รายงานวิจัยฉบับสมบูรณ์

โครงการ ตัววัดการขึ้นต่อกัน

โดย นายทรงเกียรติ สุเมธกิจการ จุฬาลงกรณ์มหาวิทยาลัย

8 กุมภาพันธ์ 2560

สัญญาเลขที่ RSA5680037

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สนับสนุนโดยสำนักงานคณะกรรมการอุดมศึกษา
สำนักงานกองทุนสนับสนุนการวิจัย และจุฬาลงกรณ์มหาวิทยาลัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย
สกว. และจุฬาลงกรณ์มหาวิทยาลัยไม่จำเป็นต้องเห็นด้วยเสมอไป)

สารบัญ

กิตติกรรมประกาศ

ข้าพเจ้า หัวหน้าโครงการวิจัยเรื่องตัววัดการขึ้นต่อกัน ขอขอบพระคุณสำนักงานกองทุนสนับสนุนการวิจัย สำนักงานคณะกรรมการอุดมศึกษา และจุฬาลงกรณ์มหาวิทยาลัย ที่ให้การสนับสนุนทุนวิจัยตลอดโครงการวิจัยนี้ ขอขอบพระคุณ รองศาสตราจารย์ ดร. พิเชฐ ชาวหา และรองศาสตราจารย์ ดร. ณัฐกาญจน์ ใจดี ที่กรุณาช่วยเหลือในการเปิดบัญชีรับทุนดังกล่าว และขอขอบพระคุณสำนักงานคณะกรรมการอุดมศึกษาและ BMWFW via Austrian Agency for Intenational Cooperation in Education and Research (OeAD-GmbH), Centre for International Cooperation and Mobility (ICM) ที่ให้การสนับสนุนทุนร่วมวิจัยผ่านโครงการ ASEA-UNINET Staff Exchange, One Month Scholarship

รองศาสตราจารย์ ดร. ทรงเกียรติ สุเมธกิจการ
ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
2560

Abstract

Project Code : RSA5680037

Project Title : Measures of dependence

Investigator : Songkiat Sumetkijakan, Chulalongkorn University

E-mail Address : songkiat.s@chula.ac.th, songkiat.weng@gmail.com

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Many aspects of bivariate copulas are investigated in the project, especially their properties closely related to dependence structure. We introduce and study two local versions of Kendall's tau conditioning on one or two random variable(s) varying less than a given distance. Unlike the Kendall's tau, these local versions are able to distinguish between complete dependence and independence copulas. A pointwise version of Kendall's tau is also proposed and shown to distinguish between comonotonicity and countermonotonicity of complete dependence copulas. Deriving from conditional variance, we introduce a class of measures of mutual complete dependence. These copula-based dependence measures are novel in the sense that they are not developed from a distance from the independence copula but rather from the *-product of the copula and its transpose.

It is well known that the dependence structure of a singular copula is encoded in its support. Motivated by a generalized closure introduced in order to study support of copulas, we developed a theory of essential closures. A typical essential closure, called a submeasure closure, collects all points that are essential with respect to a submeasure. Among many properties of essential closures, we prove that a "nice" essential closure must be a submeasure closure. Examples of submeasure closures are discussed and their applications are demonstrated, especially in the study of supports of measures and copulas. As an opposite of independence, complete dependence copulas have been well studied in the literature. A much broader type of singular copulas that deserve investigations is the implicit dependence copulas, defined as the copula of two continuous random variables X,Y for which $\alpha \circ X = \beta \circ Y$ a.s. for some Borel functions α,β. Evidently, the full mass of an implicit dependence copula is concentrated on an implicit graph, i.e. the graph of $f(x) = g(y)$ for some measure-preserving functions f,g. Our main result is the characterizations of a copula assigning full mass to an implicit graph in terms of a partial factorizability of its Markov operator and in terms of the non-atomicity of two associated sigma-algebras. As an application, we give a broad sufficient condition under which the mass of a copula with fractal support is concentrated on an implicit graph. Under extra conditions, we compute the left and right invertible factors of copulas with fractal support.

Keywords : local Kendall's tau, conditional variance, essential closures, implicit dependence copulas, non-atomic copulas

บทคัดย่อ

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ชื่อโครงการ : ตัววัดการขึ้นต่อกัน (Measures of dependence)
ชื่อนักวิจัย : นายทรงเกียรติ สุเมธกิจการ จุฬาลงกรณ์มหาวิทยาลัย
E-mail Address : songkiat.s@chula.ac.th, songkiat.weng@gmail.com
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โครงการนี้ศึกษาสมบัติต่างๆ ของคอปูล่าที่เกี่ยวกับการขึ้นต่อกันของสองตัวแปรสุ่ม เรานิยามและศึกษาค่าทาวของเคนดอลล์เฉพาะที่แบบมีเงื่อนไขให้ค่าตัวแปรสุ่มเปลี่ยนน้อยๆ ค่าทาวของเคนดอลล์เฉพาะที่เหล่านี้สามารถแยกแยะระหว่างคอปูล่าการเป็นอิสระต่อกันและคอปูล่าการขึ้นต่อกันอย่างสมบูรณ์ เรายังได้นิยามและศึกษาค่าทาวของเคนดอลล์รายจุดซึ่งสามารถแยกแยะระหว่างคอปูล่าทางเดียวกัน และคอปูล่าสวนทางกัน จากการศึกษาความแปรปรวนแบบมีเงื่อนไข เรานำเสนอตัววัดการขึ้นต่อกันอย่างสมบูรณ์คลาสหนึ่งด้วยนิยามที่เริ่มจากผลคูณของคอปูล่ากับการสลับเปลี่ยนของมัน ซึ่งแตกต่างจากตัววัดการขึ้นต่อกันอื่นๆ ที่พัฒนาจากแนวคิดของระยะทางจากคอปูล่าการเป็นอิสระต่อกัน

ดังเป็นที่ทราบดีว่า ข้อมูลของโครงสร้างการขึ้นต่อกันของคอปูล่าเอกฐานอยู่ในเซตค้ำจุนของคอปูล่า ด้วยแรงบันดาลใจจากส่วนปิดคลุมทั่วไปที่นิยามขึ้นเพื่อศึกษาเซตค้ำจุนของคอปูล่า เราได้พัฒนาทฤษฏีของส่วนปิดคลุมหลัก ส่วนปิดคลุมหลักทั่วไปซึ่งเรียกว่าส่วนปิดคลุมเมเชอร์ย่อยคือการรวบรวมจุดที่จำเป็นทั้งหมดเทียบกับเมเชอร์ย่อยหนึ่ง เราพิสูจน์สมบัติต่างๆ ของส่วนปิดคลุมหลัก โดยเฉพาะอย่างยิ่งสมบัติที่ว่าส่วนปิดคลุมหลักดีๆ ต้องเป็นส่วนปิดคลุมหลักเมเชอร์ย่อย เรายกตัวอย่างส่วมปิดคลุมหลักเมเชอร์ย่อยต่างๆ และสาธิตการนำไปใช้ โดยเฉพาะอย่างยิ่งการใช้ศึกษาเซตค้ำจุนของเมเชอร์และคอปูล่า คอปูล่าการขึ้นต่อกันอย่างสมบูรณ์ได้ผ่านการศึกษามามากในฐานะความสัมพันธ์ที่ตรงข้ามกับการเป็นอิสระต่อกัน คอปูล่าเอกฐานอีกชนิดหนึ่งที่สมควรได้รับการศึกษาบ้างคือคอปูล่าการขึ้นต่อกันโดยปริยาย ซึ่งเป็นคลาสที่กว้างขวางกว่ามากและนิยามเป็นคอปูล่าของสองตัวแปรสุ่มต่อเนื่อง X,Y ที่สอดคล้อง $\alpha\circ X=\beta\circ Y$ a.s. สำหรับบางฟังก์ชัน α,β คอปูล่า การขึ้นต่อกันโดยปริยายกระจายน้ำหนักทั้งหมดบนกราฟปริยาย นั่นคือกราฟของ $f(x)=g(y)$ สำหรับบางฟังก์ชันคงเมเชอร์ f,g ผลงานหลักคือการจำแนกคอปูล่าที่กระจายน้ำหนักทั้งหมดบนกราฟปริยายในรูปของการแยกตัวประกอบบางส่วนของตัวกระทำมาร์คอฟ และในรูปของความไร้อะตอมของพีชคณิตซิกม่าของคอปูล่า (เรียกว่าคอปูล่าไร้อะตอม) ตัวอย่างของคอปูล่าที่กระจายน้ำหนักทั้งหมดบนกราฟปริยายคือคอปูล่าที่มีเซตค้ำจุนแฟรกตอลและสอดคล้องเงื่อนไขบางประการ ภายใต้เงื่อนไขเพิ่มเติมคอปูล่าที่มีเซตค้ำจุนแฟรกตอลสามารถแยกตัวประกอบเป็นผลคูณของคอปูล่าขึ้นต่อกันอย่างสมบูรณ์

คำหลัก: ค่าทาวของเคนดอลล์เฉพาะที่ , ความแปรปรวนแบบมีเงื่อนไข , ส่วนปิดคลุมหลัก , คอปูล่าการขึ้นต่อกันโดยปริยาย , คอปูล่าไร้อะตอม

บทนำ

The joint distribution function of two random variables X and Y on a common probability space contains all information of their probabilistic dependency. However, a joint distribution function contains information on both how the random variables are depending on each other and how each random variable is distributed. When only the dependence structure of X and Y are investigated regardless of their (marginal) distributions, it is natural to study their copula instead. This is due to the fact proved by Sklar [56] that every joint distribution $F_{X,Y}$ can be written as $F_{X,Y}(x,y) = C\left(F_X(x), F_Y(y)\right)$ for some copula $C = C_{X,Y}$, where F_X and F_Y are marginal distributions of X and Y respectively, and that the copula of continuous random variables X and Y is invariant under strictly increasing transformations of the random variables. A (bivariate) *copula* is defined as a joint distribution function of two uniform [0,1] random variables.

The reason why research interests on copula theory have grown enormously in the past two decades is its successful applications in financial modeling, hydrology and in many fields [36]. Copula modeling offers much more flexibility and allows near tailor-made distributions fitting to a given situation [29]. Therefore, more theoretical studies are necessary in order to improve copula modeling.

By definition of independence, (the) copula of independent random variables is $\Pi\left(x,y\right) = xy$ and vice versa. It is a more difficult question to classify copulas of mutually completely dependent random variables, which means that the two random variables are completely dependent on each other almost surely. The support of a copula C is where quantiles correlate with positive probability and is defined as the support of the doubly stochastic measure μ_C:

$$\mu_C([u_1,u_2]\times[v_1,v_2]) = C\left(u_2,v_2\right) - C\left(u_2,v_1\right) - C\left(u_1,v_2\right) + C\left(u_1,v_1\right).$$

Though most applicable copulas have full support, certain dependencies in complicated models lead to copulas with singular components [19,36] whose supports contain crucial information. Hence study of copula supports is necessary in understanding these complicated scenarios. In fact, the supports of doubly stochastic measures have been extensively studied by mathematical analysts for decades yet the geometrical characterization problem is still open.

The Sklar's theorem and the invariance property of copulas under strictly increasing transformations of random variables [18,43] justify widespread use of copulas as a representation of dependence structure between random variables. For instance, continuous random variables X and Y are almost surely strictly increasing or almost surely strictly decreasing dependent if and only if their copula is $M(x,y) = \min(x,y)$ or $W(x,y) = \max(x+y-1,0)$, respectively. In general, following [33,55], Y is said to be *completely dependent* on X if there is a Borel measurable function f such that $Y = f(X)$. And X and Y are called *mutually completely dependent* if Y is completely dependent on X and X is completely dependent on Y. The larger class of mutual complete dependence copulas includes the shuffles of Min introduced by Mikusinski et al. [37,38]. A *shuffle of Min* is the copula of X and Y for which there exists an invertible Borel measurable function f with finitely many discontinuity points such that $Y = f(X)$ almost surely. Equivalently, f is invertible and piecewise monotonic. The opposite case is independence. Recall that X and Y are independence if and only if their copula is $\Pi(x, y) = xy$. However, most dependencies are between these two extremes. So suitable dependency levels are necessary in selecting the right copulas. Renyi [46] proposed seven postulates for which a good measure of dependence should satisfy. There are also other propositions, such as Schweizer and Wolff's [52]. Since then, many nonparametric measures of dependence have been introduced, studied and used. See [18,29,36] and references therein. Their probabilistic interpretations, invariant properties and implementations are central in our investigation.

The Kendall's tau has been a measure of concordance of choice for decades. One reason is that it depends only on the copula, not the marginal distributions. However, the independence copula is not the only copula whose Kendall's tau is zero. To remedy the incapability to distinguish independence from certain complete dependence, we propose two local versions of Kendall's tau, conditioning on the events that one or two random variables vary less than a fixed distance, hence called uni- and bi-conditional local Kendall's taus. Their formulas for various copulas are given and their limits are computed. They are shown to distinguish the independence copula from complete dependence copulas. Some preliminary results on newly defined pointwise Kendall's tau are also obtained. See [ภาคผนวก A].

The $*$-product was first introduced by Darsow, Nguyen and Olson [8] in their study of Markov processes. Their remarkable result is that if $\{X_t\}$ is a Markov process and C_{st} denotes the copula of X_s and X_t then for any $s < t < u$, $C_{su} = C_{st} * C_{tu}$. A necessary and sufficient condition of Markov process is also given in terms of a variant of the $*$-product. This gives rise to a novel method to construct a Markov process/chain by assigning desired dependence structure among states via a class of parameterized copulas satisfying some conditions. An additional benefit of this method is the freedom to choose marginal distributions of X_t without any restrictions. This was impossible before the advent of copulas and the $*$-product. Since Markov processes/chains is one of the most commonly used stochastic processes in the literature, behaviors of the $*$-product is very important and worth investigating further.

When one wants to associate a copula of two continuous random variables with certain level of mutual complete dependence, the (modified) Sobolev norm introduced by Siburg et al. [55] is the first choice. The Sobolev norm of a copula C is defined via

$$\|C\|^2 = \int_0^1\int_0^1 \left(\partial_1 C(x,y)\right)^2 + \left(\partial_2 C(x,y)\right)^2 \, dx\, dy.$$

It is derived from Darsow et al.'s study of Markov processes [8,9,44] in which they introduced and studied a binary (product) operation $*:\mathcal{C}_2\times\mathcal{C}_2\to\mathcal{C}_2$, where $\mathcal{C}_2$ is the class of all bivariate copulas, defined as

$$C*D(x,y)=\int_0^1\partial_2C(x,t)\,\partial_1D(t,y)\,dt.$$

By direct computations, for any copula C, $\Pi*C=\Pi=C*\Pi$ and $M*C=C=C*M$. Therefore, $(\mathcal{C}_2,*)$ is a semigroup with identity M and null element Π. They showed that $\|C\|=\sqrt{2/3}$ if and only if $C=\Pi$, and that the following are equivalent.

i) C is a copula of mutually completely dependent random variables.

ii) C is invertible with respect to the $*$-product.

iii) C has unit Sobolev norm.

It is evident that mutual dependence, Sobolev norm and the structure of $*$-product are strongly related. Moreover, the Sobolev norm gives rise to a measure of mutual complete dependence invariant under strictly monotonic transformations. After modification so that it is invariant under bijective transformations, Ruankong et al. [49] obtained a measure of dependence satisfying five main postulates of Renyi's. Note also that the Sobolev norm of a copula lies in the interval $\left[\sqrt{2/3},1\right]$ and that if C is left or right invertible then $\sqrt{5/6}<\|C\|\le 1$, where both bounds are sharp. These are the only known probabilistic interpretations of the Sobolev norm.

Since dependence level should be closely related to conditional variances, a careful analysis of the total conditional variance gives rise to a function of two continuous random variables that is almost a measure of mutual complete dependence. This function is based on the $*$-product of the corresponding copula and its transpose. In fixing the only missing property, we construct a class of measures of mutual complete dependence. This approach gives a fresh perspective on how one could define a measure of dependence. See [ภาคผนวก B].

In their study of relationships between the support of a complete dependence copula $C_{X,f(X)}$ and the graph of f, Ruankong and Sumetkijakan [48] found that they are equal after taking essential closure. They also showed that supports of doubly stochastic measures are essentially closed. These new concepts turn out to be a generalization of one-dimensional essential closure used in [25]. Another kind of essential closure studied in measure theory [22] is based on the notion of Radon-Nikodym derivative or density with respect to a measure. We therefore develop a general axiomatic theory of essential closures and show that all aforementioned essential closures are just special cases. A main result is that all "nice" essential closures must be defined from a sub-measure closure. We also study a more general and natural concept of essential closure operators. See [ภาคผนวก C].

All postulates of measures of dependence impose restrictions only for independence and complete dependence cases. This allows a measure of dependence to behave quite arbitrarily for dependence structure lying between the two extremes. A class of copulas that is still manageable but large enough to connect the dots between independence and complete dependence is the non-atomic copulas. Loosely speaking, a non-atomic copula is the copula of two continuous random variables that are deterministically dependent on set-level for large classes of sets. They are singular copulas and contain implicit dependence copulas, defined as the copula of two continuous random variables that are implicitly dependent. Evidently, the full mass of an implicit dependence copula is concentrated on an implicit graph. Our main result is the characterizations of a copula assigning full mass to an implicit graph in terms of a partial factorizability of its Markov operator and in terms of the non-atomicity of two associated sigma-algebras. As an application, we give a broad sufficient condition under which the mass of a copula with fractal support is concentrated on an implicit graph. Under extra conditions, we compute the left and right invertible factors of copulas with fractal support. See [ภาคผนวก D].

เนื้อหางานวิจัย

วัตถุประสงค์

1. Develop a unified theory of essential closures and find connections with other fields of mathematics.
2. Find measure(s) of dependence in terms of conditional variances or conditional distributions and study their properties and probabilistic interpretations.

วิธีทำการวิจัย

1. Find appropriate definitions of local Kendall's tau that reflect local concordance and study their properties.
2. Define pointwise Kendall's tau, derive its formula and study its properties.
3. Compute local and pointwise Kendall's tau for some copulas.
4. Investigate total conditional variance and find its formula in terms of copulas.
5. Determine whether it gives rise to norms and/or measures of dependence.
6. Study its probabilistic interpretation and connections to other measures of dependence.
7. Collect various examples of "essential closures."
8. Investigate properties that are crucial for essential closures. Choose a set of postulates that all essential closures should satisfy.
9. Derive properties from these postulates. Extra regularity conditions might be needed.
10. Find characterizations of "nice" essential closures.
11. Find connections and applications of essential closures.
12. Define non-atomic bivariate copulas to reflect the deterministicity of random variables on set levels for a large class of sets.
13. Derive properties of non-atomic copulas, especially its characterizations.
14. Study some examples of non-atomic copulas, such as copulas with fractal support, in more details.

เนื้อหางานวิจัย (ต่อ)

<u>ผลการวิจัย</u>

1. Definition and a formula of uni-conditional local Kendall's tau, as well as a formula of its limit as the variation approaches zero, are given. These values for shuffles of Min and absolutely continuous copulas are computed and shown to be different. [ภาคผนวก A. Section 2]
2. Definition and a formula of bi-conditional local Kendall's tau are given. The value and its limit as both variations approach zero are computed for simple shuffles of Min and FGM copulas. [ภาคผนวก A. Section 3]
3. Definition and a formula of pointwise Kendall's tau are given. Its limit as the variation approaches zero is computed for complete dependence copulas. [ภาคผนวก A. Section 4]
4. Show that conditional variance can be written in terms of product of the corresponding copula and its transpose, giving rise to a positive-valued function v of copulas. [ภาคผนวก B. Section 3]
5. Prove that the function v in 4 is almost a measure of mutual complete dependence. [ภาคผนวก B. Section 3]
6. Based on v, we define a measure of dependence in the sense of Renyi's and a class of measure of mutual complete dependence. [ภาคผนวก B. Section 4]
7. Definition of essential closures on a topological space equipped with an algebra is given and their properties, e.g. strong, weakly strong and sigma-non-essential, are investigated. [ภาคผนวก C. Section 2]
8. Obtain relationships between essential closures and their associated classes of non-essential sets. [ภาคผนวก C. Section 2]
9. Definition of essential closure operators is given and some properties are proved. [ภาคผนวก C. Section 3]
10. Sub-measure closures are defined and studied. Some examples, e.g. Lebesgue density closures, lower density closures and stochastic closures, are given and investigated. [ภาคผนวก C. Sections 4-5]
11. Consider the copula of X and Y, we give a definition of the associated classes of Borel sets on which X and Y are deterministically dependent and prove some fundamental properties, especially that they are sigma-algebras. [ภาคผนวก D. Section 2]
12. Definition of non-atomic copulas is given and their properties are studied. They are characterized in terms of the corresponding Markov operators and in terms of the mass distribution of the corresponding doubly stochastic measures. [ภาคผนวก D. Section 3]
13. We give a sufficient condition on a transformation matrix under which the invariant copula with fractal support is non-atomic. We also give a sufficient condition under which the invariant copula can be factored as the product of left invertible and right invertible copulas. [ภาคผนวก D. Section 4]

ข้อเสนอแนะสำหรับงานวิจัยในอนาคต

1. The definition of pointwise Kendall's tau could be developed into a measure of pointwise monotonicity. In fact, measures of pointwise monotonicity should also be studied in their own rights.
2. Find a simpler approach to construct measures of (mutual complete) dependence from conditional variances.
3. Find all, or as many as possible, norms on copulas with respect to which the non-atomic copulas and/or the implicit dependence copulas are dense.
4. Study how to obtain information on dependence structure from the associated sigma-algebras of a non-atomic copula.
5. Investigate how the information could give us a postulate on the behavior of a measure of dependence for copulas between independence and complete dependence copulas.
6. Equitability is another concept proposed to fill in the gap between independence and complete dependence. Equitability and related concepts should be investigated, especially to compare with the above approach using non-atomic copulas.

หนังสืออ้างอิง

1. A. Albanese and C. Sempi, Countably generated idempotent copulas, *Soft Methodology and Random Information Systems.* Springer (2004) 197-211.
2. R.B. Ash, *Probability and Measure Theory*, Second ed. with contributions by Catherine Doleans-Dade, Harcourt/Acad. Press, Burlington, MA, 2000.
3. I. Bairamov, S. Kotz, T.J. Kozubowski, A new measure of linear local dependence, *Statistics*. **37**(2003) no.3, 243-258.
4. H. Bauer, *Measure and Integration Theory*, Walter de Gruyter, Berlin, 2001.
5. V.I. Bogachev, *Measure Theory, vol I*, Springer Verlag, 2007.
6. Z. Buczolich and W.F. Pfeffer, Variations of additive functions, *Czech. Math. J.* **47** (1997) no.3, 525-555.
7. N. Chaidee, T. Santiwipanont and S. Sumetkijakan, Patched approximations and their convergence, Comm. Statist. Theory Methods. **45** (2016) no.9, 2654-2664.
8. W.F. Darsow, B. Nguyen, and E.T. Olsen, Copulas and Markov processes, *Illinois J. Math.* **36** (1992) no.4, 600-642.
9. W.F. Darsow and E.T. Olsen, Norms for copulas, *International Journal of Mathematics and Mathematical Sciences.* **18** (1995) no.3, 417-436.
10. W.F. Darsow and E.T. Olsen, Characterization of idempotent 2-copulas, *Note di Matematica.* **30** (2010) no.1, 147-177.
11. W.F. Darsow and E.T. Olsen, Existence of limits of analytic one-parameter semigroups of copulas, *Note di Matematica.* **30** (2010) no.2, 1-13.
12. E. de Amo, M. Dıaz Carrillo and J. Fernandez-Sanchez, Measure-preserving functions and the independence copula, *Mediterr. J. Math.* **8** (2011) no.2, 431-450.
13. H. Dette, K.F. Siburg and P.A. Stoimenov, A copula-based non-parametric measure of regression dependence, *Scandinavian Journal of Statistics*. **40** (Mar 2013) no.1, 21-41.
14. F. Durante, E.P. Klement, J.J. Quesada-Molina and P. Sarkoci, Remarks on two product-like constructions for copulas, *Kybernetika*. (Prague) **43** (2007) 235–244.
15. F. Durante, P. Sarkoci, and C. Sempi, Shuffles of copulas, *Journal of Mathematical Analysis and Applications*. **352** (2009) no.2, 914-921.
16. F. Durante and J. Fernandez-Sanchez, Multivariate shuffles and approximation of copulas, *Statistics and Probability Letters*. **80** (2010) 1827-1834.
17. F. Durante and J. Fernandez-Sanchez, On the approximation of copulas via shuffles of Min, *Statistics and Probability Letters*. **82** (2012) 1761-1767.
18. F. Durante and C. Sempi, *Principles of copula theory*, CRC Press, 2015.

19. P. Embrechts and M. Hofert, Copula Theory and Applications: Quo Vadis?, *Talk at Chiang Mai University*. January 2012.
20. K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley & Sons Ltd., West Sussex, 1990.
21. G.A. Fredricks, R.B. Nelsen and J.A. Rodrıguez-Lallena, Copulas with fractal supports, *Insurance. Math. Econom.* 37 (2005) no.1, 42-48.
22. D.H. Fremlin, *Measure Theory, Vol. 4, Topological Measure Spaces*, Biddles Short Run Books, King's Lynn, 2003.
23. D.H. Fremlin, *Measure Theory: Broad Foundations, Vol. 2*, Second ed., Torres Fremlin, Colchester, 2010.
24. H. Gebelein, Das statistische Problem der Korrelation als Variations und Eigenwertproblem und sein Zusammenhang mit der Ausgleichsrechnung, *Z. Angew. Math. Mech.* **21** (1941) 364-379.
25. F. Gesztesy, K.A. Makarov and M. Zinchenko, Essential closures and AC spectra for reflectionless CMV, Jacobi, and Schrodinger operators revisited, *Acta Applicandae Mathematicae*. **103** (2008) no.3, 315-339.
26. S. Ghosal, A. Sen, A.W. Van Der Vaart, Testing monotonicity of regression, *Ann. Stat.* **28** (2000) no.4, 1054-1082.
27. K.O. Hufthammer, Some measures of local and global dependence, Master's thesis, University of Bergen (2005).
28. B.R. Hunt, T. Sauer and J.A. Yorke, Prevalence: A translation-invariant "almost every" on infinite-dimensional spaces, *Bull. Amer. Math. Soc*. **27** (1992) no.2, 217-238.
29. H. Joe, *Multivariate models and dependence concepts*, Chapman & Hall, 1997.
30. M.C. Jones, The local dependence function, *Biometrika.* **83**(1996) no.4, 899-904.
31. N. Kamnitui, T. Santiwipanont and S. Sumetkijakan, Dependence measuring from conditional variances. *Depend. Model.* **3** (2015), 98-112.
32. G. Kimeldorf and A.R. Sampson, Monotone dependence, *Ann. Stat.* **6** (1978) no.4, 895-903.
33. H.O. Lancaster, Correlation and complete dependence of random variables, *Ann. Math. Stat.* **34** (1963) 1315-1321.
34. J. Lukes, J. Maly and L. Zajıcek, *Fine topology methods in real analysis and potential theory*, Lecture Notes in Math. Vol. 1189, Springer-Verlag, Berlin, 1986.
35. P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge Univ. Press, Cambridge, 1995.
36. A.J. McNeil, R. Frey and P. Embrechts, *Quantitative risk management*, Princeton series in finance, 2010.

37. P. Mikusinski, H. Sherwood, and M.D. Taylor, Probabilistic interpretations of copulas and their convex sums, *Advances in probability distributions with given marginals: beyond the copulas.* Springer (1991) 95-112.
38. P. Mikusinski, H. Sherwood, and M.D. Taylor, Shuffles of min, *Stochastica.* **13** (1992) no.1, 61-74.
39. P. Mikusinski, H. Sherwood, and M.D. Taylor, The Frechet bounds revisited, *Real Analysis Exchange.* **17** (1992) 759-764.
40. P. Mikusinski and M.D. Taylor, Some approximations of *n*-copulas, *Metrika.* **72** (Nov 2010) no.3, 385-414.
41. P. Mikusinski and M.D. Taylor, Markov operators and *n*-copulas, *Ann. Polon. Math.* **96** (2009) 75-95.
42. J.R. Munkres, *Topology, Second ed.*, Prentice Hall Inc., Upper Saddle River, NJ, 2000.
43. R.B. Nelsen, *An introduction to copulas*, second ed., Springer Verlag, 2006.
44. E.T. Olsen, W.F. Darsow, and B. Nguyen, Copulas and Markov operators, *Lecture Notes-Monograph Series* **28** (1996) 244-259.
45. K. Pearson and D. Heron, On Theories of Association, *Biometrika*. **9** (1913) no.1/2, 159-315.
46. A. Renyi, On measures of dependence, *Acta. Math. Acad. Sci. Hungar.* **10** (1959) 441–451.
47. H.L. Royden, *Real analysis, 3rd ed.* Macmillan, New York, 1988
48. P. Ruankong and S. Sumetkijakan, Essential closures and supports of multivariate copulas, *International Journal of Approximate Reasoning* **54** (Aug 2013) no.6, 762-768.
49. P. Ruankong, T. Santiwipanont and S. Sumetkijakan, Shuffles of copulas and a new measure of dependence, *Journal of Mathematical Analysis and Applications*. **398** (Feb 2013) no.1, 392-402.
50. Y. Sakai and T. Shimogaki, Equimeasurability of functions and doubly stochastic operators, *Kodai Math. Sem. Rep.* **24** (1972), 203-211.
51. M. Scarsini, On measures of concordance, *Stochastica***. 8** (1984) no.3, 201-218.
52. B. Schweizer and E.F. Wolff, On nonparametric measures of dependence for random variables, *Ann. Stat.* **9** (1981) no.4, 879-885.
53. C. Sempi, Conditional expectations and idempotent copulae, *Distributions with given marginals and statistical modeling.* Kluwer (2002) 223-228.
54. K.F. Siburg and P.A. Stoimenov, A scalar product for copulas, *Journal of Mathematical Analysis and Applications*. **344** (2008) no.1, 429-439.
55. K.F. Siburg and P.A. Stoimenov, A measure of mutual complete dependence, *Metrika*. **27** (2010) no.2, 239-251.
56. A. Sklar, Fonctions de repartition à *n* dimensions et leurs marges, *Publ. Inst. Statist. Univ. Paris.* **8** (1959) 229-231.

57. S. Tasena and S. Dhompongsa, Multivariate measure of mutual complete dependence, *International Journal of Approximate Reasoning* **54** (Aug 2013) no.6, 748-761.
58. W. Trutschnig, On a strong metric on the space of copulas and its induced dependence measure, *Journal of Mathematical Analysis and Applications*. **384** (2011) 690-705.
59. W. Trutschnig, On Cesaro convergence of iterates of the Star Product of copulas, *Stat. Prob. Letters*. **83** (2013) 357-365.
60. W. Trutschnig and J. Fernandez-Sanchez, Some results on shuffles of two-dimensional copulas, *Journal of Statistical Planning and Inference*. **143** (Feb 2013) no.2, 251-260.
61. W. Trutschnig, Idempotent copulas with fractal support. In: Greco, S., Bouchon-Meunier, B., Coletti, G., Fedrizzi, M., Matarazzo, B., Yager, R.R. (eds.) Advances in Computational Intelligence, Vol. 298, pp. 161–170. Springer, Berlin Heidelberg (2012)
62. W. Trutschnig and J. Fernandez-Sanchez, Idempotent and multivariate copulas with fractal support, *J. Statist. Plann. Inference*. **142** (2012), 3086-3096.
63. Z. Yanting, Y. Jingping and Z.H. Jianhua, Approximation of bivariate copulas by patched bivariate Frechet copulas, *Insurance Math. Econom.* **48** (2011), 246-256.
64. Y. Zheng, J. Yang, J.Z. Huang, Approximation of bivariate copulas by patched bivariate Fréchet copulas, *Insurance Math. Econ.* **48** (2011) 246-256.

Output จากโครงการวิจัยที่ได้รับทุนจาก สกว.

ชื่อโครงการ ตัววัดการขึ้นต่อกัน (Measures of dependence)
ระยะเวลาโครงการ 3 ปี (17 มิถุนายน 2556 – 16 มิถุนายน 2559)
ชื่อหัวหน้าโครงการวิจัยผู้รับทุน: นายทรงเกียรติ สุเมธกิจการ

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ
   A. Local Kendall's Tau (with P. Buthkhunthong, A. Junchuay, I. Ongeera and T. Santiwipanont), **published in** Econometrics of Risk, Studies in Computational Intelligence vol. 583, Van-Nam Huynh, Vladik Kreinovich, Songsak Sriboonchitta, Komsan Suriya, Eds. Springer, 2015. pp. 161–169.
      Stable URL: https://doi.org/10.1007/978-3-319-13449-9_11
   B. Dependence measuring from conditional variances (with N. Kamnitui and T. Santiwipanont), **published in** Dependence Modeling, Vol. 3, Issue 1, 2015, pp. 98-112.
      Stable URL: https://doi.org/10.1515/demo-2015-0007
   C. Essential closures (with P. Ruankong),
      published in Real Analysis Exchange, Vol. 41, No. 1, 2016, pp. 55-86.
      Stable URL: http://www.jstor.org/stable/10.14321/realanalexch.41.1.0055
   D. Non-atomic bivariate copulas and implicitly dependent random variables,
      accepted by Journal of Statistical Planning and Inference.
      Stable URL: https://doi.org/10.1016/j.jspi.2017.01.005

2. การนำผลงานวิจัยไปใช้ประโยชน์
   - เชิงพาณิชย์ (มีการนำไปผลิต/ขาย/ก่อให้เกิดรายได้ หรือมีการนำไปประยุกต์ใช้โดยภาคธุรกิจ/บุคคลทั่วไป)
     --
   - เชิงนโยบาย (มีการกำหนดนโยบายอิงงานวิจัย/เกิดมาตรการใหม่/เปลี่ยนแปลงระเบียบข้อบังคับหรือวิธีทำงาน)
     --
   - เชิงสาธารณะ (มีเครือข่ายความร่วมมือ/สร้างกระแสความสนใจในวงกว้าง)
     --
   - เชิงวิชาการ (มีการพัฒนาการเรียนการสอน/สร้างนักวิจัยใหม่)
     โครงการนี้ช่วยพัฒนานักวิจัยใหม่ที่สนใจทฤษฎีคอปูล่า 3 ท่าน รวมทั้งเป็นรากฐานในการพัฒนาความรู้เพื่อรวบรวมเขียนเป็นหนังสือเผยแพร่ในอนาคตต่อไป

3. อื่นๆ (เช่น ผลงานตีพิมพ์ในวารสารวิชาการในประเทศ การเสนอผลงานในที่ประชุมวิชาการ หนังสือ การจดสิทธิบัตร)

- การเดินทางไปทำวิจัยร่วมกับผู้เชี่ยวชาญระยะสั้น

  ทำวิจัยร่วมกับ Wolfgang Trutschnig ที่ University of Salzburg ประเทศออสเตรีย เมื่อวันที่ 16 พฤษภาคม - 15 มิถุนายน 2558 ในหัวข้อวิจัยเรื่อง Dependence measures and the product of copulas ด้วยทุนแลกเปลี่ยนคณาจารย์ระหว่างไทยกับออสเตรีย (ASEA-UNINET Staff Exchange, One-Month Scholarship)

- การเสนอผลงานในที่ประชุมวิชาการ

  1. เสนอผลงานเรื่อง Essential closures ในการประชุม The International Congress of Mathematicians 2014 (ICM2014) Session SC08-09: Analysis and its Applications ที่กรุงโซล เกาหลีใต้ เมื่อวันที่ 15 สิงหาคม 2557 ด้วยทุน NANUM จาก organizers ของการประชุมนี้
  2. เสนอผลงานเรื่อง Local Kendall’s tau ในการประชุม The 8th International Conference of the Thailand Econometric Society (TES2015) จังหวัดเชียงใหม่ เมื่อวันที่ 7-9 มกราคม 2558 ด้วยทุนเดินทางจากคณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
  3. เสนอผลงานเรื่อง Implicit dependence copulas ในการประชุม Salzburg Workshop on Dependence Models & Copulas ประเทศออสเตรีย เมื่อวันที่ 19-22 กันยายน 2559 ด้วยทุนนี้

ภาคผนวก A.

