Proof. First of all, we note again that, for imprimitivity bimodules ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ of morphisms in a commutative full C*-category, there is an explicit description of the inner products:

$$\langle x \mid y\rangle_{\mathcal{B}} := x^* y, \quad {}_{\mathcal{A}}\langle x \mid y\rangle := yx^* \quad \forall x, y \in {}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}.$$

Property (2.17) follows immediately from

$$\phi_{AA}(a) = \sum_j \langle w_j \mid az_j\rangle_{\mathcal{A}} = \sum_j w_j^* a z_j = a \sum_j \langle w_j \mid z_j\rangle_{\mathcal{A}} = a \quad \forall a \in {}_{\mathcal{A}}\mathscr{C}_{\mathcal{A}}.$$

To prove property (2.19), let w_j, z_j be finite families of elements in ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ and x_k, y_k finite families of elements in ${}_{\mathcal{B}}\mathscr{C}_{\mathcal{C}}$ such that $\sum_j \langle w_j \mid z_j\rangle_{\mathcal{B}} = 1_{\mathcal{B}}$ and $\sum_k \langle x_k \mid y_k\rangle_{\mathcal{C}} = 1_{\mathcal{C}}$. By the definition of the canonical isomorphism (2.10), we have:

$$\phi_{BA}(a) := \sum_j \langle w_j \mid az_j\rangle_{\mathcal{B}} \quad \forall a \in \mathcal{A},$$
$$\phi_{CB}(b) := \sum_k \langle x_k \mid by_k\rangle_{\mathcal{C}} \quad \forall b \in \mathcal{B}.$$

By direct calculation we see that the composition is given by:

$$\begin{aligned}\phi_{CB} \circ \phi_{BA}(a) &= \sum_k \langle x_k \mid \sum_j \langle w_j \mid az_j\rangle_{\mathcal{B}}\, y_k\rangle_{\mathcal{C}} \\ &= \sum_k \sum_j x_k^* w_j^* a z_j y_k = \sum_k \sum_j (w_j x_k)^* a (z_j y_k).\end{aligned}$$

We only need to prove that the expression above is of the form $\sum_h \langle u_h \mid av_h\rangle_{\mathcal{C}}$ for finite families of elements $u_h, v_h \in {}_{\mathcal{A}}\mathscr{C}_{\mathcal{C}}$, indexed by h, such that $\sum_h \langle u_h \mid v_h\rangle_{\mathcal{C}} = 1_{\mathcal{C}}$.
Now, the families of elements $w_j x_k$ and $z_j y_k$ satisfy exactly this property

$$\begin{aligned}\sum_k \sum_j \langle w_j x_k \mid z_j y_k\rangle_{\mathcal{C}} &= \sum_k \sum_j x_k^* w_j^* z_j y_k = \sum_k \langle x_k \mid \sum_j \langle w_j \mid z_j\rangle_{\mathcal{B}} y_k\rangle_{\mathcal{C}} \\ &= \sum_k \langle x_k \mid 1_{\mathcal{B}} y_k\rangle_{\mathcal{C}} = 1_{\mathcal{C}}\end{aligned}$$

and so we can define $u_{j,k} := w_j x_k \in {}_{\mathcal{A}}\mathscr{C}_{\mathcal{C}}$ and $v_{j,k} := z_j y_k \in {}_{\mathcal{A}}\mathscr{C}_{\mathcal{C}}$.
Property (2.18) follows by direct application of equations (2.17) and (2.19). □

Proposition 2.34. *Let $\omega : \mathscr{C} \to \mathbb{C}$ be a $*$-functor (i.e. a functor such that $\omega(x^*) = \overline{\omega(x)}$, for all $x \in \mathscr{C}$) defined on the full commutative C*-category $\mathscr{C}$. For every pair of objects $A, B \in \mathrm{Ob}_{\mathscr{C}}$, we have*

$$\omega(\phi_{BA}(a)) = \omega(a), \quad \forall a \in \mathscr{C}_{AA}.$$

Proof. Consider the imprimitivity bimodule ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ and the associated canonical isomorphism $\phi_{BA} : \mathscr{C}_{AA} \to \mathscr{C}_{BB}$. For every $a \in \mathscr{C}_{AA}$, for any given finite families $w_j, z_j \in \mathscr{C}_{AB}$ such that $\sum_j \langle w_j \mid z_j\rangle_{\mathcal{B}} = 1_{\mathcal{B}}$, we know that $\phi_{BA}(a) = \sum_j \langle w_j \mid az_j\rangle_{\mathcal{B}}$. Since $\omega : \mathscr{C} \to \mathbb{C}$ is a $*$-functor, for all $a \in \mathscr{C}_{AA}$, we have:

$$\begin{aligned}\omega(\phi_{BA}(a)) :&= \omega(\sum_j \langle w_j \mid az_j \rangle_{\mathcal{B}}) = \sum_j \omega(\langle w_j \mid az_j \rangle_{\mathcal{B}}) \\ &= \sum_j \omega(w_j^* a z_j) = \sum_j \omega(w_j^*)\omega(a)\omega(z_j) \\ &= \omega(a) \sum_j \omega(w_j^*)\omega(z_j) = \omega(a) \sum_j \omega(w_j^* z_j) \\ &= \omega(a)\omega(\sum_j \langle w_j \mid z_j \rangle_{\mathcal{B}}) = \omega(a)\omega(1_{\mathcal{B}}) = \omega(a).\end{aligned}$$

□

3 Spectral Theorem for Imprimitivity Bimodules

Let X_A and X_B be two compact Hausdorff spaces and let $R_{BA} : X_A \to X_B$ be a homeomorphism between them. To every complex bundle (E, π, R_{BA}), over the graph of the homeomorphism $R_{BA} \subset X_A \times X_B$, we can naturally associate the set $\Gamma(R_{BA}; E)$ of continuous sections of the bundle E, that turns out to be a symmetric bimodule over the commutative C*-algebra $C(R_{BA}; \mathbb{C})$ of continuous functions on the compact Hausdorff space R_{BA}.
Considering now the pair of homeomorphisms

$$\begin{aligned}&\pi_A : R_{BA} \to X_A, \quad \pi_A : (x, y) \mapsto x, \\ &\pi_B : R_{BA} \to X_B, \quad \pi_B : (x, y) \mapsto y,\end{aligned}$$

we see that the set $\Gamma(R_{BA}; E)$ becomes naturally a left module over $C(X_A; \mathbb{C})$ and a right module over $C(X_B; \mathbb{C})$ with the following left and right actions $f \cdot \sigma := (f \circ \pi_A) \cdot \sigma$ and $\sigma \cdot g := \sigma \cdot (g \circ \pi_B)$ or, in a more explicit form, for all $(x, y) \in R_{BA}$, $f \in C(X_A)$, $g \in C(X_B)$ and $\sigma \in \Gamma(R_{BA}; E)$:

$$\begin{aligned}&f \cdot \sigma(x, y) := f(x)\sigma(x, y) = (f \circ \pi_A)(x, y) \cdot \sigma(x, y), \\ &\sigma \cdot g(x, y) := \sigma(x, y)g(y) = \sigma(x, y) \cdot (g \circ \pi_B)(x, y).\end{aligned}$$

In the terminology of definition 2.10, this is the bimodule ${}_{\pi_A^\bullet}\Gamma(R_{BA}, E)_{\pi_B^\bullet}$ obtained by twisting the symmetric $C(R_{BA})$-bimodule $\Gamma(R_{BA}, E)$ by the isomorphism $\pi_A^\bullet : C(X_A) \to C(R_{BA})$ on the left and by the isomorphism $\pi_B^\bullet : C(X_B) \to C(R_{BA})$ on the right.
We say that ${}_{\pi_A^\bullet}\Gamma(R_{BA}; E)_{\pi_B^\bullet}$ is the **$C(X_A)$-$C(X_B)$-bimodule associated to the bundle (E, π, R_{BA}) over the homeomorphism** $R_{BA} : X_A \to X_B$. Note that if (E, π, R_{BA}) is a Hermitian bundle over the homeomorphism $R_{BA} : X_A \to X_B$, then the bimodule ${}_{C(R_{BA})}\Gamma(R_{BA}; E)_{C(R_{BA})}$ is a full symmetric Hilbert C*-bimodule over $C(R_{BA})$ and, as in remark 2.11, the associated bimodule ${}_{\pi_A^\bullet}\Gamma(R_{BA}; E)_{\pi_B^\bullet}$ has a natural structure as a full Hilbert C*-bimodule with inner products given by:

$$\begin{aligned}&{}_{C(X_A)}\langle \sigma \mid \rho \rangle := (\pi_A^\bullet)^{-1}(\langle \sigma \mid \rho \rangle_{C(R_{BA})}), \quad \forall \sigma, \rho \in \Gamma(R_{BA}; E), \\ &\langle \sigma \mid \rho \rangle_{C(X_B)} := (\pi_B^\bullet)^{-1}(\langle \sigma \mid \rho \rangle_{C(R_{BA})}), \quad \forall \sigma, \rho \in \Gamma(R_{BA}; E).\end{aligned}$$

Furthermore the associated bimodule ${}_{\pi_A^\bullet}\Gamma(R_{BA}; E)_{\pi_B^\bullet}$ is an imprimitivity bimodule if and only if ${}_{C(R_{BA})}\Gamma(R_{BA}; E)_{C(R_{BA})}$ is an imprimitivity bimodule and this, by Serre-Swan theorem (see e.g. [BCL1, Section 2.1.2] and references therein), happens if and only if (E, π, R_{BA}) is a Hermitian line bundle.

