รายงานวิจัยฉบับสมบูรณ์

การวิเคราะห์ความเสถียรภาพโดยวิธีการเชิงสเกลาร์ไม่เชิงเส้นสำหรับปัญหาเชิงกึ่งดุลยภาพเวกเตอร์วางนัยทั่วไปในรูปแบบพารามิเตอร์และการประยุกต์สู่ปัญหาค่าเหมาะสมและปัญหาเครือข่ายการจราจร

The Stability Analysis via Nonlinear Scalarization Methods for Parametric Generalized Vector Quasiequilibrium Problems and Applications to Optimization Problems and Traffic Network Problems

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
และมหาวิทยาลัยนเรศวร

Executive Summary

Optimization models attempt to express, in mathematical terms, the goal of solving a problem in the "best" way. That might mean running a business to maximize profit, minimize loss, maximize efficiency, or minimize risk. It might mean designing a bridge to minimize weight or maximize strength. It might mean selecting a flight plan for an aircraft to minimize time or fuel use. The desire to solve a problem in an optimal way is so common that optimization models arise in almost every area of application. They have even been used to explain the laws of nature, as in Fermat's derivation of the law of refraction for light. Optimization models have been used for centuries, since their purpose is so appealing. In recent times they have come to be essential, as businesses become larger and more complicated, and as engineering designs become more ambitious. In many circumstances it is no longer possible, or economically feasible, for decisions to be made without the aid of such models. In a large, multinational corporation, for example, a minor percentage improvement in operations might lead to a multimillion dollar increase in profit, but achieving this improvement might require analyzing all divisions of the corporation, a gargantuan task. Likewise, it would be virtually impossible to design a new computer chip involving millions of transistors without the aid of such models

In portfolio optimization, for example, we seek the best way to invest some capital in a set of n assets. The variable represents the investment in the th asset, so the vector describes the overall portfolio allocation across the set of assets. The constraints might represent a limit on the budget (i.e., a limit on the total amount to be invested), the requirement that investments are nonnegative (assuming short positions are not allowed), and a minimum acceptable value of expected return for the whole portfolio. The objective or cost function might be a measure of the overall risk or variance of the portfolio return. In this case, the optimization problem corresponds to choosing a portfolio allocation that minimizes risk, among all possible allocations that meet the firm requirements.

Another example is device sizing in electronic design, which is the task of choosing the width and length of each device in an electronic circuit. Here the variables represent the widths and lengths of the devices. The constraints represent a variety of engineering requirements, such as limits on the device sizes imposed by the manufacturing process, timing requirements that ensure that the circuit can operate reliably at a specified speed, and a limit on the total area of the circuit. A common objective in a device sizing problem is the total power consumed by the circuit. The optimization problem is to find the device sizes that satisfy the design requirements (on manufacturability, timing, and area) and are most power efficient. In data fitting, the task is to find a model, from a family of potential models, that best fits some observed data and prior information. Here the variables are the parameters in the model, and the constraints can represent prior information or required limits on the parameters (such as nonnegativity). The objective function might be a measure of misfit or prediction error between the observed data and the values predicted by the model, or a statistical measure of the unlikeliness or implausibility of the parameter values. The optimization problem is to find the model parameter values that are consistent with the prior information, and give the smallest misfit or prediction error with the observed data (or, in a statistical framework, are most likely). An amazing variety of practical problems involving decision making (or system design, analysis, and operation) can be cast in the form of a mathematical optimization problem, or some variation such as a multicriterion optimization problem. Indeed, mathematical optimization has become an important tool in many areas. It is widely used in engineering, in electronic design automation, automatic control systems, and optimal design problems arising in civil, chemical, mechanical, and aerospace engineering. Optimization is used for problems arising in network design and operation, finance, supply chain management, scheduling, and many other areas. The list of applications is still steadily expanding. For most of these applications, mathematical optimization is used as an aid to a human decision maker, system designer, or system operator, who supervises the process, checks the results, and modifies the problem (or the solution approach) when necessary. This human decision maker also carries out any actions suggested by the optimization problem, e.g., buying or selling assets to achieve the optimal portfolio.

Consider a traffic system with several cities and many roads connecting them. Suppose that the technical conditions (capacity and quality of roads, etc.) are established. Assume that we know the demands for transportation of some kind of materials or goods between each pair of two cities. The system is well functioning if all these demands are satisfied. The aim of the owner of the network is to keep the system well functioning. The users (drivers, passengers, etc.) do not behave blindly. To go from A to B they will choose one of the roads leading them from A to B with the minimum cost. This natural law is known as the user-optimizing principle or the Wardrop principle. The traffic flow satisfying demands and this law is said to be an equilibrium flow of the network. By using this principle, in most of the cases, the owner can compute or estimate the traffic flow on every road. The owner can affect on the network, for example, by requiring high fees from the users of the good roads to force them to use also some roads of lower quality. In this way, a new equilibrium flow, which is more suitable in the opinion of the owner, can be reached.

In the theory of stability and sensitivity analysis for optimization-related problems Hölder continuity of solutions plays an important role although there may be less works in the literature devoted to this property than to semicontinuity, continuity, Lipschitz continuity and (generalized) differentiability. The Lipschitz continuity of a function is more satisfactory than the continuity, since a Lipschitz function changes its values at a linear rate with respect to the change of its variables and of course is also continuous. On the other hand, Lipschitz continuity is close to differentiability by the well-known Rademacher theorem, which says that a locally Lipschitz functions on a finite dimensional space is Fréchet differentiable almost everywhere. Local Hölder continuity of degree is weaker than local Lipschitz continuity but stronger than continuity. However, as we will see in general the solution set of a quasiequilibrium problem is only Hölder continuous of degree $\alpha < 1$ although the data of the problem are Lipschitz continuous. For variational inequalities, Yen established sufficient conditions for the solution to be unique and Hölder continuous in Hilbert spaces. The subtle technique used there is with a heavy recourse to properties of metric projections in Hilbert spaces and linearity of the canonical pair involved in the variational inequality setting. Subsequently, this result is successfully generalized to various extends for equilibrium problems in metric spaces, These works constitute also a considerable contribution to the stability study for equilibrium problems, since this research field is rather new. Beside them we observe only which are devoted to various kinds of semicontinuity of solution sets. It is known that equilibrium problems were proposed in as a generalization of variational inequalities and optimization problems and include also many optimization-related problems like the fixed-point and coincidence-point problems, the complementarity problem, the traffic equilibria, the Nash equilibrium. However, in variational inequalities and equilibrium problems, the constraint sets are fixed and hence these mathematical models cannot be employed for problem settings in a number of practical situations. This was first observed by Bensoussan where the authors considered random impulse control problems and needed to use constraint sets depending on the state variables. Formulating these problems similarly as variational inequalities led to quasivariational inequalities. Nevertheless, the constraint set of the considered quasivariational inequality expresses the fulfillment of the travel demands in the traffic network and hence bears intrinsic linearity. Also, the quasivariational inequality possesses a linear nature due to the canonical pairinvolved in the problem setting. The sophisticated reasoning in Ait Mansour,M., Scrimali'results, based on these specific features of the quasivariational inequality under consideration, cannot be adapted when dealing with the generalized problem which is the parametric equilibrium problem.

บทคัดย่อ

ในงานวิจัยนี้ เราได้แนะนำการวางปัญหาที่ดีแล้วสำหรับปัญหาผนวกเชิงกึ่งแปรผันแบบทั่วไป (QVIP) นอกจากนั้นแล้วเรายังได้แนะนำการวางปัญหาที่ดีแล้วสำหรับปัญหาผนวกเชิงกึ่งแปรผันวางนัยทั่วไป และศึกษาถึงคุณลักษณะของการวางปัญหาที่ดีแล้ว ภายใต้เงื่อนไขที่เหมาะสมบางอย่างเราได้พิสูจน์ว่า การวางปัญหาที่ดีแล้วสมมูลกับการมีอยู่จริงของผลเฉลยเดียว ในส่วนของบทประยุกต์ เราได้ศึกษาการวางปัญหาที่ดีแล้วสำหรับสำหรับปัญหาผนวกเชิงกึ่งแปรผันอิงตัวแปรเสริม ปัญหากึ่งเชิงดุยภาพ

Abstract

In this project, we aim to suggest the new concept of well-posedness for the general parametric quasivariational inclusion problems (QVIP, for short). The corresponding concepts of well-posedness in the generalized sense are also introduced and investigated for (QVIP). Some metric characterizations of well-posedness for (QVIP) are given. We prove that under suiable conditions, the well-posedness is equivalent to the existence of uniqueness of solutions. As applications, we obtain immediately some results of well-posedness for the parametric quasivariational inclusion problems, parametric vector quasiequilibrium problems and parametric quasiequilibrium problems.

Contents

Chapter 1

Main Results

1.1 On the Hölder Continuity of Solution Maps to Parametric Generalized Vector Quasi-Equilibrium Problems via Nonlinear Scalarization

The generalized vector quasi-equilibrium problem is an unified model of several problems, namely, generalized vector quasi-variational inequalities, vector quasi-optimization problems, traffic network problems, fixed point and coincidence point problems, etc, see, for example [40, 41] and the references therein. It is well known that stability analysis of solution mapping for equilibrium problems is an important topic in optimization theory and applications. Stability may be understood as lower or upper semicontinuity, continuity, and Lipschitz or Hölder continuity. There have been many papers to discuss the stability of solution mapping for equilibrium problems when they are perturbed by parameters (also known the parametric (generalized) equilibrium problems). Last decade, many authors have intensively studied the sufficient conditions of upper (lower) semicontinuity of various solution mappings for parametric (generalized) equilibrium problems, see [42, 43, 44, 45, 46, 47, 48]. Let's begin now, Yen [49] obtained the Hölder continuity of the unique solution of a classic perturbed variational inequality by the metric projection method. Mansour and Riahi [50] proved the Hölder continuity of the unique solution for a parametric equilibrium problem under the concepts of strong monotonicity and Hölder continuity. Bianchi and Pini [51] introduced the concept of strong pseudomonotonicity and got the Hölder continuity of the unique solution of a parametric equilibrium problem. Anh and Khanh [52] generalized the main results of [12] to two classes of perturbed generalized equilibrium problems with set-valued mappings. Anh and Khanh [53] further discussed uniqueness and Hölder continuity of the solutions for perturbed equilibrium problems with set-valued mappings. Anh and Khanh [54] extended the results of [53] to the case of perturbed quasi-equilibrium problems with set-valued mappings and obtained the Hölder continuity of the unique solutions. Li et al. [55] introduced an assumption, which is weaker than the corresponding ones of [52], and established the Hölder continuity of the set-valued solution mappings for two classes of parametric generalized vector quasi-equilibrium problems in general metric spaces.

Among many approaches for dealing with the lower semicontinuity, continuity and Hölder continuity of the solution mapping for a parametric vector equilibrium problem in general metric spaces, the scalarization method is of considerable interest. The classical scalarization method using linear functionals has been already used for studying the lower semicontinuity of the solution mapping [57, 58, 59] and the Hölder continuity of the solution mapping to parametric vector equilibrium problems. Wang et al. [61] established the lower semicontinuity and upper semicontinuity of the solution set to a parametric generalized strong vector equilibrium problem by using a scalarization method and a density result. Recently, by using this method, Peng established the sufficient conditions for the Hölder continuity of the solution mapping to a parametric generalized vector quasi-equilibrium problem with set-valued mappings.

On the other hand, a useful approach for analyzing a vector optimization problem is to reduce it to a scalar optimization problem. Nonlinear scalarization functions play an important role in this reduction in the context of nonconvex vector optimization problems. The nonlinear scalarization function ξ_q commonly known as the Gerstewitz function in the theory of vector optimization, have been also used to studying the lower semicontinuity of the set-valued solution mapping to a parametric vector variational inequality [65]. Using this method, Bianchi and Pini [66] obtained the Hölder continuity of the single-valued solution mapping to a parametric vector equilibrium problem. Recently, Chen and Li studied Hölder continuity of the solution mapping for both set-valued and single-valued cases to parametric vector equilibrium problems. The key role in their paper is globally Lipschitz property of the Gerstewitz function. Very recently, by using the idea in Peng and Chen'results obtained Hölder continuity of the unique solution to a parametric vector quasi-equilibrium problem based on nonlinear scalarization approach, under three different kinds of monotonicity hypotheses. It is natural to raise and give an answer to the following question :
Question : Can one establish the Hölder continuity of a solution mapping to the parametric generalized vector quasi-equilibrium problem with set-valued mappings, by using a nonlinear scalarization method ?

Motivated and inspired by Peng and Chen and researches going on this direction, the aim of this paper is to give positive answers to the above question. We first establish the sufficient conditions which guarantee the Hölder continuity of a solution mapping to the parametric generalized vector quasi-equilibrium problem with set-valued mappings, by using a nonlinear scalarization method. We further study several kinds of the monotonicity conditions to obtain the Hölder continuity of the solution mapping. The main results of this paper are different from corresponding results in Peng and Chen' results. These results are improve the corresponding ones in recent literature.

Throughout the paper, unless otherwise specified, we denote by $\|\cdot\|$ and $d(.,.)$ the norm and the metric on a normed space and a metric space, respectively. A closed ball with centre $0 \in X$ and radius $\delta > 0$ is denoted by $B(0,\delta)$. We always consider X, Λ, M as metric spaces, and Y as a linear normed space with its topological dual space Y^*. For any $y^* \in Y^*$, we define $\|y^*\| := \sup\{\|\langle y^*, y\rangle\| : \|y\| = 1\}$, where $\langle y^*, y\rangle$ denotes the value of y^* at y. Let $C \subset Y$ be a pointed, closed and convex cone with int $C \neq \emptyset$, where int$C$ stands for the interior of C. Let

$$C^* := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0,\ \forall y \in C\}$$

be the dual cone of C. Since int $C \neq \emptyset$, the dual cone C^* of C has a weak* compact base. Let $e \in$ int C. Then,

$$B_e^* := \{y^* \in C^* : \langle y^*, e\rangle = 1\}$$

is a weak* compact base of C^*. Clearly, C^q is a weak*-compact base of C^*, that is, C^q is convex and weak*-compact such that $0 \notin C^q$ and $C^* = \bigcup_{t\geq 0} tC^q$.

Let $q \in$ int C, the *nonlinear scalarization function* [?, ?] $\xi_q : Y \to \mathbb{R}$ is defined by

$$\xi_q = \min\{t \in \mathbb{R} : y \in tq - C\}.$$

It is well known that ξ_q is a continuous, positively homogeneous, subadditive and convex function on Y, and it is monotone (that is, $y_2 - y_1 \in C \Rightarrow \xi_q(y_1) \leq \xi_q(y_2)$) and strictly monotone (that is, $y_2 - y_1 \in -$ int $C \Rightarrow \xi_q(y_1) < \xi_q(y_2)$) (see [?, ?]). In case, $Y = R^l$, $C = R^l_+$ and

$q = (1, 1, \dots, 1) \in$ int R^l_+, the nonlinear scalarization function can be expressed in the following equivalent form [?, Corollary 1.46]:

$$\xi_q(y) = \max_{1\le i\le l}\{y_i\}, \qquad \forall y = (y_1, y_2, \dots, y_l) \in \mathbb{R}^l. \tag{1.1.1}$$

Lemma 1. *For any fixed $q \in$ int C, $y \in Y$ and $r \in \mathbb{R}$,*

1. $\xi_q < r \Leftrightarrow y \in rq -$ int C *(that is, $\xi_q(y) \ge r \Leftrightarrow y \notin rq -$ int C);*
2. $\xi_q(y) \le r \Leftrightarrow y \in rq - C$*;*
3. $\xi_q(y) = r \Leftrightarrow y \in rq - \partial C$*, where ∂C denotes the boundary of C;*
4. $\xi_q(rq) = r$*.*

The property (i) of Lemma 1 plays an essential role in scalarization. From the definition of ξ_q, property (iv) in Lemma 1 could be strengthened as

$$\xi_q(y + rq) = \xi_q(y) + r, \qquad \forall y \in Y,\ r \in \mathbb{R}. \tag{1.1.2}$$

For any $q \in$ int C, the set C^q defined by

$$C^q := \{y^* \in C^* : \langle y^*, q\rangle = 1\}$$

is a weak*-compact set of Y^* (see [19, Lemma 5.1]). The following equivalent form of ξ_q can be deduced from [?, Corollary 2.1] or [?, Proposition 2.2] ([?, Proposition 1.53]).

Proposition 2. *Let $q \in intC$. Then for $y \in Y$, $\xi_q(y) = \max_{y^*\in C^q}\langle y^*, y\rangle$.*

Proposition 3. *ξ_q is Lipschitz on Y, and its Lipschitz constant is*

$$L := \sup_{y^*\in C^q}\|y^*\| \in \left[\frac{1}{\|q\|}, +\infty\right).$$

The following example can be found in [?, Example 2.1].

Example 4.

1. If $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, then the Lipschitz constant of ξ_q is $L = \frac{1}{q}(q > 0)$. Indeed, $|\xi_q(x) - \xi_q(y)| = \frac{1}{q}|x - y|$, for all $x, y \in \mathbb{R}$.
2. If $Y = \mathbb{R}^2$ and $C = \left\{(y_1, y_2) \in \mathbb{R}^2 : \frac{1}{4}y_1 \le y_2 \le 2y_1\right\}$. Take $q = (2, 3) \in$ int$C$, then,

$$C^q := \{(y_1, y_2) \in \mathbb{R} : 2y_1 + 3y_2 = 1,\ y_1 \in [-0.1, 2]\},$$

and the Lipschitz constant is $L = \sup_{y^*\in C^q}\|y^*\| = \|(-2, 1)\| = \sqrt{5}$. Hence,

$$|\xi_q(y) - \xi_q(y')| \le \sqrt{5}\|y - y'\|, \qquad \forall y, y' \in \mathbb{R}^2.$$

Now we recall some basic definitions and their properties which will be used in the sequel.

Definition 5 (Classical Notion)**.** Let $l \ge 0$ and $\alpha > 0$. A set-valued mapping $G : \Lambda \to 2^X$ is said to be *$l.\alpha$-Hölder continuous at λ_0 on a neighborhood $N(\lambda_0)$* of λ_0 if and only if

$$G(\lambda_1) \subseteq G(\lambda_2) + lB_X(0, d^\alpha(\lambda_1, \lambda_2)), \qquad \forall \lambda_1, \lambda_2 \in N(\lambda_0). \tag{1.1.3}$$

When X is a normed space, we say that the vector-valued mapping $g : \Lambda \to X$ is $l.\alpha$-Hölder continuous at λ_0 on a neighborhood $N(\lambda_0)$ of λ_0 iff,

$$\|g(\lambda_1) - g(\lambda_2)\| \leq l d^{\alpha}(\lambda_1, \lambda_2), \quad \forall \lambda_1, \lambda_2 \in N(\lambda_0).$$

Definition 6. Let $l_1, l_2 \geq 0$ and $\alpha_1, \alpha_2 > 0$. A set-valued mapping $G : X \times \Lambda \to 2^X$ is said to be *($l_1.\alpha_1, l_2.\alpha_2$)-Hölder continuous at x_0, λ_0 on a neighborhood* $N(x_0)$ and $N(\lambda_0)$ of x_0 and λ_0 if and only if

$$G(x_1, \lambda_1) \subseteq G(x_2, \lambda_2) + (l_1 d_X^{\alpha_1}(x_1, x_2) + l_2 d_\Lambda^{\alpha_2}(\lambda_1, \lambda_2))\, B_X(0, 1), \ \forall x_1, x_2 \in N(x_0), \ \forall \lambda_1, \lambda_2 \in N(\lambda_0). \tag{1.1.4}$$

By using a nonlinear scalarization technique, we present the sufficient conditions for Hölder continuity of the solution mapping for a parametric generalized vector quasi-equilibrium problem.

Let $N(\lambda_0) \subset \Lambda$ and $N(\mu_0) \subset M$ be neighborhoods of λ_0 and μ_0, respectively, and $K : X \times \Lambda \to 2^X$ and $F : X \times X \times M \to 2^Y$ be set-valued mappings. For each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, we consider the following parametric generalized vector quasi-equilibrium problem (**PGVQEP**):
Find $x_0 \in K(x_0, \lambda)$ such that

$$F(x_0, y, \mu) \subset Y \backslash (-\operatorname{int} C), \quad \forall y \in K(x_0, \lambda). \tag{1.1.5}$$

For each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, let

$$E(\lambda) := \{x \in X | x \in K(x, \lambda)\}.$$

The weak solution set of (1.1.5) is denoted by

$$S_W(\lambda, \mu) := \{x \in E(\lambda) : F(x, y, \mu) \subset Y \backslash (-\operatorname{int} C), \ \forall y \in K(x, \lambda)\}.$$

For each $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $q \in \operatorname{int} C$, the ξ_q-solution set of (1.1.5) is denoted by

$$S(\xi_q, \lambda, \mu) := \left\{ x \in E(\lambda) : \inf_{z \in F(x,y,\mu)} \xi_q(z) \geq 0, \ \forall y \in K(x, \lambda) \right\}.$$

We first establish the following lemmas which will be used in the sequel.

Lemma 7. *For each $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $q \in$ int C,*

$$S_W(\lambda, \mu) = S(\xi_q, \lambda, \mu).$$

Proof. Let $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $q \in \operatorname{int} C$. For any $x \in S_W(\lambda, \mu)$, we have

$$x \in E(\lambda) \text{ and } F(x, y, \mu) \subset Y \backslash (-\operatorname{int} C), \quad \forall y \in K(x, \lambda).$$

Therefore, for each $y \in K(x, \lambda)$ and each $z \in F(x, y, \mu)$, we have

$$z \notin -\operatorname{int} C = 0q - \operatorname{int} C.$$

By Lemma 1 (i), we conclude that $\xi_q(z) \geq 0$. Since z is arbitrary, we have

$$\inf_{z \in F(x,y,\mu)} \xi_q(z) \geq 0, \text{ for all } y \in K(x, \lambda),$$

which gives that $S_W(\lambda,\mu) \subseteq S(\xi_q,\lambda,\mu)$.

On the other hand, for each $x \in S(\xi_q,\lambda,\mu)$, we have that

$$x \in E(\lambda) \quad \text{and} \quad \inf_{z\in F(x,y,\mu)} \xi_q(z) \geq 0, \quad \forall y \in K(x,\lambda). \tag{1.1.6}$$

Thus, for each $y \in K(x,\lambda)$ and each $z \in F(x,y,\mu)$, we have that $\xi_q(z) \geq 0$. By Lemma 1 (i), we can obtain $z \notin -\text{ int } C$. Therefore, we have $z \in Y\backslash(-\text{ int } C)$, which implies that

$$x \in E(\lambda) \text{ and } F(x,y,\mu) \subset Y\backslash(-\text{ int } C), \quad \forall y \in K(x,\lambda).$$

Hence, $S(\xi_q,\lambda,\mu) \subseteq S_W(\lambda,\mu)$. The proof is completed. □ □

Lemma 8. *Suppose that $N(\lambda_0)$ and $N(\mu_0)$ are the given neighbourhoods of λ_0 and μ_0, respectively.*

1. *If for each $x, y \in E(N(\lambda_0))$, $F(x,y,\cdot)$ is $m_1.\gamma_1$-Hölder continuous at $\mu_0 \in M$, then for any fixed $q \in$ int C, the function*

$$\psi_{\xi_q}(x,y,\cdot) = \inf_{z\in F(x,y,\cdot)} \xi_q(z)$$

   is $Lm_1.\gamma_1$-Hölder continuous at μ_0.

2. *If for each $x \in E(N(\lambda_0))$ and $\mu \in N(E(\mu_0))$, $F(x,\cdot,\mu)$ is $m_2.\gamma_2$-Hölder continuous on $E(N(\lambda_0))$, then for any fixed $q \in$ int C, the function*

$$\psi_{\xi_q}(x,\cdot,\mu) = \inf_{z\in F(x,\cdot,\mu)} \xi_q(z)$$

   is $Lm_2.\gamma_2$-Hölder continuous on $E(N(\lambda_0))$.

Proof. (a) Let $x, y \in E(N(\lambda_0))$. The $m_1.\gamma_1$-Hölder continuity of $F(x,y,\cdot)$ implies that there exists a neighbourhood $N(\mu_0)$ of μ_0 such that for all $\mu_1,\mu_2 \in N(\mu_0)$,

$$F(x,y,\mu_1) \subset F(x,y,\mu_2) + m_1 d_M^{\gamma_1}(\mu_1,\mu_2)B_Y.$$

So, for any $z_1 \in F(x,y,\mu_1)$, there exist $z_2 \in F(x,y,\mu_2)$ and $e \in B_Y$ such that

$$z_1 = z_2 + m_1 d_M^{\gamma_1}(\mu_1,\mu_2)e.$$

By using Proposition 3, we obtain

$$\begin{aligned} |\xi_q(z_1) - \xi_q(z_2)| &\leq L\|z_1 - z_2\| \\ &= Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2)\|e\| \\ &\leq Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2), \end{aligned} \tag{1.1.7}$$

which gives that

$$-Lm_1 d^{\gamma_1}(\mu_1,\mu_2) \leq \xi_q(z_1) - \xi_q(z_2).$$

Since z_1 is arbitrary and $\xi_q(z_2) \geq \inf_{z\in F(x,y,\mu_2)} \xi_q(z)$, we have

$$-Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2) \leq \inf_{z\in F(x,y,\mu_1)} \xi_q(z) - \inf_{z\in F(x,y,\mu_2)} \xi_q(z).$$

Applying the symmetry between μ_1 and μ_2, we arrive that

$$-Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2) \le \inf_{z\in F(x,y,\mu_2)} \xi_q(z) - \inf_{z\in F(x,y,\mu_1)} \xi_q(z).$$

It follows from the last two inequalities that

$$|\psi_{\xi_q}(x,y,\mu_1) - \psi_{\xi_q}(x,y,\mu_2)| \le Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2), \quad \forall \mu_1,\mu_2 \in N(\mu_0).$$

Therefore, we conclude that $\psi_{\xi_q}(x,y,\cdot) = \inf_{z\in F(x,y,\cdot)} \xi_q(z)$ is $Lm_1.\gamma_1$-Hölder continuous at μ_0.

(b) It follows by the similar argument as in Part (a). The proof is completed. □ □

Now, by using the nonlinear scalarization technique, we propose some sufficient conditions for Hölder continuity of the solution mapping for (PGVQEP).

Theorem 9. *For each fixed $q \in$ int C, let $S(\xi_q,\lambda,\mu)$ be nonempty in a neighbourhood $N(\lambda_0)\times N(\mu_0)$ of $(\lambda_0,\mu_0) \in \Lambda \times M$. Assume that the following conditions hold.*

1. *$K(\cdot,\cdot)$ is $(l_1.\alpha_1, l_2.\alpha_2)$-Hölder continuous on $E(N(\lambda_0)) \times N(\lambda_0)$;*

2. *For each $x,y \in E(N(\lambda_0))$, $F(x,y,\cdot)$ is $m_1.\gamma_1$-Hölder continuous at $\mu_0 \in M$;*

3. *For each $x \in E(N(\lambda_0))$ and $\mu \in N(\mu_0)$, $F(x,\cdot,\mu)$ is $m_2.\gamma_2$-Hölder continuous on $E(N(\lambda_0))$;*

4. *$F(\cdot,\cdot,\mu)$ is $h.\beta$-Hölder strongly monotone with respect to ξ_q, that is, there exist constants $h>0, \beta>0$ such that for every $x,y \in E(N(\lambda_0))$, $x\neq y$,*

$$h d_X^{\beta}(x,y) \le d\left(\inf_{z\in F(x,y,\mu)} \xi_q(z), \mathbb{R}_+\right) + d\left(\inf_{z\in F(y,x,\mu)} \xi_q(z), \mathbb{R}_+\right);$$

5. *$\beta = \alpha_1\gamma_2, h > 2m_2 L l_1^{\gamma_1}$, where $L := \sup_{\lambda\in C^q} \|\lambda\| \in \left[\frac{1}{\|q\|}, +\infty\right)$ is the Lipschitz constant of ξ_q on Y.*

Then, for every $(\lambda,\mu) \in N(\lambda_0)\times N(\mu_0)$, the solution $x(\lambda,\mu)$ of (PVQGEP) *is unique, and $x(\lambda,\mu)$ as a function of λ and μ satisfies the Hölder condition: for all $(\lambda_1,\mu_1),(\lambda_2,\mu_2) \in N(\lambda_0)\times N(\mu_0)$*

$$d_X(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2)) \le \left(\frac{2m_2 L l_2^{\gamma_2}}{h-2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2/\beta}(\lambda_1,\lambda_2) + \left(\frac{m_1 L}{h-2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2),$$

where $x(\lambda_i,\mu_i) \in S_W(\lambda_i,\mu_i)$, $i=1,2$.

Proof. Let $(\lambda_1,\mu_1),(\lambda_2,\mu_2) \in N(\lambda_0)\times N(\mu_0)$. The proof is divided to the following three steps based on the fact that

$$d_X(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2)) \le d_X(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)) + d_X(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)).$$

where $x(\lambda_i,\mu_i) \in S_W(\lambda_i,\mu_i), i=1,2.$

Step 1: We prove that

$$d_1 := d_X(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)) \le \left(\frac{m_1 L}{h-2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2), \tag{1.1.8}$$

for all $x(\lambda_1,\mu_1)\in S_W(\lambda_1,\mu_1)$ and $x(\lambda_1,\mu_2)\in S_W(\lambda_1,\mu_2)$.

If $x(\lambda_1,\mu_1)=x(\lambda_1,\mu_2)$, then we are done. So, we assume that $x(\lambda_1,\mu_1)\neq x(\lambda_1,\mu_2)$. Since $x(\lambda_1,\mu_1)\in K(x(\lambda_1,\mu_1),\lambda_1)$ and $x(\lambda_1,\mu_2)\in K(x(\lambda_1,\mu_2),\lambda_1)$, by the $l_1.\alpha_1$-Hölder continuity of $K(\cdot,\lambda_1)$, there exist $x_1\in K(x(\lambda_1,\mu_1),\lambda_1)$ and $x_2\in K(x(\lambda_1,\mu_2),\lambda_1)$ such that

$$d_X(x(\lambda_1,\mu_1),x_2)\le l_1 d_X^{\alpha_1}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))=l_1 d_1^{\alpha_1}, \tag{1.1.9}$$

and

$$d_X(x(\lambda_1,\mu_2),x_1)\le l_1 d_X^{\alpha_1}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))=l_1 d_1^{\alpha_1}. \tag{1.1.10}$$

Since $x(\lambda_1,\mu_1)\in S_W(\lambda_1,\mu_1)$ and $x(\lambda_1,\mu_2)\in S_W(\lambda_1,\mu_2)$, by Lemma 7, we obtain

$$\psi_{\xi_q}(x(\lambda_1,\mu_1),x_1,\mu_1):=\inf_{z\in F(x(\lambda_1,\mu_1),x_1,\mu_1)}\xi_q(z)\ge 0 \tag{1.1.11}$$

and

$$\psi_{\xi_q}(x(\lambda_1,\mu_2),x_2,\mu_2):=\inf_{z\in F(x(\lambda_1,\mu_2),x_2,\mu_2)}\xi_q(z)\ge 0. \tag{1.1.12}$$

By the virtue of (iv), we have

$$\begin{aligned} hd_1^{\beta} &= hd_X^{\beta}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))\\ &\le d(\psi_{\xi_q}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1),\mathbb{R}_+)+d(\psi_{\xi_q}(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_1),\mathbb{R}_+). \end{aligned}$$

By combining (1.1.11) and (1.1.12) with the last inequality, we have

$$\begin{aligned} hd_1^{\beta} &\le |\psi_{\xi_q}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1)-\psi_{\xi_q}(x(\lambda_1,\mu_1),x_1,\mu_1)|\\ &\quad+|\psi_{\xi_q}(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_1)-\psi_{\xi_q}(x(\lambda_1,\mu_2),x_2,\mu_2)|\\ &\le |\psi_{\xi_q}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1)-\psi_{\xi_q}(x(\lambda_1,\mu_1),x_1,\mu_1)|\\ &\quad+|\psi_{\xi_q}(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_1)-\psi_{\xi_q}(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2)|\\ &\quad+|\psi_{\xi_q}(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2)-\psi_{\xi_q}(x(\lambda_1,\mu_2),x_2,\mu_2)|\\ &\le Lm_2 d_X^{\gamma_2}(x(\lambda_1,\mu_2),x_1)+Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2)+Lm_2 d_X^{\gamma_2}(x(\lambda_1,\mu_1),x_2)\\ &\le Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))+Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2)+Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))\\ &= 2Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))+Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2). \end{aligned} \tag{1.1.13}$$

Whence, the assumption (iv) implies that

$$d_X(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))\le\left(\frac{Lm_1}{h-Lm_2 l_1^{\gamma_2}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2).$$

Step 2: We prove that

$$d_2:=d_X(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2))\le\left(\frac{2Lm_2 l_2^{\gamma_2}}{h-2Lm_2 l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2/\beta}(\lambda_1,\lambda_2), \tag{1.1.14}$$

for all $x(\lambda_1,\mu_2)\in S_W(\lambda_1,\mu_2)$ and $x(\lambda_2,\mu_2)\in S_W(\lambda_2,\mu_2)$.

If $x(\lambda_1,\mu_2)=x(\lambda_2,\mu_2)$, then we are done. So, we assume that $x(\lambda_1,\mu_2)\neq x(\lambda_2,\mu_2)$. Since $x(\lambda_1,\mu_2)\in K(x(\lambda_1,\mu_2),\lambda_1)$ and $x(\lambda_2,\mu_2)\in K(x(\lambda_2,\mu_2),\lambda_2)$, by the $l_2.\alpha_2$-Hölder continuity of $K(x(\lambda_1,\mu_2),\cdot)$ and $K(x(\lambda_2,\mu_2),\cdot)$, there exist $x_1'\in K(x(\lambda_2,\mu_2),\lambda_1)$ and $x_2'\in K(x(\lambda_1,\mu_2),\lambda_2)$ such that

$$d_X(x(\lambda_1,\mu_2),x_2')\le l_2 d_\Lambda^{\alpha_2}(\lambda_1,\lambda_2) \tag{1.1.15}$$

and

$$d_X(x(\lambda_2,\mu_2),x_1') \leq l_2 d_\Lambda^{\alpha_2}(\lambda_1,\lambda_2). \tag{1.1.16}$$

Again, by the Hölder continuity of $K(\cdot,\cdot)$, there exist $x_1'' \in K(x(\lambda_1,\mu_2),\lambda_1)$ and $x_2'' \in K(x(\lambda_2,\mu_2),\lambda_2)$ such that

$$d_X(x_1',x_1'') \leq l_1 d_X^{\alpha_1}(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)) = l_1 d_2^{\alpha_1}, \tag{1.1.17}$$

and

$$d_X(x_2',x_2'') \leq l_1 d_X^{\alpha_1}(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)) = l_1 d_2^{\alpha_1}. \tag{1.1.18}$$

Since $x(\lambda_1,\mu_2) \in S_W(\lambda_1,\mu_2)$ and $x(\lambda_2,\mu_2) \in S_W(\lambda_2,\mu_2)$, by Lemma 7, we obtain the following

$$\psi_{\xi_q}(x(\lambda_1,\mu_2),x_1'',\mu_2) := \inf_{z\in F(x(\lambda_1,\mu_2),x_1'',\mu_2)} \xi_q(z) \geq 0, \tag{1.1.19}$$

and

$$\psi_{\xi_q}(x(\lambda_2,\mu_2),x_2'',\mu_2) := \inf_{z\in F(x(\lambda_2,\mu_2),x_2'',\mu_2)} \xi_q(z) \geq 0. \tag{1.1.20}$$

By virtue of (iv), we have

$$\begin{aligned} hd_2^\beta &= hd_X^\beta(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)) \\ &\leq d(\psi_{\xi_q}(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2),\mathbb{R}_+) + d(\psi_{\xi_q}(x(\lambda_2,\mu_2),x(\lambda_1,\mu_2),\mu_2),\mathbb{R}_+). \end{aligned}$$

By combining (1.1.19) and (1.1.20) with the last inequality, we have

$$\begin{aligned} hd_2^\beta &\leq |\psi_{\xi_q}(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2) - \psi_{\xi_q}(x(\lambda_1,\mu_2),x_1'',\mu_2)| \\ &\quad +|\psi_{\xi_q}(x(\lambda_2,\mu_2),x(\lambda_1,\mu_2),\mu_2) - \psi_{\xi_q}(x(\lambda_2,\mu_2),x_2'',\mu_2)| \\ &\leq |\psi_{\xi_q}(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2) - \psi_{\xi_q}(x(\lambda_1,\mu_2),x_1',\mu_2)| \\ &\quad +|\psi_{\xi_q}(x(\lambda_1,\mu_2),x_1',\mu_2) - \psi_{\xi_q}(x(\lambda_1,\mu_2),x_1'',\mu_2)| \\ &\quad |\psi_{\xi_q}(x(\lambda_2,\mu_2),x(\lambda_1,\mu_2),\mu_2) - \psi_{\xi_q}(x(\lambda_2,\mu_2),x_2',\mu_2)| \\ &\quad +|\psi_{\xi_q}(x(\lambda_2,\mu_2),x_2',\mu_2) - \psi_{\xi_q}(x(\lambda_2,\mu_2),x_2'',\mu_2)| \\ &\leq Lm_2 d_X^{\gamma_2}(x(\lambda_2,\mu_2),x_1') + Lm_2 d_X^{\gamma_2}(x_1',x_1'') + Lm_2 d_X^{\gamma_2}(x(\lambda_1,\mu_2),x_2',) + Lm_2 d_X^{\gamma_2}(x_2',x_2'') \end{aligned} \tag{1.1.21}$$

By virtue of (1.1.15), (1.1.16), (1.1.17) and (1.1.18), we get

$$\begin{aligned} hd_X^\beta(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)) &\leq Lm_2 l_2^{\gamma_2} d_\Lambda^{\alpha_2\gamma_2}(\lambda_1,\lambda_2) + Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)) + Lm_2 l_2^{\gamma_2} d_\Lambda^{\alpha_2\gamma_2}(\lambda_1, \\ &\quad +Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)) \\ &= 2Lm_2 l_2^{\gamma_2} d_\Lambda^{\alpha_2\gamma_2}(\lambda_1,\lambda_2) + 2Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)). \qquad (1.1 \end{aligned}$$

Whence, condition (v) implies that

$$d_X^\beta(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)) \leq \left(\frac{2Lm_2 l_2^{\gamma_2}}{h-2Lm_2 l_1^{\gamma_2}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2}(\lambda_1,\lambda_2).$$

Step 3: Let $x(\lambda_1,\mu_1) \in S_W(\lambda_1,\mu_1)$ and $x(\lambda_2,\mu_2) \in S_W(\lambda_2,\mu_2)$. It follows from (1.1.8) and (1.1.14) that

$$\begin{aligned} d(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2) &\leq d(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2) + d(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2) \\ &\leq \left(\frac{m_1 L}{h-2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2) + \left(\frac{2Lm_2 l_2^{\gamma_2}}{h-2Lm_2 l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2/\beta}(\lambda_1,\lambda_2). \end{aligned}$$

Thus,

$$\begin{aligned}
&\rho(S_W(\lambda_1,\mu_1),S_W(\lambda_2,\mu_2))\\
&= \sup_{x(\lambda_1,\mu_1)\in S_W(\lambda_1,\mu_1),x(\lambda_2,\mu_2)\in S_W(\lambda_2,\mu_2)} d_X(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2))\\
&\leq \left(\frac{m_1L}{h-2m_2Ll_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2)+\left(\frac{2Lm_2l_2^{\gamma_2}}{h-2Lm_2l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2/\beta}(\lambda_1,\lambda_2).
\end{aligned}$$

Taking $\lambda_2=\lambda_1$ and $\mu_2=\mu_1$, we see the diameter of $S(\lambda_1,\mu_1)$ is 0, that is, this set is a singleton $\{x(\lambda_1,\mu_1)\}$. This implies that the (PGVQEP) has a unique solution in a neighborhood of (λ_0,μ_0). The proof is completed. □ □

Definition 10. Let $F: X\times X\times M\to 2^Y$ be a set-valued mapping. A set-valued mapping $F(\cdot,\cdot,\mu)\mapsto 2^Y$ is said to be

1. *$h.\beta$-Hölder strongly monotone with respect to ξ_q* if there exist $q\in \text{int}C$ and $h>0, \beta>0$ such that for every $x,y\in E(N(\lambda))$ with $x\neq y$,

$$\inf_{z\in F(x,y,\mu)}\xi_q(z)+\inf_{z\in F(y,x,\mu)}\xi_q(z)+hd_X^\beta(x,y)\leq 0;$$

2. *$h.\beta$-Hölder strongly pseudomonotone with respect to* $q\in \text{int}C$ and $h>0,\beta>0$ such that for every $x,y\in E(N(\lambda_0))$ with $x\neq y$,

$$z\notin -\text{int}C,\ \exists z\in F(x,y,\mu)\ \ \Rightarrow\ \ z'+hd_X^\beta(x,y)q\in -C,\ \exists z'\in F(y,x,\mu).$$

3. *quasimonotone on* $E(N(\lambda_0))$ if $\forall x,y\in E(N(\lambda_0))$ with $x\neq y$,

$$z\in -\text{int}C,\ \exists z\in F(x,y,\mu)\ \ \Rightarrow\ \ z'\notin -\text{int}C,\ \exists z'\in F(y,x,\mu).$$

The following proposition provides the relation among monotonicity conditions defined above.

Proposition 11. *1.* $(\mathrm{A})\Rightarrow(\mathrm{iv})$

2. (B) *and* $(\mathrm{C})\Rightarrow(\mathrm{iv})$.

Proof. (i) From the definition of (A), we have

$$\begin{aligned}
hd_X^\beta(x,y) &\leq -\inf_{z\in F(x,y,\mu)}\xi_q(z)-\inf_{z\in F(y,x,\mu)}\xi_q(z)\\
&\leq d\left(\inf_{z\in F(x,y,\mu)}\xi_q(z),\mathbb{R}_+\right)+d\left(\inf_{z\in F(y,x,\mu)}\xi_q(z),\mathbb{R}_+\right).
\end{aligned}$$

(ii) Assume that F satisfies definition (B) and (C). We consider in two case.

Case 1. $z\notin -\text{int }C,\ \exists z\in F(x,y,\mu)$, then there exists $z'\in F(y,x,\mu)$ such that $z'+hd_X^\beta(x,y)q\in -C$. From Lemma 1, we have

$$\xi_q(z')+hd_X^\beta(x,y)=\xi_q(z'+hd_X^\beta(x,y)q)\leq 0,$$

which implies that $\inf_{z\in F(y,x,\mu)}\xi_q(z)\leq\xi_q(z')\leq -hd_X^\beta(x,y)$. Hence,

$$hd_X^\beta(x,y)\leq -\inf_{z\in F(y,x,\mu)}\xi_q(z)\leq d\left(\inf_{z\in F(x,y,\mu)}\xi_q(z),\mathbb{R}_+\right)+d\left(\inf_{z\in F(y,x,\mu)}\xi_q(z),\mathbb{R}_+\right).$$

Case 2. $z\in -\text{int }C,\ \exists z\in F(x,y,\mu)$, then there exists $z'\in F(y,x,\mu)$ such that $z\notin -\text{int}C$. By the similar argument as in the previous case, we have desired result. □ □

Remark 12. The converse of Proposition 11 does not hold in generally, even in the special case $X = Y = \mathbb{R}$ and $C = \mathbb{R}_+$. See, for example, Examples 1.1 and 1.2 in [53]. Therefore, Theorem 9 still holds when condition (iv) is replaced by condition (A) or (B) and (C). We can immediately obtain the following two theorems.

Theorem 13. *Theorem 9 still holds when the condition* (iv) *is replaced by condition* (A)*.*

Theorem 14. *Theorem 9 still holds when the condition* (iv) *is replaced by conditions* (B) *and* (C)*.*

Let $f : X \times X \times M \to Y$ be a vector-valued mapping. Then (**PGVQEP**) becomes to the following *parametric vector quasi-equilibrium problem* (**PVQEP**):
Find $x_0 \in K(x_0, \lambda)$ such that

$$f(x_0, y, \mu) \notin -\text{int } C, \quad \forall y \in K(x_0, \lambda). \tag{1.1.23}$$

Remark 15. In the case of vector-valued mapping, condition (iv) in Theorem 9 and condition (ii$''$) are coincide. Also, condition (A) and (B) and (C) are the same as condition (ii) and (ii$'$) in [**?**], respectively. It is obvious that Theorem 9, 13 and 14 extend Theorem 3.3, 3.1 and 3.2 in [**?**], respectively, in the case that the vector-valued mapping $f(\cdot, \cdot, \cdot)$ is extended to set-valued one.

1.1.1 Applications

Since the parametric generalized vector quasi-equilibrium problem (**PGVQEP**) contains as special cases many optimization-related problems, including quasi-variational inequalities, traffic equilibrium problems, quasi-optimization problems, fixed point and coincidence-point problems, complementarity problems, vector optimization, Nash equilibria, etc., we can derive from Theorem 9 directs consequence for such special cases. We discuss now only some applications of our results.

Quasi-variational inequalities

In this section, we assume that X is normed space. Let $K : X \times \Lambda \rightrightarrows X$ and $T : X \times M \rightrightarrows B^*(X, Y)$ be set-valued mappings, where $B^*(X, Y)$ denotes the space of all bounded linear mappings of X into Y. Setting $F(x, y, \mu) = \langle T(x, \mu), y - x\rangle := \bigcup_{t \in T(x,\mu)} \langle t, y - x\rangle$ in (1.1.5), we obtain parametric generalized vector quasi-variational inequalities (**PGVQVI**) in the case of set-valued mappings as follows:

$$\text{Find } x_0 \in K(x_0, \lambda) \text{ such that } \quad \langle T(x_0, \mu), y - x_0\rangle \subseteq Y \setminus - \text{ int } C, \quad \forall y \in K(x_0, \lambda). \tag{1.1.24}$$

For each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, let

$$E(\lambda) := \{x \in X : x \in K(x, \lambda)\}.$$

The solution set of (1.1.24) is denoted by

$$S^V_{QVI}(\lambda, \mu) := \{x \in E(\lambda) : \langle T(x, \mu), y - x\rangle \subseteq Y \setminus - \text{ int } C, \quad \forall y \in K(x, \lambda)\}.$$

For each $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $q \in$ int C, the ξ_q-solution set of (1.1.24) is

$$S^V_{QVI}(\xi_q, \lambda, \mu) := \left\{x \in E(\lambda) : \inf_{z \in \langle T(x,\mu), y - x\rangle} \xi_q(z) \geq 0, \quad \forall y \in K(x, \lambda)\right\}.$$

Theorem 16. *Assume that for each fixed $q \in int\ C$, $S^V_{QVI}(\xi_q, \lambda, \mu)$ is nonempty in a neighbourhood $N(\lambda_0) \times N(\mu_0)$ of the consider point $(\lambda_0, \mu_0) \in \Lambda \times M$. Assume further that the following conditions hold.*

1. *$K(\cdot, \cdot)$ is $(l_1 \cdot \alpha_1, l_2 \cdot \alpha_2)$-Hölder continuous on $E(N(\lambda_0)) \times N(\lambda_0)$;*
2. *For each $x \in E(N(\lambda_0))$, $T(x, \cdot)$ is $m_3 \cdot \gamma_3$-Hölder continuous at $\mu_0 \in M$;*
3. *$T(\cdot, \cdot)$ is bounded in $x \in E(N(\lambda_0))$, and $E(N(\lambda_0))$ is bounded;*
4. *$T(\cdot, \mu)$ is $h.\beta$-Hölder strongly monotone with respect to ξ_q, i.e., there exist constants $h > 0, \beta > 0$, such that for every $x, y \in E(N(\lambda_0)) : x \neq y$,*
$$h\|x - y\|^\beta \leq d\left(\inf_{z \in \langle T(x,\mu), y-x\rangle} \xi_q(z), \mathbb{R}_+\right) + d\left(\inf_{z \in \langle T(y,\mu), x-y\rangle} \xi_q(z), \mathbb{R}_+\right);$$
5. *$\beta = \alpha_1, h > 2MLl_1^{\gamma_1}$, where $L := \sup_{\lambda \in C^q} \|\lambda\| \in \left[\frac{1}{\|q\|}, +\infty\right)$ is the Lipschitz constant of ξ_q on Y.*

Then for every $(\lambda, \mu) \in N(\lambda_0) \times N(\mu_0)$, the solution of (**PGVQVI**) *is unique, $x(\lambda, \mu)$, and this function satisfies the Hölder condition: for all $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in N(\lambda_0) \times N(\mu_0)$*

$$d_X(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)) \leq \left(\frac{2MLl_2}{h - 2MLl_1^{\gamma_3}}\right)^{\frac{1}{\beta}} d_\lambda^{\alpha_2/\beta}(\lambda_1, \lambda_2) + \left(\frac{Nm_3L}{h - 2MLl_1^{\gamma_3}}\right)^{\frac{1}{\beta}} d_M^{\gamma_3/\beta}(\mu_1, \mu_2),$$

where $x(\lambda_i, \mu_i) \in S_{QVI}(\lambda_i, \mu_i), i = 1, 2.$

Proof. We verify all assumptions of Theorem 9 is fulfilled. First, (i′), (iv′) and (v′) are the same (i), (iv) and (v) in Theorem 9. We need only to verify conditions (ii) and (iii). Taking $M, \widetilde{M} > 0$ such that

$$\|T(x, \mu)\| \leq M,\ \forall (x, \mu) \in E(N(\lambda_0)) \times N(\mu_0),$$

and

$$\|x - y\| \leq \widetilde{M},\ \forall x, y \in E(N(\lambda_0)).$$

We put $m_1 = \widetilde{M} m_3$ and $\gamma_1 = \gamma_3$. For any fixed $x, y \in E(N(\lambda_0))$ by assumption (ii′), we have

$$T(x, \mu_1) \subseteq T(x, \mu_2) + m_3 d^{\gamma_3}(\mu_1, \mu_2) B_{B^*(X,Y)},\ \forall \mu_1, \mu_2 \in N(\mu_0).$$

Then,

$$\begin{aligned}
\langle T(x, \mu_1), y - x\rangle &\subseteq \left\langle T(x, \mu_2) + m_3 d^{\gamma_3}(\mu_1, \mu_2) B_{B^*(X,Y)}, y - x\right\rangle \\
&= \langle T(x, \mu_2), y - x\rangle + \langle m_3 d^{\gamma_3}(\mu_1, \mu_2) B_{B^*(X,Y)}, y - x\rangle \\
&= \langle T(x, \mu_2), y - x\rangle + m_3 d^{\gamma_3}(\mu_1, \mu_2) \langle B_{B^*(X,Y)}, y - x\rangle \\
&= \langle T(x, \mu_2), y - x\rangle + m_3 d^{\gamma_3}(\mu_1, \mu_2) \bigcup_{g \in B_{B^*(X,Y)}} \langle g, y - x\rangle \\
&\subseteq \langle T(x, \mu_2), y - x\rangle + m_3 d^{\gamma_3}(\mu_1, \mu_2) \widetilde{M} B_Y.
\end{aligned}$$

Hence

$$\langle T(x, \mu_1), y - x\rangle \subseteq \langle T(x, \mu_2), y - x\rangle + m_1 d^{\gamma_1}(\mu_1, \mu_2) \widetilde{M} B_Y.$$

Also, we put $m_2 = M$ and $\gamma_2 = 1$. We need to show that

$$\langle T(x,\mu), y_1 - x\rangle \subseteq \langle T(x,\mu), y_2 - x\rangle + \widetilde{M}\|y_1 - y_2\|B_Y.$$

For each fixed $x \in E(N(\lambda_0))$ and $\mu \in N(\mu_0)$,

$$\begin{aligned}
\langle T(x,\mu), y_1 - x\rangle &= \bigcup_{t\in T(x,\mu)} \langle t, y_1 - x\rangle \\
&= \bigcup_{t\in T(x,\mu)} \langle t, y_1 - x + y_2 - y_2\rangle \\
&= \bigcup_{t\in T(x,\mu)} \langle t, y_2 - x\rangle + \bigcup_{t\in T(x,\mu)} \langle t, y_1 - y_2\rangle \\
&\subseteq \langle T(x,\mu), y - x\rangle + M\|y_1 - y_2\|B_Y.
\end{aligned}$$

Hence, the condition (iii) is verified, and so we obtain the result. □

For (**PGVQVI**), If we put $Y = \mathbb{R}, C = [0, +\infty)$, then (1.1.24) becomes to the following parametric generalized quasi-variational inequality problem in the case of scalar valued one:

$$\text{Find } x_0 \in K(x_0,\lambda) \text{ such that } \langle t, y - x_0\rangle \geq 0, \;\; \forall y \in K(x_0,\lambda), \; \forall t \in T(x_0,\mu). \quad (1.1.25)$$

For each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, let

$$E(\lambda) := \{x \in X : x \in K(x,\lambda)\}.$$

The solution set of (1.1.25) is denoted by

$$S^S_{QVI}(\lambda,\mu) := \{x \in E(\lambda) : \langle t, y - x\rangle \geq 0, \;\; \forall y \in K(x,\lambda), \; \forall t \in T(x,\mu)\}.$$

For each $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $1 \in \text{ int } C$, the ξ_q-solution set of (1.1.24) is

$$S^S_{QVI}(\xi_1,\lambda,\mu) := \left\{x \in E(\lambda) : \inf_{z\in\langle T(x,\mu), y-x\rangle} \xi_1(z) \geq 0, \quad \forall y \in K(x,\lambda)\right\}.$$

It follows from Lemma 1 that $S^S_{QVI}(\xi_1,\lambda,\mu)$ coincides with $S^S_{QVI}(\lambda,\mu)$.

