รายงานวิจัยฉบับสมบูรณ์

โครงการ **การแก้ปัญหาอสมการแปรผันในสองแนวทาง**

Two approaches for solving variational inequality problems

(ทุนพัฒนานักวิจัย)

โดย

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สนับสนุนโดย

สำนักงานกองทุนสนับสนุนการวิจัย

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บทคัดย่อ: งานวิจัยนี้ได้ศึกษาปัญหาอสมการผันแปรและปัญหาที่เกี่ยวข้อง โดยสนใจการประมาณค่าคำตอบด้วยการลู่เข้าแบบอ่อนและแบบเข้ม ผลลัพธ์บางประการสามารถอธิบายได้ในปริภูมิเมตริกบริบูรณ์ ในระหว่างการศึกษาค้นคว้าผู้วิจัยยังได้ผลลัพธ์บางประการเกี่ยวกับเสถียรภาพของสมการจุดตรึงและสมการเชิงฟังก์ชันอีกด้วย ผลลัพธ์ที่ได้รับในโครงการนี้เป็นการพัฒนาและครอบคลุมผลงานของนักคณิตศาสตร์จำนวนมากที่มีการศึกษามาก่อนหน้านี้

คำหลัก: อสมการผันแปร ปัญหาจุดตรึง ระเบียบวิธีทำซ้ำ เสถียรภาพและเสถียรภาพแบบเกิน

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Abstract: We discuss variational inequality problem and some related problems. We are interested in both weak and strong convergence approximations of a solution. Some results are extended in the setting of a complete metric space. During our study, we also obtain some stability result of some fixed point equation and some functional equations. Our results in this project improve and unify the corresponding known results studied by many authors.

Keywords: variational inequality problem, fixed point problem, iterative method, stability and hyperstability results

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ

สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) และมหาวิทยาลัยขอนแก่น (มข.) ที่ได้ให้โอกาสผู้วิจัยได้รับทุนพัฒนานักวิจัยในการทำงานวิจัยค้นคว้าครั้งนี้

ศาสตราจารย์ ดร. สมพงษ์ ธรรมพงษา ที่อบรมสั่งสอน ถ่ายทอด ความรู้ด้านต่างๆ จนผู้วิจัยสามารถทำงานวิจัยและค้นคว้าได้

คณะผู้ประเมิน (referee) ของวารสารวิชาการต่าง ๆ ที่ได้ให้คำแนะนำ ตลอดทั้งปรับปรุงต้นฉบับของบทความที่ส่งไปเพื่อตีพิมพ์ในวารสารนั้นๆ

คณาจารย์ นักศึกษาระดับบัณฑิตศึกษาและเจ้าหน้าที่ฝ่ายสนับสนุน สาขาวิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยขอนแก่น ที่ได้มีส่วนโครงการวิจัยในครั้งนี้

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Part I

Project Summary

Project Summary

1.1 Variational inequality problems and some related formulations

Throughout this summary, we let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a closed convex subset of $\mathcal{H}$. Let us recall the following two major (nonlinear) problems:

Fixed Point Problem (FPP): Let $T : C \to C$ be a mapping. An element $u \in C$ is a *fixed point* of T if $u = Tu$. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$.

Variational Inequality (VI): Let $A : C \to \mathcal{H}$. An element $u \in C$ is a *solution* of a variational inequality for A if $\langle v - u, Au \rangle \geq 0$ for all $v \in C$. The set of all solutions of a variational inequality for A is denoted by $\mathrm{VI}(C, A)$.

These two problems are related as follows:

FPP $\Longrightarrow$ VI: For a given $T : C \to C$, we have $\mathrm{Fix}(T) = \mathrm{VI}(C, I - T)$.

VI $\Longrightarrow$ FPP: For a given $A : C \to \mathcal{H}$, we have $\mathrm{VI}(C, A) = \mathrm{Fix}(P_C \circ (I - A))$ where P_C is the metric projection from $\mathcal{H}$ onto C.

To approximate a solution of the variational inequality, we are interested in both weak and strong convergences. Recall that a sequence $\{x_n\}$ in $\mathcal{H}$ converges *strongly* (*weakly*, respectively) to $x \in \mathcal{H}$ if $\lim_{n\to\infty} \|x_n - x\| = 0$ ($\lim_{n\to\infty}\langle x_n - x, y\rangle = 0$ for all $y \in \mathcal{H}$, respectively).

In [A1], we improve three weak convergence theorems for a common fixed point of a family of firmly nonexpansive mappings with generalized parameters. We prove the same results for the class of k-demicontractive mapping where $k \leq 1$. Note that every firmly nonexpansive is k-demicontractive mapping where $k = -1$. For the case $k = 1$, we use two techniques proposed by Ishikawa[1] and by Korpelevič[2] to attack this problem. The methods and results in [A1] are extensively studied and extended to a more general algorithm via using an infinite matrix. The weak convergence are given in [A4] and the strong convergence in [A7]. In some special cases, we obtain a simple proof of Wang's method for split common fixed point problem[3] in [A8]. We also simplify the main results of Lin and Takahashi[4] and of Takahashi[5] in [A5].

1.2 Some results in a more general setting

In a more general setting than the Hilbert space setting, we discuss some results in a complete metric space. In [A2], we discuss the well known Caristi's theorem where the distance function is replaced by the w-distance. The latter notion was defined by Kada et al.[6]. We do not only obtain an approximation of a fixed point of a mapping

[1]Ishikawa, Shiro. Fixed points by a new iteration method. Proc. Amer. Math. Soc. 44 (1974), 147–150.

[2]Korpelevič, G. M. An extragradient method for finding saddle points and for other problems. (Russian) Èkonom. i Mat. Metody 12 (1976), no. 4, 747–756.

[3]Wang, Fenghui. A new method for split common fixed-point problem without priori knowledge of operator norms. J. Fixed Point Theory Appl. 19 (2017), no. 4, 2427–2436.

[4]Lin, Lai-Jiu; Takahashi, Wataru. A general iterative method for hierarchical variational inequality problems in Hilbert spaces and applications. Positivity 16 (2012), no. 3, 429–453.

[5]Takahashi, W. Strong convergence theorems for maximal and inverse-strongly monotone mappings in Hilbert spaces and applications. J. Optim. Theory Appl. 157 (2013), no. 3, 781–802.

[6]Kada, Osamu; Suzuki, Tomonari; Takahashi, Wataru. Nonconvex minimization theorems and fixed point theorems in complete metric spaces. Math. Japon. 44 (1996), no. 2, 381–391.

but also a minimizer of a certain function. We also study the similar result where the contractive condition are restricted with respect to a directed graph in [A6].

We study some geometric properties implying the existence a fixed point of a nonexpansive mapping in a Banach space in [A3]. This is a joint work with Professor Ji Gao.

1.3 Some by-product of the project

During our study in this project, we also obtain the following results. According to [A6], we can prove some new stability result in the sense of Ulam[7] for a mapping whose contractiveness is restricted with respect to a directed graph in [A10]. A classical problem of the stability result of Cauchy equation and that of the general linear equation are obtained in [A9] and [A11].

1.4 Research outputs

In this project, we published the following 11 papers.

A1: Jaipranop, Chanitnan; **Saejung, Satit**. Some improvements on weak convergence theorems of Chuang and Takahashi in Hilbert spaces. *Chamchuri J. Math.* 8 (2016), 1–17. (No impact factor)

A2: Ardsalee, Pinya; **Saejung, Satit**. On some fixed point theorems of Caristi's type via w-distance. *J. Nonlinear Convex Anal.* 17 (2016), no. 11, 2355–2364. (2017 Impact Factor: 0.56)

A3: Gao, Ji; **Saejung, Satit**. U-flatness and non-expansive mappings in Banach spaces. *J. Korean Math. Soc.* 54 (2017), no. 2, 493–506. (2017 Impact Factor: 0.684)

[7]Hyers, D. H. On the stability of the linear functional equation. Proc. Nat. Acad. Sci. U. S. A. 27, (1941). 222–224.

A4: Jaipranop, Chanitnan; **Saejung, Satit**. An explanation of over-relaxation parameters for some algorithms in Hilbert spaces. *Linear Nonlinear Anal.* 3 (2017), no. 3, 409–421. (No impact factor)

A5: Wongchan, Kanokwan; **Saejung, Satit**. Strong convergence of Browder's and Halpern's type iterations in Hilbert spaces. *Positivity* 22 (2018), no. 4, 969–982. (2017 Impact Factor: 0.92)

A6: Boonsri, Narongsuk; **Saejung, Satit**. Fixed point theorems for contractions of Reich type on a metric space with a graph. *J. Fixed Point Theory Appl.* 20 (2018), no. 2, Art. 84, 17 pp. (2017 Impact Factor: 0.971)

A7: Jaipranop, Ch.; **Saejung, Satit**. On the strong convergence of sequences of Halpern type in Hilbert spaces. *Optimization* 67 (2018), no. 11, 1895–1922. (2017 Impact Factor: 1.17)

A8: Kraikaew, Rapeepan; **Saejung, Satit**. Another look at Wang's new method for solving split common fixed-point problems without priori knowledge of operator norms. *J. Fixed Point Theory Appl.* 20 (2018), no. 2, Art. 81, 6 pp. (2017 Impact Factor: 0.971)

A9: Phochai, Theerayoot; **Saejung, Satit**. The hyperstability of general linear equation via that of Cauchy equation. *Aequationes Mathematicae* (to appear). (2017 Impact Factor: 0.644)

A10: Buakird, Apimuk; **Saejung, Satit**. Ulam stability with respect to a directed graph for some fixed point equations. *Carpathian J. Mathematics* 35 (2019), no. 1, 23–30. (2017 Impact Factor: 0.878)

A11: Phochai, Theerayoot; **Saejung, Satit**. Some notes on the Ulam stability of the general linear equation. *Acta Mathematica Hungarica* (to appear). (2017 Impact Factor: 0.481)

Part II

Reprints

A1: Jaipranop, Chanitnan; **Saejung, Satit**. Some improvements on weak convergence theorems of Chuang and Takahashi in Hilbert spaces. *Chamchuri J. Math.* 8 (2016), 1–17.

Chamchuri Journal of Mathematics
Volume 8(2016), 1–17
http://www.math.sc.chula.ac.th/cjm

Some improvements on weak convergence theorems of Chuang and Takahashi in Hilbert spaces

Chanitnan Jaipranop* and Satit Saejung†

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Abstract: Chuang and Takahashi [3] recently proved three weak convergence theorems for a family of firmly nonexpansive mappings with generalized parameters. We discuss these three results for a family of k-demicontractive mappings where $k \leq 1$. Obviously, the class of k-demicontractive mappings contains all firmly nonexpansive mappings. The situation $k = 1$ is extensively studied by means of the Ishikawa iteration and the extragradient method of Korpelevič. Some numerical results for $k = 1$ are presented and further discussed.

Keywords: fixed point, k-demicontractive mapping, Mann iteration, Ishikawa iteration, extragradient method

2000 Mathematics Subject Classification: 47H09, 47H10

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty subset of $\mathcal{H}$. An element $x \in C$ is called a *fixed point* of a mapping $T : C \to \mathcal{H}$ if $x = Tx$. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$.

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† *Corresponding author*

Our work is inspired by the recent work of Chuang and Takahashi [3]. They proved three weak convergence theorems for a family of firmly nonexpansive mappings. Their results are interesting because their iterations are established with generalized parameters. In the previous work of Mann [8], the parameter is taken in $[0,1]$ while Chuang and Takahashi's work allows the wider interval of parameters in $[0,2]$. We continue the study of these works and extend the class of firmly nonexpansive mappings to that of k-demicontractive mappings where $k \leq 1$. Note that every firmly nonexpansive mapping with a fixed point is just (-1)-demicontractive. Hence our work includes theorems of Chuang and Takahashi as a special case. We also discuss the 1-demicontractive case. This class is very interesting and beyond the scope of the work of Chuang and Takahashi. We use two techniques in the work of Kraikaew and Saejung [7] in this situation. Some numerical results are also presented and discussed.

2 Preliminaries

Throughout this paper, we use $\to$ and $\rightharpoonup$ for the strong and weak convergences, respectively. We write $x_n \equiv x$ for the statement $x_n = x$ for all $n \geq 1$.

Definition 2.1. [4] Let C be a nonempty subset of $\mathcal{H}$ and k be a real number. We say that a mapping $T : C \to \mathcal{H}$ is *k-pseudocontractive* if $\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2$ for all $x, y \in C$. If T is 1-pseudocontractive, then it is simply called *pseudocontractive*. If T is k-pseudocontractive where $k < 1$, then it is usually called *strictly pseudocontractive*. If T is 0-pseudocontractive, then it is called *nonexpansive*. If T is (-1)-pseudocontractive, then it is called *firmly nonexpansive*.

Definition 2.2. [4] Let C be a nonempty subset of $\mathcal{H}$ and k be a real number. We say that a mapping $T : C \to \mathcal{H}$ is *k-demicontractive* if $\mathrm{Fix}(T) \neq \emptyset$ and $\|Tx - p\|^2 \leq \|x-p\|^2 + k\|x - Tx\|^2$ for all $p \in \mathrm{Fix}(T), x \in C$. If T is 0-demicontractive, then it is called *quasi-nonexpansive*. If T is (-1)-demicontractive, then it is called *quasi-firmly nonexpansive*.

Remark 2.3.

1. Every k-pseudocontractive mapping with a fixed point is k-demicontractive.
2. Let C be a nonempty, closed and convex subset of $\mathcal{H}$. If $T : C \to \mathcal{H}$ is quasi-nonexpansive, then $\mathrm{Fix}(T)$ is closed and convex.

Lemma 2.4. *Let C be a nonempty subset of $\mathcal{H}$. Let $T : C \to \mathcal{H}$ be a k-demicontractive mapping. Let $S := (1-\alpha)I + \alpha T$ where α is a nonnegative real number and I is an identity mapping. Then for all $x \in C$ and $p \in \mathrm{Fix}(T)$,*

$$\|Sx - p\|^2 \leq \|x - p\|^2 - \alpha(1 - k - \alpha)\|x - Tx\|^2.$$

In addition, if $\alpha \in \,]0, 1-k[$, then $\mathrm{Fix}(S) = \mathrm{Fix}(T)$ and S is quasi-nonexpansive.

Proof. Let $x \in C$ and $p \in \mathrm{Fix}(T)$. We have

$$\begin{aligned}
\|Sx - p\|^2 &= \|(1-\alpha)(x-p) + \alpha(Tx - p)\|^2 \\
&= (1-\alpha)\|x-p\|^2 + \alpha\|Tx - p\|^2 - \alpha(1-\alpha)\|x - Tx\|^2 \\
&\leq \|x - p\|^2 - \alpha(1-k-\alpha)\|x - Tx\|^2.
\end{aligned}$$

If $\alpha \in \,]0, 1-k[$, then $\mathrm{Fix}(T) = \mathrm{Fix}(S)$ and S is quasi-nonexpansive. $\square$

The following conditions are studied in [3].

Definition 2.5. Let C be a nonempty subset of $\mathcal{H}$. Let $\{T_n : C \to \mathcal{H}\}_{n=1}^{\infty}$ be a sequence of mappings and $\mathcal{T}$ be a family of mappings from C into $\mathcal{H}$. Suppose that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$. We say that

1. $\{T_n\}_{n=1}^{\infty}$ satisfies the *resolvent property* if there exists a nonexpansive mapping $T : C \to \mathcal{H}$ and $\mathrm{Fix}(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ and there exist $n_0, k \geq 1$ such that $\|x - Tx\| \leq k\|x - T_n x\|$ for all $x \in C$ and for all $n \geq n_0$. In this situation, we also say that $\{T_n\}_{n=1}^{\infty}$ satisfies the *resolvent property with a nonexpansive mapping T*.
2. $\{T_n\}_{n=1}^{\infty}$ satisfies the *AKTT-condition* if the following two conditions are satisfied:
   (a) $\sum_{n=1}^{\infty} \sup\limits_{x \in B} \|T_{n+1}x - T_n x\| < \infty$ for each nonempty and bounded subset B of C. (In particular, the sequence $\{T_n x\}_{n=1}^{\infty}$ is Cauchy for all $x \in C$.)
   (b) The mapping $T : C \to \mathcal{H}$ given by $Tx := \lim_{n\to\infty} T_n x$ for all $x \in C$ satisfies the property $\mathrm{Fix}(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

   In this situation, we also say that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the *AKTT-condition*.
3. $(\{T_n\}_{n=1}^{\infty}, \mathcal{T})$ satisfies the *NST-condition* if

(a) $\mathrm{Fix}(\mathcal{T}) := \bigcap_{T\in\mathcal{T}} \mathrm{Fix}(T) \subset \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

(b) For each bounded sequence $\{z_n\}_{n=1}^{\infty} \subset C$, $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$ implies that $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$ for all $T \in \mathcal{T}$.

Remark 2.6. 1. If $\{T_n\}_{n=1}^{\infty}$ satisfies the resolvent property with a nonexpansive mapping T, then $(\{T_n\}_{n=1}^{\infty}, \{T\})$ satisfies the NST-condition.

2. If $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition , then $(\{T_n\}_{n=1}^{\infty}, \{T\})$ satisfies the NST-condition.

Proof. Let $\{z_n\}_{n=1}^{\infty}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$. Since $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition, $\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in \{z_n\}\} = 0$. In particular, $\lim_{n\to\infty} \|Tz_n - T_n z_n\| = 0$. This implies that

$$\limsup_{n\to\infty} \|z_n - Tz_n\| \leq \lim_{n\to\infty} \|z_n - T_n z_n\| + \lim_{n\to\infty} \|T_n z_n - Tz_n\| = 0.$$

Hence $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$. □

Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Then for each $x \in \mathcal{H}$, there is a unique element $\widehat{x} \in C$ such that

$$\|x - \widehat{x}\| = \min_{y\in C} \|x - y\|.$$

Set $P_C x = \widehat{x}$. The mapping P_C is called the *metric projection* from $\mathcal{H}$ onto C.

Lemma 2.7. *[10] Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Then, for all $x \in \mathcal{H}$ and $y \in C$, $y = P_C x$ if and only if $\langle y - x, z - y\rangle \geq 0$ for all $z \in C$.*

The following is the most general result amongst the three weak convergence theorems of Chuang and Takahashi [3].

Theorem 2.8. *[3] Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $\{T_n : C \to C\}_{n=1}^{\infty}$ be a sequence of firmly nonexpansive mappings. Let $\mathcal{T}$ be a family of nonexpansive mappings of C into itself, which satisfies NST-condition. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $]0, 2[$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen,} \\ x_{n+1} := P_C\left((1-\alpha_n)x_n + \alpha_n T_n x_n\right) \quad \forall n \geq 1. \end{cases}$$

If $\liminf_{n\to\infty} \alpha_n(2-\alpha_n) > 0$, then $x_n \rightharpoonup \overline{x}$, where $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

We recall the following facts which are of interest and play an important role in this paper.

Lemma 2.9 (Opial's property)**.** *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathcal{H}$ *such that* $x_n \rightharpoonup x \in \mathcal{H}$. *Then*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in \mathcal{H}$ *with* $y \neq x$.

Definition 2.10. Let F be a nonempty subset of $\mathcal{H}$. A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ is *Fejér monotone with respect to* F if $\|x_{n+1} - p\| \leq \|x_n - p\|$ for all $n \geq 1$ and $p \in F$.

Lemma 2.11. *[11] Let* F *be a nonempty, closed and convex subset of* $\mathcal{H}$ *and* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathcal{H}$. *If* $\{x_n\}_{n=1}^{\infty}$ *is Fejér monotone with respect to* F, *then* $\{P_F x_n\}_{n=1}^{\infty}$ *is convergent.*

Lemma 2.12. *Let* $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, *and* $\{c_n\}_{n=1}^{\infty}$ *be sequences of nonnegative real numbers such that* $a_{n+1} \leq a_n - c_n b_n$ *for all* $n \geq 1$ *and* $\liminf_{n\to\infty} c_n > 0$. *Then* $\lim_{n\to\infty} a_n$ *exists and* $\sum_{n=1}^{\infty} b_n < \infty$. *In particular,* $\lim_{n\to\infty} b_n = 0$.

Proof. The proof of this lemma is rather simple but it is given here for the sake of completeness. Note that $a_{n+1} \leq a_n$ for all $n \geq 1$. Thus $\lim_{n\to\infty} a_n$ exists. Moreover, $c_n b_n \leq a_n - a_{n+1}$. This yields $\sum_{n=1}^{k} c_n b_n \leq a_1 - a_{k+1} \leq a_1$. So $\sum_{n=1}^{\infty} c_n b_n \leq a_1 < \infty$. Since $\liminf_{n\to\infty} c_n > 0$, there are an integer $n_0 \geq 1$ and a positive real number b such that $b \leq c_n$ for all $n \geq n_0$. Thus $b \sum_{n=n_0}^{\infty} b_n \leq \sum_{n=n_0}^{\infty} c_n b_n < \infty$. Then $\sum_{n=1}^{\infty} b_n < \infty$ and hence $\lim_{n\to\infty} b_n = 0$. □

3 Results

Definition 3.1. Let C be a nonempty subset of $\mathcal{H}$. A mapping $T : C \to \mathcal{H}$ satisfies the *demiclosedness property* if $x = Tx$ whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$.

We say that a family $\mathcal{T}$ mappings from C into $\mathcal{H}$ satisfies the *demiclosedness property* if T satisfies the demiclosedness property for all $T \in \mathcal{T}$.

Lemma 3.2. *Let* C *be a nonempty, closed and convex subset of* $\mathcal{H}$. *Let* $\{T_n : C \to \mathcal{H}\}_{n=1}^{\infty}$ *be a sequence of mappings. Let* $\mathcal{T}$ *be a family of mappings of* C *into* $\mathcal{H}$ *satisfying the demiclosedness property. Assume that* $(\{T_n\}_{n=1}^{\infty}, \mathcal{T})$ *satisfies the*

NST-condition. Let $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* C. *If* $\lim_{n\to\infty}\|x_n - p\|$ *exists for all* $p \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$ *and* $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$, *then* $x_n \rightharpoonup \overline{x}$ *for some* $\overline{x} \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$.

Proof. First, we show that all weak cluster points of $\{x_n\}_{n=1}^{\infty}$ belong to the set $\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$. To see this, let $\{x_{n_k}\}_{k=1}^{\infty}$ be a weakly convergent subsequence of $\{x_n\}_{n=1}^{\infty}$. (Such a subsequence exists because $\{x_n\}_{n=1}^{\infty}$ is bounded.) We assume that $x_{n_k} \rightharpoonup u$ for some $u \in C$. Let $T \in \mathcal{T}$. Since $(\{T_n\}_{n=1}^{\infty}, \mathcal{T})$ satisfies the NST-condition, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and hence $\lim_{k\to\infty}\|x_{n_k} - Tx_{n_k}\| = 0$. Since $\mathcal{T}$ satisfies the demiclosedness property, $u \in \mathrm{Fix}(T)$. This implies that $u \in \mathrm{Fix}(\mathcal{T}) = \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$.

Finally, we show that the whole sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly to some element in the set $\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$. To see this, it suffices to prove that the set of all weak cluster points of $\{x_n\}_{n=1}^{\infty}$ is a singleton. Suppose that $\{x_{m_j}\}_{j=1}^{\infty}$ and $\{x_{p_k}\}_{k=1}^{\infty}$ are two subsequences of $\{x_n\}_{n=1}^{\infty}$ which converge weakly to u and v, respectively. From the first part of the proof, we obtain that $u, v \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$. In particular, both limits $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. Suppose that $u \neq v$. By Opial's property, we obtain the following contradiction:

$$\begin{aligned}
\liminf_{j\to\infty}\|x_{m_j} - u\| &< \lim_{j\to\infty}\|x_{m_j} - v\| \\
&= \lim_{k\to\infty}\|x_{p_k} - v\| \\
&< \lim_{k\to\infty}\|x_{p_k} - u\| \\
&= \liminf_{j\to\infty}\|x_{m_j} - u\|.
\end{aligned}$$

So $u = v$. Hence $x_n \rightharpoonup \overline{x}$ for some $\overline{x} \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$, as desired. □

3.1 k-demicontractive mappings where $k < 1$

Theorem 3.3. *Let* C *be a nonempty, closed and convex subset of* $\mathcal{H}$. *Let* $\{T_n : C \to \mathcal{H}\}_{n=1}^{\infty}$ *be a sequence of* k_n*-demicontractive mappings where* $k_n < 1$ *for all* $n \geq 1$. *Let* $\mathcal{T}$ *be a family of mappings of* C *into* $\mathcal{H}$ *satisfying the demiclosedness property. Assume that* $(\{T_n\}_{n=1}^{\infty}, \mathcal{T})$ *satisfies NST-condition. Let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence in* $]0, 1-k_n[$. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* C *defined by*

$$\begin{cases} x_1 \in C \ \text{arbitrarily chosen}, \\ x_{n+1} := P_C\left((1-\alpha_n)x_n + \alpha_n T_n x_n\right) \quad \forall n \geq 1. \end{cases}$$

If $\liminf_{n\to\infty} \alpha_n((1-k_n)-\alpha_n) > 0$, *then* $x_n \rightharpoonup \overline{x}$ *for some* $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. *Moreover,* $\overline{x} = \lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n$.

Proof. Let $p \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Let $S_n := (1-\alpha_n)I + \alpha_n T_n$ for all $n \geq 1$. By Lemma 2.4, we get the following statements:

$$\|x_{n+1} - p\|^2 \leq \|S_n x_n - p\|^2 \leq \|x_n - p\|^2 - \alpha_n((1-k_n)-\alpha_n)\|x_n - T_n x_n\|^2,$$

$\mathrm{Fix}(S_n) = \mathrm{Fix}(T_n)$, and S_n is quasi-nonexpansive for all $n \geq 1$. By Lemma 2.12 with $a_n \equiv \|x_n - p\|^2$, $b_n \equiv \|x_n - T_n x_n\|^2$ and $c_n \equiv \alpha_n((1-k_n)-\alpha_n)$, we get that $\lim_{n\to\infty} \|x_n - p\|$ exists and $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. By Lemma 3.2, we have $x_n \rightharpoonup \overline{x}$, where $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

Note that $\{x_n\}_{n=1}^{\infty}$ is Fejér monotone with respect to $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Since $\mathrm{Fix}(T_n)$ is closed and convex for all $n \geq 1$, it follows that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ is closed and convex. By Lemma 2.11, $\{P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n\}_{n=1}^{\infty}$ converges to a point q in $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. It follows from Lemma 2.7 that

$$\left\langle x_n - P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n, P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n - \overline{x} \right\rangle \geq 0.$$

Since $x_n \rightharpoonup \overline{x}$ and $P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n \to q$, we have

$$\langle x_n - P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n, P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n - \overline{x} \rangle \to \langle \overline{x} - q, q - \overline{x} \rangle = -\|\overline{x} - q\|^2 \geq 0.$$

This implies that $\overline{x} = q$. Hence $\lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n = \overline{x}$. □

Set $k_n \equiv -1$ in Theorem 3.3, we obtain the following corollary.

Corollary 3.4. *Let* C *be a nonempty, closed and convex subset of* $\mathcal{H}$. *Let* $\{T_n : C \to \mathcal{H}\}_{n=1}^{\infty}$ *be a sequence of quasi-firmly nonexpansive mappings. Let* $\mathcal{T}$ *be a family of mappings of* C *into* $\mathcal{H}$ *satisfying the demiclosedness property. Assume that* $(\{T_n\}_{n=1}^{\infty}, \mathcal{T})$ *satisfies NST-condition. Let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence in* $]0,2[$. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* C *defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen,} \\ x_{n+1} := P_C\left((1-\alpha_n)x_n + \alpha_n T_n x_n\right) \quad \forall n \geq 1. \end{cases}$$

If $\liminf_{n\to\infty} \alpha_n(2-\alpha_n) > 0$, *then* $x_n \rightharpoonup \overline{x}$ *for some* $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ *and* $\overline{x} = \lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n$.

Remark 3.5. Our Corollary 3.4 improves Theorem 3.3 of [3] in the following ways.

(a) Since every firmly nonexpansive mapping with a fixed point is quasi-firmly nonexpansive, Corollary 3.4 deals with a wider class of mappings.

(b) The family $\mathcal{T}$ in our Corollary 3.4 is more general than the family $\mathcal{T}$ of nonexpansive mappings in Theorem 3.3 of [3]. In fact, it is known that every nonexpansive mapping satisfies the demiclosedness property.

(c) All mappings in Theorem 3.3 of [3] are self-mappings while in our Corollary 3.4 they are nonself.

(d) We obtain a further information about the weak limit $\overline{x}$ of the sequence $\{x_n\}_{n=1}^{\infty}$. In fact, we can conclude that $\overline{x} = \lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n$.

From Remark 2.6 and Theorem 3.3, we obtain the following two corollaries which improve Theorems 3.1 and 3.2 of [3], respectively.

Corollary 3.6. *Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $\{T_n : C \to \mathcal{H}\}_{n=1}^{\infty}$ be a sequence of k_n-demicontractive mappings where $k_n < 1$ for all $n \geq 1$. Assume that $\{T_n\}_{n=1}^{\infty}$ satisfies the resolvent property with a nonexpansive mapping T. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $]0, 1-k_n[$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C defined by*

$$\begin{cases} x_1 \in C \ \textit{arbitrarily chosen,} \\ x_{n+1} := P_C\left((1-\alpha_n)x_n + \alpha_n T_n x_n\right) \quad \forall n \geq 1. \end{cases}$$

If $\liminf_{n\to\infty} \alpha_n((1-k_n)-\alpha_n) > 0$, then $x_n \rightharpoonup \overline{x}$ for some $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ and $\overline{x} = \lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n$.

Corollary 3.7. *Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $\{T_n : C \to \mathcal{H}\}_{n=1}^{\infty}$ be a sequence of k_n-demicontractive mappings where $k_n < 1$ for all $n \geq 1$. Assume that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition and T satisfies the demiclosedness property. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $]0, 1-k_n[$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C defined by*

$$\begin{cases} x_1 \in C \ \textit{arbitrarily chosen,} \\ x_{n+1} := P_C\left((1-\alpha_n)x_n + \alpha_n T_n x_n\right) \quad \forall n \geq 1. \end{cases}$$

If $\liminf_{n\to\infty} \alpha_n((1-k_n)-\alpha_n) > 0$, then $x_n \rightharpoonup \overline{x}$ for some $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ and $\overline{x} = \lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n$.

3.2 1-demicontractive mappings

Definition 3.8. Let C be a nonempty subset of $\mathcal{H}$ and let L be a positive real number. A mapping $T : C \to \mathcal{H}$ is *L-Lipschitzian* if $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$.

It is known that the sequence $\{x_n\}_{n=1}^{\infty}$ defined in Theorem 3.3 fails to converge even $T_n \equiv T$ where T is 1-demicontractive and L-Lipschitzian (see [2]). We modify the iteration in Theorem 3.3 to obtain two weak convergence theorems, that is, Theorems 3.10 and 3.12. We now restrict ourselves from the nonself mappings to the self ones. The first result is based on the Ishikawa iteration [5]. The following lemma is modified from [7].

Lemma 3.9. *Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $T : C \to C$ be an L-Lipschitzian and 1-demicontractive mapping. Let $\alpha, \beta \in [0, 1]$. Define the mappings S and U by $S := (1 - \alpha)I + \alpha T$ and $U := (1 - \beta)I + \beta TS$. Then for all $x \in C$ and $p \in \mathrm{Fix}(T)$,*

$$\|Ux - p\|^2 \leq \|x - p\|^2 + \alpha\beta(L^2\alpha^2 + 2\alpha - 1)\|x - Tx\|^2 + \beta(\beta - \alpha)\|x - TSx\|^2.$$

In addition, if $0 < \beta \leq \alpha < \frac{1}{\sqrt{L^2+1}+1}$, then $\mathrm{Fix}(U) = \mathrm{Fix}(T)$ and U is quasi-nonexpansive.

Proof. Let $x \in C$ and $p \in \mathrm{Fix}(T)$. Then

$$\begin{aligned}\|Ux - p\|^2 &= \|(1 - \beta)(x - p) + \beta(TSx - p)\|^2 \\ &= (1 - \beta)\|x - p\|^2 + \beta\|TSx - p\|^2 - (1 - \beta)\beta\|x - TSx\|^2.\end{aligned}$$

Since T is 1-demicontractive, $\|TSx - p\|^2 \leq \|Sx - p\|^2 + \|Sx - TSx\|^2$. Note that

$$\begin{aligned}\|Sx - p\|^2 &= \|(1 - \alpha)(x - p) + \alpha(Tx - p)\|^2 \\ &= (1 - \alpha)\|x - p\|^2 + \alpha\|Tx - p\|^2 - (1 - \alpha)\alpha\|x - Tx\|^2 \\ &\leq (1 - \alpha)\|x - p\|^2 + \alpha\|x - p\|^2 + \alpha\|x - Tx\|^2 - (1 - \alpha)\alpha\|x - Tx\|^2 \\ &= \|x - p\|^2 + \alpha^2\|x - Tx\|^2;\end{aligned}$$

and

$$\begin{aligned}\|Sx - TSx\|^2 &= \|(1-\alpha)(x - TSx) + \alpha(Tx - TSx)\|^2 \\ &= (1-\alpha)\|x - TSx\|^2 + \alpha\|Tx - TSx\|^2 - (1-\alpha)\alpha\|x - Tx\|^2 \\ &\le (1-\alpha)\|x - TSx\|^2 + \alpha L^2\|x - Sx\|^2 - (1-\alpha)\alpha\|x - Tx\|^2 \\ &= (1-\alpha)\|x - TSx\|^2 + \alpha^3 L^2\|x - Tx\|^2 - (1-\alpha)\alpha\|x - Tx\|^2 \\ &= (1-\alpha)\|x - TSx\|^2 + \alpha(L^2\alpha^2 + \alpha - 1)\|x - Tx\|^2.\end{aligned}$$

So $\|TSx - p\|^2 \le \|x - p\|^2 + \alpha(L^2\alpha^2 + 2\alpha - 1)\|x - Tx\|^2 + (1-\alpha)\|x - TSx\|^2$. We get that

$$\|Ux - p\|^2 \le \|x - p\|^2 + \alpha\beta(L^2\alpha^2 + 2\alpha - 1)\|x - Tx\|^2 + \beta(\beta - \alpha)\|x - TSx\|^2.$$

If $0 < \beta \le \alpha < \frac{1}{\sqrt{L^2+1}+1}$, then $L^2\alpha^2 + 2\alpha - 1 < 0$. This implies that $\mathrm{Fix}(T) = \mathrm{Fix}(U)$ and U is quasi-nonexpansive. $\square$

Theorem 3.10. *Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $\{T_n : C \to C\}_{n=1}^{\infty}$ be a sequence of L-Lipschitzian and 1-demicontractive mappings. Let $\mathcal{T}$ be a family of mappings of C into itself satisfying the demiclosedness property. Assume that $(\{T_n\}_{n=1}^{\infty}, \mathcal{T})$ satisfies NST-condition. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $\left]0, 1/(\sqrt{L^2+1}+1)\right[$ and $\{\beta_n\}_{n=1}^{\infty}$ be a sequence in $]0, \alpha_n]$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen,} \\ y_n := (1-\alpha_n)x_n + \alpha_n T_n x_n, \\ x_{n+1} := (1-\beta_n)x_n + \beta_n T_n y_n \quad \forall n \ge 1. \end{cases}$$

If $\liminf_{n\to\infty} \beta_n(1 - 2\alpha_n - L^2\alpha_n^2) > 0$, then $x_n \rightharpoonup \overline{x}$ for some $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ and $\overline{x} = \lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)} x_n$.

Proof. Let $p \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Let $S_n := (1-\alpha_n)I + \alpha_n T_n$ and $U_n := (1-\beta_n)I + \beta_n T_n S_n$ for all $n \ge 1$. Note that $y_n = S_n x_n$ and $x_{n+1} = U_n x_n$ for all $n \ge 1$. By Lemma 3.9, $\|U_n x_n - p\|^2 \le \|x_n - p\|^2 + \beta_n^2(L^2\alpha_n^2 + 2\alpha_n - 1)\|x_n - T_n x_n\|^2$ and U_n is quasi-nonexpansive and $\mathrm{Fix}(U_n) = \mathrm{Fix}(T_n)$. Thus

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 + \beta_n^2(L^2\alpha_n^2 + 2\alpha_n - 1)\|x_n - T_n x_n\|^2.$$

Note that $L^2\alpha_n^2 + 2\alpha_n - 1 < 0$ for all $\alpha_n \in \left]0, 1/(\sqrt{L^2+1}+1)\right[$. By Lemma 2.12, we get that $\lim_{n\to\infty}\|x_n - p\|$ exists and $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. By Lemma 3.2, we have $x_n \rightharpoonup \overline{x}$ for some $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

Since $\mathrm{Fix}(U_n) = \mathrm{Fix}(T_n)$ and U_n is quasi-nonexpansive, $\mathrm{Fix}(T_n)$ is closed and convex for all $n \geq 1$. So $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ is closed and convex. Note that $\{x_n\}_{n=1}^{\infty}$ is Fejér monotone with respect to $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. The rest of the proof is essentially the same as that of Theorem 3.3, so it is omitted. □

Next, we use the extragradient technique of Korpelevič [6] for this situation. We observe the following inequality which plays an important role in the next theorem. Note that this result is more general than the one in [7].

Lemma 3.11. *Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $T : C \to C$ be an L-Lipschitzian and 1-demicontractive mapping. Let $\alpha \in [0,1]$. Define the mappings S and U by $S := (1-\alpha)I + \alpha T$ and $U := P_C(I - \alpha S + \alpha TS)$. Then for all $x \in C$ and $p \in \mathrm{Fix}(T)$,*

$$\|Ux - p\|^2 \leq \|x - p\|^2 - (1 - \alpha^2(1+L)^2)\alpha^2 \|x - Tx\|^2.$$

In addition, if $\alpha \in \left]0, \frac{1}{1+L}\right[$, then $\mathrm{Fix}(U) = \mathrm{Fix}(T)$ and U is quasi-nonexpansive.

Proof. Let $x \in C$ and $p \in \mathrm{Fix}(T)$.

$$\begin{aligned}\|Ux - p\|^2 &\leq \|x - \alpha Sx + \alpha TSx - p\|^2 - \|x - \alpha Sx + \alpha TSx - Ux\|^2 \\ &= \|x - p - \alpha(Sx - TSx)\|^2 - \|x - Ux - \alpha(Sx - TSx)\|^2 \\ &= \|x - p\|^2 - \|x - Ux\|^2 + 2\alpha\langle p - Ux, Sx - TSx\rangle.\end{aligned}$$

Since T is 1-demicontactive, $\langle p - Sx, Sx - TSx\rangle \leq 0$. So

$$\begin{aligned}\langle p - Ux, Sx - TSx\rangle &= \langle Sx - Ux, Sx - TSx\rangle + \langle p - Sx, Sx - TSx\rangle \\ &\leq \langle Sx - Ux, Sx - TSx\rangle.\end{aligned}$$

Note that $\|x - Ux\|^2 = \|x - Sx\|^2 + 2\langle x - Sx, Sx - Ux\rangle + \|Sx - Ux\|^2$. Then

$$\begin{aligned}\|Ux - p\|^2 &\leq \|x - p\|^2 - \|x - Sx\|^2 - 2\langle x - Sx, Sx - Ux\rangle - \|Sx - Ux\|^2 \\ &\quad + 2\alpha\langle Sx - Ux, Sx - TSx\rangle \\ &= \|x - p\|^2 - \|x - Sx\|^2 - \|Sx - Ux\|^2 \\ &\quad + 2\langle Sx - Ux, \alpha(Sx - TSx) - (x - Sx)\rangle.\end{aligned}$$

We consider

$$\begin{aligned}
&2\langle Sx - Ux, \alpha(Sx - TSx) - (x - Sx)\rangle \\
&= 2\alpha\langle Sx - Ux, Sx - TSx - (x - Tx)\rangle \\
&\le 2\alpha\|Sx - Ux\|\|Sx - TSx - (x - Tx)\| \\
&\le 2\alpha\|Sx - Ux\|(\|x - Sx\| + \|Tx - TSx\|) \\
&\le 2\alpha(1 + L)\|Sx - Ux\|\|x - Sx\| \\
&\le \|Sx - Ux\|^2 + \alpha^2(1 + L)^2\|x - Sx\|^2.
\end{aligned}$$

We have that

$$\begin{aligned}
\|Ux - p\|^2 &\le \|x - p\|^2 - (1 - \alpha^2(1 + L)^2)\|x - Sx\|^2 \\
&= \|x - p\|^2 - (1 - \alpha^2(1 + L)^2)\alpha^2\|x - Tx\|^2. \qquad (1)
\end{aligned}$$

If $\alpha \in]0, 1/(1 + L)[$, then $1 - \alpha^2(1 + L)^2 \ge 0$. From (1) , we get that $\mathrm{Fix}(T) = \mathrm{Fix}(U)$ and U is quasi-nonexpansive. □

Theorem 3.12. *Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $\{T_n : C \to C\}_{n=1}^{\infty}$ be a sequence of L-Lipschitzian and 1-demicontractive mappings. Let $\mathcal{T}$ be a family of mappings of C into itself satisfying the demiclosedness property. Assume that $(\{T_n\}_{n=1}^{\infty}, \mathcal{T})$ satisfies NST-condition. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $]0, 1/(1 + L)[$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen,} \\ y_n := (1 - \alpha_n)x_n + \alpha_n T_n x_n, \\ x_{n+1} := P_C(x_n - \alpha_n y_n + \alpha_n T_n y_n) \quad \forall n \ge 1. \end{cases}$$

If $\liminf_{n\to\infty}(1 - \alpha_n(1 + L))\alpha_n > 0$, then $x_n \rightharpoonup \overline{x}$ for some $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ and $\overline{x} = \lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n$.

Proof. Let $p \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Let $S_n := (1 - \alpha_n)I + \alpha_n T_n$ and $U_n := P_C(I - \alpha_n)S_n + \alpha_n T_n S_n$ for all $n \ge 1$. Note that $y_n = S_n x_n$ and $x_{n+1} = U_n x_n$ for all $n \ge 1$. By Lemma 3.11, we get that $\|U_n x_n - p\|^2 \le \|x_n - p\|^2 - (1 - \alpha_n^2(1 + L)^2)\alpha_n^2\|x_n - T_n x_n\|^2$ and U_n is quasi-nonexpansive and $\mathrm{Fix}(U_n) = \mathrm{Fix}(T_n)$. Thus

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 - (1 - \alpha_n^2(1 + L)^2)\alpha_n^2\|x_n - T_n x_n\|^2.$$

By Lemma 2.12, we get that $\lim_{n\to\infty} \|x_n - p\|$ exists and $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. By Lemma 3.2, we have $x_n \rightharpoonup \overline{x}$ for some $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

Since $\mathrm{Fix}(U_n) = \mathrm{Fix}(T_n)$ and U_n is quasi-nonexpansive, $\mathrm{Fix}(T_n)$ is closed and convex for all $n \geq 1$. So $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ is closed and convex. Note that $\{x_n\}_{n=1}^{\infty}$ is Fejér monotone with respect to $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. The rest of the proof is essentially the same as that of Theorem 3.3, so it is omitted. □

4 Numerical Results

Finally, we show some numerical results for Theorems 3.10 and 3.12. The following example is taken from [2]. Let $\mathcal{H}$ be the two-dimensional Euclidean space $\mathbb{R}^2$. If $x = (a, b) \in \mathcal{H}$, define $x^{\perp} \in \mathcal{H}$ to be $(b, -a)$. Let $K := K_1 \cup K_2$ where

$$K_1 := \{x \in \mathcal{H} : \|x\| \leq 1/2\} \text{ and } K_2 := \{x \in \mathcal{H} : 1/2 \leq \|x\| \leq 1\}.$$

Define $T : K \to K$ by

$$Tx = \begin{cases} x + x^{\perp} & \text{if } x \in K_1, \\ \frac{x}{\|x\|} - x + x^{\perp} & \text{if } x \in K_2. \end{cases}$$

Then K is a closed and convex subset of $\mathcal{H}$. Moreover, T is 5-Lipschitzian and 1-demicontractive mapping with $\mathrm{Fix}(T) = \{(0,0)\}$. For computational purposes, it is of interest to know

(a) how the convergence behaviour of $\{x_n\}_{n=1}^{\infty}$ depends on the choice of $\{\alpha_n\}_{n=1}^{\infty}$ in Theorems 3.10 and 3.12;

(b) which of the iterations in Theorems 3.10 and 3.12 is more efficient.

To illustrate (a), we discuss Theorem 3.10 with $x_1 = (1,0)$ and $\alpha_n = \beta_n \equiv \alpha$. To guarantee the convergence of $\{x_n\}_{n=1}^{\infty}$, we are allowed to choose $\alpha \in \left]0, 1/(\sqrt{26}+1)\right[$. Figures 1 and 2 show that the larger choice α, the closer the term x_n is to the fixed point $(0,0)$. For Theorem 3.12, we set $x_1 = (1,0)$ and $\alpha_n \equiv \alpha \in \left]0, 1/6\right[$.

To illustrate (b), let $x_1 = x_1' = (0.1, 0)$ and let $\{x_n\}_{n=2}^{\infty}$ and $\{x_n'\}_{n=2}^{\infty}$ be defined by the iterations in Theorem 3.10 with $\alpha_n = \beta_n \equiv \alpha$ and Theorem 3.12 with $\alpha_n \equiv \alpha$, respectively. Note that $\left]0, 1/(\sqrt{26}+1)\right[\subset \left]0, 1/6\right[$. Figure 3 shows that in this situation the iteration in Theorem 3.12 is more efficient than the one in Theorem 3.10.

Table 1: The value of $\|x_n-(0,0)\|$ where x_n is defined by the iteration in Theorem 3.10

n	$\alpha_n=\beta_n\equiv\alpha$				
	0.004	0.007	0.080	0.160	0.163
1	1	1	1	1	1
2	9.96e−1	9.93e−1	9.23e−1	8.49e−1	8.46e−1
50	8.38e−1	7.51e−1	4.67e−1	3.61e−1	3.57e−1
100	7.26e−1	6.24e−1	4.09e−1	2.41e−1	2.36e−1
500	5.09e−1	5.00e−1	1.42e−1	9.37e−3	8.46e−3

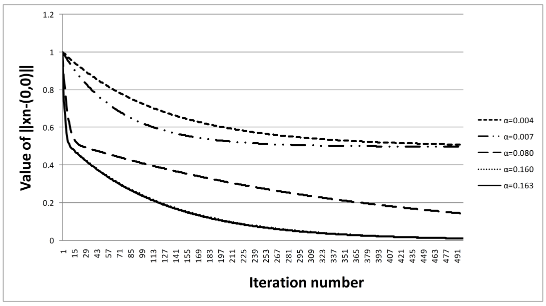

Figure 1: The behaviour of $\|x_n-(0,0)\|$ in Theorem 3.10 and the choice of $\{\alpha_n\}_{n=1}^{\infty}$

Table 2: The value of $\|x_n-(0,0)\|$ where x_n is defined by the iteration in Theorem 3.12

n	$\alpha_n\equiv\alpha$				
	0.004	0.010	0.080	0.160	0.166
1	1	1	1	1	1
2	9.96e−1	9.90e−1	9.29e−1	8.70e−1	8.66e−1
50	8.38e−1	6.88e−1	4.62e−1	3.04e−1	2.91e−1
100	7.27e−1	5.69e−1	3.94e−1	1.62e−1	1.47e−1
500	5.09e−1	5.00e−1	1.10e−1	1.03e−3	6.45e−4

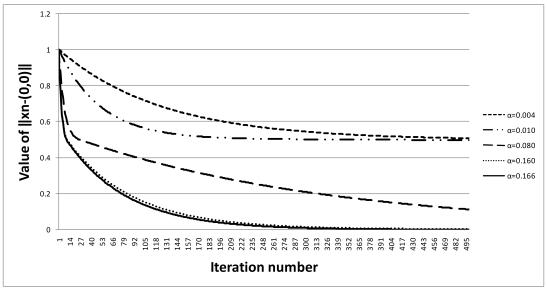

Figure 2: The behaviour of $\|x_n - (0,0)\|$ in Theorem 3.12 and the choice of $\{\alpha_n\}_{n=1}^{\infty}$

Table 3: The values of $\|x_n - (0,0)\|$ and $\|x'_n - (0,0)\|$

n	$\alpha_n \equiv \beta_n \equiv \alpha$ (Theorem 3.10)		$\alpha_n \equiv \alpha$ (Theorem 3.12)	
	0.160	0.163	0.160	0.166
1	1e−1	1e−1	1e−1	1e−1
2	9.92e−2	9.92e−2	9.87e−2	9.87e−2
50	6.72e−2	6.65e−2	5.39e−2	5.14e−2
100	4.48e−2	4.39e−2	2.86e−2	2.61e−2
500	1.74e−3	1.58e−3	1.83e−4	1.14e−4

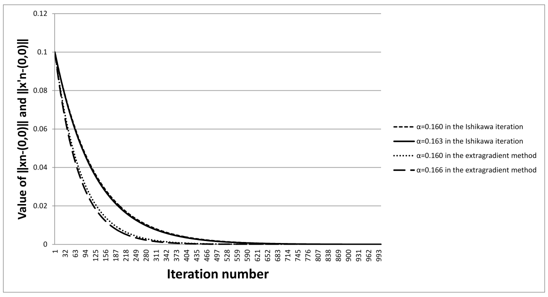

Figure 3: Comparative values of $\|x_n - (0,0)\|$ and $\|x'_n - (0,0)\|$

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ON SOME FIXED POINT THEOREMS OF CARISTI'S TYPE VIA w-DISTANCE

PINYA ARDSALEE* AND SATIT SAEJUNG†

Dedicated to Professor Tomás Domínguez Benavides on his 65th birthday

Abstract. We slightly improve two fixed point theorems proved by Kada, Suzuki and Takahashi [4]. Using our new improvement results, we extend a fixed point theorem of Chuang, Lin and Takahashi [2] by means of w-distance and we also deduce a result of Takahashi, Wong and Yao [7] from our result.

1. Introduction

In 1976, Caristi [1] proved the following fixed point theorem which is an extension of the well-known Banach fixed point theorem.

Theorem 1.1 (Caristi). *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that*

$$d(x, Tx) + f(Tx) \leq f(x) \quad \forall x \in X,$$

where $f : X \to (-\infty, \infty]$ is a proper, bounded below and lower semicontinuous function. Then there exists $u \in X$ such that $u = Tu$ and $f(u) < \infty$.

There are many generalizations of Theorem 1.1. One of them which we concern is the result using w-distances introduced by Kada, Suzuki and Takahashi [4].

Let (X, d) be a metric space. Recall that a function $p : X \times X \to [0, \infty)$ is a w-*distance* if the following conditions are satisfied:

(w1) $p(x, y) \leq p(x, z) + p(z, y)$ for all $x, y, z \in X$;
(w2) For each $x \in X$, the function $y \mapsto p(x, y)$ is lower semicontinuous;
(w3) For each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(y, z) < \varepsilon$ whenever $p(x, y) < \delta$ and $p(x, z) < \delta$.

It is obvious that d is a w-distance. Using this notion, the following two interesting results were proved in [4].

Theorem 1.2 (Kada, Suzuki and Takahashi). *Let (X, d) be a complete metric space. Let $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

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(a1) *There exist a proper and bounded below function* $f : X \to (-\infty, \infty]$ *and a* w*-distance* p *such that*

$$f(Tx) + p(x, Tx) \leq f(x) \quad \forall x \in X;$$

(a2) *The function* f *is lower semicontinuous.*

Then there exists $u \in X$ *such that* $u = Tu$ *and* $f(u) < \infty$.

Obviously, Theorem 1.2 is an extension of Theorem 1.1.

Theorem 1.3 (Kada, Suzuki and Takahashi). *Let* (X, d) *be a complete metric space. Let* $T : X \to X$ *be a mapping. Suppose that the following conditions hold:*

(b1) *There exist a number* $r \in [0, 1)$ *and a* w*-distance* p *such that*

$$p(Tx, T^2x) \leq rp(x, Tx) \quad \forall x \in X;$$

(b2) *For each* $y \in X$ *with* $y \neq Ty$,

$$\inf\{p(x, y) + p(x, Tx) : x \in X\} > 0.$$

Then there exists $u \in X$ *such that* $u = Tu$. *Moreover, if* $v = Tv$, *then* $p(v, v) = 0$.

The purpose of this paper is to present two fixed point theorems which slightly extend Theorems 1.2 and 1.3 respectively. Using our two results we discuss two interesting results. The first result proved by Chuang, Lin and Takahashi [2] is extended by replacing the metric by a w-distance. As a consequence of a particular case of this result with some additional result we deduce the second result recently proved by Takahashi, Wong and Yao [7].

Since we work on the w-distance, we also need the following lemma proved in [4].

Lemma 1.4. *Let* (X, d) *be a metric space and let* p *be a* w*-distance on* X. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences in* X, *let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0, \infty)$ *converging to* 0, *and let* $x, y, z \in X$. *Then the followings hold:*

(a) *If* $p(x_n, y_n) \leq \alpha_n$ *and* $p(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* $\{y_n\}$ *converges to* z;

(b) *If* $p(x_n, x_m) \leq \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence.*

2. Some extensions on Theorem 1.2 and Theorem 1.3

Let (X, d) be a metric space and $T : X \to X$ be any mapping. Let us compare (a1) of Theorem 1.2 and (b1) of Theorem 1.3. Suppose that (b1) holds, that is, there exist a number $r \in [0, 1)$ and a w-distance p such that

$$p(Tx, T^2x) \leq rp(x, Tx) \quad \forall x \in X.$$

It follows then that

$$f(Tx) + p(x, Tx) \leq f(x),$$

where

$$f(x) := \frac{1}{1-r} p(x, Tx).$$

It is clear that this f is proper and bounded below. However, we do not know that whether or not f is lower semicontinuous.

Now, if we assume that $f(x) = \frac{1}{1-r}p(x, Tx)$ is lower semicontinuous, then we have (b2) of Theorem 1.3. To see this, suppose that f is lower semicontinuous and let us assume that

$$\inf\{p(x, y) + p(x, Tx) : x \in X\} = 0$$

for some $y \in X$. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} p(x_n, y) = 0$ and $\lim_{n\to\infty} p(x_n, Tx_n) = 0$. It follows from (w3) of p that $\lim_{n\to\infty} d(Tx_n, y) = 0$. Since f is lower semicontinuous, we have

$$\begin{aligned} f(y) &\leq \liminf_{n\to\infty} f(Tx_n) \\ &= \frac{1}{1-r} \liminf_{n\to\infty} p(Tx_n, T^2x_n) \\ &\leq \frac{r}{1-r} \liminf_{n\to\infty} p(x_n, Tx_n) = 0. \end{aligned}$$

This implies that $p(y, Ty) = 0$. Now, we have

$$\lim_{n\to\infty} p(x_n, Ty) \leq \lim_{n\to\infty} p(x_n, y) + p(y, Ty) = 0.$$

By (w3) of p, we get $y = Ty$. So (b2) is satisfied.

