รายงานวิจัยฉบับสมบูรณ์

โครงการ

การให้สีกราฟในเชิงทฤษฎีเกม

Game Theoretical Variations of Graph Colorings

โดย

รองศาสตราจารย์ ดร.กิตติกร นาคประสิทธิ์

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หัวหน้าโครงการวิจัย
รองศาสตราจารย์ ดร.กิตติกร นาคประสิทธิ์
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยขอนแก่น

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและ
มหาวิทยาลัยขอนแก่น

Abstract

Project Code: RSA57- กิตติกร นาคประสิทธิ์

Project Title: Game Theoretical Variations of Graph Colorings

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This project investigates two parameters in graph colorings, namely, equitable vertex k-arboricity and game coloring number. The first parameters are used in the studies of equitable colorings and decomposition of graphs. Z. Guo, H. Zhao, Y. Mao found the exact values of equitable vertex 2-arboricity of balanced complete tripartite graphs except when the size of each partite is divisible by 20. This project finds a generalization of their results. More precisely, the project finds the exact values of equitable vertex 2-arboricity for all complete bipartite graphs and tripartite graphs.

The second parameter is of much interest in study of game coloring. There are many researches finding some bounds for game coloring numbers of planar graphs with various girths. For upper bounds, the planar graphs with girths 4, 5, 6, and at least 8 are studied. For lower bounds, the planar graphs with girths 4 and 5 are studied. This project finds the exact values for game coloring numbers of the game coloring numbers of planar graphs with girth 7 and 8.

Keywords: vertex k-arboricity, complete bipartite graphs, complete tripartite graphs, game coloring numbers, planar graphs, girth

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ

สำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยขอนแก่น ที่ได้ให้โอกาสผู้วิจัยได้รับทุนพัฒนานักวิจัยในการทำงานวิจัยครั้งนี้

คณะผู้ประเมินของวารสารวิชาการต่าง ๆ ที่ได้ให้คำแนะนำ ตลอดทั้งปรับปรุงต้นฉบับของบทความที่ส่งไปตีพิมพ์ในวารสารนั้นๆ

คณาจารย์ นักศึกษาและเจ้าหน้าที่ฝ่ายสนับสนุน ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยขอนแก่น ได้ร่วมศึกษาวิจัยและช่วยเหลือโครงการวิจัยในครั้งนี้

ผศ.ดร.เกียรติสุดา นาคประสิทธิ์ ที่ได้ช่วยศึกษาวิจัยและให้คำแนะนำในหลายประเด็น

รองศาสตราจารย์ ดร.กิตติกร นาคประสิทธิ์
หัวหน้าโครงการวิจัย

Contents

CHAPTER 1
Executive Summary

Graph coloring is one of the oldest and the most studied topics in graph theory. The wealth of its researches and applications can be referred from the outstanding monograph by Jensen and Toft. The origin of the topic may be dated back to 1852 from De Morgans letter to Hamilton about the observation of the number of colors need to color the map of England. The formal version of this question can be posed as follows: What is the least number of colors needed to color the map in the plane? The Four Color Problem (FCP) was presented formally by Kempe in 1879. However, the answer had eluded mathematicians for many years until 1976, when Appel and Haken demonstrated a proof of the FCP. The proof relies on a lot of computation by computer. Presently, the fundamental proof of the FCP is not known.

Coloring map in the FCP is equivalent to coloring the vertices of a certain planar graph. Graph colorings can be applied to graphs in general which can be classified into various classes. Furthermore, edges of graphs also can be colored. The problem of edge-coloring appeared in 1880. More general, colorings of graphs can be defined on various objects of graph e.g. vertices, edges, faces or combinations of these objects. Additionally, many rules are applied to graph colorings resulting in numerous types of graph colorings.

Not only that the topic of graph coloring has a long history, but it also has various applications in a real-world situations e.g. resource management, sequencing, time tabling, and schedule problems. Since there are numerous types of graph colorings, we can apply certain type of coloring to an appropriate situation. This project investigates two parameters in graph colorings, namely, equitable vertex k-arboricity and game coloring number. The first parameters are used in the studies of equitable colorings and decomposition of graphs. Z. Guo, H. Zhao, Y. Mao found the exact values of equitable vertex 2-arboricity of balanced complete tripartite graphs except when the size of each partite is divisible by 20. This project finds a generalization of their results. More precisely, the project finds the exact values of equitable vertex 2-arboricity for all complete bipartite graphs and tripartite graphs.

The second parameter is of much interest in study of game coloring. There are many researches finding some bounds for game coloring numbers of planar graphs with various girths. For upper bounds, the planar graphs with girths 4, 5, 6, and at least 8 are studied. For lower bounds, the planar graphs with girths 4 and 5 are studied. This project finds the exact values for game coloring numbers of the game coloring numbers of planar graphs with girth 7 and 8.

The first result is published in Information Processing Letters 117 (2017) 4044 with impact factor 0.546. The second result in in the process of the revise under review for Graphs and Combinatorics with impact factor 0.441.

CHAPTER 2
Main Results

2.1 The strong equitable vertex 2-arboricity of complete bipartite and tripartite graphs

Wu, Zhang, and Li introduced a (q, r)-*tree-coloring* of a graph G which is a q-coloring of vertices of G such that the subgraph induced by each color class is a forest of maximum degree at most r. A (q, ∞)-*tree-coloring* of a graph G is a q-coloring of G such that the subgraph induced by each color class is a forest. An *equitable* (q, r)-*tree-coloring* of a graph G is a (q, r)-tree-coloring such that the sizes of any two color classes differ by at most one. Thus, the result of Fan, Kierstead, Liu, Molla, Wu, and Zhang can be restated that every graph with maximum degree Δ has an equitable $(\Delta, 1)$-tree-coloring.

Let the *strong equitable vertex* k-*arboricity*, denoted by $va_r^{\equiv}(G)$, be the minimum p such that G has an equitable (q, r)-tree-coloring for every $q \geq p$. Wu, Zhang, and Li proved that $va_{\infty}^{\equiv}(G) \leq 3$ for each planar graph G with girth at least 5 and $va_{\infty}^{\equiv}(G) \leq 3$ for each planar graph G with girth at least 6 and for each outerplanar graph. Moreover, they gave a sharp upper bound for $va_1^{\equiv}(K_{n,n})$ in general case. They commented that finding the strong equitable 1-arboricity for every $K_{n,n}$ seems not to be an easy task.

Z. Guo, H. Zhao, Y. Mao found the exact values of $va_2^{\equiv}(K_{n,n,n})$ for each n except when n is divisible by 20. In this project, we find the exact value for each $va_2^{\equiv}(K_{m,n})$ and $va_2^{\equiv}(K_{l,m,n})$.

Lemma 2.1.1. *Let $m+n = 4b+c$ where b is a nonnegative integer and $0 \leq c \leq 3$. If $K_{m,n}$ has an equitable $(b, 2)$-tree-coloring, then $va_2^{\equiv}(K_{m,n}) = p(b : m, n)$, otherwise $va_2^{\equiv}(K_{m,n}) = b + 1$.*

Theorem 2.1.2. *$va_2^{\equiv}(K_{1,1}) = va_2^{\equiv}(K_{1,2}) = 1$ and $va_2^{\equiv}(K_{1,3}) = va_2^{\equiv}(K_{2,2}) = 2$. If $m + n = 4b + c$ where b is a positive integer and $0 \leq c \leq 3$, then we have the following.*

(1) For $c = 0$, if there are positive integers h and k such that $(m, n) = (4h, 4k)$, then $va_2^{\equiv}(K_{m,n}) = p(b : m, n)$, otherwise $va_2^{\equiv}(K_{m,n}) = b + 1$.
(2) For $c = 1$, if there are positive integers h and k such that $(m, n) = (4h+1, 4k)$ or $(4h, 4k + 1)$, then $va_2^{\equiv}(K_{m,n}) = p(b : m, n)$, otherwise $va_2^{\equiv}(K_{m,n}) = b + 1$.
(3) For $c = 2$, if there are positive integers h and k such that $(m, n) = (4(h + 1) + 2, 4k), (4h + 1, 4k + 1)$, or $(4h, 4(k + 1) + 2)$, then $va_2^{\equiv}(K_{m,n}) = p(b : m, n)$, otherwise $va_2^{\equiv}(K_{m,n}) = b + 1$.
(4) For $c = 3$, if $(m, n) = (5, 6)$ or there are positive integers h and k such that $(m, n) = (4(h + 2) + 3, 4k), (4(h + 1) + 2, 4k + 1)$, $(4h + 1, 4(k + 1) + 2)$, or $(4h, 4(k + 2) + 3)$, then $va_2^{\equiv}(K_{m,n}) = p(b : m, n)$, otherwise $va_2^{\equiv}(K_{m,n}) = b + 1$.

