รายงานวิจัยฉบับสมบูรณ์

โครงการ **แบบแผนทำซ้ำสำหรับปัญหาความเป็นไปได้แบบแยก**

Iterative schemes for solving split feasibility type problems

(ทุนพัฒนานักวิจัย)

โดย

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สนับสนุนโดย

สำนักงานกองทุนสนับสนุนการวิจัย

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บทคัดย่อ: งานวิจัยนี้ได้ศึกษาปัญหาความเป็นไปได้แบบแยกในรูปแบบทั่วไป กล่าวคือศึกษาปัญหาจุดตรึงร่วมแบบแยก นอกจากนี้ยังได้ระเบียบวิธีทำซ้ำในหลายรูปแบบสำหรับอสมการผันแปร ในตอนท้ายของการวิจัยได้ศึกษารูปแบบทั่วไปของตัวดำเนินการแบบตัวตัดในปริภูมิบานาค ผลลัพธ์ที่ได้รับเป็นการพัฒนาและครอบคลุมผลงานของนักคณิตศาสตร์จำนวนมากที่มีการศึกษามาก่อนหน้านี้

คำหลัก: ปัญหาความเป็นไปได้แบบแยก ปัญหาจุดตรึงร่วมแบบแยก อสมการผันแปร ระเบียบวิธีทำซ้ำ

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Abstract: We discuss the split feasibility problem in a very general form, that is, in the context of the split common fixed point problem. We also obtain various iterations for variational inequality problem. Finally, we discuss two classes of generalized cutter operators in Banach spaces. Our results improve and unify the corresponding known results studied by many authors.

Keywords: split feasibility problem, split common fixed point problem, variational inequality, iterative method

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ

สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) และมหาวิทยาลัยขอนแก่น (มข.) ที่ได้ให้โอกาสผู้วิจัยได้รับทุนพัฒนานักวิจัยในการทำงานวิจัยค้นคว้าครั้งนี้

ศาสตราจารย์ ดร. สมพงษ์ ธรรมพงษา ที่อบรมสั่งสอน ถ่ายทอด ความรู้ด้านต่างๆ จนผู้วิจัยสามารถทำงานวิจัยและค้นคว้าได้

Professor Yasunori Kimura สำหรับความร่วมมือในการศึกษาประเด็นที่น่าสนใจในโครงการ

คณะผู้ประเมิน (referee) ของวารสารวิชาการต่าง ๆ ที่ได้ให้คำแนะนำ ตลอดทั้งปรับปรุงต้นฉบับของบทความที่ส่งไปเพื่อตีพิมพ์ในวารสารนั้น ๆ

คณาจารย์ นักศึกษาระดับบัณฑิตศึกษาและเจ้าหน้าที่ฝ่ายสนับสนุน ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยขอนแก่น ที่ได้มีส่วนโครงการวิจัยในครั้งนี้

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Part I

Project Summary

Project Summary

1.1 Hilbert space setting

Throughout this summary, we let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a closed convex subset of $\mathcal{H}$. Let us recall the following two major (nonlinear) problems:

Fixed Point Problem (FPP): Let $T : C \to C$ be a mapping. An element $u \in C$ is a *fixed point* of T if $u = Tu$. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$.

Variational Inequality (VI): Let $A : C \to \mathcal{H}$. An element $u \in C$ is a *solution* of a variational inequality for A if $\langle v - u, Au \rangle \geq 0$ for all $v \in C$. The set of all solutions of a variational inequality for A is denoted by $\mathrm{VI}(C, A)$.

These two problems are related as follows:

FPP $\Longrightarrow$ VI: For a given $T : C \to C$, we have $\mathrm{Fix}(T) = \mathrm{VI}(C, I - T)$.

VI $\Longrightarrow$ FPP: For a given $A : C \to \mathcal{H}$, we have $\mathrm{VI}(C, A) = \mathrm{Fix}(P_C \circ (I - A))$ where P_C is the metric projection from $\mathcal{H}$ onto C.

However, each problem above can be solved in their own way.

In the paper A1, we introduced the concept of a "strongly quasinonexpansive sequence of mappings". This concept is very interesting and plays an important role for proving a strong convergence of Halpern type iterative sequences. This is not only a generalization of many known results in the literature but also give simple proofs of them. For example, we obtain a simple proof of the general iterative method for nonexpansive mappings which was established by Marino and Xu [Journal of Mathematical Analysis and Applications, vol. 318, no. 1, pp. 43–52, 2006.].

With the help of the results in A1, we discuss the split common fixed point problems. All results are presented in the paper A2. First, let us recall this problem. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. For a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ and two quasinonexpansive mappings $U : \mathcal{H}_1 \to \mathcal{H}_1$ and $T : \mathcal{H}_2 \to \mathcal{H}_2$, the *split common fixed point problem* is to find $u \in \mathrm{Fix}(U)$ such that $Au \in \mathrm{Fix}(T)$. This formulation is very general because with appropriate setting we can obtain the following problems as our corollaries:

- The split variational inequality problem studied by Censor, Gibali and Reich [Numer. Algorithms 59 (2012) 301–323]
- The split common null point problem studied by Bryne, Censor, Gibali and Reich [J. Nonlinear Convex Anal. 13 (2012) 759–775]
- Moudafi's split feasibility problem [Nonlinear Anal. 79 (2013) 117–121]

We next consider the problem of finding a common element of the fixed-point set of a certain mapping and the set of solutions of a variational inequality problem. The result for this problem is presented in the paper A3. The scheme in this work is inspired by the recent work of Maingé [A hybrid extragradient-viscosity method for monotone operators and fixed point problems, SIAM J. Control Optim. 47, 1499–1515 (2008)]. We also show that some assumption imposed in his result can be relaxed. Moreover, our scheme is a genuine generalization of Maingé's result because there is a class of mappings to which our scheme is applicable, but which is beyond the scope of his result.

The concept of subgradient extragradient method introduced by Censor, Gibali and Reich [J. Optim. Theory Appl. 148, 318–335 (2011)] is an improvement of that of extragradient method studied by Korpelevich [Èkon. Mat. Metody 12, 747–756 (1976) (in Russian)]. It is known that these two method provides only weak convergence. To obtain a more desirable result, that is, strong convergence, we present two variants of the modified subgradient extragradient method. These results are given in the paper A4.

1.2 Banach space setting

In this project, we also pay attention in a more general setting in Banach spaces. We are interesting in two natural generalizations of cutter operators introduced by Cegielski and Censor [Springer Optimization and its Applications No. 49 (2011)]. We present our results in the paper A5. Throughout this subsection, let E be a Banach space with the dual space E^*. Let $\langle \cdot, \cdot \rangle$ denote the dual pairing acting from $E \times E^*$ to $\mathbb{R}$ and let $J : E \to E^*$ denote the mapping defined by $x \mapsto Jx \in E^*$ where Jx is the element[1] such that

$$\langle x, Jx \rangle = \|x\|^2 = \|Jx\|^2.$$

Let C be a closed convex subset of E. A mapping $T : C \to E$ is said to be

- *a cutter operator of type (P)* if $\mathrm{Fix}(T) \neq \varnothing$ and $\langle Tx - z, J(Tx - x) \rangle \leq 0$ for all $x \in C$ and for all $z \in \mathrm{Fix}(T)$;
- *a cutter operator of type (Q)* if $\mathrm{Fix}(T) \neq \varnothing$ and $\langle Tx - z, JTx - Jx \rangle \leq 0$ for all $x \in C$ and for all $z \in \mathrm{Fix}(T)$.

We obtain two iterative schemes for approximating a common fixed point of these two operators. The first one is based on the Halpern type iteration and the second one is on shrinking projection method of Takahashi–Takeuchi–Kubota.

[1]We assume that E is *smooth*, that is, $\lim_{t \to 0} (1/t)(\|x + ty\| - 1)$ exists for all $x, y \in E$ with $\|x\| = \|y\| = 1$.

1.3 Research outputs

In this project, we published the following 5 papers.

A1: **Saejung, Satit**; Wongchan, Kanokwan. Strong convergence for a strongly quasi-nonexpansive sequence in Hilbert spaces. Abstr. Appl. Anal. 2013, Art. ID 174302, 7 pp. (No impact factor)

A2: Kraikaew, Rapeepan; **Saejung, Satit**. On split common fixed point problems. J. Math. Anal. Appl. 415 (2014), no. 2, 513–524. (Impact Factor (2014) 1.12)

A3: Kraikaew, Rapeepan; **Saejung, Satit**. On a hybrid extragradient-viscosity method for monotone operators and fixed point problems. Numer. Funct. Anal. Optim. 35 (2014), no. 1, 32–49. (Impact Factor (2014) 0.591)

A4: Kraikaew, Rapeepan; **Saejung, Satit**. Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. J. Optim. Theory Appl. 163 (2014), no. 2, 399–412. (Impact Factor (2014) 1.509)

A5: Kimura, Yasunori; **Saejung, Satit**. Strong convergence for a common fixed point of two different generalizations of cutter operators. Linear and Nonlinear Analysis, 1 (2015), no. 1, 53–65. (No impact factor)

Part II

Reprints

A1: Saejung, Satit; Wongchan, Kanokwan. Strong convergence for a strongly quasi-nonexpansive sequence in Hilbert spaces. Abstr. Appl. Anal. 2013, Art. ID 174302, 7 pp.

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Research Article

Strong Convergence for a Strongly Quasi-Nonexpansive Sequence in Hilbert Spaces

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We prove a strong convergence theorem for strongly quasi-nonexpansive sequence of mappings in Hilbert spaces. Moreover, we can improve the recent results of Tian and Jin (2011). We also give a simple proof of Marino-Xu's result (2006).

1. Introduction

Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Recall that a mapping $T : H \to H$ is said to be *L-Lipschitzian* where $L > 0$ if $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in H$. In this paper, we are interested in *nonexpansive mappings* (that is, 1-Lipschitzian ones) and *contractions* (that is, L-Lipschitzian ones with $L < 1$). The problem of finding a fixed point of such mappings plays an important role in many nonlinear equations appearing in both pure and applied sciences. The celebrated Banach's contraction principle is probably known as the major tool for the case of contraction mappings. However, for nonexpansive mappings, the situation is more difficult and different.

In 2000, Moudafi [1] introduced the viscosity approximation method, starting with an arbitrary initial $x_1 \in H$, and defined a sequence $\{x_n\}$ by

$$x_{n+1} = \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n) + \frac{1}{1+\varepsilon_n} Tx_n \quad (n \geq 1), \tag{1}$$

where T is a nonexpansive mapping, $f : H \to H$ is a contraction, and $\{\varepsilon_n\}$ is a sequence in $(0,1)$ satisfying

(M1) $\lim_{n\to\infty}\varepsilon_n = 0$;

(M2) $\sum_{n=1}^{\infty} \varepsilon_n = \infty$;

(M3) $\lim_{n\to\infty}(1/\varepsilon_n) - (1/\varepsilon_{n+1}) = 0$.

It was proved that the sequence $\{x_n\}$ generated by (1) converges to a fixed point z of T and the following inequality holds:

$$\langle f(z) - z, q - z\rangle \leq 0 \quad \forall q \in \operatorname{Fix}(T) := \{x \in H : x = Tx\}. \tag{2}$$

In the literature, Moudafi's scheme has been widely studied and extended (see [2, 3]). It should be noted that the convergence of Moudafi's scheme is equivalent to that of its special setting with a constant contraction f (see [4]). In fact, this follows from the role of the nonexpansiveness of T.

In the earlier result, the following scheme was studied by Halpern [5]; starting with an arbitrary initial $x_1 \in H$ and a given $u \in H$, he defined a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + (1-\alpha_n) Tx_n \quad (n \geq 1), \tag{3}$$

where $\{\alpha_n\}$ is a certain sequence in $(0,1)$. In fact, Halpern proved in 1967 the convergence of the iterative sequence $\{x_n\}$ where $\alpha_n = n^{-\theta}$ and $\theta \in (0,1)$. Many researchers (see, e.g., [6, 7]) have improved Halpern's result from Hilbert spaces to certain Banach spaces with the following conditions on $\{\alpha_n\}$:

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$ or $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$.

Halpern also showed that conditions (C1) and (C2) are necessary for the convergence of the sequence generated by (3) for any given $x_1, u \in H$.

On the other hand, Chidume-Chidume [8] and Suzuki [9] independently discovered that together just conditions (C1) and (C2) are sufficient for the convergence of the following iterative sequence:

$$x_1, u \in C, \quad x_{n+1} = \alpha_n u + (1-\alpha_n) T_\lambda x_n \quad (n \geq 1), \tag{4}$$

where $T_\lambda = \lambda I + (1-\lambda)T$ and $\lambda \in (0,1)$. Recently, Saejung [10] proved that the conclusion remains true if T is a strongly nonexpansive mapping. It is noted that in Hilbert spaces the mapping T_λ is strongly nonexpansive whenever $\lambda \in (0,1)$. Recall that a mapping $T : H \to H$ is *strongly nonexpansive* (see [11, 12]) if it is nonexpansive and $\lim_{n\to\infty}\|(x_n - y_n) - (Tx_n - Ty_n)\| = 0$ whenever $\{x_n\}$, $\{y_n\}$ are sequences in H such that $\{x_n - y_n\}$ is bounded and $\lim_{n\to\infty}(\|x_n - y_n\| - \|Tx_n - Ty_n\|) = 0$.

In the aforementioned results, it was assumed that T has a fixed point; that is, $\mathrm{Fix}(T) \neq \emptyset$. Now we consider the following more general settings. A mapping $T : H \to H$ is

(i) *quasi-nonexpansive* if $\mathrm{Fix}(T) \neq \emptyset$ and $\|Tx - q\| \leq \|x - q\|$ for all $x \in H$ and $q \in \mathrm{Fix}(T)$;

(ii) *strongly quasi-nonexpansive* if it is quasi-nonexpansive and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ whenever $\{x_n\}$ is a bounded sequence in H such that $\lim_{n\to\infty}(\|x_n - q\| - \|Tx_n - q\|) = 0$ for some $q \in \mathrm{Fix}(T)$;

In 2010, Maingé [2] proved the convergence of the sequence $\{x_n\}$ defined by $x_1 \in H$ and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T_\omega x_n, \tag{5}$$

where $T_\omega = (1-\omega)I + \omega T$, $\omega \in (0, 1/2)$ and T is a quasi-nonexpansive mapping under the conditions (C1) and (C2). In 2011, Wongchan and Saejung [13] improved Maingé's result by replacing T_ω with a strongly nonexpansive mapping T. Hence, the restriction $\omega \in (0,1/2)$ can be extended to $\omega \in (0,1)$.

There are also some other iterative schemes closely related to the schemes above studied by many authors. For example, inspired by the scheme studied by Yamada [14], Tian and Jin [15, 16] recently proposed the following iterative scheme, starting with an arbitrary initial $x_1 \in H$ and

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T_\omega x_n \quad (n \geq 1), \tag{6}$$

where f and T_ω are the same as Maingé's result but $F : H \to H$ is strongly monotone and Lipschitzian.

A careful reading shows that there are some connections between them. We will discuss and consolidate them into the following scheme: Started with an arbitrary initial $x_1 \in H$ and

$$x_{n+1} = \alpha_n (f(x_n) + g(T_n x_n)) + (1-\alpha_n) T_n x_n \quad (n \geq 1), \tag{7}$$

where f, g are Lipschitzian and $\{T_n\}$ is a certain sequence of quasi-nonexpansive mappings.

2. Preliminaries

In this section, we collect together some known lemmas which are our main tool in proving our results. Let C be a closed and convex subset of H. Recall that the *metric projection* $P_C : H \to C$ is defined as follows: for $x \in H$, $P_C x$ is the only one point in C satisfying

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}. \tag{8}$$

Lemma 1 (see [17]). *Let C be a nonempty closed convex subset of a Hilbert space H. Then for $x \in H$ and $y \in C$, $y = P_C x$ if and only if $\langle x - y, y - z\rangle \geq 0$ for all $z \in C$.*

Lemma 2. *Let H be a Hilbert space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \tag{9}$$

for all $x, y \in H$.

We also need the following lemma.

Lemma 3 (see [18, Lemma 2.5]). *Let $\{a_n\} \subset [0,\infty)$, $\{\alpha_n\} \subset [0,1)$, and $\{b_n\} \subset (-\infty,\infty)$, $\widehat{\alpha} \in [0,1)$ be such that*

(i) *$\{a_n\}$ is a bounded sequence;*

(ii) *$a_{n+1} \leq (1-\alpha_n)^2 a_n + 2\alpha_n \widehat{\alpha}\sqrt{a_n}\sqrt{a_{n+1}} + \alpha_n b_n$ for all $n \in \mathbb{N}$;*

(iii) *whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$, it follows that $\limsup_{k\to\infty} b_{n_k} \leq 0$;*

(iv) *$\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$.*

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 4 (see [19, Lemma 2.3]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} u_n < \infty$, and $\{t_n\}$ a sequence of real numbers with $\limsup_{n\to\infty} t_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1-\alpha_n) s_n + \alpha_n t_n + u_n \quad \forall n \in \mathbb{N}. \tag{10}$$

Then $\lim_{n\to\infty} s_n = 0$.

3. Main Results

Recall that $\{T_n : H \to H\}$ is a *strongly quasi-nonexpansive sequence* if it satisfies the following conditions:

(1) $\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$;

(2) $\|T_n x - p\| \leq \|x - p\|$ for all $x \in H$ and $p \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$ and for all $n \in \mathbb{N}$;

(3) $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$ whenever $\{x_n\}$ is a bounded sequence in H such that $\lim_{n\to\infty}(\|x_n - q\| - \|T_n x_n - q\|) = 0$ for some $q \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$.

We also say that $\{T_n\}$ satisfies the *NST-condition* if whenever $\{z_n\}$ is a bounded sequence in H such that $\lim_{n\to\infty}\|z_n - T_n z_n\| = 0$ it follows that every weak cluster point of $\{z_n\}$ belongs to $\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$.

Remark 5.

(1) Being strongly nonexpansive the sequence and NST-condition are apparently inherited by subsequences.

(2) Suppose that $T_n = T : H \to H$ for all $n \geq 1$.

(i) If T is a strongly nonexpansive mapping, then $\{T_n\}$ is a strongly nonexpansive sequence.

(ii) If $I - T$ is demiclosed at zero, then $\{T_n\}$ satisfies NST-condition.

Recall that $I - T : H \to H$ is *demiclosed at zero* if $\{x_n\}$ is a sequence in H such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $w-\lim_{n\to\infty} x_n = p$; then $p \in \text{Fix}(T)$.

We now state our main theorem.

Theorem 6. *Let $\{T_n : H \to H\}$ be a strongly quasi-nonexpansive sequence satisfying the NST-condition. Let $f, g : H \to H$ be α- and β-Lipschitzian, respectively. Suppose that $\{x_n\}$ is given by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n \left(f(x_n) + g(T_n x_n)\right) + (1-\alpha_n) T_n x_n \quad (n \geq 1), \tag{11}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the conditions (C1) and (C2). Suppose that $\alpha + \beta < 1$. Then $\{x_n\}$ converges strongly to $p = P_{\bigcap_{n=1}^{\infty}\text{Fix}(T_n)}(f+g)(p)$.

Before we give the proof, we note that $F := \bigcap_{n=1}^{\infty}\text{Fix}(T_n)$ is closed and convex. It follows from $\alpha + \beta < 1$ that $f + g$ is an $(\alpha+\beta)$-contraction. Then the mapping $P_F(f+g) : F \to F$ is a contraction. By Banach's contraction principle, there exists a unique element $p \in F$ such that $p = P_F(f+g)(p)$. It follows then from Lemma 1 that $\langle (f+g)(p) - p, z - p\rangle \leq 0$ for all $z \in F$.

Let us consider the following three lemmas first.

Lemma 7. *The sequence $\{x_n\}$ is bounded. Hence, so are the sequences $\{f(x_n)\}$, $\{T_n x_n\}$, and $\{g(T_n x_n)\}$.*

Proof. We consider the following inequality:

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) + g(T_n x_n) - p\| \\ &\quad + (1-\alpha_n)\|T_n x_n - p\|.\end{aligned} \tag{12}$$

Since each T_n is quasi-nonexpansive and $p \in \bigcap_{n=1}^{\infty}\text{Fix}(T_n)$, we have

$$\|T_n x_n - p\| \leq \|x_n - p\|. \tag{13}$$

It follows from the Lipschitzian conditions of f and g, respectively that,

$$\begin{aligned}&\alpha_n \|f(x_n) + g(T_n x_n) - p\| \\ &\quad \leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|g(T_n x_n) - g(p)\| \\ &\qquad + \alpha_n \|f(p) + g(p) - p\| \\ &\quad \leq \alpha\alpha_n \|x_n - p\| + \beta\alpha_n \|x_n - p\| \\ &\qquad + \alpha_n \|f(p) + g(p) - p\|.\end{aligned} \tag{14}$$

Then, we have

$$\begin{aligned}&\|x_{n+1} - p\| \\ &\quad \leq (1 - \alpha_n(1 - (\alpha+\beta)))\|x_n - p\| \\ &\qquad + \alpha_n (1 - (\alpha+\beta)) \frac{\|f(p) + g(p) - p\|}{1 - (\alpha+\beta)} \\ &\quad \leq \max\left\{\|x_n - p\|, \frac{\|f(p) + g(p) - p\|}{1-(\alpha+\beta)}\right\}.\end{aligned} \tag{15}$$

By induction, for all $n \geq 1$, we have

$$\|x_{n+1} - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|f(p) + g(p) - p\|}{1-(\alpha+\beta)}\right\}. \tag{16}$$

In particular, the sequence $\{x_n\}$ is bounded. □

Lemma 8. *The following inequality holds for all $n \geq 1$:*

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\quad \leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2(\alpha+\beta)\alpha_n \|x_n - p\| \\ &\qquad \times \|x_{n+1} - p\| + 2\alpha_n \langle f(p) + g(p) - p, x_{n+1} - p\rangle.\end{aligned} \tag{17}$$

Proof. It follows from Lemma 2 that

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\quad = \|\alpha_n (f(x_n) + g(T_n x_n) - p) + (1-\alpha_n)(T_n x_n - p)\|^2 \\ &\quad \leq (1-\alpha_n)^2 \|T_n x_n - p\|^2 \\ &\qquad + 2\alpha_n \langle f(x_n) + g(T_n x_n) - p, x_{n+1} - p\rangle.\end{aligned} \tag{18}$$

Since each T_n is quasi-nonexpansive and $p \in \bigcap_{n=1}^{\infty}\text{Fix}(T_n)$,

$$\|T_n x_n - p\|^2 \leq \|x_n - p\|^2. \tag{19}$$

Next, we consider

$$\begin{aligned}&\langle f(x_n) + g(T_n x_n) - p, x_{n+1} - p\rangle \\ &\quad = \langle f(x_n) - f(p), x_{n+1} - p\rangle \\ &\qquad + \langle g(T_n x_n) - g(p), x_{n+1} - p\rangle \\ &\qquad + \langle f(p) + g(p) - p, x_{n+1} - p\rangle \\ &\quad \leq \alpha \|x_n - p\| \|x_{n+1} - p\| + \beta \|x_n - p\| \\ &\qquad \times \|x_{n+1} - p\| + \langle f(p) + g(p) - p, x_{n+1} - p\rangle \\ &\quad = (\alpha+\beta)\|x_n - p\| \|x_{n+1} - p\| \\ &\qquad + \langle f(p) + g(p) - p, x_{n+1} - p\rangle.\end{aligned} \tag{20}$$

Hence, the result follows. □

Lemma 9. *If there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)\geq 0$, then*

$$\limsup_{k\to\infty}\left\langle f(p)+g(p)-p, x_{n_k+1}-p\right\rangle \leq 0. \quad (21)$$

Proof. We note that $\lim_{k\to\infty}\alpha_{n_k}=0$. We consider the following inequality:

$$\begin{aligned}
0 &\leq \liminf_{k\to\infty}\left(\left\|x_{n_k+1}-p\right\|-\left\|x_{n_k}-p\right\|\right)\\
&\leq \liminf_{k\to\infty}\left(\alpha_{n_k}\left\|f\left(x_{n_k}\right)-g\left(T_{n_k}x_{n_k}\right)-p\right\|\right.\\
&\qquad \left.+\left(1-\alpha_{n_k}\right)\left\|T_{n_k}x_{n_k}-p\right\|-\left\|x_{n_k}-p\right\|\right) \quad (22)\\
&\leq \liminf_{k\to\infty}\left(\left\|T_{n_k}x_{n_k}-p\right\|-\left\|x_{n_k}-p\right\|\right)\\
&\leq \limsup_{k\to\infty}\left(\left\|T_{n_k}x_{n_k}-p\right\|-\left\|x_{n_k}-p\right\|\right)\leq 0.
\end{aligned}$$

Then $\lim_{k\to\infty}(\|T_{n_k}x_{n_k}-p\|-\|x_{n_k}-p\|)=0$. Since $\{T_n\}$ is strongly quasi-nonexpansive, so is $\{T_{n_k}\}$. This implies that $\lim_{k\to\infty}\|x_{n_k}-T_{n_k}x_{n_k}\|=0$. Moreover,

$$\begin{aligned}
&\left\|x_{n_k+1}-x_{n_k}\right\|\\
&\quad\leq \left\|x_{n_k+1}-T_{n_k}x_{n_k}\right\|+\left\|T_{n_k}x_{n_k}-x_{n_k}\right\|\\
&\quad= \alpha_{n_k}\left\|f\left(x_{n_k}\right)+g\left(T_nx_{n_k}\right)-T_{n_k}x_{n_k}\right\| \quad (23)\\
&\qquad+\left\|T_{n_k}x_{n_k}-x_{n_k}\right\|.
\end{aligned}$$

Then $\lim_{k\to\infty}\|x_{n_k+1}-x_{n_k}\|=0$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $w-\lim_{l\to\infty}x_{n_{k_l}}=q$ and

$$\begin{aligned}
&\limsup_{k\to\infty}\left\langle f(p)+g(p)-p, x_{n_k}-p\right\rangle\\
&\quad=\lim_{l\to\infty}\left\langle f(p)+g(p)-p, x_{n_{k_l}}-p\right\rangle.
\end{aligned} \quad (24)$$

As $\lim_{k\to\infty}\|x_{n_k}-x_{n_k+1}\|=0$, we have $\limsup_{k\to\infty}\langle f(p)+g(p)-p, x_{n_k+1}-p\rangle=\langle f(p)+g(p)-p,q-p\rangle$. Since $\{T_n\}$ satisfies NST-condition, we have $q\in F$ and hence $\langle f(p)+g(p)-p,q-p\rangle\leq 0$. Therefore,

$$\limsup_{k\to\infty}\left\langle f(p)+g(p)-p, x_{n_k+1}-p\right\rangle\leq 0, \quad (25)$$

as desired. □

Proof of Theorem 6. We are ready to apply Lemma 3. Set

$$\begin{aligned}
a_n &:= \left\|x_n-p\right\|^2,\\
b_n &:= \left\langle f(p)+g(p)-p, x_{n+1}-p\right\rangle, \quad (26)\\
\widehat{\alpha} &:= \alpha+\beta.
\end{aligned}$$

It follows that

(i) $\{a_n\}$ is a bounded sequence (by Lemma 7);

(ii) $a_{n+1}\leq(1-\alpha_n)^2a_n+2\alpha_n\widehat{\alpha}\sqrt{a_n}\sqrt{a_{n+1}}+\alpha_nb_n$ for all $n\geq 1$ (by Lemma 8);

(iii) whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1}-a_{n_k})\geq 0$, it follows that $\limsup_{k\to\infty}b_{n_k}\leq 0$ (by Lemma 9).

