



รายงานโครงการวิจัยฉบับสมบูรณ์

(รายงานในช่วงตั้งแต่วันที่ 16 มิถุนายน 2559 ถึงวันที่ 15 มิถุนายน 2562)

โครงการ“การจำกัดเวลาการมีเสถียรภาพและการทำเสถียรสำหรับระบบเอกฐานเชิงเส้น
แบบไม่ต่อเนื่องทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วง”

“Finite-time stability and stabilization for linear discrete-time singular systems
with interval time-varying delays”

(ทุนพัฒนานักวิจัยรุ่นกลาง)

โดย

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มหาวิทยาลัยแม่โจ้

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ต่อเนื่องทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วง”

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สนับสนุนโดย สำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยแม่โจ้

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว.ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) ที่ได้ให้โอกาสผู้วิจัยได้รับทุนพัฒนานักวิจัยรุ่นกลาง ในการทำงานวิจัยค้นคว้าครั้งนี้ และขอขอบพระคุณ Prof. Jinde Cao ที่ให้คำแนะนำที่ดีต่าง ๆ มาโดยตลอด

รศ.ดร. เกรียงไกร ราชกิจ

หัวหน้าโครงการวิจัย

กุมภาพันธ์ 2562

บทคัดย่อ

รหัสโครงการ: RSA5980019

ชื่อโครงการ: การจำกัดเวลาการมีเสถียรภาพและการทำเสถียรสำหรับระบบเอกฐานเชิงเส้นแบบไม่ต่อเนื่องทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วง

ชื่อนักวิจัย และสถาบัน: รศ.ดร. เกรียงไกร ราชกิจ สาขาวิชาคณิตศาสตร์
คณะวิทยาศาสตร์ มหาวิทยาลัยแม่โจ้

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ระยะเวลาโครงการ: 16 มิถุนายน 2559 ถึงวันที่ 15 มิถุนายน 2562

บทคัดย่อ

ในงานวิจัยนี้ เราสนใจในการหาเงื่อนไขเพียงพอของการจำกัดเวลาการมีเสถียรภาพและการทำเสถียรสำหรับระบบเอกฐานเชิงเส้นแบบไม่ต่อเนื่องทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วง โดยทฤษฎีไลยาปูนอฟ

คำหลัก : การจำกัดเวลาการมีเสถียรภาพและการทำเสถียร/ ระบบเอกฐานเชิงเส้น/
ระบบไม่ต่อเนื่องทางเวลา/ ฟังก์ชันไลยาปูนอฟ/ อสมการเมริกซ์/
ตัวหน่วงที่แปรผันตามเวลา

Abstract

Project Code: RSA5980019

Project Title: Finite-time stability and stabilization for linear discrete-time singular systems with interval time-varying delays

Investigator: Assoc. Prof. Dr. Griengrai Rajchakit, Department of Mathematics,
Faculty of Science, Maejo University, Thailand

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Project Period: 16 June 2016 – 15 June 2019

Abstract

In this research, we are interested in finding sufficient conditions for Finite-time stability and stabilization for linear discrete-time singular systems with interval time-varying delays. By Lyapunov Theorem.

Keywords: Finite-time stability and Stabilization; singular systems; discrete-time systems; Lyapunov Function; Linear Matrix Inequalities; time-varying delays.

ความสำคัญและที่มาของปัญหาที่ทำการวิจัย

การศึกษาสมบัติเชิงคุณภาพ (Qualitative property) ของจุดสมดุล (Equilibrium point) ของสมการเชิงอนุพันธ์ (Differential Equation) และสมการเชิงผลต่าง (Difference equation) ของการจำกัดเวลาการมีเสถียรภาพและการทำเสถียรสำหรับระบบเอกฐานเชิงเส้นแบบไม่ต่อเนื่องทางเวลาที่มีตัวห้วงขึ้นกับเวลาแบบช่วง เป็นงานวิจัยที่ได้รับความสนใจจากอย่างมากในปัจจุบัน โดยเฉพาะการศึกษาระบบระบบเอกฐาน ทั้งนี้ เนื่องจากการการจำลองเชิงคณิตศาสตร์ (Mathematical Modeling) ของระบบทางกายภาพและชีวภาพที่พบได้จริงในธรรมชาติเป็นระบบไม่เชิงเส้นทั้งสิ้น อย่างไรก็ตามได้มีการพิสูจน์เชิงทฤษฎีและแล้วว่าในระบบไม่เชิงเส้นดังกล่าวหลายระบบ นั้นพบว่า สมบัติต่าง ๆ ของจุดสมดุลของการ ประมาณอันดับหนึ่ง (First approximation หรือ Linearization) ของระบบไม่เชิงเส้นดังกล่าวเหมือนกับสมบัติของผลเฉลยของระบบไม่เชิงเส้น ภายใต้เงื่อนไขบางส่วนของระบบ สมบัติที่สำคัญประการหนึ่งของจุดสมดุลของระบบที่ได้กล่าวมาแล้วคือ การจำกัดเวลาการมีเสถียรภาพและการทำเสถียรสำหรับระบบเอกฐานเชิงเส้นแบบไม่ต่อเนื่องทางเวลาที่มีตัวห้วงขึ้นกับเวลาแบบช่วง ทั้งนี้ระบบส่วนใหญ่ใน ธรรมชาติและทางกายภาพนั้นจะมีพลศาสตร์ (Dynamics) ของผลเฉลยซึ่งมีค่าเริ่มต้นใกล้กับจุดสมดุล ได้หลายประเภท เช่นผลเฉลยอาจจะลู่เข้าหาจุดสมดุล หรือลู่ออกจากจุดสมดุล หรือมีลักษณะเป็นคาบรอบ ๆ จุดสมดุล เป็นต้น แต่จุดสมดุลที่มีเสถียรภาพจะเป็นจุดสมดุลที่มีความสำคัญ และสามารถใช้ประโยชน์ได้อย่างมีประสิทธิภาพในการอธิบายหรือทำนายการเปลี่ยนแปลงของผลเฉลยในระบบต่าง ๆ เช่น จุดสมดุลเสถียรของระบบสมการที่อธิบายการอยู่ร่วมกันของประชากรของสัตว์ชนิดต่าง ๆ ระบบวงจรไฟฟ้า ระบบสลับ (Switch system) ระบบเครือข่ายประสาท (Neural network) เป็นต้น ทั้งนี้โดยทั่วไปในระบบดังกล่าวที่ได้กล่าวมาแล้วนั้นพารามิเตอร์ที่เกี่ยวข้องกับระบบส่วนใหญ่ เราจะไม่ทราบค่าที่แน่นอน เช่น ผลกระทบของคลื่นแม่เหล็กไฟฟ้าในการ ทดลองทางฟิสิกส์บางอย่าง ผลกระทบของฮอร์โมนบางชนิดต่อการทำงานของสิ่งมีชีวิตบางอย่าง เป็นต้น อย่างไรก็ตาม ปริมาณของผลกระทบเหล่านี้มักจะมีขอบเขตที่แน่นอน ดังนั้นนักวิจัยหลายท่านจึง ศึกษาการจำกัดเวลาการมีเสถียรภาพและการทำเสถียรสำหรับระบบเอกฐานเชิงเส้นแบบไม่ต่อเนื่องทางเวลาที่มีตัวห้วงขึ้นกับเวลาแบบช่วง ที่เกี่ยวข้องโดยการศึกษาลักษณะที่เราจะเรียกว่า การศึกษาเสถียรภาพทนทาน(Robust stability) เป็นต้น

วัตถุประสงค์ หรือจุดมุ่งหมายของโครงการฯ

1. เพื่อสร้างและศึกษาฟังก์ชันไลปุนอฟ
2. เพื่อศึกษาการจำกัดเวลาการมีเสถียรภาพและการทำให้เสถียรสำหรับระบบแบบเอกฐานแบบไม่ต่อเนื่องทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วง
3. เพื่อสร้างและตรวจสอบการจำกัดเวลาการมีเสถียรภาพและการทำให้เสถียรสำหรับระบบแบบเอกฐานแบบไม่ต่อเนื่องทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วง
4. เพื่อเขียนโปรแกรมคำนวณเชิงตัวเลขของการจำกัดเวลาการมีเสถียรภาพและการทำให้เสถียรสำหรับระบบแบบเอกฐานแบบไม่ต่อเนื่องทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วง
5. เพื่อสร้างเครือข่ายการวิจัยทั้งภายในและภายนอกของประเทศสำหรับการความร่วมมือในการทำวิจัยสาขา

การดำเนินงานวิจัย

1. รวบรวมเอกสารงานวิจัยทั้งหมด หนังสือและบทความทั้งหมดในวารสารต่าง ๆ ที่เกี่ยวข้องกับการจำกัดเวลาการมีเสถียรภาพและการทำให้เสถียรสำหรับระบบแบบเอกฐานแบบไม่ต่อเนื่องทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วง
2. ศึกษาวิธีการต่าง ๆ สำหรับการออกแบบฟังก์ชันไลปุนอฟ สำหรับการจำกัดเวลาการมีเสถียรภาพและการทำให้เสถียรสำหรับระบบแบบเอกฐานแบบไม่ต่อเนื่องทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วงทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วง
3. การใช้ความรู้พื้นฐานและวิธีการต่าง ๆ ในข้อ 1-2 เพื่อการศึกษาและสร้างวิธีการใหม่สำหรับการจำกัดเวลาการมีเสถียรภาพและการทำให้เสถียรสำหรับระบบแบบเอกฐานไม่เชิงเส้นแบบไม่ต่อเนื่องทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วง
4. พิสูจน์ทฤษฎีของการจำกัดเวลาการมีเสถียรภาพและการทำให้เสถียรสำหรับระบบแบบเอกฐานไม่เชิงเส้นแบบไม่ต่อเนื่องทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วง
5. เขียนงานวิจัยและส่งไปตีพิมพ์ในวารสารระดับนานาชาติ

ผลงานวิจัยที่ได้รับ

ได้เงื่อนไขที่เพียงพอสำหรับการจำกัดเวลาการมีเสถียรภาพและการทำให้เสถียรสำหรับระบบแบบเอกฐานแบบไม่ต่อเนื่องทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วงและมีโปรแกรมคำนวณทางคณิตศาสตร์ (MATLAB) ของการจำกัดเวลาการมีเสถียรภาพและการทำให้เสถียรสำหรับระบบแบบเอกฐานแบบไม่ต่อเนื่องทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วง ซึ่งสามารถนำไปประยุกต์ใช้ในระบบเครื่องกล กระบวนการ ระบบอุตสาหกรรม ยานยนต์ ระบบไฟฟ้า ระบบเครื่องบิน ระบบการจราจร การผลิต เครือข่ายการสื่อสาร วิศวกรรมยานยนต์

กระบวนการทางเคมีและสาขาอื่น ๆ อีก มากมาย และงานวิจัยนี้มีประโยชน์มากมาย เช่น ลดภาวะโลกร้อน ลดการใช้พลังงาน ลดภาวะเรือนกระจก เพิ่มประสิทธิภาพการเรียนการสอน และเนื่องจากการจำกัดเวลา การมีเสถียรภาพและการทำให้เสถียรสำหรับระบบเอกฐานแบบไม่ต่อเนื่องทางเวลาที่มีตัวหน่วงขึ้นกับเวลาแบบช่วง ซึ่งเป็นระบบสมการทางคณิตศาสตร์ ซึ่งมีเนื้อหาเกี่ยวข้องกับวิชาแคลคูลัส วิชาสมการเชิงอนุพันธ์ วิชาวิธีการเชิงตัวเลข ดังนั้นสามารถนำมาบูรณาการกับการเรียนการสอนได้

ผลงานวิจัยที่ตีพิมพ์

1 Title: Exponential stability of semi-Markovian jump generalized neural networks with interval time-varying delays

Author(s): G Rajchakit and R Saravanakumar

Source: Neural Computing and Applications

Article Number: Volume 29, Issue 2, 1 January 2018, Pages 483–492

DOI: 10.1007/s00521-016-2461-y

Published: Volume 29, Issue 2, 1 January 2018, Pages 483–492

Impact Factor: 4.213

2 Title: Improved exponential convergence result for generalized neural networks including interval time-varying delayed signals

Author(s): Rajchakit, G., Saravanakumar, R., Ahn, C.K., Karimi, H.R.

Source: Neural Networks

Article Number: Volume 86, February 2017, Pages 10–17

DOI: 10.1016/j.neunet.2016.10.009

Published: Volume 86, 1 February 2017, Pages 10–17

Impact Factor: 7.197

3 Title: Extended dissipativity of generalised neural networks including time delays

Author(s): R. Saravanakumar, Grienggrai Rajchakit, M. Syed Ali & Young Hoon Joo

Source: International Journal of Systems Science

Article Number: Volume 48, Issue 11, 18 August 2017, Pages 2311–2320

DOI: 10.1080/00207721.2017.1316882

Published: Volume 48, Issue 11, 18 August 2017, Pages 2311–2320

Impact Factor: 2.185

4 Title: Robust extended dissipativity criteria for discrete-time uncertain neural networks with time-varying delays

Author(s): R. Saravanakumar, Grienggrai Rajchakit, M. Syed Ali, Zhengrong Xiang, Young Hoon Joo

Source: Neural Computing and Applications

Article Number: Volume 30, Issue 12, 1 December 2018, Pages 3893–3904

DOI: 10.1007/s00521-017-2974-z

Published: Volume 30, Issue 12, 1 December 2018, Pages 3893–3904

Impact Factor: 4.213

5 Title: Stabilization of Linear Differential-algebraic Equations with Time-varying Delay

Author(s): Grienggrai Rajchakit, Member, IAENG

Source: Proceedings of the International MultiConference of Engineers and Computer Scientists 2017 Vol I, IMECS 2017, March 15 – 17, 2017, Hong Kong

Article Number: 2017 Vol I, IMECS 2017 **ISBN:** 978-988-14047-3-2

ISSN: 2078-0958 (Print) **ISSN:** 2078-0966 (Online)

Published: The International Association of Engineers (IAENG)

6 Title: Improved stability analysis of delayed neural networks via Wirtinger-based double integral inequality

Author(s): R. Saravanakumar, M. Syed Ali, Grienggrai Rajchakit

Published in: Inventive Computation Technologies (ICICT), International Conference on

Date of Conference: 26–27 Aug. 2016 **Date Added to IEEE Xplore:** 26 January 2017

INSPEC Accession Number: 16620614 **DOI:** 10.1109/INVENTIVE.2016.7830198

Publisher: IEEE **Conference Location:** Coimbatore, India

7 Title: Enhanced robust finite-time passivity for Markovian jumping discrete-time BAM neural networks with leakage delay

Author(s): C Sowmiya, R Raja, J Cao, G Rajchakit, A Alsaedi

Source: Advances in Difference Equations

Article Number: 2017 (1), 318

DOI: 10.1186/s13662-017-1378-9

Published: 2017 (1), 318

Impact Factor: 1.06

8 Title: Impulsive Cohen–Grossberg BAM neural networks with mixed time–delays: An exponential stability analysis issue

Author(s): C Maharajan, R Raja, J Cao, G Rajchakit, A Alsaedi

Source: Neurocomputing

Article Number: 275, pp. 2588–2602 2018

DOI: 10.1016/j.neucom.2017.11.028

Published: 275, pp. 2588–2602 2018

Impact Factor: 3.241

9 Title: Exponential dissipativity criteria for generalized BAM neural networks with variable delays

Author(s): Young Hoon Joo R. Saravanakumar, Grienggrai Rajchakit, M. Syed Ali

Source: Neural Computing and Applications

Article Number: Article in press

DOI: 10.1007/s00521-017-3224-0

Published: 11 November 2017

Impact Factor: 4.213

10 Title: Nonlinear integro–differential equations with small–unknown parameters: A controllability analysis problem

Author(s): BS Vadivoo, R Raja, RA Seadawy, G Rajchakit

Source: Mathematics and Computers in Simulation

Article Number: Article in press

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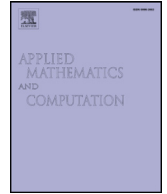
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ภาคผนวก



LMI-based results on exponential stability of BAM-type neural networks with leakage and both time-varying delays: A non-fragile state estimation approach

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ABSTRACT

In this epigrammatic, the problem of exponential stability for BAM-type neural networks (BAMNNs) with non-fragile state estimator is investigated under time-varying delays. The delays in discrete and distributed terms are assumed to be time-varying, which means that the lower and upper bounds can be derived. Without involving the time-delays or the activation functions, the non-fragile estimators are constructed in terms of simple linear formation and also the implementation of state estimators are uncomplicated. In addition, the non-fragile estimators are reduced the possible implementation errors in neural networks. For consequence, reason of energy saving, the non-fragile estimators are designed with neural networks. By fabricating a suitable LKF (Lyapunov–Krasovskii functional) and enrolling some analysis techniques, a novel sufficient conditions for exponential stability of the designated neural networks are derived in terms of Linear Matrix Inequalities (LMIs), which can be easily assessed by MATLAB LMI Control toolbox. Accordingly, the research proposed here, is advanced and less conservative than the previous one exists in the literature. Finally, two numerical examples with simulations and comparative studies are performed to substantiate the advantage and validity of our theoretical findings.

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1. Introduction

Recently, the study on neural networks (NNs) in dynamical behaviors has received considerable attention due to their potential applications in different fields, such as robot signal processing, automatic control, static image treatment, optimization problems, parallel computing, signal processing, etc [4,12,18,24]. In particular, bidirectional associative memory (BAM) neural networks is a special type of recurrent NNs, which was coined by Kosko [22,23]. A BAM neural networks is

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composed of neurons arranged in two layers: the R-layer and S-layer. The neurons in one layer are fully interconnected to the neurons in the other layer. This type of networks has been successfully applied to pattern recognition and artificial intelligence due to its generalization of the single-layer auto-associative Hebbian correlator to a two-layer pattern-matched hetero associative circuit. Over the past few decades, according to the extensive range of fruitful applications, such as pattern recognition, associative memory, parallel computing, combinational optimization, signal and image processing, BAMNNs with or without time delays have attracted much attention, see e.g. [5,7,20,42,45,60].

As is well known, the arising of time delays in neural networks is inevitable because the finite switching speed of information storage processing or of amplifiers, communication time and transmission of signals [37,40]. For sequence, in a NNs, the appearance of time delay may affect the stability performance generating unstable, divergence and swinging or chaotic behavior [9,28]. Hence, the investigation of stability analysis for time-delayed BAM neural networks has been broad interests from numerous scholars, see Refs. [2,6,14,34]. The exponential stability for stochastic BAMNNs was discussed in [2]. The delay-dependent and delay-independent criteria are two categories of time delays in neural networks, which classified by the existing time-delayed results. So far as, one can observe from available literatures, the delay-dependent case is less conserved than the delay independent ones. In [8], Cho and Park developed some time-delayed conditions for the stability analysis of neural network systems through novel approaches.

Moreover, we specify that the propagation delays can be distributed over a period of time, because the variety of axon sizes and lengths are too large. Thus, the consideration of distributed delay term is meaningful in neural networks system. So, it is significant to inspect both the time delay results in the dynamical behaviors of neural systems, see for instance [19,35,39,56]. Nie and Cao have investigated the Multistability of competitive neural networks with mixed time delays. The mixed time-delayed stability criterion for BAM neural networks under impulsive effects was discussed by the authors Zhu and Cao [59]. At the same time, to conclude the speed of neural computations, the exponential convergence rate is used. So, it generates much attention from many researchers to studied the property of exponential stability [15,29,31,36]. In [38], the researchers studied the exponential stability of time-delayed neural networks via different Lyapunov functionals. Some criteria in terms of LMIs were presented to ascertain the BAMNNs with both time-delays to be exponentially stable [26,57]. Wu et al. extensively analyzed the stability in the exponential sense of NNs in [52].

It is noteworthy point out that a different case of time delay in neural networks that which incorporated to the negative feedback term of the system, namely forgetting or leakage terms. The present of this type of delays owing to some theoretical and technical difficulties. By the issue of leakage delays, the stability of the dynamical systems may affect and it leads to be unstable. So, over the previous years, time delay in leakage term has a extensive impact on the neural networks, see for Refs. [43,49,50]. Thus, the consideration of leakage term in neural networks is necessary to stability investigation. Zhou et al. [58], investigated and obtained some results related to the problem for stability of time delayed neural networks with leakage terms. Recently, some authors discussed the neural networks with delays in the leakage terms (see [17,27,30]).

On the other hand the actually parameters in the phenomenon of estimator or controller are marginally distinct due to several reasons such as the imprecision in analog-digital conversion, limited word length of the computer and numerical roundoff errors. So, we can noticed that the drifts or variations of the controller or estimator parameters could guarantee to dramatic alterations or system instability or performance degradation of the dynamical system which fact indicates as the fragility issue [10,21]. Consequently, the non-fragility have been received much attention in the past few decades and which ensures that the dynamical system performance is unresponsive to the permissible implementation errors for the estimators [13,32,33,51]. Still, the formation of non-fragile state estimator have not been fully analyzed and, only limited research efforts have been completed with design of non-fragile estimators for the time-delayed neural network systems. Hence, the investigation of exponential stability of BAM neural networks through non-fragile technique is very complicated and interesting one. The authors in [54], non-fragile control phenomenon treated for a H_∞ problem of neural networks via uncertain parameters. To deep investigation of gain parameters of non-fragile estimator, Fang and Park [11] handled the time-delay dependent techniques and different inequality approaches in neural networks. Moreover, by the aid of Wirtinger based inequality techniques, two distinct cases of memductance functions and time-delayed uncertainties in memristive neural networks are analyzed in [46].

Motivated by the aforementioned discussions, our main motivation of this paper is to investigate the exponential stability problem for time-delayed BAM neural networks with leakage delays, via non-fragile state estimation. By employing the matrix theory, some inequality techniques and also constructing an appropriate novel Lyapunov–Krasovskii functional, some new-brand sufficient conditions for exponential stability are formulated in terms of LMIs, which can be justified easily by LMI control toolbox in MATLAB software. Also, two numerical examples with their simulations are provided to illustrate the superiority and applicability with less conservatism of the proposed method. The main contributions of this research work are outlined as follows:

- BAM-type, non-fragile state estimators, leakage, discrete & distributed time-varying delays are taken into account for the exponential stability analysis of proposed neural networks.
- In this work, a simple linear form of structured non-fragile state estimators are designed with three constant gain matrices and also the gain matrices in state estimators are permitted to accept multiplicative parameter variations with in a certain range. This multiplicative formations can excellent reveal the gain dependent perturbations.

Table 1
Nomenclature.

$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}$	the set of all real numbers
$\mathbb{R}^{n \times n}$	the set of all $n \times n$ matrices with real entries
k^T or K^T	the transpose of a vector k (or a matrix K)
K^{-1}	the inverse of a matrix K
$P > R (P \geq R)$	P - R is a positive definite (positive semi definite) matrix
$\begin{bmatrix} X & Y \\ * & Z \end{bmatrix}$	$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$
$\text{diag}\{\dots\}$	a block-diagonal matrix
$\ \alpha(t)\ $	$(\sum_{i=1}^n \alpha_i^2(t))^{\frac{1}{2}}$
$\lambda_{\max}(U)$ and $\lambda_{\min}(U)$	the maximum & minimum eigenvalues of a matrix U , respectively
I	the identity matrix with appropriate dimension

Table 2
Maximum allowable upper bounds (MAUBs)
 $\bar{\mu}_1$ & $\bar{\mu}_2$ of discrete time-varying delays.

Methods	$\bar{\mu}_1 = \bar{\mu}_2 > 0$
In [1]	0.5784
In [44]	2.1
In [41]	4.822
In [16]	5.912
In [41]	6.884
Theorem 3.1	19.297

Table 3
MAUBs of distributed time-varying delays.

Methods	[48]	[25]	[47]	Theorem 3.1
$\bar{\sigma}_1 = \bar{\sigma}_2 > 0$	0.2	0.5	0.5	12.46

- Based on contemporary Lyapunov–Krasovskii functional and integral inequality techniques, some sufficient conditions for exponential stability of BAM-type NNs are derived in the form of LMIs. In addition, compared to the existing results, the derived outcomes are different and advanced.
- In this paper, the feasibility of the obtained LMIs for exponential stability, easily solved by the aid of MATLAB LMI control toolbox.
- By handled the discrete time-varying delay and distributed time-varying delay terms in our concerned neural networks, the allowable upper bounds of discrete and distributed time-delays are maximum, when compare with some existing literatures, see Tables 2 and 3 in Example 4.1. This expressed that the approach developed in this paper is more effective and less conserved.

The designing of this paper is as follows. In Section 2, the concerned BAM-type neural networks with time-varying delays is presented and some preliminaries are introduced. By the Lyapunov–Krasovskii functionals and inequality techniques, some time-delay dependent conditions are attained for the exponential stability of proposed non-fragility dependent neural networks and the obtained results are formulated in the form of LMIs in Section 3. In Section 4, two numerical examples with comparisons are given with the help of MATLAB LMI control toolbox to validate the feasibility and effectiveness of the derived exponential stability criterions. Finally, conclusions are drawn in Section 5.

2. Model description and preliminaries

Consider the following BAM neural networks with mixed time-varying delays:

$$\begin{aligned} d\alpha(t) &= \left[-A\alpha(t - \delta_1) + Bf(\beta(t)) + Cf(\beta(t - \mu_1(t))) + \mathcal{D} \int_{t-\sigma_1(t)}^t f(\beta(s))ds \right] dt, \\ d\beta(t) &= \left[-\mathcal{L}\beta(t - \delta_2) + \mathcal{M}h(\alpha(t)) + \mathcal{N}h(\alpha(t - \mu_2(t))) + \mathcal{O} \int_{t-\sigma_2(t)}^t h(\alpha(s))ds \right] dt; \end{aligned} \quad (1)$$

where $\alpha(t) = [\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t)]^T$ and $\beta(t) = [\beta_1(t), \beta_2(t), \dots, \beta_n(t)]^T \in \mathbb{R}^n$ indicates the state vector of the neurons at time t , $f(\beta(\cdot)) = [f_1(\beta(\cdot)), f_2(\beta(\cdot)), \dots, f_n(\beta(\cdot))]^T$, $h(\alpha(\cdot)) = [h_1(\alpha(\cdot)), h_2(\alpha(\cdot)), \dots, h_n(\alpha(\cdot))]^T \in \mathbb{R}^n$ denotes the activation functions with the initial conditions $f(0) = h(0) = 0$, $A = \text{diag}\{a_1, a_2, \dots, a_n\}$ and $\mathcal{L} = \text{diag}\{l_1, l_2, \dots, l_n\}$ are the positive diagonal matrices, B , $\mathcal{M}$ stands for the connection weight matrices, C , $\mathcal{N}$ represents the discretely delayed connection

weight matrices and $\mathcal{D}$, $\mathcal{O}$ are the distributively delayed connection weight matrices, respectively. Also, $\mu_1(t)$, $\mu_2(t)$ and $\sigma_1(t)$, $\sigma_2(t)$ stands for the discrete and distributed time-varying delays, which satisfying

$$\begin{aligned} 0 &\leq \mu_1(t) \leq \bar{\mu}_1, \quad \dot{\mu}_1 \leq \mu_1, \\ 0 &\leq \mu_2(t) \leq \bar{\mu}_2, \quad \dot{\mu}_2 \leq \mu_2, \\ 0 &\leq \sigma_1(t) \leq \bar{\sigma}_1, \quad 0 \leq \sigma_2(t) \leq \bar{\sigma}_2, \end{aligned} \quad (2)$$

where $\bar{\mu}_1$, $\bar{\mu}_2$, $\bar{\sigma}_1$, $\bar{\sigma}_2$, μ_1 & μ_2 are positive scalars, and the leakage delays δ_1 and $\delta_2 \geq 0$ are constants.

Assume that the initial values of the BAMNNs (1) has the form $\alpha(t) = \psi(t)$, for $t \in [-\rho_1, 0]$ and $\beta(t) = \varphi(t)$, for $t \in [-\rho_2, 0]$, where $\psi(t)$ and $\varphi(t)$ are continuous functions, $\rho_1 = \max\{\delta_1, \bar{\mu}_1, \bar{\sigma}_1\}$ and $\rho_2 = \max\{\delta_2, \bar{\mu}_2, \bar{\sigma}_2\}$.

The concerned neural networks have the measurements in form of the following expressions:

$$\begin{aligned} u(t) &= \mathcal{E}_1 \alpha(t - \delta_1) + g(t, \beta(t)), \\ v(t) &= \mathcal{E}_2 \beta(t - \delta_2) + \tilde{g}(t, \alpha(t)), \end{aligned} \quad (3)$$

where $u(t)$ and $v(t) \in \mathbb{R}^n$ are the measurement output, $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{R}^{n \times n}$ are known matrices with constant parameters. $g(t, \beta(t))$ & $\tilde{g}(t, \alpha(t))$ denotes the nonlinear perturbation at the output of the state-dependent NNs (1) with the initial conditions $g(0, 0) = \tilde{g}(0, 0) = 0$.

Assumption 1. The neuron activation functions $f(\cdot)$ & $h(\cdot)$ satisfies the Lipschitz condition as follows:

$$\begin{aligned} |f(s_1) - f(s_2)| &\leq |\mathcal{K}_1(s_1 - s_2)|, \\ |h(s_3) - h(s_4)| &\leq |\mathcal{K}_2(s_3 - s_4)|, \end{aligned}$$

where $s_1, s_2, s_3, s_4 \in \mathbb{R}$ and $\mathcal{K}_1 = \text{diag}\{f_1, f_2, \dots, f_n\}$, $\mathcal{K}_2 = \text{diag}\{h_1, h_2, \dots, h_n\}$ are the known diagonal matrices.

Assumption 2. The nonlinear functions of the measurements satisfies the following inequalities

$$\begin{aligned} |g(t, p_1) - g(t, p_2)| &\leq |G(p_1 - p_2)|, \\ |\tilde{g}(t, p_3) - \tilde{g}(t, p_4)| &\leq |\bar{G}(p_3 - p_4)|, \end{aligned}$$

where $p_1, p_2, p_3, p_4 \in \mathbb{R}$, $G = \text{diag}\{g_1, g_2, \dots, g_n\}$ and $\bar{G} = \text{diag}\{\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n\}$ are the known constant matrices.

In this manuscript, the designed non-fragile state controllers or estimators has the following expressions:

$$\begin{aligned} d\tilde{\alpha}(t) &= [(\mathcal{B}_G + \Delta\mathcal{B}_G)\tilde{\alpha}(t) + (\mathcal{C}_G + \Delta\mathcal{C}_G)u(t) + (\mathcal{D}_G + \Delta\mathcal{D}_G)u(t)]dt, \\ d\tilde{\beta}(t) &= [(\mathcal{M}_{\tilde{G}} + \Delta\mathcal{M}_{\tilde{G}})\tilde{\beta}(t) + (\mathcal{N}_{\tilde{G}} + \Delta\mathcal{N}_{\tilde{G}})v(t) + (\mathcal{O}_{\tilde{G}} + \Delta\mathcal{O}_{\tilde{G}})v(t)]dt, \end{aligned} \quad (4)$$

where $\tilde{\alpha}$ & $\tilde{\beta}$ refers the state estimations of the neural networks (1), $\mathcal{B}_G, \mathcal{C}_G, \mathcal{D}_G, \mathcal{M}_{\tilde{G}}, \mathcal{N}_{\tilde{G}}$ and $\mathcal{O}_{\tilde{G}}$ denotes the estimator gain matrices to be designed. $\Delta\mathcal{B}_G, \Delta\mathcal{C}_G, \Delta\mathcal{D}_G, \Delta\mathcal{M}_{\tilde{G}}, \Delta\mathcal{N}_{\tilde{G}}$ and $\Delta\mathcal{O}_{\tilde{G}}$ compute the gain variations of the state estimators satisfying the norm-bounded multiplicative form as follows:

$$\begin{aligned} \Delta\mathcal{B}_G &= \mathcal{B}_G X_B \Sigma(t) U_B, \quad \Delta\mathcal{C}_G = \mathcal{C}_G X_C \Sigma(t) U_C, \\ \Delta\mathcal{D}_G &= \mathcal{D}_G X_D \Sigma(t) U_D, \quad \Delta\mathcal{M}_{\tilde{G}} = \mathcal{M}_{\tilde{G}} Y_M \Sigma^*(t) W_M, \\ \Delta\mathcal{N}_{\tilde{G}} &= \mathcal{N}_{\tilde{G}} Y_N \Sigma^*(t) W_N, \quad \Delta\mathcal{O}_{\tilde{G}} = \mathcal{O}_{\tilde{G}} Y_O \Sigma^*(t) W_O, \end{aligned}$$

where $X_B, X_C, X_D, Y_M, Y_N, Y_O, U_A, U_C, U_D, W_M, W_N$ and W_O are known matrices with suitable dimensions, $\Sigma(t)$ and $\Sigma^*(t)$ indicates unknown matrices which satisfying $\Sigma^T(t)\Sigma(t) \leq I$ & $\Sigma^{*T}(t)\Sigma^*(t) \leq I$.

Remark 2.1. It is noteworthy point out that, in Assumption 1, the activation functions are less conservative than the traditional Lipschitz condition and more general than the usual sigmoid activation functions. In addition, these functions will playing in our proposed methods to get necessary conditions for the neural networks (1) which leads to exponentially stable.

To analysis purpose, we denote $\xi(t) = [\beta^T(t)\tilde{\beta}^T(t)]^T$, $\varsigma(t) = [\alpha^T(t)\tilde{\alpha}^T(t)]^T$, $\Gamma_1(\xi(t)) = [f^T(\beta(t))f^T(\tilde{\beta}(t))]^T$, $\Gamma_2(\varsigma(t)) = [h^T(\alpha(t))h^T(\tilde{\alpha}(t))]^T$, $\Gamma_3(t, \xi(t)) = [g^T(t, \beta(t))g^T(t, \tilde{\beta}(t))]^T$, $\Gamma_4(t, \varsigma(t)) = [\tilde{g}^T(t, \alpha(t))\tilde{g}^T(t, \tilde{\alpha}(t))]^T$.

Consolidate the state estimators (4) with neural networks (1), one can obtain the following improved system as

$$\begin{aligned} d\varsigma(t) &= \left[(\bar{A}_1 + \bar{B})\varsigma(t - \delta_1) + \bar{A}_2\varsigma(t) + \bar{C}\Gamma_1(\xi(t)) + \bar{D}\Gamma_1(\xi(t - \mu_1(t))) \right. \\ &\quad \left. + \bar{E}_1 \int_{t-\sigma_1(t)}^t \Gamma_1(\xi(s))ds + \bar{E}_2\Gamma_3(t, \xi(t)) \right] dt, \\ d\xi(t) &= \left[(\bar{M}_1 + \bar{M}_3)\xi(t - \delta_2) + \bar{M}_2\xi(t) + \bar{N}_1\Gamma_2(\varsigma(t)) + \bar{N}_2\Gamma_2(\varsigma(t - \mu_2(t))) \right. \\ &\quad \left. + \bar{O}_1 \int_{t-\sigma_2(t)}^t \Gamma_2(\varsigma(s))ds + \bar{O}_2\Gamma_4(t, \varsigma(t)) \right] dt, \end{aligned} \quad (5)$$

where

$$\begin{aligned}\bar{A}_1 &= \text{diag}\{-\mathcal{A}, 0\}, \\ \bar{A}_2 &= \text{diag}\{0, (\mathcal{B}_G + \Delta\mathcal{B}_G)\}, \\ \bar{B} &= \begin{bmatrix} 0 & 0 \\ (\mathcal{C}_G + \Delta\mathcal{C}_G)\mathcal{E}_1 + (\mathcal{D}_G + \Delta\mathcal{D}_G)\mathcal{E}_1 & 0 \end{bmatrix}, \\ \bar{C} &= \text{diag}\{\mathcal{B}, 0\}, \\ \bar{D} &= \text{diag}\{\mathcal{C}, 0\}, \\ \bar{E}_1 &= \text{diag}\{\mathcal{D}, 0\}, \\ \bar{E}_2 &= \begin{bmatrix} 0 & 0 \\ (\mathcal{C}_G + \Delta\mathcal{C}_G) + \mathcal{D}_G + \Delta\mathcal{D}_G & 0 \end{bmatrix}, \\ \bar{M}_1 &= \text{diag}\{-\mathcal{L}, 0\}, \\ \bar{M}_2 &= \text{diag}\{0, \mathcal{M}_{\bar{C}} + \Delta\mathcal{M}_{\bar{C}}\}, \\ \bar{M}_3 &= \begin{bmatrix} 0 & 0 \\ (\mathcal{N}_{\bar{C}} + \Delta\mathcal{N}_{\bar{C}})\mathcal{E}_2 + (\mathcal{O}_{\bar{C}} + \Delta\mathcal{O}_{\bar{C}})\mathcal{E}_2 & 0 \end{bmatrix}, \\ \bar{N}_1 &= \text{diag}\{\mathcal{M}, 0\}, \\ \bar{N}_2 &= \text{diag}\{\mathcal{N}, 0\}, \\ \bar{O}_1 &= \text{diag}\{\mathcal{O}, 0\}, \\ \bar{O}_2 &= \begin{bmatrix} 0 & 0 \\ (\mathcal{N}_{\bar{C}} + \Delta\mathcal{N}_{\bar{C}}) + (\mathcal{O}_{\bar{C}} + \Delta\mathcal{O}_{\bar{C}}) & 0 \end{bmatrix}.\end{aligned}$$

By [Assumptions 1](#) and [2](#), we have

$$\Gamma_1^T(\xi(t))\Gamma_1(\xi(t)) \leq \xi^T(t)\bar{\mathcal{K}}_1^T\bar{\mathcal{K}}_1\xi(t), \quad (6)$$

$$\Gamma_2^T(\varsigma(t))\Gamma_2(\varsigma(t)) \leq \varsigma^T(t)\bar{\mathcal{K}}_2^T\bar{\mathcal{K}}_2\varsigma(t), \quad (7)$$

$$\Gamma_3^T(t, \xi(t))\Gamma_3(t, \xi(t)) \leq \xi^T(t)\hat{\mathcal{G}}^T\hat{\mathcal{G}}\xi(t), \quad (8)$$

$$\Gamma_4^T(t, \varsigma(t))\Gamma_4(t, \varsigma(t)) \leq \varsigma^T(t)\tilde{\mathcal{G}}^T\tilde{\mathcal{G}}\varsigma(t), \quad (9)$$

where $\bar{\mathcal{K}}_1 = \text{diag}\{\mathcal{K}_1, \mathcal{K}_1\}$, $\bar{\mathcal{K}}_2 = \text{diag}\{\mathcal{K}_2, \mathcal{K}_2\}$, $\hat{\mathcal{G}} = \text{diag}\{G, G\}$, $\tilde{\mathcal{G}} = \text{diag}\{\bar{G}, \bar{G}\}$.

Remark 2.2. By the aid of [Assumptions 1](#) and [2](#), [Theorems 3.1](#) and [3.2](#) are provides some developed sufficient conditions to guarantee the stability in exponential sense of the BAM-type neural networks with both time-varying delays [\(5\)](#). The LMIs in [Eqs. \(10\)–\(15\)](#), can be checked easily by LMI control toolbox in MATLAB software.

Definition 2.3. The enhanced neural networks [\(5\)](#) with non-fragile state estimators is said to be an exponentially stable if, there exist constants $\hat{\eta} > 0$ and $\omega > 0$ such that for $t \geq 0$

$$\|\varsigma(t)\|^2 + \|\xi(t)\|^2 \leq \hat{\eta}\{\|\varsigma(0)\|^2 + \|\xi(0)\|^2\}e^{-\omega t},$$

where $\varsigma(0)$ and $\xi(0)$ are the initial values of $\varsigma(t)$ & $\xi(t)$, respectively.

Lemma 2.4. (Schur Complement [\[3\]](#)) Given constant matrices Z_{11} , Z_{12} and Z_{22} with appropriate dimensions with $Z_{11} = Z_{11}^T$, and $Z_{22} = Z_{22}^T$, then the LMI

$$\begin{pmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{pmatrix} < 0,$$

is equivalent to $Z_{22} < 0$, and $Z_{11} - Z_{12}Z_{22}^{-1}Z_{12} < 0$.

Lemma 2.5. [\[53\]](#) Let $W = W^T$, J and H be real matrices of appropriate dimensions, with F satisfying $F^TF \leq I$, then

$$W + JFH + H^TF^TJ^T < 0$$

if and only if there exists a positive scalar $\varepsilon > 0$ such that

$$W + \frac{1}{\varepsilon}JJ^T + \varepsilon H^TH < 0,$$

(or) equivalently,

$$\begin{pmatrix} W & J & \varepsilon H^T \\ J^T & -\varepsilon I & 0 \\ \varepsilon H & 0 & -\varepsilon I \end{pmatrix} < 0.$$

Remark 2.6. Owing to programming errors, digital-analog conversation and finite resolution measuring instruments, the implementation of the designed estimator or controller is frequently inaccurate. So, it is point out that the existence of non-fragile state estimation in neural network problem is meaningful. It is notice that, the bounds of the time-varying delays, the bounds of uncertain parameters, all the variables in neural networks and the sector-bounds of neurons activation functions are reflected entirely in obtained results to guarantee the exponential stability of addressed neural networks.

Remark 2.7. In [55], Zha et al. investigates the impact of non-fragile state estimator in neural networks to find the asymptotic stability conditions via LMIs, and the authors discussed about the time-delayed linear system when the non-fragile controls applied by the aid of some inequality techniques in [33]. Yan et al. [54], the H_∞ control phenomenon in neural networks with non-fragile estimator are fully addressed and some novel criteria for robustness derived. The authors in [51], studied and obtained some new findings on problem of stability in the sense of asymptotic for the time-delayed neural networks with non-fragile state observer.

To the best of our knowledge, from the above said references, we have clearly understood the stability problem of neural networks/system is considered only with discrete time-varying delays or analysis of stability only in asymptotic sense, but the distributed time-varying delays and leakage time delays has not been taken into BAM neural networks account and also no investigates the exponential stability criteria for BAMNNs with non-fragility estimator at a time. Thus, the consideration of above facts are advanced and challenged in this research work.

3. Main results

Theorem 3.1. Under Assumptions 1 and 2, for given estimator parameters B_G , C_G , $\mathcal{D}_G$, $\mathcal{M}_{\hat{G}}$, $\mathcal{N}_{\hat{G}}$, and $\mathcal{O}_{\hat{G}}$, if there exists positive definite matrices $\mathcal{Q}_1 > 0$, $\mathcal{Q}_2 > 0$, $\mathcal{Q} > 0$, $\hat{\mathcal{Q}} > 0$, $\mathcal{P} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} > 0$, $\mathcal{R} = \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{bmatrix} > 0$, $\mathcal{S}_1 > 0$, $\mathcal{S}_2 > 0$, $\mathcal{S}_3 > 0$, $\mathcal{S}_4 > 0$, $\mathcal{S}_5 > 0$ & $\mathcal{S}_6 > 0$, matrices $\mathcal{T}_j$ ($j = 1, 2, \dots, 8$), $\mathbb{B}_1 = \begin{bmatrix} \mathbb{B}_{111} & \mathbb{B}_{112} \\ \mathbb{B}_{121} & \mathbb{B}_{122} \end{bmatrix}$, $\mathbb{B}_2 = \begin{bmatrix} \mathbb{B}_{211} & \mathbb{B}_{212} \\ \mathbb{B}_{221} & \mathbb{B}_{222} \end{bmatrix}$, $\mathbb{B}_3 = \begin{bmatrix} \mathbb{B}_{311} & \mathbb{B}_{312} \\ \mathbb{B}_{321} & \mathbb{B}_{322} \end{bmatrix}$, $\mathbb{B}_4 = \begin{bmatrix} \mathbb{B}_{411} & \mathbb{B}_{412} \\ \mathbb{B}_{421} & \mathbb{B}_{422} \end{bmatrix}$ and positive scalars γ_s ($s = 1, 2, \dots, 6$) such that the LMIs satisfied as follows :

$$\Lambda_1^* = \begin{bmatrix} \Lambda_{111} & \Lambda_{112} & \Lambda_{113} & 0 & 0 & \Lambda_{116} & \Lambda_{117} & \Lambda_{118} & \Lambda_{119} & \Lambda_{110} & \Lambda_{1111} & \tilde{\Lambda}_{1112} \\ * & \Lambda_{122} & \Lambda_{123} & 0 & 0 & \Lambda_{126} & \Lambda_{127} & \Lambda_{128} & \Lambda_{129} & \Lambda_{1210} & \Lambda_{1211} & \tilde{\Lambda}_{1212} \\ * & * & \Lambda_{133} & 0 & 0 & \Lambda_{136} & \Lambda_{137} & \Lambda_{138} & \Lambda_{139} & \Lambda_{1310} & \Lambda_{1311} & \tilde{\Lambda}_{1312} \\ * & * & * & \Lambda_{144} & \Lambda_{145} & \Lambda_{146} & 0 & 0 & \Lambda_{149} & \Lambda_{1410} & 0 & 0 \\ * & * & * & * & \Lambda_{155} & 0 & 0 & 0 & \Lambda_{159} & \Lambda_{1510} & 0 & 0 \\ * & * & * & * & * & \Lambda_{166} & \Lambda_{167} & \Lambda_{168} & \Lambda_{169} & \Lambda_{1610} & \Lambda_{1611} & \tilde{\Lambda}_{1612} \\ * & * & * & * & * & * & \Lambda_{177} & \Lambda_{178} & \Lambda_{179} & \Lambda_{1710} & \Lambda_{1711} & \tilde{\Lambda}_{1712} \\ * & * & * & * & * & * & * & \Lambda_{188} & \Lambda_{189} & \Lambda_{1810} & \Lambda_{1811} & \tilde{\Lambda}_{1812} \\ * & * & * & * & * & * & * & * & \Lambda_{199} & \Lambda_{1910} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Lambda_{11010} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & \Lambda_{11111} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Lambda_{11212}^* \end{bmatrix} < 0, \quad (10)$$

$$\Lambda_3^* = \begin{bmatrix} \Lambda_{311} & \Lambda_{312} & \Lambda_{313} & 0 & 0 & \Lambda_{316} & \Lambda_{317} & \Lambda_{318} & \Lambda_{319} & \Lambda_{310} & \Lambda_{3111} & \tilde{\Lambda}_{3112} \\ * & \Lambda_{322} & \Lambda_{323} & 0 & 0 & \Lambda_{326} & \Lambda_{327} & \Lambda_{328} & \Lambda_{329} & \Lambda_{3210} & \Lambda_{3211} & \tilde{\Lambda}_{3212} \\ * & * & \Lambda_{333} & 0 & 0 & \Lambda_{336} & \Lambda_{337} & \Lambda_{338} & \Lambda_{339} & \Lambda_{3310} & \Lambda_{3311} & \tilde{\Lambda}_{3312} \\ * & * & * & \Lambda_{344} & \Lambda_{345} & \Lambda_{346} & 0 & 0 & \Lambda_{349} & \Lambda_{3410} & 0 & 0 \\ * & * & * & * & \Lambda_{355} & 0 & 0 & 0 & \Lambda_{359} & \Lambda_{3510} & 0 & 0 \\ * & * & * & * & * & \Lambda_{366} & \Lambda_{367} & \Lambda_{368} & \Lambda_{369} & \Lambda_{3610} & \Lambda_{3611} & \tilde{\Lambda}_{3612} \\ * & * & * & * & * & * & \Lambda_{377} & \Lambda_{378} & \Lambda_{379} & \Lambda_{3710} & \Lambda_{3711} & \tilde{\Lambda}_{3712} \\ * & * & * & * & * & * & * & \Lambda_{388} & \Lambda_{389} & \Lambda_{3810} & \Lambda_{3811} & \tilde{\Lambda}_{3812} \\ * & * & * & * & * & * & * & * & \Lambda_{399} & \Lambda_{3910} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Lambda_{31010} & * & 0 \\ * & * & * & * & * & * & * & * & * & * & \Lambda_{31111} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Lambda_{31212}^* \end{bmatrix} < 0, \quad (11)$$

$$\Phi_1 = \begin{bmatrix} \mathbb{B}_1 & \mathcal{T}_1 \\ * & \mathcal{S}_1 \end{bmatrix}^T \geq 0, \quad \Phi_2 = \begin{bmatrix} \mathbb{B}_1 & \mathcal{T}_2 \\ * & \mathcal{S}_1 \end{bmatrix}^T \geq 0, \quad (12)$$

$$\Phi_3 = \begin{bmatrix} \mathbb{B}_2 & \mathcal{T}_3 \\ * & \mathcal{S}_2 \end{bmatrix}^T \geq 0, \quad \Phi_4 = \begin{bmatrix} \mathbb{B}_2 & \mathcal{T}_4 \\ * & \mathcal{S}_2 \end{bmatrix}^T \geq 0, \quad (13)$$

$$\Phi_5 = \begin{bmatrix} \mathbb{B}_3 & \mathcal{T}_5 \\ * & \mathcal{S}_5 \end{bmatrix}^T \geq 0, \quad \Phi_6 = \begin{bmatrix} \mathbb{B}_3 & \mathcal{T}_6 \\ * & \mathcal{S}_5 \end{bmatrix}^T \geq 0, \quad (14)$$

$$\Phi_7 = \begin{bmatrix} \mathbb{B}_4 & \mathcal{T}_7 \\ * & \mathcal{S}_6 \end{bmatrix}^T \geq 0, \quad \Phi_8 = \begin{bmatrix} \mathbb{B}_4 & \mathcal{T}_8 \\ * & \mathcal{S}_6 \end{bmatrix}^T \geq 0, \quad (15)$$

where

$$\begin{aligned} \Lambda_{111} &= 2\bar{A}_2\mathcal{Q}_1 - 2\mathcal{Q}_1(\bar{A}_1 + \bar{B}) + \mathcal{R}_{11} + \bar{\mu}_2\mathcal{S}_2\bar{A}_2^T\bar{A}_2 + \mathcal{S}_4 + \bar{\sigma}_2\mathcal{S}_6\bar{A}_2\bar{A}_2^T + 2\mathcal{T}_3 + 2\mathcal{T}_7 + \gamma_2\bar{N}_1\bar{K}_2^T\bar{K}_2 \\ &\quad + (\bar{O}_2 + I)\gamma_6\bar{G}^T\bar{G} + \bar{\mu}_2\mathbb{B}_2 + \gamma_2\bar{K}_2^T\bar{K}_2 + \bar{\sigma}_2\mathbb{B}_4 + \delta_1, \quad \Lambda_{137} = \bar{\mu}_2\mathcal{S}_2\bar{C}^T\bar{E}_2 + \bar{\sigma}_2\mathcal{S}_6\bar{C}^T\bar{E}_2, \\ \Lambda_{112} &= 2(\bar{B} + \bar{A}_1)\mathcal{Q}_1 + 2(\bar{A}_1 + \bar{B}) + \bar{\mu}_2\mathcal{S}_2\bar{B}^T\bar{B} + \bar{\sigma}_2\mathcal{S}_6, \quad \Lambda_{113} = 2\mathcal{Q}_1\bar{C} + \bar{\mu}_2\mathcal{S}_2\bar{C}^T\bar{C} + \bar{\sigma}_2\mathcal{S}_6\bar{C}, \\ \Lambda_{116} &= 2\mathcal{Q}_1\bar{D} + \bar{\mu}_2\mathcal{S}_2\bar{D}^T\bar{D} + \bar{\sigma}_2\mathcal{S}_6\bar{D}^T\bar{D}, \quad \Lambda_{117} = 2\mathcal{Q}_1\bar{E}_2 + \bar{\mu}_2\mathcal{S}_2\bar{E}_2^T\bar{E}_2 + \bar{\sigma}_2\mathcal{S}_6\bar{E}_2^T\bar{E}_2, \\ \Lambda_{118} &= 2\mathcal{Q}_1\bar{E}_1 + \bar{\mu}_2\mathcal{S}_2\bar{E}_1^T\bar{E}_1 + \bar{\sigma}_2\mathcal{S}_6\bar{E}_1^T\bar{E}_1, \quad \Lambda_{122} = \bar{\mu}_2\mathcal{S}_2(\bar{A}_1 + \bar{B})^T(\bar{A}_1 + \bar{B}) + \bar{\sigma}_2\mathcal{S}_6 \\ &\quad (\bar{A}_1 + \bar{B})^T(\bar{A}_1 + \bar{B}), \quad \Lambda_{123} = \bar{\mu}_2\mathcal{S}_2(\bar{A}_1 + \bar{B})^T\bar{C} + \bar{\sigma}_2\mathcal{S}_6(\bar{A}_1 + \bar{B})^T\bar{C}, \quad \Lambda_{199} = \mathcal{P}_{22}\gamma_1\bar{K}_1^T\bar{K}_1, \\ \Lambda_{126} &= \bar{\mu}_2\mathcal{S}_2(\bar{A}_2 + \bar{B})^T\bar{D} + \bar{\sigma}_2\mathcal{S}_6(\bar{A}_1 + \bar{B})^T\bar{D}, \quad \Lambda_{127} = \bar{\mu}_2\mathcal{S}_2(\bar{A}_1 + \bar{B})^T\bar{E}_2 + \bar{\sigma}_2\mathcal{S}_6(\bar{A}_1 + \bar{B})^T\bar{E}_2, \\ \Lambda_{128} &= \bar{\mu}_2\mathcal{S}_2(\bar{A}_1 + \bar{B})^T\bar{E}_1 + \bar{\sigma}_2\mathcal{S}_6(\bar{A}_1 + \bar{B})^T\bar{E}_1, \quad \Lambda_{133} = \bar{\mu}_2\mathcal{S}_2\bar{C}^T\bar{C} + \bar{\sigma}_2\mathcal{S}_6\bar{C}^T\bar{C} - \gamma_1I, \\ \Lambda_{136} &= \bar{\mu}_2\mathcal{S}_2\bar{C}^T\bar{D} + \bar{\sigma}_2\mathcal{S}_6\bar{C}^T\bar{D}, \quad \Lambda_{155} = -e^{-\nu\bar{\mu}_1}\mathcal{S}_3, \quad \Lambda_{11010} = -\mathcal{S}_5, \quad \Lambda_{31111} = -\hat{\mathcal{Q}}, \\ \Lambda_{138} &= \bar{\mu}_2\mathcal{S}_2\bar{C}^T\bar{E}_1 + \bar{\sigma}_2\mathcal{S}_6\bar{C}^T\bar{E}_1, \quad \Lambda_{144} = -e^{-\nu\bar{\mu}_1}(1 - \bar{\mu}_1)\mathcal{P}_{11} - 2(\mathcal{T}_1 - \mathcal{T}_2) - e^{-\nu\bar{\mu}_1}(1 - \mu_1) \\ &\quad \gamma_3\bar{K}_1^T\bar{K}_1\mathcal{P}_{22} + \bar{\mu}_1\mathbb{B}_1 + \gamma_3\bar{K}_1^T\bar{K}_1, \quad \Lambda_{145} = -2\mathcal{T}_2, \quad \Lambda_{146} = -e^{-\nu\bar{\mu}_1}(1 - \mu_1)(\mathcal{P}_{12} + \mathcal{P}_{21}), \\ \Lambda_{166} &= \bar{\mu}_2\mathcal{S}_2\bar{D}^T\bar{D} + \bar{\sigma}_2\mathcal{S}_6\bar{D}^T\bar{D} - \gamma_3I, \quad \Lambda_{167} = \bar{\mu}_2\mathcal{S}_2\bar{D}^T\bar{E}_2 + \bar{\sigma}_2\mathcal{S}_6\bar{D}^T\bar{E}_2, \quad \Lambda_{31010} = -\mathcal{S}_6, \\ \Lambda_{168} &= \bar{\mu}_2\mathcal{S}_2\bar{D}^T\bar{E}_1 + \bar{\sigma}_2\mathcal{S}_6\bar{D}^T\bar{E}_1, \quad \Lambda_{177} = \bar{\mu}_2\mathcal{S}_2\bar{E}_2^T\bar{E}_2 + \bar{\sigma}_2\mathcal{S}_6\bar{E}_2^T\bar{E}_2 - \gamma_5I, \quad \Lambda_{11111} = -\mathcal{Q}, \\ \Lambda_{178} &= \bar{\mu}_2\mathcal{S}_2\bar{E}_2^T\bar{E}_1 + \bar{\sigma}_2\mathcal{S}_6\bar{E}_2^T\bar{E}_1, \quad \Lambda_{188} = \bar{\mu}_2\mathcal{S}_2\bar{E}_1^T\bar{E}_1 + \bar{\sigma}_2\mathcal{S}_6\bar{E}_1^T\bar{E}_1, \quad \Lambda_{3811} = -2(\bar{M}_1 + \bar{M}_3)\mathcal{Q}_2\bar{O}_1, \\ \Lambda_{311} &= 2\bar{M}_2\mathcal{Q}_2 - 2\mathcal{Q}_2(\bar{M}_1 + \bar{M}_3) + \mathcal{P}_{11} + \bar{\mu}_1\mathcal{S}_1\bar{M}_2^T\bar{M}_2 + \mathcal{S}_3 + \bar{\sigma}_1\mathcal{S}_5\bar{M}_2^T\bar{M}_2 + 2\mathcal{T}_1 + 2\mathcal{T}_5 + \gamma_1\bar{K}_1^T\bar{K}_1\bar{C} \\ &\quad + (\bar{E}_2 + I)\gamma_5\bar{G}^T\bar{G} + \bar{\mu}_1\mathbb{B}_1 + \bar{\sigma}_1\mathbb{B}_3 + \delta_2 + \gamma_1\bar{K}_1^T\bar{K}_1, \quad \Lambda_{316} = 2\mathcal{Q}_2\bar{N}_2 + \bar{\mu}_1\mathcal{S}_1\bar{N}_2^T\bar{N}_2 + \bar{\sigma}_1\mathcal{S}_5\bar{N}_2^T\bar{N}_2, \\ \Lambda_{312} &= 2(\bar{M}_1 + \bar{M}_3)\mathcal{Q}_2 + 2(\bar{M}_1 + \bar{M}_3) + \bar{\mu}_1\mathcal{S}_1\bar{M}_3^T\bar{M}_3 + \bar{\sigma}_1\mathcal{S}_5, \quad \Lambda_{3611} = -2(\bar{M}_1 + \bar{M}_3)\mathcal{Q}_2\bar{N}_2, \\ \Lambda_{317} &= 2\mathcal{Q}_2\bar{O}_2 + \bar{\mu}_1\mathcal{S}_1\bar{O}_2^T\bar{O}_2 + \bar{\sigma}_1\mathcal{S}_5\bar{O}_2^T\bar{O}_2, \quad \Lambda_{318} = 2\mathcal{Q}_2\bar{O}_1 + \bar{\mu}_1\mathcal{S}_1\bar{O}_1^T\bar{O}_1 + \bar{\sigma}_1\mathcal{S}_5\bar{O}_1^T\bar{O}_1, \\ \Lambda_{322} &= \bar{\mu}_1\mathcal{S}_1(\bar{M}_1 + \bar{M}_3)^T(\bar{M}_1 + \bar{M}_3) + \bar{\sigma}_1\mathcal{S}_5(\bar{M}_1 + \bar{M}_3)^T(\bar{M}_1 + \bar{M}_3), \quad \Lambda_{323} = \bar{\mu}_1\mathcal{S}_1(\bar{M}_1 + \bar{M}_3)^T \\ &\quad \bar{N}_1 + \bar{\sigma}_1\mathcal{S}_5(\bar{M}_1 + \bar{M}_3)^T\bar{N}_1, \quad \Lambda_{336} = \bar{\mu}_1\mathcal{S}_1\bar{N}_1^T\bar{N}_2 + \bar{\sigma}_1\mathcal{S}_5\bar{N}_1^T\bar{N}_2, \quad \Lambda_{1311} = -2(\bar{A}_1 + \bar{B})\mathcal{Q}_1\bar{C}, \\ \Lambda_{326} &= \bar{\mu}_1\mathcal{S}_1(\bar{M}_1 + \bar{M}_3)^T\bar{N}_2 + \bar{\sigma}_1\mathcal{S}_5(\bar{M}_1 + \bar{M}_3)^T\bar{N}_2, \quad \Lambda_{338} = \bar{\mu}_1\mathcal{S}_1\bar{N}_1^T\bar{O}_1 + \bar{\sigma}_1\mathcal{S}_5\bar{N}_1^T\bar{O}_1, \\ \Lambda_{328} &= \bar{\mu}_1\mathcal{S}_1(\bar{M}_1 + \bar{M}_3)^T\bar{O}_1 + \bar{\sigma}_1\mathcal{S}_5(\bar{M}_1 + \bar{M}_3)^T\bar{O}_1, \quad \Lambda_{333} = \bar{\mu}_1\mathcal{S}_1\bar{N}_1^T\bar{N}_1 + \bar{\sigma}_1\mathcal{S}_5\bar{N}_1^T\bar{N}_1 - \gamma_2I, \\ \Lambda_{327} &= \bar{\mu}_1\mathcal{S}_1(\bar{M}_1 + \bar{M}_3)^T\bar{O}_2 + \bar{\sigma}_1\mathcal{S}_5(\bar{M}_1 + \bar{M}_3)^T\bar{O}_2, \quad \Lambda_{346} = -e^{-\nu\bar{\mu}_2}(1 - \mu_2)(\mathcal{R}_{12} + \mathcal{R}_{21}), \\ \Lambda_{399} &= \mathcal{R}_{22}\gamma_2\bar{K}_2^T\bar{K}_2, \quad \Lambda_{337} = \bar{\mu}_1\mathcal{S}_1\bar{N}_1^T\bar{O}_2 + \bar{\sigma}_1\mathcal{S}_5\bar{N}_1^T\bar{O}_2, \quad \Lambda_{377} = \bar{\mu}_1\mathcal{S}_1\bar{O}_2^T\bar{O}_2 + \bar{\sigma}_1\mathcal{S}_5\bar{O}_2^T\bar{O}_2 - \gamma_6I, \\ \Lambda_{344} &= -e^{-\nu\bar{\mu}_2}(1 - \mu_2)\mathcal{R}_{11} - 2(\mathcal{T}_3 - \mathcal{T}_4) - e^{-\nu\bar{\mu}_2}(1 - \mu_2)\gamma_4\bar{K}_2^T\bar{K}_2\mathcal{R}_{22} + \bar{\mu}_2\mathbb{B}_2 + \gamma_4\bar{K}_2^T\bar{K}_2, \\ \Lambda_{367} &= \bar{\mu}_1\mathcal{S}_1\bar{N}_2^T\bar{O}_2 + \bar{\sigma}_1\mathcal{S}_5\bar{N}_2^T\bar{O}_2, \quad \Lambda_{368} = \bar{\mu}_1\mathcal{S}_1\bar{N}_2^T\bar{O}_1 + \bar{\sigma}_1\mathcal{S}_5\bar{N}_2^T\bar{O}_1, \quad \Lambda_{3711} = -2(\bar{M}_1 + \bar{M}_3)\mathcal{Q}_2\bar{O}_2, \\ \Lambda_{1111} &= -2(\bar{A}_1 + \bar{B})\mathcal{Q}_1\bar{A}_2 + 2(\bar{A}_1 + \bar{B})^2\mathcal{Q}_1, \quad \Lambda_{1211} = -2(\bar{A}_1 + \bar{B})\mathcal{Q}_1(\bar{A}_1 + \bar{B}) - 2(\bar{A}_1 + \bar{B})^2\mathcal{Q}_1, \\ \Lambda_{1611} &= -2(\bar{A}_1 + \bar{B})\mathcal{Q}_1\bar{D}, \quad \Lambda_{1711} = -2(\bar{A}_1 + \bar{B})\mathcal{Q}_1\bar{E}_2, \quad \Lambda_{1811} = -2(\bar{A}_1 + \bar{B})\mathcal{Q}_1\bar{E}_1, \\ \Lambda_{3211} &= -2(\bar{M}_1 + \bar{M}_3)\mathcal{Q}_2(\bar{M}_1 + \bar{M}_3) - 2(\bar{M}_1 + \bar{M}_3)^2\mathcal{Q}_2, \quad \Lambda_{3311} = -2(\bar{M}_1 + \bar{M}_3)\mathcal{Q}_2\bar{N}_1, \\ \Lambda_{3111} &= -2(\bar{M}_1 + \bar{M}_3)\mathcal{Q}_2\bar{M}_2 + 2(\bar{M}_1 + \bar{M}_3)^2\mathcal{Q}_2, \quad \Lambda_{313} = 2\mathcal{Q}_2\bar{N}_1 + \bar{\mu}_1\mathcal{S}_1\bar{N}_1^T\bar{N}_1 + \bar{\sigma}_1\mathcal{S}_5\bar{N}_1, \\ \Lambda_{378} &= \bar{\mu}_1\mathcal{S}_1\bar{O}_2^T\bar{O}_1 + \bar{\sigma}_1\mathcal{S}_5\bar{O}_2^T\bar{O}_1, \quad \Lambda_{388} = \bar{\mu}_1\mathcal{S}_1\bar{O}_1^T\bar{O}_1 + \bar{\sigma}_1\mathcal{S}_5\bar{O}_1^T\bar{O}_1, \quad \Lambda_{355} = -e^{-\nu\bar{\mu}_2}\mathcal{S}_4, \\ \Lambda_{345} &= -2\mathcal{T}_4, \quad \Lambda_{366} = \bar{\mu}_1\mathcal{S}_1\bar{N}_2^T\bar{N}_2 + \bar{\sigma}_1\mathcal{S}_5\bar{N}_2^T\bar{N}_2 - \gamma_4I, \quad \tilde{\Lambda}_{3612} = \bar{\mu}_2\bar{N}_2\mathcal{Q}_2, \quad \tilde{\Lambda}_{3712} = \bar{\mu}_2\bar{O}_2\mathcal{Q}_2, \\ \Lambda_{3112} &= \bar{M}_2\bar{\mu}_2\mathcal{S}_2, \quad \Lambda_{3212} = \bar{\mu}_2(\bar{M}_1 + \bar{M}_3)\mathcal{S}_2, \quad \Lambda_{3612} = \bar{\mu}_2\bar{N}_2\mathcal{S}_2, \quad \Lambda_{3712} = \bar{\mu}_2\bar{O}_2\mathcal{S}_2, \\ \Lambda_{31212} &= -\bar{\mu}_2\mathcal{S}_2, \quad \Lambda_{1112} = \bar{\mu}_1\mathcal{S}_1\bar{A}_2, \quad \Lambda_{1212} = \bar{\mu}_1\mathcal{S}_1(\bar{A}_1 + \bar{B}), \quad \Lambda_{3312} = \bar{\mu}_2\bar{N}_1\mathcal{S}_2, \end{aligned}$$

$$\begin{aligned}
\Lambda_{1312} &= \bar{\mu}_1 \bar{C} S_1, \quad \Lambda_{1612} = \bar{\mu}_1 \bar{D} S_1, \quad \Lambda_{1712} = \bar{E}_2 \bar{\mu}_1 S_1, \quad \Lambda_{3812} = \bar{\mu}_2 \bar{O}_1 S_2, \\
\Lambda_{1812} &= \bar{\mu}_1 \bar{E}_1 S_1, \quad \Lambda_{11212} = -\bar{\mu}_1 S_1, \quad \tilde{\Lambda}_{1112} = \bar{A}_2 Q_1 \bar{\mu}_1, \quad \tilde{\Lambda}_{1212} = \bar{\mu}_1 Q_1 (\bar{A}_1 + \bar{B}), \\
\tilde{\Lambda}_{1312} &= \bar{\mu}_1 \bar{C} Q_1, \quad \tilde{\Lambda}_{1612} = \bar{\mu}_1 \bar{E}_1 Q_1, \quad \tilde{\Lambda}_{1712} = \bar{E}_2 \bar{\mu}_1 Q_1, \quad \tilde{\Lambda}_{1812} = \bar{\mu}_1 \bar{E}_1 Q_1, \\
\tilde{\Lambda}_{11212} &= -\bar{\mu}_1 Q_1 S_1^{-1} Q_1, \quad \tilde{\Lambda}_{3112} = \bar{M}_2 \bar{\mu}_2 Q_2, \quad \tilde{\Lambda}_{3212} = \bar{\mu}_2 (\bar{M}_1 + \bar{M}_3) Q_2, \quad \tilde{\Lambda}_{3812} = \bar{\mu}_2 \bar{O}_1 Q_2, \\
\tilde{\Lambda}_{31212} &= -\bar{\mu}_2 Q_2 S_2^{-1} Q_2, \quad \Lambda_{11212}^* = -\bar{\mu}_1 (2Q_1 - S_1), \quad \Lambda_{31212}^* = -\bar{\mu}_2 (2Q_2 - S_2).
\end{aligned}$$

Then the BAM neural networks with non-fragile state estimator (1) or (5) is exponentially stable.

Proof. We consider the following Lyapunov–Krasovskii functional candidate as

$$V(\zeta(t), \xi(t)) = \sum_{i=1}^5 V_i(\zeta(t), \xi(t)), \quad (16)$$

where

$$\begin{aligned}
V_1(\zeta(t), \xi(t)) &= e^{vt} \left\{ \left[\zeta(t) - (\bar{A}_1 + \bar{B}) \int_{t-\delta_1}^t \zeta(s) ds \right]^T Q_1 \left[\zeta(t) - (\bar{A}_1 + \bar{B}) \int_{t-\delta_1}^t \zeta(s) ds \right] \right. \\
&\quad + \left[\xi(t) - (\bar{M}_1 + \bar{M}_3) \int_{t-\delta_2}^t \xi(s) ds \right]^T Q_2 \left[\xi(t) - (\bar{M}_1 + \bar{M}_3) \int_{t-\delta_2}^t \xi(s) ds \right] \Big\} \\
&\quad + \int_{-\delta_1}^0 \int_{t+\theta}^t e^{vs} \zeta^T(s) Q \zeta(s) ds d\theta + \int_{-\delta_2}^0 \int_{t+\theta}^t e^{vs} \xi^T(s) \hat{Q} \xi(s) ds d\theta, \\
V_2(\zeta(t), \xi(t)) &= \int_{t-\mu_1(t)}^t e^{vs} \begin{bmatrix} \xi(s) \\ \Gamma_1(\xi(s)) \end{bmatrix}^T \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix}^T \begin{bmatrix} \xi(s) \\ \Gamma_1(\xi(s)) \end{bmatrix} ds \\
&\quad + \int_{t-\mu_2(t)}^t e^{vs} \begin{bmatrix} \zeta(s) \\ \Gamma_2(\zeta(s)) \end{bmatrix}^T \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{bmatrix}^T \begin{bmatrix} \zeta(s) \\ \Gamma_2(\zeta(s)) \end{bmatrix} ds, \\
V_3(\zeta(t), \xi(t)) &= \int_{-\bar{\mu}_1}^0 \int_{t+s}^t e^{v\theta} \xi^T(\theta) S_1 \dot{\xi}(\theta) d\theta ds + \int_{-\bar{\mu}_2}^0 \int_{t+s}^t e^{v\theta} \zeta^T(\theta) S_2 \dot{\zeta}(\theta) d\theta ds, \\
V_4(\zeta(t), \xi(t)) &= \int_{t-\bar{\mu}_1}^t e^{v\theta} \xi^T(\theta) S_3 \dot{\xi}(\theta) d\theta + \int_{t-\bar{\mu}_2}^t e^{v\theta} \zeta^T(\theta) S_4 \dot{\zeta}(\theta) d\theta, \\
V_5(\zeta(t), \xi(t)) &= \int_{-\bar{\sigma}_1}^0 \int_{t+s}^t e^{v\theta} \xi^T(\theta) S_5 \dot{\xi}(\theta) d\theta ds + \int_{-\bar{\sigma}_2}^0 \int_{t+s}^t e^{v\theta} \zeta^T(\theta) S_6 \dot{\zeta}(\theta) d\theta ds
\end{aligned}$$

Then, computing the derivative of $V_i(\zeta(t), \xi(t))$ ($i = 1, 2, 3, \dots, 6$) along the solutions of neural networks (5), one can get that

$$\begin{aligned}
\dot{V}_1(\zeta(t), \xi(t)) &= e^{vt} \left\{ 2 \left[\zeta(t) - (\bar{A}_1 + \bar{B}) \int_{t-\delta_1}^t \zeta(s) ds \right]^T Q_1 (\dot{\zeta}(t) - (\bar{A}_1 + \bar{B}) \zeta(t) \right. \right. \\
&\quad + (\bar{A}_1 + \bar{B}) \zeta(t - \delta_1)) + 2 \left[\xi(t) - (\bar{M}_1 + \bar{M}_3) \int_{t-\delta_2}^t \xi(s) ds \right]^T Q_2 \\
&\quad \times (\dot{\xi}(t) - (\bar{M}_1 + \bar{M}_3) \xi(t) + (\bar{M}_1 + \bar{M}_3) \xi(t - \delta_2)) \Big\} + \delta_1 e^{vt} \zeta^T(t) Q \\
&\quad \times \zeta(t) - \int_{t-\delta_1}^t e^{vs} \zeta^T(s) Q \zeta(s) ds + \delta_2 e^{vt} \xi^T(t) \hat{Q} \xi(t) - \int_{t-\delta_2}^t e^{vs} \xi^T(s) \\
&\quad \times \hat{Q} \xi(s) ds, \\
\dot{V}_2(\zeta(t), \xi(t)) &= e^{vt} \begin{bmatrix} \xi(t) \\ \Gamma_1(\xi(t)) \end{bmatrix}^T \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix}^T \begin{bmatrix} \xi(t) \\ \Gamma_1(\xi(t)) \end{bmatrix} - e^{v(t-\mu_1(t))} (1 - \dot{\mu}_1(t)) \\
&\quad \times \begin{bmatrix} \xi(t - \mu_1(t)) \\ \Gamma_1(\xi(t - \mu_1(t))) \end{bmatrix}^T \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix}^T \begin{bmatrix} \xi(t - \mu_1(t)) \\ \Gamma_1(\xi(t - \mu_1(t))) \end{bmatrix} + e^{vt} \\
&\quad \times \begin{bmatrix} \zeta(t) \\ \Gamma_2(\zeta(t)) \end{bmatrix}^T \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{bmatrix}^T \begin{bmatrix} \zeta(t) \\ \Gamma_2(\zeta(t)) \end{bmatrix} - e^{v(t-\mu_2(t))} (1 - \dot{\mu}_2(t))
\end{aligned}$$

$$\begin{aligned} & \times \begin{bmatrix} \zeta(t - \mu_2(t)) \\ \Gamma_2(\xi(t - \mu_2(t))) \end{bmatrix}^T \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{bmatrix}^T \begin{bmatrix} \zeta(t - \mu_2(t)) \\ \Gamma_2(\zeta(t - \mu_2(t))) \end{bmatrix} \\ & \leq e^{\nu t} \left\{ \begin{bmatrix} \xi(t) \\ \Gamma_1(\xi(t)) \end{bmatrix}^T \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix}^T \begin{bmatrix} \xi(t) \\ \Gamma_1(\xi(t)) \end{bmatrix} - e^{-\nu \bar{\mu}_1} (1 - \mu_1) \right. \\ & \quad \times \begin{bmatrix} \xi(t - \mu_1(t)) \\ \Gamma_1(\xi(t - \mu_1(t))) \end{bmatrix}^T \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix}^T \begin{bmatrix} \xi(t - \mu_1(t)) \\ \Gamma_1(\xi(t - \mu_1(t))) \end{bmatrix} \\ & \quad + \begin{bmatrix} \zeta(t) \\ \Gamma_2(\zeta(t)) \end{bmatrix}^T \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{bmatrix}^T \begin{bmatrix} \zeta(t) \\ \Gamma_2(\zeta(t)) \end{bmatrix} - e^{-\nu \mu_2} (1 - \mu_2) \\ & \quad \left. \times \begin{bmatrix} \zeta(t - \mu_2(t)) \\ \Gamma_2(\xi(t - \mu_2(t))) \end{bmatrix}^T \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{bmatrix}^T \begin{bmatrix} \zeta(t - \mu_2(t)) \\ \Gamma_2(\zeta(t - \mu_2(t))) \end{bmatrix} \right\}, \end{aligned} \quad (17)$$

$$\begin{aligned} \dot{\mathcal{V}}_3(\zeta(t), \xi(t)) &= \bar{\mu}_1 e^{\nu t} \dot{\xi}^T(t) \mathcal{S}_1 \dot{\xi}(t) - \int_{t-\bar{\mu}_1}^t e^{\nu \theta} \dot{\xi}^T(\theta) \mathcal{S}_1 \dot{\xi}(\theta) d\theta + \bar{\mu}_2 e^{\nu t} \dot{\zeta}^T(t) \\ & \quad \times \mathcal{S}_2 \dot{\zeta}(t) - \int_{t-\bar{\mu}_2}^t e^{\nu \theta} \dot{\zeta}^T(\theta) \mathcal{S}_2 \dot{\zeta}(\theta) d\theta \\ & \leq \bar{\mu}_1 e^{\nu t} \dot{\xi}^T(t) \mathcal{S}_1 \dot{\xi}(t) - \int_{t-\mu_1(t)}^t e^{\nu \theta} \dot{\xi}^T(\theta) \mathcal{S}_1 \dot{\xi}(\theta) d\theta - \int_{t-\bar{\mu}_1}^{t-\mu_1(t)} e^{\nu \theta} \\ & \quad \times \dot{\xi}^T(\theta) \mathcal{S}_1 \dot{\xi}(\theta) d\theta + \bar{\mu}_2 e^{\nu t} \dot{\zeta}^T(t) \mathcal{S}_2 \dot{\zeta}(t) - \int_{t-\mu_2(t)}^t e^{\nu \theta} \dot{\zeta}^T(\theta) \mathcal{S}_2 \\ & \quad \times \dot{\zeta}(\theta) d\theta - \int_{t-\bar{\mu}_2}^{t-\mu_2(t)} e^{\nu \theta} \dot{\zeta}^T(\theta) \mathcal{S}_2 \dot{\zeta}(\theta) d\theta, \end{aligned} \quad (18)$$

$$\begin{aligned} \dot{\mathcal{V}}_4(\zeta(t), \xi(t)) &= e^{\nu t} \left\{ \dot{\xi}^T(t) \mathcal{S}_3 \dot{\xi}(t) - e^{-\nu \bar{\mu}_1} \dot{\xi}^T(t - \bar{\mu}_1) \mathcal{S}_3 \dot{\xi}(t - \bar{\mu}_1) + \dot{\zeta}^T(t) \mathcal{S}_4 \dot{\zeta}(t) \right. \\ & \quad \left. - e^{-\nu \bar{\mu}_2} \dot{\zeta}^T(t - \bar{\mu}_2) \mathcal{S}_4 \dot{\zeta}(t - \bar{\mu}_2) \right\}, \end{aligned} \quad (19)$$

$$\begin{aligned} \dot{\mathcal{V}}_5(\zeta(t), \xi(t)) &= \bar{\sigma}_1 e^{\nu t} \dot{\xi}^T(t) \mathcal{S}_5 \dot{\xi}(t) - \int_{t-\bar{\sigma}_1}^t e^{\nu \theta} \dot{\xi}^T(\theta) \mathcal{S}_5 \dot{\xi}(\theta) d\theta + \bar{\sigma}_2 e^{\nu t} \dot{\zeta}^T(t) \mathcal{S}_6 \\ & \quad \times \dot{\zeta}(t) - \int_{t-\bar{\sigma}_2}^t e^{\nu \theta} \dot{\zeta}^T(\theta) \mathcal{S}_6 \dot{\zeta}(\theta) d\theta \\ & \leq \bar{\sigma}_1 e^{\nu t} \dot{\xi}^T(t) \mathcal{S}_5 \dot{\xi}(t) - \int_{t-\sigma_1(t)}^t e^{\nu \theta} \dot{\xi}^T(\theta) \mathcal{S}_5 \dot{\xi}(\theta) d\theta - \int_{t-\bar{\sigma}_1}^{t-\sigma_1(t)} e^{\nu \theta} \\ & \quad \times \dot{\xi}^T(\theta) \mathcal{S}_5 \dot{\xi}(\theta) d\theta + \bar{\sigma}_2 e^{\nu t} \dot{\zeta}^T(t) \mathcal{S}_6 \dot{\zeta}(t) - \int_{t-\sigma_2(t)}^t e^{\nu \theta} \dot{\zeta}^T(\theta) \mathcal{S}_6 \\ & \quad \times \dot{\zeta}(\theta) d\theta - \int_{t-\bar{\sigma}_2}^{t-\sigma_2(t)} e^{\nu \theta} \dot{\zeta}^T(\theta) \mathcal{S}_6 \dot{\zeta}(\theta) d\theta. \end{aligned} \quad (20)$$

Now, we utilize the Leibniz–Newton formula, we have the following zero equations, for matrices $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6, \mathcal{T}_7$ and $\mathcal{T}_8$ with appropriate dimensions as

$$0 = 2\chi_1^T(t) \mathcal{T}_1 \left[\xi(t) - \xi(t - \mu_1(t)) - \int_{t-\mu_1(t)}^t \dot{\xi}(\theta) d\theta \right], \quad (21)$$

$$0 = 2\chi_1^T(t) \mathcal{T}_2 \left[\xi(t - \mu_1(t)) - \xi(t - \bar{\mu}_1) - \int_{t-\bar{\mu}_1}^{t-\mu_1(t)} \dot{\xi}(\theta) d\theta \right], \quad (22)$$

$$0 = 2\chi_2^T(t) \mathcal{T}_3 \left[\zeta(t) - \zeta(t - \mu_2(t)) - \int_{t-\mu_2(t)}^t \dot{\zeta}(\theta) d\theta \right], \quad (23)$$

$$0 = 2\chi_2^T(t) \mathcal{T}_4 \left[\zeta(t - \mu_2(t)) - \zeta(t - \bar{\mu}_2) - \int_{t-\bar{\mu}_2}^{t-\mu_2(t)} \dot{\zeta}(\theta) d\theta \right], \quad (24)$$

$$0 = 2\chi_3^T(t) \mathcal{T}_5 \left[\xi(t) - \xi(t - \sigma_1(t)) - \int_{t-\sigma_1(t)}^t \dot{\xi}(\theta) d\theta \right], \quad (25)$$

$$0 = 2\chi_3^T(t)\mathcal{T}_6\left[\xi(t - \sigma_1(t)) - \xi(t - \bar{\sigma}_1) - \int_{t-\bar{\sigma}_1}^{t-\sigma_1(t)} \dot{\xi}(\theta)d\theta\right], \quad (27)$$

$$0 = 2\chi_4^T(t)\mathcal{T}_7\left[\varsigma(t) - \varsigma(t - \sigma_2(t)) - \int_{t-\sigma_2(t)}^t \dot{\varsigma}(\theta)d\theta\right], \quad (28)$$

$$0 = 2\chi_4^T(t)\mathcal{T}_8\left[\varsigma(t - \sigma_2(t)) - \varsigma(t - \bar{\sigma}_2) - \int_{t-\bar{\sigma}_2}^{t-\sigma_2(t)} \dot{\varsigma}(\theta)d\theta\right], \quad (29)$$

where $\chi_1(t) = [\xi(t)\xi(t - \mu_1(t))]^T$, $\chi_2(t) = [\varsigma(t)\varsigma(t - \mu_2(t))]^T$, $\chi_3(t) = [\xi(t)\xi(t - \sigma_1(t))]^T$, $\chi_4(t) = [\varsigma(t)\varsigma(t - \sigma_2(t))]^T$.

Further, according to Eqs. (6)–(9), for scalars $\gamma_j > 0$ ($j = 1, 2, \dots, 6$), it can be reach that

$$0 \leq \gamma_1[\xi^T(t)\bar{\mathcal{K}}_1^T\bar{\mathcal{K}}_1\xi(t) - \Gamma_1^T(\xi(t))\Gamma_1(\xi(t))], \quad (30)$$

$$0 \leq \gamma_2[\varsigma^T(t)\bar{\mathcal{K}}_2^T\bar{\mathcal{K}}_2\varsigma(t) - \Gamma_2^T(\varsigma(t))\Gamma_2(\varsigma(t))], \quad (31)$$

$$0 \leq \gamma_3[\xi^T(t - \mu_1(t))\bar{\mathcal{K}}_1^T\bar{\mathcal{K}}_1\xi(t - \mu_1(t)) - \Gamma_1^T(\xi(t - \mu_1(t)))\Gamma_1(\xi(t - \mu_1(t)))], \quad (32)$$

$$0 \leq \gamma_4[\varsigma^T(t - \mu_2(t))\bar{\mathcal{K}}_2^T\bar{\mathcal{K}}_2\varsigma(t - \mu_2(t)) - \Gamma_2^T(\varsigma(t - \mu_2(t)))\Gamma_2(\varsigma(t - \mu_2(t)))], \quad (33)$$

$$0 \leq \gamma_5[\xi^T(t)\hat{\mathcal{G}}^T\hat{\mathcal{G}}\xi(t) - \Gamma_3^T(t, \xi(t))\Gamma_3(t, \xi(t))], \quad (34)$$

$$0 \leq \gamma_6[\varsigma^T(t)\tilde{\mathcal{G}}^T\tilde{\mathcal{G}}\varsigma(t) - \Gamma_4^T(t, \varsigma(t))\Gamma_4(t, \varsigma(t))]. \quad (35)$$

For a matrix $\mathbb{B} = \mathbb{B}^T$ of any appropriate dimension, we have

$$\begin{aligned} 0 &= \int_{t-\bar{\mu}_1}^t \chi_1^T(t)\mathbb{B}_1\chi_1(t)ds - \int_{t-\bar{\mu}_1}^t \chi_1^T(t)\mathbb{B}_1\chi_1(t)ds \\ &= \bar{\mu}_1\chi_1^T(t)\mathbb{B}_1\chi_1(t) - \int_{t-\mu_1(t)}^t \chi_1^T(t)\mathbb{B}_1\chi_1(t)ds - \int_{t-\bar{\mu}_1}^{t-\mu_1(t)} \chi_1^T(t)\mathbb{B}_1\chi_1(t)ds, \end{aligned} \quad (36)$$

$$\begin{aligned} 0 &= \int_{t-\bar{\mu}_2}^t \chi_2^T(t)\mathbb{B}_2\chi_2(t)ds - \int_{t-\bar{\mu}_2}^t \chi_2^T(t)\mathbb{B}_2\chi_2(t)ds \\ &= \bar{\mu}_2\chi_2^T(t)\mathbb{B}_2\chi_2(t) - \int_{t-\mu_2(t)}^t \chi_2^T(t)\mathbb{B}_2\chi_2(t)ds - \int_{t-\bar{\mu}_2}^{t-\mu_2(t)} \chi_2^T(t)\mathbb{B}_2\chi_2(t)ds, \end{aligned} \quad (37)$$

$$\begin{aligned} 0 &= \int_{t-\bar{\sigma}_1}^t \chi_3^T(t)\mathbb{B}_3\chi_3(t)ds - \int_{t-\bar{\sigma}_1}^t \chi_3^T(t)\mathbb{B}_3\chi_3(t)ds \\ &= \bar{\sigma}_1\chi_3^T(t)\mathbb{B}_3\chi_3(t) - \int_{t-\sigma_1(t)}^t \chi_3^T(t)\mathbb{B}_3\chi_3(t)ds - \int_{t-\bar{\sigma}_1}^{t-\sigma_1(t)} \chi_3^T(t)\mathbb{B}_3\chi_3(t)ds, \end{aligned} \quad (38)$$

$$\begin{aligned} 0 &= \int_{t-\bar{\sigma}_2}^t \chi_4^T(t)\mathbb{B}_4\chi_4(t)ds - \int_{t-\bar{\sigma}_2}^t \chi_4^T(t)\mathbb{B}_4\chi_4(t)ds \\ &= \bar{\sigma}_2\chi_4^T(t)\mathbb{B}_4\chi_4(t) - \int_{t-\sigma_2(t)}^t \chi_4^T(t)\mathbb{B}_4\chi_4(t)ds - \int_{t-\bar{\sigma}_2}^{t-\sigma_2(t)} \chi_4^T(t)\mathbb{B}_4\chi_4(t)ds, \end{aligned} \quad (39)$$

The right-hand sides of (22)–(39) incorporate to the derivatives of $\mathcal{V}_i(\varsigma(t), \xi(t))$ ($i = 1, 2, \dots, 5$), then it follows that

$$\begin{aligned} \dot{\mathcal{V}}(\varsigma(t), \xi(t)) &\leq \Phi^T(t)e^{\nu t}[\Lambda_1 + \bar{\mu}_1\Lambda_2^T\mathcal{S}_1\Lambda_2]\Phi(t) + \hat{\Phi}^T(t)[\Lambda_3 + \bar{\mu}_2\Lambda_4^T\mathcal{S}_2\Lambda_4] - \int_{t-\mu_1(t)}^t \varpi_1^T(t, \theta) \\ &\quad \times \Phi_1\varpi_1(t, \theta)d\theta - \int_{t-\bar{\mu}_1}^{t-\mu_1(t)} \varpi_1^T(t, \theta)\Phi_2\varpi_1(t, \theta)d\theta - \int_{t-\mu_2(t)}^t \varpi_2^T(t, \theta)\Phi_3 \\ &\quad \times \varpi_2(t, \theta)d\theta - \int_{t-\bar{\mu}_2}^{t-\mu_2(t)} \varpi_2^T(t, \theta)\Phi_4\varpi_2(t, \theta)d\theta - \int_{t-\sigma_1(t)}^t \varpi_3^T(t, \theta)\Phi_5 \\ &\quad \times \varpi_3(t, \theta)d\theta - \int_{t-\bar{\sigma}_1}^{t-\sigma_1(t)} \varpi_3^T(t, \theta)\Phi_6\varpi_3(t, \theta)d\theta - \int_{t-\sigma_2(t)}^t \varpi_4^T(t, \theta)\Phi_7 \\ &\quad \times \varpi_4(t, \theta)d\theta - \int_{t-\bar{\sigma}_2}^{t-\sigma_2(t)} \varpi_4^T(t, \theta)\Phi_8\varpi_4(t, \theta)d\theta, \end{aligned} \quad (40)$$

where

$$\Lambda_1 = \begin{bmatrix} \Lambda_{111} & \Lambda_{112} & \Lambda_{113} & 0 & 0 & \Lambda_{116} & \Lambda_{117} & \Lambda_{118} & \Lambda_{119} & \Lambda_{1110} & \Lambda_{1111} \\ * & \Lambda_{122} & \Lambda_{123} & 0 & 0 & \Lambda_{126} & \Lambda_{127} & \Lambda_{128} & \Lambda_{129} & \Lambda_{1210} & \Lambda_{1211} \\ * & * & \Lambda_{133} & 0 & 0 & \Lambda_{136} & \Lambda_{137} & \Lambda_{138} & \Lambda_{139} & \Lambda_{1310} & \Lambda_{1311} \\ * & * & * & \Lambda_{144} & \Lambda_{145} & \Lambda_{146} & 0 & 0 & \Lambda_{149} & \Lambda_{1410} & 0 \\ * & * & * & * & \Lambda_{155} & 0 & 0 & 0 & \Lambda_{159} & \Lambda_{1510} & 0 \\ * & * & * & * & * & \Lambda_{166} & \Lambda_{167} & \Lambda_{168} & \Lambda_{169} & \Lambda_{1610} & \Lambda_{1611} \\ * & * & * & * & * & * & \Lambda_{177} & \Lambda_{178} & \Lambda_{179} & \Lambda_{1710} & \Lambda_{1711} \\ * & * & * & * & * & * & * & \Lambda_{188} & \Lambda_{189} & \Lambda_{1810} & \Lambda_{1811} \\ * & * & * & * & * & * & * & * & \Lambda_{199} & \Lambda_{1910} & 0 \\ * & * & * & * & * & * & * & * & * & \Lambda_{11010} & 0 \\ * & * & * & * & * & * & * & * & * & * & \Lambda_{11111} \end{bmatrix} < 0,$$

$$\Lambda_3 = \begin{bmatrix} \Lambda_{311} & \Lambda_{312} & \Lambda_{313} & 0 & 0 & \Lambda_{316} & \Lambda_{317} & \Lambda_{318} & \Lambda_{319} & \Lambda_{3110} & \Lambda_{3111} \\ * & \Lambda_{322} & \Lambda_{323} & 0 & 0 & \Lambda_{326} & \Lambda_{327} & \Lambda_{328} & \Lambda_{329} & \Lambda_{3210} & \Lambda_{3211} \\ * & * & \Lambda_{333} & 0 & 0 & \Lambda_{336} & \Lambda_{337} & \Lambda_{338} & \Lambda_{339} & \Lambda_{3310} & \Lambda_{3311} \\ * & * & * & \Lambda_{344} & \Lambda_{345} & \Lambda_{346} & 0 & 0 & \Lambda_{349} & \Lambda_{3410} & 0 \\ * & * & * & * & \Lambda_{355} & 0 & 0 & 0 & \Lambda_{359} & \Lambda_{3510} & 0 \\ * & * & * & * & * & \Lambda_{366} & \Lambda_{367} & \Lambda_{368} & \Lambda_{369} & \Lambda_{3610} & \Lambda_{3611} \\ * & * & * & * & * & * & \Lambda_{377} & \Lambda_{378} & \Lambda_{379} & \Lambda_{3710} & \Lambda_{3711} \\ * & * & * & * & * & * & * & \Lambda_{388} & \Lambda_{389} & \Lambda_{3810} & \Lambda_{3811} \\ * & * & * & * & * & * & * & * & \Lambda_{399} & \Lambda_{3910} & 0 \\ * & * & * & * & * & * & * & * & * & \Lambda_{31010} & 0 \\ * & * & * & * & * & * & * & * & * & * & \Lambda_{31111} \end{bmatrix} < 0,$$

$$\varpi_2(t, \theta) = [\chi_2^T(t) \dot{\zeta}^T(\theta)]^T, \quad \varpi_3(t, \theta) = [\chi_3^T(t) \dot{\xi}^T(\theta)]^T, \quad \varpi_4(t, \theta) = [\chi_4^T(t) \dot{\zeta}^T(\theta)]^T,$$

$$\varpi_1(t, \theta) = [\chi_1^T(t) \dot{\xi}^T(\theta)]^T, \quad \Lambda_2 = [\bar{A}_2 \bar{A}_1 + \bar{B} \bar{C} \ 0 \ 0 \ \bar{D} \bar{E}_2 \bar{E}_1 \ 0 \ 0 \ 0],$$

$$\Lambda_4 = [\bar{M}_2 \bar{M}_1 + \bar{M}_3 \bar{N}_1 \ 0 \ 0 \ \bar{N}_2 \bar{O}_2 \bar{O}_1 \ 0 \ 0 \ 0],$$

$$\Phi(t) = [\zeta^T(t) \zeta^T(t - \delta_1) \Gamma_1^T(\xi(t)) \xi^T(t - \mu_1(t)) \xi^T(t - \bar{\mu}_1) \Gamma_1^T(\xi(t - \mu_1(t))) \Gamma_3^T(t, \xi(t)) (\int_{t-\sigma_1(t)}^t \Gamma_1(\xi(s)) ds)^T \xi^T(t) (\int_{t-\bar{\sigma}_1}^t \dot{\xi}(\theta) d\theta)^T (\int_{t-\delta_1}^t \zeta(s) ds)^T]^T \text{ and } \hat{\Phi}(t) = [\xi^T(t) \xi^T(t - \delta_1) \Gamma_2^T(\zeta(t)) \zeta^T(t - \mu_2(t)) \zeta^T(t - \bar{\mu}_2) \Gamma_2^T(\zeta(t - \mu_2(t))) \Gamma_4^T(t, \zeta(t)) (\int_{t-\sigma_2(t)}^t \Gamma_2(\zeta(s)) ds)^T \zeta^T(t) (\int_{t-\bar{\sigma}_2}^t \dot{\zeta}(\theta) d\theta)^T (\int_{t-\delta_2}^t \xi(s) ds)^T]^T.$$

It is point out that, if $\Lambda_1 + \bar{\mu}_1 \Lambda_2^T S_1 \Lambda_2 + \bar{\sigma}_1 \Lambda_5^T S_5 \Lambda_5 < 0$, $\Lambda_3 + \bar{\mu}_2 \Lambda_4^T S_2 \Lambda_4 + \bar{\sigma}_2 \Lambda_6^T S_6 \Lambda_6 < 0$ and $\Phi_q \geq 0$ ($q = 1, 2, \dots, 8$), then the addressed NNs (5) would be stable by the Lyapunov stability theory because $\Phi_q \geq 0$ ($q = 1, 2, \dots, 8$), are true according to (12)–(15). Hence, only it remains to check that $\Lambda_1 + \bar{\mu}_1 \Lambda_2^T S_1 \Lambda_2 + \bar{\sigma}_1 \Lambda_5^T S_5 \Lambda_5 < 0$ and $\Lambda_3 + \bar{\mu}_2 \Lambda_4^T S_2 \Lambda_4 + \bar{\sigma}_2 \Lambda_6^T S_6 \Lambda_6 < 0$.

From Lemma 2.4, we can observe that $\Lambda_1 + \bar{\mu}_1 \Lambda_2^T S_1 \Lambda_2 + \bar{\sigma}_1 \Lambda_5^T S_5 \Lambda_5 < 0$ and $\Lambda_3 + \bar{\mu}_2 \Lambda_4^T S_2 \Lambda_4 + \bar{\sigma}_2 \Lambda_6^T S_6 \Lambda_6 < 0$ holds if and only if

$$\bar{\Lambda}_1 = \begin{bmatrix} \Lambda_{111} & \Lambda_{112} & \Lambda_{113} & 0 & 0 & \Lambda_{116} & \Lambda_{117} & \Lambda_{118} & \Lambda_{119} & \Lambda_{1110} & \Lambda_{1111} & \Lambda_{1112} \\ * & \Lambda_{122} & \Lambda_{123} & 0 & 0 & \Lambda_{126} & \Lambda_{127} & \Lambda_{128} & \Lambda_{129} & \Lambda_{1210} & \Lambda_{1211} & \Lambda_{1212} \\ * & * & \Lambda_{133} & 0 & 0 & \Lambda_{136} & \Lambda_{137} & \Lambda_{138} & \Lambda_{139} & \Lambda_{1310} & \Lambda_{1311} & \Lambda_{1312} \\ * & * & * & \Lambda_{144} & \Lambda_{145} & \Lambda_{146} & 0 & 0 & \Lambda_{149} & \Lambda_{1410} & 0 & 0 \\ * & * & * & * & \Lambda_{155} & 0 & 0 & 0 & \Lambda_{159} & \Lambda_{1510} & 0 & 0 \\ * & * & * & * & * & \Lambda_{166} & \Lambda_{167} & \Lambda_{168} & \Lambda_{169} & \Lambda_{1610} & \Lambda_{1611} & \Lambda_{1612} \\ * & * & * & * & * & * & \Lambda_{177} & \Lambda_{178} & \Lambda_{179} & \Lambda_{1710} & \Lambda_{1711} & \Lambda_{1712} \\ * & * & * & * & * & * & * & \Lambda_{188} & \Lambda_{189} & \Lambda_{1810} & \Lambda_{1811} & \Lambda_{1812} \\ * & * & * & * & * & * & * & * & \Lambda_{199} & \Lambda_{1910} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Lambda_{11010} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & \Lambda_{11111} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Lambda_{11212} \end{bmatrix} < 0 \quad (41)$$

$$\bar{\Lambda}_3 = \begin{bmatrix} \Lambda_{311} & \Lambda_{312} & \Lambda_{313} & 0 & 0 & \Lambda_{316} & \Lambda_{317} & \Lambda_{318} & \Lambda_{319} & \Lambda_{3110} & \Lambda_{3111} & \Lambda_{3112} \\ * & \Lambda_{322} & \Lambda_{323} & 0 & 0 & \Lambda_{326} & \Lambda_{327} & \Lambda_{328} & \Lambda_{329} & \Lambda_{3210} & \Lambda_{3211} & \Lambda_{3212} \\ * & * & \Lambda_{333} & 0 & 0 & \Lambda_{336} & \Lambda_{337} & \Lambda_{338} & \Lambda_{339} & \Lambda_{3310} & \Lambda_{3311} & \Lambda_{3312} \\ * & * & * & \Lambda_{344} & \Lambda_{345} & \Lambda_{346} & 0 & 0 & \Lambda_{349} & \Lambda_{3410} & 0 & 0 \\ * & * & * & * & \Lambda_{355} & 0 & 0 & 0 & \Lambda_{359} & \Lambda_{3510} & 0 & 0 \\ * & * & * & * & * & \Lambda_{366} & \Lambda_{367} & \Lambda_{368} & \Lambda_{369} & \Lambda_{3610} & \Lambda_{3611} & \Lambda_{3612} \\ * & * & * & * & * & * & \Lambda_{377} & \Lambda_{378} & \Lambda_{379} & \Lambda_{3710} & \Lambda_{3711} & \Lambda_{3712} \\ * & * & * & * & * & * & * & \Lambda_{388} & \Lambda_{389} & \Lambda_{3810} & \Lambda_{3811} & \Lambda_{3812} \\ * & * & * & * & * & * & * & * & \Lambda_{399} & \Lambda_{3910} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Lambda_{31010} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & \Lambda_{31111} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Lambda_{31212} \end{bmatrix} < 0 \quad (42)$$

Taking the pre and post-multiplication of the inequalities (41) and (42) by $\text{diag}\{I, I, I, I, I, I, I, I, I, I, I, I, Q_1 S_1^{-1}\}$, $\text{diag}\{I, I, I, I, I, I, I, I, I, I, I, I, S_1^{-1} Q_1\}$ and $\text{diag}\{I, I, I, I, I, I, I, I, I, I, I, I, Q_2 S_2^{-1}\}$, $\text{diag}\{I, I, I, I, I, I, I, I, I, I, I, I, S_2^{-1} Q_2\}$, one can deduced that

$$\hat{\Lambda}_1 = \begin{bmatrix} \Lambda_{111} & \Lambda_{112} & \Lambda_{113} & 0 & 0 & \Lambda_{116} & \Lambda_{117} & \Lambda_{118} & \Lambda_{119} & \Lambda_{1110} & \Lambda_{1111} & \tilde{\Lambda}_{1112} \\ * & \Lambda_{122} & \Lambda_{123} & 0 & 0 & \Lambda_{126} & \Lambda_{127} & \Lambda_{128} & \Lambda_{129} & \Lambda_{1210} & \Lambda_{1211} & \tilde{\Lambda}_{1212} \\ * & * & \Lambda_{133} & 0 & 0 & \Lambda_{136} & \Lambda_{137} & \Lambda_{138} & \Lambda_{139} & \Lambda_{1310} & \Lambda_{1311} & \tilde{\Lambda}_{1312} \\ * & * & * & \Lambda_{144} & \Lambda_{145} & \Lambda_{146} & 0 & 0 & \Lambda_{149} & \Lambda_{1410} & 0 & 0 \\ * & * & * & * & \Lambda_{155} & 0 & 0 & 0 & \Lambda_{159} & \Lambda_{1510} & 0 & 0 \\ * & * & * & * & * & \Lambda_{166} & \Lambda_{167} & \Lambda_{168} & \Lambda_{169} & \Lambda_{1610} & \Lambda_{1611} & \tilde{\Lambda}_{1612} \\ * & * & * & * & * & * & \Lambda_{177} & \Lambda_{178} & \Lambda_{179} & \Lambda_{1710} & \Lambda_{1711} & \tilde{\Lambda}_{1712} \\ * & * & * & * & * & * & * & \Lambda_{188} & \Lambda_{189} & \Lambda_{1810} & \Lambda_{1811} & \tilde{\Lambda}_{1812} \\ * & * & * & * & * & * & * & * & \Lambda_{199} & \Lambda_{1910} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Lambda_{11010} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & \Lambda_{11111} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \tilde{\Lambda}_{11212} \end{bmatrix} < 0 \quad (43)$$

$$\hat{\Lambda}_3 = \begin{bmatrix} \Lambda_{311} & \Lambda_{312} & \Lambda_{313} & 0 & 0 & \Lambda_{316} & \Lambda_{317} & \Lambda_{318} & \Lambda_{319} & \Lambda_{3110} & \Lambda_{3111} & \tilde{\Lambda}_{3112} \\ * & \Lambda_{322} & \Lambda_{323} & 0 & 0 & \Lambda_{326} & \Lambda_{327} & \Lambda_{328} & \Lambda_{329} & \Lambda_{3210} & \Lambda_{3211} & \tilde{\Lambda}_{3212} \\ * & * & \Lambda_{333} & 0 & 0 & \Lambda_{336} & \Lambda_{337} & \Lambda_{338} & \Lambda_{339} & \Lambda_{3310} & \Lambda_{3311} & \tilde{\Lambda}_{3312} \\ * & * & * & \Lambda_{344} & \Lambda_{345} & \Lambda_{346} & 0 & 0 & \Lambda_{349} & \Lambda_{3410} & 0 & 0 \\ * & * & * & * & \Lambda_{355} & 0 & 0 & 0 & \Lambda_{359} & \Lambda_{3510} & 0 & 0 \\ * & * & * & * & * & \Lambda_{366} & \Lambda_{367} & \Lambda_{368} & \Lambda_{369} & \Lambda_{3610} & \Lambda_{3611} & \tilde{\Lambda}_{3612} \\ * & * & * & * & * & * & \Lambda_{377} & \Lambda_{378} & \Lambda_{379} & \Lambda_{3710} & \Lambda_{3711} & \tilde{\Lambda}_{3712} \\ * & * & * & * & * & * & * & \Lambda_{388} & \Lambda_{389} & \Lambda_{3810} & \Lambda_{3811} & \tilde{\Lambda}_{3812} \\ * & * & * & * & * & * & * & * & \Lambda_{399} & \Lambda_{3910} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Lambda_{31010} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & \Lambda_{31111} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \tilde{\Lambda}_{31212} \end{bmatrix} < 0 \quad (44)$$

In particular, note that

$$(Q_1 - S_1)^T S_1^{-1} (Q_1 - S_1) = Q_1 S_1^{-1} Q_1 - 2Q_1 + S_1 \geq 0,$$

$$(Q_2 - S_2)^T S_2^{-1} (Q_2 - S_2) = Q_2 S_2^{-1} Q_2 - 2Q_2 + S_2 \geq 0,$$

It follows that $Q_1 S_1^{-1} Q_1 \geq 2Q_1 - S_1$ and $Q_2 S_2^{-1} Q_2 \geq 2Q_2 - S_2$.

Therefore, from (43) and (44), we have (10) and (11) is obviously true which implies that $\Lambda_1 + \bar{\mu}_1 \Lambda_2^T S_1 \Lambda_2 < 0$ & $\Lambda_3 + \bar{\mu}_2 \Lambda_4^T S_2 \Lambda_4 < 0$ are holds, (i.e., $\dot{V}(\zeta(t), \xi(t)) < 0$). Hence, it leads to the stability of the BAM neural networks.

Next, to show the convergence rate, we can obtain from the stability condition that

$$V(\zeta(t), \xi(t)) \leq V(\zeta(0), \xi(0)), \quad (45)$$

where

$$\begin{aligned} V(\zeta(0), \xi(0)) &= \left[\zeta(0) - (\bar{A}_1 + \bar{B}) \int_{-\delta_1}^0 \zeta(s) ds \right]^T Q_1 \left[\zeta(0) - (\bar{A}_1 + \bar{B}) \int_{-\delta_1}^0 \zeta(s) ds \right] \\ &\quad + \left[\xi(0) - (\bar{M}_1 + \bar{M}_3) \int_{-\delta_2}^0 \xi(s) ds \right]^T Q_2 \left[\xi(0) - (\bar{M}_1 + \bar{M}_3) \int_{-\delta_2}^0 \xi(s) ds \right] \end{aligned}$$

$$\begin{aligned}
& + \int_{-\delta_1}^0 \int_{\theta}^0 e^{\nu s} \zeta^T(s) \mathcal{Q} \zeta(s) ds d\theta + \int_{-\delta_2}^0 \int_{\theta}^0 e^{\nu s} \\
& \times \xi^T(s) \hat{\mathcal{Q}} \xi(s) ds d\theta + \int_{-\mu_1(0)}^0 e^{\nu s} \begin{bmatrix} \xi(s) \\ \Gamma_1(\xi(s)) \end{bmatrix}^T \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix}^T \\
& \times \begin{bmatrix} \xi(s) \\ \Gamma_1(\xi(s)) \end{bmatrix} ds + \int_{-\mu_2(0)}^0 e^{\nu s} \begin{bmatrix} \zeta(s) \\ \Gamma_2(\zeta(s)) \end{bmatrix}^T \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{bmatrix}^T \\
& \times \begin{bmatrix} \zeta(s) \\ \Gamma_2(\zeta(s)) \end{bmatrix} ds + \int_{-\bar{\mu}_1}^0 \int_s^0 e^{\nu \theta} \dot{\xi}^T(\theta) \mathcal{S}_1 \dot{\xi}(\theta) d\theta ds + \int_{-\bar{\mu}_2}^0 \int_s^0 e^{\nu \theta} \\
& \times \dot{\zeta}^T(\theta) \mathcal{S}_2 \dot{\zeta}(\theta) d\theta ds + \int_{-\bar{\mu}_1}^0 e^{\nu \theta} \xi^T(\theta) \mathcal{S}_3 \xi(\theta) d\theta + \int_{-\bar{\mu}_2}^0 e^{\nu \theta} \zeta^T(\theta) \mathcal{S}_4 \zeta(\theta) \\
& \times d\theta + \int_{-\bar{\sigma}_1}^0 \int_s^0 e^{\nu \theta} \dot{\xi}^T(\theta) \mathcal{S}_5 \dot{\xi}(\theta) d\theta ds + \int_{-\bar{\sigma}_2}^0 \int_s^0 e^{\nu \theta} \dot{\zeta}^T(\theta) \mathcal{S}_6 \dot{\zeta}(\theta) d\theta ds \\
& \leq \left\{ \lambda_{\max}(\mathcal{Q}_1) - 2\delta_1 \lambda_{\max}((\bar{A}_1 + \bar{B})^T \mathcal{Q}_1) + \delta_1^2 \lambda_{\max}((\bar{A}_1 + \bar{B})^T \mathcal{Q}_1 \right. \\
& \times (\bar{A}_1 + \bar{B})) + \frac{\delta_1^2}{2} \lambda_{\max}(\mathcal{Q}) + \bar{\mu}_2 [\lambda_{\max}(\mathcal{R}_{11}) + \lambda_{\max}(\mathcal{R}_{12}) \\
& + \lambda_{\max}(\mathcal{R}_{21}) + \lambda_{\max}(\mathcal{R}_{22})] + \frac{\bar{\mu}_2^2}{2} [\lambda_{\max}((\bar{A}_1 + \bar{B}) \mathcal{S}_2) \\
& + \lambda_{\max}(\bar{A}_2 \mathcal{S}_2)] + \frac{\bar{\mu}_1^2}{2} [\lambda_{\max}(\bar{N}_1 \mathcal{S}_1) + \lambda_{\max}(\bar{N}_2 \mathcal{S}_1) + \lambda_{\max}(\bar{O}_1 \mathcal{S}_1) \\
& + \lambda_{\max}(\bar{O}_2 \mathcal{S}_1)] + \bar{\mu}_2 \lambda_{\max}(\mathcal{S}_4) + \frac{\bar{\sigma}_1^2}{2} [\lambda_{\max}(\bar{N}_1 \mathcal{S}_5) \\
& + \lambda_{\max}(\bar{N}_2 \mathcal{S}_5) + \lambda_{\max}(\bar{O}_1 \mathcal{S}_5) + \lambda_{\max}(\bar{O}_2 \mathcal{S}_5)] + \frac{\bar{\sigma}_2^2}{2} [\lambda_{\max}((\bar{A}_1 \\
& + \bar{B}) \mathcal{S}_6) + \lambda_{\max}(\bar{A}_2 \mathcal{S}_6)] \left. \right\} \|\zeta(0)\|^2 + \left\{ \lambda_{\max}(\mathcal{Q}_2) - 2\delta_2 \lambda_{\max}((\bar{M}_1 \right. \\
& + \bar{M}_3)^T \mathcal{Q}_2) + \delta_2^2 \lambda_{\max}((\bar{M}_1 + \bar{M}_3)^T \mathcal{Q}_2 (\bar{M}_1 + \bar{M}_3)) + \frac{\delta_2^2}{2} \lambda_{\max}(\hat{\mathcal{Q}}) \\
& + \bar{\mu}_1 [\lambda_{\max}(\mathcal{P}_{11}) + \lambda_{\max}(\mathcal{P}_{12}) + \lambda_{\max}(\mathcal{P}_{21}) + \lambda_{\max}(\mathcal{P}_{22})] + \frac{\bar{\mu}_1^2}{2} \\
& \times [\lambda_{\max}((\bar{M}_1 + \bar{M}_3) \mathcal{S}_1) + \lambda_{\max}(\bar{M}_2 \mathcal{S}_1)] + \frac{\bar{\mu}_2^2}{2} [\lambda_{\max}(\bar{C} \mathcal{S}_2) \\
& + \lambda_{\max}(\bar{D} \mathcal{S}_2) + \lambda_{\max}(\bar{E}_1 \mathcal{S}_2) + \lambda_{\max}(\bar{E}_2 \mathcal{S}_2)] + \bar{\mu}_1 \lambda_{\max}(\mathcal{S}_3) + \frac{\bar{\sigma}_2^2}{2} \\
& \times [\lambda_{\max}(\bar{C} \mathcal{S}_6) + \lambda_{\max}(\bar{D} \mathcal{S}_6) + \lambda_{\max}(\bar{E}_1 \mathcal{S}_6) + \lambda_{\max}(\bar{E}_2 \mathcal{S}_6)] + \frac{\bar{\sigma}_1^2}{2} \\
& \times [\lambda_{\max}((\bar{M}_1 + \bar{M}_3) \mathcal{S}_5) + \lambda_{\max}(\bar{M}_2 \mathcal{S}_5)] \left. \right\} \|\xi(0)\|^2
\end{aligned} \tag{46}$$

$$\begin{aligned}
\text{Let } \zeta_1 = & \lambda_{\max}(\mathcal{Q}_1) - 2\delta_1 \lambda_{\max}((\bar{A}_1 + \bar{B})^T \mathcal{Q}_1) + \delta_1^2 \lambda_{\max}((\bar{A}_1 + \bar{B})^T \mathcal{Q}_1 \\
& \times (\bar{A}_1 + \bar{B})) + \frac{\delta_1^2}{2} \lambda_{\max}(\mathcal{Q}) + \bar{\mu}_2 [\lambda_{\max}(\mathcal{R}_{11}) + \lambda_{\max}(\mathcal{R}_{12}) \\
& + \lambda_{\max}(\mathcal{R}_{21}) + \lambda_{\max}(\mathcal{R}_{22})] + \frac{\bar{\mu}_2^2}{2} [\lambda_{\max}((\bar{A}_1 + \bar{B}) \mathcal{S}_2) + \lambda_{\max}(\bar{A}_2 \mathcal{S}_2)] \\
& + \frac{\bar{\mu}_1^2}{2} [\lambda_{\max}(\bar{N}_1 \mathcal{S}_1) + \lambda_{\max}(\bar{N}_2 \mathcal{S}_1) + \lambda_{\max}(\bar{O}_1 \mathcal{S}_1) + \lambda_{\max}(\bar{O}_2 \mathcal{S}_1)] \\
& + \bar{\mu}_2 \lambda_{\max}(\mathcal{S}_4) + \frac{\bar{\sigma}_1^2}{2} [\lambda_{\max}((\bar{M}_1 + \bar{M}_3) \mathcal{S}_5) + \lambda_{\max}(\bar{M}_2 \mathcal{S}_5) \\
& + \lambda_{\max}(\bar{N}_1 \mathcal{S}_5) + \lambda_{\max}(\bar{N}_2 \mathcal{S}_5) + \lambda_{\max}(\bar{O}_1 \mathcal{S}_5) + \lambda_{\max}(\bar{O}_2 \mathcal{S}_5)] + \frac{\bar{\sigma}_2^2}{2}
\end{aligned}$$

$$\begin{aligned}
& \times [\lambda_{\max}((\bar{A}_1 + \bar{B})S_6) + \lambda_{\max}(\bar{A}_2S_6) + \lambda_{\max}(\bar{C}S_6) + \lambda_{\max}(\bar{D}S_6) \\
& + \lambda_{\max}(\bar{E}_1S_6) + \lambda_{\max}(\bar{E}_2S_6)] \\
\zeta_2 = & \{\lambda_{\max}(Q_2) - 2\delta_2\lambda_{\max}((\bar{M}_1 + \bar{M}_3)^T Q_2) + \delta_2^2\lambda_{\max}((\bar{M}_1 + \bar{M}_3)^T Q_2 \\
& \times (\bar{M}_1 + \bar{M}_3)) + \frac{\delta^2}{2}\lambda_{\max}(\hat{Q}) + \bar{\mu}_1[\lambda_{\max}(\mathcal{P}_{11}) + \lambda_{\max}(\mathcal{P}_{12}) \\
& + \lambda_{\max}(\mathcal{P}_{21}) + \lambda_{\max}(\mathcal{P}_{22})] + \frac{\bar{\mu}_1^2}{2}[\lambda_{\max}((\bar{M}_1 + \bar{M}_3)S_1) + \lambda_{\max}(\bar{M}_2S_1)] \\
& + \frac{\bar{\mu}_2^2}{2}[\lambda_{\max}(\bar{C}S_2) + \lambda_{\max}(\bar{D}S_2) + \lambda_{\max}(\bar{E}_1S_2) + \lambda_{\max}(\bar{E}_2S_2)] \\
& + \bar{\mu}_1\lambda_{\max}(S_3) + \frac{\bar{\sigma}_2^2}{2}[\lambda_{\max}(\bar{C}S_6) + \lambda_{\max}(\bar{D}S_6) + \lambda_{\max}(\bar{E}_1S_6) \\
& + \lambda_{\max}(\bar{E}_2S_6)] + \frac{\bar{\sigma}_1^2}{2}[\lambda_{\max}((\bar{M}_1 + \bar{M}_3)S_5) + \lambda_{\max}(\bar{M}_2S_5)]\}
\end{aligned}$$

Thus, $\nu(\zeta(t), \xi(t)) \leq \zeta_1 \|\zeta(0)\|^2 + \zeta_2 \|\xi(0)\|^2$

It follows from the constructed LKF, we can reach the following inequality

$$\begin{aligned}
\nu(\zeta(t), \xi(t)) & \geq e^{\nu t} \left\{ \left[\zeta(t) - (\bar{A}_1 + \bar{B}) \int_{t-\delta_1}^t \zeta(s) ds \right]^T Q_1 \left[\zeta(t) - (\bar{A}_1 + \bar{B}) \int_{t-\delta_1}^t \zeta(s) ds \right] \right. \\
& + \left[\xi(t) - (\bar{M}_1 + \bar{M}_3) \int_{t-\delta_2}^t \xi(s) ds \right]^T Q_2 \left[\xi(t) - (\bar{M}_1 + \bar{M}_3) \int_{t-\delta_2}^t \xi(s) ds \right] \\
& + \int_{-\delta_1}^0 \int_{t+\theta}^t \zeta^T(s) Q \zeta(s) ds d\theta + \int_{-\delta_2}^0 \int_{t+\theta}^t \xi^T(s) \hat{Q} \xi(s) ds d\theta \Big\} \\
& \geq e^{\nu t} \left\{ [\lambda_{\max}(Q_1) - 2\delta_1\lambda_{\max}((\bar{A}_1 + \bar{B})^T Q_1) + \delta_1^2\lambda_{\max}((\bar{A}_1 + \bar{B})^T Q_1 (\bar{A}_1 + \bar{B}))] \|\zeta(t)\|^2 \right. \\
& + [\lambda_{\max}(Q_2) - 2\delta_2\lambda_{\max}((\bar{M}_1 + \bar{M}_3)^T Q_2) \\
& + \delta_2^2\lambda_{\max}((\bar{M}_1 + \bar{M}_3)^T Q_2 (\bar{M}_1 + \bar{M}_3))] \|\xi(t)\|^2 \Big\}, \tag{47}
\end{aligned}$$

which implies that,

$$\|\zeta(t)\|^2 + \|\xi(t)\|^2 \leq \frac{\zeta}{\lambda_{\max}(Q_a)} \{\|\zeta(0)\|^2 + \|\xi(0)\|^2\} e^{-\nu t}, \tag{48}$$

where $\zeta = \max\{\zeta_1, \zeta_2\}$ and $\lambda_{\max}(Q_a) = \min\{[\lambda_{\max}(Q_1) - 2\delta_1\lambda_{\max}((\bar{A}_1 + \bar{B})^T Q_1) + \delta_1^2\lambda_{\max}((\bar{A}_1 + \bar{B})^T Q_1 (\bar{A}_1 + \bar{B}))], [\lambda_{\max}(Q_2) - 2\delta_2\lambda_{\max}((\bar{M}_1 + \bar{M}_3)^T Q_2) + \delta_2^2\lambda_{\max}((\bar{M}_1 + \bar{M}_3)^T Q_2 (\bar{M}_1 + \bar{M}_3))]\}$. From Definition 2.3 and condition (48), we conclude that the BAMNNs (5) is exponentially stable with the convergence rate ν . This completes the proof. $\square$

Theorem 3.2. Suppose that Assumptions 1 and 2 holds, the concerned neural networks with non-fragile state estimator (1) or (5) is exponentially stable, if there are positive definite matrices $Q_1 > 0$, $Q_2 > 0$, $Q > 0$, $\hat{Q} > 0$, $\mathcal{P} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} > 0$, $\mathcal{R} = \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{bmatrix} > 0$, $S_1 > 0$, $S_2 > 0$, $S_3 > 0$, $S_4 > 0$, $S_5 > 0$ & $S_6 > 0$, matrices $\mathcal{T}_j$ ($j = 1, 2, \dots, 8$), $\mathbb{B}_1 = \begin{bmatrix} \mathbb{B}_{111} & \mathbb{B}_{112} \\ \mathbb{B}_{121} & \mathbb{B}_{122} \end{bmatrix}$, $\mathbb{B}_2 = \begin{bmatrix} \mathbb{B}_{211} & \mathbb{B}_{212} \\ \mathbb{B}_{221} & \mathbb{B}_{222} \end{bmatrix}$, $\mathbb{B}_3 = \begin{bmatrix} \mathbb{B}_{311} & \mathbb{B}_{312} \\ \mathbb{B}_{321} & \mathbb{B}_{322} \end{bmatrix}$, $\mathbb{B}_4 = \begin{bmatrix} \mathbb{B}_{411} & \mathbb{B}_{412} \\ \mathbb{B}_{421} & \mathbb{B}_{422} \end{bmatrix}$ and positive scalars γ_s ($s = 1, 2, \dots, 6$), ε_1 & $\varepsilon_2 > 0$ such that the following LMIs satisfied:

$$\Omega_1 = \begin{bmatrix} \Lambda_a & \mathcal{H}_1 & \varepsilon_1 \mathcal{Z}_1 \\ * & -\varepsilon_1 I & 0 \\ * & * & -\varepsilon_1 I \end{bmatrix} < 0, \tag{49}$$

$$\Omega_2 = \begin{bmatrix} \Lambda_b & \mathcal{H}_2 & \varepsilon_2 \mathcal{Z}_2 \\ * & -\varepsilon_2 I & 0 \\ * & * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{50}$$

$$\Phi_q \geq 0 \quad (q = 1, 2, \dots, 8), \tag{51}$$

$$\begin{aligned}
\mathcal{G}_1 &= [\text{diag}\{(U_c + U_D)\mathcal{Q}_1\mathcal{E}_1, 0\} \quad (\text{diag}\{0, 2U_B\mathcal{Q}_1\} + [\text{diag}\{0, \bar{\mu}_2 S_2 U_B\}^T \text{diag}\{0, \bar{\mu}_2 S_2 U_B\})] \\
&\quad \text{diag}\{(\mathcal{W}_N + \mathcal{W}_O)\gamma_6 \bar{G}^T \bar{G}, 0\}], \quad \mathcal{G}_9 = [\text{diag}\{2\bar{\mu}_2 S_2 (U_c + U_D), 0\} \quad \text{diag}\{2\bar{\sigma}_2 S_6 (U_c + U_D), 0\}], \\
\mathcal{G}_2 &= [\text{diag}\{(U_c + U_D)\mathcal{Q}_1\mathcal{E}_1, 0\} \quad \text{diag}\{(U_c + U_D)\mathcal{E}_1, 0\} \quad \text{diag}\{2(U_c + U_D)\bar{\mu}_2 S_2 \mathcal{E}_1, 0\}], \\
\mathcal{G}_3 &= [\text{diag}\{2(U_c + U_D)\bar{\mu}_2 S_2 \mathcal{E}_1, 0\} \quad \text{diag}\{2(U_c + U_D)\bar{\sigma}_2 S_6 \mathcal{E}_1, 0\}], \quad \mathcal{G}_{17} = [\text{diag}\{\bar{\mu}_1 S_1 \mathcal{E}_1 (U_c + U_D), 0\}], \\
\mathcal{G}_4 &= [\text{diag}\{\bar{\mu}_2 S_2^T \bar{C} (U_c + U_D) \mathcal{E}_1^T, 0\} \quad \text{diag}\{\bar{\sigma}_2 S_6^T \bar{C} (U_c + U_D) \mathcal{E}_1, 0\}], \quad \mathcal{G}_{20} = [\text{diag}\{0, \bar{\mu}_1 \mathcal{Q}_1 U_B\}], \\
\mathcal{G}_5 &= [\text{diag}\{\bar{\mu}_2 S_2^T \bar{D} (U_c + U_D) \mathcal{E}_1^T, 0\} \quad \text{diag}\{0, \bar{\mu}_2 S_2 U_B\} \quad \text{diag}\{\bar{\sigma}_2 S_6^T \bar{D} (U_c + U_D) \mathcal{E}_1, 0\}], \\
\mathcal{G}_6 &= [\text{diag}\{2\mathcal{Q}_1 (U_c + U_D), 0\} \quad \text{diag}\{2\bar{\mu}_2 S_2 (U_c + U_D), 0\} \quad \text{diag}\{2\bar{\sigma}_2 S_6^T (U_c + U_D), 0\}], \\
\mathcal{G}_7 &= [\text{diag}\{2\bar{\mu}_2 S_2^T \mathcal{E}_1^T (U_c + U_D), 0\} \quad \text{diag}\{2\bar{\sigma}_2 S_6 \mathcal{E}_1^T (U_c + U_D), 0\}], \quad \mathcal{G}_{15} = \text{diag}\{0, 0\}, \\
\mathcal{G}_8 &= \text{diag}\{(\bar{\mu}_2 S_2 + \bar{\sigma}_2 S_6)(U_c + U_D), 0\}, \quad \mathcal{G}_{14} = \text{diag}\{-2\mathcal{Q}_1 \bar{D} \mathcal{E}_1 (U_c + U_D), 0\}, \\
\mathcal{G}_{10} &= \text{diag}\{(\bar{\mu}_2 S_2^T \mathcal{E}_1^T \bar{E}_1 + \bar{\sigma}_2 S_6 \bar{E}_1 \bar{E}_1)(U_c + U_D), 0\}, \quad \mathcal{G}_{11} = \text{diag}\{(\bar{\mu}_2 S_2^T \bar{E}_1 + \bar{\sigma}_2 S_6 \bar{E}_1)(U_c + U_D), 0\}, \\
\mathcal{G}_{12} &= \text{diag}\{2\bar{A}_1 \mathcal{E}_1 (U_c + U_D) (\text{diag}\{0, -2\mathcal{Q}_1 U_B\} + U_D), 0\}, \quad \mathcal{G}_{13} = \text{diag}\{-2\mathcal{Q}_1 \bar{C} \mathcal{E}_1 (U_c + U_D), 0\}, \\
\mathcal{G}_{18} &= \text{diag}\{\bar{\mu}_1 S_1 (U_c + U_D), 0\}, \quad \mathcal{G}_{19} = [\text{diag}\{0, 0\} \quad \text{diag}\{(-4\mathcal{Q}_1^2 \mathcal{E}_1 - 4\bar{A}_1 \mathcal{Q}_1 \mathcal{E}_1)(U_c + U_D), 0\}], \\
\mathcal{G}_{16} &= \text{diag}\{-2\mathcal{Q}_1 \bar{E}_1 \mathcal{E}_1 (U_c + U_D), 0\}, \quad \mathcal{J}_1 = [\text{diag}\{(\mathcal{W}_N + \mathcal{W}_O)\mathcal{Q}_2 \mathcal{E}_2, 0\} \quad (\text{diag}\{0, 2\mathcal{W}_M \mathcal{Q}_2\} \\
&\quad + [\text{diag}\{0, \bar{\mu}_1 S_1 \mathcal{W}_M\}^T \text{diag}\{0, \bar{\mu}_1 S_1 \mathcal{W}_M\})] \quad \text{diag}\{(U_c + U_D)\gamma_5 \bar{G}^T \bar{G}, 0\}], \\
\mathcal{J}_2 &= [\text{diag}\{(\mathcal{W}_N + \mathcal{W}_O)\mathcal{Q}_2 \mathcal{E}_2, 0\} \quad \text{diag}\{(\mathcal{W}_N + \mathcal{W}_O)\mathcal{E}_2, 0\} \quad \text{diag}\{2(\mathcal{W}_N + \mathcal{W}_O)\bar{\mu}_1 S_1 \mathcal{E}_2, 0\}], \\
\mathcal{J}_3 &= [\text{diag}\{2(\mathcal{W}_N + \mathcal{W}_O)\bar{\mu}_1 S_1 \mathcal{E}_2, 0\} \quad \text{diag}\{2(\mathcal{W}_N + \mathcal{W}_O)\bar{\sigma}_1 S_5 \mathcal{E}_2, 0\}], \quad \mathcal{J}_{20} = \text{diag}\{0, \bar{\mu}_2 \mathcal{Q}_2 \mathcal{W}_M\}, \\
\mathcal{J}_4 &= [\text{diag}\{\bar{\mu}_1 S_1^T \bar{N}_1 (\mathcal{W}_N + \mathcal{W}_O) \mathcal{E}_2^T, 0\} \quad \text{diag}\{\bar{\sigma}_1 S_5^T \bar{N}_1 (\mathcal{W}_N + \mathcal{W}_O) \mathcal{E}_2, 0\}], \\
\mathcal{J}_5 &= [\text{diag}\{\bar{\mu}_1 S_1^T \bar{N}_2 (\mathcal{W}_N + \mathcal{W}_O) \mathcal{E}_2^T, 0\} \quad \text{diag}\{0, \bar{\mu}_1 S_1 \mathcal{W}_M\} \quad \text{diag}\{\bar{\sigma}_1 S_5^T \bar{N}_2 (\mathcal{W}_N + \mathcal{W}_O) \mathcal{E}_2, 0\}], \\
\mathcal{J}_6 &= [\text{diag}\{2\mathcal{Q}_2 (\mathcal{W}_N + \mathcal{W}_O), 0\} \quad \text{diag}\{2\bar{\mu}_1 S_1 (\mathcal{W}_N + \mathcal{W}_O), 0\} \quad \text{diag}\{2\bar{\sigma}_1 S_5^T (\mathcal{W}_N + \mathcal{W}_O), 0\}], \\
\mathcal{J}_{10} &= \text{diag}\{(\bar{\mu}_1 S_1^T \mathcal{E}_2^T \bar{E}_2 + \bar{\sigma}_1 S_5 \bar{E}_2 \bar{E}_2)(\mathcal{W}_N + \mathcal{W}_O), 0\}, \quad \mathbb{M}_{13} = \Xi_2, \quad \mathbb{M}_{14} = \Xi_2, \quad \mathbb{M}_{20} = \Xi_4, \\
\mathcal{J}_{12} &= \text{diag}\{2\bar{M}_1 \mathcal{E}_2 (\mathcal{W}_N + \mathcal{W}_O) (\text{diag}\{0, -2\mathcal{Q}_2 \mathcal{W}_M\} + \mathcal{W}_O), 0\}, \quad \mathcal{J}_{13} = \text{diag}\{-2\mathcal{Q}_2 \bar{N}_1 \mathcal{E}_2 (\mathcal{W}_N + \mathcal{W}_O), 0\}, \\
\mathcal{J}_{14} &= \text{diag}\{-2\mathcal{Q}_2 \bar{N}_2 \mathcal{E}_2 (\mathcal{W}_N + \mathcal{W}_O), 0\}, \quad \mathcal{G}_{15} = \text{diag}\{0, 0\}, \quad \mathcal{J}_{17} = \text{diag}\{\bar{\mu}_2 S_2 \mathcal{E}_2 (\mathcal{W}_N + \mathcal{W}_O), 0\}, \\
\mathcal{J}_{16} &= \text{diag}\{-2\mathcal{Q}_2 \bar{E}_2 \mathcal{E}_2 (\mathcal{W}_N + \mathcal{W}_O), 0\}, \quad \mathcal{J}_8 = \text{diag}\{(\bar{\mu}_1 S_1 + \bar{\sigma}_1 S_5)(\mathcal{W}_N + \mathcal{W}_O), 0\}, \\
\mathcal{J}_{18} &= \text{diag}\{\bar{\mu}_2 S_2 (\mathcal{W}_N + \mathcal{W}_O), 0\}, \quad \mathcal{J}_{19} = \text{diag}\{0, 0\} + \text{diag}\{(-4\mathcal{Q}_2^2 \mathcal{E}_2 - 4\bar{M}_1 \mathcal{Q}_2 \mathcal{E}_2)(\mathcal{W}_N + \mathcal{W}_O), 0\}, \\
\mathcal{J}_7 &= [\text{diag}\{2\bar{\mu}_1 S_1^T \mathcal{E}_2^T (\mathcal{W}_N + \mathcal{W}_O), 0\} \quad \text{diag}\{2\bar{\sigma}_1 S_5 \mathcal{E}_2^T (\mathcal{W}_N + \mathcal{W}_O), 0\}], \quad \mathbb{M}_{19} = [\Xi_2^2 + \Xi_2^2 - \Xi_2], \\
\mathcal{J}_9 &= [\text{diag}\{2\bar{\mu}_1 S_1 (\mathcal{W}_N + \mathcal{W}_O), 0\} \quad \text{diag}\{2\bar{\sigma}_1 S_5 (\mathcal{W}_N + \mathcal{W}_O), 0\}], \quad \mathbb{M}_{17} = \Xi_2, \quad \mathbb{M}_{18} = \Xi_2, \\
\mathcal{P}_1 &= [\Xi_1 \quad \Xi_3 \quad \Xi_2], \quad \mathcal{P}_2 = [\Xi_1 \quad \Xi_1 \quad \Xi_1^T], \quad \mathcal{P}_3 = [\Xi_1^T \quad \Xi_1^T \quad \Xi_1], \quad \mathcal{P}_4 = [\Xi_1^T \quad \Xi_1], \quad \mathbb{M}_{16} = \Xi_2, \\
\mathcal{P}_5 &= [\Xi_1^T \quad \Xi_3 \quad \Xi_1], \quad \mathcal{P}_6 = [\Xi_1 \quad \Xi_1^T \Xi_1 \quad \Xi_1^T \Xi_1], \quad \mathcal{P}_7 = [\Xi_1^T \Xi_1 \quad \Xi_1^T \Xi_1], \quad \mathcal{P}_8 = 2\Xi_1, \\
\mathcal{P}_9 &= [\Xi_1^T \Xi_1 \quad \Xi_1^T \Xi_1], \quad \mathcal{P}_{10} = \Xi_1^T + \Xi_1, \quad \mathcal{P}_{11} = 2\Xi_1^T, \quad \mathcal{P}_{12} = \Xi_1 \Xi_3 + \Xi_1^2 + \Xi_1, \\
\mathcal{P}_{13} &= \Xi_1, \quad \mathcal{P}_{14} = \Xi_1, \quad \mathcal{P}_{15} = \Xi_1^2, \quad \mathcal{P}_{16} = \Xi_1, \quad \mathcal{P}_{17} = \Xi_1, \quad \mathcal{P}_{18} = \Xi_1, \quad \mathbb{M}_{15} = \Xi_2^2, \\
\mathcal{P}_{19} &= [\Xi_1^2 + \Xi_1^2 - \Xi_1], \quad \mathcal{P}_{20} = \Xi_3, \quad \mathbb{M}_1 = [\Xi_2 \quad \Xi_4 \quad \Xi_1], \quad \mathbb{M}_2 = [\Xi_2 \quad \Xi_2 \quad \Xi_2^T], \\
\mathbb{M}_3 &= [\Xi_2^T \quad \Xi_2^T \quad \Xi_2], \quad \mathbb{M}_4 = [\Xi_2^T \quad \Xi_2], \quad \mathbb{M}_5 = [\Xi_2^T \quad \Xi_4 \quad \Xi_2], \quad \mathbb{M}_6 = [\Xi_2 \quad \Xi_2^T \Xi_2 \quad \Xi_2^T \Xi_2], \\
\mathbb{M}_7 &= [\Xi_2^T \Xi_2 \quad \Xi_2^T \Xi_2], \quad \mathbb{M}_8 = 2\Xi_2, \quad \mathbb{M}_9 = [\Xi_2^T \Xi_2 \quad \Xi_2^T \Xi_2], \quad \mathbb{M}_{10} = \Xi_2^T + \Xi_2, \\
\mathbb{M}_{11} &= 2\Xi_2^T, \quad \mathbb{M}_{12} = \Xi_2 \Xi_4 + \Xi_2^2 + \Xi_2, \quad \mathcal{J}_{11} = \text{diag}\{(\bar{\mu}_1 S_1^T \bar{E}_2 + \bar{\sigma}_1 S_5 \bar{E}_2)(\mathcal{W}_N + \mathcal{W}_O), 0\}, \\
\Gamma_1 &= \begin{bmatrix} 0 & 0 \\ (C_G + D_G)\mathcal{E}_1 & 0 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0 & 0 \\ (C_G + D_G) & 0 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} 0 & 0 \\ (\mathcal{N}_{\bar{G}} + \mathcal{O}_{\bar{G}})\mathcal{E}_2 & 0 \end{bmatrix}, \\
\Gamma_4 &= \begin{bmatrix} 0 & 0 \\ (\mathcal{N}_{\bar{G}} + \mathcal{O}_{\bar{G}}) & 0 \end{bmatrix}, \quad \Xi_1 = \begin{bmatrix} 0 & 0 \\ (C_G X_C + D_G X_D) & 0 \end{bmatrix}, \quad \Xi_2 = \begin{bmatrix} 0 & 0 \\ (\mathcal{N}_{\bar{G}} Y_N + \mathcal{O}_{\bar{G}} Y_O) & 0 \end{bmatrix}, \\
\Xi_3 &= \text{diag}\{0, B_C X_B\}, \quad \Xi_4 = \text{diag}\{0, \mathcal{M}_{\bar{G}} Y_M\}.
\end{aligned}$$