Lemma 2.1.3. *Let $l + m + n = 4b + c$ where b is a positive integer. If $c \leq 2$, then $K_{l,m,n}$ has an equitable $(t, 2)$-tree-coloring for each $t \geq b+1$. If $c = 3$, then $K_{l,m,n}$ has an equitable $(t, 2)$-tree-coloring for each $t \geq b + 2$.*

Lemma 2.1.4. *Assume that $l + m + n = 4b + c$ where b is a positive integer and $0 \leq c \leq 2$. If $K_{l,m,n}$ has an equitable $(b, 2)$-tree-coloring, then $va_2^{\equiv}(K_{l,m,n}) = p(b : l, m, n)$, otherwise $va_2^{\equiv}(K_{l,m,n}) = b + 1$.*

Theorem 2.1.5. *If $l + m + n = 4b + c$ where b is a positive integer and $0 \leq c \leq 2$, then we have the following.*
(1) For $c = 0$, if there are positive integers j, h, and k such that $(l, m, n) = (4j, 4h, 4k)$, then $va_2^{\equiv}(K_{l,m,n}) = p(b : l, m, n)$, otherwise $va_2^{\equiv}(K_{l,m,n}) = b + 1$.
(2) For $c = 1$, if there are positive integers j, h, and k such that $(l, m, n) = (4j+1, 4h, 4k), (4j, 4h+1, 4k)$, or $(4j, 4h, 4k+1)$, then $va_2^{\equiv}(K_{l,m,n}) = p(b : l, m, n)$, otherwise $va_2^{\equiv}(K_{l,m,n}) = b + 1$.
(3) For $c = 2$, if there are positive integers j, h, and k such that $(l, m, n) = (4(j + 1) + 2, 4h, 4k)$, $(4j, 4(h + 1) + 2, 4k), (4j, 4h, 4(k + 1) + 2)$, $(4j + 1, 4h + 1, 4k), (4j+1, 4h, 4k+1)$, or $(4j, 4h+1, 4k+1)$, then $va_2^{\equiv}(K_{l,m,n}) = p(b : l, m, n)$, otherwise $va_2^{\equiv}(K_{l,m,n}) = b + 1$.

Definition 2.1.6. *We say that (l, m, n) satisfies "Condition A" if there are positive integers j, h, and k such that $(l, m, n) = (4j, 4h, 4k - 1), (4j, 4h - 1, 4k), (4j - 1, 4h, 4k)$, $(4j, 4h-2, 4k-3), (4j, 4h-3, 4k-2), (4j-2, 4h, 4k-3)$, $(4j-2, 4h-3, 4k), (4j - 3, 4h, 4k - 2)$, or $(4j - 3, 4h - 2, 4k)$.*

Lemma 2.1.7. *Let $l + m + n = 4b + 3$ where b is a nonnegative integer. We have $K_{l,m,n}$ has an equitable $(b+1, 2)$-tree-coloring if and only if (l, m, n) satisfies condition A.*

Definition 2.1.8. *We say that* (l, m, n) *satisfies "Condition B" if there are positive integers* $j, h,$ *and* k *such that* $(l, m, n) = (4(j+2)+3, 4h, 4k), (4j, 4(h+2)+3, 4k), (4j, 4h, 4(k+2)+3,\ (4(j+1)+2, 4h+1, 4k), (4(j+1)+2, 4h, 4k+1), (4j+1, 4(h+1)+2, 4k), (4j+1, 4h, 4(k+1)+2),\ (4j, 4(h+1)+2, 4k+1), (4j, 4h+1, 4(k+1)+2),$ *or* $(4j+1, 4h+1, 4k+1)$.

Lemma 2.1.9. *Assume that* $l+m+n = 4b+3$ *where* b *is a positive integer. We have* $K_{l,m,n}$ *has an equitable* $(b, 2)$*-tree-coloring if and only if* (l, m, n) *satisfies Condition B.*

Lemma 2.1.10. *Let* $l+m+n = 4b+3$ *where* b *is a positive integer. We have the following.*
(1) $K_{l,m,n}$ *has no equitable* $(b+1, 2)$*-tree-coloring if and only if* $va_2^{\equiv}(K_{l,m,n}) = b+2$.
(2) Assume that $K_{l,m,n}$ *has an equitable* $(b+1, 2)$*-tree-coloring. If* $K_{l,m,n}$ *has an equitable* $(b, 2)$*-tree-coloring, then* $va_2^{\equiv}(K_{l,m,n}) = p(b : l, m, n)$*, otherwise* $va_2^{\equiv}(K_{l,m,n}) = b+1$.

Theorem 2.1.11. $va_2^{\equiv}(K_{1,1,1}) = 2$*. Assume that* $l+m+n = 4b+3$ *where* b *is a positive integer. Then we have the following.*
(i) If (l, m, n) *does not satisfy Condition A, then* $va_2^{\equiv}(K_{l,m,n}) = b+2$.
(ii) If (l, m, n) *satisfies Condition A but does not satisfy Condition B, then* $va_2^{\equiv}(K_{l,m,n}) = b+1$.
(iii) If (l, m, n) *satisfies Condition A and Condition B, then* $va_2^{\equiv}(K_{l,m,n}) = p(b : l, m, n)$.

2.2 The game coloring number of planar graphs with a specific girth

Let G be a simple graph with a vertex set $V(G)$ and an edge set $E(G)$. The *coloring game* is a two-person game described as follows. Two players, say Alice and Bob, with Alice playing first, alternatively colors an uncolored vertex in G with the color from the color set C so that any two adjacent vertices have distinct colors. Alice wins if all vertices are colored. The *game chromatic number* of G, denoted by $\chi_g(G)$, is the least cardinality of C in which Alice has a strategy to win the game. The game chromatic number was formally introduced by Bodlaender.

The *marking game* is also a two-person game. Two players, say Alice and Bob, with Alice playing first alternatively marks an unmarked vertex of G

until all vertices are marked. Let $b(v)$ be the number of neighbors of v that are marked before v is marked. The *game coloring number* of G, denoted by $\mathrm{col_g}(G)$, is the least s in which Alice has a strategy to keep $b(v)+1 \leq s$ for each vertex v.

The game coloring number was formally introduced by Zhu as a tool to study the game chromatic number. It is easy to see that $\chi_g(G) \leq \mathrm{col_g}(G)$. Most of the best known upper bounds for game chromatic numbers of graphs in various families are obtained from the upper bounds of game coloring numbers.

Let $\mathcal{H}$ be a family of graphs. The game chromatic number and the game coloring number of $\mathcal{H}$ are defined as $\chi_G(\mathcal{H}) := \max\{\chi_g(G) : G \in \mathcal{H}\}$ and $\mathrm{col_g}(\mathcal{H}) := \max\{\mathrm{col_g}(G) : G \in \mathcal{H}\}$.

The game coloring numbers of various families of graphs, especially planar graphs, are widely studied. Let $\mathcal{F}$ denote the family of forests, $\mathcal{I}_k$ denote the family of interval graphs with clique number k, $\mathcal{Q}$ denote the family of outerplanar graphs, $\mathcal{PT}_k$ denote the family of partial k-trees, and $\mathcal{P}$ denote the family of planar graphs. It is proved by Faigle et al. that $\chi_g(\mathcal{F}) = \mathrm{col_g}(\mathcal{F}) = 4$, by Faigle et al. and Kierstead and Yang that $\mathrm{col_g}(\mathcal{I}_k) = 3k-2$, by Guan and Zhu and Kierstead and Yang that $\mathrm{col_g}(\mathcal{Q}) = 7$, and by Zhu and Wu and Zhu that $\mathrm{col_g}(\mathcal{PT}_k) = 3k+2$ for $k \geq 2$.

