- For $\omega \in [\mathcal{C};\mathbb{C}]$, the set $\mathcal{I}_\omega := \{x \in \mathcal{C} \mid \omega(x) = 0\}$ is an ideal in $\mathcal{C}$ and $\mathcal{I}_{\omega_1} = \mathcal{I}_{\omega_2}$ if $[\omega_1] = [\omega_2]$.

- Denoting $[\omega]_{AB}$ the point $([\omega],(A,B)) \in \mathcal{X}^{\mathcal{C}}$, we define:

$$\mathcal{J}_{[\omega]_{AB}} := \mathcal{J}_\omega \cap \mathcal{C}_{AB}, \quad \mathcal{E}^{\mathcal{C}}_{[\omega]_{AB}} := \frac{\mathcal{C}_{AB}}{\mathcal{J}_{[\omega]_{AB}}}, \quad \mathcal{E}^{\mathcal{C}} := \biguplus_{[\omega]_{AB}\in\mathcal{X}^{\mathcal{C}}} \mathcal{E}^{\mathcal{C}}_{[\omega]_{AB}}.$$

Proposition 3.6 ([17]). *The map $\pi^{\mathcal{C}} : \mathcal{E}^{\mathcal{C}} \to \mathcal{X}^{\mathcal{C}}$, that sends an element $e \in \mathcal{E}^{\mathcal{C}}_{[\omega]_{AB}}$ to the point $[\omega]_{AB} \in \mathcal{X}^{\mathcal{C}}$ has a natural structure of unital rank one Fell bundle over the topological involutive category $\mathcal{X}^{\mathcal{C}}$.*

Let $\Phi : \mathcal{C} \to \mathcal{D}$ be an object-bijective $*$-functor between two small commutative full C*-categories with spaceoids $\Sigma(\mathcal{C}), \Sigma(\mathcal{D}) \in \mathscr{T}$.

We define a morphism $\Sigma^\Phi : \Sigma(\mathcal{D}) \xrightarrow{(\lambda^\Phi,\Lambda^\Phi)} \Sigma(\mathcal{C})$ in the category $\mathscr{T}$:

- $\lambda^\Phi : \mathcal{X}^{\mathcal{D}} \xrightarrow{(\lambda^\Phi_\Delta,\lambda^\Phi_{\mathcal{R}})} \mathcal{X}^{\mathcal{C}}$ where
  $\lambda^\Phi_{\mathcal{R}}(A,B) := (\Phi_o^{-1}(A), \Phi_o^{-1}(B))$, for all $(A,B) \in \mathcal{R}_{\mathrm{Ob}_{\mathcal{D}}}$;
  $\lambda^\Phi_\Delta([\omega]) := [\omega \circ \Phi] \in \Delta_{\mathrm{Sp}_b(\mathcal{C})}$, for all $[\omega] \in \Delta_{\mathrm{Sp}_b(\mathcal{D})}$.

- The bundle $\biguplus_{[\omega]_{AB}\in\mathcal{X}^{\mathcal{D}}} \frac{\mathcal{C}_{\lambda^\Phi_{\mathcal{R}}(AB)}}{\mathcal{J}_{\lambda^\Phi([\omega]_{AB})}}$ with the maps
  $\pi^\Phi : ([\omega]_{AB},\ x + \mathcal{J}_{\lambda^\Phi([\omega]_{AB})}) \mapsto [\omega]_{AB} \in \mathcal{X}^{\mathcal{D}}, \quad x \in \mathcal{C}_{\lambda^\Phi_{\mathcal{R}}(AB)}$,
  $\Phi^\pi : ([\omega]_{AB},\ x + \mathcal{J}_{\lambda^\Phi([\omega]_{AB})}) \mapsto (\lambda^\Phi([\omega]_{AB}), x + \mathcal{J}_{\lambda^\Phi([\omega]_{AB})}) \in \mathcal{E}^{\mathcal{C}}$
  is a λ^Φ-pull-back $(\lambda^\Phi)^\bullet(\mathcal{E}^{\mathcal{C}})$ of the Fell bundle $(\mathcal{E}^{\mathcal{C}}, \pi^{\mathcal{C}}, \mathcal{X}^{\mathcal{C}})$.

- Since $\Phi(\mathcal{J}_{\lambda^\Phi([\omega]_{AB})}) \subset \mathcal{J}_{[\omega]_{AB}}$ for $[\omega]_{AB} \in \mathcal{X}^{\mathcal{D}}$, we can define a map
  $\Lambda^\Phi : (\lambda^\Phi)^\bullet(\mathcal{E}^{\mathcal{C}}) \to \mathcal{E}^{\mathcal{D}}$ by
  $\Big([\omega]_{AB},\ x + \mathcal{J}_{\lambda^\Phi([\omega]_{AB})}\Big) \mapsto \Big([\omega]_{AB},\ \Phi(x) + \mathcal{J}_{[\omega]_{AB}}\Big)$.

Proposition 3.7 ([17]). *For any morphism $\mathcal{C} \xrightarrow{\Phi} \mathcal{D}$ in $\mathscr{A}$, the mapping $\Sigma(\mathcal{D}) \xrightarrow{\Sigma^\Phi} \Sigma(\mathcal{C})$ is a morphism of spectral spaceoids. The pair of maps $\Sigma : \mathcal{C} \mapsto \Sigma(\mathcal{C})$ and $\Sigma : \Phi \mapsto \Sigma^\Phi$ give a contravariant functor $\Sigma : \mathscr{A} \to \mathscr{T}$, from the category $\mathscr{A}$ of object-bijective $*$-functors between small commutative full C*-categories to the category $\mathscr{T}$ of spaceoids.*

We can now state our main duality theorem for commutative full C*-categories:

Theorem 3.8 (P. Bertozzini-R.Conti-W. Lewkeeratiyutkul [17]). *There exists a duality (Γ, Σ) between the category $\mathscr{T}$ of object-bijective morphisms between spaceoids and the category $\mathscr{A}$ of object-bijective $*$-functors between small commutative full C*-categories, where*

- Γ *is the functor that to every spaceoid* $(\mathcal{E},\pi,\mathcal{X}) \in \mathrm{Ob}_{\mathscr{T}}$ *associates the small commutative full C*-category* $\Gamma(\mathcal{E})$ *and that to every morphism between spaceoids* $(f,\mathcal{F}) : (\mathcal{E}_1,\pi_1,\mathcal{X}_1) \to (\mathcal{E}_2,\pi_2,\mathcal{X}_2)$ *associates the* $*$*-functor* $\Gamma_{(f,\mathcal{F})}$*;*
- Σ *is the functor that to every small commutative full C*-category* $\mathcal{C}$ *associates its spectral spaceoid* $\Sigma(\mathcal{C})$ *and that to every object-bijective* $*$*-functor* $\Phi : \mathcal{C} \to \mathcal{D}$ *of C*-categories in* $\mathscr{A}$ *associates the morphism* $\Sigma^{\Phi} : \Sigma(\mathcal{D}) \to \Sigma(\mathcal{C})$ *between spaceoids.*

The natural isomorphism $\mathfrak{G} : \mathcal{I}_{\mathscr{A}} \to \Gamma\circ\Sigma$ is provided by the **horizontally categorified Gel'fand transforms** $\mathfrak{G}_{\mathcal{C}} : \mathcal{C} \to \Gamma(\Sigma(\mathcal{C}))$ defined by

$$\mathfrak{G}_{\mathcal{C}} : \mathcal{C} \to \Gamma(\mathcal{E}^{\mathcal{C}}), \quad \mathfrak{G}_{\mathcal{C}} : x \mapsto \hat{x} \quad \text{where}$$
$$\hat{x}^{AB}_{[\omega]} := x + \mathcal{J}_{[\omega]_{AB}}, \quad \forall x \in \mathcal{C}_{AB}.$$

Proposition 3.9 ([17]). *The functor* $\Gamma : \mathscr{T} \to \mathscr{A}$ *is representative i.e. given a commutative full C*-category* $\mathcal{C}$*, the Gel'fand transform* $\mathfrak{G}_{\mathcal{C}} : \mathcal{C} \to \Gamma(\Sigma(\mathcal{C}))$ *is a full isometric (hence faithful)* $*$*-functor.*

The natural isomorphism $\mathfrak{E} : \mathcal{I}_{\mathscr{T}} \to \Sigma\circ\Gamma$ is provided by the **horizontally categorified "evaluation" transforms** $\mathfrak{E}_{\mathcal{E}} : (\mathcal{E},\pi,\mathcal{X}) \xrightarrow{(\eta^{\mathcal{E}},\Omega^{\mathcal{E}})} \Sigma(\Gamma(\mathcal{E}))$, defined as follows:

- $\eta^{\mathcal{E}}_{\mathcal{R}}(A,B) := (A,B), \quad \forall (A,B) \in \mathcal{R}_{\mathcal{O}}$.
- $\eta^{\mathcal{E}}_{\Delta} : \Delta_X \to \Delta_{\mathrm{Sp}_b(\Gamma(\mathcal{E}))}$, $p \mapsto [\gamma\circ \mathrm{ev}_p]$, where $\mathrm{ev}_p : \Gamma(\mathcal{E}) \to \uplus_{(AB)\in\mathcal{R}_{\mathcal{O}}} \mathcal{E}_{p_{AB}}$ is the evaluation map given by $\sigma \mapsto \sigma^{AB}_p$ that is a $*$-functor with values in a one dimensional C*-category that actually determines[31] a unique point $[\gamma\circ \mathrm{ev}_p] \in \Delta_{\mathrm{Sp}_b(\Gamma(\mathcal{E}))}$.
- $\biguplus_{p_{AB}\in\mathcal{X}} \Gamma(\mathcal{E})_{\eta^{\mathcal{E}}_{\mathcal{R}}(AB)}/\mathcal{J}_{\eta^{\mathcal{E}}(p_{AB})}$ when equipped with the natural projection map $(p_{AB},\ \sigma + \mathcal{J}_{\eta^{\mathcal{E}}(p_{AB})}) \mapsto p_{AB}$, and with the $\mathcal{E}^{\Gamma(\mathcal{E})}$-valued function $(p_{AB},\ \sigma + \mathcal{J}_{\eta^{\mathcal{E}}(p_{AB})}) \mapsto \sigma + \mathcal{J}_{\eta^{\mathcal{E}}(p_{AB})}$, is a $\eta^{\mathcal{E}}$-pull-back $(\eta^{\mathcal{E}})^{\bullet}(\mathcal{E}^{\Gamma(\mathcal{E})})$ of $\Sigma(\Gamma(\mathcal{E}))$.
- $\Omega^{\mathcal{E}} : (\eta^{\mathcal{E}})^{\bullet}(\mathcal{E}^{\Gamma(\mathcal{E})}) \to \mathcal{E}$ is defined by
  $\Omega^{\mathcal{E}} : (p_{AB},\ \sigma + \mathcal{J}_{\eta^{\mathcal{E}}(p_{AB})}) \mapsto \sigma^{AB}_p, \quad \forall \sigma \in \Gamma(\mathcal{E})_{AB}, \quad p_{AB} \in \mathcal{X}$.

In particular, with such definitions we can prove:

Proposition 3.10 ([17]). *The functor* $\Sigma : \mathscr{A} \to \mathscr{T}$ *is representative i.e. given a spaceoid* $(\mathcal{E},\pi,\mathcal{X})$*, the evaluation transform* $\mathfrak{E}_{\mathcal{E}} : (\mathcal{E},\pi,\mathcal{X}) \to \Sigma(\Gamma(\mathcal{E}))$ *is an isomorphism in the category of spaceoids.*

[31] There is always a $\mathbb{C}$ valued $*$-functor $\gamma : \uplus_{(AB)\in\mathcal{R}_{\mathcal{O}}} \mathcal{E}_{p_{AB}} \to \mathbb{C}$ and any two compositions of ev_p with such $*$-functors are unitarily equivalent because they coincide on the diagonal C*-algebras $\mathcal{E}_{p_{AA}}$.

We are now working on a number of generalizations and extensions of our horizontal categorified Gel'fand duality:

- The first immediate possibility is to extend Gel'fand duality to include the case of categories of general ∗-functors between full commutative C*-categories. This will necessarily require the consideration of categories of ∗-relators (see [15]) between C*-categories.
- Our duality theorem is for now limited to the case of full commutative C*-categories and further work is necessary in order to extend the result to a Gel'fand duality for non-full C*-categories.
- Very interesting is the possibility to generalize our duality to a full spectral theory for non-commutative C*-categories in term of endofunctors in the category of Fell bundles. In particular we would like to explore if our approach will allow to develope categorifications of Dauns-Hofmann [98] and Cirelli-Manià-Pizzocchero [56] spectral theorems for general non-commutative C*-algebras.
- In the same order of ideas, motivated by a general spectral theory for C*-categories, it is worth investigating in the non-commutative case the connection between C*-categories, spectral spaceoids and categorified notions of (locale) quantale already developed for (commutative) C*-algebras (see D. Kruml-J. Pelletier-P. Resende-J. Rosicky [153], L. Crane [86], D. Kruml-P. Resende [154], P. Resende [200] and references therein for details).
- The existence of a horizontal categorified Gel'fand transform might be relevant for the study of harmonic analysis on commutative groupoids. In this direction it is natural to investigate the implications for a Pontrjagin duality for commutative groupoids and later, in a fully non-commutative context, the relations with the theory of C*-pseudo-multiplicative unitaries that has been recently developed by T. Timmermann [221, 222].
- Extremely intriguing for its possible physical implications in algebraic quantum field theory is the appearance of a natural "local gauge structure" on the spectra: the spectrum is no more just a (topological) space, but a special fiber bundle. Possible relations with the work of E. Vasselli [224] on continous fields of C*-categories in the theory of superselection sectors and especially with the recent work on net bundles and gauge theory by J. Roberts-G. Ruzzi-E. Vasselli [204] remain to be explored.

3.2.2 Higher C*-categories.

In our last forthcoming work, we proceed to further extend the categorification process of Gel'fand duality theorem to a full "vertical categorification" [4].

For this purpose we first provide, via globular sets (see T. Leinster [166]), a suitable definition of "strict" n-C*-category.

In practice, without entering here in further technical details, a strict higher C*-category $\mathcal{C}$ (or more generally a higher Fell bundle over a higher $*$-category $\mathcal{X}$), is provided by a strict higher $*$-category $\mathcal{C}$ fibered over a strict higher $*$-category $\mathcal{X}$ whose compositions and involutions satisfy, fiberwise at all levels, "appropriate versions" of all the properties listed in the definition of a Fell bundle.

In the special case of commutative full strict n-C*-categories, we develope a spectral Gel'fand theorem in term of n-spaceoids i.e. rank-one n-C*-Fell bundles over a "particular" n-$*$-category (that is given by the direct product of the diagonal equivalence relation of a compact Hausdorff space and the quotient n-$*$-category $\mathcal{C}/\mathcal{C}$ of an n-C*-category $\mathcal{C}$).

- Unfortunately our definition is for now limited to the case of strict higher C*-categories. Of course, as always the case in higher category theory, an even more interesting problem will be the characterization of suitable axioms for "weak higher C*-categories". This is one of the main obstacles in the development of a full categorification of the notion of spectral triple and of A. Connes non-commutative geometry.

- Note that several examples and definitions of 2-C*-categories are already available in the literature (see for example R. Longo-J. Roberts [168] and P. Zito [228]). In general such cases will not exactly fit with the strict version of our axioms for n-C*-categories. Actually we expect to have a complete hierarchy of definitions of higher C*-categories depending on the "depth" at which some axioms are required to be satisfied (i.e. some properties can be required to hold only for p-arrows with p higher that a certain depth).

- In our work, we define (Hilbert C*) modules over strict n-C*-categories and in this way we can provide interesting definitions of n-Hilbert spaces and start a development of "higher functional analysis".

3.3 Categorical Non-commutative Geometry and Non-commutative Topoi.

One of the main goals of our investigation is to discuss the interplay between ideas of categorification and non-commutative geometry. Here there is still much to be done and we can present only a few suggestions. Work is in progress.

- Every isomorphism class of a full commutative C*-category can be identified with an equivalence relation in the Picard-Morita 1-category of Abelian unital C*-algebras. In practice a C*-category is just a "strict implementation" of an equivalence relation subcategory of Picard-Morita.

Since morphism of spectral triples (more generally morphisms of non-commutative spaces) are essentially "special cases" of Morita morphisms, we started the study of "spectral triples over C*-categories" and we are now trying to develop a notion of horizontal categorification of spectral triples (and of other spectral geometries) in order to identify a correct definition of morphism of spectral triples that supports a duality with a suitable spectrum (in the commutative case).

The general picture that is emerging is that a correct notion of metric morphism between spectral triples is given by a kind of "bivariant version" of spectral triple i.e. a bimodule over two different algebras that is equipped with a left/right action of "Dirac-like" operators.

- As a very first step in the direction of a full "higher non-commutative geometry"[32] we plan to start the study of a strict version of "higher spectral triples" i.e. spectral triples over strict higher C*-categories. As in the case of horizontal categorification, this will provide some hints for a correct definition of "higher spectral triples".
- Although at the moment it is only a speculative idea, it is very interesting to explore the possible relation between such "higher spectra" (higher spaceoids) and the notions of stacks and gerbes already used in higher gauge theory. The recent work by C. Daenzer [94] in the context of T-duality discuss a Pontryagin duality between commutative principal bundles and gerbes that might be connected with our categorified Gel'fand transform for commutative C*-categories.
- Extremely intriguing is the possible connection between the notions of (category of) spectral triples and A. Grothendieck topoi. Speculations in this direction have been given by P. Cartier [38] and are also discussed by A. Connes [64]. A full (categorical) notion of non-commutative space (non-commutative Klein program / non-commutative Grothendieck topos) is still waiting to be defined.

Actually some interesting proposal for a definition of a "quantum topos" is already available in the recent work by L. Crane [86] based on the notion of "quantaloids", a categorification of the notion of quantale (see P. Resende [200] and references therein).

At this level of generality, it is important to emphasize that our discussion of non-commutative geometry has been essentially confined to the consideration of A. Connes' approach. In the field of algebraic geometry (see V. Ginzburg [116], M. Kontsevich-Y. Soibelman [147, 148] and S. Mahanta [172, 173] as recent

[32] On this topic the reader is strongly advised to read the interesting discussions on the "n-category café" `http://golem.ph.utexas.edu/category/` and in particular: U. Schreiber, Connes Spectral Geometry and the Standard Model II, 06 September 2006.

references), many other people have been trying to propose definitions of non-commutative schemes and non-commutative spaces (see for example A. Rosenberg [205] and M. Kontsevich-A. Rosenberg [137]) as "spectra" of Abelian categories (or generalization of Abelian categories such as triangulated, dg, or A_∞ categories). Since every Abelian category is essentially a category of modules, it is in fact usually assumed that an Abelian category should be considered as a topos of sheaves over a non-commutative space.

- It is worth noting that the categories naturally arising in the theory of self-adjoint operator algebras and in A. Connes' non-commutative geometry are $*$-monoidal categories (see [17] for detailed definitions). The monoidal property is perfectly in line with the recent proposal by T. Maszczyk [179] to construct a theory of algebraic non-commutative geometry based on Abelian categories equipped with a monoidal structure.

  At this point it is actually tempting (in our opinion) to think that also the involutive structures (and other properties strictly related to the existence of an involution including Tomita-Takesaki modular theory are going to play some vital role in the correct definition of a non-commutative generalization of space. But this is still speculation in progress!

- Finally, there are strong indications (V. Dolgushev-D. Tamarkin-B. Tsygan [99])[33] coming again from "algebraic non-commutative geometry" that a proper categorification of non-commutative geometry might actually be possible only considering ∞-categories. The implications for a program of categorification of A. Connes' spectral triples is not yet clear to us.

4 Applications to Physics.

In this final section we would like to spend some time to introduce (in a non-technical way) the mathematical readers to the consideration of some extremely important topics in quantum physics that are essentially motivating the construction of non-commutative spaces, the use of categorical ideas and the eventual merging of these two lines of thought.

The two main subjects of our discussion, non-commutative geometry and category theory, have been separately used and applied in theoretical physics (although not as widely as we would have liked to see). Anyway, our feeling is that the most important input to physics will come from a kind of "combined" approach where non-commutative and categorical structures are applied in a "synergic way" in an "algebraic theory of quantum gravity". A concrete proposal in this direction is presented in section 4.4.

[33] See also the very detailed discussion on the blog "*n*-category café": J. Baez, Infinitely Categorified Calculus, 09 February 2007.

4.1 Categorical Covariance.

Covariance of physical theories has been always discussed in the limited domain of groups acting on spaces: the group of rotations in Aristotles' physics; Galilei's and Poincaré groups and the diffeomorphism groups of Lorentzian manifolds in Einstein's general relativity. Different observers are "related" through transformations in the given covariance group.

$\looparrowright$ There is no deep physical or operational reason to think that only groups (or quantum groups) might be the right mathematical structure to capture the "translation" between different observers and actually, in our opinion, categories provide a much more suitable environment in which also the discussion of "partial translations" between observers can be described. Work is in progress on these issues.

As an example of the relevance of categorical covariance, we mention the works by R. Brunetti-K. Fredenhagen-R. Verch [27]. Similar ideas are used in the non-commutative versions of the axioms recently proposed by M. Paschke and R. Verch [190, 191].

4.2 Non-commutative and Spectral Space-Time.

There are three main reasons for the introduction of non-commutative space-time structures in physics and for the deep interest developed by physicists for "non-commutative geometry" (not only A. Connes'one):

- The awareness that quantum effects (Heisenberg uncertainty principle), coupled to the general relativistic effect of the energy-momentum tensor on the curvature of space-time (Einstein equation), entail that at very small scales the space-time manifold structure might be "unphysical".

- The belief that modification to the short scale structure of space-time might help to resolve the problems of "ultraviolet divergences" in quantum field theory (that arise, by Heisenberg uncertainty, from the arbitrary high momentum associated with arbitrary small length scales) and of "singularities" in general relativity.

- The intuition that in order to include the remaining physical forces (nuclear and electromagnetic) in a "geometrization" program, going beyond the one realized for gravity by A. Einstein's general relativity, it might be necessary to make use of geometrical environments more sophisticated than those provided by usual Riemannian/Lorentzian geometry.

What we call here "spectral space-time" is the idea that space-time (commutative or not) has to be "reconstructed a posteriori", from other operationally defined degrees of freedom, in a spectral way.

Space-time as a "relational" a posteriori entity originate from ideas of G.W. Leibnitz, G. Berkeley, E. Mach. Although pregeometrical speculations, in western philosophy, probably date as far back as Pythagoras, their first modern incarnation probably starts with J. Wheeler's "pregeometries" and "it from bit" proposals.

R. Geroch [115], with his Einstein algebras, was the first to suggest a "transition" from spaces to algebras in order to solve the problem of "singularities" in general relativity.

The fundamental idea that space-time can be recovered from the specification of suitable states of the system, has been the subject of scattered speculations in algebraic quantum field theory in the past by A. Ocneanu [34], S. Doplicher [100], U. Bannier [8] and, in the "modular localization program" (see R. Brunetti-D. Guido-R. Longo [26] and references therein), have been conjectured by N. Pinamonti [193].

Extremely important rigorous results including a complete reconstruction of Minkowski space-time [216] have been achieved in the "geometric modular action" program developed by D. Buchholz-S. Summers (see [215] for an excellent review and references).

- That non-commutative geometry provides a suitable environment for the implementation of the spectral reconstruction of space-time from states and observables in quantum physics has been the main motivating idea of one us (P.B.) since 1990. The idea that space-time might be spectrally reconstructed, via non-commutative geometry, from Tomita-Takesaki modular theory applied to the algebra of physical observables was first elaborated in 1995 by P.B. and independently (motivated by the possibility to obtain cyclic cocycles in algebraic quantum field theory from modular theory) by R. Longo [167]. Since then this conjecture is still the main subject and motivation of our investigation [14].

  Similar speculations on the interplay between modular theory and (some aspects of) space-time geometry have been suggested by S. Lord [169, Section VII.3] and by M. Paschke-R. Verch [190, Section 6].

- One of the authors (R.C.) has raised the somehow puzzling question whether it is possible to reinterpret the one parameter group of modular automorphisms as a renormalization (semi-)group in physics. The connection with P. Cartier's idea of a "universal Galois group" [38], currently developed by A. Connes-M. Marcolli, is extremely intriguing.

[34] As reported in A. Jadczyk [132].

4.3 A. Connes' Non-commutative Geometry and Gravity

It is often claimed that non-commutative geometry will be a key ingredient (a kind of quantum version of Riemannian geometry) for the formulation of a fundamental theory of quantum gravity (see for example L. Smolin [212] and P. Martinetti [178]) and actually non-commutative geometry is often listed among the current alternative approaches to quantum gravity.

In reality, with the only notable exceptions of the extremely interesting programs outlined in M. Paschke [187] and in A. Connes-M. Marcolli [75], a foundational approach to quantum physics based on A. Connes' non-commutative geometry has never been proposed. So far, most of the current applications of A. Connes' non-commutative geometry to (quantum) gravity have been limited to:

- the study of some "quantized" example: C. Rovelli [208], F, Besnard [19],
- the use of its mathematical framework for the reformulation of classical (Euclidean) general relativity: D. Kastler [139], A. Chamseddine-G. Felder-J. Fröhlich [52], W. Kalau-M. Walze [136], C. Rovelli-G. Landi [161, 162, 158],
- attempts to use its mathematical framework "inside" some already established theories such as strings (A. Connes-M. Douglas-A. Schwarz [71], J. Fröhlich, O. Grandjean, A. Recknagel [112], J. Brodzki, V. Mathai, J. Rosenberg, R. Szabo [25]) and loop gravity (J. Aastrup-J. Grimstrup [1, 2], F. Girelli-E. Livine [117]),
- the formulation of Hamiltonian theories of gravity on globally hyperbolic cases, where only the "spacial-slides" are described by non-commutative geometries: E. Hawkins [128], T. Kopf-M. Paschke [150, 151, 149].

4.4 A Proposal for (Modular) Algebraic Quantum Gravity.

Our ongoing research project is aiming at the construction of an **algebraic theory of quantum gravity** in which "non-commutative" space-time is spectrally reconstructed from Tomita-Takesaki modular theory.

What we propose is to develop an approach to the foundations of quantum physics technically based on algebraic quantum theory (operator algebras) and A. Connes' non-commutative geometry. The research is building on the experience already gained in our previous/current mathematics research plans on "modular spectral triples in non-commutative geometry and physics" [14] and on "categorical non-commutative geometry" and is conducted in the standard of mathematical rigour typical of the tradition of mathematical physics' research in algebraic quantum field theory [3, 127].

In the mathematical framework of A. Connes' non-commutative geometry, we are addressing the problem of the "spectral reconstruction" of "geometries" from the underlying operational data defined by "states" over "observables' C*-algebras" of physical systems. More specifically:

- Building on our previous research on "modular spectral-triples" and on recent results on semi-finite spectral triples recently developed by A. Carey-J. Phillips-A. Rennie-F. Sukhocev [30], we make use of Tomita-Takesaki modular theory of operator algebras to associate non-commutative geometrical objects (that are only formally similar to A. Connes' spectral-triples) to suitable states over involutive normed algebras.

- We are now developing an "event" interpretation of the formalism of states and observables in algebraic quantum physics that is in line with C. Isham's "history projection operator theory" [130] and C. Rovelli's "relational/relativistic quantum mechanics" [207, 206].

- Making contact with our current research project on "categorical non-commutative geometry" and with other projects in categorical quantum gravity (J. Baez [5, 6] and L. Crane [85, 86]), we plan to generalize the diffeomorphism covariance group of general relativity in a categorical context and use it to "identify" the degrees of freedom related to the spatio-temporal structure of the physical system.

- Techniques from "decoherence/einselection" (H.Zeh [227], W.Zurek [229]) and/or "emergence/noiseless subsystems" (for example O. Dreyer [106], F. Markopoulou [177]), superselection (I. Ojima [185]) and the "cooling" procedure developed by A. Connes-M. Marcolli [75] are expected to be relevant in order to extract from our spectrally defined non-commutative geometries, a macroscopic space-time for the pair state/system and its "classical residue".

- Possible reproduction of quantum geometries already defined in the context of loop quantum gravity (see T. Thiemann [220], J. Aastrup-J. Grimstrup [1, 2]) and/or S. Doplicher-J. Roberts-K. Fredenhagen models [105] will be investigated.

If partially successful, the project will have a significant fallout: a background-independent powerful approach to "quantum relativity" that is suitable for the purpose of unification of physics, geometry and information theory that lies ahead.

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A horizontal categorification of Gel'fand duality

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A Horizontal Categorification of Gel'fand Duality

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This paper is dedicated to J. E. Roberts, the "pioneer" of C-categories.*

Abstract

In the setting of C*-categories, we provide a definition of "spectrum" of a commutative full C*-category as a one-dimensional unital saturated Fell bundle over a suitable groupoid (equivalence relation) and prove a categorical Gel'fand duality theorem generalizing the usual Gel'fand duality between the categories of commutative unital C*-algebras and compact Hausdorff spaces. Although many of the individual ingredients that appear along the way are well-known, the somehow unconventional way we "glue" them together seems to shed some new light on the subject.

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1 Introduction

There is no need to explain why the notions of "geometry" and "space" are fundamental both in mathematics and in physics. Typically, a rigorous way to encode at least some basic geometrical content into a mathematical framework makes use of the notion of a "topological space", i.e. a set equipped with a topological structure. Although being just a preliminary step in the process of developing a more sophisticated apparatus, this way of thinking has been very fruitful for both abstract and concrete purposes.
In a very important development, I. M. Gel'fand looked not at the topological space itself but rather at the space of all continuous functions on it, and realized that these seemingly different structures are in fact essentially the same. In slightly more precise terms, he found a basic example of anti-equivalence between certain categories of spaces and algebras (see for example [Bl, Theorems II.2.2.4, II.2.2.6] or [L, Section 6]). Since on the

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analytic side $C(X;\mathbb{C})$ is a special type of a Banach algebra called a C*-algebra, the study of possibly non-commutative C*-algebras has been often regarded as a good framework for "non-commutative topology".
The "duality" aspect has been later enforced by the Serre-Swan equivalence [K, Theorem 6.18] between vector bundles and suitable modules (see also [FGV] for a Hermitian version of the theorem and [T1, T2, W] for generalizations involving Hilbert bundles). By then, breakthrough results have continued to emerge both in geometry and functional analysis, based on Gel'fand's original intuition, for about four decades.
In connection with physical ideas, L. Crane-D. Yetter [CY] and J. Baez-J. Dolan [BD] have recently proposed a process of categorification of mathematical structures, in which sets and functions are replaced by categories and functors.
From this perspective, in this paper, we wish to discuss a categorification of the notion of space extending and merging together Gel'fand duality and Serre-Swan equivalence.
On one side of the extended duality we have a "horizontal categorification" of the notion of commutative C*-algebra, namely a "commutative C*-category" (or commutative C*-algebroid) whilst the corresponding replacement of spaces, the "spaceoids", are supposed to parametrize their spectra. Spaceoids could be described in several different albeit equivalent ways. In this paper we have decided to focus on a characterization based on the notion of Fell bundle. Originally Fell bundles were introduced in connection with the study of representations of locally compact groups, but we argue that they come to life naturally on the basis of purely "topological" principles.
Rather surprisingly, to the best of our knowledge, the notions of commutative C*-category and its spectrum have not been discussed before, despite the fact that (mostly highly non-commutative) C*-categories have been somehow intensively exploited over the last 30 years in several areas of research, including Mackey induction, superselection structure in quantum field theory, abstract group duality, subfactors and the Baum-Connes conjecture. At any rate, we make frequent contact with the related notions that can be found in the literature, hoping that our approach sheds new light on the subject by approaching the matter from a kind of unconventional viewpoint.
Of course, once we have a running definition, it seems quite challenging in the next step to look for some natural occurrence of the notion of spaceoid in other contexts. For instance, we are not aware of any connection with the powerful concepts that have been introduced in algebraic topology to date. Also, the appearence of bundles in the structure of the spectrum suggests an intriguing connection to local gauge theory but we have not developed these ideas yet. Some of our considerations have been motivated by a categorical approach to non-commutative geometry [BCL2], and it is rewarding that some of its relevant tools (e.g., Serre-Swan theorem, Morita equivalence) appear naturally in our context. More structure is expected to emerge when our categories are equipped with a differentiable structure. In the case of usual spaces, in the setting of A. Connes' non-commutative geometry [C], this has been achieved by means of a Dirac operator, and then axiomatized using the concept of "spectral triple".