Proof. To prove the exponential stability of NNs (5) through non-fragile state estimators, we substitute the non-fragile gain parameters of $\bar{A}_2$, $\bar{B}$, $\bar{E}_2$, $\bar{M}_2$, $\bar{M}_3$, and $\bar{O}_2$, in (10) and (11), one can see that

$$\Lambda_1^* = \Lambda_a + \Delta \Lambda_a, \quad (52)$$

$$\Lambda_3^* = \Lambda_b + \Delta \Lambda_b, \quad (53)$$

where

$$\Lambda_a = \begin{bmatrix} \mathcal{U}_{11} & \mathcal{U}_{12} & \Lambda_{113} & 0 & 0 & \Lambda_{116} & \mathcal{U}_{17} & \Lambda_{118} & \Lambda_{119} & \Lambda_{1110} & \mathcal{U}_{111} & \mathcal{U}_{112} \\ * & \mathcal{U}_{22} & \mathcal{U}_{23} & 0 & 0 & \mathcal{U}_{26} & \mathcal{U}_{27} & \mathcal{U}_{28} & \Lambda_{129} & \Lambda_{1210} & \mathcal{U}_{211} & \mathcal{U}_{212} \\ * & * & \Lambda_{133} & 0 & 0 & \Lambda_{136} & \mathcal{U}_{37} & \Lambda_{138} & \Lambda_{139} & \Lambda_{1310} & \mathcal{U}_{311} & \Lambda_{1312} \\ * & * & * & \Lambda_{144} & \Lambda_{145} & \Lambda_{146} & 0 & 0 & \Lambda_{149} & \Lambda_{1410} & 0 & 0 \\ * & * & * & * & \Lambda_{155} & 0 & 0 & 0 & \Lambda_{159} & \Lambda_{1510} & 0 & 0 \\ * & * & * & * & * & \Lambda_{166} & \Lambda_{167} & \Lambda_{168} & \Lambda_{169} & \Lambda_{1610} & \mathcal{U}_{611} & \Lambda_{1612} \\ * & * & * & * & * & * & \mathcal{U}_{77} & \mathcal{U}_{78} & \Lambda_{179} & \Lambda_{1710} & \mathcal{U}_{711} & \mathcal{U}_{712} \\ * & * & * & * & * & * & * & \Lambda_{188} & \Lambda_{189} & \Lambda_{1810} & \mathcal{U}_{811} & \Lambda_{1812} \\ * & * & * & * & * & * & * & * & \Lambda_{199} & \Lambda_{1910} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Lambda_{11010} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & \Lambda_{1111} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Lambda_{11212} \end{bmatrix} < 0,$$

$$\Lambda_b = \begin{bmatrix} \hat{\mathcal{U}}_{11} & \hat{\mathcal{U}}_{12} & \Lambda_{313} & 0 & 0 & \Lambda_{316} & \hat{\mathcal{U}}_{17} & \Lambda_{318} & \Lambda_{319} & \Lambda_{3110} & \hat{\mathcal{U}}_{111} & \hat{\mathcal{U}}_{112} \\ * & \hat{\mathcal{U}}_{22} & \hat{\mathcal{U}}_{23} & 0 & 0 & \hat{\mathcal{U}}_{26} & \hat{\mathcal{U}}_{27} & \hat{\mathcal{U}}_{28} & \Lambda_{329} & \Lambda_{3210} & \hat{\mathcal{U}}_{211} & \hat{\mathcal{U}}_{212} \\ * & * & \Lambda_{333} & 0 & 0 & \Lambda_{336} & \hat{\mathcal{U}}_{37} & \Lambda_{338} & \Lambda_{339} & \Lambda_{3310} & \hat{\mathcal{U}}_{311} & \Lambda_{3312} \\ * & * & * & \Lambda_{344} & \Lambda_{345} & \Lambda_{346} & 0 & 0 & \Lambda_{349} & \Lambda_{3410} & 0 & 0 \\ * & * & * & * & \Lambda_{355} & 0 & 0 & 0 & \Lambda_{359} & \Lambda_{3510} & 0 & 0 \\ * & * & * & * & * & \Lambda_{366} & \Lambda_{367} & \Lambda_{368} & \Lambda_{369} & \Lambda_{3610} & \hat{\mathcal{U}}_{611} & \Lambda_{3612} \\ * & * & * & * & * & * & \hat{\mathcal{U}}_{77} & \hat{\mathcal{U}}_{78} & \Lambda_{379} & \Lambda_{3710} & \hat{\mathcal{U}}_{711} & \hat{\mathcal{U}}_{712} \\ * & * & * & * & * & * & * & \Lambda_{388} & \Lambda_{389} & \Lambda_{3810} & \hat{\mathcal{U}}_{811} & \Lambda_{3812} \\ * & * & * & * & * & * & * & * & \Lambda_{399} & \Lambda_{3910} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Lambda_{31010} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & \Lambda_{3111} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Lambda_{31212} \end{bmatrix} < 0,$$

$$\begin{aligned} \mathcal{U}_{11} &= 2\text{diag}\{0, \mathcal{B}_G\} \mathcal{Q}_1 - 2\mathcal{Q}_1(\bar{\mathcal{A}}_1)\Gamma_1 + \mathcal{R}_{11} + \bar{\mu}_2 \mathcal{S}_2[\text{diag}\{0, \mathcal{B}_G\}^T \text{diag}\{0, \mathcal{B}_G\}] + \mathcal{S}_4 \\ &\quad + \sigma_2 \mathcal{S}_6[\text{diag}\{0, \mathcal{B}_G\}^T \text{diag}\{0, \mathcal{B}_G\}] + 2\mathcal{T}_3 + 2\mathcal{T}_7 + \gamma_2 \bar{\mathcal{N}}_1 \bar{\mathcal{K}}_2^T \bar{\mathcal{K}}_2 + (\Gamma_4 + I) \gamma_6 \bar{\mathcal{G}}^T \bar{\mathcal{G}} \\ &\quad + \bar{\mu}_2 \mathbb{B}_2 + \gamma_2 \bar{\mathcal{K}}_2^T \bar{\mathcal{K}}_2 + \bar{\sigma}_2 \mathbb{B}_4 + \delta_1, \quad \mathcal{U}_{12} = 2(\Gamma_1 + \bar{\mathcal{A}}_1) \mathcal{Q}_1 + 2(\bar{\mathcal{A}}_1 + \Gamma_1) + \bar{\mu}_2 \mathcal{S}_2(\Gamma_1^T \Gamma_1) \\ &\quad + \bar{\sigma}_2 \mathcal{S}_6, \quad \mathcal{U}_{17} = 2\mathcal{Q}_1 \Gamma_2 + \bar{\mu}_2 \mathcal{S}_2(\Gamma_2^T \Gamma_2) + \bar{\sigma}_2 \mathcal{S}_6(\Gamma_2^T \Gamma_2), \quad \mathcal{U}_{8,11} = -2(\bar{\mathcal{A}}_1 + \Gamma_1) \mathcal{Q}_1 \bar{\mathcal{E}}_1, \\ \mathcal{U}_{22} &= \bar{\mu}_2 \mathcal{S}_2(\bar{\mathcal{A}}_1 + \Gamma_1)^T (\bar{\mathcal{A}}_1 + \Gamma_1) + \bar{\sigma}_2 \mathcal{S}_6(\bar{\mathcal{A}}_1 + \Gamma_1)^T (\bar{\mathcal{A}}_1 + \Gamma_1), \quad \mathcal{U}_{712} = \bar{\mu}_1 \mathcal{Q}_1 \Gamma_2, \\ \mathcal{U}_{23} &= \bar{\mu}_2 \mathcal{S}_2(\bar{\mathcal{A}}_1 + \Gamma_1)^T \bar{\mathcal{C}} + \bar{\sigma}_2 \mathcal{S}_6(\bar{\mathcal{A}}_1 + \Gamma_1)^T \bar{\mathcal{C}}, \quad \mathcal{U}_{37} = \bar{\mu}_2 \mathcal{S}_2(\bar{\mathcal{C}}^T \Gamma_2) + \bar{\sigma}_2 \mathcal{S}_6(\bar{\mathcal{C}}^T \Gamma_2), \\ \mathcal{U}_{26} &= \bar{\mu}_2 \mathcal{S}_2(\text{diag}\{0, \mathcal{B}_G\} + \Gamma_1)^T (\bar{\mathcal{A}}_1 + \Gamma_1) \bar{\mathcal{D}} + \bar{\sigma}_2 \mathcal{S}_6(\bar{\mathcal{A}}_1 + \Gamma_1)^T (\bar{\mathcal{A}}_1 + \Gamma_1) \bar{\mathcal{D}}, \\ \mathcal{U}_{27} &= \bar{\mu}_2 \mathcal{S}_2(\bar{\mathcal{A}}_1 + \Gamma_1)^T \Gamma_2 + \bar{\sigma}_2 \mathcal{S}_6(\bar{\mathcal{A}}_1 + \Gamma_1)^T \Gamma_2, \quad \mathcal{U}_{78} = \bar{\mu}_2 \mathcal{S}_2(\Gamma_2^T \bar{\mathcal{E}}_1) + \bar{\sigma}_2 \mathcal{S}_6(\Gamma_2^T \bar{\mathcal{E}}_1), \\ \mathcal{U}_{28} &= \bar{\mu}_2 \mathcal{S}_2(\bar{\mathcal{A}}_1 + \Gamma_1)^T \bar{\mathcal{E}}_1 + \bar{\sigma}_2 \mathcal{S}_6(\bar{\mathcal{A}}_1 + \Gamma_1)^T \bar{\mathcal{E}}_1, \quad \mathcal{U}_{77} = \bar{\mu}_2 \mathcal{S}_2(\Gamma_2^T \Gamma_2) + \bar{\sigma}_2 \mathcal{S}_6(\Gamma_2^T \Gamma_2) - \gamma_5 I, \\ \mathcal{U}_{111} &= -2(\bar{\mathcal{A}}_1 + \Gamma_1) \mathcal{Q}_1 (\text{diag}\{0, \mathcal{B}_G\}) + 2(\bar{\mathcal{A}}_1 + \Gamma_1)^2 \mathcal{Q}_1, \quad \mathcal{U}_{7,11} = -2(\bar{\mathcal{A}}_1 + \Gamma_1) \mathcal{Q}_1 \Gamma_2, \\ \mathcal{U}_{211} &= -2(\bar{\mathcal{A}}_1 + \Gamma_1) \mathcal{Q}_1 (\bar{\mathcal{A}}_1 + \Gamma_1) - 2(\bar{\mathcal{A}}_1 + \Gamma_1)^2 \mathcal{Q}_1, \quad \mathcal{U}_{6,11} = -2(\bar{\mathcal{A}}_1 + \Gamma_1) \mathcal{Q}_1 \bar{\mathcal{D}}, \\ \mathcal{U}_{3,11} &= -2(\bar{\mathcal{A}}_1 + \Gamma_1) \mathcal{Q}_1 \bar{\mathcal{C}}, \quad \mathcal{U}_{112} = \bar{\mu}_1 \mathcal{Q}_1 [\text{diag}\{0, \mathcal{B}_G\}], \quad \mathcal{U}_{212} = \bar{\mu}_1 (\bar{\mathcal{A}}_1 + \Gamma_1) \mathcal{Q}_1, \\ \hat{\mathcal{U}}_{11} &= 2\text{diag}\{0, \mathcal{M}_{\bar{\mathcal{C}}}\} \mathcal{Q}_2 - 2\mathcal{Q}_2(\bar{\mathcal{M}}_1) \Gamma_3 + \mathcal{P}_{11} + \bar{\mu}_1 \mathcal{S}_1[\text{diag}\{0, \mathcal{M}_{\bar{\mathcal{C}}}\}^T \text{diag}\{0, \mathcal{M}_{\bar{\mathcal{C}}}\}] \\ &\quad + \mathcal{S}_3 + \sigma_1 \mathcal{S}_5[\text{diag}\{0, \mathcal{M}_{\bar{\mathcal{C}}}\}^T \text{diag}\{0, \mathcal{M}_{\bar{\mathcal{C}}}\}] + 2\mathcal{T}_1 + 2\mathcal{T}_5 + \gamma_1 \bar{\mathcal{C}} \bar{\mathcal{K}}_1^T \bar{\mathcal{K}}_1 + (\Gamma_2 + I) \\ &\quad \times \gamma_5 \bar{\mathcal{G}}^T \bar{\mathcal{G}} + \bar{\mu}_1 \mathbb{B}_1 + \gamma_1 \bar{\mathcal{K}}_1^T \bar{\mathcal{K}}_1 + \bar{\sigma}_1 \mathbb{B}_3 + \delta_2, \quad \hat{\mathcal{U}}_{37} = \bar{\mu}_1 \mathcal{S}_1 \bar{\mathcal{N}}_1^T \Gamma_4 + \bar{\sigma}_1 \mathcal{S}_5 \bar{\mathcal{N}}_1^T \Gamma_4, \\ \hat{\mathcal{U}}_{12} &= 2(\Gamma_3 + \bar{\mathcal{M}}_1) \mathcal{Q}_2 + 2(\bar{\mathcal{M}}_1 + \Gamma_3) + \bar{\mu}_1 \mathcal{S}_1(\Gamma_3^T \Gamma_3) + \bar{\sigma}_1 \mathcal{S}_5, \quad \hat{\mathcal{U}}_{712} = \bar{\mu}_2 \mathcal{Q}_2 \Gamma_4, \\ \hat{\mathcal{U}}_{17} &= 2\mathcal{Q}_2 \Gamma_4 + \bar{\mu}_1 \mathcal{S}_1(\Gamma_4^T \Gamma_4) + \bar{\sigma}_1 \mathcal{S}_5(\Gamma_4^T \Gamma_4), \quad \hat{\mathcal{U}}_{78} = \bar{\mu}_1 \mathcal{S}_1(\Gamma_4^T \bar{\mathcal{E}}_2) + \bar{\sigma}_1 \mathcal{S}_5(\Gamma_4^T \bar{\mathcal{E}}_2), \\ \hat{\mathcal{U}}_{22} &= \bar{\mu}_1 \mathcal{S}_1(\bar{\mathcal{M}}_1 + \Gamma_3)^T (\bar{\mathcal{M}}_1 + \Gamma_3) + \bar{\sigma}_1 \mathcal{S}_5(\bar{\mathcal{M}}_1 + \Gamma_3)^T (\bar{\mathcal{M}}_1 + \Gamma_3), \\ \hat{\mathcal{U}}_{23} &= \bar{\mu}_1 \mathcal{S}_1(\bar{\mathcal{M}}_1 + \Gamma_3)^T \bar{\mathcal{N}}_1 + \bar{\sigma}_1 \mathcal{S}_5(\bar{\mathcal{M}}_1 + \Gamma_3)^T \bar{\mathcal{N}}_1, \quad \hat{\mathcal{U}}_{112} = \bar{\mu}_2 \mathcal{Q}_2 [\text{diag}\{0, \mathcal{M}_{\bar{\mathcal{C}}}\}], \\ \hat{\mathcal{U}}_{26} &= \bar{\mu}_1 \mathcal{S}_1(\text{diag}\{0, \mathcal{M}_{\bar{\mathcal{C}}}\} + \Gamma_3)^T (\bar{\mathcal{M}}_1 + \Gamma_3) \bar{\mathcal{N}}_2 + \bar{\sigma}_1 \mathcal{S}_5(\bar{\mathcal{M}}_1 + \Gamma_3)^T (\bar{\mathcal{M}}_1 + \Gamma_3) \bar{\mathcal{N}}_2, \\ \hat{\mathcal{U}}_{27} &= \bar{\mu}_1 \mathcal{S}_1(\bar{\mathcal{M}}_1 + \Gamma_3)^T \Gamma_4 + \bar{\sigma}_1 \mathcal{S}_5(\bar{\mathcal{M}}_1 + \Gamma_3)^T \Gamma_4, \quad \hat{\mathcal{U}}_{6,11} = -2(\bar{\mathcal{M}}_1 + \Gamma_3) \mathcal{Q}_2 \bar{\mathcal{N}}_2, \\ \hat{\mathcal{U}}_{28} &= \bar{\mu}_1 \mathcal{S}_1(\bar{\mathcal{M}}_1 + \Gamma_3)^T \bar{\mathcal{E}}_2 + \bar{\sigma}_1 \mathcal{S}_5(\bar{\mathcal{M}}_1 + \Gamma_3)^T \bar{\mathcal{E}}_2, \quad \hat{\mathcal{U}}_{7,11} = -2(\bar{\mathcal{M}}_1 + \Gamma_3) \mathcal{Q}_2 \Gamma_4, \\ \hat{\mathcal{U}}_{77} &= \bar{\mu}_1 \mathcal{S}_1(\Gamma_4^T \Gamma_4) + \bar{\sigma}_1 \mathcal{S}_5(\Gamma_4^T \Gamma_4) - \gamma_6 I, \quad \hat{\mathcal{U}}_{3,11} = -2(\bar{\mathcal{M}}_1 + \Gamma_3) \mathcal{Q}_2 \bar{\mathcal{N}}_1, \\ \hat{\mathcal{U}}_{111} &= -2(\bar{\mathcal{M}}_1 + \Gamma_3) \mathcal{Q}_2 (\text{diag}\{0, \mathcal{M}_{\bar{\mathcal{C}}}\}) + 2(\bar{\mathcal{M}}_1 + \Gamma_3)^2 \mathcal{Q}_2, \quad \hat{\mathcal{U}}_{212} = \bar{\mu}_2 (\bar{\mathcal{M}}_1 + \Gamma_3) \mathcal{Q}_2, \\ \hat{\mathcal{U}}_{211} &= -2(\bar{\mathcal{M}}_1 + \Gamma_3) \mathcal{Q}_2 (\bar{\mathcal{M}}_1 + \Gamma_3) - 2(\bar{\mathcal{M}}_1 + \Gamma_3)^2 \mathcal{Q}_2, \quad \hat{\mathcal{U}}_{811} = -2(\bar{\mathcal{M}}_1 + \Gamma_3) \mathcal{Q}_2 \bar{\mathcal{E}}_2. \end{aligned}$$

Then, it follows that

$$\Lambda_1^* = \Lambda_a + \Delta \Lambda_a = \Lambda_a + \mathcal{H}_1 \Sigma(t) \mathcal{Z}_1 + \mathcal{Z}_1^T \Sigma^T(t) \mathcal{H}_1^T < 0, \quad (54)$$

$$\Lambda_3^* = \Lambda_b + \Delta \Lambda_b = \Lambda_b + \mathcal{H}_2 \Sigma^*(t) \mathcal{Z}_2 + \mathcal{Z}_2^T \Sigma^{*T}(t) \mathcal{H}_2^T < 0, \quad (55)$$

where

$$\begin{aligned} \mathcal{H}_1 &= [\hat{\mathcal{B}}_1 \ \hat{\mathcal{B}}_2 \ \hat{\mathcal{B}}_3 \ 0 \ 0 \ \hat{\mathcal{B}}_4 \ \hat{\mathcal{B}}_5 \ \hat{\mathcal{B}}_6 \ 0 \ 0 \ \hat{\mathcal{B}}_7 \ \hat{\mathcal{B}}_8], \\ \mathcal{Z}_1 &= [\hat{\mathcal{G}}_1 \ \hat{\mathcal{G}}_2 \ \hat{\mathcal{G}}_3 \ 0 \ 0 \ \hat{\mathcal{G}}_4 \ \hat{\mathcal{G}}_5 \ \hat{\mathcal{G}}_6 \ 0 \ 0 \ \hat{\mathcal{G}}_7 \ \hat{\mathcal{G}}_8], \\ \mathcal{H}_2 &= [\hat{\mathcal{A}}_1 \ \hat{\mathcal{A}}_2 \ \hat{\mathcal{A}}_3 \ 0 \ 0 \ \hat{\mathcal{A}}_4 \ \hat{\mathcal{A}}_5 \ \hat{\mathcal{A}}_6 \ 0 \ 0 \ \hat{\mathcal{A}}_7 \ \hat{\mathcal{A}}_8], \\ \mathcal{Z}_2 &= [\hat{\mathcal{J}}_1 \ \hat{\mathcal{J}}_2 \ \hat{\mathcal{J}}_3 \ 0 \ 0 \ \hat{\mathcal{J}}_4 \ \hat{\mathcal{J}}_5 \ \hat{\mathcal{J}}_6 \ 0 \ 0 \ \hat{\mathcal{J}}_7 \ \hat{\mathcal{J}}_8], \\ \hat{\mathcal{B}}_1 &= [\mathcal{P}_1^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{B}}_2 = [\mathcal{P}_2^T \ \mathcal{P}_3^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{B}}_3 &= [0 \ \mathcal{P}_4^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{B}}_4 = [0 \ \mathcal{P}_5^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{B}}_5 &= [\mathcal{P}_6^T \ \mathcal{P}_7^T \ \mathcal{P}_8^T \ 0 \ 0 \ 0 \ \mathcal{P}_9^T \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{B}}_6 = [0 \ \mathcal{P}_{10}^T \ 0 \ 0 \ 0 \ 0 \ \mathcal{P}_{11}^T \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{B}}_7 &= [\mathcal{P}_{12}^T \ \mathcal{P}_{19}^T \ \mathcal{P}_{13}^T \ 0 \ 0 \ \mathcal{P}_{14}^T \ \mathcal{P}_{15}^T \ \mathcal{P}_{16}^T \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{B}}_8 = [\mathcal{P}_{20}^T \ \mathcal{P}_{17}^T \ 0 \ 0 \ 0 \ 0 \ \mathcal{P}_{18}^T \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{G}}_1 &= [\mathcal{G}_1^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{G}}_2 = [\mathcal{G}_2^T \ \mathcal{G}_3^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{G}}_3 &= [0 \ \mathcal{G}_4^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{G}}_4 = [0 \ \mathcal{G}_5^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{G}}_5 &= [\mathcal{G}_6^T \ \mathcal{G}_7^T \ \mathcal{G}_8^T \ 0 \ 0 \ 0 \ \mathcal{G}_9^T \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{G}}_6 = [0 \ \mathcal{G}_{10}^T \ 0 \ 0 \ 0 \ 0 \ \mathcal{G}_{11}^T \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{G}}_7 &= [\mathcal{G}_{12}^T \ \mathcal{G}_{19}^T \ \mathcal{G}_{13}^T \ 0 \ 0 \ \mathcal{G}_{14}^T \ \mathcal{G}_{15}^T \ \mathcal{G}_{16}^T \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{G}}_8 = [\mathcal{G}_{20}^T \ \mathcal{G}_{17}^T \ 0 \ 0 \ 0 \ 0 \ \mathcal{G}_{18}^T \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{A}}_1 &= [\mathcal{M}_1^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{A}}_2 = [\mathcal{M}_2^T \ \mathcal{M}_3^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{A}}_3 &= [0 \ \mathcal{M}_4^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{A}}_4 = [0 \ \mathcal{M}_5^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{A}}_5 &= [\mathcal{M}_6^T \ \mathcal{M}_7^T \ \mathcal{M}_8^T \ 0 \ 0 \ 0 \ \mathcal{M}_9^T \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{A}}_6 = [0 \ \mathcal{M}_{10}^T \ 0 \ 0 \ 0 \ 0 \ \mathcal{M}_{11}^T \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{A}}_7 &= [\mathcal{M}_{12}^T \ \mathcal{M}_{19}^T \ \mathcal{M}_{13}^T \ 0 \ 0 \ \mathcal{M}_{14}^T \ \mathcal{M}_{15}^T \ \mathcal{M}_{16}^T \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{A}}_8 = [\mathcal{M}_{20}^T \ \mathcal{M}_{17}^T \ 0 \ 0 \ 0 \ 0 \ \mathcal{M}_{18}^T \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{J}}_1 &= [\mathcal{J}_1^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{J}}_2 = [\mathcal{J}_2^T \ \mathcal{J}_3^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{J}}_3 &= [0 \ \mathcal{J}_4^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{J}}_4 = [0 \ \mathcal{J}_5^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{J}}_5 &= [\mathcal{J}_6^T \ \mathcal{J}_7^T \ \mathcal{J}_8^T \ 0 \ 0 \ 0 \ \mathcal{J}_9^T \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{J}}_6 = [0 \ \mathcal{J}_{10}^T \ 0 \ 0 \ 0 \ 0 \ \mathcal{J}_{11}^T \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \hat{\mathcal{J}}_7 &= [\mathcal{J}_{12}^T \ \mathcal{J}_{19}^T \ \mathcal{J}_{13}^T \ 0 \ 0 \ \mathcal{J}_{14}^T \ \mathcal{J}_{15}^T \ \mathcal{J}_{16}^T \ 0 \ 0 \ 0 \ 0]^T, \quad \hat{\mathcal{J}}_8 = [\mathcal{J}_{20}^T \ \mathcal{J}_{17}^T \ 0 \ 0 \ 0 \ 0 \ \mathcal{J}_{18}^T \ 0 \ 0 \ 0 \ 0 \ 0]^T, \end{aligned}$$

With the aid of Lemma 2.5, clearly we know that (49) and (50) holds if and only if (54) and (55) satisfied. Therefore, it reach that the derivative of LKF proposed in this paper is less than zero. Remaining proof of this theorem is similar to the procedure of Theorem 3.1 and we acquire the neural networks (1) or (5) is exponentially stable. $\square$

Remark 3.3. In this research work, the connection between time-varying delays and their allowable upper bounds is fully addressed. Further, several inequality techniques are handled to obtain the maximum upper bounds of mixed time-varying delays, see Example 4.1. Therefore, the designed methods in this paper, may lead to novelty.

4. Numerical examples with comparisons

In this section, two numerical examples will be provided to testify the benefits of the proposed results.

Example 4.1. Consider the BAM-type neural networks (5) with the following associated parameters:

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0.02 & 0.01 \\ -0.02 & 0.01 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0.04 & 0.03 \\ 0.01 & 0.03 \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} 0.03 & -0.02 \\ 0.03 & 0.02 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \\ \mathcal{M} &= \begin{pmatrix} 0.01 & 0.04 \\ 0.02 & 0.06 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} 0.03 & -0.02 \\ 0.05 & 0.02 \end{pmatrix}, \quad \mathcal{O} = \begin{pmatrix} 0.02 & -0.03 \\ 0.02 & 0.04 \end{pmatrix}, \quad \bar{\mathcal{A}}_1 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \bar{\mathcal{B}} &= \begin{pmatrix} 0 & 0 & 0 \\ 0.02 & 0 & 0 \\ 0 & 0.05 & 0 \end{pmatrix}, \quad \bar{\mathcal{C}} = \begin{pmatrix} 0.02 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\mathcal{A}}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.02 \end{pmatrix}, \quad \bar{\mathcal{D}} = \begin{pmatrix} 0.04 & 0 & 0 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \bar{\mathcal{E}}_1 &= \begin{pmatrix} 0.03 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\mathcal{E}}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0.2 & 0.04 & 0 \\ 0.01 & 0.03 & 0 \end{pmatrix}, \quad \bar{\mathcal{M}}_1 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\mathcal{M}}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.04 \end{pmatrix}, \\ \bar{\mathcal{M}}_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0.03 & 0.01 & 0 \\ 0.04 & 0.02 & 0 \end{pmatrix}, \quad \bar{\mathcal{N}}_1 = \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.06 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\mathcal{N}}_2 = \begin{pmatrix} 0.03 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\mathcal{O}}_1 = \begin{pmatrix} 0.02 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

$$\bar{O}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0.5 & 0.1 & 0 \\ 0.03 & 0.02 & 0 \end{pmatrix}, \quad \bar{K}_1 = \begin{pmatrix} 0.04 & 0 & 0 \\ 0 & 0.06 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}, \quad \bar{K}_2 = \begin{pmatrix} 0.06 & 0 & 0 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0.05 \end{pmatrix},$$

$$\hat{G} = \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.06 \end{pmatrix}, \quad \bar{G} = \begin{pmatrix} 0.04 & 0 & 0 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0.07 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\bar{\mu}_1 = \bar{\mu}_2 = 17.412$, $\mu_1 = 0.4$, $\mu_2 = 0.7$, $\bar{\sigma}_1 = \bar{\sigma}_2 = 10.73$, $\delta_1 = 0.3$, $\mu_1(t) = 0.2 - 0.2 * \cos(t)$, $\mu_2(t) = 0.4 - 0.4 * \cos(t)$, $\sigma_1(t) = \sinh(t)$, $\sigma_2(t) = 0.5 * |t|$, $\delta_2 = 0.5$, $v = 0.8$,

The following activation functions are playing in neural networks (5):

$$f(\beta(t)) = \tanh(\beta(t)), \quad h(\alpha(t)) = \sin(\alpha(t)).$$

By solving the LMIs in Theorem 3.1 using the MATLAB LMI control toolbox, one can obtain the feasible solutions as follows:

$$\mathcal{Q}_1 = \begin{pmatrix} 17.8242 & 0.4331 & 0.1010 \\ 0.4331 & 6.2824 & 0.4337 \\ 0.1010 & 0.4337 & 66.3108 \end{pmatrix}, \quad \mathcal{Q}_2 = \begin{pmatrix} 17.9548 & 0.3650 & 0.4248 \\ 0.3650 & 3.7615 & 0.0991 \\ 0.4248 & 0.0991 & 63.7982 \end{pmatrix},$$

$$\mathcal{Q} = 10^3 \times \begin{pmatrix} 3.8895 & -0.0149 & 0.0117 \\ -0.0149 & 3.4211 & 0.1867 \\ 0.0117 & 0.1867 & 0.7098 \end{pmatrix}, \quad \hat{\mathcal{Q}} = 10^3 \times \begin{pmatrix} 3.9200 & -0.0195 & 0.0742 \\ -0.0195 & 3.8997 & 0.0748 \\ 0.0742 & 0.0748 & 0.7004 \end{pmatrix},$$

$$\mathcal{P}_{11} = \begin{pmatrix} 697.0254 & -0.0631 & 0.0079 \\ -0.0631 & 712.8302 & -0.2982 \\ 0.0079 & -0.2982 & 706.7658 \end{pmatrix}, \quad \mathcal{P}_{12} = \begin{pmatrix} 697.0256 & 0.0000 & 0.0000 \\ 0.0000 & 697.0256 & 0.0000 \\ 0.0000 & 0.0000 & 697.0256 \end{pmatrix},$$

$$\mathcal{P}_{21} = \begin{pmatrix} 697.0256 & 0.0000 & 0.0000 \\ 0.0000 & 697.0256 & 0.0000 \\ 0.0000 & 0.0000 & 697.0256 \end{pmatrix}, \quad \mathcal{P}_{22} = 10^3 \times \begin{pmatrix} 1.3282 & -0.0000 & 0.0000 \\ -0.0000 & 1.2541 & -0.0000 \\ 0.0000 & -0.0000 & 1.3899 \end{pmatrix},$$

$$\mathcal{R}_{11} = 10^3 \times \begin{pmatrix} 1.0425 & -0.0025 & -0.0059 \\ -0.0025 & 1.0180 & 0.0028 \\ -0.0059 & 0.0028 & 0.9521 \end{pmatrix}, \quad \mathcal{R}_{12} = \begin{pmatrix} 697.0256 & 0.0000 & 0.0000 \\ 0.0000 & 697.0256 & 0.0000 \\ 0.0000 & 0.0000 & 697.0256 \end{pmatrix},$$

$$\mathcal{R}_{21} = \begin{pmatrix} 697.0256 & 0.0000 & 0.0000 \\ 0.0000 & 697.0256 & 0.0000 \\ 0.0000 & 0.0000 & 697.0256 \end{pmatrix}, \quad \mathcal{R}_{22} = \begin{pmatrix} 42.2444 & -0.0000 & -0.0000 \\ -0.0000 & 42.2444 & 0.0000 \\ -0.0000 & 0.0000 & 42.2444 \end{pmatrix},$$

$$\mathcal{S}_1 = \begin{pmatrix} 4.1231 & 0.0936 & 0.4625 \\ 0.0936 & 1.0039 & 2.5739 \\ 0.4625 & 2.5739 & 74.3078 \end{pmatrix}, \quad \mathcal{S}_2 = \begin{pmatrix} 4.1571 & 0.0795 & 1.3945 \\ 0.0795 & 0.6147 & 0.9375 \\ 1.3945 & 0.9375 & 71.4743 \end{pmatrix},$$

$$\mathcal{S}_3 = 10^3 \times \begin{pmatrix} 1.4272 & -0.0092 & 0.0015 \\ -0.0092 & 1.4217 & -0.0223 \\ 0.0015 & -0.0223 & 1.1917 \end{pmatrix}, \quad \mathcal{S}_4 = 10^3 \times \begin{pmatrix} 1.1699 & -0.0033 & -0.0072 \\ -0.0033 & 1.1467 & 0.0033 \\ -0.0072 & 0.0033 & 1.0539 \end{pmatrix},$$

$$\mathcal{S}_5 = \begin{pmatrix} 0.2110 & 0.0056 & 0.2141 \\ 0.0056 & 0.0263 & 0.5676 \\ 0.2141 & 0.5676 & 144.2893 \end{pmatrix}, \quad \mathcal{S}_6 = \begin{pmatrix} 0.2136 & 0.0066 & 0.5906 \\ 0.0066 & 0.0365 & 0.2876 \\ 0.5906 & 0.2876 & 131.4575 \end{pmatrix},$$

$$\mathcal{T}_1 = \begin{pmatrix} -0.0180 & -0.0008 & 0.0027 \\ -0.0033 & -0.0067 & -0.1162 \\ -0.0090 & -0.0525 & -1.4808 \end{pmatrix}, \quad \mathcal{T}_2 = 10^3 \times \begin{pmatrix} 0.2527 & -0.0054 & -0.0025 \\ -0.0072 & 0.1983 & 0.0094 \\ -0.0023 & 0.0067 & 0.0938 \end{pmatrix},$$

$$\mathcal{T}_3 = \begin{pmatrix} -0.0921 & -0.0026 & -0.0838 \\ -0.0027 & -0.0140 & 0.0022 \\ -0.0500 & -0.0314 & -2.4078 \end{pmatrix}, \quad \mathcal{T}_4 = 10^3 \times \begin{pmatrix} 0.1715 & -0.0006 & 0.0013 \\ -0.0014 & 0.1716 & -0.0035 \\ 0.0009 & -0.0013 & 0.0405 \end{pmatrix},$$

$$\mathcal{T}_5 = \begin{pmatrix} 0.0276 & 0.0008 & 0.0346 \\ 0.0005 & 0.0028 & -0.0585 \\ 0.0258 & 0.0684 & 17.3788 \end{pmatrix}, \quad \mathcal{T}_6 = \mathcal{T}_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{T}_7 = \begin{pmatrix} 0.0251 & 0.0006 & -0.0080 \\ 0.0009 & 0.0043 & 0.0690 \\ 0.0646 & 0.0315 & 14.4092 \end{pmatrix}, \quad \mathbb{B}_1 = \begin{pmatrix} -77.1649 & 0.2455 & -0.0436 \\ -0.6163 & -76.9077 & -0.6567 \\ 0.0981 & -0.3117 & -82.3579 \end{pmatrix},$$

$$\mathbb{B}_2 = \begin{pmatrix} -82.9791 & -0.2112 & -0.0126 \\ 0.0065 & -83.5082 & -0.1326 \\ -0.4845 & 0.3548 & -86.9947 \end{pmatrix}, \quad \mathbb{B}_3 = \begin{pmatrix} -206.4306 & -1.2703 & 0.1772 \\ -1.2940 & -203.6287 & -3.6119 \\ 0.1779 & -3.3211 & -245.7505 \end{pmatrix},$$

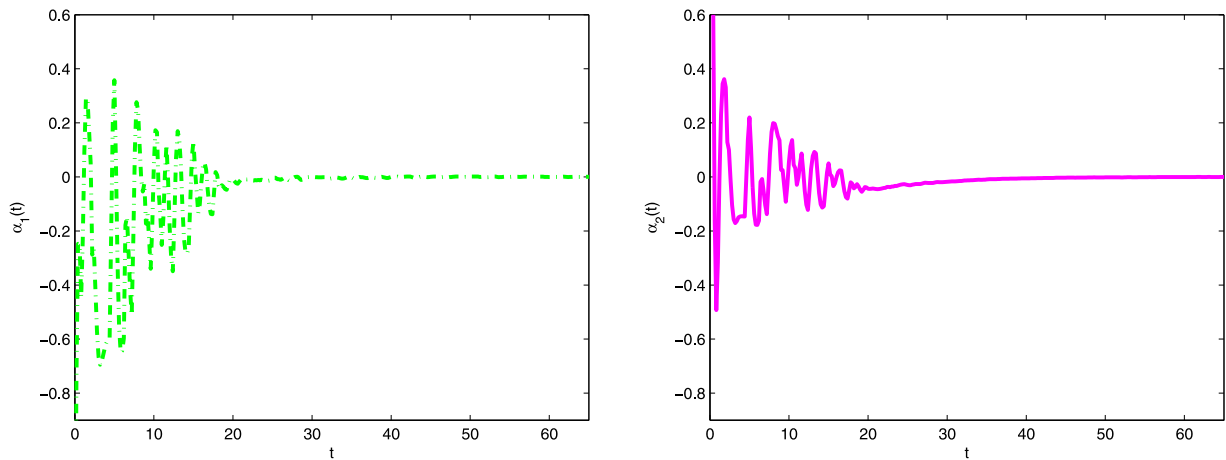


Fig. 1. The state responses $\alpha_1(t)$ and $\alpha_2(t)$ for BAMNNs (1).

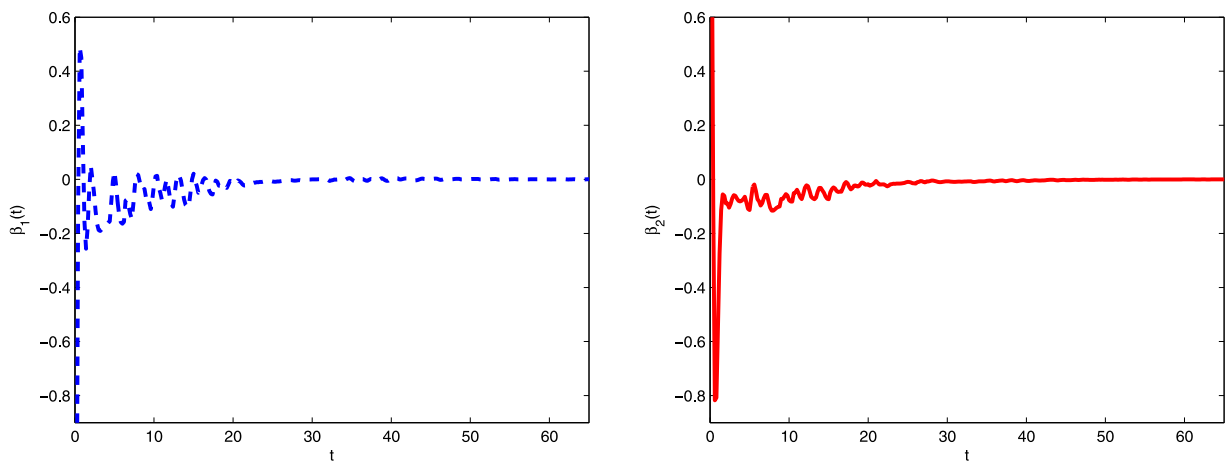


Fig. 2. The state trajectories $\beta_1(t)$ and $\beta_2(t)$ for (1).

Table 4

MAUBs of discrete time-varying delays for different μ_1 and μ_2 .

$0 < \mu_1 = \mu_2 < 1$	0.2	0.5	0.8	1
Theorem 3.1	17.164	11.022	6.391	0.905

$$\mathbb{B}_4 = \begin{pmatrix} -249.7155 & -0.9582 & -2.1999 \\ -0.8055 & -253.2492 & 1.0892 \\ -2.2718 & 0.8952 & -286.7763 \end{pmatrix}, \quad \gamma_1 = 499.6382, \quad \gamma_2 = 501.7781, \quad \gamma_3 = 507.2264,$$

$$\gamma_4 = 498.6319, \quad \gamma_5 = 526.4222, \quad \gamma_6 = 824.7407.$$

Therefore, via LMI control toolbox in MATLAB software, the state trajectories of proposed neural networks (1) are obtained, which are depicted in Figs. 1–3. Moreover, by solving the LMIs (10)–(15) in Theorem 3.1 the feasible solutions can be easily obtained. The achieved upper bounds of mixed time-varying delays for neural networks (5) are maximum, which are listed in Tables 2–4. According to Theorem 3.1, we can conclude that the BAM neural networks (1) or (5) is exponentially stable. Hence, this experiment demonstrate the contributions of this paper is more extensive and less conservative than some available literatures.

Example 4.2. Consider the form of non-fragile state estimator (4) with the following parameters:

$$\mathcal{C}_G = \begin{pmatrix} 1.2 & -0.3 \\ 0.2 & 1.5 \end{pmatrix}, \quad \mathcal{D}_G = \begin{pmatrix} 0.3 & 0.2 \\ -0.3 & 0.4 \end{pmatrix}, \quad \mathcal{M}_{\bar{G}} = \begin{pmatrix} 0.3 & 0.2 \\ -0.1 & 0.4 \end{pmatrix}, \quad \mathcal{N}_{\bar{G}} = \begin{pmatrix} 0.3 & -0.2 \\ 0.1 & -0.5 \end{pmatrix},$$

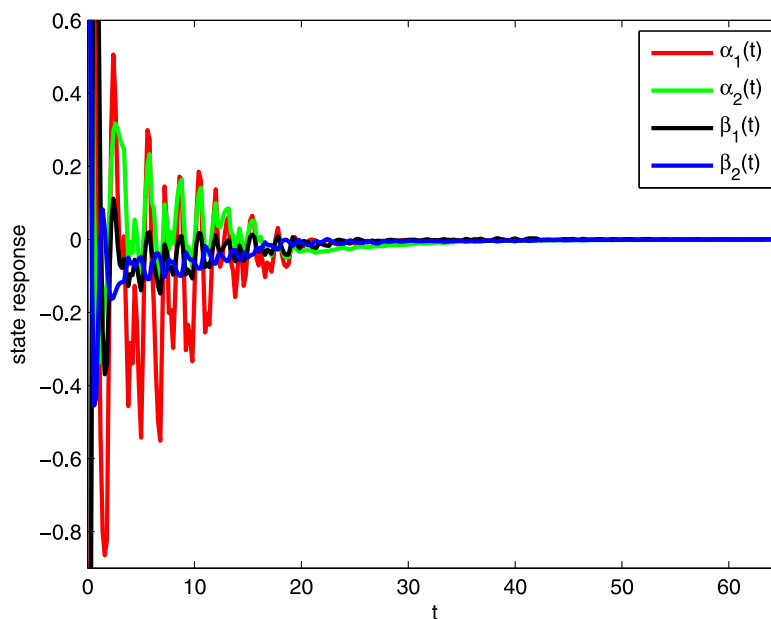


Fig. 3. State responses of developed BAMNNs (1).

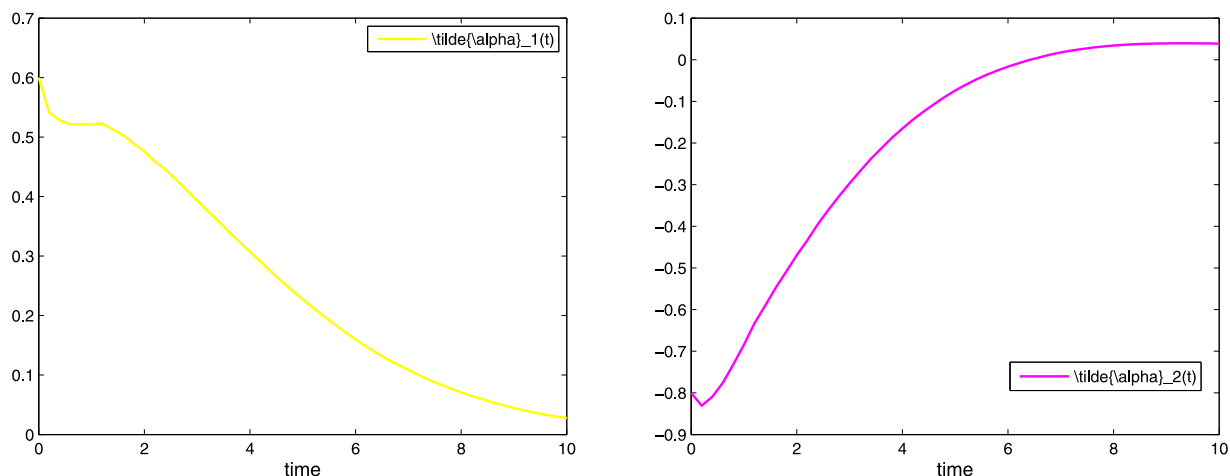


Fig. 4. The time responses of non-fragile state estimators for NNs (1).

$$\begin{aligned}
 \mathcal{O}_{\bar{G}} &= \begin{pmatrix} 0.1 & -0.3 \\ 0.1 & -0.2 \end{pmatrix}, \quad U_B = \begin{pmatrix} 0.2 & -0.1 \\ 0.3 & -0.2 \end{pmatrix}, \quad U_C = \begin{pmatrix} 0.1 & -0.5 \\ 0.2 & -0.3 \end{pmatrix}, \quad U_D = \begin{pmatrix} 0.3 & -0.1 \\ 0.3 & -0.4 \end{pmatrix}, \\
 \mathcal{W}_{\mathcal{M}} &= \begin{pmatrix} 0.1 & -0.4 \\ 0.2 & -0.5 \end{pmatrix}, \quad \mathcal{W}_{\mathcal{N}} = \begin{pmatrix} 0.3 & -0.4 \\ 0.1 & -0.1 \end{pmatrix}, \quad \mathcal{W}_{\mathcal{O}} = \begin{pmatrix} 0.4 & -0.2 \\ 0.1 & -0.2 \end{pmatrix}, \quad \mathcal{B}_G = \begin{pmatrix} 0.3 & 0.2 \\ -0.3 & 0.4 \end{pmatrix}, \\
 X_C &= \begin{pmatrix} 0.05 & -0.01 \\ -0.03 & 0.02 \end{pmatrix}, \quad X_B = \begin{pmatrix} 0.03 & 0.02 \\ 0.01 & 0.06 \end{pmatrix}, \quad X_D = \begin{pmatrix} 0.02 & 0.03 \\ -0.01 & 0.03 \end{pmatrix}, \quad Y_{\mathcal{M}} = \begin{pmatrix} 0.01 & 0.05 \\ 0.01 & 0.01 \end{pmatrix}, \\
 Y_{\mathcal{N}} &= \begin{pmatrix} 0.04 & 0.01 \\ 0.02 & 0.01 \end{pmatrix}, \quad Y_{\mathcal{O}} = \begin{pmatrix} 0.02 & 0.03 \\ -0.02 & 0.01 \end{pmatrix}, \quad \varepsilon_1 = \begin{pmatrix} 0.06 & 0.02 \\ 0.3 & 0.02 \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} 0.02 & 0.04 \\ 0.1 & 0.05 \end{pmatrix},
 \end{aligned}$$

The neuron-state-dependent nonlinear perturbation $g(t, \beta(t)) = 0.5 * \tanh(\beta(t))$, $\bar{g}(t, \alpha(t)) = 0.3 * |\alpha(t)|$. By utilizing the MATLAB LMI control toolbox, we can get the following state trajectories

Thus, Figs. 4–6 narrates the time responses of non-fragile state estimators (4) for concerned neural networks (5). From Examples 4.1 and 4.2, we conclude that the BAM neural networks with non-fragile state estimators (5) is exponentially stable.

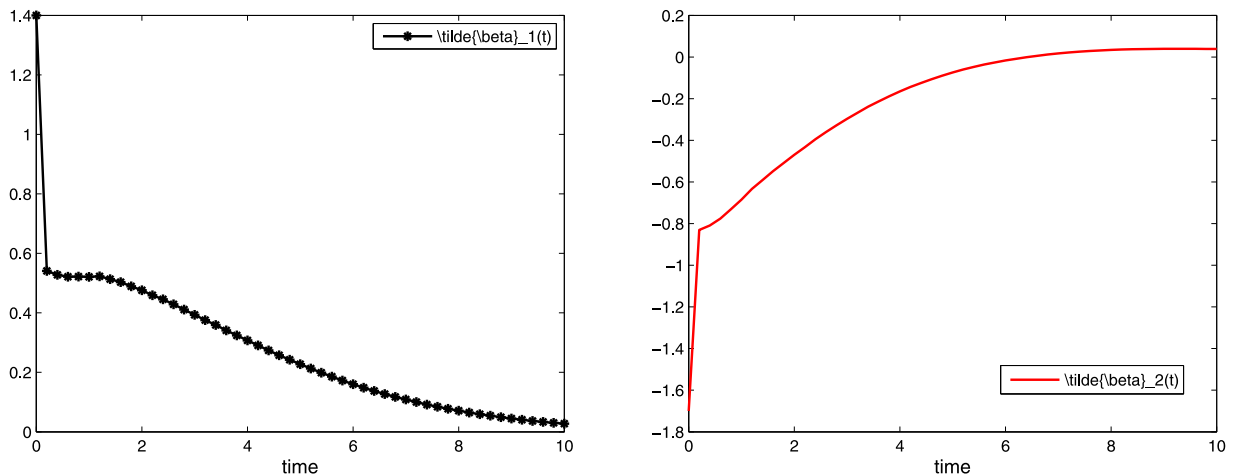


Fig. 5. The non-fragile state trajectories for (1).

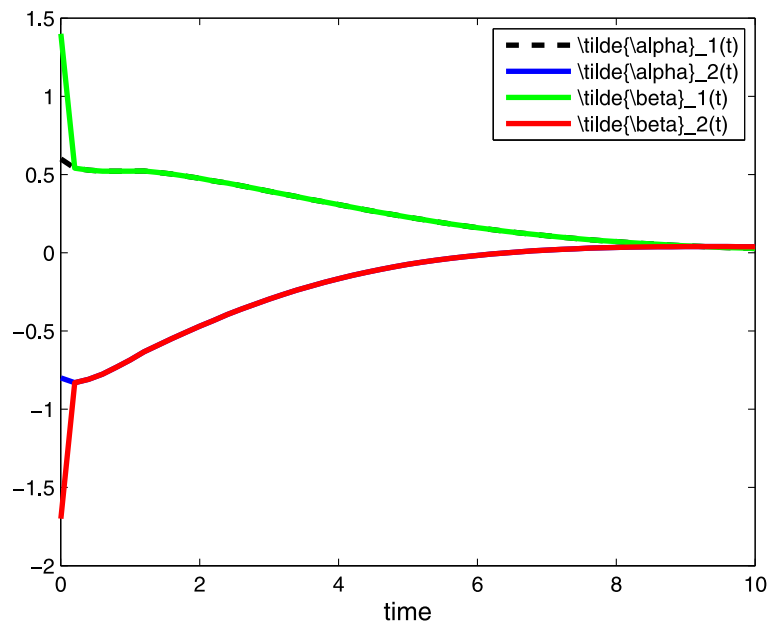


Fig. 6. Time responses of full-order non-fragile state estimators (4).

5. Conclusions

This paper has developed a new LMI approach to investigate the exponential stability criteria for a class of BAM-type neural networks with leakage, mixed time-varying delays via non-fragile state estimator. By a key role of Lyapunov theory, inequality techniques and matrix theory, some novel sufficient conditions were derived for the addressed neural networks to ensure the exponential stability. Additionally, the upper bounds of the both time-varying delays for our concerned design is maximum when compared with some existing results, which may possess highly important significance in the concerned neural networks. The obtained theoretical outcomes are less conserved and it can be testified by two numerical examples with time responses of non-fragile state estimators, which exhibit that the effectiveness of this research work.

To the best of our knowledge, there are no results on the Lagrange stability of time-delayed Cohen–Grossberg neural networks with non-fragile state estimators via Wirtinger based inequality, which might be our future research work.

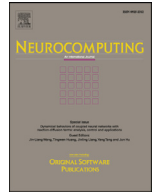
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Further synchronization in finite time analysis for time-varying delayed fractional order memristive competitive neural networks with leakage delay[☆]

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ABSTRACT

This article mainly concerns the synchronization in finite-time to the time-varying delay fractional order memristive competitive neural networks (TMFCNNs) with leakage delay. By means of Fillipov's theory, Gronwall–Bellman integral inequality, Hölder's inequality, and the Caputo derivative properties, the novel algebraic sufficient conditions are proposed to guarantee the synchronization in finite time of addressing TMFCNNs with non-integer order: $0 < p < \frac{1}{2}$ and $\frac{1}{2} \leq p < 1$ via finite-time output feedback controller. Up to now, there are no relevant results in fractional order competitive-type neural networks, and this article makes filling up for this gap. The obtained results are improved to some existing results on integer-order memristive competitive neural networks. Finally, two numerical examples with simulations are also designed to confirm the merits of the proposed theoretical results, while we estimate the upper bound of the settling time for the synchronization errors.

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1. Introduction

Over the past few days, the fractional order dynamical system attracts many researchers in different branches, especially science and engineering. Comparing to the traditional integer order dynamical system, the main distinguished influence of fractional order is that infinite memory and more degrees of freedom because it has nonlocal and weakly singular kernels [7]. As a very suitable instrument, fractional order calculus acted as the description of memory and hereditary properties of various materials and processes [3], and it takes the main role in all manners in different areas such as fluid mechanics [7], system identification [12], viscoelasticity [30], robotics [31] and so on. Moreover, the fractional order dynamical system provides fundamental and general computational ability for one neuron, which can help to signal processing, frequency-independent phase shifts of oscillatory neuronal firing and simulates to human brain [20]. When a network model involves at least one fractional order derivative or fractional integral, it is called a fractional order network model. Recently, the incorporation of neural network models into its fractional derivative is the main area of research in the field concerned, because it permits a long term memory and it describes the linkage between input-output signals accurately. Additionally, the superiority of the Caputo fractional order derivative is that the initial conditions for fractional order differential equations with Caputo derivatives take on the similar form as those for integer-order differentiation. Above all, the quantitative behavior of the fractional order dynamical system is an important step in the practical design for Caputo sense fractional-order neural networks. As we know, some excellent results of fractional order neural networks have been done in the literature, see Refs. [18,21,35,36,38,47,49].

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Nowadays, the dynamics of fractional order neural networks have attracted the attention of many researchers and various results on stability, dissipativity, pseudo-almost periodic solution, synchronization have been founded in literature, among which the fractional order synchronization dynamical system with respect to neural networks has been heated and challenging research areas because it has been successively applied to many areas such as associative memory, secure communications and so on. The prevailing synchronization results can be separated into two classifications as per the convergence time, specifically infinite time synchronization together with asymptotic synchronization, Mittag-Leffler synchronization, and generalized Mittag-Leffler synchronization, and finite time synchronization. Numerous outcomes as for infinite time synchronization have been done in the literature, see [8,29,48]. In real life, asymptotic synchronization is not convenient, for example, in human beings, the lifespan of machines are finite, so finite time synchronization is much better than asymptotic(or infinite time) synchronization [15,44]. Different from infinite time synchronization, finite-time synchronization requires the master-slave system to remain completely identical after some finite time, which is known as the settling time. For a finite time master-slave synchronization of neural network many aspects have been extensively studied under various control techniques. But, there are few consequences of finite-time synchronization in the fractional order sense, see [39,41].

At present, various kinds of neural network model exist, which include competitive type-neural networks [1], Cohen–Grossberg-type neural networks [19], cellular-type neural networks [24], recurrent-type neural networks [32,34], BAM-type neural networks [35], Hopfield-type neural networks [49] and so on. Competitive type-neural networks (CNN) were firstly deliberated in [6]. In after effect, Meyer-Base et al. developed two different time scales on CNNs, which contains twin kinds of memory such as long-term and short-term. Two classes of state variables are considered in this paper, one is short-term memory, which depicts fast neuronal activity, another one is long-term memory, which depicts the slow unsupervised synaptic modifications. Additionally, delays are unavoidable in real life practical situations and the several results in dynamical behaviors of the delayed differential system have been extensively studied in [4,5,13,14,27,51,52]. So far, there are numerous kinds of delays taken in the existing literature, for example, constant delay, time-varying delay, neutral delay, leakage delay, distributed delay and so on [4,42,53]. Time-varying delays have occurred in almost every neural network model and it affects the behaviors of stability and synchronization process. The negative feedback term in state vector is called leakage delay(or forgetting delay), which leads to instability of the neural network system. At that same time, these two delays are not easy to handle, because they occur in some technical complications. As a consequence, it is mandatory and significant to analyze the time-varying delay for competitive neural network dynamical systems with leakage effect. Through the review of literature, several results have been available on the integer order dynamics of competitive neural networks, in stability and, synchronization with respect to finite or infinite time, see Refs. [1,9,10,25,45,46]. But it is regrettable that extremely a few results in synchronization of fractional order competitive neural networks is reported in [51].

Memristor, which means the memory resistor was originally introduced by Prof. Chua in 1971, reflect the linkage between flux and charge. Most of the research areas such as signal processing, image processing, associative memories, high-speed low-power processors, cell phone batteries takes a significant role in memristive neural networks. It is known that, the characteristic of memristor(memory property) is similar to the one exhibited by biological synapses, and it is used to easily understand the neuron process in the human brain. Generally, our human brain contains more than hundred billion of neurons, and each neuron contains more than two thousand synapses. In synaptic circuit implementation is the main role to build a brain like machine. In artificial neural network model, the resistor can be replaced by the memristor, then memristive neural networks can be formulated. Owing to this feature, it is a suitable candidate to intimate the functionalities of human brain [37]. Since the memristive neural networks are nonlinear system because these memristances heavily depend on the whole charge that has passed over the device from the past to the present. Otherwise, the resistor, capacitor, inductor is applied to the theory of linear-time invariant. Several studies have investigated the non-integer memristive neural networks with synchronization sense, see the reference therein [2,43,50]. Recently, many researchers have hammered to do research with fractional order synchronization in finite time about memristive neural networks by means of a linear feedback control and a succession of few results have been established in the literature, as shown by Mingwen et al. [23], Stamova et al. [24], Velmurugan et al. [33], and Xiao et al. [40]. In real world systems, it is sometimes not able to acquire the real-time inner states of the associated systems; however, the output state variables are measurable. Therefore, in order to fill this gap, it's miles interesting for us to design an output feedback controller to finite time synchronization of fractional order memristor based competitive neural networks with time-varying delay and leakage delay.

Motivated by the above aforementioned discussion, we consider a class of synchronization analysis in finite time for time-varying delayed fractional order memristive competitive neural networks with leakage delay effect. The main contribution of this manuscript lies in the following aspects:

- This is the first work that investigates the TMFCNNs with leakage delay, which is to generalize and improve those existing works.
- By endowing the concept of Filippov's solutions and differential inclusion theory, the master-slave synchronization system will be designed and transformed into differential inclusion systems with time-varying delay and leakage delay by means of finite-time output feedback control.
- In the meanwhile according to the value of p , the proposed models can be divided into two classes such as $0 < p < \frac{1}{2}$ and $\frac{1}{2} \leq p < 1$. Then the proposed fractional order differential equations are converted into the Volterra-integral equation, which simplifies our proof procedure of our main results.
- By utilizing some inequality techniques such as the Cauchy–Schwartz inequality, Holder's inequality and Gronwall's inequality and several new sufficient conditions are derived to ensure the finite synchronization of the considered neural network system.
- Furthermore, the upper bound of the settling time of the finite time synchronization are explicitly estimated and this work is less conservative compared to some existing works.

The remaining structure of this proposal is organized as follows: Basic definitions including fractional order properties, Lemma's and the assumption will be presented in Section 2. In Section 3, the synchronization in finite-time criteria of the considered model are derived with two different kinds of orders: $0 < p < \frac{1}{2}$ and $\frac{1}{2} \leq p < 1$. In Section 4, two numerical simulations are demonstrated by effectiveness of theoretical analysis. At last, the conclusion of the proposal is drawn in Section 5.

2. Preliminaries

Notations: Throughout this article, $\mathbb{R}$ is the space of real number, $\mathbb{R}^n$ denotes the n -D Euclidean space and $\mathbb{Z}^+$ stands for the set of nonnegative integers. For any $q = (q_1, \dots, q_n)^T \in \mathbb{R}^n$, we shall use the vector norm of q is $\|q\| = \sum_{j=1}^n |q_j|$, while we chosen the matrix norm is $\|V\| = \max_{1 \leq i \leq n} \sum_{j=1}^n |v_{ij}|$. $\text{co}[\cdot]$ represent the closure of the convex hull. $C^n([t_0, +\infty), \mathbb{R})$ stands for the space consisting of n -order continuous differentiable functions from $[t_0, +\infty)$ into $\mathbb{R}$.

In order to the subsequent analysis, we will display the following basic definition's, basic Lemma's and assumption.

Definition 2.1. [28]. The fractional order integral of order p for an integral function

$q : [t_0, +\infty) \rightarrow \mathbb{R}$ is defined as

$$\mathcal{I}^p q(t) = \frac{1}{\Gamma(p)} \int_{t_0}^t (t - \omega)^{p-1} q(\omega) d\omega,$$

where $p > 0$ and $\Gamma(\cdot)$ is the Gamma function which is defined by

$$\Gamma(p) = \int_0^{+\infty} e^{-t} t^{p-1} dt, \quad (\text{Re}(p) > 0),$$

where $\text{Re}(p)$ is the real part of p .

Definition 2.2. [16]. The Caputo fractional-order derivative of order p for a function $q(t) \in C^n([t_0, +\infty))$ is defined as

$$D^p q(t) = \frac{1}{\Gamma(n-p)} \int_{t_0}^t \frac{q^{(n)}(\omega)}{(t - \omega)^{p-n+1}} d\omega,$$

where $t \geq t_0$ and $n \in \mathbb{Z}^+$ with $p \in (n-1, n)$.

Particularly, when $0 < p < 1$,

$$D^p q(t) = \frac{1}{\Gamma(1-p)} \int_{t_0}^t \frac{q'(\omega)}{(t - \omega)^p} d\omega.$$

In addition, the following properties about Caputo fractional-order derivative are given.

Property 1. [49] $D^p \gamma = 0$ holds, where γ is any constant.

Property 2. [49] For any constant ζ and ς , the linearity of Caputo fractional order derivative gives

$$\mathcal{D}^p (\zeta f(t) + \varsigma g(t)) = \zeta \mathcal{D}^p f(t) + \varsigma \mathcal{D}^p g(t).$$

Lemma 2.3. [22] If $q(t) \in C^n[0, +\infty)$ and $n-1 < p < n$, then

$$\mathcal{D}^{-p} \mathcal{D}^p q(t) = q(t) - \sum_{l=1}^{n-1} \frac{t^l}{l!} q^{(l)}(0), \quad p \geq 0,$$

Lemma 2.4. [7] (Gronwall–Bellman inequality) Suppose the function $h(t)$ satisfying $h(t) \leq u(t) + \int_0^t v(\omega) h(\omega) d\omega$ on the interval $[0, T]$, where $h(t)$, $u(t)$, $v(t) \in C[0, T]$, $T \leq +\infty$ and $v(t) > 0$, then

$$h(t) \leq u(t) + \int_0^t v(\omega) u(\omega) \exp \left[\int_0^t v(k) dk \right] d\omega, \quad t \in [0, T], \quad T \leq +\infty,$$

if, moreover $u(t)$ is nondecreasing, then

$$h(t) \leq u(t) \exp \left(\int_0^t v(\omega) d\omega \right), \quad t \in [0, T].$$

We will consider the following time-varying delay fractional order memristive competitive neural networks (TMFCNNs) with leakage delay effect:

$$\begin{aligned} \mathcal{D}^p q_i(t) &= -a_i q_i(t - \lambda) + \sum_{j=1}^n b_{ij}(q_i(t)) h_j(q_j(t)) + \sum_{j=1}^n c_{ij}(q_i(t)) h_j(q_j(t - \mu(t))) + d_i \sum_{k=1}^m s_{ik}(t) \rho_k, \\ \mathcal{D}^p s_{ik}(t) &= -u_i s_{ik}(t - \delta) + \rho_k v_i h_i(q_i(t)), \quad i = 1, 2, \dots, n, \quad k = 1, 2, \dots, m, \end{aligned} \quad (1)$$

where $\mathcal{D}^p$ denotes Caputo fractional derivative of order p with $0 < p < 1$; $q_i(t)$ is the corresponding state variable of the number of n neurons at time t ; $a_i > 0$ and $u_i > 0$ represent the self feedback connection weight matrices, $b_{ij}(q_i(t))$ and $c_{ij}(q_i(t))$ are, respectively, memristive connection weights, which is associated between i th neurons and j th neurons; $\lambda > 0$ and $\delta > 0$ denotes the leakage delays; $h_j(q_j(t))$ and $h_j(q_j(t - \mu(t)))$ are referred the bounded neuron output activation, where the time varying delay $\mu(t)$ is bounded and differentiable on $0 \leq t \leq T$ and it satisfies $0 \leq \mu_2 \leq \mu(t) \leq \mu_1$, $0 \leq \mu'(t) < 1$, $\mu_3 = \sup_{0 \leq \omega \leq T} \frac{1}{|1 - \mu'(\omega)|}$; $s_j(t) = (s_{j1}(t), \dots, s_{jn}(t))^T$ stands for the synaptic efficiency; d_k denotes the strengths of the external stimulus, while ρ_k means the constant external stimulus; $v_i > 0$ represent disposable scaling constant; Based on the memristive model, the connection weights satisfy the following conditions:

$$b_{ij}(q_i(t)) = \begin{cases} b_{ij}^+, & |q_i(t)| > T_i, \\ b_{ij}^-, & |q_i(t)| \leq T_i, \end{cases} \quad c_{ij}(q_i(t)) = \begin{cases} c_{ij}^+, & |q_i(t)| > T_i, \\ c_{ij}^-, & |q_i(t)| \leq T_i, \end{cases}$$

where $T_i > 0$ is the switching jumps and b_{ij}^+ , b_{ij}^- , c_{ij}^+ and c_{ij}^- are all scalars, for $i, j = 1, 2, \dots, n$.

After setting $r_i(t) = \sum_{k=1}^m s_{ik}(t)\rho_k = s_i^T(t)\rho$, where $s_i(t) = (s_{i1}, \dots, s_{im})^T$, $\rho = (\rho_1, \dots, \rho_m)^T$. Without loss of generality, ρ is assumed to be normalized with unit magnitude $|\rho|^2 = 1$, where ρ is the input stimulus. Then the model (1) can be simplified to the following state-space form such as

$$\begin{aligned} \mathcal{D}^p q_i(t) &= -a_i q_i(t - \lambda) + \sum_{j=1}^n b_{ij}(q_i(t)) h_j(q_j(t)) + \sum_{j=1}^n c_{ij}(q_i(t)) h_j(q_j(t - \mu(t))) + d_i r_i(t), \\ \mathcal{D}^p r_i(t) &= -u_i r_i(t - \delta) + v_i h_i(q_i(t)), \quad i = 1, 2, \dots, n. \end{aligned} \quad (2)$$

The initial conditions of the model (2) is described as:

$$q_i(\omega) = \phi_i(\omega), \quad r_i(\omega) = \psi_i(\omega), \quad \omega \in [-\varpi, 0], \quad i = 1, 2, \dots, n,$$

where $\phi_i(\omega), \psi_i(\omega) \in \mathcal{C}([-\varpi, 0], \mathbb{R}^n)$, $\varpi = \max\{\lambda, \delta, \mu_1\}$. And norm of an element in $\mathcal{C}([-\varpi, 0], \mathbb{R}^n)$ is given by

$$\|\phi(\omega)\| = \sum_{i=1}^n \sup_{\omega \in [-\varpi, 0]} \{e^{-\omega} |\phi_i(\omega)|\}, \quad \|\psi(\omega)\| = \sum_{i=1}^n \sup_{\omega \in [-\varpi, 0]} \{e^{-\omega} |\psi_i(\omega)|\}.$$

By means of set valued maps and differential inclusions theory, please see the reference [11] to system (2), it can be expressed in the following fractional order differential inclusion forms:

$$\begin{aligned} \mathcal{D}^p q_i(t) &\in -a_i q_i(t - \lambda) + \sum_{j=1}^n \text{co}[b_{ij}(q_i(t))] h_j(q_j(t)) + \sum_{j=1}^n \text{co}[c_{ij}(q_i(t))] h_j(q_j(t - \mu(t))) + d_i r_i(t), \\ \mathcal{D}^p r_i(t) &= -u_i r_i(t - \delta) + v_i h_i(q_i(t)), \quad i = 1, 2, \dots, n, \end{aligned}$$

where

$$\text{co}[b_{ij}(q_i(t))] = \begin{cases} b_{ij}^+, & |q_i(t)| > T_i, \\ [b_{ij}^+, b_{ij}^-], & |q_i(t)| = T_i, \\ b_{ij}^-, & |q_i(t)| < T_i, \end{cases} \quad \text{co}[c_{ij}(q_i(t))] = \begin{cases} c_{ij}^+, & |q_i(t)| > T_i, \\ [c_{ij}^+, c_{ij}^-], & |q_i(t)| = T_i, \\ c_{ij}^-, & |q_i(t)| < T_i, \end{cases}$$

or equivalently, there exist a measurable function $\hat{b}_{ij}(t) \in \text{co}[b_{ij}(q_i(t))]$, $\hat{c}_{ij}(t) \in \text{co}[c_{ij}(q_i(t))]$ such that

$$\begin{aligned} \mathcal{D}^p q_i(t) &= -a_i q_i(t - \lambda) + \sum_{j=1}^n \hat{b}_{ij}(t) h_j(q_j(t)) + \sum_{j=1}^n \hat{c}_{ij}(t) h_j(q_j(t - \mu(t))) + d_i r_i(t), \\ \mathcal{D}^p r_i(t) &= -u_i r_i(t - \delta) + v_i h_i(q_i(t)), \quad t \in [0, T], \quad i = 1, 2, \dots, n. \end{aligned} \quad (3)$$

In this paper, we consider the master-slave synchronization, the corresponding slave synchronization of (2) is given as the following form:

$$\begin{aligned} \mathcal{D}^p \hat{q}_i(t) &= -a_i \hat{q}_i(t - \lambda) + \sum_{j=1}^n b_{ij}(\hat{q}_i(t)) h_j(\hat{q}_j(t)) + \sum_{j=1}^n c_{ij}(\hat{q}_i(t)) h_j(\hat{q}_j(t - \mu(t))) + d_i \hat{r}_i(t) + \varsigma_i(t), \\ \mathcal{D}^p \hat{r}_i(t) &= -u_i \hat{r}_i(t - \delta) + v_i h_i(\hat{q}_i(t)) + \varrho_i(t), \quad i = 1, 2, \dots, n. \end{aligned} \quad (4)$$

where $\varsigma_i(t)$ and $\varrho_i(t)$ are suitable controller, which is designed to be for achieving synchronization goal of the master-slave system, while the memristive connection weights $b_{ij}(\hat{q}_i(t))$ and $c_{ij}(\hat{q}_i(t))$ can be given by:

$$b_{ij}(\hat{q}_i(t)) = \begin{cases} b_{ij}^+, & |\hat{q}_i(t)| > T_i, \\ [b_{ij}^+, b_{ij}^-], & |\hat{q}_i(t)| = T_i, \\ b_{ij}^-, & |\hat{q}_i(t)| < T_i, \end{cases} \quad c_{ij}(\hat{q}_i(t)) = \begin{cases} c_{ij}^+, & |\hat{q}_i(t)| > T_i, \\ [c_{ij}^+, c_{ij}^-], & |\hat{q}_i(t)| = T_i, \\ c_{ij}^-, & |\hat{q}_i(t)| < T_i. \end{cases}$$

Similarly, by applying differential inclusion theory and set valued map analysis to slave system (4), it can be expressed in the following fractional order differential inclusion forms:

$$\begin{aligned} \mathcal{D}^p \hat{q}_i(t) &\in -a_i \hat{q}_i(t - \lambda) + \sum_{j=1}^n \text{co}[b_{ij}(\hat{q}_i(t))] h_j(\hat{q}_j(t)) + \sum_{j=1}^n \text{co}[c_{ij}(\hat{q}_i(t))] h_j(\hat{q}_j(t - \mu(t))) + d_i \hat{r}_i(t) + \varsigma_i(t), \\ \mathcal{D}^p \hat{r}_i(t) &= -u_i \hat{r}_i(t - \delta) + v_i h_i(\hat{q}_i(t)) + \varrho_i(t), \quad i = 1, 2, \dots, n, \end{aligned}$$

where

$$\text{co}[b_{ij}(\hat{q}_i(t))] = \begin{cases} b_{ij}^+, & |\hat{q}_i(t)| > T_j, \\ [b_{ij}^+, b_{ij}^-], & |\hat{q}_i(t)| = T_j, \\ b_{ij}^-, & |\hat{q}_i(t)| < T_j, \end{cases} \quad \text{co}[c_{ij}(\hat{q}_i(t))] = \begin{cases} c_{ij}^+, & |\hat{q}_i(t)| > T_j, \\ [c_{ij}^+, c_{ij}^-], & |\hat{q}_i(t)| = T_j, \\ c_{ij}^-, & |\hat{q}_i(t)| < T_j, \end{cases}$$

or equivalently, there exist a measurable function $\hat{b}_{ij}(t) \in \text{co}[b_{ij}(\hat{q}_i(t))]$, $\hat{c}_{ij}(t) \in \text{co}[c_{ij}(\hat{q}_i(t))]$ such that

$$\begin{aligned} \mathcal{D}^p \hat{q}_i(t) &= -a_i \hat{q}_i(t - \lambda) + \sum_{j=1}^n \hat{b}_{ij}(t) h_j(\hat{q}_j(t)) + \sum_{j=1}^n \hat{c}_{ij}(t) h_j(\hat{q}_j(t - \mu(t))) + d_i \hat{r}_i(t) + \varsigma_i(t), \\ \mathcal{D}^p \hat{r}_i(t) &= -u_i \hat{r}_i(t - \delta) + v_i h_i(\hat{q}_i(t)) + \varrho_i(t), \quad t \in [0, T], \quad i = 1, 2, \dots, n. \end{aligned} \quad (5)$$

The initial conditions of the slave system (4) is described by:

$$\hat{q}_i(\omega) = \hat{\phi}_i(\omega), \quad \hat{r}_i(\omega) = \hat{\psi}_i(\omega), \quad \omega \in [-\varpi, 0], \quad i = 1, 2, \dots, n,$$

where $\hat{\phi}_i(\omega), \hat{\psi}_i(\omega) \in \mathcal{C}([-\varpi, 0], \mathbb{R}^n)$. And norm of an element in $\mathcal{C}([-\varpi, 0], \mathbb{R}^n)$ is given by

$$\|\hat{\phi}(\omega)\| = \sum_{i=1}^n \sup_{\omega \in [-\varpi, 0]} \{e^{-\omega} |\hat{\phi}_i(\omega)|\}, \quad \|\hat{\psi}(\omega)\| = \sum_{i=1}^n \sup_{\omega \in [-\varpi, 0]} \{e^{-\omega} |\hat{\psi}_i(\omega)|\}.$$

Next, we design a finite-time output feedback controller as follows:

$$\zeta_i(t) = \mathcal{G}_i[\hat{h}_i(\hat{q}_i(t)) - h_i(q_i(t))], \quad \varrho_i(t) = \mathcal{H}_i[\hat{h}_i(\hat{r}_i(t)) - h_i(r_i(t))], \quad (6)$$

where each $\mathcal{G}_i > 0, \mathcal{H}_i > 0$ are control gains.

Definition 2.5. The master system (2) is said to be synchronized in finite time with the slave (4) based on the controller (6), if there exist positive constant $\{\epsilon, \varepsilon, T\}, 0 < \epsilon < \varepsilon$ such that $[\|\hat{\phi}_i - \phi_i\| + \|\hat{\psi}_i - \psi_i\|] < \epsilon$ implies $[\|\hat{q}_i(t) - q_i(t)\| + \|\hat{r}_i(t) - r_i(t)\|] < \varepsilon, t \in \mathcal{M} = [0, T]$.

To ensure the existence and uniqueness of solutions of system (1) or (3), the following assumption is given.

Assumption[A₁]. For any $q_j, \hat{q}_j \in \mathbb{R}$, the neuron nonlinear activation function $h_j, h_j(\pm T_j) = 0$ satisfies the Lipschitz conditions if there exist a constant $\mathcal{R}_j > 0$ such that

$$|h_j(\hat{q}_j) - h_j(q_j)| \leq \mathcal{R}_j |\hat{q}_j - q_j|,$$

for $j = 1, 2, \dots, n$, or $\forall p, \hat{p} \in \mathbb{R}^n$,

$$\|h(\hat{p}) - h(p)\| \leq \|\mathcal{R}\| \|\hat{p} - p\|,$$

where $\mathcal{R} = \text{diag}\{\mathcal{R}_1, \dots, \mathcal{R}_n\}$.

Lemma 2.6. [4]. Under assumption [A₁], if $h_j(\pm T_j) = 0$, then

$$|\text{co}[b_{ij}(\hat{q}_i(t))]h_j(\hat{q}_j) - \text{co}[b_{ij}(q_i(t))]h_j(q_j)| \leq \tilde{b}_{ij}\mathcal{R}_j|\hat{q}_j - q_j|,$$

that is, for any $\hat{b}_{ij}(t) \in \text{co}[b_{ij}(\hat{q}_i(t))], \hat{b}_{ij}(t) \in \text{co}[b_{ij}(q_i(t))]$, one has

$$|\hat{b}_{ij}(t)h_j(\hat{q}_j) - \hat{b}_{ij}(t)h_j(q_j)| \leq \tilde{b}_{ij}\mathcal{R}_j|\hat{q}_j - q_j|,$$

for $1 \leq i, j \leq n$, where $\tilde{b}_{ij} = \max\{|b_{ij}^+|, |b_{ij}^-|\}$.

Lemma 2.7. [17]. Let $\eta_1, \dots, \eta_m \geq 0, 0 < \tau \leq 1, \nu > 1$ then we have

$$\left(\sum_{j=1}^m \eta_j\right)^\tau \leq \sum_{j=1}^m \eta_j^\tau, \quad \left(\sum_{j=1}^m \eta_j\right)^\nu \leq m^{\nu-1} \sum_{j=1}^m \eta_j^\nu.$$

Lemma 2.8. [26]. Assume that $\theta, \vartheta > 1$, and $\frac{1}{\theta} + \frac{1}{\vartheta} = 1$, if $|f_1(\cdot)|^\theta, |f_2(\cdot)|^\vartheta \in \mathcal{L}^1(E)$, then $f_1(\cdot)f_2(\cdot) \in \mathcal{L}^1(E)$ and

$$\int_E |f_1(q)f_2(q)|dq \leq \left(\int_E |f_1(q)|^\theta dq\right)^{\frac{1}{\theta}} \left(\int_E |f_2(q)|^\vartheta dq\right)^{\frac{1}{\vartheta}},$$

where $\mathcal{L}^1(E)$ be the Banach space of all Lebesgue measurable function $f_1 : E \rightarrow \mathbb{R}$ with $\int_E |f_1(q)|dq < +\infty$. It is called Hölder inequality. When $\theta = \vartheta = 2$, this inequality reduces to the following form:

$$\int_E |f_1(q)f_2(q)|dq \leq \left(\int_E |f_1(q)|^2 dq\right)^{\frac{1}{2}} \left(\int_E |f_2(q)|^2 dq\right)^{\frac{1}{2}}.$$

It is called Cauchy–Schwartz inequality.

We define the synchronization error between (3) and (5) can be described as: $\alpha_i(t) = \hat{q}_i(t) - q_i(t)$ and $\beta_i(t) = \hat{r}_i(t) - r_i(t)$:

$$\begin{aligned} \mathcal{D}^p \alpha_i(t) &= -a_i \alpha_i(t - \lambda) + \sum_{j=1}^n [\hat{b}_{ij}(t)h_j(\hat{q}_j(t)) - \hat{b}_{ij}(t)h_j(q_j(t))] + \sum_{j=1}^n [\hat{c}_{ij}(t)h_j(\hat{q}_j((t - \mu(t)))) \\ &\quad - \hat{c}_{ij}((t - \mu(t)))h_j(q_j(t))] + d_i \beta_i(t) + \mathcal{G}_i[h_i(\hat{q}_i(t)) - h_i(q_i(t))], \\ \mathcal{D}^p \beta_i(t) &= -u_i \beta_i(t - \delta) + v_i[h_i(\hat{q}_i(t)) - h_i(q_i(t))] + \mathcal{H}_i[h_i(\hat{r}_i(t)) - h_i(r_i(t))]. \end{aligned} \quad (7)$$

The initial conditions of the error system (7) is given by:

$$\alpha_i(\omega) = \hat{\phi}_i(\omega) - \phi_i(\omega) = \xi_i(\omega), \quad \beta_i(\omega) = \hat{\psi}_i(\omega) - \psi_i(\omega) = \zeta_i(\omega), \quad \omega \in [-\varpi, 0], \quad i = 1, 2, \dots, n.$$

For sake simplicity, we denote the following notations are used in later analysis:

$$\alpha(t) = (\alpha_1(t), \dots, \alpha_n(t))^T, \quad \beta(t) = (\beta_1(t), \dots, \beta_n(t))^T, \quad A = \text{diag}\{a_1, \dots, a_n\}, \quad D = \text{diag}\{d_1, \dots, d_n\}, \quad U = \text{diag}\{u_1, \dots, u_n\}, \quad V = \text{diag}\{v_1, \dots, v_n\}, \quad \hat{B} = (\hat{b}_{ij})_{n \times n}, \quad \tilde{B} = (\tilde{b}_{ij})_{n \times n}, \quad \hat{C} = (\hat{c}_{ij})_{n \times n}, \quad \tilde{C} = (\tilde{c}_{ij})_{n \times n}, \quad \tilde{C} = (\tilde{c}_{ij})_{n \times n}, \quad \mathcal{H} = \text{diag}\{\mathcal{H}_1, \dots, \mathcal{H}_n\}, \quad \mathcal{G} = \text{diag}\{\mathcal{G}_1, \dots, \mathcal{G}_n\}.$$

Now, the transformation error system (7) in the vector form is given by:

$$\begin{aligned} \mathcal{D}^p \alpha(t) &= -A\alpha(t - \lambda) + \hat{B}(t)h(\hat{q}(t)) - \tilde{B}(t)h(q(t)) + \hat{C}(t)h(\hat{q}((t - \mu(t)))) - \tilde{C}(t)h(q(t - \mu(t))) \\ &\quad + D\beta(t) + \mathcal{G}[h(\hat{q}(t)) - h(q(t))], \end{aligned}$$

$$D^p \beta(t) = -U\beta(t - \delta) + V[h(\hat{q}(t)) - h(q(t))] + \mathcal{H}[h(\hat{r}(t)) - h(r(t))]. \quad (8)$$

By using Lemma 2.3 and the above error system (8), we have

$$\begin{aligned} \alpha(t) &= \alpha(0) + \frac{1}{\Gamma(p)} \int_0^t (t - \kappa)^{p-1} \left[-A\alpha(\kappa - \lambda) + [\hat{B}(\kappa)h(\hat{q}(\kappa)) - \hat{B}(\kappa)h(q(\kappa))] \right. \\ &\quad \left. + [\hat{C}(\kappa)h(\hat{q}(\kappa - \mu(\kappa))) - \hat{C}(\kappa)h(q(\kappa - \mu(\kappa)))] + D\beta(\kappa) + \mathcal{G}[h(\hat{q}(\kappa)) - h(q(\kappa))] \right] d\kappa \\ \beta(t) &= \beta(0) + \frac{1}{\Gamma(p)} \int_0^t (t - \kappa)^{p-1} \left[-U\beta(\kappa - \delta) + V[h(\hat{q}(\kappa)) - h(q(\kappa))] + \mathcal{H}[h(\hat{r}(\kappa)) - h(r(\kappa))] \right] d\kappa \end{aligned} \quad (9)$$

Applying appropriate norm on both sides of system (9), we obtain

$$\begin{aligned} \|\alpha(t)\| &\leq \|\alpha(0)\| + \frac{1}{\Gamma(p)} \int_0^t (t - \kappa)^{p-1} \left[\|A\| \|\alpha(\kappa - \lambda)\| + (\|\tilde{B}\| + \|\mathcal{G}\|) \|\mathcal{R}\| \|\alpha(\kappa)\| \right. \\ &\quad \left. + \|\tilde{C}\| \|\mathcal{R}\| \|\alpha(\kappa - \mu(\kappa))\| + \|D\| \|\beta(\kappa)\| \right] d\kappa \\ \|\beta(t)\| &\leq \|\beta(0)\| + \frac{1}{\Gamma(p)} \int_0^t (t - \kappa)^{p-1} \left[\|U\| \|\beta(\kappa - \delta)\| + \|V\| \|\mathcal{R}\| \|\alpha(\kappa)\| + \|\mathcal{H}\| \|\mathcal{R}\| \|\beta(\kappa)\| \right] d\kappa \end{aligned} \quad (10)$$

where assumption $[A_1]$ and Lemma 2.6 has been used.

Remark 2.9. Little while back, [43] investigated Quasi-uniform synchronization of fractional-order memristor-based neural networks with delay. However, they focused on quasi-uniform synchronization of neural networks through an output feedback control and not finite time synchronization of TMFCNNs with leakage delay. Moreover, it is useful for us to prove our required synchronization criterion from the results obtained in the above said reference [43].

3. Synchronization results

In this section, we will demonstrate the two sufficient conditions for the the synchronization in finite time of TFMCCNs with leakage delay with two different order such as $0 < p < \frac{1}{2}$ and $\frac{1}{2} \leq p < 1$.

Theorem 3.1. When $0 < p < \frac{1}{2}$, if assumption $[A_1]$ holds, and

$$\sqrt[\eta]{\frac{\eta 6^{\eta-1} + [6^{\eta-1} \mathcal{L} + \mathcal{J}(\eta + \mathcal{L})] e^{(\eta+\mathcal{L})t}}{[\eta + \mathcal{L}]}} < \frac{\varepsilon}{\epsilon}, \quad t \in \mathcal{M}, \quad (11)$$

then the synchronization of error dynamical system (8) of the master system (3) and the slave system (5) can reach synchronization goal in finite time under the controller (6), where

$$\begin{aligned} \mathcal{J} &= \max \left\{ 6^{\eta-1} \left[\Psi_A \left(\frac{1 - e^{-\eta\lambda}}{\eta} \right)^{\frac{1}{\eta}} + \Psi_C \left(\mu_3 e^{-\eta\mu_2} \left[\frac{e^{\eta\mu_1} - 1}{\eta} \right]^{\frac{1}{\eta}} \right)^{\frac{1}{\eta}} \right]^{\eta}, 6^{\eta-1} \left[\Psi_U \left(\frac{1 - e^{-\eta\delta}}{\eta} \right)^{\frac{1}{\eta}} \right]^{\eta} \right\}, \\ \mathcal{L} &= \max \left\{ 6^{\eta-1} \left[\Psi_A e^{-\lambda} + \Psi_B + \Psi_C \left(\mu_3 e^{-\eta\mu_2} \right)^{\frac{1}{\eta}} + \Psi_V \right]^{\eta}, 6^{\eta-1} \left[\Psi_D + \Psi_U e^{-\delta} + \Psi_H \right]^{\eta} \right\}, \end{aligned}$$

$$\begin{aligned} \Psi_A &= \frac{\|A\|}{\Gamma(p)} \left[\frac{\Gamma(\theta(p-1)+1)}{\theta^{\theta(p-1)+1}} \right]^{\frac{1}{\theta}}, \quad \Psi_B = \frac{(\|\tilde{B}\| + \|\mathcal{G}\|) \|\mathcal{R}\|}{\Gamma(p)} \left[\frac{\Gamma(\theta(p-1)+1)}{\theta^{\theta(p-1)+1}} \right]^{\frac{1}{\theta}}, \quad \theta = 1 + p, \quad \eta = 1 + \frac{1}{p}, \\ \Psi_C &= \frac{\|\tilde{C}\| \|\mathcal{R}\|}{\Gamma(p)} \left[\frac{\Gamma(\theta(p-1)+1)}{\theta^{\theta(p-1)+1}} \right]^{\frac{1}{\theta}}, \quad \Psi_D = \frac{\|D\|}{\Gamma(p)} \left[\frac{\Gamma(\theta(p-1)+1)}{\theta^{\theta(p-1)+1}} \right]^{\frac{1}{\theta}}, \quad \Psi_U = \frac{\|U\|}{\Gamma(p)} \left[\frac{\Gamma(\theta(p-1)+1)}{\theta^{\theta(p-1)+1}} \right]^{\frac{1}{\theta}}, \\ \Psi_V &= \frac{\|V\| \|\mathcal{R}\|}{\Gamma(p)} \left[\frac{\Gamma(\theta(p-1)+1)}{\theta^{\theta(p-1)+1}} \right]^{\frac{1}{\theta}}, \quad \Psi_H = \frac{\|\mathcal{H}\| \|\mathcal{R}\|}{\Gamma(p)} \left[\frac{\Gamma(\theta(p-1)+1)}{\theta^{\theta(p-1)+1}} \right]^{\frac{1}{\theta}}. \end{aligned}$$

Proof. For presentation convenience, we take $\Phi_A = \|A\|$, $\Phi_B = (\|\tilde{B}\| + \|\mathcal{G}\|) \|\mathcal{R}\|$, $\Phi_C = \|\tilde{C}\| \|\mathcal{R}\|$, $\Phi_D = \|D\|$, $\Phi_U = \|U\|$, $\Phi_V = \|V\| \|\mathcal{R}\|$, $\Phi_H = \|\mathcal{H}\| \|\mathcal{R}\|$, then by follows from Lemma 2.8 and Eq. (10), one has obtain that

$$\begin{aligned} \|\alpha(t)\| &\leq \|\alpha(0)\| + \frac{1}{\Gamma(p)} \int_0^t (t - \kappa)^{p-1} \left[\Phi_A \|\alpha(\kappa - \lambda)\| + \Phi_B \|\alpha(\kappa)\| + \Phi_C \|\alpha(\kappa - \mu(\kappa))\| + \Phi_D \|\beta(\kappa)\| \right] d\kappa \\ &= \|\alpha(0)\| + \frac{\Phi_A}{\Gamma(p)} \int_0^t (t - \kappa)^{p-1} e^{\kappa-\lambda} e^{-(\kappa-\lambda)} \|\alpha(\kappa - \lambda)\| d\kappa + \frac{\Phi_B}{\Gamma(p)} \int_0^t (t - \kappa)^{p-1} e^{\kappa} e^{-\kappa} \|\alpha(\kappa)\| d\kappa \\ &\quad + \frac{\Phi_C}{\Gamma(p)} \int_0^t (t - \kappa)^{p-1} e^{\kappa} e^{-\kappa} \|\alpha(\kappa - \mu(\kappa))\| d\kappa + \frac{\Phi_D}{\Gamma(p)} \int_0^t (t - \kappa)^{p-1} e^{\kappa} e^{-\kappa} \|\beta(\kappa)\| d\kappa \\ &\leq \|\xi\| + \frac{\Phi_A}{\Gamma(p)} \left[\int_0^t (t - \kappa)^{\theta(p-1)} e^{\theta(\kappa-\lambda)} d\kappa \right]^{\frac{1}{\theta}} \left[\int_0^t e^{-\eta(\kappa-\lambda)} \|\alpha(\kappa - \lambda)\|^{\eta} d\kappa \right]^{\frac{1}{\eta}} \end{aligned}$$

$$\begin{aligned}
& + \frac{\Phi_B}{\Gamma(p)} \left[\int_0^t (t-\kappa)^{\theta(p-1)} e^{\theta\kappa} d\kappa \right]^{\frac{1}{\theta}} \left[\int_0^t e^{-\eta\kappa} \|\alpha(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} + \frac{\Phi_C}{\Gamma(p)} \left[\int_0^t (t-\kappa)^{\theta(p-1)} e^{\theta\kappa} d\kappa \right]^{\frac{1}{\theta}} \\
& \times \left[\int_0^t e^{-\eta\kappa} \|\alpha(\kappa - \mu(\kappa))\|^\eta d\kappa \right]^{\frac{1}{\eta}} + \frac{\Phi_D}{\Gamma(p)} \left[\int_0^t (t-\kappa)^{\theta(p-1)} e^{\theta\kappa} d\kappa \right]^{\frac{1}{\theta}} \left[\int_0^t e^{-\eta\kappa} \|\beta(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}}
\end{aligned} \quad (12)$$

It is pointed that

$$\begin{aligned}
\int_0^t (t-\kappa)^{\theta(p-1)} e^{\theta\kappa} d\kappa &= \int_0^t \omega^{\theta(p-1)} e^{\theta(1-\omega)} d\omega = e^{\theta t} \int_0^t \omega^{\theta p - \theta} e^{-\omega\theta} d\omega \\
&= \frac{e^{\theta t}}{\theta^{\theta p - \theta + 1}} \int_0^{\theta t} \sigma^{\theta(p-1)} e^{-\sigma} d\sigma \leq \frac{e^{\theta t}}{\theta^{\theta p - \theta + 1}} \Gamma(\theta(p-1) + 1).
\end{aligned} \quad (13)$$

Substituting the inequality (13) into (12), one has

$$\begin{aligned}
\|\alpha(t)\| &\leq \|\xi\| + \frac{\Phi_A}{\Gamma(p)} \left[\frac{e^{\theta t} \Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \left[e^{-\eta\lambda} \int_{-\lambda}^{t-\lambda} e^{-\eta\kappa} \|\alpha(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} \\
&+ \frac{\Phi_B}{\Gamma(p)} \left[\frac{e^{\theta t} \Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \left[\int_0^t e^{-\eta\kappa} \|\alpha(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} \\
&+ \frac{\Phi_C}{\Gamma(p)} \left[\frac{e^{\theta t} \Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \left[\int_0^t e^{-\eta(\kappa - \mu(\kappa))} e^{-\eta\mu(\kappa)} \frac{\|\alpha(\kappa - \mu(\kappa))\|^\eta}{|1 - \mu'(\kappa)|} d(\kappa - \mu(\kappa)) \right]^{\frac{1}{\eta}} \\
&+ \frac{\Phi_D}{\Gamma(p)} \left[\frac{e^{\theta t} \Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \left[\int_0^t e^{-\eta\kappa} \|\beta(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} \\
&\leq \|\xi\| + \frac{\Phi_A}{\Gamma(p)} \left[\frac{e^{\theta t} \Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \left[e^{-\eta\lambda} \int_{-\lambda}^0 e^{-\eta\kappa} \|\xi\|^\eta d\kappa + e^{-\eta\lambda} \int_0^{t-\lambda} e^{-\eta\kappa} \|\alpha(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} \\
&+ \frac{\Phi_B}{\Gamma(p)} \left[\frac{e^{\theta t} \Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \left[\int_0^t e^{-\eta\kappa} \|\alpha(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} + \frac{\Phi_C}{\Gamma(p)} \left[\frac{e^{\theta t} \Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \\
&\times \left[\int_{-\mu_1}^0 \mu_3 e^{-\eta\kappa} e^{-\eta\mu_2} \|\xi\|^\eta d\kappa + \int_0^t \mu_3 e^{-\eta\kappa} e^{-\eta\mu_2} \|\alpha(k)\|^\eta d\kappa \right]^{\frac{1}{\eta}} \\
&+ \frac{\Phi_D}{\Gamma(p)} \left[\frac{e^{\theta t} \Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \left[\int_0^t e^{-\eta\kappa} \|\beta(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} \\
&\leq \|\xi\| + \frac{\Phi_A}{\Gamma(p)} \left[\frac{e^{\theta t} \Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \left[e^{-\eta\lambda} \left[\frac{e^{\eta\lambda} - 1}{\eta} \right] \|\xi\|^\eta + e^{-\eta\lambda} \int_0^t e^{-\eta\kappa} \|\alpha(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} \\
&+ \frac{\Phi_B}{\Gamma(p)} \left[\frac{e^{\theta t} \Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \left[\int_0^t e^{-\eta\kappa} \|\alpha(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} + \frac{\Phi_C}{\Gamma(p)} \left[\frac{e^{\theta t} \Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \\
&\times \left[\mu_3 e^{-\eta\mu_2} \left(\frac{e^{\eta\mu_1} - 1}{\eta} \right) \|\xi\|^\eta + \int_0^t \mu_3 e^{-\eta\kappa} e^{-\eta\mu_2} \|\alpha(k)\|^\eta d\kappa \right]^{\frac{1}{\eta}} \\
&+ \frac{\Phi_D}{\Gamma(p)} \left[\frac{e^{\theta t} \Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \left[\int_0^t e^{-\eta\kappa} \|\beta(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} \\
&\leq \|\xi\| + \frac{\Phi_A e^t}{\Gamma(p)} \left[\frac{\Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \left[\frac{1 - e^{-\eta\lambda}}{\eta} \right]^{\frac{1}{\eta}} \|\xi\| + \frac{\Phi_A e^t}{\Gamma(p)} \left[\frac{\Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \\
&\times e^{-\lambda} \left[\int_0^t e^{-\eta\kappa} \|\alpha(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} + \frac{\Phi_B e^t}{\Gamma(p)} \left[\frac{\Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \left[\int_0^t e^{-\eta\kappa} \|\alpha(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} \\
&+ \frac{\Phi_C e^t}{\Gamma(p)} \left[\frac{\Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \left[\mu_3 e^{-\eta\mu_2} \left(\frac{e^{\eta\mu_1} - 1}{\eta} \right) \right]^{\frac{1}{\eta}} \|\xi\| + \frac{\Phi_C e^t}{\Gamma(p)} \left[\frac{\Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \\
&\times \left[\int_0^t \mu_3 e^{-\eta\kappa} e^{-\eta\mu_2} \|\alpha(k)\|^\eta d\kappa \right]^{\frac{1}{\eta}} + \frac{\Phi_D e^t}{\Gamma(p)} \left[\frac{\Gamma(\theta(p-1) + 1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}} \left[\int_0^t e^{-\eta\kappa} \|\beta(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}}.
\end{aligned}$$

That is,

$$\begin{aligned} \|\alpha(t)\| \leq & \|\xi\| + \left[e^t \Psi_A \left[\frac{1 - e^{-\eta\lambda}}{\eta} \right]^{\frac{1}{\eta}} + e^t \Psi_C \left[\mu_3 e^{-\eta\mu_2} \left(\frac{e^{\eta\mu_1} - 1}{\eta} \right) \right]^{\frac{1}{\eta}} \right] \|\xi\| + e^t \left[\Psi_A e^{-\lambda} + \Psi_B \right. \\ & \left. + \Psi_C [\mu_3 e^{-\eta\mu_2}]^{\frac{1}{\eta}} \right] \left[\int_0^t e^{-\eta\kappa} \|\alpha(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} + e^t \Psi_D \left[\int_0^t e^{-\eta\kappa} \|\beta(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}}. \end{aligned} \quad (14)$$

Similarly, by the same procedures as the above inequality, we obtains

$$\|\beta(t)\| \leq \|\zeta\| + \left[e^t \Psi_U \left[\frac{1 - e^{-\eta\delta}}{\eta} \right]^{\frac{1}{\eta}} \right] \|\zeta\| + e^t \Psi_V \left[\int_0^t e^{-\eta\kappa} \|\alpha(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} + e^t \left[\Psi_U e^{-\delta} + \Psi_H \right] \left[\int_0^t e^{-\eta\kappa} \|\beta(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}}, \quad (15)$$

where

$$\Psi_i = \frac{\Phi_i}{\Gamma(p)} \left[\frac{\Gamma(\theta(p-1)+1)}{\theta^{\theta p - \theta + 1}} \right]^{\frac{1}{\theta}}, \quad i = A, B, C, D, U, V, H.$$

Adding Eqs. (14) and (15), one obtains

$$\begin{aligned} \|\alpha(t)\| + \|\beta(t)\| \leq & \|\xi\| + e^t \left[\Psi_A \left[\frac{1 - e^{-\eta\lambda}}{\eta} \right]^{\frac{1}{\eta}} + \Psi_C \left[\mu_3 e^{-\eta\mu_2} \left(\frac{e^{\eta\mu_1} - 1}{\eta} \right) \right]^{\frac{1}{\eta}} \right] \|\xi\| \\ & + e^t \left[\Psi_A e^{-\lambda} + \Psi_B + \Psi_C [\mu_3 e^{-\eta\mu_2}]^{\frac{1}{\eta}} + \Psi_V \right] \left[\int_0^t e^{-\eta\kappa} \|\alpha(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} + \|\zeta\| \\ & + \left[e^t \Psi_U \left[\frac{1 - e^{-\eta\delta}}{\eta} \right]^{\frac{1}{\eta}} \right] \|\zeta\| + e^t \left[\Psi_D + \Psi_U e^{-\delta} + \Psi_H \right] \left[\int_0^t e^{-\eta\kappa} \|\beta(\kappa)\|^\eta d\kappa \right]^{\frac{1}{\eta}} \end{aligned} \quad (16)$$

According to Lemma 2.7 and let $m = 6$, $\nu = \eta$, one has obtains

$$\begin{aligned} \left[\|\alpha(t)\| + \|\beta(t)\| \right]^\eta \leq & 6^{\eta-1} \|\xi\|^\eta + 6^{\eta-1} e^{\eta t} \left[\Psi_A \left[\frac{1 - e^{-\eta\lambda}}{\eta} \right]^{\frac{1}{\eta}} + \Psi_C \left[\mu_3 e^{-\eta\mu_2} \left(\frac{e^{\eta\mu_1} - 1}{\eta} \right) \right]^{\frac{1}{\eta}} \right]^\eta \|\xi\|^\eta \\ & + 6^{\eta-1} e^{\eta t} \left[\Psi_A e^{-\lambda} + \Psi_B + \Psi_C [\mu_3 e^{-\eta\mu_2}]^{\frac{1}{\eta}} + \Psi_V \right]^\eta \left[\int_0^t e^{-\eta\kappa} \|\alpha(\kappa)\|^\eta d\kappa \right] \\ & + 6^{\eta-1} \|\zeta\|^\eta + 6^{\eta-1} e^{\eta t} \left[\Psi_U \left[\frac{1 - e^{-\eta\delta}}{\eta} \right]^{\frac{1}{\eta}} \right]^\eta \|\zeta\|^\eta + 6^{\eta-1} e^{\eta t} \left[\Psi_D + \Psi_U e^{-\delta} + \Psi_H \right]^\eta \\ & \times \left[\int_0^t e^{-\eta\kappa} \|\beta(\kappa)\|^\eta d\kappa \right]. \end{aligned} \quad (17)$$

Multiply $e^{-\eta t}$ on both sides of inequality (17), which implies

$$\begin{aligned} & \left[\|\alpha(t)\| + \|\beta(t)\| \right]^\eta e^{-\eta t} \\ & \leq 6^{\eta-1} e^{-\eta t} \left[\|\xi\|^\eta + \|\zeta\|^\eta \right] + \max \left\{ 6^{\eta-1} \left[\Psi_A \left[\frac{1 - e^{-\eta\lambda}}{\eta} \right]^{\frac{1}{\eta}} + \Psi_C \left[\mu_3 e^{-\eta\mu_2} \left(\frac{e^{\eta\mu_1} - 1}{\eta} \right) \right]^{\frac{1}{\eta}} \right]^\eta, \right. \\ & \quad 6^{\eta-1} \left[\Psi_U \left[\frac{1 - e^{-\eta\delta}}{\eta} \right]^{\frac{1}{\eta}} \right]^\eta \left. \right\} \left[\|\xi\|^\eta + \|\zeta\|^\eta \right] + \max \left\{ 6^{\eta-1} \left[\Psi_A e^{-\lambda} + \Psi_B + \Psi_C [\mu_3 e^{-\eta\mu_2}]^{\frac{1}{\eta}} + \Psi_V \right]^\eta, \right. \\ & \quad 6^{\eta-1} \left[\Psi_D + \Psi_U e^{-\delta} + \Psi_H \right]^\eta \left. \right\} \left[\int_0^t e^{-\eta\kappa} \left[\|\alpha(\kappa)\|^\eta + \|\beta(\kappa)\|^\eta \right] d\kappa \right] \\ & \leq \left[6^{\eta-1} e^{-\eta t} + \mathcal{J} \right] \left[\|\xi\| + \|\zeta\| \right]^\eta + \mathcal{L} \int_0^t e^{-\eta\kappa} \left[\|\alpha(\kappa)\| + \|\beta(\kappa)\| \right]^\eta d\kappa. \end{aligned} \quad (18)$$

Then by means of Gronwall inequality in Lemma 2.4, one has

$$\begin{aligned} & \left[\|\alpha(t)\| + \|\beta(t)\| \right]^\eta e^{-\eta t} \\ & \leq \left[6^{\eta-1} e^{-\eta t} + \mathcal{J} \right] \left[\|\xi\| + \|\zeta\| \right]^\eta + \int_0^t \left[6^{\eta-1} e^{-\eta\omega} + \mathcal{J} \right] \left[\|\xi\| + \|\zeta\| \right]^\eta \mathcal{L} \exp \left[\int_\omega^t \mathcal{L} d\omega \right] d\omega \\ & = \left[6^{\eta-1} e^{-\eta t} + \mathcal{J} + \mathcal{L} \int_0^t \left[6^{\eta-1} e^{-\eta\omega} + \mathcal{J} \right] \exp \left[\mathcal{L} t - \mathcal{L} \omega \right] d\omega \right] \left[\|\xi\| + \|\zeta\| \right]^\eta \\ & = \left[6^{\eta-1} e^{-\eta t} + \mathcal{J} + 6^{\eta-1} \mathcal{L} e^{\mathcal{L} t} \int_0^t e^{-(\eta+\mathcal{L})\omega} d\omega + \mathcal{J} \mathcal{L} e^{\mathcal{J} t} \int_0^t e^{-\mathcal{L}\omega} d\omega \right] \left[\|\xi\| + \|\zeta\| \right]^\eta \end{aligned}$$

$$= \frac{\eta 6^{\eta-1} e^{-\eta t} + \left[6^{\eta-1} \mathcal{L} + \mathcal{J}(\eta + \mathcal{L}) \right] e^{\mathcal{L}t}}{[\eta + \mathcal{L}]} \left[\|\xi\| + \|\zeta\| \right]^\eta.$$

That is

$$\left[\|\alpha(t)\| + \|\beta(t)\| \right] \leq \sqrt[\eta]{\frac{\eta 6^{\eta-1} + \left[6^{\eta-1} \mathcal{L} + \mathcal{J}(\eta + \mathcal{L}) \right] e^{(\eta+\mathcal{L})t}}{[\eta + \mathcal{L}]}} \left[\|\xi\| + \|\zeta\| \right].$$

By using definition 2.5, it known that $\left[\|\xi\| + \|\zeta\| \right] < \epsilon$, then $\left[\|\alpha(t)\| + \|\beta(t)\| \right] < \epsilon$, which implies the synchronization of error dynamical system (8) of the master system (3) and the slave system (5) can reach synchronization goal in finite time via controller (6).

When the system (3) and (5) has without delays under the controller (6), we have the following result readily can be obtained from Theorem 3.1. $\square$

Corollary 3.2. When $0 < p < \frac{1}{2}$, if assumption $[A_1]$ holds, and

$$\sqrt[\eta]{\frac{\eta 6^{\eta-1} + \left[6^{\eta-1} \mathcal{L}_1 + \mathcal{J}_1(\eta + \mathcal{L}_1) \right] e^{(\eta+\mathcal{L}_1)t}}{[\eta + \mathcal{L}_1]}} < \frac{\epsilon}{\epsilon}, \quad t \in \mathcal{M},$$

then the synchronization of error dynamical system (8) of the master system (3) and the slave system (5) can reach synchronization goal in finite time under the controller (6), where

$$\mathcal{J}_1 = \max \left\{ 6^{\eta-1} \left[\Psi_A \left(\frac{1-e^{-\eta\lambda}}{\eta} \right)^{\frac{1}{\eta}} \right]^\eta, 6^{\eta-1} \left[\Psi_U \left(\frac{1-e^{-\eta\delta}}{\eta} \right)^{\frac{1}{\eta}} \right]^\eta \right\},$$

$$\mathcal{L}_1 = \max \left\{ 6^{\eta-1} \left[\Psi_A e^{-\lambda} + \Psi_B + \Psi_V \right]^\eta, 6^{\eta-1} \left[\Psi_D + \Psi_U e^{-\delta} + \Psi_H \right]^\eta \right\},$$

and $\Psi_A, \Psi_B, \Psi_D, \Psi_U, \Psi_V, \Psi_H, \theta, \eta$ are already defined in Theorem 3.1.

Theorem 3.3. When $\frac{1}{2} \leq p < 1$, if assumption $[A_1]$ holds, and

$$\sqrt{\frac{12 + \left[6\mathcal{N} + \mathcal{K}(2 + \mathcal{N}) \right] e^{(2+\mathcal{N})t}}{[2 + \mathcal{N}]}} < \frac{\epsilon}{\epsilon}, \quad t \in \mathcal{M}, \quad (19)$$

then the synchronization of error dynamical system (8) of the master system (3) and the slave system (5) can reach synchronization goal in finite time under the controller (6), where

$$\mathcal{K} = \max \left\{ 6 \left[\check{\Psi}_A \left(\frac{1-e^{-2\lambda}}{2} \right)^{\frac{1}{2}} + \check{\Psi}_C \sqrt{\mu_3} e^{-\mu_2} \left(\frac{e^{2\mu_1} - 1}{2} \right)^{\frac{1}{2}} \right]^2, 6 \left[\check{\Psi}_U \left(\frac{1-e^{-2\delta}}{2} \right)^{\frac{1}{2}} \right]^2 \right\},$$

$$\mathcal{N} = \max \left\{ 6 \left[\check{\Psi}_A e^{-\lambda} + \check{\Psi}_B + \check{\Psi}_C \sqrt{\mu_3} e^{-\mu_2} + \check{\Psi}_V \right]^2, 6 \left[\check{\Psi}_D + \check{\Psi}_U e^{-\delta} + \check{\Psi}_H \right]^2 \right\},$$

$$\check{\Psi}_A = \frac{\|A\|}{\Gamma(p)} \left[\frac{\Gamma(2p-1)}{2^{2p-1}} \right]^{\frac{1}{2}}, \quad \check{\Psi}_B = \frac{(\|\tilde{B}\| + \|\mathcal{G}\|)\|\mathcal{R}\|}{\Gamma(p)} \left[\frac{\Gamma(2p-1)}{2^{2p-1}} \right]^{\frac{1}{2}}, \quad \check{\Psi}_C = \frac{\|\tilde{C}\|\|\mathcal{R}\|}{\Gamma(p)} \left[\frac{\Gamma(2p-1)}{2^{2p-1}} \right]^{\frac{1}{2}},$$

$$\check{\Psi}_D = \frac{\|D\|}{\Gamma(p)} \left[\frac{\Gamma(2p-1)}{2^{2p-1}} \right]^{\frac{1}{2}}, \quad \check{\Psi}_U = \frac{\|U\|}{\Gamma(p)} \left[\frac{\Gamma(2p-1)}{2^{2p-1}} \right]^{\frac{1}{2}}, \quad \check{\Psi}_V = \frac{\|V\|\|\mathcal{R}\|}{\Gamma(p)} \left[\frac{\Gamma(2p-1)}{2^{2p-1}} \right]^{\frac{1}{2}},$$

$$\check{\Psi}_H = \frac{\|\tilde{H}\|\|\mathcal{R}\|}{\Gamma(p)} \left[\frac{\Gamma(2p-1)}{2^{2p-1}} \right]^{\frac{1}{2}}.$$

Proof. Let $\Phi_A = \|A\|$, $\Phi_B = (\|\tilde{B}\| + \|\mathcal{G}\|)\|\mathcal{R}\|$, $\Phi_C = \|\tilde{C}\|\|\mathcal{R}\|$, $\Phi_D = \|D\|$, $\Phi_U = \|U\|$, $\Phi_V = \|V\|\|\mathcal{R}\|$, $\Phi_H = \|\tilde{H}\|\|\mathcal{R}\|$, then by follows from Lemma 2.8 and Eq. (10), one has obtain that

$$\begin{aligned} \|\alpha(t)\| &\leq \|\alpha(0)\| + \frac{1}{\Gamma(p)} \int_0^t (t-\kappa)^{p-1} \left[\Phi_A \|\alpha(\kappa-\lambda)\| + \Phi_B \|\alpha(\kappa)\| + \Phi_C \|\alpha(\kappa-\mu(\kappa))\| + \Phi_D \|\beta(\kappa)\| \right] d\kappa \\ &= \|\alpha(0)\| + \frac{\Phi_A}{\Gamma(p)} \int_0^t (t-\kappa)^{p-1} e^{\kappa-\lambda} e^{-(\kappa-\lambda)} \|\alpha(\kappa-\lambda)\| d\kappa + \frac{\Phi_B}{\Gamma(p)} \int_0^t (t-\kappa)^{p-1} e^{\kappa} e^{-\kappa} \|\alpha(\kappa)\| d\kappa \\ &\quad + \frac{\Phi_C}{\Gamma(p)} \int_0^t (t-\kappa)^{p-1} e^{\kappa} e^{-\kappa} \|\alpha(\kappa-\mu(\kappa))\| d\kappa + \frac{\Phi_D}{\Gamma(p)} \int_0^t (t-\kappa)^{p-1} e^{\kappa} e^{-\kappa} \|\beta(\kappa)\| d\kappa \\ &\leq \|\xi\| + \frac{\Phi_A}{\Gamma(p)} \left[\int_0^t (t-\kappa)^{2(p-1)} e^{2(\kappa-\lambda)} d\kappa \right]^{\frac{1}{2}} \left[\int_0^t e^{-2(\kappa-\lambda)} \|\alpha(\kappa-\lambda)\|^2 d\kappa \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
& + \frac{\Phi_B}{\Gamma(p)} \left[\int_0^t (t-\kappa)^{2(p-1)} e^{2\kappa} d\kappa \right]^{\frac{1}{2}} \left[\int_0^t e^{-2\kappa} \|\alpha(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}} + \frac{\Phi_C}{\Gamma(p)} \left[\int_0^t (t-\kappa)^{2(p-1)} e^{2\kappa} d\kappa \right]^{\frac{1}{2}} \\
& \times \left[\int_0^t e^{-2\kappa} \|\alpha(\kappa - \mu(\kappa))\|^2 d\kappa \right]^{\frac{1}{2}} + \frac{\Phi_D}{\Gamma(p)} \left[\int_0^t (t-\kappa)^{2(p-1)} e^{2\kappa} d\kappa \right]^{\frac{1}{2}} \left[\int_0^t e^{-2\kappa} \|\beta(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}}
\end{aligned} \quad (20)$$

Similar to Eq. (13), one has

$$\int_0^t (t-\kappa)^{2(p-1)} e^{2(\kappa-\lambda)} d\kappa \leq \frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1). \quad (21)$$

Substituting the inequality (21) into (20), we can get

$$\begin{aligned}
\|\alpha(t)\| & \leq \|\xi\| + \frac{\Phi_A}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}} \left[e^{-2\lambda} \int_{-\lambda}^{t-\lambda} e^{-2(\kappa-\lambda)} \|\alpha(\kappa-\lambda)\|^2 d\kappa \right]^{\frac{1}{2}} \\
& + \frac{\Phi_B}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}} \left[\int_0^t e^{-2\kappa} \|\alpha(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}} \\
& + \frac{\Phi_C}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}} \left[\int_0^t e^{-2(\kappa-\mu(\kappa))} e^{-2\mu(\kappa)} \frac{\|\alpha(\kappa-\mu(\kappa))\|^2}{|1-\mu'(\kappa)|} d(\kappa-\mu(\kappa)) \right]^{\frac{1}{2}} \\
& + \frac{\Phi_D}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}} \left[\int_0^t e^{-2\kappa} \|\beta(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}} \\
& \leq \|\xi\| + \frac{\Phi_A}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}} \left[e^{-2\lambda} \left[\frac{e^{2\lambda}-1}{2} \right] \|\xi\|^2 + e^{-2\lambda} \int_0^t e^{-2\kappa} \|\alpha(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}} \\
& + \frac{\Phi_B}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}} \left[\int_0^t e^{-2\kappa} \|\alpha(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}} \\
& + \frac{\Phi_C}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}} \left[\mu_3 e^{-2\mu_2} \left(\frac{e^{2\mu_1}-1}{2} \right) \|\xi\|^2 + \int_0^t \mu_3 e^{-2\kappa} e^{-2\mu_2} \|\alpha(k)\|^2 d\kappa \right]^{\frac{1}{2}} \\
& + \frac{\Phi_D}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}} \left[\int_0^t e^{-2\kappa} \|\beta(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}} \\
& \leq \|\xi\| + \frac{\Phi_A}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}} \left[\frac{1-e^{-2\lambda}}{2} \right]^{\frac{1}{2}} \|\xi\| + \frac{\Phi_A}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}} \\
& \times \left[e^{-2\lambda} \int_0^t e^{-2\kappa} \|\alpha(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}} + \frac{\Phi_B}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}} \left[\int_0^t e^{-2\kappa} \|\alpha(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}} \\
& + \frac{\Phi_C}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}} \left[\mu_3 e^{-2\mu_2} \left(\frac{e^{2\mu_1}-1}{2} \right) \right]^{\frac{1}{2}} \|\xi\| + \frac{\Phi_C}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}} \\
& \times \left[\int_0^t \mu_3 e^{-2\kappa} e^{-2\mu_2} \|\alpha(k)\|^2 d\kappa \right]^{\frac{1}{2}} + \frac{\Phi_D}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}} \left[\int_0^t e^{-2\kappa} \|\beta(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}}.
\end{aligned}$$

That is,

$$\begin{aligned}
\|\alpha(t)\| & \leq \|\xi\| + \left[e^t \tilde{\Psi}_A \left(\frac{1-e^{-2\lambda}}{2} \right)^{\frac{1}{2}} + e^t \tilde{\Psi}_C \sqrt{\mu_3} e^{-\mu_2} \left(\frac{e^{2\mu_1}-1}{2} \right)^{\frac{1}{2}} \right] \|\xi\| + e^t \left[\tilde{\Psi}_A e^{-\lambda} + \tilde{\Psi}_B \right. \\
& \left. + \tilde{\Psi}_C \sqrt{\mu_3} e^{-\mu_2} \right] \left[\int_0^t e^{-2\kappa} \|\alpha(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}} + e^t \tilde{\Psi}_D \left[\int_0^t e^{-2\kappa} \|\beta(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}}.
\end{aligned} \quad (22)$$

Similarly, by the same procedures as the above inequality, we obtains

$$\|\beta(t)\| \leq \|\zeta\| + \left[e^t \tilde{\Psi}_U \left(\frac{1-e^{-2\delta}}{2} \right)^{\frac{1}{2}} \right] \|\xi\| + e^t \tilde{\Psi}_V \left[\int_0^t e^{-2\kappa} \|\alpha(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}} + e^t \left[\tilde{\Psi}_U e^{-\delta} + \tilde{\Psi}_H \right] \left[\int_0^t e^{-2\kappa} \|\beta(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}}, \quad (23)$$

where

$$\tilde{\Psi}_i = \frac{\Phi_i}{\Gamma(p)} \left[\frac{e^{2t}}{2^{2p-1}} \Gamma(2p-1) \right]^{\frac{1}{2}}, \quad i = A, B, C, D, U, V, H.$$

Adding Eqs. (22) and (23), one obtains

$$\begin{aligned} \|\alpha(t)\| + \|\beta(t)\| &\leq \|\xi\| + e^t \left[\check{\Psi}_A \left(\frac{1-e^{-2\lambda}}{2} \right)^{\frac{1}{2}} + \check{\Psi}_C \sqrt{\mu_3} e^{-\mu_2} \left(\frac{e^{2\mu_1}-1}{2} \right)^{\frac{1}{2}} \right] \|\xi\| \\ &\quad + e^t \left[\check{\Psi}_A e^{-\lambda} + \check{\Psi}_B + \check{\Psi}_C \sqrt{\mu_3} e^{-\mu_2} + \check{\Psi}_V \right] \left[\int_0^t e^{-2\kappa} \|\alpha(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}} + \|\zeta\| \\ &\quad + e^t \left[\check{\Psi}_U \left(\frac{1-e^{-2\delta}}{2} \right)^{\frac{1}{2}} \right] \|\zeta\| + e^t \left[\check{\Psi}_D + \check{\Psi}_U e^{-\delta} + \check{\Psi}_H \right] \left[\int_0^t e^{-2\kappa} \|\beta(\kappa)\|^2 d\kappa \right]^{\frac{1}{2}}. \end{aligned} \quad (24)$$

According to Lemma 2.7 and let $m = 6$, $\nu = 2$, one can obtains

$$\begin{aligned} \left[\|\alpha(t)\| + \|\beta(t)\| \right]^2 &\leq 6\|\xi\|^2 + 6e^{2t} \left[\check{\Psi}_A \left(\frac{1-e^{-2\lambda}}{2} \right)^{\frac{1}{2}} + \check{\Psi}_C \sqrt{\mu_3} e^{-\mu_2} \left(\frac{e^{2\mu_1}-1}{2} \right)^{\frac{1}{2}} \right]^2 \|\xi\|^2 \\ &\quad + 6e^{2t} \left[\check{\Psi}_A e^{-\lambda} + \check{\Psi}_B + \check{\Psi}_C \sqrt{\mu_3} e^{-\mu_2} + \check{\Psi}_V \right]^2 \left[\int_0^t e^{-2\kappa} \|\alpha(\kappa)\|^2 d\kappa \right] \\ &\quad + 6\|\zeta\|^2 + 6e^{2t} \left[\check{\Psi}_U \left(\frac{1-e^{-2\delta}}{2} \right)^{\frac{1}{2}} \right]^2 \|\zeta\|^2 + 6e^{2t} \left[\check{\Psi}_D + \check{\Psi}_U e^{-\delta} + \check{\Psi}_H \right]^2 \\ &\quad \times \left[\int_0^t e^{-2\kappa} \|\beta(\kappa)\|^2 d\kappa \right]. \end{aligned} \quad (25)$$

Multiply e^{-2t} on both sides of inequality (25), which implies

$$\begin{aligned} &\left[\|\alpha(t)\| + \|\beta(t)\| \right]^2 e^{-2t} \\ &\leq 6e^{-2t} \left[\|\xi\|^2 + \|\zeta\|^2 \right] + \max \left\{ 6 \left[\check{\Psi}_A \left(\frac{1-e^{-2\lambda}}{2} \right)^{\frac{1}{2}} + \check{\Psi}_C \sqrt{\mu_3} e^{-\mu_2} \left(\frac{e^{2\mu_1}-1}{2} \right)^{\frac{1}{2}} \right]^2, \right. \\ &\quad \left. 6 \left[\check{\Psi}_U \left(\frac{1-e^{-2\delta}}{2} \right)^{\frac{1}{2}} \right]^2 \right\} \left[\|\xi\|^2 + \|\zeta\|^2 \right] + \max \left\{ 6 \left[\check{\Psi}_A e^{-\lambda} + \check{\Psi}_B + \check{\Psi}_C \sqrt{\mu_3} e^{-\mu_2} + \check{\Psi}_V \right]^2, \right. \\ &\quad \left. 6 \left[\check{\Psi}_D + \check{\Psi}_U e^{-\delta} + \check{\Psi}_H \right]^2 \right\} \left[\int_0^t e^{-2\kappa} \left[\|\alpha(\kappa)\|^2 + \|\beta(\kappa)\|^2 \right] d\kappa \right] \\ &\leq \left[6e^{-2t} + \mathcal{K} \right] \left[\|\xi\| + \|\zeta\| \right]^2 + \mathcal{N} \int_0^t e^{-2\kappa} \left[\|\alpha(\kappa)\| + \|\beta(\kappa)\| \right]^2 d\kappa. \end{aligned} \quad (26)$$

Then by means of Gronwall inequality in Lemma 2.4, one has

$$\begin{aligned} &\left[\|\alpha(t)\| + \|\beta(t)\| \right]^2 e^{-2t} \\ &\leq \left[6e^{-2t} + \mathcal{K} \right] \left[\|\xi\| + \|\zeta\| \right]^2 + \int_0^t \left[6e^{-2\omega} + \mathcal{K} \right] \left[\|\xi\| + \|\zeta\| \right]^2 \mathcal{N} \exp \left[\int_\omega^t \mathcal{N} d\omega \right] d\omega \\ &= \left[6e^{-2t} + \mathcal{K} + \mathcal{N} \int_0^t \left[6e^{-2\omega} + \mathcal{K} \right] \exp \left[\mathcal{N}t - \mathcal{N}\omega \right] d\omega \right] \left[\|\xi\| + \|\zeta\| \right]^2 \\ &= \left[6e^{-2t} + \mathcal{K} + 6\mathcal{N}e^{\mathcal{N}t} \int_0^t e^{-(2+\mathcal{N})\omega} d\omega + \mathcal{K}\mathcal{N}e^{\mathcal{N}t} \int_0^t e^{-\mathcal{N}\omega} d\omega \right] \left[\|\xi\| + \|\zeta\| \right]^2 \\ &= \frac{12e^{-2t} + [6\mathcal{N} + \mathcal{K}(2 + \mathcal{N})]e^{\mathcal{N}t}}{[2 + \mathcal{N}]} \left[\|\xi\| + \|\zeta\| \right]^2. \end{aligned}$$

That is

$$\left[\|\alpha(t)\| + \|\beta(t)\| \right] \leq \sqrt{\frac{12 + [6\mathcal{N} + \mathcal{K}(2 + \mathcal{N})]e^{(2+\mathcal{N})t}}{[2 + \mathcal{N}]}} \left[\|\xi\| + \|\zeta\| \right].$$

By using definition 2.5, it known that $\left[\|\xi\| + \|\zeta\| \right] < \epsilon$, then $\left[\|\alpha(t)\| + \|\beta(t)\| \right] < \varepsilon$, which implies the synchronization of error dynamical system (8) of the master system (3) and the slave system (5) can reach synchronization goal in finite time via controller (6).

When the system (3) and (5) has without delays under the controller (6), we have the following result readily can be obtained from Theorem 3.3. $\square$

Corollary 3.4. When $\frac{1}{2} \leq p < 1$, if assumption $[A_1]$ holds, and

$$\sqrt{\frac{12 + [6\mathcal{N}_1 + \mathcal{K}_1(2 + \mathcal{N}_1)]e^{(2+\mathcal{N}_1)t}}{[2 + \mathcal{N}_1]}} < \frac{\varepsilon}{\epsilon}, \quad t \in \mathcal{M},$$

then the synchronization of error dynamical system (8) of the master system (3) and the slave system (5) can reach synchronization goal in finite time under the controller (6), where

$$\mathcal{K}_1 = \max \left\{ 6 \left[\tilde{\Psi}_A \left(\frac{1 - e^{-2\lambda}}{2} \right)^{\frac{1}{2}} \right]^2, 6 \left[\tilde{\Psi}_U \left(\frac{1 - e^{-2\delta}}{2} \right)^{\frac{1}{2}} \right]^2 \right\},$$

$$\mathcal{N}_1 = \max \left\{ 6 \left[\tilde{\Psi}_A e^{-\lambda} + \tilde{\Psi}_B + \tilde{\Psi}_V \right]^2, 6 \left[\tilde{\Psi}_D + \tilde{\Psi}_U e^{-\delta} + \tilde{\Psi}_H \right]^2 \right\},$$

and $\tilde{\Psi}_A, \tilde{\Psi}_B, \tilde{\Psi}_D, \tilde{\Psi}_U, \tilde{\Psi}_V, \tilde{\Psi}_H$ are already defined in Theorem 3.3.

Remark 3.5. The finite-time synchronization results about TMFCNNs with leakage delay have not yet tackled, hence Theorems 3.1, 3.3, Corollaries 3.2 and 3.4 are new to this field of study. Moreover, the concerned result is still true for finite-time synchronization analysis of integer order competitive neural networks and there is no need to MATLAB-LMI toolbox for a computational purpose, this leads to the main advantage of the proposed results.

Remark 3.6. In [33], the authors addressed finite time synchronization of fractional order memristive neural networks by using the Laplace transform, Mittag-Leffler function and simple linear feedback controller. In [51], the authors deliberated the synchronization control of Riemann–Liouville sense fractional order competitive neural networks by employing linear feedback controller, Lyapunov-functional associated with fractional integral terms, LMI techniques and delay-partitioning method. Differing with the techniques in [33,51], we dealt with some inequality techniques to study the finite-time synchronization criteria in Caputo sense fractional order time-varying delayed competitive neural networks with leakage delay by using output feedback controller.

Remark 3.7. The increase in the number of neurons will increase the computational complexity. In spite of that, Lyapunov-functional approaches to deal with synchronization criteria for TMFCNNs will greatly increase the computational complexity of the system solutions, because of these results are fully in the form of matrix elements or in terms of system matrices. In order to triumph over the difficulties, we adopt some inequality techniques such as Gronwall–Bellman integral inequality, Hölder's inequality and Cauchy–Schwartz inequality to drive the finite-time synchronization criteria for the considered system. However, in our proposed works, we used very less number of matrices and elements for computation. What is more, contrast with the Lyapunov-functional approach, the algebraic computations of this paper are much simpler and MATLAB control toolbox is not necessarily required for computation, which leads to less conservatism.

Remark 3.8. When the connection weights satisfies $b_{ij}^+ = b_{ij}^-$ and $c_{ij}^+ = c_{ij}^-$, then our problem becomes finite-time synchronization of time-varying delay fractional order competitive neural networks. There is no results which is not yet discussed when $r_i(t) = 0$, hence it motivates the finite-time synchronization of time-varying delayed fractional order neural networks(FSTFNNs). However to the best our knowledge, the obtained results are less conservative.

4. Numerical simulation

In this section, we justify the obtained theoretical results via two numerical simulation example.