Combining a lower bound from and an upper bound from gives $11 \leq \mathrm{col_g}(\mathcal{P}) \leq 17$. Let $\mathcal{P}_k$ be the family of planar graphs of girth at least k. It is proved by Sekiguchi that $\mathrm{col_g}(\mathcal{P}_4) \leq 13$, by He et al. that $\mathrm{col_g}(\mathcal{P}_5) \leq 8$, by Kleitman that $\mathrm{col_g}(\mathcal{P}_6) \leq 6$, by Wang and Zhang that $\mathrm{col_g}(\mathcal{P}_8) \leq 5$, and by Borodin et al. that $\mathrm{col_g}(G) \leq 9$ if G is a quadrangle-free planar graph. For lower bounds, it was proved by Sekiguchi that $\mathrm{col_g}(\mathcal{P}_4) \geq 7$ and $\mathrm{col_g}(\mathcal{P}_5) \geq 6$.

In this project, we show that $\mathrm{col_g}(\mathcal{P}_7) \leq 5$. This result extends a result about the coloring number by Wang and Zhang. We also show that this bound is sharp by constructing a graph G where $G \in \mathcal{P}_k$ for each $k \leq 8$ such that $\mathrm{col_g}(G) = 5$. As a consequence, $\mathrm{col_g}(\mathcal{P}_k) = 5$ for $k = 7, 8$.

Theorem 2.2.1. *If G is a planar graph with girth at least 7, then $\mathrm{col_g}(G) \leq 5$.*

Theorem 2.2.2. *For $3 \leq k \leq 8$, there is a planar graph G with girth k such that $\mathrm{col_g}(G) = 5$. As a consequence, $\mathrm{col_g}(\mathcal{P}_k) \geq 5$.*

Corollary 2.2.3. $\mathrm{col_g}(\mathcal{G}_k) = 5$ *for* $k = 7, 8$.

Output จากโครงการวิจัยที่ได้รับทุนจาก สกว.

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ (ระบุชื่อผู้แต่ง ชื่อเรื่อง ชื่อวารสาร ปี เล่มที่ เลขที่ และหน้า) หรือผลงานตามที่คาดไว้ในสัญญาโครงการ

   1.1 Keaitsuda Maneeruk Nakprasit, Kittikorn Nakprasit, Equitable vertex 2-arboricity of complete bipartite and tripartite graphs, Information Processing Letters 117 (2017) 40–44.

   1.2 Keaitsuda Maneeruk Nakprasit, Kittikorn Nakprasit, The game coloring number of planar graphs with a specific girth, Graphs and Combinatorics, Revised

2. การนำผลงานวิจัยไปใช้ประโยชน์

   ผลงานวิจัยที่ได้มามีการนำไปใช้ประโยชน์ทั้งเชิงวิชาการ และเชิงสาธารณะโดยทำให้มีการพัฒนาการเรียนการสอนและมีเครือข่ายความร่วมมือการวิจัยสร้างความสนใจในวงกว้าง

Appendix

A1 Keaitsuda Maneeruk Nakprasit, Kittikorn Nakprasit, Equitable vertex 2-arboricity of complete bipartite and tripartite graphs, Information Processing Letters 117 (2017) 40–44.

A2 Keaitsuda Maneeruk Nakprasit, Kittikorn Nakprasit, The game coloring number of planar graphs with a specific girth, Graphs and Combinatorics, Revised

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A1 Keaitsuda Maneeruk Nakprasit, Kittikorn Nakprasit, Equitable vertex 2-arboricity of complete bipartite and tripartite graphs, Information Processing Letters 117 (2017) 40–44.

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A1 Keaitsuda Maneeruk Nakprasit, Kittikorn Nakprasit, Equitable vertex 2-arboricity of complete bipartite and tripartite graphs, Information Processing Letters 117 (2017) 40–44

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The strong equitable vertex 2-arboricity of complete bipartite and tripartite graphs

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ABSTRACT

A (q,r)-*tree-coloring* of a graph G is a q-coloring of vertices of G such that the subgraph induced by each color class is a forest of maximum degree at most r. An *equitable* (q,r)-*tree-coloring* of a graph G is a (q,r)-tree-coloring such that the sizes of any two color classes differ by at most one. Let the *strong equitable vertex r-arboricity* of G, denoted by $va_r^{\equiv}(G)$, be the minimum p such that G has an equitable (q,r)-tree-coloring for every $q \geq p$.

Z. Guo, H. Zhao, Y. Mao [4] found the exact values of $va_2^{\equiv}(K_{n,n,n})$ for each n except when n is divisible by 20. In this paper, we find the exact value for each $va_2^{\equiv}(K_{m,n})$ and $va_2^{\equiv}(K_{l,m,n})$.

1. Introduction

Throughout this paper, all graphs are finite, undirected, and simple. We use $V(G)$ and $E(G)$, respectively, to denote the vertex set and the edge set of a graph G. For a complete bipartite graph $K_{m,n}$ where $m \leq n$, we let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ be the partite sets of $K_{m,n}$. For a complete tripartite graph $K_{l,m,n}$ where $l \leq m \leq n$, we have $X = \{x_1, \ldots, x_l\}$, $Y = \{y_1, \ldots, y_m\}$, and $Z = \{z_1, \ldots, z_n\}$, to be the partite sets of $K_{l,m,n}$.

An *equitable k-coloring* of a graph is a proper vertex k-coloring such that the sizes of every two color classes differ by at most 1.

It is known [5] that determining if a planar graph with maximum degree 4 is 3-colorable is NP-complete. For a given n-vertex planar graph G with maximum degree 4, let G' be the graph obtained from G by adding $2n$ isolated vertices. Then G has a 3-coloring if and only if G' has an equitable 3-coloring. Thus, finding the minimum number of colors needed to color a graph equitably, even for a planar graph, is an NP-complete problem.

Hajnal and Szemerédi [6] settled a conjecture of Erdős by proving that every graph G with maximum degree at most Δ has an equitable k-coloring for every $k \geq 1 + \Delta$. This result is now known as Hajnal and Szemerédi Theorem. Later, Kierstead and Kostochka [7] gave a simpler proof of Hajnal and Szemerédi Theorem. The bound of the Hajnal–Szemerédi theorem is sharp, but it can be improved for some important classes of graphs. In fact, Chen, Lih, and Wu [1] put forth the following conjecture.

Conjecture 1. *Every connected graph G with maximum degree $\Delta \geq 2$ has an equitable coloring with Δ colors, except when G is a complete graph or an odd cycle or Δ is odd and $G = K_{\Delta,\Delta}$.*

Lih and Wu [10] proved the conjecture for bipartite graphs. Meyer [11] proved that every forest with maximum degree Δ has an equitable k-coloring for each $k \geq 1 + \lceil \Delta/2 \rceil$. This result implies the conjecture holds for forests. Yap and Zhang [16] proved that the conjecture holds for outerplanar graphs. Later Kostochka [8] improved the result by proving that every outerplanar graph with

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maximum degree Δ has an equitable k-coloring for each $k \geq 1 + \lceil \Delta/2 \rceil$.

In [18], Zhang and Yap essentially proved that the conjecture holds for planar graphs with maximum degree at least 13. Later Nakprasit [12] extended the result to all planar graphs with maximum degree at least 9. Some related results are about planar graphs without some restricted cycles [9,13,19].

Moreover, the conjecture has been confirmed for other classes of graphs, such as graphs with degree at most 3 [1, 2] and series-parallel graphs [17].

In contrast with ordinary coloring, a graph may have an equitable k-coloring but has no equitable $(k+1)$-coloring. For example, $K_{7,7}$ has an equitable k-coloring for $k = 2, 4, 6$ and $k \geq 8$, but has no equitable k-coloring for $k = 3, 5$ and 7. This leads to the definition of the *equitable chromatic threshold* which is the minimum p such that G has an equitable q-coloring for every $q \geq p$.

In [3], Fan, Kierstead, Liu, Molla, Wu, and Zhang considered an equitable relaxed coloring. They proved that every graph with maximum degree Δ has an equitable Δ-coloring, such that each color class induces a forest with maximum degree at most one.