Here below we present a short description of the content of the paper.
In section 2 we mention, mainly for the purpose of fixing our notation, some basic definitions on C*-categories. Section 3 opens recalling the notion of a Fell bundle in the case of involutive inverse base categories and then proceeds to introduce the definition of the category of spaceoids that will eventually "subsume" that of compact Hausdorff spaces in our duality theorem. The construction of a small commutative full C*-category starting from a spaceoid is undertaken in section 4, while the spectral analysis of a commutative full C*-category is the subject of the more technical section 5 where a "spectrum functor" from the category

of full commutative C*-categories to our category of spaceoids is defined.
Section 6 presents the main result of this paper in the form of a duality between a certain category of commutative full C*-categories and the category of their spectra (spaceoids). A "categorified version" of Gel'fand transform is introduced and used to prove a Gel'fand spectral reconstruction theorem for full commutative C*-categories. Similarly a "categorified evaluation transform" is defined for the purpose of proving the representativity of the spectum functor.
While in the usual Gel'fand duality theory a spectrum is just a compact topological space, in the situation under consideration it comes up equipped with a natural bundle structure. In particular, the spectrum of a commutative full C*-category is identified with a kind of "groupoid of Hermitian line bundles" that can be conveniently described using the language of Fell bundles or equivalently as a continuous field of one-dimensional full C*-categories).
Along the way, we also discuss several categorical versions of well-known concepts like the Gel'fand transform that we think are of independent interest. Notice that a notion of Fourier transform in the setting of compact groupoids has been discussed by M. Amini [A].
Our duality is reminiscent of an interesting but widely ignored duality result of A. Takahashi [T1, T2]. Takahashi's duality can be essentially understood as a duality of weak monoidal categories, although he does not explicitly examine the natural monoidal structure on the categories of Hilbert bundles and Hilbert C*-modules. The duality considered in this paper is essentially a "strict $*$-monoidal" version of the former, where we consider C*-categories ("strict" equivalence relations in the Picard groupoid) and Fell bundles ("strict subcategories" of the monoidal category of Hilbert bundles) instead of C*-modules and fields of Hilbert spaces (certain Banach bundles).
Most of the results presented here have been announced in our survey paper [BCL2] and have been presented in several seminars in Thailand, Australia, Italy, UK since May 2006.

Note added in proof. When the present work was under preparation, we became aware of some related results in T. Timmermann's Ph.D. dissertation [Ti] where, in the context of Hopf algebraic quantum groupoids, a very general non-commutative Pontryagin duality theory is developed by means of pseudomultiplicative unitaries in C*-modules; and also in V. Deaconu-A. Kumjian-B. Ramazan [DKR], where a notion of Abelian Fell bundle (which contains our commutative C*-categories as a special case) is introduced and a structure theorem for them (in terms of "twisted coverings of groupoids") is proved. In the framework of T-duality, a Pontryagin type duality between commutative principal bundles and gerbes has been proposed by C. Daenzer [D]; while a generalization of Pontryagin duality for locally compact Abelian group bundles has been provided by G. Goehle [G]

2 Category $\mathscr{A}$ of full commutative C*-categories

The notion of C*-category, introduced by J. Roberts (see P. Ghez-R. Lima-J. Roberts [GLR] and also P. Mitchener [M]) has been extensively used in algebraic quantum field theory:

Definition 2.1. *A* ***C*-category*** *is a category $\mathcal{C}$ such that: the sets $\mathcal{C}_{AB} := \mathrm{Hom}_{\mathcal{C}}(B, A)$ are complex Banach spaces; the compositions are bilinear maps such that $\|xy\| \leq \|x\| \cdot \|y\|$, for all $x \in \mathcal{C}_{AB}$, $y \in \mathcal{C}_{BC}$; there is an involutive antilinear contravariant functor $* : \mathcal{C} \to \mathcal{C}$, acting identically on the objects, such that $\|x^*x\| = \|x\|^2$, $\forall x \in \mathcal{C}_{BA}$ and such that x^*x is a positive element in the C*-algebra $\mathcal{C}_{AA}$, for every $x \in \mathcal{C}_{BA}$ (i.e. $x^*x = y^*y$ for some $y \in \mathcal{C}_{AA}$).*

In a C*-category $\mathcal{C}$, the "diagonal blocks" $\mathcal{C}_{AA} := \mathrm{Hom}_{\mathcal{C}}(A, A)$ are unital C*-algebras and the "off-diagonal blocks" $\mathcal{C}_{AB} := \mathrm{Hom}_{\mathcal{C}}(B, A)$ are unital Hilbert C*-bimodules on the

C*-algebras $\mathcal{C}_{AA}$ and $\mathcal{C}_{BB}$. We say that $\mathcal{C}$ is **full** if all the bimodules $\mathcal{C}_{AB}$ are imprimitivity bimodules. In practice, every full C*-category is a "strict-ification" of an equivalence relation in the Picard-Morita groupoid of unital C*-algebras. It is also very useful to see a C*-category as an involutive category fibered over the equivalence relation of its objects: in this way, a (full) C*-category becomes a special case of a (saturated) unital Fell bundle over an involutive (discrete) base category as described in definition 3.1 below. We say that $\mathcal{C}$ is **one-dimensional** if all the bimodules $\mathcal{C}_{AB}$ are one-dimensional and hence Hilbert spaces.
The first problem that we have to face is how to select a suitable full subcategory $\mathscr{A}$ of "commutative" full C*-categories playing the role of horizontal categorification of the category of commutative unital C*-algebras. Since we are working in a completely strict categorical environment, our choice is to define a C*-category $\mathcal{C}$ to be **commutative** if all its diagonal blocks $\mathcal{C}_{AA}$ are commutative C*-algebras.
If $\mathcal{C}, \mathcal{D} \in \mathscr{A}$ are two full commutative small C*-categories (with the same cardinality of the set of objects), a **morphism** in the category $\mathscr{A}$ is an object bijective $*$-functor $\Phi : \mathcal{C} \to \mathcal{D}$.

For later usage, recall from [GLR, Definition 1.6] and [M, Section 4] that a closed two-sided ideal $\mathcal{J}$ in a C*-category $\mathcal{C}$ is always a $*$-ideal and that the quotient $\mathcal{C}/\mathcal{J}$ has a natural structure as a C*-category with a natural **quotient functor** $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{J}$. We have this "first isomorphism theorem", whose proof is standard.

Theorem 2.2. *Let $\Phi : \mathcal{C} \to \mathcal{D}$ be a $*$-functor between C*-categories. The* ***kernel*** *of Φ defined by $\ker \Phi := \{x \in \mathcal{C} \mid \Phi(x) = 0\}$ is a closed two-sided ideal in $\mathcal{C}$ and there exists a unique $*$-functor $\check{\Phi} : \mathcal{C}/\ker \Phi \to \mathcal{D}$ such that $\check{\Phi} \circ \pi = \Phi$. The functor $\check{\Phi}$ is faithful if and only if the functor Φ is injective on the objects and it is full if and only if Φ is full.*

Recall (see [GLR, Definition 1.8]) that a **representation** of a C*-category $\mathcal{C}$ is a $*$-functor $\Phi : \mathcal{C} \to \mathcal{H}$ with values in the C*-category $\mathcal{H}$ of bounded linear maps between Hilbert spaces.

Lemma 2.3. *A one-dimensional C*-category $\mathcal{C}$, admits at least a $*$-functor $\gamma : \mathcal{C} \to \mathbb{C}$.*

Proof. Fix an object $A \in \mathrm{Ob}_{\mathcal{C}}$ and the representation $\Phi : \mathcal{C} \to \mathcal{H}$ given by "left composition": $\Phi_B := \mathcal{C}_{BA}$, $B \in \mathrm{Ob}_{\mathcal{C}}$, and for all $x \in \mathcal{C}_{CD}$, $\Phi_D \xrightarrow{\Phi_x} \Phi_C$ given by $\Phi_x(\xi) := x\xi$, for all $\xi \in \Phi_D$. The Hilbert spaces $\Phi_B = \mathcal{C}_{BA}$ are one-dimensional and choosing normalized vectors $\xi_B \in \Phi_B$, with $\xi_A := \iota_A$, provides isomorphisms $T_B : \Phi_B \to \mathbb{C}$.
The map $\gamma : x \mapsto \det(T_C \circ \Phi_x \circ T_D^{-1})$ for all $x \in \mathcal{C}_{CD}$ is the required $*$-functor. □

3 Category $\mathscr{T}$ of full topological spaceoids

We now proceed to the identification of a good category $\mathscr{T}$ of "spaceoids" playing the role of horizontal categorification of the category of continuous maps between compact Hausdorff topological spaces. Making use of Gel'fand duality (see e.g. [L, Section 6]) for the diagonal blocks $\mathcal{C}_{AA}$ and (Hermitian) Serre-Swan equivalence (see e.g. [BCL2, Section 2.1.2] and references therein) for the off-diagonal blocks $\mathcal{C}_{AB}$ of a commutative full C*-category $\mathcal{C}$, we see that the spectrum of $\mathcal{C}$ identifies an equivalence relation embedded in the Picard groupoid of Hermitian line bundles over the Gel'fand spectra of the diagonal C*-algebras $\mathcal{C}_{AA}$. Finally, reassembling such block-data, we recognize that, globally, the spectrum of a commutative full C*-category can be described as a very special kind of a Fell bundle that we call a full topological spaceoid. Fell bundles over topological groups were first introduced by J. Fell [FD, Section II.16] and later generalized to the case of groupoids by S. Yamagami (see A. Kumjian [Ku] and references therein) and to the case of inverse semigroups by N. Sieben (see R. Exel [E, Section 2]). These notions admit a natural extension to that of a

Fell bundle over an involutive inverse category[1] that we systematically adopt below, see [BCL2, Section 4.2.1] for more details. For the definition of a Banach bundle we refer to J. Fell-R. Doran [FD, Section I.13].

Definition 3.1. *A **Fell bundle** $(\mathcal{E},\pi,\mathcal{X})$ over an involutive inverse category $\mathcal{X}$ is a Banach bundle that is also an involutive category $\mathcal{E}$ fibered over the involutive category $\mathcal{X}$ with continuous fiberwise bilinear compositions and fiberwise conjugate-linear involutions such that $\|ef\| \leq \|e\|\cdot\|f\|$ for all composable $e,f\in\mathcal{E}$, $\|e^*e\| = \|e\|^2$ for all $e\in\mathcal{E}$ and e^*e is a positive element in the C*-algebra $\mathcal{E}_{\pi(e^*e)} := \{f \mid \pi(f) = \pi(e^*e)\}$.*

Definition 3.2. *A **topological spaceoid** (or simply a spaceoid, for short) $(\mathcal{E},\pi,\mathcal{X})$ is a unital rank-one Fell bundle over the product involutive topological category $\mathcal{X} := \Delta_X \times \mathcal{R}_\mathcal{O}$ where $\Delta_X := \{(p,p) \mid p \in X\}$ is the minimal equivalence relation of a compact Hausdorff space X and $\mathcal{R}_\mathcal{O} := \mathcal{O}\times\mathcal{O}$ is the maximal equivalence relation of a discrete space $\mathcal{O}$.*

With a slight abuse of notation, the points of the base involutive category $\mathcal{X}$ of a full spaceoid will simply be denoted by $p_{AB} := ((p,p),(A,B)) \in \Delta_X \times \mathcal{R}_\mathcal{O}$.
Note that, since a constant finite-rank Banach bundle over a locally compact Hausdorff space is locally trivial [FD, Remark I.13.9] and hence a vector bundle, a topological spaceoid is a Hermitian line bundle over $\mathcal{X}$ and is a disjoint union of the Hermitian line bundles $\mathcal{E}_{AA} := \pi^{-1}(\Delta_X \times \{AA\})$. Furthermore a topological spaceoid is always a one-dimensional C*-category that is a disjoint union of the "continuous field" of the full one-dimensional C*-categories $\mathcal{E}_p := \pi^{-1}(\{(p,p)\}\times\mathcal{R}_\mathcal{O})$ for all $p\in X$.

A **morphism of spaceoids**[2] $(f,\mathcal{F})\colon (\mathcal{E}_1,\pi_1,\mathcal{X}_1) \to (\mathcal{E}_2,\pi_2,\mathcal{X}_2)$ is a pair $(f,\mathcal{F})$ where:

- $f := (f_\Delta, f_\mathcal{R})$ with $f_\Delta : \Delta_1 \to \Delta_2$ being a continuous map of topological spaces and $f_\mathcal{R} : \mathcal{R}_1 \to \mathcal{R}_2$ an isomorphism of equivalence relations;
- $\mathcal{F} : f^\bullet(\mathcal{E}_2) \to \mathcal{E}_1$ is a fiberwise linear continuous $*$-functor such that $\pi_1 \circ \mathcal{F} = \pi_2^f$, where $(f^\bullet(\mathcal{E}_2), \pi_2^f, \mathcal{X}_1)$ denotes the standard f-pull-back[3] of $(\mathcal{E}_2,\pi_2,\mathcal{X}_2)$.

Topological spaceoids constitute a category if compositions and identities are given by

$$(g,\mathcal{G})\circ(f,\mathcal{F}) := (g\circ f, \mathcal{F}\circ f^\bullet(\mathcal{G})\circ\Theta^{\mathcal{E}_3}_{g,f}) \qquad \text{and} \qquad \iota(\mathcal{E},\pi,\mathcal{X}) := (\iota_\mathcal{X}, \iota^\pi_\mathcal{X}),$$

where $\Theta^{\mathcal{E}_3}_{g,f} : (g\circ f)^\bullet(\mathcal{E}_3) \to f^\bullet(g^\bullet(\mathcal{E}_3))$ is the natural isomorphisms between standard pull-backs given by $\Theta^{\mathcal{E}_3}_{g,f}(x_1,e_3) := (x_1,(f(x_1),e_3))$, for all $(x_1,e_3)\in(g\circ f)^\bullet(\mathcal{E}_3)$.

4 The section functor Γ

Here we are going to define a **section functor** $\Gamma : \mathscr{T} \to \mathscr{A}$ that to every spaceoid $(\mathcal{E},\pi,\mathcal{X})$, with $\mathcal{X} := \Delta_X \times \mathcal{R}_\mathcal{O}$, associates a commutative full C*-category $\Gamma(\mathcal{E})$ as follows:

- $\mathrm{Ob}_{\Gamma(\mathcal{E})} := \mathcal{O}$;

[1] By **involutive category** we mean a category $\mathcal{X}$ equipped with an involution i.e. an object preserving contravariant functor $* : \mathcal{X} \to \mathcal{X}$ such that $(x^*)^* = x$ for all $x\in\mathcal{X}$. If $\mathcal{X}$ has a topology we also require composition and involution to be continuous. $\mathcal{X}$ is an involutive **inverse category** if $xx^*x = x$ for all $x\in\mathcal{X}$.

[2] Morphisms of spaceoids can be seen as examples of J. Baez notion of spans (in this case, a span of the Fell bundles of the spaceoids).

[3] Recall that $f^\bullet(\mathcal{E}_2) := \{(p_{AB},e)\in\mathcal{X}_1\times\mathcal{E}_2 \mid f(p_{AB}) = \pi_2(e)\}$ with $f\circ\pi_2^f = \pi_2\circ f^{\pi_2}$ where $\pi_2^f(p_{AB},e) := p_{AB}$ and $f^{\pi_2}(p_{AB},e) := e$. If $\mathcal{E}_2$ is a Fell bundle over $\mathcal{X}_2$, $f^\bullet(\mathcal{E}_2)$ is a Fell bundle over $\mathcal{X}_1$.

- $\forall A, B \in \mathrm{Ob}_{\Gamma(\mathcal{E})}$, $\quad \mathrm{Hom}_{\Gamma(\mathcal{E})}(B, A) := \Gamma(\Delta_X \times \{(A, B)\}; \mathcal{E})$, where $\Gamma(\Delta_X \times \{(A, B)\}; \mathcal{E})$ denotes the set of continuous sections $\sigma : \Delta_X \times \{(A, B)\} \to \mathcal{E}$, $\sigma : p_{AB} \mapsto \sigma_p^{AB} \in \mathcal{E}_{p_{AB}}$ of the restriction of $\mathcal{E}$ to the base space $\Delta_X \times \{(A, B)\} \subset \mathcal{X}$.

- for all $\sigma \in \mathrm{Hom}_{\Gamma(\mathcal{E})}(A, B)$ and $\rho \in \mathrm{Hom}_{\Gamma(\mathcal{E})}(B, C)$:

$$\rho \circ \sigma : p_{AC} \mapsto (\rho \circ \sigma)_p^{AC} := \rho_p^{AB} \circ \sigma_p^{BC},$$
$$\sigma^* : p_{BA} \mapsto (\sigma^*)_p^{BA} := (\sigma_p^{AB})^*,$$
$$\|\sigma\| := \sup_{p \in \Delta_X} \|\sigma_p^{AB}\|_{\mathcal{E}},$$

  with operations taken in the total space $\mathcal{E}$ of the Fell bundle.

We extend now the definition of Γ to the morphism of $\mathscr{T}$. Let $(f, \mathcal{F})$ be a morphism in $\mathscr{T}$ from $(\mathcal{E}_1, \pi_1, \mathcal{X}_1)$ to $(\mathcal{E}_2, \pi_2, \mathcal{X}_2)$. Given $\sigma \in \Gamma(\mathcal{E}_2)$, we consider the unique section $f^\bullet(\sigma) : \mathcal{X}_1 \to f^\bullet(\mathcal{E}_2)$ such that $f^{\pi_2} \circ f^\bullet(\sigma) = \sigma \circ f$ and the composition $\mathcal{F} \circ f^\bullet(\sigma)$. In this way we have a map

$$\Gamma_{(f,\mathcal{F})} : \Gamma(\mathcal{E}_2) \to \Gamma(\mathcal{E}_1), \quad \Gamma_{(f,\mathcal{F})} : \sigma \mapsto \mathcal{F} \circ f^\bullet(\sigma), \quad \forall \sigma \in \Gamma(\mathcal{E}_2).$$

Proposition 4.1. *For any morphism* $(\mathcal{E}_1, \pi_1, \mathcal{X}_1) \xrightarrow{(f,\mathcal{F})} (\mathcal{E}_2, \pi_2, \mathcal{X}_2)$ *in the category* $\mathscr{T}$*, the map* $\Gamma_{(f,\mathcal{F})} : \Gamma(\mathcal{E}_2) \to \Gamma(\mathcal{E}_1)$ *is a morphism in the category* $\mathscr{A}$*.*
The pair of maps $\Gamma : (\mathcal{E}, \pi, \mathcal{X}) \mapsto \Gamma(\mathcal{E})$ *and* $\Gamma : (f, \mathcal{F}) \mapsto \Gamma_{(f,\mathcal{F})}$ *gives a contravariant functor from the category* $\mathscr{T}$ *of spaceoids to the category* $\mathscr{A}$ *of small full commutative C*-categories.*

Proof. Let $(\mathcal{E}_1, \pi_1, \mathcal{X}_1) \xrightarrow{(f,\mathcal{F})} (\mathcal{E}_2, \pi_2, \mathcal{X}_2)$ and $(\mathcal{E}_2, \pi_2, \mathcal{X}_2) \xrightarrow{(g,\mathcal{G})} (\mathcal{E}_3, \pi_3, \mathcal{X}_3)$ be two composable morphisms in the category $\mathscr{T}$ and let $(\mathcal{E}, \pi, \mathcal{X}) \xrightarrow{(\iota_\mathcal{X}, \iota_\mathcal{X}^\pi)} (\mathcal{E}, \pi, \mathcal{X})$ be the identity morphism of $(\mathcal{E}, \pi, \mathcal{X})$. To complete the proof we must show that

$$\Gamma_{(g,\mathcal{G})\circ(f,\mathcal{F})} = \Gamma_{(f,\mathcal{F})} \circ \Gamma_{(g,\mathcal{G})}, \quad \Gamma_{(\iota_\mathcal{X}, \iota_\mathcal{X}^\pi)} = \iota_{\Gamma(\mathcal{E})},$$

and these are obtained by tedious but straightforward calculations. □

5 The spectrum functor Σ

This section is devoted to the construction of a **spectrum functor** $\Sigma : \mathscr{A} \to \mathscr{T}$ that to every commutative full C*-category $\mathcal{C}$ associates its **spectral spaceoid** $\Sigma(\mathcal{C})$.

Let $\mathcal{C}$ be a C*-category, we denote by $\mathcal{R}^\mathcal{C}$ the topologically discrete $*$-category $\mathcal{C}/\mathcal{C} \simeq \mathcal{R}_{\mathrm{Ob}_\mathcal{C}}$ and by $\mathbb{C}\mathcal{R}^\mathcal{C} := \rho^\bullet(\mathbb{C})$ the one-dimensional C*-category pull-back of $\mathbb{C}$ (considered as a C*-category with only one object $\bullet$) under the constant map $\rho : \mathcal{R}^\mathcal{C} \to \{\bullet\}$. Note that from the defining property of pull-backs there is a bijective map $\omega \mapsto \widetilde{\omega}$ between the set of $\mathbb{C}$-valued $*$-functors $[\mathcal{C}; \mathbb{C}]$ and the set of $\mathbb{C}\mathcal{R}^\mathcal{C}$-valued $*$-functors $[\mathcal{C}; \mathbb{C}\mathcal{R}^\mathcal{C}]$.
By definition two $*$-functors ω_1, ω_2 in $[\mathcal{C}; \mathbb{C}]$ are **unitarily equivalent** if there exists a "unitary" natural transformation $A \mapsto \nu_A \in \mathbb{T}$ between them.
Note that the set $\mathcal{I}_\omega := \{x \in \mathcal{C} \mid \omega(x) = 0\}$, which is also equal to $\{x \in \mathcal{C} \mid \omega(x^*x) = 0\}$, is an ideal in $\mathcal{C}$ and $\mathcal{I}_{\omega_1} = \mathcal{I}_{\omega_2}$ if (and only if) the equivalence classes $[\omega_1]$ and $[\omega_2]$ coincide.

We also need the following lemmas whose routine proof are omitted:[4]

Lemma 5.1. *If $\omega, \omega' \in [\mathcal{C}; \mathbb{C}]$ are unitarily equivalent, there is a unique map $\psi : \mathcal{R}^{\mathcal{C}} \to \mathbb{T}$ such that $\omega'_{AB} = \psi_{AB} \cdot \omega_{AB}$ for all $AB \in \mathcal{R}^{\mathcal{C}}$ and the map $\psi : AB \mapsto \psi_{AB}$ is a $*$-morphism:*

$$\psi_{AB}\psi_{BC} = \psi_{AC}, \quad \psi_{AB} = \psi_{BA}^{-1}, \quad \psi_{AA} = 1_{\mathbb{C}}. \tag{5.1}$$

Conversely, given a $*$-morphism $\psi \in [\mathcal{R}^{\mathcal{C}}; \mathbb{T}]$, two $*$-functors ω, ω' such that $\omega'_{AB} = \psi_{AB}\omega_{AB}$ are unitarily equivalent.

Lemma 5.2. *Every object preserving $*$-automorphism γ of the C^*-category $\mathbb{C}\mathcal{R}^{\mathcal{C}}$ is given by the multiplication by an element $\psi \in [\mathcal{R}^{\mathcal{C}}; \mathbb{T}]$ i.e. $\gamma(x) = \psi_{AB} \cdot x$ for all $x \in (\mathbb{C}\mathcal{R}^{\mathcal{C}})_{AB}$.*

Proposition 5.3. *Two $*$-functors ω, $\omega' \in [\mathcal{C}; \mathbb{C}]$ are unitarily equivalent if and only if $\omega_{AA} = \omega'_{AA}$ for all $A \in \mathrm{Ob}_{\mathcal{C}}$.*

Proof. By lemma 5.1, if $[\omega] = [\omega']$, then $\omega'_{AA} = \psi_{AA} \cdot \omega_{AA} = \omega_{AA}$, for all objects A.
Let $\omega, \omega' \in [\mathcal{C}; \mathbb{C}]$ and suppose that $\omega_{AA} = \omega'_{AA}$, for all $A \in \mathrm{Ob}_{\mathcal{C}}$. Consider the corresponding $\mathbb{C}\mathcal{R}^{\mathcal{C}}$-valued $*$-functors $\widetilde{\omega}, \widetilde{\omega}' \in [\mathcal{C}; \mathbb{C}\mathcal{R}^{\mathcal{C}}]$. Note that $\mathrm{Ker}(\widetilde{\omega}) = \mathcal{I}_{\omega} = \mathcal{I}_{\omega'} = \mathrm{Ker}(\widetilde{\omega}')$ and hence, $\omega_{AB}, \widetilde{\omega}_{AB}$ are nonzero if and only if $\omega'_{AB}, \widetilde{\omega}'_{AB}$ are nonzero. If ω_{AB} is nonzero for all $AB \in \mathcal{R}^{\mathcal{C}}$, by theorem 2.2 we have two $*$-isomorphisms $\mathcal{C}/\mathrm{Ker}(\omega) \xrightarrow{\hat{\omega}} \mathbb{C}\mathcal{R}^{\mathcal{C}} \xleftarrow{\hat{\omega}'} \mathcal{C}/\mathrm{Ker}(\omega')$. From lemma 5.2 there is a $\psi \in [\mathcal{R}^{\mathcal{C}}; \mathbb{T}]$ such that $\hat{\omega}' = \psi \cdot \hat{\omega}$ and hence also $\omega' = \psi \cdot \omega$ so that the proposition follows from lemma 5.1.
To eliminate the restriction ω_{AB} is nonzero for all $AB \in \mathcal{R}^{\mathcal{C}}$, note that by Zorn's lemma, every object of $\mathcal{C}$ is contained in family $\mathscr{S} \subset \mathrm{Ob}_{\mathcal{C}}$, maximal under inclusion, such that ω_{AB} is nonzero for every pair of $A, B \in \mathscr{S}$. Any pair $\mathscr{S}_1, \mathscr{S}_2$ of such maximal subfamilies are "disjoint" i.e. for any pair of objects $A \in \mathscr{S}_1$ and $B \in \mathscr{S}_2$, we have that $\omega(\mathcal{C}_{AB}) = \{0\}$. Each maximal subfamily $\mathscr{S}$ determines a full subcategory of $\mathcal{C}$ and from above we can choose phases ν_A for all $A \in \mathscr{S}$ such that $\psi_{AB} = \nu_A\nu_B^{-1}$ for all $A, B \in \mathscr{S}$. Now for every pair $A \in \mathscr{S}_1$, $B \in \mathscr{S}_2$ in disjoint maximal subfamilies, defining $\psi_{AB} = \nu_A\nu_B^{-1}$ is a perfectly compatible choice since $\omega'(x) = \psi_{AB}\omega(x)$ for all $x \in \mathcal{C}_{AB}$. □

Proposition 5.4. *The set $[\mathcal{C}; \mathbb{C}]$ of $\mathbb{C}$-valued $*$-functors $\omega : \mathcal{C} \to \mathbb{C}$, with the weakest topology making all evaluations continuous, is a compact Hausdorff topological space.*

Proof. Note that for all $\omega \in [\mathcal{C}; \mathbb{C}]$ and for all $x \in \mathcal{C}_{AB}$,

$$|\omega(x)| = \sqrt{\overline{\omega(x)}\omega(x)} = \sqrt{\omega(x^*x)} = \sqrt{\omega_{AA}(x^*x)} \leq \sqrt{\|x^*x\|} = \sqrt{\|x\|^2} = \|x\|,$$

because ω_{AA} is a state over the C*-algebra $\mathcal{C}_{AA}$. Hence $[\mathcal{C}; \mathbb{C}]$ is a subspace of the compact Hausdorff space $\prod_{x \in \mathcal{C}} D_{\|x\|}$, where $D_{\|x\|}$ is the closed ball in $\mathbb{C}$ of radius $\|x\|$. The rest of the proof follows from the same argument for the Banach-Alaoglu theorem. □

Let $\mathrm{Sp}_b(\mathcal{C}) := \{[\omega] \mid \omega \in [\mathcal{C}; \mathbb{C}]\}$ denote the **base spectrum** of $\mathcal{C}$, defined as the set of unitary equivalence classes of $*$-functors in $[\mathcal{C}; \mathbb{C}]$. It is a compact space with the quotient topology induced by the map $\omega \mapsto [\omega]$. To show that $\mathrm{Sp}_b(\mathcal{C})$ is Hausdorff it is enough to note that, by proposition 5.3, if $[\omega] \neq [\omega']$, there exists at least one object A such that $\omega_{AA} \neq \omega'_{AA}$ and so there exists at least one evaluation ev_x with $x \in \mathcal{C}_{AA}$ such that $\mathrm{ev}_x(\omega) \neq \mathrm{ev}_x(\omega')$. Since, for $x \in \mathcal{C}_{AA}$, ev_x is well-defined on the quotient space $\mathrm{Sp}_b(\mathcal{C})$, the result follows.

[4] Note that, for $\omega \in [\mathcal{C}; \mathbb{C}]$ and $A, B \in \mathrm{Ob}_{\mathcal{C}}$, we denote by ω_{AB} the restriction of ω to $\mathcal{C}_{AB}$.

Proposition 5.5. *Let $\mathcal{C}$ be a full commutative C*-category. For all $A \in \mathrm{Ob}_{\mathcal{C}}$, there exists a natural bijective map, between the base spectrum of $\mathcal{C}$ and the usual Gel'fand spectrum $\mathrm{Sp}(\mathcal{C}_{AA})$ of the C*-algebra $\mathcal{C}_{AA}$, given by the restriction $|_{AA} : \omega \mapsto \omega|_{\mathcal{C}_{AA}}$.*
In particular, for all objects $A \in \mathrm{Ob}_{\mathcal{C}}$, one has $\mathrm{Sp}_b(\mathcal{C})|_{AA} = \mathrm{Sp}(\mathcal{C}_{AA}) \simeq \mathrm{Sp}_b(\mathcal{C}_{AA})$.