It is interesting to discuss a fixed point theorem in the presence of conditions (a1) and (b2). The following theorem is referred as Theorem 2.1. It is an extension of Theorem 1.3.

Theorem 2.1. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

(a1) *There exist a proper and bounded below function $f : X \to (-\infty, \infty]$ and a w-distance p such that*

$$f(Tx) + p(x, Tx) \leq f(x) \quad \forall x \in X;$$

(b2) *For each $y \in X$ with $y \neq Ty$,*

$$\inf\{p(x, y) + p(x, Tx) : x \in X\} > 0.$$

Then, for each $x \in X$ with $f(x) < \infty$, the Picard sequence $\{T^n x\}$ converges to a fixed point of T.

Lemma 2.2. *Let (X, d) be a complete metric space and let p be a w-distance on X. Let $\{x_n\}$ be a sequence in X. If $\sum_{n=0}^{\infty} p(x_n, x_{n+1}) < \infty$, then $\{x_n\}$ is a Cauchy sequence and hence $\lim_{n\to\infty} d(x_n, z) = 0$ for some $z \in X$. Moreover, we also have $\lim_{n\to\infty} p(x_n, z) = 0$.*

Proof. For each natural numbers n and k, we have

$$p(x_n, x_{n+k}) \leq \sum_{j=n}^{n+k-1} p(x_j, x_{j+1}) \leq \sum_{j=n}^{\infty} p(x_j, x_{j+1}).$$

Since $\sum_{n=0}^{\infty} p(x_n, x_{n+1}) < \infty$, it follows from Lemma 1 with $\alpha_n = \sum_{j=n}^{\infty} p(x_j, x_{j+1})$ that $\{x_n\}$ is a Cauchy sequence. So there exists an element $z \in X$ such that $\lim_{n\to\infty} d(x_n, z) = 0$. Moreover, for each natural number n, we have

$$p(x_n, z) \leq \liminf_{k\to\infty} p(x_n, x_{n+k}) \leq \sum_{j=n}^{\infty} p(x_j, x_{j+1}).$$

Again, it follows from $\sum_{n=0}^{\infty} p(x_n, x_{n+1}) < \infty$ that $\lim_{n\to\infty} p(x_n, z) = 0$. □

Proof of Theorem 2.1. Let $x_0 \in X$ be such that $f(x_0) < \infty$. It follows from (a1) that

$$f(x_{n+1}) + p(x_n, x_{n+1}) \le f(x_n)$$

for each $n \ge 0$ where $x_{n+1} = Tx_n$ for each $n \ge 0$. Hence $\{f(x_n)\}$ is a nonincreasing sequence. Since f is bounded below, the limit $\lim_{n\to\infty} f(x_n)$ exists (and finite). Consequently, $\sum_{n=0}^{\infty} p(x_n, x_{n+1}) < \infty$. By Lemma 2.2, there exists an element $u \in X$ such that $\lim_{n\to\infty} p(x_n, u) = 0$. Moreover, we have

$$\inf\{p(x, u) + p(x, Tx) : x \in X\} \le \lim_{n\to\infty} p(x_n, u) + p(x_n, x_{n+1}) = 0.$$

It follows from (b2) that $u = Tu$. This completes the proof. □

We slightly extend Theorem 1.2 by weakening the lower semicontinuity of f.

Definition 2.3. Let (X, d) be a metric space and let p be a w-distance. We say that a mapping $f : X \to (-\infty, \infty]$ is lower semicontinuous type if

$$\lim_{n\to\infty} d(y_n, z) = \lim_{n\to\infty} p(y_n, z) = 0 \implies f(z) \le \liminf_{n\to\infty} f(y_n)$$

whenever $\{y_n\}$ is a sequence in X and $z \in X$.

Remark 2.4. Obviously, lower semicontinuity $\implies$ lower semicontinuity type.

Theorem 2.5. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

(a1) *There exist a proper and bounded below function $f : X \to (-\infty, \infty]$ and a w-distance p such that for all $x \in X$ satisfying*

$$f(Tx) + p(x, Tx) \le f(x);$$

(a2*) *f is lower semicontinuous type.*

Then, there exists an element $u \in X$ such that $u = Tu$ and $f(u) < \infty$. In particular, $p(u, u) = 0$.

Proof. The proof is very similar to the original one but it is presented here for the sake of completeness. Let $S : X \to 2^X$ be a mapping defined by

$$S(y) := \{w \in X : f(w) + p(y, w) \le f(y)\} \quad \forall y \in X.$$

It is noted that $Ty \in S(y)$ for all $y \in X$. Let $x_0 \in X$ be such that $f(x_0) < \infty$. Let $\{x_n\}$ be an iterative sequence such that $x_{n+1} \in S(x_n)$ and

$$f(x_{n+1}) \le \inf_{u \in S(x_n)} f(u) + \frac{1}{n+1} \quad \forall n \ge 0.$$

Since $x_{n+1} \in S(x_n)$, we obtain $f(x_{n+1}) + p(x_n, x_{n+1}) \le f(x_n)$. Since f is bounded below and $\{f(x_n)\}$ is nonincreasing, the limit $\lim_{n\to\infty} f(x_n)$ exists (and finite) which implies that $\sum_{n=0}^{\infty} p(x_n, x_{n+1}) < \infty$. It follows from Lemma 2.2 that there is $u \in X$ such that $\lim_{n\to\infty} d(x_n, u) = \lim_{n\to\infty} p(x_n, u) = 0$. By (a2*), we have

$$f(u) \le \liminf_{n\to\infty} f(x_n).$$

Note that if $x \in S(y)$ and $y \in S(z)$, then $x \in S(z)$. Hence $x_{n+k} \in S(x_n)$ for all $n \geq 0$ and $k \geq 1$, that is,

$$f(x_{n+k}) + p(x_n, x_{n+k}) \leq f(x_n).$$

In particular, we have

$$\begin{aligned} f(u) + p(x_n, u) &\leq \liminf_{k\to\infty} f(x_{n+k}) + p(x_n, x_{n+k}) \\ &\leq f(x_n) \quad \forall n \geq 0. \end{aligned}$$

This implies that $u \in \cap_{n\geq 0} S(x_n)$ and $f(u) < \infty$. Suppose that there is an element $z \in \cap_{n\geq 0} S(x_n)$. Note that $f(z) < \infty$ and

$$f(z) + p(x_{n+1}, z) \leq f(x_{n+1}) \leq \inf_{u\in S(x_n)} f(u) + \frac{1}{n+1} \leq f(z) + \frac{1}{n+1} \quad \forall n \geq 0.$$

Then $\lim_{n\to\infty} p(x_n, z) = 0$. Since $\lim_{n\to\infty} p(x_n, u) = 0$, we have $u = z$. Hence $\{u\} = \cap_{n\geq 0} S(x_n)$. Since $Tu \in S(u) \subset \cap_{n\geq 0} S(x_n)$, we get $u = Tu$. □

3. Discussions on some recent fixed point theorems which are deduced from our Theorems 2.5 and 2.1

We recall the following concept first. Let l^∞ be the Banach space of bounded real sequences with the supremum norm. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. For $x = (x_1, x_2, x_3, \ldots)$, the value $\mu(x)$ is also denoted by $\mu_n(x_n)$.

The following result was recently proved by Chuang, Lin and Takahashi (see [2, Theorem 3.1]).

Theorem 3.1 (Chuang, Lin and Takahashi)**.** *Let (X, d) be a complete metric space, let μ be a mean on l^∞, let $\{x_n\}$ be a bounded sequence in X, and let $\psi : X \to (-\infty, \infty]$ be a proper, bounded below, and lower semicontinuous function. Let $T : X \to X$ be a mapping. Suppose that there exists $m \in \mathbb{N} \cup \{0\}$ such that*

$$\mu_n d(x_n, T^m y) + \psi(Ty) \leq \psi(y), \quad \forall y \in X. \tag{3.1}$$

Then there exists an element $u \in X$ such that

(a) *$\mu_n d(x_n, u) = 0$;*
(b) *$u = \lim_{k\to\infty} T^k y$ for all $y \in X$ with $\psi(y) < \infty$;*
(c) *$\psi(u) = \inf\{\psi(v) : v \in X\}$;*
(d) *$u = Tu$;*
(e) *If there exists $v \in X$ with $v = Tv$ and $\psi(v) < \infty$, then $u = v$.*

Remark 3.2. It should be noted that Theorem 3.1 of [2] is not stated as above. In fact, in place of (d) and (e) above, it was stated that

(d') u is a unique fixed point of T.

We find that (d') is not correct.

Remark 3.3. Theorem 3.1 is very interesting because:

- The contractive condition is expressed in terms of means. Various mappings recently introduced in [3, 5, 6] satisfy this condition.

- The conclusion of Theorem 3.1 simultaneously presents the existence a common solution of the problem of finding a fixed point of a given mapping and that of finding a minimizer of a given lower semicontinuous function.
- The solution above is unique and can be approximated by a simple iterative sequence.

We will generalize Theorem 3.1 by replacing a metric d in (3.1) by a symmetric w-distance p. We start with the following lemma which does not require the completeness of the space.

Lemma 3.4. *Let (X, d) be a metric space and p be a w-distance. Let $T : X \to X$ be a mapping and let $\psi : X \to (-\infty, \infty]$ be a proper and bounded below function. Suppose that there exist $u \in X$ and $m \in \mathbb{N} \cup \{0\}$ such that $p(u, u) = 0$ and*

$$p(u, T^m y) + \psi(Ty) \leq \psi(y) \quad \forall y \in X.$$

Then

(a) $\lim_{k\to\infty} T^k y = u$ *for all* $y \in X$ *with* $\psi(y) < \infty$;
(b) *If there exists* $v \in X$ *with* $v = Tv$ *and* $\psi(v) < \infty$, *then* $u = v$;
(c) *If* $m \neq 0$, $u = T^m u$ *and* $\psi(u) < \infty$, *then* $u = Tu$;
(d) *If* ψ *is lower semicontinuous, then* $\psi(u) = \inf\{\psi(v) : v \in X\}$.

Proof. Let $y \in X$ be such that $\psi(y) < \infty$. So we obtain $p(u, T^{m+k} y) \leq \psi(T^k y) - \psi(T^{k+1} y)$ for all $k \geq 0$. Hence

$$\sum_{k=0}^{\infty} p(u, T^{m+k} y) \leq \psi(y) - \lim_{k\to\infty} \psi(T^k y) < \infty.$$

So $\lim_{k\to\infty} p(u, T^k y) = 0$. It follows from Lemma 1.4 and $p(u, u) = 0$ that $\lim_{k\to\infty} T^k y = u$, that is, (a) holds.

(b) and (c) follow immediately from (a).

To see (d), we assume that ψ is lower semicontinuous. Note that $\psi(Ty) \leq \psi(y)$ for all $y \in X$. It follows from (a) that

$$\psi(u) \leq \liminf_{k\to\infty} \psi(T^k y) \leq \psi(y), \quad \forall y \in X \quad \text{with} \quad \psi(y) < \infty.$$

That is, $\psi(u) = \inf\{\psi(y) : y \in X\}$. □

3.1. An improvement of Theorem 3.1 where d is replaced by a symmetric w-distance p.

Lemma 3.5. *Let (X, d) be a metric space and p be a w-distance. Let $\{x_n\}$ be a sequence in X such that $\{p(x_n, x)\}$ is bounded for some $x \in X$. Let μ be a mean on l^∞. Then*

$$\lim_{k\to\infty} p(y_k, y) = \lim_{k\to\infty} p(y, y_k) = 0 \Rightarrow \lim_{k\to\infty} \mu_n p(x_n, y_k) = \mu_n p(x_n, y)$$

whenever $\{y_k\}$ is a sequence in X and $y \in X$.

Proof. Since $\{p(x_n, x)\}$ is bounded for some $x \in X$, it follows that $\mu_n p(x_n, z)$ is well-defined for all $z \in X$. Let $\{y_k\}$ be a sequence in X and $y \in X$ such that $\lim_{k\to\infty} p(y_k, y) = \lim_{k\to\infty} p(y, y_k) = 0$. It follows that

$$\begin{aligned}
\mu_n p(x_n, y) &\le \liminf_{k\to\infty}(\mu_n p(x_n, y_k) + p(y_k, y)) \\
&= \liminf_{k\to\infty} \mu_n p(x_n, y_k) \\
&\le \limsup_{k\to\infty} \mu_n p(x_n, y_k) \\
&\le \mu_n p(x_n, y) + \limsup_{k\to\infty} p(y, y_k) \\
&= \mu_n p(x_n, y).
\end{aligned}$$

□

We are ready to prove the following result via Theorem 2.5.

Theorem 3.6. *Let (X, d) be a complete metric space, let p be a symmetric w-distance, let μ be a mean on l^∞, let $\{x_n\}$ be a sequence in X such that $\{p(x_n, x)\}$ is bounded for some $x \in X$ and let $\psi : X \to (-\infty, \infty]$ be a proper, bounded below and lower semicontinuous function. Let $T : X \to X$ be a mapping satisfying that there exists $m \in \mathbb{N} \cup \{0\}$ such that*

$$\mu_n p(x_n, T^m y) + \psi(Ty) \le \psi(y), \quad \forall y \in X. \tag{3.2}$$

Then there exists an element $u \in X$ such that

- (a) *$\mu_n p(x_n, u) = 0$;*
- (b) *$u = \lim_{k\to\infty} T^k y$ for all $y \in X$ with $\psi(y) < \infty$;*
- (c) *$\psi(u) = \inf\{\psi(v) : v \in X\}$;*
- (d) *$u = Tu$;*
- (e) *If there exists $v \in X$ with $v = Tv$ and $\psi(v) < \infty$, then $u = v$.*

Proof of Theorem 3.6 where $m \neq 0$. Since $\psi(Ty) \le \psi(y)$ for all $y \in X$, we have $\psi(T^m y) \le \psi(Ty)$. Then

$$\mu_n p(x_n, T^m y) + \psi(T^m y) \le \psi(y) \quad \forall y \in X. \tag{3.3}$$

We set

$$\widehat{p}(x, y) = \frac{1}{2}p(x, y) \text{ and } \widehat{\psi}(y) = \psi(y) + \frac{1}{2}\mu_n p(x_n, y) \quad \forall x, y \in X.$$

Obviously, $\widehat{p}$ is a w-distance and $\widehat{\psi}$ is lower semicontinuous type with respect to $\widehat{p}$. Note that, for each $y \in X$, we have

$$\begin{aligned}
\widehat{p}(y, T^m y) + \widehat{\psi}(T^m y) &= \frac{1}{2}p(y, T^m y) + \psi(T^m y) + \frac{1}{2}\mu_n p(x_n, T^m y) \\
&\le \frac{1}{2}\mu_n p(x_n, y) + \frac{1}{2}\mu_n p(x_n, T^m y) + \psi(Ty) + \frac{1}{2}\mu_n p(x_n, T^m y) \\
&= \frac{1}{2}\mu_n p(x_n, y) + \mu_n p(x_n, T^m y) + \psi(Ty) \\
&\le \frac{1}{2}\mu_n p(x_n, y) + \psi(y) \\
&= \widehat{\psi}(y).
\end{aligned}$$

Using Theorem 2.5, there is $u \in X$ such that $T^m u = u$ and $\widehat{\psi}(u) < \infty$. Thus $\psi(u) < \infty$. By the inequality (3.3), we get

$$\mu_n p(x_n, u) + \psi(u) \le \psi(u).$$

Hence $\mu_n p(x_n, u) = 0$, that is, (a) holds. Moreover, $p(u, u) \le 2\mu_n p(x_n, u) = 0$. Then

$$p(u, T^m y) \le \mu_n p(x_n, u) + \mu_n p(x_n, T^m y) \le \mu_n p(x_n, T^m y) \quad \forall y \in X$$

which implies that

$$p(u, T^m y) + \psi(Ty) \le \psi(y) \quad \forall y \in X.$$

Using Lemma 3.4, we get that (b), (c), (d) and (e) hold. The proof is complete. □

Proof of Theorem 3.6 where $m = 0$. We put

$$\widehat{p}(x, y) = \frac{1}{2} p(x, y) \quad \text{and} \quad \widehat{\psi}(y) = \psi(y) - \frac{1}{2}\mu_n p(x_n, y)$$

for all $x, y \in X$. Obviously, $\widehat{p}$ is a w-distance and $\widehat{\psi}$ is lower semicontinuous type with respect to $\widehat{p}$.

Then we have the following

$$\begin{aligned}
\widehat{p}(y, Ty) + \widehat{\psi}(Ty) &= \frac{1}{2} p(y, Ty) + \psi(Ty) - \frac{1}{2}\mu_n p(x_n, Ty) \\
&\le \frac{1}{2}\mu_n p(x_n, y) + \frac{1}{2}\mu_n p(x_n, Ty) + \psi(Ty) - \frac{1}{2}\mu_n p(x_n, Ty) \\
&= \mu_n p(x_n, y) + \psi(Ty) - \frac{1}{2}\mu_n p(x_n, y) \\
&\le \psi(y) - \frac{1}{2}\mu_n p(x_n, y) = \widehat{\psi}(y).
\end{aligned}$$

Using Theorem 2.5, there is u such that $u = Tu$ and $\widehat{\psi}(u) < \infty$ which imply that $\psi(u) < \infty$, that is, (d) holds. Moreover, it follows from (3.2) that

$$\mu_n d(x_n, u) \le \psi(u) - \psi(Tu) = 0.$$

We obtain that $\mu_n d(x_n, u) = 0$, that is, (a) holds and then

$$d(u, y) \le \mu_n d(x_n, u) + \mu_n d(x_n, y) = \mu_n d(x_n, y) \quad \forall y \in X.$$

Consequently, we obtain

$$d(u, y) + \psi(Ty) \le \psi(y) \quad \forall y \in X.$$

By using Lemma 3.4, we get that (b), (c) and (e) hold. □

3.2. A supplement to Theorem 3.6 in the absence of the lower semicontinuity of ψ.

Theorem 3.7. *Let (X, d) be a complete metric space, let p be a symmetric w-distance, let μ be a mean on l^∞, let $\{x_n\}$ be a sequence in X such that $\{p(x_n, x)\}$ is bounded for some $x \in X$ and let $\psi : X \to (-\infty, \infty)$ be a bounded below function. Let $T : X \to X$ be a mapping. Assume that one of the following statements is true.*

(♠) *There exists $m \in \mathbb{N}$ such that*
- $\mu_n p(x_n, T^m y) + \psi(Ty) \le \psi(y)\ \forall y \in X$;

- $\inf\{p(x,z)+p(x,T^m x) : x \in X\} > 0 \ \forall z \in X$ *with* $z \neq T^m z$.

($\heartsuit$) T *satisfies the following conditions*

- $\mu_n p(x_n, y) + \psi(Ty) \leq \psi(y) \ \forall y \in X$;
- $\inf\{p(x,z)+p(x,Tx) : x \in X\} > 0 \ \forall z \in X$ *with* $z \neq Tz$.

Then there exists an element $u \in X$ *such that*

(a) $\mu_n p(x_n, u) = 0$;
(b) $u = \lim_{k\to\infty} T^k y$ *for all* $y \in X$;
(c) u *is a unique fixed point of* T.

Proof. Note that $\psi(y) < \infty$ for all $y \in X$. First, we assume that the condition ($\spadesuit$) holds. As in the proof of Theorem 3.6 where $m \neq 0$, we have

$$\widehat{p}(y, T^m y) + \widehat{\psi}(T^m y) \leq \widehat{\psi}(y) \quad \forall y \in X.$$

By using Theorem 2.1, there is $u \in X$ such that $u = T^m u$. Then we can follow the result of the proof of Theorem 3.6 where $m \neq 0$ and obtain the conclusions via Lemma 3.4.

The conclusions for the condition ($\heartsuit$) can be obtained by following the proof of Theorem 3.6 where $m = 0$ and applying Theorem 2.1 and Lemma 3.4. So it is omitted. $\square$

We can deduce a recent result proved by Takahashi, Wong and Yao [7, Theorem 3.2] from our Theorem 3.7.

Theorem 3.8. *Let* (X, d) *be a complete metric space, let* p *be a symmetric* w*-distance and let* $\{x_n\}$ *be a sequence in* X *such that* $\{p(x_n, x)\}$ *is bounded for some* $x \in X$. *Let* T *be a mapping of* X *into itself. Suppose that there exist a real number* $r \in [0,1)$ *and a mean* μ *on* l^∞ *such that*

$$\mu_n p(x_n, Ty) \leq r \mu_n p(x_n, y), \forall y \in X.$$

Then, the followings hold:

(a) T *has a unique fixed point* u *in* X;
(b) $u = \lim_{k\to\infty} T^k y$ *for all* $y \in X$.

Proof. To prove this theorem by using Theorem 3.7, we show that the condition ($\heartsuit$) holds. Let $y \in X$. We have

$$(1-r)\mu_n p(x_n, y) + \mu_n p(x_n, Ty) \leq r\mu_n p(x_n, y) + (1-r)\mu_n p(x_n, y)$$

which implies that

$$\mu_n p(x_n, y) + \frac{1}{1-r}\mu_n p(x_n, Ty) \leq \frac{1}{1-r}\mu_n p(x_n, y),$$

that is, $\mu_n p(x_n, y) + \varphi(Ty) \leq \varphi(y)$ where $\varphi(y) = \frac{1}{1-r}\mu_n p(x_n, y)$. Next, we assume that there exist an element $z \in X$ and a sequence $\{y_m\}$ in X such that

$$\lim_{m\to\infty} p(y_m, z) = \lim_{m\to\infty} p(y_m, Ty_m) = 0.$$

It follows from

$$\lim_{m\to\infty} p(Ty_m, z) \leq \lim_{m\to\infty} (p(y_m, Ty_m) + p(y_m, z)) = 0$$

and Lemma 3.5 that

$$\lim_{m\to\infty} \mu_n p(x_n, y_m) = \mu_n p(x_n, z) = \lim_{m\to\infty} \mu_n p(x_n, Ty_m).$$

Therefore,

$$\mu_n p(x_n, z) = \lim_{m\to\infty} \mu_n p(x_n, Ty_m) \leq r \lim_{m\to\infty} \mu_n p(x_n, y_m) = r\mu_n p(x_n, z)$$

which implies that

$$\mu_n p(x_n, z) = 0.$$

In particular, $p(z, Tz) \leq \mu_n p(x_n, z) + \mu_n p(x_n, Tz) \leq (1 + r)\mu_n p(x_n, z) = 0$. Moreover, $p(z, z) \leq 2p(z, Tz) = 0$. It follows from Lemma 1.4 that $z = Tz$. □

Remark 3.9. It is worth mentioning that the mean μ in Theorems 3.6, 3.7 and 3.8 can be replaced by $\limsup$.

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U-FLATNESS AND NON-EXPANSIVE MAPPINGS IN BANACH SPACES

Ji Gao and Satit Saejung

Abstract. In this paper, we define the modulus of n-dimensional U-flatness as the determinant of an $(n+1)\times(n+1)$ matrix. The properties of the modulus are investigated and the relationships between this modulus and other geometric parameters of Banach spaces are studied. Some results on fixed point theory for non-expansive mappings and normal structure in Banach spaces are obtained.

1. Introduction

Let X be a real Banach space with the dual space X^*. Denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. Recall that $\nabla_x \subset S_{X^*}$ denotes the set of norm 1 supporting functionals of $x \in S_X$.

Brodskiĭ and Mil'man [2] introduced the following geometric concepts in 1948:

Definition 1.1. Let X be a Banach space. A nonempty bounded and convex subset K of X is said to have *normal structure* if for every convex subset C of K that contains more than one point there is a point $x_0 \in C$ such that

$$\sup\{\|x_0 - y\| : y \in C\} < \operatorname{diam} C.$$

A Banach space X is said to have

- *normal structure* if every bounded convex subset of X has normal structure;
- *weak normal structure* if every weakly compact convex set K of X has normal structure;
- *uniform normal structure* if there exists $0 < c < 1$ such that for every bounded closed convex subset C of K that contains more than one point there is a point $x_0 \in C$ such that

$$\sup\{\|x_0 - y\| : y \in C\} < c \cdot \operatorname{diam} C.$$

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Remark 1.2. The following facts are known.

- uniform normal structure $\Longrightarrow$ normal structure $\Longrightarrow$ weak normal structure.
- In the setting of reflexive spaces, normal structure $\Longleftrightarrow$ weak normal structure.

Kirk [9] proved that if a Banach space X has weak normal structure, then it has weak fixed point property, that is, every non-expansive mapping from a weakly compact and convex subset of X into itself has a fixed point.

Let $\mathbb{N}$ be the set of all natural numbers and $n \in \mathbb{N}$.

For two sets of vectors $\{x_i\}_{i=1}^{n+1} \subseteq X$ and $\{f_i\}_{i=2}^{n+1} \subseteq X^*$, the following $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \langle x_1, f_2 \rangle & \langle x_2, f_2 \rangle & \cdots & \langle x_{n+1}, f_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_1, f_{n+1} \rangle & \langle x_2, f_{n+1} \rangle & \cdots & \langle x_{n+1}, f_{n+1} \rangle \end{bmatrix}$$

is denoted by $m(x_1, x_2, \ldots, x_{n+1}; f_2, f_3, \ldots, f_{n+1})$ [6].

Gao and Saejung [6] introduced the concept of volume by the convex hull of $x_1, x_2, \ldots, x_{n+1}$ in X of

$$v(x_1, x_2, \ldots, x_{n+1}) := \sup\{\det m(x_1, x_2, \ldots, x_{n+1}; f_2, f_3, \ldots, f_{n+1})\},$$

where the supremum is taken over all $f_i \in \nabla_{x_i}$, where $i = 2, 3, \ldots, n+1$.

Definition 1.3 ([6]). Let $\nu_X^n = \sup\{v(x_1, x_2, \ldots, x_{n+1}) : x_1, x_2, \ldots x_{n+1} \in S_X\}$ be the upper bound of all n-dimensional volume in X.

Definition 1.4 ([6]). Let X be a Banach space. Define

$$U_X^n(\varepsilon) = \inf\left\{1 - \frac{1}{n+1}\|x_1 + x_2 + \cdots + x_{n+1}\| : \begin{array}{c} x_1, x_2, \ldots, x_{n+1} \in S_X, \\ v(x_1, x_2, \ldots, x_{n+1}) \geq \varepsilon \end{array}\right\},$$

where $0 \leq \varepsilon \leq \nu_X^n$ to be the modulus of n-dimensional U-convexity of X.

The following results were proved [6]:

Proposition 1.5. *For a Banach space X with $\dim(X) > n$, we have $\nu_X^n \geq 2$.*

Lemma 1.6. *$U_X^n(\varepsilon)$ is a continuous function in $[0, \nu_X^n)$.*

Theorem 1.7. *If X is a Banach space with $U_X^n(1) > 0$ for some $n \in \mathbb{N}$, then X is reflexive.*

Theorem 1.8. *If X is a Banach space with $U_X^n(1) > 0$ for some $n \in \mathbb{N}$, then X has normal structure.*

2. Main results

We introduce the concept of the modulus of n-dimensional flatness as follows:

Definition 2.1. Let X be a Banach space and $0 \le \varepsilon \le \nu_X^n$. Then the modulus of n-dimensional U-flatness of X is defined as follows:

$$W_X^n(\varepsilon) = \sup\left\{1 - \frac{1}{n+1}\|x_1 + x_2 + \cdots + x_{n+1}\|\right\},$$

where the supremum is taken over all $\{x_i\}_{i=1}^{n+1} \subseteq S_X$ such that there exist $\{f_i\}_{i=2}^{n+1} \subseteq S_{X^*}$ with $f_i \in \nabla_{x_i}$ for all $i = 2, \ldots, n+1$ and $\det m(x_1, x_2, \ldots, x_{n+1}; f_2, f_3, \ldots, f_{n+1}) \le \varepsilon$.

Remark 2.2. $W_X^n(\varepsilon)$ is an increasing and continuous function on $[0, \nu_X^n)$.

Proof. The proof is the same as that of Corollary 5 of [10]. □

Remark 2.3. The name of the modulus, U-flatness, is defined by comparing with Definition 1.4.

Lemma 2.4 (Bishop-Phelps-Bollobás [1])**.** *Let X be a Banach space, and let $0 < \varepsilon < 1$. Given $z \in B_X$ and $h \in S_{X^*}$ with $1 - \langle z, h\rangle < \frac{\varepsilon^2}{4}$, then there exist $y \in S_X$ and $g \in \nabla_y$ such that $\|y - z\| < \varepsilon$ and $\|g - h\| < \varepsilon$.*

Lemma 2.5. *Let $A_{n\times n}$ be the following $n \times n$ matrix*

$$A_{n\times n} := \begin{bmatrix} 1 & -1 & 1 & \cdots & (-1)^{n-1} & (-1)^{n-2} & (-1)^{n-1} \\ -\frac{1}{2} & 1 & -1 & \cdots & (-1)^{n+1} & (-1)^{n-1} & (-1)^n \\ 0 & -\frac{1}{2} & 1 & \cdots & (-1)^{n-1} & (-1)^n & (-1)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & 1 \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

Then $\det(A_{n\times n}) = \frac{1}{2^{n-1}}$.

Proof. It follows from mathematical induction:

By repeatedly using add $\frac{1}{2}$ times the first row to second row, then use the first row to estimate the determinant, we get the result. □

Lemma 2.6. *Let $B_{(n+1)\times(n+1)}$ be the following $(n+1) \times (n+1)$ matrix*

$$B_{(n+1)\times(n+1)} := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -\frac{1}{2} & 1 & -1 & \cdots & (-1)^{n-1} & (-1)^n & (-1)^{n+1} \\ 0 & -\frac{1}{2} & 1 & \cdots & (-1)^n & (-1)^{n+1} & (-1)^{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & 1 \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

Then $\det(B_{(n+1)\times(n+1)}) = \frac{2n+1}{2^n}$.

Proof. It follows from mathematical induction and the preceding lemma:

Let $n = 1, B_{2\times 2} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & 1 \end{bmatrix}, \det(B_{2\times 2}) = \frac{3}{2}$.

If for n, $\det(B_{n\times n}) = \frac{2n-1}{2^{n-1}}$, then for $n+1$, by using the first column to estimate the matrix, we have

$$\begin{aligned} \det(B_{(n+1)\times(n+1)}) &= \det(A_{n\times n}) + \frac{1}{2}\det(B_{n\times n}) \\ &= \frac{1}{2^{n-1}} + \frac{2n-1}{2^n} = \frac{2n+1}{2^n}. \end{aligned}$$

□

Theorem 2.7 ([7]). *Let X be a Banach space. Then X is not reflexive if and only if for any $0 < \delta < 1$ there are a sequence $\{x_n\} \subseteq S_X$ and a sequence $\{f_n\} \subseteq S_{X^*}$ such that*

(a) $\langle x_m, f_n\rangle = \delta$ *whenever* $n \le m$; *and*
(b) $\langle x_m, f_n\rangle = 0$ *whenever* $n > m$.

Theorem 2.8. *If X is a Banach space with $W_X^n(\frac{2n+1}{2^n}) < 1 - \frac{1}{n+1}$ for some $n \in \mathbb{N}$, then X is reflexive.*

Proof. Suppose that X is not reflexive. Let $0 < \delta < 1$ be given. Let $\{x_i\} \subseteq S_X$ and $\{f_i\} \subseteq S_{X^*}$ be two sequences satisfying the two conditions in Theorem 2.7.

Let $n \in \mathbb{N}$ be given. Let $y_i = (-1)^{i+1}\frac{x_i + x_{i+1}}{2}$ for $i = 1, \dots, n+1$ and $g_i = (-1)^{i+1} f_i \in S_{X^*}$ for $i = 2, \dots, n+1$. Then, we have

$$\delta \le \langle y_i, g_i\rangle = \left\langle (-1)^{i+1}\frac{x_i + x_{i+1}}{2}, (-1)^{i+1} f_i \right\rangle \le \frac{1}{2}\|x_i + x_{i+1}\| = \|y_i\| \le 1,$$

and

$$\begin{aligned} &\det m(y_1, y_2, y_3 \dots, y_{n-1}, y_n, y_{n+1}; g_2, g_3, g_4 \dots, g_{n-1}, g_n, g_{n+1}) \\ &= \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ \langle y_1, g_2\rangle & \langle y_2, g_2\rangle & \langle y_3, g_2\rangle & \cdots & \langle y_{n-1}, g_2\rangle & \langle y_n, g_2\rangle & \langle y_{n+1}, g_2\rangle \\ \langle y_1, g_3\rangle & \langle y_2, g_3\rangle & \langle y_3, g_3\rangle & \cdots & \langle y_{n-1}, g_3\rangle & \langle y_n, g_3\rangle & \langle y_{n+1}, g_3\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \langle y_1, g_{n-1}\rangle & \langle y_2, g_{n-1}\rangle & \langle y_3, g_{n-1}\rangle & \cdots & \langle y_{n-1}, g_{n-1}\rangle & \langle y_n, g_{n-1}\rangle & \langle y_{n+1}, g_{n-1}\rangle \\ \langle y_1, g_n\rangle & \langle y_2, g_n\rangle & \langle y_3, g_n\rangle & \cdots & \langle y_{n-1}, g_n\rangle & \langle y_n, g_n\rangle & \langle y_{n+1}, g_n\rangle \\ \langle y_1, g_{n+1}\rangle & \langle y_2, g_{n+1}\rangle & \langle y_3, g_{n+1}\rangle & \cdots & \langle y_2, g_{n+1}\rangle & \langle y_n, g_{n+1}\rangle & \langle y_{n+1}, g_{n+1}\rangle \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -\frac{\delta}{2} & \delta & -\delta & \cdots & (-1)^{n-1}\delta & (-1)^n\delta & (-1)^{n+1}\delta \\ 0 & -\frac{\delta}{2} & \delta & \cdots & (-1)^{n-2}\delta & (-1)^{n-1}\delta & (-1)^n\delta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \delta & -\delta & \delta \\ 0 & 0 & 0 & \cdots & -\frac{\delta}{2} & \delta & -\delta \\ 0 & 0 & 0 & \cdots & 0 & -\frac{\delta}{2} & \delta \end{bmatrix} \end{aligned}$$

$$= \delta^n \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -\frac{1}{2} & 1 & -1 & \cdots & (-1)^{n-1} & (-1)^n & (-1)^{n+1} \\ 0 & -\frac{1}{2} & 1 & \cdots & (-1)^{n-2} & (-1)^{n-1} & (-1)^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & 1 \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

By Lemmas 2.5 and 2.6, we have

$$\det m(y_1, y_2, \ldots, y_{n+1}; g_2, g_3, \ldots, g_{n+1}) = \delta^n \frac{2n+1}{2^n}.$$

On the other hand, since

$$\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} = \frac{\|(-1)^{n+2} x_{n+2} + x_1\|}{2(n+1)} \leq \frac{1}{n+1},$$

we have

$$1 - \frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} \geq 1 - \frac{1}{n+1}.$$

Since δ can be chosen arbitrarily closed to 1, let $\delta = 1 - \frac{\varepsilon^2}{4}$ where ε can be chosen arbitrarily closed to 0.

Let $z_1 = y_1$. Next, let $i = 2, 3, \ldots, n+1$. From Bishop-Phelps-Bollobás result (Lemma 2.4), there exist $z_i \in S_X$ and $h_i \in \nabla_{z_i}$ such that $\|y_i - z_i\| < \varepsilon$ and $\|g_i - h_i\| < \varepsilon$.

This implies that

$$|\langle z_i, h_j \rangle - \langle y_i, g_j \rangle| \leq |\langle z_i - y_i, g_j \rangle| + |\langle y_i, h_j - g_j \rangle| + |\langle z_i - y_i, h_j - g_j \rangle| \leq 3\varepsilon.$$

It follows then that

$$\det m(z_1, z_2, \ldots, z_{n+1}; h_2, h_3, \ldots, h_{n+1}) = \left(1 - \frac{\varepsilon^2}{4}\right)^n \frac{2n+1}{2^n} + c\varepsilon,$$

where c is a bounded constant. Moreover,

$$1 - \frac{\|z_1 + z_2 + \cdots + z_{n+1}\|}{n+1} \geq 1 - \frac{1+\varepsilon}{n+1}.$$

From the definition of $W_X^n(\varepsilon)$, we have

$$W_X^n\left(\left(1 - \frac{\varepsilon^2}{4}\right)^n \frac{2n+1}{2^n} + c\varepsilon\right) \geq 1 - \frac{1+\varepsilon}{n+1}.$$

Since ε can be arbitrarily close to 0, the theorem is proved. □

Let $C_{(n+1)\times(n+1)}$ be the following $(n+1)\times(n+1)$ matrix:

$$C_{(n+1)\times(n+1)} := \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -\frac{2}{3} & 1 & -1 & 1 & \cdots & (-1)^{n-1} & (-1)^{n} & (-1)^{n+1} \\ \frac{1}{3} & -\frac{2}{3} & 1 & -1 & \cdots & (-1)^{n} & (-1)^{n+1} & (-1)^{n+2} \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 & \cdots & (-1)^{n+1} & (-1)^{n+2} & (-1)^{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{2}{3} & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix}.$$

Then $\det(C_{2\times 2}) = \frac{5}{3}$, and $\det(C_{3\times 3}) = \frac{7}{9}$.

Theorem 2.9. *If X is a Banach space with $W_X^n(\det C_{(n+1)\times(n+1)}) < \frac{2}{3}$ for some $n \in \mathbb{N}$, then X is reflexive. In particular, for $n = 1$ we have if $W_X^1(\frac{5}{3}) < \frac{2}{3}$, then X is reflexive; and for $n = 2$ we have if $W_X^2(\frac{7}{9}) < \frac{2}{3}$, then X is reflexive.*

Proof. Suppose that X is not reflexive. Let $0 < \delta < 1$ be given. Let $\{x_i\} \subseteq S_X$ and $\{f_i\} \subseteq S_{X^*}$ be two sequences satisfying the two conditions in Theorem 2.7.

Let $n \in \mathbb{N}$ be given. Let $y_i = (-1)^{i+1}\frac{x_i+x_{i+1}+x_{i+2}}{3}$ for $i = 1, \ldots, n+1$ and $g_i = (-1)^{i+1} f_i \in S_{X^*}$ for $i = 2, \ldots, n+1$. Then, we have

$$\begin{aligned} \delta \le \langle y_i, g_i \rangle &= \left\langle (-1)^{i+1}\frac{x_i + x_{i+1} + x_{i+2}}{3}, (-1)^{i+1} f_i \right\rangle \\ &\le \frac{1}{3}\|x_i + x_{i+1} + x_{i+2}\| = \|y_i\| \le 1, \end{aligned}$$

and

$$\begin{aligned} & m(y_1, y_2, y_3, y_4, \ldots, y_{n-1}, y_n, y_{n+1}; g_2, g_3, g_4, \ldots, g_{n-1}, g_n, g_{n+1}) \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ \langle y_1, g_2\rangle & \langle y_2, g_2\rangle & \langle y_3, g_2\rangle & \langle y_4, g_2\rangle & \cdots & \langle y_{n-1}, g_2\rangle & \langle y_n, g_2\rangle & \langle y_{n+1}, g_2\rangle \\ \langle y_1, g_3\rangle & \langle y_2, g_3\rangle & \langle y_3, g_3\rangle & \langle y_4, g_3\rangle & \cdots & \langle y_{n-1}, g_3\rangle & \langle y_n, g_3\rangle & \langle y_{n+1}, g_3\rangle \\ \langle y_1, g_4\rangle & \langle y_2, g_4\rangle & \langle y_3, g_4\rangle & \langle y_4, g_4\rangle & \cdots & \langle y_{n-1}, g_4\rangle & \langle y_n, g_4\rangle & \langle y_{n+1}, g_4\rangle \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \langle y_1, g_{n-1}\rangle & \langle y_2, g_{n-1}\rangle & \langle y_3, g_{n-1}\rangle & \langle y_4, g_{n-1}\rangle & \cdots & \langle y_{n-1}, g_{n-1}\rangle & \langle y_n, g_{n-1}\rangle & \langle y_{n+1}, g_{n-1}\rangle \\ \langle y_1, g_n\rangle & \langle y_2, g_n\rangle & \langle y_3, g_n\rangle & \langle y_4, g_n\rangle & \cdots & \langle y_{n-1}, g_n\rangle & \langle y_n, g_n\rangle & \langle y_{n+1}, g_n\rangle \\ \langle y_1, g_{n+1}\rangle & \langle y_2, g_{n+1}\rangle & \langle y_3, g_{n+1}\rangle & \langle y_4, g_{n+1}\rangle & \cdots & \langle y_{n-1}, g_{n+1}\rangle & \langle y_n, g_{n+1}\rangle & \langle y_{n+1}, g_{n+1}\rangle \end{bmatrix} \\ &= \delta^n \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -\frac{2}{3} & 1 & -1 & 1 & \cdots & (-1)^{n-1} & (-1)^{n} & (-1)^{n+1} \\ \frac{1}{3} & -\frac{2}{3} & 1 & -1 & \cdots & (-1)^{n} & (-1)^{n+1} & (-1)^{n+2} \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 & \cdots & (-1)^{n+1} & (-1)^{n+2} & (-1)^{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{2}{3} & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix}. \end{aligned}$$

We have

$$\det m(y_1, y_2, y_3, y_4, \ldots, y_{n-1}, y_n, y_{n+1}; g_2, g_3, g_4, \ldots, g_{n-1}, g_n, g_{n+1}) \\ = \delta^n \det C_{(n+1)\times(n+1)}.$$

On the other hand, for $n \geq 2$,

$$\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} = \frac{\|x_1 + x_3 - x_4 + \cdots + (-1)^n x_{n+1} + (-1)^{n+2} x_{n+3}\|}{3(n+1)} \\ \leq \frac{n+1}{3(n+1)}\delta = \frac{1}{3}\delta,$$

and for $n = 1$,

$$\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} = \frac{\|x_1 - x_4\|}{6} \leq \frac{1}{3}\delta.$$

We have

$$1 - \frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} \geq 1 - \frac{1}{3}\delta \geq \frac{2}{3}\delta$$

for all $n \in \mathbb{N}$.

The theorem can be proved by using the Bishop-Phelps-Bollobás result (Lemma 2.4), and same idea in the proof of Theorem 2.8. □

We consider $n = 1$.

Theorem 2.10. *If X is a Banach space with $W_X^1(\frac{2m+1}{m+1}) < \frac{m}{m+1}$ for some $m \in \mathbb{N}$, then X is reflexive. In particular, for $m = 2$ we have if $W_X^1(\frac{5}{3}) < \frac{2}{3}$, then X is reflexive.*

Proof. Suppose that X is not reflexive. Let $0 < \delta < 1$ be given. Let $\{x_i\} \subseteq S_X$ and $\{f_i\} \subseteq S_{X^*}$ be two sequences satisfying the two conditions in Theorem 2.7.

Let $m \in \mathbb{N}$ be given. Let

$$y_1 = \frac{x_1 + x_2 + \cdots + x_m + x_{m+1}}{m+1}, y_2 = -\frac{x_2 + x_3 + \cdots + x_{m+1} + x_{m+2}}{m+1}$$

and $g_2 = -f_2 \in S_{X^*}$.

Consider the 2-dimensional subspace of X spanned by y_1 and y_2.

We have

$$\det m(y_1, y_2; g_2) = \det \begin{bmatrix} 1 & 1 \\ \langle y_1, g_2\rangle & \langle y_2, g_2\rangle \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ -\frac{m}{m+1} & 1 \end{bmatrix} \delta = \frac{2m+1}{m+1}\delta,$$

and

$$\left\|\frac{y_1 + y_2}{2}\right\| = \left\|\frac{x_1 - x_{m+2}}{2(m+1)}\right\| \leq \frac{1}{m+1}\delta.$$

This is

$$1 - \left\|\frac{y_1 + y_2}{2}\right\| \geq \frac{m}{m+1}\delta.$$

Similar to the proof of Theorem 2.8 we have

$$W_X^1\left(\frac{2m+1}{m+1}\right) \geq \frac{m}{m+1}.$$

This completes the proof. $\square$

In 2008, Saejung proved the following result:

Lemma 2.11 ([11]). *If X is a Banach space with B_{X^*} is weak* sequentially compact and it fails to have weak normal structure, then for any $\varepsilon > 0$ and $n \in \mathbb{N}$ there are $\{x_1, x_2, \ldots, x_n\} \subseteq S_X$ and $\{f_1, f_2, \ldots, f_n\} \subseteq S_{X^*}$ such that*

(a) $|\|x_i - x_j\| - 1| < \varepsilon$ *for all* $i \neq j$;
(b) $\langle x_i, f_i\rangle = 1$ *for all* $1 \leq i \leq n$; *and*
(c) $|\langle x_i, f_j\rangle| < \varepsilon$ *for all* $i \neq j$.

Theorem 2.12. *If X is a Banach space with B_{X^*} is weak* sequentially compact and $W_X^n(1) < 1 - \frac{1}{n+1}$ for some $n \in \mathbb{N}$, then X has weak normal structure.*

Proof. Suppose that X does not have weak normal structure. Let $0 < \varepsilon < 1$ be given. Then there are $\{x_i\}_{i=1}^{n+1} \subseteq S_X$ and $\{f_i\}_{i=1}^{n+1} \subseteq S_{X^*}$ satisfying the three conditions in Lemma 2.11.

For convenience, let $|\langle x_i, f_j\rangle| = \varepsilon_{i,j}$. Then $\varepsilon_{i,j} \leq \varepsilon$ for all $i \neq j$.

Let $y_i = \frac{x_{i+1} - x_i}{\|x_{i+1} - x_i\|} \in S_X$ for $i = 1, \ldots, n+1$ and $g_i = f_{i+1} \in S_{X^*}$ for $i = 2, \ldots, n+1$. Then

$$\|y_i - (x_{i+1} - x_i)\| \leq \varepsilon$$

for $i = 1, \ldots, n+1$. Moreover,

$$\begin{aligned}
&\|y_1 + y_2 + \cdots + y_i + \cdots + y_{n+1}\| \\
&\leq \|(x_2 - x_1) + (x_3 - x_2) + \cdots + (x_{i+1} - x_i) + \cdots + (x_{n+2} - x_{n+1})\| + (n+1)\varepsilon \\
&= \|x_{n+2} - x_1\| + (n+1)\varepsilon.
\end{aligned}$$

Next, we consider the following matrix:

$$\begin{aligned}
&m(y_1, y_2, \ldots, y_{n+1}; g_2, g_3, \ldots, g_{n+1}) \\
&= \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
\langle y_1, g_2\rangle & \langle y_2, g_2\rangle & \langle y_3, g_2\rangle & \cdots & \langle y_n, g_2\rangle & \langle y_{n+1}, g_2\rangle \\
\langle y_1, g_3\rangle & \langle y_2, g_3\rangle & \langle y_3, g_3\rangle & \cdots & \langle y_n, g_3\rangle & \langle y_{n+1}, g_3\rangle \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\langle y_1, g_n\rangle & \langle y_2, g_n\rangle & \langle y_3, g_n\rangle & \cdots & \langle y_n, g_n\rangle & \langle y_{n+1}, g_n\rangle \\
\langle y_1, g_{n+1}\rangle & \langle y_2, g_{n+1}\rangle & \langle y_3, g_{n+1}\rangle & \cdots & \langle y_n, g_{n+1}\rangle & \langle y_{n+1}, g_{n+1}\rangle
\end{bmatrix} \\
&= \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
\frac{\varepsilon_{2,3} - \varepsilon_{1,3}}{\|x_2 - x_1\|} & \frac{1 - \varepsilon_{2,3}}{\|x_3 - x_2\|} & \frac{\varepsilon_{4,3} - 1}{\|x_4 - x_3\|} & \cdots & \frac{\varepsilon_{n+1,3} - \varepsilon_{n,3}}{\|x_{n+1} - x_n\|} & \frac{\varepsilon_{n+2,3} - \varepsilon_{n+1,3}}{\|x_{n+2} - x_{n+1}\|} \\
\frac{\varepsilon_{2,4} - \varepsilon_{1,4}}{\|x_2 - x_1\|} & \frac{\varepsilon_{3,4} - \varepsilon_{2,4}}{\|x_3 - x_2\|} & \frac{1 - \varepsilon_{3,4}}{\|x_4 - x_3\|} & \cdots & \frac{\varepsilon_{n+1,4} - \varepsilon_{n,4}}{\|x_{n+1} - x_n\|} & \frac{\varepsilon_{n+2,4} - \varepsilon_{n+1,4}}{\|x_{n+2} - x_{n+1}\|} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\varepsilon_{2,n+1} - \varepsilon_{1,n+1}}{\|x_2 - x_1\|} & \frac{\varepsilon_{3,n+1} - \varepsilon_{2,n+1}}{\|x_3 - x_2\|} & \frac{\varepsilon_{4,n+1} - \varepsilon_{3,n+1}}{\|x_4 - x_3\|} & \cdots & \frac{1 - \varepsilon_{n,n+1}}{\|x_{n+1} - x_n\|} & \frac{\varepsilon_{n+2,n+1} - 1}{\|x_{n+2} - x_{n+1}\|} \\
\frac{\varepsilon_{2,n+2} - \varepsilon_{1,n+2}}{\|x_2 - x_1\|} & \frac{\varepsilon_{3,n+2} - \varepsilon_{2,n+2}}{\|x_3 - x_2\|} & \frac{\varepsilon_{4,n+2} - \varepsilon_{3,n+2}}{\|x_4 - x_3\|} & \cdots & \frac{\varepsilon_{n+1,n+2} - \varepsilon_{n,n+2}}{\|x_{n+1} - x_n\|} & \frac{1 - \varepsilon_{n+1,n+2}}{\|x_{n+2} - x_{n+1}\|}
\end{bmatrix}.
\end{aligned}$$

It follows then that

$$\det m(y_1, y_2, \ldots, y_{n+1}; g_2, g_3, \ldots, g_{n+1}) = 1 + c\varepsilon,$$

where c is a bounded constant.

On the other hand, since

$$\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} \le \frac{\|x_{n+2} - x_1\|}{n+1} + \varepsilon \le \frac{1+\varepsilon}{n+1} + \varepsilon,$$

we have

$$1 - \frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} \ge 1 - \frac{1+\varepsilon}{n+1} - \varepsilon.$$

Let $z_1 = y_1$. Next, let $i = 2, 3, \ldots, n+1$.

From Bishop-Phelps-Bollobás result (Lemma 2.4), there exist $z_i \in S_X$ and $h_i \in \nabla_{z_i}$ such that

$$\|y_i - z_i\| < \varepsilon \text{ and } \|g_i - h_i\| < \varepsilon.$$

In particular,

$$|\langle z_i, h_j\rangle - \langle y_i, g_j\rangle| \le |\langle z_i - y_i, g_j\rangle| + |\langle y_i, h_j - g_j\rangle| + |\langle z_i - y_i, h_j - g_j\rangle| \le 3\varepsilon.$$

This implies that

$$\begin{aligned} &\det m(z_1, z_2, \ldots, z_{n+1}, h_2, h_3, \ldots, h_{n+1}) \\ &= \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \langle z_1, h_2\rangle & \langle z_2, h_2\rangle & \langle z_3, h_2\rangle & \cdots & \langle z_n, h_2\rangle & \langle z_{n+1}, h_2\rangle \\ \langle z_1, h_3\rangle & \langle z_2, h_3\rangle & \langle z_3, h_3\rangle & \cdots & \langle z_n, h_3\rangle & \langle z_{n+1}, h_3\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle z_1, h_n\rangle & \langle z_2, h_n\rangle & \langle z_3, h_n\rangle & \cdots & \langle z_n, h_n\rangle & \langle z_{n+1}, h_n\rangle \\ \langle z_1, h_{n+1}\rangle & \langle z_2, h_{n+1}\rangle & \langle z_3, h_{n+1}\rangle & \cdots & \langle z_n, h_{n+1}\rangle & \langle z_{n+1}, h_{n+1}\rangle \end{bmatrix} \\ &= 1 + d\varepsilon, \end{aligned}$$

where d is a bounded constant. Hence

$$1 - \frac{\|z_1 + z_2 + \cdots + z_{n+1}\|}{n+1} \ge 1 - \frac{1+\varepsilon}{n+1} - 2\varepsilon.$$

Since ε can be arbitrarily small, it follows from the definition of $W_X^n(\cdot)$ that

$$W_X^n(1) \ge 1 - \frac{1}{n+1}.$$

This completes the proof. □

Theorem 2.13. *If X is a Banach space satisfying one of the following two conditions:*

- $W_X^n(1) < 1 - \frac{1}{n+1}$ *for some $n \in \mathbb{N}$ with $n \ge 2$; or*
- $W_X^1(1) < \frac{1}{2}$ *and* $W_X^1(\frac{5}{3}) < \frac{2}{3}$ *for $n = 1$.*

Then X has normal structure.

Proof. Since X is reflexive, it follows that B_{X^*} is weak* sequentially compact. Moreover, $\frac{2n+1}{2^n} < 1$ for $n \in \mathbb{N}$ and $n \ge 3$, and $\frac{7}{9} < 1$ for $n = 2$. The first result is a direct consequence of Theorems 2.8, 2.9 and 2.12. The second result is a direct consequence of Theorems 2.10 and 2.12. □

Definition 2.14 ([4, 5])**.** Let X and Y be Banach spaces. We say that Y is *finitely representable in* X if for any $\varepsilon > 0$ and any finite dimensional subspace $N \subseteq Y$ there is an isomorphism $T : N \to X$ such that for any $y \in N$,

$$(1-\varepsilon)\|y\| \leq \|Ty\| \leq (1+\varepsilon)\|y\|.$$

We say that X is *super-reflexive* if any space Y which is finitely representable in X is reflexive.

Theorem 2.15. *If X is a Banach space with $W_X^n(\frac{2n+1}{2^n}) < 1 - \frac{1}{n+1}$ for some $n \in \mathbb{N}$ and $n \geq 2$, or $W_X^1(\frac{2m+1}{m+1}) < \frac{m}{m+1}$ for $n = 1$ and some $m \in \mathbb{N}$, then X is super-reflexive. In particular, for $m = 2$ we have if $W_X^1(\frac{5}{3}) < \frac{2}{3}$, then X is super-reflexive.*

Proof. We only prove the first part (for $n \geq 2$). The proof of second part (for $n = 1$) is same.