Example 4.1. Consider a four-state TMFCNNs with leakage delay:

$$\begin{aligned} \mathcal{D}^p q_i(t) &= -a_i q_i(t - \lambda) + \sum_{j=1}^4 b_{ij}(q_i(t)) h_j(q_j(t)) + \sum_{j=1}^4 c_{ij}(q_i(t)) h_j(q_j(t - \mu(t))) + d_i r_i(t), \\ \mathcal{D}^p r_i(t) &= -u_i r_i(t - \delta) + v_i h_i(q_i(t)), \quad i = 1, 2, 3, 4 \end{aligned} \quad (27)$$

where $A = \text{diag}[0.75, 0.75, 0.75, 0.75]$, $D = \text{diag}[0.5, 0.5, 0.5, 0.5]$, $U = \text{diag}[0.5, 0.5, 0.5, 0.5]$, $V = \text{diag}[2.7, 2.7, 2.7, 2.7]$, $h_j(q) = 0.1 \tanh(q)$,

$j = 1, 2, 3, 4$. According to assumption $[A_1]$, we take $\mathcal{R} = \text{diag}[0.05, 0.05, 0.05, 0.05]$, while we set $\lambda = 2.5$, $\delta = 3.5$, $\mu(t) = \frac{1}{8} |\cos t|$, $\varepsilon = 10$, $\epsilon = 0.2$. The memristive connection parameters are given by:

$$\begin{aligned} b_{11}(q_1(t)) &= \begin{cases} -\frac{1}{7}, & |q_1(t)| > 1, \\ \frac{1}{7}, & |q_1(t)| < 1, \end{cases} & b_{12}(q_2(t)) &= \begin{cases} -\frac{2}{7}, & |q_2(t)| > 1, \\ \frac{2}{7}, & |q_2(t)| < 1, \end{cases} & b_{13}(q_3(t)) &= \begin{cases} -\frac{3}{10}, & |q_3(t)| > 1, \\ \frac{3}{10}, & |q_3(t)| < 1, \end{cases} \\ b_{14}(q_4(t)) &= \begin{cases} -\frac{2}{9}, & |q_4(t)| > 1, \\ \frac{2}{9}, & |q_4(t)| < 1, \end{cases} & b_{21}(q_1(t)) &= \begin{cases} -\frac{6}{19}, & |q_1(t)| > 1, \\ \frac{6}{19}, & |q_1(t)| < 1, \end{cases} & b_{22}(q_2(t)) &= \begin{cases} -\frac{9}{16}, & |q_2(t)| > 1, \\ \frac{9}{16}, & |q_2(t)| < 1, \end{cases} \\ b_{23}(q_3(t)) &= \begin{cases} -\frac{2}{9}, & |q_3(t)| > 1, \\ \frac{2}{9}, & |q_3(t)| < 1, \end{cases} & b_{24}(q_4(t)) &= \begin{cases} -\frac{7}{17}, & |q_4(t)| > 1, \\ \frac{7}{17}, & |q_4(t)| < 1, \end{cases} & b_{31}(q_1(t)) &= \begin{cases} -\frac{9}{19}, & |q_1(t)| > 1, \\ \frac{9}{19}, & |q_1(t)| < 1, \end{cases} \end{aligned}$$

$$\begin{aligned}
b_{32}(q_2(t)) &= \begin{cases} -\frac{3}{11}, & |q_2(t)| > 1, \\ \frac{3}{11}, & |q_2(t)| < 1, \end{cases} & b_{33}(q_3(t)) &= \begin{cases} -\frac{2}{13}, & |q_3(t)| > 1, \\ \frac{2}{13}, & |q_3(t)| < 1, \end{cases} & b_{34}(q_4(t)) &= \begin{cases} -\frac{2}{7}, & |q_4(t)| > 1, \\ \frac{2}{7}, & |q_4(t)| < 1, \end{cases} \\
b_{41}(q_1(t)) &= \begin{cases} -\frac{6}{19}, & |q_1(t)| > 1, \\ \frac{6}{19}, & |q_1(t)| < 1, \end{cases} & b_{42}(q_2(t)) &= \begin{cases} -\frac{12}{17}, & |q_2(t)| > 1, \\ \frac{12}{17}, & |q_2(t)| < 1, \end{cases} & b_{43}(q_3(t)) &= \begin{cases} -\frac{3}{10}, & |q_3(t)| > 1, \\ \frac{3}{10}, & |q_3(t)| < 1, \end{cases} \\
b_{44}(q_4(t)) &= \begin{cases} -\frac{8}{15}, & |q_4(t)| > 1, \\ \frac{8}{15}, & |q_4(t)| < 1, \end{cases} & c_{11}(q_1(t)) &= \begin{cases} -\frac{2}{9}, & |q_1(t)| > 1, \\ \frac{2}{9}, & |q_1(t)| < 1, \end{cases} & c_{12}(q_2(t)) &= \begin{cases} -\frac{2}{13}, & |q_2(t)| > 1, \\ \frac{2}{13}, & |q_2(t)| < 1, \end{cases} \\
c_{13}(q_3(t)) &= \begin{cases} -\frac{2}{12}, & |q_3(t)| > 1, \\ \frac{2}{11}, & |q_3(t)| < 1, \end{cases} & c_{14}(q_4(t)) &= \begin{cases} -\frac{3}{16}, & |q_4(t)| > 1, \\ \frac{3}{16}, & |q_4(t)| < 1, \end{cases} & c_{21}(q_1(t)) &= \begin{cases} -\frac{4}{9}, & |q_1(t)| > 1, \\ \frac{4}{9}, & |q_1(t)| < 1, \end{cases} \\
c_{22}(q_2(t)) &= \begin{cases} -\frac{3}{13}, & |q_2(t)| > 1, \\ \frac{3}{13}, & |q_2(t)| < 1, \end{cases} & c_{23}(q_3(t)) &= \begin{cases} -\frac{1}{11}, & |q_3(t)| > 1, \\ \frac{1}{11}, & |q_3(t)| < 1, \end{cases} & c_{24}(q_4(t)) &= \begin{cases} -\frac{5}{16}, & |q_4(t)| > 1, \\ \frac{5}{16}, & |q_4(t)| < 1, \end{cases} \\
c_{31}(q_1(t)) &= \begin{cases} -\frac{7}{9}, & |q_1(t)| > 1, \\ \frac{7}{9}, & |q_1(t)| < 1, \end{cases} & c_{32}(q_2(t)) &= \begin{cases} -\frac{1}{13}, & |q_2(t)| > 1, \\ \frac{1}{13}, & |q_2(t)| < 1, \end{cases} & c_{33}(q_3(t)) &= \begin{cases} -\frac{3}{11}, & |q_3(t)| > 1, \\ \frac{3}{11}, & |q_3(t)| < 1, \end{cases} \\
c_{34}(q_4(t)) &= \begin{cases} -\frac{7}{16}, & |q_4(t)| > 1, \\ \frac{7}{16}, & |q_4(t)| < 1, \end{cases} & c_{41}(q_1(t)) &= \begin{cases} -\frac{1}{9}, & |q_1(t)| > 1, \\ \frac{1}{9}, & |q_1(t)| < 1, \end{cases} & c_{42}(q_2(t)) &= \begin{cases} -\frac{3}{13}, & |q_2(t)| > 1, \\ \frac{3}{13}, & |q_2(t)| < 1, \end{cases} \\
c_{43}(q_3(t)) &= \begin{cases} -\frac{3}{11}, & |q_3(t)| > 1, \\ \frac{3}{11}, & |q_3(t)| < 1, \end{cases} & c_{44}(q_4(t)) &= \begin{cases} -\frac{9}{16}, & |q_4(t)| > 1, \\ \frac{9}{16}, & |q_4(t)| < 1, \end{cases}
\end{aligned}$$

According to the finite-time controller (6), we select the control gain matrices $\mathcal{G}$ and $\mathcal{H}$ such that $\mathcal{G} = \text{diag}[4.25, 4.25, 4.25, 4.25]$ and $\mathcal{H} = \text{diag}[4.5, 4.5, 4.5, 4.5]$.

When, $p = 0.47$, after by the simple manipulation, it easily to get the above parameters, then $\mathcal{L} = 32.75$, $\mathcal{J} = 3.79$. Based on the inequality (11), we get the upper bound of settling time value t is $t < 0.102$, while we take $t = 0.1$, then

$$\sqrt[\eta]{\frac{\eta 6^{\eta-1} + [6^{\eta-1}\mathcal{L} + \mathcal{J}(\eta + \mathcal{L})]e^{(\eta+\mathcal{L})t}}{[\eta + \mathcal{L}]}} = 10.673, \quad \frac{\varepsilon}{\epsilon} = 50, \quad t \in \mathcal{M}. \quad (28)$$

That is $t \in \mathcal{M} = [0, 0.102]$, obviously the inequality (11) holds.

On the other hand, when $p = 0.98$, it simply to verify that $\mathcal{K} = 0.448$, $\mathcal{N} = 0.832$. According to the inequality (19), we get the upper bound of settling time value t is $t < 1.718$, while we take $t = 1.5$, then

$$\sqrt{\frac{12 + [6\mathcal{N} + \mathcal{K}(2 + \mathcal{N})]e^{(2+\mathcal{N})t}}{[2 + \mathcal{N}]}} = 2.793, \quad \frac{\varepsilon}{\epsilon} = 50, \quad t \in \mathcal{M}. \quad (29)$$

That is $t \in \mathcal{M} = [0, 1.718]$, obviously the inequality (11) holds. Based on the above numerical simulation results, definition 2.5 and Theorem 3.1 that the master system (3) and the slave system (5) can reach finite-time synchronization based on the controller (6). In the numerical simulations, the following initial values are taken: $(q_1(t), q_2(t), q_3(t), q_4(t), \hat{q}_1(t), \hat{q}_2(t), \hat{q}_3(t), \hat{q}_4(t)) = (1.5, 1.3, -2.7, -3.5, 2.4, 0.75, 0.25, -0.4)$ and $(r_1(t), r_2(t), r_3(t), r_4(t), \hat{r}_1(t), \hat{r}_2(t), \hat{r}_3(t), \hat{r}_4(t)) = (3.5, -0.3, 2.7, -2.5, 1.4, -2.75, 0.25, 1.4)$. The synchronization error evolution of state variables $\alpha(t)$ and $\beta(t)$ are presented in Figs. 1–2.

Example 4.2. Consider a three-state TMFCNNs with leakage delay:

$$\begin{aligned}
\mathcal{D}^p q_i(t) &= -a_i q_i(t - \lambda) + \sum_{j=1}^3 b_{ij}(q_i(t))h_j(q_j(t)) + \sum_{j=1}^3 c_{ij}(q_i(t))h_j(q_j(t - \mu(t))) + d_i r_i(t), \\
\mathcal{D}^p r_i(t) &= -u_i r_i(t - \delta) + v_i h_i(q_i(t)), \quad i = 1, 2, 3,
\end{aligned} \quad (30)$$

where $A = \text{diag}[0.5, 0.5, 0.5]$, $D = \text{diag}[0.4, 0.4, 0.4]$, $U = \text{diag}[0.4, 0.2, 0.3]$, $V = \text{diag}[1.5, 1.5, 1.5]$, $h_j(q) = 0.7 \tanh(q)$, $j = 1, 2, 3$. By using assumption $[A_1]$, we select $\mathcal{R} = \text{diag}[0.1, 0.1, 0.1, 0.1]$, while we set $\lambda = 1.5$, $\delta = 1$, $\mu(t) = \frac{1}{8} \cos t$, $\varepsilon = 10$, $\epsilon = 0.2$. The memristive connection parameters are given by:

$$\begin{aligned}
b_{11}(q_1(t)) &= \begin{cases} -\frac{1}{6}, & |q_1(t)| > 1, \\ \frac{1}{6}, & |q_1(t)| < 1, \end{cases} & b_{12}(q_2(t)) &= \begin{cases} -\frac{1}{11}, & |q_2(t)| > 1, \\ \frac{1}{11}, & |q_2(t)| < 1, \end{cases} & b_{13}(q_3(t)) &= \begin{cases} -\frac{2}{7}, & |q_3(t)| > 1, \\ \frac{2}{7}, & |q_3(t)| < 1, \end{cases} \\
b_{21}(q_1(t)) &= \begin{cases} -\frac{2}{9}, & |q_1(t)| > 1, \\ \frac{2}{9}, & |q_1(t)| < 1, \end{cases} & b_{22}(q_2(t)) &= \begin{cases} -\frac{3}{11}, & |q_2(t)| > 1, \\ \frac{3}{11}, & |q_2(t)| < 1, \end{cases} & b_{23}(q_3(t)) &= \begin{cases} -\frac{1}{7}, & |q_3(t)| > 1, \\ \frac{1}{7}, & |q_3(t)| < 1, \end{cases}
\end{aligned}$$

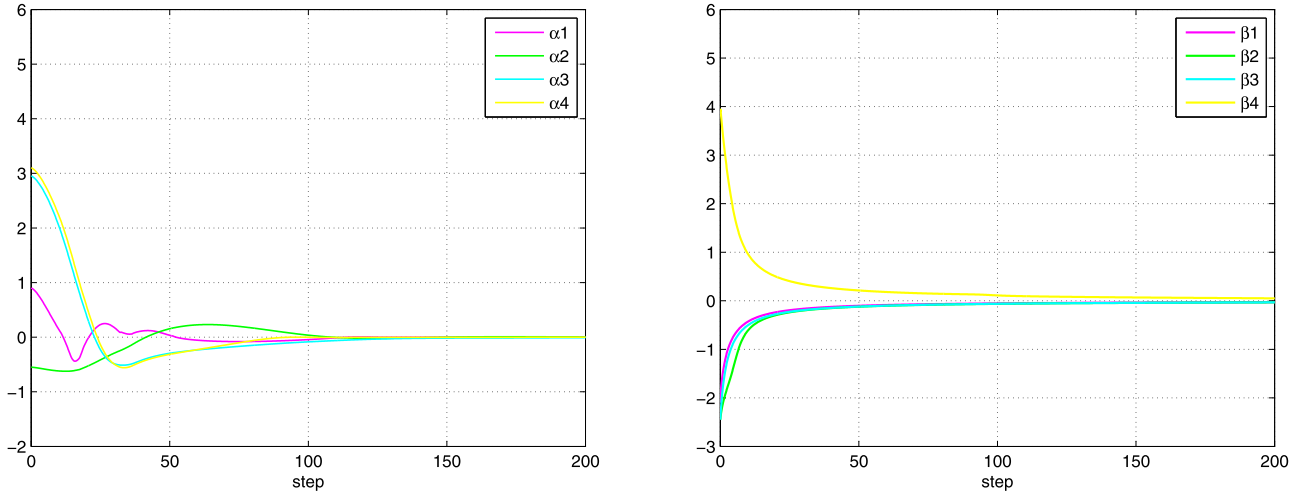


Fig. 1. Synchronization error evolution of state variables $\alpha(t)$ and $\beta(t)$ with order 0.47 in Example 4.1.

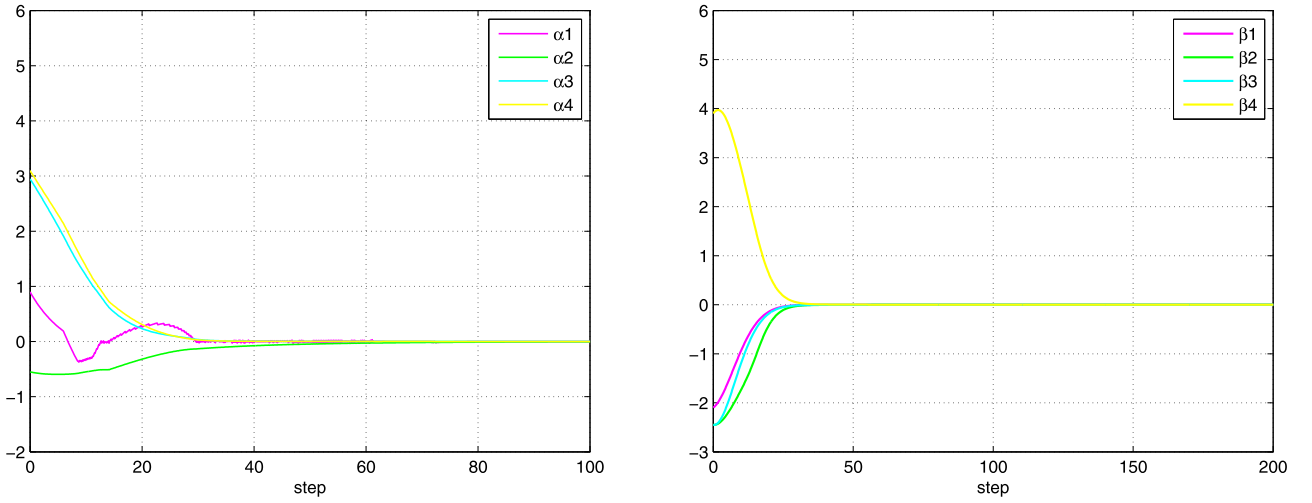


Fig. 2. The synchronization error evolution of state variables $\alpha(t)$ and $\beta(t)$ with order 0.98 in Example 4.1.

$$\begin{aligned}
 b_{31}(q_1(t)) &= \begin{cases} -\frac{2}{13}, & |q_1(t)| > 1, \\ \frac{2}{13}, & |q_1(t)| < 1, \end{cases} & b_{32}(q_2(t)) &= \begin{cases} -\frac{5}{11}, & |q_2(t)| > 1, \\ \frac{5}{11}, & |q_2(t)| < 1, \end{cases} & b_{33}(q_3(t)) &= \begin{cases} -\frac{5}{9}, & |q_3(t)| > 1, \\ \frac{5}{9}, & |q_3(t)| < 1, \end{cases} \\
 c_{11}(q_1(t)) &= \begin{cases} -\frac{2}{9}, & |q_1(t)| > 1, \\ \frac{2}{9}, & |q_1(t)| < 1, \end{cases} & c_{12}(q_2(t)) &= \begin{cases} -\frac{3}{10}, & |q_2(t)| > 1, \\ \frac{3}{10}, & |q_2(t)| < 1, \end{cases} & c_{13}(q_3(t)) &= \begin{cases} -\frac{1}{11}, & |q_3(t)| > 1, \\ \frac{1}{11}, & |q_3(t)| < 1, \end{cases} \\
 c_{21}(q_1(t)) &= \begin{cases} -\frac{2}{11}, & |q_1(t)| > 1, \\ \frac{2}{11}, & |q_1(t)| < 1, \end{cases} & c_{22}(q_2(t)) &= \begin{cases} -\frac{5}{16}, & |q_2(t)| > 1, \\ \frac{5}{16}, & |q_2(t)| < 1, \end{cases} & c_{23}(q_3(t)) &= \begin{cases} -\frac{2}{15}, & |q_3(t)| > 1, \\ \frac{2}{15}, & |q_3(t)| < 1, \end{cases} \\
 c_{31}(q_1(t)) &= \begin{cases} -\frac{2}{13}, & |q_1(t)| > 1, \\ \frac{2}{11}, & |q_1(t)| < 1, \end{cases} & c_{32}(q_2(t)) &= \begin{cases} -\frac{1}{14}, & |q_2(t)| > 1, \\ \frac{1}{14}, & |q_2(t)| < 1, \end{cases} & c_{33}(q_3(t)) &= \begin{cases} -\frac{3}{13}, & |q_3(t)| > 1, \\ \frac{3}{13}, & |q_3(t)| < 1, \end{cases}
 \end{aligned}$$

By using the finite-time controller (6), we take the control gain matrices $\mathcal{G}$ and $\mathcal{H}$ such that $\mathcal{G} = \text{diag}[1.5, 1.5, 1.5]$ and $\mathcal{H} = \text{diag}[1, 1, 1]$.

When, $p = 0.46$, after by the simple calculation, it easily to get the above parameters, then $\mathcal{L} = 21.99$, $\mathcal{J} = 3.623$. Based on the inequality (11), we get the upper bound of settling time value t is $t < 0.148$, while we take $t = 0.1$, then

$$\sqrt[\eta]{\frac{\eta 6^{\eta-1} + [6^{\eta-1} \mathcal{L} + \mathcal{J}(\eta + \mathcal{L})]e^{(\eta+\mathcal{L})t}}{[\eta + \mathcal{L}]}} = 7.433, \quad \frac{\varepsilon}{\varepsilon} = 50, \quad t \in \mathcal{M}. \quad (31)$$

That is $t \in \mathcal{M} = [0, 0.148]$, obviously the inequality (11) holds.

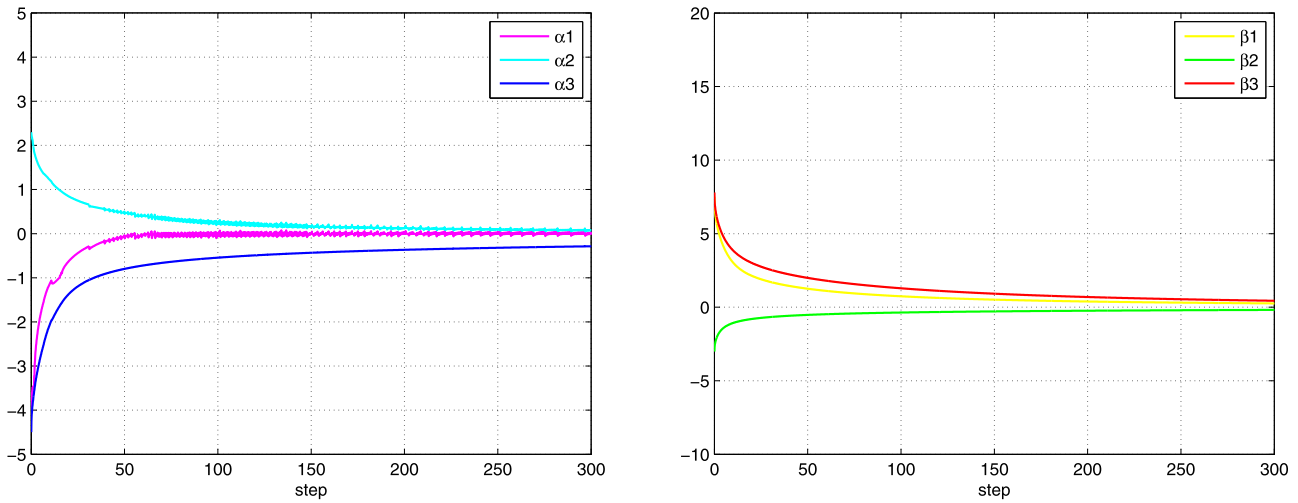


Fig. 3. The synchronization error evolution of state variables $\alpha(t)$ and $\beta(t)$ with order 0.45 in Example 4.2.

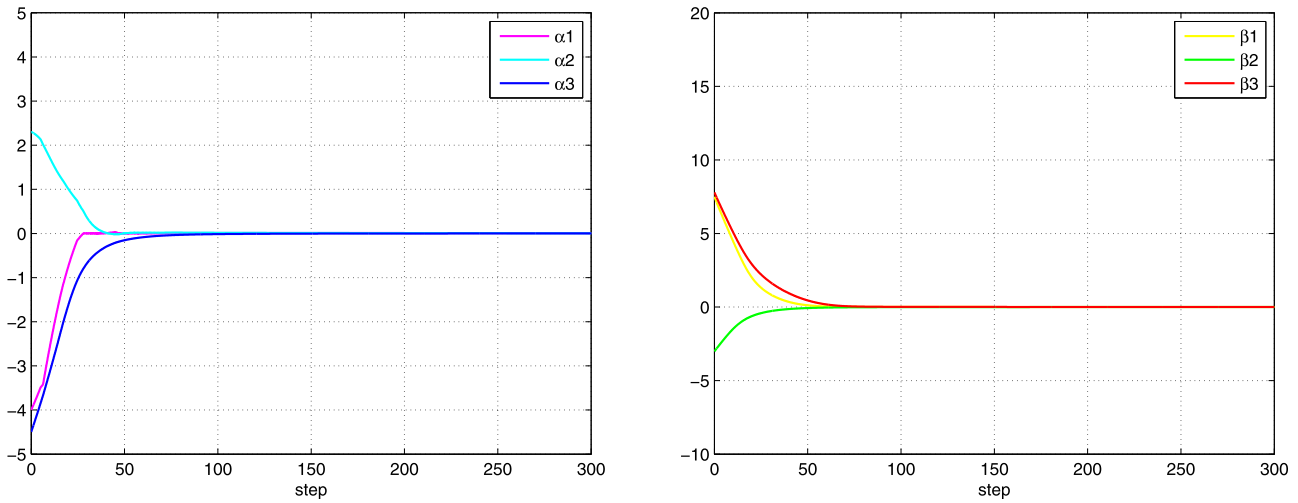


Fig. 4. The synchronization error evolution of state variables $\alpha(t)$ and $\beta(t)$ with order 0.97 in Example 4.2.

On the other hand, when $p = 0.97$, it simply to verify that $\mathcal{K} = 0.215$, $\mathcal{N} = 1.59$. Based on the inequality (19), we obtain the upper bound of settling time value t is $t < 0.818$, while we take $t = 0.5$, then

$$\sqrt{\frac{12 + [6\mathcal{N} + \mathcal{K}(2 + \mathcal{N})]e^{(2+\mathcal{N})t}}{[2 + \mathcal{N}]}} = 4.542, \quad \frac{\varepsilon}{\epsilon} = 50, \quad t \in \mathcal{M}. \quad (32)$$

That is $t \in \mathcal{M} = [0, 0.818]$, obviously the inequality holds. Based on the above numerical simulation results, definition 2.5 and Theorem 3.4 that the master system (3) and the slave system (5) can reach finite-time synchronization based on the controller (6). In the numerical simulations, the following initial values are taken: $(q_1(t), q_2(t), q_3(t), \hat{q}_1(t), \hat{q}_2(t), \hat{q}_3(t)) = (2.5, -1.3, 0.75, -1.5, 1, -3.75)$ and $(r_1(t), r_2(t), r_3(t), \hat{r}_1(t), \hat{r}_2(t), \hat{r}_3(t)) = (-3.15, 0.92, -4.7, 4.35, -2.1, 3.1)$. The synchronization error evolution of state variables $\alpha(t)$ and $\beta(t)$ are presented in Figs. 3–4.

Remark 4.3. The authors in [48] stated that the convergence rate is affected by the order of the fractional dynamical system. The convergence rate receives quicker to the aspect of growing fractional order p . In Fig 3 and Fig 4, the convergence speed of $p = 0.97$ is faster than that of $p = 0.46$, which coincides with their outcomes.

5. Conclusion

In this article, the problem on finite-time synchronization of time-varying delay fractional order memristor-based competitive neural networks (TFMCNNs) with leakage delay has been extensively studied. Compared to existing works, both two delay terms such as time varying delay and leakage delay are taken into consideration, which is more challenging to investigate the master-slave finite-time synchronization issue of different fractional order $0 < p < \frac{1}{2}$ and $\frac{1}{2} \leq p < 1$ and it leads to meaningful in practice. As a view finite time controller, a series of new conditions are obtained to assure the finite-time synchronization of TMFCNNs with leakage delay by using the Caputo properties, under the framework of Fillipov's solution and differential inclusion theory, and some inequality techniques. Additionally, two numerical examples are given to testify the effectiveness of proposed results, and we may estimate the upper bounds of settling

time. At the same time, our obtained results can be easily improved to the BAM-type of neural network model, and these works will be considered in the future direction.

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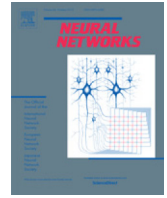
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Improved exponential convergence result for generalized neural networks including interval time-varying delayed signals

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ABSTRACT

This article examines the exponential stability analysis problem of generalized neural networks (GNNs) including interval time-varying delayed states. A new improved exponential stability criterion is presented by establishing a proper Lyapunov–Krasovskii functional (LKF) and employing new analysis theory. The improved reciprocally convex combination (RCC) and weighted integral inequality (WII) techniques are utilized to obtain new sufficient conditions to ascertain the exponential stability result of such delayed GNNs. The superiority of the obtained results is clearly demonstrated by numerical examples.

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1. Introduction

In the past two decades, neural networks (NNs) have been extensively studied, and they have been applied and extended to numerous engineering systems, including image processing, communication, fault diagnosis, fixed-point computations, parallel computations, and industrial automation (Haykin, 1994; Pawlus, Karimi, & Robbersmyr, 2014). During the implementation and application of NNs, time-delayed signals are unavoidable because of the inherent finite signal transfer time between the neurons. The existence of time-delayed signals in systems may cause divergence, instability, and the gradual degradation of the system performance. Thus, time-delayed NNs have received a great deal of research attention (Ahn, 2013, 2014; Ahn, Shi, & Wu, 2015; Choi, Ahn, Shi, Wu, & Lim, 2016; Karimi & Maass, 2009; Liu, Ma, & Mahmoud, 2012; Syed Ali, Saravanakumar, & Cao, 2016; Zhang, Shi, Nguang, Zhang, & Karimi, 2014; Zhou, Li, Wu, Wang, & Ahn, 2016). Dynamic, passivity, and dissipativity analysis, filtering and control design (Ahn, Shi, & Basin, 2015), and the state estimation (Ahn, Shi, & Basin, 2016) of NNs have been widely investigated in the past few decades (Ahn, 2013; Ahn, Shi, Agarwal, & Xu, 2016; Choi, Ahn, Shi, Lim, & Song, 2015; Liu et al., 2012; Saravanakumar, Syed Ali, & Hua, 2016; Syed Ali et al., 2016). As noted in

Xu, Qiao, Peng, and Zhang (2004), the NN model can be classified into two categories: static NNs (Kwon, Park, Park, Lee, & Cha, 2014) and local field NNs (Niamsup, Ratchagit, & Phat, 2015). Through some assumptions, the above two NN models can be equivalently transferred from one to another. However, in many applications, these assumptions do not hold (Liang & Cao, 2006). Thus, static NNs and local field NNs cannot be equivalent, and these two models have been studied separately. In Zhang and Han (2011), Zhang et al. constructed a unified NN model called a generalized neural network (GNN). Very recently, some stability problems for delayed GNNs have been established in Liu, Lee, Kwon, and Park (2015), Samidurai, Manivannan, Ahn, and Karimi (2016), Saravanakumar, Syed Ali, Ahn, Karimi, and Shi (2016), Zeng, He, Wu, and Xiao (2015) and Zhang, He, Jiang, Wu, and Wu (2014). In particular, the stability analysis problem of GNNs including two additive delayed signals was studied in Zhang, He et al. (2014). In Zeng et al. (2015), the stability analysis of a delayed GNN was proposed. In Liu et al. (2015), the improved stability criteria for delayed GNNs were studied. Very recently, the stability analysis problem was revisited for Markovian jump delayed GNNs in Saravanakumar, Syed Ali, Ahn et al. (2016).

The stability analysis problems for NNs have been the main focus of various research studies. In recent years, asymptotic, complete, and global exponential stability problems have been investigated for NNs (Weera & Niamsup, 2016; Zhu & Cao, 2011). As discussed in Liao, Chen, and Sanchez (2002), when investigating the exponential stability criteria, the exponential convergence rate can be utilized in ascertaining the speed of neural

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calculations. For this reason, studying the exponential stability has practical significance, and it is useful to estimate and ensure the exponential convergence rate of NNs. Hence, analyzing the exponential stability problems for delayed NNs has some value (Hua, Yang, Yan, & Guan, 2011; Kao, Shi, Xie, & Karimi, 2015; Karimi & Gao, 2010; Mahmoud & Xia, 2011; Rajchakit, Niamsup, & Rajchakit, 2013; Weera & Niamsup, 2016). To the best of our knowledge, the exponential stability problem of delayed GNNs has not yet been completely considered in the literature. Hence, the main goal of this work is to analyze the exponential stability of GNNs including interval time-varying delays.

From the previous statements, the exponential stability analysis of delayed GNNs is considered. Suitable Lyapunov–Krasovskii functionals (LKFs) are constructed, and then improved reciprocally convex combination (RCC) and weighted integral inequality (WII) techniques are used to obtain the linear matrix inequalities (LMIs). A novel procedure using the lower and higher bound information of the delayed signal is used to obtain sufficient conditions of the exponential stability for such GNNs. Additionally, numerical examples are added to demonstrate the usefulness and superiority of the obtained results.

Notations: $\mathbb{R}^n$ denotes the n -dimensional Euclidean space, and $\mathbb{R}^{n \times n}$ denotes the set of all $n \times n$ real matrices. $*$ denotes the term that is induced by symmetry. X^T denotes the matrix transpose of X . $Y > 0$ denotes that Y is a real symmetric positive definite matrix. I denotes the identity matrix with appropriate dimensions and the Kronecker product is denoted by $\otimes$.

2. Preliminaries

Consider the delayed GNNs:

$$\begin{aligned} \dot{h}(t) &= -Dh(t) + W_0g(W_2h(t)) + W_1g(W_2h(t - \chi(t))) + \mathbb{J}, \\ x(t) &= \phi(t), \quad \forall t \in [-\chi_2, 0], \end{aligned} \quad (1)$$

where $h(t) = [h_1(t), h_2(t), \dots, h_n(t)]^T \in \mathbb{R}^n$, $h_j(t)$ is the neuron state vector signal of the j th neuron, and $g(W_2h(t)) = [g(W_2h_1(t)), \dots, g(W_2h_n(t))]^T \in \mathbb{R}^n$ represents the neuron activation function vector. $D = \text{diag}\{d_1, \dots, d_n\}$ is a positive diagonal matrix with $d_j > 0, j = 1, 2, \dots, n$. Matrices W_0, W_1 , and W_2 are the connection weighting matrices. $\mathbb{J} = [\mathbb{J}_1, \dots, \mathbb{J}_n]^T \in \mathbb{R}^n$ is a bias vector. $\chi(t)$ is a time-varying delayed signal and satisfies

$$0 < \chi_1 \leq \chi(t) \leq \chi_2, \quad \dot{\chi}(t) \leq \ell, \quad (2)$$

where χ_1, χ_2 and ℓ are positive constants.

Assumption (A1) (Liu, Wang, & Liu, 2006). $g_j(t) (j = 1, 2, \dots, n)$ is continuous and bounded function, and satisfies:

$$z_j^- \leq \frac{g_j(W_{2j}u) - g_j(W_{2j}v)}{W_{2j}u - W_{2j}v} \leq z_j^+, \quad (3)$$

where $W_{2j} = [W_{21}, \dots, W_{2n}]$, $g_j(0) = 0$, $L_1 = \text{diag}(z_1^-, z_2^-, \dots, z_n^-)$, and $L_2 = \text{diag}(z_1^+, z_2^+, \dots, z_n^+)$ are some constants and $u, v \in \mathbb{R}, u \neq v$.

Remark 2.1. The above assumption was first introduced in Liu et al. (2006). The constants z^- and z^+ in the assumption can be negative, positive, or zero. Thus, the activation functions could be non-monotonic, bounded, and continuous. Currently, many results on the stability of NNs have such lower restrictions. Therefore, this assumption is weaker than the assumptions in Mahmoud and Xia (2011), Niamsup et al. (2015) and Rajchakit et al. (2013).

Assume $h^* = [h_1^*, h_2^*, \dots, h_n^*]^T$ is an equilibrium point of GNN (1). Now we derive the following delayed GNN by using the

transformation $x(t) = h(t) - h^*$:

$$\begin{aligned} \dot{x}(t) &= -Dx(t) + W_0f(W_2x(t)) + W_1f(W_2x(t - \chi(t))), \\ x(t) &= \phi_x(t), \quad \forall t \in [-\chi_2, 0], \end{aligned} \quad (4)$$

where $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathbb{R}^n$, $x_j(t)$ is the state vector signal of the transformed system and $f(W_2x(\cdot)) = [f_1(W_{21}x_1(\cdot)), \dots, f_n(W_{2n}x_n(\cdot))]^T$, $f(W_2x(t)) = g(W_2(x(t) + h^*))g(W_2h^*)$, $\phi_x(t) = \phi(t) - \phi^*$. The following inequality is obtained from Assumption (A1)

$$z_j^- \leq \frac{f_j(W_{2j}(u))}{W_{2j}u} \leq z_j^+, \quad (5)$$

where $u \in \mathbb{R}, u \neq 0$.

Remark 2.2. Many important benefits of NNs, such as convergence issues, oscillation, and qualitative properties of stability, were investigated in Hou and Qian (1998) and Li, Michel, and Porod (1988). These applications depend on a unique equilibrium point or equilibrium point existence and its stability. Thus, the stability analysis of NNs has received the attention of many researchers. NNs with time delays have been a highly studied subject in the literature, and different sufficient conditions for stability of the equilibrium point NNs have been developed (see, for example, Arik, 2010).

Lemma 2.3 (Hien & Trinh, 2016). Given scalar $\alpha > 0$, matrix $R > 0$ with $n \times n$ dimensions, and function $\phi \in C([a, b], \mathbb{R}^n)$, satisfies the following relation:

$$\begin{aligned} &\int_a^b e^{\alpha(s-b)} \phi^T(s) R \phi(s) ds \\ &\geq \frac{\alpha}{\gamma_0} \left(\int_a^b \phi(s) ds \right)^T R \left(\int_a^b \phi(s) ds \right) \\ &\quad + \frac{\alpha}{\rho_0} \zeta^T (K_1^T K_1 \otimes R) \zeta \end{aligned}$$

where

$$\begin{aligned} \bar{\rho}_0 &= \frac{\bar{\gamma}_0}{\bar{\gamma}_1^2} [\bar{\gamma}_0^2 - (\alpha l)^2 e^{\alpha l}], \quad \bar{\gamma}_0 = e^{\alpha l} - 1, \\ \bar{\gamma}_1 &= \bar{\gamma}_0 - \alpha l, \quad l = b - a, \\ \zeta &= \left[\left(\int_a^b \phi(s) ds \right)^T \quad \left(\int_a^b \int_s^b \phi(u) du ds \right)^T \right]^T, \\ K_1 &= \begin{bmatrix} 1 & -\alpha \bar{\gamma}_0 / \bar{\gamma}_1 \end{bmatrix}. \end{aligned}$$

Remark 2.4. Based on the exponential function for Taylor series, the following facts can be found:

$$\lim_{\alpha \rightarrow 0^+} \frac{\alpha}{\bar{\rho}_0} = \frac{3}{l}, \quad \lim_{\alpha \rightarrow 0^+} K_1 = \bar{K}_1 = \begin{bmatrix} 1 & -\frac{2}{l} \end{bmatrix}.$$

Hence, we can obtain the following results when α tends to zero

$$\int_a^b \phi^T(s) R \phi(s) ds \geq \frac{1}{l} \zeta^T (K_0^T K_0 + 3 \bar{K}_1^T \bar{K}_1) \otimes R \zeta, \quad (6)$$

where $K_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Lemma 2.5 (Improved RCC (Seuret, Gouaisbaut, & Fridman, 2013)). Given matrices $R_1 \in \mathbb{R}_{>0}^{n \times n}$, $R_2 \in \mathbb{R}_{>0}^{r \times r}$, if there exists an appropriately dimensioned matrix $Y \in \mathbb{R}^{n \times r}$ satisfying $\begin{bmatrix} R_1 & Y \\ * & R_2 \end{bmatrix} \geq 0$ such that the

$$\begin{aligned}\eta(t) &\triangleq \begin{bmatrix} x^T(t) & \int_{t-\chi_1}^t x^T(s)ds & \int_{t-\chi_2}^{t-\chi_1} x^T(s)ds \end{bmatrix}^T, \\ \xi(t) &\triangleq \begin{bmatrix} x^T(t) & x^T(t-\chi_1) & x^T(t-\chi(t)) & x^T(t-\chi_2) & f^T(W_2x(t)) & f^T(W_2x(t-\chi(t))) \\ \frac{1}{\chi_1} \int_{t-\chi_1}^t x^T(s)ds & \frac{1}{\chi(t)-\chi_1} \int_{t-\chi(t)}^{t-\chi_1} x^T(s)ds & \frac{1}{\chi_2-\chi(t)} \int_{t-\chi_2}^{t-\chi(t)} x^T(s)ds \end{bmatrix}^T, \\ e_r &\triangleq [0_{n \times (r-1)n} \quad I_n \quad 0_{n \times (9-r)n}]^T, \quad r = 1, 2, \dots, 9.\end{aligned}$$

Box I.

following inequality

$$\begin{bmatrix} \frac{1}{\delta} R_1 & 0 \\ * & \frac{1}{1-\delta} R_2 \end{bmatrix} \geq \begin{bmatrix} R_1 & Y \\ * & R_2 \end{bmatrix}$$

holds for all $\delta \in (0, 1)$.

Definition 2.6 (Ji, He, Wu, & Zhang, 2015). Consider GNN (1). If there exist some positive constants $\kappa \geq 1, \alpha > 0$

$$\|x(t)\| \leq \kappa \psi e^{-\alpha t}, \quad \forall t \geq 0, \quad \psi = \sup_{-\chi_2 \leq \theta \leq 0} \|x(\theta)\|,$$

then the origin of GNN (1) is said to be exponentially stable, where α can be called the exponential convergence rate.

Remark 2.7. GNN (1) contains both local field NNs and static NNs. The following NN models are different special cases derived from GNN (4):

(a) when $W_0 = 0, W_1 = I$, we have the following static NN:

$$\dot{x}(t) = -Dx(t) + f(W_2x(t - \chi(t))), \quad (7)$$

(b) when $W_2 = I$, we have the following local field NN:

$$\dot{x}(t) = -Dx(t) + W_0f(x(t)) + W_1f(x(t - \chi(t))). \quad (8)$$

3. Main results

We introduce the definitions in Box I for simplicity.

Based on the above definitions, we present the main result of this paper in the next theorem.

Theorem 3.1. Given scalars $0 < \chi_1 < \chi_2, \ell > 0$, GNN (1) is exponentially stable if there exist positive definite matrices $P \in \mathbb{R}^{3n \times 3n}, Q_1, Q_2, Q_3, R, S \in \mathbb{R}^{n \times n}, \Gamma_p = \text{diag}\{\gamma_{p1}, \gamma_{p2}, \dots, \gamma_{pn}\} \in \mathbb{R}^{n \times n}, (p = 1, 2, 3), \Lambda_q = \text{diag}\{\Lambda_{q1}, \Lambda_{q2}, \dots, \Lambda_{qn}\} \in \mathbb{R}^{n \times n} (q = 1, 2)$ and any matrix $X \in \mathbb{R}^{2n \times 2n}$ satisfying

$$\Upsilon(\chi) < 0, \quad (9)$$

$$\Pi = \begin{bmatrix} \tilde{S} & X \\ 0 & \tilde{S} \end{bmatrix} \geq 0,$$

where

$$\Upsilon(\chi) = \Upsilon_0(\chi) + \sum_{z=1}^5 \Upsilon_z - e^{-\alpha\chi_{21}} \Psi^T \Pi \Psi,$$

$$\Upsilon_0(\chi) = \Phi^T(\chi) P \Phi_0 + \Phi_0^T P \Phi(\chi) + \alpha \Phi^T(\chi) P \Phi(\chi),$$

$$\Phi(\chi) = \begin{bmatrix} e_1 \\ \chi_1 e_7 \\ (\chi(t) - \chi_1) e_8 + (\chi_2 - \chi(t)) e_9 \end{bmatrix},$$

$$\Phi_0 = \begin{bmatrix} \hat{A} \\ e_1 - e_2 \\ e_2 - e_4 \end{bmatrix},$$

$$\Phi_1 = \begin{bmatrix} e_1 - e_2 \\ \chi_1(e_1 - e_7) \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_8 \end{bmatrix},$$

$$\Phi_3 = \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_9 \end{bmatrix},$$

$$\Upsilon_1 = [2[e_5 - L_1 W_2 e_1]^T \Lambda_1 \hat{A} + 2[L_2 W_2 e_1 - e_5]^T \Lambda_2 \hat{A}],$$

$$\Upsilon_2 = e_1^T Q_1 e_1 - e^{-\alpha\chi_1} e_2^T Q_1 e_2 + e^{-\alpha\chi_1} e_2^T Q_2 e_2 - e^{-\alpha\chi_1} e_4^T Q_2 e_4 + e_5^T Q_3 e_5 - (1 - \ell) e^{-\alpha\chi_2} e_6^T Q_3 e_6,$$

$$\Upsilon_3 = \hat{A}^T [\chi_1^2 R + \chi_{21}^2 e^{\alpha\chi_{21}} S] \hat{A},$$

$$\Upsilon_4 = -\left(\frac{\alpha\chi_1}{\gamma_0} (e_1 - e_2)^T R (e_1 - e_2) + \frac{\alpha\chi_1}{\rho_0} \Phi_1^T (K^T K \otimes R) \Phi_1\right),$$

$$\begin{aligned}\Upsilon_5 &= -2e_1^T W_2^T L_1 \Gamma_1 L_2 W_2 e_1 + e_1^T W_2^T (L_1 + L_2) \Gamma_1 e_5 \\ &\quad + e_5^T \Gamma_1 (L_1 + L_2) W_2 e_1 - 2e_5^T \Gamma_1 e_5 \\ &\quad - 2e_2^T W_2^T L_1 \Gamma_2 L_2 W_2 e_2 + e_2^T W_2^T (L_1 + L_2) \Gamma_2 e_6 \\ &\quad + e_6^T \Gamma_2 (L_1 + L_2) W_2 e_2 - 2e_6^T \Gamma_2 e_6 \\ &\quad - 2[e_5 - e_6 - L_1 W_2 (e_1 - e_2)]^T \Gamma_3 \\ &\quad \times [e_5 - e_6 - L_2 W_2 (e_1 - e_2)],\end{aligned}$$

$$\rho_0 = \frac{\gamma_0}{\gamma_1^2} [\gamma_0^2 - (\alpha\chi_1)^2 e^{\alpha\chi_1}], \quad \gamma_0 = e^{\alpha\chi_1} - 1,$$

$$\gamma_1 = \gamma_0 - \alpha\chi_1, \quad K = \begin{bmatrix} 1 & -\alpha\gamma_0/\gamma_1 \end{bmatrix},$$

$$\Psi = [\Phi_2^T \quad \Phi_3^T]^T, \quad \tilde{S} = \text{diag}\{S, 3S\},$$

$$\hat{A} = -Ae_1 + W_0 e_5 + W_1 e_6, \quad \chi_{21} = \chi_2 - \chi_1.$$

Proof. Choose the following LKF candidate:

$$\begin{aligned}V(x_t) &= \eta^T(t) P \eta(t) + 2 \sum_{j=1}^n \lambda_{1j} \int_0^{W_{2j}x(t)} (f_j(s) - z_j^- s) ds \\ &\quad + 2 \sum_{j=1}^n \lambda_{2j} \int_0^{W_{2j}x(t)} (z_j^+ s - f_j(s)) ds \\ &\quad + \int_{t-\chi_1}^t e^{\alpha(s-t)} x^T(s) Q_1 x(s) ds \\ &\quad + \int_{t-\chi_2}^{t-\chi_1} e^{\alpha(s-t)} x^T(s) Q_2 x(s) ds \\ &\quad + \int_{t-\chi(t)}^t e^{\alpha(s-t)} f^T(W_2x(s)) Q_3 f(W_2x(s)) ds \\ &\quad + \chi_1 \int_{-\chi_1}^0 \int_{t+s}^t e^{\alpha(\theta-t)} \dot{x}^T(\theta) R \dot{x}(\theta) d\theta ds \\ &\quad + \chi_{21} \int_{-\chi_2}^{-\chi_1} \int_{t+s}^t e^{\alpha(\theta+\chi_1-t)} \dot{x}^T(\theta) S \dot{x}(\theta) d\theta ds.\end{aligned} \quad (10)$$

From (10), we obtain $V(x_t) \geq \lambda_{\min}(P) \|x(t)\|^2$, which leads to the time derivative of $V(x_t)$ as

$$\begin{aligned}\dot{V}(x_t) + \alpha V(x_t) &\leq \xi^T(t) \Upsilon_0(\chi) \xi(t) + \xi^T [2[e_5 - L_1 W_2 e_1]^T \Lambda_1 \hat{A} \\ &\quad + 2[L_2 W_2 e_1 - e_5]^T \Lambda_2 \hat{A}] \xi(t) + \xi^T [e_1^T Q_1 e_1\end{aligned}$$

$$\begin{aligned}
& -e^{-\alpha\chi_1}e_2^T Q_1 e_2 + e^{-\alpha\chi_1}e_2^T Q_2 e_2 \\
& -e^{-\alpha\chi_1}e_4^T Q_2 e_4 + e_5^T Q_3 e_5 \\
& - (1-\ell)e^{-\alpha\chi_2}e_6^T Q_3 e_6] \xi(t) + \xi^T(t) \widehat{A}^T \\
& \times [\chi_1^2 R + \chi_{21}^2 e^{\alpha\chi_{21}} S] \widehat{A} \xi(t) \\
& - \chi_1 \int_{t-\chi_1}^t e^{\alpha(s-t)} \dot{x}^T(s) R \dot{x}(s) ds \\
& - \chi_{21} \int_{t-\chi_2}^{t-\chi(t)} e^{\alpha(s+\chi_1-t)} \dot{x}^T(s) S \dot{x}(s) ds \\
& - \chi_{21} \int_{t-\chi(t)}^{t-\chi_1} e^{\alpha(s+\chi_1-t)} \dot{x}^T(s) S \dot{x}(s) ds. \quad (11)
\end{aligned}$$

Through Lemma 2.3, we obtain

$$- \chi_1 \int_{t-\chi_1}^t e^{\alpha(s-t)} \dot{x}^T(s) R \dot{x}(s) ds \leq \xi^T(t) \Upsilon_4 \xi(t).$$

Through Lemma 2.5 and relation (6), we obtain

$$\begin{aligned}
& - \chi_{21} \int_{t-\chi_2}^{t-\chi(t)} e^{\alpha(s+\chi_1-t)} \dot{x}^T(s) S \dot{x}(s) ds \\
& - \chi_{21} \int_{t-\chi(t)}^{t-\chi_1} e^{\alpha(s+\chi_1-t)} \dot{x}^T(s) S \dot{x}(s) ds \\
& \leq - \frac{\chi_{21} e^{-\alpha\chi_{21}}}{\chi(t) - \chi_1} \zeta_1^T(t) \widetilde{S} \zeta_1(t) - \frac{\chi_{21} e^{-\alpha\chi_{21}}}{\chi_2 - \chi(t)} \zeta_2^T(t) \widetilde{S} \zeta_2(t) \\
& \leq -e^{-\alpha\chi_{21}} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}^T \begin{bmatrix} \frac{\chi_{21}}{\chi(t) - \chi_1} \widetilde{S} & 0 \\ 0 & \frac{\chi_{21}}{\chi_2 - \chi(t)} \widetilde{S} \end{bmatrix} \\
& \times \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} \\
& = -e^{-\alpha\chi_{21}} \xi^T(t) \Psi^T \Pi \Psi \xi(t),
\end{aligned}$$

where $\zeta_1(t) = \Phi_2 \xi(t)$, $\zeta_2(t) = \Phi_3 \xi(t)$. From (A1), we have

$$0 \leq 2[f(W_2 x(t)) - L_1 W_2 x(t)]^T \Gamma_1 [L_2 W_2 x(t) - f(W_2 x(t))], \quad (12)$$

$$\begin{aligned}
0 & \leq 2[f(W_2 x(t - \chi(t))) - L_1 W_2 x(t - \chi(t))]^T \Gamma_2 \\
& \times [L_2 W_2 x(t - \chi(t)) - f(W_2 x(t - \chi(t)))], \quad (13)
\end{aligned}$$

$$\begin{aligned}
0 & \leq 2[f(W_2 x(t)) - f(W_2 x(t - \chi(t)))]^T \Gamma_3 [f(W_2 x(t)) - f(W_2 x(t - \chi(t)))] \\
& - L_2 (W_2 x(t) - W_2 x(t - \chi(t))). \quad (14)
\end{aligned}$$

Combining (11)–(14), we obtain

$$\dot{V}(x_t) + \alpha V(x_t) \leq \xi^T(t) \Upsilon(\chi) \xi(t), \quad (15)$$

where $\Upsilon(\chi)$ is defined in (9). From (9) we obtain $\dot{V}(x_t) + \alpha V(x_t) \leq 0$. By integrating both sides, we obtain $V(x_t) \leq V(\phi) e^{-\alpha t}$. Then, by Definition 2.6, it shows that the equilibrium point is exponentially stable. $\square$

In Theorem 3.1, the exponential stability for delayed GNNs is studied. Now, we establish the next corollary for showing the reduced conservatism of the asymptotic stability condition. In Corollary 3.2, the asymptotic stability problem is considered for delayed GNN (1), and some examples are given to show the reduced conservatism.

Corollary 3.2. Given scalars $0 < \chi_1 < \chi_2, \ell > 0$, GNN (1) is asymptotically stable when there exist positive definite matrices $P \in \mathbb{R}^{3n \times 3n}$, $Q_1, Q_2, Q_3, R, S \in \mathbb{R}^{n \times n}$, $\Gamma_p = \text{diag}\{\gamma_{p1}, \gamma_{p2}, \dots, \gamma_{pn}\} \in$

$\mathbb{R}^{n \times n}$, ($p = 1, 2, 3$), $\Lambda_q = \text{diag}\{\Lambda_{q1}, \Lambda_{q2}, \dots, \Lambda_{qn}\} \in \mathbb{R}^{n \times n}$ ($q = 1, 2$), and a matrix $X \in \mathbb{R}^{2n \times 2n}$ satisfying

$$\begin{aligned}
& \Sigma(\chi) < 0, \\
& \Pi = \begin{bmatrix} \widetilde{S} & X \\ 0 & \widetilde{S} \end{bmatrix} \geq 0, \quad (16)
\end{aligned}$$

where

$$\begin{aligned}
& \Sigma(\chi) = \Sigma_0(\chi) + \sum_{z=1}^5 \Sigma_z - \Phi^T \Pi \Phi, \\
& \Sigma_0(\chi) = \Omega^T(\chi) P \Omega_0 + \Omega_0^T P \Omega(\chi), \\
& \Omega(\chi) = \begin{bmatrix} e_1 \\ \chi_1 e_7 \\ (\chi(t) - \chi_1) e_8 + (\chi_2 - \chi(t)) e_9 \end{bmatrix}, \\
& \Omega_0 = \begin{bmatrix} \widehat{A} \\ e_1 - e_2 \\ e_2 - e_4 \end{bmatrix}, \\
& \Omega_1 = \begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_7 \end{bmatrix}, \\
& \Omega_2 = \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_8 \end{bmatrix}, \quad \Omega_3 = \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_9 \end{bmatrix}, \\
& \Sigma_1 = [2[e_5 - L_1 W_2 e_1]^T \Lambda_1 \widehat{A} + 2[L_2 W_2 e_1 - e_5]^T \Lambda_2 \widehat{A}], \\
& \Sigma_2 = e_1^T Q_1 e_1 - e_2^T Q_1 e_2 + e_2^T Q_2 e_2 - e_4^T Q_2 e_4 \\
& \quad + e_5^T Q_3 e_5 - (1-\ell) e_6^T Q_3 e_6, \\
& \Sigma_3 = \widehat{A}^T [\chi_1^2 R + \chi_{21}^2 S] \widehat{A}, \\
& \Sigma_4 = -\Omega_1^T \widetilde{R} \Omega_1, \quad \widetilde{R} = \text{diag}\{R, 3R\}, \\
& \Sigma_5 = -2e_1^T W_2^T L_1 \Gamma_1 L_2 W_2 e_1 + e_1^T W_2^T (L_1 + L_2) \Gamma_1 e_5 \\
& \quad + e_5^T \Gamma_1 (L_1 + L_2) W_2 e_1 - 2e_5^T \Gamma_1 e_5 \\
& \quad - 2e_2^T W_2^T L_1 \Gamma_2 L_2 W_2 e_2 + e_2^T W_2^T (L_1 + L_2) \Gamma_2 e_6 \\
& \quad + e_6^T \Gamma_2 (L_1 + L_2) W_2 e_2 - 2e_6^T \Gamma_2 e_6 \\
& \quad - 2[e_5 - e_6 - L_1 W_2 (e_1 - e_2)]^T \\
& \quad \times \Gamma_3 [e_5 - e_6 - L_2 W_2 (e_1 - e_2)], \\
& \Phi = [\Omega_2^T \quad \Omega_3^T]^T, \quad \widetilde{S} = \text{diag}\{S, 3S\}, \\
& \widehat{A} = -Ae_1 + W_0 e_5 + W_1 e_6, \quad \chi_{21} = \chi_2 - \chi_1.
\end{aligned}$$

Proof. Choose the LKF candidate:

$$\begin{aligned}
V(x_t) &= \eta^T(t) P \eta(t) + 2 \sum_{j=1}^n \lambda_{1j} \int_0^{W_{2j} x(t)} (f_j(s) - z_j^- s) ds \\
&+ 2 \sum_{j=1}^n \lambda_{2j} \int_0^{W_{2j} x(t)} (z_j^+ s - f_j(s)) ds \\
&+ \int_{t-\chi_1}^t x^T(s) Q_1 x(s) ds + \int_{t-\chi_2}^{t-\chi_1} x^T(s) Q_2 x(s) ds \\
&+ \int_{t-\chi(t)}^t f^T(W_2 x(s)) Q_3 f(W_2 x(s)) ds \\
&+ \chi_1 \int_{-\chi_1}^0 \int_{t+s}^t \dot{x}^T(\theta) R \dot{x}(\theta) d\theta ds \\
&+ \chi_{21} \int_{-\chi_2}^{-\chi_1} \int_{t+s}^t \dot{x}^T(\theta) S \dot{x}(\theta) d\theta ds. \quad (17)
\end{aligned}$$

Now, we obtain the time derivative of $V(x_t)$,

$$\begin{aligned} \dot{V}(x_t) &\leq \xi^T(t) \Sigma_0(\chi) \xi(t) + \xi^T [2e_5 - L_1 W_2 e_1]^T A_1 \hat{A} \\ &\quad + 2[L_2 W_2 e_1 - e_5]^T A_2 \hat{A} \xi(t) \\ &\quad + \xi^T [e_1^T Q_1 e_1 - e_2^T Q_1 e_2 + e_2^T Q_2 e_2 \\ &\quad - e_4^T Q_2 e_4 + e_5^T Q_3 e_5 - (1 - \ell) e_6^T Q_3 e_6] \xi(t) \\ &\quad + \xi^T(t) \hat{A}^T [\chi_1^2 R + \chi_{21}^2 S] \hat{A} \xi(t) \\ &\quad - \chi_1 \int_{t-\chi_1}^t \dot{x}^T(s) R \dot{x}(s) ds \\ &\quad - \chi_{21} \int_{t-\chi_2}^{t-\chi(t)} \dot{x}^T(s) S \dot{x}(s) ds \\ &\quad - \chi_{21} \int_{t-\chi(t)}^{t-\chi_1} \dot{x}^T(s) S \dot{x}(s) ds. \end{aligned} \quad (18)$$

Using the inequality in (6), we obtain

$$-\chi_1 \int_{t-\chi_1}^t e^{\alpha(s-t)} \dot{x}^T(s) R \dot{x}(s) ds \leq \xi^T(t) \Sigma_4 \xi(t).$$

Through inequality (6) and Lemma 2.5, we obtain

$$\begin{aligned} &-\chi_{21} \int_{t-\chi_2}^{t-\chi(t)} \dot{x}^T(s) S \dot{x}(s) ds - \chi_{21} \int_{t-\chi(t)}^{t-\chi_1} \dot{x}^T(s) S \dot{x}(s) ds \\ &\leq -\frac{\chi_{21}}{\chi(t) - \chi_1} \zeta_1^T(t) \tilde{S} \zeta_1(t) - \frac{\chi_{21}}{\chi_2 - \chi(t)} \zeta_2^T(t) \tilde{S} \zeta_2(t) \\ &\leq -\begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}^T \begin{bmatrix} \frac{\chi_{21}}{\chi(t) - \chi_1} \tilde{S} & 0 \\ 0 & \frac{\chi_{21}}{\chi_2 - \chi(t)} \tilde{S} \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} \\ &= -\xi^T(t) \Phi^T \Pi \Phi \xi(t), \end{aligned}$$

where $\zeta_1(t) = \Omega_2 \xi(t)$, $\zeta_2(t) = \Omega_3 \xi(t)$. From (A1), we have

$$0 \leq 2[f(W_2 x(t)) - L_1 W_2 x(t)]^T \Gamma_1 [L_2 W_2 x(t) - f(W_2 x(t))], \quad (19)$$

$$\begin{aligned} 0 &\leq 2[f(W_2 x(t - \chi(t))) - L_1 W_2 x(t - \chi(t))]^T \\ &\quad \times \Gamma_2 [L_2 W_2 x(t - \chi(t)) - f(W_2 x(t - \chi(t)))], \end{aligned} \quad (20)$$

$$\begin{aligned} 0 &\leq 2[f(W_2 x(t)) - f(W_2 x(t - \chi(t)))] \\ &\quad - L_1 (W_2 x(t) - W_2 x(t - \chi(t)))^T \Gamma_3 \\ &\quad \times [f(W_2 x(t)) - f(W_2 x(t - \chi(t)))] \\ &\quad - L_2 (W_2 x(t) - W_2 x(t - \chi(t))). \end{aligned} \quad (21)$$

Combining (18)–(21), we obtain

$$\dot{V}(x_t) \leq \xi^T(t) \Sigma(\chi) \xi(t), \quad (22)$$

where $\Sigma(\chi)$ is defined in (16). This completes the proof of Corollary 3.2. $\square$

Remark 3.3. LKFs (10) and (17) are not conservative. To avoid numerical computation complexity, LKFs (10) and (17) are constructed using a smaller number of decision variables (NDV) compared with the methods in Lee, Lee, and Park (2015), Raja, Zhu, Senthilraj, and Samidurai (2015), Saravanakumar, Syed Ali, Ahn et al. (2016), Sun and Chen (2013), Syed Ali, Arik, and Saravanakumar (2015), Wang, Ma, and Wen (2015), Wu, Lam, Su, and Chu (2012) and Yang, Ren, and Xie (2014). In particular, the NDVs in Lee et al. (2015), Raja et al. (2015), Saravanakumar, Syed Ali, Ahn et al. (2016), Sun and Chen (2013), Syed Ali et al. (2015), Wang et al. (2015), Wu et al. (2012) and Yang et al. (2014) are respectively, $\frac{(26n(n+1))}{2} + 22n^2$, $\frac{(11n(n+1))}{2} + 7n^2$, $\frac{(24n(n+1))}{2} + 12n^2$, $\frac{(20n(n+1))}{2} + 6n^2$, $\frac{(21n(n+1))}{2} + 4n^2$, $\frac{(23n(n+1))}{2} + 33n^2$, $\frac{(11n(n+1))}{2} + 23n^2$, and $\frac{(38n(n+1))}{2} + 22n^2$. The NDV in this paper is $\frac{(13n(n+1))}{2} + 3n^2$. In

Table 1

$\alpha_{\max}$ for various values of ℓ .

Methods	0	0.6	0.8	1.5	Unknown ℓ
Theorem 3.1	1.5234	1.3654	1.2157	1.1034	0.9871
Ref. Liu et al. (2012)	1.1609	0.9428	0.8759	0.8739	0.9426
Ref. Mahmoud and Xia (2011)	1.1605	0.9064	0.8381	0.8371	0.8359

Table 2

MAUB χ_2 for different values of χ_1 and α .

Methods	α	1	2	4	6
Ref. Liu et al. (2012)	0.05	1.8642	2.9492	4.8409	6.7416
Theorem 3.1		2.7698	3.8962	5.7865	7.5847
Ref. Liu et al. (2012)	0.1	1.6314	2.7981	4.6218	6.4718
Theorem 3.1		2.5234	3.6154	5.5257	7.2632
Ref. Liu et al. (2012)	0.2	1.4632	2.5712	4.3200	6.1321
Theorem 3.1		2.3984	3.4561	5.2157	6.9854

addition, through the new WII technique, the maximum allowable upper bounds (MAUBs) are obtained. Thus, our presented results are significantly less conservative than those of the existing approaches in Lee et al. (2015), Raja et al. (2015), Saravanakumar, Syed Ali, Ahn et al. (2016), Sun and Chen (2013), Syed Ali et al. (2015), Wang et al. (2015), Wu et al. (2012) and Yang et al. (2014).

Remark 3.4. In recent years, some researchers have used the integral inequality techniques, such as the RCC and Wirtinger's inequality techniques. These two inequalities are the best techniques to reduce conservatism, in which Jensen's inequality is one of the special cases. However, these techniques may not be accurate techniques for exponential stability problems. In this paper, we utilized the new WII for both single and double integral terms.

4. Examples

Example 4.1. Consider GNN (1) with the matrix parameters given in Mahmoud and Xia (2011):

$$D = \begin{bmatrix} 2.5 & 0 \\ 0 & 3.5 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & -1 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} -0.5 & -0.1 \\ 0.6 & 0.8 \end{bmatrix}, \quad W_2 = 0.8I_2,$$

$$L_1 = 0.998I_2, \quad L_2 = I_2.$$

In order to obtain exponential stability, we take $\ell = 1.2$, $\chi_1 = 1$, and $\chi_2 = 5$ and apply Theorem 3.1 to GNN (4), which results in feasible solutions to LMI (9) as follows:

$$P = \begin{bmatrix} 64.8237 & -16.6141 & 7.2589 & 0.5330 & 0.1672 & -0.0499 \\ -16.6141 & 224.2116 & -21.7835 & 24.1900 & -0.0993 & -0.0573 \\ 7.2589 & -21.7835 & 26.6280 & -6.3558 & 0.1544 & -0.0059 \\ 0.5330 & 24.1900 & -6.3558 & 61.6231 & -0.0297 & 0.1055 \\ 0.1672 & -0.0993 & 0.1544 & -0.0297 & 0.1277 & 0.0069 \\ -0.0499 & -0.0573 & -0.0059 & 0.1055 & 0.0069 & 0.1155 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 293.3602 & -96.9010 \\ -96.9010 & 795.0035 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 100.2312 & -13.4243 \\ -13.4243 & 295.8030 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 37.8974 & -71.2210 \\ -71.2210 & 272.3378 \end{bmatrix}, \quad R = \begin{bmatrix} 21.0194 & -0.5564 \\ -0.5564 & 16.2392 \end{bmatrix},$$

$$S = \begin{bmatrix} 6.2195 & 0.2961 \\ 0.2961 & 5.3239 \end{bmatrix}.$$

Therefore, equilibrium point h^* is unique, and GNN (1) is exponentially stable and $\alpha = 0.5638$.

Moreover, when $\chi_1 = 0.8$, $\chi_2 = 1$, and $W_2 = I_2$, $\alpha_{\max}$ is calculated for various values of ℓ , which is listed in Table 1. If $\chi(t)$ is not differentiable, then Table 2 shows the MAUB χ_2 for various values of χ_1 and α .

Table 3
MAUB χ_2 for different values of ℓ and α .

Methods	α	0.8	0.9
Ref. Liu et al. (2012) Theorem 3.1	0.03	36.5796 42.6321	25.0271 31.3562
Ref. Liu et al. (2012) Theorem 3.1	0.05	21.3416 28.3452	14.4101 21.5461
Ref. Liu et al. (2012) Theorem 3.1	0.642	16.2771 22.3849	10.8788 17.4156

Table 4
MAUB χ_2 for various values of ℓ when $\chi_1 = 1$.

ℓ	0.95	0.99
χ_2 by Syed Ali et al. (2015)	12.5249	9.5445
χ_2 by Raja et al. (2015)	13.8548	11.1142
χ_2 by Saravanakumar, Syed Ali, Ahn et al. (2016)	18.4786	14.6548
χ_2 by Corollary 3.2	19.3928	16.1321

Table 5
MAUB χ_2 for different values of ℓ when $\chi_1 = 1$.

ℓ	0.8	0.9	Unknown ℓ
χ_2 by Wang et al. (2015) ($N = 2$)	1.6201	1.3955	1.2363
χ_2 by Wang et al. (2015) ($N = 3$)	1.6895	1.4385	1.2617
χ_2 by Saravanakumar, Syed Ali, Ahn et al. (2016)	1.7732	1.6457	1.4765
χ_2 by Corollary 3.2	1.9238	1.7134	1.6528

Example 4.2. Consider NN (7) with the following matrix parameters:

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.1 & -0.2 \\ 0 & 0.1 \end{bmatrix}, \quad W_2 = I_2, \quad L_1 = O_2, \quad L_2 = I_2.$$

When $\chi_1 = 2$ and $W_2 = I_2$, the MAUB χ_2 for various values of ℓ and α are listed in Table 3. Thus, the developed result is better than that in Liu et al. (2012).

Example 4.3. Choose NN (8) and matrix values in Raja et al. (2015):

$$D = \begin{bmatrix} 0.8 & 0 \\ 0 & 5.3 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.1 \end{bmatrix},$$

$$W_0 = O_2, \quad z_1^- = z_2^- = 0, \quad z_1^+ = z_2^+ = 1.$$

By solving Example 4.3 through Corollary 3.2, when $\chi_1 = 1$, one can reach the MAUB χ_2 for various values of ℓ (see Table 4), which demonstrates the effectiveness of the proposed work. The results established in this paper are clearly better than those of existing approaches (Raja et al., 2015; Saravanakumar, Syed Ali, Ahn et al., 2016; Syed Ali et al., 2015).

Example 4.4. Choose NN (8) and matrix values in Wang et al. (2015), where

$$D = I_2, \quad W_0 = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1.5 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -2 & 0.5 \\ 0.5 & -2 \end{bmatrix},$$

$$z_1^- = z_2^- = 0, \quad z_1^+ = 0.4, \quad z_2^+ = 0.8.$$

When $\chi_1 = 1$, applying Corollary 3.2, we get the MAUB χ_2 for various values of ℓ , which is given in Table 5. It shows the superiority of our work.

Table 6
MAUB χ_2 for various values of ℓ when $\chi_1 = 0.5$.

ℓ	0.2	0.3	0.9	Unknown ℓ
Wu et al. (2012)	0.6551	0.5886	0.5885	0.5885
Sun and Chen (2013)	0.7515	0.6021	0.6021	0.6021
Yang et al. (2014)	0.7578	0.6053	0.6053	0.6053
Lee et al. (2015)	0.8006	0.6344	–	0.6169
Saravanakumar, Syed Ali, Ahn et al. (2016)	0.8413	0.7542	0.6547	0.6547
Corollary 3.2	0.9143	0.8429	0.7542	0.7542

Example 4.5. Consider the NN in (7) and the matrix values in Wu et al. (2012):

$$D = \text{diag}\{7.3458, 6.9987, 5.5949\},$$

$$W_2 = \begin{bmatrix} 13.6014 & -2.9616 & -0.6936 \\ 7.4736 & 21.6810 & 3.2100 \\ 0.7290 & -2.6334 & -20.1300 \end{bmatrix},$$

$$z_1^- = z_2^- = z_3^- = 0, \quad z_1^+ = 0.3680,$$

$$z_2^+ = 0.1795, \quad z_3^+ = 0.2876.$$

Solving Corollary 3.2, one can reach the MAUB χ_2 for various values of ℓ when $\chi_1 = 0.5$ (see Table 6). Hence, this method is much better than the methods in Lee et al. (2015), Saravanakumar, Syed Ali, Ahn et al. (2016), Sun and Chen (2013), Wu et al. (2012) and Yang et al. (2014).

Example 4.6. A real-time application for a biological NN is considered in this example. The biological NN has been introduced in *Escherichia coli* (Elowitz & Leibler, 2000) as a repressor (synthetic genetic regulatory network) mathematical design. Consider the time-delayed genetic regulatory network (GRN) as follows (see Wu et al., 2012):

$$\begin{cases} \dot{a}(t) = \mathcal{U}a(t) + \mathcal{V}g(b(t - \chi(t))) + I', \\ \dot{b}(t) = \mathcal{X}b(t) + \mathcal{Y}a(t - \chi(t)), \end{cases} \quad (23)$$

where $a(t) = [a_1(t), a_2(t), \dots, a_n(t)]^T$, $b(t) = [b_1(t), b_2(t), \dots, b_n(t)]^T$, $\mathcal{U} = \text{diag}\{q_1, \dots, q_n\} > 0$, $\mathcal{X} = \text{diag}\{m_1, \dots, m_n\} > 0$, $\mathcal{Y} = \text{diag}\{w_1, \dots, w_n\} > 0$, and $\mathcal{V}$ are known matrices. $I' = [I'_1, \dots, I'_n]^T$ and $g(b(t)) = [g_1(b_1(t)), \dots, g_n(b_n(t))]^T$ satisfy

$$0 \leq \frac{g_i(n_1) - g_i(n_2)}{n_1 - n_2} \leq u_p^+, \quad n_1 \neq n_2 \in \mathcal{R}. \quad (24)$$

Now consider the network (with $n = 3$). For any p , we select $g(v) = v^2/(1 + v^2)$, which implies $u_i^+ = 0.65 (i = 1, 2, 3)$, and

$$\mathcal{U} = \mathcal{X} = I_3, \quad \mathcal{Y} = \text{diag}\{0.6, 0.4, 0.5\},$$

$$\mathcal{V} = \begin{bmatrix} 0 & 0 & -0.8 \\ -0.6 & 0 & 0 \\ 0 & -0.6 & 0 \end{bmatrix}, \quad I' = \begin{bmatrix} 0.8 \\ 0.6 \\ 0.6 \end{bmatrix}.$$

The equilibrium point for (23) is assumed to be (a^*, b^*) . Using $\hat{a}(t) = a(t) - a^*$, $\hat{b}(t) = b(t) - b^*$, and $h(\hat{b}(t)) = g(\hat{b}(t) + b^*) - g(b^*)$, GRN (23) can be transformed into

$$\begin{cases} \dot{\hat{a}}(t) = \mathcal{U}\hat{a}(t) + \mathcal{V}h(\hat{b}(t - \chi(t))), \\ \dot{\hat{b}}(t) = \mathcal{X}\hat{b}(t) + \mathcal{Y}\hat{a}(t - \chi(t)), \end{cases} \quad (25)$$

letting

$$y(t) = \begin{bmatrix} \hat{a}(t) \\ \hat{b}(t) \end{bmatrix}, \quad f(y(t - \chi(t))) = \begin{bmatrix} \hat{a}(t - \chi(t)) \\ h(\hat{b}(t - \chi(t))) \end{bmatrix}, \quad (26)$$

$$D = \begin{bmatrix} \mathcal{U} & 0 \\ 0 & \mathcal{X} \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & \mathcal{V} \\ \mathcal{Y} & 0 \end{bmatrix}, \quad (27)$$

and $x(t) = W_2^{-1}y(t)$. Thus, GRN (23) can be transformed into GNN (4) with $W_0 = 0$, $W_1 = I$. By using Theorem 3.1 we obtain the

Table 7
Upper bounds χ_2 for various values of ℓ .

ℓ	0.5	0.8	0.9	Unknown ℓ
Theorem 3.1	2.5134	2.1354	1.6352	1.3458

upper bounds χ_2 for various values of ℓ when $\chi_1 = 1$ (see Table 7). Thus, GRN (23) is stable.

Remark 4.1. It is worth mentioning that the Refs. Liu et al. (2012) and Mahmoud and Xia (2011) require the strong assumption that the time-varying delay is differentiable, and its time derivative should be bounded or smaller than one. However, note that this assumption is omitted in our paper.

Remark 4.2. In Zhang, Shen, Yin, and Sun (2015), Zhang et al. studied the issue of passivity for memristor-based NNs with mixed delays. It is worth mentioning that this work can be extended to Zhang et al. (2015). Thus, it is possible that the issue of passivity for memristor-based GNNs can be solved in the future.

Remark 4.3. In Rakkiyappan, Velmurugan, and Li (2015), complete stability for impulsive complex-valued neural networks (CNNs) is considered. In Liu and Chen (2016), exponential stability of CNNs with asynchronous delays is presented. In Song, Yan, Zhao, and Liu (2016a), Song et al. established the global exponential stability of CNNs with impulsive effects. Song et al. studied the global exponential stability of impulsive CNNs with mixed delays in Song, Yan, Zhao, and Liu (2016b). The method in this paper is valid for CNNs.

5. Conclusions

This article examined the exponential stability problem for delayed GNNs. Improved exponential stability conditions were established by choosing a suitable LKF. All the proposed conditions were expressed in terms of LMIs by using a new WII technique. In order to demonstrate the usefulness of the presented results, five numerical examples and one practical example were given. The established results were compared with those of recent methods existing in the literature. It is difficult to use the WII technique to solve stochastic exponential stability problems. Thus, we expect to extend this work to passivity and dissipativity analysis problems of GNNs with Markovian jumping parameters.

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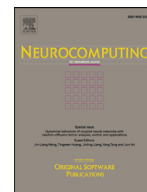
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Impulsive Cohen–Grossberg BAM neural networks with mixed time-delays: An exponential stability analysis issue[☆]

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ABSTRACT

This paper investigates, the globally exponential stability analysis problem for a class of markovian jumping Cohen–Grossberg BAM-type neural networks (CGBAMNNs) with mixed time delays and impulsive effects. Here the jumping parameters are considered, which are governed by a markov process with discrete & finite state space. The mixed time delays carries both discrete time-varying and distributed delays, which means the lower and upper bounds of discrete time delays are available. By fabricating an appropriate Lyapunov–Krasovskii functional (LKF), some new sufficient conditions are obtained in terms of linear matrix inequalities (LMIs) to guarantee the globally exponential stability for the labeled neural networks. The obtained conditions are expressed in terms of LMIs whose feasibility can be checked easily by MATLAB LMI control toolbox. Furthermore, we have collated our effort with foregoing one in the available literatures and showed that it is less conserved. Finally, three numerical examples with their simulative reactions are provided to demonstrate the viability of the notional outcomes.

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1. Introduction

Cohen–Grossberg neural networks (CGNNs) was first coined by Cohen–Grossberg [6] in 1983 and Bi-directional associative memory (BAM) neural networks was proposed by Kosko in [23,24]. A BAM neural networks is composed of neurons arranged in two layers: the A-layer and B-layer. The neurons in one layer are fully interconnected to the neurons in the other layer. This class of networks has been successfully applied to pattern recognition and artificial intelligence due to its generalization of the single-

layer auto-associative Hebbian correlator to a two-layer pattern-matched hetero associative circuit. Over the past few decades, according to the extensive range of successful applications, such as pattern recognition, associative memory, parallel computing, combinational optimization, signal and image processing, Cohen–Grossberg neural networks (CGNNs) and bidirectional associative memory (BAM) neural networks with or without time delays have attracted considerable attention, see e.g. [4,5,16–20,41]. It is well known that the theory on the dynamics of this networks, including global asymptotic stability (GAS) and global exponential stability (GES), had been extensively studied, see [8,57]. In [27], a new type of model which combines CGNNs with BAM neural networks, called Cohen–Grossberg-type bidirectional associative memory (CGBAMNNs) neural networks, is proposed and sufficient conditions are proposed ensuring global exponential stability of the zero solution by means of the inequality techniques and the properties of an M-matrix [28]. The authors in [54] investigated, the existence of periodic solution and stability responses

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of an equilibrium point for the BAM-type Cohen–Grossberg neural networks.

On the other hand, time delays are commonly encountered in various physical, engineering and neural based systems which cause the poor performance and instability of dynamic systems, see [7,11,21,22,51]. Accordingly, time delay can be classified into two types: a Discrete and Distributed delays. Here, we have taken both the delays into account while model our network system, because the length of the axon sizes are too large. Although it has been recognized that discrete time-delays can be introduced into communication channels since they are ubiquitous in both the neural processing and signal transmission, a neural network also has a special nature due to the presence of an amount of parallel pathways with a variety of axon sizes and lengths. Such an inherent nature can be suitably modeled by distributed delays [15,35,43], because the signal propagation is distributed during a certain time period. For example, in [31], a neural circuit has been designed with distributed delays, which solves a general problem of recognizing patterns in a time-dependent signal. As a matter of fact, a realistic neural network should involve both discrete and distributed delays [46,49]. Recently, the stability analysis problems for Cohen–Grossberg neural networks with distributed time-delays have begun to receive some attention from some researchers; see, e.g., [10,44,49]. In [56], the problems of the exponential stability and the almost sure exponential stability have been studied for a class of fuzzy stochastic Cohen–Grossberg neural networks with time-invariant delays. Moreover, the input-to-state stability problem has been addressed in [9] for a class of stochastic impulsive Cohen–Grossberg neural networks with mixed delays.

In nature, some times a neural networks has finite state representations (also called modes, patterns or clusters), and the modes may switch (or jump) from one to another at certain moments of times [35]. Recently, it has been revealed in [52,59] that, jumping between different neural networks modes can be controlled by a Markovian chain. Specifically, the class of neural networks with Markovian jump parameters has two components in the state vector. In [1], researchers discussed about the existence of Markovian jumping in CGNNs and exploit the LKF approach, the new sufficient criteria were derived for the globally exponential stability in the mean square. The authors in [14,38], developed some conditions for the stability analysis of neural network systems with integral inequalities approach. The criteria to obtained the stability result of neural networks with time varying delays has been checked in [3]. Conversely, besides delay, impulsive effect is likely to exist in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time in the fields such as medicine and biology, economics, electronics, and telecommunications. Neural networks are often subject to impulsive perturbation that in turn affects dynamical behaviors of systems [32,33]. In this regard, impulsive neural networks have gained increasing research attention in the past few years, see [13,34] and references therein. It should be noted that, in all the consequences reported in the above literatures, are concerned only with Markovian jumping CGNNs with time-varying delays. To the best of our knowledge, up to now, very little attention has been paid to the problem of the global exponential stability of Markovian jumping CGBAMNNs with mixed time delays and impulsive effects, which frequently appear in realistic neural networks. This situation motivates our present research work.

Our main motivation of the current research is to analyze the globally exponential stability problem for Markovian jumping Cohen–Grossberg BAM type neural networks with mixed time-delays & impulsive effects has not been fully addressed. By utilizing the Lyapunov stability theory and inequality technique, the exponential stability problem of the addressed neural networks is converted into the feasibility problem of a set of linear matrix

Table 1

Maximum allowable upper bounds of discrete time delays.

Methods	$\bar{\sigma}_i$	System status
In Ref [39]	0.33	Stable
In Ref [37]	0.5	Stable
In Ref [26]	1	Stable
In Ref [55]	1	Stable
In Ref [25]	1.01	Stable
In Ref [42]	4.6818	Stable
In Ref [40]	5	Stable
In this paper	$\bar{\sigma}_1 = \bar{\sigma}_2 = 17.4386$	Stable

inequalities. By constructing a new Lyapunov–Krasovskii functional, some sufficient criteria are given such that the considered Cohen–Grossberg BAM-type neural networks are global exponential stable in mean square if a set of linear matrix inequalities is feasible. It is worthy point out that the proposed results are delay-dependent and can be easily solved by using the MATLAB LMI control toolbox. The main contributions of this research work as follows:

- (1) BAM-type, discrete & distributed time-delays, markovian jumps and impulsive effects are taken into account in the stability analysis of proposed Cohen–Grossberg neural networks.
- (2) Based on some new Lyapunov–Krasovskii functional, the sufficient conditions for globally exponential stability of Cohen–Grossberg BAM-type neural networks are derived.
- (3) By considering the mixed-time delay terms in our Markovian jumping Cohen–Grossberg BAM neural networks, the allowable upper bounds of discrete time-delay is maximum, when compare with the previous literatures, see Table 1 in Example 4.1. This shows that the approach developed in this paper is effective and less conservative than some existing results.
- (4) Two special cases of main results are derived in Corollary 1, 2 and also by using MATLAB LMI control toolbox checked its LMIs are feasible as well as its simulations are given in Examples 4.2 & 4.3, which ensures the advantages of this paper.

The organization of this paper is as follows. The considered markovian jumping Cohen–Grossberg BAM neural networks model is presented and some preliminaries are introduced in Section 2. In Section 3, by the Lyapunov–Krasovskii functionals and integral inequality techniques, some time delay-dependent conditions are obtained for the globally exponential stability of addressed neural networks. The obtained results are formulated in the form of linear matrix inequalities (LMIs), that are easily solvable by using MATLAB LMI control toolbox. Then, three numerical examples with their simulations are provided to demonstrate the feasibility and effectiveness of the derived globally exponential stability criterions in Section 4. Finally, conclusions are drawn in Section 5.

2. Model description and preliminaries

Notations. Throughout this paper, $\mathbb{R}^n$ denotes the n -dimensional Euclidean space and $\mathbb{N}$ be set of all natural numbers. For a matrix B , B^T & B^{-1} represents its transpose and inverse, respectively. The notation $B > 0$ ($B \geq 0$) represents that matrix B is symmetric and positive definite (positive semi-definite). I and 0 represent the identity matrix and the zero matrix with appropriate dimensions, respectively. $C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{M}; \mathbb{R}^+)$ is the family of all non negative functions $V(t, x(t), i)$ on $\mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{M}$ which are continuously twice differentiable in x and once differentiable in t ; $(\mathbb{A}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ a complete probability space, where $\mathbb{A}$ is the sample space, $\mathcal{F}$ is the σ -algebra of subsets of the sample space, $\mathbb{P}$ is the probability measure on $\mathcal{F}$ and $\{\mathcal{F}_t\}_{t \geq 0}$ denotes the filtration; The symbol “*” denotes a block that is readily inferred by

symmetry. $C([- \rho, 0]; \mathbb{R}^n)$ denotes the family of continuous functions ϕ from $[- \rho, 0]$ to $\mathbb{R}^n$ with norm $\|\phi\| = \sup_{-\rho \leq \theta \leq 0} |\phi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. Denote by $\mathcal{L}_{\mathcal{F}_0}^p([-\rho, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$ measurable $C([-\rho, 0]; \mathbb{R}^n)$ -valued random variables $\xi = \{\xi(\theta) : -\rho \leq \theta \leq 0\}$ such that $\sup_{-\rho \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^p < \infty$ where $\mathbb{E}\{x\}$ stands for the mathematical expectation operator of x with respect to the given probability measure $\mathcal{P}$ and $\mathbb{E}\{x|y\}$ is the mathematical expectation of x conditional on y . $\text{diag}\{\dots\}$ denotes a block-diagonal matrix. For a matrix A , $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and minimum eigenvalues of matrix A , respectively.

In this paper, we consider the Markovian jumping Cohen–Grossberg BAM-type neural networks with mixed time delays and impulses is described by the following integro-differential equation system:

$$\begin{aligned} \dot{x}_i(t) = & -c_i(x_i(t), r(t)) \left[b_i(x_i(t), r(t)) - \sum_{j=1}^n d_{ij}(r(t)) f_j(y_j(t)) \right. \\ & - \sum_{j=1}^n e_{ij}(r(t)) g_j(y_j(t - \sigma_{1,r(t)}(t))) \\ & \left. - \sum_{j=1}^n a_{ij}(r(t)) \int_{t-\tau_{1,r(t)}}^t h_j(y_j(s)) ds \right], \\ i = & 1, 2, \dots, n, t > 0, t \neq t_k, \end{aligned}$$

$$\Delta x_i(t) = x_i(t^+) - x_i(t^-); \quad t = t_k, \quad k \in \mathbb{Z}_+$$

$$\begin{aligned} \dot{y}_j(t) = & -l_j(y_j(t), \tilde{r}(t)) \left[m_j(y_j(t), \tilde{r}(t)) - \sum_{i=1}^n w_{0ji}(\tilde{r}(t)) \tilde{f}_i(x_i(t)) \right. \\ & - \sum_{i=1}^n w_{1ji}(\tilde{r}(t)) \tilde{g}_i(x_i(t - \sigma_{2,\tilde{r}}(t))) \\ & \left. - \sum_{i=1}^n w_{2ji}(\tilde{r}(t)) \int_{t-\tau_{2,\tilde{r}(t)}}^t \tilde{h}_i(x_i(s)) ds \right], \\ j = & 1, 2, \dots, n, t > 0, t \neq t_k, \end{aligned}$$

$$\Delta y_j(t) = y_j(t^+) - y_j(t^-); \quad t = t_k, \quad k \in \mathbb{Z}_+. \quad (1)$$

Then the neural networks (1) can be rewritten in the following compact form:

$$\begin{aligned} \dot{x}(t) = & -C(x(t), i) \left[B(x(t), i) - D(i) f(y(t)) - E(i) g(y(t - \sigma_{1,i}(t))) \right. \\ & \left. - A(i) \int_{t-\tau_{1,i}}^t h(y(s)) ds \right], \quad i = 1, 2, \dots, n, t > 0, t \neq t_k, \\ \Delta x(t) = & x(t^+) - x(t^-); \quad t = t_k, \quad k \in \mathbb{Z}_+. \\ \dot{y}(t) = & -L(y(t), j) \left[M(y(t), j) - W_0(j) \tilde{f}(x(t)) \right. \\ & - W_1(j) \tilde{g}(x(t - \sigma_{2,j}(t))) \\ & \left. - W_2(j) \int_{t-\tau_{2,j}}^t \tilde{h}(x(s)) ds \right], \quad j = 1, 2, \dots, n, t > 0, t \neq t_k, \\ \Delta y(t) = & y(t^+) - y(t^-); \quad t = t_k, \quad k \in \mathbb{Z}_+. \end{aligned} \quad (2)$$

where $x(t)$ is the state of the neurons at time t , $C(x(t), i)$, $L(y(t), j)$ are the amplification functions and $B(x(t), i)$, $M(y(t), j)$ are the behaved functions respectively and $f(\cdot)$, $\tilde{f}(\cdot)$, $g(\cdot)$, $\tilde{g}(\cdot)$, $h(\cdot)$, & $\tilde{h}(\cdot)$ are activation functions. $D(i)$, $W_0(j)$ represents the connection weight matrices, $E(i)$, $W_1(j)$ indicates the discretely delayed connection weight matrices and $A(i)$, $W_2(j)$ denotes the distributively delayed connection weight matrices. $\sigma_{1,i}(t)$ and $\sigma_{2,j}(t)$ are the discrete time varying delays which are bounded with $0 < \sigma_{1,i}(t) < \bar{\sigma}_{1,i}$, $\bar{\sigma}_{1,i}(t) \leq \sigma_{1,i} < 1$ and $0 < \sigma_{2,j}(t) < \bar{\sigma}_{2,j}$, $\bar{\sigma}_{2,j}(t) \leq \sigma_{2,j} < 1$, respectively; $\tau_{1,i}$ and $\tau_{2,j}$ are constant delays; $\Delta x(t_k)$

$= I_k^*(x(t_k))$, $\Delta y(t_k) = I_k^*(y(t_k))$ are the impulsive effects at moments t_k , and the impulsive times t_k satisfy $0 = t_0 < t_1 < \dots < t_k \rightarrow \infty$, (ie, $\lim_{k \rightarrow \infty} t_k = +\infty$) and $\inf_{k \in \mathbb{Z}_+} \{t_k - t_{k-1}\} > 0$. Assume that $x(t_k)$ is right continuous, ie, $x(t_k) = x(t_k^+)$. Hence, the solutions of (2) are piecewise right-hand continuous functions with discontinuities at $t = t_k$.

For presentation convenience, denote $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$, $y(t) = [y_1(t), y_2(t), \dots, y_n(t)]^T$, $f(y(t)) = [f_1(y(t)), f_2(y(t)), \dots, f_n(y(t))]^T$, $g(y(t - \sigma_{1i}(t))) = [g_1(y(t - \sigma_{1i}(t))), g_2(y(t - \sigma_{1i}(t))), \dots, g_n(y(t - \sigma_{1i}(t)))]^T$, $h(y(t)) = [h_1(y(t)), h_2(y(t)), \dots, h_n(y(t))]^T$, $\tilde{f}(x(t)) = [\tilde{f}_1(x(t)), \tilde{f}_2(x(t)), \dots, \tilde{f}_n(x(t))]^T$, $\tilde{g}(x(t - \sigma_{2j}(t))) = [\tilde{g}_1(x(t - \sigma_{2j}(t))), \tilde{g}_2(x(t - \sigma_{2j}(t))), \dots, \tilde{g}_n(x(t - \sigma_{2j}(t)))]^T$, $\tilde{h}(y(t)) = [\tilde{h}_1(y(t)), \tilde{h}_2(y(t)), \dots, \tilde{h}_n(y(t))]^T$, $C(x(t), k) = \text{diag}\{c_1(x(t), k), c_2(x(t), k), \dots, c_n(x(t), k)\}$, $L(y(t), \tilde{k}) = \text{diag}\{l_1(y(t), \tilde{k}), l_2(y(t), \tilde{k}), \dots, l_n(y(t), \tilde{k})\}$, $B(x(t), k) = [b_1(x(t), k), b_2(x(t), k), \dots, b_n(x(t), k)]^T$, $M(y(t), \tilde{k}) = [m_1(y(t), \tilde{k}), m_2(y(t), \tilde{k}), \dots, m_n(y(t), \tilde{k})]^T$, $D(k) = (d_{ij}(k))_{n \times n}$, $E(k) = (e_{ij}(k))_{n \times n}$, $A(k) = (a_{ij}(k))_{n \times n}$, $W_0(\tilde{k}) = (W_{0ji}(\tilde{k}))_{n \times n}$, $W_1(\tilde{k}) = (W_{1ji}(\tilde{k}))_{n \times n}$, $W_2(\tilde{k}) = (W_{2ji}(\tilde{k}))_{n \times n}$.

Let $\{r(t), t \geq 0\}$ be a right continuous Markov chain on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ taking values in a finite state space $\mathfrak{M} = \{1, 2, \dots, N\}$ with generator $\Pi = (\pi_{ij})_{N \times N}$ given by

$$P\{r(t + \Delta t) = j | r(t) = i\} = \begin{cases} \pi_{ij} \Delta t + O(\Delta t), & \text{if } i \neq j, \\ 1 + \pi_{ii} \Delta t + O(\Delta t), & \text{if } i = j, \end{cases}$$

where $\Delta t > 0$ and $\lim_{\Delta t \rightarrow 0} \frac{O(\Delta t)}{\Delta t} = 0$. Here $\pi_{ij} \geq 0$ is the transition probability rate from i to j if $i \neq j$, while $\pi_{ii} = -\sum_{j=1}^N \pi_{ij}$.

Remark 2.1. The neural network (2) is quite general. It encompasses many well-known Cohen–Grossberg neural networks as special cases, see e.g. [20].

To proceed, we make the following assumptions on the amplification function, behaved function and neuron activation functions.

Assumption 1. For $1 \leq i, j \leq n$, $k, \tilde{k} \in \mathfrak{M} = \{1, 2, 3, \dots, m\}$, $m \in \mathbb{N}$, the behaved functions $b_i(\cdot, k)$ & $m_j(\cdot, \tilde{k})$ are continuous and satisfies that $b_i(0, k) = m_j(0, \tilde{k}) = 0$. In addition, there exist known constants $\varrho_i(k)$ & $\tilde{\varrho}_j(\tilde{k})$ such that

$$p_1 b_i(p_1, k) \geq \varrho_i(k) p_1^2, \quad p_2 m_j(p_2, \tilde{k}) \geq \tilde{\varrho}_j(\tilde{k}) p_2^2, \quad \forall p_1, p_2 \in \mathbb{R}.$$

Assumption 2. For the amplification functions $c_i(\cdot, k)$, $l_j(\cdot, \tilde{k})$ and $l_j(\cdot, \tilde{k})$, $1 \leq i, j \leq n$, $k, \tilde{k} \in \mathfrak{M} = \{1, 2, 3, \dots, m\}$, $m \in \mathbb{N}$, there exist known constants $\underline{c}_i(k)$, $\bar{c}_i(k)$, $\underline{l}_j(\tilde{k})$ and $\bar{l}_j(\tilde{k})$ such that $0 < \underline{c}_i(k) \leq c_i(\cdot, k) \leq \bar{c}_i(k)$ & $0 < \underline{l}_j(\tilde{k}) \leq l_j(\cdot, \tilde{k}) \leq \bar{l}_j(\tilde{k})$.

Assumption 3. The neuron activation functions $f(\cdot)$, $\tilde{f}(\cdot)$, $g(\cdot)$, $\tilde{g}(\cdot)$, $h(\cdot)$ & $\tilde{h}(\cdot)$ satisfy $f(0) = \tilde{f}(0) = g(0) = \tilde{g}(0) = h(0) = \tilde{h}(0) = 0$. For $1 \leq i, j \leq n$ and $p_1, p_2 \in \mathbb{R}$, there exist constants $\alpha_i > 0$, $\tilde{\alpha}_j > 0$, $\delta_i > 0$, $\tilde{\delta}_j > 0$, $\varsigma_{1i} > 0$ and $\varsigma_{2j} > 0$ such that

$$\begin{aligned} |f_i(p_1) - f_i(p_2)| & \leq \alpha_i |p_1 - p_2|, \quad |\tilde{f}_j(p_1) - \tilde{f}_j(p_2)| \\ & \leq \tilde{\alpha}_j |p_1 - p_2|, \quad |g_i(p_1) - g_i(p_2)| \leq \delta_i |p_1 - p_2|, \\ |\tilde{g}_j(p_1) - \tilde{g}_j(p_2)| & \leq \tilde{\delta}_j |p_1 - p_2|, \quad |h_i(p_1) - h_i(p_2)| \\ & \leq \varsigma_{1i} |p_1 - p_2|, \quad |\tilde{h}_j(p_1) - \tilde{h}_j(p_2)| \leq \varsigma_{2j} |p_1 - p_2|. \end{aligned}$$

To derive the main results we denote some parameters

$$\begin{aligned} \bar{\pi} &= \max_{1 \leq i \leq p} \{|\pi_{ii}|\}, \quad \tilde{\pi} = \max_{1 \leq j \leq p} \{|\tilde{\pi}_{jj}|\}, \\ \bar{\sigma}_1 &= \max_{1 \leq i \leq p} \{\bar{\sigma}_{1,i}\}, \quad \underline{\sigma}_1 = \min_{1 \leq i \leq p} \{\bar{\sigma}_{1,i}\}, \end{aligned}$$

$$\begin{aligned}
\bar{\sigma}_2 &= \max_{1 \leq j \leq q} \{\bar{\sigma}_{2,j}\}, \quad \underline{\sigma}_2 = \min_{1 \leq j \leq q} \{\bar{\sigma}_{2,j}\}, \\
\bar{\tau}_1 &= \max_{1 \leq i \leq p} \{\tau_{1,i}\}, \quad \underline{\tau}_1 = \min_{1 \leq i \leq p} \{\tau_{1,i}\}, \\
\bar{\tau}_2 &= \max_{1 \leq j \leq q} \{\tau_{2,j}\}, \quad \underline{\tau}_2 = \min_{1 \leq j \leq q} \{\tau_{2,j}\}, \\
\sigma &= \max\{\bar{\sigma}_1, \bar{\sigma}_2\} \quad \tau = \max\{\bar{\tau}_1, \bar{\tau}_2\}, \\
G &= \text{diag}\{\alpha_1, \alpha_2, \dots, \alpha_n\}, \quad \tilde{G} = \text{diag}\{\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n\}, \\
\Delta &= \text{diag}\{\delta_1, \delta_2, \dots, \delta_n\}, \\
\tilde{\Delta} &= \text{diag}\{\tilde{\delta}_1, \tilde{\delta}_2, \dots, \tilde{\delta}_n\}, \quad \Lambda_1 = \text{diag}\{\varsigma_{11}, \varsigma_{12}, \dots, \varsigma_{1n}\}, \\
\Lambda_2 &= \text{diag}\{\varsigma_{21}, \varsigma_{22}, \dots, \varsigma_{2n}\}, \\
\tilde{\varsigma}_1 &= \max_{1 \leq i \leq n} \{\varsigma_{1i}\}, \quad \Gamma_1(i) = \text{diag}\{\varrho_1(i), \varrho_2(i), \dots, \varrho_n(i)\}, \\
\tilde{\varsigma}_2 &= \max_{1 \leq j \leq n} \{\varsigma_{2j}\}, \\
\Gamma_2(j) &= \text{diag}\{\tilde{\varrho}_1(j), \tilde{\varrho}_2(j), \dots, \tilde{\varrho}_n(j)\}.
\end{aligned}$$

Remark 2.2. Firstly, the description given in [Assumption 3](#) is less conservative than the traditional Lipschits condition and more general than the usual sigmoid activation functions. Secondly, the form of the Cohen–Grossberg BAM neural networks (2) is more general than those ones in [\[29\]](#).

Remark 2.3. Under [Assumptions 1, 2, 3, Theorems 3.1 and 3.2](#) are provides some sufficient conditions ensuring the asymptotic stability and globally exponential stability, respectively, of the BAM type Cohen–Grossberg neural networks with both time delays (2). The LMIs in [Eqs. \(3\)–\(6\)](#) can be checked easily by resorting to recently developed algorithms in solving LMIs.

Consider a general system $dx(t) = f(x(t), t, r(t)) dt$, $t \geq 0$ with initial value $x(0) = x_0 \in \mathbb{R}^n$, where $f: \mathbb{R}^n \times \mathbb{R}^+ \times \mathfrak{M} \rightarrow \mathbb{R}^n$ and $r(t)$ is the Markov chain. Let $\mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}^+ \times \mathfrak{M}; \mathbb{R}^+)$ denote a family of all non - negative functions V on $\mathbb{R}^n \times \mathbb{R}^+ \times \mathfrak{M}$ which are twice continuously differentiable in x and once differentiable in t . For any $V \in \mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}^+ \times \mathfrak{M}; \mathbb{R}^+)$ define $\mathcal{LV}: \mathbb{R}^n \times \mathbb{R}^+ \times \mathfrak{M} \rightarrow \mathbb{R}$ by

$$\begin{aligned}
\mathcal{LV}(x(t), t, i) &= V_t(x(t), t, i) + V_x(x(t), t, i)f(x(t), t, i) \\
&\quad + V_{xx}(x(t), t, i)f(x(t), t, i)) + \sum_{j=1}^N \gamma_{ij}V(x(t), t, j),
\end{aligned}$$

where

$$\begin{aligned}
V_t(x(t), t, i) &= \frac{\partial V(x(t), t, i)}{\partial t}, \\
V_x(x(t), t, i) &= \left(\frac{\partial V(x(t), t, i)}{\partial x_1}, \frac{\partial V(x(t), t, i)}{\partial x_2}, \dots, \frac{\partial V(x(t), t, i)}{\partial x_n} \right), \\
V_{xx}(x(t), t, i) &= \frac{\partial^2 V(x(t), t, i)}{\partial x_j \partial x_k}.
\end{aligned}$$

By the generalized Ito's formula, one can see that

$$\begin{aligned}
\mathbb{E}V(x(t), y(t), t, r(t)) &= \mathbb{E}V(x(0), y(0), 0, r(0)) \\
&\quad + \mathbb{E} \int_0^t \mathcal{LV}(x(s), y(s), s, r(s)) ds.
\end{aligned}$$

Before ending this section, we introduce some definitions and lemmas, which will play an important role in proving our main theorems as mentioned below:

Definition 2.4. The equilibrium point of the neural networks (2) is said to be asymptotically stable in the mean square sense if for any solution $(x(t), y(t))$ of the Cohen–Grossberg BAM neural networks satisfies the following condition $\lim_{t \rightarrow \infty} \mathbb{E}\{x(t)\} + \mathbb{E}\{y(t)\} = 0$. Also the equilibrium point of NNs (2) is said to be globally exponential stable in the mean square if there exists scalars $\chi > 0$ and $\eta > 0$ such that

$$\begin{aligned}
\mathbb{E}\{|x(t)|^2\} + \mathbb{E}\{|y(t)|^2\} &\leq \chi e^{-\eta t} \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|x(s)|^2\} \\
&\quad + \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|y(s)|^2\}.
\end{aligned}$$

Definition 2.5. We introduce the stochastic Lyapunov–Krasovskii functional $V \in \mathcal{C}^{2,1}(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathfrak{M}; \mathbb{R}^+)$ of the system (2), the weak infinitesimal generator of random process $\mathcal{LV}$ from $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathfrak{M}$ to $\mathbb{R}^+$ defined by

$$\begin{aligned}
&\mathcal{LV}(t, u(t), v(t), i) \\
&= \lim_{\Delta t \rightarrow 0^+} \frac{1}{\Delta t} [\mathbb{E}\{V((t + \Delta t), u(t + \Delta t), v(t + \Delta t), r(t + \Delta t)) | \\
&\quad u(t), v(t), r(t) = i\} - V(t, u(t), v(t), r(t) = i)]
\end{aligned}$$

Lemma 2.6 ([58]). For any real matrix $M > 0$, scalars a and b with $0 \leq a < b$, vector function $x(\alpha)$ such that the following integrals are well defined, we have

$$\int_{-a}^{-b} \int_{t-a}^t x(\alpha)^T M x(\alpha) d\alpha d\beta \leq (b-a) \int_{t-b}^t x(\alpha)^T M x(\alpha) d\alpha.$$

Lemma 2.7 ([12]). Let $Q \in \mathbb{R}^{n \times n}$ be a positive semi-definite matrix, scalars e and f satisfy $e < f$ and $w(\cdot): [e, f] \rightarrow \mathbb{R}^n$ be a vector function. If the integrations concerned are well defined, then the following inequality holds:

$$\left(\int_e^f w(\alpha) d\alpha \right)^T Q \left(\int_e^f w(\alpha) d\alpha \right) \leq (f-e) \int_e^f w^T(\alpha) Q w(\alpha) d\alpha.$$

Lemma 2.8 (Schur Complement [58]). Given constant symmetric matrices Y_1, Y_2 and Y_3 with appropriate dimensions, where $\Upsilon_1^T = Y_1$ and $\Upsilon_2^T = Y_2 > 0$, $\Upsilon_1 + \Upsilon_3^T \Upsilon_2^{-1} \Upsilon_3 < 0$ if and only if

$$\begin{pmatrix} \Upsilon_1 & \Upsilon_3^T \\ \Upsilon_3 & -\Upsilon_2 \end{pmatrix} < 0 \text{ [or]} \begin{pmatrix} \Upsilon_1 & \Upsilon_3 \\ \Upsilon_3^T & -\Upsilon_2 \end{pmatrix} < 0.$$

3. Main results

Theorem 3.1. Suppose that [Assumptions 1–3](#) holds, the Cohen–Grossberg neural networks (2) is asymptotically stable in the mean square if there exists a set of diagonal matrices $S_i > 0$ & $\tilde{S}_j > 0$, two matrices $T, \tilde{T} > 0$ and $R, \tilde{R} > 0$, and three sets of scalars $\xi_1(i), \xi_2(i), \xi_3(i)$ and $\varepsilon_1(j), \varepsilon_2(j), \varepsilon_3(j)$ such that the following LMIs are satisfied:

$$\xi_2(i) \Delta^2 < T, \quad \varepsilon_2(j) \tilde{\Delta}^2 < \tilde{T}, \quad (3)$$

$$\xi_3(i) \tau_{1,i} \tilde{\varsigma}_1^2 I < R, \quad \varepsilon_3(j) \tau_{2,j} \tilde{\varsigma}_2^2 I < \tilde{R}, \quad (4)$$

$$\Theta_i = \begin{pmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} \\ * & \Theta_{22} & \Theta_{23} & \Theta_{24} \\ * & * & \Theta_{33} & \Theta_{34} \\ * & * & * & \Theta_{44} \end{pmatrix} < 0, \quad (5)$$

$$\Upsilon_i = \begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & \Upsilon_{14} \\ * & \Upsilon_{22} & \Upsilon_{23} & \Upsilon_{24} \\ * & * & \Upsilon_{33} & \Upsilon_{34} \\ * & * & * & \Upsilon_{44} \end{pmatrix} < 0, \quad (6)$$

where

$$\Theta_{11} = -2\Gamma_1(i) S_i \tilde{C}(i) + \xi_1^{-1}(i) \lambda_{\max}(D(i) D^T(i)) S_i \tilde{C}^2(i) S_i + \vartheta_1;$$

$$\Theta_{12} = \Theta_{13} = \Theta_{14} = \Theta_{23} = 0;$$

$$\vartheta_1 = \xi_2^{-1}(i) \lambda_{\max}(E(i) E^T(i)) S_i \tilde{C}^2(i) S_i$$

$$+ \xi_3^{-1}(i) \lambda_{\max}(A(i) A^T(i)) S_i \tilde{C}^2(i) S_i + \sum_{j=1}^m \pi_{ij} S_i;$$

$$\begin{aligned}
\Theta_{33} &= \xi_2(i) \Delta^2 - T; \quad \Theta_{44} = \xi_3(i) \tau_{1,i} \tilde{\zeta}_1^2 - R; \\
\Upsilon_{33} &= \varepsilon_2(j) \tilde{\Delta}^2 - \tilde{T}; \quad \Upsilon_{44} = \varepsilon_3(j) \tau_{2,j} \tilde{\zeta}_2^2 - \tilde{R}; \\
\Upsilon_{11} &= -2\Gamma_2(j) \tilde{S}_j \tilde{L}(j) + \varepsilon_1^{-1}(j) \mu_{\max}(W_0(j) W_0^T(j)) \tilde{S}_j \tilde{L}^2(j) \tilde{S}_j \\
&\quad + \vartheta_2; \quad \Upsilon_{23} = 0 \quad \Upsilon_{24} = 0; \\
\vartheta_2 &= \varepsilon_2^{-1}(j) \mu_{\max}(W_1(j) W_1^T(j)) \tilde{S}_j \tilde{L}^2(j) \tilde{S}_j \\
&\quad + \varepsilon_3^{-1}(j) \mu_{\max}(W_2(j) W_2^T(j)) \tilde{S}_j \tilde{L}^2(j) \tilde{S}_j + \sum_{i=1}^m \tilde{\pi}_{ji} \tilde{S}_j; \\
\Upsilon_{22} &= \varepsilon_1(j) \tilde{G}^2 + \tilde{T} + \tilde{\pi}(\bar{\sigma}_2 - \underline{\sigma}_2) \tilde{T} + \tau_{2,j} \tilde{R} + \frac{1}{2} \tilde{\pi}(\tilde{\tau}_2^2 - \underline{\tau}_2^2) \tilde{R}; \\
\Upsilon_{12} &= 0; \quad \Upsilon_{13} = 0; \quad \Upsilon_{14} = 0; \\
\Theta_{22} &= \xi_1(i) G^2 + T + \tilde{\pi}(\bar{\sigma}_1 - \underline{\sigma}_1) T + \tau_{1,i} R + \frac{1}{2} \pi(\tilde{\tau}_1^2 - \underline{\tau}_1^2) R; \\
\Theta_{24} &= 0; \quad \Upsilon_{34} = 0; \quad \Theta_{34} = 0;
\end{aligned}$$

Proof. Define a new markov process $\{x_t, y_t, r(t), \tilde{r}(t), t \geq 0\}$ with $x_t(s) = x(t+s)$, $\varrho \geq s \geq 0$ & $y_t(s) = y(t+s)$, $\tilde{\varrho} \geq s \geq 0$.

In order to establish the stability conditions for neural networks (2), we construct the Lyapunov–Krosovskii functional candidate as follows:

$$\begin{aligned}
V(x_t, y_t, t, r(t), \tilde{r}(t)) &= x^T(t) S_r(t) x(t) + \int_{t-\sigma_{1,r(t)}}^t y^T(s) T y(s) ds \\
&\quad + \tilde{\pi} \int_{\underline{\sigma}_1}^{\bar{\sigma}_1} \int_{t-s}^t y^T(\theta) T y(\theta) d\theta ds \\
&\quad + \int_0^{\tau_{1,r(t)}} \int_{t-s}^t y^T(\theta) R y(\theta) d\theta ds \\
&\quad + \tilde{\pi} \int_{-\underline{\tau}_1}^{\bar{\tau}_1} \int_0^u \int_{t-s}^t y^T(\theta) R y(\theta) d\theta ds du \\
&\quad + y^T(t) \tilde{S}_{\tilde{r}(t)} y(t) + \int_{t-\sigma_{2,\tilde{r}(t)}}^t x^T(s) \tilde{T} x(s) ds \\
&\quad + \tilde{\pi} \int_{\underline{\sigma}_2}^{\bar{\sigma}_2} \int_{t-s}^t x^T(\theta) \tilde{T} x(\theta) \\
&\quad \times d\theta ds + \int_0^{\tau_{2,\tilde{r}(t)}} \int_{t-s}^t x^T(\theta) \tilde{R} x(\theta) d\theta ds \\
&\quad + \tilde{\pi} \int_{-\underline{\tau}_2}^{\bar{\tau}_2} \int_0^v \int_{t-s}^t x^T(\theta) \tilde{R} x(\theta) \\
&\quad \times d\theta ds dv
\end{aligned} \quad (7)$$

Let $\mathcal{L}$ be the weak infinitesimal generator of the random processes $\{x_t, y_t, r(t), \tilde{r}(t), t \geq 0\}$ along the neural networks (2). Then we have

$$\begin{aligned}
\mathcal{L}V(x_t, y_t, t, i, j) &= \lim_{\Delta t \rightarrow 0^+} \frac{1}{\Delta t} [\mathbb{E}\{V(x_{t+\Delta t}, y_{t+\Delta t}, t+\Delta t, r(t+\Delta t), \tilde{r}(t+\Delta t)) | x_t, y_t, \\
&\quad r(t) = i, \tilde{r}(t) = j\} - V(x_t, y_t, t, i, j)] \\
&= -2x^T(t) S_i \left(C(x(t), i) [B(x(t), i) - D(i) f(y(t)) \right. \\
&\quad \left. - E(i) g(y(t - \sigma_{1,i})) - A(i) \times \int_{t-\tau_{1,i}}^t h(y(s)) ds] \right) \\
&\quad + \sum_{j=1}^m \pi_{ij} x^T(t) S_j x(t) - 2y^T(t) \tilde{S}_j \left(L(y(t), j) [M(y(t), j) \right. \\
&\quad \left. - W_0(j) \tilde{f}(x(t)) - W_1(j) \tilde{g}(x(t - \sigma_{2,j})) \right. \\
&\quad \left. - W_2(j) \int_{t-\tau_{2,j}}^t \tilde{h}(x(s)) ds] \right) + \sum_{i=1}^m \tilde{\pi}_{ij} \times y^T(t) \tilde{S}_i y(t) \\
&\quad + y^T(t) T y(t) - y^T(t - \sigma_{1,i}) T y(t - \sigma_{1,i}) + \sum_{j=1}^m \pi_{ij}
\end{aligned}$$

$$\begin{aligned}
&\times \int_{t-\sigma_{1,i}}^t y^T(s) T y(s) ds + x^T(t) \tilde{T} x(t) - x^T(t - \sigma_{2,j}) \tilde{T} x(t - \sigma_{2,j}) \\
&\quad + \sum_{i=1}^m \tilde{\pi}_{ij} \times \int_{t-\sigma_{2,j}}^t x^T(s) \tilde{T} x(s) ds + \tilde{\pi}(\bar{\sigma}_1 - \underline{\sigma}_1) y^T(t) T y(t) \\
&\quad - \tilde{\pi} \int_{\underline{\sigma}_1}^{\bar{\sigma}_1} y^T(t-s) T y(t-s) \times ds + \tilde{\pi}(\bar{\sigma}_2 - \underline{\sigma}_2) x^T(t) \tilde{T} x(t) \\
&\quad - \tilde{\pi} \int_{\underline{\sigma}_2}^{\bar{\sigma}_2} x^T(t-s) \tilde{T} x(t-s) ds + \tau_{1,i} y^T(t) R y(t) \\
&\quad - \int_0^{\tau_{1,i}} y^T(t-s) R y(t-s) ds + \sum_{j=1}^m \pi_{ij} \int_0^{\tau_{1,i}} \int_{t-s}^t y^T(\theta) R y(\theta) d\theta ds \\
&\quad + \tau_{2,j} \times x^T(t) \tilde{R} x(t) - \int_0^{\tau_{2,j}} x^T(t-s) \tilde{R} x(t-s) ds \\
&\quad + \sum_{i=1}^m \tilde{\pi}_{ij} \int_0^{\tau_{2,j}} \int_{t-s}^t x^T(\theta) \tilde{R} x(\theta) \times d\theta ds \\
&\quad + \tilde{\pi} \int_{\underline{\tau}_1}^{\bar{\tau}_1} \int_0^u y^T(t) R y(t) ds du - \tilde{\pi} \int_{\underline{\tau}_1}^{\bar{\tau}_1} \int_0^u y^T(t-s) R y(t-s) ds du \\
&\quad + \tilde{\pi} \int_{\underline{\tau}_2}^{\bar{\tau}_2} \int_0^v x^T(t) \tilde{R} x(t) ds dv - \tilde{\pi} \int_{\underline{\tau}_2}^{\bar{\tau}_2} \int_0^v x^T(t-s) \tilde{R} x(t-s) ds dv.
\end{aligned} \quad (8)$$

Let $S_i = \text{diag}\{s_{i1}, s_{i2}, \dots, s_{in}\}$ & $\tilde{S}_j = \text{diag}\{\tilde{s}_{j1}, \tilde{s}_{j2}, \dots, \tilde{s}_{jn}\}$. Then, by Assumptions 1 and 2, we get

$$\begin{aligned}
-2x^T(t) S_i C(x(t), i) B(x(t), i) &= -2 \sum_{j=1}^n x_i^T(t) s_{ij} c_i(x_i(t), i) b_i(x_i(t), i) \\
&\leq -2 \sum_{j=1}^n \varrho_j(i) s_{ij} c_i(i) x_i^2(t) \\
&= -2x^T(t) \Gamma_1(i) S_i \underline{C}(i) x(t)
\end{aligned} \quad (9)$$

$$\begin{aligned}
-2y^T(t) \tilde{S}_j L(y(t), j) M(y(t), j) &= -2 \sum_{i=1}^n y_j^T(t) \tilde{s}_{ji} l_j(y_j(t), j) m_j(y_j(t), j) \\
&\leq -2 \sum_{i=1}^n \tilde{\varrho}_i(j) \tilde{s}_{ji} l_j(j) y_j^2(t) \\
&= -2y^T(t) \Gamma_2(j) \tilde{S}_j \underline{L}(j) y(t)
\end{aligned} \quad (10)$$

On the other hand, it is easy to see that

$$\begin{aligned}
2x^T(t) S_i C(x(t), i) D(i) f(y(t)) &\leq \xi_1^{-1}(i) x^T(t) S_i C(x(t), i) D(i) D^T(i) C(x(t), i) S_i x(t) \\
&\quad + \xi_1(i) f^T(y(t)) f(y(t)) \\
&\leq \xi_1^{-1}(i) \lambda_{\max}(D(i) D^T(i)) x^T(t) S_i C^2(x(t), i) S_i x(t) \\
&\quad + \xi_1(i) f^T(y(t)) f(y(t)) \leq \xi_1^{-1}(i) \lambda_{\max}(D(i) D^T(i)) x^T(t) S_i \bar{C}^2 S_i x(t) \\
&\quad + \xi_1(i) \times y^T(t) G^2 y(t)
\end{aligned} \quad (11)$$

$$\begin{aligned}
2y^T(t) \tilde{S}_j L(y(t), j) W_0(j) \tilde{f}(x(t)) &\leq \varepsilon_1^{-1}(j) y^T(t) \tilde{S}_j L(y(t), j) W_0(j) W_0^T(j) L(y(t), j) \tilde{S}_j y(t) \\
&\quad + \varepsilon_1(j) \tilde{f}^T(x(t)) \tilde{f}(x(t)) \\
&\leq \varepsilon_1^{-1}(j) \mu_{\max}(W_0(j) W_0^T(j)) y^T(t) \tilde{S}_j L^2(y(t), j) \tilde{S}_j y(t) \\
&\quad + \varepsilon_1(j) \tilde{f}^T(x(t)) \tilde{f}(x(t)) \\
&\leq \varepsilon_1^{-1}(j) \mu_{\max}(W_0(j) W_0^T(j)) y^T(t) \tilde{S}_j \tilde{L}^2 \tilde{S}_j y(t) + \varepsilon_1(j) \\
&\quad \times x^T(t) \tilde{G}^2 x(t).
\end{aligned} \quad (12)$$

Similarly, one has

$$\begin{aligned}
 & 2x^T(t)S_iC(x(t), i)E(i)g(y(t - \sigma_{1,i})) \\
 & \leq \xi_2^{-1}(i)x^T(t)S_iC(x(t), i)E(i)E^T(i)C(x(t), i)S_ix(t) \\
 & + \xi_2(i)g^T(y(t - \sigma_{1,i}))g(y(t - \sigma_{1,i})) \\
 & \leq \xi_2^{-1}(i)\lambda_{\max}(E(i)E^T(i))x^T(t)S_iC^2(x(t), i)S_ix(t) \\
 & + \xi_2(i)g^T(y(t - \sigma_{1,i}))g(y(t - \sigma_{1,i})) \\
 & \leq \xi_2^{-1}(i)\lambda_{\max}(E(i)E^T(i))x^T(t)S_i\bar{C}^2(i)S_ix(t) + \xi_2(i) \\
 & \times y^T(t - \sigma_{1,i})\Delta^2y(t - \sigma_{1,i})
 \end{aligned} \quad (13)$$

$$\begin{aligned}
 & 2y^T(t)\tilde{S}_jL(y(t), j)W_1(j)\tilde{g}(x(t - \sigma_{2,j})) \\
 & \leq \varepsilon_2^{-1}(j)y^T(t)\tilde{S}_jL(y(t), j)W_1(j)W_1^T(j)L(y(t), j)\tilde{S}_j \\
 & \times y(t) + \varepsilon_2(j)\tilde{g}^T(x(t - \sigma_{2,j}))\tilde{g}(x(t - \sigma_{2,j})) \\
 & \leq \varepsilon_2^{-1}(j)\mu_{\max}(W_1(j)W_1^T(j))y^T(t)\tilde{S}_jL^2(y(t), j)\tilde{S}_jy(t) \\
 & + \varepsilon_2(j)\tilde{g}^T(x(t - \sigma_{2,j}))\tilde{g}(x(t - \sigma_{2,j})) \\
 & \leq \varepsilon_2^{-1}(j)\mu_{\max}(W_1(j)W_1^T(j))y^T(t)\tilde{S}_j\bar{L}^2(j)\tilde{S}_jy(t) \\
 & + \varepsilon_2(j)x^T(t - \sigma_{2,j})\tilde{\Delta}^2x(t - \sigma_{2,j})
 \end{aligned} \quad (14)$$

and

$$\begin{aligned}
 & 2x^T(t)S_iC(x(t), i)A(i)\int_{t-\tau_{1,i}}^t h(y(s))ds \\
 & \leq \xi_3^{-1}(i)x^T(t)S_iC(x(t), i)A(i)A^T(i)C(x(t), i)S_ix(t) \\
 & + \xi_3(i)\left[\int_{t-\tau_{1,i}}^t h(y(s))ds\right]^T\left[\int_{t-\tau_{1,i}}^t h(y(s))ds\right] \\
 & \leq \xi_3^{-1}(i)\lambda_{\max}(A(i)A^T(i))x^T(t)S_i\bar{C}^2(i)S_ix(t) + \xi_3(i) \\
 & \times \left[\int_{t-\tau_{1,i}}^t h(y(s))ds\right]^T\left[\int_{t-\tau_{1,i}}^t h(y(s))ds\right] \\
 & \leq \xi_3^{-1}(i)\lambda_{\max}(A(i)A^T(i))x^T(t)S_i\bar{C}^2(i)S_ix(t) + \xi_3(i) \\
 & \times \xi_{\tau_{1,i}}\int_{t-\tau_{1,i}}^t h^T(y(s))h(y(s))ds \quad (\text{by Lemma 2.7}) \\
 & \leq \xi_3^{-1}(i)\lambda_{\max}(A(i)A^T(i))x^T(t)S_i\bar{C}^2(i)S_ix(t) + \xi_3(i) \\
 & \times \tau_{1,i}\bar{\xi}_1^2\int_{t-\tau_{1,i}}^t y^T(s)y(s)ds
 \end{aligned} \quad (15)$$

$$\begin{aligned}
 & 2y^T(t)\tilde{S}_jL(y(t), j)W_2(j)\int_{t-\tau_{2,j}}^t \tilde{h}(x(s))ds \\
 & \leq \varepsilon_3^{-1}(j)y^T(t)\tilde{S}_jL(y(t), j)W_2(j)W_2^T(j)L(y(t), j)\tilde{S}_jy(t) \\
 & + \varepsilon_3(j)\left[\int_{t-\tau_{2,j}}^t \tilde{h}(x(s))ds\right]^T\left[\int_{t-\tau_{2,j}}^t \tilde{h}(x(s))ds\right] \\
 & \leq \varepsilon_3^{-1}(j)\mu_{\max}(W_2(j)W_2^T(j))y^T(t)\tilde{S}_j\bar{L}^2(j)\tilde{S}_jy(t) + \varepsilon_3(j) \\
 & \times \left[\int_{t-\tau_{2,j}}^t \tilde{h}(y(s))ds\right]^T\left[\int_{t-\tau_{2,j}}^t \tilde{h}(y(s))ds\right] \\
 & \leq \varepsilon_3^{-1}(j)\mu_{\max}(W_2(j)W_2^T(j))y^T(t)\tilde{S}_j\bar{L}^2(j)\tilde{S}_jy(t) + \varepsilon_3(j) \\
 & \times \tau_{2,j}\int_{t-\tau_{2,j}}^t \tilde{h}^T(x(s))\tilde{h}(x(s))ds \quad (\text{by Lemma 2.7}) \\
 & \leq \varepsilon_3^{-1}(j)\mu_{\max}(W_2(j)W_2^T(j))y^T(t)\tilde{S}_j\bar{L}^2(j)\tilde{S}_jy(t) + \varepsilon_3(j) \\
 & \times \tau_{2,j}\bar{\xi}_2^2\int_{t-\tau_{2,j}}^t x^T(s)x(s)ds
 \end{aligned} \quad (16)$$

Next, it is easy to see that

$$-\bar{\pi}\int_{\underline{\sigma}_1}^{\bar{\sigma}_1}y^T(t-s)Ty(t-s)ds = -\bar{\pi}\int_{t-\bar{\sigma}_1}^{t-\underline{\sigma}_1}y^T(s)Ty(s)ds, \quad (17)$$

$$-\bar{\tilde{\pi}}\int_{\underline{\sigma}_2}^{\bar{\sigma}_2}x^T(t-s)\tilde{T}x(t-s)ds = -\bar{\tilde{\pi}}\int_{t-\bar{\sigma}_2}^{t-\underline{\sigma}_2}x^T(s)\tilde{T}x(s)ds, \quad (18)$$

$$-\int_0^{\tau_{1,i}}y^T(t-s)Ry(t-s)ds = -\int_{t-\tau_{1,i}}^ty^T(s)Ry(s)ds, \quad (19)$$

$$-\int_0^{\tau_{2,j}}x^T(t-s)\tilde{R}x(t-s)ds = -\int_{t-\tau_{2,j}}^tx^T(s)\tilde{R}x(s)ds, \quad (20)$$

$$\bar{\pi}\int_{\underline{\tau}_1}^{\bar{\tau}_1}\int_0^uy^T(t)Ry(t)dsdu = \frac{1}{2}\bar{\pi}(\bar{\tau}_1^2 - \underline{\tau}_1^2)y^T(t)Ry(t), \quad (21)$$

$$\bar{\tilde{\pi}}\int_{\underline{\tau}_2}^{\bar{\tau}_2}\int_0^vx^T(t)\tilde{R}x(t)dsdv = \frac{1}{2}\bar{\tilde{\pi}}(\bar{\tau}_2^2 - \underline{\tau}_2^2)x^T(t)\tilde{R}x(t), \quad (22)$$

$$-\bar{\pi}\int_{\underline{\tau}_1}^{\bar{\tau}_1}\int_0^uy^T(t-s)Ry(t-s)dsdu = -\bar{\pi}\int_{\underline{\tau}_1}^{\bar{\tau}_1}\int_{t-s}^ty^T(\theta)Ry(\theta)d\theta ds, \quad (23)$$

$$-\bar{\tilde{\pi}}\int_{\underline{\tau}_2}^{\bar{\tau}_2}\int_0^vx^T(t-s)\tilde{R}x(t-s)dsdv = -\bar{\tilde{\pi}}\int_{\underline{\tau}_2}^{\bar{\tau}_2}\int_{t-s}^tx^T(\theta)\tilde{R}x(\theta)d\theta ds. \quad (24)$$

Moreover, one has

$$\begin{aligned}
 & \sum_{j=1}^m\pi_{ij}\int_{t-\sigma_{1,i}}^ty^T(s)Ty(s)ds \\
 & = \sum_{j \neq i}^m\pi_{ij}\int_{t-\sigma_{1,i}}^ty^T(s)Ty(s)ds + \pi_{ii}\int_{t-\sigma_{1,i}}^ty^T(s)Ty(s)ds \\
 & \leq \sum_{j \neq i}^m\pi_{ij}\int_{t-\bar{\sigma}_1}^ty^T(s)Ty(s)ds + \pi_{ii}\int_{t-\underline{\sigma}_1}^ty^T(s)Ty(s)ds \\
 & = -\pi_{ii}\int_{t-\bar{\sigma}_1}^ty^T(s)Ty(s)ds + \pi_{ii}\int_{t-\underline{\sigma}_1}^ty^T(s)Ty(s)ds \\
 & \leq \bar{\pi}\int_{t-\bar{\sigma}_1}^{t-\underline{\sigma}_1}y^T(s)Ty(s)ds,
 \end{aligned} \quad (25)$$

$$\begin{aligned}
 & \sum_{i=1}^m\tilde{\pi}_{ji}\int_{t-\sigma_{2,j}}^tx^T(s)\tilde{T}x(s)ds \\
 & = \sum_{i \neq j}^m\tilde{\pi}_{ji}\int_{t-\sigma_{2,j}}^tx^T(s)\tilde{T}x(s)ds + \tilde{\pi}_{jj}\int_{t-\sigma_{2,j}}^tx^T(s)\tilde{T}x(s)ds \\
 & \leq \sum_{i \neq j}^m\tilde{\pi}_{ji}\int_{t-\bar{\sigma}_2}^tx^T(s)\tilde{T}x(s)ds + \tilde{\pi}_{jj}\int_{t-\underline{\sigma}_2}^tx^T(s)\tilde{T}x(s)ds \\
 & = -\tilde{\pi}_{jj}\int_{t-\bar{\sigma}_2}^tx^T(s)\tilde{T}x(s)ds + \tilde{\pi}_{jj}\int_{t-\underline{\sigma}_2}^tx^T(s)\tilde{T}x(s)ds \\
 & \leq \bar{\tilde{\pi}}\int_{t-\bar{\sigma}_2}^{t-\underline{\sigma}_2}x^T(s)\tilde{T}x(s)ds
 \end{aligned} \quad (26)$$

and also

$$\sum_{j=1}^m \pi_{ij} \int_0^{\tau_{1,i}} \int_{t-s}^t y^T(\theta) R y(\theta) d\theta ds \leq \bar{\pi} \int_{\underline{\tau}_1}^{\bar{\tau}_1} \int_{t-s}^t y^T(\theta) R y(\theta) d\theta ds, \quad (27)$$

$$\sum_{i=1}^m \bar{\pi}_{ji} \int_0^{\tau_{2,j}} \int_{t-s}^t x^T(\theta) \tilde{R} x(\theta) d\theta ds \leq \bar{\pi} \int_{\underline{\tau}_2}^{\bar{\tau}_2} \int_{t-s}^t x^T(\theta) \tilde{R} x(\theta) d\theta ds, \quad (28)$$

Substituting (10)–(28) into (8) and employing (3) & (4), we have

$$\begin{aligned} \mathcal{LV}(x_t, y_t, t, i, j) &\leq -2x^T(t) \Gamma_1(i) S_i \bar{C}(i) x(t) \\ &\quad + \xi_1^{-1}(i) \lambda_{\max}(D(i) D^T(i)) x^T(t) S_i \bar{C}^2(i) S_i x(t) + \xi_1(i) y^T(t) G^2 y(t) \\ &\quad + \xi_2^{-1}(i) \lambda_{\max}(E(i) E^T(i)) x^T(t) S_i \bar{C}^2(i) S_i x(t) \\ &\quad + \xi_2(i) y^T(t - \sigma_{1,i}) \Delta^2 y(t - \sigma_{1,i}) \\ &\quad + \xi_3^{-1}(i) \lambda_{\max}(A(i) A^T(i)) x^T(t) S_i \bar{C}^2(i) \\ &\quad \times S_i x(t) - \xi_3(i) \tau_{1,i} \bar{S}_1^{-2} \left(- \int_{t-\tau_{1,i}}^t y^T(s) y(s) ds \right) + \sum_{j=1}^m \pi_{ij} x^T(t) S_i x(t) \\ &\quad + y^T(t) T y(t) - y^T(t - \sigma_{1,i}) T y(t - \sigma_{1,i}) + \bar{\pi} \int_{t-\bar{\sigma}_1}^{t-\underline{\sigma}_1} y^T(s) T y(s) ds + \bar{\pi} \\ &\quad \times (\bar{\sigma}_1 - \underline{\sigma}_1) y^T(t) T y(t) - \bar{\pi} \int_{t-\bar{\sigma}_1}^{t-\underline{\sigma}_1} y^T(s) T y(s) ds + \tau_{1,i} y^T R y(t) \\ &\quad - \int_{t-\tau_{1,i}}^t y^T(s) R y(s) ds + \bar{\pi} \int_{\underline{\tau}_1}^{\bar{\tau}_1} \int_{t-s}^t y^T(\theta) R y(\theta) d\theta ds \\ &\quad + \frac{1}{2} \pi (\bar{\tau}_1^2 - \underline{\tau}_1^2) \times y^T(t) R y(t) - \bar{\pi} \int_{\underline{\tau}_1}^{\bar{\tau}_1} \int_{t-s}^t y^T(\theta) R y(\theta) d\theta ds \\ &\quad - 2y^T(t) \Gamma_2(j) \tilde{S}_j \bar{L}(j) y(t) \\ &\quad + \varepsilon_1^{-1}(j) \mu_{\max}(W_0(j) W_0^T(j)) y^T(t) \tilde{S}_j \bar{L}^2(j) \tilde{S}_j y(t) \\ &\quad + \varepsilon_1(j) x^T(t) \tilde{G}^2 x(t) \\ &\quad + \varepsilon_2^{-1}(j) \mu_{\max}(W_1(j) W_1^T(j)) y^T(t) \tilde{S}_j \bar{L}^2(j) \tilde{S}_j y(t) + \varepsilon_2(j) x^T(t - \sigma_{2,j}) \\ &\quad \times \tilde{\Delta}^2 x(t - \sigma_{2,j}) + \varepsilon_3^{-1}(j) \mu_{\max}(W_2(j) W_2^T(j)) y^T(t) \tilde{S}_j \bar{L}^2(j) \tilde{S}_j y(t) \\ &\quad - \varepsilon_3(j) \tau_{2,j} \bar{S}_2^{-2} \left(- \int_{t-\tau_{2,j}}^t x^T(s) x(s) ds \right) + \sum_{i=1}^m \bar{\pi}_{ji} y^T(t) \tilde{S}_j y(t) + x^T(t) \tilde{T} \\ &\quad \times x(t) - x^T(t - \sigma_{2,j}) \tilde{T} x(t - \sigma_{2,j}) + \bar{\pi} \int_{t-\bar{\sigma}_2}^{t-\underline{\sigma}_2} x^T(s) \tilde{T} x(s) ds + \bar{\pi} (\bar{\sigma}_2 \\ &\quad - \underline{\sigma}_2) x^T(t) \tilde{T} x(t) - \bar{\pi} \int_{t-\bar{\sigma}_2}^{t-\underline{\sigma}_2} x^T(s) \tilde{T} x(s) ds + \tau_{2,j} x^T \tilde{R} x(t) \\ &\quad - \int_{t-\tau_{2,j}}^t x^T(s) \tilde{R} x(s) ds + \bar{\pi} \int_{\underline{\tau}_2}^{\bar{\tau}_2} \int_{t-s}^t x^T(\theta) \tilde{R} x(\theta) d\theta ds \\ &\quad + \frac{1}{2} \bar{\pi} (\bar{\tau}_2^2 - \underline{\tau}_2^2) \times x^T(t) \tilde{R} x(t) - \bar{\pi} \int_{\underline{\tau}_2}^{\bar{\tau}_2} \int_{t-s}^t x^T(\theta) \tilde{R} x(\theta) d\theta ds \quad (29) \end{aligned}$$

It is easy to see that,

$$- \int_{t-\tau_{1,i}}^t y^T(s) y(s) ds \leq - \left(\int_{t-\tau_{1,i}}^t y(s) ds \right)^T \left(\int_{t-\tau_{1,i}}^t y(s) ds \right) \quad (30)$$

$$\& - \int_{t-\tau_{1,i}}^t y^T(s) R y(s) ds \leq - \left(\int_{t-\tau_{1,i}}^t y(s) ds \right)^T R \left(\int_{t-\tau_{1,i}}^t y(s) ds \right) \quad (31)$$

$$- \int_{t-\tau_{2,j}}^t x^T(s) x(s) ds \leq - \left(\int_{t-\tau_{2,j}}^t x(s) ds \right)^T \left(\int_{t-\tau_{2,j}}^t x(s) ds \right) \quad (32)$$

$$\& - \int_{t-\tau_{2,j}}^t x^T(s) \tilde{R} x(s) ds \leq - \left(\int_{t-\tau_{2,j}}^t x(s) ds \right)^T \tilde{R} \left(\int_{t-\tau_{2,j}}^t x(s) ds \right) \quad (33)$$

$$\mathcal{LV}(x_t, y_t, t, i, j) \leq \Psi^T(t) \Theta_i \Psi(t) + \Phi^T(t) \Upsilon_j \Phi(t), \quad (34)$$

where $\Psi^T(t) = [x^T(t), y^T(t), y^T(t - \sigma_{1,i}), (\int_{t-\tau_{1,i}}^t y(s) ds)^T]^T$ & $\Phi^T(t) = [y^T(t), x^T(t), x^T(t - \sigma_{2,j}), (\int_{t-\tau_{2,j}}^t x(s) ds)^T]^T$. Then, it follows from Lemma 2.8 that (5) and (6) are equivalent to Θ_i & $\Upsilon_j < 0$.