Proof. By proposition 5.3, the correspondence $[\omega] \mapsto \omega_{AA}$ is functional.
We show that the map $[\omega] \mapsto \omega_{AA}$ is injective. Given $\omega, \omega' \in [\mathcal{C}; \mathbb{C}]$ with $\omega_{AA} = \omega'_{AA}$, we know from [BCL3, Proposition 2.30], that $\omega_{BB}(x) = \omega_{AA}(\phi_{AB}(x))$, for all $x \in \mathcal{C}_{BB}$, for all $B \in \mathrm{Ob}_{\mathcal{C}}$, where $\phi_{AB} : \mathcal{C}_{BB} \to \mathcal{C}_{AA}$ is the canonical isomorphism associated to the imprimitivity bimodule $\mathcal{C}_{BA}$. It follows that $\omega_{BB} = \omega_{AA} \circ \phi_{AB} = \omega'_{AA} \circ \phi_{AB} = \omega'_{BB}$, for all $B \in \mathrm{Ob}_{\mathcal{C}}$ and, by proposition 5.3, we see that $[\omega] = [\omega']$.

We show that the function $[\omega] \mapsto \omega_{AA}$ is surjective.
Given $\omega^o \in \mathrm{Sp}(\mathcal{C}_{AA})$, consider the set $\mathcal{J} := \cup_{BC \in \mathcal{R}^{\mathcal{C}}} \mathcal{J}_{BC}$, with $\mathcal{J}_{BB} := \phi_{BA}(\mathrm{Ker}(\omega^o))$ and $\mathcal{J}_{BC} := \mathcal{J}_{BB}\mathcal{C}_{BC} := \{\sum_{j=1}^{N} b_j x_j \mid b_j \in \mathcal{I}_{BB},\ x_j \in \mathcal{C}_{BC},\ N \in \mathbb{N}_0\}$ where, as defined in [BCL3, Sections 2.2-2.3], $\phi_{BA} : \mathcal{C}_{AA} \to \mathcal{C}_{BB}$ is the canonical isomorphism induced by the imprimitivity bimodule $\mathcal{C}_{AB}$. Making use of [BCL3, Theorem 2.24] and [BCL3, Proposition 2.29], we have $\sum b_j x_j = \sum x_j \phi_{CB}(b_j) = \sum x_j \phi_{CA}(\phi_{AB}(b_j)) \in \mathcal{C}_{BC}\mathcal{J}_{CC}$ and hence $\mathcal{J}_{BB}\mathcal{C}_{BC} = \mathcal{C}_{BC}\mathcal{J}_{CC}$, for all $B, C \in \mathrm{Ob}_{\mathcal{C}}$. Clearly, it follows that, for all $B, C, D \in \mathrm{Ob}_{\mathcal{C}}$, $\mathcal{J}_{BC}\mathcal{J}_{CD} = \mathcal{J}_{BB}\mathcal{C}_{BC}\mathcal{C}_{CD}\mathcal{J}_{DD} \subset \mathcal{J}_{CC}\mathcal{C}_{BD}\mathcal{J}_{DD} \subset \mathcal{J}_{BD}$, $\mathcal{J}^*_{BC} = (\mathcal{J}_{BB}\mathcal{C}_{BC})^* = \mathcal{C}_{CB}\mathcal{J}_{BB} = \mathcal{J}_{CB}$ and hence $\mathcal{J}$ is an involutive ideal in $\mathcal{C}$ (actually the ideal generated by $\mathrm{Ker}(\omega^o)$).
The quotient C*-category $\mathcal{C}/\mathcal{J}$ is one-dimensional. In fact, by [BCL3, Proposition 2.27], $\mathcal{C}_{BC}/\mathcal{J}_{BC}$ is an imprimitivity bimodule over the one-dimensional C*-algebras $\mathcal{C}_{BB}/\mathcal{J}_{BB} \simeq \mathbb{C} \simeq \mathcal{C}_{CC}/\mathcal{J}_{CC}$ and hence it becomes a Hilbert space that is necessarily one-dimensional because, if this is not the case, we can find two different orthonormal vectors $x, z \in H$ and then the imprimitivity, with $y := x + z$, implies the contradiction $z = \langle x \mid y \rangle z = x \langle y \mid z \rangle = x$.
By lemma 2.3 there exists at least one $\mathbb{C}$-valued $*$-functor $\gamma : \mathcal{C}/\mathcal{J} \to \mathbb{C}$ whose restriction to $\mathcal{C}_{AA}/\mathcal{J}_{AA}$ is the canonical isomorphism with $\mathbb{C}$ (since $\xi_A := \iota_A$).
Composing the quotient $*$-functor $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{J}$ with the chosen $*$-functor $\gamma : \mathcal{C}/\mathcal{J} \to \mathbb{C}$, we obtain a $\mathbb{C}$-valued $*$-functor $\omega := \gamma \circ \pi : \mathcal{C} \to \mathbb{C}$. Clearly ω_{AA} coincides with ω^o because they are two states on the unital C*-algebra $\mathcal{C}_{AA}$, with the same kernel ideal $\mathcal{J}_{AA}$.
Since $|_{AA}$: $[\omega] \mapsto \omega^o$, the surjectivity of the map $|_{AA}$ is proved. □

Theorem 5.6. *Let $\mathcal{C}$ be a full commutative C*-category. For every $A \in \mathrm{Ob}_{\mathcal{C}}$, the bijective map $|_{AA} : \mathrm{Sp}_b(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C}_{AA})$ given by $[\omega] \mapsto \omega_{AA}$ is a homeomorphism between $\mathrm{Sp}_b(\mathcal{C})$ and the Gel'fand spectrum $\mathrm{Sp}(\mathcal{C}_{AA})$ of the unital C*-algebra $\mathcal{C}_{AA}$.*

Proof. Since both $\mathrm{Sp}_b(\mathcal{C})$ and $\mathrm{Sp}(\mathcal{C}_{AA})$ are compact Hausdorff spaces, and the map $|_{AA}$ is bijective, it is enough to show that $|_{AA}$: $\mathrm{Sp}_b(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C}_{AA})$ is continuous. Since $\mathrm{Sp}_b(\mathcal{C})$ is equipped with the quotient topology induced by the projection map $\pi : [\mathcal{C}; \mathbb{C}] \to \mathrm{Sp}_b(\mathcal{C})$, the map $|_{AA}$ is continuous if and only if $|_{AA} \circ \pi : [\mathcal{C}; \mathbb{C}] \to \mathrm{Sp}(\mathcal{C}_{AA})$ is continuous. The spaces $[\mathcal{C}; \mathbb{C}]$ and $\mathrm{Sp}(\mathcal{C}_{AA})$ are equipped with the weakest topology making the evaluation maps continuous. It follows that the continuity of $|_{AA} \circ \pi$ is equivalent to the continuity of $\mathrm{ev}_x = \mathrm{ev}_x \circ |_{AA} \circ \pi : [\mathcal{C}; \mathbb{C}] \to \mathbb{C}$ for all $x \in \mathcal{C}_{AA}$. Since $\mathrm{ev}_x : [\mathcal{C}; \mathbb{C}] \to \mathbb{C}$ is continuous, the result is established. □

Let $\mathcal{X}^{\mathcal{C}} := \Delta^{\mathcal{C}} \times \mathcal{R}^{\mathcal{C}}$ be the direct product equivalence relation of the compact Hausdorff $*$-category $\Delta^{\mathcal{C}} := \Delta_{\mathrm{Sp}_b(\mathcal{C})}$ and the topologically discrete $*$-category $\mathcal{R}^{\mathcal{C}} := \mathcal{C}/\mathcal{C} \simeq \mathcal{R}_{\mathrm{Ob}_{\mathcal{C}}}$.

With a slight abuse of notation, we write $AB \in \mathcal{R}^{\mathcal{C}}$ for the point $\mathcal{C}_{AB}/\mathcal{C}_{AB}$ in $\mathcal{R}^{\mathcal{C}}$. Denoting by $[\omega]_{AB}$ the point $([\omega], AB) = ([\omega], \mathcal{C}_{AB}/\mathcal{C}_{AB}) \in \mathcal{X}^{\mathcal{C}}$, we define:

$$\mathcal{I}_{[\omega]_{AB}} := \mathcal{I}_\omega \cap \mathcal{C}_{AB}, \quad \mathcal{E}^{\mathcal{C}}_{[\omega]_{AB}} := \frac{\mathcal{C}_{AB}}{\mathcal{I}_{[\omega]_{AB}}}, \quad \mathcal{E}^{\mathcal{C}} := \biguplus_{[\omega]_{AB} \in \mathcal{X}^{\mathcal{C}}} \mathcal{E}^{\mathcal{C}}_{[\omega]_{AB}}.$$

Proposition 5.7. *The map $\pi^{\mathcal{C}} \colon \mathcal{E}^{\mathcal{C}} \to \mathcal{X}^{\mathcal{C}}$, sending an element $e \in \mathcal{E}^{\mathcal{C}}_{[\omega]_{AB}}$ to the point $[\omega]_{AB} \in \mathcal{X}^{\mathcal{C}}$, has a natural structure of a unital rank-one Fell bundle over the topological involutive inverse category $\mathcal{X}^{\mathcal{C}}$.*

Proof. The topology on $\mathcal{E}^{\mathcal{C}}$ whose fundamental system of neighbourhoods are the sets $U^{O,\varepsilon}_{e_0} := \{e \in \mathcal{E}^{\mathcal{C}} \mid \exists x, x_0 \in \mathcal{C} : \hat{x}(\pi^{\mathcal{C}}(e)) = e, \hat{x}_0(\pi^{\mathcal{C}}(e_0)) = e_0, \ \forall p \in O, \|\hat{x}(p) - \hat{x}_0(p)\| < \varepsilon\}$, where $e_0 \in \mathcal{E}^{\mathcal{C}}$, O is open in $\mathcal{X}^{\mathcal{C}}$, $\varepsilon > 0$ and $\hat{x}$ denotes the Gel'fand transform of x defined in section 6.1, entails that a net (e_μ) is convergent to the point e in $\mathcal{E}^{\mathcal{C}}$ if and only if the net $\pi^{\mathcal{C}}(e_\mu)$ converges to $\pi^{\mathcal{C}}(e)$ in $\mathcal{X}^{\mathcal{C}}$ and there exists a net of Gel'fand transforms $\hat{x}_\mu$, "passing" in e_μ, that uniformly converges, on a neighbourhood of $\pi^{\mathcal{C}}(e_0)$, to a Gel'fand transform $\hat{x}$ "passing" in e_0.
With such a topology the (partial) operations on $\mathcal{E}$ i.e. sum, scalar multiplication, product, involution, inner product (and hence norm) become continuous and $(\mathcal{E}^{\mathcal{C}}, \pi^{\mathcal{C}}, \mathcal{X}^{\mathcal{C}})$ becomes a Banach bundle.
Since every equivalence relation in $\mathcal{X}^{\mathcal{C}}$ is a disjoint union of "grids" $\{[\omega]\} \times \mathcal{R}^{\mathcal{C}}$ whose inverse image under $\pi^{\mathcal{C}}$ is the one-dimensional C*-category $\mathcal{C}/\operatorname{Ker}(\omega)$, $(\mathcal{E}^{\mathcal{C}}, \pi^{\mathcal{C}}, \mathcal{X}^{\mathcal{C}})$ is a rank-one unital Fell bundle over the equivalence relation $\mathcal{X}^{\mathcal{C}}$ and hence a spaceoid. □

To a commutative full C*-category $\mathcal{C}$ we have associated a topological **spectral spaceoid** $\Sigma(\mathcal{C}) := (\mathcal{E}^{\mathcal{C}}, \pi^{\mathcal{C}}, \mathcal{X}^{\mathcal{C}})$. We extend now the definition of Σ to the morphism of $\mathscr{A}$. Let $\Phi : \mathcal{C} \to \mathcal{D}$ be an object-bijective $*$-functor between two small commutative full C*-categories with spaceoids $\Sigma(\mathcal{C}), \Sigma(\mathcal{D}) \in \mathscr{T}$ and define a morphism $\Sigma^\Phi : \Sigma(\mathcal{D}) \xrightarrow{(\lambda^\Phi, \Lambda^\Phi)} \Sigma(\mathcal{C})$ in the category $\mathscr{T}$ as follows.

$\lambda^\Phi : \mathcal{X}^{\mathcal{D}} \xrightarrow{(\lambda^\Phi_\Delta, \lambda^\Phi_{\mathcal{R}})} \mathcal{X}^{\mathcal{C}}$ where $\lambda^\Phi_{\mathcal{R}} : \mathcal{R}^{\mathcal{D}} \to \mathcal{R}^{\mathcal{C}}$ is the isomorphism of equivalence relations given by $\lambda^\Phi_{\mathcal{R}}(AB) := \Phi^{-1}(A)\Phi^{-1}(B)$, for $AB \in \mathcal{R}^{\mathcal{D}}$, and where $\lambda^\Phi_\Delta : \Delta^{\mathcal{D}} \to \Delta^{\mathcal{C}}$ (since $\omega \mapsto \omega \circ \Phi$ is continuous and preserves equivalence by unitary natural transformations) is the well-defined continuous map given by $\lambda^\Phi_\Delta([\omega]) := [\omega \circ \Phi] \in \Delta^{\mathcal{C}}$, for all $[\omega] \in \Delta^{\mathcal{D}}$.

The bundle $\biguplus_{[\omega]_{AB} \in \mathcal{X}^{\mathcal{D}}} \frac{\mathcal{C}_{\lambda^\Phi_{\mathcal{R}}(AB)}}{\mathcal{I}_{\lambda^\Phi([\omega]_{AB})}}$ with the maps

$$\pi^\Phi : ([\omega]_{AB}, \ x + \mathcal{I}_{\lambda^\Phi([\omega]_{AB})}) \mapsto [\omega]_{AB} \in \mathcal{X}^{\mathcal{D}}, \quad x \in \mathcal{C}_{\lambda^\Phi_{\mathcal{R}}(AB)},$$
$$\Phi^\pi : ([\omega]_{AB}, \ x + \mathcal{I}_{\lambda^\Phi([\omega]_{AB})}) \mapsto (\lambda^\Phi([\omega]_{AB}), x + \mathcal{I}_{\lambda^\Phi([\omega]_{AB})}) \in \mathcal{E}^{\mathcal{C}}$$

is the standard λ^Φ-pull-back $(\lambda^\Phi)^\bullet(\mathcal{E}^{\mathcal{C}})$ of the Fell bundle $(\mathcal{E}^{\mathcal{C}}, \pi^{\mathcal{C}}, \mathcal{X}^{\mathcal{C}})$.
Since $\Phi(\mathcal{I}_{\lambda^\Phi([\omega]_{AB})}) \subset \mathcal{I}_{[\omega]_{AB}}$ for $[\omega]_{AB} \in \mathcal{X}^{\mathcal{D}}$, we define $\Lambda^\Phi : (\lambda^\Phi)^\bullet(\mathcal{E}^{\mathcal{C}}) \to \mathcal{E}^{\mathcal{D}}$ by

$$\Lambda^\Phi\Big([\omega]_{AB}, \ x + \mathcal{I}_{\lambda^\Phi([\omega]_{AB})}\Big) := \Big([\omega]_{AB}, \ \Phi(x) + \mathcal{I}_{[\omega]_{AB}}\Big).$$

Proposition 5.8. *For any morphism $\mathcal{C} \xrightarrow{\Phi} \mathcal{D}$ in $\mathscr{A}$, the map $\Sigma(\mathcal{D}) \xrightarrow{\Sigma^\Phi} \Sigma(\mathcal{C})$ is a morphism of spectral spaceoids. The pair of maps $\Sigma : \mathcal{C} \mapsto \Sigma(\mathcal{C})$ and $\Sigma : \Phi \mapsto \Sigma^\Phi$ give a contravariant functor $\Sigma : \mathscr{A} \to \mathscr{T}$, from the category $\mathscr{A}$ of object-bijective $*$-functors between small commutative full C*-categories to the category $\mathscr{T}$ of spaceoids.*

Proof. We have to prove that Σ is antimultiplicative and preserves the identities.
If $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ and $\Psi : \mathcal{C}_2 \to \mathcal{C}_3$ are two $*$-functors in $\mathscr{A}$, by definition,

$$\Sigma^{\Psi\circ\Phi} = (\lambda^{\Psi\circ\Phi}, \Lambda^{\Psi\circ\Phi}) = \left(\lambda^{\Phi}\circ\lambda^{\Psi},\ \Lambda^{\Psi}\circ(\lambda^{\Psi})^{\bullet}(\Lambda^{\Phi})\circ\Theta^{\mathcal{E}^{\mathcal{C}_1}}_{\lambda^{\Phi},\lambda^{\Psi}}\right) = (\lambda^{\Phi},\Lambda^{\Phi})\circ(\lambda^{\Psi},\Lambda^{\Psi}) = \Sigma^{\Phi}\circ\Sigma^{\Psi}.$$

Also, if $\iota_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the identity functor of the C*-category $\mathcal{C}$, then the morphism $\Sigma^{\iota_{\mathcal{C}}} = (\lambda^{\iota_{\mathcal{C}}}, \Lambda^{\iota_{\mathcal{C}}})$ is the identity morphism of the spaceoid $\Sigma(\mathcal{C})$. □

6 Horizontal Categorification of Gel'fand Duality

6.1 Gel'fand Transform

For a given C*-category $\mathcal{C}$ in $\mathscr{A}$, we define a horizontally categorified version of Gel'fand transform as $\mathfrak{G}_{\mathcal{C}} : \mathcal{C} \to \Gamma(\Sigma(\mathcal{C}))$ given by $\mathfrak{G}_{\mathcal{C}} : x \mapsto \hat{x}$ where $\hat{x}^{AB}_{[\omega]} := x + \mathcal{I}_{\omega_{AB}}, \quad \forall x \in \mathcal{C}_{BA}$. Clearly $\mathfrak{G}_{\mathcal{C}} : \mathcal{C} \to \Gamma(\Sigma(\mathcal{C}))$ is an object bijective $*$-functor.

Proposition 6.1. *The Gel'fand transform $\mathfrak{G}_{\mathcal{C}} : \mathcal{C} \to \Gamma(\mathcal{E}^{\mathcal{C}})$ of a commutative full C*-category $\mathcal{C}$ is an isometric (hence faithful) $*$-functor.*

Proof. The maps $x + \mathcal{I}_{\omega_{BB}} \mapsto \|x + \mathcal{I}_{\omega_{BB}}\|$ and $x + \mathcal{I}_{\omega_{BB}} \mapsto |\omega(x)|$ coincide (because they are two norms on the same one-dimensional Banach space that coincide on the element $\iota_B + \mathcal{I}_{\omega_{BB}}$). If $x \in \mathcal{C}_{AB}$ with $A \neq B$, then

$$\|x + \mathcal{I}_{\omega_{AB}}\| = \sqrt{\|(x + \mathcal{I}_{\omega_{AB}})^*(x + \mathcal{I}_{\omega_{AB}})\|} = \sqrt{\|x^*x + \mathcal{I}_{\omega_{BB}}\|} = \sqrt{|\omega(x^*x)|} = |\omega(x)|. \quad (6.1)$$

Furthermore, since $C(\mathrm{Sp}(\mathcal{C}_{BB}); \mathbb{C})$ is canonically isomorphic to $\Gamma(\Sigma(\mathcal{C}_{BB}))$, by the usual Gel'fand theorem applied to the commutative unital C*-algebra $\mathcal{C}_{BB}$, we know that

$$\|\widehat{x^*x}\|_{\Gamma(\Sigma(\mathcal{C}_{BB}))} = \|\widehat{x^*x}\|_{C(\mathrm{Sp}(\mathcal{C}_{BB}))} = \|x^*x\|_{\mathcal{C}_{BB}} = \|x^*x\|_{\mathcal{C}}. \quad (6.2)$$

The isometry of $\mathfrak{G}_{\mathcal{C}}$ is obtained from the following computation for all $x \in \mathcal{C}_{AB}$:

$$\begin{aligned}
\|\hat{x}\|_{\Gamma(\Sigma(\mathcal{C}))} &= \sup_{[\omega]\in\Delta^{\mathcal{C}}} \|\hat{x}^{AB}_{[\omega]}\|_{\mathcal{E}^{\mathcal{C}}} = \sup_{[\omega]\in\Delta^{\mathcal{C}}} \|x + \mathcal{I}_{\omega_{AB}}\|_{\mathcal{E}^{\mathcal{C}}} = \sup_{[\omega]\in\Delta^{\mathcal{C}}} |\omega(x)| \\
&= \Big(\sup_{[\omega]\in\Delta^{\mathcal{C}}} \omega(x^*x)\Big)^{1/2} = \Big(\sup_{\omega\in\mathrm{Sp}(\mathcal{C}_{BB})} \widehat{x^*x}(\omega)\Big)^{1/2} \quad \text{by proposition 5.5} \\
&= \|\widehat{x^*x}\|^{1/2}_{C(\mathrm{Sp}(\mathcal{C}_{BB}))} = \|x^*x\|^{1/2}_{\mathcal{C}} = \|x\|_{\mathcal{C}}.
\end{aligned}$$

□

Lemma 6.2. *Let $\mathcal{C}$ and $\mathcal{C}^o$ be full commutative C*-categories and suppose that $\mathcal{C}^o$ is a subcategory of $\mathcal{C}$ such that $\mathcal{C}^o_{AA} = \mathcal{C}_{AA}$ for all $A \in \mathrm{Ob}_{\mathcal{C}} = \mathrm{Ob}_{\mathcal{C}^o}$. Then $\mathcal{C}^o_{AB} = \mathcal{C}_{AB}$ for all $A, B \in \mathrm{Ob}_{\mathcal{C}}$.*

Proof. By the fullness of the bimodule ${}_{\mathcal{A}}\mathcal{C}^o_{\mathcal{B}}$ there is a sequence of pairs $u_j, v_j \in {}_{\mathcal{A}}\mathcal{C}^o_{\mathcal{B}}$ such that $\iota_{\mathcal{B}} = \sum_{j=1}^{\infty} u_j^* v_j$. We have $x = x\iota_{\mathcal{B}} = x\sum_{j=1}^{\infty} u_j^* v_j = \sum_{j=1}^{\infty}(xu_j^*)v_j \in {}_{\mathcal{A}}\mathcal{C}^o_{\mathcal{B}}$ for all $x \in {}_{\mathcal{A}}\mathcal{C}_{\mathcal{B}}$, because $xu_j^* \in {}_{\mathcal{A}}\mathcal{C}_{\mathcal{A}} = {}_{\mathcal{A}}\mathcal{C}^o_{\mathcal{A}}$ and so $(xu_j^*)v_j \in {}_{\mathcal{A}}\mathcal{C}^o_{\mathcal{B}}$ for all j. □

Theorem 6.3. *The Gel'fand transform $\mathfrak{G}_{\mathcal{C}} : \mathcal{C} \to \Gamma(\Sigma(\mathcal{C}))$ of a commutative full C*-category $\mathcal{C}$ is a full isometric (hence faithful) $*$-functor.*

Proof. The isometry (and faithfulness) of the $*$-functor $\mathfrak{G}_{\mathcal{C}}$ is proved in proposition 6.1. The "image" $\mathfrak{G}_{\mathcal{C}}(\mathcal{C})$ of $\mathfrak{G}_{\mathcal{C}}$ is a subcategory of the commutative full C*-category $\Gamma(\mathcal{E})$ that is clearly a commutative full C*-category on its own. By lemma 6.2, the $*$-functor $\mathfrak{G}_{\mathcal{C}}$ is full as long as $\mathfrak{G}_{\mathcal{C}}(\mathcal{C}_{AA}) = \Gamma(\mathcal{E})_{AA}$, for all objects $A \in \mathrm{Ob}_{\mathcal{C}}$. The last statement follows from the fact that the Gel'fand transform $\mathfrak{G}_{\mathcal{C}}$, when restricted to any "diagonal" commutative unital C*-algebra $\mathcal{C}_{AA}$ can be "naturally identified" with the usual Gel'fand transform of $\mathcal{C}_{AA}$ via the homeomorphism $[\omega] \mapsto \omega|_{AA}$ (see proposition 5.5 and theorem 5.6). $\square$

6.2 Evaluation Transform

For every topological spaceoid $(\mathcal{E}, \pi, \mathcal{X})$ we define a horizontal categorified version of **evaluation transform** $\mathfrak{E}_{\mathcal{E}} : (\mathcal{E}, \pi, \mathcal{X}) \xrightarrow{(\eta^{\mathcal{E}}, \Omega^{\mathcal{E}})} \Sigma(\Gamma(\mathcal{E}))$ as follows:

- $\eta^{\mathcal{E}}_{\mathcal{R}} : \mathcal{R}_{\mathcal{O}} \to \mathcal{R}^{\Gamma(\mathcal{E})}$ is the canonical isomorphism $\mathcal{R}_{\mathcal{O}} = \mathcal{R}_{\mathrm{Ob}_{\Gamma(\mathcal{E})}} \simeq \Gamma(\mathcal{E})/\Gamma(\mathcal{E})$, explicitly: $\eta^{\mathcal{E}}_{\mathcal{R}}(AB) := \Gamma(\mathcal{E})_{AB}/\Gamma(\mathcal{E})_{AB}, \quad \forall AB \in \mathcal{R}_{\mathcal{O}}$ that is, according to the running notation, written as an identity map $\eta^{\mathcal{E}}_{\mathcal{R}}(AB) = AB \in \mathcal{R}^{\Gamma(\mathcal{E})}$.
- $\eta^{\mathcal{E}}_{\Delta} : \Delta_X \to \Delta^{\Gamma(\mathcal{E})}$ is given by $\eta^{\mathcal{E}}_{\Delta} : p \mapsto [\gamma_p \circ \mathrm{ev}_p] \quad \forall p \in \Delta_X$, where the evaluation map $\mathrm{ev}_p : \Gamma(\mathcal{E}) \to \uplus_{(AB)\in\mathcal{R}_{\mathcal{O}}} \mathcal{E}_{p_{AB}}$ given by $\mathrm{ev}_p : \sigma \mapsto \sigma^{AB}_p$ is a $*$-functor with values in a one-dimensional C*-category that determines[5] a unique point $[\gamma_p \circ \mathrm{ev}_p] \in \Delta_{\mathrm{Sp}_b(\Gamma(\mathcal{E}))}$.
- $\biguplus_{p_{AB}\in\mathcal{X}} \Gamma(\mathcal{E})_{\eta^{\mathcal{E}}_{\mathcal{R}}(AB)}/\mathcal{I}_{\eta^{\mathcal{E}}(p_{AB})}$ with the natural projection $(p_{AB},\ \sigma + \mathcal{I}_{\eta^{\mathcal{E}}(p_{AB})}) \mapsto p_{AB}$, and with the $\mathcal{E}^{\Gamma(\mathcal{E})}$-valued map $(p_{AB},\ \sigma + \mathcal{I}_{\eta^{\mathcal{E}}(p_{AB})}) \mapsto \sigma + \mathcal{I}_{\eta^{\mathcal{E}}(p_{AB})}$, is the standard $\eta^{\mathcal{E}}$-pull-back $(\eta^{\mathcal{E}})^{\bullet}(\mathcal{E}^{\Gamma(\mathcal{E})})$ of $\Sigma(\Gamma(\mathcal{E}))$.
- $\Omega^{\mathcal{E}} : (\eta^{\mathcal{E}})^{\bullet}(\mathcal{E}^{\Gamma(\mathcal{E})}) \to \mathcal{E}$ is defined by $\Omega^{\mathcal{E}} : (p_{AB},\ \sigma + \mathcal{I}_{\eta^{\mathcal{E}}(p_{AB})}) \mapsto \sigma^{AB}_p,\ \forall \sigma \in \Gamma(\mathcal{E})_{AB}$, $\forall p_{AB} \in \mathcal{X}$.

In particular, with such definitions we can prove:

Theorem 6.4. *The functor $\Sigma : \mathscr{A} \to \mathscr{T}$ is representative i.e. given a spaceoid $(\mathcal{E}, \pi, \mathcal{X})$, the evaluation transform $\mathfrak{E}_{\mathcal{E}} : (\mathcal{E}, \pi, \mathcal{X}) \to \Sigma(\Gamma(\mathcal{E}))$ is an isomorphism in the category of spaceoids.*

Proof. Note that $(\mathcal{E}_{AA}, \pi, X)$ is naturally isomorphic to the trivial $\mathbb{C}$-bundle over X and thus there is an isomorphism of the C*-algebras $\Gamma(\mathcal{E})_{AA}$ and $C(X)$ that "preserves" evaluations. The map $\eta^{\mathcal{E}}_{\Delta}$ is injective. In fact, if $p \neq q$, by Urysohn's lemma, there is a section $\sigma \in \Gamma(\mathcal{E})_{AA}$ such that $\gamma_p(\sigma^{AA}_p) \neq \gamma'_q(\sigma^{AA}_q)$ for some (and thus for all) $A \in \mathcal{O}$, which implies $\eta^{\mathcal{E}}_{\Delta}(p) \neq \eta^{\mathcal{E}}_{\Delta}(q)$ by proposition 5.3. To see that $\eta^{\mathcal{E}}_{\Delta}$ is surjective, let $[\omega] \in \mathrm{Sp}_b(\Gamma(\mathcal{E}))$. Then its restriction $\omega_{AA} : \Gamma(\mathcal{E})_{AA} \to \mathbb{C}$ does not depend on the choice of the representative $\omega \in [\omega]$. Any pure state on $C(X)$ coincides with an evaluation at a point $p \in X$, so that $\omega_{AA}(\sigma) = \gamma_p(\sigma(p)) = \gamma_p \circ \mathrm{ev}_p(\sigma)$, which implies $\eta^{\mathcal{E}}_{\Delta}(p) = [\omega]$.

Since $\eta^{\mathcal{E}}_{\Delta} : \Delta_X \to \Delta^{\Gamma(\mathcal{E})}$ is a bijective map between compact Hausdorff spaces, to prove that $\eta^{\mathcal{E}}_{\Delta}$ is a homeomorphism, it is enough to show that $\eta^{\mathcal{E}}_{\Delta}$ is continuous.

For this purpose, consider the set $[\Gamma(\mathcal{E})_{AA}; \mathcal{E}_{AA}]$ of fiberwise linear $*$-functors from the C*-algebra $\Gamma(\mathcal{E})_{AA}$ to the total space $\mathcal{E}_{AA}$ of the block AA of the spaceoid and consider on it the weakest topology making the evaluation maps $\mathrm{ev}_\sigma : [\Gamma(\mathcal{E})_{AA}; \mathcal{E}_{AA}] \to \mathcal{E}_{AA}$ continuous,

[5] By lemma 2.3, there is always a $\mathbb{C}$-valued $*$-functor $\gamma_p : \mathcal{E}_p \to \mathbb{C}$ and by proposition 5.3 any two compositions of ev_p with such $*$-functors are unitarily equivalent because they coincide on the diagonal C*-algebras $\mathcal{E}_{p_{AA}}$.

for all $\sigma \in \Gamma(\mathcal{E})_{AA}$. With this topology, the map $\mathrm{ev}_{AA} : \Delta_X \to [\Gamma(\mathcal{E})_{AA}; \mathcal{E}_{AA}]$, given by $\mathrm{ev}_{AA} : p \mapsto \mathrm{ev}_{p_{AA}}$, is continuous. Let $\gamma_{AA} : \mathcal{E}_{AA} \to \mathbb{C}$ be the disjoint union of the canonical isomorphisms of one-dimensional C*-algebras $\gamma_{p_{AA}} : \mathcal{E}_{p_{AA}} \to \mathbb{C}$ and note that it is continuous. The map $L_{\gamma_{AA}} : [\Gamma(\mathcal{E})_{AA}; \mathcal{E}_{AA}] \to [\Gamma(\mathcal{E})_{AA}; \mathbb{C}]$, $\quad L_{\gamma_{AA}} : \Phi \mapsto \gamma_{AA} \circ \Phi$ is continuous because, for all $\sigma \in \Gamma(\mathcal{E})_{AA}$ and all $\Phi \in [\Gamma(\mathcal{E})_{AA}; \mathcal{E}_{AA}]$, $\mathrm{ev}_\sigma \circ L_{\gamma_{AA}}(\Phi) = \gamma_{AA} \circ \Phi(\sigma) = \gamma_{AA} \circ \mathrm{ev}_\sigma(\Phi)$ and $\gamma_{AA} \circ \mathrm{ev}_\sigma$ is a continuous function of Φ. Clearly the map $\zeta_A : \Delta_X \to \mathrm{Sp}(\Gamma(\mathcal{E})_{AA})$ given by $\zeta_A(p) := |_{AA} \circ \eta^{\mathcal{E}}_\Delta(p) = \gamma_{p_{AA}} \circ \mathrm{ev}_{p_{AA}} = L_{\gamma_{AA}} \circ \mathrm{ev}_{AA}(p)$ is continuous and so is $\eta^{\mathcal{E}}_\Delta = |^{-1}_{AA} \circ \zeta_A$.