Hence, $\lim_{n\to\infty}\|x_n-p\|=\lim_{n\to\infty}a_n=0$. This completes the proof. □

4. Deduced Results

4.1. Wongchan and Saejung's Result. Setting $g\equiv 0$ and $T_n\equiv T$ for all $n\in\mathbb{N}$ in the proof of Theorem 6, we immediately have the following result of Wongchan and Saejung ([13, Theorem 6]).

Corollary 10. *Let C be a closed convex subset of a Hilbert space H and $T:C\to C$ a strongly quasi-nonexpansive mapping such that $I-T$ is demiclosed at zero. Suppose that $f:C\to C$ is a contraction and a sequence $\{x_n\}$ is generated by $x_1\in C$ and*

$$x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n)Tx_n, \quad (27)$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the conditions (C1) and (C2). Then $\{x_n\}$ converges strongly to $p=P_{\mathrm{Fix}(T)}f(p)$.

4.2. Tian and Jin's Result I. Recall that a mapping $F:H\to H$ is *η-strongly monotone* if $\langle x-y,Fx-Fy\rangle\geq\eta\|x-y\|^2$ for all $x,y\in H$.

Lemma 11. *Let $F:H\to H$ be an η-strongly monotone and κ-Lipschitzian mapping. Then $\|(I-\mu F)x-(I-\mu F)y\|\leq\sqrt{1-2\tau}\|x-y\|$ where $\tau=\mu(\eta-(\mu\kappa^2/2))$ for all $x,y\in H$. In particular, if $0<\mu<2\eta/\kappa^2$, then $I-\mu F$ is a contraction.*

Proof. Let $x,y\in H$. Then

$$\begin{aligned}
&\left\|(I-\mu F)x-(I-\mu F)y\right\|^2\\
&\quad=\left\|(x-y)-\mu(Fx-Fy)\right\|^2\\
&\quad=\left\|x-y\right\|^2-2\mu\left\langle x-y,Fx-Fy\right\rangle\\
&\qquad+\mu^2\left\|Fx-Fy\right\|^2 \quad (28)\\
&\quad\leq\left\|x-y\right\|^2-2\mu\eta\left\|x-y\right\|^2+\mu^2\kappa^2\left\|x-y\right\|^2\\
&\quad=\left(1-2\mu\left(\eta-\frac{\mu\kappa^2}{2}\right)\right)\left\|x-y\right\|^2\\
&\quad=(1-2\tau)\left\|x-y\right\|^2.
\end{aligned}$$

□

Theorem 12. *Let $T : H \to H$ be a strongly quasi-nonexpansive mapping such that $I-T$ is demiclosed at zero. Let $F : H \to H$ be an η-strongly monotone and κ-Lipschitzian mapping. Let $f : H \to H$ be an L-Lipschitzian mapping and let a sequence $\{x_n\}$ be generated by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T x_n \quad (n \geq 1), \tag{29}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies the conditions (C1) and (C2). Suppose that $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma L < 1 - \sqrt{1-2\tau}$, where $\tau = \mu(\eta - (\mu\kappa^2/2))$. Then $\{x_n\}$ converges to $p = P_{\mathrm{Fix}(T)}(I - \mu F + \gamma f)p$.

Proof. First we rewrite the iteration (29) as follows:

$$x_{n+1} = \alpha_n \left(\hat{f}(x_n) + \hat{g}(Tx_n)\right) + (1-\alpha_n) T x_n, \tag{30}$$

where $\hat{f} = \gamma f$ and $\hat{g} = I - \mu F$. Note that $\hat{f}$ is a γL-Lipschitzian and $\hat{g}$ is a $\sqrt{1-2\tau}$-Lipschitzian. Using $\gamma L + \sqrt{1-2L} < 1$ and putting $T_n = T$ for all $n \in \mathbb{N}$ in Theorem 6 imply that $\{x_n\}$ converges to $p \in \mathrm{Fix}(T)$, where

$$p = P_{\mathrm{Fix}(T)}\left(\hat{f} + \hat{g}\right)(p) = P_{\mathrm{Fix}(T)}(I - \mu F + \gamma f)(p). \tag{31}$$

□

Lemma 13 (see [12]). *If $T : H \to H$ is a quasi-nonexpansive mapping, then the mapping $T_\omega := (1-\omega)I + \omega T$ is strongly quasi-nonexpansive wherever $\omega \in (0,1)$.*

Using Theorem 12 and Lemma 13, we immediately have the following result which is an improvement of Tian and Jin's result ([15, Theorem 3.1]).

Theorem 14. *Let $T : H \to H$ be a quasi-nonexpansive mapping such that $I-T$ is demiclosed at zero. Let $F : H \to H$ be an η-strongly monotone and κ-Lipschitzian mapping. Let $f : H \to H$ be an L-Lipschitzian mapping and let the sequence $\{x_n\}$ be generated by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T_\omega x_n \quad (n \geq 1), \tag{32}$$

where $T_\omega = (1-\omega)I + \omega T$, $\omega \in (0,1)$ and the sequence $\{\alpha_n\} \subset (0,1)$ satisfies the conditions (C1) and (C2). Suppose that $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma L < 1 - \sqrt{1-2\tau}$ where $\tau = \mu(\eta - (\mu\kappa^2/2))$. Then $\{x_n\}$ converges to $p = P_{\mathrm{Fix}(T)}(I - \mu F + \gamma f)(p)$.

Remark 15. Theorem 14 improves the result of Tian and Jin ([15, Theorem 3.1]) in the following ways.

(i) We assume that $\gamma L < 1 - \sqrt{1-2\tau}$ while [15, Theorem 3.1] is proved under the assumptions $\gamma L < \tau$. We note that $\tau < 1 - \sqrt{1-2\tau}$.

(ii) Our result allows us to choose ω in the wider interval $(0,1)$ while [15, Theorem 3.1] is proved under the assumptions $\omega \in (0, 1/2)$.

4.3. Tian and Jin's Result II. Recall that a mapping $A : H \to H$ is *strongly positive* with the coefficient $\overline{\gamma} > 0$ if

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2 \tag{33}$$

for all $x \in H$.

Lemma 16 (see [20]). *Let A be a strongly positive self-adjoint linear bounded operator with coefficient $\overline{\gamma} > 0$ on H and $0 < \rho \leq \|A\|^{-1}$. Then $\| I - \rho A \| \leq 1 - \rho\overline{\gamma}$.*

Theorem 17. *Let $T : H \to H$ be a strongly quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero. Let $A : H \to H$ be a bounded linear self-adjoint operator and strongly positive with the coefficient $\overline{\gamma}$. Let $f : H \to H$ be an α-contraction mapping and let a sequence $\{x_n\}$ be generated by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n \quad (n \geq 1), \tag{34}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies the conditions (C1) and (C2). Suppose that $0 < \gamma\alpha < \overline{\gamma}$. Then $\{x_n\}$ converges to $p = P_{Fix(T)}(I - A + \gamma f)p$.

Proof. By Lemma 16, we can choose $t \in (0,1)$ such that $\| I - tA \| \leq 1 - t\overline{\gamma}$. Rewrite the iteration (34) as follows:

$$x_{n+1} = \hat{\alpha}_n \left(\hat{f}(x_n) + \hat{g}(Tx_n)\right) + (1 - \hat{\alpha}_n) T x_n, \tag{35}$$

where $\hat{f} := t\gamma f$, $\hat{g} := I - tA$ and $\hat{\alpha}_n \equiv \alpha_n/t$ for all $n \in \mathbb{N}$. Note that $\hat{f}$ is $t\gamma\alpha$-Lipschitzian and $\hat{g}$ is $(1-t\overline{\gamma})$-Lipschitzian. It follows from $0 < \gamma\alpha < \overline{\gamma}$ that

$$t\gamma\alpha + 1 - t\overline{\gamma} = 1 - t\left(\overline{\gamma} - \alpha\gamma\right) < 1. \tag{36}$$

Setting $T_n \equiv T$ for all $n \in \mathbb{N}$ in Theorem 6 implies that $\{x_n\}$ converges to $p \in \mathrm{Fix}(T)$ such that $p = P_{\mathrm{Fix}(T)}(\hat{f} + \hat{g})p = P_{\mathrm{Fix}(T)}(t\gamma f + I - tA)p$; that is, $\langle t\gamma f(p) + p - tAp - p, p - w\rangle \geq 0$ for all $w \in \mathrm{Fix}(T)$. This implies that $\langle \gamma f(p) - Ap, p - w\rangle \geq 0$ for all $w \in \mathrm{Fix}(T)$; that is, $p = P_{\mathrm{Fix}(T)}(\gamma f + I - A)p$. This completes the proof. □

Using Lemma 13 and Theorem 17, we immediately have the following result which is an improvement of Tian and Jin's result ([16, Theorem 3.1]).

Theorem 18. *Let $T : H \to H$ be a quasi-nonexpansive mapping such that $I-T$ is demiclosed at zero. Let $A : H \to H$ be a bounded linear self-adjoint operator and strongly positive with the coefficient $\overline{\gamma}$. Let $f : H \to H$ be an α-contraction mapping, and let the sequence $\{x_n\}$ be generated by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T_\omega x_n \quad (n \geq 1), \tag{37}$$

where $T_\omega = (1-\omega)I + \omega T$, $\omega \in (0,1)$ and the sequence $\{\alpha_n\} \subset (0,1)$ satisfies the conditions (C1) and (C2). Suppose that $0 < \gamma\alpha < \overline{\gamma}$. Then $\{x_n\}$ converges to $p = P_{\mathrm{Fix}(T)}(I - A + \gamma f)p$.

Remark 19. Theorem 18 improves the result of Tian and Jin ([16, Theorem 3.1]). In fact, their result was proved under the assumption $\omega \in (0, 1/2)$ while our result allows us to choose ω in the wider interval $(0,1)$.

5. A Discussion on Marino-Xu's Result

The following theorem is studied by many authors; for example, see [3].

Theorem 20. *Let C be a closed convex subset of a Hilbert space H. Suppose that*

(i) $T : C \to C$ *is a nonexpansive mapping and* $\mathrm{Fix}(T) \neq \emptyset$;

(ii) $\{\alpha_n\} \subset (0,1)$ *is a sequence satisfying the conditions (C1), (C2), and (C3).*

Define the following iterative sequence:

$$u, x_1 \in C, \tag{38}$$

$$x_{n+1} = \alpha_n u + (1-\alpha_n) T x_n. \tag{39}$$

Then $\{x_n\}$ converges to $P_{\mathrm{Fix}(T)} u$.

Using the technique in [4], we can give a simple proof of the following result proved by Marino and Xu [20].

Theorem 21. *Suppose that*

(i) $A : H \to H$ *is a bounded linear self-adjoint operator and it is strongly positive with the coefficient* $\overline{\gamma}$;

(ii) $T : H \to H$ *is a nonexpansive mapping and* $\mathrm{Fix}(T) \neq \emptyset$;

(iii) $f : H \to H$ *is an α-contraction;*

(iv) γ *is a positive number such that* $0 < \gamma\alpha < \overline{\gamma}$;

(v) $\{\alpha_n\} \subset (0,1)$ *is a sequence satisfying the conditions (C1), (C2), and (C3).*

Define the following iterative sequence:

$$z_1 \in H \tag{40}$$

$$z_{n+1} = \alpha_n \gamma f(z_n) + (I - \alpha_n A) T z_n. \tag{41}$$

Then $\{z_n\}$ converges to $\widehat{z} \in \mathrm{Fix}(T)$ and $\langle A\widehat{z} - \gamma f(\widehat{z}), \widehat{z} - w\rangle \le 0$ for all $w \in \mathrm{Fix}(T)$.

Proof. Choose $t \in (0,1)$ such that $\| I - tA \| \le 1 - t\overline{\gamma}$. First we show that $I - tA + t\gamma f$ is a contraction. To see this, let $x, y \in H$. Then

$$\begin{aligned} &\|(I - tA + t\gamma f)x - (I - tA + t\gamma f)y\| \\ &\quad\le \|(I - tA)x - (I - tA)y\| + t\gamma \|f(x) - f(y)\| \\ &\quad\le \|I - tA\| \|x - y\| + t\gamma \|f(x) - f(y)\| \\ &\quad\le (1 - t\overline{\gamma})\|x - y\| + t\gamma\alpha \|x - y\| \\ &\quad= (1 - t(\overline{\gamma} - \gamma\alpha))\|x - y\|. \end{aligned} \tag{42}$$

It follows from $\gamma\alpha < \overline{\gamma}$ that $I - tA + t\gamma f$ is a contraction. Note that $P_{\mathrm{Fix}(T)}$ is nonexpansive and hence $P_{\mathrm{Fix}(T)}(I - tA + t\gamma f)$ is a contraction from $\mathrm{Fix}(T)$ into itself. It follows from the closedness of $\mathrm{Fix}(T)$ and the Banach's contraction principle that there exists a unique element $\widehat{z} \in \mathrm{Fix}(T)$ such that

$$\widehat{z} = P_{\mathrm{Fix}(T)}(I - tA + t\gamma f)(\widehat{z}). \tag{43}$$

Therefore,

$$\langle A\widehat{z} - \gamma f(\widehat{z}), \widehat{z} - w\rangle \le 0 \quad \forall w \in \mathrm{Fix}(T). \tag{44}$$

Now we define the following iterative sequence:

$$\begin{aligned} x_1 &= z_1, \\ x_{n+1} &= \frac{\alpha_n}{t}\left((I - tA)T\widehat{z} + t\gamma f(\widehat{z})\right) + \left(1 - \frac{\alpha_n}{t}\right) T x_n. \end{aligned} \tag{45}$$

It follows from Theorem 20 that the sequence $\{x_n\}$ converges to $\widehat{z} = P_{\mathrm{Fix}(T)}(I - tA + t\gamma f)(\widehat{z})$. Observe that

$$z_{n+1} = \frac{\alpha_n}{t}\left((I - tA)Tz_n + t\gamma f(z_n)\right) + \left(1 - \frac{\alpha_n}{t}\right) T z_n. \tag{46}$$

We next consider the following expression:

$$\begin{aligned} &\|z_{n+1} - x_{n+1}\| \\ &= \left\| \left(1 - \frac{\alpha_n}{t}\right)(Tz_n - Tx_n) + \frac{\alpha_n}{t}(I - tA)(Tz_n - T\widehat{z}) \right. \\ &\qquad \left. + \frac{\alpha_n}{t} t\gamma \left(f(z_n) - f(\widehat{z})\right) \right\| \\ &\le \left(1 - \frac{\alpha_n}{t}\right)\|z_n - x_n\| + \frac{\alpha_n}{t}(1 - t\overline{\gamma})\|z_n - \widehat{z}\| + \alpha_n\gamma\alpha\|z_n - \widehat{z}\| \\ &= \left(1 - \frac{\alpha_n}{t}\right)\|z_n - x_n\| + \left(\frac{\alpha_n}{t} - \alpha_n(\overline{\gamma} - \gamma\alpha)\right)\|z_n - \widehat{z}\| \\ &\le (1 - \alpha_n(\overline{\gamma} - \gamma\alpha))\|z_n - x_n\| + \left(\frac{\alpha_n}{t} - \alpha_n(\overline{\gamma} - \gamma\alpha)\right)\|x_n - \widehat{z}\| \\ &= (1 - \alpha_n(\overline{\gamma} - \gamma\alpha))\|z_n - x_n\| + \alpha_n(\overline{\gamma} - \gamma\alpha) \\ &\qquad \times \left(\frac{(1/t) - (\overline{\gamma} - \gamma\alpha)}{\overline{\gamma} - \gamma\alpha}\right)\|x_n - \widehat{z}\|. \end{aligned} \tag{47}$$

It follows from Lemma 4 that $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Therefore, we conclude that $\{z_n\}$ converges to $\widehat{z} \in \mathrm{Fix}(T)$ and $\langle A\widehat{z} - \gamma f(\widehat{z}), \widehat{z} - w\rangle \le 0$ for all $w \in \mathrm{Fix}(T)$. This completes the proof. □

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On split common fixed point problems

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Based on the convergence theorem recently proved by the second author, we modify the iterative scheme studied by Moudafi for quasi-nonexpansive operators to obtain strong convergence to a solution of the split common fixed point problem. It is noted that Moudafi's original scheme can conclude only weak convergence. As a consequence, we obtain strong convergence theorems for split variational inequality problems for Lipschitz continuous and monotone operators, split common null point problems for maximal monotone operators, and Moudafi's split feasibility problem.

1. Introduction

Let C and Q be closed convex subsets of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The *split feasibility problem* (SFP) which was first introduced by Censor and Elfving [4] is to find

$$\widehat{x} \in C \quad \text{such that} \quad A\widehat{x} \in Q. \tag{1}$$

Suppose that P_C and P_Q are the (orthogonal) projections onto the sets C and Q, respectively. Assuming that SFP is consistent (i.e., (1) has a solution), it is not difficult to see that $\widehat{x} \in \mathcal{H}_1$ solves (1) if and only if it solves the fixed-point equation

$$\widehat{x} = P_C\big(I + \gamma A^*(P_Q - I)A\big)\widehat{x},$$

where $\gamma > 0$ is any positive constant, I is the identity operator and A^* denotes the adjoint of A. To solve (1), in the setting of the finite dimensional case, Byrne [2] proposed the following so-called CQ algorithm:

$$x_{n+1} = P_C\big(x_n + \gamma A^t(P_Q - I)Ax_n\big), \quad n \in \mathbb{N},$$

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where $\gamma \in]0, \frac{2}{L}[$, with L being the largest eigenvalue of the matrix A^tA (t stands for matrix transposition). SFP is important and has been widely studied because it plays a prominent role in the signal processing and image reconstruction problem. Initiated by SFP, several "split type" problems have been investigated and studied, for example, the split variational inequality problem (SVIP) and the split null point problem (SCNP). We will consolidate these problems. Let $U : \mathcal{H}_1 \to \mathcal{H}_1$ and $T : \mathcal{H}_2 \to \mathcal{H}_2$ be two operators with nonempty fixed point sets $\mathrm{Fix}(U) := \{x \in \mathcal{H}_1\colon\ x = Ux\}$ and $\mathrm{Fix}(T)$, respectively. *The split common fixed point problem* (SCFP) is to find

$$\widehat{x} \in \mathrm{Fix}(U) \quad \text{such that} \quad A\widehat{x} \in \mathrm{Fix}(T).$$

If $U := P_C$ and $T := P_Q$, then $\mathrm{Fix}(U) = C$ and $\mathrm{Fix}(T) = Q$ and hence SCFP immediately reduces to SFP. In the case that U and T are directed operators, Censor and Segal [5] proposed and proved, still in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{n+1} = U\big(x_n + \gamma A^t(T - I)Ax_n\big), \quad n \in \mathbb{N},$$

where γ and L are as mentioned before. Note that a class of directed operators includes the metric projections. Hence the result of Censor et al. recovers Byrne's CQ algorithm.

Moudafi [9] recently studied the convergence properties of a relaxed algorithm for SCFP for a class of quasi-nonexpansive operators T such that $I - T$ is demiclosed at zero. He also proved a weak convergence theorem as shown below.

Theorem 1.1. *Given a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$, let $U : \mathcal{H}_1 \to \mathcal{H}_1$ and $T : \mathcal{H}_2 \to \mathcal{H}_2$ be two quasi-nonexpansive operators with nonempty sets* $\mathrm{Fix}(U) = C$ *and* $\mathrm{Fix}(T) = Q$. *Assume that $I - U$ and $I - T$ are demiclosed at zero. Suppose $\Gamma := \{x \in C\colon\ Ax \in Q\} \neq \varnothing$ and define an iterative sequence $\{x_n\}$ by*

$$\begin{cases} x_0 \in \mathcal{H}_1, \\ u_n = x_n + \gamma\beta A^*(T - I)Ax_n, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U(u_n), \end{cases}$$

where $\beta \in]0, 1[$, $\alpha_n \in]0, 1[$ and $\gamma \in]0, \frac{1}{\lambda\beta}[$ with $\lambda = \|A^*A\|$. Then $\{x_n\}$ converges weakly to $\widehat{x} \in \Gamma$ provided that $\alpha_n \in]\delta, 1 - \delta[$ for a small enough $\delta > 0$.

Note that, in the setting of finite dimensional spaces, weak and strong convergences are equivalent. Differently, in infinite dimensional cases, they are not the same. Furthermore, Moudafi's result [9] can guarantee only weak convergence. In most cases, strong convergence is more desirable than weak convergence. In this paper, we slightly modify the algorithm to obtain a strong convergence.

2. Definitions and preliminaries

Throughout, let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. We denote the strong and weak convergence of a sequence $\{x_n\}$ in $\mathcal{H}$ to an element $x \in \mathcal{H}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. For a closed convex subset C of $\mathcal{H}$, the (*metric*) *projection* $P_C : \mathcal{H} \to C$ is defined for each $x \in \mathcal{H}$ as the unique element $P_Cx \in C$ such that

$$\|x - P_Cx\| = \inf\big\{\|x - z\|\colon\ z \in C\big\}.$$

For $x \in \mathcal{H}$ and $y \in C$, it is known that

$$y = P_Cx \quad \Longleftrightarrow \quad \langle y - x, z - y\rangle \geqslant 0 \quad \text{for all } z \in C.$$

In this paper, the fixed-point set of an operator $T : \mathcal{H} \to \mathcal{H}$ is denoted by $\mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) = \{x \in \mathcal{H}\colon\ x = Tx\}$.

Let us recall some definitions of operators involved in our study.

Definition 2.1. An operator $T : \mathcal{H} \to \mathcal{H}$ is called:

- *L-Lipschitzian* if
$$\|Tx - Ty\| \leqslant L\|x - y\| \quad \text{for all } x, y \in \mathcal{H};$$
- *a contraction* if it is α-Lipschitzian with $\alpha \in [0, 1[$, and in this case, we also say that T is a contraction with the coefficient α;
- *nonexpansive* if T is 1-Lipschitzian;
- *quasi-nonexpansive* if $\mathrm{Fix}(T) \neq \varnothing$ and
$$\|Tx - p\| \leqslant \|x - p\| \quad \text{for all } x \in \mathcal{H},\ p \in \mathrm{Fix}(T);$$
equivalently, for all $x \in \mathcal{H}$ and $p \in \mathrm{Fix}(T)$,
$$\langle x - Tx, p - x\rangle \leqslant -\frac{1}{2}\|x - Tx\|^2;$$
- *strongly quasi-nonexpansive* if T is quasi-nonexpansive and
$$x_n - Tx_n \to 0$$
whenever $\{x_n\}$ is a bounded sequence in $\mathcal{H}$ and $\|x_n - p\| - \|Tx_n - p\| \to 0$ for some $p \in \mathrm{Fix}(T)$;
- *monotone* if
$$\langle Tx - Ty, x - y\rangle \geqslant 0 \quad \text{for all } x, y \in \mathcal{H}.$$

Proposition 2.2. *If $T : \mathcal{H} \to \mathcal{H}$ is a nonexpansive operator, then the following inequality holds for all $x, y \in \mathcal{H}$*
$$\big\langle x - y, (I - T)x - (I - T)y\big\rangle \geqslant \frac{1}{2}\big\|(I - T)x - (I - T)y\big\|^2.$$

Proof. Since T is nonexpansive, we have
$$\begin{aligned}\|x - y\|^2 &\geqslant \|Tx - Ty\|^2 \\ &= \big\|(I - T)x - (I - T)y - (x - y)\big\|^2 \\ &= \big\|(I - T)x - (I - T)y\big\|^2 - 2\big\langle x - y, (I - T)x - (I - T)y\big\rangle + \|x - y\|^2.\end{aligned}$$
Therefore we get
$$\big\langle x - y, (I - T)x - (I - T)y\big\rangle \geqslant \frac{1}{2}\big\|(I - T)x - (I - T)y\big\|^2. \qquad \square$$

Corollary 2.3. *Let $S : \mathcal{H} \to \mathcal{H}$ be a quasi-nonexpansive operator and*
$$T := (1 - \alpha)I + \alpha S,$$

for some $\alpha \in]0,1]$. Then, for all $x \in \mathcal{H}$ and $p \in \mathrm{Fix}(T)$, we have the following inequality

$$\langle x - Tx, p - x\rangle \leqslant -\frac{1}{2\alpha}\|x - Tx\|^2.$$

Proof. Obviously, $\mathrm{Fix}(T) = \mathrm{Fix}(S)$. It follows from Proposition 2.2 that

$$\langle x - Tx, p - x\rangle = \alpha\langle x - Sx, p - x\rangle \leqslant -\frac{\alpha}{2}\|x - Sx\|^2 = -\frac{1}{2\alpha}\|x - Tx\|^2.$$

The proof is finished. □

3. Main results

Let us recall first the result proved by the second author.