Reprint of

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Local Kendall's Tau

P. Buthkhunthong, A. Junchuay, I. Ongeera, T. Santiwipanont and S. Sumetkijakan

Abstract We introduce two local versions of Kendall's tau conditioning on one or two random variable(s) varying less than a fixed distance. Some basic properties are proved. These local Kendall's taus are computed for some shuffles of Min and the Farlie-Gumbel-Morgenstern copulas and shown to distinguish between complete dependence and independence copulas. A pointwise version of Kendall's tau is also proposed and shown to distinguish between comonotonicity and countermonotonicity for complete dependence copulas.

1 Introduction and Preliminaries

Let (X, Y) and (X', Y') be independent and identically distributed random vectors. Then the population version of the Kendall's tau of X, Y is defined as

$$\tau(X, Y) \equiv P\left((X' - X)(Y' - Y) > 0\right) - P\left((X' - X)(Y' - Y) < 0\right).$$

If X and Y are continuous random variables with joint distribution function $F_{X,Y}$ then there exists a unique copula $C = C_{X,Y}$ such that $F_{X,Y}(u, v) = C(F_X(u), F_Y(v))$ for all $u, v \in [0, 1]$ where F_X and F_Y are the distribution functions of X and Y, respectively. A copula can be defined as the restriction onto I^2 of a joint distribution

P. Buthkhunthong · A. Junchuay · I. Ongeera ·
T. Santiwipanont · S. Sumetkijakan (✉)
Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok, Thailand
e-mail: songkiat.s@chula.ac.th

P. Buthkhunthong
e-mail: pitchaya.bu@gmail.com

A. Junchuay
e-mail: anusorn-j@outlook.com

I. Ongeera
e-mail: ilada.ogr@gmail.com

T. Santiwipanont
e-mail: tippawan.s@chula.ac.th

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of two uniform (0, 1) random variables. See [6] for a systematic treatment of the theory of copulas. In this setting, the Kendall's tau can be written in terms of the copula C as

$$\tau(C) = \tau(X, Y) = 4 \iint_{I^2} C(u, v)\, dC(u, v) - 1.$$

Kendall's tau is a measure of concordance in the sense of Scarsini [8], i.e. $\tau(X, Y)$ is defined for all continuous random variables X and Y; τ attains value in $[-1, 1]$; τ is equal to 0 if X and Y are independent; τ is symmetric ($\tau(Y, X) = \tau(X, Y)$); τ is coherence ($C_{XY} \leq C_{X'Y'}$ implies $\tau(X, Y) \leq \tau(X', Y')$); $\tau(-X, Y) = -\tau(X, Y)$; and τ is continuous with respect to convergence in distribution.

However, there are copulas of dependent random variables whose Kendall's tau is zero. An extreme example is the copula $S_{1/2}$, defined in Example 2.1, of continuous random variables that are completely dependent by an injective function which is strictly increasing on two disjoint intervals separated at the median. Its Kendall's tau is zero because of cancellation between local concordance and global discordance. Local dependence has been studied in [1, 3, 4]. To bring more local dependence into focus, we propose two local versions of Kendall's tau called uni- and bi-conditional local Kendall's taus in Sects. 2 and 3. Their formulas for shuffles of Min and FGM copulas are given and their basic properties are proved. In particular, $S_{1/2}$ has non-zero local Kendall's taus.

Both uni- and bi-conditional local Kendall's taus are conditioning on one or two random variables varying less than a fixed small distance. They are measures of local concordance/discordance between two random variables without restriction on the range of the conditioning random variable(s). In a sense, they detect local dependence globally. In order to detect true local dependence, we introduce a pointwise Kendall's tau in Sect. 4. Its empirical version was first introduced in [2] and shown to detect monotonicity of two random variables. We show that the pointwise Kendall's tau can distinguish between comonotonicity and countermonotonicity for complete dependence copulas.

2 Uni-conditional Local Kendall's Tau

Let X and Y be continuous random variables with the copula C. We will consider the difference between the probabilities of concordance and discordance conditioning on the event that X is varying less than a fixed distance regardless of the value of X. Since the same amount of variation of X could reflect different interpretations depending on the X-value, we assume that X and Y are uniformly distributed on $[0, 1]$. Let $0 < \varepsilon \leq 1$. The *uni-conditional local Kendall's tau* of X and Y given that X varies less than ε is defined as

$$\tau_\varepsilon(C) = \tau_\varepsilon(X,Y) = P\left((X-X')(Y-Y')>0 \,\|X-X'|<\varepsilon\right) \\ - P\left((X-X')(Y-Y')<0 \,\|X-X'|<\varepsilon\right)$$

where (X', Y') is an independent copy of (X, Y). The following quantity shall be needed in computing local Kendall's tau.

$$T_C(a,b) = \iint_{I^2} C(u-a, v-b)\, dC(u,v) \quad \text{for } -1 \leq a, b \leq 1. \tag{1}$$

Note that $T_C(a,b) = P(X'-X > a, Y'-Y > b)$. By uniform continuity of C, the function T_C is continuous on $[-1,1]^2$ and in particular $T_C(\cdot,0)\colon a \mapsto T_C(a,0)$ is continuous on $[-1,1]$.

Proposition 2.1 1. *The uni-conditional local Kendall's tau of a copula C can be computed by the formula*

$$\tau_\varepsilon(C) = \frac{4T_C(0,0) - 2\left[T_C(-\varepsilon,0) + T_C(\varepsilon,0)\right]}{\varepsilon(2-\varepsilon)}. \tag{2}$$

2. *The mapping $\varepsilon \mapsto \tau_\varepsilon(C)$ is continuous on* $(0,1]$.

Proof 1. It is straightforward to verify that $P(|X-X'|<\varepsilon) = 2\varepsilon - \varepsilon^2$,

$$P\left(-\varepsilon < X-X' < 0, Y-Y' < 0\right) = P\left(0 < X'-X < \varepsilon, 0 < Y'-Y\right) \\ = T_C(0,0) - T_C(\varepsilon,0), \quad \text{and} \\ P\left(-\varepsilon < X-X' < 0, 0 < Y-Y'\right) = T_C(-\varepsilon,0) - T_C(0,0).$$

Therefore,

$$P\left((X-X')(Y-Y')>0, |X-X'|<\varepsilon\right) = 2\left(T_C(0,0) - T_C(\varepsilon,0)\right), \\ P\left((X-X')(Y-Y')<0, |X-X'|<\varepsilon\right) = 2\left(T_C(-\varepsilon,0) - T_C(0,0)\right),$$

and the formula follows.

2. This is clear from Eq. (2) and the continuity of $T_C(\cdot,0)$.

Recall the definition of three important copulas Π, M and W: for $u, v \in [0,1]$, $\Pi(u,v) = uv$, $M(u,v) = \min(u,v)$ and $W(u,v) = \max(u+v-1,0)$. Then it can be shown via the Eq. (2) that $\tau_\varepsilon(\Pi) = 0$, $\tau_\varepsilon(M) = 1$, $\tau_\varepsilon(W) = -1$ for all $\varepsilon \in (0,1]$.

Example 2.1 Let us consider simple shuffles of Min introduced in [5]. Let S_α be the shuffle of Min whose support is illustrated in Fig. 1 and defined by

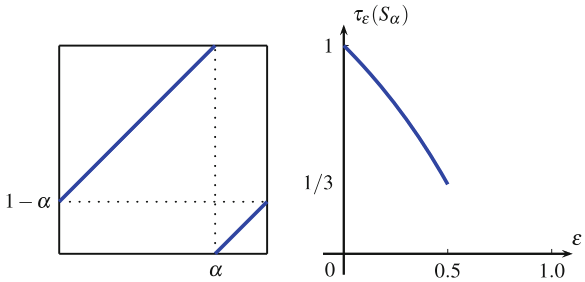

Fig. 1 The support of S_α and the uni-conditional local Kendall's tau of S_α

$$S_\alpha(x,y)=\begin{cases}0 & \text{if } 0\le x\le\alpha, 0\le y\le 1-\alpha,\\ \min(x,y-(1-\alpha)) & \text{if } 0\le x\le\alpha, 1-\alpha<y\le 1,\\ \min(x-\alpha,y) & \text{if } \alpha<x\le 1, 0\le y\le 1-\alpha,\\ x+y-1 & \text{if } \alpha<x\le 1, 1-\alpha<y\le 1.\end{cases} \tag{3}$$

Since S_α is supported on the lines $\ell_1\colon y=x+(1-\alpha), 0\le x\le\alpha$ and $\ell_2\colon y=x-\alpha, \alpha\le x\le 1$, we have

$$\iint_{I^2} f(x,y)\,dS_\alpha(x,y)=\int_0^\alpha f(x,x+1-\alpha)\,dx+\int_\alpha^1 f(x,x-\alpha)\,dx. \tag{4}$$

Because $S_\alpha(x-a,y-b)$ has positive value on rectangles $(a,b)+R_i$ where $R_1\equiv[\alpha,1]\times[0,1-\alpha]$, $R_2\equiv[0,\alpha]\times[1-\alpha,1]$, $R_3\equiv[\alpha,1]\times[1-\alpha,1]$, $R_4\equiv[1,\infty]\times[0,1]$, $R_5\equiv[0,1]\times[1,\infty]$, and $R_6\equiv[1,\infty]\times[1,\infty]$, it can be derived that

$$T_{S_\alpha}(a,b)=\int_{L_1} S_\alpha(x-a,(x+1-\alpha)-b)\,dx+\int_{L_2} S_\alpha(x-a,(x-\alpha)-b)\,dx \tag{5}$$

where each L_1 and L_2 is a union of six non-overlapping possibly empty intervals.[1] For $\frac{1}{2}\le\alpha<1$ and $0<\varepsilon\le\min(\alpha,1-\alpha)$, $T_{S_\alpha}(0,0)=\frac{1}{2}-\alpha(1-\alpha)$, $T_{S_\alpha}(\varepsilon,0)=\frac{1}{2}-\alpha-\varepsilon+\alpha^2+\varepsilon^2$, $T_{S_\alpha}(-\varepsilon,0)=\frac{1}{2}-\alpha+\alpha^2+\frac{\varepsilon^2}{2}$, and hence by (2),

$$\tau_\varepsilon(S_\alpha)=\frac{2-3\varepsilon}{2-\varepsilon}$$

[1] $L_1=[\max(0,\alpha+a,\alpha-1+b),\min(\alpha,1+a,b)]\cup[\max(0,a,b),\min(\alpha,\alpha+a,\alpha+b)]\cup[\max(0,\alpha+a,b),\min(\alpha,1+a,b+\alpha)]\cup[\max(0,1+a,\alpha-1+b),\min(\alpha,b+\alpha)]\cup[\max(0,\alpha+b,a),\min(\alpha,1+a)]\cup[\max(0,\alpha+b,1+a),\alpha]$ and $L_2=[\max(1-\alpha,\alpha+a,\alpha+b),\min(1,1+a,1+b)]\cup[\max(1-\alpha,a,1+b),\min(1,\alpha+a,\alpha+1+b)]\cup[\max(1-\alpha,\alpha+a,1+b),\min(1,1+a,\alpha+1+b)]\cup[\max(1-\alpha,1+a,\alpha+b),\min(1,\alpha+1+b)]\cup[\max(1-\alpha,a,\alpha+1+b),\min(1,1+a)]\cup[\max(1-\alpha,1+a,\alpha+1+b),1]$.

as shown in Fig. 1. Surprisingly, $\tau_\varepsilon(S_\alpha)$ is independent of α when ε is sufficiently small. But it is not unexpected that we obtain $\lim_{\varepsilon\to 0^+} \tau_\varepsilon(S_\alpha) = 1$.

For any given copula C, we then investigate the limit of $\tau_\varepsilon(C)$ as ε goes down to 0, denoted by $\tau_{\text{loc}}(C)$:

$$\tau_{\text{loc}}(C) = \lim_{\varepsilon\to 0^+} \tau_\varepsilon(C)$$

wherever the limit exists. The left-hand and right-hand derivatives are denoted respectively by

$$\partial_1^- C(u,v) = \lim_{\varepsilon\to 0^-} \frac{C(u+\varepsilon,v) - C(u,v)}{\varepsilon} \text{ and}$$
$$\partial_1^+ C(u,v) = \lim_{\varepsilon\to 0^+} \frac{C(u+\varepsilon,v) - C(u,v)}{\varepsilon};$$

and $\partial_1 C(u,v) = \partial_1^+ C(u,v) = \partial_1^- C(u,v)$ wherever the one-sided derivatives exist and are equal. Let μ_C denote the doubly stochastic measure on $[0,1]^2$ induced by C.

Theorem 2.1 *Let* $0 < \varepsilon < 1$ *and* C *be a copula. Then*

$$\tau_{\text{loc}}(C) = \iint_{I^2} \left(\partial_1^- C(u,v) - \partial_1^+ C(u,v)\right) dC(u,v)$$

provided that the set of points (u,v) *where the left and right partial derivatives* $\partial_1^- C(u,v)$ *and* $\partial_1^+ C(u,v)$ *exist has* C*-volume one.*

Proof By Proposition 2.1, $\tau_\varepsilon(C)$ can be reformulated as

$$\tau_\varepsilon(C) = \frac{2}{2-\varepsilon} \iint_{I^2} \left[\frac{C(u,v) - C(u-\varepsilon,v)}{\varepsilon} - \frac{C(u+\varepsilon,v) - C(u,v)}{\varepsilon}\right] dC(u,v).$$

Note that both quotients are bounded by 1 and the integral is with respect to a finite measure μ_C. Applying the dominated convergence theorem on the set of points (u,v) where the first quotient converges to $\partial_1^- C(u,v)$ and the second quotient converges to $\partial_1^+ C(u,v)$, we have the desired identity.

Corollary 2.1 *Let* C *be a copula. If* $\partial_1 C$ *exists for* μ_C*-almost everywhere on* I^2 *then* $\tau_{\text{loc}}(C) = 0$.

Example 2.2 Let C be a copula. If $\partial_1 C$ exists everywhere then $\tau_{\text{loc}}(C) = 0$. In particular, if C_θ is a Farlie-Gumbel-Morgenstern (FGM) copula, then $\tau_{\text{loc}}(C_\theta) = 0$ for all θ.

Example 2.3 We show that $\tau_{\text{loc}}(S) = 1$ for all straight shuffles of Min S. Let (u,v) be in the support of S. From the assumption that S is a straight shuffle of Min, there exists $\delta > 0$ such that $\partial_1 S(t, v) = 1$ for $u - \delta < t < u$ and $\partial_1 S(t, v) = 0$ for $u < t < u + \delta$. Since S is continuous, we have $\partial_1^- S(u, v) = 1$ and $\partial_1^+ S(u, v) = 0$ at all (u, v) in the support of S. Hence

$$\tau_{\text{loc}}(S) = \iint_{I^2} \left(\partial_1^- S(u, v) - \partial_1^+ S(u, v)\right) dS(u, v) = \mu_S(\text{supp } S) = 1.$$

Similar arguments show that if S is a flipped shuffle of Min then $\tau_{\text{loc}}(S) = -1$ and for any shuffle of Min S,

$$\tau_{\text{loc}}(S) = \lambda(I_S) - \lambda(D_S)$$

where I_S (D_S) is the set of points u for which the support of S is a line of slope 1 (-1) in a neighbourhood of $(u, v) \in S$.

3 Bi-conditional Local Kendall's Tau

For uniform (0, 1) random variables X and Y with copula C and any $0 < \varepsilon \leq 1$, the *bi-conditional local Kendall's tau* of X and Y, or equivalently of C, given that both X and Y vary less than ε is defined as

$$\begin{aligned} \tau_{[\varepsilon]}(C) = {} & P\left((X_1 - X_2)(Y_1 - Y_2) > 0 \,\|X_1 - X_2| < \varepsilon, |Y_1 - Y_2| < \varepsilon\right) \\ & - P\left((X_1 - X_2)(Y_1 - Y_2) < 0 \,\|X_1 - X_2| < \varepsilon, |Y_1 - Y_2| < \varepsilon\right). \end{aligned}$$

Theorem 3.1 *The bi-conditional local Kendall's tau can be computed by*

$$\tau_{[\varepsilon]}(C) = \frac{2\left[2T_C(0,0) - 3T_C(\varepsilon,0) + T_C(\varepsilon,\varepsilon) - T_C(-\varepsilon,0) + T_C(\varepsilon,-\varepsilon)\right]}{T_C(\varepsilon,\varepsilon) - 2T_C(\varepsilon,-\varepsilon) + T_C(-\varepsilon,-\varepsilon)}.$$

Proof This results from a long calculation similar to the proof of Proposition 2.1 but much more tedious.

Theorem 3.2 $\tau_{[\varepsilon]}(C)$ *is a continuous function of* $\varepsilon \in (0, 1]$.

Proof This is because T_C is continuous on $[-1, 1]^2$.

Let us define $\tau_{[\text{loc}]}(C) = \lim_{\varepsilon \to 0^+} \tau_{[\varepsilon]}(C)$.

Example 3.1 $\tau_{[\varepsilon]}(\Pi) = 0$, $\tau_{[\varepsilon]}(M) = 1$, $\tau_{[\varepsilon]}(W) = -1$ for all $\varepsilon \in [0, 1]$.

Example 3.2 For $\alpha \geq \frac{1}{2}$ and $0 < \varepsilon < \frac{1}{2}\min(\alpha, 1 - \alpha)$, lengthy computations give $T_{S_\alpha}(\varepsilon, \varepsilon) = \frac{1}{2} - \alpha - \varepsilon + \alpha^2 + \varepsilon^2$, $T_{S_\alpha}(\varepsilon, -\varepsilon) = \frac{1}{2} - \alpha - \varepsilon + \alpha^2 + 3\varepsilon^2/2$ and $T_{S_\alpha}(-\varepsilon, -\varepsilon) = \frac{1}{2} - \alpha + \varepsilon + \alpha^2 + \varepsilon^2/2$ and hence

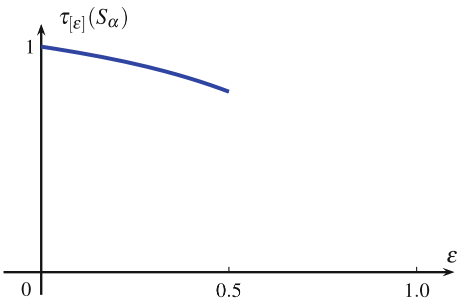

Fig. 2 The bi-conditional local Kendall's tau of S_α

$$\tau_{[\varepsilon]}(S_\alpha) = \frac{4 - 4\varepsilon}{4 - 3\varepsilon},$$

which is again independent of α. Its graph is shown in Fig. 2. So $\tau_{[\text{loc}]}(S_\alpha) = \lim_{\varepsilon \to 0^+} \tau_{[\varepsilon]}(S_\alpha) = 1$.

Example 3.3 Let C_θ be the FGM copulas. Then

$$\tau_{[\varepsilon]}(C_\theta) = \frac{2(3 - 2\varepsilon)^2 \varepsilon^2 \theta}{9(2 - \varepsilon)^2 + (1 - \varepsilon)^4 (2 + \varepsilon)\theta^2} \quad \text{and} \quad \tau_{[\text{loc}]}(C_\theta) = 0.$$

4 Pointwise Kendall's Tau

Definition 4.1 Let X and Y be continuous random variables on a common sample space with marginal distributions F and G, respectively. Let (X_1, Y_1) and (X_2, Y_2) be independent random vectors with identical joint distribution as (X, Y). Let $t \in (0, 1)$ and $r \in (0, \min(t, 1-t))$. Then *a population version of the local Kendall's tau around a point t for X and Y* is defined as

$$\begin{aligned} \tau_{X,Y,r}(t) = &P\left[(X_1 - X_2)(Y_1 - Y_2) > 0 \mid -\underline{r} < X_i - F^{-1}(t) < \overline{r},\ \forall i = 1, 2\right] \\ &- P\left[(X_1 - X_2)(Y_1 - Y_2) < 0 \mid -\underline{r} < X_i - F^{-1}(t) < \overline{r},\ \forall i = 1, 2\right] \end{aligned}$$

where $-\underline{r} = \triangle F_{-r}^{-1}(t) = F^{-1}(t - r) - F^{-1}(t)$ and $\overline{r} = \triangle F_r^{-1}(t) = F^{-1}(t + r) - F^{-1}(t)$. Note that

$$\begin{aligned} \tau_{X,Y,r} &= \tau_{F(X),G(Y),r} \\ &= P\left[\text{Conc} \mid |FX_i - t| < r,\ \forall i = 1, 2\right] \\ &\quad - P\left[\text{Disc} \mid |FX_i - t| < r,\ \forall i = 1, 2\right] \end{aligned}$$

where

$$\text{Conc} = \{(FX_1 - FX_2)(GY_1 - GY_2) > 0\} \text{ and}$$
$$\text{Disc} = \{(FX_1 - FX_2)(GY_1 - GY_2) < 0\}.$$

The following theorem shows that local Kendall's tau around a point depends only on the copula of continuous random variables X and Y. It can be proved straightforwardly.

Theorem 4.1 *Let X and Y be continuous random variables with copula C. Let t be in $(0, 1)$ and r be in $(0, \min(t, 1 - t))$. Then a population version of the local Kendall's tau around a point t for X and Y is given by*

$$\tau_{X,Y,r}(t) = \frac{1}{r^2} \iint\limits_{(t-r,t+r)\times[0,1]} \left(C(x, y) - \frac{C(t-r, y) + C(t+r, y)}{2} \right) dC(x, y).$$

Since $\tau_{X,Y,r}$ depends only on the copula C, it is also called *the local Kendall's tau around a point of C* and denoted by $\tau_{C,r}$. The *pointwise Kendall's tau of C at t* is given by

$$\tau_C(t) = \lim_{r\to 0^+} \tau_{C,r}(t). \tag{6}$$

Similar to its empirical counterpart introduced in [2], $\tau_C(t) = \tau_{X,Y}(t)$ can detect monotonicity at t at least in the case when Y is completely dependent on X.

Theorem 4.2 *If C is the complete dependence copula $C_{U,f(U)}$ for some measure preserving function f and uniform $(0, 1)$ random variable U, then for every continuity point t of f, $\tau_C(t) = \operatorname{sgn}(f'(t))$.*

Proof Since f is measure preserving and continuous on $(t - \delta, t + \delta)$ for some $\delta > 0$, it must be affine on $(t - \delta, t + \delta)$ with slope $m \neq 0$. Assume without loss of generality that $m > 0$. Put $s = f(t)$ so that $f(t + \triangle t) = s + m\triangle t$ if $|\triangle t| < \delta$. By a theorem in [7], $\partial_1 C(x, y) = 0$ for $y < f(x)$ and $\partial_1 C(x, y) = 1$ for $y > f(x)$. So, for $r < \delta$, $x \mapsto C(x, y)$ is constant on $[t - r, t + r]$ whenever $y \geq s + mr$ or $y \leq s - mr$.

Let $r < \delta$. Then

$$\begin{aligned}
\tau_{C,r}(t) &= \frac{1}{r^2} \int_{s-mr}^{s+mr} \int_{t-r}^{t+r} \left(C(x, y) - \frac{C(t-r, y) + C(t+r, y)}{2} \right) C(dx, dy) \\
&= \frac{1}{r^2} \int_{t-r}^{t+r} \Bigg(C(x, s + m(x - t)) \\
&\qquad - \frac{C(t-r, s + m(x-t)) + C(t+r, s + m(x-t))}{2} \Bigg) dx
\end{aligned}$$

where, in the last equality, we use the fact that $\iint_{I\times J} g(x, y)\, dC(x, y) = \int_I g(x, f(x))dx$ if C is supported on the line $y = f(x)$. Since $C(\cdot, s + m(x - t))$ is constant on $[x, t + r]$ and affine of slope 1 on $[t - r, x]$, we have

$$\tau_{C,r}(t) = \frac{1}{2r^2}\int_{t-r}^{t+r} C(x, s + m(x - t)) - C(t - r, s + m(x - t))\, dx$$

$$= \frac{1}{2r^2}\int_{t-r}^{t+r} (x - t + r)\, dx = 1.$$

Hence $\tau_C(t) = 1$. Derivation for the case $m < 0$ gives $\tau_C(t) = -1$.

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References

1. Bairamov, I., Kotz, S., Kozubowski, T.J.: A new measure of linear local dependence. Statistics **37**(3), 243–258 (2003)
2. Ghosal, S., Sen, A., Van Der Vaart, A.W.: Testing monotonicity of regression. Ann. Stat. **28**(4), 1054–1082 (2000)
3. Hufthammer, K.O.: Some measures of local and global dependence. Master's thesis, University of Bergen (2005)
4. Jones, M.C.: The local dependence function. Biometrika **83**(4), 899–904 (1996)
5. Mikusinski, P., Sherwood, H., Taylor, M.D.: Shuffles of min. Stochastica **13**, 61–74 (1992)
6. Nelsen, R.B.: An Introduction to Copulas, 2nd edn. Springer Series in Statistics. Springer, New York (2006)
7. Ruankong, P., Santiwipanont, T., Sumetkijakan, S.: Shuffles of copulas and a new measure of dependence. J. Math. Anal. Appl. **398**(1), 398–402 (2013)
8. Scarsini, M.: On measures of concordance. Stochastica **8**(3), 201–218 (1984)

ภาคผนวก B.

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Noppadon Kamnitui, Tippawan Santiwipanont, and Songkiat Sumetkijakan*

Dependence Measuring from Conditional Variances

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Abstract: A conditional variance is an indicator of the level of independence between two random variables. We exploit this intuitive relationship and define a measure ν which is almost a measure of mutual complete dependence. Unsurprisingly, the measure attains its minimum value for many pairs of non-independent random variables. Adjusting the measure so as to make it invariant under all Borel measurable injective transformations, we obtain a copula-based measure of dependence $\nu_\star$ satisfying A. Rényi's postulates. Finally, we observe that every nontrivial convex combination of ν and $\nu_\star$ is a measure of mutual complete dependence.

Keywords: conditional variances; measures of dependence; copulas; mutual complete dependence; shuffles of Min

MSC: 60A10, 62H20

1 Introduction

The problem of how to assign the level of dependence between two random variables in a consistent manner can never be solved completely by using only a single measure of dependence. There are many attributes to consider in choosing the "right" measure of dependence in a given situation. Among them are the nature of dependence (linear, monotone, or other types of dependence), a reference to the normal correlation coefficient and other specific purposes. Many measures of dependence have been proposed and studied since the beginning of the twentieth century. See [10, 14, 16, 18, 20, 22]. But it is not until the seminal paper of A. Rényi [16] that this problem attracted much wider attention. He proposed the following set of seven properties that should be valid for a generic measure of dependence δ. To the best of our knowledge, the only measure satisfying all of these properties is the maximal correlation coefficient [8]: $R(X, Y) = \sup_{f,g} \gamma(f(X), g(Y))$, where the supremum is taken over all Borel measurable functions f and g such that the correlation coefficient $\gamma(f(X), g(Y))$ can be defined.

- R0. $\delta(X, Y)$ is defined for all random variables X and Y, neither of them being constant almost surely (a.s.).
- R1. $\delta(X, Y) = \delta(Y, X)$.
- R2. $0 \le \delta(X, Y) \le 1$.
- R3. $\delta(X, Y) = 0$ if and only if X and Y are independent.
- R4. $\delta(X, Y) = 1$ if X and Y are completely dependent, i.e. X is almost surely a Borel measurable function of Y or vice versa.
- R5. $\delta(f(X), g(Y)) = \delta(X, Y)$ for all Borel measurable injective transformations f and g.
- R6. $\delta(X, Y) = |\rho|$ if X and Y are jointly normal with correlation coefficient ρ.

***Corresponding Author: Songkiat Sumetkijakan:** Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand, E-mail: songkiat.s@chula.ac.th
Noppadon Kamnitui, Tippawan Santiwipanont: Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn university, Thailand, E-mails: noppadon.kam@gmail.com, tippawan.s@chula.ac.th

Since the discovery of copulas and the famous Sklar's theorem [13, 21], many measures of dependence defined via copulas - called *copula-based measures of dependence* - have been introduced. As a pair of random variables X and Y has a unique copula when they are continuous, a copula-based measure of dependence is guaranteed to be well-defined only for continuous random variables and hence R0 may not hold. For the copula-based measures of dependence, R6 is usually replaced by a weaker postulate that $\delta(X, Y)$ is a strictly increasing function of $|\rho|$. Two such measures are Schweizer and Wolff's σ [18],

$$\sigma(X, Y) = 12 \int_{I^2} |C - \Pi| \, d\lambda_2,$$

and Siburg and Stoimenov's ω [20],

$$\omega(X, Y) = \left(3 \int_{I^2} \left((\partial_1 C)^2 + (\partial_2 C)^2 \right) d\lambda_2 - 2 \right)^{1/2},$$

where C is the copula of X and Y, denoted by $C_{X,Y}$. Recall that Π, defined by $\Pi(x, y) = xy$, is the copula of independent continuous random variables; $M(x, y) = \min(x, y)$ is the copula of comonotonic random variables; and $W(x, y) = \max(x + y - 1, 0)$ is the copula of countermonotonic random variables. Both σ and ω are defined for continuous random variables and satisfy R1-R3. σ is called a *measure of monotone dependence* because its maximum value detects strict monotone dependence:

σ4. $\sigma(X, Y) = 1$ if and only if X and Y are a.s. strictly monotonically dependent, i.e. $C_{X,Y} = M$ or W.
σ5. $\sigma(f(X), g(Y)) = \sigma(X, Y)$ for all a.s. strictly monotonic measurable transformations f and g.

The measure ω is called a *measure of mutual complete dependence* because its maximum value is attained exactly when the random variables are mutually completely dependent, i.e. they are completely dependent on each other:

ω4. $\omega(X, Y) = 1$ if and only if X and Y are mutually completely dependent.
ω5. $\omega(f(X), g(Y)) = \omega(X, Y)$ for all a.s. strictly monotonic measurable transformations f and g.

Observe that the properties R4-R5 need to be adjusted according to which types of dependence the measures aim to detect. Historically, σ and ω have their roots in the Spearman's ρ and the (modified) Sobolev norm of copulas [13, 18–20].

The conditional variance of Y given X is an indicator of how weakly Y is dependent on X. We make this relationship more explicit as follows. For the uniform $[0, 1]$ random variables X and Y with joint distribution function or copula C, we observe that the L^1-norm of the conditional variance, called the *total conditional variance*,

$$\sigma^2_{Y|X} = \int_0^1 Var(Y|X = x) \, dx$$

is equal to the L^1-norm of the difference $(M - C^T * C)$. This suggests that the L^1-norms of $C^T * C$ and $C * C^T$ might possibly give rise to new measures of dependence with close ties to conditional variances. It turns out that the sum of these two L^1-norms give a "measure of mutual complete dependence" ν that satisfies R1-R5 except that in R3 it holds only that $\nu(X, Y) = 0$ if X and Y are independent. Moreover, $\nu(X, Y)$ is very close to $|\rho|$ in the case when X and Y are jointly normal with correlation coefficient ρ. All of the above are developed in section 3. In section 4, we overcome the inability of ν in classifying the independence and define a new measure $\nu_\star$. It is proved to satisfy R1-R5. We also show that the converse of R4 does not hold for $\nu_\star$. Finally, a class of measures of mutual complete dependence is given by all nontrivial convex combinations of ν and $\nu_\star$. Note also that, by computation, both ν and $\nu_\star$ are increasing functions of $|\rho|$. Let us begin with a section summarizing all the necessary backgrounds on copulas including their properties and constructions.

2 Background on copulas

Denote $I = [0, 1]$, $\mathcal{B}(I)$ the Borel σ-algebra on I and let λ and λ_2 denote the Lebesgue measures on I and I^2 respectively. The Lebesgue integral on I is denoted simply by $\int_0^1 dx$. The symbol $\partial_i C$ denotes the partial derivative of C with respect to the ith variable.

A function $C\colon I^2 \to I$ is called a (bivariate) *copula* if for all $u, v \in I$,

(1) $C(0, v) = 0 = C(u, 0)$;
(2) $C(1, v) = v$, $C(u, 0) = u$; and
(3) $C(u', v') - C(u', v) - C(u, v') + C(u, v) \geq 0$ for all $[u, u'] \times [v, v'] \subseteq I^2$.

Every copula C can be extended to a joint distribution function of uniform $[0, 1]$ random variables in a unique way. Let X and Y be any random variables whose distribution functions are F and G, respectively. Sklar's theorem states that every joint distribution function H of X and Y can be written as

$$H(x, y) = C(F(x), G(y)), \tag{1}$$

for some copula C. If F and G are continuous, then C is uniquely determined by (1) and called the *copula of X and Y*. Conversely, putting an arbitrary copula C into (1) always yields a joint distribution function H. A copula C is said to be *symmetric* if its *transpose* C^T, given by $C^T(u, v) = C(v, u)$, is equal to C. For more details on the theory of copulas, see [13].

In a series of papers [3, 4, 15], Darsow, Nguyen and Olsen introduce a binary operation on the class of bivariate copulas, called the *-*product*, defined by

$$C * D(u, v) = \int_0^1 \partial_2 C(u, t)\partial_1 D(t, v)\, dt.$$

M is the identity ($M * C = C = C * M$) while Π is the zero ($\Pi * C = \Pi = C * \Pi$). We say that C is *left invertible* (*right invertible*) if $C^T * C = M$ ($C * C^T = M$). It was shown that a copula C is left invertible (right invertible) if and only if C is the copula of X and $f(X)$ ($f(X)$ and X) for some continuous random variable X and Borel measurable transformation f. Random variables X and Y are said to be *completely dependent* if $Y = f(X)$ a.s. or $X = f(Y)$ a.s. for some Borel measurable f. They are said to be *mutually completely dependent* if $Y = f(X)$ a.s. for some Borel measurable injection f. A mutual complete dependence copula is the copula of two continuous random variables which are mutually completely dependent. Note that the invertible copulas, whose class is denoted by $\mathcal{I}$, are exactly the mutual complete dependence copulas.

Shuffles of Min are the copulas of random variables X and $f(X)$ for which f is a piecewise continuous injection. They are simple mutual complete dependence copulas in the sense that they can be constructed by cutting I^2 into a finite number of vertical stripes and shuffling the masses of $M(u, v) = \min(u, v)$ on the main diagonal with possible flipping of the stripes. See [12, 13] for more details on shuffles of Min. Note that the *-product of shuffles of Min is a shuffle of Min.

The *ordinal sum* of copulas $C_1, C_2, \ldots, C_n$ with respect to a partition $\{[a_i, b_i]\}_{i=1}^n$ of I is the copula C given by

$$C(u, v) = \begin{cases} a_i + (b_i - a_i)C_i\left(\dfrac{u - a_i}{b_i - a_i}, \dfrac{v - a_i}{b_i - a_i}\right) & \text{for } (u, v) \in [a_i, b_i]^2, \\ M(u, v) & \text{otherwise.} \end{cases}$$

The mass of C is spread in each square $[a_i, b_i]^2$ according to the copula C_i. So the ordinal sum of shuffles of Min is still a shuffle of Min. Recall from [3, Theorem 8.3] that:

Lemma 2.1. *If C and D are the ordinal sums of $\{C_n\}$ and $\{D_n\}$, respectively, with respect to the same partition of $[0, 1]$, then $C * D$ is the ordinal sum of $\{C_n * D_n\}$ with respect to that partition.*

3 Conditional variances and dependence measuring

Let C be the copula of uniform $[0, 1]$ random variables X and Y on a common probability space and $x \in I$. Recall that the conditional distribution of Y given $X = x$ satisfies $F_{Y|X=x}(y) \equiv P(Y \leq y|X = x) = \partial_1 C(x, y)$ a.s. and so the conditional expectation of Y given $X = x$ is given by

$$\mu_{Y|X}(x) \equiv E(Y|X = x) = \int_0^1 y \partial_1 C(x, dy).$$

Denote the conditional variance of Y given $X = x$ by

$$\sigma^2_{Y|X}(x) \equiv Var(Y|X = x) = E((Y - \mu_{Y|X}(x))^2|X = x) = E(Y^2|X = x) - \mu^2_{Y|X}(x) \tag{2}$$

and the total conditional variance of Y given X by $\sigma^2_{Y|X} \equiv \int_0^1 \sigma^2_{Y|X}(x)\,dx$.