Letting $\delta = \max_{1 \leq i \leq m} \lambda_{\max}(\Theta_i)$ & $\hat{\delta} = \max_{1 \leq j \leq m} \mu_{\max}(\Upsilon_j)$, we have δ & $\hat{\delta} < 0$ and

$$\mathcal{LV}(x_t, y_t, t, i, j) \leq \delta \Psi^T(t) \Theta_i \Psi(t) + \hat{\delta} \Phi^T(t) \Upsilon_j \Phi(t) \quad (35)$$

For any scalar $T > 0$, integrating on both sides of the above inequality from 0 to T yield

$$\begin{aligned} \mathbb{E}\{V(x_T, y_T, T, r(T), \tilde{r}(T))\} - \mathbb{E}\{V(x_0, y_0, 0, r(0), \tilde{r}(0))\} \\ \leq \delta^* \int_0^T \mathbb{E}\{x^T(t) x(t) + y^T(t) y(t)\} dt \quad (\text{since } \delta^* = \max\{\delta, \hat{\delta}\}) \\ \leq \delta^* \left[\int_0^T \mathbb{E}\{x^T(t) x(t)\} dt + \int_0^T \mathbb{E}\{y^T(t) y(t)\} dt \right] \quad (36) \end{aligned}$$

Since $\mathbb{E}\{V(x_T, y_T, T, r(T), \tilde{r}(T))\} \geq 0$, we have

$$\begin{aligned} \int_0^T \mathbb{E}\{x^T(t) x(t)\} dt + \int_0^T \mathbb{E}\{y^T(t) y(t)\} dt \\ \leq -\frac{1}{\delta^*} \mathbb{E}\{V(x_0, y_0, 0, r(0), \tilde{r}(0))\}, \quad (37) \end{aligned}$$

which implies that

$$\begin{aligned} \int_0^{+\infty} \mathbb{E}\{|x(t)|^2\} dt + \int_0^{+\infty} \mathbb{E}\{|y(t)|^2\} dt \\ \leq -\frac{1}{\delta^*} \mathbb{E}\{V(x_0, y_0, 0, r(0), \tilde{r}(0))\} \quad (38) \end{aligned}$$

According to Barbalat's Lemma in [45], we obtain

$$\lim_{t \rightarrow \infty} \{\mathbb{E}\{|x(t)|^2\} + \mathbb{E}\{|y(t)|^2\}\} = 0; \text{ for } t \neq t_k \quad (39)$$

For $t = t_k$, we have

$$\begin{aligned} V(x_{t_k}, y_{t_k}, t_k, i, j) - V(x_{t_k^-}, y_{t_k^-}, t_k^-, i, j) \\ = [x^T(t_k) S_i x(t_k) + y^T(t_k) \tilde{S}_j y(t_k)] - [x^T(t_k^-) S_i \\ \times x(t_k^-) + y^T(t_k^-) \tilde{S}_j y(t_k^-)] \\ \leq x^T(t_k^-) [P_1^T S_i P_1 - S_i] x(t_k^-) \\ + y^T(t_k^-) [P_2^T \tilde{S}_j P_2 - \tilde{S}_j] y(t_k^-) \\ \leq 0. \end{aligned}$$

therefore $V(x_{t_k}, y_{t_k}, t_k, i, j) \leq V(x_{t_k^-}, y_{t_k^-}, t_k^-, i, j)$, $k \in \mathbb{Z}_+$. (40)

From (39) & (40), the neural networks under consideration is asymptotically stable in the mean square sense. Hence, the proof of this theorem is complete.

Up to now, a sufficient condition has been established to guarantee the asymptotic stability in the mean square sense for the addressed neural networks (2). Next, we aim to discuss about the globally exponential stability problem for the CGMBAMNNs (2). □

Theorem 3.2. Under the same Assumptions and conditions as in Theorem 3.1, the Cohen–Grossberg neural networks (2) is globally exponential stable in the mean square.

Proof. To deal with the globally exponential stability of the neural networks (2), we choose the following Lyapunov–Krasovskii functional candidate:

$$\begin{aligned}
V^*(x_t, y_t, t, r(t), \tilde{r}(t)) &= V(x_t, y_t, t, r(t), \tilde{r}(t)) - \frac{\delta}{2\sigma} \int_0^\sigma \int_{t-s}^t y^T(\theta) y(\theta) d\theta ds \\
&\quad - \frac{\delta}{2\tau} \int_0^\tau \int_{t-s}^t y^T(\theta) \\
&\quad \times y(\theta) d\theta ds - \frac{\hat{\delta}}{2\sigma} \int_0^\sigma \int_{t-s}^t x^T(\theta) x(\theta) d\theta ds \\
&\quad - \frac{\hat{\delta}}{2\tau} \int_0^\tau \int_{t-s}^t x^T(\theta) x(\theta) \times d\theta ds, \quad (41)
\end{aligned}$$

where $V(x_t, y_t, t, r(t), \tilde{r}(t))$ is defined as in (7). Then, according to (35), we have

$$\begin{aligned}
\mathcal{L}V^*(x_t, y_t, t, i, j) &= \mathcal{L}V(x_t, y_t, t, i, j) - \frac{\delta}{2} y^T(t) y(t) + \frac{\delta}{2\sigma} \int_0^\sigma y^T(t-s) y(t-s) \\
&\quad \times ds - \frac{\hat{\delta}}{2} x^T(t) x(t) + \frac{\hat{\delta}}{2\tau} \int_0^\tau x^T(t-s) x(t-s) ds - \frac{\delta}{2} \\
&\quad \times y^T(t) y(t) + \frac{\delta}{2\tau} \int_0^\tau y^T(t-s) y(t-s) ds - \frac{\hat{\delta}}{2} x^T(t) x(t) \\
&\quad + \frac{\hat{\delta}}{2\sigma} \int_0^\sigma x^T(t-s) x(t-s) ds \\
&\leq \frac{\delta}{2} y^T(t) y(t) + \frac{\hat{\delta}}{2} x^T(t) x(t) + \frac{\delta}{2\sigma} \int_{t-\sigma}^t y^T(s) y(s) ds + \frac{\hat{\delta}}{2\tau} \\
&\quad \times \int_{t-\tau}^t x^T(s) x(s) ds + \frac{\delta}{2} y^T(t) y(t) + \frac{\hat{\delta}}{2} x^T(t) x(t) + \frac{\delta}{2\tau} \\
&\quad \times \int_{t-\tau}^t y^T(s) y(s) ds + \frac{\hat{\delta}}{2\sigma} \int_{t-\sigma}^t x^T(s) x(s) ds. \quad (42)
\end{aligned}$$

From (7) and (41), we have

$$\begin{aligned}
V^*(x_t, y_t, t, i, j) &\leq x^T(t) S_i x(t) + \int_{t-\sigma}^t y^T(s) T y(s) ds + \tilde{\pi} \int_{\underline{\sigma}_1}^{\bar{\sigma}_1} \int_{t-\sigma}^t y^T(\theta) T y(\theta) \\
&\quad \times d\theta ds + \int_0^{\tau_{1,i}} \int_{t-\tau}^t y^T(\theta) R y(\theta) d\theta ds + \tilde{\pi} \int_{\underline{\tau}_1}^{\bar{\tau}_1} \int_0^t \int_{t-\tau}^t y^T(\theta) \\
&\quad \times R y(\theta) d\theta ds + y^T(t) \tilde{S}_j y(t) + \int_{t-\sigma}^t x^T(s) \tilde{T} x(s) ds \\
&\quad + \tilde{\pi} \int_{\underline{\sigma}_2}^{\bar{\sigma}_2} \int_{t-\sigma}^t x^T(\theta) \tilde{T} x(\theta) d\theta ds + \int_0^{\tau_{2,j}} \int_{t-\tau}^t x^T(\theta) \tilde{R} x(\theta) d\theta ds \\
&\quad + \tilde{\pi} \int_{\underline{\tau}_2}^{\bar{\tau}_2} \int_0^t \int_{t-\tau}^t x^T(\theta) \tilde{R} x(\theta) d\theta ds + \frac{\delta}{2\sigma} \int_0^\sigma \int_{t-\sigma}^t y^T(\theta) y(\theta) \\
&\quad \times d\theta ds - \frac{\delta}{2\tau} \int_0^\tau \int_{t-\tau}^t y^T(\theta) y(\theta) d\theta ds - \frac{\hat{\delta}}{2\sigma} \int_0^\sigma \int_{t-\sigma}^t x^T(\theta) x(\theta) \\
&\quad \times d\theta ds - \frac{\hat{\delta}}{2\tau} \int_0^\tau \int_{t-\tau}^t x^T(\theta) x(\theta) d\theta ds \\
&\leq v_0 x^T(t) x(t) + \tilde{v}_0 y^T(t) y(t) + v_1 \int_{t-\sigma}^t y^T(s) y(s) ds + \tilde{v}_1 \int_{t-\sigma}^t x^T(s) x(s) \\
&\quad \times x(s) ds + v_2 \int_{t-\tau}^t y^T(s) y(s) ds + \tilde{v}_2 \int_{t-\tau}^t x^T(s) x(s) ds, \quad (43)
\end{aligned}$$

where $v_0 = \max_{1 \leq i \leq m} \{\lambda_{\max}(S_i)\}$, $\tilde{v}_0 = \max_{1 \leq j \leq m} \{\mu_{\max}(\tilde{S}_j)\}$, $v_1 = \max_{1 \leq i \leq m} \{(1 + \tilde{\pi}(\bar{\sigma}_1 - \underline{\sigma}_1)) \lambda_{\max}(T) - \frac{\delta}{2}\}$, $\tilde{v}_1 = \max_{1 \leq j \leq m} \{(1 + \tilde{\pi}(\bar{\sigma}_2 - \underline{\sigma}_2)) \mu_{\max}(\tilde{T}) - \frac{\hat{\delta}}{2}\}$, $v_2 = (\tau_{1,i} + \frac{1}{2} \tilde{\pi}(\bar{\tau}_1^2 - \underline{\tau}_1^2)) \lambda_{\max}(R) - \frac{\delta}{2}$, $\tilde{v}_2 = (\tau_{2,j} + \frac{1}{2} \tilde{\pi}(\bar{\tau}_2^2 - \underline{\tau}_2^2)) \mu_{\max}(\tilde{R}) - \frac{\hat{\delta}}{2}$.

Subsequently, let β_0 be a constant to be determined later and consider

$$\begin{aligned}
\mathcal{L}[e^{\beta_0 t} V^*(x_t, y_t, t, i, j)] &= \beta_0 e^{\beta_0 t} V^*(x_t, y_t, t, i, j) \\
&\quad + e^{\beta_0 t} \mathcal{L}[V^*(x_t, y_t, t, i, j)] \quad (44)
\end{aligned}$$

It follows from (42) to (44) that

$$\begin{aligned}
\mathcal{L}[e^{\beta_0 t} V^*(x_t, y_t, t, i, j)] &= e^{\beta_0 t} \left[(\beta_0 v_0 + \hat{\delta}) x^T(t) x(t) \right. \\
&\quad + (\beta_0 \tilde{v}_0 + \delta) y^T(t) y(t) + \left(\beta_0 v_1 + \frac{\delta}{2\sigma} \right) \\
&\quad \times \int_{t-\sigma}^t y^T(s) y(s) ds + \left(\beta_0 \tilde{v}_1 + \frac{\hat{\delta}}{2\sigma} \right) \\
&\quad \times \int_{t-\sigma}^t x^T(s) x(s) ds + (\beta_0 v_2 \\
&\quad + \frac{\delta}{2\tau}) \int_{t-\tau}^t y^T(s) y(s) ds + \left(\beta_0 \tilde{v}_2 + \frac{\hat{\delta}}{2\tau} \right) \\
&\quad \times \int_{t-\tau}^t x^T(s) x(s) ds \left. \right] \quad (45)
\end{aligned}$$

Now, we choose β_0 satisfying $\beta_0 v_0 + \hat{\delta}$, $\beta_0 \tilde{v}_0 + \delta$, $\beta_0 v_1 + \frac{\delta}{2\sigma}$, $\beta_0 \tilde{v}_1 + \frac{\hat{\delta}}{2\sigma}$, $\beta_0 v_2 + \frac{\delta}{2\tau}$ & $\beta_0 \tilde{v}_2 + \frac{\hat{\delta}}{2\tau}$ are < 0 . Then, we obtain

$$\mathcal{L}[e^{\beta_0 t} V^*(x_t, y_t, t, i, j)] \leq 0 \quad (46)$$

By integration, we have

$$\begin{aligned}
\mathbb{E}\{e^{\beta_0 t} V^*(x_t, y_t, t, r(t), \tilde{r}(t))\} &= \mathbb{E}\{V^*(x_0, y_0, 0, r(0), \tilde{r}(0))\} \\
&\quad + \mathbb{E}\left\{\int_0^t \mathcal{L}[e^{\beta_0 s} V^*(x_s, y_s, s, r(s), \tilde{r}(s))] ds\right\} \\
&\leq \mathbb{E}\{V^*(x_0, y_0, 0, r(0), \tilde{r}(0))\} \quad (47)
\end{aligned}$$

Together with (43), one has

$$\begin{aligned}
\mathbb{E}\{V^*(x_0, y_0, 0, r(0), \tilde{r}(0))\} &\leq (v_0 + \tilde{v}_1 \sigma + \tilde{v}_2 \tau) \\
&\quad \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|x(s)|^2\} + (\tilde{v}_0 + v_1 \sigma + v_2 \tau) \\
&\quad \times \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|y(s)|^2\} \quad (48)
\end{aligned}$$

Furthermore, it is easy to see that

$$\mathbb{E}\{e^{\beta_0 t} V^*(x_t, y_t, t, r(t), \tilde{r}(t))\} \geq \lambda_0 e^{\beta_0 t} \mathbb{E}\{|x(t)|^2\} + \mu_0 e^{\beta_0 t} \mathbb{E}\{|y(t)|^2\}, \quad (49)$$

where $\lambda_0 = \max_{1 \leq i \leq m} \lambda_{\max}(S_i)$ & $\mu_0 = \max_{1 \leq j \leq m} \mu_{\max}(\tilde{S}_j)$. Then, it follows from (47)–(49) that:

$$\begin{aligned}
\lambda_0 \mathbb{E}\{|x(t)|^2\} + \mu_0 \mathbb{E}\{|y(t)|^2\} &\leq e^{-\beta_0 t} \left[M \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|x(s)|^2\} \right. \\
&\quad \left. + N \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|y(s)|^2\} \right], \quad (50)
\end{aligned}$$

where $M = v_0 + \tilde{v}_1 \sigma + \tilde{v}_2 \tau > 0$ and $N = \tilde{v}_0 + v_1 \sigma + v_2 \tau > 0$.

Put $\Lambda = \max\{M, N\}$ & $\eta = \min\{\lambda_0, \mu_0\}$.

$$\begin{aligned}
\eta \{\mathbb{E}\{|x(t)|^2\} + \mathbb{E}\{|y(t)|^2\}\} &\leq \Lambda e^{-\beta_0 t} \left\{ \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|x(s)|^2\} \right. \\
&\quad \left. + \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|y(s)|^2\} \right\} \quad (51)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\{|x(t)|^2\} + \mathbb{E}\{|y(t)|^2\} &\leq \frac{\Lambda}{\eta} e^{-\beta_0 t} \left\{ \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|x(s)|^2\} \right. \\
&\quad \left. + \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|y(s)|^2\} \right\}, \text{ for } t \neq t_k. \quad (52)
\end{aligned}$$

For $t = t_k$,

Now consider a new Lyapunov candidate as follows:

$$V'(x(t_k), y(t_k), t_k, i, j) = e^{s^*t} V(x(t_k), y(t_k), t_k, i, j), \quad (53)$$

where $V(x(t_k), y(t_k), t_k, i, j)$ is previously defined in Theorem 3.1.

From (40) in previous theorem, which implies that,

$$V(x(t_k), y(t_k), t_k, i, j) \leq V(x(t_{k-}), y(t_{k-}), t_{k-}, i, j), \quad k \in \mathbb{Z}_+.$$

Similarly, we have

$$V'(x(t_k), y(t_k), t_k, i, j) \leq e^{s^*t} V(x(t_{k-}), y(t_{k-}), t_{k-}, i, j), \quad k \in \mathbb{Z}_+.$$

Using mathematical induction, we have that for all $1 \leq i, j \leq m$ and $k \geq 1$;

$$\begin{aligned} \mathbb{E}V'(x(t_k), y(t_k), t_k, i, j) &\leq e^{s^*t} \mathbb{E}V(x(t_{k-}), y(t_{k-}), t_{k-}, i, j) \\ &\leq e^{s^*t} \mathbb{E}V(x(t_{k-1}), y(t_{k-1}), t_{k-1}, r(t_{k-1}), \tilde{r}(t_{k-1})) \\ &\leq e^{s^*t} \mathbb{E}V(x(t_{k-1}^-), y(t_{k-1}^-), t_{k-1}^-, r(t_{k-1}^-), \tilde{r}(t_{k-1}^-)) \\ &\leq \dots \\ &\leq e^{s^*t} \mathbb{E}V(x(t_0), y(t_0), t_0, r(t_0), \tilde{r}(t_0)) \end{aligned}$$

Since $t > t'$, it follows from Dynkin's formula, we have:

$$\begin{aligned} \mathbb{E}V'(x(t), y(t), t, i, j) &\leq e^{s^*t} \mathbb{E}V(x(t'), y(t'), t', r(t'), \tilde{r}(t')) \\ \mathbb{E}V'(x(t), y(t), t, i, j) &\leq \mathbb{E}V(x(0), y(0), 0, r(0), \tilde{r}(0)) \\ &\leq \mathbb{E}x^T(0)S_{r(0)}x(0) + \mathbb{E}y^T(0)\tilde{S}_{\tilde{r}(0)}y(0) \\ &\leq \max_{1 \leq i \leq m}(\lambda_i) \sup_{-\tau, \sigma \leq s \leq 0} \mathbb{E}\{|x(s)|^2\} \\ &\quad + \max_{1 \leq j \leq m}(\mu_j) \sup_{-\tau, \sigma \leq s \leq 0} \mathbb{E}\{|y(s)|^2\} \end{aligned} \quad (54)$$

By the definition of $V'(x(t), y(t), t, i, j)$, we obtain

$$\begin{aligned} V'(x(t), y(t), t, i, j) &= e^{s^*t} [x^T(t)S_i x(t) + y^T(t)\tilde{S}_j y(t)] \\ &\geq e^{s^*t} [\lambda_{\min}(S_i)|x(t)|^2 + \mu_{\min}(\tilde{S}_j)|y(t)|^2] \\ &\geq e^{s^*t} \min\{\lambda_{\min}(S_i), \mu_{\min}(\tilde{S}_j)\} [|x(t)|^2 \\ &\quad + |y(t)|^2] \end{aligned} \quad (55)$$

$\Lambda^* = \max_{1 \leq i \leq m}(\lambda_i)$ and $\Upsilon^* = \max_{1 \leq j \leq m}(\mu_j)$. Then combining (54) and (55), we have

$$\begin{aligned} \mathbb{E}\{|x(t)|^2\} + \mathbb{E}\{|y(t)|^2\} &\leq \\ &\frac{e^{-s^*t} [\Lambda^* \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|x(s)|^2\} + \Upsilon^* \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|y(s)|^2\}]}{\min\{\lambda_{\min}(S_i), \mu_{\min}(\tilde{S}_j)\}} \end{aligned}$$

Take $\Omega = \max\{\Lambda^*, \Upsilon^*\}$, we get

$$\begin{aligned} \mathbb{E}\{|x(t)|^2\} + \mathbb{E}\{|y(t)|^2\} &\leq \\ &\frac{e^{-s^*t} \Omega [\sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|x(s)|^2\} + \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|y(s)|^2\}]}{\min\{\lambda_{\min}(S_i), \mu_{\min}(\tilde{S}_j)\}} \\ \text{therefore } \mathbb{E}\{|x(t)|^2\} + \mathbb{E}\{|y(t)|^2\} &\leq \\ \chi e^{-s^*t} \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|x(s)|^2\} + \sup_{-\tau, -\sigma \leq s \leq 0} \mathbb{E}\{|y(s)|^2\}. \end{aligned} \quad (56)$$

where $\chi = \frac{\Omega}{\min\{\lambda_{\min}(S_i), \mu_{\min}(\tilde{S}_j)\}}$, for $t = t_k$.

Therefore From (52) & (56), we have the equilibrium point of the neural networks (2) is globally exponential stable in the mean square.

This completes the proof of this theorem. $\square$

Remark 3.1. Suppose, in neural networks (2) the Markovian jumps are absent, then the mixed time delayed CGBAMNNs (2) reduces to

$$\begin{aligned} \dot{x}(t) &= -C(x(t)) \left[B(x(t)) - Df(y(t)) - Eg(y(t - \sigma_1(t))) \right. \\ &\quad \left. - A \int_{t-\tau_1}^t h(y(s))ds \right], \quad t > 0, t \neq t_k, \\ \Delta x(t) &= x(t^+) - x(t^-); \quad t = t_k, \quad k \in \mathbb{Z}_+. \\ \dot{y}(t) &= -L(y(t)) \left[M(y(t)) - W_0 \tilde{f}(x(t)) - W_1 \tilde{g}(x(t - \sigma_2(t))) \right. \\ &\quad \left. - W_2 \int_{t-\tau_2}^t \tilde{h}(x(s))ds \right], \quad t > 0, t \neq t_k, \\ \Delta y(t) &= y(t^+) - y(t^-); \quad t = t_k, \quad k \in \mathbb{Z}_+ \end{aligned} \quad (57)$$

Then by Theorem 3.2, it is easy to have the following Corollary 3.2.

Corollary 3.2. Suppose that Assumptions 1–3 holds, the Cohen–Grossberg neural networks (2) is globally exponential stable in the mean square sense if there exists a set of diagonal matrices $S > 0$ & $\tilde{S} > 0$, two matrices $T, \tilde{T} > 0$ and $R, \tilde{R} > 0$, and three sets of scalars ξ_1, ξ_2, ξ_3 and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that the following LMIs are satisfied:

$$\xi_2 \Delta^2 < T, \quad \varepsilon_2 \tilde{\Delta}^2 < \tilde{T}, \quad (58)$$

$$\xi_3 \tau_1 \tilde{\zeta}_1^2 I < R, \quad \varepsilon_3 \tau_2 \tilde{\zeta}_2^2 I < \tilde{R}, \quad (59)$$

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} \\ * & \Theta_{22} & \Theta_{23} & \Theta_{24} \\ * & * & \Theta_{33} & \Theta_{34} \\ * & * & * & \Theta_{44} \end{pmatrix} < 0, \quad (60)$$

$$\Upsilon = \begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & \Upsilon_{14} \\ * & \Upsilon_{22} & \Upsilon_{23} & \Upsilon_{24} \\ * & * & \Upsilon_{33} & \Upsilon_{34} \\ * & * & * & \Upsilon_{44} \end{pmatrix} < 0, \quad (61)$$

where

$$\begin{aligned} \Theta_{11} &= -2\Gamma_1 \tilde{S} \underline{C} + \xi_1^{-1} \lambda_{\max}(DD^T) \tilde{S} \tilde{C}^2 S + \vartheta_1; \quad \Theta_{12} = \Theta_{13} = \Theta_{14} \\ &= \Theta_{23} = 0; \quad \vartheta_1 = \xi_2^{-1} \lambda_{\max}(EE^T) \tilde{S} \tilde{C}^2 S + \xi_3^{-1} \lambda_{\max}(AA^T) \tilde{S} \tilde{C}^2(i)S; \\ \Upsilon_{34} &= 0; \quad \Theta_{34} = 0; \quad \Theta_{33} = \xi_2 \Delta^2 - T; \quad \Theta_{44} = \xi_3 \tau_1 \tilde{\zeta}_1^2 - R; \\ \Upsilon_{33} &= \varepsilon_2 \tilde{\Delta}^2 - \tilde{T}; \quad \Upsilon_{44} = \varepsilon_3 \tau_2 \tilde{\zeta}_2^2 - \tilde{R}; \end{aligned}$$

$$\begin{aligned} \Upsilon_{11} &= -2\Gamma_2 \tilde{S} \underline{L} + \varepsilon_1^{-1} \mu_{\max}(W_0 W_0^T) \tilde{S} \tilde{L}^2 \tilde{S} + \vartheta_2; \quad \Upsilon_{23} = 0 \quad \Upsilon_{24} = 0; \\ \vartheta_2 &= \varepsilon_2^{-1} \mu_{\max}(W_1 W_1^T) \tilde{S} \tilde{L}^2 \tilde{S} + \varepsilon_3^{-1} \mu_{\max}(W_2 W_2^T) \tilde{S} \tilde{L}^2 \tilde{S}; \quad \Upsilon_{14} = 0; \\ \Theta_{24} &= 0; \quad \Upsilon_{22} = \varepsilon_1 \tilde{G}^2 + \tilde{T} + \tau_2 \tilde{R}; \quad \Upsilon_{12} = 0; \\ \Upsilon_{13} &= 0; \quad \Theta_{22} = \xi_1 G^2 + T + \tau_1 R. \end{aligned}$$

Remark 3.3. If $C(x(t), i) = L(y(t), j) = 1$, $B(x(t), i) = A(u(t))$ & $M(y(t), j) = Bh(t)$ and $A(i) = W_2(j) = 0$, for all $x(t), y(t) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $i, j \in \mathcal{N}$ in the neural networks (2), Yang et. al [53] have derived many sufficient conditions for the exponential stability of the simplified neural networks. So, the obtained results in this paper improve and more generalize a large class of earlier results.

Remark 3.4. When $\tau_{1,i} = \tau_{2,j} = 0$, that is, the time delayed CG-BAMNNs (2) reduces to

$$\begin{aligned} \dot{x}(t) &= -C(x(t), i) \left[B(x(t), i) - D(i)f(y(t)) - E(i)g(y(t) \right. \\ &\quad \left. - \sigma_{1,i}(t))) \right], \quad i = 1, 2, \dots, n, \quad t > 0, t \neq t_k, \\ \Delta x(t) &= x(t^+) - x(t^-); \quad t = t_k, \quad k \in \mathbb{Z}_+. \\ \dot{y}(t) &= -L(y(t), j) \left[M(y(t), j) \right. \end{aligned}$$

$$\begin{aligned} & -W_0(j)\tilde{f}(x(t)) - W_1(j)\tilde{g}(x(t - \sigma_{2,j}(t))) \Big], \\ & j = 1, 2, \dots, n, t > 0, t \neq t_k, \\ \Delta y(t) = y(t^+) - y(t^-); t = t_k, k \in \mathbb{Z}_+ \end{aligned} \quad (62)$$

Then, according to Theorem 3.2, it is easy to have the following Corollary 3.5.

Corollary 3.5. Suppose that Assumptions 1–3 holds, the Cohen–Grossberg neural networks (2) is globally exponential stable in the mean square sense if there exists a set of diagonal matrices $S_i > 0$ & $\tilde{S}_j > 0$, two matrices $T, \tilde{T} > 0$ and $R, \tilde{R} > 0$, and three sets of scalars $\xi_1(i), \xi_2(i), \xi_3(i)$ and $\varepsilon_1(j), \varepsilon_2(j), \varepsilon_3(j)$ such that the following LMIs are satisfied:

$$\xi_2(i)\Delta^2 < T, \quad \varepsilon_2(j)\tilde{\Delta}^2 < \tilde{T}, \quad (63)$$

$$\Theta_i = \begin{pmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} \\ * & \Theta_{22} & \Theta_{23} & \Theta_{24} \\ * & * & \Theta_{33} & \Theta_{34} \\ * & * & * & \Theta_{44} \end{pmatrix} < 0, \quad (64)$$

$$\Upsilon_i = \begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & \Upsilon_{14} \\ * & \Upsilon_{22} & \Upsilon_{23} & \Upsilon_{24} \\ * & * & \Upsilon_{33} & \Upsilon_{34} \\ * & * & * & \Upsilon_{44} \end{pmatrix} < 0, \quad (65)$$

where

$$\begin{aligned} \Theta_{11} &= -2\Gamma_1(i)S_i\bar{C}(i) + \xi_1^{-1}(i)\lambda_{\max}(D(i)D^T(i))S_i\bar{C}^2(i)S_i + \vartheta_1; \\ \Theta_{12} &= \Theta_{13} = \Theta_{14} = 0; \vartheta_1 = \xi_2^{-1}(i)\lambda_{\max}(E(i)E^T(i))S_i\bar{C}^2(i)S_i \\ &+ \sum_{j=1}^m \pi_{ij}S_i; \Upsilon_{12} = 0; \quad \Upsilon_{13} = 0; \quad \Upsilon_{14} = 0; \Theta_{33} = \xi_2(i)\Delta^2 - T; \\ \Theta_{44} &= -R; \Upsilon_{33} = \varepsilon_2(j)\tilde{\Delta}^2 - \tilde{T}; \quad \Upsilon_{44} = -\tilde{R}; \quad \Theta_{23} = 0; \\ \Upsilon_{11} &= -2\Gamma_2(j)\tilde{S}_j\bar{L}(j) + \varepsilon_1^{-1}(j)\mu_{\max}(W_0(j)W_0^T(j))\tilde{S}_j\bar{L}^2(j)\tilde{S}_j + \vartheta_2; \\ \Upsilon_{23} &= 0; \quad \Upsilon_{24} = 0; \vartheta_2 = \varepsilon_2^{-1}(j)\mu_{\max}(W_1(j)W_1^T(j))\tilde{S}_j\bar{L}^2(j)\tilde{S}_j \\ &+ \sum_{i=1}^m \tilde{\pi}_{ji}\tilde{S}_j; \quad \Theta_{24} = 0; \Upsilon_{34} = 0; \quad \Theta_{34} = 0; \\ \Upsilon_{22} &= \varepsilon_1(j)\tilde{G}^2 + \tilde{T} + \tilde{\pi}(\tilde{\sigma}_2 - \underline{\sigma}_2)\tilde{T}; \\ \Theta_{22} &= \xi_1(i)G^2 + T + \tilde{\pi}(\tilde{\sigma}_1 - \underline{\sigma}_1)T. \end{aligned}$$

Proof. Similar to the proof of Theorems 3.1 and 3.2, let $A(i) = W_2(j) = 0$, we can easily derive this Corollary. Its proof is straightforward and hence omitted. $\square$

Remark 3.6. As for the special cases of system (2), Corollaries 3.2 & 3.5 presents the sufficient conditions ensuring the globally exponential stability of neural networks (57) & (62). The LMIs in (58)–(61) and (63)–(65), respectively, can also solved easily checked by using the MATLAB LMI control toolbox.

4. Numerical examples

In this section, we will present three numerical examples with their simulations to guarantee the superiority and validity of our theoretical results.

Example 4.1. Consider a two dimensional Markovian jumping Cohen–Grossberg BAM neural networks with impulses (2) with $x(t) = (x_1(t), x_2(t))^T$, $y(t) = (y_1(t), y_2(t))^T$; $r(t), \tilde{r}(t)$ denotes

right-continuous markovian chains taking values in $\mathfrak{M} = \{1, 2\}$ with generators

$$\Pi = \begin{pmatrix} 0.3 & 0.6 \\ 0.4 & 0.1 \end{pmatrix} \quad \tilde{\Pi} = \begin{pmatrix} 0.3 & 0.5 \\ 0.6 & 0.2 \end{pmatrix}$$

The associated parameters of neural networks (2) are follows:

$$D_1 = \begin{pmatrix} 0.02 & -0.01 \\ 0.02 & 0.01 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0.01 & -0.02 \\ 0.02 & 0.03 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 0.01 & 0.02 \\ -0.01 & 0.01 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0.02 & -0.02 \\ 0.01 & 0.01 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0.01 & 0.01 \\ 0.03 & 0.01 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.02 & 0.01 \\ 0.02 & 0.02 \end{pmatrix},$$

$$W_{01} = \begin{pmatrix} 0.02 & -0.02 \\ 0.01 & 0.03 \end{pmatrix}, \quad W_{02} = \begin{pmatrix} 0.03 & 0.01 \\ -0.03 & 0.02 \end{pmatrix},$$

$$W_{11} = \begin{pmatrix} 0.01 & 0.02 \\ -0.2 & 0.01 \end{pmatrix}, \quad W_{12} = \begin{pmatrix} 0.02 & 0.01 \\ -0.02 & 0.03 \end{pmatrix},$$

$$W_{21} = \begin{pmatrix} 0.02 & -0.01 \\ 0.02 & 0.01 \end{pmatrix}, \quad W_{22} = \begin{pmatrix} 0.02 & -0.02 \\ 0.01 & 0.01 \end{pmatrix},$$

Taking the values

$$\begin{aligned} \tau_{11} &= 0.7, \quad \tau_{12} = 0.6, \quad \tau_{21} = 0.5, \quad \tau_{22} = 1.3, \quad \epsilon_{11} = 13, \quad \epsilon_{12} = 17, \\ \epsilon_{21} &= 16, \quad \epsilon_{22} = 13, \quad \epsilon_{31} = 15, \quad \epsilon_{32} = 15, \quad \bar{c}_1 = 0.2, \quad \bar{c}_1 = 0.3, \quad \bar{c}_2 = 0.4, \\ \bar{c}_2 &= 0.3, \quad \bar{b}_1 = 0.2, \quad \bar{b}_1 = 0.4, \quad \bar{b}_2 = 0.1, \quad \bar{b}_2 = 0.5, \quad \bar{l}_1 = 0.2, \quad \bar{l}_1 = 0.4, \\ \bar{l}_2 &= 0.3, \quad \bar{l}_2 = 0.5, \quad \bar{m}_1 = 0.2, \quad \bar{m}_1 = 0.6, \quad \bar{m}_2 = 0.1, \quad \bar{m}_2 = 0.6, \\ \pi &= 0.02, \quad \bar{\pi} = 0.2, \quad \bar{\pi} = 0.03, \quad \bar{\pi} = 0.1, \quad \underline{\sigma}_1 = 0.2, \quad \bar{\sigma}_1 = 17.4386, \\ \underline{\sigma}_2 &= 0.1, \quad \bar{\sigma}_2 = 17.4386, \quad \underline{\tau}_1 = 0.4, \quad \bar{\tau}_1 = 0.7, \quad \underline{\tau}_2 = 0.3, \quad \bar{\tau}_2 = 0.6, \\ \xi_{11} &= 15, \quad \xi_{12} = 13, \quad \xi_{21} = 13, \quad \xi_{22} = 12, \quad \xi_{31} = 11, \quad \xi_{32} = 12, \quad \sigma_{11} = 0.5, \\ \sigma_{12} &= 0.3, \quad \sigma_{21} = 0.4, \quad \sigma_{22} = 0.4, \quad \Gamma_{11} = 3, \quad \Gamma_{12} = 2, \quad \Gamma_{21} = 2, \quad \Gamma_{22} = 1, \\ \lambda_{\max}(D_1 D_1^T) &= 0.02, \quad \lambda_{\max}(D_1 D_2^T) = 0.03, \quad \lambda_{\max}(D_2 D_1^T) = 0.04, \\ \lambda_{\max}(D_2 D_2^T) &= 0.01, \quad \lambda_{\max}(E_1 E_1^T) = 0.02, \quad \lambda_{\max}(E_1 E_2^T) = 0.03, \\ \lambda_{\max}(E_2 E_1^T) &= 0.03, \quad \lambda_{\max}(E_2 E_2^T) = 0.01, \quad \lambda_{\max}(A_1 A_1^T) = 0.02, \\ \lambda_{\max}(A_1 A_2^T) &= 0.04, \quad \lambda_{\max}(A_2 A_1^T) = 0.03, \quad \lambda_{\max}(A_2 A_2^T) = 0.02, \\ \mu_{\max}(W_{01} W_{01}^T) &= 0.02, \quad \mu_{\max}(W_{01} W_{02}^T) = 0.03, \quad \mu_{\max}(W_{02} W_{01}^T) = \\ &0.01, \quad \mu_{\max}(W_{02} W_{02}^T) = -0.04, \quad \mu_{\max}(W_{11} W_{11}^T) = 0.02, \\ \mu_{\max}(W_{11} W_{12}^T) &= 0.03, \quad \mu_{\max}(W_{12} W_{11}^T) = 0.01, \quad \mu_{\max}(W_{12} W_{12}^T) = \\ &-0.03, \quad \mu_{\max}(W_{21} W_{21}^T) = 0.04, \quad \mu_{\max}(W_{21} W_{22}^T) = 0.01, \\ \mu_{\max}(W_{22} W_{21}^T) &= 0.02, \quad \mu_{\max}(W_{22} W_{22}^T) = -0.03, \quad C_1(x(t)) = C_2(x(t)) \\ &= 1 + \cos(x(t)), \quad B_1(x(t)) = B_2(x(t)) = 0.5 * x(t), \quad L_1(y(t)) = \\ &L_2(y(t)) = 1 + \sin(y(t)), \quad M_1(y(t)) = M_2(y(t)) = 0.5 * y(t), \end{aligned}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$I_k^*(x(t_k)) = \exp(0.05) * \sin(x(t_k)), \quad J_k^*(y(t_k)) = \tan(0.98 * |y(t_k)|).$$

The following activation functions are playing in neural network system (2):

$$f(y(t)) = g(y(t)) = h(y(t)) = \tanh(y(t)), \quad \tilde{f}(x(t)) = \tilde{g}(x(t)) = \tilde{h}(x(t)) = \tanh(x(t)).$$

By solving the LMIs in Theorem 3.2 using the Matlab LMI Toolbox, one can obtain the feasible solutions as follows:

$$S_1 = \begin{pmatrix} 42.4974 & 0 \\ 0 & 42.4974 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 25.4965 & 0 \\ 0 & 25.4965 \end{pmatrix},$$

$$\tilde{S}_1 = \begin{pmatrix} 76.8619 & 0 \\ 0 & 76.8619 \end{pmatrix},$$

$$\tilde{S}_2 = \begin{pmatrix} 94.0088 & 0 \\ 0 & 94.0088 \end{pmatrix}, \quad T = \begin{pmatrix} 36.0206 & 0 \\ 0 & 36.0206 \end{pmatrix},$$

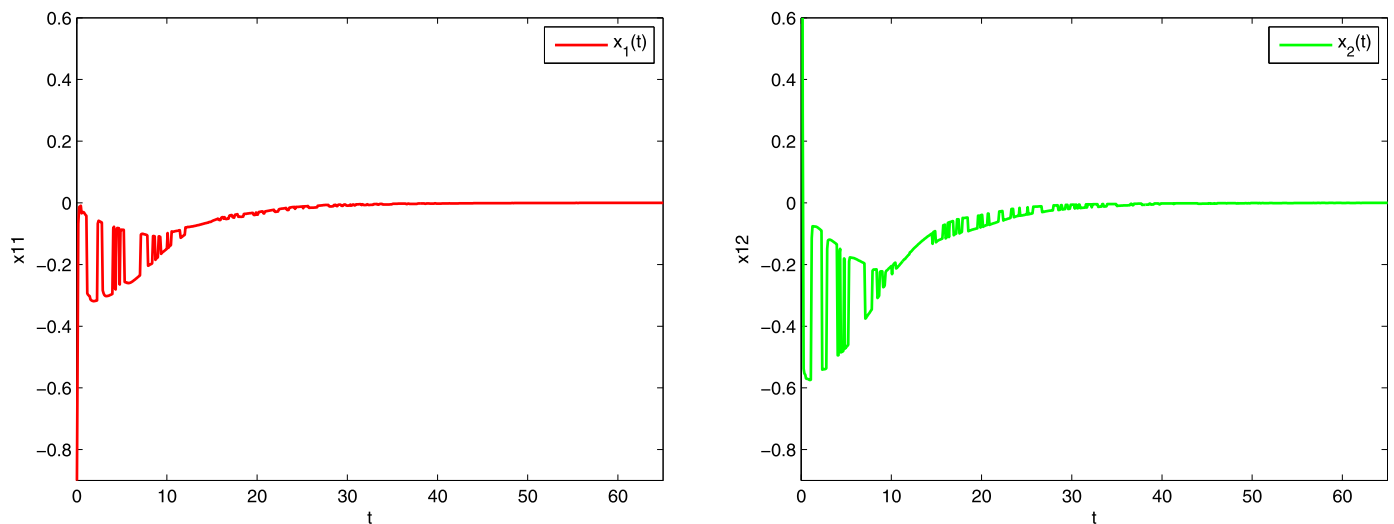


Fig. 1. The state responses $x_1(t)$ and $x_2(t)$ of (2) with impulsive effects.

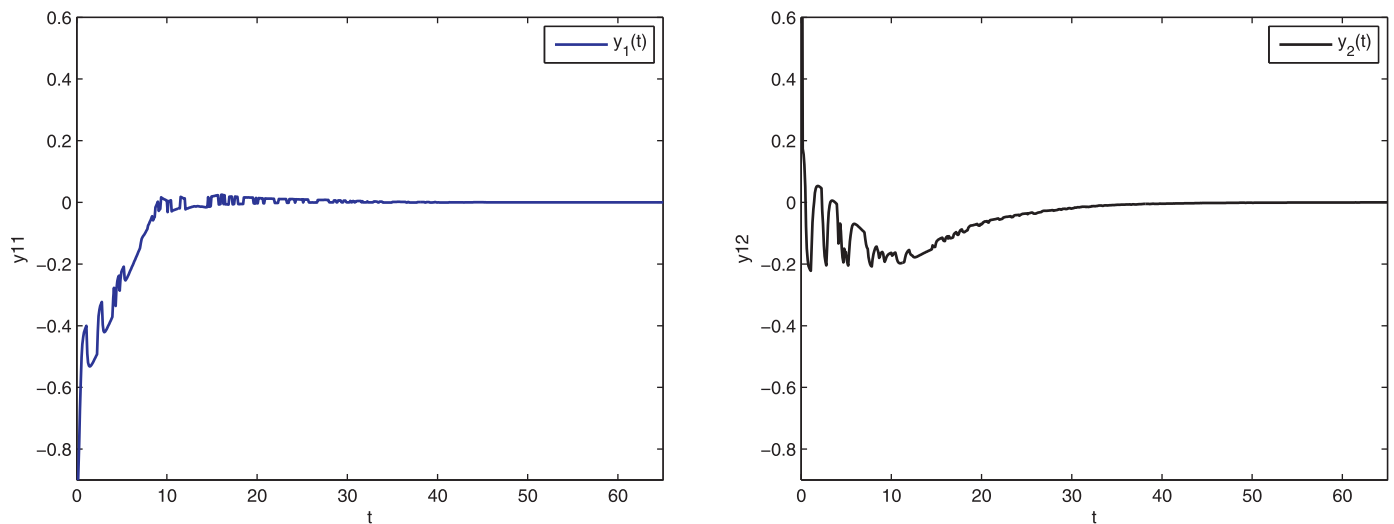


Fig. 2. The state responses $y_1(t)$ and $y_2(t)$ of (2) with impulsive effects.

$$\tilde{T} = \begin{pmatrix} 37.1292 & 0 \\ 0 & 37.1292 \end{pmatrix},$$

$$R = \begin{pmatrix} 58.8508 & 0 \\ 0 & 58.8508 \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} 44.6937 & 0 \\ 0 & 44.6937 \end{pmatrix},$$

$$G = \begin{pmatrix} 1.8722 & 0 \\ 0 & 1.8722 \end{pmatrix},$$

$$\Delta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\Delta} = 10^{-3} \times \begin{pmatrix} 0.1706 & 0 \\ 0 & 0.1706 \end{pmatrix},$$

$$\bar{\varsigma}_1 = 0.2127, \quad \bar{\varsigma}_2 = 0.0268$$

Thus the state responses of the Cohen–Grossberg BAM-type NNs (2) are shown in Figs. 1–3 and the Markovian jumps of neural networks (2) is shown in Fig. 4. By solving the LMIs (3)–(6) in Theorem 3.2 we can find the feasible solutions. The obtained discrete time-delay upper bounds of $\bar{\sigma}_1$ and $\bar{\sigma}_2$ for neural networks (2) are very large, which are given in Table 1. Therefore by Theorem 3.2, we conclude that the neural networks (2) is globally exponentially stable for the maximum allowable upper bounds $\bar{\sigma}_1 = \bar{\sigma}_2 = 17.4386$. This shows that the contributions of

this research work is more effective and less conservative than some existing literatures.

Remark 4.1. Based on the above Example 4.1, it is easy to see that the obtained results are better than the ones in [18,36,48]. Hence, the proposed method is an improvement over the previous ones.

Example 4.2. Consider a two dimensional CGBAMNNs (57) with $x(t) = (x_1(t), x_2(t))^T$, $y(t) = (y_1(t), y_2(t))^T$; The associated parameters of neural networks (57) are follows:

$$D = \begin{pmatrix} 0.03 & -0.03 \\ 0.03 & 0.01 \end{pmatrix}, \quad E = \begin{pmatrix} 0.02 & 0.01 \\ -0.02 & -0.04 \end{pmatrix},$$

$$A = \begin{pmatrix} 0.02 & -0.03 \\ 0.02 & 0.02 \end{pmatrix}, \quad W_0 = \begin{pmatrix} 0.03 & -0.01 \\ 0.02 & 0.01 \end{pmatrix},$$

$$W_1 = \begin{pmatrix} 0.02 & 0.03 \\ -0.02 & 0.02 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0.01 & -0.02 \\ 0.01 & 0.02 \end{pmatrix},$$

Take the parameters of LMIs in Corollary 3.2 as follows:

$$\xi_1 = 23, \xi_2 = 16, \xi_3 = 15, \epsilon_1 = 13, \epsilon_2 = 12, \epsilon_3 = 11, \tau_1 = 6.462, \tau_2 = 6.462, \underline{C} = 0.2, \bar{C} = 0.3, \underline{L} = 0.3, \bar{L} = 0.5, \Gamma_1 = 3, \Gamma_2 = 2, \lambda_{\max}(DD^T) = 0.02, \lambda_{\max}(EE^T) = 0.03, \lambda_{\max}(AA^T) = 0.02,$$

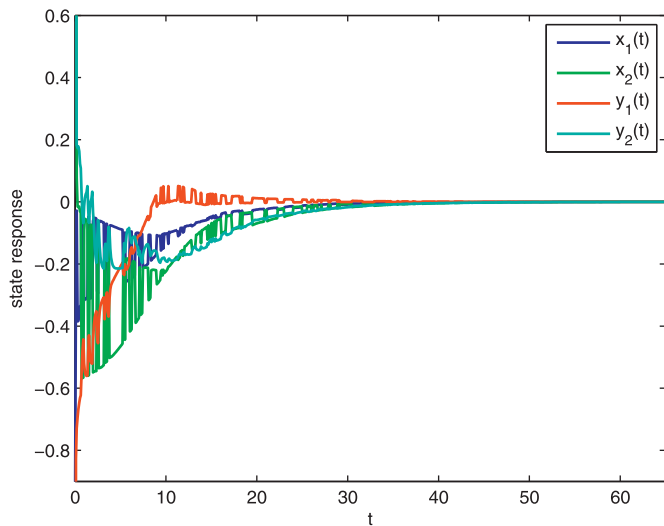


Fig. 3. State trajectories of concerned Cohen–Grossberg BAM neural networks (2).

$$\mu_{\max}(W_0 W_0^T) = 0.02, \quad \mu_{\max}(W_1 W_1^T) = 0.01, \quad \mu_{\max}(W_2 W_2^T) = 0.03, \\ C_1(x(t)) = C_2(x(t)) = 1 + \cos(x(t)), \quad B_1(x(t)) = B_2(x(t)) = 0.5 * x(t), \\ L_1(y(t)) = L_2(y(t)) = 1 + \sin(y(t)), \quad M_1(y(t)) = M_2(y(t)) = 0.5 * y(t),$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The same activation functions in Example 4.1, are also playing in NNs (57):

Then by using the MATLAB LMI control toolbox, it can be concluded that the system (57) is globally exponential stable and the solutions to the LMIs in Corollary 3.2 are

$$S = 10^8 \times \begin{pmatrix} 1.9276 & 0 \\ 0 & 1.9276 \end{pmatrix}, \quad \tilde{S} = 10^8 \times \begin{pmatrix} 1.9311 & 0 \\ 0 & 1.9311 \end{pmatrix},$$

$$T = \tilde{T} = 10^7 \times \begin{pmatrix} 1.2154 & 0 \\ 0 & 1.2154 \end{pmatrix},$$

$$R = \tilde{R} = 10^7 \times \begin{pmatrix} 1.2154 & 0 \\ 0 & 1.2154 \end{pmatrix}, \quad \Delta = \tilde{\Delta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

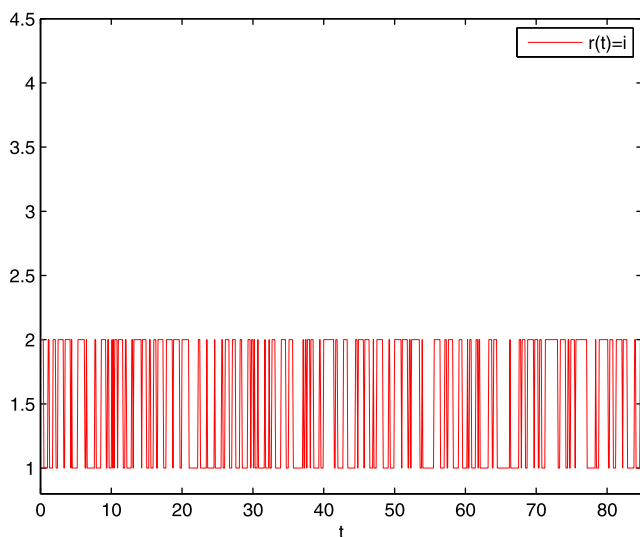


Fig. 4. The state response $r(t) = i$ and $\tilde{r}(t) = j$ denotes markovian jumps in system (2).

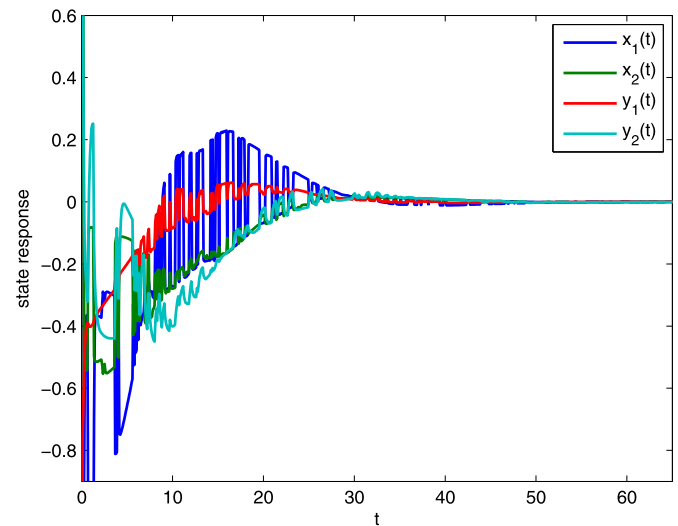


Fig. 5. State trajectories of concerned Cohen–Grossberg BAM neural networks (57).

$$\bar{\zeta}_1 = \bar{\zeta}_2 = 0.$$

Thus the CGBAMNNs (57) is globally exponential stable in the mean square sense. The response of the state dynamics of the NNs (57) is shown in Fig. 5.

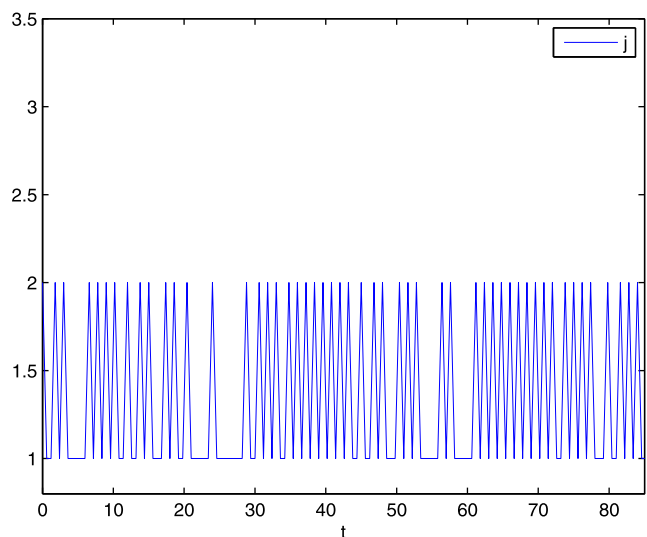
Example 4.3. Consider a two dimensional addressed neural networks (62) with $x(t) = (x_1(t), x_2(t))^T$, $y(t) = (y_1(t), y_2(t))^T$; $r(t)$, $\tilde{r}(t)$ denotes right-continuous markovian chains taking values in $\mathfrak{M} = \{1, 2\}$ with different generators

$$\Pi = \begin{pmatrix} 0.3 & 0.4 \\ 0.5 & 0.2 \end{pmatrix}, \quad \tilde{\Pi} = \begin{pmatrix} 0.1 & 0.2 \\ 0.4 & 0.3 \end{pmatrix}$$

The neural networks (62) have the following associated parameters:

$$D_1 = \begin{pmatrix} 0.5 & -0.1 \\ 0.2 & 0.3 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0.3 & -0.1 \\ 0.2 & 0.3 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 0.2 & 0.2 \\ -0.3 & -0.3 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0.2 & -0.2 \\ 0.3 & 0.2 \end{pmatrix},$$



$$A_1 = \begin{pmatrix} 0.3 & -0.2 \\ 0.2 & 0.1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.4 & 0.1 \\ 0.4 & 0.1 \end{pmatrix},$$

$$W_{01} = \begin{pmatrix} 0.3 & -0.1 \\ 0.1 & 0.2 \end{pmatrix}, \quad W_{02} = \begin{pmatrix} 0.2 & 0.1 \\ -0.3 & 0.2 \end{pmatrix},$$

$$W_{11} = \begin{pmatrix} 0.2 & 0.2 \\ -0.3 & 0.1 \end{pmatrix}, \quad W_{12} = \begin{pmatrix} 0.1 & 0.3 \\ -0.2 & 0.3 \end{pmatrix}$$

Taking the values

$$\begin{aligned} \epsilon_{11} = 12, \quad \epsilon_{12} = 14, \quad \epsilon_{21} = 15, \quad \epsilon_{22} = 14, \quad \epsilon_{31} = 12, \quad \epsilon_{32} = 13, \\ \zeta_1 = 0.3, \quad \bar{\zeta}_1 = 0.4, \quad \zeta_2 = 0.2, \quad \bar{\zeta}_2 = 0.3, \quad \bar{b}_1 = 0.2, \quad \bar{b}_1 = 0.5, \\ \bar{b}_2 = 0.2, \quad \bar{b}_2 = 0.5, \quad \bar{l}_1 = 0.1, \quad \bar{l}_1 = 0.3, \quad \bar{l}_2 = 0.3, \quad \bar{l}_2 = 0.4, \\ \bar{m}_1 = 0.2, \quad \bar{m}_1 = 0.4, \quad \bar{m}_2 = 0.1, \quad \bar{m}_2 = 0.5, \quad \pi = 0.03, \quad \bar{\pi} = 0.3, \\ \bar{\pi} = 0.02, \quad \bar{\pi} = 0.2, \quad \sigma_1 = 0.5, \quad \bar{\sigma}_1 = 58, \quad \sigma_2 = 0.6, \quad \bar{\sigma}_2 = 58, \\ \xi_{11} = 15, \quad \xi_{12} = 16, \quad \xi_{21} = 14, \quad \xi_{22} = 14, \quad \xi_{31} = 13, \quad \xi_{32} = 12, \\ \sigma_{11} = 0.6, \quad \sigma_{12} = 0.2, \quad \sigma_{21} = 0.3, \quad \sigma_{22} = 0.4, \quad \Gamma_{11} = 4, \quad \Gamma_{12} = 3, \\ \Gamma_{21} = 2, \quad \Gamma_{22} = 2, \quad \lambda_{\max}(D_1 D_1^T) = 0.2, \quad \lambda_{\max}(D_1 D_2^T) = 0.3, \\ \lambda_{\max}(D_2 D_1^T) = 0.4, \quad \lambda_{\max}(D_2 D_2^T) = 0.1, \quad \lambda_{\max}(E_1 E_1^T) = 0.2, \\ \lambda_{\max}(E_1 E_2^T) = 0.3, \quad \lambda_{\max}(E_2 E_1^T) = 0.3, \quad \lambda_{\max}(E_2 E_2^T) = 0.1, \\ \lambda_{\max}(A_1 A_1^T) = 0.2, \quad \lambda_{\max}(A_1 A_2^T) = 0.4, \quad \lambda_{\max}(A_2 A_1^T) = 0.3, \\ \lambda_{\max}(A_2 A_2^T) = 0.2, \quad \mu_{\max}(W_{01} W_{01}^T) = 0.2, \quad \mu_{\max}(W_{01} W_{02}^T) = 0.3, \\ \mu_{\max}(W_{02} W_{01}^T) = 0.1, \quad \mu_{\max}(W_{02} W_{02}^T) = -0.4, \quad \mu_{\max}(W_{11} W_{11}^T) = 0.2, \\ \mu_{\max}(W_{11} W_{12}^T) = 0.3, \quad \mu_{\max}(W_{12} W_{11}^T) = 0.1, \quad \mu_{\max}(W_{12} W_{12}^T) = -0.3, \\ \mu_{\max}(W_{21} W_{21}^T) = 0.4, \quad \mu_{\max}(W_{21} W_{22}^T) = 0.1, \\ \mu_{\max}(W_{22} W_{21}^T) = 0.2, \quad \mu_{\max}(W_{22} W_{22}^T) = -0.3, \quad C_1(x(t)) = \\ C_2(x(t)) = 1 + \cos(x(t)), \quad B_1(x(t)) = B_2(x(t)) = 0.5 * x(t), \quad L_1(y(t)) = \\ L_2(y(t)) = 1 + \sin(y(t)), \quad M_1(y(t)) = M_2(y(t)) = 0.5 * y(t) \text{ and} \\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Here also the same activation functions in Example 4.1, are used.

Then by using MATLAB LMI control toolbox, we can find that (62) is globally exponential stable with the solutions to the LMIs in Corollary 3.5 as follows:

$$S_1 = \begin{pmatrix} 4.4873 & 0 \\ 0 & 4.4873 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 9.4435 & 0 \\ 0 & 9.4435 \end{pmatrix},$$

$$\tilde{S}_1 = \begin{pmatrix} 33.2027 & 0 \\ 0 & 33.2027 \end{pmatrix},$$

$$\tilde{S}_2 = \begin{pmatrix} 9.9211 & 0 \\ 0 & 9.9211 \end{pmatrix}, \quad T = \begin{pmatrix} 5.1897 & 0 \\ 0 & 5.1897 \end{pmatrix},$$

$$\tilde{T} = \begin{pmatrix} 6.2054 & 0 \\ 0 & 6.2054 \end{pmatrix},$$

$$R = \begin{pmatrix} 9.3930 & 0 \\ 0 & 9.3930 \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} 9.3930 & 0 \\ 0 & 9.3930 \end{pmatrix},$$

$$G = \begin{pmatrix} 0.7677 & 0 \\ 0 & 0.7677 \end{pmatrix},$$

$$\Delta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\Delta} = 10^{-3} \times \begin{pmatrix} 0.1605 & 0 \\ 0 & 0.1605 \end{pmatrix},$$

$$\bar{\zeta}_1 = 0, \quad \bar{\zeta}_2 = 0.$$

Then by MATLAB LMI control toolbox, we get that $t_{\min} = -0.1311 < 0$. Therefore, according to Corollary 3.5, the equilibrium point of neural networks (62) is globally exponential stable. Moreover, its numerical simulation is shown in Fig. 6.

Remark 4.2. Based on Examples 4.1, 4.3, it is easy to see that the obtained results are better than the ones in [2,30,47,50]. Hence, the proposed method is an improvement over the existing literatures.

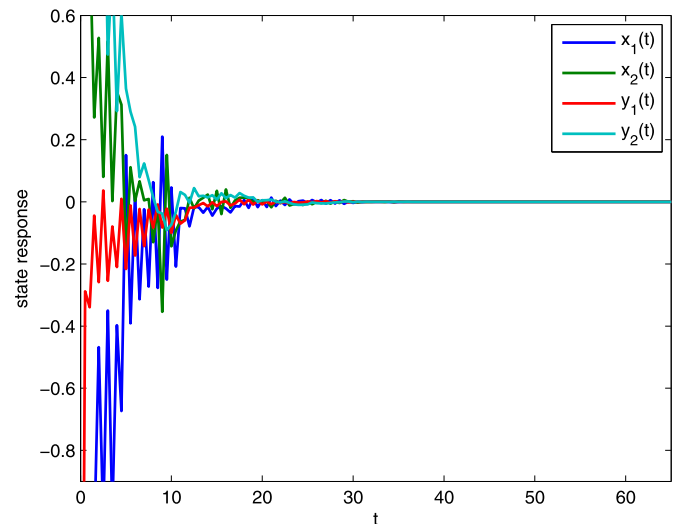


Fig. 6. State trajectories of concerned Cohen–Grossberg BAM neural networks (62).

5. Conclusions

This paper mainly focuses on exploring a LMI approach to studying globally exponential stability for a class of markovian jumping Cohen–Grossberg BAM neural networks with impulsive effects and mixed time delays. By utilizing the new Lyapunov–Krasovskii functionals, M-matrix theory, Stability theory, Inequality techniques and Schur complement lemma to sufficient conditions derived for the globally exponential stability of Cohen–Grossberg BAM neural networks in the mean square sense. In addition, our concerned method provide the allowable upper bounds of the discrete time varying delays is very large when compared with the existing results, which may possess highly important significance in the design of time-delayed addressed neural networks with markovian jumps and impulsive effects. The rest of this paper is proposed three numerical examples with simulations to illustrate the effectiveness of this proposed research work.

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A state estimation H_∞ issue for discrete-time stochastic impulsive genetic regulatory networks in the presence of leakage, multiple delays and Markovian jumping parameters[☆]

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Abstract

In this work, we probe the stability results of H_∞ state estimation for discrete-time stochastic genetic regulatory networks with leakage, distributed delays, Markovian jumping parameters and impulsive effects. Here, we focus to evaluate the true absorption of mRNAs and proteins by calculating the H_∞ estimator in such a way that the estimation error dynamics is stochastically stable during the completion of the prescribed H_∞ disturbance attenuation level. In favor of decreasing the data communion in trouble, the H_∞ system accept and evaluate the outputs that are only transferred to the estimator when a certain case is acrosses. Further, few sufficient conditions are formulated, by utilizing the Lyapunov–Krasovskii functional under which the estimation error system is stochastically stable and also satisfied the H_∞

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attainment constraint. The estimator is obtained in terms of linear matrix inequalities (LMIs) and these LMIs are attainable, only if the estimator gains can be absolutely given. In addition to that, two numerical examples are exposed to establish the efficiency of our obtained results.

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1. Introduction

The study of genetic regulatory networks (GRNs) has become a focus of interest in the past few years. It has been proved that many diseases are actually originated from the sicknesses of GRNs of the corresponding cell lineage, which mainly includes cancer, diabetes and AIDS. It is crucial to examine the dynamical behaviors of the GRNs. The stability has been analyzed too few terms that are reported in [4,32,40–43,50–58]. The research that depicts on GRNs can elucidate the exchange of information linking genes and proteins which is a mixed one, and it develops the biochemical system dynamics. So, GRNs can have the ability to accommodate implementations that have differentiated to control such gene expression. GRNs illustrate the reciprocities among products of mRNA, genes and proteins, in terms of transcriptional and translational processes. Especially, a RNA copy of a gene progression is generated in the transcription process through the duplication exclaimed mRNAs that are in translation, and mRNA particle adapts amino acid into protein, for a period in progression.

Thus, the analysis of GRNs spread the research interest between scientists and researchers from entire world and thus many notable research works have been proposed as various types of models to define GRNs. Until now, the Boolean model and the differential equation model [1,5,31,46–48] are the basic models related to GRNs. In the first highlighted model, the genes that express from the network are simulated to be either ON or OFF and there is no interposed action levels are usually taken into deliberation, and gene state is decided by the Boolean function of the other related gene states [5,19–29,32]. Whereas, the variables involve, which defines the change rate of the concentration of mRNAs and proteins, are continuous values in the differential equation model. Henceforth, the Boolean model is more efficiency than the differential equation model.

On the other hand, time delays emerge routinely in several dynamical systems like economic systems, chemical processes, and neural networks [[2,3,6–17]. Note that, GRNs effected by transcription, translation, and translocation action with the time delays in view of the idle action may affect oscillation, divergence, and instability of the networks. In this regard, the study of time delays in GRNs have been largely deliberated, and many significant and fascinating results have been proposed in the available literature of all class of patterns [28,34,36,49]. Generally, GRNs, as an indicative network, consist of many of genes, so it is difficult to find the state estimation of GRNs, and it may take huge number of computing assets until the particular mRNA or protein states can be evaluated. Accordingly, the conceptual consequence and pragmatic concernment is to evolve the H_∞ performance of the approximation of state vectors to access the absorption of the desired mRNA and protein. The Markovian jumping problems has been studied in [30–33,35,38,44]. Here, the best of the author's ability, yet, the H_∞ state estimation problem with the existence of Markovian jumping parameters, impulses and mixed-delays has not been still studied for GRNs. So, here we have considered all the above facts simultaneously. This comprises the foremost incentive of this paper.

Induced by the beyond deliberation, we will design the H_∞ complication for a discrete-interval stochastic impulsive GRNs with leakage, multiple delays and Markovian jumping parameters. The contribution of this paper is given as follows:

- (1) The GRNs covered by the deliberation are absolute that implicate the discrete-time case, stochastic disturbances, Markov jumps, distributed delays, time-varying delays and impulses;
- (2) We will propose the H_∞ system towards the state estimation problem because the GRNs transmitted, that completely diminish the enlightenment communion trouble;
- (3) We will determine the process of absorption linking mRNAs and protein in GRNs by applying an appropriate Lyapunov–Krasovskii functional and linear matrix inequality (LMI) approach, the H_∞ state estimator is devised.
- (4) We will analyze the stability criterion for discrete-time GRNs with leakage delays, multiple delays and impulsive effects by using the discrete-type Wirtinger inequality. Lastly, the theoretical results are justified with two suitable examples in numeric sense and simulations are given to expo the capability of our stability conditions.

Notations: $\mathbb{R}^n$ denotes the n dimensional Euclidean space and $\mathbb{R}^{n \times m}$ denotes the set of all $n \times m$ real matrices. The superscript “ T ” means that the matrix transposition and the symbol “ $*$ ” is denoted by the symmetric term. $\mathfrak{K}$ denotes the discrete-time in GRNs. I means that the identity matrix of appropriate dimensions and $\|\cdot\|$ is the Euclidean norm. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are real symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). $l_2[0, \infty)$ is the space of square summable infinite sequence. If A is a square matrix, $\lambda_{\max}(A)$ is the largest eigenvalue of A and $\lambda_{\min}(A)$ is the smallest eigenvalue of A . $\mathbb{E}\{x\}$ is the mathematical expectation of a stochastic variable x and $\mathbb{E}\{x|y\}$ is the expectation of x conditional on y .

2. Model description and preliminaries

In this paper, we consider the discrete-time GRN with leakage delays, distributed delays, impulses and time-varying delays of the form:

$$\begin{aligned} \mathbb{M}(\mathfrak{K} + 1) &= \mathfrak{A}\mathbb{M}(\mathfrak{K} - \rho_1) + \mathfrak{B}g(\mathbb{P}(\mathfrak{K} - d(\mathfrak{K}))) + \mathfrak{C}_m v(\mathfrak{K}) + \mathfrak{F}_m \sum_{\mathfrak{M}=1}^{\infty} \mu_{\mathfrak{M}} h(y(\mathfrak{K} - \mathfrak{M})) + L \\ \mathbb{M}(\mathfrak{K}_r) &= c_r \mathbb{M}(\mathfrak{K}_r - 1), \quad r = 1, 2, \dots, \\ \mathbb{P}(\mathfrak{K} + 1) &= \mathfrak{C}\mathbb{P}(\mathfrak{K} - \rho_2) + \mathfrak{D}\mathbb{M}(\mathfrak{K} - \tau(\mathfrak{K})) + \mathfrak{E}_p v(\mathfrak{K}) + \mathfrak{F}_p \sum_{\mathfrak{P}=1}^{\infty} \mu_{\mathfrak{P}} x(\mathfrak{K} - \mathfrak{P}) \\ \mathbb{P}(\mathfrak{K}_r) &= \hat{c}_r \mathbb{P}(\mathfrak{K}_r - 1), \quad r = 1, 2, \dots, \end{aligned} \quad (1)$$

where $\mathbb{M}(\mathfrak{K} - \rho_1) = [\mathbb{M}_1(\mathfrak{K} - \rho_1), \mathbb{M}_2(\mathfrak{K} - \rho_1), \dots, \mathbb{M}_n(\mathfrak{K} - \rho_1)]^T \in \mathbb{R}^n$ is the concentrations of mRNA at time $\mathfrak{K}$; $\mathbb{P}(\mathfrak{K} - \rho_2) = [\mathbb{P}_1(\mathfrak{K} - \rho_2), \mathbb{P}_2(\mathfrak{K} - \rho_2), \dots, \mathbb{P}_n(\mathfrak{K} - \rho_2)]^T \in \mathbb{R}^n$ represents the protein absorption at time $\mathfrak{K}$; $\mathfrak{A} = \text{diag}\{a_1, a_2, \dots, a_n\}$ is the degradation rate of mRNA with entries $|a_i| < 1$ and it is a constant matrix, $\mathfrak{C} = \text{diag}\{c_1, c_2, \dots, c_n\}$ is the degradation rate of proteins with entries $|c_i| < 1$ and the translation rate is $\mathfrak{D} = \text{diag}\{d_1, d_2, \dots, d_n\}$; The coupling matrix of the genetic regulatory network is defined as $\mathfrak{B} = (b_{ij})_{n \times n}$; The non-linear function $g(\mathbb{P}(\mathfrak{K} - d(\mathfrak{K}))) = [g_1(\mathbb{P}(\mathfrak{K} - d(\mathfrak{K}))), g_2(\mathbb{P}(\mathfrak{K} - d(\mathfrak{K}))), \dots, g_n(\mathbb{P}(\mathfrak{K} -$

$d(\mathfrak{K}))^T \in \mathbb{R}^n$ constitutes the regulation of protein from their feedback in transcription period; $d(\mathfrak{K})$ denotes the delay that regulates the feedback; $\tau(\mathfrak{K})$ denotes the translation delay satisfying the conditions $0 \leq d_m \leq d(\mathfrak{K}) \leq d_M$ and $0 \leq \tau_m \leq \tau(\mathfrak{K}) \leq \tau_M$ respectively. $\mathfrak{F}_m = \text{diag}\{f_{m1}, f_{m2}, \dots, f_{mn}\}$ and $\mathfrak{F}_p = \text{diag}\{f_{p1}, f_{p2}, \dots, f_{pn}\}$ are the weight matrices. $h(y(\mathfrak{K})) = [h_1(y(\mathfrak{K})), \dots, h_n(y(\mathfrak{K}))]^T \in \mathbb{R}^n$ are denoted the activation function. $L = [l_1, \dots, l_n]^T$ denotes a constant vector with l_i . Moreover, it symbolizes the transcription rate of each and every gene (i.e. $i = 1, 2, \dots, n$) from the repressor; The exogenous disturbance signal $v(\mathfrak{K}) \in \mathbb{R}^s$ belongs to $l_2([0, \infty), \mathbb{R}^s)$; The intensities of the exogenous disturbances $\mathfrak{E}_m = [e_{m1}, \dots, e_{mn}]^T$ and $\mathfrak{E}_p = [e_{p1}, \dots, e_{pn}]^T$ are characterizing to the mRNA and the protein, respectively.

Where $\mathbb{M}(\mathfrak{K}_r)$ and $\mathbb{P}(\mathfrak{K}_r)$ are the impulses and $\mathbb{M}(\mathfrak{K}_r^-) = \lim_{\mathfrak{K} \rightarrow \mathfrak{K}_r} \mathbb{M}(\mathfrak{K}_r)$, $\mathbb{P}(\mathfrak{K}_r^-) = \lim_{\mathfrak{K} \rightarrow \mathfrak{K}_r} \mathbb{P}(\mathfrak{K}_r)$, $c_r, \hat{c}_r; \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous over $\mathbb{R}^n$, and the impulse instant $\mathfrak{K}_r$ are assumed to satisfy that $0 = \mathfrak{K}_0 < \mathfrak{K}_1 < \mathfrak{K}_2 < \dots < \mathfrak{K}_{r-1} < \mathfrak{K}_r < \dots$ and $\mathfrak{K}_r - \mathfrak{K}_{r-1} > 0$, for $(r = 1, 2, \dots)$.

Assumption 1. The non-linear activation function $g_i(\cdot)$ ($i = 1, 2, \dots, n$) satisfies

$$0 \leq \frac{g_i(t_1) - g_i(t_2)}{t_1 - t_2} \leq v_i, \quad (2)$$

$\forall t_1, t_2 \in \mathbb{R}$ with $t_1 \neq t_2$, where v_i is a known positive constant.

Assume that $\mathbb{P}^* = [\mathbb{P}_1^*, \dots, \mathbb{P}_n^*]^T$ and $\mathbb{M}^* = [\mathbb{M}_1^*, \dots, \mathbb{M}_n^*]^T$ are the equilibrium point of system (1). Taking $\tilde{\mathbb{M}}(\mathfrak{K} - \rho_1) = \mathbb{M}(\mathfrak{K} - \rho_1) - \mathbb{M}^*$ and $\tilde{\mathbb{P}}(\mathfrak{K} - \rho_2) = \mathbb{P}(\mathfrak{K} - \rho_2) - \mathbb{P}^*$, system (1) can be converted toward,

$$\begin{aligned} \tilde{\mathbb{M}}(\mathfrak{K} + 1) &= \mathfrak{A}\tilde{\mathbb{M}}(\mathfrak{K} - \rho_1) + \mathfrak{B}\tilde{f}(\tilde{\mathbb{P}}(\mathfrak{K} - d(\mathfrak{K}))) + \mathfrak{E}_m v(\mathfrak{K}) + \mathfrak{F}_m \sum_{\mathfrak{M}=1}^{\infty} \mu_{\mathfrak{M}} h(y(\mathfrak{K} - \mathfrak{M})) + L \\ \tilde{\mathbb{M}}(\mathfrak{K}_r) &= c_r \tilde{\mathbb{M}}(\mathfrak{K}_r - 1), \quad r = 1, 2, \dots, \\ \tilde{\mathbb{P}}(\mathfrak{K} + 1) &= \mathfrak{C}\tilde{\mathbb{P}}(\mathfrak{K} - \rho_2) + \mathfrak{D}\tilde{\mathbb{M}}(\mathfrak{K} - \tau(\mathfrak{K})) + \mathfrak{E}_p v(\mathfrak{K}) + \mathfrak{F}_p \sum_{\mathfrak{P}=1}^{\infty} \mu_{\mathfrak{P}} x(\mathfrak{K} - \mathfrak{P}) \\ \tilde{\mathbb{P}}(\mathfrak{K}_r) &= \hat{c}_r \tilde{\mathbb{P}}(\mathfrak{K}_r - 1), \quad r = 1, 2, \dots, \end{aligned} \quad (3)$$

where $\tilde{g}(\tilde{\mathbb{P}}(\mathfrak{K} - d(\mathfrak{K}))) = g(\tilde{\mathbb{P}}(\mathfrak{K} - d(\mathfrak{K})) + \mathbb{P}^*) - g(\mathbb{P}^*)$. With the help of Assumption 1, we can change the non-linear activation function as $\tilde{g}(\cdot)$ which satisfies the following condition,

$$\tilde{g}^T(\tilde{\mathbb{P}})(\tilde{g}(\tilde{\mathbb{P}} - W\tilde{\mathbb{P}})) \leq 0, \quad (4)$$

$\forall \tilde{\mathbb{P}} \in \mathbb{R}^n$, where $W = \text{diag}\{W_1, \dots, W_n\}$. Let $\tau(\mathfrak{K})(\mathfrak{K} \geq 0)$ be a homogeneous Markovian chain on the probability space transmittable ideals in a finite state space $N = \{1, 2, \dots, p\}$ with generator $\Lambda = [\Phi_{ji}]_{p \times p}$ is stated as:

$$\mathbf{P}\{\tau(\mathfrak{K} + 1) = i | \tau(\mathfrak{K}) = j\} = \Phi_{ji}, \quad j, i \in N. \quad (5)$$

where $\Phi_{ji} \geq 0$ and $\sum_{i=1}^p \Phi_{ji} = 1$.

The following complete GRNs is of the form:

$$\begin{aligned} \tilde{\mathbb{M}}(\mathfrak{K} + 1) &= \mathfrak{A}(\tau(\mathfrak{K}))\tilde{\mathbb{M}}(\mathfrak{K} - \rho_1) + \mathfrak{B}(\tau(\mathfrak{K}))\tilde{g}(\tilde{\mathbb{P}}(\mathfrak{K} - d(\mathfrak{K}))) + \mathfrak{E}_m(\tau(\mathfrak{K}))v(\mathfrak{K}) \\ &\quad + \mathfrak{F}_m(\tau(\mathfrak{K})) \sum_{\mathfrak{M}=1}^{\infty} \mu_{\mathfrak{M}} h(y(\mathfrak{K} - \mathfrak{M})) + \delta_m(\tilde{\mathbb{M}}(\mathfrak{K} - \rho_1), \tilde{\mathbb{P}}(\mathfrak{K} - \rho_2))U_m(\mathfrak{K}) \end{aligned}$$

$$\begin{aligned}
\tilde{\mathbb{M}}(\mathfrak{K}_r) &= c_r \tilde{\mathbb{M}}(\mathfrak{K}_r - 1), \quad r = 1, 2, \dots, \\
\tilde{\mathbb{P}}(\mathfrak{K} + 1) &= \mathfrak{C}(\mathfrak{r}(\mathfrak{K})) \tilde{\mathbb{P}}(\mathfrak{K} - \rho_2) + \mathfrak{D}(\mathfrak{r}(\mathfrak{K})) \tilde{\mathbb{M}}(\mathfrak{K} - \tau(\mathfrak{K})) + \mathfrak{E}_p(\mathfrak{r}(\mathfrak{K})) v(\mathfrak{K}) \\
&\quad + \mathfrak{F}_p(\mathfrak{r}(\mathfrak{K})) \sum_{\mathfrak{P}=1}^{\infty} \mu_{\mathfrak{P}} x(\mathfrak{K} - \mathfrak{P}) + \delta_p(\tilde{\mathbb{M}}(\mathfrak{K} - \rho_1), \tilde{\mathbb{P}}(\mathfrak{K} - \rho_2)) U_p(\mathfrak{K}) \\
\tilde{\mathbb{P}}(\mathfrak{K}_r) &= \hat{c}_r \tilde{\mathbb{P}}(\mathfrak{K}_r - 1), \quad r = 1, 2, \dots,
\end{aligned} \tag{6}$$

Here the independent stochastic perturbations $U_m(\mathfrak{K}) \in \mathbb{R}$ and $U_p(\mathfrak{K}) \in \mathbb{R}$ are satisfying

$$\begin{aligned}
\mathbf{E}\{U_m(\mathfrak{K})\} &= 0, \quad \mathbf{E}\{U_p(\mathfrak{K})\} = 0, \quad \mathbf{E}\{U_m(\mathfrak{K})U_m(l)\} = 0, k \neq l, \\
\mathbf{E}\{U_m^2(\mathfrak{K})\} &= 1, \quad \mathbf{E}\{U_p^2(\mathfrak{K})\} = 1, \quad \mathbf{E}\{U_p(\mathfrak{K})U_p(l)\} = 0, k \neq l.
\end{aligned} \tag{7}$$

Also, the potency of noise function vectors $\delta_m(., .)$ and $\delta_p(., .)$ satisfy,

$$\begin{aligned}
\delta_m^T(\tilde{\mathbb{M}}(\mathfrak{K} - \rho_1), \tilde{\mathbb{P}}(\mathfrak{K} - \rho_2)) \delta_m(\tilde{\mathbb{M}}(\mathfrak{K} - \rho_1), \tilde{\mathbb{P}}(\mathfrak{K} - \rho_2)) &\leq \tilde{\mathbb{M}}^T(\mathfrak{K} - \rho_1) \varpi_{m1} \tilde{\mathbb{M}}(\mathfrak{K} - \rho_1) \\
&\quad + \tilde{\mathbb{P}}^T(\mathfrak{K} - \rho_2) \varpi_{m2} \tilde{\mathbb{P}}(\mathfrak{K} - \rho_2), \\
\delta_p^T(\tilde{\mathbb{M}}(\mathfrak{K} - \rho_1), \tilde{\mathbb{P}}(\mathfrak{K} - \rho_2)) \delta_p(\tilde{\mathbb{M}}(\mathfrak{K} - \rho_1), \tilde{\mathbb{P}}(\mathfrak{K} - \rho_2)) &\leq \tilde{\mathbb{M}}^T(\mathfrak{K} - \rho_1) \varpi_{p1} \tilde{\mathbb{M}}(\mathfrak{K} - \rho_1) \\
&\quad + \tilde{\mathbb{P}}^T(\mathfrak{K} - \rho_2) \varpi_{p2} \tilde{\mathbb{P}}(\mathfrak{K} - \rho_2)
\end{aligned} \tag{8}$$

respectively, where $\varpi_{mi} \geq 0$ and $\varpi_{pi} \geq 0$ ($i=1, 2$) which is familiar matrices. If the Markovian chain $\mathfrak{r}(\mathfrak{K})$ does not depends on the stochastic perturbations $U_m(\mathfrak{K})$ and $U_p(\mathfrak{K})$, then by the method used in [39], we reframe the system as:

$$y(\mathfrak{K}) = L_m(\mathfrak{r}(\mathfrak{K})) \tilde{\mathbb{M}}(\mathfrak{K} - \rho_1) + L_p(\mathfrak{r}(\mathfrak{K})) \tilde{\mathbb{P}}(\mathfrak{K} - \rho_2), \tag{9}$$

Here the evaluation output is $y(\mathfrak{K}) \in \mathbb{R}^q$ ($q < n$) at time $\mathfrak{K}$ and the known constant matrices are $L_m(\mathfrak{r}(\mathfrak{K}))$, $L_p(\mathfrak{r}(\mathfrak{K}))$ with compatible dimensions.

By setting the terms $x(\mathfrak{K}) = [\tilde{\mathbb{M}}^T(\mathfrak{K} - \rho_1) \quad \tilde{\mathbb{P}}^T(\mathfrak{K} - \rho_2)]^T$, $x(\mathfrak{K}_r) = [\tilde{\mathbb{M}}^T(\mathfrak{K}_r) \quad \tilde{\mathbb{P}}^T(\mathfrak{K}_r)]^T$ in Eqs. (6) and (9), we get the regenerated system as:

$$\begin{aligned}
x(\mathfrak{K} + 1) &= \bar{\mathfrak{A}}(\mathfrak{r}(\mathfrak{K})) x(\mathfrak{K}) + \bar{\mathfrak{B}}(\mathfrak{r}(\mathfrak{K})) \tilde{g}(N(x(\mathfrak{K} - d(\mathfrak{K})))) + \bar{\mathfrak{D}}(\mathfrak{r}(\mathfrak{K})) x(\mathfrak{K} - \tau(\mathfrak{K})) \\
&\quad + \bar{\mathfrak{E}}(\mathfrak{r}(\mathfrak{K})) v(\mathfrak{K}) + \bar{\mathfrak{F}}(\mathfrak{r}(\mathfrak{K})) \mu(\mathfrak{K}) + \bar{\delta}(x(\mathfrak{K})) U(\mathfrak{K}) \\
y(\mathfrak{K}) &= \bar{L}(\mathfrak{r}(\mathfrak{K})) x(\mathfrak{K}) \\
x(\mathfrak{K}_r) &= c_r x(\mathfrak{K}_r - 1), \quad r = 1, 2, \dots
\end{aligned} \tag{10}$$

and the restraint conditions (4) and (8) develop into,

$$\tilde{g}^T(N(x(\mathfrak{K} - d(\mathfrak{K})))) \tilde{g}(N(x(\mathfrak{K} - d(\mathfrak{K})))) \leq \tilde{g}^T(N(x(\mathfrak{K} - d(\mathfrak{K})))) \bar{W} x(\mathfrak{K} - d(\mathfrak{K})), \tag{11}$$

$$\text{trace}(\bar{\delta}^T(x(\mathfrak{K})) \bar{\delta}(x(\mathfrak{K}))) \leq x^T(\mathfrak{K}) \bar{\omega} x(\mathfrak{K}), \tag{12}$$

where $\bar{L}(\mathfrak{r}(\mathfrak{K})) = [L_m(\mathfrak{r}(\mathfrak{K})) \quad L_p(\mathfrak{r}(\mathfrak{K}))]$, $\bar{W} = [0 \quad W]$, $N = [0 \quad I]$,

$$\bar{\mathfrak{D}}(\mathfrak{r}(\mathfrak{K})) = \begin{bmatrix} 0 & 0 \\ \mathfrak{D}(\mathfrak{r}(\mathfrak{K})) & 0 \end{bmatrix}, \quad \bar{\mathfrak{A}}(\mathfrak{r}(\mathfrak{K})) = \begin{bmatrix} \mathfrak{A}(\mathfrak{r}(\mathfrak{K})) & 0 \\ 0 & \mathfrak{C}(\mathfrak{r}(\mathfrak{K})) \end{bmatrix}, \quad \bar{\mathfrak{B}}(\mathfrak{r}(\mathfrak{K})) = \begin{bmatrix} \mathfrak{B}(\mathfrak{r}(\mathfrak{K})) \\ 0 \end{bmatrix}$$

$$\bar{\mathfrak{E}}(\mathfrak{r}(\mathfrak{K})) = \begin{bmatrix} \mathfrak{E}_m(\mathfrak{r}(\mathfrak{K})) \\ \mathfrak{E}_p(\mathfrak{r}(\mathfrak{K})) \end{bmatrix}, \quad U(\mathfrak{K}) = \begin{bmatrix} U_m(\mathfrak{K}) \\ U_p(\mathfrak{K}) \end{bmatrix}, \quad \bar{\mathfrak{F}}(\mathfrak{r}(\mathfrak{K})) = \begin{bmatrix} \mathfrak{F}_m(\mathfrak{r}(\mathfrak{K})) & 0 \\ 0 & \mathfrak{F}_p(\mathfrak{r}(\mathfrak{K})) \end{bmatrix}$$

$$\mu(\mathfrak{K}) = \left[\frac{\sum_{\mathfrak{M}=1}^{\infty} \mu_{\mathfrak{M}} h(y(\mathfrak{K} - \mathfrak{M}))}{\sum_{\mathfrak{P}=1}^{\infty} \mu_{\mathfrak{P}} x(\mathfrak{K} - \mathfrak{P})} \right], \quad \tilde{\omega} = \begin{bmatrix} \omega_{m1} + \omega_{p1} & 0 \\ 0 & \omega_{m2} + \omega_{p2} \end{bmatrix}$$

$$\bar{\delta}(x(\mathfrak{K})) = \begin{bmatrix} \delta_m(\tilde{M}(\mathfrak{K} - \rho_1), \tilde{P}(\mathfrak{K} - \rho_2)) & 0 \\ 0 & \delta_p(\tilde{M}(\mathfrak{K} - \rho_1), \tilde{P}(\mathfrak{K} - \rho_2)) \end{bmatrix}.$$

In this paper, we define $\{r_0, r_1, \dots\}$ as the sequence of time, that occur instantly and the outcomes can be communicated to the estimator. The following condition can be described for each time instant r_l .

$$\varsigma^T(\mathfrak{K})\varsigma(\mathfrak{K}) > \sigma z^T(\mathfrak{K})z(\mathfrak{K}), \quad \sigma > 0, \quad (13)$$

Here the particular error is structured as $\varsigma(\mathfrak{K}) = z(\mathfrak{K}) - z(r_f)$ for $\mathfrak{K} \in [r_1, r_{f+1})$, $f = 0, 1, 2, \dots, \infty$. Then, by evaluating Eq. (13), it can be directed to the states of estimation. Now, we interpret the following equation:

$$\hat{x}(\mathfrak{K} + 1) = \hat{\mathfrak{A}}(\mathfrak{r}(\mathfrak{K}))\hat{x}(\mathfrak{K}) + \hat{G}(\mathfrak{r}(\mathfrak{K}))z(r_l) \quad (14)$$

for $\mathfrak{K} \in [r_1, r_{l+1})$, $l = 0, 1, 2, \dots, \infty$, where $\hat{x}(\mathfrak{K}) \in \mathbb{R}^{2n}$ is the estimate for $x(t)$; The estimator parameters are $\hat{\mathfrak{A}}(\mathfrak{r}(\mathfrak{K})) \in \mathbb{R}^{2n \times 2n}$ and $\hat{G}(\mathfrak{r}(\mathfrak{K})) \in \mathbb{R}^{2n \times p}$ can be resolved. By setting, the estimator initial rate as $\hat{x}(\mathfrak{K}) = 0$ for $\mathfrak{K} \leq 0$. Here, we mark that, the entire feasible value $\mathfrak{r}(\mathfrak{K}) = i$, ($i \in S$),

$$\begin{aligned} \bar{\mathfrak{A}}(\mathfrak{r}(\mathfrak{K}) = i) &= \mathfrak{A}_i, \quad \bar{\mathfrak{B}}(\mathfrak{r}(\mathfrak{K}) = i) = \mathfrak{B}_i, \quad \bar{\mathfrak{D}}(\mathfrak{r}(\mathfrak{K}) = i) = \mathfrak{D}_i, \quad \bar{\mathfrak{E}}(\mathfrak{r}(\mathfrak{K}) = i) = \mathfrak{E}_i, \\ \hat{\mathfrak{A}}(\mathfrak{r}(\mathfrak{K}) = i) &= \hat{\mathfrak{A}}_i, \quad \hat{G}(\mathfrak{r}(\mathfrak{K}) = i) = \hat{G}_i, \quad \hat{L}(\mathfrak{r}(\mathfrak{K}) = i) = L_i, \quad \hat{\mathfrak{F}}(\mathfrak{r}(\mathfrak{K}) = i) = \mathfrak{F}_i. \end{aligned} \quad (15)$$

Assume that the estimation error vector $e(\mathfrak{K}) = x(\mathfrak{K}) - \hat{x}(\mathfrak{K})$, for each $i \in S$, then the error system can be described to be of the form:

$$\begin{aligned} x(\mathfrak{K} + 1) &= \mathfrak{A}_i x(\mathfrak{K}) + \mathfrak{B}_i \tilde{g}(N(x(\mathfrak{K} - d(\mathfrak{K})))) + \mathfrak{D}_i x(\mathfrak{K} - \tau(\mathfrak{K})) + \mathfrak{E}_i v(\mathfrak{K}) \\ &\quad + \mathfrak{F}_i \mu(\mathfrak{K}) + \bar{\delta}(x(\mathfrak{K}))U(\mathfrak{K}) \\ e(\mathfrak{K} + 1) &= (\mathfrak{A}_i - \hat{\mathfrak{A}}_i + L_i \hat{G}_i)x(\mathfrak{K}) + \hat{\mathfrak{A}}_i e(\mathfrak{K}) + \hat{G}_i \varsigma(\mathfrak{K}) + \mathfrak{B}_i \tilde{g}(N(x(\mathfrak{K} - d(\mathfrak{K})))) \\ &\quad + \mathfrak{D}_i x(\mathfrak{K} - \tau(\mathfrak{K})) + \mathfrak{E}_i v(\mathfrak{K}) + \mathfrak{F}_i \mu(\mathfrak{K}) + \bar{\delta}(x(\mathfrak{K}))U(\mathfrak{K}) \\ x(\mathfrak{K}_r) &= c_r x(\mathfrak{K}_r - 1), \quad r = 1, 2, \dots \end{aligned} \quad (16)$$

Remark 2.1. In [18] S. Hu et al. have developed the problem the Event-based H_∞ filtering problem for networked system with communication delay. However, the inclusion of discrete-interval GRNs with leakage delays, distributed delays and impulsive effects have not taken into account. So, the prime intention of this work is to elucidate the H_∞ problem for the improved estimation error system (16) with impulsive effects is stochastically stable.

The following definitions and lemmas are needed:

Definition 2.2. The exogenous disturbance signal $v(\mathfrak{K}) = 0$ with system (16) is called stochastically stable, whether the initial value of every states $\mathfrak{R}_0 = \{\check{y}(\mathfrak{K}), \check{E}(\mathfrak{K}), \mathfrak{K} = -\delta, -\delta + 1, \dots, 0\}$ with $\delta = \max\{\tau_M, \theta_M\}$, satisfies the following condition:

$$\mathbb{E} \left\{ \sum_{\mathfrak{K}=0}^{\infty} (\|\check{y}(\mathfrak{K})\|^2 + \|\check{E}(\mathfrak{K})\|^2) | \mathfrak{R}_0, r(0) \right\} < \infty. \quad (17)$$

In light with above definition, the main contribution is to find the following criteria:

- (1) The exogenous disturbance signal $v(\mathfrak{K})$ is zero with the estimation error system (16) is stochastically stable.
- (2) The estimation error fascinates the following H_∞ attainment compulsion via the zero-initial condition:

$$\mathbb{E} \left\{ \sum_{\mathfrak{K}=0}^{\infty} \|\check{E}(\mathfrak{K})\|^2 \right\} < \gamma^2 \sum_{\mathfrak{K}=0}^{\infty} \|v(\mathfrak{K})\|^2 \quad (18)$$

for all non-zero $v(\mathfrak{K})$, here the disturbance attenuation level $\gamma > 0$ is given.

Lemma 2.3. Let $\mathbb{M} \in \mathbb{R}^{n \times n}$ be a positive semi-definite matrix, $x_i \in \mathbb{R}^n$, and $a_i \geq 0$, ($i = 1, 2, \dots$). If the series perturbed are convergent, the following inequality holds,

$$\left(\sum_{i=1}^{+\infty} a_i x_i \right)^T \mathbb{M} \left(\sum_{i=1}^{+\infty} a_i x_i \right) \leq \left(\sum_{i=1}^{+\infty} a_i \right) \sum_{i=1}^{+\infty} a_i x_i^T \mathbb{M} x_i.$$

Lemma 2.4. (Schur complement, Wan and Fang [45])

Given constant symmetric matrices $\hat{\Sigma}_1, \hat{\Sigma}_2, \hat{\Sigma}_3$ where $\hat{\Sigma}_1^T = \hat{\Sigma}_1$ and $0 < \hat{\Sigma}_2 = \hat{\Sigma}_2^T$, then $\hat{\Sigma}_1 + \hat{\Sigma}_3^T \hat{\Sigma}_2^{-1} \hat{\Sigma}_3 < 0$ iff

$$\begin{bmatrix} \hat{\Sigma}_1 & \hat{\Sigma}_3^T \\ \hat{\Sigma}_3 & -\hat{\Sigma}_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\hat{\Sigma}_2 & \hat{\Sigma}_3 \\ \hat{\Sigma}_3^T & \hat{\Sigma}_1 \end{bmatrix} < 0.$$

Lemma 2.5. (Writinger-based inequality in discrete sense [52])

For a given positive-definite matrix S and the non-negative integers c, d and $\mathfrak{K}$ satisfying $c \leq d \leq \mathfrak{K}$, let us denote

$$\Pi(\mathfrak{K}, c, d) = \begin{cases} \frac{1}{d-c} [2 \sum_{i=\mathfrak{K}-d}^{\mathfrak{K}-c-1} x(i) + x(\mathfrak{K}-c) - x(\mathfrak{K}-d)], & (c < d), \\ 2x(\mathfrak{K}-c), & (c = d). \end{cases}$$

Then, we have

$$-(b-a) \sum_{i=\mathfrak{K}-d}^{\mathfrak{K}-c-1} y^T(i) S y(i) \leq - \begin{bmatrix} \Lambda_0 \\ \Lambda_1 \end{bmatrix}^T \begin{bmatrix} S & 0 \\ 0 & 3S \end{bmatrix} \begin{bmatrix} \Lambda_0 \\ \Lambda_1 \end{bmatrix},$$

where $y(\mathfrak{K}) = x(\mathfrak{K}+1) - x(\mathfrak{K})$, $\Lambda_0 = x(\mathfrak{K}-c) - x(\mathfrak{K}-d)$, $\Lambda_1 = x(\mathfrak{K}-c) + x(\mathfrak{K}-d) - \Pi(\mathfrak{K}, c, d)$.

Lemma 2.6. For given a scalar $\beta \in (0, 1)$ and positive integers n, m , an $n \times n$ -matrix $S > 0$, two $n \times m$ -matrices W_1, W_2 . Define, for all vector $\Omega \in S^m$, the function $\Xi(\beta, S)$ given by

$$\Xi(\beta, S) = \frac{1}{\alpha} \Omega^T T_1^T S T_1 \Omega + \frac{1}{1-\alpha} \Omega^T T_2^T S T_2 \Omega.$$

If there is a matrix $X \in S^m$ such that

$$\begin{bmatrix} S & X \\ * & S \end{bmatrix} > 0$$

then the following inequality holds:

$$\min \Xi(\beta, S) \geq \begin{bmatrix} T_1 \Omega \\ T_2 \Omega \end{bmatrix}^T \begin{bmatrix} S & X \\ * & S \end{bmatrix} \begin{bmatrix} T_1 \Omega \\ T_2 \Omega \end{bmatrix}.$$

3. Main results

In this portion, we early accord a sufficient condition under which the disturbance level $v(\mathfrak{K}) = 0$ with the augmented system (16) is stochastically stable with the estimation error satisfying a prescribed H_∞ estimator.

Theorem 3.1. *The estimator parameters $\hat{\mathfrak{A}}_i$ and $\hat{G}_i (i \in S)$ be given. Then, the disturbance signal $v(\mathfrak{K}) = 0$ with the estimation error system (16) is stochastically stable if there exist positive definite matrices $R_{1i}, R_{2i}, S_1, S_2, N_1, N_2$ and a positive scalar θ such that*

$$\tilde{R}_{1i} + \tilde{R}_{2i} \leq \theta I, \quad (19)$$

and let $c_r^T R_{1i} c_r = d_r, \quad r = 1, 2, \dots$, and constant $\xi (\xi \leq 1)$, such that the following conditions hold:

Condition 1:

$$\Lambda_i = \begin{bmatrix} \Psi_{11i} & \Psi_{12i}^T & \Psi_{13i}^T \\ * & -\tilde{R}_{1i}^{-1} & 0 \\ * & * & -\tilde{R}_{2i}^{-1} \end{bmatrix} < 0, \quad (20)$$

Where

$$\Psi_{11i} = \begin{bmatrix} \Gamma_{1i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -S_1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \bar{W}^T & 0 & 0 & 0 & 0 \\ * & * & -S_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Gamma_{4i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -N_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -N_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \bar{\mu} S_1 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -\bar{\mu} S_1 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -\bar{\xi} S_2 \end{bmatrix},$$

$$\tilde{R}_{1i} = \sum_{j=1}^p \Pi_{ij} R_{1j}, \quad \tilde{R}_{2i} = \sum_{j=1}^p \Pi_{ij} R_{2j}; \quad \bar{\mu} = \sum_{s=1}^{\infty} \mu_s, \quad \bar{\xi} = \sum_{n=1}^{\infty} \xi_n,$$

$$\Gamma_{1i} = -R_{1i} + (1 + d_M - d_m)(S_1 + S_2) + \theta \tilde{w} + \sigma L_i^T L_i,$$

$$\Gamma_{4i} = -R_{2i} + (1 + \tau_M - \tau_m)(N_1 + N_2),$$

$$\Psi_{12i} = [\mathfrak{A}_i \ 0 \ \mathfrak{D}_i \ 0 \ 0 \ 0 \ 0 \ \mathfrak{B}_i \ \mathfrak{F}_i \ 0 \ 0 \ 0],$$

$$\Psi_{13i} = [\mathfrak{A}_i - \tilde{\mathfrak{A}}_i - \tilde{G}_i L_i \ 0 \ \mathfrak{D}_i \ \tilde{\mathfrak{A}}_i \ 0 \ 0 \ \tilde{G}_i \ \mathfrak{B}_i \ \mathfrak{F}_i \ 0 \ 0 \ 0].$$

Condition 2. $\sum_{i=1}^r d_i + \mathfrak{K} \ln(1 - \xi) \leq \Phi(\mathfrak{K}), \quad \text{for any } \mathfrak{K} \in [\mathfrak{K}_r, \mathfrak{K}_{r+1}), \quad \text{then}$
 $\lim_{\mathfrak{K} \rightarrow \infty} \Phi(\mathfrak{K}) = +\infty.$

Proof. By formulating an appropriate LKF for the augmented GRNs Eq. (16) and denote $\mathfrak{R}_{\mathfrak{K}} = \{x(\mathfrak{K} - d), e(\mathfrak{K} - d), \dots, x(\mathfrak{K}), e(\mathfrak{K})\}$

$$\begin{aligned} \mathbb{V}(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) &= \mathbb{V}_1(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) + \mathbb{V}_2(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) + \mathbb{V}_3(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) \\ &\quad + \mathbb{V}_4(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) + \mathbb{V}_5(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) \\ &\quad + \mathbb{V}_6(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})), \end{aligned} \quad (21)$$

Where

$$\begin{aligned} \mathbb{V}_1(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) &= x^T(\mathfrak{K})R_{1r(\mathfrak{K})}x(\mathfrak{K}) + e^T(\mathfrak{K})R_{2r(\mathfrak{K})}e(\mathfrak{K}) \\ \mathbb{V}_2(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) &= \sum_{\mu=\mathfrak{K}-d(\mathfrak{K})}^{\mathfrak{K}-1} x^T(\mu)S_1x(\mu) + \sum_{\mu=\mathfrak{K}-\tau(\mathfrak{K})}^{\mathfrak{K}-1} x^T(\mu)S_2x(\mu) \\ \mathbb{V}_3(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) &= \sum_{\mu=\mathfrak{K}-d(\mathfrak{K})}^{\mathfrak{K}-1} e^T(\mu)N_1e(\mu) + \sum_{\mu=\mathfrak{K}-\tau(\mathfrak{K})}^{\mathfrak{K}-1} e^T(\mu)N_2e(\mu) \\ \mathbb{V}_4(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) &= \sum_{v=-d_m-1}^{-d_m-1} \sum_{\mu=\mathfrak{K}+v}^{\mathfrak{K}-1} x^T(\mu)S_1x(\mu) + \sum_{v=-d_m}^{-1} \sum_{\mu=\mathfrak{K}+v}^{\mathfrak{K}-1} x^T(\mu)S_2x(\mu) \\ \mathbb{V}_5(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) &= \sum_{v=-\tau_m-1}^{-\tau_m-1} \sum_{\mu=\mathfrak{K}+v}^{\mathfrak{K}-1} e^T(\mu)N_1e(\mu) + \sum_{v=-\tau_m}^{-1} \sum_{\mu=\mathfrak{K}+v}^{\mathfrak{K}-1} e^T(\mu)N_2e(\mu) \\ \mathbb{V}_6(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) &= \sum_{i=1}^{\infty} \mu_i \sum_{j=\mathfrak{K}-i}^{\mathfrak{K}-1} h^T(y(j))S_1h(y(j)) + \sum_{i=1}^{\infty} \xi_i \sum_{j=\mathfrak{K}-i}^{\mathfrak{K}-1} x^T(i)S_2x(i). \end{aligned}$$

For $\mathfrak{r}(\mathfrak{K}) = i$, $\mathfrak{r}(\mathfrak{K} + 1) = j$, and by using Eq. (5) the time difference $V(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K}))$ can be calculated beside the trajectories of model Eq. (16) with $v(\mathfrak{K}) = 0$, it can be clear to see that,

$$\begin{aligned} \mathbb{E}[\Delta \mathbb{V}_1(\mathfrak{K}) | \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})] &= \mathbb{E}[\mathbb{V}_1(\mathfrak{K} + 1, \mathfrak{R}_{\mathfrak{K}+1}, \mathfrak{r}(\mathfrak{K} + 1)) - \mathbb{V}_1(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) | \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})] \\ &= \zeta^T(\mathfrak{K})(\Psi_{12i}^T \tilde{R}_{1i} \Psi_{12i} + \Psi_{13i}^T \tilde{R}_{2i} \Psi_{13i}) \zeta(\mathfrak{K}) - x^T(\mathfrak{K})R_{1i}x(\mathfrak{K}) \\ &\quad - e^T(\mathfrak{K})R_{2i}e(\mathfrak{K}) + \text{trace}[\tilde{\delta}^T(x(\mathfrak{K}))(\tilde{R}_{1i} + \tilde{R}_{2i})\tilde{\delta}(x(\mathfrak{K}))], \end{aligned} \quad (22)$$

$$\begin{aligned} \mathbb{E}[\Delta \mathbb{V}_2(\mathfrak{K}) | \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})] &= \mathbb{E}[\mathbb{V}_2(\mathfrak{K} + 1, \mathfrak{R}_{\mathfrak{K}+1}, \mathfrak{r}(\mathfrak{K} + 1)) - \mathbb{V}_2(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) | \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})] \\ &\leq x^T(\mathfrak{K})S_1x(\mathfrak{K}) - x^T(\mathfrak{K} - d(\mathfrak{K}))S_1x(\mathfrak{K} - d(\mathfrak{K})) \\ &\quad + \sum_{\mu=\mathfrak{K}-d_m+1}^{\mathfrak{K}-d_m} x^T(\mu)S_1x(\mu) + x^T(\mathfrak{K})S_2x(\mathfrak{K}) \\ &\quad - x^T(\mathfrak{K} - \tau(\mathfrak{K}))S_2x(\mathfrak{K} - \tau(\mathfrak{K})) + \sum_{\mu=\mathfrak{K}-\tau_m+1}^{\mathfrak{K}-\tau_m} x^T(\mu)S_2x(\mu) \end{aligned} \quad (23)$$

$$\begin{aligned} \mathbb{E}[\Delta \mathbb{V}_3(\mathfrak{K}) | \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})] &= \mathbb{E}[\mathbb{V}_3(\mathfrak{K} + 1, \mathfrak{R}_{\mathfrak{K}+1}, \mathfrak{r}(\mathfrak{K} + 1)) - \mathbb{V}_3(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})) | \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})] \\ &\leq e^T(\mathfrak{K})N_1e(\mathfrak{K}) - e^T(\mathfrak{K} - d(\mathfrak{K}))N_1e(\mathfrak{K} - d(\mathfrak{K})) + \sum_{\mu=\mathfrak{K}-d_m+1}^{\mathfrak{K}-d_m} e^T(\mu)N_1e(\mu) \end{aligned}$$

$$\begin{aligned}
& + e^T(\mathfrak{K})N_2e(\mathfrak{K}) - e^T(\mathfrak{K} - \tau(\mathfrak{K}))N_2e(\mathfrak{K} - \tau(\mathfrak{K})) \\
& + \sum_{\mu=\mathfrak{K}-\tau_M+1}^{\mathfrak{K}-\tau_m} e^T(\mu)N_2e(\mu)
\end{aligned} \quad (24)$$

$$\begin{aligned}
\mathbb{E}\{\Delta V_4(\mathfrak{K})|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} &= \mathbb{E}\{[\mathbb{V}_4(\mathfrak{K}+1, \mathfrak{R}_{\mathfrak{K}+1}, \mathfrak{r}(\mathfrak{K}+1)) - \mathbb{V}_4(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K}))]| \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} \\
&= (d_M - d_m)x^T(\mathfrak{K})S_1x(\mathfrak{K}) + (d_M - d_m)x^T(\mathfrak{K})S_2x(\mathfrak{K}) \\
&\quad - \sum_{\mu=\mathfrak{K}-d_M}^{\mathfrak{K}-d_m-1} x^T(\mu)S_1x(\mu) - \sum_{\mu=\mathfrak{K}-d_m}^{\mathfrak{K}-1} x^T(\mu)S_2x(\mu)
\end{aligned} \quad (25)$$

$$\begin{aligned}
\mathbb{E}\{\Delta V_5(\mathfrak{K})|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} &= \mathbb{E}\{[\mathbb{V}_5(\mathfrak{K}+1, \mathfrak{R}_{\mathfrak{K}+1}, \mathfrak{r}(\mathfrak{K}+1)) - \mathbb{V}_5(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K}))]| \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} \\
&= (\tau_M - \tau_m)e^T(\mathfrak{K})N_1e(\mathfrak{K}) + (\tau_M - \tau_m)e^T(\mathfrak{K})N_2e(\mathfrak{K}) \\
&\quad - \sum_{\mu=\mathfrak{K}-\tau_M}^{\mathfrak{K}-\tau_m-1} e^T(\mu)N_1e(\mu) - \sum_{\mu=\mathfrak{K}-\tau_m}^{\mathfrak{K}-1} e^T(\mu)N_2e(\mu)
\end{aligned} \quad (26)$$

$$\begin{aligned}
\mathbb{E}\{\Delta V_6(\mathfrak{K})|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} &= \mathbb{E}\{[\mathbb{V}_6(\mathfrak{K}+1, \mathfrak{R}_{\mathfrak{K}+1}, \mathfrak{r}(\mathfrak{K}+1)) - \mathbb{V}_6(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K}))]| \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} \\
&= \sum_{i=1}^{\infty} \mu_i \sum_{j=\mathfrak{K}+1-i}^{\mathfrak{K}+1-1} h^T(y(j))S_1h(y(j)) + \sum_{i=1}^{\infty} \xi_i \sum_{j=\mathfrak{K}+1-i}^{\mathfrak{K}+1-1} x^T(i)S_2x(i) \\
&\quad - \sum_{i=1}^{\infty} \mu_i \sum_{j=\mathfrak{K}-i}^{\mathfrak{K}-1} h^T(y(j))S_1h(y(j)) - \sum_{i=1}^{\infty} \xi_i \sum_{j=\mathfrak{K}-i}^{\mathfrak{K}-1} x^T(i)S_2x(i) \\
&= \sum_{i=1}^{\infty} \mu_i [h^T(y(\mathfrak{K}))S_1h(y(\mathfrak{K})) - h^T(y(\mathfrak{K}-i))S_1h(y(\mathfrak{K}-i))] \\
&\quad + \sum_{i=1}^{\infty} \xi_i [x^T(\mathfrak{K})S_2x(\mathfrak{K}) - x^T(\mathfrak{K}-i)S_2x(\mathfrak{K}-i)]
\end{aligned}$$

Using [Lemma 2.3](#) we get,

$$\begin{aligned}
&\leq \bar{\mu}h^T(y(\mathfrak{K}))S_1h(y(\mathfrak{K})) - \bar{\mu}[\bar{\mu}h(y(\mathfrak{K}-s))]^T S_1[\bar{\mu}h(y(\mathfrak{K}-s))] \\
&\quad + \bar{\xi}x^T(\mathfrak{K})S_2x(\mathfrak{K}) - \bar{\xi}[\bar{\xi}x(\mathfrak{K}-n)]^T S_2[\bar{\xi}x(\mathfrak{K}-n)],
\end{aligned} \quad (27)$$

where

$$\begin{aligned}
\Omega(\mathfrak{K}) &= [x^T(\mathfrak{K}), x^T(\mathfrak{K}-d(\mathfrak{K})), x^T(\mathfrak{K}-\tau(\mathfrak{K})), e^T(\mathfrak{K}), e^T(\mathfrak{K}-d(\mathfrak{K})), e^T(\mathfrak{K}-\tau(\mathfrak{K})), \varsigma^T(\mathfrak{K}), \\
&\quad \tilde{g}^T(N(x(\mathfrak{K}-d(\mathfrak{K})))), \mu^T(\mathfrak{K}), h^T(y(\mathfrak{K})), [\bar{\mu}h(y(\mathfrak{K}-s))]^T, [\bar{\xi}x(\mathfrak{K}-n)]^T]^T.
\end{aligned} \quad (28)$$

From [Eqs. \(12\)](#) and [\(42\)](#), we get

$$\text{trace}[\bar{\delta}^T(x(\mathfrak{K}))(\tilde{R}_{1i} + \tilde{R}_{2i})\bar{\delta}(x(\mathfrak{K}))] \leq \theta x^T(\mathfrak{K})\tilde{\omega}x(\mathfrak{K}). \quad (29)$$

By summing Eqs. (11), (13), (20), (22)–(29), we get

$$\begin{aligned}\mathbb{E}\{\Delta\mathbb{V}(\mathfrak{K})|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} &= \mathbb{E}\{[\mathbb{V}(\mathfrak{K}+1, \mathfrak{R}_{\mathfrak{K}+1}, \mathfrak{r}(\mathfrak{K}+1)) - \mathbb{V}(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K}))]| \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} \\ &\leq \Omega^T(\mathfrak{K})\Lambda_i\Omega(\mathfrak{K}) \\ &\leq \lambda_{\max}(\Lambda_i)\|\Omega(\mathfrak{K})\|^2, \quad (i = 1, 2, \dots, 5).\end{aligned}\quad (30)$$

From Eq. (29), one can observe that,

$$\Delta\mathbb{V}(x(\mathfrak{K})) = \mathbb{V}(x(\mathfrak{K}+1)) - \mathbb{V}(x(\mathfrak{K})) < 0. \quad (31)$$

Then there exist a positive constant $\xi \leq 1$ such that,

$$\mathbb{V}(x(\mathfrak{K}+1)) \leq (1-\xi)\mathbb{V}(x(\mathfrak{K})), \quad \text{for } \mathfrak{K} \neq \mathfrak{K}_r$$

Then,

$$\mathbb{V}(x(\mathfrak{K}_0+1)) \leq (1-\xi)\mathbb{V}(x(\mathfrak{K}_0))$$

$$\mathbb{V}(x(\mathfrak{K}_0+2)) \leq (1-\xi)^2\mathbb{V}(x(\mathfrak{K}_0))$$

$$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$$

$$\mathbb{V}(x(\mathfrak{K}_1)) = \mathbb{V}(x(\mathfrak{K}_0 + \mathfrak{K}_1 - \mathfrak{K}_0)) \leq (1-\xi)^{\mathfrak{K}_1 - \mathfrak{K}_0}\mathbb{V}(x(\mathfrak{K}_0))$$

Similarly,

$$\begin{aligned}\mathbb{V}(x(\mathfrak{K}_2)) &= \mathbb{V}(x(\mathfrak{K}_1 + \mathfrak{K}_2 - \mathfrak{K}_1)) \leq (1-\xi)^{\mathfrak{K}_2 - \mathfrak{K}_1}\mathbb{V}(x(\mathfrak{K}_1)) \\ &\leq (1-\xi)^{\mathfrak{K}_2 - \mathfrak{K}_1}(1-\xi)^{\mathfrak{K}_1 - \mathfrak{K}_0}\mathbb{V}(x(\mathfrak{K}_0)) \\ &= (1-\xi)^{\mathfrak{K}_2 - \mathfrak{K}_0}\mathbb{V}(x(\mathfrak{K}_0))\end{aligned}$$

and therefore

$$\mathbb{V}(x(\mathfrak{K})) \leq (1-\xi)^{\mathfrak{K} - \mathfrak{K}_r}\mathbb{V}(x(\mathfrak{K}_r)), \quad \text{for } \mathfrak{K} \in [\mathfrak{K}_r, \mathfrak{K}_{r+1}). \quad (32)$$

Here we note that,

$$\begin{aligned}\mathbb{V}(x(\mathfrak{K}_r)) &= x^T(\mathfrak{K}_r)R_{1i}x(\mathfrak{K}_r) + e^T(\mathfrak{K}_r)R_{2i}e(\mathfrak{K}_r) + \sum_{\mu=\mathfrak{K}_r-d(\mathfrak{K}_r)}^{\mathfrak{K}_r-1} x^T(\mu)S_1x(\mu) \\ &+ \sum_{\mu=\mathfrak{K}_r-\tau(\mathfrak{K}_r)}^{\mathfrak{K}_r-1} x^T(\mu)S_2x(\mu) + \sum_{\mu=\mathfrak{K}_r-d(\mathfrak{K}_r)}^{\mathfrak{K}_r-1} e^T(\mu)N_1e(\mu) \\ &+ \sum_{\mu=\mathfrak{K}_r-\tau(\mathfrak{K}_r)}^{\mathfrak{K}_r-1} e^T(\mu)N_2e(\mu) + \sum_{v=-d_M+1}^{-d_m} \sum_{\mu=\mathfrak{K}_r+v}^{\mathfrak{K}_r-1} x^T(\mu)S_1x(\mu) \\ &+ \sum_{v=-\tau_M+1}^{-\tau_m} \sum_{\mu=\mathfrak{K}_r+v}^{\mathfrak{K}_r-1} x^T(\mu)S_2x(\mu) + \sum_{v=-d_M+1}^{-d_m} \sum_{\mu=\mathfrak{K}_r+v}^{\mathfrak{K}_r-1} e^T(\mu)N_1e(\mu)\end{aligned}$$

$$\begin{aligned}
& + \sum_{v=-\tau_M+1}^{-\tau_m} \sum_{\mu=\mathfrak{K}_r+v}^{\mathfrak{K}_r-1} e^T(\mu) N_2 e(\mu) + \sum_{i=1}^{\infty} \mu_i \sum_{j=\mathfrak{K}_r-i}^{\mathfrak{K}_r-1} h^T(y(j)) S_1 h(y(j)) \\
& + \sum_{i=1}^{\infty} \xi_i \sum_{j=\mathfrak{K}_r-i}^{\mathfrak{K}_r-1} x^T(i) S_2 x(i) \\
& = [c_r x(\mathfrak{K}_r - 1)]^T R_{1i} [c_r x(\mathfrak{K}_r - 1)] + e^T(\mathfrak{K}_r^-) R_{2i} e(\mathfrak{K}_r^-) + \sum_{\mu=\mathfrak{K}_r-d(\mathfrak{K}_r^-)}^{\mathfrak{K}_r^- - 1} x^T(\mu) S_1 x(\mu) \\
& + \sum_{\mu=\mathfrak{K}_r-\tau(\mathfrak{K}_r^-)}^{\mathfrak{K}_r^- - 1} x^T(\mu) S_2 x(\mu) + \sum_{\mu=\mathfrak{K}_r-d(\mathfrak{K}_r^-)}^{\mathfrak{K}_r^- - 1} e^T(\mu) N_1 e(\mu) \\
& + \sum_{\mu=\mathfrak{K}_r-\tau(\mathfrak{K}_r^-)}^{\mathfrak{K}_r^- - 1} e^T(\mu) N_2 e(\mu) + \sum_{v=-d_M+1}^{-d_m} \sum_{\mu=\mathfrak{K}_r+v}^{\mathfrak{K}_r^- - 1} x^T(\mu) S_1 x(\mu) \\
& + \sum_{v=-\tau_M+1}^{-\tau_m} \sum_{\mu=\mathfrak{K}_r+v}^{\mathfrak{K}_r^- - 1} x^T(\mu) S_2 x(\mu) + \sum_{v=-d_M+1}^{-d_m} \sum_{\mu=\mathfrak{K}_r+v}^{\mathfrak{K}_r^- - 1} e^T(\mu) N_1 e(\mu) \\
& + \sum_{v=-\tau_M+1}^{-\tau_m} \sum_{\mu=\mathfrak{K}_r+v}^{\mathfrak{K}_r^- - 1} e^T(\mu) N_2 e(\mu) + \sum_{i=1}^{\infty} \mu_i \sum_{j=\mathfrak{K}_r-i}^{\mathfrak{K}_r^- - 1} h^T(y(j)) S_1 h(y(j)) \\
& + \sum_{i=1}^{\infty} \xi_i \sum_{j=\mathfrak{K}_r-i}^{\mathfrak{K}_r^- - 1} x^T(i) S_2 x(i) \\
& = x^T(\mathfrak{K}_r - 1) [c_r^T R_{1i} c_r] x(\mathfrak{K}_r - 1) + e^T(\mathfrak{K}_r^-) R_{2i} e(\mathfrak{K}_r^-) + \sum_{\mu=\mathfrak{K}_r-d(\mathfrak{K}_r^-)}^{\mathfrak{K}_r^- - 1} x^T(\mu) S_1 x(\mu) \\
& + \sum_{\mu=\mathfrak{K}_r-\tau(\mathfrak{K}_r^-)}^{\mathfrak{K}_r^- - 1} x^T(\mu) S_2 x(\mu) + \sum_{\mu=\mathfrak{K}_r-d(\mathfrak{K}_r^-)}^{\mathfrak{K}_r^- - 1} e^T(\mu) N_1 e(\mu) \\
& + \sum_{\mu=\mathfrak{K}_r-\tau(\mathfrak{K}_r^-)}^{\mathfrak{K}_r^- - 1} e^T(\mu) N_2 e(\mu) + \sum_{v=-d_M+1}^{-d_m} \sum_{\mu=\mathfrak{K}_r+v}^{\mathfrak{K}_r^- - 1} x^T(\mu) S_1 x(\mu) \\
& + \sum_{v=-\tau_M+1}^{-\tau_m} \sum_{\mu=\mathfrak{K}_r+v}^{\mathfrak{K}_r^- - 1} x^T(\mu) S_2 x(\mu) + \sum_{v=-d_M+1}^{-d_m} \sum_{\mu=\mathfrak{K}_r+v}^{\mathfrak{K}_r^- - 1} e^T(\mu) N_1 e(\mu) \\
& + \sum_{v=-\tau_M+1}^{-\tau_m} \sum_{\mu=\mathfrak{K}_r+v}^{\mathfrak{K}_r^- - 1} e^T(\mu) N_2 e(\mu) + \sum_{i=1}^{\infty} \mu_i \sum_{j=\mathfrak{K}_r-i}^{\mathfrak{K}_r^- - 1} h^T(y(j)) S_1 h(y(j)) \\
& + \sum_{i=1}^{\infty} \xi_i \sum_{j=\mathfrak{K}_r-i}^{\mathfrak{K}_r^- - 1} x^T(i) S_2 x(i)
\end{aligned}$$

$$\begin{aligned}
&\leq c_r^T R_{1i} c_r \mathbb{V}(x(\mathfrak{K}_r - 1)) \\
&\leq d_r \mathbb{V}(x(\mathfrak{K}_r - 1)).
\end{aligned} \tag{33}$$

Using Eqs. (31) and (33) successively in each interval $[\mathfrak{K}_r, \mathfrak{K}_{r+1})$ yields the following inequalities when $\mathfrak{K} \in [\mathfrak{K}_0, \mathfrak{K}_1)$.

$$\begin{aligned}
\mathbb{V}(x(\mathfrak{K})) &\leq (1 - \xi)^{\mathfrak{K} - \mathfrak{K}_0} \mathbb{V}(x(\mathfrak{K}_0)) \\
&\text{and} \\
\mathbb{V}(x(\mathfrak{K}_1)) &= d_1 \mathbb{V}(x(\mathfrak{K}_1 - 1)) \\
&\leq d_1 (1 - \xi)^{\mathfrak{K}_1 - 1 - \mathfrak{K}_0} \mathbb{V}(x(\mathfrak{K}_0)) \\
\mathbb{V}(x(\mathfrak{K}_2)) &= d_2 \mathbb{V}(x(\mathfrak{K}_2 - 1)) \\
&\leq d_2 d_1 (1 - \xi)^{\mathfrak{K}_2 - 2 - \mathfrak{K}_0} \mathbb{V}(x(\mathfrak{K}_0))
\end{aligned}$$

and

$$\mathbb{V}(x(\mathfrak{K}_3)) \leq d_3 d_2 d_1 (1 - \xi)^{\mathfrak{K}_3 - 3 - \mathfrak{K}_0} \mathbb{V}(x(\mathfrak{K}_0)).$$

Therefore by induction, for any $\mathfrak{K} \in [\mathfrak{K}_r, \mathfrak{K}_{r+1})$, for $r = 0, 1, 2, \dots$

$$\begin{aligned}
\mathbb{V}(x(\mathfrak{K})) &\leq (1 - \xi)^{\mathfrak{K} - \mathfrak{K}_r} \mathbb{V}(x(\mathfrak{K}_r - 1)) \\
&\leq (\prod_{i=1}^r d_i) (1 - \xi)^{\mathfrak{K}} \mathbb{V}(x(\mathfrak{K}_0)) \\
&= \mathbb{V}(x(\mathfrak{K}_0)) e^{[\sum_{i=1}^r d_i + \mathfrak{K} \ln(1 - \xi)]} \\
\mathbb{V}(x(\mathfrak{K})) &\leq \mathbb{V}(x(\mathfrak{K}_0)) e^{\phi(\mathfrak{K})}.
\end{aligned}$$

Then it follows from Condition 2 that the origination of discrete-interval GRNs with impulsive effect is stochastically stable and the labeled system (16) with $v(\mathfrak{K}) = 0$ is assured, henceforth Theorem 3.1 is complete. $\square$

Theorem 3.2. Assume that the estimator parameters $\hat{\mathfrak{A}}_i$, $\hat{\mathfrak{G}}_i (i \in S)$ and the H_∞ performance index γ are given. Then, the disturbance signal $v(\mathfrak{K}) = 0$ of system (16) is stochastically stable and $e(\mathfrak{K})$ often refers as estimation error satisfies the H_∞ attainment constraint (18) under the zero initial condition for all non-zero $v(\mathfrak{K})$ if there exist positive definite matrices R_{1i} , R_{2i} , S_1 , S_2 , N_1 , N_2 and a positive scalar θ such that

$$\tilde{R}_{1i} + \tilde{R}_{2i} \leq \theta I, \tag{34}$$

and let $c_r^T R_{1i} c_r = d_r$, $r = 1, 2, \dots$ and constant $\xi (\xi \leq 1)$, such that the following conditions hold.

Condition 1.

$$\tilde{\Lambda}_i = \begin{bmatrix} \tilde{\Psi}_{11i} & \tilde{\Psi}_{12i}^T & \tilde{\Psi}_{13i}^T \\ * & -\tilde{R}_{1i}^{-1} & 0 \\ * & * & -\tilde{R}_{2i}^{-1} \end{bmatrix} < 0, \quad (35)$$

Where

$$\tilde{\Psi}_{11i} = \begin{bmatrix} \Gamma_{1i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -S_1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\bar{W}^T & 0 & 0 & 0 & 0 & 0 \\ * & * & -S_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \tilde{\Gamma}_{4i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -N_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -N_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \bar{\mu}S_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -\bar{\mu}S_1 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -\bar{\xi}S_2 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & -\gamma^2 I \end{bmatrix}$$

$$\tilde{\Gamma}_{4i} = -R_{2i} + (1 + d_M - d_m)N_1 + (1 + \tau_M - \tau_m)N_2 + I,$$

$$\tilde{\Psi}_{12i} = [\mathfrak{A}_i \ 0 \ \mathfrak{D}_i \ 0 \ 0 \ 0 \ 0 \ \mathfrak{B}_i \ \mathfrak{F}_i \ 0 \ 0 \ 0 \ \mathfrak{E}_i],$$

$$\tilde{\Psi}_{13i} = [\mathfrak{A}_i - \tilde{\mathfrak{A}}_i - \tilde{G}_i L_i \ 0 \ \mathfrak{D}_i \ \tilde{\mathfrak{A}}_i \ 0 \ 0 \ \tilde{G}_i \ \mathfrak{B}_i \ \mathfrak{F}_i \ 0 \ 0 \ 0 \ \mathfrak{E}_i].$$

Condition 2. $\sum_{i=1}^r d_i + \mathfrak{K} \ln(1 - \xi) \leq \Phi(\mathfrak{K})$, for any $\mathfrak{K} \in [\mathfrak{K}_r, \mathfrak{K}_{r+1})$, then $\lim_{\mathfrak{K} \rightarrow \infty} \Phi(\mathfrak{K}) = +\infty$.

Proof. It is clear to see that Eq. (35) implies Eq. (20). Therefore, in order to Theorem 3.1, the stochastic stability of the system (16) with the disturbance signal $v(\mathfrak{K}) = 0$ is assured. Considering the H_∞ attainment investigation, we model into an indicator assignment stated as below:

$$J = \mathbb{E} \left\{ \sum_{\mathfrak{K}=0}^M [E^T(\mathfrak{K})E(\mathfrak{K}) - \gamma^2 v^T(\mathfrak{K})v(\mathfrak{K})] | \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K}) \right\}, \quad (36)$$

Where M is a non-negative integer. The main objective is to obtain $J < 0$ for which the initial condition to be zero.

Consider the LKF and employing the same steps in the above theorem, we derive the $\mathbb{E}\{\Delta V(\mathfrak{K}) | \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\}$ as:

$$\begin{aligned} \mathbb{E}\{\Delta V(\mathfrak{K}) | \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} &= \mathbb{E}\{[V(\mathfrak{K} + 1, \mathfrak{R}_{\mathfrak{K}+1}, \mathfrak{r}(\mathfrak{K} + 1)) - V(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K}))] | \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} \\ &= \mathbb{E}\{[\Delta V_1(\mathfrak{K}) + \Delta V_2(\mathfrak{K}) + \Delta V_3(\mathfrak{K}) + \Delta V_4(\mathfrak{K}) \\ &\quad + \Delta V_5(k) + \Delta V_6(k)] | \mathfrak{R}_k, \mathfrak{r}(k)\} \\ &\leq \tilde{\Omega}^T(\mathfrak{K}) \tilde{\Lambda}_i \tilde{\Omega}(\mathfrak{K}), \end{aligned} \quad (37)$$

Where $\tilde{\Omega}(\mathfrak{K}) = [\Omega^T(\mathfrak{K}) \ v^T(\mathfrak{K})]^T$ and

$$\tilde{\Lambda}_i = \begin{bmatrix} \tilde{\Psi}_{11i} & \tilde{\Psi}_{12i}^T & \tilde{\Psi}_{13i}^T \\ * & -\tilde{R}_{1i}^{-1} & 0 \\ * & * & -\tilde{R}_{2i}^{-1} \end{bmatrix}$$

Where

$$\tilde{\Psi}_{11i} = \begin{bmatrix} \Gamma_{1i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -S_1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\tilde{W}^T & 0 & 0 & 0 & 0 \\ * & * & -S_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Gamma_{4i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -N_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -N_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \bar{\mu}S_1 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -\bar{\mu}S_1 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -\bar{\xi}S_2 \end{bmatrix}$$

Next, the result follows from Eq. (37), by substituting the zero initial condition

$$\begin{aligned} J &= \mathbb{E} \left\{ \sum_{\mathfrak{K}=0}^M [E^T(\mathfrak{K})E(\mathfrak{K}) - \gamma^2 v^T(\mathfrak{K})v(\mathfrak{K}) + \Delta \mathbb{V}(\mathfrak{K})] | \mathfrak{R}_{\mathfrak{K}}, \tau(\mathfrak{K}) \right\} - \mathbb{E} \left\{ \sum_{\mathfrak{K}=0}^M \Delta \mathbb{V}(\mathfrak{K}) | \mathfrak{R}_{\mathfrak{K}}, \tau(\mathfrak{K}) \right\} \\ &\leq \mathbb{E} \left\{ \sum_{\mathfrak{K}=0}^M [E^T(\mathfrak{K})E(\mathfrak{K}) - \gamma^2 v^T(\mathfrak{K})v(\mathfrak{K}) + \Delta \mathbb{V}(\mathfrak{K})] | \mathfrak{R}_{\mathfrak{K}}, \tau(\mathfrak{K}) \right\} \\ &= \sum_{\mathfrak{K}=0}^M \tilde{\Omega}^T(\mathfrak{K}) \tilde{\Lambda}_i \tilde{\Omega}(\mathfrak{K}). \end{aligned} \quad (38)$$

Also, it can be completed from Eq. (35) that $J < 0$. Lastly, by letting $M \rightarrow \infty$, it is clear to view that

$$\mathbb{E} \left\{ \sum_{\mathfrak{K}=0}^M E^T(\mathfrak{K})E(\mathfrak{K}) \right\} < \sum_{\mathfrak{K}=0}^{\infty} \gamma^2 v^T(\mathfrak{K})v(\mathfrak{K}), \quad (39)$$

Which arrives the proof of Theorem 3.2. The proof of Condition 2 is similar to that of Theorem 3.1 and hence it is omitted here. $\square$

Remark 3.3. Here Theorem 3.2 states that, the estimation error system is stochastically stable so that the sufficient condition for proving the H_{∞} consumption constraint is imitative and it may be fleck conventional. Further, the preservation could be diminished by presuming other leading mechanization to conception with time-delays acting as delay-fractioning approach.

The following theorem is proposed by the beyond investigation.

Theorem 3.4. The H_{∞} attainment indicator γ is given and the H_{∞} estimation problem is solvent for the GRN Eq. (16), if there exist matrices $R_{1i} > 0$, $R_{2i} > 0$, $S_1 > 0$, $S_2 > 0$, $N_1 > 0$, $N_2 > 0$, X_i , Y_i , ($i \in S$) and a positive scalar θ such that,

$$\tilde{R}_{1i} + \tilde{R}_{2i} \leq \theta I, \quad (40)$$

and let $c_r^T R_{1i} c_r = d_r$, $r = 1, 2, \dots$, and constant $\xi (\xi \leq 1)$, such that the following conditions hold.

Condition 1.

$$\tilde{\Lambda}_i = \begin{bmatrix} \tilde{\Psi}_{11i} & \tilde{\Psi}_{12i}^T & \tilde{\Psi}_{13i}^T \\ * & -\tilde{R}_{1i}^{-1} & 0 \\ * & * & -\tilde{R}_{2i}^{-1} \end{bmatrix} < 0, \quad (41)$$

Where

$$\begin{aligned} \tilde{\Psi}_{12i} &= [\tilde{R}_{1i} A_i \ 0 \ \tilde{R}_{1i} D_i \ 0 \ 0 \ 0 \ 0 \ \tilde{R}_{1i} B_i \ \tilde{R}_{1i} F_i \ 0 \ 0 \ 0 \ \tilde{R}_{1i} E_i] \\ \tilde{\Psi}_{13i} &= [\tilde{R}_{2i} A_i - X_i - Y_i L_i \ 0 \ \tilde{R}_{2i} D_i \ X_i \ 0 \ 0 \ Y_i \ \tilde{R}_{2i} B_i \ \tilde{R}_{2i} F_i \ 0 \ 0 \ 0 \ \tilde{R}_{2i} E_i] \end{aligned}$$

Moreover, we attain the feasible solution for the LMIs Eqs. (40) and (41), then the gain matrices can be derived from the estimator and is given by,

$$X_i = \tilde{R}_{2i} \tilde{A}_i, \quad Y_i = \tilde{R}_{2i} \tilde{G}_i.$$

Condition 2 $\sum_{i=1}^r d_i + \mathfrak{K} \ln(1 - \xi) \leq \Phi(\mathfrak{K})$, for any $\mathfrak{K} \in [\mathfrak{K}_r, \mathfrak{K}_{r+1})$, then $\lim_{\mathfrak{K} \rightarrow \infty} \Phi(\mathfrak{K}) = +\infty$.

Proof. The proof of Theorem 3.4 immediately follows from Theorem 3.2. $\square$

The following theorem is based on the Wirtinger inequality in discrete sense.