On the basis of the aforementioned research, Wu, Zhang, and Li [15] introduced a (q,r)*-tree-coloring* of a graph G which is a q-coloring of vertices of G such that the subgraph induced by each color class is a forest of maximum degree at most r. A (q,∞)*-tree-coloring* of a graph G is a q-coloring of G such that the subgraph induced by each color class is a forest. An *equitable* (q,r)*-tree-coloring* of a graph G is a (q,r)-tree-coloring such that the sizes of any two color classes differ by at most one. Thus, the result of Fan, Kierstead, Liu, Molla, Wu, and Zhang can be restated that every graph with maximum degree Δ has an equitable $(\Delta, 1)$-tree-coloring.

Let the *strong equitable vertex k-arboricity*, denoted by $va_r^{\equiv}(G)$, be the minimum p such that G has an equitable (q,r)-tree-coloring for every $q \geq p$. Wu, Zhang, and Li [15] proved that $va_\infty^{\equiv}(G) \leq 3$ for each planar graph G with girth at least 5 and $va_\infty^{\equiv}(G) \leq 3$ for each planar graph G with girth at least 6 and for each outerplanar graph. Moreover, they gave a sharp upper bound for $va_1^{\equiv}(K_{n,n})$ in general case. They commented that finding the strong equitable 1-arboricity for every $K_{n,n}$ seems not to be an easy task.

In [4], Z. Guo, H. Zhao, Y. Mao found the exact values of $va_2^{\equiv}(K_{n,n,n})$ for each n except when n is divisible by 20. In this paper, we find the exact value for each $va_2^{\equiv}(K_{m,n})$ and $va_2^{\equiv}(K_{l,m,n})$.

In Section 2, we introduce lemmas which are useful for finding $va_2^{\equiv}(K_{m,n})$ in Section 3 and $va_2^{\equiv}(K_{l,m,n})$ in Section 4.

2. Useful lemmas

We introduce the notion of $p(q : n_1, \ldots, n_k)$ which can be computed in linear-time.

Definition 1. Assume that $G = K_{n_1,\ldots,n_k}$ has an equitable q-coloring, and d is the minimum value greater than $\lfloor (n_1 + \cdots + n_k)/q \rfloor$ such that (i) there are distinct i and j in which n_i and n_j are not divisible by d, or (ii) there is n_j with $n_j/\lfloor n_j/d \rfloor > d + 1$. Define $p(q : n_1, \ldots, n_k) = \lceil n_1/d \rceil + \cdots + \lceil n_k/d \rceil$.

Theorem 1. *[14] Assume that $G = K_{n_1,\ldots,n_k}$ has an equitable q-coloring. Then $p(q : n_1, \ldots, n_k)$ is the minimum p such that G is equitable r-colorable for each r satisfying $p \leq r \leq q$.*

Lemma 2. *Let G be a complete multipartite graph with n vertices. If the size of a color class from a $(q,2)$-tree-coloring of G is at least 4, then the color class is independent. Consequently, each equitable $(q,2)$-tree-coloring of G such that $n/q \geq 4$ is a proper equitable coloring.*

Proof. Suppose to the contrary that a color class C of size $k \geq 4$ is not an independent set. Then C induces $K_{1,k-1}$ or a graph with a cycle, a contradiction. The remaining of the Lemma follows immediately. □

Lemma 3. *Let $G = K_{n_1,\ldots,n_k}$ and $N = n_1 + \cdots + n_k$. Assume that G has an equitable q-coloring where $N/(q-1) \geq 4$ and G has an equitable $(r,2)$-tree-coloring for each $r \geq q$. Then $va_2^{\equiv}(G) = p(q : n_1, \ldots, n_k)$.*

Proof. Let $p = p(q : n_1, \ldots, n_k)$. From the definition of p and the condition of q, the graph G has an equitable $(r,1)$-tree-coloring for each $r \geq p$. To complete the proof, it suffices to show that G has no equitable $(p-1,2)$-tree-coloring. Suppose to the contrary that G has an equitable $(p-1,2)$-tree-coloring. Since $p - 1 \leq q - 1$, each color class has size at least $N/(p-1) \geq N/(q-1) \geq 4$. Lemma 2 yields that G has a proper equitable $(p-1)$-coloring. But this contradicts to Theorem 1. □

Let $G = K_{m,n}$ or $K_{l,m,n}$. We introduce an algorithm to construct a $(q,2)$-tree-coloring of G. The first key idea is that we arrange vertices of G in a way that vertices in the same partite set are consecutively ordered. Then we partition $V(G)$ in a way that each of q partitioned sets (color classes) contains k or $k+1$ consecutive vertices (where $k = \lfloor |V(G)|/q \rfloor$) from the arrangement. By this method, there are at most one non-independent color class in $K_{m,n}$, and at most two non-independent color classes in $K_{l,m,n}$.

The second key idea is that we want each non-independent color class to have size at most 3. The final key idea is that we want vertices in each non-independent color class to come from exactly two partite sets. To achieve this objective for any $K_{l,m,n}$ except $K_{1,1,1}$, we have vertices in the partite set Z (a largest partite set) in the middle of the arrangement.

If a coloring satisfies all of these three key ideas, then each non-independent color class induces a tree of maximum degree at most 2. If the sizes of any two color classes differ by at most one, then we have an equitable $(q,2)$-tree-coloring. Now we summarize a desired algorithm to obtain an equitable $(q,2)$-tree-coloring as follows.

Definition 2 *(Algorithm A).* Let $N = |V(G)|$ and $k = \lfloor N/q \rfloor \leq 3$. Assume $q > N/4$ for $G = K_{m,n}$ and $q > (N+1)/4$ for $G = K_{l,m,n}$.

Arrange vertices of G in a way that vertices in the same partite set are consecutively ordered. Partition $V(G)$ in a way that each of q equitable partitioned sets (color classes) contains k or $k+1$ consecutive vertices from the arrangement. Furthermore, if we need to have a color class span two partite sets, we always use a class of size at most 3.

3. $va_2^{\equiv}(K_{m,n})$

Lemma 4. *Let $m+n=4b+c$ where b is a nonnegative integer and $0 \le c \le 3$. Then $K_{m,n}$ has an equitable $(t,2)$-tree-coloring for each $t \ge b+1$.*

Proof. Let $m+n=4b+c=rk+s(k+1)$ where r is a positive integer, s and k are nonnegative integers. First, consider the case $r+s=b+1$. Then $k \le 3$. If $k \le 2$, then each color class from Algorithm A is an independent set or induces K_2, or $K_{1,2}$. Thus, we obtain an equitable $(b+1,2)$-tree-coloring.

Now, we assume $k=3$. Consequently, $c=0,1,2$, or 3. If $c=0$, then $r=4$, $s=b-3$, and $b \ge 3$. If $c=1$, then $r=3$, $s=b-2$, and $b \ge 2$. If $c=2$, then $r=2$, $s=b-1$, and $b \ge 1$. If $c=3$, then $r=1$ and $s=b$.

We show that Algorithm A yields an equitable $(b+1,2)$-tree-coloring. By Algorithm A, a non-independent color class (if exists) consists of three vertices from two partite sets. Then each color class is independent set or induces $K_{1,2}$. Thus we obtain an equitable $(b+1,2)$-tree-coloring.

Finally, consider the case that $r+s \ge b+2$. Again we have (i) $k \le 2$ or (ii) $k=3$ and $r \ge 2$. Similar to the case of $r+s=b+1$, we can use Algorithm A to obtain an equitable $(r+s,2)$-tree-coloring. This completes the proof. □

Lemma 5. *Let $m+n=4b+c$ where b is a nonnegative integer and $0 \le c \le 3$. If $K_{m,n}$ has an equitable $(b,2)$-tree-coloring, then $va_2^{\equiv}(K_{m,n})=p(b:m,n)$, otherwise $va_2^{\equiv}(K_{m,n})=b+1$.*

Proof. From Lemma 4, $K_{m,n}$ has an equitable $(t,2)$-tree-coloring for each $t \ge b+1$.

If $K_{m,n}$ has no equitable $(b,2)$-tree-coloring, then $va_2^{\equiv}(K_{m,n})=b+1$ by definition of $va_2^{\equiv}(K_{m,n})$.