Theorem 3.1. *(See [13].) Let C be a closed and convex subset of a Hilbert space $\mathcal{H}$ and let $T : C \to C$ be a strongly quasi-nonexpansive operator such that $I - T$ is demiclosed at zero. Suppose that $x_0 \in C$ and $\{x_n\}$ is a sequence generated iteratively by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n,$$

where $\{\alpha_n\}$ is a sequence in $]0,1[$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a fixed point $P_{\mathrm{Fix}(T)}x_0$ of T.

Recall that an operator T is *demiclosed at zero* [15] if

$$Tx = 0 \quad \text{whenever } x_n \rightharpoonup x \text{ and } Tx_n \to 0.$$

3.1. The split common fixed point problem

Throughout this paper, let $\Gamma := \{x \in \mathrm{Fix}(U)\colon\ Ax \in \mathrm{Fix}(T)\}$. It is clear that Γ is closed and convex.

Theorem 3.2. *Let $U : \mathcal{H}_1 \to \mathcal{H}_1$ be a strongly quasi-nonexpansive operator and $T : \mathcal{H}_2 \to \mathcal{H}_2$ be a quasi-nonexpansive operator such that both $I - U$ and $I - T$ are demiclosed at zero. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with $L = \|A^*A\|$. Suppose that $\Gamma \neq \varnothing$. Let $\{x_n\} \subset \mathcal{H}_1$ be a sequence generated by*

$$\begin{cases} x_0 \in \mathcal{H}_1, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)U\big(x_n + \gamma A^*(T - I)Ax_n\big), \end{cases}$$

where the parameter γ and the sequence $\{\alpha_n\}$ satisfy the following conditions:

(a) $\gamma \in]0, \frac{1}{L}[$,
(b) $\{\alpha_n\} \subset]0,1[$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $x_n \to P_\Gamma x_0$.

The following lemma is extracted from Lemma 6.2 of [6] which is needed for proving our main result.

Lemma 3.3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Let $T : \mathcal{H}_2 \to \mathcal{H}_2$ be a nonexpansive operator and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with $L = \|A^*A\|$. For a positive real number γ, define the operator $W : \mathcal{H}_1 \to \mathcal{H}_1$ by*

$$W := I + \gamma A^*(T - I)A.$$

Then the following hold:

- *For all $x, y \in \mathcal{H}_1$,*

$$\|Wx - Wy\|^2 \leqslant \|x - y\|^2 + \gamma(\gamma L - 1)\big\|(T - I)Ax - (T - I)Ay\big\|^2.$$

  In addition, if $T := (1 - \alpha)I + \alpha S$ where $S : \mathcal{H}_2 \to \mathcal{H}_2$ is a nonexpansive operator, then

$$\|Wx - Wy\|^2 \leqslant \|x - y\|^2 + \gamma\left(\gamma L - \frac{1}{\alpha}\right)\big\|(T - I)Ax - (T - I)Ay\big\|^2.$$

- *If $Ax \in \operatorname{Fix}(T)$, then $x \in \operatorname{Fix}(W)$ and the converse holds provided that $\gamma \in \left]0, \frac{1}{L}\right[$.*

Proof. • Let $x, y \in \mathcal{H}_1$. Then we have

$$\begin{aligned}
\|Wx - Wy\|^2 &= \big\|\big(x + \gamma A^*(T - I)Ax\big) - \big(y + \gamma A^*(T - I)Ay\big)\big\|^2 \\
&= \big\|(x - y) + \gamma A^*\big((T - I)Ax - (T - I)Ay\big)\big\|^2 \\
&= \|x - y\|^2 + 2\gamma\big\langle x - y, A^*\big((T - I)Ax - (T - I)Ay\big)\big\rangle + \gamma^2\big\|A^*\big((T - I)Ax - (T - I)Ay\big)\big\|^2 \\
&= \|x - y\|^2 + 2\gamma\big\langle Ax - Ay, (T - I)Ax - (T - I)Ay\big\rangle \\
&\quad + \gamma^2\big\langle A^*\big((T - I)Ax - (T - I)Ay\big), A^*\big((T - I)Ax - (T - I)Ay\big)\big\rangle \\
&= \|x - y\|^2 + 2\gamma\big\langle Ax - Ay, (T - I)Ax - (T - I)Ay\big\rangle \\
&\quad + \gamma^2\big\langle AA^*\big((T - I)Ax - (T - I)Ay\big), (T - I)Ax - (T - I)Ay\big\rangle \\
&\leqslant \|x - y\|^2 + 2\gamma\big\langle Ax - Ay, (T - I)Ax - (T - I)Ay\big\rangle \\
&\quad + \gamma^2\big\|AA^*\big\|\big\|(T - I)Ax - (T - I)Ay\big\|^2.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\|Wx - Wy\|^2 \leqslant{}& \|x - y\|^2 + 2\gamma\big\langle Ax - Ay, (T - I)Ax - (T - I)Ay\big\rangle \\
&+ \gamma^2 L\big\|(T - I)Ax - (T - I)Ay\big\|^2.
\end{aligned} \tag{2}$$

It follows from Proposition 2.2 that

$$\|Wx - Wy\|^2 \leqslant \|x - y\|^2 + \gamma(\gamma L - 1)\big\|(T - I)Ax - (T - I)Ay\big\|^2.$$

Furthermore, if $T := (1 - \alpha)I + \alpha S$ where S is a nonexpansive operator, then

$$\begin{aligned}
\big\langle Ax - Ay, (T - I)Ax - (T - I)Ay\big\rangle &= \alpha\big\langle Ax - Ay, (S - I)Ax - (S - I)Ay\big\rangle \\
&\leqslant \frac{-\alpha}{2}\big\|(I - S)Ay - (I - S)Ax\big\|^2 \\
&= -\frac{1}{2\alpha}\big\|(I - T)Ay - (I - T)Ax\big\|^2.
\end{aligned}$$

Hence from (2) and Proposition 2.2, we obtain

$$\|Wx - Wy\|^2 \leqslant \|x-y\|^2 + \gamma\left(\gamma L - \frac{1}{\alpha}\right)\big\|(T-I)Ax - (T-I)Ay\big\|^2.$$

- It is obvious that $Ax \in \operatorname{Fix}(T)$ implies $x \in \operatorname{Fix}(W)$. To see the converse, let $\gamma \in]0, \frac{1}{L}[$. Let $x \in \operatorname{Fix}(W)$ and $z \in \mathcal{H}_1$ be such that $Az \in \operatorname{Fix}(T)$. It follows that $z \in \operatorname{Fix}(W)$ and hence we get

$$\|x-z\|^2 = \|Wx - Wz\|^2 \leqslant \|x-z\|^2 + \gamma(\gamma L - 1)\big\|(T-I)Ax\big\|^2.$$

Since $\gamma L < 1$, we have $(T-I)Ax = 0$, that is, $Ax \in \operatorname{Fix}(T)$. $\square$

Corollary 3.4. *Let $T : \mathcal{H}_2 \to \mathcal{H}_2$ be a quasi-nonexpansive operator and A, W be operators defined as in Lemma 3.3. Then*

$$\|Wx - z\|^2 \leqslant \|x-z\|^2 + \gamma(\gamma L - 1)\big\|(T-I)Ax\big\|^2,$$

for all $x \in \mathcal{H}_1$ and $z \in \mathcal{H}_1$ such that $Az \in \operatorname{Fix}(T)$.

Lemma 3.5. *Let $U : \mathcal{H}_1 \to \mathcal{H}_1$ be a strongly quasi-nonexpansive operator and $T : \mathcal{H}_2 \to \mathcal{H}_2$ be a quasi-nonexpansive operator. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with $L = \|A^*A\|$. Define the operator $W : \mathcal{H}_1 \to \mathcal{H}_1$ as in Lemma 3.3 where $\gamma L < 1$. Suppose that $\operatorname{Fix}(U) \cap \operatorname{Fix}(W) \neq \varnothing$ and $\{x_n\}$ is a bounded sequence in $\mathcal{H}_1$. Then the following are equivalent:*

(a) $UWx_n - Wx_n \to 0$ *and* $Wx_n - x_n \to 0$;
(b) $UWx_n - x_n \to 0$;
(c) $\|x_n - p\| - \|UWx_n - p\| \to 0$ *for some* $p \in \operatorname{Fix}(U) \cap \operatorname{Fix}(W)$.

Proof. It is obvious that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). We now show that (c) $\Rightarrow$ (a). Suppose that $\|x_n - p\| - \|UWx_n - p\| \to 0$ for some $p \in \operatorname{Fix}(U) \cap \operatorname{Fix}(W)$. By using Corollary 3.4 and the quasi-nonexpansiveness of U, we get

$$\|UWx_n - p\| \leqslant \|Wx_n - p\| \leqslant \|x_n - p\|.$$

Therefore we have $\|Wx_n - p\| - \|UWx_n - p\| \to 0$. Since U is strongly quasi-nonexpansive, we have $UWx_n - Wx_n \to 0$. Notice that $\|x_n - p\|^2 - \|UWx_n - p\|^2 \to 0$. Using Corollary 3.4 again gives

$$\gamma(1-\gamma L)\big\|(T-I)Ax_n\big\|^2 \leqslant \|x_n - p\|^2 - \|UWx_n - p\|^2 \to 0.$$

Since $\gamma L < 1$, we get $Wx_n - x_n = \gamma A^*(T-I)Ax_n \to 0$. Then (a) is satisfied and the proof is finished. $\square$

Proof of Theorem 3.2. To conclude the result, by using Theorem 3.1, it suffices to show that:

(♠) the operator UW is strongly quasi-nonexpansive, where $W := I + \gamma A^*(T-I)A$;
(♡) $I - UW$ is demiclosed at zero.

We first note that $\Gamma = \operatorname{Fix}(U) \cap \operatorname{Fix}(W) = \operatorname{Fix}(UW)$. Indeed, it follows from Lemma 3.3 that

$$\begin{aligned}
\Gamma &= \big\{x \in \mathcal{H}_1\colon\ x \in \operatorname{Fix}(U) \text{ and } Ax \in \operatorname{Fix}(T)\big\} \\
&= \big\{x \in \mathcal{H}_1\colon\ x \in \operatorname{Fix}(U) \text{ and } x \in \operatorname{Fix}(W)\big\} \\
&= \operatorname{Fix}(U) \cap \operatorname{Fix}(W).
\end{aligned}$$

Then $\mathrm{Fix}(U) \cap \mathrm{Fix}(W) \neq \varnothing$. We next show that $\mathrm{Fix}(U) \cap \mathrm{Fix}(W) = \mathrm{Fix}(UW)$. To see this, it suffices to show $\mathrm{Fix}(UW) \subset \mathrm{Fix}(U) \cap \mathrm{Fix}(W)$. Then let $p \in \mathrm{Fix}(U) \cap \mathrm{Fix}(W)$ and $x \in \mathrm{Fix}(UW)$. By using Lemma 3.5 with $x_n \equiv x$, we get that $Wx = x$ and $UWx = Wx$, that is, $x \in \mathrm{Fix}(U) \cap \mathrm{Fix}(W)$. So our assertion is obtained. Combining this fact with Lemma 3.5, we have (♠). To prove (♡), let $\{x_n\}$ be a sequence such that $x_n - UWx_n \to 0$ and $x_n \rightharpoonup x$ for some $x \in \mathcal{H}_1$. It follows from Lemma 3.5 that $x_n - Wx_n \to 0$ and $y_n - Uy_n \to 0$ where $y_n \equiv Wx_n$. Notice that $y_n \rightharpoonup x$. Since $I - U$ and $I - T$ are demiclosed at zero, we have $x \in \mathrm{Fix}(W) \cap \mathrm{Fix}(U) = \mathrm{Fix}(UW)$. □

4. Another split problems deduced from SCFP

4.1. The split variational inequality problem

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Given operators $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$, a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ and nonempty closed convex subsets $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$, the *split variational inequality problem* (SVIP) is the problem of finding a point $\widehat{x} \in \mathrm{VIP}(C, f)$ such that $A\widehat{x} \in \mathrm{VIP}(Q, g)$, that is,

$$\begin{cases} \widehat{x} \in C \quad \text{such that} \quad \langle f(\widehat{x}), x - \widehat{x} \rangle \geqslant 0 \quad \text{for all } x \in C, \\ \widehat{y} := A\widehat{x} \in Q \quad \text{such that} \quad \langle g(\widehat{y}), y - \widehat{y} \rangle \geqslant 0 \quad \text{for all } y \in Q. \end{cases}$$

This is equivalent to the problem of finding $\widehat{x} \in \mathrm{Fix}(P_C(I - \lambda f))$ such that $A\widehat{x} \in \mathrm{Fix}(P_Q(I - \lambda g))$ where $\lambda > 0$. We denote the set of solutions by $\mathrm{SVIP}(A, C, Q, f, g)$. Therefore SVIP can be viewed as SCFP. Under appropriate conditions of the operators f and g, we can apply our result for SVIP.

In the work of Censor et al. [6], the operators f and g are assumed to be *α-inverse strongly monotone* where $\alpha > 0$, that is,

$$\langle x - y, f(x) - f(y) \rangle \geqslant \alpha \|f(x) - f(y)\|^2 \quad \text{and} \quad \langle u - v, g(u) - g(v) \rangle \geqslant \alpha \|g(u) - g(v)\|^2,$$

for all $x, y \in \mathcal{H}_1$ and $u, v \in \mathcal{H}_2$. It is known that if f is α-inverse strongly monotone and $\lambda \in]0, 2\alpha[$ then $P_C(I - \lambda f)$ is strongly quasi-nonexpansive and $I - P_C(I - \lambda f)$ is demiclosed at zero. Hence their result becomes a special case of ours. However, since every α-inverse strongly monotone operator is monotone and Lipschitz continuous, the latter class of operators is then more general. It is worth noting that there exists a monotone Lipschitz continuous operator f such that $P_C(I - \lambda f)$ fails to be quasi-nonexpansive [7]. Thanks to the extragradient method introduced by Korpelevič [8], we obtain a slight modification of such operators and prove a strong convergence theorem for SVIP in the case when f and g are monotone and Lipschitz continuous. More precisely, the following corollary is established.

Corollary 4.1. *Let C and Q be nonempty closed convex subsets of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ be monotone and κ-Lipschitz continuous operators on C and Q, respectively and $A : \mathcal{H}_1 \to \mathcal{H}_2$ a bounded linear operator with $\|A^*A\| = L$. Suppose that* $\mathrm{SVIP}(A, C, Q, f, g) \neq \varnothing$. *Define an iterative sequence $\{x_n\} \subset \mathcal{H}_1$ by*

$$\begin{cases} x_0 \in \mathcal{H}_1, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) U(x_n + \gamma A^*(T - I)Ax_n), \end{cases}$$

where $\gamma \in]0, \frac{1}{L}[$,

$$\begin{aligned} U &:= P_C(I - \lambda f P_C(I - \lambda f)), \\ T &:= P_Q(I - \lambda g P_Q(I - \lambda g)), \end{aligned} \tag{3}$$

and $\lambda \in]0, \frac{1}{\kappa}[$. Then the sequence $\{x_n\}$ converges strongly to $\widehat{x} \in \mathrm{SVIP}(A, C, Q, f, g)$.

Before giving a proof, we present the following two lemmas.

Lemma 4.2. *Let $f : \mathcal{H} \to \mathcal{H}$ be monotone and κ-Lipschitz continuous on C. Let $S := P_C(I - \lambda f)$ where $\lambda > 0$. If $\{x_n\}$ is a sequence in C satisfying $x_n \rightharpoonup \widehat{x}$ and $x_n - Sx_n \to 0$, then $\widehat{x} \in \mathrm{VIP}(C, f)$.*

Proof. Since f is monotone and continuous, we have (see e.g., [14])

$$\widehat{x} \in \mathrm{VIP}(C, f) \quad \iff \quad \langle f(x), x - \widehat{x} \rangle \geqslant 0 \quad \text{for all } x \in C.$$

Let $x \in C$. Note that

$$\langle x_n - \lambda f(x_n) - Sx_n, Sx_n - x \rangle \geqslant 0 \quad \text{for all } n \in \mathbb{N}.$$

Next, we consider

$$\begin{aligned}
\langle \lambda f(x), x_n - x \rangle &\leqslant \langle \lambda f(x_n), x_n - x \rangle \\
&= \langle \lambda f(x_n), x_n - Sx_n \rangle + \langle \lambda f(x_n), Sx_n - x \rangle \\
&= \langle \lambda f(x_n), x_n - Sx_n \rangle - \langle x_n - \lambda f(x_n) - Sx_n, Sx_n - x \rangle + \langle x_n - Sx_n, Sx_n - x \rangle \\
&\leqslant \langle \lambda f(x_n), x_n - Sx_n \rangle + \langle x_n - Sx_n, Sx_n - x \rangle \\
&\leqslant \lambda \| f(x_n) \| \| x_n - Sx_n \| + \| x_n - Sx_n \| \| Sx_n - x \|.
\end{aligned}$$

Hence

$$\langle f(x), x_n - x \rangle \leqslant \| f(x_n) \| \| x_n - Sx_n \| + \frac{1}{\lambda} \| x_n - Sx_n \| \| Sx_n - x \|.$$

Since $\{f(x_n)\}$ is bounded, $x_n - Sx_n \to 0$ and $x_n \rightharpoonup \widehat{x}$, we have

$$\langle f(x), \widehat{x} - x \rangle = \lim_{n \to \infty} \langle f(x), x_n - x \rangle \leqslant 0.$$

The proof is finished. □

The following lemma is extracted from [12].

Lemma 4.3. *Let $f : \mathcal{H} \to \mathcal{H}$ be a monotone and κ-Lipschitz operator on C and λ be a positive number. Let $V := P_C(I - \lambda f)$ and $S := P_C(I - \lambda f V)$. Then, for all $q \in \mathrm{VIP}(C, f)$, we have*

$$\| Sx - q \|^2 \leqslant \| x - q \|^2 - (1 - \lambda^2 \kappa^2) \| x - Vx \|^2.$$

In particular, if $\kappa\lambda < 1$, then S is a strongly quasi-nonexpansive operator and $\mathrm{Fix}(S) = \mathrm{Fix}(V) = \mathrm{VIP}(C, f)$.

Proof. Let $q \in \mathrm{VIP}(C, f)$. Note that

$$\begin{aligned}
\| Sx - q \|^2 &\leqslant \| (x - \lambda f(Vx)) - q \|^2 - \| (x - \lambda f(Vx)) - Sx \|^2 \\
&= \| x - q \|^2 + 2\lambda \langle q - Sx, f(Vx) \rangle - \| x - Sx \|^2 \\
&= \| x - q \|^2 + 2\lambda \langle q - Vx, f(Vx) - f(q) \rangle \\
&\quad + 2\lambda \langle q - Vx, f(q) \rangle + 2\lambda \langle Vx - Sx, f(Vx) \rangle - \| x - Sx \|^2 \\
&\leqslant \| x - q \|^2 + 2\lambda \langle Vx - Sx, f(Vx) \rangle - \| x - Sx \|^2
\end{aligned}$$

$$= \|x-q\|^2 + 2\lambda\big\langle Vx - Sx, f(Vx)\big\rangle - \|x - Vx\|^2 - 2\langle x - Vx, Vx - Sx\rangle - \|Vx - Sx\|^2$$
$$= \|x-q\|^2 - \|x - Vx\|^2 - \|Vx - Sx\|^2 + 2\big\langle x - \lambda f(Vx) - Vx, Sx - Vx\big\rangle.$$

Now we estimate the last term of the preceding expression

$$\begin{aligned}\big\langle x - \lambda f(Vx) - Vx, Sx - Vx\big\rangle &= \big\langle x - \lambda f(x) - Vx, Sx - Vx\big\rangle + \big\langle \lambda f(x) - \lambda f(Vx), Sx - Vx\big\rangle \\ &\leqslant \big\langle \lambda f(x) - \lambda f(Vx), Sx - Vx\big\rangle \\ &\leqslant \lambda\kappa\|x - Vx\|\|Sx - Vx\|.\end{aligned}$$

So we have

$$\begin{aligned}\|Sx - q\|^2 &\leqslant \|x-q\|^2 - \|x - Vx\|^2 - \|Vx - Sx\|^2 + 2\lambda\kappa\|x - Vx\|\|Sx - Vx\| \\ &\leqslant \|x-q\|^2 - \|x - Vx\|^2 - \|Vx - Sx\|^2 + \lambda^2\kappa^2\|x - Vx\|^2 + \|Sx - Vx\|^2 \\ &= \|x-q\|^2 - \big(1 - \lambda^2\kappa^2\big)\|x - Vx\|^2.\end{aligned}$$

Assume further that $\kappa\lambda < 1$ and let $\{x_n\}$ be a sequence in $\mathcal{H}$ such that $\|Sx_n - q\| - \|x_n - q\| \to 0$ for some $q \in \mathrm{Fix}(S)$. It follows from the above inequality that $x_n - Vx_n \to 0$ which can be easily deduced to $x_n - Sx_n \to 0$. Therefore S is strongly quasi-nonexpansive and it is not difficult to see that $\mathrm{Fix}(S) = \mathrm{Fix}(V) = \mathrm{VIP}(C, f)$. □

Proof of Corollary 4.1. It follows from Lemma 4.3 that both operators U and T defined in (3) are strongly quasi-nonexpansive. We next show that $I - U$ is demiclosed at zero. Let $\{x_n\}$ be a sequence in $\mathcal{H}_1$ such that $x_n - Ux_n \to 0$ and $x_n \rightharpoonup x$. Notice that $\|x_n - q\|^2 - \|Ux_n - q\|^2 \to 0$ for some $q \in \mathrm{VIP}(C, f)$. Using Lemma 4.3, we get

$$\big(1 - \lambda^2\kappa^2\big)\big\|x_n - P_C(I - \lambda f)x_n\big\|^2 \leqslant \|x_n - q\|^2 - \|Ux_n - q\|^2 \to 0.$$

Thus $x_n - P_C(I - \lambda f)x_n \to 0$. Therefore, by Lemma 4.2, we get $x \in \mathrm{VIP}(C, f) = \mathrm{Fix}(U)$. Similarly, $I - T$ is also demiclosed at zero. Then the result follows from Theorem 3.2. □

4.2. The split common null point problem

Given two set-valued operators $B_1 \subset \mathcal{H}_1 \times \mathcal{H}_1$ and $B_2 \subset \mathcal{H}_2 \times \mathcal{H}_2$ and a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$, the *split common null point problem* (SCNP) is the problem of finding

$$\widehat{x} \in \mathcal{H}_1 \quad \text{such that} \quad 0 \in B_1(\widehat{x}) \text{ and } 0 \in B_2(A\widehat{x}). \tag{4}$$

Recently, Byrne et al. [3] proposed a strong convergence theorem for finding such a solution $\widehat{x}$ when B_1 and B_2 are maximal monotone. Recall that $B \subset \mathcal{H} \times \mathcal{H}$ is:

- *monotone* if $\langle x - y, u - v\rangle \geqslant 0$ for all $(x, u) \in B_1$ and $(y, v) \in B_2$;
- *maximal monotone* if it is monotone and its graph is not properly contained in the graph of any other monotone operator.

For a maximal monotone operator $B \subset \mathcal{H} \times \mathcal{H}$ and $\lambda > 0$, we can define a single-valued operator

$$J_\lambda^B =: (I + \lambda B)^{-1} : \mathcal{H} \to \mathcal{H}.$$

It is known that J_λ^B is *firmly nonexpansive*, that is, for all $x, y \in \mathcal{H}$,

$$\langle x - y, J_\lambda^B x - J_\lambda^B y \rangle \geqslant \left\| J_\lambda^B x - J_\lambda^B y \right\|^2,$$

and

$$0 \in B(\widehat{x}) \quad \Longleftrightarrow \quad \widehat{x} \in \operatorname{Fix}\bigl(J_\lambda^B\bigr).$$

Therefore, the problem (4) is equivalent to the problem of finding

$$\widehat{x} \in \mathcal{H}_1 \quad \text{such that} \quad \widehat{x} \in \operatorname{Fix}\bigl(J_\lambda^{B_1}\bigr) \text{ and } A\widehat{x} \in \operatorname{Fix}\bigl(J_\lambda^{B_2}\bigr),$$

where λ is a positive real number, that is, the SCNP reduces to the SCFP.

The result of Byrne et al. [3] is a consequence of our Theorem 3.2.

Corollary 4.4. *(See [3].) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Given two set-valued maximal monotone operators $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ and a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ with $L = \|A^*A\|$, we define an iterative sequence $\{x_n\}$ by*

$$\begin{cases} x_0 \in \mathcal{H}_1, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) J_\lambda^{B_1}\bigl(x_n + \gamma A^*\bigl(J_\lambda^{B_2} - I\bigr)Ax_n\bigr), \end{cases} \tag{5}$$

where the parameters λ, γ and the sequence $\{\alpha_n\}$ satisfy the following conditions:

(a) $\lambda > 0$, $\gamma \in \,]0, \frac{2}{L}[$,
(b) $\{\alpha_n\} \subset \,]0, 1[$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^\infty \alpha_n = \infty$.

Suppose that the solution set of (4), says Γ, is nonempty. Then $x_n \to \widehat{x} \in \Gamma$.

Remark 4.5.

(1) Notice that Corollary 4.4 can be viewed as a corollary of our Theorem 3.2 for the following reasons.
  (a) For a maximal monotone B and $\lambda > 0$, it is known that J_λ^B is firmly nonexpansive and hence nonexpansive. Moreover, $I - J_\lambda^B$ is demiclosed at zero [1] and

$$J_\lambda^B = \frac{1}{2}I + \frac{1}{2}S,$$

  for some nonexpansive operator $S : \mathcal{H} \to \mathcal{H}$.
  (b) For B_2 and A defined as in Corollary 4.4, it follows from Lemma 3.3 with $\alpha = \frac{1}{2}$ that

$$\|Wx - y\|^2 \leqslant \|x - y\|^2 + \gamma(\gamma L - 2)\left\|\bigl(J_\lambda^{B_2} - I\bigr)Ax\right\|^2,$$

  for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$ such that $Ay \in \operatorname{Fix}(J_\lambda^{B_2})$ where $W := I + \gamma A^*(J_\lambda^{B_2} - I)A$. So, in this case, the parameter γ can be relaxed, that is, $\gamma \in \,]0, \frac{2}{L}[$ instead of $]0, \frac{1}{L}[$.
(2) Our Theorem 3.2 allows the parameter λ for $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ in Corollary 4.4 to be chosen differently.
(3) The strong limit $\widehat{x}$ of the sequence $\{x_n\}$ generated by (5) is indeed the nearest point projection of x_0 onto the solution set Γ.