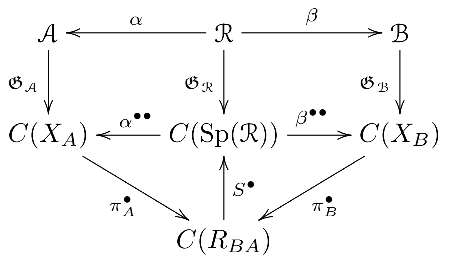
In this section, making use of the results in section 2.2, we prove, in the case of imprimitivity bimodules, a converse to the previous construction i.e. that (up to isomorphism of bimodules) every imprimitivity Hilbert C*-bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ over unital commutative C*-algebras $\mathcal{A}$ and $\mathcal{B}$ actually arises as the bimodule associated to a Hermitian line bundle over a homeomorphism between the compact Hausdorff spaces $\mathrm{Sp}(\mathcal{A})$ and $\mathrm{Sp}(\mathcal{B})$.

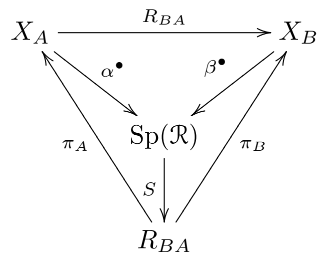
Theorem 3.1. *Given an imprimitivity C*-bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ over two commutative unital C*-algebras $\mathcal{A}, \mathcal{B}$, there exists a Hermitian line bundle (E, π, R_{BA}), over the graph of a homeomorphism $R_{BA} : X_A \to X_B$ between the two compact Hausdorff spaces $X_A := \mathrm{Sp}(\mathcal{A})$, $X_B := \mathrm{Sp}(\mathcal{B})$, whose associated $C(X_A)$-$C(X_B)$-bimodule ${}_{\pi_A^\bullet}\Gamma(R_{BA}; E)_{\pi_B^\bullet}$, when twisted on the left by the Gel'fand transform isomorphism $\mathfrak{G}_{\mathcal{A}} : \mathcal{A} \to C(\mathrm{Sp}(\mathcal{A}))$ and on the right by the Gel'fand isomorphism $\mathfrak{G}_{\mathcal{B}} : \mathcal{B} \to C(\mathrm{Sp}(\mathcal{B}))$, becomes a bimodule ${}_{\pi_A^\bullet \circ \mathfrak{G}_{\mathcal{A}}}\Gamma(R_{BA}; E)_{\pi_B^\bullet \circ \mathfrak{G}_{\mathcal{B}}}$ that is isomorphic, as an $\mathcal{A}$-$\mathcal{B}$-bimodule, to the initial Hilbert C*-bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$.*

Proof. By theorem 2.24, we have a canonical isomorphism $\phi_{\mathcal{M}} : \mathcal{A} \to \mathcal{B}$. Using Gel'fand theorem, applied to the isomorphism $\phi_{\mathcal{M}}^{-1} : \mathcal{B} \to \mathcal{A}$ of unital C*-algebras, we recover a homeomorphism $R_{BA} := (\phi_M^{-1})^\bullet : X_A \to X_B$ between the two compact Hausdorff spaces $X_A := \mathrm{Sp}(\mathcal{A})$ and $X_B := \mathrm{Sp}(\mathcal{B})$. Furthermore we know that the Gel'fand transforms $\mathfrak{G}_{\mathcal{A}} : \mathcal{A} \to C(X_A; \mathbb{C})$, $\mathfrak{G}_{\mathcal{B}} : \mathcal{B} \to C(X_B; \mathbb{C})$ provide two isomorphisms of C*-algebras.
Consider now the set $\mathcal{R} \subset \mathcal{A} \times \mathcal{B}$ defined by $\mathcal{R} := \{(a, b) \in \mathcal{A} \times \mathcal{B} \mid b = \phi_{\mathcal{M}}(a)\}$ and note that $\mathcal{R}$ has a natural structure of unital C*-algebra with componentwise multiplication and norm defined by $\|(a, b)\|_{\mathcal{R}} := \max\{\|a\|, \|b\|\} = \|a\| = \|b\|$. There are natural isomorphisms $\alpha : \mathcal{R} \to \mathcal{A}$ and $\beta : \mathcal{R} \to \mathcal{B}$ given by

$$\alpha : (a, b) \mapsto a, \quad \beta : (a, b) \mapsto b, \quad \forall (a, b) \in \mathcal{R},$$

and they satisfy $\phi_{\mathcal{M}} = \beta \circ \alpha^{-1}$.
Note also that the topological space $\mathrm{Sp}(\mathcal{R})$ is canonically homeomorphic to R_{BA}. In fact, since $R_{BA} \circ (\alpha^{-1})^\bullet = (\phi_{\mathcal{M}}^{-1})^\bullet \circ (\alpha^{-1})^\bullet = (\alpha \circ \beta^{-1})^\bullet \circ (\alpha^{-1})^\bullet = (\beta^{-1})^\bullet$, the function $S : \omega \mapsto ((\alpha^{-1})^\bullet(\omega), (\beta^{-1})^\bullet(\omega))$, for $\omega \in \mathrm{Sp}(\mathcal{R})$, takes values in R_{BA} and being bijective continuous between compact Hausdorff spaces it is a homeomorphism.
We summarize the situation with the following commutative diagrams that might come helpful to visualize the several isomorphisms and homeomorphisms involved:





Twisting (see definition 2.10) the bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ by α on the left and β on the right, we obtain a Hilbert C*-bimodule ${}_{\alpha}\mathcal{M}_{\beta}$ over $\mathcal{R}$ that is symmetric because

$$(a, b) \cdot x = \alpha(a, b)x = ax = x\phi_{\mathcal{M}}(a) = x\beta(a, b) = x \cdot (a, b), \forall (a, b) \in \mathcal{R}.$$

Twisting one more time ${}_{\alpha}\mathcal{M}_{\beta}$ with the isomorphism

$$\gamma := \mathfrak{G}_{\mathcal{R}}^{-1} \circ S^\bullet : C(R_{BA}) \to \mathcal{R},$$

we get a symmetric Hilbert C*-bimodule ${}_{\alpha\circ\gamma}\mathcal{M}_{\beta\circ\gamma}$ over the C*-algebra $C(R_{BA})$. By a direct application of Serre-Swan theorem (see e.g. [BCL1, Theorem 2.2]), we see that there exists a Hermitian bundle (E, π, R_{BA}) over the compact Hausdorff space R_{BA} such that there exists an isomorphism of $C(R_{BA})$-bimodules $\Phi : {}_{\alpha\circ\gamma}\mathcal{M}_{\beta\circ\gamma} \to \Gamma(R_{BA}; E)$. Since ${}_{\mathcal{A}}M_{\mathcal{B}}$ is an imprimitivity bimodule, so is ${}_{\alpha\circ\gamma}\mathcal{M}_{\beta\circ\gamma}$ and hence (E, π, R_{BA}) is a Hermitian line bundle. Making use of proposition 2.12, we have that the map Φ also becomes an isomorphism $\Phi : {}_{\mathcal{A}}M_{\mathcal{B}} \to {}_{(\alpha\circ\gamma)^{-1}}\Gamma(R_{BA}; E)_{(\beta\circ\gamma)^{-1}}$ of Hilbert C*-bimodules over $\mathcal{A}$ and $\mathcal{B}$. Since, by the diagram above, we have $(\alpha \circ \gamma)^{-1} = \pi_A^\bullet \circ \mathfrak{G}_A$ and $(\beta \circ \gamma)^{-1} = \pi_B^\bullet \circ \mathfrak{G}_B$, we finally obtain an isomorphism of left $\mathcal{A}$, right $\mathcal{B}$ Hilbert C*-bimodules

$$\Phi : {}_{\mathcal{A}}M_{\mathcal{B}} \to {}_{\pi_A^\bullet\circ\mathfrak{G}_A}\Gamma(R_{BA}; E)_{\pi_B^\bullet\circ\mathfrak{G}_B}.$$

□

Note that the theorem says that for an imprimitivity bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$ over commutative unital C*-algebras, the triple $(\mathfrak{G}_{\mathcal{A}}, \Phi, \mathfrak{G}_{\mathcal{B}})$ provides an isomorphism, in the category of Hilbert C*-bimodules, from the bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$ to the $C(X_A)$-$C(X_B)$-bimodule ${}_{\pi_A^\bullet}\Gamma(R_{BA}; E)_{\pi_B^\bullet}$ associated to the Hermitian line bundle (E, π, R_{BA}) over the homeomorphism $R_{BA} : X_A \to X_B$. This means that $\Phi(axb) = \mathfrak{G}_{\mathcal{A}}(a)\Phi(x)\mathfrak{G}_{\mathcal{B}}(b)$, for all $x \in \mathcal{M}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$. The map Φ is essentially a "canonical extension" of the Gel'fand transform of the C*-algebras $\mathcal{A}$ and $\mathcal{B}$ to the imprimitivity bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$ over them.