Theorem 3.5. The estimator parameters $\hat{\mathfrak{A}}_i, \hat{G}_i (i \in S)$, non-negative scales d_m, d_M, τ_m and τ_M be given. Then, the disturbance signal $v(\mathfrak{K}) = 0$ with the estimation error system (16) is asymptotically stable if there exist positive definite matrices $R_{1i}, R_{2i}, S_1, S_2, N_1, N_2, Y_1, Y_2$ and a positive scalar θ such that

$$\tilde{R}_{1i} + \tilde{R}_{2i} \leq \theta I, \quad (42)$$

and let $c_r^T R_{1i} c_r = d_r$, $r = 1, 2, \dots$, and constant $\xi (\xi \leq 1)$, such that the following conditions hold:

Condition 1:

$$\Lambda_i = \begin{bmatrix} \Psi_{11i} & \Psi_{12i}^T & \Psi_{13i}^T \\ * & -\tilde{R}_{1i}^{-1} & 0 \\ * & * & -\tilde{R}_{2i}^{-1} \end{bmatrix} < 0, \quad (43)$$

Where

$$\Psi_{11i} = \begin{bmatrix} \Gamma_{1i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -S_1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\bar{W}^T & 0 & 0 & 0 & 0 \\ * & * & -S_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Gamma_{4i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -N_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -N_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \bar{\mu}S_1 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -\bar{\mu}S_1 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -\bar{\xi}S_2 \end{bmatrix},$$

$$\tilde{R}_{1i} = \sum_{j=1}^p \Pi_{ij} R_{1j}, \quad \tilde{R}_{2i} = \sum_{j=1}^p \Pi_{ij} R_{2j}; \quad \bar{\mu} = \sum_{s=1}^{\infty} \mu_s, \quad \bar{\xi} = \sum_{n=1}^{\infty} \xi_n,$$

$$\Gamma_{1i} = -R_{1i} + (1 + d_M - d_m)(S_1 + S_2) + \theta \tilde{w} + \sigma L_i^T L_i,$$

$$\Gamma_{4i} = -R_{2i} + (1 + \tau_M - \tau_m)(N_1 + N_2),$$

$$\Psi_{12i} = [\mathfrak{A}_i \ 0 \ \mathfrak{D}_i \ 0 \ 0 \ 0 \ 0 \ \mathfrak{B}_i \ \mathfrak{F}_i \ 0 \ 0 \ 0],$$

$$\Psi_{13i} = [\mathfrak{A}_i - \tilde{\mathfrak{A}}_i - \tilde{G}_i L_i \ 0 \ \mathfrak{D}_i \ \tilde{\mathfrak{A}}_i \ 0 \ 0 \ \tilde{G}_i \ \mathfrak{B}_i \ \mathfrak{F}_i \ 0 \ 0 \ 0].$$

Condition 2: $\sum_{i=1}^2 \Upsilon_i < 0$ where

$$\Upsilon_1 = -\frac{1}{d_m} \Theta_1 \tilde{S}_2 \Theta_1^T - \frac{1}{d_M - d_m} \Theta_7 \mathcal{Q}_1 \Theta_7^T$$

$$\Upsilon_2 = -\frac{1}{\tau_m} \Theta_4 \tilde{N}_2 \Theta_4^T - \frac{1}{\tau_M - \tau_m} \Theta_8 \mathcal{Q}_2 \Theta_8^T$$

$$\Theta_1 = \begin{bmatrix} e_1 - e_{13} \\ e_1 + e_{13} - \Pi_1(\mathfrak{K}) \end{bmatrix}, \quad \Theta_2 = \begin{bmatrix} e_2 - e_{14} \\ e_2 + e_{14} - \Pi_2(\mathfrak{K}) \end{bmatrix}, \quad \Theta_3 = \begin{bmatrix} e_{13} - e_2 \\ e_{13} + e_2 - \Pi_3(\mathfrak{K}) \end{bmatrix},$$

$$\Theta_4 = \begin{bmatrix} e_4 - e_{15} \\ e_4 + e_{15} - \Pi_4(\mathfrak{K}) \end{bmatrix}, \quad \Theta_5 = \begin{bmatrix} e_6 - e_{16} \\ e_6 + e_{16} - \Pi_5(\mathfrak{K}) \end{bmatrix}, \quad \Theta_6 = \begin{bmatrix} e_{15} - e_6 \\ e_{15} + e_6 - \Pi_6(\mathfrak{K}) \end{bmatrix},$$

$$e_i = [0_{n \times (i-1)n} \ I_n \ 0_{n \times (16-i)n}]^T, \quad i = 1, 2, \dots, 22, \quad \Theta_7 = [\Theta_2 \ \Theta_3], \quad \Theta_8 = [\Theta_5 \ \Theta_6],$$

$$\tilde{S}_i = \begin{bmatrix} S_i & 0 \\ 0 & 3S_i \end{bmatrix}, \quad i = 1, 2 \quad \tilde{N}_i = \begin{bmatrix} N_i & 0 \\ 0 & 3N_i \end{bmatrix}, \quad i = 1, 2 \quad \mathcal{Q}_1 = \begin{bmatrix} \tilde{S}_1 & Y_1 \\ * & \tilde{S}_1 \end{bmatrix}, \quad \mathcal{Q}_2 = \begin{bmatrix} \tilde{N}_1 & Y_2 \\ * & \tilde{N}_1 \end{bmatrix}.$$

Condition 3. $\sum_{i=1}^r d_i + \mathfrak{K} \ln(1 - \xi) \leq \Phi(\mathfrak{K})$, for any $\mathfrak{K} \in [\mathfrak{K}_r, \mathfrak{K}_{r+1})$, then $\lim_{\mathfrak{K} \rightarrow \infty} \Phi(\mathfrak{K}) = +\infty$.

Proof. Choose the same LKF in [Theorem 3.1](#) then we derive,

$$\begin{aligned}\mathbb{E}\{\Delta \mathbb{V}_1(\mathfrak{K})|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} &= \mathbb{E}\{[\mathbb{V}_1(\mathfrak{K}+1, \mathfrak{R}_{\mathfrak{K}+1}, \mathfrak{r}(\mathfrak{K}+1)) - \mathbb{V}_1(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K}))]|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} \\ &= \zeta^T(\mathfrak{K})(\Psi_{12i}^T \tilde{R}_{1i} \Psi_{12i} + \Psi_{13i}^T \tilde{R}_{2i} \Psi_{13i}) \zeta(\mathfrak{K}) - x^T(\mathfrak{K}) R_{1i} x(\mathfrak{K}) \\ &\quad - e^T(\mathfrak{K}) R_{2i} e(\mathfrak{K}) + \text{trace}[\tilde{\delta}^T(x(\mathfrak{K}))(\tilde{R}_{1i} + \tilde{R}_{2i})\tilde{\delta}(x(\mathfrak{K}))],\end{aligned}\quad (44)$$

$$\begin{aligned}\mathbb{E}\{\Delta \mathbb{V}_2(\mathfrak{K})|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} &= \mathbb{E}\{[\mathbb{V}_2(\mathfrak{K}+1, \mathfrak{R}_{\mathfrak{K}+1}, \mathfrak{r}(\mathfrak{K}+1)) - \mathbb{V}_2(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K}))]|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} \\ &\leq x^T(\mathfrak{K}) S_1 x(\mathfrak{K}) - x^T(\mathfrak{K} - d(\mathfrak{K})) S_1 x(\mathfrak{K} - d(\mathfrak{K})) + \sum_{\mu=\mathfrak{K}-d_M+1}^{\mathfrak{K}-d_m} x^T(\mu) S_1 x(\mu) \\ &\quad + x^T(\mathfrak{K}) S_2 x(\mathfrak{K}) - x^T(\mathfrak{K} - \tau(\mathfrak{K})) S_2 x(\mathfrak{K} - \tau(\mathfrak{K})) \\ &\quad + \sum_{\mu=\mathfrak{K}-\tau_M+1}^{\mathfrak{K}-\tau_m} x^T(\mu) S_2 x(\mu)\end{aligned}\quad (45)$$

$$\begin{aligned}\mathbb{E}\{\Delta \mathbb{V}_3(\mathfrak{K})|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} &= \mathbb{E}\{[\mathbb{V}_3(\mathfrak{K}+1, \mathfrak{R}_{\mathfrak{K}+1}, \mathfrak{r}(\mathfrak{K}+1)) - \mathbb{V}_3(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K}))]|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} \\ &\leq e^T(\mathfrak{K}) N_1 e(\mathfrak{K}) - e^T(\mathfrak{K} - d(\mathfrak{K})) N_1 e(\mathfrak{K} - d(\mathfrak{K})) + \sum_{\mu=\mathfrak{K}-d_M+1}^{\mathfrak{K}-d_m} e^T(\mu) N_1 e(\mu) \\ &\quad + e^T(\mathfrak{K}) N_2 e(\mathfrak{K}) - e^T(\mathfrak{K} - \tau(\mathfrak{K})) N_2 e(\mathfrak{K} - \tau(\mathfrak{K})) \\ &\quad + \sum_{\mu=\mathfrak{K}-\tau_M+1}^{\mathfrak{K}-\tau_m} e^T(\mu) N_2 e(\mu)\end{aligned}\quad (46)$$

$$\begin{aligned}\mathbb{E}\{\Delta \mathbb{V}_4(\mathfrak{K})|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} &= \mathbb{E}\{[\mathbb{V}_4(\mathfrak{K}+1, \mathfrak{R}_{\mathfrak{K}+1}, \mathfrak{r}(\mathfrak{K}+1)) - \mathbb{V}_4(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K}))]|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} \\ &= (d_M - d_m) x^T(\mathfrak{K}) S_1 x(\mathfrak{K}) + (d_M - d_m) x^T(\mathfrak{K}) S_2 x(\mathfrak{K}) \\ &\quad - \sum_{\mu=\mathfrak{K}-d_M}^{\mathfrak{K}-d_m-1} x^T(\mu) S_1 x(\mu) - \sum_{\mu=\mathfrak{K}-d_m}^{\mathfrak{K}-1} x^T(\mu) S_2 x(\mu)\end{aligned}\quad (47)$$

$$\begin{aligned}\mathbb{E}\{\Delta \mathbb{V}_5(\mathfrak{K})|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} &= \mathbb{E}\{[\mathbb{V}_5(\mathfrak{K}+1, \mathfrak{R}_{\mathfrak{K}+1}, \mathfrak{r}(\mathfrak{K}+1)) - \mathbb{V}_5(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K}))]|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} \\ &= (\tau_M - \tau_m) e^T(\mathfrak{K}) N_1 e(\mathfrak{K}) + (\tau_M - \tau_m) e^T(\mathfrak{K}) N_2 e(\mathfrak{K}) \\ &\quad - \sum_{\mu=\mathfrak{K}-\tau_M}^{\mathfrak{K}-\tau_m-1} e^T(\mu) N_1 e(\mu) - \sum_{\mu=\mathfrak{K}-\tau_m}^{\mathfrak{K}-1} e^T(\mu) N_2 e(\mu)\end{aligned}\quad (48)$$

$$\begin{aligned}\mathbb{E}\{\Delta \mathbb{V}_6(\mathfrak{K})|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} &= \mathbb{E}\{[\mathbb{V}_6(\mathfrak{K}+1, \mathfrak{R}_{\mathfrak{K}+1}, \mathfrak{r}(\mathfrak{K}+1)) - \mathbb{V}_6(\mathfrak{K}, \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K}))]|\mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} \\ &= \sum_{i=1}^{\infty} \mu_i \sum_{j=\mathfrak{K}+1-i}^{\mathfrak{K}+1-1} h^T(y(j)) S_1 h(y(j)) + \sum_{i=1}^{\infty} \xi_i \sum_{j=\mathfrak{K}+1-i}^{\mathfrak{K}+1-1} x^T(i) S_2 x(i) \\ &\quad - \sum_{i=1}^{\infty} \mu_i \sum_{j=\mathfrak{K}-i}^{\mathfrak{K}-1} h^T(y(j)) S_1 h(y(j)) - \sum_{i=1}^{\infty} \xi_i \sum_{j=\mathfrak{K}-i}^{\mathfrak{K}-1} x^T(i) S_2 x(i) \\ &= \sum_{i=1}^{\infty} \mu_i [h^T(y(\mathfrak{K})) S_1 h(y(\mathfrak{K})) - h^T(y(\mathfrak{K}-i)) S_1 h(y(\mathfrak{K}-i))] \\ &\quad + \sum_{i=1}^{\infty} \xi_i [x^T(i) S_2 x(i) - x^T(i) S_2 x(i)]\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^{\infty} \xi_i [x^T(\mathfrak{K}) S_2 x(\mathfrak{K}) - x^T(\mathfrak{K} - i) S_2 x(\mathfrak{K} - i)] \\
& \leq \bar{\mu} h^T(y(\mathfrak{K})) S_1 h(y(\mathfrak{K})) - \bar{\mu} [\bar{\mu} h(y(\mathfrak{K} - s))]^T S_1 [\bar{\mu} h(y(\mathfrak{K} - s))] \\
& \quad + \bar{\xi} x^T(\mathfrak{K}) S_2 x(\mathfrak{K}) - \bar{\xi} [\bar{\xi} x(\mathfrak{K} - n)]^T S_2 [\bar{\xi} x(\mathfrak{K} - n)],
\end{aligned} \tag{49}$$

From Eqs. (12) and (42), we get

$$\text{trace}[\bar{\delta}^T(x(\mathfrak{K}))(\tilde{R}_{1i} + \tilde{R}_{2i})\bar{\delta}(x(\mathfrak{K}))] \leq \theta x^T(\mathfrak{K}) \tilde{\omega} x(\mathfrak{K}). \tag{50}$$

In Lemma 2.5 $\Pi(\mathfrak{K}, c, d)$ is defined then we define the following notations:

$$\begin{aligned}
\Pi_1(\mathfrak{K}) &= \Pi(\mathfrak{K}, 0, d_m), \quad \Pi_2(\mathfrak{K}) = \Pi(\mathfrak{K}, d(\mathfrak{K}), d_M), \\
\Pi_3(\mathfrak{K}) &= \Pi(\mathfrak{K}, d_m, d(\mathfrak{K})), \quad \Pi_4(\mathfrak{K}) = \Pi(\mathfrak{K}, 0, \tau_m), \\
\Pi_5(\mathfrak{K}) &= \Pi(\mathfrak{K}, \tau(\mathfrak{K}), \tau_M), \quad \Pi_6(\mathfrak{K}) = \Pi(\mathfrak{K}, \tau_m, \tau(\mathfrak{K})).
\end{aligned}$$

where

$$\begin{aligned}
\Omega(\mathfrak{K}) &= [x^T(\mathfrak{K}), x^T(\mathfrak{K} - d(\mathfrak{K})), x^T(\mathfrak{K} - \tau(\mathfrak{K})), e^T(\mathfrak{K}), e^T(\mathfrak{K} - d(\mathfrak{K})), e^T(\mathfrak{K} - \tau(\mathfrak{K})), \varsigma^T(\mathfrak{K}), \\
& \quad \tilde{g}^T(N(x(\mathfrak{K} - d(\mathfrak{K})))), \mu^T(\mathfrak{K}), h^T(y(\mathfrak{K})), [\bar{\mu} h(y(\mathfrak{K} - s))]^T, [\bar{\xi} x(\mathfrak{K} - n)]^T, \\
& \quad x^T(\mathfrak{K} - d_m), x^T(\mathfrak{K} - d_M), e^T(\mathfrak{K} - \tau_m), e^T(\mathfrak{K} - \tau_M), \Pi_1^T(\mathfrak{K}), \\
& \quad \Pi_2^T(\mathfrak{K}), \Pi_3^T(\mathfrak{K}), \Pi_4^T(\mathfrak{K}), \Pi_5^T(\mathfrak{K}), \Pi_6^T(\mathfrak{K}),]^T.
\end{aligned}$$

We can find the following terms by using Lemmas 2.5,

$$\begin{aligned}
- \sum_{\mu=\mathfrak{K}-d_m}^{\mathfrak{K}-1} x^T(\mu) S_2 x(\mu) & \leq - \frac{1}{d_m} \begin{bmatrix} x(\mathfrak{K}) - x(\mathfrak{K} - d_m) \\ x(\mathfrak{K}) + x(\mathfrak{K} - d_m) - \Pi_1(\mathfrak{K}) \end{bmatrix}^T \begin{bmatrix} S_2 & 0 \\ 0 & 3S_2 \end{bmatrix} \\
& \quad \begin{bmatrix} x(\mathfrak{K}) - x(\mathfrak{K} - d_m) \\ x(\mathfrak{K}) + x(\mathfrak{K} - d_m) - \Pi_1(\mathfrak{K}) \end{bmatrix} \\
& = - \frac{1}{d_m} \Omega^T(\mathfrak{K}) \Theta_1 \tilde{S}_2 \Theta_1^T \Omega(\mathfrak{K}).
\end{aligned} \tag{51}$$

The following term follows from Lemmas 2.5 and 2.6 that

$$\begin{aligned}
- \sum_{\mu=\mathfrak{K}-d_M}^{\mathfrak{K}-d_m-1} x^T(\mu) S_1 x(\mu) & = - \sum_{\mu=\mathfrak{K}-d_M}^{\mathfrak{K}-d_k-1} x^T(\mu) S_1 x(\mu) - \sum_{\mu=\mathfrak{K}-d(\mathfrak{K})}^{\mathfrak{K}-d_m-1} x^T(\mu) S_1 x(\mu) \\
& = - \frac{d_M - d(\mathfrak{K})}{d_M - d_m} \sum_{\mu=\mathfrak{K}-d_M}^{\mathfrak{K}-d(\mathfrak{K})-1} x^T(\mu) S_1 x(\mu) - \frac{d(\mathfrak{K}) - d_m}{d_M - d_m} \\
& \quad \times \sum_{\mu=\mathfrak{K}-d_M}^{\mathfrak{K}-d(\mathfrak{K})-1} x^T(\mu) S_1 x(\mu) - \frac{d(\mathfrak{K}) - d_m}{d_M - d_m} \sum_{\mu=\mathfrak{K}-d(\mathfrak{K})}^{\mathfrak{K}-d_m-1} x^T(\mu) S_1 x(\mu) \\
& \quad - \frac{d_M - d(\mathfrak{K})}{d_M - d_m} \sum_{\mu=\mathfrak{K}-d(\mathfrak{K})}^{\mathfrak{K}-d_m-1} x^T(\mu) S_1 x(\mu) \\
& \leq - \frac{1}{d_M - d_m} \Omega^T(\mathfrak{K}) \Theta_2 \tilde{S}_1 \Theta_2^T \Omega(\mathfrak{K}) - \frac{d(\mathfrak{K}) - d_m}{d_M - d_m} \frac{1}{d_M - d(\mathfrak{K})}
\end{aligned}$$

Table 1
Comparisons of upper bounds of time-delays.

Methods	Lower bounds	Upper bounds
In ref [37]	$d_m = \tau_m = 2$	$d_M = \tau_M = 2$
In ref [52]	$d_m = \tau_m = 2$	$d_M = \tau_M = 20$
In ref [55]	$d_m = \tau_m = 3$	$d_M = \tau_M = 5$
In Theorem 3.5	$d_m = \tau_m = 4$	$d_M = \tau_M > 0$

$$\begin{aligned}
 & \times \Omega^T(\mathfrak{K}) \Theta_2 \tilde{S}_1 \Theta_2^T \Omega(\mathfrak{K}) \frac{1}{d_M - d_m} \Omega^T(\mathfrak{K}) \Theta_3 \tilde{S}_1 \Theta_3^T \Omega(\mathfrak{K}) \\
 & - \frac{d_M - d(\mathfrak{K})}{d_M - d_m} \frac{1}{d(\mathfrak{K}) - d_m} \Omega^T(\mathfrak{K}) \Theta_3 \tilde{S}_1 \Theta_3^T \Omega(\mathfrak{K}) \\
 & = \frac{1}{d_M - d_m} \left[\frac{d_M - d_m}{d_M - d(\mathfrak{K})} \Omega^T(\mathfrak{K}) \Theta_2 \tilde{S}_1 \Theta_2^T \Omega(\mathfrak{K}) + \frac{d_M - d_m}{d(\mathfrak{K}) - d_m} \right. \\
 & \quad \left. \Omega^T(\mathfrak{K}) \Theta_3 \tilde{S}_1 \Theta_3^T \Omega(\mathfrak{K}) \right] \\
 & \leq - \frac{1}{d_M - d_m} \Omega^T(\mathfrak{K}) \Theta_7 \mathcal{Q}_1 \Theta_7^T \Omega(\mathfrak{K}). \tag{52}
 \end{aligned}$$

Combining Eqs. (48), (51) and (52), we get

$$\mathbb{E}\{\Delta \mathbb{V}_4(\mathfrak{K}) | \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} \leq \Omega^T(\mathfrak{K}) \Upsilon_1 \Omega(\mathfrak{K}). \tag{53}$$

Using the same method, we derive

$$\mathbb{E}\{\Delta \mathbb{V}_5(\mathfrak{K}) | \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} \leq \Omega^T(\mathfrak{K}) \Upsilon_2 \Omega(\mathfrak{K}). \tag{54}$$

From Eqs. (53) and (54), we get

$$\mathbb{E}\{\Delta \mathbb{V}(\mathfrak{K}) | \mathfrak{R}_{\mathfrak{K}}, \mathfrak{r}(\mathfrak{K})\} \leq \Omega^T(\mathfrak{K}) \sum_{i=1}^2 \Upsilon_i \Omega(\mathfrak{K}). \tag{55}$$

Therefore the condition 2 holds. The remaining proof of this theorem is same as in Theorem 3.1. Based on the Wirtinger inequality, GRN Eq. (16) is asymptotically stable. Hence the proof is completed. $\square$

Remark 3.6. In this research work, the discrete-type Wirtinger-based inequality technique and the reciprocally convex inequality methods are used. Because, Wirtinger-based inequality technique and the reciprocally convex inequality helps us to reduce the less conservativeness of our stability results (Table 1).

Remark 3.7. Theorem 3.1 analyze the H_∞ problem for the impulsive GRN Eq. (16) with leakage delays, multiple delays and Markovian jumping parameters is stochastically stable. Theorem 3.5 investigate the discrete-time GRN Eq. (16) with leakage delays, multiple delays, Markovian jumping parameters and impulsive effects is asymptotically stable by using the discrete-type Wirtinger-based inequality. Moreover, Theorem 3.5 is less conservatism than Theorem 3.1, which is mentioned in Table 2.

Table 2

Comparisons of upper bounds of time-delays.

Methods	Lower bounds	Upper bounds
In Theorem 3.1	$d_m = \tau_m = 2$	$d_M = \tau_M = 30$
In Theorem 3.5	$d_m = \tau_m = 4$	$d_M = \tau_M > 0$

4. Numerical examples

In this portion, we give two numerical examples and simulative responses to expose the capability of the advanced H_∞ state estimation.

Example 4.1. Consider the discrete-interval GRNs [Eq. \(1\)](#) with parameters given by,

$$A = \begin{pmatrix} 0.12 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad B = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad C = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.7 \end{pmatrix},$$

$$D = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.9 \end{pmatrix}, \quad F_m = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad F_p = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.8 \end{pmatrix},$$

$$E_m = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.3 \end{pmatrix}, \quad E_p = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.13 \end{pmatrix}, \quad \mu_m = \mu_p = \exp(-1)$$

and the leakage delays $\rho_1 = \rho_2 = 2$. The regulatory function is taken as $f(s) = \frac{s^2}{1+s^2}$. We can choose the time-varying delays as $\tau(\mathfrak{K}) = 1 + \sin(\mathfrak{K} * \pi)/2$ and $d(\mathfrak{K}) = 1 + \cos(\mathfrak{K} * \pi)$, respectively. The exogenous disturbance input is selected as $v(\mathfrak{K}) = \sin(6\mathfrak{K}) * \exp(-0.1\mathfrak{K})$.

Now, we have to evaluate the transmittable ideals from the Markov chain which is a finite one in the state space $N = \{1, 2\}$ with $\Lambda = [\Phi_{ji}]_{p \times p}$ given by

$$\Lambda = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.1 \end{pmatrix}.$$

Now, consider the estimation error system [\(16\)](#) and Markov chain that extracts the utility of the state space in finite sense as $N = \{1, 2\}$ with the parameters are given by,

$$A_1 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad F_1 = \begin{pmatrix} -0.01 & 0 \\ 0 & 0.12 \end{pmatrix},$$

$$F_2 = \begin{pmatrix} 0.18 & 0 \\ 0 & 0.15 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0.08 & 0 \\ 0 & 0.9 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.14 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} -0.2 & 0 \\ 0 & -0.02 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -0.03 & 0 \\ 0 & -0.03 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0.13 & 0 \\ 0 & 0.01 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 0.28 & 0 \\ 0 & 0.05 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0.1 & 0 \\ 0 & \sin(k\pi)/2 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.02 \end{pmatrix}$$

The parameter of measurement outputs are given by,

$$L_{x1} = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad L_{x2} = \begin{pmatrix} -0.01 & 0 \\ 0 & 0.2 \end{pmatrix}$$

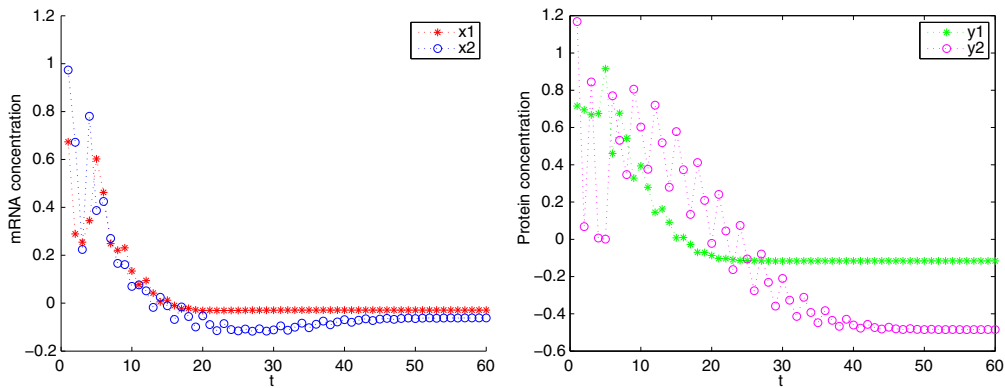


Fig. 1. mRNA and protein concentration.

and the leakage delays $\rho_1 = \rho_2 = 2$. The regulatory function is taken as $f(s) = \frac{s^2}{1+s^2}$. $d_m = 2$, $d_M = 30$, $\tau_m = 2$ and $\tau_M = 30$ are the time-varying delays and the exogenous disturbance input is selected as $v(\mathfrak{K}) = \sin(2\mathfrak{K}) * \exp(-0.01\mathfrak{K})$ and the event error $\zeta(\mathfrak{K}) = \cos(\mathfrak{K}\pi)$.

Now, by using the toolbox in Matlab, the desired LMIs in Eqs. (40) and (41) get the feasible solutions given as follows:

$$X_1 = \begin{pmatrix} 1.6388 & -0.0557 \\ -0.0557 & 1.6753 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1.6721 & -0.0012 \\ -0.0012 & 1.6738 \end{pmatrix}, \quad R_{11} = \begin{pmatrix} 1.5144 & -0.0211 \\ -0.0211 & 1.5232 \end{pmatrix}$$

$$R_{21} = \begin{pmatrix} 1.6896 & 0.0542 \\ 0.0542 & 1.6964 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 1.6716 & 0.0209 \\ 0.0209 & 0.4300 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 1.6717 & 0.0003 \\ 0.0003 & 1.6752 \end{pmatrix}$$

The state estimator gain matrices can be calculated as follows

$$A_{x1} = \begin{pmatrix} 0.9699 & -1.0276 \\ -1.0276 & 0.9875 \end{pmatrix}, \quad A_{x2} = \begin{pmatrix} 0.9813 & -0.1001 \\ -0.1001 & 0.9845 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} 0.9893 & 0.3856 \\ 0.3856 & 0.2540 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0.9811 & 0.0251 \\ 0.0251 & 0.9853 \end{pmatrix}$$

In the simulation, the exogenous disturbance inputs are selected as $v(\mathfrak{K}) = \sin(2\mathfrak{K})\exp(-0.01\mathfrak{K})$ and the initial values are $x(0) = \{0.76, 0.69\}$, $y(0) = \{0.69, 0.29\}$. The Fig. 2 reveals, the estimation errors of the concentrations of the mRNAs and the estimation errors of the concentrations of proteins, respectively (Fig. 1–4).

Example 4.2. Consider the discrete-interval GRNs (1) with parameters given by,

$$A = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad B = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad C = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.9 \end{pmatrix}, \quad D = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad F_m = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix}$$

$$F_p = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.7 \end{pmatrix}, \quad E_m = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad E_p = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.3 \end{pmatrix}, \quad \mu_m = \mu_p = \exp(-2)$$

and the leakage delays $\rho_1 = \rho_2 = 2.5$. The regulatory function is taken as $f(s) = \frac{s^2}{1+s^2}$. We can choose the time-varying delays as $\tau(\mathfrak{K}) = 1 + \sin(\mathfrak{K} * \pi)/2$ and $d(\mathfrak{K}) = 1 + \cos(\mathfrak{K} * \pi)$, respectively. The exogenous disturbance input is selected as $v(\mathfrak{K}) = \sin(5\mathfrak{K}) * \exp(-0.2\mathfrak{K})$.

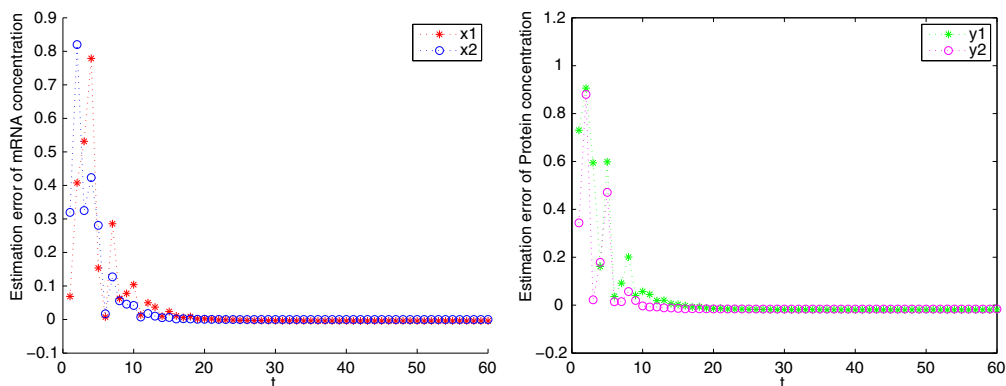


Fig. 2. Estimation error for mRNA and protein concentration.

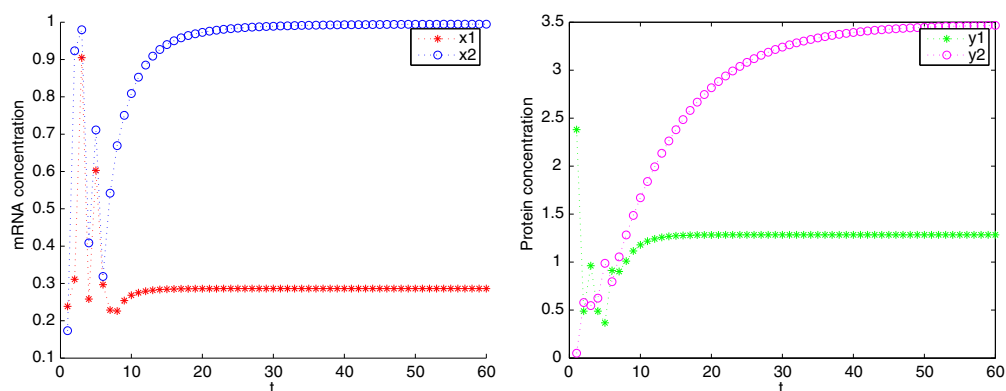


Fig. 3. mRNA and protein concentration.

Now, we have to evaluate the transmittable ideals from the Markov chain which is a finite one in the state space $N = \{1, 2\}$ with $\Lambda = [\Phi_{ji}]_{p \times p}$ given by

$$\Lambda = \begin{pmatrix} 0.6 & 0.5 \\ 0.2 & 0.2 \end{pmatrix}.$$

Now, consider the estimation error system (16) and Markov chain that extracts the utility of the state space in finite sense as $N = \{1, 2\}$ with the parameters are given by,

$$A_1 = \begin{pmatrix} 0.23 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.12 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0.31 & 0 \\ 0 & 0.16 \end{pmatrix},$$

$$F_2 = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.11 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.9 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.4 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} -0.23 & 0 \\ 0 & -0.21 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -0.3 & 0 \\ 0 & -0.3 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0.13 & 0 \\ 0 & 0.12 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 0.38 & 0 \\ 0 & 0.15 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & \sin(k\pi)/2 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}$$

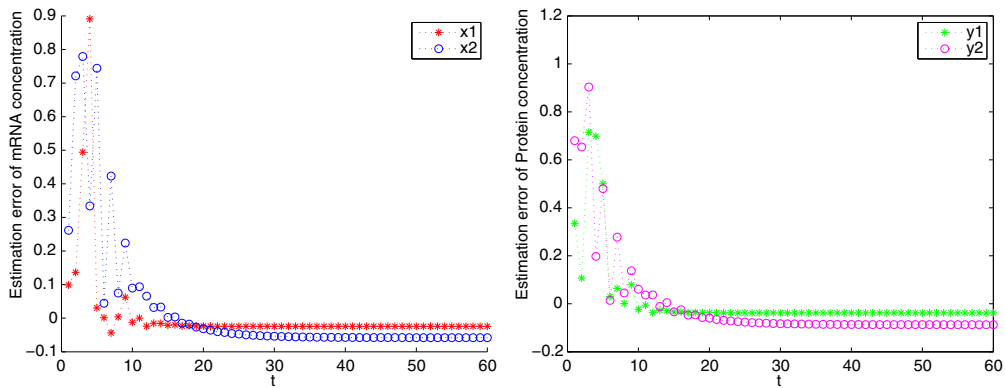


Fig. 4. Estimation error for mRNA and protein concentration.

The parameter of measurement outputs are given by,

$$L_{x1} = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad L_{x2} = \begin{pmatrix} -0.01 & 0 \\ 0 & 0.6 \end{pmatrix}$$

and the leakage delays $\rho_1 = \rho_2 = 2$. The regulatory function is taken as $f(s) = \frac{s^2}{1+s^2}$. $d_m = 4$, $d_M = 9$, $\tau_m = 4$ and $\tau_M = 9$ are the time-varying delays and the exogenous disturbance input is selected as $v(\mathfrak{K}) = \sin(2\mathfrak{K}) * \exp(-0.01\mathfrak{K})$ and the event error $\zeta(\mathfrak{K}) = \cos(\mathfrak{K}\pi)$.

By applying the Matlab LMI toolbox, we get the feasible solutions. Due to the space consideration part of the solutions are given as follows:

$$X_2 = \begin{pmatrix} 1.1252 & -0.0928 \\ -0.0928 & 1.2984 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0.7684 & -0.0007 \\ -0.0007 & 0.7645 \end{pmatrix}, \quad R_{11} = \begin{pmatrix} 1.1155 & 0.0607 \\ 0.0607 & 1.0873 \end{pmatrix}$$

$$R_{21} = \begin{pmatrix} 1.4331 & 0.2079 \\ 0.2079 & 1.2489 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 1.4686 & 0.0889 \\ 0.0889 & 0.5718 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 1.0395 & -0.0045 \\ -0.1019 & 0.4586 \end{pmatrix}$$

In the simulation, the exogenous disturbance inputs are selected as $v(\mathfrak{K}) = \sin(6\mathfrak{K})\exp(-0.1\mathfrak{K})$ and the initial values are $x(0) = \{0.29, 0.34\}$, $y(0) = \{0.75, 0.01\}$. The Fig. 4 reveals, the estimation errors of the concentrations of the mRNAs and the estimation errors of the concentrations of proteins, respectively.

According to Theorem 3.5, we can obtain that the genetic regulatory networks system (16) with the above given parameters are asymptotically stable with the help of Lyapunov functionals and Wirtinger-based inequality method.

5. Conclusions

In this research, the stability analysis of H_∞ state estimation for discrete-time stochastic genetic regulatory networks with leakage delays, distributed delays, Markovian jumping parameters, time-varying delays and impulsive effects has been investigated. By constructing a suitable Lyapunov functional and using some analysis techniques, several sufficient conditions have been derived for the stability of stochastic GRNs. By using the discrete-type Wirtinger-based inequality together with the reciprocally convex combination inequality, we

obtain a less conservative stability criterion. The derived conditions which are in the form of LMIs explore the real GRNs in the presence of time delays and it can be solved through some known software packages. Two numerical examples are finally presented to illustrate the application and effectiveness of the obtained results.

In the near future, we plan to work with stabilization of stochastic genetic regulatory networks with leakage and impulsive effects in switching sense. Also, we will try to present a real life model to justify our theoretical concepts for the considered GRN.

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Novel global robust exponential stability criterion for uncertain inertial-type BAM neural networks with discrete and distributed time-varying delays via Lagrange sense[☆]

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Abstract

In this paper, the global robust exponential stability problem for a class of uncertain inertial-type BAM neural networks with both time-varying delays is focused through Lagrange sense. The existence of time-varying delays in discrete and distributed terms is explored with the availability of lower and upper bounds of time-varying delays. Firstly, we transform the proposed inertial BAM neural networks to usual one. Secondly, by the aid of LKF, stability theory, integral inequality, some novel sufficient conditions for the global robust exponential stability of the addressed neural networks are obtained in terms of linear matrix inequalities, which can be easily tested in practice by utilizing LMI control toolbox in MATLAB software. Furthermore, many comparisons of proposed work are listed with some existing literatures to get less conservatism. Finally, two numerical examples are provided to demonstrate the advantages and superiority of our theoretical outcomes.

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1. Introduction

Owing to the widespread applications of neural networks in various applied fields, such as combinatorial optimization, automatic control, computer science, information science, parallel computations, systems engineering, neural networks have been received considerable research attention from many researchers [1,4,5,10,13,38]. Kosko was initially proposed the BAM neural networks in [18,19], which contains two layers, one is the A-layer and the other one is B-layer. In both layers, the neurons are fully interconnected from one layer to other layer. This class of neural networks has been successfully applied to pattern recognition and artificial intelligence due to its generalization of the single-layer auto-associative Hebbian correlator to a two-layer pattern-matched hetero associative circuit. According to the extensive range of fruitful applications, the BAM neural networks with or without time delays leads a great impact over the past few decades, see [4,8,22,28,49].

The arising of time delays in neural networks is inevitable, since the finite switching speed of information storage processing or of amplifiers, communication time and transmission of signals. As we know, the appearance of time delay may affect the stability performance of a system and generates unstable, divergence, and oscillation or chaotic behavior [5,14,20,25,45,46,55]. Accordingly, the delay-dependent and delay-independent criteria are two categories of time delays in neural networks, which classified by the existing time delayed results. So far as, one can observe from available literatures, the delay-dependent case is less conserved than the delay independent ones. On the other hand, based on the huge length of the axon sizes, the time delay can be deviated into two types: discrete and distributed time delays. So, the consideration of both time delays into account when model our neural networks system is meaningful. So, it is noteworthy to inspect both the time delay result in the dynamical behaviors of neural systems, see for instance [15,32,43,44,57,65].

Different from usual neural systems, a second order or third order inertial neural networks initially formulated by Wheeler and Schieve in [53] and its bifurcation, stability, chaos phenomenon also investigated. It is noted that lots of existing studies on neural networks mainly focused only with first order derivative of the states, so it is also of significant importance to propose an inertial term into the artificial neural networks. As far as, very little attention has been paid to investigate the inertial-type neural networks, refer to [3,16,47,62]. The inertial-type neural networks are designed in such a way that, for a given external input, they can reveal only one global asymptotic or exponential stable equilibrium point. Thus, some methods have been proposed to ensure the asymptotic stability or exponential stability for inertial neural networks with time delays, see [27,61]. For example, Miao and Ke in [34] examined exponential stability of inertial neural networks with time delays by suitable LKF. In [60], Zhang et al. briefly discussed the stability of state variable in the sense of exponential for a BAM neural networks with inertial type. Tu et al. have investigated the dissipativity of inertial neural networks [48], and the circuit implementation of inertial neural networks was also provided.

In real nervous system, the connection weights of the neuron depend on certain resistance and capacitance values that include uncertain parameters (modeling errors). Moreover, the uncertainties are inevitable while modeling a neural networks [9]. So, it is point out that the parameter fluctuation and errors during their implementation, which also outcomes in poor performance and instability of the neural networks. Hence the uncertain parameters of a system are only due to the deviations and perturbations of its parameters. Recently, several robust stability conditions are obtained for the neural networks with uncertain parameters via

different approaches, see for example, Raja et al. in [39] investigated the stability performance of neutral-type uncertain neural networks with the perturbations of leakage and impulsive effects. Also, time-delayed stability conditions for uncertain Hopfield neural networks have been derived by Wang et al. in [52]. It is well known that the theory on the dynamics of this networks, including GAS or GES, had been extensively studied, for instance, Cao and Bao in [2] studied the exponential stability for BAM networks with discrete and distributed delays. In [63], Zhou and Zhang investigated the exponential stability conditions for recurrent neural networks with the help of fixed point theory. Further in [54], Wen et al. analyzed and obtained some findings related to the problem for asymptotic stability of time-delayed BAM neural networks. The less conserved criteria for exponential stability of time delayed neural networks is obtained and investigated by Zhou in [64]. Through some novel approaches, time-delayed sufficient conditions for the exponential stability of neural network system with uncertain parameters is developed by Li et al. in [24].

Prosperously, distinct from stability in the Lyapunov sense, the Lagrange stability regards to the constancy of a entire system. Also, a system stable in Lagrange sense is likely to obtain multi-stable dynamics, since it concerning the solutions boundedness and also the estimation for globally attractive set. Therefore, the analysis of Lagrange stability in neural networks is more important one. In [30,31,50], some developed results reported regarding Lagrange stability of time delayed neural network system. In [51], Wang et al. investigates the delay dependent stability criteria for neural networks in the sense of Lagrange through some different LKF. Luo et al. in [33] extensively analyzed the Lagrange stability in the way of exponential for time-delayed neural networks.

However, inspired by the aforementioned discussions, the problem of robust exponential stability of uncertain inertial-type BAM neural networks with both time delays via Lagrange sense has not been fully studied yet, which motivates our current research work. With this aim to fill this gap, in this paper, we handled the global robust exponential stability problem for a class of UIBAMNNs with mixed time delays through Lagrange sense. By utilizing the stability theory, matrix theory and several inequality techniques, some new-brand time dependent sufficient criteria for global robust exponential stability in Lagrange sense are formulated, which can be easily validated by the LMI control toolbox in MATLAB software. Also, two numerical examples are given to illustrate the applicability and superiority of the designed methods. The novelty and main contributions of this research work are highlighted as follows

- Uncertain parameters, BAM-type, discrete and distributed time-delays, are taken into account in the stability analysis of proposed inertial-type neural networks.
- By employing some different LKF, variable transformation techniques and integral inequality techniques, the sufficient conditions for global robust exponential stability of inertial-type BAM neural networks are derived by the way of Lagrange sense.
- By reflect of both time-varying delay terms in our inertial-type BAM neural networks, the allowable upper bounds of discrete and distributed time-delays are large, when compare with the existing literatures, see Tables 2, 3 and 4 in Examples 4.1 and 4.2. This explore that the approach developed in this paper is effective and less conserved than some available results.
- A special case of main results is derived in Corollary 3.3 and also by using MATLAB LMI control toolbox verified its LMIs are feasible as well as its simulations are given in Example 4.2, which ensures the advantages of this paper.

Table 2

MAUBs of discrete time-varying delays.

Methods	$\eta_1 = \eta_2 > 0$
In Ref [40]	2.1
In Ref [37]	4.822
In Ref [35]	5
In Ref [12]	5.912
Theorem 3.1	17.429
Corollary 3.3	22.517

Table 3

The MAUBs of discrete time-varying delays $\eta_1 = \eta_2$ for various $\eta = \tilde{\eta}$.

Methods	$\eta = \tilde{\eta}$			
	0.2	0.5	0.7	0.9
Theorem 3.1	16.183	9.51	8.35	3.27
Corollary 3.3	20.073	15.928	11.261	6.208

Table 4

The MAUBs $\mu_1 = \mu_2$ of distributed time-varying delays (Example 4.2).

MAUBs	Methods				
	[21]	[41]	[42]	Theorem 3.1	Corollary 3.3
$\mu_1 = \mu_2 > 0$	0.5	0.5	0.2	8.212	13.604

The reminder of this paper is outlined as follows. In Section 2, a brief account of the inertial BAM neural networks model is formulated and some preliminaries are provided. By the aid of LKF and integral inequality techniques, several novel conditions are obtained for the Lagrange sense global robust exponential stability of concerned neural networks in Section 3. The obtained results are formulated in the form of LMIs, that are easily checked by using the MATLAB LMI control toolbox. Then, two numerical examples with their simulations are equipped to show the feasibility and advantages of the derived stability criteria in Section 4. Finally, the conclusions of this paper is drawn in Section 5.

2. Model description and preliminaries

Consider the following inertial-type uncertain BAM neural networks with time-varying delays:

$$\begin{aligned}
 \mathcal{D}^2(u_i(t)) &= (e_i + \Delta e_i(t))\mathcal{D}(u_i(t)) - (c_i + \Delta c_i(t))u_i(t) + \sum_{i=1}^n (a_{ij} + \Delta a_{ij}(t))g_i(v_i(t)) \\
 &\quad + \sum_{i=1}^n (b_{ij} + \Delta b_{ij}(t))g_i(v_i(t - \eta_{1i}(t))) + \sum_{i=1}^n (l_{ij} + \Delta l_{ij}(t)) \int_{t-\mu_{1i}(t)}^t g_i(v_i(s))ds + \hat{f}_i(t), \\
 \mathcal{D}^2(v_j(t)) &= (m_j + \Delta m_j(t))\mathcal{D}(v_j(t)) - (n_j + \Delta n_j(t))v_j(t) + \sum_{j=1}^n (p_{ji} + \Delta p_{ji}(t))h_j(u_j(t))
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^n (r_{ji} + \Delta r_{ji}(t)) h_j(u_j(t - \eta_{2j}(t))) + \sum_{j=1}^n (o_{ji} + \Delta o_{ji}(t)) \int_{t-\mu_{2j}(t)}^t h_j(u_j(s)) \\
& \times ds + \hat{I}_j(t), \quad i, j \in \mathcal{S}^*,
\end{aligned} \tag{1}$$

where the second derivative indicates an inertial term of neural networks (1), $u_i(t)$ and $v_j(t)$ represents the state vector of the i th and j th neurons at time t , the positive constants e_i , c_i , m_j and n_j are stands for the connection weights of the NNs (1), the activation functions of the i th and j th neurons at time ' t ' denoted by $g_i(\cdot)$ and $h_j(\cdot)$ with $g_i(0) = h_j(0) = 0$ ($i, j \in \mathcal{S}^*$), sign Δ indicates the uncertain term and the description of uncertain parameter is given in Assumption 2, the time-varying delays $\eta_{1i}(t)$, $\mu_{1i}(t)$, $\eta_{2j}(t)$ and $\mu_{2j}(t)$ satisfies the following conditions

$$0 \leq \eta_{1i}(t) \leq \eta_1, \quad \dot{\eta}_{1i}(t) \leq \eta < 1, \tag{2}$$

$$0 \leq \eta_{2j}(t) \leq \eta_2, \quad \dot{\eta}_{2j}(t) \leq \tilde{\eta} < 1, \tag{3}$$

$$0 \leq \mu_{1i}(t) \leq \mu_1, \quad 0 \leq \mu_{2j}(t) \leq \mu_2, \tag{4}$$

$\hat{J}_i(t)$ and $\hat{I}_j(t)$ denotes the external input on the i th and j th neurons at time t , and also $|\hat{J}_i(t)| \leq \hat{J}_i$, $|\hat{I}_j(t)| \leq \hat{I}_j$, $i, j \in \mathcal{S}^*$.

The inertial BAM neural networks (1) has the following initial conditions

$$u_i(s) = \theta_i(s), \quad \mathcal{D}u_i(s) = \vartheta_i(s), \tag{5}$$

$$v_j(s) = \psi_j(s), \quad \mathcal{D}v_j(s) = \varphi_j(s), \quad s \in [-\rho, 0], \quad i, j \in \mathcal{S}^*, \tag{6}$$

where $\theta_i(s)$, $\vartheta_i(s)$, $\psi_j(s)$ and $\varphi_j(s)$ are continuous functions on $[-\rho, 0] \in \mathbb{R}$, $\rho = \max\{\eta_1, \eta_2, \mu_1, \mu_2\}$.

To derive the main results for neural networks (1), we make the following assumptions

Assumption 1. The activation functions in neural networks (1) satisfy the following inequalities, that is, there exist four diagonal matrices $W_1 = \text{diag}\{w_1^-, w_2^-, \dots, w_n^-\}$ and $W_2 = \text{diag}\{w_1^+, w_2^+, \dots, w_n^+\}$, $W_3 = \text{diag}\{\hat{w}_1^-, \hat{w}_2^-, \dots, \hat{w}_n^-\}$ and $W_4 = \text{diag}\{\hat{w}_1^+, \hat{w}_2^+, \dots, \hat{w}_n^+\}$ such that

$$\begin{aligned}
w_i^- & \leq \frac{g_i(z_1) - g_i(z_2)}{z_1 - z_2} \leq w_i^+, \\
\hat{w}_j^- & \leq \frac{h_j(z_3) - h_j(z_4)}{z_3 - z_4} \leq \hat{w}_j^+, \quad i, j \in \mathcal{S}^*,
\end{aligned}$$

for any z_1, z_2, z_3 and $z_4 \in \mathbb{R}$.

Remark 2.1. From Assumption 1, one can simplified as

$$v^T(t) \mathcal{K} \mathcal{L}_1 \mathcal{K} v(t) - g^T(v(t)) \mathcal{L}_1 g(v(t)) \geq 0, \tag{7}$$

$$u^T(t) \hat{\mathcal{K}} \mathcal{L}_2 \hat{\mathcal{K}} u(t) - h^T(u(t)) \mathcal{L}_2 h(u(t)) \geq 0, \tag{8}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are diagonal matrices with positive entries, $\mathcal{K} = \text{diag}\{k_{11}, k_{12}, \dots, k_{1n}\}$, $\hat{\mathcal{K}} = \text{diag}\{\hat{k}_{11}, \hat{k}_{12}, \dots, \hat{k}_{1n}\}$, $k_{1i} = \max\{|w_i^-|, |w_i^+|\}$, $\hat{k}_{1j} = \max\{|\hat{w}_j^-|, |\hat{w}_j^+|\}$, $i, j \in \mathcal{S}^*$.

Remark 2.2. From [Assumption 1](#), it is point out that the proposed activation function is better than the both sigmoid as well as Lipschitz-type activation functions. Because, the constants in Assumption 1 are permitted to be positive, zero or negative which leads to less conservatism, see for instance [\[6,7,23,36\]](#).

To reduce the complexity of stability conditions in Lagrange sense for addressed neural networks [\(1\)](#), we take the following variable transformation

$$\alpha_i(t) = \mathcal{D}u_i(t) + \zeta_i u_i(t), \quad (9)$$

$$\beta_j(t) = \mathcal{D}v_j(t) + \varrho_j v_j(t), \quad i, j \in \mathcal{S}^* \quad (10)$$

then [Eq. \(1\)](#) can be revised as

$$\begin{aligned} \mathcal{D}u_i(t) &= -\zeta_i u_i(t) + \alpha_i(t) \\ \mathcal{D}v_j(t) &= -\varrho_j v_j(t) + \beta_j(t) \\ \mathcal{D}\alpha_i(t) &= -\gamma_{1i}\alpha_i(t) + \gamma_{2i}u_i(t) + \sum_{i=1}^n (a_{ij} + \Delta a_{ij}(t))g_i(v_i(t)) + \sum_{i=1}^n (b_{ij} + \Delta b_{ij}(t)) \\ &\quad \times g_i(v_i(t - \eta_{1i}(t))) + \sum_{i=1}^n (l_{ij} + \Delta l_{ij}(t)) \int_{t-\mu_{1i}(t)}^t g_i(v_i(s))ds - \Delta e_i(t) \\ &\quad \times [-\zeta_i u_i(t) + \alpha_i(t)] - \Delta c_i(t)u_i(t) + \hat{J}_i(t) \\ \mathcal{D}\beta_j(t) &= -\gamma_{3j}\beta_j(t) + \gamma_{4j}v_j(t) + \sum_{j=1}^n (p_{ji} + \Delta p_{ji}(t))h_j(u_j(t)) + \sum_{j=1}^n (r_{ji} + \Delta r_{ji}(t)) \\ &\quad \times h_j(u_j(t - \eta_{2j}(t))) + \sum_{j=1}^n (o_{ji} + \Delta o_{ji}(t)) \int_{t-\mu_{2j}(t)}^t h_j(u_j(s))ds - \Delta m_j(t) \\ &\quad \times [-\varrho_j v_j(t) + \beta_j(t)] - \Delta n_j(t)v_j(t) + \hat{I}_j(t), \end{aligned} \quad (11)$$

where $\gamma_{1i} = e_i - \zeta_i$, $\gamma_{2i} = \gamma_{1i}\zeta_i - c_i$, $\gamma_{3j} = m_j - \varrho_j$, $\gamma_{4j} = \gamma_{3j}\varrho_j - n_j$, and the initial conditions of [Eq. \(1\)](#) can be written as

$$\begin{cases} u_i(s) = \theta_i(s), \\ \alpha_i(s) = \zeta_i \theta_i(s) + \vartheta_i(s), \\ v_j(s) = \psi_j(s), \\ \beta_j(s) = \varrho_j \psi_j(s) + \varphi_j(s), \quad s \in [-\rho, 0], \quad i, j \in \mathcal{S}^* \end{cases} \quad (12)$$

Thus, the neural networks [\(11\)](#) can be formulated as

$$\begin{aligned} \mathcal{D}u(t) &= -\Lambda_1 u(t) + \alpha(t) \\ \mathcal{D}v(t) &= -\Lambda_2 v(t) + \beta(t) \\ \mathcal{D}\alpha(t) &= -\Gamma_1 \alpha(t) + \Gamma_2 u(t) + (\mathcal{A} + \Delta \mathcal{A}(t))g(v(t)) + (\mathcal{B} + \Delta \mathcal{B}(t))g(v(t - \eta_1(t))) \\ &\quad + (\mathcal{L} + \Delta \mathcal{L}(t)) \int_{t-\mu_1(t)}^t g(v(s))ds + \Delta \mathcal{E}(t)[\Lambda_1 u(t) - \alpha(t)] - \Delta \mathcal{C}(t)u(t) + \hat{J}(t) \end{aligned}$$

$$\begin{aligned} \mathcal{D}\beta(t) = & -\Gamma_3\beta(t) + \Gamma_4v(t) + (\mathcal{P} + \Delta\mathcal{P}(t))h(u(t)) + (\mathcal{R} + \Delta\mathcal{R}(t))h(u(t - \eta_2(t))) \\ & + (\mathcal{O} + \Delta\mathcal{O}(t)) \int_{t-\mu_2(t)}^t h(u(s))ds + \Delta\mathcal{M}(t)[\Lambda_2v(t) - \beta(t)] - \Delta\mathcal{N}(t)v(t) + \hat{I}(t), \end{aligned} \quad (13)$$

where $u(t) = [u_1(t), u_2(t), \dots, u_n(t)]^T$, $v(t) = [v_1(t), v_2(t), \dots, v_n(t)]^T$, $\alpha(t) = [\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t)]^T$, $\beta(t) = [\beta_1(t), \beta_2(t), \dots, \beta_n(t)]^T \in \mathbb{R}^n$, $\Lambda_1 = \text{diag}\{\zeta_1, \zeta_2, \dots, \zeta_n\}$, $\Lambda_2 = \text{diag}\{\varrho_1, \varrho_2, \dots, \varrho_n\}$, $\Gamma_1 = \text{diag}\{\gamma_{11}, \gamma_{12}, \dots, \gamma_{1n}\}$, $\Gamma_2 = \text{diag}\{\gamma_{21}, \gamma_{22}, \dots, \gamma_{2n}\}$, $\Gamma_3 = \text{diag}\{\gamma_{31}, \gamma_{32}, \dots, \gamma_{3n}\}$, $\Gamma_4 = \text{diag}\{\gamma_{41}, \gamma_{42}, \dots, \gamma_{4n}\}$, $\mathcal{A} = (a_{ij})_{n \times n}$, $\mathcal{B} = (b_{ij})_{n \times n}$, $\mathcal{L} = (l_{ij})_{n \times n}$, $\mathcal{P} = (p_{ji})_{n \times n}$, $\mathcal{R} = (r_{ji})_{n \times n}$, $\mathcal{O} = (o_{ji})_{n \times n}$, $\eta_1(t) = (\eta_{11}(t), \eta_{12}(t), \dots, \eta_{1n}(t))$, $\eta_2(t) = (\eta_{21}(t), \eta_{22}(t), \dots, \eta_{2n}(t))$, $\mu_1(t) = (\mu_{11}(t), \mu_{12}(t), \dots, \mu_{1n}(t))$, $\mu_2(t) = (\mu_{21}(t), \mu_{22}(t), \dots, \mu_{2n}(t))$, $\hat{J}(t) = (\hat{J}_1(t), \hat{J}_2(t), \dots, \hat{J}_n(t))$, $\hat{I}(t) = (\hat{I}_1(t), \hat{I}_2(t), \dots, \hat{I}_n(t))$, $\hat{J} = (\hat{J}_1, \hat{J}_2, \dots, \hat{J}_n)$, $\hat{I} = (\hat{I}_1, \hat{I}_2, \dots, \hat{I}_n)$.

Assumption 2. The perturbed uncertain matrices $\Delta\mathcal{E}(t)$, $\Delta\mathcal{C}(t)$, $\Delta\mathcal{A}(t)$, $\Delta\mathcal{B}(t)$, $\Delta\mathcal{L}(t)$, $\Delta\mathcal{M}(t)$, $\Delta\mathcal{N}(t)$, $\Delta\mathcal{P}(t)$, $\Delta\mathcal{R}(t)$ and $\Delta\mathcal{O}(t)$ are time-varying functions satisfying:

$\Delta\mathcal{E}(t) = \Delta\mathcal{C}(t) = \Delta\mathcal{A}(t) = \Delta\mathcal{B}(t) = \Delta\mathcal{L}(t) = \mathcal{N}^*\mathcal{Y}(t) [Z_1 Z_2 Z_3 Z_4 Z_5]$ and $\Delta\mathcal{M}(t) = \Delta\mathcal{N}(t) = \Delta\mathcal{P}(t) = \Delta\mathcal{R}(t) = \Delta\mathcal{O}(t) = \mathcal{X}\Sigma(t)[\mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \mathcal{H}_4 \mathcal{H}_5]$, where $\mathcal{N}^*$, Z_1, Z_2, Z_3, Z_4, Z_5 , $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5$ and $\mathcal{X}$ are given constant matrices with real entries, and $\mathcal{Y}(\cdot)$, $\Sigma(\cdot) \in \mathbb{R}^{n \times m}$ are unknown time-varying matrices which have the structure as follows: $\mathcal{Y}^T(t)\mathcal{Y}(t) \leq I$ and $\Sigma^T(t)\Sigma(t) \leq I$, $\forall t \geq 0$.

Remark 2.3. Based on [Assumptions 1, 2](#), [Theorem 3.1](#) will be provide some novelty conditions ensuring the global robust exponential stability in Lagrange sense for the inertial-type BAM neural networks with both time-varying delays (1). Also, the LMI (15) is verified easily by LMI control toolbox in Matlab software in [Section 4](#).

Remark 2.4. Liao et al. in [\[31\]](#) investigated the Lagrange-stability conditions for time delayed recurrent neural networks with the help of different LKF. In [\[58\]](#), Yu et al. conversed the impact of time delays in inertial-type neural networks to ensure the stability performance in the sense of exponential. Also, some new LMI criterions for stability have been derived for inertial neural networks by Ke and Miao [\[17\]](#).

From all the above revealed references the stability problem for inertial-type neural networks is considered only for the mixed time-delays, but uncertain parameters and BAM-type has not been taken into account and also no one investigates the globally robustly exponentially stable in the sense of Lagrange at a time. So, the consideration of the above facts is more difficult and a challenging one, and this motivates the present paper.

Before ending this section, need some definitions and lemmas to proving our main results.

Definition 2.5. The set $\Omega^* \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is called an attractive set of neural networks (13), if for any solution $(u(t), v(t)) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \Omega^*$ with initial condition (12), $\lim_{t \rightarrow +\infty} \varpi((u(t), v(t)), \Omega^*) = 0$ holds, where the complement of Ω^* is denoted by $\mathbb{R} \times \mathbb{R} \setminus \Omega^*$, and $\varpi((u, v), \Omega^*) = \inf_{(u^*, v^*) \in \Omega^*} \|(u, v) - (u^*, v^*)\|$ is the distance between (u, v) and Ω^* .

Definition 2.6. A compact set $\Omega^* = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n | \mathcal{V}(u(t), v(t)) \leq \lambda^*\}$ is said to be a GEA set of [Eq. \(13\)](#), if there exist a nonnegative continuous function $U(\cdot)$, a radially unbounded and positive definite function $\mathcal{V}(\cdot)$, and scalars λ^* , $\nu > 0$ such that for every solution $(u(t), v(t))$

of Eq. (13), $\mathcal{V}(u(t), v(t)) > \lambda^* \Rightarrow$

$$\mathcal{V}(u(t), v(t)) - \lambda^* \leq U(\theta, \psi) \exp\{-\nu t\}, \quad t \geq 0 \quad (14)$$

The neural networks (13) or (1) is said to be globally robustly exponentially stable in the sense of Lagrange, if it has a GEA set.

Lemma 2.7 [56]. Let $R = R^T$, H and E be real matrices of appropriate dimensions, with F satisfying $F^T F \leq I$, then

$$R + HFE + E^T F^T H^T < 0$$

if and only if there exists a positive scalar $\varepsilon > 0$ such that

$$R + \frac{1}{\varepsilon} HH^T + \varepsilon E^T E < 0,$$

(or) equivalently,

$$\begin{pmatrix} R & H & \varepsilon E^T \\ H^T & -\varepsilon I & 0 \\ \varepsilon E & 0 & -\varepsilon I \end{pmatrix} < 0.$$

Lemma 2.8 [11]. Let $Q \in \mathbb{R}^{n \times n}$ be a positive semi-definite matrix, scalars e and f satisfy $e < f$ and $w(\cdot) : [e, f] \rightarrow \mathbb{R}^n$ be a vector function. If the integrations concerned are well defined, then the following inequality holds:

$$\left(\int_e^f w(\alpha) d\alpha \right)^T Q \left(\int_e^f w(\alpha) d\alpha \right) \leq (f - e) \int_e^f w^T(\alpha) Q w(\alpha) d\alpha.$$

Lemma 2.9 [29]. Let $u, v \in \mathbb{R}^n$, and G is a positive definite matrix, then

$$2u^T v \leq u^T G u + v^T G^{-1} v.$$

Lemma 2.10 [47]. Let the constant matrices $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ and $\mathcal{G}_4 \in \mathbb{R}^{n \times n}$, and appropriate matrices P, Q with reversible then the following condition satisfied:

$$\begin{bmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \end{bmatrix} P^{-1} \begin{bmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \end{bmatrix}^T + \begin{bmatrix} \mathcal{G}_3 \\ \mathcal{G}_4 \end{bmatrix} Q^{-1} \begin{bmatrix} \mathcal{G}_3 \\ \mathcal{G}_4 \end{bmatrix}^T = \begin{bmatrix} \mathcal{G}_1 & \mathcal{G}_3 \\ \mathcal{G}_2 & \mathcal{G}_4 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} \mathcal{G}_1 & \mathcal{G}_3 \\ \mathcal{G}_2 & \mathcal{G}_4 \end{bmatrix}^T$$

Lemma 2.11. (Schur Complement [59]) Given constant symmetric matrices Y_1, Y_2 and Y_3 with appropriate dimensions, where $\Upsilon_1^T = Y_1$ and $\Upsilon_2^T = Y_2 > 0$, $\Upsilon_1 + \Upsilon_3^T \Upsilon_2^{-1} \Upsilon_3 < 0$ if and only if

$$\begin{pmatrix} \Upsilon_1 & \Upsilon_3^T \\ \Upsilon_3 & -\Upsilon_2 \end{pmatrix} < 0 \quad [\text{or}] \quad \begin{pmatrix} \Upsilon_1 & \Upsilon_3 \\ \Upsilon_3^T & -\Upsilon_2 \end{pmatrix} < 0.$$

Lemma 2.12 [26]. Given symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times n}$ is a symmetric matrix, then

$$\lambda_{\min}(A^{-1}B) z^T A z \leq z^T B z \leq \lambda_{\max}(A^{-1}B) z^T A z, \quad z \in \mathbb{R}^n.$$

3. Main results

Theorem 3.1. Suppose that Assumptions 1 and 2 holds, the inertial-type BAM neural networks (1) or (13) is globally robustly exponentially stable in Lagrange sense, if there exist four

positive diagonal matrices $\mathcal{L}_1, \mathcal{L}_2, \Xi, \Pi$, positive definite matrices $\mathcal{Q}_l (l = 1, 2, 3, \dots, 14)$ and $\mathcal{T}_p (1, 2, 3, 4)$ such that the LMIs satisfied as follows:

$$\Theta = \begin{bmatrix} \Upsilon^* & \Theta_1 \\ * & \Theta_2 \end{bmatrix} < 0, \quad (15)$$

$$\Upsilon^* = \begin{bmatrix} \Upsilon_{11}^* & \Upsilon_{12} & \Upsilon_{13} & 0 & \Upsilon_{15} & \Upsilon_{16} \\ * & \Upsilon_{22}^* & 0 & 0 & 0 & \Upsilon_{26} \\ * & * & \Upsilon_{33}^* & \Upsilon_{34} & \Upsilon_{35} & \Upsilon_{36} \\ * & * & * & \Upsilon_{44}^* & \Upsilon_{45} & 0 \\ * & * & * & * & \Upsilon_{55}^* & 0 \\ * & * & * & * & * & \Upsilon_{66}^* \end{bmatrix},$$

$$\Theta_1 = \begin{bmatrix} \Upsilon_{17} & \Upsilon_{18} & 0 & 0 & \Upsilon_{111} & \Upsilon_{112} & \Upsilon_{113} & \Upsilon_{114} & \Upsilon_{115} & \Upsilon_{116} & 0 & 0 \\ 0 & \Upsilon_{28} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Upsilon_{39} & \Upsilon_{310} & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_{317} & \Upsilon_{318} \\ 0 & 0 & 0 & \Upsilon_{410} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Upsilon_{58} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Upsilon_{610} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} \Theta_2 &= \text{diag}\{-\varepsilon_1 I, -\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_2 I, -\varepsilon_3 I, -\varepsilon_3 I, -\varepsilon_4 I, -\varepsilon_4 I, -\varepsilon_5 I, -\varepsilon_5 I, -\varepsilon_6 I, -\varepsilon_6 I\}; \\ \Upsilon_{11} &= -2\mathcal{Q}_2\Gamma_1 + 2\mathcal{Q}_2\mathcal{B}[(1-\eta)\mathcal{Q}_5\eta_1]^{-1} + 2\mathcal{Q}_2\mathcal{L}\mathcal{Q}_6^{-1} + 2\mathcal{Q}_2\mathcal{T}_1^{-1}; \\ \Upsilon_{12} &= 2[\mathcal{Q}_1 + \mathcal{Q}_2\Gamma_2 - \mathcal{W}_3\Pi]; \\ \Upsilon_{22} &= -2[\Lambda_1\mathcal{Q}_1 - \mathcal{W}_3\Pi\Lambda_1] + \hat{\mathcal{K}}\mathcal{L}_2\hat{\mathcal{K}}; \quad \Upsilon_{26} = -2\Pi\Lambda_1; \quad \Upsilon_{33} = -2\mathcal{Q}_4\Gamma_3; \quad \Upsilon_{15} = 2\eta_2\mathcal{Q}_7\mathcal{A}; \\ \Upsilon_{34} &= 2[\mathcal{Q}_3 + \mathcal{Q}_4\Gamma_4 - \mathcal{W}_1\Xi] \quad \Upsilon_{35} = 4\Xi; \quad \Upsilon_{36} = 2\mathcal{P}\mathcal{Q}_4; \quad \Upsilon_{66} = \eta_2\mathcal{Q}_7 + \mu_2^2\mathcal{Q}_8 + \mu_2\mathcal{T}_4 - \mathcal{L}_2; \\ \Upsilon_{44} &= -2[\Lambda_2\mathcal{Q}_3 - \mathcal{W}_1\Xi\Lambda_2] + \mathcal{K}\mathcal{L}_1\mathcal{K}; \quad \Upsilon_{45} = -\Xi\Lambda_2; \quad \Upsilon_{55} = \eta_1\mathcal{Q}_5 + \mu_1^2\mathcal{Q}_6 + \mu_1\mathcal{T}_3 - \mathcal{L}_1; \\ \Upsilon_{13} &= 2\mathcal{Q}_4\mathcal{R}[(1-\tilde{\eta})\mathcal{Q}_7\eta_2]^{-1} + 2\mathcal{Q}_4\mathcal{O}\mathcal{Q}_8^{-1} + 2\mathcal{Q}_4\mathcal{T}_2^{-1}; \quad \Upsilon_{66}^* = \mathcal{Q}_{14} + \eta_2\mathcal{Q}_7 + \mu_2^2\mathcal{Q}_8 + \mu_2\mathcal{T}_4 - \mathcal{L}_2; \\ \Upsilon_{17} &= \mathcal{Q}_2\mathcal{N}^*; \quad \Upsilon_{18} = -\varepsilon_1\mathcal{Z}_1^T; \quad \Upsilon_{111} = 2\mathcal{T}_3^{-1}\mathcal{Q}_2\mathcal{N}^*; \quad \Upsilon_{112} = \varepsilon_3\mathcal{Z}_5^T; \quad \Upsilon_{113} = 2\mathcal{T}_4^{-1}\mathcal{Q}_4\mathcal{X}; \\ \Upsilon_{39} &= \mathcal{Q}_4\mathcal{X}; \quad \Upsilon_{310} = -\varepsilon_2\mathcal{H}_1^T; \quad \Upsilon_{317} = 2[(1-\tilde{\eta})\mathcal{Q}_7\eta_2]^{-1}\mathcal{Q}_4\mathcal{X}; \quad \Upsilon_{318} = \varepsilon_6\mathcal{H}_4^T; \quad \Upsilon_{410} = \varepsilon_2\omega_2^T; \\ \Upsilon_{58} &= \varepsilon_1\mathcal{Z}_3^T; \quad \Upsilon_{610} = \varepsilon_2\mathcal{H}_3^T; \quad \Upsilon_{114} = \varepsilon_4\mathcal{H}_5^T; \quad \Upsilon_{115} = 2[(1-\eta)\mathcal{Q}_5\eta_1]^{-1}\mathcal{Q}_2\mathcal{N}^*; \quad \Upsilon_{116} = \varepsilon_5\mathcal{Z}_4^T; \\ \Upsilon_{11}^* &= \mathcal{Q}_9 - 2\mathcal{Q}_2\Gamma_1 + 2\mathcal{Q}_2\mathcal{B}[(1-\eta)\mathcal{Q}_5\eta_1]^{-1} + 2\mathcal{Q}_2\mathcal{L}\mathcal{Q}_6^{-1} + 2\mathcal{Q}_2\mathcal{T}_1^{-1}; \quad \Upsilon_{16} = 2\Pi; \quad \Upsilon_{28} = \varepsilon_1\omega_1^T; \\ \Upsilon_{22}^* &= \mathcal{Q}_{10} + 2\Pi(\mathcal{W}_4 - \mathcal{W}_3) - 2[\Lambda_1\mathcal{Q}_1 - \mathcal{W}_3\Pi\Lambda_1] + \hat{\mathcal{K}}\mathcal{L}_2\hat{\mathcal{K}}; \quad \Upsilon_{33}^* = \mathcal{Q}_{11} - 2\mathcal{Q}_4\Gamma_3; \\ \Upsilon_{44}^* &= \mathcal{Q}_{12} + 2\Xi(\mathcal{W}_2 - \mathcal{W}_1) - 2[\Lambda_2\mathcal{Q}_3 - \mathcal{W}_1\Xi\Lambda_2] + \mathcal{K}\mathcal{L}_1\mathcal{K}; \quad \Upsilon_{55}^* = \mathcal{Q}_{13} + \eta_1\mathcal{Q}_5 + \mu_1^2\mathcal{Q}_6 + \mu_1\mathcal{T}_3 - \mathcal{L}_1, \end{aligned}$$

and the set $\Omega^* = \{(u(t), v(t)) \in \mathbb{R}^n \times \mathbb{R}^n | u^T(t)\mathcal{Q}_1 u(t) + v^T(t)\mathcal{Q}_3 v(t) \leq \frac{\tilde{J}^T \mathcal{T}_1 \tilde{J} + \tilde{I}^T \mathcal{T}_2 \tilde{I}}{\tilde{\sigma}}\}$ is a globally exponentially attractive set of Eq. (13), where the positive number $\tilde{\sigma} = \min\{1, \hat{\sigma}\}$, $\hat{\sigma}$ is the supremum of the solution of the set inequalities $\sigma\lambda_{\max}(\mathcal{Q}_1) \leq \lambda_{\min}(\mathcal{Q}_{10})$, $\sigma\lambda_{\max}(\mathcal{Q}_2) \leq \lambda_{\min}(\mathcal{Q}_9)$, $\sigma\lambda_{\max}(\mathcal{Q}_3) \leq \lambda_{\min}(\mathcal{Q}_{12})$, $\sigma\lambda_{\max}(\mathcal{Q}_4) \leq \lambda_{\min}(\mathcal{Q}_{11})$, $2\sigma\eta_1^2 e^{\sigma\eta_1} \lambda_{\max}(\mathcal{Q}_5) + \sigma\mu_1^3 e^{\sigma\mu_1} \lambda_{\max}(\mathcal{Q}_6) + \sigma\mu_1^2 e^{\sigma\mu_1} \lambda_{\max}(\mathcal{T}_3) \leq \lambda_{\min}(\mathcal{Q}_{13})$, $2\sigma\eta_2^2 e^{\sigma\eta_2} \lambda_{\max}(\mathcal{Q}_7) + \sigma\mu_2^3 e^{\sigma\mu_2} \lambda_{\max}(\mathcal{Q}_8) + \sigma\mu_2^2 e^{\sigma\mu_2} \lambda_{\max}(\mathcal{T}_4) \leq \lambda_{\min}(\mathcal{Q}_{14})$.

Proof. To establish the Lagrange exponential stability conditions, construct the LKF with positive definite matrices $\mathcal{Q}_h (h = 1, 2, 3, \dots, 8)$, $\mathcal{T}_3, \mathcal{T}_4$ and positive diagonal matrices $\Xi = \text{diag}\{\xi_1, \xi_2, \dots, \xi_n\}$, $\Pi = \text{diag}\{\pi_1, \pi_2, \dots, \pi_n\}$ as follows $\square$

$$\begin{aligned}
\mathcal{V}(u(t), v(t)) = & u^T(t) \mathcal{Q}_1 u(t) + \alpha^T(t) \mathcal{Q}_2 \alpha(t) + v^T(t) \mathcal{Q}_3 v(t) + \beta^T(t) \mathcal{Q}_4 \beta(t) + 2 \sum_{i=1}^n \xi_i \\
& \times \int_0^{v_i(t)} (g_i(\varsigma) - w_i^- \varsigma) d\varsigma + 2\eta_1 \int_{t-\eta_1(t)}^t g^T(v(\varsigma)) \mathcal{Q}_5 g(v(\varsigma)) d\varsigma + \mu_1 \\
& \times \int_{t-\mu_1}^t \int_{\varsigma}^t g^T(v(s)) \mathcal{Q}_6 g(v(s)) ds d\varsigma + 2 \sum_{j=1}^n \pi_j \int_0^{u_j(t)} (h_j(\varsigma) - \hat{w}_j^- \varsigma) \\
& \times d\varsigma + 2\eta_2 \int_{t-\eta_2(t)}^t h^T(u(\varsigma)) \mathcal{Q}_7 h(u(\varsigma)) d\varsigma + \int_{t-\mu_1(t)}^t \int_{t-\varsigma}^t g^T(v(s)) \mathcal{T}_3 \\
& \times g(v(s)) ds d\varsigma + \int_{t-\mu_2(t)}^t \int_{t-\varsigma}^t h^T(u(s)) \mathcal{T}_4 h(u(s)) ds d\varsigma + \mu_2 \\
& \times \int_{t-\mu_2}^t \int_{\varsigma}^t h^T(u(s)) \mathcal{Q}_8 h(u(s)) ds d\varsigma
\end{aligned} \tag{16}$$

Calculating the derivative of $\mathcal{V}(u(t), v(t))$ along the trajectories of Eq. (13), we obtain

$$\begin{aligned}
\mathcal{DV}(u(t), v(t)) = & 2u^T(t) \mathcal{Q}_1 \dot{u}(t) + 2\alpha^T(t) \mathcal{Q}_2 \dot{\alpha}(t) + 2v^T(t) \mathcal{Q}_3 \dot{v}(t) + 2\beta^T(t) \mathcal{Q}_4 \dot{\beta}(t) \\
& + 2(g(v(t)) - W_1 v(t))^T \Xi \dot{v}(t) + 2\eta_1 [g^T(v(t)) \mathcal{Q}_5 g(v(t)) \\
& - g^T(v(t - \eta_1(t))) \mathcal{Q}_5 g(v(t - \eta_1(t)))] (1 - \dot{\eta}_1(t)) + \mu_1^2 g^T(v(t)) \mathcal{Q}_6 \\
& \times g(v(t)) - \mu_1 \int_{t-\mu_1}^t g^T(v(\varsigma)) \mathcal{Q}_6 g(v(\varsigma)) d\varsigma + 2(h(u(t)) - W_3 u(t))^T \\
& \times \Pi \dot{u}(t) + 2\eta_2 [h^T(u(t)) \mathcal{Q}_7 h(u(t)) - h^T(u(t - \eta_2(t))) \mathcal{Q}_7 h(u(t - \eta_2(t)))] \\
& \times (1 - \dot{\eta}_2(t)) + \mu_2^2 h^T(u(t)) \mathcal{Q}_8 h(u(t)) - \mu_2 \int_{t-\mu_2}^t h^T(v(\varsigma)) \mathcal{Q}_8 h(v(\varsigma)) \\
& \times d\varsigma + \mu_1(t) g^T(v(t)) \mathcal{T}_3 g(v(t)) - \int_{t-\mu_1(t)}^t g^T(v(\varsigma)) \mathcal{T}_3 g(v(\varsigma)) d\varsigma + \mu_2(t) \\
& \times h^T(u(t)) \mathcal{T}_4 h(u(t)) - \int_{t-\mu_2(t)}^t h^T(u(\varsigma)) \mathcal{T}_4 h(u(\varsigma)) d\varsigma \\
\leq & -2u^T(t) [\Lambda_1 \mathcal{Q}_1 - W_3 \Pi \Lambda_1] u(t) + 2u^T(t) [\mathcal{Q}_1 + \mathcal{Q}_2 \Gamma_2 - W_3 \Pi] \alpha(t) \\
& - 2\alpha^T(t) (\mathcal{Q}_2 \Gamma_1) \alpha(t) + 2\alpha^T(t) (\mathcal{Q}_7 \mathcal{A}) g(v(t)) + 2\alpha^T(t) (\mathcal{Q}_2 \mathcal{B}) \\
& \times g(v(t - \eta_1(t))) + 2\alpha^T(t) (\mathcal{Q}_2 \mathcal{L}) \int_{t-\mu_1(t)}^t g(v(\varsigma)) d\varsigma + 2\alpha^T(t) \mathcal{Q}_2 \hat{\mathcal{J}} \\
& - 2v^T(t) [\Lambda_2 \mathcal{Q}_3 - W_1 \Xi \Lambda_2] v(t) + 2v^T(t) [\mathcal{Q}_3 + \mathcal{Q}_4 \Gamma_4 - W_1 \Xi] \beta(t) \\
& - 2\beta^T(t) (\mathcal{Q}_4 \Gamma_3) \beta(t) + 2\beta^T(t) (\mathcal{Q}_4 \mathcal{P}) h(u(t)) + 2\beta^T(t) (\mathcal{Q}_4 \mathcal{R}) \\
& \times h(u(t - \eta_2(t))) + 2\beta^T(t) (\mathcal{Q}_4 \mathcal{O}) \int_{t-\mu_2(t)}^t h(u(\varsigma)) d\varsigma + 2\beta^T(t) \mathcal{Q}_4 \hat{\mathcal{I}} \\
& - 2g^T(v(t)) \Xi \Lambda_2 v(t) + 2g^T(v(t)) \Xi \beta(t) + g^T(v(t)) \Xi \Lambda_2 v(t) \\
& + 2g^T(v(t)) \Xi \beta(t) + g^T(v(t)) [2\eta_1 \mathcal{Q}_5 + \mu_1^2 \mathcal{Q}_6] g(v(t)) - 2(1 - \eta) \eta_1 \mathcal{Q}_5 \\
& \times g^T(v(t - \eta_1(t))) g(v(t - \eta_1(t))) - \mu_1 \int_{t-\mu_1}^t g^T(v(\varsigma)) \mathcal{Q}_6 g(v(\varsigma)) d\varsigma
\end{aligned}$$

$$\begin{aligned}
& -2h^T(u(t))\Pi\Lambda_1u(t) + 2h^T(u(t))\Pi\alpha(t) + h^T(u(t))[2\eta_2\mathcal{Q}_7 + \mu_2^2\mathcal{Q}_8] \\
& \times h(u(t)) - 2\eta_2(1 - \tilde{\eta})\mathcal{Q}_7h^T(u(t - \eta_2(t)))h(u(t - \eta_2(t))) - \mu_2 \\
& \times \int_{t-\mu_2}^t h^T(u(\varsigma))\mathcal{Q}_8h(u(\varsigma))d\varsigma + \mu_1g^T(v(t))\mathcal{T}_3g(v(t)) \\
& - \int_{t-\mu_1(t)}^t g^T(v(\varsigma))\mathcal{T}_3g(v(\varsigma))d\varsigma + \mu_2h^T(u(t))\mathcal{T}_4h(u(t)) \\
& - \int_{t-\mu_2(t)}^t h^T(u(\varsigma))\mathcal{T}_4h(u(\varsigma))d\varsigma + 2\alpha^T(t)\mathcal{Q}_2(\mathcal{N}^*\mathcal{Y}(t)\mathcal{Z}_3)g(v(t)) \\
& + 2\alpha^T(t)\mathcal{Q}_2(\mathcal{N}^*\mathcal{Y}(t)\mathcal{Z}_4)g(v(t - \eta_1(t))) + 2\alpha^T(t)\mathcal{Q}_2(\mathcal{N}^*\mathcal{Y}(t)\mathcal{Z}_5) \\
& \times \int_{t-\mu_1(t)}^t g(v(\varsigma))d\varsigma + 2\beta^T(t)\mathcal{Q}_4(\mathcal{X}\Sigma(t)\mathcal{H}_3)h(u(t)) + 2\beta^T(t)\mathcal{Q}_4 \\
& \times (\mathcal{X}\Sigma(t)\mathcal{H}_4)h(u(t - \eta_2(t))) + 2\beta^T(t)\mathcal{Q}_4(\mathcal{X}\Sigma(t)\mathcal{H}_5) \int_{t-\mu_2(t)}^t h(u(\varsigma)) \\
& \times d\varsigma + 2\alpha^T(t)[\mathcal{Q}_2\mathcal{N}^*\mathcal{Y}(t)\omega_1]u(t) + 2\beta^T(t)[\mathcal{Q}_4\mathcal{X}\Sigma(t)\omega_2]v(t) - 2\alpha^T(t) \\
& \times [\mathcal{Q}_2(\mathcal{N}^*\mathcal{Y}(t)\mathcal{Z}_1)]\alpha(t) - 2\beta^T(t)[\mathcal{Q}_4(\mathcal{X}\Sigma(t)\mathcal{H}_1)]\beta(t), \tag{17}
\end{aligned}$$

$\omega_1 = \Lambda_1\mathcal{Z}_1 - \mathcal{Z}_2$ and $\omega_2 = \Lambda_2\mathcal{H}_1 - \mathcal{H}_2$. From [Lemma 2.8](#) and [2.9](#) one can get

$$\begin{aligned}
& 2\alpha^T(t)\mathcal{Q}_2\mathcal{B}g(v(t - \eta_1(t))) - (1 - \eta)\eta_1g^T(v(t - \eta_1(t)))\mathcal{Q}_5g(v(t - \eta_1(t))) \\
& \leq \alpha^T(t)\mathcal{Q}_2\mathcal{B}[(1 - \eta)\eta_1\mathcal{Q}_5]^{-1}\mathcal{B}^T\mathcal{Q}_2\alpha(t) \\
& = \begin{bmatrix} \alpha(t) \\ \beta(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \begin{bmatrix} \mathcal{Q}_2\mathcal{B} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} [(1 - \eta)\eta_1\mathcal{Q}_5]^{-1} \begin{bmatrix} \mathcal{Q}_2\mathcal{B}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \alpha(t) \\ \beta(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \tag{18}
\end{aligned}$$

$$\begin{aligned}
& 2\beta^T(t)\mathcal{Q}_4\mathcal{R}h(u(t - \eta_2(t))) - (1 - \tilde{\eta})\eta_2h^T(u(t - \eta_2(t)))\mathcal{Q}_7h(u(t - \eta_2(t))) \\
& \leq \beta^T(t)\mathcal{Q}_4\mathcal{R}[(1 - \tilde{\eta})\eta_2\mathcal{Q}_7]^{-1}\mathcal{R}^T\mathcal{Q}_4\beta(t) \\
& = \begin{bmatrix} \alpha(t) \\ \beta(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \mathcal{Q}_4\mathcal{R} \\ 0 \\ 0 \\ 0 \end{bmatrix} [(1 - \tilde{\eta})\eta_2\mathcal{Q}_7]^{-1} \begin{bmatrix} 0 \\ 0 \\ \mathcal{Q}_4\mathcal{R} \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \alpha(t) \\ \beta(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \tag{19}
\end{aligned}$$

$$\begin{aligned}
& 2\alpha^T(t)\mathcal{Q}_2\mathcal{L} \int_{t-\mu_1(t)}^t g(v(\varsigma))d\varsigma - \mu_1 \int_{t-\mu_1}^t \mathcal{Q}_6g^T(v(\varsigma))g(v(\varsigma))d\varsigma \\
& \leq 2\alpha^T(t)\mathcal{Q}_2\mathcal{L} \int_{t-\mu_1(t)}^t g(v(\varsigma))d\varsigma - \left[\int_{t-\mu_1(t)}^t g(v(\varsigma))d\varsigma \right]^T \\
& \times \mathcal{Q}_6 \left[\int_{t-\mu_1(t)}^t g(v(\varsigma))d\varsigma \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha^T(t) \mathcal{Q}_2 \mathcal{L} \mathcal{Q}_6^{-1} \mathcal{L}^T \mathcal{Q}_2 \alpha(t) \\
&= \begin{bmatrix} \alpha(t) \\ \beta(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \begin{bmatrix} \mathcal{Q}_2 \mathcal{L} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathcal{Q}_6^{-1} \begin{bmatrix} \mathcal{Q}_2 \mathcal{L} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \alpha(t) \\ \beta(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \quad (20)
\end{aligned}$$

$$\begin{aligned}
&2\beta^T(t) \mathcal{Q}_4 \mathcal{O} \int_{t-\mu_2(t)}^t h(u(\varsigma)) d\varsigma - \mu_2 \int_{t-\mu_2}^t h^T(u(\varsigma)) \mathcal{Q}_8 h(u(\varsigma)) d\varsigma \\
&\leq 2\beta^T(t) \mathcal{Q}_4 \mathcal{O} \int_{t-\mu_2(t)}^t h(u(\varsigma)) d\varsigma - \left[\int_{t-\mu_2(t)}^t h(u(\varsigma)) d\varsigma \right]^T \\
&\quad \times \mathcal{Q}_8 \left[\int_{t-\mu_2(t)}^t h(u(\varsigma)) d\varsigma \right] \\
&\leq \beta^T(t) \mathcal{Q}_4 \mathcal{O} \mathcal{Q}_8^{-1} \mathcal{O}^T \mathcal{Q}_4 \beta(t) \\
&= \begin{bmatrix} \alpha(t) \\ \beta(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \mathcal{Q}_4 \mathcal{O} \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathcal{Q}_8^{-1} \begin{bmatrix} 0 \\ 0 \\ \mathcal{Q}_4 \mathcal{O} \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \alpha(t) \\ \beta(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \quad (21)
\end{aligned}$$

$$\begin{aligned}
2\alpha^T(t) \mathcal{Q}_2 \hat{J} &\leq \alpha^T(t) \mathcal{Q}_2 \mathcal{T}_1^{-1} \mathcal{Q}_2 \alpha(t) + \hat{J}^T \mathcal{T}_1 \hat{J} \\
&= \begin{bmatrix} \alpha(t) \\ \beta(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \begin{bmatrix} \mathcal{Q}_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathcal{T}_1^{-1} \begin{bmatrix} \mathcal{Q}_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \alpha(t) \\ \beta(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} + \hat{J}^T \mathcal{T}_1 \hat{J} \quad (22)
\end{aligned}$$

$$\begin{aligned}
2\beta^T(t) \mathcal{Q}_4 \hat{I} &\leq \beta^T(t) \mathcal{Q}_4 \mathcal{T}_2^{-1} \mathcal{Q}_4 \beta(t) + \hat{I}^T \mathcal{T}_2 \hat{I} \\
&= \begin{bmatrix} \alpha(t) \\ \beta(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \mathcal{Q}_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathcal{T}_2^{-1} \begin{bmatrix} 0 \\ 0 \\ \mathcal{Q}_4 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \alpha(t) \\ \beta(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} + \hat{I}^T \mathcal{T}_2 \hat{I} \quad (23)
\end{aligned}$$

$$\begin{aligned}
2\alpha^T(t) \mathcal{Q}_2 \mathcal{N}^* \mathcal{Y}(t) \mathcal{Z}_5 &\int_{t-\mu_1(t)}^t g(v(\varsigma)) d\varsigma - \int_{t-\mu_1(t)}^t g^T(v(\varsigma)) \mathcal{T}_3 g(v(\varsigma)) d\varsigma \\
&\leq \alpha^T(t) \chi_1 \mathcal{T}_3^{-1} \chi_1^T \alpha(t)
\end{aligned}$$

$$\leq \begin{bmatrix} \alpha(t) \\ \beta(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \chi_1 \mathcal{T}_3^{-1} \chi_1 \begin{bmatrix} \alpha(t) \\ \beta(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \quad (24)$$

$$2\alpha^T \mathcal{Q}_2(\mathcal{N}^* \mathcal{Y}(t) \mathcal{Z}_4) g(v(t - \eta_1(t))) - (1 - \eta) \eta_1 g^T(v(t - \eta_1(t))) \mathcal{Q}_5 g(v(t - \eta_1(t))) \\ \leq \alpha^T(t) \chi_3 [\mathcal{Q}_5(1 - \eta) \eta_1]^{-1} \chi_3^T \alpha(t), \quad (25)$$

$$2\beta^T \mathcal{Q}_4(\mathcal{X} \Sigma(t) \mathcal{H}_4) h(u(t - \eta_2(t))) - (1 - \tilde{\eta}) \eta_2 h^T(u(t - \eta_2(t))) \mathcal{Q}_7 h(u(t - \eta_2(t))) \\ \leq \beta^T(t) \chi_4 [\mathcal{Q}_7(1 - \tilde{\eta}) \eta_2]^{-1} \chi_4^T \beta(t), \quad (26)$$

where $\chi_1 = \mathcal{Q}_2 \mathcal{N}^* \mathcal{Y}(t) \mathcal{Z}_5$, $\chi_2 = \mathcal{Q}_4 \mathcal{X} \Sigma(t) \mathcal{H}_5$, $\chi_3 = \mathcal{Q}_2 \mathcal{N}^* \mathcal{Y}(t) \mathcal{Z}_4$, $\chi_4 = \mathcal{Q}_4 \mathcal{X} \Sigma(t) \mathcal{H}_4$.

$$2\beta^T(t) \mathcal{Q}_4 \mathcal{X} \Sigma(t) \mathcal{H}_5 \int_{t-\mu_2(t)}^t h(u(\varsigma)) d\varsigma - \int_{t-\mu_2(t)}^t h^T(u(\varsigma)) \mathcal{T}_4 h(u(\varsigma)) d\varsigma \leq \beta^T(t) \chi_2 \mathcal{T}_4^{-1} \chi_2^T \beta(t) \\ \leq \begin{bmatrix} \alpha(t) \\ u(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \chi_2 \mathcal{T}_4^{-1} \chi_2 \begin{bmatrix} \alpha(t) \\ u(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \quad (27)$$

Adding the Eqs. (7), (8) and (17)–(27) and Lemma 2.10, we can infer that

$$\mathcal{DV}(u(t), v(t)) \leq -2u^T(t) [\Lambda_1 \mathcal{Q}_1 - \mathcal{W}_3 \Pi \Lambda_1] u(t) + 2u^T(t) [\mathcal{Q}_1 + \mathcal{Q}_2 \Gamma_2 - \mathcal{W}_3 \Pi] \alpha(t) \\ - 2\alpha^T(t) (\mathcal{Q}_2 \Gamma_1) \alpha(t) + 2\alpha^T(t) (\mathcal{Q}_7 \mathcal{A}) g(v(t)) - 2v^T(t) [\Lambda_2 \mathcal{Q}_3 \\ - \mathcal{W}_1 \Xi \Lambda_2] v(t) + 2v^T(t) [\mathcal{Q}_3 + \mathcal{Q}_4 \Gamma_4 - \mathcal{W}_1 \Xi] \beta(t) - 2\beta^T(t) (\mathcal{Q}_4 \Gamma_3) \\ \times \beta(t) + 2\beta^T(t) (\mathcal{Q}_4 \mathcal{P}) h(u(t)) - 2g^T(v(t)) \Xi \Lambda_2 v(t) + 2g^T(v(t)) \Xi \beta(t) \\ + g^T(v(t)) \Xi \Lambda_2 v(t) + 2g^T(v(t)) \Xi \beta(t) + g^T(v(t)) [\eta_1 \mathcal{Q}_5 + \mu_1^2 \mathcal{Q}_6] g(v(t)) \\ - 2h^T(u(t)) \Pi \Lambda_1 u(t) + 2h^T(u(t)) \Pi \alpha(t) + h^T(u(t)) [\eta_2 \mathcal{Q}_7 + \mu_2^2 \mathcal{Q}_8] \\ \times h(u(t)) + \mu_1 g^T(v(t)) \mathcal{T}_3 g(v(t)) + \mu_2 h^T(u(t)) \mathcal{T}_4 h(u(t)) + \alpha^T(t) \mathcal{Q}_2 \\ \times \mathcal{B}[(1 - \eta) \eta_1 \mathcal{Q}_5]^{-1} \mathcal{B}^T \mathcal{Q}_2 \alpha(t) + \beta^T(t) \mathcal{Q}_4 \mathcal{R}[(1 - \tilde{\eta}) \eta_2 \mathcal{Q}_7]^{-1} \mathcal{Q}_4 \mathcal{R}^T \beta(t) \\ + \alpha^T(t) \mathcal{Q}_2 \mathcal{L} \mathcal{Q}_6^{-1} \mathcal{L}^T \mathcal{Q}_2 \alpha(t) + \beta^T(t) \mathcal{Q}_4 \mathcal{O} \mathcal{Q}_8^{-1} \mathcal{Q}_4 \mathcal{O}^T \beta(t) + \alpha^T(t) \mathcal{Q}_2 \\ \times \mathcal{T}_1^{-1} \mathcal{Q}_2 \alpha(t) + \hat{J}^T \mathcal{T}_1 \hat{J} + \beta^T(t) \mathcal{Q}_4 \mathcal{T}_2^{-1} \mathcal{Q}_4 \beta(t) + \hat{I}^T \mathcal{T}_2 \hat{I} + v^T(t) \mathcal{K} \mathcal{L}_1 \\ \times \mathcal{K} v(t) - g^T(v(t)) \mathcal{L}_1 g(v(t)) + u^T(t) \hat{\mathcal{K}} \mathcal{L}_2 \hat{\mathcal{K}} u(t) - h^T(u(t)) \mathcal{L}_2 h(u(t)) \\ + \begin{bmatrix} \mathcal{Q}_2 \mathcal{N}^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathcal{Y}(t) \begin{bmatrix} -Z_1^T \\ \omega_1^T \\ 0 \\ 0 \\ Z_3^T \\ 0 \end{bmatrix}^T + \begin{bmatrix} -Z_1^T \\ \omega_1^T \\ 0 \\ 0 \\ Z_3^T \\ 0 \end{bmatrix} \mathcal{Y}^T(t) \begin{bmatrix} \mathcal{Q}_2 \mathcal{N}^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

$$\begin{aligned}
& + \begin{bmatrix} 0 \\ 0 \\ Q_4 \mathcal{X} \\ 0 \\ 0 \\ 0 \end{bmatrix} \Sigma(t) \begin{bmatrix} 0 \\ 0 \\ -\mathcal{H}_1^T \\ \omega_2 \\ 0 \\ \mathcal{H}_3^T \end{bmatrix}^T + \begin{bmatrix} 0 \\ 0 \\ -\mathcal{H}_1^T \\ \omega_2 \\ 0 \\ \mathcal{H}_3^T \end{bmatrix} \Sigma^T(t) \begin{bmatrix} 0 \\ 0 \\ Q_4 \mathcal{X} \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \\
& + \mathcal{T}_3^{-1} \left\{ \begin{bmatrix} Q_2 \mathcal{N}^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathcal{Y}(t) \begin{bmatrix} -Z_5^T \\ \omega_1^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} -Z_5^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \mathcal{Y}^T(t) \begin{bmatrix} Q_2 \mathcal{N}^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \right\} \\
& + \mathcal{T}_4^{-1} \left\{ \begin{bmatrix} 0 \\ 0 \\ Q_4 \mathcal{X} \\ 0 \\ 0 \\ 0 \end{bmatrix} \Sigma(t) \begin{bmatrix} 0 \\ 0 \\ \mathcal{H}_5^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} 0 \\ 0 \\ \mathcal{H}_5^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \Sigma^T(t) \begin{bmatrix} 0 \\ 0 \\ Q_4 \mathcal{X} \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \right\} \\
& + [\mathcal{Q}_5(1 - \eta)\eta_1]^{-1} \left\{ \begin{bmatrix} Q_2 \mathcal{N}^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathcal{Y}(t) \begin{bmatrix} Z_4^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} Z_4^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \mathcal{Y}^T(t) \begin{bmatrix} Q_2 \mathcal{N}^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \right\} \\
& + [\mathcal{Q}_7(1 - \tilde{\eta})\eta_2]^{-1} \left\{ \begin{bmatrix} 0 \\ 0 \\ Q_4 \mathcal{X} \\ 0 \\ 0 \\ 0 \end{bmatrix} \Sigma(t) \begin{bmatrix} 0 \\ 0 \\ \mathcal{H}_4^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} 0 \\ 0 \\ \mathcal{H}_4^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \Sigma^T(t) \begin{bmatrix} 0 \\ 0 \\ Q_4 \mathcal{X} \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \right\} \quad (28) \\
& = \begin{bmatrix} \alpha(t) \\ u(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} \Upsilon \begin{bmatrix} \alpha(t) \\ u(t) \\ \beta(t) \\ v(t) \\ g(v(t)) \\ h(u(t)) \end{bmatrix} + \hat{J}^T \mathcal{T}_1 \hat{J} + \hat{I}^T \mathcal{T}_2 \hat{I} + \varepsilon_1 \begin{bmatrix} -Z_1^T \\ \omega_1^T \\ 0 \\ 0 \\ 0 \\ Z_3^T \\ 0 \end{bmatrix}^T \begin{bmatrix} -Z_1^T \\ \omega_1^T \\ 0 \\ 0 \\ 0 \\ Z_3^T \\ 0 \end{bmatrix} \\
& + \varepsilon_1^{-1} \begin{bmatrix} Q_2 \mathcal{N}^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} Q_2 \mathcal{N}^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \varepsilon_2 \begin{bmatrix} 0 \\ 0 \\ -\mathcal{H}_1 \\ \omega_2^T \\ 0 \\ \mathcal{H}_3^T \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ -\mathcal{H}_1 \\ \omega_2^T \\ 0 \\ \mathcal{H}_3^T \end{bmatrix} + \varepsilon_2^{-1} \begin{bmatrix} 0 \\ 0 \\ Q_4 \mathcal{X} \\ 0 \\ 0 \\ 0 \end{bmatrix}^T
\end{aligned}$$

$$\begin{aligned}
& \times \begin{bmatrix} 0 \\ 0 \\ Q_4 \mathcal{X} \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2\mathcal{T}_3^{-1} \left\{ \varepsilon_3 \begin{bmatrix} Z_5^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} Z_5^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \varepsilon_3^{-1} \begin{bmatrix} Q_2 \mathcal{N}^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} Q_2 \mathcal{N}^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \\
& + 2\mathcal{T}_4^{-1} \left\{ \varepsilon_4 \begin{bmatrix} 0 \\ 0 \\ \mathcal{H}_5^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ \mathcal{H}_5^T \\ 0 \\ 0 \\ 0 \end{bmatrix} + \varepsilon_4^{-1} \begin{bmatrix} 0 \\ 0 \\ Q_4 \mathcal{X} \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ Q_4 \mathcal{X} \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} + 2[Q_5(1-\eta)\eta_1]^{-1} \\
& \times \left\{ \varepsilon_5 \begin{bmatrix} Z_4^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} Z_4^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \varepsilon_5^{-1} \begin{bmatrix} Q_2 \mathcal{N}^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} Q_2 \mathcal{N}^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} + 2[Q_7(1-\tilde{\eta})\eta_2]^{-1} \\
& \times \left\{ \varepsilon_6 \begin{bmatrix} 0 \\ 0 \\ \mathcal{H}_4^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ \mathcal{H}_4^T \\ 0 \\ 0 \\ 0 \end{bmatrix} + \varepsilon_6^{-1} \begin{bmatrix} 0 \\ 0 \\ Q_4 \mathcal{X} \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ Q_4 \mathcal{X} \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, \\
& = \Theta + \hat{J}^T \mathcal{T}_1 \hat{J} + \hat{I}^T \mathcal{T}_2 \hat{I} \tag{28}
\end{aligned}$$

where

$$\Upsilon = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & 0 & \Upsilon_{15} & \Upsilon_{16} \\ * & \Upsilon_{22} & 0 & 0 & 0 & \Upsilon_{26} \\ * & * & \Upsilon_{33} & \Upsilon_{34} & \Upsilon_{35} & \Upsilon_{36} \\ * & * & * & \Upsilon_{44} & \Upsilon_{45} & 0 \\ * & * & * & * & \Upsilon_{55} & 0 \\ * & * & * & * & * & \Upsilon_{66} \end{bmatrix}, \tag{29}$$

Sequentially, it follows from Eq. (15) and Lemma 2.11 that $\Theta < 0$, and also it is equal to

$$\Upsilon < \begin{bmatrix} -Q_9 & 0 & 0 & 0 & 0 & 0 \\ * & -Q_{10} - 2\Pi(W_4 - W_3) & 0 & 0 & 0 & 0 \\ * & * & -Q_{11} & 0 & 0 & 0 \\ * & * & * & -Q_{12} - 2\Xi(W_2 - W_1) & 0 & 0 \\ * & * & * & * & -Q_{13} & 0 \\ * & * & * & * & * & -Q_{14} \end{bmatrix}, \tag{30}$$

By Eq. (28), (30), and Lemma 2.12, one can obtain that

$$\begin{aligned}
 \mathcal{DV}(u(t), v(t)) &\leq -\alpha^T(t) \mathcal{Q}_9 \alpha(t) - u^T(t) [\mathcal{Q}_{10} + 2\Pi(W_4 - W_3)]u(t) - \mathcal{Q}_{11} \beta^T(t) \beta(t) \\
 &\quad - [\mathcal{Q}_{12} + 2(W_2 - W_1)]v^T(t)v(t) - \mathcal{Q}_{13} g^T(v(t))g(v(t)) - \mathcal{Q}_{14} \\
 &\quad \times h^T(u(t))h(u(t)) + \hat{J}^T \mathcal{T}_1 \hat{J} + \hat{I}^T \mathcal{T}_2 \hat{I} \\
 &\leq -\lambda_{\min}(\mathcal{Q}_9) \alpha^T(t) \alpha(t) - \lambda_{\min}(\mathcal{Q}_{10}) u^T(t) u(t) - \lambda_{\min}(\mathcal{Q}_{11}) \beta^T(t) \beta(t) \\
 &\quad - \lambda_{\min}(\mathcal{Q}_{12}) v^T(t) v(t) - \lambda_{\min}(\mathcal{Q}_{13}) g^T(v(t))g(v(t)) - \lambda_{\min}(\mathcal{Q}_{14}) \\
 &\quad \times h^T(u(t))h(u(t)) - 2u^T(t) \Pi(W_4 - W_3)u(t) - 2v^T(t) \Xi(W_2 - W_1) \\
 &\quad \times v(t) + \hat{J}^T \mathcal{T}_1 \hat{J} + \hat{I}^T \mathcal{T}_2 \hat{I}
 \end{aligned} \tag{31}$$

Let $\delta_1 = \hat{J}^T \mathcal{T}_1 \hat{J}$ and $\delta_2 = \hat{I}^T \mathcal{T}_2 \hat{I}$ and consider the LKF $\mathcal{V}^*(u(t), v(t)) = \left(\mathcal{V}(u(t), v(t)) - \frac{\delta_1 + \delta_2}{\sigma} \right) \times e^{\sigma t}$ with $0 < \sigma \leq 1$. Take the derivative of $\mathcal{V}^*(u(t), v(t))$ along the solutions of Eq. (13), it follows that

$$\begin{aligned}
 \mathcal{DV}^*(u(t), v(t)) &\leq \sigma e^{\sigma t} \left[u^T(t) \mathcal{Q}_1 u(t) + \alpha^T(t) \mathcal{Q}_2 \alpha(t) + v_n^T(t) \mathcal{Q}_3 v(t) + \beta^T(t) \mathcal{Q}_4 \beta(t) + 2 \sum_{i=1}^n \xi_i \right. \\
 &\quad \times \int_0^{v_i(t)} (g_i(\varsigma) - w_i^- \varsigma) d\varsigma + 2\eta_1 \int_{t-\eta_1(t)}^t g^T(v(\varsigma)) \mathcal{Q}_5 g(v(\varsigma)) d\varsigma + \mu_1 \\
 &\quad \times \int_{t-\mu_1}^t \int_{\varsigma}^t g^T(v(s)) \mathcal{Q}_6 g(v(s)) ds d\varsigma + 2 \sum_{j=1}^n \pi_j \int_0^{u_j(t)} (h_j(\varsigma) - \hat{w}_j^- \varsigma) d\varsigma \\
 &\quad + 2\eta_2 \int_{t-\eta_2(t)}^t h^T(u(\varsigma)) \mathcal{Q}_7 h(u(\varsigma)) d\varsigma + \int_{t-\mu_1(t)}^t \int_{t-\varsigma}^t g^T(v(s)) \mathcal{T}_3 g(v(s)) ds d\varsigma \\
 &\quad + \int_{t-\mu_2(t)}^t \int_{t-\varsigma}^t h^T(u(s)) \mathcal{T}_4 h(u(s)) ds d\varsigma + \mu_1 \int_{t-\mu_2}^t \int_{\varsigma}^t h^T(u(s)) \mathcal{Q}_8 \\
 &\quad \times h(u(s)) ds d\varsigma \left. \right] + e^{\sigma t} [-\lambda_{\min}(\mathcal{Q}_9) \alpha^T(t) \alpha(t) - \lambda_{\min}(\mathcal{Q}_{10}) u^T(t) u(t) \\
 &\quad - \lambda_{\min}(\mathcal{Q}_{11}) \beta^T(t) \beta(t) - \lambda_{\min}(\mathcal{Q}_{12}) v^T(t) v(t) - \lambda_{\min}(\mathcal{Q}_{13}) g^T(v(t)) \\
 &\quad \times g(v(t)) - \lambda_{\min}(\mathcal{Q}_{14}) h^T(u(t)) h(u(t)) - 2u^T(t) \Pi(W_4 - W_3)u(t) \\
 &\quad - 2v^T(t) \Xi(W_2 - W_1)v(t) + \delta_1 + \delta_2] - (\delta_1 + \delta_2) e^{\sigma t} \\
 &\leq e^{\sigma t} \left\{ (\sigma \lambda_{\max}(\mathcal{Q}_1) - \lambda_{\min}(\mathcal{Q}_{10})) u^T(t) u(t) + (\sigma \lambda_{\max}(\mathcal{Q}_2) - \lambda_{\min}(\mathcal{Q}_9)) \alpha^T(t) \right. \\
 &\quad \times \alpha(t) + (\sigma \lambda_{\max}(\mathcal{Q}_3) - \lambda_{\min}(\mathcal{Q}_{12})) v^T(t) v(t) + (\sigma \lambda_{\max}(\mathcal{Q}_4) \\
 &\quad - \lambda_{\min}(\mathcal{Q}_{11})) \beta^T(t) \beta(t) + 2\sigma \sum_{i=1}^n \xi_i \int_0^{v_i(t)} (g_i(\varsigma) - w_i^- \varsigma) d\varsigma + 2\sigma \eta_1 \\
 &\quad \times \int_{t-\eta_1(t)}^t g^T(v(\varsigma)) \mathcal{Q}_5 g(v(\varsigma)) d\varsigma + \sigma \mu_1 \int_{t-\mu_1}^t \int_{\varsigma}^t g^T(v(s)) \mathcal{Q}_6 g(v(s)) ds \\
 &\quad \times d\varsigma + 2\sigma \sum_{j=1}^n \pi_j \int_0^{u_j(t)} (h_j(\varsigma) - \hat{w}_j^- \varsigma) d\varsigma + 2\sigma \eta_2 \int_{t-\eta_2(t)}^t h^T(u(\varsigma)) \mathcal{Q}_7
 \end{aligned}$$

$$\begin{aligned}
& \times h(u(\varsigma))d\varsigma + \sigma\mu_2 \int_{t-\mu_2}^t \int_{\varsigma}^t h^T(u(s))\mathcal{Q}_8 h(u(s))dsd\varsigma + \sigma \\
& \times \int_{t-\mu_1(t)}^t \int_{\varsigma}^t g^T(v(s))\mathcal{T}_3 g(v(s))dsd\varsigma + \sigma \int_{t-\mu_2(t)}^t \int_{\varsigma}^t h^T(u(s))\mathcal{T}_4 \\
& \times h(u(s))dsd\varsigma - \lambda_{\min}(\mathcal{Q}_{13})g^T(v(t))g(v(t)) - \lambda_{\min}(\mathcal{Q}_{14})h^T(u(t))h(u(t)) \\
& - 2u^T(t)\Pi(W_4 - W_3)u(t) - 2v^T(t)\Xi(W_2 - W_1)v(t) \Big\} \\
& \leq e^{\sigma t} \Big[2\sigma\eta_1 \int_{t-\eta_1(t)}^t g^T(v(\varsigma))g(v(\varsigma))\mathcal{Q}_5 d\varsigma \\
& + 2\sigma\eta_2 \int_{t-\eta_2(t)}^t h^T(u(\varsigma))h(u(\varsigma))\mathcal{Q}_7 d\varsigma \\
& + \sigma\mu_1 \int_{t-\mu_1}^t \int_{\varsigma}^t g^T(v(s))\mathcal{Q}_6 g(v(s))dsd\varsigma + \sigma\mu_2 \int_{t-\mu_2}^t \int_{\varsigma}^t h^T(u(s))\mathcal{Q}_8 \\
& \times h(u(s))dsd\varsigma + \sigma \int_{t-\mu_1(t)}^t \int_{\varsigma}^t g^T(v(s))\mathcal{T}_3 g(v(s))dsd\varsigma + \sigma \\
& \times \int_{t-\mu_2(t)}^t \int_{\varsigma}^t h^T(u(s))\mathcal{T}_4 h(u(s))dsd\varsigma - \lambda_{\min}(\mathcal{Q}_{13})g^T(v(t))g(v(t)) \\
& - \lambda_{\min}(\mathcal{Q}_{14})h^T(u(t))h(u(t)) \Big] \tag{32}
\end{aligned}$$

Then, integrate the both sides of Eq. (32) from 0 to t^* ($t^* > 0$), we have

$$\begin{aligned}
e^{\sigma t^*} \left(\mathcal{V}(t^*) - \frac{\delta_1 + \delta_2}{\sigma} \right) & \leq \mathcal{V}(0) - \frac{\delta_1 + \delta_2}{\sigma} - \lambda_{\min}(\mathcal{Q}_{13}) \int_0^{t^*} e^{\sigma t} g^T(v(s))g(v(s))ds - \lambda_{\min}(\mathcal{Q}_{14}) \\
& \times \int_0^{t^*} e^{\sigma t} h^T(u(s))h(u(s))ds + 2\sigma\eta_1 \int_0^{t^*} e^{\sigma t} \int_{t-\eta_1(t)}^t g^T(v(\varsigma))\mathcal{Q}_5 \\
& \times g(v(\varsigma))d\varsigma dt + 2\sigma\eta_2 \int_0^{t^*} e^{\sigma t} \int_{t-\eta_2(t)}^t h^T(u(\varsigma))\mathcal{Q}_7 h(u(\varsigma))d\varsigma dt + \sigma\mu_1 \\
& \times \int_0^{t^*} e^{\sigma t} \int_{t-\mu_1}^t \int_{\varsigma}^t g^T(v(s))\mathcal{Q}_6 g(v(s))dsd\varsigma dt + \sigma\mu_2 \int_0^{t^*} e^{\sigma t} \\
& \times \int_{t-\mu_2}^t \int_{\varsigma}^t h^T(u(s))\mathcal{Q}_8 h(u(s))dsd\varsigma dt + \sigma \int_0^{t^*} \int_{t-\mu_1(t)}^t \int_{\varsigma}^t g^T(v(s)) \\
& \times \mathcal{T}_3 g(v(s))dsd\varsigma dt + \sigma \int_0^{t^*} \int_{t-\mu_2(t)}^t \int_{\varsigma}^t h^T(u(s))\mathcal{T}_4 h(u(s))dsd\varsigma dt \tag{33}
\end{aligned}$$

From the integral inequality techniques, the double integral terms in Eq. (33) are scaled as follows:

$$\begin{aligned}
2\sigma\eta_1 \int_0^{t^*} e^{\sigma t} \int_{t-\eta_1(t)}^t g^T(v(\varsigma))\mathcal{Q}_5 g(v(\varsigma))d\varsigma dt & \leq 2\sigma\eta_1 \int_{-\eta_1}^{t^*} \int_{\max\{\varsigma, 0\}}^{\min\{\varsigma+\eta_1, t^*\}} e^{\sigma t} g^T(v(\varsigma)) \\
& \times \mathcal{Q}_5 g(v(\varsigma))d\varsigma dt
\end{aligned}$$

$$\begin{aligned}
&\leq 2\sigma\eta_1^2 \int_{-\eta_1}^{t^*} e^{\sigma(\varsigma+\eta_1)} g^T(v(\varsigma)) \mathcal{Q}_5 g(v(\varsigma)) d\varsigma \\
&\leq 2\sigma\eta_1^2 e^{\sigma\eta_1} \left[\int_0^{t^*} e^{\sigma\varsigma} g^T(v(\varsigma)) \mathcal{Q}_5 g(v(\varsigma)) d\varsigma \right. \\
&\quad \left. + \int_{-\eta_1}^0 g^T(v(\varsigma)) \mathcal{Q}_5 g(v(\varsigma)) d\varsigma \right], \quad (34)
\end{aligned}$$

$$\begin{aligned}
2\sigma\eta_2 \int_0^{t^*} e^{\sigma t} \int_{t-\eta_2(t)}^t h^T(u(\varsigma)) \mathcal{Q}_7 h(u(\varsigma)) d\varsigma dt &\leq 2\sigma\eta_2 \int_{-\eta_2}^{t^*} \int_{\max\{\varsigma, 0\}}^{\min\{\varsigma+\eta_2, t^*\}} e^{\sigma t} h^T(u(\varsigma)) \\
&\quad \times \mathcal{Q}_7 h(u(\varsigma)) d\varsigma dt \\
&\leq 2\sigma\eta_2^2 e^{\sigma\eta_2} \left[\int_0^{t^*} e^{\sigma\varsigma} h^T(u(\varsigma)) \mathcal{Q}_7 h(u(\varsigma)) d\varsigma \right. \\
&\quad \left. + \int_{-\eta_2}^0 h^T(u(\varsigma)) \mathcal{Q}_7 h(u(\varsigma)) d\varsigma \right], \quad (35)
\end{aligned}$$

$$\int_{t-\mu_1}^t \int_{\varsigma}^t g^T(v(s)) \mathcal{Q}_6 g(v(s)) ds d\varsigma \leq \mu_1 \int_{t-\mu_1}^t g^T(v(s)) \mathcal{Q}_6 g(v(s)) ds, \quad (36)$$

$$\int_{t-\mu_2}^t \int_{\varsigma}^t h^T(u(s)) \mathcal{Q}_8 h(u(s)) ds d\varsigma \leq \mu_2 \int_{t-\mu_2}^t h^T(u(s)) \mathcal{Q}_8 h(u(s)) ds, \quad (37)$$

$$\int_{t-\mu_1(t)}^t \int_{\varsigma}^t g^T(v(s)) \mathcal{T}_3 g(v(s)) ds d\varsigma \leq \mu_1 \int_{t-\mu_1(t)}^t g^T(v(s)) \mathcal{T}_3 g(v(s)) ds, \quad (38)$$

$$\int_{t-\mu_2(t)}^t \int_{\varsigma}^t h^T(u(s)) \mathcal{T}_4 h(u(s)) ds d\varsigma \leq \mu_2 \int_{t-\mu_2(t)}^t h^T(u(s)) \mathcal{T}_4 h(u(s)) ds, \quad (39)$$

Proceed the similar method of Eqs. (34) and (35), the triple integral terms in Eq. (32) are scaled as

$$\begin{aligned}
\sigma\mu_1 \int_0^{t^*} e^{\sigma t} \int_{t-\mu_1}^t \int_{\varsigma}^t g^T(v(s)) \mathcal{Q}_6 g(v(s)) ds d\varsigma dt &\leq \sigma\mu_1^2 \int_0^{t^*} e^{\sigma t} \int_{t-\mu_1}^t g^T(v(\varsigma)) \mathcal{Q}_6 g(v(\varsigma)) d\varsigma dt \\
&\leq \sigma\mu_1^3 e^{\sigma\mu_1} \left[\int_0^{t^*} e^{\sigma\varsigma} g^T(v(\varsigma)) \mathcal{Q}_6 g(v(\varsigma)) d\varsigma \right. \\
&\quad \left. + \int_{-\mu_1}^0 g^T(v(\varsigma)) \mathcal{Q}_6 g(v(\varsigma)) d\varsigma \right], \quad (40)
\end{aligned}$$

$$\sigma \mu_2 \int_0^{t^*} e^{\sigma t} \int_{t-\mu_2}^t \int_{\varsigma}^t h^T(u(s)) \mathcal{Q}_8 h(u(s)) ds d\varsigma dt \leq \sigma \mu_2^3 e^{\sigma \mu_2} \left[\int_0^{t^*} e^{\sigma \varsigma} h^T(u(\varsigma)) \mathcal{Q}_8 h(u(\varsigma)) d\varsigma + \int_{-\mu_2}^0 h^T(u(\varsigma)) \mathcal{Q}_8 h(u(\varsigma)) d\varsigma \right], \quad (41)$$

$$\sigma \int_0^{t^*} e^{\sigma t} \int_{t-\mu_1(t)}^t \int_{t-\varsigma}^t g^T(v(s)) \mathcal{T}_3 g(v(s)) ds d\varsigma dt \leq \sigma \mu_1^2 e^{\sigma \mu_1} \left[\int_0^{t^*} e^{\sigma \varsigma} g^T(v(\varsigma)) \mathcal{T}_3 g(v(\varsigma)) d\varsigma + \int_{-\mu_1}^0 g^T(v(\varsigma)) \mathcal{T}_3 g(v(\varsigma)) d\varsigma \right], \quad (42)$$

$$\sigma \int_0^{t^*} e^{\sigma t} \int_{t-\mu_2(t)}^t \int_{t-\varsigma}^t h^T(u(s)) \mathcal{T}_4 h(u(s)) ds d\varsigma dt \leq \sigma \mu_2^2 e^{\sigma \mu_2} \left[\int_0^{t^*} e^{\sigma \varsigma} h^T(u(\varsigma)) \mathcal{T}_4 h(u(\varsigma)) d\varsigma + \int_{-\mu_2}^0 h^T(u(\varsigma)) \mathcal{T}_4 h(u(\varsigma)) d\varsigma \right]. \quad (43)$$

According to Eq. (34), (35) and (40)–(43), it can be seen that

$$\begin{aligned} e^{\sigma t^*} \left(\mathcal{V}(t^*) - \frac{\delta_1 + \delta_2}{\sigma} \right) &\leq \mathcal{V}(0) - \frac{\delta_1 + \delta_2}{\sigma} + \int_0^{t^*} e^{\sigma t} \{ -\lambda_{\min}(\mathcal{Q}_{13}) + 2\sigma \eta_1^2 e^{\sigma \eta_1} \lambda_{\max}(\mathcal{Q}_5) \\ &\quad + \sigma \mu_1^3 e^{\sigma \mu_1} \lambda_{\max}(\mathcal{Q}_6) + \sigma \mu_1^2 e^{\sigma \mu_1} \lambda_{\max}(\mathcal{T}_3) \} g^T(v(s)) g(v(s)) ds \\ &\quad + \int_0^{t^*} e^{\sigma t} \{ -\lambda_{\min}(\mathcal{Q}_{14}) + 2\sigma \eta_2^2 e^{\sigma \eta_2} \lambda_{\max}(\mathcal{Q}_7) + \sigma \mu_2^3 e^{\sigma \mu_2} \\ &\quad \times \lambda_{\max}(\mathcal{Q}_8) + \sigma \mu_2^2 e^{\sigma \mu_2} \lambda_{\max}(\mathcal{T}_4) \} h^T(u(s)) h(u(s)) ds + 2\sigma \eta_1^2 \\ &\quad \times e^{\sigma \eta_1} \int_{-\eta_1}^0 g^T(v(\varsigma)) \mathcal{Q}_5 g(v(\varsigma)) d\varsigma + 2\sigma \eta_2^2 e^{\sigma \eta_2} \int_{-\eta_2}^0 h^T(u(\varsigma)) \\ &\quad \times \mathcal{Q}_7 h(u(\varsigma)) d\varsigma + \sigma \mu_1^3 e^{\sigma \mu_1} \int_{-\mu_1}^0 g^T(v(\varsigma)) \mathcal{Q}_6 g(v(\varsigma)) d\varsigma + \sigma \mu_2^3 \\ &\quad \times e^{\sigma \mu_2} \int_{-\mu_2}^0 h^T(u(\varsigma)) \mathcal{Q}_8 h(u(\varsigma)) d\varsigma + \sigma \mu_1^2 e^{\sigma \mu_1} \int_{-\mu_1}^0 g^T(v(\varsigma)) \\ &\quad \times \mathcal{T}_3 g(v(\varsigma)) d\varsigma + \sigma \mu_2^2 e^{\sigma \mu_2} \int_{-\mu_2}^0 h^T(u(\varsigma)) \mathcal{T}_4 h(u(\varsigma)) d\varsigma \\ &\leq \mathcal{V}(0) - \frac{\delta_1 + \delta_2}{\sigma} + \Omega, \end{aligned} \quad (44)$$

where $\Omega = 2\sigma \eta_1^2 e^{\sigma \eta_1} \int_{-\eta_1}^0 g^T(v(\varsigma)) \mathcal{Q}_5 g(v(\varsigma)) d\varsigma + 2\sigma \eta_2^2 e^{\sigma \eta_2} \int_{-\eta_2}^0 h^T(u(\varsigma)) \mathcal{Q}_7 h(u(\varsigma)) d\varsigma + \sigma \mu_1^3 e^{\sigma \mu_1} \int_{-\mu_1}^0 g^T(v(\varsigma)) \mathcal{Q}_6 g(v(\varsigma)) d\varsigma + \sigma \mu_2^3 e^{\sigma \mu_2} \int_{-\mu_2}^0 h^T(u(\varsigma)) \mathcal{Q}_8 h(u(\varsigma)) d\varsigma + \sigma \mu_1^2 e^{\sigma \mu_1} \int_{-\mu_1}^0 g^T(v(\varsigma)) \mathcal{T}_3 g(v(\varsigma)) d\varsigma + \sigma \mu_2^2 e^{\sigma \mu_2} \int_{-\mu_2}^0 h^T(u(\varsigma)) \mathcal{T}_4 h(u(\varsigma)) d\varsigma$.

Hence,

$$\mathcal{V}(t^*) - \frac{\delta_1 + \delta_2}{\sigma} \leq (\mathcal{V}(0) + \Omega) e^{-\sigma t^*} \quad (45)$$

Since t^* is arbitrary, we compute that

$$\begin{aligned}
[u^T(t)\mathcal{Q}_1u(t) + v^T(t)\mathcal{Q}_3v(t)] - \frac{\delta_1 + \delta_2}{\sigma} &\leq \mathcal{V}(t) - \frac{\delta_1 + \delta_2}{\sigma} \\
\mathcal{V}(u(t), v(t)) - \frac{\delta_1 + \delta_2}{\sigma} &\leq (\mathcal{V}(0) + \Omega)e^{-\sigma t}, t > 0.
\end{aligned} \tag{46}$$

Therefore, by Eq. (46) and Definition 2.6, we conclude that the neural networks (1) is globally robustly exponentially stable in sense of Lagrange, since the inertial neural networks (1) is equivalent to the neural networks (13), and the set Ω^* is a GEA set of Eq. (13).

Remark 3.2. Suppose, the uncertain parameters are not admit in neural networks (13), then one can obtain that the remodified BAM neural networks as follows

$$\begin{aligned}
\mathcal{D}u(t) &= -\Lambda_1u(t) + \alpha(t) \\
\mathcal{D}v(t) &= -\Lambda_2v(t) + \beta(t) \\
\mathcal{D}\alpha(t) &= -\Gamma_1\alpha(t) + \Gamma_2u(t) + \mathcal{A}g(v(t)) + \mathcal{B}g(v(t - \eta_1(t))) + \mathcal{L} \int_{t-\mu_1(t)}^t g(v(s))ds + \hat{J}(t) \\
\mathcal{D}\beta(t) &= -\Gamma_3\beta(t) + \Gamma_4v(t) + \mathcal{P}h(u(t)) + \mathcal{R}h(u(t - \eta_2(t))) + \mathcal{O} \int_{t-\mu_2(t)}^t h(u(s))ds + \hat{I}(t),
\end{aligned} \tag{47}$$

Then by Theorem 3.1, it is easy to have the following Corollary 3.3.

Corollary 3.3. Assume that Assumption 1 holds, the inertial-type BAMNNs (47) is globally exponentially stable in Lagrange sense, if there exist four positive diagonal matrices $\mathcal{L}_1, \mathcal{L}_2, \Xi, \Pi$, positive definite matrices $\mathcal{Q}_l$ ($l = 1, 2, 3, \dots, 14$) and $\mathcal{T}_p$ ($1, 2, 3, 4$) such that the following LMIs are satisfied:

$$\Upsilon^* = \begin{bmatrix} \Upsilon_{11}^* & \Upsilon_{12} & \Upsilon_{13} & 0 & \Upsilon_{15} & \Upsilon_{16} \\ * & \Upsilon_{22}^* & 0 & 0 & 0 & \Upsilon_{26} \\ * & * & \Upsilon_{33}^* & \Upsilon_{34} & \Upsilon_{35} & \Upsilon_{36} \\ * & * & * & \Upsilon_{44}^* & \Upsilon_{45} & 0 \\ * & * & * & * & \Upsilon_{55}^* & 0 \\ * & * & * & * & * & \Upsilon_{66}^* \end{bmatrix} < 0, \tag{48}$$

where

$$\begin{aligned}
\Upsilon_{12} &= 2[\mathcal{Q}_1 + \mathcal{Q}_2\Gamma_2 - \mathcal{W}_3\Pi]; \quad \Upsilon_{44}^* = \mathcal{Q}_{12} + 2\Xi(\mathcal{W}_2 - \mathcal{W}_1) - 2[\Lambda_2\mathcal{Q}_3 - \mathcal{W}_1\Xi\Lambda_2] + \mathcal{K}\mathcal{L}_1\mathcal{K}; \\
\Upsilon_{26} &= -2\Pi\Lambda_1; \quad \Upsilon_{33} = -2\mathcal{Q}_4\Gamma_3; \quad \Upsilon_{15} = 2\eta_2\mathcal{Q}_7\mathcal{A}; \\
\Upsilon_{55}^* &= \mathcal{Q}_{13} + \eta_1\mathcal{Q}_5 + \mu_1^2\mathcal{Q}_6 + \mu_1\mathcal{T}_3 - \mathcal{L}_1, \\
\Upsilon_{45} &= -\Xi\Lambda_2; \quad \Upsilon_{22}^* = \mathcal{Q}_{10} + 2\Pi(\mathcal{W}_4 - \mathcal{W}_3) - 2[\Lambda_1\mathcal{Q}_1 - \mathcal{W}_3\Pi\Lambda_1] + \hat{\mathcal{K}}\mathcal{L}_2\hat{\mathcal{K}}; \\
\Upsilon_{33}^* &= \mathcal{Q}_{11} - 2\mathcal{Q}_4\Gamma_3; \\
\Upsilon_{13} &= 2\mathcal{Q}_4\mathcal{R}[(1 - \tilde{\eta})\mathcal{Q}_7\eta_2]^{-1} + 2\mathcal{Q}_4\mathcal{O}\mathcal{Q}_8^{-1} + 2\mathcal{Q}_4\mathcal{T}_2^{-1}; \\
\Upsilon_{66}^* &= \mathcal{Q}_{14} + \mathcal{Q}_7 + \mu_2^2\mathcal{Q}_8 + \mu_2\mathcal{T}_4 - \mathcal{L}_2; \\
\Upsilon_{11}^* &= \mathcal{Q}_9 - 2\mathcal{Q}_2\Gamma_1 + 2\mathcal{Q}_2\mathcal{B}[(1 - \eta)\mathcal{Q}_5\eta_1]^{-1} + 2\mathcal{Q}_2\mathcal{L}\mathcal{Q}_6^{-1} + 2\mathcal{Q}_2\mathcal{T}_1^{-1}; \quad \Upsilon_{16} = 2\Pi; \\
\Upsilon_{36} &= 2\mathcal{P}\mathcal{Q}_4; \\
\Upsilon_{34} &= 2[\mathcal{Q}_3 + \mathcal{Q}_4\Gamma_4 - \mathcal{W}_1\Xi] \quad \Upsilon_{35} = 4\Xi,
\end{aligned}$$

and the set $\Omega^* = \{(u(t), v(t)) \in \mathbb{R}^n \times \mathbb{R}^n \mid u^T(t)Q_1u(t) + v^T(t)Q_3v(t) \leq \frac{\hat{f}^T T_1 \hat{f} + \hat{f}^T T_2 \hat{f}}{\check{\sigma}}\}$ is a GEA set of Eq. (47), where the positive number $\check{\sigma} = \min\{1, \hat{\sigma}\}$, $\hat{\sigma}$ is the supremum of the solution of the set inequalities $\sigma \lambda_{\max}(Q_1) \leq \lambda_{\min}(Q_{10})$, $\sigma \lambda_{\max}(Q_2) \leq \lambda_{\min}(Q_9)$, $\sigma \lambda_{\max}(Q_3) \leq \lambda_{\min}(Q_{12})$, $\sigma \lambda_{\max}(Q_4) \leq \lambda_{\min}(Q_{11})$, $2\sigma \eta_1 e^{\sigma \eta_1} \lambda_{\max}(Q_5) + \sigma \mu_1^3 e^{\sigma \mu_1} \lambda_{\max}(Q_6) + \sigma \mu_1^2 e^{\sigma \mu_1} \lambda_{\max}(T_3) \leq \lambda_{\min}(Q_{13})$, $2\sigma \eta_2 e^{\sigma \eta_2} \lambda_{\max}(Q_7) + \sigma \mu_2^3 e^{\sigma \mu_2} \lambda_{\max}(Q_8) + \sigma \mu_2^2 e^{\sigma \mu_2} \lambda_{\max}(T_4) \leq \lambda_{\min}(Q_{14})$.

Remark 3.4. In this research work, the relationship between time-varying delays and their maximum allowable upper bounds are fully investigated. For sequence, several inequality techniques are used to attain the maximum upper bound of discrete and distributed time-delays, see Tables 2, 3 and 4 in Examples 4.1 and 4.2. Therefore, this techniques and methods may lead to less conservative criteria.

4. Numerical examples

In this section, we will present two numerical examples with their simulations to guarantee the superiority and validity of our theoretical findings.