The proof is similar to that of Theorem 2.12 in [6]. Suppose that X is not super-reflexive. Then there exists a nonreflexive Banach space Y such that Y can be finitely representable. From Remark 2.2 and Theorem 2.8, for each n there exists some positive function $f(\varepsilon)$ which goes to 0 as ε goes to 0, satisfies $W_Y^n(\frac{2n+1}{2^n} - \varepsilon) > 1 - \frac{1}{n+1} - f(\varepsilon)$. Therefore, there exist $\{y_i\}_{i=1}^{n+1} \subseteq S_Y$ and $\{g_i\} \in \nabla_{y_i} \subseteq S_{Y^*}$ for $i = 2, \ldots, n+1$ such that

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \langle y_1, g_2\rangle & \langle y_2, g_2\rangle & \cdots & \langle y_n, g_2\rangle & \langle y_{n+1}, g_2\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle y_1, g_{n+1}\rangle & \langle y_2, g_{n+1}\rangle & \cdots & \langle y_n, g_{n+1}\rangle & \langle y_{n+1}, g_{n+1}\rangle \end{bmatrix} \leq \frac{2n+1}{2^n} - \varepsilon,$$

and

$$1 - \frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} > 1 - \frac{1}{n+1} - f(\varepsilon).$$

Let $N = \operatorname{span}\{y_1, y_2, \ldots, y_{n+1}\}$, and $T : N \to M \subseteq X$ be an isomorphism with range M.

Let us consider the conjugate mapping T^* of T. Let $g_{i|N}$ be the restricting g_i on N. Then $\langle Ty_j, (T^*)^{-1}g_{i|N}\rangle = \langle y_j, g_i\rangle$ for $1 \leq i, j \leq n+1$.

We have

$$1 - \varepsilon \leq \|T\| \leq 1 + \varepsilon,$$

$$1 - \varepsilon \leq \|T^*\| \leq 1 + \varepsilon,$$

and

$$1 - \varepsilon \leq \|(T^*)^{-1}\| \leq 1 + \varepsilon.$$

Let $x_i = Ty_i$ and $f_i = (T^*)^{-1}g_{i|N}$ for $i = 1, \ldots, n+1$. Then

$$\langle x_j, f_i\rangle = \langle Ty_j, (T^*)^{-1}g_{i|N}\rangle = \langle y_j, g_i\rangle.$$

If $i = j$, then $\langle x_i, f_i\rangle = \langle y_i, g_i\rangle = 1$, so $f_i \in \nabla_{x_i}$ and we have

$$\begin{aligned}
&\det\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \langle x_1, f_2\rangle & \langle x_2, f_2\rangle & \cdots & \langle x_n, f_2\rangle & \langle x_{n+1}, f_2\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle x_1, f_{n+1}\rangle & \langle x_2, f_{n+1}\rangle & \cdots & \langle x_n, f_{n+1}\rangle & \langle x_{n+1}, f_{n+1}\rangle \end{bmatrix} \\
= &\det\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \langle y_1, g_2\rangle & \langle y_2, g_2\rangle & \cdots & \langle y_n, g_2\rangle & \langle y_{n+1}, g_2\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle y_1, g_{n+1}\rangle & \langle y_2, g_{n+1}\rangle & \cdots & \langle y_n, g_{n+1}\rangle & \langle y_{n+1}, g_{n+1}\rangle \end{bmatrix} \\
\le &\frac{2n+1}{2^n} - \varepsilon.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\|x_1 + x_2 + \cdots + x_{n+1}\|}{n+1} &= \frac{\|T(y_1 + y_2 + \cdots + y_{n+1})\|}{n+1} \\
&\le (1+\varepsilon)\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} \\
&\le \frac{1+\varepsilon}{n+1} + (1+\varepsilon)f(\varepsilon).
\end{aligned}$$

This implies that

$$1 - \frac{\|x_1 + x_2 + \cdots + x_{n+1}\|}{n+1} \ge 1 - \frac{1+\varepsilon}{n+1} - (1+\varepsilon)f(\varepsilon).$$

Since $f(\varepsilon)$ can be arbitrarily small, we have

$$W_X^n\left(\frac{2n+1}{2^n}\right) \ge 1 - \frac{1}{n+1}.$$

This completes the proof. □

We consider the uniform normal structure. To discuss this result, let us recall the concept of the "ultra"-technique.

Let $\mathcal{F}$ be a filter of an index set I, and let $\{x_i\}_{i\in I}$ be a subset in a Hausdorff topological space X, $\{x_i\}_{i\in I}$ is said to *converge to x with respect to $\mathcal{F}$*, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood V of x, $\{i \in I : x_i \in V\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an *ultrafilter* if it is maximal with respect to the ordering of the set inclusion. An ultrafilter is called *trivial* if it is of the form $\{A : A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$. We will use the fact that if $\mathcal{U}$ is an ultrafilter, then

(i) for any $A \subseteq I$, either $A \subseteq \mathcal{U}$ or $I - A \subseteq \mathcal{U}$;
(ii) if $\{x_i\}_{i\in I}$ has a cluster point x, then $\lim_{\mathcal{U}} x_i$ exists and equals to x.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i\in I}\|x_i\| < \infty$.

Definition 2.16 ([3, 12])**.** Let $\mathcal{U}$ be an ultrafilter on I and let $N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}$. The *ultra-product* of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm.

We will use $(x_i)_{\mathcal{U}}$ to denote the element of the ultra-product. It follows from remark (ii) above, and the definition of quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|. \tag{2.1}$$

In the following we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X, i \in \mathbb{N}$ for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we use $X_{\mathcal{U}}$ to denote the ultra-product. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $X_{\mathcal{U}}$ isometrically.

Lemma 2.17 ([12])**.** *Suppose that $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and X is a Banach space. Then $(X^*)_{\mathcal{U}} \cong (X_{\mathcal{U}})^*$ if and only if X is super-reflexive; and in this case, the mapping J defined by*

$$\langle (x_i)_{\mathcal{U}}, J((f_i)_{\mathcal{U}})\rangle = \lim_{\mathcal{U}} \langle x_i, f_i\rangle \quad \text{for all } (x_i)_{\mathcal{U}} \in X_{\mathcal{U}}$$

is the canonical isometric isomorphism from $(X^*)_{\mathcal{U}}$ onto $(X_{\mathcal{U}})^*$.

Theorem 2.18. *Let X be a super-reflexive Banach space. Then for any nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and for all $n \in \mathbb{N}$ and $\varepsilon > 0$, we have $W^n_{X_{\mathcal{U}}}(\varepsilon) = W^n_X(\varepsilon)$.*

Proof. Since X can be embedded into $X_{\mathcal{U}}$ isometrically, we may consider X as a subspace of $X_{\mathcal{U}}$. From the definition of $W^n_X(\varepsilon)$, we have $W^n_{X_{\mathcal{U}}}(\varepsilon) \geq W^n_X(\varepsilon)$.

We prove the reverse inequality.

For any very small $\eta > 0$, from the definition of $W^n_{X_{\mathcal{U}}}(\varepsilon)$, let $(x^1_i)_{\mathcal{U}}$, $(x^2_i)_{\mathcal{U}}$, $\ldots$, $(x^n_i)_{\mathcal{U}}$, $(x^{n+1}_i)_{\mathcal{U}}$ belong to $S_{X_{\mathcal{U}}}$, and let $(f^2_i)_{\mathcal{U}} \in \nabla_{(x^2_i)_{\mathcal{U}}}$, $(f^3_i)_{\mathcal{U}} \in \nabla_{(x^3_i)_{\mathcal{U}}}$, $\ldots$, $(f^n_i)_{\mathcal{U}} \in \nabla_{(x^n_i)_{\mathcal{U}}}$, $(f^{n+1}_i)_{\mathcal{U}} \in \nabla_{(x^{n+1}_i)_{\mathcal{U}}}$ be such that

$$m((x^1_i)_{\mathcal{U}}, (x^2_i)_{\mathcal{U}}, \ldots, (x^n_i)_{\mathcal{U}}, (x^{n+1}_i)_{\mathcal{U}}; (f^2_i)_{\mathcal{U}}, (f^3_i)_{\mathcal{U}}, \ldots, (f^n_i)_{\mathcal{U}}, (f^{n+1}_i)_{\mathcal{U}}) \leq \varepsilon,$$

and

$$1 - \frac{\|(x^1_i)_{\mathcal{U}} + (x^2_i)_{\mathcal{U}} + \cdots + (x^n_i)_{\mathcal{U}} + (x^{n+1}_i)_{\mathcal{U}}\|}{n+1} > W^n_{X_{\mathcal{U}}}(\varepsilon) - \eta.$$

Without loss of generality, we may assume by (2.1) that

$$1 - \eta < \|(x^j_i)_{\mathcal{U}}\| < 1 + \eta \text{ for } j = 1, \ldots, n+1,$$

$$1 - \eta < \|(f^j_i)_{\mathcal{U}}\| < 1 + \eta \text{ for } j = 2, \ldots, n+1,$$

and

$$1 - \eta < \langle (x^j_i)_{\mathcal{U}}, (f^j_i)_{\mathcal{U}}\rangle < 1 + \eta \text{ for } j = 2, \ldots, n+1.$$

From the property of ultra-product, we know the subsets

$$P = \{i : m((x^1_i)_{\mathcal{U}}, (x^2_i)_{\mathcal{U}}, \ldots, (x^n_i)_{\mathcal{U}}, (x^{n+1}_i)_{\mathcal{U}}; (f^2_i)_{\mathcal{U}}, (f^3_i)_{\mathcal{U}}, \ldots, (f^n_i)_{\mathcal{U}}, (f^{n+1}_i)_{\mathcal{U}}) \leq \varepsilon\}$$

and

$$Q = \left\{ i : 1 - \frac{\|(x_i^1)_{\mathcal{U}} + (x_i^2)_{\mathcal{U}} + \cdots + (x_i^n)_{\mathcal{U}} + (x_i^{n+1})_{\mathcal{U}}\|}{n+1} > W^n_{X_{\mathcal{U}}}(\varepsilon) - \eta \right\}$$

are all in $\mathcal{U}$. So the intersection $P \cap Q$ is in $\mathcal{U}$ too, and hence is not empty.

Let $i \in P \cap Q$ be fixed. Then

$$1 - \eta < \|x_i^j\| < 1 + \eta \text{ for } j = 1, \ldots, n+1;$$
$$1 - \eta < \|f_i^j\| < 1 + \eta \text{ for } j = 2, \ldots, n+1;$$
$$1 - \eta < \langle x_i^j, f_i^j \rangle < 1 + \eta \text{ for } j = 2, \ldots, n+1;$$
$$m(x_i^1, x_i^2, \ldots, x_i^n, x_i^{n+1}; f_i^2, f_i^3, \ldots, f_i^n, f_i^{n+1}) \leq \varepsilon;$$

and

$$1 - \frac{\|x_i^1 + x_i^2 + \cdots + x_i^n + x_i^{n+1}\|}{n+1} > W^n_{X_U}(\varepsilon) - \eta.$$

From Lemma 2.4, for $0 < \eta < 1$ (since η can be arbitrarily small, if necessary we can normalize vectors x_i^j and f_i^j to use Lemma 2.4) there are $\{y_j\}_{j=1}^{n+1} \subseteq S_X$ and $\{g_j\}_{j=2}^{n+1} \subseteq S_{X^*}$ such that

- $g_j \in \nabla_{y_j}$ for all $j = 2, \ldots, n+1$;
- $\|x_i^j - y_j\| < \eta$ for all $j = 1, \ldots, n+1$;
- $\|f_i^j - g_j\| < \eta$ for $j = 2, \ldots, n+1$.

Similar to the proof of Theorem 2.8, we have

$$\det m(y_1, y_2, \ldots, y_n, y_{n+1}; g_2, g_3, \ldots, g_n, g_{n+1}) \leq \varepsilon + c\eta,$$

and $1 - \frac{\|y_1 + y_2 + \cdots + y_n + y_{n+1}\|}{n+1} > W^n_{X_{\mathcal{U}}}(\varepsilon) - d\eta$, where c and d are constants.

Since $\eta > 0$ can be arbitrarily small, we have $W^n_X(\varepsilon) \geq W^n_{X_{\mathcal{U}}}(\varepsilon)$. □

Lemma 2.19 ([8]). *If X is a super-reflexive Banach space, then X has uniform normal structure if and only if $X_{\mathcal{U}}$ has normal structure.*

Theorem 2.20. *Suppose that X is a Banach space satisfying one of the following conditions*:

- $W^n_X(1) < 1 - \frac{1}{n+1}$ *for some* $n \in \mathbb{N}$ *with* $n \geq 2$; *or*
- $W^1_X(1) < \frac{1}{2}$ *and* $W^1_X(\frac{5}{3}) < \frac{2}{3}$ *for* $n = 1$.

Then X has uniform normal structure.

Proof. It follows directly from Theorems 2.13, 2.15, 2.18 and Lemma 2.19. □

Example. Let H be a Hilbert space. We have $W^1_H(\varepsilon) = \frac{2-\sqrt{4-2\varepsilon}}{2}$ for $0 \leq \varepsilon \leq 2$.

Since $W^1_H(1) = \frac{2-\sqrt{2}}{2} = 0.29289\cdots < \frac{1}{2}$, and $W^1_H(\frac{5}{3}) = \frac{2-\sqrt{\frac{2}{3}}}{2} = 0.59175\cdots < \frac{2}{3}$, from Theorem 2.20, H has uniform normal structure.

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AN EXPLANATION OF OVER-RELAXATION PARAMETERS FOR SOME ALGORITHMS IN HILBERT SPACES

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ABSTRACT. Many known algorithms concerning mappings of firmly nonexpansive type have been proposed with the relaxation parameters in the interval $[0,2]$. Using a fact from theory of nonexpansive mappings, we show that such an over-relaxation can be deduced from the usual relaxation the interval $[0,1]$. In this paper we discuss a more general form algorithms than the recent works of Chuang and Takahashi [4]. This is inspired by the one studied by Combettes and Pennanen [7]. Finally, we use the same technique to explain the over-relaxation of the contraction-proximal point algorithm of Wang and Cui [17].

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. We focus on the following two interesting problems.

Fixed Point Problem: Let C be a nonempty subset of $\mathcal{H}$ and $T : C \to \mathcal{H}$ be a given mapping. An element $x \in C$ is a *fixed point* of T if $x = Tx$. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$.

Zeros of (Multivalued) Operator: Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a given multivalued operator. An element $x \in \mathcal{H}$ is a *zero* of A if $0 \in Ax$. Denote by $A^{-1}0$ the set of all zeros of A.

Many investigations of iterative sequences for finding a solution of these problems have been made by many mathematicians (see, for example, [3] and the references therein). In this paper, we consider the Mann type algorithms for the first problem and the contraction-proximal point algorithm for the second one. Several weak convergence theorems for the Mann type algorithms were proved by Chuang and Takahashi [4]. As mentioned in their Remark 3.1, the relaxation parameters $\{\alpha_n\}_{n=0}^{\infty}$ is taken in the interval $[0,2]$. Note that the usual Mann type algorithm makes essential use of the parameters in $[0,1]$. For the second problem, we consider the strong convergence theorem of contraction-proximal point algorithm. This result was recently proved by Wang and Cui [17]. They discussed the algorithm when the relaxation

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parameters are taken in $[0,2]$. The previous related works are restricted for only parameters in $[0,1]$. This improvement as discussed by Eckstein and Ferris [10] may speed up the convergence to a solution.

It is our purpose to use some facts from the theory of nonexpansive mappings to explain why such over-relaxations can be made. Moreover, we show that how some known results can be used to immediately deduce these results.

2. Results

The sets of nonnegative real numbers and of nonnegative integers are denoted by $\mathbb{R}_+$ and $\mathbb{N}_+$, respectively. We use $\to$ and $\rightharpoonup$ for the strong and weak convergence, respectively. For a given sequence $\{x_n\}_{n=0}^\infty$, let $\mathfrak{W}\{x_n\}_{n=0}^\infty$ denote the set of all weak cluster points of $\{x_n\}_{n=0}^\infty$, that is,

$$\mathfrak{W}\{x_n\}_{n=0}^\infty := \{z : x_{n_k} \rightharpoonup z \text{ for some subsequence } \{x_{n_k}\}_{k=0}^\infty \text{ of } \{x_n\}_{n=0}^\infty\}.$$

2.1. **On weak convergence theorem of Chuang and Takahashi.** Throughout this subsection, let $A := [\alpha_{n,j}]_{n,j=0}^\infty$ be an infinite matrix satisfying the following conditions:

(A1) $\alpha_{n,j} \geq 0$ for all $n, j \in \mathbb{N}_+$.
(A2) $\alpha_{n,j} = 0$ for all n, j with $j > n$.
(A3) $\sum_{j=0}^n \alpha_{n,j} = 1$ for all $n \in \mathbb{N}_+$.
(A4) $\lim_{n\to\infty} \alpha_{n,j} = 0$ for all $j \in \mathbb{N}_+$.

Algorithm 1 ([7]). Let $\{T_n : \mathcal{H} \to \mathcal{H}\}_{n=0}^\infty$ be a sequence of mappings. Let $\{\alpha_n\}_{n=0}^\infty$ be a sequence in $\mathbb{R}$. Let $\{x_n\}_{n=0}^\infty$ be an arbitrary sequence generated by

$$\begin{cases} x_0 \in \mathcal{H} \text{ arbitrarily chosen,} \\ \overline{x}_n := \sum_{j=0}^n \alpha_{n,j} x_j, \\ x_{n+1} := \overline{x}_n + \alpha_n (T_n \overline{x}_n + e_n - \overline{x}_n), \quad \forall n \in \mathbb{N}_+, \end{cases}$$

where $e_n \in \mathcal{H}$ for all $n \in \mathbb{N}_+$.

Note that each e_n is regarded as the error made in the computation of $T_n\overline{x}_n$. This provides a more realistic model of the actual implementation of the algorithm. Before moving on, let us mention some advantage of the element x_{n+1} incorporating the past iterates $\{x_j\}_{j=0}^n$. This method can mitigate the zig-zagging ([5, 15]) and spiralling ([8, 9]) of sequences reported in some applications.

The following algorithm was studied by Chuang and Takahashi [4].

Algorithm 2. Let C be a nonempty closed and convex subset of a Hilbert space $\mathcal{H}$. Let $\{T_n : C \to \mathcal{H}\}_{n=0}^\infty$ be a sequence of mappings. Let $\{\alpha_n\}_{n=0}^\infty$ be a sequence in $\mathbb{R}$. Let $\{x_n\}_{n=0}^\infty$ be a sequence in C defined by

$$\begin{cases} x_0 \in C \text{ is chosen arbitrarily,} \\ x_{n+1} := P_C\left((1-\alpha_n)x_n + \alpha_n T_n x_n\right), \quad \forall n \in \mathbb{N}_+, \end{cases}$$

where P_C is the metric projection onto C.

Inspired by the preceding two algorithms, we are interested in the following one.

Algorithm 3. Let C be a nonempty closed and convex subset of a Hilbert space $\mathcal{H}$. Let $\{T_n : C \to \mathcal{H}\}_{n=0}^{\infty}$ be a sequence of mappings. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in C defined by

$$\begin{cases} x_0 \in C \text{ is chosen arbitrarily,} \\ \overline{x}_n := \sum_{j=0}^{n} \alpha_{n,j} x_j, \\ x_{n+1} := P_C\left(\overline{x}_n + \alpha_n(T_n\overline{x}_n + e_n - \overline{x}_n)\right), \quad \forall n \in \mathbb{N}_+, \end{cases}$$

where $e_n \in \mathcal{H}$ for all $n \in \mathbb{N}_+$.

Remark 2.1. In Algorithm 3, if $C = \mathcal{H}$, then Algorithm 3 can be reduced to Algorithm 1.

Remark 2.2. In Algorithm 3, if $\alpha_{n,n} = 1$ for all $n \in \mathbb{N}_+$, then $\overline{x}_n = x_n$ and hence Algorithm 3 can be reduced Algorithm 2.

Definition 2.3 ([7]). Let $A = [\alpha_{n,j}]_{n,j=0}^{\infty}$ be an infinite matrix satisfying the conditions (A1)-(A4). We say that A is *concentrating* if whenever $\{\xi_n\}_{n=0}^{\infty}$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ are sequences in $[0,\infty)$ such that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and

$$\begin{cases} \overline{\xi}_n = \sum_{j=0}^{n} \alpha_{n,j}\xi_j \\ \xi_{n+1} \le \overline{\xi}_n + \varepsilon_n \end{cases}$$

for all $n \in \mathbb{N}_+$, it follows that $\{\xi_n\}_{n=0}^{\infty}$ is convergent.

The following are interesting examples of concentrating matrices.

Example 2.4. An infinite matrix $A = [\alpha_{n,j}]_{n,j=0}^{\infty}$ is concentrating if one the following conditions is satisfied (see [7,14]).

(1) $\alpha_{n,n} = 1$ for all $n \in \mathbb{N}_+$.
(2) $\liminf_{n\to\infty} \alpha_{n,n} > 0$ and $\sum_{n=0}^{\infty} \tau_n < \infty$, where $\tau_n := \sum_{j=0}^{n} |\alpha_{n+1,j} - (1 - \alpha_{n+1,n+1})\alpha_{n,j}|$ for all $n \in \mathbb{N}_+$.
(3) Let $\tau_j := \max\{0, \sum_{n\ge j} \alpha_{n,j} - 1\}$ for all $j \in \mathbb{N}_+$ and $J_n := \{j \in \mathbb{N}_+ \mid \alpha_{n,j} > 0\}$ for all $n \in \mathbb{N}_+$. Suppose that the following conditions hold:
   (a) $\sum_{j=0}^{\infty} \tau_j < \infty$,
   (b) $J_{n+1} \subset J_n \cup \{n+1\}$ for all $n \in \mathbb{N}_+$,
   (c) that there exists $\underline{\alpha} \in (0,1)$ such that
   $$\alpha_{n,j} \ge \underline{\alpha} \quad \text{for all } n \in \mathbb{N}_+ \text{ and for all } j \in J_n.$$
(4) Suppose that there exist numbers $\{a_i\}_{0\le i\le m}$ in $\mathbb{R}_+$ such that $\sum_{i=0}^{m} a_i = 1$ and the roots of the polynomial
   $$p(z) = z^{m+1} - (a_0 z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m)$$
   are all within the unit disc, with exactly one root on its boundary. The entries $\alpha_{n,j}$ are defined as follows:
   $$\begin{cases} \forall n \in \{0,\ldots,m-1\} \quad \alpha_{n,j} = \begin{cases} 0 & \text{if } 0 \le j < n, \\ 1 & \text{if } j = n, \end{cases} \\ \forall n \ge m \quad \alpha_{n,j} = \begin{cases} 0 & \text{if } 0 \le j < n-m, \\ a_{n-j} & \text{if } n-m \le j \le n. \end{cases} \end{cases}$$

Definition 2.5. Let C be a nonempty subset of a Hilbert space $\mathcal{H}$. A mapping $T : C \to \mathcal{H}$ is

(1) *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$.
(2) *quasi-nonexpansive* if $\mathrm{Fix}(T) \neq \varnothing$ and $\|Tx - p\| \le \|x - p\|$ for all $x \in C$ and $p \in \mathrm{Fix}(T)$.
(3) *firmly nonexpansive* if $\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(x - Tx) - (y - Ty)\|^2$ for all $x, y \in C$.
(4) *firmly quasi-nonexpansive* if $\mathrm{Fix}(T) \neq \varnothing$ and $\|Tx - p\|^2 \le \|x - p\|^2 - \|x - Tx\|^2$ for all $x \in C$ and $p \in \mathrm{Fix}(T)$.

Remark 2.6. (1) Every nonexpansive (firmly nonexpansive, resp.) mapping with a fixed point is quasi-nonexpansive (firmly quasi-nonexpansive, resp.).
(2) T is firmly nonexpansive (firmly quasi-nonexpansive, resp.) if and only if $2T - I$ is nonexpansive (quasi-nonexpansive, resp.), where I is the identity mapping. In particular, if $T : C \to \mathcal{H}$ is firmly nonexpansive (firmly quasi-nonexpansive, resp.), then $T = \frac{1}{2}I + \frac{1}{2}S$ for some nonexpansive (quasi-nonexpansive, resp.) mapping $S : C \to \mathcal{H}$.
(3) If $T : C \to \mathcal{H}$ is a nonexpansive mapping and $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $\varnothing \neq \mathfrak{W}\{x_n\}_{n=0}^{\infty} \subset \mathrm{Fix}(T)$.

Definition 2.7. Let C be a nonempty subset of a Hilbert space $\mathcal{H}$. Let $\{T_n : C \to \mathcal{H}\}_{n=0}^{\infty}$ be a sequence of mappings such that

$$\mathbf{S} := \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n) \neq \varnothing$$

and let $\mathcal{T}$ be a family of mappings from C into $\mathcal{H}$.

(♠) $\{T_n\}_{n=0}^{\infty}$ satisfies the *resolvent property* [4] if there exists a nonexpansive mapping $T : C \to \mathcal{H}$ with $\mathrm{Fix}(T) = \mathbf{S}$ and there exist $n_0, k \in \mathbb{N}$ such that $\|x - Tx\| \le k\|x - T_n x\|$ for all $x \in C$ and for all $n \ge n_0$. In this situation, we also say that $\{T_n\}_{n=0}^{\infty}$ satisfies *the resolvent property with a nonexpansive mapping* T.

(♡) $\{T_n\}_{n=0}^{\infty}$ satisfies the *AKTT-condition* [1] if the following two conditions are satisfied:
  (a) $\sum_{n=0}^{\infty} \sup_{x\in B} \|T_{n+1}x - T_n x\| < \infty$ for each bounded subset B of C. (In particular, $\{T_n x\}_{n=0}^{\infty}$ is a Cauchy sequence for all $x \in C$ and hence $\lim_{n\to\infty} T_n x$ exists for all $x \in C$.)
  (b) The fixed point set of the mapping $T : C \to \mathcal{H}$ defined by $Tx := \lim_{n\to\infty} T_n x$ for all $x \in C$ coincides with $\mathbf{S}$, that is, $\mathrm{Fix}(T) = \mathbf{S}$.

In this situation, we also say that $(\{T_n\}_{n=0}^{\infty}, T)$ satisfies the *AKTT-condition*.

(♢) $(\{T_n\}_{n=0}^{\infty}, \mathcal{T})$ satisfies the *NST-condition* [13] if the following two conditions hold.
  (a) $\mathrm{Fix}(\mathcal{T}) \subset \mathbf{S}$.
  (b) For each bounded sequence $\{z_n\}_{n=0}^{\infty} \subset C$, the following implication holds

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0 \implies \lim_{n\to\infty} \|z_n - Tz_n\| = 0 \quad \forall T \in \mathcal{T}.$$

(♣) $\{T_n\}_{n=0}^{\infty}$ satisfies the *CP-condition* [7] if for each bounded sequence $\{z_n\}_{n=0}^{\infty}$ in C, the following implication holds

$$z_n - T_n z_n \to 0 \implies \mathfrak{W}\{z_n\}_{n=0}^{\infty} \subset \mathbf{S}.$$

Remark 2.8. (1.) Either (♠) or (♡) implies (♢).

(2.) If $\mathcal{T}$ is a family of nonexpansive mappings, then (♢) ⇒ (♣).

(3.) (♠) ⇎ (♡). In fact, Example 3.1 in [4] shows that (♠) ⇏ (♡).

Example 2.9. ((♡) ⇏ (♠)) Let $\mathcal{H} = \mathbb{R}$ and $C = [0,1]$. For each $n \in \mathbb{N}$, let $T_n : C \to \mathcal{H}$ be defined by

$$T_n x = \begin{cases} x, & \text{if } x \in \left[0, \frac{1}{2^n}\right], \\ \frac{1}{2^n}, & \text{if } x \in \left(\frac{1}{2^n}, 1\right]. \end{cases}$$

Then $\sum_{n=1}^{\infty} \sup_{x\in C} \|T_{n+1}x - T_n x\| \le \sum_{n=1}^{\infty} \left|\frac{1}{2^{n+1}} - \frac{1}{2^n}\right| = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} < \infty$. Note that $Tx := \lim_{n\to\infty} T_n x = 0$ for all $x \in C$ and $\mathrm{Fix}(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) = \{0\}$. So $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition.

We show that $\{T_n\}_{n=1}^{\infty}$ does not satisfy the resolvent property. Suppose that there are a nonexpansive mapping $T : C \to \mathcal{H}$ and constants $n_0, k \in \mathbb{N}$ such that $\mathrm{Fix}(T) = \{0\}$ and $\|x - Tx\| \le k\|x - T_n x\|$ for all $x \in C$ and $n \ge n_0$. For any $x \in [0, 1/2^{n_0}]$, we have $T_{n_0}x = x$. Hence $Tx = x$. This implies that $[0, 1/2^{n_0}] \subset \mathrm{Fix}(T) = \{0\}$ which is a contradiction.

Note that each T_n is firmly nonexpansive because

$$(2T_n - I)x = \begin{cases} x, & \text{if } x \in \left[0, \frac{1}{2^n}\right], \\ -x + \frac{1}{2^{n-1}}, & \text{if } x \in \left(\frac{1}{2^n}, 1\right] \end{cases}$$

is nonexpansive.

Lemma 2.10 ([6]). *Let $\{\xi_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ be sequences in $\mathbb{R}_+$ such that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and*

$$\xi_{n+1} \le \xi_n - \beta_n + \varepsilon_n \quad \textit{for all } n \in \mathbb{N}_+.$$

Then $\{\xi_n\}_{n=0}^{\infty}$ is convergent and $\sum_{n=0}^{\infty} \beta_n < \infty$.

Lemma 2.11 ([12]). *Let A be an infinite matrix satisfying the conditions (A1)-(A4). Let $\{\xi_n\}_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$ and $\overline{\xi}_n = \sum_{j=0}^{n} \alpha_{n,j}\xi_j$ for all $n \in \mathbb{N}_+$. If $\xi_n \to \xi$, then $\overline{\xi}_n \to \xi$.*

Lemma 2.12 ([6]). *Let S be a nonempty closed and convex subset of a Hilbert space $\mathcal{H}$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in $\mathcal{H}$ and let $\{\varepsilon_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers such that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and*

$$\|x_{n+1} - p\| \le \|x_n - p\| + \varepsilon_n$$

for all $n \in \mathbb{N}_+$ and all $p \in S$. Then $\{P_S x_n\}_{n=0}^{\infty}$ converges to a point in S.

Lemma 2.13 (Opial's Theorem). *Let S be a nonempty subset of $\mathcal{H}$ and $\{x_n\}_{n=0}^{\infty}$ be a sequence in $\mathcal{H}$. If the following two conditions hold:*

(*a*) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all $p \in S$,*

(*b*) $\mathfrak{W}\{x_n\}_{n=0}^{\infty} \subset S$, *then* $\{x_n\}_{n=0}^{\infty}$ *converges weakly to a point in* S.

We now state our first weak convergence theorem for Algorithm 3.

Theorem 2.14. *Let* $\{T_n : C \to \mathcal{H}\}_{n=0}^{\infty}$ *be a sequence of quasi-nonexpansive mappings with the common fixed point set* $\mathbf{S}$. *Let* $\{x_n\}_{n=0}^{\infty}$ *be an arbitrary sequence generated by Algorithm 3 where* A *is concentrating,* $\{\alpha_n\}_{n=0}^{\infty}$ *is a sequence in* $[0,1]$, *and* $\sum_{n=0}^{\infty} \|e_n\| < \infty$. *If* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\{T_n\}_{n=0}^{\infty}$ *satisfies the CP-condition, then* $x_n \rightharpoonup u \in \mathbf{S}$.

Proof. Let $p \in \mathbf{S}$ and let $\xi_n := \|x_n - p\|$ for all $n \in \mathbb{N}_+$. Now we have

$$\begin{aligned}
\xi_{n+1} = \|x_{n+1} - p\| &= \|P_C(\overline{x}_n + \alpha_n(T_n\overline{x}_n + e_n - \overline{x}_n)) - P_C(p)\| \\
&\le \|(1-\alpha_n)\overline{x}_n + \alpha_n T_n\overline{x}_n + \alpha_n e_n - p\| \\
&= \|(1-\alpha_n)(\overline{x}_n - p) + \alpha_n(T_n\overline{x}_n - p) + \alpha_n e_n\| \\
&\le (1-\alpha_n)\|\overline{x}_n - p\| + \alpha_n\|T_n\overline{x}_n - p\| + \alpha_n\|e_n\| \\
&\le \|\overline{x}_n - p\| + \alpha_n\|e_n\| \\
&\le \sum_{j=0}^{n} \alpha_{n,j}\|x_j - p\| + \alpha_n\|e_n\| \\
&= \overline{\xi}_n + \alpha_n\|e_n\|.
\end{aligned}$$

Since A is concentrating and $\sum_{n=0}^{\infty}\|e_n\| < \infty$, the limit $\lim_{n\to\infty}\|x_n - p\| := a$ exists. It follows from Lemma 2.11 that

$$\lim_{n\to\infty} \overline{\xi}_n = \lim_{n\to\infty} \sum_{j=0}^{n} \alpha_{n,j}\|x_j - p\| = a.$$

Note that $\xi_{n+1} \le \|\overline{x}_n - p\| + \alpha_n\|e_n\| \le \overline{\xi}_n + \alpha_n\|e_n\|$ for all $n \in \mathbb{N}_+$. Consequently,

$$\lim_{n\to\infty} \|\overline{x}_n - p\| = a.$$

Let $\varepsilon_n := 2\|(1-\alpha_n)(\overline{x}_n - p) + \alpha_n(T_n\overline{x}_n - p)\|\|e_n\| + \|e_n\|^2$ for all $n \in \mathbb{N}_+$. Note that

$$\|(1-\alpha_n)(\overline{x}_n - p) + \alpha_n(T_n\overline{x}_n - p)\| \le \|\overline{x}_n - p\|.$$

So we have $\sum_{n=0}^{\infty} \varepsilon_n < \infty$. Now we consider

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (\|(1-\alpha_n)(\overline{x}_n - p) + \alpha_n(T_n\overline{x}_n - p)\| + \|\alpha_n e_n\|)^2 \\
&= \|(1-\alpha_n)(\overline{x}_n - p) + \alpha_n(T_n\overline{x}_n - p)\|^2 + \alpha_n^2\|e_n\|^2 \\
&\quad + 2\alpha_n\|(1-\alpha_n)(\overline{x}_n - p) + \alpha_n(T_n\overline{x}_n - p)\|\|e_n\| \\
&= (1-\alpha_n)\|\overline{x}_n - p\|^2 + \alpha_n\|T_n\overline{x}_n - p\|^2 - (1-\alpha_n)\alpha_n\|\overline{x}_n - T_n\overline{x}_n\|^2 \\
&\quad + 2\alpha_n\|(1-\alpha_n)(\overline{x}_n - p) + \alpha_n(T_n\overline{x}_n - p)\|\|e_n\| + \alpha_n^2\|e_n\|^2 \\
&\le \|\overline{x}_n - p\|^2 - (1-\alpha_n)\alpha_n\|\overline{x}_n - T_n\overline{x}_n\|^2 + \varepsilon_n \\
&\le \sum_{j=0}^{n} \alpha_{n,j}\|x_j - p\|^2 - (1-\alpha_n)\alpha_n\|\overline{x}_n - T_n\overline{x}_n\|^2 + \varepsilon_n.
\end{aligned}$$

So $(1-\alpha_n)\alpha_n\|\overline{x}_n - T_n\overline{x}_n\|^2 \le \sum_{j=0}^n \alpha_{n,j}\|x_j - p\|^2 - \|x_{n+1}-p\|^2 + \varepsilon_n$. Note that

$$\lim_{n\to\infty}\|x_n - p\|^2 = \lim_{n\to\infty}\sum_{j=0}^{n}\alpha_{n,j}\|x_j - p\|^2 = a^2.$$

Thus $\lim_{n\to\infty}(1-\alpha_n)\alpha_n\|\overline{x}_n - T_n\overline{x}_n\|^2 = 0$. Since $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, we have

$$\overline{x}_n - T_n\overline{x}_n \to 0.$$

Moreover, since $\{T_n\}_{n=0}^\infty$ satisfies the CP-condition, we have $\mathfrak{W}\{\overline{x}_n\}_{n=0}^\infty \subset \mathbf{S}$. In the proof above, we can infer that $\lim_{n\to\infty}\|\overline{x}_n - p\|$ exists for all $p \in \mathbf{S}$. It follows then from Opial's Theorem that $\overline{x}_n \rightharpoonup u \in \mathbf{S}$. Note that

$$\begin{aligned}\|x_{n+1} - \overline{x}_n\| &= \|P_C\left(\overline{x}_n + \alpha_n(T_n\overline{x}_n + e_n - \overline{x}_n)\right) - P_C(\overline{x}_n)\| \\ &\le \|(1-\alpha_n)\overline{x}_n + \alpha_n T_n\overline{x}_n + \alpha_n e_n - \overline{x}_n\| \\ &= \alpha_n\|T_n\overline{x}_n + e_n - \overline{x}_n\| \\ &\le \alpha_n(\|T_n\overline{x}_n - \overline{x}_n\| + \|e_n\|).\end{aligned}$$

So $x_{n+1} - \overline{x}_n \to 0$. Hence $x_n \rightharpoonup u \in \mathbf{S}$. This completes the proof. □

Letting $C = \mathcal{H}$ in the preceding theorem gives the following result which is due to Combettes and Pennanen [7].

Corollary 2.15 ([7]). *Let $\{T_n : \mathcal{H} \to \mathcal{H}\}_{n=0}^\infty$ be a sequence of firmly quasi-nonexpansive mappings. Let $\{x_n\}_{n=0}^\infty$ be an arbitrary sequence generated by Algorithm 1 where A is concentrating and $\{\alpha_n\}_{n=0}^\infty$ lies in $[\delta, 2-\delta]$ for some $\delta \in (0,1)$, and $\sum_{n=0}^\infty \|e_n\| < \infty$. If $\{T_n\}_{n=0}^\infty$ satisfies the CP-condition, then $x_n \rightharpoonup u \in \mathbf{S}$.*

Proof. Since each $T_n : \mathcal{H} \to \mathcal{H}$ is a firmly quasi-nonexpansive mapping, there is a quasi-nonexpansive mapping $S_n : \mathcal{H} \to \mathcal{H}$ such that $T_n = \frac{1}{2}I + \frac{1}{2}S_n$. In particular, $I - T_n = \frac{1}{2}(I - S_n)$. This implies that $\mathbf{S} = \bigcap_{n=0}^\infty \mathrm{Fix}(S_n)$ and $\{S_n\}_{n=0}^\infty$ satisfies the CP-condition. Moreover,

$$\begin{aligned}x_{n+1} &= \overline{x}_n + \alpha_n(T_n\overline{x}_n + e_n - \overline{x}_n) \\ &= \overline{x}_n + \frac{\alpha_n}{2}(S_n\overline{x}_n + 2e_n - \overline{x}_n).\end{aligned}$$

Note that $\alpha_n/2 \in [0,1]$, $\liminf_{n\to\infty}(1-\alpha_n/2)\alpha_n/2 \ge \delta^2/4 > 0$, and $\sum_{n=0}^\infty \|2e_n\| < \infty$. By Theorem 2.14, we get the result. □

In Theorem 2.14 and Corollary 2.15, we do not know much about the weak limit u of the sequence $\{x_n\}_{n=0}^\infty$. We now discuss a certain subclass of concentrating matrices which is taken from Example 2.5 in [7] and obtain some information about the weak limit u. More precisely, we show that

$$u = \lim_{n\to\infty} P_{\mathbf{S}}x_n.$$

Theorem 2.16. *Let $\{T_n : C \to \mathcal{H}\}_{n=0}^\infty$ be a sequence of quasi-nonexpansive mappings. Let $\tau_n := \sum_{j=0}^n |\alpha_{n+1,j} - (1-\alpha_{n+1,n+1})\alpha_{n,j}|$. Suppose that $\sum_{n=0}^\infty \tau_n < \infty$ and $\liminf_{n\to\infty}\alpha_{n,n} > 0$. Let $\{x_n\}_{n=0}^\infty$ be an arbitrary sequence generated by Algorithm 3 where $\{\alpha_n\}_{n=0}^\infty$ is a sequence in $[0,1]$, and $\sum_{n=0}^\infty \|e_n\| < \infty$. Assume that $\{T_n\}_{n=0}^\infty$ satisfies the CP-condition. Then*

(a) *If* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, *then* $x_n \rightharpoonup u = \lim_{n\to\infty} P_{\mathbf{S}} x_n$.
(b) *Suppose that* $\{T_n : C \to C\}_{n=0}^{\infty}$ *is a sequence of nonexpansive mappings and* $(\{T_n\}_{n=0}^{\infty}, T)$ *satisfies the AKTT-condition. If* $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, *then* $x_n \rightharpoonup u = \lim_{n\to\infty} P_{\mathbf{S}} x_n$.

Proof. It was proved in [7, Example 2.5] that A is concentrating. As in the proof of our Theorem 2.14, both limits $\lim_{n\to\infty} \|x_n - p\|$ and $\lim_{n\to\infty} \|\overline{x}_n - p\|$ exist and they are equal and $x_n \rightharpoonup u \in \mathbf{S}$. In particular, $\{x_n\}_{n=0}^{\infty}$ is bounded. We show that $\lim_{n\to\infty} P_{\mathbf{S}} x_n = u$. Since we also proved that $x_{n+1} - \overline{x}_n \to 0$, it suffices to show that

$$\lim_{n\to\infty} P_{\mathbf{S}} \overline{x}_n = u.$$

We will apply Lemma 2.12. Let $p \in \mathbf{S}$. As in the proof of Theorem 2.14, we have $\|x_{n+1} - p\| \le \|\overline{x}_n - p\| + \alpha_n \|e_n\|$. Let $y_n := \sum_{j=0}^{n} (\alpha_{n+1,j} - (1-\alpha_{n+1,n+1})\alpha_{n,j}) x_j$. Since $\sum_{n=0}^{\infty} \tau_n < \infty$ and $\{x_n\}_{n=0}^{\infty}$ is bounded, we have $\sum_{n=0}^{\infty} \|y_n\| \le \sum_{n=0}^{\infty} \tau_n \|x_n\| < \infty$. Note that

$$\begin{aligned} \overline{x}_{n+1} &= \alpha_{n+1,n+1} x_{n+1} + \sum_{j=0}^{n} \alpha_{n+1,j} x_j \\ &= \overline{x}_n + \sum_{j=0}^{n} (\alpha_{n+1,j} - (1-\alpha_{n+1,n+1})\alpha_{n,j}) x_j - \alpha_{n+1,n+1}(\overline{x}_n - x_{n+1}) \\ &= (1-\alpha_{n+1,n+1})\overline{x}_n + \alpha_{n+1,n+1} x_{n+1} + y_n. \end{aligned} \tag{2.1}$$

It follows from (2.1) that

$$\begin{aligned} \|\overline{x}_{n+1} - p\| &\le (1-\alpha_{n+1,n+1})\|\overline{x}_n - p\| + \alpha_{n+1,n+1}\|x_{n+1} - p\| + \|y_n\| \\ &\le \|\overline{x}_n - p\| + \alpha_{n+1,n+1}\alpha_n \|e_n\| + \|y_n\| \\ &\le \|\overline{x}_n - p\| + \|e_n\| + \|y_n\|. \end{aligned}$$

By Lemma 2.12, $\{P_{\mathbf{S}} \overline{x}_n\}_{n=0}^{\infty}$ converges to some element $a \in \mathbf{S}$. Note that $u \in \mathbf{S}$. It follow then that $0 \le \langle \overline{x}_n - P_{\mathbf{S}}\overline{x}_n, P_{\mathbf{S}}\overline{x}_n - u \rangle$ for all $n \in \mathbb{N}_+$. Since $\overline{x}_n - P_{\mathbf{S}}\overline{x}_n \rightharpoonup u - a$ and $P_{\mathbf{S}}\overline{x}_n - u \to a - u$, we have $\langle \overline{x}_n - P_{\mathbf{S}}\overline{x}_n, P_{\mathbf{S}}\overline{x}_n - u \rangle \to -\|u-a\|^2$. Then $-\|u-a\|^2 \ge 0$ and hence $u = a$. This implies that

$$\lim_{n\to\infty} P_{\mathbf{S}} \overline{x}_n = u.$$

The proof of Part (a) is complete.

Part (b). Suppose that $\{T_n : C \to C\}_{n=0}^{\infty}$ is a sequence of nonexpansive mappings and $(\{T_n\}_{n=0}^{\infty}, T)$ satisfies the AKTT-condition and $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Let $p \in \mathbf{S}$. As in the proof of Theorem 2.14, we have $\lim_{n\to\infty} \|\overline{x}_n - p\|$ exists. Then

$$\begin{aligned} \overline{x}_{n+1} &= (1-\alpha_{n+1,n+1})\overline{x}_n + \alpha_{n+1,n+1} x_{n+1} + y_n \\ &= (1-\alpha_{n+1,n+1}\alpha_n)\overline{x}_n + \alpha_{n+1,n+1}\alpha_n T_n \overline{x}_n \\ &\quad + \alpha_{n+1,n+1}(x_{n+1} - ((1-\alpha_n)\overline{x}_n + \alpha_n T_n \overline{x}_n)) + y_n. \\ &= (1-\beta_n)\overline{x}_n + \beta_n T_n \overline{x}_n + w_n, \end{aligned}$$

where $\beta_n := \alpha_{n+1,n+1}\alpha_n$ and $w_n := \alpha_{n+1,n+1}(x_{n+1} - ((1-\alpha_n)\overline{x}_n + \alpha_n T_n \overline{x}_n)) + y_n$. Since $T_n : C \to C$, we have $\|w_n\| \le \alpha_{n+1,n+1}\alpha_n \|e_n\| + \|y_n\|$. Then $\sum_{n=0}^{\infty} \|w_n\| <$

∞. Let $\gamma_n := 2\|(1-\beta_n)\overline{x}_n + \beta_n T_n \overline{x}_n - p\|\|w_n\| + \|w_n\|^2$ for all $n \in \mathbb{N}_+$. Then $\sum_{n=0}^{\infty} \gamma_n < \infty$. We now consider

$$\begin{aligned}\|\overline{x}_{n+1} - p\|^2 &\le \|(1-\beta_n)\overline{x}_n + \beta_n T_n \overline{x}_n - p\|^2 + \gamma_n \\ &= (1-\beta_n)\|\overline{x}_n - p\|^2 + \beta_n \|T_n \overline{x}_n - p\|^2 - (1-\beta_n)\beta_n \|\overline{x}_n - T_n \overline{x}_n\|^2 + \gamma_n \\ &\le \|\overline{x}_n - p\|^2 - (1-\beta_n)\beta_n \|\overline{x}_n - T_n \overline{x}_n\|^2 + \gamma_n.\end{aligned}$$

By Lemma 2.10, $\sum_{n=0}^{\infty}(1-\beta_n)\beta_n\|\overline{x}_n - T_n\overline{x}_n\|^2 < \infty$. Since $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\liminf_{n\to\infty} \alpha_{n,n} > 0$, we have $\sum_{n=0}^{\infty} \beta_n(1-\beta_n) = \infty$. Thus

$$\liminf_{n\to\infty} \|\overline{x}_n - T_n \overline{x}_n\| = 0.$$

We prove that $\lim_{n\to\infty} \|\overline{x}_n - T_n\overline{x}_n\|$ exists. Now, we have

$$\begin{aligned}&\|\overline{x}_{n+1} - T_{n+1}\overline{x}_{n+1}\| \\ &= \|(1-\beta_n)(\overline{x}_n - T_n\overline{x}_n) + (T_n\overline{x}_n - T_n\overline{x}_{n+1}) + (T_n\overline{x}_{n+1} - T_{n+1}\overline{x}_{n+1}) + w_n\| \\ &\le (1-\beta_n)\|\overline{x}_n - T_n\overline{x}_n\| + \|\overline{x}_n - \overline{x}_{n+1}\| + \|T_n\overline{x}_{n+1} - T_{n+1}\overline{x}_{n+1}\| + \|w_n\| \\ &\le \|\overline{x}_n - T_n\overline{x}_n\| + \sup\{\|T_n x - T_{n+1}x\| : x \in \{\overline{x}_n\}\} + 2\|w_n\|.\end{aligned}$$

Since $(\{T_n\}_{n=0}^{\infty}, T)$ satisfies the AKTT-condition, $\sum_{n=0}^{\infty} \sup_{x\in\{\overline{x}_n\}} \|T_{n+1}x - T_n x\| < \infty$. Thus $\lim_{n\to\infty} \|\overline{x}_n - T_n\overline{x}_n\|$ exists and hence $\lim_{n\to\infty} \|\overline{x}_n - T_n\overline{x}_n\| = 0$. We follow the proof of Theorem 2.14 and we have $x_n \rightharpoonup u \in \mathbf{S}$. Moreover, it follows from Part (a) that $u = \lim_{n\to\infty} P_{\mathbf{S}} x_n$. The proof of Part (b) is completed. □

Using the same technique as in the proof of Corollary 2.15, we immediately obtain the following two corollaries from our Theorem 2.16(a) and (b), respectively.

Corollary 2.17. *Let $\{T_n : C \to \mathcal{H}\}_{n=0}^{\infty}$ be a sequence of firmly quasi-nonexpansive mappings. Let $\tau_n := \sum_{j=0}^{n} |\alpha_{n+1,j} - (1-\alpha_{n+1,n+1})\alpha_{n,j}|$. Suppose that $\sum_{n=0}^{\infty} \tau_n < \infty$ and $\liminf_{n\to\infty} \alpha_{n,n} > 0$. Let $\{x_n\}_{n=0}^{\infty}$ be an arbitrary sequence generated by Algorithm 3 where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,2]$, and $\sum_{n=0}^{\infty} \|e_n\| < \infty$. If $\liminf_{n\to\infty} \alpha_n(2-\alpha_n) > 0$ and $\{T_n\}_{n=0}^{\infty}$ satisfies the CP-condition, then $x_n \rightharpoonup u = \lim_{n\to\infty} P_{\mathbf{S}} x_n$.*

Corollary 2.18. *Let $\{T_n : \mathcal{H} \to \mathcal{H}\}_{n=0}^{\infty}$ be a sequence of firmly nonexpansive mappings. Let $\tau_n := \sum_{j=0}^{n} |\alpha_{n+1,j} - (1-\alpha_{n+1,n+1})\alpha_{n,j}|$. Suppose that $\sum_{n=0}^{\infty} \tau_n < \infty$ and $\liminf_{n\to\infty} \alpha_{n,n} > 0$. Let $\{x_n\}_{n=0}^{\infty}$ be an arbitrary sequence generated by Algorithm 3 where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,2]$, and $\sum_{n=0}^{\infty} \|e_n\| < \infty$. If $\sum_{n=0}^{\infty} \alpha_n(2-\alpha_n) = \infty$ and $(\{T_n\}_{n=0}^{\infty}, T)$ satisfies the AKTT-condition, then $x_n \rightharpoonup u = \lim_{n\to\infty} P_{\mathbf{S}} x_n$.*

Remark 2.19. Our Corollary 2.18 gives a weak convergence theorem for a sequence of firmly nonexpansive mappings from $\mathcal{H}$ into $\mathcal{H}$ with the condition $\sum_{n=0}^{\infty} \alpha_n(2-\alpha_n) = \infty$. This is a generalization of [11].

The following corollary gives a generalization of Chuang and Takahashi's weak convergence theorem [4].

Corollary 2.20. *Let $\{T_n : C \to \mathcal{H}\}_{n=0}^{\infty}$ be a sequence of firmly nonexpansive mappings. Let $\tau_n := \sum_{j=0}^{n} |\alpha_{n+1,j} - (1-\alpha_{n+1,n+1})\alpha_{n,j}|$. Suppose that $\sum_{n=0}^{\infty} \tau_n < \infty$ and $\liminf_{n\to\infty} \alpha_{n,n} > 0$. Let $\mathcal{T}$ be a family of nonexpansive mappings of C into*

$\mathcal{H}$. Assume that $(\{T_n\}_{n=0}^{\infty}, \mathcal{T})$ satisfies the NST-condition. Let $\{x_n\}_{n=0}^{\infty}$ be an arbitrary sequence generated by Algorithm 3 where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,2]$, and $\sum_{n=0}^{\infty}\|e_n\| < \infty$. If $\liminf_{n\to\infty}\alpha_n(2-\alpha_n) > 0$, then $x_n \rightharpoonup u = \lim_{n\to\infty} P_{\mathbf{S}}x_n$.

Corollary 2.21. *Let $\{T_n : C \to \mathcal{H}\}_{n=0}^{\infty}$ be a sequence of firmly nonexpansive mappings. Let $\mathcal{T}$ be a family of nonexpansive mappings of C into $\mathcal{H}$. Assume that $(\{T_n\}_{n=0}^{\infty}, \mathcal{T})$ satisfies the NST-condition. Let $\{x_n\}_{n=0}^{\infty}$ be an arbitrary sequence generated by Algorithm 2 where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,2]$. If $\liminf_{n\to\infty}\alpha_n(2-\alpha_n) > 0$, then $x_n \rightharpoonup u = \lim_{n\to\infty} P_{\mathbf{S}}x_n$.*

Proof. Let $e_n = 0$ and $\alpha_{n,n} = 1$ for all $n \in \mathbb{N}_+$ in Corollary 2.20. Then $\overline{x}_n = x_n$, $\sum_{n=0}^{\infty}\tau_n = 0$, and $\liminf_{n\to\infty}\alpha_{n,n} = 1 > 0$. Then Corollary 2.21 follows. □

Remark 2.22. Our Corollary 2.21 gives an informative conclusion about the weak limit of an iterative sequence in Theorem 3.3 of [4]. In fact, we have $u = \lim_{n\to\infty} P_{\mathbf{S}}x_n$.

2.2. **On generalized contraction-proximal point algorithm of Wang and Cui.** Recall that a multivalued operator $A : \mathcal{H} \to 2^{\mathcal{H}}$ is *monotone* if $\langle x-y, u-v\rangle \geq 0$ for all $x, y \in \mathcal{H}$ and $u \in Ax, v \in Ay$. A monotone operator A is *maximal monotone* if the graph $\mathrm{Graph}(A) := \{(x,y) \in \mathcal{H}\times\mathcal{H} : y \in Ax\}$ is not perperly contained in the graph of any other monotone mapping. We refer the reader to [2] for more details. It is known that if A is maximal monotone, then for each $c > 0$ and for each $x \in \mathcal{H}$ there exists a unique element $z \in \mathcal{H}$ such that

$$x \in z + cAz.$$

In this case, we write $J_c x = z$. Consequently, J_c is a mapping from $\mathcal{H}$ into $\mathcal{H}$. It is also known that

(a) $\mathrm{Fix}(J_c) = A^{-1}0$.
(b) J_c is firmly nonexpansive.
(c) If $c_1, c_2 > 0$, then $J_{c_1}x = J_{c_2}\left(\frac{c_2}{c_1}x + \frac{c_1 - c_2}{c_1}J_{c_1}x\right)$ for all $x \in \mathcal{H}$.