For every element $e \in \mathcal{E}$, we have $\pi(e) \in \Delta_X \times \mathcal{R}_{\mathcal{O}}$ and, since a spaceoid is actually a vector bundle, it is always possible to find a section $\sigma \in \Gamma(\mathcal{E})$ such that $\sigma_{\pi(e)} = e$. For any such section we consider the element $\sigma + \mathcal{I}_{\eta^{\mathcal{E}}(\pi(e))} \in \Gamma(\mathcal{E})/\mathcal{I}_{\eta^{\mathcal{E}}(\pi(e))} =: \mathcal{E}^{\Gamma(\mathcal{E})}_{\eta^{\mathcal{E}}(\pi(e))}$ (note that the element does not depend on the choice of $\sigma \in \Gamma(\mathcal{E})$ such that $\sigma_{\pi(e)} = e$) and in this way we have a map $\Theta : \mathcal{E} \to \mathcal{E}^{\Gamma(\mathcal{E})}$ by $\Theta : e \mapsto \sigma + \mathcal{I}_{\eta^{\mathcal{E}}(\pi(e))}$. The map Θ uniquely induces a morphism of Fell bundles $\Xi^{\mathcal{E}} : \mathcal{E} \to (\eta^{\mathcal{E}})^\bullet(\mathcal{E}^{\Gamma(\mathcal{E})})$ with the standard $\eta^{\mathcal{E}}$-pull-back of $\mathcal{E}^{\Gamma(\mathcal{E})}$ given by $\Xi^{\mathcal{E}}(e) := (\pi(e), \Theta(e))$. By direct computation the map $\Xi^{\mathcal{E}}$ is an isomorphism of Fell bundles whose inverse is $\Omega^{\mathcal{E}}$ and hence the evaluation transform $\mathfrak{E}^{\mathcal{E}} := (\eta^{\mathcal{E}}, \Omega^{\mathcal{E}})$ is an isomorphism of spaceoids. The continuity of $\Omega^{\mathcal{E}}$ is equivalent to that of $\widetilde{\Omega}^{\mathcal{E}} : \mathcal{E}^{\Gamma(\mathcal{E})} \to \mathcal{E}$, $\widetilde{\Omega}^{\mathcal{E}}(\sigma + \mathcal{I}_{\eta^{\mathcal{E}}(p_{AB})}) := \sigma^{AB}_p$, with $\sigma \in \Gamma(\mathcal{E})_{AB}$. Given a net $j \to \sigma^j + \mathcal{I}_{\eta^{\mathcal{E}}(p^j_{AB^j})}$ in $\mathcal{E}^{\Gamma(\mathcal{E})}$ converging to the point $\sigma + \mathcal{I}_{\eta^{\mathcal{E}}(p_{AB})}$ in the topology defined in proposition 5.7, without loss of generality we can assume that $j \to \sigma^j$ is uniformly convergent to σ in a neighbourhood U of $\eta^{\mathcal{E}}(p_{AB})$. This means that, for all $\epsilon > 0$, eventually in j, $\|\sigma^j([\omega]_{AB}) - \sigma([\omega]_{AB})\| < \epsilon$ for $[\omega]_{AB} \in U$. Since $\mathcal{R}^{\Gamma(\mathcal{E})}$ is discrete, the net AB^j is eventually equal to AB and since $\eta^{\mathcal{E}}$ is a homeomorphism, p^j_{AB} eventually lies in any neighbourhood of p_{AB} and hence the net $\widetilde{\Omega}^{\mathcal{E}}(\sigma^j + \mathcal{I}_{\eta^{\mathcal{E}}(p^j_{AB^j})}) = (\sigma^j)^{AB^j}_{p^j}$ converges to $\widetilde{\Omega}(\sigma + \mathcal{I}_{\eta^{\mathcal{E}}(p_{AB})}) = \sigma^{AB}_p$ in the Banach bundle topology of $\mathcal{E}$. Since $\Omega^{\mathcal{E}}$ is an isometry, it follows from [FD, Proposition 13.17] that its inverse is continuous too. □

6.3 Duality

Theorem 6.5. *The pair of functors (Γ, Σ) provides a duality between the category $\mathscr{T}$ of object-bijective morphisms between spaceoids and the category $\mathscr{A}$ of object-bijective $*$-functors between small commutative full C*-categories.*

Proof. To see that the map $\mathfrak{G} : \mathcal{C} \mapsto \mathfrak{G}_{\mathcal{C}}$ (that to every $\mathcal{C} \in \mathrm{Ob}_{\mathscr{A}}$ associates the Gel'fand transform of $\mathcal{C}$) is a natural isomorphism between the identity endofunctor $\mathcal{I}_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$ and the functor $\Gamma \circ \Sigma : \mathscr{A} \to \mathscr{A}$ we have to show that, given an object-bijective $*$-functor $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$, the identity $\Gamma_{\Sigma^\Phi}(\mathfrak{G}_{\mathcal{C}_1}(x)) = \mathfrak{G}_{\mathcal{C}_2}(\Phi(x))$ holds for any $x \in \mathcal{C}_1$.

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\mathfrak{G}_{\mathcal{C}_1}} & \Gamma(\Sigma(\mathcal{C}_1)) \\ \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \Gamma_{\Sigma^\Phi}} \\ \mathcal{C}_2 & \xrightarrow{\mathfrak{G}_{\mathcal{C}_2}} & \Gamma(\Sigma(\mathcal{C}_2)). \end{array}$$

$$\Gamma_{\Sigma^\Phi}(\mathfrak{G}_{\mathcal{C}_1}(x))^{A_2B_2}_{[\omega_2]} = \Lambda^\Phi\Big((\lambda^\Phi)^\bullet(\hat{x})^{A_2B_2}_{[\omega_2]}\Big) = \Lambda^\Phi\Big([\omega_2]_{A_2B_2}, \, \hat{x}(\lambda^\Phi([\omega_2]_{A_2B_2}))\Big)$$
$$= \Lambda^\Phi\Big([\omega_2]_{A_2B_2}, \, x + \mathcal{I}_{\lambda^\Phi([\omega_2]_{A_2B_2})}\Big) = \Big([\omega_2]_{A_2B_2}, \, \Phi(x) + \mathcal{I}_{[\omega_2]_{A_2B_2}}\Big) = \mathfrak{G}_{\mathcal{C}_2}(\Phi(x))^{A_2B_2}_{[\omega_2]}.$$

To see that the map $\mathfrak{E} : \mathcal{E} \mapsto \mathfrak{E}_{\mathcal{E}}$ (that to every spaceoid $(\mathcal{E}, \pi, \mathcal{X})$ associates its evaluation transform $\mathfrak{E}_{\mathcal{E}}$) is a natural isomorphism between the identity endofunctor $\mathcal{I}_{\mathscr{T}} : \mathscr{T} \to \mathscr{T}$ and the functor $\Sigma \circ \Gamma : \mathscr{T} \to \mathscr{T}$ we must provide, for any given morphism of spaceoids $(f, \mathcal{F})$ from $(\mathcal{E}_1, \pi_1, \mathcal{X}_1)$ to $(\mathcal{E}_2, \pi_2, \mathcal{X}_2)$, the commutativity of the diagram:

$$\begin{array}{ccc}
(\mathcal{E}_1, \pi_1, \mathcal{X}_1) & \xrightarrow{\mathfrak{E}_{\mathcal{E}_1} = (\eta^{\mathcal{E}_1}, \Omega^{\mathcal{E}_1})} & \Sigma(\Gamma(\mathcal{E}_1)) \\
{\scriptstyle (f,\mathcal{F})} \downarrow & & \downarrow {\scriptstyle \Sigma^{\Gamma(f,\mathcal{F})} = (\lambda^{\Gamma(f,\mathcal{F})}, \Lambda^{\Gamma(f,\mathcal{F})})} \\
(\mathcal{E}_2, \pi_2, \mathcal{X}_2) & \xrightarrow{\mathfrak{E}_{\mathcal{E}_2} = (\eta^{\mathcal{E}_2}, \Omega^{\mathcal{E}_2})} & \Sigma(\Gamma(\mathcal{E}_2)).
\end{array}$$

The proof amounts to showing the equalities

$$\lambda^{\Gamma(f,\mathcal{F})} \circ \eta^{\mathcal{E}_1} = \eta^{\mathcal{E}_2} \circ f, \quad \Omega^{\mathcal{E}_1} \circ (\eta^{\mathcal{E}_1})^\bullet(\Lambda^{\Gamma(f,\mathcal{F})}) \circ \Theta_1 = \mathcal{F} \circ f^\bullet(\Omega^{\mathcal{E}_2}) \circ \Theta_2, \tag{6.3}$$

where $\Theta_1 := \Theta^{\mathcal{E}^{\Gamma(\mathcal{E}_2)}}_{\lambda^{\Gamma(f,\mathcal{F})}, \eta^{\mathcal{E}_1}}$, $\Theta_2 := \Theta^{\mathcal{E}^{\Gamma(\mathcal{E}_2)}}_{\eta^{\mathcal{E}_2}, f}$.
Since for every point $p_{AB} \in \mathcal{X}_1$, we have $\lambda^{\Gamma(f,\mathcal{F})} \circ \eta^{\mathcal{E}_1}(p_{AB}) = ([\gamma_p \circ \mathrm{ev}_p \circ \Gamma_{(f,\mathcal{F})}], f_{\mathcal{R}}(AB))$ and $\eta^{\mathcal{E}_2} \circ f(p_{AB}) = ([\gamma_{f(p)} \circ \mathrm{ev}_{f(p)}], f_{\mathcal{R}}(AB))$, the first equation is a consequence of proposition 5.3. The second equation is then proved by a lengthy but elementary calculation. □

The usual Gel'fand theorem is easily recovered identifying a compact Hausdorff topological space X with the trivial spaceoid $(\Delta_X \times \{(\bullet, \bullet)\}) \times \mathbb{C}$.

7 Outlook

We have introduced commutative C*-categories and started a program for their "topological description" in terms of their spectra, here called spaceoids.
In particular, we have obtained a Gel'fand-type theorem for full commutative C*-categories. Although the statement of the main result (theorem 6.5) looks extremely natural, our proofs mostly rely on a "brute force" exploitation of the underlying structure and more streamlined arguments are likely to be found. Also, the result by itself is not as general as possible and certainly it leaves room for extensions in several directions, still hopefully we have provided some insight about how to achieve them.
For instance, we have only considered the case of $*$-functors between (full, commutative) C*-categories that are bijective on the objects. (Of course, this trivially includes morphisms between C*-algebras). In the next step, one would like to treat the case of $*$-functors that are not bijective on the objects. We believe this should not require significant modifications of our treatment and possibly it could be dealt with using relators (that we introduced in [BCL1]).
Perhaps a more important point would be to remove the condition of fullness. At present we have not discussed the issue in detail, but certainly the information that we have already acquired should significantly simplify the task.
Also, along the way, we have somehow taken advantage of our prior knowledge of the Gel'fand and Serre-Swan theorems. Eventually one would like to provide more intrinsic proofs directly in the framework of C*-categories (possibly unifying and extending both Gel'fand and Serre-Swan theorems in a "strict $*$-monoidal" version of Takahashi theorem [T1, T2]). In this respect, it looks promising to work directly with module categories. Besides, it is somehow disappointing that to date, for X and Y compact Hausdorff spaces, there seems to be no available general classification result for $C(X)$-$C(Y)$-bimodules.

The case of non-imprimitivity C*-bimodules should definitely play a role when discussing a classification result for generally non-commutative C*-categories, possibly along the lines of a generalization to C*-categories of the Dauns-Hofmann theorem for C*-algebras [DH]. One might also explore possible connections with the non-commutative Gel'fand spectral theorem of R. Cirelli-A. Manià-L. Pizzocchero [CMP] and the subsequent non-commutative Serre-Swan duality by E. Elliott-K. Kawamura [Ka, EK]. Similarly, it might be very interesting to investigate the connections between our spectral spaceoids and other spectral notions such as locales and topoi already used in the spectral theorems by B. Banachewki-C. Mulvey [BM] and C. Heunen-K. Landsmann-B. Spitters [HLS].

In the long run, one would like to (define and) classify commutative Fell bundles over suitable involutive categories. The notion of a Fell bundle could be even generalized to that of a fibered category enriched over another ($*$-monoidal) category.

Needless to say, one should analyze more closely the mathematical structure of spaceoids, introduce suitable topological invariants, study their symmetries, . . . , and investigate relations to other concepts that are widely used in other branches of mathematics, e.g. in algebraic topology/geometry as well as in gauge theories. Some geometric structures could become apparent when considering the representation of spaceoids as continuous fields of (one-dimensional commutative) C*-categories as discussed by E. Vasselli in [V].

The Gel'fand transform for general commutative C*-categories raises several questions (undoubtedly it could be defined for more general Banach categories, leading to a wide range of possibilities for further studies).
In particular, an immediate application would yield a Fourier transform and accordingly a reasonable concrete duality theory for "commutative" discrete groupoids (see M. Amini [A] for another approach that applies to compact but-not-necessarily-commutative-groupoids and T. Timmermann [Ti] for a more abstract setup).

As far as we are concerned, our main motivation to work with C*-categories came from analysing the categorical structure of non-commutative geometry (where morphisms of "non-commutative spaces" are given by bimodules) and one is naturally led to speculate about the possible evolution of the notion of spectra and morphism in A. Connes' non-commutative geometry (cf. [BCL1, BCL2, CCM]). In this direction, some of the first questions that come to mind are:

> Is there a suitable notion of spectral triple over a C*-category?
>
> Is it possible to consider a horizontal categorification of a spectral triple?

Of course this represents only the starting point for a much more ambitious program aiming at a "vertical categorification" of the notion of spectral triple[6] and from several fronts (see for example [DTT] and also the very detailed discussion by J. Baez [B] on the weblog "The n-category café") it is mounting the evidence that a suitable notion of non-commutative calculus necessarily require a higher (actually ∞) categorical setting.
In this respect, it seems reasonable to look for a Gel'fand theorem that applies to (strict) commutative higher categories (cf. [Ko]). A suitable definition of strict n-C*-categories (cf. [Z] for the case $n = 2$) and the proof of a categorical Gel'fand duality (at least for "commutative" full strict n-C*-categories) are topics that have recently attracted our attention [BCLS].

Finally, in this line of thoughts, one could envisage potential applications of a notion of Gromov-Hausdorff distance (cf. [R]) for C*-categories.

[6]The need for a notion of "higher spectral triple" has been already advocated by U. Schreiber [S].

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A spectral theorem for imprimitivity C*-bimodule

(submitted to Expositiones Mathematicae)

A Spectral Theorem for Imprimitivity C*-bimodules

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Abstract
After recalling in detail some basic definitions on Hilbert C*-bimodules, Morita equivalence and imprimitivity, we discuss a spectral reconstruction theorem for imprimitivity Hilbert C*-bimodules over commutative unital C*-algebras and consider some of its applications in the theory of commutative full C*-categories.

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Keywords: Imprimitivity C*-bimodule, Hermitian Line Bundle, C*-category, Non-commutative Geometry.

1 Introduction

A. Connes' non-commutative geometry [C] is the most powerful incarnation of R. Descartes' idea of trading "geometrical spaces" with commutative "algebras of coordinates" and it is based on the existence of suitable dualities between categories constructed from commutative algebras and categories of their "spectra". The most celebrated example is I. Gel'fand-M. Naĭmark theorem (see e.g. [B, Theorem II.2.2.4]) asserting that, via Gel'fand transform, a unital commutative C*-algebra $\mathcal{A}$ is isomorphic to the algebra of continuous complex-valued functions on a compact Hausdorff topological space, namely the spectrum of $\mathcal{A}$. In this way a commutative unital C*-algebra can be reconstructed (up to isomorphism) from its spectrum.
The equally famous Serre-Swan theorem (see e.g. [K, Theorem 6.18]) permits the reconstruction, up to isomorphism, of a finite projective module over a commutative unital C*-algebra from a spectrum that turns out to be a finite-rank complex vector bundle over the Gel'fand spectrum of the C*-algebra. When we restrict to the case of Hilbert C*-modules over commutative unital C*-algebras, Serre-Swan theorem admits a more powerful formulation, Takahashi theorem [T1, T2, W], with spectra given by Hilbert bundles over compact Hausdorff spaces.

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The purpose of this paper is to start the development of a spectral reconstruction theorem for suitable bimodules over commutative unital C*-algebras, i.e. a "bivariant version" of Takahashi and Serre-Swan results, considering for now the case of imprimitivity Hilbert C*-bimodules.
In order to make the result almost completely self-contained, we precede the discussion of our spectral theorem with a detailed treatment of basic facts on imprimitivity C*-bimodules and Morita equivalence including an explicit construction of a natural isomorphism between a pair of C*-algebras associated to a given imprimitivity Hilbert C*-bimodule over them.
Our main result is that the spectrum of an imprimitivity Hilbert C*-bimodule over two commutative unital C*-algebras is described by a Hermitian line bundle over a compact Hausdorff space that is the graph of a canonical homeomorphism between the Gel'fand spectra of the two unital C*-algebras i.e. every imprimitivity Hilbert C*-bimodule is isomorphic to a suitably twisted bimodule of sections of this "spectral" Hermitian line bundle.
We will also collect together some facts about imprimitivity C*-bimodules in the setting of C*-categories that provide a useful background for our study of a categorical Gel'fand duality [BCL2] and that cannot be easily found in the literature.

The content of the paper is as follows.
In section 2, for the benefit of the readers, we recall the basic definitions and properties of Hilbert C*-modules. In subsection 2.3 we explore some specific properties of imprimitivity bimodules arising from C*-categories that will be crucial in the study of the categorification of Gel'fand duality that will be undergone in [BCL2]. Section 3 contains the proof of the spectral reconstruction theorem for imprimitivity Hilbert C*-bimodules as well as some relevant bibliographical references to other available spectral results for C*-modules.
The complete construction of a bivariant duality, between categories of "bivariant Hermitian (line) bundles" and categories of (imprimitivity) Hilbert C*-bimodules over commutative unital C*-algebras, will not be completed here (in particular there is no discussion of the appropriate classes of morphisms and no construction of the section/spectrum functors supporting such a duality), but it is our intention to return later to this topic.
Part of the results presented here have been announced in our survey paper [BCL1] and have been presented in several seminars in Thailand, Australia, Italy, UK since May 2006.

2 Preliminaries on Hilbert C*-Modules

For convenience of the reader and in order to establish notation and terminology, we provide here some background material on the theory of Hilbert C*-modules. General references are the books by N. Wegge-Olsen [WO], C. Lance [L] and B. Blackadar [B, Section II.7].

In the following, $\mathcal{A}, \mathcal{B}, \dots$ denote unital C*-algebras and $\mathcal{A}_+ := \{a^*a \in \mathcal{A} \mid a \in \mathcal{A}\}$ is the positive part of the C*-algebra $\mathcal{A}$.

Definition 2.1. *a* ***right pre-Hilbert C*-module*** $M_\mathcal{B}$ *over a unital C*-algebra* $\mathcal{B}$ *is a unital right module over the unital ring* $\mathcal{B}$ *that is equipped with a* $\mathcal{B}$*-valued inner product* $(x,y) \mapsto \langle x \mid y\rangle_\mathcal{B}$ *such that:*

$$\langle z \mid x+y\rangle_\mathcal{B} = \langle z \mid x\rangle_\mathcal{B} + \langle z \mid y\rangle_\mathcal{B} \quad \forall x,y,z \in M,$$
$$\langle z \mid x\cdot b\rangle_\mathcal{B} = \langle z \mid x\rangle_\mathcal{B} b \quad \forall x,y \in M, \qquad \forall b \in \mathcal{B},$$
$$\langle y \mid x\rangle_\mathcal{B} = \langle x \mid y\rangle_\mathcal{B}^* \quad \forall x,y \in M,$$
$$\langle x \mid x\rangle_\mathcal{B} \in \mathcal{B}_+ \quad \forall x \in M,$$
$$\langle x \mid x\rangle_\mathcal{B} = 0_\mathcal{B} \Rightarrow x = 0_M.$$

Analogously, a ***left pre-Hilbert C*-module*** ${}_{\mathcal{A}}M$ *over a unital C*-algebra* $\mathcal{A}$ *is a unital left module* M *over the unital ring* $\mathcal{A}$*, that is equipped with an* $\mathcal{A}$*-valued inner product* $M \times M \to \mathcal{A}$ *denoted by* $(x, y) \mapsto {}_{\mathcal{A}}\langle x \mid y\rangle$*. Here the* $\mathcal{A}$*-linearity in on the first variable.*

Remark 2.2. *A right (respectively left) pre-Hilbert C*-module* $M_{\mathcal{B}}$ *over the C*-algebra* $\mathcal{B}$ *is naturally equipped with a norm (for a proof see for example [FGV, Lemma 2.14 and Corollary 2.15]):*

$$\|x\|_M := \sqrt{\|\langle x \mid x\rangle_{\mathcal{B}}\|_{\mathcal{B}}}, \quad \forall x \in M.$$

Definition 2.3. *A right (resp. left)* ***Hilbert C*-module*** *is a right (resp. left) pre-Hilbert C*-module over a C*-algebra* $\mathcal{B}$ *that is a Banach space with respect to the previous norm* $\|\cdot\|_M$ *(resp.* ${}_M\|\cdot\|$*).*

Definition 2.4. *A right Hilbert C*-module* $M_{\mathcal{B}}$ *is said to be* ***full*** *if*

$$\langle M_{\mathcal{B}} \mid M_{\mathcal{B}}\rangle_{\mathcal{B}} := \overline{\operatorname{span}\{\langle x \mid y\rangle_{\mathcal{B}} \mid x, y \in M_{\mathcal{B}}\}} = \mathcal{B},$$

where the closure is in the norm topology of the C-algebra* $\mathcal{B}$*. A similar definition holds for a left Hilbert C*-module.*

We recall the following well-known result (see [FGV, p. 65]), whose proof is included here:

Lemma 2.5. *Let* $M_{\mathcal{B}}$ *be a right Hilbert C*-module over a unital C*-algebra* $\mathcal{B}$*. Then* $M_{\mathcal{B}}$ *is full if and only if* $\operatorname{span}\{\langle x \mid y\rangle_{\mathcal{B}} \mid x, y \in M_{\mathcal{B}}\} = \mathcal{B}$*.*

Proof. If $M_{\mathcal{B}}$ is full, for any $\epsilon > 0$, we can find a natural number $n \in \mathbb{N}_0$ and elements $x_j, y_j \in M$, with $j = 1, \dots, n$, such that

$$\|\sum_{j=1}^{n} \langle x_j \mid y_j\rangle_{\mathcal{B}} - 1_{\mathcal{B}}\|_{\mathcal{B}} < \epsilon.$$

Taking $\epsilon \leq 1$, we see that $\sum_{j=1}^{n} \langle x_j \mid y_j\rangle_{\mathcal{B}}$ is invertible i.e. there exists an element b_ϵ in $\mathcal{B}$ such that $(\sum_{j=1}^{n} \langle x_j \mid y_j\rangle_{\mathcal{B}}) b_\epsilon = 1_{\mathcal{B}}$. Hence $\sum_{j=1}^{n} \langle x_j \mid y_j b_\epsilon\rangle_{\mathcal{B}} = 1_{\mathcal{B}}$, i.e. $1_{\mathcal{B}}$ is in the ideal $\operatorname{span}\{\langle x \mid y\rangle_{\mathcal{B}} \mid x, y \in M_{\mathcal{B}}\}$ that therefore coincides with $\mathcal{B}$. □

We note that the notion of Hilbert C*-modules behaves naturally under quotients:

Proposition 2.6. *Let* $M_{\mathcal{A}}$ *be a right Hilbert C*-module over a unital C*-algebra* $\mathcal{A}$ *and* $\mathcal{I} \subset \mathcal{A}$ *an involutive ideal in* $\mathcal{A}$*. Then the set* $M\mathcal{I} := \{\sum_{j=1}^{N} x_j a_j \mid x_j \in M,\ a_j \in \mathcal{I},\ N \in \mathbb{N}_0\}$ *is a submodule of* M*. The quotient module* $M/(M\mathcal{I})$ *has a natural structure as a right Hilbert C*-module over the quotient C*-algebra* $\mathcal{A}/\mathcal{I}$*. If* M *is full over* $\mathcal{A}$*, also* $M/(M\mathcal{I})$ *is full over* $\mathcal{A}/\mathcal{I}$*. A similar statement holds for a left Hilbert C*-module.*

Proof. Clearly $M\mathcal{I}$ is a submodule of the right $\mathcal{A}$-module M. It is immediately checked that the operation of right multiplication by elements of $\mathcal{A}/\mathcal{I}$ and the $\mathcal{A}/\mathcal{I}$-valued inner product given by:

$$(x + M\mathcal{I}) \cdot (a + \mathcal{I}) := xa + M\mathcal{I}, \quad \forall x + M\mathcal{I} \in M/(M\mathcal{I})\ \forall a + \mathcal{I} \in \mathcal{A}/\mathcal{I},$$
$$\langle x + M\mathcal{I} \mid y + M\mathcal{I}\rangle_{\mathcal{A}/\mathcal{I}} := \langle x \mid y\rangle_{\mathcal{A}} + \mathcal{I}, \quad \forall x + M\mathcal{I},\ y + M\mathcal{I} \in M/(M\mathcal{I}),$$

are well-defined so that $M/(M\mathcal{I})$ becomes a right Hilbert C*-module over $\mathcal{A}/\mathcal{I}$. Of course if $\langle M \mid M\rangle = \mathcal{A}$, also $\langle M/(M\mathcal{I}) \mid M/(M\mathcal{I})\rangle = \mathcal{A}/\mathcal{I}$. □

Definition 2.7. *A* ***morphism of right Hilbert C*-modules****, from* $(M_{\mathcal{B}}, \langle \cdot \mid \cdot \rangle_{\mathcal{B}})$ *into* $(N_{\mathcal{B}}, \langle \cdot \mid \cdot \rangle'_{\mathcal{B}})$ *is an adjointable map i.e. a function* $T : M_{\mathcal{B}} \to N_{\mathcal{B}}$ *such that*

$$\exists S : N \to M, \quad \langle S(x) \mid y \rangle_{\mathcal{B}} = \langle x \mid T(y) \rangle'_{\mathcal{B}}, \quad \forall x \in N, \ \forall y \in M.$$

Remark 2.8. *It is well-known, see e.g. N. Landsman [La, Theorem 3.2.5], that an adjointable map* $T \colon M_{\mathcal{B}} \to N_{\mathcal{B}}$ *between Hilbert C*-modules is necessarily continuous and* $\mathcal{B}$*-linear:*

$$T(xa + yb) = T(x)a + T(y)b, \quad \forall x, y \in M, \ \forall a, b \in \mathcal{B}.$$

Furthermore, the family $\operatorname{End}(M_{\mathcal{B}})$ *of morphisms on* $\mathcal{M}_{\mathcal{B}}$ *has a natural structure of a unital C*-algebra.*

Given $x, y \in M_{\mathcal{B}}$, an operator $\theta_{x,y} : M_{\mathcal{B}} \to M_{\mathcal{B}}$ of the form

$$\theta_{x,y} : z \mapsto x \cdot \langle y \mid z \rangle_{\mathcal{B}} \tag{2.1}$$

is clearly a morphism of the right Hilbert C*-module $M_{\mathcal{B}}$ with adjoint given by $\theta_{y,x}$.

Definition 2.9. *A* ***finite-rank*** *operator of the Hilbert C*-module* $M_{\mathcal{B}}$ *is a finite linear combination of operators of the form* $\theta_{x,y}$*,* $x, y \in M_{\mathcal{B}}$*, as described in* (2.1)*.*
The family $\mathcal{K}(M_{\mathcal{B}})$ *of* ***compact*** *operators of the right Hilbert C*-module* $M_{\mathcal{B}}$ *is by definition the C*-subalgebra of* $\operatorname{End}(M_{\mathcal{B}})$ *generated by the finite-rank operators.*

Definition 2.10. *Let* $M_{\mathcal{B}}$ *be a right unital module over a unital ring* $\mathcal{B}$ *and let* $\alpha : \mathcal{A} \to \mathcal{B}$ *be a unital homomorphism of rings. The* ***right twisted module*** *of* $M_{\mathcal{B}}$ *by the homomorphism* α *is the right unital module* M_α *over the unital ring* $\mathcal{A}$ *with the right action defined by:*

$$x \cdot a := x \cdot \alpha(a), \quad \forall x \in M, \ \forall a \in \mathcal{A}.$$

The ***left twisted module*** *of* ${}_{\mathcal{B}}M$ *by the homomorphism* $\alpha : \mathcal{A} \to \mathcal{B}$ *is analogously defined.*

Remark 2.11. *If* $M_{\mathcal{B}}$ *is a right (pre-)Hilbert C*-module and* $\alpha : \mathcal{A} \to \mathcal{B}$ *is an isomorphism of unital C*-algebras, then the right* $\mathcal{A}$*-module* M_α *obtained by right twisting* $M_{\mathcal{B}}$ *by the isomorphism* α *has a natural structure as a (pre-)Hilbert C*-module over* $\mathcal{A}$ *with the inner product given by* $\langle x \mid y \rangle_{\mathcal{A}} := \alpha^{-1}(\langle x \mid y \rangle_{\mathcal{B}})$*.*

Proposition 2.12. *Let* $\alpha \colon \mathcal{A} \to \mathcal{B}$ *be a unital isomorphism of unital rings. Let* $M_{\mathcal{A}}$ *and* $N_{\mathcal{B}}$ *be unital right modules over* $\mathcal{A}$ *and respectively* $\mathcal{B}$*. Then* $\Phi \colon M_{\mathcal{A}} \to N_\alpha$ *is a morphism of right modules over* $\mathcal{A}$ *if and only if* $\Phi \colon M_{\alpha^{-1}} \to N_{\mathcal{B}}$ *is a morphism of right* $\mathcal{B}$*-modules.*
The result holds true also when $M_{\mathcal{A}}$ *and* $N_{\mathcal{B}}$ *are (pre-)Hilbert C*-modules and* $\Phi : M_{\mathcal{A}} \to N_\alpha$ *is a morphism of (pre-)Hilbert C*-modules over* $\mathcal{A}$*.*

Proof. Clearly $\Phi(x \cdot a) = \Phi(x) \cdot \alpha(a)$ if and only if $\Phi(x \cdot \alpha^{-1}(b)) = \Phi(x) \cdot b$. Also $\Phi : M_{\mathcal{A}} \to N_\alpha$ is adjointable, with adjoint Ψ, if and only $\Phi : M_{\alpha^{-1}} \to N_{\mathcal{B}}$ is adjointable with the same adjoint: $\alpha^{-1}(\langle x \mid \Phi(y) \rangle_{\mathcal{B}}) = \langle \Psi(x) \mid y \rangle_{\mathcal{A}}$ if and only if $\langle x \mid \Phi(y) \rangle_{\mathcal{B}} = \alpha(\langle \Psi(x) \mid y \rangle_{\mathcal{A}})$, for all $x \in N$, $y \in M$. □

2.1 Hilbert C*-bimodules and Morita Equivalence

Recall that a unital bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$ over two unital rings $\mathcal{A}$ and $\mathcal{B}$ is a left unital $\mathcal{A}$-module and a right unital $\mathcal{B}$-module such that $(a \cdot x) \cdot b = a \cdot (x \cdot b)$, for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $x \in M$.