4.3. Moudafi's split feasibility problem

Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be Hilbert spaces and $C \subset \mathcal{H}_1$, $Q \subset \mathcal{H}_2$ be nonempty closed convex sets. Let $A : \mathcal{H}_1 \to \mathcal{H}_3$, $B : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators. *Moudafi's split feasibility problem* [10,11] is the problem of finding

$$x \in C \text{ and } y \in Q \quad \text{such that} \quad Ax = By. \tag{6}$$

We will transform this problem into the original SFP. Let us denote

$$\begin{aligned}
&\mathbf{H_1} := \mathcal{H}_1 \times \mathcal{H}_2,\\
&\mathbf{H_2} := \mathcal{H}_3 \times \mathcal{H}_3,\\
&\mathbf{C} := C \times Q \subset \mathbf{H_1},\\
&\mathbf{Q} := \bigl\{(z, w) \in \mathbf{H_2}\colon\ z = w\bigr\}.
\end{aligned}$$

Define a linear operator $\mathbf{A} : \mathbf{H_1} \to \mathbf{H_2}$ by

$$\mathbf{A}(x, y) = (Ax, By) \quad \text{for all } (x, y) \in \mathbf{H_1}.$$

If the set $\Gamma := \{(x, y) \in \mathbf{C}\colon\ \mathbf{A}(x, y) \in \mathbf{Q}\}$ is nonempty, then $(x, y) \in \mathbf{H_1}$ solves (6) if and only if

$$(x, y) = P_{\mathbf{C}}\bigl(I + \gamma \mathbf{A}^*(P_{\mathbf{Q}} - I)\mathbf{A}\bigr)(x, y).$$

Note that:

- $P_{\mathbf{C}}(x, y) = (P_C x, P_Q y)$ for all $(x, y) \in \mathbf{H_1}$;
- $P_{\mathbf{Q}}(z, w) = (\frac{z+w}{2}, \frac{z+w}{2})$ for all $(z, w) \in \mathbf{H_2}$;
- $\mathbf{A}^*(z, w) = (A^* z, B^* w)$ for all $(z, w) \in \mathbf{H_2}$.

As a consequence of our Theorem 3.2, the following iterative sequence $\{(x_n, y_n)\}$ defined by

$$\begin{cases}
x_0 \in \mathcal{H}_1,\\
y_0 \in \mathcal{H}_2,\\
x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) P_C\left(x_n + \dfrac{\gamma}{2} A^*(By_n - Ax_n)\right),\\
y_{n+1} = \alpha_n y_0 + (1 - \alpha_n) P_Q\left(y_n + \dfrac{\gamma}{2} B^*(Ax_n - By_n)\right),
\end{cases}$$

converges strongly to $(\widehat{x}, \widehat{y})$ which simultaneously solves Moudafi's split feasibility problem (6) and is nearest to the initial guess (x_0, y_0).

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ON A HYBRID EXTRAGRADIENT-VISCOSITY METHOD FOR MONOTONE OPERATORS AND FIXED POINT PROBLEMS

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□ *The purpose of this article is to give a more general scheme for approximating a common element of the fixed-point set of a certain mapping and the set of solutions of a variational inequality problem. This scheme is inspired by the recent work of Maingé [A hybrid extragradient-viscosity method for monotone operators and fixed point problems, SIAM J. Control Optim. 47, 1499–1515 (2008)]. We also show that some assumption imposed in his result can be relaxed. Moreover, our scheme is a genuine generalization of Maingé's result because there is a class of mappings to which our scheme is applicable, but which is beyond the scope of his result.*

Keywords Extragradient method; Fixed-point problem; Hybrid steepest descent; Monotone mapping; Variational inequality.

Mathematics Subject Classification 47H05; 47H10; 90C25.

1. INTRODUCTION

Variational inequality problems for monotone mappings play an important role in many branches in pure and applied sciences. To solve these problems, various iterative methods have been proposed and studied by many authors in the literature. The purpose of this article is to give a short and simple proof of the recent method proposed by Maingé [12].

In this article, let $\mathscr{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let C be a closed and convex subset of $\mathscr{H}$. The *variational inequality problem* for a given mapping $A : C \to \mathscr{H}$ is the problem of finding an element $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0 \quad \forall x \in C. \tag{1}$$

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We denote the solution set of the problem above by $\mathrm{VIP}(C, A)$, that is, $x^* \in \mathrm{VIP}(C, A)$ if and only if $x^* \in C$ and (1) holds. Usually, the variational inequality problem above is treated as the *fixed point problem* for a certain mapping $T : C \to C$, that is, the problem of finding an element $x^* \in C$ such that $x^* = Tx^*$. In fact, it is noted that for a number $\lambda > 0$

$$
\begin{aligned}
&\langle Ax^*, x - x^* \rangle \geq 0 \, \forall x \in C \\
&\quad \Leftrightarrow \langle x^* - (x^* - \lambda Ax^*), x - x^* \rangle \geq 0 \quad \forall x \in C \\
&\quad \Leftrightarrow x^* = P_C(I - \lambda A)x^*,
\end{aligned}
$$

where P_C is the metric projection from $\mathscr{H}$ onto C and I is the identity mapping. To simplify the notation, let $\mathrm{Fix}(T)$ denote the set of fixed points of T. It follows then that

$$
\mathrm{VIP}(C, A) = \mathrm{Fix}(P_C(I - \lambda A)) \quad \forall \lambda > 0.
$$

Let us recall the following three interesting methods for variational inequality problems and fixed point problems which have been studied by many researchers in the literature. The related definitions and notions will be given in Section 2.

1.1. Korpelevich's Extragradient Method

Suppose that the mapping $A : C \to \mathscr{H}$ is

- *monotone*, that is, $\langle Ax - Ay, x - y \rangle \geq 0$ for all $x, y \in C$;
- *κ-Lipschitz continuous* where $\kappa > 0$, that is, $\|Ax - Ay\| \leq \kappa \|x - y\|$ for all $x, y \in C$.

Korpelevich [11] proposed the following so-called extragradient method for finding an element in $\mathrm{VIP}(C, A)$:

$$
\begin{cases}
x_0 \in C \\
y_n = P_C(x_n - \lambda Ax_n) \\
x_{n+1} = P_C(x_n - \lambda Ay_n),
\end{cases}
\tag{2}
$$

where the stepsize $\lambda \in (0, 1/\kappa)$. It is known that if $\mathrm{VIP}(C, A) \neq \varnothing$, the sequence $\{x_n\}$ generated by (2) converges *weakly* to an element in $\mathrm{VIP}(C, A)$. Recently, Censor et al. had made great progress in this method (see [4–6]).

1.2. Moudafi's Viscosity Method

Let $T : C \to C$ be a *nonexpansive* mapping, that is, it is 1-Lipschitz continuous. Moudafi [14] proposed the so-called viscosity method for the fixed point problem for T as follows:

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \end{cases} \tag{3}$$

where $f : C \to C$ is a *strict contraction*, that is, f is L-Lipschitz continuous mapping with $0 \le L < 1$, and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C2) $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$.

It is proved that if $\mathrm{Fix}(T) \neq \varnothing$, then the sequence $\{x_n\}$ generated by (3) converges *strongly* to an element $z \in \mathrm{VIP}(\mathrm{Fix}(T), I - f)$, that is, $z \in \mathrm{Fix}(T)$ and $\langle z - f(z), q - z\rangle \ge 0$ for all $q \in \mathrm{Fix}(T)$. In this setting, it is known that $\mathrm{Fix}(T)$ is closed and convex and $\mathrm{VIP}(\mathrm{Fix}(T), I - f) = \{z\}$. Let us note that Moudafi's viscosity method is a variant of Halpern's method [9].

1.3. Yamada's Hybrid Steepest Descent Method

Let $T : \mathscr{H} \to \mathscr{H}$ be a nonexpansive mapping and $A : \mathscr{H} \to \mathscr{H}$ be an L-Lipschitz continuous and *strongly monotone* mapping, that is, there exists an $\eta > 0$ such that $\langle Ax - Ay, x - y\rangle \ge \eta\|x - y\|^2$ for all $x, y \in \mathscr{H}$ (in this case, we also say that A is η-strongly monotone). Yamada [20] proposed the so-called hybrid steepest descent method for the fixed point problem for T as follows:

$$\begin{cases} x_0 \in \mathscr{H} \\ x_{n+1} = Tx_n - \alpha_n ATx_n, \end{cases} \tag{4}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the conditions (C1) and (C2) as in Moudafi's method. It is proved that if $\mathrm{Fix}(T) \neq \varnothing$, then the sequence $\{x_n\}$ generated by (4) converges *strongly* to an element $z \in \mathrm{VIP}(\mathrm{Fix}(T), A)$.

1.4. Maingé's Recent Result

Recall the following concept: A mapping $T : \mathscr{H} \to \mathscr{H}$ is called *β-demicontractive*, where $\beta \in [0, 1)$ if $\mathrm{Fix}(T) \neq \varnothing$ and for all $x \in \mathscr{H}$ and $q \in \mathrm{Fix}(T)$

$$\|Tx - q\|^2 \le \|x - q\|^2 + \beta\|x - Tx\|^2,$$

which is equivalent to

$$\langle x - Tx, x - q\rangle \geq \frac{1-\beta}{2}\|x - Tx\|^2.$$

Remark that if $T : \mathscr{H} \to \mathscr{H}$ is nonexpansive, then

- $I - T$ is *demiclosed at zero*, that is, $Tx = 0$ whenever $\{x_n\} \subset \mathscr{H}$ converges weakly to $x \in \mathscr{H}$ and $\{Tx_n\}$ converges strongly to zero.
- T is 0-demicontractive if $\mathrm{Fix}(T) \neq \varnothing$.

Inspired by the preceding three methods and Nadezhkina–Takahshi's result [15], Maingé proved the following result.

Theorem 1. *Let C be a closed convex subset of $\mathscr{H}$, $\beta \in [0,1)$ and $\kappa, \eta, L > 0$. Suppose that $A : \mathscr{H} \to \mathscr{H}$ is monotone on C and κ-Lipschitz continuous on $\mathscr{H}$, $T : \mathscr{H} \to \mathscr{H}$ is β-demicontractive such that $I - T$ is demiclosed at zero and $\mathrm{VIP}(C, A) \cap \mathrm{Fix}(T) \neq \varnothing$. Suppose that $\mathscr{F} : \mathscr{H} \to \mathscr{H}$ is L-Lipschitz continuous and η-strongly monotone on C. Let $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ be sequences in $\mathscr{H}$ generated by*

$$\begin{cases} x_0 \in \mathscr{H} \\ y_n = P_C(x_n - \lambda_n A x_n) \\ t_n = P_C(x_n - \lambda_n A y_n) \\ v_n = t_n - \alpha_n \mathscr{F} t_n \\ x_{n+1} = (1-\omega) v_n + \omega T v_n, \end{cases} \tag{5}$$

where the parameters $\{\lambda_n\}$, $\{\alpha_n\}$ and ω satisfy the following conditions:

(a) $\lambda_n \in [a, b] \subset (0, 1/\kappa)$ *for some* $a, b \in (0, 1/\kappa)$*;*
(b) $\alpha_n \in [0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*
(c) $\omega \in (0, \frac{1-\beta}{2}]$.

Then all sequences $\{x_n\}$, $\{y_n\}$, and $\{t_n\}$ converge strongly to $x^* \in \mathrm{VIP}(C, A) \cap \mathrm{Fix}(T)$ and this x^* is the only element such that

$$\langle \mathscr{F} x^*, x - x^*\rangle \geq 0 \quad \forall x \in \mathrm{VIP}(C, A) \cap \mathrm{Fix}(T).$$

We will give a short and simple proof of Maingé's theorem and we can show that the result remains true under weaker assumptions. The proof technique is based on the recent result of the author [16] and the elegant observation of Aoyama and Kimura [1].

2. DEFINITIONS AND PRELIMINARIES

Let $\mathscr{H}$ be a real Hilbert space. We denote the strong and weak convergence of a sequence $\{x_n\}$ in $\mathscr{H}$ to an element $x \in \mathscr{H}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

The following inequalities are known in a Hilbert space $\mathscr{H}$.

Lemma 1 [18]. *For $x, y \in \mathscr{H}$, we have the following statements:*

- $|\langle x, y\rangle| \leq \|x\|\|y\|$;
- $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$. *(the subdifferential inequality)*

For a closed convex subset C of $\mathscr{H}$, the *metric projection* $P_C : \mathscr{H} \to C$ is defined for each $x \in \mathscr{H}$ as the unique element $P_C x \in C$ such that

$$\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}.$$

Lemma 2 [8, 18]. *Let C be a nonempty closed convex subset of $\mathscr{H}$. Then, for all $x \in \mathscr{H}$ and $y \in C$, the following are satisfied:*

- $y = P_C x$ *if and only if* $\langle y - x, z - y\rangle \geq 0$ *for all* $z \in C$,
- $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|P_C x - x\|^2$.

Lemma 3 [13]. *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, \ldots, k\}$ such that $a_n < a_{n+1}$ holds.

Lemma 4 ([17, 19]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{b_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} b_n < \infty$ and $\{\gamma_n\}$ a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that the following inequality*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\gamma_n + b_n$$

holds for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 5 (Lemma 7.1.7 of [17]). *Let C be a nonempty closed convex subset of $\mathscr{H}$ and $x^* \in C$. Let $A : C \to \mathscr{H}$ be a monotone and hemicontinuous mapping, that is, for any $x, y \in C$ and $z \in \mathscr{H}$, the function*

$$t \mapsto \langle z, A(tx + (1-t)y)\rangle$$

of $[0, 1]$ *into* $\mathbb{R}$ *is continuous. Then*

$$x^* \in VIP(C, A) \Leftrightarrow \langle Ax, x - x^*\rangle \geq 0 \quad \textit{for all } x \in C.$$

3. MAIN RESULT

Recall that a mapping $T : C \to C$ is called

- *quasi-nonexpansive* if $\mathrm{Fix}(T) \neq \varnothing$ and $\|Tx - q\| \leq \|x - q\|$ for all $x \in C$ and $q \in \mathrm{Fix}(T)$;
- *strongly quasi-nonexpansive* [3] if it is quasi-nonexpansive and $x_n - Tx_n \to 0$ whenever $\{x_n\}$ is a bounded sequence in C such that $\|x_n - q\| - \|Tx_n - q\| \to 0$ for some $q \in \mathrm{Fix}(T)$.

The concept of strong quasi-nonexpansiveness was introduced by Bruck and Reich in 1977 [3]. Inspired by this, Aoyama et al. [2] introduced the following natural generalization. A sequence $\{T_n : C \to C\}$ of quasi-nonexpansive mappings such that $F := \cap_{n=0}^{\infty}\mathrm{Fix}(T_n) \neq \varnothing$ is called a *strongly quasi-nonexpansive sequence* if $x_n - T_n x_n \to 0$ whenever $\{x_n\}$ is a bounded sequence in C such that $\|x_n - q\| - \|T_n x_n - q\| \to 0$ for some $q \in F$.

Theorem 2. *Suppose that $\{T_n : \mathscr{H} \to \mathscr{H}\}$ is a strongly quasi-nonexpansive sequence such that $F := \cap_{n=0}^{\infty}\mathrm{Fix}(T_n) \neq \varnothing$. Suppose that $f : \mathscr{H} \to \mathscr{H}$ is a contraction. Let $\{x_n\}$ be a sequence in $\mathscr{H}$ defined by*

$$\begin{cases} x_1 = x \in \mathscr{H} \ \textit{arbitrarily chosen;} \\ x_{n+1} = \alpha_n f(T_n x_n) + (1 - \alpha_n) T_n x_n, \end{cases} \tag{6}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Suppose that $\{T_n\}$ satisfies the condition (Z), that is,

$$\omega_w\{z_n\} \subset F \quad \textit{whenever } \{z_n\} \subset \mathscr{H} \quad \textit{is bounded and } z_n - T_n z_n \to 0.$$

Here $\omega_w\{z_n\}$ denotes the set of all weak cluster points of the sequence $\{z_n\}$. Then the sequence $\{x_n\}$ converges to an element $z \in F$ and the following inequality holds

$$\langle f(z) - z, q - z\rangle \le 0$$

for all $q \in F$.

Let us assume that $\|f(x) - f(y)\| \le \alpha\|x - y\|$ for all $x, y \in C$, where α is a real number in $[0, 1)$. We split the proof into several lemmas.

Lemma 6. *The sequence $\{x_n\}$ is bounded.*

Proof. We consider the following inequality

$$\begin{aligned}
\|x_{n+1} - z\| &\le \alpha_n\|f(T_nx_n) - z\| + (1 - \alpha_n)\|T_nx_n - z\| \\
&\le \alpha_n\|f(T_nx_n) - f(z)\| + \alpha_n\|f(z) - z\| + (1 - \alpha_n)\|T_nx_n - z\| \\
&\le \alpha_n\alpha\|T_nx_n - z\| + \alpha_n\|f(z) - z\| + (1 - \alpha_n)\|T_nx_n - z\| \\
&\le (\alpha_n\alpha + 1 - \alpha_n)\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&= (1 - \alpha_n(1 - \alpha))\|x_n - z\| + \alpha_n(1 - \alpha)\frac{\|f(z) - z\|}{1 - \alpha} \\
&\le \max\left\{\|x_n - z\|, \frac{\|f(z) - z\|}{1 - \alpha}\right\}.
\end{aligned}$$

By induction, we conclude that the sequence $\{\|x_n - z\|\}$ is bounded and, hence, so is the sequence $\{x_n\}$. □

Lemma 7. *The following inequality holds for all $n \in \mathbb{N}$:*

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\quad \le (1 - \alpha_n)^2\|x_n - z\|^2 + 2\alpha_n\alpha\|x_n - z\|\|x_{n+1} - z\| + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle.
\end{aligned}$$

Proof. It follows from the subdifferential inequality that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n(f(T_nx_n) - z) + (1 - \alpha_n)(T_nx_n - z)\|^2 \\
&\le (1 - \alpha_n)^2\|T_nx_n - z\|^2 + 2\alpha_n\langle f(T_nx_n) - z, x_{n+1} - z\rangle \\
&\le (1 - \alpha_n)^2\|x_n - z\|^2 + 2\alpha_n\langle f(T_nx_n) - f(z), x_{n+1} - z\rangle \\
&\quad + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\le (1 - \alpha_n)^2\|x_n - z\|^2 + 2\alpha_n\|f(T_nx_n) - f(z)\|\|x_{n+1} - z\| \\
&\quad + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle
\end{aligned}$$

$$\leq (1-\alpha_n)^2\|x_n - z\|^2 + 2\alpha_n\alpha\|x_n - z\|\|x_{n+1} - z\| + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle. \qquad \square$$

Lemma 8. *If there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\liminf_{k\to\infty}(\|x_{n_k+1} - z\| - \|x_{n_k} - z\|) \geq 0$, *then* $\limsup_{k\to\infty}\langle f(z) - z, x_{n_k+1} - z\rangle \leq 0$.

Proof. First, we note that $\alpha_{n_k} \to 0$ and let us consider the following inequality

$$\begin{aligned}
0 &\leq \liminf_{k\to\infty}(\|x_{n_k+1} - z\| - \|x_{n_k} - z\|) \\
&\leq \liminf_{k\to\infty}(\alpha_{n_k}\|f(T_{n_k}x_{n_k}) - z\| + (1-\alpha_{n_k})\|T_{n_k}x_{n_k} - z\| - \|x_{n_k} - z\|) \\
&= \liminf_{k\to\infty}(\|T_{n_k}x_{n_k} - z\| - \|x_{n_k} - z\|) \\
&\leq \limsup_{k\to\infty}(\|T_{n_k}x_{n_k} - z\| - \|x_{n_k} - z\|) \\
&\leq 0.
\end{aligned}$$

This implies that $\lim_{k\to\infty}(\|x_{n_k} - z\| - \|T_{n_k}x_{n_k} - z\|) = 0$. Since $\{T_n\}$ is a strongly quasi-nonexpansive sequence, $x_{n_k} - T_{n_k}x_{n_k} \to 0$. In particular, $x_{n_k} - x_{n_k+1} \to 0$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \rightharpoonup q$ and

$$\lim_{l\to\infty}\langle f(z) - z, x_{n_{k_l}} - z\rangle = \limsup_{k\to\infty}\langle f(z) - z, x_{n_k} - z\rangle.$$

It follows from property (Z) that $q \in F$. Then

$$\lim_{l\to\infty}\langle f(z) - z, x_{n_{k_l}} - z\rangle = \langle f(z) - z, q - z\rangle \leq 0.$$

Hence, $\limsup_{k\to\infty}\langle f(z) - z, x_{n_k+1} - z\rangle = \limsup_{k\to\infty}\langle f(z) - z, x_{n_k} - z\rangle \leq 0$, as desired. $\square$

Proof of Theorem 2. Let us consider the following two cases.

Case 1: There exists an $N \in \mathbb{N}$ such that $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $n \geq N$. It follows then that $\lim_{n\to\infty}\|x_n - z\|$ exists and, hence, $\liminf_{n\to\infty}(\|x_{n+1} - z\| - \|x_n - z\|) = 0$. This implies that $\limsup_{n\to\infty}\langle f(z) - z, x_{n+1} - z\rangle \leq 0$. By Lemma 7, for all $n \geq N$,

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1-\alpha_n)^2\|x_n - z\|^2 + 2\alpha_n\alpha\|x_n - z\|\|x_{n+1} - z\| \\
&\quad + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\leq (1 - 2\alpha_n + 2\alpha_n\alpha)\|x_n - z\|^2 + \alpha_n^2\|x_n - z\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ 2\alpha_n \langle f(z) - z, x_{n+1} - z\rangle \\
&= (1 - 2\alpha_n(1-\alpha))\|x_n - z\|^2 \\
&\quad + 2\alpha_n(1-\alpha)\left(\frac{\alpha_n\|x_n - z\|^2}{2(1-\alpha)} + \frac{\langle f(z) - z, x_{n+1} - z\rangle}{1-\alpha}\right).
\end{aligned}
$$

Notice that $\sum_{n=N}^{\infty} 2\alpha_n(1-\alpha) = \infty$ and

$$
\limsup_{n\to\infty}\left(\frac{\alpha_n\|x_n - z\|^2}{2(1-\alpha)} + \frac{\langle f(z) - z, x_{n+1} - z\rangle}{1-\alpha}\right) \le 0.
$$

By Lemma 4, we have $\lim_{n\to\infty}\|x_n - z\|^2 = 0$.

Case 2: There exists a subsequence $\{\|x_{n_j} - z\|\}$ of $\{\|x_n - z\|\}$ such that $\|x_{n_j} - z\| < \|x_{n_j+1} - z\|$ for all $j \in \mathbb{N}$. In this case, it follows from Lemma 3 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$
\|x_{m_k} - z\| \le \|x_{m_k+1} - z\| \quad \text{and} \quad \|x_k - z\| \le \|x_{m_k+1} - z\|.
$$

It follows from $\liminf_{k\to\infty}(\|x_{m_k+1} - z\| - \|x_{m_k} - z\|) \ge 0$ that $\limsup_{k\to\infty}$ $\langle f(z) - z, x_{m_k+1} - z\rangle \le 0$. By discarding the repeated terms of $\{m_k\}$ but still denoted by $\{m_k\}$, we can view $\{x_{m_k}\}$ as a subsequence of $\{x_n\}$. Hence, by Lemma 7, we have

$$
\begin{aligned}
\|x_{m_k+1} - z\|^2 &\le (1-\alpha_{m_k})^2\|x_{m_k} - z\|^2 + 2\alpha_{m_k}\alpha\|x_{m_k} - z\|\|x_{m_k+1} - z\| \\
&\quad + 2\alpha_{m_k}\langle f(z) - z, x_{m_k+1} - z\rangle \\
&\le (1-\alpha_{m_k})^2\|x_{m_k+1} - z\|^2 + 2\alpha_{m_k}\alpha\|x_{m_k+1} - z\|^2 \\
&\quad + 2\alpha_{m_k}\langle f(z) - z, x_{m_k+1} - z\rangle.
\end{aligned}
$$

In particular, it follows that

$$
(2 - \alpha_{m_k} - 2\alpha)\|x_{m_k+1} - z\|^2 \le 2\langle f(z) - z, x_{m_k+1} - z\rangle.
$$

This implies that

$$
(2-2\alpha)\limsup_{k\to\infty}\|x_{m_k+1} - z\|^2 \le \limsup_{k\to\infty} 2\langle f(z) - z, x_{m_k+1} - z\rangle \le 0.
$$

Hence,

$$
\limsup_{k\to\infty}\|x_k - z\|^2 \le \limsup_{k\to\infty}\|x_{m_k+1} - z\|^2 = 0.
$$

Then $\lim_{k\to\infty}\|x_k - z\|^2 = 0$. This completes the proof. $\square$

The following corollary recovers Yamada's hybrid steepest descent method.