Assume $K_{m,n}$ has an equitable $(b,2)$-tree-coloring. Then each color class has size at least 4. By Lemma 2, such equitable $(b,2)$-tree-coloring is a proper equitable b-coloring. Thus $va_2^{\equiv}(K_{m,n})=p(b:m,n)$ by Lemma 3. □

Theorem 6. *$va_2^{\equiv}(K_{1,1})=va_2^{\equiv}(K_{1,2})=1$ and $va_2^{\equiv}(K_{1,3})=va_2^{\equiv}(K_{2,2})=2$. If $m+n=4b+c$ where b is a positive integer and $0 \le c \le 3$, then we have the following.*

(1) For $c=0$, if there are positive integers h and k such that $(m,n)=(4h,4k)$, then $va_2^{\equiv}(K_{m,n})=p(b:m,n)$, otherwise $va_2^{\equiv}(K_{m,n})=b+1$.
(2) For $c=1$, if there are positive integers h and k such that $(m,n)=(4h+1,4k)$ or $(4h,4k+1)$, then $va_2^{\equiv}(K_{m,n})=p(b:m,n)$, otherwise $va_2^{\equiv}(K_{m,n})=b+1$.
(3) For $c=2$, if there are positive integers h and k such that $(m,n)=(4(h+1)+2,4k)$, $(4h+1,4k+1)$, or $(4h,4(k+1)+2)$, then $va_2^{\equiv}(K_{m,n})=p(b:m,n)$, otherwise $va_2^{\equiv}(K_{m,n})=b+1$.
(4) For $c=3$, if $(m,n)=(5,6)$ or there are positive integers h and k such that $(m,n)=(4(h+2)+3,4k)$, $(4(h+1)+2,4k+1)$, $(4h+1,4(k+1)+2)$, or $(4h,4(k+2)+3)$, then $va_2^{\equiv}(K_{m,n})=p(b:m,n)$, otherwise $va_2^{\equiv}(K_{m,n})=b+1$.

Proof. It is easy to see that $va_2^{\equiv}(K_{1,1})=va_2^{\equiv}(K_{1,2})=1$ and $va_2^{\equiv}(K_{1,3})=va_2^{\equiv}(K_{2,2})=2$. Now consider the part $m+n=4b+c$ where b is a positive integer and $0 \le c \le 3$. Since $(m+n)/b \ge 4$, Lemma 2 yields that $K_{m,n}$ has an equitable $(b,2)$-tree-coloring if and only if $K_{m,n}$ has a proper equitable b-coloring. Thus each color class from an equitable $(b,2)$-tree-coloring of $K_{m,n}$ is an independent set. Moreover, each color class is in X or Y.

Case 1: $c=0$. An equitable $(b,2)$-tree-coloring of $K_{m,n}$ yields b color classes of size 4. This can happen if and only if there are positive integers h and k such that $(m,n)=(4h,4k)$.

Case 2: $c=1$. An equitable $(b,2)$-tree-coloring of $K_{m,n}$ yields $b-1$ color classes of size 4 and 1 color class of size 5. This can happen if and only if there are positive integers h and k such that $(m,n)=(4h+1,4k)$ or $(4h,4k+1)$.

Case 3: $c=2$.

Subcase 3.1: $b=1$. Then $m+n=6$. One can see that $va_2^{\equiv}(K_{m,n})=2$.

Subcase 3.2: $b \ge 2$. An equitable $(b,2)$-tree-coloring of $K_{m,n}$ yields $b-2$ color classes of size 4 and 2 color classes of size 5. This can happen if and only if there are positive integers h and k such that $(m,n)=(4(h+1)+2,4k)$, $(4h+1,4k+1)$, or $(4h,4(k+1)+2)$.

Case 4: $c=3$.

Subcase 4.1: $b=1$. Then $m+n=7$. One can see that $va_2^{\equiv}(K_{m,n})=2=b+1$.

Subcase 4.2: $b=2$. Then $m+n=11$. Lemma 4 yields that $K_{m,n}$ has an equitable $(q,2)$-tree-coloring for every $q \ge b+1=3$. On the other hand, an equitable $(b,2)$-tree-coloring (that is an equitable $(2,2)$-tree-coloring) of $K_{m,n}$ yields 1 color class of size 5 and 1 color class of size 6. By Lemma 2, each color class is independent. This can happen if and only if $(m,n)=(5,6)$.

Subcase 4.3: $b \ge 3$. An equitable $(b,2)$-tree-coloring of $K_{m,n}$ has $b-3$ color classes of size 4 and 3 color classes of size 5. By Lemma 2, each color class is independent. This can happen if and only if there are positive integers h and k such that $(m,n)=(4(h+2)+3,4k)$, $(4(h+1)+2,4k+1)$, $(4h+1,4(k+1)+2)$, or $(4h,4(k+2)+3)$.

Combining these facts with Lemma 5, we complete the proof. □

4. $va_2^{\equiv}(K_{l,m,n})$

Lemma 7. *Let $l+m+n=4b+c$ where b is a positive integer. If $c \le 2$, then $K_{l,m,n}$ has an equitable $(t,2)$-tree-coloring for each $t \ge b+1$. If $c=3$, then $K_{l,m,n}$ has an equitable $(t,2)$-tree-coloring for each $t \ge b+2$.*

Proof. For $c \le 2$, the proof is similar to that of Lemma 4. Now we assume $c=3$. Let $l+m+n=4b+3=rk+s(k+1)$ where r is a positive integer, s and k are nonnegative integers. First consider the case $r+s=b+2$. Then (i) $k \le 2$

or (ii) $r = 5$, $s = b - 3$, $k = 3$ and $b \geq 3$. Again we can use Algorithm A to obtain an equitable $(b + 2, 2)$-tree-coloring.

Finally, consider the case that $r + s \geq b + 3$. Then (i) $k \leq 2$ or (ii) $k = 3$ and $r \geq 5$. Again we can use Algorithm A to obtain an equitable $(r + s, 2)$-tree-coloring. This completes the proof. $\square$

Lemma 8. *Assume that $l + m + n = 4b + c$ where b is a positive integer and $0 \leq c \leq 2$. If $K_{l,m,n}$ has an equitable $(b, 2)$-tree-coloring, then $va_2^{\equiv}(K_{l,m,n}) = p(b : l, m, n)$, otherwise $va_2^{\equiv}(K_{l,m,n}) = b + 1$.*

Proof. From Lemma 7, $K_{l,m,n}$ has an equitable $(t, 2)$-tree-coloring for each $t \geq b + 1$. By definition of $va_2^{\equiv}(K_{l,m,n})$, we have $K_{l,m,n}$ has no equitable $(b, 2)$-tree-coloring if and only if $va_2^{\equiv}(K_{l,m,n}) = b + 1$.

Assume $K_{l,m,n}$ has an equitable $(b, 2)$-tree-coloring. Then each color class has size at least 4. By Lemma 2, such equitable $(b, 2)$-tree-coloring is a proper equitable b-coloring. Thus $va_2^{\equiv}(K_{l,m,n}) = p(b : l, m, n)$ by Lemma 3.

Assume $K_{l,m,n}$ has no equitable $(b, 2)$-tree-coloring. Using Lemma 8, we have $va_2^{\equiv}(K_{l,m,n}) = b + 1$. This completes the proof. $\square$

Theorem 9. *If $l + m + n = 4b + c$ where b is a positive integer and $0 \leq c \leq 2$, then we have the following.*

(1) For $c = 0$, if there are positive integers j, h, and k such that $(l, m, n) = (4j, 4h, 4k)$, then $va_2^{\equiv}(K_{l,m,n}) = p(b : l, m, n)$, otherwise $va_2^{\equiv}(K_{l,m,n}) = b + 1$.
(2) For $c = 1$, if there are positive integers j, h, and k such that $(l, m, n) = (4j + 1, 4h, 4k), (4j, 4h + 1, 4k)$, or $(4j, 4h, 4k + 1)$, then $va_2^{\equiv}(K_{l,m,n}) = p(b : l, m, n)$, otherwise $va_2^{\equiv}(K_{l,m,n}) = b + 1$.
(3) For $c = 2$, if there are positive integers j, h, and k such that $(l, m, n) = (4(j + 1) + 2, 4h, 4k), (4j, 4(h + 1) + 2, 4k), (4j, 4h, 4(k + 1) + 2), (4j + 1, 4h + 1, 4k), (4j + 1, 4h, 4k + 1)$, or $(4j, 4h + 1, 4k + 1)$, then $va_2^{\equiv}(K_{l,m,n}) = p(b : l, m, n)$, otherwise $va_2^{\equiv}(K_{l,m,n}) = b + 1$.