Proposition 3.1. *Suppose random variables X and Y are uniformly distributed on* $[0, 1]$ *with copula C. Then*

$$\sigma^2_{Y|X} = \frac{1}{3} - \int_0^1 \int_0^1 C^T \star C(x, y)\,dx\,dy.$$

Our proof will use some identities collected in the following Lemma.

Lemma 3.2. *Let C be a copula and f be a nonnegative bounded measurable function on* $[0, 1]$. *Then for almost all $x \in I$,*

$$\int_0^1 f(y)\partial_1 C(x, dy) = \frac{d}{dx}\left(\int_0^1 f(y)\partial_2 C(x, y)\,dy\right) \quad \text{and} \tag{3}$$

$$\int_0^1 \partial_1 C(x, y)\,dy = \frac{d}{dx}\int_0^1 C(x, y)\,dy. \tag{4}$$

Proof. (3) can be proved by repeating the arguments in the proof of Lemma 3.1 in [3].

To prove (4), let $x \in I$ be such that $\frac{d}{dx}\int_0^1 C(x, y)\,dy$ exists and $\partial_1 C(x, y)$ exists a.e. y. Note that almost every x possesses these properties. Consider a sequence $\{x_n\}$ converging to x. By the Lipschitz condition of copula, $\left|\frac{C(x,y)-C(x_n,y)}{x-x_n}\right| \leq 1$ for all $y \in I$. So, by the dominated convergence theorem,

$$\frac{d}{dx}\int_0^1 C(x, y)\,dy = \int_0^1 \lim_{n\to\infty} \frac{C(x, y) - C(x_n, y)}{x - x_n}\,dy = \int_0^1 \partial_1 C(x, y)\,dy. \qquad \square$$

Proof of Proposition 3.1. By (2), $\sigma^2_{Y|X}(x) = \int_0^1 y^2\,\partial_1 C(x,dy) - \mu^2_{Y|X}(x)$. So

$$\begin{aligned}
\sigma^2_{Y|X} &= \int_0^1\int_0^1 y^2\,\partial_1 C(x,dy)\,dx - \int_0^1\left[\int_0^1 t\,\partial_1 C(x,dt)\right]^2 dx \\
&= \int_0^1 \frac{d}{dx}\left(\int_0^1 y^2\partial_2 C(x,y)\,dy\right)dx - \int_0^1\left[\frac{d}{dx}\left(\int_0^1 t\partial_2 C(x,t)\,dt\right)\right]^2 dx \\
&= \int_0^1 y^2\left[\partial_2 C(x,y)\right]_{x=0}^1 dy - \int_0^1\left[\frac{d}{dx}\left(x - \int_0^1 C(x,t)\,dt\right)\right]^2 dx \\
&= \frac{1}{3} - \int_0^1\left[1 - \int_0^1 \partial_1 C(x,t)\,dt\right]^2 dx.
\end{aligned}$$

We have used (3) twice in the second line, the first fundamental theorem of calculus and the method of integration by parts in the third line, and (4) in the last line. Applying Tonelli's theorem, the second integral in the last line equals

$$1 - 2\int_0^1\int_0^1 \partial_1 C(x,t)\,dx\,dt + \int_0^1\int_0^1\int_0^1 \partial_2 C^T(s,x)\partial_1 C(x,t)\,dx\,ds\,dt$$

which clearly equals $\int_0^1\int_0^1 C^T * C(s,t)\,ds\,dt$ as desired. □

The total conditional variance $\sigma^2_{X|Y}$ of X given Y is defined similarly and can be proved to satisfy

$$\sigma^2_{X|Y} = \frac{1}{3} - \int_0^1\int_0^1 C * C^T(x,y)\,dx\,dy.$$

Motivated by this relationship, we define $[\![C]\!]$ for every bivariate copula C by $[\![C]\!] = [\![C]\!]_1 + [\![C]\!]_2$ where

$$[\![C]\!]_1 = \int_{I^2} C^T * C\,d\lambda_2 \quad \text{and} \quad [\![C]\!]_2 = \int_{I^2} C * C^T\,d\lambda_2.$$

Recall that if C is the copula of mutually completely dependent continuous random variables then C is invertible with inverse C^T, i.e. $C * C^T = C^T * C = M$. See [19, 20]. So

$$[\![C]\!] = \int_{I^2} C^T * C + C * C^T\,d\lambda_2 = 2\int_{I^2} M\,d\lambda_2 = \frac{2}{3}.$$

For the independence case, if $C = \Pi$, then $[\![C]\!] = 2\int_0^1\int_0^1 uv\,du\,dv = \frac{1}{2}$. Let us note here that every idempotent copula C is symmetric [5, 23]. So $[\![C]\!] = 2\int_{I^2} C\,d\lambda_2$ and

$$[\![C]\!] = 2\int_{I^2}(C - \Pi)\,d\lambda_2 + \frac{1}{2} = \frac{1}{6}\rho(C) + \frac{1}{2}$$

where ρ denotes the Spearman's rho.

Theorem 3.3. *Let C be a bivariate copula.*

(i) $\frac{1}{4} \le [\![C]\!]_i \le \frac{1}{3}$ *for* $i = 1, 2$.

(ii) $[\![C]\!]_1 = \frac{1}{3}$ *if and only if C is left invertible.*

(iii) $[\![C]\!]_2 = \dfrac{1}{3}$ *if and only if C is right invertible.*

(iv) $[\![C]\!]_1 = \dfrac{1}{4}$ *if and only if* $\int_0^1 C(u,v)\,dv = \dfrac{u}{2}$ *for all* $u \in [0,1]$.

(v) $[\![C]\!]_2 = \dfrac{1}{4}$ *if and only if* $\int_0^1 C(u,v)\,du = \dfrac{v}{2}$ *for all* $v \in [0,1]$.

Proof. Let C be a bivariate copula. We only prove the statements for $[\![\cdot]\!]_1$.

(i): Since $C^T \star C \le M$, $[\![C]\!]_1 \le \int_{I^2} M\,d\lambda_2 = \frac{1}{3}$. By Cauchy-Schwarz inequality,

$$\int_0^1 \left[\int_0^1 \partial_1 C(t,u)\,du\right] dt \le \left(\int_0^1 \left[\int_0^1 \partial_1 C(t,u)\,du\right]^2 dt\right)^{1/2} \left(\int_0^1 1^2\,dt\right)^{1/2}.$$

Therefore, by Tonelli's theorem and the fundamental theorem of calculus,

$$\begin{aligned}\int_0^1\int_0^1 C^T \star C(u,v)\,du\,dv &= \int_0^1\int_0^1\int_0^1 \partial_1 C(t,u)\partial_1 C(t,v)\,dt\,du\,dv \\ &= \int_0^1 \left[\int_0^1 \partial_1 C(t,u)\,du\right]^2 dt \\ &\ge \left(\int_0^1 \left[\int_0^1 \partial_1 C(t,u)\,du\right] dt\right)^2 = \left(\int_0^1 u\,du\right)^2 = \frac{1}{4}.\end{aligned}$$

(ii): C is left invertible, i.e. $C^T \star C = M$, if and only if $\int_{I^2} C^T \star C\,d\lambda_2 = \int_{I^2} M\,d\lambda_2 = \frac{1}{3}$.

(iv): By Cauchy-Schwarz inequality,

$$\left(\int_0^1 \left[\int_0^1 \partial_1 C(t,u)\,du\right] dt\right)^2 = \int_0^1 \left[\int_0^1 \partial_1 C(t,u)\,du\right]^2 dt$$

only if $\int_0^1 \partial_1 C(t,u)\,du = K$, a constant function of t. Then

$$K = \int_0^1\int_0^1 \partial_1 C(t,u)\,dt\,du = \int_0^1 u\,du = \frac{1}{2}$$

and hence $\int_0^1 C(t,u)\,du = \frac{t}{2}$ for all $t \in [0,1]$. The converse is clear. □

Therefore, $[\![C]\!]$ takes value in the range $\left[\frac{1}{2}, \frac{2}{3}\right]$ where the maximum is attained if and only if C is invertible. However, the minimum value of $[\![\cdot]\!]$ cannot identify independence of random variables because $\left|\left[\frac{M+W}{2}\right]\right| = \frac{1}{2} = [\![\Pi]\!]$. In fact, the minimum value of $[\![\cdot]\!]$ is attained for every copula of continuous random variables which are jointly symmetric about $(0.5, 0.5)$. Recall [13] that the *jointly symmetric copulas* are precisely the copulas whose associated doubly stochastic measures are symmetric with respect to the line $x = 0.5$ and the line $y = 0.5$.

Proposition 3.4. *For every jointly symmetric copula C,* $[\![C]\!] = \frac{1}{2}$.

Proof. A copula C which is symmetric with respect to the line $x = 0.5$ and the line $y = 0.5$ satisfies $C(x,y) = y - C(1-x,y)$ and $C(x,y) = x - C(x,1-y)$, for all $x, y \in I$. So

$$\int_0^1 C(x,y)\,dx = y - \int_0^1 C(x,y)\,dx \quad \text{and} \quad \int_0^1 C(x,y)\,dy = x - \int_0^1 C(x,y)\,dy.$$

Consequently, $\int_0^1 C(x,y)\,dx = \frac{y}{2}$ and $\int_0^1 C(x,y)\,dy = \frac{x}{2}$ for all $x, y \in I$. By Theorem 3.3 *(iv)* and *(v)*, $[\![C]\!]_2 = \frac{1}{4} = [\![C]\!]_1$. Thus, $[\![C]\!] = \frac{1}{2}$. □

However, there are some non-jointly-symmetric copulas C for which $[\![C]\!] = \frac{1}{2}$ as demonstrated in Example 3.6. The following lemma will be useful in computing $[\![C]\!]$.

Lemma 3.5. *Let F_1 and F_2 denote the uniform distributions on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively. If $A_{11}, A_{12}, A_{21}, A_{22}$ are copulas for which $\int_{I^2} A_{ij}\, d\lambda_2 = \frac{1}{4}$ for all $i, j = 1, 2$ then the function $A\colon I^2 \to I$ defined by*

$$A(u, v) = \frac{1}{4} \sum_{i,j=1}^{2} A_{ij}(F_i(u), F_j(v))$$

is a copula satisfying $\int_{I^2} A\, d\lambda_2 = \frac{1}{4}$.

Proof. By considering all pertinent cases, it can be verified straightforwardly that A is a copula. In fact, A is called the *uniform* $\{A_{ij}\}_{i,j=1}^{2}$*-patched copula*. See [2, 24]. Next, let us denote $I_1 = [0, \frac{1}{2}]$ and $I_2 = [\frac{1}{2}, 1]$. By appropriate linear changes of variables, the integrals of A over each of the four squares $I_i \times I_j$, $i, j = 1, 2$, can be written in terms of the integrals of A_{ij}'s over I^2 as follows:

$$\int_{I_1\times I_1} A\, d\lambda_2 = \frac{1}{16}\int_{I^2} A_{11}\, d\lambda_2, \qquad \int_{I_2\times I_2} A\, d\lambda_2 = \frac{1}{8} + \frac{1}{16}\int_{I^2} A_{22}\, d\lambda_2,$$

$$\int_{I_1\times I_2} A\, d\lambda_2 = \frac{1}{32} + \frac{1}{16}\int_{I^2} A_{12}\, d\lambda_2, \qquad \int_{I_2\times I_1} A\, d\lambda_2 = \frac{1}{32} + \frac{1}{16}\int_{I^2} A_{21}\, d\lambda_2.$$

Summing the four integrals gives $\int_{I^2} A\, d\lambda_2 = \frac{3}{16} + \frac{1}{16}\sum_{i,j}\int_{I^2} A_{ij}\, d\lambda_2 = \frac{1}{4}$. □

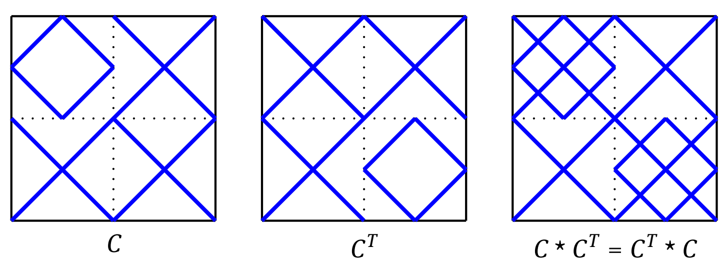

Figure 1: The supports of C, C^T and $C \star C^T$

Example 3.6. Let C be the copula whose mass is spread uniformly on the line segments shown in Figure 1. It follows that $C \star C^T$ and $C^T \star C$ are the uniform $\{A_{ij}\}$-patched copula where $A_{11} = A_{22} = E_0 \equiv \frac{M+W}{2}$ and $A_{12} = A_{21} = E_1$, the uniform $\{E_0\}_{i,j=1}^{2}$-patched copula. See Figure 1. The integral of each A_{ij} is $\frac{1}{4}$. By Lemma 3.5, $\int_{I^2} C \star C^T\, d\lambda_2 = \int_{I^2} C^T \star C\, d\lambda_2 = \frac{1}{4}$ and hence $[\![C]\!] = \frac{1}{2}$.

Observe that the uniform patched copula of four copies of the same copula with minimum $[\![\cdot]\!]$ still has minimum $[\![\cdot]\!]$. As an example, starting from $E_0 = \frac{M+W}{2}$, the iterative uniform patching gives a sequence $\{E_n\}$ for which E_n is the uniform patched copula of four copies of E_{n-1} and $[\![E_n]\!] = \frac{1}{2}$.

Proposition 3.7. *Recall that F_1 and F_2 denote the uniform distributions on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively. If D_0 is an idempotent copula with $[\![D_0]\!] = \frac{1}{2}$ then the uniform $\{D_0\}_{i,j=1}^{2}$-patched copula D_1 given by*

$$D_1(u, v) = \frac{1}{4} \sum_{i,j=1}^{2} D_0(F_i(u), F_j(v))$$

is idempotent with $[\![D_1]\!] = \frac{1}{2}$. *Note that* D_1 *may not be jointly symmetric if* D_0 *is not.*

Proof. By a straightforward but tedious computation, we obtain

$$D_1 * D_1(u,v) = \frac{1}{4}\sum_{i,j=1}^{2} D_0 * D_0(F_i(u), F_j(v)) = \frac{1}{4}\sum_{i,j=1}^{2} D_0(F_i(u), F_j(v)) = D_1(u,v).$$

So D_1 is idempotent. By Lemma 3.5, $[\![D_1]\!] = 2\int_{I^2} D_1\, d\lambda_2 = \frac{1}{2}$. □

Since $[\![C]\!]$ is defined in terms of $C^T * C$ and $C * C^T$, let us investigate further some properties of the self-map Ψ on the class of copulas $\mathcal{C}$ defined by $\Psi(C) = C^T * C$. The mapping $C \mapsto C * C^T$ has the analogous properties.

Proposition 3.8. *Let* C *be a copula.*

1. Ψ *is neither one-to-one nor onto.*
2. $\Psi(C)$ *is symmetric and hence* $\|[\Psi(C)]\|_1 = \|[\Psi(C)]\|_2$.
3. $\|[\Psi(C)]\|_1 = \frac{1}{4}$ *if and only if* $[\![C]\!]_1 = \frac{1}{4}$.
4. Ψ *is a continuous function from* $\mathcal{C}$ *endowed with the Sobolev norm into itself. That is, if a sequence of copulas* $\{C_n\}$ *converges to a copula* C *in the Sobolev norm, then* $\{\Psi(C_n)\}$ *converges to* $\Psi(C)$ *in the Sobolev norm.*

Proof. 2: This is clear as $(C^T * C)^T = C^T * (C^T)^T = C^T * C$.

1: Since all the left invertible copulas map to M, Ψ is not one-to-one. Ψ is not onto because all the left invertible copulas $S \neq M$ are not in the range of Ψ. For if a left invertible copula $S \neq M$ were of the form $C^T * C$ for some copula C, then C could not be left invertible but

$$M = S^T * S = C^T * C * C^T * C,$$

which means that C is left invertible, a contradiction.

3: ($\Leftarrow$) Since $[\![C]\!]_1 = \frac{1}{4}$, it follows from Theorem 3.3 (3.3) that $\int_0^1 C(t,v)dv = \frac{t}{2}$ for all $t \in I$. Then for all $u \in I$, by (4),

$$\begin{aligned}\int_0^1 C^T * C(u,v)\,dv &= \int_0^1\int_0^1 \partial_2 C^T(u,t)\partial_1 C(t,v)\,dt\,dv \\ &= \int_0^1 \partial_2 C^T(u,t)\frac{d}{dt}\left(\int_0^1 C(t,v)\,dv\right)dt \\ &= \int_0^1 \partial_2 C^T(u,t)\frac{d}{dt}\left(\frac{t}{2}\right)dt \\ &= \frac{1}{2}\int_0^1 \partial_2 C^T(u,t)\,dt = \frac{1}{2}C^T(u,1) = \frac{u}{2}.\end{aligned}$$

Again, by Theorem 3.3 *(iv)*, $\|[\Psi(C)]\|_1 = \left\|\left[C^T * C\right]\right\|_1 = \frac{1}{4}$.

($\Rightarrow$) If $\|[\Psi(C)]\|_1 = \frac{1}{4}$, then by Theorem 3.3 *(iv)*, $\int_0^1 C^T * C(u,v)\,dv = \frac{u}{2}$ for all u. So $[\![C]\!]_1 = \int_0^1 \frac{u}{2}\,du = \frac{1}{4}$.

4: This follows from the fact that the *-product is jointly continuous with respect to the Sobolev norm. See Theorem 4.2 in [4]. □

We are now ready to define the first candidate for a measure of dependence ν. For all continuous random variables X and Y, let

$$\nu(X,Y) = \sqrt{6\,\|[C_{X,Y}]\| - 3}. \tag{5}$$

Theorem 3.9. *The measure ν satisfies the following properties:*

ν1. $\nu(X, Y) = \nu(Y, X)$.
ν2. $0 \le \nu(X, Y) \le 1$.
ν3. $\nu(X, Y) = 0$ *if X and Y are independent.*
ν4. $\nu(X, Y) = 1$ *if and only if X and Y are mutually completely dependent.*
ν5. $\nu(f(X), g(Y)) = \nu(X, Y)$ *for all strictly monotonic transformations f and g.*

Proof. *ν1-ν4* follow directly from the definitions of $[\![\cdot]\!]$ and ν and Theorem 3.3. To prove *ν5*, let $C_{X,Y}$ be the copula of X and Y and consider the following four cases. If f and g are strictly increasing, then $C_{f(X),g(Y)} = C_{X,Y}$ [13, Theorem 2.4.3]. So $\nu(f(X), g(Y)) = \nu(X, Y)$. If f is strictly increasing and g is strictly decreasing, then $C_{f(X),g(Y)}(x, y) = x - C_{X,Y}(x, 1 - y) = C_{X,Y} * W(x, y)$ [13, Theorem 2.4.4 and Example 6.7]. Consequently, $C_{f(X),g(Y)} * C^T_{f(X),g(Y)} = C_{X,Y} * W * W^T * C^T_{X,Y} = C_{X,Y} * C^T_{X,Y}$ and so $\|[C_{f(X),g(Y)}]\|_2 = \|[C_{X,Y}]\|_2$. We also have

$$\begin{aligned}
\|[C_{f(X),g(Y)}]\|_1 &= \int_0^1\int_0^1 (C_{X,Y} * W)^T * (C_{X,Y} * W)(x, y)\,dx\,dy \\
&= \int_0^1\int_0^1 W * C^T_{X,Y} * C_{X,Y} * W(x, y)\,dx\,dy \\
&= \int_0^1\int_0^1 \left(x + y - 1 + C^T_{X,Y} * C_{X,Y}(1 - x, 1 - y)\right) dx\,dy \\
&= \int_0^1\int_0^1 C^T_{X,Y} * C_{X,Y}(x, y)\,dx\,dy = \|[C_{X,Y}]\|_1 .
\end{aligned}$$

Thus, $\|[C_{f(X),g(Y)}]\| = \|[C_{X,Y}]\|$, i.e. $\nu(f(X), g(Y)) = \nu(X, Y)$. The case where f is strictly decreasing and g is strictly increasing follows from the symmetry of ν. The last case when f and g are strictly decreasing can be proved using the fact that $C_{f(X),g(Y)} = W * C_{X,Y} * W$. □

Note that the property *ν5* in Theorem 3.9 is not valid for all Borel measurable injections where we utilize the same counterexample as the one in page 109.

Example 3.10. Consider jointly normal random variables X and Y with correlation coefficient ρ. Then $\nu(X, Y)$ is a strictly increasing function of $|\rho|$. Its graph obtained from a Matlab implementation is shown in Figure 2. Note the small difference between them whose graph is shown in Figure 3.

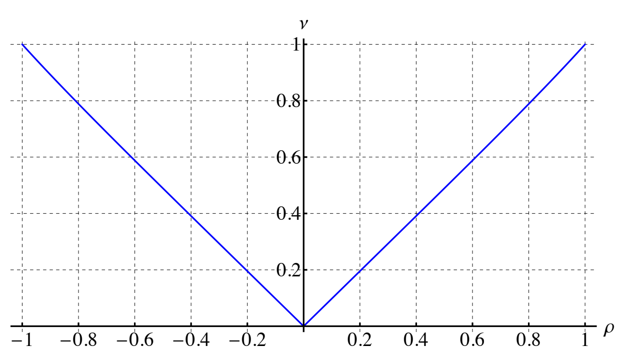

Figure 2: $\nu(X, Y)$ as a function of ρ for jointly normal X, Y

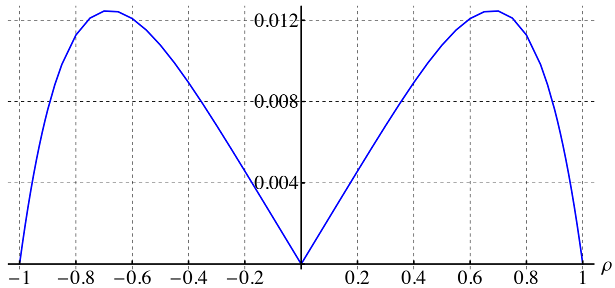

Figure 3: The difference $|\rho| - \nu(X, Y)$ for jointly normal X, Y

4 A measure of dependence from ν

In Section 3, we show that ν, defined in (5), is almost a measure of dependence. In fact, ν is almost a measure of mutual complete dependence as it can classify the mutual complete dependence: $\nu(X, Y) = 1$ if and only if X and Y are mutually completely dependent. However, it lacks the ability to classify the independence, as there are infinitely many copulas whose measure ν is zero. This is due to the fact proved in Theorem 3.3 that the minimum value of ν-measure is attained exactly when

$$\int_0^1 (C - \Pi)(u, v)\, dv = 0 \quad \text{and} \quad \int_0^1 (C - \Pi)(u, v)\, du = 0 \qquad \text{for all } u, v \in I. \tag{6}$$

These equalities reflect that the copula C spreads the probability mass in an almost uniform way albeit in a weaker sense than having the uniform mass distribution.

For a copula satisfying (6) but having a non-uniform mass distribution, shuffling masses of the horizontal or vertical stripes can alter the copula values in such a way that (6) no longer holds. Recall from [17] that shuffling a bivariate copula amounts to transforming one of the two random variables by a Borel measurable piecewise continuous injection. More generally, $C_{f(X),X} \star C_{X,Y} = C_{f(X),Y}$ for any Borel measurable injection f. See [17, Theorem 4.1]. Such a copula $C_{f(X),X}$ is invertible with inverse $C^T_{f(X),X} = C_{X,f(X)}$: $C_{X,f(X)} \star C_{f(X),X} = M = C_{f(X),X} \star C_{X,f(X)}$. In light of this observation, given continuous random variables X and Y with copula $C_{X,Y}$, we define

$$\left\|[C_{X,Y}]\right\|_\star = \sup_{f,g} \left\|[C_{f(X),g(Y)}]\right\| \quad \text{and} \quad \nu_\star(X, Y) = \sqrt{6\left\|[C_{X,Y}]\right\|_\star - 3},$$

where the supremum is taken over all Borel measurable injective transformations f and g. Using the facts that $C_{f(X),g(Y)} = C_{f(X),X} \star C_{X,Y} \star C_{Y,g(Y)}$ (see [17, Corollary 4.6]) and that $C_{f(X),X}$ and $C_{Y,g(Y)}$ are invertible, we obtain

$$\begin{aligned}\left\|[C_{f(X),g(Y)}]\right\|_1 &= \int_{I^2} C^T_{Y,g(Y)} \star C^T_{X,Y} \star C^T_{f(X),X} \star C_{f(X),X} \star C_{X,Y} \star C_{Y,g(Y)}\, d\lambda_2 \\ &= \int_{I^2} C^T_{Y,g(Y)} \star C^T_{X,Y} \star C_{X,Y} \star C_{Y,g(Y)}\, d\lambda_2 = \left\|[C_{X,Y} \star C_{Y,g(Y)}]\right\|_1 .\end{aligned}$$

Likewise, we can show that $\left\|[C_{f(X),g(Y)}]\right\|_2 = \left\|[C_{f(X),X} \star C_{X,Y}]\right\|_2$. Therefore,

$$[\![C]\!]_\star = \sup_{S\in\mathfrak{I}} [\![C \star S]\!]_1 + \sup_{S\in\mathfrak{I}} [\![S \star C]\!]_2 .$$

The following theorem shows that $\nu_\star$ is a copula-based measure of dependence in the sense of A. Rényi [16], i.e. all R1-R5 are satisfied.

Theorem 4.1. *The measure $\nu_\star$ satisfies the following properties:*

$\nu_\star 1$. $\nu_\star(X, Y) = \nu_\star(Y, X)$.
$\nu_\star 2$. $0 \le \nu_\star(X, Y) \le 1$.
$\nu_\star 3$. $\nu_\star(X, Y) = 0$ *if and only if* X *and* Y *are independent.*
$\nu_\star 4$. $\nu_\star(X, Y) = 1$ *if* X *and* Y *are completely dependent.*
$\nu_\star 5$. $\nu_\star(f(X), g(Y)) = \nu_\star(X, Y)$ *for all Borel measurable injective transformations* f *and* g.

Proof. Properties $\nu_\star 1$ and $\nu_\star 5$ follow immediately from the definitions of $[\![\cdot]\!]$ and $[\![\cdot]\!]_\star$. The bounds of $[\![\cdot]\!]$ in Theorem 3.3 give $\nu_\star 2$.

In order to prove $\nu_\star 4$, we will show only that $\nu_\star(X, Y) = 1$ if Y is a Borel measurable function of X as the other case is similar. This is equivalent to proving that $[\![C]\!]_\star = \frac{2}{3}$ when C is left invertible. Suppose a copula C is left invertible. Then $C^T * C = M$ and $[\![C * S]\!]_1 = \int_{I^2} S^T * C^T * C * S \, d\lambda_2 = \int_{I^2} M \, d\lambda_2 = \frac{1}{3}$ for every invertible copula S. For $\sup_{S\in\mathcal{I}} [\![S * C]\!]_2$, let us start from the fact, see [5], that $E = C * C^T$ is an idempotent copula whose invariant sets form a nonatomic σ-algebra $\mathcal{E} \subseteq \mathcal{B}(I)$.[1] As a consequence, by Corollary 1.12.10 in [1], for every integer $n = 1, 2, \dots$, there exists an *essential partition* consisting of sets $P_1, \dots, P_n$ in $\mathcal{E}$ in the sense that

$$\lambda\left(\bigcup_{i=1}^{n} P_i\right) = 1, \ \lambda(P_i) = \frac{1}{n} \text{ for all } i \text{ and } \lambda(P_i \cap P_j) = 0 \text{ for } i \neq j.$$

Since each P_i is an invariant set of E, we have

$$\mu_E([a, b] \times P_i) = \int_a^b T_E\chi_{P_i} \, d\lambda = \int_a^b \chi_{P_i} \, d\lambda = \lambda([a, b] \cap P_i),$$

where the first equality follows from a standard measure theoretic argument starting from P_i being intervals. By the symmetry of the idempotent E, we get $\mu_E(P_i \times [a, b]) = \lambda([a, b] \cap P_i)$. So E has zero mass in $[(I \setminus P_i) \times P_i] \cup [P_i \times (I \setminus P_i)]$ for $i = 1, \dots, n$.

We will then use $\{P_i\}$ to construct an invertible copula S_n for which $S_n * E * S_n^T$ is supported in $\bigcup_{i=1}^n I_i^2$, where $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right]$. In fact, by Lemma 4.2 in [5], there exist measure preserving Borel functions $h_i \colon P_i \to I_i$ and measure preserving Borel functions $g_i \colon I_i \to P_i$ such that $h_i \circ g_i(x) = x$ and $g_i \circ h_i(x) = x$ for almost every x. The functions g_i and h_i are called *essential inverses* of each other. Now, define $R_n, T_n \colon I \to I$ by

$$R_n(x) = g_i(x) \quad \text{if } x \in I_i \quad \text{and} \quad T_n(x) = h_i(x) \quad \text{if } x \in P_i.$$

Since $\{I_i\}$ and $\{P_i\}$ are essential partitions of I, the self-maps R_n and T_n are measure preserving. Moreover, $R_n \circ T_n(x) = x$ and $T_n \circ R_n(x) = x$ for almost all $x \in I$. Finally, let S_n be the copula of U and $R_n(U)$ where U is a uniform $[0, 1]$ random variable. Equivalently, $S_n = C_{e,R_n} = C_{T_n,e}$ where e denotes the identity map and $C_{k,\ell}(x, y) = \lambda(k^{-1}([0, x]) \cap \ell^{-1}([0, y]))$ for $x, y \in I$. Hence, S_n is invertible. Recall that the left multiplication of E by S_n amounts to shuffling (moving) masses of E on the vertical stripes $P_i \times I$ to $I_i \times I$ for $i = 1, \dots, n$. The right multiplication by S_n^T shuffles the horizontal stripes. From the fact shown above that E has no mass in $\bigcup_i[(I \setminus P_i) \times P_i] \cup [P_i \times (I \setminus P_i)]$, $S_n * E * S_n^T$ is supported in $\bigcup_{i=1}^n I_i^2$. So $S_n * E * S_n^T$ converges to M pointwise and thus $\sup_{S\in\mathcal{I}} [\![S * C]\!]_2 = \frac{1}{3}$.

Clearly, $[\![\Pi]\!]_\star = [\![\Pi]\!] = \frac{1}{2}$. To prove the opposite direction of $\nu_\star 3$, or equivalently that $[\![C]\!]_\star = \frac{1}{2}$ implies that $C = \Pi$, we define $S_{\alpha,\beta}$, for $0 \le \alpha \le \beta \le 1$, as the shuffle of Min whose support consists of at most three line

1 A measurable set S is called an *invariant set* of a copula A if the characteristic function χ_S is a fixed point of the Markov operator $T_A \colon L^\infty(I) \to L^\infty(I)$ defined by $T_A\psi(x) = \frac{d}{dx}\int_I \frac{\partial A}{\partial t}(x, t)\psi(t)\, dt$. It also holds that $A(x, y) = \int_0^x T_A\chi_{[0,y]} \, d\lambda$. A σ-algebra $\mathcal{E}$ is said to be *nonatomic* if for every $S \in \mathcal{E}$ there exists a subset S' of S in $\mathcal{E}$ such that $0 < \lambda(S') < \lambda(S)$.

segments of slope 1 shown in Figure 4, i.e.

$$S_{\alpha,\beta}(x,y) = \begin{cases} 0 & \text{if } 0 \le x \le \alpha, 0 \le y \le \beta-\alpha, \\ \min(x, y-(\beta-\alpha)) & \text{if } 0 \le x \le \alpha, \beta-\alpha < y \le \beta, \\ \min(x-\alpha, y) & \text{if } \alpha < x \le \beta, 0 \le y \le \beta-\alpha, \\ x+y-\beta & \text{if } \alpha < x \le \beta, \beta-\alpha < y \le \beta, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

For every copula C and $0 \le \alpha \le \beta \le 1$, a direct computation gives

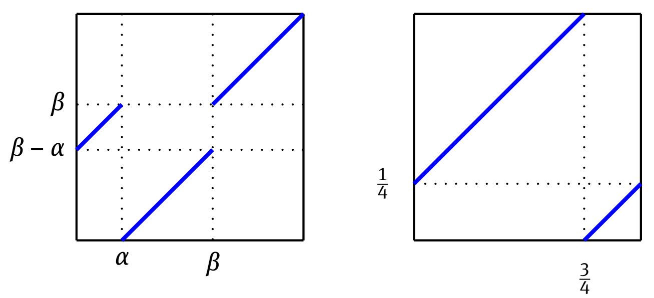

Figure 4: The supports of $S_{\alpha,\beta}$ (left) and $S_1 \equiv S_{\frac{3}{4},1}$ (right)

$$S_{\alpha,\beta} \star C(x,y) = \begin{cases} C(x+\beta-\alpha, y) - C(\beta-\alpha, y) & \text{if } 0 \le x \le \alpha, \\ C(x-\alpha, y) + C(\beta, y) - C(\beta-\alpha, y) & \text{if } \alpha < x \le \beta, \\ C(x,y) & \text{if } \beta < x \le 1. \end{cases}$$

Integrating $S_{\alpha,\beta} \star C$ with respect to x and making suitable changes of variables yield

$$\int_0^1 S_{\alpha,\beta} \star C(x,y)\,dx = \int_0^1 C(x,y)\,dx + (\beta-\alpha)\,C(\beta,y) - \beta C(\beta-\alpha,y). \tag{7}$$

Then $[\![C]\!]_\star = \frac{1}{2}$ implies that both $\sup_{S\in\mathcal{J}} [\![C \star S]\!]_1$ and $\sup_{S\in\mathcal{J}} [\![S \star C]\!]_2$ attain the minimum value $\frac{1}{4}$. So $[\![C \star S]\!]_1 = \frac{1}{4} = [\![S \star C]\!]_2$ for all $S \in \mathcal{J}$. Let $n \in \mathbb{N}$, $i \in \{0, 1, 2, \dots, 2^n - 1\}$ and $y \in [0,1]$. Applying Theorem 3.3 *(v)* to C and $S_{\frac{1}{2^n},\frac{i+1}{2^n}} \star C$ yields

$$\int_0^1 C(x,y)\,dx = \frac{y}{2} \quad \text{and} \quad \int_0^1 S_{\frac{1}{2^n},\frac{i+1}{2^n}} \star C(x,y)\,dx = \frac{y}{2}.$$

By (7), $\frac{i+1}{2^n} C\left(\frac{i}{2^n}, y\right) = \frac{i}{2^n} C\left(\frac{i+1}{2^n}, y\right)$. Since i is arbitrary, repeated use of this equation gives

$$C\left(\frac{i}{2^n}, y\right) = \frac{i}{i+1} C\left(\frac{i+1}{2^n}, y\right) = \cdots = \frac{i}{2^n} C(1,y) = \frac{i}{2^n} y.$$

By the continuity of C and the denseness of the dyadic rationals in $[0,1]$, we have $C = \Pi$ as desired. □

Remark. We then give an example to demonstrate that the converse of $\nu_\star 4$ is not true. By the proof of $\nu_\star 4$ above, any nonatomic idempotent copula gives maximum $[\![\cdot]\!]_\star$. However, in order to illustrate the shuffling in the proof, let us consider $E_0 = \frac{M+W}{2}$ which is neither left nor right invertible. Equivalently, if E_0 is the copula of X and Y then they cannot be completely dependent. We will show that $[\![E_0]\!]_\star = \frac{2}{3}$ and so $\nu_\star(X, Y) = 1$. Since E_0 is idempotent and hence symmetric, for every $S \in \mathfrak{I}$

$$\left\|\left[E_0 * S^T\right]\right\|_1 = [\![S * E_0]\!]_2 = \int_{I^2} (S * E_0 * E_0 * S^T)\, d\lambda_2 = \int_{I^2} (S * E_0 * S^T)\, d\lambda_2.$$

So it suffices to show that $\sup_{S\in\mathfrak{I}} \int_{I^2} (S * E_0 * S^T)\, d\lambda_2 = \frac{1}{3}$. We shall acquire this by constructing a sequence $\{S_1, S_2, \ldots\}$ of shuffles of Min such that $\int_{I^2} (S_n * E_0 * S_n^T)\, d\lambda_2$ converges to $\frac{1}{3}$.