Example 4.1. Consider the two dimensional inertial-type uncertain BAM neural networks (13) with the associated parameters as follows:

$$A = \begin{pmatrix} 0.3000 & 0.2000 \\ -0.3000 & 0.4000 \end{pmatrix}, \quad B = \begin{pmatrix} -0.2000 & 0.4000 \\ 0.3000 & 0.2000 \end{pmatrix}, \quad P = \begin{pmatrix} 0.3000 & 0.2000 \\ -0.1000 & 0.4000 \end{pmatrix},$$

$$L = \begin{pmatrix} 0.1000 & 0.5000 \\ -0.4000 & 0.3000 \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} 1.0000 & 0 \\ 0 & 1.5000 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0.7000 & 0 \\ 0 & 0.6000 \end{pmatrix},$$

$$\Gamma_1 = \begin{pmatrix} 1.0000 & 0 \\ 0 & 0.5000 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 1.2000 & 0 \\ 0 & 1.0000 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0.5000 & 0 \\ 0 & 0.3000 \end{pmatrix},$$

$$\Gamma_4 = \begin{pmatrix} 0.8000 & 0 \\ 0 & 0.8000 \end{pmatrix}, \quad K = \begin{pmatrix} 0.3000 & 0 \\ 0 & 0.5000 \end{pmatrix}, \quad \hat{K} = \begin{pmatrix} 0.7000 & 0 \\ 0 & 0.4000 \end{pmatrix},$$

$$\mathcal{N}^* = \begin{pmatrix} 0.5000 & 0.2000 \\ 0.4000 & 0.1000 \end{pmatrix}, \quad \mathcal{X} = \begin{pmatrix} -0.4000 & 0.2000 \\ 0.5000 & -0.2000 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} 0.1000 & 0.1000 \\ 0.4000 & 0.3000 \end{pmatrix}.$$

$$Z_2 = \begin{pmatrix} 0.3000 & 0.2000 \\ -0.3000 & 0.4000 \end{pmatrix}, \quad Z_3 = \begin{pmatrix} 1.2000 & 0.4000 \\ 0.7000 & 0.2000 \end{pmatrix}, \quad Z_4 = \begin{pmatrix} 0.1000 & 0.2000 \\ -0.5000 & 0.4000 \end{pmatrix},$$

$$Z_5 = \begin{pmatrix} 0.3000 & 0.2000 \\ -0.5000 & 0.1000 \end{pmatrix}, \quad \mathcal{H}_1 = \begin{pmatrix} 0.1000 & 0.2000 \\ 0.2000 & 0.2000 \end{pmatrix}, \quad \mathcal{H}_2 = \begin{pmatrix} -0.3000 & 0.2000 \\ -0.3000 & 0.4000 \end{pmatrix},$$

$$\mathcal{H}_3 = \begin{pmatrix} 0.2000 & -0.1000 \\ 0.3000 & 0.3000 \end{pmatrix}, \quad \mathcal{H}_4 = \begin{pmatrix} 0.5000 & 0.2000 \\ 0.2000 & 0.5000 \end{pmatrix}, \quad \mathcal{H}_5 = \begin{pmatrix} 0.4000 & 0.2000 \\ 0.4000 & 0.5000 \end{pmatrix},$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W_1 = \begin{pmatrix} 0.2000 & 0 \\ 0 & 0.2000 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 2.0000 & 0 \\ 0 & 1.5000 \end{pmatrix},$$

$$W_3 = \begin{pmatrix} 0.3000 & 0 \\ 0 & 0.3000 \end{pmatrix}, \quad W_4 = \begin{pmatrix} 1.5000 & 0 \\ 0 & 1.5000 \end{pmatrix}, \quad \mathcal{O} = \begin{pmatrix} 0.4000 & -0.2000 \\ 0.1000 & -0.3000 \end{pmatrix},$$

$$\mathcal{R} = \begin{pmatrix} 0.2000 & -0.5000 \\ 0.2000 & 0.4000 \end{pmatrix}, \quad \mu_1 = \mu_2 = 3.7, \quad \eta = 0.6, \quad \eta_1 = \eta_2 = 4.02, \quad \tilde{\eta} = 0.2,$$

$$\eta_1(t) = \eta_2(t) = 0.5(t) + 0.02\sin(t) + 4, \quad \mu_1(t) = \mu_2(t) = 1.7\sin(4t) + 2.$$

Taking the following activation functions are used in neural network system (13), which are expressed by

$$g(v(t)) = 0.3 * [\tanh(v(t)) + 0.2 * (\tan(v(t)))], \quad h(u(t)) = \tan(u(t)).$$

By solving the LMIs in Theorem 3.1 with the help of the Matlab LMI control toolbox, one can obtain the following feasible solutions as

$$\mathcal{Q}_1 = \begin{pmatrix} 0.4228 & 0.0946 \\ 0.0946 & 0.2600 \end{pmatrix}, \quad \mathcal{Q}_2 = \begin{pmatrix} 0.0985 & -0.0743 \\ -0.0743 & 0.0685 \end{pmatrix}, \quad \mathcal{Q}_3 = \begin{pmatrix} 0.0441 & -0.0112 \\ -0.0112 & 0.0464 \end{pmatrix},$$

$$\mathcal{Q}_4 = \begin{pmatrix} 0.1065 & 0.0379 \\ 0.0379 & 0.1607 \end{pmatrix}, \quad \mathcal{Q}_5 = \begin{pmatrix} 0.3906 & -0.0118 \\ -0.0118 & 0.4651 \end{pmatrix}, \quad \mathcal{Q}_6 = \begin{pmatrix} 0.3640 & -0.0019 \\ -0.0019 & 0.3768 \end{pmatrix},$$

$$\mathcal{Q}_7 = \begin{pmatrix} 0.0342 & -0.0038 \\ -0.0038 & 0.0140 \end{pmatrix}, \quad \mathcal{Q}_8 = \begin{pmatrix} 0.1187 & 0.0068 \\ 0.0068 & 0.2478 \end{pmatrix}, \quad \mathcal{Q}_9 = \begin{pmatrix} 0.1025 & -0.0529 \\ -0.0529 & 0.0652 \end{pmatrix},$$

$$\mathcal{Q}_{10} = \begin{pmatrix} 0.1418 & -0.0391 \\ -0.0391 & 0.1928 \end{pmatrix}, \quad \mathcal{Q}_{11} = \begin{pmatrix} 0.0215 & -0.0009 \\ -0.0009 & 0.0104 \end{pmatrix}, \quad \mathcal{Q}_{12} = \begin{pmatrix} 0.4536 & -0.0347 \\ -0.0347 & 0.4509 \end{pmatrix},$$

$$\mathcal{Q}_{13} = \begin{pmatrix} 0.3640 & -0.0019 \\ -0.0019 & 0.3768 \end{pmatrix}, \quad \mathcal{Q}_{14} = \begin{pmatrix} 0.1890 & 0.0063 \\ 0.0063 & 0.2990 \end{pmatrix}, \quad \mathcal{T}_1 = \begin{pmatrix} 0.3570 & 0 \\ 0 & 0.3570 \end{pmatrix},$$

$$\mathcal{T}_2 = \begin{pmatrix} 0.3570 & 0 \\ 0 & 0.3570 \end{pmatrix}, \quad \mathcal{T}_3 = \begin{pmatrix} 0.3640 & -0.0019 \\ -0.0019 & 0.3768 \end{pmatrix}, \quad \mathcal{T}_4 = \begin{pmatrix} 0.1528 & 0.0068 \\ 0.0068 & 0.2763 \end{pmatrix},$$

$$\mathcal{L}_1 = \begin{pmatrix} 3.7076 & 0 \\ 0 & 3.9573 \end{pmatrix}, \quad \mathcal{L}_2 = \begin{pmatrix} 2.4424 & 0 \\ 0 & 3.1534 \end{pmatrix}, \quad \Xi = \begin{pmatrix} 0.4061 & 0 \\ 0 & 0.8339 \end{pmatrix},$$

$$\Pi = \begin{pmatrix} 0.6489 & 0 \\ 0 & 0.3963 \end{pmatrix}, \quad \check{\sigma} = 0.14, \quad \varepsilon_1 = 0.1375, \quad \varepsilon_2 = 0.2382, \quad \varepsilon_3 = 0.2172, \quad \varepsilon_4 = 0.1250.$$

By solving the LMIs (15) in Theorem 3.1, we can find the solutions with feasibility. That is, all the conditions in Theorem 3.1 are satisfied. Further, the obtained discrete time-varying delay upper bounds of $\eta_1 = \eta_2$ for neural networks (13) are very large when compared with some existing literatures, which are listed in Tables 2 and 3 to demonstrate the less conservativeness of the proposed research method. Therefore, by Theorem 3.1 and Definition 2.6, we conclude that the neural networks (13) is globally robustly exponentially stable in the Lagrange sense and the set $\Omega^* = \{(u(t), v(t)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid 0.4228u_1^2(t) + 0.00894u_1(t)u_2(t) + 0.2600u_2^2(t) +$

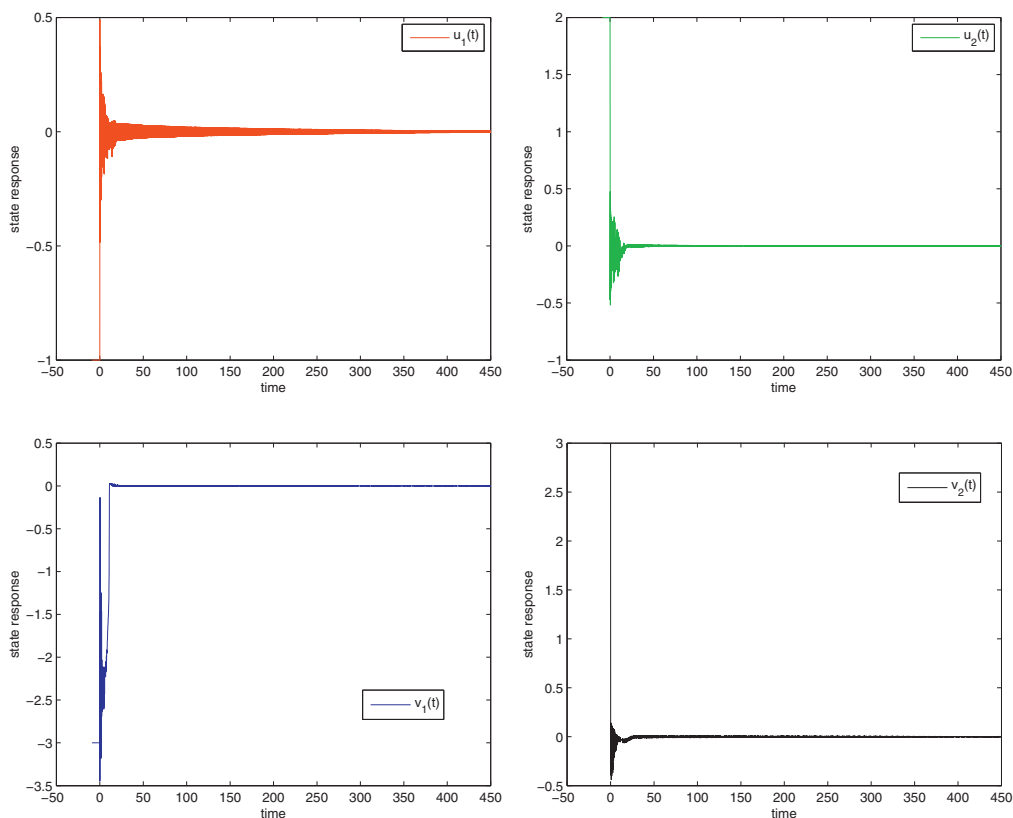


Fig. 1. The state responses of the neural networks (13) in Example 4.1.

$0.0441v_1^2(t) + 0.000125v_1(t)v_2(t) + 0.0464v_2^2(t) \leq 1.82\}$ is a GEA set of Eq. (13). The state responses of the inertial-type BAM neural networks (13) are depicted in Figs. 1 and 2 under the initial conditions as $[u^T(0)v^T(0)] = [26, -15, -25, 17]$. As seen in Figs. 1 and 2, all the system states converge to the equilibrium point, which indicates the sensitivity of neural networks (13).

Example 4.2. Consider the following two dimensional inertial-type BAM neural networks (47) without uncertain parameters. The associated parameters and the activation functions of neural networks (47) are same as in Example 4.1. Then, taking the input scalar values as follows

$$\mu_1 = \mu_2 = 5.91, \quad \eta = 0.5, \quad \eta_1 = \eta_2 = 1.66, \quad \tilde{\eta} = 0.4,$$

$$\eta_1(t) = \eta_2(t) = 1.6 + 0.06\sin(t), \quad \mu_1(t) = \mu_2(t) = 0.8\exp(2t).$$

By utilizing the MATLAB LMI control toolbox the following feasible solutions for the LMIs in Corollary 3.3 obtained as

$$\mathcal{Q}_1 = \begin{pmatrix} 40.7148 & 5.8889 \\ 5.8889 & 22.6268 \end{pmatrix}, \quad \mathcal{Q}_2 = \begin{pmatrix} 13.2929 & -6.8937 \\ -6.8937 & 8.2047 \end{pmatrix}, \quad \mathcal{Q}_3 = \begin{pmatrix} 5.9839 & 0.2462 \\ 0.2462 & 8.3412 \end{pmatrix},$$

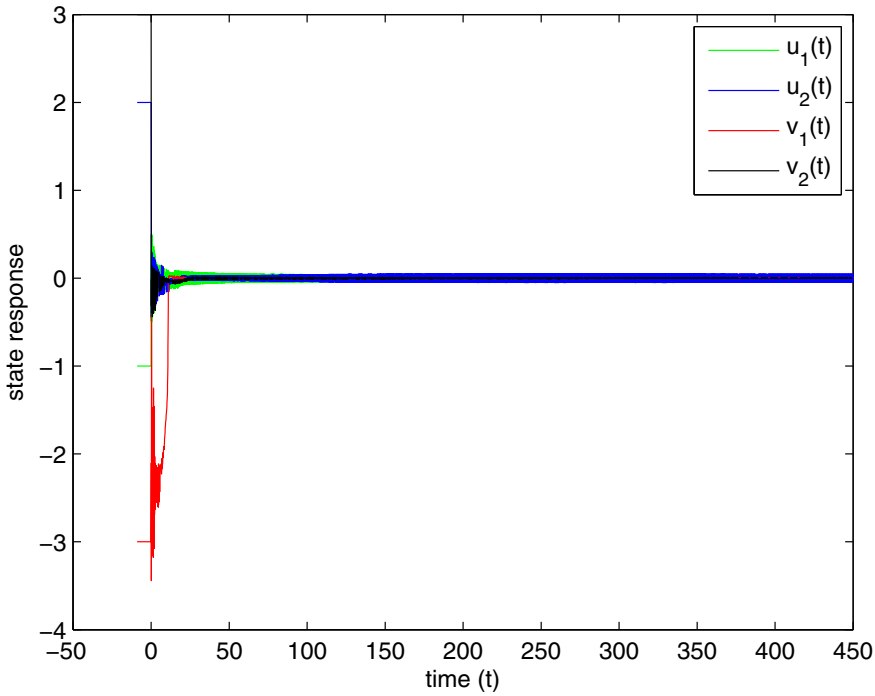


Fig. 2. Trajectory response of state variables $u(t)$ and $v(t)$ of the neural networks (13) in Example 4.1.

$$\begin{aligned}
 Q_4 &= \begin{pmatrix} 16.2366 & -0.7252 \\ -0.7252 & 15.2053 \end{pmatrix}, & Q_5 &= \begin{pmatrix} 48.4271 & 0.2802 \\ 0.2802 & 46.0385 \end{pmatrix}, & Q_6 &= \begin{pmatrix} 49.1771 & 0.3856 \\ 0.3856 & 45.7323 \end{pmatrix}, \\
 Q_7 &= \begin{pmatrix} 4.0942 & -0.6915 \\ -0.6915 & 2.7255 \end{pmatrix}, & Q_8 &= \begin{pmatrix} 9.5696 & -1.4024 \\ -1.4024 & 23.3863 \end{pmatrix}, & Q_9 &= \begin{pmatrix} 18.4333 & -5.4479 \\ -5.4479 & 8.8944 \end{pmatrix}, \\
 Q_{10} &= \begin{pmatrix} 15.5460 & -4.8933 \\ -4.8933 & 19.6965 \end{pmatrix}, & Q_{11} &= \begin{pmatrix} 4.8831 & -0.2589 \\ -0.2589 & 3.0805 \end{pmatrix}, & Q_{12} &= \begin{pmatrix} 54.5461 & 0.7277 \\ 0.7277 & 45.7273 \end{pmatrix}, \\
 Q_{13} &= \begin{pmatrix} 47.1793 & 0.0970 \\ 0.0970 & 46.4302 \end{pmatrix}, & Q_{14} &= \begin{pmatrix} 21.1562 & -1.4664 \\ -1.4664 & 35.6322 \end{pmatrix}, & T_1 &= \begin{pmatrix} 46.5456 & 0 \\ 0 & 46.5456 \end{pmatrix}, \\
 T_2 &= \begin{pmatrix} 46.5456 & 0 \\ 0 & 46.5456 \end{pmatrix}, & T_3 &= \begin{pmatrix} 47.8287 & 0.1936 \\ 0.1936 & 46.2487 \end{pmatrix}, & T_4 &= \begin{pmatrix} 14.7510 & -1.5492 \\ -1.5492 & 30.0138 \end{pmatrix}, \\
 L_1 &= \begin{pmatrix} 551.1488 & 0 \\ 0 & 537.5967 \end{pmatrix}, & L_2 &= \begin{pmatrix} 255.0939 & 0 \\ 0 & 338.1630 \end{pmatrix}, & \Xi &= \begin{pmatrix} 52.3320 & 0 \\ 0 & 91.2100 \end{pmatrix}, \\
 \Pi &= \begin{pmatrix} 67.6852 & 0 \\ 0 & 37.0405 \end{pmatrix}.
 \end{aligned}$$

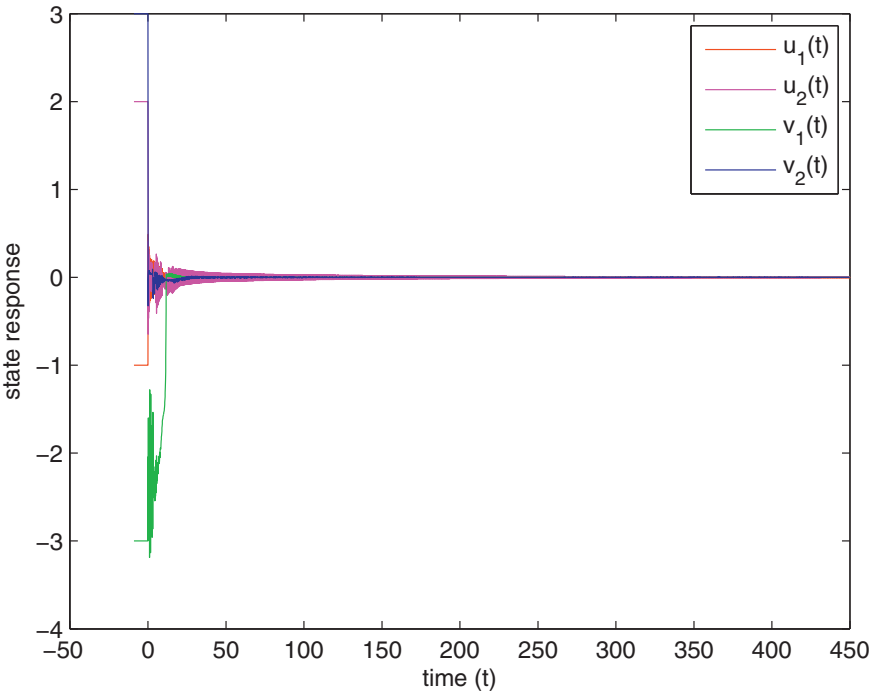


Fig. 3. State curves of neural networks (47) in Example 4.2.

Table 1
Nomenclature and abbreviations.

$\mathbb{R}$	The set of all real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathcal{S}^*$	$\{1, 2, 3, \dots, n\}$
$\mathbb{R}^{n \times n}$	The set of all $n \times n$ matrices with real entries
$\mathcal{D}x$	$\frac{d}{dt}(x)$
$M > N (M \geq N)$	M-N is a positive definite (positive semi definite) matrix
E^T	The transpose of a matrix E
E^{-1}	The inverse of a matrix E
$\begin{bmatrix} P & R \\ * & S \end{bmatrix}$	$\begin{bmatrix} P & R \\ R^T & S \end{bmatrix}$
$diag\{\dots\}$	A block-diagonal matrix
I	The identity matrix with appropriate dimension
BAM	Bi-directional associative memory
BAMNNs	Bi-directional associative memory neural networks
UIBAMNNs	Uncertain inertial-type BAM neural networks
MAUBs	Maximum allowable upper bounds
LKF	Lyapunov–Krasovskii functional
LMI	Linear matrix inequality
LMIs	Linear matrix inequalities
GEA	Globally exponentially attractive
GES	Global exponential stability
GAS	Global asymptotic stability

Therefore, by solving the LMIs (48) in Corollary 3.3, we can find the feasible solutions. In addition, the allowable maximum upper bounds of distributed time-varying delays for neural networks (47) compared with some available research works, see Table 4. Hence, from Corollary 3.3, we can obtain that the system (47) is globally exponentially stable in the Lagrange sense. For the neural networks (47) with the above parameters, we randomly choose the initial value $[u^T(0)v^T(0)] = [12, -8, -20, 16]$, and the simulation result in Fig. 3 shows that the state of the networks approaches the equilibrium point, which matches well with the theoretical findings. Hence, this experiment illustrates the novelty of the designed methods.

5. Conclusions

In this paper, the Lagrange sense robust exponential stability problem has been investigated for UIBAMNNs with mixed time-varying delays. By a key role of the Lyapunov technique, stability analysis procedure, M-matrix theory and Inequality techniques, some new-brand LMI conditions are established to illustrate the proposed neural networks is globally robustly exponentially stable in the Lagrange sense. Moreover, our proposed method provides the allowable upper bounds of the discrete and distributed time-varying delays for a very large value when compared with some available previous literature, which may possess highly important significance in the design of time-delayed addressed neural networks. Finally, two numerical examples with simulations are given to show the effectiveness of the proposed results.

To the best of our knowledge, there are no results on Lagrange sense exponential stability of inertial-type BAM neural networks with mixed-time delays and stochastic noise via non-fragile state estimation approach, which might be our future research work.

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Discrete-time stochastic impulsive BAM neural networks with leakage and mixed time delays: An exponential stability problem[☆]

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Abstract

In this paper, the stability analysis of impulsive discrete-time stochastic BAM neural networks with leakage and mixed time delays is investigated via some novel Lyapunov–Krasovskii functional terms and effective techniques. For the target model, stochastic disturbances are described by Brownian motion. Then the result is further extended to address the problem of robust stability of uncertain discrete-time BAM neural networks. The conditions obtained here are expressed in terms of Linear Matrix Inequalities (LMIs), which can be easily checked by MATLAB LMI control toolbox. Finally, few numerical examples are presented to substantiate the effectiveness of the derived LMI-based stability conditions.

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1. Introduction

Over the past few decades, the research areas such as signal processing, pattern recognition, associative memories and optimization solvers takes a major role in delayed neural

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networks. Above all, quantitative behavior of dynamical applications is an important step for the practical design of neural networks [2]. In practice, the discrete-time neural network models are more applicable in many reality problems. The physical or biological reality of the continuous-time neural networks with mild restrictions can ideally keep the dynamic characteristics functional similarity in discrete-time neural networks. Thus, some stability criteria for discrete-time neural networks have been proposed and its dynamic characteristic have extensively investigated in [17,24,27].

Due to the fruitful and wide applications, the study of bidirectional associative memory (BAM) neural networks have attracted many researchers. Among which, some interesting stability properties of BAM neural networks were considered in [16,18–20]. Kosko [13,14] proposed a BAM neural networks, which is a two layered non-linear feedback network model and it was described that the neurons in one layer always interconnected with neurons in the other layer.

Time delays, which is unavoidable in the implementation of neural networks when considering the finite speed of signal propagation and the finite switching speed of neuron amplifiers [37–43]. According to the way it occurs, time delay can be classified as two types: Discrete and Distributed delays. when the axon size is small, there exists a possibility for delay in the process of storing or sending information (i.e discrete time delay). Generally speaking, if the size of the axon length is much larger, then it is not easier to compute the information size, there occurs delay in the process (i.e. distributed time delay). Neural networks have been assumed to act in continuous time manner. However, in implementing and applications of neural networks, discrete-time system plays an more important role than the continuous-time neural networks, because the computer-based simulation, computation or experimentation is usual to discretize the continuous-time networks [21–23]. Moreover, discrete-time neural networks could be more opted in modeling digitally transmitted signals. Discrete-time delay is easier to be identified in practice and hence the stability analysis for BAM neural networks with discrete delays has been an attractive subject of research in the past few years [1,4,5].

Among the stability of neural networks, systems with leakage delays becomes one of the important topic. While carrying information to the axon by the state vectors, a delay will emerge often called as leakage delay. Leakage delay also called as forgetting delay is the negative term in the state vector which can be drawn back to 1992. The author in [13] noticed that the leakage delay had great impact on neural networks system. Since then, some interesting results have been proposed and much attention have paid to the NNs with leakage delay [25,34]. For instance, Gopalsamy deliberated a population dynamics model with leakage delay in [11] and also found the destabilization of NNs. The author in [12] initially explored the BAM neural networks with leakage delay as follows:

$$\begin{aligned}\frac{dx_i(t)}{dt} &= -a_i x_i(t - \tau_i) + \sum_{j=1}^n a_{ij} f_j(y_j(t - \sigma_j(t))) + I_i \\ \frac{dy_i(t)}{dt} &= -b_i y_i(t - \tau_i) + \sum_{j=1}^n b_{ij} g_j(x_j(t - \sigma_j(t))) + J_i\end{aligned}$$

An abrupt change at certain moments of time is called impulse and this phenomenon is a wide variety of evolutionary process which exists universally [26]. We extend this phenomenon in neural networks. For example, the electrical impulses will be conveyed to the neural net and impulsive phenomenon when a stimulus from the external environment or the body is received by receptors during the implementation of electronic network which is called

impulsive perturbations [27,29,31]. Moreover, impulsive perturbations as well as time delays can affect the dynamical behaviors of neural networks [46,47].

Practically, noises are ubiquitous both in nature and in man-made systems. So, the stochastic influence exists undoubtedly. Therefore, the study of stochastic neural networks is also necessary. In real neural networks, synaptic transmission can be viewed as noisy process introduced by random fluctuations from the release of neuro transmitters and other probabilistic causes [45]. Thus, we can say that the stochastic disturbances also exist in neural networks. However, there has been very little extant works pay attention to the neural networks with time delay in the leakage(or forgetting)term, which owing to some theoretical and technical difficulties [34]. Time delay in leakage term is also has great impact on the dynamics of neural networks [35]. Sometimes, leakage causes instability to the neural networks.

However, in [10], Gao and Cui discussed discrete-time BAM neural networks with time-varying delays leads to robust stability or not. In [17] authors investigated the stability for discrete-time BAM neural networks which is exponential. In [30] authors discussed discrete-time neural networks with stochastic, mixed time delays and impulsive effects. Authors Sun and Zhang investigated impulsive discrete-time BAM neural networks via exponential stability approach in [36]. Dong et al., studied the stability analysis for time-delayed stochastic neural networks in [9]. In [3], global asymptotic stability criterion for recurrent neural networks(RNNs) with multiple time-varying delays in the leakage term was investigated. Also, exponential stability of discrete-time RNNs was analyzed in [16]. From the above analysis, the stability of BAM neural networks is considered only with discrete delay or impulse or leakage delay or stochastic disturbances or uncertain parameters in separate way. But, no one considered all the above factors with distributed time delay in BAM neural networks at that time. So, this motivates our present research work.

Motivated by the above discussions, the exponential stability problem for a new class of impulsive stochastic BAM neural networks with leakage and mixed time delays is considered in this manuscript. By utilizing a Lyapunov–Krasovskii functional and some well known inequalities, a unified linear matrix inequality (LMI) approach is developed to establish sufficient conditions for the target model which ensures the globally exponentially stable. Finally, simulation examples that reveal the effectiveness of the proposed theoretical concepts are given along with the comparative results.

The main contributions of this manuscript lies in four aspects

- Along with mixed time-varying delays, stochastic and constant leakage delays are proposed for the first time in the impulsive discrete-time BAM neural networks to investigate the exponential stability result. Moreover, uncertain discrete-time BAM neural networks are fully addressed with numerical examples.
- With the assistance of Lyapunov–Krasovskii functionals, exponential stability conditions are obtained in terms of LMIs which can be expressed in the example section in the form of feasibility and trajectory with the help of LMI control toolbox in MATLAB software.
- Two special cases of main results are derived in [Corollaries 3.3](#) and [3.4](#) which ensures the advantages of the proposed work.
- From [Table 1](#), one can observe that the allowable upper bounds are large when compared with the existing literature which leads to less conservatism of the designed work.

The rest of this paper can be well organized as follows. In [Section 2](#), we have provided the problem formulation and preliminaries. The essences of the main results are devoted in

Table 1
Comparisons of upper bounds of time delays.

Methods	$\sigma_m = \tau_m$	$\sigma_M = \tau_M$
In Ref. [10]	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M = 4$
In Ref. [23]	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M = 4$
In Ref. [28]	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M = 4$
In Ref. [31]	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M > 0$
In Ref. [32]	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M = 2$
In this paper	$\sigma_m = \tau_m > 0$	$\sigma_M = \tau_M > 0$

Section 3. Uncertain discrete-time neural networks are designed and also the stability results are obtained in **Section 4**. **Section 5** contains some examples with simulation results which helps to illustrate the effectiveness of the proposed method. Finally, conclusions are drawn in **Section 6**.

Notations. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote, respectively, the n -dimensional Euclidean space and the set of all $n \times m$ real matrices. $\mathbb{R}$ be the real numbers and $\mathbb{Z}$ be the positive integers. I denotes the identity matrix with appropriate dimensions and $\text{diag}(\cdot)$ denotes the diagonal matrix. A^T denotes the transpose of matrix A . k denotes the set of positive integers. For real symmetric matrices X and Y , the notation $X \geq Y$ (resp., $X > Y$) means that the matrix $X - Y$ is positive semi definite (resp., positive definite). $\mathbb{N} = \{1, 2, \dots, n\}$ and $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^n$. $\lambda_{\max}(X)$ (resp., $\lambda_{\min}(X)$) stands for the maximum (resp., minimum) eigenvalue of the matrix X . The symbol $*$ within a matrix represents the symmetric term of the matrix.

2. Model description and preliminaries

Consider the following impulsive discrete-time stochastic BAM neural networks with leakage and mixed time delays:

$$\begin{aligned}
 x_i(k+1) = & \left[a_{1i}x_i(k - \xi_1) + \sum_{j=1}^n a_{2ji}\hat{f}_j(y_j(k)) + \sum_{j=1}^n m_{ji}\hat{g}_j(y_j(k - d_1(k))) \right. \\
 & \left. + \sum_{j=1}^n s_{ji} \sum_{s=1}^{\infty} \mu_s \hat{h}_j(y_j(k - s)) + I_i^* \right] + \hat{\delta}_{ji}(x_i(k - \xi_1), \\
 & \times y_j(k - d_1(k)), k) \hat{w}_1(k), \text{ where } k \neq \tau_m, i \in \mathbb{N}; \\
 x_i(\tau_m + 1) = & x_i(\tau_m) + I_{im}(x_i(\tau_m)), \text{ where } k = \tau_m; \\
 y_j(k+1) = & \left[b_{1j}y_j(k - \xi_2) + \sum_{i=1}^n b_{2ij}\hat{f}_i(x_i(k)) + \sum_{i=1}^n n_{ij}\hat{g}_i(x_i(k - d_2(k))) \right. \\
 & \left. + \sum_{i=1}^n w_{ij} \sum_{w=1}^{\infty} \rho_w \hat{h}_i(x_i(k - w)) + J_j^* \right] + \hat{\sigma}_{ij}(y_j(k - \xi_2), \\
 & \times x_i(k - d_2(k)), k) \hat{w}_2(k), \text{ where } k \neq \tau_m, j \in \mathbb{N}; \\
 y_j(\tau_m + 1) = & y_j(\tau_m) + J_{jm}(y_j(\tau_m)), \text{ where } k = \tau_m.
 \end{aligned} \tag{1}$$

or, in a vector form:

$$\begin{aligned}
 x(k+1) = & \left[A_1 x(k - \xi_1) + A_2 \hat{f}(y(k)) + M \hat{g}(y(k - d_1(k))) \right. \\
 & \left. + S \sum_{s=1}^{\infty} \mu_s \hat{h}(y(k-s)) + I^* \right] + \hat{\delta}(x(k - \xi_1), \\
 & \times y(k - d_1(k)), k) \hat{w}_1(k), \text{ where } k \neq \tau_m; \\
 x(\tau_m + 1) = & x(\tau_m) + I_m(x(\tau_m)), \text{ where } k = \tau_m; \\
 y(k+1) = & \left[B_1 y(k - \xi_2) + B_2 \hat{f}(x(k)) + N \hat{g}(x(k - d_2(k))) \right. \\
 & \left. + W \sum_{w=1}^{\infty} \rho_w \hat{h}(x(k-w)) + J^* \right] + \hat{\sigma}(y(k - \xi_2), \\
 & \times x(k - d_2(k)), k) \hat{w}_2(k), \text{ where } k \neq \tau_m; \\
 y(\tau_m + 1) = & y(\tau_m) + J_m(y(\tau_m)), \text{ where } k = \tau_m; \tag{2}
 \end{aligned}$$

for $k = 1, 2, 3, \dots$ and $m \in \mathbb{Z}^+$, where $x(k)$ and $y(k)$ are the neural state vectors, $A_1 = \text{diag}\{a_1, a_2, \dots, a_n\}$, $|a_i| < 1$ and $B_1 = \text{diag}\{b_1, b_2, \dots, b_n\}$, $|b_j| < 1$ are the state feedback coefficient matrices; $A_2 = [a_{ij}]_{n \times n}$, $B_2 = [b_{ij}]_{n \times n}$, $M = [m_{ij}]_{n \times n}$, $N = [n_{ij}]_{n \times n}$, $S = [s_{ij}]_{n \times n}$ and $W = [w_{ij}]_{n \times n}$ are respectively, the connection weight matrices, the discretely delayed connection weight matrices and distributed delayed connection weight matrices. $d_1(k)$ and $d_2(k)$ denote discrete time-varying delays satisfies $d_{1m} \leq d_1(k) \leq d_{1M}$, $d_{2m} \leq d_2(k) \leq d_{2M}$ where $d_{1m}, d_1(k), d_{1M}, d_{2m}, d_2(k), d_{2M}$ are known positive integers; S and W denote the distributed time-varying delays. Then $\hat{f}(y(k)) = [\hat{f}_1(y_1(k)), \hat{f}_2(y_2(k)), \dots, \hat{f}_n(y_n(k))]^T$, $\hat{g}(y(k)) = [\hat{g}_1(y_1(k)), \hat{g}_2(y_2(k)), \dots, \hat{g}_n(y_n(k))]^T$, $\hat{h}(y(k)) = [\hat{h}_1(y_1(k)), \hat{h}_2(y_2(k)), \dots, \hat{h}_n(y_n(k))]^T$, $\hat{f}(y(k)) = [\hat{f}_1(y_1(k)), \hat{f}_2(y_2(k)), \dots, \hat{f}_n(y_n(k))]^T$, $\hat{g}(y(k)) = [\hat{g}_1(y_1(k)), \hat{g}_2(y_2(k)), \dots, \hat{g}_n(y_n(k))]^T$, $\hat{h}(y(k)) = [\hat{h}_1(y_1(k)), \hat{h}_2(y_2(k)), \dots, \hat{h}_n(y_n(k))]^T$ denotes the neuron activation functions. The leakage delays ξ_1 and ξ_2 are constants. τ_m are the impulse instants satisfying $0 = \tau_0 < \tau_1 < \dots, \tau_m < \tau_{m+1} < \dots$ and $\tau_{m+1} - \tau_m > 1$; $I_m, J_m : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ($m = 1, 2, \dots$) denotes a sequence of jump operators, The constant vectors $J = [J_1, J_2, \dots, J_n]^T$ and $I = [I_1, I_2, \dots, I_n]^T$ are external input from outside the system; μ_s ($s = 1, 2, \dots$) and ρ_w ($w = 1, 2, \dots$) are scalar constants, where $w_1(k)$ and $w_2(k)$ wiener process on probability space $[\Omega, \mathcal{F}, \beta]$ with

$$\begin{aligned}
 \mathbb{E}[w_1(k)] &= 0, \quad \mathbb{E}[w_1^2(k)] = 1, \quad \mathbb{E}[w_1(i)w_1(j)] = 0 \quad (i \neq j), \\
 \mathbb{E}[w_2(k)] &= 0, \quad \mathbb{E}[w_2^2(k)] = 1, \quad \mathbb{E}[w_2(i)w_2(j)] = 0 \quad (i \neq j), \tag{3}
 \end{aligned}$$

with $\mathbb{E}(\cdot)$ being the mathematical expectation operator, $\hat{\delta} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ and $\hat{\sigma} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ are non-linear vector functions representing the disturbance intensities.

We make the following assumptions for neuron activation functions.

Assumption I. For $i, j \in \{1, 2, \dots, n\}$ the neuron activation functions $\hat{f}_j(\cdot)$, $\hat{f}_i(\cdot)$, $\hat{g}_j(\cdot)$, $\hat{g}_i(\cdot)$, $\hat{h}_j(\cdot)$, $\hat{h}_i(\cdot)$ in Eq. (2) are continuous on $\mathbb{R}$.

Assumption II. For $i, j \in \{1, 2 \dots n\}$ the neuron activation functions in Eq. (2) satisfy

$$\begin{aligned} l_j^- &\leq \frac{\hat{f}_j(p_1) - \hat{f}_j(p_2)}{p_1 - p_2} \leq l_j^+, \forall p_1, p_2 \in \mathbb{R}, & u_i^- &\leq \frac{\hat{f}_i(q_1) - \hat{f}_i(q_2)}{q_1 - q_2} \leq u_i^+, \forall q_1, q_2 \in \mathbb{R}, \\ m_j^- &\leq \frac{\hat{g}_j(p_1) - \hat{g}_j(p_2)}{p_1 - p_2} \leq m_j^+, \forall p_1, p_2 \in \mathbb{R}, & n_i^- &\leq \frac{\hat{g}_i(q_1) - \hat{g}_i(q_2)}{q_1 - q_2} \leq n_i^+, \forall q_1, q_2 \in \mathbb{R}, \\ s_j^- &\leq \frac{\hat{h}_j(p_1) - \hat{h}_j(p_2)}{p_1 - p_2} \leq s_j^+, \forall p_1, p_2 \in \mathbb{R}, & w_i^- &\leq \frac{\hat{h}_i(q_1) - \hat{h}_i(q_2)}{q_1 - q_2} \leq w_i^+, \forall q_1, q_2 \in \mathbb{R}, \end{aligned} \quad (4)$$

where $l_j^-, l_j^+, m_j^-, m_j^+, s_j^-, s_j^+, u_i^-, u_i^+, n_i^-, n_i^+, w_i^-, w_i^+$ are some constants.

Remark 2.1. Non-monotonic functions can be more suitable to describe the neuron activation functions. Because the input-output functions of amplifiers may be neither monotonically increasing nor continuously differentiable in many electronic circuits. The constants in Assumption II are positive, negative or zero. so, activation function may be non monotonic and more widespread than usual sigmoid functions and Lipschitz functions. such conditions are discourteous in qualifying the lower and upper bounds of the activation functions, Therefore, we use generalized activation functions, Because it is considerate for using LMI-based technique to reduce the possible conservatism.

In order to guarantee the global exponential stability of the system (2), we assume that the system tending to its equilibrium point and the stochastic noise contribution vanishes (i.e.)

Assumption A. $\hat{\delta}(x, y, k) = 0$ and $\hat{\sigma}(y, x, k) = 0$.

For such deterministic BAM neural networks, we have the following system of equations:

$$\begin{aligned} x(k+1) &= \left[A_1 x(k - \xi_1) + A_2 \hat{f}(y(k)) + M \hat{g}(y(k - d_1(k))) \right. \\ &\quad \left. + S \sum_{s=1}^{\infty} \mu_s \hat{h}(y(k - s)) + I^* \right], \text{ where } k \neq \tau_m; \\ x(\tau_m + 1) &= x(\tau_m) + I_m(x(\tau_m)), \text{ where } k = \tau_m; \\ y(k+1) &= \left[B_1 y(k - \xi_2) + B_2 \hat{f}(x(k)) + N \hat{g}(x(k - d_2(k))) \right. \\ &\quad \left. + W \sum_{w=1}^{\infty} \rho_w \hat{h}(x(k - w)) + J^* \right], \text{ where } k \neq \tau_m; \\ y(\tau_m + 1) &= y(\tau_m) + J_m(y(\tau_m)), \text{ where } k = \tau_m; \end{aligned} \quad (5)$$

For the deterministic stochastic BAM neural network system, We shift the equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ and $y^* = (y_1^*, y_2^*, \dots, y_n^*)^T$ of (2) to be origin. Let $u(k) = x(k) - x^*$ and $v(k) = y(k) - y^*$, then the system (2) can be transformed to:

$$\begin{aligned} u(k+1) &= \left[A_1 u(k - \xi_1) + A_2 f(v(k)) + M g(v(k - d_1(k))) + S \sum_{s=1}^{\infty} \mu_s h(v(k - s)) \right] \\ &\quad + \delta(u(k - \xi_1), v(k - d_1(k)), k) \bar{w}_1(k), \text{ } k \neq \tau_m; \end{aligned}$$

$$u(\tau_m + 1) = u(\tau_m) + I_m(u(\tau_m)), \quad k = \tau_m;$$

$$v(k + 1) = \left[B_1 v(k - \xi_2) + B_2 \bar{f}(u(k)) + N \bar{g}(u(k - d_2(k))) + W \sum_{w=1}^{\infty} \rho_w \bar{h}(u(k - w)) \right] \\ + \bar{\sigma}(v(k - \xi_2), u(k - d_2(k)), k) \bar{w}_2(k), \quad k \neq \tau_m;$$

$$v(\tau_m + 1) = v(\tau_m) + I_m(v(\tau_m)), \quad k = \tau_m; \quad (6)$$

where

$$\begin{aligned} u(k) &= (u_1(k), u_2(k), \dots, u_n(k))^T, \\ v(k) &= (v_1(k), v_2(k), \dots, v_n(k))^T, \\ u(k - \xi_1) &= (u_1(k - \xi_1), u_2(k - \xi_1), \dots, u_n(k - \xi_1))^T, \\ v(k - \xi_2) &= (v_1(k - \xi_2), v_2(k - \xi_2), \dots, v_n(k - \xi_2))^T, \\ f(v(k)) &= [f_1(v_1(k)), f_2(v_2(k)), \dots, f_n(v_n(k))]^T, \\ \bar{f}(u(k)) &= [\bar{f}_1(u_1(k)), \bar{f}_2(u_2(k)), \dots, \bar{f}_n(u_n(k))]^T, \\ g(v(k - d_1(k))) &= [g_1(v_1(k - d_1(k))), g_2(v_2(k - d_1(k))), \dots, \\ &\quad g_n(v_n(k - d_1(k)))]^T, \\ \bar{g}(u(k - d_2(k))) &= [\bar{g}_1(u_1(k - d_2(k))), \bar{g}_2(u_2(k - d_2(k))), \dots, \\ &\quad \bar{g}_n(u_n(k - d_2(k)))]^T, \\ h(v(k)) &= [h_1(v_1(k)), \bar{h}_2(v_2(k)), \dots, \bar{h}_n(v_n(k))]^T, \\ \bar{h}(u(k)) &= [\bar{h}_1(u_1(k)), \bar{h}_2(u_2(k)), \dots, \bar{h}_n(u_n(k))]^T, \\ f(v(k)) &= f(y(k) + y^*) - f(y^*), \\ \bar{f}(u(k)) &= \bar{f}(x(k) + x^*) - \bar{f}(x^*), \\ g(v(k - d_1(k))) &= g(y(k - d_1(k))) + y^* - g(y^*), \\ \bar{g}(u(k - d_2(k))) &= \bar{g}_j(x(k - d_2(k)) + x^*) - \bar{g}(x^*), \\ h(v(k)) &= h(y(k) + y^*) - h(y^*), \\ \bar{\delta}(u(k - \xi_1), v(k), v(k - d_1), k) &= \delta(u(k - \xi_1) + x^*, v(k) + y^*, v(k - d_1(k)) + y^*, k), \\ \bar{\sigma}(v(k - \xi_2), u(k), u(k - d_2), k) &= \sigma(v(k - \xi_2) + y^*, u(k) + x^*, u(k - d_2(k)) + x^*, k), \\ \bar{\delta}(u(k - \xi_1), v(k), v(k - d_1), k) &= (\bar{\delta}_1(u_1(k - \xi_1), v_1(k), v_1(k - d_1(k)), k), \dots, \\ &\quad \bar{\delta}_n(u_n(k - \xi_1), v_n(k), v_n(k - d_1(k)), k))^T, \\ \bar{\sigma}(v(k - \xi_2), u(k), u(k - d_2), k) &= (\bar{\sigma}_1(k_1(k - \xi_2), u_1(k), u_1(k - d_2(k)), k), \dots, \\ &\quad \bar{\sigma}_n(v_n(k - \xi_2), u_n(k), u_n(k - d_2(k)), k))^T. \end{aligned}$$

Assumption III. Obviously the activation functions $f_i, \bar{f}_j, g_i, \bar{g}_j, h_i, \bar{h}_j (i, j \in \mathbb{N})$ satisfy the following conditions:

$$\begin{aligned} l_i^- &\leq \frac{f_i(p)}{p} \leq l_i^+, & u_i^- &\leq \frac{\bar{f}_j(q)}{q} \leq u_j^+, \\ m_i^- &\leq \frac{g_i(p)}{p} \leq m_i^+, & n_i^- &\leq \frac{\bar{g}_j(q)}{q} \leq n_j^+, \\ s_i^- &\leq \frac{h_i(p)}{p} \leq s_i^+, & w_i^- &\leq \frac{\bar{h}_j(q)}{q} \leq w_j^+. \end{aligned} \quad (7)$$

Assumption IV. The constants $\mu_s, \rho_w \geq 0$ satisfy the following conditions:

$$\begin{aligned} \sum_{s=1}^{\infty} \mu_s < +\infty, \quad \sum_{s=1}^{\infty} s\mu_s < +\infty, \\ \sum_{w=1}^{\infty} \rho_w < +\infty, \quad \sum_{w=1}^{\infty} w\rho_w < +\infty. \end{aligned} \quad (8)$$

Remark 2.2. Above assumption assures that the terms $S \sum_{s=1}^{\infty} \mu_s h(v(k-s))$ and $W \sum_{w=1}^{\infty} \rho_w \bar{h}(u(k-w))$ are convergent and bounded.

Assumption V. There exists constant matrices G and K such that

$$\begin{aligned} \delta^T(x, y, k) \delta^T(x, y, k) &\leq |Gx|^2, \quad \forall x, y \in \mathbb{R}^n, \\ \bar{\sigma}^T(x, y, k) \bar{\sigma}^T(x, y, k) &\leq |Ky|^2, \quad \forall x, y \in \mathbb{R}^n. \end{aligned} \quad (9)$$

Assumption VI. There exist constants $h_i^-, h_i^+ (i = 1, 2, \dots, n)$ and $h_j^-, h_j^+ (j = 1, 2, \dots, n)$ such that

$$\begin{aligned} h_i^- &\leq \frac{I_{im}(\alpha_1) - I_{im}(\alpha_2)}{\alpha_1 - \alpha_2} \leq h_i^+, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}, \alpha_1 \neq \alpha_2, r = 1, 2, \dots; \\ h_j^- &\leq \frac{J_{jm}(\alpha_1) - J_{jm}(\alpha_2)}{\alpha_1 - \alpha_2} \leq h_j^+, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}, \alpha_1 \neq \alpha_2, r = 1, 2, \dots \end{aligned} \quad (10)$$

The following definition and lemmas will be vital in engaging the exponential stability conditions.

Definition 2.3. The delayed discrete-time impulsive stochastic BAM neural networks (5) is said to be globally exponentially stable, if there exists two positive scalars $\gamma \geq 0$ and $\eta \geq 0$ such that

$$\mathbb{E}\{\|u(k)\|^2 + \|v(k)\|^2\} \leq \gamma e^{-\eta k} \left[\sup_{-d_{1M} \leq c \leq 0} \mathbb{E}\{\|u(c)\|^2\} + \sup_{-d_{2M} \leq c \leq 0} \mathbb{E}\{\|v(c)\|^2\} \right]. \quad (11)$$

Lemma 2.4. Ren and Cao [33] Let X and Y be any n -dimensional real vectors and let P be any n -dimensional real vectors and let P be any $n \times n$ positive semidefinite matrix. Then the following inequality holds:

$$2X^T PY \leq X^T PX + Y^T PY.$$

Lemma 2.5. (Khargonekar et al. [15]) Let $S \in \mathbb{R}^n$ be a positive semi definite matrix, $x_i \in \mathbb{R}^n$ and $a_i \geq 0, (i=1,2,\dots)$.

If series concerned are convergent, the following inequality holds:

$$\left(\sum_{i=1}^{\infty} a_i x_i \right)^T S \left(\sum_{i=1}^{\infty} a_i x_i \right) \leq \sum_{i=1}^{\infty} a_i \sum_{i=1}^{\infty} a_i x_i^T S x_i.$$

Lemma 2.6. (Khargonekar et al. [15]). Let P, Q, R and F be a real matrices of appropriate dimensions with P satisfying $P^T = P$, then

$$P + QFR + R^T F^T Q^T < 0, F^T F \leq I$$

if and only if there exists a scalar $\epsilon > 0$ such that

$$P + \epsilon^{-1} QQ^T + \epsilon RR^T < 0.$$

3. Main results

Theorem 3.1. Under *Assumptions I and II*, the solution of neural networks (5) exists on $(0, +\infty) \times (0, +\infty)$.

Proof. Since the activation functions are bounded, there exists a constant $\mathcal{T} > 0$ such that $|\hat{f}(a^*)| \leq \mathcal{T}$, $|\hat{f}(b^*)| \leq \mathcal{T}$, $|\hat{g}(a^*)| \leq \mathcal{T}$, $|\hat{g}(b^*)| \leq \mathcal{T}$, $|\hat{h}(a^*)| \leq \mathcal{T}$, $|\hat{h}(b^*)| \leq \mathcal{T}$, $\forall a^*, b^* \in \mathbb{R}^n$. $\square$

It follows from $|a_i| < 1$, $|b_j| < 1$ that $I - A_1$ and $I - B_1$ are invertible.

Denote $\mathcal{D}(s) = \{(a^*, b^*) \in C((0, +\infty), \mathbb{R}^n) \times C((0, +\infty), \mathbb{R}^n) / |(a^*, b^*)| \leq s\}$ with $s = \|(I - A_1)^{-1}\|(\|A_2\|\mathcal{T} + \|M\|\mathcal{T} + \|S\|\|\mathfrak{S}_1\|\mathcal{T} + |I^*| + \|(I - B_1)^{-1}\|(\|B_2\|\mathcal{T} + \|N\|\mathcal{T} + \|W\|\|\mathfrak{S}_2\|\mathcal{T} + |J^*|)$, and define the map $\mathcal{H} : \mathcal{D} \rightarrow \mathbb{R}^n$ by

$$\begin{aligned} \mathcal{H}((a^*, b^*)) = & (I - A_1)^{-1} \left(A_2 \hat{f}(b^*) + M \hat{g}(b^*) + S \sum_{s=1}^{\infty} \mu_s \hat{h}(b^*) + I^* \right) \\ & + (I - B_1)^{-1} \left(B_2 \hat{f}(a^*) + N \hat{g}(a^*) + W \sum_{w=1}^{\infty} \rho_w \hat{h}(a^*) + J^* \right) \end{aligned}$$

Obviously, $\mathcal{H}$ is continuous map and it follows that

$$\begin{aligned} |\mathcal{H}(a^*, b^*)| & \leq \|(I - A_1)^{-1}\| \left(\|A_2\| |\hat{f}(b^*)| + \|M\| |\hat{g}(b^*)| + \|S\| \left| \sum_{s=1}^{\infty} \mu_s |\hat{h}(b^*)| + |I^*| \right| \right) \\ & + \|(I - B_1)^{-1}\| \left(\|B_2\| |\hat{f}(a^*)| + \|N\| |\hat{g}(a^*)| + \|W\| \left| \sum_{w=1}^{\infty} \rho_w |\hat{h}(a^*)| + |J^*| \right| \right) \\ & \leq \|(I - A_1)^{-1}\| (\|A_2\|\mathcal{T} + \|M\|\mathcal{T} + \|S\|\|\mathfrak{S}_1\|\mathcal{T} + |I^*|) \\ & + \|(I - B_1)^{-1}\| (\|B_2\|\mathcal{T} + \|N\|\mathcal{T} + \|W\|\|\mathfrak{S}_2\|\mathcal{T} + |J^*|) \\ & \leq s \end{aligned}$$

Hence, $\mathcal{H}$ maps $\mathcal{D}$ of s into itself (i.e) $\mathcal{H} : \mathcal{D}(s) \rightarrow \mathcal{D}(s)$.

By Brower's fixed point theorem, one can infer that there exists a fixed point $(\hat{a}^*, \hat{b}^*)$ of $\mathcal{H}$.

Namely,

$$\begin{aligned} (I - A_1)^{-1} \left(A_2 \hat{f}(b^*) + M \hat{g}(b^*) + S \sum_{s=1}^{\infty} \mu_s \hat{h}(b^*) + I^* \right) & = \hat{a}^*, \\ (I - B_1)^{-1} \left(B_2 \hat{f}(a^*) + N \hat{g}(a^*) + W \sum_{w=1}^{\infty} \rho_w \hat{h}(a^*) + J^* \right) & = \hat{b}^*. \end{aligned}$$

are equivalently

$$\begin{aligned} \hat{a}^* & = A_1 \hat{a}^* + A_2 \hat{f}(\hat{b}^*) + M \hat{g}(\hat{b}^*) + S \sum_{s=1}^{\infty} \mu_s \hat{h}(\hat{b}^*) + I^* \\ \hat{b}^* & = I - B_1 \hat{b}^* + B_2 \hat{f}(\hat{a}^*) + N \hat{g}(\hat{a}^*) + W \sum_{w=1}^{\infty} \rho_w \hat{h}(\hat{a}^*) + J^* \end{aligned}$$

Thus, it is easy to verify that $(\hat{a}^*, \hat{b}^*)$ is the solution of neural networks (5) defined on $(0, +\infty) \times (0, +\infty)$.

Now we are ready to state our main results of this paper in the following theorems.

Theorem 3.2. Under Assumptions I–VI the system (6) is globally exponentially stable in the mean square, if there exists constant $\lambda_0 > 0$ and $\theta_0 > 0$ if there exists a diagonal matrices $\Lambda_1 = \text{diag}\{\lambda_1^{(1)}, \lambda_2^{(1)}, \dots, \lambda_n^{(1)}\} > 0$ and $\Lambda_2 = \text{diag}\{\lambda_1^{(2)}, \lambda_2^{(2)}, \dots, \lambda_n^{(2)}\} > 0$, $\Gamma_1 = \text{diag}\{\gamma_1^{(1)}, \gamma_2^{(1)}, \dots, \gamma_n^{(1)}\} > 0$ and $\Gamma_2 = \text{diag}\{\gamma_1^{(2)}, \gamma_2^{(2)}, \dots, \gamma_n^{(2)}\} > 0$, $\Omega_1 = \text{diag}\{\omega_1^{(1)}, \omega_2^{(1)}, \dots, \omega_n^{(1)}\} > 0$ and $\Omega_2 = \text{diag}\{\omega_1^{(2)}, \omega_2^{(2)}, \dots, \omega_n^{(2)}\} > 0$ and positive definite $P_1 > 0$, $P_2 > 0$ and $Q_1 > 0$ and $Q_2 > 0$, $R_1 > 0$, $R_2 > 0$ such that following LMIs hold:

$$P_1 < \lambda_0 I$$

$$P_2 < \theta_0 I$$

$$\Xi_1 = \begin{pmatrix} \pi_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \pi_{22} & 0 & \Lambda_2 U_2 & \Gamma_2 N_2 & 0 & \Omega_2 W_2 & 0 \\ * & * & -Q_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \pi_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Gamma_2 & 0 & 0 & 0 \\ * & * & * & * & * & \pi_{66} & 0 & 0 \\ * & * & * & * & * & * & -\Omega_2 & 0 \\ * & * & * & * & * & * & * & \pi_{88} \end{pmatrix} < 0 \quad (12)$$

$$\Xi_2 = \begin{pmatrix} \Theta_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \Theta_{22} & 0 & \Lambda_1 L_2 & \Gamma_1 S_2 & 0 & \Omega_1 M_2 & 0 \\ * & * & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Theta_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Gamma_1 & 0 & 0 & 0 \\ * & * & * & * & * & \Theta_{66} & 0 & 0 \\ * & * & * & * & * & * & -\Omega_1 & 0 \\ * & * & * & * & * & * & * & \Theta_{88} \end{pmatrix} < 0 \quad (13)$$

$$\Xi_3 = \begin{pmatrix} \phi_{11} & 0 & \Lambda_2 U_2 & \Gamma_2 N_2 & 0 & \Omega_2 W_2 & 0 \\ * & -Q_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & \phi_{33} & 0 & 0 & 0 & 0 \\ * & * & * & -\Gamma_2 & 0 & 0 & 0 \\ * & * & * & * & P_2 & 0 & 0 \\ * & * & * & * & * & -\Omega_2 & 0 \\ * & * & * & * & * & * & \phi_{77} \end{pmatrix} < 0 \quad (14)$$

$$\Xi_4 = \begin{pmatrix} \psi_{11} & 0 & \Lambda_1 L_2 & \Gamma_1 S_2 & 0 & \Omega_1 M_2 & 0 \\ * & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & \psi_{33} & 0 & 0 & 0 & 0 \\ * & * & * & -\Gamma_1 & 0 & 0 & 0 \\ * & * & * & * & P_1 & 0 & 0 \\ * & * & * & * & * & -\Omega_1 & 0 \\ * & * & * & * & * & * & \psi_{77} \end{pmatrix} < 0 \quad (15)$$

where

$$\begin{aligned}
 \pi_{11} &= A_1^T P_1 A_1 - 2P_1 + \Lambda_0 G^T G, \\
 \pi_{22} &= (1 + d_{2M} - d_{2m})Q_2 - \Lambda_2 U_1 - \Gamma_2 N_1 - \Omega_2 W_1 + \bar{\rho} R_2, \\
 \pi_{44} &= B_1 B_2 P_1 B_2^T B_1^T - \Lambda_2 + 2P_2, \\
 \pi_{66} &= B_1 N P_2 N^T B_1^T + B_2 N P_2 B_2^T + P_2, \\
 \pi_{88} &= B_1 W P_2 W^T B_1^T + B_2 W P_2 W^T B_2^T + N W P_2 W^T N^T - \rho^{-1} R_2, \\
 \Theta_{11} &= B_1^T P_2 B_1 - 2P_2 + \theta_0 K^T K, \\
 \Theta_{22} &= (1 + d_{1M} - d_{1m})Q_1 - \Lambda_1 L_1 - \Gamma_1 S_1 - \Omega_1 M_1 + \bar{\mu} R_1, \\
 \Theta_{44} &= A_1 A_2 P_2 A_2^T A_1^T - \Lambda_1 + 2P_1, \\
 \Theta_{66} &= A_1 M P_1 M^T A_1^T + A_2 M P_1 M^T A_2^T + P_1, \\
 \Theta_{88} &= A_1 S P_1 S^T A_1^T + A_2 S P_1 S^T A_2^T + M S P_1 S^T M^T - \mu^{-1} R_1, \\
 \Phi_{11} &= H_1 P_1 + P_1^T H_1 + H_2 P_1 H_2 - \Lambda_2 U_1 - \Gamma_2 N_1 - \Omega_2 W_1 + (1 + d_{2M} - d_{2m})Q_2 \\
 &\quad \bar{\rho} R_2 + \Lambda_0 G^T G, \\
 \Phi_{33} &= -\Lambda_2, \\
 \Phi_{77} &= -\rho^{-1} R_2, \\
 \psi_{11} &= H_3 P_2 + P_2^T H_3 + H_4 P_2 H_4 - \Lambda_1 L_1 - \Gamma_1 S_1 - \Omega_1 M_1 + (1 + d_{1M} - d_{1m})Q_1 \\
 &\quad + \bar{\mu} R_1 + \theta_0 K^T K, \\
 \psi_{33} &= -\Lambda_1, \\
 \psi_{77} &= -\mu^{-1} R_1, \\
 L_1 &= \text{diag}\{l_1^+ l_1^-, l_2^+ l_2^-, \dots, l_n^+ l_n^-\}, \quad L_2 = \text{diag}\left\{\frac{l_1^+ + l_1^-}{2}, \frac{l_2^+ + l_2^-}{2}, \dots, \frac{l_n^+ + l_n^-}{2}\right\}, \\
 M_1 &= \text{diag}\{m_1^+ m_1^-, m_2^+ m_2^-, \dots, m_n^+ m_n^-\}, \quad M_2 = \text{diag}\left\{\frac{m_1^+ + m_1^-}{2}, \frac{m_2^+ + m_2^-}{2}, \dots, \frac{m_n^+ + m_n^-}{2}\right\}, \\
 S_1 &= \text{diag}\{s_1^+ s_1^-, s_2^+ s_2^-, \dots, s_n^+ s_n^-\}, \quad S_2 = \text{diag}\left\{\frac{s_1^+ + s_1^-}{2}, \frac{s_2^+ + s_2^-}{2}, \dots, \frac{s_n^+ + s_n^-}{2}\right\}, \\
 U_1 &= \text{diag}\{u_1^+ u_1^-, u_2^+ u_2^-, \dots, u_n^+ u_n^-\}, \quad U_2 = \text{diag}\left\{\frac{u_1^+ + u_1^-}{2}, \frac{u_2^+ + u_2^-}{2}, \dots, \frac{u_n^+ + u_n^-}{2}\right\}, \\
 N_1 &= \text{diag}\{n_1^+ n_1^-, n_2^+ n_2^-, \dots, n_n^+ n_n^-\}, \quad N_2 = \text{diag}\left\{\frac{n_1^+ + n_1^-}{2}, \frac{n_2^+ + n_2^-}{2}, \dots, \frac{n_n^+ + n_n^-}{2}\right\}, \\
 W_1 &= \text{diag}\{w_1^+ w_1^-, w_2^+ w_2^-, \dots, w_n^+ w_n^-\}, \quad W_2 = \text{diag}\left\{\frac{w_1^+ + w_1^-}{2}, \frac{w_2^+ + w_2^-}{2}, \dots, \frac{w_n^+ + w_n^-}{2}\right\}, \\
 H_1 &= \text{diag}(h_1^+, h_2^+, \dots, h_n^+) = H_3. \\
 H_2 &= \text{diag}(\max\{|h_1^-|, |h_1^+|\}, \max\{|h_2^-|, |h_2^+|\}, \dots, \max\{|h_n^-|, |h_n^+|\}) = H_4, \\
 \bar{\mu} &= \sum_{k=1}^{\infty} \mu_k, \quad \bar{\rho} = \sum_{k=1}^{\infty} \rho_k.
 \end{aligned}$$

Proof. Let us choose the following Lyapunov–Krasovskii functional

$$\begin{aligned} V_1(k) &= e^{\eta k} \left[u^T(k - \xi_1) P_1 u(k - \xi_1) + v^T(k - \xi_2) P_2 v(k - \xi_2) \right], \\ V_2(k) &= e^{\eta k} \left[\sum_{i=k-d_1(k)}^{k-1} v^T(i) Q_1 v(i) + \sum_{i=k-d_2(k)}^{k-1} u^T(i) Q_2 u(i) \right], \\ V_3(k) &= e^{\eta k} \left[\sum_{j=k-d_{1M}}^{k-d_{1m}} \sum_{i=j}^{k-1} v^T(i) Q_1 v(i) + \sum_{j=k-d_{2M}+1}^{k-d_{2m}} \sum_{i=j}^{k-1} u^T(i) Q_2 u(i) \right], \\ V_4(k) &= e^{\eta k} \left[\sum_{i=j}^{\infty} \mu_i \sum_{j=k-i}^{k-1} v^T(i) R_1 v(i) + \sum_{i=1}^{k-1} \rho_i \sum_{j=k-i}^{k-1} u^T(i) R_2 u(i) \right]. \end{aligned} \quad (16)$$

Then, calculating differences $\Delta V(k)$ of Lyapunov function $V(k)$ along with trajectories of BAM neural network (6), yields

$$\Delta V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + \Delta V_4(k), \quad (17)$$

□

Case 1. When $k \neq \tau_m$.

$$\mathbb{E}\{\Delta V_1(k)\} = \mathbb{E}\{u^T(k+1)P_1 u(k+1) - u^T(k)P_1 u(k) + v^T(k+1)P_2 v(k+1) - v^T(k)P_2 v(k)\} \quad (18)$$

$$\begin{aligned} &= e^{\eta k} \mathbb{E} \left\{ \left[A_1 u(k - \xi_1) + A_2 f(v(k)) + M g(v(k - d_1(k))) + S \sum_{s=1}^{\infty} \mu_s h(v(k - s)) \right]^T \right. \\ &\quad \times P_1 \left[A_1 u(k - \xi_1) + A_2 f(v(k)) + M g(v(k - d_1(k))) + S \sum_{s=1}^{\infty} \mu_s h(v(k - s)) \right] \\ &\quad - u^T(k - \xi_1) P_1 u(k - \xi_1) + \left[B_1 v(k - \xi_2) + B_2 \bar{f}(u(k)) + N \bar{g}(u(k - d_2(k))) \right. \\ &\quad \left. + W \sum_{w=1}^{\infty} \rho_w \bar{h}(u(k - w)) \right]^T \times P_2 \left[B_1 v(k - \xi_2) + B_2 \bar{f}(u(k)) \right. \\ &\quad \left. + N \bar{g}(u(k - d_2(k))) + W \sum_{w=1}^{\infty} \rho_w \bar{h}(u(k - w)) \right] - v^T(k - \xi_2) P_2 v(k - \xi_2) \\ &\quad \left. + \delta^T(u, v, k) P_1 \delta(u, v, k) + \bar{\sigma}^T(u, v, k) P_2 \bar{\sigma}(u, v, k) \right\} \end{aligned} \quad (19)$$

$$= e^{\eta k} \mathbb{E} \left\{ u^T(k - \xi_1) (A_1 P_1 A_1 - P_1) u(k - \xi_1) + 2u^T(k - \xi_1) A_1 P_1 A_2 f(v(k)) \right. \quad (20)$$

$$\begin{aligned} &\quad + 2u^T(k - \xi_1) A_1 P_1 M g(v(k - d_1(k))) + 2u^T(k - \xi_1) A_1 P_1 S \left(\sum_{s=1}^{\infty} \mu_s h(v(k - s)) \right) \\ &\quad \left. + f^T(v(k)) A_2 P_1 A_2 f(v(k)) + 2f^T(v(k)) A_2 P_1 M g(v(k - d_1(k))) \right\} \end{aligned} \quad (21)$$

$$\begin{aligned}
& +2f^T(v(k))A_2P_1S\left(\sum_{s=1}^{\infty}\mu_sh(v(k-s))\right)+g^T(v(k-d_1(k)))MP_1 \\
& \times Mg(v(k-d_1(k)))+2g^T(v(k-d_1(k)))MP_1S\left(\sum_{s=1}^{\infty}\mu_sh(v(k-s))\right) \\
& +\left(\sum_{s=1}^{\infty}\mu_sh(v(k-s))\right)^TSP_1S\left(\sum_{s=1}^{\infty}\mu_sh(v(k-s))\right)+v^T(k-\xi_2)(B_1P_2B_1-P_2) \\
& \times v(k-\xi_2)+2v^T(k-\xi_2)B_1P_2B_2f(u(k))+2v^T(k-\xi_2)B_1P_2N \\
& \times \bar{f}(u(k-d_2(k)))+2v^T(k-\xi_2)B_1P_2W\left(\sum_{w=1}^{\infty}\bar{h}(u(k-w))\right)+\bar{f}^T(u(k))B_2P_2B_2 \\
& \times \bar{f}(u(k))+2\bar{f}^T(u(k))B_2P_2N\bar{g}(u(k-d_2(k)))+2f^T(k)B_2P_2 \\
& \times W\left(\sum_{w=1}^{\infty}\bar{h}(u(k-w))\right)+\bar{g}^T(u(k-d_2(k)))NP_2N\bar{g}^T(u(k-d_2(k))) \\
& +2g^T(k-d_2(k))NP_2W\left(\sum_{w=1}^{\infty}\rho_w\bar{h}(u(k-w))\right)+\left(\sum_{w=1}^{\infty}\rho_w\bar{h}(u(k-w))\right)W \\
& \times P_2W\left(\sum_{w=1}^{\infty}\rho_w\bar{h}(u(k-w))\right)+u^T(k-\xi_1)\lambda_0G^TGu(k-\xi_1) \\
& +v^T(k-\xi_2)\theta_0K^Tkv(k-\xi_2)\Big\}. \tag{22}
\end{aligned}$$

By using [Lemma 2.4](#), we have

$$\begin{aligned}
2u^T(k-\xi_1)A_1P_1A_2f(v(k)) & \leq u^T(k-\xi_1)P_1u(k-\xi_1)+f^T(v(k)) \\
& \times A_1A_2P_1A_2^TA_1^Tf(v(k)), \\
2u^T(k-\xi_1)A_1P_1Mg(v(k-d_1(k))) & \leq u^T(k-\xi_1)P_1u(k-\xi_1)+g^T(v(k-d_1(k))) \\
& \times A_1MP_1M^TA_1^Tg(v(k-d_1(k))), \\
2u^T(k-\xi_1)A_1P_1S\left(\sum_{s=1}^{\infty}\mu_sh(v(k-s))\right) & \leq u^T(k-\xi_1)P_1u(k-\xi_1) \\
& +S\left(\sum_{s=1}^{\infty}\mu_sh(v(k-s))\right)^TA_1SP_1S^TA_1^T \\
& \times S\left(\sum_{s=1}^{\infty}\mu_sh(v(k-s))\right), \\
2f^T(v(k))A_2P_1M(v(k-d_1(k))) & \leq f^T(v(k))P_1f(v(k))+g^T(v(k-d_1(k))) \\
& \times A_2MP_1M^TA_2^Tg(v(k-d_1(k))), \\
2f^T(v(k-\xi_1))A_2P_1S\left(\sum_{s=1}^{\infty}\mu_sh(v(k-s))\right) & \leq f^T(v(k))P_1f(v(k))+\left(\sum_{s=1}^{\infty}\mu_sh(v(k-s))\right)^T \\
& \times A_2SP_1S^TA_2^T\left(\sum_{s=1}^{\infty}\mu_sh(v(k-s))\right),
\end{aligned}$$

$$2g^T(v(k-d_1(k)))MP_1S\left(\sum_{s=1}^{\infty}\mu_sh(v(k-s))\right)\leq g^T(v(k-d_1(k)))P_1g(v(k-d_1(k)))$$

$$+\left(\sum_{s=1}^{\infty}\mu_sh(v(k-s))\right)^T MSP_1S^T M^T$$

$$\left(\sum_{s=1}^{\infty}\mu_sh(v(k-s))\right),$$

$$2v^T(k-\xi_2)B_1P_2N\bar{g}(u(k-d_2(k)))\leq v^T(k-\xi_2)P_2v(k-\xi_2)+\bar{g}^T(u(k-d_2(k)))$$

$$\times B_1NP_2N^TB_1^T\bar{g}(u(k-d_2(k))),$$

$$2v^T(k-\xi_2)B_1P_2B_2\bar{f}(u(k))\leq v^T(k-\xi_2)P_2v(k-\xi_2)+\bar{f}^T(u(k))B_1B_2P_2$$

$$\times B_2^TB_1^T\bar{f}(u(k)),$$

$$2v^T(k-\xi_2)B_1P_2W\left(\sum_{w=1}^{\infty}\rho_w\bar{h}(v(k-w))\right)\leq v^T(k-\xi_2)P_2v(k-\xi_2)$$

$$+\left(\sum_{w=1}^{\infty}\rho_w\bar{h}(u(k-w))\right)^TB_1WP_2W^T$$

$$\times B_1^TW\left(\sum_{w=1}^{\infty}\rho_w\bar{h}(u(k-w))\right),$$

$$2\bar{f}^T(u(k))B_2P_2N\bar{g}(u(k-d_2(k)))\leq \bar{f}^T(u(k))P_2f(u(k))+\bar{g}^T(u(k-d_2(k)))B_2$$

$$\times NP_2N^TB_2^T\bar{g}(u(k-d_2(k))),$$

$$2\bar{f}^T(u(k))B_2P_2W\left(\sum_{w=1}^{\infty}\rho_w\bar{h}(u(k-w))\right)\leq \bar{f}^T(u(k))P_2\bar{f}(v(k))+\left(\sum_{w=1}^{\infty}\rho_w\bar{h}(u(k-w))\right)^T$$

$$\times B_2WP_2W^TB_2^T\left(\sum_{w=1}^{\infty}\rho_w\bar{h}(u(k-w))\right),$$

$$2\bar{g}^T(u(k-d_2(k)))NP_2W\left(\sum_{w=1}^{\infty}\rho_w\bar{h}(u(k-w))\right)\leq \bar{g}^T(u(k-d_2(k)))P_2\bar{g}(u(k-d_2(k)))$$

$$+\left(\sum_{w=1}^{\infty}\rho_w\bar{h}(u(k-w))\right)^TNWP_2W^TB_2^T$$

$$\times N^T\left(\sum_{w=1}^{\infty}\rho_w\bar{h}(u(k-w))\right).$$

$$\mathbb{E}\left\{\Delta V_2(k)\right\}=\mathbb{E}\left\{V_2(k+1)-V_2(k)\right\}$$

$$=e^{\eta k}\mathbb{E}\left[\sum_{i=k+1-d_1(k+1)}^kv^T(i)Q_1v(i)-\sum_{i=k-d_1(k)}^{k-1}v^T(i)Q_1v(i)+\sum_{i=k+1-d_2(k+1)}^ku^T(i)\right.$$

$$\left.\times Q_2u(i)-\sum_{i=k-d_2(k)}^{k-1}u^T(i)Q_2u(i)\right]$$

$$\begin{aligned}
&= e^{\eta k} \mathbb{E} \left[v^T(k) Q_1 v(k) - v^T(k - d_1(k)) Q_1 v(k - d_1(k)) \sum_{i=k+1-d_1(k)+1}^{k-1} v^T(i) Q_1 v(i) \right. \\
&\quad - \sum_{i=k-d_1(k)+1}^{k-1} v^T(i) Q_1 v(i) + u^T(k) Q_2 u(k) - u^T(k - d_2(k)) Q_2 v(k - d_2(k)) \\
&\quad \left. + \sum_{i=k+1-d_2(k)+1}^{k-1} u^T(i) Q_2 u(i) - \sum_{i=k-d_2(k)+1}^{k-1} u^T(i) Q_2 u(i) \right] \\
&= e^{\eta k} \mathbb{E} \left[v^T(k) Q_1 v(k) - v^T(k - d_1(k)) Q_1 v(k - d_1(k)) \sum_{i=k-d_{1M}+1}^{k-d_{1M}} v^T(i) Q_1 v(i) \right. \\
&\quad \left. + u^T(k) Q_2 u(k) - u^T(k - d_2(k)) Q_2 v(k - d_2(k)) + \sum_{i=k-d_{2M}+1}^{k-d_{2M}} u^T(i) Q_2 u(i) \right].
\end{aligned}$$

$$\mathbb{E} \{ \Delta V_3(k) \} = \mathbb{E} \{ V_3(k+1) - V_3(k) \}$$

$$\begin{aligned}
&= e^{\eta k} \mathbb{E} \left[\sum_{j=k-d_{1M}+2}^{k-d_{1M}+1} \sum_{i=j}^k v^T(i) Q_1 v(i) - \sum_{j=k-d_{1M}+1}^{k-d_{1M}} \sum_{i=j}^k v^T(i) Q_1 v(i) \right. \\
&\quad \left. + \sum_{j=k-d_{2M}+2}^{k-d_{2M}+1} \sum_{i=j}^k u^T(i) Q_2 u(i) - \sum_{j=k-d_{2M}+1}^{k-d_{2M}} \sum_{i=j}^k u^T(i) Q_2 u(i) \right] \\
&= e^{\eta k} \mathbb{E} \left[\sum_{j=k-d_{1M}+1}^{k-d_{1M}} \sum_{i=j}^k v^T(i) Q_1 v(i) - \sum_{j=k-d_{1M}+1}^{k-d_{1M}} \sum_{i=j}^{k-1} v^T(i) Q_1 v(i) \right. \\
&\quad \left. + \sum_{j=k-d_{2M}+1}^{k-d_{2M}} \sum_{i=j}^k u^T(i) Q_2 u(i) - \sum_{j=k-d_{2M}+1}^{k-d_{2M}} \sum_{i=j}^{k-1} u^T(i) Q_2 u(i) \right] \quad (23)
\end{aligned}$$

$$\begin{aligned}
&= e^{\eta k} \mathbb{E} \left[\sum_{j=k-d_{1M}+1}^{k-d_{1M}} (v^T(k) Q_1 v(k) - v^T(j) Q_1 v(j)) + \sum_{j=k-d_{2M}+1}^{k-d_{2M}} \right. \\
&\quad \left. (u^T(k) Q_2 u(k) - u^T(j) Q_2 u(j)) \right] \\
&= e^{\eta k} \mathbb{E} \left[(d_{1M} - d_{1m}) v^T(k) Q_1 v(k) - \sum_{j=k-d_{1M}+1}^{k-d_{1M}} v^T(i) Q_1 v(i) d_{2M} \right. \\
&\quad \left. - d_{2m} u^T(k) Q_2 u(k) - \sum_{j=k-d_{2M}+1}^{k-d_{2M}} u^T(i) Q_2 u(i) \right]. \quad (24)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E} \{\Delta V_4(k)\} &= \mathbb{E}\{V_4(k+1) - V_4(k)\} \\
&= e^{\eta k} \mathbb{E} \left[\sum_{i=1}^{\infty} \mu_i \sum_{j=k-i+1}^k v^T(i) R_1 v(i) - \sum_{i=1}^{\infty} \mu_i \sum_{j=k-i}^{k-1} v^T(i) R_1 v(i) + \sum_{i=1}^{\infty} \rho_i \right. \\
&\quad \left. \sum_{j=k-i+1}^k u^T(i) R_2 u(i) - \sum_{i=1}^{\infty} \rho_i \sum_{j=k-i}^{k-1} u^T(i) R_2 u(i) \right] \\
&= e^{\eta k} \mathbb{E} \left[\sum_{i=1}^{\infty} \mu_i (v^T(k) R_1 v(k) - v^T(k-i) R_1 v(k-i)) + \sum_{i=1}^{\infty} \rho_i (u^T(k) R_2 u(k) \right. \\
&\quad \left. - u^T(k-i) R_2 u(k-i)) \right] \\
&= e^{\eta k} \mathbb{E} \left[\bar{\mu} v^T(k) R_1 v(k) - \frac{1}{\bar{\mu}} \left(\sum_{s=1}^{\infty} \mu_s h(v(k-s)) \right)^T R_1 \left(\sum_{s=1}^{\infty} \mu_s h(v(k-s)) \right) + \bar{\rho} u^T(k) \right. \\
&\quad \left. \times R_2 u(k) - \frac{1}{\bar{\rho}} \left(\sum_{w=1}^{\infty} \rho_w \hat{h}(u(k-w)) \right)^T R_2 \left(\sum_{w=1}^{\infty} \rho_w \hat{h}(u(k-w)) \right) \right]. \quad (25)
\end{aligned}$$

From [Assumption III](#), it is clear that

$$\begin{aligned}
(f_j(v_j(k)) - l_j^+ v_j(k))(f_j(v_j(k)) - l_j^- v_j(k)) &\leq 0, \\
(g_j(v_j(k)) - m_j^+ v_j(k))(g_j(v_j(k)) - m_j^- v_j(k)) &\leq 0, \\
(h_j(v_j(k)) - s_j^+ v_j(k))(h_j(v_j(k)) - s_j^- v_j(k)) &\leq 0, \\
(\bar{f}_i(u_i(k)) - u_i^+ u_i(k))(\bar{f}_i(u_i(k)) - u_i^- v_i(k)) &\leq 0, \\
(\bar{g}_i(u_i(k)) - n_i^+ u_i(k))(\bar{g}_i(n_i(k)) - v_i^- u_i(k)) &\leq 0, \\
(\bar{h}_i(u_i(k)) - w_i^+ u_i(k))(\bar{f}_i(u_i(k)) - w_i^- u_i(k)) &\leq 0, \quad (26)
\end{aligned}$$

which is equivalent to

$$\begin{pmatrix} v(k) \\ f(v(k)) \end{pmatrix}^T \begin{pmatrix} l_j^+ l_j^- e_j e_j^T & \frac{-l_j^+ + l_j^-}{2} e_j e_j^T \\ \frac{-l_j^+ + l_j^-}{2} e_j e_j^T & e_j e_j^T \end{pmatrix} \begin{pmatrix} v(k) \\ f(v(k)) \end{pmatrix} < 0, \text{ where } k = 1, 2, \dots, n \quad (27)$$

where e_k denotes the unit column vector having '1' element on its k th row and zeros elsewhere. Consequently,

$$\sum_{j=1}^n \lambda_j^1 \begin{pmatrix} v(k) \\ f(v(k)) \end{pmatrix}^T \begin{pmatrix} l_j^+ l_j^- e_j e_j^T & \frac{-l_j^+ + l_j^-}{2} e_j e_j^T \\ \frac{-l_j^+ + l_j^-}{2} e_j e_j^T & e_j e_j^T \end{pmatrix} \begin{pmatrix} v(k) \\ f(v(k)) \end{pmatrix} \leq 0, \quad (28)$$

$$\Rightarrow \begin{pmatrix} v(k) \\ f(v(k)) \end{pmatrix}^T \begin{pmatrix} \Lambda_1 L_1 & -\Lambda_1 L_2 \\ -\Lambda_1 L_2 & \Lambda_1 \end{pmatrix} \begin{pmatrix} v(k) \\ f(v(k)) \end{pmatrix} \leq 0. \quad (29)$$

Similarly from [Eq. \(26\)](#) we have,

$$\begin{pmatrix} v(k) \\ g(v(k)) \end{pmatrix}^T \begin{pmatrix} \Gamma_1 M_1 & -\Gamma_1 M_2 \\ -\Gamma_1 M_2 & \Gamma_1 \end{pmatrix} \begin{pmatrix} v(k) \\ g(v(k)) \end{pmatrix} \leq 0,$$

$$\begin{aligned}
& \begin{pmatrix} v(k) \\ h(v(k)) \end{pmatrix}^T \begin{pmatrix} \Omega_1 S_1 & -\Omega_1 S_2 \\ -\Omega_1 S_2 & \Omega_1 \end{pmatrix} \begin{pmatrix} v(k) \\ h(v(k)) \end{pmatrix} \leq 0, \\
& \begin{pmatrix} u(k) \\ \bar{f}(u(k)) \end{pmatrix}^T \begin{pmatrix} \Lambda_2 U_1 & -\Lambda_2 U_2 \\ -\Lambda_2 U_2 & \Lambda_2 \end{pmatrix} \begin{pmatrix} u(k) \\ \bar{f}(u(k)) \end{pmatrix} \leq 0, \\
& \begin{pmatrix} u(k) \\ \bar{g}(u(k)) \end{pmatrix}^T \begin{pmatrix} \Gamma_2 N_1 & -\Gamma_2 N_2 \\ -\Gamma_2 N_2 & \Gamma_2 \end{pmatrix} \begin{pmatrix} u(k) \\ \bar{g}(u(k)) \end{pmatrix} \leq 0, \\
& \begin{pmatrix} u(k) \\ \bar{h}(u(k)) \end{pmatrix}^T \begin{pmatrix} \Omega_2 W_1 & -\Omega_2 W_2 \\ -\Omega_2 W_2 & \Omega_2 \end{pmatrix} \begin{pmatrix} u(k) \\ \bar{h}(u(k)) \end{pmatrix} \leq 0.
\end{aligned} \tag{30}$$

Then, from Eqs. (22)–(25) and (30), we obtain

$$\begin{aligned}
& \mathbb{E}\{\Delta V(k)\} \\
&= e^{\eta k} \mathbb{E} \left\{ u^T(k - \xi_1) \left[A_1 P_1 A_1 - 2P_1 + \lambda_0 G^T G \right] u(k - \xi_1) + u^T(k) \left[(1 + d_{2M} - d_{1m}) \right. \right. \\
&\quad \times Q_2 - \Lambda_2 U_1 - \Gamma_2 N_1 - \Omega_2 W_1 + \bar{\rho} R_2 \left. \right] u(k) + u^T(k) \left[\Lambda_2 U_2 \right] \bar{f}(u(k)) + u^T(k) \left[\Gamma_2 N_2 \right] \\
&\quad \times \bar{g}(u(k)) + u^T(k) \left[\Omega_2 W_2 \right] \bar{h}(u(k)) + u^T(k - d_2(k)) \left[(-Q_2) \right] u(k - d_2(k)) + \bar{f}^T(u(k)) \\
&\quad \times \left[B_1 B_2 P_1 B_2^T B_1^T - \Lambda_2 + 2P_2 \right] \bar{f}(u(k)) + \bar{g}^T(u(k - d_2(k))) \left[B_1 N P_2 N^T B_1^T + B_2 N P_2 \right. \\
&\quad \times \left. N^T B_2^T + P_2 \right] \bar{g}(u(k - d_2(k))) + \left(\sum_{w=1}^{\infty} \rho_w \bar{h}(u(k - w)) \right)^T \left[B_1 W P_2 B_1^T W^T + B_2 W P_2 \right. \\
&\quad \times \left. W^T B_2^T + N W P_2 W^T N^T - \bar{\rho}^{-1} R_2 \right] \left(\sum_{w=1}^{\infty} \rho_w \bar{h}(u(k - w)) \right) + v^T(k - \xi_2) \left[P_2 B_1 - 2P_2 \right. \\
&\quad + \theta_0 K^T K \left. \right] v(k - \xi_2) + v^T(k) \left[(1 + d_{1M} - d_{1m}) Q_1 - \Lambda_1 L_1 - \Gamma_1 S_1 - \Omega_1 M_1 + \bar{\mu} R_1 \right] \\
&\quad \times v(k) + v^T(k) \left[\Lambda_1 L_2 \right] f(v(k)) + v^T(k) \left[\Gamma_1 S_2 \right] g(v(k)) + v^T(k) \left[\Omega_1 N_2 \right] h(v(k)) \\
&\quad + v^T(k - d_1(k)) \left[(-Q_1) \right] v(k - d_1(k)) + f^T(v(k)) \left[A_1 A_2 P_1 A_2^T A_1^T - \Lambda_2 + 2P_1 \right] f(v(k)) \\
&\quad + g^T(v(k - d_1(k))) \left[A_1 M P_1 M^T A_1^T + A_2 M P_1 M^T A_2^T + P_1 \right] g(v(k - d_1(k))) \\
&\quad + \left(\sum_{s=1}^{\infty} \mu_s h(v(k - s)) \right)^T \left[A_1 S P_1 S^T A^T + A_2 S P_1 S^T A_2^T + M S P_1 S^T M^T - \mu^{-1} R_1 \right] \\
&\quad \times \left(\sum_{s=1}^{\infty} \mu_s h(v(k - s)) \right) - \begin{pmatrix} v(k) \\ f(v(k)) \end{pmatrix}^T \begin{pmatrix} \Lambda_1 L_1 & -\Lambda_1 L_2 \\ -\Lambda_1 L_2 & \Lambda_1 \end{pmatrix} \begin{pmatrix} v(k) \\ f(v(k)) \end{pmatrix} \\
&\quad - \begin{pmatrix} v(k) \\ g(v(k)) \end{pmatrix}^T \begin{pmatrix} \Gamma_1 M_1 & -\Gamma_1 M_2 \\ -\Gamma_1 M_2 & \Gamma_1 \end{pmatrix} \begin{pmatrix} v(k) \\ g(v(k)) \end{pmatrix} \\
&\quad - \begin{pmatrix} v(k) \\ h(v(k)) \end{pmatrix}^T \begin{pmatrix} \Omega_1 N_1 & -\Omega_1 N_2 \\ -\Omega_1 N_2 & \Omega_1 \end{pmatrix} \begin{pmatrix} v(k) \\ h(v(k)) \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& - \begin{pmatrix} u(k) \\ \bar{f}(u(k)) \end{pmatrix}^T \begin{pmatrix} \Lambda_2 U_1 & -\Lambda_2 U_2 \\ -\Lambda_2 U_2 & \Lambda_2 \end{pmatrix} \begin{pmatrix} u(k) \\ \bar{f}(u(k)) \end{pmatrix} \\
& - \begin{pmatrix} u(k) \\ \bar{g}(u(k)) \end{pmatrix}^T \begin{pmatrix} \Gamma_2 V_1 & -\Gamma_2 V_2 \\ -\Gamma_1 V_2 & \Gamma_2 \end{pmatrix} \begin{pmatrix} u(k) \\ \bar{g}(v(k)) \end{pmatrix} \\
& - \begin{pmatrix} u(k) \\ \bar{h}(u(k)) \end{pmatrix}^T \begin{pmatrix} \Omega_2 W_1 & -\Omega_2 W_2 \\ -\Omega_2 W_2 & \Omega_2 \end{pmatrix} \begin{pmatrix} u(k) \\ \bar{h}(u(k)) \end{pmatrix} \\
& = e^{\eta k} \mathbb{E} \left\{ \left[\beta^T(k) \Xi_1 \beta(k) + \eta^T(k) \Xi_2 \eta(k) \right] \right\},
\end{aligned}$$

where

$$\begin{aligned}
\beta^T(k) &= e^{\eta k} \mathbb{E} \left[u^T(k - \xi_1), u^T(k), u^T(k - d_2(k)), \bar{f}^T(u(k)), \bar{g}^T(u(k)), \bar{g}^T(u(k - d_2(k))), \right. \\
&\quad \left. \bar{h}^T(u(k)) \times \left(\sum_{w=1}^{\infty} \rho_w \bar{h}^T(u(k - w)) \right) \right]. \\
\eta^T(k) &= e^{\eta k} \mathbb{E} \left[v^T(k - \xi_2), v^T(k), v^T(k - d_1(k)), f^T(v(k)), g^T(v(k)), g^T(v(k - d_1(k))), \right. \\
&\quad \left. h^T(v(k)) \times \left(\sum_{s=1}^{\infty} \mu_s h^T(v(k - s)) \right) \right].
\end{aligned}$$

Therefore it can be concluded that $\Delta V(k) \leq 0$. It follows that $V(k) \leq V(0)$

Case 2. When $k = \tau_m$. Note that the inequalities from Eqs. (22) to (30) except Eq. (22) hold for $k = \tau_m$ calculating $\Delta V_1(\tau_m)$, we have

$$\begin{aligned}
\Delta V_1(\tau_m) &= e^{\eta k} \mathbb{E} \left\{ u^T(\tau_m + 1) P_1 u(\tau_m + 1) - u^T(\tau_m) P_1 u(\tau_m) + v^T(\tau_m + 1) P_2 v(\tau_m + 1) \right. \\
&\quad \left. - v^T(\tau_m) P_2 v(\tau_m) \right\}, \\
&= e^{\eta k} \mathbb{E} \left\{ \left[u(\tau_m) + I_m(u(\tau_m)) \right]^T P_1 \left[u(\tau_m) + I_m(u(\tau_m)) \right] - u^T(\tau_m) P_1 u(\tau_m) \right. \\
&\quad \left. + \left[v(\tau_m) + J_m(v(\tau_m)) \right]^T P_2 \left[v(\tau_m) + J_m(v(\tau_m)) \right] - v^T(\tau_m) P_2 v(\tau_m) \right\}, \\
&\leq e^{\eta k} \mathbb{E} \left\{ u^T(\tau_m) (2H_1 P_1 + H_2 P_1 H_2) u(\tau_m) + v^T(\tau_m) (2H_3 P_2 + H_4 P_2 H_4) v(\tau_m) \right\}. \tag{31}
\end{aligned}$$

By using similar method in Case 1, we obtain that

$$\mathbb{E} \left\{ \Delta V(k) \right\} = e^{\eta k} \mathbb{E} \left\{ \beta^T(k) \Xi_3 \beta(k) + \eta^T(k) \Xi_4 \eta(k) \right\} \tag{32}$$

Combining the above discussion in Case 1 and Case 2, we obtain from Eq. (31) and above

$$\mathbb{E} \left\{ \Delta V(k) \right\} \leq -\alpha e^{\eta k} \mathbb{E} \left\{ (\|u(k)\|^2 + \|v(k)\|^2) \right\}. \tag{33}$$

From the definition of $V(k)$

$$\begin{aligned}
 \mathbb{E} \{V(k)\} &\leq e^{\eta k} \mathbb{E} \left[\lambda_{\max}(P_1) \|u(k)\|^2 + \lambda_{\max}(P_2) \|v(k)\|^2 + \lambda_{\max}(Q_1) \right. \\
 &\quad \times \sum_{i=k-d_{1M}}^{k-1} \|v(i)\|^2 + \lambda_{\max}(Q_2) \sum_{i=k-d_{2M}}^{k-1} \|u(i)\|^2 + (d_{1M} - d_{2M}) \\
 &\quad \left. + \lambda_{\max}(Q_1) \sum_{i=k-d_{1M}}^{k-1} \|v(i)\|^2 + (d_{2M} - d_{2m}) + \lambda_{\max}(Q_2) \sum_{i=k-d_{2M}}^{k-1} \|u(i)\|^2 \right] \\
 &= e^{\eta k} \mathbb{E} \left[\lambda_{\max}(P_1) \|u(k)\|^2 + \lambda_{\max}(P_2) \|v(k)\|^2 + \alpha_1 \sum_{i=k-d_2(M)}^{k-1} \|u(i)\|^2 \right. \\
 &\quad \left. + \alpha_2 \sum_{i=k-d_1(M)}^{k-1} \|v(i)\|^2 \right], \tag{34}
 \end{aligned}$$

Where

$$\alpha_1 = (1 + (d_{2M} - d_{2m}) + \lambda_{\max}(Q_1)).$$

$$\alpha_2 = (1 + (d_{1M} - d_{1m}) + \lambda_{\max}(Q_2)).$$

Choose a scalar $\chi > 1$, satisfying

$$-\beta\chi + (\chi - 1)(\lambda_{\max}(P_1) + \lambda_{\max}(P_2)) + (\chi - 1)(\alpha_1 d_{1M} \chi^{d_{1M}} + \alpha_2 d_{2M} \chi^{d_{2M}}) = 0. \tag{35}$$

By Eqs. (34) and (35), we have

$$\begin{aligned}
 \mathbb{E} \{ \chi^{k+1} V(k+1) - \chi^k V(k) \} &= \mathbb{E} \{ \chi^{k+1} V(k+1) - \chi^{k+1} V(k) + \chi^{k+1} V(k) - \chi^k V(k) \} \\
 &\leq \chi^{k+1} \mathbb{E} \{ V(k+1) - V(k) \} + \chi^k (\chi - 1) \mathbb{E} \{ V(k) \} \\
 &= \chi^{k+1} \mathbb{E} \{ \Delta V(k) \} + \chi^k (\chi - 1) \mathbb{E} \{ V(k) \} \\
 &\leq e^{\eta k} \left[\alpha_3 \chi^k \mathbb{E} \left\{ \left(\|u(k)\|^2 + \|v(k)\|^2 \right) \right\} \right. \\
 &\quad \left. + \alpha_4 \chi^k \mathbb{E} \left\{ \left(\sum_{i=k-d_2(M)}^{k-1} \|u(i)\|^2 + \sum_{i=k-d_1(M)}^{k-1} \|v(i)\|^2 \right) \right\} \right], \tag{36}
 \end{aligned}$$

where $\alpha_3 = -\alpha\chi + (\chi - 1)(\lambda_{\max}(P_1) + \lambda_{\max}(P_2))$ and

$$\alpha_4 = (\chi - 1)(\alpha_1 + \alpha_2).$$

Therefore, for any integer $N \geq d_{1M} + 1$ and $N \geq d_{2M} + 1$, Summing up both sides of Eq. (37) from 0 to $N - 1$ with respect to k , we have

$$\begin{aligned}
 \mathbb{E} \{ \chi^N V(N) - V(0) \} &\leq e^{\eta k} \left[\alpha_3 \sum_{k=0}^{N-1} \chi^k \mathbb{E} \left\{ \left(\|u(k)\|^2 + \|v(k)\|^2 \right) \right\} + \alpha_4 \sum_{k=0}^{N-1} \chi^k \right. \\
 &\quad \left. \mathbb{E} \left\{ \left(\sum_{i=k-d_{2M}}^{k-1} \|u(i)\|^2 + \sum_{i=k-d_{1M}}^{k-1} \|v(i)\|^2 \right) \right\} \right]. \tag{37}
 \end{aligned}$$

Here, we note that for $d_{2M} \geq 1$, $d_{1M} \geq 1$, we have

$$\begin{aligned}
\mathbb{E}\left\{\sum_{k=0}^{N-1}\chi^k\left(\sum_{i=k-d_{2M}}^{k-1}\|u(i)\|^2+\sum_{i=k-d_{1M}}^{k-1}\|v(i)\|^2\right)\right\} &= d_{2M}(d_{2M}+1)\chi^{d_{2M}} \\
&\times\mathbb{E}\left\{\sup_{-d_{2M}\leq i\leq 0}\|u(i)\|^2\right\} \\
&+d_{1M}(d_{1M}+1)\chi^{d_{1M}} \\
&\times\mathbb{E}\left\{\sup_{-d_{1M}\leq i\leq 0}\|v(i)\|^2\right\}+d_{2M}\chi^{d_{2M}} \\
&\times\sum_{k=0}^{N-1}\chi^k\mathbb{E}\left\{\|u(k)\|^2\right\}+d_{1M}\chi^{d_{1M}} \\
&\sum_{k=0}^{N-1}\chi^k\mathbb{E}\left\{\|v(k)\|^2\right\}. \tag{38}
\end{aligned}$$

$$\begin{aligned}
\chi^N\mathbb{E}\{V(N)\} &\leq e^{\eta k}\left[\alpha_3+\alpha_4(d_{2M}\chi^{d_{2M}}+d_{1M}\chi^{d_{1M}})\sum_{k=0}^{T-1}\chi^k\mathbb{E}\left\{(\|u(k)\|^2+\|v(k)\|^2)\right\}\right. \\
&\quad +\alpha_4\left(d_{2M}(d_{2M}+1)\chi^{d_{2M}}+d_{1M}(d_{1M}+1)\chi^{d_{1M}}\right) \\
&\quad \times\left(\sup_{-d_{2M}\leq i\leq 0}\mathbb{E}\{\|u(i)\|^2\}+\sup_{-d_{1M}\leq i\leq 0}\mathbb{E}\{\|v(i)\|^2\}\right)\Bigg]+\mathbb{E}\{V(0)\}. \tag{39}
\end{aligned}$$

Substituting Eq. (39) in Eq. (38), We can observe that

$$\mathbb{E}\{V(N)\}\geq\left\{\lambda_{\min}(P_1),\lambda_{\min}(P_2)\right\}e^{\eta k}\mathbb{E}\left\{(\|u(N)\|^2+\|v(N)\|^2)\right\}. \tag{40}$$

It follows easily from Eq. (35) that

$$\mathbb{E}\{V(0)\}\leq e^{\eta k}\left[\alpha_1d_{2M}\sup_{-d_{2M}\leq i\leq 0}\mathbb{E}\{\|u(i)\|^2\}+\alpha_2d_{1M}\sup_{-d_{1M}\leq i\leq 0}\mathbb{E}\{\|v(i)\|^2\}\right]. \tag{41}$$

Then it follows from Eq. (37), Eq. (41) that

$$\mathbb{E}\{\|u(N)\|^2+\|v(N)\|^2\}\leq\gamma e^{-\eta k}\left[\sup_{-d_{2M}\leq i\leq 0}\mathbb{E}\{\|u(i)\|^2\}+\sup_{-d_{1M}\leq i\leq 0}\mathbb{E}\{\|v(i)\|^2\}\right]. \tag{42}$$

$$\text{where } \gamma = \frac{\alpha_1d_{2M}+\alpha_2d_{1M}+\alpha_4\left[d_{2M}(d_{2M}+1)\chi^{d_{2M}}+d_{1M}(d_{1M}+1)\chi^{d_{1M}}\right]}{\lambda_{\min}(P_1),\lambda_{\min}(P_2)}$$

Therefore, by definition Eqs. (11) and (42), we see that equilibrium point of the neural networks (6) is globally exponentially stable for $k\geq 0$ in the mean square.

Remark 3.2. In [6–8,44], the authors analyzed the time dependent conditions for different dynamical systems with control inputs and also more comparisons with some previous results are listed to show the effectiveness of designed methods. By the above motivation, the impact of time delays in discrete and distributed terms of our addressed neural networks system, novel sufficient conditions for stability in the sense of exponential is generated. Further, the

correctness of our theoretical outcomes is solved and the obtained upper bounds of discrete delays is better when compared with some available literature.

Remark 3.3. The term $x(k - \xi_1)$, $y(k - \xi_2)$ in system (1) corresponds to a stabilizing negative feedback of the system which acts instantaneously with time delay. The term is variously known as leakage (or forgetting) term. As pointed out by Gopalasamy [5] leakage time delay in the stabilizing negative feedback term has a tendency to destabilize a system [12].

Remark 3.4. In Theorem 3.1, the time delay-dependent global exponential stability criterion for the system (5) is established. For consequence, a new brand Lyapunov–Krasovskii functional handled in the following Theorem 4.1 to guarantee the uncertain neural-networks (50) is globally exponentially stable.

Remark 3.5. As much as know, all the existing results concerning the dynamical behaviors of neural networks [1,3,19,20] have not considered the globally exponential stability performance and time-varying delayed situation, which are investigated via LMI approach in our paper. Therefore, our conclusions are brand-new when compared to previous results.

Remark 3.6. This paper fully considers the relationship between time-varying delays and their upper bounds. Also, some inequality techniques are used to obtain the maximum upper bounds of discrete & distributed time-varying delays, see Table 1 in Example 5.1. Hence this techniques and methods helps us to lead the proposed work to novel criterions.

Remark 3.7. Based on Examples 5.1–5.4, it is easy to see that the conclude results are better than the ones in [10,23,28,31]. Hence, the proposed method is an improvement over the existing literatures.

For a BAM Neural Network we have the following system of equations:

$$\begin{aligned}
 x(k+1) &= \left[A_1 x(k - \xi_1) + A_2 f(y(k)) + M g(y(k - d_1(k))) + S \sum_{s=1}^{\infty} \mu_s h(y(k - s)) + I \right], \\
 &\quad \text{where } k \neq \tau_m; \\
 x(\tau_m + 1) &= x(\tau_m) + I_m(x(\tau_m)), \text{ where } k = \tau_m. \\
 y(k+1) &= \left[B_1 y(k - \xi_2) + B_2 \bar{f}(x(k)) + N \bar{g}(x(k - d_2(k))) + W \sum_{w=1}^{\infty} \rho_w \bar{h}(x(k - w)) + J \right], \\
 &\quad \text{where } k \neq \tau_m; \\
 y(\tau_m + 1) &= y(\tau_m) + J_m(y(\tau_m)), \text{ where } k = \tau_m.
 \end{aligned} \tag{43}$$

Corollary 3.3. Under Assumptions I–IV & VI, the system (43) is globally exponentially stable in the mean square, if there exists constant $\lambda_0 > 0$ and $\theta_0 > 0$ if there exists a diagonal matrices $\Lambda_1 = \text{diag}\{\lambda_1^{(1)}, \lambda_2^{(1)}, \dots, \lambda_n^{(1)}\} > 0$ and $\Lambda_2 = \text{diag}\{\lambda_1^{(2)}, \lambda_2^{(2)}, \dots, \lambda_n^{(2)}\} > 0$, $\Gamma_1 = \text{diag}\{\gamma_1^{(1)}, \gamma_2^{(1)}, \dots, \gamma_n^{(1)}\} > 0$ and $\Gamma_2 = \text{diag}\{\gamma_1^{(2)}, \gamma_2^{(2)}, \dots, \gamma_n^{(2)}\} > 0$, $\Omega_1 = \text{diag}\{\omega_1^{(1)}, \omega_2^{(1)}, \dots, \omega_n^{(1)}\} > 0$ and $\Omega_2 = \text{diag}\{\omega_1^{(2)}, \omega_2^{(2)}, \dots, \omega_n^{(2)}\} > 0$ and positive definite $P_1 > 0$, $P_2 > 0$ and $Q_1 > 0$ and $Q_2 > 0$, $R_1 > 0$, $R_2 > 0$ such that following LMIs hold:

$$\Xi_5 = \begin{pmatrix} \varpi_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \pi_{22} & 0 & \Lambda_2 U_2 & \Gamma_2 N_2 & 0 & \Omega_2 M_2 & 0 \\ * & * & -Q_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \pi_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Gamma_2 & 0 & 0 & 0 \\ * & * & * & * & * & \pi_{66} & 0 & 0 \\ * & * & * & * & * & * & -\Omega_2 & 0 \\ * & * & * & * & * & * & * & \pi_{88} \end{pmatrix} < 0 \quad (44)$$

$$\Xi_6 = \begin{pmatrix} \varrho_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \Theta_{22} & 0 & \Lambda_1 L_2 & \Gamma_1 S_2 & 0 & \Omega_1 W_2 & 0 \\ * & * & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Theta_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Gamma_1 & 0 & 0 & 0 \\ * & * & * & * & * & \Theta_{66} & 0 & 0 \\ * & * & * & * & * & * & -\Omega_1 & 0 \\ * & * & * & * & * & * & * & \Theta_{88} \end{pmatrix} < 0 \quad (45)$$

$$\Xi_7 = \begin{pmatrix} \phi_{11} & 0 & \Lambda_2 U_2 & \Gamma_2 N_2 & 0 & \Omega_2 M_2 & 0 \\ * & -Q_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & \phi_{33} & 0 & 0 & 0 & 0 \\ * & * & * & -\Gamma_2 & 0 & 0 & 0 \\ * & * & * & * & \phi_{55} & 0 & 0 \\ * & * & * & * & * & -\Omega_2 & 0 \\ * & * & * & * & * & * & \phi_{77} \end{pmatrix} < 0 \quad (46)$$

$$\Xi_8 = \begin{pmatrix} \psi_{11} & 0 & \Lambda_1 L_2 & \Gamma_1 S_2 & 0 & \Omega_1 W_2 & 0 \\ * & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & \psi_{33} & 0 & 0 & 0 & 0 \\ * & * & * & -\Gamma_1 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & -\Omega_1 & 0 \\ * & * & * & * & * & * & \psi_{77} \end{pmatrix} < 0 \quad (47)$$

where

$$\varpi_{11} = A_1^T P_1 A_1 - 2P_1,$$

$$\varrho_{11} = B_1^T P_2 B_1 - 2P_2,$$

and $\pi_{22}, \pi_{44}, \pi_{66}, \pi_{88}, \Theta_{22}, \Theta_{44}, \Theta_{66}, \Theta_{88}, \Phi_{11}, \Phi_{33}, \Phi_{77}, \psi_{11}, \psi_{33}, \psi_{77}$ are defined in above theorem.

For a deterministic BAM neural networks we have the following system of equations:

$$\begin{aligned} u(k+1) &= \left[A_1 u(k - \xi_1) + A_2 f(v(k)) + M g(v(k - d_1(k))) + S \sum_{s=1}^{\infty} \mu_s h(v(k - s)) \right] \\ &\quad + \delta(u(k - \xi_1), v(k - d_1(k)), k) \bar{w}_1(k); \quad k \neq \tau_m, \\ v(k+1) &= \left[B_1 v(k - \xi_2) + B_2 \bar{f}(u(k)) + N \bar{g}(u(k - d_2(k))) + W \sum_{w=1}^{\infty} \rho_w \bar{h}(u(k - w)) \right] \\ &\quad + \bar{\sigma}(v(k - \xi_2), u(k - d_1(k)), k) \bar{w}_2(k); \quad k \neq \tau_m. \end{aligned} \quad (48)$$

Corollary 3.4. Under Assumptions I–V) the system (48) is globally exponentially stable in the mean square, if there exists constant $\lambda_0 > 0$ and $\theta_0 > 0$ if there exists a diagonal matrices $\Lambda_1 = \text{diag}\{\lambda_1^{(1)}, \lambda_2^{(1)}, \dots, \lambda_n^{(1)}\} > 0$ and $\Lambda_2 = \text{diag}\{\lambda_1^{(2)}, \lambda_2^{(2)}, \dots, \lambda_n^{(2)}\} > 0$, $\Gamma_1 = \text{diag}\{\gamma_1^{(1)}, \gamma_2^{(1)}, \dots, \gamma_n^{(1)}\} > 0$ and $\Gamma_2 = \text{diag}\{\gamma_1^{(2)}, \gamma_2^{(2)}, \dots, \gamma_n^{(2)}\} > 0$, $\Omega_1 = \text{diag}\{\omega_1^{(1)}, \omega_2^{(1)}, \dots, \omega_n^{(1)}\} > 0$ and $\Omega_2 = \text{diag}\{\omega_1^{(2)}, \omega_2^{(2)}, \dots, \omega_n^{(2)}\} > 0$ and positive definite $P_1 > 0$, $P_2 > 0$ and $Q_1 > 0$ and $Q_2 > 0$, $R_1 > 0$, $R_2 > 0$ such that following LMIs hold:

$$P_1 < \lambda_0 I$$

$$P_2 < \theta_0 I$$

$$\Xi_9 = \begin{pmatrix} \pi_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \pi_{22} & 0 & \Lambda_2 U_2 & \Gamma_2 N_2 & 0 & \Omega_2 M_2 & 0 \\ * & * & -Q_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \pi_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Gamma_2 & 0 & 0 & 0 \\ * & * & * & * & * & \pi_{66} & 0 & 0 \\ * & * & * & * & * & * & -\Omega_2 & 0 \\ * & * & * & * & * & * & * & \pi_{88} \end{pmatrix} < 0 \quad (49)$$

$$\Xi_{10} = \begin{pmatrix} \Theta_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \Theta_{22} & 0 & \Lambda_1 L_2 & \Gamma_1 S_2 & 0 & \Omega_1 W_2 & 0 \\ * & * & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Theta_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Gamma_1 & 0 & 0 & 0 \\ * & * & * & * & * & \Theta_{66} & 0 & 0 \\ * & * & * & * & * & * & -\Omega_1 & 0 \\ * & * & * & * & * & * & * & \Theta_{88} \end{pmatrix} < 0 \quad (50)$$

where $\pi_{11}, \pi_{22}, \pi_{44}, \pi_{66}, \pi_{88}, \Theta_{11}, \Theta_{22}, \Theta_{44}, \Theta_{66}, \Theta_{88}$ are defined in above theorem.

4. Uncertain discrete-time neural networks

Uncertainties are unavoidable in neural networks because of the existence of modeling errors and external disturbances in practical implementations. To manifest such a reality, inevitable parameter uncertainties should be taken into account, while framing the network. Therefore, the following discrete time stochastic BAM neural networks with impulses and parameter uncertainties are considered:

$$\begin{aligned} u(k+1) &= \left[A_1(k)u(k - \xi_1) + A_2(k)f(v(k)) + M(k)g(v(k - d_1(k))) + S(k) \sum_{s=1}^{\infty} \mu_s h(v(k-s)) \right] \\ &\quad + \delta(u(k - \xi_1), v(k - d_1(k)), k)w_1(k), \quad k \neq \tau_m; \\ u(\tau_m + 1) &= u(\tau_m) + I_m(u(\tau_m)), \quad k = \tau_m; \\ v(k+1) &= \left[B_1(k)v(k - \xi_2) + B_2(k)\bar{f}(u(k)) + N(k)\bar{g}(u(k - d_2(k))) + W(k) \sum_{w=1}^{\infty} \rho_w \bar{h}(u(k-w)) \right] \\ &\quad + \bar{\sigma}(v(k - \xi_2), u(k - d_1(k)), k)\bar{w}_2(k), \quad k \neq \tau_m; \\ v(\tau_m + 1) &= v(\tau_m) + I_m(v(\tau_m)), \quad k = \tau_m; \end{aligned} \quad (51)$$

where $A_1(k)=A_1+\Delta A_1(k)$, $A_2(k)=A_2+\Delta A_2(k)$, $M(k)=M+\Delta M(k)$, $S(k)=S+\Delta S(k)$ and $B_1(k)=B_1+\Delta B_1(k)$, $B_2(k)=B_2+\Delta B_2(k)$, $N(k)=N+\Delta N(k)$, $W(k)=W+\Delta W(k)$; $A_1, A_2, B_1, B_2, M, S, W, N \in \mathbb{R}^{n \times n}$ represents the weight matrices; $\Delta A_1(k)$, $\Delta A_2(k)$, $\Delta M(k)$, $\Delta S(k)$ and $\Delta B_1(k)$, $\Delta B_2(k)$, $\Delta N(k)$, $\Delta W(k)$ denote the time-varying structured uncertainties, which have the following form,

$$\begin{aligned} & [\Delta A_1(k), \Delta A_2(k), \Delta M(k), \Delta S(k), \Delta B_1(k), \Delta B_2(k), \Delta N(k), \Delta W(k)] \\ & = LE(k)[N_{a1} N_{a2} N_m N_s N_{b1} N_{b2} N_n N_w], \end{aligned}$$

where $L, N_{a1}, N_{a2}, N_m, N_s, N_{b1}, N_{b2}, N_n, N_w$ are known real constant matrices of appropriate dimensions; $E(k)$ is unknown time-varying matrix function satisfying $E^T(k)E(k) \leq I$, $k \in \mathbb{N}^+$.

In next theorem, we inaugurate the required conditions for the stability of uncertain discrete-time stochastic neural networks (51) with impulses.

Theorem 4.1. Under Assumptions I–VI, the system (50) is globally exponentially stable in the mean square, if there exists constant ρ , ρ^{-1} , $\lambda_0 > 0$ and $\theta_0 > 0$ if there exists a diagonal matrices $\Lambda_1 = \text{diag}\{\lambda_1^{(1)}, \lambda_2^{(1)}, \dots, \lambda_n^{(1)}\} > 0$ and $\Lambda_2 = \text{diag}\{\lambda_1^{(2)}, \lambda_2^{(2)}, \dots, \lambda_n^{(2)}\} > 0$, $\Gamma_1 = \text{diag}\{\gamma_1^{(1)}, \gamma_2^{(1)}, \dots, \gamma_n^{(1)}\} > 0$ and $\Gamma_2 = \text{diag}\{\gamma_1^{(2)}, \gamma_2^{(2)}, \dots, \gamma_n^{(2)}\} > 0$, $\Omega_1 = \text{diag}\{\omega_1^{(1)}, \omega_2^{(1)}, \dots, \omega_n^{(1)}\} > 0$ and $\Omega_2 = \text{diag}\{\omega_1^{(2)}, \omega_2^{(2)}, \dots, \omega_n^{(2)}\} > 0$ and positive definite $P_1 > 0$, $P_2 > 0$ and $Q_1 > 0$ and $Q_2 > 0$, $R_1 > 0$, $R_2 > 0$ such that following LMIs hold:

$$P_1 < \lambda_0 I$$

$$P_2 < \theta_0 I$$

$$\Xi_{11} = \begin{pmatrix} \pi_{11} & 0 & 0 & \rho N_{a1}^T N_{a2} & 0 & \rho N_{a1}^T N_m & 0 & \rho N_{a1}^T N_s \\ * & \pi_{22} & 0 & \Lambda_2 U_2 & \Gamma_2 N_2 & 0 & \Omega_2 W_2 & 0 \\ * & * & -Q_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \pi_{44} & 0 & \rho N_{a2}^T N_m & 0 & \rho N_{a2}^T N_s \\ * & * & * & * & -\Gamma_2 & 0 & 0 & 0 \\ * & * & * & * & * & \pi_{66} & 0 & \rho N_m^T N_s \\ * & * & * & * & * & * & -\Omega_2 & 0 \\ * & * & * & * & * & * & * & \pi_{88} \end{pmatrix} < 0 \quad (52)$$

$$\Xi_{12} = \begin{pmatrix} \Theta_{11} & 0 & 0 & \rho N_{b1}^T N_{b2} & 0 & \rho N_{b1}^T N_n & 0 & \rho N_{b1}^T N_w \\ * & \Theta_{22} & 0 & \Lambda_1 L_2 & \Gamma_1 S_2 & 0 & \Omega_1 M_2 & 0 \\ * & * & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Theta_{44} & 0 & \rho N_{b2}^T N_n & 0 & \rho N_{b2}^T N_w \\ * & * & * & * & -\Gamma_1 & 0 & 0 & 0 \\ * & * & * & * & * & \Theta_{66} & 0 & \rho N_n^T N_w \\ * & * & * & * & * & * & -\Omega_1 & 0 \\ * & * & * & * & * & * & * & \Theta_{88} \end{pmatrix} < 0 \quad (53)$$

where

$$\pi_{11} = A_1^T P_1 A_1 - 2P_1 + \Lambda_0 G^T G + \rho N_{a1}^T N_{a1} + \rho^{-1} P_1 L L^T P_1^T,$$

$$\pi_{22} = (1 + d_{2M} - d_{2m})Q_2 - \Lambda_2 U_1 - \Gamma_2 N_1 - \Omega_2 W_1 + \bar{\rho} R_2,$$

$$\pi_{44} = B_1 B_2 P_1 B_2^T B_1^T - \Lambda_2 + 2P_2 + \rho N_{a2}^T N_{a2},$$

$$\pi_{66} = B_1 N P_2 N^T B_1^T + B_2 N P_2 B_2^T + P_2 + \rho N_m^T N_m,$$

$$\begin{aligned}
\pi_{88} &= B_1 W P_2 W^T B_1^T + B_2 W P_2 W^T B_2^T + N W P_2 W^T N^T - \rho^{-1} R_2 + \rho N_s^T N_s, \\
\Theta_{11} &= B_1^T P_2 B_1 - 2P_2 + \theta_0 K^T K + \rho N_{b1}^T N_{b1} + \rho^{-1} P_2 L L^T P_2^T, \\
\Theta_{22} &= (1 + d_{1M} - d_{1m}) Q_1 - \Lambda_1 L_1 - \Gamma_1 S_1 - \Omega_1 M_1 + \bar{\mu} R_1, \\
\Theta_{44} &= A_1 A_2 P_2 A_2^T A_1^T - \Lambda_1 + 2p_1 + \rho N_{b2}^T N_{b2}, \\
\Theta_{66} &= A_1 M P_1 M^T A_1^T + A_2 M P_1 M^T A_2^T + P_1 + \rho N_n^T N_n, \\
\Theta_{88} &= A_1 S P_1 S^T A_1^T + A_2 S P_1 S^T A_2^T + M S P_1 S^T M^T - \mu^{-1} R_1 + \rho N_w^T N_w.
\end{aligned}$$

Proof. Replace A_1 , A_2 , M , S and B_1 , B_2 , N , W in Eq. (6) with $A_1 + \Delta A_1(k)$, $A_2 + \Delta A_2(k)$, $M + \Delta M(k)$, $S + \Delta S(k)$ and $B_1 + \Delta B_1(k)$, $B_2 + \Delta B_2(k)$, $N + \Delta N(k)$, $W + \Delta W(k)$, by using the same Lyapunov–Krasovskii functional as defined in Eq. (16), we calculate the difference among them, which yields the following inequality:

$$\begin{aligned}
\Xi_7 &= \Xi_1 + \begin{pmatrix} P_1 L \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + E(k) \begin{pmatrix} N_{a1} & 0 & 0 & N_{a2} & 0 & N_m & 0 & N_s \end{pmatrix} + \begin{pmatrix} N_{a1} \\ 0 \\ 0 \\ N_{a2} \\ 0 \\ N_m \\ 0 \\ N_s \end{pmatrix}^T E^T(k) \\
&\quad \times \begin{pmatrix} P_1 L & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\Xi_8 &= \Xi_2 + \begin{pmatrix} P_2 L \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + E(k) \begin{pmatrix} N_{b1} & 0 & 0 & N_{b2} & 0 & N_n & 0 & N_w \end{pmatrix} + \begin{pmatrix} N_{b1} \\ 0 \\ 0 \\ N_{b2} \\ 0 \\ N_n \\ 0 \\ N_w \end{pmatrix}^T E^T(k) \\
&\quad \times \begin{pmatrix} P_2 L & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \Xi_1 + \rho^{-1} \begin{pmatrix} P_1 L \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} P_1 L & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T + \rho \begin{pmatrix} N_{a1} \\ 0 \\ 0 \\ N_{a2} \\ 0 \\ N_m \\ 0 \\ N_s \end{pmatrix}^T \\
&\quad \times \begin{pmatrix} N_{a1} & 0 & 0 & N_{a2} & 0 & N_m & 0 & N_s \end{pmatrix}
\end{aligned}$$

and

$$= \Xi_2 + \rho^{-1} \begin{pmatrix} P_2 L \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} (P_2 L \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T + \rho \begin{pmatrix} N_{b1} \\ 0 \\ 0 \\ N_{b2} \\ 0 \\ N_n \\ 0 \\ N_w \end{pmatrix}^T \\ \times (N_{b1} \quad 0 \quad 0 \quad N_{b2} \quad 0 \quad N_n \quad 0 \quad N_w).$$

The remaining proof of this theorem is similar to that of as in [Theorem 3.2](#). Thus under conditions (52) and (53), system (51) is globally exponentially stable with uncertain parameters $\Delta A_1(k)$, $\Delta A_2(k)$, $\Delta M(k)$, $\Delta S(k)$, $\Delta B_1(k)$, $\Delta B_2(k)$, $\Delta N(k)$, $\Delta W(k)$. $\square$

5. Numerical examples

In this section, four numerical examples with simulation results are presented to show the effectiveness and applicability of the obtained theory.

Example 5.1. Consider the nominal discrete time stochastic impulsive BAM Neural Networks (6) with the following parameters.

$$A_1 = \begin{pmatrix} 0.07 & 0 \\ 0 & 0.03 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.6 & 0.3 \\ -0.8 & 0.5 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.05 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.6 & 0.09 \\ -0.7 & 0.8 \end{pmatrix},$$

$$M = \begin{pmatrix} -0.4 & 0.4 \\ 0.02 & -0.3 \end{pmatrix}, \quad S = \begin{pmatrix} 0.5 & -0.8 \\ 0.4 & 0.01 \end{pmatrix}, \quad N = \begin{pmatrix} 0.5 & 0.02 \\ 0.3 & -0.1 \end{pmatrix}, \quad W = \begin{pmatrix} 0.5 & 0.3 \\ 0.8 & 0.04 \end{pmatrix}$$

$$\delta = \begin{pmatrix} 0.04(\cos(\frac{\pi}{2})) & 0 \\ 0 & 0.06(\sin(\frac{\pi}{2}) - 2) \end{pmatrix}, \quad \bar{\sigma} = \begin{pmatrix} 0.08(\sin(\frac{\pi}{2})) & 0 \\ 0 & 0.05(\cos(\frac{\pi}{2}) - 3) \end{pmatrix}$$

$I_m = -2$, $J_m = -2$, $\xi_1 = \xi_2 = 5$, $\mu = \exp(-0.8)$, $\rho = \exp(-0.7)$, $\omega_1 = 0.7$, $\bar{\omega}_2 = 0.6$. The activation functions are taken as

$$f(y(k)) = g(y(k)) = h(y(k)) = \begin{pmatrix} \tan h(0.7y) \\ \tan h(0.7y) \\ \tan h(0.7y) \end{pmatrix},$$

$$\bar{f}(x(k)) = \bar{g}(x(k)) = \bar{h}(x(k)) = \begin{pmatrix} \tan h(0.5x) \\ \tan h(0.5x) \\ \tan h(0.5x) \end{pmatrix}$$

Our main aim in this example is to estimate the maximum allowable upper bound delay d_{1M} , d_{2M} , for given lower bound d_{1m} , d_{1m} . By solving MATLAB LMI toolbox, one can easily obtain the feasible solution for any time delay satisfying $0 < d_1(k) < d_{1M}$ for any large finite value and $0 < d_2(k) < d_{2M}$ for any large finite value. For example, if we take

$$d_{1m} = d_{2m} = 2, \quad d_{1M} = d_{2M} = 7, \quad l_1^+ = 2, \quad l_1^- = -4, \quad l_2^+ = 1, \quad l_2^- = -1, \quad s_1^+ = 0.2, \quad s_1^- = 0.2, \\ s_2^+ = 0.2, \quad s_2^- = 0.2, \quad m_1^+ = 3, \quad m_1^- = -0.3, \quad m_2^+ = 3, \quad m_2^- = 1.65, \quad u_1^+ = 4, \quad u_1^- = -0.2, \quad u_2^+ = 4,$$

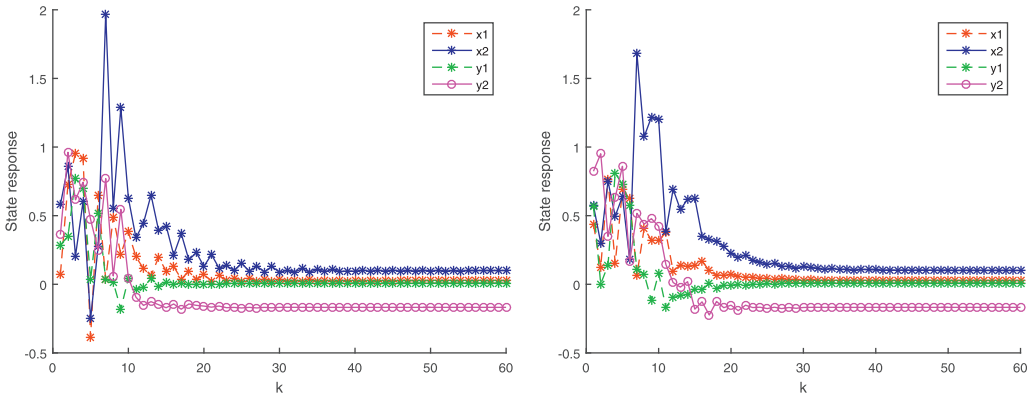


Fig. 1. For $I_m = -2J_m = -2$, the state response $x(k)$, $y(k)$ of Eq. (6) with impulsive and non-impulsive effects.

$u_2^- = 0.2, n_1^+ = 0.2, n_1^- = 0.2, n_2^+ = 0.2, n_2^- = 0.2, w_1^+ = 3, w_1^- = 0.2, w_2^+ = 3, w_2^- = -0.2$. with

$$L_1 = \begin{pmatrix} -8 & 0 \\ 0 & -1 \end{pmatrix}, L_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, S_1 = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.04 \end{pmatrix}, S_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} -0.9 & 0 \\ 0 & 0.9 \end{pmatrix}, M_2 = \begin{pmatrix} 1.35 & 0 \\ 0 & 1.65 \end{pmatrix}, U_1 = \begin{pmatrix} -0.8 & 0 \\ 0 & 0.8 \end{pmatrix}, U_2 = \begin{pmatrix} 1.9 & 0 \\ 0 & 2.1 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.04 \end{pmatrix}, N_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, W_1 = \begin{pmatrix} 0.6 & 0 \\ 0 & -0.6 \end{pmatrix}, W_2 = \begin{pmatrix} 1.6 & 0 \\ 0 & 1.4 \end{pmatrix}.$$

According to Theorem 3.2, we can obtain that system (6) with above given parameters are exponentially stable with the help of Lyapunov functions and state trajectories $x_1(k)$, $x_2(k)$, $y_1(k)$, $y_2(k)$ of above discrete-time stochastic impulsive BAM neural networks are depicted in Fig. 1.

Example 5.2. Consider the nominal discrete-time BAM neural networks (43) with the following parameters.

$$A_1 = \begin{pmatrix} 0.07 & 0 \\ 0 & 0.06 \end{pmatrix}, A_2 = \begin{pmatrix} 0.8 & -0.7 \\ -0.1 & -0.6 \end{pmatrix}, B_1 = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.08 \end{pmatrix}, B_2 = \begin{pmatrix} 0.06 & 1.35 \\ -0.2 & -0.5 \end{pmatrix},$$

$$M = \begin{pmatrix} -0.2 & -0.8 \\ 0 & -0.9 \end{pmatrix}, S = \begin{pmatrix} 0.25 & -0.8 \\ -0.4 & -0.5 \end{pmatrix}, N = \begin{pmatrix} -0.9 & 0.2 \\ 0.3 & 1.22 \end{pmatrix}, W = \begin{pmatrix} 0.3 & -0.3 \\ 0.8 & -0.4 \end{pmatrix}$$

$I_m = -2, J_m = -2, \xi_1 = \xi_2 = 4, \mu = \exp(-9), \rho = \exp(-8)$ Take the activation functions as,

$$f(y(k)) = g(y(k)) = h(y(k)) = \begin{pmatrix} \sinh(-0.5y) \\ \sinh(-0.5y) \\ \sinh(-0.5y) \end{pmatrix}$$

$$\bar{f}(x(k)) = \bar{g}(x(k)) = \bar{h}(x(k)) = \begin{pmatrix} \sinh(-0.5x) \\ \sinh(-0.5x) \\ \sinh(-0.5x) \end{pmatrix}$$

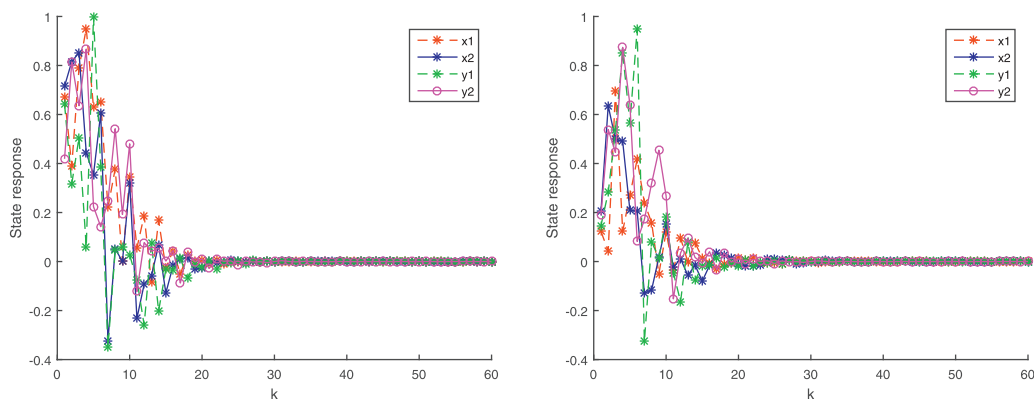


Fig. 2. For $I_m = -2J_m = -2$, the state response $x(k)$, $y(k)$ of Eq. (43) with impulsive and non-impulsive effects.

According to Corollary 3.3, we can obtain that system (43) with above given parameters without stochastic are exponentially stable with the help of Lyapunov functions and state trajectories $x_1(k)$, $x_2(k)$, $y_1(k)$, $y_2(k)$ of above discrete-time impulsive BAM neural networks are depicted in Fig. 2.

Example 5.3. Consider the nominal discrete-time stochastic BAM neural networks (48) with the following parameters.

$$A_1 = \begin{pmatrix} 0.09 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.5 & 0.6 \\ -0.9 & 1.7 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.06 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1.6 & 0.9 \\ -0.2 & 1.6 \end{pmatrix},$$

$$M = \begin{pmatrix} 0.7 & -0.4 \\ 0.02 & -0.02 \end{pmatrix}, \quad S = \begin{pmatrix} -0.5 & -0.8 \\ 0.4 & 1.1 \end{pmatrix}, \quad N = \begin{pmatrix} 0.05 & 0.02 \\ 0.03 & -0.1 \end{pmatrix}, \quad W = \begin{pmatrix} 0.05 & 1.3 \\ 0.8 & 0.04 \end{pmatrix}$$

$$\omega_1 = \begin{pmatrix} 0.9(\cos(\frac{\pi}{2})) & 0 \\ 0 & 0.4(\sin(\frac{\pi}{2}) - 1) \end{pmatrix}, \quad \bar{\omega}_2 = \begin{pmatrix} 0.4(\sin(\frac{\pi}{2})) & 0 \\ 0 & 0.5(\cos(\frac{\pi}{2}) - 2) \end{pmatrix}$$

$$\xi_1 = \xi_2 = 4 \mu = \exp(-1.7) \quad \rho = \exp(-1.9) \quad \omega_1 = \bar{\omega}_2 = -2.6.$$

Take the activation functions as,

$$f(y(k)) = g(y(k)) = h(y(k)) = \begin{pmatrix} \tanh(-0.5y) \\ \tanh(-0.5y) \\ \tanh(-0.5y) \end{pmatrix},$$

$$\bar{f}(x(k)) = \bar{g}(x(k)) = \bar{h}(x(k)) = \begin{pmatrix} \tanh(-0.5x) \\ \tanh(-0.5x) \\ \tanh(-0.5x) \end{pmatrix}$$

According to Corollary 3.4, we can obtain that system (48) with above given parameters without impulse are exponentially stable with the help of Lyapunov functions and state trajectories $x_1(k)$, $x_2(k)$, $y_1(k)$, $y_2(k)$ of above discrete-time stochastic BAM neural networks are depicted in Fig. 3.

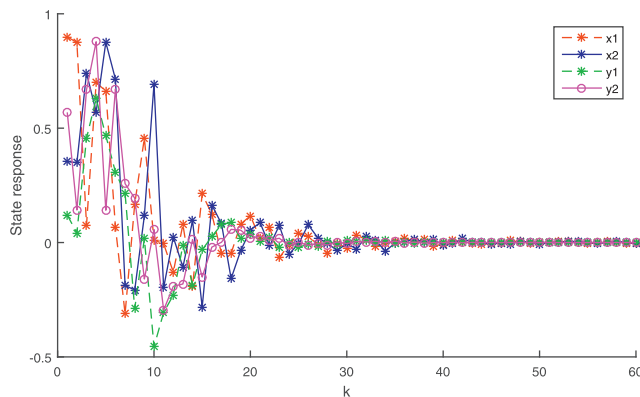


Fig. 3. For the state response $x(k)$, $y(k)$ of Eq. (48) with non-impulsive effects.

Example 5.4. Consider the uncertain discrete-time stochastic BAM neural networks (51) with the following parameters.

$$A_1 = \begin{pmatrix} 0.08 & 0 \\ 0 & 0.9 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1.7 & 0.9 \\ -0.6 & 0.4 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.08 & 0 \\ 0 & 0.7 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1.8 & 0.6 \\ -0.7 & 0.09 \end{pmatrix},$$

$$M = \begin{pmatrix} 0.6 & 0.5 \\ 0.2 & 0.1 \end{pmatrix}, \quad S = \begin{pmatrix} 0.5 & -0.8 \\ 0.4 & 0.1 \end{pmatrix}, \quad N = \begin{pmatrix} -0.5 & 0.09 \\ 0.3 & -0.1 \end{pmatrix}, \quad W = \begin{pmatrix} 0.5 & -0.3 \\ 0.8 & 0.4 \end{pmatrix}$$

$$N_{a1} = \begin{pmatrix} -0.8 & 0.2 \\ 0.3 & -0.9 \end{pmatrix}, \quad N_{a2} = \begin{pmatrix} -1.4 & 0.1 \\ 0.4 & 0.3 \end{pmatrix}, \quad N_m = \begin{pmatrix} -0.2 & 0.9 \\ 0.8 & 1.3 \end{pmatrix}, \quad N_s = \begin{pmatrix} -1.3 & 0.7 \\ 0.6 & 0.5 \end{pmatrix},$$

$$N_{b1} = \begin{pmatrix} -2.3 & 0.5 \\ 0.9 & 0.3 \end{pmatrix}, \quad N_{b2} = \begin{pmatrix} 1.3 & 0.8 \\ -0.09 & 1.3 \end{pmatrix}, \quad N_n = \begin{pmatrix} 1.3 & -0.9 \\ -0.8 & 0.3 \end{pmatrix}, \quad N_w = \begin{pmatrix} 2.3 & 0.67 \\ 0.78 & -0.4 \end{pmatrix}$$

$$\delta = \begin{pmatrix} 0.4(\cos(\frac{\pi}{2})) & 0 \\ 0 & 0.6(\sin(\frac{\pi}{2}) - 2) \end{pmatrix}, \quad \bar{\sigma} = \begin{pmatrix} 0.8(\sin(\frac{\pi}{2})) & 0 \\ 0 & 0.5(\cos(\frac{\pi}{2}) - 3) \end{pmatrix}$$

$$N^* = \begin{pmatrix} 0.09 & 0 \\ 0 & 0.09 \end{pmatrix} G = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, \quad \xi_1 = \xi_2 = 2 \quad \mu = \rho = \exp(-1.6)$$

Take the activation functions as,

$$f(y(k)) = g(y(k)) = h(y(k)) = \begin{pmatrix} y^5/(1+y^3) \\ y^2/(1+y^3) \\ y^2/(1+y^3) \end{pmatrix},$$

$$\bar{f}(x(k)) = \bar{g}(x(k)) = \bar{h}(x(k)) = \begin{pmatrix} x^2/(1+x^3) \\ x^2/(1+x^3) \\ x^2/(1+x^3) \end{pmatrix}$$

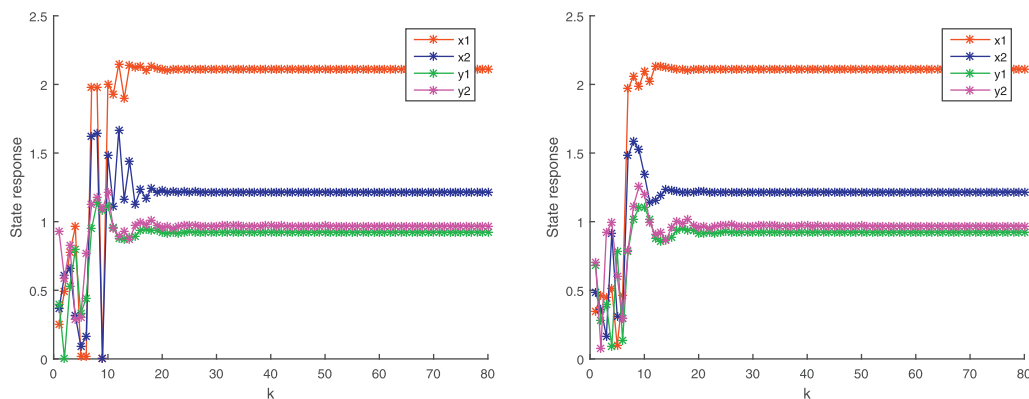


Fig. 4. For $I_m = -2J_m = -2$, the state response $x(k)$, $y(k)$ of Eq. (51) with impulsive and non-impulsive effects.