Proof. Since $(l + m + n)/b \geq 4$, Lemma 2 yields that $K_{l,m,n}$ has an equitable $(b, 2)$-tree-coloring if and only if $K_{l,m,n}$ has a proper equitable b-coloring. Thus each color class from an equitable $(b, 2)$-tree-coloring of $K_{l,m,n}$ is an independent set.

Case 1: $c = 0$. An equitable $(b, 2)$-tree-coloring of $K_{l,m,n}$ yields b color classes of size 4. By Lemma 2, each color class is independent. That is each color class is in a partite set. This can happen if and only if there are positive integers j, h, and k such that $(l, m, n) = (4j, 4h, 4k)$.

Case 2: $c = 1$. An equitable $(b, 2)$-tree-coloring of $K_{l,m,n}$ yields $b - 1$ color classes of size 4 and 1 color class of size 5. By Lemma 2, each color class is independent. That is each color class is in a partite set. This can happen if and only if there are positive integers j, h, and k such that $(l, m, n) = (4j + 1, 4h, 4k), (4j, 4h + 1, 4k)$, or $(4j, 4h, 4k + 1)$.

Case 3: $c = 2$.

Subcase 3.1: $b = 1$. Then $l + m + n = 6$. One can see that $va_2^{\equiv}(K_{l,m,n}) = 2$.

Subcase 3.2: $b \geq 2$. An equitable $(b, 2)$-tree-coloring of $K_{l,m,n}$ yields $b - 2$ color classes of size 4 and 2 color classes of size 5. By Lemma 2, each color class is independent. That is each color class is in a partite set. This can happen if and only if there are positive integers j, h, and k such that $(l, m, n) = (4(j + 1) + 2, 4h, 4k), (4j, 4(h + 1) + 2, 4k), (4j, 4h, 4(k + 1) + 2), (4j + 1, 4h + 1, 4k), (4j + 1, 4h, 4k + 1)$, or $(4j, 4h + 1, 4k + 1)$.

Combining these facts with Lemma 8, we complete the proof. $\square$

Definition 3. We say that (l, m, n) satisfies "Condition A" if there are positive integers j, h, and k such that $(l, m, n) = (4j, 4h, 4k - 1), (4j, 4h - 1, 4k), (4j - 1, 4h, 4k), (4j, 4h - 2, 4k - 3), (4j, 4h - 3, 4k - 2), (4j - 2, 4h, 4k - 3), (4j - 2, 4h - 3, 4k), (4j - 3, 4h, 4k - 2)$, or $(4j - 3, 4h - 2, 4k)$.

Lemma 10. *Let $l + m + n = 4b + 3$ where b is a nonnegative integer. We have $K_{l,m,n}$ has an equitable $(b + 1, 2)$-tree-coloring if and only if (l, m, n) satisfies condition A.*

Proof. Consider an equitable $(b + 1, 2)$-tree-coloring of G. Then there are b color classes of size 4 and 1 color class of size 3. Lemma 2 yields that each color class of size 4 is independent. By definition of a $(q, 2)$-tree-coloring, a color class of size 3, say C, is an independent set or C induces $K_{1,2}$.

The case that C is an independent set can happen if and only if there are positive integers j, h, and k such that $(l, m, n) = (4j, 4h, 4k - 1), (4j, 4h - 1, 4k)$, or $(4j - 1, 4h, 4k)$.

The case that C induces $K_{1,2}$ can happen if and only if one element of C is in one partite set and two other elements are in a different partite set. Thus the case that C induces $K_{1,2}$ can happen if and only if there are positive integers j, h, and k such that $(4j, 4h - 2, 4k - 3), (4j, 4h - 3, 4k - 2), (4j - 2, 4h, 4k - 3), (4j - 2, 4h - 3, 4k), (4j - 3, 4h, 4k - 2)$, or $(4j - 3, 4h - 2, 4k)$. This completes the proof. $\square$

Definition 4. We say that (l, m, n) satisfies "Condition B" if there are positive integers j, h, and k such that $(l, m, n) = (4(j + 2) + 3, 4h, 4k), (4j, 4(h + 2) + 3, 4k), (4j, 4h, 4(k + 2) + 3, (4(j + 1) + 2, 4h + 1, 4k), (4(j + 1) + 2, 4h, 4k + 1), (4j + 1, 4(h + 1) + 2, 4k), (4j + 1, 4h, 4(k + 1) + 2), (4j, 4(h + 1) + 2, 4k + 1), (4j, 4h + 1, 4(k + 1) + 2)$, or $(4j + 1, 4h + 1, 4k + 1)$.

Lemma 11. *Assume that $l + m + n = 4b + 3$ where b is a positive integer. We have $K_{l,m,n}$ has an equitable $(b, 2)$-tree-coloring if and only if (l, m, n) satisfies Condition B.*

Proof. For $b = 1$ or 2, one can check that G does not have an equitable $(b, 2)$-tree-coloring and (l, m, n) does not satisfy Condition B. Now assume $b \geq 3$. Consider an equitable $(b, 2)$-tree-coloring of G. Then there are $(b - 3)$ color classes of size 4 and 3 color classes of size 5. Lemma 2 yields that each color class is independent. This can happen if and only if (l, m, n) satisfies Condition B. $\square$

Lemma 12. *Let $l+m+n=4b+3$ where b is a positive integer. We have the following.*

(1) $K_{l,m,n}$ has no equitable $(b+1,2)$-tree-coloring if and only if $va_2^{\equiv}(K_{l,m,n})=b+2$.
(2) Assume that $K_{l,m,n}$ has an equitable $(b+1,2)$-tree-coloring. If $K_{l,m,n}$ has an equitable $(b,2)$-tree-coloring, then $va_2^{\equiv}(K_{l,m,n})=p(b:l,m,n)$, otherwise $va_2^{\equiv}(K_{l,m,n})=b+1$.

Proof. From Lemma 7, $K_{l,m,n}$ has an equitable $(t,2)$-tree-coloring for each $t\geq b+2$. By definition of $va_2^{\equiv}(K_{l,m,n})$, $K_{l,m,n}$ has no equitable $(b+1,2)$-tree-coloring if and only if $va_2^{\equiv}(K_{l,m,n})=b+2$.

Now assume that $K_{l,m,n}$ has an equitable $(b+1,2)$-tree-coloring. Thus $K_{l,m,n}$ has an equitable $(t,2)$-tree-coloring for each $t\geq b+1$. If $K_{l,m,n}$ has no equitable $(b,2)$-tree-coloring, then $va_2^{\equiv}(K_{l,m,n})=b+1$ by the definition. Consider the case that $K_{l,m,n}$ has an equitable $(b,2)$-tree-coloring. Then each color class has size at least 4. Lemma 2 yields that such a coloring is a proper equitable b-coloring. Lemma 3 yields $va_2^{\equiv}(K_{l,m,n})=p(b:l,m,n)$. □

Theorem 13. *$va_2^{\equiv}(K_{1,1,1})=2$. Assume that $l+m+n=4b+3$ where b is a positive integer. Then we have the following.*

(i) If (l,m,n) does not satisfy Condition A, then $va_2^{\equiv}(K_{l,m,n})=b+2$.
(ii) If (l,m,n) satisfies Condition A but does not satisfy Condition B, then $va_2^{\equiv}(K_{l,m,n})=b+1$.
(iii) If (l,m,n) satisfies Condition A and Condition B, then $va_2^{\equiv}(K_{l,m,n})=p(b:l,m,n)$.