The above theorem is just the starting point for the development of a "bivariant Serre-Swan equivalence" and, more generally, a bivariant "Takahashi duality" (see e.g. [BCL1, Section 2.1.2] and references therein) for the category of Hilbert C*-bimodules over commutative C*-algebras. This will be done elsewhere.
Our spectral theorem, for imprimitivity bimodules over Abelian C*-algebras, is dealing only with the representativity of a potential functor that, to every Hermitian line bundle (E, π, R_{BA}) over the graph of a homeomorphism $R_{BA} : X_A \to X_B$ between compact Hausdorff spaces, associates the imprimitivity bimodule ${}_{\pi_A^\bullet}\Gamma(R_{BA}; E)_{\pi_B^\bullet}$ over the commutative C*-algebras $C(X_A)$ and $C(X_B)$. To proceed further we have to provide a suitable notion of morphisms and define our functor on them.
The above result is for now stated in the case of imprimitivity bimodules and hence it does not provide neither an answer to the problem of classifying, nor a geometric interpretation of general $C(X)$-$C(Y)$-bimodules for given compact Hausdorff spaces X and Y. Warning the reader to take due care of some differences in notations and definitions, for some related results on the "spectral theory" of Hilbert C*-bimodules, one may consult B. Abadie-R. Exel [AE], H. Bursztyn-S. Waldmann [BW], A. Hopenwasser-J. Peters-J. Powers [HPP], A. Hopenwasser [H], T. Kajiwara-C. Pinzari-Y. Watatani [KPW], P. Muhly-B.Solel [MS].
In particular, B. Abadie and R. Exel [AE, Proposition 1.9] proved that every imprimitivity C*-bimodule over a commutative C*-algebra $\mathcal{A}$ is always obtained from its right symmetrization by twisting on one side with a given automorphism θ and, in a more algebraic setting, a result of H. Burzstyn-S. Waldmann [BW, Proposition 2.3] assures that if two imprimitivity bimodules ${}_{\mathcal{A}}M_{\mathcal{B}}$ and ${}_{\mathcal{A}}N_{\mathcal{B}}$ over the same commutative algebras are isomorphic as right modules, there is a unique isomorphism of the C*-algebra $\mathcal{B}$ such that the bimodule M is isomorphic to the twisting of N.
Gathering together the above facts, in the special case of commutative full C*-categories, we obtain the following result.

Theorem 3.2. *Let $\mathscr{C}$ be a full commutative C*-category. Then for every pair of objects A and B, one has:*

- ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ *is an imprimitivity* ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{A}}$*-*${}_{\mathcal{B}}\mathscr{C}_{\mathcal{B}}$ *bimodule. That is,* ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{A}}$ *and* ${}_{\mathcal{B}}\mathscr{C}_{\mathcal{B}}$ *are Morita equivalent and thus there is a canonical* $*$*-isomorphism implemented by* $x^*y \mapsto yx^*$*,* $x, y \in {}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$*.*
- ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{B}}$ *is the (non-symmetric)* ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{A}}$*-*${}_{\mathcal{B}}\mathscr{C}_{\mathcal{B}}$*-bimodule of continuous sections of a Hermitian line bundle over the graph of the corresponding homeomorphism between the Gel'fand spectra of* ${}_{\mathcal{A}}\mathscr{C}_{\mathcal{A}}$ *and* ${}_{\mathcal{B}}\mathscr{C}_{\mathcal{B}}$*.*

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A remark on Gel’fand duality for spectral triples

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A Remark on Gel'fand Duality for Spectral Triples

Paolo Bertozzini *@, Roberto Conti *†, Wicharn Lewkeeratiyutkul *‡
@ e-mail: paolo.th@gmail.com
† *Mathematics, School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia*
e-mail: Roberto.Conti@newcastle.edu.au
‡ *Department of Mathematics, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand*
e-mail: Wicharn.L@chula.ac.th

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Abstract

We present a duality between the category of compact Riemannian spin manifolds (equipped with a given spin bundle and charge conjugation) with isometries as morphisms and a suitable "metric" category of spectral triples over commutative pre-C*-algebras. We also construct an embedding of a "quotient" of the category of spectral triples introduced in [BCL1] into the latter metric category. Finally we discuss a further related duality in the case of orientation and spin-preserving maps between manifolds of fixed dimension.

keywords: Spectral Triple, Spin Manifold, Category.

MSC-2000: 46L87, 46M15, 18F99, 15A66.

1 Introduction.

Although the main strength of non-commutative geometry is a full treatment of non-commutative algebras as "duals of geometric spaces", the foundation of the theory relies on the construction of suitable categorical equivalences, resp. anti-equivalences (i.e. covariant, resp. contravariant functors that are isomorphisms of categories "up to natural transformations") between categories of "geometric spaces" and categories of commutative algebras of functions over these spaces.[1]

Typical examples of such (anti-)equivalences are listed below, itemized by the name of the people who worked them out:

- **Hilbert:** between algebraic sets and finitely generated algebras over an algebraically closed field [H];

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[1] For the elementary background in "category theory" the reader can refer to the on-line introduction by J. Baez [B] and the classical books by S. McLane [M] and M. Barr-C. Wells [BW].

- **Stone:** between totally disconnected compact Hausdorff topological spaces and Boolean algebras [St1, St2];
- **Gel'fand-Naĭmark:** between the category of continuous maps of compact Hausdorff topological spaces and the category of unital involutive homomorphisms of unital commutative C*-algebras[2] [G, GN];
- **Halmos-von Neumann:** between the category of measurable maps of measure spaces and the category of unital involutive homomorphisms of commutative von Neumann algebras;[3]
- **Serre-Swan:** between the category of vector bundle maps of finite-dimensional locally trivial vector bundles over a compact Hausdorff topological space and the category of homomorphisms of finite projective modules over a commutative unital C*-algebra [Se, Sw];
- **Cartier-Grothendieck:** between the category of commutative schemes (ringed spaces) in algebraic geometry and the category of topoi (sheaves over topological spaces);[4]
- **Takahashi:** between the category of Hilbert bundles on (different) compact Hausdorff spaces and the category of Hilbert C*-modules over (different) commutative unital C*-algebras [T1, T2];

Even more dualities arise when the spaces in question are equipped with additional structure, most notably a group structure or the like (see Pontryagin-Van Kampen [Po, VK], Tannaka-Kreĭn [Ta, Kr] and Doplicher-Roberts [DR]).
In this paper we will focus our attention on the Gel'fand-Naĭmark duality, to which the other dualities are related in significant way. In short, the fundamental message that can be read off from the celebrated Gel'fand-Naĭmark theorem on commutative C*-algebras is that, at the "topological level", the information on a "space" can be completely encoded in (and recovered from) a suitable "algebraic structure".
In applications to physics (at least for those branches that are dealing with "metric structures" such as general relativity), it would be important to "tune" Gel'fand-Naĭmark's correspondence in order to embrace classes of spaces with more detailed geometric structures (e.g. differential, metric, connection, curvature).
In recent times, Connes' non-commutative geometry [C1, FGV] has emerged as the most outstanding proposal in this direction, based on the notion of spectral triple.
In this short note we provide a simple further example of categorical anti-equivalence between Riemannian spin manifolds and commutative Connes' spectral triples (see theorem 3.2). This line of thought is expected to play an important role in future developments of the categorical structure of non-commutative geometry, and spectral triples in particular (see [BCL2]), as well as in the study of (geometric) quantization, where the construction

[2]Or more generally between the category of proper continuous maps of locally compact Hausdorff spaces and the category of involutive homomorphisms of commutative C*-algebras.

[3]The origin of a dual treatment of measure theory (at least for locally compact Hausdorff spaces) can be traced back to F. Riesz-A. Makov-S. Kakutani-A. Weil theorem [Rie, Ma, K, W], but the proof that a measure space can be recovered from a commutative von Neumann algebra is due to P. Halmos-J. von Neumann [HvN].

[4]As reported by I. Dolgachev in his useful historical review [D, Section 1], the idea of P. Cartier (1957) that affine schemes are in duality with ringed spaces of the form $\mathrm{Sp}(\mathcal{A})$ was developed by Grothendieck in the full theory of schemes.

of functorial relations between "commutative" and "quantum" spaces are central points of investigation.
Although the idea of reconstructing a smooth manifold out of a commutative spectral triple has been latent for some time, (see [C3, C4, R, RV1, C5, C6]) the point to promote it to a categorical level seems to be new. Our main tool is the notion of metric morphisms of spectral triples, namely those preserving Connes' distance on the state space.
In the second part of the paper, we examine some connection between the category of "metric spectral triples" (on which the equivalence result is based) and our previous work on morphism of spectral triples [BCL1]. It should be possible to provide other equivalence results in terms of categories of spectral triples based on different notions of morphism (at least for some classes of Riemannian manifolds); some of these issues are presently under investigation (see [BCL2, Section 4.1] for an overview).