Henceforth, for a copula A, let us denote by $\mathrm{ord}_n(A)$ the ordinal sum of n copies of A with respect to the partition $\{[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \ldots, [\frac{n-1}{n}, 1]\}$ of $[0, 1]$. Observe that $\mathrm{ord}_m\left(\mathrm{ord}_n(A)\right) = \mathrm{ord}_{mn}(A)$ for any $m, n \in \mathbb{N}$.

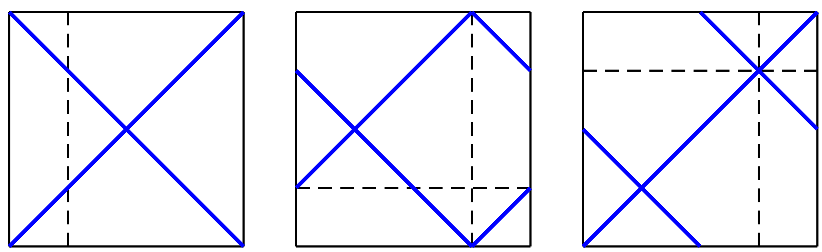

Figure 5: The supports of E_0, $S_1 * E_0$ and $E_1 \equiv S_1 * E_0 * S_1^T$ respectively

Denote $S_1 \equiv S_{\frac{3}{4},1}$ which is the shuffle of Min supported on the main diagonals of the squares $[0, \frac{3}{4}] \times [\frac{1}{4}, 1]$ and $[\frac{3}{4}, 1] \times [0, \frac{1}{4}]$ as shown in Figure 4. Our proof hinges on an observation that the product $S_1 * E_0$ is the result of (horizontally) shuffling the mass of E_0 on the rectangles $[0, \frac{1}{4}] \times I$ and $[\frac{1}{4}, 1] \times I$ and that $S_1 * E_0 * S_1^T$ can be obtained by (vertically) shuffling the mass of $S_1 * E_0$ on the rectangles $I \times [0, \frac{1}{4}]$ and $I \times [\frac{1}{4}, 1]$. Therefore, $E_1 \equiv S_1 * E_0 * S_1^T$ is equal to the ordinal sum of $\{E_0, E_0\}$ with respect to $\{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$, i.e. $E_1 = \mathrm{ord}_2(E_0)$. Their supports are shown in Figure 5. For more details on shuffles of copulas, see [7, 11], specifically [17] for their relations with the $*$-product used here. This observation can be made more rigorous but probably less transparent by using the conditional probability to decompose E_0 according to the partition $\{[0, \frac{1}{4}], [\frac{1}{4}, \frac{3}{4}], [\frac{3}{4}, 1]\}$ of $[0, 1]$ on both axes. Both horizontal and vertical shuffles of this so-called patched decomposition, introduced and studied in [2, 24], can then be conveniently manipulated. For details, see [9].

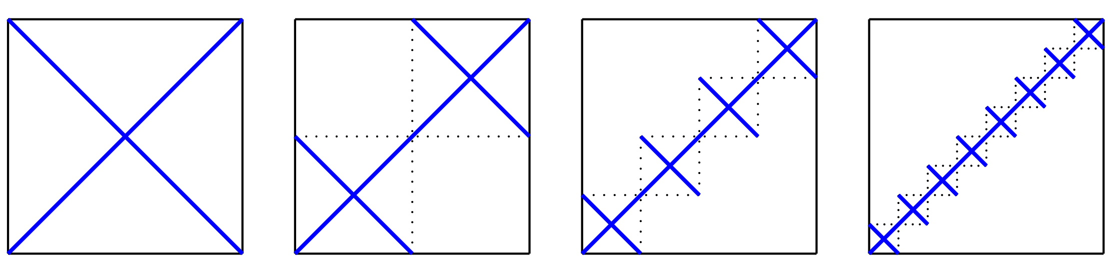

Figure 6: The supports of E_0, E_1, E_2 and E_3 respectively

For each $n \geq 2$, we put $E_n = S_n * E_0 * S_n^T$ where

$$S_n = \mathrm{ord}_{2^{n-1}}(S_1) * \mathrm{ord}_{2^{n-2}}(S_1) * \cdots * \mathrm{ord}_2(S_1) * S_1.$$

It follows that S_n is a shuffle of Min, $S_n = \mathrm{ord}_{2^{n-1}}(S_1) * S_{n-1}$ and

$$E_n = \mathrm{ord}_{2^{n-1}}(S_1) * E_{n-1} * \mathrm{ord}_{2^{n-1}}(S_1)^T. \tag{8}$$

Using the recursive relation (8), it can be shown by induction on n that $E_n = \mathrm{ord}_{2^n}(E_0)$. The first few E_n's are illustrated in Figure 6. Since the value of this ordinal sum agrees with M except possibly on the union $\bigcup_{i=1}^{2^n}[\frac{i-1}{2^n}, \frac{i}{2^n}]^2$ whose area is $\frac{1}{2^n} \to 0$, $\int_{I^2} S_n * E_0 * S_n^T \, d\lambda_2$ converges to $\int_{I^2} M \, d\lambda_2 = \frac{1}{3}$.

Owing to one of the anonymous referees, we are pleased to propose a class of measures of mutual complete dependence ν_α defined as a nontrivial convex combination $\nu_\alpha = \alpha\nu + (1-\alpha)\nu_\star$ for $0 < \alpha < 1$.

Theorem 4.2. *The measure ν_α satisfies the following properties:*

ν_α1. $\nu_\alpha(X, Y) = \nu_\alpha(Y, X)$.
ν_α2. $0 \le \nu_\alpha(X, Y) \le 1$.
ν_α3. $\nu_\alpha(X, Y) = 0$ *if and only if X and Y are independent.*
ν_α4. $\nu_\alpha(X, Y) = 1$ *if and only if X and Y are mutually completely dependent.*
ν_α5. $\nu_\alpha(f(X), g(Y)) = \nu_\alpha(X, Y)$ *for all strictly monotonic transformations f and g.*

Proof. The proof uses corresponding properties of ν and $\nu_\star$ in Theorems 3.9 and 4.1. $\nu_\alpha 1$ and $\nu_\alpha 2$ clearly follow from the same properties of ν and $\nu_\star$.

If X and Y are independent, then $\nu(X, Y) = \nu_\star(X, Y) = 0$ and hence $\nu_\alpha(X, Y) = 0$. Conversely, if $\nu_\alpha(X, Y) = 0$ then $\nu_\star(X, Y)$ must be zero and so X, Y are independent.

If X, Y are mutually completely dependent, then $\nu(X, Y) = \nu_\star(X, Y) = 1$ and hence $\nu_\alpha(X, Y) = 1$. Conversely, if $\nu_\alpha(X, Y) = 1$ then $\nu(X, Y)$ must be one and so X, Y are mutually completely dependent.

$\nu_\alpha 5$ is a result of $\nu 5$ and $\nu_\star 5$. □

5 Conclusion

We show that $\left\|C^T * C + C * C^T\right\|_{L^1}$ gives rise to a $[0, 1]$-valued function ν of continuous random variables which is almost a measure of mutual complete dependence as it cannot identify independence. We then prove that the measure $\nu_\star$, modified from ν in such a way that it is invariant under all one-to-one transformations, satisfies the five essential properties in Rényi's postulates for measures of dependence. Finally, every nontrivial convex combination of ν and $\nu_\star$ is a measure of mutual complete dependence.

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References

[1] V.I. Bogachev, Measure Theory, vol I, Springer Verlag, 2007.
[2] N. Chaidee, T. Santiwipanont, S. Sumetkijakan, Patched approximations and their convergence, Comm. Statist. Theory Methods, in press, http://dx.doi.org/10.1080/03610926.2014.887112.
[3] W.F. Darsow, B. Nguyen, E.T. Olsen, Copulas and Markov processes, Illinois J. Math. 36 (1992) 600–642.
[4] W.F. Darsow, E.T. Olsen, Norms for copulas, Int. J. Math. Math. Sci. 18 (1995) 417–436.
[5] W.F. Darsow, E.T. Olsen, Characterization of idempotent 2-copulas, Note Mat. 30 (2010) 147–177.

[6] F. Durante, E.P. Klement, J.J. Quesada-Molina, P. Sarkoci, Remarks on two product-like constructions for copulas, Kybernetika (Prague) 43 (2007) 235–244.
[7] F. Durante, P. Sarkoci, C. Sempi, Shuffles of copulas, J. Math. Anal. Appl. 352 (2009) 914–921.
[8] H. Gebelein, Das statistische Problem der Korrelation als Variations und Eigenwertproblem und sein Zusammenhang mit der Ausgleichsrechnung, Z. Angew. Math. Mech. 21 (1941) 364–379.
[9] N. Kamnitui, New Measure of Dependence from Conditional Variances, Master thesis, 2015.
[10] H.O. Lancaster, Correlation and complete dependence of random variables, Ann. Math. Statist. 34 (1963) 1315–1321.
[11] P. Mikusiński, H. Sherwood, M.D. Taylor, Probabilistic interpretations of copulas and their convex sums, in: G. Dall'Aglio, S. Kotz, G. Salinetti (Eds.), Advances in Probability Distributions with Given Marginals: Beyond the Copulas, Kluwer Dordrecht. 67 (1991) 95–112.
[12] P. Mikusiński, H. Sherwood, M.D. Taylor, Shuffles of min, Stochastica 13 (1992) 61–74.
[13] R.B. Nelsen, An Introduction to Copulas, second ed., Springer Verlag, 2006.
[14] K. Pearson, D. Heron, On theories of association, Biometrika 9(1/2) (1913) 159–315.
[15] E.T. Olsen, W.F. Darsow, B. Nguyen, Copulas and Markov operators, Lecture Notes-Monograph Series 28 (1996) 244–259.
[16] A. Rényi, On measures of dependence, Acta. Math. Acad. Sci. Hungar. 10 (1959) 441–451.
[17] P. Ruankong, T. Santiwipanont, S. Sumetkijakan, Shuffles of copulas and a new measure of dependence, J. Math. Anal. Appl. 398(1) (2013) 398–402.
[18] B. Schweizer, E.F. Wolff, On nonparametric measures of dependence for random variables, Ann. Statist. 9 (1981) 879–885.
[19] K.F. Siburg, P.A. Stoimenov, A scalar product for copulas, J. Math. Anal. Appl. 344 (2008) 429–439.
[20] K.F. Siburg, P.A. Stoimenov, A measure of mutual complete dependence, Metrika 71 (2009) 239–251.
[21] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, Publ. Inst. Statist. Univ. Paris 8 (1959) 229–231.
[22] W. Trutschnig, On a strong metric on the space of copulas and its induced dependence measure, J. Math. Anal. Appl. 384 (2011) 690–705.
[23] W. Trutschnig, On Cesaro convergence of iterates of the star product of copulas, Stat. Prob. Letters 83 (2013) 357–365.
[24] Y. Zheng, J. Yang, J.Z. Huang, Approximation of bivariate copulas by patched bivariate Fréchet copulas, Insurance Math. Econ. 48 (2011) 246–256.

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RESEARCH

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Pongpol Ruankong, Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand. email: fscippru@ku.ac.th

Songkiat Sumetkijakan,* Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand. email: songkiat.s@chula.ac.th

ESSENTIAL CLOSURES

Abstract

Based on the Zermelo-Fraenkel system of axioms ZF, we introduce a theory of essential closures. It is a generalization of the concept of topological closures in which a set may not be contained in its essential closure. A typical essential closure collects all points that are essential with respect to a submeasure; hence it is called a submeasure closure. One of our main results states that a "nice" essential closure must be a submeasure closure. Many examples of known and new submeasure closures are discussed and their applications are demonstrated, especially in the study of the supports of measures.

1 Introduction

It was suggested in [1, Proposition 1] and [17, Lemma 10] that the probability mass of a complete dependence copula $C = C_{U,f(U)}$ is concentrated on the graph of f in the sense that $V_C(\operatorname{gr} f) = 1$. Here, the random variable U is uniformly distributed on $[0,1]$, $f\colon [0,1] \to [0,1]$ is measure-preserving and V_C denotes the Borel probability measure on $[0,1]^2$ induced by C. However, to the best of our knowledge, due probably to the lack of a suitable tool, no one

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had determined an explicit formula of the support of V_C in terms of the graph of f. Recently in [16], a formula of $\operatorname{supp} V_C$ in terms of $\operatorname{gr} f$ was obtained via a "new" tool called an *essential closure*. In $\mathbb{R}^2$, the essential closure $\widetilde{A}$ of a set A is the set of points $x \in \mathbb{R}^2$ for which the projection of each open neighborhood of x in A onto a coordinate axis has a positive Lebesgue outer measure. It was derived in [16, Theorem 3.3.3] that

$$\operatorname{supp} V_C = \widetilde{\operatorname{gr} f}$$

given that f is *essentially refined*, of which some examples are piecewise linear functions.

The adjective "essential" is quite ubiquitous in mathematical analysis and is often used to indicate that a defining condition holds outside of a negligible set. As such, taking essential closure should mean taking closure by ignoring "small" sets. For instance, in the above essential closure on $\mathbb{R}^2$, small sets are precisely those sets whose coordinate projections each have Lebesgue outer measure zero. As a tool in their study of absolutely continuous spectra for some linear operators, Gesztesy et al. [9] defined an essential closure on $\mathbb{R}$, with respect to which small sets are sets of Lebesgue measure zero. Both essential closures are our prototypes of general essential closures and share many satisfying properties. Thus far, there seems to be no systematic treatment of essential closures.

In this article, our aim is to propose a set of postulates for general essential closures and to develop a theory of essential closures. Among many results, the concept of non-essential or "small" sets is (re)introduced and proved to be closely related to essential closures. Submeasure closures, defined as the essential closures whose non-essential sets are sets of submeasure zero, and their examples are investigated. They are shown to be useful in the study of the supports of measures. An interesting result is that the class of submeasure closures is large enough to contain all "nice" essential closures.

In section 2, we develop a theory of essential closures starting with a set of four postulates. In section 3, we present a motivation behind the set of postulates of essential closures from a topological point of view. In section 4, we construct concrete examples of essential closures via submeasures and demonstrate their applications. Finally, we discuss some existing concepts related to essential closures in the last section.

2 Essential closures

In sections 2 and 3, we denote both essential closures and essential closure operators by $\mathcal{E}$. Likewise, we use the notations $A \mapsto \overline{A}$ and cl to denote both topological closures and topological closure operators. In addition, for a given topological space X, $\mathfrak{N}(x)$ denotes the collection of open neighborhoods of $x \in X$ and $\mathcal{P}(X)$ denotes the collection of subsets of X.

2.1 Postulates for essential closures

After experimenting with various potential sets of postulates for essential closures, we have come to a conclusion that the following set of postulates seems the most natural.

Postulate 1. Let (X, τ) be a topological space equipped with an algebra Ω over X. We say that a unary operation $\mathcal{E}\colon \Omega \to \Omega$ is an *essential closure* if for every $A, B \in \Omega$, the following hold:

1. $\mathcal{E}(A)$ is a closed set;
2. $\mathcal{E}(A) \subseteq \overline{A}$;
3. $\mathcal{E}(A \cup B) = \mathcal{E}(A) \cup \mathcal{E}(B)$; and
4. $\mathcal{E}$ is idempotent (i.e., $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$).

Remark. It follows directly from 2 and 3 of Postulate 1 that $\mathcal{E}(\emptyset) = \emptyset$ and that $\mathcal{E}$ is monotonic with respect to the set inclusion, respectively.

Definition 2. A unary operation $\mathcal{E}$ on a topological space (X, τ) equipped with an algebra Ω over X is said to be

1. *strong* if $\mathcal{E}(A \setminus \mathcal{E}(A)) = \emptyset$ for every $A \in \Omega$; and
2. *weakly strong* if for each $A \in \Omega$ and $x \notin \mathcal{E}(A)$, there is $G \in \mathfrak{N}(x) \cap \Omega$ such that $\mathcal{E}(A \cap G) = \emptyset$.

Remark.

1. Let $\mathcal{E}$ be a unary operation on an algebra Ω satisfying 1, 2 and 3 of Postulate 1. If $\mathcal{E}(A \setminus \mathcal{E}(A)) = \emptyset$ for each $A \in \Omega$, then $\mathcal{E}$ is idempotent as

$$\mathcal{E}(A) = \mathcal{E}(A \setminus \mathcal{E}(A)) \cup \mathcal{E}(A \cap \mathcal{E}(A)) \subseteq \mathcal{E}(\mathcal{E}(A)) \subseteq \overline{\mathcal{E}(A)} = \mathcal{E}(A).$$

   Thus, if $\mathcal{E}$ satisfies 1, 2 and 3 and is strong then it is an essential closure. However, a weakly strong unary operation satisfying 1, 2 and 3 need not be an essential closure. See Example 8.

2. A strong unary operation satisfying 1 of Postulate 1 is also weakly strong.

Example 1. Let $X = \{0,1\}$, $\tau = \{\emptyset, \{0\}, X\}$ and $\Omega = \mathcal{P}(X)$. Define a unary operation $\mathcal{E}\colon \Omega \to \Omega$ by $\mathcal{E}(\emptyset) = \emptyset$ and $\mathcal{E}(A) = \{1\}$ if A is not empty. It is easy to check that $\mathcal{E}$ is an essential closure. Moreover,

$$\mathcal{E}(X \setminus \mathcal{E}(X)) = \mathcal{E}(\{1\}^c) = \{1\} \neq \emptyset.$$

Hence $\mathcal{E}$ is not strong. Note that $\mathcal{E}(X) \neq X$.

Proposition 1. *Let $\mathcal{E}$ be an essential closure on an algebra Ω over a topological space X and suppose that $\mathcal{E}(X) = X$. Then the following hold:*

1. $\mathcal{E}(A)^c \subseteq \mathcal{E}(A^c)$ *for each* $A \in \Omega$*;*

2. $\mathcal{E}(G) = \overline{G}$ *for every open set* $G \in \Omega$*; and*

3. $\overline{\operatorname{int} A} \subseteq \mathcal{E}(A)$ *for each* $A \in \Omega$ *such that* $\operatorname{int} A \in \Omega$.

Proof. Recall the properties of essential closures in Postulate 1.

1. Observe that $X = \mathcal{E}(X) = \mathcal{E}(A \cup A^c) = \mathcal{E}(A) \cup \mathcal{E}(A^c)$. Hence we have $\mathcal{E}(A)^c \subseteq \mathcal{E}(A^c)$.

2. If $G \in \Omega$ is open, then $\mathcal{E}(G)^c \subseteq \mathcal{E}(G^c) \subseteq \overline{G^c} = G^c$. Thus $G \subseteq \mathcal{E}(G) \subseteq \overline{G}$. Since $\mathcal{E}(G)$ is closed, $\mathcal{E}(G) = \overline{G}$.

3. Since $\operatorname{int} A \in \Omega$ is open, $\overline{\operatorname{int} A} = \mathcal{E}(\operatorname{int} A) \subseteq \mathcal{E}(A)$. □

2.2 Non-essential sets

In this section, we introduce one of the most important concepts related to essential closures, namely the concept of essential and non-essential sets. This concept is at the core of the theory of essential closures. Non-essential sets can be viewed as small sets with respect to an essential closure.

Definition 3. Let $\mathcal{E}$ be an essential closure on Ω. Then a set $A \in \Omega$ is said to be *non-essential* if $\mathcal{E}(A) = \emptyset$; otherwise, A is said to be *essential*. The collection of non-essential sets is denoted by $\mathcal{N}_\Omega(\mathcal{E})$.

Theorem 2. *Let $\mathcal{E}$ be an essential closure on Ω. Then $\mathcal{E}$ is weakly strong if and only if, for each $A \in \Omega$, $\mathcal{E}(A)$ is the intersection of the closed sets $F \in \Omega$ such that $A \setminus F$ is non-essential.*

Proof. Assume that $\mathcal{E}$ is weakly strong. For each $A \in \Omega$, if $x \notin \mathcal{E}(A)$, then there exists $G \in \mathfrak{N}(x) \cap \Omega$ such that $\mathcal{E}(A \cap G) = \emptyset$. Therefore,

$$\mathcal{E}(A)^c \subseteq \bigcup \{G \in \Omega \colon G \text{ is open and } \mathcal{E}(A \cap G) = \emptyset\}.$$

So $\mathcal{E}(A) \supseteq \bigcap \{F \in \Omega \colon F \text{ is closed and } \mathcal{E}(A \setminus F) = \emptyset\}$. For the other inclusion, it suffices to show that any closed set $F \in \Omega$ with $\mathcal{E}(A \setminus F) = \emptyset$ necessarily contains $\mathcal{E}(A)$. Observe that for such a set F,

$$\mathcal{E}(A) = \mathcal{E}(A \cap F) \cup \mathcal{E}(A \setminus F) = \mathcal{E}(A \cap F) \subseteq \mathcal{E}(F) \subseteq \overline{F} = F.$$

To prove the converse, let $A \in \Omega$ and suppose $x \notin \mathcal{E}(A)$. Then, by the assumption, $x \in G$ for some open set $G \in \Omega$ such that $\mathcal{E}(A \cap G) = \emptyset$. In other words, $\mathcal{E}$ is weakly strong. □

According to Theorem 2, one can see that the collection of non-essential sets acts as a generator of its corresponding weakly strong essential closure. To study weakly strong essential closures, it suffices to study their non-essential sets.

Corollary 3. *Suppose $\mathcal{E}_1$ and $\mathcal{E}_2$ are weakly strong essential closures on Ω such that $\mathcal{N}_\Omega(\mathcal{E}_1) = \mathcal{N}_\Omega(\mathcal{E}_2)$. Then the two essential closures coincide.*

Definition 4. Let $\mathcal{E}$ be an essential closure on an algebra Ω over a topological space X. Then a set $A \in \Omega$ is said to be *locally essential* if $\mathcal{E}(G \cap A) \neq \emptyset$ for every open set $G \in \Omega$ such that $G \cap A \neq \emptyset$.

Proposition 4. *Let $\mathcal{E}$ be an essential closure on an algebra Ω over X and suppose $\mathcal{E}(X) = X$. Then every open set $O \in \Omega$ is locally essential.*

Proof. Let $G \in \Omega$ be an open set such that $G \cap O \neq \emptyset$. By Proposition 1(2), $\mathcal{E}(G \cap O) = \overline{G \cap O} \supseteq G \cap O \neq \emptyset$. □

Definition 5. An essential closure on Ω is said to be *σ-non-essential* if Ω is a σ-algebra and the union of every countable collection of non-essential sets is non-essential.

Lemma 5. *Let $\mathcal{E}$ be an essential closure on an algebra Ω and $x \in \mathcal{E}(A)$. Then for any $G \in \mathfrak{N}(x) \cap \Omega$, $G \cap A$ is essential.*

Proof. Suppose there exists $G \in \mathfrak{N}(x) \cap \Omega$ with $\mathcal{E}(G \cap A) = \emptyset$. Then

$$\mathcal{E}(A) = \mathcal{E}(A \cap G^c) \subseteq \mathcal{E}(A) \cap \mathcal{E}(G^c) \subseteq \mathcal{E}(A) \cap \overline{G^c} = \mathcal{E}(A) \setminus G,$$

which contradicts the fact that $\mathcal{E}(A) \setminus G$ is a proper subset of $\mathcal{E}(A)$. □

Definition 6. A *topological measurable space* is a triple (X, τ, Ω) where (X, τ) is a topological space and Ω is a σ-algebra over X containing the topology τ.

The following result requires a technical assumption that for each point $x \in X$ and each $G \in \mathfrak{N}(x)$, there is $O \in \mathfrak{N}(x)$ with $\overline{O} \subseteq G$. A topological space with such a property is called *regular*. More information on regular spaces can be found in Munkres' book [13]. A *regular measurable space* is a topological measurable space where the topology is regular.

Theorem 6. *Let $\mathcal{E}$ be a σ-non-essential essential closure on a regular measurable space. Then for every sequence of sets A_i in Ω,*

$$\mathcal{E}\left(\bigcup_{i=1}^{\infty} A_i\right) = \overline{\bigcup_{i=1}^{\infty} \mathcal{E}(A_i)}.$$

Proof. If $x \in \mathcal{E}\left(\bigcup_{i=1}^{\infty} A_i\right)$ and $G \in \mathfrak{N}(x)$, then there exists $O \in \mathfrak{N}(x)$ such that $\overline{O} \subseteq G$. By Lemma 5, $\mathcal{E}\left(\bigcup_{i=1}^{\infty}(O \cap A_i)\right) = \mathcal{E}\left(O \cap \bigcup_{i=1}^{\infty} A_i\right) \neq \emptyset$. Since the essential closure is σ-non-essential, there exists A_j with $\mathcal{E}(O \cap A_j) \neq \emptyset$. Hence

$$\emptyset \neq \mathcal{E}(O \cap A_j) \subseteq \mathcal{E}(O) \cap \mathcal{E}(A_j) \subseteq \overline{O} \cap \mathcal{E}(A_j) \subseteq G \cap \bigcup_{i=1}^{\infty} \mathcal{E}(A_i).$$

This implies that $x \in \overline{\bigcup_{i=1}^{\infty} \mathcal{E}(A_i)}$. The other inclusion follows trivially from the fact that the essential closure of a set is closed. ☐

In what follows, we will consider various essential closures defined in the same manner. Given an algebra Ω over X, $\mathcal{I} \subseteq \Omega$ is an *ideal* if (1) $\emptyset \in \mathcal{I}$; (2) for every $A \in \mathcal{I}$, if $B \in \Omega$ is such that $B \subseteq A$, then $B \in \mathcal{I}$; and (3) if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. Given an ideal $\mathcal{I}$ in Ω and a set $F \in \Omega \setminus \mathcal{I}$, it is straightforward to verify that the unary operation $\mathcal{E} = \mathcal{E}_{\mathcal{I},F}$ on Ω defined by

$$\mathcal{E}(A) = \begin{cases} \emptyset & \text{if } A \in \mathcal{I}, \\ F & \text{otherwise} \end{cases} \tag{1}$$

is an essential closure with respect to the topology $\tau = \{\emptyset, F^c, X\}$.

The following example shows that an essential closure on a σ-algebra is not necessarily σ-non-essential.

Example 2. Let $X = \mathbb{N}$, $\tau = \{\emptyset, \{1\}^c, X\}$ and $\Omega = \mathcal{P}(X)$. Consider an essential closure $\mathcal{E} = \mathcal{E}_{\mathcal{I},\{1\}}$ where $\mathcal{I} = \{A \in \Omega \colon A \text{ is finite and } 1 \notin A\}$. Observe that $\mathcal{E}(X \setminus \mathcal{E}(X)) = \{1\} \neq \emptyset = \bigcup_{x \neq 1} \mathcal{E}(\{x\})$. Hence there exists an essential closure on a σ-algebra that is neither strong nor σ-non-essential.

The following two examples show that the concepts of strong essential closures and σ-non-essential essential closures are not related in an obvious way, that is one does not imply the other.

Example 3. Let $X = \mathbb{N}$, $\tau = \{\emptyset, X\}$, $\Omega = \mathcal{P}(X)$ and $\mathcal{E} = \mathcal{E}_{\mathcal{I},X}$ where $\mathcal{I} = \{A \in \Omega \colon A \text{ is finite}\}$. It is easy to check that the essential closure $\mathcal{E}$ is strong. However, $\mathcal{E}(X) = X \neq \emptyset = \bigcup_{x \in X} \mathcal{E}(\{x\})$. Hence, there is a strong essential closure on a σ-algebra that is not σ-non-essential.

Example 4. Let $X = \mathbb{N}$, $\tau = \{\emptyset, \{1\}^c, X\}$, $\Omega = \mathcal{P}(X)$ and $\mathcal{E} = \mathcal{E}_{\{\emptyset\},\{1\}}$. Clearly, $\mathcal{E}$ is σ-non-essential. However, observe that

$$\mathcal{E}(X \setminus \mathcal{E}(X)) = \mathcal{E}(\{1\}^c) = \{1\} \neq \emptyset.$$

Hence, there is a σ-non-essential essential closure that is not strong.

Clearly, the non-essential sets of the essential closure $\mathcal{E}_{\mathcal{I},F}$ defined by (1) are exactly the sets in the ideal $\mathcal{I}$. Observe also that the non-essential sets of any given essential closure form an ideal. Conversely, if one has in mind which sets should be considered small, then there always exists an essential closure with respect to which the pre-assigned small sets are non-essential. We will be more interested in σ-non-essential essential closures.

Definition 7. Let $\emptyset \neq S \subseteq \Omega$, where Ω is a σ-algebra over X. Define $\mathcal{N}_\Omega(S)$ to be the smallest collection that satisfies the following conditions for all $B \in \Omega$ and $A, A_1, A_2, \dots \in \mathcal{N}_\Omega(S)$:

1. $S \subseteq \mathcal{N}_\Omega(S) \subseteq \Omega$;
2. $B \subseteq A$ implies $B \in \mathcal{N}_\Omega(S)$; and
3. $\bigcup_{n=1}^{\infty} A_n \in \mathcal{N}_\Omega(S)$.

Remark. Notice that $\mathcal{N}_\Omega(S)$ is the smallest σ-ideal of Ω containing S; see page 13 in Bauer's book [3].

In the sequel, we often require that every subset of the space is Lindelöf. Such a topological space is called *hereditarily Lindelöf.*

Theorem 7. *Let (X, τ, Ω) be a Lindelöf measurable space and S be a non-empty subcollection of Ω. Then there exists a unique σ-non-essential weakly strong essential closure whose collection of non-essential sets is exactly the collection $\mathcal{N}_\Omega(S)$. In fact, it is defined by*

$$\mathcal{E}(A) = \bigcap \{F \in \Omega \colon F \text{ is closed and } A \setminus F \in \mathcal{N}_\Omega(S)\} \quad \text{for } A \in \Omega. \tag{2}$$

Proof. It is straightforward to verify that the unary operation $\mathcal{E}$ defined by (2) is an essential closure on Ω. By the definition of $\mathcal{E}$ in (2), $A \in \mathcal{N}_\Omega(S)$ implies $\mathcal{E}(A) = \emptyset$. Conversely, suppose $\mathcal{E}(A) = \emptyset$. Then for each $x \in X$, there exists $G \in \mathfrak{N}(x) \cap \Omega$ such that $A \cap G \in \mathcal{N}_\Omega(S)$. Let $\mathcal{G}$ be the collection of open sets $G \in \Omega$ such that $A \cap G \in \mathcal{N}_\Omega(S)$. Hence $\mathcal{G}$ covers X, which is Lindelöf. Let $\{G_n\}_{n\in\mathbb{N}}$ be a countable subcover of $\mathcal{G}$. Since $A \cap G_n \in \mathcal{N}_\Omega(S)$ for all $n \in \mathbb{N}$, $A = \bigcup_{n=1}^{\infty}(A \cap G_n) \in \mathcal{N}_\Omega(S)$ by property 3 in Definition 7. Hence the collections $\mathcal{N}_\Omega(\mathcal{E})$ and $\mathcal{N}_\Omega(S)$ coincide.

Let $A \in \Omega$ and $x \notin \mathcal{E}(A)$. Then $x \in G$ for some open set $G \in \Omega$ such that $A \cap G \in \mathcal{N}_\Omega(S)$, which implies that $\mathcal{E}(A \cap G) = \emptyset$. Thus $\mathcal{E}$ is weakly strong. Moreover, since the collection $\mathcal{N}_\Omega(S)$ is closed under countable union, the induced essential closure is σ-non-essential. The uniqueness part follows from Corollary 3. □

In the previous theorem, a similar result also holds if we replace Lindelöf and weakly strong with hereditarily Lindelöf and strong, respectively.

Theorem 8. *Let (X, τ, Ω) be a hereditarily Lindelöf measurable space and S be a non-empty subcollection of Ω. Then there exists a unique σ-non-essential strong essential closure, defined by* (2)*, whose collection of non-essential sets is exactly the collection $\mathcal{N}_\Omega(S)$.*

Proof. In view of Theorem 7, $\mathcal{E}$ is an essential closure and it suffices to show that $\mathcal{E}$ is strong. Let $A \in \Omega$. Since $\mathcal{E}(A)^c$ is Lindelöf,

$$\begin{aligned} A \setminus \mathcal{E}(A) &= A \setminus \bigcap \{F \in \Omega \colon F \text{ is closed and } A \setminus F \in \mathcal{N}_\Omega(S)\} \\ &= A \setminus \bigcap_{n=1}^{\infty} \{F_n \in \Omega \colon F_n \text{ is closed and } A \setminus F_n \in \mathcal{N}_\Omega(S)\} \\ &= \bigcup_{n=1}^{\infty} \{A \setminus F_n \colon F_n \in \Omega \text{ is closed and } A \setminus F_n \in \mathcal{N}_\Omega(S)\} \end{aligned}$$

for some countable subcollection $\{F_n\}_{n\in\mathbb{N}}$ of Ω. Hence, $A \setminus \mathcal{E}(A) \in \mathcal{N}_\Omega(S)$ by property 3 in Definition 7. In consequence, $\mathcal{E}$ is a strong essential closure. The uniqueness of $\mathcal{E}$ follows clearly from Theorem 7. □

In Theorems 7 and 8, since (X, τ, Ω) is a topological measurable space, Ω is assumed to contain the topology τ. If the σ-algebra does not contain the topology, the theorems may fail to hold. This is demonstrated in the following example.

Example 5. Choose pairwise distinct elements a, b and c. Put $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $S = \{\emptyset\}$ and $\Omega = \{\emptyset, \{a\}, \{b, c\}, X\}$. Notice that Ω does not contain τ and $\mathcal{N}_\Omega(S) = \{\emptyset\}$. Using the same construction as in Theorems 7 and 8, we have $\mathcal{E}(\{a\}) = X$ while $\overline{\{a\}} = \{a, c\}$. Hence the induced mapping is not an essential closure since it violates the second property of essential closures in Postulate 1.

2.3 Essential closedness

In this section, we introduce another important concept related to essential closures, namely the concept of essential closedness.

Definition 8. Let $\mathcal{E}$ be an essential closure on Ω. A set $F \in \Omega$ is said to be *essentially closed* if and only if $\mathcal{E}(F) = F$. We denote the collection of essentially closed sets by $\mathcal{C}_\Omega(\mathcal{E})$.