Definition 2.13. *A* ***pre-Hilbert C*-bimodule*** ${}_{\mathcal{A}}M_{\mathcal{B}}$ *over a pair of unital C*-algebras* $\mathcal{A}, \mathcal{B}$ *is a left pre-Hilbert C*-module over* $\mathcal{A}$ *and a right pre-Hilbert C*-module over* $\mathcal{B}$ *such that:*

$$(a \cdot x) \cdot b = a \cdot (x \cdot b) \quad \forall a \in \mathcal{A},\ x \in M,\ b \in \mathcal{B}, \tag{2.2}$$

$$\langle x \mid ay\rangle_{\mathcal{B}} = \langle a^* x \mid y\rangle_{\mathcal{B}} \quad \forall x, y \in M,\ \forall a \in \mathcal{A}, \tag{2.3}$$

$$_{\mathcal{A}}\langle xb \mid y\rangle = {}_{\mathcal{A}}\langle x \mid yb^*\rangle \quad \forall x, y \in M,\ \forall b \in \mathcal{B}. \tag{2.4}$$

A ***correspondence from*** $\mathcal{A}$ ***to*** $\mathcal{B}$ *is an* $\mathcal{A}$*-*$\mathcal{B}$*-bimodule that is also a right Hilbert C*-module over* $\mathcal{B}$ *whose* $\mathcal{B}$*-valued inner product satisfies property* (2.3).
A ***Hilbert C*-bimodule*** ${}_{\mathcal{A}}M_{\mathcal{B}}$ *is a pre-Hilbert C*-bimodule over* $\mathcal{A}$ *and* $\mathcal{B}$ *that is simultaneously a left Hilbert C*-module over* $\mathcal{A}$ *and a right Hilbert C*-module over* $\mathcal{B}$*.*
A Hilbert C-bimodule is* ***full*** *if it is full as a right and also as a left module.*
A full Hilbert C-bimodule over the C*-algebras* $\mathcal{A}$*-*$\mathcal{B}$ *is said to be an* ***imprimitivity bimodule*** *or an* ***equivalence bimodule*** *if:*

$$_{\mathcal{A}}\langle x \mid y\rangle \cdot z = x \cdot \langle y \mid z\rangle_{\mathcal{B}}, \quad \forall x, y, z \in M. \tag{2.5}$$

Remark 2.14. *Note that our definitions of pre-Hilbert and Hilbert C*-bimodule are not necessarily in line with often conflicting similar definitions available in the literature: for example, H. Figueroa-J. Gracia-Bondia-J. Varilly [FGV, Definition 4.7] and B. Abadie-R. Exel [AE] require pre-Hilbert C*-bimodules to satisfy condition* (2.5)*; A. Connes [C, Page 159] calls Hilbert C*-bimodules what we call here correspondences (in this case, only one inner product is assumed). In an* $\mathcal{A}$*-*$\mathcal{B}$ *pre-Hilbert C*-bimodule there are two, usually different, norms:*

$$_{M}\|x\| := \sqrt{\|_{\mathcal{A}}\langle x \mid x\rangle\|_{\mathcal{A}}}, \quad \|x\|_M := \sqrt{\|\langle x \mid x\rangle_{\mathcal{B}}\|_{\mathcal{B}}}, \quad \forall x \in M.$$

The two norms coincide for an imprimitivity bimodule or, more generally, for a pre-Hilbert C-bimodule* ${}_{\mathcal{A}}M_{\mathcal{B}}$ *such that* ${}_{\mathcal{A}}\langle x \mid x\rangle x = x\langle x \mid x\rangle_{\mathcal{B}}$*, for all* $x \in M$*. In fact*

$$\begin{aligned} _{M}\|x\|^4 &= \|_{\mathcal{A}}\langle x \mid x\rangle\|^2_{\mathcal{A}} = \|_{\mathcal{A}}\langle x \mid x\rangle_{\mathcal{A}}\langle x \mid x\rangle\|_{\mathcal{A}} = \|_{\mathcal{A}}\langle x\langle x \mid x\rangle_{\mathcal{B}} \mid x\rangle\|_{\mathcal{A}} \\ &\leq \|\langle x \mid x\rangle_{\mathcal{B}}\|_{\mathcal{B}} \cdot \|_{\mathcal{A}}\langle x \mid x\rangle\|_{\mathcal{A}} = \|x\|^2_M \cdot {}_M\|x\|^2. \end{aligned}$$

Definition 2.15. *A* ***morphism of correspondences*** *from* $\mathcal{A}$ *to* $\mathcal{B}$ *is a morphism of right Hilbert C*-modules over* $\mathcal{B}$ *that further satisfies:*

$$T(ax) = aT(x), \quad \forall x \in M,\ \forall a \in \mathcal{A}. \tag{2.6}$$

A ***morphism of (pre-)Hilbert C*-bimodules*** *is just a morphism of right and left (pre-) Hilbert C*-bimodules.*

Remark 2.16. *Morphisms of correspondences are just morphisms of bimodules that are adjointable for the right C*-module structure.*
Note that in a (pre-)Hilbert C-bimodule there are in general two different notions of left and of right adjoint of a morphism. The left and right adjoints of a morphism coincide if and only if* ${}_{\mathcal{A}}\langle x \mid y\rangle = 0_{\mathcal{A}} \Leftrightarrow \langle x \mid y\rangle_{\mathcal{B}} = 0_{\mathcal{B}}$*, for all* $x, y \in M$*. This condition is true for all full (pre)-Hilbert C*-bimodules such that*

$$_{\mathcal{A}}\langle x \mid y\rangle x = x\langle y \mid x\rangle_{\mathcal{B}}, \quad \forall x, y \in {}_{\mathcal{A}}M_{\mathcal{B}}. \tag{2.7}$$

Proposition 2.17. *If $_{\mathcal{A}}M_{\mathcal{B}}$ is an imprimitivity bimodule over the unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$, the map $T : \mathcal{A} \to \mathcal{K}(M_{\mathcal{B}})$ given by $\alpha \mapsto T_\alpha$, where we define $T_\alpha(x) := \alpha \cdot x$, is an isomorphism of C*-algebras. Furthermore the C*-algebra of compact operators $\mathcal{K}(M_{\mathcal{B}})$ coincides with the family of finite-rank operators.*

Proof. Clearly T_α is a morphism of the Hilbert C*-module $M_{\mathcal{B}}$ with adjoint given by T_{α^*}. The map $\alpha \mapsto T_\alpha$ is a unital involutive homomorphism from $\mathcal{A}$ to $\mathrm{End}(M_{\mathcal{B}})$ and so its image is a unital C*-subalgebra of the C*-algebra $\mathrm{End}(M_{\mathcal{B}})$. Furthermore, from the fullness of $M_{\mathcal{B}}$, we see that $\alpha \mapsto T_\alpha$ is injective so that $\mathcal{A}$ is isomorphic to its image under T in $\mathrm{End}(M_{\mathcal{B}})$.
The image of T contains all the finite-rank operators, for if $S = \sum_k \theta_{x_k,y_k}$, with $x_k, y_k \in M_{\mathcal{B}}$, then for all $z \in M_{\mathcal{B}}$,

$$S(z) = \sum_k \theta_{x_k,y_k}(z) = \sum_k x_k \langle y_k \mid z\rangle_{\mathcal{B}} = \sum_k {}_{\mathcal{A}}\langle x_k \mid y_k\rangle z = T_\alpha(z),$$

where $\alpha := \sum_k {}_{\mathcal{A}}\langle x_k \mid y_k\rangle$. Since, by lemma 2.5, every $\alpha \in \mathcal{A}$ can always be written as a finite combination $\alpha = \sum_k {}_{\mathcal{A}}\langle x_k \mid y_k\rangle$, we see that T_α is always a finite-rank operator, and hence the image of T coincides with the family of finite-rank operators.
Since the closure of the finite-rank operators is the C*-algebra of compact operators $\mathcal{K}(M_{\mathcal{B}})$, we see that T is an isomorphism of C*-algebras from $\mathcal{A}$ onto $\mathcal{K}(M_{\mathcal{B}})$ and that $\mathcal{K}(M_{\mathcal{B}})$ coincides with the family of finite-rank operators. □

There is a natural notion of **Rieffel interior tensor product** between Hilbert C*-modules and correspondences [R2]:

Proposition 2.18. *Given two unital C*-algebras $\mathcal{A}, \mathcal{B}$, let $M_{\mathcal{A}}$ be a right Hilbert C*-module over $\mathcal{A}$ and let $_{\mathcal{A}}N_{\mathcal{B}}$ be a correspondence from $\mathcal{A}$ to $\mathcal{B}$. The algebraic tensor product $M \otimes_{\mathcal{A}} N$ of the right $\mathcal{A}$-module M with the $\mathcal{A}$-$\mathcal{B}$-bimodule N is naturally a right Hilbert C*-module over $\mathcal{B}$ with the unique $\mathcal{B}$-valued inner product such that:*

$$\langle x_1 \otimes y_1 \mid x_2 \otimes y_2\rangle_{\mathcal{B}} = \langle y_1 \mid \langle x_1 \mid x_2\rangle_{\mathcal{A}} \cdot y_2\rangle_{\mathcal{B}}, \quad \forall x_1, x_2 \in M,\ \forall y_1, y_2 \in N.$$

Similarly, the algebraic tensor product $M \otimes_{\mathcal{B}} N$, of a pair of (pre-)Hilbert C-bimodules $_{\mathcal{A}}M_{\mathcal{B}}$, $_{\mathcal{B}}N_{\mathcal{C}}$ has a natural structure of (pre-)Hilbert C*-bimodule on the unital C*-algebras $\mathcal{A}$-$\mathcal{C}$ where the "left-action" of $\mathcal{A}$ satisfies:*

$$a(x \otimes y) := (ax) \otimes y, \quad \forall a \in \mathcal{A},\ \forall x \in M,\ y \in N.$$

There is also a natural notion of **Rieffel dual** of a (pre-)Hilbert C*-bimodule [R2] that is uniquely defined (up to isomorphism) via the following proposition:

Proposition 2.19. *Let $_{\mathcal{B}}M_{\mathcal{A}}$ be a (pre-)Hilbert C*-bimodule. Then there exist a (pre-) Hilbert C*-bimodule $_{\mathcal{A}}M^*_{\mathcal{B}}$ and an anti-homomorphism of bimodules $\iota : {}_{\mathcal{B}}M_{\mathcal{A}} \to {}_{\mathcal{A}}M^*_{\mathcal{B}}$, i.e. a map such that $\iota(bxa) = a^*\iota(x)b^*$ $\forall x \in M$ $\forall a \in \mathcal{A}$ $\forall b \in \mathcal{B}$, satisfying the following universal property: for every (pre-)Hilbert C*-bimodule $_{\mathcal{A}}N_{\mathcal{B}}$ and any anti-homomorphism of bimodules $\Phi : {}_{\mathcal{B}}M_{\mathcal{A}} \to {}_{\mathcal{A}}N_{\mathcal{B}}$ there exists a unique homomorphism of bimodules $\Phi' : {}_{\mathcal{A}}M^*_{\mathcal{B}} \to {}_{\mathcal{A}}N_{\mathcal{B}}$ such that $\Phi = \Phi' \circ \iota$.*

Proof. We take $M^* := M$ as sets, but we define on M^* the following bimodule structure:

$$a \cdot x := xa^*, \quad \forall x \in M^* = M, \quad \forall a \in \mathcal{A},$$
$$x \cdot b := b^*x, \quad \forall x \in M^* = M, \quad \forall b \in \mathcal{B}.$$

It is easily checked that ${}_{\mathcal{A}}M^*_{\mathcal{B}}$ is a bimodule and that it becomes a (pre-)Hilbert C*-bimodule if the inner products on M^* are defined as follows:

$$\langle x \mid y\rangle'_{\mathcal{B}} := {}_{\mathcal{B}}\langle x \mid y\rangle^*, \quad \forall x, y \in M^*,$$
$${}_{\mathcal{A}}\langle x \mid y\rangle' := \langle x \mid y\rangle^*_{\mathcal{A}}, \quad \forall x, y \in M^*,$$

where ${}_{\mathcal{A}}\langle x \mid y\rangle'$ and $\langle x \mid y\rangle'_{\mathcal{B}}$ denote the inner products on ${}_{\mathcal{A}}M^*_{\mathcal{B}}$.
Clearly the identity map $\iota : M \to M^*$ is an anti-homomorphism of bimodules and for any anti-homomorphism of bimodules $\Phi : {}_{\mathcal{B}}M_{\mathcal{A}} \to {}_{\mathcal{A}}N_{\mathcal{B}}$, $\Phi' := \Phi$ is the unique homomorphism of bimodules $\Phi' : {}_{\mathcal{A}}M^*_{\mathcal{B}} \to {}_{\mathcal{A}}N_{\mathcal{B}}$ such that $\Phi = \Phi' \circ \iota$. □

The pair $(\iota, {}_{\mathcal{A}}M^*_{\mathcal{B}})$ is unique up to isomorphism (as for any concept defined through a universal property) and is called the dual of the (pre-)Hilbert C*-bimodule ${}_{\mathcal{B}}M_{\mathcal{A}}$.

Definition 2.20. *The* ***Morita category*** *is the involutive category*[1] *with objects the unital associative rings, with morphisms the isomorphism classes of bimodules, with composition the isomorphism classes of the tensor product of bimodules, and with involution given by isomorphism classes of the dual bimodules. The* ***(algebraic) Picard groupoid*** *is the nerve of the Morita category*[2]*. Two unital associative rings are* ***Morita equivalent*** *if they are in the same orbit of the Picard groupoid.*

Here we are interested only in the full subcategory of the Morita category whose objects are unital C*-algebras. In this case, it is usually better to "restrict" also the family of allowed arrows as long as the new category preserves the notion of Morita equivalence i.e. its nerve has the same orbits of the Picard groupoid.[3]
The category described in the following definition is the **Morita-Rieffel category** of unital C*-algebras and it plays a key role in the discussion of the horizontal categorification of Gel'fand Theorem [BCL2].

Definition 2.21. *The* ***Morita-Rieffel*** *category is the subcategory of the Morita category whose objects are unital C*-algebras, whose arrows are the isomorphism classes of correspondences and whose composition is the Rieffel tensor product of correspondences. The nerve of this category is the (algebraic)* ***Picard-Rieffel*** *groupoid. Two C*-algebras in the the same orbit of the Picard-Rieffel groupoid are said to be* ***strongly Morita equivalent*** *[R1].*

Remark 2.22. *Note that the Morita-Rieffel category is not an involutive category (the substitution of bimodules with correspondences "breaks the symmetry" between left and right module structures). It is possible to eliminate this problem considering other subcategories of the Morita category. Two possible natural choices are the involutive subcategory of the Morita category consisting of isomorphism classes of (pre-)Hilbert C*-bimodules or (whenever it is necessary to have a unique Banach norm and a unique notion of adjoint of a morphism of the bimodules involved) the subcategory consisting of full Hilbert C*-bimodules such that property* (2.7) *is satisfied. In these cases the involution is given by the Rieffel dual of the bimodules.*

The following proposition is a well-known result (see e.g. [GMS, Section 8.8] for a review).

[1] By an **involutive category** we mean a category $\mathscr{C}$ equipped with an involutive contravariant endofunctor acting identically on the objects of $\mathscr{C}$ i.e. a map $* : \mathscr{C} \to \mathscr{C}$ such that $(x^*)^* = x$ and $(x \circ y)^* = y^* \circ x^*$ for all $x, y \in \mathscr{C}$.

[2] The nerve of a category is its class of invertible arrows.

[3] There are also interesting versions of Morita theory for involutive unital algebras (see P. Ara [A] and H. Bursztyn-S. Waldmann [BW]).

Proposition 2.23. *Two unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent if and only if there exists an imprimitivity bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$. The Picard-Rieffel groupoid consists of isomorphism classes of imprimitivity Hilbert C*-bimodules. Moreover, the notions of Morita equivalence and strong Morita equivalence coincide.*

Proof. If $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, there exists bimodules ${}_{\mathcal{A}}M_{\mathcal{B}}$ and ${}_{\mathcal{B}}N_{\mathcal{A}}$ such that $M \otimes_{\mathcal{B}} N \simeq \mathcal{A}$ and $N \otimes_{\mathcal{A}} M \simeq \mathcal{B}$. Any bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$ with the previous properties is necessarily finite projective [GMS, Theorem 10.4.3]. Any finite projective right module can be equipped with an inner product that makes it a correspondence from $\mathcal{A}$ to $\mathcal{B}$ and hence ${}_{\mathcal{A}}M_{\mathcal{B}}$ must be an imprimitivity bimodule. □

2.2 Imprimitivity Bimodules on Abelian C*-algebras.

It is well-known that in some cases imprimitivity bimodules can be used to construct explicit isomorphisms between the associated C*-algebras, see e.g. [Bo, Lemma 10.19]. In this subsection we follow a similar route, recovering and further elaborating on a "classical" result [R3, Theorem 3.1 and Corollary 3.3] that is certainly folklore among specialists. For the sake of self-containment we present a full account of the situation at hand.

The following theorem is motivated by P. Ara [A, Theorem 4.2].

Theorem 2.24. *Let ${}_{\mathcal{A}}M_{\mathcal{B}}$ be an $\mathcal{A}$-$\mathcal{B}$ imprimitivity bimodule, where $\mathcal{A}$ and $\mathcal{B}$ are commutative unital C*-algebras. Then there exists a unique canonical isomorphism $\phi_M : \mathcal{A} \to \mathcal{B}$ such that:*

$$\phi_M({}_{\mathcal{A}}\langle x \mid y\rangle) = \langle y \mid x\rangle_{\mathcal{B}}, \quad \forall x, y \in M. \tag{2.8}$$

Moreover the canonical isomorphism ϕ_M satisfies the following property:

$$a \cdot x = x \cdot \phi_M(a), \quad \forall x \in M, \ \forall a \in \mathcal{A}. \tag{2.9}$$

Proof. The uniqueness of the map follows from the fullness of the left Hilbert C*-module ${}_{\mathcal{A}}M$. By the fullness of the right Hilbert C*-module $M_{\mathcal{B}}$ we can write $1_{\mathcal{B}}$ as a finite sum $1_{\mathcal{B}} = \sum_{j=1}^n \langle w_j \mid z_j\rangle_{\mathcal{B}}$, where $w_j, z_j \in M$, $j = 1, \dots, n$. For any $a \in \mathcal{A}$, define

$$\phi_M(a) = \sum_{j=1}^n \langle w_j \mid a z_j\rangle_{\mathcal{B}}, \tag{2.10}$$

where $w_j, z_j \in M$ are such that $\sum_{j=1}^n \langle w_j \mid z_j\rangle_{\mathcal{B}} = 1_{\mathcal{B}}$.
To show that ϕ_M is well-defined, let w_j, z_j and x_k, y_k be two pairs of finite sequences such that $\sum_j \langle w_j \mid z_j\rangle_{\mathcal{B}} = 1_{\mathcal{B}}$ and $\sum_k \langle x_k \mid y_k\rangle_{\mathcal{B}} = 1_{\mathcal{B}}$. Write $b = \sum_j \langle w_j \mid a z_j\rangle_{\mathcal{B}}$. Then

$$\begin{aligned}
\langle x_k \mid y_k\rangle_{\mathcal{B}}\, b &= \langle x_k \mid y_k\rangle_{\mathcal{B}} \sum_j \langle w_j \mid a z_j\rangle_{\mathcal{B}} \\
&= \sum_j \langle x_k \mid y_k \langle w_j \mid a z_j\rangle_{\mathcal{B}}\rangle_{\mathcal{B}} = \sum_j \langle x_k \mid {}_{\mathcal{A}}\langle y_k \mid w_j\rangle\, a z_j\rangle_{\mathcal{B}} \\
&= \sum_j \langle x_k \mid a\, {}_{\mathcal{A}}\langle y_k \mid w_j\rangle z_j\rangle_{\mathcal{B}} = \sum_j \langle x_k \mid a y_k \langle w_j \mid z_j\rangle_{\mathcal{B}}\rangle_{\mathcal{B}} \\
&= \langle x_k \mid a y_k\rangle_{\mathcal{B}}.
\end{aligned}$$

It follows that $b = \sum_k \langle x_k \mid a y_k\rangle_{\mathcal{B}}$, which shows that $\phi_M(a)$ is well-defined.

We now show that ϕ_M is a homomorphism of algebras. Clearly ϕ_M is additive and $\mathbb{C}$-linear. The multiplicativity follows from:

$$\begin{aligned}\phi_M(a)\cdot\phi_M(a') &= \sum_j \langle w_j \mid az_j\rangle_{\mathcal{B}} \sum_k \langle w'_k \mid a'z'_k\rangle_{\mathcal{B}} \\ &= \sum_{j,k} \langle w_j \mid az_j\langle w'_k \mid a'z'_k\rangle_{\mathcal{B}}\rangle_{\mathcal{B}} = \sum_{j,k}\langle w_j \mid a\,{}_{\mathcal{A}}\langle z_j \mid w'_k\rangle a'z'_k\rangle_{\mathcal{B}} \\ &= \sum_{j,k}\langle w_j \mid {}_{\mathcal{A}}\langle z_j \mid w'_k\rangle aa'z'_k\rangle_{\mathcal{B}} = \sum_{j,k}\langle w_j \mid z_j\langle w'_k \mid aa'z'_k\rangle_{\mathcal{B}}\rangle_{\mathcal{B}} \\ &= \sum_{j,k}\langle w_j \mid z_j\rangle_{\mathcal{B}}\langle w'_k \mid aa'z'_k\rangle_{\mathcal{B}} = \sum_k \langle w'_k \mid aa'z'_k\rangle_{\mathcal{B}} = \phi_M(aa').\end{aligned}$$

Of course ϕ_M is unital: $\phi_M(1_{\mathcal{A}}) = \sum_j \langle w_j \mid 1_{\mathcal{A}} z_j\rangle_{\mathcal{B}} = \sum_j \langle w_j \mid z_j\rangle_{\mathcal{B}} = 1_{\mathcal{B}}$. To prove the involutivity of ϕ_M, note that if $\sum_j \langle w_j \mid z_j\rangle_{\mathcal{B}} = 1_{\mathcal{B}}$, taking the adjoints, we also have $\sum_j \langle z_j \mid w_j\rangle_{\mathcal{B}} = 1_{\mathcal{B}}$. Hence

$$\phi_M(a^*) = \sum_j \langle w_j \mid a^* z_j\rangle_{\mathcal{B}} = \sum_j \langle aw_j \mid z_j\rangle_{\mathcal{B}} = \sum_j \langle z_j \mid aw_j\rangle^*_{\mathcal{B}} = \phi_M(a)^*.$$

Similarly, there is a canonical homomorphism $\psi_M : \mathcal{B} \to \mathcal{A}$ defined by:

$$\psi_M(b) := \sum_i {}_{\mathcal{A}}\langle t_i b \mid u_i\rangle \quad \forall b \in \mathcal{B},$$

where $t_i, u_i \in M$ is a pair of finite sequences such that $\sum_i {}_{\mathcal{A}}\langle t_i \mid u_i\rangle = 1_{\mathcal{A}}$. Then

$$\begin{aligned}\psi_M(\phi_M(a)) &= \sum_i {}_{\mathcal{A}}\langle t_i\phi_M(a) \mid u_i\rangle \\ &= \sum_{i,j} {}_{\mathcal{A}}\langle t_i\langle w_j \mid az_j\rangle_{\mathcal{B}} \mid u_i\rangle = \sum_{i,j} {}_{\mathcal{A}}\langle {}_{\mathcal{A}}\langle t_i \mid w_j\rangle az_j \mid u_i\rangle \\ &= \sum_{i,j} a\,{}_{\mathcal{A}}\langle t_i\langle w_j \mid z_j\rangle_{\mathcal{B}} \mid u_i\rangle = \sum_i a\,{}_{\mathcal{A}}\langle t_i \mid u_i\rangle = a.\end{aligned}$$

By the same argument, we can show that $\phi_M(\psi_M(b)) = b$ for all $b \in \mathcal{B}$. Hence ψ_M is the inverse of ϕ_M, which implies that ϕ_M is an isomorphism.
To establish (2.8), let $w_j, z_j \in M$ be finite sequences such that $\sum_j \langle w_j \mid z_j\rangle_{\mathcal{B}} = 1_{\mathcal{B}}$. Define $\alpha := \sum_j {}_{\mathcal{A}}\langle z_j \mid w_j\rangle$ and note that

$$\phi_M({}_{\mathcal{A}}\langle x \mid y\rangle) = \langle y \mid \alpha x\rangle_{\mathcal{B}}, \quad \forall x, y \in M, \tag{2.11}$$

which follows from this computation:

$$\begin{aligned}\phi_M({}_{\mathcal{A}}\langle x \mid y\rangle) &= \sum_j \langle w_j \mid {}_{\mathcal{A}}\langle x \mid y\rangle z_j\rangle_{\mathcal{B}} = \sum_j \langle w_j \mid x\langle y \mid z_j\rangle_{\mathcal{B}}\rangle_{\mathcal{B}} \\ &= \sum_j \langle w_j \mid x\rangle_{\mathcal{B}}\langle y \mid z_j\rangle_{\mathcal{B}} = \sum_j \langle y \mid z_j\rangle_{\mathcal{B}}\langle w_j \mid x\rangle_{\mathcal{B}} \\ &= \sum_j \langle y \mid z_j\langle w_j \mid x\rangle_{\mathcal{B}}\rangle_{\mathcal{B}} = \sum_j \langle y \mid {}_{\mathcal{A}}\langle z_j \mid w_j\rangle x\rangle_{\mathcal{B}} \\ &= \langle y \mid \sum_j {}_{\mathcal{A}}\langle z_j \mid w_j\rangle x\rangle_{\mathcal{B}} = \langle y \mid \alpha x\rangle_{\mathcal{B}}.\end{aligned}$$

The element $\alpha \in \mathcal{A}$ is independent from the choice of the finite sequences $w_j, z_j \in M$ such that $\sum_j \langle w_j \mid z_j \rangle_{\mathcal{B}} = 1_{\mathcal{B}}$. In fact, given another pair of finite sequences $w'_i, z'_i \in M$ such that $\sum_i \langle w'_i \mid z'_i \rangle_{\mathcal{B}} = 1_{\mathcal{B}}$, we see that $\phi_M({}_{\mathcal{A}}\langle x \mid y \rangle) = \langle y \mid \alpha' x \rangle_{\mathcal{B}}$, where $\alpha' := \sum_i {}_{\mathcal{A}}\langle z'_i \mid w'_i \rangle$ so that $\langle y \mid \alpha x \rangle_{\mathcal{B}} = \langle y \mid \alpha' x \rangle_{\mathcal{B}}$ for all $x, y \in M$ that implies immediately $(\alpha - \alpha')x = 0_M$ that (by the fulless of the module ${}_{\mathcal{A}}M$) implies $\alpha' = \alpha$.
We see that α is Hermitian because for all $x, y \in M$:

$$\begin{aligned}\langle x \mid \alpha y \rangle_{\mathcal{B}} &= \phi_M({}_{\mathcal{A}}\langle y \mid x \rangle) = \phi_M({}_{\mathcal{A}}\langle x \mid y \rangle^*) \\ &= \phi_M({}_{\mathcal{A}}\langle x \mid y \rangle)^* = \langle y \mid \alpha x \rangle_{\mathcal{B}}^* = \langle \alpha x \mid y \rangle_{\mathcal{B}} = \langle x \mid \alpha^* y \rangle_{\mathcal{B}},\end{aligned}$$

which implies that $\alpha = \alpha^*$.
We can actually prove that $\alpha \in \mathcal{A}$ is positive. Since $\phi_M : \mathcal{A} \to \mathcal{B}$ is an isomorphism, the map $(x, y) \mapsto \phi_M({}_{\mathcal{A}}\langle x \mid y \rangle) = \langle y \mid \alpha x \rangle_{\mathcal{B}}$ is a $\mathcal{B}$-valued inner product on M. Hence $\phi_M({}_{\mathcal{A}}\langle x \mid x \rangle) = \langle x \mid \alpha x \rangle_{\mathcal{B}}$ is a positive element in $\mathcal{B}$ for all $x \in M$. Considering the positive and negative parts of the Hermitian element α, i.e. the unique pair of positive elements $\alpha_+, \alpha_- \in \mathcal{A}_+$ such that $\alpha = \alpha_+ - \alpha_-$ with $\alpha_+ \alpha_- = 0_{\mathcal{A}}$, we see that

$$\langle x \mid \alpha_+ x \rangle_{\mathcal{B}} - \langle x \mid \alpha_- x \rangle_{\mathcal{B}} \in \mathcal{B}_+, \quad \forall x \in M.$$

From the calculation below,

$$\begin{aligned}\langle x \mid \alpha_+ x \rangle_{\mathcal{B}} \langle x \mid \alpha_- x \rangle_{\mathcal{B}} &= \langle x \mid \alpha_+ x \langle x \mid \alpha_- x \rangle_{\mathcal{B}} \rangle_{\mathcal{B}} \\ &= \langle x \mid \alpha_+ {}_{\mathcal{A}}\langle x \mid x \rangle \alpha_- x \rangle_{\mathcal{B}} = \langle x \mid \alpha_+ \alpha_- {}_{\mathcal{A}}\langle x \mid x \rangle x \rangle_{\mathcal{B}} \\ &= \langle x \mid 0_{\mathcal{A}} {}_{\mathcal{A}}\langle x \mid x \rangle x \rangle_{\mathcal{B}} = 0_{\mathcal{B}},\end{aligned}$$

it follows that the positive terms $\langle x \mid \alpha_\pm x \rangle_{\mathcal{B}} = \langle \alpha_\pm^{1/2} x \mid \alpha_\pm^{1/2} x \rangle_{\mathcal{B}}$ are the positive and negative parts of the positive element $\langle x \mid \alpha x \rangle_{\mathcal{B}}$. Therefore $\langle x \mid \alpha_- x \rangle_{\mathcal{B}} = 0_{\mathcal{B}}$ for all $x \in M$, and thus $\alpha_- = 0_{\mathcal{A}}$, and so α is positive.
Next we prove that $\|\alpha\|_{\mathcal{A}} \leq 1$. Consider the operator $T_\alpha : M_{\mathcal{B}} \to M_{\mathcal{B}}$ given by

$$T_\alpha(x) := \alpha \cdot x, \quad \forall x \in M$$

and note that $\|T_\alpha\| \leq 1$ because, for all $x \in M$,

$$\begin{aligned}\|T_\alpha(x)\|^2 &= \|\langle T_\alpha(x) \mid T_\alpha(x) \rangle_{\mathcal{B}}\| = \|\langle T_\alpha(x) \mid \alpha x \rangle_{\mathcal{B}}\| = \|\phi_M({}_{\mathcal{A}}\langle T_\alpha(x) \mid x \rangle)\| \\ &= \|{}_{\mathcal{A}}\langle T_\alpha(x) \mid x \rangle\| \leq \|T_\alpha(x)\| \cdot \|x\|.\end{aligned}$$