It should be remarked that Connes' distance formula has been systematically adopted by M. Rieffel as the backbone of his notion of quantum compact metric space (see [Ri] and references therein). Although we present our result in the framework of Connes' spectral triples, it is likely that our ideas might find some application also in Rieffel's framework.

In order to keep the length of this note as short as possible, we will refer to the literature for all the background material and only recall the basic definitions.

1.1 Spectral Triples.

Following A. Connes' axiomatization (see [C1, FGV, C5] for all the details), a **compact spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ consists of

a) a unital pre-C*-algebra $\mathcal{A}$ (that is sometimes required to be closed under holomorphic functional calculus),

b) a (faithful) representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of the algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and

c) a **Dirac operator**, i.e. a (generally unbounded) self-adjoint operator D, with compact resolvent $(D - \lambda)^{-1}$ for every $\lambda \in \mathbb{C} - \mathbb{R}$ and such that[5] $[D, \pi(a)]_- \in \mathcal{B}(\mathcal{H})$, $\forall a \in \mathcal{A}$.

A spectral triple is called **even** if it is equipped with a grading operator, i.e. a bounded self-adjoint operator $\Gamma \in \mathcal{B}(\mathcal{H})$ such that:

$$\Gamma^2 = \mathrm{Id}_{\mathcal{H}}; \quad [\Gamma, \pi(a)]_- = 0 \quad \forall a \in \mathcal{A}; \quad [\Gamma, D]_+ = 0.$$

A spectral triple without grading is called **odd**.

A spectral triple is **regular** if the functions $\Xi_x : t \mapsto \exp(it|D|)x\exp(-it|D|)$ are "smooth" i.e. $\Xi_x \in \mathrm{C}^\infty(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ for every $x \in \Omega_D(\mathcal{A})$, where we define [6]

$$\Omega_D(\mathcal{A}) := \mathrm{span}\{\pi(a_0)[D, \pi(a_1)]_- \cdots [D, \pi(a_n)]_- \mid n \in \mathbb{N},\ a_0, \dots, a_n \in \mathcal{A}\}.$$

This regularity condition can be equivalently expressed requiring that, for all $a \in \mathcal{A}$, $\pi(a)$ and $[D, \pi(a)]_-$ are contained in $\cap_{m=1}^{\infty} \mathrm{Dom}\, \delta^m$, where δ is the derivation given by $\delta(x) := [|D|, x]_-$.

The spectral triple is n**-dimensional** iff there exists an integer n such that the Dixmier trace of $|D|^{-n}$ is finite non-zero. A spectral triple is θ**-summable** if $\exp(-tD^2)$ is a trace-class operator for every $t > 0$.

[5] Here $[x, y]_\pm := xy \pm yx$ denote respectively the anticommutator and the commutator of $x, y \in \mathcal{B}(\mathcal{H})$.
[6] We assume that for $n = 0 \in \mathbb{N}$ the term in the formula simply reduces to $\pi(a_0)$.

A spectral triple is **real** if it is equipped with a real structure i.e. an antiunitary operator $J : \mathcal{H} \to \mathcal{H}$ such that:

$$[\pi(a), J\pi(b^*)J^{-1}]_- = 0 \quad \forall a, b \in \mathcal{A};$$
$$[[D, \pi(a)]_-, J\pi(b^*)J^{-1}]_- = 0 \quad \forall a, b \in \mathcal{A}, \quad \textbf{first-order condition};$$
$$J^2 = \pm \mathrm{Id}_{\mathcal{H}}; \quad [J, D]_\pm = 0;$$
$$\text{and, only in the even case,} \quad [J, \Gamma]_\pm = 0,$$

where the choice of $\pm$ in the last three formulas depends on the "dimension" n of the spectral triple modulo 8 according to the following table:

n	0	1	2	3	4	5	6	7
$J^2 = \pm \mathrm{Id}_{\mathcal{H}}$	+	+	−	−	−	−	+	+
$[J, D]_\pm = 0$	−	+	−	−	−	+	−	−
$[J, \Gamma]_\pm = 0$	−		+		−		+	

A spectral triple is **finite** if $\mathcal{H}_\infty := \cap_{k=1}^\infty \mathrm{Dom}\, D^k$ is a finite projective $\mathcal{A}$-bimodule and **absolutely continuous** if, there exists a Hermitian form $(\xi, \eta) \mapsto (\xi \mid \eta)$ on $\mathcal{H}_\infty$ such that, for all $a \in \mathcal{A}$, $\langle \xi \mid \pi(a)\eta \rangle$ is the Dixmier trace of $\pi(a)(\xi \mid \eta)|D|^{-n}$.

An n-dimensional spectral triple is said to be **orientable** if there is a Hochschild cycle $c = \sum_{j=1}^m a_0^{(j)} \otimes a_1^{(j)} \otimes \cdots \otimes a_n^{(j)}$ such that its "representation" on the Hilbert space $\mathcal{H}$, $\pi(c) = \sum_{j=1}^m \pi(a_0^{(j)})[D, \pi(a_1^{(j)})]_- \cdots [D, \pi(a_n^{(j)})]_-$ is the grading operator in the even case or the identity operator in the odd case[7].

A real spectral triple is said to satisfy **Poincaré duality** if its fundamental class in the KR-homology of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ induces (via Kasparov intersection product) an isomorphism between the K-theory $\mathrm{K}_\bullet(\mathcal{A})$ and the K-homology $\mathrm{K}^\bullet(\mathcal{A})$ of $\mathcal{A}$.[8]

A spectral triple will be called **commutative** whenever $\mathcal{A}$ is commutative.

Finally a spectral triple is **irreducible** if there is no non-trivial closed subspace in $\mathcal{H}$ that is invariant for $\pi(\mathcal{A}), D, J, \Gamma$.

1.2 Reconstruction Theorem (Commutative Case).

Let M be a real compact orientable Riemannian m-dimensional spin C^∞ manifold with a given volume form μ_M. Let us denote (see [S] for details) by $S(M)$ a given irreducible complex spinor bundle over M i.e. a bundle over M equipped with a left action $c : \mathbb{C}\mathrm{l}^{(+)}(T(M)) \otimes S(M) \to S(M)$ of the "Clifford" bundle[9] $\mathbb{C}\mathrm{l}^{(+)}(T(M))$ inducing a bundle isomorphism between $\mathbb{C}\mathrm{l}^{(+)}(T(M))$ and $\mathrm{End}(S(M))$. Let $[S(M)]$ be the spin^c structure[10] of M determined by $S(M)$.

[7] In the following, in order to simplify the discussion, we will always refer to a "grading operator" Γ that actually coincides with the grading operator in the even case and that is by definition the identity operator in the odd case.

[8] In [RV1] some of the axioms are reformulated in a different form, in particular this condition is replaced by the requirement that the C*-module completion of $\mathcal{H}_\infty$ is a Morita equivalence bimodule between (the norm completions of) $\mathcal{A}$ and $\Omega_D(\mathcal{A})$.

[9] Following [FGV, Page 373], we denote by $\mathbb{C}\mathrm{l}^{(+)}(T(M))$ the complexified Clifford bundle of M if $\dim M$ is even and respectively its even subalgebra bundle $\mathbb{C}\mathrm{l}^+(T(M))$ if $\dim M$ is odd.

[10] An orientable Riemannian manifold is spin^c if it admits a complex irreducible spinor bundle [S, Definition 7]. Recall that a spin^c manifold usually admits several inequivalent spin^c structures and that for a given spin^c structure, a complex irreducible spinor bundle over M is determined only up to (Hermitian) bundle isomorphism.

Let C_M be a given "spinorial" charge conjugation[11] on $S(M)$ i.e. an antilinear Hermitian bundle morphism such that $C_M \circ C_M = \pm \mathrm{Id}_{S(M)}$ (signs depending on $\dim M$ modulo 8 as in the table in section 1.1) that is "compatible" with the charge conjugation[12] κ in $\mathbb{C}\mathrm{l}^{(+)}(T(M))$ i.e. $C_M(\beta(p)\cdot\sigma(p)) = \kappa(\beta(p))\cdot C_M(\sigma(p))$, for any section $\beta \in \Gamma(\mathbb{C}\mathrm{l}^{(+)}(T(M)))$ of the Clifford bundle and any section $\sigma \in \Gamma(S(M))$ of the spinor bundle. We denote by $[(S(M), C_M)]$ the spin structure on M determined by C_M.