It can be verified that $d_{1m} = d_{2m} = 2$, $d_{1M} = d_{2M} = 7$, $l_1^+ = 2$, $l_1^- = -4$, $l_2^+ = 1$, $l_2^- = -1$, $s_1^+ = 0.2$, $s_1^- = 0.2$, $s_2^+ = 0.2$, $s_2^- = 0.2$, $m_1^+ = 3$, $m_1^- = -0.3$, $m_2^+ = 3$, $m_2^- = 1.65$, $u_1^+ = 4$, $u_1^- = -0.2$, $u_2^+ = 4$, $u_2^- = 0.2$, $n_1^+ = 0.2$, $n_1^- = 0.2$, $n_2^+ = 0.2$, $n_2^- = 0.2$, $w_1^+ = 3$, $w_1^- = 0.2$, $w_2^+ = 3$, $w_2^- = -0.2$. with

$$L_1 = \begin{pmatrix} -8 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.04 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} -0.9 & 0 \\ 0 & 0.9 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1.35 & 0 \\ 0 & 1.65 \end{pmatrix}, \quad U_1 = \begin{pmatrix} -0.8 & 0 \\ 0 & 0.8 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1.9 & 0 \\ 0 & 2.1 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.04 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad W_1 = \begin{pmatrix} 0.6 & 0 \\ 0 & -0.6 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 1.6 & 0 \\ 0 & 1.4 \end{pmatrix}.$$

According to Theorem 4.1, we can obtain that system (51) with above given parameters are exponentially stable with the help of Lyapunov functions and state trajectories $x_1(k)$, $x_2(k)$, $y_1(k)$, $y_2(k)$ of above uncertain discrete-time stochastic impulsive BAM neural networks are depicted in Fig. 4.

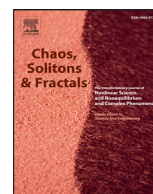
6. Conclusion

The stability analysis of impulsive discrete-time stochastic BAM neural networks with delays and uncertain terms have been investigated. By employing Lyapunov–Krasovskii functional, linear matrix inequality approach, some sufficient conditions for the SBAMNNs to be globally exponentially stable and robust exponentially stable are derived. The derived occlusions can be easily checked by using LMI control toolbox in MATLAB software. Finally, four numerical examples and their simulations are given to show the superiority of our proposed stability conditions. It should be worth noting that the idea given in this paper can be widely applicable in many cases. It would be interesting to extend the proposed main results in this paper to switched neural networks with mode leakage time-varying and Markovian jumping parameters by using delay partitioning approaches and corresponding results will appear in near future.

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Novel results on passivity and exponential passivity for multiple discrete delayed neutral-type neural networks with leakage and distributed time-delays

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ABSTRACT

This paper investigates the problem of passivity and exponential passivity for neutral-type neural networks (NNNs) with leakage, multiple discrete delay and distributed time-delay, via some novel sufficient conditions. Based on an appropriate Lyapunov-Krasovskii functional (LKF), free weighting matrix approach and some inequality techniques, enhanced passivity criteria for the concerned neural networks is established in the form of Linear matrix inequalities (LMIs). The feasibility of the attained passivity and exponential passivity criterions easily verified by the aid of LMI control toolbox in MATLAB software. Furthermore, we have compared our method with previous one in the existing literature, which depicts its less conservativeness. To substantiate the superiority and effectiveness of our analytical design, two examples with their numerical simulations are provided.

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1. Introduction

In current scenario, extensive attention has been paid on neural networks (NNs) due to their fruitful applications in many fields, such as intelligent robot, signal processing, associative memories, fixed-point computations, automatic control, artificial intelligence, and so on [13,16,34,44]. It is well known that the time delays often occur from finite switching speed of the communication time, amplifiers and faults in the electrical circuits when the implementation of neural networks. Thus, the existence of time delays is inevitable in dynamical systems, which may affects the passiveness generating instability, bad performance, chaotic behavior

and swinging or oscillation. Hence, the study on neural networks with time-delays have been considerable attention from many researchers, see for references [6,31,36]. In [43,46], the authors investigates the passivity criteria for time delayed neural networks with the help of LKF.

In accordance, the delay-dependent and delay-independent criteria are two categories of time delays in neural networks, which classified by the existing time delayed results. So far as, one can observe from available literatures, the delay-dependent case is less conserved than the delay independent ones. Further, time delay can be characterized into two types: a Discrete and Distributed delays. Here, we have taken both time delays, that is multiple discrete time delays and the distributed delays, into account while model our network system, because the length of the axon sizes are too large. For sequence, the passivity of various classes of neural networks with time delay has become an interesting area and different passivity conditions have been established for such NNs for multiple delays [53,57] and distributed time-delays [8,23,63]. As a result, it is noteworthy to inspect both the time-delay

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results in the passivity behaviors of systems, see for reference [18].

It is pointed out that a special case of time delay in neural networks that which incorporated to the negative feedback term of the system, namely forgetting or leakage terms. The present of this type of delays owing to some theoretical and technical difficulties. By the result of leakage delays the stability of the dynamical systems may affect and it leads to be unstable. So, over the past few years, time delay in leakage term has a enormous impact on the neural networks, see for references [12,24,26,38]. Thus, the consideration of leakage term in neural networks is necessary to passivity investigation [28,45]. The authors in [3], studied the stability of recurrent NNs when arise the leakage term in state variable via linear matrix inequalities. Mala et al., in [33], analyzed the problem of passiveness for neural networks, in which some suitable LKF are handled with leakage terms. Thus, the authors in [64], studied and obtained some results related to the problem for stability of time delayed neural networks with leakage terms.

Moreover, neutral time delay is another type of delay which has drawn a lot of attention nowadays. Because, a neutral-type delay phenomenon contains delays both in its derivatives and state variables. Practically, such phenomenon can be established in several fields including mechanics, automatic control, vibrating masses attached to an elastic bar, ecology, lossless transmission lines, robots in contact with rigid environments, heat exchangers, etc. Consequently, the delayed neural networks with neutral terms has received much attention in recent years and a enormous number of findings on this issue have been reported [2,10,27,37]. As for as, in [4], H. Bao and J. Cao discussed about the stability performance of neutral type neural networks with time delays. The less conserved criteria for stability of time delayed networks is obtained and investigated by Y. He, G. P. Liu, D. Rees in [14].

On the other hand, from the circuit theory, the concept of passivity was investigated and it plays a key role in the design and analysis of both linear and nonlinear systems, particularly for high-order systems. Also, the analysis of passivity has received a great attention [5,32,62], because it serves as a wonderful tool for studying the properties of nonlinear systems, such as signal processing, synchronization and stability. The core of the theory states that the properties of passiveness can ideally keep the system internally stable. This means that, the product of input and output in the passive neural networks is used as the energy provision and embodies the energy attenuation character which is a noteworthy feature of passivity. By the reason of its significance, the results on analysis of passivity for neural networks has been widespread studied in the available literature [9,25]. In [54], the authors examined the time dependent criteria for passivity in neural networks. In addition, P. Balasubramaniam, G. Nagamani and R. Rakkiyappan in [1], analyzed and obtained the LMI criteria for passivity, when the neutral and leakage delays are taken into account simultaneously. In current decades, the problem of passivity in the exponential sense for neural networks has been analyzed by many researchers. In [51], a new criterion were obtained for the addressed NNs to ensure the exponential passivity. It is note worthy to point out that, owing to its mathematical complexity, the exponential passivity condition has not been studied in the existing literature for neutral type neural networks with leakage, multiple discrete time-varying delay and distributed delay. Also, Du et al., investigates the existence of time delays in neural networks and exploit the LKF approach, the new sufficient criteria were derived for exponential passivity in [9]. So, the aim of this paper is meaningful and this motivates the present research problem.

Motivated by the earlier discussions, the problem for passivity and exponential passivity investigation of the neutral-type neural networks with leakage, multiple discrete delays and distributed time delays has not been fully studied yet, which motivates our

Table 1

Maximum allowable upper bounds (h_k^*) of discrete delay for various τ_k .

Methods	$\tau_k = 0$	$\tau_k = 0.3$	$\tau_k = 0.5$	$\tau_k = 0.9$
In Ref [11]	-	0.5763	0.5679	0.5273
In Ref [22]	-	1.1921	1.1590	1.1081
In Ref [58]	-	-	1.4693	1.4243
In Ref [42]	5.346	-	4.592	-
In Ref [7]	8.7788	-	0.8912	0.3463
Theorem 3.1	28.71	21.627	19.39	14.4311

Table 2

Maximum allowable upper bounds (η_k^*) of neutral delay for different λ_k .

λ_k	0.2	0.5	0.9
In Ref [35]	-	4.5306	4.1632
In Ref [20]	-	4.7388	1.5708
Theorem 3.1	26.83	17.291	13.045

Table 3

Maximum allowable upper bounds (MAUB) h_k^* in Example 5.2 for different τ_k .

Methods	0.3	0.7	0.8	0.9
In Ref [30]	-	2.3368	1.9109	1.5523
In Ref [60]	-	2.0205	1.6377	1.2632
In Ref [56]	-	2.9436	2.5630	2.2150
In Ref [41]	1.4021	1.3916	-	1.3734
Theorem 4.1	22.59	17.104	10.221	7.28

current research work. With this aim to fill this gap, in this paper, we consider the passivity as well as exponential passivity problem for NNNs with leakage delays, multiple discrete delays and distributed delays. By utilizing the Lyapunov-Krasovskii functional, matrix theory and some inequality techniques, brand-new sufficient conditions for passivity and exponential passivity are formulated in the form of Linear matrix inequalities, which can be substantiated easily by LMI control toolbox in MATLAB software. As well, two numerical examples with their simulations are provided to illustrate the superiority and applicability of the proposed method. The main contributions of this work are highlighted as follows

- (1) The neutral term, leakage time delay, multiple discrete delay & distributed time-delay are taken into account in the passivity and exponential passivity analysis of proposed neural networks.
- (2) Based on different novel Lyapunov-Krasovskii functional, some sufficient conditions for passivity and exponential passivity of neutral-type NNs are derived in the form of LMIs. Furthermore, compared to the existing results, the derived conclusions are different and advanced.
- (3) In this paper, the feasibility of the obtained LMIs for passivity and exponential passivity, easily solved by the aid of LMI control toolbox in MATLAB software.
- (4) By handled the multiple discrete time-varying delay and neutral time-varying delay terms in our addressed neural networks, the allowable upper bounds of multiple discrete time-delay and neutral delay are maximum, when compare with the existing literatures, see Table 1, Table 2 and Table 3 in Example 5.1 & 5.2. This shows that the approach developed in this paper is effective and less conservative than some available results.

The organization of this paper is as follows. The considered time-delayed neutral-type neural network model is presented and some preliminaries are introduced in Section 2. In Section 3 & Section 4, by the Lyapunov-Krasovskii functionals and integral inequality techniques, some time delay dependent conditions are

obtained for the passivity and exponential passivity of concerned neural networks. The obtained results are formulated in the form of linear matrix inequalities, that are easily solvable by using MATLAB LMI control toolbox. Then, two numerical examples with their simulations are provided to demonstrate the feasibility and effectiveness of the derived passivity and exponential passivity criterions in Section 5. Finally, conclusions are drawn in Section 6.

Nomenclature: Throughout this paper, $\mathbb{R}^n$ denotes the n -dimensional Euclidean space and $\mathbb{R}$ be set of all real numbers. $\mathbb{R}^{n \times n}$ is the set of all $n \times n$ matrices with real entries. For a matrix Q , $Q > 0$ ($Q \geq 0$) represents that matrix Q is symmetric and positive definite (positive semi-definite). The notations Q^T & Q^{-1} denotes its transpose and inverse, respectively. $C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ is the family of all non negative functions $V(t, u)$ on $\mathbb{R}^+ \times \mathbb{R}^n$ which are twice continuously differentiable in u and once differentiable in t ; The symbol ** indicates symmetric term in a symmetric matrix. $C([-m, 0]; \mathbb{R}^n)$ stands for the family of continuous functions ϕ from $[-m, 0]$ to $\mathbb{R}^n$ with norm $\|\phi\| = \sup_{-m \leq \theta \leq 0} |\phi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. $\text{diag}\{\dots\}$ denotes a block-diagonal matrix. For a matrix U , $\lambda_{\max}(U)$ and $\lambda_{\min}(U)$ denote the maximum and minimum eigenvalues of matrix U , respectively.

2. Model description and preliminaries

Consider the neutral-type neural networks with multiple discrete time-varying delay, distributed and leakage delays as follows

$$\begin{aligned} \dot{u}_i(t) &= \sum_{j=1}^n W_{2ij} \dot{u}_j(t - \eta_k(t)) \\ &= -a_i u_i(t - \delta) + \sum_{j=1}^n W_{0ij} g_j(u(t)) + \sum_{j=1}^n \sum_{k=1}^c W_{1ij}^{(k)} g_j(u(t - h_k(t))) \\ &\quad + \sum_{j=1}^n W_{3ij} \int_{t-\sigma}^t g_j(u(s)) ds + \mathcal{H}_i(t), \\ v_i(t) &= \sum_{j=1}^n g_j(u(t)) + \sum_{j=1}^n g_j(u(t - h_k(t))) + \sum_{j=1}^n \int_{t-\sigma}^t g_j(u(s)) ds \\ &\quad + \mathcal{H}_i(t), \quad i = 1, 2, 3, \dots, n; \quad c \in \mathbb{N} \end{aligned} \quad (1)$$

It can be remodified as

$$\begin{aligned} \dot{u}(t) &= W_2 \dot{u}(t - \eta_k(t)) \\ &= -Au(t - \delta) + W_0 g(u(t)) + \sum_{k=1}^c W_1^{(k)} g(u(t - h_k(t))) + W_3 \\ &\quad \times \int_{t-\sigma}^t g(u(s)) ds + \mathcal{H}(t), \\ v(t) &= g(u(t)) + g(u(t - h_k(t))) + \int_{t-\sigma}^t g(u(s)) ds + \mathcal{H}(t), \end{aligned} \quad (2)$$

where $u(t) = [u_1(t), u_2(t), \dots, u_n(t)]^T \in \mathbb{R}^n$ be the neurons state vector at time t . $A = \text{diag}[a_1, a_2, \dots, a_n] > 0$ represents the passive decay rate; $g(u(t)) = [g_1(u(t)), g_2(u(t)), \dots, g_n(u(t))]^T$ is the activation function of neurons, W_0 denotes the connection weight matrix and $W_1^{(k)}$, W_2 & W_3 are the delayed connection weight matrices; $\mathcal{H}(t) = [\mathcal{H}_1(t), \mathcal{H}_2(t), \dots, \mathcal{H}_n(t)]^T$ indicates the external inputs at time t , the output of the neural networks represent by $v(t)$. The function $v^T(t)\mathcal{H}(t)$ indicates a supply rate if it is locally integrable; i.e., for all input-output pairs $\mathcal{H}(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^n$ it satisfies $\int_0^t |v^T(\xi)\mathcal{H}(\xi)| d\xi \leq \infty$, for all $t \geq 0$. $h_k(t)$ denotes the discrete transmission delay with $0 \leq h_k(t) \leq h_k^*$; $\dot{h}_k(t) \leq \tau_k < 1$, where $h_k^* = \max\{h_k(t)\}$ and σ is the constant delay, δ represent leakage delay, which is a constant. $\eta_k(t)$ denotes the discrete transmission neutral delay with $0 \leq \eta_k(t) \leq \eta_k^*$, $\dot{\eta}_k(t) \leq \lambda_k < 1$, where $\eta_k^* = \max\{\eta_k(t)\}$.

Furthermore, the initial condition of neural networks (2) can be described by

$$u(t) = \phi(t), \quad \dot{u}(t) = \varphi(t), \quad t \in [-m, 0], \quad m = \max\{h_k^*, \eta_k^*\}, \quad (3)$$

where $\phi(t)$ & $\varphi(t)$ are continuous functions on $[-m, 0]$.

Hence, by the aid of well-known Brouwer's fixed-point theorem, neural networks (2) has at least one equilibrium point, say u^* .

Remark 2.1. With the motivation of function of a human brain, artificial neural networks are able to learn from experience. These powerful problem solvers are highly effective where traditional, formal analysis would be complexity or impossible. Their strength lies in their ability to make sense out of difficult, noisy, or nonlinear data. Neural networks can provide robust solutions to problems in a wide range of disciplines, particularly areas involving filtering, pattern recognition, classification, prediction, function approximation and optimization. Naturally, the passivity issue is internally stable for neural networks. Time delay is unavoidable in real-world neural networks. In practical implementations by electrical circuits, time-delays occur in the finite switching speed of amplifiers and the signal transmissions among neurons. As is well known that, the existence of time-delays in multiple discrete, distributed, leakage and neutral terms causes poor performance, oscillation and instability of the concerned neural networks. Thus, by the help of different Lyapunov-Krasovskii functional, the oscillations and instability are controlled while checking the passivity and also the exponential convergence rate increases the speed to reach the system stability while checking the exponential passivity.

Throughout this paper, to obtain sufficient criteria for passivity as well as exponential passivity, we made the following assumptions.

Assumption 1. The activation function g in (2) is continuously differentiable, bounded and satisfies

$$\begin{aligned} 0 &\leq \frac{g_j(s)}{s} \leq \beta_j, \quad \forall s \in \mathbb{R}, \\ g_j(0) &= 0, \quad (j = 1, 2, \dots, n), \end{aligned} \quad (4)$$

where $\beta_j > 0$.

Remark 2.2. It is point out that, in Assumption 1, the activation functions are less restrictive than the descriptions on sigmoid activation functions. These functions will playing in our proposed methods to get necessary conditions for the neural networks (2) which leads to passivity as well as exponential passivity. By the Assumption 1 we can easily observe that the functions $g_j(\cdot)$ are monotonic increasing and $g_j(s)s \geq 0$, $(j = 1, 2, \dots, n)$.

Definition 2.3. The neural networks (2) with input $\mathcal{H}(t)$ & output $v(t)$ is said to be passive if $\exists$ a positive scalar μ such that

$$2 \int_{t_0}^{t_1} v^T(t)\mathcal{H}(t) dt \geq -\mu \int_{t_0}^{t_1} \mathcal{H}^T(t)\mathcal{H}(t) dt,$$

$\forall t_1 \geq 0$ and for all solutions of (2) under zero initial condition.

Definition 2.4. Neural networks (2) is called passive in the sense exponential with input $\mathcal{H}(t)$ and output $v(t)$, if there exist an Lyapunov function in the sense of exponential (or, called the exponential storage function) V defined on $\mathbb{R}^n$ and a scalar $\gamma > 0$ such that the inequality is satisfied as follows :

$$\dot{V}(u(t), t) + \gamma V(u(t), t) \leq 2v^T(t)\mathcal{H}(t), \quad t \geq 0,$$

for all $\mathcal{H}(t)$, all initial conditions $u(0)$, all $t \geq t_0$, where $\dot{V}(u(t), t)$ indicates the total derivative of $V(u(t), t)$ along the trajectories $u(t)$, $t \geq 0$, of (2).

Lemma 2.5. ([61]) Let B , D , and P be matrices with real entries of appropriate dimensions with $P > 0$. Then the following statement holds:

For any vectors a_1 and b_1 with appropriate dimensions

$$2a_1^T B D b_1 \leq a_1^T B P B^T a_1 + b_1^T D^T P^{-1} D b_1$$

By using the Leibniz-Newton formula, for any matrices K_i , A_i , ($i = 1, 2, 3, \dots, 8$) and M_j ($j = 1, 2, 3$) with appropriate dimensions, we can get the following equations:

$$\begin{aligned} & 2[u^T(t-\delta)K_1 + u^T(t)K_2 + u^T(t-h_k(t))K_3 + u^T(t-\sigma)K_4 \\ & + \dot{u}^T(t-\eta_k(t))K_5 + g^T(u(t))K_6 + g(u(t-h_k(t)))K_7 \\ & + \dot{u}^T(t)K_8] \left[u(t) - u(t-h_k(t)) - \int_{t-h_k(t)}^t \dot{u}(s)ds \right] = 0 \end{aligned} \quad (5)$$

$$\begin{aligned} & 2[u^T(t-\delta)A_1 + u^T(t)A_2 + u^T(t-h_k(t))A_3 + u^T(t-\sigma)A_4 \\ & + \dot{u}^T(t-\eta_k(t))A_5 + g^T(u(t))A_6 + g(u(t-h_k(t)))A_7 \\ & + \dot{u}^T(t)A_8] \left[u(t) - u(t-\sigma) - \int_{t-\sigma}^t \dot{u}(s)ds \right] = 0 \end{aligned} \quad (6)$$

$$\begin{aligned} & 2[u^T(t-\delta)M_1 + u^T(t)M_2 + \dot{u}^T(t)M_3 + \mathcal{H}^T(t)\mathcal{B}_1^*] \\ & \times \left\{ -Au(t-\delta) + W_0g(u(t)) + W_1^k g(u(t-h_k(t))) \right. \\ & \left. + W_3 \int_{t-\sigma}^t g(u(s))ds + \mathcal{H}(t) - \dot{u}(t) + W_2\dot{u}(t-\eta_k(t)) \right\} = 0 \end{aligned} \quad (7)$$

Remark 2.6. In [40,50], the output fully depends on the state vectors and controls the input of a system. From the aforementioned motivations, the output of NNs in our developed method which is obtained by the activation function of discrete as well as distributed time-delayed state variable terms, which generates advances of passivity conditions.

Remark 2.7. The enhanced neural networks proposed with constant value (or time invariant) of discrete & distributed delays in [21] and the constant time-delay in neutral term is derived in the system of [17,19] to guarantee the stochastic stability. In this paper, we contribute to the further improvement of passivity and exponential passivity criteria for neural networks with discrete, neutral and distributed time-varying delays with some bound conditions, which exposes the novelty of our designed method.

3. Passivity results

Theorem 3.1. Under Assumption 1, for given $\tau_k < 1$ and $\lambda_k < 1$, if there exists matrices $Q > 0$, $S > 0$, $R > 0$, $Y > 0$, L_1 , L_2 , H_1 , $H_2 > 0$ and diagonal matrix $T > 0$ and any matrices K_i ($i = 1, 2, \dots, 8$), $\mathcal{B}_1^*$, and M_j ($j = 1, 2, 3$) with appropriate dimension such that the following LMI satisfied

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} & \Xi_{17} & \Xi_{18} & 0 & \Xi_{110} & 0 & \Xi_{112} & 0 & \Xi_{114} & \Xi_{115} \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} & \Xi_{26} & \Xi_{27} & \Xi_{28} & 0 & \Xi_{210} & 0 & \Xi_{212} & 0 & \Xi_{214} & \Xi_{215} \\ * & * & \Xi_{33} & 0 & 0 & 0 & \Xi_{37} & \Xi_{38} & 0 & \Xi_{310} & 0 & \Xi_{312} & 0 & \Xi_{314} & \Xi_{315} \\ * & * & * & \Xi_{44} & 0 & 0 & \Xi_{47} & \Xi_{48} & 0 & 0 & 0 & \Xi_{412} & 0 & \Xi_{414} & \Xi_{415} \\ * & * & * & * & \Xi_{55} & 0 & 0 & \Xi_{58} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Xi_{66} & \Xi_{67} & \Xi_{68} & 0 & \Xi_{610} & 0 & \Xi_{612} & 0 & \Xi_{614} & \Xi_{615} \\ * & * & * & * & * & * & \Xi_{77} & \Xi_{78} & 0 & \Xi_{710} & 0 & \Xi_{712} & 0 & \Xi_{714} & 0 \\ * & * & * & * & * & * & * & \Xi_{88} & 0 & \Xi_{810} & 0 & \Xi_{812} & 0 & \Xi_{814} & \Xi_{815} \\ * & * & * & * & * & * & * & * & \Xi_{99} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Xi_{1010} & 0 & \Xi_{1012} & 0 & \Xi_{1014} & 0 \\ * & * & * & * & * & * & * & * & * & * & \Xi_{1111} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Xi_{1212} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & \Xi_{1313} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & * & \Xi_{1414} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & \Xi_{1515} \end{bmatrix} < 0, \quad (8)$$

where

$$\begin{aligned} \Xi_{11} &= QW_3\Theta_2^{-1} + Y + H_1 + K_2 + A_2, \\ \Xi_{12} &= -2QA + 2K_1 + 2A_1 - 2AM_2, \quad \Xi_{1212} = -K_8\Theta_3, \\ \Xi_{13} &= 2W_0Q + 2K_6 + 2A_6 + 2W_0M_2 + 2T\Upsilon, \\ \Xi_{14} &= 2QW_1^{(k)} + 2K_7 + 2A_7 + 2M_2W_1^{(k)}, \\ \Xi_{15} &= 2QW_3 + 2W_3M_2, \quad \Xi_{16} = 2QW_2 + 2K_5 + 2W_2M_2, \\ \Xi_{17} &= 2K_3 - 2K_2 + 2A_3, \quad \Xi_{1313} = L_2, \quad \Xi_{18} = 2K_8 + 2A_8 - 2M_2, \\ \Xi_{110} &= 2K_4 + 2A_4 - 2A_2, \quad \Xi_{112} = -2K_2, \\ \Xi_{114} &= -2A_2, \quad \Xi_{23} = 2W_0M_1, \\ \Xi_{24} &= 2W_1^{(k)}M_1, \quad \Xi_{25} = 2M_1W_3, \quad \Xi_{26} = 2W_2M_1, \\ \Xi_{27} &= -2K_1, \quad \Xi_{28} = -2AM_3 - 2M_1, \\ \Xi_{37} &= -2K_6, \quad \Xi_{212} = -2K_1, \quad \Xi_{214} = -2A_1, \quad \Xi_{210} = -2A_1, \\ \Xi_{38} &= 2W_0M_3, \quad \Xi_{310} = -2A_6, \\ \Xi_{47} &= -2K_7, \quad \Xi_{314} = -2A_6, \quad \Xi_{312} = -2K_6, \\ \Xi_{48} &= 2W_1^{(k)}M_3, \quad \Xi_{412} = -2K_7, \quad \Xi_{414} = -2A_7, \\ \Xi_{58} &= 2W_3M_3, \quad \Xi_{67} = -2K_5, \quad \Xi_{68} = 2W_2M_3, \\ \Xi_{610} &= -2A_5, \quad \Xi_{612} = -2K_5, \quad \Xi_{614} = -2A_5, \\ \Xi_{78} &= -2K_8, \quad \Xi_{710} = -2K_4 - 2A_3, \quad \Xi_{712} = -2K_3, \\ \Xi_{714} &= -2A_3, \quad \Xi_{810} = -2A_8, \quad \Xi_{812} = -2K_8, \\ \Xi_{22} &= -2AM_1, \quad \Xi_{1014} = -2A_4, \quad \Xi_{1012} = -2K_4, \\ \Xi_{33} &= S + H_2 - 2T, \quad \Xi_{44} = -(1 - \tau_k)S, \\ \Xi_{55} &= QW_3\Theta_2 + \mathcal{B}_1^*W_3\Theta_4, \\ \Xi_{66} &= -(1 - \lambda_k)R, \quad \Xi_{77} = -(1 - \tau_k)Y - K_3, \quad \Xi_{814} = -2A_8, \\ \Xi_{88} &= -A_8\Theta_1^{-1} - K_8\Theta_3^{-1} - h_k^*L_1 - \eta_k^*L_2 - M_3, \\ \Xi_{99} &= -H_2, \quad \Xi_{1010} = -A_4, \quad \Xi_{1111} = L_1, \\ \Xi_{115} &= M_2, \quad \Xi_{215} = M_1 - AB_1^*, \\ \Xi_{315} &= W_0\mathcal{B}_1^*, \quad \Xi_{1414} = -A_8\Theta_1, \\ \Xi_{615} &= \mathcal{B}_1^*W_2, \quad \Xi_{415} = W_1^{(k)}\mathcal{B}_1^*, \\ \Xi_{815} &= M_3 - \mathcal{B}_1^*, \quad \Xi_{1515} = \mathcal{B}_1^* + \mathcal{B}_1^*W_3\Theta_4^{-1}. \end{aligned}$$

then the addressed neural networks (2) is passive.

Proof. Consider the following Lyapunov-Krasovskii functional for NNs (2) as

$$V(u(t), t) = \sum_{p=1}^8 V_p(u(t), t), \quad (9)$$

where

$$V_1(u(t), t) = u^T(t)Qu(t),$$

$$V_2(u(t), t) = \int_{t-h_k(t)}^t u^T(s)Yu(s)ds,$$

$$V_3(u(t), t) = \int_{t-h_k(t)}^t g^T(u(s))Sg(u(s)),$$

$$V_4(u(t), t) = \int_{t-\eta_k(t)}^t \dot{u}^T(s)R\dot{u}(s)ds,$$

$$V_5(u(t), t) = \int_{-h_k^*}^0 \int_{t+\theta}^t \dot{u}^T(s)L_1\dot{u}(s)dsd\theta,$$

$$V_6(u(t), t) = \int_{-\eta_k^*}^0 \int_{t+\theta}^t \dot{u}^T(s)L_2\dot{u}(s)dsd\theta,$$

$$V_7(u(t), t) = \int_{t-\sigma}^t u^T(s)H_1u(s)ds,$$

$$V_8(u(t), t) = \int_{t-\sigma}^t g^T(u(s))H_2g(u(s))ds.$$

Taking the derivative of $V(u(t), t)$ along the trajectories of (2), we have

$$\begin{aligned} \dot{V}(u(t), t) &= 2u^T(t)Q\dot{u}(t) + u^T(t)Yu(t) - (1 - \dot{h}_k(t))u^T(t - h_k(t))Yu(t - h_k(t)) + g^T(u(t)) \\ &\quad \times Sg(u(t)) - (1 - \dot{h}_k(t))g^T(u(t - h_k(t)))Sg(u(t - h_k(t))) + \dot{u}^T(t)R\dot{u}(t) \\ &\quad - (1 - \dot{\eta}_k(t))\dot{u}^T(t - \eta_k(t))R\dot{u}(t - \eta_k(t)) + h_k^*\dot{u}^T(t)L_1\dot{u}(t) - \int_{t-h_k^*}^t \dot{u}^T(s)L_1 \\ &\quad \times \dot{u}(s)ds + \eta_k^*\dot{u}^T(t)L_2\dot{u}(t) - \int_{t-\eta_k^*}^t \dot{u}^T(s)L_2\dot{u}(s)ds + u^T(t)H_1u(t) - u^T(t - \sigma) \\ &\quad \times H_1u(t - \sigma) + g^T(u(t))H_2g(u(t)) - g^T(u(t - \sigma))H_2g(u(t - \sigma)) \end{aligned} \quad (10)$$

Then, from (5) to (7), one can obtain that

(See Appendix)

According to Lemma 2.5, we obtain that

$$\begin{aligned} -2\dot{u}^T(t)A_8 \int_{t-\sigma}^t \dot{u}(\gamma)d\gamma &\leq -\dot{u}^T(t)A_8\Theta_1^{-1}\dot{u}(t) - \left(\int_{t-\sigma}^t \dot{u}(\gamma)d\gamma \right)^T A_8\Theta_1 \\ &\quad \times \left(\int_{t-\sigma}^t \dot{u}(\gamma)d\gamma \right) \end{aligned} \quad (11)$$

$$\begin{aligned} 2u^T(t)QW_3 \int_{t-\sigma}^t g(u(s))ds &\leq u^T(t)QW_3\Theta_2^{-1}u(t) + \left(\int_{t-\sigma}^t g(u(s))ds \right)^T QW_3 \\ &\quad \times \Theta_2 \left(\int_{t-\sigma}^t g(u(s))ds \right) \end{aligned} \quad (12)$$

$$\begin{aligned} -2\dot{u}^T(t)K_8 \int_{t-h_k(t)}^t \dot{u}(\alpha)d\alpha &\leq -\dot{u}^T(t)K_8\Theta_3^{-1}\dot{u}(t) - \left(\int_{t-h_k(t)}^t \dot{u}(\alpha)d\alpha \right)^T K_8 \\ &\quad \times \Theta_3 \left(\int_{t-h_k(t)}^t \dot{u}(\alpha)d\alpha \right) \end{aligned} \quad (13)$$

$$\begin{aligned} \mathcal{H}^T(t)\mathcal{B}_1^*W_3 \left(\int_{t-\sigma}^t g(u(s))ds \right) &\leq \mathcal{H}^T(t)\mathcal{B}_1^*W_3\Theta_4^{-1}\mathcal{H}(t) + \left(\int_{t-\sigma}^t g(u(s))ds \right)^T \mathcal{B}_1^*W_3 \\ &\quad \times \Theta_4 \left(\int_{t-\sigma}^t g(u(s))ds \right) \end{aligned} \quad (14)$$

Substitute the equations (11)–(14) in (28), we have

$$\dot{V}(u(t), t) \leq e^T(t)\Xi e(t), \quad (15)$$

where $e(t) = \begin{bmatrix} u^T(t) & u^T(t - \delta) & g^T(u(t)) & g^T(u(t - h_k(t))) & (\int_{t-\sigma}^t g(u(s))ds)^T & \dot{u}^T(t - \eta_k(t)) & u^T(t - h_k(t)) & \dot{u}^T(t) & g^T(u(t - \sigma)) & u^T(t - \sigma) \\ (\int_{t-h_k^*}^t \dot{u}(s)ds)^T & (\int_{t-h_k(t)}^t \dot{u}(\alpha)d\alpha)^T & (\int_{t-\eta_k^*}^t \dot{u}(s)ds)^T & (\int_{t-\sigma}^t \dot{u}(\gamma)d\gamma)^T & \mathcal{H}^T(t) \end{bmatrix}$

Let $v(t) = g(u(t)) + g(u(t - h_k(t))) + \int_{t-\sigma}^t g(u(s))ds$ be the output of the neural networks. Hence, by (15) we can estimate that

$$\dot{V}(u(t), t) - 2v^T(t)\mathcal{H}(t) - \mu\mathcal{H}^T(t)\mathcal{H}(t) \leq e^T(t)\Xi e(t) \quad (16)$$

By applying the Schur complement lemma, it concludes that $\Xi < 0$ is equivalent to (8). If $\Xi < 0$, then $\dot{V}(u(t), t) - 2v^T(t)\mathcal{H}(t) - \mu\mathcal{H}^T(t)\mathcal{H}(t) < 0$ and from which it follows that

$$2 \int_{t_0}^{t_1} v^T(t)\mathcal{H}(t)dt > V(u(t_1), t_1) - V(u(t_0), t_0) - \mu \int_{t_0}^{t_1} \mathcal{H}^T(t)\mathcal{H}(t)dt \quad (17)$$

Since $V(u(t), t) > 0$ for $u(t) \neq 0$ and $V(u(t)) = 0$ for $u(t) = 0$, which implies that $t_0 \rightarrow t_1$, the time delayed neural networks (2) is passive in Definition 2.3. Therefore, the proof of this theorem is completed. $\square$

Remark 3.2. Suppose, the neutral term in neural networks (2), does not exist then we have the transformed neural networks as follows

$$\begin{aligned} \dot{u}(t) &= -Au(t - \delta) + W_0g(u(t)) + \sum_{k=1}^c W_1^{(k)}g(u(t - h_k(t))) + W_3 \\ &\quad \times \int_{t-\sigma}^t g(u(s))ds + \mathcal{H}(t), \\ v(t) &= g(u(t)) + g(u(t - h_k(t))) + \int_{t-\sigma}^t g(u(s))ds + \mathcal{H}(t), \end{aligned} \quad (18)$$

Then by Theorem 3.1, it is easy to have the following Corollary 3.3, respectively.

Corollary 3.3. Under Assumption 1, for given $\tau_k < 1$ and $\lambda_k < 1$, if there exist matrices $Q > 0$, $S > 0$, $Y > 0$, $H_1, H_2 > 0$ and diagonal matrix $T > 0$ and any matrices $K_i (i = \{1, 2, \dots, 8\} \setminus \{5\})$, B_1^* , and $M_j (j = 1, 2, 3)$ with appropriate dimension such that the LMI holds as follows

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16}^* & \Xi_{17}^* & 0 & \Xi_{19}^* & \Xi_{110}^* & \Xi_{111}^* & \Xi_{112}^* \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} & \Xi_{26}^* & \Xi_{27}^* & 0 & \Xi_{29}^* & \Xi_{210}^* & \Xi_{211}^* & \Xi_{212}^* \\ * & * & \Xi_{33} & 0 & 0 & \Xi_{36}^* & \Xi_{37}^* & 0 & \Xi_{39}^* & \Xi_{310}^* & \Xi_{311}^* & \Xi_{312}^* \\ * & * & * & \Xi_{44} & 0 & \Xi_{46}^* & \Xi_{47}^* & 0 & 0 & \Xi_{410}^* & \Xi_{411}^* & \Xi_{412}^* \\ * & * & * & * & \Xi_{55} & 0 & \Xi_{57}^* & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Xi_{66}^* & 0 & 0 & \Xi_{69}^* & \Xi_{610}^* & \Xi_{611}^* & \Xi_{612}^* \\ * & * & * & * & * & * & \Xi_{77}^* & 0 & \Xi_{79}^* & \Xi_{710}^* & \Xi_{711}^* & \Xi_{712}^* \\ * & * & * & * & * & * & * & \Xi_{88} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Xi_{99}^* & \Xi_{910}^* & \Xi_{911}^* & 0 \\ * & * & * & * & * & * & * & * & * & \Xi_{1010}^* & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & \Xi_{1111}^* & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Xi_{1212}^* \end{bmatrix} < 0, \quad (19)$$

where

$$\begin{aligned} \Xi_{11} &= QW_3\Theta_2^{-1} + Y + H_1 + K_2 + A_2, \quad \Xi_{12} = -2QA + 2K_1 + 2A_1 - 2AM_2, \quad \Xi_{1010}^* = -K_8\Theta_3, \\ \Xi_{13} &= 2W_0Q + 2K_6 + 2A_6 + 2W_0M_2 + 2T\Upsilon, \quad \Xi_{14} = 2QW_1^{(k)} + 2K_7 + 2A_7 + 2M_2W_1^{(k)}, \\ \Xi_{15} &= 2QW_3 + 2W_3M_2, \quad \Xi_{16}^* = 2K_3 - 2K_2 + 2A_3, \quad \Xi_{712}^* = M_3 - B_1^*, \quad \Xi_{1212}^* = B_1^* + B_1^*W_3\Theta_4^{-1}, \end{aligned}$$

$$\begin{aligned} \Xi_{17}^* &= 2K_8 + 2A_8 - 2M_2, \quad \Xi_{19}^* = 2K_4 + 2A_4 - 2A_2, \quad \Xi_{110}^* = -2K_2, \quad \Xi_{111}^* = -2A_2, \quad \Xi_{23} = 2W_0M_1, \\ \Xi_{24} &= 2W_1^{(k)}M_1, \quad \Xi_{25} = 2M_1W_3, \quad \Xi_{26}^* = -2K_1, \quad \Xi_{27}^* = -2AM_3 - 2M_1, \quad \Xi_{88} = -H_2, \\ \Xi_{36}^* &= -2K_6, \quad \Xi_{210}^* = -2K_1, \quad \Xi_{211}^* = -2A_1, \quad \Xi_{29}^* = -2A_1, \quad \Xi_{37}^* = 2W_0M_3, \quad \Xi_{39}^* = -2A_6, \\ \Xi_{46}^* &= -2K_7, \quad \Xi_{311}^* = -2A_6, \quad \Xi_{310}^* = -2K_6, \quad \Xi_{47}^* = 2W_1^{(k)}M_3, \quad \Xi_{410}^* = -2K_7, \quad \Xi_{411}^* = -2A_7, \\ \Xi_{67}^* &= -2K_8, \quad \Xi_{69}^* = -2K_4 - 2A_3, \quad \Xi_{610}^* = -2K_3, \quad \Xi_{611}^* = -2A_3, \quad \Xi_{79}^* = -2A_8, \quad \Xi_{710}^* = -2K_8, \\ \Xi_{22} &= -2AM_1, \quad \Xi_{911}^* = -2A_4, \quad \Xi_{910}^* = -2K_4, \quad \Xi_{33} = S + H_2 - 2T, \quad \Xi_{44} = -(1 - \tau_k)S, \\ \Xi_{55} &= QW_3\Theta_2 + B_1^*W_3\Theta_4, \quad \Xi_{66}^* = -(1 - \tau_k)Y - K_3, \quad \Xi_{711}^* = -2A_8, \quad \Xi_{57}^* = 2W_3M_3, \quad \Xi_{99}^* = -A_4, \\ \Xi_{112}^* &= M_2, \quad \Xi_{212}^* = M_1 - AB_1^*, \quad \Xi_{312}^* = W_0B_1^*, \quad \Xi_{1111}^* = -A_8\Theta_1, \quad \Xi_{412}^* = W_1^{(k)}B_1^*, \\ \Xi_{77}^* &= -A_8\Theta_1^{-1} - K_8\Theta_3^{-1} - M_3, \end{aligned}$$

then the neural networks (18) is passive.

Remark 3.4. The authors in [15], investigated the impact of multiple time-delays in neural networks to find the stability conditions via LMIs, and M. V. Thuan, H. Trinh, L. V. Hien, discussed about the passivity study on time-delayed neural networks by using some inequality techniques in [48]. The neutral terms in neural networks were fully addressed by R. Rakkiyappan & P. Balasubramaniam in [39] to verify the stability criteria in the sense of asymptotic. The authors in [29], studied and obtained some new results on problem of stability for the time delayed neural networks with leakage terms. Also, in [52], Wu et al., conversed the exponential passivity performance of neural networks through different Lyapunov-Krasovskii functionals.

To the best of our knowledge, from the above said references, we have clearly understood the passivity problem of neural networks which is considered only with discrete time-varying delays and external inputs, but the distributed delays, multiple discrete time-delays, neutral time-delays and leakage time delays has not been taken into account and also no one investigates the passivity as well as exponential passivity criteria for NNs at a time. Although the considered facts are complicated, we took this as a challenging task which admits us to do the proposed research work.

4. Exponential passivity results

Theorem 4.1. Suppose that [Assumption 1](#) holds. For a given scalars $\tau_k < 1$ and $\lambda_k < 1$, then the time-delayed neural networks (2) is passive in the sense of exponential, if there exists positive definite matrices $\mathcal{C}_r$ ($r = 1, 2, 3, \dots, 10$), $\mathcal{F}^*$, $\mathcal{D}_1$, positive diagonal matrices $\mathcal{O}$, $\mathcal{J}_1$ and $\mathcal{E}_i^T = [\varepsilon_{i1}^T, \varepsilon_{i2}^T, \varepsilon_{i3}^T, \varepsilon_{i4}^T, \varepsilon_{i5}^T, \varepsilon_{i6}^T, \varepsilon_{i7}^T, \varepsilon_{i8}^T, \varepsilon_{i9}^T, \varepsilon_{i10}^T, \varepsilon_{i11}^T, \varepsilon_{i12}^T, \varepsilon_{i13}^T]$, $\mathcal{E}_2^T = [\varepsilon_{21}^T, \varepsilon_{22}^T, \varepsilon_{23}^T, \varepsilon_{24}^T, \varepsilon_{25}^T, \varepsilon_{26}^T, \varepsilon_{27}^T, \varepsilon_{28}^T, \varepsilon_{29}^T, \varepsilon_{210}^T, \varepsilon_{211}^T, \varepsilon_{212}^T, \varepsilon_{213}^T]$, $\mathcal{E}_3^T = [\varepsilon_{31}^T, \varepsilon_{32}^T, \varepsilon_{33}^T, \varepsilon_{34}^T, \varepsilon_{35}^T, \varepsilon_{36}^T, \varepsilon_{37}^T, \varepsilon_{38}^T, \varepsilon_{39}^T, \varepsilon_{310}^T, \varepsilon_{311}^T, \varepsilon_{312}^T, \varepsilon_{313}^T]$, $\mathcal{E}_4^T = [\varepsilon_{41}^T, \varepsilon_{42}^T, \varepsilon_{43}^T, \varepsilon_{44}^T, \varepsilon_{45}^T, \varepsilon_{46}^T, \varepsilon_{47}^T, \varepsilon_{48}^T, \varepsilon_{49}^T, \varepsilon_{410}^T, \varepsilon_{411}^T, \varepsilon_{412}^T, \varepsilon_{413}^T]$ such that the LMI's are satisfied as follows

$$\Lambda^* = \begin{bmatrix} \sum & \sqrt{h_k^*} \varepsilon_1 & \sqrt{h_k^*} \varepsilon_2 & \sqrt{\sigma} \varepsilon_3 & \sqrt{\sigma} \varepsilon_4 W_3 \\ * & -\mathcal{C}_9 & 0 & 0 & 0 \\ * & * & -\mathcal{C}_9 & 0 & 0 \\ * & * & * & -\mathcal{D}_1 & 0 \\ * & * & * & * & -\mathcal{D}_1 \end{bmatrix} < 0, \quad (20)$$

$$\Sigma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} & \Gamma_{16} & \Gamma_{17} & \Gamma_{18} & \Gamma_{19} & \Gamma_{110} & \Gamma_{111} & 0 & \Gamma_{113} \\ * & \Gamma_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{210} & 0 & 0 & \Gamma_{213} \\ * & * & \Gamma_{33} & 0 & \Gamma_{35} & 0 & 0 & 0 & 0 & \Gamma_{310} & 0 & 0 & \Gamma_{313} \\ * & * & * & \Gamma_{44} & \Gamma_{45} & \Gamma_{46} & \Gamma_{47} & \Gamma_{48} & 0 & \Gamma_{410} & 0 & 0 & \Gamma_{413} \\ * & * & * & * & \Gamma_{55} & \Gamma_{56} & 0 & 0 & 0 & \Gamma_{510} & 0 & 0 & \Gamma_{513} \\ * & * & * & * & * & \Gamma_{66} & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{613} \\ * & * & * & * & * & * & \Gamma_{77} & \Gamma_{78} & \Gamma_{79} & \Gamma_{710} & \Gamma_{711} & \Gamma_{712} & \Gamma_{713} \\ * & * & * & * & * & * & * & \Gamma_{88} & 0 & \Gamma_{810} & 0 & 0 & \Gamma_{813} \\ * & * & * & * & * & * & * & * & \Gamma_{99} & \Gamma_{910} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Gamma_{1010} & 0 & 0 & \Gamma_{1013} \\ * & * & * & * & * & * & * & * & * & * & \Gamma_{1111} & 0 & \Gamma_{1113} \\ * & * & * & * & * & * & * & * & * & * & * & \Gamma_{1212} & \Gamma_{1213} \\ * & * & * & * & * & * & * & * & * & * & * & * & \Gamma_{1313} \end{bmatrix} < 0,$$

where

$$\begin{aligned} \Gamma_{11} &= \mathcal{C}_3 + \mathcal{C}_4 + \mathcal{C}_6 + \gamma \mathcal{C}_1 + h_k^* \mathcal{C}_8 + \eta_k^* \mathcal{F}^* + 2\mathcal{E}_{11}, & \Gamma_{12} &= 2\mathcal{E}_{12} - \varepsilon_{42} A^T - A \varepsilon_{42}^T, & \Gamma_{111} &= 2\mathcal{E}_{111}, \\ \Gamma_{13} &= 2\mathcal{C}_1 + 2\mathcal{E}_{13} - \varepsilon_{43}^T - \varepsilon_{43}, & \Gamma_{14} &= 2\mathcal{E}_{14}, & \Gamma_{15} &= 2\mathcal{J}_1 \Upsilon + 2\mathcal{E}_{15} + \gamma \mathcal{O} + W_0 \varepsilon_{45}^T + \varepsilon_{45} W_0^T, \\ \Gamma_{16} &= 2\mathcal{E}_{16} + W_1^{(k)} \varepsilon_{46}^T + \varepsilon_{46} W_1^{(k)T}, & \Gamma_{17} &= 2\mathcal{E}_{17}, & \Gamma_{18} &= 2\mathcal{E}_{18}, & \Gamma_{19} &= 2\mathcal{E}_{19}, \\ \Gamma_{110} &= 2\mathcal{E}_{110} + \varepsilon_{410}^T + \varepsilon_{410} - 2\mathcal{E}_{310}, & \Gamma_{213} &= -2\mathcal{E}_{313}, & \Gamma_{35} &= 2\mathcal{O} - \varepsilon_{45}^T - \varepsilon_{45} + W_0 \varepsilon_{45}^T + W_0^T \varepsilon_{45}, \\ \Gamma_{1010} &= \varepsilon_{410}^T + \varepsilon_{410}, & \Gamma_{33} &= \mathcal{C}_5 + \mathcal{C}_7 + 2h_k^* \mathcal{C}_9 + \eta_k^* \mathcal{C}_{10} - \varepsilon_{43}^T - \varepsilon_{43}, & \Gamma_{1213} &= -2\mathcal{E}_{313}, & \Gamma_{1313} &= -2\mathcal{E}_{313}, \\ \Gamma_{313} &= -2\mathcal{E}_{313} - \varepsilon_{413}^T - \varepsilon_{413}, & \Gamma_{44} &= -(1 - \tau_k) \mathcal{C}_3 - 2\mathcal{E}_{14} + 2\mathcal{E}_{24}, & \Gamma_{113} &= 2(\mathcal{E}_{113} - \varepsilon_{313}), \\ \Gamma_{46} &= -2\mathcal{E}_{16} + 2\mathcal{E}_{26} + 2\mathcal{J}_1 \Upsilon + W_1^{(k)} \varepsilon_{46}^T + \varepsilon_{46} W_1^{(k)T}, & \Gamma_{47} &= -2\mathcal{E}_{17} + 2\mathcal{E}_{27}, & \Gamma_{513} &= \varepsilon_{413} W_0^T + \varepsilon_{413}^T W_0, \\ \Gamma_{413} &= -2\mathcal{E}_{113} + 2\mathcal{E}_{213} - 2\mathcal{E}_{313}, & \Gamma_{55} &= \mathcal{C}_2 + 3\sigma \mathcal{D}_1 + 2\mathcal{E}_{35} - 2\mathcal{J}_1 + \varepsilon_{45} W_0^T + \varepsilon_{45}^T W_0, & \Gamma_{1113} &= -2\mathcal{E}_{313}, \\ \Gamma_{56} &= \varepsilon_{46}^T W_0 + W_0^T \varepsilon_{46} + W_1^{(k)} \varepsilon_{46}^T + \varepsilon_{46} W_1^{(k)T}, & \Gamma_{66} &= -e^{-\gamma h_k^*} (1 - \tau_k) \mathcal{C}_2 - 2\mathcal{J}_1 - W_1^{(k)} \varepsilon_{46}^T - \varepsilon_{46} W_1^{(k)T}, \\ \Gamma_{613} &= -2\mathcal{E}_{313} - W_1^{(k)} \varepsilon_{413}^T - \varepsilon_{413} W_1^{(k)T}, & \Gamma_{77} &= -\mathcal{C}_4 - 2\mathcal{E}_{27}, & \Gamma_{48} &= W_2^T \varepsilon_{48} + \varepsilon_{48}^T W_2 - 2\mathcal{E}_{18} + 2\mathcal{E}_{28}, \\ \Gamma_{79} &= -2\mathcal{E}_{29}, & \Gamma_{710} &= -2\mathcal{E}_{210}, & \Gamma_{711} &= -2\mathcal{E}_{211}, & \Gamma_{712} &= -2\mathcal{E}_{212}, & \Gamma_{78} &= -2\mathcal{E}_{28} + W_2 \varepsilon_{48}^T + \varepsilon_{48} W_2^T, \\ \Gamma_{713} &= -2\mathcal{E}_{213} - 2\mathcal{E}_{313}, & \Gamma_{88} &= -e^{-\gamma \eta_k^*} (1 - \lambda_k) \mathcal{C}_5 + W_2 \varepsilon_{48}^T + W_2^T \varepsilon_{48}, & \Gamma_{813} &= -2\mathcal{E}_{313} + W_2 \varepsilon_{413}^T \\ & & & + W_2^T \varepsilon_{413}, & \Gamma_{99} &= -2\mathcal{E}_{39}, & \Gamma_{1013} &= -2\mathcal{E}_{313} + \varepsilon_{413} + \varepsilon_{413}^T, & \Gamma_{1111} &= -e^{-\gamma \eta_k^*} (1 - \lambda_k) \mathcal{C}_6, \\ \Gamma_{18} &= 2\mathcal{E}_{18} + W_2^T \varepsilon_{48} + \varepsilon_{48} W_2, & \Gamma_{1212} &= -e^{-\gamma \eta_k^*} \mathcal{C}_7, & \Gamma_{a10} &= -2\mathcal{E}_{310} + \varepsilon_{410} + \varepsilon_{410}^T, & a &= 1, 2, 3, \dots, 9; \\ \Gamma_{22} &= -A \varepsilon_{42}^T - A^T \varepsilon_{42}, & \Gamma_{45} &= -2\mathcal{E}_{15} + 2\mathcal{E}_{25} + \varepsilon_{45} W_0^T + W_0 \varepsilon_{45}^T. \end{aligned}$$

Proof. Consider the following Lyapunov-Krasovskii functional for neural networks (2) as

$$V(u(t), t) = \sum_{l=1}^6 V_l(u(t), t), \quad (21)$$

where

$$\begin{aligned} V_1(u(t), t) &= u^T(t) \mathcal{C}_1 u(t) + 2 \sum_{k=1}^m o_k \int_0^{u_k(t)} g_k(\xi) d\xi, \\ V_2(u(t), t) &= \int_{t-h_k(t)}^t e^{-\gamma(t-\xi)} (g^T(u(\xi)) \mathcal{C}_2 g(u(\xi)) + u^T(\xi) \mathcal{C}_3 u(\xi)) d\xi \\ &\quad + \int_{t-h_k^*}^t e^{-\gamma(t-\xi)} u^T(\xi) \mathcal{C}_4 u(\xi) d\xi, \end{aligned}$$

$$\begin{aligned}
V_3(u(t), t) &= \int_{t-\eta_k(t)}^t e^{-\gamma(t-\xi)} (\dot{u}^T(\xi) C_5 \dot{u}(\xi) + u^T(\xi) C_6 u(\xi)) d\xi + \int_{t-\eta_k^*}^t e^{-\gamma(t-\xi)} \dot{u}^T(\xi) C_7 \dot{u}(\xi) d\xi, \\
V_4(u(t), t) &= \int_{-h_k^*}^0 \int_{t+\alpha}^t e^{-\gamma(t-\alpha)} (u^T(\xi) C_8 u(\xi) + 2\dot{u}^T(\xi) C_9 \dot{u}(\xi)) d\xi d\alpha, \\
V_5(u(t), t) &= \int_{-\eta_k^*}^0 \int_{t+\alpha}^t e^{-\gamma(t-\alpha)} (\dot{u}^T(\xi) C_{10} \dot{u}(\xi) + u^T(\xi) \mathcal{F}^* u(\xi)) d\xi d\alpha, \\
V_6(u(t), t) &= 3 \int_{-\sigma}^0 \int_{t+\theta}^t e^{-\gamma(t-\theta)} g^T(u(\xi)) \mathcal{D}_1 g(u(\xi)) d\xi d\theta
\end{aligned}$$

Calculating the derivative of $V_l(u(t), t)$, along the trajectory $u(t)$ of (2) are

$$\dot{V}(u(t), t) = \sum_{l=1}^6 \dot{V}_l(u(t), t), \quad (22)$$

where

$$\begin{aligned}
\dot{V}_1(u(t), t) &\leq 2u^T(t) C_1 \dot{u}(t) + 2g^T(u(t)) O \dot{u}(t), \\
\dot{V}_2(u(t), t) &\leq -\gamma V_2 + g^T(u(t)) C_2 g(u(t)) - e^{-\gamma h_k^*} (1 - \tau_k) g^T(u(t - h_k(t))) C_2 g(u(t - h_k(t))) \\
&\quad + u^T(t) C_3 u(t) - (1 - \tau_k) u^T(t - h_k(t)) C_3 u(t - h_k(t)) + u^T(t) C_4 u(t) - e^{-\gamma h_k^*} u^T(t - h_k^*) C_4 u(t - h_k^*) \\
\dot{V}_3(u(t), t) &\leq -\gamma V_3 + \dot{u}^T(t) C_5 \dot{u}(t) - e^{-\gamma \eta_k^*} (1 - \lambda_k) \dot{u}^T(t - \eta_k(t)) C_5 \dot{u}(t - \eta_k(t)) + u^T(t) C_6 u(t) \\
&\quad - e^{-\gamma \eta_k^*} (1 - \lambda_k) u^T(t - \eta_k(t)) C_6 u(t - \eta_k(t)) + \dot{u}^T(t) C_7 \dot{u}(t) - e^{-\gamma \eta_k^*} \dot{u}^T(t - \eta_k^*) C_7 \dot{u}(t - \eta_k^*) \\
\dot{V}_4(u(t), t) &\leq -\gamma V_4 + h_k^* (u^T(t) C_8 u(t) + 2\dot{u}^T(t) C_9 \dot{u}(t)) - e^{-\gamma h_k^*} \int_{t-h_k^*}^t (u^T(\xi) C_8 u(\xi) + 2\dot{u}^T(\xi) C_9 \dot{u}(\xi)) d\xi \\
\dot{V}_5(u(t), t) &\leq -\gamma V_5 + \eta_k^* (\dot{u}^T(t) C_{10} \dot{u}(t) + u^T(t) \mathcal{F}^* u(t)) - e^{-\gamma \eta_k^*} \int_{t-\eta_k^*}^t (\dot{u}^T(\xi) C_{10} \dot{u}(\xi) + u^T(\xi) \mathcal{F}^* u(\xi)) d\xi \\
\dot{V}_6(u(t), t) &\leq -\gamma V_6 + 3\sigma g^T(u(t)) \mathcal{D}_1 g(u(t)) - 3e^{-\gamma \sigma} \left(\int_{t-\sigma}^t g(u(\xi)) d\xi \right)^T \mathcal{D}_1 \left(\int_{t-\sigma}^t g(u(\xi)) d\xi \right)
\end{aligned}$$

By utilizing the Newton-Leibnitz formula and (2), for any matrices $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ and $\mathcal{E}_4$ with appropriate dimensions, one can get that (See Appendix B)

By the Schur complement lemma, the LMI (20) is equivalent to $\Lambda^* < 0$. By (41), we get that

$$\dot{V}(u(t), t) + \gamma V(u(t), t) - 2v^T(t) \mathcal{H}(t) \leq \zeta^T(t) \Lambda^* \zeta(t) \leq 0 \quad (23)$$

So,

$$\dot{V}(u(t), t) + \gamma V(u(t), t) \leq 2v^T(t) \mathcal{H}(t) \quad (24)$$

According to (24) and Definition 2.4, the delayed neural networks (2) is exponentially passive. $\square$

Corollary 4.2. Suppose that Assumption 1 holds. For a given scalars $\tau_k < 1$ and $\lambda_k < 1$, then the time-delayed neural networks (18) is passive in the sense of exponential, if there exists positive definite matrices $C_r (r = 1, 2, 3, \dots, 10)$, $\mathcal{F}^*$, $\mathcal{D}_1$, positive diagonal matrices $\mathcal{O}$, $\mathcal{J}_1$ and $\mathcal{E}_1^T = [\mathcal{E}_{11}^T, \mathcal{E}_{12}^T, \mathcal{E}_{13}^T, \mathcal{E}_{14}^T, \mathcal{E}_{15}^T, \mathcal{E}_{16}^T, \mathcal{E}_{17}^T, \mathcal{E}_{18}^T, \mathcal{E}_{19}^T, \mathcal{E}_{20}^T, \mathcal{E}_{21}^T, \mathcal{E}_{22}^T, \mathcal{E}_{23}^T, \mathcal{E}_{24}^T, \mathcal{E}_{25}^T, \mathcal{E}_{26}^T, \mathcal{E}_{27}^T, \mathcal{E}_{28}^T, \mathcal{E}_{29}^T, \mathcal{E}_{30}^T, \mathcal{E}_{31}^T, \mathcal{E}_{32}^T, \mathcal{E}_{33}^T, \mathcal{E}_{34}^T, \mathcal{E}_{35}^T, \mathcal{E}_{36}^T, \mathcal{E}_{37}^T, \mathcal{E}_{38}^T, \mathcal{E}_{39}^T, \mathcal{E}_{40}^T, \mathcal{E}_{41}^T, \mathcal{E}_{42}^T, \mathcal{E}_{43}^T, \mathcal{E}_{44}^T, \mathcal{E}_{45}^T, \mathcal{E}_{46}^T, \mathcal{E}_{47}^T, \mathcal{E}_{48}^T, \mathcal{E}_{49}^T, \mathcal{E}_{50}^T, \mathcal{E}_{51}^T, \mathcal{E}_{52}^T, \mathcal{E}_{53}^T, \mathcal{E}_{54}^T, \mathcal{E}_{55}^T, \mathcal{E}_{56}^T, \mathcal{E}_{57}^T, \mathcal{E}_{58}^T, \mathcal{E}_{59}^T, \mathcal{E}_{60}^T, \mathcal{E}_{61}^T, \mathcal{E}_{62}^T, \mathcal{E}_{63}^T, \mathcal{E}_{64}^T, \mathcal{E}_{65}^T, \mathcal{E}_{66}^T, \mathcal{E}_{67}^T, \mathcal{E}_{68}^T, \mathcal{E}_{69}^T, \mathcal{E}_{70}^T, \mathcal{E}_{71}^T, \mathcal{E}_{72}^T, \mathcal{E}_{73}^T, \mathcal{E}_{74}^T, \mathcal{E}_{75}^T, \mathcal{E}_{76}^T, \mathcal{E}_{77}^T, \mathcal{E}_{78}^T, \mathcal{E}_{79}^T, \mathcal{E}_{80}^T, \mathcal{E}_{81}^T, \mathcal{E}_{82}^T, \mathcal{E}_{83}^T, \mathcal{E}_{84}^T, \mathcal{E}_{85}^T, \mathcal{E}_{86}^T, \mathcal{E}_{87}^T, \mathcal{E}_{88}^T, \mathcal{E}_{89}^T, \mathcal{E}_{90}^T, \mathcal{E}_{91}^T, \mathcal{E}_{92}^T, \mathcal{E}_{93}^T, \mathcal{E}_{94}^T, \mathcal{E}_{95}^T, \mathcal{E}_{96}^T, \mathcal{E}_{97}^T, \mathcal{E}_{98}^T, \mathcal{E}_{99}^T, \mathcal{E}_{100}^T, \mathcal{E}_{101}^T, \mathcal{E}_{102}^T, \mathcal{E}_{103}^T, \mathcal{E}_{104}^T, \mathcal{E}_{105}^T, \mathcal{E}_{106}^T, \mathcal{E}_{107}^T, \mathcal{E}_{108}^T, \mathcal{E}_{109}^T, \mathcal{E}_{110}^T, \mathcal{E}_{111}^T, \mathcal{E}_{112}^T, \mathcal{E}_{113}^T, \mathcal{E}_{114}^T, \mathcal{E}_{115}^T, \mathcal{E}_{116}^T, \mathcal{E}_{117}^T, \mathcal{E}_{118}^T, \mathcal{E}_{119}^T, \mathcal{E}_{120}^T, \mathcal{E}_{121}^T, \mathcal{E}_{122}^T, \mathcal{E}_{123}^T, \mathcal{E}_{124}^T, \mathcal{E}_{125}^T, \mathcal{E}_{126}^T, \mathcal{E}_{127}^T, \mathcal{E}_{128}^T, \mathcal{E}_{129}^T, \mathcal{E}_{130}^T, \mathcal{E}_{131}^T, \mathcal{E}_{132}^T, \mathcal{E}_{133}^T, \mathcal{E}_{134}^T, \mathcal{E}_{135}^T, \mathcal{E}_{136}^T, \mathcal{E}_{137}^T, \mathcal{E}_{138}^T, \mathcal{E}_{139}^T, \mathcal{E}_{140}^T, \mathcal{E}_{141}^T, \mathcal{E}_{142}^T, \mathcal{E}_{143}^T, \mathcal{E}_{144}^T, \mathcal{E}_{145}^T, \mathcal{E}_{146}^T, \mathcal{E}_{147}^T, \mathcal{E}_{148}^T, \mathcal{E}_{149}^T, \mathcal{E}_{150}^T, \mathcal{E}_{151}^T, \mathcal{E}_{152}^T, \mathcal{E}_{153}^T, \mathcal{E}_{154}^T, \mathcal{E}_{155}^T, \mathcal{E}_{156}^T, \mathcal{E}_{157}^T, \mathcal{E}_{158}^T, \mathcal{E}_{159}^T, \mathcal{E}_{160}^T, \mathcal{E}_{161}^T, \mathcal{E}_{162}^T, \mathcal{E}_{163}^T, \mathcal{E}_{164}^T, \mathcal{E}_{165}^T, \mathcal{E}_{166}^T, \mathcal{E}_{167}^T, \mathcal{E}_{168}^T, \mathcal{E}_{169}^T, \mathcal{E}_{170}^T, \mathcal{E}_{171}^T, \mathcal{E}_{172}^T, \mathcal{E}_{173}^T, \mathcal{E}_{174}^T, \mathcal{E}_{175}^T, \mathcal{E}_{176}^T, \mathcal{E}_{177}^T, \mathcal{E}_{178}^T, \mathcal{E}_{179}^T, \mathcal{E}_{180}^T, \mathcal{E}_{181}^T, \mathcal{E}_{182}^T, \mathcal{E}_{183}^T, \mathcal{E}_{184}^T, \mathcal{E}_{185}^T, \mathcal{E}_{186}^T, \mathcal{E}_{187}^T, \mathcal{E}_{188}^T, \mathcal{E}_{189}^T, \mathcal{E}_{190}^T, \mathcal{E}_{191}^T, \mathcal{E}_{192}^T, \mathcal{E}_{193}^T, \mathcal{E}_{194}^T, \mathcal{E}_{195}^T, \mathcal{E}_{196}^T, \mathcal{E}_{197}^T, \mathcal{E}_{198}^T, \mathcal{E}_{199}^T, \mathcal{E}_{200}^T]$ such that the following LMIs holds

$$\Lambda^* = \begin{bmatrix} \sum^* & \sqrt{\sigma} \mathcal{E}_3 & \sqrt{\sigma} \mathcal{E}_4 W_3 \\ * & -\mathcal{D}_1 & 0 \\ * & * & -\mathcal{D}_1 \end{bmatrix} < 0, \quad (25)$$

$$\sum^* = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} & \Gamma_{16} & \Gamma_{17} & \Gamma_{19} & \Gamma_{110} & \Gamma_{111} & \Gamma_{113} \\ * & \Gamma_{22} & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{210} & 0 & \Gamma_{213} \\ * & * & \Gamma_{33}^* & 0 & \Gamma_{35} & 0 & 0 & 0 & \Gamma_{310} & 0 & \Gamma_{313} \\ * & * & * & \Gamma_{44} & \Gamma_{45} & \Gamma_{46} & \Gamma_{47} & 0 & \Gamma_{410} & 0 & \Gamma_{413} \\ * & * & * & * & \Gamma_{55} & \Gamma_{56} & 0 & 0 & \Gamma_{510} & 0 & \Gamma_{513} \\ * & * & * & * & * & \Gamma_{66} & 0 & 0 & 0 & 0 & \Gamma_{613} \\ * & * & * & * & * & * & \Gamma_{77} & \Gamma_{79} & \Gamma_{710} & \Gamma_{711} & \Gamma_{713} \\ * & * & * & * & * & * & * & \Gamma_{99} & \Gamma_{910} & 0 & 0 \\ * & * & * & * & * & * & * & * & \Gamma_{1010} & 0 & \Gamma_{1013} \\ * & * & * & * & * & * & * & * & * & \Gamma_{1111} & \Gamma_{1113} \\ * & * & * & * & * & * & * & * & * & * & \Gamma_{1313} \end{bmatrix} < 0,$$

where $\Gamma_{33}^* = -\mathcal{E}_{43}^T - \mathcal{E}_{43}$. Also, the columns and rows of 8th and 12th terms in Theorem 4.1 are zero, the remaining values of this corollary are same as in Theorem 4.1.

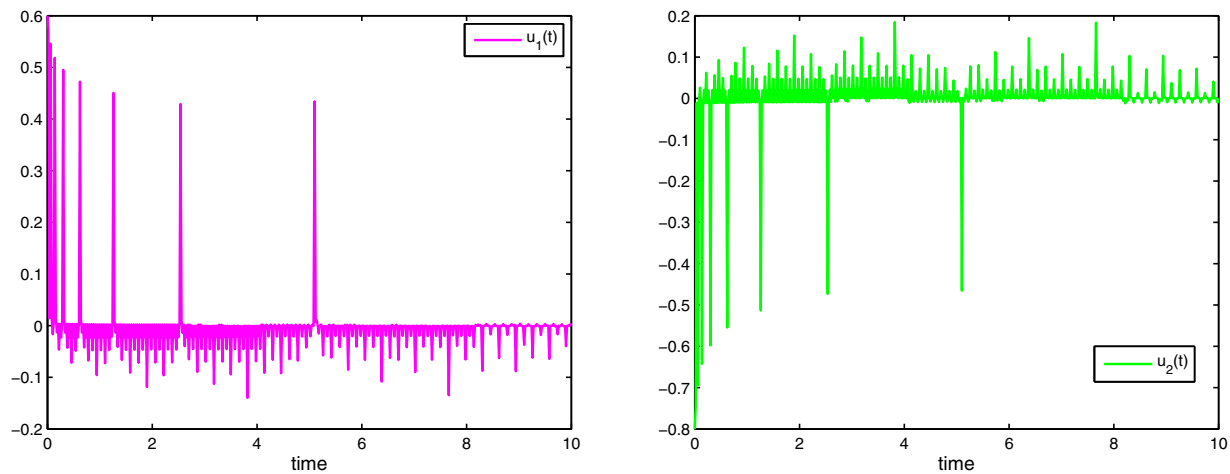


Fig. 1. The state responses $u_1(t)$ and $u_2(t)$ of (2) in Example 5.1.

Remark 4.3. There are no passivity and exponential passivity results on NNNs with multiple time delays (2), established via LMIs in available literature. Hence, the above gap is filled by Theorem 3.1 and 4.1 in this paper. For consequence, a suitable Lyapunov-Krasovskii functional handled to guarantee the neutral-type neural networks (2) is passive and exponentially passive, which means that our results generalize the previous ones.

Remark 4.4. Based on the Examples 5.1–5.2, it is easy to see that the conclude results are less conserved than the existing ones in the references cited [47,49,55,59]. Therefore, the proposed method is an improvement over the available sources.

5. Numerical examples.

In this section, we provide two numerical examples with their simulations to demonstrate the superiority and benefits of the proposed criteria.

Example 5.1. Consider a two dimensional multiple delayed neutral-type neural networks (2) with the following associated parameters :

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad W_0 = \begin{pmatrix} -0.5 & 0.4 \\ 0.3 & 0.1 \end{pmatrix}, \quad W_1^{(1)} = \begin{pmatrix} 0.1 & 0.2 \\ -0.08 & 0.4 \end{pmatrix}, \quad W_1^{(2)} = \begin{pmatrix} 0.3 & 0.05 \\ 0.01 & 0.03 \end{pmatrix},$$

$$W_2 = \begin{pmatrix} 0.09 & -0.2 \\ 0.6 & -0.1 \end{pmatrix}, \quad W_3 = \begin{pmatrix} -0.6 & 0.3 \\ 0.5 & 0.2 \end{pmatrix}, \quad \Theta_1 = \begin{pmatrix} 0.2 & 0.1 \\ 0.06 & 0.3 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 0.1 & 0.5 \\ 0.4 & 0.7 \end{pmatrix},$$

$$\Theta_3 = \begin{pmatrix} 0.4 & 0.6 \\ 0.03 & 0.2 \end{pmatrix}, \quad \Theta_4 = \begin{pmatrix} 0.1 & 0.08 \\ 0.03 & 0.6 \end{pmatrix}, \quad \Upsilon = \begin{pmatrix} 0.5219 & 0 \\ 0 & 1.8993 \end{pmatrix},$$

$$\delta = 2, \quad \sigma = 0.5, \quad h_k^* = 4.5, \quad \eta_k^* = 2.7, \quad \tau_k = 0.3, \quad \lambda_k = 0.5,$$

The following activation functions are playing in neural networks (2):

$$g(u(t)) = 0.3 * (|(u(t) + 1)| - |(u(t) - 1)|).$$

Taking the external input $\mathcal{H}(t) = |(0.7 * t)|$, and the LMI in Theorem 3.1 is solved by using the MATLAB LMI control toolbox.

Due to page limitation, the output solutions are ignored.

Hence the state trajectories of the neutral-type neural networks (2) are depicted in Fig. 1, Fig. 2. By solving the LMI (8), in Theorem 3.1 we can find the feasible solutions. The obtained upper bounds of multiple discrete time-delay & neutral delay h_k^* and η_k^* for neural networks (2) are maximum, which are given in Table 1 and Table 2, respectively. Therefore by Theorem 3.1, we conclude that the neural networks (2) is passive. This demonstrate that the contributions of this work is more effective and less conservative than some existing literatures.

Example 5.2. Consider a two dimensional multiple discrete delayed neutral-type neural networks (2) with the following associated parameters :

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \quad W_0 = \begin{pmatrix} 0.05 & 0.1 \\ 0.2 & 0.07 \end{pmatrix}, \quad W_1^{(1)} = \begin{pmatrix} 0.3 & 0.5 \\ 0.1 & 0.4 \end{pmatrix}, \quad W_1^{(2)} = \begin{pmatrix} 0.5 & 0.04 \\ 0.03 & 0.1 \end{pmatrix},$$

$$W_2 = \begin{pmatrix} 0.4 & -0.1 \\ 1 & -0.3 \end{pmatrix}, \quad W_3 = \begin{pmatrix} -0.2 & 0.3 \\ -0.2 & 0.8 \end{pmatrix}, \quad \Upsilon = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.03 \end{pmatrix},$$

$$\delta = 1, \quad h_k^* = 2.5, \quad \eta_k^* = 1.9, \quad \tau_k = 0.3, \quad \lambda_k = 0.5, \quad \sigma = 0.3,$$

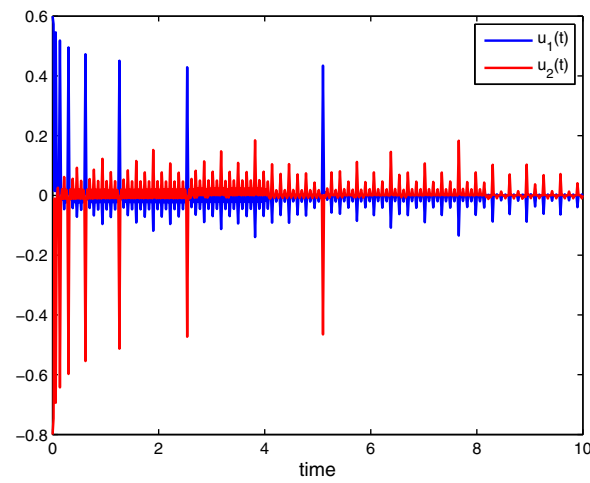


Fig. 2. State trajectories of addressed neutral-type neural networks (2) in Example 5.1.

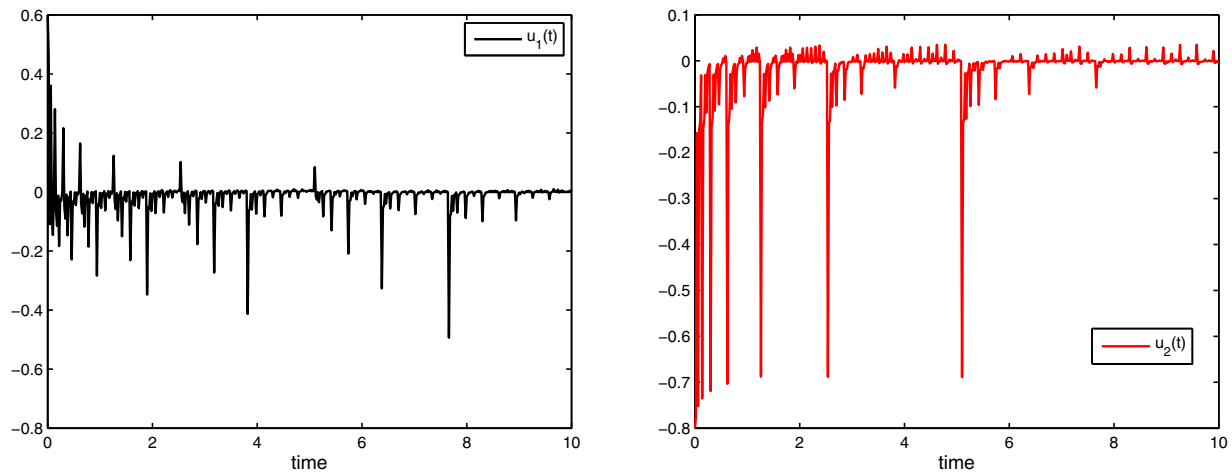


Fig. 3. The state trajectories of neural networks (2) in Example 5.2.

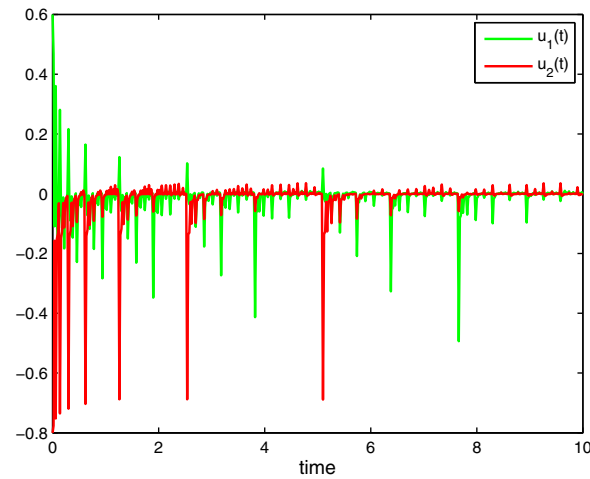


Fig. 4. State performance of concerned neural networks (2) in Example 5.2.

The following activation functions are playing in neural networks (2):

$$g(u(t)) = 0.05 * \sin(u(t)).$$

Taking the external input $\mathcal{H}(t) = \sin(-0.4) * (|(0.5 * t)|)$, and the LMI in Theorem 4.1 is solved by utilizing the LMI control toolbox in Matlab software.

Due to page limitation, the output solutions are omitted.

By using Matlab LMI control toolbox, we have the comparison of upper bounds of multiple discrete delay as listed in Table 3 when the passivity investigation. Moreover, the state responses of (2) are explore in Fig. 3 and Fig. 4. Thus, according to Theorem 4.1, we conclude that neural network system (2) is exponentially passive.

6. Conclusions

This paper mainly focuses on exploring a LMI approach to studying passivity and exponential passivity for a class of neutral-type neural networks with leakage, multiple discrete delay and distributed time delay. By employing the Lyapunov theory, inequality techniques and matrix theory, some new-brand sufficient conditions were derived for the addressed neural networks to guarantee the passivity and exponential passivity. Additionally, the upper bounds of the multiple discrete time-varying delay for our concerned design is very large when compared with the previous results, which may possess highly important significance in the proposed time-delayed neural networks. The obtained theoretical findings are less conserved and it can be verified by two numerical examples with their simulations, which shows that the validity of this research work.

Appendix A

$$\begin{aligned}
 \dot{V}(u(t), t) \leq & 2u^T(t)Q\dot{u}(t) + u^T(t)Yu(t) - (1 - \tau_k)u^T(t - h_k(t))Yu(t - h_k(t)) + g^T(u(t))S \\
 & \times g(u(t)) - (1 - \tau_k)g^T(u(t - h_k(t)))Sg(u(t - h_k(t))) + \dot{u}^T(t)R\dot{u}(t) - (1 - \lambda_k) \\
 & \times (\dot{u}^T(t - \eta_k(t)))R\dot{u}(t - \eta_k(t)) + u^T(t)H_1u(t) - u^T(t - \sigma)H_1u(t - \sigma) \\
 & + g^T(u(t))H_2g(u(t)) - g^T(u(t - \sigma))H_2g(u(t - \sigma)) + 2[u^T(t - \delta)K_1 + u^T(t)K_2 \\
 & + u^T(t - h_k(t))K_3 + u^T(t - \sigma)K_4 + \dot{u}^T(t - \eta_k(t))K_5 + g^T(u(t))K_6 \\
 & + g^T(u(t - h_k(t)))K_7 + \dot{u}^T(t)K_8] \left[u(t) - u(t - h_k(t)) - \int_{t-h_k(t)}^t \dot{u}(s)ds \right] \\
 & + 2[u^T(t - \delta)K_1 + u^T(t)K_2 + u^T(t - h_k(t))K_3 + u^T(t - \sigma)K_4 + \dot{u}^T(t - \eta_k(t)) \\
 & \times K_5 + g^T(u(t))K_6 + g^T(u(t - h_k(t)))K_7 + \dot{u}^T(t)K_8] \left[u(t) - u(t - \sigma) \right. \\
 & \left. - \int_{t-\sigma}^t \dot{u}(s)ds \right] + 2[u^T(t - \delta)M_1 + u^T(t)M_2 + \dot{u}^T(t)M_3 + \mathcal{H}^T(t)\mathcal{B}_1^*] \left\{ -Au(t - \delta) \right. \\
 & + W_0g(u(t)) + W_1^{(k)}g(u(t - h_k(t))) + W_3 \int_{t-\sigma}^t g(u(s))ds + \mathcal{H}(t) - \dot{u}(t) + W_2 \\
 & \times \dot{u}(t - \eta_k(t)) \left. \right\} - 2g^T(u(t))Tg(u(t)) + 2g^T(u(t))Tg(u(t)) + h_k^* \dot{u}^T(t)L_1\dot{u}(t) \\
 & - \int_{t-h_k^*}^t \dot{u}^T(s)L_1\dot{u}(s)ds + \eta_k^* \dot{u}^T(t)L_2\dot{u}(t) - \int_{t-\eta_k^*}^t \dot{u}^T(s)L_2\dot{u}(s)ds, \tag{26}
 \end{aligned}$$

where the diagonal matrix $T > 0$. By Assumption 1, condition (4) implies that

$$g^T(u(t))Tg(u(t)) \leq u^T(t)T\Upsilon g(u(t)) \tag{27}$$

where $\Upsilon = \text{diag}(\beta_1, \beta_2, \dots, \beta_n)$, $\beta_j > 0$, $j = 1, 2, \dots, n$ are given in Assumption 1.

Then it follows from (5) to (7) that

$$\begin{aligned}
 \dot{V}(u(t), t) \leq & 2u^T(t)Q \left[-Au(t - \delta) + W_0g(u(t)) + W_1^{(k)}g(u(t - h_k(t))) + W_3 \int_{t-\sigma}^t g(u(s))ds \right. \\
 & \left. + \mathcal{H}(t) + W_2\dot{u}(t - \eta_k(t)) \right] + u^T(t)Yu(t) - (1 - \tau_k)u^T(t - h_k(t))Yu(t - h_k(t)) \\
 & + g^T(u(t))Sg(u(t)) - (1 - \tau_k)g^T(u(t - h_k(t)))Sg(u(t - h_k(t))) + \dot{u}^T(t)R\dot{u}(t) \\
 & - (1 - \lambda_k)\dot{u}^T(t - \eta_k(t))R\dot{u}(t - \eta_k(t)) + u^T(t)H_1u(t) + g^T(u(t))H_2g(u(t)) \\
 & - g^T(u(t - \sigma))H_2g(u(t - \sigma)) + 2\{[u^T(t - \delta)K_1u(t) + u^T(t)K_2u(t) \\
 & + u^T(t - h_k(t))K_3u(t) + u^T(t - \sigma)K_4u(t) + \dot{u}^T(t - \eta_k(t))K_5u(t) + g^T(u(t)) \\
 & \times K_6u(t) + g^T(u(t - h_k(t)))K_7u(t) + \dot{u}^T(t)K_8u(t) - u^T(t - \delta)K_1u(t - h_k(t)) \\
 & - u^T(t)K_2u(t - h_k(t)) - u^T(t - h_k(t))K_3u(t - h_k(t)) - u^T(t - \sigma)K_4u(t \\
 & - h_k(t)) - \dot{u}^T(t - \eta_k(t))K_5u(t - h_k(t)) - g^T(u(t))K_6u(t - h_k(t)) \\
 & - g^T(u(t - h_k(t)))K_7u(t - h_k(t)) - \dot{u}^T(t)K_8u(t - h_k(t))\} - h_k^* \dot{u}^T(t)L_1\dot{u}(t) \\
 & + \int_{t-h_k^*}^t \dot{u}^T(s)L_1\dot{u}(s)ds - 2[u^T(t - \delta)K_1 + u^T(t)K_2 + u^T(t - h_k(t))K_3 \\
 & + u^T(t - \sigma)K_4 + \dot{u}^T(t - \eta_k(t))K_5 + g^T(u(t))K_6 + g^T(u(t - h_k(t)))K_7 \\
 & + \dot{u}^T(t)K_8] \int_{t-h_k(t)}^t \dot{u}(\alpha)d\alpha + 2\{[u^T(t - \delta)A_1u(t) + u^T(t)A_2u(t)
 \end{aligned}$$

$$\begin{aligned}
& + u^T(t - h_k(t))A_3u(t) + u^T(t - \sigma)A_4u(t) + \dot{u}^T(t - \eta_k(t))A_5u(t) + g^T(u(t)) \\
& \times A_6u(t) + g^T(u(t - h_k(t)))A_7u(t) + \dot{u}^T(t)A_8u(t) - u^T(t - \delta)A_1u(t - \sigma) \\
& - u^T(t)A_2u(t - \sigma) - u^T(t - h_k(t))A_3u(t - \sigma) - u^T(t - \sigma)A_4u(t - \sigma) \\
& - \dot{u}^T(t - \eta_k(t))A_5u(t - \sigma) - g^T(u(t))A_6u(t - \sigma) - g^T(u(t - h_k(t)))A_7u(t - \sigma) \\
& - \dot{u}^T(t)A_8u(t - \sigma)] - \eta_k^* \dot{u}^T(t)L_2\dot{u}(t) + \int_{t-\eta_k^*}^t \dot{u}^T(s)L_2\dot{u}(s)ds - 2[u^T(t - \delta)A_1 \\
& + u^T(t)A_2 + u^T(t - h_k(t))A_3 + u^T(t - \sigma)A_4 + \dot{u}^T(t - \eta_k(t))A_5 + g^T(u(t))A_6 \\
& + g^T(u(t - h_k(t)))A_7 + \dot{u}^T(t)A_8] \int_{t-\sigma}^t \dot{u}(\gamma)d\gamma + 2 \left\{ -u^T(t - \delta)AM_1u(t - \delta) \right. \\
& - AM_2u^T(t)u(t - \delta) - AM_3\dot{u}^T(t)u(t - \delta) + W_0M_1u^T(t - \delta)g(u(t)) + W_0M_2 \\
& \times u^T(t)g(u(t)) + W_0M_3\dot{u}^T(t)g(u(t)) + W_1^{(k)}M_1u^T(t - \delta)g(u(t - h_k(t))) \\
& + M_2W_1^{(k)}u^T(t)g(u(t - h_k(t))) + W_1^{(k)}M_3\dot{u}^T(t)g(u(t - h_k(t))) + M_1W_3 \\
& \times u^T(t - \delta) \int_{t-\sigma}^t g(u(s))ds + W_3M_2u^T(t) \int_{t-\sigma}^t g(u(s))ds + W_3M_3\dot{u}^T(t) \\
& \times \int_{t-\sigma}^t g(u(s))ds + u^T(t - \delta)M_1\mathcal{H}(t) + u^T(t)M_2\mathcal{H}(t) + \dot{u}^T(t)M_3\mathcal{H}(t) + \mathcal{H}^T(t) \\
& \times B_1^*\mathcal{H}(t) - u^T(t - \delta)M_1\dot{u}(t) - u^T(t)M_2\dot{u}(t) - \dot{u}^T(t)M_3\dot{u}(t) + W_2M_1u^T(t - \delta) \\
& \times \dot{u}(t - \eta_k(t)) + W_2M_2u^T(t)\dot{u}(t - \eta_k(t)) + W_2M_3\dot{u}^T(t)\dot{u}(t - \eta_k(t)) - \mathcal{H}^T(t) \\
& \times AB_1^*u(t - \delta) + \mathcal{H}^T(t)W_0B_1^*g(u(t)) + \mathcal{H}^T(t)W_1^{(k)}B_1^*g(u(t - h_k(t))) + \mathcal{H}^T(t)W_3 \\
& \times B_1^* \left(\int_{t-\sigma}^t g(u(s))ds \right) - \mathcal{H}^T(t)B_1^*\dot{u}(t) + \mathcal{H}^T(t)B_1^*W_2\dot{u}(t - \eta_k(t)) \left. \right\} - 2g^T(u(t))T \\
& \times g(u(t)) + 2u^T(t)T\Upsilon g(u(t))
\end{aligned} \tag{28}$$

Appendix B

$$\varrho_1 = 2\zeta^T(t)\mathcal{E}_1 \left[u(t) - u(t - h_k(t)) - \int_{t-h_k(t)}^t \dot{u}(\xi)d\xi \right] = 0, \tag{29}$$

$$\varrho_2 = 2\zeta^T(t)\mathcal{E}_2 \left[u(t - h_k(t)) - u(t - h_k^*) - \int_{t-h_k^*}^{t-h_k(t)} \dot{u}(\xi)d\xi \right] = 0, \tag{30}$$

$$\varrho_3 = 2\zeta^T(t)\mathcal{E}_3 \left[g(u(t)) - g(u(t - \sigma)) - \int_{t-\sigma}^t g(u(\xi))d\xi \right] = 0, \tag{31}$$

$$\begin{aligned}
\varrho_4 = 2\zeta^T(t)\mathcal{E}_4 \left[-Au(t - \delta) + W_0g(u(t)) + W_1^{(k)}g(u(t - h_k(t))) + W_3 \int_{t-\sigma}^t g(u(s))ds \right. \\
\left. + W_2\dot{u}(t - \eta_k(t)) + \mathcal{H}(t) - \dot{u}(t) \right] = 0,
\end{aligned} \tag{32}$$

where

$$\zeta^T(t) = \begin{bmatrix} u(t) \\ u(t - \delta) \\ \dot{u}(t) \\ u(t - h_k(t)) \\ g(u(t)) \\ g(u(t - h_k(t))) \\ u(t - h_k^*) \\ \dot{u}(t - \eta_k(t)) \\ u(t - \sigma) \\ \mathcal{H}(t) \\ u(t - \eta_k(t)) \\ \dot{u}(t - \eta_k^*) \\ g(u(t - \sigma)) \end{bmatrix}$$

From the elementary inequality, for any matrix $\mathcal{C}_9$, $\mathcal{D}_1 > 0$, we obtain

$$-2\zeta^T(t)\mathcal{E}_1 \int_{t-h_k(t)}^t \dot{u}(\xi)d\xi \leq h_k^* \zeta^T(t)\mathcal{E}_1 \mathcal{C}_9^{-1} \mathcal{E}_1^T \zeta(t) + \int_{t-h_k(t)}^t \dot{u}^T(\xi)\mathcal{C}_9 \dot{u}(\xi)d\xi \quad (33)$$

$$-2\zeta^T(t)\mathcal{E}_2 \int_{t-h_k^*}^{t-h_k(t)} \dot{u}(\xi)d\xi \leq h_k^* \zeta^T(t)\mathcal{E}_2 \mathcal{C}_9^{-1} \mathcal{E}_2^T \zeta(t) + \int_{t-h_k^*}^{t-h_k(t)} \dot{u}^T(\xi)\mathcal{C}_9 \dot{u}(\xi)d\xi, \quad (34)$$

$$-2\zeta^T(t)\mathcal{E}_3 \int_{t-\sigma}^t g(u(\xi))d\xi \leq \sigma \zeta^T(t)\mathcal{E}_3 \mathcal{D}_1^{-1} \mathcal{E}_3^T \zeta(t) + \int_{t-\sigma}^t g^T(u(\xi))\mathcal{D}_1 g(u(\xi))d\xi, \quad (35)$$

$$2\zeta^T(t)\mathcal{E}_4 W_3 \int_{t-\sigma}^t g(u(s))ds \leq \sigma \zeta^T(t)(\mathcal{E}_4 W_3)\mathcal{D}_1^{-1}(\mathcal{E}_4 W_3)^T \zeta(t) + \int_{t-\sigma}^t g^T(u(s)) \times \mathcal{D}_1 g(u(s))ds \quad (36)$$

$$\mathcal{Q}_4 = \zeta^T(t)\{\mathcal{E}_4 \mathcal{M}^* + \mathcal{M}^{*T} \mathcal{E}_4\} \zeta(t), \quad (37)$$

where $\mathcal{M}^* = [0, -A, -I, 0, W_0, W_1^{(k)}, 0, W_2, 0, I, 0, 0, 0]$.

Then, from [Assumption 1](#), it can be deduced that for any positive diagonal matrix $\mathcal{J}_1$,

$$\mathcal{Q}_5 = 2g^T(u(t))\mathcal{J}_1[\Upsilon u(t) - g(u(t))] \geq 0, \quad (38)$$

$$\mathcal{Q}_6 = 2g^T(u(t-h_k(t)))\mathcal{J}_1[\Upsilon u(t-h_k(t)) - g(u(t-h_k(t)))] \geq 0, \quad (39)$$

where $\Upsilon = \text{diag}(\beta_1, \beta_2, \dots, \beta_n)$, $\beta_j > 0$, $j = 1, 2, 3, \dots, n$ are provided in [Assumption 1](#).

From (22)–(39), it follows that

$$\begin{aligned} \dot{V}(u(t), t) - 2v^T(t)\mathcal{H}(t) &\leq 2u^T(t)\mathcal{C}_1 \dot{u}(t) + 2g^T(u(t))\mathcal{C} \dot{u}(t) - \gamma V_2 + g^T(u(t))\mathcal{C}_2 g(u(t)) \\ &\quad + u^T(t)\mathcal{C}_3 u(t) - e^{-\gamma h_k^*}(1 - \tau_k)g^T(u(t-h_k(t)))\mathcal{C}_2 g(u(t-h_k(t))) \\ &\quad - (1 - \tau_k)e^{-\gamma h_k^*}u^T(t-h_k(t))\mathcal{C}_3 u(t-h_k(t)) + u^T(t)\mathcal{C}_4 u(t) \\ &\quad - e^{-\gamma h_k^*}u^T(t-h_k^*)\mathcal{C}_4 u(t-h_k^*) - \gamma V_3 + \dot{u}^T(t)\mathcal{C}_5 \dot{u}(t) + u^T(t)\mathcal{C}_6 \\ &\quad \times u(t) - e^{-\gamma \eta_k^*}(1 - \lambda_k)\dot{u}^T(t-\eta_k(t))\mathcal{C}_5 \dot{u}(t-\eta_k(t)) - e^{-\gamma \eta_k^*} \\ &\quad \times (1 - \lambda_k)u^T(t-\eta_k(t))\mathcal{C}_6 u(t-\eta_k(t)) + \dot{u}^T(t)\mathcal{C}_7 \dot{u}(t) - e^{-\gamma \eta_k^*} \\ &\quad \times \dot{u}^T(t-\eta_k^*)\mathcal{C}_7 \dot{u}(t-\eta_k^*) - \gamma V_4 + h_k^*(u^T(t)\mathcal{C}_8 u(t) + 2\dot{u}^T(t)\mathcal{C}_9 \dot{u}(t)) \\ &\quad - e^{-\gamma h_k^*} \int_{t-h_k^*}^t (u^T(\xi)\mathcal{C}_8 u(\xi) + 2\dot{u}^T(\xi)\mathcal{C}_9 \dot{u}(\xi))d\xi - \gamma V_5 + \eta_k^*(\dot{u}^T(t) \\ &\quad \times \mathcal{C}_{10} \dot{u}(t) + u^T(t)\mathcal{F}^* u(t)) - e^{-\gamma \eta_k^*} \int_{t-\eta_k^*}^t (\dot{u}^T(\xi)\mathcal{C}_{10} \dot{u}(\xi) + u^T(\xi)\mathcal{F}^* \\ &\quad \times u(\xi))d\xi - \gamma V_6 + 3\sigma g^T(u(t))\mathcal{D}_1 g(u(t)) - 3e^{-\gamma \sigma} \left(\int_{t-\sigma}^t g(u(\xi))d\xi \right)^T \\ &\quad \times \mathcal{D}_1 \left(\int_{t-\sigma}^t g(u(\xi))d\xi \right) - 2v^T(t)\mathcal{H}(t) + \sum_{q=1}^6 \mathcal{Q}_q \end{aligned} \quad (40)$$

From the inequality (40), we have

$$\begin{aligned} \dot{V}(u(t), t) - 2v^T(t)\mathcal{H}(t) &\leq \zeta^T(t)\Lambda^* \zeta(t) - \gamma V(u(t), t) - e^{-\gamma h_k^*} \int_{t-h_k^*}^t u^T(\xi)\mathcal{C}_8 u(\xi)d\xi - 2e^{-\gamma h_k^*} \\ &\quad \times \int_{t-h_k^*}^t \dot{u}^T(\xi)\mathcal{C}_9 \dot{u}(\xi)d\xi - e^{-\gamma \eta_k^*} \int_{t-\eta_k^*}^t \dot{u}^T(\xi)\mathcal{C}_{10} \dot{u}(\xi)d\xi - e^{-\gamma \eta_k^*} \\ &\quad \times \int_{t-\eta_k^*}^t u^T(\xi)\mathcal{F}^* u(\xi)d\xi - 3e^{-\gamma \sigma} \left(\int_{t-\sigma}^t g(u(\xi))d\xi \right)^T \mathcal{D}_1 \\ &\quad \times \left(\int_{t-\sigma}^t g(u(\xi))d\xi \right), \end{aligned} \quad (41)$$

where $\Lambda^* = \Sigma + h_k^* \mathcal{E}_1 \mathcal{C}_9^{-1} \mathcal{E}_1^T + h_k^* \mathcal{E}_2 \mathcal{C}_9^{-1} \mathcal{E}_2^T + \sigma \mathcal{E}_3 \mathcal{D}_1^{-1} \mathcal{E}_3^T + \sigma (\mathcal{E}_4 W_3) \mathcal{D}_1^{-1} (\mathcal{E}_4 W_3)^T$.

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Original articles

Nonlinear integro-differential equations with small unknown parameters: A controllability analysis problem

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Abstract

This manuscript is perturbed with a controllability problem of nonlinear fractional dynamical systems with delay in the state variable. By employing Laplace transformation technique and using Mittag-Leffler function, solution representation of the examined fractional delay differential equations can be devised. Besides, we build the necessary as well as sufficient condition, in order to prove the controllability of linear fractional delay dynamical structures. Especially, the sufficiency part for the controllability results is obtained by using the fixed point argument. In addition to that, we have provided three numerical examples to illustrate the essence of our obtained theoretical statements.

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1. Introduction

The inclusion of fractional order in differential equations of diverse kinds plays a crucial role not only in Mathematics but also in physics, control systems, dynamical systems and engineering and also it is a valuable tool in modeling of many physical phenomena, see for instance [18,21]. Basically, the fractional order differentiation is the generalization of classical integer order differentiation. Numerous writers have stretched the solutions from integer order to a fractional sense, see the references [1,2,4,6,13,22] that owed to its signification. Moreover, the fractional differential equation is considered to be a substitute model for nonlinear differential equation [8]. So, it increases the research interest among the scientists and researchers from around the world, see the references [5,14]. Furthermore, in numerous medical and physical phenomena the involvement of delay features arises in the fractional

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differential equation. The aroused delay may be with the state-dependent one or in the non-constant delay. Basically, delay differential equations are solved by using numerical methods, asymptotic solutions and graphical tools. Bellman Cooke [7], Smith [23], Halanay [11], Oguztoreli [20] and Hale [12] deliberated the analytic form of solutions to delay differential equations. As yet there is no relevant work dealing with the explicit solution of representations of fractional delay differential equations with small parameters. The variation of parameter formula for integer order delay differential equations is acquired by employing the Lambert W function [24–26], construction of Lambert 'W' function is impossible in fractional sense.

On the other hand, one of the most cardinal qualitative responses of a dynamical structure is controllability. Controllability is a process of attainable to steer any starting state of the system to any terminal state in some finite time employing an admissible control [15,16,17]. Generally, controllability for nonlinear dynamical systems is not as consistent and associated as in the case of linear dynamical structures. As a result, there is a lack in solving nonlinear differential or functional differential equations from the existing general methods. In order to overcome this situation, the authors Balachandran et al. in [1,2] used a fixed point technique to acquire the controllability results for nonlinear dynamical structures. Hence this method play a key role to find the controllability analysis for nonlinear systems. Also, the utilization of fractional-order derivatives and integrals in control theory steers to some quality results than the integer-order approaches. So, many physicists, mathematicians and engineers made impressed contributions to the evolution of both the theory and applications of fractional differential equations. This inspires our present work.

Our main contributions are highlighted as follows:

- This manuscript investigates the problem on controllability of nonlinear fractional dynamical systems with delay in the state variable.
- By employing Laplace transformation technique and using Mittag-Leffler function, the solution representation for the examined fractional delay differential equations can be devised.
- In our proposed work, we utilized the fixed point techniques to prove the controllability results, whereas in most of the available literature, they used the semigroup theory which is not applicable to obtain the controllability criteria. So, this shows the novelty of our work.
- Additionally, we have provided three numerical examples, in order to justify our theoretical concepts.

This paper is organized as follows. In Section 2, we present some basic definitions and preliminary facts. In Section 3, we give the solution representation of fractional delay differential equations by following the work of Hale [12]. By defining the Grammian matrix we have studied the controllability of linear fractional delay dynamical structures. Sufficient conditions for the controllability of nonlinear fractional delay systems have been established by using Schauder's fixed point theorem. In Section 4, three examples are given to illustrate the effectiveness and validity of controllability criteria. Finally conclusions are drawn in Section 5.

2. Preliminaries

In this component, we first provide some rudimentary definitions employed as the same in [14,19] and then we furnish few preliminary concepts to help us to prove the controllability criteria for the inscribed structure.

Definition 2.1. The Riemann–Liouville fractional integral operator of order $\alpha > 0$ with the lower limit zero for a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is defined as

$$(I_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^{\alpha} (t-s)^{\alpha-1} f(s) ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. The Riemann–Liouville fractional derivative of order $\alpha > 0$, with the lower limit zero for a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^n$, $n-1 < \alpha < n$, $n \in \mathbb{N}$ is defined as

$$(D_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Definition 2.3. The Caputo fractional derivative of order $\alpha > 0, n - 1 < \alpha < n$ is defined as

$$({}^C D_{0+}^\alpha f)(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} f^n(s) ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n - 1)$. If $0 < \alpha < 1$, then

$$\begin{aligned}({}^C D_{0+}^\alpha f)(t) &= \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{f'(s)}{(t - s)^\alpha} ds \\({}^C D_{0+}^\alpha f)(t) &= (I_{0+}^{n-\alpha} D^n f)(t)\end{aligned}$$

In particular $I_{0+}^\alpha f(t) = f(t) - f(0)$, $(0 < \alpha < 1)$.

Some special properties which is useful in solving our required criteria of the above mentioned operators are as:

- (i) $({}^C D_{0+}^\alpha 1) = 0$.
- (ii) $(D_{0+}^\alpha 1) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}$.
- (iii) $I_{0+}^\alpha(f(t) + g(t)) = I_{0+}^\alpha f(t) + I_{0+}^\alpha g(t)$.
- (iv) $I_{0+}^\alpha I_{0+}^\beta f(t) = I_{0+}^{\alpha+\beta} f(t) = I_{0+}^\beta I_{0+}^\alpha f(t)$.
- (v) $D_{0+}^\alpha I_{0+}^\alpha f(t) = f(t)$.
- (vi) In general, $D_{0+}^\alpha D_{0+}^\beta f(t) \neq D_{0+}^{\alpha+\beta} f(t)$ and $D_{0+}^\alpha D_{0+}^\beta f(t) \neq D_{0+}^\beta D_{0+}^\alpha f(t)$.
- (vii) The general formula of the Laplace transform using Caputo fractional derivative is

$$L {}^C D_{0+}^\alpha f(t)(s) = s^\alpha F(s) - \sum_{k=0}^{n-1} f^k(0^+) s^{\alpha-1-k}.$$

Definition 2.4. The Mittag-Leffler function in two parameters is defined as

$$\begin{aligned}E_\alpha(z) &= E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad z \in \mathbb{C}, \quad \operatorname{Re}(\alpha) > 0 \\E_{\alpha,\beta}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad z, \beta \in \mathbb{C}, \quad \operatorname{Re}(\alpha) > 0 \\E_{\alpha,\beta}^\gamma(-\lambda t^\alpha) &= \sum_{k=0}^{\infty} \frac{\gamma_k(-\lambda)^k}{k! \Gamma(k\alpha + \beta)} t^{\alpha k},\end{aligned}$$

where $(\gamma)_n$ is a Pochhammer symbol is defined as $\gamma(\gamma + 1) \dots (\gamma + n - 1)$ and $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$. The Laplace transforms of the above Mittag-Leffler functions are defined to be

$$\begin{aligned}L[E_{\alpha,1}(\pm \lambda t^\alpha)](s) &= \frac{s^{\alpha-1}}{s^\alpha \pm \lambda}, \quad \operatorname{Re}(\alpha) > 0, \\L[t^{\beta-1} E_{\alpha,\beta}(\pm \lambda t^\alpha)](s) &= \frac{s^{\alpha-\beta}}{s^\alpha \pm \lambda}, \quad \operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\beta) > 0 \\L[t^{\beta-1} E_{\alpha,\beta}^\gamma(\pm \lambda t^\alpha)](s) &= \frac{s^{\alpha\gamma-\beta}}{(s^\alpha \pm \lambda)^\gamma}, \quad \operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\beta) > 0.\end{aligned}$$

The general Mittag-Leffler function satisfies

$$\int_0^\infty e^{-t} t^{\beta-1} E_{\alpha,\beta}(t^\alpha z) dt = \frac{1}{1-z}, \quad \text{for } |z| < 1.$$

Definition 2.5. The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L(f(t)) = \int_0^{+\infty} e^{-st} f(t) dt, \quad s \in \mathbb{C},$$

where $f(t)$ is n -dimensional vector valued function. The delayed unit step function is defined by

$$u_m(t) = \begin{cases} 1, & t \geq m \\ 0, & t < m, \end{cases}$$

and its Laplace transformation is given by

$$L[u_m(t)](s) = \frac{e^{-ms}}{s}, \quad s > 0$$

$$\{If \quad F(s) = L[f(t)](s) \text{ for } \operatorname{Re}(s) > 0, \text{ then } ([22]) \quad F(s - m) = L[e^{mt} f(t)](s)\},$$

$$\text{and } L[u_m(t)f(t - m)](s) = e^{-ms} F(s), \quad m \geq 0,$$

and also we have

$$L^{-1}[e^{-ms} F(s)](t) = u_m(t)f(t - m).$$

Lemma 2.6 ([21]). Let $\mathbb{C}$ be complex plane, for any $\alpha > 0$, $\beta > 0$ and $A \in \mathbb{C}^{n \times n}$ then

$$L[t^{\beta-1} E_{\alpha, \beta}(At^\alpha)] = s^{\alpha-\beta} (s^\alpha I - A)^{-1}, \quad \Re(s) > \|A\|^{\frac{1}{\alpha}}$$

holds, where $\Re(s)$ denotes the real part of the complex numbers s and I represents the identity matrix. In particular, for $\beta = 1$,

$$E_{\alpha, 1}(\lambda z^\alpha) = E_\alpha(\lambda z^\alpha) = \sum_{k=0}^{\infty} \frac{\lambda^k z^{k\alpha}}{\Gamma(\alpha k + 1)}, \quad \lambda, z \in \mathbb{C}$$

have the interesting property

$${}^C D^\alpha E_\alpha(\lambda t^\alpha) = \lambda E_\alpha(\lambda t^\alpha)$$

$$\text{and } L E_\alpha(\pm a t^\alpha)(s) = \frac{s^{\alpha-1}}{s^\alpha \pm a}.$$

3. Solution representation

Before proving the controllability criteria in this module, firstly, we acquire the existence of solutions of the considered system.

Now, consider a fractional delay integro differential equation of the form

$${}^C D^\alpha x(t) = A(\xi)x(t) + Ex(t - \tau(t, \xi)) + \int_0^t K(t-s)x(s)ds + f(t), \quad 0 < \alpha < 1, t \in [0, T] = J,$$

$$x(t) = \phi(t), \quad t \in [-\tau, 0], \quad (1)$$

where $x \in \mathbb{R}^n$, A and E are $n \times n$ matrices $x \in \mathbb{R}^n$, K is an $n \times n$ continuous matrix, $\phi(t)$ is a continuous function on $[-\tau, 0]$ and f is a real continuous function in $\mathbb{R}^n$ and ξ is a small parameter.

By enrolling a Laplace transform to equation (1), we have

$$\begin{aligned} S^\alpha X(s) - S^{\alpha-1} \phi(0) &= A(\xi) \int_0^\infty x(t)e^{-st} dt + E \int_0^\infty x(t - \tau(t, \xi))e^{-st} dt \\ &\quad + \int_0^\infty x(t)e^{-st} dt \int_0^\infty K(t-s)x(s)ds + F(s) \\ S^\alpha X(s) - S^{\alpha-1} \phi(0) &= A(\xi)X(s) + L(K)X(s) \\ &\quad + Ee^{-s\tau(t, \xi)}X(s) + Ee^{-s\tau(t, \xi)} \int_{-\tau}^0 e^{-s\eta} \phi(\eta) d\eta + F(s) \\ X(s)(S^\alpha I - A(\xi) - Ee^{-s\tau(t, \xi)} - L(K)) &= S^{\alpha-1} \phi(0) + Ee^{-s\tau(t, \xi)} \int_{-\tau}^0 e^{-s\eta} \phi(\eta) d\eta + F(s) \\ X(s) &= \frac{S^{\alpha-1}}{S^\alpha I - A(\xi) - Ee^{-s\tau(t, \xi)} - L(K)} \phi(0) \\ &\quad + \frac{F(s)}{S^\alpha I - A(\xi) - Ee^{-s\tau(t, \xi)} - L(K)} \\ &\quad + E \frac{e^{-s\tau(t, \xi)}}{S^\alpha I - A(\xi) - Ee^{-s\tau(t, \xi)} - L(K)} \int_{-\tau}^0 e^{-s\eta} \phi(\eta) d\eta. \end{aligned}$$

By employing inverse Laplace transform on both sides, we have

$$\begin{aligned} x(t) &= L^{-1}\{S^{\alpha-1}(S^{\alpha}I - A(\xi) - Ee^{-s\tau(t,\xi)} - L(K))^{-1}\}\phi(0) + L^{-1}\{F(s)\}(t) \\ &\quad \times L^{-1}\{S^{\alpha-1}(S^{\alpha}I - A(\xi) - Ee^{-s\tau(t,\xi)} - L(K))^{-1}\}(t) \\ &\quad + EL^{-1}\{e^{-s\tau(t,\xi)} \int_{-\tau}^0 e^{-s\eta} \phi(\eta) d\eta\}(t) \\ &\quad \times L^{-1}\{S^{\alpha-1}(S^{\alpha}I - A(\xi) - Ee^{-s\tau(t,\xi)} - L(K))^{-1}S^{1-\alpha}\}(t). \end{aligned}$$

Let

$$X_{\alpha}(t) = L^{-1}[S^{\alpha-1}(S^{\alpha}I - A(\xi) - Ee^{-s\tau(t,\xi)} - L(K))^{-1}](t)$$

and

$$\begin{aligned} L^{-1}\left[\frac{S^{\alpha-1}}{S^{\alpha}I - A(\xi) - Ee^{-s\tau(t,\xi)} - L(K)}S^{1-\alpha}\right](t) &= X_{\alpha}(t) * \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \\ &= \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} X_{\alpha}(s) ds \\ &= t^{\alpha-1} X_{\alpha,\alpha}(t), \end{aligned}$$

where

$$X_{\alpha,\alpha}(t) = t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} X_{\alpha}(s) ds.$$

Define $\psi(t) : [-\tau, \infty] \rightarrow [0, 1]$ by $\psi(t) = 0$ for $t \geq 0$ and $\psi(t) = 1$ for $t < 0$. If $\phi(t)$ is prolonged to $(-\tau, \infty)$ by taking $\phi(t) = \phi(0)$ for $t \geq 0$, then

$$\begin{aligned} e^{-s\tau(t,\xi)} \int_{-\tau}^0 e^{-s\eta} \phi(\eta) d\eta &= e^{-s\tau(t,\xi)} \int_{-\tau}^0 e^{-s(-\tau+\lambda)} \phi(-\tau+\lambda) d\lambda \\ &= e^{(-s\lambda)} \phi(-\tau+\lambda) \psi(-\tau+\lambda) d\lambda \\ &= L[\phi(-\tau + \cdot) \psi(-\tau + \cdot)](t). \end{aligned}$$

Thus

$$\begin{aligned} x(t) &= X_{\alpha}(t)\phi(0) + E \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \phi(s-\tau) \psi(s-\tau) ds + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) f(s) ds \\ x(t) &= X_{\alpha}(t)\phi(0) + E \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \phi(s-\tau) ds + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) f(s) ds. \end{aligned}$$

Hence

$$x(t) = X_{\alpha}(t)\phi(0) + E \int_{-\tau}^0 (t-s-\tau)^{\alpha-1} X_{\alpha,\alpha}(t-s-\tau) \phi(s) ds + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) f(s) ds.$$

After inspecting the linear fractional delay dynamical structure in the following form

$${}^C D^{\alpha} x(t) = A(\xi)x(t) + Ex(t-\tau(t,\xi)) + \int_0^t K(t-s)x(s) ds + Bu(t), \quad 0 < \alpha < 1, \quad t \in [0, T] = J,$$

$$x(t) = \phi(t), \quad t \in [-\tau, 0] \tag{2}$$

now, we construct the following nonlinear dynamical system

$$\begin{aligned} {}^C D^{\alpha} x(t) &= A(\xi)x(t) + Ex(t-\tau(t,\xi)) + \int_0^t K(t-s)x(s) ds + Bu(t) \\ &\quad + f(t, x(t-\tau(t,\xi)), {}^C D^{\alpha} x(t-\tau(t,\xi)), u(t)), \quad 0 < \alpha < 1, \quad t \in [0, T] = J, \end{aligned}$$

$$x(t) = \phi(t), \quad t \in [-\tau, 0]. \tag{3}$$

The solution of (2) is obtained to be

$$x(t) = X_{\alpha}(t)\phi(0) + E \int_{-\tau}^0 (t-s-\tau)^{\alpha-1} X_{\alpha,\alpha}(t-s-\tau) \phi(s) ds + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) Bu(s) ds.$$

Further, the solution of (3) is termed to be of the following form:

$$x(t) = X_{\alpha}(t)\phi(0) + E \int_{-\tau}^0 (t-s-\tau)^{\alpha-1} X_{\alpha,\alpha}(t-s-\tau)\phi(s)ds + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \\ \times [Bu(s) + f(s, x(s-\tau(s, \xi)), {}^C D^{\alpha}x(s-\tau(s, \xi)), u(s))]ds.$$

Definition 3.1. The system (2) or (3) is said to be completely controllable on J if for every initial function $\phi(t)$ and $x_1 \in \mathbb{R}^n$, there exists a continuous control function u such that the solution of (2) or (3) satisfy $x(T) = x_1$.

Here the solution of Eq.(2) related to u is

$$x_M(t) = x_M(t; \phi) + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s)Bu(s)ds, \quad t \in J, \quad (4)$$

where

$$x_M(t; \phi) = X_{\alpha}(t)\phi(0) + E \int_{-\tau}^0 (t-s-\tau)^{\alpha-1} X_{\alpha,\alpha}(t-s-\tau)\phi(s)ds.$$

Similarly, the solution $x(t)$ of the structure (3) related to u takes the form

$$x(t) = x_M(t; \phi) + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s)Bu(s)ds + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \\ \times f(s, x(s-\tau(s, \xi)), {}^C D^{\alpha}x(s-\tau(s, \xi)), u(s))ds, \quad \text{for } t \in J$$

and $x(t) = \phi(t)$ for $t \in [-\tau, 0]$. The Grammian matrix is defined as

$$W = \int_0^T (X_{\alpha,\alpha}(T-s)B)(X_{\alpha,\alpha}(T-s)B)^*ds, \quad (5)$$

where $*$ represents the matrix transpose.

Theorem 3.2. The structure (2) is completely controllable on J if and only if W is nonsingular.

Proof. Suppose that W is nonsingular. Let $\phi(t)$ be continuous on $[-\tau, 0]$ and let $x_1 \in \mathbb{R}^n$. The control function u is defined by

$$u(t) = (T-t)^{1-\alpha}(X_{\alpha,\alpha}(T-t)B)^*W^{-1}(x_1 - x_M(t; \phi)), \quad t \in J. \quad (6)$$

Substituting $t = T$ in the solution of (4), we get

$$x_M(T) = x_M(T; \phi) + \int_0^T (T-s)^{\alpha-1} X_{\alpha,\alpha}(T-s)Bu(s)ds, \quad t \in J \quad (7)$$

and using (6) in (7), we have

$$x_M(T) = x_M(T; \phi) + \int_0^T (T-s)^{\alpha-1} X_{\alpha,\alpha}(T-s)B(T-s)^{1-\alpha}(X_{\alpha,\alpha}(T-s)B)^*W^{-1}(x_1 - x_M(s; \phi))ds \\ = x_1.$$

Now, suppose that W is nonsingular. There is a vector $l \neq 0$ such that $l^*Wl = 0$.

$$\int_0^T l^*(X_{\alpha,\alpha}(T-s)B)(l^*(X_{\alpha,\alpha}(T-s)B))^*ds = 0.$$

Therefore $l^*(X_{\alpha,\alpha}(T-s)B) = 0$ for $s \in J$.

Consider the zero initial function $\phi = 0$ and the terminal point $x_1 = l$. Since the system is controllable there is a control $u(t)$ on J that steers the response to $x_1 = l$ at $t = T$, that is $x_M(T) = l$. From $\phi = 0$, $x_M(T; \phi) = 0$ and $l^*l \neq 0$ for $l \neq 0$.

On the other hand

$$l = x_M(T) = \int_0^T (T-s)^{\alpha-1} X_{\alpha,\alpha}(T-s)Bu(s)ds,$$

and hence

$$l^* = \int_0^T (T-s)^{\alpha-1} X_{\alpha,\alpha}(T-s) B u(s) ds = 0$$

we get a contradiction to $l \neq 0$. Therefore W is nonsingular.

Theorem 3.3. Let f be the continuous function that satisfies the condition $\lim_{|p| \rightarrow \infty} \frac{|f(t,p)|}{|p|} = 0$ uniformly in $t \in J$ and assume that the system (2) is completely controllable on J . Then the system (3) is completely controllable on J , where $p = (x, l, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ and let $|p| = |x| + |l| + |u|$.

Proof. Assume that $\phi(t)$ be continuous on $[-\tau, 0]$ and let $x_1 \in \mathbb{R}^n$. Let Q be the Banach space of all continuous functions $(x, u) : [-\tau, T] \times [0, T] \rightarrow \mathbb{R}^n \times \mathbb{R}^m$, with the norm $\|(x, u)\| = \|x\| + \|u\|$ where $\|x\| = \{\sup |x(t)| \text{ for } t \in [-\tau, T]\}$ and $\|u\| = \{\sup |u(t)| \text{ for } t \in [0, T]\}$.

An operator $\Omega : Q \rightarrow Q$ is defined by $\Psi(x, u) = (l, v)$. Here

$$\begin{aligned} v(t) &= (T-t)^{1-\alpha} (X_{\alpha,\alpha}(T-t) B)^* W^{-1} (x_1 - x_M(T; \phi) - \int_0^t (T-t)^{\alpha-1} X_{\alpha,\alpha}(T-t) \\ &\quad \times f(t, x(t-\tau(t, \xi)), {}^C D^\alpha x(t-\tau(t, \xi)), u(t)) dt \\ l(t) &= x_M(t; \phi) + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) B v(s) ds + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \\ &\quad \times f(s, x(s-\tau(s, \xi)), {}^C D^\alpha x(s-\tau(s, \xi)), u(s)) ds \end{aligned}$$

if $t \in J$ and $l(t) = \phi(t)$, $t \in [-\tau, 0]$. Let

$$\begin{aligned} a_1 &= \sup_{t \in J} |x_M(t; \phi)| + |x_1|; a_2 = \sup_{t,s \in J} |X_{\alpha,\alpha}(t-s)| \\ a_3 &= \sup_{0 \leq s \leq t \leq T} |X_{\alpha,\alpha}(t-s) B|; a_4 = |W^{-1}| \\ b &= \max(\alpha^{-1} T^\alpha a_3, 1) c_1 = 6a_2 T^\alpha \alpha^{-1}; c_2 = 6ba_2 a_3 a_4 T^\alpha \alpha^{-1} \\ d_1 &= 6a_1; d_2 = 6ba_1(a_3 a_2); c = \max\{c_1, c_2\}; d = \max\{d_1, d_2\} \\ \sup |f| &= \sup_{s \in J} \{|f(s, x(s-\tau(s, \xi)), {}^C D^\alpha x(s-\tau(s, \xi)), u(s))|\}. \end{aligned}$$

Then

$$\begin{aligned} |v(t)| &\leq a_3 a_4 (a_1 + a_2 \frac{T^\alpha}{\alpha} \sup |f|) \\ &\leq \frac{d_2}{6b} + \frac{c_2}{6b} \sup |f| \leq \frac{1}{6b} (d + c \sup |f|) \\ |l(t)| &\leq a_1 + \frac{T^\alpha}{\alpha} \|v\| a_3 + a_2 \frac{T^\alpha}{\alpha} \sup |f| \\ &\leq \frac{d}{6} + b \|v\| + \frac{c}{6} \sup |f| \end{aligned}$$

By hypothesis the function f satisfies the following condition [9]. For each pair of constants c and $d > 0$, there is a constant $r > 0$ such that, if $|(x, u)| \leq r$, then

$$c|f(t, p)| + d \leq r, \text{ for all } t \in J. \quad (8)$$

Therefore the inequality (8) is satisfied, $r < r_1$ for any r_1 will fulfill (8). Now $\sup_{-1 \leq t \leq 0} |\phi(t)| \leq \frac{r}{3}$. Therefore, if $\|x\| \leq \frac{r}{3}$ and $\|u\| \leq \frac{r}{3}$, then $|x(s-\tau(s, \xi))| + |{}^C D^\alpha x(s-\tau(s, \xi))| + |u(s)| \leq r$, $s \in J$. Therefore $d + c + \sup |f| \leq r$, for $s \in J$. So, $|v(t)| \leq \frac{r}{6b}$ for all $t \in J$ and thus $\|v(t)\| \leq \frac{r}{6b}$, we get $\|l(t)\| \leq \frac{r}{3}$. Hence Ω maps $Q(r)$ into itself, if $Q(r) = \{(x, u) \in Q : \|x\| \leq \frac{r}{3} \text{ and } \|u\| \leq \frac{r}{3}\}$. The operator Ω is continuous since f is continuous. Suppose that M_0 is a bounded subset of Q , then we take a sequence (l_j, v_j) contained in $\Omega(M)$, put $(l_j, v_j) = \Omega(x_j, u_j)$, for some $(x_j, u_j) \in M_0$, for $j = 1, 2, \dots$ since f is continuous $|f(s, x(s-\tau(s, \xi)), {}^C D^\alpha x(s-\tau(s, \xi)), u(s))|$ is uniformly bounded for all $s \in J$, and $j = 1, 2, 3, \dots$. Therefore $\{(l_j, v_j)\}$ is a bounded sequence in Q . Hence $v_j(t)$ is equi-continuous and uniformly bounded sequence on $[0, T]$.

Therefore $\Omega(M_0)$ is sequentially compact, since $\Omega(M_0)$ is sequentially compact. Hence Ω is completely continuous. Since $Q(r)$ that is closed, bounded and convex, using Schauder fixed point theorem Ω has a fixed point $(x, u) \in Q(r)$, such that $(l, v) = \Omega(x, u) = (x, u)$. Therefore

$$x(t) = x_M(t; \phi) + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) Bu(s) ds + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \\ \times f(s, x(s-\tau(s, \xi)), {}^C D^\alpha x(s-\tau(s, \xi)), u(s)) ds, \text{ for } t \in J$$

$$\text{and } x(t) = \varphi(t), t \in [-\tau, 0].$$

Hence, $x(t)$ is a solution of the system and

$$x(T) = x_M(T; \phi) + \int_0^T (T-s)^{\alpha-1} (X_{\alpha,\alpha}(T-s)B)(X_{\alpha,\alpha}(T-s)B)^* ds \\ \times W^{-1} [x_1 - x_M(T; \phi) - \int_0^T (T-s)^{\alpha-1} X_{\alpha,\alpha}(T-s) \\ \times f(s, x(s-\tau(s, \xi)), {}^C D^\alpha x(s-\tau(s, \xi)), u(s)) ds \\ + \int_0^T (T-s)^{\alpha-1} X_{\alpha,\alpha}(T-s) f(s, x(s-\tau(s, \xi)), {}^C D^\alpha x(s-\tau(s, \xi)), u(s)) ds] \\ x(w) = x_w.$$

Therefore the structure (4) is completely controllable.

Theorem 3.4. Let f be the function that satisfies the condition $\|f(t, g)\| \leq \sum_{i=1}^k h_i(t) \theta_i(g)$, where $\theta_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_+$ are measurable functions and $h_i P : J \rightarrow \mathbb{R}_+$ are L_1 functions for $i = 1, 2, 3, \dots, k$. Suppose that the linear system (2) is controllable and the following condition holds

$$\limsup_{r \rightarrow \infty} (r - \sum_{i=1}^p c_i \sup \theta_i(g) : \|g\| \leq r) = +\infty. \quad (9)$$

Therefore the nonlinear fractional delay dynamical structure (3) is controllable on J .

Proof. The proof is homogeneous to Theorems 3.2 and 3.3 in [3]. Therefore it is eliminated.

Remark 3.5. Zhou et al. [27] developed the controllability of fractional linear time-invariant neutral dynamical system. In [10], Debbouche and Baleanu investigated the controllability of fractional nonlocal integro-differential equations via implicit evolution system. In all the above analysis, the authors have proved the controllability results for the fractional differential system with the influence of semigroup theory. It should be pointed out that the semigroup theory is not a proper tool for solving the abstract fractional differential equations. More recently, Balachandran et al. [1,2] studied the controllability results for nonlinear dynamical systems using fixed-point technique. With the motivation from the above works, in this paper, we propose a controllability criteria for nonlinear fractional integro-differential systems with state delay.

4. Examples

In order to justify our theoretical concepts derived in the main section, we have included three numerical examples that shows the validity of our proposed work.

Example 4.1. Examine the linear fractional delay structure with delay $\tau = 1$

$${}^C D^{\frac{1}{2}} x(t) = A(\xi)x(t) + x(t-1) + \int_0^t 2x(s) ds + u(t), 0 < \alpha < 1, t \in [0, T] = J, \\ x(t) = 1, t \in [-1, 0]. \quad (10)$$

where

$$A(\xi) = \begin{pmatrix} 0.6 + 0.4\xi & 0.81 + 0.2\cos 2\xi \\ 0.99 + \sin \xi & 0.5 + 0.5\xi \end{pmatrix}$$

when $\xi = 1$ then

$$A(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Applying property (vii) to Eq. (10), it becomes

$$\begin{aligned} s^{\frac{1}{2}}X(s) - s^{\frac{1}{2}-1}x(0) &= -\int_0^\infty A(\xi)x(t)e^{-st}dt + \int_0^\infty e^{-st}x(t-1)dt + \int_0^\infty x(t)e^{-st} \\ &\quad \times (2\int_0^t x(s)ds)dt + \int_0^\infty e^{-st}u(t)dt \\ s^{\frac{1}{2}}X(s) - s^{\frac{1}{2}-1}x(0) &= -X(s) + e^{-s}\int_{-1}^0 e^{-s\tau}x(\tau)d\tau + e^{-s}X(s) + L(2)X(s) + U(s), \end{aligned}$$

where $U(s) = \int_0^\infty e^{-st}u(t)dt$, then

$$(s^{\frac{1}{2}} + 1 - e^{-s} - 2)X(s) = s^{-\frac{1}{2}} + e^{-s}\int_{-1}^0 e^{-s\tau}d\tau + U(s).$$

Then

$$\begin{aligned} x(t) &= L^{-1}\left(\frac{s^{-\frac{1}{2}}}{s^{\frac{1}{2}}-1-e^{-s}}\right)(t) + L^{-1}\left(\frac{s^{-\frac{1}{2}}}{s^{\frac{1}{2}}-1-e^{-s}}s^{\frac{1}{2}}\right)(t) \\ &\quad * L^{-1}(e^{-s}\int_{-1}^0 e^{-s\tau}d\tau)(t) + L^{-1}(U(s))(t) * L^{-1}\left(\frac{s^{-\frac{1}{2}}}{s^{\frac{1}{2}}-1-e^{-s}}s^{\frac{1}{2}}\right)(t). \end{aligned}$$

But

$$\begin{aligned} L^{-1}\left(\frac{s^{-\frac{1}{2}}}{s^{\frac{1}{2}}-1-e^{-s}}\right)(t) &= L^{-1}\left(\frac{s^{-\frac{1}{2}}}{s^{\frac{1}{2}}-1}\sum_{m=0}^{\infty}\frac{e^{-ns}}{(s^{\frac{1}{2}}-1)^n}\right)(t) \\ &= \sum_{m=0}^{\infty}L^{-1}\left(\frac{s^{-\frac{1}{2}}e^{-ms}}{(s^{\frac{1}{2}}-1)^{m+1}}\right)(t) \\ &= u_m(t)(t-m)^{\frac{1}{2}m}E_{\frac{1}{2},\frac{1}{2}m+1}^{m+1}(t-m)^{\frac{1}{2}} \\ &= \begin{cases} (t-m)^{\frac{1}{2}m}E_{\frac{1}{2},\frac{1}{2}m+1}^{m+1}(t-m)^{\frac{1}{2}}, & t \geq m \\ 0, & t < m. \end{cases} \end{aligned}$$

Let

$$X_{\frac{1}{2}}(t) = (t-m)^{\frac{1}{2}m}E_{\frac{1}{2},\frac{1}{2}m+1}^{m+1}(t-m)^{\frac{1}{2}}, t \geq m, \quad (11)$$

then

$$\begin{aligned} L^{-1}\left(\frac{s^{-\frac{1}{2}}}{s^{\frac{1}{2}}-1-e^{-s}s^{\frac{1}{2}}}\right)(t) &= L^{-1}\left(\frac{s^{-\frac{1}{2}}}{s^{\frac{1}{2}}-1-e^{-s}}\right)(t) * L^{-1}\left(\frac{1}{s^{-\frac{1}{2}}}\right)(t) \\ &= \int_t^m \frac{(t-m)^{-\frac{3}{2}}}{\Gamma(-\frac{1}{2})}(s-m)^{\frac{1}{2}m}E_{\frac{1}{2},\frac{1}{2}m+1}^{m+1}(s-m)^{\frac{1}{2}}ds \\ &= (t-m)^{\frac{1}{2}m-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}m+1}^{m+1}(t-m)^{\frac{1}{2}}. \end{aligned}$$

Therefore

$$X_{\frac{1}{2},\frac{1}{2}}(t) = (t-m)^{\frac{1}{2}m-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}m+1}^{m+1}(t-m)^{\frac{1}{2}}, t \geq m. \quad (12)$$

Define $\psi(t) : [-1, \infty) \rightarrow [0, 1]$ by $\psi(t) = 0$ for $t \geq 0$ and $\psi(t) = 1$ for $t < 0$. The function $\phi(t) = 1$ is prolonged to $[-1, \infty)$ by taking $\phi(t) = \phi(0) = 1$ for $t \geq 0$, then

$$e^{-s} \int_{-1}^0 e^{-st} dt = e^{-s} \int_0^1 e^{-s(-1+\lambda)} d\lambda = \int_0^\infty e^{-s\lambda} w(-1+\lambda) d\lambda = L[w(-1+.)](t).$$

Using Eqs. (11) and (12) we get

$$\begin{aligned} x(t) &= \sum_{m=0}^{[t]} (t-m)^{\frac{1}{2}m} E_{\frac{1}{2}, \frac{1}{2}m+1}^{m+1} ((t-m)^{\frac{1}{2}}) \\ &\quad + \sum_{m=0}^{t-m} (t-s-m)^{\frac{1}{2}m-\frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}m+1}^{m+1} ((t-s-m)^{\frac{1}{2}}) w(-1+s) ds \\ &\quad + \sum_{m=0}^{t-m} (t-s-m)^{\frac{1}{2}m-\frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}m+\frac{1}{2}}^{m+1} ((t-s-m)^{\frac{1}{2}}) u(s) ds \\ &= \sum_{m=0}^{[t]} (t-m)^{\frac{1}{2}m} E_{\frac{1}{2}, \frac{1}{2}m+1}^{m+1} ((t-m)^{\frac{1}{2}}) \\ &\quad + \sum_{m=0}^{t-m} (t-s-m-1)^{\frac{1}{2}m-\frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}m+1}^{m+1} ((t-s-m-1)^{\frac{1}{2}}) ds \\ &\quad + \sum_{m=0}^{t-m} (t-s-m)^{\frac{1}{2}m-\frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}m+\frac{1}{2}}^{m+1} ((t-s-m)^{\frac{1}{2}}) u(s) ds. \end{aligned}$$

Here $[.]$ is the integral part of the function. Let us examine the controllability on $[0, 1]$ with starting point $x(0) = 1$ and the terminal point $x(1) = 5$. In this case $[t] = 0$, hence the result of (10) on $[0, 1]$ is

$$\begin{aligned} x(t) &= E_{\frac{1}{2}}(t^{\frac{1}{2}}) + \int_{-1}^0 (t-s-1)^{-\frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}}^{m+1} ((t-s-1)^{\frac{1}{2}}) ds \\ &\quad + \int_0^t (t-s)^{-\frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}}^{m+1} ((t-s)^{\frac{1}{2}}) u(s) ds. \end{aligned} \quad (13)$$

Therefore

$$x(t) = E_{\frac{1}{2}}(t^{\frac{1}{2}}) + (t)^{\frac{1}{2}} E_{\frac{1}{2}, \frac{3}{2}}((t)^{\frac{1}{2}}) + \int_0^t (t-s)^{-\frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}}^{m+1} ((t-s)^{\frac{1}{2}}) u(s) ds.$$

Here $X_{\frac{1}{2}}(t) = E_{\frac{1}{2}}(t^{\frac{1}{2}})$ and $X_{\frac{1}{2}, \frac{1}{2}}(t^{\frac{1}{2}})$. The controllability Grammian matrix is defined by

$$W = \int_0^1 (E_{\frac{1}{2}, \frac{1}{2}}((1-s)^{\frac{1}{2}}))(E_{\frac{1}{2}, \frac{1}{2}}((1-s)^{\frac{1}{2}}))^* ds, \quad (14)$$

$W > 0$. Hence the system (10) is controllable on $[0, 1]$.

Example 4.2. Inspect the following linear fractional delay system with delay $\tau = 1$

$$\begin{aligned} {}^C D^{\frac{1}{2}} x(t) &= A(\xi)x(t) + x(t-1) + \int_0^t 2x(s)ds + u(t) + \frac{x(t)}{1+x^2(t)+u(t)} + \frac{x(t-1)}{1+x^2(t-1)}, \\ x(t) &= 1, \quad t \in [-1, 0], \end{aligned} \quad (15)$$

where

$$A(\xi) = \begin{pmatrix} 0.6 + 0.4\xi & 0.81 + 0.2\cos 2\xi \\ 0.99 + \sin \xi & 0.5 + 0.5\xi \end{pmatrix}.$$

Let us examine the controllability on $[0, 1]$ with the starting point $x(0) = 1$ and the terminal point $x(1) = 5$.