Corollary 17. *Assume that $S^S_{QVI}(\lambda,\mu)$ is nonempty in a neighbourhood $N(\lambda_0) \times N(\mu_0)$ of the consider point $(\lambda_0,\mu_0) \in \Lambda \times M$. Assume further that the conditions* (i′)-(iii′) *and* (v′) *in Corollary 16 hold. Replace* (iv′) *by* (iv″).

(iv″) *$T(\cdot,\mu)$ is $h.\beta$-Hölder strongly monotone, i.e., there exist constants $h > 0, \beta > 0$, such that for every $x, y \in E(N(\lambda_0)) : x \neq y$,*

$$\langle u - v, x - y\rangle \geq h\|x - y\|^\beta, \; \forall u \in T(x), \forall v \in T(y).$$

Then for every $(\lambda,\mu) \in N(\lambda_0) \times N(\mu_0)$, the solution of (**PGVQVI**) *is unique, $x(\lambda,\mu)$, and this function satisfies the Hölder condition: for all $(\lambda_1,\mu_1), (\lambda_2,\mu_2) \in N(\lambda_0) \times N(\mu_0)$*

$$d_X(x(\lambda_1,\mu_1), x(\lambda_2,\mu_2)) \leq \left(\frac{2Ml_2}{h - 2Ml_1^{\gamma_3}}\right)^{\frac{1}{\beta}} d_\lambda^{\alpha_2/\beta}(\lambda_1,\lambda_2) + \left(\frac{Nm_3}{h - 2Ml_1^{\gamma_3}}\right)^{\frac{1}{\beta}} d_M^{\gamma_3/\beta}(\mu_1,\mu_2),$$

where $x(\lambda_i,\mu_i) \in S^S_{QVI}(\lambda_i,\mu_i), i = 1, 2.$

Proof. It is not hard to show that (iv″) implies (iv′). Indeed, for any $x, y \in E(N(\lambda_0))$ with $x \neq y$,

$$\begin{aligned} h\|x-y\|^\beta &\leq \langle u-v, x-y\rangle \\ &= \langle u, x-y\rangle + \langle v, y-x\rangle \\ &\leq \sup_{u\in T(x)} \langle u, x-y\rangle + \sup_{v\in T(y)} \langle v, y-x\rangle \\ &= \sup_{u\in T(x)} -\langle u, y-x\rangle + \sup_{v\in T(y)} -\langle v, x-y\rangle \\ &= -\inf_{u\in T(x)} \langle u, y-x\rangle - \inf_{v\in T(y)} \langle v, x-y\rangle \\ &\leq d\left(\inf_{u\in T(x)} \langle u, y-x\rangle, \mathbb{R}_+\right) + d\left(\inf_{v\in T(y)} \langle v, x-y\rangle, \mathbb{R}_+\right). \end{aligned}$$

Therefore, (iv′) is satisfied □

Remark 18. Corollary 17 extends Corollary 3.1 in [**?**], since the mapping T is a multivalued mapping.

1.1.2 Traffic equilibrium problems

The foundation of the study of traffic network problems goes back to Wardrop [**?**], who stated the basic equilibrium principle in 1952. Over the past decades, a large number of efforts have been devoted to the study of traffic assignment models, with emphasis on efficiency, and optimality, in order to improve practicability, reduce gas emissions, and contribute to the welfare of the community. The variational inequality approach to such problems begins with the seminal work of Smith [**?**], who proved that the user-optimized equilibrium can be expressed in terms of a variational inequality. Thus, the possibility of exploiting the powerful tools of variational analysis has led to deal with a large variety of models, reaching valuable theoretical results and providing applications in practical situations. In this paper, we are concerned with a class of equilibrium problems which can be studied in the framework of quasi-variational inequalities, see [**?**, **?**].

Let a set N of nodes, a set L of links, a set $W := (W_1, \dots, W_l)$ of origin-destination pairs (O/D pairs for short) be given. Assume that there are $r_j \geq 1$ paths connecting the pairs $W_j, j = 1, \dots, l$, whose set is denoted by P_j. Set $m := r_1 + \dots + r_l$; *i.e.*, there are in whole m paths in the traffic network. Let $F := (F_1, \dots, F_m)$ stand for the path flow vector. Assume that the travel cost of the path $R_s, s = 1, \dots, m$, is a set $T_s(F) \subset \mathbb{R}_+$. So, we have a multifunction $T : \mathbb{R}^m_+ \rightrightarrows \mathbb{R}^m_+$ with $T(F) := (T_1(F), \dots, T_m(F))$. Let the capacity restriction be

$$F \in A := \{F \in \mathbb{R}^m_+ : F_s \leq \Gamma_s, s = 1, \dots, m\},$$

where Γ_s are given real numbers. Extending the Wardrop definition to the case of multivalued costs, we propose the following definition.
A path flow vector H is said to be a ***weak*** *equilibrium flow* vector if

$$\forall W_j,\ \forall R_q \in P_j,\ R_s \in P_j,\ \text{there exists } t \in T(H) \text{ such that } t_q < t_s \Rightarrow H_q = \Gamma_q \text{ or } H_s = 0, \tag{1.1.26}$$

where $j = 1, \dots, l$ and $q, s \in \{1, \dots, m\}$ are among r_j indices corresponding to P_j.

A path flow vector H is said to be a ***strong*** *equilibrium flow* vector if

$$\forall W_j,\ \forall R_q \in P_j,\ R_s \in P_j,\ \text{for all } t \in T(H) \text{ such that } t_q < t_s \Rightarrow H_q = \Gamma_q \text{ or } H_s = 0. \tag{1.1.27}$$

Suppose the travel demand ρ_j of the O/D pair $W_j, j = 1, \ldots, l$, depends on the weak(or strong) equilibrium problem flow H. So, considering all the O/D pairs, we have a mapping $\rho : \mathbb{R}^m_+ \to \mathbb{R}^l_+$. We use the Kronecker notation

$$\phi_{js} = \begin{cases} 1 & \text{if } s \in P_j, \\ 0 & \text{if } s \notin P_j. \end{cases}$$

Then, the matrix

$$\phi = \{\phi_{js}\}, j = 1, \ldots, l, s = 1, \ldots, m,$$

is called an O/D pair/path incidence matrix. The path flow vectors meeting the travel demands are called the feasible path flow vectors and form the constraint set, for a given weak(or strong) equilibrium flow H,

$$K(H, \lambda) := \{F \in A : \phi F = \rho(H, \lambda)\}.$$

Assume further that the path costs are also perturbed, *i.e.*, depend on a perturbation parameter μ of a metric space M: $T_s(F, \mu), s = 1, \ldots, m$.

Our traffic equilibrium problem is equivalent to a quasi-variational inequality as follows (see [?]).

Lemma 19. *A path vector flow $H \in K(H, \lambda)$ is a **weak** equilibrium flow if and only if it is a solution of the following quasi-variational inequality:*

find $H \in K(H, \lambda)$ such that there exists $t \in T(H, \lambda)$ satisfying , $\langle t, F - H \rangle \geq 0, \ \forall F \in K(H, \lambda)$.

Lemma 20. *A path vector flow $H \in K(H, \lambda)$ is a **strong** equilibrium flow if and only if it is a solution of the following quasi-variational inequality:*

find $H \in K(H, \lambda)$ such that for all $t \in T(H, \lambda)$ satisfying , $\langle t, F - H \rangle \geq 0, \ \forall F \in K(H, \lambda)$.

Corollary 21. *Assume that solutions of the traffic network equilibrium problem exist and all assumptions of Corollary 17 are satisfied. Then, in a neighborhood of (λ_0, μ_0), the solution is unique and satisfies the same Hölder condition as in Corollary 17.*

1.1.3 Quasioptimization Problem

For the normed linear space Y and pointed, closed and convex cone C with nonempty interior, we denote the ordering induced by C as follows:

$$x \leq y \text{ iff } y - x \in C;$$
$$x < y \text{ iff } y - x \in \operatorname{int} C.$$

The ordering $\geq$ and $>$ are defined similarly. Let $g : X \times M \to Y$ be a vector valued mapping. For each $(\lambda, \mu) \in \Lambda \times M$, consider the problem of parametric quasi-optimization problem (**PQOP**) finding $x_0 \in K(x_0, \lambda)$ such that

$$g(x_0, \mu) = \min_{y \in K(x_0, \lambda)} g(y, \mu). \tag{1.1.28}$$

Since the constraint set depends on the minimizer x_0, this is a quasi-optimization problem. Setting $f(x, y, \mu) = g(y, \mu) - g(x, \mu)$, (**PVQEP**) becomes a special case of (**PQOP**).

The following results are derived from Theorem 14 (Theorem 9 cannot be applied since $f(x, y, \mu) + f(y, x, \mu) = 0, \ \forall x, y \in A$ and $\mu \in M$).

Theorem 22. *For* (**PQOP**)*, assume that the solution exists in a neighbourhood $N(\lambda_0) \times N(\mu_0)$ of the consider point $(\lambda_0, \mu_0) \in \Lambda \times M$. Assume further that the following conditions hold.*

1. $K(\cdot,\cdot)$ *is* $(l_1 \cdot \alpha_1, l_2 \cdot \alpha_2)$*-Hölder continuous on* $E(N(\lambda_0)) \times N(\lambda_0)$*;*
2. *For each* $x, y \in E(N(\lambda_0))$*,* $F(x, y, \cdot)$ *is* $m_1 \cdot \gamma_1$*-Hölder continuous at* $\mu_0 \in M$*;*
3. *For each* $x \in E(N(\lambda_0))$ *and* $\mu \in N(\mu_0)$*,* $F(x, \cdot, \mu)$ *is* $m_2 \cdot \gamma_2$*-Hölder continuous on* $E(N(\lambda_0))$*;*
4. $F(\cdot, \cdot, \mu)$ *is* $h.\beta$*-Hölder strongly monotone with respect to* ξ_q*, i.e., there exist constants* $h > 0, \beta > 0$*, such that for every* $x, y \in E(N(\lambda_0)) : x \neq y$*,*
$$hd_X^{\beta}(x, y) \leq d\left(\inf_{z \in F(x,y,\mu)} \xi_q(z), \mathbb{R}_+\right) + d\left(\inf_{z \in F(y,x,\mu)} \xi_q(z), \mathbb{R}_+\right);$$
5. $\beta = \alpha_1\gamma_2, h > 2m_2Ll_1^{\gamma_1}$*, where* $L := \sup_{\lambda \in C^q} \|\lambda\| \in \left[\frac{1}{\|q\|}, +\infty\right)$ *is the Lipschitz constant of* ξ_q *on* Y*.*

Then for every $(\lambda, \mu) \in N(\lambda_0) \times N(\mu_0)$*, the solution of* (**PVQGEP**) *is unique,* $x(\lambda, \mu)$*, and this function satisfies the Hölder condition: for all* $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in N(\lambda_0) \times N(\mu_0)$

$$d_X(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)) \leq \left(\frac{2m_2Ll_2^{\gamma_2}}{h - 2m_2Ll_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2/\beta}(\lambda_1, \lambda_2) + \left(\frac{m_1L}{h - 2m_2Ll_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1, \mu_2),$$

where $x(\lambda_i, \mu_i) \in S_W(\lambda_i, \mu_i), i = 1, 2.$

1.2 Lower semicontinuity of approximate solution mappings for parametric generalized vector equilibrium problems

The vector equilibrium problem is a unified model of several problems, for example, the vector optimization problem, the vector variational inequality problem, the vector complementarity problem and the vector saddle point problem. In the literature, existence results for various types of vector equilibrium problems have been investigated intensively, e.g., see [79, 80, 81, 82] and the references therein. The stability analysis of the solution mappings for VEP is an important topic in vector equilibrium theory. Recently, the semicontinuity, especially the lower semicontinuity, of solution mappings to parametric vector equilibrium problems has been studied in the literature, such as [83, 84, 85, 86, 90, 91, 93, 96, 94, 97, 98, 99]. In the mentioned results, the lower semicontinuity of solution mapping to parametric generalized strong vector equilibrium problems are established under the the assumptions of monotonicity and compactness. Very recently, Han and Gong [95] studied the lower semicontinuity of solution mapping to parametric generalized strong vector equilibrium problems without the assumptions of monotonicity and compactness.

On the other hand, exact solutions of the problems may not exist in many practical problems because the data of the problems are not sufficiently "regular". Moreover, these mathematical models are solved usually by numerical methods which produce approximations to the exact solutions. So it is impossible to obtain an exact solution of many practical problems. Naturally, investigating approximate solutions of parametric equilibrium problems is of interest in both practical applications and computations. Anh and Khanh [87] considered two kinds of approximate solution mappings to parametric generalized vector quasiequilibrium problems and established the sufficient conditions for their Hausdorff semicontinuity (or Berge semicontinuity). Among many approaches for dealing with the lower semicontinuity and continuity of solution mappings for parametric vector variational inequalities and parametric vector equilibrium problems, the scalarization method is of considerable interest. By using a scalarization method, Li and Li [101] discussed the Berge lower semicontinuity and Berge continuity of a approximate solution mapping for a parametric vector equilibrium problem.

Motivated by the work reported in [87, 101, 95], this paper aims to establish the efficient conditions for the lower semicontinuity of an approximate solution mapping for a parametric generalized vector equilibrium problem involving set-valued mappings. By using a scalarization method, we obtain the lower semicontinuity of an approximate solution mapping for a such problem without the assumptions of monotonicity and compactness.

Throughout this paper, let X and Y be real Hausdorff topological vector spaces, and let Z be a real topological space. We also assume that C is a pointed closed convex cone in Y with its interior $\operatorname{int} C \neq \emptyset$. Let Y^* be the topological dual space of Y. Let $C^* := \{\xi \in Y^* : \langle \xi, y\rangle \geq 0, \forall y \in C\}$ be the dual cone of C, where $\langle \xi, y\rangle$ denotes the value of ξ at y. Since $\operatorname{int} C \neq \emptyset$, the dual cone C^* of C has a weak* compact base. Let $e \in \operatorname{int} C$. Then, $B_e^* := \{\xi \in C^* : \langle \xi, e\rangle = 1\}$ is a weak* compact base of C^*.

Suppose that K is a nonemmpty subset of X and $F : K \times K \to 2^Y\backslash\{\emptyset\}$ is a set-valued mapping. We consider the following generalized vector equilibrium problem (GVEP) of finding $x_0 \in K$ such that

$$F(x_0, y) \subset Y\backslash - \operatorname{int} C\ , \quad \forall y \in K. \tag{1.2.1}$$

When the set K and the mapping F are perturbed by a parameter μ which varies over a set M of Z, we consider the following parametric generalized vector equilibrium problem (PGVEP) of finding $x_0 \in K(\mu)$ such that

$$F(x_0, y, \mu) \subset Y\backslash - \operatorname{int} C\ , \quad \forall y \in K(\mu), \tag{1.2.2}$$

where $K : M \to 2^X\backslash\{\emptyset\}$ is a set-valued mapping, $F : B \times B \times M \subset X \times X \times Z \to 2^Y\backslash\{\emptyset\}$ is a set-valued mapping with $K(M) = \bigcup_{\mu\in M} K(\mu) \subset B$. For each $\varepsilon > 0$, and $\mu \in M$, the approximate solution set of (PGVEP) is defined by

$$\widetilde{S}(\varepsilon, \mu) := \{x \in K(\mu) : F(x, y, \mu) + \varepsilon e\ \subset Y\backslash - \operatorname{int} C, \quad \forall y \in K(\mu)\},$$

where $e \in \operatorname{int} C$. For each $\xi \in B_e^*$ and $(\varepsilon, \mu) \in \mathbb{R}^+ \times M$, by $\widetilde{S}_\xi(\varepsilon, \mu)$ we denote the ξ-approximate solution set of (PGVEP), i.e.,

$$\widetilde{S}_\xi(\varepsilon, \mu) := \left\{x \in K(\mu) : \inf_{z\in F(x,y,\mu)} \xi(z) + \varepsilon \geq 0, \quad \forall y \in K(\mu)\right\}.$$

Definition 23. Let D be a nonempty convex subset of X. A set-valued mapping $G : X \to 2^Y$ is said to be :

1. *C-convex* on D if, for any $x_1, x_2 \in D$ and for any $t \in [0, 1]$, we have

$$tG(x_1) + (1 - t)G(x_2) \subseteq G(tx_1 + (1 - t)x_2) + C.$$

2. *C-concave* on D if, for any $x_1, x_2 \in D$ and for any $t \in [0, 1]$, we have

$$G(tx_1 + (1 - t)x_2) \subseteq tG(x_1) + (1 - t)G(x_2) + C.$$

Definition 24. [95] Let M and M_1 be topological vector spaces. Let D be a nonempty subset of M . A set-valued mapping $G : M \to 2^{M_1}$ is said to be *uniformly continuous* on D if, for any neighborhood V of $0 \in M_1$, there exists a neighborhood U_0 of $0 \in M$ such that $G(x_1) \subseteq G(x_2) + V$ for any $x_1, x_2 \in D$ with $x_1 - x_2 \in U_0$.

Definition 25. [88] Let M and M_1 be topological vector spaces. A set-valued mapping $G : M \to 2^{M_1}$ is said to be:

(i) *Hausdorff upper semicontinuous (H-u.s.c)* at $u_0 \in M$ if, for any neighborhood V of $0 \in M_1$, there exists a neighborhood $U(u_0)$ of u_0 such that

$$G(u) \subseteq G(u_0) + V, \text{ for every } u \in U(u_0).$$

(ii) *Lower semicontinuous (l.s.c)* at $u_0 \in M$ if, for any $x \in G(u_0)$ and any neighborhood V of x, there exists a neighborhood $U(u_0)$ of u_0 such that

$$G(u) \cap V \neq \emptyset, \text{ for every } u \in U(u_0).$$

The following lemma plays an important role in the proof of the lower semicontinuity of the solution mapping $\widetilde{S}(\cdot,\cdot)$.

Lemma 26. [89, Theorem 2] *The union $\Gamma = \bigcup_{i\in I} \Gamma_i$ of a family of l.s.c set-valued mappings Γ_i from a topological space X into a topological space Y is also an l.s.c set-valued mapping from X into Y, where I is an index set.*

1.2.1 Lower semicontinuity of the approximate solution mapping for (PGVEP)

In this section, we establish the lower semicontinuity of the approximate solution mapping for (PGVEP) at the considered point $(\varepsilon_0, \mu_0) \in \mathbb{R}^+ \times M$ with $\varepsilon_0 > 0$.

Firstly, using the same argument as in the proof given in [100] Lemma 3.1, we can prove the following useful result.

Lemma 27. *For each $\varepsilon > 0, \mu \in M$, if for each $x \in K(\mu), F(x, K(\mu), \mu) + C$ is a convex set, then*

$$\widetilde{S}(\varepsilon,\mu) = \bigcup_{\xi\in C^*\backslash\{0\}} \widetilde{S}_\xi(\varepsilon,\mu) = \bigcup_{\xi\in B_e^*} \widetilde{S}_\xi(\varepsilon,\mu).$$

Proof. For any $x \in \bigcup_{\xi\in C^*\backslash\{0\}} \widetilde{S}_\xi(\varepsilon,\mu)$, there exists $\xi' \in C^*\backslash\{0\}$ such that $x \in \widetilde{S}_{\xi'}(\varepsilon,\mu)$. Thus, we can obtain that $x \in K(\mu)$ and $\inf_{z\in F(x,y,\mu)} \xi'(z) + \varepsilon \geq 0, \forall y \in K(\mu)$. Then, for each $y \in K(\mu)$ and $z \in F(x, y, \mu), \xi'(z) + \varepsilon \geq 0$, which arrives that $z \notin -\text{int}C$. It then follows that, for each $z \in F(x, y, \mu)$,

$$F(x, y, \mu) + \varepsilon e \subseteq Y\backslash - \text{int}C, \quad \forall y \in K(\mu).$$

which gives that $x \in \widetilde{S}(\varepsilon,\mu)$. Hence, $\bigcup_{\xi\in C^*\backslash\{0\}} \widetilde{S}_\xi(\varepsilon,\mu) \subseteq \widetilde{S}(\varepsilon,\mu)$. Conversely, let $x \in \widetilde{S}(\varepsilon,\mu)$ be arbitrary. Then $x \in K(\mu)$ and $F(x, y, \mu) + \varepsilon e \subseteq Y\backslash - \text{int}C, \quad \forall y \in K(\mu)$. Thus, we have

$$F(x, K(\mu), \mu) \cap (-\text{int}C) = \emptyset,$$

and hence,

$$(F(x, K(\mu), \mu) + C) \cap (-\text{int}C) = \emptyset.$$

Because $F(x, K(\mu), \mu) + C$ is a convex set, by the well-known Edidelheit's separation theorem (see[92], Theorem 3.16), there exists a continuous linear functional $\xi \in Y^*\backslash\{0\}$ and a real number γ such that

$$\xi(\hat{c}) < \gamma \leq \xi(z + c),$$

for all $z \in (F(x, K(\mu), \mu), c \in C$ and $\hat{c} \in -\text{int}C$. Since C is a cone, we have $\xi(\hat{c}) \leq 0$ for all $\hat{c} \in -\text{int}C$. Thus, $\xi(\hat{c}) \geq 0$ for all $\hat{c} \in C$, that is, $\xi \in C^*$. Moreover, it follows from $c \in C, \hat{c} \in -\text{int}C$ and the continuity of ξ that $\xi(z) + \varepsilon \geq 0$ for all $z \in F(x, K(\mu), \mu)$. Thus, for all $y \in K(\mu)$, we have $\inf_{z\in F(x,y,\mu)}\xi(z) + \varepsilon \geq 0$, i.e., $x \in \widetilde{S}_\xi(\varepsilon, \mu) \subseteq \bigcup_{\xi\in C^*\backslash\{0\}} \widetilde{S}_\xi(\varepsilon, \mu)$.
□ □

Theorem 28. *We assume that for any given $\xi \in B_e^*$, there exists $\delta > 0$ such that the ξ-approximate solution set $\widetilde{S}_\xi(\cdot, \cdot)$ exists in $[\varepsilon_0, \delta) \times N(\mu_0)$, where $N(\mu_0)$ is a neighborhood of μ_0. Assume further that the following conditions are satisfied :*

1. *$K(\mu_0)$ is nonempty convex;*
2. *K is H-u.s.c at μ_0 and l.s.c at μ_0;*
3. *for any $y \in K(\mu_0), F(\cdot, y, \mu_0)$ is C-concave on $K(\mu_0)$;*
4. *$F(\cdot, \cdot, \cdot)$ is uniformly continuous on $K(M) \times K(M) \times N(\mu_0)$.*

Then, the ξ-approximate solution mapping $\widetilde{S}_\xi : [\varepsilon_0, \delta) \times N(\mu_0) \to 2^X$ is l.s.c at (ε_0, μ_0).

Proof. Suppose to the contrary that $\widetilde{S}_\xi(\cdot, \cdot)$ is not l.s.c at (ε_0, μ_0), then there exist $x_0 \in \widetilde{S}_\xi(\varepsilon_0, \mu_0)$ and a neighborhood W_0 of $0_X \in X$, for any neighborhoods $J(\varepsilon_0)$ and $U(\mu_0)$ of ε_0 and μ_0, respectively, there exist $\varepsilon' \in J(\varepsilon_0) \cap [\varepsilon_0, \delta)$ and $\mu' \in U(\mu_0)$ such that $(x_0 + W_0) \cap \widetilde{S}_\xi(\varepsilon', \mu') = \emptyset$. In particular, there exist sequences $\{\varepsilon_n\} \downarrow \varepsilon_0$ and $\{\mu_n\} \to \mu_0$ such that

$$(x_0 + W_0) \cap \widetilde{S}_\xi(\varepsilon_n, \mu_n) = \emptyset, \quad \forall n \in \mathbb{N}. \tag{1.2.3}$$

For the above W_0, there exists a neighborhood W_1 of $0_X \in X$ such that

$$W_1 + W_1 \subseteq W_0. \tag{1.2.4}$$

We define a ξ-set-valued mapping $H_\xi : [0, \delta) \to 2^X$ by

$$H_\xi(\varepsilon) = \{x \in K(\mu_0) : \inf_{z\in F(x,y,\mu_0)} \xi(z) + \varepsilon + \varepsilon_0 \geq 0, \ \forall y \in K(\mu_0)\}, \ \varepsilon \in [0, \delta).$$

Notice that $H_\xi(0) = \widetilde{S}_\xi(\varepsilon_0, \mu_0) \neq \emptyset$. Next, we claim that H_ξ is l.s.c at 0. Suppose to the contrary that H_ξ is not l.s.c at 0, then there exist $\bar{x} \in H_\xi(0)$ and a neighborhood O_0 of $0_X \in X$, for any neighborhood U of 0, there exists $\varepsilon \in U$ such that $(\bar{x} + O_0) \cap H_\xi(\varepsilon) = \emptyset$. In particular, there exists a nonnegative sequence $\{\varepsilon'_n\} \downarrow 0$ such that

$$(\bar{x} + O_0) \cap H_\xi(\varepsilon'_n) = \emptyset, \quad \forall n \in \mathbb{N}. \tag{1.2.5}$$

Since $H_\xi(0) \neq \emptyset$, we choose $x^* \in H_\xi(0)$. Since $\varepsilon'_n \to 0$, there exists ε'_{n_0} such that

$$\frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}x^* = \bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}(x^* - \bar{x}) \in \bar{x} + O_0. \tag{1.2.6}$$

We claim that $\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x}+\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^* \in H_\xi(\varepsilon'_{n_0})$. In fact, since $\bar{x} \in H_\xi(0)$ and $x^* \in H_\xi(0)$, for any $y \in K(\mu_0)$, we have $\inf_{t\in F(\bar{x},y,\mu_0)}\xi(t)+\varepsilon_0 \geq 0$ and $\inf_{k\in F(x^*,y,\mu_0)}\xi(k)+\varepsilon_0 \geq 0$. Then, for any $u \in F(\bar{x},y,\mu_0)$

$$\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\xi(u)+\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\varepsilon_0 \geq 0, \tag{1.2.7}$$

and for any $v \in F(x^*,y,\mu_0)$

$$\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}\xi(v)+\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}\varepsilon_0 \geq 0. \tag{1.2.8}$$

By the C-concavity of $F(\cdot,y,\mu_0)$, we have that

$$F\left(\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x}+\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^*,y,\mu_0\right) \subseteq \frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}F(\bar{x},y,\mu_0)+\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}F(x^*,y,\mu_0)+C.$$

It follows that, for any $w \in F\left(\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x}+\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^*,y,\mu_0\right)$, there exist $\bar{z} \in F(\bar{x},y,\mu_0)$, $z^* \in F(x^*,y,\mu_0)$ and $c' \in C$ such that $w = \frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{z}+\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}z^*+c'$. It follows from linearlity of ξ that $\xi(w)-\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\xi(\bar{z})-\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}\xi(z^*)=\xi(c')\geq 0$, which gives that $\xi(w) \geq \frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\xi(\bar{z})+\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}\xi(z^*)$. For all $w \in F\left(\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x}+\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^*,y,\mu_0\right)$, by (1.2.7) and (1.2.8), we have,

$$\xi(w) \geq -\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\varepsilon_0-\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}\varepsilon_0=-\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}(\varepsilon'_{n_0}+\varepsilon_0)\geq-(\varepsilon'_{n_0}+\varepsilon_0).$$

This implies that $\inf_{z\in F\left(\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x}+\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^*,y,\mu_0\right)}\xi(z)+\varepsilon'_{n_0}+\varepsilon_0 \geq 0$, that is $\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x}+\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^* \in H_\xi(\varepsilon'_{n_0})$. By (1.2.6), we get that $\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x}+\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^* \in (\bar{x}+O_0)\cap H_\xi(\varepsilon'_{n_0})$, which contradicts (1.2.5). Therefore, H_ξ is l.s.c at 0. Since H_ξ is l.s.c at 0, for above $x_0 \in \widetilde{S}_\xi(\varepsilon_0,\mu_0)=H_\xi(0)$ and for above W_1, there exists a balanced neighborhood V_0 of 0 such that $(x_0+W_1)\cap H_\xi(\varepsilon)\neq\emptyset,\ \ \forall\varepsilon\in V_0$. In particular, from $\{\varepsilon_n\}\downarrow\varepsilon_0$, there exits $N_0\in\mathbb{N}$ such that $(x_0+W_1)\cap H_\xi(\varepsilon_{N_0}-\varepsilon_0)\neq\emptyset$. Let $x'\in(x_0+W_1)\cap H_\xi(\varepsilon_{N_0}-\varepsilon_0)$.

For any $\bar{\varepsilon}>0$, since $e\in\text{int}C$, there exists $\delta_0>0$ such that

$$\delta_0 B_Y+\bar{\varepsilon}e\subseteq C. \tag{1.2.9}$$

Since $F(\cdot,\cdot,\cdot)$ is uniformly continuous on $K(M)\times K(M)\times N(\mu_0)$, for above $\delta_0 B_Y$, there exists a neighborhood V_1 of $0\in B$, a neighborhood U_1 of $0\in B$ and a neighborhood N_1 of $0\in M$, for any $(x_1,y_1,\mu_1),(x_2,y_2,\mu_2)\in K(M)\times K(M)\times N(\mu_0)$ with $x_1-x_2\in V_1, y_1-y_2\in U_1$ and $\mu_1-\mu_2\in N_1$, we have

$$F(x_1,y_1,\mu_1)\subseteq\delta_0 B_Y+F(x_2,y_2,\mu_2). \tag{1.2.10}$$

Since K is H-u.s.c. at μ_0, for above U_1, there exists a neighborhood $U_1(\mu_0)$ of μ_0 such that

$$K(\mu)\subseteq K(\mu_0)+U_1,\ \ \forall\mu\in U_1(\mu_0). \tag{1.2.11}$$

We see that $x' \in K(\mu_0)$. Since K is l.s.c. at μ_0, for $V_1 \cap W_1$, there exists a neighborhood $U_2(\mu_0)$ of μ_0 such that

$$(x' + V_1 \cap W_1) \cap K(\mu) \neq \emptyset, \quad \forall \mu \in U_2(\mu_0). \tag{1.2.12}$$

It follows from $\mu_n \to \mu_0$ that there exists a positive integer $N_0' \geq N_0$ such that $\mu_{N_0'} \in U_1(\mu_0) \cap U_2(\mu_0) \cap U(\mu_0) \cap (\mu_0 + N_1)$. Noting that (1.2.11) and (1.2.12), we obtain

$$K(\mu_{N_0'}) \subseteq K(\mu_0) + U_1, \tag{1.2.13}$$

and

$$(x' + V_1 \cap W_1) \cap K(\mu_{N_0'}) \neq \emptyset. \tag{1.2.14}$$

By (1.2.14), we choose

$$x'' \in (x' + V_1 \cap W_1) \cap K(\mu_{N_0'}). \tag{1.2.15}$$

Next, we prove that $x'' \in \widetilde{S}_\xi(\varepsilon_{N_0'}, \mu_{N_0'})$. For any $y' \in K(\mu_{N_0'})$, by (1.2.13), there exists $y_0 \in K(\mu_0)$ such that $y' - y_0 \in U_1$. It follows from (1.2.15) that $x'' - x' \in V_1$. Noting that $\mu_{N_0'} \in U(\mu_0) \cap (\mu_0 + N_1)$ and (1.2.10), we have

$$F(x'', y', \mu_{N_0'}) \subseteq \delta_0 B_Y + F(x', y_0, \mu_0).$$

By (1.2.9), we have

$$F(x'', y', \mu_{N_0'}) \subseteq C - \bar{\varepsilon} e + F(x', y_0, \mu_0). \tag{1.2.16}$$

Hence, for any $y \in K(\mu_{N_0'})$ and $z'' \in F(x'', y', \mu_{N_0'})$, there exist $c'' \in C$ and $z' \in F(x', y, \mu_0)$ such that

$$z'' = c'' - \bar{\varepsilon} e + z'.$$

It follows from the linearity of ξ that $\xi(z'') + \bar{\varepsilon} \geq \xi(z')$ for all $\bar{\varepsilon} > 0$. This leads to $\xi(z'') \geq \xi(z')$. Thus

$$\xi(z'') + \varepsilon_{N_0'} \geq \xi(z') + \varepsilon_{N_0'} = \xi(z') + (\varepsilon_{N_0'} - \varepsilon_0) + \varepsilon_0 \geq 0.$$

Hence $x'' \in \widetilde{S}_\xi(\varepsilon_{N_0'}, \mu_{N_0'})$. Also, since $x' \in (x_0 + W_1)$ and by (1.2.4) and (1.2.15), we have

$$x'' \in x' + V_1 \cap W_1 \subseteq x_0 + W_1 + W_1 \subseteq x_0 + W_0.$$

This mean that $(x_0 + W_0) \cap \widetilde{S}_\xi(\varepsilon_{N_0'}, \mu_{N_0'}) \neq \emptyset$, which contradicts with (1.2.3). This completes the proof. □ □

Theorem 29. *We assume that for any given $\xi \in B_e^*$, there exists $\delta > 0$ such that the approximate solution set $\widetilde{S}_\xi(\cdot, \cdot)$ exists in $[\varepsilon_0, \delta) \times N(\mu_0)$. Suppose that the conditions (i)-(iv) as in Theorem 1 are satisfied. Assume further that for each $x \in K(\mu_0)$, $F(x, K(\mu_0), \mu_0) + C$ is a convex set. Then, the approximate solution mapping $\widetilde{S} : [\varepsilon_0, \delta) \times N(\mu_0) \to 2^X$ is l.s.c at (ε_0, μ_0).*

Proof. Since $F(x, K(\mu_0), \mu_0) + C$ is a convex set for each $x \in K(\mu_0)$, by virtue of Lemma 27, it holds that $\widetilde{S}(\varepsilon_0, \mu_0) = \bigcup_{\xi \in B_e^*} \widetilde{S}_\xi(\varepsilon_0, \mu_0)$. It follows from Theorem 1 that for each $\xi \in B_e^*$, $\widetilde{S}_\xi(\cdot, \cdot)$ is l.s.c at (ε_0, μ_0). Thus, in view of Lemma 26, we obtain that $\widetilde{S}(\cdot, \cdot)$ is l.s.c at (ε_0, μ_0). □ □

The following example illustrates all of the assumptions in Theorem 29.

Example 30. Let $Y = \mathbb{R}^2, C = \mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ and $Z = X = \mathbb{R}$. Let $B\left(0, \frac{1}{2}\right)$ be the closed ball of radius $1/2$ in $\mathbb{R}^2$. Let $B = [-2, 2]$, $M = [-1, 1]$ and the set-valued mapping $F : B \times B \times M \to 2^Y$ be defined by

$$F(x, y, \mu) = (w(x, y, \mu), v(x, y, \mu)) + B\left(0, 1/2\right),$$

where $w(x, y, \mu) := y^2(2^\mu - 1) + x(y - x + 1) - 3y + 2$ and $v(x, y, \mu) := y^2(2^\mu - 1) - x^2 + 2xy + 3$. Define a set-valued mapping $K : M \to 2^X$, for all $\mu \in M$, by $K(\mu) := [-2 + \mu, 2 + \mu] \cap [-2, 2]$. We choose $e = (1, 1) \in \text{int}C$, $\varepsilon_0 = 2.5$, $\mu_0 = 0$ and $\xi = (1, 0)$. We can see that $B^*_{(1,1)} = \{(x_1, x_2) : x_1 + x_2 = 1, x_1, x_2 \geq 0\}$ and $1 \in \widetilde{S}_{(1,0)}(\varepsilon_0, 0)$. Further, for any $\mu \in (-1, 1)$ there exists $\varepsilon \in [2.5, 4.5)$ such that $1 \in \widetilde{S}_{(1,0)}(\varepsilon, \mu)$. Hence, $\widetilde{S}_{(1,0)}(\cdot, \cdot)$ exists in $[2.5, 4.5) \times [-1, 1]$. It is easy to observe that for any $y \in K(0), F(\cdot, y, 0)$ is C-concave on $K(0)$. Clearly, the condition (ii) is true. It is obvious that $K(M) = [-2, 2]$. Let $N(\mu_0) = [-1, 1]$, we can see that $F(\cdot, \cdot, \cdot)$ is uniformly continuous on $K(M) \times K(M) \times N(\mu_0)$. Finally, we can check that for each $x \in [-2, 2]$, $F(x, [-2, 2], 0) + C$ is a convex set. Applying Theorem 29, we obtain that $\widetilde{S}$ is l.s.c at $(2.5, 0)$. □

The following example illustrate that the concavity of F cannot be dropped.

Example 31. Let $Y = \mathbb{R}^2, C = \mathbb{R}^2_+$ and $Z = X = \mathbb{R}$. Let $B = [-2, 2]$, $M = [-1, 1]$ and the set-valued mapping $F : B \times B \times M \to 2^Y$ be defined by

$$F(x, y, \mu) = [\mu x(x - y) - 0.5, 2] \times \{x(x - y) - 0.5\}.$$

Define a set-valued mapping $K : M \to 2^X$, for all $\mu \in M$, by $K(\mu) := [0, 1]$. We choose $e = (1, 1) \in \text{int}C$, $\varepsilon_0 = 0.5$, $\mu_0 = 0$. Then, all assumption of Theorem 29 are satisfied except (iii). Indeed, taking $y = 1$, $x_1 = 0, x_2 = 1$ and $t = 0.5$, we have

$$\begin{aligned}
(-2.5, -0.25) &= (-0.5, -0.75) - 0.5(2, -0.5) - 0.5(2, -0.5) \\
&\in [-0.5, 2] \times \{-0.75\} - 0.5\left([-0.5, 2] \times \{-0.5\}\right) \\
&\quad -0.5\left([-0.5, 2] \times \{-0.5\}\right) \\
&\in F(0.5(0) + 0.5(1), 1, 0) - 0.5F(0, 1, 0) - 0.5F(1, 1, 0) \\
&= F(0.5, 1, 0) - 0.5F(0, 1, 0) - 0.5F(1, 1, 0),
\end{aligned}$$

but $(-2.5, -0.25) \notin C$. The direct computation shows that

$$\tilde{S}(\varepsilon_0, \mu) = \begin{cases} \{0, 1\}, & \text{if } \mu \in (0, 1], \\ [0, 1], & \text{if } \mu = 0, \\ \{0\}, & \text{if } \mu \in [-1, 0). \end{cases} \tag{1.2.17}$$

Clearly, we see that $\tilde{S}(\cdot, \cdot)$ is even not l.s.c at (ε_0, μ_0), since $F(\cdot, y, \mu_0)$ is not C-concave on $K(\mu_0)$.

1.3 Well-posedness by perturbations for the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term

It is well known that the well-posedness is very important for both optimization theory and numerical methods of optimization problems, which guarantees that, for approximating solution sequences, there is a subsequence which converges to a solution. The study of well-posedness originates from Tikhonov [137], which means the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Levitin-Polyak [122] introduced a new notion of well-posedness that strengthened Tykhonov's concept as it required the convergence to the optimal solution of each sequence belonging to a larger set of minimizing sequences.

Another important notion of well-posedness for a minimization problem is the well-posedness by perturbations or extended well-posedness due to Zolezzi [142, 143]. The notion of well-posedness by perturbations establishes a form of continuous dependence of the solutions upon a parameter. There are many other notions of well-posedness in optimization problems. For more details, see, e.g., [142, 143, 103, 107, 112, 116, 119, 128, 133, 138, 140]. Meanwhile, the concept of well-posedness has been generalized to other varia- tional problems such as variational inequalities [106, 111, 113, 114, 125, 126, 127, 128], saddle point problems [104], Nash equilibrium problems [127, 129, 130, 131, 132, 134], equilibrium problems [115], inclusion problems [123, 124] ,and fixed point problems [123, 124, 141]

Lucchetti and Patrone [128] introduced the notion of well-posedness for variational inequalities and proved some related results by means of Ekeland's variational principle. From then on, many papers have been devoted to the extensions of well-posedness of minimization problems to various variational inequalities. Lignola and Morgan [126] generalized the notion of well-posedness by perturbations to a variational inequality and established the equivalence between the well-posedness by perturbations of a variational inequality and the well-posedness by perturbations of the corresponding minimization problem. Lignola and Morgan [127] investigated the concepts of α-well-posedness for variational inequalities. Del Prete et al. [111] further proved that the α-well-posedness of variational inequalities is closely related to the well-posedness of minimization problems. Recently, Fang et al. [117] generalized the notions of well-posedness and α-well-posedness to a mixed variational inequality. In the setting of Hilbert spaces, Fang et al. [117] proved that under suitable conditions the well-posedness of a mixed variational inequality is equivalent to the existence and uniqueness of its solution. They also showed that the well-posedness of a mixed variational inequality has close links with the well-posedness of the corresponding inclusion problem and corresponding fixed point problem in the setting of Hilbert spaces. Very recently, Fang et al. [116] generalized the notion of well-posedness by perturbations to a mixed variational inequality in Banach spaces. In the setting of Banach spaces, they established some metric characterizations, and showed that the well-posedness by perturbations of a mixed variational inequality is closely related to the well-posedness by perturbations of the corresponding inclusion problem and corresponding fixed point problem. They also derived some conditions under which the well-posedness by perturbations of the mixed variational inequality is equivalent to the existence and uniqueness of its solution.

On the other hand, the notion of hemivariational inequality was introduced by Panagiotopoulos [135, 136] at the beginning of the 1980s as a variational formulation for several classes of mechanical problems with nonsmooth and nonconvex energy super-potentials. In the case of convex super-potentials, hemivariational inequalities reduce to variational inequalities which were studied earlier by many authors (see e.g. Fichera [118] or Hartman and Stampacchia [120]). Wangkeeree and Preechasilp [139] also introduced and studied some existence results for the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term in reflexive Banach spaces. Recently Ceng et al. [105] considered an extension of the notion of well-posedness by perturbations, introduced by Zolezzi for a minimization problem, to a class of variational-hemivariational inequalities with perturbations in Banach spaces. Under very mild conditions, they established some metric characterizations for the well-posed variational-hemivariational inequality, and proved that the well-posedness by perturbations of a variational hemivariational inequality is closely related to the well-posedness by perturbations of the corresponding inclusion problem. Furthermore, in the setting of finite-dimensional spaces they also derived some conditions under which the variational-hemivariational inequality is strongly generalized well-posed-like by perturbations.

The aim of this paper is to introduce the new notion of well-posedness by perturbations to the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term (HVIMN) in Banach spaces. Under very suitable conditions, we establish some metric characterizations for the well-posed (HVIMN). In the setting of finite-dimensional spaces, the strongly generalized well-posedness by perturbations for (HVIMN) are established. The example illustrating main results is established. Our results are new and improve recent existing ones in the literature.

Let K be a nonempty, closed and convex subset of a real reflexive Banach space E with its dual E^*, $F : K \rightrightarrows 2^{E^*}$ a multivalued mapping. Let Ω be a bounded open set in $\mathbb{R}^N$, $T : E \to L^q(\Omega; \mathbb{R}^k)$ a linear continuous mapping, where $1 < q < \infty, k \geq 1$ and $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ a function. We shall denote $\hat{u} := Tu, j^\circ(x, y; h)$ denotes the Clarke's generalized directional derivative of a locally Lipschitz mapping $j(x, \cdot)$ at the point $y \in \mathbb{R}^k$ with respect to direction $h \in \mathbb{R}^k$, where $x \in \Omega$.

For the given bifunction $f : K \times K \to [-\infty, +\infty]$ imposed the condition that the set $\mathcal{D}_1(f) = \{u \in K : \ f(u, v) \neq -\infty, \ \forall v \in K\}$ is nonempty, Wangkeeree and Preechasilp [139] introduced and studied the existence of a solution for the following hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term

$$\text{(HVIMN)} \begin{cases} \text{Find } u \in \mathcal{D}_1(f) \text{ and } u^* \in F(u) \text{ such that} \\ \langle u^*, v - u\rangle + f(u, v) + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \\ \forall v \in K. \end{cases} \tag{1.3.1}$$

Now, let us consider some special cases of the problem (1.3.1). If $f(u, v) = \phi(v) - \phi(u)$, where $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function such that $K_\phi = K \cap \mathrm{dom}\phi \neq \emptyset$, then $\mathcal{D}_1(f) = K_\phi$ and (1.3.1) is reduced to the following *variational-hemivariational inequality problem*: Find $u \in K_\phi$ such that

$$\langle u^*, v - u\rangle + \phi(v) - \phi(u) + \int_\Omega j(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \quad \forall v \in K. \tag{1.3.2}$$

The problem (1.3.2) was studied by Costea and Lupu [109] by assuming that F is monotone and lower hemicontinuous and several existence results were obtained. Furthermore, if $F \equiv 0$ and $f(u, v) = \Lambda(u, v) - \langle g^*, v - u\rangle$, where $\Lambda : K \times K \to \mathbb{R}$ and $g^* \in X^*$, then (1.3.1) reduces to the problem: Find $u \in K$ such that

$$\Lambda(u, v) + \int_\Omega j(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq \langle g^*, v - u\rangle, \quad \forall v \in K. \tag{1.3.3}$$

The problem (1.3.3) was studied by Costea and Radulescu [110] and it was called nonlinear hemivariational inequality (see also Andrei and Costea [102] for some applications of nonlinear hemivariational inequalities to Nonsmooth Mechanics).

Now, suppose that L is a parametric normed space, $P \subset L$ is a closed ball with positive radius $p^* \in P$ is a fixed point. Let $\tilde{F} : P \times K \to 2^{E^*}$ be multivalued mapping. Let $\tilde{T} : P \times E \to L^p(\Omega; \mathbb{R}^k)$ be a linear continuous mapping, where $1 < p < \infty, k \geq 1$ and $\tilde{j} : P \times \Omega \times \mathbb{R}^k \to \mathbb{R}$ a function. We denote $\tilde{j}^\circ_p(x, y; h)$ denotes the Clarke's generalized directional derivative of a locally Lipschitz mapping $\tilde{j}(p, x, \cdot)$ at the point $y \in \mathbb{R}^k$ with respect to direction $h \in \mathbb{R}^k$. For the given bifunction $\tilde{f} : P \times K \times K \to [-\infty, +\infty]$, we assume the condition

$$\tilde{\mathcal{D}}_1(\tilde{f}) = \{u \in K | \tilde{f}(p^*, u, v) \neq -\infty, \forall v \in K\} \neq \emptyset.$$

The perturbed problem of the HVIMN (1.3.1) is given by

$$(\text{HVIMN}_{p^*}) \begin{cases} \text{Find } u \in \tilde{\mathcal{D}}_1(\tilde{f}) \text{ and } u^* \in \tilde{F}(p^*, u) \text{ such that} \\ \langle u^*, v - u \rangle + \tilde{f}(p^*, u, v) + \int_\Omega \tilde{j}^\circ_{p*}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x)) dx \geq 0, \\ \forall v \in K. \end{cases} \tag{1.3.4}$$

Let $\bar{\partial} j : E \to 2^{E^*} \setminus \{0\}$ denote the Clarke's generalized gradient of locally Lipschitz functional j (see [108]). That is

$$\bar{\partial} j(x) = \{\xi \in E^* : \langle \xi, v \rangle \leq j^0(x, y), \forall y \in E\}.$$

The following useful results can be found in [108].