Let $\mathcal{A}_M := \mathrm{C}^\infty(M;\mathbb{C})$ be the commutative pre-C*-algebra of smooth complex valued functions on M. We denote by π_M its representation by pointwise multiplication on the space $\mathcal{H}_M := \mathrm{L}^2(M, S(M))$, the completion of the space $\Gamma^\infty(M, S(M))$ of smooth sections of the spinor bundle $S(M)$ equipped with the inner product $\langle \sigma \mid \tau \rangle := \int_M \langle \sigma(p) \mid \tau(p) \rangle_p \, \mathrm{d}\mu_M$, where $\langle \cdot \mid \cdot \rangle_p$ is the unique inner product on $S_p(M)$ compatible with the Clifford action and the Clifford product. Note that the spinorial charge conjugation C_M (being unitary on the fibers) has a unique antilinear unitary extension $J_M : \mathcal{H}_M \to \mathcal{H}_M$ determined by $(J_M\sigma)(p) := C_M(\sigma(p))$ for $\sigma \in \Gamma^\infty(S(M))$ and $p \in M$.

Let Γ_M be the unique unitary extension on $\mathcal{H}_M$ of the operator Λ_M on $\Gamma(S(M))$ acting by left action of the chirality element $\gamma \in \Gamma(\mathbb{C}\mathrm{l}^{(+)}(T(M)))$, that implements the grading χ of $\Gamma(\mathbb{C}\mathrm{l}^{(+)}(T(M)))$ as inner automorphism.[13]

Denote by D_M the Atiyah-Singer Dirac operator on the Hilbert space $\mathcal{H}_M$, i.e. the closure of the operator that on $\Gamma^\infty(S(M))$ is obtained by "contracting" the unique spin covariant derivative ∇^S (induced on $\Gamma^\infty(S(M))$ by the Levi-Civita covariant derivative of M, see [FGV, Theorem 9.8]) with the Clifford multiplication. For a detailed discussion on Atiyah-Singer Dirac operators we refer to [BGV, LM, S].

We have the following fundamental results:

Theorem 1.1 (Connes, see e.g. [C1, C2] and Section 11.1 in [FGV])**.** *Given an orientable compact spin Riemannian m-dimensional differentiable manifold M, with a given complex spinor bundle $S(M)$, a given spinorial charge conjugation C_M and a given volume form μ_M,[14] the data $(\mathcal{A}_M, \mathcal{H}_M, D_M)$ defines a commutative regular finite absolutely continuous m-dimensional spectral triple that is real, with real structure J_M, orientable, with grading Γ_M, and satisfies Poincaré duality.*

Theorem 1.2 (Connes [C3, C5])**.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be an irreducible commutative real (with real structure J and grading Γ) strongly regular[15] m-dimensional finite absolutely continuous orientable spectral triple, with totally antisymmetric (in the last m entries) Hochschild cycle, and satisfying Poincaré duality. The spectrum of (the norm closure of) $\mathcal{A}$ can be endowed, in a unique way, with the structure of an m-dimensional connected compact orientable spin Riemannian manifold M with an irreducible complex spinor bundle $S(M)$, a charge conjugation J_M and a grading Γ_M such that:*

$$\mathcal{A} \simeq C^\infty(M;\mathbb{C}), \quad \mathcal{H} \simeq L^2(M, S(M)), \quad D \simeq D_M, \quad J \simeq J_M, \quad \Gamma \simeq \Gamma_M.$$

[11] A spin^c manifold is spin if and only if it admits a complex spinor bundle with a charge conjugation [S, Definition 8]. Recall that a spin manifold usually admits several inequivalent spin structures even for the same spin^c structure and that for a given spin structure a conjugation operator is determined only up to intertwining with (Hermitian) bundle isomorphisms.

[12] κ is the composition of the natural grading operator and the canonical conjugation.

[13] The grading is actually the identity in odd dimension.

[14] Remember that an orientable manifolds admits two different orientations and that, on a Riemannian manifold, the choice of an orientation canonically determines a volume form μ_M.

[15] In the sense of [C5, Definition 6.1].

A. Connes first proved the previous theorem 1.2 under the additional condition that $\mathcal{A}$ is already given as the algebra of smooth complex-valued functions over a differentiable manifold M, namely $\mathcal{A} = C^\infty(M;\mathbb{C})$ (for a detailed proof see e.g. [FGV, Theorem 11.2]), and conjectured [C3], [C4, Theorem 6, Remark (a)] the result for general commutative pre-C*-algebras $\mathcal{A}$.
A tentative proof of this last fact has been published by A. Rennie [R]; some gaps were pointed out in the original argument, a different revised, but still incorrect, proof appears in [RV1] (see also [RV2]) under some additional technical conditions. Recently A. Connes [C5] finally provided the missing steps in the proof of the result.
As a consequence, there exists a one-to-one correspondence between unitary equivalence classes of spectral triples and connected compact oriented Riemannian spin manifolds up to spin-preserving isometric diffeomorphisms.
Similar results should also be available for spin^c manifolds [C4, Theorem 6, Remark (e)].

1.3 Connes' Distance Formula.

Given a spectral triple $(\mathcal{A},\mathcal{H},D)$, let us denote by $\mathcal{S}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A})$ the sets of **states** and **pure states** of the pre-C*-algebra $\mathcal{A}$, respectively. If $\mathcal{A} := \mathrm{C}^\infty(M;\mathbb{C})$, for all $p \in M$ we denote by $\mathrm{ev}_p : x \mapsto x(p)$ the "evaluation functional" in p of the functions $x \in \mathcal{A}$ and note that $\mathrm{ev}_p \in \mathcal{P}(\mathcal{A})$. Actually in this case $\mathcal{P}(\mathcal{A})$ coincides with the set of all evaluation functionals.
Going back to the general case, the **Connes' distance** d_D on $\mathcal{P}(\mathcal{A})$ is the function on $\mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{A})$ given by

$$d_D(\omega_1,\omega_2) := \sup\{|\omega_1(x) - \omega_2(x)| \mid x \in \mathcal{A},\ \|[D,\pi(x)]\| \leq 1\}.$$

Strictly speaking, without imposing other conditions, d_D could also take the value $+\infty$ as in the case of non-connected manifolds. In turn, one can use the same formula to define a "distance" on the set of all the states of $\mathcal{A}$.

Theorem 1.3 (Connes's distance formula). *[FGV, Proposition 9.12] If the spectral triple $(\mathcal{A},\mathcal{H},D)$ is obtained as in theorem 1.1 from a compact finite-dimensional oriented Riemannian spin manifold M equipped with a spinor bundle $S(M)$ and a spinorial charge conjugation C_M, then for every $p,q \in M$, $d_D(\mathrm{ev}_p,\mathrm{ev}_q)$ coincides with the geodesic distance*

$$d_M(p,q) := \inf\left\{ \int_a^b \|\gamma'(t)\|\, dt \mid \gamma \text{ is a geodesic with } \gamma(a) = p,\ \gamma(b) = q \right\}.$$

Of course, given a unital $*$-morphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ there is a pull-back $\phi^\bullet : \mathcal{S}(\mathcal{A}_2) \to \mathcal{S}(\mathcal{A}_1)$ defined by $\phi^\bullet(\omega) := \omega \circ \phi$ for all $\omega \in \mathcal{S}(\mathcal{A}_2)$.

2 A Metric Category of Spectral Triples.

The objects of all of our categories will be compact spectral triples $(\mathcal{A},\mathcal{H},D)$.
Given two spectral triples $(\mathcal{A}_j,\mathcal{H}_j,D_j)$, with $j = 1,2$, a **metric morphism** of spectral triples $(\mathcal{A}_1,\mathcal{H}_1,D_1) \xrightarrow{\phi} (\mathcal{A}_2,\mathcal{H}_2,D_2)$ is by definition a unital epimorphism[16] $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ of pre-C*-algebras whose pull-back $\phi^\bullet : \mathcal{P}(\mathcal{A}_2) \to \mathcal{P}(\mathcal{A}_1)$ is an isometry, i.e.

$$d_{D_1}(\phi^\bullet(\omega_1),\phi^\bullet(\omega_2)) = d_{D_2}(\omega_1,\omega_2),\ \forall \omega_1,\omega_2 \in \mathcal{P}(\mathcal{A}_2).$$

Spectral triples with metric morphisms form a category $\mathscr{S}^m$.

[16]Note that if ϕ is an epimorphism, its pull-back $\phi^\bullet$ maps pure states into pure states.

Remark 2.1. *A unitary equivalence of spectral triples gives an isomorphism in the category* $\mathscr{S}^m$.