Proof. It is easy to see that $va_2^{\equiv}(K_{1,1,1})=2$. Now consider the part $l+m+n=4b+3$ with a positive integer b. Using Lemmas 10 and 12 (1), we have (i). Using Lemmas 10, 11, and 12 (2), we have (ii) and (iii). This completes the proof. □

Corollary 14. *Let t be a nonnegative integer. Let d be the minimum integer greater than 4 such that $4t$ is not divisible by d, then $va_2^{\equiv}(K_{4t,4t,4t})=3\lceil 4t/d\rceil$.*

Moreover, $va_2^{\equiv}(K_{4t+1,4t+1,4t+1})=va_2^{\equiv}(K_{4t+2,4t+2,4t+2})=3t+2$ and $va_2^{\equiv}(K_{4t+3,4t+3,4t+3})=3t+3$.

Proof. Using Definition 1 and Theorem 9 (1), we have the desired result for $va_2^{\equiv}(K_{4t,4t,4t})$.

Using Theorem 13, we have $va_2^{\equiv}(K_{4t+1,4t+1,4t+1})=3t+2$. Using Theorem 9, we have $va_2^{\equiv}(K_{4t+2,4t+2,4t+2})=3t+2$ and $va_2^{\equiv}(K_{4t+3,4t+3,4t+3})=3t+3$. □

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The game coloring number of planar graphs with a specific girth

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Abstract

Let $\mathrm{col_g}(G)$ be the game coloring number of a given graph G. Define the game coloring number of a family of graphs $\mathcal{H}$ as $\mathrm{col_g}(\mathcal{H}) := \max\{\mathrm{col_g}(G) : G \in \mathcal{H}\}$. Let $\mathcal{P}_k$ be the family of planar graphs of girth at least k. We show that $\mathrm{col_g}(\mathcal{P}_7) \leq 5$. This result extends a result about the coloring number by Wang and Zhang [10] ($\mathrm{col_g}(\mathcal{P}_8) \leq 5$). We also show that these bounds are sharp by constructing a graph G where $G \in \mathcal{P}_k \geq 5$ for each $k \leq 8$ such that $\mathrm{col_g}(G) = 5$. As a consequence, $\mathrm{col_g}(\mathcal{P}_k) = 5$ for $k = 7, 8$.

1 Introduction

Let G be a simple graph with a vertex set $V(G)$ and an edge set $E(G)$. The *coloring game* is a two-person game described as follows. Two players, say Alice and Bob, with Alice playing first alternatively colors an uncolored vertex in G with the color from the color set C so that any two adjacent vertices have distinct colors. Alice wins if all vertices are colored. The *game chromatic number* of G, denoted by $\chi_g(G)$, is the least cardinality of C in which Alice has a strategy to win the game. The game chromatic number was formally introduced by Bodlaender [1].

The *marking game* is also a two-person game. Two players, say Alice and Bob, with Alice playing first alternatively marks an unmarked vertex of G until all vertices are marked. Let $b(v)$ be the number of neighbors of v that are marked before v is marked. The *game coloring number* of G, denoted by $\mathrm{col_g}(G)$, is the least s in which Alice has a strategy to obtain $b(v) + 1$ to be at most s for each vertex v.

The game coloring number was formally introduced by Zhu [12] as a tool to study the game chromatic number. It is easy to see that $\chi_g(G) \leq \mathrm{col_g}(G)$. The best known upper bounds for game chromatic numbers of graphs in various families are obtained from the upper bounds of game coloring numbers.

Let $\mathcal{H}$ be a family of graphs. The game chromatic number and the game coloring number of $\mathcal{H}$ are defined as $\chi_G(\mathcal{H}) := \max\{\chi_g(G) : G \in \mathcal{H}\}$ and $\mathrm{col_g}(\mathcal{H}) := \max\{\mathrm{col_g}(G) : G \in \mathcal{H}\}$.

The game coloring numbers of various families of graphs, especially planar graphs, are widely studied. Let $\mathcal{F}$ denote the family of forests, $\mathcal{I}_k$ denote the family of interval graphs with clique number k, $\mathcal{Q}$ denote the family of outerplanar graphs, $\mathcal{PT}_k$ denote the family of partial k-trees, and $\mathcal{P}$ denote the family of planar graphs. It is proved by Faigle et al. [3] that $\chi_g(\mathcal{F}) = \mathrm{col_g}(\mathcal{F}) = 4$, by Faigle et al. [3] and Kierstead and Yang [7] that $\mathrm{col_g}(\mathcal{I}_k) = 3k-2$, by Guan and Zhu [4] and Kierstead and Yang [7] that $\mathrm{col_g}(\mathcal{Q}) = 7$, and by Zhu [13] and Wu and Zhu [11] that $\mathrm{col_g}(\mathcal{PT}_k) = 3k+2$ for $k \geq 2$.

Combining a lower bound from [11] and an upper bound from [14] gives $11 \leq \mathrm{col_g}(\mathcal{P}) \leq 17$. Let $\mathcal{P}_k$ be the family of planar graphs of girth at least k. It is proved by Sekiguchi [9] that $\mathrm{col_g}(\mathcal{P}_4) \leq 13$, by He et al. [5] that $\mathrm{col_g}(\mathcal{P}_5) \leq 8$, by Kleitman [8] that $\mathrm{col_g}(\mathcal{P}_6) \leq 6$, by Wang and Zhang [10] that $\mathrm{col_g}(\mathcal{P}_8) \leq 5$, and by Borodin et al. [2] that $\mathrm{col_g}(G) \leq 9$ if G is a quadrangle-free planar graph. For lower bounds, it was proved by Sekiguchi [9] that $\mathrm{col_g}(\mathcal{P}_4) \geq 7$ and $\mathrm{col_g}(\mathcal{P}_5) \geq 6$.

In this paper, we show that $\mathrm{col_g}(\mathcal{P}_7) \leq 5$. This result extends a result about the coloring number by Wang and Zhang [10]. We also show that these bounds are sharp by constructing a graph G where $G \in \mathcal{P}_k \geq 5$ for each $k \leq 8$ such that $\mathrm{col_g}(G) = 5$. As a consequence, $\mathrm{col_g}(\mathcal{P}_k) = 5$ for $k = 7, 8$.

2 Upper bounds for the game coloring number of planar graphs with girth at least 7

For a graph G, let $\Pi(G)$ be the set of linear orderings of $V(G)$. The digraph G_L with respect to $L \in \Pi(G)$ is obtained from G by orienting an edge uv in G with $u >_L v$ into an arc (u, v).

For a vertex u, we denote the set of neighbors of u in G by $N_G(u)$, the set of out-neighbors of u in G_L by $N^+_{G_L}(u)$, and the set of in-neighbors of u in G_L by $N^-_{G_L}(u)$. Let $d_G(u) := |N_G(u)|$, $d^+_{G_L}(u) := |N^+_{G_L}(u)|$, and $d^-_{G_L}(u) := |N^-_{G_L}(u)|$.

We define $V^+_{G_L}(u) := \{v \in V(G_L) : v <_L u\}$ and $V^-_{G_L}(u) := \{v \in V(G_L) : v >_L u\}$. Moreover, $N^+_{G_L}[u] := N^+_{G_L}(u) \cup \{u\}$, $N^-_{G_L}[u] := N^-_{G_L}(u) \cup \{u\}$, $V^+_{G_L}[u] := V^+_{G_L}(u) \cup \{u\}$, and $V^-_{G_L}[u] := V^-_{G_L}(u) \cup \{u\}$. The subscripts maybe omitted if G or G_L is clear from the context.

In [6], Kierstead introduces the activation strategy which gives the bound for $\mathrm{col_g}(G)$ as follows.

Definition 1 *For a given graph G, we say that M is a matching from A to B if M covers all vertices in A and each edge in M joins a vertex in A and a vertex in $B - A$.*

For a vertex u in G_L, we define $m(u, L, G)$ to be the size of a largest Z such that $Z \subseteq N^-[u]$ with a partition $Z = X \cup Y$ and there exist matchings M from $X \subseteq N^-[u]$ to $V^+(u)$ and N from $Y \subseteq N^-(u)$ to $V^+[u]$.

Let

$$r(u, L, G) := d^+_{G_L}(u) + m(u, L, G),$$
$$r(L, G) := \max_{u \in V(G)} r(u, L, G),$$
$$r(G) := \min_{L \in \Pi(G)} r(L, G).$$

Theorem 1 *(Kierstead [6])* $\mathrm{col_g}(G) \leq 1 + r(G)$.

Now we are ready to prove the following theorem.