Wang and Cui [17] proposed the contraction-proximal point algorithm with the over-relaxed parameters. We explain how the over-relaxed parameter simply works in this situation.

Definition 2.23 ([16]). $\{T_n : \mathcal{H} \to \mathcal{H}\}_{n=0}^{\infty}$ is said to be a *strongly quasi-nonexpansive sequence* if it satisfies the following conditions:

(1) $\bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n) \neq \varnothing$ and $\|T_n x - p\| \leq \|x - p\|$ for all $x \in \mathcal{H}$ and $p \in \bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n)$ and for all $n \in \mathbb{N}_+$.
(2) $\lim_{n\to\infty}\|y_n - T_n y_n\| = 0$ whenever $\{y_n\}_{n=0}^{\infty}$ is a bounded sequence in $\mathcal{H}$ such that

$$\lim_{n\to\infty}\left(\|y_n - p\| - \|T_n y_n - p\|\right) = 0 \quad \text{for some } p \in \bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n).$$

Lemma 2.24 ([16]). *Let $\{T_n : \mathcal{H} \to \mathcal{H}\}_{n=0}^{\infty}$ be a strongly quasi-nonexpansive sequence satisfying the CP-condition. Suppose that $\{x_n\}_{n=0}^{\infty}$ is a given by $x_0, u \in \mathcal{H}$ and*

$$x_{n+1} = \lambda_n u + (1-\lambda_n)T_n x_n \quad \textit{for all } n \in \mathbb{N}_+$$

where $\{\lambda_n\}_{n=0}^{\infty}$ *is a sequence in* $(0,1)$ *satisfying the conditions* $\lim_{n\to\infty}\lambda_n = 0$ *and* $\sum_{n=0}^{\infty}\lambda_n = \infty$. *Then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_{\bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n)}u$.

Theorem 2.25 ([17]). *Suppose that* $A : \mathcal{H} \to 2^{\mathcal{H}}$ *is a maximal monotone operator such that* $A^{-1}0 \neq \varnothing$. *Suppose that* $\{\lambda_n\}_{n=0}^{\infty} \subset (0,1)$, $\{\gamma_n\}_{n=0}^{\infty} \subset (-1,1)$, $\{\delta_n\}_{n=0}^{\infty} \subset (0,2)$, *and* $\lambda_n + \gamma_n + \delta_n = 1$ *for all* $n \in \mathbb{N}_+$. *Assume that the following conditions hold:*

(1) $\liminf_{n\to\infty} c_n > 0$;
(2) $\lim_{n\to\infty}\lambda_n = 0$, $\sum_{n=0}^{\infty}\lambda_n = \infty$;
(3) $0 < \liminf_{n\to\infty}\delta_n \leq \limsup_{n\to\infty}\delta_n < 2$.

Then, for any initial guess $x_0, u \in \mathcal{H}$, *the sequence* $\{x_n\}_{n=0}^{\infty}$ *generated by*

$$x_{n+1} := \lambda_n u + \gamma_n x_n + \delta_n J_{c_n} x_n \text{ for all } n \in \mathbb{N}_+,$$

converges strongly to $P_{A^{-1}0}(u)$.

A simple proof of Theorem 2.25. Using the same technique as in the proof of Corollary 2.15, we can write

$$J_{c_n} = \frac{1}{2}I + \frac{1}{2}S_n$$

for some nonexpansive mapping $S_n : \mathcal{H} \to \mathcal{H}$. Moreover, we have the following expression:

$$\begin{aligned} x_{n+1} &= \lambda_n u + \gamma_n x_n + \delta_n\Big(\frac{1}{2}I + \frac{1}{2}S_n\Big)x_n \\ &= \lambda_n u + (\gamma_n + (\delta_n/2))\,x_n + (\delta_n/2)S_n x_n \\ &= \lambda_n u + (1-\lambda_n)T_n x_n, \end{aligned}$$

where

$$T_n := \frac{(\gamma_n + \delta_n/2)I + (\delta_n/2)S_n}{1-\lambda_n}.$$

We obtain the conclusion from Lemma 2.24 by showing that:

(♠) $\{T_n\}_{n=0}^{\infty}$ is a strongly quasi-nonexpansive sequence;
(♡) $\{T_n\}_{n=0}^{\infty}$ satisfies the CP-condition.

To prove (♠), we first note that

$$\varnothing \neq A^{-1}0 = \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n) = \bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n).$$

Let $x \in \mathcal{H}$ and $p \in \bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n)$. We consider

$$\begin{aligned} \|T_n x - p\|^2 &= \Big\|\frac{\gamma_n + \delta_n/2}{1-\lambda_n}(x-p) + \frac{\delta_n/2}{1-\lambda_n}(S_n x - p)\Big\|^2 \\ &= \frac{\gamma_n + \delta_n/2}{1-\lambda_n}\|x-p\|^2 + \frac{\delta_n/2}{1-\lambda_n}\|S_n x - p\|^2 \\ &\quad - \Big(\frac{\gamma_n + \delta_n/2}{1-\lambda_n}\Big)\Big(\frac{\delta_n/2}{1-\lambda_n}\Big)\|x - S_n x\|^2 \\ &\leq \|x-p\|^2 - \Big(\frac{\gamma_n + \delta_n/2}{1-\lambda_n}\Big)\Big(\frac{\delta_n/2}{1-\lambda_n}\Big)\|x - S_n x\|^2. \end{aligned}$$

In particular, $\|T_n x - p\| \leq \|x - p\|$ for all $x \in \mathcal{H}$ and $p \in \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ and for all $n \in \mathbb{N}_+$. Next, let $\{y_n\}_{n=0}^{\infty}$ be a bounded sequence in $\mathcal{H}$ such that

$$\lim_{n\to\infty} \Big(\|y_n - p\| - \|T_n y_n - p\| \Big) = 0 \text{ for some } p \in \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n).$$

It follows from the boundedness of $\{y_n\}_{n=0}^{\infty}$ that $\lim_{n\to\infty} \Big(\|y_n - p\|^2 - \|T_n y_n - p\|^2 \Big) = 0$. In particular,

$$\lim_{n\to\infty} \Big(\frac{\gamma_n + \delta_n/2}{1-\lambda_n} \Big) \Big(\frac{\delta_n/2}{1-\lambda_n} \Big) \|y_n - S_n y_n\|^2 = 0.$$

Note that $\liminf_{n\to\infty} \frac{\delta_n}{1-\lambda_n} > 0$ and $\liminf_{n\to\infty} \frac{\gamma_n+\delta_n/2}{1-\lambda_n} > 0$. This implies that $\lim_{n\to\infty} \|y_n - S_n y_n\| = 0$ and hence

$$\lim_{n\to\infty} \|y_n - T_n y_n\| = \lim_{n\to\infty} \frac{\delta_n/2}{1-\lambda_n} \|y_n - S_n y_n\| = 0.$$

To prove ($\heartsuit$), let $\{z_n\}_{n=0}^{\infty}$ be a bounded sequence in $\mathcal{H}$ such that $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$. Note that

$$\|z_n - T_n z_n\| = \frac{\delta_n/2}{1-\lambda_n} \|z_n - S_n z_n\| = \frac{\delta_n/2}{1-\lambda_n} \|z_n - (2J_{c_n} - I) z_n\| = \frac{\delta_n}{1-\lambda_n} \|z_n - J_{c_n} z_n\|.$$

Since $\liminf_{n\to\infty} \delta_n > 0$ and $\lim_{n\to\infty} \lambda_n = 0$, we have $\liminf_{n\to\infty} \frac{\delta_n}{1-\lambda_n} > 0$ and hence $\lim_{n\to\infty} \|z_n - J_{c_n} z_n\| = 0$. Note that $J_{c_n} x = J_1\big(\frac{1}{c_n} x + \frac{c_n - 1}{c_n} J_{c_n} x\big)$ for all $x \in \mathcal{H}$. Then

$$\|J_{c_n} x_n - J_1 x_n\| \leq \Big\| \Big(\frac{1}{c_n} x_n + \frac{c_n - 1}{c_n} J_{c_n} x_n \Big) - x_n \Big\| = \Big| 1 - \frac{1}{c_n} \Big| \|x_n - J_{c_n} x_n\|.$$

It follows from $\liminf_{n\to\infty} c_n > 0$ and $\lim_{n\to\infty} \|z_n - J_{c_n} z_n\| = 0$ that $\lim_{n\to\infty} \|z_n - J_1 z_n\| = 0$. Since J_1 is nonexpansive, it follows from Remark 2.6(3) that $\mathfrak{W}\{z_n\}_{n=0}^{\infty} \subset \mathrm{Fix}(J_1) = A^{-1}0$, that is, ($\heartsuit$) holds. $\square$

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Strong convergence of Browder's and Halpern's type iterations in Hilbert spaces

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Abstract We use results from theory of nonexpansive mappings to unify and deduce the recent results of Lin and Takahashi (Positivity 16:429–453, 2012) and of Takahashi (J Optim Theory Appl 157:781–802, 2013).

Keywords Fixed point · Maximal monotone operator · Resolvent · Inverse strongly monotone mapping

Mathematics Subject Classification 47H05 · 47H10 · 58E35

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. For a mapping $T : C \to \mathcal{H}$ where $C \subset \mathcal{H}$, we write $\mathrm{Fix}(T)$ for the *fixed point set* of T, that is, $\mathrm{Fix}(T) := \{x \in C : x = Tx\}$. It is known that many problems can be reformulated to the problem of finding a fixed point of an associated mapping. In this paper, we use a fixed point algorithm approach to the problem of finding a zero of maximal monotone operator. Recall that an operator $A \subset \mathcal{H} \times \mathcal{H}$ is *monotone* if $\langle x - y, u - v \rangle \geq 0$ for all $(x, u), (y, v) \in A$. We say that A is *maximal monotone* if it is monotone and it cannot be properly included in any other monotone operator. Minty [5] proved that a monotone operator $A \subset \mathcal{H} \times \mathcal{H}$ is maximal monotone if and only if the operator

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$I + A$ is surjective, that is, the range of $I + A$ is entirely $\mathcal{H}$ (see also [7]). Using this result, the concept of *resolvent* can be defined as follows: For each $x \in \mathcal{H}$ and $r > 0$, there exists a unique element $z \in \mathcal{H}$ such that $z \in x + rAx$. In this case, we write $z = J_{rA}x := (I + rA)^{-1}x$. It is easy to see that $\mathrm{Fix}(J_{rA})$ coincides with the set of zeros of A, that is, $\mathrm{Fix}(J_{rA}) = A^{-1}0 := \{x \in \mathcal{H} : 0 \in Ax\}$. In other word, the problem of finding a zero of a maximal monotone operator is equivalent to that of finding a fixed point of its resolvent.

Recently, Lin and Takahashi [3] and Takahashi [6] proposed an iterative sequence to approximate a common zero of two maximal monotone operators. More precisely, let C be a closed convex subset of a Hilbert space $\mathcal{H}$. Suppose that $A : C \to \mathcal{H}$ is inverse strongly monotone and $B, F \subset \mathcal{H} \times \mathcal{H}$ are maximal monotone such that $\mathrm{dom}(F) := \{x \in \mathcal{H} : Fx \neq \varnothing\} \subset C$. The problem discussed in [3] and [6] is to approximate an element in $(A + B)^{-1}0 \cap F^{-1}0$. The purpose of this paper is to use theory of nonexpansive mappings to give a concise and short proof of both results. The main results of this paper are presented in two subsections, that is, the Browder's type and the Halpern's type iterations.

2 Some definitions and preliminaries

Let C be a subset of a Hilbert space $\mathcal{H}$. Recall that a mapping $T : C \to \mathcal{H}$ is called *L-Lipschitzian* if $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$. An L-Lipschitzian mapping with $L < 1$ and $L = 1$ is called a *contraction* and a *nonexpansive mapping*, respectively. The theory of nonexpansive mappings is closely connected to theory of maximal monotone operators. In fact, every resolvent of a maximal monotone operator is a nonexpansive mapping.

For a closed convex subset C of a Hilbert space $\mathcal{H}$, the projection of an element $x \in \mathcal{H}$ onto C is the unique point $P_C x$ in C such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

In particular, we have $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $y \in C$.

We recall two recent interesting results. The first one is from the work of Takahashi [8] and the second one from the work of Aoyama et al. [1].

Theorem T *Suppose that $\{T_n : \mathcal{H} \to \mathcal{H}\}$ is a countable family of nonexpansive mappings and $T : \mathcal{H} \to \mathcal{H}$ is a nonexpansive mapping such that $\varnothing \neq \mathrm{Fix}(T) \subset \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Suppose that $\{T_n\}$ satisfies the NST-condition (I) with T, that is, $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$ whenever $\{z_n\}$ is a bounded sequence in $\mathcal{H}$ with $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$. Let $u \in \mathcal{H}$ and $\{x_n\}$ be a sequence in $\mathcal{H}$ defined by*

$$x_n = \alpha_n u + (1 - \alpha_n)T_n x_n, \quad \textit{for all } n \geq 1$$

where $\{\alpha_n\} \subset (0, 1)$ and $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ converges to $P_{\mathrm{Fix}(T)}u$.

Theorem AKTT *Suppose that $\{T_n : \mathcal{H} \to \mathcal{H}\}$ is a countable family of nonexpansive mappings such that $F := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \varnothing$. Suppose that $\{T_n\}$ satisfies the AKTT-condition, that is, the following two conditions are satified:*

- *$\sum_{n=1}^{\infty} \sup\{\|T_n z - T_{n+1} z\| : z \in B\} < \infty$ whenever B is a bounded subset of $\mathcal{H}$ (and hence $\{T_n x\}$ is a Cauchy sequence for all $x \in \mathcal{H}$);*
- *The mapping $T : \mathcal{H} \to \mathcal{H}$ defined by $Tx := \lim_{n\to\infty} T_n x$ for all $x \in \mathcal{H}$ satisfies $\mathrm{Fix}(T) = F$.*

Let $u \in \mathcal{H}$ and $\{x_n\}$ be a sequence in $\mathcal{H}$ defined by $x_1 \in \mathcal{H}$ arbitrarily chosen and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_n x_n, \quad \textit{for all } n \geq 1$$

where $\{\alpha_n\} \subset [0, 1]$ satisfies the following conditions

$$\lim_{n\to\infty} \alpha_n = 0; \quad \sum_{n=1}^{\infty} \alpha_n = \infty; \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

Then the sequence $\{x_n\}$ converges to $P_F u$.

In this paper, we modify Theorems T and AKTT to Theorems 1 and 2, respectively. Moreover, we show that the following four results are deduced from our Theorems 1 and 2.

The following two results are Theorems 7 and 8 of [3].

Theorem LT1 *Let C be a nonempty closed convex subset of $\mathcal{H}$. Let $\alpha > 0$ and $A : C \to \mathcal{H}$ be α-inverse strongly monotone. Let B, F be a maximal monotone operator on $\mathcal{H}$ such that domain of F included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ and $T_r = (I + rF)^{-1}$ be the resolvent of B for $\lambda > 0$ and F for $r > 0$, respectively. Let $k \in (0, 1)$ and let g be a k-contraction of $\mathcal{H}$ into itself. Let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian with $\overline{\gamma} > 0$ and $L > 0$. Suppose that μ and γ are two real numbers such that*

$$0 < \mu < \frac{2\overline{\gamma}}{L^2} \textit{ and } 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2 \mu}{2}}{k}.$$

Suppose that $(A + B)^{-1}0 \cap F^{-1}0 \neq \varnothing$. Assume that $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, \infty)$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, 0 < a \leq \lambda_n \leq 2\alpha \textit{ and } \liminf_{n\to\infty} r_n > 0.$$

Then the following hold:

(1) *For each $n \geq 1$, define*

$$T_n x := \alpha_n \gamma g(x) + (I - \alpha_n V) J_{\lambda_n} (I - \lambda_n A) T_{r_n} x \quad \textit{for all } x \in \mathcal{H}.$$

Then T_n has a unique fixed point $x_n \in \mathcal{H}$ and $\{x_n\}$ is well-defined and bounded.

(2) *The sequence* $\{x_n\}$ *converges strongly to a unique element* $z_0 \in (A+B)^{-1}0 \cap F^{-1}0$ *and*

$$\langle (V - \gamma g)z_0, q - z_0 \rangle \geq 0 \text{ for all } q \in (A+B)^{-1}0 \cap F^{-1}0.$$

Recall that $V : \mathcal{H} \to \mathcal{H}$ is $\overline{\gamma}$*-strongly monotone* if $\langle x - y, Vx - Vy \rangle \geq \overline{\gamma}\|x - y\|^2$ for all $x, y \in \mathcal{H}$.

Theorem LT2 *In the setting of Theorem LT1, let* $x_1 = x \in \mathcal{H}$ *and*

$$x_{n+1} := \alpha_n \gamma g(x_n) + (I - \alpha_n V)J_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n \quad \text{for all } n \geq 1,$$

where $\{\alpha_n\} \subset (0,1)$*,* $\{\lambda_n\} \subset (0,\infty)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_n - \alpha_n| < \infty, 0 < a \leq \lambda_n \leq 2\alpha,$$

$$\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty, \liminf_{n\to\infty} r_n > 0, \text{ and } \sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty.$$

Then the sequence $\{x_n\}$ *converges strongly to a unique element* $z_0 \in (A+B)^{-1}0 \cap F^{-1}0$ *and*

$$\langle (V - \gamma g)z_0, q - z_0 \rangle \geq 0 \text{ for all } q \in (A+B)^{-1}0 \cap F^{-1}0.$$

The following two results are Theorems 3.1 and 3.2 of [6].

Theorem T1 *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$*. Let* $\alpha > 0$ *and* $A : C \to \mathcal{H}$ *be* α*-inverse strongly monotone. Let* B, F *be a maximal monotone operator on* $\mathcal{H}$ *such that domain of* F *included in* C*. Let* $J_\lambda = (I + \lambda B)^{-1}$ *and* $T_r = (I + rF)^{-1}$ *be the resolvent of* B *for* $\lambda > 0$ *and* F *for* $r > 0$*, respectively. Let* $k \in (0,1)$ *and let* g *be a* k*-contraction of* $\mathcal{H}$ *into itself. Let* G *be a strongly positive bounded linear self-adjoint operator* $\mathcal{H}$ *with coefficient* $\overline{\gamma} > 0$*. Suppose that* $0 < \gamma < \overline{\gamma}/k$ *and* $(A+B)^{-1}0 \cap F^{-1}0 \neq \varnothing$*. Assume that* $\{\alpha_n\} \subset (0,1)$*,* $\{\lambda_n\} \subset (0,\infty)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, 0 < a \leq \lambda_n \leq 2\alpha \text{ and } \liminf_{n\to\infty} r_n > 0.$$

Then the following hold:

(1) *For sufficiently large* $n \geq 1$*, define*

$$T_n x := \alpha_n \gamma g(x) + (I - \alpha_n G)J_{\lambda_n}(I - \lambda_n A)T_{r_n}x \quad \text{for all } x \in \mathcal{H}.$$

Then T_n *has a unique fixed point* $x_n \in \mathcal{H}$ *and* $\{x_n\}$ *is well-defined and bounded.*

(2) *The sequence* $\{x_n\}$ *converges strongly to a unique element* $z_0 \in (A+B)^{-1}0 \cap F^{-1}0$ *and*

$$\langle (G - \gamma g)z_0, q - z_0 \rangle \geq 0 \text{ for all } q \in (A+B)^{-1}0 \cap F^{-1}0.$$

Theorem T2 *In the setting of Theorem T1, let* $x_1 = x \in \mathcal{H}$ *and*

$$x_{n+1} := \alpha_n \gamma g(x_n) + (I - \alpha_n G) J_{\lambda_n}(I - \lambda_n A) T_{r_n} x_n \quad \text{for all } n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, \infty)$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_n - \alpha_n| < \infty, 0 < a \leq \lambda_n \leq 2\alpha,$$

$$\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty, \liminf_{n\to\infty} r_n > 0, \text{ and } \sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty.$$

Then the sequence $\{x_n\}$ *converges strongly to a unique element* $z_0 \in (A+B)^{-1}0 \cap F^{-1}0$ *and*

$$\langle (G - \gamma g)z_0, q - z_0 \rangle \geq 0 \text{ for all } q \in (A+B)^{-1}0 \cap F^{-1}0.$$

In this paper, we need the following results.

Lemma 1 *Suppose that* $\{s_n\}$, $\{t_n\}$, *and* $\{\alpha_n\}$ *are sequences of real numbers such that* $s_n \geq 0$, $\alpha_n \in [0, 1]$, *and* $s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n$ *for all* $n \geq 1$. *If* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\limsup_{n\to\infty} t_n \leq 0$, *then* $\lim_{n\to\infty} s_n = 0$.

Lemma 2 *Suppose that* $B \subset \mathcal{H} \times \mathcal{H}$ *is a maximal monotone operator and* J_r *is the resolvent of* B *for* $r > 0$. *Then*

$$\|J_r x - J_s x\| \leq \left|1 - \frac{s}{r}\right| \|x - J_r x\|$$

for all $x \in \mathcal{H}$ *and* $r, s > 0$.

3 Main results

3.1 Browder's type iterations

We first modify Theorem T in the following way.

Theorem 1 *Suppose that* $\{T_n : \mathcal{H} \to \mathcal{H}\}$ *is a countable family of nonexpansive mappings and* $T : \mathcal{H} \to \mathcal{H}$ *is a nonexpansive mapping such that* $\emptyset \neq \mathrm{Fix}(T) \subset \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. *Suppose that* $\{T_n\}$ *satisfies the NST-condition (I) with* T. *Suppose that*

$f, g : \mathcal{H} \to \mathcal{H}$ is α- and β-Lipschitzian, respectively and $\alpha + \beta < 1$. Let $\{x_n\}$ be a sequence in $\mathcal{H}$ defined by

$$x_n = \alpha_n(f(x_n) + g(T_n x_n)) + (1 - \alpha_n)T_n x_n, \quad \textit{for all } n \geq 1$$

where $\{\alpha_n\} \subset (0, 1)$ and $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ converges to $z_0 = P_{\mathrm{Fix}(T)}(f + g)(z_0)$.

Proof Obviously, the sequence $\{x_n\}$ is well-defined. Note that $P_{\mathrm{Fix}(T)} \circ (f + g)$ is an $(\alpha + \beta)$-contraction. By the Banach contraction principle, there exists a unique element $z_0 \in \mathrm{Fix}(T)$ such that

$$z_0 = P_{\mathrm{Fix}(T)}(f + g)(z_0).$$

Define the following iterative sequence

$$z_n = \alpha_n(f + g)(z_0) + (1 - \alpha_n)T_n z_n$$

for all $n \geq 1$. It follows from Theorem T that the sequence $\{z_n\}$ converges to $z_0 = P_{\mathrm{Fix}(T)}(f + g)(z_0)$. Now we consider

$$\begin{aligned} \|f(x_n) - f(z_0)\| &\leq \|f(x_n) - f(z_n)\| + \|f(z_n) - f(z_0)\| \\ &\leq \alpha\|x_n - z_n\| + \alpha\|z_n - z_0\|; \end{aligned}$$

and

$$\begin{aligned} \|g(Tx_n) - g(z_0)\| &= \|g(Tx_n) - g(Tz_0)\| \\ &\leq \|g(Tx_n) - g(Tz_n)\| + \|g(Tz_n) - g(Tz_0)\| \\ &\leq \beta\|Tx_n - Tz_n\| + \beta\|Tz_n - Tz_0\| \\ &\leq \beta\|x_n - z_n\| + \beta\|z_n - z_0\|. \end{aligned}$$

This implies that

$$\begin{aligned} &\|x_n - z_n\| \\ &\quad \leq \alpha_n\|f(x_n) - f(z_0)\| + \alpha_n\|g(T_n x_n) - g(z_0)\| + (1 - \alpha_n)\|T_n x_n - T_n z_n\| \\ &\quad \leq \alpha_n\alpha\|x_n - z_n\| + \alpha_n\alpha\|z_n - z_0\| \\ &\qquad + \alpha_n\beta\|x_n - z_n\| + \alpha_n\beta\|z_n - z_0\| + (1 - \alpha_n)\|x_n - z_n\|. \end{aligned}$$

In particular,

$$(1 - (\alpha + \beta))\|x_n - z_n\| \leq (\alpha + \beta)\|z_n - z_0\|.$$

It follows from $\lim_{n\to\infty} \|z_n - z_0\| = 0$ and $\alpha + \beta < 1$ that $\lim_{n\to\infty} \|x_n - z_n\| = 0$. We conclude that $\{x_n\}$ converges to $z_0 = P_{\mathrm{Fix}(T)}(f + g)(z_0)$. This completes the proof. □

Theorem 2 *Theorem 1 implies Theorem LT1.*

Theorem 3 *Theorem 1 implies Theorem T1.*

Before we prove the preceding two theorems, we show the following result.

Lemma 3 *Suppose that C is a closed convex subset of $\mathcal{H}$. Let $A : C \to \mathcal{H}$ be α-inverse strongly monotone where $\alpha > 0$. Let B, F be a maximal monotone operator on $\mathcal{H}$ such that domain of F included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ and $T_r = (I + rF)^{-1}$ be the resolvent of B for $\lambda > 0$ and F for $r > 0$, respectively. Assume that $\{\lambda_n\} \subset (0, \infty)$ and $\{r_n\} \subset (0, \infty)$ satisfy*

$$0 < a \leq \lambda_n \leq 2\alpha \text{ and } \liminf_{n\to\infty} r_n > 0.$$

Suppose that $(A+B)^{-1}0 \cap F^{-1}0 \neq 0$. Then $\{J_{\lambda_n}(I-\lambda_n A)T_{r_n}\}$ satisfies NST-condition with $J_\alpha(I - \alpha A)T_\alpha$.

Proof First we note that $\mathrm{Fix}(J_{\lambda_n}(I - \lambda_n A)) = (A + B)^{-1}0$ and $\mathrm{Fix}(T_{r_n}) = F^{-1}0$ for all $n \geq 1$. Let $\{x_n\}$ be a bounded sequence in H such that

$$\lim_{n\to\infty} \|x_n - J_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n\| = 0.$$

Let $z \in (A + B)^{-1}0 \cap F^{-1}0$. Then $z = J_{\lambda_n}(I - \lambda_n A)z = T_{r_n}z$ for all $n \geq 1$. In particular,

$$\|J_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n - z\| \leq \|T_{r_n}x_n - z\| \leq \|x_n - z\|.$$

This implies that $\lim_{n\to\infty}(\|x_n - z\| - \|T_{r_n}x_n - z\|) = 0$. Since each T_{r_n} is firmly nonexpansive,

$$\|x_n - T_{r_n}x_n\|^2 + \|T_{r_n}x_n - z\|^2 \leq \|x_n - z\|^2.$$

We obtain that

$$\lim_{n\to\infty} \|x_n - T_{r_n}x_n\| = 0. \tag{1}$$

Note that

$$\begin{aligned}
&\|x_n - J_{\lambda_n}(I - \lambda_n A)x_n\| \\
&\quad \leq \|x_n - J_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n\| + \|J_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n - J_{\lambda_n}(I - \lambda_n A)x_n\| \\
&\quad \leq \|x_n - J_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n\| + \|T_{r_n}x_n - x_n\|.
\end{aligned}$$

It follows from (1) that

$$\lim_{n\to\infty} \|x_n - J_{\lambda_n}(I - \lambda_n A)x_n\| = 0. \tag{2}$$

Let $\{\lambda_{n_k}\}$ be a subsequence $\{\lambda_n\}$ such that $\lim_{k\to\infty}\lambda_{n_k} = \lambda$ for some $\lambda \in [a, 2\alpha]$. Next, we observe that

$$\begin{aligned}
&\|J_{\lambda_{n_k}}(I-\lambda_{n_k}A)x_{n_k} - J_\lambda(I-\lambda_{n_k}A)x_{n_k}\| \\
&\quad\le \left|1-\frac{\lambda}{\lambda_{n_k}}\right| \|(I-\lambda_{n_k}A)x_{n_k} - J_{\lambda_{n_k}}(I-\lambda_{n_k}A)x_{n_k}\| \\
&\quad\le \left|1-\frac{\lambda}{\lambda_{n_k}}\right| \Big(\|x_{n_k} - J_{\lambda_{n_k}}(I-\lambda_{n_k}A)x_{n_k}\| + \lambda_{n_k}\|Ax_{n_k}\|\Big).
\end{aligned}$$

Therefore, since $\{Ax_{n_k}\}$ is bounded, we have

$$\lim_{k\to\infty}\|J_{\lambda_{n_k}}(I-\lambda_n A)x_{n_k} - J_\lambda(I-\lambda_{n_k}A)x_{n_k}\| = 0. \tag{3}$$

Moreover,

$$\begin{aligned}
&\|x_{n_k} - J_\lambda(I-\lambda A)x_{n_k}\| \\
&\quad\le \|x_{n_k} - J_\lambda(I-\lambda_{n_k}A)x_{n_k}\| + \|J_\lambda(I-\lambda_{n_k}A)x_{n_k} - J_\lambda(I-\lambda A)x_{n_k}\| \\
&\quad\le \|x_{n_k} - J_\lambda(I-\lambda_{n_k}A)x_{n_k}\| + |\lambda_{n_k}-\lambda|\|Ax_{n_k}\| \\
&\quad\le \|x_{n_k} - J_{\lambda_{n_k}}(I-\lambda_{n_k}A)x_{n_k}\| + \|J_{\lambda_{n_k}}(I-\lambda_{n_k}A)x_{n_k} - J_\lambda(I-\lambda_{n_k}A)x_{n_k}\| \\
&\qquad + |\lambda_{n_k}-\lambda|\|Ax_{n_k}\|.
\end{aligned}$$

It follows from (2) and (3) that

$$\lim_{k\to\infty}\|x_{n_k} - J_\lambda(I-\lambda A)x_{n_k}\| = 0. \tag{4}$$

Next, we consider

$$\begin{aligned}
&\|x_{n_k} - J_\alpha(I-\alpha A)x_{n_k}\| \\
&\quad\le \|x_{n_k} - J_\lambda(I-\lambda A)x_{n_k}\| + \|J_\lambda(I-\lambda A)x_{n_k} - J_\alpha(I-\alpha A)x_{n_k}\| \\
&\quad\le \|x_{n_k} - J_\lambda(I-\lambda A)x_{n_k}\| + \left|1-\frac{\alpha}{\lambda}\right|\|x_{n_k} - J_\lambda(I-\lambda A)x_{n_k}\|.
\end{aligned}$$

Using (4), we obtain that

$$\lim_{k\to\infty}\|x_{n_k} - J_\alpha(I-\alpha A)x_{n_k}\| = 0.$$

By the double extract subsequence principle, we conclude that

$$\lim_{n\to\infty}\|x_n - J_\alpha(I-\alpha A)x_n\| = 0. \tag{5}$$

Now we consider the following inequality

$$
\begin{aligned}
&\|x_n - J_\alpha(I-\alpha A)T_\alpha x_n\| \\
&\quad \le \|x_n - J_\alpha(I-\alpha A)x_n\| + \|J_\alpha(I-\alpha A)x_n - J_\alpha(I-\alpha A)T_\alpha x_n\| \\
&\quad \le \|x_n - J_\alpha(I-\alpha A)x_n\| + \|x_n - T_\alpha x_n\| \\
&\quad \le \|x_n - J_\alpha(I-\alpha A)x_n\| + \|x_n - T_{r_n} x_n\| + \|T_{r_n} x_n - T_\alpha x_n\| \\
&\quad \le \|x_n - J_\alpha(I-\alpha A)x_n\| + \left(1 + \left|1 - \frac{\alpha}{r_n}\right|\right)\|x_n - T_{r_n} x_n\|.
\end{aligned}
$$

It follows from (1), (5), and $\liminf_{n\to\infty} r_n > 0$ that

$$\lim_{n\to\infty} \|x_n - J_\alpha(I-\alpha A)T_\alpha x_n\| = 0.$$

This completes the proof. □

Proof of Theorem 2 Put $S := J_\alpha(I-\alpha A)T_\alpha$ and $S_n := J_{\lambda_n}(I-\lambda_n A)T_{r_n}$ for all $n \ge 1$. It follows from Lemma 3 that $\{S_n\}$ satisfies NST-condition with S. Moreover, $\mathrm{Fix}(S) = (A+B)^{-1}0 \cap F^{-1}0$. Rewrite the formula of each x_n in Theorem LT1 as follows:

$$x_n = \widehat{\alpha}_n\big(\widehat{f}(x) + \widehat{g}(S_n x)\big) + (1-\widehat{\alpha}_n)S_n x_n$$

where $\widehat{\alpha}_n := \frac{\alpha_n}{\mu}$, $\widehat{f} = \mu\gamma g$ and $\widehat{g} = I - \mu V$ for all $n \ge 1$. Note that $\widehat{f}$ is a $\mu\gamma k$-contraction and $\widehat{g}$ is a $\sqrt{1-2\mu\tau}$-contraction. Moreover, $\gamma\mu k + \sqrt{1-2\mu\tau} < 1$. It follows from our Theorem 1 that $\{x_n\}$ converges to

$$z_0 = P_{\mathrm{Fix}(S)}(\widehat{f} + \widehat{g})z_0.$$

Moreover,

$$\langle (V-\gamma g)z_0, q - z_0\rangle \ge 0 \text{ for all } q \in \mathrm{Fix}(S) = (A+B)^{-1}0 \cap F^{-1}0.$$

The proof is finished. □

Remark 1 In Theorem LT1, the constants μ and γ are chosen such that

$$0 < \mu < \frac{2\overline{\gamma}}{L^2} \text{ and } 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

It should be noted that the conclusion of Theorem LT1 remains true if the following more general condition is satisfied:

$$0 < \mu < \frac{2\overline{\gamma}}{L^2} \text{ and } 0 < \gamma\mu k < 1 - \sqrt{1 - 2\mu\left(\overline{\gamma} - \frac{L^2\mu}{2}\right)}.$$

Proof of Theorem 3 Put $S := J_\alpha(I - \alpha A)T_\alpha$ and $S_n := J_{\lambda_n}(I - \lambda_n A)T_{r_n}$ for all $n \geq 1$. It follows from Lemma 3 that $\{S_n\}$ satisfies NST-condition with S. Moreover, $\mathrm{Fix}(S) = (A + B)^{-1}0 \cap F^{-1}0$. Rewrite the formula of each x_n in Theorem LT1 as follows:

$$x_n = \widehat{\alpha}_n\big(\widehat{f}(x) + \widehat{g}(S_n x)\big) + (1 - \widehat{\alpha}_n)S_n x_n$$

where $\widehat{\alpha}_n := \frac{\alpha_n}{t}$, $\widehat{f} = t\gamma g$ and $\widehat{g} = I - tG$ for all $n \geq 1$. Note that $\widehat{f}$ is a γtk-contraction and $\widehat{g}$ is a $1-t\overline{\gamma}$-contraction. Moreover, $\gamma tk+(1-t\overline{\gamma}) = 1-(\overline{\gamma}-\gamma k)t < 1$. It follows from our Theorem 1 that $\{x_n\}$ converges to

$$z_0 = P_{\mathrm{Fix}(S)}(\widehat{f} + \widehat{g})z_0.$$

Moreover,

$$\langle (G - \gamma g)z_0, q - z_0\rangle \geq 0 \text{ for all } q \in \mathrm{Fix}(S) = (A + B)^{-1}0 \cap F^{-1}0.$$

The proof is finished. □

3.2 Halpern's type iterations

We modify Theorem AKTT in the following way.

Theorem 4 *Suppose that $\{T_n : \mathcal{H} \to \mathcal{H}\}$ is a countable family of nonexpansive mappings such that $F := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$. Suppose that $\{T_n\}$ satisfies the AKTT-condition. Suppose that $f, g : \mathcal{H} \to \mathcal{H}$ is α- and β-Lipschitzian, respectively. Let $\{x_n\}$ be a sequence in $\mathcal{H}$ defined by $x_1 \in \mathcal{H}$ arbitrarily chosen and*

$$x_{n+1} := \alpha_n(f(x_n) + g(T_n x_n)) + (1 - \alpha_n)T_n x_n, \quad \text{for all } n \geq 1,$$

where $\{\alpha_n\} \subset [0, 1]$ satisfies the following conditions

$$\lim_{n\to\infty} \alpha_n = 0; \quad \sum_{n=1}^{\infty} \alpha_n = \infty; \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

If $\alpha + \beta < 1$, then the sequence $\{x_n\}$ converges to $P_F u$.

Proof Note that $P_F \circ (f + g)$ is an $(\alpha + \beta)$-contraction. By the Banach contraction principle, there exists a unique element $z_0 \in F$ such that

$$z_0 = P_{\mathrm{Fix}(T)}(f + g)(z_0).$$

Define the following iterative sequence

$$z_{n+1} := \alpha_n(f + g)(z_0) + (1 - \alpha_n)T_n z_n$$

for all $n \in \mathbb{N}$. It follows from Theorem AKTT that the sequence $\{z_n\}$ converges to $z_0 = P_F(f+g)(z_0)$. Now we consider

$$\begin{aligned}\|f(x_n) - f(z_0)\| &\le \|f(x_n) - f(z_n)\| + \|f(z_n) - f(z_0)\|\\ &\le \alpha\|x_n - z_n\| + \alpha\|z_n - z_0\|;\end{aligned}$$

and

$$\begin{aligned}\|g(Tx_n) - g(z_0)\| &= \|g(Tx_n) - g(Tz_0)\|\\ &\le \|g(Tx_n) - g(Tz_n)\| + \|g(Tz_n) - g(Tz_0)\|\\ &\le \beta\|Tx_n - Tz_n\| + \beta\|z_n - z_0\|\\ &\le \beta\|x_n - z_n\| + \beta\|z_n - z_0\|.\end{aligned}$$

This implies that

$$\begin{aligned}&\|x_{n+1} - z_{n+1}\|\\ &\quad\le \alpha_n\|f(x_n) - f(z_0)\| + \alpha_n\|g(T_n x_n) - g(z_0)\| + (1-\alpha_n)\|T_n x_n - T_n z_n\|\\ &\quad\le \alpha_n\alpha\|x_n - z_n\| + \alpha_n\alpha\|z_n - z_0\|\\ &\qquad + \alpha_n\beta\|x_n - z_n\| + \alpha_n\beta\|z_n - z_0\| + (1-\alpha_n)\|x_n - z_n\|\\ &\quad= (1 - \alpha_n(1 - (\alpha+\beta)))\|x_n - z_n\| + \alpha_n(\alpha+\beta)\|z_n - z_0\|.\end{aligned}$$

Note that $\sum_{n=1}^{\infty} \alpha_n(1 - (\alpha+\beta)) = \infty$ and $\lim_{n\to\infty} \frac{(\alpha+\beta)\|z_n - z_0\|}{1-(\alpha+\beta)} = 0$. It follows from Lemma 1 that $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Hence, we conclude that $\{x_n\}$ converges to $z_0 = P_F(f+g)(z_0)$. This completes the proof. □

Theorem 5 *Theorem 4 implies Theorem LT2.*

Theorem 6 *Theorem 4 implies Theorem T2.*

Before we prove the preceding two theorems, we show the following result.

Lemma 4 *Suppose that C is a closed convex subset of $\mathcal{H}$. Let $A : C \to \mathcal{H}$ be α-inverse strongly monotone where $\alpha > 0$. Let B, F be a maximal monotone operator on H such that domain of F included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ and $T_r = (I + rF)^{-1}$ be the resolvent of B for $\lambda > 0$ and F for $r > 0$, respectively. Assume that $\{\lambda_n\} \subset (0, \infty)$ and $\{r_n\} \subset (0, \infty)$ satisfy*

$$0 < a \le \lambda_n \le 2\alpha, \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty, \liminf_{n\to\infty} r_n > 0, \text{ and } \sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty.$$

Suppose that $(A + B)^{-1}0 \cap F^{-1}0 \neq \emptyset$. Then $\{J_{\lambda_n}(I - \lambda_n A)T_{r_n}\}$ satisfies AKTT-condition.

Proof First, we note that $\sum_{n=1}^{\infty} |1 - \frac{\lambda_{n+1}}{\lambda_n}| < \infty$ and $\sum_{n=1}^{\infty} |1 - \frac{r_{n+1}}{r_n}| < \infty$. Let B be a bounded subset of $\mathcal{H}$ and $x \in B$. Let $p \in (A + B)^{-1}0 \cap F^{-1}0$. It follows that $p = J_{\lambda_n}(I - \lambda_n A)p = T_{r_n} p$ for all $n \geq 1$. Note that all the mappings J_{λ_n}, $I - \lambda_n A$, and T_{r_n} are nonexpansive. We note that

$$\begin{aligned}
&\|J_{\lambda_n}(I - \lambda_n A)T_{r_n}x - p\| \\
&\quad = \|J_{\lambda_n}(I - \lambda_n A)T_{r_n}x - J_{\lambda_n}(I - \lambda_n A)p\| \\
&\quad \leq \|(I - \lambda_n A)T_{r_n}x - (I - \lambda_n A)p\| \\
&\quad \leq \|T_{r_n}x - p\| \\
&\quad \leq \|x - p\|.
\end{aligned}$$

For each $n \geq 1$, we have the following three suprema are finite:

$$\begin{aligned}
&\sup\{\|J_{\lambda_n}(I - \lambda_n A)T_{r_n}x\| : x \in B\}, \quad \sup\{\|T_{r_n}x\| : x \in B\}, \\
&\quad \text{and } \sup\{\|AT_{r_n}x\| : x \in B\}.
\end{aligned}$$

Moreover, we consider the following three estimates:

$$\begin{aligned}
&\|J_{\lambda_{n+1}}(I - \lambda_{n+1}A)T_{r_n}x - J_{\lambda_{n+1}}(I - \lambda_{n+1}A)T_{r_{n+1}}x\| \\
&\quad \leq \|T_{r_n}x - T_{r_{n+1}}x\| \\
&\quad \leq \left|1 - \frac{r_{n+1}}{r_n}\right| \|x - T_{r_n}x\|; \\
&\|J_{\lambda_{n+1}}(I - \lambda_n A)T_{r_n}x - J_{\lambda_{n+1}}(I - \lambda_{n+1}A)T_{r_n}x\| \\
&\quad \leq \|(I - \lambda_n A)T_{r_n}x - (I - \lambda_{n+1}A)T_{r_n}x\| \\
&\quad = |\lambda_n - \lambda_{n+1}| \|AT_{r_n}x\|;
\end{aligned}$$

and

$$\begin{aligned}
&\|J_{\lambda_n}(I - \lambda_n A)T_{r_n}x - J_{\lambda_{n+1}}(I - \lambda_n A)T_{r_n}x\| \\
&\leq \left|1 - \frac{\lambda_{n+1}}{\lambda_n}\right| \|(I - \lambda_n A)T_{r_n}x - J_{\lambda_n}(I - \lambda_n A)T_{r_n}x\|.
\end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty} \sup\{\|J_{\lambda_n}(I - \lambda_n A)T_{r_n}x - J_{\lambda_{n+1}}(I - \lambda_{n+1}A)T_{r_{n+1}}x\| : x \in B\} < \infty.$$

Finally, we assume that $\lambda := \lim_{n\to\infty} \lambda_n$ and $r := \lim_{n\to\infty} r_n$. It is obvious that

$$\lim_{n\to\infty} J_{\lambda_n}(I - \lambda_n A)T_{r_n}x = J_\lambda(I - \lambda A)T_r x$$

for all $x \in \mathcal{H}$ and

$$\mathrm{Fix}(J_{\lambda_n}(I-\lambda_n A)T_{r_n}) = \mathrm{Fix}(J_\lambda(I-\lambda A)T_r)$$

for all $n \geq 1$. This completes the proof. □

Proof of Theorem 5 Put $S := J_\alpha(I-\alpha A)T_\alpha$ and $S_n := J_{\lambda_n}(I-\lambda_n A)T_{r_n}$ for all $n \geq 1$. It follows from Lemma 4 that $\{S_n\}$ satisfies AKTT-condition. Moreover, $\mathrm{Fix}(S) = (A+B)^{-1}0 \cap F^{-1}0$. Rewrite the formula of each x_n in Theorem LT2 as follows:

$$x_{n+1} = \widehat{\alpha}_n\big(\widehat{f}(x_n) + \widehat{g}(S_n x_n)\big) + (1-\widehat{\alpha}_n)S_n x_n$$

where $\widehat{\alpha}_n := \frac{\alpha_n}{\mu}$, $\widehat{f} = \mu\gamma g$ and $\widehat{g} = I - \mu V$ for all $n \geq 1$. Note that $\widehat{f}$ is a $\mu\gamma k$-contraction and $\widehat{g}$ is a $\sqrt{1-2\mu\tau}$-contraction. Moreover, $\gamma\mu k + \sqrt{1-2\mu\tau} < 1$. It follows from our Theorem 4 that $\{x_n\}$ converges to

$$z_0 = P_{\mathrm{Fix}(S)}(\widehat{f} + \widehat{g})z_0.$$

Moreover,

$$\langle (V-\gamma g)z_0, q - z_0\rangle \geq 0 \text{ for all } q \in \mathrm{Fix}(S) = (A+B)^{-1}0 \cap F^{-1}0.$$

The proof is finished. □

Remark 2 In Theorem LT2, the constants μ and γ are chosen such that

$$0 < \mu < \frac{2\overline{\gamma}}{L^2} \text{ and } 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

It should be noted that the conclusion of Theorem LT1 remains true if the following more general condition is satisfied:

$$0 < \gamma\mu k < 1 - \sqrt{1 - 2\mu\Big(\overline{\gamma} - \frac{L^2\mu}{2}\Big)}.$$

Proof of Theorem 6 Put $S := J_\alpha(I-\alpha A)T_\alpha$ and $S_n := J_{\lambda_n}(I-\lambda_n A)T_{r_n}$ for all $n \geq 1$. It follows from Lemma 4 that $\{S_n\}$ satisfies AKTT-condition with S. Moreover, $\mathrm{Fix}(S) = (A+B)^{-1}0 \cap F^{-1}0$. Rewrite the formula of each x_{n+1} in Theorem T2 as follows:

$$x_{n+1} = \widehat{\alpha}_n\big(\widehat{f}(x_n) + \widehat{g}(S_n x_n)\big) + (1-\widehat{\alpha}_n)S_n x_n$$

where $\widehat{\alpha}_n := \frac{\alpha_n}{t}$, $\widehat{f} = t\gamma g$ and $\widehat{g} = I - tG$ for all $n \geq 1$. Note that $\widehat{f}$ is a $\gamma t k$-contraction and $\widehat{g}$ is a $1-t\overline{\gamma}$-contraction. Moreover, $\gamma t k + (1-t\overline{\gamma}) = 1 - t(\overline{\gamma} - \gamma k) < 1$. It follows from our Theorem 4 that $\{x_n\}$ converges to

$$z_0 = P_{\mathrm{Fix}(S)}(\widehat{f} + \widehat{g})z_0.$$

Moreover,

$$\langle (G - \gamma g)z_0, q - z_0 \rangle \geq 0 \text{ for all } q \in \mathrm{Fix}(S) = (A + B)^{-1}0 \cap F^{-1}0.$$

The proof is finished. □

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Fixed point theorems for contractions of Reich type on a metric space with a graph

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Abstract. In a metric space with a directed graph G, Jachymski (Proc Am Math Soc 1(136):1359–1373, 2008) introduced the concept of Banach G-contraction and proved two fixed point theorems for such mappings. Bojor (Nonlinear Anal 75:3895–3901, 2012) generalized this concept to Reich G-contraction and obtain a fixed point theorem. Note that Bojor's theorem is established under the additional type of connectedness of G and it does not include Jachymski's results as a special case. Moreover, there are some mistakes in several corollaries. Some examples and counterexamples are illustrated. It is our purpose to improve Bojor's theorem and to present two fixed point theorems for Reich G-contractions. Our results are extensions of the two Jachymski's theorems. Finally, we also discuss some priori error estimates.

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Keywords. Fixed point, Reich G-contraction, Banach G-contraction, metric space with a directed graph.

1. Introduction

For a mapping T from a nonempty set X into itself, the set of fixed points of T is denoted by $\mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) = \{x \in X : x = Tx\}$. If (X, d) is a metric space, we say that $T : X \to X$ is a *Picard operator* (abbr., PO) if $\mathrm{Fix}(T) = \{x^*\}$ and $\lim_{n\to\infty} d(T^n x, x^*) = 0$ for all $x \in X$. We also say that T is a *weakly Picard operator* (abbr., WPO) if for every $x \in X$, there exists $x^* \in \mathrm{Fix}(T)$, such that $\lim_{n\to\infty} d(T^n x, x^*) = 0$. Obviously, every PO is a WPO, but the converse is not true. One of many interesting results in the literature giving a sufficient condition for being a PO is the Banach Contraction Principle.

Theorem B. *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction, that is, there exists $\alpha \in (0, 1)$, such that $d(Tx, Ty) \le \alpha d(x, y)$ for all $x, y \in X$. Then, T is a PO.*

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It was Reich [11,12] who generalized Theorem B by introducing the so-called *Reich's contraction*: there are $\alpha, \beta, \gamma \in [0, 1)$, such that $\alpha + \beta + \gamma < 1$ and $d(Tx, Ty) \le \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty)$ for all $x, y \in X$. He also proved that if (X, d) is complete, then every Reich's contraction $T\colon X \to X$ is a PO.

On the other hand, Ran and Reurings [10] introduced a new type of contractions with respect to a given partial order in a metric space and they proposed a fixed point theorem which is another generalization of Theorem B. Ran–Reurings's results were discussed further by Turinici [13,14]. Nieto and Rodríguez-López [8,9] gave some more general sufficient conditions for the mapping to be a PO. Later on, Jachymski [7] replaced a partial order using a more general relation, that is, a directed graph. He also proved a fixed point theorem which includes the results of Nieto and Rodríguez-López as a special case (see Theorems J1 and J2 below). To be precise, for a metric space X, let G be a directed graph, where the vertex set $V(G)$ of G is X and the edge set $E(G)$ of G is a subset of the Cartesian product $X \times X$. Throughout this paper, we assume that $(x, x) \in E(G)$ for all $x \in X$. He also introduced the following mapping: $T\colon X \to X$ is a *Banach* G*-contraction* if there exists $\alpha \in (0, 1)$, such that for all $(x, y) \in E(G)$ the following two conditions hold:

- $(Tx, Ty) \in E(G)$;
- $d(Tx, Ty) \le \alpha d(x, y)$.

It is clear that if $E(G) = X \times X$, then a Banach G-contraction reduces to a contraction in Theorem B. Before passing, we note that some fixed point theorems obtained from the combination of a notion of directed graphs and that of Reich's contractions were investigated in [1–3]. However, our results in this paper are not deduced from these papers.

For a given directed graph $G = (V(G), E(G))$ and for $x, y \in V(G)$, a G*-path from* x *to* y is a finite sequence $\{x_i\}_{i=0}^N$ in $V(G)$, such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for all $i = 1, 2, \ldots, N$. For $x \in X$, we write

$$[x]_G = \{y \in X : \text{ there exists a } G\text{-path from } x \text{ to } y\}.$$

We say that G is *connected* if $V(G) = [x]_G$ for all $x \in X$, that is, there exists a G-path from x to y for every pair $x, y \in V(G)$.

For a directed graph G, let G^{-1} be the directed graph obtained from G by reversing the direction of edges, that is, $V(G^{-1}) := V(G)$ and

$$E(G^{-1}) := \{(y, x) : (x, y) \in E(G)\}.$$

We also interested in the undirected graph $\widetilde{G}$ obtained from G by ignoring the direction of edges, that is, $V(\widetilde{G}) := V(G)$ and

$$E(\widetilde{G}) := E(G) \cup E(G^{-1}).$$

Due to this notation, we say that G is *weakly connected* if $V(G) = [x]_{\widetilde{G}}$ for some (and hence, for all) $x \in X$, that is, there exists a $\widetilde{G}$-path from x to y for every pair $x, y \in V(G)$.

For a metric space (X, d) endowed with a directed graph G and for a mapping $T\colon X \to X$, we write $X_T = \{x \in X : (x, Tx) \in E(G)\}$.

Remark 1. Let (X, d) be a metric space endowed with a directed graph G and $T\colon X \to X$ be a mapping. If $X = \bigcup\{[x]_G : x \in X_T\}$ and there exists $x_0 \in X_T$, such that $X_T \subset [x_0]_G$, then $X = [x_0]_G$.

The following two fixed point theorems were proved by Jachymski.

Theorem J1. [7, Theorem 3.2] *Let (X, d) be a complete metric space endowed with a directed graph G and $T\colon X \to X$ be a Banach G-contraction. Suppose that the following condition holds*:

(J-1) *For any sequence $\{x_n\}$ in X, if $\lim_{n\to\infty} x_n = x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.*

Then, the following statements are true.

1. $\operatorname{card}\operatorname{Fix}(T) = \operatorname{card}\{[x]_{\widetilde{G}} : x \in X_T\}$.
2. $\operatorname{Fix}(T) \neq \varnothing$ *if and only if* $X_T \neq \varnothing$.
3. *T has a unique fixed point if and only if there exists $x_0 \in X_T$, such that $X_T \subset [x_0]_{\widetilde{G}}$.*
4. *$T|_{[x_0]_{\widetilde{G}}}$ is a PO for all $x_0 \in X_T$.*
5. *If $X_T \neq \varnothing$ and G is weakly connected, then T is a PO.*
6. *$T|_Y$ is a WPO, where $Y = \bigcup\{[x]_{\widetilde{G}} : x \in X_T\}$.*
7. *If $X_T = X$, then T is a WPO.*

Theorem J2. [7, Theorem 3.3] *Let (X, d) be a complete metric space endowed with a directed graph G and $T\colon X \to X$ be a Banach G-contraction. Suppose that the following condition holds*:

(J-2) *T is orbitally G-continuous, that is, for all $x, y \in X$ and for any subsequence $\{T^{n_k}x\}$ of $\{T^n x\}$, if $\lim_{k\to\infty} T^{n_k}x = y$ and $(T^{n_k}x, T^{n_k+1}x) \in E(G)$ for all $k \in \mathbb{N}$, then $\lim_{k\to\infty} T(T^{n_k}x) = Ty$.*

Then, the following statements are true.