Proposition 32. *Let X be a Banach space, $x, y \in X$ and J be a locally Lipschitz functional defined on X. Then*

(i) The function $y \mapsto j^\circ(x, y)$ is finite, positively homogeneous, subadditive and then convex on X;

(ii) $j^\circ(x, y)$ is upper semicontinuous as a function of (x, y), as a function of y alone, is Lipschitz continuous on X;

(iii) $j^\circ(x, -y) = (-j)^\circ(x, y)$;

(iv) $\bar{\partial} j(x)$ is a nonempty, convex, bounded, weak-compact subset of X^*;*

(v) For every $y \in X$, one has

$$j^\circ(x, y) = \max\{\langle \xi, y \rangle : \xi \in \bar{\partial} j(x)\}.$$

Definition 33. *The set-valued map F is said to be*

1. upper semicontinuous (usc) *at $x \in$ dom F if for any open set U satisfying $F(x) \subset U$, there exists a $\delta > 0$ such that $F(y) \subset U$, for every $y \in B(x, \delta)$;*

2. lower semicontinuous (lsc) *at $x \in$ dom F if for any open set U satisfying $F(x) \cap U \neq \emptyset$, there exists a $\delta > 0$ such that $F(y) \cap U \neq \emptyset$, for every $y \in B(x, \delta)$;*

3. closed *at $x \in$ dom F if for each sequence $\{x_n\}$ in X converging to x and $\{y_n\}$ in Y converging to y such that $y_n \in F(x_n)$, we have $y \in F(x)$.*

If $S \subseteq X$, then F is said to be usc (lsc, closed respectively) on the set S if F is usc (lsc, closed respectively) at every $x \in$ dom $F \cap S$.

Remark 34. An equivalent formulation of Definition 33(ii) is as follows: F is said to be lsc at $x \in$ dom F if for each sequence $\{x_n\}$ in dom F converging to x and for any $y \in F(x)$, there exists a sequence $\{y_n\}$ in $F(x_n)$ converging to y.

Definition 35. (see [121]) Let S be a nonempty subset of X. The measure, say μ, of noncompactness for the set S is defined by

$$\mu(S) := \inf\{\varepsilon > 0 : S \subset \cup_{i=1}^{n} S_i,\ \text{diam}|S_i| < \varepsilon, i = 1, 2, \dots, n,\ \text{for some integer } n \geq 1\},$$

where diam$|S_i|$ means the diameter of set S_i.

Definition 36. (see[121]) Let A, B be nonempty subsets of X. The Hausdorff metric $H(\cdot,\cdot)$ between A and B is defined by

$$H(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) := \sup_{a\in A} d(a, B)$ with $d(a, B) = \inf_{b\in B} \|a - b\|$.

Let $\{A_n\}$ be a sequence of nonempty subsets of X. We say that A_n converges to A in the sense of Hausdorff metric if $H(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all section $a_n \in A_n$. For more details on this topic, we refer the readers to [121].

1.4 Well-posedness by perturbations and metric characterizations

In this section, we generalize the concepts of well-posedness by perturbations to the variationalhemivariational inequality and establish their metric characterizations. In the sequel we always denote by $\to$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively. Let $\alpha \geq 0$ be a fixed number.

Definition 37. Let $\{p_n\} \subset P$ be such that $p_n \to p^*$. A sequence $\{u_n\} \subset E$ is called an α-approximating sequence corresponding to $\{p_n\}$ for HVIMN (1.3.1) if there exist a sequence $\{\varepsilon_n\}$ of nonnegative numbers with $\varepsilon_n \to 0$, $u_n^* \in \tilde{F}(p_n, u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$, and

$$\begin{aligned}&\langle u_n^*, v - u_n\rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx\\ &\geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon_n, \qquad \forall v \in K.\end{aligned}$$

for each $n \geq 1$. Whenever $\alpha = 0$, we say that $\{u_n\}$ is an approximating sequence corresponding to $\{p_n\}$ for HVIMN (1.3.1). Clearly, every α_2-approximating sequence corresponding to $\{p_n\}$ is α_1-approximating sequence corresponding to $\{p_n\}$ whenever $\alpha_1 > \alpha_2 \geq 0$.

Definition 38. We say that HVIMN (1.3.1) is strongly (resp., weakly) $\alpha-$well-posed by perturbations if

(i) HVIMN (1.3.1) has a unique solution

(ii) for any $\{p_n\} \subset P$ with $p_n \to p^*$, every $\alpha-$approximating sequence corresponding to $\{p_n\}$ converges strongly (resp., weakly) to the unique solution.

In the sequel, strong (resp., weak) $0-$well-posedness by perturbations is always called as strong (resp., weak) well-posedness by perturbations. If $\alpha_1 > \alpha_2 \geq 0$, then strong (resp., weak) α_1-well-posedness by perturbations implies strong (resp., weak) α_2-well-posedness by perturbations.

Definition 39. We say that HVIMN (1.3.1) is strongly (resp., weakly) generalized $\alpha-$well-posed by perturbations if

(i) HVIMN (1.3.1) has a nonempty solution set S

(ii) for any $\{p_n\} \subset P$ with $p_n \to p^*$, every $\alpha-$approximating sequence corresponding to $\{p_n\}$ has some subsequence which converges strongly (resp., weakly) to some point of S

In the sequel, strong (resp., weak) generalized $0-$well-posedness by perturbations is always called as strong (resp., weak) generalized well-posedness by perturbations.

If $\alpha_1 > \alpha_2 \geq 0$, then strong (resp., weak) generalized α_1-well-posedness by perturbations implies strong (resp., weak) generalized α_2-well-posedness by perturbations.

To derive the metric characterizations of α-well-posedness by perturbations, we consider the following approximating solution set of HVIMN (1.3.1):

$$\begin{aligned}\Omega_\alpha(\varepsilon) \quad = \quad & \bigcup_{p\in B(p^*,\varepsilon)} \{u \in \tilde{D}_1(\tilde{f}), u^* \in \tilde{F}(p,u) : \langle u^*, v-u\rangle + \tilde{f}(p,u,v) \\ & + \int_\Omega \tilde{j}_p^\circ(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))dx \geq -\frac{\alpha}{2}\|v-u\|^2 - \varepsilon, \forall v \in K.\}\end{aligned}$$

when $B(p^*,\varepsilon)$ denotes the closed ball centered at p^* with radius ε. In this section, we assume that $\bar{u}$ is a fixed solution of HVIMN (1.3.1). Define

$$\theta(\varepsilon) = \sup\{\|u-\bar{u}\| : u \in \Omega_\alpha(\varepsilon)\}, \ \ \forall\varepsilon \geq 0.$$

It is easy to see that $\theta(\varepsilon)$ is the radius of the smallest closed ball centered at $\bar{u}$ containing $\Omega_\alpha(\varepsilon)$. Now, we give a metric characterization of strong α-well-posedness by perturbations by considering the behavior of $\theta(\varepsilon)$ when $\varepsilon \to 0$.

Theorem 40. *HVIMN (1.3.1) is strongly $\alpha-$well-posed by perturbations if and only if $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Proof. Assume that HVIMN (1.3.1) is strongly $\alpha-$well-posed by perturbations. Then $\bar{u} \in E$ is the unique solution of HVIMN (1.3.1). Suppose to the contrary that $\theta(\varepsilon) \not\to 0$ as $\varepsilon \to 0$. There exist $\delta > 0$ and $0 < \varepsilon_n \to 0$ such that

$$\theta(\varepsilon_n) > \delta > 0.$$

By the definition of θ, there exists $u_n \in \Omega_\alpha(\varepsilon_n)$ such that

$$\|u_n - \bar{u}\| > \delta. \tag{1.4.1}$$

Since $u_n \in \Omega_\alpha(\varepsilon_n)$, there exist $p_n \in B(p^*,\varepsilon_n), u_n^* \in \tilde{F}(p_n,u_n)$ such that

$$\langle u_n^*, v-u_n\rangle + \tilde{f}(p_n,u_n,v) + \int_\Omega \tilde{j}_{p_n}^\circ(x,\hat{u}_n(x);\hat{v}(x)-\hat{u}_n(x))dx \geq -\frac{\alpha}{2}\|v-u_n\|^2 - \varepsilon,$$

for all $v \in K$ and $n \geq 1$. Since $p_n \in B(p^*,\varepsilon_n)$, we have $p_n \to p^*$. Then $\{u_n\}$ is an α approximating sequence corresponding to $\{p_n\}$ for HVIMN (1.3.1). Since HVIMN (1.3.1) is strongly $\alpha-$well-posed by perturbations, we can get that $\|u_n - \bar{u}\| \to 0$, which leads to a contradiction with (1.4.1).

Conversely, suppose that $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then $\bar{u} \in E$ is the unique solution of HVIMN (1.3.1). Indeed, if $\hat{u}$ is another solution of HVIMN (1.3.1) with $\hat{u} \neq \bar{u}$, then by definition,

$$\theta(\varepsilon) \geq \|\bar{u} - \hat{u}\| > 0, \ \ \forall\varepsilon \geq 0,$$

a contradiction. Let $p_n \in P$ be such that $p_n \to p^*$ and let $\{u_n\}$ be an $\alpha-$approximating sequence corresponding to $\{p_n\}$ for HVIMN (1.3.1). Then there exist $0 < \varepsilon_n \to 0, u_n^* \in \tilde{F}(p_n, u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$ and

$$\begin{aligned}&\langle u_n^*, v - u_n\rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \\ &\geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon_n,\end{aligned}$$

for all $v \in K$ and $n \geq 1$. Take $\delta_n = \|p_n - p^*\|$ and $\varepsilon_n' = \max\{\delta_n, \varepsilon_n\}$. It is easy to verify that $u_n \in \Omega_\alpha(\varepsilon_n')$ with $\varepsilon_n' \to 0$. Put

$$t_n = \|u_n - \bar{u}\|,$$

by definition of θ, we can get that

$$\theta(\varepsilon_n') \geq t_n = \|u_n - \bar{u}\|.$$

Since $\theta(\varepsilon_n') \to 0$, we have $\|u_n - \bar{u}\| \to 0$ as $n \to \infty$. So, HVIMN (1.3.1) is strongly $\alpha-$well-posed by perturbations. □ □

Now, we give an example to illustrate Theorem 40.

Example 41. Let $E = \mathbb{R}, P = [-1, 1], K = \mathbb{R}, p^* = 0, \alpha = 2, \tilde{F}(p, u) = \{2u\}, \tilde{j} = 0, \tilde{f}(p, u, v) = (1 - \frac{(p^2+1)^2}{4})u^2$ for all $p \in P, u, v \in K$. Clearly $u = 0$ is a solution of HVIMN (1.3.1). For any $\varepsilon > 0$, it follows that

$$\begin{aligned}\Omega^p_\alpha(\varepsilon) &= \{u \in \tilde{D}_1(\tilde{f}), u^* \in \tilde{F}(p) : \langle u^*, v - u\rangle + u^2 - \frac{(p^2+1)^2}{4}u^2 \geq -(v-u)^2 - \varepsilon, \ \ \forall v \in K\} \\ &= \{u \in \mathbb{R} : 2u(v-u) + u^2 - \frac{(p^2+1)^2}{4}u^2 \geq -(v-u)^2 - \varepsilon, \ \ \forall v \in \mathbb{R}\} \\ &= \{u \in \mathbb{R} : -u^2 + 2uv - \frac{(p^2+1)^2}{3}u^2 \geq -(v-u)^2 - \varepsilon, \ \ \forall v \in \mathbb{R}\} \\ &= \{u \in \mathbb{R} : v^2 - (v-u)^2 - \frac{(p^2+1)^2}{4}u^2 \geq -(v-u)^2 - \varepsilon, \ \ \forall v \in \mathbb{R}\} \\ &= \{u \in \mathbb{R} : -v^2 + \frac{(p^2+1)^2}{4}u^2 \leq +\varepsilon, \ \ \forall v \in \mathbb{R}\} \\ &= \Big[-\frac{2\sqrt{\varepsilon}}{p^2+1}, \frac{2\sqrt{\varepsilon}}{p^2+1}\Big].\end{aligned}$$

Therefore,

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(0,\varepsilon)} \Omega^p_\alpha(\varepsilon) = \Big[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}\Big],$$

for sufficiently small $\varepsilon > 0$. By trivial computation, we have

$$\theta(\varepsilon) = \sup\{u - u^* : u \in \Omega_\alpha(\varepsilon)\} = 2\sqrt{\varepsilon} \to 0 \text{ as } \varepsilon \to 0.$$

By Theorem 40, HVIMN (1.3.1) is 2-well-posed by perturbations

To derive a characterization of strong generalized $\alpha-$well-posedness by perturbations, we need another function q which is defined by

$$q(\varepsilon) = e(\Omega_\alpha(\varepsilon), S), \ \ \forall \varepsilon \geq 0,$$

where S is the solution set of HVIMN (1.3.1) and e is defined as in definition 36.

Theorem 42. *HVIMN (1.3.1) is strongly generalized $\alpha-$well-posed by perturbations if and only if S is nonempty compact and $q(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Proof. Assume that HVIMN (1.3.1) is strongly generalized $\alpha-$well-posed by perturbations. Clearly, S is nonempty. Let $\{u_n\}$ be any sequence in S and $\{p_n\} \subset P$ be such that $p_n = p^*$. Then $\{u_n\}$ is an α-approximating sequence corresponding to $\{p_n\}$ for HVIMN (1.3.1). Since HVIMN (1.3.1) is strongly generalized $\alpha-$well-posed by perturbations, we have $\{u_n\}$ has a subsequence which converges strongly to some point of S. Thus S is compact. Next, we suppose that $q(\varepsilon) \not\to 0$ as $\varepsilon \to 0$, then there exist $l > 0, 0 < \varepsilon_n \to 0$ and $u_n \in \Omega_\alpha(\varepsilon_n)$ such that

$$u_n \notin S + B(0,l), \quad \forall n \geq 1. \tag{1.4.2}$$

Since $u_n \in \Omega_\alpha(\varepsilon_n)$, there exist $p_n \in B(p^*, \varepsilon), u_n^* \in \tilde{F}(p_n, u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$ and

$$\langle u_n^*, v - u_n \rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon,$$

for all $v \in K$ and $n \geq 1$. Since $p_n \in B(p^*, \varepsilon_n)$, we have $p_n \to p^*$. Then $\{u_n\}$ is an α approximating sequence corresponding to $\{p_n\}$ for HVIMN (1.3.1). Since HVIMN (1.3.1) is strongly generalized $\alpha-$well-posed by perturbations, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging strongly to some point of S, which leads to a contradiction with (1.4.2) and so $q(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Conversely, we assume that S is nonempty compact and $q(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let $\{p_n\} \subset P$ be such that $p_n \to p^*$ and let $\{u_n\}$ be an $\alpha-$approximating sequence corresponding to $\{p_n\}$. Take $\varepsilon'_n = \max\{\varepsilon_n, \|p_n - p^*\|\}$. Thus $\varepsilon'_n \to 0$ and $x_n \in \Omega_\alpha(\varepsilon'_n)$. It follows that

$$d(u_n, S) \geq e(\Omega_\alpha(\varepsilon'_n), S) = q(\varepsilon'_n) \to 0.$$

Since S is compact, there exists $\bar{u}_n \in S$ such that

$$\|u_n - \bar{u}_n\| = d(x_n, S) \to 0.$$

Again from the compactness of S, $\{\bar{u}_n\}$ has a subsequence $\{\bar{u}_{n_k}\}$ which converges to $\bar{u}$. Thus HVIMN (1.3.1) is strongly generalized $\alpha-$well-posed by perturbations. □ □

The following example is shown for illustrating the metric characterizations in Theorem 42.

Example 43. Let $E = \mathbb{R}, P = [-1,1], K = \mathbb{R}, p^* = 0, \alpha = 2, \tilde{F}(p,u) = \{2u\}, \tilde{j} = 0, \tilde{f}(p,u,v) = (1 - \frac{(p^2+1)^2}{4})u^2$ for all $p \in P, u, v \in K$. It is easy to see that $u = 0$ is a solution of HVIMN (1.3.1). Repeating the same argument as in Example 41, we obtain that

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(0,\varepsilon)} \Omega^p_\alpha(\varepsilon) = \Big[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}\Big],$$

for sufficiently small $\epsilon > 0$. By trivial computation, we have

$$q(\epsilon) = e(\Omega_\alpha(\epsilon), S) = \sup_{u(\epsilon) \in \Omega_\alpha(\epsilon)} d(u(\epsilon), S) \to 0 \text{ as } \epsilon \to 0.$$

By Theorem 42, HVIMN (1.3.1) is strongly generalized $\alpha-$well-posed by perturbations.

The strong generalized α-well-posedness by perturbations can be also characterized by the behavior of the noncompactness measure $\mu(\Omega_\alpha(\epsilon))$.

Theorem 44. *Let L be finite-dimensional, $\tilde{j}_p^\circ(x,y)$ be upper semicontinuous as a functional of $(p,x,y) \in P \times E \times E$ and f is convex. Let $\tilde{F}$ is closed on $P \times K$ and $\tilde{f}$ be continuous on $P \times K \times K$. Then HVIMN (1.3.1) is strongly generalized $\alpha-$well-posed by perturbations if and only if $\Omega_\alpha(\varepsilon) \neq \emptyset, \forall \varepsilon > 0$ and $\mu(\Omega_\alpha(\varepsilon)) \to 0$ as $\varepsilon \to 0$.*

Proof. First, we will prove that $\Omega_\alpha(\varepsilon)$ is closed for all $\varepsilon \geq 0$. Let $\{u_n\} \subset \Omega_\alpha(\varepsilon)$ with $u_n \to \bar{u}$. Then there exist $p_n \in B(p^*,\varepsilon), u_n^* \in \tilde{F}(p_n,u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$ and

$$\langle u_n^*, v-u_n\rangle + \tilde{f}(p_n,u_n,v) + \int_\Omega \tilde{j}_{p_n}^\circ(x,\hat{u}_n(x);\hat{v}(x)-\hat{u}_n(x))dx \geq -\frac{\alpha}{2}\|v-u_n\|^2 - \varepsilon, \tag{1.4.3}$$

for all $v \in K$ and $n \geq 1$. Without loss of generality, we may assume that $p_n \to \bar{p} \in B(p^*,\varepsilon)$ because L is finite dimensional. Since $\tilde{j}_p(x,y)$ is upper semicontinuous as a functional of $(p,x,y) \in P \times E \times E$. Hence it follows from (1.4.3) and the continuity of $\tilde{f}$ that

$$\begin{aligned}
&\langle u^*, v-\bar{u}\rangle + \tilde{f}(\bar{p},\bar{u},v) + \int_\Omega \tilde{j}_{\bar{p}}^\circ(x,\hat{u}_n(x);\hat{v}(x)-\hat{u}_n(x))dx \\
&\geq \limsup_{n\to\infty} \langle u_n^*, v-u_n\rangle + \tilde{f}(p_n,u_n,v) + \int_\Omega \tilde{j}_{p_n}^\circ(x,\hat{u}_n(x);\hat{v}(x)-\hat{u}_n(x))dx \\
&\geq \limsup_{n\to\infty} -\frac{\alpha}{2}\|v-u_n\|^2 - \varepsilon, \\
&= -\frac{\alpha}{2}\|v-\bar{u}\|^2 - \varepsilon \ \ \forall v \in K.
\end{aligned}$$

Thus $\bar{u} \in \Omega_\alpha(\varepsilon)$. Hence $\Omega_\alpha(\varepsilon)$ is closed.

Next, we show that

$$S = \bigcap_{\varepsilon>0} \Omega_\alpha(\varepsilon). \tag{1.4.4}$$

It is easy to see that $S \subseteq \cap_{\varepsilon>0}\Omega_\alpha(\varepsilon)$. Thus, we show that $\cap_{\varepsilon>0}\Omega_\alpha(\varepsilon) \subseteq S$. Let $\bar{u} \in \cap_{\varepsilon>0}\Omega_\alpha(\varepsilon)$. Let $\{\varepsilon_n\}$ be a sequence of positive real numbers such that $\varepsilon_n \to 0$. Thus

$$\bar{u} \in \Omega_\alpha(\varepsilon_n)$$

and so there exist $p_n \in B(p^*,\varepsilon_n)$ and $u^* \in \tilde{F}(p_n,\bar{u})$ such that $\bar{u} \in \tilde{D}_1(\tilde{f})$ and

$$\langle u^*, v-\bar{u}\rangle + \tilde{f}(p_n,\bar{u},v) + \int_\Omega \tilde{j}_{p_n}^\circ(x,\hat{\bar{u}}(x);\hat{v}(x)-\hat{\bar{u}}(x))dx \geq -\frac{\alpha}{2}\|v-\bar{u}\|^2 - \varepsilon_n, \tag{1.4.5}$$

for all $v \in K$ and $n \geq 1$. It is easy to verify that $p_n \to p^*$. Taking limit as $n \to \infty$, we can get that

$$\begin{aligned}
&\langle u^*, v-\bar{u}\rangle + f(\bar{u},v) + \int_\Omega j^\circ(x,\hat{\bar{u}}(x);\hat{v}(x)-\hat{\bar{u}}(x))dx \\
&= \langle u^*, v-\bar{u}\rangle + \tilde{f}(p^*,\bar{u},v) + \int_\Omega \tilde{j}_{p^*}^\circ(x,\hat{\bar{u}}(x);\hat{v}(x)-\hat{\bar{u}}(x))dx \\
&\geq -\frac{\alpha}{2}\|v-\bar{u}\|^2, \quad \forall v \in K
\end{aligned} \tag{1.4.6}$$

Since $\tilde{F}$ is closed on $P \times K$, we have $u^* \in F(\bar{u})$ and for any $z \in K$ and $t \in (0,1)$, letting $v = \bar{u} + t(z - \bar{u})$ in (1.4.6), we can get from T is linear, f is convex and definition of j° that

$$\begin{aligned}
&t\langle u^*, z - \bar{u}\rangle + tf(\bar{u}, z) + \int_\Omega j^\circ(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \\
&\geq t\langle u^*, z - \bar{u}\rangle + f(\bar{u}, \bar{u} + t(z - \bar{u})) + \int_\Omega j^\circ(x, \hat{\bar{u}}(x); \hat{z}(x) - \hat{\bar{u}}(x))dx \\
&\geq -\frac{\alpha t^2}{2}\|z - \bar{u}\|^2.
\end{aligned}$$

This implies that

$$\langle u^*, z - \bar{u}\rangle + tf(\bar{u}, z) + \int_\Omega j^\circ(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \geq -\frac{\alpha t}{2}\|z - \bar{u}\|^2 \quad \forall z \in K.$$

As $t \to 0$ in the last inequality, we get

$$\langle u^*, z - \bar{u}\rangle + tf(\bar{u}, z) + \int_\Omega j^\circ(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \geq 0 \quad \forall z \in K.$$

Hence $\bar{u} \in S$ and thus (1.4.4) is proved. Next, we suppose that HVIMN (1.3.1) is strongly generalized $\alpha-$well-posed by perturbations. By Theorem 42, we can get that S is nonempty compact and $q(\varepsilon) \to 0$. Since $S \subset \Omega_\alpha(\varepsilon)$ for all $\varepsilon > 0$, we have

$$\Omega_\alpha(\varepsilon) \neq \emptyset, \quad \forall \varepsilon > 0.$$

We observe that for each $\varepsilon > 0$,

$$H(\Omega_\alpha(\varepsilon), S) = \max\{e(\Omega_\alpha(\varepsilon), S), e(S, \Omega_\alpha(\varepsilon))\} = e(\Omega_\alpha(\varepsilon), S).$$

By the compactness of S, we have

$$\mu(\Omega_\alpha(\varepsilon)) \leq 2H(\Omega_\alpha(\varepsilon), S) = 2q(\varepsilon) \to 0.$$

Conversely, we suppose that $\Omega_\alpha(\varepsilon) \neq \emptyset, \quad \forall \varepsilon > 0$ and $\mu(\Omega_\alpha(\varepsilon)) \to 0$ as $\varepsilon \to 0$. Since $\Omega_\alpha(\cdot)$, by the Kuratowski theorem, we can get from (1.4.4) that

$$q(\varepsilon) = H(\Omega_\alpha(\varepsilon), S) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and S is nonempty compact. Hence HVIMN (1.3.1) is strongly generalized $\alpha-$well-posed by perturbations by Theorem 42. $\square$ $\square$

The following example is given for illustrating the measure in Theorem 44.

Example 45. Let $E = \mathbb{R}, P = [-1,1], K = \mathbb{R}, p^* = 0, \alpha = 2, \tilde{F}(p,u) = \{2u\}, \tilde{j} = 0, \tilde{f}(p,u,v) = (1 - \frac{(p^2+1)^2}{4})u^2$ for all $p \in P, u, v \in K$. It is easy to see that $u = 0$ is a solution of HVIMN (1.3.1). Repeating the same argument as in Example 41, we obtain that

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(0,\varepsilon)} \Omega_\alpha^p(\varepsilon) = \Big[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon} \Big].$$

We will show that $\mu(\Omega_\alpha(\epsilon)) = 0$ for each $\epsilon > 0$. Let $\epsilon > 0$. Consider

$$\mu(\Omega_\alpha(\epsilon)) = \inf\{\lambda > 0 : [-2\sqrt{\epsilon}, 2\sqrt{\epsilon}] \subseteq \bigcup_{k=1}^{n} [a_k, b_k], \text{ with } \operatorname{diam}[a_k, b_k] < \lambda, \forall i = 1, \dots, n, \exists n \in \mathbb{N}\}.$$

For every $\lambda > 0$, we can find $n \in \mathbb{N}$ with $a_1 = -2\sqrt{\epsilon}, b_n = 2\sqrt{\epsilon}$ such that

$$[-2\sqrt{\epsilon}, 2\sqrt{\epsilon}] \subseteq \bigcup_{k=1}^{n} [a_k, b_k] \quad \text{and} \quad \operatorname{diam}[a_k, b_k] < \lambda.$$

This implies that $\mu(\Omega_\alpha(\epsilon)) = 0$ for each $\epsilon > 0$. Then HVIMN (1.3.1) is strongly generalized $\alpha-$well-posed by perturbations.

Remark 46. Any solution of HVIMN (1.3.1) is a solution of the α problem: find $u \in D_1(f)$ and $u^* \in F(u)$ such that

$$\langle u^*, v - u \rangle + f(u, v) + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq -\frac{\alpha}{2}\|y - x\|^2, \quad \forall v \in K,$$

but the converse is not true in general. To show this, let $K = \mathbb{R}$,

$$F(u) = \{u\}, f(u,v) = 2u^2 - v \text{ and } j = 0,$$

for all $u, v \in K$. It is easy to see that the solution set of HVIMN (1.3.1) is empty and $u^* = u = 0$ is the unique solution of the corresponding α problem with $\alpha = 2$.

1.5 Levitin-Polyak Well-posedness for Lexicographic Vector Equilibrium Problems

Equilibrium problems first considered by Blum and Oettli [151] have been playing an important role in optimization theory with many striking applications particularly in transportation, mechanics, economics, etc. Equilibrium models incorporate many other important problems such as: optimization problems, variational inequalities, complementarity problems, saddlepoint/minimax problems, and fixed points. Equilibrium problems with scalar and vector objective functions have been widely studied. The crucial issue of solvability (the existence of solutions) has attracted the most considerable attention of researchers, see, e.g., [157, 161, 164, 183].

On the other hand, well-posedness plays an important role in the stability analysis and numerical methods for optimization theory and applications. Since any algorithm can generate only an approximating solution sequence which is meaningful only if the problem is well-posed under consideration. The first and oldest well-posedness is Hadamard well-posedness

[163], which means existence, uniqueness and continuous dependence of the optimal solution and optimal value from perturbed data. The second is Tikhonov well-posedness [184], which means the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Well-posedness properties have been intensively studied and the two classical well-posedness notions have been extended and blended. For parametric problems, well-posedness is closely related to stability. Up to now, there have been many works dealing with well-posedness of optimization-related problems as mathematical programming [182, 165], constrained minimization [155, **?**, **?**, 160] variational inequalities [155, 153, 159, 173, 185], Nash equilibria [185, 177], and equilibrium problems [160, 145, 167]. A fundamental requirement in Tykhonov well-posedness is that every minimizing sequence is from within the feasible region. However, in several numerical methods such as exterior penalty methods and augmented Lagrangian methods, the minimizing sequence generated may not be feasible. Taking this into account, Levitin and Polyak [171] introduced another notion of well-posedness which does not necessarily require the feasibility of the minimizing sequence. However, it requires the distance of the minimizing sequence from the feasible set to approach to zero eventually. Since then, many authors investigated the well-posedness and well-posedness in the gener- alized sense for optimization, variational inequalities and equilibrium problems. The study of Levitin-Polyak type well-posedness for scalar convex optimization probiems with functional constraints was initiated by Konsulova and Revalski [169]. In 1981, Lucchetti and Patrone [176] introduced and studied the well-posedness for variational inequalities, which is a generalization of the Tykhonov well-posedness of minimization problems. Long et al. [174] introduced and studied four types of Levitin-Polyak well-posedness of equilibrium problems with abstract set constraints and functional constraints. Li and Li [172] introduced and researched two types of Levitin-Polyak well-posedness of vector equilibrium problems with abstract set constraints. Peng et al. [179] introduced and studied four types of Levitin-Polyak well-posedness of vector equilibrium problems with abstract set constraints and functional constraints. Peng, Wu and Wang [180] introduced several types of Levitin-Polyak well-posedness for a generalized vector quasi-equilibrium problem with functional constraints and abstract set constraints. Chen, Wan and Cho [154] studied the Levitin-Polyak well-posedness by perturbations for a class of general systems of set-valued vector quasi-equilibrium problems in Hausdorff topological vector spaces. Very recently Lalitha and Bhatia [170] studied the LP well-posedness for a parametric quasivariational inequality problem of the Minty type.

With regard to vector equilibrium problems, most of existing results correspond to the case when the order is induced by a closed convex cone in a vector space. Thus, they cannot be applied to lexicographic cones, which are neither closed nor open. These cones have been extensively investigated in the framework of vector optimization, see, e.g., [146, 149, 150, 152, 158, 162, 168, 166]. For instance, Konnov and Ali [168] studied sequential problems, especially exploiting its relation with regularization methods. Bianchi et al. in [149] analyzed lexicographic equilibrium problems on a topological Hausdorff vector space, and their relationship with some other vector equilibrium problems. They obtained the existence results for the tangled lexicographic problem via the study of a related sequential problem. However, for equilibrium problems, the main emphasis has been on the issue of solvability/existence. To the best of the knowledge, very recently, Anh et al. in [146] studied the Tikhonov well-posedness for lexicographic vector equilibrium problems in metric spaces and gave the sufficient conditions for a family of such problems to be well-posed and uniquely well-posed at the considered point. Furthermore, they derived several results on well-posedness for a class of variational inequalities.

In this paper, we first introduce the new notions of Levitin-Polyak(LP) well-posedness and LP well-posedness in the generalized sense for the Lexicographic vector equilibrium problems.

Then, we establish some sufficient conditions for this problems to be LP well-posedness at the reference point. Furthermore, we give numerous examples to explain that all the imposed assumptions are very relaxed and cannot be dropped.

The layout of the paper is as follows. In Sect. 2, we introduce the notions of LP well-posedness and LP well-posedness in the generalized sense for the Lexicographic vector equilibrium problems. In Sect. 3, we establish some sufficient conditions for this problems to be LP well-posedness at the reference point. Section 4 is devoted to LP well-posedness in the generalized sense for the Lexicographic vector equilibrium problems. Some concluding remarks are included in the end of this paper.

We first recall the concept of lexicographic cone in finite dimensional spaces and models of equilibrium problems with the order induced by such a cone. The lexicographic cone of $\mathbb{R}^n$, denoted C_l , is the collection of zero and all vectors in $\mathbb{R}^n$ with the first nonzero coordinate being positive, i.e.,

$$C_l := \{0\} \cup \{x \in \mathbb{R}^n | \exists i \in \{1, 2, \ldots, n\} : x_i > 0 \text{ and } x_j = 0, \quad \forall j < i\}.$$

This cone is convex and pointed, and induces the total order as follow:

$$x \geq_l y \Leftrightarrow x - y \in C_l.$$

We also observe that it is neither closed nor open. Indeed, when comparing with the cone $C_1 := \{x \in \mathbb{R}^n | x_1 \geq 0\}$, we see that $\text{int}C_1 \subsetneq C_l \subsetneq C_1$, while

$$\text{int}C_l = \text{int}C_1 \text{ and } \text{cl}C_l = C_1.$$

Throughout this paper, if not other specified, X be a metric space and Λ denote the metric space. Let $X_0 \subset X$ be nonempty and closed sets . Let $f := (f_1, f_2, \ldots, f_n) : X \times X \times \Lambda \to \mathbb{R}^n$ be vector-valued function and $K : \Lambda \to 2^X$ being a closed valued map. The lexicographic vector quasiequilibrium problem consists of, for each $\lambda \in \Lambda$,

(LEP$_\lambda$) finding $\bar{x} \in K(\lambda)$ such that

$$f(\bar{x}, y, \lambda) \geq_l 0 \quad , \forall y \in K(\lambda).$$

Instead of writing $\{(\text{LEP}_\lambda) | \lambda \in \Lambda\}$ for the family of lexicographic vector equilibrium problem, i.e., the lexicographic parametric problem, we will simply write (LEP) in the sequel. Let $S : \Lambda \to 2^X$ be the solution map of (LEP); that is, for each $\lambda \in \Lambda$,

$$S(\bar{\lambda}) := \{x \in K(\bar{\lambda}) | f(x, y, \bar{\lambda}) \geq_l 0, \ \forall y \in K(\bar{\lambda})\}. \tag{1.5.1}$$

Following the lines of investigating ε-solutions to vector optimization problems initiated by Loridan [175], we consider, for each $\lambda \in \Lambda$ and each $\varepsilon \in [0, \infty)$, the following approximate problem:

(LEP$_{\lambda,\varepsilon}$) find $\bar{x} \in K(\lambda)$ such that

$$d(\bar{x}, K(\lambda)) \leq \epsilon \text{ and } f(\bar{x}, y, \lambda) + \varepsilon e \geq_l 0, \ \ \forall y \in K(\lambda),$$

where $e := (\underbrace{0, 0, \ldots, 0}_{n-1}, 1) \in \mathbb{R}^n$. The solution set of (LEP$_{\lambda,\varepsilon}$) is denoted by $\tilde{S}(\lambda, \varepsilon)$; that is the set valued-map $\tilde{S} : \Lambda \times \mathbb{R} \to 2^X$ is defined by

$$\tilde{S}(\lambda, \varepsilon) = \{x \in X | d(x, K(\lambda)) \leq \epsilon \text{ and } f(x, y, \lambda) + \varepsilon e \geq_l 0, \ \forall y \in K(\lambda)\}, \tag{1.5.2}$$

for all $(\lambda, \varepsilon) \in \Lambda \times \mathbb{R}$.

Now we introduce the concept of LP well-posedness for LEP. For this purpose, we require the the following notions of an LP approximating sequence.

Definition 47. *Let $\{\lambda_n\}$ be a sequence in Λ such that $\lambda_n \to \bar{\lambda}$. A sequence $\{x_n\}$ is said to be an LP approximating sequence for LEP with respect to $\{\lambda_n\}$ if there is a sequence $\{\epsilon_n\}$ in $(0, \infty)$ satisfying $\epsilon_n \to 0$ as $n \to \infty$, such that*

1. $d(x_n, K(\lambda_n)) \leq \epsilon_n$, for all $n \in \mathbb{N}$;
2. $f(x_n, y_n, \lambda_n) + \epsilon_n e \geq_l 0,\ \forall y_n \in K(\lambda_n)$.

Definition 48. The problem (LEP) is *LP well-posed* at $\bar{\lambda}$ if

1. there exists a unique solution $\bar{x}$ of LEP;
2. for any sequence $\{\lambda_n\}$ converging to $\bar{\lambda}$, every LP approximating sequence $\{x_n\}$ with respect to $\{\lambda_n\}$ converges to $\bar{x}$.

Definition 49. [147] *Let $Q : X \rightrightarrows Y$ be a set-valued mapping between metric spaces*

1. *Q is upper semicontinuous (usc) at $\bar{x}$ if for any open set $U \supseteq Q(\bar{x})$, there is a neighborhood N of $\bar{x}$ such that $Q(N) \subseteq U$.*
2. *Q is lower semicontinuous (lsc) at $\bar{x}$ if for any open subset U of Y with $Q(\bar{x}) \cap U \neq \emptyset$, there is a neighborhood N of $\bar{x}$ such that $Q(x) \cap U \neq \emptyset$ for all $x \in N$.*
3. *Q is closed at $\bar{x}$ if for any sequences $x_k \to \bar{x}$ and $y_k \to \bar{y}$ with $y_k \in Q(x_k)$, it holds $\bar{y} \in Q(\bar{x})$.*

Lemma 50. [147]

1. *If Q is usc at $\bar{x}$ and $Q(\bar{x})$ is compact, then for any sequence $x_n \to \bar{x}$, every sequence $\{y_n\}$ with $y_n \in Q(x_n)$ has a subsequence converging to some point in $Q(\bar{x})$. If, in addition, $Q(\bar{x}) = \{\bar{y}\}$ is a singleton, then such a sequence $\{y_n\}$ must converge to $\bar{y}$.*
2. *Q is lsc at $\bar{x}$ if and only if for any sequence $x_n \to \bar{x}$ and any point $y \in Q(\bar{x})$, there is a sequence $\{y_n\}$ with $y_n \in Q(x_n)$ converging to y.*

Definition 51. [146, 144] Let g be an extended real-valued function on a metric space X and ε be a real number.

1. g is *upper ε-level closed* at $\bar{x} \in X$ if for any sequence $x_n \to \bar{x}$,
$$[g(x_n) \geq \varepsilon,\ \forall n] \Rightarrow [g(\bar{x}) \geq \varepsilon].$$
2. g *is strongly upper ε-level closed* at $\bar{x} \in X$ if for any sequences $x_n \to \bar{x}$ and $\{v_n\} \subset [0, \infty)$ converging to 0,
$$[g(x_n) + v_n \geq \varepsilon,\ \forall n] \Rightarrow [g(\bar{x}) \geq \varepsilon].$$

Let A, B be two subsets of metric space X. The Hausdorff distance between A and B is defined as follows
$$H(A, B) = \max\{H^*(A, B), H^*(B, A)\},$$
where $H^*(A, B) = \sup_{a \in A} d(a, B)$, and $d(x, A) = \inf_{y \in A} d(x, y)$.

1.6 LP well-posedness for Lexicographic vector Equilibrium Problems

In this section, we shall give some neccessary and/or sufficient conditions for (LEP) to be LP well-posed at the reference point $\bar{\lambda} \in \Lambda$. To simplify the presentation, in the sequel, the results will be formulated for the case $n = 2$. For any two positive numbers α, ϵ, the solution set of approximation solutions for the problem (LEP$_{\lambda,\varepsilon}$) is denoted by

$$\Gamma(\bar{\lambda}, \alpha, \epsilon) = \bigcup_{\lambda \in B(\bar{\lambda},\alpha)\cap\Lambda} \{x \in X | d(x, K(\lambda)) \leq \epsilon \text{ and } f(x, y, \lambda) + \varepsilon e \geq_l 0,\ \forall y \in K(\lambda)\}, \tag{1.6.1}$$

where $B(\bar{\lambda}, \alpha)$ denote the closed ball centered at $\bar{\lambda}$ with radius α. The set-valued mapping $Z : \Lambda \times X \to 2^X$ next defined will play an important role our analysis

$$Z(\lambda, x) = \begin{cases} \{z \in K(\lambda) | f_1(x, z, \lambda) = 0\} & \text{if } (\lambda, x) \in \text{ gr } Z_1; \\ X \text{ otherwise,} \end{cases}$$

where $Z_1 : \Lambda \to 2^X$ denotes the solution mapping of the scalar equilibrium problem determined by the real-valued function f_1 :

$$Z_1(\lambda) = \{x \in K(\lambda) | f_1(x, y, \lambda) \geq 0,\ \forall y \in K(\lambda)\}.$$

Then (1.6.1) is equivalent to

$$\Gamma(\bar{\lambda}, \alpha, \epsilon)$$

$$= \bigcup_{\lambda \in B(\bar{\lambda},\alpha)\cap\Lambda} \{x \in X | d(x, K(\lambda)) \leq \epsilon, f_1(x, y, \lambda) \geq 0, \forall y \in K(\lambda) \text{ and } f_2(x, z, \lambda) + \epsilon \geq 0, \forall z \in Z(\lambda, x)\}$$

$$= \bigcup_{\lambda \in B(\bar{\lambda},\alpha)\cap\Lambda} \tilde{S}(\lambda, \varepsilon),$$

where $\tilde{S}$ is the solution map for (LEP$_{\lambda,\varepsilon}$) defined by (1.5.2). For the solution map $S : \Lambda \to 2^X$ of (LEP), in general, we observe that

$$\Gamma(\bar{\lambda}, 0, 0) = S(\bar{\lambda}) \text{ and } S(\bar{\lambda}) \subseteq \Gamma(\bar{\lambda}, \alpha, \epsilon),\ \forall \alpha, \epsilon > 0,$$

and hence

$$S(\bar{\lambda}) \subseteq \bigcap_{\alpha,\epsilon>0} \Gamma(\bar{\lambda}, \alpha, \epsilon).$$

Next, we provide the sufficient conditions for the two sets to coincide.

Proposition 52. *Suppose that the following conditions are satisfied :*

1. *K is closed and lsc on Λ;*
2. *Z is lsc on $\Lambda \times X$;*
3. *f_1 is upper 0-level closed on $X \times X \times \Lambda$;*
4. *f_2 is strongly upper 0-level closed on $X \times X \times \Lambda$;*

then

$$\bigcap_{\alpha,\epsilon>0} \Gamma(\bar{\lambda}, \alpha, \epsilon) = S(\bar{\lambda}).$$

Proof. Let $\bar{x} \in \bigcap_{\alpha,\epsilon>0} \Gamma(\bar{\lambda},\alpha,\epsilon)$, then without loss of generality, there exist sequences $\alpha_n > 0, \epsilon_n > 0$ with $\alpha_n \to 0, \epsilon_n \to 0$, such that $\bar{x} \in \Gamma(\bar{\lambda},\alpha_n,\epsilon_n)$. Hence, it follows that there exists a sequence $\lambda_n \in B(\bar{\lambda},\alpha_n) \cap \Lambda$, such that, for all $n \in \mathbb{N}$,

$$d(\bar{x}, K(\lambda_n)) \leq \epsilon_n, \tag{1.6.2}$$

and

$$f_1(\bar{x},y,\lambda_n) \geq 0,\ \forall y \in K(\lambda_n) \text{ and } f_2(\bar{x},z,\lambda_n) + \epsilon_n \geq 0,\ \forall z \in Z(\lambda_n,\bar{x}). \tag{1.6.3}$$

Since $K(\bar{\lambda})$ is a closed set in X, it follows from (1.6.2) that we can choose $x_n \in K(\lambda_n)$, such that

$$d(\bar{x}, x_n) \leq \epsilon_n,\ \forall n \in \mathbb{N}. \tag{1.6.4}$$

Thus $x_n \to \bar{x}$ as $n \to \infty$. Clearly $\lambda_n \to \bar{\lambda}$ as $n \to \infty$ and also as K is closed at $\bar{\lambda}$, it follows that $\bar{x} \in K(\bar{\lambda})$. As K is lsc at $\bar{\lambda}$ and $\lambda_n \to \bar{\lambda}$ for any $y \in K(\bar{\lambda})$ there exists $y_n \in K(\lambda_n)$ such that $y_n \to y$. Also Z is lsc at $(\bar{\lambda},\bar{x})$ and $(\lambda_n,x_n) \to (\bar{\lambda},\bar{x})$, it is clear that for any $z \in Z(\bar{\lambda},\bar{x})$ there exists a sequence $z_n \in Z(\lambda_n,x_n)$ such that $z_n \to z$. This implies by assumption (iii),(iv), and (1.6.3) that $f_1(\bar{x},y,\bar{\lambda}) \geq 0,\ f_2(\bar{x},z,\bar{\lambda}) \geq 0$ and hence, $\bar{x} \in S(\bar{\lambda})$. □ □

Theorem 1. *Suppose that the conditions (i)-(iv) in Proposition 52 are satisfied. Then* (LEP) *is LP well-posed at* $\bar{\lambda} \in \Lambda$ *if and only if* $\Gamma(\bar{\lambda},\alpha,\epsilon) \neq \emptyset, \forall \alpha,\epsilon > 0$ *and* diam $\Gamma(\bar{\lambda},\alpha,\epsilon) \to 0$ *as* $(\alpha,\epsilon) \to (0,0)$.

Proof. Suppose that the problem (LEP) is LP well-posed. Hence, it has a unique solution $\bar{x} \in S(\bar{\lambda})$ and hence $\Gamma(\bar{\lambda},\alpha,\epsilon) \neq \emptyset, \forall \alpha,\epsilon > 0$ as $S(\bar{\lambda}) \subseteq \Gamma(\bar{\lambda},\alpha,\epsilon)$. Suppose on the contrary that diam $\Gamma(\bar{\lambda},\alpha,\epsilon) \nrightarrow 0$ as $(\alpha,\epsilon) \to (0,0)$. Then there are positive numbers r, m and sequences $\{\alpha_n\}, \{\epsilon_n\}$ in $(0,\infty)$ with $(\alpha_n,\epsilon_n) \to (0,0)$ and $x_n, x'_n \in \Gamma(\bar{\lambda},\alpha_n,\epsilon_n)$ such that

$$d(x_n, x'_n) > r,\ \forall n \geq m. \tag{1.6.5}$$

By $x_n, x'_n \in \Gamma(\bar{\lambda},\alpha_n,\epsilon_n)$, there exist $\lambda_n, \lambda'_n \in B(\bar{\lambda},\alpha_n) \cap \Lambda$ such that

$$d(x_n, K(\lambda_n)) \leq \epsilon_n,$$

$$f_1(x_n,y,\lambda_n) \geq 0,\ \forall y \in K(\lambda_n) \text{ and } f_2(x_n,z,\lambda_n) + \varepsilon_n \geq 0,\ \forall z \in Z(\lambda_n,x_n) \tag{1.6.6}$$

and

$$d(x'_n, K(\lambda'_n)) \leq \epsilon_n,$$

$$f_1(x'_n,y,\lambda'_n) \geq 0,\ \forall y \in K(\lambda'_n), f_2(x'_n,z,\lambda'_n) + \varepsilon_n \geq 0,\ \forall z \in Z(\lambda'_n,x_n). \tag{1.6.7}$$

The sequence $\{x_n\}$ and $\{x'_n\}$ are LP approximating sequences for (LEP) corresponding to sequences $\lambda_n \to \bar{\lambda}$ and $\lambda'_n \to \bar{\lambda}'$, respectively. Since (LEP) is LP well-posed, we have that $\{x_n\}$ and $\{x'_n\}$ converse to the unique solution $\bar{x}$, which arrives a contradiction to (1.6.5). Hence, diam $\Gamma(\bar{\lambda},\alpha,\epsilon) \to 0$ as $(\alpha,\epsilon) \to (0,0)$.

Conversely, let $\{\lambda_n\}$ be a sequence in Λ converging to $\bar{\lambda}$ and $\{x_n\}$ be a LP approximating sequence with respect to $\{\lambda_n\}$. Then there exists a sequence $\{\epsilon_n\}$ in $(0,\infty)$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$d(x_n, K(\lambda_n)) \leq \epsilon_n,$$

$$f_1(x_n,y,\lambda_n) \geq 0,\ \forall y \in K(\lambda_n) \text{ and } f_2(x_n,z,\lambda_n) + \varepsilon_n \geq 0,\ \forall z \in Z(\lambda_n,x_n). \tag{1.6.8}$$

If we choose $\alpha_n = d(\lambda_n, \bar{\lambda})$, then $\alpha_n \to 0$ and $x_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$. Since diam $\Gamma(\bar{\lambda}, \alpha_n, \epsilon_n) \to 0$ as $n \to \infty$, it follows that $\{x_n\}$ is a Cauchy sequence in X and hence it converges to $\bar{x} \in X$. For each positive integer n, $K(\lambda_n)$ is compact. Thus, there exists $x_n' \in K(\lambda_n)$ such that

$$d(x_n, x_n') \leq \epsilon_n, \text{ for all } n \in \mathbb{N},$$

which implies that $x_n' \to \bar{x}$. Since K is closed at $\bar{\lambda}$, it follows that $\bar{x} \in K(\bar{\lambda})$. Suppose on the contrary $\bar{x} \notin S(\bar{\lambda})$, that is, there exist $\bar{y} \in K(\bar{\lambda})$ and $\bar{z} \in Z(\bar{\lambda}, \bar{x})$ such that

$$f_1(\bar{x}, \bar{y}, \bar{\lambda}) < 0 \text{ or } f_2(\bar{x}, \bar{z}, \bar{\lambda}) + \epsilon < 0. \tag{1.6.9}$$

Since K is lsc at $\bar{\lambda}$ and $\lambda_n \to \bar{\lambda}$, it is clear that for any $y \in K(\bar{\lambda})$ there exists a sequence $y_n \in K(\lambda_n)$ such that $y_n \to \bar{y}$. Again, since Z is lsc at $(\bar{\lambda}, \bar{x})$ and $(\lambda_n, x_n) \to (\bar{\lambda}, \bar{x})$ there exists a sequence $z_n \in Z(\lambda_n, x_n)$ such that $z_n \to \bar{z}$. Hence, we obtain by assumption (iv), (v) and (1.6.8) that,

$$f_1(\bar{x}, \bar{y}, \bar{\lambda}) \geq 0 \text{ and } f_2(\bar{x}, \bar{z}, \bar{\lambda}) \geq 0.$$

This yields a contradiction to (1.6.9). Hence, we conclude that $\bar{x} \in S(\bar{\lambda})$.