2.1 A Local Metric Category of Spectral Triples.

For convenience of the reader, we recall here the definitions of morphisms of spectral triples proposed in our previous work [BCL1, Sections 2.2-2.3].
A **morphism** in the category $\mathscr{S}$, between spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, $j = 1, 2$ of the same dimension, is a pair (ϕ, Φ), where $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a $*$-morphism between the pre-C*-algebras $\mathcal{A}_1, \mathcal{A}_2$ and $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear map in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $\pi_2(\phi(x)) \circ \Phi = \Phi \circ \pi_1(x)$, $\forall x \in \mathcal{A}_1$ and $D_2 \circ \Phi(\xi) = \Phi \circ D_1(\xi)$ $\forall \xi \in \mathrm{Dom}\, D_1$.
In a similar way, a **morphism of real spectral triples** $(\mathcal{A}_j, \mathcal{H}_j, D_j, J_j)$ with $j = 1, 2$, in the category of real spectral triples $\mathscr{S}_r$, is a morphism in $\mathscr{S}$ such that Φ also satisfies $J_2 \circ \Phi = \Phi \circ J_1$. Finally a **morphism of even spectral triples** $(\mathcal{A}_j, \mathcal{H}_j, D_j, \Gamma_j)$ with $j = 1, 2$, in the category of even spectral triples $\mathscr{S}_e$, is a morphism in $\mathscr{S}$ such that $\Gamma_2 \circ \Phi = \Phi \circ \Gamma_1$. We will denote by $\mathscr{S}_I$ (respectively $\mathscr{S}_{Ir}, \mathscr{S}_{Ire}$) the subcategory of $\mathscr{S}$ (respectively $\mathscr{S}_r, \mathscr{S}_{re}$) consisting of "isometric" morphisms of spectral triples, i.e. pairs (ϕ, Φ) with ϕ surjective and Φ co-isometric. We have the following inclusion of non-full subcategories: $\mathscr{S}_{re} := \mathscr{S}_r \cap \mathscr{S}_e \subset \mathscr{S}$.

3 The Metric Connes-Rennie-Varilly Functor.

Let us consider the class $\mathscr{M}$ of C^∞ metric isometries[17] of compact finite-dimensional C^∞ orientable Riemannian spin manifolds M equipped with a fixed spinor bundle $S(M)$, a given spinorial charge conjugation C_M and a volume form μ_M. The class $\mathscr{M}$ with the usual composition of functions forms a category.

Proposition 3.1. *There is a contravariant functor* $\mathfrak{C}$ *from the category* $\mathscr{M}$ *to the category* $\mathscr{S}^m$ *that to every triple* $(M, S(M), C_M) \in \mathscr{M}$ *associates the spectral triple* $(\mathcal{A}, \mathcal{H}, D) \in \mathscr{S}^m$ *given as in theorem 1.1 and that to every smooth metric isometry* $f : M_1 \to M_2$ *associates its pull-back* $f^\bullet : \mathcal{A}_2 \to \mathcal{A}_1$.

Proof. Every smooth metric isometry $f : M_1 \to M_2$ in $\mathscr{M}$ is a Riemannian isometry of M_1 onto a closed embedded submanifold $f(M_1)$ of M_2. Since every smooth function on a closed embedded submanifold is the restriction of a smooth function on M_2, the pull-back $\phi := f^\bullet$ is a unital epimorphism of the pre-C*-algebras $\phi : \mathcal{A}_2 \to \mathcal{A}_1$ and, by theorem 1.3, $\phi^\bullet : \mathcal{P}(\mathcal{A}_1) \to \mathcal{P}(\mathcal{A}_2)$ is metric-preserving:

$$\begin{aligned} d_{D_2}(\phi^\bullet(\omega_1), \phi^\bullet(\omega_2)) &= d_{D_2}(\phi^\bullet(\mathrm{ev}_p), \phi^\bullet(\mathrm{ev}_q)) = d_{D_2}(\mathrm{ev}_{f(p)}, \mathrm{ev}_{f(q)}) \\ &= d_{M_2}(f(p), f(q)) = d_{M_1}(p, q) = d_{D_1}(\mathrm{ev}_p, \mathrm{ev}_q) \\ &= d_{D_1}(\omega_1, \omega_2), \end{aligned}$$

where $p, q \in M_1$ are the unique points such that $\omega_1 = \mathrm{ev}_p$ and $\omega_2 = \mathrm{ev}_q$.
Of course $\mathfrak{C}(g \circ f) = (g \circ f)^\bullet = f^\bullet \circ g^\bullet = \mathfrak{C}_f \circ \mathfrak{C}_g$ and $\mathfrak{C}_{\iota_M} = \iota_{\mathfrak{C}(M)}$. □

We will call the functor $\mathfrak{C}$ the **metric Connes-Rennie-Varilly functor**.

Here we present the main result of this paper. We denote by ab-$\mathscr{S}^m$ the full subcategory of $\mathscr{S}^m$ of **direct sums of irreducible Abelian spectral triples**[18].

[17] Note that in general a Riemannian isometry is not necessarily a metric isometry.

[18] In a completely similar way we will denote by ab-$\mathscr{S}$ the full subcategory of direct sums of irreducible Abelian spectral triples in $\mathscr{S}$.

Theorem 3.2. *The metric Connes-Rennie-Varilly functor is an anti-equivalence between the categories $\mathscr{M}$ and ab-$\mathscr{S}^m$.*

Proof. The functor $\mathfrak{C}$ is faithful: if $\mathfrak{C}_f = \mathfrak{C}_g$ for two smooth isometries $f, g : M_1 \to M_2$, then $f^\bullet = g^\bullet$ as morphisms of pre-C*-algebras and hence they coincide also when uniquely extend to morphisms of C*-algebras of continuous functions and the result $f = g$ follows from Gel'fand duality theorem.
The functor $\mathfrak{C}$ is full: if $\phi : \mathfrak{C}(M_2) \to \mathfrak{C}(M_1)$ is a metric morphism in $\mathscr{S}^m$, as a homomorphisms of pre-C* algebras of smooth functions, ϕ extends uniquely to a morphism of C*-algebras of continuous functions and, from Gel'fand duality theorem, there exists a unique continuous function $f : M_1 \to M_2$ such that $f^\bullet = \phi$. From the fact that $f^\bullet$ maps smooth functions on M_2 to smooth functions on M_1 it follows that f is a smooth function between manifolds. Since ϕ also preserves the spectral distances, it follows that f is a smooth metric isometry hence a Riemannian isometry.
The functor $\mathfrak{C}$ is representative: for when restricted to the subcategory of connected manifolds with target the subcategory of irreducible spectral triples, this is actually a restatement of the reconstruction theorem 1.2 and remark 2.1. Since the Connes-Rennie-Varilly functor $\mathfrak{C}$ maps disjoint unions of connected components into direct sums of spectral triples, the result follows. $\square$

Unfortunately, at this stage, we cannot present a statement involving the category of all Abelian spectral triples. The above result raises naturally the issue of decomposing (Abelian) spectral triples in terms of irreducible components.

Remark 3.3. *In restriction to the subcategory $\mathscr{M}_d$ of **dimension-preserving** smooth isometries (i.e. isometric immersions with fiberwise isomorphic tangent maps), the metric Connes-Rennie-Varilly functor $\mathfrak{C}$ is an anti-equivalence between $\mathscr{M}_d$ and the subcategory ab-$\mathscr{S}^m_d$ of metric morphisms of direct sums of irreducible Abelian spectral triples with the same dimension. In a similar way, denoting by $\mathcal{N}(\mathscr{C})$ the nerve of the category $\mathscr{C}$, i.e. the groupoid of isomorphisms of $\mathscr{C}$, we have that $\mathfrak{C}|_{\mathcal{N}(\mathscr{M})}$ is an anti-equivalence between $\mathcal{N}(\mathscr{M})$[19] and the nerve $\mathcal{N}($ab-$\mathscr{S}^m)$.*

4 Metric and Spin Categories.

We now proceed to establish a connection between the category $\mathscr{S}^m$ of metric spectral triples and the categories of spectral triples $\mathscr{S}$ (respectively real spectral triples $\mathscr{S}_r$) introduced in [BCL1, Section 2.2-2.3] and briefly recalled in section 2.1.
Denote by $\mathscr{S}^0$ (respectively $\mathscr{S}^0_{Ired}$) the category of spectral triples whose morphisms are those homomorphisms of algebras ϕ for which there exists at least one Φ such that the pair (ϕ, Φ) is a morphism in $\mathscr{S}$ (respectively $\mathscr{S}_{Ired}$). We have a "forgetful" full functor $\mathfrak{F} : \mathscr{S} \to \mathscr{S}^0$ that to every morphism (ϕ, Φ) in $\mathscr{S}$ associates ϕ as a morphism in $\mathscr{S}^0$.