Corollary 1. *Suppose that* $\{T_n : \mathcal{H} \to \mathcal{H}\}$ *is a strongly quasi-nonexpansive sequence such that* $F := \cap_{n=0}^{\infty} \mathrm{Fix}(T_n) \neq \varnothing$. *Suppose that* $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ *is* L-*Lipschitz continuous and* η-*strongly monotone where* $L^2 < 2\eta$. *Let* $\{x_n\}$ *be a sequence in* $\mathcal{H}$ *generated by*

$$\begin{cases} x_0 \in \mathcal{H} \\ x_{n+1} = T_n x_n - \alpha_n \mathcal{F} T_n x_n, \end{cases} \tag{7}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Suppose that* $\{T_n\}$ *satisfies the condition (Z). Then* $\{x_n\}$ *converges strongly to* $x^* \in F$ *and this* x^* *is the only element such that*

$$\langle \mathcal{F} x^*, x - x^* \rangle \geq 0 \quad \forall x \in F.$$

Proof. The proof is inspired by Aoyama and Kimura's result but the method given here is totally different. It is worth noting that we do assume that each mapping T_n is quasi-nonexpansive and hence it is not necessarily continuous. As mentioned by Aoyama and Kimura, the condition $L^2 < 2\eta$ is not restrictive because we may replace $\mathcal{F}$ by $\widehat{\mathcal{F}} := t\mathcal{F}$ where $t \in (0, 2\eta/L^2)$ and it is easy to see that $\mathrm{VIP}(F, \mathcal{F}) = \mathrm{VIP}(F, \widehat{\mathcal{F}})$.

We now rewrite the scheme (7) as follows:

$$\begin{cases} x_0 \in \mathcal{H} \\ x_{n+1} = \alpha_n f(T_n x_n) + (1 - \alpha_n) T_n x_n, \end{cases} \tag{8}$$

where $f := I - \mathcal{F}$ is a contraction. The conclusion follows immediately from our Theorem 2. □

Here we give a useful lemma that is needed for showing how we obtain Maingé's result from the preceding corollary.

Lemma 9. *Let* $A : \mathcal{H} \to \mathcal{H}$ *be monotone and* κ-*Lipschitz continuous on* C. *Let* $S := P_C(I - \tau A)$ *where* $\tau > 0$. *If* $\{x_n\}$ *is a sequence in* C *satisfying* $x_n \rightharpoonup \widehat{x}$ *and* $x_n - Sx_n \to 0$, *then* $\widehat{x} \in \mathrm{VIP}(C, A) = \mathrm{Fix}(S)$. *In particular, if* $\{z_n\}$ *is a bounded sequence in* C *such that* $z_n - P_C(z_n - \tau_n A z_n) \to 0$, *where* $\{\tau_n\} \subset [a, b] \subset (0, \infty)$, *then* $\omega_w\{z_n\} \subset \mathrm{VIP}(C, A)$.

Proof. Since A is monotone and hemicontinuous, it suffices to show that $\langle Ax, x - \widehat{x} \rangle \geq 0$ for all $x \in C$. Let $x \in C$ and $\tau > 0$. Note that $\langle x_n - \tau A x_n - Sx_n, Sx_n - x \rangle \geq 0$ for all $n \in \mathbb{N}$. Next, we consider

$$\begin{aligned} \langle \tau A x_n, x_n - x \rangle &= \langle \tau A x_n, x_n - Sx_n \rangle + \langle \tau A x_n, Sx_n - x \rangle \\ &= \langle \tau A x_n, x_n - Sx_n \rangle - \langle x_n - \tau A x_n - Sx_n, Sx_n - x \rangle \\ &\quad + \langle x_n - Sx_n, Sx_n - x \rangle \end{aligned}$$

$$\leq \langle \tau Ax_n, x_n - Sx_n \rangle + \langle x_n - Sx_n, Sx_n - x \rangle$$
$$\leq \tau \|Ax_n\| \|x_n - Sx_n\| + \|x_n - Sx_n\| \|Sx_n - x\|.$$

Since $\{Ax_n\}$ is bounded and $x_n - Sx_n \to 0$, it follows from the monotonicity of A that

$$\langle Ax, \widehat{x} - x \rangle = \frac{1}{\tau} \limsup_{n\to\infty} \langle \tau Ax, x_n - x \rangle \leq \frac{1}{\tau} \limsup_{n\to\infty} \langle \tau Ax_n, x_n - x \rangle \leq 0.$$

The proof is finished. □

The following estimate plays an important role in this article. In fact, it is extracted from Nadezhkina–Takahshi's article [15] and included here for the reader's convenience.

Lemma 10. *Let $A : \mathscr{H} \to \mathscr{H}$ be a monotone and κ-Lipschitz mapping on C and λ be a positive number such that $\kappa\lambda < 1$. Let $T := P_C(I - \lambda A)$ and $S := P_C(I - \lambda AT)$. Then, for all $q \in \mathrm{VIP}(C, A)$, we have*

$$\|Sx - q\|^2 \leq \|x - q\|^2 - (1 - \lambda^2\kappa^2)\|x - Tx\|^2. \tag{9}$$

Proof. Note that

$$\begin{aligned} \|Sx - q\|^2 &\leq \|(x - \lambda ATx) - q\|^2 - \|(x - \lambda ATx) - Sx\|^2 \\ &= \|(x - q) - \lambda ATx\|^2 - \|(x - Sx) - \lambda ATx\|^2 \\ &= \|x - q\|^2 - \|x - Sx\|^2 + 2\lambda\langle q - Sx, ATx \rangle. \end{aligned}$$

It follows from the monotonicity of A and $q \in \mathrm{VIP}(C, A)$ that $\langle q - Tx, ATx - Aq \rangle \leq 0$ and $\langle q - Tx, Aq \rangle \leq 0$, respectively. Hence, we have the following estimation,

$$\begin{aligned} \langle q - Sx, ATx \rangle &= \langle q - Tx, ATx - Aq \rangle + \langle q - Tx, Aq \rangle \\ &\quad + \langle Tx - Sx, ATx \rangle \\ &\leq \langle Tx - Sx, ATx \rangle. \end{aligned}$$

Moreover,

$$\begin{aligned} \|x - Sx\|^2 &= \|(x - Tx) + (Tx - Sx)\|^2 \\ &= \|x - Tx\|^2 + 2\langle Tx - Sx, x - Tx \rangle + \|Tx - Sx\|^2. \end{aligned}$$

Then we get

$$\begin{aligned}\|Sx - q\|^2 &\le \|x - q\|^2 - \|x - Tx\|^2 - \|Tx - Sx\|^2 \\ &\quad - 2\langle Tx - Sx, x - Tx\rangle + 2\lambda\langle Tx - Sx, ATx\rangle \\ &= \|x-q\|^2 - \|x-Tx\|^2 - \|Tx-Sx\|^2 + 2\langle Tx-Sx, Tx-(x-\lambda ATx)\rangle.\end{aligned}$$

Now we will estimate the last term of the preceding expression. It follows from $Tx = P_C(x - \lambda Ax)$ and Lemma 2 that $\langle Tx - Sx, Tx - (x - \lambda Ax)\rangle \le 0$. Hence,

$$\begin{aligned}&2\langle Tx - Sx, Tx - (x - \lambda ATx)\rangle \\ &\quad = 2\langle Tx - Sx, Tx - (x - \lambda Ax)\rangle + 2\langle Tx - Sx, \lambda ATx - \lambda Ax\rangle \\ &\quad \le 2\langle Tx - Sx, \lambda ATx - \lambda Ax\rangle \\ &\quad \le 2\lambda\|Tx - Sx\|\|Ax - ATx\| \\ &\quad \le 2\lambda\kappa\|Tx - Sx\|\|x - Tx\| \\ &\quad \le \|Tx - Sx\|^2 + \lambda^2\kappa^2\|x - Tx\|^2.\end{aligned}$$

So we have

$$\begin{aligned}\|Sx-q\|^2 &\le \|x-q\|^2 - \|x-Tx\|^2 - \|Tx-Sx\|^2 + \|Tx-Sx\|^2 + \lambda^2\kappa^2\|x - Tx\|^2 \\ &= \|x - q\|^2 - (1 - \lambda^2\kappa^2)\|x - Tx\|^2,\end{aligned}$$

as desired. □

Corollary 2. *Let C be a closed convex subset of $\mathcal{H}$, $\beta \in [0,1)$. Suppose that $A : \mathcal{H} \to \mathcal{H}$ is monotone on C and κ-Lipschitz continuous on $\mathcal{H}$, $T : \mathcal{H} \to \mathcal{H}$ is β-demicontractive such that $I - T$ is demiclosed at zero and* $\mathrm{VIP}(C, A) \cap \mathrm{Fix}(T) \neq \varnothing$. *Suppose that $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is L-Lipschitz continuous and η-strongly monotone on C. Let $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ be sequences in $\mathcal{H}$ generated by*

$$\begin{cases} x_0 \in \mathcal{H} \\ y_n = P_C(x_n - \lambda_n Ax_n) \\ t_n = P_C(x_n - \lambda_n Ay_n) \\ v_n = t_n - \alpha_n \mathcal{F} t_n \\ x_{n+1} = (1 - \omega)v_n + \omega Tv_n, \end{cases} \tag{10}$$

where the parameters $\{\lambda_n\}$, $\{\alpha_n\}$ and ω satisfy the following conditions:

(a) $\lambda_n \in [a, b] \subset (0, 1/\kappa)$ *for some* $a, b \in (0, 1/\kappa)$*;*
(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*
(c) $\omega \in (0, 1 - \beta)$.

Then all sequences $\{x_n\}$, $\{y_n\}$ *and* $\{t_n\}$ *converge strongly to* $x^* \in \mathrm{VIP}(C,A) \cap \mathrm{Fix}(T)$ *and this* x^* *is the only element such that*

$$\langle \mathscr{F}x^*, x - x^* \rangle \geq 0 \quad \forall x \in \mathrm{VIP}(C,A) \cap \mathrm{Fix}(T).$$

Proof. Set

$$\begin{aligned} &S_n := P_C(I - \lambda_n A(P_C(I - \lambda_n A))), \quad S := (1-\omega)I + \omega T, \\ &T_n := S_n \circ S, \quad f := I - t\mathscr{F}, \quad x_n^* := v_{n-1}, \end{aligned}$$

where $t \in (0, 2\eta/L^2)$. Then f is a contraction and

$$x_{n+1}^* = \alpha_n f(T_n x_n^*) + (1-\alpha_n) T_n x_n^*.$$

To complete the proof, it suffices to show that

(♠) $\{T_n\}$ is a strongly quasi-nonexpansive sequence;

(♡) $\{T_n\}$ satisfies property (Z).

For simplicity, we let $F := \mathrm{VIP}(C,A) \cap \mathrm{Fix}(T)$. We first observe the following two inequalities: for $x \in \mathscr{H}$ and $q \in F$,

$$\begin{aligned} \|S_n x - q\|^2 &\leq \|x-q\|^2 - (1 - \lambda_n^2\kappa^2)\|x - P_C(x - \lambda_n Ax)\|^2, \\ \|Sx - q\|^2 &= \|(1-\omega)x + \omega Tx - q\|^2 \\ &= \|x-q\|^2 - 2\omega\langle x-q, x-Tx\rangle + \omega^2\|Tx - x\|^2 \\ &\leq \|x-q\|^2 - \omega(1-\beta-\omega)\|x - Tx\|^2. \end{aligned}$$

It follows then that

$$\begin{aligned} \|T_n x - q\|^2 &= \|S_n S x - q\|^2 \\ &\leq \|Sx - q\|^2 - (1-\lambda_n^2\kappa^2)\|Sx - P_C(Sx - \lambda_n ASx)\|^2 \\ &\leq \|x-q\|^2 - \omega(1-\beta-\omega)\|x-Tx\|^2 \\ &\quad - (1-\lambda_n^2\kappa^2)\|Sx - P_C(Sx - \lambda_n ASx)\|^2. \end{aligned} \tag{11}$$

First noting that $\mathrm{Fix}(T_n) = F$. It is obvious that $F \subset \mathrm{Fix}(T_n)$. To see the converse, let $x \in \mathrm{Fix}(T_n)$ and $q \in F$. Then by (11) we have

$$\begin{aligned} \|x-q\|^2 &= \|T_n x - q\|^2 \\ &\leq \|x-q\|^2 - \omega(1-\beta-\omega)\|x-Tx\|^2 \\ &\quad - (1-\lambda_n^2\kappa^2)\|Sx - P_C(Sx - \lambda_n ASx)\|^2. \end{aligned}$$

Hence, $x = Tx$ and $Sx = P_C(Sx - \lambda_n ASx)$. It follows that $x = Sx$ and therefore $x = P_C(x - \lambda_n Ax)$, that is, $x \in F$. We now show (♠). It can be easily seen, by using (11), that each T_n is a quasi-nonexpansive mapping. Next, let $\{z_n\}$ be a bounded sequence in C such that $\|z_n - q\| - \|T_n z_n - q\| \to 0$ for some $q \in F$. Hence, replacing x by z_n in (11) gives $z_n - Tz_n \to 0$ and $Sz_n - P_C(I - \lambda_n A)Sz_n \to 0$. It follows that $z_n - Sz_n = \omega(z_n - Tz_n) \to 0$ and, therefore, $z_n - P_C(I - \lambda_n A)Sz_n \to 0$. So we get

$$\begin{aligned}
\|T_n z_n - z_n\| &\le \|T_n z_n - P_C(I - \lambda_n A)Sz_n\| + \|P_C(I - \lambda_n A)Sz_n - z_n\| \\
&= \|P_C(I - \lambda_n AP_C(I - \lambda_n A))Sz_n - P_C(I - \lambda_n A)Sz_n\| \\
&\quad + \|P_C(I - \lambda_n A)Sz_n - z_n\| \\
&\le \|Sz_n - \lambda_n AP_C(I - \lambda_n A)Sz_n - (Sz_n - \lambda_n ASz_n)\| \\
&\quad + \|P_C(I - \lambda_n A)Sz_n - z_n\| \\
&= \|\lambda_n AP_C(I - \lambda_n A)Sz_n - \lambda_n ASz_n\| + \|P_C(I - \lambda_n A)Sz_n - z_n\| \\
&\le \lambda_n \kappa \|P_C(I - \lambda_n A)Sz_n - Sz_n\| + \|P_C(I - \lambda_n A)Sz_n - z_n\| \to 0.
\end{aligned}$$

This shows that $\{T_n\}$ is a strongly quasi-nonexpansive sequence and, hence, (♠) is asserted.

To show (♡), let $\{z_n\}$ be a bounded sequence in $\mathscr{H}$ such that $z_n - T_n z_n \to 0$. We show that $\omega_w\{z_n\} \subset F$. It follows from the assumption that $\|z_n - q\| - \|T_n z_n - q\| \to 0$ for some $q \in F$. As a consequence of the proof in (♠), we have $z_n - Tz_n \to 0$ and $z_n - P_C(z_n - \lambda_n Az_n) \to 0$. Now assume that $\{z_{n_k}\}$ is a subsequence of $\{z_n\}$ such that $z_{n_k} \rightharpoonup z$. Hence, we immediately get that $z_{n_k} - Tz_{n_k} \to 0$ and $z_{n_k} - P_C(z_{n_k} - \lambda_{n_k} Az_{n_k}) \to 0$. By Lemma 9 and the demiclosedness of the mapping $I - T$, we conclude that $z \in F = \mathrm{Fix}(T_n)$. Therefore $\omega_w\{z_n\} \subset F$, that is, the sequence $\{T_n\}$ satisfies condition (Z) as desired. □

The limitation on the use of Maingé's result and our Corollary 2 happens when dealing with β-demicontractive mappings where $\beta = 1$. We will modify the preceding construction of a strongly quasi-nonexpansive mappings from a 1-demicontractive mapping which is L-Lipschitzian by using Ishikawa's idea (see [10]).

Proposition 1. *Let $T : \mathscr{H} \to \mathscr{H}$ be a γ-Lipschitzian and 1-demicontractive mapping. Define the mappings S and U for some positive valued α by*

$$S := (1 - \alpha)I + \alpha T,$$
$$U := (1 - \alpha)I + \alpha TS.$$

Then

$$\|Ux - q\|^2 \le \|x - q\|^2 + \alpha^2(\alpha^2\gamma^2 + 2\alpha - 1)\|x - Tx\|^2,$$
$$\textit{for all } (x, q) \in \mathscr{H} \times \mathrm{Fix}(T).$$

In addition, if $\alpha \in (0, \frac{-1+\sqrt{\gamma^2+1}}{\gamma^2})$, *then* U *is a strongly quasi-nonexpansive mapping.*

Proof. Let $(x, q) \in \mathscr{H} \times \mathrm{Fix}(T)$. Note that

$$\begin{aligned}\|Ux - q\|^2 &= \|(1-\alpha)(x-q) + \alpha(TSx - q)\|^2 \\ &= (1-\alpha)\|x-q\|^2 + \alpha\|TSx - q\|^2 - \alpha(1-\alpha)\|TSx - x\|^2. \quad (12)\end{aligned}$$

Since T is 1-demicontractive, we have

$$\|TSx - q\|^2 \le \|Sx - q\|^2 + \|Sx - TSx\|^2. \quad (13)$$

Next we estimate two terms on the right of the preceding inequality:

$$\begin{aligned}&\|Sx - q\|^2 \\ &\quad = \|(1-\alpha)(x-q) + \alpha(Tx - q)\|^2 \\ &\quad = (1-\alpha)\|x-q\|^2 + \alpha\|Tx - q\|^2 - \alpha(1-\alpha)\|Tx - x\|^2 \\ &\quad \le \|x-q\|^2 + \alpha\|x - Tx\|^2 - \alpha(1-\alpha)\|Tx - x\|^2 \\ &\quad = \|x-q\|^2 + \alpha^2\|x - Tx\|^2; \quad (14)\end{aligned}$$

$$\begin{aligned}&\|Sx - TSx\|^2 \\ &\quad = \|(1-\alpha)(x - TSx) + \alpha(Tx - TSx)\|^2 \\ &\quad = (1-\alpha)\|x - TSx\|^2 + \alpha\|Tx - TSx\|^2 - \alpha(1-\alpha)\|Tx - x\|^2 \\ &\quad \le (1-\alpha)\|x - TSx\|^2 + \alpha\gamma^2\|x - Sx\|^2 - \alpha(1-\alpha)\|Tx - x\|^2 \\ &\quad = (1-\alpha)\|x - TSx\|^2 + \alpha\gamma^2\alpha^2\|x - Tx\|^2 - \alpha(1-\alpha)\|Tx - x\|^2 \\ &\quad = (1-\alpha)\|x - TSx\|^2 + \alpha(\gamma^2\alpha^2 + \alpha - 1)\|x - Tx\|^2. \quad (15)\end{aligned}$$

From (13), (14), and (15), we obtain

$$\begin{aligned}\|TSx - q\|^2 \le \|x-q\|^2 &+ \alpha(\gamma^2\alpha^2 + 2\alpha - 1)\|x - Tx\|^2 \\ &+ (1-\alpha)\|x - TSx\|^2. \quad (16)\end{aligned}$$

It follows from (12) and (16) that

$$\|Ux - q\|^2 \le \|x - q\|^2 + \alpha^2(\gamma^2\alpha^2 + 2\alpha - 1)\|Tx - x\|^2.$$

This proves the first assertion.

Finally, we prove the last assertion. Note that $\alpha^2(\gamma^2\alpha^2 + 2\alpha - 1) < 0$ for all $\alpha \in (0, \frac{-1+\sqrt{\gamma^2+1}}{\gamma^2})$. It follows from the inequality obtained in the first part that $\mathrm{Fix}(U) = \mathrm{Fix}(T)$. Then it is clear that U is a quasi-nonexpansive mapping. To prove that U is a strongly quasi-nonexpansive mapping, let $\{x_n\}$ be a bounded sequence in $\mathscr{H}$ such that $\|x_n - q\| - \|Ux_n - q\| \to 0$, for some $q \in \mathrm{Fix}(U)(= \mathrm{Fix}(T))$. From the last inequality, we have

$$0 \le -\alpha^2(\gamma^2\alpha^2 + 2\alpha - 1)\|Tx_n - x_n\|^2 \le \|x_n - q\|^2 - \|Ux_n - q\|^2 \to 0.$$

This implies that $x_n - Sx_n = \alpha(x_n - Tx_n) \to 0$. By the continuity of T, we also get $Tx_n - TSx_n \to 0$. Then $x_n - TSx_n \to 0$ and hence $x_n - Ux_n = \alpha(x_n - TSx_n) \to 0$. The proof is finished. $\square$

Corollary 3. *Let C be a closed convex subset of $\mathscr{H}$ and $\kappa, \gamma, \eta, L > 0$. Suppose that $A : \mathscr{H} \to \mathscr{H}$ is monotone and κ-Lipschitz continuous, $T : \mathscr{H} \to \mathscr{H}$ is 1-demicontractive γ-Lipschitz continuous such that $I - T$ is demiclosed at zero and* $\mathrm{VIP}(C, A) \cap \mathrm{Fix}(T) \ne \varnothing$. *Suppose that $\mathscr{F} : \mathscr{H} \to \mathscr{H}$ is L-Lipschitz continuous and η-strongly monotone. Let $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ be sequences in $\mathscr{H}$ generated by the following scheme:*

$$\begin{cases} x_0 \in \mathscr{H} \\ y_n = P_C(x_n - \lambda_n A x_n) \\ t_n = P_C(x_n - \lambda_n A y_n) \\ v_n = t_n - \alpha_n \mathscr{F} t_n \\ x_{n+1} = Uv_n, \end{cases} \tag{17}$$

where $U = (1 - \xi)I + \xi T((1 - \xi)I + \xi T)$ and the parameters $\{\lambda_n\}$, $\{\alpha_n\}$, and ξ satisfy the following conditions:

(a) $\lambda_n \in [a, b] \subset (0, 1/\kappa)$ *for some* $a, b \in (0, 1/\kappa)$;
(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(c) $\xi \in (0, \frac{-1+\sqrt{\gamma^2+1}}{\gamma^2})$.

Then all sequences $\{x_n\}$, $\{y_n\}$, and $\{t_n\}$ converge strongly to $x^* \in$ $\mathrm{VIP}(C, A) \cap \mathrm{Fix}(T)$ *and this x^* is the only element such that*

$$\langle \mathscr{F} x^*, x - x^* \rangle \ge 0 \quad \forall x \in \mathrm{VIP}(C, A) \cap \mathrm{Fix}(T).$$

Finally, we remark that our scheme is a *genuine generalization* of Maingé's result because there is a 1-demicontractive and Lipschitzian mapping which is not β-demicontractive for all $\beta \in [0, 1)$. The following mapping is introduced by Chidume and Mutangadura [7].

Example 1. Let $\mathscr{H} = \mathbb{R}^2, B = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}, B_1 = \{x \in B : \|x\| \leq \frac{1}{2}\}$ and $B_2 = \{x \in B : \frac{1}{2} \leq \|x\| \leq 1\}$. For $x = (a, b) \in \mathscr{H}$, let $x^{\perp} = (b, -a)$. Define the mapping $T : B \to B$ by

$$Tx = \begin{cases} x + x^{\perp} & \text{if } x \in B_1; \\ \dfrac{x}{\|x\|} - x + x^{\perp} & \text{if } x \in B_2. \end{cases}$$

It is easy to see that $\mathrm{Fix}(T) = \{0\}$. It was proved in [7] that T is 5-Lipschitzian and

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|x - Tx - (y - Ty)\|^2 \tag{18}$$

for all $x, y \in B$. In particular, T is 1-demicontractive. Moreover, the inequality (18) becomes an equality whenever $x \in B_1$ and $y = 0$, that is,

$$\|Tx\|^2 = \|x + x^{\perp}\|^2 = \|x\|^2 + \|x^{\perp}\|^2 = \|x\|^2 + \|x - Tx\|^2.$$

This shows that T cannot be a β-demicontractive mapping where $\beta \in [0, 1)$.

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Strong Convergence of the Halpern Subgradient Extragradient Method for Solving Variational Inequalities in Hilbert Spaces

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Abstract Building upon the subgradient extragradient method proposed by Censor et al., we prove the strong convergence of the iterative sequence generated by a modification of this method by means of the Halpern method. We also consider the problem of finding a common element of the solution set of a variational inequality and the fixed-point set of a quasi-nonexpansive mapping with a demiclosedness property.

Keywords Subgradient extragradient method · Halpern method · Variational inequality · Quasi-nonexpansive mapping · Fixed point

1 Introduction

Many problems in science and engineering can be recast as variational inequalities (see, for example, [1–8]). Iterative methods for solving these problems have been proposed and analyzed by many authors (see, for example, [9–12] and references therein). In this paper, we are interested in the extragradient method proposed by Korpelevič [13] and the modified one by Censor et al. [14]. For the former method, two calculations of the projection onto a closed and convex subset are needed. As mentioned in [14] this may affect the efficiency of the method, and Censor et al. modified Korpelevič's method by replacing the second projection onto the closed and convex subset with the one onto the subgradient half-space. So the latter method is called the subgradient extragradient method. The same authors continued the study of this method in [15, 16]. Under some appropriate setting, the subgradient extragradient method [14] converges *weakly* to a solution of a variational inequality. The purpose

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of this paper is to develop the method of Censor et al. [14] together with the Halpern method [17, 18] to obtain the *strong* convergence. The original result of Halpern [17] deals with the problem of finding a fixed point of a single operator and Bauschke [19] extended to that of finding a common fixed point of finitely many operators. We refer the readers to Cegielski's book [20] for the recent progress on this subject. It is worth reminding the reader that our approach and the one in [15] are different. More precisely, Algorithm 3.6 (and hence 5.1) of [15] seems to be difficult to use in practice because the computation of the next iterate becomes a subproblem of finding a point in the intersection of two additional half-spaces. Our method does not involve this subproblem but, as mentioned by the reviewer, our method may cause some numerical instabilities.

The paper is organized as follows. In Sect. 2, we collect together definitions and some preliminaries that pertain the argument of the paper with corresponding references. Our main results are presented in Sects. 3 and 4. Finally, we summarize our results in Sect. 5.

2 Definitions and Preliminaries

Throughout, let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. For a closed and convex subset C of $\mathcal{H}$, the *(metric) projection* $P_C:\mathcal{H}\to C$ is defined, for each $x\in\mathcal{H}$, as the unique element $P_C x\in C$ such that

$$\|x-P_C x\|=\inf\bigl\{\|x-z\|:z\in C\bigr\}.$$

For $x\in\mathcal{H}$ and $y\in C$, it is known that

$$y=P_C x \quad\Longleftrightarrow\quad \langle y-x,z-y\rangle\ge 0 \quad \text{for all } z\in C.$$

For other relevant properties of the projection, the interested readers are referred to, for example, Chap. 3 in the book by Goebel and Reich [21].