By proposition 2.17, the map $T \colon \mathcal{A} \to \mathcal{K}(M_{\mathcal{B}})$, $\alpha \mapsto T_\alpha$, is an isomorphism from $\mathcal{A}$ onto the C*-algebra of compact operators $\mathcal{K}(M_{\mathcal{B}})$. Thus

$$\|\alpha\| = \|T_\alpha\| \leq 1, \quad \forall \alpha \in \mathcal{A}.$$

In a completely similar way, we can find a positive Hermitian element $\beta \in \mathcal{B}$ such that $\|\beta\| \leq 1$ and that

$$\psi_M(\langle x \mid y \rangle_{\mathcal{B}}) = {}_{\mathcal{A}}\langle y\beta \mid x \rangle, \quad \forall x, y \in M. \tag{2.12}$$

The two elements α and β are related by $\phi_M(\alpha)\beta = 1_{\mathcal{B}}$ and $\psi_M(\beta)\alpha = 1_{\mathcal{A}}$. In order to prove this, we first note that

$$x \cdot \phi_M(a) = a \cdot x, \quad \forall x \in M, \quad \forall a \in \mathcal{A}. \tag{2.13}$$

In fact, if $w_j, z_j \in M$ is a pair of sequences such that $\sum_j \langle w_j \mid z_j \rangle_{\mathcal{B}} = 1_{\mathcal{B}}$, equation (2.13) follows from this direct computation:

$$\begin{aligned} x \cdot \phi_M(a) &= x \sum_j \langle w_j \mid a z_j \rangle_{\mathcal{B}} = \sum_j {}_{\mathcal{A}}\langle x \mid w_j \rangle \, a z_j \\ &= \sum_j a \, {}_{\mathcal{A}}\langle x \mid w_j \rangle z_j = \sum_j a x \langle w_j \mid z_j \rangle_{\mathcal{B}} = a \cdot x. \end{aligned}$$

Next we see that

$$\alpha \cdot x \cdot \beta = x, \quad \forall x \in M. \tag{2.14}$$

To see this, we apply (2.11) and (2.12) to the following calculation:

$$\begin{aligned} \langle \alpha \cdot x \cdot \beta \mid y \rangle_{\mathcal{B}} = \langle x \cdot \beta \mid \alpha \cdot y \rangle_{\mathcal{B}} &= \phi_M({}_{\mathcal{A}}\langle y \mid x \cdot \beta \rangle) = \phi_M({}_{\mathcal{A}}\langle y \cdot \beta \mid x \rangle) \\ &= \phi_M(\psi_M(\langle x \mid y \rangle_{\mathcal{B}}) = \langle x \mid y \rangle_{\mathcal{B}}. \end{aligned}$$

From (2.13) and (2.14), we obtain $x\phi_M(\alpha)\beta = x$ for all $x \in M$, which implies $\phi_M(\alpha)\beta = 1_{\mathcal{B}}$, by the fullness of the module $M_{\mathcal{B}}$. Similarly, we have $\psi_M(\beta)\alpha = 1_{\mathcal{A}}$.
It follows that α and β are invertible and $\|\alpha^{-1}\| = \|\psi_M(\beta)\| = \|\beta\| \leq 1$. Since α and α^{-1} are positive elements with norm no larger than one in the commutative C*-algebra $\mathcal{A}$, we have $\alpha = 1_{\mathcal{A}}$. □

Definition 2.25. *Let ${}_{\mathcal{A}}M$ be a left module over an algebra $\mathcal{A}$ and denote by $\mathcal{A}^\circ$ the opposite algebra*[4] *of $\mathcal{A}$. The* ***right symmetrized bimodule*** *of ${}_{\mathcal{A}}M$ is the $\mathcal{A}$-$\mathcal{A}^\circ$ bimodule ${}_{\mathcal{A}}M^s_{\mathcal{A}^\circ}$ with right multiplication defined by:*

$$x \cdot a := ax, \quad \forall x \in M, \ \forall a \in \mathcal{A}.$$

In a similar way, given a right module $M_{\mathcal{A}}$, we define its ***left symmetrized bimodule*** *${}_{\mathcal{A}^\circ}{}^sM_{\mathcal{A}}$ via the left multiplication given by $a \cdot x := xa$ for all $x \in M$ and $a \in \mathcal{A}$.*

In the case of a commutative algebra $\mathcal{A}$, the opposite algebra $\mathcal{A}^\circ$ coincides with $\mathcal{A}$ and the left (respectively right) symmetrized of a module is clearly a symmetric bimodule over $\mathcal{A}$.

Proposition 2.26. *Suppose that ${}_{\mathcal{A}}M_{\mathcal{B}}$ is an imprimitivity $\mathcal{A}$-$\mathcal{B}$-bimodule over two unital commutative C*-algebras $\mathcal{A}$ and $\mathcal{B}$. Let $\phi_M : \mathcal{A} \to \mathcal{B}$ be the canonical isomorphism defined in theorem 2.24.*
The bimodule ${}_{\mathcal{A}}M_{\phi_M}$ coincides with the right symmetrized bimodule ${}_{\mathcal{A}}M^s_{\mathcal{A}}$.
The bimodule ${}_{\phi_M^{-1}}M_{\mathcal{B}}$ coincides with the left symmetrized bimodule ${}_{\mathcal{B}}{}^sM_{\mathcal{B}}$.

Proof. Take $x \in M$ and $a \in \mathcal{A}$. We already proved in (2.13) that $x \cdot \phi_M(a) = a \cdot x$, for all $x \in M$ and for all $a \in \mathcal{A}$.
The second part of the proposition $x \cdot b = \phi_M^{-1}(b) \cdot x$ is completed with an exactly similar argument.
In order to complete the proof, we have to show that the inner products on the right ϕ_M-twisted bimodule ${}_{\mathcal{A}}M_{\phi_M}$ coincides with the inner products of the right symmetrized bimodule ${}_{\mathcal{A}}M^s_{\mathcal{A}}$ and this is precisely equation (2.8).
A similar argument applies to the case of the left symmetrized bimodule ${}_{\mathcal{B}}{}^sM_{\mathcal{B}}$ and the left ϕ_M-twisted bimodule ${}_{\psi_M}M_{\mathcal{B}}$. □

[4]Recall that the opposite algebra $\mathcal{A}^\circ$ of an algebra $\mathcal{A}$ is just the vector space $\mathcal{A}$ equipped with the multiplication $a \cdot_{\mathcal{A}^\circ} b := b \cdot_{\mathcal{A}} a$.

The imprimitivity condition also behaves naturally under quotients.

Proposition 2.27. *Let ${}_{\mathcal{A}}M_{\mathcal{B}}$ be an imprimitivity bimodule over the unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{I}$ be an involutive ideal in the C*-algebra $\mathcal{A}$. Then $M/(\mathcal{I}M)$ is an imprimitivity bimodule over $\mathcal{A}/\mathcal{I}$ and $\mathcal{B}/\phi_M(\mathcal{I})$.*

Proof. Since $\phi_M : \mathcal{A} \to \mathcal{B}$ is an isomorphism of C*-algebras, if $\mathcal{I}$ is an involutive ideal in $\mathcal{A}$, also $\phi_M(\mathcal{I}) \subset \mathcal{B}$ is an involutive ideal in $\mathcal{B}$. Note that property (2.14) implies that $\mathcal{I}M = M\phi_M(\mathcal{I})$ and so, by proposition 2.6, $M/(\mathcal{I}M) = M/(M\phi_M(\mathcal{I}))$ is a full left Hilbert C*-module over $\mathcal{A}/\mathcal{I}$ and a full right Hilbert C*-module over $\mathcal{B}/\phi_M(\mathcal{I})$. Finally, by direct computation, we have:

$$\begin{aligned}
{}_{\mathcal{A}/\mathcal{I}}\langle x + \mathcal{I}M \mid y + \mathcal{I}M\rangle(x + \mathcal{I}M) &= ({}_{\mathcal{A}}\langle x \mid y\rangle + \mathcal{I})(z + \mathcal{I}M) \\
&= {}_{\mathcal{A}}\langle x \mid y\rangle z + \mathcal{I}M \\
&= x\langle y \mid z\rangle_{\mathcal{B}} + \mathcal{I}M \\
&= (x + \mathcal{I}M)(\langle y \mid z\rangle_{\mathcal{B}} + \mathcal{I}) \\
&= (x + \mathcal{I}M)\langle y + \mathcal{I}M \mid z + \mathcal{I}M\rangle_{\mathcal{B}/\phi_M(\mathcal{I})}.
\end{aligned}$$

□

2.3 Imprimitivity Bimodules in Commutative C*-categories.

Following P. Ghez-R. Lima-J. Roberts [GLR] and P. Mitchener [M] we recall the following basic definition.

Definition 2.28. *A **C*-category** is a category $\mathscr{C}$ such that: for all $A, B \in \mathrm{Ob}_{\mathscr{C}}$, the sets $\mathscr{C}_{AB} := \mathrm{Hom}_{\mathscr{C}}(B, A)$ are complex Banach spaces; the compositions are bilinear maps such that $\|xy\| \leq \|x\| \cdot \|y\|$ $\forall x \in \mathscr{C}_{AB}$ $\forall y \in \mathscr{C}_{BC}$; there is an involutive antilinear contravariant functor $* : \mathrm{Hom}_{\mathscr{C}} \to \mathrm{Hom}_{\mathscr{C}}$, acting identically on the objects, such that $\|x^*x\| = \|x\|^2$ $\forall x \in \mathscr{C}_{BA}$ and such that x^*x is a positive element in the C*-algebra $\mathscr{C}_{AA}$, for every $x \in \mathscr{C}_{BA}$ (i.e. $x^*x = y^*y$ for some $y \in \mathscr{C}_{AA}$).*

Every C*-algebra can be seen as a C*-category with only one object.

In a C*-category $\mathscr{C}$, the "diagonal blocks" $\mathscr{C}_{AA}$ are unital C*-algebras and the "off-diagonal blocks" $\mathscr{C}_{AB}$ are unital Hilbert C*-bimodules on the C*-algebras $\mathcal{A} := \mathscr{C}_{AA}$ and $\mathcal{B} := \mathscr{C}_{BB}$. For short, we often write ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}} := {}_{\mathscr{C}_{AA}}\mathscr{C}_{AB\mathscr{C}_{BB}}$ when we want to consider $\mathscr{C}_{AB}$ as a bimodule.

We say that $\mathscr{C}$ is **full** if all the bimodules $\mathscr{C}_{AB}$ are imprimitivity bimodules. Clearly [GLR, Remark 7.10] in a full C*-category, for all $A, B \in \mathrm{Ob}_{\mathscr{C}}$, $\mathcal{A} := \mathscr{C}_{AA}$ and $\mathcal{B} := \mathscr{C}_{BB}$ are always Morita-Rieffel equivalent C*-algebras with the imprimitivity bimodule ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ as an equivalence bimodule.

Lemma 2.29. *A C*-category $\mathscr{C}$ is full if and only if it satisfies the following property*

$$\overline{\mathscr{C}_{AB} \circ \mathscr{C}_{BC}} = \mathscr{C}_{AC}, \quad \forall A, B, C \in \mathrm{Ob}_{\mathscr{C}}\,. \tag{2.15}$$

Proof. Clearly property (2.15) is stronger than fullness.
The fullness of $\mathscr{C}$ tells us that $\mathscr{C}_{AA} = \overline{\mathscr{C}_{AB} \circ \mathscr{C}_{BA}}$. The continuity of composition implies $\overline{\mathscr{C}_{AB} \circ \mathscr{C}_{BA}} \circ \mathscr{C}_{AC} \subset \overline{\mathscr{C}_{AB} \circ \mathscr{C}_{BA} \circ \mathscr{C}_{AC}}$. From the following computation

$$\begin{aligned}
\mathscr{C}_{AC} &= \mathscr{C}_{AA} \circ \mathscr{C}_{AC} = \overline{\mathscr{C}_{AB} \circ \mathscr{C}_{BA}} \circ \mathscr{C}_{AC} \\
&\subset \overline{\mathscr{C}_{AB} \circ \mathscr{C}_{BA} \circ \mathscr{C}_{AC}} \subset \overline{\mathscr{C}_{AB} \circ \mathscr{C}_{BC}} \subset \overline{\mathscr{C}_{AC}} = \mathscr{C}_{AC}
\end{aligned}$$

we obtain $\mathscr{C}_{AC} = \overline{\mathscr{C}_{AB} \circ \mathscr{C}_{BC}}$. □

We use the previous lemma to show that in a full C*-category the maps

$$ {}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}} \otimes {}_{\mathcal{B}}\mathscr{C}_{\mathcal{C}} \to {}_{\mathcal{A}}\mathscr{C}_{\mathcal{C}}, \quad \text{given by} \quad x \otimes y \mapsto x \circ y $$

are isomorphisms of $\mathcal{A}$-$\mathcal{C}$-bimodules, for all $A, B, C \in \mathrm{Ob}_{\mathscr{C}}$.

Proposition 2.30. *If $\mathscr{C}$ is a full C*-category, for all $A, B, C \in \mathrm{Ob}_{\mathscr{C}}$, $({}_{\mathcal{A}}\mathscr{C}_{\mathcal{C}}, \circ)$ is a Rieffel interior tensor product for the pair of bimodules ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathscr{C}_{\mathcal{C}}$.*

Proof. We show that there exists an isomorphism $T : {}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}} \otimes {}_{\mathcal{B}}\mathscr{C}_{\mathcal{C}} \to {}_{\mathcal{A}}\mathscr{C}_{\mathcal{C}}$ of Hilbert C*-bimodules such that $T(x \otimes y) = x \circ y$ for all $x \in {}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ and for all $y \in {}_{\mathcal{B}}\mathscr{C}_{\mathcal{C}}$.
Consider the composition map $\circ : {}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}} \times {}_{\mathcal{B}}\mathscr{C}_{\mathcal{C}} \to {}_{\mathcal{A}}\mathscr{C}_{\mathcal{C}}$ and note that it is a bilinear map of Hilbert C*-bimodules and hence (by the universal factorization property for tensor products of Hilbert C*-bimodules) there exists one and only one Hilbert C*-bimodule morphism $T : {}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}} \otimes {}_{\mathcal{B}}\mathscr{C}_{\mathcal{C}} \to {}_{\mathcal{A}}\mathscr{C}_{\mathcal{C}}$ such that $T(x \otimes y) = x \circ y$.
Now we show that, under the fullness condition, the map T is an isomorphism.
First of all we note that T is an isometric map on the dense sub-bimodule generated by simple tensors:

$$
\begin{aligned}
\langle T(\sum_j x_j \otimes y_j) \,|T(\sum_k x_k \otimes y_k)\rangle_{\mathcal{C}} &= \sum_{j,k} \langle x_j \circ y_j \mid x_k \circ y_k\rangle_{\mathcal{C}} \\
&= \sum_{j,k} (x_j \circ y_j)^* \circ (x_k \circ y_k) = \sum_{j,k} y_j^* \circ x_j^* \circ x_k \circ y_k \\
&= \sum_{j,k} \langle y_j \mid \langle x_j \mid x_k\rangle_{\mathcal{B}} y_k\rangle_{\mathcal{C}} = \sum_{j,k} \langle x_j \otimes y_j \mid x_k \otimes y_k\rangle_{\mathcal{C}} \\
&= \langle \sum_j x_j \otimes y_j \mid \sum_k x_k \otimes y_k\rangle_{\mathcal{C}}.
\end{aligned}
$$

By continuity T extends to an isometry on all of ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}} \otimes {}_{\mathcal{B}}\mathscr{C}_{\mathcal{C}}$. Finally T is surjective because it is an isometry that, from lemma 2.29, has a dense image in ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{C}}$. □

Apart from a strictly associative (tensor) product (with partial identities given by ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{A}}$), the family of imprimitivity bimodules of a full C*-category $\mathscr{C}$ is naturally equipped with a strictly antimultiplicative notion of involution given by Rieffel duality (see definition 2.19).

Proposition 2.31. *If $\mathscr{C}$ is a full C*-category, $({}_{\mathcal{B}}\mathscr{C}_{\mathcal{A}}, *)$ is a Rieffel dual of the bimodule ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$, for all $A, B \in \mathrm{Ob}_{\mathscr{C}}$.*

Proof. Note that the map $* : {}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}} \to {}_{\mathcal{B}}\mathscr{C}_{\mathcal{A}}$ is conjugate-linear, it is an anti-isomorphism of Hilbert C*-bimodules[5] and it is isometric. We need to prove that $({}_{\mathcal{B}}\mathscr{C}_{\mathcal{A}}, *)$ satisfies the universal factorization property for conjugate-linear anti-homomorphisms of bimodules.
Clearly every conjugate-linear map $\Phi : {}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}} \to {}_{\mathcal{B}}M_{\mathcal{A}}$, with values in a Hilbert C*-bimodule ${}_{\mathcal{B}}M_{\mathcal{A}}$, such that $\Phi(axb) = b^*\Phi(x)a^*$ for all $x \in M$, $a \in \mathcal{A}$, $b \in \mathcal{B}$, factorizes as $\Phi = (\Phi \circ *) \circ *$ via a unique morphism $\Phi \circ * : {}_{\mathcal{B}}\mathscr{C}_{\mathcal{A}} \to {}_{\mathcal{B}}M_{\mathcal{A}}$ of $\mathcal{B}$-$\mathcal{A}$-bimodules. □

Every full C*-category $\mathscr{C}$ determines a subgroupoid, actually a total equivalence relation, in the (algebraic) Picard-Rieffel groupoid, with objects given by the diagonal C*-algebras $\mathscr{C}_{AA}$, for all $A \in \mathrm{Ob}_{\mathscr{C}}$, and morphisms given by the equivalence classes, under isomorphism of bimodules, of ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$. Such an association is functorial as specified by the following result, whose proof is now elementary.

[5]Recall that by an anti-homomorphism $\Phi : {}_{\mathcal{A}}M_{\mathcal{B}} \to {}_{\mathcal{B}}M_{\mathcal{A}}$ between unital Hilbert C*-bimodules M, N, we mean a conjugate-linear map that satisfies $\Phi(axb) = b^*\Phi(x)a^*$ for all $x \in M$, $a \in \mathcal{A}$, $b \in \mathcal{B}$.

Theorem 2.32. *Let $\mathscr{C}$ be a full C*-category. Denote by $[\mathscr{C}_{AB}]$ the equivalence class of Hilbert C*-bimodules that are isomorphic to the imprimitivity bimodule ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$. Consider $[\mathscr{C}_{AB}]$, for all $A, B \in \mathrm{Ob}_{\mathscr{C}}$, as arrows in the (algebraic) Picard-Rieffel groupoid. The family*

$$\mathrm{Pic}_{\mathscr{C}} := \{[\mathscr{C}_{AB}] \mid A, B \in \mathrm{Ob}_{\mathscr{C}}\},$$

is a total equivalence relation (i.e. a subgroupoid with one and only one arrow for every pair of objects) contained in the algebraic Picard-Rieffel groupoid.
A $*$-functor[6] *$\Phi : \mathscr{C} \to \mathscr{D}$ between full C*-categories that is bijective on objects uniquely determines an isomorphism $\mathrm{Pic}(\Phi) : \mathrm{Pic}_{\mathscr{C}} \to \mathrm{Pic}_{\mathscr{D}}$ of equivalence relations given by:*

$$\mathrm{Pic}(\Phi) : [\mathscr{C}_{AB}] \mapsto [\mathscr{D}_{\Phi_A \Phi_B}], \quad \forall A, B \in \mathrm{Ob}_{\mathscr{C}},$$

where $\Phi : A \mapsto \Phi_A \in \mathrm{Ob}_{\mathscr{D}}$ denotes the bijective action of the functor on the objects of $\mathscr{C}$. The map Pic is a functor from the category of object-bijective $*$-functors between small full C-categories into the category of (object bijective) groupoid homomorphisms between total equivalence relations contained in the algebraic Picard-Rieffel groupoid.*

An important tool related to these considerations is the "linking algebra" $\begin{bmatrix} \mathcal{A} & {}_{\mathcal{A}}M_{\mathcal{B}} \\ {}_{\mathcal{B}}M^*_{\mathcal{A}} & \mathcal{B} \end{bmatrix}$ of an imprimitivity bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$ as defined in L. Brown-P. Green-M. Rieffel [BGR], that could be seen as the enveloping C*-algebra (see [GLR]) of a C*-category with two objects.
Since by [BGR, Theorem 1.1] two unital C*-algebras $\mathcal{A}, \mathcal{B}$ are Morita equivalent if and only if there exists another unital C*-algebra $\mathcal{C}$ and two projections $p, q \in \mathcal{C}$ such that:

$$p + q = 1, \quad p\mathcal{C}p \simeq \mathcal{A}, \quad q\mathcal{C}q \simeq \mathcal{B}, \quad \overline{\mathcal{C}p\mathcal{C}} = \mathcal{C}, \quad \overline{\mathcal{C}q\mathcal{C}} = \mathcal{C},$$

and in this case there is a natural C*-category with two objects with linking algebra $\begin{bmatrix} p\mathcal{C}p & q\mathcal{C}p \\ p\mathcal{C}q & q\mathcal{C}q \end{bmatrix}$, it is likely that every full C*-category can be seen as a "strictification" of a total equivalence relation in the "weak" Picard-Rieffel groupoid and hence that the functor Pic in theorem 2.32 is surjective on objects. We will return to these considerations elsewhere.

Following now [BCL1, BCL2], we say that a C*-category $\mathscr{C}$ is **commutative** if all its diagonal blocks $\mathscr{C}_{AA}$ are commutative C*-algebras.

When an imprimitivity bimodule is actually the bimodule ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ of morphisms $\mathrm{Hom}_{\mathscr{C}}(B, A)$ in a full commutative C*-category $\mathscr{C}$, much more can be said about the properties of the canonical isomorphisms of theorem 2.24

$$\phi_{BA} := \phi_{{}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}} : \mathcal{A} \to \mathcal{B}. \tag{2.16}$$

Proposition 2.33. *Let $\mathscr{C}$ be a full commutative C*-category, the family of canonical isomorphisms $(A, B) \mapsto \phi_{BA}$ associated to the imprimitivity bimodules ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ satisfies the following compatibility conditions:*

$$\phi_{AA} = \iota_{\mathcal{A}}, \quad \forall A \in \mathrm{Ob}_{\mathscr{C}}, \tag{2.17}$$
$$\phi_{BA} = \phi^{-1}_{AB}, \quad \forall A, B \in \mathrm{Ob}_{\mathscr{C}}, \tag{2.18}$$
$$\phi_{CB} \circ \phi_{BA} = \phi_{CA}, \quad \forall A, B, C \in \mathrm{Ob}_{\mathscr{C}}. \tag{2.19}$$

[6] A $*$-functor $\Phi : \mathscr{C} \to \mathscr{D}$ between C*-categories is just a functor (linear on each block $\mathscr{C}_{AB}$, $A, B \in \mathrm{Ob}_{\mathscr{C}}$) such that $\Phi(x^*) = \Phi(x)^*$ for all $x \in \mathrm{Hom}_{\mathscr{C}}$.

Proof. First of all, we note again that, for imprimitivity bimodules ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ of morphisms in a commutative full C*-category, there is an explicit description of the inner products:

$$\langle x \mid y\rangle_{\mathcal{B}} := x^* y, \quad {}_{\mathcal{A}}\langle x \mid y\rangle := y x^* \quad \forall x, y \in {}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}.$$

Property (2.17) follows immediately from

$$\phi_{AA}(a) = \sum_j \langle w_j \mid a z_j\rangle_{\mathcal{A}} = \sum_j w_j^* a z_j = a \sum_j \langle w_j \mid z_j\rangle_{\mathcal{A}} = a \quad \forall a \in {}_{\mathcal{A}}\mathscr{C}_{\mathcal{A}}.$$

To prove property (2.19), let w_j, z_j be finite families of elements in ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ and x_k, y_k finite families of elements in ${}_{\mathcal{B}}\mathscr{C}_{\mathcal{C}}$ such that $\sum_j \langle w_j \mid z_j\rangle_{\mathcal{B}} = 1_{\mathcal{B}}$ and $\sum_k \langle x_k \mid y_k\rangle_{\mathcal{C}} = 1_{\mathcal{C}}$. By the definition of the canonical isomorphism (2.10), we have:

$$\phi_{BA}(a) := \sum_j \langle w_j \mid a z_j\rangle_{\mathcal{B}} \quad \forall a \in \mathcal{A},$$
$$\phi_{CB}(b) := \sum_k \langle x_k \mid b y_k\rangle_{\mathcal{C}} \quad \forall b \in \mathcal{B}.$$

By direct calculation we see that the composition is given by:

$$\begin{aligned}\phi_{CB} \circ \phi_{BA}(a) &= \sum_k \langle x_k \mid \sum_j \langle w_j \mid a z_j\rangle_{\mathcal{B}}\, y_k\rangle_{\mathcal{C}} \\ &= \sum_k \sum_j x_k^* w_j^* a z_j y_k = \sum_k \sum_j (w_j x_k)^* a (z_j y_k).\end{aligned}$$

We only need to prove that the expression above is of the form $\sum_h \langle u_h \mid a v_h\rangle_{\mathcal{C}}$ for finite families of elements $u_h, v_h \in {}_{\mathcal{A}}\mathscr{C}_{\mathcal{C}}$, indexed by h, such that $\sum_h \langle u_h \mid v_h\rangle_{\mathcal{C}} = 1_{\mathcal{C}}$.
Now, the families of elements $w_j x_k$ and $z_j y_k$ satisfy exactly this property

$$\begin{aligned}\sum_k \sum_j \langle w_j x_k \mid z_j y_k\rangle_{\mathcal{C}} &= \sum_k \sum_j x_k^* w_j^* z_j y_k = \sum_k \langle x_k \mid \sum_j \langle w_j \mid z_j\rangle_{\mathcal{B}} y_k\rangle_{\mathcal{C}} \\ &= \sum_k \langle x_k \mid 1_{\mathcal{B}} y_k\rangle_{\mathcal{C}} = 1_{\mathcal{C}}\end{aligned}$$

and so we can define $u_{j,k} := w_j x_k \in {}_{\mathcal{A}}\mathscr{C}_{\mathcal{C}}$ and $v_{j,k} := z_j y_k \in {}_{\mathcal{A}}\mathscr{C}_{\mathcal{C}}$.
Property (2.18) follows by direct application of equations (2.17) and (2.19). □

Proposition 2.34. *Let $\omega : \mathscr{C} \to \mathbb{C}$ be a $*$-functor (i.e. a functor such that $\omega(x^*) = \overline{\omega(x)}$, for all $x \in \mathscr{C}$) defined on the full commutative C*-category $\mathscr{C}$. For every pair of objects $A, B \in \mathrm{Ob}_{\mathscr{C}}$, we have*

$$\omega(\phi_{BA}(a)) = \omega(a), \quad \forall a \in \mathscr{C}_{AA}.$$

Proof. Consider the imprimitivity bimodule ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ and the associated canonical isomorphism $\phi_{BA} : \mathscr{C}_{AA} \to \mathscr{C}_{BB}$. For every $a \in \mathscr{C}_{AA}$, for any given finite families $w_j, z_j \in \mathscr{C}_{AB}$ such that $\sum_j \langle w_j \mid z_j\rangle_{\mathcal{B}} = 1_{\mathcal{B}}$, we know that $\phi_{BA}(a) = \sum_j \langle w_j \mid a z_j\rangle_{\mathcal{B}}$. Since $\omega : \mathscr{C} \to \mathbb{C}$ is a $*$-functor, for all $a \in \mathscr{C}_{AA}$, we have:

$$\begin{aligned}\omega(\phi_{BA}(a)) : &= \omega(\sum_j \langle w_j \mid a z_j\rangle_{\mathcal{B}}) = \sum_j \omega(\langle w_j \mid a z_j\rangle_{\mathcal{B}}) \\ &= \sum_j \omega(w_j^* a z_j) = \sum_j \omega(w_j^*)\omega(a)\omega(z_j) \\ &= \omega(a)\sum_j \omega(w_j^*)\omega(z_j) = \omega(a)\sum_j \omega(w_j^* z_j) \\ &= \omega(a)\omega(\sum_j \langle w_j \mid z_j\rangle_{\mathcal{B}}) = \omega(a)\omega(1_{\mathcal{B}}) = \omega(a).\end{aligned}$$

□

3 Spectral Theorem for Imprimitivity Bimodules

Let X_A and X_B be two compact Hausdorff spaces and let $R_{BA} : X_A \to X_B$ be a homeomorphism between them. To every complex bundle (E, π, R_{BA}), over the graph of the homeomorphism $R_{BA} \subset X_A \times X_B$, we can naturally associate the set $\Gamma(R_{BA}; E)$ of continuous sections of the bundle E, that turns out to be a symmetric bimodule over the commutative C*-algebra $C(R_{BA}; \mathbb{C})$ of continuous functions on the compact Hausdorff space R_{BA}.
Considering now the pair of homeomorphisms

$$\begin{aligned}&\pi_A : R_{BA} \to X_A, \quad \pi_A : (x, y) \mapsto x,\\ &\pi_B : R_{BA} \to X_B, \quad \pi_B : (x, y) \mapsto y,\end{aligned}$$

we see that the set $\Gamma(R_{BA}; E)$ becomes naturally a left module over $C(X_A; \mathbb{C})$ and a right module over $C(X_B; \mathbb{C})$ with the following left and right actions $f \cdot \sigma := (f \circ \pi_A) \cdot \sigma$ and $\sigma \cdot g := \sigma \cdot (g \circ \pi_B)$ or, in a more explicit form, for all $(x, y) \in R_{BA}$, $f \in C(X_A)$, $g \in C(X_B)$ and $\sigma \in \Gamma(R_{BA}; E)$:

$$\begin{aligned}&f \cdot \sigma(x, y) := f(x)\sigma(x, y) = (f \circ \pi_A)(x, y) \cdot \sigma(x, y),\\ &\sigma \cdot g(x, y) := \sigma(x, y)g(y) = \sigma(x, y) \cdot (g \circ \pi_B)(x, y).\end{aligned}$$

In the terminology of definition 2.10, this is the bimodule ${}_{\pi_A^\bullet}\Gamma(R_{BA}, E)_{\pi_B^\bullet}$ obtained by twisting the symmetric $C(R_{BA})$-bimodule $\Gamma(R_{BA}, E)$ by the isomorphism $\pi_A^\bullet : C(X_A) \to C(R_{BA})$ on the left and by the isomorphism $\pi_B^\bullet : C(X_B) \to C(R_{BA})$ on the right.
We say that ${}_{\pi_A^\bullet}\Gamma(R_{BA}; E)_{\pi_B^\bullet}$ is the **$C(X_A)$-$C(X_B)$-bimodule associated to the bundle (E, π, R_{BA}) over the homeomorphism** $R_{BA} : X_A \to X_B$. Note that if (E, π, R_{BA}) is a Hermitian bundle over the homeomorphism $R_{BA} : X_A \to X_B$, then the bimodule ${}_{C(R_{BA})}\Gamma(R_{BA}; E)_{C(R_{BA})}$ is a full symmetric Hilbert C*-bimodule over $C(R_{BA})$ and, as in remark 2.11, the associated bimodule ${}_{\pi_A^\bullet}\Gamma(R_{BA}; E)_{\pi_B^\bullet}$ has a natural structure as a full Hilbert C*-bimodule with inner products given by:

$$\begin{aligned}&{}_{C(X_A)}\langle \sigma \mid \rho\rangle := (\pi_A^\bullet)^{-1}(\langle \sigma \mid \rho\rangle_{C(R_{BA})}), \quad \forall \sigma, \rho \in \Gamma(R_{BA}; E),\\ &\langle \sigma \mid \rho\rangle_{C(X_B)} := (\pi_B^\bullet)^{-1}(\langle \sigma \mid \rho\rangle_{C(R_{BA})}), \quad \forall \sigma, \rho \in \Gamma(R_{BA}; E).\end{aligned}$$

Furthermore the associated bimodule ${}_{\pi_A^\bullet}\Gamma(R_{BA}; E)_{\pi_B^\bullet}$ is an imprimitivity bimodule if and only if ${}_{C(R_{BA})}\Gamma(R_{BA}; E)_{C(R_{BA})}$ is an imprimitivity bimodule and this, by Serre-Swan theorem (see e.g. [BCL1, Section 2.1.2] and references therein), happens if and only if (E, π, R_{BA}) is a Hermitian line bundle.