Finally, we will show that $\bar{x}$ is the only solution of (LEP). Let x^* be another point in $S(\bar{\lambda})$ $(x^* \neq \bar{x})$. It is clear that they both belong to $\Gamma(\bar{\lambda}, \alpha, \epsilon)$ for any $\alpha, \epsilon > 0$. Then, it follows that

$$0 \leq d(\bar{x}, x^*) \leq \text{ diam } \Gamma(\bar{\lambda}, \alpha, \epsilon) \downarrow 0 \text{ as } (\alpha, \epsilon) \downarrow (0, 0).$$

This is impossible and, therefore, we are done. The proof is completed. □ □

The following examples show that none of the assumptions in Theorem 1 can be dropped.

Example 53. (**Lower semicontinuity of** K) Let $X = \Lambda = [0, 2]$ and K and f be defined by

$$K(\lambda) = \begin{cases} [0,1] & \text{if } \lambda \neq 0; \\ [0,2] & \text{if } \lambda = 0, \end{cases}$$

$$f(x, y, \lambda) = (x - y, \lambda).$$

One can check that K is closed but not lsc at $\bar{\lambda} = 0$ and

$$S(\lambda) = Z_1(\lambda) = \begin{cases} \{1\} & \text{if } \lambda \neq 0; \\ \{2\} & \text{if } \lambda = 0, \end{cases}$$

$$Z(\lambda, x) = \{x\}, \ \forall (\lambda, x) \in \text{ gr } Z_1.$$

Thus, assumption (iii)-(v) hold true. However, (LEP) is not LP well-posed at $\bar{\lambda}$. Indeed, let $\lambda_n := \frac{1}{n}$ and $x_n := 1 + \frac{1}{2n}$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an LP approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to $\{\lambda_n\}$ with $\epsilon_n := \frac{1}{n}$, while $x_n \to 1 \notin S(0)$.

Example 54. (**Closedness** of K) Let $X = \Lambda = [-2, 2]$, $K(\lambda) = (0, 1]$ (continuous), and a function $f := (f_1, f_2) : X \times X \times \Lambda \to \mathbb{R}^2$ be defined by, for all $x, y \in X$ and $\lambda \in \Lambda$,

$$f(x, y, \lambda) = (x - \frac{y}{2}, \frac{1}{2} - x).$$

It can be calculated that

$$Z(\lambda, x) = \begin{cases} \{1\} & \text{if } x = \frac{1}{2}; \\ \emptyset & \text{if } x \in (\frac{1}{2}, 1]; \\ X & \text{otherwise.} \end{cases}$$

Then, we can conclude that

$$\Gamma(\lambda,\alpha,\epsilon) = \Big[\frac{1}{2}, \frac{1}{2} + \min\{\varepsilon, \frac{3}{2}\}\Big]$$

and

$$\text{diam } \Gamma(\lambda,\alpha,\varepsilon) \to 0 \text{ as } (\alpha,\varepsilon) \to (0,0).$$

One can check that,

$$S(\lambda) = \Big\{\frac{1}{2}\Big\}.$$

We observe that (LEP) is not LP well-posed. Indeed, put $\lambda_n := \frac{1}{n}$, $x_n := 1 + \frac{\varepsilon_n}{n}$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an LP approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to $\{\lambda_n\}$ with $\epsilon_n := \frac{1}{n}$, while $x_n \to 1 \notin S(\lambda)$.

Example 55. (**Lower semicontinuity of** Z) Let $X = \Lambda = [0,1]$, $K(\lambda) = [0,1]$ (continuous and closed), $\bar{\lambda} = 0$ and $f(x,y,\lambda) = (\lambda x(x-y), y-x)$. One can check that

$$Z_1(\lambda) = \begin{cases} [0,1] & \text{if } \lambda = 0; \\ \{0,1\} & \text{if } \lambda \neq 0. \end{cases}$$

and, for each $(\lambda, x) \in$ gr Z_1,

$$Z(\lambda,x) = \begin{cases} [0,1] & \text{if } \lambda = 0 \text{ or } x = 0; \\ \{1\} & \text{if } \lambda \neq 0 \text{ and } x \neq 0. \end{cases}$$

Z is not lsc at $(0,1)$. Indeed, taking $\lambda_n := \frac{1}{2n}$ and $x_n := 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$, we have $(\lambda_n, x_n) \to (0,1)$ and $Z(\lambda_n, x_n) = \{1\}$ for all n, while $Z(0,1) = [0,1]$. Assumption (iv) and (v) are obviously satisfied. By calculating the solution mapping S explicitly as follows:

$$S(\lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0; \\ \{0,1\} & \text{if } \lambda \neq 0. \end{cases}$$

We observe that (LEP) is not LP well-posed at $\bar{\lambda}$. Indeed, let $\lambda_n := \frac{1}{2n}$ and $x_n := 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to $\{\lambda_n\}$ with $\epsilon_n := \frac{1}{n}$, while $x_n \to 1 \notin S(0)$.

Example 56. (**Upper 0-level closedness of** f_1) Let $X = \Lambda = [0,1]$, $K(\lambda) = [0,1]$ (continuous and closed), $\bar{\lambda} = 0$ and

$$f(x,y,\lambda) = \begin{cases} (x-y, \lambda) & \text{if } \lambda = 0; \\ (y-x, \lambda) & \text{if } \lambda \neq 0. \end{cases}$$

One can check that

$$S(\lambda) = Z_1(\lambda) = \begin{cases} \{1\} & \text{if } \lambda = 0; \\ \{0\} & \text{if } \lambda \neq 0. \end{cases}$$

$$Z(\lambda,x) = \{x\}, \quad \forall (\lambda,x) \in \text{ gr } Z_1.$$

Hence, all the assumption except number (iv) hold true. However, (LEP) is not LP well-posed at $\bar{\lambda}$. Indeed, take sequences $\lambda_n := \frac{1}{n+1}$ and $x_n := 0$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an LP approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to $\{\lambda_n\}$ with $\epsilon_n := \frac{1}{n}$, while $x_n \to 0 \notin S(0)$.

Finally, we show that assumption 4 is not satisfied. Indeed, take $\{x_n\}$ and $\{\lambda_n\}$ as above and $\{y_n := 1\}$, we have $(x_n, y_n, \lambda_n) \to (0,1,0)$ and $f_1(x_n, y_n, \lambda_n) = 1 > 0$ for all n, while $f_1(0,1,0) = -1 < 0$.

Example 57. (**Strongly upper 0-level closedness of f_2**) Let X, Λ, K be as in Example 56 and

$$f(x,y,\lambda) = \begin{cases} (0, x-y) & \text{if } \lambda = 0; \\ (0, x(x-y)) & \text{if } \lambda \neq 0. \end{cases}$$

One can check that

$$Z_1(\lambda) = Z(\lambda, x) = [0,1], \ \ \forall x, \lambda \in [0,1],$$

$$S(\lambda) = \begin{cases} \{1\} & \text{if } \lambda = 0; \\ \{0,1\} & \text{if } \lambda \neq 0. \end{cases}$$

Thus, all the assumptions of Theorem 1 except (v) are satisfied. However, (LEP) is not LP well-posed at $\bar{\lambda}$. Indeed, take sequences $\lambda_n := \frac{1}{n+1}$ and $x_n := 0$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an LP approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to $\{\lambda_n\}$, while $x_n \to 0 \notin S(0)$. Finally, we show that assumption (iv) is not satisfied. Indeed, take sequences $x_n := 0, y_n := 1, \lambda_n := \frac{1}{n+1}$ and $\epsilon_n := \frac{1}{n}$ for all $n \in \mathbb{N}$, we have $(x_n, y_n, \lambda_n, \epsilon_n) \to (0,1,0,0)$ and $f_2(x_n, y_n, \lambda_n) + \epsilon_n > 0$ for all n, while $f_2(0,1,0)$.

Corollary 58. *If the conditions of the previous theorem hold then (LEP) is LP well-posed if and only if $S(\bar{\lambda}) \neq \emptyset$ and*

$$\textit{diam } \Gamma(\bar{\lambda}, \alpha, \epsilon) \to 0 \textit{ as } (\alpha, \epsilon) \to (0,0).$$

Then (LEP) is LP well-posed if and only if $\Gamma(\bar{\lambda}, \alpha, \epsilon) \neq \emptyset, \forall \alpha, \epsilon > 0$ and diam $\Gamma(\bar{\lambda}, \alpha, \epsilon) \to 0$ as $(\alpha, \epsilon) \to (0,0)$.

Theorem 2. *Suppose that the conditions (i)-(iv) in Proposition 52 are satisfied. Then (LEP) is LP well-posed if and only if it has a unique solution.*

Proof. By the definition, we know that LP well-posedness for (LEP) implies it has a unique solution. For the converse, suppose that the problem (LEP) has a unique solution x'. Let $\{\lambda_n\}$ be a sequence in Λ converging to $\bar{\lambda}$ and $\{x_n\}$ an LP approximating sequence with respect to $\{\lambda_n\}$. Then, there exists a sequence $\{\epsilon_n\}$ in $(0, \infty)$ with $\epsilon_n \to 0$, as $n \to \infty$, such that

$$d(x_n, K(\lambda_n)) \leq \epsilon_n, \ \text{for all } n \in \mathbb{N}, \tag{1.6.10}$$

and

$$f_1(x_n, y, \lambda_n) \geq 0, \ \forall y \in K(\lambda_n), \ \ f_2(x_n, z, \lambda_n) + \epsilon_n \geq 0, \ \ \forall z \in Z(\lambda_n, x_n). \tag{1.6.11}$$

By (1.6.10) and the closedness of $K(\lambda_n)$ in X, for each positive integer n, we can choose $x'_n \in K(\lambda_n)$ such that

$$d(x_n, x'_n) \leq \epsilon_n. \tag{1.6.12}$$

Since X is a compact set, the sequence $\{x'_n\}$ has a subsequence $\{x'_{n_k}\}$ which converges to a point $\bar{x} \in X$. Using (1.6.12), we conclude that the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $\bar{x}$. Again as K is closed at $\bar{\lambda}$, it follows that $\bar{x} \in K(\bar{\lambda})$. Proceeding along the lines of converse part in the proof of Theorem 1, we can show that $\bar{x} \in S(\bar{\lambda})$. Consequently, $\bar{x}$ coincides with $x'(\bar{x} = x')$. Again, by the uniqueness of the solution, it is obvious that every possible subsequence converges to the unique solution x' and hence the whole sequence $\{x_n\}$ converges to x', thus yielding the LP well-posedness of (LEP). □ □

To weaken the assumption of LP well-posednes in Theorem 1, we are going to use the notions of measures of noncompactness in a metric space X.

Definition 59. Let M be a nonempty subset of a metric space X.
(i) The *Kuratowski measure* of M is

$$\mu(M) = \inf\Big\{\varepsilon > 0 | M \subseteq \bigcup_{k=1}^{n} M_k \text{ and diam } M_k \leq \varepsilon, k = 1, \ldots, n, \ \exists n \in \mathbb{N}\Big\}.$$

(ii) The *Hausdorff measure* of M is

$$\eta(M) = \inf\Big\{\varepsilon > 0 | M \subseteq \bigcup_{k=1}^{n} B(x_k, \varepsilon), x_k \in X, \text{ for some } n \in \mathbb{N}\Big\}.$$

(iii) The *Istr□tescu measure* of M is

$$\iota(M) = \inf\Big\{\varepsilon > 0 | M \text{ have no infinite } \varepsilon - \text{ discrete subset } \Big\}.$$

Daneš [156] obtained the following inequalities:

$$\eta(M) \leq \iota(M) \leq \mu(M) \leq 2\eta(M). \tag{1.6.13}$$

The measures μ, η and ι share many common properties and we will use γ in the sequel to denote either one of them. γ is a regular measure (see [148, 181]), i.e., it enjoys the following properties.

Lemma 60. *Let M be a nonempty subset of a metric space X.*

1. $\gamma(M) = +\infty$ *if and only if the set M is unbounded;*
2. $\gamma(M) = \gamma(clM)$;
3. *from $\gamma(M) = 0$ it follows that M is totally bounded set;*
4. *if X is a complete space and if $\{A_n\}$ is a sequence of closed subsets of X such that $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to+\infty} \gamma(A_n) = 0$, then $K := \bigcap_{n\in\mathbb{N}} A_n$ is a nonempty compact set and $\lim_{n\to+\infty} H(A_n, K) = 0$, where H is the Hausdorff metric;*
5. *from $M \subseteq N$ it follows that $\gamma(M) \leq \gamma(N)$.*

In terms of a measure $\gamma \in \{\mu, \eta, \iota\}$ of noncompactness, we have the following result.

Theorem 3. *Let X and Λ be metric spaces.*

1. *If* LEP *is LP well-posed at $\bar{\lambda}$, then $\gamma(\Gamma(\bar{\lambda}, \alpha, \varepsilon)) \downarrow 0$ as $(\alpha, \varepsilon) \downarrow (0, 0)$.*
2. *Conversely, suppose that $S(\bar{\lambda})$ has a unique point and $\gamma(\Gamma(\bar{\lambda}, \alpha, \varepsilon)) \downarrow 0$ as $(\alpha, \varepsilon) \downarrow (0, 0)$, and the following conditions hold*
   (a) *X is complete and Λ is compact or a finite dimensional normed space;*
   (b) *K is continuous, closed and compact-valued on Λ;*
   (c) *Z is lsc on $\Lambda \times X$;*
   (d) *f_1 is upper 0-level closed on $X \times X \times \Lambda$;*
   (e) *f_2 is upper b-level closed on $X \times X \times \Lambda$ for every negative b close to zero.*

Then LEP *is LP well-posed at* $\bar{\lambda}$.

Proof. By the relationship (1.6.13) the proof is similar for the three mentioned measures of non-compactness. We discuss only the case $\gamma = \mu$, the Kuratowski measure.

(i) Suppose that (LEP) be LP-well posed at $\bar{\lambda}$.

Applying Proposition 62, we can conclude that $S(\bar{\lambda})$ is compact, and hence $\mu(S(\bar{\lambda})) = 0$. Let $\epsilon > 0$ and assume that

$$S(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} M_k \text{ with } \operatorname{diam} M_k \leq \epsilon \text{ for all } k = 1, \ldots, n.$$

We set

$$N_k = \{y \in X | d(y, M_k) \leq H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda}))\}$$

and want to show that $\Gamma(\bar{\lambda}, \alpha, \epsilon) \subseteq \bigcup_{k=1}^{n} N_k$. For any $x \in \Gamma(\bar{\lambda}, \alpha, \epsilon)$, we have

$$d(x, S(\bar{\lambda})) \leq H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})).$$

Due to $S(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} M_k$, one has

$$d(x, \bigcup_{k=1}^{n} M_k) \leq H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})).$$

Then, there exists $\bar{k} \in \{1, 2, \ldots, n\}$ such that

$$d(x, M_{\bar{k}}) \leq H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})),$$

i.e., $x \in N_{\bar{k}}$. Thus, $\Gamma(\bar{\lambda}, \alpha, \epsilon) \subseteq \bigcup_{k=1}^{n} N_k$. Because $\mu(S(\bar{\lambda})) = 0$ and

$$\operatorname{diam} N_k = \operatorname{diam} M_k + 2H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})) \leq \epsilon + 2H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})),$$

it holds

$$\mu(\Gamma(\bar{\lambda}, \alpha, \epsilon)) \leq 2H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})).$$

Note that $H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})) = H^*(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda}))$ since $S(\bar{\lambda}) \subseteq \Gamma(\bar{\lambda}, \alpha, \epsilon)$ for all $\alpha, \epsilon > 0$. Now, we claim that $H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})) \downarrow 0$ as $\alpha, \epsilon \downarrow 0$ and . Indeed, if otherwise, we can assume that there exist $r > 0$ and sequences $\alpha_n, \epsilon_n \downarrow 0$, and $\{x_n\}$ with $x_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$ such that

$$d(x_n, S(\bar{x})) \geq r, \quad \forall n. \tag{1.6.14}$$

Since $\{x_n\}$ is an approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to some $\{\lambda_n\}$ with $\lambda_n \in B(\bar{\lambda}, \alpha_n) \cap \Lambda$, it has a subsequence $\{x_{n_k}\}$ converging to some $x \in S(\bar{\lambda})$, which gives a contradiction with (1.6.14). Therefore, we conclude that $\mu(\Gamma(\bar{\lambda}, \alpha, \epsilon))$ as $\xi \downarrow 0$ and $\varepsilon \downarrow 0$.

(ii) Suppose that $\mu(\Gamma(\bar{\lambda}, \alpha, \epsilon)) \to 0$ as $(\alpha, \epsilon) \to (0, 0)$ First, we show that $\Gamma(\bar{\lambda}, \alpha, \epsilon)$ is closed for any $\alpha, \epsilon > 0$. Let $\{x_n\} \subseteq \Gamma(\bar{\lambda}, \alpha, \epsilon)$, with $x_n \to \bar{x}$. Then for each $n \in \mathbb{N}$, there exists $\lambda_n \in B(\bar{\lambda}, \alpha) \cap \Lambda$ such that

$$d(x_n, K(\lambda_n)) \leq \epsilon$$

and

$$f_1(x_n, y, \lambda_n) \geq 0, \quad \forall y \in K(\lambda_n) \text{ and } f_2(x_n, z, \lambda_n) + \epsilon \geq 0, \quad \forall z \in Z(\lambda_n, x_n), \text{ for all } n \in \mathbb{N}.$$

By the assumption of Λ, this implies that $B(\bar{\lambda},\alpha)$ is compact. We can assume $\{\lambda_n\}$ converges to some $\lambda \in B(\bar{\lambda},\alpha)\cap\Lambda$. First, we claim that $d(\bar{x},K(\lambda)) \leq \epsilon$. Since $K(\lambda_n)$ is compact, there exists $x'_n \in K(\lambda_n)$ such that $d(x_n,x'_n) \leq \epsilon$ for all $n \in \mathbb{N}$. By the upper continuity and compactness of K, there exists a subsequence $\{x'_{n_j}\}$ of $\{x'_n\}$ such that $x'_{n_j} \to x' \in K(\lambda)$. Consequently,

$$d(\bar{x},K(\lambda)) \leq d(\bar{x},x') = \lim_{n\to\infty} d(x_n,x'_n) \leq \epsilon. \tag{1.6.15}$$

For each $y \in K(\lambda)$, the lower semicontinuity of K at λ, there exists a sequence $\{y_n\} \subseteq K(\lambda_n)$ such that $y_n \to y$. It follows from the upper 0-level closedness of f_1 that

$$f_1(\bar{x},y,\lambda) \geq 0;$$

that is

$$f_1(\bar{x},y,\lambda) \geq 0, \ \forall y \in K(\lambda). \tag{1.6.16}$$

Next, we show that

$$f_2(\bar{x},z,\lambda)+\epsilon \geq 0, \ \ \forall z \in Z(\lambda,\bar{x}). \tag{1.6.17}$$

Suppose to the contrary that there exists $\bar{z} \in Z(\lambda,\bar{x})$ such that

$$f_2(\bar{x},\bar{z},\lambda)+\epsilon < 0.$$

Since Z is lower semicontinuous at $(\lambda,\bar{x})$, we have for all n, there is $z_n \in Z(\lambda_n,x_n)$ such that $z_n \to \bar{z}$ as $n \to \infty$. It follows from the upper $(-\epsilon)$-level closedness f_2 at $(\bar{x},\bar{z},\lambda)$ that

$$f_2(x_n,z_n,\lambda_n) < -\epsilon$$

when n is sufficiently large which leads to a contradiction. By (1.6.15), (1.6.16) and (1.6.17), we can conclude that $\bar{x} \in \tilde{S}(\lambda,\epsilon)$, and so $\bar{x} \in \Gamma(\bar{\lambda},\alpha,\epsilon)$. Therefore $\Gamma(\bar{\lambda},\alpha,\epsilon)$ is closed for any $\alpha,\epsilon > 0$. Now we show that

$$S(\bar{\lambda}) = \bigcap_{\alpha,\epsilon>0} \Gamma(\bar{\lambda},\alpha,\epsilon).$$

It is clear that, $S(\bar{\lambda}) \subseteq \bigcap_{\alpha,\epsilon>0}\Gamma(\bar{\lambda},\alpha,\epsilon)$. Next, we first check that, for each $\varepsilon > 0$,

$$\bigcap_{\alpha>0} \Gamma(\bar{\lambda},\alpha,\epsilon) \subseteq \tilde{S}(\bar{\lambda},\epsilon).$$

For any $x \in \bigcap_{\alpha>0}\Gamma(\bar{\lambda},\alpha,\epsilon)$. Then for each $\{\alpha_n\} \downarrow 0$, there exists a sequence $\{\lambda_n\}$ with $\lambda_n \in B(\bar{\lambda},\alpha_n)\cap\Lambda$ such that $x \in \tilde{S}(\lambda_n,\epsilon)$ for all $n \in \mathbb{N}$, which gives that

$$d(x,K(\lambda_n)) \leq \epsilon,$$

$$f_1(x,y,\lambda_n) \geq 0, \, \forall y \in K(\lambda_n), \ \text{ and } \ f_2(x,z,\lambda_n)+\epsilon \geq 0, \ \forall z \in Z(\lambda_n,x).$$

Since $K(\lambda_n)$ is compact , we can choose $x_n \in K(\lambda_n)$ such that

$$d(x,x_n) \leq \epsilon, \ \forall n \in \mathbb{N}.$$

By the upper continuity and compactness of K, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x' \in K(\lambda)$, which arrives that

$$d(x,K(\bar{\lambda})) \leq d(x,x') = \lim_{n\to\infty} d(x,x_n) \leq \epsilon. \tag{1.6.18}$$

By assumptions on K and f_1 again, we have $x \in Z_1(\bar{\lambda})$; that is

$$f_1(x, y, \bar{\lambda}) \geq 0. \tag{1.6.19}$$

Next, for each $z \in Z(\bar{\lambda}, x)$, there exists $z_n \in Z(\lambda_n, x)$ such that $z_n \to z$ since Z is lsc at $(\bar{\lambda}, x)$. As $x \in \tilde{S}(\lambda_n, \epsilon)$, it holds

$$f_2(x, z_n, \lambda_n) + \epsilon \geq 0, \quad \forall n \in \mathbb{N}.$$

Since f_2 is upper -ϵ-level closed at $(x, z, \bar{\lambda})$, we have

$$f_2(x, z, \bar{\lambda}) + \epsilon \geq 0. \tag{1.6.20}$$

From (1.6.18)-(1.6.20), we get that $x \in \tilde{S}(\bar{\lambda}, \epsilon)$. We obtain that $\bigcap_{\alpha>0} \Gamma(\bar{\lambda}, \alpha, \epsilon) \subseteq \tilde{S}(\bar{\lambda}, \epsilon)$ for every $\varepsilon > 0$. Consequently,

$$\bigcap_{\alpha,\epsilon>0} \Gamma(\bar{\lambda}, \alpha, \epsilon) \subseteq \bigcap_{\epsilon>0} \tilde{S}(\bar{\lambda}, \epsilon) = S(\bar{\lambda}).$$

Therefore, we obtain that $S(\bar{\lambda}) = \bigcap_{\alpha,\epsilon>0} \Gamma(\bar{\lambda}, \alpha, \epsilon)$. Further, since $\mu(\Gamma(\bar{\lambda}, \alpha, \epsilon)) \to 0$ as $(\alpha, \epsilon) \to (0, 0)$. Applying Lemma 60 (iv), we get that $S(\bar{\lambda})$ is compact and $H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})) \longrightarrow 0$ as $(\alpha, \epsilon) \to (0, 0)$.

Finally, we prove that LEP is LP well-posedness. Indeed, let $\{x_n\}$ be an LP-approximating sequence of $(\mathrm{LEP}_{\bar{\lambda}})$ corresponding to some $\lambda_n \longrightarrow \bar{\lambda}$. Then there exists a sequence $\{\epsilon_n\}$ in $(0, \infty)$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$d(x_n, K(\lambda_n)) \leq \epsilon_n,$$

$$f_1(x_n, y, \lambda_n) \geq 0, \ \forall y \in K(\lambda_n) \ \text{ and } \ f_2(x_n, z, \lambda_n) + \varepsilon_n \geq 0, \ \forall z \in Z(\lambda_n, x_n). \tag{1.6.21}$$

If we choose $\alpha_n = d(\lambda_n, \bar{\lambda})$, then $\alpha_n \to 0$ and $x_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$. We see that

$$d(x_n, S(\bar{\lambda})) \leq H(\Gamma(\bar{\lambda}, \alpha_n, \epsilon_n), S(\bar{\lambda})) \longrightarrow 0 \text{ as } n \to \infty.$$

Hence, there exist a sequence $\{\bar{x}_n\}$ in $S(\bar{\lambda})$ such that $d(x_n, \bar{x}_n) \to 0$ as $n \to \infty$. By the compactness of $S(\bar{\lambda})$, there is a subsequence $\{\bar{x}_{n_j}\}$ of $\{\bar{x}_n\}$ converging to a point $\bar{x}$ in $S(\bar{\lambda})$. Consequently, the corresponding subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converses to $\bar{x}$. Hence, LEP is LP well-posedness. The proof is completed. □

1.7 LP well-posedness in the generalized sense

In many practical situations, the problem (LEP) may not always possess a unique solution. Hence, in this section, we introduce a generalization of LP well-posedness for (LEP).

Definition 61. The problem (LEP) is said to be LP well-posed in the generalized sense at $\bar{\lambda}$ if

1. the solution set $S(\bar{\lambda})$ is nonempty;

2. for any sequence $\{\lambda_n\}$ converging to $\bar{\lambda}$, every LP approximating sequence $\{x_n\}$ with respect to $\{\lambda_n\}$ has a subsequence converging to some point of $S(\bar{\lambda})$.

Proposition 62. *If (LEP) is LP well-posed in the generalized sense at $\bar{\lambda}$, then its solution set $S(\bar{\lambda})$ is a nonempty compact set.*

Proof. Let $\{x_n\}$ be any sequence in $S(\bar{\lambda})$. Then, of course, it is an LP approximating sequence with respect to sequences $\lambda_n := \bar{\lambda}$ and $\epsilon_n := \frac{1}{n}$, for every $n \in \mathbb{N}$. The generalized LP well-posedness of (LEP) ensures the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to a point of in $S(\bar{\lambda})$. Therefore, we conclude that $S(\bar{\lambda})$ is a nonempty compact set. The proof is completed.
□ □

Next, we present a metric characterization for the generalized LP well-posedness of (LEP) in terms of the upper semicontinuity of the approximate solution set.

Theorem 4. *(LEP) is LP well-posed in the generalized sense if and only if $S(\bar{\lambda})$ is a nonempty, compact set and $\Gamma(\bar{\lambda}, \cdot, \cdot)$ is usc at $(\alpha, \epsilon) := (0, 0)$.*

Proof. Suppose that (LEP) is LP well-posed in the generalized sense. Therefore, $S(\bar{\lambda}) \neq \emptyset$ and further on using Proposition 62, we have $S(\bar{\lambda})$ is compact. Next, we assume, on the contrary, that $\Gamma(\bar{\lambda}, \alpha, \epsilon)$ is not usc at $(0, 0)$. Consequently, there exist an open set U containing $\Gamma(\bar{\lambda}, 0, 0) = S(\bar{\lambda})$ and positive sequences $\{\alpha_n\}$ and $\{\epsilon_n\}$ satisfying $\alpha_n \to 0$ and $\epsilon_n \to 0$ such that

$$\Gamma(\bar{\lambda}, \alpha_n, \epsilon_n) \subsetneq U, \text{ for all } n \in \mathbb{N}.$$

Thus, there exists a sequence $\{x_n\}$ in $\Gamma(\bar{\lambda}, \alpha_n, \epsilon_n) \backslash S(\bar{\lambda})$. Therefore, of course, $\{x_n\}$ is an LP approximating sequence for (LEP), such that none of its subsequence converges to a point of $S(\bar{\lambda})$, which is a contradiction.

Conversely, let $\{\lambda_n\}$ be a sequence in Λ converging to $\bar{\lambda}$ and $\{x_n\}$ be an LP approximating sequence with respect to $\{\lambda_n\}$. If we choose a sequence $\alpha_n = d(\lambda_n, \bar{\lambda})$ then $\alpha_n \to 0$ and $x_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$. As $\Gamma(\bar{\lambda}, \alpha, \epsilon)$ is usc at $(\alpha, \epsilon) = (0, 0)$ and $S(\bar{\lambda}) \neq \emptyset$, it follows that for every $\delta > 0, \Gamma(\bar{\lambda}, \delta_n, \epsilon_n) \subset S(\bar{\lambda}) + B(0, \delta)$ for n sufficiently large. Thus $x_n \in S(\bar{\lambda}) + B(0, \delta)$, for n sufficiently large and hence there exists a sequence $\bar{x}_n \in S(\bar{\lambda})$, such that

$$d(x_n, \bar{x}_n) \leq \delta. \tag{1.7.1}$$

Since $S(\bar{\lambda})$ is compact, there exists a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ converging to $\bar{x} \in S(\bar{\lambda})$. Using (1.7.1), we conclude that the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $\bar{x} \in S(\bar{\lambda})$.
□ □

The following result illustrates the fact that LP well-posedness in the generalized sense of LEP ensures the stability, in terms of the upper semi-continuity of the solution set S.

Theorem 5. *If (LEP) is LP well-posed in the generalized sense, then the solution mapping S is usc at $\bar{\lambda}$.*

Proof. Suppose on the contrary, S is not usc at $\bar{\lambda}$. Then there exists an open set U containning $S(\bar{\lambda})$ such that for every sequence $\lambda_n \to \bar{\lambda}$, there exists $x_n \in S(\lambda_n)$ such that $x_n \notin U$, for every n. Since $\lambda_n \to \bar{\lambda}$, $\{x_n\}$ is an LP approximating sequence for (LEP) and none of its subsequnces converge to a point of $S(\bar{\lambda})$, hence we have a contradiction to the fact that (LEP) is LP well-posed in the generalized sence. □ □

1.8 Bibliography

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Chapter 2

Output

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On the Hölder continuity of solution maps to parametric generalized vector quasi-equilibrium problems via nonlinear scalarization

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This paper is dedicated to Professor Shih-sen Chang on his 80th birthday

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Abstract
In this paper, by using a nonlinear scalarization technique, we obtain sufficient conditions for Hölder continuity of the solution mapping for a parametric generalized vector quasi-equilibrium problem with set-valued mappings. The results are different from the recent ones in the literature.

Keywords: parametric generalized vector quasi-equilibrium problem; solution mapping; Hölder continuity; nonlinear scalarization

1 Introduction

The generalized vector quasi-equilibrium problem is a unified model of several problems, namely generalized vector quasi-variational inequalities, vector quasi-optimization problems, traffic network problems, fixed point and coincidence point problems, *etc.* (see, for example, [1, 2] and the references therein). It is well known that the stability analysis of a solution mapping for equilibrium problems is an important topic in optimization theory and applications. Stability may be understood as lower or upper semicontinuity, continuity, and Lipschitz or Hölder continuity. There have been many papers to discuss the stability of solution mapping for equilibrium problems when they are perturbed by parameters (also known the parametric (generalized) equilibrium problems). Last decade, many authors intensively studied the sufficient conditions of upper (lower) semicontinuity of various solution mappings for parametric (generalized) equilibrium problems, see [3–10]. Let us begin now, Yen [11] obtained the Hölder continuity of the unique solution of a classic perturbed variational inequality by the metric projection method. Mansour and Riahi [12] proved the Hölder continuity of the unique solution for a parametric equilibrium problem under the concepts of strong monotonicity and Hölder continuity. Bianchi and Pini [13] introduced the concept of strong pseudomonotonicity and got the Hölder continuity of the unique solution of a parametric equilibrium problem. Anh and Khanh [14] generalized the main results of [13] to two classes of perturbed generalized equilibrium problems with set-valued mappings. Anh and Khanh [15] further discussed the uniqueness and Hölder continuity of the solutions for perturbed equilibrium problems with set-valued mappings. Anh and Khanh [16] extended the results of [15] to the case of perturbed quasi-equilibrium problems with set-valued mappings and obtained the Hölder continuity of the unique solutions. Li *et al.* [17] introduced an assumption, which is weaker than the corresponding ones of [13, 14], and established the Hölder continuity of the set-valued solution mappings for two classes of parametric generalized vector quasi-equilibrium problems in general metric spaces. Li *et al.* [18] extended the results of [17] to perturbed generalized vector quasi-equilibrium problems.

Among many approaches for dealing with the lower semicontinuity, continuity and Hölder continuity of the solution mapping for a parametric vector equilibrium problem in general metric spaces, the scalarization method is of considerable interest. The classical scalarization method using linear functionals has been already used for studying the lower semicontinuity of the solution mapping [19–21] and the Hölder continuity [22] of the solution mapping to parametric vector equilibrium problems. Wang *et al.* [23] established the lower semicontinuity and upper semicontinuity of the solution set to a parametric generalized strong vector equilibrium problem by using a scalarization method and a density result. Recently, by using this method, Peng [24] established the sufficient conditions for the Hölder continuity of the solution mapping to a parametric generalized vector quasi-equilibrium problem with set-valued mappings.

On the other hand, a useful approach for analyzing a vector optimization problem is to reduce it to a scalar optimization problem. Nonlinear scalarization functions play an important role in this reduction in the context of nonconvex vector optimization problems. The nonlinear scalarization function ξ_q, commonly known as the Gerstewitz function in the theory of vector optimization [25, 26], has been also used to study the lower semicontinuity of the set-valued solution mapping to a parametric vector variational inequality [27]. Using this method, Bianchi and Pini [28] obtained the Hölder continuity of the single-valued solution mapping to a parametric vector equilibrium problem. Recently, Chen and Li [29] studied Hölder continuity of the solution mapping for both set-valued and single-valued cases to parametric vector equilibrium problems. The key role in their paper is a globally Lipschitz property of the Gerstewitz function. Very recently, by using the idea in [29], Chen [30] obtained Hölder continuity of the unique solution to a parametric vector quasi-equilibrium problem based on nonlinear scalarization approach under three different kinds of monotonicity hypotheses. It is natural to raise and give an answer to the following question.

Question Can one establish the Hölder continuity of a solution mapping to the parametric generalized vector quasi-equilibrium problem with set-valued mappings by using a nonlinear scalarization method?

Motivated and inspired by Peng [24] and Chen [30] and research going on in this direction, in this paper we aim to give positive answers to the above question. We first establish the sufficient conditions which guarantee the Hölder continuity of a solution mapping to the parametric generalized vector quasi-equilibrium problem with set-valued mappings by using a nonlinear scalarization method. We further study several kinds of the monotonicity conditions to obtain the Hölder continuity of the solution mapping. The main results of this paper are different from the corresponding results in Peng [24] and Chen [30]. These results improve the corresponding ones in recent literature.

The structure of the paper is as follows. Section 2 presents the parametric generalized vector quasi-equilibrium problem and materials used in the rest of this paper. We establish, in Section 3, a sufficient condition for the Hölder continuity of the solution mapping to a parametric generalized vector quasi-equilibrium problem.

2 Preliminaries

Throughout the paper, unless otherwise specified, we denote by $\|\cdot\|$ and $d(\cdot,\cdot)$ the norm and the metric on a normed space and a metric space, respectively. A closed ball with center $0 \in X$ and radius $\delta > 0$ is denoted by $B(0,\delta)$. We always consider X, Λ, M as metric spaces, and Y as a linear normed space with its topological dual space Y^*. For any $y^* \in Y^*$, we define $\|y^*\| := \sup\{\|\langle y^*, y\rangle\| : \|y\| = 1\}$, where $\langle y^*, y\rangle$ denotes the value of y^* at y. Let $C \subset Y$ be a pointed, closed and convex cone with $\operatorname{int} C \neq \emptyset$, where $\operatorname{int} C$ stands for the interior of C. Let

$$C^* := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \forall y \in C\}$$

be the dual cone of C. Since $\operatorname{int} C \neq \emptyset$, the dual cone C^* of C has a weak* compact base. Let $e \in \operatorname{int} C$. Then

$$B_e^* := \{y^* \in C^* : \langle y^*, e\rangle = 1\}$$

is a weak*-compact base of C^*. Clearly, C^q is a weak*-compact base of C^*, that is, C^q is convex and weak*-compact such that $0 \notin C^q$ and $C^* = \bigcup_{t\geq 0} tC^q$.

Let $q \in \operatorname{int} C$, the *nonlinear scalarization function* [25, 26] $\xi_q : Y \to \mathbb{R}$ is defined by

$$\xi_q = \min\{t \in \mathbb{R} : y \in tq - C\}.$$

It is well known that ξ_q is a continuous, positively homogeneous, subadditive and convex function on Y, and it is monotone (that is, $y_2 - y_1 \in C \Rightarrow \xi_q(y_1) \leq \xi_q(y_2)$) and strictly monotone (that is, $y_2 - y_1 \in -\operatorname{int} C \Rightarrow \xi_q(y_1) < \xi_q(y_2)$) (see [25, 26]). In case, $Y = R^l$, $C = R^l_+$ and $q = (1, 1, \ldots, 1) \in \operatorname{int} R^l_+$, the nonlinear scalarization function can be expressed in the following equivalent form [25, Corollary 1.46]:

$$\xi_q(y) = \max_{1\leq i\leq l}\{y_i\}, \quad \forall y = (y_1, y_2, \ldots, y_l) \in \mathbb{R}^l. \tag{1}$$

Lemma 2.1 [25, Proposition 1.43] *For any fixed $q \in \operatorname{int} C$, $y \in Y$ and $r \in \mathbb{R}$,*

(i) $\xi_q < r \Leftrightarrow y \in rq - \operatorname{int} C$ (*that is*, $\xi_q(y) \geq r \Leftrightarrow y \notin rq - \operatorname{int} C$);
(ii) $\xi_q(y) \leq r \Leftrightarrow y \in rq - C$;
(iii) $\xi_q(y) = r \Leftrightarrow y \in rq - \partial C$, *where ∂C denotes the boundary of C*;
(iv) $\xi_q(rq) = r$.

The property (i) of Lemma 2.1 plays an essential role in scalarization. From the definition of ξ_q, property (iv) in Lemma 2.1 could be strengthened as

$$\xi_q(y + rq) = \xi_q(y) + r, \quad \forall y \in Y, r \in \mathbb{R}. \tag{2}$$

For any $q \in \operatorname{int} C$, the set C^q defined by

$$C^q := \{y^* \in C^* : \langle y^*, q\rangle = 1\}$$

is a weak*-compact set of Y^* (see [19, Lemma 5.1]). The following equivalent form of ξ_q can be deduced from [31, Corollary 2.1] or [32, Proposition 2.2] ([25, Proposition 1.53]).

Proposition 2.2 [30, Proposition 2.2] *Let $q \in \operatorname{int} C$. Then, for $y \in Y$,*

$$\xi_q(y) = \max_{y^* \in C^q} \langle y^*, y\rangle.$$

Proposition 2.3 [30, Proposition 2.3] *ξ_q is Lipschitz on Y, and its Lipschitz constant is*

$$L := \sup_{y^* \in C^q} \|y^*\| \in \left[\frac{1}{\|q\|}, +\infty\right).$$

The following example can be found in [30, Example 2.1].

Example 2.4

(i) If $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, then the Lipschitz constant of ξ_q is $L = \frac{1}{q}$ $(q > 0)$. Indeed, $|\xi_q(x) - \xi_q(y)| = \frac{1}{q}|x - y|$ for all $x, y \in \mathbb{R}$.

(ii) If $Y = \mathbb{R}^2$ and $C = \{(y_1, y_2) \in \mathbb{R}^2 : \frac{1}{4}y_1 \le y_2 \le 2y_1\}$. Take $q = (2, 3) \in \operatorname{int} C$, then

$$C^q := \{(y_1, y_2) \in \mathbb{R} : 2y_1 + 3y_2 = 1, y_1 \in [-0.1, 2]\},$$

and the Lipschitz constant is $L = \sup_{y^* \in C^q} \|y^*\| = \|(-2, 1)\| = \sqrt{5}$. Hence,

$$|\xi_q(y) - \xi_q(y')| \le \sqrt{5}\|y - y'\|, \quad \forall y, y' \in \mathbb{R}^2.$$

Now we recall some basic definitions and their properties which will be used in the sequel.

Definition 2.5 (Classical notion) Let $l \ge 0$ and $\alpha > 0$. A set-valued mapping $G : \Lambda \to 2^X$ is said to be *$l \cdot \alpha$-Hölder continuous at λ_0 on a neighborhood* $N(\lambda_0)$ of λ_0 if and only if

$$G(\lambda_1) \subseteq G(\lambda_2) + lB_X(0, d^\alpha(\lambda_1, \lambda_2)), \quad \forall \lambda_1, \lambda_2 \in N(\lambda_0). \tag{3}$$

When X is a normed space, we say that the vector-valued mapping $g : \Lambda \to X$ is $l \cdot \alpha$-Hölder continuous at λ_0 on a neighborhood $N(\lambda_0)$ of λ_0 iff

$$\|g(\lambda_1) - g(\lambda_2)\| \le ld^\alpha(\lambda_1, \lambda_2), \quad \forall \lambda_1, \lambda_2 \in N(\lambda_0). \tag{4}$$

Definition 2.6 Let $l_1, l_2 \ge 0$ and $\alpha_1, \alpha_2 > 0$. A set-valued mapping $G : X \times \Lambda \to 2^X$ is said to be *$(l_1 \cdot \alpha_1, l_2 \cdot \alpha_2)$-Hölder continuous at x_0, λ_0 on neighborhoods* $N(x_0)$ and $N(\lambda_0)$ of x_0 and λ_0 if and only if

$$G(x_1, \lambda_1) \subseteq G(x_2, \lambda_2) + (l_1 d_X^{\alpha_1}(x_1, x_2) + l_2 d_\Lambda^{\alpha_2}(\lambda_1, \lambda_2))B_X(0, 1) \tag{5}$$

for all $x_1, x_2 \in N(x_0)$, $\forall \lambda_1, \lambda_2 \in N(\lambda_0)$.

3 Main results

By using a nonlinear scalarization technique, we present the sufficient conditions for Hölder continuity of the solution mapping for a parametric generalized vector quasi-equilibrium problem.

Let $N(\lambda_0) \subset \Lambda$ and $N(\mu_0) \subset M$ be neighborhoods of λ_0 and μ_0, respectively, and let $K : X \times \Lambda \to 2^X$ and $F : X \times X \times M \to 2^Y$ be set-valued mappings. For each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, we consider the following parametric generalized vector quasi-equilibrium problem (PGVQEP):

Find $x_0 \in K(x_0, \lambda)$ such that

$$F(x_0, y, \mu) \subset Y \backslash (-\operatorname{int} C), \quad \forall y \in K(x_0, \lambda). \tag{6}$$

For each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, let

$$E(\lambda) := \{x \in X | x \in K(x, \lambda)\}.$$

The weak solution set of (6) is denoted by

$$S_W(\lambda, \mu) := \{x \in E(\lambda) : F(x, y, \mu) \subset Y \backslash (-\operatorname{int} C), \forall y \in K(x, \lambda)\}.$$

For each $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $q \in \operatorname{int} C$, the ξ_q-solution set of (6) is denoted by

$$S(\xi_q, \lambda, \mu) := \Big\{x \in E(\lambda) : \inf_{z \in F(x,y,\mu)} \xi_q(z) \geq 0, \forall y \in K(x, \lambda)\Big\}.$$

We first establish the following lemmas which will be used in the sequel.

Lemma 3.1 *For each $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $q \in \operatorname{int} C$,*

$$S_W(\lambda, \mu) = S(\xi_q, \lambda, \mu).$$

Proof Let $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $q \in \operatorname{int} C$. For any $x \in S_W(\lambda, \mu)$, we have

$$x \in E(\lambda) \quad \text{and} \quad F(x, y, \mu) \subset Y \backslash (-\operatorname{int} C), \quad \forall y \in K(x, \lambda).$$

Therefore, for each $y \in K(x, \lambda)$ and each $z \in F(x, y, \mu)$, we have

$$z \notin -\operatorname{int} C = 0q - \operatorname{int} C.$$

By Lemma 2.1(i), we conclude that $\xi_q(z) \geq 0$. Since z is arbitrary, we have

$$\inf_{z \in F(x,y,\mu)} \xi_q(z) \geq 0 \quad \text{for all } y \in K(x, \lambda),$$

which gives that $S_W(\lambda, \mu) \subseteq S(\xi_q, \lambda, \mu)$.

On the other hand, for each $x \in S(\xi_q, \lambda, \mu)$, we have that

$$x \in E(\lambda) \quad \text{and} \quad \inf_{z \in F(x,y,\mu)} \xi_q(z) \geq 0, \quad \forall y \in K(x, \lambda). \tag{7}$$

Thus, for each $y \in K(x,\lambda)$ and each $z \in F(x,y,\mu)$, we have that $\xi_q(z) \geq 0$. By Lemma 2.1(i), we can obtain $z \notin -\operatorname{int} C$. Therefore, we have $z \in Y \setminus (-\operatorname{int} C)$, which implies that

$$x \in E(\lambda) \quad \text{and} \quad F(x,y,\mu) \subset Y \setminus (-\operatorname{int} C), \quad \forall y \in K(x,\lambda).$$

Hence, $S(\xi_q,\lambda,\mu) \subseteq S_W(\lambda,\mu)$. The proof is completed. □

Lemma 3.2 *Suppose that $N(\lambda_0)$ and $N(\mu_0)$ are the given neighborhoods of λ_0 and μ_0, respectively.*

(a) *If for each $x,y \in E(N(\lambda_0))$, $F(x,y,\cdot)$ is $m_1 \cdot \gamma_1$-Hölder continuous at $\mu_0 \in M$, then for any fixed $q \in \operatorname{int} C$, the function*

$$\psi_{\xi_q}(x,y,\cdot) = \inf_{z \in F(x,y,\cdot)} \xi_q(z)$$

is $Lm_1 \cdot \gamma_1$-Hölder continuous at μ_0.

(b) *If for each $x \in E(N(\lambda_0))$ and $\mu \in N(E(\mu_0))$, $F(x,\cdot,\mu)$ is $m_2 \cdot \gamma_2$-Hölder continuous on $E(N(\lambda_0))$, then for any fixed $q \in \operatorname{int} C$, the function*

$$\psi_{\xi_q}(x,\cdot,\mu) = \inf_{z \in F(x,\cdot,\mu)} \xi_q(z)$$

is $Lm_2 \cdot \gamma_2$-Hölder continuous on $E(N(\lambda_0))$.

Proof (a) Let $x,y \in E(N(\lambda_0))$. The $m_1 \cdot \gamma_1$-Hölder continuity of $F(x,y,\cdot)$ implies that there exists a neighborhood $N(\mu_0)$ of μ_0 such that for all $\mu_1,\mu_2 \in N(\mu_0)$,

$$F(x,y,\mu_1) \subset F(x,y,\mu_2) + m_1 d_M^{\gamma_1}(\mu_1,\mu_2) B_Y.$$

So, for any $z_1 \in F(x,y,\mu_1)$, there exist $z_2 \in F(x,y,\mu_2)$ and $e \in B_Y$ such that

$$z_1 = z_2 + m_1 d_M^{\gamma_1}(\mu_1,\mu_2) e.$$

By using Proposition 2.3, we obtain

$$\begin{aligned} \left|\xi_q(z_1) - \xi_q(z_2)\right| &\leq L\|z_1 - z_2\| \\ &= Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2)\|e\| \\ &\leq Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2), \end{aligned} \tag{8}$$

which gives that

$$-Lm_1 d^{\gamma_1}(\mu_1,\mu_2) \leq \xi_q(z_1) - \xi_q(z_2).$$

Since z_1 is arbitrary and $\xi_q(z_2) \geq \inf_{z \in F(x,y,\mu_2)} \xi_q(z)$, we have

$$-Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2) \leq \inf_{z \in F(x,y,\mu_1)} \xi_q(z) - \inf_{z \in F(x,y,\mu_2)} \xi_q(z).$$

Applying the symmetry between μ_1 and μ_2, we arrive at

$$-Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2) \le \inf_{z\in F(x,y,\mu_2)} \xi_q(z) - \inf_{z\in F(x,y,\mu_1)} \xi_q(z).$$

It follows from the last two inequalities that

$$\left|\psi_{\xi_q}(x,y,\mu_1) - \psi_{\xi_q}(x,y,\mu_2)\right| \le Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2), \quad \forall \mu_1,\mu_2 \in N(\mu_0).$$

Therefore, we conclude that $\psi_{\xi_q}(x,y,\cdot) = \inf_{z\in F(x,y,\cdot)} \xi_q(z)$ is $Lm_1 \cdot \gamma_1$-Hölder continuous at μ_0.

(b) It follows by a similar argument as in part (a). The proof is completed. □

Now, by using the nonlinear scalarization technique, we propose some sufficient conditions for Hölder continuity of the solution mapping for (PGVQEP).