Theorem 2 *If G is a planar graph with girth at least 7, then* $\mathrm{col_g}(G) \leq 5$.

Proof. Fix a planar graph embedding of G. It suffices to construct a linear ordering L such that $r(L,(G)) \leq 4$ as follows. Initially, we have a set of chosen vertices $C := \emptyset$ and a set of unchosen vertices $U := V(G)$. At each stage of the construction, we choose a vertex $u \in U$, and replace U by $U = \{u\}$ and C by $C = C \cup \{u\}$. Define a linear order L by $u <_L v$ if we choose v before u.

It is well known that for a planar graph G with girth at least 4, there exists a vertex u with degree at most 3. If C is empty, then choose a vertex of degree at most 3 as u. Suppose that C is not empty. Let H be the graph obtained from G by
(i) deleting all edges between vertices in C,
(ii) deleting each vertex $x \in C$ such that $|N_G(x) \cap U| \leq 3$,
(iii) if $x \in C$ and $|N_G(x) \cap U| = 2$, then we add an edge between two vertices in $N_G(x) \cap U$,
(iv) if $x \in C$ and $|N_G(x) \cap U| = 3$, then we add two new edges between three vertices in $N_G(x) \cap U$ to form a path.
Clearly, H is a planar graph. Since the girth of G is at least 7, a new edge in H joins two vertices that are not adjacent in G and the girth of H is at least 4. Note that each $v \in C$ has $d_H(v) \geq 4$. Thus there is a vertex $u \in U$ with $d_H(u) \leq 3$. Choose such u.

Let
$$S := \{v \in U : uv \in E(G)\},$$
$$S' := \{v \in U : uv \in E(H) - E(G)\},$$
$$A := \{x \in C : ux \in E(H) \text{ and } d_H(x) \geq 4\}.$$
Let σ, σ', and α be the cardinalities of S, S', and A, respectively.

Then
$$d_H(u) = \sigma + \sigma' + \alpha \leq 3.$$

Consider a set $Z \subseteq N^-[u]$ with $|Z| = m(u, L, G)$ such that there exists a partition $Z = X \cup Y$ and there exist matchings M from $X \subseteq N^-[u]$ to $V^+(u)$ and N from $Y \subseteq N^-(u)$ to $V^+[u]$. Let $Z_1 := Z \cap A$, $Z_2 = Z \cap \{x \in C : d_H(x) \leq 3\}$, and $Z_3 = Z \cap \{a\}$. Then the sets Z_i for $i = 1, 2, 3$ are mutually disjoint but some sets maybe empty.

Clearly, $d^+(u) = \sigma$ and $m(u, L, G) = |Z_1| + |Z_2| + |Z_3|$. We aim show that $|Z_1| + |Z_2| + |Z_3| \leq \alpha + \sigma' + 1$. From definitions, $|Z_1| \leq \alpha$ and $|Z_3| \leq 1$. It remains to show that $|Z_2| \leq \sigma'$.

Define $B(M) := \{v \in V^+(u) : \exists x \in Z_2 \text{ such that } xv \in M\}$ and $B(N) := \{v \in V^+(u) : \exists v \in Z_2 \text{ such that } yv \in N\}$. Assume furthermore that M is a matchings from X to $V^+(u)$ with maximum $|B(M)|$ and N is matching from Y to $V^+[u]$ with maximum $|B(N)|$. First, we claim that $B(M) \cup B(N) \subseteq S'$. Suppose to the contrary that there is $w \in B(M) - S'$. Let $xw \in M$. Note that $w \notin S$, otherwise $uwxu$ is C_3 in G, a contradiction. By construction, there is $v \in S'$ such that w and v are neighbors or x. We have $M \cup \{xv\} - \{xw\}$ is a matching from X to $V^+(u)$, otherwise there is $x'v \in M$ implying $uxvx'u$ is C_4 in G, a contradiction. But $|B(M \cup \{xv\} - \{xw\})| = B(M) + 1$ which contradicts the maximality of $|B(M)|$. Thus $B(M) \subseteq S'$. Similarly, $B(N) \subseteq S'$.

Suppose to the contrary that there is $v \in B(M) \cap B(N)$. This means there is $xv \in M$ and $yv \in N$. But then $uxvyu$ is C_4 in G, a contradiction. Altogether, we have $|Z_2| = |B(M)| + |B(N)| \leq |S'| = \sigma'$.

Thus $r(u, L, G) = d^+(u) + m(u, L, G) \leq \sigma + \alpha + \sigma' + 1 \leq 4$. Hence $\mathrm{col_g}(G) \leq 5$. $\square$

3 Lower bounds for the game coloring number of planar graphs with girth at most 8

Theorem 3 *For $3 \leq k \leq 8$, there is a planar graph G with girth k such that $\mathrm{col_g}(G) = 5$. As a consequence, $\mathrm{col_g}(\mathcal{P}_k) \geq 5$.*

Proof. Let H_1 be a hexagon (a cycle of length 6). We construct H_n ($n \geq 2$) from H_{n-1} by adding $6(n-1)$ hexagon around H_{n-1}.

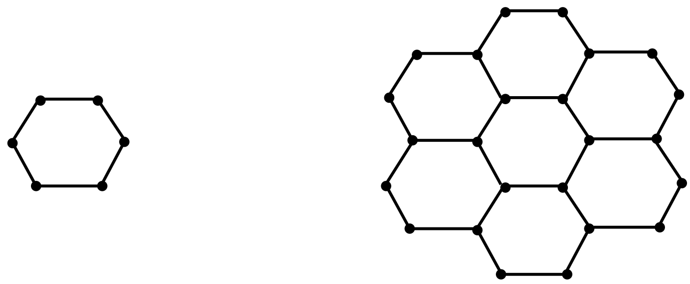

Figure 1: H_1 and H_2

We construct G_n from H_n as follows. For each $v \in V(H_n)$, we add a vertex a_v and an edge between a_v and v. Furthermore, we subdivide each horizontal edge of H_n. The resulting graph G_n is a planar graph with girth 8.

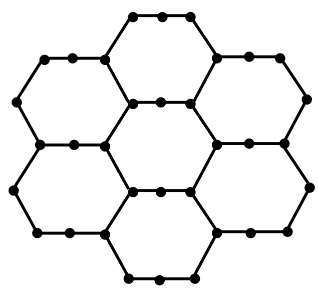

Figure 2: G_2 (we omit the vertices of A)

Let

$$A := \{a_v \in V(H_n) : v \in V(H_n)\},$$

$$B := V(G_n) - (V(H_n) \cup A),$$
$$V_I := \{v \in V(H_n) : v \text{ is not incident to the unbounded face of } H_n\},$$
$$V_O := \{v \in V(H_n) : v \text{ is incident to the unbounded face of } H_n\}.$$

Then we have

$$|B| = 3n^2 - n,$$
$$|V_I| = 6n^2 - 12n + 6,$$
$$|V_O| = 12n - 6.$$

First, Bob marks all the vertices of B and V_O. Note that $|B|+|V_O|+1 < |V_I|$ for $n \geq 9$. It follows that when all the vertices of B and V_O are marked, there are at least two vertices of V_I are unmarked. Consider Bob's turn immediately after all the vertices of $B \cup V_O$ are marked. Suppose $v_1, v_2, \ldots, v_t$ are all unmarked vertices in V_I. Bob can force all $a_{v_1}, a_{v_2}, \ldots, a_{v_t}$ to be marked before all $v_1, v_2, \ldots, v_t$ are marked. Thus the last unmarked vertex, say v, in V_I satisfies $b(v) + 1 = 5$. Therefore $\mathrm{col_g}(G_n) \geq 5$. Since $\Delta(G_n) = 4$, we have $\mathrm{col_g}(G_n) = 5$.

For $k \leq 8$, one can see that $\mathrm{col_g}(G_n \cup C_k) = 5$. It follows that $\mathrm{col_g}(\mathcal{P}_k) \geq 5$. □

Corollary 4 $\mathrm{col_g}(\mathcal{G}_k) = 5$ *for* $k = 7, 8$.

Proof. From Theorems 2, 3, and the fact that $\mathrm{col_g}(\mathcal{P}_8) \leq 5$ [10], Corollary 4 follows. □

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