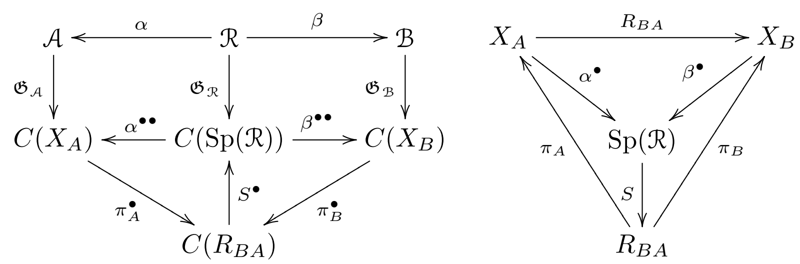
In this section, making use of the results in section 2.2, we prove, in the case of imprimitivity bimodules, a converse to the previous construction i.e. that (up to isomorphism of bimodules) every imprimitivity Hilbert C*-bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ over unital commutative C*-algebras $\mathcal{A}$ and $\mathcal{B}$ actually arises as the bimodule associated to a Hermitian line bundle over a homeomorphism between the compact Hausdorff spaces $\mathrm{Sp}(\mathcal{A})$ and $\mathrm{Sp}(\mathcal{B})$.

Theorem 3.1. *Given an imprimitivity C*-bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ over two commutative unital C*-algebras $\mathcal{A}, \mathcal{B}$, there exists a Hermitian line bundle (E, π, R_{BA}), over the graph of a homeomorphism $R_{BA} : X_A \to X_B$ between the two compact Hausdorff spaces $X_A := \mathrm{Sp}(\mathcal{A})$, $X_B := \mathrm{Sp}(\mathcal{B})$, whose associated $C(X_A)$-$C(X_B)$-bimodule ${}_{\pi_A^\bullet}\Gamma(R_{BA}; E)_{\pi_B^\bullet}$, when twisted on the left by the Gel'fand transform isomorphism $\mathfrak{G}_{\mathcal{A}} : \mathcal{A} \to C(\mathrm{Sp}(\mathcal{A}))$ and on the right by the Gel'fand isomorphism $\mathfrak{G}_{\mathcal{B}} : \mathcal{B} \to C(\mathrm{Sp}(\mathcal{B}))$, becomes a bimodule ${}_{\pi_A^\bullet \circ \mathfrak{G}_{\mathcal{A}}}\Gamma(R_{BA}; E)_{\pi_B^\bullet \circ \mathfrak{G}_{\mathcal{B}}}$ that is isomorphic, as an $\mathcal{A}$-$\mathcal{B}$-bimodule, to the initial Hilbert C*-bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$.*

Proof. By theorem 2.24, we have a canonical isomorphism $\phi_{\mathcal{M}} : \mathcal{A} \to \mathcal{B}$. Using Gel'fand theorem, applied to the isomorphism $\phi_{\mathcal{M}}^{-1} : \mathcal{B} \to \mathcal{A}$ of unital C*-algebras, we recover a homeomorphism $R_{BA} := (\phi_M^{-1})^\bullet : X_A \to X_B$ between the two compact Hausdorff spaces $X_A := \mathrm{Sp}(\mathcal{A})$ and $X_B := \mathrm{Sp}(\mathcal{B})$. Furthermore we know that the Gel'fand transforms $\mathfrak{G}_{\mathcal{A}} : \mathcal{A} \to C(X_A; \mathbb{C})$, $\mathfrak{G}_{\mathcal{B}} : \mathcal{B} \to C(X_B; \mathbb{C})$ provide two isomorphisms of C*-algebras.
Consider now the set $\mathcal{R} \subset \mathcal{A} \times \mathcal{B}$ defined by $\mathcal{R} := \{(a, b) \in \mathcal{A} \times \mathcal{B} \mid b = \phi_{\mathcal{M}}(a)\}$ and note that $\mathcal{R}$ has a natural structure of unital C*-algebra with componentwise multiplication and norm defined by $\|(a, b)\|_{\mathcal{R}} := \max\{\|a\|, \|b\|\} = \|a\| = \|b\|$. There are natural isomorphisms $\alpha : \mathcal{R} \to \mathcal{A}$ and $\beta : \mathcal{R} \to \mathcal{B}$ given by

$$\alpha : (a, b) \mapsto a, \quad \beta : (a, b) \mapsto b, \quad \forall (a, b) \in \mathcal{R},$$

and they satisfy $\phi_{\mathcal{M}} = \beta \circ \alpha^{-1}$.
Note also that the topological space $\mathrm{Sp}(\mathcal{R})$ is canonically homeomorphic to R_{BA}. In fact, since $R_{BA} \circ (\alpha^{-1})^\bullet = (\phi_{\mathcal{M}}^{-1})^\bullet \circ (\alpha^{-1})^\bullet = (\alpha \circ \beta^{-1})^\bullet \circ (\alpha^{-1})^\bullet = (\beta^{-1})^\bullet$, the function $S : \omega \mapsto ((\alpha^{-1})^\bullet(\omega), (\beta^{-1})^\bullet(\omega))$, for $\omega \in \mathrm{Sp}(\mathcal{R})$, takes values in R_{BA} and being bijective continuous between compact Hausdorff spaces it is a homeomorphism.
We summarize the situation with the following commutative diagrams that might come helpful to visualize the several isomorphisms and homeomorphisms involved:



Twisting (see definition 2.10) the bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ by α on the left and β on the right, we obtain a Hilbert C*-bimodule ${}_{\alpha}\mathcal{M}_{\beta}$ over $\mathcal{R}$ that is symmetric because

$$(a, b) \cdot x = \alpha(a, b)x = ax = x\phi_{\mathcal{M}}(a) = x\beta(a, b) = x \cdot (a, b), \forall (a, b) \in \mathcal{R}.$$

Twisting one more time ${}_{\alpha}\mathcal{M}_{\beta}$ with the isomorphism

$$\gamma := \mathfrak{G}_{\mathcal{R}}^{-1} \circ S^\bullet : C(R_{BA}) \to \mathcal{R},$$

we get a symmetric Hilbert C*-bimodule ${}_{\alpha\circ\gamma}\mathcal{M}_{\beta\circ\gamma}$ over the C*-algebra $C(R_{BA})$. By a direct application of Serre-Swan theorem (see e.g. [BCL1, Theorem 2.2]), we see that there exists a Hermitian bundle (E, π, R_{BA}) over the compact Hausdorff space R_{BA} such that there exists an isomorphism of $C(R_{BA})$-bimodules $\Phi : {}_{\alpha\circ\gamma}\mathcal{M}_{\beta\circ\gamma} \to \Gamma(R_{BA}; E)$. Since ${}_{\mathcal{A}}M_{\mathcal{B}}$ is an imprimitivity bimodule, so is ${}_{\alpha\circ\gamma}\mathcal{M}_{\beta\circ\gamma}$ and hence (E, π, R_{BA}) is a Hermitian line bundle. Making use of proposition 2.12, we have that the map Φ also becomes an isomorphism $\Phi : {}_{\mathcal{A}}M_{\mathcal{B}} \to {}_{(\alpha\circ\gamma)^{-1}}\Gamma(R_{BA}; E)_{(\beta\circ\gamma)^{-1}}$ of Hilbert C*-bimodules over $\mathcal{A}$ and $\mathcal{B}$. Since, by the diagram above, we have $(\alpha \circ \gamma)^{-1} = \pi^\bullet_A \circ \mathfrak{G}_A$ and $(\beta \circ \gamma)^{-1} = \pi^\bullet_B \circ \mathfrak{G}_B$, we finally obtain an isomorphism of left $\mathcal{A}$, right $\mathcal{B}$ Hilbert C*-bimodules

$$\Phi : {}_{\mathcal{A}}M_{\mathcal{B}} \to {}_{\pi^\bullet_A\circ\mathfrak{G}_A}\Gamma(R_{BA}; E)_{\pi^\bullet_B\circ\mathfrak{G}_B}.$$

□

Note that the theorem says that for an imprimitivity bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$ over commutative unital C*-algebras, the triple $(\mathfrak{G}_{\mathcal{A}}, \Phi, \mathfrak{G}_{\mathcal{B}})$ provides an isomorphism, in the category of Hilbert C*-bimodules, from the bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$ to the $C(X_A)$-$C(X_B)$-bimodule ${}_{\pi^\bullet_A}\Gamma(R_{BA}; E)_{\pi^\bullet_B}$ associated to the Hermitian line bundle (E, π, R_{BA}) over the homeomorphism $R_{BA} : X_A \to X_B$. This means that $\Phi(axb) = \mathfrak{G}_{\mathcal{A}}(a)\Phi(x)\mathfrak{G}_{\mathcal{B}}(b)$, for all $x \in \mathcal{M}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$. The map Φ is essentially a "canonical extension" of the Gel'fand transform of the C*-algebras $\mathcal{A}$ and $\mathcal{B}$ to the imprimitivity bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$ over them.

The above theorem is just the starting point for the development of a "bivariant Serre-Swan equivalence" and, more generally, a bivariant "Takahashi duality" (see e.g. [BCL1, Section 2.1.2] and references therein) for the category of Hilbert C*-bimodules over commutative C*-algebras. This will be done elsewhere.
Our spectral theorem, for imprimitivity bimodules over Abelian C*-algebras, is dealing only with the representativity of a potential functor that, to every Hermitian line bundle (E, π, R_{BA}) over the graph of a homeomorphism $R_{BA} : X_A \to X_B$ between compact Hausdorff spaces, associates the imprimitivity bimodule ${}_{\pi^\bullet_A}\Gamma(R_{BA}; E)_{\pi^\bullet_B}$ over the commutative C*-algebras $C(X_A)$ and $C(X_B)$. To proceed further we have to provide a suitable notion of morphisms and define our functor on them.
The above result is for now stated in the case of imprimitivity bimodules and hence it does not provide neither an answer to the problem of classifying, nor a geometric interpretation of general $C(X)$-$C(Y)$-bimodules for given compact Hausdorff spaces X and Y. Warning the reader to take due care of some differences in notations and definitions, for some related results on the "spectral theory" of Hilbert C*-bimodules, one may consult B. Abadie-R. Exel [AE], H. Bursztyn-S. Waldmann [BW], A. Hopenwasser-J. Peters-J. Powers [HPP], A. Hopenwasser [H], T. Kajiwara-C. Pinzari-Y. Watatani [KPW], P. Muhly-B.Solel [MS].
In particular, B. Abadie and R. Exel [AE, Proposition 1.9] proved that every imprimitivity C*-bimodule over a commutative C*-algebra $\mathcal{A}$ is always obtained from its right symmetrization by twisting on one side with a given automorphism θ and, in a more algebraic setting, a result of H. Burzstyn-S. Waldmann [BW, Proposition 2.3] assures that if two imprimitivity bimodules ${}_{\mathcal{A}}M_{\mathcal{B}}$ and ${}_{\mathcal{A}}N_{\mathcal{B}}$ over the same commutative algebras are isomorphic as right modules, there is a unique isomorphism of the C*-algebra $\mathcal{B}$ such that the bimodule M is isomorphic to the twisting of N.
Gathering together the above facts, in the special case of commutative full C*-categories, we obtain the following result.

Theorem 3.2. *Let $\mathscr{C}$ be a full commutative C*-category. Then for every pair of objects A and B, one has:*

- ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ *is an imprimitivity* ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{A}}$-${}_{\mathcal{B}}\mathscr{C}_{\mathcal{B}}$ *bimodule. That is,* ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{A}}$ *and* ${}_{\mathcal{B}}\mathscr{C}_{\mathcal{B}}$ *are Morita equivalent and thus there is a canonical* $*$*-isomorphism implemented by* $x^*y \mapsto yx^*$, $x, y \in {}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$.
- ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ *is the (non-symmetric)* ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{A}}$-${}_{\mathcal{B}}\mathscr{C}_{\mathcal{B}}$*-bimodule of continuous sections of a Hermitian line bundle over the graph of the corresponding homeomorphism between the Gel'fand spectra of* ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{A}}$ *and* ${}_{\mathcal{B}}\mathscr{C}_{\mathcal{B}}$.

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A remark on Gel'fand duality for spectral triples

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A Remark on Gel'fand Duality for Spectral Triples

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Abstract

We present a duality between the category of compact Riemannian spin manifolds (equipped with a given spin bundle and charge conjugation) with isometries as morphisms and a suitable "metric" category of spectral triples over commutative pre-C*-algebras. We also construct an embedding of a "quotient" of the category of spectral triples introduced in [BCL1] into the latter metric category. Finally we discuss a further related duality in the case of orientation and spin-preserving maps between manifolds of fixed dimension.

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1 Introduction.

Although the main strength of non-commutative geometry is a full treatment of non-commutative algebras as "duals of geometric spaces", the foundation of the theory relies on the construction of suitable categorical equivalences, resp. anti-equivalences (i.e. covariant, resp. contravariant functors that are isomorphisms of categories "up to natural transformations") between categories of "geometric spaces" and categories of commutative algebras of functions over these spaces.[1]

Typical examples of such (anti-)equivalences are listed below, itemized by the name of the people who worked them out:

- **Hilbert:** between algebraic sets and finitely generated algebras over an algebraically closed field [H];

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[1] For the elementary background in "category theory" the reader can refer to the on-line introduction by J. Baez [B] and the classical books by S. McLane [M] and M. Barr-C. Wells [BW].

- **Stone:** between totally disconnected compact Hausdorff topological spaces and Boolean algebras [St1, St2];
- **Gel'fand-Naĭmark:** between the category of continuous maps of compact Hausdorff topological spaces and the category of unital involutive homomorphisms of unital commutative C*-algebras[2] [G, GN];
- **Halmos-von Neumann:** between the category of measurable maps of measure spaces and the category of unital involutive homomorphisms of commutative von Neumann algebras;[3]
- **Serre-Swan:** between the category of vector bundle maps of finite-dimensional locally trivial vector bundles over a compact Hausdorff topological space and the category of homomorphisms of finite projective modules over a commutative unital C*-algebra [Se, Sw];
- **Cartier-Grothendieck:** between the category of commutative schemes (ringed spaces) in algebraic geometry and the category of topoi (sheaves over topological spaces);[4]
- **Takahashi:** between the category of Hilbert bundles on (different) compact Hausdorff spaces and the category of Hilbert C*-modules over (different) commutative unital C*-algebras [T1, T2];

Even more dualities arise when the spaces in question are equipped with additional structure, most notably a group structure or the like (see Pontryagin-Van Kampen [Po, VK], Tannaka-Kreĭn [Ta, Kr] and Doplicher-Roberts [DR]).
In this paper we will focus our attention on the Gel'fand-Naĭmark duality, to which the other dualities are related in significant way. In short, the fundamental message that can be read off from the celebrated Gel'fand-Naĭmark theorem on commutative C*-algebras is that, at the "topological level", the information on a "space" can be completely encoded in (and recovered from) a suitable "algebraic structure".
In applications to physics (at least for those branches that are dealing with "metric structures" such as general relativity), it would be important to "tune" Gel'fand-Naĭmark's correspondence in order to embrace classes of spaces with more detailed geometric structures (e.g. differential, metric, connection, curvature).
In recent times, Connes' non-commutative geometry [C1, FGV] has emerged as the most outstanding proposal in this direction, based on the notion of spectral triple.
In this short note we provide a simple further example of categorical anti-equivalence between Riemannian spin manifolds and commutative Connes' spectral triples (see theorem 3.2). This line of thought is expected to play an important role in future developments of the categorical structure of non-commutative geometry, and spectral triples in particular (see [BCL2]), as well as in the study of (geometric) quantization, where the construction

[2] Or more generally between the category of proper continuous maps of locally compact Hausdorff spaces and the category of involutive homomorphisms of commutative C*-algebras.

[3] The origin of a dual treatment of measure theory (at least for locally compact Hausdorff spaces) can be traced back to F. Riesz-A. Makov-S. Kakutani-A. Weil theorem [Rie, Ma, K, W], but the proof that a measure space can be recovered from a commutative von Neumann algebra is due to P. Halmos-J. von Neumann [HvN].

[4] As reported by I. Dolgachev in his useful historical review [D, Section 1], the idea of P. Cartier (1957) that affine schemes are in duality with ringed spaces of the form $\mathrm{Sp}(\mathcal{A})$ was developed by Grothendieck in the full theory of schemes.

of functorial relations between "commutative" and "quantum" spaces are central points of investigation.
Although the idea of reconstructing a smooth manifold out of a commutative spectral triple has been latent for some time, (see [C3, C4, R, RV1, C5, C6]) the point to promote it to a categorical level seems to be new. Our main tool is the notion of metric morphisms of spectral triples, namely those preserving Connes' distance on the state space.
In the second part of the paper, we examine some connection between the category of "metric spectral triples" (on which the equivalence result is based) and our previous work on morphism of spectral triples [BCL1]. It should be possible to provide other equivalence results in terms of categories of spectral triples based on different notions of morphism (at least for some classes of Riemannian manifolds); some of these issues are presently under investigation (see [BCL2, Section 4.1] for an overview).

It should be remarked that Connes' distance formula has been systematically adopted by M. Rieffel as the backbone of his notion of quantum compact metric space (see [Ri] and references therein). Although we present our result in the framework of Connes' spectral triples, it is likely that our ideas might find some application also in Rieffel's framework.

In order to keep the length of this note as short as possible, we will refer to the literature for all the background material and only recall the basic definitions.

1.1 Spectral Triples.

Following A. Connes' axiomatization (see [C1, FGV, C5] for all the details), a **compact spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ consists of

a) a unital pre-C*-algebra $\mathcal{A}$ (that is sometimes required to be closed under holomorphic functional calculus),

b) a (faithful) representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of the algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and

c) a **Dirac operator**, i.e. a (generally unbounded) self-adjoint operator D, with compact resolvent $(D - \lambda)^{-1}$ for every $\lambda \in \mathbb{C} - \mathbb{R}$ and such that[5] $[D, \pi(a)]_- \in \mathcal{B}(\mathcal{H})$, $\forall a \in \mathcal{A}$.

A spectral triple is called **even** if it is equipped with a grading operator, i.e. a bounded self-adjoint operator $\Gamma \in \mathcal{B}(\mathcal{H})$ such that:

$$\Gamma^2 = \mathrm{Id}_{\mathcal{H}}; \quad [\Gamma, \pi(a)]_- = 0 \quad \forall a \in \mathcal{A}; \quad [\Gamma, D]_+ = 0.$$

A spectral triple without grading is called **odd**.

A spectral triple is **regular** if the functions $\Xi_x : t \mapsto \exp(it|D|)x\exp(-it|D|)$ are "smooth" i.e. $\Xi_x \in \mathrm{C}^\infty(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ for every $x \in \Omega_D(\mathcal{A})$, where we define [6]

$$\Omega_D(\mathcal{A}) := \mathrm{span}\{\pi(a_0)[D, \pi(a_1)]_- \cdots [D, \pi(a_n)]_- \mid n \in \mathbb{N},\ a_0, \ldots, a_n \in \mathcal{A}\}.$$

This regularity condition can be equivalently expressed requiring that, for all $a \in \mathcal{A}$, $\pi(a)$ and $[D, \pi(a)]_-$ are contained in $\cap_{m=1}^{\infty} \mathrm{Dom}\, \delta^m$, where δ is the derivation given by $\delta(x) := [|D|, x]_-$.

The spectral triple is **n-dimensional** iff there exists an integer n such that the Dixmier trace of $|D|^{-n}$ is finite non-zero. A spectral triple is **θ-summable** if $\exp(-tD^2)$ is a trace-class operator for every $t > 0$.

[5]Here $[x, y]_\pm := xy \pm yx$ denote respectively the anticommutator and the commutator of $x, y \in \mathcal{B}(\mathcal{H})$.
[6]We assume that for $n = 0 \in \mathbb{N}$ the term in the formula simply reduces to $\pi(a_0)$.

A spectral triple is **real** if it is equipped with a real structure i.e. an antiunitary operator $J : \mathcal{H} \to \mathcal{H}$ such that:

$$[\pi(a), J\pi(b^*)J^{-1}]_- = 0 \quad \forall a, b \in \mathcal{A};$$
$$[[D, \pi(a)]_-, J\pi(b^*)J^{-1}]_- = 0 \quad \forall a, b \in \mathcal{A}, \quad \textbf{first-order condition};$$
$$J^2 = \pm \mathrm{Id}_{\mathcal{H}}; \quad [J, D]_\pm = 0;$$
$$\text{and, only in the even case,} \quad [J, \Gamma]_\pm = 0,$$

where the choice of $\pm$ in the last three formulas depends on the "dimension" n of the spectral triple modulo 8 according to the following table:

n	0	1	2	3	4	5	6	7
$J^2 = \pm\mathrm{Id}_{\mathcal{H}}$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
$[J, D]_\pm = 0$	$-$	$+$	$-$	$-$	$-$	$+$	$-$	$-$
$[J, \Gamma]_\pm = 0$	$-$		$+$		$-$		$+$	

A spectral triple is **finite** if $\mathcal{H}_\infty := \cap_{k=1}^\infty \mathrm{Dom}\, D^k$ is a finite projective $\mathcal{A}$-bimodule and **absolutely continuous** if, there exists a Hermitian form $(\xi, \eta) \mapsto (\xi \mid \eta)$ on $\mathcal{H}_\infty$ such that, for all $a \in \mathcal{A}$, $\langle \xi \mid \pi(a)\eta \rangle$ is the Dixmier trace of $\pi(a)(\xi \mid \eta)|D|^{-n}$.

An n-dimensional spectral triple is said to be **orientable** if there is a Hochschild cycle $c = \sum_{j=1}^m a_0^{(j)} \otimes a_1^{(j)} \otimes \cdots \otimes a_n^{(j)}$ such that its "representation" on the Hilbert space $\mathcal{H}$, $\pi(c) = \sum_{j=1}^m \pi(a_0^{(j)})[D, \pi(a_1^{(j)})]_- \cdots [D, \pi(a_n^{(j)})]_-$ is the grading operator in the even case or the identity operator in the odd case[7].

A real spectral triple is said to satisfy **Poincaré duality** if its fundamental class in the KR-homology of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ induces (via Kasparov intersection product) an isomorphism between the K-theory $\mathrm{K}_\bullet(\mathcal{A})$ and the K-homology $\mathrm{K}^\bullet(\mathcal{A})$ of $\mathcal{A}$.[8]

A spectral triple will be called **commutative** whenever $\mathcal{A}$ is commutative.

Finally a spectral triple is **irreducible** if there is no non-trivial closed subspace in $\mathcal{H}$ that is invariant for $\pi(\mathcal{A}), D, J, \Gamma$.

1.2 Reconstruction Theorem (Commutative Case).

Let M be a real compact orientable Riemannian m-dimensional spin C^∞ manifold with a given volume form μ_M. Let us denote (see [S] for details) by $S(M)$ a given irreducible complex spinor bundle over M i.e. a bundle over M equipped with a left action $c : \mathbb{C}\mathrm{l}^{(+)}(T(M)) \otimes S(M) \to S(M)$ of the "Clifford" bundle[9] $\mathbb{C}\mathrm{l}^{(+)}(T(M))$ inducing a bundle isomorphism between $\mathbb{C}\mathrm{l}^{(+)}(T(M))$ and $\mathrm{End}(S(M))$. Let $[S(M)]$ be the spin^c structure[10] of M determined by $S(M)$.

[7] In the following, in order to simplify the discussion, we will always refer to a "grading operator" Γ that actually coincides with the grading operator in the even case and that is by definition the identity operator in the odd case.

[8] In [RV1] some of the axioms are reformulated in a different form, in particular this condition is replaced by the requirement that the C*-module completion of $\mathcal{H}_\infty$ is a Morita equivalence bimodule between (the norm completions of) $\mathcal{A}$ and $\Omega_D(\mathcal{A})$.

[9] Following [FGV, Page 373], we denote by $\mathbb{C}\mathrm{l}^{(+)}(T(M))$ the complexified Clifford bundle of M if $\dim M$ is even and respectively its even subalgebra bundle $\mathbb{C}\mathrm{l}^{+}(T(M))$ if $\dim M$ is odd.

[10] An orientable Riemannian manifold is spin^c if it admits a complex irreducible spinor bundle [S, Definition 7]. Recall that a spin^c manifold usually admits several inequivalent spin^c structures and that for a given spin^c structure, a complex irreducible spinor bundle over M is determined only up to (Hermitian) bundle isomorphism.

Let C_M be a given "spinorial" charge conjugation[11] on $S(M)$ i.e. an antilinear Hermitian bundle morphism such that $C_M \circ C_M = \pm \mathrm{Id}_{S(M)}$ (signs depending on $\dim M$ modulo 8 as in the table in section 1.1) that is "compatible" with the charge conjugation[12] κ in $\mathbb{C}\mathrm{l}^{(+)}(T(M))$ i.e. $C_M(\beta(p)\cdot\sigma(p)) = \kappa(\beta(p))\cdot C_M(\sigma(p))$, for any section $\beta \in \Gamma(\mathbb{C}\mathrm{l}^{(+)}(T(M)))$ of the Clifford bundle and any section $\sigma \in \Gamma(S(M))$ of the spinor bundle. We denote by $[(S(M), C_M)]$ the spin structure on M determined by C_M.

Let $\mathcal{A}_M := \mathrm{C}^\infty(M;\mathbb{C})$ be the commutative pre-C*-algebra of smooth complex valued functions on M. We denote by π_M its representation by pointwise multiplication on the space $\mathcal{H}_M := \mathrm{L}^2(M, S(M))$, the completion of the space $\Gamma^\infty(M, S(M))$ of smooth sections of the spinor bundle $S(M)$ equipped with the inner product $\langle \sigma \mid \tau \rangle := \int_M \langle \sigma(p) \mid \tau(p)\rangle_p \, \mathrm{d}\mu_M$, where $\langle \cdot \mid \cdot \rangle_p$ is the unique inner product on $S_p(M)$ compatible with the Clifford action and the Clifford product. Note that the spinorial charge conjugation C_M (being unitary on the fibers) has a unique antilinear unitary extension $J_M : \mathcal{H}_M \to \mathcal{H}_M$ determined by $(J_M\sigma)(p) := C_M(\sigma(p))$ for $\sigma \in \Gamma^\infty(S(M))$ and $p \in M$.

Let Γ_M be the unique unitary extension on $\mathcal{H}_M$ of the operator Λ_M on $\Gamma(S(M))$ acting by left action of the chirality element $\gamma \in \Gamma(\mathbb{C}\mathrm{l}^{(+)}(T(M)))$, that implements the grading χ of $\Gamma(\mathbb{C}\mathrm{l}^{(+)}(T(M)))$ as inner automorphism.[13]

Denote by D_M the Atiyah-Singer Dirac operator on the Hilbert space $\mathcal{H}_M$, i.e. the closure of the operator that on $\Gamma^\infty(S(M))$ is obtained by "contracting" the unique spin covariant derivative ∇^S (induced on $\Gamma^\infty(S(M))$ by the Levi-Civita covariant derivative of M, see [FGV, Theorem 9.8]) with the Clifford multiplication. For a detailed discussion on Atiyah-Singer Dirac operators we refer to [BGV, LM, S].

We have the following fundamental results:

Theorem 1.1 (Connes, see e.g. [C1, C2] and Section 11.1 in [FGV])**.** *Given an orientable compact spin Riemannian m-dimensional differentiable manifold M, with a given complex spinor bundle $S(M)$, a given spinorial charge conjugation C_M and a given volume form μ_M,[14] the data $(\mathcal{A}_M, \mathcal{H}_M, D_M)$ defines a commutative regular finite absolutely continuous m-dimensional spectral triple that is real, with real structure J_M, orientable, with grading Γ_M, and satisfies Poincaré duality.*

Theorem 1.2 (Connes [C3, C5])**.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be an irreducible commutative real (with real structure J and grading Γ) strongly regular[15] m-dimensional finite absolutely continuous orientable spectral triple, with totally antisymmetric (in the last m entries) Hochschild cycle, and satisfying Poincaré duality. The spectrum of (the norm closure of) $\mathcal{A}$ can be endowed, in a unique way, with the structure of an m-dimensional connected compact orientable spin Riemannian manifold M with an irreducible complex spinor bundle $S(M)$, a charge conjugation J_M and a grading Γ_M such that:*

$$\mathcal{A} \simeq C^\infty(M;\mathbb{C}), \quad \mathcal{H} \simeq L^2(M, S(M)), \quad D \simeq D_M, \quad J \simeq J_M, \quad \Gamma \simeq \Gamma_M.$$

[11] A spin^c manifold is spin if and only if it admits a complex spinor bundle with a charge conjugation [S, Definition 8]. Recall that a spin manifold usually admits several inequivalent spin structures even for the same spin^c structure and that for a given spin structure a conjugation operator is determined only up to intertwining with (Hermitian) bundle isomorphisms.

[12] κ is the composition of the natural grading operator and the canonical conjugation.

[13] The grading is actually the identity in odd dimension.

[14] Remember that an orientable manifolds admits two different orientations and that, on a Riemannian manifold, the choice of an orientation canonically determines a volume form μ_M.

[15] In the sense of [C5, Definition 6.1].