Proposition 9. *Let $\mathcal{E}$ be a strong essential closure on Ω. Then for any $A \in \Omega$, $\mathcal{E}(A) = \bigcap\{F \in \mathcal{C}_\Omega(\mathcal{E})\colon A \setminus F \in \mathcal{N}_\Omega(\mathcal{E})\}$.*

PROOF. Since essentially closed sets are closed, it follows from Theorem 2 that for any $A \in \Omega$, $\mathcal{E}(A) \subseteq \bigcap\{F \in \mathcal{C}_\Omega(\mathcal{E})\colon A \setminus F \in \mathcal{N}_\Omega(\mathcal{E})\}$. The other inclusion follows from the fact that $\mathcal{E}(A)$ is essentially closed and $\mathcal{E}$ is strong. $\square$

Example 6. The above result does not generally hold for weakly strong essential closures. For example, let

$$X = \{0\} \cup \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}$$

be equipped with the subspace topology inherited from the standard topology of $\mathbb{R}$ and let $\Omega = \mathcal{P}(X)$. For each $A \in \Omega$, define $\mathcal{E}(A) = \emptyset$ if A is finite and $0 \notin A$; otherwise, $\mathcal{E}(A) = \{0\}$.

It is straightforward to verify that $\mathcal{E}$ is a weakly strong essential closure and $\mathcal{C}_\Omega(\mathcal{E}) = \{\emptyset, \{0\}\}$. One can see that $\mathcal{E}(\{0\}^c) = \{0\}$, but on the other hand, there is no essentially closed set F such that $\mathcal{E}(\{0\}^c \setminus F) = \emptyset$. Therefore, $X = \bigcap\{F \in \mathcal{C}_\Omega(\mathcal{E})\colon \mathcal{E}(\{0\}^c \setminus F) = \emptyset\}$ as it is the empty intersection. Thus $\mathcal{E}(\{0\}^c) \neq \bigcap\{F \in \mathcal{C}_\Omega(\mathcal{E})\colon \{0\}^c \setminus F \in \mathcal{N}_\Omega(\mathcal{E})\}$.

Proposition 10. *Let $\mathcal{E}$ be an essential closure on Ω and $F \in \Omega$. If F is essentially closed, then F is closed and locally essential.*

PROOF. Assume F is essentially closed. Then F is closed. Hence for any open set $G \in \Omega$ such that $G \cap F \neq \emptyset$, $\mathcal{E}(F \setminus G) \subseteq \overline{F \setminus G} = F \setminus G \subsetneq F$. Moreover, $F = \mathcal{E}(F) = \mathcal{E}(F \setminus G) \cup \mathcal{E}(F \cap G)$. Thus $\mathcal{E}(F \cap G) \neq \emptyset$. $\square$

Proposition 11. *Let $\mathcal{E}$ be a weakly strong essential closure on Ω and $F \in \Omega$. If F is closed and locally essential, then F is essentially closed.*

Proof. Since F is closed, $\mathcal{E}(F) \subseteq \overline{F} = F$. Suppose that $F \setminus \mathcal{E}(F) \neq \emptyset$ and let $x \in F \setminus \mathcal{E}(F)$. Then there exists $G \in \mathfrak{N}(x) \cap \Omega$ such that $\mathcal{E}(G \cap F) = \emptyset$. Since F is locally essential and $\mathcal{E}(G \cap F) = \emptyset$, $G \cap F = \emptyset$. This contradicts the fact that $x \in G \cap F$. Therefore, F is essentially closed. □

Together, Propositions 10 and 11 give a characterization of essential closedness for weakly strong essential closures.

Corollary 12. *Let $\mathcal{E}$ be a weakly strong essential closure on an algebra Ω and $F \in \Omega$. Then F is essentially closed if and only if F is closed and locally essential.*

3 Essential closure operators

In this section, we provide an alternative approach to postulating the concept of essential closures. An advantage of this approach is that we need not assume any a priori topological structure. Recall that a *topological closure operator* on a set X is defined as a unary operation $\mathrm{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the following properties for all $A, B \subseteq X$:

1. $\mathrm{cl}(\emptyset) = \emptyset$;
2. $A \subseteq \mathrm{cl}(A)$;
3. $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$; and
4. cl is idempotent.

It is well known that there is a one-to-one correspondence between the collection of topological closure operators and the collection of topological closures (equivalently, the collection of topologies) on a common space. If we want to add the prefix "essential," then the property that $A \subseteq \mathrm{cl}(A)$ should be excluded. We propose a set of postulates for essential closure operators accordingly.

Postulate 9. Let X be a non-empty set and Ω an algebra over X. An *essential closure operator* on (X, Ω) is a unary operation $\mathcal{E}\colon \Omega \to \Omega$ that satisfies the following properties for all sets $A, B \in \Omega$:

1. $\mathcal{E}(\emptyset) = \emptyset$;

2. $\mathcal{E}(A \cup B) = \mathcal{E}(A) \cup \mathcal{E}(B)$; and

3. $\mathcal{E}$ is idempotent.

Remark. A topological closure operator restricted to any algebra is an essential closure operator. Moreover, it is the unique essential closure operator with the property that $A \subseteq \mathcal{E}(A)$ for each A in the algebra.

Next, we demonstrate a relationship between essential closures and essential closure operators. First, we need the following two technical lemmas.

Lemma 13. *Let X be a non-empty set and Ω an algebra over X. Assume that $A \mapsto \overline{A} \colon \Omega \to \Omega$ satisfies the following properties for all $A, B \in \Omega$:*

1. $\overline{\emptyset} = \emptyset$;

2. $A \subseteq \overline{A}$;

3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$; *and*

4. $A \mapsto \overline{A}$ *is idempotent.*

Then $A \mapsto \overline{A}$ can be extended to a topological closure operator on X.

PROOF. Define $\mathrm{cl} \colon \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$\mathrm{cl}(A) = \bigcap_{\overline{C} \supseteq A} \overline{C},$$

where C ranges over all sets in Ω. First, we verify that cl is indeed an extension. Suppose $A \in \Omega$. Then we have

$$\mathrm{cl}(A) = \bigcap_{\overline{C} \supseteq A} \overline{C} \subseteq \overline{A} \subseteq \bigcap_{\overline{C} \supseteq \overline{A}} \overline{C} \subseteq \bigcap_{\overline{C} \supseteq A} \overline{C} = \mathrm{cl}(A),$$

where the first inclusion follows from property 2 and the last inclusion follows from the fact that $A \subseteq \overline{C}$ implies $\overline{A} \subseteq \overline{C}$. Hence the unary operation cl is an extension of $A \mapsto \overline{A}$. Moreover, observe the following properties of cl.

- $\mathrm{cl}(\emptyset) = \overline{\emptyset} = \emptyset$ since $\emptyset \in \Omega$.

- $A \subseteq \displaystyle\bigcap_{\overline{C} \supseteq A} \overline{C} = \mathrm{cl}(A)$.

- Observe that

$$\bigcap_{\overline{C}\supseteq A\cup B}\overline{C}\subseteq\bigcap_{\overline{D}\supseteq A,\overline{E}\supseteq B}\overline{D}\cup\overline{E}=\Big(\bigcap_{\overline{D}\supseteq A}\overline{D}\Big)\cup\Big(\bigcap_{\overline{E}\supseteq B}\overline{E}\Big).$$

Hence $\mathrm{cl}(A\cup B)\subseteq\mathrm{cl}(A)\cup\mathrm{cl}(B)$. Moreover, the other inclusion follows from the fact that cl is monotonic with respect to the set inclusion.

- If $A\subseteq\overline{C}$, then $\mathrm{cl}(A)\subseteq\mathrm{cl}(\overline{C})=\overline{C}$ since $\overline{C}\in\Omega$. Hence

$$\mathrm{cl}(\mathrm{cl}(A))=\bigcap_{\overline{C}\supseteq\mathrm{cl}(A)}\overline{C}\subseteq\bigcap_{\overline{C}\supseteq A}\overline{C}=\mathrm{cl}(A).$$

The opposite inclusion holds.

Therefore, cl is a topological closure operator on X. □

Lemma 14. *Let $\mathcal{E}$ be an essential closure operator on an algebra Ω over X. Then there exists a topology τ on X such that $\mathcal{E}(A)$ is closed and $\mathcal{E}(A)\subseteq\mathrm{cl}(A)$ for every $A\in\Omega$.*

Proof. Define $\overline{A}=A\cup\mathcal{E}(A)$ for each $A\in\Omega$. It is straightforward to check that $A\mapsto\overline{A}\colon\Omega\to\Omega$ satisfies the properties in Lemma 13. Let cl be a topological closure operator extended from $A\mapsto\overline{A}\colon\Omega\to\Omega$ and let τ be the topology induced by cl. Observe that $\mathcal{E}(A)\subseteq A\cup\mathcal{E}(A)=\mathrm{cl}(A)$ for each $A\in\Omega$. Moreover, $\mathrm{cl}(\mathcal{E}(A))=\mathcal{E}(A)\cup\mathcal{E}(\mathcal{E}(A))=\mathcal{E}(A)$. Hence $\mathcal{E}(A)$ is closed with respect to the topology τ for each $A\in\Omega$. □

Given an essential closure, if we take out its underlying topological structure, what we obtain is an essential closure operator. The following result, which is one of our main results, shows that there is a natural way to induce an underlying topology for a given essential closure operator. However, it is not guaranteed that the induced topology coincides with the given topology.

Theorem 15. *Let $\mathcal{E}$ be an essential closure operator on Ω. Define $\tau_{\mathcal{E}}=\bigcap\tau_\alpha$, where the non-empty intersection is taken over all topologies τ_α on X satisfying the properties in Lemma 14, and let $\mathrm{cl}_{\mathcal{E}}$ be the topological closure relative to $\tau_{\mathcal{E}}$. Then $\mathcal{E}\colon\Omega\to\Omega$ satisfies the following properties for all $A\in\Omega$:*

1. *$\mathcal{E}(A)$ is closed in $(X,\tau_{\mathcal{E}})$; and*
2. *$\mathcal{E}(A)\subseteq\mathrm{cl}_{\mathcal{E}}(A)$.*

In other words, $\mathcal{E}$ is an essential closure on $(X,\tau_{\mathcal{E}},\Omega)$. Furthermore, $\tau_{\mathcal{E}}$ is generated by the collection $\{\mathcal{E}(A)^c\}_{A\in\Omega}$.

Proof. Let $A \in \Omega$. Observe that $\mathcal{E}(A)$ is closed in $(X, \tau_{\mathcal{E}})$ because $\mathcal{E}(A)^c \in \tau_\alpha$ for all α. Moreover, $\mathcal{E}(A) \subseteq \mathrm{cl}_\alpha(A) \subseteq \mathrm{cl}_{\mathcal{E}}(A)$ because $\tau_{\mathcal{E}} \subseteq \tau_\alpha$. Therefore, $\mathcal{E}$ is an essential closure on $(X, \tau_{\mathcal{E}}, \Omega)$.

Let τ be the topology generated by the collection $\{\mathcal{E}(A)^c\}_{A\in\Omega}$. Since $\mathcal{E}(A)$ is closed in $(X, \tau_{\mathcal{E}})$ for all $A \in \Omega$, $\tau \subseteq \tau_{\mathcal{E}}$. Consequently, $\mathrm{cl}_{\mathcal{E}}(A) \subseteq \mathrm{cl}_\tau(A)$ for all $A \in \Omega$. Since $\mathcal{E}$ is an essential closure, $\mathcal{E}(A) \subseteq \mathrm{cl}_{\mathcal{E}}(A) \subseteq \mathrm{cl}_\tau(A)$ for all $A \in \Omega$. Moreover, $\mathcal{E}(A)$ is closed in (X, τ) for all $A \in \Omega$ since τ is generated by $\{\mathcal{E}(A)^c\}_{A\in\Omega}$. Hence τ is a topology satisfying the properties in Lemma 14, which implies that $\tau_{\mathcal{E}} \subseteq \tau$. Thus the two topologies coincide. □

Remark. Let $A \mapsto \overline{A}$ be a topological closure operator, hence an essential closure operator. One can see that the topology induced by a topological closure operator $A \mapsto \overline{A}$, as an essential closure operator, coincides with the topology induced by $A \mapsto \overline{A}$ as a topological closure operator.

Given an essential closure operator $\mathcal{E}$ on (X, Ω), any topology τ containing $\tau_{\mathcal{E}}$ with the property that $\mathcal{E}(A) \subseteq \mathrm{cl}_\tau(A)$ for all $A \in \Omega$ is said to be *compatible* with $\mathcal{E}$. Notice that if τ is a compatible topology, then $(X, \tau, \Omega, \mathcal{E})$ is an essential closure space.

On a given essential closure operator space, there can be several compatible topologies, among which the topology $\tau_{\mathcal{E}}$ is the smallest. The induced topology $\tau_{\mathcal{E}}$ is called the *canonical topology.* The following result gives a characterization of the canonical topologies.

Theorem 16. *Let $\mathcal{E}$ be an essential closure on (X, τ, Ω). Then τ is the canonical topology $\tau_{\mathcal{E}}$ if and only if there exists a subbase of τ whose elements are of the form $\mathcal{E}(A)^c$ where $A \in \Omega$.*

Proof. If τ is canonical, then it is generated by the collection $\{\mathcal{E}(A)^c\}_{A\in\Omega}$. On the other hand, assume that τ is generated by a subcollection of $\{\mathcal{E}(A)^c\}_{A\in\Omega}$. Then $\tau \subseteq \tau_{\mathcal{E}}$. Moreover, for each $A \in \Omega$, $\mathcal{E}(A)$ is closed with respect to τ since $\mathcal{E}$ is an essential closure. Thus $\tau_{\mathcal{E}} \subseteq \tau$. □

4 Submeasure closures

In this section, we construct concrete examples of essential closures and demonstrate some of their applications, especially in the study of the supports of measures. Let us remark that, even though we can avoid the argument of the Axiom of Choice in all of our proofs, many (if not most) existing concepts and results used below are so relevant to the axiom that it cannot be completely disregarded. An example is the countability of the union of a countable collection of countable sets, which is required in defining Lebesgue measures. As such, in the sequel, we additionally assume the Axiom of Choice, hence ZFC (the Zermelo-Fraenkel system of axioms with the Axiom of Choice).

4.1 Definition and properties

Definition 10. Let Ω be a σ-algebra over X. A *submeasure* on (X, Ω) is a set function $\mu\colon \Omega \to [0, \infty]$ satisfying

1. $\mu(\emptyset) = 0$;
2. $\mu(A) \leq \mu(B)$ for any $A, B \in \Omega$ such that $A \subseteq B$; and
3. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$ for any $A_1, A_2, \ldots \in \Omega$.

Remark. Let us note a few facts about our submeasures.

1. Our submeasures are defined on σ-algebras and are countably subadditive, unlike classical submeasures, which are defined on algebras and are finitely subadditive.
2. Every submeasure on a σ-algebra can be extended, perhaps not uniquely, to an outer measure. In other words, every submeasure is a restriction of some outer measure. The reason we do not simply call it an outer measure or a restriction of an outer measure is for convenience in stating our results.

Definition 11. A topological submeasure space is a quadruple (X, τ, Ω, μ) where (X, τ, Ω) is a topological measurable space and μ is a submeasure on (X, Ω).

Definition 12. Let (X, τ, Ω, μ) be a topological submeasure space. For any measurable set $A \in \Omega$, we say that $x \in \overline{A}^{\mu}$ if $\mu(G \cap A) > 0$ for every $G \in \mathfrak{N}(x)$. The set $\overline{A}^{\mu}$ is called the *μ-closure* of A.

Remark. The following are immediate results from the definition.

1. If $\mu(A) = 0$, then $\overline{A}^{\mu} = \emptyset$.
2. Every submeasure closure is weakly strong.
3. If μ and ν are submeasures on a common measurable space such that $\mu \ll \nu$, then $\overline{A}^{\mu} \subseteq \overline{A}^{\nu}$ for every measurable set A.

Example 7. Recall the definition of the submeasure closure ($A \mapsto \overline{A}^e$) on the real line defined in [9]. It is an essential closure (with respect to the standard topology τ_s) on the Lebesgue σ-algebra $\mathfrak{L}(\mathbb{R})$. Now, we temporarily remove the topology and view the essential closure as an essential closure operator on $\mathfrak{L}(\mathbb{R})$. We will show that the canonical topology (i.e., the induced topology $\tau_{A \mapsto \overline{A}^e}$ in Theorem 15) is, in fact, the given standard topology.

Recall that the collection of non-empty open intervals forms a subbase for the standard topology. Moreover, notice that each non-empty open interval (a, b) can be written as $\left(\overline{A}^e\right)^c$ where $A = (-\infty, a] \cup [b, \infty) \in \mathfrak{L}(\mathbb{R})$. Hence by Theorem 16, the canonical topology is the standard topology.

Example 8. Let $X = (-\infty, 0]$ and τ be the topology on X generated by the collection of singletons $\{x\}$ where $x \in (-\infty, 0)$. Notice that $\{x\}$ is a neighborhood of x for every point $x \in (-\infty, 0)$. However, the only neighborhood of 0 is X.

Define a measure μ on $\mathcal{P}(X)$ by setting $\mu(A) = 0$ if A is countable and $\mu(A) = \infty$ otherwise. Observe that $\overline{X}^\mu = \{0\}$ while $\overline{\overline{X}^\mu}^\mu = \emptyset$. Thus the μ-closure is not idempotent. Hence it is not an essential closure.

According to Example 8, a submeasure closure need not be idempotent. Nevertheless, it is easy to verify that every submeasure closure satisfies the other three properties in Postulate 1. As a result,

$$\overline{\overline{A}^\mu}^\mu \subseteq \overline{\overline{A}^\mu} = \overline{A}^\mu$$

for every μ-measurable set A.

Two sufficient conditions for a submeasure closure to be an essential closure are given in the following result.

Theorem 17. *Assume that (X, τ, Ω, μ) is either a hereditarily Lindelöf submeasure space or an inner regular measure space. Then $A \mapsto \overline{A}^\mu$ is a strong essential closure.*

Proof. Let A be a measurable set and $\mathcal{G}$ be the collection of the open sets G such that $\mu(G \cap (A \setminus \overline{A}^\mu)) = 0$. Observe that $x \in A \setminus \overline{A}^\mu$ implies $x \notin \overline{A \setminus \overline{A}^\mu}^\mu$.

If (X, τ, Ω, μ) is a hereditarily Lindelöf submeasure space, then for each $x \in A \setminus \overline{A}^\mu$, there is $G \in \mathfrak{N}(x)$ such that $\mu(G \cap (A \setminus \overline{A}^\mu)) = 0$. Thus $\mathcal{G}$ covers $A \setminus \overline{A}^\mu$. Hence there exists a countable subcover $\{G_1, G_2, \dots\}$ of $\mathcal{G}$. Consequently,

$$\mu(A \setminus \overline{A}^\mu) \leq \sum_{i=1}^{\infty} \mu\left(G_i \cap (A \setminus \overline{A}^\mu)\right) = 0.$$

Therefore, $\overline{A \setminus \overline{A}^{\mu}}^{\mu} = \emptyset$.

If (X, τ, Ω, μ) is an inner regular measure space, consider any compact set $K \subseteq A \setminus \overline{A}^{\mu}$. Then $x \in K$ implies $x \notin \overline{A \setminus \overline{A}^{\mu}}^{\mu}$. Therefore, for each $x \in K$, there is $G \in \mathfrak{N}(x)$ such that $\mu(G \cap (A \setminus \overline{A}^{\mu})) = 0$. Thus $\mathcal{G}$ covers K. Hence there exists a finite subcover $\{G_1, \dots, G_n\}$ of $\mathcal{G}$. Consequently,

$$\mu(K) \leq \sum_{i=1}^{n} \mu(G_i \cap K) \leq \sum_{i=1}^{n} \mu\left(G_i \cap (A \setminus \overline{A}^{\mu})\right) = 0.$$

Therefore, $\mu(A \setminus \overline{A}^{\mu}) = 0$ by inner regularity. Thus $\overline{A \setminus \overline{A}^{\mu}}^{\mu} = \emptyset$. □

Observe that the two parts of the proof of Theorem 17 are very similar. One proof uses a countable subcover while the other uses a finite subcover. So in the sequel, if there are twin results like these, we shall omit the proof for the case of inner regular measure spaces.

Theorem 18. *Assume that (X, τ, Ω, μ) is either a Lindelöf submeasure space or an inner regular measure space. Then $\overline{A}^{\mu} = \emptyset$ if and only if $\mu(A) = 0$.*

Proof. Assume that (X, τ, Ω, μ) is a Lindelöf submeasure space and $\overline{A}^{\mu} = \emptyset$. Let $\mathcal{G}$ be the collection of the open sets G such that $\mu(G \cap A) = 0$. Since $\overline{A}^{\mu} = \emptyset$, $\mathcal{G}$ covers X. Thus there is a countable subcover $\{G_1, G_2, \dots\}$ of $\mathcal{G}$. Therefore,

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(G_i \cap A) = 0.$$

The converse is an immediate result from the definition of submeasure closures. The case of inner regular measure spaces can be proved similarly. □

Corollary 19. *Assume that (X, τ, Ω, μ) is either a Lindelöf submeasure space or an inner regular measure space.*

1. *If $A \mapsto \overline{A}^{\mu}$ is an essential closure, then it is σ-non-essential.*
2. *If $A \mapsto \overline{A}^{\mu}$ is a strong essential closure, then $\mu(\overline{A}^{\mu}) \geq \mu(A)$ for every measurable set A.*

Proof. 1. Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of non-essential sets. By Theorem 18, $\mu(A_i) = 0$. Consequently, $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 0$. Again, by Theorem 18, $\bigcup_{i=1}^{\infty} A_i$ is a non-essential set.

2. Observe that

$$\begin{aligned}\mu(A) &\leq \mu(A \cap \overline{A}^{\mu}) + \mu(A \setminus \overline{A}^{\mu}) \\ &= \mu(A \cap \overline{A}^{\mu}) \\ &\leq \mu(\overline{A}^{\mu}).\end{aligned}$$

This completes the proof. □

The following result gives a characterization of the σ-non-essential strong essential closures on a hereditarily Lindelöf measurable space.

Theorem 20. *Assume that (X, τ, Ω) is a hereditarily Lindelöf measurable space. Then an essential closure $\mathcal{E}$ on Ω is strong and σ-non-essential if and only if it is a submeasure closure on (X, τ, Ω).*

Proof. Since the case $\mathcal{E}(X) = \emptyset$ is trivial, assume that $\mathcal{E}(X) \neq \emptyset$. Suppose $\mathcal{E}$ is a σ-non-essential strong essential closure on Ω. Define $\mu\colon \Omega \to [0, \infty]$ by $\mu(A) = 0$ if $\mathcal{E}(A) = \emptyset$; otherwise, $\mu(A) = 1$. We show that μ is a submeasure on (X, Ω). Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of measurable sets.

- Since $\emptyset$ is non-essential, $\mu(\emptyset) = 0$.
- Suppose $A_1 \subseteq A_2$. Then $\mathcal{E}(A_1) \subseteq \mathcal{E}(A_2)$. If A_2 is non-essential, then A_1 is also non-essential. Hence $\mu(A_1) = 0 = \mu(A_2)$. If A_2 is essential, then $\mu(A_1) \leq 1 = \mu(A_2)$.
- If there is an essential set A_j in $\{A_i\}$ then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq 1 \leq \sum_{i=1}^{\infty} \mu(A_i)$. If every A_i is non-essential, then $\bigcup_{i=1}^{\infty} A_i$ is also non-essential. So

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 0 = \sum_{i=1}^{\infty} \mu(A_i).$$

Therefore, μ is a submeasure on (X, Ω) and the μ-closure on (X, τ, Ω) is a σ-non-essential strong essential closure by Theorem 17 and Corollary 19(1). Moreover, the fact that $\mathcal{E}$ and the μ-closure coincide follows directly from Corollary 3 and Theorem 18. Finally, the converse follows from Theorem 17 and Corollary 19(1). □

A submeasure is said to be *trivial* if the space is of submeasure zero and is said to be *normalized* if the space is of submeasure one.

Remark. An essential closure induces a normalized submeasure if the space is essential. Otherwise, it induces the trivial submeasure.

Example 9. Consider a μ_1-closure and a μ_2-closure on a common topological measurable space. One can verify that the set function $\mathcal{E}$ defined as

$$\mathcal{E}(A) = \overline{A}^{\mu_1} \cup \overline{A}^{\mu_2}$$

is, in fact, the $(\mu_1 + \mu_2)$-closure. Moreover, if both the μ_1-closure and the μ_2-closure are essential closures, then so is the $(\mu_1 + \mu_2)$-closure.

Example 10. If a μ-closure is a strong essential closure, then $A \setminus \overline{A}^{\mu}$ is a non-essential set. However, $\overline{A}^{\mu} \setminus A$ can be an essential set. For example, take $\mu = \lambda_1$, the 1-dimensional Lebesgue measure on $[0,1]$, and $A = [0,1] \setminus C$, where C is a positive Lebesgue measure Cantor set on $[0,1]$. Then for each $x \in [0,1]$ and $G \in \mathfrak{N}(x)$, $G \cap A$ contains a non-empty open interval. Hence $\lambda_1(G \cap A) > 0$. Therefore, $\overline{A}^{\lambda_1} = [0,1]$. As a result,

$$\overline{A}^{\lambda_1} \setminus A = [0,1] \setminus A = C,$$

which is of positive Lebesgue measure.

4.2 Applications

In this section, we demonstrate some applications of submeasure closures, especially the study of the supports of measures. An essential closure can be viewed as a tool to eliminate the non-essential part of a set. In the case of an essential closure defined via a measure, one can expect that eliminating the non-essential part of the space should give the support of that measure.

4.2.1 The supports of measures

The support of a submeasure is defined analogously to the definition of the support of a measure.

Theorem 21. *Let μ be a submeasure on (X, τ, Ω). Then*

$$\operatorname{supp} \mu = \overline{A}^{\mu}$$

for any measurable set A such that $\mu(A^c) = 0$. In particular, if the μ-closure is an essential closure, then $\operatorname{supp} \mu$ is μ-essentially closed.

Proof. If $x \notin \operatorname{supp} \mu$, then there exists $G \in \mathfrak{N}(x)$ such that $\mu(G) = 0$. Thus $\mu(G \cap A) = 0$ for any measurable set A. Hence $x \notin \overline{A}^{\mu}$. Conversely, if $x \notin \overline{A}^{\mu}$ then there exists $G \in \mathfrak{N}(x)$ such that $\mu(G \cap A) = 0$. Since $\mu(A^c) = 0$, we have $\mu(G) \leq \mu(G \cap A) + \mu(G \cap A^c) = 0$. Therefore, $x \notin \operatorname{supp} \mu$. □

Theorem 22. *Let (X, τ, Ω, μ) be a hereditarily Lindelöf measure space. Then a set $A \in \Omega$ is μ-essentially closed if and only if there is an absolutely continuous measure $\nu \ll \mu$ such that* $\operatorname{supp} \nu = A$.

PROOF. For each measurable set B, define $\nu(B) = \mu(A \cap B)$. Clearly, $\nu \ll \mu$. It is left to show that $\operatorname{supp} \nu = A$. First of all, observe that A is closed and $\nu(A^c) = 0$. Hence $\operatorname{supp} \nu \subseteq A$. We prove the opposite inclusion by first noting that

$$\mu((\operatorname{supp} \nu)^c \cap A) = \nu((\operatorname{supp} \nu)^c) = 0.$$

Suppose $(\operatorname{supp} \nu)^c \cap A \neq \emptyset$. Let $x \in (\operatorname{supp} \nu)^c \cap A$. Then $(\operatorname{supp} \nu)^c \in \mathfrak{N}(x)$ and $x \in A = \overline{A}^{\mu}$. Therefore, $\mu((\operatorname{supp} \nu)^c \cap A) > 0$, a contradiction. Hence $(\operatorname{supp} \nu)^c \cap A = \emptyset$. In other words, $A \subseteq \operatorname{supp} \nu$.

Conversely, it suffices to show that $\operatorname{supp} \nu \subseteq \overline{\operatorname{supp} \nu}^{\mu}$. If $x \notin \overline{\operatorname{supp} \nu}^{\mu}$, then there exists $G \in \mathfrak{N}(x)$, $\mu(G \cap \operatorname{supp} \nu) = 0$. By the absolute continuity, we have $\nu(G) = \nu(G \cap \operatorname{supp} \nu) = 0$. So $x \notin \operatorname{supp} \nu$. □

The following result can be proved similarly. Notice the difference in the inner regularity of the measure ν.

Corollary 23. *Let (X, τ, Ω, μ) be a topological inner regular measure space. Then a set $A \in \Omega$ is μ-essentially closed if and only if there is an inner regular measure $\nu \ll \mu$ such that* $\operatorname{supp} \nu = A$.

Theorem 24. *Let (X, τ, Ω, μ) be a hereditarily Lindelöf measure space where μ is σ-finite. For any σ-finite measure η on (X, τ, Ω) with Lebesgue decomposition $\eta = \eta_a + \eta_s$ with respect to μ, if $\mu(\operatorname{supp} \eta_s) = 0$, then* $\operatorname{supp} \eta_a = \overline{\operatorname{supp} \eta}^{\mu}$.

PROOF. It is straightforward to verify that $\operatorname{supp} \eta = \operatorname{supp} \eta_a \cup \operatorname{supp} \eta_s$. If $x \notin \operatorname{supp} \eta_a = \overline{\operatorname{supp} \eta_a}^{\mu}$ (since $\eta_a \ll \mu$, $\operatorname{supp} \eta_a$ is μ-essentially closed), then there exists $G \in \mathfrak{N}(x)$ such that $\mu(G \cap \operatorname{supp} \eta_a) = 0$. Thus

$$\mu(G \cap \operatorname{supp} \eta) \leq \mu(G \cap \operatorname{supp} \eta_a) + \mu(G \cap \operatorname{supp} \eta_s) = 0.$$

Hence $x \notin \overline{\operatorname{supp} \eta}^{\mu}$. Conversely, if $x \notin \overline{\operatorname{supp} \eta}^{\mu}$, then there exists $G \in \mathfrak{N}(x)$ such that $\mu(G \cap \operatorname{supp} \eta) = 0$. Therefore,

$$\mu(G \cap \operatorname{supp} \eta_a) \leq \mu(G \cap \operatorname{supp} \eta) = 0.$$

Hence $x \notin \overline{\operatorname{supp} \eta_a}^{\mu} = \operatorname{supp} \eta_a$. □

4.2.2 The essential supports of functions

In this section, we introduce the concept of the essential supports of functions, which is partly motivated by the study of the supports of Radon-Nikodym derivatives; see Chapter 23 in Fremlin's book [8]. We are particularly interested in the study of Radon-Nikodym derivatives via techniques from geometric measure theory.

For any pair of Radon measures (see Definition 1.5 and Corollary 1.11 in Mattila's book [12]) ν and μ on $\mathbb{R}^n$, equipped with a σ-algebra containing the Borel sets, such that $\nu \ll \mu$, it was shown in [12, Theorem 2.12] that the function

$$D_{\nu,\mu}(x) = \lim_{\epsilon\to 0^+} \frac{\nu(B(x,\epsilon))}{\mu(B(x,\epsilon))} \tag{3}$$

is defined μ-almost everywhere on $\mathbb{R}^n$ and coincides μ-almost everywhere with the Radon-Nikodym derivative of ν with respect to μ.

Similarly, for any locally finite measure ν defined on the Borel σ-algebra over $\mathbb{R}^n$ such that $\nu \ll \lambda_n$, it was shown in [2, Theorem 2.3.8] that the function D_{ν,λ_n} is defined Lebesgue almost everywhere on $\mathbb{R}^n$ and coincides Lebesgue almost everywhere with the Radon-Nikodym derivative of ν with respect to λ_n.

Definition 13. Let ν and μ be σ-finite measures on a metric measurable space (X, d, Ω). We say that ν is *differentiable* with respect to μ if $\nu \ll \mu$ and $D_{\nu,\mu}$ defined in (3) exists μ-almost everywhere and coincides μ-almost everywhere with the Radon-Nikodym derivative of ν with respect to μ.

Proposition 25. *Let ν and μ be σ-finite measures on a metric measurable space such that ν is differentiable with respect to μ. Then* $\operatorname{supp} D_{\nu,\mu} = \operatorname{supp} \nu$.

Proof. If $x \notin \operatorname{supp}\nu$, then there exists $\epsilon > 0$ such that $\nu(B(x,\epsilon)) = 0$. Hence $D_{\nu,\mu}(x) = 0$. So $\{x \colon D_{\nu,\mu}(x) \neq 0\} \subseteq \operatorname{supp}\nu$. Therefore, $\operatorname{supp} D_{\nu,\mu} \subseteq \operatorname{supp}\nu$. Conversely, if $x \notin \operatorname{supp} D_{\nu,\mu}$, then there is $G \in \mathfrak{N}(x)$ such that $D_{\nu,\mu} \equiv 0$ on G. Observe that $\nu(G) = \int_G D_{\nu,\mu}\, d\mu = 0$. Thus $x \notin \operatorname{supp}\nu$. Hence $\operatorname{supp}\nu \subseteq \operatorname{supp} D_{\nu,\mu}$. Therefore, $\operatorname{supp} D_{\nu,\mu} = \operatorname{supp}\nu$. □

Radon-Nikodym derivatives are unique up to a set of measure zero. As a result, the concept of topological supports fails to detect the essential parts of such functions. We demonstrate an extreme case in the following example.

Example 11. Consider the trivial measure $\nu \equiv 0$ on the Lebesgue σ-algebra $\mathfrak{L}(\mathbb{R})$, which is absolutely continuous with respect to the Lebesgue measure. Observe that both $f \equiv 0$ and $g = \chi_{\mathbb{Q}}$ are the Radon-Nikodym derivatives of ν with respect to λ_1. However, $\operatorname{supp} f = \emptyset$ while $\operatorname{supp} g = \mathbb{R}$.

In the above example, even though $\mathbb{Q}$ is negligible in the sense that it has Lebesgue measure zero, it is dense in $\mathbb{R}$.

Definition 14. Let f be an extended real-valued measurable function on a topological submeasure space (X, τ, Ω, μ). Define the *essential support* of f with respect to μ by $\operatorname{ess\,supp}_\mu f = \overline{\{x \in X : f(x) \neq 0\}}^{\mu}$.

Remark. In Ondreját's work [15], the essential support of a function f on a set $D \subseteq \mathbb{R}^n$ is defined to be the intersection of all closed subsets F in D such that $f = 0$ Lebesgue almost everywhere on the complement of F. It is straightforward to verify that this existing concept agrees with Definition 14.

Similarly to the concept of almost everywhere for measures, for the case of submeasures, we say that a property holds *almost everywhere* if the set of elements for which the property does not hold is a subset of a submeasure zero set.

Proposition 26. *Let f and g be extended real-valued measurable functions on a topological submeasure space (X, τ, Ω, μ). If f and g are equal μ-almost everywhere, then* $\operatorname{ess\,supp}_\mu f = \operatorname{ess\,supp}_\mu g$.

Proof. Since $f = g$ μ-almost everywhere, we have

$$\mu(\{x \in X : g(x) \neq 0\}) = \mu(\{x \in X : f(x) \neq 0\}),$$

which implies that the essential supports of f and g coincide. □

Theorem 27. *Assume that (X, τ, Ω, μ) is either a hereditarily Lindelöf submeasure space or an inner regular measure space. For each extended real-valued measurable function f, let $[f]_\mu$ denote the class of extended real-valued measurable functions on X which are equal to f μ-almost everywhere. Then there exists $f_0 \in [f]_\mu$ such that*

$$\operatorname{supp} f_0 = \operatorname{ess\,supp}_\mu f,$$

which is μ-essentially closed.

Proof. Define f_0 to be the function that coincides with f on $\operatorname{ess\,supp}_\mu f$ and vanishes elsewhere. Since the μ-closure is a strong essential closure,

$$\begin{aligned}\{x \in X : f(x) \neq f_0(x)\} &= \{x \in X : f(x) \neq 0\} \setminus \operatorname{ess\,supp}_\mu f \\ &= \{x \in X : f(x) \neq 0\} \setminus \overline{\{x \in X : f(x) \neq 0\}}^{\mu}\end{aligned}$$

is μ-non-essential. Hence $\mu(\{x \in X : f(x) \neq f_0(x)\}) = 0$ by Theorem 18. Thus f and f_0 are equal μ-almost everywhere (i.e., $f_0 \in [f]_\mu$). By Proposition 26, we have $\operatorname{ess\,supp}_\mu f = \operatorname{ess\,supp}_\mu f_0$.