Lemma 4.1. *A metric isometry of Riemannian manifolds with the same dimension is a smooth Riemannian isometry onto a union of connected components.*

Proof. Let $f : M \to N$ be a metric isometry. Since $\dim M = \dim N$, by Brouwer's theorem, we see that f is open and maps each connected component of M onto a unique connected

[19]The nerve of $\mathscr{M}$ (always a subcategory of $\mathscr{M}_d$) is actually the "disjoint union" of denuberable "connected components" consisting of the categories of smooth bijective isometries of n-dimensional spin manifolds.

component of N. By the Myers-Steenrod theorem (see for example [P, Section 5.9, Theorem 9.1]), any such bijective map between connected components is a smooth Riemannian surjective isometry; hence $f : M \to N$ is a smooth Riemannian isometry onto $f(M)$, a union of connected components of N. □

Let $f : (M, S(M), C_M) \to (N, S(N), C_N)$ be a morphism in $\mathscr{M}_d$. Thanks to the last lemma, we can consider the differential $Df : T(M) \to T(N)$. It is a monomorphism of Euclidean bundles and induces a unique Bogoljubov morphism $\mathbb{C}\mathrm{l}_{Df} : \mathbb{C}\mathrm{l}^{(+)}(T(M)) \to \mathbb{C}\mathrm{l}^{(+)}(T(N))$ of the Clifford bundles that is actually an isomorphism of $\mathbb{C}\mathrm{l}^{(+)}(T(M))$ with subbundle $\mathbb{C}\mathrm{l}^{(+)}(T(f(M)))$, the Clifford bundle of the submanifold $f(M)$.[20] This isomorphism can be used to "transfer" the irreducible Clifford action of $\mathbb{C}\mathrm{l}^{(+)}(T(f(M)))$ on the bundle $S(f(M)) := S(N)|_{f(M)}$ to an irreducible action of $\mathbb{C}\mathrm{l}^{(+)}(T(M))$ and, since the bundle $f^\bullet(S(N)) = f^\bullet(S(f(M)))$ is naturally isomorphic to $S(f(M))$, the bundle $f^\bullet(S(N))$ becomes an irreducible complex spinor bundle on M. By a similar argument, $f^\bullet(S(N))$ comes equipped with a spinorial charge conjugation $f^\bullet(C_N)$ obtained by "pull-back" of (the restriction to $S(f(M))$ of) C_N through the isomorphism $f^\bullet(S(N)) \simeq S(f(M))$.

We say that f is **spin-preserving** if the spin structure $[(f^\bullet(S(N)), f^\bullet(C_N))]$ determined by $f^\bullet(S(N))$ with spinorial charge conjugation $f^\bullet(C_N)$ coincides with the spin structure of M i.e. if there exists an isomorphism of Hermitian bundles $U : f^\bullet(S(N)) \to S(M)$ that intertwines the charge conjugations: $U \circ f^\bullet(C_N) = C_M \circ U$ and the Clifford actions. Note that if f is orientation-preserving, the isomorphism U also intertwines the grading operators of the spinor bundles.
Let us denote by $\mathscr{M}_d$-spin the subcategory of spin and orientation-preserving maps in $\mathscr{M}_d$. The following result, that we report for completeness, is certainly well-known although we could not find any suitable reference. Note that ab-$\mathscr{S}_{Ired}$ denotes the full subcategory of $\mathscr{S}_{Ired}$ whose objects are direct sums of irreducible Abelian spectral triples.

Proposition 4.2. *Let M, N be two compact orientable Riemannian spin manifolds in the category $\mathscr{M}$. If $f : M \to N$ is a spin-preserving isomorphism of Riemannian manifolds, the spectral triples $(\mathcal{A}_M, \mathcal{H}_M, D_M)$ and $(\mathcal{A}_N, \mathcal{H}_N, D_N)$ are isomorphic in the category ab-$\mathscr{S}_{Ired}$.*

Proof. The pull-back $\phi := f^\bullet$ is a $*$-isomorphism $\phi : \mathcal{A}_N \to \mathcal{A}_M$ of pre-C*-algebras.

Consider the "pull-back of spinor fields" given by the invertible map $\Psi := \sigma \mapsto \sigma \circ f$, for all $\sigma \in \mathcal{H}_N$. Since f is an orientation-preserving Riemannian isometry, it leaves invariant the volume forms $f^\bullet(\mu_N) = \mu_M$ and so we obtain

$$\int_M \langle \Psi(\sigma)(x) \mid \Psi(\tau)(x) \rangle \, d\mu_M(x) = \int_N \langle \sigma(y) \mid \tau(y) \rangle \, d\mu_N(y)$$

that implies that the map $\Psi : \mathcal{H}_N \to \mathrm{L}^2(M, f^\bullet(S(N))) =: \mathcal{H}^\bullet$ is a unitary operator.
Since $f^\bullet(S(N))$ is a Hermitian bundle over M, $\mathcal{H}^\bullet$ carries a natural representation $\pi^\bullet$ of the algebra $\mathcal{A}_M$ given by pointwise multiplication. Ψ intertwines π_N and $\pi^\bullet \circ \phi$, i.e. $\Psi(\pi_N(a)\sigma) = \pi^\bullet(\phi(a))\Psi(\sigma)$ for $a \in \mathcal{A}_N$ and $\sigma \in \mathcal{H}_N$.

[20] From this we see that the subalgebra $\mathbb{C}\mathrm{l}^{(+)}(f(M)) \subset \mathbb{C}\mathrm{l}^{(+)}(N)$ of sections of the Clifford bundle of N with support in $f(M)$ is naturally isomorphic with the algebra $\mathbb{C}\mathrm{l}^{(+)}(M)$ of sections of the Clifford bundle of M. Since the restriction to $f(M)$ is a natural epimorphism $\rho : \mathbb{C}\mathrm{l}^{(+)}(N) \to \mathbb{C}\mathrm{l}^{(+)}(f(M))$, ($\rho$ acts on Clifford fields by multiplication with the characteristic function of $f(M)$), there is a natural unital epimorphism of algebras $\psi : \mathbb{C}\mathrm{l}^{(+)}(N) \to \mathbb{C}\mathrm{l}^{(+)}(M)$ that becomes an isomorphism when restricted to $\mathbb{C}\mathrm{l}^{(+)}(f(M))$.

Let $U : f^\bullet(S(N)) \to S(M)$ be a (noncanonical) isomorphism of Hermitian bundles induced by the spin-preserving condition on f. Since we know that U is unitary on the fibers, we have $\int_M \langle U\sigma(p) \mid U\tau(p)\rangle_{S_p(M)} \, \mathrm{d}\mu_M(p) = \int_M \langle \sigma(p) \mid \tau(p)\rangle_{f_p^\bullet(S(N))} \, \mathrm{d}\mu_M(p)$, for all $\sigma, \tau \in \Gamma^\infty(f^\bullet(S(N)))$. Hence U uniquely extends to a unitary map $\Theta_U : \mathcal{H}^\bullet \to \mathcal{H}_M$. Note that Θ_U is $\mathcal{A}_M$-linear: $\Theta_U(a \cdot \sigma) = a \cdot \Theta_U(\sigma)$, for $a \in \mathcal{A}_M$ and $\sigma \in \mathcal{H}^\bullet$.

Now it is not difficult to check that the pair $(\phi, \Theta_U \circ \Psi)$ is an isomorphism in the category $\mathscr{S}_{Ired}$ from the spectral triple $(\mathcal{A}_N, \mathcal{H}_N, D_N)$ to $(\mathcal{A}_M, \mathcal{H}_M, D_M)$. □

Proposition 4.3. *The Connes-Rennie-Varilly functor is an embedding of the category $\mathscr{M}_d$-spin into ab-$\mathscr{S}^0_{Ired}$.*

Proof. Let $f : M \to N$ be a spin-preserving metric isometry in $\mathscr{M}_d$-spin. By Lemma 4.1 $f : M \to N$ is a smooth Riemannian isometry onto the closed submanifold $f(M)$, a union of connected components of N.

We denote by $\rho : \mathcal{A}_N \to \mathcal{A}_{f(M)}$ the restriction epimorphism.

The Hilbert space $\mathcal{H}_N = \mathrm{L}^2(N, S(N))$ decomposes as the direct sum $\oplus_{j\in\pi^0(N)}\mathcal{H}_j$ of Hilbert spaces (one for each connected component $j \in \pi^0(N)$ of N) and the multiplication operator P by the characteristic function $\chi_{f(M)}$ is the projection operator onto the subspace $\mathcal{H}_{f(M)} := P(\mathcal{H}_N) = \oplus_{j\in\pi^0(f(M))}\mathcal{H}_j$ (cf. [FGV, Page 491]). Note that, since the Dirac operator D_N is "local" (i.e. it preserves the support of the spinor fields), the subspace $\mathcal{H}_{f(M)}$ is invariant for D_N. In the same way, since J_N and Γ_N acts fiberwise, $\mathcal{H}_{f(M)}$ is invariant for the charge conjugation and grading of N.