1. $\operatorname{Fix}(T) \neq \varnothing$ *if and only if* $X_T \neq \varnothing$.
2. *For any $x \in X_T$ and $y \in [x]_{\widetilde{G}}$, $\{T^n y\}$ converges to a fixed point of T and $\lim_{n\to\infty} T^n y$ does not depend on y.*
3. *If $X_T \neq \varnothing$ and G is weakly connected, then T is a PO.*
4. *If $X_T = X$, then T is a WPO.*

Note that the key assumption of Theorem Ji is the condition (J-i) where $i = 1, 2$. It is clear that (J-1) $\not\Leftrightarrow$ (J-2).

Recently, Bojor [4] proved a fixed point theorem for Reich's mappings in the setting of a complete metric space with a directed graph. To state Bojor's result, we recall the following type of connectedness introduced in [4].

Definition 2. Let (X, d) be a metric space endowed with a directed graph G and $T\colon X \to X$. We say that the graph G is *T-connected* if for all vertices x, y of G with $(x, y) \notin E(G)$, there exists a G-path $\{x_i\}_{i=0}^N$ from x to y, such that $x_0 = x$, $x_N = y$, and $(x_i, Tx_i) \in E(G)$ for all $i = 1, 2, \dots, N-1$. We say that G is *weakly T-connected* if $\widetilde{G}$ is T-connected.

Remark 3. It is clear that if G is T-connected, then G is connected and $X_T \neq \varnothing$. However, the converse is not true.

The following type of mappings was studied by Bojor [4].

Definition 4. [4] Let (X,d) be a metric space endowed with a directed graph G. A mapping $T\colon X \to X$ is said to be a *Reich G-contraction* if there exist $\alpha, \beta, \gamma \in [0,1)$, such that $\alpha + \beta + \gamma < 1$, and for all $(x,y) \in E(G)$, the following two conditions hold:

- $(Tx, Ty) \in E(G)$;
- $d(Tx, Ty) \le \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty)$.

In this case, we also say that T is a *Reich G-contraction with parameters* α, β, γ.

Remark 5. (1) If T is a Reich G-contraction and $E(G) = X \times X$, then T is a Reich contraction.
(2) If T is a Reich G-contraction with parameters α, β, γ and $\beta = \gamma = 0$, then it is a Banach G-contraction.
(3) If T is a Reich G-contraction with parameters α, β, γ and $\beta = \gamma$, then it is a Reich $\widetilde{G}$-contraction.

Remark 6. Suppose that T is a Reich G-contraction with parameters α, β, γ. Suppose that the following condition hold:

$$(x,y) \in E(G) \iff (y,x) \in E(G). \tag{$\spadesuit$}$$

Then, T is a Reich G-contraction with parameters α', β', γ', where $\alpha' = \alpha$ and $\beta' = \gamma' = \frac{1}{2}(\beta + \gamma)$. In fact, we assume that $(x,y) \in E(G)$. Then, $(y,x) \in E(G)$. It follows from the definition of Reich G-contraction that:

$$\begin{aligned} d(Tx,Ty) &\le \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) \\ d(Ty,Tx) &\le \alpha d(y,x) + \beta d(y,Ty) + \gamma d(x,Tx). \end{aligned}$$

This implies that

$$d(Tx,Ty) \le \alpha' d(x,y) + \beta' d(x,Tx) + \gamma' d(y,Ty),$$

where $\alpha' = \alpha$ and $\beta' = \gamma' = \frac{1}{2}(\beta+\gamma)$. Note that the conclusion above fails if ($\spadesuit$) is not satisfied.

Remark 7. It follows from Remarks 5(1) and 6 that if T is a Reich contraction with parameters α, β, γ, then T is a Reich contraction with parameters α', β', and γ', where $\alpha' = \alpha$ and $\beta' = \gamma' = \frac{1}{2}(\beta + \gamma)$.

The following example shows that the class of Reich G-contractions is different from that of Reich contractions and that of Banach G-contractions.

Example 8. Let $X = \mathbb{R}$ equipped with the usual metric d. Define a directed graph G on X by $E(G) := \{(x,x) : x \in X\} \cup \{(0,x) : x \neq 0\}$. Define $T\colon X \to X$ by $Tx = -x$ for all $x \in X$. It is clear that if $(x,y) \in E(G)$, then $(Tx,Ty) \in E(G)$. Let $(x,y) \in E(G)$. We may assume that $x = 0$ and $y \neq 0$. Note that $d(Tx,Ty) = d(x,y) = |y|$, $d(x,Tx) = 0$, and $d(y,Ty) = 2|y|$. It follows that T is a Reich G-contraction with parameters $\alpha = \frac{1}{4}$, $\beta = 0$, and $\gamma = \frac{1}{2}$. We show that T is not a Reich contraction. Suppose that T is a Reich contraction with parameters α', β', and γ', where $\beta' = \gamma'$. Note that $T0 = 0$,

$T1 = -1$ and $1 = d(T0, T1) \leq \alpha' d(0,1) + \beta' d(0, T0) + \gamma' d(1, T1) = \alpha' + 2\gamma'$. In particular,

$$1 + \beta' \leq (\alpha' + \beta' + \gamma') + \gamma' < 1 + \gamma'.$$

This implies that $\beta' < \gamma'$ which is a contradiction. Moreover, it is obvious that T is not a Banach G-contraction.

Bojor [4] proposed the following theorem [4, Theorem 6] which is related to Theorem J1.

Theorem Bo. *Let (X, d) be a complete metric space endowed with a directed graph G and $T \colon X \to X$ be a Reich G-contraction. Suppose that the following condition holds*:

(J-1) *For any sequence $\{x_n\}$ in X, if $\lim_{n\to\infty} x_n = x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.*

If G is T-connected, then T is a PO.

Due to Remark 3, we cannot conclude that Theorem Bo is a generalization of Theorem J1. It is worth mentioning that Theorem Bo was extended from the single-valued mapping to the multi-valued one by Alfuraidan and Khamsi (see [3, Theorem 4.4]).

It is our purpose to give two generalizations of Theorem J1 and of Theorem J2 for Reich G-contraction. In fact, the concept of T-connectedness as was the key assumption of Theorem Bo can be replaced by the condition originally discussed in Theorems J1 and J2. The paper is organized as follows: In main results, we prepare several sufficient conditions for the two sequences $\{T^n x\}$ and $\{T^n y\}$ to be Cauchy equivalent. The extensions of Theorems J1 and J2 are presented in Sects. 2.1 and 2.2, respectively. Note that the extension of Theorem J2 obtained in this paper is a nice application of the result of Hicks and Rhoades [6]. In the last subsection, if we assume that a Reich G-contraction has a fixed point in place of the completeness of a space and the conditions (J-1) and (J-2), some priori error estimates are presented. We consider the priori error estimate when the graph is defined from a partial order. We also discuss some gaps in Bojor's results and present some related examples and counterexamples to his results.

2. Main results

The following result was implicitly proved in [4].

Lemma 9. [4] *Let (X, d) be a metric space endowed with a directed graph G and $T \colon X \to X$ be a Reich G-contraction with parameters α, β, γ. Let $\delta = \frac{\alpha+\beta}{1-\gamma}$. If $(x, Tx) \in E(G)$, then*

$$d(T^{n+1}x, T^{n+2}x) \leq \delta d(T^n x, T^{n+1}x) \quad \textit{for all } n \geq 1,$$

and hence, $\{T^n x\}$ is a Cauchy sequence.

Definition 10. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers. We write $a_n = O(b_n)$ for the situation that there exist a positive number $C > 0$ and an integer $N \geq 1$, such that $a_n \leq Cb_n$ for all $n \geq N$.

Lemma 11. *Let $\delta \in (0, 1)$ and k be a nonnegative integer. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers, such that $a_{n+1} \leq \delta a_n + b_n$ for all $n \geq 1$. If $b_n = O(n^k\delta^n)$, then $a_n = O(n^{k+1}\delta^n)$.*

Proof. Since $b_n = O(n^k\delta^n)$, there exist a positive number $D > 0$ and an integer $N \geq 1$, such that $b_n \leq Dn^k\delta^n$ for all $n \geq N$. Then, for all $n \geq N$, we have

$$\frac{a_{n+1}}{D} \leq \delta\frac{a_n}{D} + n^k\delta^n.$$

Without loss of generality, we assume that $D = N = 1$. That is,

$$a_{n+1} \leq \delta a_n + n^k\delta^n \quad \text{for all } n \geq 1.$$

Let $n \geq 2$. Then

$$\begin{aligned}
a_n &\leq \delta a_{n-1} + (n-1)^k\delta^{n-1}\\
&\leq \delta(\delta a_{n-2} + (n-2)^k\delta^{n-2}) + (n-1)^k\delta^{n-1}\\
&= \delta^2 a_{n-2} + (n-2)^k\delta^{n-1} + (n-1)^k\delta^{n-1}\\
&\ \vdots\\
&\leq \delta^{n-1}a_1 + 1^k\delta^{n-1} + 2^k\delta^{n-1} + \cdots + (n-1)^k\delta^{n-1}\\
&= \delta^{n-1}a_1 + (1^k + 2^k + \cdots + (n-1)^k)\delta^{n-1}\\
&\leq \delta^{n-1}a_1 + (n-1)^{k+1}\delta^{n-1}\\
&= \frac{1}{\delta}\left(\frac{a_1}{n^{k+1}} + \left(\frac{n-1}{n}\right)^{k+1}\right)n^{k+1}\delta^n\\
&\leq Cn^{k+1}\delta^n,
\end{aligned}$$

where $C = \frac{1}{\delta}\left(\frac{a_1}{2^{k+1}} + 1\right)$. Hence, $a_n = O(n^{k+1}\delta^n)$. The proof is finished. □

Lemma 12. *Let (X, d) be a metric space endowed with a directed graph G and $T: X \to X$ be a Reich G-contraction with parameters α, β, γ. Let $\delta := \max\left\{\frac{\alpha+\beta}{1-\gamma}, \frac{\alpha+\gamma}{1-\beta}\right\}$. Suppose that $(x, y) \in E(G)$. Then, the following two statements are true.*

(1) *If $\beta \geq \gamma$ and $d(T^nx, T^{n+1}x) = O(n^k\delta^n)$ for some nonnegative integer k, then $d(T^nx, T^ny) = O(n^{k+1}\delta^n)$ and $d(T^ny, T^{n+1}y) = O(n^{k+1}\delta^n)$.*
(2) *If $\beta \leq \gamma$ and $d(T^ny, T^{n+1}y) = O(n^k\delta^n)$ for some nonnegative integer k, then $d(T^nx, T^ny) = O(n^{k+1}\delta^n)$ and $d(T^nx, T^{n+1}x) = O(n^{k+1}\delta^n)$.*

Proof. (1) Assume that $\beta \geq \gamma$ and $d(T^n x, T^{n+1}x) = O(n^k \delta^n)$ for some non-negative integer k. Let $n \geq 1$. Note that $(T^n x, T^n y) \in E(G)$ and

$$\begin{aligned} d(T^{n+1}x, T^{n+1}y) &= d(T(T^n x), T(T^n y)) \\ &\leq \alpha d(T^n x, T^n y) + \beta d(T^n x, T^{n+1}x) + \gamma d(T^n y, T^{n+1}y) \\ &\leq \alpha d(T^n x, T^n y) + \beta d(T^n x, T^{n+1}x) + \gamma d(T^n y, T^n x) \\ &\quad + \gamma d(T^n x, T^{n+1}x) + \gamma d(T^{n+1}x, T^{n+1}y). \end{aligned}$$

In particular,

$$d(T^{n+1}x, T^{n+1}y) \leq \frac{\alpha+\gamma}{1-\gamma} d(T^n x, T^n y) + \frac{\beta+\gamma}{1-\gamma} d(T^n x, T^{n+1}x).$$

Since $\beta \geq \gamma$, we have $\frac{\alpha+\gamma}{1-\gamma} \leq \frac{\alpha+\beta}{1-\gamma} \leq \delta$. Hence,

$$d(T^{n+1}x, T^{n+1}y) \leq \delta d(T^n x, T^n y) + \frac{\beta+\gamma}{1-\gamma} d(T^n x, T^{n+1}x).$$

Note that $\frac{\beta+\gamma}{1-\gamma} d(T^n x, T^{n+1}x) = O(n^k \delta^n)$. By Lemma 11, we have

$$d(T^n x, T^n y) = O(n^{k+1}\delta^n).$$

Since $d(T^n y, T^{n+1}y) \leq d(T^n y, T^n x) + d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+1}y)$, we have $d(T^n y, T^{n+1}y) = O(n^{k+1}\delta^n)$.

(2) The proof is similar to that of (1), so it is omitted. □

Remark 13. The assumption $\beta \geq \gamma$ in Lemma 12(1) cannot be omitted as shown in the following example.

Example 14. Let X, d, G and T be defined as in Example 8. Note that T is a Reich G-contraction with parameters $\alpha = \frac{1}{4}$, $\beta = 0$, and $\gamma = \frac{1}{2}$. Finally, we observe that $(0, y) \in E(G)$ for all $y \neq 0$ and $d(T^n 0, T^{n+1}0) = 0 = O(1/2^n)$. However, the sequence $\{T^n y\}$ is not Cauchy.

Definition 15. Let (X, d) be a metric space. We say that the sequences $\{x_n\}$ and $\{y_n\}$ in X are *Cauchy equivalent* if one of them (hence, all of them) is a Cauchy sequence and $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Definition 16. Let G be a directed graph. For each pair $x, y \in V(G)$ with $x \neq y$, we define $e(x, y) := \infty$ if $y \notin [x]_G$; and

$$e(x, y) := \min \{k \in \mathbb{N} : \{z_j\}_{j=0}^k \text{ is a } G\text{-path from } x \text{ to } y\}$$

if $y \in [x]_G$. Moreover, we define $e(x, x) = 0$ for all $x \in V(G)$.

The following result gives a sufficient condition for the two sequences $\{T^n x\}$ and $\{T^n y\}$ to be Cauchy equivalent.

Lemma 17. *Let (X, d) be a metric space endowed with a directed graph G and $T \colon X \to X$ be a Reich G-contraction with parameters α, β, γ. Let $\delta := \max\left\{\frac{\alpha+\beta}{1-\gamma}, \frac{\alpha+\gamma}{1-\beta}\right\}$ and let $x \in X_T$. Then, the following statements are true.*

(1) *If $\beta \geq \gamma$ and $y \in [x]_G$, then $d(T^n x, T^n y) = O(n^k \delta^n)$, where $k := e(x, y)$. In particular, $\{T^n x\}$ and $\{T^n y\}$ are Cauchy equivalent.*

(2) *If $\beta \leq \gamma$ and $y \in [x]_{G^{-1}}$, then $d(T^n x, T^n y) = O(n^k \delta^n)$, where $k := e(y, x)$. In particular, $\{T^n x\}$ and $\{T^n y\}$ are Cauchy equivalent.*
(3) *If $y \in X_T$ and $y \in [x]_G$, then $\{T^n x\}$ and $\{T^n y\}$ are Cauchy equivalent.*

Proof. Let $x \in X_T$. It follows from Lemma 9 that $d(T^n x, T^{n+1}x) = O(\delta^n)$.

(1) Assume that $\beta \geq \gamma$ and $y \in [x]_G$ and let $k := e(x, y)$. Then, there exists a G-path $\{z_j\}_{j=0}^k$ from x to y, such that $z_0 = x$, $z_k = y$ and $(z_j, z_{j+1}) \in E(G)$ for all $j = 0, 1, 2, \ldots, k-1$. Since $(z_0, z_1) \in E(G)$ and $d(T^n z_0, T^{n+1} z_0) = d(T^n x, T^{n+1} x) = O(\delta^n)$, it follows from Lemma 12(1) that

$$d(T^n z_0, T^n z_1) = O(n\delta^n) \quad \text{and} \quad d(T^n z_1, T^{n+1} z_1) = O(n\delta^n).$$

Similarly, since $(z_1, z_2) \in E(G)$ and $d(T^n z_1, T^{n+1} z_1) = O(n\delta^n)$, it follows from Lemma 12(1) that

$$d(T^n z_1, T^n z_2) = O(n^2\delta^n) \quad \text{and} \quad d(T^n z_2, T^{n+1} z_2) = O(n\delta^n).$$

Continuing this process gives $d(T^n z_{k-1}, T^n y) = d(T^n z_{k-1}, T^n z_k) = O(n^k \delta^n)$ and $d(T^n y, T^{n+1} y) = d(T^n z_k, T^{n+1} z_k) = O(n^k \delta^n)$. Note that $\sum_{n=0}^{\infty} n^k \delta^n < \infty$ and $\{T^n x\}$ is a Cauchy sequence. Hence, $\{T^n x\}$ and $\{T^n y\}$ are Cauchy equivalent.

(2) We can follow the proof of (1) but with an application of Lemma 12(2).

(3) is a consequence of (1) and (2). □

Corollary 18. *Let (X, d) be a metric space endowed with a directed graph G and $T\colon X \to X$ be a Reich G-contraction with parameters α, β, γ, such that $\beta \geq \gamma$. Let $\delta = \frac{\alpha+\beta}{1-\gamma}$. Assume that there exists $x_0 \in X_T$, such that $X = [x_0]_G$. Then, the following statements hold.*

(1) *For each $x \in X$, $d(T^n x_0, T^n x) = O(n^k \delta^n)$ where $k := e(x_0, x)$. In particular, $\{T^n x\}$ and $\{T^n x_0\}$ are Cauchy equivalent.*
(2) *$\{T^n x\}$ and $\{T^n y\}$ are Cauchy equivalent for all $x, y \in X$.*

Proof. (1) and (2) follow from Lemma 17(1). □

The following lemma can be proved using the same technique as in Lemma 12 and Corollary 18 so its proof is omitted.

Lemma 19. *Let (X, d) be a metric space endowed with a directed graph G and $T\colon X \to X$ be a Reich G-contraction with parameters α, β, γ, such that $\beta = \gamma$. Let $\delta = \frac{\alpha+\beta}{1-\gamma}$. Then, the following statements hold:*

(1) *If $(x, y) \in E(\widetilde{G})$ and $d(T^n x, T^{n+1} x) = O(n^k \delta^n)$ for some nonnegative integer k, then $d(T^n x, T^n y) = O(n^{k+1} \delta^n)$ and $d(T^n y, T^{n+1} y) = O(n^{k+1} \delta^n)$.*
(2) *If $x \in X_T$ and $y \in [x]_{\widetilde{G}}$, then $\{T^n x\}$ and $\{T^n y\}$ are Cauchy equivalent.*
(3) *If $X_T \neq \varnothing$ and G is weakly connected, then the sequences $\{T^n x\}$ and $\{T^n y\}$ are Cauchy equivalent for all $x, y \in X$.*

2.1. An extension of Theorems J1 and Bo

Lemma 20. *Let (X, d) be a complete metric space endowed with a directed graph G and $T\colon X \to X$ be a Reich G-contraction. Suppose that Condition* (J-1) *holds. If $x \in X$ satisfies the condition $(x, Tx) \in E(G)$, then $\lim_n T^n x = z$ for some $z \in \mathrm{Fix}(T)$.*

Proof. Suppose that $T\colon X \to X$ is a Reich G-contraction with parameters α, β, γ. Let $x \in X$ be such that $(x, Tx) \in E(G)$. It follows that $(T^n x, T^{n+1}x) \in E(G)$ for all $n \in \mathbb{N}$. Moreover, it follows from Lemma 9 that $\{T^n x\}$ is a Cauchy sequence. Since X is complete, $\lim_n T^n x = z$ for some $z \in X$. It follows from Condition (J-1) that there is a subsequence $\{T^{n_k}x\}$ of $\{T^n x\}$, such that $(T^{n_k}x, z) \in E(G)$ for all $k \in \mathbb{N}$. This implies that

$$\begin{aligned} d(z, Tz) &= \lim_{k\to\infty} d(T^{n_k+1}x, Tz) \\ &= \lim_{k\to\infty} d(T(T^{n_k}x), Tz) \\ &\le \lim_{k\to\infty} (\alpha d(T^{n_k}x, z) + \beta d(T^{n_k}x, T^{n_k+1}x) + \gamma d(z, Tz)) \\ &= \gamma d(z, Tz). \end{aligned}$$

Hence, $z \in \mathrm{Fix}(T)$ as desired. □

Theorem 21. *Let (X, d) be a complete metric space endowed with a directed graph G and $T\colon X \to X$ be a Reich G-contraction with parameters α, β, γ. Suppose that Condition* (J-1) *holds. Then, the following statements are true.*

(1) $\mathrm{Fix}(T) \neq \varnothing$ *if and only if* $X_T \neq \varnothing$.
(2) $T|_{X_T}$ *is a PO if and only if there exists $x_0 \in X_T$, such that $X_T \subset [x_0]_G \cup [x_0]_{G^{-1}}$.*

Suppose, in addition, that $\beta \ge \gamma$. Then, the following statements are true.

(3) $T|_{[x_0]_G}$ *is a PO for all $x_0 \in X_T$.*
(4) *If there exists $x_0 \in X_T$, such that $X = [x_0]_G$, then T is a PO. In particular, if $X = \bigcup\{[x]_G : x \in X_T\}$ and there exists $x_0 \in X_T$, such that $X_T \subset [x_0]_G$, then T is a PO.*
(5) $T|_Y$ *is a WPO, where $Y = \bigcup\{[x]_G : x \in X_T\}$.*
(6) *If $X_T = X$, then T is a WPO.*

Proof. We prove (2). Note that $T(X_T) \subset X_T$. ($\Rightarrow$) Assume that $T|_{X_T}$ is a PO. Then, $\mathrm{Fix}(T|_{X_T}) = \{x_0\}$ for some $x_0 \in X_T$. Let $x \in X_T$. Then, $(T^n x, T^{n+1}x) \in E(G)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} T^n x = x_0$. By (J-1), there exists $k \in \mathbb{N}$, such that $(T^k x, x_0) \in E(G)$. Then, $\{x, Tx, T^2x, \ldots, T^k x, x_0\}$ is a G-path from x to x_0. Hence, $x \in [x_0]_{G^{-1}} \subset [x_0]_G \cup [x_0]_{G^{-1}}$.

($\Leftarrow$) Assume that there exists $x_0 \in X_T$, such that $X_T \subset [x_0]_G \cup [x_0]_{G^{-1}}$. It follows from Lemma 20 $\lim_n T^n x_0 = z$ for some $z \in \mathrm{Fix}(T)$. Obviously, $z \in X_T$. Now, we prove that $T|_{X_T}$ is a PO. To see this, let $x \in X_T \subseteq [x_0]_G \cup [x_0]_{G^{-1}}$. It follows that either $x \in [x_0]_G$ or $x_0 \in [x]_G$. By Lemma 17(3), the sequences $\{T^n x\}$ and $\{T^n x_0\}$ are Cauchy equivalent. This implies that $\lim_n T^n x = z$. Therefore, $T|_{X_T}$ is a PO.

From now on, we assume that $\beta \ge \gamma$.

(3) Let $x_0 \in X_T$. We first observe that $T([x_0]_G) \subset [x_0]_G$. It follows from Lemma 20 that $\lim_n T^n x_0 = z$ for some $z \in \mathrm{Fix}(T)$. To show that $T|_{[x_0]_G}$ is a PO, let $x \in [x_0]_G$. It follows from Lemma 17(1) that $\{T^n x_0\}$ and $\{T^n x\}$ are Cauchy equivalent. Hence, $\lim_n T^n x = z$ and $T|_{[x_0]_G}$ is a PO.

(4) is a direct consequence of (3) and Remark 1.

(5) follows from (3).

(6) follows from Lemma 20. □

The following result is similar to the preceding theorem but for the situation that $\beta \leq \gamma$. The proof is left for the reader to verify.

Theorem 22. *Let (X, d) be a complete metric space endowed with a directed graph G and $T : X \to X$ be a Reich G-contraction with parameters α, β, γ. Suppose that Condition* (J-1) *holds. Then, the following statements are true.*

(1) $\mathrm{Fix}(T) \neq \varnothing$ *if and only if* $X_T \neq \varnothing$.
(2) $T|_{X_T}$ *is a PO if and only if there exists* $x_0 \in X_T$*, such that* $X_T \subset [x_0]_G \cup [x_0]_{G^{-1}}$.

Suppose, in addition, that $\beta \leq \gamma$. Then, the following statements are true.

(3) $T|_{[x_0]_{G^{-1}}}$ *is a PO for all* $x_0 \in X_T$.
(4) *If there exists $x_0 \in X_T$, such that $X = [x_0]_{G^{-1}}$, then T is a PO. In particular, if $X = \bigcup\{[x]_{G^{-1}} : x \in X_T\}$ and there exists $x_0 \in X_T$, such that $X_T \subset [x_0]_{G^{-1}}$, then T is a PO.*
(5) *$T|_Y$ is a WPO, where $Y = \bigcup\{[x]_{G^{-1}} : x \in X_T\}$.*
(6) *If $X_T = X$, then T is a WPO.*

Corollary 23. *Let (X, d) be a complete metric space endowed with a directed graph G and $T : X \to X$ be a Reich G-contraction with parameters α, β, γ where $\beta = \gamma$. Suppose that Condition* (J-1) *holds. Then, the following hold*:

(1) $\operatorname{card}\mathrm{Fix}(T) = \operatorname{card}\{[x]_{\widetilde{G}} : x \in X_T\}$;
(2) $\mathrm{Fix}(T) \neq \varnothing$ *if and only if* $X_T \neq \varnothing$;
(3) *T has a unique fixed point if and only if there exists $x_0 \in X_T$, such that $X_T \subset [x_0]_{\widetilde{G}}$*;
(4) *$T|_{[x_0]_{\widetilde{G}}}$ is a PO for all $x_0 \in X_T$*;
(5) *If $X_T \neq \varnothing$ and G is weakly connected, then T is a PO. In particular, if $X = \bigcup\{[x]_{\widetilde{G}} : x \in X_T\}$ and there exists $x_0 \in X_T$, such that $X_T \subset [x_0]_{\widetilde{G}}$, then T is a PO*;
(6) *$T|_Y$ is a WPO, where $Y = \bigcup\{[x]_{\widetilde{G}} : x \in X_T\}$*;
(7) *If $X_T = X$, then T is a WPO.*

Proof. The proof of (1) is similar to that Theorem 3.2(1) of [7]. Observe that (3) is a consequence of (1). The rest follows directly from the proof of Theorem 21 and Lemma 19. □

Remark 24. We note that the T-connectedness of the graph G which is a requirement of Theorem Bo is weaken. Moreover, the following example is applicable for our result but beyond the scope of Theorem Bo.

Example 25. Let $X = \mathbb{N}$ and $E(G) = \{(n, n) : n \in X\} \cup \{(n, n+1) : n \in \mathbb{N}\} \cup \{(2, 1)\}$. Define a mapping $T : X \to X$ by $T1 = T2 = T3 = 1$ and $Tn = n - 2$

for all $n \geq 4$. Then, T is a Reich G-contraction with parameters $\alpha = \frac{1}{2}$, $\beta = \gamma = \frac{1}{5}$. Note that all the conditions of Theorem 21 are satisfied, and hence, we obtain that 1 is the unique fixed point of T and $\lim_{n\to\infty} T^n x = 1$ for all $x \in X$. To see that G is not T-connected, we consider $x = 3$ and $y = 5$. Note that $(x, y) \notin E(G)$ and the sequence $\{x, z, y\}$ where $z = 4$ is the only G-path from x to y. Obviously, $(z, Tz) \notin E(G)$. Hence, this example is not applicable in Theorem Bo.

The following example show that the condition "there exists $x_0 \in X_T$, such that $X_T \subset [x_0]_{\widetilde{G}}$" in Corollary 23(5) cannot be dropped.

Example 26. We modify the preceding example as follows. Let X, G, and T be the same as Example 25. Let $X^* = X \cup X'$, where $X' := \{-n : n \in X\}$. Let G^* be a directed graph, such that $V(G^*) = X^*$ and $E(G^*) = E(G) \cup E(G')$, where

$$(-x, -y) \in E(G') \iff (x, y) \in E(G).$$

Define $T' \colon X' \to X'$ by $T'(-x) = -Tx$ for all $x \in X$. Now, we define $T^* \colon X^* \to X^*$ by $T^* x = Tx$ if $x \in X$ and $T^* x = T' x$ if $x \in X'$. Then, T^* is a Reich G^*-contraction with parameters $\alpha = \frac{1}{2}$, $\beta = \gamma = \frac{1}{5}$. It follows from Corollary 23(5) that T is a WPO. In fact, $\mathrm{Fix}(T^*) = \{-1, 1\}$. Observe that the condition "there exists $x_0 \in X^*_{T^*}$, such that $X^*_{T^*} \subset [x_0]_{\widetilde{G^*}}$" is not satisfied.

To point out some error in Bojor's results, we quote his three results as follows.

Corollary Bo1. *Let (X, d) be a complete metric space endowed with a directed graph G. Suppose that Condition* (J-1) *holds. Suppose that $T \colon X \to X$ satisfies one of the following conditions*:

- *T is a Reich G-contraction with parameters α, β, γ, such that $\beta = \gamma$* [4, Corollary 1];
- *T is a Banach G-contraction* [4, Corollary 2];
- *T is a Kannan G-contraction, that is, T is a Reich G-contraction with parameters $\alpha = 0$, $\beta = \gamma$* [4, Corollary 3].

If G is weakly T-connected, then T is a PO.

The preceding results of Bojor are not true. In fact, it follows from Corollary 23 that to guarantee the existence of a fixed point of T, it is necessary and sufficient that $X_T \neq \varnothing$. However, the condition $X_T \neq \varnothing$ is not assumed. Moreover, the weak T-connectedness does not imply that $X_T \neq \varnothing$ as shown in the following example.

Example 27. Let $X = [0, 1]$ be a usual metric space. Define a directed graph G on X by $E(G) = \{(x, x) : x \in X\} \cup \{(x, y) : x, y \in (0, 1] \text{ and } x \leq y\} \cup \{(1, 0)\}$. Define a mapping $T \colon X \to X$ by $Tx = \frac{x}{4}$ for all $x > 0$ and $T0 = 1$. It follows that

- T is a Kannan G-contraction with a parameter $\beta = \frac{3}{7}$;
- T is a Banach G-contraction with a parameter $\alpha = \frac{3}{4}$.

Note that G is weakly T-connected and satisfies the condition (J-1). It is easy to see that $\mathrm{Fix}(T) = \varnothing$.

2.2. An extension of Theorem J2

In this subsection, we show that an extension of Theorem J2 via the fixed point theorem of Hicks and Rhoades [6]. We first recall the following concepts.

Definition 28. Let (X, d) be a metric space. Let $T\colon X \to X$ and $x_0 \in X$. Let $\mathrm{Orb}(x_0, T) = \{x_0, Tx_0, T^2x_0, \ldots\}$ be an orbit of x_0 under T. A function $g\colon X \to [0, \infty)$ is said to be *T-orbitally lower semicontinuous at x_0* if $\{y_n\}$ is a sequence in $\mathrm{Orb}(x_0, T)$ and $\lim_{n\to\infty} y_n = y$ implies $g(y) \le \liminf_{n\to\infty} g(y_n)$.

The following fixed point theorem was proved by Hicks and Rhoades [6].

Theorem HR. *Let (X, d) be a complete metric space and $\delta \in [0, 1)$. Suppose that $T\colon X \to X$ is a mapping and there exists let $x_0 \in X$, such that*

$$d(Ty, T^2y) \le \delta d(y, Ty) \quad \textit{for all } y \in \mathrm{Orb}(x_0, T).$$

Then, the following statements are true.

1. *$\lim_n T^n x_0 := z$ exists.*
2. *The element z in (1) is a fixed point of T if and only if $x \mapsto d(x, Tx)$ is T-orbitally lower semicontinuous at x_0.*

The idea of the following lemma is taken from our recent work [5].

Lemma 29. *Let (X, d) be a metric space endowed with a directed graph G. Suppose that $T\colon X \to X$ is a Reich G-contraction. If Condition (J-2) is satisfied, then the function $x \mapsto d(x, Tx)$ is T-orbitally lower semicontinuous at x_0 for all $x_0 \in X_T$.*

Proof. Let $x_0 \in X_T$. Then, $(T^n x_0, T^{n+1}x_0) \in E(G)$ for all $n \in \mathbb{N}$. We show that $g(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at x_0. Let $\{y_n\}$ be a sequence in $\mathrm{Orb}(x_0, T)$ and $\lim_{n\to\infty} y_n = y \in X$. For each $n \in \mathbb{N}$, let $m(n)$ be the smallest number k, such that $T^k x_0 = y_n$. We consider the set $\mathbb{K} = \{m(n) : n \in \mathbb{N}\}$.

Case 1. $\mathbb{K}$ is an infinite set. Therefore, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$, such that $m(n_k) < m(n_{k+1})$ for all $k \in \mathbb{N}$. Hence, $\{y_{n_k}\}$ is a subsequence of $\{T^n x_0\}$. In particular, there is a strictly increasing sequence $\{p_k\}$ of natural numbers, such that $y_{n_k} = T^{p_k}x_0$ for all $k \in \mathbb{N}$. Then, $\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} T^{p_k}x_0 = y$. It follows from $(T^{p_k}x_0, T^{p_k+1}x_0) \in E(G)$ and Condition (J-2) that

$$\lim_{k\to\infty} Ty_{n_k} = \lim_{k\to\infty} T^{p_k+1}x_0 = Ty.$$

Since $x_0 \in X_T$, it follows from Lemma 9 that $\lim_{k\to\infty} d(T^{p_k}x_0, T^{p_k+1}x_0) = 0$. Hence, $g(y) = d(y, Ty) = 0$. Therefore, $g(y) = 0 \le \liminf_{n\to\infty} g(y_n)$.

Case 2. $\mathbb{K}$ is a finite set. Since $\{y_n\}$ is a sequence in a finite set $\{T^j x_0 : j \in \mathbb{K}\}$ and $\lim_n y_n = y$, there exist $k \in \mathbb{K}$ and $N \in \mathbb{N}$, such that $y_n = T^k x_0$ for all $n \geq N$. Hence, $y = T^k x_0$, that is, $y_n = y$ for all $n \geq N$. Then

$$g(y) = d(y, Ty) = \lim_{n\to\infty} d(y_n, Ty_n) = \liminf_{n\to\infty} g(y_n).$$

As considered in the preceding two cases, the function $g(x)$ is T-orbitally lower semicontinuous at x_0. □

The following result is our extension of Theorem J2 for Reich G-contractions.

Theorem 30. *Let (X, d) be a complete metric space endowed with a directed graph G and $T\colon X \to X$ be a Reich G-contraction with parameters α, β, γ, such that $\beta \geq \gamma$. Suppose that Condition* (J-2) *holds. Then, the following statements hold.*

(1) $\mathrm{Fix}(T) \neq \varnothing$ *if and only if* $X_T \neq \varnothing$.
(2) *For each $x \in X_T$ and for each $y \in [x]_G$, $\{T^n y\}$ converges to a fixed point of T and $\lim_{n\to\infty} T^n y$ does not depend on y.*
(3) *If there exists $x_0 \in X_T$, such that $X = [x_0]_G$, then T is a PO. In particular, if $X = \bigcup\{[x]_G : x \in X_T\}$ and there exists $x_0 \in X_T$, such that $X_T \subset [x_0]_G$,, then T is a PO.*
(4) *$T|_Y$ is a WPO, where $Y = \bigcup\{[x]_G : x \in X_T\}$.*
(5) *If there exists $x_0 \in X_T$, such that $X_T \subset [x_0]_G \cup [x_0]_{G^{-1}}$, then T has a unique fixed point.*
(6) *If $X_T = X$, then T is a WPO.*

Proof. (1) ($\Rightarrow$) is obvious, because $\mathrm{Fix}(T) \subset X_T$.

($\Leftarrow$) Let $x_0 \in X_T$. Then, $(T^n x_0, T^{n+1} x_0) \in E(G)$ for all $n \in \mathbb{N}$. Put $\delta = \frac{\alpha+\beta}{1-\gamma}$. Then, by Lemma 9, we have

$$d(T^{n+1}x_0, T^{n+2}x_0) \leq \delta d(T^n x_0, T^{n+1}x_0) \quad \text{for all } n \in \mathbb{N}. \tag{$\heartsuit$}$$

Let $y \in O(x_0, \infty)$. Then, $y = T^n x_0$ for some $n \geq 0$. Hence, by ($\heartsuit$), we have

$$\begin{aligned} d(Ty, T^2 y) &= d(T^{n+1}x_0, T^{n+2}x_0) \\ &\leq \delta d(T^n x_0, T^{n+1}x_0) \\ &= \delta d(y, Ty). \end{aligned}$$

Then, by Theorem HR(1), there exists an element $z \in X$, such that $z = \lim_{n\to\infty} T^n x_0$. It follows from Condition (J-2) and Lemma 29 that $x \mapsto d(x, Tx)$ is T-orbitally lower semicontinuous at x_0. Then, by Theorem HR(2), we have $z = Tz$. Hence, $z \in \mathrm{Fix}(T) \neq \varnothing$.

We show (2). Let $x \in X_T$ and let $y \in [x]_G$. It follows from (1) that $\lim_n T^n x = z$ for some $z \in \mathrm{Fix}(T)$. From Lemma 17(1), $\{T^n x\}$ and $\{T^n y\}$ are Cauchy equivalent. This implies that $\{T^n y\}$ converges to a fixed point of T.

(3) follows from (2) and Remark 1.
(4) follows from (2).
(5) follows from (1) and Lemma 17(3).
(6) follows from (1). □

2.3. Some priori error estimates

In the preceding two subsection, it is shown that the conditions (J-1) and (J-2) are sufficient for the existence of a fixed point of Reich G-contractions in a complete metric space. in this subsection, we consider the situation that a Reich G-contraction has a fixed point and we study some priori error estimates. Note that the following result requires neither the completeness nor Jachymski's conditions.

Theorem 31. *Let (X, d) be a metric space endowed with a directed graph G. Suppose that $T\colon X \to X$ is a Reich G-contraction with parameters α, β, γ, such that $\mathrm{Fix}(T) \neq \varnothing$. Then, the following statements are true.*

(1) *If $\beta = \gamma$ and G is weakly connected, then T is a PO.*

(2) *Let $\delta := \max\{\frac{\alpha+\beta}{1-\gamma}, \frac{\alpha+\gamma}{1-\beta}\}$. Suppose that G is connected. Then, T is a PO with a unique fixed point z. Moreover, the following statements are true.*
   - (a) *$d(T^n x, z) = O(\delta^n)$ for all $x \in X_T$. In fact, $d(T^n x, z) \leq \frac{\delta^n}{1-\delta} d(x, Tx)$ for all $n \in \mathbb{N}$.*
   - (b) *If $\beta \geq \gamma$ and $x \in X \backslash X_T$, then $d(T^n x, z) = O(n^l \delta^n)$, where $l := e(z, x)$.*
   - (c) *If $\beta \leq \gamma$ and $x \in X \backslash X_T$, then $d(T^n x, z) = O(n^r \delta^n)$, where $r := e(x, z)$.*

Proof. Pick $z \in \mathrm{Fix}(T)$. Note that $z \in X_T$.

(1) Assume that $\beta = \gamma$ and G is weakly connected. It follows Lemma 19(2) that $\{T^n x\}$ and $\{T^n z = z\}$ are Cauchy equivalent for all $x \in X$. Then, T is a PO.

To see (2), we assume first that $\beta \geq \gamma$ and G is connected. Let $x \in X$. Note that $X = [z]_G$. It follows from Corollary 18(1) that $d(T^n x, z) = d(T^n x, T^n z) = O(n^l \delta^n)$, where $l := e(z, x)$. Therefore, (b) holds. In particular, $\lim_n T^n x = z$ for all $x \in X$. This implies that T is PO. Now, we consider the case $x \in X_T$. For all $n, m \in \mathbb{N}$, it follows from Lemma 9 that

$$\begin{aligned}
& d(T^n x, T^{n+m} x) \\
& \quad \leq d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+2} x) + \cdots + d(T^{n+m-1} x, T^{n+m} x) \\
& \quad \leq \delta^n d(x, Tx) + \delta^{n+1} d(x, Tx) + \cdots + \delta^{n+m} d(x, Tx) \\
& \quad \leq \left(\sum_{j=n}^{\infty} \delta^j \right) d(x, Tx) \\
& \quad = \frac{\delta^n}{1-\delta} d(x, Tx).
\end{aligned}$$

Letting $m \to \infty$ gives $d(T^n x, z) \leq \frac{\delta^n}{1-\delta} d(x, Tx)$ for all $n \in \mathbb{N}$. This proves (a).

For the case $\beta \leq \gamma$ and G is connected, it can be proved analogously, so the proof is omitted. □

We now restrict ourselves to a certain directed graph. Let $(X, \preceq)$ be a partially ordered set. Let G be a directed graph on X, such that

$$E(G) := \{(x, y) \in X \times X : x \preceq y\}.$$

Then, for $x_0 \in X$, the condition $X = [x_0]_G$ is equivalent to $x_0 \preceq x$ for all $x \in X$. In this case, $e(x_0, x) = 1$ for all $x \neq x_0$.

Corollary 32. *Let $(X, \preceq)$ be a partially ordered set and let (X, d) be a complete metric space. Let $T\colon X \to X$ be a nondecreasing mapping (with respect to $\preceq$), that is, $Tx \preceq Ty$ whenever $x \preceq y$. Assume that there exist nonnegative real numbers α, β, γ, such that $\alpha + \beta + \gamma < 1$ and*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \quad \textit{for all } x \preceq y.$$

Assume that one the following conditions holds:

(N-1) *For any nondecreasing sequence $\{x_n\}$ in X, if $\lim_{n\to\infty} x_n = x \in X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;*

(N-2) *For all $x, y \in X$ and for any subsequence $\{T^{n_k}x\}$ of $\{T^n x\}$, if $\lim_{k\to\infty} T^{n_k}x = y$ and $T^{n_k}x \preceq T^{n_k+1}x$ for all $k \in \mathbb{N}$, then $\lim_{k\to\infty} T(T^{n_k}x) = Ty$.*

Then, the following statements hold.

1. $\operatorname{Fix}(T) \neq \varnothing$ *if and only if there exists $x_0 \in X$, such that $x_0 \preceq Tx_0$.*
2. *Suppose that $\beta \geq \gamma$ and there exists $x_0 \in X$, such that $x_0 \preceq x$ for all $x \in X$. Then, T is a PO with a unique fixed point z. Moreover, if $\delta = \frac{\alpha+\beta}{1-\gamma}$, then the following statements are true.*
   - *$d(T^n x, z) = O(\delta^n)$ for all $x \preceq Tx$.*
   - *$d(T^n x, z) = O(n\delta^n)$ for all $x \not\preceq Tx$.*
3. *Suppose that $\beta = \gamma$ and there exists $x_0 \in X$, such that $x_0 \preceq Tx_0$. Suppose that every pair of elements of X has either an upper bound or a lower bound. Then, T is a PO with a unique fixed point z. Moreover, if $\delta = \frac{\alpha+\beta}{1-\gamma}$, then the following statements are true.*
   - *$d(T^n x, z) = O(\delta^n)$ for all $x \preceq Tx$.*
   - *$d(T^n x, z) = O(n^2\delta^n)$ for all $x \not\preceq Tx$.*
   - *If $\beta = \gamma = 0$, then $d(T^n x, z) = O(\alpha^n)$ for all $x \neq z$.*

Finally, we discuss the following result of Bojor [4, Corollary 4].

Corollary Bo2. *Let $(X, \preceq)$ be a partially ordered set and let (X, d) be a complete metric space. Let $T\colon X \to X$ be a nondecreasing mapping. Assume that there exist nonnegative real numbers α, β, γ, such that $\alpha + \beta + \gamma < 1$ and*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \quad \textit{for all } x \preceq y.$$

Assume that the following conditions hold:

1. *For any nondecreasing sequence $\{x_n\}$ in X, if $\lim_{n\to\infty} x_n = x \in X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*
2. *For each $x, y \in X$, incomparable elements of $(X, \preceq)$, there exists $z \in X$, such that $x \preceq z$, $y \preceq z$ and $z \preceq Tz$.*

Then, T is a PO.

It follows from our Corollary 32(1) that $\operatorname{Fix}(T) \neq \varnothing$ if and only if there exists $x_0 \in X$, such that $x_0 \preceq Tx_0$. In particular, if T is a PO, then there exists $x_0 \in X$, such that $x_0 \preceq Tx_0$. However, even we assume that there exists $x_0 \in X$, such that $x_0 \preceq Tx_0$, we do not have the conclusion as shown in the following example.

Example 33. Let X, G, and T be defined in Example 8. For $x, y \in X$, we define

$$x \preceq y \iff (y, x) \in E(G).$$

Then

$$d(Tx, Ty) \leq \frac{1}{4}d(x, y) + \frac{1}{2}d(x, Tx) \quad \text{for all } x \preceq y.$$

It is obvious that T is not a PO.

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On the strong convergence of sequences of Halpern type in Hilbert spaces

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ABSTRACT

In this paper, we introduce a concept of *A*-sequences of Halpern type where *A* is an averaging infinite matrix. If *A* is the identity matrix, this notion become the well-know sequence generated by Halpern's iteration. A necessary and sufficient condition for the strong convergence of *A*-sequences of Halpern type is given whenever the matrix *A* satisfies some certain concentrating conditions. This class of matrices includes two interesting classes of matrices considered by Combettes and Pennanen [J. Math. Anal. Appl. 2002;275:521–536]. We deduce all the convergence theorems studied by Cianciaruso et al. [Optimization. 2016;65:1259–1275] and Muglia et al. [J. Nonlinear Convex Anal. 2016;17:2071–2082] from our result. Moreover, these results are established under the weaker assumptions. We also show that the same conclusion remains true under a new condition.

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, that is, $\|x\|^2 = \langle x,x\rangle$ for all $x \in \mathcal{H}$. Recall that an element $x \in \mathcal{H}$ is a *fixed point* of a mapping $T : \mathcal{H} \to \mathcal{H}$ if $x = Tx$ and the set of all fixed points of T is denoted by $\mathrm{Fix}(T)$. We use $\to$ and $\rightharpoonup$ for the strong and weak convergence, respectively. For a given sequence $\{x_n\}_{n=1}^{\infty}$, let $\mathfrak{W}\{x_n\}_{n=1}^{\infty}$ denote the set of all weak cluster points of $\{x_n\}_{n=1}^{\infty}$, that is,

$$\mathfrak{W}\{x_n\}_{n=1}^{\infty} := \{z \in \mathcal{H} : x_{n_k} \rightharpoonup z \text{ for some subsequence } \{x_{n_k}\}_{k=1}^{\infty} \text{ of } \{x_n\}_{n=1}^{\infty}\}.$$

In this paper, we are interested in the approximation of a fixed point of a mapping via an iteration if such a fixed point exists.

The following type of mappings was introduced by Aoyama et al. [1].

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Definition 1.1: A mapping $T : \mathcal{H} \to \mathcal{H}$ is *L-hybrid* where $L \geq 0$ if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + L\langle x - Tx, y - Ty\rangle$$

for all $x, y \in \mathcal{H}$. Every L-hybrid mapping where $L = 0$ and $L = 2$ is called a *nonexpansive* and *nonspreading* mapping, respectively.

The concept of nonspreading mappings was introduced by Kohsaka and Takahashi [2]. Let us summarize several fact about L-hybrid mappings.

Remark 1.1: (1) If $T : \mathcal{H} \to \mathcal{H}$ is firmly nonexpansive, that is, $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle$ for all $x, y \in \mathcal{H}$, then T is an L-hybrid mapping for all $L \in [0, 2]$ (see [1]).
(2) Not every L-hybrid mapping is continuous. In fact, for each $L > 0$ there exists a noncontinuous L-hybrid mapping (see [1]).
(3) If $T : \mathcal{H} \to \mathcal{H}$ is L-hybrid, then $I - T$ is *demiclosed at zero*, that is, $x = Tx$ whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{H}$ such that $x_n \rightharpoonup x \in \mathcal{H}$ and $x_n - Tx_n \to 0$ (see [3]).
(4) Every L-hybrid mapping with a fixed point is quasi-nonexpansive. Recall that a mapping $T : \mathcal{H} \to \mathcal{H}$ is *quasi-nonexpansive* if $\mathrm{Fix}(T) \neq \varnothing$ and $\|Tx - p\| \leq \|x - p\|$ for all $x \in \mathcal{H}$ and $p \in \mathrm{Fix}(T)$. Note that the fixed point set of a quasi-nonexpansive (in particular, an L-hybrid) mapping is closed and convex (see [4]).

In this paper, the following mappings are also studied.

Definition 1.2: A mapping $D : \mathcal{H} \to \mathcal{H}$ is said to be

(1) *β-strongly monotone* where $\beta > 0$ if

$$\langle Dx - Dy, x - y\rangle \geq \beta\|x - y\|^2 \quad \text{for all } x, y \in \mathcal{H};$$

(2) *δ-Lipschizian* where $\delta > 0$ if

$$\|Dx - Dy\| \leq \delta\|x - y\| \quad \text{for all } x, y \in \mathcal{H}.$$

Every δ-Lipschitzian mapping where $\delta < 1$ is specifically called a *δ-contraction* or a *contraction*.

Remark 1.2: If $D : \mathcal{H} \to \mathcal{H}$ is a β-strongly monotone and δ-Lipschitzian mapping and $0 < \mu < 2\beta/\delta^2$, then the mapping $I - \mu D$ is an η-contraction where $\eta := (1 - \mu(2\beta - \mu\delta^2))^{1/2}$. Roughly speaking, if μ is not too large, then $I - \mu D$ is a contraction.

The iterations in this paper are defined by using averaging matrices. Recall that an infinite real matrix $[a_{n,k}]_{n,k=1}^{\infty}$ is *averaging* if the following conditions are satisfied:

(A1) $a_{n,k} \geq 0$ for all $n, k \geq 1$ and $a_{n,k} = 0$ for all $n \geq 1$ and $k > n$;
(A2) $\sum_{k=1}^{n} a_{n,k} = 1$ for all $n \geq 1$;
(A3) $\lim_{n\to\infty} a_{n,k} = 0$ for all $k \geq 1$.

Lemma 1.3 ([5]): *Suppose that $[a_{n,k}]_{n,k=1}^{\infty}$ is an averaging matrix. Let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of real numbers and $\overline{\xi}_n := \sum_{k=1}^{n} a_{n,k}\xi_k$ for all $n \geq 1$. If $\xi_n \to \xi$ for some real number ξ, then $\overline{\xi}_n \to \xi$.*

Recall that for a real number a, the *positive part of a*, denoted by a^+, is defined by $a^+ := \max\{a, 0\}$.

Definition 1.4: An averaging matrix $[a_{n,k}]_{n,k=1}^{\infty}$ satisfies

- *BB1-condition* [6] if $\lim_{n\to\infty} \sum_{k=1}^{n-1}(a_{n,k+1} - a_{n,k})^+ = 0$;
- *BB2-condition* [6] if $\lim_{n\to\infty} \sum_{k=1}^{n-1}|a_{n,k+1} - a_{n,k}| = 0$;
- *CMMX-condition* [7] if $a_{n,1} \geq a_{n,2} \geq \cdots \geq a_{n,n}$ for all $n \geq 1$.

Remark 1.3: Obviously,

$$\text{CMMX} - \text{condition} \implies \text{BB2-condition} \implies \text{BB1-condition}.$$

Moreover, none of the implication above can be reversed.

Remark 1.4: If an averaging matrix $[a_{n,k}]_{n,k=1}^{\infty}$ satisfies the condition BB2, then $\lim_{n\to\infty} a_{n,n} = 0$. In fact, since $[a_{n,k}]_{n,k=1}^{\infty}$ is averaging, we have $\lim_{n\to\infty} a_{n,1} = 0$. This implies that $\lim_{n\to\infty} |a_{n,n}| \leq \lim_{n\to\infty}(|a_{n,1}| + \sum_{k=1}^{n-1}|a_{n,k+1} - a_{n,k}|) = 0$ and hence $\lim_{n\to\infty} a_{n,n} = 0$.

The following three strong convergence theorems for a fixed point of an L-hybrid mapping are our starting point. The first one was proved by Cianciaruso et al. [7] and the second and the third ones were proved by Muglia et al. [8].

Theorem 1.5 ([7, Theorem 3.5]): *Let $D : \mathcal{H} \to \mathcal{H}$ be a β-strongly monotone and δ-Lipschitzian operator and let $T : \mathcal{H} \to \mathcal{H}$ be an L-hybrid mapping such that* $\mathrm{Fix}(T) \neq \emptyset$. *Let $[a_{n,k}]_{n,k=1}^{\infty}$ and $[b_{n,k}]_{n,k=1}^{\infty}$ be averaging matrices. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$ defined by*

$x_1 \in \mathcal{H}$ arbitrarily chosen,

$$\overline{x}_n := \sum_{k=1}^{n} a_{n,k} x_k,$$

$$x_{n+1} := \gamma_n x_n + (1-\gamma_n)(I - \mu_n D) \sum_{k=0}^{n-1} b_{n,k+1} T^k \overline{x}_n \quad \textit{for all } n \geq 1,$$

where

(a) $\{\mu_n\}_{n=1}^{\infty} \subset (0, \mu)$ *with* $\mu < 2\beta/\delta^2$ *and* $\lim_{n\to\infty} \mu_n = 0$ *and* $\sum_{n=1}^{\infty} \mu_n = \infty$;
(b) $[b_{n,k}]_{n,k=1}^{\infty}$ *satisfies CMMX-condition*;
(c) $\lim_{n\to\infty}(1 - a_{n,n})/\mu_n = 0$;
(d) $\{\gamma_n\}_{n=1}^{\infty} \subset [0, \alpha) \subset [0, 1)$ *and* $\lim_{n\to\infty} \gamma_n = 0$.

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in \mathrm{Fix}(T)$ *and this element* p *is the unique solution of the variational inequality* $\langle Dp, y - p\rangle \geq 0$ *for all* $y \in \mathrm{Fix}(T)$.