In this paper, the fixed-point set of a mapping $T:\mathcal{H}\to\mathcal{H}$ is denoted by $\mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T):=\{x\in\mathcal{H}:x=Tx\}$. For a given closed and convex subset $C\subset\mathcal{H}$, we are interested in the so-called *variational inequality* [4, 5] for a mapping $f:\mathcal{H}\to\mathcal{H}$, that is, the problem of finding an element $\widehat{x}\in C$ such that

$$\bigl\langle f(\widehat{x}),x-\widehat{x}\bigr\rangle\ge 0 \quad \text{for all } x\in C.$$

We denote the set of all such $\widehat{x}$ by $\mathrm{VI}(C,f)$. Hence

$$\begin{aligned}
&\widehat{x}\in\mathrm{VI}(C,f)\\
&\overset{\text{def}}{\Longleftrightarrow}\quad \bigl\langle f(\widehat{x}),x-\widehat{x}\bigr\rangle\ge 0 \quad \text{for all } x\in C\\
&\Longleftrightarrow\quad \bigl\langle \widehat{x}-\bigl(\widehat{x}-\lambda f(\widehat{x})\bigr),x-\widehat{x}\bigr\rangle\ge 0 \quad \text{for all } x\in C, \text{for all } \lambda>0\\
&\Longleftrightarrow\quad \widehat{x}=P_C\bigl(\widehat{x}-\lambda f(\widehat{x})\bigr)=P_C(I-\lambda f)\widehat{x} \quad \text{for all } \lambda>0.
\end{aligned}$$

Let us recall some definitions of mappings involved in our study.

Definition 2.1 [6] A mapping $T : \mathcal{H} \to \mathcal{H}$ is called

- L-*Lipschitzian*, where $L > 0$, iff

$$\|Tx - Ty\| \le L\|x - y\| \quad \text{for all } x, y \in \mathcal{H};$$

- *nonexpansive* iff T is 1-Lipschitzian;
- *quasi-nonexpansive* iff $\mathrm{Fix}(T) \ne \varnothing$ and

$$\|Tx - p\| \le \|x - p\| \quad \text{for all } x \in \mathcal{H},\ p \in \mathrm{Fix}(T);$$

- *monotone* iff

$$\langle Tx - Ty, x - y\rangle \ge 0 \quad \text{for all } x, y \in \mathcal{H}.$$

For solving the variational inequality for $f : \mathcal{H} \to \mathcal{H}$, which is monotone and L-Lipschitzian continuous on C, a well-known algorithm is the extragradient method proposed by Korpelevich [13] for the Euclidean case and by Censor et al. for the Hilbert space case [14]. More precisely, this method generates the following iterative sequence $\{x_n\}$:

$$\begin{cases} x_0 \in C \\ y_n = P_C(x_n - \tau f(x_n)) \\ x_{n+1} = P_C(x_n - \tau f(y_n)), \end{cases} \tag{1}$$

where the stepsize $\tau \in]0, \frac{1}{L}[$. It was proved that, if $\mathrm{VI}(C, f) \ne \varnothing$, then $\{x_n\}$ converges *weakly* to an element in $\mathrm{VI}(C, f)$.

Inspired by the extragradient method, Censor et al. [14, 16] recently modified this algorithm and called it the subgradient extragradient method. Since the computation of the projection onto a general closed and convex set C is rather complicated, the purpose of this modification is to replace two projections onto C to one projection onto C and one onto a half-space. Let us note that the latter projection (onto a half-space) is easier to compute. We summarize their result as follows.

Theorem 2.1 *Let $f : \mathcal{H} \to \mathcal{H}$ be a monotone and L-Lipschitz mapping on C and $\mathrm{VI}(C, f) \ne \varnothing$. Let $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \tau f(x_n)), \\ T_n = \{w \in \mathcal{H} : \langle x_n - \tau f(x_n) - y_n, w - y_n\rangle \le 0\}, \\ x_{n+1} = P_{T_n}(x_n - \tau f(y_n)), \end{cases} \tag{2}$$

where $\tau \in]0, \frac{1}{L}[$. Then $\{x_n\}$ and $\{y_n\}$ converge weakly to $\widehat{u}$, where $\widehat{u} \in \mathrm{VI}(C, f)$ and moreover, $\widehat{u} = \lim_{n\to\infty} P_{\mathrm{VI}(C,f)}(x_n)$.

Here we recall some known results with the corresponding references.

Lemma 2.1 [6] *For $x, y \in \mathcal{H}$, we have the following statements*:

- $|\langle x, y\rangle| \le \|x\|\|y\|$;
- $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$ (*the subdifferential inequality*).

Lemma 2.2 [22] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{a_{m_j}\}$ of $\{a_n\}$ such that $a_{m_j} < a_{m_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{n_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} n_k = \infty$ and the following properties are satisfied by all* (*sufficiently large*) *number $k \in \mathbb{N}$*:

$$a_{n_k} \le a_{n_k+1} \quad \text{and} \quad a_k \le a_{n_k+1}.$$

In fact, n_k is the largest number n in the set $\{1, \dots, k\}$ such that $a_n < a_{n+1}$ holds.

Lemma 2.3 [23, 24] *Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\alpha_n\}$ a sequence in $]0, 1[$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{b_n\}$ a sequence of non-negative real numbers with $\sum_{n=1}^{\infty} b_n < \infty$ and $\{\gamma_n\}$ a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \le 0$. Suppose that the following inequality*:

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\gamma_n + b_n$$

holds for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n = 0$.

A special case of this lemma already appears in the proof of Theorem 1 of [25].

Let C be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and $A : C \to \mathcal{H}$ be a mapping. Then A is called *hemicontinuous* [26] iff, for any $x, y \in C$ and $z \in \mathcal{H}$, the function

$$t \mapsto \big\langle z, A\big(tx + (1-t)y\big)\big\rangle$$

of $[0, 1]$ into $\mathbb{R}$ is continuous.

Lemma 2.4 (See, e.g., [27, Lemma 7.1.7]) *Let C be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$. Let $A : C \to \mathcal{H}$ be a monotone and hemicontinuous mapping and $\widehat{x} \in C$. Then*

$$\widehat{x} \in \mathrm{VI}(C, A) \quad \Leftrightarrow \quad \langle Ax, x-\widehat{x}\rangle \ge 0 \quad \textit{for all } x \in C.$$

3 The Subgradient Extragradient Algorithm

Inspired by Halpern's result [17], we introduce the subgradient extragradient algorithm which finds a solution of the variational inequality and we also prove a strong convergence theorem. Our strong convergence theorem is quite different from the scheme proposed by Censor et al. [15]. In fact, we do not need to calculate the projections onto the constructible sets C_n and Q_n as in [15]. It seems to us that we simplify their result with the same conclusion.

The following lemma is extracted from Lemma 5.2 of [14].

Lemma 3.1 *Let $f:\mathcal{H}\to\mathcal{H}$ be a monotone and L-Lipschitz mapping on C and τ be a positive number and suppose that* $\mathrm{VI}(C,f)$ *is nonempty. Let $x\in\mathcal{H}$. Define*

$$\begin{aligned}U(x)&:=P_C\bigl(x-\tau f(x)\bigr)\\ T^x&:=\bigl\{w\in\mathcal{H}:\bigl\langle x-\tau f(x)-U(x),w-U(x)\bigr\rangle\le 0\bigr\}\\ V(x)&:=P_{T^x}\bigl(x-\tau f\bigl(U(x)\bigr)\bigr).\end{aligned}$$

Then, for all $u\in\mathrm{VI}(C,f)$, we have

$$\bigl\|V(x)-u\bigr\|^2\le\|x-u\|^2-(1-\tau L)\bigl\|x-U(x)\bigr\|^2-(1-\tau L)\bigl\|V(x)-U(x)\bigr\|^2. \quad (3)$$

In particular, if $\tau L\le 1$, we have $\|V(x)-u\|\le\|x-u\|$.

Proof First, we consider

$$\begin{aligned}\bigl\|V(x)-u\bigr\|^2&\le\bigl\|\bigl(x-\tau f\bigl(U(x)\bigr)\bigr)-u\bigr\|^2-\bigl\|\bigl(x-\tau f\bigl(U(x)\bigr)\bigr)-V(x)\bigr\|^2\\ &=\|x-u\|^2+2\tau\bigl\langle u-V(x),f\bigl(U(x)\bigr)\bigr\rangle-\bigl\|x-V(x)\bigr\|^2\\ &=\|x-u\|^2+2\tau\bigl\langle u-U(x),f\bigl(U(x)\bigr)-f(u)\bigr\rangle\\ &\quad+2\tau\bigl\langle u-U(x),f(u)\bigr\rangle+2\tau\bigl\langle U(x)-V(x),f\bigl(U(x)\bigr)\bigr\rangle-\bigl\|x-V(x)\bigr\|^2\\ &\le\|x-u\|^2+2\tau\bigl\langle U(x)-V(x),f\bigl(U(x)\bigr)\bigr\rangle-\bigl\|x-V(x)\bigr\|^2\\ &=\|x-u\|^2+2\tau\bigl\langle U(x)-V(x),f\bigl(U(x)\bigr)\bigr\rangle-\bigl\|x-U(x)\bigr\|^2\\ &\quad-2\tau\bigl\langle x-U(x),U(x)-V(x)\bigr\rangle-\bigl\|U(x)-V(x)\bigr\|^2\\ &=\|x-u\|^2-\bigl\|x-U(x)\bigr\|^2-\bigl\|U(x)-V(x)\bigr\|^2\\ &\quad+2\bigl\langle x-\tau f\bigl(U(x)\bigr)-U(x),V(x)-U(x)\bigr\rangle.\end{aligned}$$

Now we estimate

$$\begin{aligned}&\bigl\langle x-\tau f\bigl(U(x)\bigr)-U(x),V(x)-U(x)\bigr\rangle\\ &\quad=\bigl\langle x-\tau f(x)-U(x),V(x)-U(x)\bigr\rangle+\bigl\langle\tau f(x)-\tau f\bigl(U(x)\bigr),V(x)-U(x)\bigr\rangle\\ &\quad\le\bigl\langle\tau f(x)-\tau f\bigl(U(x)\bigr),V(x)-U(x)\bigr\rangle\\ &\quad\le\tau L\bigl\|x-U(x)\bigr\|\bigl\|V(x)-U(x)\bigr\|.\end{aligned}$$

So we have

$$\begin{aligned}\bigl\|V(x)-u\bigr\|^2&\le\|x-u\|^2-\bigl\|x-U(x)\bigr\|^2-\bigl\|U(x)-V(x)\bigr\|^2\\ &\quad+2\tau L\bigl\|x-U(x)\bigr\|\bigl\|V(x)-U(x)\bigr\|\\ &=\|x-u\|^2-(1-\tau L)\bigl\|x-U(x)\bigr\|^2-(1-\tau L)\bigl\|U(x)-V(x)\bigr\|^2\\ &\quad-\tau L\bigl(\bigl\|x-U(x)\bigr\|-\bigl\|V(x)-U(x)\bigr\|\bigr)^2\\ &\le\|x-u\|^2-(1-\tau L)\bigl\|x-U(x)\bigr\|^2-(1-\tau L)\bigl\|U(x)-V(x)\bigr\|^2.\end{aligned}$$ □

We denote the strong and weak convergence of a sequence $\{x_n\}$ in $\mathcal{H}$ to an element $x \in \mathcal{H}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Recall that the mapping $I - T$ is *demiclosed at zero* [28] iff $x \in \mathrm{Fix}(T)$ whenever $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$. It is known that, if $T : \mathcal{H} \to \mathcal{H}$ is nonexpansive, then $I - T$ is demiclosed at zero.

The next result is the demiclosedness-like property for the mapping $P_C(I - \tau f)$. Note that we do not use the concept of the maximal monotonicity of $f + N_C$, where N_C is the normal cone of C, as was the case in other papers (see, e.g., [9–11]).

Lemma 3.2 *Let $f : \mathcal{H} \to \mathcal{H}$ be a monotone and L-Lipschitz mapping on C. Let $U := P_C(I - \tau f)$ where $\tau > 0$. If $\{x_n\}$ is a sequence in C satisfying $x_n \rightharpoonup \widehat{x}$ and $x_n - U(x_n) \to 0$, then $\widehat{x} \in \mathrm{VI}(C, f) = \mathrm{Fix}(U)$.*

Proof Since f is monotone and hemicontinuous, it suffices to show that

$$\langle f(x), x - \widehat{x} \rangle \geq 0 \quad \text{for all } x \in C.$$

Let $x \in C$ and $\tau > 0$. Note that

$$\langle x_n - \tau f(x_n) - U(x_n), U(x_n) - x \rangle \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

Next, we consider

$$\begin{aligned}
\langle \tau f(x_n), x_n - x \rangle &= \langle \tau f(x_n), x_n - U(x_n) \rangle + \langle \tau f(x_n), U(x_n) - x \rangle \\
&= \langle \tau f(x_n), x_n - U(x_n) \rangle - \langle x_n - \tau f(x_n) - U(x_n), U(x_n) - x \rangle \\
&\quad + \langle x_n - U(x_n), U(x_n) - x \rangle \\
&\leq \langle \tau f(x_n), x_n - U(x_n) \rangle + \langle x_n - U(x_n), U(x_n) - x \rangle \\
&\leq \tau \| f(x_n) \| \| x_n - U(x_n) \| + \| x_n - U(x_n) \| \| U(x_n) - x \|.
\end{aligned}$$

Since $\{f(x_n)\}$ is bounded and $x_n - U(x_n) \to 0$, $\limsup_{n\to\infty} \langle \tau f(x_n), x_n - x \rangle \leq 0$. It follows from the monotonicity of f that

$$\langle f(x), \widehat{x} - x \rangle = \frac{1}{\tau} \limsup_{n\to\infty} \langle \tau f(x), x_n - x \rangle \leq \frac{1}{\tau} \limsup_{n\to\infty} \langle \tau f(x_n), x_n - x \rangle \leq 0.$$

The proof is finished. □

Now we study the following algorithm. For a mapping $f : \mathcal{H} \to \mathcal{H}$ and a closed and convex subset C of $\mathcal{H}$, define two iterative sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{cases}
x_0 \in \mathcal{H}, \\
y_n := P_C(x_n - \tau f(x_n)), \\
T_n := \{ w \in \mathcal{H} : \langle x_n - \tau f(x_n) - y_n, w - y_n \rangle \leq 0 \}, \\
x_{n+1} := \alpha_n x_0 + (1 - \alpha_n) P_{T_n}(x_n - \tau f(y_n)),
\end{cases} \tag{4}$$

where $\{\alpha_n\}$ is a sequence in $]0, 1[$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Note that T_n in (4) is just T^{x_n} in Lemma 3.1.

Throughout this paper, we assume that $\mathrm{VI}(C, f)$ is nonempty and we denote $\omega_w\{z_n\}$ the set of all weak cluster points of the sequence $\{z_n\}$.

Lemma 3.3 *Let $f : \mathcal{H} \to \mathcal{H}$ be a monotone and L-Lipschitz mapping on C and τ be a positive real number such that $\tau L \leq 1$. Then the sequence $\{x_n\}$ generated by (4) satisfies the following inequality*:

$$\|x_{n+1} - z\| \leq \alpha_n \|x_0 - z\| + (1 - \alpha_n)\|x_n - z\|,$$

for all $z \in \mathrm{VI}(C, f)$. In particular, $\{x_n\}$ is bounded.

Proof Let $z \in \mathrm{VI}(C, f)$. For convenience, write

$$w_n = P_{T_n}\big(I - \tau f P_C(I - \tau f)\big)x_n.$$

Hence $x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)w_n$. It follows from Lemma 3.1 that $\|w_n - z\| \leq \|x_n - z\|$ and hence

$$\begin{aligned}\|x_{n+1} - z\| &\leq \alpha_n \|x_0 - z\| + (1 - \alpha_n)\|w_n - z\| \\ &\leq \alpha_n \|x_0 - z\| + (1 - \alpha_n)\|x_n - z\|.\end{aligned}$$

In particular,

$$\|x_{n+1} - z\| \leq \max\big\{\|x_0 - z\|, \|x_n - z\|\big\}.$$

By induction, we have

$$\|x_n - z\| \leq \|x_0 - z\| \quad \text{for all } n \in \mathbb{N}.$$

Hence, the sequence $\{x_n\}$ is bounded. □

Theorem 3.1 *Let $f : \mathcal{H} \to \mathcal{H}$ be a monotone and L-Lipschitz mapping on C and τ be a positive real number such that $\tau L < 1$. Let $\{x_n\} \subset \mathcal{H}$ be a sequence generated by (4). Then $x_n \to P_{\mathrm{VI}(C,f)}x_0$.*

Proof Recall that $x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)w_n$. Put $z = P_{\mathrm{VI}(C,f)}x_0$. Let us start from the following inequalities, which are consequences of (4) and the subdifferential inequality:

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1 - \alpha_n)^2\|w_n - z\|^2 + 2\alpha_n\langle x_0 - z, x_{n+1} - z\rangle \\ &\leq (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle x_0 - z, x_{n+1} - z\rangle. \qquad (5)\end{aligned}$$

Let us consider the following two cases.

Case 1: There exists an $n_0 \in \mathbb{N}$ such that $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $n \geq n_0$. Then $\lim_{n\to\infty} \|x_n - z\|$ exists. It follows from (5) that

$$\|w_n - z\|^2 - \|x_n - z\|^2 \to 0.$$

By Lemma 3.1, we conclude that

$$x_n - P_C\big(x_n - \tau f(x_n)\big) \to 0.$$

Using Lemma 3.2, we have $\omega_w\{x_n\} \subset \mathrm{VI}(C, f)$. Passing to a suitable subsequence $\{x_{p_i}\}$, we assume that

$$\limsup_{n\to\infty} \langle x_0 - z, x_{n+1} - z\rangle = \lim_{i\to\infty} \langle x_0 - z, x_{p_i} - z\rangle$$

and

$$x_{p_i} \rightharpoonup z' \quad \text{for some } z' \in \mathrm{VI}(C, f).$$

Consequently,

$$\limsup_{n\to\infty} \langle x_0 - z, x_{n+1} - z\rangle = \big\langle x_0 - P_{\mathrm{VI}(C,f)}x_0, z' - P_{\mathrm{VI}(C,f)}x_0\big\rangle \le 0.$$

By Lemma 2.3, we have $\lim_{n\to\infty} \|x_n - z\|^2 = 0$, that is, $x_n \to z$.

Case 2: There exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that

$$\|x_{m_j} - z\| < \|x_{m_j+1} - z\| \quad \text{for all } j \in \mathbb{N}.$$

From Lemma 2.2, there exists a nondecreasing sequence $\{n_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} n_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{n_k} - z\| \le \|x_{n_k+1} - z\| \quad \text{and} \quad \|x_k - z\| \le \|x_{n_k+1} - z\|. \tag{6}$$

Note that

$$\begin{aligned}\|x_{n_k} - z\| \le \|x_{n_k+1} - z\| &\le \alpha_{n_k}\|x_0 - z\| + (1 - \alpha_{n_k})\|w_{n_k} - z\| \\ &\le \alpha_{n_k}\|x_0 - z\| + (1 - \alpha_{n_k})\|x_{n_k} - z\|.\end{aligned}$$

It follows from $\lim_{n\to\infty} \alpha_n = 0$ that

$$\|w_{n_k} - z\| - \|x_{n_k} - z\| \to 0.$$

By discarding the repeated terms of $\{n_k\}$, but still denoted by $\{n_k\}$, we can view $\{x_{n_k}\}$ as a subsequence of $\{x_n\}$. Hence, by Lemma 3.1 and Lemma 3.2, we have

$$x_{n_k} - P_C\big(x_{n_k} - \tau f(x_{n_k})\big) \to 0 \quad \text{and} \quad \omega_w\{x_{n_k}\} \subset \mathrm{VI}(C, f).$$

Note that $x_{n_k} - x_{n_k+1} \to 0$. In fact, it follows from Lemma 3.1 with the same notion U that $\|w_{n_k} - U(x_{n_k})\| \to 0$, $\|U(x_{n_k}) - x_{n_k}\| \to 0$ and

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &= \big\|\alpha_{n_k}x_0 + (1 - \alpha_{n_k})w_{n_k} - x_{n_k}\big\| \\ &\le \alpha_{n_k}\|x_0 - x_{n_k}\| + (1 - \alpha_{n_k})\|w_{n_k} - x_{n_k}\| \\ &\le \alpha_{n_k}\|x_0 - x_{n_k}\| + (1 - \alpha_{n_k})\big(\big\|w_{n_k} - U(x_{n_k})\big\| + \big\|U(x_{n_k}) - x_{n_k}\big\|\big) \\ &\to 0.\end{aligned}$$

As proved in the first case, we can conclude that

$$\limsup_{k\to\infty}\langle x_0 - z, x_{n_k+1} - z\rangle = \limsup_{k\to\infty}\langle x_0 - z, x_{n_k} - z\rangle \le 0 \tag{7}$$

It follows then from (5) and (6) that

$$\begin{aligned}\|x_{n_k+1} - z\|^2 &\le (1-\alpha_{n_k})\|x_{n_k} - z\|^2 + 2\alpha_{n_k}\langle x_0 - z, x_{n_k+1} - z\rangle\\ &\le (1-\alpha_{n_k})\|x_{n_k+1} - z\|^2 + 2\alpha_{n_k}\langle x_0 - z, x_{n_k+1} - z\rangle.\end{aligned}$$

In particular, since $\alpha_{n_k} > 0$,

$$\|x_k - z\|^2 \le \|x_{n_k+1} - z\|^2 \le 2\langle x_0 - z, x_{n_k+1} - z\rangle.$$

Hence, by (7), we have

$$\limsup_{k\to\infty}\|x_k - z\|^2 \le 2\limsup_{k\to\infty}\langle x_0 - z, x_{n_k+1} - z\rangle \le 0.$$

Therefore, $x_k \to z$. □

4 The Modified Subgradient Extragradient Algorithm

Inspired by the second main result of Censor et al. [14], we present a modified subgradient extragradient algorithm for finding a solution of the variational inequality which is also a fixed point of a given nonexpansive mapping. Our algorithm is as follows.

For mappings $f, S : \mathcal{H} \to \mathcal{H}$ and a closed and convex subset C of $\mathcal{H}$, define three iterative sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by

$$\begin{cases} x_0 \in \mathcal{H},\\ y_n := P_C(x_n - \tau f(x_n)),\\ T_n := \{w \in \mathcal{H} : \langle x_n - \tau f(x_n) - y_n, w - y_n\rangle \le 0\},\\ z_n := \alpha_n x_0 + (1-\alpha_n)P_{T_n}(x_n - \tau f(y_n)),\\ x_{n+1} := \beta_n x_n + (1-\beta_n)Sz_n, \end{cases} \tag{8}$$

where $\{\beta_n\} \subset [a,b] \subset]0,1[$ for some $a, b \in]0,1[$ and $\{\alpha_n\}$ is a sequence in $]0,1[$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Theorem 4.1 *Let $S : \mathcal{H} \to \mathcal{H}$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $f : \mathcal{H} \to \mathcal{H}$ a monotone and L-Lipschitz mapping on C. Let τ be a positive real number such that $\tau L < 1$. Suppose that $\mathrm{VI}(C, f) \cap \mathrm{Fix}(S)$ is nonempty. Let $\{x_n\} \subset \mathcal{H}$ be a sequence generated by (8). Then $x_n \to P_{\mathrm{VI}(C,f)\cap\mathrm{Fix}(S)}x_0$.*

We split the proof into several lemmas.

Lemma 4.1 *The sequence* $\{x_n\}$ *is bounded.*

Proof Let $u \in \mathrm{VI}(C, f) \cap \mathrm{Fix}(S)$. Then we have

$$\begin{aligned}
\|x_{n+1} - u\| &\le \beta_n \|x_n - u\| + (1 - \beta_n)\big\|S(z_n) - u\big\| \\
&\le \beta_n \|x_n - u\| + (1 - \beta_n)\|z_n - u\| \\
&= \beta_n \|x_n - u\| + (1 - \beta_n)\big\|\alpha_n x_0 + (1 - \alpha_n) w_n - u\big\| \\
&\le \beta_n \|x_n - u\| + (1 - \beta_n)\big(\alpha_n \|x_0 - u\| + (1 - \alpha_n)\|w_n - u\|\big) \\
&\le \beta_n \|x_n - u\| + (1 - \beta_n)\big(\alpha_n \|x_0 - u\| + (1 - \alpha_n)\|x_n - u\|\big) \\
&\le \max\big\{\|x_0 - u\|, \|x_n - u\|\big\}.
\end{aligned}$$

By induction, the sequence $\{x_n\}$ is bounded. □

Lemma 4.2 *The following inequality holds for all* $u \in \mathrm{VI}(C, f) \cap \mathrm{Fix}(S)$ *and* $n \in \mathbb{N}$,

$$\begin{aligned}
\|x_{n+1} - u\|^2 \le{}& \big(1 - \alpha_n(1 - \beta_n)\big)\|x_n - u\|^2 + 2\alpha_n(1 - \beta_n)\langle x_0 - u, z_n - u\rangle \\
&- \beta_n(1 - \beta_n)\big\|x_n - S(z_n)\big\|^2.
\end{aligned} \tag{9}$$

Proof Let $u \in \mathrm{VI}(C, f) \cap \mathrm{Fix}(S)$ and put $w_n := P_{T_n}(I - \tau f(P_C(I - \tau f)))x_n$. It follows from Lemma 3.1 with $\tau L < 1$ and the subdifferential inequality that

$$\begin{aligned}
\|x_{n+1} - u\|^2 &= \big\|\beta_n(x_n - u) + (1 - \beta_n)\big(S(z_n) - u\big)\big\|^2 \\
&= \beta_n \|x_n - u\|^2 + (1 - \beta_n)\big\|S(z_n) - u\big\|^2 - \beta_n(1 - \beta_n)\big\|x_n - S(z_n)\big\|^2 \\
&\le \beta_n \|x_n - u\|^2 + (1 - \beta_n)\|z_n - u\|^2 - \beta_n(1 - \beta_n)\big\|x_n - S(z_n)\big\|^2 \\
&= \beta_n \|x_n - u\|^2 + (1 - \beta_n)\big\|\alpha_n x_0 + (1 - \alpha_n) w_n - u\big\|^2 \\
&\quad - \beta_n(1 - \beta_n)\big\|x_n - S(z_n)\big\|^2 \\
&\le \beta_n \|x_n - u\|^2 - \beta_n(1 - \beta_n)\big\|x_n - S(z_n)\big\|^2 \\
&\quad + (1 - \beta_n)\big((1 - \alpha_n)^2\|w_n - u\|^2 + 2\alpha_n\langle x_0 - u, z_n - u\rangle\big) \\
&\le \beta_n \|x_n - u\|^2 - \beta_n(1 - \beta_n)\big\|x_n - S(z_n)\big\|^2 \\
&\quad + (1 - \beta_n)\big((1 - \alpha_n)\|x_n - u\|^2 + 2\alpha_n\langle x_0 - u, z_n - u\rangle\big) \\
&= \big(1 - \alpha_n(1 - \beta_n)\big)\|x_n - u\|^2 + 2\alpha_n(1 - \beta_n)\langle x_0 - u, z_n - u\rangle \\
&\quad - \beta_n(1 - \beta_n)\big\|x_n - S(z_n)\big\|^2.
\end{aligned}$$
□

Lemma 4.3 *Let* $u \in \mathrm{VI}(C, f) \cap \mathrm{Fix}(S)$. *If there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\liminf_{k\to\infty}(\|x_{n_k+1} - u\| - \|x_{n_k} - u\|) \ge 0$, *then* $\omega_w\{x_{n_k}\} \subset \mathrm{VI}(C, f) \cap \mathrm{Fix}(S)$.