If $f_0(x) \neq 0$, then $x \in \operatorname{ess\,supp}_\mu f$ by the construction. Therefore, we have that $\{x \in X\colon f_0(x) \neq 0\} \subseteq \operatorname{ess\,supp}_\mu f$. Hence

$$\operatorname{supp} f_0 \subseteq \operatorname{ess\,supp}_\mu f = \operatorname{ess\,supp}_\mu f_0 \subseteq \operatorname{supp} f_0.$$

Thus $\operatorname{supp} f_0 = \operatorname{ess\,supp}_\mu f_0 = \operatorname{ess\,supp}_\mu f$. As a consequence, $\operatorname{supp} f_0$ is μ-essentially closed. □

Proposition 28. *Let ν and μ be σ-finite measures on (X, τ, Ω) with $\nu \ll \mu$, and let $\dfrac{d\nu}{d\mu}$ denote the Radon-Nikodym derivative. Then* $\operatorname{ess\,supp}_\mu \dfrac{d\nu}{d\mu} = \operatorname{supp} \nu$.

Proof. Let f denote $\frac{d\nu}{d\mu}$. If $x \notin \operatorname{supp}\nu$, then there exists $G \in \mathfrak{N}(x)$ such that $\nu(G) = 0$. Thus $f = 0$ μ-almost everywhere on G. Therefore, we have that $\mu(G \cap \{x \in X\colon f(x) \neq 0\}) = 0$. Hence $x \notin \operatorname{ess\,supp}_\mu f$. Conversely, if $x \notin \operatorname{ess\,supp}_\mu f$, then $x \notin \overline{\{x \in X\colon f(x) \neq 0\}}^{\mu}$. Therefore, there exists $G \in \mathfrak{N}(x)$ such that $\mu(G \cap \{x \in X\colon f(x) \neq 0\}) = 0$. Thus $f = 0$ μ-almost everywhere on G. Hence $\nu(G) = 0$. So $x \notin \operatorname{supp}\nu$. □

Corollary 29. *Let ν and μ be σ-finite measures on a metric measurable space such that ν is differentiable with respect to μ. Then* $\operatorname{ess\,supp}_\mu D_{\nu,\mu} = \operatorname{supp} D_{\nu,\mu}$.

Proof. This follows directly from Propositions 25 and 28. □

Example 12. There exists an absolutely continuous measure $\nu \ll \mu$ with full support such that μ is not absolutely continuous with respect to ν. To see this, let μ be the 1-dimensional Lebesgue measure on $[0, 1]$ and let ν be a measure on $[0, 1]$ defined, for each Lebesgue measurable set $B \subseteq [0, 1]$, by $\nu(B) = \lambda_1(B \cap A^c)$, where A is a positive Lebesgue measure Cantor set on $[0, 1]$. Obviously, $\nu \ll \lambda_1$ by construction. Moreover, by Proposition 28,

$$\operatorname{supp}\nu = \operatorname{ess\,supp}_{\lambda_1} \chi_{A^c} = \overline{A^c}^{\lambda_1} = [0, 1].$$

Therefore, ν has full support. However, $\nu(A) = 0$ while $\lambda_1(A) > 0$.

Example 13. Let (X, τ, Ω, μ) be a hereditarily Lindelöf measure space and let f be an extended real-valued measurable function. We already know that

$$\int_X f \, d\mu = \int_{\operatorname{supp} f} f \, d\mu.$$

Let $\mathcal{G}$ be the collection of the open sets G such that $\mu(G \cap \{f \neq 0\}) = 0$. Since for each $x \notin \text{ess supp}_\mu f$, there is $G \in \mathfrak{N}(x)$ with $\mu(G \cap \{f \neq 0\}) = 0$, $\mathcal{G}$ covers $(\text{ess supp}_\mu f)^c$. Thus there is a countable subcover $\{G_1, G_2, \dots\}$ of $\mathcal{G}$.

By the countable additivity of measures, it is straightforward to show that $\mu((\text{ess supp}_\mu f)^c \cap \{f \neq 0\}) = 0$. Thus

$$\int_X f \, d\mu = \int_{\text{ess supp}_\mu f} f \, d\mu = \int_{\text{supp} f_0} f_0 \, d\mu,$$

where f_0 is a representative of the class $[f]_\mu$ in Theorem 27. Also note that

$$\text{supp} f_0 = \text{ess supp}_\mu f \subseteq \text{supp} f.$$

In this case, we see that f_0 is indeed a good representative of the class $[f]_\mu$.

4.2.3 Local Hausdorff dimension

In the sequel, let $\mathcal{H}^s$ denote the s-dimensional Hausdorff measure. More details on the Hausdorff measures and Hausdorff dimension $\dim_H$ can be found, for example, in Falconer's book [5] and Fremlin's book [8].

Definition 15. Let (X, d) be a metric space and τ_d denote the topology induced by the metric d. The *s-Hausdorff closure* is defined to be the submeasure closure on $(X, \tau_d, \mathcal{P}(X))$ induced by $\mathcal{H}^s$.

Lemma 30. *If a set A is s-Hausdorff essentially closed, then it has local Hausdorff dimension at least s.*

Proof. Suppose there exist $x \in A$ and $G \in \mathfrak{N}(x)$ such that $\dim_H(G \cap A) < s$, where $\dim_H$ denotes the Hausdorff dimension. See [5]. Then $\mathcal{H}^s(G \cap A) = 0$, contradicting the fact that $x \in A = \overline{A}^{\mathcal{H}^s}$. □

Theorem 31. *Let ν be an n-stochastic measure on $[0,1]^n$. Then $\text{supp}\,\nu$ is 1-Hausdorff essentially closed. In particular, by Lemma 30, $\text{supp}\,\nu$ has local Hausdorff dimension at least one.*

Proof. It suffices to show that $\text{supp}\,\nu \subseteq \overline{\text{supp}\,\nu}^{\mathcal{H}^1}$. If $x \notin \overline{\text{supp}\,\nu}^{\mathcal{H}^1}$, then there exists $G \in \mathfrak{N}(x)$, $\mathcal{H}^1(G \cap \text{supp}\,\nu) = 0$. Note that $\nu(G) = \nu(G \cap \text{supp}\,\nu)$. Suppose $\nu(G \cap \text{supp}\,\nu) > 0$. Then

$$\mathcal{H}^1(\pi_1(G \cap \text{supp}\,\nu)) = \lambda_1(\pi_1(G \cap \text{supp}\,\nu)) > 0,$$

where π_1 denotes the orthogonal projection onto the first variable. Thus $\mathcal{H}^1(G \cap \text{supp}\,\nu) > 0$, a contradiction. So $\nu(G) = \nu(G \cap \text{supp}\,\nu) = 0$, which implies $x \notin \text{supp}\,\nu$. Therefore, $\text{supp}\,\nu = \overline{\text{supp}\,\nu}^{\mathcal{H}^1}$. Hence $\text{supp}\,\nu$ is 1-Hausdorff essentially closed. □

It is well known that there is a one-to-one correspondence between the collection of n-stochastic measures and the collection of n-copulas. More information on n-copulas can be found in Nelsen's book [14].

Example 14. In [6, Theorem 1], Fredricks et al. show that for each $s \in (1,2)$, there is a copula with a fractal support of Hausdorff dimension s. Also, there are copulas with supports of Hausdorff dimensions 1 and 2, examples of which include the Fréchet-Hoeffding bounds and the independence copula, respectively. Moreover, Theorem 31 implies that the support of a copula is of Hausdorff dimension at least 1. Together with the result of Fredricks et al., it follows that the supports of copulas are of Hausdorff dimension at least 1 and for each possible value $s \in [1,2]$, there is a copula whose support is of Hausdorff dimension s.

5 Existing and related concepts

In this section, we discuss various concepts that are related to the concept of essential closures. Most of them are related to measures and submeasures as expected.

5.1 Lebesgue closure

Recall the definition of the essential closure on $\mathbb{R}$ introduced in [9] and called by us *Lebesgue closure*. It is easy to see that the Lebesgue closure coincides with the λ_1-closure defined in the previous section. Also recall from [9] the definition of the Lebesgue closure defined on S^1, the unit circle with center at the origin in $\mathbb{R}^2$.

According to [8, Theorem 265E], the pushforward Lebesgue measure on S^1 through the canonical map $(\theta \mapsto e^{i\theta})$ coincides with the Hausdorff measure $\mathcal{H}^1$ on S^1. As a result, the Lebesgue closure on S^1 coincides with the $\mathcal{H}^1$-closure on S^1.

5.2 Lebesgue density closures

To avoid confusion, the essential closures cl^* in Buczolich and Pfeffer's work [4] and in Fremlin's book [7], defined for each Lebesgue measurable set $A \subseteq \mathbb{R}^n$ by

$$\text{cl}^* A = \left\{ x \in \mathbb{R}^n : \limsup_{\epsilon \to 0^+} \frac{\lambda_n(B(x,\epsilon) \cap A)}{\lambda_n(B(x,\epsilon))} > 0 \right\},$$

will be called *Lebesgue density closures*. Note that, with respect to the standard topology on $\mathbb{R}^n$, the Lebesgue density closure fails to satisfy at least the first property of essential closures in Postulate 1.

For each λ_n-density closure cl^* on the Lebesgue σ-algebra $\mathfrak{L}(\mathbb{R}^n)$, we define the *modified* λ_n*-density closure* of $A \in \mathfrak{L}(\mathbb{R}^n)$ by $\mathcal{E}(A) = \overline{\mathrm{cl}^* A}$. As a consequence of taking the topological closure of $\mathrm{cl}^* A$, $\mathcal{E}$ is forced to satisfy the first property of essential closures. Surprisingly, not only that $\mathcal{E}$ is an essential closure, but it can also be shown that $\mathcal{E}$ coincides with the λ_n-closure defined on $\mathfrak{L}(\mathbb{R}^n)$.

Firstly, we show that the modified λ_n-density closure and the λ_n-closure coincide on the Borel σ-algebra $\mathfrak{B}(\mathbb{R}^n)$. Let $A \subseteq \mathbb{R}^n$ be Borel measurable. For each Borel measurable set $B \subseteq \mathbb{R}^n$, define $\lambda_A(B) = \lambda_n(B \cap A)$. It is clear that λ_A is σ-finite and $\lambda_A \ll \lambda_n$ on the Borel σ-algebra. According to Theorem 2.3.8 in Ash's book [2], λ_A is differentiable with respect to λ_n. As a result,

$$D_{\lambda_A,\lambda_n}(x) = \lim_{\epsilon\to 0^+} \frac{\lambda_A(B(x,\epsilon))}{\lambda_n(B(x,\epsilon))} = \limsup_{\epsilon\to 0^+} \frac{\lambda_n(B(x,\epsilon)\cap A)}{\lambda_n(B(x,\epsilon))}$$

defines the Radon-Nikodym derivative of λ_A with respect to λ_n. By Proposition 25 and Theorem 21, we have

$$\mathcal{E}(A) = \overline{\mathrm{cl}^* A} = \operatorname{supp} D_{\lambda_A,\lambda_n} = \operatorname{supp} \lambda_A = \overline{A}^{\lambda_A}.$$

Moreover, it is straightforward to verify that $\overline{A}^{\lambda_A} = \overline{A}^{\lambda_n}$. Hence $\mathcal{E}(A) = \overline{A}^{\lambda_n}$ for each Borel measurable set $A \subseteq \mathbb{R}^n$.

Finally, we extend the result to the Lebesgue σ-algebra $\mathfrak{L}(\mathbb{R}^n)$. Let $A \subseteq \mathbb{R}^n$ be Lebesgue measurable. There is a Borel measurable set $B \subseteq \mathbb{R}^n$ such that $A \subseteq B$ and $\lambda_n(B \setminus A) = 0$. According to Lemma 475C in Fremlin's book [7], cl^* is distributive over finite unions and $\mathrm{cl}^*(E) = \emptyset$ if $\lambda_n(E) = 0$. As a result,

$$\mathrm{cl}^*(B) = \mathrm{cl}^*(A) \cup \mathrm{cl}^*(B \setminus A) = \mathrm{cl}^*(A).$$

Similarly, $\overline{A}^{\lambda_n} = \overline{B}^{\lambda_n}$. Thus $\mathcal{E}(A) = \overline{\mathrm{cl}^*(A)} = \overline{\mathrm{cl}^*(B)} = \mathcal{E}(B) = \overline{B}^{\lambda_n} = \overline{A}^{\lambda_n}$ for each Lebesgue measurable set $A \subseteq \mathbb{R}^n$.

5.3 Lower density operators

The essential interiors int^* in Buczolich and Pfeffer's work [4] and in Fremlin's book [7] are lower density operators. In general, lower density operators are defined as follows.

Let Ω be a σ-algebra over a set X and $\mathcal{P} \subseteq \Omega$ be a σ-ideal. For $A, B \in \Omega$, we denote $A \sim B$ when the symmetric difference $A \triangle B$ is in the σ-ideal $\mathcal{P}$.

Definition 16 ([11, p. 207])**.** A *lower density operator* on $(X, \Omega, \mathcal{P})$ is a unary operation $\Phi\colon \Omega \to \Omega$ satisfying the following conditions for all $A, B \in \Omega$:

1. If $A \sim B$, then $\Phi(A) = \Phi(B)$;
2. $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$;
3. $\Phi(\emptyset) = \emptyset$ and $\Phi(X) = X$;
4. $A \sim \Phi(A)$.

For more details on lower density operators, see the classical book of Lukeš, Malý and Zajíček [11]. According to Lemma 475C in Fremlin's book [7],

$$\mathrm{cl}^*(A) = \mathrm{int}^*(A^c)^c, \tag{4}$$

for each measurable set A. Motivated by the above relation, we derive a result on the essential closure operators induced by lower density operators.

Theorem 32. *Let Φ and $\mathcal{E}$ be unary operations on a σ-algebra Ω over X satisfying*

$$\mathcal{E}(A) = \Phi(A^c)^c \quad \textit{for all } A \in \Omega. \tag{5}$$

Then Φ is a lower density operator on $(X, \Omega, \mathcal{P})$ if and only if

1. *$\mathcal{E}$ is a σ-non-essential essential closure operator on (X, Ω),*
2. *$A \sim \mathcal{E}(A)$ for all $A \in \Omega$,*
3. *$\mathcal{P} = \mathcal{N}_\Omega(\mathcal{E})$,*
4. *$\mathcal{E}(X) = X$.*

Proof. Assume that Φ is a lower density operator on $(X, \Omega, \mathcal{P})$. By (5), we have

- $\mathcal{E}(\emptyset) = \Phi(X)^c = \emptyset$,
- $\mathcal{E}(A \cup B) = \Phi(A^c \cap B^c)^c = \mathcal{E}(A) \cup \mathcal{E}(B)$ for all $A, B \in \Omega$, and
- $\mathcal{E}(\mathcal{E}(A)) = \Phi(\Phi(A^c))^c = \Phi(A^c)^c = \mathcal{E}(A)$ for all $A \in \Omega$.

Hence $\mathcal{E}$ is an essential closure operator.

For each $A \in \Omega$, $A \sim \Phi(A^c)^c = \mathcal{E}(A)$ because $A^c \sim \Phi(A^c)$. Consequently, if A_n is non-essential for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^\infty A_n \sim \bigcup_{n=1}^\infty \mathcal{E}(A_n) = \emptyset$. Thus $\mathcal{E}(\bigcup_{n=1}^\infty A_n) = \Phi(X)^c = \emptyset$. Hence $\mathcal{E}$ is σ-non-essential.

If $A \in \mathcal{P}$, then $A \sim \emptyset$. Therefore, $\mathcal{E}(A) = \emptyset$. Conversely, if $\mathcal{E}(A) = \emptyset$, then $A \sim \Phi(A^c)^c = \emptyset$, which implies that $A \in \mathcal{P}$. Hence $\mathcal{P} = \mathcal{N}_\Omega(\mathcal{E})$. Finally, $\mathcal{E}(X) = \Phi(\emptyset)^c = X$.

To prove the converse, assume that conditions 1, 2, 3 and 4 hold. Since $\mathcal{E}$ is σ-non-essential, $\mathcal{P}$ is a σ-ideal. Let $A, B \in \Omega$.

- Since $\mathcal{E}(A) = \Phi(A^c)^c$, $\Phi(A) = \mathcal{E}(A^c)^c$.
- If $A \sim B$, then $A \setminus B$ and $B \setminus A$ are non-essential since $\mathcal{P} = \mathcal{N}_\Omega(\mathcal{E})$. Thus $\mathcal{E}(A \setminus B) = \emptyset = \mathcal{E}(B \setminus A)$. Therefore,

$$\mathcal{E}(A^c) = \mathcal{E}(A^c \cap B^c) \cup \mathcal{E}(B \setminus A) = \mathcal{E}(A^c \cap B^c) \cup \mathcal{E}(A \setminus B) = \mathcal{E}(B^c).$$

  Hence $\Phi(A) = \Phi(B)$.
- That $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ follows directly from the assumption that $\mathcal{E}(A \cup B) = \mathcal{E}(A) \cup \mathcal{E}(B)$.
- Obviously, $\Phi(\emptyset) = \emptyset$ and $\Phi(X) = X$.
- Since $A^c \sim \mathcal{E}(A^c)$, $\Phi(A) = \mathcal{E}(A^c)^c \sim A$.

Hence Φ is a lower density operator on $(X, \Omega, \mathcal{P})$. □

Corollary 33. *Let Φ be a lower density operator on $(X, \Omega, \mathcal{P})$. Define $\mathcal{E}$ by equation* (5)*. Then $\mathcal{E}$ is an essential closure operator on (X, Ω). Moreover, the induced topology τ_Φ is a compatible topology for $\mathcal{E}$.*

Proof. According to Theorem 32, $\mathcal{E}$ is an essential closure operator on (X, Ω). Recall from [11, Proposition 6.37] that $\mathcal{B}_\Phi = \{A \in \Omega \colon A \subseteq \Phi(A)\}$ is an open base for τ_Φ. To show that τ_Φ is a compatible topology for $\mathcal{E}$, it suffices to show that, with respect to τ_Φ, i) $\mathcal{E}(A)$ is closed and ii) $\mathcal{E}(A) \subseteq \mathrm{cl}(A)$ for all $A \in \Omega$.

Observe that $\Phi(A)$ is open in τ_Φ since $\Phi(A) \in \mathcal{B}_\Phi$ for all $A \in \Omega$. Thus $\mathcal{E}(A)$ is closed in τ_Φ. Moreover, for each $A \in \Omega$,

$$\begin{aligned}
\mathrm{int}(A) &= \bigcup\{O \in \tau_\Phi \colon O \subseteq A\} \\
&= \bigcup\{G \in \Omega \colon G \subseteq \Phi(G) \text{ and } G \subseteq A\} \\
&\subseteq \bigcup\{G \in \Omega \colon G \subseteq \Phi(G) \subseteq \Phi(A)\} \\
&\subseteq \bigcup\{G \in \Omega \colon G \subseteq \Phi(A)\} \\
&\subseteq \Phi(A).
\end{aligned}$$

Consequently, $\mathcal{E}(A) \subseteq \mathrm{cl}(A)$ for each $A \in \Omega$. Therefore, τ_Φ is a compatible topology for $\mathcal{E}$. □

The following corollary is an immediate application. Recall that each cl^* is not an essential closure with respect to the standard topology. However, with a suitable topology, it turns into one.

Corollary 34. *Each Lebesgue density closure* cl^* *is an essential closure on* $(\mathbb{R}^n, \tau_{\mathrm{int}^*}, \mathfrak{L}(\mathbb{R}^n))$.

PROOF. This follows from Corollary 33 and equation (4). □

5.4 Stochastic closures

We will call essential closures defined in [16] *stochastic closures* to avoid confusion. It has been verified that these stochastic closures are indeed essential closures. The next question is whether these essential closures are strong and σ-non-essential. And if they are, what are their corresponding submeasures?

For each integer $1 \le d \le n$, define $\mathcal{S}_d\colon \mathcal{P}([0,1]^n) \to [0,\infty]$ as follows:

$$\mathcal{S}_d(A) = \sum_W \lambda_d^*(\pi_W(A))$$

where the sum is taken over all d-dimensional standard subspaces (i.e., subspaces spanned by a collection of standard basis elements) W of $\mathbb{R}^n$. It is easy to verify that $\mathcal{S}_d$ is an outer measure, hence a submeasure, on $[0,1]^n$. Moreover, it is easy to see that for each $d \in \mathbb{N}$, the d-stochastic closure coincides with the $\mathcal{S}_d$-closure, hence strong and σ-non-essential.

5.5 Prevalence

The concept of prevalent sets is a measure-theoretic approach to defining what it means for a statement to hold "almost everywhere" in a possibly infinite-dimensional complete metric vector space. It was observed in [10] that the concept of prevalent sets extends the concept of Lebesgue almost everywhere in finite-dimensional Euclidean spaces. It is well known that there is no non-trivial translation-invariant measure in infinite-dimensional spaces. So we ask whether there is something weaker, for example, a non-trivial translation-invariant submeasure whose submeasure zero sets are exactly the shy sets (i.e., the complements of the prevalent sets). Via the theory of essential closures, such a submeasure can be constructed. Let us recall some basic properties of shy sets. Let $A, A_1, A_2, \ldots$ be shy sets and v be a vector. Then the following hold:

1. $A + v$ is shy;
2. $B \subseteq A$ implies B is shy;

3. $\bigcup_{n=1}^{\infty} A_n$ is shy.

Observe that, with a suitable underlying σ-algebra, the collection of shy sets satisfies the properties in Definition 7. In the sequel, let V be a hereditarily Lindelöf complete metric vector space.

Theorem 35. *There exists a finite non-trivial translation-invariant submeasure on V whose submeasure zero sets are exactly the shy sets.*

Proof. The σ-algebra generated by the open subsets and the shy subsets of V will be called the *prevalence σ-algebra* and denoted by $\mathfrak{L}(V)$. According to Hunt et al. [10], the collection of shy sets on V satisfies the properties in Definition 7 with respect to $\mathfrak{L}(V)$. By Theorem 8, there exists a unique σ-non-essential strong essential closure whose collection of non-essential sets is exactly the collection of shy sets. We call the induced essential closure the *prevalence closure.*

By Theorem 20, the prevalence closure induces a submeasure on $\mathfrak{L}(V)$. Note that an induced submeasure is not unique. We call such a submeasure a *prevalence submeasure.* Moreover, by Theorem 18, the collection of non-essential sets, which is the collection of shy sets, is exactly the collection of prevalence submeasure zero sets. In addition, it is worth mentioning that the space V is essentially closed with respect to the prevalence closure. This is due to the fact that non-empty open sets are not shy, hence are of positive prevalence submeasure.

To conclude, we have a prevalence submeasure on $\mathfrak{L}(V)$ whose prevalence submeasure zero sets are exactly the shy sets on V. Moreover, it is straightforward to verify that the prevalence closure commutes with the translations. However, a prevalence submeasure is generally not translation-invariant. Nevertheless, there is a special prevalence submeasure that is translation-invariant.

In the proof of Theorem 20, the normalized submeasure obtained from the prevalence closure will be called the *normalized prevalence submeasure* and denoted by μ_p. For each vector $v \in V$, observe that $\mu_p(A+v) = 0$ if and only if $A + v$ is shy, which in turn is valid if and only if A is shy. Equivalently, $\mu_p(A) = 0$. Since μ_p assumes the value of either 0 or 1, the normalized prevalence submeasure μ_p is translation-invariant. □

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References

[1] E. de Amo, M. Díaz Carrillo and J. Fernández-Sánchez, *Measure-preserving functions and the independence copula*, Mediterr. J. Math., **8(3)** (2011), 431–450.

[2] R. B. Ash, *Probability and Measure Theory*, Second ed. With contributions by Catherine Doléans-Dade, Harcourt/Acad. Press, Burlington, MA, 2000.

[3] H. Bauer, *Measure and Integration Theory*, Walter de Gruyter, Berlin, 2001.

[4] Z. Buczolich and W. F. Pfeffer, *Variations of additive functions*, Czech. Math. J., **47(3)** (1997), 525–555.

[5] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley & Sons Ltd., West Sussex, 1990.

[6] G. A. Fredricks, R. B. Nelsen and J. A. Rodríguez-Lallena, *Copulas with fractal supports*, Insurance. Math. Econom., **37(1)** (2005), 42–48.

[7] D. H. Fremlin, *Measure Theory: Topological Measure Spaces*, Vol. 4, Torres Fremlin, Colchester, 2006.

[8] D. H. Fremlin, *Measure Theory: Broad Foundations*, Vol. 2, Second ed., Torres Fremlin, Colchester, 2010.

[9] F. Gesztesy, K. A. Makarov and M. Zinchenko, *Essential closures and AC spectra for reflectionless CMV, Jacobi, and Schrödinger operators revisited*, Acta Appl. Math., **103(3)** (2008), 315–339.

[10] B. R. Hunt, T. Sauer and J. A. Yorke, *Prevalence: A translation-invariant "almost every" on infinite-dimensional spaces*, Bull. Amer. Math. Soc., **27(2)** (1992), 217–238.

[11] J. Lukeš, J. Malý and L. Zajíček, *Fine topology methods in real analysis and potential theory*, Lecture Notes in Math., Vol. 1189, Springer-Verlag, Berlin, 1986.

[12] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge Univ. Press, Cambridge, 1995.

[13] J. R. Munkres, *Topology*, Second ed., Prentice Hall Inc., Upper Saddle River, NJ, 2000.

[14] R. B. Nelsen, *An Introduction to Copulas*, Second ed., Lecture Notes in Stat., Vol. 139, Springer, New York, 2006.

[15] M. Ondreját, *Existence of global martingale solutions to stochastic hyperbolic equations driven by a spatially homogeneous Wiener process*, Stoch. Dyn., **6(1)** (2006), 23–52.

[16] P. Ruankong and S. Sumetkijakan, *Essential closures and supports of multivariate copulas*, Int. J. Approx. Reason., **54(6)** (2013), 762–768.

[17] W. Trutschnig, *On a strong metric on the space of copulas and its induced dependence measure*, J. Math. Anal. Appl., **384(2)** (2011), 690–705.

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Non-atomic bivariate copulas and implicitly dependent random variables

Songkiat Sumetkijakan

Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Phayathai Road, Bangkok, Thailand

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ABSTRACT

Two (continuous) random variables X and Y are *implicitly dependent* if there exist Borel functions α and β such that $\alpha \circ X = \beta \circ Y$ almost surely. The copulas of such random variables are exactly the copulas that are factorizable as the $*$-product of a left invertible copula and a right invertible copula. Consequently, every implicit dependence copula assigns full mass to the graph of $f(x) = g(y)$ for some measure-preserving functions f and g but the converse is not true in general.

We obtain characterizations of a copula C assigning full mass to the graph of $f(x) = g(y)$ in terms of a partial factorizability of its Markov operator T_C and in terms of the non-atomicity of two newly defined associated σ-algebras σ_C and σ_C^*, in which case C is called *non-atomic*. As an application, we give a broad sufficient condition under which a copula with fractal support has an implicit dependence support. Under certain extra conditions, we explicitly compute the left invertible and right invertible factors of the copula with fractal support.

1. Introduction

It is well-known that the bivariate copula of two continuous random variables completely captures their dependence structure. Notable examples are the *independence copula* $\Pi(u, v) = uv$, which corresponds to independent random variables, and the copulas of completely dependent random variables, called *complete dependence copulas*. Since it was discovered (Mikusiński et al., 1992, 1991; Kimeldorf and Sampson, 1978) that there are complete dependence copulas arbitrarily closed to Π in the uniform norm, many norms have been introduced and investigated in the literature (Darsow and Olsen, 1995) giving rise to measures of dependence such as ω in Siburg and Stoimenov (2009) and ζ_1 in Trutschnig (2011). These dependence measures defined in terms of the copula's first partial derivatives attain the maximum value 1 at least for complete dependence copulas and the minimum value 0 when and only when the copula is Π. However, with respect to these dependence measures, the independence copula can still be approximated by *implicit dependence copulas* (Chaidee et al., 2016), defined as copulas of random variables X and Y which are implicitly dependent in the sense that $\alpha \circ X = \beta \circ Y$ a.s. for some Borel measurable functions α and β. For some Rényi-type measures of dependence (Rényi, 1959) such as ω_* in Ruankong et al. (2013) and ν_* in Kamnitui et al. (2015), with respect to which all complete dependence copulas have measure 1, all implicit dependence copulas also attain the maximum measure (Ruankong et al., 2013, Corollary 4.12). It is then evident that implicit dependence copulas play a crucial role in understanding as well as comparing and contrasting measures of MCD and Rényi-type dependence measures. Every implicit dependence copulas assigns its full mass to the graph

E-mail address: songkiat.s@chula.ac.th.

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of $f(x) = g(y)$, called an implicit graph, for some measure-preserving functions f and g. Closely related and constituting a much larger class than the implicit dependence copulas are the copulas whose mass is concentrated on an implicit graph.

Motivated by the concept of invariant sets in Darsow and Olsen's study of idempotent copulas (Darsow and Olsen, 2010), we shall investigate copulas C assigning full mass to implicit graphs via their corresponding Markov operators T_C and associated σ-algebras σ_C and σ_C^*. For the copula C of $\mathscr{U}[0,1]$-random variables X and Y, S belongs to σ_C and R belongs to σ_C^* if and only if $Y \in S$ with probability one given that $X \in R$. We derive some fundamental properties of these associated σ-algebras. Intuitively, the larger the σ_C is, the stronger the dependence of Y on X is. We then obtain characterizations of a copula C assigning its full mass to an implicit graph in terms of the factorizability of the corresponding Markov operator T_C on a subclass of the Borel functions and in terms of the size (non-atomicity) of the associated σ-algebras σ_C and σ_C^*. Naturally, these characterizations could be useful in investigating singular copulas. Our main results find an application in copulas with fractal supports introduced by Fredricks, Nelsen and Rodríguez-Lallena (Fredricks et al., 2005). Given a transformation matrix A, there is a unique copula C_A such that $[A](C_A) = C_A$, where $[A]$ maps the class of bivariate copulas into itself according to the weights given by the entries in A. As a consequence, we obtain a broad sufficient condition on a transformation matrix A under which the copula C_A is non-atomic and hence assigns its full mass to an implicit graph. Working directly with the transformation matrix A, a sufficient condition under which C_A is an implicit dependence copula is also given. Our ongoing research is to find a characterization of general implicit dependence copulas via behaviors of their σ-algebras. Such a characterization would be beneficial in the study of products of implicit dependence copulas.

The manuscript is organized as follows. Section 2 lays the necessary background on copulas and Markov operators for the rest of the paper. We then define the associated σ-algebras of a copula and prove their basic properties in Section 3. Section 4 gives a definition of non-atomic copulas and some of their fundamental properties summarizing in characterizations of non-atomic copulas. In the final section, the characterizations are used in an investigation of copulas with fractal support. We also give a sufficient condition on a transformation matrix under which the induced invariant copula can be written as the product of a left invertible copula and a right invertible copula.

2. Background on copulas and Markov operators

Let λ denote the Lebesgue measure on $\mathbb{R}$, $\mathbf{I} \equiv [0,1]$ and $\mathscr{B} \equiv \mathscr{B}(\mathbf{I})$ the Borel σ-algebra on $\mathbf{I}$. Since we always consider λ-integrable functions on $\mathbf{I}$ that are measurable with respect to various sub-σ-algebras $\mathscr{M}$ of $\mathscr{B}$, we will denote by $L^1(\mathscr{M})$ the class of λ-integrable $\mathscr{M}$-measurable functions on $\mathbf{I}$. $\mathscr{B}$-measurable functions are called *Borel functions*. For $A \in \mathscr{B}$, $\mathbf{1}_A$ denotes the indicator function of A and $\mathbf{1} \equiv \mathbf{1}_{\mathbf{I}}$.

A (bivariate) *copula* C is a function from $\mathbf{I}^2$ to $\mathbf{I}$ which is the joint distribution function of two random variables uniformly distributed on $[0,1]$. For random variables X and Y with joint distribution $F_{X,Y}$ and continuous marginal distributions F_X and F_Y, there exists, by the Sklar's theorem, a unique copula C, called the *copula of X and Y*, for which $F_{X,Y}(x,y) = C(F_X(x), G_Y(y))$ for all x, y. The *independence copula* is the product copula $\Pi(u,v) = uv$. *Complete dependence copulas* are either the copulas $C_{ef} \equiv C_{e,f}$ or $C_{fe} \equiv C_{f,e}$ where $e(x) = x$ and f is a measure-preserving function on $\mathbf{I}$ in the sense that $\lambda(f^{-1}(B)) = \lambda(B)$ for every $B \in \mathscr{B}$. Here, $C_{f,g}(u,v) = \lambda(f^{-1}([0,u]) \cap g^{-1}([0,v]))$ for $u, v \in \mathbf{I}$. The *comonotonic* and *countermonotonic* copulas are $M = C_{e,e}$ and $W = C_{e,\mathbf{1}-e}$, respectively.

Definition 1. Two random variables X and Y are said to be *implicitly dependent* if there exist Borel functions α and β such that $\alpha \circ X = \beta \circ Y$ almost surely. The copula of two implicitly dependent continuous random variables is called an *implicit dependence copula*.

It is evident that all implicit dependence copulas are of the form $C_{ef} * C_{ge}$ for some measure-preserving functions f and g on $\mathbf{I}$.

Each copula C induces a *doubly stochastic measure* μ_C by $\mu_C((a,b] \times (c,d]) = C(b,d) - C(b,c) - C(a,d) + C(a,c)$. The *support* of C is then defined as the support of the induced measure μ_C, i.e. the complement of the union of all open sets having zero μ_C-measure. One can construct a new copula by taking any convex combinations of two or more copulas. Any two copulas C, D also give rise to a new copula via the $*$-*product*: $(C * D)(u,v) = \int_0^1 \partial_2 C(u,t)\partial_1 D(t,v)\,dt$. The binary operation $*$ makes the class of copulas a monoid with null element Π and identity M. If $C * D = M$ then C is a left inverse of D and D is a right inverse of C. The left invertible copulas are exactly the complete dependence copulas C_{ef}, while the right invertible copulas are exactly the complete dependence copulas C_{fe}. See Nelsen (2006) and Durante and Sempi (2015) for comprehensive introductions to many aspects of copulas.

A linear operator T on $L^1(\mathscr{B})$ is called a *Markov operator* if

i. $T\mathbf{1} = \mathbf{1}$,
ii. $\int_0^1 T\psi\,d\lambda = \int_0^1 \psi\,d\lambda$ for every $\psi \in L^1$, which is equivalent to $T^*\mathbf{1} = \mathbf{1}$, and
iii. $T\psi \geq 0$ for every $\psi \geq 0$, which means T is positive.

So a Markov operator must be a bounded linear operator on L^1 (and L^∞). From Olsen et al. (1996), for each copula C, there corresponds a Markov operator T_C defined by

$$(T_C\psi)(x) = \frac{d}{dx}\int_0^1 \partial_2 C(x,t)\psi(t)\,dt.$$

In fact, the mapping $\Phi : C \mapsto T_C$ is an isomorphism from the set of copulas endowed with the $*$-product onto the set of Markov operators under the composition. In particular, $T_{C*D} = T_C \circ T_D$. Define also $(T_C^*\varphi)(y) = \frac{d}{dy}\int_0^1 \partial_1 C(t, y)\varphi(t)\,dt$. The copula C and the Markov operators T_C and T_C^* are also related by the identities

$$C(x, y) = \int_0^x T_C \mathbf{1}_{[0,y]}(t)\,dt \quad \text{and} \quad C(x, y) = \int_0^y T_C^* \mathbf{1}_{[0,x]}(t)\,dt. \tag{1}$$

In fact, it is a good exercise in functional analysis to verify that the Markov operator T_C^* coincides with the extension of the adjoint of T_C to a Markov operator on L^1, i.e. $T_C^* = (T_C)^*$.