Theorem 3.3 *For each fixed $q \in \operatorname{int} C$, let $S(\xi_q,\lambda,\mu)$ be nonempty in a neighborhood $N(\lambda_0)\times N(\mu_0)$ of $(\lambda_0,\mu_0) \in \Lambda \times M$. Assume that the following conditions hold.*

(i) *$K(\cdot,\cdot)$ is $(l_1\cdot\alpha_1, l_2\cdot\alpha_2)$-Hölder continuous on $E(N(\lambda_0))\times N(\lambda_0)$;*

(ii) *For each $x,y \in E(N(\lambda_0))$, $F(x,y,\cdot)$ is $m_1\cdot\gamma_1$-Hölder continuous at $\mu_0 \in M$;*

(iii) *For each $x \in E(N(\lambda_0))$ and $\mu \in N(\mu_0)$, $F(x,\cdot,\mu)$ is $m_2\cdot\gamma_2$-Hölder continuous on $E(N(\lambda_0))$;*

(iv) *$F(\cdot,\cdot,\mu)$ is $h\cdot\beta$-Hölder strongly monotone with respect to ξ_q, that is, there exist constants $h>0$, $\beta>0$ such that for every $x,y \in E(N(\lambda_0))$, $x \neq y$,*

$$h d_X^{\beta}(x,y) \le d\Bigl(\inf_{z\in F(x,y,\mu)} \xi_q(z), \mathbb{R}_+\Bigr) + d\Bigl(\inf_{z\in F(y,x,\mu)} \xi_q(z), \mathbb{R}_+\Bigr);$$

(v) *$\beta = \alpha_1\gamma_2$, $h > 2m_2 L l_1^{\gamma_1}$, where $L := \sup_{\lambda\in C^q} \|\lambda\| \in [\frac{1}{\|q\|}, +\infty)$ is the Lipschitz constant of ξ_q on Y.*

Then, for every $(\lambda,\mu) \in N(\lambda_0)\times N(\mu_0)$, the solution $x(\lambda,\mu)$ of (PVQGEP) *is unique, and $x(\lambda,\mu)$ as a function of λ and μ satisfies the Hölder condition: for all $(\lambda_1,\mu_1),(\lambda_2,\mu_2) \in N(\lambda_0)\times N(\mu_0)$,*

$$\begin{aligned} d_X\bigl(x(\lambda_1,\mu_1), x(\lambda_2,\mu_2)\bigr) \le{}& \left(\frac{2m_2 L l_2^{\gamma_2}}{h - 2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2/\beta}(\lambda_1,\lambda_2) \\ &+ \left(\frac{m_1 L}{h - 2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2), \end{aligned}$$

where $x(\lambda_i,\mu_i) \in S_W(\lambda_i,\mu_i)$, $i = 1,2$.

Proof Let $(\lambda_1,\mu_1),(\lambda_2,\mu_2) \in N(\lambda_0)\times N(\mu_0)$. The proof is divided into the following three steps based on the fact that

$$d_X\bigl(x(\lambda_1,\mu_1), x(\lambda_2,\mu_2)\bigr) \le d_X\bigl(x(\lambda_1,\mu_1), x(\lambda_1,\mu_2)\bigr) + d_X\bigl(x(\lambda_1,\mu_2), x(\lambda_2,\mu_2)\bigr),$$

where $x(\lambda_i,\mu_i) \in S_W(\lambda_i,\mu_i)$, $i = 1,2$.

Step 1: We prove that

$$d_1 := d_X\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) \leq \left(\frac{m_1 L}{h-2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2) \tag{9}$$

for all $x(\lambda_1,\mu_1)\in S_W(\lambda_1,\mu_1)$ and $x(\lambda_1,\mu_2)\in S_W(\lambda_1,\mu_2)$.

If $x(\lambda_1,\mu_1)=x(\lambda_1,\mu_2)$, then we are done. So, we assume that $x(\lambda_1,\mu_1)\neq x(\lambda_1,\mu_2)$. Since $x(\lambda_1,\mu_1)\in K(x(\lambda_1,\mu_1),\lambda_1)$ and $x(\lambda_1,\mu_2)\in K(x(\lambda_1,\mu_2),\lambda_1)$, by the $l_1\cdot\alpha_1$-Hölder continuity of $K(\cdot,\lambda_1)$, there exist $x_1\in K(x(\lambda_1,\mu_1),\lambda_1)$ and $x_2\in K(x(\lambda_1,\mu_2),\lambda_1)$ such that

$$d_X\big(x(\lambda_1,\mu_1),x_2\big) \leq l_1 d_X^{\alpha_1}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) = l_1 d_1^{\alpha_1} \tag{10}$$

and

$$d_X\big(x(\lambda_1,\mu_2),x_1\big) \leq l_1 d_X^{\alpha_1}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) = l_1 d_1^{\alpha_1}. \tag{11}$$

Since $x(\lambda_1,\mu_1)\in S_W(\lambda_1,\mu_1)$ and $x(\lambda_1,\mu_2)\in S_W(\lambda_1,\mu_2)$, by Lemma 3.1, we obtain

$$\psi_{\xi_q}\big(x(\lambda_1,\mu_1),x_1,\mu_1\big) := \inf_{z\in F(x(\lambda_1,\mu_1),x_1,\mu_1)} \xi_q(z) \geq 0 \tag{12}$$

and

$$\psi_{\xi_q}\big(x(\lambda_1,\mu_2),x_2,\mu_2\big) := \inf_{z\in F(x(\lambda_1,\mu_2),x_2,\mu_2)} \xi_q(z) \geq 0. \tag{13}$$

By virtue of (iv), we have

$$\begin{aligned} hd_1^{\beta} &= hd_X^{\beta}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) \\ &\leq d\big(\psi_{\xi_q}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1\big),\mathbb{R}_+\big) + d\big(\psi_{\xi_q}\big(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_1\big),\mathbb{R}_+\big). \end{aligned}$$

By combining (12) and (13) with the last inequality, we have

$$\begin{aligned} hd_1^{\beta} &\leq \big|\psi_{\xi_q}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1\big) - \psi_{\xi_q}\big(x(\lambda_1,\mu_1),x_1,\mu_1\big)\big| \\ &\quad + \big|\psi_{\xi_q}\big(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_1\big) - \psi_{\xi_q}\big(x(\lambda_1,\mu_2),x_2,\mu_2\big)\big| \\ &\leq \big|\psi_{\xi_q}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1\big) - \psi_{\xi_q}\big(x(\lambda_1,\mu_1),x_1,\mu_1\big)\big| \\ &\quad + \big|\psi_{\xi_q}\big(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_1\big) - \psi_{\xi_q}\big(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2\big)\big| \\ &\quad + \big|\psi_{\xi_q}\big(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2\big) - \psi_{\xi_q}\big(x(\lambda_1,\mu_2),x_2,\mu_2\big)\big| \\ &\leq Lm_2 d_X^{\gamma_2}\big(x(\lambda_1,\mu_2),x_1\big) + Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2) + Lm_2 d_X^{\gamma_2}\big(x(\lambda_1,\mu_1),x_2\big) \\ &\leq Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) \\ &\quad + Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2) + Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) \\ &= 2Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) + Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2). \end{aligned} \tag{14}$$

Whence, assumption (iv) implies that

$$d_X\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) \le \left(\frac{Lm_1}{h-Lm_2l_1^{\gamma_2}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2).$$

Step 2: We prove that

$$d_2 := d_X\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\big) \le \left(\frac{2Lm_2l_2^{\gamma_2}}{h-2Lm_2l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2/\beta}(\lambda_1,\lambda_2) \tag{15}$$

for all $x(\lambda_1,\mu_2)\in S_W(\lambda_1,\mu_2)$ and $x(\lambda_2,\mu_2)\in S_W(\lambda_2,\mu_2)$.

If $x(\lambda_1,\mu_2) = x(\lambda_2,\mu_2)$, then we are done. So, we assume that $x(\lambda_1,\mu_2) \neq x(\lambda_2,\mu_2)$. Since $x(\lambda_1,\mu_2) \in K(x(\lambda_1,\mu_2),\lambda_1)$ and $x(\lambda_2,\mu_2) \in K(x(\lambda_2,\mu_2),\lambda_2)$, by the $l_2\cdot\alpha_2$-Hölder continuity of $K(x(\lambda_1,\mu_2),\cdot)$ and $K(x(\lambda_2,\mu_2),\cdot)$, there exist $x_1'\in K(x(\lambda_2,\mu_2),\lambda_1)$ and $x_2'\in K(x(\lambda_1,\mu_2),\lambda_2)$ such that

$$d_X\big(x(\lambda_1,\mu_2),x_2'\big) \le l_2 d_\Lambda^{\alpha_2}(\lambda_1,\lambda_2) \tag{16}$$

and

$$d_X\big(x(\lambda_2,\mu_2),x_1'\big) \le l_2 d_\Lambda^{\alpha_2}(\lambda_1,\lambda_2). \tag{17}$$

Again, by the Hölder continuity of $K(\cdot,\cdot)$, there exist $x_1''\in K(x(\lambda_1,\mu_2),\lambda_1)$ and $x_2''\in K(x(\lambda_2,\mu_2),\lambda_2)$ such that

$$d_X\big(x_1',x_1''\big) \le l_1 d_X^{\alpha_1}\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\big) = l_1 d_2^{\alpha_1} \tag{18}$$

and

$$d_X\big(x_2',x_2''\big) \le l_1 d_X^{\alpha_1}\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\big) = l_1 d_2^{\alpha_1}. \tag{19}$$

Since $x(\lambda_1,\mu_2)\in S_W(\lambda_1,\mu_2)$ and $x(\lambda_2,\mu_2)\in S_W(\lambda_2,\mu_2)$, by Lemma 3.1, we obtain the following:

$$\psi_{\xi_q}\big(x(\lambda_1,\mu_2),x_1'',\mu_2\big) := \inf_{z\in F(x(\lambda_1,\mu_2),x_1'',\mu_2)} \xi_q(z) \ge 0 \tag{20}$$

and

$$\psi_{\xi_q}\big(x(\lambda_2,\mu_2),x_2'',\mu_2\big) := \inf_{z\in F(x(\lambda_2,\mu_2),x_2'',\mu_2)} \xi_q(z) \ge 0. \tag{21}$$

By virtue of (iv), we have

$$\begin{aligned} hd_2^\beta &= hd_X^\beta\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\big) \\ &\le d\big(\psi_{\xi_q}\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2\big),\mathbb{R}_+\big) + d\big(\psi_{\xi_q}\big(x(\lambda_2,\mu_2),x(\lambda_1,\mu_2),\mu_2\big),\mathbb{R}_+\big). \end{aligned}$$

By combining (20) and (21) with the last inequality, we have

$$\begin{aligned} hd_2^\beta &\le \bigl|\psi_{\xi_q}\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2\bigr)-\psi_{\xi_q}\bigl(x(\lambda_1,\mu_2),x_1'',\mu_2\bigr)\bigr| \\ &\quad+\bigl|\psi_{\xi_q}\bigl(x(\lambda_2,\mu_2),x(\lambda_1,\mu_2),\mu_2\bigr)-\psi_{\xi_q}\bigl(x(\lambda_2,\mu_2),x_2'',\mu_2\bigr)\bigr| \\ &\le \bigl|\psi_{\xi_q}\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2\bigr)-\psi_{\xi_q}\bigl(x(\lambda_1,\mu_2),x_1',\mu_2\bigr)\bigr| \\ &\quad+\bigl|\psi_{\xi_q}\bigl(x(\lambda_1,\mu_2),x_1',\mu_2\bigr)-\psi_{\xi_q}\bigl(x(\lambda_1,\mu_2),x_1'',\mu_2\bigr)\bigr| \\ &\quad+\bigl|\psi_{\xi_q}\bigl(x(\lambda_2,\mu_2),x(\lambda_1,\mu_2),\mu_2\bigr)-\psi_{\xi_q}\bigl(x(\lambda_2,\mu_2),x_2',\mu_2\bigr)\bigr| \\ &\quad+\bigl|\psi_{\xi_q}\bigl(x(\lambda_2,\mu_2),x_2',\mu_2\bigr)-\psi_{\xi_q}\bigl(x(\lambda_2,\mu_2),x_2'',\mu_2\bigr)\bigr| \\ &\le Lm_2 d_X^{\gamma_2}\bigl(x(\lambda_2,\mu_2),x_1'\bigr)+Lm_2 d_X^{\gamma_2}\bigl(x_1',x_1''\bigr) \\ &\quad+Lm_2 d_X^{\gamma_2}\bigl(x(\lambda_1,\mu_2),x_2',\bigr)+Lm_2 d_X^{\gamma_2}\bigl(x_2',x_2''\bigr). \end{aligned} \tag{22}$$

By virtue of (16), (17), (18) and (19), we get

$$\begin{aligned} &hd_X^\beta\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\bigr) \\ &\quad\le Lm_2 l_2^{\gamma_2} d_\Lambda^{\alpha_2\gamma_2}(\lambda_1,\lambda_2)+Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\bigr) \\ &\qquad+Lm_2 l_2^{\gamma_2} d_\Lambda^{\alpha_2\gamma_2}(\lambda_1,\lambda_2)+Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\bigr) \\ &\quad=2Lm_2 l_2^{\gamma_2} d_\Lambda^{\alpha_2\gamma_2}(\lambda_1,\lambda_2)+2Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\bigr). \end{aligned} \tag{23}$$

Whence, condition (v) implies that

$$d_X^\beta\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\bigr)\le\left(\frac{2Lm_2 l_2^{\gamma_2}}{h-2Lm_2 l_1^{\gamma_2}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2}(\lambda_1,\lambda_2).$$

Step 3: Let $x(\lambda_1,\mu_1)\in S_W(\lambda_1,\mu_1)$ and $x(\lambda_2,\mu_2)\in S_W(\lambda_2,\mu_2)$. It follows from (9) and (15) that

$$\begin{aligned} &d\bigl(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2)\bigr) \\ &\quad\le d\bigl(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\bigr)+d\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\bigr) \\ &\quad\le\left(\frac{m_1 L}{h-2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2)+\left(\frac{2Lm_2 l_2^{\gamma_2}}{h-2Lm_2 l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2/\beta}(\lambda_1,\lambda_2). \end{aligned}$$

Thus,

$$\begin{aligned} &\rho\bigl(S_W(\lambda_1,\mu_1),S_W(\lambda_2,\mu_2)\bigr) \\ &\quad=\sup_{x(\lambda_1,\mu_1)\in S_W(\lambda_1,\mu_1),x(\lambda_2,\mu_2)\in S_W(\lambda_2,\mu_2)} d_X\bigl(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2)\bigr) \\ &\quad\le\left(\frac{m_1 L}{h-2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2)+\left(\frac{2Lm_2 l_2^{\gamma_2}}{h-2Lm_2 l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2/\beta}(\lambda_1,\lambda_2). \end{aligned}$$

Taking $\lambda_2=\lambda_1$ and $\mu_2=\mu_1$, we see that the diameter of $S(\lambda_1,\mu_1)$ is 0, that is, this set is a singleton $\{x(\lambda_1,\mu_1)\}$. This implies that the (PGVQEP) has a unique solution in a neighborhood of (λ_0,μ_0). The proof is completed. □

Definition 3.4 Let $F : X \times X \times M \to 2^Y$ be a set-valued mapping. A set-valued mapping $F(\cdot, \cdot, \mu) \mapsto 2^Y$ is said to be

(A) *$h \cdot \beta$-Hölder strongly monotone with respect to ξ_q* if there exist $q \in \operatorname{int} C$ and $h > 0$, $\beta > 0$ such that for every $x, y \in E(N(\lambda))$ with $x \neq y$,

$$\inf_{z \in F(x,y,\mu)} \xi_q(z) + \inf_{z \in F(y,x,\mu)} \xi_q(z) + hd_X^{\beta}(x, y) \leq 0;$$

(B) *$h \cdot \beta$-Hölder strongly pseudomonotone with respect to $q \in \operatorname{int} C$* and $h > 0$, $\beta > 0$ such that for every $x, y \in E(N(\lambda_0))$ with $x \neq y$,

$$z \notin -\operatorname{int} C, \quad \exists z \in F(x, y, \mu) \quad \Rightarrow \quad z' + hd_X^{\beta}(x, y)q \in -C, \quad \exists z' \in F(y, x, \mu).$$

(C) *quasi-monotone on $E(N(\lambda_0))$* if $\forall x, y \in E(N(\lambda_0))$ with $x \neq y$,

$$z \in -\operatorname{int} C, \quad \exists z \in F(x, y, \mu) \quad \Rightarrow \quad z' \notin -\operatorname{int} C, \quad \exists z' \in F(y, x, \mu).$$

The following proposition provides the relation among monotonicity conditions defined above.

Proposition 3.5

(i) (A) $\Rightarrow$ (iv).

(ii) (B) *and* (C) $\Rightarrow$ (iv).

Proof (i) From the definition of (A), we have

$$\begin{aligned} hd_X^{\beta}(x, y) &\leq - \inf_{z \in F(x,y,\mu)} \xi_q(z) - \inf_{z \in F(y,x,\mu)} \xi_q(z) \\ &\leq d\Bigl(\inf_{z \in F(x,y,\mu)} \xi_q(z), \mathbb{R}_+\Bigr) + d\Bigl(\inf_{z \in F(y,x,\mu)} \xi_q(z), \mathbb{R}_+\Bigr). \end{aligned}$$

(ii) Assume that F satisfies definitions (B) and (C). We consider two cases.

Case 1. $z \notin -\operatorname{int} C$, $\exists z \in F(x, y, \mu)$, then there exists $z' \in F(y, x, \mu)$ such that $z' + hd_X^{\beta}(x, y)q \in -C$. From Lemma 2.1, we have

$$\xi_q(z') + hd_X^{\beta}(x, y) = \xi_q\bigl(z' + hd_X^{\beta}(x, y)q\bigr) \leq 0,$$

which implies that $\inf_{z \in F(y,x,\mu)} \xi_q(z) \leq \xi_q(z') \leq -hd_X^{\beta}(x, y)$. Hence,

$$hd_X^{\beta}(x, y) \leq - \inf_{z \in F(y,x,\mu)} \xi_q(z) \leq d\Bigl(\inf_{z \in F(x,y,\mu)} \xi_q(z), \mathbb{R}_+\Bigr) + d\Bigl(\inf_{z \in F(y,x,\mu)} \xi_q(z), \mathbb{R}_+\Bigr).$$

Case 2. $z \in -\operatorname{int} C$, $\exists z \in F(x, y, \mu)$, then there exists $z' \in F(y, x, \mu)$ such that $z \notin -\operatorname{int} C$. By a similar argument as in the previous case, we have the desired result. □

Remark 3.6 The converse of Proposition 3.5 does not hold in general, even in the special case $X = Y = \mathbb{R}$ and $C = \mathbb{R}_+$. See, for example, Examples 1.1 and 1.2 in [15]. Therefore, Theorem 3.3 still holds when condition (iv) is replaced by condition (A) or conditions (B) and (C). We can immediately obtain the following two theorems.

Theorem 3.7 *Theorem* 3.3 *still holds when condition* (iv) *is replaced by condition* (A).

Theorem 3.8 *Theorem* 3.3 *still holds when condition* (iv) *is replaced by conditions* (B) *and* (C).

Let $f : X \times X \times M \to Y$ be a vector-valued mapping. Then (PGVQEP) becomes the following *parametric vector quasi-equilibrium problem* (PVQEP):

Find $x_0 \in K(x_0, \lambda)$ such that

$$f(x_0, y, \mu) \notin -\operatorname{int} C, \quad \forall y \in K(x_0, \lambda). \tag{24}$$

Remark 3.9 In the case of a vector-valued mapping, condition (iv) in Theorem 3.3 and condition (ii″) coincide. Also, condition (A) and conditions (B) and (C) are the same as conditions (ii) and (ii′) in [30], respectively. It is obvious that Theorems 3.3, 3.7 and 3.8 extend Theorems 3.3, 3.1 and 3.2 in [30], respectively, in the case that the vector-valued mapping $f(\cdot,\cdot,\cdot)$ is extended to a set-valued one.

4 Applications

Since the parametric generalized vector quasi-equilibrium problem (PGVQEP) contains as special cases many optimization-related problems, including quasi-variational inequalities, traffic equilibrium problems, quasi-optimization problems, fixed point and coincidence point problems, complementarity problems, vector optimization, Nash equilibria, *etc.*, we can derive from Theorem 3.3 a direct consequence for such special cases. We discuss now only some applications of our results.

4.1 Quasi-variational inequalities

In this section, we assume that X is a normed space. Let $K : X \times \Lambda \rightrightarrows X$ and $T : X \times M \rightrightarrows B^*(X, Y)$ be set-valued mappings, where $B^*(X, Y)$ denotes the space of all bounded linear mappings of X into Y. Setting $F(x, y, \mu) = \langle T(x, \mu), y - x\rangle := \bigcup_{t \in T(x,\mu)} \langle t, y - x\rangle$ in (6), we obtain parametric generalized vector quasi-variational inequalities (PGVQVI) in the case of set-valued mappings as follows:

$$\text{Find } x_0 \in K(x_0, \lambda) \text{ such that } \bigl\langle T(x_0, \mu), y - x_0\bigr\rangle \subseteq Y \setminus -\operatorname{int} C, \quad \forall y \in K(x_0, \lambda). \tag{25}$$

For each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, let

$$E(\lambda) := \bigl\{x \in X : x \in K(x, \lambda)\bigr\}.$$

The solution set of (25) is denoted by

$$S^V_{QVI}(\lambda, \mu) := \bigl\{x \in E(\lambda) : \bigl\langle T(x, \mu), y - x\bigr\rangle \subseteq Y \setminus -\operatorname{int} C, \forall y \in K(x, \lambda)\bigr\}.$$

For each $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $q \in \operatorname{int} C$, the ξ_q-solution set of (25) is

$$S^V_{QVI}(\xi_q, \lambda, \mu) := \Bigl\{x \in E(\lambda) : \inf_{z \in \langle T(x,\mu), y - x\rangle} \xi_q(z) \geq 0, \forall y \in K(x, \lambda)\Bigr\}.$$

Theorem 4.1 *Assume that for each fixed $q \in \operatorname{int} C$, $S^V_{QVI}(\xi_q, \lambda, \mu)$ is nonempty in a neighborhood $N(\lambda_0) \times N(\mu_0)$ of the considered point $(\lambda_0, \mu_0) \in \Lambda \times M$. Assume further that the following conditions hold.*

(i′) *$K(\cdot,\cdot)$ is $(l_1 \cdot \alpha_1, l_2 \cdot \alpha_2)$-Hölder continuous on $E(N(\lambda_0)) \times N(\lambda_0)$;*

(ii′) *For each $x \in E(N(\lambda_0))$, $T(x,\cdot)$ is $m_3 \cdot \gamma_3$-Hölder continuous at $\mu_0 \in M$;*

(iii′) *$T(\cdot,\cdot)$ is bounded in $x \in E(N(\lambda_0))$, and $E(N(\lambda_0))$ is bounded;*

(iv′) *$T(\cdot,\mu)$ is $h \cdot \beta$-Hölder strongly monotone with respect to ξ_q, i.e., there exist constants $h > 0$, $\beta > 0$ such that for every $x, y \in E(N(\lambda_0))$: $x \neq y$,*

$$h\|x-y\|^{\beta} \le d\Big(\inf_{z\in\langle T(x,\mu),y-x\rangle} \xi_q(z), \mathbb{R}_+\Big) + d\Big(\inf_{z\in\langle T(y,\mu),x-y\rangle} \xi_q(z), \mathbb{R}_+\Big);$$

(v′) *$\beta = \alpha_1$, $h > 2MLl_1^{\gamma_1}$, where $L := \sup_{\lambda\in C^q} \|\lambda\| \in [\frac{1}{\|q\|}, +\infty)$ is the Lipschitz constant of ξ_q on Y.*

Then, for every $(\lambda,\mu) \in N(\lambda_0) \times N(\mu_0)$, the solution of (PGVQVI) *is unique, $x(\lambda,\mu)$, and this function satisfies the Hölder condition: for all $(\lambda_1,\mu_1), (\lambda_2,\mu_2) \in N(\lambda_0) \times N(\mu_0)$,*

$$\begin{aligned} d_X\big(x(\lambda_1,\mu_1), x(\lambda_2,\mu_2)\big) \le{}& \left(\frac{2MLl_2}{h-2MLl_1^{\gamma_3}}\right)^{\frac{1}{\beta}} d_{\lambda}^{\alpha_2/\beta}(\lambda_1,\lambda_2) \\ &+ \left(\frac{Nm_3L}{h-2MLl_1^{\gamma_3}}\right)^{\frac{1}{\beta}} d_M^{\gamma_3/\beta}(\mu_1,\mu_2), \end{aligned}$$

where $x(\lambda_i,\mu_i) \in S_{QVI}(\lambda_i,\mu_i)$, $i = 1,2$.

Proof We verify that all the assumptions of Theorem 3.3 are fulfilled. First, (i′), (iv′) and (v′) are the same as (i), (iv) and (v) in Theorem 3.3. We need only to verify conditions (ii) and (iii). Taking $M, \widetilde{M} > 0$ such that

$$\|T(x,\mu)\| \le M, \quad \forall (x,\mu) \in E\big(N(\lambda_0)\big) \times N(\mu_0)$$

and

$$\|x-y\| \le \widetilde{M}, \quad \forall x,y \in E\big(N(\lambda_0)\big).$$

We put $m_1 = \widetilde{M}m_3$ and $\gamma_1 = \gamma_3$. For any fixed $x, y \in E(N(\lambda_0))$, by assumption (ii′), we have

$$T(x,\mu_1) \subseteq T(x,\mu_2) + m_3 d^{\gamma_3}(\mu_1,\mu_2) B_{B^*(X,Y)}, \quad \forall \mu_1,\mu_2 \in N(\mu_0).$$

Then

$$\begin{aligned} \big\langle T(x,\mu_1), y-x\big\rangle &\subseteq \big\langle T(x,\mu_2) + m_3 d^{\gamma_3}(\mu_1,\mu_2) B_{B^*(X,Y)}, y-x\big\rangle \\ &= \big\langle T(x,\mu_2), y-x\big\rangle + \big\langle m_3 d^{\gamma_3}(\mu_1,\mu_2) B_{B^*(X,Y)}, y-x\big\rangle \\ &= \big\langle T(x,\mu_2), y-x\big\rangle + m_3 d^{\gamma_3}(\mu_1,\mu_2) \langle B_{B^*(X,Y)}, y-x\rangle \\ &= \big\langle T(x,\mu_2), y-x\big\rangle + m_3 d^{\gamma_3}(\mu_1,\mu_2) \bigcup_{g\in B_{B^*(X,Y)}} \langle g, y-x\rangle \\ &\subseteq \big\langle T(x,\mu_2), y-x\big\rangle + m_3 d^{\gamma_3}(\mu_1,\mu_2) \widetilde{M} B_Y. \end{aligned}$$

Hence

$$\langle T(x,\mu_1), y-x\rangle \subseteq \langle T(x,\mu_2), y-x\rangle + m_1 d^{\gamma_1}(\mu_1,\mu_2)\widetilde{M}B_Y.$$

Also, we put $m_2 = M$ and $\gamma_2 = 1$. We need to show that

$$\langle T(x,\mu), y_1-x\rangle \subseteq \langle T(x,\mu), y_2-x\rangle + \widetilde{M}\|y_1-y_2\|B_Y.$$

For each fixed $x \in E(N(\lambda_0))$ and $\mu \in N(\mu_0)$,

$$\begin{aligned}
\langle T(x,\mu), y_1-x\rangle &= \bigcup_{t\in T(x,\mu)} \langle t, y_1-x\rangle \\
&= \bigcup_{t\in T(x,\mu)} \langle t, y_1-x+y_2-y_2\rangle \\
&= \bigcup_{t\in T(x,\mu)} \langle t, y_2-x\rangle + \bigcup_{t\in T(x,\mu)} \langle t, y_1-y_2\rangle \\
&\subseteq \langle T(x,\mu), y-x\rangle + M\|y_1-y_2\|B_Y.
\end{aligned}$$

Hence, condition (iii) is verified, and so we obtain the result. □

For (PGVQVI), if we put $Y = \mathbb{R}$, $C = [0,+\infty)$, then (25) becomes the following parametric generalized quasi-variational inequality problem in the case of scalar-valued one:

$$\text{Find } x_0 \in K(x_0,\lambda) \text{ such that } \langle t, y-x_0\rangle \geq 0, \quad \forall y \in K(x_0,\lambda), \forall t \in T(x_0,\mu). \tag{26}$$

For each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, let

$$E(\lambda) := \{x \in X : x \in K(x,\lambda)\}.$$

The solution set of (26) is denoted by

$$S^S_{QVI}(\lambda,\mu) := \{x \in E(\lambda) : \langle t, y-x\rangle \geq 0, \forall y \in K(x,\lambda), \forall t \in T(x,\mu)\}.$$

For each $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $1 \in \operatorname{int} C$, the ξ_q-solution set of (25) is

$$S^S_{QVI}(\xi_1,\lambda,\mu) := \Big\{x \in E(\lambda) : \inf_{z\in\langle T(x,\mu),y-x\rangle} \xi_1(z) \geq 0, \forall y \in K(x,\lambda)\Big\}.$$

It follows from Lemma 2.1 that $S^S_{QVI}(\xi_1,\lambda,\mu)$ coincides with $S^S_{QVI}(\lambda,\mu)$.

Corollary 4.2 *Assume that $S^S_{QVI}(\lambda,\mu)$ is nonempty in a neighborhood $N(\lambda_0) \times N(\mu_0)$ of the considered point $(\lambda_0,\mu_0) \in \Lambda \times M$. Assume further that conditions* (i′)-(iii′) *and* (v′) *in Corollary* 4.1 *hold. Replace* (iv′) *by* (iv″).

(iv″) *$T(\cdot,\mu)$ is $h \cdot \beta$-Hölder strongly monotone, i.e., there exist constants $h > 0$, $\beta > 0$, such that for every $x, y \in E(N(\lambda_0))$: $x \neq y$,*

$$\langle u-v, x-y\rangle \geq h\|x-y\|^{\beta}, \quad \forall u \in T(x), \forall v \in T(y).$$

Then, for every $(\lambda,\mu)\in N(\lambda_0)\times N(\mu_0)$, *the solution of* (PGVQVI) *is unique*, $x(\lambda,\mu)$, *and this function satisfies the Hölder condition*: *for all* $(\lambda_1,\mu_1),(\lambda_2,\mu_2)\in N(\lambda_0)\times N(\mu_0)$,

$$d_X\bigl(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2)\bigr)\le\left(\frac{2Ml_2}{h-2Ml_1^{\gamma_3}}\right)^{\frac{1}{\beta}}d_\lambda^{\alpha_2/\beta}(\lambda_1,\lambda_2)+\left(\frac{Nm_3}{h-2Ml_1^{\gamma_3}}\right)^{\frac{1}{\beta}}d_M^{\gamma_3/\beta}(\mu_1,\mu_2),$$

where $x(\lambda_i,\mu_i)\in S^S_{QVI}(\lambda_i,\mu_i)$, $i=1,2$.

Proof It is not hard to show that (iv″) implies (iv′). Indeed, for any $x,y\in E(N(\lambda_0))$ with $x\neq y$,

$$\begin{aligned}
h\|x-y\|^\beta &\le \langle u-v,x-y\rangle\\
&=\langle u,x-y\rangle+\langle v,y-x\rangle\\
&\le\sup_{u\in T(x)}\langle u,x-y\rangle+\sup_{v\in T(y)}\langle v,y-x\rangle\\
&=\sup_{u\in T(x)}-\langle u,y-x\rangle+\sup_{v\in T(y)}-\langle v,x-y\rangle\\
&=-\inf_{u\in T(x)}\langle u,y-x\rangle-\inf_{v\in T(y)}\langle v,x-y\rangle\\
&\le d\Bigl(\inf_{u\in T(x)}\langle u,y-x\rangle,\mathbb{R}_+\Bigr)+d\Bigl(\inf_{v\in T(y)}\langle v,x-y\rangle,\mathbb{R}_+\Bigr).
\end{aligned}$$

Therefore, (iv′) is satisfied. □

Remark 4.3 Corollary 4.2 extends Corollary 3.1 in [33] since the mapping T is a multi-valued mapping.

4.2 Traffic equilibrium problems

The foundation of the study of traffic network problems goes back to Wardrop [34], who stated the basic equilibrium principle in 1952. Over the past decades, a large number of efforts have been devoted to the study of traffic assignment models, with emphasis on efficiency and optimality, in order to improve practicability, reduce gas emissions and contribute to the welfare of the community. The variational inequality approach to such problems begins with the seminal work of Smith [35] who proved that the user-optimized equilibrium can be expressed in terms of a variational inequality. Thus, the possibility of exploiting the powerful tools of variational analysis has led to dealing with a large variety of models, reaching valuable theoretical results and providing applications in practical situations. In this paper, we are concerned with a class of equilibrium problems which can be studied in the framework of quasi-variational inequalities, see [36, 37].

Let a set N of nodes, a set L of links, a set $W:=(W_1,\dots,W_l)$ of origin-destination pairs (O/D pairs for short) be given. Assume that there are $r_j\ge 1$ paths connecting the pairs W_j, $j=1,\dots,l$, whose set is denoted by P_j. Set $m:=r_1+\cdots+r_l$; *i.e.*, there are in whole m paths in the traffic network. Let $F:=(F_1,\dots,F_m)$ stand for the path flow vector. Assume that the travel cost of the path R_s, $s=1,\dots,m$, is a set $T_s(F)\subset\mathbb{R}_+$. So, we have a multifunction $T:\mathbb{R}^m_+\rightrightarrows\mathbb{R}^m_+$ with $T(F):=(T_1(F),\dots,T_m(F))$. Let the capacity restriction be

$$F\in A:=\bigl\{F\in\mathbb{R}^m_+:F_s\le\Gamma_s,s=1,\dots,m\bigr\},$$

where Γ_s are given real numbers. Extending the Wardrop definition to the case of multi-valued costs, we propose the following definition.

A path flow vector H is said to be a ***weak*** *equilibrium flow* vector if

$$\begin{aligned}&\forall W_j, \forall R_q \in P_j, R_s \in P_j, \text{ there exists } t \in T(H) \text{ such that}\\ &\quad t_q < t_s \Rightarrow H_q = \Gamma_q \text{ or } H_s = 0,\end{aligned} \tag{27}$$

where $j = 1,\dots,l$ and $q,s \in \{1,\dots,m\}$ are among r_j indices corresponding to P_j.

A path flow vector H is said to be a ***strong*** *equilibrium flow* vector if

$$\forall W_j, \forall R_q \in P_j, R_s \in P_j, \text{ for all } t \in T(H) \text{ such that } t_q < t_s \Rightarrow H_q = \Gamma_q \text{ or } H_s = 0. \tag{28}$$

Suppose that the travel demand ρ_j of the O/D pair W_j, $j = 1,\dots,l$, depends on the weak (or strong) equilibrium problem flow H. So, considering all the O/D pairs, we have a mapping $\rho : \mathbb{R}^m_+ \to \mathbb{R}^l_+$. We use the Kronecker notation

$$\phi_{js} = \begin{cases} 1 & \text{if } s \in P_j, \\ 0 & \text{if } s \notin P_j. \end{cases}$$

Then the matrix

$$\phi = \{\phi_{js}\}, \quad j = 1,\dots,l, s = 1,\dots,m,$$

is called an O/D pair/path incidence matrix. The path flow vectors meeting the travel demands are called the feasible path flow vectors and form the constraint set, for a given weak (or strong) equilibrium flow H,

$$K(H,\lambda) := \{F \in A : \phi F = \rho(H,\lambda)\}.$$

Assume further that the path costs are also perturbed, *i.e.*, depend on a perturbation parameter μ of a metric space M: $T_s(F,\mu)$, $s = 1,\dots,m$.

Our traffic equilibrium problem is equivalent to a quasi-variational inequality as follows (see [38]).

Lemma 4.4 *A path vector flow $H \in K(H,\lambda)$ is a* ***weak*** *equilibrium flow if and only if it is a solution of the following quasi-variational inequality*:

$$\begin{aligned}&\textit{Find } H \in K(H,\lambda) \textit{ such that there exists } t \in T(H,\lambda) \textit{ satisfying } \langle t, F - H\rangle \geq 0,\\ &\quad \forall F \in K(H,\lambda).\end{aligned}$$

Lemma 4.5 *A path vector flow $H \in K(H,\lambda)$ is a* ***strong*** *equilibrium flow if and only if it is a solution of the following quasi-variational inequality*:

$$\begin{aligned}&\textit{Find } H \in K(H,\lambda) \textit{ such that for all } t \in T(H,\lambda) \textit{ it satisfies } \langle t, F - H\rangle \geq 0,\\ &\quad \forall F \in K(H,\lambda).\end{aligned}$$

Corollary 4.6 *Assume that solutions of the traffic network equilibrium problem exist and all the assumptions of Corollary* 4.2 *are satisfied. Then, in a neighborhood of* (λ_0, μ_0), *the solution is unique and satisfies the same Hölder condition as in Corollary* 4.2.

4.3 Quasi-optimization problem

For the normed linear space Y and pointed, closed and convex cone C with nonempty interior, we denote the ordering induced by C as follows:

$$x \leq y \quad \text{iff} \quad y - x \in C;$$
$$x < y \quad \text{iff} \quad y - x \in \operatorname{int} C.$$

The orderings $\geq$ and $>$ are defined similarly. Let $g : X \times M \to Y$ be a vector-valued mapping. For each $(\lambda, \mu) \in \Lambda \times M$, consider the problem of parametric quasi-optimization problem (PQOP) finding $x_0 \in K(x_0, \lambda)$ such that

$$g(x_0, \mu) = \min_{y \in K(x_0, \lambda)} g(y, \mu). \tag{29}$$

Since the constraint set depends on the minimizer x_0, this is a quasi-optimization problem. Setting $f(x, y, \mu) = g(y, \mu) - g(x, \mu)$, (PVQEP) becomes a special case of (PQOP).

The following results are derived from Theorem 3.8 (Theorem 3.3 cannot be applied since $f(x, y, \mu) + f(y, x, \mu) = 0$, $\forall x, y \in A$ and $\mu \in M$).

Theorem 4.7 *For* (PQOP), *assume that the solution exists in a neighborhood* $N(\lambda_0) \times N(\mu_0)$ *of the considered point* $(\lambda_0, \mu_0) \in \Lambda \times M$. *Assume further that the following conditions hold*.

(i) $K(\cdot, \cdot)$ *is* $(l_1 \cdot \alpha_1, l_2 \cdot \alpha_2)$-*Hölder continuous on* $E(N(\lambda_0)) \times N(\lambda_0)$;
(ii) *For each* $x, y \in E(N(\lambda_0))$, $F(x, y, \cdot)$ *is* $m_1 \cdot \gamma_1$-*Hölder continuous at* $\mu_0 \in M$;
(iii) *For each* $x \in E(N(\lambda_0))$ *and* $\mu \in N(\mu_0)$, $F(x, \cdot, \mu)$ *is* $m_2 \cdot \gamma_2$-*Hölder continuous on* $E(N(\lambda_0))$;
(iv) $F(\cdot, \cdot, \mu)$ *is* $h \cdot \beta$-*Hölder strongly monotone with respect to* ξ_q, *i.e., there exist constants* $h > 0$, $\beta > 0$ *such that for every* $x, y \in E(N(\lambda_0))$: $x \neq y$,

$$hd_X^\beta(x, y) \leq d\Bigl(\inf_{z \in F(x, y, \mu)} \xi_q(z), \mathbb{R}_+\Bigr) + d\Bigl(\inf_{z \in F(y, x, \mu)} \xi_q(z), \mathbb{R}_+\Bigr);$$

(v) $\beta = \alpha_1 \gamma_2$, $h > 2m_2 L l_1^{\gamma_1}$, *where* $L := \sup_{\lambda \in C^q} \|\lambda\| \in [\frac{1}{\|q\|}, +\infty)$ *is the Lipschitz constant of* ξ_q *on* Y.

Then, for every $(\lambda, \mu) \in N(\lambda_0) \times N(\mu_0)$, *the solution of* (PVQGEP) *is unique*, $x(\lambda, \mu)$, *and this function satisfies the Hölder condition*:

for all $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in N(\lambda_0) \times N(\mu_0)$,

$$\begin{aligned} d_X\bigl(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)\bigr) \leq{}& \left(\frac{2m_2 L l_2^{\gamma_2}}{h - 2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2 \gamma_2 / \beta}(\lambda_1, \lambda_2) \\ &+ \left(\frac{m_1 L}{h - 2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1 / \beta}(\mu_1, \mu_2), \end{aligned}$$

where $x(\lambda_i, \mu_i) \in S_W(\lambda_i, \mu_i)$, $i = 1, 2$.

5 Conclusions

In this paper, by using a nonlinear scalarization technique, we obtain sufficient conditions for Hölder continuity of the solution mapping for a parametric generalized vector quasi-equilibrium problem in the case where the mapping F is a general set-valued one. As applications, we derived this Hölder continuity for some quasi-variational inequalities, traffic network problems and quasi-optimization problems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

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Lower semicontinuity of approximate solution mappings for parametric generalized vector equilibrium problems

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Abstract

In this paper, we obtain sufficient conditions for the lower semicontinuity of an approximate solution mapping for a parametric generalized vector equilibrium problem involving set-valued mappings. By using a scalarization method, we obtain the lower semicontinuity of an approximate solution mapping for such a problem without the assumptions of monotonicity and compactness.

Keywords: lower semicontinuity; approximate solution mapping; parametric generalized vector equilibrium problems; scalarization method

1 Introduction

The vector equilibrium problem is a unified model of several problems, for example, the vector optimization problem, the vector variational inequality problem, the vector complementarity problem and the vector saddle point problem. In the literature, existence results for various types of vector equilibrium problems have been investigated intensively, *e.g.*, see [1–4] and the references therein. The stability analysis of the solution mappings for VEP is an important topic in vector equilibrium theory. Recently, the semicontinuity, especially the lower semicontinuity, of solution mappings to parametric vector equilibrium problems has been studied in the literature, see [5–16]. In the mentioned results, the lower semicontinuity of solution mappings to parametric generalized strong vector equilibrium problems is established under the assumptions of monotonicity and compactness. Very recently, Han and Gong [17] studied the lower semicontinuity of solution mappings to parametric generalized strong vector equilibrium problems without the assumptions of monotonicity and compactness.

On the other hand, exact solutions of the problems may not exist in many practical problems because the data of the problems are not sufficiently 'regular'. Moreover, these mathematical models are solved usually by numerical methods which produce approximations to the exact solutions. So it is impossible to obtain an exact solution of many practical problems. Naturally, investigating approximate solutions of parametric equilibrium problems is of interest in both practical applications and computations. Anh and Khanh [18] considered two kinds of approximate solution mappings to parametric generalized vector quasiequilibrium problems and established the sufficient conditions for their Hausdorff semicontinuity (or Berge semicontinuity). Among many approaches for dealing with the lower semicontinuity and continuity of solution mappings for parametric vector variational inequalities and parametric vector equilibrium problems, the scalarization method is of considerable interest. By using a scalarization method, Li and Li [19] discussed the Berge lower semicontinuity and Berge continuity of an approximate solution mapping for a parametric vector equilibrium problem.

Motivated by the work reported in [17–19], in this paper we aim to establish efficient conditions for the lower semicontinuity of an approximate solution mapping for a parametric generalized vector equilibrium problem involving set-valued mappings. By using a scalarization method, we obtain the lower semicontinuity of an approximate solution mapping for such a problem without the assumptions of monotonicity and compactness.

2 Preliminaries

Throughout this paper, let X and Y be real Hausdorff topological vector spaces, and let Z be a real topological space. We also assume that C is a pointed closed convex cone in Y with its interior $\operatorname{int} C \neq \emptyset$. Let Y^* be the topological dual space of Y. Let $C^* := \{\xi \in Y^* : \langle \xi, y\rangle \geq 0, \forall y \in C\}$ be the dual cone of C, where $\langle \xi, y\rangle$ denotes the value of ξ at y. Since $\operatorname{int} C \neq \emptyset$, the dual cone C^* of C has a weak* compact base. Let $e \in \operatorname{int} C$. Then $B_e^* := \{\xi \in C^* : \langle \xi, e\rangle = 1\}$ is a weak* compact base of C^*.

Suppose that K is a nonempty subset of X and $F : K \times K \to 2^Y \setminus \{\emptyset\}$ is a set-valued mapping. We consider the following generalized vector equilibrium problem (GVEP) of finding $x_0 \in K$ such that

$$F(x_0, y) \subset Y \setminus -\operatorname{int} C, \quad \forall y \in K. \tag{2.1}$$

When the set K and the mapping F are perturbed by a parameter μ which varies over a set M of Z, we consider the following parametric generalized vector equilibrium problem (PGVEP) of finding $x_0 \in K(\mu)$ such that

$$F(x_0, y, \mu) \subset Y \setminus -\operatorname{int} C, \quad \forall y \in K(\mu), \tag{2.2}$$

where $K : M \to 2^X \setminus \{\emptyset\}$ is a set-valued mapping, $F : B \times B \times M \subset X \times X \times Z \to 2^Y \setminus \{\emptyset\}$ is a set-valued mapping with $K(M) = \bigcup_{\mu \in M} K(\mu) \subset B$. For each $\varepsilon > 0$ and $\mu \in M$, the approximate solution set of (PGVEP) is defined by

$$\widetilde{S}(\varepsilon, \mu) := \bigl\{x \in K(\mu) : F(x, y, \mu) + \varepsilon e \subset Y \setminus -\operatorname{int} C, \forall y \in K(\mu)\bigr\},$$

where $e \in \operatorname{int} C$. For each $\xi \in B_e^*$ and $(\varepsilon, \mu) \in \mathbb{R}^+ \times M$, by $\widetilde{S}_\xi(\varepsilon, \mu)$ we denote the ξ-approximate solution set of (PGVEP), *i.e.*,

$$\widetilde{S}_\xi(\varepsilon, \mu) := \Bigl\{x \in K(\mu) : \inf_{z \in F(x,y,\mu)} \xi(z) + \varepsilon \geq 0, \forall y \in K(\mu)\Bigr\}.$$

Definition 2.1 Let D be a nonempty convex subset of X. A set-valued mapping $G : X \to 2^Y$ is said to be:

(i) *C-convex* on D if, for any $x_1, x_2 \in D$ and for any $t \in [0, 1]$, we have

$$tG(x_1) + (1 - t)G(x_2) \subseteq G\bigl(tx_1 + (1 - t)x_2\bigr) + C.$$

(ii) *C-concave* on D if, for any $x_1, x_2 \in D$ and for any $t \in [0,1]$, we have

$$G\big(tx_1 + (1-t)x_2\big) \subseteq tG(x_1) + (1-t)G(x_2) + C.$$

Definition 2.2 [17] Let M and M_1 be topological vector spaces. Let D be a nonempty subset of M. A set-valued mapping $G : M \to 2^{M_1}$ is said to be *uniformly continuous* on D if, for any neighborhood V of $0 \in M_1$, there exists a neighborhood U_0 of $0 \in M$ such that $G(x_1) \subseteq G(x_2) + V$ for any $x_1, x_2 \in D$ with $x_1 - x_2 \in U_0$.

Definition 2.3 [20] Let M and M_1 be topological vector spaces. A set-valued mapping $G : M \to 2^{M_1}$ is said to be:

(i) *Hausdorff upper semicontinuous* (*H-u.s.c.*) at $u_0 \in M$ if, for any neighborhood V of $0 \in M_1$, there exists a neighborhood $U(u_0)$ of u_0 such that

$$G(u) \subseteq G(u_0) + V \quad \text{for every } u \in U(u_0).$$

(ii) *Lower semicontinuous* (*l.s.c.*) at $u_0 \in M$ if, for any $x \in G(u_0)$ and any neighborhood V of x, there exists a neighborhood $U(u_0)$ of u_0 such that

$$G(u) \cap V \neq \emptyset \quad \text{for every } u \in U(u_0).$$

The following lemma plays an important role in the proof of the lower semicontinuity of the solution mapping $\widetilde{S}(\cdot,\cdot)$.

Lemma 2.4 [21, Theorem 2] *The union* $\Gamma = \bigcup_{i\in I} \Gamma_i$ *of a family of l.s.c. set-valued mappings* Γ_i *from a topological space* X *into a topological space* Y *is also an l.s.c. set-valued mapping from* X *into* Y, *where* I *is an index set.*

3 Lower semicontinuity of the approximate solution mapping for (PGVEP)

In this section, we establish the lower semicontinuity of the approximate solution mapping for (PGVEP) at the considered point $(\varepsilon_0, \mu_0) \in \mathbb{R}^+ \times M$ with $\varepsilon_0 > 0$.

Firstly, using the same argument as in the proof given in [22, Lemma 3.1], we can prove the following useful result.