Proof Observe that, whenever $\liminf_{k\to\infty}(\|x_{n_k+1}-u\|-\|x_{n_k}-u\|)\geq 0$, we get

$$\begin{aligned}
0 &\leq \liminf_{k\to\infty}\bigl(\|x_{n_k+1}-u\|-\|x_{n_k}-u\|\bigr)\\
&\leq \liminf_{k\to\infty}\bigl(\beta_{n_k}\|x_{n_k}-u\|+(1-\beta_{n_k})\bigl\|S\bigl(\alpha_{n_k}x_0+(1-\alpha_{n_k})w_{n_k}\bigr)-u\bigr\|-\|x_{n_k}-u\|\bigr)\\
&\leq \liminf_{k\to\infty}(1-\beta_{n_k})\bigl(\alpha_{n_k}\|x_0-u\|+(1-\alpha_{n_k})\|w_{n_k}-u\|-\|x_{n_k}-u\|\bigr)\\
&= \liminf_{k\to\infty}(1-\beta_{n_k})\bigl(\|w_{n_k}-u\|-\|x_{n_k}-u\|\bigr)\\
&\leq (1-a)\liminf_{k\to\infty}\bigl(\|w_{n_k}-u\|-\|x_{n_k}-u\|\bigr)\\
&\leq (1-a)\limsup_{k\to\infty}\bigl(\|w_{n_k}-u\|-\|x_{n_k}-u\|\bigr)\\
&\leq 0.
\end{aligned}$$

Hence $\|w_{n_k}-u\|-\|x_{n_k}-u\|\to 0$. It follows from Lemma 3.1 and Lemma 3.2 that

$$x_{n_k}-w_{n_k}\to 0 \quad\text{and}\quad \omega_w\{x_{n_k}\}\subset \mathrm{VI}(C,f). \tag{10}$$

We next show that $\omega_w\{x_{n_k}\}\subset \mathrm{Fix}(S)$. By (9), we have

$$\begin{aligned}
0 &\leq \liminf_{k\to\infty}\bigl(\|x_{n_k+1}-u\|^2-\|x_{n_k}-u\|^2\bigr)\\
&\leq \liminf_{k\to\infty}\bigl(-\alpha_{n_k}(1-\beta_{n_k})\|x_{n_k}-u\|^2+2\alpha_{n_k}(1-\beta_{n_k})\langle x_0-u,z_{n_k}-u\rangle\\
&\quad -\beta_{n_k}(1-\beta_{n_k})\bigl\|x_{n_k}-S(z_{n_k})\bigr\|^2\bigr)\\
&= -\limsup_{k\to\infty}\beta_{n_k}(1-\beta_{n_k})\bigl\|x_{n_k}-S(z_{n_k})\bigr\|^2\\
&\leq -a(1-b)\limsup_{k\to\infty}\bigl\|x_{n_k}-S(z_{n_k})\bigr\|^2.
\end{aligned}$$

Hence $x_{n_k}-S(z_{n_k})\to 0$. It follows from (10) that

$$z_{n_k}-x_{n_k}=\alpha_n(x_0-x_{n_k})+(1-\alpha_n)(w_{n_k}-x_{n_k})\to 0. \tag{11}$$

Therefore

$$\bigl\|z_{n_k}-S(z_{n_k})\bigr\|\leq\bigl\|z_{n_k}-x_{n_k}\bigr\|+\bigl\|x_{n_k}-S(z_{n_k})\bigr\|\to 0.$$

By (11) and the demiclosedness of the mapping $I-S$, we get

$$\omega_w\{x_{n_k}\}=\omega_w\{z_{n_k}\}\subset \mathrm{Fix}(S).$$

Then $\omega_w\{x_{n_k}\}\subset \mathrm{VI}(C,f)\cap \mathrm{Fix}(S)$. □

Proof of Theorem 4.1 Let $z:=P_{\mathrm{VI}(C,f)\cap\mathrm{Fix}(S)}x_0$. Since $\beta_n<1$ for all $n\in\mathbb{N}$, it follows from (9) that

$$\|x_{n+1}-z\|^2\leq\bigl(1-\alpha_n(1-\beta_n)\bigr)\|x_n-u\|^2+2\alpha_n(1-\beta_n)\langle x_0-u,z_n-u\rangle. \tag{12}$$

Case 1: There exists an $n_0 \in \mathbb{N}$ such that $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $n \geq n_0$. Then $\lim_{n\to\infty} \|x_n - z\|$ exists. In particular, $\liminf_{n\to\infty}(\|x_{n+1} - z\| - \|x_n - z\|) = 0$. It follows from Lemma 4.3 that $\omega_w\{x_n\} \subset \mathrm{VI}(C, f) \cap \mathrm{Fix}(S)$ and $w_n - x_n \to 0$. Since $z_n - x_n = \alpha_n(x_0 - x_n) + (1 - \alpha_n)(w_n - x_n) \to 0$, we have $\omega_w\{z_n\} = \omega_w\{x_n\}$ Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \widehat{x}$ and

$$\lim_{k\to\infty} \langle x_0 - z, x_{n_k} - z\rangle = \limsup_{n\to\infty}\langle x_0 - z, x_n - z\rangle = \limsup_{n\to\infty}\langle x_0 - z, z_n - z\rangle.$$

Because $\omega_w\{x_n\} \subset \mathrm{VI}(C, f)$, we have

$$\lim_{k\to\infty} \langle x_0 - z, x_{n_k} - z\rangle = \langle x_0 - z, \widehat{x} - z\rangle \leq 0.$$

Hence $\limsup_{n\to\infty}\langle x_0 - z, z_n - z\rangle \leq 0$. By applying Lemma 2.3 to (12), we have $\|x_n - z\| \to 0$, that is, $x_n \to z$.

Case 2: There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\|x_{m_j} - z\| < \|x_{m_j+1} - z\| \quad \text{for all } j \in \mathbb{N}.$$

From Lemma 2.2, there exists a nondecreasing sequence $\{n_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} n_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{n_k} - z\| \leq \|x_{n_k+1} - z\| \quad \text{and} \quad \|x_k - z\| \leq \|x_{n_k+1} - z\|. \tag{13}$$

By discarding the repeated terms of $\{n_k\}$, but still denoted by $\{n_k\}$, we can view $\{x_{n_k}\}$ as a subsequence of $\{x_n\}$. In this case, we have $\liminf_{k\to\infty}(\|x_{n_k+1} - z\| - \|x_{n_k} - z\|) \geq 0$. Hence $\omega_w\{x_{n_k}\} \subset \mathrm{VI}(C, f) \cap \mathrm{Fix}(S)$ and, by the same argument as in the first case, $\omega_w\{z_{n_k}\} = \omega_w\{x_{n_k}\}$. It follows from the boundedness of $\{x_{n_k}\}$ that there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \rightharpoonup \widehat{x}$ and

$$\lim_{l\to\infty} \langle x_0 - z, x_{n_{k_l}} - z\rangle = \limsup_{k\to\infty}\langle x_0 - z, x_{n_k} - z\rangle = \limsup_{k\to\infty}\langle x_0 - z, z_{n_k} - z\rangle.$$

Because $\omega_w\{x_{n_k}\} \subset \mathrm{VI}(C, f)$, we have

$$\limsup_{k\to\infty}\langle x_0 - z, z_{n_k} - z\rangle = \lim_{l\to\infty} \langle x_0 - z, x_{n_{k_l}} - z\rangle = \langle x_0 - z, \widehat{x} - z\rangle \leq 0.$$

It follows from (12) and (13) that

$$\begin{aligned}\|x_{n_k+1} - z\|^2 &\leq \big(1 - \alpha_{n_k}(1 - \beta_{n_k})\big)\|x_{n_k} - u\|^2 + 2\alpha_{n_k}(1 - \beta_{n_k})\langle x_0 - u, z_{n_k} - u\rangle \\ &\leq \big(1 - \alpha_{n_k}(1 - \beta_{n_k})\big)\|x_{n_k+1} - u\|^2 + 2\alpha_{n_k}(1 - \beta_{n_k})\langle x_0 - u, z_{n_k} - u\rangle.\end{aligned}$$

In particular, since $\alpha_{n_k}(1 - \beta_{n_k}) > 0$ for all $k \in \mathbb{N}$,

$$\|x_k - z\|^2 \leq \|x_{n_k+1} - z\|^2 \leq 2\langle x_0 - z, x_{n_k+1} - z\rangle.$$

Consequently,

$$\limsup_{k\to\infty} \|x_{n_k} - z\|^2 \leq \limsup_{k\to\infty} 2\langle x_0 - z, z_{n_k} - z\rangle \leq 0.$$

Therefore $x_k \to z$. □

We next introduce another algorithm, which is a slight modification of (8). The iteration is obtained in the theorem below by using the same restrictions on parameters as in Theorem 4.1. Under some appropriate conditions, this new iterative sequence not only converges to a common solution of a variational inequality and a fixed point of a given quasi-nonexpansive mapping, but it also includes the algorithm (4) when S is the identity mapping. Since the proof of this result is very similar to that of Theorem 4.1, we leave the proof for the reader to verify.

Theorem 4.2 *Let $S:\mathcal{H}\to\mathcal{H}$ be a quasi-nonexpansive mapping such that $I-S$ is demiclosed at zero and $f:\mathcal{H}\to\mathcal{H}$ a monotone and L-Lipschitz mapping on C. Let τ be a positive real number such that $\tau L<1$. Suppose that $\mathrm{VI}(C,f)\cap\mathrm{Fix}(S)$ is nonempty. Let $\{x_n\},\{y_n\},\{z_n\}\subset\mathcal{H}$ be sequences generated by*

$$\begin{cases} x_0\in\mathcal{H},\\ y_n:=P_C(x_n-\tau f(x_n)),\\ T_n:=\{w\in\mathcal{H}:\langle x_n-\tau f(x_n)-y_n, w-y_n\rangle\le 0\},\\ z_n:=\alpha_n x_0+(1-\alpha_n)P_{T_n}(x_n-\tau f(y_n)),\\ x_{n+1}:=\beta_n z_n+(1-\beta_n)Sz_n,\end{cases}$$

where $\{\beta_n\}\subset[a,b]\subset]0,1[$ for some $a,b\in]0,1[$ and $\{\alpha_n\}$ is a sequence in $]0,1[$ satisfying $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$. Then $x_n\to P_{\mathrm{VI}(C,f)\cap\mathrm{Fix}(S)}x_0$.

5 Conclusions

The subgradient extragradient method initiated by Censor et al. [14] provides a *weak* convergence theorem for variational inequalities of monotone and Lipschitz continuous operators in Hilbert spaces. In this paper, we modified this method to obtain *strong* convergence by means of Halpern method [17, 18]. It should be noted that our strong convergence theorem is different from the one studied in [15]. We also presented two iterative methods for the problem of finding a common element of the solution set of a variational inequality and of the set of fixed point of a quasi-nonexpansive mapping with a demiclosedness property. Since every nonexpansive mapping with a fixed point is quasi-nonexpansive and satisfies a demiclosedness property, it follows that our two methods improve and extend the corresponding result of Censor et al. [14].

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STRONG CONVERGENCE FOR A COMMON FIXED POINT OF TWO DIFFERENT GENERALIZATIONS OF CUTTER OPERATORS

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ABSTRACT. We propose two iterative methods for finding a common fixed point of two different generalizations of cutter mappings in Banach spaces. The results obtained in this paper extend the recent results announced by Kimura et al.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. We say that $T : H \to H$ is a *cutter operator* if $\mathrm{Fix}(T) := \{x \in H : x = Tx\} \neq \varnothing$ and

$$\langle Tx - z, Tx - x\rangle \leq 0 \quad \text{for all } x \in H \text{ and } z \in \mathrm{Fix}(T).$$

This type of operators was studied by Bauschke and Combettes [5] and Combettes [9]. The term cutter operator was proposed by Cegielski and Censor [7]. These operators play an important and interesting role in various nonlinear problems. The purpose of this paper is to continue the study of these operators in Banach space setting.

Let E be a real Banach space with the norm $\|\cdot\|$. We say that E is

- *smooth* if the limit $\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in E$ with $\|x\| = \|y\| = 1$;
- *Fréchet smooth* if the limit above does not only exists but is also attained uniformly for all $\|y\| = 1$ whenever x is fixed and $\|x\| = 1$;
- *uniformly smooth* if the limit above does not only exists but is also attained uniformly for all $x, y \in E$ with $\|x\| = \|y\| = 1$.

For more details on the geometry of Banach spaces we refer the reader to [18].

Throughout the paper, we denote by E^* the dual space of E and denote by $\langle\cdot,\cdot\rangle$ the dual pairing acting from $E \times E^*$ into $\mathbb{R}$, that is, whenever $x \in E$ and $x^* \in E^*$, $\langle x, x^*\rangle$ denote the value of x^* at x. We use the notions $\to$ and $\rightharpoonup$ for strong and

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weak convergences, respectively. For a bounded sequence $\{x_n\}$, let

$$\omega_w\{x_n\} = \{z : \exists\{x_{n_k}\} \subset \{x_n\} \text{ such that } x_{n_k} \rightharpoonup z \text{ as } k \to \infty\}.$$

The *duality mapping* $J : E \to 2^{E^*}$ is the point-to-set mapping defined by

$$x \mapsto Jx := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

The following facts are known and referred in the paper.

- If E is smooth, then Jx is a singleton for all $x \in E$, and hence we treat J as a single-valued mapping from E into E^*.
- If E is Fréchet smooth, then $J : E \to E^*$ is norm-to-norm continuous.
- If E is uniformly smooth, then $J : E \to E^*$ is uniformly norm-to-norm continuous on bounded subsets of E.
- If E^* is Fréchet smooth and $\{x_n\}$ is a sequence in E such that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

In a similar way, we consider the duality mapping $J^* : E^* \to 2^{E^{**}}$. It is not hard to see that if E and E^* are smooth and E is reflexive, then $J : E \to E^*$ is bijective and $J^* = J^{-1}$. We refer the readers to [8] and its review [21] for further information on duality mappings.

Let C be a closed and convex subset of a smooth Banach space E. The following mappings are two different generalizations of cutter operators in Banach space setting. A mapping $T : C \to E$ is said to be

- *cutter mapping of type (P)* if $\mathrm{Fix}(T) \neq \varnothing$ and $\langle Tx - z, J(Tx - x) \rangle \leq 0$ for all $x \in C$ and $z \in \mathrm{Fix}(T)$;
- *cutter mapping of type (Q)* if $\mathrm{Fix}(T) \neq \varnothing$ and $\langle Tx - z, JTx - Jx \rangle \leq 0$ for all $x \in C$ and $z \in \mathrm{Fix}(T)$.

The notations (P) and (Q) are from the recent paper of Aoyama et al. (see [3]). This definition of mappings is a particular case of the quasi-Bregman firmly nonexpansive mappings which was introduced first in 2003 by Bauschke, Borwein and Combettes in [4]. This class and several more class of operators with respect to Bregman distances were studied intensively during the last ten years (see, for instance, [4, 17, 24]).

We recall the concept of the distance-like function in a smooth Banach space E. Let $\varphi : E \times E \to \mathbb{R}$ be defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \text{for all } x, y \in E.$$

It is clear that $(\|x\| - \|y\|)^2 \leq \varphi(x, y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$. If E is a Hilbert space, then $\varphi(x; y) = \|x - y\|^2$. It is also known that if E and E^* are smooth spaces, then

$$\varphi(x, y) = 0 \iff x = y.$$

Due to this function φ, Alber [1] introduced the following type of projection. Suppose that E is a reflexive Banach space such that E and E^* are smooth, and C is a nonempty, closed and convex subset of E. It is known that for each $x \in E$ there exists a unique element z in C, denoted by $\Pi_C x$, such that

$$\varphi(\Pi_C x, x) = \inf\{\varphi(y, x) : y \in C\}.$$

Moreover, the relation above can be characterized by the following inequalities: for $z \in C$,

$$\begin{aligned} z = \Pi_C x &\iff \langle y - z, Jx - Jz\rangle \le 0 \quad \text{for all } y \in C \\ &\iff \varphi(y,z) + \varphi(z,x) \le \varphi(y,x) \quad \text{for all } y \in C. \end{aligned}$$

It is not hard to see that $\Pi_C : E \to C$ is a cutter mapping of type (Q).

In this paper, we also deal with the metric projection. For a closed and convex subset C and for $x \in E$, there exists a unique element z in C, denoted by $P_C x$, such that

$$\|P_C x - x\| = \inf\{\|y - x\| : y \in C\}.$$

It is also not hard to see that $P_C : E \to C$ is a cutter mapping of type (P) (for example, see [28]).

The following result shows a relation between convergences in the sense of φ and of the norm.

Lemma 1.1 (Kamimura and Takahashi [12]). *Suppose that E is a smooth Banach space and E^* is uniformly smooth. If $\{x_n\}$ and $\{y_n\}$ are sequences in E such that one of them is bounded and $\varphi(x_n, y_n) \to 0$, then $\|x_n - y_n\| \to 0$.*

We also need the following lemma proved by Maingé.

Lemma 1.2 ([16]). *Let $\{\gamma_n\}$ be a sequence of real numbers such that there exists a subsequence $\{\gamma_{n_j}\}$ of $\{\gamma_n\}$ such that $\gamma_{n_j} < \gamma_{n_j+1}$ for all $j \ge 1$. Then there exists a nondecreasing sequence $\{m_k\}$ of positive integers such that $\lim_{k\to\infty} m_k = \infty$ and the following two inequalities:*

$$\gamma_{m_k} \le \gamma_{m_k+1} \quad \text{and} \quad \gamma_k \le \gamma_{m_k+1}$$

hold for all (sufficiently large) numbers k. In fact, m_k is the largest number n in the set $\{1, 2, \ldots, k\}$ such that the condition $\gamma_n < \gamma_{n+1}$ holds.

2. Main results

2.1. Strong convergence via a new averaged projection method of Halpern type. Recall that a mapping $U : C \to E$ is *closed at zero* if whenever $\{x_n\}$ is a sequence in C such that $x_n \to p \in C$ and $Ux_n \to 0$ it follows that $Up = 0$.

Theorem 2.1. *Suppose that E and E^* are uniformly smooth spaces. Let C be a closed and convex subset of E. Suppose that $S, T : C \to C$ are two mappings such that the following properties are satisfied:*

- *S is a cutter mapping of type (P);*
- *T is a cutter mapping of type (Q);*
- *$F := \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \varnothing$;*
- *$I - S$ and $I - T$ are closed at zero.*

Define an iterative sequence $\{x_n\}$ by the following way:

$$\begin{cases} x_1 = \widehat{x} \in C \text{ is arbitrarily chosen;} \\ A_n = \{z \in C : \langle Sx_n - z, J(Sx_n - x_n)\rangle \leq 0\}; \\ B_n = \{z \in C : \langle Tx_n - z, JTx_n - Jx_n\rangle \leq 0\}; \\ C_n = A_n \cap B_n; \\ y_n^* = \alpha_n J\widehat{x} + (1-\alpha_n)\left(\sum_{k=1}^{n} \beta_n^k J\Pi_{C_k} x_n\right); \\ x_{n+1} = \Pi_C J^{-1} y_n^*; \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n^k\}_{n,k}$ are sequences in $(0,1)$ such that

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(2) $\sum_{k=1}^{n} \beta_n^k = 1$ *for all* n;
(3) $\lim_{n\to\infty} \beta_n^k = \beta^k \in (0,1)$ *for all* k *and* $\lim_{n\to\infty} \sum_{k=1}^{n} |\beta_n^k - \beta^k| = 0$.

Then the sequence $\{x_n\}$ converges to $\Pi_F \widehat{x}$.

Remark 2.2. It follows from the assumptions of the theorem that $\sum_{k=1}^{\infty} \beta^k = 1$.

We split the proof of Theorem 2.1 into the following six lemmas.

Lemma 2.3. *If the element x_n is defined, then C_n is a closed and convex subset containing F.*

Denote $z := \Pi_{\bigcap_{n=1}^{\infty} C_n} \widehat{x}$ and $U_n := J^{-1}\left(\sum_{k=1}^{n} \beta_n^k J\Pi_{C_k}\right)$.

Lemma 2.4. *For each $n \geq 1$, the following inequalities hold:*

$$\begin{aligned} \varphi(z, x_{n+1}) &\leq \alpha_n \varphi(z, \widehat{x}) + (1-\alpha_n)\varphi(z, U_n x_n) \\ &\leq \alpha_n \varphi(z, \widehat{x}) + (1-\alpha_n)\left(\varphi(z, x_n) - \sum_{k=1}^{n} \beta_n^k \varphi(\Pi_{C_k} x_n, x_n)\right) \\ &\leq \alpha_n \varphi(z, \widehat{x}) + (1-\alpha_n)\varphi(z, x_n). \end{aligned}$$

In particular, the sequence $\{x_n\}$ is bounded.

Lemma 2.5. *For each $n \geq 1$, the following inequality holds:*

$$\varphi(z, x_{n+1}) \leq (1-\alpha_n)\varphi(z, x_n) + 2\alpha_n \langle J^{-1} y_n^* - z, J\widehat{x} - Jz\rangle.$$

Proof. We first observe the following inequality

$$\varphi(u, J^{-1}(\gamma Jv + (1-\gamma)Jw)) \leq (1-\gamma)\varphi(u, w) + 2\gamma\langle J^{-1}(\gamma Jv + (1-\gamma)Jw) - u, Jv - Ju\rangle$$

whenever $u, v, w \in E$ and $\gamma \in (0,1)$. In fact, it follows from the subdifferential inequality of $\|\cdot\|^2$ on E^*. Consequently,

$$\begin{aligned} &\varphi(z, J^{-1} y_n^*) \\ &= \varphi\left(z, J^{-1}\left(\alpha_n J\widehat{x} + (1-\alpha_n)\left(\sum_{k=1}^{n} \beta_n^k J\Pi_{C_k} x_n\right)\right)\right) \end{aligned}$$

$$\leq (1-\alpha_n)\varphi\left(z, J^{-1}\left(\sum_{k=1}^{n}\beta_n^k J\Pi_{C_k}x_n\right)\right) + 2\alpha_n\langle J^{-1}y_n^* - z, J\widehat{x} - Jz\rangle.$$

Note that $z \in \bigcap_{k=1}^{\infty} C_k$. Hence

$$\varphi(z, \Pi_{C_k}x_n) \leq \varphi(z, \Pi_{C_k}x_n) + \varphi(\Pi_{C_k}x_n, x_n) \leq \varphi(z, x_n).$$

It follows then that

$$\varphi\left(z, J^{-1}\left(\sum_{k=1}^{n}\beta_n^k J\Pi_{C_k}x_n\right)\right) \leq \sum_{k=1}^{n}\beta_n^k\varphi(z, \Pi_{C_k}x_n) \leq \varphi(z, x_n).$$

Therefore, since $z \in C$, we have

$$\varphi(z, x_{n+1}) \leq \varphi(z, J^{-1}y_n^*) \leq (1-\alpha_n)\varphi(z, x_n) + 2\alpha_n\langle J^{-1}y_n^* - z, J\widehat{x} - Jz\rangle. \quad \square$$

The following result can be easily obtained by the recent result of Nilsrakoo and Saejung [20].

Lemma 2.6. *Suppose that*

$$U = J^{-1}\left(\sum_{k=1}^{\infty}\beta^k J\Pi_{C_k}\right)$$

and that $\{z_m\}$ *is a bounded sequence in* C. *Then the following are equivalent:*

- $z_m - \Pi_{C_n}z_m \to 0$ *as* $m \to \infty$ *for all* $n \in \mathbb{N}$;
- $z_m - Uz_m \to 0$.