Let us quote a very useful result from Darsow and Olsen (2010, Theorem 2.11) where, for brevity, we denote $T_{C_{fg}} = T_{fg}$.

Theorem 2.1. *Let f be a measure-preserving Borel function and $\psi \in L^1(\mathscr{B})$. Then $[T_{ef}\psi](x) = \psi \circ f(x)$ and $[T_{fe}(\psi \circ f)](x) = \psi(x)$ for almost every $x \in [0, 1]$.*

3. Associated σ-algebras

Unless stated otherwise, all equalities of two functions hold λ-almost everywhere and all equalities of two sets mean that their symmetric difference has Lebesgue measure zero. The integral on the whole unit interval $\mathbf{I}$ is denoted simply by $\int$.

Let C be the copula of random variables X and Y which are uniformly distributed on $[0, 1]$. Then for any Borel sets $R, S \subseteq [0, 1]$, we have

$$T_C\mathbf{1}_S(x) = P(Y \in S|X = x) \quad \text{and} \quad T_C^*\mathbf{1}_R(y) = P(X \in R|Y = y).$$

See Darsow et al. (1992). We also have $T_C^* = T_{C^t}$ and $\left(T_C^*\right)^* = T_C$. Roughly speaking, if $T_C\mathbf{1}_S = \mathbf{1}_R$, then it happens with probability one that if $X \in R$ then $Y \in S$. For each copula C or Markov operator $T = T_C$, let us define

$$\sigma_C = \sigma_T \equiv \{S \in \mathscr{B} : T\mathbf{1}_S = \mathbf{1}_R \text{ for some } R\} \text{ and}$$
$$\sigma_C^* = \sigma_T^* \equiv \{R \in \mathscr{B} : T\mathbf{1}_S = \mathbf{1}_R \text{ for some } S\}.$$

Example 1. Let us explicitly compute the associated σ-algebras σ_C and σ_C^* for $C = \Pi$, $C =$ some complete dependence copulas and $C = \frac{M+W}{2}$.

1. $T_\Pi\mathbf{1}_S = \mathbf{1}_R$ is equivalent to $\mathbf{1}_R(x) = \frac{d}{dx}\int_S \partial_2\Pi(x, t)\,dt = \lambda(S)$ for a.e. $x \in [0, 1]$. So $\lambda(S) = 0$ or 1 and hence $\lambda(R) = \lambda(S) = 0$ or 1. Thus, $\sigma_\Pi = \sigma_\Pi^* = \{S \in \mathscr{B} : \lambda(S) = 0 \text{ or } 1\}$.
2. With essentially the same arguments as that in Darsow and Olsen (2010), σ_M, σ_M^*, σ_W and σ_W^* are the Borel sets.
3. For $C = \frac{M+W}{2}$, $T_C\mathbf{1}_S(x) = \frac{1}{2}\mathbf{1}_S(x) + \frac{1}{2}\mathbf{1}_S(1 - x)$ a.e. x. If S is symmetric with respect to $\frac{1}{2}$, i.e. $x \in S$ if and only if $1 - x \in S$, then $T_C\mathbf{1}_S = \mathbf{1}_S$. Conversely, $T_C\mathbf{1}_S(x) = \mathbf{1}_R(x)$ implies that $x \in S$ if and only if $1 - x \in S$. Moreover, for such a symmetric set S, $(\mathbf{1}_R - \mathbf{1}_S)(x) = (T_C\mathbf{1}_S - \mathbf{1}_S)(x) = \frac{1}{2}(\mathbf{1}_S(1 - x) - \mathbf{1}_S(x)) = 0$ a.e. x. That is, $R = S$. Hence, $\sigma_{\frac{M+W}{2}} = \sigma^*_{\frac{M+W}{2}} = \{S \subseteq [0, 1] : x \in S \Leftrightarrow 1 - x \in S\}$.
4. For $0 < \alpha < 1$, let L_α denote the complete dependence copula whose support consists of the line segments $y = \frac{x}{\alpha}$, $0 \le x \le \alpha$, and $y = \frac{x-\alpha}{1-\alpha}$, $\alpha < x \le 1$. Then a direct computation yields $T_{L_\alpha}\mathbf{1}_S = \mathbf{1}_{(\alpha S)\cup(\alpha+(1-\alpha)S)}$ and hence $\sigma_{L_\alpha} = \mathscr{B}$ and $\sigma_{L_\alpha}^* = \{\alpha S \cup (\alpha + (1 - \alpha)S) : S \in \mathscr{B}\}$.

Listed below are basic properties of sets S and R linked by T.

Lemma 3.1. *Let C be a copula with associated Markov operator T_C and doubly stochastic measure μ_C and $R, S \subseteq \mathscr{B}$. $T_C\mathbf{1}_S = \mathbf{1}_R$ if and only if $\mu_C(R \times S) = \lambda(R) = \lambda(S)$.*

Proof. Note the fact that $\mu_C(R \times S) = \int_R T_C\mathbf{1}_S\,d\lambda$. If $T_C\mathbf{1}_S = \mathbf{1}_R$ then $\mu_C(R \times S) = \lambda(R)$ and, by the property of T_C, $\lambda(R) = \int \mathbf{1}_R\,d\lambda = \int T_C\mathbf{1}_S\,d\lambda = \int \mathbf{1}_S\,d\lambda = \lambda(S)$. Conversely, if $\mu_C(R \times S) = \lambda(R) = \lambda(S)$ then $\int_R T_C\mathbf{1}_S\,d\lambda = \mu_C(R \times S) = \lambda(S) = \int \mathbf{1}_S\,d\lambda = \int T_C\mathbf{1}_S\,d\lambda$ and $\int_R T_C\mathbf{1}_S\,d\lambda = \mu_C(R \times S) = \lambda(R) = \int_R \mathbf{1}\,d\lambda$. Since $T_C\mathbf{1}_S(t) \in [0, 1]$, $T_C\mathbf{1}_S(t) = 0$ for every $t \notin R$ and $T_C\mathbf{1}_S(t) = 1$ for every $t \in R$, i.e. $T_C\mathbf{1}_S = \mathbf{1}_R$. □

Proposition 3.2. *Let T be a Markov operator and $S_1, S_2, R_1, R_2 \in \mathscr{B}$. Then*

1. *if $T\mathbf{1}_{S_1} = \mathbf{1}_{R_1}$ then $T^*\mathbf{1}_{R_1} = \mathbf{1}_{S_1}$;*
2. *if $T\mathbf{1}_{S_1} = \mathbf{1}_{R_1}$ then $T\mathbf{1}_{S_1^c} = \mathbf{1}_{R_1^c}$;*
3. *if $T\mathbf{1}_{S_i} = \mathbf{1}_{R_i}$ for $i = 1, 2$, then $T\mathbf{1}_{S_1\cap S_2} = \mathbf{1}_{R_1\cap R_2}$;*
4. *the classes σ_T and σ_T^* are σ-algebras; and*
5. *$\sigma_T^* = \sigma_{T^*}$, that is $\sigma_C^* = \sigma_{C^t}$.*

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Proof. 1. By the definition of $T^*\mathbf{1}_R$ applied to $\mathbf{1}_S$ with the canonical identification between $(L^1)^*$ and L^∞, $\int_S T^*\mathbf{1}_R\,d\lambda = \int \mathbf{1}_S T^*\mathbf{1}_R\,d\lambda = \int \mathbf{1}_R\mathbf{1}_R\,d\lambda = \lambda(R) = \int_S \mathbf{1}_S\,d\lambda$. Since $0 \le T^*\mathbf{1}_R \le 1$ (a.e.), $T^*\mathbf{1}_R = \mathbf{1}_S$ as desired.
2. If $T\mathbf{1}_S = \mathbf{1}_R$ then it follows from $T\mathbf{1}_S + T\mathbf{1}_{S^c} = T(\mathbf{1}_S + \mathbf{1}_{S^c}) = T\mathbf{1} = \mathbf{1} = \mathbf{1}_R + \mathbf{1}_{R^c}$ that $T\mathbf{1}_{S^c} = \mathbf{1}_{R^c}$.
3. Suppose $T\mathbf{1}_{S_i} = \mathbf{1}_{R_i}$ for $i = 1, 2$ and let $S = S_1 \cap S_2$, $R = R_1 \cap R_2$. Since $S \subseteq S_i$, $\mathbf{1}_{S_i} - \mathbf{1}_S \ge 0$ and so $\mathbf{1}_{R_i} - T\mathbf{1}_S = T(\mathbf{1}_{S_i} - \mathbf{1}_S) \ge 0$. Therefore $T\mathbf{1}_S \le \min(\mathbf{1}_{R_1}, \mathbf{1}_{R_2}) = \mathbf{1}_{R_1\cap R_2} = \mathbf{1}_R$. But $\int (\mathbf{1}_R - T\mathbf{1}_S)\,d\lambda = \int \mathbf{1}_R\,d\lambda - \int \mathbf{1}_S\,d\lambda = \lambda(R) - \lambda(S) = 0$, where the last equality follows from considering $T^*\mathbf{1}_{R_2}(\mathbf{1}_{S_1}) = \mathbf{1}_{R_2}(T\mathbf{1}_{S_1})$. Hence, $T\mathbf{1}_S = \mathbf{1}_R$.
4. Suppose $T\mathbf{1}_{S_i} = \mathbf{1}_{R_i}$ for every $i \in \mathbb{N}$. By 3, if $S_1, S_2, \ldots$ are mutually disjoint then so is the sequence $R_1, R_2, \ldots$ because $\sum_{i=1}^\infty \mathbf{1}_{R_i} = \sum_{i=1}^\infty T\mathbf{1}_{S_i} = T\left(\sum_{i=1}^\infty \mathbf{1}_{S_i}\right) = T\left(\mathbf{1}_{\dot\cup_i S_i}\right) \le \mathbf{1}$. Thus, $T\left(\mathbf{1}_{\dot\cup_i S_i}\right) = \mathbf{1}_{\dot\cup_i R_i}$. Generally, we write $\bigcup_i S_i$ as the disjoint union $\cup_i \tilde{S}_i$, where $\tilde{S}_i = S_i \setminus \cup_{j<i}\tilde{S}_j$. Letting $T\mathbf{1}_{\tilde{S}_i} = \mathbf{1}_{\tilde{R}_i}$, it follows from 2, 3 and the disjoint union case above that $T\mathbf{1}_{S_i\setminus\cup_{j<i}\tilde{S}_j} = \mathbf{1}_{R_i\setminus\cup_{j<i}\tilde{R}_j}$. Hence, $\tilde{R}_i = R_i \setminus \cup_{j<i}\tilde{R}_j$ by induction. Consequently, $T\left(\mathbf{1}_{\cup_i S_i}\right) = \mathbf{1}_{\cup_i R_i}$.
5. This clearly follows from 1 and the fact that $T^{**} = T$. □

Remark. By Theorem 2.1, $T_L^* \circ T_L = id$, the identity map, on $L^1(\mathscr{B})$ if the copula L is left invertible. However, it follows from the above proposition that for *every copula* C, $T_C^* \circ T_C = id$ on $\{\mathbf{1}_S : S \in \sigma_C\}$, or equivalently on the class of all integrable σ_C-measurable functions.

Theorem 3.3. *Let T be a Markov operator with associated σ-algebras σ_T and σ_T^*. If ψ is σ_T-measurable then $T\psi$ is σ_T^*-measurable.*

Proof. By the linearity of T and the definition of σ_T and σ_T^*, if ψ is a simple σ_T-measurable function then $T\psi$ is simple and σ_T^*-measurable. The case when ψ is σ_T-measurable follows from the continuity of T. □

4. Non-atomic copulas

Definition 2. Let $\mathscr{S}$ be a sub-σ-algebra of $\mathscr{B}$. A set S in $\mathscr{S}$ is called an *atom* in $\mathscr{S}$ if $\lambda(S) > 0$ and for every $E \in \mathscr{S}$, either $\lambda(S \cap E) = \lambda(S)$ or $\lambda(S \cap E) = 0$. The σ-algebra $\mathscr{S}$ is said to be *non-atomic* if there are no atoms in $\mathscr{S}$; otherwise, it is called *atomic*. $\mathscr{S}$ is *totally atomic* if there is a (countable) collection of essentially disjoint atoms $E_1, E_2, \ldots$ in $\mathscr{S}$ such that $\sum_i \lambda(E_i) = 1$. We say that a bivariate copula C is *non-atomic* if both σ-algebras σ_C and σ_C^* are non-atomic. And C is called *totally atomic* if both σ_C and σ_C^* are totally atomic.

In fact, one can verify the non-atomicity of a copula via only one of its two associated σ-algebras.

Proposition 4.1. *For every copula C, σ_C is non-atomic if and only if σ_C^* is non-atomic.*

Proof. By Proposition 3.2(5), it suffices to prove only that if σ_C is non-atomic then so is σ_C^*. Let $R \in \sigma_C^*$ with $\lambda(R) > 0$. Then $T_C\mathbf{1}_S = \mathbf{1}_R$ for some $S \in \sigma_C$ and $\lambda(S) = \lambda(R) > 0$. By the non-atomicity of σ_C, there exists $S' \in \sigma_C$ for which $S' \subset S$ and $0 < \lambda(S') < \lambda(S)$. Then $T_C\mathbf{1}_{S'} = \mathbf{1}_{R'}$ for some $R' \in \sigma_C^*$ and $\lambda(R') = \lambda(S') \in (0, \lambda(R))$. It is only left to show that $R' \subset R$ in the sense that $\mathbf{1}_{R'} \le \mathbf{1}_R$ a.e. But this clearly follows from the positivity of T_C. □

Note that the non-atomicity is a generalization of the notion of the same name in Darsow and Olsen (2010), which is defined only for idempotent copulas C via their invariant sets defined as Borel sets S for which $T_C\mathbf{1}_S = \mathbf{1}_S$. However, the two notions agree for idempotent copulas.

Proposition 4.2. *If a copula C is non-atomic and idempotent then $\sigma_C = \sigma_C^*$ is the σ-algebra of invariant sets.*

Proof. Since an idempotent copula is symmetric, $T_C^* = T_{C^t} = T_C$. Consequently, $T_C^* \circ T_C = T_C \circ T_C = T_{C*C} = T_C$. Now, if $T_C\mathbf{1}_S = \mathbf{1}_R$ then $T_C^*\mathbf{1}_R = \mathbf{1}_S$ and so $T_C^* \circ T_C\mathbf{1}_S = \mathbf{1}_S$. Hence S is an invariant set of C and so is R as $\mathbf{1}_R = T_C\mathbf{1}_S = \mathbf{1}_S$. That is, σ_C and σ_C^* are subsets of the class of invariant sets. The converse inclusions are clear. □

Example 2. It is evident from the computations of σ_C and σ_C^* in Example 1 that Π is totally atomic but M, W, $\frac{M+W}{2}$ and L_α are non-atomic. In fact, every complete dependence copula is non-atomic.

Proposition 4.3. *Let C be a copula.*

1. *$\sigma_C = \mathscr{B}$ if and only if C is left invertible.*
2. *$\sigma_C^* = \mathscr{B}$ if and only if C is right invertible.*

Proof. We shall prove only 1 as 2 can be proved in a similar manner. If $C = C_{eg}$ then $T_C = T_{eg}$ maps ψ to $\psi \circ g$. So for every $B \in \mathscr{B}$, $T_C\mathbf{1}_B = \mathbf{1}_B \circ g = \mathbf{1}_{g^{-1}(B)}$. Hence, $\sigma_C = \mathscr{B}$. Conversely, Theorem 5 in Sakai and Shimogaki (1972) implies in particular that T is multiplicative, meaning $T(\psi \cdot \phi) = T\psi \cdot T\phi$ for every $\psi, \phi \in L^\infty$, if and only if $T = T_{eg}$ for some measure-preserving function g. By the linearity and continuity of $T = T_C$, it then suffices to show that $T(\mathbf{1}_B \cdot \mathbf{1}_E) = T\mathbf{1}_B \cdot T\mathbf{1}_E$ for all $B, E \in \mathscr{B}$. In fact, if $T\mathbf{1}_B = \mathbf{1}_{B'}$ and $T\mathbf{1}_E = \mathbf{1}_{E'}$ then $T\mathbf{1}_{B\cap E} = \mathbf{1}_{B'\cap E'}$ (by Proposition 3.2(3)) and the desired equality follows. □

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Proposition 4.4. *Let T be a Markov operator and f and g be measure-preserving functions on* $[0, 1]$. *Then the following are equivalent.*

1. $T\mathbf{1}_{g^{-1}(B)} = \mathbf{1}_{f^{-1}(B)}$ *for every* $B \in \mathscr{B}$.
2. $T(\theta \circ g) = \theta \circ f$ *for every Borel function* $\theta \in L^1$.
3. $T\psi = T_{ef} \circ T_{ge}\psi$ *for every* $g^{-1}(\mathscr{B})$*-measurable function* ψ.

Proof. $1 \Rightarrow 2$: For every Borel set $B \subseteq \mathbf{I}$, $T(\mathbf{1}_B \circ g) = T\mathbf{1}_{g^{-1}(B)} = \mathbf{1}_{f^{-1}(B)} = \mathbf{1}_B \circ f$. So, $T(\psi \circ g) = \psi \circ f$ for all simple Borel functions ψ. By the standard measure-theoretic argument, $T(\theta \circ g) = \theta \circ f$ for every Borel function θ.

$2 \Rightarrow 3$: Using Theorem 2.1, 2 implies that $T_{fe} \circ T \circ T_{eg}\theta = T_{fe}(T(\theta \circ g)) = T_{fe}(\theta \circ f) = \theta$ for every Borel function θ, and hence $T_{fe} \circ T \circ T_{eg} = id = T_M$, that is $C_{fe} * C * C_{eg} = M$. This means that $C_{fe} * C$ is the unique left inverse of C_{eg}, or $C_{fe} * C = C_{ge}$. Left multiplying by C_{ef} yields $C_{ef} * C_{fe} * C = C_{ef} * C_{ge}$, i.e. $T_{ef} \circ T_{fe} \circ T = T_{ef} \circ T_{ge}$. On the other hand, by Theorem 2.1, for any Borel function θ, $T_{ef} \circ T_{fe}(\theta \circ f) = (\theta \circ f)$ which gives $T_{ef} \circ T_{fe} \circ T(\theta \circ g) = T(\theta \circ g)$. Since $\{\theta \circ g : \theta \text{ is a Borel function}\}$ coincides with the class of $g^{-1}(\mathscr{B})$-measurable functions, $T_{ef} \circ T_{fe} \circ T = T$ on the class of L^1-functions which are $g^{-1}(\mathscr{B})$-measurable. Therefore, $T\psi = (T_{ef} \circ T_{ge})\psi$ for every $g^{-1}(\mathscr{B})$-measurable $\psi \in L^1$.

$3 \Rightarrow 1$: This is clear from taking $\psi = \mathbf{1}_{g^{-1}(B)}$. □

The equivalence relation $\approx$ on $\mathscr{B}$ is defined as follows: $E \approx F$ if and only if the symmetric difference $E \triangle F$ has Lebesgue measure zero. Of course, $\approx$ is still an equivalence relation on any $\mathscr{S} \subseteq \mathscr{B}$. The equivalence class of S in $\mathscr{S}$ is denoted by $[S]_{\mathscr{S}}$ or just $[S]$ if no confusion can arise. The collection of equivalence classes in $\mathscr{S}$ is denoted by $[\mathscr{S}]$. $[\mathscr{S}]$ is in fact a measure algebra induced by the Lebesgue measure λ. That is, $[\mathscr{S}]$ is a Boolean σ-algebra with respect to the operations $[S] \vee [R] = [S \cup R]$ and $[S] \wedge [R] = [S \cap R]$ together with $\lambda : [\mathscr{S}] \to [0, 1]$ defined by $\lambda([S]) = \lambda(S)$ and satisfying $\lambda\left(\bigvee_{i=1}^{\infty}[A_i]\right) = \sum_{i=1}^{\infty}\lambda([A_i])$ if $[A_i] \wedge [A_j] = [\emptyset]$ for $i \neq j$. See Royden (1988, p.398).

T_C induces a well-defined equivalence class function $\Upsilon_C : [\sigma_C] \to [\sigma_C^*]$ mapping $[S]$ to $[R]$ if and only if $T_C\mathbf{1}_S = \mathbf{1}_R$. It follows from the defining property ii of T that $\lambda(R) = \lambda(S)$ and hence Υ_C is measure-preserving. Statements 1 and 2 in the following lemma are quoted from Theorem 3.2 and Theorem 3.3 in Darsow and Olsen (2010), respectively, except for the additional claim in 2 that the measure-preserving Borel function h is unique a.e. We therefore include its proof, part of which is the construction procedure of h taken from Darsow and Olsen (2010, Theorem 3.3).

Lemma 4.5 (*Darsow and Olsen, 2010*). *Let $\mathscr{S} \subseteq \mathscr{B}$ be a non-atomic σ-algebra. Then*

1. *there exists a surjective isomorphism $\Psi : [\mathscr{S}] \to [\mathscr{B}]$ which means $\Psi([S]^c) = \Psi([S])^c$, $\Psi([S_1] \vee [S_2]) = \Psi([S_1]) \vee \Psi([S_2])$, $\lambda([S]) = \lambda(\Psi([S]))$ and Ψ is onto (it follows that Ψ is one-to-one and an isometry with respect to the metric $\rho([S], [R]) = \lambda([S]\triangle[R])$ and preserves countable unions and intersections); and*
2. *for any surjective isomorphism $\Psi : [\mathscr{S}] \to [\mathscr{B}]$, there exists a unique (a.e.) measure-preserving Borel function $h : [0, 1] \to [0, 1]$ such that $h^{-1}(\mathscr{B}) \subseteq \mathscr{S}$ (in fact, they are essentially equivalent,) and that $h^{-1}(B) \in \Psi^{-1}([B])$ for every $B \in \mathscr{B}$.*

Proof. We prove only 2. Enumerate $\mathbb{Q} \cap [0, 1] = \{r_n\}_{n\in\mathbb{N}}$ and set $I_n = [0, r_n]$. For each $n \in \mathbb{N}$, choose S_n in the equivalence class $\Psi^{-1}([I_n])$ so that $S_k = [0, 1]$ if $r_k = 1$ and $S_k \subset S_l$ whenever $r_k < r_l$. Define a Borel measure-preserving function $h(x) = \inf\{r_k : x \in S_k\}$. For $S \in \mathscr{S}$, choose $B_0 \in \Psi([S])$ and set $S_0 = h^{-1}(B_0)$. Since Ψ is one-to-one, $[S] = \Psi^{-1}([B_0])$. We prove that $h^{-1}(B_0) \in \Psi^{-1}([B_0])$ by considering $\mathscr{M} = \left\{B \in \mathscr{B} : h^{-1}(B) \in \Psi^{-1}([B])\right\}$. Since $\mathscr{M}$ is a monotone class containing every $[0, r_k]$, it contains all Borel sets and we have the claim. To prove uniqueness, if $k : [0, 1] \to [0, 1]$ is a measure-preserving Borel function such that $k^{-1}(B) \in \Psi^{-1}([B])$ for every $B \in \mathscr{B}$, then $k^{-1}([0, r_n]) \in \Psi^{-1}([\![0, r_n]\!])$. So $\lambda\left(h^{-1}([0, r_n])\triangle k^{-1}([0, r_n])\right) = 0$ for every n. By Proposition 10 in Royden (1988, Chapter 11 (p. 261)), $h = k$ a.e. □

Theorem 4.6. *Let C be a copula with associated Markov operator T_C and doubly stochastic measure μ_C. For measure-preserving functions f and g on $\mathbf{I}$, the following are equivalent.*

1. $\mu_C\left(\Gamma_{f,g}\right) = 1$, *where* $\Gamma_{f,g} = \left\{(x, y) \in \mathbf{I}^2 : f(x) = g(y)\right\}$.
2. $T_C = T_{ef} \circ T_{ge}$ *on the class of* $g^{-1}(\mathscr{B})$*-measurable functions.*
3. *C is non-atomic with $\sigma_C \supseteq g^{-1}(\mathscr{B})$ and $\sigma_C^* \supseteq f^{-1}(\mathscr{B})$.*

Proof. $1 \Rightarrow 2$: By Proposition 4.4, Lemma 3.1 and the measure-preserving property of f and g, it suffices to show that $\mu_C(f^{-1}(B) \times g^{-1}(B)) = \lambda(f^{-1}(B))$ for every $B \in \mathscr{B}$. This follows from $\mu_C\left(f^{-1}(B) \times g^{-1}(B)\right) \le \mu_C\left(f^{-1}(B) \times \mathbf{I}\right) = \lambda\left(f^{-1}(B)\right)$ and

$$
\begin{aligned}
\mu_C\left(f^{-1}(B) \times g^{-1}(B)\right) &\ge \mu_C\left(\{(x, y) : f(x) = g(y) \in B\}\right) \\
&= \mu_C\left(\{(x, y) : f(x) = g(y)\} \cap \{(x, y) : f(x) \in B\}\right) \\
&= \lambda\left(f^{-1}(B)\right).
\end{aligned}
$$

$2 \Rightarrow 1$: For each $n \in \mathbb{N}$ and $i = 1, 2, \ldots, 2^n$, put $I_{i,n} = \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]$ and $B_n = \bigcup_i f^{-1}(I_{i,n}) \times g^{-1}(I_{i,n})$. Then $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$ and

$$
\mu_C(B_n) = \sum_{i=1}^{2^n}\mu_C\left(f^{-1}(I_{i,n}) \times g^{-1}(I_{i,n})\right) = \sum_{i=1}^{2^n}\lambda\left(f^{-1}(I_{i,n})\right) = \sum_{i=1}^{2^n}\lambda\left(I_{i,n}\right) = 1,
$$

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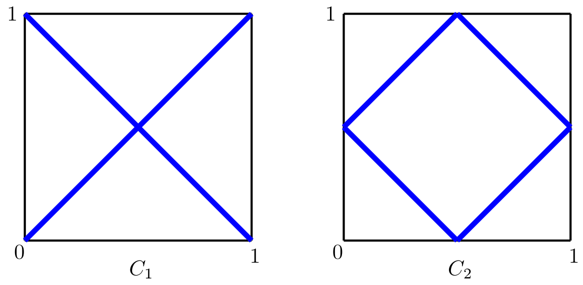

Fig. 1. Supports of copulas whose both associated σ-algebras are $\wedge^{-1}(\mathscr{B})$.

where we have used Lemma 3.1 in the second equality and the measure-preserving property of f in the third. Set $B = \bigcap_{n=1}^{\infty} B_n$. We have $\mu_C(B) = \lim_{n\to\infty} \mu_C(B_n) = 1$. It then suffices to show that $B = \Gamma_{f,g}$. First, if $f(x) \neq g(y)$ then there exists $n \in \mathbb{N}$ such that $\frac{1}{2^n} < |f(x) - g(y)|$ and hence $(x, y) \notin B_n$. Conversely, it is clear that $\Gamma_{f,g} = \bigcup_i \left\{(x, y) : f(x) = g(y) \in I_{i,n}\right\} \subseteq B_n$ for every n.

2 ⇒ 3: By Proposition 4.4, $T_C \mathbf{1}_{g^{-1}(B)} = \mathbf{1}_{f^{-1}(B)}$ for every $B \in \mathscr{B}$, which implies that $g^{-1}(\mathscr{B}) \subseteq \sigma_C$ and $f^{-1}(\mathscr{B}) \subseteq \sigma_C^*$. But both $g^{-1}(\mathscr{B})$ and $f^{-1}(\mathscr{B})$ are non-atomic because g and f are measure-preserving. Therefore the finer σ-algebras σ_C and σ_C^* are non-atomic as well.

3 ⇒ 2: Our proof is in three steps. (i) By applying Lemma 4.5(1) to σ_C and σ_C^*, it follows that $\Upsilon_C : [\sigma_C] \to [\sigma_C^*]$ defined earlier is one-to-one, onto, measure-preserving and preserves order, complementation and the lattice operation on equivalence classes corresponding to countable unions of monotonic sequence of sets, i.e. Υ_C is a surjective isomorphism. (ii) By Lemma 4.5(2), there exists $\Xi_C : [\sigma_C^*] \to [\mathscr{B}]$ and a unique (a.e.) Borel function $f : \mathbf{I} \to \mathbf{I}$ such that $f^{-1}(B) \in \Xi_C^{-1}([B])$ for every $B \in \mathscr{B}$. So the composition $\Xi_C \circ \Upsilon_C : [\sigma_C] \to [\mathscr{B}]$ satisfies the same properties and induces a unique Borel function $g : \mathbf{I} \to \mathbf{I}$ such that

$$g^{-1}(B) \in (\Xi_C \circ \Upsilon_C)^{-1}([B]) \quad \text{for every } B \in \mathscr{B}.$$

As a consequence, $\Upsilon_C\left([g^{-1}(B)]\right) = \Xi_C^{-1}([B])$ and, by the definition of Υ_C, $T_C \mathbf{1}_{g^{-1}(B)} = \mathbf{1}_{f^{-1}(B)}$ for every $B \in \mathscr{B}$. And the proof is done by Proposition 4.4. □

In particular, by taking $C = C_{ef} * C_{ge}$ where f and g are measure-preserving functions on $\mathbf{I}$, 2 holds and hence μ_C assigns full mass to $\Gamma_{f,g}$, that is $\mu_C(\Gamma_{f,g}) = 1$. By the Remark after Proposition 3.2, Theorem 4.6 also implies that $T_C^* \circ T_C = id$ on the family of all integrable and $g^{-1}(\mathscr{B})$-measurable functions. Even though it holds for simple (piecewise linear) measure-preserving functions f and g that $\sigma_C = g^{-1}(\mathscr{B})$ and $\sigma_C^* = f^{-1}(\mathscr{B})$, we could not prove these for arbitrary measure-preserving functions. It is quite probable that they are only essentially equivalent. Note that σ_C and σ_C^* do not characterize the copula C or even its support as shown in the following example.

Example 3. Consider the copulas $C_1 = \frac{M+W}{2}$ and C_2 whose mass is distributed uniformly along the four line segments shown in Fig. 1. They are both symmetric with associated σ-algebras $\sigma_{C_1} = \sigma_{C_1}^* = \sigma_{C_2} = \sigma_{C_2}^* = \wedge^{-1}(\mathscr{B})$ where $\wedge(x) = 2\min\{x, 1-x\}$ for $x \in \mathbf{I}$. Observe that $\Gamma_{\wedge,\wedge} = \{(x, y) : x = y \text{ or } x = 1 - y\}$, which coincides with the support of C_1. Note also that for any measure-preserving function f, the set $\Gamma_{f,f}$ contains the diagonal $\{(x, x) : x \in \mathbf{I}\}$ and hence M assigns full mass to $\Gamma_{f,f}$.

Example 4. For a fixed $\alpha \in (0, 1)$, consider $C_\alpha = \alpha M + (1-\alpha)W$ with Markov operator T_α. It is readily verified that $\sigma_{C_\alpha} = \sigma_{C_\alpha}^* = \{S \in \mathscr{B} : S = 1 - S\}$ and $T_\alpha \mathbf{1}_S = \mathbf{1}_S$ for every $S \in \sigma_{C_\alpha}$. But only $T_{\frac{1}{2}}$ has the property that $T_{\frac{1}{2}} \mathbf{1}_B$ is $\sigma_{C_{\frac{1}{2}}}$-measurable for every $B \in \mathscr{B}$. Note also that $C_{\frac{1}{2}} = L_{\frac{1}{2}} * L_{\frac{1}{2}}^t$.

Proposition 4.7. *For any given measure-preserving functions f and g, every copula C fulfilling $\mu_C\left(\Gamma_{f,g}\right) = 1$ is singular.*

Proof. By Fubini's theorem, the 2-dimensional Lebesgue measure of $\Gamma_{f,g}$ equals

$$\int_0^1 \int_0^1 \mathbf{1}_{\Gamma_{f,g}}(x, y)\, dy\, dx = \int_0^1 \lambda\left(g^{-1}(\{f(x)\})\right) dx = \int_0^1 \lambda\left(\{f(x)\}\right) dx = 0,$$

where we have used the measure-preserving property of g. □

With respect to the uniform norm, since the shuffles of Min are already dense in the class of copulas, so is the class of implicit dependence copulas. A natural question, by one of the referees, is whether the class of implicit dependence copulas is dense with respect to stronger norms such as the modified Sobolev norm. Considering the modified Sobolev norm, though the independence copula can be approximated by implicit dependence copulas (see Chaidee et al., 2016, Theorem 4.4), its proof for an arbitrary copula still escapes us. However, Theorem 4.3 in the same paper (Chaidee et al., 2016) implies the denseness of the class of copulas assigning full mass to an implicit graph.

5. Copulas with fractal support

Let us recall the construction of copulas with fractal supports in Fredricks et al. (2005) put in the context of patched copulas (Yanting et al., 2011; Chaidee et al., 2016).

Definition 3. A *transformation matrix* is a matrix A with nonnegative entries, for which the sum of all entries is 1 and every row and column has at least one non-zero entry.

Given a transformation matrix $A = [a_{ij}]_{k\times\ell}$ where the first index (i) is the column number from left $(i = 1)$ to right $(i = k)$ and the second index (j) is the row number from bottom $(j = 1)$ to top $(j = \ell)$, the matrix multiplication of A and $B = [b_{jm}]_{\ell\times n}$ is defined by $[a_{ij}][b_{jm}] = \left[\sum_{j=1}^{\ell} a_{ij}b_{jm}\right]$. $A^t = [a_{ji}]_{\ell\times k}$. This unconventional entry arrangement syncs well with the *-product of copulas.

Let p_i denote the sum of the entries in the first i columns of A and let q_j denote the sum of the entries in the first j rows of A where $p_0 = 0$ and $q_0 = 0$. The partitions $\{p_i\}_{i=0}^{k}$ and $\left\{q_j\right\}_{j=0}^{\ell}$ of $[0, 1]$ yield a rectangular partition of $[0, 1]^2$ consisting of $k\cdot\ell$ rectangles with vertices (p_i, q_j). More precisely, the rectangular partition consists of $R_{ij} = R_{i,} \times R_{,j}$ where $R_{1,} = [0, p_1]$, $R_{,1} = [0, q_1]$, $R_{i,} = (p_{i-1}, p_i]$ for $1 < i \le k$ and $R_{,j} = (q_{j-1}, q_j]$ for $1 < j \le \ell$. Given a copula C, we construct a new copula denoted by $[A](C)$ by placing a scaled copy of C in each R_{ij} weighted according to the mass given by the corresponding entry in A. It is defined for $(u, v) \in R_{ij}$ by

$$[A](C)(u, v) = \sum_{i'<i,j'<j} a_{i'j'} + \frac{u - p_{i-1}}{p_i - p_{i-1}}\sum_{j'<j} a_{ij'} + \frac{v - q_{j-1}}{q_j - q_{j-1}}\sum_{i'<i} a_{i'j} + a_{ij}C\left(\frac{u - p_{i-1}}{p_i - p_{i-1}}, \frac{v - q_{j-1}}{q_j - q_{j-1}}\right).$$

Here, empty sums are zero by convention. See Fredricks et al. (2005) for more details. It will be more convenient to view $[A](C)$ as the so-called *patched copula* (Yanting et al., 2011; Chaidee et al., 2016) defined as

$$[A](C)(u, v) = \sum_{i=1}^{k}\sum_{j=1}^{\ell} a_{ij}C\left(F_i(u), G_j(v)\right)$$

where $F_i(u) = \min\left(1, \frac{u-p_{i-1}}{p_i-p_{i-1}}\right)\mathbf{1}_{(p_{i-1},\infty)}(u)$ is the uniform distribution on $[p_{i-1}, p_i]$ and $G_j(v) = \min\left(1, \frac{v-q_{j-1}}{q_j-q_{j-1}}\right)\mathbf{1}_{(q_{j-1},\infty)}(v)$ is the uniform distribution on $[q_{j-1}, q_j]$. Denote $\triangle p_i = p_i - p_{i-1}$ and $\triangle q_j = q_j - q_{j-1}$.