Lemma 3.1 *For each* $\varepsilon > 0$, $\mu \in M$, *if for each* $x \in K(\mu)$, $F(x, K(\mu), \mu) + C$ *is a convex set, then*

$$\widetilde{S}(\varepsilon,\mu) = \bigcup_{\xi\in C^*\setminus\{0\}} \widetilde{S}_\xi(\varepsilon,\mu) = \bigcup_{\xi\in B_e^*} \widetilde{S}_\xi(\varepsilon,\mu).$$

Proof For any $x \in \bigcup_{\xi\in C^*\setminus\{0\}} \widetilde{S}_\xi(\varepsilon,\mu)$, there exists $\xi' \in C^*\setminus\{0\}$ such that $x \in \widetilde{S}_{\xi'}(\varepsilon,\mu)$. Thus, we can obtain that $x \in K(\mu)$ and $\inf_{z\in F(x,y,\mu)} \xi'(z) + \varepsilon \geq 0$, $\forall y \in K(\mu)$. Then, for each $y \in K(\mu)$ and $z \in F(x,y,\mu)$, $\xi'(z) + \varepsilon \geq 0$, which arrives at $z \notin -\operatorname{int} C$. It then follows that, for each $z \in F(x,y,\mu)$,

$$F(x,y,\mu) + \varepsilon e \subseteq Y \setminus -\operatorname{int} C, \quad \forall y \in K(\mu),$$

which gives that $x \in \widetilde{S}(\varepsilon, \mu)$. Hence, $\bigcup_{\xi \in C^* \setminus \{0\}} \widetilde{S}_\xi(\varepsilon, \mu) \subseteq \widetilde{S}(\varepsilon, \mu)$. Conversely, let $x \in \widetilde{S}(\varepsilon, \mu)$ be arbitrary. Then $x \in K(\mu)$ and $F(x, y, \mu) + \varepsilon e \subseteq Y \setminus -\operatorname{int} C$, $\forall y \in K(\mu)$. Thus, we have

$$F\bigl(x, K(\mu), \mu\bigr) \cap (-\operatorname{int} C) = \emptyset,$$

and hence

$$\bigl(F\bigl(x, K(\mu), \mu\bigr) + C\bigr) \cap (-\operatorname{int} C) = \emptyset.$$

Because $F(x, K(\mu), \mu) + C$ is a convex set, by the well-known Edidelheit separation theorem (see [23], Theorem 3.16), there exist a continuous linear functional $\xi \in Y^* \setminus \{0\}$ and a real number γ such that

$$\xi(\hat{c}) < \gamma \leq \xi(z + c)$$

for all $z \in F(x, K(\mu), \mu)$, $c \in C$ and $\hat{c} \in -\operatorname{int} C$. Since C is a cone, we have $\xi(\hat{c}) \leq 0$ for all $\hat{c} \in -\operatorname{int} C$. Thus, $\xi(\hat{c}) \geq 0$ for all $\hat{c} \in C$, that is, $\xi \in C^*$. Moreover, it follows from $c \in C$, $\hat{c} \in -\operatorname{int} C$ and the continuity of ξ that $\xi(z) + \varepsilon \geq 0$ for all $z \in F(x, K(\mu), \mu)$. Thus, for all $y \in K(\mu)$, we have $\inf_{z \in F(x,y,\mu)} \xi(z) + \varepsilon \geq 0$, *i.e.*, $x \in \widetilde{S}_\xi(\varepsilon, \mu) \subseteq \bigcup_{\xi \in C^* \setminus \{0\}} \widetilde{S}_\xi(\varepsilon, \mu)$. □

Theorem 3.2 *We assume that for any given $\xi \in B_e^*$, there exists $\delta > 0$ such that the ξ-approximate solution set $\widetilde{S}_\xi(\cdot, \cdot)$ exists in $[\varepsilon_0, \delta) \times N(\mu_0)$, where $N(\mu_0)$ is a neighborhood of μ_0. Assume further that the following conditions are satisfied*:

(i) *$K(\mu_0)$ is nonempty convex*;
(ii) *K is H-u.s.c. at μ_0 and l.s.c. at μ_0*;
(iii) *for any $y \in K(\mu_0)$, $F(\cdot, y, \mu_0)$ is C-concave on $K(\mu_0)$*;
(iv) *$F(\cdot, \cdot, \cdot)$ is uniformly continuous on $K(M) \times K(M) \times N(\mu_0)$.*

Then the ξ-approximate solution mapping $\widetilde{S}_\xi : [\varepsilon_0, \delta) \times N(\mu_0) \to 2^X$ is l.s.c. at (ε_0, μ_0).

Proof Suppose to the contrary that $\widetilde{S}_\xi(\cdot, \cdot)$ is not l.s.c. at (ε_0, μ_0), then there exist $x_0 \in \widetilde{S}_\xi(\varepsilon_0, \mu_0)$ and a neighborhood W_0 of $0_X \in X$. For any neighborhoods $J(\varepsilon_0)$ and $U(\mu_0)$ of ε_0 and μ_0, respectively, there exist $\varepsilon' \in J(\varepsilon_0) \cap [\varepsilon_0, \delta)$ and $\mu' \in U(\mu_0)$ such that $(x_0 + W_0) \cap \widetilde{S}_\xi(\varepsilon', \mu') = \emptyset$. In particular, there exist sequences $\{\varepsilon_n\} \downarrow \varepsilon_0$ and $\{\mu_n\} \to \mu_0$ such that

$$(x_0 + W_0) \cap \widetilde{S}_\xi(\varepsilon_n, \mu_n) = \emptyset, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

For the above W_0, there exists a neighborhood W_1 of $0_X \in X$ such that

$$W_1 + W_1 \subseteq W_0. \tag{3.2}$$

We define a ξ-set-valued mapping $H_\xi : [0, \delta) \to 2^X$ by

$$H_\xi(\varepsilon) = \Bigl\{x \in K(\mu_0) : \inf_{z \in F(x,y,\mu_0)} \xi(z) + \varepsilon + \varepsilon_0 \geq 0, \forall y \in K(\mu_0)\Bigr\}, \quad \varepsilon \in [0, \delta).$$

Notice that $H_\xi(0) = \widetilde{S}_\xi(\varepsilon_0, \mu_0) \neq \emptyset$. Next, we claim that H_ξ is l.s.c. at 0. Suppose to the contrary that H_ξ is not l.s.c. at 0, then there exist $\bar{x} \in H_\xi(0)$ and a neighborhood O_0 of

$0_X \in X$. For any neighborhood U of 0, there exists $\varepsilon \in U$ such that $(\bar{x} + O_0) \cap H_\xi(\varepsilon) = \emptyset$. In particular, there exists a nonnegative sequence $\{\varepsilon'_n\} \downarrow 0$ such that

$$(\bar{x} + O_0) \cap H_\xi(\varepsilon'_n) = \emptyset, \quad \forall n \in \mathbb{N}. \tag{3.3}$$

Since $H_\xi(0) \neq \emptyset$, we choose $x^* \in H_\xi(0)$. Since $\varepsilon'_n \to 0$, there exists ε'_{n_0} such that

$$\frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}x^* = \bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}(x^* - \bar{x}) \in \bar{x} + O_0. \tag{3.4}$$

We claim that $\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^* \in H_\xi(\varepsilon'_{n_0})$. In fact, since $\bar{x} \in H_\xi(0)$ and $x^* \in H_\xi(0)$, for any $y \in K(\mu_0)$, we have $\inf_{t\in F(\bar{x},y,\mu_0)} \xi(t) + \varepsilon_0 \geq 0$ and $\inf_{k\in F(x^*,y,\mu_0)} \xi(k) + \varepsilon_0 \geq 0$. Then, for any $u \in F(\bar{x}, y, \mu_0)$,

$$\frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}\xi(u) + \frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}\varepsilon_0 \geq 0, \tag{3.5}$$

and for any $v \in F(x^*, y, \mu_0)$,

$$\frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}\xi(v) + \frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}\varepsilon_0 \geq 0. \tag{3.6}$$

By the C-concavity of $F(\cdot, y, \mu_0)$, we have that

$$F\left(\frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}x^*, y, \mu_0\right) \subseteq \frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}F(\bar{x}, y, \mu_0) + \frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}F(x^*, y, \mu_0) + C.$$

It follows that, for any $w \in F(\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^*, y, \mu_0)$, there exist $\bar{z} \in F(\bar{x}, y, \mu_0)$, $z^* \in F(x^*, y, \mu_0)$ and $c' \in C$ such that $w = \frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{z} + \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}z^* + c'$. It follows from the linearity of ξ that $\xi(w) - \frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\xi(\bar{z}) - \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}\xi(z^*) = \xi(c') \geq 0$, which gives that $\xi(w) \geq \frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\xi(\bar{z}) + \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}\xi(z^*)$. For all $w \in F(\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^*, y, \mu_0)$, by (3.5) and (3.6), we have

$$\xi(w) \geq -\frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}\varepsilon_0 - \frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}\varepsilon_0 = -\frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}(\varepsilon'_{n_0} + \varepsilon_0) \geq -(\varepsilon'_{n_0} + \varepsilon_0).$$

This implies that $\inf_{z\in F(\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x}+\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^*,y,\mu_0)} \xi(z) + \varepsilon'_{n_0} + \varepsilon_0 \geq 0$, that is, $\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^* \in H_\xi(\varepsilon'_{n_0})$. By (3.4), we get that $\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^* \in (\bar{x} + O_0) \cap H_\xi(\varepsilon'_{n_0})$, which contradicts (3.3). Therefore, H_ξ is l.s.c. at 0. Since H_ξ is l.s.c. at 0, for above $x_0 \in \widetilde{S}_\xi(\varepsilon_0, \mu_0) = H_\xi(0)$ and for above W_1, there exists a balanced neighborhood V_0 of 0 such that $(x_0 + W_1) \cap H_\xi(\varepsilon) \neq \emptyset$, $\forall \varepsilon \in V_0$. In particular, from $\{\varepsilon_n\} \downarrow \varepsilon_0$, there exits $N_0 \in \mathbb{N}$ such that $(x_0 + W_1) \cap H_\xi(\varepsilon_{N_0} - \varepsilon_0) \neq \emptyset$. Let $x' \in (x_0 + W_1) \cap H_\xi(\varepsilon_{N_0} - \varepsilon_0)$.

For any $\bar{\varepsilon} > 0$, since $e \in \operatorname{int} C$, there exists $\delta_0 > 0$ such that

$$\delta_0 B_Y + \bar{\varepsilon} e \subseteq C. \tag{3.7}$$

Since $F(\cdot,\cdot,\cdot)$ is uniformly continuous on $K(M) \times K(M) \times N(\mu_0)$, for above $\delta_0 B_Y$, there exists a neighborhood V_1 of $0 \in B$, a neighborhood U_1 of $0 \in B$ and a neighborhood N_1 of $0 \in M$, for any $(x_1, y_1, \mu_1), (x_2, y_2, \mu_2) \in K(M) \times K(M) \times N(\mu_0)$ with $x_1 - x_2 \in V_1$, $y_1 - y_2 \in U_1$ and $\mu_1 - \mu_2 \in N_1$, we have

$$F(x_1, y_1, \mu_1) \subseteq \delta_0 B_Y + F(x_2, y_2, \mu_2). \tag{3.8}$$

Since K is H-u.s.c. at μ_0, for above U_1, there exists a neighborhood $U_1(\mu_0)$ of μ_0 such that

$$K(\mu) \subseteq K(\mu_0) + U_1, \quad \forall \mu \in U_1(\mu_0). \tag{3.9}$$

We see that $x' \in K(\mu_0)$. Since K is l.s.c. at μ_0, for $V_1 \cap W_1$, there exists a neighborhood $U_2(\mu_0)$ of μ_0 such that

$$\bigl(x' + V_1 \cap W_1\bigr) \cap K(\mu) \neq \emptyset, \quad \forall \mu \in U_2(\mu_0). \tag{3.10}$$

It follows from $\mu_n \to \mu_0$ that there exists a positive integer $N_0' \geq N_0$ such that $\mu_{N_0'} \in U_1(\mu_0) \cap U_2(\mu_0) \cap U(\mu_0) \cap (\mu_0 + N_1)$. Noting that (3.9) and (3.10), we obtain

$$K(\mu_{N_0'}) \subseteq K(\mu_0) + U_1 \tag{3.11}$$

and

$$\bigl(x' + V_1 \cap W_1\bigr) \cap K(\mu_{N_0'}) \neq \emptyset. \tag{3.12}$$

By (3.12), we choose

$$x'' \in \bigl(x' + V_1 \cap W_1\bigr) \cap K(\mu_{N_0'}). \tag{3.13}$$

Next, we prove that $x'' \in \widetilde{S}_\xi(\varepsilon_{N_0'}, \mu_{N_0'})$. For any $y' \in K(\mu_{N_0'})$, by (3.11), there exists $y_0 \in K(\mu_0)$ such that $y' - y_0 \in U_1$. It follows from (3.13) that $x'' - x' \in V_1$. Noting that $\mu_{N_0'} \in U(\mu_0) \cap (\mu_0 + N_1)$ and (3.8), we have

$$F\bigl(x'', y', \mu_{N_0'}\bigr) \subseteq \delta_0 B_Y + F\bigl(x', y_0, \mu_0\bigr).$$

By (3.7), we have

$$F\bigl(x'', y', \mu_{N_0'}\bigr) \subseteq C - \bar{\varepsilon} e + F\bigl(x', y_0, \mu_0\bigr). \tag{3.14}$$

Hence, for any $y \in K(\mu_{N_0'})$ and $z'' \in F(x'', y', \mu_{N_0'})$, there exist $c'' \in C$ and $z' \in F(x', y, \mu_0)$ such that

$$z'' = c'' - \bar{\varepsilon} e + z'.$$

It follows from the linearity of ξ that $\xi(z'') + \bar{\varepsilon} \geq \xi(z')$ for all $\bar{\varepsilon} > 0$. This leads to $\xi(z'') \geq \xi(z')$. Thus

$$\xi\bigl(z''\bigr) + \varepsilon_{N_0'} \geq \xi\bigl(z'\bigr) + \varepsilon_{N_0'} = \xi\bigl(z'\bigr) + (\varepsilon_{N_0'} - \varepsilon_0) + \varepsilon_0 \geq 0.$$

Hence $x'' \in \widetilde{S}_\xi(\varepsilon_{N'_0}, \mu_{N'_0})$. Also, since $x' \in (x_0 + W_1)$ and by (3.2) and (3.13), we have

$$x'' \in x' + V_1 \cap W_1 \subseteq x_0 + W_1 + W_1 \subseteq x_0 + W_0.$$

This means that $(x_0 + W_0) \cap \widetilde{S}_\xi(\varepsilon_{N'_0}, \mu_{N'_0}) \neq \emptyset$, which contradicts (3.1). This completes the proof. □

Theorem 3.3 *We assume that for any given* $\xi \in B_e^*$, *there exists* $\delta > 0$ *such that the approximate solution set* $\widetilde{S}_\xi(\cdot,\cdot)$ *exists in* $[\varepsilon_0, \delta) \times N(\mu_0)$. *Suppose that conditions* (i)-(iv) *as in Theorem* 3.2 *are satisfied. Assume further that for each* $x \in K(\mu_0)$, $F(x, K(\mu_0), \mu_0) + C$ *is a convex set. Then the approximate solution mapping* $\widetilde{S} : [\varepsilon_0, \delta) \times N(\mu_0) \to 2^X$ *is l.s.c. at* (ε_0, μ_0).

Proof Since $F(x, K(\mu_0), \mu_0) + C$ is a convex set for each $x \in K(\mu_0)$, by virtue of Lemma 3.1, it holds that $\widetilde{S}(\varepsilon_0, \mu_0) = \bigcup_{\xi \in B_e^*} \widetilde{S}_\xi(\varepsilon_0, \mu_0)$. It follows from Theorem 3.2 that for each $\xi \in B_e^*$, $\widetilde{S}_\xi(\cdot,\cdot)$ is l.s.c. at (ε_0, μ_0). Thus, in view of Lemma 2.4, we obtain that $\widetilde{S}(\cdot,\cdot)$ is l.s.c. at (ε_0, μ_0). □

The following example illustrates all of the assumptions in Theorem 3.3.

Example 3.4 Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ and $Z = X = \mathbb{R}$. Let $B(0, \frac{1}{2})$ be the closed ball of radius 1/2 in $\mathbb{R}^2$. Let $B = [-2, 2]$, $M = [-1, 1]$ and the set-valued mapping $F : B \times B \times M \to 2^Y$ be defined by

$$F(x, y, \mu) = \bigl(w(x, y, \mu), v(x, y, \mu)\bigr) + B(0, 1/2),$$

where $w(x, y, \mu) := y^2(2^\mu - 1) + x(y - x + 1) - 3y + 2$ and $v(x, y, \mu) := y^2(2^\mu - 1) - x^2 + 2xy + 3$. Define a set-valued mapping $K : M \to 2^X$ for all $\mu \in M$, by $K(\mu) := [-2 + \mu, 2 + \mu] \cap [-2, 2]$. We choose $e = (1, 1) \in \operatorname{int} C$, $\varepsilon_0 = 2.5$, $\mu_0 = 0$ and $\xi = (1, 0)$. We can see that $B^*_{(1,1)} = \{(x_1, x_2) : x_1 + x_2 = 1, x_1, x_2 \geq 0\}$ and $1 \in \widetilde{S}_{(1,0)}(\varepsilon_0, 0)$. Further, for any $\mu \in (-1, 1)$, there exists $\varepsilon \in [2.5, 4.5)$ such that $1 \in \widetilde{S}_{(1,0)}(\varepsilon, \mu)$. Hence, $\widetilde{S}_{(1,0)}(\cdot,\cdot)$ exists in $[2.5, 4.5) \times [-1, 1]$. It is easy to observe that for any $y \in K(0)$, $F(\cdot, y, 0)$ is C-concave on $K(0)$. Clearly, condition (ii) is true. It is obvious that $K(M) = [-2, 2]$. Let $N(\mu_0) = [-1, 1]$, we can see that $F(\cdot,\cdot,\cdot)$ is uniformly continuous on $K(M) \times K(M) \times N(\mu_0)$. Finally, we can check that for each $x \in [-2, 2]$, $F(x, [-2, 2], 0) + C$ is a convex set. Applying Theorem 3.3, we obtain that $\widetilde{S}$ is l.s.c. at $(2.5, 0)$.

The following example illustrates that the concavity of F cannot be dropped.

Example 3.5 Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and $Z = X = \mathbb{R}$. Let $B = [-2, 2]$, $M = [-1, 1]$ and the set-valued mapping $F : B \times B \times M \to 2^Y$ be defined by

$$F(x, y, \mu) = \bigl[\mu x(x - y) - 0.5, 2\bigr] \times \bigl\{x(x - y) - 0.5\bigr\}.$$

Define a set-valued mapping $K : M \to 2^X$ for all $\mu \in M$, by $K(\mu) := [0, 1]$. We choose $e = (1, 1) \in \operatorname{int} C$, $\varepsilon_0 = 0.5$, $\mu_0 = 0$. Then, all the assumptions of Theorem 3.3 are satisfied except (iii). Indeed, taking $y = 1$, $x_1 = 0$, $x_2 = 1$ and $t = 0.5$, we have

$$\begin{aligned}(-2.5,-0.25) &= (-0.5,-0.75) - 0.5(2,-0.5) - 0.5(2,-0.5)\\ &\in [-0.5,2]\times\{-0.75\} - 0.5\big([-0.5,2]\times\{-0.5\}\big)\\ &\quad - 0.5\big([-0.5,2]\times\{-0.5\}\big)\\ &\in F\big(0.5(0)+0.5(1),1,0\big) - 0.5F(0,1,0) - 0.5F(1,1,0)\\ &= F(0.5,1,0) - 0.5F(0,1,0) - 0.5F(1,1,0),\end{aligned}$$

but $(-2.5,-0.25) \notin C$. The direct computation shows that

$$\tilde{S}(\varepsilon_0,\mu) = \begin{cases}\{0,1\} & \text{if } \mu \in (0,1],\\ [0,1] & \text{if } \mu = 0,\\ \{0\} & \text{if } \mu \in [-1,0).\end{cases} \tag{3.15}$$

Clearly, we see that $\tilde{S}(\cdot,\cdot)$ is even not l.s.c. at (ε_0,μ_0) since $F(\cdot,y,\mu_0)$ is not C-concave on $K(\mu_0)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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WELL-POSEDNESS BY PERTURBATIONS FOR THE HEMIVARIATIONAL INEQUALITY GOVERNED BY A MULTI-VALUED MAP PERTURBED WITH A NONLINEAR TERM

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Abstract: In this paper, we introduce the notion of well-posedness by perturbations to the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term (HVIMN) in Banach spaces. Under very suitable conditions, we establish some metric characterizations for the well-posed (HVIMN). In the setting of finite-dimensional spaces, the strongly generalized well-posedness by perturbations for (HVIMN) are established. Our results are new and improve recent existing ones in the literature.

Key words: *well-posedness by perturbations, hemivariational inequality, multi-valued map, Banace space*

Mathematics Subject Classification: *49J40, 49K40, 90C31*

1 Introduction

It is well known that the well-posedness is very important for both optimization theory and numerical methods of optimization problems, which guarantees that, for approximating solution sequences, there is a subsequence which converges to a solution. The study of well-posedness originates from Tikhonov [36], which means the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Levitin-Polyak [21] introduced a new notion of well-posedness that strengthened Tykhonov's concept as it required the convergence to the optimal solution of each sequence belonging to a larger set of minimizing sequences.

Another important notion of well-posedness for a minimization problem is the well-posedness by perturbations or extended well-posedness due to Zolezzi [41, 42]. The notion of well-posedness by perturbations establishes a form of continuous dependence of the solutions upon a parameter. There are many other notions of well-posedness in optimization problems. For more details, see, e.g., [41, 42, 2, 6, 11, 15, 18, 27, 32, 37, 39]. Meanwhile, the concept of well-posedness has been generalized to other varia- tional problems such as variational inequalities [5, 10, 12, 13, 24, 25, 26, 27], saddle point problems [3], Nash equilibrium problems [26, 28, 29, 30, 31, 33], equilibrium problems [14], inclusion problems [22, 23] ,and fixed point problems [22, 23, 40]

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Lucchetti and Patrone [27] introduced the notion of well-posedness for variational inequalities and proved some related results by means of Ekeland ' s variational principle. From then on, many papers have been devoted to the extensions of well-posedness of minimization problems to various variational inequalities. Lignola and Morgan [25] generalized the notion of well-posedness by perturbations to a variational inequality and established the equivalence between the well-posedness by perturbations of a variational inequality and the well-posedness by perturbations of the corresponding minimization problem. Lignola and Morgan [26] investigated the concepts of α-well-posedness for variational inequalities. Del Prete et al. [10] further proved that the α-well-posedness of variational inequalities is closely related to the well-posedness of minimization problems. Recently, Fang et al. [16] generalized the notions of well-posedness and α-well-posedness to a mixed variational inequality. In the setting of Hilbert spaces, Fang et al. [16] proved that under suitable conditions the well-posedness of a mixed variational inequality is equivalent to the existence and uniqueness of its solution. They also showed that the well-posedness of a mixed variational inequality has close links with the well-posedness of the corresponding inclusion problem and corresponding fixed point problem in the setting of Hilbert spaces. Very recently, Fang et al. [15] generalized the notion of well-posedness by perturbations to a mixed variational inequality in Banach spaces. In the setting of Banach spaces, they established some metric characterizations, and showed that the well-posedness by perturbations of a mixed variational inequality is closely related to the well-posedness by perturbations of the corresponding inclusion problem and corresponding fixed point problem. They also derived some conditions under which the well-posedness by perturbations of the mixed variational inequality is equivalent to the existence and uniqueness of its solution.

On the other hand, the notion of hemivariational inequality was introduced by Panagiotopoulos [34, 35] at the beginning of the 1980s as a variational formulation for several classes of mechanical problems with nonsmooth and nonconvex energy super-potentials. In the case of convex super-potentials, hemivariational inequalities reduce to variational inequalities which were studied earlier by many authors (see e.g. Fichera [17] or Hartman and Stampacchia [19]). Wangkeeree and Preechasilp [38] also introduced and studied some existence results for the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term in reflexive Banach spaces. Recently Ceng et al. [4] considered an extension of the notion of well-posedness by perturbations, introduced by Zolezzi for a minimization problem, to a class of variational-hemivariational inequalities with perturbations in Banach spaces. Under very mild conditions, they established some metric characterizations for the well-posed variational-hemivariational inequality, and proved that the well-posedness by perturbations of a variational hemivariational inequality is closely related to the well-posedness by perturbations of the corresponding inclusion problem. Furthermore, in the setting of finite-dimensional spaces they also derived some conditions under which the variational-hemivariational inequality is strongly generalized well-posed-like by perturbations.

The aim of this paper is to introduce the new notion of well-posedness by perturbations to the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term (HVIMN) in Banach spaces. Under very suitable conditions, we establish some metric characterizations for the well-posed (HVIMN). In the setting of finite-dimensional spaces, the strongly generalized well-posedness by perturbations for (HVIMN) are established. The example illustrating main results is established. Our results are new and improve recent existing ones in the literature.

2 Preliminaries

Let K be a nonempty, closed and convex subset of a real reflexive Banach space E with its dual E^*, $F : K \rightrightarrows 2^{E^*}$ a multivalued mapping. Let Ω be a bounded open set in $\mathbb{R}^N$, $T : E \to L^q(\Omega; \mathbb{R}^k)$ a linear continuous mapping, where $1 < q < \infty$, $k \geq 1$ and $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ a function. We shall denote $\hat{u} := Tu$, $j^\circ(x, y; h)$ denotes the Clarke's generalized directional derivative of a locally Lipschitz mapping $j(x, \cdot)$ at the point $y \in \mathbb{R}^k$ with respect to direction $h \in \mathbb{R}^k$, where $x \in \Omega$.

For the given bifunction $f : K \times K \to [-\infty, +\infty]$ imposed the condition that the set $\mathcal{D}_1(f) = \{u \in K : \ f(u, v) \neq -\infty, \ \forall v \in K\}$ is nonempty, Wangkeeree and Preechasilp [38] introduced and studied the existence of a solution for the following hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term

$$\text{(HVIMN)} \begin{cases} \text{Find } u \in \mathcal{D}_1(f) \text{ and } u^* \in F(u) \text{ such that} \\ \langle u^*, v - u \rangle + f(u, v) + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \\ \forall v \in K. \end{cases} \tag{2.1}$$

Now, let us consider some special cases of the problem (2.1). If $f(u, v) = \phi(v) - \phi(u)$, where $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function such that $K_\phi = K \cap \mathrm{dom}\phi \neq \emptyset$, then $\mathcal{D}_1(f) = K_\phi$ and (2.1) is reduced to the following *variational-hemivariational inequality problem*: Find $u \in K_\phi$ such that

$$\langle u^*, v - u \rangle + \phi(v) - \phi(u) + \int_\Omega j(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \quad \forall v \in K. \tag{2.2}$$

The problem (2.2) was studied by Costea and Lupu [8] by assuming that F is monotone and lower hemicontinuous and several existence results were obtained. Furthermore, if $F \equiv 0$ and $f(u, v) = \Lambda(u, v) - \langle g^*, v - u \rangle$, where $\Lambda : K \times K \to \mathbb{R}$ and $g^* \in X^*$, then (2.1) reduces to the problem: Find $u \in K$ such that

$$\Lambda(u, v) + \int_\Omega j(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq \langle g^*, v - u \rangle, \quad \forall v \in K. \tag{2.3}$$

The problem (2.3) was studied by Costea and Radulescu [9] and it was called nonlinear hemivariational inequality (see also Andrei and Costea [1] for some applications of nonlinear hemivariational inequalities to Nonsmooth Mechanics).

Now, suppose that L is a parametric normed space, $P \subset L$ is a closed ball with positive radius $p^* \in P$ is a fixed point. Let $\tilde{F} : P \times K \to 2^{E^*}$ be multivalued mapping. Let $\tilde{T} : P \times E \to L^p(\Omega; \mathbb{R}^k)$ be a linear continuous mapping, where $1 < p < \infty, k \geq 1$ and $\tilde{j} : P \times \Omega \times \mathbb{R}^k \to \mathbb{R}$ a function. We denote $\tilde{j}_p^\circ(x, y; h)$ denotes the Clarke's generalized directional derivative of a locally Lipschitz mapping $\tilde{j}(p, x, \cdot)$ at the point $y \in \mathbb{R}^k$ with respect to direction $h \in \mathbb{R}^k$. For the given bifunction $\tilde{f} : P \times K \times K \to [-\infty, +\infty]$, we assume the condition

$$\tilde{\mathcal{D}_1}(\tilde{f}) = \{u \in K | \tilde{f}(p^*, u, v) \neq -\infty, \forall v \in K\} \neq \emptyset.$$

The perturbed problem of the HVIMN (2.1) is given by

$$(\text{HVIMN}_{p^*}) \begin{cases} \text{Find } u \in \tilde{\mathcal{D}_1}(\tilde{f}) \text{ and } u^* \in \tilde{F}(p^*, u) \text{ such that} \\ \langle u^*, v - u \rangle + \tilde{f}(p^*, u, v) + \int_\Omega \tilde{j}_{p*}^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \\ \forall v \in K. \end{cases} \tag{2.4}$$

Let $\bar{\partial} j : E \to 2^{E^*} \setminus \{0\}$ denote the Clarke ' s generalized gradient of locally Lipschitz functional j (see [7]). That is

$$\bar{\partial} j(x) = \{\xi \in E^* : \langle \xi, v\rangle \leq j^0(x,y), \forall y \in E\}.$$

The following useful results can be found in [7].

Proposition 2.1. *Let X be a Banach space, $x, y \in X$ and J be a locally Lipschitz functional defined on X. Then*

(i) *The function $y \mapsto j^\circ(x,y)$ is finite, positively homogeneous, subadditive and then convex on X;*

(ii) *$j^\circ(x,y)$ is upper semicontinuous as a function of (x,y), as a function of y alone, is Lipschitz continuous on X;*

(iii) *$j^\circ(x,-y) = (-j)^\circ(x,y)$;*

(iv) *$\bar{\partial} j(x)$ is a nonempty, convex, bounded, weak*-compact subset of X^*;*

(v) *For every $y \in X$, one has*

$$j^\circ(x,y) = \max\{\langle \xi, y\rangle : \xi \in \bar{\partial} j(x)\}.$$

Definition 2.2. The set-valued map F is said to be

(i) upper semicontinuous (usc) at $x \in \text{dom } F$ if for any open set U satisfying $F(x) \subset U$, there exists a $\delta > 0$ such that $F(y) \subset U$, for every $y \in B(x,\delta)$;

(ii) lower semicontinuous (lsc) at $x \in \text{dom } F$ if for any open set U satisfying $F(x) \cap U \neq \emptyset$, there exists a $\delta > 0$ such that $F(y) \cap U \neq \emptyset$, for every $y \in B(x,\delta)$;

(iii) closed at $x \in \text{dom } F$ if for each sequence $\{x_n\}$ in X converging to x and $\{y_n\}$ in Y converging to y such that $y_n \in F(x_n)$, we have $y \in F(x)$.

If $S \subseteq X$, then F is said to be usc (lsc, closed respectively) on the set S if F is usc (lsc, closed respectively) at every $x \in \text{dom } F \cap S$.

Remark 2.3. An equivalent formulation of Definition 2.2(ii) is as follows: F is said to be lsc at $x \in \text{dom } F$ if for each sequence $\{x_n\}$ in dom F converging to x and for any $y \in F(x)$, there exists a sequence $\{y_n\}$ in $F(x_n)$ converging to y.

Definition 2.4 (see [20])**.** Let S be a nonempty subset of X. The measure, say μ, of noncompactness for the set S is defined by

$$\mu(S) := \inf\{\varepsilon > 0 : S \subset \cup_{i=1}^n S_i, \ \text{diam}|S_i| < \varepsilon, i = 1, 2, \ldots, n, \ \text{for some integer } n \geq 1\},$$

where diam$|S_i|$ means the diameter of set S_i.

Definition 2.5 (see[20])**.** Let A, B be nonempty subsets of X. The Hausdorff metric $H(\cdot,\cdot)$ between A and B is defined by

$$H(A,B) = \max\{e(A,B), e(B,A)\},$$

where $e(A,B) := \sup_{a\in A} d(a,B)$ with $d(a,B) = \inf_{b\in B} \|a-b\|$.

Let $\{A_n\}$ be a sequence of nonempty subsets of X. We say that A_n converges to A in the sense of Hausdorff metric if $H(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all section $a_n \in A_n$. For more details on this topic, we refer the readers to [20].

$\boxed{3}$ Well-Posedness by Perturbations and Metric Characterizations

In this section, we generalize the concepts of well-posedness by perturbations to the variationalhemivariational inequality and establish their metric characterizations. In the sequel we always denote by $\to$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively. Let $\alpha \geq 0$ be a fixed number.

Definition 3.1. Let $\{p_n\} \subset P$ be such that $p_n \to p^*$. A sequence $\{u_n\} \subset E$ is called an α-approximating sequence corresponding to $\{p_n\}$ for HVIMN (2.1) if there exist a sequence $\{\varepsilon_n\}$ of nonnegative numbers with $\varepsilon_n \to 0$, $u_n^* \in \tilde{F}(p_n, u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$, and

$$\begin{aligned}&\langle u_n^*, v - u_n\rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \\ &\geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon_n, \qquad \forall v \in K.\end{aligned}$$

for each $n \geq 1$. Whenever $\alpha = 0$, we say that $\{u_n\}$ is an approximating sequence corresponding to $\{p_n\}$ for HVIMN (2.1). Clearly, every α_2-approximating sequence corresponding to $\{p_n\}$ is α_1-approximating sequence corresponding to $\{p_n\}$ whenever $\alpha_1 > \alpha_2 \geq 0$.

Definition 3.2. We say that HVIMN (2.1) is strongly (resp., weakly) $\alpha-$well-posed by perturbations if

(i) HVIMN (2.1) has a unique solution

(ii) for any $\{p_n\} \subset P$ with $p_n \to p^*$, every $\alpha-$approximating sequence corresponding to $\{p_n\}$ converges strongly (resp., weakly) to the unique solution.

In the sequel, strong (resp., weak) $0-$well-posedness by perturbations is always called as strong (resp., weak) well-posedness by perturbations. If $\alpha_1 > \alpha_2 \geq 0$, then strong (resp., weak) α_1-well-posedness by perturbations implies strong (resp., weak) α_2-well-posedness by perturbations.

Definition 3.3. We say that HVIMN (2.1) is strongly (resp., weakly) generalized $\alpha-$well-posed by perturbations if

(i) HVIMN (2.1) has a nonempty solution set S

(ii) for any $\{p_n\} \subset P$ with $p_n \to p^*$, every $\alpha-$approximating sequence corresponding to $\{p_n\}$ has some subsequence which converges strongly (resp., weakly) to some point of S

In the sequel, strong (resp., weak) generalized $0-$well-posedness by perturbations is always called as strong (resp., weak) generalized well-posedness by perturbations.

If $\alpha_1 > \alpha_2 \geq 0$, then strong (resp., weak) generalized α_1-well-posedness by perturbations implies strong (resp., weak) generalized α_2-well-posedness by perturbations.

To derive the metric characterizations of α-well-posedness by perturbations, we consider the following approximating solution set of HVIMN (2.1):

$$\begin{aligned}\Omega_\alpha(\varepsilon) \quad = \quad &\bigcup_{p\in B(p^*,\varepsilon)} \{u \in \tilde{D}_1(\tilde{f}), u^* \in \tilde{F}(p, u) : \langle u^*, v - u\rangle + \tilde{f}(p, u, v) \\ &+ \int_\Omega \tilde{j}^\circ_p(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq -\frac{\alpha}{2}\|v - u\|^2 - \varepsilon, \forall v \in K.\}\end{aligned}$$

when $B(p^*, \varepsilon)$ denotes the closed ball centered at p^* with radius ε. In this section, we assume that $\bar{u}$ is a fixed solution of HVIMN (2.1). Define

$$\theta(\varepsilon) = \sup\{\|u - \bar{u}\| : u \in \Omega_\alpha(\varepsilon)\}, \ \ \forall \varepsilon \geq 0.$$

It is easy to see that $\theta(\varepsilon)$ is the radius of the smallest closed ball centered at $\bar{u}$ containing $\Omega_\alpha(\varepsilon)$. Now, we give a metric characterization of strong α-well-posedness by perturbations by considering the behavior of $\theta(\varepsilon)$ when $\varepsilon \to 0$.

Theorem 3.4. *HVIMN (2.1) is strongly $\alpha-$well-posed by perturbations if and only if $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Proof. Assume that HVIMN (2.1) is strongly $\alpha-$well-posed by perturbations. Then $\bar{u} \in E$ is the unique solution of HVIMN (2.1). Suppose to the contrary that $\theta(\varepsilon) \not\to 0$ as $\varepsilon \to 0$. There exist $\delta > 0$ and $0 < \varepsilon_n \to 0$ such that

$$\theta(\varepsilon_n) > \delta > 0.$$

By the definition of θ, there exists $u_n \in \Omega_\alpha(\varepsilon_n)$ such that

$$\|u_n - \bar{u}\| > \delta. \tag{3.1}$$

Since $u_n \in \Omega_\alpha(\varepsilon_n)$, there exist $p_n \in B(p^*, \varepsilon_n), u_n^* \in \tilde{F}(p_n, u_n)$ such that

$$\langle u_n^*, v - u_n\rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon,$$

for all $v \in K$ and $n \geq 1$. Since $p_n \in B(p^*, \varepsilon_n)$, we have $p_n \to p^*$. Then $\{u_n\}$ is an α approximating sequence corresponding to $\{p_n\}$ for HVIMN (2.1). Since HVIMN (2.1) is strongly $\alpha-$well-posed by perturbations, we can get that $\|u_n - \bar{u}\| \to 0$, which leads to a contradiction with (3.1).

Conversely, suppose that $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then $\bar{u} \in E$ is the unique solution of HVIMN (2.1). Indeed, if $\hat{u}$ is another solution of HVIMN (2.1) with $\hat{u} \neq \bar{u}$, then by definition,

$$\theta(\varepsilon) \geq \|\bar{u} - \hat{u}\| > 0, \ \ \forall \varepsilon \geq 0,$$

a contradiction. Let $p_n \in P$ be such that $p_n \to p^*$ and let $\{u_n\}$ be an $\alpha-$approximating sequence corresponding to $\{p_n\}$ for HVIMN (2.1). Then there exist $0 < \varepsilon_n \to 0, u_n^* \in \tilde{F}(p_n, u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$ and

$$\langle u_n^*, v - u_n\rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon_n,$$

for all $v \in K$ and $n \geq 1$. Take $\delta_n = \|p_n - p^*\|$ and $\varepsilon'_n = \max\{\delta_n, \varepsilon_n\}$. It is easy to verify that $u_n \in \Omega_\alpha(\varepsilon'_n)$ with $\varepsilon'_n \to 0$. Put

$$t_n = \|u_n - \bar{u}\|,$$

by definition of θ, we can get that

$$\theta(\varepsilon'_n) \geq t_n = \|u_n - \bar{u}\|.$$

Since $\theta(\varepsilon'_n) \to 0$, we have $\|u_n - \bar{u}\| \to 0$ as $n \to \infty$. So, HVIMN (2.1) is strongly $\alpha-$well-posed by perturbations. $\square$

Now, we give an example to illustrate Theorem 3.4.

Example 3.5. Let $E = \mathbb{R}, P = [-1, 1], K = \mathbb{R}, p^* = 0, \alpha = 2, \tilde{F}(p, u) = \{2u\}, \tilde{j} = 0, \tilde{f}(p, u, v) = (1 - \frac{(p^2+1)^2}{4})u^2$ for all $p \in P, u, v \in K$. Clearly $u = 0$ is a solution of HVIMN (2.1). For any $\varepsilon > 0$, it follows that

$$\begin{aligned}
\Omega^p_\alpha(\varepsilon) &= \Big\{u \in \tilde{D}_1(\tilde{f}), u^* \in \tilde{F}(p) : \langle u^*, v - u\rangle + u^2 - \frac{(p^2+1)^2}{4}u^2 \geq -(v-u)^2 - \varepsilon,\ \forall v \in K\Big\} \\
&= \Big\{u \in \mathbb{R} : 2u(v-u) + u^2 - \frac{(p^2+1)^2}{4}u^2 \geq -(v-u)^2 - \varepsilon, \quad \forall v \in \mathbb{R}\Big\} \\
&= \Big\{u \in \mathbb{R} : -u^2 + 2uv - \frac{(p^2+1)^2}{3}u^2 \geq -(v-u)^2 - \varepsilon, \quad \forall v \in \mathbb{R}\Big\} \\
&= \Big\{u \in \mathbb{R} : v^2 - (v-u)^2 - \frac{(p^2+1)^2}{4}u^2 \geq -(v-u)^2 - \varepsilon, \quad \forall v \in \mathbb{R}\Big\} \\
&= \Big\{u \in \mathbb{R} : -v^2 + \frac{(p^2+1)^2}{4}u^2 \leq +\varepsilon, \quad \forall v \in \mathbb{R}\Big\} \\
&= \Big[-\frac{2\sqrt{\varepsilon}}{p^2+1}, \frac{2\sqrt{\varepsilon}}{p^2+1}\Big].
\end{aligned}$$

Therefore,

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(0,\varepsilon)} \Omega^p_\alpha(\varepsilon) = \Big[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}\Big],$$

for sufficiently small $\varepsilon > 0$. By trivial computation, we have

$$\theta(\varepsilon) = \sup\{u - u^* : u \in \Omega_\alpha(\varepsilon)\} = 2\sqrt{\varepsilon} \to 0 \text{ as } \varepsilon \to 0.$$

By Theorem 3.4, HVIMN (2.1) is 2-well-posed by perturbations

To derive a characterization of strong generalized $\alpha-$well-posedness by perturbations, we need another function q which is defined by

$$q(\varepsilon) = e(\Omega_\alpha(\varepsilon), S), \ \ \forall \varepsilon \geq 0,$$

where S is the solution set of HVIMN (2.1) and e is defined as in definition 2.5.

Theorem 3.6. *HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations if and only if S is nonempty compact and $q(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Proof. Assume that HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations. Clearly, S is nonempty. Let $\{u_n\}$ be any sequence in S and $\{p_n\} \subset P$ be such that $p_n = p^*$. Then $\{u_n\}$ is an α-approximating sequence corresponding to $\{p_n\}$ for HVIMN (2.1). Since HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations, we have $\{u_n\}$ has a subsequence which converges strongly to some point of S. Thus S is compact. Next, we suppose that $q(\varepsilon) \not\to 0$ as $\varepsilon \to 0$, then there exist $l > 0, 0 < \varepsilon_n \to 0$ and $u_n \in \Omega_\alpha(\varepsilon_n)$ such that

$$u_n \notin S + B(0, l), \ \ \forall n \geq 1. \tag{3.2}$$

Since $u_n \in \Omega_\alpha(\varepsilon_n)$, there exist $p_n \in B(p^*, \varepsilon), u^*_n \in \tilde{F}(p_n, u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$ and

$$\langle u^*_n, v - u_n\rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon,$$

for all $v \in K$ and $n \geq 1$. Since $p_n \in B(p^*, \varepsilon_n)$, we have $p_n \to p^*$. Then $\{u_n\}$ is an α approximating sequence corresponding to $\{p_n\}$ for HVIMN (2.1). Since HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging strongly to some point of S, which leads to a contradiction with (3.2) and so $q(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Conversely, we assume that S is nonempty compact and $q(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let $\{p_n\} \subset P$ be such that $p_n \to p^*$ and let $\{u_n\}$ be an $\alpha-$approximating sequence corresponding to $\{p_n\}$. Take $\varepsilon'_n = \max\{\varepsilon_n, \|p_n - p^*\|\}$. Thus $\varepsilon'_n \to 0$ and $x_n \in \Omega_\alpha(\varepsilon'_n)$. It follows that

$$d(u_n, S) \geq e(\Omega_\alpha(\varepsilon'_n), S) = q(\varepsilon'_n) \to 0.$$

Since S is compact, there exists $\bar{u}_n \in S$ such that

$$\|u_n - \bar{u}_n\| = d(x_n, S) \to 0.$$

Again from the compactness of S, $\{\bar{u}_n\}$ has a subsequence $\{\bar{u}_{n_k}\}$ which converges to $\bar{u}$. Thus HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations. □

The following example is shown for illustrating the metric characterizations in Theorem 3.6.

Example 3.7. Let $E = \mathbb{R}, P = [-1, 1], K = \mathbb{R}, p^* = 0, \alpha = 2, \tilde{F}(p, u) = \{2u\}, \tilde{j} = 0, \tilde{f}(p, u, v) = (1 - \frac{(p^2+1)^2}{4})u^2$ for all $p \in P, u, v \in K$. It is easy to see that $u = 0$ is a solution of HVIMN (2.1). Repeating the same argument as in Example 3.5, we obtain that

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(0,\varepsilon)} \Omega^p_\alpha(\varepsilon) = \Big[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}\Big],$$

for sufficiently small $\epsilon > 0$. By trivial computation, we have

$$q(\epsilon) = e(\Omega_\alpha(\epsilon), S) = \sup_{u(\epsilon) \in \Omega_\alpha(\epsilon)} d(u(\epsilon), S) \to 0 \text{ as } \epsilon \to 0.$$

By Theorem 3.6, HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations.

The strong generalized α-well-posedness by perturbations can be also characterized by the behavior of the noncompactness measure $\mu(\Omega_\alpha(\epsilon))$.

Theorem 3.8. *Let L be finite-dimensional, $\tilde{j}^\circ_p(x, y)$ be upper semicontinuous as a functional of $(p, x, y) \in P \times E \times E$ and f is convex. Let $\tilde{F}$ is closed on $P \times K$ and $\tilde{f}$ be continuous on $P \times K \times K$. Then HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations if and only if $\Omega_\alpha(\varepsilon) \neq \emptyset, \forall \varepsilon > 0$ and $\mu(\Omega_\alpha(\varepsilon)) \to 0$ as $\varepsilon \to 0$.*

Proof. First, we will prove that $\Omega_\alpha(\varepsilon)$ is closed for all $\varepsilon \geq 0$. Let $\{u_n\} \subset \Omega_\alpha(\varepsilon)$ with $u_n \to \bar{u}$. Then there exist $p_n \in B(p^*, \varepsilon), u^*_n \in \tilde{F}(p_n, u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$ and

$$\langle u^*_n, v - u_n\rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon, \quad (3.3)$$

for all $v \in K$ and $n \geq 1$. Without loss of generality, we may assume that $p_n \to \bar{p} \in B(p^*, \varepsilon)$ because L is finite dimensional. Since $\tilde{j}_p(x, y)$ is upper semicontinuous as a functional of $(p, x, y) \in P \times E \times E$. Hence it follows from (3.3) and the continuity of $\tilde{f}$ that

$$\langle u^*, v - \bar{u}\rangle + \tilde{f}(\bar{p}, \bar{u}, v) + \int_\Omega \tilde{j}^\circ_{\bar{p}}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx$$

$$\begin{aligned}
&\geq \limsup_{n\to\infty}\langle u_n^*, v-u_n\rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \\
&\geq \limsup_{n\to\infty} -\frac{\alpha}{2}\|v-u_n\|^2 - \varepsilon, \\
&= -\frac{\alpha}{2}\|v-\bar{u}\|^2 - \varepsilon \ \ \forall v \in K.
\end{aligned}$$

Thus $\bar{u} \in \Omega_\alpha(\varepsilon)$. Hence $\Omega_\alpha(\varepsilon)$ is closed.

Next, we show that

$$S = \bigcap_{\varepsilon>0} \Omega_\alpha(\varepsilon). \tag{3.4}$$

It is easy to see that $S \subseteq \cap_{\varepsilon>0}\Omega_\alpha(\varepsilon)$. Thus, we show that $\cap_{\varepsilon>0}\Omega_\alpha(\varepsilon) \subseteq S$. Let $\bar{u} \in \cap_{\varepsilon>0}\Omega_\alpha(\varepsilon)$. Let $\{\varepsilon_n\}$ be a sequence of positive real numbers such that $\varepsilon_n \to 0$. Thus

$$\bar{u} \in \Omega_\alpha(\varepsilon_n)$$

and so there exist $p_n \in B(p^*, \varepsilon_n)$ and $u^* \in \tilde{F}(p_n, \bar{u})$ such that $\bar{u} \in \tilde{D}_1(\tilde{f})$ and

$$\langle u^*, v-\bar{u}\rangle + \tilde{f}(p_n, \bar{u}, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \geq -\frac{\alpha}{2}\|v-\bar{u}\|^2 - \varepsilon_n, \tag{3.5}$$

for all $v \in K$ and $n \geq 1$. It is easy to verify that $p_n \to p^*$. Taking limit as $n \to \infty$, we can get that

$$\begin{aligned}
&\langle u^*, v-\bar{u}\rangle + f(\bar{u}, v) + \int_\Omega j^\circ(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \\
&= \langle u^*, v-\bar{u}\rangle + \tilde{f}(p^*, \bar{u}, v) + \int_\Omega \tilde{j}^\circ_{p^*}(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \\
&\geq -\frac{\alpha}{2}\|v-\bar{u}\|^2, \quad \forall v \in K
\end{aligned} \tag{3.6}$$

Since $\tilde{F}$ is closed on $P \times K$, we have $u^* \in F(\bar{u})$ and for any $z \in K$ and $t \in (0,1)$, letting $v = \bar{u} + t(z-\bar{u})$ in (3.6), we can get from T is linear, f is convex and definition of j° that

$$\begin{aligned}
&t\langle u^*, z-\bar{u}\rangle + tf(\bar{u}, z) + \int_\Omega j^\circ(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \\
&\geq t\langle u^*, z-\bar{u}\rangle + f(\bar{u}, \bar{u}+t(z-\bar{u})) + \int_\Omega j^\circ(x, \hat{\bar{u}}(x); \hat{z}(x) - \hat{\bar{u}}(x))dx \\
&\geq -\frac{\alpha t^2}{2}\|z-\bar{u}\|^2.
\end{aligned}$$

This implies that

$$\langle u^*, z-\bar{u}\rangle + tf(\bar{u}, z) + \int_\Omega j^\circ(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \geq -\frac{\alpha t}{2}\|z-\bar{u}\|^2 \ \ \forall z \in K.$$

As $t \to 0$ in the last inequality, we get

$$\langle u^*, z-\bar{u}\rangle + tf(\bar{u}, z) + \int_\Omega j^\circ(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \geq 0 \ \ \forall z \in K.$$

Hence $\bar{u} \in S$ and thus (3.4) is proved. Next, we suppose that HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations. By Theorem 3.6, we can get that S is nonempty compact and $q(\varepsilon) \to 0$. Since $S \subset \Omega_\alpha(\varepsilon)$ for all $\varepsilon > 0$, we have

$$\Omega_\alpha(\varepsilon) \neq \emptyset, \quad \forall \varepsilon > 0.$$

We observe that for each $\varepsilon > 0$,

$$H(\Omega_\alpha(\varepsilon), S) = \max\{e(\Omega_\alpha(\varepsilon), S), e(S, \Omega_\alpha(\varepsilon))\} = e(\Omega_\alpha(\varepsilon), S).$$

By the compactness of S, we have

$$\mu(\Omega_\alpha(\varepsilon)) \leq 2H(\Omega_\alpha(\varepsilon), S) = 2q(\varepsilon) \to 0.$$

Conversely, we suppose that $\Omega_\alpha(\varepsilon) \neq \emptyset, \ \ \forall\varepsilon > 0$ and $\mu(\Omega_\alpha(\varepsilon)) \to 0$ as $\varepsilon \to 0$. Since $\Omega_\alpha(\cdot)$, by the Kuratowski theorem, we can get from (3.4) that

$$q(\varepsilon) = H(\Omega_\alpha(\varepsilon), S) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and S is nonempty compact. Hence HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations by Theorem 3.6. □

The following example is given for illustrating the measure in Theorem 3.8.