A. Connes first proved the previous theorem 1.2 under the additional condition that $\mathcal{A}$ is already given as the algebra of smooth complex-valued functions over a differentiable manifold M, namely $\mathcal{A} = C^\infty(M;\mathbb{C})$ (for a detailed proof see e.g. [FGV, Theorem 11.2]), and conjectured [C3], [C4, Theorem 6, Remark (a)] the result for general commutative pre-C*-algebras $\mathcal{A}$.
A tentative proof of this last fact has been published by A. Rennie [R]; some gaps were pointed out in the original argument, a different revised, but still incorrect, proof appears in [RV1] (see also [RV2]) under some additional technical conditions. Recently A. Connes [C5] finally provided the missing steps in the proof of the result.
As a consequence, there exists a one-to-one correspondence between unitary equivalence classes of spectral triples and connected compact oriented Riemannian spin manifolds up to spin-preserving isometric diffeomorphisms.
Similar results should also be available for spin^c manifolds [C4, Theorem 6, Remark (e)].

1.3 Connes' Distance Formula.

Given a spectral triple $(\mathcal{A},\mathcal{H},D)$, let us denote by $\mathcal{S}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A})$ the sets of **states** and **pure states** of the pre-C*-algebra $\mathcal{A}$, respectively. If $\mathcal{A} := \mathrm{C}^\infty(M;\mathbb{C})$, for all $p \in M$ we denote by $\mathrm{ev}_p : x \mapsto x(p)$ the "evaluation functional" in p of the functions $x \in \mathcal{A}$ and note that $\mathrm{ev}_p \in \mathcal{P}(\mathcal{A})$. Actually in this case $\mathcal{P}(\mathcal{A})$ coincides with the set of all evaluation functionals.
Going back to the general case, the **Connes' distance** d_D on $\mathcal{P}(\mathcal{A})$ is the function on $\mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{A})$ given by

$$d_D(\omega_1,\omega_2) := \sup\{|\omega_1(x) - \omega_2(x)| \mid x \in \mathcal{A},\ \|[D,\pi(x)]\| \leq 1\}.$$

Strictly speaking, without imposing other conditions, d_D could also take the value $+\infty$ as in the case of non-connected manifolds. In turn, one can use the same formula to define a "distance" on the set of all the states of $\mathcal{A}$.

Theorem 1.3 (Connes's distance formula). *[FGV, Proposition 9.12] If the spectral triple $(\mathcal{A},\mathcal{H},D)$ is obtained as in theorem 1.1 from a compact finite-dimensional oriented Riemannian spin manifold M equipped with a spinor bundle $S(M)$ and a spinorial charge conjugation C_M, then for every $p, q \in M$, $d_D(\mathrm{ev}_p, \mathrm{ev}_q)$ coincides with the geodesic distance*

$$d_M(p,q) := \inf\left\{\int_a^b \|\gamma'(t)\|\, dt \mid \gamma \text{ is a geodesic with } \gamma(a) = p,\ \gamma(b) = q\right\}.$$

Of course, given a unital $*$-morphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ there is a pull-back $\phi^\bullet : \mathcal{S}(\mathcal{A}_2) \to \mathcal{S}(\mathcal{A}_1)$ defined by $\phi^\bullet(\omega) := \omega \circ \phi$ for all $\omega \in \mathcal{S}(\mathcal{A}_2)$.

2 A Metric Category of Spectral Triples.

The objects of all of our categories will be compact spectral triples $(\mathcal{A},\mathcal{H},D)$.
Given two spectral triples $(\mathcal{A}_j,\mathcal{H}_j,D_j)$, with $j = 1,2$, a **metric morphism** of spectral triples $(\mathcal{A}_1,\mathcal{H}_1,D_1) \xrightarrow{\phi} (\mathcal{A}_2,\mathcal{H}_2,D_2)$ is by definition a unital epimorphism[16] $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ of pre-C*-algebras whose pull-back $\phi^\bullet : \mathcal{P}(\mathcal{A}_2) \to \mathcal{P}(\mathcal{A}_1)$ is an isometry, i.e.

$$d_{D_1}(\phi^\bullet(\omega_1),\phi^\bullet(\omega_2)) = d_{D_2}(\omega_1,\omega_2),\ \forall \omega_1,\omega_2 \in \mathcal{P}(\mathcal{A}_2).$$

Spectral triples with metric morphisms form a category $\mathscr{S}^m$.

[16]Note that if ϕ is an epimorphism, its pull-back $\phi^\bullet$ maps pure states into pure states.

Remark 2.1. *A unitary equivalence of spectral triples gives an isomorphism in the category* $\mathscr{S}^m$.

2.1 A Local Metric Category of Spectral Triples.

For convenience of the reader, we recall here the definitions of morphisms of spectral triples proposed in our previous work [BCL1, Sections 2.2-2.3].
A **morphism** in the category $\mathscr{S}$, between spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, $j = 1, 2$ of the same dimension, is a pair (ϕ, Φ), where $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a $*$-morphism between the pre-C*-algebras $\mathcal{A}_1, \mathcal{A}_2$ and $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear map in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $\pi_2(\phi(x)) \circ \Phi = \Phi \circ \pi_1(x)$, $\forall x \in \mathcal{A}_1$ and $D_2 \circ \Phi(\xi) = \Phi \circ D_1(\xi)$ $\forall \xi \in \operatorname{Dom} D_1$.
In a similar way, a **morphism of real spectral triples** $(\mathcal{A}_j, \mathcal{H}_j, D_j, J_j)$ with $j = 1, 2$, in the category of real spectral triples $\mathscr{S}_r$, is a morphism in $\mathscr{S}$ such that Φ also satisfies $J_2 \circ \Phi = \Phi \circ J_1$. Finally a **morphism of even spectral triples** $(\mathcal{A}_j, \mathcal{H}_j, D_j, \Gamma_j)$ with $j = 1, 2$, in the category of even spectral triples $\mathscr{S}_e$, is a morphism in $\mathscr{S}$ such that $\Gamma_2 \circ \Phi = \Phi \circ \Gamma_1$. We will denote by $\mathscr{S}_I$ (respectively $\mathscr{S}_{Ir}, \mathscr{S}_{Ire}$) the subcategory of $\mathscr{S}$ (respectively $\mathscr{S}_r, \mathscr{S}_{re}$) consisting of "isometric" morphisms of spectral triples, i.e. pairs (ϕ, Φ) with ϕ surjective and Φ co-isometric. We have the following inclusion of non-full subcategories: $\mathscr{S}_{re} := \mathscr{S}_r \cap \mathscr{S}_e \subset \mathscr{S}$.

3 The Metric Connes-Rennie-Varilly Functor.

Let us consider the class $\mathscr{M}$ of C^∞ metric isometries[17] of compact finite-dimensional C^∞ orientable Riemannian spin manifolds M equipped with a fixed spinor bundle $S(M)$, a given spinorial charge conjugation C_M and a volume form μ_M. The class $\mathscr{M}$ with the usual composition of functions forms a category.

Proposition 3.1. *There is a contravariant functor* $\mathfrak{C}$ *from the category* $\mathscr{M}$ *to the category* $\mathscr{S}^m$ *that to every triple* $(M, S(M), C_M) \in \mathscr{M}$ *associates the spectral triple* $(\mathcal{A}, \mathcal{H}, D) \in \mathscr{S}^m$ *given as in theorem 1.1 and that to every smooth metric isometry* $f : M_1 \to M_2$ *associates its pull-back* $f^\bullet : \mathcal{A}_2 \to \mathcal{A}_1$.

Proof. Every smooth metric isometry $f : M_1 \to M_2$ in $\mathscr{M}$ is a Riemannian isometry of M_1 onto a closed embedded submanifold $f(M_1)$ of M_2. Since every smooth function on a closed embedded submanifold is the restriction of a smooth function on M_2, the pull-back $\phi := f^\bullet$ is a unital epimorphism of the pre-C*-algebras $\phi : \mathcal{A}_2 \to \mathcal{A}_1$ and, by theorem 1.3, $\phi^\bullet : \mathcal{P}(\mathcal{A}_1) \to \mathcal{P}(\mathcal{A}_2)$ is metric-preserving:

$$\begin{aligned} d_{D_2}(\phi^\bullet(\omega_1), \phi^\bullet(\omega_2)) &= d_{D_2}(\phi^\bullet(\mathrm{ev}_p), \phi^\bullet(\mathrm{ev}_q)) = d_{D_2}(\mathrm{ev}_{f(p)}, \mathrm{ev}_{f(q)}) \\ &= d_{M_2}(f(p), f(q)) = d_{M_1}(p, q) = d_{D_1}(\mathrm{ev}_p, \mathrm{ev}_q) \\ &= d_{D_1}(\omega_1, \omega_2), \end{aligned}$$

where $p, q \in M_1$ are the unique points such that $\omega_1 = \mathrm{ev}_p$ and $\omega_2 = \mathrm{ev}_q$.
Of course $\mathfrak{C}(g \circ f) = (g \circ f)^\bullet = f^\bullet \circ g^\bullet = \mathfrak{C}_f \circ \mathfrak{C}_g$ and $\mathfrak{C}_{\iota_M} = \iota_{\mathfrak{C}(M)}$. □

We will call the functor $\mathfrak{C}$ the **metric Connes-Rennie-Varilly functor**.

Here we present the main result of this paper. We denote by ab-$\mathscr{S}^m$ the full subcategory of $\mathscr{S}^m$ of **direct sums of irreducible Abelian spectral triples**[18].

[17] Note that in general a Riemannian isometry is not necessarily a metric isometry.
[18] In a completely similar way we will denote by ab-$\mathscr{S}$ the full subcategory of direct sums of irreducible Abelian spectral triples in $\mathscr{S}$.

Theorem 3.2. *The metric Connes-Rennie-Varilly functor is an anti-equivalence between the categories $\mathscr{M}$ and ab-$\mathscr{S}^m$.*

Proof. The functor $\mathfrak{C}$ is faithful: if $\mathfrak{C}_f = \mathfrak{C}_g$ for two smooth isometries $f, g : M_1 \to M_2$, then $f^\bullet = g^\bullet$ as morphisms of pre-C*-algebras and hence they coincide also when uniquely extend to morphisms of C*-algebras of continuous functions and the result $f = g$ follows from Gel'fand duality theorem.
The functor $\mathfrak{C}$ is full: if $\phi : \mathfrak{C}(M_2) \to \mathfrak{C}(M_1)$ is a metric morphism in $\mathscr{S}^m$, as a homomorphisms of pre-C* algebras of smooth functions, ϕ extends uniquely to a morphism of C*-algebras of continuous functions and, from Gel'fand duality theorem, there exists a unique continuous function $f : M_1 \to M_2$ such that $f^\bullet = \phi$. From the fact that $f^\bullet$ maps smooth functions on M_2 to smooth functions on M_1 it follows that f is a smooth function between manifolds. Since ϕ also preserves the spectral distances, it follows that f is a smooth metric isometry hence a Riemannian isometry.
The functor $\mathfrak{C}$ is representative: for when restricted to the subcategory of connected manifolds with target the subcategory of irreducible spectral triples, this is actually a restatement of the reconstruction theorem 1.2 and remark 2.1. Since the Connes-Rennie-Varilly functor $\mathfrak{C}$ maps disjoint unions of connected components into direct sums of spectral triples, the result follows. □

Unfortunately, at this stage, we cannot present a statement involving the category of all Abelian spectral triples. The above result raises naturally the issue of decomposing (Abelian) spectral triples in terms of irreducible components.

Remark 3.3. *In restriction to the subcategory $\mathscr{M}_d$ of* ***dimension-preserving*** *smooth isometries (i.e. isometric immersions with fiberwise isomorphic tangent maps), the metric Connes-Rennie-Varilly functor $\mathfrak{C}$ is an anti-equivalence between $\mathscr{M}_d$ and the subcategory ab-$\mathscr{S}^m_d$ of metric morphisms of direct sums of irreducible Abelian spectral triples with the same dimension. In a similar way, denoting by $\mathcal{N}(\mathscr{C})$ the nerve of the category $\mathscr{C}$, i.e. the groupoid of isomorphisms of $\mathscr{C}$, we have that $\mathfrak{C}|_{\mathcal{N}(\mathscr{M})}$ is an anti-equivalence between $\mathcal{N}(\mathscr{M})$[19] and the nerve $\mathcal{N}$(ab-$\mathscr{S}^m$).*

4 Metric and Spin Categories.

We now proceed to establish a connection between the category $\mathscr{S}^m$ of metric spectral triples and the categories of spectral triples $\mathscr{S}$ (respectively real spectral triples $\mathscr{S}_r$) introduced in [BCL1, Section 2.2-2.3] and briefly recalled in section 2.1.
Denote by $\mathscr{S}^0$ (respectively $\mathscr{S}^0_{Ired}$) the category of spectral triples whose morphisms are those homomorphisms of algebras ϕ for which there exists at least one Φ such that the pair (ϕ, Φ) is a morphism in $\mathscr{S}$ (respectively $\mathscr{S}_{Ired}$). We have a "forgetful" full functor $\mathfrak{F} : \mathscr{S} \to \mathscr{S}^0$ that to every morphism (ϕ, Φ) in $\mathscr{S}$ associates ϕ as a morphism in $\mathscr{S}^0$.

Lemma 4.1. *A metric isometry of Riemannian manifolds with the same dimension is a smooth Riemannian isometry onto a union of connected components.*

Proof. Let $f : M \to N$ be a metric isometry. Since $\dim M = \dim N$, by Brouwer's theorem, we see that f is open and maps each connected component of M onto a unique connected

[19]The nerve of $\mathscr{M}$ (always a subcategory of $\mathscr{M}_d$) is actually the "disjoint union" of denuberable "connected components" consisting of the categories of smooth bijective isometries of n-dimensional spin manifolds.

component of N. By the Myers-Steenrod theorem (see for example [P, Section 5.9, Theorem 9.1]), any such bijective map between connected components is a smooth Riemannian surjective isometry; hence $f : M \to N$ is a smooth Riemannian isometry onto $f(M)$, a union of connected components of N. □

Let $f : (M, S(M), C_M) \to (N, S(N), C_N)$ be a morphism in $\mathscr{M}_d$. Thanks to the last lemma, we can consider the differential $Df : T(M) \to T(N)$. It is a monomorphism of Euclidean bundles and induces a unique Bogoljubov morphism $\mathbb{C}l_{Df} : \mathbb{C}l^{(+)}(T(M)) \to \mathbb{C}l^{(+)}(T(N))$ of the Clifford bundles that is actually an isomorphism of $\mathbb{C}l^{(+)}(T(M))$ with subbundle $\mathbb{C}l^{(+)}(T(f(M)))$, the Clifford bundle of the submanifold $f(M)$.[20] This isomorphism can be used to "transfer" the irreducible Clifford action of $\mathbb{C}l^{(+)}(T(f(M)))$ on the bundle $S(f(M)) := S(N)|_{f(M)}$ to an irreducible action of $\mathbb{C}l^{(+)}(T(M))$ and, since the bundle $f^\bullet(S(N)) = f^\bullet(S(f(M)))$ is naturally isomorphic to $S(f(M))$, the bundle $f^\bullet(S(N))$ becomes an irreducible complex spinor bundle on M. By a similar argument, $f^\bullet(S(N))$ comes equipped with a spinorial charge conjugation $f^\bullet(C_N)$ obtained by "pull-back" of (the restriction to $S(f(M))$ of) C_N through the isomorphism $f^\bullet(S(N)) \simeq S(f(M))$.

We say that f is **spin-preserving** if the spin structure $[(f^\bullet(S(N)), f^\bullet(C_N))]$ determined by $f^\bullet(S(N))$ with spinorial charge conjugation $f^\bullet(C_N)$ coincides with the spin structure of M i.e. if there exists an isomorphism of Hermitian bundles $U : f^\bullet(S(N)) \to S(M)$ that intertwines the charge conjugations: $U \circ f^\bullet(C_N) = C_M \circ U$ and the Clifford actions. Note that if f is orientation-preserving, the isomorphism U also intertwines the grading operators of the spinor bundles.
Let us denote by $\mathscr{M}_d$-spin the subcategory of spin and orientation-preserving maps in $\mathscr{M}_d$. The following result, that we report for completeness, is certainly well-known although we could not find any suitable reference. Note that ab-$\mathscr{S}_{Ired}$ denotes the full subcategory of $\mathscr{S}_{Ired}$ whose objects are direct sums of irreducible Abelian spectral triples.

Proposition 4.2. *Let M, N be two compact orientable Riemannian spin manifolds in the category $\mathscr{M}$. If $f : M \to N$ is a spin-preserving isomorphism of Riemannian manifolds, the spectral triples $(\mathcal{A}_M, \mathcal{H}_M, D_M)$ and $(\mathcal{A}_N, \mathcal{H}_N, D_N)$ are isomorphic in the category ab-$\mathscr{S}_{Ired}$.*

Proof. The pull-back $\phi := f^\bullet$ is a $*$-isomorphism $\phi : \mathcal{A}_N \to \mathcal{A}_M$ of pre-C*-algebras.

Consider the "pull-back of spinor fields" given by the invertible map $\Psi := \sigma \mapsto \sigma \circ f$, for all $\sigma \in \mathcal{H}_N$. Since f is an orientation-preserving Riemannian isometry, it leaves invariant the volume forms $f^\bullet(\mu_N) = \mu_M$ and so we obtain

$$\int_M \langle \Psi(\sigma)(x) \mid \Psi(\tau)(x) \rangle \, d\mu_M(x) = \int_N \langle \sigma(y) \mid \tau(y) \rangle \, d\mu_N(y)$$

that implies that the map $\Psi : \mathcal{H}_N \to \mathrm{L}^2(M, f^\bullet(S(N))) =: \mathcal{H}^\bullet$ is a unitary operator.
Since $f^\bullet(S(N))$ is a Hermitian bundle over M, $\mathcal{H}^\bullet$ carries a natural representation $\pi^\bullet$ of the algebra $\mathcal{A}_M$ given by pointwise multiplication. Ψ intertwines π_N and $\pi^\bullet \circ \phi$, i.e. $\Psi(\pi_N(a)\sigma) = \pi^\bullet(\phi(a))\Psi(\sigma)$ for $a \in \mathcal{A}_N$ and $\sigma \in \mathcal{H}_N$.

[20] From this we see that the subalgebra $\mathbb{C}l^{(+)}(f(M)) \subset \mathbb{C}l^{(+)}(N)$ of sections of the Clifford bundle of N with support in $f(M)$ is naturally isomorphic with the algebra $\mathbb{C}l^{(+)}(M)$ of sections of the Clifford bundle of M. Since the restriction to $f(M)$ is a natural epimorphism $\rho : \mathbb{C}l^{(+)}(N) \to \mathbb{C}l^{(+)}(f(M))$, ($\rho$ acts on Clifford fields by multiplication with the characteristic function of $f(M)$), there is a natural unital epimorphism of algebras $\psi : \mathbb{C}l^{(+)}(N) \to \mathbb{C}l^{(+)}(M)$ that becomes an isomorphism when restricted to $\mathbb{C}l^{(+)}(f(M))$.

Let $U : f^\bullet(S(N)) \to S(M)$ be a (noncanonical) isomorphism of Hermitian bundles induced by the spin-preserving condition on f. Since we know that U is unitary on the fibers, we have $\int_M \langle U\sigma(p) \mid U\tau(p)\rangle_{S_p(M)}\, \mathrm{d}\mu_M(p) = \int_M \langle \sigma(p) \mid \tau(p)\rangle_{f_p^\bullet(S(N))}\, \mathrm{d}\mu_M(p)$, for all $\sigma, \tau \in \Gamma^\infty(f^\bullet(S(N)))$. Hence U uniquely extends to a unitary map $\Theta_U : \mathcal{H}^\bullet \to \mathcal{H}_M$. Note that Θ_U is $\mathcal{A}_M$-linear: $\Theta_U(a \cdot \sigma) = a \cdot \Theta_U(\sigma)$, for $a \in \mathcal{A}_M$ and $\sigma \in \mathcal{H}^\bullet$.

Now it is not difficult to check that the pair $(\phi, \Theta_U \circ \Psi)$ is an isomorphism in the category $\mathscr{S}_{Ired}$ from the spectral triple $(\mathcal{A}_N, \mathcal{H}_N, D_N)$ to $(\mathcal{A}_M, \mathcal{H}_M, D_M)$. □

Proposition 4.3. *The Connes-Rennie-Varilly functor is an embedding of the category $\mathscr{M}_d$-spin into ab-$\mathscr{S}^0_{Ired}$.*

Proof. Let $f : M \to N$ be a spin-preserving metric isometry in $\mathscr{M}_d$-spin. By Lemma 4.1 $f : M \to N$ is a smooth Riemannian isometry onto the closed submanifold $f(M)$, a union of connected components of N.

We denote by $\rho : \mathcal{A}_N \to \mathcal{A}_{f(M)}$ the restriction epimorphism.

The Hilbert space $\mathcal{H}_N = \mathrm{L}^2(N, S(N))$ decomposes as the direct sum $\oplus_{j\in\pi^0(N)}\mathcal{H}_j$ of Hilbert spaces (one for each connected component $j \in \pi^0(N)$ of N) and the multiplication operator P by the characteristic function $\chi_{f(M)}$ is the projection operator onto the subspace $\mathcal{H}_{f(M)} := P(\mathcal{H}_N) = \oplus_{j\in\pi^0(f(M))}\mathcal{H}_j$ (cf. [FGV, Page 491]). Note that, since the Dirac operator D_N is "local" (i.e. it preserves the support of the spinor fields), the subspace $\mathcal{H}_{f(M)}$ is invariant for D_N. In the same way, since J_N and Γ_N acts fiberwise, $\mathcal{H}_{f(M)}$ is invariant for the charge conjugation and grading of N.

Defining $D_{f(M)} := P \circ D_N \circ P$, $J_{f(M)} := P \circ J_N \circ P$ and $\Gamma_{f(M)} := P \circ \Gamma_N \circ P$, it is immediate that $(\mathcal{A}_{f(M)}, \mathcal{H}_{f(M)}, D_{f(M)})$ is a real (even) spectral triple and it follows that the "restriction" map $P : \mathcal{H}_N \to \mathcal{H}_{f(M)}$ satisfies $\forall a \in \mathcal{A}_N$, $\sigma \in \mathcal{H}_N$, $P(a\sigma) = \rho(a)P(\sigma)$, $P \circ D_N = D_{f(M)} \circ P$, $P \circ J_N = J_{f(M)} \circ P$, $P \circ \Gamma_N = \Gamma_{f(M)} \circ P$. This means that the pair (ρ, P) is a morphism in the category $\mathscr{S}_{Ired}$ from $(\mathcal{A}_N, \mathcal{H}_N, D_N)$ to the triple $(\mathcal{A}_{f(M)}, \mathcal{H}_{f(M)}, D_{f(M)})$, which is nothing but the spectral triple obtained from the manifold $f(M)$. By Proposition 4.2, there exists an isomorphism from $(\mathcal{A}_{f(M)}, \mathcal{H}_{f(M)}, D_{f(M)})$ to $(\mathcal{A}_N, \mathcal{H}_N, D_N)$ in the category $\mathscr{S}_{Ired}$, and the conclusion follows by composition with the previous (ρ, P). □

Lemma 4.4. *If M and N are two orientable compact Riemannian spin manifolds in the category $\mathscr{M}$ and (u, U) is an isomorphism from $(\mathcal{A}_N, \mathcal{H}_N, D_N)$ to $(\mathcal{A}_M, \mathcal{H}_M, D_M)$ in the category ab-$\mathscr{S}_{Ire}$, then there is a spin-preserving orientation-preserving Riemannian isometry (metric isometry) $f : M \to N$ such that $f^\bullet = u$.*

Proof. The map $u : \mathcal{A}_N \to \mathcal{A}_M$ naturally extends to a $*$-isomorphisms of C*-algebras and by Gel'fand theorem there exists a homeomorphism $f : M \to N$ such that $f^\bullet = u$. Since $f^\bullet$ maps smooth functions onto smooth functions, f is a diffeomorphism.
The filtered algebra $\Omega_M(\mathcal{A}_M)$ (respectively $\Omega_N(\mathcal{A}_N)$) coincides with the filtered algebra of smooth sections of the Clifford bundle $\mathbb{C}\mathrm{l}^{(+)}(T(M))$ (respectively $\mathbb{C}\mathrm{l}^{(+)}(T(N))$) and the map $\mathrm{Ad}_U : \Omega_{D_N}(\mathcal{A}_N) \to \Omega_{D_M}(\mathcal{A}_M)$ is a filtered isomorphisms (extending $f^\bullet$). Therefore its restricition $\mathrm{Ad}_U : \Omega^1_{D_N}(\mathcal{A}_N) \to \Omega^1_{D_M}(\mathcal{A}_M)$ is an isomorphism between the Hermitian modules of sections of the complexification of the tangent bundles $T(M)$ and $T(N)$.
From Serre-Swan theorem, $Df : T(M) \to T(N)$ is an isomorphism of Euclidean bundles which implies that f is a Riemannian isometry.
Since $\mathrm{Ad}_U(J_N) = J_M$ and $\mathrm{Ad}_U(\Gamma_N) = \Gamma_M$, f is orientation and spin-preserving. □

Proposition 4.5. *The identity functor is an inclusion of the category ab-$\mathscr{S}^0_{Ired}$ into the category ab-$\mathscr{S}^m_d$.*

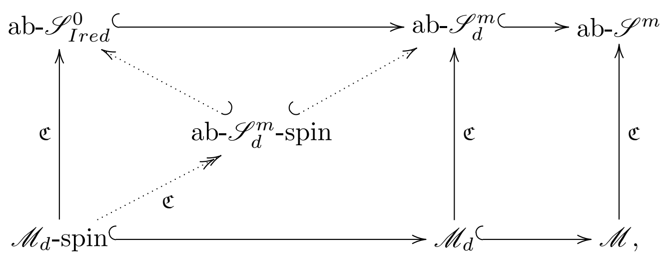
Proof. Let $\phi : (\mathcal{A}_1, \mathcal{H}_1, D_1) \to (\mathcal{A}_2, \mathcal{H}_2, D_2)$ be an isomorphism in the category ab-$\mathscr{S}^0_{Ired}$. By the reconstruction theorem 1.2, there are two manifolds M and N in the category $\mathscr{M}$ such that $(\mathcal{A}_N, \mathcal{H}_N, D_N)$ is isomorphic to $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $(\mathcal{A}_M, \mathcal{H}_M, D_M)$ is isomorphic to $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ with isomorphisms (ϕ_N, U_N) and (ϕ_M, U_M), respectively, in the category $\mathscr{S}_{Ired}$.
By lemma 4.4, $\phi_M \circ \phi \circ \phi_N^{-1} \in \mathscr{S}^0_{Ired}$ is the image under $\mathfrak{C}$ of a spin-preserving Riemannian isometry f that (for manifolds of the same dimension) is a metric isometry in $\mathscr{M}_d$.
Since ϕ_M, ϕ_N are isomorphisms in ab-$\mathscr{S}^0_{Ired}$ and hence, by remark 2.1, isomorphisms also in ab-$\mathscr{S}^m_d$, it follows that $\phi = \phi_M^{-1} \circ \mathfrak{C}(f) \circ \phi_N \in$ ab-$\mathscr{S}^m_d$. □

We can now state the promised equivalence result.

Theorem 4.6. *The Connes-Rennie-Varilly functor is an equivalence between the category $\mathscr{M}_d$-spin and the category ab-$\mathscr{S}^0_{Ired}$.*

Proof. The Connes-Rennie-Varilly functor is already faithful because of proposition 4.3 and representative because of proposition 4.5. We need only to show its fullness.
Let M and N be manifolds in the category $\mathscr{M}_d$-spin and let $\phi : \mathfrak{C}(N) \to \mathfrak{C}(M)$ be a morphism in the category $\mathscr{S}^0_{Ired}$. By proposition 4.5 ϕ is a morphism in the category $\mathscr{S}^m_d$ and from remark 3.3 there exists a metric isometry $f : M \to N$ in the category $\mathscr{M}_d$ such that $\mathfrak{C}(f) = \phi$. Since ϕ defines an isomorphism between $\mathfrak{C}(f(M))$ and $\mathfrak{C}(M)$ in $\mathscr{S}^0_{Ired}$ then, by lemma 4.4, $f : M \to f(M)$ is (orientation and) spin-preserving and we are done. □

Let us summarize the categorical "relations" now available with the commutative diagram of functors



where ab-$\mathscr{S}^m_d$-spin := $\mathfrak{C}(\mathscr{M}_d$-spin). The left and right vertical inclusion functors correspond respectively to the embedding in theorem 4.3 and to the Connes-Rennie-Varilly anti-equivalence in theorem 3.2; the horizontal top-left arrow is the inclusion functor described in proposition 4.5.

Loosely speaking, one would expect a similar structure to carry over to the general non-commutative setting, relating subcategories of "spin-preserving" morphisms in $\mathscr{S}^m$ and "metric-preserving" morphisms in $\mathscr{S}^0_{Ire}$. However, in general things might be more complicated. For the time being, we just mention the following result, omitting the (easy) details of the proof.

Proposition 4.7. *Let $(\mathcal{A}_1, \mathcal{H}_1, D_1) \xrightarrow{(\phi,\Phi)} (\mathcal{A}_2, \mathcal{H}_2, D_2)$ be a morphism of the spectral triples in the category $\mathscr{S}$, where Φ is a coisometry. Then*

$$d_{D_1}(\omega_1 \circ \phi, \omega_2 \circ \phi) \leq d_{D_2}(\omega_1, \omega_2), \quad \forall \omega_1, \omega_2 \in \mathcal{S}(\mathcal{A}_2).$$

We have discussed only the case of spin manifolds. We also expect analogous statements to hold true for spin^c manifolds.

5 Final comments

The main result presented in this paper is nothing more than a simple observation on how Gel'fand-Naĭmark duality can be reformulated in the light of Connes' reconstruction theorem for spin Riemannian manifolds. However, it seems to us that the functoriality of the Connes-Rennie-Varilly correspondence has some intriguing appeal and one could ask to which extent it is possible to "lift" it to some of the other main objects entering the scene, notably the Dirac operators. This issue is presently under investigation.
From the perspective of this work, the use of the spin structure has been only instrumental in recasting Gel'fand-Naĭmark theorem in the light of the Connes' reconstruction theorem, and actually it might appear as an unnecessary complication: it introduces some redundancy in the main result and, when incorporated tout-court in the setup, it does not lead to a genuine categorical anti-equivalence.
This might suggest that in a successive step one could try to get rid of such a structure, thus obtaining a different kind of categorical duality between a metric category of (isometries of) Riemannian manifolds and suitable categories of spectral data (for example considering spectral triples arising from the signature Dirac operator in place of those arising from the usual Atiyah-Singer Dirac operator). Although several variants of morphisms can be introduced between spectral triples (see [BCL2, Section 4.1] for details), corresponding to the "rigidity" imposed on the maps between manifolds (totally geodesic isometries, Riemannian isometries, ...), this line of thought does not require significant structural modifications in the definitions of morphisms for the categories of spectral geometries involved (as a pair of maps at the algebra and the Hilbert space level) and will be pursued elsewhere (see [Be] for more details).
The actual construction of functors (and dualities) from categories of spin Riemannian manifolds (with different dimensions) to "suitable" categories of spectral triples (of the Atiyah-Singer "type") is a more interesting goal whose main obstruction is the lack of a sufficiently general notion of pull-back of spinor fields. In order to solve this problem it will be necessary to construct "relational categories" of spectral triples, via "spectral conguences" and/or "spectral spans" following the lines already announced in the seminar slides [Be]. We will return to these topics in forthcoming papers.

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