In particular, $\mathrm{Fix}(U) = \bigcap_{n=1}^{\infty} C_n$. *Moreover,* $JU_n \to JU$ *uniformly on bounded sets.*

Proof. We prove only the last assertion. Let B be a bounded set and let M be a number such that $\|x\| \leq M$ for all $x \in B$. It follows from $z \in \bigcap_{k=1}^{\infty} C_k$ that $(\|z\| - \|\Pi_{C_k}x\|)^2 \leq \varphi(z, \Pi_{C_k}x) \leq \varphi(z, x) \leq (\|z\| + \|x\|)^2 \leq (\|z\| + M)^2$ for all $x \in B$ and $k \in \mathbb{N}$. Hence $\|\Pi_{C_k}x\| \leq 2\|z\| + M$ for all $x \in B$ and $k \in \mathbb{N}$. Consequently, for $x \in B$, we get

$$\begin{aligned}
\|JU_nx - JUx\| &= \left\|\sum_{k=1}^{n}(\beta_n^k - \beta^k)J\Pi_{C_k}x + \sum_{k=n+1}^{\infty}\beta^k J\Pi_{C_k}x\right\| \\
&\leq \sum_{k=1}^{n}|\beta_n^k - \beta^k|\|J\Pi_{C_k}x\| + \sum_{k=n+1}^{\infty}\beta^k\|J\Pi_{C_k}x\| \\
&\leq \left(\sum_{k=1}^{n}|\beta_n^k - \beta^k| + \sum_{k=n+1}^{\infty}\beta^k\right)(2\|z\| + M).
\end{aligned}$$

It follows that $\lim_{n\to\infty}\sup\{\|JU_nx - JUx\| : x \in B\} = 0$. $\square$

Lemma 2.7. *If there exists a subsequence* $\{x_{m_j}\}$ *of* $\{x_n\}$ *such that*

$$\liminf_{j\to\infty}(\varphi(z, x_{m_j+1}) - \varphi(z, x_{m_j})) \geq 0,$$

then $\omega_w\{x_{m_j}\}_{j=1}^{\infty} \subset \bigcap_{n=1}^{\infty} C_n$. *Moreover,* $\limsup_{j\to\infty}\langle J^{-1}y_{m_j}^* - z, J\widehat{x} - Jz\rangle \leq 0$.

Proof. It follows from Lemma 2.4 and $\lim_{n\to\infty}\alpha_n = 0$ that

$$\lim_{j\to\infty}\sum_{k=1}^{m_j}\beta_{m_j}^k\varphi(\Pi_{C_k}x_{m_j}, x_{m_j}) = 0.$$

In particular, for each k, we have

$$\beta^k\lim_{j\to\infty}\varphi(\Pi_{C_k}x_{m_j}, x_{m_j}) = \lim_{j\to\infty}\beta_{m_j}^k\varphi(\Pi_{C_k}x_{m_j}, x_{m_j}) = 0.$$

This implies that $x_{m_j} - \Pi_{C_k}x_{m_j} \to 0$ as $j\to\infty$ because E^* is uniformly smooth. Consequently, $\omega_w\{x_{m_j}\}\subset C_k$. Since the last inclusion holds for all $k\in\mathbb{N}$, we have $\omega_w\{x_{m_j}\}\subset\bigcap_{k=1}^\infty C_k$.

Finally, to prove the "Moreover" part, we claim that $J^{-1}y_{m_j}^* - x_{m_j}\to 0$ as $j\to\infty$. If this is so, then it follows from $\omega_w\{x_{m_j}\}_{j=1}^\infty\subset\bigcap_{k=1}^\infty C_k$ that

$$\begin{aligned}&\limsup_{j\to\infty}\langle J^{-1}y_{m_j}^* - z, J\widehat{x} - Jz\rangle\\ &= \limsup_{j\to\infty}\langle x_{m_j} - \Pi_{\bigcap_{n=1}^\infty C_n}\widehat{x}, J\widehat{x} - J\Pi_{\bigcap_{n=1}^\infty C_n}\widehat{x}\rangle \le 0.\end{aligned}$$

To prove the last claim, let us note from the first part that $x_{m_j} - \Pi_{C_k}x_{m_j}\to 0$ as $j\to\infty$ for all $k = 1, 2, \dots$. In virtue of Lemma 2.6, we have $x_{m_j} - Ux_{m_j}\to 0$ as $j\to\infty$ and hence $Jx_{m_j} - JUx_{m_j}\to 0$ as $j\to\infty$. Note that $JU_n\to JU$ uniformly on bounded sets. It follows then that $Jx_{m_j} - y_{m_j}^* = Jx_{m_j} - JU_{m_j}x_{m_j}\to 0$ as $j\to\infty$, that is, $J^{-1}y_{m_j}^* - x_{m_j}\to 0$ as $j\to\infty$. □

The following lemma also plays an important role in this subsection. However, its proof given there is not quite accurate.

Lemma 2.8 (Saejung and Yotkaew [26]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^\infty\alpha_n = \infty$, and $\{t_n\}$ be a sequence of real numbers. Suppose that*

$$s_{n+1}\le(1-\alpha_n)s_n + \alpha_n t_n\quad\textit{for all } n\ge 1.$$

If $\limsup_{j\to\infty}t_{m_j}\le 0$ for every subsequence $\{s_{m_j}\}$ of $\{s_n\}$ satisfying

$$\liminf_{j\to\infty}(s_{m_j+1} - s_{m_j})\ge 0,$$

then $\lim_{n\to\infty}s_n = 0$.

Proof. The proof is split into two cases.

Case 1: There exists an $n_0\in\mathbb{N}$ such that $s_{n+1}\le s_n$ for all $n\ge n_0$. It follows then that $\lim_{n\to\infty}s_n = s$ for some $s\ge 0$. In particular, $\liminf_{n\to\infty}(s_{n+1} - s_n) = 0$ and hence $\limsup_{n\to\infty}t_n\le 0$. On the other hand, for $n\ge n_0$, we have

$$\alpha_n(s_n - t_n)\le s_n - s_{n+1}.$$

Let $\varepsilon > 0$ be given. Then there exists an integer $n_1\ge n_0$ such that $s_n\ge s-\varepsilon$ and $t_n\le\varepsilon$ for all $n\ge n_1$. For any $n\ge n_1$, we have

$$\alpha_n(s-2\varepsilon)\le\alpha_n(s_n - t_n)\le s_n - s_{n+1}.$$

In particular,

$$(s-2\varepsilon)\sum_{n=n_1}^{\infty}\alpha_n \le s_{n_1}-s<\infty.$$

It follows from $\sum_{n=1}^{\infty}\alpha_n=\infty$ that $s\le 2\varepsilon$. Since $\varepsilon>0$ is arbitrary, we conclude that $s=0$.

Case 2: There exists a subsequence $\{s_{n_j}\}$ of $\{s_n\}$ such that $s_{n_j}<s_{n_j+1}$ for all $j\in\mathbb{N}$. In this case, we can apply Lemma 1.2 to find a nondecreasing sequence $\{m_k\}$ of positive integers such that $\lim_{k\to\infty}m_k=\infty$ and the following two inequalities:

$$s_{m_k}\le s_{m_k+1} \quad \text{and} \quad s_k\le s_{m_k+1}$$

hold for all (sufficiently large) numbers k. Note that $\{s_{m_k}\}$ is not necessarily a subsequence of $\{s_n\}$. Let $\{p_j\}$ be the subsequence of $\{m_k\}$ such that $\{p_j\}$ is strictly increasing and each term in $\{m_k\}$ belongs to $\{p_j\}$. Now $\{s_{p_j}\}$ is a subsequence of $\{s_n\}$. It follows from the first inequality that $\liminf_{j\to\infty}(s_{p_j+1}-s_{p_j})\ge 0$ and hence $\limsup_{j\to\infty}t_{p_j}\le 0$. Moreover, by the first inequality again, we have

$$s_{p_j+1}\le(1-\alpha_{p_j})s_{p_j}+\alpha_{p_j}t_{p_j}\le(1-\alpha_{p_j})s_{p_j+1}+\alpha_{p_j}t_{p_j}.$$

In particular, since each $\alpha_{p_j}>0$, we have $s_{p_j+1}\le t_{p_j}$. Finally, it follows from the second inequality that

$$\limsup_{k\to\infty}s_k\le\limsup_{k\to\infty}s_{m_k+1}=\limsup_{j\to\infty}s_{p_j+1}\le\limsup_{j\to\infty}t_{p_j}\le 0.$$

Hence $\lim_{k\to\infty}s_k=0$.

This completes the proof. □

We now give the proof of the main result.

Proof of Theorem 2.1. Denote $s_n:=\varphi(z,x_n)$ and $t_n:=2\langle J^{-1}y_n^*-z, J\widehat{x}-Jz\rangle$. It follows from Lemma 2.5 that

$$s_{n+1}\le(1-\alpha_n)s_n+\alpha_n t_n \quad \text{for all } n\ge 1.$$

All prerequisites of Lemma 2.8 are satisfied. Then $x_n\to z$.

We are going to make use of the closedness of $I-S$ and $I-T$ at zero. Since $z=\Pi_{\bigcap_{k=1}^{\infty}C_k}\widehat{x}\in\bigcap_{k=1}^{\infty}A_k\subset A_n$ for all n and $Sx_n=P_{A_n}x_n$, we have

$$\|Sx_n-x_n\|\le\|z-x_n\|\to 0.$$

It follows then that $z=Sz$. Similarly, since $z=\Pi_{\bigcap_{n=1}^{\infty}C_n}\widehat{x}\in\bigcap_{n=1}^{\infty}B_n\subset B_n$ for all n and $Tx_n=\Pi_{B_n}x_n$, we have

$$\varphi(Tx_n,x_n)\le\varphi(z,x_n)\to 0.$$

In particular, $x_n-Tx_n\to 0$ by Lemma 1.1 and hence $z=Tz$.

Moreover, it follows from $z=\Pi_{\bigcap_{n=1}^{\infty}C_n}\widehat{x}$ and $F\subset\bigcap_{n=1}^{\infty}C_n$ that $\varphi(z,\widehat{x})\le\varphi(\Pi_F\widehat{x},\widehat{x})$. Because $z\in F$, so $z=\Pi_F\widehat{x}$. The proof is finished. □

Using the same proof (with a slight modification) as the preceding result, we also have the following:

Theorem 2.9. *Suppose that E and E^* are uniformly smooth. Let C be a closed convex subset of E. Suppose that $T : C \to C$ and $S : C \to C$ are two mappings such that the following properties are satisfied:*

- *S is a cutter mapping of type (P);*
- *T is relatively quasi-nonexpansive, that is, $\mathrm{Fix}(T) \neq \varnothing$ and $\varphi(z, Tx) \leq \varphi(z, x)$ for all $x \in C$ and $z \in \mathrm{Fix}(T)$;*
- *$F := \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \varnothing$;*
- *$I - S$ and $I - T$ are closed at zero.*

Define an iterative sequence $\{x_n\}$ by the following way:

$$\begin{cases} x_1 = \widehat{x} \in C \text{ is arbitrarily chosen;} \\ A_n = \{z \in C : \langle Sx_n - z, J(Sx_n - x_n)\rangle \leq 0\}; \\ B_n = \{z \in C : \varphi(z, Tx_n) \leq \varphi(z, x_n)\}; \\ C_n = A_n \cap B_n; \\ y_n^* = \alpha_n J\widehat{x} + (1 - \alpha_n)\left(\displaystyle\sum_{k=1}^{n} \beta_n^k J\Pi_{C_k} x_n\right); \\ x_{n+1} = \Pi_C J^{-1} y_n^*; \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n^k\}_{n,k}$ are sequences in $(0, 1)$ such that

(1) *$\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;*
(2) *$\sum_{k=1}^{n} \beta_n^k = 1$ for all n;*
(3) *$\lim_{n\to\infty} \beta_n^k = \beta^k \in (0, 1)$ for all k and $\lim_{n\to\infty} \sum_{k=1}^{n} |\beta_n^k - \beta^k| = 0$.*

Then the sequence $\{x_n\}$ converges to $\Pi_F \widehat{x}$.

Remark 2.10. Theorem 2.9 can be viewed as an extension of the recent result of Kimura et al. [15]. It is worth mentioning that our assumption on the sequence $\{\beta_n^k\}_{n,k}$ is strictly weaker than that of the aforementioned result. In fact, if $\{\beta_n^k\}_{n,k}$ is a sequence in $(0, 1)$ such that $\sum_{k=1}^{n} \beta_n^k = 1$ for all n and $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\beta_n^k - \beta_{n+1}^k| < \infty$ and $\lim_{n\to\infty} \beta_n^k = \beta^k \in (0, 1)$ for all k, then $\lim_{n\to\infty} \sum_{k=1}^{n} |\beta_n^k - \beta^k| = 0$.

Remark 2.11. Theorem 2.1 itself can be regarded as an extension of Kimura et al. In fact, let $T' : C \to H$ be a quasi-nonexpansive mapping. It is easy to see that

$$\{z \in C : \|z - T'x\| \leq \|z - x\|\} = \{z \in C : \langle Tx - z, Tx - x\rangle \leq 0\}$$

where $T = \frac{1}{2}(I + T')$. Moreover, T is a cutter mapping.

2.2. **Strong convergence via the shrinking projection method.** In this subsection, we present another strong convergence theorem without assuming the uniform smoothness of E and E^*.

Let us recall the concept of Mosco convergence [19] for a sequence of closed and convex sets in a Banach space. Suppose that E is a reflexive Banach space and $\{C_n\}$ is a sequence of nonempty closed and convex subsets of E. We consider the following two sets:

$$x \in \text{s-liminf}_{n\to\infty} C_n \iff \exists\{x_n\} \subset E \text{ such that } x_n \to x \text{ and } x_n \in C_n \text{ for all } n;$$
$$x \in \text{w-limsup}_{n\to\infty} C_n \iff \exists\{n_k\} \subset \{n\}\ \exists\{x_k\} \subset E \text{ such that } x_k \rightharpoonup x$$

Theorem 2.9. *Suppose that E and E^* are uniformly smooth. Let C be a closed convex subset of E. Suppose that $T : C \to C$ and $S : C \to C$ are two mappings such that the following properties are satisfied:*

- *S is a cutter mapping of type (P);*
- *T is relatively quasi-nonexpansive, that is, $\mathrm{Fix}(T) \neq \varnothing$ and $\varphi(z, Tx) \leq \varphi(z, x)$ for all $x \in C$ and $z \in \mathrm{Fix}(T)$;*
- *$F := \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \varnothing$;*
- *$I - S$ and $I - T$ are closed at zero.*

Define an iterative sequence $\{x_n\}$ by the following way:

$$\begin{cases} x_1 = \widehat{x} \in C \text{ is arbitrarily chosen;} \\ A_n = \{z \in C : \langle Sx_n - z, J(Sx_n - x_n)\rangle \leq 0\}; \\ B_n = \{z \in C : \varphi(z, Tx_n) \leq \varphi(z, x_n)\}; \\ C_n = A_n \cap B_n; \\ y_n^* = \alpha_n J\widehat{x} + (1 - \alpha_n)\left(\displaystyle\sum_{k=1}^{n} \beta_n^k J\Pi_{C_k} x_n\right); \\ x_{n+1} = \Pi_C J^{-1} y_n^*; \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n^k\}_{n,k}$ are sequences in $(0, 1)$ such that

(1) *$\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;*
(2) *$\sum_{k=1}^{n} \beta_n^k = 1$ for all n;*
(3) *$\lim_{n\to\infty} \beta_n^k = \beta^k \in (0, 1)$ for all k and $\lim_{n\to\infty} \sum_{k=1}^{n} |\beta_n^k - \beta^k| = 0$.*

Then the sequence $\{x_n\}$ converges to $\Pi_F \widehat{x}$.

Remark 2.10. Theorem 2.9 can be viewed as an extension of the recent result of Kimura et al. [15]. It is worth mentioning that our assumption on the sequence $\{\beta_n^k\}_{n,k}$ is strictly weaker than that of the aforementioned result. In fact, if $\{\beta_n^k\}_{n,k}$ is a sequence in $(0, 1)$ such that $\sum_{k=1}^{n} \beta_n^k = 1$ for all n and $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\beta_n^k - \beta_{n+1}^k| < \infty$ and $\lim_{n\to\infty} \beta_n^k = \beta^k \in (0, 1)$ for all k, then $\lim_{n\to\infty} \sum_{k=1}^{n} |\beta_n^k - \beta^k| = 0$.

Remark 2.11. Theorem 2.1 itself can be regarded as an extension of Kimura et al. In fact, let $T' : C \to H$ be a quasi-nonexpansive mapping. It is easy to see that

$$\{z \in C : \|z - T'x\| \leq \|z - x\|\} = \{z \in C : \langle Tx - z, Tx - x\rangle \leq 0\}$$

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$$x \in \text{s-liminf}_{n\to\infty} C_n \iff \exists\{x_n\} \subset E \text{ such that } x_n \to x \text{ and } x_n \in C_n \text{ for all } n;$$
$$x \in \text{w-limsup}_{n\to\infty} C_n \iff \exists\{n_k\} \subset \{n\}\ \exists\{x_k\} \subset E \text{ such that } x_k \rightharpoonup x$$

$$\text{and } x_k \in C_{n_k} \text{ for all } k.$$

If there exists a subset $C_0 \subset E$ such that $C_0 = \text{s-liminf}_{n\to\infty} C_n = \text{w-limsup}_{n\to\infty} C_n$, then we say that $\{C_n\}$ *converges to* C_0 *in the sense of Mosco* and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. The proof of the following main result makes use of the so-called Tsukada's Theorem.

Lemma 2.12 (Tsukada [30]). *Suppose that E is a smooth Banach space and E^* is Fréchet smooth. If $\{C_n\}$ is a sequence of nonempty closed and convex subsets of E such that $C_0 := \text{M-lim}_{n\to\infty} C_n \neq \varnothing$, then $P_{C_n}x \to P_C x$ for all $x \in E$.*

We also need the following lemma.

Lemma 2.13. *Suppose that E and E^* are Fréchet smooth. If $\{x_n\}$ and $\{y_n\}$ are two sequences in E such that $\varphi(x_n, y_n) \to 0$ and $y_n \to z \in E$, then $x_n \to z$.*

Proof. Note that $\{x_n\}$ and $\{y_n\}$ are bounded, $\varphi(y_n, z) \to 0$, and $Jy_n \to Jz$. Consequently,

$$\varphi(x_n, z) = \varphi(x_n, y_n) + \varphi(y_n, z) + 2\langle x_n - y_n, Jy_n - Jz\rangle \to 0.$$

Next, we show that $\omega_w\{x_n\} = \{z\}$. Suppose that $x_{n_k} \rightharpoonup z'$ for some $\{x_{n_k}\} \subset \{x_n\}$. It follows then that

$$\varphi(z', z) \leq \liminf_{k\to\infty}(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jz\rangle + \|z\|^2) = \liminf_{k\to\infty} \varphi(x_{n_k}, z) = 0.$$

In particular, $z' = z$. Hence, $x_n \rightharpoonup z$. It follows from $\|x_n\| \to \|z\|$ that $x_n \to z$. $\square$

Theorem 2.14. *Let E be a Banach space such that both E and its dual space E^* are Fréchet smooth. Let C be a closed and convex subset of E. Suppose that $S, T : C \to C$ are two mappings such that the following properties are satisfied:*

- *S is a cutter mapping of type (P);*
- *T is a cutter mapping of type (Q);*
- *$F := \text{Fix}(S) \cap \text{Fix}(T) \neq \varnothing$;*
- *$I - S$ and $I - T$ are closed at zero.*

Define an iterative sequence $\{x_n\}$ by the following way:

$$\begin{cases} x_1 = \widehat{x} \in C_1 := C \text{ is arbitrarily chosen;} \\ A_n = \{z \in C : \langle Sx_n - z, J(Sx_n - x_n)\rangle \leq 0\}; \\ B_n = \{z \in C : \langle Tx_n - z, JTx_n - Jx_n\rangle \leq 0\}; \\ C_{n+1} = A_n \cap B_n \cap C_n; \\ x_{n+1} = P_{C_{n+1}}\widehat{x}. \end{cases}$$

Then the sequence $\{x_n\}$ converges to $P_F\widehat{x}$.

Proof. It is clear from the assumption that $F \subset A_n \cap B_n$ for all n and hence $F \subset \bigcap_{n=1}^{\infty} C_n$. In particular, each C_n is a nonempty closed and convex subset of E. Thus $\{x_n\}$ is well-defined. Note that $C_n \supset C_{n+1}$ for all n. This implies that

$$C_0 := \text{M-}\lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n \neq \varnothing.$$

It follows from Lemma 2.12 that $x_n \to P_{C_0}\widehat{x} =: x'$. It is clear from the iteration that

$$Sx_n = P_{A_n}x_n \quad \text{and} \quad Tx_n = \Pi_{B_n}x_n.$$

As $x_{n+1} \in C_{n+1} \subset A_n \cap B_n$, we have

$$\|Sx_n - x_n\| \leq \|x_{n+1} - x_n\| \quad \text{and} \quad \varphi(Tx_n, x_n) \leq \varphi(x_{n+1}, x_n).$$

We will prove that

(1) $x' \in \mathrm{Fix}(S)$;
(2) $x' \in \mathrm{Fix}(T)$.

To see (1), we will make use of the closedness of $I - S$ at zero. It is clear that $Sx_n \to x'$ and hence (1) holds.

To see (2), let us note from Lemma 2.13 and $\varphi(Tx_n, x_n) \to 0$ that $Tx_n \to x'$. It follows from the closedness of $I - T$ at zero that (2) holds.

Finally, it follows from $F \subset \bigcap_{n=1}^{\infty} C_n$ and $x' \in F$ that $x' = P_F\widehat{x}$. □

Remark 2.15. This type of iterative scheme called the shrinking projection method was first proposed by Takahashi et al. [29]. The technique of the proof using Mosco convergence is due to Kimura and Takahashi [14]; see also [13].

The following result can be obtained with a slight modification of the preceding proof so its proof is omitted.

Theorem 2.16. *Let E be a Banach space such that both E and its dual space E^* are Fréchet smooth. Let C be a closed and convex subset of E. Suppose that $T : C \to C$ and $S : C \to C$ are two mappings such that the following properties are satisfied:*

- *S is a cutter mapping of type (P);*
- *T is relatively quasi-nonexpansive;*
- *$F := \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \varnothing$;*
- *$I - S$ and $I - T$ are closed at zero.*

Define an iterative sequence $\{x_n\}$ by the following way:

$$\begin{cases} x_1 = \widehat{x} \in C := C_1 \text{ is arbitrarily chosen;} \\ A_n = \{z \in C : \langle Sx_n - z, J(Sx_n - x_n)\rangle \leq 0\}; \\ B_n = \{z \in C : \varphi(z, Tx_n) \leq \varphi(z, x_n)\}; \\ C_{n+1} = A_n \cap B_n \cap C_n; \\ x_{n+1} = P_{C_{n+1}}\widehat{x}. \end{cases}$$

Then the sequence $\{x_n\}$ converges to $P_F\widehat{x}$.

Remark 2.17. Let us note that the metric projection involved in our iterations in the preceding two theorems can be replaced by Alber's generalized projections. To prove this, we just invoke the analogue of Tsukada's Theorem for generalized projections. In fact, in the same setting as Tsukada's theorem, Ibaraki et al. [11] proved that $\Pi_{C_n}x \to \Pi_{C_0}x$ for all $x \in E$.

Finally, we present a related result which is deduced from our Theorem 2.14 where T is the identity mapping.

Theorem 2.18. *Let E be a smooth Banach space such that E^* is Fréchet smooth. Let C be a closed and convex subset of E. Suppose that $f : C \times C \to \mathbb{R}$ satisfies the following conditions:*

- $f(x, x) = 0$ *for all* $x \in C$;
- $f(x, y) + f(y, x) \leq 0$ *for all* $x, y \in C$;
- $f(x, \cdot)$ *is convex and lower semicontinuous for all* $x \in C$;
- *for every* $x \in C$ *and* $x^* \in E^*$ *the following implication holds:*

$$f(z, x) + \langle x - z, x^* \rangle \leq 0 \quad \forall z \in C \implies f(x, y) + \langle y - x, x^* \rangle \geq 0 \quad \forall y \in C.$$

Define an iterative sequence $\{x_n\}$ *by the following way:*

$$\begin{cases} x_1 = \widehat{x} \in C =: C_1 \text{ is arbitrarily chosen;} \\ C_{n+1} = \{z \in C : \langle F_{r_n} x_n - z, J(F_{r_n} x_n - x_n) \rangle \leq 0\} \cap C_n; \\ x_{n+1} = P_{C_{n+1}} \widehat{x}, \end{cases}$$

where $\{r_n\}$ *is a sequence of positive real numbers such that* $\liminf_{n\to\infty} r_n > 0$. *If* $\mathrm{EP}(f) \neq \varnothing$, *then the sequence* $\{x_n\}$ *converges to* $P_{\mathrm{EP}(f)}\widehat{x}$. *Here for each* $x \in E$ *and* $r > 0$, *the element* $F_r x$ *is a unique element in* C *such that*

$$f(F_r x, y) + \frac{1}{r}\langle y - F_r x, J(F_r x - x) \rangle \geq 0 \quad \forall y \in C.$$

Remark 2.19. The preceding theorem is proved in [27, Theorem 3.2] under the assumption that E^* is uniformly smooth. It is noted that F_r is a cutter mapping of type (P) and $\mathrm{Fix}(F_r) = \mathrm{EP}(f)$. Moreover, the proof of Theorem 2.14 does not alter if we can replace a single mapping S with a sequence of mappings $\{S_n\}$ such that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \neq \varnothing$ and the following condition holds:

$$\{z_n\} \subset C,\ z_n \to z,\ S_n z_n \to z \implies z \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n).$$

3. Concluding remarks

We propose a new alternative iterative scheme for approximation of a common fixed point of two different types of generalizations of cutters mappings. This appears as the first theoretical framework dealing with two different types of mappings in just only one scheme. Let us consider the convex feasibility problem, that is, the problem of fining a common element in the intersection of two (or more) closed and convex subsets of a certain Banach space. As already mentioned that there are two types of projections for these two sets, we can choose the easier calculated projection on each set. If these two projections are different, the schemes in this paper will generates an appropriate sequence for the problem.

The calculation of the projection onto general closed and convex sets is a hard task. However, if C in our theorems is the whole space E, the closed and convex set we are dealing with is a half space. To calculate such a projection, we refer to a formula proposed by Butnariu and Resmerita (see [6, Theorem 4.7] with $p = 2$).

In the recent works of Reich and Sabach (see [22, 23, 24, 25, 17]), they considered the classes of operators containing the cutter mappings of type (Q). It is very interesting to extend our results to these classes.

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