We then investigate how this rectangular patching of a copula C according to the transformation matrix A affects C in terms of their Markov operators.

$$\begin{aligned} T_{[A](C)}f(x) &= \sum_{i=1}^{k}\sum_{j=1}^{\ell} \frac{a_{ij}}{\triangle q_j}\frac{d}{dx}\int_{q_{j-1}}^{q_j} \partial_2 C\left(F_i(x), G_j(t)\right) f(t)\,dt \\ &= \sum_{i=1}^{k}\sum_{j=1}^{\ell} a_{ij}\frac{d}{dx}\int_0^1 \partial_2 C\left(F_i(x), s\right) f\circ G_j^{-1}(s)\,ds \\ &= \sum_{i=1}^{k}\sum_{j=1}^{\ell} \frac{a_{ij}}{\triangle p_i} T_C(f\circ G_j^{-1})(F_i(x)). \end{aligned} \tag{2}$$

Consequently, $T_{[A](C)}(f)(F_i^{-1}(x)) = \sum_{j=1}^{\ell}\frac{a_{ij}}{\triangle p_i}T_C(f\circ G_j^{-1})(x)$, which can be written in matrix form as

$$\left[T_{[A](C)}(f)(F_i^{-1}(x))\right]_{k\times 1} = \left[\frac{a_{ij}}{\triangle p_i}\right]_{k\times\ell}\left[T_C(f\circ G_j^{-1})(x)\right]_{\ell\times 1}.$$

See Trutschnig (2012) and Trutschnig and Fernández-Sánchez (2012) for essentially the same identity in terms of Markov kernels.

Define inductively $[A]^n(C) = [A]\left([A]^{n-1}(C)\right)$ for $n \ge 2$. Fredricks et al. (2005) showed that for any transformation matrix $A \ne [1]$ and copula C, $[A]^n(C)$ converges (pointwise and hence uniformly) to a unique copula C_A, as $n \to \infty$. Moreover, C_A is the fixed point of $[A]$, i.e. $[A](C_A) = C_A$. However, since uniform convergence will not suffice for our purposes, we shall investigate the convergence of $[A]^k(D)$ with respect to a stronger norm with respect to which the $*$-product is jointly continuous. We choose the modified Sobolev norm defined as $\|C\|_S^2 = \|C\|_1^2 + \|C\|_2^2$ where $\|C\|_i^2 = \int_0^1\int_0^1 |\partial_i C(u, v)|^2\,du\,dv$. See Darsow and Olsen (1995). Our proof of the following proposition is quite the same as the analysis of the mapping V, corresponding to our mapping $[A]$, in Trutschnig (2011, pp. 703–704). So we give only a sketch of proof.

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Proposition 5.1. *Let A be a transformation matrix whose dimension is at least* 2×2 *and let C and D be copulas. Then*

$$\|[A](C)-[A](D)\|_S \le r\,\|C-D\|_S$$

where $r^2 = \max\left(\sum_{j=1}^{\ell}\sum_{i=1}^{k} a_{ij}^2 \frac{q_j-q_{j-1}}{p_i-p_{i-1}}, \sum_{j=1}^{\ell}\sum_{i=1}^{k} a_{ij}^2 \frac{p_i-p_{i-1}}{q_j-q_{j-1}}\right) < 1$ *and* $\|\cdot\|_S$ *is the modified Sobolev norm.*

Proof. Denote $A=[a_{ij}]_{k\times\ell}$ with $k,\ell \ge 2$ and let $\{p_i\}_{i=0}^k$, $\left\{q_j\right\}_{j=0}^{\ell}$ be the induced partitions of $[0,1]$ on the x-axis and y-axis, respectively. Then, for $u \in (p_{i-1},p_i)$ and $v \in (q_{j-1},q_j)$,

$$|\partial_1[A](C)(u,v)-\partial_1[A](D)(u,v)| = \frac{a_{ij}}{p_i-p_{i-1}}\left|\partial_1 C\left(F_i(u),G_j(v)\right)-\partial_1 D\left(F_i(u),G_j(v)\right)\right|,$$

which implies that $\|[A](C)-[A](D)\|_1^2 = \left(\sum_{j=1}^{\ell}\sum_{i=1}^{k} a_{ij}^2\frac{q_j-q_{j-1}}{p_i-p_{i-1}}\right)\|C-D\|_1^2$. Using the relations $a_{ij} \le p_i-p_{i-1}$, $\sum_i a_{ij} = q_j-q_{j-1}$ and the assumption $\ell \ge 2$, the constant must be less than 1. A similar proof yields an analogous bound for $\|[A](C)-[A](D)\|_2^2$. □

We now have that the mapping $[A]$ is a contraction on the class of copulas $\mathcal{C}_2$ with respect to the modified Sobolev norm. Using the fact (Darsow and Olsen, 1995) that $\mathcal{C}_2$ is complete with respect to the modified Sobolev norm, it follows from the Contraction-Mapping Theorem that:

Theorem 5.2. *For each transformation matrix A of dimension at least* 2×2*, and for any initial copula D, the sequence* $\{[A]^r(D)\}_r$ *converges to the copula* C_A *in the modified Sobolev norm.*

Evidently, C_A depends only on the matrix A. So, in order to investigate the non-atomicity of C_A, we are led to introduce a completely analogous concept of invariant pairs on the transformation matrix A which automatically yields a pair (R,S) of unions of intervals satisfying $T_{[A](D)}\mathbf{1}_S = \mathbf{1}_R$.

Definition 4. Let $A=[a_{ij}]_{k\times\ell}$ be a transformation matrix, $\emptyset \ne I \subseteq \{1,2,\ldots,k\}$ and $\emptyset \ne J \subseteq \{1,2,\ldots,\ell\}$. The pair (I,J) is called an *invariant pair* of A if $a_{ij}=0$ for every $(i,j)\in(I\times J^c)\cup(I^c\times J)$. If (I,J) is an invariant pair of a transformation matrix A, then there must be some $(i,j)\in I\times J$ for which $a_{ij}>0$.

Two invariant pairs (I_1,J_1) and (I_2,J_2) of A are called *disjoint* if $I_1\cap I_2=\emptyset$ and $J_1\cap J_2=\emptyset$. This is equivalent to the condition that $(I_1\times J_1)\cap(I_2\times J_2)=\emptyset$ simply because $(I_1\times J_1)\cap(I_2\times J_2)=(I_1\cap I_2)\times(J_1\cap J_2)$. If (I_1,J_1) and (I_2,J_2) are not disjoint then both $I_1\cap I_2$ and $J_1\cap J_2$ are not empty.

We say that A is *disjointly decomposable* if A has a finite number of pairwise disjoint invariant pairs $(I_1,J_1),(I_2,J_2),\ldots,(I_N,J_N)$ such that $N\ge 2$, $\bigcup_{n=1}^N I_n=\{1,2,\ldots,k\}$ and $\bigcup_{n=1}^N J_n=\{1,2,\ldots,\ell\}$. For each $n=1,2,\ldots,N$, let us denote by A_n the $k\times\ell$ matrix whose (i,j)th entry is a_{ij} if $i\in I_n$ and $j\in J_n$ and is equal to zero otherwise. Observe that $A=\sum_{n=1}^N A_n$ and in particular, every non-zero entry in A appears in exactly one A_n. We also say that A is *disjointly decomposable* as the sum $\sum_{n=1}^N A_n$ or *disjointly decomposable* by N invariant pairs. It is worth mentioning that disjointly decomposable transformation matrices are exactly the transformation matrices which, after possible rearrangements of columns and rows, can be written in a diagonal block form.

Such a disjoint decomposition of A gives rise to two partitions of $[0,1]$: $\{Q_n\}_{n=1}^N$ and $\{P_n\}_{n=1}^N$ defined by $Q_n=\bigcup_{j\in J_n}(q_{j-1},q_j)$ and $P_n=\bigcup_{i\in I_n}(p_{i-1},p_i)$. Then $\lambda(Q_n)>0$ and $\lambda(P_n)>0$ for all $n=1,\ldots,N$. Each pair (I_n,J_n) induces set functions $\mathscr{G}_n$ and $\mathscr{F}_n$ mapping $B\in\mathscr{B}$ to

$$\mathscr{G}_n(B)\equiv\bigcup_{j\in J_n}G_j^{-1}(B)\subseteq Q_n \quad\text{and}\quad \mathscr{F}_n(B)\equiv\bigcup_{i\in I_n}F_i^{-1}(B)\subseteq P_n.$$

Note that $G_j^{-1}((0,1))=(q_{j-1},q_j)$ and $F_i^{-1}((0,1))=(p_{i-1},p_i)$.

Remark. If A has N invariant pairs $(I_1,J_1),(I_2,J_2),\ldots,(I_N,J_N)$, not necessarily disjoint, such that $\bigcup_{n=1}^N I_n=\{1,2,\ldots,k\}$ and $\bigcup_{n=1}^N J_n=\{1,2,\ldots,\ell\}$ then A is disjointly decomposable by $N'\ge N$ invariant pairs. In fact, without loss of generality, if (I_1,J_1) and (I_2,J_2) are not disjoint then $I'\equiv I_1\cap I_2\ne\emptyset$ and $J'\equiv J_1\cap J_2\ne\emptyset$ and it can be shown that $(I_1\setminus I_2,J_1\setminus J_2),(I_2\setminus I_1,J_2\setminus J_1),(I',J')$ are pairwise disjoint invariant pairs. Replacing $(I_1,J_1),(I_2,J_2)$ with these three pairs gives us a finer list of invariant pairs. Then repeat the process until all invariant pairs are pairwise disjoint.

Lemma 5.3. *If* (I,J) *is an invariant pair of a transformation matrix A and* $S,R\in\mathscr{B}$ *are such that* $T_C\mathbf{1}_S=\mathbf{1}_R$*, then* $T_{[A](C)}\mathbf{1}_{\bigcup_{j\in J}G_j^{-1}(S)}=\mathbf{1}_{\bigcup_{i\in I}F_i^{-1}(R)}$.

Proof. For $j\in J$, we have $\mathbf{1}_{\bigcup_{j'\in J}G_{j'}^{-1}(S)}\circ G_j^{-1}(x)=\mathbf{1}_S(x)$; otherwise, $\mathbf{1}_{\bigcup_{j'\in J}G_{j'}^{-1}(S)}\circ G_j^{-1}(x)=0$. So, by (2),

$$T_{[A](C)}\mathbf{1}_{\bigcup\limits_{j\in J}G_j^{-1}(S)}=\sum_{i=1}^{k}\sum_{j=1}^{\ell}\frac{a_{ij}}{\triangle p_i}T_C\left(\mathbf{1}_{\bigcup\limits_{j'\in J}G_{j'}^{-1}(S)}\circ G_j^{-1}\right)(F_i(x))$$

$$= \sum_{i=1}^{k} \sum_{j \in J} \frac{a_{ij}}{\triangle p_i} T_C(\mathbf{1}_S)(F_i(x))$$
$$= \sum_{i \in I} \frac{\sum_{j \in J} a_{ij}}{\triangle p_i} T_C(\mathbf{1}_S)(F_i(x)),$$

where we have used the assumption that $a_{ij} = 0$ if $i \notin I$ and $j \in J$. Since for each $i \in I$, $a_{ij} = 0$ for every $j \notin J$, the sum $\sum_{j \in J} a_{ij}$ is equal to $\triangle p_i$ and

$$T_{[A](C)} \mathbf{1}_{\bigcup_{j \in J} G_j^{-1}(S)} = \sum_{i \in I} T_C(\mathbf{1}_S)(F_i(x)) = \sum_{i \in I} \mathbf{1}_R(F_i(x)) = \mathbf{1}_{\bigcup_{i \in I} F_i^{-1}(R)}(x). \quad \square$$

Lemma 5.4. *Let C be a copula. Suppose a transformation matrix A is disjointly decomposable by N invariant pairs. If $T_{[A](C)} \mathbf{1}_S = \mathbf{1}_R$ then for $n = 1, \ldots, N$, $T_{[A](C)} \mathbf{1}_{S_n} = \mathbf{1}_{R_n}$ where $S_n \equiv S \cap Q_n$ and $R_n \equiv R \cap P_n$.*

Proof. For each $n = 1, \ldots, N$, the positivity of $T_{[A](C)}$ implies that $T_{[A](C)} \mathbf{1}_{S_n} \leq T_{[A](C)} \mathbf{1}_S = \mathbf{1}_R$. If it holds that $T_{[A](C)} \mathbf{1}_{S_n} = 0$ for a.e. x not in P_n, then $T_{[A](C)} \mathbf{1}_{S_n} \leq \mathbf{1}_{R \cap P_n} = \mathbf{1}_{R_n}$. Summing over every n gives

$$\mathbf{1}_R = T_{[A](C)} \mathbf{1}_S = \sum_n T_{[A](C)} \mathbf{1}_{S_n} \leq \sum_n \mathbf{1}_{R_n} = \mathbf{1}_R$$

and hence $T_{[A](C)} \mathbf{1}_{S_n} = \mathbf{1}_{R_n}$ for every n. We then prove the claim. If $x \notin P_n \equiv \bigcup_{i' \in I_n} (p_{i'-1}, p_{i'})$, then $F_i(x) = 0$ or 1 for every $i \in I_n$ and

$$T_{[A](C)} \mathbf{1}_{S_n}(x) = \sum_{i \in I_n} \sum_{j \in J_n} \frac{a_{ij}}{\triangle p_i} T_C \left(\sum_{j' \in J_n} \mathbf{1}_{S \cap (q_{j'-1}, q_{j'})} \circ G_j^{-1} \right) (F_i(x))$$
$$= \sum_{i \in I_n} \sum_{j \in J_n} \frac{a_{ij}}{\triangle p_i} T_C \left(\mathbf{1}_{S \cap (q_{j-1}, q_j)} \circ G_j^{-1} \right) (F_i(x)) = 0.$$

We use the convention that 0 and 1 are not in the support of all functions. $\square$

Theorem 5.5. *Suppose a transformation matrix A is disjointly decomposable by $N \geq 2$ invariant pairs. Then the invariant copula C_A is non-atomic.*

Proof. Let $S, R \in \mathscr{B}$ be such that $T_{C_A} \mathbf{1}_S = \mathbf{1}_R$ and $\lambda(S) = \lambda(R) > 0$. Since $C_A = [A](C_A)$, by Lemma 5.4, $T_{C_A} \mathbf{1}_{S_n} = \mathbf{1}_{R_n}$ for all $n = 1, 2, \ldots, N$ where $S_n \equiv \mathscr{G}_n(S) \subseteq Q_n$ and $R_n \equiv \mathscr{F}_n(R) \subseteq P_n$ are such that $S = \bigcup_n S_n$ and $R = \bigcup_n R_n$. If one of the $\lambda(S_n)$'s is strictly between 0 and $\lambda(S)$ then S and R are not atoms of σ_{C_A} and $\sigma_{C_A}^*$, respectively. Otherwise, there is an n_1 such that $\lambda(S_{n_1}) = \lambda(S)$ and we repeat the process by applying Lemma 5.4 to $T_{C_A} \mathbf{1}_{S_{n_1}} = \mathbf{1}_{R_{n_1}}$ and obtain $S_{n_1,n} \equiv \mathscr{G}_n(S_{n_1}) \subseteq \mathscr{G}_n(Q_{n_1})$ and $R_{n_1,n} \equiv \mathscr{F}_n(R_{n_1}) \subseteq \mathscr{F}_n(P_{n_1})$ are such that $S_{n_1} = \bigcup_n S_{n_1,n}$, $R_{n_1} = \bigcup_n R_{n_1,n}$ and $T_{C_A} \mathbf{1}_{S_{n_1,n}} = \mathbf{1}_{R_{n_1,n}}$ for all $n = 1, 2, \ldots, N$. If some $\lambda(S_{n_1,n})$ lies between 0 and $\lambda(S)$ then we are done. Otherwise, there must be an n_2 such that $\lambda(S_{n_1,n_2}) = \lambda(S)$. This process will certainly stop because $\lambda(\mathscr{G}_{n_k}(\cdots(\mathscr{G}_{n_2}(Q_{n_1})))) = \lambda(Q_{n_k}) \cdots \lambda(Q_{n_2}) \lambda(Q_{n_1}) \to 0$ as $k \to \infty$ and $\lambda(S) > 0$. $\square$

Example 5. Let $A_1 = K_0 + K_1$ and $A_2 = K_0 + K_2$ where

$$K_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 1/12 & 0 & 1/4 \\ 0 & 0 & 0 \\ 1/4 & 0 & 1/12 \end{bmatrix} \quad \text{and} \quad K_2 = \begin{bmatrix} 1/6 & 0 & 1/6 \\ 0 & 0 & 0 \\ 1/6 & 0 & 1/6 \end{bmatrix}.$$

Then both transformation matrices A_1 and A_2 are disjointly decomposable by 2 invariant pairs $(\{1, 3\}, \{1, 3\})$ and $(\{2\}, \{2\})$. By Theorem 5.5, the invariant copulas C_{A_1} and C_{A_2} are non-atomic and hence, by Theorem 4.6, they assign full mass to implicit graphs, which can be shown to coincide. See Fig. 2. Note also that both copulas are symmetric as the corresponding matrices are. We will see later that (only) C_{A_2} can be factored as $L_1 * L_2^t$ for some left invertible copulas L_1 and L_2.

A transformation matrix $L = [\lambda_{in}]_{k \times N}$ is said to be a *left complete dependence matrix* if there is exactly one nonzero entry in each column, i.e. for each i, there exists a positive integer $n_i \leq N$ such that $\lambda_{in} = 0$ for $n \neq n_i$. *Right complete dependence matrices* are defined similarly.

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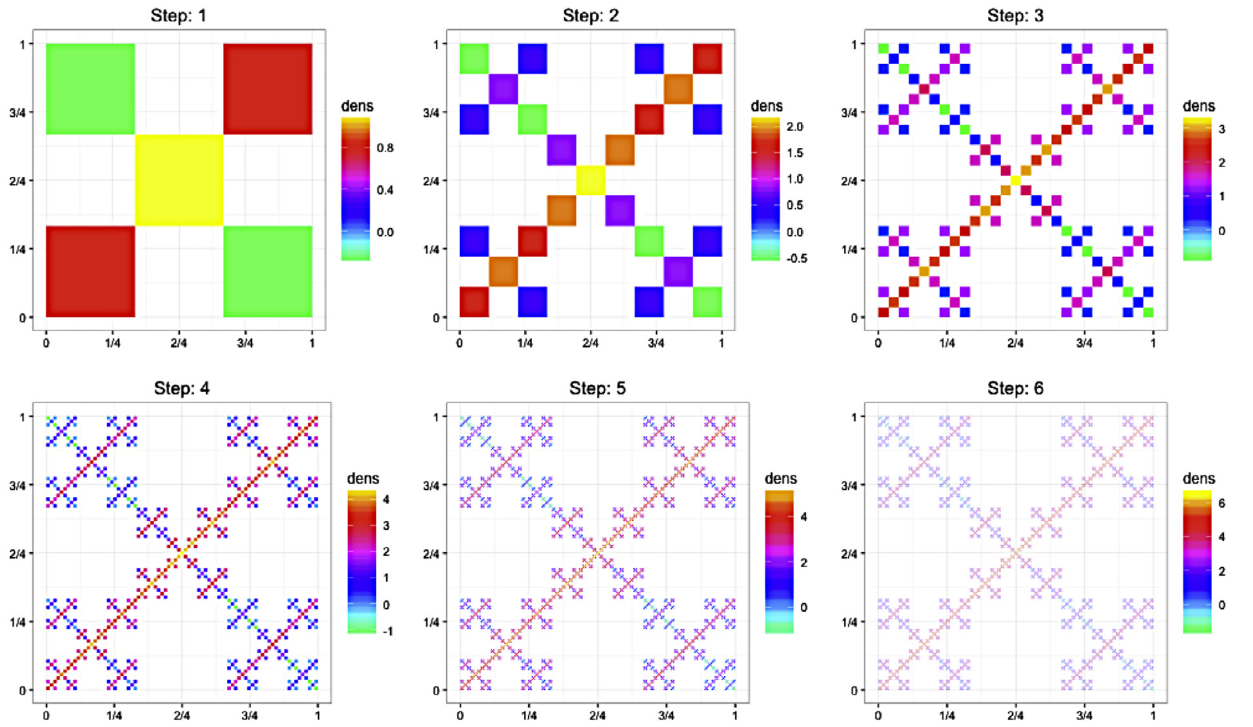

Fig. 2. Supports of $[A_1]^r(\Pi)$, $r = 1, 2, \ldots, 5$. Colors indicate the average density of the rectangles (log scale). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Lemma 5.6. *Let L be a left complete dependence matrix with dimension at least 2×2 and C be a left invertible copula. Then $[L](C), [L]^2(C), \ldots, [L]^r(C), \ldots$ are left invertible copulas converging to a left invertible copula C_L. The same statement also holds if all occurrences of "left" are replaced by "right".*

Proof. Since the general case can be proved by induction on r, it suffices to show that $[L](C)$ is left invertible. By definition, the x-partition of $[L](C)$ is determined by the only non-zero entry in each column: $p_0 = 0$ and $p_i = \sum_{i' \le i} \lambda_{i' n_{i'}}$, $i = 1, \ldots, k$. So $[L](C)(u, v) = \sum_{i=1}^{k} \lambda_{i n_i} C\left(F_i(u), G_{n_i}(v)\right)$ and

$$
\begin{aligned}
[L](C)^t * [L](C)(u, v) &= \sum_{i'=1}^{k} \sum_{i=1}^{k} \lambda_{i n_i} \lambda_{i' n_{i'}} \int_0^1 \frac{\partial}{\partial t} C\left(F_{i'}(t), G_{n_{i'}}(u)\right) \frac{\partial}{\partial t} C\left(F_i(t), G_{n_i}(v)\right)\, dt \\
&= \sum_{i=1}^{k} \lambda_{i n_i}^2 \int_0^1 \partial_1 C\left(F_i(t), G_{n_i}(u)\right) \partial_1 C\left(F_i(t), G_{n_i}(v)\right) F_i'(t)^2\, dt \\
&= \sum_{i=1}^{k} \lambda_{i n_i} C^t * C(G_{n_i}(u), G_{n_i}(v)) \\
&= \sum_{i=1}^{k} \lambda_{i n_i} G_{n_i}(\min(u, v)) = M(u, v).
\end{aligned}
$$

Therefore, $[L](C)$ is left invertible. In general, we have $[L]^r(C)^t * [L]^r(C) = M$ and so, by taking the limit as $r \to \infty$ with respect to the modified Sobolev norm (see Theorem 5.2), $C_L^t * C_L = M$. □

Theorem 5.7. *Let A be a $k \times \ell$ transformation matrix, with $k, \ell \ge 2$, which is disjointly decomposable by $N \ge 2$ invariant pairs. If all $A_1, A_2, \ldots, A_N$ have rank one, then $C_A = L * R$ for some left invertible copula L and right invertible copula R.*

Proof. If A_n has rank one, then there exist a row matrix $L_n = [\lambda_{in}]_{k \times 1}$ and a column matrix $R_n = [\rho_{nj}]_{1 \times \ell}$ such that $|L_n| = |A_n| = |R_n|$ and

$$
A_n = \frac{1}{|A_n|} L_n R_n, \tag{3}
$$

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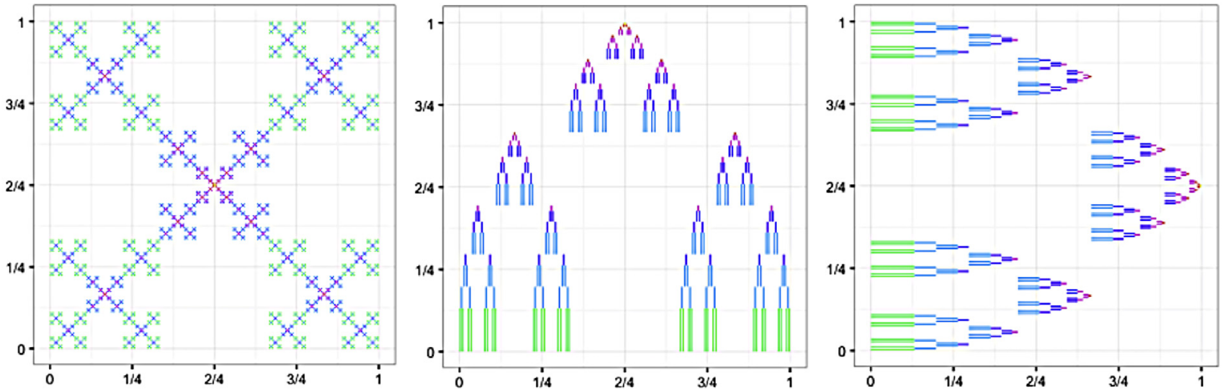

Fig. 3. Supports of $[A_2]^5(\Pi)$, $[L_2]^5(\Pi)$ and $[R_2]^5(\Pi)$ in Example 6.

where $|A|$ denotes the sum of the absolute values of all entries in a matrix A. Stacking up the L_n's vertically and the R_n's horizontally, we obtain transformation matrices

$$L = [\lambda_{in}]_{k\times N} = \begin{bmatrix} L_N \\ \vdots \\ L_1 \end{bmatrix} \quad \text{and} \quad R = [\rho_{nj}]_{N\times \ell} = \begin{bmatrix} R_1 & \cdots & R_N \end{bmatrix}.$$

Since the I_n's are disjoint, each column of L has exactly one non-zero entry. Similarly, each row of R has exactly one non-zero entry.

We then show that $[L](C_1) * [R](C_2) = [A](C_1 * C_2)$ for any copulas C_1 and C_2. For $m = 1, 2, \ldots, N$, denote by H_m the uniform distribution on $\left[\sum_{n=1}^{m-1} |A_n|, \sum_{n=1}^{m} |A_n|\right]$. So $H_m' = \frac{1}{|A_m|}$ on its support. For $u, v \in [0, 1]$,

$$\begin{aligned}
&[L](C_1) * [R](C_2)(u, v) \\
&= \int_0^1 \left(\sum_{i=1}^{k} \sum_{n=1}^{N} \lambda_{in} \partial_2 C_1 \left(F_i(u), H_n(t)\right) \right) \left(\sum_{j=1}^{\ell} \sum_{m=1}^{N} \rho_{mj} \partial_1 C_2 \left(H_m(t), G_j(v)\right) \right) dt \\
&= \sum_{i=1}^{k} \sum_{j=1}^{\ell} \left(\sum_{n=1}^{N} \frac{\lambda_{in}\rho_{nj}}{|A_n|} \right) C_1 * C_2 \left(F_i(u), G_j(v)\right) \qquad (4) \\
&= [A](C_1 * C_2)(u, v). \qquad (5)
\end{aligned}$$

It is left to verify that the sum over n in (4) is equal to the (i, j)th-element in A. Since $\sum_{n=1}^{N} A_n$ is a disjoint decomposition of $A = [a_{ij}]_{k\times\ell}$, Eq. (3) implies that if $(i, j) \in I_m \times J_m$ for some (unique) m then

$$a_{ij} = \frac{\text{the } (i, j)\text{th element of } L_m R_m}{|A_m|} = \frac{\lambda_{im}\rho_{mj}}{|A_m|} = \sum_{n=1}^{N} \frac{\lambda_{in}\rho_{nj}}{|A_n|},$$

where the last equality follows from the fact that $\lambda_{in}\rho_{nj} = 0$ if $(i, j) \notin I_n \times J_n$. It also clearly follows from this fact that if $(i, j) \notin I_m \times J_m$ for every m then $\sum_{n=1}^{N} \frac{\lambda_{in}\rho_{nj}}{|A_n|} = 0 = a_{ij}$.

By (5), a proof by induction on $r = 1, 2, \ldots$ yields $[L]^r(C_1)*[R]^r(C_2) = [A]\left([L]^{r-1}(C_1) * [R]^{r-1}(C_2)\right) = \cdots = [A]^r(C_1*C_2)$. So $[L]^r(E) * [R]^r(E) = [A]^r(E)$ for every $r \geq 1$ if E is an idempotent copula, e.g. $E = M$ or Π. By Lemma 5.6, $[L]^r(M)$ are left invertible copulas and $[R]^r(M)$ are right invertible copulas.

By Theorem 5.2, with respect to the modified Sobolev norm, $[L]^r(M)$, $[R]^r(M)$ and $[A]^r(M)$ converge respectively to a left invertible copula C_L, a right invertible copula C_R and a copula C_A satisfying $[L](C_L) = C_L$, $[R](C_R) = C_R$ and $[A](C_A) = C_A$. As a consequence of the joint continuity of the $*$-product with respect to $\|\cdot\|_S$, we obtain $C_L * C_R = C_A$ as desired. □

Example 6. Fix $r \in (0, 1/2)$ and consider the transformation matrix

$$\begin{bmatrix} r/2 & 0 & r/2 \\ 0 & 1-2r & 0 \\ r/2 & 0 & r/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1-2r & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} r/2 & 0 & r/2 \\ 0 & 0 & 0 \\ r/2 & 0 & r/2 \end{bmatrix}.$$

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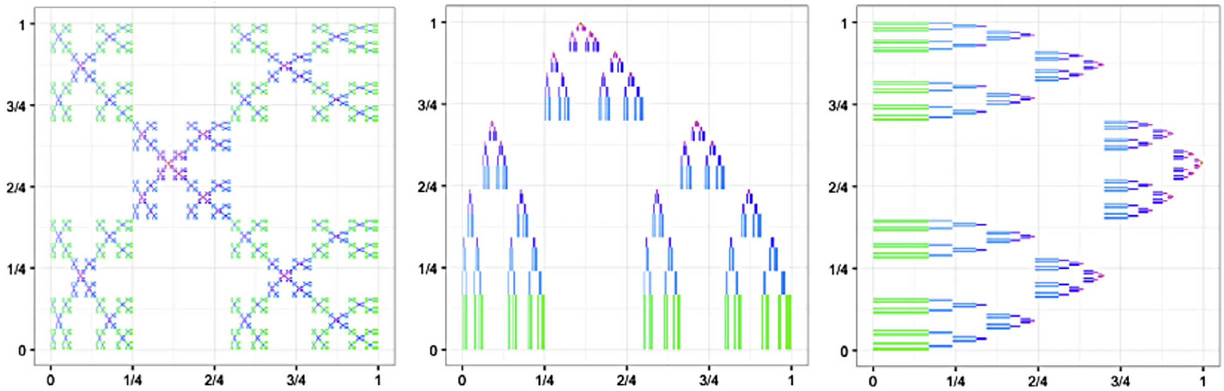

Fig. 4. Supports of $[A_3]^5(\Pi)$, $[L_3]^5(\Pi)$ and $[R_3]^5(\Pi)$ in Example 7.

Both matrices on the right hand side have rank one which implies by Theorem 5.7 that the invariant copula can be factored as the product of a left invertible copula and a right invertible copula. When $r = 1/3$, the transformation matrix is A_2 in Example 5. The copula factors, C_{L_2} and C_{R_2}, of C_{A_2} are shown (approximately) in Fig. 3, where $L_2 = \begin{bmatrix} 0 & 1/3 & 0 \\ 1/3 & 0 & 1/3 \end{bmatrix} = R_2^t$.

Example 7. Let us also consider the factorization of a non-symmetric invariant copula by first setting

$$K_3 = \begin{bmatrix} 3/28 & 0 & 27/140 \\ 0 & 0 & 0 \\ 1/7 & 0 & 9/35 \end{bmatrix}, \qquad L_3 = \begin{bmatrix} 0 & 3/10 & 0 \\ 1/4 & 0 & 9/20 \end{bmatrix} \quad \text{and} \quad R_3 = \begin{bmatrix} 3/10 & 0 \\ 0 & 3/10 \\ 4/10 & 0 \end{bmatrix}.$$

Then the non-symmetric transformation matrix $A_3 = K_0 + K_3$ is disjointly decomposable by 2 invariant pairs $(\{1, 3\}, \{1, 3\})$ and $(\{2\}, \{2\})$. Since K_0 and K_3 have rank one, it follows from Theorem 5.7 and its proof that $C_{A_3} = C_{L_3} * C_{R_3}$. Approximations of their supports are illustrated in Fig. 4.

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References

Chaidee, N., Santiwipanont, T., Sumetkijakan, S., 2016. Patched approximations and their convergence. Comm. Statist. Theory Methods 45, 2654–2664.
Darsow, W.F., Nguyen, B., Olsen, E.T., 1992. Copulas and Markov processes. Illinois J. Math. 36, 600–642.
Darsow, W.F., Olsen, E.T., 1995. Norms for copulas. Int. J. Math. Math. Sci. 18, 417–436.
Darsow, W.F., Olsen, E.T., 2010. Characterization of idempotent 2-copulas. Note Mat. 30, 147–177.
Durante, F., Sempi, C., 2015. Principles of Copula Theory. CRC Press.
Fredricks, G.A., Nelsen, R.B., Rodríguez-Lallena, J.A., 2005. Copulas with fractal supports. Insurance Math. Econom. 37, 42–48.
Kamnitui, N., Santiwipanont, T., Sumetkijakan, S., 2015. Dependence measuring from conditional variances. Depend. Model. 3, 98–112.
Kimeldorf, G., Sampson, A.R., 1978. Monotone dependence. Ann. Statist. 6, 895–903.
Mikusiński, P., Sherwood, H., Taylor, M.D., 1991. Probabilistic interpretations of copulas and their convex sums. In: Dall'Aglio, G., Kotz, S., Salinetti, G. (Eds.), Advances in Probability Distributions with Given Marginals: Beyond the Copulas. Kluwer Academic Publishers, Dordrecht, pp. 111–128.
Mikusiński, P., Sherwood, H., Taylor, M.D., 1992. Shuffles of min. Stochastica 13, 61–74.
Nelsen, R.B., 2006. An Introduction to Copulas, second ed. Springer-Verlag, New York.
Olsen, E.T., Darsow, W.F., Nguyen, B., 1996. Copulas and Markov operators. In: Rüschendorf, L., Schweizer, B., Taylor, M.D. (Eds.), Distributions with Fixed Marginals and Related Topics. In: Lecture Notes-Monograph Series, vol. 28. Institute of Mathematical Statistics, Hayward, CA, pp. 244–259.
Rényi, A., 1959. On measures of dependence. Acta Math. Acad. Sci. Hungar. 10, 441–451.
Royden, H.L., 1988. Real Analysis, third ed. Macmillan, New York.
Ruankong, P., Santiwipanont, T., Sumetkijakan, S., 2013. Shuffles of copulas and a new measure of dependence. J. Math. Anal. Appl. 398, 392–402.
Sakai, Y., Shimogaki, T., 1972. Equimeasurability of functions and doubly stochastic operators. Kodai Math. Sem. Rep. 24, 203–211.
Siburg, K.F., Stoimenov, P.A., 2009. A measure of mutual complete dependence. Metrika 71, 239–251.
Trutschnig, W., 2011. On a strong metric on the space of copulas and its induced dependence measure. J. Math. Anal. Appl. 384, 690–705.
Trutschnig, W., 2012. Idempotent copulas with fractal support. In: Greco, S., Bouchon-Meunier, B., Coletti, G., Fedrizzi, M., Matarazzo, B., Yager, R.R. (Eds.), Advances in Computational Intelligence, Vol. 298. Springer, Berlin, Heidelberg, pp. 161–170.
Trutschnig, W., Fernández-Sánchez, J., 2012. Idempotent and multivariate copulas with fractal support. J. Statist. Plann. Inference 142, 3086–3096.
Yanting, Z., Jingping, Y., Jianhua, Z.H., 2011. Approximation of bivariate copulas by patched bivariate Fréchet copulas. Insurance Math. Econom. 48, 246–256.