Theorem 1.6 ([8, Theorem 2.3]): *Let* $D : \mathcal{H} \to \mathcal{H}$ *be a* β*-strongly monotone and* δ*-Lipschitzian operator and let* $T : \mathcal{H} \to \mathcal{H}$ *be a nonspreading mapping such that* $\mathrm{Fix}(T) \neq \varnothing$. *Let* $[b_{n,k}]_{n,k=1}^{\infty}$ *be an averaging matrix. Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathcal{H}$ *defined by*

$$x_1 \in \mathcal{H} \text{ arbitrarily chosen,}$$

$$x_{n+1} := \alpha_n (I - \mu_n D) x_n + (1-\alpha_n) \sum_{k=0}^{n-1} b_{n,k+1} T^k x_n \quad \text{for all } n \geq 1,$$

where

(a) $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1)$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\{\mu_n\}_{n=1}^{\infty} \subset (0, \mu)$ *with* $\mu < 2\beta/\delta^2$ *and* $\sum_{n=1}^{\infty} \alpha_n \mu_n = \infty$;
(b) $[b_{n,k}]_{n,k=1}^{\infty}$ *satisfies CMMX-condition.*

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in \mathrm{Fix}(T)$ *and this element* p *is the unique solution of the variational inequality* $\langle Dp, y - p\rangle \geq 0$ *for all* $y \in \mathrm{Fix}(T)$.

Theorem 1.7 ([8, Corollary 2.6]): *Let* $f : \mathcal{H} \to \mathcal{H}$ *be an* α*-contraction and let* $T : \mathcal{H} \to \mathcal{H}$ *be a nonspreading mapping such that* $\mathrm{Fix}(T) \neq \varnothing$. *Let* $[b_{n,k}]_{n,k=1}^{\infty}$ *be an averaging matrix. Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathcal{H}$ *defined by*

$$x_1 \in \mathcal{H} \text{ arbitrarily chosen,}$$

$$x_{n+1} := \alpha_n f(x_n) + (1-\alpha_n) \sum_{k=0}^{n-1} b_{n,k+1} T^k x_n \quad \text{for all } n \geq 1,$$

where

(a) $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $[b_{n,k}]_{n,k=1}^{\infty}$ *satisfies CMMX-condition.*

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in \mathrm{Fix}(T)$ *and this element* p *is the unique solution of the variational inequality* $\langle (I-f)p, y-p\rangle \geq 0$ *for all* $y \in \mathrm{Fix}(T)$.

It is worth mentioning that there are two averaging matrices $[a_{n,k}]_{n,k=1}^{\infty}$ and $[b_{n,k}]_{n,k=1}^{\infty}$ involving the iterative scheme mentioned above. The first matrix $[a_{n,k}]_{n,k=1}^{\infty}$ is exploited to update x_{n+1} from the past iterates $\{x_k\}_{k=1}^{n}$. This approach motivated by the work of Combettes and Pennanen [9] mitigates the zig-zagging [10, 11] and spiraling [12, 13] of sequences reported in some applications. The second matrix $[b_{n,k}]_{n,k=1}^{\infty}$ is motivated from the work of Brézis and Browder [6]. It can be viewed as an extension of the usual Cesàro mean of ergodic theory.

It is our purpose to introduce two concepts: (1) A-sequences of Halpern type where A is an averaging infinite matrix, and (2) concentrating matrices in the sense of Halpern; to simultaneously unify and generalize the preceding three results. The paper is organized as follows: In Section 2, we first prove some auxiliary results which is a refinement of Xu's lemma. The definition of concentrating matrices in the sense of Halpern is introduced in Subsection 2.1. This kind of matrices is inspired by the work of Combettes and Pennanen [9]. In Subsection 2.2, we provide some tools used extensively in this paper. Our main convergence theorem is presented in Subsection 2.3 after the introduction of A-sequences of Halpern type where A is an averaging infinite matrix. A necessary and sufficient condition for the convergence of an A-sequence of Halpern type is given in terms of some properties of the set of all weak cluster points of some sequences defined from this sequence. In Subsection 2.4, we show that all the results of Cianciaruso et al. [7] and of Muglia et al. [8] are easily deduced from our result with weaker assumptions and with some new conditions. Finally, in Section 3, we discuss two interesting examples of concentrating matrices in the sense of Halpern. More precisely, we show that averaging matrices satisfying either the generalized segmenting or the generalized moving average condition are concentrating in the sense of Halpern.

2. Main results

We start our main result by refining the result which is known as Xu's lemma [14].

Lemma 2.1: *Let* $\{s_n\}_{n=1}^{\infty}$ *be a sequence in* $[0,\infty)$, $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence in* $[0,1]$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\{t_n\}_{n=1}^{\infty}$ *be a sequence of real numbers. Assume that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n t_n$$

for all $n \geq 1$. *Then* $\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$. *In particular, Xu's lemma follows, that is, if* $\limsup_{n\to\infty} t_n \leq 0$, *then* $\lim_{n\to\infty} s_n = 0$.

Proof: Without loss of generality, we can assume that $t_{n+1} \leq t_n$ for all $n \geq 1$. (In fact, we can replace t_n by $\widehat{t_n} := \sup\{t_k : k \geq n\}$ and we see that $\lim_{n\to\infty} \widehat{t_n} = \limsup_{n\to\infty} t_n$.) Note that

$$0 \leq \max\{s_{n+1}, t_{n+1}\} \leq \max\{s_n, t_n\} \quad \text{for all } n \geq 1.$$

In particular, $\lim_{n\to\infty} \max\{s_n, t_n\}$ exists. Observe that $\alpha_n(s_n - t_n) \leq s_n - s_{n+1}$. Then $\sum_{n=1}^{m} \alpha_n(s_n - t_n) \leq s_1 - s_{m+1} \leq s_1$ for all $m \geq 1$. It follows from $\sum_{n=1}^{\infty} \alpha_n = \infty$ that

$$\liminf_{n\to\infty}(s_n - t_n) \leq 0.$$

Now we consider the following two cases.

Case 1: $\lim_{n\to\infty} s_n$ exists. It follows that

$$\lim_{n\to\infty} s_n \leq \liminf_{n\to\infty}(s_n - t_n) + \limsup_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n.$$

Case 2: $\lim_{n\to\infty} s_n$ does not exist. This case is broken into two subcases.

Subcase 2.1: There is an integer N such that $s_n \geq t_n$ for all $n \geq N$. Then $s_n = \max\{s_n, t_n\}$ for all $n \geq N$. So $\lim_{n\to\infty} s_n$ exists, which is a contradiction.

Subcase 2.2: The inequality $s_{n_k} < t_{n_k}$ holds for infinitely many n_k. Then

$$\limsup_{n\to\infty} s_n \leq \lim_{n\to\infty} \max\{s_n, t_n\} = \lim_{k\to\infty} \max\{s_{n_k}, t_{n_k}\} = \lim_{k\to\infty} t_{n_k} \leq \limsup_{n\to\infty} t_n.$$

This completes the proof. ■

Lemma 2.2: *Let $\{s_n\}_{n=1}^{\infty}$ and $\{\varepsilon_n\}_{n=1}^{\infty}$ be sequences in $[0,\infty)$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{t_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Assume that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n + \varepsilon_n$$

for all $n \geq 1$. Then $\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$. In particular, Xu's lemma follows, that is, if $\limsup_{n\to\infty} t_n \leq 0$, then $\lim_{n\to\infty} s_n = 0$.

Proof: Note that

$$s_{n+1} + \sum_{k=n+1}^{\infty} \varepsilon_k \leq (1 - \alpha_n)\left(s_n + \sum_{k=n}^{\infty} \varepsilon_k\right) + \alpha_n\left(t_n + \sum_{k=n}^{\infty} \varepsilon_k\right).$$

By Lemma 2.1 and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, we have

$$\limsup_{n\to\infty} s_n = \limsup_{n\to\infty}\left(s_n + \sum_{k=n}^{\infty} \varepsilon_k\right) \leq \limsup_{n\to\infty}\left(t_n + \sum_{k=n}^{\infty} \varepsilon_k\right) = \limsup_{n\to\infty} t_n.$$

Then the result follows. ■

2.1. Concentrating matrices in the sense of Halpern

Inspired by the concentrating matrices in the sense of Combettes and Pennanen [9], we introduced the following matrices.

Definition 2.3: An averaging matrix $[a_{n,k}]_{n,k=1}^{\infty}$ is *concentrating in the sense of Halpern*, (*H-concentrating*, in short) if whenever $\{s_n\}_{n=1}^{\infty}$, $\{\varepsilon_n\}_{n=1}^{\infty}$ are sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{t_n\}_{n=1}^{\infty}$ is a sequence of real numbers with $\limsup_{n\to\infty} t_n \leq 0$, and

$$\bar{s}_n := \sum_{k=1}^{n} a_{n,k} s_k$$
$$s_{n+1} \leq (1-\alpha_n)\bar{s}_n + \alpha_n t_n + \varepsilon_n$$

for all $n \geq 1$, it follows that $\lim_{n\to\infty} s_n = 0$. It is clear from Lemma 2.2 that the identity matrix is H-concentrating. Some interesting examples of H-concentrating matrices are discussed in Section 3.2.

2.2. Auxiliary results

Lemma 2.4: *Let C be a nonempty convex subset of $\mathcal{H}$ and let $T : C \to C$ be a mapping. Let $n \geq 1$ and let $\{b_k\}_{k=1}^{n}$ be a finite sequence of nonnegative numbers with $\sum_{k=1}^{n} b_k = 1$. Let $z \in C$ and let $\{y_k\}_{k=1}^{n+1}$ be a sequence in C and $\{\xi_k\}_{k=1}^{n}$ be a sequence of real numbers such that*

$$\|y_{k+1} - Tz\|^2 \leq \|y_k - z\|^2 + \xi_k$$

for all $k = 1, \ldots, n$. Then

$$\|z - Tz\|^2 \leq 2\left\langle \sum_{k=1}^{n} b_k y_k - z, Tz - z \right\rangle + \sum_{k=1}^{n-1} (b_{k+1} - b_k)\|y_{k+1} - Tz\|^2$$
$$+ b_1\|y_1 - Tz\|^2 - b_n\|y_{n+1} - Tz\|^2 + \sum_{k=1}^{n} b_k \xi_k.$$

Proof: Note that

$$\begin{aligned}
\|y_{k+1} - Tz\|^2 &\leq \|y_k - z\|^2 + \xi_k \\
&= \|(y_k - Tz) + (Tz - z)\|^2 + \xi_k \\
&= \|y_k - Tz\|^2 + 2\langle y_k - Tz, Tz - z\rangle + \|Tz - z\|^2 + \xi_k \\
&= \|y_k - Tz\|^2 + 2\langle y_k - z, Tz - z\rangle - \|Tz - z\|^2 + \xi_k.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{k=1}^{n} b_k\|y_{k+1} - Tz\|^2 \\
&\quad \leq \sum_{k=1}^{n} b_k\|y_k - Tz\|^2 + 2\left\langle \sum_{k=1}^{n} b_k y_k - z, Tz - z \right\rangle - \|Tz - z\|^2 + \sum_{k=1}^{n} b_k\xi_k.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|z - Tz\|^2 \leq 2\left\langle \sum_{k=1}^{n} b_k y_k - z, Tz - z \right\rangle &+ \sum_{k=1}^{n-1}(b_{k+1} - b_k)\|y_{k+1} - Tz\|^2 \\
&+ b_1\|y_1 - Tz\|^2 - b_n\|y_{n+1} - Tz\|^2 + \sum_{k=1}^{n} b_k\xi_k.
\end{aligned}$$

This completes the proof. ■

Lemma 2.5: *Let C be a nonempty closed and convex subset of $\mathcal{H}$ and let $T : C \to C$ be a mapping. Let $\{y_{n,k}\}_{n,k=1}^{\infty}$ be a bounded double sequence in C and $\{\xi_{n,k}\}_{n,k=1}^{\infty}$ be a bounded double sequence of real numbers. Let $[b_{n,k}]_{n,k=1}^{\infty}$ be an averaging matrix satisfying the BB1-condition. Suppose that $z_n := \sum_{k=1}^{n} b_{n,k}y_{n,k}$ and*

$$\|y_{n,k+1} - Tz_n\|^2 \leq \|y_{n,k} - z_n\|^2 + \xi_{n,k}$$

for all $n \geq 1$ and for all $k = 1, 2, \ldots, n$. If $\lim_{n\to\infty} \sum_{k=1}^{n} b_{n,k}\xi_{n,k} = 0$, then

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0.$$

Proof: It follows from Lemma 2.4 and $z_n := \sum_{k=1}^{n} b_{n,k}y_{n,k}$ that

$$\begin{aligned}
&\|z_n - Tz_n\|^2 \\
&\quad \leq 2\left\langle \sum_{k=1}^{n} b_{n,k}y_{n,k} - z_n, Tz_n - z_n \right\rangle + \sum_{k=1}^{n-1}(b_{n,k+1} - b_{n,k})\|y_{n,k+1} - Tz_n\|^2 \\
&\qquad + b_{n,1}\|y_{n,1} - Tz_n\|^2 - b_{n,n}\|y_{n,n+1} - Tz_n\|^2 + \sum_{k=1}^{n} b_{n,k}\xi_{n,k} \\
&\quad \leq \sum_{k=1}^{n-1}(b_{n,k+1} - b_{n,k})^+\|y_{n,k+1} - Tz_n\|^2 + b_{n,1}\|y_{n,1} - Tz_n\|^2 + \sum_{k=1}^{n} b_{n,k}\xi_{n,k}.
\end{aligned}$$

Note that $\{z_n\}$ is bounded and hence so are the sequence $\{y_{n,1} - Tz_n\}_{n=1}^{\infty}$ and the double sequences $\{y_{n,k+1} - Tz_n\}_{n,k=1}^{\infty}$. The conclusion follows from the BB1-condition of $[b_{n,k}]_{n,k=1}^{\infty}$ and $\lim_{n\to\infty} \sum_{k=1}^{n} b_{n,k}\xi_{n,k} = 0$. ■

Lemma 2.6: *Let C be a nonempty closed and convex subset of $\mathcal{H}$ and let $T : C \to C$ be L-hybrid with a fixed point. Let $B := [b_{n,k}]_{n,k=1}^{\infty}$ be an averaging matrix. Suppose that that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in C. If one of the following conditions is satisfied:*

(a) *B satisfies BB1-condition and $L = 0$;*
(b) *B satisfies BB2-condition and $L > 0$,*

then

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{n} b_{n,k} T^{k-1} x_n - T\Big(\sum_{k=1}^{n} b_{n,k} T^{k-1} x_n \Big) \right\| = 0$$

and hence $\mathfrak{W}\{\sum_{k=1}^{n} b_{n,k} T^{k-1} x_n\}_{n=1}^{\infty} \subset \mathrm{Fix}(T)$.

Proof: Let $y_{n,k} := T^{k-1}x_n$, $z_n := \sum_{k=1}^{n} b_{n,k} y_{n,k}$ and $\xi_{n,k} := L\langle T^{k-1}x_n - T^k x_n, z_n - Tz_n\rangle$ for all $n, k \geq 1$. Since T is L-hybrid,

$$\|y_{n,k+1} - Tz_n\|^2 \leq \|y_{n,k} - z_n\|^2 + \xi_{n,k}.$$

We assume that (a) holds. It follows that $\lim_{n\to\infty} \sum_{k=1}^{n} b_{n,k}\xi_{n,k} = 0$ and the result follows from Lemma 2.5.

We assume that (b) holds. Note that T is quasi-nonexpansive. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, the sequence $\{z_n - Tz_n\}_{n=1}^{\infty}$ and the double sequence $\{T^k x_n\}_{n,k=1}^{\infty}$ are bounded. It follows from the BB2-condition of B that

$$\lim_{n\to\infty} b_{n,1}x_n = \lim_{n\to\infty} \sum_{k=1}^{n-1} (b_{n,k+1} - b_{n,k}) T^k x_n = \lim_{n\to\infty} b_{n,n} T^n x_n = 0.$$

In particular,

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{k=1}^{n} b_{n,k}\xi_{n,k} \\
&\quad = L \lim_{n\to\infty} \sum_{k=1}^{n} b_{n,k} \langle T^{k-1}x_n - T^k x_n, z_n - Tz_n\rangle \\
&\quad = L \lim_{n\to\infty} \left\langle b_{n,1}x_n + \sum_{k=1}^{n-1} (b_{n,k+1} - b_{n,k}) T^k x_n - b_{n,n} T^n x_n, z_n - Tz_n \right\rangle = 0.
\end{aligned}$$

The conclusion follows again from Lemma 2.5. ■

2.3. Convergence theorems

We first define the following notion which plays a key role in this paper.

Definition 2.7: Let F be a nonempty closed and convex subset of $\mathcal{H}$ and $A := [a_{n,k}]_{n,k=1}^{\infty}$ be an averaging matrix. We say that a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$ is of *A-Halpern type with respect to F* if there exist a contraction $f : \mathcal{H} \to \mathcal{H}$; two sequences $\{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$ in $\mathcal{H}$; and two sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ in $[0, 1]$ such that the following conditions are satisfied:

(a) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\liminf_{n\to\infty}(1 - \beta_n) > 0$;
(b) $\|u_n - p\| \leq \sum_{k=1}^{n} a_{n,k}\|x_k - p\|$ and $\|v_n - p\| \leq \sum_{k=1}^{n} a_{n,k}\|x_k - p\|$ for all $n \geq 1$ and for all $p \in F$;
(c) $x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n f(u_n) + (1 - \alpha_n)v_n)$ for all $n \geq 1$.

Remark 2.1: Suppose that F is a nonempty closed and convex subset of $\mathcal{H}$ and A is an averaging matrix. Every A-Halpern type sequence with respect to F is bounded.

Proof: Let $A := [a_{n,k}]_{n,k=1}^{\infty}$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is of A-Halpern type with respect to F where $f : \mathcal{H} \to \mathcal{H}$, $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are given by Definition 2.7. Let $p \in F$. Suppose that f is an α-contraction. For each $n \geq 1$, set

$$M_n := \max\left\{\|x_1 - p\|, \ldots, \|x_n - p\|, \frac{1}{1-\alpha}\|f(p) - p\|\right\}.$$

Note that

$$\|u_n - p\| \leq \sum_{k=1}^{n} a_{n,k}\|x_k - p\| \leq M_n$$

and

$$\|v_n - p\| \leq \sum_{k=1}^{n} a_{n,k}\|x_k - p\| \leq M_n.$$

Moreover,

$$\begin{aligned} \|f(u_n) - p\| &\leq \|f(u_n) - f(p)\| + \|f(p) - p\| \\ &\leq \alpha\|u_n - p\| + \|f(p) - p\| \\ &\leq \alpha M_n + (1 - \alpha)M_n = M_n. \end{aligned}$$

This implies that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \beta_n\|x_n - p\| + (1 - \beta_n)(\alpha_n\|f(u_n) - p\| + (1 - \alpha_n)\|v_n - p\|) \\ &\leq \beta_n M_n + (1 - \beta_n)\,(\alpha_n M_n + (1 - \alpha_n)M_n) = M_n. \end{aligned}$$

By induction, we get that $\|x_n - p\| \leq \max\{\|x_1 - p\|, (1/(1 - \alpha))\|f(p) - p\|\}$ for all $n \geq 1$ and hence $\{x_n\}_{n=1}^{\infty}$ is bounded. ■

We now give a necessary and sufficient condition for the convergence of a sequence of A-Halpern type with respect to F.

Theorem 2.8: *Let F be a nonempty closed and convex subset of $\mathcal{H}$ and $A := [a_{n,k}]_{n,k=1}^{\infty}$ be an averaging matrix. Suppose that a sequence $\{x_n\}_{n=1}^{\infty}$ is of A-Halpern type with respect to F where $f : \mathcal{H} \to \mathcal{H}$, $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are given by Definition 2.7. Suppose that $y_n := \alpha_n f(u_n) + (1-\alpha_n)v_n$ for all $n \geq 1$. Suppose in addition that one of the following conditions is satisfied*:

(a) *A is H-concentrating and $\sum_{n=1}^{\infty} \beta_n(1-a_{n,n}) < \infty$;*
(b) $\lim_{n\to\infty}(1-a_{n,n})/\alpha_n = 0$.

Then $x_n \to z = P_F f(z)$ if and only if $\mathfrak{W}\{y_n\} \subset F$.

Proof: We may assume that f is an α-contraction where $\alpha \in (0,1)$. In particular, the composition $P_F \circ f$ is also an α-contraction on F and hence there exists a unique element $z \in F$ such that $z = P_F f(z)$.

($\Rightarrow$) Assume that $x_n \to z$. Note that $x_{n+1} - x_n = (1-\beta_n)(y_n - x_n)$. It follows from $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\liminf_{n\to\infty}(1-\beta_n) > 0$ that $\lim_{n\to\infty} \|y_n - x_n\| = 0$ and hence $\mathfrak{W}\{y_n\} = \mathfrak{W}\{x_n\} = \{z\} \subset F$.

($\Leftarrow$) We assume that $\mathfrak{W}\{y_n\} \subset F$. Then $\limsup_{n\to\infty}\langle f(z) - z, y_n - z\rangle \leq 0$. It follows from Remark 2.1 that $\{x_n\}_{n=1}^{\infty}$ is bounded and so are the sequences $\{u_n\}$ and $\{v_n\}$. Set $M := \sup\{\|x_n - z\|^2 : n \geq 1\}$. We consider the following estimates:

$$
\begin{aligned}
&\|y_n - z\|^2 \\
&\quad = \|\alpha_n(f(u_n) - f(z)) + \alpha_n(f(z) - z) + (1-\alpha_n)(v_n - z)\|^2 \\
&\quad \leq \|\alpha_n(f(u_n) - f(z)) + (1-\alpha_n)(v_n - z)\|^2 + 2\alpha_n \left\langle f(z) - z, y_n - z\right\rangle \\
&\quad \leq \alpha_n\alpha^2\|u_n - z\|^2 + (1-\alpha_n)\|v_n - z\|^2 + 2\alpha_n \left\langle f(z) - z, y_n - z\right\rangle \\
&\quad \leq (1-\alpha_n(1-\alpha^2))\sum_{k=1}^{n} a_{n,k}\|x_k - z\|^2 + 2\alpha_n \left\langle f(z) - z, y_n - z\right\rangle.
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 \\
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)(1-\alpha_n(1-\alpha^2))\Big(\sum_{k=1}^{n} a_{n,k}\|x_k - z\|^2\Big) \\
&\quad + 2(1-\beta_n)\alpha_n\langle f(z) - z, y_n - z\rangle.
\end{aligned}
$$

We now discuss the following two cases.

Case 1: We assume that A is H-concentrating and $\sum_{n=1}^{\infty}\beta_n(1-a_{n,n})<\infty$. In this case, we start by refining the preceding estimate:

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\quad\le \beta_n\|x_n-z\|^2+(1-\beta_n-(1-\beta_n)\alpha_n(1-\alpha^2))\Big(\sum_{k=1}^{n}a_{n,k}\|x_k-z\|^2\Big)\\
&\qquad+2(1-\beta_n)\alpha_n\left\langle f(z)-z,y_n-z\right\rangle\\
&\quad=\beta_n\Big(\|x_n-z\|^2-\sum_{k=1}^{n}a_{n,k}\|x_k-z\|^2\Big)\\
&\qquad+(1-(1-\beta_n)\alpha_n(1-\alpha^2))\Big(\sum_{k=1}^{n}a_{n,k}\|x_k-z\|^2\Big)\\
&\qquad+2(1-\beta_n)\alpha_n\Big\langle f(z)-z,y_n-z\Big\rangle\\
&\quad\le \beta_n(1-a_{n,n})\|x_n-z\|^2+(1-(1-\beta_n)\alpha_n(1-\alpha^2))\Big(\sum_{k=1}^{n}a_{n,k}\|x_k-z\|^2\Big)\\
&\qquad+2(1-\beta_n)\alpha_n\Big\langle f(z)-z,y_n-z\Big\rangle\\
&\quad\le \beta_n(1-a_{n,n})M+(1-(1-\beta_n)\alpha_n(1-\alpha^2))\Big(\sum_{k=1}^{n}a_{n,k}\|x_k-z\|^2\Big)\\
&\qquad+(1-\beta_n)\alpha_n(1-\alpha^2)\frac{2\langle f(z)-z,y_n-z\rangle}{1-\alpha^2}.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\liminf_{n\to\infty}(1-\beta_n)>0$, we have $\sum_{n=1}^{\infty}(1-\beta_n)\alpha_n(1-\alpha^2)=\infty$. Since A is H-concentrating and $\sum_{n=1}^{\infty}\beta_n(1-a_{n,n})<\infty$, we have $\lim_{n\to\infty}\|x_n-z\|^2=0$, that is, $x_n\to z$.

Case 2: We assume that $\lim_{n\to\infty}(1-a_{n,n})/\alpha_n=0$. In this case, we follow the idea from [7]. Note that $\sum_{k=1}^{n-1}a_{n,k}\|x_k-z\|^2\le(1-a_{n,n})M$ for all $n\ge 2$. We now consider the following estimate:

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\quad\le \beta_n\|x_n-z\|^2+(1-\beta_n)(1-\alpha_n(1-\alpha^2))\left((1-a_{n,n})M+a_{n,n}\|x_n-z\|^2\right)\\
&\qquad+2(1-\beta_n)\alpha_n\left\langle f(z)-z,y_n-z\right\rangle\\
&\quad=\left(\beta_n+(1-\beta_n)(1-\alpha_n(1-\alpha^2))a_{n,n}\right)\|x_n-z\|^2\\
&\qquad+(1-\beta_n)(1-\alpha_n(1-\alpha^2))(1-a_{n,n})M\\
&\qquad+2(1-\beta_n)\alpha_n\left\langle f(z)-z,y_n-z\right\rangle\\
&\quad\le (1-(1-\beta_n)\alpha_n(1-\alpha^2))\|x_n-z\|^2+(1-\beta_n)(1-a_{n,n})M\\
&\qquad+2(1-\beta_n)\alpha_n\left\langle f(z)-z,y_n-z\right\rangle
\end{aligned}$$

$$= \Big(1 - (1-\beta_n)\alpha_n(1-\alpha^2)\Big)\|x_n - z\|^2$$
$$+ (1-\beta_n)\alpha_n(1-\alpha^2)\left(\frac{1-a_{n,n}}{\alpha_n(1-\alpha^2)}M + \frac{2\langle f(z) - z, y_n - z\rangle}{1-\alpha^2}\right).$$

Note that $\sum_{n=1}^{\infty}(1-\beta_n)\alpha_n(1-\alpha^2) = \infty$ and $\lim_{n\to\infty}(1-a_{n,n})/\alpha_n = 0$. This implies that $\lim_{n\to\infty}\|x_n - z\|^2 = 0$, that is, $x_n \to z$. ∎

2.4. Deduced results

We now present the first deduced result which is an improvement of Theorem 1.5.

Theorem 2.9: *Let $D : \mathcal{H} \to \mathcal{H}$ be a β-strongly monotone and δ-Lipschitzian operator and $T : \mathcal{H} \to \mathcal{H}$ be an L-hybrid mapping such that $\mathrm{Fix}(T) \neq \emptyset$. Let $A := [a_{n,k}]_{n,k=1}^{\infty}$ and $B := [b_{n,k}]_{n,k=1}^{\infty}$ be averaging matrices. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$ defined by*

$$x_1 \in \mathcal{H} \text{ arbitrarily chosen,}$$
$$\bar{x}_n := \sum_{k=1}^{n} a_{n,k}x_k,$$
$$x_{n+1} := \gamma_n x_n + (1-\gamma_n)(I - \mu_n D)\sum_{k=0}^{n-1} b_{n,k+1}T^k\bar{x}_n \quad \textit{for all } n \geq 1,$$

where

(a) $\{\mu_n\}_{n=1}^{\infty} \subset (0,\mu)$ *with* $\mu < 2\beta/\delta^2$ *and* $\lim_{n\to\infty}\mu_n = 0$ *and* $\sum_{n=1}^{\infty}\mu_n = \infty$;
(b) *one of the following two conditions is satisfied*:
  (1) *B satisfies BB1-condition and $L = 0$*;
  (2) *B satisfies BB2-condition and $L > 0$*;
(c) *one of the following two conditions is satisfied*:
  (1) *A is H-concentrating and* $\sum_{n=1}^{\infty}\gamma_n(1-a_{n,n}) < \infty$;
  (2) $\lim_{n\to\infty}(1-a_{n,n})/\mu_n = 0$;
(d) $\{\gamma_n\}_{n=1}^{\infty} \subset [0,1]$ *with* $\limsup_{n\to\infty}\gamma_n < 1$.

Then $x_n \to z = P_{\mathrm{Fix}(T)}(I - D)z$.

Proof: We apply our Theorem 2.8 to prove this result by showing first that the sequence $\{x_n\}_{n=1}^{\infty}$ is of A-Halpern type with respect to $\mathrm{Fix}(T)$. Set

$$\alpha_n := \frac{\mu_n}{\mu}, \quad \beta_n := \gamma_n, \quad f := I - \mu D, \quad u_n = v_n := \sum_{k=0}^{n-1} b_{n,k+1}T^k\bar{x}_n.$$

Note that $\|u_n - p\| = \|v_n - p\| = \| \sum_{k=0}^{n-1} b_{n,k+1}(T^k\overline{x}_n - p)\| \leq \sum_{k=0}^{n-1} b_{n,k+1}\| T^k\overline{x}_n - p\| \leq \|\overline{x}_n - p\| = \| \sum_{k=1}^{n} a_{n,k}(x_k - p)\| \leq \sum_{k=1}^{n} a_{n,k}\|x_k - p\|$ for all $p \in \mathrm{Fix}(T)$. Moreover,

$$\begin{aligned} x_{n+1} &= \gamma_n x_n + (1-\gamma_n)\left(\frac{\mu_n}{\mu}(I-\mu D)(u_n) + \left(1 - \frac{\mu_n}{\mu}\right)v_n\right) \\ &= \beta_n x_n + (1-\beta_n)(\alpha_n f(u_n) + (1-\alpha_n)v_n). \end{aligned}$$

Finally, we prove that $\mathfrak{W}\{y_n\}_{n=1}^{\infty} \subset \mathrm{Fix}(T)$ where $y_n := \alpha_n f(u_n) + (1-\alpha_n)v_n$. To see this, we note from Remark 2.1 that $\{x_n\}_{n=1}^{\infty}$ is bounded and hence so is the sequence $\{\overline{x}_n\}_{n=1}^{\infty}$. By Lemma 2.6, we have $\mathfrak{W}\{v_n\}_{n=1}^{\infty} \subset \mathrm{Fix}(T)$. Since $\lim_{n\to\infty} \|y_n - v_n\| = \lim_{n\to\infty} \alpha_n \|f(u_n) - v_n\| = 0$, we have $\mathfrak{W}\{y_n\}_{n=1}^{\infty} = \mathfrak{W}\{v_n\}_{n=1}^{\infty} \subset \mathrm{Fix}(T)$. ∎

Remark 2.2: The conditions (b), (c), and (d) of Theorem 2.9 are weaker than the conditions (b), (c), and (d) of Theorem 1.5, respectively.

Remark 2.3:

(1) The matrix $A' = [a_{n,k}]_{n,k=1}^{\infty}$ in Example 3.13 is H-concentrating but $\lim_{n\to\infty}(1 - a_{n,n})/\alpha_n = \infty$. Hence our Theorem 2.9 is established under a new condition and it cannot be applicable by Theorem 1.5.

(2) Let $B := [b_{n,k}]_{n,k=1}^{\infty}$ be defined by

$$b_{n,k} := \begin{cases} 1 & \text{if } n = k = 1; \\ 0 & \text{if } n \geq 1 \quad \text{and} \quad k > n; \\ \dfrac{1}{n+1} & \text{if } n \geq 2 \quad \text{and} \quad k = 1, 2, \ldots, n-1; \\ \dfrac{2}{n+1} & \text{if } n \geq 2 \quad \text{and} \quad k = n. \end{cases}$$

That is,

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1/3 & 2/3 & 0 & 0 & 0 & \cdots \\ 1/4 & 1/4 & 2/4 & 0 & 0 & \cdots \\ 1/5 & 1/5 & 1/5 & 2/5 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then B satisfies the BB2-condition but it does not satisfy the CMMX-condition.

Theorem 2.10: *Let $D : \mathcal{H} \to \mathcal{H}$ be a β-strongly monotone and δ-Lipschitzian operator and $T : \mathcal{H} \to \mathcal{H}$ be L-hybrid such that* $\mathrm{Fix}(T) \neq \varnothing$. *Let $A := [a_{n,k}]_{n,k=1}^{\infty}$ and $B := [b_{n,k}]_{n,k=1}^{\infty}$ be averaging matrices. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$ defined by*

$$
\begin{aligned}
&x_1 \in \mathcal{H} \text{ arbitrarily chosen,}\\
&\overline{x}_n := \sum_{k=1}^{n} a_{n,k} x_k,\\
&x_{n+1} := \alpha_n (I - \mu_n D)\overline{x}_n + (1-\alpha_n) \sum_{k=0}^{n-1} b_{n,k+1} T^k \overline{x}_n \quad \text{for all } n \geq 1
\end{aligned}
$$

where

(a) $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\{\mu_n\}_{n=1}^{\infty} \subset (0,\mu)$ *with* $\mu < 2\beta/\delta^2$ *and* $\sum_{n=1}^{\infty} \alpha_n \mu_n = \infty$;
(b) *one of the following two conditions is satisfied*:
  (1) *B satisfies BB1-condition and $L = 0$*;
  (2) *B satisfies BB2-condition and $L > 0$*;
(c) *one of the following two conditions is satisfied*:
  (1) *A is H-concentrating*;
  (2) $\lim_{n\to\infty} (1 - a_{n,n})/\alpha_n \mu_n = 0$.

Then $x_n \to z = P_{\mathrm{Fix}(T)}(I - D)z$.

Proof: We apply our Theorem 2.8 to prove this result by showing first that the sequence $\{x_n\}_{n=1}^{\infty}$ is of A-Halpern type with respect to $\mathrm{Fix}(T)$. To see this, set

$$\widehat{\alpha}_n := \frac{\alpha_n \mu_n}{\mu}, \quad \beta_n := 0, \quad f := I - \mu D, \quad u_n := \overline{x}_n,$$

and

$$v_n := \Big(1 - \frac{(1-\alpha_n)\mu}{\mu - \alpha_n \mu_n}\Big)\overline{x}_n + \frac{(1-\alpha_n)\mu}{\mu - \alpha_n \mu_n} \sum_{k=0}^{n-1} b_{n,k+1} T^k \overline{x}_n.$$

It follows that

$$
\begin{aligned}
x_{n+1} &= \alpha_n (I - \mu_n D)\overline{x}_n + (1-\alpha_n) \sum_{k=0}^{n-1} b_{n,k+1} T^k \overline{x}_n\\
&= \frac{\alpha_n \mu_n}{\mu}(I - \mu D)\overline{x}_n + \alpha_n \Big(1 - \frac{\mu_n}{\mu}\Big)\overline{x}_n + (1-\alpha_n) \sum_{k=0}^{n-1} b_{n,k+1} T^k \overline{x}_n
\end{aligned}
$$

$$= \frac{\alpha_n \mu_n}{\mu}(I - \mu D)\overline{x}_n$$

$$+ \left(1 - \frac{\alpha_n \mu_n}{\mu}\right)\left(\frac{\alpha_n\left(1 - \frac{\mu_n}{\mu}\right)}{1 - \frac{\alpha_n \mu_n}{\mu}}\overline{x}_n + \frac{1 - \alpha_n}{1 - \frac{\alpha_n \mu_n}{\mu}} \sum_{k=0}^{n-1} b_{n,k+1} T^k \overline{x}_n\right)$$

$$= \widehat{\alpha}_n f(u_n) + (1 - \widehat{\alpha}_n) v_n.$$

It is clear that

$$\|u_n - p\| \le \sum_{k=1}^{n} a_{n,k}\|x_k - p\| \quad \text{and} \quad \|v_n - p\| \le \sum_{k=1}^{n} a_{n,k}\|x_k - p\|$$

for all $n \ge 1$ and for all $p \in \mathrm{Fix}(T)$.

Finally, we prove that $\mathfrak{W}\{x_n\}_{n=1}^\infty \subset \mathrm{Fix}(T)$. To see this, we note from Remark 2.1 that $\{x_n\}_{n=1}^\infty$ is bounded and hence so is the sequence $\{\overline{x}_n\}_{n=1}^\infty$. By Lemma 2.6, we have $\mathfrak{W}\{\sum_{k=0}^{n-1} b_{n,k+1} T^k \overline{x}_n\}_{n=1}^\infty \subset \mathrm{Fix}(T)$. Note that

$$\lim_{n\to\infty} \left\| x_{n+1} - \sum_{k=0}^{n-1} b_{n,k+1} T^k \overline{x}_n \right\| = \lim_{n\to\infty} \alpha_n \left\| (I - \mu_n D)\overline{x}_n - \sum_{k=0}^{n-1} b_{n,k+1} T^k \overline{x}_n \right\| = 0.$$

This implies that $\mathfrak{W}\{x_n\}_{n=1}^\infty = \mathfrak{W}\{\sum_{k=0}^{n-1} b_{n,k+1} T^k \overline{x}_n\}_{n=1}^\infty \subset \mathrm{Fix}(T)$. Hence the conclusion follows. ■

Remark 2.4: We compare Theorem 1.6 and our Theorem 2.10.

(1) It is obvious that the condition (b) of our Theorem 2.10 is more general than the condition (b) of Theorem 1.6.
(2) The iteration in our Theorem 2.10 where A is the identity matrix is just the one studied in Theorem 1.6. Note that in the iteration studied in Theorem 1.6 the update element x_{n+1} involves only the current element x_n and the past elements $x_1, x_2, \ldots, x_{n-1}$ are not exploited. As mentioned by Combettes and Pennanen [9], acting on an average of the past elements naturally centers the iterative sequence and mitigates zigzagging and spiraling.
(3) The mapping in our Theorem 2.10 includes the one studied in Theorem 1.6. In fact, every nonspreading mapping is 2-hybrid.

Theorem 2.11: *Let $f : \mathcal{H} \to \mathcal{H}$ be α-contractive and $T : \mathcal{H} \to \mathcal{H}$ be L-hybrid such that* $\mathrm{Fix}(T) \neq \varnothing$. *Let $A := [a_{n,k}]_{n,k=1}^\infty$ and $B := [b_{n,k}]_{n,k=1}^\infty$ be averaging* *matrices. Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathcal{H}$ *defined by*

$$\begin{aligned} &x_1 \in \mathcal{H} \text{ arbitrarily chosen,} \\ &\overline{x}_n := \sum_{k=1}^{n} a_{n,k} x_k, \\ &x_{n+1} := \alpha_n f(\overline{x}_n) + (1-\alpha_n) \sum_{k=0}^{n-1} b_{n,k+1} T^k \overline{x}_n \quad \text{ for all } n \geq 1 \end{aligned}$$

where

(a) $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) *one of the following two conditions is satisfied*:
  (1) *B satisfies BB1-condition and* $L = 0$;
  (2) *B satisfies BB2-condition and* $L > 0$;
(c) *one of the following two conditions is satisfied*:
  (1) *A is H-concentrating*;
  (2) $\lim_{n\to\infty} (1-a_{n,n})/\alpha_n = 0$.

Then $x_n \to z = P_{\mathrm{Fix}(T)} f(z)$.

Proof: We apply our Theorem 2.8 to prove this result by showing first that the sequence $\{x_n\}_{n=1}^{\infty}$ is of A-Halpern type with respect to $\mathrm{Fix}(T)$. Set

$$\beta_n := 0, \quad u_n := \overline{x}_n, v_n := \sum_{k=0}^{n-1} b_{n,k+1} T^k \overline{x}_n.$$

Then $x_{n+1} = \alpha_n f(u_n) + (1-\alpha_n) v_n$. It is clear that

$$\|u_n - p\| \leq \sum_{k=1}^{n} a_{n,k} \|x_k - p\| \quad \text{ and } \quad \|v_n - p\| \leq \sum_{k=1}^{n} a_{n,k} \|x_k - p\|$$

for all $n \geq 1$ and for all $p \in \mathrm{Fix}(T)$.

Finally, we prove that $\mathfrak{W}\{x_n\}_{n=1}^{\infty} \subset \mathrm{Fix}(T)$. To see this, we note from Remark 2.1 that $\{x_n\}_{n=1}^{\infty}$ is bounded and hence so is the sequence $\{\overline{x}_n\}_{n=1}^{\infty}$. By Lemma 2.6, we have $\mathfrak{W}\{v_n\}_{n=1}^{\infty} \subset \mathrm{Fix}(T)$. Note that the sequences $\{f(u_n)\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are bounded, so

$$\lim_{n\to\infty} \|x_{n+1} - v_n\| = \lim_{n\to\infty} \alpha_n \left\| f(u_n) - v_n \right\| = 0.$$

This implies that $\mathfrak{W}\{x_n\}_{n=1}^{\infty} = \mathfrak{W}\{v_n\}_{n=1}^{\infty} \subset \mathrm{Fix}(T)$. Hence the conclusion follows. ■

Remark 2.5: As discussed in Remark 2.4, our Theorem 2.11 significantly improves Theorem 1.7.

3. Examples of concentrating matrices in the sense of Halpern

In this section, we give some examples of H-concentrating matrices.

3.1. CP-concentrating matrices: definitions and examples

We first recall the following concept introduced by Combettes and Pannanen [9].

Definition 3.1: An averaging matrix $[a_{n,k}]_{n,k=1}^{\infty}$ is *concentrating in the sense of Combettes and Pannanen*, (*CP-concentrating*, in short) if whenever $\{s_n\}_{n=1}^{\infty}$, $\{\varepsilon_n\}_{n=1}^{\infty}$ are sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and

$$\bar{s}_n := \sum_{k=1}^{n} a_{n,k} s_k,$$
$$s_{n+1} \leq \bar{s}_n + \varepsilon_n,$$

for all $n \geq 1$, it follows that $\{s_n\}_{n=1}^{\infty}$ converges.

To mention some interesting examples of CP-concentrating matrices, we define some notations. For a given matrix $A := [a_{n,k}]_{n,k=1}^{\infty}$, let

$$a'_{n,k} := a_{n+1,k} - (1 - a_{n+1,n+1}) a_{n,k};$$
$$\rho_k := \left(\sum_{n=k}^{\infty} a_{n,k} - 1 \right)^{+};$$
$$J_n := \{k : a_{n,k} > 0\}.$$

Definition 3.2 (Generalized segmenting condition): We say that an averaging matrix $[a_{n,k}]_{n,k=1}^{\infty}$ satisfies the *generalized segmenting condition* if

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} |a'_{n,k}| < \infty.$$

In particular, if $a'_{n,k} = 0$ for all $n, k \geq 1$, then $[a_{n,k}]_{n,k=1}^{\infty}$ is said to satisfy the *segmenting condition* [15].

Proposition 3.3: *Every averaging matrix $[a_{n,k}]_{n,k=1}^{\infty}$ satisfying the generalized segmenting condition and $\liminf_{n\to\infty} a_{n,n} > 0$ is CP-concentrating.* (*See* [9, Example 2.5]) *In particular, the identity matrix is CP-concentrating.*

The following result tells us that the condition $\liminf_{n\to\infty} a_{n,n} > 0$ is not only sufficient but also necessary for averaging matrix $[a_{n,k}]_{n,k=1}^{\infty}$ satisfying the generalized segmenting condition to be CP-concentrating.

Proposition 3.4: *If* $[a_{n,k}]_{n,k=1}^{\infty}$ *is a CP-concentrating matrix satisfying the generalized segmenting condition, then* $\liminf_{n\to\infty} a_{n,n} > 0$.

Before proving this result, let us observe the following fact from the generalized segmenting condition.

Remark 3.1: If an averaging matrix $[a_{n,k}]_{n,k=1}^{\infty}$ satisfies the generalized segmenting condition and $\{s_n\}_{n=1}^{\infty}$ is a sequence of nonnegative real numbers such that $s_{n+1} \le \bar{s}_n$ for all $n \ge 1$, then $\{s_n\}$ is bounded above and $\lim_{n\to\infty} \bar{s}_n$ exists. The first assertion follows easily and we assume that there exists $M > 0$ such that $s_n \le M$ for all $n \ge 1$. To see the second assertion, we first note that

$$\begin{aligned}
\bar{s}_{n+1} &= \sum_{k=1}^{n} a_{n+1,k}s_k + a_{n+1,n+1}s_{n+1} \\
&= a_{n+1,n+1}s_{n+1} + \sum_{k=1}^{n}(a_{n+1,k} - (1 - a_{n+1,n+1})a_{n,k})s_k + (1 - a_{n+1,n+1})\bar{s}_n \\
&\le \bar{s}_n - a_{n+1,n+1}(\bar{s}_n - s_{n+1}) + M\sum_{k=1}^{n}|a'_{n,k}| \\
&\le \bar{s}_n + M\sum_{k=1}^{n}|a'_{n,k}|.
\end{aligned}$$

Since the identity matrix is CP-concentrating and $\sum_{n=1}^{\infty}\sum_{k=1}^{n}|a'_{n,k}| < \infty$, we conclude that $\lim_{n\to\infty} \bar{s}_n$ exists.

Proof of Proposition 3.4: Suppose that $[a_{n,k}]_{n,k=1}^{\infty}$ is a CP-concentrating matrix satisfying the generalized segmenting condition but $\liminf_{n\to\infty} a_{n,n} = 0$. Passing to a suitable subsequence $\{n_k\}$ we may assume that

$$\sum_{j=1}^{\infty} a_{n_j+1,n_j+1} \le 1 \quad \text{and} \quad \sum_{n=n_1+1}^{\infty}\sum_{k=1}^{n}|a'_{n,k}| \le 1/2.$$

Define a sequence $\{\delta_n\} \subset \{0,1\}$ by

$$\delta_n := \begin{cases} 1 & \text{if } n = n_k + 1 \quad \text{for some } k \ge 1; \\ 0 & \text{otherwise.} \end{cases}$$

We also define a sequence $\{s_n\}$ by

$$s_1 := 4 \quad \text{and} \quad s_{n+1} := \bar{s}_n - \delta_{n+1} \quad \text{for all } n \ge 1.$$

Note that

- $\sum_{j=1}^{\infty} a_{j,j}\delta_j \leq 1$;
- $s_1 = s_2 = \cdots = s_{n_1}$ and $s_n \leq s_1$ for all $n \geq n_1 + 1$;
- $\sum_{k=1}^{l} a'_{l,k}s_k = 0$ for all $l = 1, 2, \ldots, n_1$.

This implies that

$$\begin{aligned}
\bar{s}_{n+1} &= a_{n+1,n+1}s_{n+1} + \sum_{k=1}^{n} a'_{n,k}s_k + (1 - a_{n+1,n+1})\bar{s}_n \\
&= \bar{s}_n - a_{n+1,n+1}\delta_{n+1} + \sum_{k=1}^{n} a'_{n,k}s_k.
\end{aligned}$$

In particular, for $n \geq n_1 + 1$, we have

$$\begin{aligned}
\bar{s}_{n+1} &= \bar{s}_1 - \sum_{j=1}^{n} a_{j+1,j+1}\delta_{j+1} + \sum_{j=1}^{n}\sum_{k=1}^{j} a'_{j,k}s_k \\
&= \bar{s}_1 - \sum_{j=1}^{n} a_{j+1,j+1}\delta_{j+1} + \sum_{j=n_1+1}^{n}\sum_{k=1}^{j} a'_{j,k}s_k \\
&\geq \bar{s}_1 - \sum_{j=1}^{\infty} a_{j+1,j+1}\delta_{j+1} + \sum_{j=n_1+1}^{n}\sum_{k=1}^{j} a'_{j,k}s_k \\
&\geq \bar{s}_1 - 1 + \sum_{j=n_1+1}^{n}\sum_{k=1}^{j} a'_{j,k}s_k.
\end{aligned}$$

Note that $s_1 = s_2 = \cdots = s_{n_1} \geq 0$ and $s_{n_1+1} = \bar{s}_{n_1} - \delta_{n_1+1} = 3 \geq 0$. We prove by induction that $s_n \geq 0$ for $n \geq n_1$. Suppose that there exists $n \geq n_1$ such that $s_k \geq 0$ for all $k = 1, 2, \ldots, n+1$. We show that $s_{n+2} \geq 0$. To see this, we consider

$$\begin{aligned}
\bar{s}_{n+1} &\geq \bar{s}_1 - 1 + \sum_{j=n_1+1}^{n}\sum_{k=1}^{j} a'_{j,k}s_k \\
&\geq s_1 - 1 - \sum_{j=n_1+1}^{n}\sum_{k=1}^{j} |a'_{j,k}|s_1 \\
&\geq 3 - 4\sum_{j=n_1+1}^{\infty}\sum_{k=1}^{j} |a'_{j,k}| \geq 1.
\end{aligned}$$

This implies that $s_{n+2} \geq \bar{s}_{n+1} - 1 \geq 0$. By induction, we conclude that $s_n \geq 0$ for all $n \geq 1$. It follows from Remark 3.1 that $\lim_{n\to\infty} \bar{s}_n$ exists. However, it is easy to see that $\lim_{n\to\infty} s_n$ does not exist which is a contradiction. This completes the proof. ■

Definition 3.5 (Generalized moving average condition): We say that an averaging matrix $[a_{n,k}]_{n,k=1}^{\infty}$ satisfies the *generalized moving average condition* if the following conditions hold:

(a) $\sum_{k=1}^{\infty} \rho_k < \infty$,
(b) $J_{n+1} \subset J_n \cup \{n+1\}$ for all $n \geq 1$,
(c) there exists $\underline{a} \in (0,1)$ such that $a_{n,k} \geq \underline{a}$ for all $n \geq 1$ and for all $k \in J_n$.

Proposition 3.6: *Every averaging matrix satisfying the generalized moving average condition is CP-concentrating.* (*See* [9, Example 2.6])

3.2. H-concentrating matrices and some examples

To show that an averaging matrix is H-concentrating, we use the following easier characterization.

Proposition 3.7: *An averaging matrix $[a_{n,k}]_{n,k=1}^{\infty}$ is H-concentrating if and only if whenever $\{s_n\}_{n=1}^{\infty}$ is a sequence of nonnegative real numbers, $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{t_n\}_{n=1}^{\infty}$ is a sequence of real numbers with $\limsup_{n\to\infty} t_n \leq 0$ and*

$$\bar{s}_n := \sum_{k=1}^{n} a_{n,k} s_k$$

$$s_{n+1} \leq (1-\alpha_n)\bar{s}_n + \alpha_n t_n$$

for all $n \geq 1$, it follows that $\lim_{n\to\infty} s_n = 0$.

Proof: The necessity is trivial. To prove the sufficiency, we assume that the latter statement holds. Suppose that $\{s_n\}_{n=1}^{\infty}$, $\{\varepsilon_n\}_{n=1}^{\infty}$ are sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{t_n\}_{n=1}^{\infty}$ is a sequence of real numbers with $\limsup_{n\to\infty} t_n \leq 0$ and

$$\bar{s}_n := \sum_{k=1}^{n} a_{n,k} s_k$$

$$s_{n+1} \leq (1-\alpha_n)\bar{s}_n + \alpha_n t_n + \varepsilon_n$$

for all $n \geq 1$. Set $r_n := s_n + \sum_{i=n}^{\infty} \varepsilon_i$ for all $n \geq 1$. Observe that

$$\bar{s}_n + \sum_{i=n}^{\infty} \varepsilon_i \leq \sum_{k=1}^{n} a_{n,k}\Big(s_k + \sum_{i=k}^{\infty} \varepsilon_i\Big) = \sum_{k=1}^{n} a_{n,k} r_k = \bar{r}_n.$$

It follows that

$$\begin{aligned} r_{n+1} &= s_{n+1} + \sum_{j=n+1}^{\infty} \varepsilon_j \\ &\leq (1-\alpha_n)\bar{s}_n + \alpha_n t_n + \sum_{j=n}^{\infty} \varepsilon_i \\ &= (1-\alpha_n)\Big(\bar{s}_n + \sum_{j=n}^{\infty} \varepsilon_j\Big) + \alpha_n\Big(t_n + \sum_{j=n}^{\infty} \varepsilon_j\Big) \\ &\leq (1-\alpha_n)\bar{r}_n + \alpha_n\Big(t_n + \sum_{j=n}^{\infty} \varepsilon_j\Big). \end{aligned}$$

Note that $\limsup_{n\to\infty}(t_n + \sum_{j=n}^{\infty} \varepsilon_j) = \limsup_{n\to\infty} t_n \leq 0$. By the assumption of this part, we have $\lim_{n\to\infty} r_n = 0$. In particular,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\Big(r_n - \sum_{j=n}^{\infty} \varepsilon_j\Big) = 0.$$

This implies that $[a_{n,k}]_{n,k=1}^{\infty}$ is H-concentrating. ∎

Remark 3.2: Suppose that $[a_{n,k}]_{n,k=1}^{\infty}$ is an averaging matrix. Suppose that $\{s_n\}_{n=1}^{\infty}$ is a sequence of nonnegative real numbers, $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$, $\{t_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers, and

$$\bar{s}_n := \sum_{k=1}^{n} a_{n,k}s_k$$
$$s_{n+1} \leq (1-\alpha_n)\bar{s}_n + \alpha_n t_n$$

for all $n \geq 1$. Then $\{s_n\}_{n=1}^{\infty}$ is bounded.

Proof: Suppose that $M := \sup\{s_1, t_1, t_2, \ldots\}$. Obviously, $s_1 \leq M$. Assume that $s_k \leq M$ for all $k = 1, 2, \ldots, n$. Then $\bar{s}_n = \sum_{k=1}^{n} a_{n,k}s_k \leq M$ and $s_{n+1} \leq (1-\alpha_n)\bar{s}_n + \alpha_n t_n \leq M$. By induction, the sequence $\{s_n\}_{n=1}^{\infty}$ is bounded. ∎

3.2.1. Generalized segmenting condition

We prove that every averaging matrix $[a_{n,k}]_{n,k=1}^{\infty}$ satisfying $\liminf_{n\to\infty} a_{n,n} > 0$ is H-concentrating. In fact, we have the following result.

Proposition 3.8: *Suppose that $A := [a_{n,k}]_{n,k=1}^{\infty}$ is an averaging matrix satisfying the generalized segmenting condition. Then the following two statements are equivalent.*

(a) *A is H-concentrating*;
(b) $\liminf_{n\to\infty} a_{n,n} > 0$.