Example 3.9. Let $E = \mathbb{R}, P = [-1, 1], K = \mathbb{R}, p^* = 0, \alpha = 2, \tilde{F}(p, u) = \{2u\}, \tilde{j} = 0, \tilde{f}(p, u, v) = (1 - \frac{(p^2+1)^2}{4})u^2$ for all $p \in P, u, v \in K$. It is easy to see that $u = 0$ is a solution of HVIMN (2.1). Repeating the same argument as in Example 3.5, we obtain that

$$\Omega_\alpha(\varepsilon) = \bigcup_{p\in B(0,\varepsilon)} \Omega^p_\alpha(\varepsilon) = \Big[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}\Big].$$

We will show that $\mu(\Omega_\alpha(\epsilon)) = 0$ for each $\epsilon > 0$. Let $\epsilon > 0$. Consider

$$\mu(\Omega_\alpha(\epsilon)) = \inf\{\lambda > 0 : [-2\sqrt{\epsilon}, 2\sqrt{\epsilon}] \subseteq \bigcup_{k=1}^{n}[a_k, b_k], \text{ with } \operatorname{diam}[a_k, b_k] < \lambda, \forall i = 1, \dots, n, \exists n \in \mathbb{N}\}.$$

For every $\lambda > 0$, we can find $n \in \mathbb{N}$ with $a_1 = -2\sqrt{\epsilon}, b_n = 2\sqrt{\epsilon}$ such that

$$[-2\sqrt{\epsilon}, 2\sqrt{\epsilon}] \subseteq \bigcup_{k=1}^{n}[a_k, b_k] \quad \text{and} \quad \operatorname{diam}[a_k, b_k] < \lambda.$$

This implies that $\mu(\Omega_\alpha(\epsilon)) = 0$ for each $\epsilon > 0$. Then HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations.

Remark 3.10. Any solution of HVIMN (2.1) is a solution of the α problem: find $u \in D_1(f)$ and $u^* \in F(u)$ such that

$$\langle u^*, v - u\rangle + f(u, v) + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq -\frac{\alpha}{2}\|y - x\|^2, \ \ \forall v \in K,$$

but the converse is not true in general. To show this, let $K = \mathbb{R}$,

$$F(u) = \{u\}, f(u, v) = 2u^2 - v \text{ and } j = 0,$$

for all $u, v \in K$. It is easy to see that the solution set of HVIMN (2.1) is empty and $u^* = u = 0$ is the unique solution of the corresponding α problem with $\alpha = 2$.

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Semicontinuity and closedness of parametric generalized lexicographic quasiequilibrium problems

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Abstract
This paper is mainly concerned with the upper semicontinuity, closedness, and the lower semicontinuity of the set-valued solution mapping for a parametric lexicographic equilibrium problem where both two constraint maps and the objective bifunction depend on both the decision variable and the parameters. The sufficient conditions for the upper semicontinuity, closedness, and the lower semicontinuity of the solution map are established. Many examples are provided to ensure the essentialness of the imposed assumptions.

Keywords: parametric generalized lexicographic quasiequilibrium problem; upper semicontinuity; closedness and lower semicontinuity

1 Introduction

Equilibrium problems first considered by Blum and Oettli [1] have been playing an important role in optimization theory with many striking applications particularly in transportation, mechanics, economics, *etc.* Equilibrium models incorporate many other important problems such as: optimization problems, variational inequalities, complementarity problems, saddlepoint/minimax problems, and fixed points. Equilibrium problems with scalar and vector objective functions have been widely studied. The crucial issue of solvability (the existence of solutions) has attracted most considerable attention of researchers; see, *e.g.*, [2–5].

With regard to vector equilibrium problems, most of the existing results correspond to the case when the order is induced by a closed convex cone in a vector space. Thus, they cannot be applied to lexicographic cones, which are neither closed nor open. These cones have been extensively investigated in the framework of vector optimization; see, *e.g.*, [6–13]. For instance, Konnov and Ali [12] studied sequential problems, especially exploiting its relation with regularization methods. Bianchi *et al.* in [7] analyzed lexicographic equilibrium problems on a topological Hausdorff vector space, and their relationship with some other vector equilibrium problems. They obtained the existence results for the tangled lexicographic problem via the study of a related sequential problem.

As a unified model of vector optimization problems, vector variational inequality problems, variational inclusion problems and vector complementarity problems, vector equilibrium problems have been intensively studied. The stability analysis of the solution map-

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ping for these problems is an important topic in vector optimization theory. Recently, a great deal of research has been devoted to the semicontinuity of the solution mapping for a parametric vector equilibrium problem. Based on the assumption of the (strong) C-inclusion property of a function, Anh and Khanh [14] obtained the upper and lower semicontinuity of the solution set map of parametric multivalued (strong) vector quasiequilibrium problems. Anh and Khanh [15] obtained the semicontinuity of a class of parametric quasiequilibrium problems by a generalized concavity assumption and a closedness of the level set of functions. Wangkeeree *et al.* [16] established the continuity of the efficient solution mappings to a parametric generalized strong vector equilibrium problem involving a set-valued mapping under the Holder relation assumption. Recently, Wangkeeree *et al.* [17] obtained the sufficient conditions for the lower semicontinuity of an approximate solution mapping for a parametric generalized vector equilibrium problem involving set-valued mappings. By using a scalarization method, they obtained the lower semicontinuity of an approximate solution mapping for such a problem without the assumptions of monotonicity and compactness. For other qualitative stability results on parametric generalized vector equilibrium problems, see [14–20] and the references therein.

It is well known that partial order plays an important role in vector optimization theory. The vector optimization problems in the previous references are studied in the partial order induced by a closed or open cone. But in some situations, the cone is neither open nor closed, such as the lexicographic cone. On the other hand, since the lexicographic order induced by the lexicographic cone is a total order, it can refine the optimal solution points to make it smaller in the theory of vector optimization. Thus, it is valuable to investigate the vector optimization problems in the lexicographic order. To the best of our knowledge, the first lower stability results of the solution set map based on the density of the solution set mapping for a parametric lexicographic vector equilibrium problem have been established by Shi-miao *et al.* [21]. Recently, Anh *et al.* [22] established the sufficient conditions for the upper semicontinuity, closedness, and continuity of the solution maps for a parametric lexicographic equilibrium problem. However, to the best of our knowledge, there is no work to study the stability analysis for a parametric lexicographic equilibrium problem where both two constraint maps and the objective bifunction depend on both the decision variable and the parameters. We observe that quasiequilibrium models are the important general models including as special cases quasivariational inequalities, complementarity problems, vector minimization problems, Nash equilibria, fixed-point and coincidence-point problems, traffic networks, *etc.* A quasioptimization problem is more general than an optimization one as constraint sets depend on the decision variable as well.

Motivated by the mentioned works, this paper is devoted to the study of closedness upper and lower of the solution map for a parametric lexicographic equilibrium problem where both two constraint maps and the objective bifunction depend on both the decision variable and the parameters. The sufficient conditions for the upper semicontinuity, closedness, and the lower semicontinuity of the solution map are established. Many examples are provided to ensure the essentialness of the imposed assumptions.

The paper is organized as follows. In Section 2, we first introduce the parametric lexicographic equilibrium problem where both two constraint maps and the objective bifunction depend on both the decision variable and the parameters, and we recall some basic definitions on semicontinuity of set-valued maps. Section 3 establishes the sufficient conditions for the upper semicontinuity and closedness of the solution map. Many examples are provided to ensure the essentialness of the imposed assumptions. Section 4 establishes the sufficient conditions for the lower semicontinuity of the solution map. Furthermore, we give also many examples ensuring the essentialness of the imposed assumptions.

2 Preliminaries

Throughout this paper, if not otherwise specified, let X and Λ be Hausdorff topological vector spaces. Let $A \subseteq X$ be nonempty. Let $K_1, K_2 : A \times \Lambda \to 2^X$ be two multivalued constraint maps and $f := (f_1, f_2, \dots, f_n) : A \times A \times \Lambda \to \mathbb{R}^n$ a vector-valued function where, for each $i \in I_n := \{1, 2, \dots, n\}$, $f_i : A \times A \times \Lambda \to \mathbb{R}$ is a real valued function. We assume that, for every $x \in X$ and $i \in I_n$, $f_i(x, x, \lambda) = 0$, *i.e.*, f_i is an equilibrium function. Set $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_- = -\mathbb{R}_+$ and $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. For a subset A of X, $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{bd} A$ stand for the interior, closure, and boundary of A, respectively. For any given $\alpha \in \mathbb{R}$, *the upper* α*-level set* and *the lower* α*-level set* of the function $f : X \to \bar{\mathbb{R}}$ are denoted, respectively, by

$$\operatorname{lev}_{\geq \alpha} f := \{x \in X | f(x) \geq \alpha\}$$

and

$$\operatorname{lev}_{\leq \alpha} f := \{x \in X | f(x) \leq \alpha\}.$$

Recall that the lexicographic cone of $\mathbb{R}^n$, denoted by C_L, is defined as

$$C_L := \{0\} \cup \{x \in \mathbb{R}^n | \exists i \in I_n : x_i > 0, \forall j < i, x_j = 0\}.$$

We observe that it is neither closed nor open. Indeed, when comparing with the cone $C_1 := \{x \in \mathbb{R}^n | x_1 \geq 0\}$, we have

$$\operatorname{int} C_1 \subsetneq C_L \subsetneq C_1, \qquad \operatorname{int} C_L = \operatorname{int} C_1 \quad \text{and} \quad \operatorname{cl} C_L = C_1.$$

However, it is worth noticing that the lexicographic cone is convex, pointed, and total ('total' means that $C_L \cup (-C_L) = \mathbb{R}^n$). The lexicographic order, $\geq_L$, in C_L is defined by

$$x \geq_L y \quad \Longleftrightarrow \quad x - y \in C_L.$$

This is a total (called also linear) order, *i.e.*, any pair of elements is comparable. In [23], it was shown that, for a fixed orthogonal base, the lexicographic order is the unique total order. We will see later that this causes difficulties in studies of many topics related to ordering cones.

Next, we shall introduce and study a problem where both the two constraint maps and the bifunction depend on parameters. For a given $\lambda \in \Lambda$, *the parametric generalized lexicographic quasiequilibrium problem*, denoted by GLQEP_λ, is

$$(\mathrm{GLQEP}_\lambda) \begin{cases} \text{finding } \bar{x} \in K_1(\bar{x}, \lambda) \text{ such that, for all } y \in K_2(\bar{x}, \lambda), \\ f(\bar{x}, y, \lambda) \geq_L 0. \end{cases}$$

Remark 2.1 When $K_1 = K_2 := K : \Lambda \to 2^X$, the (GLQEP$_\lambda$) collapses to the lexicographic vector quasiequilibrium problem (LEP$_\lambda$): for each $\lambda \in \Lambda$,

$$(\mathrm{LEP}_\lambda)\begin{cases}\text{finding } \bar{x} \in K(\lambda) \text{ such that}\\ f(\bar{x}, y, \lambda) \geq_L 0, \forall y \in K(\lambda).\end{cases}$$

The stability analysis of the set-valued solution mapping for (LEP$_\lambda$) are studied in Anh *et al.* [6] and Shi-miao *et al.* [21].

Let the set-valued mappings $E : \Lambda \to 2^X$ and $S_{f_1} : \Lambda \to 2^X$ be defined by

$$E(\lambda) = \{x \in A : x \in K_1(x, \lambda)\}$$

and

$$S_{f_1}(\lambda) = \{x \in E(\lambda) : f_1(x, y, \lambda) \geq 0, \forall y \in K_2(x, \lambda)\}.$$

Furthermore, let a mapping $Z : S_{f_1}(\lambda) \times \Lambda \to 2^X$ be given by

$$Z(x, \lambda) := \{y \in K_2(x, \lambda) \mid f_1(x, y, \lambda) = 0\}.$$

For the sake of simplicity, we consider the case $n = 2$, since the general case is similar. Then GLQEP$_\lambda$ collapses to: find $\bar{x} \in K_1(\bar{x}, \lambda)$ such that

$$\begin{cases} f_1(\bar{x}, y, \lambda) \geq 0, & \forall y \in K_2(\bar{x}, \lambda),\\ f_2(\bar{x}, z, \lambda) \geq 0, & \forall z \in Z(\bar{x}, \lambda).\end{cases}$$

Thus, GLQEP$_\lambda$ can be rewritten as

$$\text{find } \bar{x} \in S_{f_1}(\lambda) \text{ such that } f_2(\bar{x}, y, \lambda) \geq 0, \text{ for all } y \in Z(\bar{x}, \lambda). \tag{2.1}$$

The solution mapping for GLQEP$_\lambda$ is denoted by S_f. We denote the whole family of problems, say of GLQEP$_\lambda$, for $\lambda \in \Lambda$, by $(\mathrm{GLQEP}_\lambda)_{\lambda \in \Lambda}$. We first observe some basic facts about lexicographic equilibrium problems. The lexicographic cone C_L contains clearly all pointed closed and convex cones C included in the closed half space $\{x \in \mathbb{R}^n : x_1 \geq 0\}$. Then, for an ordering cone C, we consider some kinds of parametric equilibrium problems: *the parametric generalized quasiequilibrium problem* [23], denoted by GQEP$_\lambda$, is

$$(\mathrm{GQEP}_\lambda)\begin{cases}\text{finding } \bar{x} \in K_1(\bar{x}, \lambda) \text{ such that, for all } y \in K_2(\bar{x}, \lambda),\\ f(\bar{x}, y, \lambda) \in C.\end{cases}$$

The solution mapping for GQEP$_\lambda$ is denoted by S_{GQEP}. Therefore, for any pointed closed and convex cones C included in the closed half space $\{x \in \mathbb{R}^n : x_1 \geq 0\}$, we can get the following fact: $S_{GQEP} \subseteq C_L$. Hence, the existence results of solutions for *GLQEP* can be obtained by the nonemptiness of S_{GQEP}. Next, we need to recall some well-known definitions.

Definition 2.2 [24] Let $\{A_n\}$ be a sequence of subsets of X. Then

(i) the *upper limit* or *outer limit* of the sequence $\{A_n\}$ is a subset of X given by

$$\limsup_{n\to\infty} A_n = \Bigl\{x \in X \Bigm| \liminf_{n\to\infty} \operatorname{dist}(x, A_n) = 0\Bigr\};$$

(ii) the *lower limit* or *inner limit* of the sequence $\{A_n\}$ is a subset of X given by

$$\liminf_{n\to\infty} A_n = \Bigl\{x \in X \Bigm| \lim_{n\to\infty} \operatorname{dist}(x, A_n) = 0\Bigr\};$$

(iii) if $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$, then we say that the *limit of* $\{A_n\}$ exist and

$$\lim_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n.$$

Consequently, we have the following result.

Proposition 2.3 *Let* $\{A_n\}$ *be a sequence of subsets of* X. *Then*

(i) $\limsup_{n\to\infty} A_n = \{x \in A | x_{n_k} \in A_{n_k} : x_{n_k} \to x\}$;

(ii) $\liminf_{n\to\infty} A_n = \{x \in A | x_n \in A_n : x_n \to x\}$.

Definition 2.4 [25] Let X and Y be Hausdorff topological vector spaces and $S : X \to 2^Y$ a given set-valued map.

(i) S is said to be *upper semicontinuous* (usc, for short) at $x_0 \in X$ iff for any open set $V \subset Y$, where $S(x_0) \subset V$, there exists a neighborhood $U \subset X$ of x_0 such that

$$S(x) \subset V, \quad \forall x \in U.$$

The map $S(\cdot)$ is said to be u.s.c. on X if it is u.s.c. at every $x \in X$.

(ii) S is said to be *lower semicontinuous* (lsc, for short) at $x_0 \in X$ iff for any open set $V \subset Y$ such that $S(x_0) \cap V \neq \emptyset$, there exists a neighborhood $U \subset X$ of x_0 such that

$$S(x) \cap V \neq \emptyset, \quad \forall x \in U.$$

The map $S(\cdot)$ is said to be l.s.c. on X if $S(\cdot)$ is l.s.c. at every $x \in X$.

(iii) S is said to be *closed* at x_0 if from (x_n, y_n) in the graph $\operatorname{gr} S := \{(x, y) \in X \times Y \mid y \in S(x)\}$ of S and tends to (x_0, y_0) it follows that $(x_0, y_0) \in \operatorname{gr} S$.

We will often use the well-known fact: if $S(x)$ is compact, then S is usc at x if and only if for any sequence $\{x_n\}$ in X converging to x and $y_n \in Q(x_n)$, there is a subsequence of $\{y_n\}$ converging to a point $y \in Q(x)$. Next we give equivalent forms of the lower semicontinuity of S.

For a set-valued map $Q : X \to 2^Y$ between two linear spaces, Q is called *concave* [15] on a convex subset $A \subseteq X$ if, for each $x_1, x_2 \in A$ and $t \in [0, 1]$,

$$Q\bigl((1-t)x_1 + tx_2\bigr) \subseteq tQ(x_1) + (1-t)Q(x_2).$$

Lemma 2.5 *Let $S : X \to 2^Y$ be a given set-valued map. The following are equivalent*:

(i) *S is lsc at x_0*;

(ii) *if $\{x_n\}$ is any sequence such that $x_n \to x_0$ and $V \subset Y$ an open subset such that $S(x_0) \cap V \neq \emptyset$, then*

$$\exists N \geq 1 : S(x_n) \cap V \neq \emptyset, \quad \forall n \geq N;$$

(iii) *if $\{x_n\}$ is a sequence such that $x_n \to x_0$ and $y_0 \in S(x_0)$ arbitrary, then there is a sequence $\{y_n\}$ with $y_n \in S(x_n)$ such that $y_n \to y_0$ as $n \to \infty$.*

From Proposition 2.3 and Lemma 2.5 we can obtain the following lemma immediately.

Lemma 2.6 *Let $S : X \to 2^Y$ be a given set-valued map. Then S is lsc at x_0 iff for any sequence $\{x_n\} \subseteq X$ converging to x_0,*

$$S(x_0) \subset \liminf_{n\to\infty} S(x_n).$$

The following relaxed continuity properties are also needed and can be found in [26].

Definition 2.7 ([26]) Let X be a topological space and $g : X \to \bar{\mathbb{R}}$ be a function on X.

(i) g is said to be (*sequentially*) *upper pseudocontinuous* at $x_0 \in X$ if for any sequence $\{x_n\}$ in X converging to x_0 and for each $x \in X$ such that $g(x) > g(x_0)$,

$$g(x) > \limsup_{n\to\infty} g(x_n).$$

(ii) g is called (*sequentially*) *lower pseudocontinuous* at $x_0 \in X$ if for any sequence $\{x_n\}$ in X converging to x_0 and for each $x \in X$ such that $g(x) < g(x_0)$,

$$g(x) < \liminf_{n\to\infty} g(x_n).$$

(iii) g is *pseudocontinuous* at $x_0 \in X$ if it is both lower and upper pseudocontinuous at this point.

The class of the pseudocontinuous functions strictly contains that of the semicontinuous functions as shown by the following.

Example 2.8 The function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -2, & \text{if } x < 0, \end{cases}$$

is pseudocontinuous, but neither upper nor lower semicontinuous at 0.

Lemma 2.9 ([16]) *Let X be a topological space. Then $g : X \to \overline{\mathbb{R}}$ is pseudocontinuous in X if and only if, for all sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$*

and $g(y) < g(x)$,

$$\limsup_{n\to\infty} g(y_n) < \liminf_{n\to\infty} g(x_n).$$

The following important definition can be found in [15].

Definition 2.10 Let $g : X \times Z \to \mathbb{R}$ and $\Delta \subset \mathbb{R}$, where $\operatorname{int}\Delta \neq \emptyset$. g is called generalized Δ-concave in a convex set $A \subset Z$, if for each $x \in X$ and $z_1, z_2 \in A$ satisfying $g(x, z_1) \in \Delta$ and $g(x, z_2) \in \operatorname{int}\Delta$, it follows that, for all $t \in (0, 1)$,

$$g\bigl(x, (1-t)z_1 + tz_2\bigr) \in \operatorname{int}\Delta.$$

3 The upper semicontinuouity and closedness of S_f

In this section, we discuss the upper semicontinuity and closedness of the solution mapping S_f. Since there have been a number of contributions to existence issues, focusing on stability we always assume that $S_{f_1}(\lambda)$ and $S_f(\lambda)$ are nonempty for all λ in a neighborhood of the considered point $\bar{\lambda}$. First of all, we shall establish the upper semicontinuity and closedness of the solution mapping S_{f_1}.

Lemma 3.1 *For* $(\mathrm{GLQEP}_\lambda)_{\lambda\in\Lambda}$ *assume that*
- (i) *E is usc at $\bar{\lambda}$ and $E(\bar{\lambda})$ is compact*;
- (ii) *K_2 is lsc in $K_1(A, \Lambda) \times \{\bar{\lambda}\}$*;
- (iii) *$\mathrm{lev}_{\geq 0} f_i(\cdot, \cdot, \bar{\lambda})$ is closed in $K_1(A, \Lambda) \times K_2(A, \Lambda) \times \{\bar{\lambda}\}$ for $i = 1, 2$.*

Then the solution map S_{f_1} is both usc and closed at $\bar{\lambda}$.

Proof We first prove that the solution map S_{f_1} is usc at $\bar{\lambda}$. Suppose on the contrary that there exists an open set $U \supseteq S_{f_1}(\bar{\lambda})$ such that for any neighborhood $N(\bar{\lambda})$ of $\bar{\lambda}$, there exists $\lambda \in N(\bar{\lambda})$ such that $S_{f_1}(\lambda) \nsubseteq U$. In particular, for each $n \in \mathbb{N}$, there exist sequences $\{\lambda_n\} \subseteq \Lambda$ converging to $\bar{\lambda}$ and $\{x_n\} \subseteq S_{f_1}(\lambda_n) \subseteq E(\lambda_n)$ with $x_n \notin U$. By the upper semicontinuity of E and the compactness of $E(\bar{\lambda})$, one can assume that $x_n \to x_0$, for some $x_0 \in E(\bar{\lambda})$. Next, we claim that $x_0 \in S_{f_1}(\bar{\lambda})$. Again suppose on the contrary that there exists $y_0 \in K_2(x_0, \bar{\lambda})$ such that $f_1(x_0, y_0, \bar{\lambda}) < 0$. The lower semicontinuity of K_2 at $(x_0, \bar{\lambda})$, by Lemma 2.5, implies that there exists a sequence $\{y_n\}$ in $K_2(x_n, \lambda_n)$ such that $y_n \to y_0$ as $n \to \infty$. For each $n \in \mathbb{N}$, since $x_n \in S_{f_1}(\lambda_n)$, we have

$$f_1(x_n, y_n, \lambda_n) \geq 0.$$

It follows from the closedness of $\mathrm{lev}_{\geq 0} f_i(\cdot, \cdot, \bar{\lambda})$ that $f_1(x_0, y_0, \bar{\lambda}) \geq 0$, which leads to a contradiction. Therefore, $x_0 \in S_{f_1}(\bar{\lambda}) \subseteq U$, again a contradiction, since $x_n \notin U$ for all n. Thus, S_{f_1} is usc at $\bar{\lambda}$.

Next, we prove that S_{f_1} is closed at $\bar{\lambda}$. We suppose on the contrary that S_{f_1} is not closed at $\bar{\lambda}$, *i.e.*, there a sequence $\{\lambda_n\}$ converging to $\bar{\lambda}$ and $\{x_n\} \subseteq S_{f_1}(\lambda_n)$ with $x_n \to x_0$ but $x_0 \notin S_{f_1}(\bar{\lambda})$. The same argument as above ensures that $x_0 \in S_{f_1}(\bar{\lambda})$, which gives a contradiction. Therefore, we can conclude that S_{f_1} is closed at $\bar{\lambda}$. □

Now, we are in the position to discuss the upper semicontinuity and closedness of the solution mapping S_f.

Theorem 3.2 *For* $(\mathrm{GLQEP}_\lambda)_{\lambda\in\Lambda}$ *assume that*

(i) *E is usc at $\bar{\lambda}$ and $E(\bar{\lambda})$ is compact*;
(ii) *K_2 is lsc in $K_1(A,\Lambda)\times\{\bar{\lambda}\}$*;
(iii) *$\mathrm{lev}_{\geq 0}f_i(\cdot,\cdot,\bar{\lambda})$ is closed in $K_1(A,\Lambda)\times K_2(A,\Lambda)\times\{\bar{\lambda}\}$ for $i=1,2$*;
(iv) *Z is lsc in $S_{f_1}(\bar{\lambda})\times\{\bar{\lambda}\}$.*

Then the solution map S_f is both usc and closed at $\bar{\lambda}$.

Proof We first claim that the solution map S_f is usc at $\bar{\lambda}$. Suppose there exist an open set $U\supseteq S_f(\bar{\lambda})$, $\{\lambda_n\}\to\bar{\lambda}$, and $\{x_n\}\subseteq S_f(\lambda_n)$ such that $x_n\notin U$ for all n. By the upper semicontinuity of S_{f_1} at $\bar{\lambda}$ and the compactness of $S_{f_1}(\bar{\lambda})$, without loss of generality we can assume that $x_n\to x_0$ as $n\to\infty$ for some $x_0\in S_{f_1}(\bar{\lambda})$. If $x_0\notin S_f(\bar{\lambda})$, there exists $y_0\in Z(x_0,\bar{\lambda})$ such that $f_2(x_0,y_0,\bar{\lambda})<0$. The lower semicontinuity of Z in turn yields $y_n\in Z(x_n,\lambda_n)$ tending to y_0. Notice that for each $n\in\mathbb{N}$, $f_2(x_n,y_n,\lambda_n)\geq 0$. This together with the closedness of $\mathrm{lev}_{\geq 0}f_2(\cdot,\cdot,\bar{\lambda})$ in $K_1(A,\Lambda)\times K_2(A,\Lambda)\times\{\bar{\lambda}\}$ implies that $f_2(x_0,y_0,\bar{\lambda})\geq 0$, which gives a contradiction. If $x_0\in S_f(\bar{\lambda})\subseteq U$, one has another contradiction, since $x_n\notin U$ for all n. Thus, S_f is usc at $\bar{\lambda}$.

Now we prove that S_f is closed at $\bar{\lambda}$. Suppose on the contrary that there exists a sequence $\{(\lambda_n,x_n)\}$ converging to $(\bar{\lambda},x_0)$ with $x_n\in S_f(\lambda_n)$ but $x_0\notin S_f(\bar{\lambda})$. Then $f_2(x_0,y_0,\bar{\lambda})<0$ for some $y_0\in Z(x_0,\bar{\lambda})$. Due to the lower semicontinuity of Z, there is $y_n\in Z(x_n,\lambda_n)$ such that $y_n\to y_0$. Since $x_n\in S_f(\lambda_n)$, $f_2(x_n,y_n,\lambda_n)\geq 0$. By the closedness of the set $\mathrm{lev}_{\geq 0}f_2$, $f_2(x_0,y_0,\bar{\lambda})\geq 0$, which is impossible since $f_2(x_0,y_0,\bar{\lambda})<0$. Therefore, S_f is closed at $\bar{\lambda}$. □

Corollary 3.3 *For GLQEP, suppose that the conditions* (i), (ii), *and* (iv) *given in Theorem* 3.2 *are satisfied. Further, for each $i=1,2$, assume that f_i is upper pseudocontinuous in $K_1(A,\Lambda)\times K_2(A,\Lambda)\times\{\bar{\lambda}\}$. Then the solution map S_f is both usc and closed at $\bar{\lambda}$.*

Proof It is suffice to derive the condition (iii) that given in Theorem 3.2. For $i=1,2$, suppose $\{(x_n,y_n,\lambda_n)\}$ is any sequence in $\mathrm{lev}_{\geq 0}f_i(\cdot,\cdot,\bar{\lambda})$ such that $(x_n,y_n,\lambda_n)\to(\bar{x},\bar{y},\bar{\lambda})$ as $n\to\infty$. Assume that on the contrary that $(\bar{x},\bar{y},\bar{\lambda})\notin\mathrm{lev}_{\geq 0}f_i(\cdot,\cdot,\bar{\lambda})$, which implies that $f_i(\bar{x},\bar{y},\bar{\lambda})<0=f_i(\bar{x},\bar{x},\bar{\lambda})$. The upper pseudocontinuity of f_i at $(\bar{x},\bar{y},\bar{\lambda})$ implies that

$$0=f_i(\bar{x},\bar{x},\bar{\lambda})>\limsup_{n\to\infty}f_i(x_n,y_n,\lambda_n)\geq 0,$$

which gives a contradiction. Hence, we can conclude that $(\bar{x},\bar{y},\bar{\lambda})\in\mathrm{lev}_{\geq 0}f_i(\cdot,\cdot,\bar{\lambda})$. Now, the closedness of $\mathrm{lev}_{\geq 0}f_i(\cdot,\cdot,\bar{\lambda})$ is proved. Applying Theorem 3.2, we get the desired result. □

The following examples show that all assumptions imposed in Theorem 3.2 are very essential and cannot be relaxed.

Example 3.4 (The upper semicontinuity and compactness in (i) are crucial) Let $A=X=\mathbb{R}$, $\Lambda=[0,1]$, $\bar{\lambda}=0$. Define the mappings K_1, K_2, and f by

$$K_1(x,\lambda)=(\lambda,1+\lambda]\quad\text{and}\quad K_2(x,\lambda)=(0,1]$$

and

$$f(x,y,\lambda)=\bigl(x(x-y)^2(1+\lambda),2^{\lambda+xy}x(x-y)\bigr).$$

Then we have

$$E(0) = (0,1] \quad \text{and} \quad E(\lambda) = (\lambda, 1+\lambda], \quad \forall \lambda \in (0,1].$$

Hence, $E(0)$ is not compact and E is not usc. Indeed, we choose an open set $U := (0,3/2) \supseteq E(0) = (0,1]$. We observe that, for any $\varepsilon > 0$, we can choose $\lambda' = -\varepsilon/2 \in N(0,\varepsilon)$ such that

$$E(\lambda') = (-\varepsilon/2, \varepsilon/2] \nsubseteq U.$$

Clearly, the conditions (ii) and (iii) are all satisfied. Easy calculations yield

$$S_{f_1}(\lambda) = \begin{cases} (0,1], & \text{if } \lambda = 0, \\ (\lambda, 1+\lambda], & \text{if } \lambda \neq 0, \end{cases}$$

and $Z(x,\lambda) = \{x\}$. Hence, assumption (iv) is satisfied. Direct computations give

$$S_f(\lambda) = \begin{cases} (0,1], & \text{if } \lambda = 0, \\ (\lambda, 1+\lambda], & \text{if } \lambda \neq 0. \end{cases}$$

It is evident that S_f is neither usc nor closed at $\bar{\lambda} = 0$. This is caused by the fact that E is neither upper semicontinuous nor compact-valued at $\bar{\lambda} = 0$.

Example 3.5 (The lower semicontinuity of K_2 in $K_1(A,\Lambda) \times \{\bar{\lambda}\}$ is essential) Let $A = X = \mathbb{R}$, $\Lambda = [0,1]$, $\bar{\lambda} = 0$. Define the mappings K_1, K_2, and f by

$$K_1(x,\lambda) = K_2(x,\lambda) = \begin{cases} [-1,0], & \text{if } \lambda = 0, \\ [-\frac{1}{2}, 0], & \text{if } \lambda \neq 0, \end{cases}$$

and

$$f(x,y,\lambda) = \bigl((1+\lambda)(y-x), 2^{\lambda y}(y-x)\bigr).$$

Then we have

$$E(\lambda) = \begin{cases} [-1,0], & \text{if } \lambda = 0, \\ [-\frac{1}{2}, 0], & \text{if } \lambda \neq 0, \end{cases}$$

which shows that E is usc at 0 and $E(0)$ is compact, that is, (i) is satisfied. Clearly, the condition (iii) in Theorem 3.2 is satisfied. Furthermore, easy calculations yield

$$S_{f_1}(\lambda) = \begin{cases} \{-1\}, & \text{if } \lambda = 0, \\ \{-\frac{1}{2}\}, & \text{if } \lambda \neq 0, \end{cases}$$

and $Z(x,\lambda) = \{x\}$, which is lsc in $S_{f_1}(\bar{\lambda}) \times \{\bar{\lambda}\}$. Direct calculation gives

$$S_f(\lambda) = \begin{cases} \{-1\}, & \text{if } \lambda = 0, \\ \{-\frac{1}{2}\}, & \text{if } \lambda \neq 0. \end{cases}$$

It is evident that S_f is neither usc nor closed at $\bar{\lambda} = 0$. This is caused by the fact that K_2 is not lsc at $\bar{\lambda} = 0$. Indeed, we observe that $(0,0) \in K_1(A,\Lambda) \times \{0\}$ and $\{(\frac{1}{n}, \frac{1}{n})\} \to (0,0)$. We choose $y := -1 \in K_2(0,0) = [-1,0]$ such that there is not any sequence $\{y_n\}$ in $K_2(\frac{1}{n}, \frac{1}{n}) = [-\frac{1}{2}, 0]$ converging to y.

Example 3.6 (The lower semicontinuity of Z in $S_{f_1}(\bar{\lambda}) \times \{\bar{\lambda}\}$ are crucial) Let $A = X = \mathbb{R}$, $\Lambda = [0,1]$, $\bar{\lambda} = 0$. Define the mappings K_1, K_2, and f by

$$K_1(x,\lambda) = K_2(x,\lambda) = [0,1]$$

and

$$f(x,y,\lambda) = \bigl(\lambda(x-y), 2^{\lambda y}(y-x)\bigr).$$

Hence Conditions (i), (ii), and (iii) clearly hold. By direct calculations, we can get

$$S_{f_1}(\lambda) = \begin{cases} [0,1], & \text{if } \lambda = 0, \\ \{1\}, & \text{if } \lambda \neq 0, \end{cases}$$

$$Z(x,\lambda) = \begin{cases} [0,1], & \text{if } \lambda = 0, \\ \{x\}, & \text{if } \lambda \neq 0. \end{cases}$$

Hence Z is not lsc in $[0,1] \times \{0\}$. Further

$$S_f(\lambda) = \begin{cases} \{0\}, & \text{if } \lambda = 0, \\ \{1\}, & \text{if } \lambda \neq 0. \end{cases}$$

It is evident that S_f is neither usc nor closed at $\bar{\lambda} = 0$.

4 The lower semicontinuouity of S_f

For investigation the lower semicontinuity of the solution mapping S_f, as an auxiliary problem we consider, for a given $\lambda \in \Lambda$, *an auxiliary parametric generalized lexicographic quasiequilibrium problem*, denoted by AGLQEP_λ:

$$(\mathrm{AGLQEP}_\lambda) \begin{cases} \text{finding } \bar{x} \in K_1(\bar{x},\lambda) \text{ such that} \\ f_1(\bar{x},y,\lambda) > 0, \text{ for all } y \in K_2(\bar{x},\lambda). \end{cases}$$

Let the set-valued mappings $E : \Lambda \to 2^X$ and $S_{AGQEP} : \Lambda \to 2^X$ be defined by

$$E(\lambda) = \bigl\{x \in A : x \in K_1(x,\lambda)\bigr\},$$

and the solution mapping for AGLQEP_λ is denoted by $S_{AGLQEP}(\lambda)$, *i.e.*

$$S_{AGQEP}(\lambda) = \bigl\{x \in E(\lambda) : f_1(x,y,\lambda) > 0, \forall y \in K_2(x,\lambda)\bigr\}.$$

First, we establish the lower semicontinuity of the solution mapping S_{AGQEP}.

Lemma 4.1 *For AGLQEP, assume that the following conditions are satisfied*:

(i) *E is lsc at $\bar{\lambda}$*;

(ii) *K_2 is usc and compact-valued in $K_1(A,\Lambda)\times\{\bar{\lambda}\}$*;

(iii) $\mathrm{lev}_{\leq 0}f_1(\cdot,\cdot,\bar{\lambda})$ *is closed in $K_1(A,\Lambda)\times K_2(A,\Lambda)\times\{\bar{\lambda}\}$*.

Then the solution map S_{AGQEP} is lsc at $\bar{\lambda}$.

Proof Suppose to the contrary that S_{AGQEP} is not lsc at $\bar{\lambda}$, *i.e.*, there a sequence $\{\lambda_n\}\subseteq\Lambda$ with $\lambda_n\to\bar{\lambda}$ and there exists $\bar{x}\in S_{AGQEP}(\bar{\lambda})\subseteq E(\bar{\lambda})$ such that,

$$\text{for all sequence } \{y_n\}\subseteq S_{AGQEP}(\lambda_n)\subseteq E(\lambda_n),\quad y_n\nrightarrow\bar{x}. \tag{4.1}$$

Since E is lsc at $\bar{\lambda}$, there exists a sequence $\{x_n\}$ in $E(\lambda_n)$ with $x_n\to\bar{x}$ as $n\to\infty$. It follows from (4.1) that there exists $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\notin S_{AGQEP_1}(\lambda_{n_k})$ for all $k\in\mathbb{N}$. This implies that there $y_{n_k}\in K_2(x_{n_k},\lambda_{n_k})$ satisfying

$$f_1(x_{n_k},y_{n_k},\lambda_{n_k})\leq 0,\quad \text{for each } k\in\mathbb{N}.$$

As K_2 is usc at $(\bar{x},\bar{\lambda})$ and $K_2(\bar{x},\bar{\lambda})$ is compact, there exists $\bar{y}\in K_2(\bar{x},\bar{\lambda})$ such that

$$y_{n_k}\to\bar{y}\quad \text{as } k\to\infty \text{ (taking a subsequence if necessary)}.$$

It follows from the closedness of $\mathrm{lev}_{\leq 0}f_1(\cdot,\cdot,\bar{\lambda})$ that $f_1(\bar{x},\bar{y},\bar{\lambda})\leq 0$, which is impossible since $\bar{x}\in S_{AGQEP}(\bar{\lambda})$. The proof is completed. □

Now, we establish the lower semicontinuity of the solution mapping S_f.

Theorem 4.2 *For* (GLQEP) *let the following conditions be satisfied*:

(i) *E is lsc at $\bar{\lambda}$ and $E(\bar{\lambda})$ is convex*;

(ii) *$K_2(\cdot,\bar{\lambda})$ is usc and compact-valued in $K_1(A,\Lambda)\times\{\bar{\lambda}\}$*; *$K_2(\cdot,\bar{\lambda})$ is concave in $E(\bar{\lambda})$*;

(iii) $\mathrm{lev}_{\leq 0}f_1(\cdot,\cdot,\bar{\lambda})$ *is closed in $K_1(A,\Lambda)\times K_2(A,\Lambda)\times\{\bar{\lambda}\}$*;

(iv) *$f_1(\cdot,\cdot,\bar{\lambda})$ is generalized $\mathbb{R}_+$-concave in $E(\bar{\lambda})\times K_2(A,\bar{\lambda})$*.

Then the solution map S_f is lsc at $\bar{\lambda}$.

Proof First, we notice that $S_f(\bar{\lambda})\subseteq \mathrm{cl}\,S_{AGLQEP}(\bar{\lambda})$. Indeed, let $\bar{x}\in S_f(\bar{\lambda})$ be arbitrary. For any $\bar{x}_A\in S_{AGLQEP}(\bar{\lambda})$ and $t\in(0,1)$, define $x_t=(1-t)\bar{x}+t\bar{x}_A$. Clearly, $x_t\to\bar{x}$ as $t\downarrow 0$ and by the virtue of the convexity of $E(\bar{\lambda})$, $x_t\in K_1(x_t,\bar{\lambda})$. Further, for all $y\in K_2(x_t,\bar{\lambda})$, the concavity of $K_2(\cdot,\bar{\lambda})$ implies that there exist $\bar{y}\in K_2(\bar{x},\bar{\lambda})$ and $\bar{y}_A\in K_2(\bar{x}_A,\bar{\lambda})$ such that $y=(1-t)\bar{y}+t\bar{y}_A$. It is clear that $f_1(\bar{x},\bar{y},\bar{\lambda})\geq 0$ and $f_1(\bar{x}_A,\bar{y}_A,\bar{\lambda})>0$. It follows from the generalized $\mathbb{R}_+$-concavity of $f_1(\cdot,\cdot,\bar{\lambda})$ that $f_1(x_t,y,\bar{\lambda})>0$, *i.e.*, $x_t\in S_{AGQEP}(\bar{\lambda})$. Therefore, we conclude that $\bar{x}\in \mathrm{cl}\,S_{AGQEP}(\bar{\lambda})$, which shows that $S_f(\bar{\lambda})\subseteq\mathrm{cl}\,S_{AGQEP}(\bar{\lambda})$. Next, for any sequence $\{\lambda_n\}$ satisfying $\lambda_n\to\bar{\lambda}$ as $n\to\infty$, by the lower semicontinuity of S_{AGQEP} at $\bar{\lambda}$ given in Lemma 4.1, we have

$$S_f(\bar{\lambda})\subseteq\mathrm{cl}\,S_{AGQEP}(\bar{\lambda})\subseteq\mathrm{cl}\liminf_{n\to\infty}S_{AGQEP}(\lambda_n)\subseteq\mathrm{cl}\liminf_{n\to\infty}S_f(\lambda_n),$$

which gives the lower semicontinuity of S_f at $\bar{\lambda}$ since Lemma 2.6. The proof is completed. □

Corollary 4.3 *For GLQEP, suppose that the conditions* (i), (ii), *and* (iv) *given in Theorem* 4.2 *are satisfied. Further, assume that* f_1 *is lower pseudocontinuous in* $K_1(A,\Lambda) \times K_2(A,\Lambda) \times \{\bar{\lambda}\}$. *Then the solution map* S_f *is lsc at* $\bar{\lambda}$.

Proof It is suffice to derive the condition (iii) that imposed in Theorem 4.2. Let $\{(x_n, y_n, \lambda_n)\}$ be any sequence in $\mathrm{lev}_{\leq 0} f_1(\cdot,\cdot,\bar{\lambda})$ such that $(x_n, y_n, \lambda_n) \to (\bar{x}, \bar{y}, \bar{\lambda})$ as $n \to \infty$. Assume that on the contrary that $(\bar{x}, \bar{y}, \bar{\lambda}) \notin \mathrm{lev}_{\leq 0} f_1$, which implies that $f_1(\bar{x}, \bar{y}, \bar{\lambda}) > 0 = f_1(\bar{x}, \bar{x}, \bar{\lambda})$. It follows from the lower pseudocontinuity of f_1 at $(\bar{x}, \bar{y}, \bar{\lambda})$ that

$$0 = f_1(\bar{x}, \bar{x}, \bar{\lambda}) < \liminf_{n\to\infty} f_1(x_n, y_n, \lambda_n) \leq 0,$$

which gives a contradiction. Hence, we can conclude that $(\bar{x}, \bar{y}, \bar{\lambda}) \in \mathrm{lev}_{\leq 0} f_1$. Now, the closedness of $\mathrm{lev}_{\leq 0} f_1(\cdot,\cdot,\bar{\lambda})$ is proved. Applying Theorem 4.2 we obtain the desired result. □

The following example illustrates that the lower semicontinuity assumption for the set E cannot be relaxed in Theorem 4.2.

Example 4.4 (The lower semicontinuity of E at $\bar{\lambda}$ is crucial) Let $A = X = \mathbb{R}$, $\Lambda = [0,1]$, $\bar{\lambda} = 0$. Define the mappings K_1, K_2, and f by

$$K_1(x,\lambda) = \begin{cases} [-1,1], & \text{if } \lambda = 0, \\ [-1,0] \cup \{1\}, & \text{if } \lambda \neq 0, \end{cases} \qquad K_2(x,\lambda) = [-1,0],$$

and

$$f(x,y,\lambda) = \big((1+\lambda)(x-y), 2^{\lambda y}(x-2y)\big).$$

Hence conditions (ii)-(iv) clearly hold. However, E is not lsc at $\bar{\lambda} = 0$. Indeed, we choose a sequence $\{1/n\} \subseteq \Lambda$ such that $1/n \to 0$ and $1/2 \in E(0) = [-1,1]$. We can see that, for all sequences $\{y_n\} \subseteq E(1/n) := [-1,0] \cup \{1\}$, $y_n \nrightarrow 1/2$ as $n \to \infty$. By direct calculations, we can get

$$S_{AGQEP}(\lambda) = S_f(\lambda) = \begin{cases} (0,1], & \text{if } \lambda = 0, \\ \{1\}, & \text{if } \lambda \neq 0. \end{cases}$$

Hence S_f is not lsc at $\bar{\lambda} = 0$, indeed, we choose $\lambda_n = \frac{1}{n} \to 0$ and $\frac{1}{2} \in S_f(0)$ but we cannot find a sequence in $S_f(\frac{1}{n})$ which converges to $\frac{1}{2}$.

The next example indicates the essential role of the upper semicontinuity assumption for the set K_2 in Theorem 4.2.

Example 4.5 (The upper semicontinuity of K_2 is crucial) Let $A = X = \mathbb{R}$, $\Lambda = [0,1]$, $\bar{\lambda} = 0$. Define the mappings K_1, K_2, and f by

$$K_1(x,\lambda) = [0,1], \qquad K_2(x,\lambda) = \begin{cases} \{-\frac{1}{2}\} \cup [0,1], & \text{if } \lambda = 0, \\ [-\frac{2}{3}, \frac{2}{3}], & \text{if } \lambda \neq 0, \end{cases}$$

and

$$f(x,y,\lambda) = (x+y, x-y).$$

It is clear that the upper semicontinuity of K_2 is not satisfied. Indeed, for each $n \in \mathbb{N}$, we choose

$$(x_n, \lambda_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \quad \text{and} \quad y_n = -\frac{2}{3} \in K_2(x_n, \lambda_n).$$

It is obvious that there is not any subsequence of $\{y_n\}$ converging to an element in $\{-\frac{1}{2}\} \cup [0,1] := K(0,0)$. However, all conditions (i), (iii)-(v) of Theorem 4.2 are satisfied. By direct calculations, we have

$$S_{AGQEP}(\lambda) = S_f(\lambda) = \begin{cases} (\frac{1}{2}, 1], & \text{if } \lambda = 0, \\ (\frac{2}{3}, 1], & \text{if } \lambda \neq 0. \end{cases}$$

Hence S_f is not lsc at $\bar{\lambda} = 0$. The cause is that (ii) is not fulfilled.

5 Conclusion

We presented the upper semicontinuity, closedness, and the lower semicontinuity of the set-valued solution mapping for a parametric lexicographic equilibrium problem where both two constraint maps and the objective bifunction depend on both the decision variable and the parameters. The sufficient conditions for the upper semicontinuity, closedness, and the lower semicontinuity of the solution map are established. Many examples are provided to ensure the essentialness of the imposed assumptions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

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Levitin-Polyak Well-posedness for Lexicographic Vector Equilibrium Problems

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Abstract

We introduce the notions of Levitin-Polyak(LP) well-posedness and LP well-posedness in the generalized sense for the Lexicographic vector equilibrium problems. Then, we establish some sufficient conditions for Lexicographic vector equilibrium problems to be LP well-posedness at the reference point. Numerous examples are provided to explain that all the assumptions we impose are very relaxed and cannot be dropped. The results in this paper unify, generalize and extend some known results in the literature.

Keywords: Levitin-polyak well-posedness, lexicographic vector equilibrium problems, metric spaces .
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1. Introduction and Preliminaries

Equilibrium problems first considered by Blum and Oettli [8] have been playing an important role in optimization theory with many striking applications particularly in transportation, mechanics, economics, etc. Equilibrium models incorporate many other important problems such as: optimization problems, variational inequalities, complementarity problems, saddlepoint/minimax problems, and fixed points. Equilibrium problems with scalar and vector objective functions have been widely studied. The crucial issue of solvability (the existence of solutions) has attracted the most considerable attention of researchers, see, e.g., [14, 18, 21, 40].

On the other hand, well-posedness plays an important role in the stability analysis and numerical methods for optimization theory and applications. Since any algorithm can generate only an approximating solution sequence which is meaningful only if the problem is well-posed under consideration. The first and oldest well-posedness is Hadamard well-posedness [20], which means existence, uniqueness and continuous

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dependence of the optimal solution and optimal value from perturbed data. The second is Tikhonov well-posedness [41], which means the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Well-posedness properties have been intensively studied and the two classical well-posedness notions have been extended and blended. For parametric problems, well-posedness is closely related to stability. Up to now, there have been many works dealing with well-posedness of optimization-related problems as mathematical programming [39, 22], constrained minimization [12, 44, 43, 17] variational inequalities [12, 10, 16, 30, 42], Nash equilibria [42, 34], and equilibrium problems [17, 2, 24]. A fundamental requirement in Tykhonov well-posedness is that every minimizing sequence is from within the feasible region. However, in several numerical methods such as exterior penalty methods and augmented Lagrangian methods, the minimizing sequence generated may not be feasible. Taking this into account, Levitin and Polyak [28] introduced another notion of well-posedness which does not necessarily require the feasibility of the minimizing sequence. However, it requires the distance of the minimizing sequence from the feasible set to approach to zero eventually. Since then, many authors investigated the well-posedness and well-posedness in the gener- alized sense for optimization, variational inequalities and equilibrium problems. The study of Levitin-Polyak type well-posedness for scalar convex optimization probiems with functional constraints was initiated by Konsulova and Revalski [26]. In 1981, Lucchetti and Patrone [33] introduced and studied the well-posedness for variational inequalities, which is a generalization of the Tykhonov well-posedness of minimization problems. Long et al. [31] introduced and studied four types of Levitin-Polyak well-posedness of equilibrium problems with abstract set constraints and functional constraints. Li and Li [29] introduced and researched two types of Levitin-Polyak well-posedness of vector equilibrium problems with abstract set constraints. Peng et al. [36] introduced and studied four types of Levitin-Polyak well-posedness of vector equilibrium problems with abstract set constraints and functional constraints. Peng, Wu and Wang [37] introduced several types of Levitin-Polyak well-posedness for a generalized vector quasi-equilibrium problem with functional constraints and abstract set constraints. Chen, Wan and Cho [11] studied the Levitin-Polyak well-posedness by perturbations for a class of general systems of set-valued vector quasi-equilibrium problems in Hausdorff topological vector spaces. Very recently Lalitha and Bhatia [27] studied the LP well-posedness for a parametric quasivariational inequality problem of the Minty type.