Defining $D_{f(M)} := P \circ D_N \circ P$, $J_{f(M)} := P \circ J_N \circ P$ and $\Gamma_{f(M)} := P \circ \Gamma_N \circ P$, it is immediate that $(\mathcal{A}_{f(M)}, \mathcal{H}_{f(M)}, D_{f(M)})$ is a real (even) spectral triple and it follows that the "restriction" map $P : \mathcal{H}_N \to \mathcal{H}_{f(M)}$ satisfies $\forall a \in \mathcal{A}_N$, $\sigma \in \mathcal{H}_N$, $P(a\sigma) = \rho(a)P(\sigma)$, $P \circ D_N = D_{f(M)} \circ P$, $P \circ J_N = J_{f(M)} \circ P$, $P \circ \Gamma_N = \Gamma_{f(M)} \circ P$. This means that the pair (ρ, P) is a morphism in the category $\mathscr{S}_{Ired}$ from $(\mathcal{A}_N, \mathcal{H}_N, D_N)$ to the triple $(\mathcal{A}_{f(M)}, \mathcal{H}_{f(M)}, D_{f(M)})$, which is nothing but the spectral triple obtained from the manifold $f(M)$. By Proposition 4.2, there exists an isomorphism from $(\mathcal{A}_{f(M)}, \mathcal{H}_{f(M)}, D_{f(M)})$ to $(\mathcal{A}_N, \mathcal{H}_N, D_N)$ in the category $\mathscr{S}_{Ired}$, and the conclusion follows by composition with the previous (ρ, P). □

Lemma 4.4. *If M and N are two orientable compact Riemannian spin manifolds in the category $\mathscr{M}$ and (u, U) is an isomorphism from $(\mathcal{A}_N, \mathcal{H}_N, D_N)$ to $(\mathcal{A}_M, \mathcal{H}_M, D_M)$ in the category ab-$\mathscr{S}_{Ire}$, then there is a spin-preserving orientation-preserving Riemannian isometry (metric isometry) $f : M \to N$ such that $f^\bullet = u$.*

Proof. The map $u : \mathcal{A}_N \to \mathcal{A}_M$ naturally extends to a $*$-isomorphisms of C*-algebras and by Gel'fand theorem there exists a homeomorphism $f : M \to N$ such that $f^\bullet = u$. Since $f^\bullet$ maps smooth functions onto smooth functions, f is a diffeomorphism.
The filtered algebra $\Omega_M(\mathcal{A}_M)$ (respectively $\Omega_N(\mathcal{A}_N)$) coincides with the filtered algebra of smooth sections of the Clifford bundle $\mathbb{C}\mathrm{l}^{(+)}(T(M))$ (respectively $\mathbb{C}\mathrm{l}^{(+)}(T(N))$) and the map $\mathrm{Ad}_U : \Omega_{D_N}(\mathcal{A}_N) \to \Omega_{D_M}(\mathcal{A}_M)$ is a filtered isomorphisms (extending $f^\bullet$). Therefore its restricition $\mathrm{Ad}_U : \Omega^1_{D_N}(\mathcal{A}_N) \to \Omega^1_{D_M}(\mathcal{A}_M)$ is an isomorphism between the Hermitian modules of sections of the complexification of the tangent bundles $T(M)$ and $T(N)$.
From Serre-Swan theorem, $Df : T(M) \to T(N)$ is an isomorphism of Euclidean bundles which implies that f is a Riemannian isometry.
Since $\mathrm{Ad}_U(J_N) = J_M$ and $\mathrm{Ad}_U(\Gamma_N) = \Gamma_M$, f is orientation and spin-preserving. □

Proposition 4.5. *The identity functor is an inclusion of the category ab-$\mathscr{S}^0_{Ired}$ into the category ab-$\mathscr{S}^m_d$.*

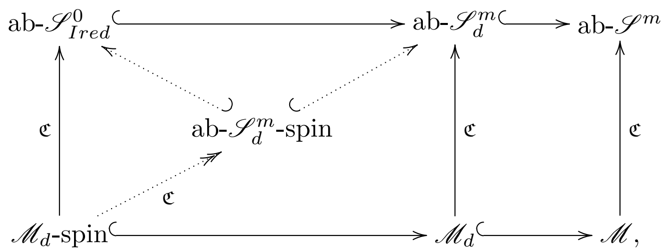
Proof. Let $\phi : (\mathcal{A}_1, \mathcal{H}_1, D_1) \to (\mathcal{A}_2, \mathcal{H}_2, D_2)$ be an isomorphism in the category ab-$\mathscr{S}^0_{Ired}$. By the reconstruction theorem 1.2, there are two manifolds M and N in the category $\mathscr{M}$ such that $(\mathcal{A}_N, \mathcal{H}_N, D_N)$ is isomorphic to $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $(\mathcal{A}_M, \mathcal{H}_M, D_M)$ is isomorphic to $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ with isomorphisms (ϕ_N, U_N) and (ϕ_M, U_M), respectively, in the category $\mathscr{S}_{Ired}$.
By lemma 4.4, $\phi_M \circ \phi \circ \phi_N^{-1} \in \mathscr{S}^0_{Ired}$ is the image under $\mathfrak{C}$ of a spin-preserving Riemannian isometry f that (for manifolds of the same dimension) is a metric isometry in $\mathscr{M}_d$.
Since ϕ_M, ϕ_N are isomorphisms in ab-$\mathscr{S}^0_{Ired}$ and hence, by remark 2.1, isomorphisms also in ab-$\mathscr{S}^m_d$, it follows that $\phi = \phi_M^{-1} \circ \mathfrak{C}(f) \circ \phi_N \in$ ab-$\mathscr{S}^m_d$. □

We can now state the promised equivalence result.

Theorem 4.6. *The Connes-Rennie-Varilly functor is an equivalence between the category $\mathscr{M}_d$-spin and the category ab-$\mathscr{S}^0_{Ired}$.*

Proof. The Connes-Rennie-Varilly functor is already faithful because of proposition 4.3 and representative because of proposition 4.5. We need only to show its fullness.
Let M and N be manifolds in the category $\mathscr{M}_d$-spin and let $\phi : \mathfrak{C}(N) \to \mathfrak{C}(M)$ be a morphism in the category $\mathscr{S}^0_{Ired}$. By proposition 4.5 ϕ is a morphism in the category $\mathscr{S}^m_d$ and from remark 3.3 there exists a metric isometry $f : M \to N$ in the category $\mathscr{M}_d$ such that $\mathfrak{C}(f) = \phi$. Since ϕ defines an isomorphism between $\mathfrak{C}(f(M))$ and $\mathfrak{C}(M)$ in $\mathscr{S}^0_{Ired}$ then, by lemma 4.4, $f : M \to f(M)$ is (orientation and) spin-preserving and we are done. □

Let us summarize the categorical "relations" now available with the commutative diagram of functors



where ab-$\mathscr{S}^m_d$-spin := $\mathfrak{C}(\mathscr{M}_d$-spin). The left and right vertical inclusion functors correspond respectively to the embedding in theorem 4.3 and to the Connes-Rennie-Varilly anti-equivalence in theorem 3.2; the horizontal top-left arrow is the inclusion functor described in proposition 4.5.

Loosely speaking, one would expect a similar structure to carry over to the general non-commutative setting, relating subcategories of "spin-preserving" morphisms in $\mathscr{S}^m$ and "metric-preserving" morphisms in $\mathscr{S}^0_{Ire}$. However, in general things might be more complicated. For the time being, we just mention the following result, omitting the (easy) details of the proof.

Proposition 4.7. *Let $(\mathcal{A}_1, \mathcal{H}_1, D_1) \xrightarrow{(\phi,\Phi)} (\mathcal{A}_2, \mathcal{H}_2, D_2)$ be a morphism of the spectral triples in the category $\mathscr{S}$, where Φ is a coisometry. Then*

$$d_{D_1}(\omega_1 \circ \phi, \omega_2 \circ \phi) \le d_{D_2}(\omega_1, \omega_2), \quad \forall \omega_1, \omega_2 \in \mathcal{S}(\mathcal{A}_2).$$

We have discussed only the case of spin manifolds. We also expect analogous statements to hold true for spin^c manifolds.

5 Final comments

The main result presented in this paper is nothing more than a simple observation on how Gel'fand-Naĭmark duality can be reformulated in the light of Connes' reconstruction theorem for spin Riemannian manifolds. However, it seems to us that the functoriality of the Connes-Rennie-Varilly correspondence has some intriguing appeal and one could ask to which extent it is possible to "lift" it to some of the other main objects entering the scene, notably the Dirac operators. This issue is presently under investigation.
From the perspective of this work, the use of the spin structure has been only instrumental in recasting Gel'fand-Naĭmark theorem in the light of the Connes' reconstruction theorem, and actually it might appear as an unnecessary complication: it introduces some redundancy in the main result and, when incorporated tout-court in the setup, it does not lead to a genuine categorical anti-equivalence.
This might suggest that in a successive step one could try to get rid of such a structure, thus obtaining a different kind of categorical duality between a metric category of (isometries of) Riemannian manifolds and suitable categories of spectral data (for example considering spectral triples arising from the signature Dirac operator in place of those arising from the usual Atiyah-Singer Dirac operator). Although several variants of morphisms can be introduced between spectral triples (see [BCL2, Section 4.1] for details), corresponding to the "rigidity" imposed on the maps between manifolds (totally geodesic isometries, Riemannian isometries, ...), this line of thought does not require significant structural modifications in the definitions of morphisms for the categories of spectral geometries involved (as a pair of maps at the algebra and the Hilbert space level) and will be pursued elsewhere (see [Be] for more details).
The actual construction of functors (and dualities) from categories of spin Riemannian manifolds (with different dimensions) to "suitable" categories of spectral triples (of the Atiyah-Singer "type") is a more interesting goal whose main obstruction is the lack of a sufficiently general notion of pull-back of spinor fields. In order to solve this problem it will be necessary to construct "relational categories" of spectral triples, via "spectral conguences" and/or "spectral spans" following the lines already announced in the seminar slides [Be]. We will return to these topics in forthcoming papers.

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