Proof: (b)⇒(a) Suppose that $\liminf_{n\to\infty} a_{n,n} > 0$. Suppose that $\{s_n\}_{n=1}^{\infty}$ is a sequence of nonnegative real numbers, $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{t_n\}_{n=1}^{\infty}$ is a sequence of real numbers with $\limsup_{n\to\infty} t_n \le 0$ and

$$\bar{s}_n := \sum_{k=1}^{n} a_{n,k} s_k$$
$$s_{n+1} \le (1-\alpha_n)\bar{s}_n + \alpha_n t_n$$

for all $n \ge 1$. By Remark 3.2, we assume that $s_n \le M < \infty$ for all $n \ge 1$. Note that

$$\begin{aligned}
\bar{s}_{n+1} &= a_{n+1,n+1}s_{n+1} + (1-a_{n+1,n+1})\bar{s}_n + \sum_{k=1}^{n} a'_{n,k}s_k \\
&\le a_{n+1,n+1}((1-\alpha_n)\bar{s}_n + \alpha_n t_n) + (1-a_{n+1,n+1})\bar{s}_n + M\sum_{k=1}^{n}|a'_{n,k}| \\
&= (1-\alpha_n a_{n+1,n+1})\bar{s}_n + \alpha_n a_{n+1,n+1}t_n + M\sum_{k=1}^{n}|a'_{n,k}|.
\end{aligned}$$

Since $\liminf_{n\to\infty} a_{n,n} > 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, we have $\sum_{n=1}^{\infty} \alpha_n a_{n+1,n+1} = \infty$. By Lemma 2.2, we obtain $\lim_{n\to\infty} \bar{s}_n = 0$. Since $s_{n+1} \le \max\{\bar{s}_n, t_n\}$ and $\limsup_{n\to\infty} t_n \le 0$, we have $\lim_{n\to\infty} s_n = 0$. This implies that A is H-concentrating.

(a) ⇒ (b) Suppose that $[a_{n,k}]_{n,k=1}^{\infty}$ is an H-concentrating matrix satisfying the generalized segmenting condition but $\liminf_{n\to\infty} a_{n,n} = 0$. Passing to a suitable subsequence $\{n_k\}$ we may assume that

$$a_{n_k+1,n_k+1} \le \frac{1}{k+1} \quad \text{for all } k \ge 1 \quad \text{and} \quad \sum_{n=n_1}^{\infty}\sum_{k=1}^{n}|a'_{n,k}| \le \frac{1}{4}.$$

Define a sequence $\{\alpha_n\} \subset [0,1]$ by

$$\alpha_n := \begin{cases} \dfrac{1}{k+1} & \text{if } n = n_k + 1 \quad \text{for some } k \ge 1; \\ 0 & \text{otherwise.} \end{cases}$$

We also define a sequence $\{s_n\}$ by

$$s_1 > 0 \quad \text{and} \quad s_{n+1} := (1-\alpha_{n+1})\bar{s}_n \quad \text{for all } n \ge 1.$$

Note that

- $\sum_{j=1}^{\infty} \alpha_j = \infty$ and $\sum_{j=1}^{\infty} a_{j,j}\alpha_j \leq \sum_{j=1}^{\infty} 1/j^2 < \infty$;
- $\prod_{l=n_1}^{\infty}(1 - a_{l+1,l+1}\alpha_{l+1}) \geq \prod_{k=1}^{\infty}(1 - 1/(k+1)^2) = 1/2$;
- $s_1 = s_2 = \cdots = s_{n_1}$ and $0 \leq s_n \leq s_1$ for all $n \geq n_1 + 1$;
- $\bar{s}_{n_1} = s_1$.

This implies that

$$\begin{aligned}\bar{s}_{n+1} &= a_{n+1,n+1}s_{n+1} + \sum_{k=1}^{n} a'_{n,k}s_k + (1 - a_{n+1,n+1})\bar{s}_n \\ &= (1 - \alpha_{n+1}a_{n+1,n+1})\bar{s}_n + \sum_{k=1}^{n} a'_{n,k}s_k.\end{aligned}$$

In particular, we have

$$\begin{aligned}\bar{s}_{n+1} &= (1 - \alpha_{n+1}a_{n+1,n+1})\bar{s}_n + \sum_{k=1}^{n} a'_{n,k}s_k \\ &= \prod_{l=n-1}^{n} (1 - \alpha_{l+1}a_{l+1,l+1})\bar{s}_{n-1} \\ &\quad + (1 - \alpha_{n+1}a_{n+1,n+1})\sum_{k=1}^{n-1} a'_{n-1,k}s_k + \sum_{k=1}^{n} a'_{n,k}s_k \\ &\geq \prod_{l=n-1}^{n} (1 - \alpha_{l+1}a_{l+1,l+1})\bar{s}_{n-1} - \sum_{l=n-1}^{n}\left(\sum_{k=1}^{l} |a'_{l,k}|s_k\right) \\ &\vdots \\ &\geq \prod_{l=n_1}^{n}(1 - \alpha_{l+1}a_{l+1,l+1})\bar{s}_{n_1} - \sum_{l=n_1}^{n}\left(\sum_{k=1}^{l} |a'_{l,k}|s_k\right) \\ &\geq \prod_{l=n_1}^{n}(1 - \alpha_{l+1}a_{l+1,l+1})s_1 - \sum_{l=n_1}^{n}\left(\sum_{k=1}^{l} |a'_{l,k}|\right)s_1 \\ &\geq (1/2)s_1 - (1/4)s_1 = (1/2)s_1.\end{aligned}$$

It follows from $s_{n+1} \leq \bar{s}_n$ for all $n \geq 1$ and Remark 3.1 that $\lim_{n\to\infty} \bar{s}_n$ exists. Thus $\lim_{n\to\infty} \bar{s}_n \geq (1/2)s_1 > 0$. Since $\lim_{n\to\infty} \alpha_n = 0$, we have $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \bar{s}_n > 0$ which is a contradiction. This completes the proof. ■

3.2.2. Generalized moving average condition

We prove that every averaging matrix satisfying the generalized moving average condition is H-concentrating.

Lemma 3.9: *Suppose that $[a_{n,k}]_{n,k=1}^{\infty}$ is CP-concentrating. Suppose that $\{s_n\}_{n=1}^{\infty}$ is a sequence of nonnegative real numbers, $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$ with $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\{t_n\}_{n=1}^{\infty}$ is a sequence of real numbers with $\limsup_{n\to\infty} t_n\le 0$ and*

$$\bar{s}_n := \sum_{k=1}^{n} a_{n,k}s_k$$
$$s_{n+1}\le(1-\alpha_n)\bar{s}_n+\alpha_n t_n$$

for all $n\ge 1$. Then $\lim_{n\to\infty} s_n$ exists. Furthermore, if $\liminf_{n\to\infty} s_n=0$, then $\lim_{n\to\infty} s_n=0$.

Proof: Without loss of generality, we can assume that $0\le t_{n+1}\le t_n$ for all $n\ge 1$. (Otherwise, we replace t_n by $\widehat{t_n}:=\sup\{0,t_n,t_{n+1},\ldots\}$ and it is clear that $s_{n+1}\le(1-\alpha_n)\bar{s}_n+\alpha_n\widehat{t_n}$.) Hence $\lim_{n\to\infty} t_n=0$. Set $\bar{t}_n:=\sum_{k=1}^{n}a_{n,k}t_k$. Note $\bar{t}_n\ge t_n\ge t_{n+1}$. Then

$$s_{n+1}\le(1-\alpha_n)\bar{s}_n+\alpha_n\bar{t}_n\le\max\{\bar{s}_n,\bar{t}_n\}.$$

Moreover, $t_{n+1}\le\max\{\bar{s}_n,\bar{t}_n\}$. Set $\xi_n:=\max\{s_n,t_n\}$ and $\bar{\xi}_n:=\sum_{k=1}^{n}a_{n,k}\xi_k$. Then

$$\xi_{n+1}\le\max\{\bar{s}_n,\bar{t}_n\}=\max\left\{\sum_{k=1}^{n}a_{n,k}s_k,\sum_{k=1}^{n}a_{n,k}t_k\right\}$$
$$\le\sum_{k=1}^{n}a_{n,k}\max\{s_k,t_k\}=\sum_{k=1}^{n}a_{n,k}\xi_k=\bar{\xi}_n.$$

Since $[a_{n,k}]_{n,k=1}^{\infty}$ is CP-concentrating, $\lim_{n\to\infty}\xi_n$ exists. Since $\lim_{n\to\infty}t_n=0$, we get that $\lim_{n\to\infty}s_n$ exists. ■

Lemma 3.10: *Suppose that $[a_{n,k}]_{n,k=1}^{\infty}$ is an averaging matrix satisfying the generalized moving average condition. Let $c_{n,k}:=\sum_{i=n}^{\infty}a_{i,k}$ and $u_n:=\sum_{k=1}^{n}c_{n+1,k}$ for all $n,k\ge 1$. Then $\{u_n\}_{n=1}^{\infty}$ is bounded above.*

Proof: Since $\underline{a}|J_n|\le\sum_{k=1}^{n}a_{n,k}=1$, there is an integer $m\ge 1$ such that $|J_n|\le 1/\underline{a}\le m$ for all $n\ge 1$. It follows from $\sum_{k=1}^{\infty}\rho_k<\infty$ that $\{\rho_n\}_{n=1}^{\infty}$ is bounded. There is a real number M such that $c_{n,k}\le c_{k,k}\le\rho_k+1\le M$ for all $n,k\ge 1$. Then

$$0\le u_n=\sum_{k=1}^{n}c_{n+1,k}=\sum_{k\in J_{n+1}\setminus\{n+1\}}c_{n+1,k}\le mM$$

for all $n\ge 1$. ■

Proposition 3.11: *Every averaging matrix satisfying the generalized moving average condition is H-concentrating.*

Proof: Suppose that $A := [a_{n,k}]_{n,k=1}^{\infty}$ is an averaging matrix satisfying the generalized moving average condition. Suppose that $\{s_n\}_{n=1}^{\infty}$ is a sequence of nonnegative real numbers, $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{t_n\}_{n=1}^{\infty}$ is a sequence of real numbers with $\limsup_{n\to\infty} t_n \le 0$ and

$$\begin{aligned}\bar{s}_n &:= \sum_{k=1}^{n} a_{n,k}s_k \\ s_{n+1} &\le (1-\alpha_n)\bar{s}_n + \alpha_n t_n\end{aligned}$$

for all $n \ge 1$. By Remark 3.2, we assume that $s_n \le M < \infty$ for all $n \ge 1$. Let $c_{n,k} := \sum_{i=n}^{\infty} a_{i,k}$ and $u_n := \sum_{k=1}^{n} c_{n+1,k}s_k$. By Lemma 3.10, we have $\{c_{n,k}\}_{n,k=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are bounded above. Obviously, $c_{n,k} - c_{n+1,k} = a_{n,k}$ and $c_{n,n} - 1 \le \rho_n$. Since

$$\begin{aligned}s_{n+1} &\le (1-\alpha_n)\Big(\sum_{k=1}^{n} a_{n,k}s_k\Big) + \alpha_n t_n \\ &= (1-\alpha_n)\Big(\sum_{k=1}^{n} (c_{n,k} - c_{n+1,k})s_k\Big) + \alpha_n t_n,\end{aligned}$$

we obtain

$$\begin{aligned}s_{n+1} + u_n &= s_{n+1} + \sum_{k=1}^{n} c_{n+1,k}s_k \\ &\le (1-\alpha_n)\Big(\sum_{k=1}^{n} c_{n,k}s_k\Big) + \alpha_n\Big(t_n + \sum_{k=1}^{n} c_{n+1,k}s_k\Big) \\ &= (1-\alpha_n)(s_n + u_{n-1}) + \alpha_n(t_n + u_n) + (1-\alpha_n)(c_{n,n} - 1)s_n \\ &\le (1-\alpha_n)(s_n + u_{n-1}) + \alpha_n(t_n + u_n) + \rho_n M.\end{aligned}$$

Since $\sum_{k=1}^{\infty} \rho_k < \infty$ and $\limsup_{n\to\infty} t_n \le 0$, it follows from Lemma 2.2 that

$$\begin{aligned}\liminf_{n\to\infty} s_n + \limsup_{n\to\infty} u_n &= \liminf_{n\to\infty} s_n + \limsup_{n\to\infty} u_{n-1} \\ &\le \limsup_{n\to\infty}(s_n + u_{n-1}) \\ &\le \limsup_{n\to\infty}(t_n + u_n) \le \limsup_{n\to\infty} u_n.\end{aligned}$$

In particular, $\liminf_{n\to\infty} s_n = 0$. It follows from Lemma 3.9 that $\lim_{n\to\infty} s_n = 0$. Hence A is H-concentrating. This completes the proof. ■

3.2.3. Some concrete examples

Note that the generalized segmenting condition and the generalized moving average condition are independent as shown in the following two examples.

Example 3.12: Let $A := [a_{n,k}]_{n,k=1}^{\infty}$ be defined by

$$a_{n,k} := \begin{cases} 1 & \text{if } n = k = 1; \\ 0 & \text{if } n \geq 1 \quad \text{and} \quad k > n; \\ a_{n-1,k}/2 & \text{if } n \geq 2 \quad \text{and} \quad k < n; \\ 1/2 & \text{if } n \geq 2 \quad \text{and} \quad k = n. \end{cases}$$

That is,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 1/4 & 1/4 & 1/2 & 0 & 0 & \cdots \\ 1/8 & 1/8 & 1/4 & 1/2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then A is an averaging matrix satisfying the generalized segmenting condition but not the generalized moving average condition.

Example 3.13: Let $A' := [a_{n,k}]_{n,k=1}^{\infty}$ be defined by

$$a_{n,k} := \begin{cases} 1 & \text{if } n = k = 1; \\ 0 & \text{if } n \geq 1 \quad \text{and} \quad k > n; \\ 0 & \text{if } n \geq 2 \quad \text{and} \quad k = 1, 2, \ldots, n-2; \\ 1/2 & \text{if } n \geq 2 \quad \text{and} \quad k = n-1, n. \end{cases}$$

That is,

$$A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 0 & 1/2 & 1/2 & 0 & 0 & \cdots \\ 0 & 0 & 1/2 & 1/2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then A' is an averaging matrix satisfying the generalized moving average condition but not the generalized segmenting condition.

Remark 3.3: The question naturally arises whether the classes of CP-concentrating matrices and H-concentrating matrices are equal.

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Another look at Wang's new method for solving split common fixed-point problems without priori knowledge of operator norms

Rapeepan Kraikaew and Satit Saejung

Abstract. In this paper, we give a simple proof of Wang's recent result concerning split common fixed-point problems (F. Wang, J Fixed Point Theory Appl 19(4): 2427–2436, 2017). Moreover, we provide a more general sufficient condition than Wang's for the weak convergence to a solution of a split common fixed-point problem.

Mathematics Subject Classification. Primary 47J25; Secondary 47J20, 49N45, 65J15.

Keywords. Split common fixed-point problem, firmly nonexpansive mapping, Hilbert space.

1. Introduction

The split feasibility problem (SFP) which was first introduced by Censor and Elfving [3] is to find

$$\widehat{x} \in C \quad \text{such that} \quad L\widehat{x} \in Q,$$

where C and Q are closed convex subsets of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $L : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. The SFP has received much attention, due to its applications in signal processing and image reconstruction.

Suppose that P_C and P_Q are the (orthogonal) projections onto the sets C and Q, respectively. Assuming that SFP has a solution, it is not difficult to see that $\widehat{x} \in \mathcal{H}_1$ solves the SFP if and only if it solves the fixed-point equation

$$\widehat{x} = P_C(I - \gamma L^*(I - P_Q)L)\widehat{x},$$

where $\gamma > 0$ is any constant, I is the identity operator, and L^* is the adjoint of L. Byrne [1] proposed the following algorithm: $x_1 \in \mathcal{H}_1$ is arbitrarily chosen

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and for all $n \geq 1$,

$$x_{n+1} = P_C(x_n - \gamma L^*(I - P_Q)Lx_n),$$

to approximate a solution of the SFP. In fact, it was proved that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional, then $\{x_n\}$ converges to a solution of the SFP provided that the SFP is consistent and $\gamma \in \big(0, 2/\|L\|^2\big)$.

We now reformulate the SFP into the so-called split common fixed-point problem (SCFP), that is, the problem of finding

$$\widehat{x} \in \operatorname{Fix}(U) \text{ such that } L\widehat{x} \in \operatorname{Fix}(T),$$

where $U : \mathcal{H}_1 \to \mathcal{H}_1$ and $T : \mathcal{H}_2 \to \mathcal{H}_2$ are two mappings with nonempty fixed-point sets $\operatorname{Fix}(U) := \{x \in \mathcal{H}_1 : x = Ux\}$ and $\operatorname{Fix}(T) := \{y \in \mathcal{H}_2 : y = Ty\}$, respectively, and $L : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. If $U := P_C$ and $T := P_Q$, then $\operatorname{Fix}(U) = C$ and $\operatorname{Fix}(T) = Q$ and hence the SCFP immediately reduces to the SFP. It should be noted here that Byrne et al. [2] showed that the SFP can be reformulated into the split common null point problem. Censor and Segal [4] proposed and proved the convergence of the following algorithm: $x_1 \in \mathcal{H}_1$ is arbitrarily chosen and for all $n \geq 1$,

$$x_{n+1} = U(x_n - \gamma L^*(I - T)Lx_n)$$

where U and T are firmly nonexpansive mappings (see the definition in Sect. 2) in the finite dimensional setting and $\gamma \in \big(0, 2/\|L\|^2\big)$. To implement this algorithm, we have to know or estimate the operator norm of the bounded linear operator L. However, the computation (or the estimate) of $\|L\|$ is not an easy task. To overcome this drawback, many variable step sizes without the prior information about $\|L\|$ have been constructed (see [7]). Following the idea of Yang [9], Wang [8] proposed the following method for the SCFP.

Theorem 1.1. *Let $U : \mathcal{H}_1 \to \mathcal{H}_1$, $T : \mathcal{H}_2 \to \mathcal{H}_2$ be firmly nonexpansive mappings and $L : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Define*

$$\begin{cases} x_1 \in \mathcal{H}_1 \\ x_{n+1} = (I - \rho_n(I - U + L^*(I - T)L))x_n, \end{cases}$$

where $\{\rho_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \rho_n = \infty$ and $\sum_{n=1}^{\infty} \rho_n^2 < \infty$. If $\Omega := \{x \in \mathcal{H}_1 : x \in \operatorname{Fix}(U)$ and $Lx \in \operatorname{Fix}(T)\} \neq \varnothing$, then $x_n \rightharpoonup z \in \Omega$. Furthermore, $z = \lim_{n\to\infty} P_\Omega x_n$.

The purpose of this paper is to give a short and simple proof of Wang's result and provide a weaker sufficient condition on $\{\rho_n\}$ which does not require the computation of the operator norms.

2. Results

We first recall some definitions concerning our result. Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. For a closed convex subset C of $\mathcal{H}$, the (orthogonal) projection $P_C : \mathcal{H} \to C$ is defined for each $x \in \mathcal{H}$ as the unique element $P_C x \in C$ such that

$$\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}.$$

Definition 2.1. An operator $T : \mathcal{H} \to \mathcal{H}$ is called

- firmly nonexpansive if, for all $x, y \in \mathcal{H}$,
$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2;$$
- α-inverse strongly monotone where $\alpha > 0$ if αT is firmly nonexpansive, that is, for all $x, y \in \mathcal{H}$,
$$\langle x - y, Tx - Ty\rangle \geq \alpha\|Tx - Ty\|^2.$$

Remark 2.2. Every projection is firmly nonexpansive.

More information concerning firmly nonexpansive mappings and, in particular, nearest point projections can be found in the book by Goebel and Reich [5].

Let us recall the following result of Groetsch [6].

Theorem 2.3. *Let $\mathcal{H}$ be a Hilbert space and let $S : \mathcal{H} \to \mathcal{H}$ be a firmly nonexpansive mapping such that* $\mathrm{Fix}(S) := \{x \in \mathcal{H} : x = Sx\} \neq \varnothing$. *Define*

$$\begin{cases} x_1 \in \mathcal{H} \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S x_n. \end{cases}$$

If $\{\alpha_n\}$ is a sequence in $[0, 2]$ such that

$$\sum_{n=1}^{\infty} \alpha_n(2 - \alpha_n) = \infty,$$

then the sequence $\{x_n\}$ converges weakly to an element $z \in \mathrm{Fix}(S)$ and $z = \lim_{n\to\infty} P_{\mathrm{Fix}(S)} x_n$.

Lemma 2.4. *Let $\mathcal{H}$ be a Hilbert space. A mapping $A : \mathcal{H} \to \mathcal{H}$ is α-inverse strongly monotone if and only if $I - \alpha A$ is firmly nonexpansive.*

Lemma 2.5. *If a, b, c are positive real numbers, then*

$$a^2 + \frac{b^2}{c} \geq \frac{(a + b)^2}{1 + c}.$$

Lemma 2.6. *Let $U : \mathcal{H}_1 \to \mathcal{H}_1$, $T : \mathcal{H}_2 \to \mathcal{H}_2$ be firmly nonexpansive mappings and let $L : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Suppose that $\alpha := \frac{1}{1+\|L\|^2}$ and $A := I - U + L^*(I - T)L$. The following statements are true.*

(a) *$I - \alpha A$ is firmly nonexpansive;*
(b) *if $\Omega := \{x \in \mathcal{H}_1 : x \in \mathrm{Fix}(U)$ and $Lx \in \mathrm{Fix}(T)\} \neq \varnothing$, then $\mathrm{Fix}(I - \alpha A) = \Omega$.*

Proof. (a) By Lemma 2.4, it suffices to show that A is α-inverse strongly monotone. Suppose that $x, y \in \mathcal{H}_1$. It follows from $\|L^*\| = \|L\|$ and Lemma 2.5 that

$$\begin{aligned}\langle x - y, Ax - Ay\rangle &= \langle x - y, (I - U)x - (I - U)y\rangle + \langle Lx - Ly, (I - T)Lx - (I - T)Ly\rangle \\ &\geq \|(I - U)x - (I - U)y\|^2 + \|(I - T)Lx - (I - T)Ly\|^2\end{aligned}$$

$$\begin{aligned}
&\geq \|(I-U)x-(I-U)y\|^2+\frac{\|L^*(I-T)Lx-L^*(I-T)Ly\|^2}{\|L^*\|^2}\\
&\geq \frac{(\|(I-U)x-(I-U)y\|+\|L^*(I-T)Lx-L^*(I-T)Ly\|)^2}{1+\|L\|^2}\\
&\geq \alpha\|Ax-Ay\|^2.
\end{aligned}$$

(b) Obviously, $\Omega \subset \mathrm{Fix}(I-\alpha A)$. Suppose that $\Omega \neq \varnothing$. To prove that $\mathrm{Fix}(I-\alpha A) \subset \Omega$, let $z \in \mathrm{Fix}(I-\alpha A)$ and $p \in \Omega$. This implies that $Az = Ap = 0$. It follows from (a) that

$$\begin{aligned}
0 &= \langle z-p, Az-Ap\rangle\\
&\geq \|(I-U)z-(I-U)p\|^2+\|(I-T)Lz-(I-T)Lp\|^2\\
&= \|(I-U)z\|^2+\|(I-T)Lz\|^2.
\end{aligned}$$

In particular, $z \in \mathrm{Fix}(U)$ and $Lz \in \mathrm{Fix}(T)$. This means that $z \in \Omega$ and the proof is finished. □

Now we present the following result which is an improvement of Theorem 1.1.

Theorem 2.7. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Let $U : \mathcal{H}_1 \to \mathcal{H}_1$, $T : \mathcal{H}_2 \to \mathcal{H}_2$ be firmly nonexpansive mappings and $L : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Define*

$$\begin{cases} x_1 \in \mathcal{H} \\ x_{n+1} = (I-\rho_n(I-U+L^*(I-T)L))x_n, \end{cases}$$

where $\{\rho_n\}$ is a sequence in $\left[0, 2/(1+\|L\|^2)\right]$ such that

$$\sum_{n=1}^{\infty} \rho_n(2-(1+\|L\|^2)\rho_n) = \infty.$$

Suppose that $\Omega := \{x \in \mathcal{H}_1 : x \in \mathrm{Fix}(U) \text{ and } Lx \in \mathrm{Fix}(T)\} \neq \varnothing$. Then the sequence $\{x_n\}$ converges weakly to $z \in \Omega$. Furthermore, $z = \lim_{n\to\infty} P_\Omega x_n$.

Proof. We write $A := I-U+L^*(I-T)L$. It follows from Lemma 2.6 that $I-\alpha A$ is firmly nonexpansive where $\alpha := 1/(1+\|L\|^2)$. Moreover,

$$x_{n+1} = (I-\rho_n A)x_n = \left(1-\frac{\rho_n}{\alpha}\right)x_n+\frac{\rho_n}{\alpha}(I-\alpha A)x_n.$$

As a consequence of Theorem 2.3, we obtain the result. □

Remark 2.8. Our Theorem 2.7 recovers the result of Wang in the following ways:

1. If $\{\rho_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \rho_n = \infty$ and $\sum_{n=1}^{\infty} \rho_n^2 < \infty$, then the sequence $\{\rho_n\}$ ultimately sits in the interval $\left[0, 2/(1+\|L\|^2)\right]$ and it satisfies the condition $\sum_{n=1}^{\infty} \rho_n(2 - (1+\|L\|^2)\rho_n) = \infty$. Hence, our Theorem 2.7 implies Wang's result (Theorem 1.1).
2. Wang's result remains true if $\{\rho_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \rho_n = \infty$ and $\lim_{n\to\infty} \rho_n = 0$. This condition does not require the computation of $\|L\|$ and it is strictly weaker than Wang's condition.

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Aequationes Mathematicae

The hyperstability of general linear equation via that of Cauchy equation

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Abstract. In this paper, we show that the hyperstability of the general linear equation recently proved by Piszczek (Aequationes Math 88:163–168, 2014) is a direct consequence of that of the Cauchy equation proved earlier by Brzdęk (Acta Math Hung 141:58–67, 2013).

Mathematics Subject Classification. Primary 39B82, 39B62; Secondary 47H14, 47J20.

Keywords. Hyperstability, General linear equation, Cauchy equation.

1. Introduction

One of the many interesting questions in the theory of functional equations is the hyperstability problem: When is it true that a function which approximately satisfies a functional equation must also be a solution of the functional equation?

In this paper, we assume that $\mathbb{F}, \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ where $\mathbb{R}$ and $\mathbb{C}$ are the sets of all real numbers and complex numbers, respectively; and we assume that $\mathbb{N}$ is the set of all positive integers. Suppose that X and Y are normed spaces over the scalar fields $\mathbb{F}$ and $\mathbb{K}$, respectively. We are interested in the following functional equation. A function $f : X \to Y$ satisfies the *general linear equation* if there exist constants $a, b \in \mathbb{F} \setminus \{0\}$ and $A, B \in \mathbb{K}$ such that

$$f(ax + by) = Af(x) + Bf(y) \quad \text{for all } x, y \in X. \tag{1.1}$$

In particular, we say that f satisfies the *Cauchy equation* if (1.1) holds where $a = b = A = B = 1$; the *Jensen equation* if (1.1) holds where $a = b = A = B = 1/2$. It is known that a function $f : X \to Y$ with $f(0) = 0$ satisfies the Jensen equation if and only if it satisfies the Cauchy equation [7,8]. In particular, it follows that if a function $f : X \to Y$ satisfies the Jensen equation, then the odd part f_o of f satisfies the Cauchy equation and the even part f_e of f is a

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constant. Recall that $f_o(x) := \frac{1}{2}(f(x) - f(-x))$ and $f_e(x) := \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$.

Piszczek [9] recently proved the following hyperstability result for general linear equations.

Theorem P. [9, Theorem 2] *Suppose that X is a normed space and Y is Banach space over the scalar fields $\mathbb{F}$ and $\mathbb{K}$, respectively. Suppose that $a, b \in \mathbb{F} \setminus \{0\}$ and $A, B \in \mathbb{K}$. Suppose that $c \geq 0$, $p < 0$, and $f : X \to Y$ satisfies the following condition:*

$$\|f(ax + by) - Af(x) - Bf(y)\| \leq c(\|x\|^p + \|y\|^p)$$

for all $x, y \in X \setminus \{0\}$. Then f satisfies the equation

$$f(ax + by) = Af(x) + Bf(y) \quad \text{for all} \quad x, y \in X \setminus \{0\}.$$

Remark 1.1. Note that the completeness of Y in Theorem P can be relaxed, that is, the conclusion of Theorem P remains true if Y is a normed space.

The method of the proof of Theorem P given in [9] is based on the fixed point theorem of Brzdęk [4]. In this paper, we provide a simple and direct proof of Theorem P via the following hyperstability of the Cauchy equation which was proved by Brzdęk [5].

Theorem B. [5, Theorem 1.2] *Let X and Y be normed spaces, $c \geq 0$, and $p < 0$. Suppose that $E := X \setminus \{0\}$. If a function $g : X \to Y$ satisfies*

$$\|g(x + y) - g(x) - g(y)\| \leq c(\|x\|^p + \|y\|^p),$$

for all $x, y \in E$ with $x + y \in E$, then g satisfies the Cauchy equation on E, that is,

$$g(x + y) = g(x) + g(y)$$

for all $x, y \in E$ with $x + y \in E$.

2. Main results

The following result tells us that if a function *approximately* satisfies the general linear equation, then its odd part *approximately* satisfies the Cauchy equation and its even part is a constant.

Theorem 2.1. *Let X and Y be normed spaces over the scalar fields $\mathbb{F}$ and $\mathbb{K}$, respectively. Let $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, $c \geq 0$, and $p < 0$. Suppose that $f : X \to Y$ satisfies the inequality*

$$\|f(ax + by) - Af(x) - Bf(y)\| \leq c(\|x\|^p + \|y\|^p)$$

for all $x, y \in X \setminus \{0\}$. Then the odd and even parts $f_o, f_e : X \to Y$ of f satisfy the following conditions:

(a) $f(x) = f_o(x) + f_e(x)$, $f_o(-x) = -f_o(x)$, *and* $f_e(-x) = f_e(x)$ *for all* $x \in X$;

(b) $f_o(0) = 0$ *and* $f_e(0) = f(0)$;

(c) $\left\|f_o(x) - 2Af_o\left(\frac{x}{2a}\right)\right\| \leq 2\alpha\|x\|^p$ *and* $\left\|f_o(x) - 2Bf_o\left(\frac{x}{2b}\right)\right\| \leq 2\alpha\|x\|^p$ *for all* $x \in X \setminus \{0\}$ *where* $\alpha := c\left(\frac{1}{|2a|^p} + \frac{1}{|2b|^p}\right)$;

(d) $\|f_o(x+y) - f_o(x) - f_o(y)\| \leq C(\|x\|^p + \|y\|^p)$ *for all* $x, y \in X \setminus \{0\}$ *where* $C := 2\alpha(1 + 2^{p+1} + 3^p + 4^p)$;

(e) $f_e(x) = f(0)$ *for all* $x \in X$; *and* $f_e(0) = Af_e(0) + Bf_e(0)$.

Proof. It follows immediately that (a) and (b) hold.

We now prove (c) and (d). Note that

$$\begin{aligned}
&\|f_o(ax + by) - Af_o(x) - Bf_o(y)\| \\
&= \frac{1}{2}\|f(ax+by)-f(-ax - by) - A(f(x) - f(-x)) - B(f(y) - f(-y))\| \\
&\leq \frac{1}{2}\|f(ax+by)-Af(x) - Bf(y)\|+\frac{1}{2}\|f(-ax - by) - Af(-x) - Bf(-y)\| \\
&\leq \frac{c}{2}(\|x\|^p + \|y\|^p) + \frac{c}{2}(\|x\|^p + \|y\|^p) \\
&= c(\|x\|^p + \|y\|^p) \qquad (2.1)
\end{aligned}$$

for all $x, y \in X \setminus \{0\}$. Let $x \in X \setminus \{0\}$. Replacing x by $\frac{x}{2a}$ and y by $\frac{x}{2b}$ in (2.1), we get

$$\left\|f_o(x) - Af_o\left(\frac{x}{2a}\right) - Bf_o\left(\frac{x}{2b}\right)\right\| \leq \alpha\|x\|^p \qquad (2.2)$$

where $\alpha := c\left(\frac{1}{|2a|^p} + \frac{1}{|2b|^p}\right)$. Next, replacing x by $\frac{x}{2a}$ and y by $-\frac{x}{2b}$ in (2.1), it follows from (2.1), (a), and (b) that

$$\left\|-Af_o\left(\frac{x}{2a}\right) + Bf_o\left(\frac{x}{2b}\right)\right\| \leq \alpha\|x\|^p. \qquad (2.3)$$

Then (2.2) and (2.3) imply that

$$\begin{aligned}
\left\|f_o(x) - 2Af_o\left(\frac{x}{2a}\right)\right\| &\leq 2\alpha\|x\|^p; \\
\left\|f_o(x) - 2Bf_o\left(\frac{x}{2b}\right)\right\| &\leq 2\alpha\|x\|^p
\end{aligned}$$

for all $x \in X \setminus \{0\}$ and so we get (c). If $x, y \in X \setminus \{0\}$, then it follows from (2.1) and (c) that

$$\begin{aligned}
&\left\|f_o\left(\frac{1}{2}x + \frac{1}{2}y\right) - \frac{1}{2}f_o(x) - \frac{1}{2}f_o(y)\right\| \\
&\leq \left\|f_o\left(a\frac{x}{2a} + b\frac{y}{2b}\right) - Af_o\left(\frac{x}{2a}\right) - Bf_o\left(\frac{y}{2b}\right)\right\| \\
&\quad + \left\|Af_o\left(\frac{x}{2a}\right) - \frac{1}{2}f_o(x)\right\| + \left\|Bf_o\left(\frac{y}{2b}\right) - \frac{1}{2}f_o(y)\right\|
\end{aligned}$$

$$\le c\left(\left\|\frac{x}{2a}\right\|^p + \left\|\frac{y}{2b}\right\|^p\right) + \alpha\|x\|^p + \alpha\|y\|^p$$
$$\le 2\alpha(\|x\|^p + \|y\|^p). \tag{2.4}$$

Let $x \in X \setminus \{0\}$. Replacing x by $2x$ and y by $4x$ in (2.4), we get

$$\left\|f_o(3x) - \frac{1}{2}f_o(2x) - \frac{1}{2}f_o(4x)\right\| \le 2\alpha(2^p + 4^p)\|x\|^p. \tag{2.5}$$

Next, replacing x by $-2x$ and y by $4x$ in (2.4), we get

$$\left\|f_o(x) + \frac{1}{2}f_o(2x) - \frac{1}{2}f_o(4x)\right\| \le 2\alpha(2^p + 4^p)\|x\|^p. \tag{2.6}$$

Then (2.5) and (2.6) imply

$$\|f_o(3x) - f_o(x) - f_o(2x)\| \le 4\alpha(2^p + 4^p)\|x\|^p. \tag{2.7}$$

Note that (2.4) with $y = 3x$ becomes

$$\left\|f_o(2x) - \frac{1}{2}f_o(x) - \frac{1}{2}f_o(3x)\right\| \le 2\alpha(1 + 3^p)\|x\|^p. \tag{2.8}$$

Then (2.7) and (2.8) imply

$$\|f_o(2x) - 2f_o(x)\| \le 4\alpha(1 + 2^p + 3^p + 4^p)\|x\|^p.$$

Consequently, if $x, y \in X \setminus \{0\}$, then it follows from (2.4) that

$$\begin{aligned}
&\|f_o(x + y) - f_o(x) - f_o(y)\| \\
&\le \left\|f_o\left(\frac{1}{2}2x + \frac{1}{2}2y\right) - \frac{1}{2}f_o(2x) - \frac{1}{2}f_o(2y)\right\| \\
&\quad + \left\|\frac{1}{2}f_o(2x) - f_o(x)\right\| + \left\|\frac{1}{2}f_o(2y) - f_o(y)\right\| \\
&\le 2^{p+1}\alpha(\|x\|^p + \|y\|^p) + 2\alpha(1 + 2^p + 3^p + 4^p)\|x\|^p \\
&\quad + 2\alpha(1 + 2^p + 3^p + 4^p)\|y\|^p \\
&= C(\|x\|^p + \|y\|^p)
\end{aligned}$$

where $C := 2\alpha(1 + 2^{p+1} + 3^p + 4^p)$. Hence (d) is proved.

Next we prove (e). By the definition of f_e, we get

$$\begin{aligned}
&\|f_e(ax + by) - Af_e(x) - Bf_e(y)\| \\
&= \frac{1}{2}\|f(ax + by) + f(-ax - by) - A(f(x) + f(-x)) - B(f(y) + f(-y))\| \\
&\le \frac{1}{2}\|f(ax+by) - Af(x) - Bf(y)\| + \frac{1}{2}\|f(-ax - by) - Af(-x) - Bf(-y)\| \\
&\le \frac{c}{2}(\|x\|^p + \|y\|^p) + \frac{c}{2}(\|x\|^p + \|y\|^p) \\
&= c(\|x\|^p + \|y\|^p)
\end{aligned} \tag{2.9}$$

for all $x, y \in X \setminus \{0\}$. It follows from (2.9) that

$$\begin{aligned}
&\|f_e(x) - f_e(0)\| \\
&\quad \leq \left\| f_e(x) - Af_e\left(\frac{x}{2a}\right) - Bf_e\left(\frac{x}{2b}\right) \right\| + \left\| f_e(0) - Af_e\left(-\frac{x}{2a}\right) - Bf_e\left(\frac{x}{2b}\right) \right\| \\
&\quad \leq 2\alpha \|x\|^p
\end{aligned}$$

for all $x \in X \setminus \{0\}$. In particular, for each $n \in \mathbb{N}$ and for each $x \in X \setminus \{0\}$, we have

$$\|f_e(nx) - f_e(0)\| \leq 2\alpha \|nx\|^p = 2\alpha n^p \|x\|^p.$$

Since $p < 0$, we have

$$\lim_{n\to\infty} f_e(nx) = f_e(0)$$

for all $x \in X \setminus \{0\}$. Let $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$. Replacing x by $\frac{(n+2)x}{2a}$ and y by $-\frac{nx}{2b}$ in (2.9), we get

$$\left\| f_e(x) - Af_e\left(\frac{(n+2)x}{2a}\right) - Bf_e\left(\frac{-nx}{2b}\right) \right\| \leq c\left(\left|\frac{n+2}{2a}\right|^p + \left|\frac{n}{2b}\right|^p \right) \|x\|^p.$$

In particular,

$$\lim_{n\to\infty} \left\| f_e(x) - Af_e\left(\frac{(n+2)x}{2a}\right) - Bf_e\left(\frac{-nx}{2b}\right) \right\| = 0.$$

Since $\lim_{n\to\infty} f_e\left(\frac{(n+2)x}{2a}\right) = \lim_{n\to\infty} f_e\left(\frac{-nx}{2b}\right) = f_e(0)$, we have $f_e(x) = Af_e(0) + Bf_e(0)$. Moreover, (2.9) with $x = \frac{nx}{2a}$ and $y = \frac{-nx}{2b}$, gives

$$\left\| f_e(0) - Af_e\left(\frac{nx}{2a}\right) - Bf_e\left(\frac{-nx}{2b}\right) \right\| \leq \alpha n^p \|x\|^p.$$

Letting $n \to \infty$, we obtain that $f_e(0) = Af_e(0) + Bf_e(0)$. Then $f_e(x) = f_e(0) = Af_e(0) + Bf_e(0)$ for all $x \in X$ and hence (e) is proved. $\square$

It is known [6] that if X and Y are normed spaces and $g : X \to Y$ is a function satisfying the Cauchy equation for all $x, y \in X \setminus \{0\}$, then it satisfies the Cauchy equation. In fact, this result is true even in a more general setting (see [1–3]).

Remark 2.2. Let $X, Y, \mathbb{F}, \mathbb{K}, a, b, A, B, c, p$, and f be the same as in Theorem 2.1. Then f_o satisfies the following conditions:

(a) $f_o(x+y) = f_o(x) + f_o(y)$ for all $x, y \in X$;
(b) $f_o(ax) = Af_o(x)$ and $f_o(bx) = Bf_o(x)$ for all $x \in X$.

Proof. (a) Theorem 2.1(d) and Theorem B imply that

$$f_o(x+y) = f_o(x) + f_o(y)$$

for all $x, y \in X \setminus \{0\}$ with $x + y \in X \setminus \{0\}$. Since $f_o(0) = 0$, the function f_o satisfies the Cauchy equation for all x and y different from zero. In particular, the statement (a) holds.

To prove (b), let $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$. By condition (a) of f_o and Theorem 2.1(c), we get $f_o(nx) = nf_o(x)$ and

$$n \left\| f_o(x) - 2Af_o\left(\frac{x}{2a}\right) \right\| = \left\| f_o(nx) - 2Af_o\left(\frac{nx}{2a}\right) \right\| \leq 2\alpha\|nx\|^p = 2\alpha n^p \|x\|^p.$$

That is,

$$\left\| f_o(x) - 2Af_o\left(\frac{x}{2a}\right) \right\| \leq 2\alpha n^{p-1}\|x\|^p.$$

Letting $n \to \infty$, we get that

$$f_o(x) = 2Af_o\left(\frac{x}{2a}\right).$$

Moreover,

$$f_o(ax) = Af_o\left(\frac{x}{2}\right) + Af_o\left(\frac{x}{2}\right) = A\left(f_o\left(\frac{x}{2} + \frac{x}{2}\right)\right) = Af_o(x)$$

for all $x \in X \setminus \{0\}$. Since $f_o(0) = 0$, we now conclude that $f_o(ax) = Af_o(x)$ for all $x \in X$. Similarly, we can prove that $f_o(bx) = Bf_o(x)$ for all $x \in X$. Therefore, (b) is proved. □

By using Theorem 2.1 and Remark 2.2, we prove Theorem P via the hyperstability of the Cauchy equation.

A simple proof of Theorem P. By Theorem 2.1 and Remark 2.2, the odd and even parts $f_o, f_e : X \to Y$ of f satisfy the following conditions

- $f_o(x + y) = f_o(x) + f_o(y)$ for all $x, y \in X$;
- $f(x) = f_o(x) + f_e(x)$, $f_e(x) = Af_e(0) + Bf_e(0)$, $f_o(ax) = Af_o(x)$, and $f_o(bx) = Bf_o(x)$ for all $x \in X$.

Let $x, y \in X$. Then

$$\begin{aligned} f(ax + by) &= f_o(ax + by) + f_e(ax + by) \\ &= f_o(ax) + f_o(by) + f_e(0) \\ &= Af_o(x) + Bf_o(y) + Af_e(0) + Bf_e(0) \\ &= Af_o(x) + Bf_o(y) + Af_e(x) + Bf_e(y) \\ &= A(f_o(x) + f_e(x)) + B(f_o(y) + f_e(y)) \\ &= Af(x) + Bf(y). \end{aligned}$$

This completes the proof. □

3. Some further remarks

We end the paper with the following remark which explains a relation between the approximate general linear equation and the approximate Cauchy equation.

Theorem 3.1. *Let X and Y be normed spaces over the scalar fields $\mathbb{F}$ and $\mathbb{K}$, respectively. Let $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, $p \in \mathbb{R}$, and $\beta \in Y$. Suppose that $\beta = A\beta + B\beta$ and $g : X \to Y$ is a function such that there exists a constant $C \geq 0$ so that the following statements are true.*

(a) $\|g(x+y) - g(x) - g(y)\| \leq C(\|x\|^p + \|y\|^p)$ *for all* $x, y \in X \setminus \{0\}$.

(b) $\left\|g(x) - 2Ag\left(\frac{x}{2a}\right)\right\| \leq C\|x\|^p$ *and* $\left\|g(x) - 2Bg\left(\frac{x}{2b}\right)\right\| \leq C\|x\|^p$ *for all* $x \in X \setminus \{0\}$.

If $f : X \to Y$ is defined by $f(x) = g(x) + \beta$ for all $x \in X$, then there exists a constant $c \geq 0$ such that

$$\|f(ax+by) - Af(x) - Bf(y)\| \leq c(\|x\|^p + \|y\|^p)$$

for all $x, y \in X \setminus \{0\}$.

Proof. Let $x, y \in X \setminus \{0\}$. Note that

$$\begin{aligned}\|g(ax) - Ag(x)\| &\leq \left\|g(ax) - 2Ag\left(\frac{x}{2}\right)\right\| + |A|\left\|2g\left(\frac{x}{2}\right) - g(x)\right\| \\ &\leq C\|ax\|^p + 2|A|C\left\|\frac{x}{2}\right\|^p \\ &= C\left(|a|^p + \frac{2|A|}{2^p}\right)\|x\|^p.\end{aligned}$$

Similarly, we have

$$\|g(by) - Bg(y)\| \leq C\left(|b|^p + \frac{2|B|}{2^p}\right)\|y\|^p.$$

It follows that

$$\begin{aligned}&\|f(ax+by) - Af(x) - Bf(y)\| \\ &\quad = \|g(ax+by) + \beta - A(g(x)+\beta) - B(g(y)+\beta)\| \\ &\quad = \|g(ax+by) - Ag(x) - Bg(y)\| \\ &\quad \leq \|g(ax+by) - g(ax) - g(by)\| + \|g(ax) - Ag(x)\| + \|g(by) - Bg(y)\| \\ &\quad \leq C(\|ax\|^p + \|by\|^p) + C\left(|a|^p + \frac{2|A|}{2^p}\right)\|x\|^p + C\left(|b|^p + \frac{2|B|}{2^p}\right)\|y\|^p \\ &\quad \leq c(\|x\|^p + \|y\|^p)\end{aligned}$$

where $c := 2C\left(|a|^p + \frac{|A|}{2^p} + |b|^p + \frac{|B|}{2^p}\right)$. □

Note that if we set $c := 0$ in Theorem 2.1 and $C := 0$ in Theorem 3.1, then we immediately obtain the following result of Piszczek [10].

Corollary 3.2. [10, Theorem 1.2] *Let X and Y be normed spaces over the scalar fields $\mathbb{F}$ and $\mathbb{K}$, respectively. Let $a, b \in \mathbb{F}\setminus\{0\}$ and $A, B \in \mathbb{K}$. Then the following statements are true.*

(a) *If a function $f : X \to Y$ satisfies*

$$f(ax + by) = Af(x) + Bf(y) \quad \text{for all } x, y \in X \setminus \{0\},$$

then there exists a function $g : X \to Y$ satisfying the Cauchy equation and

$$g(ax) = Ag(x) \text{ and } g(bx) = Bg(x) \quad \text{for all } x \in X$$

and there exists $\beta \in Y$ where $\beta = A\beta + B\beta$ such that

$$f(x) = g(x) + \beta \quad \text{for all } x \in X.$$

(b) *If a function $g : X \to Y$ satisfies the Cauchy equation and*

$$g(ax) = Ag(x) \text{ and } g(bx) = Bg(x) \quad \text{for all } x \in X;$$

and $\beta \in Y$ is a scalar such that $\beta = A\beta+B\beta$, and if a function $f : X \to Y$ is defined by

$$f(x) := g(x) + \beta \quad \text{for all } x \in X,$$

then f satisfies the general linear equation.

Proof. (a) It follows from our Theorem 2.1 where $c := 0$ that the desired function g is the odd part of f and β is the even part of f. Hence (a) holds.

(b) is obvious. □

Remark 3.3. The proof of Corollary 3.2(a) is slightly different from the one originally given in [10, Theorem 1.2].

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Ulam stability with respect to a directed graph for some fixed point equations

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ABSTRACT. In this paper, we introduce a new concept of Ulam stability of fixed point equation with respected to a directed graph. Two fixed point theorems of Matkowski and of Jachymski are discussed further in the sense of this stability concept. Some examples about the validity of our notion are given. Finally, we discuss the vagueness of the recent stability results of Sintunavarat [Sintunavarat, W., *A new approach to α-ψ-contractive mappings and generalized Ulam–Hyers stability, well-posedness and limit shadowing results*, Carpathian J. Math., **31** (2015), 395–401].

1. INTRODUCTION

Ulam posed the following interesting question in 1940 (see also [19]).

> Suppose that $G_1 := (G_1, *)$ and $G_2 := (G_2, \diamond)$ are two groups and $d : G_2 \times G_2 \to [0, \infty)$ is a metric. For a given $\varepsilon > 0$ does there exist a number $\delta := \delta(\varepsilon) > 0$ such that if $f : G_1 \to G_2$ satisfies
> $$d(f(x * y), f(x) \diamond f(y)) \le \delta \text{ for all } x, y \in G_1,$$
> then one can find a homomorphism $F : G_1 \to G_2$ such that $d(f(x), F(x)) \le \varepsilon$ for all $x \in G_1$?

Hyers [6] gave a partial answer to Ulam's question in 1941 where G_1 and G_2 are Banach spaces. In this setting, he also obtained that $\delta(\varepsilon) \le \varepsilon$ for all $\varepsilon > 0$.

There are strict connections between Ulam stability and fixed point theory and for further information we refer to the survey by Brzdęk et al. (see [2]). Ulam's question was reformulated in the context of fixed point equation as follows. For more detail, we refer to the excellent survey by Rus and Şerban [14].

> Suppose that $X := (X, d)$ is a metric space and $T : X \to X$ is given with a fixed point set $\mathrm{Fix}(T) := \{p \in X : p = Tp\}$. For a given $\varepsilon > 0$ does there exist a number $\delta := \delta(\varepsilon) > 0$ such that if $w \in X$ satisfies
> $$d(w, Tw) \le \delta,$$
> then one can find a fixed point $p \in \mathrm{Fix}(T)$ such that $d(p, w) \le \varepsilon$?

If the preceding is true for the mapping T, then we say that the fixed point equation $x = Tx$ is *Ulam stable*. For simplicity from now on, we simply say that T is *Ulam stable* if the fixed point equation $x = Tx$ is Ulam stable. If there exists a constant $c > 0$ such that $\delta(\varepsilon) \le c\varepsilon$ for all $\varepsilon > 0$, then we say that T is *Ulam–Hyers stable*. That is, T is Ulam–Hyers stable if and only if there exists $c > 0$ such that for any pair $(w, \varepsilon) \in X \times (0, \infty)$ with $d(w, Tw) \le \varepsilon$ there exists a fixed point $p \in \mathrm{Fix}(T)$ such that $d(p, w) \le c\varepsilon$. In the literature, the following generalization of Ulam–Hyers stability was introduced.

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Definition 1.1. Let $X := (X, d)$ be a metric space and $T : X \to X$ be given. We say that T is *generalized Ulam–Hyers stable* [14] if there exists an increasing function $\xi : [0, \infty) \to [0, \infty)$ such that $\xi(0) = 0$, ξ is continuous at zero, and for any pair $(w, \varepsilon) \in X \times (0, \infty)$ with $d(w, Tw) \leq \varepsilon$ there exists a fixed point $p \in \mathrm{Fix}(T)$ such that $d(p, w) \leq \xi(\varepsilon)$.

Remark 1.1. If T is generalized Ulam–Hyers stable, then it is Ulam stable. To see this, let $\varepsilon > 0$ be given. Since T is generalized Ulam–Hyers stable, there exists an increasing function $\xi : [0, \infty) \to [0, \infty)$ such that $\xi(0) = 0$, ξ is continuous at zero, and for any pair $(w, \eta) \in X \times (0, \infty)$ with $d(w, Tw) \leq \eta$ there exists a fixed point $p \in \mathrm{Fix}(T)$ such that $d(p, w) \leq \xi(\eta)$. We choose $\eta := \xi^{-1}(\varepsilon) > 0$. It follows that if $w \in X$ satisfies

$$d(w, Tw) \leq \eta,$$

then one can find a fixed point $p \in \mathrm{Fix}(T)$ such that $d(p, w) \leq \xi(\eta) = \varepsilon$. In particular, we have the following implications.

$$\begin{array}{c} \text{Ulam–Hyers} \\ \text{stability} \end{array} \implies \begin{array}{c} \text{generalized} \\ \text{Ulam–Hyers} \\ \text{stability} \end{array} \implies \begin{array}{c} \text{Ulam} \\ \text{stability} \end{array}$$

There are several other types of stability, for more detail we refer to [10] and [4]. It is easy to see that every Banach contraction defined on a complete metric space is Ulam–Hyers stable. Recall that a mapping $T : X \to X$ is a *Banach's contraction* if there exists a constant $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) \text{ for all } x, y \in X.$$

In the literature, there are many generalizations of a Banach's contraction. We are mainly interested in the following two types of generalizations due to Matkowski [9] and to Jachymski [7], respectively.

Matkowski's contractions.

Definition 1.2. Let $X := (X, d)$ be a metric space and $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing function such that $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. A mapping $T : X \to X$ is a *Matkowski's contraction* or *ψ-contraction* if

$$d(Tx, Ty) \leq \psi(d(x, y)) \text{ for all } x, y \in X.$$

Remark 1.2. *If $\psi(t) = \alpha t$ where $\alpha \in (0, 1)$, then a ψ-contraction becomes a Banach's contraction.*

Jachymski's contractions. Recently, Jachymski introduced a class of mappings including all Banach's contractions and proved a fixed point theorem for mappings in this class. From now on, by saying that X is a metric space with a directed graph G on X, we mean that the vertex set $V(G)$ of G is X and the edge set $E(G)$ of G is a subset of the Cartesian product $X \times X$ such that $(x, x) \in E(G)$ for all $x \in X$.

Definition 1.3. Let $X := (X, d)$ be a metric space with a directed graph G on X. A mapping $T : X \to X$ is a *Banach G-contraction* if there exists $\alpha \in (0, 1)$ such that for all $(x, y) \in E(G)$ the following two conditions hold:

- $(Tx, Ty) \in E(G)$;
- $d(Tx, Ty) \leq \alpha d(x, y)$.

Remark 1.3. It is clear that if $E(G) = X \times X$, then a Banach G-contraction becomes a Banach's contraction.

Remark 1.4. We note that not every Banach G-contraction is Ulam stable. See Example 2.1.

In this paper, we introduce a new type of Ulam stability to explain the stability of Banach G-contractions. In fact, our result includes a wider class of mappings whose contractiveness in the sense of Matkowski is given with respect to a directed graph. We also discuss some vague result proved by Sintunavarat [17] concerning the generalized Ulam–Hyers stability. As pointed out by the referee, the results of the paper are related to some simplified versions of outcomes in several other papers (which can be found in the references of the survey [2]) such as Corollary 3.2 in [3].

2. Main results

In this paper we introduce the following concept.

Definition 2.4. Let (X, d) be a metric space with a directed graph G on X and $T : X \to X$ be a mapping such that $\mathrm{Fix}(T) \neq \varnothing$. We say that a mapping $T : X \to X$ is *Ulam stable with respect to G* if for each $\varepsilon > 0$ there is a $\delta := \delta(\varepsilon) > 0$ such the following implication holds:

$$d(w, Tw) \leq \delta \text{ and } (w, Tw) \in E(G) \implies \text{there exists } p \in \mathrm{Fix}(T) \text{ such that } d(p, w) \leq \varepsilon.$$

In the preceding notion, if there exists a constant $c > 0$ such that $\delta(\varepsilon) \leq c\varepsilon$ for all $\varepsilon > 0$, then we say that T is *Ulam–Hyers stable with respect to G*.