With regard to vector equilibrium problems, most of existing results correspond to the case when the order is induced by a closed convex cone in a vector space. Thus, they cannot be applied to lexicographic cones, which are neither closed nor open. These cones have been extensively investigated in the framework of vector optimization, see, e.g., [3, 6, 7, 9, 15, 19, 25, 23]. For instance, Konnov and Ali [25] studied sequential problems, especially exploiting its relation with regularization methods. Bianchi et al. in [6] analyzed lexicographic equilibrium problems on a topological Hausdorff vector space, and their relationship with some other vector equilibrium problems. They obtained the existence results for the tangled lexicographic problem via the study of a related sequential problem. However, for equilibrium problems, the main emphasis has been on the issue of solvability/existence. To the best of the knowledge, very recently, Anh et al. in [3] studied the Tikhonov well-posedness for lexicographic vector equilibrium problems in metric spaces and gave the sufficient conditions for a family of such problems to be well-posed and uniquely well-posed at the considered point. Furthermore, they derived several results on well-posedness for a class of variational inequalities.

In this paper, we first introduce the new notions of Levitin-Polyak(LP) well-posedness and LP well-posedness in the generalized sense for the Lexicographic vector equilibrium problems. Then, we establish some sufficient conditions for this problems to be LP well-posedness at the reference point. Furthermore, we give numerous examples to explain that all the imposed assumptions are very relaxed and cannot be dropped.

The layout of the paper is as follows. In Sect. 2, we introduce the notions of LP well-posedness and LP well-posedness in the generalized sense for the Lexicographic vector equilibrium problems. In Sect. 3, we establish some sufficient conditions for this problems to be LP well-posedness at the reference point. Section 4 is devoted to LP well-posedness in the generalized sense for the Lexicographic vector equilibrium problems. Some concluding remarks are included in the end of this paper.

We first recall the concept of lexicographic cone in finite dimensional spaces and models of equilibrium problems with the order induced by such a cone. The lexicographic cone of $\mathbb{R}^n$, denoted C_l , is the collection of zero and all vectors in $\mathbb{R}^n$ with the first nonzero coordinate being positive, i.e.,

$$C_l := \{0\} \cup \{x \in \mathbb{R}^n | \exists i \in \{1, 2, \dots, n\} : x_i > 0 \text{ and } x_j = 0, \quad \forall j < i\}.$$

This cone is convex and pointed, and induces the total order as follow:

$$x \geq_l y \Leftrightarrow x - y \in C_l.$$

We also observe that it is neither closed nor open. Indeed, when comparing with the cone $C_1 := \{x \in \mathbb{R}^n | x_1 \geq 0\}$, we see that $\text{int}C_1 \subsetneq C_l \subsetneq C_1$, while

$$\text{int}C_l = \text{int}C_1 \text{ and } \text{cl}C_l = C_1.$$

Throughout this paper, if not other specified, X be a metric space and Λ denote the metric space. Let $X_0 \subset X$ be nonempty and closed sets . Let $f := (f_1, f_2, \dots, f_n) : X \times X \times \Lambda \to \mathbb{R}^n$ be vector-valued function and $K : \Lambda \to 2^X$ being a closed valued map. The lexicographic vector quasiequilibrium problem consists of, for each $\lambda \in \Lambda$,

(LEP$_\lambda$) finding $\bar{x} \in K(\lambda)$ such that

$$f(\bar{x}, y, \lambda) \geq_l 0 \quad , \forall y \in K(\lambda).$$

Instead of writing $\{(\text{LEP}_\lambda) | \lambda \in \Lambda\}$ for the family of lexicographic vector equilibrium problem, i.e., the lexicographic parametric problem, we will simply write (LEP) in the sequel. Let $S : \Lambda \to 2^X$ be the solution map of (LEP); that is, for each $\lambda \in \Lambda$,

$$S(\bar{\lambda}) := \{x \in K(\bar{\lambda}) | f(x, y, \bar{\lambda}) \geq_l 0, \ \forall y \in K(\bar{\lambda})\}. \tag{1.1}$$

Following the lines of investigating ε-solutions to vector optimization problems initiated by Loridan [32], we consider, for each $\lambda \in \Lambda$ and each $\varepsilon \in [0, \infty)$, the following approximate problem:

(LEP$_{\lambda,\varepsilon}$) find $\bar{x} \in K(\lambda)$ such that

$$d(\bar{x}, K(\lambda)) \leq \epsilon \text{ and } f(\bar{x}, y, \lambda) + \varepsilon e \geq_l 0, \ \forall y \in K(\lambda),$$

where $e := (\underbrace{0, 0, \dots, 0}_{n-1}, 1) \in \mathbb{R}^n$. The solution set of (LEP$_{\lambda,\varepsilon}$) is denoted by $\tilde{S}(\lambda, \varepsilon)$; that is the set valued-map $\tilde{S} : \Lambda \times \mathbb{R} \to 2^X$ is defined by

$$\tilde{S}(\lambda, \varepsilon) = \{x \in X | d(x, K(\lambda)) \leq \epsilon \text{ and } f(x, y, \lambda) + \varepsilon e \geq_l 0, \ \forall y \in K(\lambda)\}, \tag{1.2}$$

for all $(\lambda, \varepsilon) \in \Lambda \times \mathbb{R}$.

Now we introduce the concept of LP well-posedness for LEP. For this purpose, we require the the following notions of an LP approximating sequence.

Definition 1.1. *Let $\{\lambda_n\}$ be a sequence in Λ such that $\lambda_n \to \bar{\lambda}$. A sequence $\{x_n\}$ is said to be an LP approximating sequence for LEP with respect to $\{\lambda_n\}$ if there is a sequence $\{\epsilon_n\}$ in $(0, \infty)$ satisfying $\epsilon_n \to 0$ as $n \to \infty$, such that*

(i) $d(x_n, K(\lambda_n)) \leq \epsilon_n$, for all $n \in \mathbb{N}$;
(ii) $f(x_n, y_n, \lambda_n) + \epsilon_n e \geq_l 0, \ \forall y_n \in K(\lambda_n)$.

Definition 1.2. The problem (LEP) is *LP well-posed* at $\bar{\lambda}$ if

(i) there exists a unique solution $\bar{x}$ of LEP;

(ii) for any sequence $\{\lambda_n\}$ converging to $\bar{\lambda}$, every LP approximating sequence $\{x_n\}$ with respect to $\{\lambda_n\}$ converges to $\bar{x}$.

Definition 1.3. [4] *Let $Q : X \rightrightarrows Y$ be a set-valued mapping between metric spaces*

(i) *Q is upper semicontinuous (usc) at $\bar{x}$ if for any open set $U \supseteq Q(\bar{x})$, there is a neighborhood N of $\bar{x}$ such that $Q(N) \subseteq U$.*

(ii) *Q is lower semicontinuous (lsc) at $\bar{x}$ if for any open subset U of Y with $Q(\bar{x}) \cap U \neq \emptyset$, there is a neighborhood N of $\bar{x}$ such that $Q(x) \cap U \neq \emptyset$ for all $x \in N$.*

(iii) *Q is closed at $\bar{x}$ if for any sequences $x_k \to \bar{x}$ and $y_k \to \bar{y}$ with $y_k \in Q(x_k)$, it holds $\bar{y} \in Q(\bar{x})$.*

Lemma 1.4. [4]

(i) *If Q is usc at $\bar{x}$ and $Q(\bar{x})$ is compact, then for any sequence $x_n \to \bar{x}$, every sequence $\{y_n\}$ with $y_n \in Q(x_n)$ has a subsequence converging to some point in $Q(\bar{x})$. If, in addition, $Q(\bar{x}) = \{\bar{y}\}$ is a singleton, then such a sequence $\{y_n\}$ must converge to $\bar{y}$.*

(ii) *Q is lsc at $\bar{x}$ if and only if for any sequence $x_n \to \bar{x}$ and any point $y \in Q(\bar{x})$, there is a sequence $\{y_n\}$ with $y_n \in Q(x_n)$ converging to y.*

Definition 1.5. [3, 1] Let g be an extended real-valued function on a metric space X and ε be a real number.

(i) g is *upper ε-level closed* at $\bar{x} \in X$ if for any sequence $x_n \to \bar{x}$,

$$[g(x_n) \geq \varepsilon,\ \forall n] \Rightarrow [g(\bar{x}) \geq \varepsilon].$$

(ii) g *is strongly upper ε-level closed* at $\bar{x} \in X$ if for any sequences $x_n \to \bar{x}$ and $\{v_n\} \subset [0, \infty)$ converging to 0,

$$[g(x_n) + v_n \geq \varepsilon,\ \forall n] \Rightarrow [g(\bar{x}) \geq \varepsilon].$$

Let A, B be two subsets of metric space X. The Hausdorff distance between A and B is defined as follows

$$H(A, B) = \max\{H^*(A, B), H^*(B, A)\},$$

where $H^*(A, B) = \sup_{a \in A} d(a, B)$, and $d(x, A) = \inf_{y \in A} d(x, y)$.

2. LP well-posedness for Lexicographic vector Equilibrium Problems

In this section, we shall give some neccessary and/or sufficient conditions for (LEP) to be LP well-posed at the reference point $\bar{\lambda} \in \Lambda$. To simplify the presentation, in the sequel, the results will be formulated for the case $n = 2$. For any two positive numbers α, ϵ, the solution set of approximation solutions for the problem ($\mathrm{LEP}_{\lambda,\varepsilon}$) is denoted by

$$\Gamma(\bar{\lambda}, \alpha, \epsilon) = \bigcup_{\lambda \in B(\bar{\lambda}, \alpha) \cap \Lambda} \{x \in X | d(x, K(\lambda)) \leq \epsilon \text{ and } f(x, y, \lambda) + \varepsilon e \geq_l 0,\ \forall y \in K(\lambda)\}, \tag{2.1}$$

where $B(\bar{\lambda}, \alpha)$ denote the closed ball centered at $\bar{\lambda}$ with radius α. The set-valued mapping $Z : \Lambda \times X \to 2^X$ next defined will play an important role our analysis

$$Z(\lambda, x) = \begin{cases} \{z \in K(\lambda) | f_1(x, z, \lambda) = 0\} & \text{if } (\lambda, x) \in \text{ gr } Z_1; \\ X \text{ otherwise}, \end{cases}$$

where $Z_1 : \Lambda \to 2^X$ denotes the solution mapping of the scalar equilibrium problem determined by the real-valued function f_1 :

$$Z_1(\lambda) = \{x \in K(\lambda) | f_1(x, y, \lambda) \geq 0,\ \forall y \in K(\lambda)\}.$$

Then (2.1) is equivalent to

$$\Gamma(\bar{\lambda},\alpha,\epsilon)$$

$$= \bigcup_{\lambda\in B(\bar{\lambda},\alpha)\cap\Lambda} \{x\in X | d(x,K(\lambda))\leq\epsilon, f_1(x,y,\lambda)\geq 0, \forall y\in K(\lambda) \text{ and } f_2(x,z,\lambda)+\epsilon\geq 0, \forall z\in Z(\lambda,x)\}$$

$$= \bigcup_{\lambda\in B(\bar{\lambda},\alpha)\cap\Lambda} \tilde{S}(\lambda,\varepsilon),$$

where $\tilde{S}$ is the solution map for (LEP$_{\lambda,\varepsilon}$) defined by (1.2). For the solution map $S:\Lambda\to 2^X$ of (LEP), in general, we observe that

$$\Gamma(\bar{\lambda},0,0)=S(\bar{\lambda}) \text{ and } S(\bar{\lambda})\subseteq\Gamma(\bar{\lambda},\alpha,\epsilon),\ \forall\alpha,\epsilon>0,$$

and hence

$$S(\bar{\lambda})\subseteq\bigcap_{\alpha,\epsilon>0}\Gamma(\bar{\lambda},\alpha,\epsilon).$$

Next, we provide the sufficient conditions for the two sets to coincide.

Proposition 2.1. *Suppose that the following conditions are satisfied :*

(i) *K is closed and lsc on Λ;*
(ii) *Z is lsc on $\Lambda\times X$;*
(iii) *f_1 is upper 0-level closed on $X\times X\times\Lambda$;*
(iv) *f_2 is strongly upper 0-level closed on $X\times X\times\Lambda$;*

then

$$\bigcap_{\alpha,\epsilon>0}\Gamma(\bar{\lambda},\alpha,\epsilon)=S(\bar{\lambda}).$$

Proof. Let $\bar{x}\in\bigcap_{\alpha,\epsilon>0}\Gamma(\bar{\lambda},\alpha,\epsilon)$, then without loss of generality, there exist sequences $\alpha_n>0,\epsilon_n>0$ with $\alpha_n\to 0,\epsilon_n\to 0$, such that $\bar{x}\in\Gamma(\bar{\lambda},\alpha_n,\epsilon_n)$. Hence, it follows that there exists a sequence $\lambda_n\in B(\bar{\lambda},\alpha_n)\cap\Lambda$, such that, for all $n\in\mathbb{N}$,

$$d(\bar{x},K(\lambda_n))\leq\epsilon_n, \tag{2.2}$$

and

$$f_1(\bar{x},y,\lambda_n)\geq 0,\ \forall y\in K(\lambda_n) \text{ and } f_2(\bar{x},z,\lambda_n)+\epsilon_n\geq 0,\ \forall z\in Z(\lambda_n,\bar{x}). \tag{2.3}$$

Since $K(\bar{\lambda})$ is a closed set in X, it follows from (2.2) that we can choose $x_n\in K(\lambda_n)$, such that

$$d(\bar{x},x_n)\leq\epsilon_n,\ \forall n\in\mathbb{N}. \tag{2.4}$$

Thus $x_n\to\bar{x}$ as $n\to\infty$. Clearly $\lambda_n\to\bar{\lambda}$ as $n\to\infty$ and also as K is closed at $\bar{\lambda}$, it follows that $\bar{x}\in K(\bar{\lambda})$. As K is lsc at $\bar{\lambda}$ and $\lambda_n\to\bar{\lambda}$ for any $y\in K(\bar{\lambda})$ there exists $y_n\in K(\lambda_n)$ such that $y_n\to y$. Also Z is lsc at $(\bar{\lambda},\bar{x})$ and $(\lambda_n,x_n)\to(\bar{\lambda},\bar{x})$, it is clear that for any $z\in Z(\bar{\lambda},\bar{x})$ there exists a sequence $z_n\in Z(\lambda_n,x_n)$ such that $z_n\to z$. This implies by assumption (iii),(iv), and (2.3) that $f_1(\bar{x},y,\bar{\lambda})\geq 0$, $f_2(\bar{x},z,\bar{\lambda})\geq 0$ and hence, $\bar{x}\in S(\bar{\lambda})$. □ □

Theorem 2.2. *Suppose that the conditions (i)-(iv) in Proposition 2.1 are satisfied. Then* (LEP) *is LP well-posed at $\bar{\lambda}\in\Lambda$ if and only if $\Gamma(\bar{\lambda},\alpha,\epsilon)\neq\emptyset,\forall\alpha,\epsilon>0$ and* diam $\Gamma(\bar{\lambda},\alpha,\epsilon)\to 0$ *as* $(\alpha,\epsilon)\to(0,0)$.

Proof. Suppose that the problem (LEP) is LP well-posed. Hence, it has a unique solution $\bar{x} \in S(\bar{\lambda})$ and hence $\Gamma(\bar{\lambda}, \alpha, \epsilon) \neq \emptyset, \forall \alpha, \epsilon > 0$ as $S(\bar{\lambda}) \subseteq \Gamma(\bar{\lambda}, \alpha, \epsilon)$. Suppose on the contrary that diam $\Gamma(\bar{\lambda}, \alpha, \epsilon) \nrightarrow 0$ as $(\alpha, \epsilon) \to (0, 0)$. Then there are positive numbers r, m and sequences $\{\alpha_n\}, \{\epsilon_n\}$ in $(0, \infty)$ with $(\alpha_n, \epsilon_n) \to (0, 0)$ and $x_n, x'_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$ such that

$$d(x_n, x'_n) > r, \ \forall n \geq m. \tag{2.5}$$

By $x_n, x'_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$, there exist $\lambda_n, \lambda'_n \in B(\bar{\lambda}, \alpha_n) \cap \Lambda$ such that

$$d(x_n, K(\lambda_n)) \leq \epsilon_n,$$

$$f_1(x_n, y, \lambda_n) \geq 0, \ \forall y \in K(\lambda_n) \text{ and } f_2(x_n, z, \lambda_n) + \varepsilon_n \geq 0, \ \forall z \in Z(\lambda_n, x_n) \tag{2.6}$$

and

$$d(x'_n, K(\lambda'_n)) \leq \epsilon_n,$$

$$f_1(x'_n, y, \lambda'_n) \geq 0, \ \forall y \in K(\lambda'_n), f_2(x'_n, z, \lambda'_n) + \varepsilon_n \geq 0, \ \forall z \in Z(\lambda'_n, x_n). \tag{2.7}$$

The sequence $\{x_n\}$ and $\{x'_n\}$ are LP approximating sequences for (LEP) corresponding to sequences $\lambda_n \to \bar{\lambda}$ and $\lambda'_n \to \bar{\lambda}'$, respectively. Since (LEP) is LP well-posed, we have that $\{x_n\}$ and $\{x'_n\}$ converse to the unique solution $\bar{x}$, which arrives a contradiction to (2.5). Hence, diam $\Gamma(\bar{\lambda}, \alpha, \epsilon) \to 0$ as $(\alpha, \epsilon) \to (0, 0)$.

Conversely, let $\{\lambda_n\}$ be a sequence in Λ converging to $\bar{\lambda}$ and $\{x_n\}$ be a LP approximating sequence with respect to $\{\lambda_n\}$. Then there exists a sequence $\{\epsilon_n\}$ in $(0, \infty)$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$d(x_n, K(\lambda_n)) \leq \epsilon_n,$$

$$f_1(x_n, y, \lambda_n) \geq 0, \ \forall y \in K(\lambda_n) \ \text{ and } \ f_2(x_n, z, \lambda_n) + \varepsilon_n \geq 0, \ \forall z \in Z(\lambda_n, x_n). \tag{2.8}$$

If we choose $\alpha_n = d(\lambda_n, \bar{\lambda})$, then $\alpha_n \to 0$ and $x_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$. Since diam $\Gamma(\bar{\lambda}, \alpha_n, \epsilon_n) \to 0$ as $n \to \infty$, it follows that $\{x_n\}$ is a Cauchy sequence in X and hence it converges to $\bar{x} \in X$. For each positive integer n, $K(\lambda_n)$ is compact. Thus, there exists $x'_n \in K(\lambda_n)$ such that

$$d(x_n, x'_n) \leq \epsilon_n, \ \text{for all } n \in \mathbb{N},$$

which implies that $x'_n \to \bar{x}$. Since K is closed at $\bar{\lambda}$, it follows that $\bar{x} \in K(\bar{\lambda})$. Suppose on the contrary $\bar{x} \notin S(\bar{\lambda})$, that is, there exist $\bar{y} \in K(\bar{\lambda})$ and $\bar{z} \in Z(\bar{\lambda}, \bar{x})$ such that

$$f_1(\bar{x}, \bar{y}, \bar{\lambda}) < 0 \ \text{ or } \ f_2(\bar{x}, \bar{z}, \bar{\lambda}) + \epsilon < 0. \tag{2.9}$$

Since K is lsc at $\bar{\lambda}$ and $\lambda_n \to \bar{\lambda}$, it is clear that for any $y \in K(\bar{\lambda})$ there exists a sequence $y_n \in K(\lambda_n)$ such that $y_n \to \bar{y}$. Again, since Z is lsc at $(\bar{\lambda}, \bar{x})$ and $(\lambda_n, x_n) \to (\bar{\lambda}, \bar{x})$ there exists a sequence $z_n \in Z(\lambda_n, x_n)$ such that $z_n \to \bar{z}$. Hence, we obtain by assumption (iv), (v) and (2.8) that,

$$f_1(\bar{x}, \bar{y}, \bar{\lambda}) \geq 0 \ \text{ and } \ f_2(\bar{x}, \bar{z}, \bar{\lambda}) \geq 0.$$

This yields a contradiction to (2.9). Hence, we conclude that $\bar{x} \in S(\bar{\lambda})$.

Finally, we will show that $\bar{x}$ is the only solution of (LEP). Let x^* be another point in $S(\bar{\lambda})$ ($x^* \neq \bar{x}$). It is clear that they both belong to $\Gamma(\bar{\lambda}, \alpha, \epsilon)$ for any $\alpha, \epsilon > 0$. Then, it follows that

$$0 \leq d(\bar{x}, x^*) \leq \text{ diam } \Gamma(\bar{\lambda}, \alpha, \epsilon) \downarrow 0 \text{ as } (\alpha, \epsilon) \downarrow (0, 0).$$

This is impossible and, therefore, we are done. The proof is completed. □ □

The following examples show that none of the assumptions in Theorem 2.2 can be dropped.

Example 2.3. (**Lower semicontinuity of** K) Let $X = \Lambda = [0,2]$ and K and f be defined by

$$K(\lambda) = \begin{cases} [0,1] & \text{if } \lambda \neq 0; \\ [0,2] & \text{if } \lambda = 0, \end{cases}$$

$$f(x,y,\lambda) = (x-y,\lambda).$$

One can check that K is closed but not lsc at $\bar{\lambda} = 0$ and

$$S(\lambda) = Z_1(\lambda) = \begin{cases} \{1\} & \text{if } \lambda \neq 0; \\ \{2\} & \text{if } \lambda = 0, \end{cases}$$

$$Z(\lambda,x) = \{x\}, \ \ \forall(\lambda,x) \in \text{ gr } Z_1.$$

Thus, assumption (iii)-(v) hold true. However, (LEP) is not LP well-posed at $\bar{\lambda}$. Indeed, let $\lambda_n := \frac{1}{n}$ and $x_n := 1 + \frac{1}{2n}$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an LP approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to $\{\lambda_n\}$ with $\epsilon_n := \frac{1}{n}$, while $x_n \to 1 \notin S(0)$.

Example 2.4. (**Closedness** of K) Let $X = \Lambda = [-2,2]$, $K(\lambda) = (0,1]$ (continuous), and a function $f := (f_1, f_2) : X \times X \times \Lambda \to \mathbb{R}^2$ be defined by, for all $x, y \in X$ and $\lambda \in \Lambda$,

$$f(x,y,\lambda) = (x - \frac{y}{2}, \frac{1}{2} - x).$$

It can be calculated that

$$Z(\lambda,x) = \begin{cases} \{1\} & \text{if } x = \frac{1}{2}; \\ \emptyset & \text{if } x \in (\frac{1}{2},1]; \\ X & \text{otherwise.} \end{cases}$$

Then, we can conclude that

$$\Gamma(\lambda,\alpha,\epsilon) = \left[\frac{1}{2}, \frac{1}{2} + \min\{\varepsilon, \frac{3}{2}\}\right]$$

and

$$\text{diam } \Gamma(\lambda,\alpha,\varepsilon) \to 0 \text{ as } (\alpha,\varepsilon) \to (0,0).$$

One can check that,

$$S(\lambda) = \left\{\frac{1}{2}\right\}.$$

We observe that (LEP) is not LP well-posed. Indeed, put $\lambda_n := \frac{1}{n}$, $x_n := 1 + \frac{\varepsilon_n}{n}$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an LP approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to $\{\lambda_n\}$ with $\epsilon_n := \frac{1}{n}$, while $x_n \to 1 \notin S(\lambda)$.

Example 2.5. (**Lower semicontinuity of** Z) Let $X = \Lambda = [0,1]$, $K(\lambda) = [0,1]$ (continuous and closed), $\bar{\lambda} = 0$ and $f(x,y,\lambda) = (\lambda x(x-y), y-x)$. One can check that

$$Z_1(\lambda) = \begin{cases} [0,1] & \text{if } \lambda = 0; \\ \{0,1\} & \text{if } \lambda \neq 0. \end{cases}$$

and, for each $(\lambda,x) \in$ gr Z_1,

$$Z(\lambda,x) = \begin{cases} [0,1] & \text{if } \lambda = 0 \text{ or } x = 0; \\ \{1\} & \text{if } \lambda \neq 0 \text{ and } x \neq 0. \end{cases}$$

Z is not lsc at $(0,1)$. Indeed, taking $\lambda_n := \frac{1}{2n}$ and $x_n := 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$, we have $(\lambda_n, x_n) \to (0,1)$ and $Z(\lambda_n, x_n) = \{1\}$ for all n, while $Z(0,1) = [0,1]$. Assumption (iv) and (v) are obviously satisfied. By calculating the solution mapping S explicitly as follows:

$$S(\lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0; \\ \{0,1\} & \text{if } \lambda \neq 0. \end{cases}$$

We observe that (LEP) is not LP well-posed at $\bar{\lambda}$. Indeed, let $\lambda_n := \frac{1}{2n}$ and $x_n := 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to $\{\lambda_n\}$ with $\epsilon_n := \frac{1}{n}$, while $x_n \to 1 \notin S(0)$.

Example 2.6. (**Upper 0-level closedness of** f_1) Let $X = \Lambda = [0,1]$, $K(\lambda) = [0,1]$ (continuous and closed), $\bar{\lambda} = 0$ and

$$f(x,y,\lambda) = \begin{cases} (x-y,\lambda) & \text{if } \lambda = 0; \\ (y-x,\lambda) & \text{if } \lambda \neq 0. \end{cases}$$

One can check that

$$S(\lambda) = Z_1(\lambda) = \begin{cases} \{1\} & \text{if } \lambda = 0; \\ \{0\} & \text{if } \lambda \neq 0. \end{cases}$$

$$Z(\lambda, x) = \{x\}, \ \ \forall(\lambda, x) \in \text{ gr } Z_1.$$

Hence, all the assumption except number (iv) hold true. However, (LEP) is not LP well-posed at $\bar{\lambda}$. Indeed, take sequences $\lambda_n := \frac{1}{n+1}$ and $x_n := 0$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an LP approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to $\{\lambda_n\}$ with $\epsilon_n := \frac{1}{n}$, while $x_n \to 0 \notin S(0)$.

Finally, we show that assumption 4 is not satisfied. Indeed, take $\{x_n\}$ and $\{\lambda_n\}$ as above and $\{y_n := 1\}$, we have $(x_n, y_n, \lambda_n) \to (0,1,0)$ and $f_1(x_n, y_n, \lambda_n) = 1 > 0$ for all n, while $f_1(0,1,0) = -1 < 0$.

Example 2.7. (**Strongly upper 0-level closedness of** f_2) Let X, Λ, K be as in Example 2.6 and

$$f(x,y,\lambda) = \begin{cases} (0, x-y) & \text{if } \lambda = 0; \\ (0, x(x-y)) & \text{if } \lambda \neq 0. \end{cases}$$

One can check that

$$Z_1(\lambda) = Z(\lambda, x) = [0,1], \ \ \forall x, \lambda \in [0,1],$$

$$S(\lambda) = \begin{cases} \{1\} & \text{if } \lambda = 0; \\ \{0,1\} & \text{if } \lambda \neq 0. \end{cases}$$

Thus, all the assumptions of Theorem 2.2 except (v) are satisfied. However, (LEP) is not LP well-posed at $\bar{\lambda}$. Indeed, take sequences $\lambda_n := \frac{1}{n+1}$ and $x_n := 0$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an LP approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to $\{\lambda_n\}$, while $x_n \to 0 \notin S(0)$. Finally, we show that assumption (iv) is not satisfied. Indeed, take sequences $x_n := 0, y_n := 1, \lambda_n := \frac{1}{n+1}$ and $\epsilon_n := \frac{1}{n}$ for all $n \in \mathbb{N}$, we have $(x_n, y_n, \lambda_n, \epsilon_n) \to (0,1,0,0)$ and $f_2(x_n, y_n, \lambda_n) + \epsilon_n > 0$ for all n, while $f_2(0,1,0)$.

Corollary 2.8. *If the conditions of the previous theorem hold then (LEP) is LP well-posed if and only if* $S(\bar{\lambda}) \neq \emptyset$ *and*

$$diam\ \Gamma(\bar{\lambda}, \alpha, \epsilon) \to 0 \ \ as \ (\alpha, \epsilon) \to (0,0).$$

Then (LEP) is LP well-posed if and only if $\Gamma(\bar{\lambda}, \alpha, \epsilon) \neq \emptyset, \forall \alpha, \epsilon > 0$ and $\text{diam } \Gamma(\bar{\lambda}, \alpha, \epsilon) \to 0$ as $(\alpha, \epsilon) \to (0,0)$.

Theorem 2.9. *Suppose that the conditions (i)-(iv) in Proposition 2.1 are satisfied. Then (LEP) is LP well-posed if and only if it has a unique solution.*

Proof. By the definition, we know that LP well-posedness for (LEP) implies it has a unique solution. For the converse, suppose that the problem (LEP) has a unique solution x'. Let $\{\lambda_n\}$ be a sequence in Λ converging to $\bar{\lambda}$ and $\{x_n\}$ an LP approximating sequence with respect to $\{\lambda_n\}$. Then, there exists a sequence $\{\epsilon_n\}$ in $(0, \infty)$ with $\epsilon_n \to 0$, as $n \to \infty$, such that

$$d(x_n, K(\lambda_n)) \leq \epsilon_n, \ \text{ for all } n \in \mathbb{N}, \tag{2.10}$$

and

$$f_1(x_n, y, \lambda_n) \geq 0, \ \forall y \in K(\lambda_n), \ \ f_2(x_n, z, \lambda_n) + \epsilon_n \geq 0, \ \ \forall z \in Z(\lambda_n, x_n). \tag{2.11}$$

By (2.10) and the closedness of $K(\lambda_n)$ in X, for each positive integer n, we can choose $x'_n \in K(\lambda_n)$ such that

$$d(x_n, x'_n) \leq \epsilon_n. \tag{2.12}$$

Since X is a compact set, the sequence $\{x'_n\}$ has a subsequence $\{x'_{n_k}\}$ which converges to a point $\bar{x} \in X$. Using (2.12), we conclude that the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $\bar{x}$. Again as K is closed at $\bar{\lambda}$, it follows that $\bar{x} \in K(\bar{\lambda})$. Proceeding along the lines of converse part in the proof of Theorem 2.2, we can show that $\bar{x} \in S(\bar{\lambda})$. Consequently, $\bar{x}$ coincides with $x'(\bar{x} = x')$. Again, by the uniqueness of the solution, it is obvious that every possible subsequence converges to the unique solution x' and hence the whole sequence $\{x_n\}$ converges to x', thus yielding the LP well-posedness of (LEP). □ □

To weaken the assumption of LP well-posednes in Theorem 2.2, we are going to use the notions of measures of noncompactness in a metric space X.

Definition 2.10. Let M be a nonempty subset of a metric space X.

(i) The *Kuratowski measure* of M is

$$\mu(M) = \inf\Big\{\varepsilon > 0 | M \subseteq \bigcup_{k=1}^{n} M_k \text{ and } \operatorname{diam} M_k \leq \varepsilon, k = 1, \ldots, n, \ \exists n \in \mathbb{N}\Big\}.$$

(ii) The *Hausdorff measure* of M is

$$\eta(M) = \inf\Big\{\varepsilon > 0 | M \subseteq \bigcup_{k=1}^{n} B(x_k, \varepsilon), x_k \in X, \text{ for some } n \in \mathbb{N}\Big\}.$$

(iii) The *Istrătescu measure* of M is

$$\iota(M) = \inf\Big\{\varepsilon > 0 | M \text{ have no infinite } \varepsilon - \text{ discrete subset } \Big\}.$$

Daneš [13] obtained the following inequalities:

$$\eta(M) \leq \iota(M) \leq \mu(M) \leq 2\eta(M). \tag{2.13}$$

The measures μ, η and ι share many common properties and we will use γ in the sequel to denote either one of them. γ is a regular measure (see [5, 38]), i.e., it enjoys the following properties.

Lemma 2.11. *Let M be a nonempty subset of a metric space X.*

(i) $\gamma(M) = +\infty$ *if and only if the set M is unbounded;*
(ii) $\gamma(M) = \gamma(clM)$;
(iii) *from $\gamma(M) = 0$ it follows that M is totally bounded set;*
(iv) *if X is a complete space and if $\{A_n\}$ is a sequence of closed subsets of X such that $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to+\infty} \gamma(A_n) = 0$, then $K := \bigcap_{n\in\mathbb{N}} A_n$ is a nonempty compact set and $\lim_{n\to+\infty} H(A_n, K) = 0$, where H is the Hausdorff metric;*
(v) *from $M \subseteq N$ it follows that $\gamma(M) \leq \gamma(N)$.*

In terms of a measure $\gamma \in \{\mu, \eta, \iota\}$ of noncompactness, we have the following result.

Theorem 2.12. *Let X and Λ be metric spaces.*

(i) *If* LEP *is LP well-posed at $\bar{\lambda}$, then $\gamma(\Gamma(\bar{\lambda}, \alpha, \varepsilon)) \downarrow 0$ as $(\alpha, \varepsilon) \downarrow (0, 0)$.*
(ii) *Conversely, suppose that $S(\bar{\lambda})$ has a unique point and $\gamma(\Gamma(\bar{\lambda}, \alpha, \varepsilon)) \downarrow 0$ as $(\alpha, \varepsilon) \downarrow (0, 0)$, and the following conditions hold*
  (a) *X is complete and Λ is compact or a finite dimensional normed space;*
  (b) *K is continuous, closed and compact-valued on Λ;*
  (c) *Z is lsc on $\Lambda \times X$;*
  (d) *f_1 is upper 0-level closed on $X \times X \times \Lambda$;*

(e) f_2 *is upper b-level closed on* $X \times X \times \Lambda$ *for every negative b close to zero.*

Then LEP *is LP well-posed at* $\bar{\lambda}$.

Proof. By the relationship (2.13) the proof is similar for the three mentioned measures of noncompactness. We discuss only the case $\gamma = \mu$, the Kuratowski measure.

(i) Suppose that (LEP) be LP-well posed at $\bar{\lambda}$.

Applying Proposition 3.2, we can conclude that $S(\bar{\lambda})$ is compact, and hence $\mu(S(\bar{\lambda})) = 0$. Let $\epsilon > 0$ and assume that

$$S(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} M_k \text{ with } \operatorname{diam} M_k \leq \epsilon \text{ for all } k = 1, \dots, n.$$

We set

$$N_k = \{y \in X | d(y, M_k) \leq H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda}))\}$$

and want to show that $\Gamma(\bar{\lambda}, \alpha, \epsilon) \subseteq \bigcup_{k=1}^{n} N_k$. For any $x \in \Gamma(\bar{\lambda}, \alpha, \epsilon)$, we have

$$d(x, S(\bar{\lambda})) \leq H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})).$$

Due to $S(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} M_k$, one has

$$d(x, \bigcup_{k=1}^{n} M_k) \leq H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})).$$

Then, there exists $\bar{k} \in \{1, 2, \dots, n\}$ such that

$$d(x, M_{\bar{k}}) \leq H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})),$$

i.e., $x \in N_{\bar{k}}$. Thus, $\Gamma(\bar{\lambda}, \alpha, \epsilon) \subseteq \bigcup_{k=1}^{n} N_k$. Because $\mu(S(\bar{\lambda})) = 0$ and

$$\operatorname{diam} N_k = \operatorname{diam} M_k + 2H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})) \leq \epsilon + 2H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})),$$

it holds

$$\mu(\Gamma(\bar{\lambda}, \alpha, \epsilon)) \leq 2H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})).$$

Note that $H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})) = H^*(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda}))$ since $S(\bar{\lambda}) \subseteq \Gamma(\bar{\lambda}, \alpha, \epsilon)$ for all $\alpha, \epsilon > 0$. Now, we claim that $H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})) \downarrow 0$ as $\alpha, \epsilon \downarrow 0$ and . Indeed, if otherwise, we can assume that there exist $r > 0$ and sequences $\alpha_n, \epsilon_n \downarrow 0$, and $\{x_n\}$ with $x_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$ such that

$$d(x_n, S(\bar{x})) \geq r, \ \ \forall n. \tag{2.14}$$

Since $\{x_n\}$ is an approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to some $\{\lambda_n\}$ with $\lambda_n \in B(\bar{\lambda}, \alpha_n) \cap \Lambda$, it has a subsequence $\{x_{n_k}\}$ converging to some $x \in S(\bar{\lambda})$, which gives a contradiction with (2.14). Therefore, we conclude that $\mu(\Gamma(\bar{\lambda}, \alpha, \epsilon))$ as $\xi \downarrow 0$ and $\varepsilon \downarrow 0$.

(ii) Suppose that $\mu(\Gamma(\bar{\lambda}, \alpha, \epsilon)) \to 0$ as $(\alpha, \epsilon) \to (0, 0)$ First, we show that $\Gamma(\bar{\lambda}, \alpha, \epsilon)$ is closed for any $\alpha, \epsilon > 0$. Let $\{x_n\} \subseteq \Gamma(\bar{\lambda}, \alpha, \epsilon)$, with $x_n \to \bar{x}$. Then for each $n \in \mathbb{N}$, there exists $\lambda_n \in B(\bar{\lambda}, \alpha) \cap \Lambda$ such that

$$d(x_n, K(\lambda_n)) \leq \epsilon$$

and

$$f_1(x_n, y, \lambda_n) \geq 0, \ \ \forall y \in K(\lambda_n) \text{ and } f_2(x_n, z, \lambda_n) + \epsilon \geq 0, \ \ \forall z \in Z(\lambda_n, x_n), \text{ for all } n \in \mathbb{N}.$$

By the assumption of Λ, this implies that $B(\bar{\lambda}, \alpha)$ is compact. We can assume $\{\lambda_n\}$ converges to some $\lambda \in B(\bar{\lambda}, \alpha) \cap \Lambda$. First, we claim that $d(\bar{x}, K(\lambda)) \leq \epsilon$. Since $K(\lambda_n)$ is compact, there exists $x_n' \in K(\lambda_n)$ such that $d(x_n, x'_n) \leq \epsilon$ for all $n \in \mathbb{N}$. By the upper continuity and compactness of K, there exists a subsequence $\{x'_{n_j}\}$ of $\{x'_n\}$ such that $x'_{n_j} \to x' \in K(\lambda)$. Consequently,

$$d(\bar{x}, K(\lambda)) \leq d(\bar{x}, x') = \lim_{n\to\infty} d(x_n, x'_n) \leq \epsilon. \tag{2.15}$$

For each $y \in K(\lambda)$, the lower semicontinuity of K at λ, there exists a sequence $\{y_n\} \subseteq K(\lambda_n)$ such that $y_n \to y$. It follows from the upper 0-level closedness of f_1 that

$$f_1(\bar{x}, y, \lambda) \geq 0;$$

that is

$$f_1(\bar{x}, y, \lambda) \geq 0, \ \forall y \in K(\lambda). \tag{2.16}$$

Next, we show that

$$f_2(\bar{x}, z, \lambda) + \epsilon \geq 0, \ \forall z \in Z(\lambda, \bar{x}). \tag{2.17}$$

Suppose to the contrary that there exists $\bar{z} \in Z(\lambda, \bar{x})$ such that

$$f_2(\bar{x}, \bar{z}, \lambda) + \epsilon < 0.$$

Since Z is lower semicontinuous at $(\lambda, \bar{x})$, we have for all n, there is $z_n \in Z(\lambda_n, x_n)$ such that $z_n \to \bar{z}$ as $n \to \infty$. It follows from the upper $(-\epsilon)$-level closedness f_2 at $(\bar{x}, \bar{z}, \lambda)$ that

$$f_2(x_n, z_n, \lambda_n) < -\epsilon$$

when n is sufficiently large which leads to a contradiction. By (2.15), (2.16) and (2.17), we can conclude that $\bar{x} \in \tilde{S}(\lambda, \epsilon)$, and so $\bar{x} \in \Gamma(\bar{\lambda}, \alpha, \epsilon)$. Therefore $\Gamma(\bar{\lambda}, \alpha, \epsilon)$ is closed for any $\alpha, \epsilon > 0$. Now we show that

$$S(\bar{\lambda}) = \bigcap_{\alpha,\epsilon>0} \Gamma(\bar{\lambda}, \alpha, \epsilon).$$

It is clear that, $S(\bar{\lambda}) \subseteq \bigcap_{\alpha,\epsilon>0} \Gamma(\bar{\lambda}, \alpha, \epsilon)$. Next, we first check that, for each $\varepsilon > 0$,

$$\bigcap_{\alpha>0} \Gamma(\bar{\lambda}, \alpha, \epsilon) \subseteq \tilde{S}(\bar{\lambda}, \epsilon).$$

For any $x \in \bigcap_{\alpha>0} \Gamma(\bar{\lambda}, \alpha, \epsilon)$. Then for each $\{\alpha_n\} \downarrow 0$, there exists a sequence $\{\lambda_n\}$ with $\lambda_n \in B(\bar{\lambda}, \alpha_n) \cap \Lambda$ such that $x \in \tilde{S}(\lambda_n, \epsilon)$ for all $n \in \mathbb{N}$, which gives that

$$d(x, K(\lambda_n)) \leq \epsilon,$$

$$f_1(x, y, \lambda_n) \geq 0, \forall y \in K(\lambda_n), \ \text{ and } \ f_2(x, z, \lambda_n) + \epsilon \geq 0, \ \forall z \in Z(\lambda_n, x).$$

Since $K(\lambda_n)$ is compact , we can choose $x_n \in K(\lambda_n)$ such that

$$d(x, x_n) \leq \epsilon, \ \forall n \in \mathbb{N}.$$

By the upper continuity and compactness of K, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x' \in K(\lambda)$, which arrives that

$$d(x, K(\bar{\lambda})) \leq d(x, x') = \lim_{n\to\infty} d(x, x_n) \leq \epsilon. \tag{2.18}$$

By assumptions on K and f_1 again, we have $x \in Z_1(\bar{\lambda})$; that is

$$f_1(x, y, \bar{\lambda}) \geq 0. \tag{2.19}$$

Next, for each $z \in Z(\bar{\lambda}, x)$, there exists $z_n \in Z(\lambda_n, x)$ such that $z_n \to z$ since Z is lsc at $(\bar{\lambda}, x)$. As $x \in \tilde{S}(\lambda_n, \epsilon)$, it holds

$$f_2(x, z_n, \lambda_n) + \epsilon \geq 0, \quad \forall n \in \mathbb{N}.$$

Since f_2 is upper -ϵ-level closed at $(x, z, \bar{\lambda})$, we have

$$f_2(x, z, \bar{\lambda}) + \epsilon \geq 0. \tag{2.20}$$

From (2.18)-(2.20), we get that $x \in \tilde{S}(\bar{\lambda}, \epsilon)$. We obtain that $\bigcap_{\alpha>0} \Gamma(\bar{\lambda}, \alpha, \epsilon) \subseteq \tilde{S}(\bar{\lambda}, \epsilon)$ for every $\varepsilon > 0$. Consequently,

$$\bigcap_{\alpha,\epsilon>0} \Gamma(\bar{\lambda}, \alpha, \epsilon) \subseteq \bigcap_{\epsilon>0} \tilde{S}(\bar{\lambda}, \epsilon) = S(\bar{\lambda}).$$

Therefore, we obtain that $S(\bar{\lambda}) = \bigcap_{\alpha,\epsilon>0} \Gamma(\bar{\lambda}, \alpha, \epsilon)$. Further, since $\mu(\Gamma(\bar{\lambda}, \alpha, \epsilon)) \to 0$ as $(\alpha, \epsilon) \to (0, 0)$. Applying Lemma 2.11 (iv), we get that $S(\bar{\lambda})$ is compact and $H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})) \longrightarrow 0$ as $(\alpha, \epsilon) \to (0, 0)$.

Finally, we prove that LEP is LP well-posedness. Indeed, let $\{x_n\}$ be an LP-approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to some $\lambda_n \longrightarrow \bar{\lambda}$. Then there exists a sequence $\{\epsilon_n\}$ in $(0, \infty)$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$d(x_n, K(\lambda_n)) \leq \epsilon_n,$$

$$f_1(x_n, y, \lambda_n) \geq 0, \ \forall y \in K(\lambda_n) \ \text{ and } \ f_2(x_n, z, \lambda_n) + \varepsilon_n \geq 0, \ \forall z \in Z(\lambda_n, x_n). \tag{2.21}$$

If we choose $\alpha_n = d(\lambda_n, \bar{\lambda})$, then $\alpha_n \to 0$ and $x_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$. We see that

$$d(x_n, S(\bar{\lambda})) \leq H(\Gamma(\bar{\lambda}, \alpha_n, \epsilon_n), S(\bar{\lambda})) \longrightarrow 0 \text{ as } n \to \infty.$$

Hence, there exist a sequence $\{\bar{x}_n\}$ in $S(\bar{\lambda})$ such that $d(x_n, \bar{x}_n) \to 0$ as $n \to \infty$. By the compactness of $S(\bar{\lambda})$, there is a subsequence $\{\bar{x}_{n_j}\}$ of $\{\bar{x}_n\}$ converging to a point $\bar{x}$ in $S(\bar{\lambda})$. Consequently, the corresponding subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converses to $\bar{x}$. Hence, LEP is LP well-posedness. The proof is completed. □

3. LP well-posedness in the generalized sense

In many practical situations, the problem (LEP) may not always possess a unique solution. Hence, in this section, we introduce a generalization of LP well-posedness for (LEP).

Definition 3.1. The problem (LEP) is said to be LP well-posed in the generalized sense at $\bar{\lambda}$ if

(i) the solution set $S(\bar{\lambda})$ is nonempty;
(ii) for any sequence $\{\lambda_n\}$ converging to $\bar{\lambda}$, every LP approximating sequence $\{x_n\}$ with respect to $\{\lambda_n\}$ has a subsequence converging to some point of $S(\bar{\lambda})$.

Proposition 3.2. *If (LEP) is LP well-posed in the generalized sense at $\bar{\lambda}$, then its solution set $S(\bar{\lambda})$ is a nonempty compact set.*

Proof. Let $\{x_n\}$ be any sequence in $S(\bar{\lambda})$. Then, of course, it is an LP approximating sequence with respect to sequences $\lambda_n := \bar{\lambda}$ and $\epsilon_n := \frac{1}{n}$, for every $n \in \mathbb{N}$. The generalized LP well-posedness of (LEP) ensures the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to a point of in $S(\bar{\lambda})$. Therefore, we conclude that $S(\bar{\lambda})$ is a nonempty compact set. The proof is completed. □ □

Next, we present a metric characterization for the generalized LP well-posedness of (LEP) in terms of the upper semicontinuity of the approximate solution set.

Theorem 3.3. *(LEP) is LP well-posed in the generalized sense if and only if $S(\bar{\lambda})$ is a nonempty, compact set and $\Gamma(\bar{\lambda}, \cdot, \cdot)$ is usc at $(\alpha, \epsilon) := (0, 0)$.*

Proof. Suppose that (LEP) is LP well-posed in the generalized sense. Therefore, $S(\bar{\lambda}) \neq \emptyset$ and further on using Proposition 3.2, we have $S(\bar{\lambda})$ is compact. Next, we assume, on the contrary, that $\Gamma(\bar{\lambda}, \alpha, \epsilon)$ is not usc at $(0,0)$. Consequently, there exist an open set U containing $\Gamma(\bar{\lambda}, 0, 0) = S(\bar{\lambda})$ and positive sequences $\{\alpha_n\}$ and $\{\epsilon_n\}$ satisfying $\alpha_n \to 0$ and $\epsilon_n \to 0$ such that

$$\Gamma(\bar{\lambda}, \alpha_n, \epsilon_n) \subsetneq U, \text{ for all } n \in \mathbb{N}.$$

Thus, there exists a sequence $\{x_n\}$ in $\Gamma(\bar{\lambda}, \alpha_n, \epsilon_n) \backslash S(\bar{\lambda})$. Therefore, of course, $\{x_n\}$ is an LP approximating sequence for (LEP), such that none of its subsequence converges to a point of $S(\bar{\lambda})$, which is a contradiction.

Conversely, let $\{\lambda_n\}$ be a sequence in Λ converging to $\bar{\lambda}$ and $\{x_n\}$ be an LP approximating sequence with respect to $\{\lambda_n\}$. If we choose a sequence $\alpha_n = d(\lambda_n, \bar{\lambda})$ then $\alpha_n \to 0$ and $x_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$. As $\Gamma(\bar{\lambda}, \alpha, \epsilon)$ is usc at $(\alpha, \epsilon) = (0, 0)$ and $S(\bar{\lambda}) \neq \emptyset$, it follows that for every $\delta > 0, \Gamma(\bar{\lambda}, \delta_n, \epsilon_n) \subset S(\bar{\lambda}) + B(0, \delta)$ for n sufficiently large. Thus $x_n \in S(\bar{\lambda}) + B(0, \delta)$, for n sufficiently large and hence there exists a sequence $\bar{x}_n \in S(\bar{\lambda})$, such that

$$d(x_n, \bar{x}_n) \leq \delta. \tag{3.1}$$

Since $S(\bar{\lambda})$ is compact, there exists a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ converging to $\bar{x} \in S(\bar{\lambda})$. Using (3.1), we conclude that the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $\bar{x} \in S(\bar{\lambda})$. □ □

The following result illustrates the fact that LP well-posedness in the generalized sense of LEP ensures the stability, in terms of the upper semi-continuity of the solution set S.

Theorem 3.4. *If (LEP) is LP well-posed in the generalized sense, then the solution mapping S is usc at $\bar{\lambda}$.*

Proof. Suppose on the contrary, S is not usc at $\bar{\lambda}$. Then there exists an open set U containning $S(\bar{\lambda})$ such that for every sequence $\lambda_n \to \bar{\lambda}$, there exists $x_n \in S(\lambda_n)$ such that $x_n \notin U$, for every n. Since $\lambda_n \to \bar{\lambda}$, $\{x_n\}$ is an LP approximating sequence for (LEP) and none of its subsequnces converge to a point of $S(\bar{\lambda})$, hence we have a contradiction to the fact that (LEP) is LP well-posed in the generalized sence. □ □

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