Remark 2.5. In particular, if $E(G) := X \times X$, then the Ulam stability with respect to G (Ulam–Hyers stability with respect to G, respectively) becomes the Ulam stability (Ulam–Hyers stability, respectively).

Inspired by the works of Matkowski [9] and of Jachymski [7], we introduce the following mappings.

Definition 2.5. Let (X, d) be a metric space with a directed graph G on X. Define $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function. We say that $T : X \to X$ is

(i) a *(ψ, G)-contraction of type I* if the following conditions hold
- $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$;
- $(Tx, T^2x) \in E(G)$ whenever $(x, Tx) \in E(G)$;
- $d(Tx, Ty) \leq \psi(d(x, y))$ whenever $(x, y) \in E(G)$;

(ii) a *(ψ, G)-contraction of type II* if the following conditions hold
- $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$;
- $(Tx, T^2x) \in E(G)$ whenever $(x, Tx) \in E(G)$;
- $d(Tx, Ty) \leq \psi(d(x, y))$ whenever $(x, y) \in E(G)$.

Our stability results rely on the following two additional assumptions (see [7]).

(J1) If $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$ and $x_n \to p$ for some $p \in X$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, p) \in E(G)$ for all $k \geq 1$.

(J2) For each $x, y \in X$ if $T^{n_k}x \to y$ and $(T^{n_k}x, T^{n_k+1}x) \in E(G)$ for all $k \geq 1$, then $T(T^{n_k}x) \to Ty$.

Remark 2.6.
- Since we will mention some fixed point theorems proved under a bit stronger assumption than the condition (J1), we refer to this assumption as (J1*). More precisely, it is defined as follows.

  (J1*) If $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$ and $x_n \to p$ for some $p \in X$, then $(x_n, p) \in E(G)$ for all $n \geq 1$.
- The condition (J2) is sometimes referred as the *orbital G-continuity of T* [7].

2.1. (ψ, G)-contractions of type I.

2.1.1. *Fixed point theorem.*

Theorem 2.1. *Let* (X, d) *be a complete metric space with a directed graph* G *on* X*. Suppose that* $T : X \to X$ *is a* (ψ, G)*-contraction of type I and suppose that either the condition (J1) or (J2) holds. If there exists an element* $x_0 \in X$ *such that* $(x_0, Tx_0) \in E(G)$*, then* $T^n x_0 \to p$ *for some* $p \in \mathrm{Fix}(T)$*, that is,* $\mathrm{Fix}(T) \neq \varnothing$*.*

Proof. Set $x_1 := x_0$ and $x_{n+1} := Tx_n$ for all $n \geq 1$. If $d(x_1, x_2) = 0$, then $x_1 = Tx_1$ and we are done. We now assume that $d(x_1, x_2) > 0$. Note that $(x_n, x_{n+1}) \in E(G)$ and $d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1}))$ for all $n \geq 1$. In particular,

$$d(x_{n+1}, x_{n+2}) \leq \psi^n(d(x_1, x_2))$$

for all $n \geq 1$. Note that $\sum_{n=1}^{\infty} \psi^n(d(x_1, x_2)) < \infty$, and hence $\sum_{n=1}^{\infty} d(x_{n+1}, x_{n+2}) < \infty$. This implies that $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is an element $p \in X$ such that $x_n \to p$. We now show that $p = Tp$. The proof is divided into 2 cases.

Case 1: We assume the condition (J1). Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, p) \in E(G)$ for all $k \geq 1$. We consider the following

$$\begin{aligned} d(p, Tp) &\leq d(p, x_{n_k+1}) + d(x_{n_k+1}, Tp) \\ &= d(p, x_{n_k+1}) + d(Tx_{n_k}, Tp) \\ &\leq d(p, x_{n_k+1}) + \psi(d(x_{n_k}, p)) \end{aligned}$$

for all $k \geq 1$. Letting $k \to \infty$ gives $p = Tp$, that is, p is a fixed point of T.

Case 2: We assume the condition (J2). In this case, we have

$$p = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T^n x_1 = \lim_{n\to\infty} T(T^n x_1) = Tp.$$

Then p is a fixed point of T.

This completes the proof. □

2.1.2. *Ulam stability with respect to* G*.*

Lemma 2.1. *Let* $\psi : [0, \infty) \to [0, \infty)$ *be a nondecreasing function such that* $\sum_{k=1}^{\infty} \psi^k(t) < \infty$ *for all* $t > 0$*. Then for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\delta + \sum_{k=1}^{\infty} \psi^k(\delta) \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. Then $\sum_{k=1}^{\infty} \psi^k(\varepsilon) < \infty$. In particular, there exists a positive integer N such that $\sum_{k=N}^{\infty} \psi^k(\varepsilon) \leq \varepsilon$. We now choose $\delta := \psi^N(\varepsilon)$. Hence, $\delta + \sum_{k=1}^{\infty} \psi^k(\delta) = \sum_{k=N}^{\infty} \psi^k(\varepsilon) \leq \varepsilon$ as desired. □

We present two Ulam stability results with respect to G for (ψ, G)-contractions of type I.

Theorem 2.2. *Let* (X, d) *be a complete metric space with a directed graph* G *on* X*. Suppose that* $T : X \to X$ *is a* (ψ, G)*-contraction of type I. Suppose that either the condition (J1) or (J2) holds. If* $\mathrm{Fix}(T) \neq \varnothing$*, then* T *is Ulam stable with respect to* G*.*

Proof. Suppose that $T : X \to X$ is a (ψ, G)-contraction of type I. Let $\varepsilon > 0$. By Lemma 2.1, there exists a $\delta > 0$ such that $\delta + \sum_{k=1}^{\infty} \psi^k(\delta) \leq \varepsilon$. Let w be an element in X such that $(w, Tw) \in E(G)$ and $d(w, Tw) \leq \delta$. (Note that such an element w exists because $\{(x, x) : x \in X\} \subset E(G)$ and $\mathrm{Fix}(T) \neq \varnothing$.) Set $x_1 := w$ and define $x_{n+1} := Tx_n$ for all

$n \geq 1$. It follows from Theorem 2.1 that $x_n \to p$ for some $p \in \mathrm{Fix}(T)$. Furthermore, since $d(x_k, x_{k+1}) \leq \psi^k(d(x_1, x_2)) \leq \psi^k(\delta)$, we also have

$$d(w,p) = \lim_{n\to\infty} d(x_1, x_{n+1}) \leq \lim_{n\to\infty} \sum_{k=1}^{n} d(x_k, x_{k+1}) \leq \delta + \lim_{n\to\infty} \sum_{k=1}^{n} \psi^k(\delta) = \delta + \sum_{k=1}^{\infty} \psi^k(\delta) \leq \varepsilon.$$

Thus, T is Ulam stable with respect to G. □

The following example shows that our concept of Ulam stability with respect to G is more suitable for (ψ, G)-contractions than the classical Ulam stability.

Example 2.1. Let $X := [0,1]$ be a metric space with a usual metric d. Define $T : X \to X$ by

$$Tx := \begin{cases} x/2 & \text{if} \quad x \in [0,1/2] \cup \{1\} \\ 1 & \text{if} \quad x \in (1/2, 1). \end{cases}$$

Then the following statements are true.

(a) T is a (ψ, G)-contraction of type I where $\psi(t) = t/2$ for all $t \geq 0$ and $E(G) := [0,1/2]^2 \cup \{(x,x) : x \in (1/2, 1]\}$.
(b) T is not Ulam stable.
(c) T is Ulam stable with respect to G.

Proof. (a) It is clear that ψ is nondecreasing and $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$. Moreover, if $(x,y) \in E(G)$, then $(Tx, Ty) \in E(G)$ and $d(Tx, Ty) \leq \psi(d(x,y))$. Hence T is a (ψ, G)-contraction of type I.

(b) To see this, we choose $\varepsilon := 1/2$ and set $x_n := (n+1)/(n+2)$ for all $n \geq 1$. It follows that $d(x_n, Tx_n) = d((n+1)/(n+2), 1) \to 0$. Note that 0 is the only one fixed point of T and $d(x_n, 0) \geq 1/2$ for all $n \geq 1$. Hence T is not Ulam stable.

(c) We conclude the result by using Theorem 2.2. In fact, we show that the condition (J1) is satisfied. Suppose that $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$ and $x_n \to p$ for some $p \in X$. The result follows easily if $p \neq 1/2$. We now consider the case $p = 1/2$. If there exists an integer N such that $x_n > p$ for all $n \geq N$, then $x_n = x_N$ for all $n \geq N$ which is impossible. Hence there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \leq p$ for all $k \geq 1$. In particular, $(x_{n_k}, p) \in E(G)$ for all $k \geq 1$. So T is Ulam stable with respect to G. □

2.2. (ψ, G)**-contractions of type II.**

2.2.1. *Fixed point theorem.*

Lemma 2.2 ([1])**.** *Suppose that $\{x_n\}$ is a sequence in a metric space (X,d). If $\{x_n\}$ is not a Cauchy sequence, then there exist a constant $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that the following two conditions hold: for all $k \geq 1$ one has*

$$n_k < m_k \quad \textit{and} \quad d(x_{n_k}, x_{m_k - 1}) < \varepsilon \leq d(x_{n_k}, x_{m_k}).$$

Theorem 2.3. *Let (X,d) be a complete metric space with a directed graph G on X. Suppose that G is transitive, that is, $(x,z) \in E(G)$ whenever $(x,y) \in E(G)$ and $(y,z) \in E(G)$. Suppose that $T : X \to X$ is a (ψ, G)-contraction of type II and suppose that either the condition (J1) or (J2) holds. If there exists an element $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, then $T^n x_0 \to p$ for some $p \in \mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) \neq \varnothing$.*

Proof. Set $x_1 := x_0$ and $x_{n+1} := Tx_n$ for all $n \geq 1$. If $d(x_1, x_2) = 0$, then $x_1 = Tx_1$ and we are done. We now assume that $d(x_1, x_2) > 0$. Note that $(x_n, x_{n+1}) \in E(G)$ and $d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1}))$ for all $n \geq 1$. In particular,

$$d(x_{n+1}, x_{n+2}) \leq \psi^n(d(x_1, x_2))$$

for all $n \geq 1$. In particular, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. We show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. There exist an $\eta > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of $\{n\}$ such that $k \leq n_k \leq m_k$ and $d(x_{n_k}, x_{m_k-1}) < \eta \leq d(x_{n_k}, x_{m_k})$ for all $k \geq 1$. We note from the transitivity of G that $(x_{n_k}, x_{m_k-1}) \in E(G)$ and we obtain the following

$$\begin{aligned}\eta - d(x_{n_k}, x_{n_k+1}) &\leq d(x_{n_k}, x_{m_k}) - d(x_{n_k}, x_{n_k+1})\\ &\leq d(x_{n_k+1}, x_{m_k})\\ &\leq \psi(d(x_{n_k}, x_{m_k-1}))\\ &\leq \psi(\eta).\end{aligned}$$

Letting $k \to \infty$ gives $\eta \leq \psi(\eta)$, that is, $\eta = 0$ which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is an element p in X such that $x_n \to p$. The proof that p is a fixed point of T follows exactly as the same as the proof of Theorem 2.1 so it is left for the reader to verify. □

2.2.2. *Ulam stability with respect to G.*

Lemma 2.3 ([15, 18])**.** *If $\psi : [0,\infty) \to [0,\infty)$ is nonexpansive, that is, $|\psi(s) - \psi(t)| \leq |s - t|$ for all $s, t \geq 0$, then $I - \psi$ is nondecreasing where I is the identity mapping.*

We present the Ulam stability result with respect to G for (ψ, G)-contractions of type II.

Theorem 2.4. *Let (X, d) be a complete metric space with a directed graph G on X. Suppose that $T : X \to X$ is a (ψ, G)-contraction of type II where G is transitive and ψ is nonexpansive. Suppose that either the condition (J1) or (J2) holds. If $\mathrm{Fix}(T) \neq \varnothing$, then T is Ulam stable with respect to G.*

Proof. Suppose that $T : X \to X$ is a (ψ, G)-contraction mapping of type II where ψ is a nonexpansive mapping and G is transitive. Let $\varepsilon > 0$. We choose $\delta := (\varepsilon - \psi(\varepsilon))/2$. Let w be an element in X such that $(w, Tw) \in E(G)$ and $d(w, Tw) \leq \delta$. Set $x_1 := w$ and define $x_{n+1} := Tx_n$ for all $n \geq 1$. It follows from Theorem 2.3 that $x_n \to p$ where $p \in \mathrm{Fix}(T)$. We consider

$$\begin{aligned}d(w, p) = d(x_1, p) &\leq d(x_1, x_2) + d(x_2, p)\\ &= d(x_1, Tx_1) + d(Tx_1, Tp)\\ &\leq d(x_1, Tx_1) + \psi(d(x_1, p))\\ &\leq \delta + \psi(d(w, p)).\end{aligned}$$

In particular, $(I - \psi)(d(w, p)) \leq \delta$. Suppose that $\varepsilon < d(w, p)$. Then $(I - \psi)(\varepsilon) \leq (I - \psi)(d(w, p)) \leq \delta = (\varepsilon - \psi(\varepsilon))/2$ which is a contradiction. Hence, $d(w, p) \leq \varepsilon$. Thus, T is Ulam stable with respect to G. □

There exists a nondecreasing and nonexpansive function $\psi : [0,\infty) \to [0,\infty)$ such that $\lim_{n\to\infty} \psi^n(t) = 0$ and $\sum_{n=1}^{\infty} \psi^n(t) = \infty$ for all $t > 0$. In particular, this reveals the importance of Theorem 2.4.

Example 2.2. Define $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t/(1 + t)$ for all $t \geq 0$. Then ψ is nonexpansive, $\lim_{n\to\infty} \psi^n(t) = 0$, and $\sum_{n=1}^{\infty} \psi^n(t) = \infty$ for all $t > 0$. In fact, for each $n \geq 1$, we note that $\psi^n(t) = \frac{t}{1+nt}$ for all $t \geq 0$.

3. Deduced results and some remarks

3.1. **Ulam–Hyers stability with respect to G of Banach G-contractions.** By Theorem 2.2, we obtain the following result which supplements the result of Jachymski [7].

Corollary 3.1. *Suppose that (X, d) is a complete metric space with a directed graph G on X. Suppose that $T : X \to X$ is a Banach G-contraction with a fixed point. If either the condition (J1) or (J2) holds, then T is Ulam–Hyers stable with respect to G.*

Proof. Suppose that there exists a constant $\alpha \in (0, 1)$ such that $(Tx, Ty) \in E(G)$ and $d(Tx, Ty) \leq \alpha d(x, y)$ for all $(x, y) \in E(G)$. Put $\psi(t) := \alpha t$ for all $t \geq 0$. Let $\varepsilon > 0$ be given. Note that if $\delta = (1-\alpha)\varepsilon$, then $\delta+\sum_{k=1}^{\infty} \psi^k(\delta) = \delta+\sum_{k=1}^{\infty} \alpha^k\delta = \varepsilon$. It follows from the proof of Theorem 2.2 that for each $w \in X$ with $(w, Tw) \in E(G)$ and $d(w, Tw) \leq \delta = (1 - \alpha)\varepsilon$ there exists a fixed point p of T such that $d(p, w) \leq \varepsilon$. This completes the proof. □

3.2. **Remarks on Sintunavarat's results.** We will discuss some vague statements in the recent result of Sintunavarat [17]. His results are established in a different context but it will be seen later that it is equivalent to the setting with a directed graph (see Remark 3.7(1)). We first recall some concepts.

Definition 3.6. Let (X, d) be a metric space. Suppose that $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$. Suppose that $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$.

- T is *weakly α-admissible* if $\alpha(Tx, T^2x) \geq 1$ whenever $\alpha(x, Tx) \geq 1$.
- T is an *(α, ψ)-contraction* if $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$.
- X is *α-regular* if whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 1$ and $x_n \to p$ for some $p \in X$ it follows that $\alpha(x_n, p) \geq 1$ for all $n \geq 1$.

Definition 3.7. Let (X, d) be a metric space and $\varepsilon > 0$. A point $p \in X$ is an *ε-fixed point of a mapping* $T : X \to X$ if $d(p, Tp) \leq \varepsilon$.

We carefully restate the following result from [17, Theorems 2.1, 2.2, 2.3, and 3.4].

Theorem 3.5. *Suppose that (X, d) is a complete metric space. Suppose that $T : X \to X$ is an (α, ψ)-contraction and it is weakly α-admissible with $\alpha(x_0, Tx_0) \geq 1$ for some $x_0 \in X$. Suppose in addition that either T is continuous or X is α-regular. Then the following statements are true.*

(a) $\mathrm{Fix}(T) \neq \varnothing$.
(b) *If $\alpha(p, q) \geq 1$ for all $p, q \in \mathrm{Fix}(T)$, then $\mathrm{Fix}(T)$ is a singleton.*
(c) *Suppose that $I - \psi$ is strictly increasing and onto. If $\alpha(p', q') \geq 1$ for all ε-fixed points p' and q' of T, then T is generalized Ulam–Hyers stable.*

The remarks for the preceding theorem are as follows.

Remark 3.7. (1) Suppose that $T : X \to X$ is an (α, ψ)-contraction and it is weakly α-admissible. We define a directed graph G on X by letting $E(G) := \{(x, y) : \alpha(x, y) \geq 1\}$. It follows that T is a (ψ, G)-contraction of type I. In fact, if $(x, y) \in E(G)$, then $\alpha(x, y) \geq 1$ and hence

$$d(Tx, Ty) \leq \alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)).$$

The continuity of T can be replaced by the G-orbital continuity of T, that is, the condition (J2). The α-regularity of X becomes the condition (J1*) which is a stronger assumption than the condition (J1). On the other hand, suppose that T is a (ψ, G)-contraction of type I. Now, we define $\alpha(x, y) := 1$ if $(x, y) \in E(G)$ and $\alpha(x, y) := 0$ if $(x, y) \notin E(G)$. It follows that T is an (α, ψ)-contraction and it is weakly α-admissible.

(2) Our result for (ψ, G)-contractions of type II also provides a new information which is beyond the scope of the work of [17].

(3) No quantifier about ε is given in the statement (c) of Theorem 3.5. (The same patterns of vague statements are in [8, 11, 5, 16, 12, 13].) Moreover, in the proof of [17, Theorem 3.4] (page 400 line 7), the given $\varepsilon > 0$ is not arbitrary as required in the definition of the generalized Ulam–Hyers stability. Finally, we discuss the validity of the assumption: $\alpha(p', q') \geq 1$ for all ε-fixed points p' and q' of T. Note that if we set $X_\varepsilon := \{x : d(x, Tx) \leq \varepsilon\}$, then it follows from the continuity of T or the α-regularity X that the subset X_ε is closed and hence complete. It is clear that $T : X_\varepsilon \to X_\varepsilon$ is a ψ-contraction. From this point, the function α plays no role in the study.

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SOME NOTES ON THE ULAM STABILITY OF THE GENERAL LINEAR EQUATION

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Abstract. We discuss the hyperstability outcome given by Aiemsomboon and Sintunavarat [1] and concerning the general linear equation. We give a simple proof of it via the hyperstability result for the Cauchy equation. Our proof is based on Brzdęk's fixed point theorem. Moreover, we use a weaker assumption.

1. Introduction

Throughout the paper, we assume that $\mathbb{F}, \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ where $\mathbb{R}$ and $\mathbb{C}$ are the set of all real numbers and complex numbers, respectively; and we assume that $\mathbb{N}$ and $\mathbb{R}_+$ are the set of all positive integers and nonnegative real numbers, respectively. Suppose that X and Y are normed spaces over the scalar fields $\mathbb{F}$ and $\mathbb{K}$, respectively. Hyers [10] was the first one who answered the well known Ulam's problem [17] concerning approximate homomorphisms of groups as follows.

THEOREM H. *Suppose that X and Y are two Banach spaces. If $\delta > 0$ is a real number and $f \colon X \to Y$ satisfies the condition*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \quad \text{for all } x, y \in X,$$

then there exists a unique function $f \colon X \to Y$ such that

(a) *F satisfies the Cauchy equation, that is, $F(x+y) = F(x) + F(y)$ for all $x, y \in X$;*

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(b) $\|F(x) - f(x)\| \le \delta$ *for all* $x \in X$.

Rassias and Tabor [16] proposed to study the stability of the following generalized form of Cauchy equation:

$$f(ax + by) = Af(x) + Bf(y) \quad \text{for all } x, y \in X \tag{1}$$

for some $a, b \in \mathbb{F} \setminus \{0\}$ and $A, B \in \mathbb{K}$. In this case, we say that f satisfies the *general linear equation* with constants a, b, A, B. Some authors [1–3,7,9,11,14,15] proved several stability results of these equations. For more details on this subject, we refer the readers to the recent monograph of Brzdęk et al. [8]. In particular, we say that $f: X \to Y$ satisfies the *Jensen equation* if (1) holds with $a = b = A = B = 1/2$. Clearly, if f satisfies the general linear equation with $a = b = A = B = 1$, then f satisfies the Cauchy equation.

Piszczek [15] recently proved the following hyperstability result for general linear equation (1).

THEOREM P [15, Theorem 2]. *Suppose that X is a normed space and Y is a Banach space over the scalar fields $\mathbb{F}$ and $\mathbb{K}$, respectively. Suppose that $a, b \in \mathbb{F} \setminus \{0\}$ and $A, B \in \mathbb{K}$. Suppose that $c \ge 0$, $p < 0$, and $f: X \to Y$ satisfies the condition*

$$\big\| f(ax + by) - Af(x) - Bf(y)\big\| \le c(\|x\|^p + \|y\|^p)$$

for all $x, y \in X \setminus \{0\}$. Then f satisfies the general linear equation.

Using some idea from Brzdęk's result [5], Aiemsomboon and Sintunavarat [1] proved the following result which is a generalization of Theorem P.

THEOREM AS [1, Theorem 2.1]. *Suppose that X is a normed space and Y is a Banach space over the scalar fields $\mathbb{F}$ and $\mathbb{K}$, respectively. Suppose that $a, b \in \mathbb{F} \setminus \{0\}$ and $A, B \in \mathbb{K}$. Suppose that $h: X \setminus \{0\} \to \mathbb{R}_+$ and $f: X \to Y$ satisfies the condition*

$$\big\| f(ax + by) - Af(x) - Bf(y)\big\| \le h(x) + h(y)$$

for all $x, y \in X \setminus \{0\}$. Suppose that

$$M_0 := \Big\{ n \in \mathbb{N} : |A| s\Big(\frac{1}{a}(n+1)\Big) + |B| s\Big(-\frac{1}{b} n\Big) < 1 \Big\}$$

is an infinite set where $s(\alpha) := \inf\big\{ t \ge 0 : h(\alpha x) \le t h(x) \text{ for all } x \in X \setminus \{0\}\big\}$ for $\alpha \in \mathbb{F} \setminus \{0\}$; and

$$\lim_{\alpha \to \infty} s(\alpha) = \lim_{\alpha \to \infty} s(-\alpha) = 0.$$

Then f satisfies the general linear equation.

REMARK 1. Note that the completeness of Y in Theorem P (and Theorem AS) can be relaxed, that is, the conclusion of Theorem P (and Theorem AS) remains true if Y is a normed space because without loss of generality we can replace Y by its completion.

REMARK 2. If we set $h(x) := c\|x\|^p$ where $p < 0$ and $c \geq 0$, then all the conditions of Theorem AS concerning h are satisfied.

The aim of this paper is to give a simple proof of Theorem AS via the hyperstability of Cauchy equation. Moreover, we show that the conclusion of Theorem AS remains true under a weaker assumption. In particular, we also remark that some assumptions of Theorem AS are superfluous.

2. Preliminaries and some notes

The key ingredient of our proof is based on Brzdęk's fixed point result [6]. In the following result, we write Y^X for the set of functions from a nonempty set X into a nonempty set Y.

THEOREM B [6, Theorem 1]. *Let X be a nonempty set and Y be a Banach space. Let $f_1, f_2 \in X^X$ be given. Let $\mathcal{T}\colon Y^X \to Y^X$ be an operator satisfying the inequality*

$$\big\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\big\| \leq \sum_{i=1}^{2} \big\|\xi(f_i(x)) - \mu(f_i(x))\big\| \quad \text{for all } \xi, \mu \in Y^X \text{ and } x \in X.$$

Let $\Lambda\colon \mathbb{R}_+^X \to \mathbb{R}_+^X$ be an operator defined by

$$\Lambda\delta(x) := \sum_{i=1}^{2} \delta(f_i(x)) \quad \text{for all } \delta \in \mathbb{R}_+^X \text{ and } x \in X.$$

Suppose that $\varepsilon \in \mathbb{R}_+^X$ and $\varphi \in Y^X$ satisfy the conditions

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \varepsilon(x) \quad \text{and} \quad \varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty \text{ for all } x \in X.$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ such that $\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x)$ for all $x \in X$. Moreover, $\psi(x) = \lim_{n\to\infty} \mathcal{T}^n\varphi(x)$ for all $x \in X$.

It is known that a function $f\colon X \to Y$ with $f(0) = 0$ satisfies the Jensen equation if and only if it satisfies the Cauchy equation [12,13]. In particular, it follows that if a function $f\colon X \to Y$ satisfies the Jensen equation, then the odd part f_o of f satisfies the Cauchy equation and the even part f_e of f is a constant. Recall that $f_o(x) := \frac{1}{2}(f(x) - f(-x))$ and $f_e(x) := \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$.

Remark 3. Suppose that $f\colon X \to Y$ is given and f_o, f_e are odd and even parts of f, respectively. Note that $f_o(0) = 0$, $f_o(-x) = -f_o(x)$, $f_e(-x) = f_e(x)$, and $f(x) = f_o(x) + f_e(x)$ for all $x \in X$. In particular, $f_e(0) = f(0)$.

3. Main results

We first note the following easy observation.

Lemma 4. *Let X be a normed space over the scalar field $\mathbb{F}$ and $h\colon X \setminus \{0\} \to \mathbb{R}_+$ be given. For each $n \in \mathbb{N}$, define*

$$s(n) := \inf\bigl\{ t \geq 0 : h(nx) \leq th(x) \text{ for all } x \in X \setminus \{0\} \bigr\}.$$

Suppose that $a_1, a_2, \dots, a_m \in \mathbb{F} \setminus \{0\}$ where $m \in \mathbb{N}$. If $\mathcal{H}\colon X \setminus \{0\} \to \mathbb{R}_+$ is defined by

$$\mathcal{H}(x) := \sum_{i=1}^{m} h(a_i x) \quad \text{for all } x \in X \setminus \{0\}.$$

Then

$$\mathcal{H}(nx) \leq s(n)\mathcal{H}(x) \quad \text{for all } x \in X \setminus \{0\}.$$

In particular, $h(nx) \leq s(n)h(x)$ for all $x \in X \setminus \{0\}$.

We establish the following hyperstability result of Cauchy equation.

Lemma 5. *Let X and Y be normed spaces over the scalar fields $\mathbb{F}$ and $\mathbb{K}$, respectively. Suppose that $\mathcal{H}\colon X \setminus \{0\} \to \mathbb{R}_+$ and $g\colon X \to Y$ are given such that $g(0) = 0$, $g(-x) = -g(x)$ for all $x \in X$ and $\mathcal{H}(-x) = \mathcal{H}(x)$ for all $x \in X \setminus \{0\}$. For each $n \in \mathbb{N}$, define*

$$s(n) := \inf\bigl\{ t \geq 0 : \mathcal{H}(nx) \leq t\mathcal{H}(x) \text{ for all } x \in X \setminus \{0\} \bigr\}.$$

Suppose that g satisfies the inequality

$$\|g(x+y) - g(x) - g(y)\| \leq \mathcal{H}(x) + \mathcal{H}(y) \tag{2}$$

for all $x, y \in X \setminus \{0\}$ and $\inf\{s(n+1) + s(n) : n \in \mathbb{N}\} = 0$. Then the following two statements are true.

(a) *g satisfies the Cauchy equation.*

(b) *If, in addition, there exist $a \in \mathbb{F} \setminus \{0\}$ and $A \in \mathbb{K}$ such that*

$$\Bigl\| g(x) - 2Ag\Bigl(\frac{x}{2a}\Bigr) \Bigr\| \leq \mathcal{H}(x) \tag{3}$$

for all $x \in X \setminus \{0\}$, then

$$g(ax) = Ag(x) \quad \text{for all } x \in X.$$

Proof. (a) We may assume that Y is complete (otherwise, we replace Y by its completion). Let $m \in \mathbb{N}$ be such that $s(m+1)+s(m)<1$. Such an integer m exists because $\inf\{s(n+1)+s(n): n \in \mathbb{N}\}=0$. Let $x \in X \setminus \{0\}$. Replacing x by $(m+1)x$ and y by $-mx$ in (2), we get

$$\left\| g(x)-g((m+1)x)-g(-mx)\right\| \le \mathcal{H}((m+1)x)+\mathcal{H}(mx). \tag{4}$$

Define $\mathcal{T}_m\colon Y^{X\setminus\{0\}} \to Y^{X\setminus\{0\}}$ by

$$\mathcal{T}_m\xi(x) := \xi((m+1)x)+\xi(-mx) \tag{5}$$

for all $x \in X\setminus\{0\}$ and $\xi \in Y^{X\setminus\{0\}}$. Next, we define $\varepsilon_m\colon X\setminus\{0\} \to \mathbb{R}_+$ by

$$\varepsilon_m(x) := \mathcal{H}((m+1)x)+\mathcal{H}(mx) \le \alpha\mathcal{H}(x) \tag{6}$$

for all $x \in X\setminus\{0\}$ where $\alpha := s(m+1)+s(m)$. Then it follows from (4) that

$$\|\mathcal{T}_m g(x)-g(x)\| \le \varepsilon_m(x)$$

for all $x \in X\setminus\{0\}$. Define $\Lambda_m\colon \mathbb{R}_+^{X\setminus\{0\}} \to \mathbb{R}_+^{X\setminus\{0\}}$ by

$$\Lambda_m\eta(x) := \eta((m+1)x)+\eta(-mx)$$

for all $\eta \in \mathbb{R}_+^{X\setminus\{0\}}$ and $x \in X\setminus\{0\}$. Then Λ_m satisfies the condition of Theorem B with $f_1(x) := (m+1)x$ and $f_2(x) := -mx$. Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$ and $x \in X\setminus\{0\}$, we have

$$\begin{aligned}\left\| \mathcal{T}_m\xi(x)-\mathcal{T}_m\mu(x)\right\| &= \left\| \xi((m+1)x)+\xi(-mx)-\mu((m+1)x)-\mu(-mx)\right\| \\ &\le \left\| \xi((m+1)x)-\mu((m+1)x\right\| + \|\xi(-mx)-\mu(-mx)\| \\ &= \sum_{i=1}^{2} \|\xi(f_i(x))-\mu(f_i(x))\|.\end{aligned}$$

We show by induction that for each $n \in \mathbb{N}\cup\{0\}$,

$$\Lambda_m^n\varepsilon_m(x) \le \alpha^{n+1}\mathcal{H}(x) \quad \text{for all } x \in X\setminus\{0\}. \tag{7}$$

The inequality (7) holds for $n=0$ because of (6). Suppose that the inequality (7) holds for $n=k$ where $k \in \mathbb{N}\cup\{0\}$. Let $x \in X\setminus\{0\}$. Then

$$\begin{aligned}\Lambda_m^{k+1}\varepsilon_m(x) &= \Lambda_m(\Lambda_m^k\varepsilon_m(x)) = \Lambda_m^k\varepsilon_m((m+1)x)+\Lambda_m^k\varepsilon_m(-mx) \\ &\le \alpha^{k+1}\mathcal{H}((m+1)x)+\alpha^{k+1}\mathcal{H}(mx) \\ &\le \alpha^{k+1}(s(m+1)+s(m))\mathcal{H}(x) = \alpha^{k+2}\mathcal{H}(x).\end{aligned}$$

That is, the inequality (7) holds for $n = k+1$. By induction, (7) holds for all $n \in \mathbb{N} \cup \{0\}$. Thus

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \le \sum_{n=0}^{\infty} \left(\alpha^{n+1}\right) \mathcal{H}(x) = \frac{s(m+1)+s(m)}{1-s(m+1)-s(m)} \mathcal{H}(x)$$

for all $x \in X \setminus \{0\}$. By Theorem B, there exists a unique fixed point $G_m \in Y^{X \setminus \{0\}}$ of $\mathcal{T}_m$ such that

$$\|g(x) - G_m(x)\| \le \frac{s(m+1)+s(m)}{1-s(m+1)-s(m)} \mathcal{H}(x) \quad \text{for all } x \in X \setminus \{0\}. \tag{8}$$

Moreover,

$$G_m(x) = \lim_{n\to\infty} (\mathcal{T}_m^n g)(x)$$

for all $x \in X \setminus \{0\}$.

Next, we show by induction that for each $n \in \mathbb{N} \cup \{0\}$,

$$\left\| \mathcal{T}_m^n g(x+y) - \mathcal{T}_m^n g(x) - \mathcal{T}_m^n g(y) \right\| \le \alpha^n (\mathcal{H}(x) + \mathcal{H}(y)) \tag{9}$$

for all $x, y \in X \setminus \{0\}$ with $x+y \in X \setminus \{0\}$. The inequality (9) holds for $n = 0$ because of (2). Suppose that (9) holds for $n = k$ where $k \in \mathbb{N} \cup \{0\}$. Let $x, y \in X \setminus \{0\}$ with $x+y \in X \setminus \{0\}$. Then

$$\begin{aligned}
&\left\| \mathcal{T}_m^{k+1} g(x+y) - \mathcal{T}_m^{k+1} g(x) - \mathcal{T}_m^{k+1} g(y) \right\| \\
&\le \left\| \mathcal{T}_m^k g((m+1)(x+y)) - \mathcal{T}_m^k g((m+1)x) - \mathcal{T}_m^k g((m+1)y) \right\| \\
&\quad + \left\| \mathcal{T}_m^k g(-m(x+y)) - \mathcal{T}_m^k g(-mx) - \mathcal{T}_m^k g(-my) \right\| \\
&\le \alpha^k \big(\mathcal{H}((m+1)x) + \mathcal{H}((m+1)y) \big) + \alpha^k \big(\mathcal{H}(mx) + \mathcal{H}(my) \big) \\
&\le \alpha^k (s(m+1) + s(m))(\mathcal{H}(x) + \mathcal{H}(y)) = \alpha^{k+1}(\mathcal{H}(x) + \mathcal{H}(y)).
\end{aligned}$$

That is, the inequality (9) holds for $n = k+1$. By induction, (9) holds for all $n \in \mathbb{N} \cup \{0\}$. Letting $n \to \infty$ in (9) gives

$$G_m(x+y) = G_m(x) + G_m(y) \text{ for all } x, y \in X \setminus \{0\} \text{ with } x+y \in X \setminus \{0\}. \tag{10}$$

We now prove that for each $x, y \in X \setminus \{0\}$ with $x + y \neq 0$

$$\left\| g(x+y) - g(x) - g(y) \right\| \le \frac{s(m+1)+s(m)}{1-s(m+1)-s(m)} (\mathcal{H}(x+y) + \mathcal{H}(x) + \mathcal{H}(y)). \tag{11}$$

To see this, let $x, y \in X \setminus \{0\}$ with $x + y \neq 0$. Then, by (10) and (8), we have

$$\begin{aligned} \|g(x+y) - g(x) - g(y)\| &\le \|g(x+y) - G_m(x+y)\| \\ + \|G_m(x+y) - G_m(x) - G_m(y)\| &+ \|G_m(x) - g(x)\| + \|G_m(y) - g(y)\| \\ &\le \frac{s(m+1)+s(m)}{1-s(m+1)-s(m)} \big(\mathcal{H}(x+y) + \mathcal{H}(x) + \mathcal{H}(y)\big). \end{aligned}$$

In particular, we have

$$\big\| g(x+y) - g(x) - g(y)\big\| \le s_0\big(\mathcal{H}(x+y) + \mathcal{H}(x) + \mathcal{H}(y)\big)$$

for all $x, y \in X \setminus \{0\}$ with $x + y \neq 0$ where

$$s_0 := \inf\Big\{\frac{s(m+1)+s(m)}{1-s(m+1)-s(m)} : m \in \mathbb{N} \text{ and } s(m+1)+s(m) < 1\Big\}.$$

Since $\inf\{s(n+1) + s(n) : n \in \mathbb{N}\} = 0$, we have $s_0 = 0$ and hence

$$g(x+y) = g(x) + g(y) \quad \text{for all } x, y \in X \setminus \{0\} \text{ with } x + y \in X \setminus \{0\}.$$

Since $g(0) = 0$ and $g(-x) = -g(x)$ for all $x \in X$, the function g satisfies the Cauchy equation.

(b) Now, we assume further that there exist $a \in \mathbb{F} \setminus \{0\}$ and $A \in \mathbb{K}$ such that the inequality (3) holds for all $x \in X \setminus \{0\}$. Let $x \in X \setminus \{0\}$ and let $n \in \mathbb{N}$. Note that $g(nx) = ng(x)$ and $g\left(\frac{nx}{2a}\right) = ng\Big(\frac{x}{2a}\Big)$. Then

$$n\Big\|g(x) - 2Ag\Big(\frac{x}{2a}\Big)\Big\| = \Big\|g(nx) - 2Ag\Big(\frac{nx}{2a}\Big)\Big\| \le s(n)\,\mathcal{H}(x).$$

That is,

$$\Big\|g(x) - 2Ag\Big(\frac{x}{2a}\Big)\Big\| \le \frac{s(n)}{n}\,\mathcal{H}(x).$$

This implies that

$$\Big\|g(x) - 2Ag\Big(\frac{x}{2a}\Big)\Big\| \le t_0\,\mathcal{H}(x),$$

where

$$t_0 := \inf\Big\{\frac{s(n)}{n} : n \in \mathbb{N}\Big\}.$$

It follows from $\inf\{s(n+1) + s(n) : n \in \mathbb{N}\} = 0$ that $\inf\big\{\frac{s(n)}{n} : n \in \mathbb{N}\big\} = 0$ and hence $g(x) = 2Ag(\frac{x}{2a})$. In particular,

$$g(ax) = Ag\Big(\frac{x}{2}\Big) + Ag\Big(\frac{x}{2}\Big) = Ag\Big(\frac{x}{2} + \frac{x}{2}\Big) = Ag(x).$$

Since $g(0)=0$, we now conclude that $g(ax)=Ag(x)$ for all $x\in X$. $\square$

We now show that if f satisfies the general linear equation approximately, then its odd part f_o satisfies the Cauchy equation approximately.

THEOREM 6. *Let X and Y be normed spaces over the scalar fields $\mathbb{F}$ and $\mathbb{K}$, respectively, $a,b\in\mathbb{F}\setminus\{0\}$ and $A,B\in\mathbb{K}$. Suppose that $h\colon X\setminus\{0\}\to\mathbb{R}_+$ and $f\colon X\to Y$ are given. Suppose that f satisfies the inequality*

$$\|f(ax+by)-Af(x)-Bf(y)\|\le h(x)+h(y)$$

for all $x,y\in X\setminus\{0\}$. Then there exists a function $\mathcal{H}\colon X\setminus\{0\}\to\mathbb{R}_+$ such that $\mathcal{H}(-x)=\mathcal{H}(x)$ for all $x\in X\setminus\{0\}$ and the odd and even parts $f_o,f_e\colon X\to Y$ of f satisfy the following conditions:

(a) $\big\|f_o(x)-2Af_o(\frac{x}{2a})\big\|\le\mathcal{H}(x)$ *and* $\big\|f_o(x)-2Bf_o(\frac{x}{2b})\big\|\le\mathcal{H}(x)$ *for all* $x\in X\setminus\{0\}$;

(b) $\big\|f_o(x+y)-f_o(x)-f_o(y)\big\|\le\mathcal{H}(x)+\mathcal{H}(y)$ *for all* $x,y\in X\setminus\{0\}$.

If $\inf\{s(n+1)+s(n):n\in\mathbb{N}\}=0$ *where*

$$s(n):=\inf\big\{t\ge0:h(nx)\le th(x)\ \textit{for all}\ x\in X\setminus\{0\}\big\}\quad(n\in\mathbb{N}),$$

then

(i) $f_o(x+y)=f_o(x)+f_o(y)$ *for all* $x,y\in X$;

(ii) $f_o(ax)=Af_o(x)$ *and* $f_o(bx)=Bf_o(x)$ *for all* $x\in X$;

(iii) $f_e(x)=f_e(0)=f(0)$ *for all* $x\in X$; *and* $f_e(0)=Af_e(0)+Bf_e(0)$.

PROOF. Define

$$h'(x):=\frac12(h(x)+h(-x)),\quad H'(x):=h'\Big(\frac{x}{2a}\Big)+h'\Big(\frac{x}{2b}\Big),$$
$$\mathcal{H}(x):=2(H'(x)+H'(2x)+H'(3x))$$

for all $x\in X\setminus\{0\}$. Then $h'(-x)=h'(x)$, $H'(-x)=H'(x)$, and $\mathcal{H}(-x)=\mathcal{H}(x)$ for all $x\in X\setminus\{0\}$. Note that

$$\big\|f_o(ax+by)-Af_o(x)-Bf_o(y)\big\| \tag{12}$$
$$=\frac12\big\|f(ax+by)-f(-ax-by)-A(f(x)-f(-x))-B(f(y)-f(-y))\big\|$$
$$\le\frac12\big\|f(ax+by)-Af(x)-Bf(y)\big\|+\frac12\big\|f(-ax-by)-Af(-x)-Bf(-y)\big\|$$
$$\le\frac12(h(x)+h(y))+\frac12(h(-x)+h(-y))=h'(x)+h'(y)$$

for all $x,y\in X\setminus\{0\}$. Let $x\in X\setminus\{0\}$. Replacing x by $\frac{x}{2a}$ and y by $\frac{x}{2b}$ in (12), we get

$$\Big\|f_o(x)-Af_o\Big(\frac{x}{2a}\Big)-Bf_o\Big(\frac{x}{2b}\Big)\Big\|\le H'(x). \tag{13}$$

Next, replacing x by $\frac{x}{2a}$ and y by $-\frac{x}{2b}$ in (12), we get

$$\Big\| - Af_o\Big(\frac{x}{2a}\Big) + Bf_o\Big(\frac{x}{2b}\Big)\Big\| \le H'(x). \tag{14}$$

Then (13) and (14) imply that

$$\begin{cases} \Big\|f_o(x) - 2Af_o\Big(\frac{x}{2a}\Big)\Big\| \le 2H'(x) \le \mathcal{H}(x), \\ \Big\|f_o(x) - 2Bf_o\Big(\frac{x}{2b}\Big)\Big\| \le 2H'(x) \le \mathcal{H}(x), \end{cases} \tag{15}$$

for all $x \in X \setminus \{0\}$ and so we get (a). If $x, y \in X \setminus \{0\}$, then it follows from (12) and (15) that

$$\begin{aligned} &\Big\|f_o\Big(\frac{1}{2}x + \frac{1}{2}y\Big) - \frac{1}{2}f_o(x) - \frac{1}{2}f_o(y)\Big\| \\ &\le \Big\|f_o\Big(a\frac{x}{2a} + b\frac{y}{2b}\Big) - Af_o\Big(\frac{x}{2a}\Big) - Bf_o\Big(\frac{y}{2b}\Big)\Big\| \\ &+ \Big\|Af_o\Big(\frac{x}{2a}\Big) - \frac{1}{2}f_o(x)\Big\| + \Big\|Bf_o\Big(\frac{y}{2b}\Big) - \frac{1}{2}f_o(y)\Big\| \\ &\le h'\Big(\frac{x}{2a}\Big) + h'\Big(\frac{y}{2b}\Big) + H'(x) + H'(y) \le 2H'(x) + 2H'(y). \end{aligned} \tag{16}$$

Let $x \in X \setminus \{0\}$. Replacing x by $3x$ and y by $-x$ in (16), we get

$$\Big\|\frac{3}{2}f_o(x) - \frac{1}{2}f_o(3x)\Big\| \le 2H'(3x) + 2H'(x). \tag{17}$$

Next, replacing x by $3x$ and y by x in (16), we get

$$\Big\|f_o(2x) - \frac{1}{2}f_o(3x) - \frac{1}{2}f_o(x)\Big\| \le 2H'(3x) + 2H'(x). \tag{18}$$

Then (17) and (18) imply

$$\|f_o(2x) - 2f_o(x)\| \le 4H'(x) + 4H'(3x). \tag{19}$$

Consequently, let $x, y \in X \setminus \{0\}$, then it follows from (16) and (19) that

$$\begin{aligned} \|f_o(x+y) - f_o(x) - f_o(y)\| &\le \Big\|f_o\Big(\frac{1}{2}2x + \frac{1}{2}2y\Big) - \frac{1}{2}f_o(2x) - \frac{1}{2}f_o(2y)\Big\| \\ &+ \Big\|\frac{1}{2}f_o(2x) - f_o(x)\Big\| + \Big\|\frac{1}{2}f_o(2y) - f_o(y)\Big\| \le \mathcal{H}(x) + \mathcal{H}(y). \end{aligned}$$

So we get (b).

Now we assume further that $\inf\{s(n+1)+s(n) : n \in \mathbb{N}\} = 0$. Note that, by Lemma 4, $\mathcal{H}(nx) \le s(n)\mathcal{H}(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Then all the conditions of Lemma 5 are satisfied and hence we obtain the statements (i) and (ii).

Next we prove (iii). By the definition of f_e, we get

$$\begin{aligned}
(20) \qquad & \| f_e(ax+by) - Af_e(x) - Bf_e(y)\| \\
&= \frac{1}{2}\| f(ax+by) + f(-ax-by) - A(f(x)+f(-x)) - B(f(y)+f(-y))\| \\
&\le \frac{1}{2}\| f(ax+by) - Af(x) - Bf(y)\| + \frac{1}{2}\| f(-ax-by) - Af(-x) - Bf(-y)\| \\
&\le \frac{1}{2}(h(x)+h(y)) + \frac{1}{2}(h(-x)+h(-y)) = h'(x)+h'(y)
\end{aligned}$$

for all $x, y \in X \setminus \{0\}$. It follows from (20) that

$$\begin{aligned}
\|f_e(x) - f_e(0)\| &\le \Big\|f_e(x) - Af_e\Big(\frac{x}{2a}\Big) - Bf_e\Big(\frac{x}{2b}\Big)\Big\| \\
&+ \Big\|f_e(0) - Af_e\Big(-\frac{x}{2a}\Big) - Bf_e\Big(\frac{x}{2b}\Big)\Big\| \le 2H'(x) \le \mathcal{H}(x)
\end{aligned}$$

for all $x \in X \setminus \{0\}$. For each $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$, we have

$$\|f_e(nx) - f_e(0)\| \le \mathcal{H}(nx) \le s(n)\mathcal{H}(x).$$

Since $\inf\{s(n+1)+s(n) : n \in \mathbb{N}\} = 0$, there exists an increasing sequence $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ such that $\lim\limits_{k\to\infty}(s(n_k+1)+s(n_k)) = 0$. Note that $\lim\limits_{k\to\infty} s(n_k) = \lim\limits_{k\to\infty} s(n_k+1) = 0$. In particular,

$$\lim_{k\to\infty} f_e((n_k+1)x) = \lim_{k\to\infty} f_e(n_k x) = f_e(0) \quad \text{for all } x \in X \setminus \{0\}.$$

Let $x \in X \setminus \{0\}$. Replacing x by $\frac{(n_k+1)x}{a}$ and y by $-\frac{n_k x}{b}$ in (20), we get

$$\Big\|f_e(x) - Af_e\Big(\frac{(n_k+1)x}{a}\Big) - Bf_e\Big(\frac{-n_k x}{b}\Big)\Big\| \le s(n_k+1)h'\Big(\frac{x}{a}\Big) + s(n_k)h'\Big(\frac{x}{b}\Big).$$

In particular,

$$\lim_{k\to\infty}\Big\|f_e(x) - Af_e\Big(\frac{(n_k+1)x}{a}\Big) - Bf_e\Big(\frac{-n_k x}{b}\Big)\Big\| = 0.$$

Since $\lim_{k\to\infty} f_e\big(\frac{(n_k+1)x}{a}\big) = \lim_{k\to\infty} f_e\big(\frac{-n_k x}{b}\big) = f_e(0)$, we have

$$f_e(x) = Af_e(0) + Bf_e(0).$$

Moreover, (20) with $x = \frac{n_k x}{a}$ and $y = \frac{-n_k x}{b}$, we get

$$\Big\| f_e(0) - Af_e\Big(\frac{n_k x}{a}\Big) - Bf_e\Big(\frac{-n_k x}{b}\Big)\Big\| \le s(n_k)h'\Big(\frac{x}{a}\Big) + s(n_k)h'\Big(\frac{x}{b}\Big).$$

Letting $k \to \infty$, we obtain that

$$f_e(0) = Af_e(0) + Bf_e(0).$$

Then $f_e(x) = f_e(0) = Af_e(0) + Bf_e(0)$ for all $x \in X$ and hence we get (iii). □

By using Theorem 6, we immediately obtain the following hyperstability result of the general linear equation which is related to Theorem AS.

Theorem 7. *Let X and Y be normed spaces over the scalar fields $\mathbb{F}$ and $\mathbb{K}$, respectively, $a, b \in \mathbb{F} \setminus \{0\}$ and $A, B \in \mathbb{K}$. Suppose that $h\colon X \setminus \{0\} \to \mathbb{R}_+$ and $f\colon X \to Y$ are given. For each $n \in \mathbb{N}$, define*

$$s(n) := \inf\big\{ t \ge 0 : h(nx) \le th(x) \text{ for all } x \in X \setminus \{0\}\big\}.$$

Suppose that f satisfies the inequality

$$\big\| f(ax+by) - Af(x) - Bf(y)\big\| \le h(x) + h(y)$$

for all $x, y \in X \setminus \{0\}$. If $\inf\{s(n+1) + s(n) : n \in \mathbb{N}\} = 0$, then f satisfies the equation

$$f(ax+by) = Af(x) + Bf(y)$$

for all $x, y \in X$.

Proof. Assume that $\inf\{s(n+1) + s(n) : n \in \mathbb{N}\} = 0$. By Theorem 6, the odd and even parts $f_o, f_e\colon X \to Y$ of f satisfy the following conditions:

- $f_o(x+y) = f_o(x) + f_o(y)$ for all $x, y \in X$;
- $f(x) = f_o(x) + f_e(x)$, $f_e(x) = f_e(0) = Af_e(0) + Bf_e(0)$, $f_o(ax) = Af_o(x)$, and $f_o(bx) = Bf_o(x)$ for all $x \in X$.

Let $x, y \in X$. Then

$$f(ax+by) = f_o(ax+by) + f_e(ax+by) = f_o(ax) + f_o(by) + f_e(0)$$
$$= Af_o(x) + Bf_o(y) + Af_e(0) + Bf_e(0) = Af_o(x) + Bf_o(y) + Af_e(x) + Bf_e(y)$$
$$= A(f_o(x) + f_e(x)) + B(f_o(y) + Bf_e(y)) = Af(x) + Bf(y). \quad \square$$

Remark 8. (1) For Theorem AS, we note that $\lim_{\alpha\to\infty} s(\alpha) = 0 \Leftrightarrow \lim_{\alpha\to\infty} s(-\alpha) = 0$. Moreover, it follows from $\lim_{\alpha\to\infty} s(\alpha) = 0$ that M_0 is an infinite set. Hence the latter condition is superfluous.

(2) Our hyperstability result in Theorem 7 follows from a weaker assumption. In fact, it is easy to see that the condition $\lim_{\alpha\to\infty} s(\alpha) = 0$ implies $\inf\{s(n+1) + s(n) : n \in \mathbb{N}\} = 0$.

By using Theorem 7, we obtain the following corollary concerning the inhomogeneous version of the general linear equation. It generalizes [1, Corollary 2.5]. For more details about stability of the inhomogeneous functional equations, we refer to the work of Brzdęk [4].

Corollary 9. *Let X and Y be normed spaces over the scalar fields $\mathbb{F}$ and $\mathbb{K}$, respectively, $a, b \in \mathbb{F} \setminus \{0\}$ and $A, B \in \mathbb{K}$. Suppose that $h\colon X \setminus \{0\} \to \mathbb{R}_+$ and $d\colon X \times X \to Y$ are given. For each $n \in \mathbb{N}$, define*

$$s(n) := \inf\big\{ t \geq 0 : h(nx) \leq th(x) \text{ for all } x \in X \setminus \{0\}\big\}.$$

Suppose that $f\colon X \to Y$ satisfies the inequality

$$\big\| f(ax+by) - Af(x) - Bf(y) - d(x,y)\big\| \leq h(x) + h(y) \quad \text{for all } x, y \in X \setminus \{0\}$$

and there exists a function $f_0\colon X \to Y$ such that

$$f_0(ax+by) = Af_0(x) + Bf_0(y) + d(x,y) \quad \text{for all } x, y \in X.$$

If $\inf\{s(n+1) + s(n) : n \in \mathbb{N}\} = 0$, then

$$f(ax+by) = Af(x) + Bf(y) + d(x,y) \quad \text{for all } x, y \in X.$$

Proof. Assume that $\inf\{s(n+1) + s(n) : n \in \mathbb{N}\} = 0$. Set $g(x) := f(x) - f_0(x)$ for all $x \in X$. It follows that

$$\begin{aligned} &\big\| g(ax+by) - Ag(x) - Bg(y)\big\| \\ &= \big\| f(ax+by) - Af(x) - Bf(y) - (f_0(ax+by) - Af_0(x) - Bf_0(y))\big\| \\ &= \big\| f(ax+by) - Af(x) - Bf(y) - d(x,y)\big\| \leq h(x) + h(y) \end{aligned}$$

f or all $x, y \in X \setminus \{0\}$. By Theorem 7, we get

$$g(ax+by) = Ag(x) + Bg(y) \quad \text{for all } x, y \in X.$$

In particular, we have

$$f(ax+by) = Af(x) + Bf(y) + d(x,y) \quad \text{for all } x, y \in X. \qquad \square$$

Remark 10. Note that in the case $A + B \neq 1$ and d is a constant function, that is, $d(x,y) := c$ for all $x, y \in X$, the function $f_0\colon X \to Y$ defined by

$$f_0(x) = \frac{c}{1 - A - B} \quad \text{for all } x \in X,$$

satisfies the equation

$$f_0(ax+by) = Af_0(x) + Bf_0(y) + d(x,y) \quad \text{for all } x \in X.$$

Therefore, our Corollary 9 also generalizes [1, Corollary 2.6].

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