Let

$$f(t, x(t-1), {}^C D^\alpha x(t-\tau(t, \xi)), u(t)) = \frac{x(t)}{1+x^2(t)+u(t)} + \frac{x(t-1)}{1+x^2(t-1)}$$

assures the hypothesis of [Theorem 3.3](#). Therefore the structure (15) is controllable on $[0, 1]$.

Example 4.3. Inspect the following fractional delay differential structure as:

$${}^C D^{\frac{1}{2}} x(t) = 2x(t) + 4x(t-1) + \int_0^t 2x(s)ds + u(t) + \frac{x(t)}{1+x(t)+u(t)} + \frac{x(t-1)}{1+x(t-1)},$$

$$x(t) = 1, \quad t \in [-1, 0], \quad (16)$$

where $A(\xi) = 2$, $E = 4$ and $B = 2$, $\phi(t) = 1$, $\tau = 1$ and

$$f(t, x(t-1), {}^C D^\alpha x(t-\tau(t, \xi)), u(t)) = \frac{x(t)}{1+x(t)+u(t)} + \frac{x(t-1)}{1+x(t-1)}.$$

Applying Laplace transform, the solution of Eq. (16) is

$$x(t) = \sum_{m=0}^{[t]} 4^m (t-m)^{\frac{1}{2}m} E_{\frac{1}{2}, \frac{1}{2}m+1}^{m+1} (2(t-m)^{\frac{1}{2}}) + 4 \sum_{m=0}^t \int_{-1}^0 4^m (t-s-m-1)^{\frac{1}{2}m-\frac{1}{2}} \\ \times E_{\frac{1}{2}, \frac{1}{2}m+1}^{m+1} (2(t-s-m-1)^{\frac{1}{2}}) ds + 2 \sum_{m=0}^t \int_0^{t-m} 4^m (t-s-m)^{\frac{1}{2}m-\frac{1}{2}} \\ \times E_{\frac{1}{2}, \frac{1}{2}m+1}^{m+1} (2(t-s-m)^{\frac{1}{2}}) u(s) ds.$$

Let us examine the controllability on $[0, 1]$ with starting point $x(0) = 1$ and the terminal point $x(1) = 10$. In this case $[t] = 0$, hence the result of (16) on $[0, 1]$ is

$$x(t) = E_{\frac{1}{2}}(2t^{\frac{1}{2}}) + 4 \int_{-1}^0 (t-s-1)^{-\frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}}(2(t-s-1)^{\frac{1}{2}}) ds \\ + 2 \int_0^t (t-s)^{-\frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}}(2(t-s)^{\frac{1}{2}}) u(s) ds. \quad (17)$$

Therefore

$$x(t) = E_{\frac{1}{2}}(2t^{\frac{1}{2}}) + 4(t)^{\frac{1}{2}} E_{\frac{1}{2}, \frac{3}{2}}(2(t)^{\frac{1}{2}}) + 2 \int_0^t (t-s)^{-\frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}}(2(t-s)^{\frac{1}{2}}) u(s) ds.$$

The controllability Grammian matrix is defined by

$$W = \int_0^1 (E_{\frac{1}{2}, \frac{1}{2}}(2(1-s)^{\frac{1}{2}}))(E_{\frac{1}{2}, \frac{1}{2}}(2(1-s)^{\frac{1}{2}}))^* ds, \quad (18)$$

$W > 0$. For the structure (16) $u(t)$ is constructed by

$$u(t) = (1-t)^{\frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}}(2((1-t))^{\frac{1}{2}})[10 - E_{\frac{1}{2}}(2) - 4E_{\frac{1}{2}, \frac{1}{2}}(2) \\ - \int_0^1 (1-t)^{-\frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}}(2(1-t)^{\frac{1}{2}})(\frac{x(1)}{1+x(1)+u(t)} + \frac{\phi(0)}{1+\phi(0)})]dt$$

because $W > 0$, the linear system is controllable and the nonlinear function

$$f(t, x(t-1), {}^C D^\alpha x(t-\tau(t, \xi)), u(t)) = \frac{x(t)}{1+x(t)+u(t)} + \frac{x(t-1)}{1+x(t-1)}.$$

Have not satisfied the condition in [Theorem 3.3](#). But, it satisfies [Theorem 3.4](#) under the conditions

$$\|f(t, x(t-1), {}^C D^\alpha x(t-\tau(t, \xi)), u(t))\| \leq \frac{1}{x(t)} + \frac{1}{x(t-1)}$$

take $k = 1$, $h(t) = 1$, $\phi(p) = \frac{1}{x(t)} + \frac{1}{x(t-1)}$. Therefore $\lim_{r \rightarrow \infty} \sup(r - \sup c \frac{1}{x(t)} + \frac{1}{x(t-1)}) = +\infty$. Thus, the hypothesis of [Theorem 3.4](#) are fulfilled. Therefore the structure (16) is controllable on $[0, 1]$.

5. Conclusion

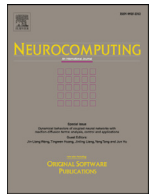
This paper implements the idea of controllability criterion of linear and nonlinear fractional delay dynamical structures. Employing the Laplace transform method and Mittag-Leffler matrix function we find the solution representation for the observed fractional dynamical system. We proposed the necessary as well as sufficient conditions of the controllability results for the labeled structure. Additionally two numerical illustrations were presented to justify the obtained theoretical results. In our future plan, we try to concentrate with the problem of controllability results for non linear fractional differential systems with delays and impulses using small parameter.

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Fractional delay segments method on time-delayed recurrent neural networks with impulsive and stochastic effects: An exponential stability approach[☆]

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ABSTRACT

This paper concerns with the problem of exponential stability analysis of time-delayed recurrent neural networks with impulsive and stochastic effects under fractional segments or intervals in delays. The delays in discrete term are assumed to be time-varying and different from existing literature, the discrete delay interval has been separated into fractional segments, which guarantees the availability of lower and upper bounds for feasibility with accuracy. By constructing a suitable Lyapunov–Krasovskii functional (LKF), with the aid of stability theory and inequality techniques, several novel criteria are originated via linear matrix inequalities (LMIs) to ensure the exponential stability of addressed neural networks in the mean square sense. Finally, two numerical examples are presented to substantiate the superiority and effectiveness of our theoretical outcomes.

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1. Introduction

Over the past few decades, the analysis of recurrent neural networks (RNNs) in dynamical behaviors has received great attention due to their potential applications in different fields, such as static image treatment, optimization problems, parallel computing, signal processing, etc [3,23,78]. In recurrent neural networks the arising of time delays is inevitable, due to the finite switching speed of information storage processing or of amplifiers, communication time and transmission of signals. For sequence, in a neural networks, the appearance of time delay may affect the stability performance generating unstable, divergence and swinging or chaotic behavior. Hence, the study of stability analysis for time-delayed neural net-

works has been broad interests from numerous scholars, see references [7,8,21,62]. In accordance, the delay-dependent and delay-independent criteria are two categories of time delays in neural networks (NNs), which classified by the existing time delayed results. So far as, one can observe from available literatures, the delay-dependent case is less conserved than the delay independent ones. In [4], Cao and Song discussed about the time-delay criteria for stability performance of BAM neural networks. The authors in [20], through novel approaches, developed some time-delayed conditions for the stability analysis of neural network systems.

On the other hand, we notice that the propagation delays can be distributed over a period of time, because the variety of axon sizes and lengths are too large. Thus, the consideration of distributed delay term is meaningful in neural network system. So, it is significant to inspect both the time delay results in the dynamical behaviors of neural systems, see for instance [2,34,51,70,77]. Some criteria in terms of LMIs were presented to ascertain the stochastic BAM neural networks with mixed delays to be exponential stable [2]. Park investigated the impact of mixed time delays in neural networks when check the stability through LKF [38]. Also,

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the act of state variables of recurrent neural networks with both time-delays are studied [48]. Meanwhile, to conclude the speed of neural computations, the exponential convergence rate is used. So, it generates much attention from many researchers to studied the property of exponential stability [5,30,57]. Further, Li in [22] studied and obtain the some findings related to the problem for exponential stability of stochastic delayed neural networks. In [49], the authors extensively analyzed the stability in the exponential sense of NNs. Furthermore, the exponential stability criteria for neural networks have been investigated by He in [14].

Additionally, several dynamical processes, specifically some biological fields such as bursting rhythm models in pathology, biological neural networks, as well as flying object motions, frequency-modulated signal processing systems and optimal control models in economics are distinguished by abrupt changes of state of the networks at certain moments of time, see [11,32,58] and references therein. This changes indicates the impulsive phenomena. As is well known that, the impulses can damage the properties of stability, which possibly leads to oscillation and instability [11,32,46]. Therefore, in RNNs, the incorporate of impulsive perturbations is essentially one. Raja et al., in [41], widely discussed about the impulsive behavior of neutral-type NNs and derive its stability criteria by using some inequality techniques.

On the other hand, the noisy perturbations have received intensive interest in the area of neural networks. Besides, the stochastic phenomenon in neural networks constructed when random variations introducing into the networks, either by giving them stochastic weights, or by giving the network's neurons stochastic transfer functions. Practically, the random fluctuations helps to escape from instability, so it makes them valuable tool for neural networks. In accordance to the above reasons, stochastic neural networks have found fruitful applications in oncology, bio informatics, risk management and other similar fields. Therefore, the stochastic perturbations in neural networks become a fixate issue for stability analysis of recurrent neural networks, and recently the stability analysis problems for time-delayed stochastic neural networks have attracted broad interest, see for instants [15,40,59,76]. The authors in [45], studied the stochastic effects of state behavior for time-delayed neural networks. The delayed condition to attain the stability result of neural networks with stochastic disturbances analyzed in [35]. Liao and Mao in [28], investigates stability criteria for neural networks when the stochastic noise affects.

Inspired by the aforementioned discussions, our main motivation of this paper is to investigate the fractional delay segments on the exponential stability problem for impulsive recurrent neural networks with stochastic effects and mixed time-delays. By employing the stochastic analysis theory, stability theory and some inequality techniques, the exponential stability problem of the concerned neural networks is transformed into the feasibility problem of a set of linear matrix inequalities. Consequently, by constructing an appropriate novel Lyapunov-Krasovskii functional, some new-brand sufficient criteria for exponential stability are formulated in terms of LMIs, which can be justified easily by LMI control toolbox in MATLAB software. Also, two numerical examples with their simulations are provided to demonstrate the superiority and feasibility with less conservatism of the proposed results. The main contributions of this research work are highlighted as follows:

- (1) Impulsive effects, stochastic noises and discrete and distributed time delays are taken into account in the stability analysis of proposed recurrent neural networks.
- (2) By the implementation, Lyapunov-Krasovskii functionals, stability theory, bounds lemma and some novel inequality techniques, a brand-new sufficient conditions for exponential stability of impulsive RNNs are obtained in terms of LMIs.

Table 1

Maximum allowable upper bounds $\bar{\mu}$ when $0 \leq \mu(t) \leq \frac{\bar{\mu}}{2}$.

Methods	$\bar{\mu} > 0$	Status
In [43]	0.3733	Feasible
In [53]	0.4265	Feasible
In [65]	0.4602	Feasible
In [1]	0.8411	Feasible
In [47]	1	Feasible
In [78]	1.3323	Feasible
In [17]	1.5575	Feasible
In [10]	1.7685	Feasible
In [13]	2.3534	Feasible
In [72]	2.6784	Feasible
In [39]	2.882	Feasible
In [16]	3.2775	Feasible
In [25]	3.2819	Feasible
In [69]	3.3574	Feasible
In [68]	3.5204	Feasible
In [67]	3.5423	Feasible
In [55]	3.5546	Feasible
In [63]	3.7665	Feasible
In [12]	3.8428	Feasible
In [66]	3.8739	Feasible
In [64]	4.1903	Feasible
In [46]	4.6141	Feasible
In [50]	11.9128	Feasible
In [29]	12.9895	Feasible
Theorem 3.1	24.9731	Feasible

Table 2

Maximum allowable upper bounds $\bar{\mu}$ when $\frac{\bar{\mu}}{2} \leq \mu(t) \leq \frac{2}{3}\bar{\mu}$.

Methods	$\bar{\mu} > 0$	Status
In [71]	0.25	Feasible
In [60]	1.00	Feasible
In [36]	1.88	Feasible
In [54]	1.9528	Feasible
In [73]	2.1564	Feasible
In [42]	2.2448	Feasible
In [31]	2.5191	Feasible
In [9]	2.5680	Feasible
In [14]	6.247	Feasible
In [56]	21.1573	Feasible
Theorem 3.1	26.425	Feasible

Table 3

Maximum allowable upper bounds $\bar{\mu}$ when $\frac{2}{3}\bar{\mu} \leq \mu(t) \leq \bar{\mu}$.

Methods	$\bar{\mu} > 0$	Status
In [61]	0.2642	Feasible
In [49]	0.8643	Feasible
In [18]	0.8696	Feasible
In [19]	0.8784	Feasible
In [44]	1.96	Feasible
In [74]	3.81	Feasible
In [20]	4.17	Feasible
Theorem 3.1	26.742	Feasible

- (3) By handled both time-delay terms in our stochastic recurrent neural networks with impulsive effects, the allowable upper bounds of discrete time-varying delay is very huge, when compared with the previous results, see Tables 1,2 and 3 in Example 4.1. Also, the comparisons of upper bounds for various μ , ς are shown in Tables 4, 5 and 6 respectively. This explored that the approach developed in this paper is more effective and less conserved than some existing works in the literature, which ensures the novelty of this research work.
- (4) By separating the delay intervals into fractional segments, we obtain the allowable upper bounds of discrete time-varying de-

lay is very large with accuracy and the average of allowable upper bounds given in Table 7.

- (5) Two special cases of the main results are derived in Corollary 3.3, 3.5 and also by using MATLAB LMI control toolbox, we have checked that the LMIs of Corollary 3.3 are feasible as well as its simulations are given in Examples 4.2.

The remainder of this paper is systematized as follows. In Section 2, the stochastic recurrent neural networks model is formulated and some preliminaries are provided. By the aid of Lyapunov–Krasovskii functionals and integral inequality techniques, some sufficient conditions are attained for the exponential stability of addressed neural networks in Section 3. The obtained results are formulated in the form of linear matrix inequalities (LMIs), that are easily solved by using the MATLAB LMI control toolbox. Then, two numerical examples with their simulations are equipped to demonstrate the feasibility and advantages of the derived exponential stability criteria in Section 4. Finally, in Section 5 conclusions are drawn.

Notations : Throughout this paper, $\mathbb{R}^n$ denotes the n -dimensional Euclidean space and $\mathbb{Z}^+$ be set of all positive integers. $\mathbb{R}^{n \times n}$ is the set of all $n \times n$ matrices with real entries. For a matrix $\mathcal{R}$, $\mathcal{R} > 0$ ($\mathcal{R} \geq 0$) represents that matrix $\mathcal{R}$ is symmetric and positive definite (positive semi-definite). The notations $\mathcal{R}^T$ and $\mathcal{R}^{-1}$ denotes its transpose and inverse, respectively. I represents the identity matrix with appropriate dimensions. $C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ is the family of all non negative functions $V(t, x(t))$ on $\mathbb{R}^+ \times \mathbb{R}^n$ which are continuously twice differentiable in x and once differentiable in t ; $(\mathbb{A}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ a complete probability space, where $\mathbb{A}$ is the sample space, $\mathcal{F}$ is the σ -algebra of subsets of the sample space, $\mathcal{P}$ is the probability measure on $\mathcal{F}$ and $\{\mathcal{F}_t\}_{t \geq 0}$ denotes the Filtration; The symbol “ $\bullet$ ” denotes a block that is readily inferred by symmetry. $C([-v, 0]; \mathbb{R}^n)$ denotes the family of continuous functions ϕ from $[-v, 0]$ to $\mathbb{R}^n$ with norm $\|\phi\| = \sup_{-v \leq \theta \leq 0} |\phi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. Denote by $\mathcal{L}_{\mathcal{F}_0}^2([-v, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$ measurable $C([-v, 0]; \mathbb{R}^n)$ -valued random variables $\varphi = \{\varphi(\theta) : -v \leq \theta \leq 0\}$ such that $\sup_{-v \leq \theta \leq 0} \mathbb{E}|\varphi(\theta)|^2 < \infty$ where $\mathbb{E}\{x\}$ stands for the mathematical expectation operator of x with respect to the given probability measure $\mathcal{P}$. $\text{diag}\{\dots\}$ denotes a block-diagonal matrix. For a matrix $\mathcal{B}$, $\lambda_{\max}(\mathcal{B})$ and $\lambda_{\min}(\mathcal{B})$ denote the maximum and minimum eigenvalues of matrix $\mathcal{B}$, respectively.

2. Problem formulation and preliminaries

In this paper, a class of recurrent neural networks with time-varying delays described by :

$$\begin{aligned} dx(t) &= \left[-Ax(t) + W_0 f(x(t)) + W_1 g(x(t - \mu(t))) \right. \\ &\quad \left. + W_2 \int_{t-\sigma}^t h(x(s)) ds + J \right] dt \\ &\quad + \rho(x(t), x(t - \mu(t)), t) d\omega(t); t > 0, t \neq t_k, \\ x(t_k) &= \mathcal{G}_k x(t_k^-); t = t_k, k \in \mathbb{Z}^+, \\ x(t) &= \psi(t); t \in [-v, 0], \end{aligned} \quad (1)$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$ denotes neuron state vector at time t ; The diagonal matrix $A = \text{diag}\{a_1, a_2, \dots, a_n\} > 0$ have positive entries $a_i > 0$ ($i = 1, 2, \dots, n$); $f(x(t)) = (f_1(x(t)), f_2(x(t)), \dots, f_n(x(t)))^T$, $g(x(t - \mu(t))) = (g_1(x(t - \mu(t))), g_2(x(t - \mu(t))), \dots, g_n(x(t - \mu(t))))^T$, and $h(x(t)) = (h_1(x(t)), h_2(x(t)), \dots, h_n(x(t)))^T$, are the activation functions of the neurons with $f(0) = g(0) = h(0) = 0$; $W_0 \in \mathbb{R}^{n \times n}$ is the connection weight matrix; W_1 and W_2 are the discrete and distributed delayed connection weight matrices respectively; $J = [J_1, J_2, \dots, J_n]^T$ represents external constant input vector; $\mu(t)$ is the discrete time varying delays which are bounded with $0 < \mu(t) < \bar{\mu}$, $\dot{\mu}(t) \leq \mu < 1$;

σ is distributed constant delay; $\rho : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ denotes the stochastic disturbance; $\omega(t) = (\omega_1(t), \omega_2(t), \dots, \omega_n(t))^T$ is n -dimensional Brownian motion defined on a complete probability space $(\mathbb{A}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets) and $\mathbb{E}\{d\omega(t)\} = 0$, $\mathbb{E}\{d\omega^2(t)\} = dt$; $\mathcal{G}_k$ is impulsive gain matrix; The discrete set $\{t_k\}$ satisfies $0 = t_0 < t_1 < t_2 < \dots < t_k < \dots$, $\lim_{k \rightarrow \infty} t_k = \infty$; $x(t_k^-)$ denote the left-hand limits at t_k ; Similarly $x(t_k^+)$ denote the right-hand limits at time t_k ; $\psi(t)$ is continuous function; $v = \max(\bar{\mu}, \sigma)$. Assume that $x(t)$ is right-continuous, that is $x(t_k^+) = x(t_k)$.

On the neuron activation functions the following assumptions are made to derive the main results.

Assumption 1. The activation functions $f_i(t)$, $g_i(t)$ and $h_i(t)$ in (1) satisfies the following conditions with bounded property :

$$\begin{aligned} \alpha_i^- &\leq \frac{f_i(\xi_1) - f_i(\xi_2)}{\xi_1 - \xi_2} \leq \alpha_i^+, \\ \beta_i^- &\leq \frac{g_i(\xi_1) - g_i(\xi_2)}{\xi_1 - \xi_2} \leq \beta_i^+, \\ \gamma_i^- &\leq \frac{h_i(\xi_1) - h_i(\xi_2)}{\xi_1 - \xi_2} \leq \gamma_i^+, \end{aligned}$$

for all $\xi_1, \xi_2 \in \mathbb{R}$, $\xi_1 \neq \xi_2$, $i = 1, 2, \dots, n$. The values $\alpha_i^-, \alpha_i^+, \beta_i^-, \beta_i^+, \gamma_i^-, \gamma_i^+$, ($i = 1, 2, \dots, n$) in this assumption are scalars.

Remark 2.1. In Assumption 1, the scalars $\alpha_i^-, \alpha_i^+, \beta_i^-, \beta_i^+, \gamma_i^-, \gamma_i^+$ are permitted to be greater than zero, less than zero or equal to zero. Thus, the outcome of the activation functions may be non-monotonic and more general than the Lipschitz conditions and usual sigmoid activation functions as point out in [33,34]. This type of activation functions are very supportive in utilizing an LMI-based approach and also it leads to less conservatism.

The equilibrium point of (1) is represented by $x^* = [x_1^*, x_2^*, \dots, x_n^*]^T$. By the shifting $u(\cdot) = x(\cdot) - x^*$, one can transform (1) into the error system as follows

$$\begin{aligned} du(t) &= \left[-Au(t) + W_0 \tilde{f}(u(t)) + W_1 \tilde{g}(u(t - \mu(t))) \right. \\ &\quad \left. + W_2 \int_{t-\sigma}^t \tilde{h}(u(s)) ds \right] dt \\ &\quad + \tilde{\rho}(u(t), u(t - \mu(t)), t) d\tilde{\omega}(t); t > 0, t \neq t_k, \\ u(t_k) &= \tilde{\mathcal{G}}_k u(t_k^-); t = t_k, k \in \mathbb{Z}^+, \\ u(t) &= \varphi(t); t \in [-v, 0], \end{aligned} \quad (2)$$

where $u(t) = [u_1(t), u_2(t), \dots, u_n(t)]^T$, $u(t - \mu(t)) = [u_1(t - \mu(t)), u_2(t - \mu(t)), \dots, u_n(t - \mu(t))]^T$, $\tilde{f}(u(t)) = [\tilde{f}_1(u(t)), \tilde{f}_2(u(t)), \dots, \tilde{f}_n(u(t))]^T$, $\tilde{g}(u(t - \mu(t))) = [\tilde{g}_1(u(t - \mu(t))), \tilde{g}_2(u(t - \mu(t))), \dots, \tilde{g}_n(u(t - \mu(t)))]^T$, $\tilde{h}(u(t)) = [\tilde{h}_1(u(t)), \tilde{h}_2(u(t)), \dots, \tilde{h}_n(u(t))]^T$, $\tilde{\rho}(u(t), u(t - \mu(t)), t) = [\tilde{\rho}_1(u(t), u(t - \mu(t)), t), \tilde{\rho}_2(u(t), u(t - \mu(t)), t), \dots, \tilde{\rho}_n(u(t), u(t - \mu(t)), t)]^T$, $\tilde{f}(u(t)) = f(u(t) + x^*) - f(x^*)$, $\tilde{g}(u(t - \mu(t))) = g(u(t - \mu(t)) + x^*) - g(x^*)$, $\tilde{h}(u(t)) = h(u(t) + x^*) - h(x^*)$, $\tilde{\rho}(u(t), u(t - \mu(t)), t) = \rho(u(t) + x^*, u(t - \mu(t)) + x^*, t) - \rho(x^*, x^*, t)$.

Now, one can obtain that

$$\alpha_i^- \leq \frac{\tilde{f}_i(\xi_1)}{\xi_1} \leq \alpha_i^+, \quad \tilde{f}_i(0) = 0, \quad (3)$$

$$\beta_i^- \leq \frac{\tilde{g}_i(\xi_1)}{\xi_1} \leq \beta_i^+, \quad \tilde{g}_i(0) = 0, \quad (4)$$

$$\gamma_i^- \leq \frac{\tilde{h}_i(\xi_1)}{\xi_1} \leq \gamma_i^+, \quad \tilde{h}_i(0) = 0, \quad (5)$$

where $i = 1, 2, 3, \dots, n$.

Assume that $\tilde{\rho} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ are Locally Lipschitz continuous and satisfies the following assumption,

Assumption 2. There exists positive definite matrices S_1, S_2 with appropriate dimensions such that

$$\begin{aligned} & \text{trace}[\tilde{\rho}^T(u_1(t), u_2(t - \mu(t)), t)\tilde{\rho}(u_1(t), u_2(t - \mu(t)), t)] \\ & \leq u_1^T(t)S_1u_1(t) + u_2^T(t - \mu(t))S_2u_2(t - \mu(t)), \end{aligned}$$

for all $u_1(t), u_2(t - \mu(t)) \in \mathbb{R}^n$.

Consider a general stochastic system $dz(t) = f(z(t), t) + g(z(t), t)d\omega(t)$, $t \geq 0$ with initial condition $z(0) = z_0 \in \mathbb{R}^n$, where $f : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$. Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ denote a collection of all non-negative functions V on $\mathbb{R}^n \times \mathbb{R}^+$ which are twice continuously differentiable in z and once differentiable in t . For any $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ define $\mathcal{L}V : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}$ by

$$\begin{aligned} \mathcal{L}V(z(t), t) &= V_t(z(t), t) + V_z(z(t), t)f(z(t), t) \\ &+ \frac{1}{2}\text{trace}(g^T(z(t), t)V_{zz}(z(t), t)g(z(t), t)), \end{aligned}$$

where

$$\begin{aligned} V_t(z(t), t) &= \frac{\partial V(z(t), t)}{\partial t}, \\ V_z(z(t), t) &= \left(\frac{\partial V(z(t), t)}{\partial z_1}, \frac{\partial V(z(t), t)}{\partial z_2}, \dots, \frac{\partial V(z(t), t)}{\partial z_n} \right), \\ V_{zz}(z(t), t) &= \frac{\partial^2 V(z(t), t)}{\partial z \partial z}. \end{aligned}$$

By the generalized Ito's formula, we get that

$$\mathbb{E}V(u(t), t) = \mathbb{E}V(u(0), 0) + \mathbb{E} \int_0^t \mathcal{L}V(u(s), s)ds.$$

Let $u(t)$ denote the state trajectory from the initial data value $u(0) = \varphi(0)$ on $-\nu \leq \theta \leq 0$ in $L^2_{\mathcal{F}_0}([-\nu, 0]; \mathbb{R}^n)$. Clearly, the neural networks (2) admits a trivial solution $u(t, 0) \equiv 0$ corresponding to the initial data $\varphi = 0$. For simplicity, we can write $u(t; \varphi) = u(t)$.

Remark 2.2. As specified in [6], the existence of an equilibrium point for neural networks (2) can be proved by Brouwer's fixed-point theorem and Assumption 1. In the consequence, we shall investigate the exponential stability of the trivial solution, which implies that the equilibrium point to be unique.

Before ending this section, we introduce some basic definitions and lemmas, which will play an main role in proving the proofs of our main theorems below.

Definition 2.3. The trivial solution of neural networks (2) is said to be exponentially stable in the mean square sense if for any $\varphi \in L^2_{\mathcal{F}_0}([-\nu, 0]; \mathbb{R}^n)$ there exist constants $\eta \geq 0$ and $\chi > 0$ such that

$$\mathbb{E}\{u(t)\} \leq \chi e^{-\eta t} \sup_{-\nu \leq s \leq 0} \mathbb{E}\{\|\varphi(s)\|^2\}, \quad \forall t > 0.$$

Definition 2.4. We introduce the stochastic Lyapunov–Krasovskii functional $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ of the system (2), the weak infinitesimal generator of random process $\mathcal{L}V$ from $\mathbb{R}^n \times \mathbb{R}^+$ to $\mathbb{R}^+$ defined by

$$\begin{aligned} \mathcal{L}V(t, u(t)) &= \lim_{\Delta t \rightarrow 0^+} \frac{1}{\Delta t} [\mathbb{E}\{V((t + \Delta t), u(t + \Delta t)) | u(t)\} - V(t, u(t))] \end{aligned}$$

Lemma 2.5 [27]. For all real vectors a, b and all matrix $Q > 0$ with appropriate dimensions, it follows that :

$$2a^T b \leq a^T Q a + b^T Q^{-1} b$$

Lemma 2.6 [36]. For any scalars $q_1 > q_2 > 0$, and any constant matrix $M \in \mathbb{R}^{n \times n}$, $M = M^T > 0$, vector function $\varpi : [q_2, q_1] \rightarrow \mathbb{R}^n$ such

that the integrations concerned are well defined, then the following inequality holds:

$$(q_1 - q_2) \int_{q_2}^{q_1} \varpi^T(s) M \varpi(s) ds \geq \left(\int_{q_2}^{q_1} \varpi(s) ds \right)^T M \left(\int_{q_2}^{q_1} \varpi(s) ds \right)$$

Lemma 2.7. From Eqs. (3)–(5) the inequalities satisfied as follows :

$$0 \leq \int_0^{x_i(t)} [f_i(s) - \alpha_i^- s] ds \leq [f_i(x_i(t)) - \alpha_i^- x_i(t)] x_i(t) \quad (6)$$

$$0 \leq \int_0^{x_i(t)} [\alpha_i^+ s - f_i(s)] ds \leq [\alpha_i^+ x_i(t) + f_i(x_i(t))] x_i(t) \quad (7)$$

$$0 \leq \int_0^{x_i(t)} [g_i(s) - \beta_i^- s] ds \leq [g_i(x_i(t)) - \beta_i^- x_i(t)] x_i(t) \quad (8)$$

$$0 \leq \int_0^{x_i(t)} [\beta_i^+ s - g_i(s)] ds \leq [\beta_i^+ x_i(t) + g_i(x_i(t))] x_i(t) \quad (9)$$

$$0 \leq \int_0^{x_i(t)} [h_i(s) - \gamma_i^- s] ds \leq [h_i(x_i(t)) - \gamma_i^- x_i(t)] x_i(t) \quad (10)$$

$$0 \leq \int_0^{x_i(t)} [\gamma_i^+ s - h_i(s)] ds \leq [\gamma_i^+ x_i(t) + h_i(x_i(t))] x_i(t) \quad (11)$$

Proof. To prove this lemma, let $F_i(s) = f_i(s) - \alpha_i^- s$, we obtain $\frac{F_i(s)}{s} = \frac{f_i(s)}{s} - \alpha_i^- \geq 0$. Thus,

$$\int_0^{x_i(t)} [f_i(s) - \alpha_i^- s] ds \geq 0 \quad (12)$$

From Assumption 1,

$$\begin{aligned} D^+ F_i(s) &= \lim_{\kappa \rightarrow 0^+} \frac{F_i(s + \kappa) - F_i(s)}{\kappa} \\ &= \lim_{\kappa \rightarrow 0^+} \frac{f_i(s + \kappa) - \alpha_i^-(s + \kappa) - f_i(s) + \alpha_i^- s}{\kappa} \\ &= \lim_{\kappa \rightarrow 0^+} \left(\frac{f_i(s + \kappa) - f_i(s)}{\kappa} - \alpha_i^- \right) \\ &\geq 0 \end{aligned}$$

then the monotone increasing function $F_i(s)$ is attained. So, finally we have the inequality as given below

$$0 \leq \int_0^{x_i(t)} [f_i(s) - \alpha_i^- s] ds \leq [f_i(x_i(t)) - \alpha_i^- x_i(t)] x_i(t)$$

Similarly, the proof of (7)–(11) have the same procedure of (6), hence ignore here. $\square$

Remark 2.8. To establish the time-delayed exponential stability criteria for recurrent neural networks (2), a different LKF handled in the following Theorem 3.1. Further, fractional segments in time-varying delays are analyzed, which helps to obtain the maximum allowable upper bounds is large when compare with the previous results.

Remark 2.9. Li and Feng in [24] analyzed the impact of delay interval on recurrent neural networks for ensuring the stability performance. In [52], Wu et al., discussed about the exponential convergence rate of RNNs with the existence of both time-delays. Moreover, the comparisons for maximum allowable upper bounds of discrete time-varying delays has been listed. The authors in [26] investigates the stability behavior in the sense of exponential for recurrent neural networks with mixed time delays. Further, in [26], the contribution of activation functions is very important to find the stability criteria with less conservatism. The impulsive effects on RNNs and its stability criteria is conversed by Zhou and Zhang in [75] via Lyapunov–Krasovskii functionals.

In all the above mentioned references, the stability problem for recurrent neural networks is considered only with discrete time-varying delays or mixed time delays or impulses, but both the time-delays, stochastic noise and impulsive effects has not been taken into account and also no one investigates the exponential stability, via delays in fractional segments or intervals at a time. So, by considering all the above facts into account, we have proposed this research work which is a challenging task.

3. Exponential stability results for deterministic systems

In this part, we will develop some novel exponential stability criterion for the time-delayed recurrent neural networks (2).

Theorem 3.1. Suppose that Assumptions 1 and 2 holds, for given scalars $Y_1 = \text{diag}\{\alpha_1^-, \alpha_2^-, \dots, \alpha_n^-\}$, $Y_2 = \text{diag}\{\alpha_1^+, \alpha_2^+, \dots, \alpha_n^+\}$, $Y_3 = \text{diag}\{\beta_1^-, \beta_2^-, \dots, \beta_n^-\}$, $Y_4 = \text{diag}\{\beta_1^+, \beta_2^+, \dots, \beta_n^+\}$, $Y_5 = \text{diag}\{\gamma_1^-, \gamma_2^-, \dots, \gamma_n^-\}$, $Y_6 = \text{diag}\{\gamma_1^+, \gamma_2^+, \dots, \gamma_n^+\}$, $\mu \leq 1$, if there exists symmetric positive

definite matrices K , L_i ($i = 1, 2, 3, 4$), $\tilde{L}_j$ ($j = 1, 2$), M_1 , M_2 , M_4 , M_5 , M_6 and $M_3 = \begin{bmatrix} M_{311} & M_{312} & M_{313} \\ * & M_{322} & M_{323} \\ * & * & M_{333} \end{bmatrix}$, positive diagonal matrices Λ_p

$= \text{diag}\{\lambda_{p1}, \lambda_{p2}, \dots, \lambda_{pn}\}$ ($p = 1, 2, 3$), $\Theta_l = \text{diag}\{\theta_{l1}, \theta_{l2}, \dots, \theta_{ln}\}$ ($l = 1, 2, 3$), N_q ($q = 1, 2, 3$), such that the following LMIs hold :

$$K \leq \bar{d}I, \quad (13)$$

$$\bar{G}_k^T K \bar{G}_k - K \leq 0, \quad (14)$$

Case (i) When $0 \leq \mu(t) \leq \frac{\bar{\mu}}{3}$

$$\Xi_1 = \begin{bmatrix} \Gamma_{1,1} & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & \Gamma_{1,5} & 0 & 0 & 0 & 0 & \Gamma_{1,10} & \Gamma_{1,11} & 0 & 0 \\ * & \Gamma_{2,2} & 0 & \Gamma_{2,4} & \Gamma_{2,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Gamma_{3,3} & \Gamma_{3,4} & 0 & 0 & 0 & \Gamma_{3,8} & 0 & \Gamma_{3,10} & 0 & 0 & 0 \\ * & * & * & \Gamma_{4,4} & 0 & 0 & 0 & \Gamma_{4,8} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Gamma_{5,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Gamma_{6,6} & 0 & 0 & \Gamma_{6,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Gamma_{7,7} & 0 & \Gamma_{7,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Gamma_{8,8} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Gamma_{9,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Gamma_{10,10} & \Gamma_{10,11} & \Gamma_{10,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & \Gamma_{11,11} & \Gamma_{11,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Gamma_{12,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & \Gamma_{13,13} \end{bmatrix} < 0, \quad (15)$$

Case (ii) When $\frac{\bar{\mu}}{3} \leq \mu(t) \leq \frac{2}{3}\bar{\mu}$

$$\Xi_2 = \begin{bmatrix} \Gamma_{1,1} & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & \Gamma_{1,5} & 0 & 0 & 0 & 0 & \gamma_{1,10} & \Gamma_{1,11} & 0 & 0 \\ * & \Gamma_{2,2} & 0 & \Gamma_{2,4} & \Gamma_{2,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \gamma_{3,3} & \Gamma_{3,4} & 0 & 0 & 0 & \Gamma_{3,8} & 0 & \gamma_{3,10} & \gamma_{3,11} & 0 & 0 \\ * & * & * & \Gamma_{4,4} & 0 & 0 & 0 & \Gamma_{4,8} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Gamma_{5,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Gamma_{6,6} & 0 & 0 & \Gamma_{6,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Gamma_{7,7} & 0 & \Gamma_{7,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Gamma_{8,8} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Gamma_{9,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Gamma_{10,10} & \gamma_{10,11} & \Gamma_{10,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & \Gamma_{11,11} & \Gamma_{11,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Gamma_{12,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & \Gamma_{13,13} \end{bmatrix} < 0, \quad (16)$$

Case (iii) When $\frac{2}{3}\bar{\mu} \leq \mu(t) \leq \bar{\mu}$

$$\Xi_3 = \begin{bmatrix} \Gamma_{1,1} & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & \Gamma_{1,5} & 0 & 0 & 0 & 0 & \delta_{1,10} & \Gamma_{1,11} & 0 & 0 \\ * & \Gamma_{2,2} & 0 & \Gamma_{2,4} & \Gamma_{2,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \delta_{3,3} & \Gamma_{3,4} & 0 & 0 & 0 & \Gamma_{3,8} & 0 & \Gamma_{3,10} & \delta_{3,11} & \delta_{3,12} & 0 \\ * & * & * & \Gamma_{4,4} & 0 & 0 & 0 & \Gamma_{4,8} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Gamma_{5,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Gamma_{6,6} & 0 & 0 & \Gamma_{6,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Gamma_{7,7} & 0 & \Gamma_{7,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Gamma_{8,8} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Gamma_{9,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Gamma_{10,10} & \Gamma_{10,11} & \Gamma_{10,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & \Gamma_{11,11} & \delta_{11,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Gamma_{12,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & \Gamma_{13,13} \end{bmatrix} < 0, \quad (17)$$

where

$$\Gamma_{1,1} = 2\zeta K - KA - KA + \delta S_1 + 4\zeta \Upsilon_1 A \Theta_1 - 4\zeta \Upsilon_2 \Lambda_1 A - A(\Upsilon_2 \Lambda_1 - \Upsilon_1 \Theta_1) - (\Upsilon_2 \Lambda_1 - \Upsilon_1 \Theta_1)A + \eta_1$$

$$\eta_1 = \sum_{i=1}^4 L_i + \sum_{j=1}^2 \tilde{L}_j + M_1 + M_{3,11} + \left(\frac{\bar{\mu}}{3}\right)^2 A[M_4 + M_5 + M_6]A - 2\Upsilon_1 N_1 \Upsilon_2 - e^{-2\frac{\bar{\mu}}{3}} M_6,$$

$$\Gamma_{1,2} = KW_0 - 2(A - W_0 \Upsilon_1) \Theta_1 + 2\zeta (\Upsilon_2 W_0 + A) \Lambda_1 + (\Theta_1 - \Lambda_1) + (\Upsilon_2 \Lambda_1 - \Upsilon_1 \Theta_1) W_0 + \eta_2$$

$$\eta_2 = -AW_0 \left(\sum_{j=1}^2 \tilde{L}_j \right) - \left(\frac{\bar{\mu}}{3}\right)^2 [M_4 + M_5 + M_6]AW_0 + N_1(\Upsilon_1 + \Upsilon_2), \quad \Gamma_{1,3} = \frac{3}{2} e^{-2\frac{\bar{\mu}}{3}} M_6,$$

$$\Gamma_{1,4} = KW_1 - 2\zeta \Upsilon_1 W_1 + 2\zeta \Upsilon_2 W_1 \Lambda_1 + (\Upsilon_2 \Lambda_1 - \Upsilon_1 \Theta_1) W_1 - AW_1 \left(\sum_{j=1}^2 \tilde{L}_j \right) - \left(\frac{\bar{\mu}}{3}\right)^2 [M_4 + M_5 + M_6]AW_1,$$

$$\Gamma_{1,10} = M_{3,12}, \quad \Gamma_{1,11} = M_{3,13}, \quad \Gamma_{2,2} = 4\zeta \Theta_1 W_0 - 4\zeta \Lambda_1 W_0 + \eta_3, \quad \delta_{1,10} = M_{3,12} + e^{-2\zeta \frac{\bar{\mu}}{3}} M_6,$$

$$\eta_3 = 2(\Theta_1 - \Lambda_1)W_0 + W_0 \left(\sum_{j=1}^2 \tilde{L}_j \right) W_0 + \left(\frac{\bar{\mu}}{3}\right)^2 W_0(M_4 + M_5 + M_6)W_0 - 2N_1, \quad \Gamma_{4,4} = -2N_2,$$

$$\Gamma_{3,3} = \delta S_2 - e^{-2\zeta \bar{\mu}} (1 - \mu) L_1 - e^{-2\zeta \bar{\mu}} L_4 - 3e^{-2\zeta \frac{\bar{\mu}}{3}} M_6 - 2\Upsilon_3 N_2 \Upsilon_4, \quad \gamma_{10,11} = M_{3,23} - e^{-2\zeta \frac{\bar{\mu}}{3}} M_{3,12},$$

$$\Gamma_{5,5} = \left(\frac{\bar{\mu}}{3}\right)^2 W_2[M_4 + M_5 + M_6]W_2 - \frac{1}{\sigma} M_2 + W_2 \left(\sum_{j=1}^2 \tilde{L}_j \right) W_2, \quad \Gamma_{6,6} = -e^{-2\zeta \sigma} M_1 - 2\Upsilon_5 N_3 \Upsilon_6,$$

$$\Gamma_{7,7} = -2N_3, \quad \Gamma_{8,8} = -e^{-2\zeta \bar{\mu}} \tilde{L}_1 (1 - \mu), \quad \Gamma_{9,9} = -e^{-2\zeta \sigma} \tilde{L}_2, \quad \Gamma_{10,10} = -e^{-2\zeta \frac{\bar{\mu}}{3}} M_6 - e^{-2\zeta \frac{2}{3}\bar{\mu}} M_5 + \eta_4$$

$$\eta_4 = M_{3,22} - e^{-2\zeta \frac{\bar{\mu}}{3}} M_{3,11} - e^{-2\zeta \frac{\bar{\mu}}{3}} L_2, \quad \Gamma_{12,12} = -e^{-2\zeta \bar{\mu}} M_4 - e^{-2\zeta \frac{\bar{\mu}}{3}} M_{3,33} - e^{-2\zeta \bar{\mu}} L_4, \quad \Gamma_{13,13} = \sigma M_2,$$

$$\Gamma_{2,4} = 2\zeta \Theta_1 W_1 - 2\zeta \Lambda_1 W_1 + (\Theta_1 - \Lambda_1)W_1 + W_0 \left(\frac{\bar{\mu}}{3}\right)^2 [M_4 + M_5 + M_6]W_1, \quad \Gamma_{3,10} = e^{-2\zeta \frac{\bar{\mu}}{3}} M_6,$$

$$\Gamma_{11,11} = -e^{-4\zeta \frac{\bar{\mu}}{3}} L_3 + M_{3,33} - e^{-2\zeta \frac{\bar{\mu}}{3}} M_{3,22} - e^{-2\zeta \bar{\mu}} M_4 - e^{-4\zeta \frac{\bar{\mu}}{3}} M_5, \quad \Gamma_{11,12} = -e^{-2\zeta \frac{\bar{\mu}}{3}} M_{3,23} + e^{-2\zeta \bar{\mu}} M_4,$$

$$\Gamma_{2,5} = 2\zeta \Theta_1 W_2 - 2\zeta \Lambda_1 W_2 + (\Theta_1 - \Lambda_1)W_2 + W_0 \left(\frac{\bar{\mu}}{3}\right)^2 [M_4 + M_5 + M_6]W_2, \quad \Gamma_{6,7} = N_3(\Upsilon_5 + \Upsilon_6),$$

$$\Gamma_{3,8} = -2\zeta \Upsilon_3 \Theta_2 + 2\zeta \Upsilon_4 + (\Upsilon_4 \Lambda_2 - \Upsilon_3 \Theta_2), \quad \Gamma_{4,8} = 2\zeta \Theta_2 - 2\zeta \Lambda_2 + (\Theta_2 - \Lambda_2), \quad \delta_{3,12} = e^{-2\zeta \bar{\mu}} M_4,$$

$$\Gamma_{6,9} = -2\zeta \Upsilon_5 \Theta_3 + 2\zeta \Upsilon_6 + (\Upsilon_6 \Lambda_3 - \Upsilon_5 \Theta_3), \quad \Gamma_{7,9} = 2\zeta \Theta_3 - 2\zeta \Lambda_3 + (\Theta_3 - \Lambda_3), \quad \delta_{3,11} = e^{-2\zeta \bar{\mu}} M_4,$$

$$\Gamma_{10,11} = M_{3,23} - e^{-2\zeta \frac{\bar{\mu}}{3}} M_{3,12} + e^{-4\zeta \frac{\bar{\mu}}{3}} M_5, \quad \Gamma_{10,12} = -e^{-2\zeta \frac{\bar{\mu}}{3}} M_{3,13}, \quad \Gamma_{3,4} = N_2(\Upsilon_3 + \Upsilon_4),$$

$$\gamma_{3,3} = -e^{-2\zeta \bar{\mu}} (1 - \mu) L_1 + \delta S_2 - e^{-2\zeta \bar{\mu}} L_4 - \frac{3}{2} e^{-4\zeta \frac{\bar{\mu}}{3}} M_5, \quad \gamma_{3,10} = e^{-4\zeta \frac{\bar{\mu}}{3}} M_5, \quad \delta_{11,12} = -e^{-2\zeta \frac{\bar{\mu}}{3}} M_{3,23},$$

$$\gamma_{3,11} = e^{-4\zeta \frac{\bar{\mu}}{3}} M_5, \quad \gamma_{1,10} = M_{3,12} + e^{-2\zeta \frac{\bar{\mu}}{3}} M_6, \quad \delta_{3,3} = -e^{-2\zeta \bar{\mu}} (1 - \mu) L_1 + \delta S_2 - e^{-2\zeta \bar{\mu}} L_4 - 3M_4,$$

$$\Gamma_{1,5} = KW_2 - 2\zeta \Upsilon_1 \Theta_1 W_2 + 2\zeta \Upsilon_2 \Lambda_1 W_2 + (\Upsilon_2 \Lambda_1 - \Upsilon_1 \Theta_1)W_2 + \left(\sum_{j=1}^2 \tilde{L}_j \right) W_2.$$

Then the equilibrium point of recurrent neural networks (2) is exponentially stable in the mean square sense with the exponential convergence rate ζ .

Proof. Consider the following Lyapunov–Krasovskii functional :

$$V(u(t)) = \sum_{i=1}^7 V_i(u(t)), \quad (18)$$

where

$$V_1(u(t)) = e^{2\zeta t} u^T(t) K u(t)$$

$$V_2(u(t)) = 2e^{2\zeta s} \sum_{i=1}^n \left[\left(\int_0^{u_i(t)} \vartheta(\tilde{f}_i(s) - \alpha_i^- s) ds + \int_0^{u_i(t)} \lambda_i(\alpha_i^+ s - \tilde{f}_i(s)) ds \right) + \left(\int_0^{u_i(t)} \vartheta(\tilde{g}_i(s) - \beta_i^- s) ds + \int_0^{u_i(t)} \lambda_i(\beta_i^+ s - \tilde{g}_i(s)) ds \right) \right. \\ \left. + \left(\int_0^{u_i(t)} \vartheta(\tilde{h}_i(s) - \gamma_i^- s) ds + \int_0^{u_i(t)} \lambda_i(\gamma_i^+ s - \tilde{h}_i(s)) ds \right) \right]$$

$$V_3(u(t)) = \int_{t-\mu(t)}^t e^{2\zeta s} u^T(s) L_1 u(s) ds + \int_{t-\frac{\tilde{\mu}}{3}}^t e^{2\zeta s} u^T(s) L_2 u(s) ds + \int_{t-\frac{2}{3}\tilde{\mu}}^t e^{2\zeta s} u^T(s) L_3 u(s) ds + \int_{t-\tilde{\mu}}^t e^{2\zeta s} u^T(s) L_4 u(s) ds \\ + \int_{t-\mu(t)}^t e^{2\zeta s} \dot{u}^T(s) \tilde{L}_1 \dot{u}(s) ds + \int_{t-\sigma}^t e^{2\zeta s} \dot{u}^T(s) \tilde{L}_2 \dot{u}(s) ds$$

$$V_4(u(t)) = \int_{t-\sigma}^t e^{2\zeta s} u^T(s) M_1 u(s) ds$$

$$V_5(u(t)) = \int_{-\sigma}^0 \int_{t+s}^t e^{2\zeta s} \tilde{h}^T(u(s)) M_2 \tilde{h}(u(s)) ds$$

$$V_6(u(t)) = \int_{t-\frac{\tilde{\mu}}{3}}^t e^{2\zeta s} \begin{pmatrix} u(s) \\ u\left(s - \frac{\tilde{\mu}}{3}\right) \\ u\left(s - \frac{2}{3}\tilde{\mu}\right) \end{pmatrix}^T M_3 \begin{pmatrix} u(s) \\ u\left(s - \frac{\tilde{\mu}}{3}\right) \\ u\left(s - \frac{2}{3}\tilde{\mu}\right) \end{pmatrix} ds$$

$$V_7(u(t)) = \frac{\tilde{\mu}}{3} \int_{-\tilde{\mu}}^{-\frac{2}{3}\tilde{\mu}} \int_{t+\theta}^t e^{2\zeta s} \dot{u}^T(s) M_4 \dot{u}(s) ds d\theta + \frac{\tilde{\mu}}{3} \int_{-\frac{2}{3}\tilde{\mu}}^{-\frac{\tilde{\mu}}{3}} \int_{t+\theta}^t e^{2\zeta s} \dot{u}^T(s) M_5 \dot{u}(s) ds d\theta + \frac{\tilde{\mu}}{3} \int_{-\frac{\tilde{\mu}}{3}}^0 \int_{t+\theta}^t e^{2\zeta s} \dot{u}^T(s) M_6 \dot{u}(s) ds d\theta$$

For $t \neq t_k$, $t > 0$, for all $k \in \mathbb{Z}^+$. Then, by [Assumption 2](#) and the condition (13), we have

$$\text{trace}[\tilde{\rho}^T(u_1, u_2, t) K \tilde{\rho}(u_1, u_2, t)] \leq \delta[u_1^T S_1 u_1 + u_2^T S_2 u_2]$$

Hence, the aid of [Lemmas 2.5–2.7](#), it follows from (2) and [Definition 2.4](#) (weak infinitesimal operator $\mathcal{L}V$) that

$$\mathcal{L}V(u(t)) = \sum_{i=1}^7 \mathcal{L}V_i(u(t)),$$

where

$$\mathcal{L}V_1(u(t)) = 2\zeta e^{2\zeta t} u^T(t) K u(t) + 2e^{2\zeta t} u^T(t) K \left[-Au(t) + W_0 \tilde{f}(u(t)) + W_1 \tilde{g}(u(t - \mu(t))) + W_2 \int_{t-\sigma}^t \tilde{h}(u(s)) ds \right] \\ + e^{2\zeta t} \text{trace}(\tilde{\rho}^T(u(t), u(t - \mu(t)), t) K \tilde{\rho}(u(t), u(t - \mu(t)), t)) \\ \leq e^{2\zeta t} u^T(t) [2\zeta K - 2KA + \delta S_1] u(t) + e^{2\zeta t} u^T(t) [2KW_0] \tilde{f}(u(t)) + e^{2\zeta t} u^T(t) [2KW_1] \tilde{g}(u(t - \mu(t))) \\ + e^{2\zeta t} u^T(t) [2KW_2] \int_{t-\sigma}^t \tilde{h}(u(s)) ds + e^{2\zeta t} u^T(t - \mu(t)) \delta S_2 u(t - \mu(t)) \quad (19)$$

$$\mathcal{L}V_2(u(t)) = 4\zeta e^{2\zeta t} \sum_{i=1}^n \left[\int_0^{u_i(t)} \vartheta_{1i}(\tilde{f}_i(s) - \alpha_i^- s) ds + \int_0^{u_i(t)} \lambda_{1i}(\alpha_i^+ s - \tilde{f}_i(s)) ds \right] + 2e^{2\zeta t} [(\tilde{f}^T(u(t))) \\ - u^T(t) \Upsilon_1) \Theta_1 \dot{u}(t) + (u^T(t) \Upsilon_2 - \tilde{f}^T(u(t))) \Lambda_1 \dot{u}(t)] + 4\zeta e^{2\zeta t} \sum_{i=1}^n \left[\int_0^{u_i(t-\mu(t))} \vartheta_{2i}(\tilde{g}_i(s) - \beta_i^- s) ds \right. \\ \left. + \int_0^{u_i(t-\mu(t))} \lambda_{2i}(\beta_i^+ s - \tilde{g}_i(s)) ds \right] + 2e^{2\zeta t} [(\tilde{g}^T(u(t - \mu(t)))) - u^T(t - \mu(t)) \Upsilon_3) \Theta_2 \dot{u}(t - \mu(t)) (1 - \bar{\mu}) \\ + (u^T(t - \mu(t)) \Upsilon_4 - \tilde{g}^T(u(t - \mu(t)))) \Lambda_2 \dot{u}(t - \mu(t)) (1 - \mu)] + 4\zeta e^{2\zeta t} \sum_{i=1}^n \left[\int_0^{u_i(t-\sigma)} \vartheta_{3i}(\tilde{h}_i(s) - \gamma_i^- s) ds \right. \\ \left. + \int_0^{u_i(t-\sigma)} \lambda_{3i}(\gamma_i^+ s - \tilde{h}_i(s)) ds \right] + 2e^{2\zeta t} [(\tilde{h}^T(u(t - \sigma))) - u^T(t - \sigma) \Upsilon_5) \Theta_3 \dot{u}(t - \sigma) + (u^T(t - \sigma) \Upsilon_6 \\ - \tilde{h}^T(u(t - \sigma))) \Lambda_3 \dot{u}(t - \sigma)] \\ \leq 4\zeta e^{2\zeta t} [(\tilde{f}^T(u(t))) - u^T(t) \Upsilon_1) \Theta_1 \dot{u}(t) + (u^T(t) \Upsilon_2 - \tilde{f}^T(u(t))) \Lambda_1 \dot{u}(t)] + 2e^{2\zeta t} [\tilde{f}^T(u(t)) (\Theta_1 - \Lambda_1) \\ + u^T(t) (\Upsilon_2 \Lambda_1 - \Upsilon_1 \Theta_1)] \dot{u}(t) + 4\zeta e^{2\zeta t} [(\tilde{g}^T(u(t - \mu(t)))) - u^T(t - \mu(t)) \Upsilon_3) \Theta_2 \dot{u}(t - \mu(t)) \\ + u^T(t - \mu(t)) (\Upsilon_4 \Lambda_2 - \Upsilon_2 \Theta_2)] \dot{u}(t - \mu(t)) + 4\zeta e^{2\zeta t} [(\tilde{h}^T(u(t - \sigma))) - u^T(t - \sigma) \Upsilon_5) \Theta_3 \dot{u}(t - \sigma) \\ + u^T(t - \sigma) (\Upsilon_6 \Lambda_3 - \Upsilon_3 \Theta_3)] \dot{u}(t - \sigma) + 2e^{2\zeta t} [\tilde{h}^T(u(t - \sigma)) (\Theta_3 - \Lambda_3) \\ + u^T(t - \sigma) (\Upsilon_5 \Lambda_3 - \Upsilon_3 \Theta_3)] \dot{u}(t - \sigma)]$$

$$\begin{aligned}
& + (u^T(t - \mu(t))\Upsilon_4 - \tilde{g}^T(u(t - \mu(t))))\Lambda_2\dot{u}(t - \mu(t)) + 2e^{2\varsigma t}[\tilde{g}^T(u(t - \mu(t)))(\Theta_2 - \Lambda_2) \\
& + u^T(t - \mu(t))(\Upsilon_4\Lambda_2 - \Upsilon_3\Theta_2)]\dot{u}(t - \mu(t)) + 4\varsigma e^{2\varsigma t}[\tilde{h}^T(u(t - \sigma)) - u^T(t - \sigma)\Upsilon_5)\Theta_3\dot{u}(t - \sigma) \\
& + (u^T(t - \sigma)\Upsilon_6 - \tilde{h}^T(u(t - \sigma)))\Lambda_3\dot{u}(t - \sigma) + 2e^{2\varsigma t}[\tilde{h}^T(u(t - \sigma))(\Theta_3 - \Lambda_3) + u^T(t - \sigma)(\Upsilon_6\Lambda_3 - \Upsilon_5\Theta_3)]\dot{u}(t - \sigma) \quad (20)
\end{aligned}$$

$$\begin{aligned}
\mathcal{LV}_3(u(t)) & \leq e^{2\varsigma t}u^T(t)\left(\sum_{i=1}^4 L_i\right)u(t) + e^{2\varsigma t}\dot{u}^T(t)\left(\sum_{j=1}^2 \tilde{L}_j\right)\dot{u}(t) - e^{2\varsigma(t-\bar{\mu})}(1-\mu)u^T(t-\mu(t))L_1u(t-\mu(t)) \\
& - e^{2\varsigma(t-\frac{\bar{\mu}}{3})}u^T\left(t-\frac{\bar{\mu}}{3}\right)L_2u\left(t-\frac{\bar{\mu}}{3}\right) - e^{2\varsigma(t-\frac{2}{3}\bar{\mu})}u^T\left(t-\frac{2}{3}\bar{\mu}\right)L_3u\left(t-\frac{2}{3}\bar{\mu}\right) - e^{2\varsigma(t-\bar{\mu})}u^T(t-\bar{\mu})L_4u(t-\bar{\mu}) \\
& - e^{2\varsigma(t-\bar{\mu})}\dot{u}^T(t-\mu(t))\tilde{L}_1\dot{u}(t-\mu(t))(1-\mu) - e^{2\varsigma(t-\sigma)}\dot{u}^T(t-\sigma)\tilde{L}_2\dot{u}(t-\sigma) \quad (21)
\end{aligned}$$

$$\mathcal{LV}_4(u(t)) = e^{2\varsigma t}u^T(t)M_1u(t) - e^{2\varsigma(t-\sigma)}u^T(t-\sigma)M_1u(t-\sigma) \quad (22)$$

$$\begin{aligned}
\mathcal{LV}_5(u(t)) & \leq \sigma e^{2\varsigma t}\tilde{h}^T(u(t))M_2\tilde{h}(u(t)) - \int_{t-\sigma}^t e^{2\varsigma s}\tilde{h}^T(u(s))M_2\tilde{h}(u(s))ds \\
& \leq \sigma e^{2\varsigma t}\tilde{h}^T(u(t))M_2\tilde{h}(u(t)) - \frac{1}{\sigma}e^{2\varsigma t}\left(\int_{t-\sigma}^t \tilde{h}^T(u(s))ds\right)^T M_2\left(\int_{t-\sigma}^t \tilde{h}(u(s))ds\right) \quad (23)
\end{aligned}$$

$$\mathcal{LV}_6(u(t)) = e^{2\varsigma t}\begin{pmatrix} u(t) \\ u(t-\frac{\bar{\mu}}{3}) \\ u(t-\frac{2}{3}\bar{\mu}) \end{pmatrix}^T M_3 \begin{pmatrix} u(t) \\ u(t-\frac{\bar{\mu}}{3}) \\ u(t-\frac{2}{3}\bar{\mu}) \end{pmatrix} - e^{2\varsigma(t-\frac{\bar{\mu}}{3})}\begin{pmatrix} u(t-\frac{\bar{\mu}}{3}) \\ u(t-\frac{2}{3}\bar{\mu}) \\ u(t-\bar{\mu}) \end{pmatrix}^T M_3 \begin{pmatrix} u(t-\frac{\bar{\mu}}{3}) \\ u(t-\frac{2}{3}\bar{\mu}) \\ u(t-\bar{\mu}) \end{pmatrix} \quad (24)$$

$$\begin{aligned}
\mathcal{LV}_7(u(t)) & = \left(\frac{\bar{\mu}}{3}\right)^2 e^{2\varsigma t}\dot{u}^T(t)[M_4 + M_5 + M_6]\dot{u}(t) - \frac{\bar{\mu}}{3}\int_{t-\frac{\bar{\mu}}{3}}^t e^{2\varsigma s}\dot{u}^T(s)M_6\dot{u}(s)ds - \frac{\bar{\mu}}{3}\int_{t-\frac{2}{3}\bar{\mu}}^{t-\frac{\bar{\mu}}{3}} e^{2\varsigma s}\dot{u}^T(s)M_5\dot{u}(s)ds \\
& - \frac{\bar{\mu}}{3}\int_{t-\bar{\mu}}^{t-\frac{2}{3}\bar{\mu}} e^{2\varsigma s}\dot{u}^T(s)M_4\dot{u}(s)ds \quad (25)
\end{aligned}$$

Case (i) When $0 \leq \mu(t) \leq \frac{\bar{\mu}}{3}$

By utilizing the bounds lemma of [37], we can get that

$$-\frac{\bar{\mu}}{3}\int_{t-\frac{\bar{\mu}}{3}}^t e^{2\varsigma s}\dot{u}^T(s)M_6\dot{u}(s)ds \leq e^{2\varsigma(t-\frac{\bar{\mu}}{3})}\begin{pmatrix} u(t) \\ u(t-\mu(t)) \\ u(t-\frac{\bar{\mu}}{3}) \end{pmatrix}^T \begin{pmatrix} -M_6 & 2M_6 & 0 \\ M_6 & -3M_6 & 2M_6 \\ 0 & 2M_6 & -M_6 \end{pmatrix} \begin{pmatrix} u(t) \\ u(t-\mu(t)) \\ u(t-\frac{\bar{\mu}}{3}) \end{pmatrix} \quad (26)$$

$$-\frac{\bar{\mu}}{3}\int_{t-\bar{\mu}}^{t-\frac{2}{3}\bar{\mu}} e^{2\varsigma s}\dot{u}^T(s)M_4\dot{u}(s)ds \leq -e^{2\varsigma(t-\bar{\mu})}\left[u(t-\frac{2}{3}\bar{\mu}) - u(t-\bar{\mu})\right]^T M_4 \left[u(t-\frac{2}{3}\bar{\mu}) - u(t-\bar{\mu})\right] \quad (27)$$

$$-\frac{\bar{\mu}}{3}\int_{t-\frac{2}{3}\bar{\mu}}^{t-\frac{\bar{\mu}}{3}} e^{2\varsigma s}\dot{u}^T(s)M_5\dot{u}(s)ds \leq -e^{2\varsigma(t-\frac{2}{3}\bar{\mu})}\left[u(t-\frac{\bar{\mu}}{3}) - u(t-\frac{2}{3}\bar{\mu})\right]^T M_5 \left[u(t-\frac{\bar{\mu}}{3}) - u(t-\frac{2}{3}\bar{\mu})\right] \quad (28)$$

Case (ii) When $\frac{\bar{\mu}}{3} \leq \mu(t) \leq \frac{2}{3}\bar{\mu}$. Then, we obtain

$$-\frac{\bar{\mu}}{3}\int_{t-\frac{2}{3}\bar{\mu}}^{t-\frac{\bar{\mu}}{3}} e^{2\varsigma s}\dot{u}^T(s)M_5\dot{u}(s)ds \leq e^{2\varsigma(t-\frac{2}{3}\bar{\mu})}\begin{pmatrix} u(t-\frac{\bar{\mu}}{3}) \\ u(t-\mu(t)) \\ u(t-\frac{2}{3}\bar{\mu}) \end{pmatrix}^T \begin{pmatrix} -M_5 & 2M_5 & 0 \\ M_5 & -3M_5 & 2M_5 \\ 0 & 2M_5 & -M_5 \end{pmatrix} \begin{pmatrix} cu(t-\frac{\bar{\mu}}{3}) \\ u(t-\mu(t)) \\ u(t-\frac{2}{3}\bar{\mu}) \end{pmatrix} \quad (29)$$

$$-\frac{\bar{\mu}}{3}\int_{t-\bar{\mu}}^{t-\frac{2}{3}\bar{\mu}} e^{2\varsigma s}\dot{u}^T(s)M_4\dot{u}(s)ds \leq -e^{2\varsigma(t-\bar{\mu})}\left[u(t-\frac{2}{3}\bar{\mu}) - u(t-\bar{\mu})\right]^T M_4 \left[u(t-\frac{2}{3}\bar{\mu}) - u(t-\bar{\mu})\right] \quad (30)$$

$$-\frac{\bar{\mu}}{3}\int_{t-\frac{\bar{\mu}}{3}}^t e^{2\varsigma s}\dot{u}^T(s)M_6\dot{u}(s)ds \leq -e^{2\varsigma(t-\frac{\bar{\mu}}{3})}\left[u(t) - u(t-\frac{\bar{\mu}}{3})\right]^T M_6 \left[u(t) - u(t-\frac{\bar{\mu}}{3})\right] \quad (31)$$

Case (iii) When $\frac{2}{3}\bar{\mu} \leq \mu(t) \leq \bar{\mu}$.

From the bounds lemma of [37], one can obtain that

$$-\frac{\bar{\mu}}{3}\int_{t-\bar{\mu}}^{t-\frac{2}{3}\bar{\mu}} e^{2\varsigma s}\dot{u}^T(s)M_4\dot{u}(s)ds \leq e^{2\varsigma(t-\bar{\mu})}\begin{pmatrix} u(t-\frac{2}{3}\bar{\mu}) \\ u(t-\mu(t)) \\ u(t-\bar{\mu}) \end{pmatrix}^T \begin{pmatrix} -M_4 & 2M_4 & 0 \\ M_4 & -3M_4 & 2M_4 \\ 0 & 2M_4 & -M_4 \end{pmatrix} \begin{pmatrix} u(t-\frac{2}{3}\bar{\mu}) \\ u(t-\mu(t)) \\ u(t-\bar{\mu}) \end{pmatrix} \quad (32)$$

$$-\frac{\bar{\mu}}{3} \int_{t-\frac{\bar{\mu}}{3}}^t e^{2\zeta s} \dot{u}^T(s) M_6 \dot{u}(s) ds \leq -e^{2\zeta(t-\frac{\bar{\mu}}{3})} \left[u(t) - u(t-\frac{\bar{\mu}}{3}) \right]^T M_6 \left[u(t) - u(t-\frac{\bar{\mu}}{3}) \right] \quad (33)$$

$$-\frac{\bar{\mu}}{3} \int_{t-\frac{2}{3}\bar{\mu}}^{t-\frac{\bar{\mu}}{3}} e^{2\zeta s} \dot{u}^T(s) M_5 \dot{u}(s) ds \leq -e^{2\zeta(t-\frac{2}{3}\bar{\mu})} \left[u(t-\frac{\bar{\mu}}{3}) - u(t-\frac{2}{3}\bar{\mu}) \right]^T M_5 \left[u(t-\frac{\bar{\mu}}{3}) - u(t-\frac{2}{3}\bar{\mu}) \right] \quad (34)$$

Additionally, to derive main results with reduced conservatism, as a result of (3)–(5), we insert the inequalities with positive diagonal matrices N_1 , N_2 and N_3 , as follows

$$e^{2\zeta t} [-2\tilde{f}^T(u(t)) N_1 \tilde{f}(u(t)) + 2u^T(t) N_1 (\Upsilon_1 + \Upsilon_2) \tilde{f}(u(t)) - 2u^T(t) \Upsilon_1 N_1 \Upsilon_2 u(t)] \geq 0 \quad (35)$$

$$e^{2\zeta t} [-2\tilde{g}^T(u(t-\mu(t))) N_2 \tilde{g}(u(t-\mu(t))) + 2u^T(t-\mu(t)) N_2 (\Upsilon_3 + \Upsilon_4) \tilde{g}(u(t-\mu(t))) - 2u^T(t-\mu(t)) \Upsilon_3 N_2 \Upsilon_4 u(t-\mu(t))] \geq 0 \quad (36)$$

$$e^{2\zeta t} [-2\tilde{h}^T(u(t-\sigma)) N_3 \tilde{h}(u(t-\sigma)) + 2u^T(t-\sigma) N_3 (\Upsilon_5 + \Upsilon_6) \tilde{h}(u(t-\sigma)) - 2u^T(t-\sigma) \Upsilon_5 N_3 \Upsilon_6 u(t-\sigma)] \geq 0 \quad (37)$$

Case (i) When $0 \leq \mu(t) \leq \frac{\bar{\mu}}{3}$. In accordance to (15), from the Eqs. (19)–(28), (35)–(37), we have

$$\mathcal{L}V(u(t)) \leq e^{2\zeta t} \zeta^T(t) \Xi_1 \zeta(t) \leq 0, \quad (38)$$

where

$$\zeta^T(t) = \left[u^T(t) \tilde{f}^T(u(t)) u^T(t-\mu(t)) \tilde{g}^T(u(t-\mu(t))) \left(\int_{t-\sigma}^t \tilde{h}(u(s)) ds \right)^T u^T(t-\sigma) \tilde{h}^T(u(t-\sigma)) \right. \\ \left. \dot{u}^T(t-\mu(t)) \dot{u}^T(t-\sigma) u^T(t-\frac{\bar{\mu}}{3}) u^T(t-\frac{2}{3}\bar{\mu}) u^T(t-\bar{\mu}) \tilde{h}^T(u(t)) \right]$$

Case (ii) When $\frac{\bar{\mu}}{3} \leq \mu(t) \leq \frac{2}{3}\bar{\mu}$. In accordance to (16), from the Eqs. (19)–(25), (29)–(31), (35)–(37), we get

$$\mathcal{L}V(u(t)) \leq e^{2\zeta t} \zeta^T(t) \Xi_2 \zeta(t) \leq 0 \quad (39)$$

Case (iii) When $\frac{2}{3}\bar{\mu} \leq \mu(t) \leq \bar{\mu}$. In accordance to (17), by the Eqs. (19)–(25), (32)–(37), we obtain that

$$\mathcal{L}V(u(t)) \leq e^{2\zeta t} \zeta^T(t) \Xi_3 \zeta(t) \leq 0 \quad (40)$$

Now, For $t = t_k$, by the condition (14), it can be easily get that

$$\begin{aligned} V_1(u(t_k)) - V_1(u(t_{k-})) &= e^{2\zeta t_k} u^T(t_k) K u(t_k) - e^{2\zeta t_{k-}} u^T(t_{k-}) K u(t_{k-}) \\ &= e^{2\zeta t_k} u^T(t_{k-}) \tilde{\mathcal{G}}_k^T K \tilde{\mathcal{G}}_k u(t_{k-}) - e^{2\zeta t_k} u^T(t_{k-}) K u(t_{k-}) \\ &= e^{2\zeta t_k} u^T(t_{k-}) [\tilde{\mathcal{G}}_k^T K \tilde{\mathcal{G}}_k - K] u(t_{k-}) \\ &\leq 0 \end{aligned}$$

Thus,

$$V_1(u(t_k)) \leq V_1(u(t_{k-})), \quad k \in \mathbb{Z}^+ \quad (41)$$

It follows that

$$V(u(t_k)) \leq V(u(t_{k-})), \quad k \in \mathbb{Z}^+ \quad (42)$$

From (38)–(40) and (42), we can easily obtain

$$V(u(t)) \leq V(u(0)) \quad (43)$$

Taking expectation on both sides of (43), then

$$\mathbb{E}\{V(u(t))\} \leq \mathbb{E}\{V(u(0))\} \quad (44)$$

Alternatively,

$$\begin{aligned} V_1(u(0)) &\leq \lambda_{\max}(K) \|u(0)\|^2 \\ &\leq \lambda_{\max}(K) \sup_{-v \leq s \leq 0} \{\|\varphi(s)\|^2\} \end{aligned} \quad (45)$$

$$\begin{aligned} V_2(u(0)) &\leq 2[\tilde{f}(u(0)) - \Upsilon_1 u(0)]^T \Theta_1 u(0) + 2[u(0) \Upsilon_2 - \tilde{f}(u(0))]^T \Lambda_1 u(0) + 2[\tilde{g}(u(-\mu(0))) - \Upsilon_3 u(-\mu(0))]^T \Theta_2 u(-\mu(0)) \\ &\quad + 2[u(-\mu(0)) \Upsilon_4 - \tilde{g}(u(-\mu(0)))]^T \Lambda_2 u(-\mu(0)) + 2[\tilde{h}(u(-\sigma)) - \Upsilon_5 u(-\sigma)]^T \Theta_3 u(-\sigma) \\ &\quad + 2[\Upsilon_6 u(-\sigma) - \tilde{h}(u(-\sigma))]^T \Lambda_3 u(-\sigma) \\ &\leq 2\lambda_{\max}(\Upsilon_2 - \Upsilon_1) [\lambda_{\max}(\Theta_1) + \lambda_{\max}(\Lambda_1)] \sup_{-v \leq s \leq 0} \{\|\varphi(s)\|^2\} + 2\lambda_{\max}(\Upsilon_4 - \Upsilon_3) [\lambda_{\max}(\Theta_2) + \lambda_{\max}(\Lambda_2)] \sup_{-v \leq s \leq 0} \{\|\varphi(s)\|^2\} \\ &\quad + 2\lambda_{\max}(\Upsilon_6 - \Upsilon_5) [\lambda_{\max}(\Theta_3) + \lambda_{\max}(\Lambda_3)] \sup_{-v \leq s \leq 0} \{\|\varphi(s)\|^2\} \end{aligned} \quad (46)$$

$$V_3(u(0)) \leq \left(\frac{\bar{\mu}}{3} \lambda_{\max}(L_1) + \frac{\bar{\mu}}{3} \lambda_{\max}(L_2) + \frac{2\bar{\mu}}{3} \lambda_{\max}(L_3) + \bar{\mu} \lambda_{\max}(L_4) + \frac{\bar{\mu}}{3} \lambda_{\max}(\tilde{L}_1) + \sigma \lambda_{\max}(\tilde{L}_2) \right) \sup_{-v \leq s \leq 0} \{\|\varphi(s)\|^2\} \quad (47)$$

$$V_4(u(0)) \leq \sigma \lambda_{\max}(M_1) \sup_{-v \leq s \leq 0} \{\|\varphi(s)\|^2\} \quad (48)$$

$$V_5(u(0)) \leq \sigma^2 \gamma \lambda_{\max}(M_2) \sup_{-v \leq s \leq 0} \{\|\varphi(s)\|^2\} \quad (49)$$

$$\begin{aligned} V_6(u(0)) &\leq \int_{-\frac{\bar{\mu}}{3}}^0 e^{2\zeta s} \begin{pmatrix} u(s) \\ u(s - \frac{\bar{\mu}}{3}) \\ u(s - \frac{2\bar{\mu}}{3}) \end{pmatrix}^T M_3 \begin{pmatrix} u(s) \\ u(s - \frac{\bar{\mu}}{3}) \\ u(s - \frac{2\bar{\mu}}{3}) \end{pmatrix} ds \\ &\leq \int_{-\frac{\bar{\mu}}{3}}^0 [u^T(s) M_{3_{11}} u(s) + 2u^T(s) M_{3_{12}} u(s - \frac{\bar{\mu}}{3}) + 2u^T(s) M_{3_{13}} u(s - \frac{2\bar{\mu}}{3}) + u^T(s - \frac{\bar{\mu}}{3}) M_{3_{23}} u(s - \frac{2\bar{\mu}}{3}) M_{3_{33}} u(s - \frac{2\bar{\mu}}{3})] ds \end{aligned}$$

By Lemma 2.6, we have

$$V_6(u(0)) \leq \frac{\bar{\mu}}{3} \left[\lambda_{\max}(M_{3_{11}}) + 2\lambda_{\max}(M_{3_{12}}) + 2\lambda_{\max}(M_{3_{13}}) + \lambda_{\max}(M_{3_{22}}) + 2\lambda_{\max}(M_{3_{23}}) + \lambda_{\max}(M_{3_{33}}) \right] \sup_{-v \leq s \leq 0} \{\|\varphi(s)\|^2\} \quad (50)$$

$$\begin{aligned} V_7(u(0)) &\leq \frac{\bar{\mu}}{3} \lambda_{\max}(M_4) \int_{-\bar{\mu}}^{-\frac{2\bar{\mu}}{3}} \int_{\theta}^0 \dot{u}^T(s) \dot{u}(s) ds d\theta + \frac{\bar{\mu}}{3} \lambda_{\max}(M_5) \int_{-\frac{2\bar{\mu}}{3}}^{-\frac{\bar{\mu}}{3}} \int_{\theta}^0 \dot{u}^T(s) \dot{u}(s) ds d\theta + \frac{\bar{\mu}}{3} \lambda_{\max}(M_6) \int_{-\frac{\bar{\mu}}{3}}^0 \int_{\theta}^0 \dot{u}^T(s) \dot{u}(s) ds d\theta \\ &\leq \left\{ \frac{\bar{\mu}}{3} \left[\lambda_{\max}(M_4) \left(\frac{5\bar{\mu}^2}{18} \right) \right] + \frac{\bar{\mu}}{3} \left[\lambda_{\max}(M_5) \left(\frac{3\bar{\mu}^2}{18} \right) \right] + \frac{\bar{\mu}}{3} \left[\lambda_{\max}(M_6) \left(\frac{\bar{\mu}^2}{18} \right) \right] \right\} \left[\lambda_{\max}(AA^T) + \alpha^2 \lambda_{\max}(W_0 W_0^T) + \beta^2 \lambda_{\max}(W_1 W_1^T) \right. \\ &\quad \left. + \gamma^2 \lambda_{\max}(W_2 W_2^T) \right] \sup_{-v \leq s \leq 0} \{\|\varphi(s)\|^2\} \end{aligned} \quad (51)$$

From the Eqs. (45)–(51), we get the following inequalities:

$$\mathbb{E}\{V(u(0))\} \leq \mathcal{M}^* \sup_{-v \leq s \leq 0} \mathbb{E}\{\|\varphi(s)\|^2\}, \quad (52)$$

where

$$\begin{aligned} \mathcal{M}^* &= \lambda_{\max}(K) + 2\lambda_{\max}(\Upsilon_2 - \Upsilon_1)(\lambda_{\max}(\Theta_1) + \lambda_{\max}(\Lambda_1)) + 2\lambda_{\max}(\Upsilon_4 - \Upsilon_3)[\lambda_{\max}(\Theta_2) + \lambda_{\max}(\Lambda_2)] \\ &\quad + 2\lambda_{\max}(\Upsilon_6 - \Upsilon_5)[\lambda_{\max}(\Theta_3) + \lambda_{\max}(\Lambda_3)] + \frac{\bar{\mu}}{3} \lambda_{\max}(L_1) + \frac{\bar{\mu}}{3} \lambda_{\max}(L_2) + \frac{2\bar{\mu}}{3} \lambda_{\max}(L_3) + \bar{\mu} \lambda_{\max}(L_4) + \frac{\bar{\mu}}{3} \lambda_{\max}(\tilde{L}_1) + \sigma \lambda_{\max}(L_2) \\ &\quad + \sigma \lambda_{\max}(M_1) + \sigma^2 \gamma \lambda_{\max}(M_2) + \frac{\bar{\mu}}{3} [\lambda_{\max}(M_{3_{11}}) + 2\lambda_{\max}(M_{3_{12}}) + 2\lambda_{\max}(M_{3_{13}}) + \lambda_{\max}(M_{3_{22}}) + 2\lambda_{\max}(M_{3_{23}}) + \lambda_{\max}(M_{3_{33}})] \\ &\quad + \left[\frac{5\bar{\mu}^3}{54} \lambda_{\max}(M_4) + \frac{3\bar{\mu}^3}{54} \lambda_{\max}(M_5) + \frac{\bar{\mu}^3}{54} \lambda_{\max}(M_6) \right] \left[\lambda_{\max}(AA^T) + \alpha^2 \lambda_{\max}(W_0 W_0^T) + \beta^2 \lambda_{\max}(W_1 W_1^T) + \gamma^2 \lambda_{\max}(W_2 W_2^T) \right] \end{aligned}$$

On the other hand, we can conclude that

$$\mathbb{E}\{V(u(t))\} \geq e^{2\zeta t} \lambda_{\min}(K) \mathbb{E}\{\|u(t)\|^2\} \quad (53)$$

Hence,

$$\mathbb{E}\{\|u(t)\|\} \leq \sqrt{\frac{\mathcal{M}^*}{\lambda_{\min}(K)}} e^{-\zeta t} \sup_{-v \leq s \leq 0} \mathbb{E}\{\|\varphi(s)\|\} \quad (54)$$

Therefore, by (54) and Definition 2.3, the neural networks (2) is exponentially stable in mean square sense. $\square$

Remark 3.1 The integral terms $2e^{2\zeta s} \sum_{i=1}^n \int_0^{u_i(t)} \vartheta(\tilde{f}_i(s) - \alpha_i^- s) ds$, $2e^{2\zeta s} \sum_{i=1}^n \int_0^{u_i(t)} \lambda_i(\alpha_i^+(s) - \tilde{f}_i(s)) ds$, $2e^{2\zeta s} \sum_{i=1}^n \int_0^{u_i(t)} \vartheta(\tilde{g}_i(s) - \beta_i^- s) ds$, $2e^{2\zeta s} \sum_{i=1}^n \int_0^{u_i(t)} \lambda_i(\beta_i^+(s) - \tilde{g}_i(s)) ds$, $2e^{2\zeta s} \sum_{i=1}^n \int_0^{u_i(t)} \vartheta(\tilde{h}_i(s) - \gamma_i^- s) ds$ and $2e^{2\zeta s} \sum_{i=1}^n \int_0^{u_i(t)} \lambda_i(\gamma_i^+(s) - \tilde{h}_i(s)) ds$, in Lyapunov–Krasovskii functional $V_2(u(t))$ are involved and may lead to attain enhanced feasibility for exponential stability criteria.

Remark 3.2. Suppose the stochastic disturbances are not appear in neural networks (2), then the system can be remodified as

$$\begin{aligned} du(t) &= \left[-Au(t) + W_0 \tilde{f}(u(t)) + W_1 \tilde{g}(u(t - \mu(t))) + W_2 \int_{t-\sigma}^t \tilde{h}(u(s)) ds \right] dt; \\ t &> 0, t \neq t_k, \\ u(t_k) &= \tilde{G}_k u(t_k^-); t = t_k, k \in \mathbb{Z}^+, \\ u(t) &= \varphi(t); t \in [-v, 0] \end{aligned} \quad (55)$$

Then by Theorem 3.1, it is easy to have the following Corollary 3.3.

Corollary 3.3. Suppose that Assumptions 1 and 2 holds, for given scalars $Y_1 = \text{diag}\{\alpha_1^-, \alpha_2^-, \dots, \alpha_n^-\}$, $Y_2 = \text{diag}\{\alpha_1^+, \alpha_2^+, \dots, \alpha_n^+\}$, $Y_3 = \text{diag}\{\beta_1^-, \beta_2^-, \dots, \beta_n^-\}$, $Y_4 = \text{diag}\{\beta_1^+, \beta_2^+, \dots, \beta_n^+\}$, $Y_5 = \text{diag}\{\gamma_1^-, \gamma_2^-, \dots, \gamma_n^-\}$, $Y_6 = \text{diag}\{\gamma_1^+, \gamma_2^+, \dots, \gamma_n^+\}$, $\mu \leq 1$, if there exists symmetric positive

definite matrices K , L_i ($i = 1, 2, 3, 4$), $\tilde{L}_j$ ($j = 1, 2$), M_1, M_2, M_4, M_5, M_6 and $M_3 = \begin{bmatrix} M_{311} & M_{312} & M_{313} \\ * & M_{322} & M_{323} \\ * & * & M_{333} \end{bmatrix}$, positive diagonal matrices $\Lambda_p = \text{diag}\{\lambda_{p1}, \lambda_{p2}, \dots, \lambda_{pn}\}$ ($p = 1, 2, 3$), $\Theta_l = \text{diag}\{\theta_{l1}, \theta_{l2}, \dots, \theta_{ln}\}$ ($l = 1, 2, 3$), N_q ($q = 1, 2, 3$), such that the following LMIs hold :

$$\tilde{G}_k^T K \tilde{G}_k - K \leq 0 \quad (56)$$

Case (i) When $0 \leq \mu(t) \leq \frac{\bar{\mu}}{3}$

$$\Xi_1^* = \begin{bmatrix} \Gamma_{1,1}^* & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & \Gamma_{1,5} & 0 & 0 & 0 & 0 & \Gamma_{1,10} & \Gamma_{1,11} & 0 & 0 \\ * & \Gamma_{2,2} & 0 & \Gamma_{2,4} & \Gamma_{2,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Gamma_{3,3}^* & \Gamma_{3,4} & 0 & 0 & 0 & \Gamma_{3,8} & 0 & \Gamma_{3,10} & 0 & 0 & 0 \\ * & * & * & \Gamma_{4,4} & 0 & 0 & 0 & \Gamma_{4,8} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Gamma_{5,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Gamma_{6,6} & 0 & 0 & \Gamma_{6,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Gamma_{7,7} & 0 & \Gamma_{7,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Gamma_{8,8} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Gamma_{9,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Gamma_{10,10} & \Gamma_{10,11} & \Gamma_{10,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & \Gamma_{11,11} & \Gamma_{11,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Gamma_{12,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & \Gamma_{13,13} \end{bmatrix} < 0, \quad (57)$$

Case (ii) When $\frac{\bar{\mu}}{3} \leq \mu(t) \leq \frac{2}{3}\bar{\mu}$

$$\Xi_2^* = \begin{bmatrix} \Gamma_{1,1}^* & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & \Gamma_{1,5} & 0 & 0 & 0 & 0 & \gamma_{1,10} & \Gamma_{1,11} & 0 & 0 \\ * & \Gamma_{2,2} & 0 & \Gamma_{2,4} & \Gamma_{2,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \gamma_{3,3}^* & \Gamma_{3,4} & 0 & 0 & 0 & \Gamma_{3,8} & 0 & \gamma_{3,10} & \gamma_{3,11} & 0 & 0 \\ * & * & * & \Gamma_{4,4} & 0 & 0 & 0 & \Gamma_{4,8} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Gamma_{5,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Gamma_{6,6} & 0 & 0 & \Gamma_{6,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Gamma_{7,7} & 0 & \Gamma_{7,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Gamma_{8,8} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Gamma_{9,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Gamma_{10,10} & \gamma_{10,11} & \Gamma_{10,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & \Gamma_{11,11} & \Gamma_{11,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Gamma_{12,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & \Gamma_{13,13} \end{bmatrix} < 0, \quad (58)$$

Case (iii) When $\frac{2}{3}\bar{\mu} \leq \mu(t) \leq \bar{\mu}$

$$\Xi_3^* = \begin{bmatrix} \Gamma_{1,1}^* & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & \Gamma_{1,5} & 0 & 0 & 0 & 0 & \delta_{1,10} & \Gamma_{1,11} & 0 & 0 \\ * & \Gamma_{2,2} & 0 & \Gamma_{2,4} & \Gamma_{2,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \delta_{3,3}^* & \Gamma_{3,4} & 0 & 0 & 0 & \Gamma_{3,8} & 0 & \Gamma_{3,10} & \delta_{3,11} & \delta_{3,12} & 0 \\ * & * & * & \Gamma_{4,4} & 0 & 0 & 0 & \Gamma_{4,8} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Gamma_{5,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Gamma_{6,6} & 0 & 0 & \Gamma_{6,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Gamma_{7,7} & 0 & \Gamma_{7,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Gamma_{8,8} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Gamma_{9,9} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Gamma_{10,10} & \Gamma_{10,11} & \Gamma_{10,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & \Gamma_{11,11} & \delta_{11,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Gamma_{12,12} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & \Gamma_{13,13} \end{bmatrix} < 0, \quad (59)$$

where

$$\Gamma_{1,1}^* = 2\zeta K - KA - KA + 4\zeta \Upsilon_1 A \Theta_1 - 4\zeta \Upsilon_2 \Lambda_1 A - A(\Upsilon_2 \Lambda_1 - \Upsilon_1 \Theta_1) - (\Upsilon_2 \Lambda_1 - \Upsilon_1 \Theta_1)A + \eta_1,$$

$$\eta_1 = \sum_{i=1}^4 L_i + \sum_{j=1}^2 \tilde{L}_j + M_1 + M_{311} + \left(\frac{\bar{\mu}}{3}\right)^2 A[M_4 + M_5 + M_6]A - 2\Upsilon_1 N_1 \Upsilon_2 - e^{-2\frac{\bar{\mu}}{3}} M_6,$$

$$\gamma_{3,3}^* = -e^{-2\zeta\bar{\mu}}(1-\mu)L_1 - e^{-2\zeta\bar{\mu}}L_4 - \frac{3}{2}e^{-4\zeta\frac{\bar{\mu}}{3}}M_5, \quad \delta_{3,3}^* = -e^{-2\zeta\bar{\mu}}(1-\mu)L_1 - e^{-2\zeta\bar{\mu}}L_4 - 3M_4,$$

$$\Gamma_{3,3}^* = -e^{-2\zeta\bar{\mu}}(1-\mu)L_1 - e^{-2\zeta\bar{\mu}}L_4 - 3e^{-2\zeta\frac{\bar{\mu}}{3}}M_6 - 2\Upsilon_3 N_2 \Upsilon_4.$$

The remaining values of Ξ_1^* , Ξ_2^* and Ξ_3^* are same as in Theorem 3.1. Then the recurrent neural networks (55) is exponentially stable with the exponential convergence rate ζ .

Proof. Similar to the proof of Theorem 3.1, let $\tilde{\rho}(u(t), u(t - \mu(t)), t) d\tilde{\omega}(t) = 0$, we can easily derive this Corollary. Its proof is straightforward and hence omitted. $\square$

Remark 3.4. In case, the impulsive effect and distributed delay term are absent in system (2), then we get the following neural networks

$$\begin{aligned} du(t) &= \left[-Au(t) + W_0 \tilde{f}(u(t)) + W_1 \tilde{g}(u(t - \mu(t))) \right] dt + \tilde{\rho}(u(t), u(t - \mu(t)), t) d\tilde{\omega}(t); t > 0, \\ u(t) &= \varphi(t); t \in [-v, 0], \end{aligned} \quad (60)$$

Then by Theorem 3.1, it is easy to have the following Corollary.

Corollary 3.5. Under the Assumptions 1 and 2, for given scalars $Y_1 = \text{diag}\{\alpha_1^-, \alpha_2^-, \dots, \alpha_n^-\}$, $Y_2 = \text{diag}\{\alpha_1^+, \alpha_2^+, \dots, \alpha_n^+\}$, $Y_3 = \text{diag}\{\beta_1^-, \beta_2^-, \dots, \beta_n^-\}$, $Y_4 = \text{diag}\{\beta_1^+, \beta_2^+, \dots, \beta_n^+\}$, $Y_5 = \text{diag}\{\gamma_1^-, \gamma_2^-, \dots, \gamma_n^-\}$, $Y_6 = \text{diag}\{\gamma_1^+, \gamma_2^+, \dots, \gamma_n^+\}$, $\mu \leq 1$, if there exists symmetric positive definite

matrices K , L_i ($i = 1, 2, 3, 4$), $\tilde{L}_j$ ($j = 1, 2$), M_1, M_2, M_4, M_5, M_6 and $M_3 = \begin{bmatrix} M_{311} & M_{312} & M_{313} \\ * & M_{322} & M_{323} \\ * & * & M_{333} \end{bmatrix}$, positive diagonal matrices $\Lambda_p = \text{diag}\{\lambda_{p1}, \lambda_{p2}, \dots, \lambda_{pn}\}$ ($p = 1, 2, 3$), $\Theta_l = \text{diag}\{\theta_{l1}, \theta_{l2}, \dots, \theta_{ln}\}$ ($l = 1, 2, 3$), N_q ($q = 1, 2, 3$), such that the following LMIs hold :

$$K \leq \delta I, \quad (61)$$

Case (i) When $0 \leq \mu(t) \leq \frac{\bar{\mu}}{3}$

$$\Xi_1 = \begin{bmatrix} \Gamma_{1,1} & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & 0 & \Omega_{1,6}^* & \Omega_{1,7} & 0 \\ * & \Gamma_{2,2} & 0 & \Gamma_{2,4} & 0 & 0 & 0 & 0 \\ * & * & \Gamma_{3,3} & \Gamma_{3,4} & \Omega_{3,5} & \Omega_{3,6} & 0 & 0 \\ * & * & * & \Gamma_{4,4} & \Omega_{4,5} & 0 & 0 & 0 \\ * & * & * & * & \Omega_{5,5} & 0 & 0 & 0 \\ * & * & * & * & * & \Omega_{6,6} & \Omega_{6,7} & \Omega_{6,8} \\ * & * & * & * & * & * & \Omega_{7,7} & \Omega_{7,8} \\ * & * & * & * & * & * & * & \Omega_{8,8} \end{bmatrix} < 0, \quad (62)$$

Case (ii) When $\frac{\bar{\mu}}{3} \leq \mu(t) \leq \frac{2}{3}\bar{\mu}$

$$\Xi_2 = \begin{bmatrix} \Gamma_{1,1} & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & 0 & \gamma_{1,6}^* & \Omega_{1,7} & 0 \\ * & \Gamma_{2,2} & 0 & \Gamma_{2,4} & 0 & 0 & 0 & 0 \\ * & * & \gamma_{3,3} & \Gamma_{3,4} & \Omega_{3,5} & \gamma_{3,6}^* & \gamma_{3,7}^* & 0 \\ * & * & * & \Gamma_{4,4} & \Omega_{4,5} & 0 & 0 & 0 \\ * & * & * & * & \Omega_{5,5} & 0 & 0 & 0 \\ * & * & * & * & * & \Omega_{6,6} & \gamma_{6,7}^* & \Omega_{6,8} \\ * & * & * & * & * & * & \Omega_{7,7} & \Omega_{7,8} \\ * & * & * & * & * & * & * & \Omega_{8,8} \end{bmatrix} < 0, \quad (63)$$

Case (iii) When $\frac{2}{3}\bar{\mu} \leq \mu(t) \leq \bar{\mu}$

$$\Xi_3 = \begin{bmatrix} \Gamma_{1,1} & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & 0 & \delta_{1,6}^* & \Omega_{1,7} & 0 \\ * & \Gamma_{2,2} & 0 & \Gamma_{2,4} & 0 & 0 & 0 & 0 \\ * & * & \delta_{3,3} & \Gamma_{3,4} & \Omega_{3,5} & \Omega_{3,6} & \delta_{3,7}^* & \delta_{3,8}^* \\ * & * & * & \Gamma_{4,4} & \Omega_{4,5} & 0 & 0 & 0 \\ * & * & * & * & \Omega_{5,5} & 0 & 0 & 0 \\ * & * & * & * & * & \Omega_{6,6} & \Omega_{6,7} & \Omega_{6,8} \\ * & * & * & * & * & * & \Omega_{7,7} & \delta_{7,8}^* \\ * & * & * & * & * & * & * & \Omega_{8,8} \end{bmatrix} < 0, \quad (64)$$

where

$$\Gamma_{1,1} = 2\zeta K - KA - KA + \delta S_1 + 4\zeta Y_1 A \Theta_1 - 4\zeta Y_2 \Lambda_1 A - A(Y_2 \Lambda_1 - Y_1 \Theta_1) - (Y_2 \Lambda_1 - Y_1 \Theta_1)A + \eta_1$$

$$\eta_1 = \sum_{i=1}^4 L_i + \sum_{j=1}^2 \tilde{L}_j + M_1 + M_{311} + \left(\frac{\mu}{3}\right)^2 A[M_4 + M_5 + M_6]A - 2Y_1 N_1 Y_2 - e^{-2\frac{\bar{\mu}}{3}} M_6,$$

$$\Gamma_{1,2} = KW_0 - 2(A - W_0 Y_1) \Theta_1 + 2\zeta(Y_2 W_0 + A) \Lambda_1 + (\Theta_1 - \Lambda_1) + (Y_2 \Lambda_1 - Y_1 \Theta_1)W_0 + \eta_2,$$

$$\eta_2 = -AW_0 \left(\sum_{j=1}^2 \tilde{L}_j \right) - \left(\frac{\bar{\mu}}{3} \right)^2 [M_4 + M_5 + M_6]AW_0 + N_1(Y_1 + Y_2), \quad \Gamma_{1,3} = \frac{3}{2}e^{-2\frac{\bar{\mu}}{3}} M_6,$$

$$\Gamma_{1,4} = KW_1 - 2\zeta Y_1 W_1 + 2\zeta Y_2 W_1 \Lambda_1 + (Y_2 \Lambda_1 - Y_1 \Theta_1)W_1 - AW_1 \left(\sum_{j=1}^2 \tilde{L}_j \right) - \left(\frac{\bar{\mu}}{3} \right)^2 [M_4 + M_5 + M_6]AW_1,$$

$$\Omega_{1,6} = M_{312}, \quad \Omega_{6,7} = M_{323} - e^{-2\zeta\frac{\bar{\mu}}{3}} M_{312} + e^{-4\zeta\frac{\bar{\mu}}{3}} M_5, \quad \Omega_{6,8} = -e^{-2\zeta\frac{\bar{\mu}}{3}} M_{313}, \quad \Gamma_{3,4} = N_2(Y_3 + Y_4),$$

$$\Gamma_{2,2} = 4\zeta \Theta_1 W_0 - 4\zeta \Lambda_1 W_0 + \eta_3 + 2(\Theta_1 - \Lambda_1)W_0 + W_0 \left(\sum_{j=1}^2 \tilde{L}_j \right) W_0 + \left(\frac{\bar{\mu}}{3} \right)^2 W_0(M_4 + M_5 + M_6)W_0 - 2N_1,$$

$$\delta_{1,6}^* = M_{312} + e^{-2\zeta\frac{\bar{\mu}}{3}} M_6, \quad \delta_{3,7}^* = e^{-2\zeta\bar{\mu}} M_4, \quad \delta_{7,8}^* = -e^{-2\zeta\frac{\bar{\mu}}{3}} M_{323}, \quad \gamma_{1,6}^* = M_{312} + e^{-2\zeta\frac{\bar{\mu}}{3}} M_6,$$

$$\Gamma_{3,3} = \delta S_2 - e^{-2\zeta\bar{\mu}}(1-\mu)L_1 - e^{-2\zeta\bar{\mu}}L_4 - 3e^{-2\zeta\frac{\bar{\mu}}{3}}M_6 - 2\Upsilon_3N_2\Upsilon_4, \quad \gamma_{6,7}^* = M_{3,23} - e^{-2\zeta\frac{\bar{\mu}}{3}}M_{3,12},$$

$$\Omega_{5,5} = -e^{-2\zeta\bar{\mu}}\tilde{L}_1(1-\mu), \quad \Omega_{6,6} = -e^{-2\zeta\frac{\bar{\mu}}{3}}M_6 - e^{-2\zeta\frac{2}{3}\bar{\mu}}M_5 + \eta_4, \quad \Gamma_{4,4} = -2N_2, \quad \Omega_{1,7} = M_{3,13},$$

$$\eta_4 = M_{3,22} - e^{-2\zeta\frac{\bar{\mu}}{3}}M_{3,11} - e^{-2\zeta\frac{\bar{\mu}}{3}}L_2, \quad \Omega_{8,8} = -e^{-2\zeta\bar{\mu}}M_4 - e^{-2\zeta\frac{\bar{\mu}}{3}}M_{3,33} - e^{-2\zeta\bar{\mu}}L_4,$$

$$\Gamma_{2,4} = 2\zeta\Theta_1W_1 - 2\zeta\Lambda_1W_1 + (\Theta_1 - \Lambda_1)W_1 + W_0\left(\frac{\bar{\mu}}{3}\right)^2[M_4 + M_5 + M_6]W_1, \quad \Omega_{3,6} = e^{-2\zeta\frac{\bar{\mu}}{3}}M_6,$$

$$\Omega_{7,7} = -e^{-4\zeta\frac{\bar{\mu}}{3}}L_3 + M_{3,33} - e^{-2\zeta\frac{\bar{\mu}}{3}}M_{3,22} - e^{-2\zeta\bar{\mu}}M_4 - e^{-4\zeta\frac{\bar{\mu}}{3}}M_5, \quad \Omega_{7,8} = -e^{-2\zeta\frac{\bar{\mu}}{3}}M_{3,23} + e^{-2\zeta\bar{\mu}}M_4,$$

$$\Omega_{3,5} = -2\zeta\Upsilon_3\Theta_2 + 2\zeta\Upsilon_4 + (\Upsilon_4\Lambda_2 - \Upsilon_3\Theta_2), \quad \Omega_{4,5} = 2\zeta\Theta_2 - 2\zeta\Lambda_2 + (\Theta_2 - \Lambda_2), \quad \delta_{3,8}^* = e^{-2\zeta\bar{\mu}}M_4.$$

Then the system (60) is exponentially stable in the mean square sense with the exponential convergence rate index ζ .

Remark 3.6. In this paper, the discrete delay interval $[0, b]$ is divided into $[0, \frac{b}{3}]$, $[\frac{b}{3}, \frac{2b}{3}]$ and $[\frac{2b}{3}, b]$. For a sequence, each segments handled in different Lyapunov–Krasovskii functionals to obtain variety of LMIs, which yields the novelty of this paper.

Remark 3.7. The method enhanced in this paper, exposed the relationship between time-varying delays and their upper bounds. In addition, some inequality techniques and fractional segments in LKF, are used to obtain the maximum upper bounds for discrete time-varying delays, see Examples 4.1 and 4.2. And also the state responses of (60) are depicted in Figs. 7 and 8. Hence the approaches and methods used in our proposed work may reduced the conservative criterions.

Remark 3.8. In general, the delayed fractional order differential equations are used to explicit the absolute value of rate of change for any dynamical systems at a fractional time variations. So, we can calculate the rate of change of dynamical systems in any fractional situations and also the delays are analyzed in an accurate manner. Similarly in this paper, the delay intervals are divided into fractional segments which leads the exact upper bound for time delay in each fractional parts. This shows that the advantages of our designed work consisting of fractional delay segments.

Remark 3.9. In neural networks for time-delay systems Lyapunov–Krasovskii functionals and Lyapunov–Razumikhin are the two Lypunov methods which results in linear matrix inequality conditions. Additionally, Halanay's inequality that extends the Razumikhin method to the exponential stability which can also lead delay independent conditions for systems with fast varying delays. Therefore, in our proposed method, the parameters of designed neural networks are considered via Lyapunov–Krasovskii functional and also with the help of bounds lemma the novel based LMIs are formulated. By using the MATLAB software, the upper bounds are calculated and reached a very large value when compared with the previous available literatures.

4. Numerical examples with comparisons

In this section, providing two numerical examples with their simulations to testify the superiority and advantage of the above findings.

Example 4.1. Consider a two-neuron time-delayed recurrent neural networks (2) with stochastic disturbances, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1.5 \end{pmatrix}, \quad W_0 = \begin{pmatrix} -1 & 0.5 \\ 0.5 & -1 \end{pmatrix}, \quad W_1 = \begin{pmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix}, \quad W_2 = \begin{pmatrix} -1.5 & 0.3 \\ 0.3 & -1.5 \end{pmatrix},$$

$$\Upsilon_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Upsilon_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Upsilon_3 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \quad \Upsilon_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Upsilon_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Upsilon_6 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0.3 & 0.7 \\ 0.7 & 0.4 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0.2 & 0.8 \\ 0.5 & 0.3 \end{pmatrix},$$

$$\bar{G}_k = \begin{pmatrix} 0.3 & 0.5 \\ 0.7 & 0.2 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\tilde{\rho}(u(t), u(t - \mu(t)), t) = \begin{pmatrix} 0.3 * u(t) & 0.2 * u(t - \mu(t)) \\ 0.1 * u(t - \mu(t)) & 0.4 * (u(t) + u(t - \mu(t))) \end{pmatrix}$$

$$\sigma = 0.2, \quad \mu = 0.3, \quad \bar{\mu} = 1.5, \quad \zeta = 1, \quad \mu(t) = 0.05 * \text{cost} + 0.08.$$

The following activation functions in neural networks (2) is expressed by

$$\tilde{f}(u(.)) = \tilde{g}(u(.)) = \tilde{h}(u(.)) = 0.01 * \tan(.).$$

By utilizing the MATLAB LMI control toolbox, we solve the LMIs (13)–(17) in Theorem 3.1, one can obtain the following feasible solutions

Case (i) When $0 \leq \mu(t) \leq \frac{\bar{\mu}}{3}$

$$K = \begin{pmatrix} 297.4327 & 43.1135 \\ 43.1135 & 248.2842 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 129.6604 & 12.5487 \\ 12.5487 & 145.4912 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 241.2601 & -15.2993 \\ -15.2993 & 239.7575 \end{pmatrix}.$$

$$\begin{aligned}
L_3 &= \begin{pmatrix} 195.2024 & -1.1016 \\ -1.1016 & 200.5533 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 149.0607 & 11.9117 \\ 11.9117 & 161.2766 \end{pmatrix}, \quad \tilde{L}_1 = \begin{pmatrix} 126.2069 & -2.8498 \\ -2.8498 & 183.4237 \end{pmatrix}, \\
\tilde{L}_2 &= \begin{pmatrix} 101.3926 & -2.4908 \\ -2.4908 & 178.8969 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 133.6347 & 6.6002 \\ 6.6002 & 144.7483 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 166.6933 & -52.3475 \\ -52.3475 & 296.2633 \end{pmatrix}, \\
M_{3_{11}} &= \begin{pmatrix} 241.2601 & -15.2993 \\ -15.2993 & 239.7575 \end{pmatrix}, \quad M_{3_{12}} = \begin{pmatrix} 117.9187 & 3.3894 \\ 3.3894 & 123.0750 \end{pmatrix}, \quad M_{3_{13}} = \begin{pmatrix} 64.7960 & 7.6215 \\ 7.6215 & 74.7864 \end{pmatrix}, \\
M_{3_{22}} &= \begin{pmatrix} 163.0136 & 18.5317 \\ 18.5317 & 175.6056 \end{pmatrix}, \quad M_{3_{23}} = \begin{pmatrix} 58.3407 & 8.2808 \\ 8.2808 & 68.0220 \end{pmatrix}, \quad M_{3_{33}} = \begin{pmatrix} 65.4190 & 10.7266 \\ 10.7266 & 79.0804 \end{pmatrix}, \\
M_4 &= \begin{pmatrix} 188.2816 & 76.8249 \\ 76.8249 & 316.1668 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 192.3344 & 71.5522 \\ 71.5522 & 316.0494 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 217.7163 & 145.2992 \\ 145.2992 & 280.8532 \end{pmatrix}, \\
N_1 &= \begin{pmatrix} 498.1179 & 0 \\ 0 & 631.9291 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 23.4118 & 0 \\ 0 & 19.8698 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 70.8987 & 0 \\ 0 & 69.7910 \end{pmatrix}, \\
\Theta_1 &= \begin{pmatrix} 188.3165 & 0 \\ 0 & 360.3329 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 0.1429 & 0 \\ 0 & 0.1429 \end{pmatrix}, \quad \Theta_3 = \begin{pmatrix} 0.2222 & 0 \\ 0 & 0.2222 \end{pmatrix}, \\
\Lambda_1 &= \begin{pmatrix} 195.9231 & 0 \\ 0 & 339.1446 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0.1429 & 0 \\ 0 & 0.1429 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0.2222 & 0 \\ 0 & 0.2222 \end{pmatrix},
\end{aligned}$$

$$\delta = 457.8393.$$

Case (ii) When $\frac{\bar{\mu}}{3} \leq \mu(t) \leq \frac{2}{3}\bar{\mu}$

$$\begin{aligned}
K &= \begin{pmatrix} 184.3284 & 2.1622 \\ 2.1622 & 175.5190 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 195.5362 & -30.6835 \\ -30.6835 & 232.2150 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 360.1819 & -113.5741 \\ -113.5741 & 395.5414 \end{pmatrix}, \\
L_3 &= \begin{pmatrix} 273.4868 & -97.8512 \\ -97.8512 & 310.3006 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 249.6840 & -38.0942 \\ -38.0942 & 275.7278 \end{pmatrix}, \quad \tilde{L}_1 = \begin{pmatrix} 176.5499 & -47.0087 \\ -47.0087 & 320.3038 \end{pmatrix}, \\
\tilde{L}_2 &= \begin{pmatrix} 158.3050 & -38.5343 \\ -38.5343 & 314.5785 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 182.6822 & -30.6795 \\ -30.6795 & 213.2553 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 310.6333 & -76.8329 \\ -76.8329 & 534.4909 \end{pmatrix}, \\
M_{3_{11}} &= \begin{pmatrix} 360.1819 & -113.5741 \\ -113.5741 & 395.5414 \end{pmatrix}, \quad M_{3_{12}} = \begin{pmatrix} 170.1319 & -2.5641 \\ -2.5641 & 182.3840 \end{pmatrix}, \quad M_{3_{13}} = \begin{pmatrix} 89.8729 & -4.5600 \\ -4.5600 & 103.2997 \end{pmatrix}, \\
M_{3_{22}} &= \begin{pmatrix} 200.9706 & -49.5120 \\ -49.5120 & 226.3328 \end{pmatrix}, \quad M_{3_{23}} = \begin{pmatrix} 75.9153 & -11.5321 \\ -11.5321 & 89.3217 \end{pmatrix}, \quad M_{3_{33}} = \begin{pmatrix} 93.4015 & 4.7941 \\ 4.7941 & 119.6738 \end{pmatrix}, \\
M_4 &= \begin{pmatrix} 188.9308 & 79.1984 \\ 79.1984 & 386.3980 \end{pmatrix}, \quad M_5 = 10^3 \times \begin{pmatrix} 0.8680 & 0.9300 \\ 0.9300 & 1.5139 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 101.6559 & -1.4483 \\ -1.4483 & 132.7352 \end{pmatrix}, \\
N_1 &= 10^3 \times \begin{pmatrix} 0.9489 & 0 \\ 0 & 1.0973 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 7.8172 & 0 \\ 0 & 12.8242 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 110.7583 & 0 \\ 0 & 107.3554 \end{pmatrix}, \\
\Theta_1 &= \begin{pmatrix} 302.3383 & 0 \\ 0 & 718.0815 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 0.1429 & 0 \\ 0 & 0.1429 \end{pmatrix}, \quad \Theta_3 = \begin{pmatrix} 0.2222 & 0 \\ 0 & 0.2222 \end{pmatrix}, \\
\Lambda_1 &= \begin{pmatrix} 312.4878 & 0 \\ 0 & 695.7133 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0.1429 & 0 \\ 0 & 0.1429 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0.2222 & 0 \\ 0 & 0.2222 \end{pmatrix},
\end{aligned}$$

$$\delta = 252.3612.$$

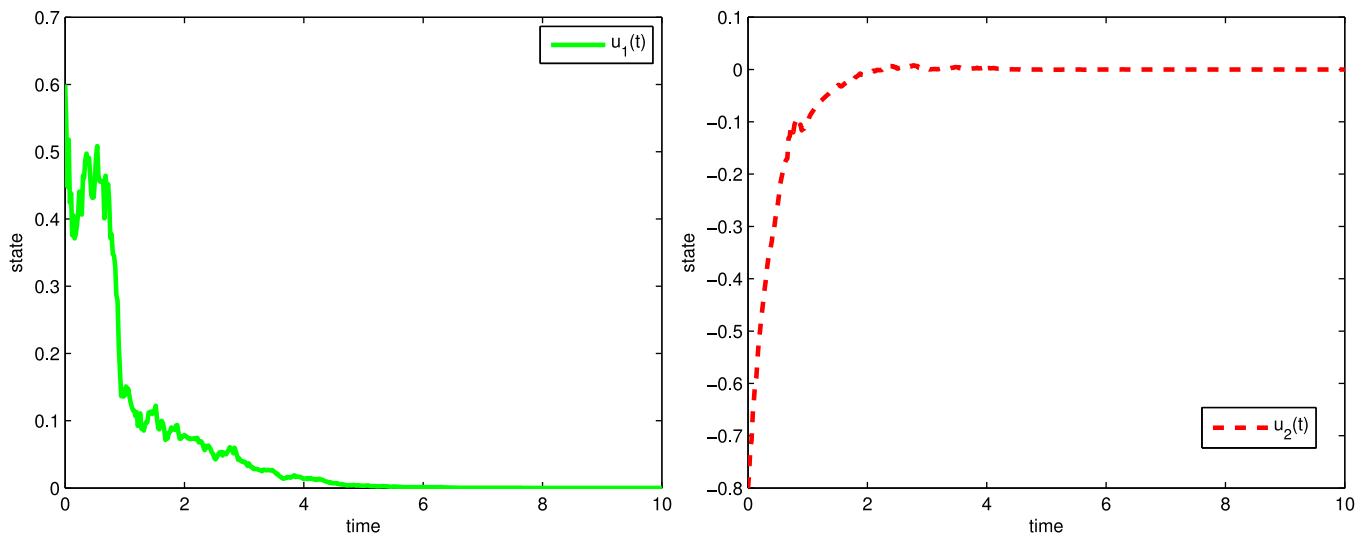


Fig. 1. The state responses $u_1(t)$ and $u_2(t)$ of (2) with stochastic noises.

Case (iii) When $\frac{2}{3}\bar{\mu} \leq \mu(t) \leq \bar{\mu}$

$$K = \begin{pmatrix} 286.1838 & 35.9863 \\ 35.9863 & 271.6308 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 84.8003 & 13.5585 \\ 13.5585 & 132.1982 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 160.6892 & 8.0090 \\ 8.0090 & 206.1634 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 133.9874 & 13.6528 \\ 13.6528 & 200.7249 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 100.0023 & 13.7172 \\ 13.7172 & 152.7700 \end{pmatrix}, \quad \tilde{L}_1 = \begin{pmatrix} 58.8833 & 1.5770 \\ 1.5770 & 50.3468 \end{pmatrix},$$

$$\tilde{L}_2 = \begin{pmatrix} 47.1206 & 3.7692 \\ 3.7692 & 50.6134 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 77.6690 & 5.7908 \\ 5.7908 & 99.0707 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 72.4023 & -13.2490 \\ -13.2490 & 36.1766 \end{pmatrix},$$

$$M_{3_{11}} = \begin{pmatrix} 160.6892 & 8.0090 \\ 8.0090 & 206.1634 \end{pmatrix}, \quad M_{3_{12}} = \begin{pmatrix} 56.2838 & 1.7891 \\ 1.7891 & 64.0922 \end{pmatrix}, \quad M_{3_{13}} = \begin{pmatrix} 33.5049 & 3.0056 \\ 3.0056 & 43.3442 \end{pmatrix},$$

$$M_{3_{22}} = \begin{pmatrix} 89.7724 & 7.9031 \\ 7.9031 & 122.0923 \end{pmatrix}, \quad M_{3_{23}} = \begin{pmatrix} 30.4692 & 3.9849 \\ 3.9849 & 41.6014 \end{pmatrix}, \quad M_{3_{33}} = \begin{pmatrix} 35.4952 & 5.5363 \\ 5.5363 & 53.0075 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 60.4425 & 27.1723 \\ 27.1723 & 79.0333 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 68.0277 & 25.2940 \\ 25.2940 & 118.5780 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 43.9607 & 7.3005 \\ 7.3005 & 59.8167 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} 170.2128 & 0 \\ 0 & 110.9905 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 24.5334 & 0 \\ 0 & 18.6373 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 31.2106 & 0 \\ 0 & 29.9248 \end{pmatrix},$$

$$\Theta_1 = \begin{pmatrix} 117.7489 & 0 \\ 0 & 92.4943 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 0.4000 & 0 \\ 0 & 0.2500 \end{pmatrix}, \quad \Theta_3 = \begin{pmatrix} 0.2222 & 0 \\ 0 & 0.2222 \end{pmatrix},$$

$$\Lambda_1 = \begin{pmatrix} 121.6330 & 0 \\ 0 & 85.0721 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0.4000 & 0 \\ 0 & 0.2500 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0.2222 & 0 \\ 0 & 0.2222 \end{pmatrix},$$

$$\delta = 423.0894.$$

Therefore, via MATLAB LMI control toolbox, the state trajectories of proposed neural networks (2) are obtained, which are illustrated in Figs. 1,2 and also the impulsive effects of NNs (2) is shown in Figs. 3 and 4. Furthermore, by solving the LMIs (13)–(17) in Theorem 3.1 the feasible solutions can be easily obtained. From the above three cases, the achieved discrete time-delay upper

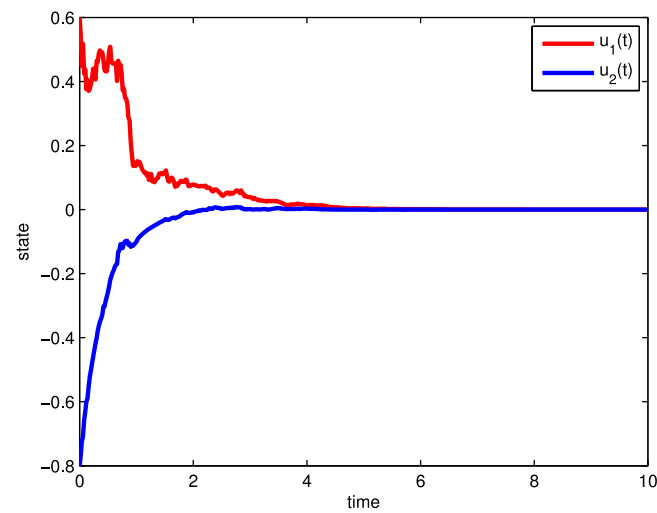


Fig. 2. State trajectories of concerned neural networks (2) with stochastic noises.

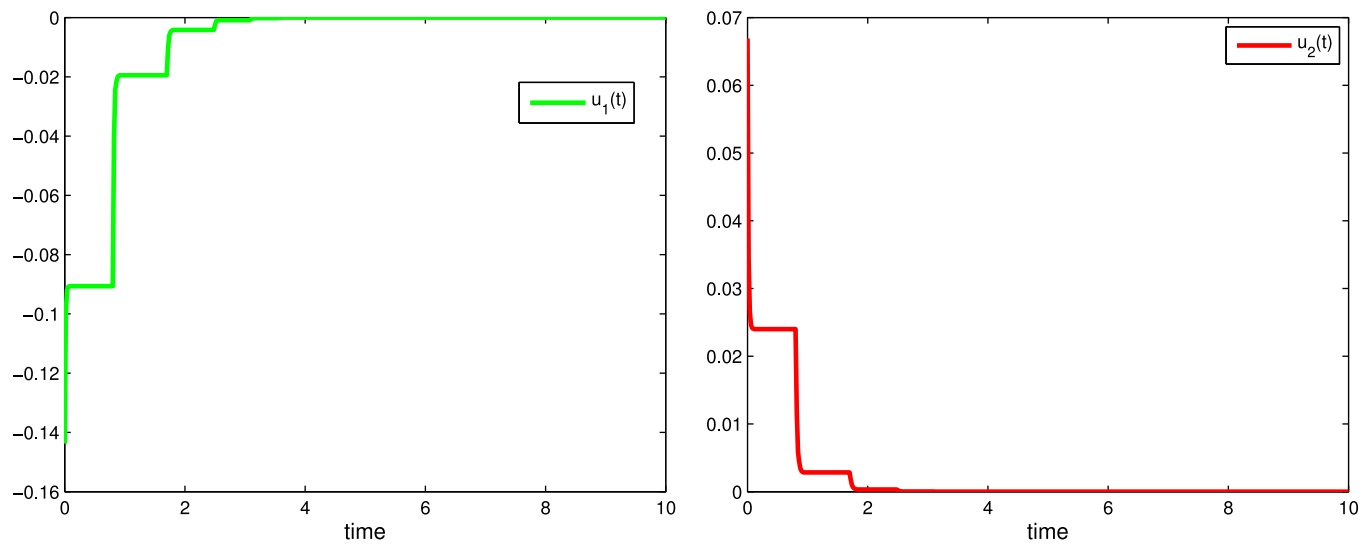


Fig. 3. Impulsive effects of $u_1(t)$ and $u_2(t)$ for the system (2).

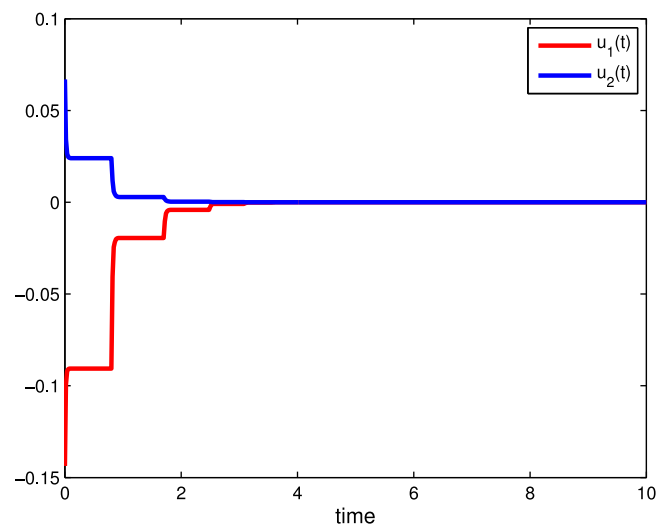


Fig. 4. Impulsive effects of concerned neural networks (2).

bounds $\bar{\mu}$ of neural networks (2) are very large, which are listed in Tables 1–3. According to Theorem 3.1, we can conclude that the recurrent neural networks (2) is exponentially stable in the mean square sense for the maximum allowable bounds $\bar{\mu} = 26.0467$. Hence, this experiment demonstrate the contributions of this paper is more extensive and less conservative than some available literatures.

Example 4.2. Consider a two-neuron time-delayed recurrent neural networks (55) with stochastic disturbances, where

$$A = \begin{pmatrix} 2.5 & 0 \\ 0 & 3 \end{pmatrix}, \quad W_0 = \begin{pmatrix} -0.5 & 0.4 \\ 0.3 & -1.2 \end{pmatrix}, \quad W_1 = \begin{pmatrix} -0.8 & 0.2 \\ -0.6 & 1 \end{pmatrix}, \quad W_2 = \begin{pmatrix} -0.9 & 0.7 \\ 0.1 & -0.6 \end{pmatrix},$$

$$\Upsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1.5 \end{pmatrix}, \quad \Upsilon_2 = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad \Upsilon_3 = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \quad \Upsilon_4 = \begin{pmatrix} 1.2 & 0 \\ 0 & 0.7 \end{pmatrix},$$

$$\Upsilon_5 = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad \Upsilon_6 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0.2 & 0.4 \\ 0.6 & 0.2 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.1 \end{pmatrix},$$

$$\bar{G}_k = \begin{pmatrix} 0.1 & 0.6 \\ 0.3 & 0.3 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\sigma = 0.2, \quad \mu = 0.3, \quad \bar{\mu} = 1.5, \quad \varsigma = 1, \quad \mu(t) = 0.05 * cost + 0.08.$$

The following activation functions in neural networks (55) is expressed by

$$\tilde{f}(u(.)) = \tilde{g}(u(.)) = \tilde{h}(u(.)) = 0.3 * \sin(.).$$

By utilizing the MATLAB LMI control toolbox, we solve the LMIs (56)–(59) in Corollary 3.3, one can obtain the following feasible solutions

Case (i) When $0 \leq \mu(t) \leq \frac{\bar{\mu}}{3}$

$$K = \begin{pmatrix} 93.5135 & 19.2460 \\ 19.2460 & 103.9725 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 26.8410 & 4.8582 \\ 4.8582 & 26.5179 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 53.9558 & 6.3805 \\ 6.3805 & 55.2295 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 46.0929 & 8.1690 \\ 8.1690 & 46.3342 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 32.3370 & 5.6483 \\ 5.6483 & 32.4489 \end{pmatrix}, \quad \tilde{L}_1 = \begin{pmatrix} 10.6419 & 2.8659 \\ 2.8659 & 8.9497 \end{pmatrix},$$

$$\tilde{L}_2 = \begin{pmatrix} 10.4774 & 2.7760 \\ 2.7760 & 10.2738 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 25.6914 & 2.4699 \\ 2.4699 & 24.0643 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 22.8331 & -5.6401 \\ -5.6401 & 21.7681 \end{pmatrix},$$

$$M_{3_{11}} = \begin{pmatrix} 53.9558 & 6.3805 \\ 6.3805 & 55.2295 \end{pmatrix}, \quad M_{3_{12}} = \begin{pmatrix} 18.2685 & 1.2872 \\ 1.2872 & 17.8207 \end{pmatrix}, \quad M_{3_{13}} = \begin{pmatrix} 10.0242 & 1.0701 \\ 1.0701 & 9.3571 \end{pmatrix},$$

$$M_{3_{22}} = \begin{pmatrix} 27.2997 & 3.7845 \\ 3.7845 & 26.7441 \end{pmatrix}, \quad M_{3_{23}} = \begin{pmatrix} 8.7194 & 1.0519 \\ 1.0519 & 7.9528 \end{pmatrix}, \quad M_{3_{33}} = \begin{pmatrix} 9.3444 & 1.5642 \\ 1.5642 & 8.7334 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 9.8921 & 3.1483 \\ 3.1483 & 7.3991 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 9.9322 & 3.1432 \\ 3.1432 & 7.4134 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 8.4759 & 2.3627 \\ 2.3627 & 6.8976 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} 25.5791 & 0 \\ 0 & 16.6159 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 5.2974 & 0 \\ 0 & 3.5579 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 9.2733 & 0 \\ 0 & 9.1459 \end{pmatrix},$$

$$\Theta_1 = \begin{pmatrix} 16.6177 & 0 \\ 0 & 3.6253 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 0.4500 & 0 \\ 0 & 0.1037 \end{pmatrix}, \quad \Theta_3 = \begin{pmatrix} 0.3529 & 0 \\ 0 & 0.6857 \end{pmatrix},$$

$$\Lambda_1 = \begin{pmatrix} 19.0729 & 0 \\ 0 & 3.7997 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0.4500 & 0 \\ 0 & 0.1037 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0.3529 & 0 \\ 0 & 0.6857 \end{pmatrix},$$

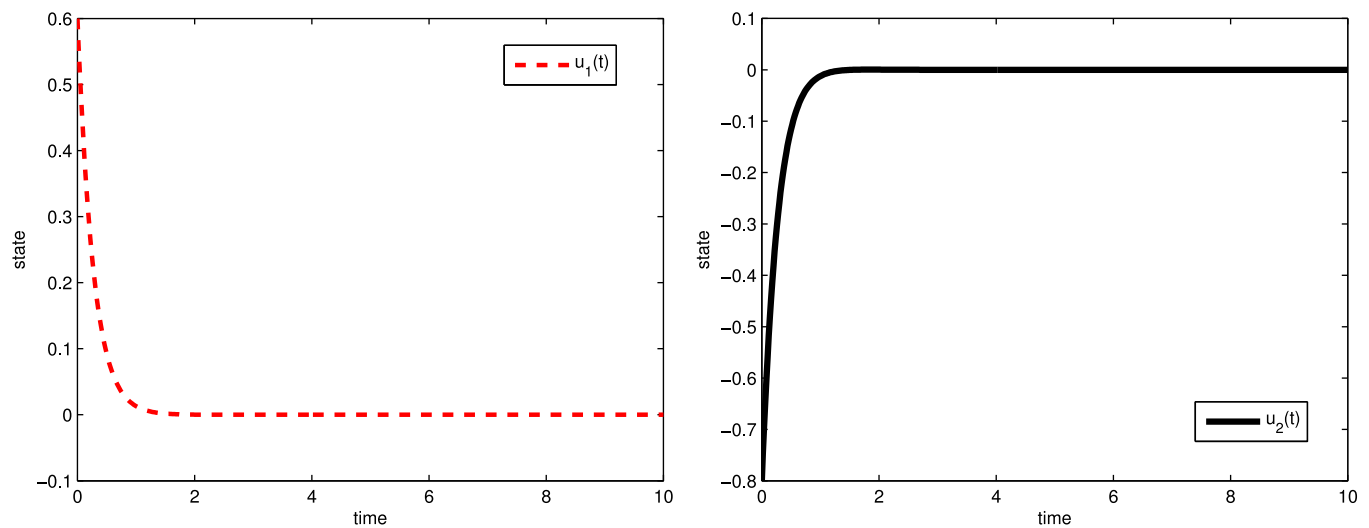


Fig. 5. The trajectories $u_1(t)$ and $u_2(t)$ of (55) without stochastic noises.

Case (ii) When $\frac{\bar{\mu}}{3} \leq \mu(t) \leq \frac{2}{3}\bar{\mu}$

$$\begin{aligned}
 K &= 10^3 \times \begin{pmatrix} 1.9530 & 0.3595 \\ 0.3595 & 1.7720 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 525.0336 & 154.0803 \\ 154.0803 & 657.0035 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 808.2075 & 179.6285 \\ 179.6285 & 916.2269 \end{pmatrix}, \\
 L_3 &= \begin{pmatrix} 685.8293 & 218.8894 \\ 218.8894 & 794.3283 \end{pmatrix}, \quad L_4 = 10^3 \times \begin{pmatrix} 0.8122 & 0.1932 \\ 0.1932 & 1.1322 \end{pmatrix}, \quad \tilde{L}_1 = \begin{pmatrix} 207.5894 & 51.8928 \\ 51.8928 & 169.6937 \end{pmatrix}, \\
 \tilde{L}_2 &= \begin{pmatrix} 201.4030 & 49.7873 \\ 49.7873 & 186.6060 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 378.3844 & 62.7872 \\ 62.7872 & 383.7466 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 446.4879 & -128.4703 \\ -128.4703 & 389.8791 \end{pmatrix}, \\
 M_{3_{11}} &= \begin{pmatrix} 808.2075 & 179.6285 \\ 179.6285 & 916.2269 \end{pmatrix}, \quad M_{3_{12}} = \begin{pmatrix} 263.3187 & 25.6429 \\ 25.6429 & 275.4882 \end{pmatrix}, \quad M_{3_{13}} = \begin{pmatrix} 149.9989 & 19.4029 \\ 19.4029 & 156.8935 \end{pmatrix}, \\
 M_{3_{22}} &= \begin{pmatrix} 394.4401 & 87.5801 \\ 87.5801 & 427.9593 \end{pmatrix}, \quad M_{3_{23}} = \begin{pmatrix} 131.4958 & 22.4961 \\ 22.4961 & 138.5714 \end{pmatrix}, \quad M_{3_{33}} = \begin{pmatrix} 132.4279 & 31.6712 \\ 31.6712 & 135.2703 \end{pmatrix}, \\
 M_4 &= \begin{pmatrix} 117.4310 & 37.9103 \\ 37.9103 & 82.8754 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 212.6124 & 35.9029 \\ 35.9029 & 128.2797 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 108.7480 & 21.0485 \\ 21.0485 & 75.0654 \end{pmatrix}, \\
 N_1 &= \begin{pmatrix} 289.7910 & 0 \\ 0 & 298.7472 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 28.1988 & 0 \\ 0 & 12.7275 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 143.3759 & 0 \\ 0 & 140.4321 \end{pmatrix}, \\
 \Theta_1 &= \begin{pmatrix} 190.6153 & 0 \\ 0 & 47.6090 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 0.4500 & 0 \\ 0 & 0.1037 \end{pmatrix}, \quad \Theta_3 = \begin{pmatrix} 0.3529 & 0 \\ 0 & 0.6857 \end{pmatrix}, \\
 \Lambda_1 &= \begin{pmatrix} 140.3728 & 0 \\ 0 & 55.7249 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0.4500 & 0 \\ 0 & 0.1037 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0.3529 & 0 \\ 0 & 0.6857 \end{pmatrix},
 \end{aligned}$$

Case (iii) When $\frac{2}{3}\bar{\mu} \leq \mu(t) \leq \bar{\mu}$

$$K = \begin{pmatrix} 117.6003 & 11.7177 \\ 11.7177 & 127.0201 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 47.7436 & 8.1934 \\ 8.1934 & 48.7013 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 91.4727 & 9.7156 \\ 9.7156 & 94.6644 \end{pmatrix},$$

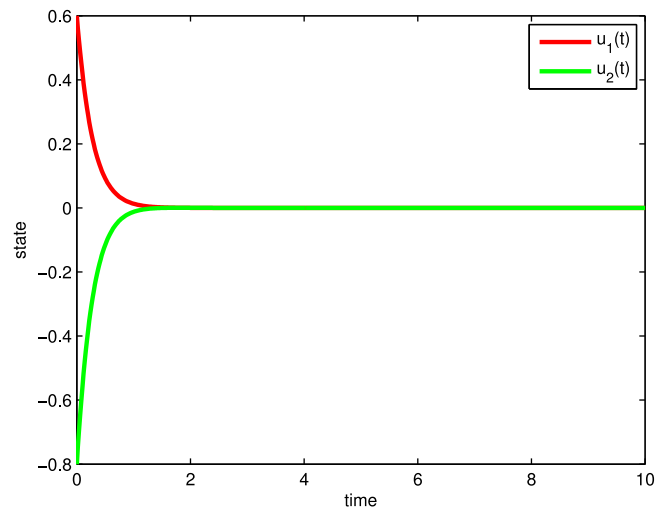


Fig. 6. State responses of delayed neural networks (55) without stochastic noises.

$$L_3 = \begin{pmatrix} 84.0982 & 14.0082 \\ 14.0082 & 87.2643 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 58.8446 & 9.8299 \\ 9.8299 & 60.5210 \end{pmatrix}, \quad \tilde{L}_1 = \begin{pmatrix} 12.7105 & 3.6301 \\ 3.6301 & 8.8938 \end{pmatrix},$$

$$\tilde{L}_2 = \begin{pmatrix} 12.3936 & 3.8481 \\ 3.8481 & 9.4193 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 41.2564 & 3.3948 \\ 3.3948 & 39.5715 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 34.0296 & -8.8205 \\ -8.8205 & 28.9844 \end{pmatrix},$$

$$M_{3_{11}} = \begin{pmatrix} 91.4727 & 9.7156 \\ 9.7156 & 94.6644 \end{pmatrix}, \quad M_{3_{12}} = \begin{pmatrix} 26.6132 & 1.0716 \\ 1.0716 & 26.7316 \end{pmatrix}, \quad M_{3_{13}} = \begin{pmatrix} 15.6119 & 1.2809 \\ 1.2809 & 15.4248 \end{pmatrix},$$

$$M_{3_{22}} = \begin{pmatrix} 45.6132 & 5.3903 \\ 5.3903 & 46.0531 \end{pmatrix}, \quad M_{3_{23}} = \begin{pmatrix} 13.9882 & 1.3106 \\ 1.3106 & 13.6292 \end{pmatrix}, \quad M_{3_{33}} = \begin{pmatrix} 15.9298 & 2.2444 \\ 2.2444 & 15.6328 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 13.5425 & 1.1062 \\ 1.1062 & 13.8552 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 16.9581 & 4.5459 \\ 4.5459 & 12.5997 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 13.5213 & 3.0100 \\ 3.0100 & 11.0290 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} 27.0460 & 0 \\ 0 & 8.9605 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 9.9041 & 0 \\ 0 & 6.9596 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 13.5187 & 0 \\ 0 & 13.2036 \end{pmatrix},$$

$$\Theta_1 = \begin{pmatrix} 22.2287 & 0 \\ 0 & 1.2546 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 0.4500 & 0 \\ 0 & 0.1037 \end{pmatrix}, \quad \Theta_3 = \begin{pmatrix} 0.3529 & 0 \\ 0 & 0.6857 \end{pmatrix},$$

$$\Lambda_1 = \begin{pmatrix} 20.7083 & 0 \\ 0 & 13.7955 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0.4500 & 0 \\ 0 & 0.1037 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0.3529 & 0 \\ 0 & 0.6857 \end{pmatrix}$$

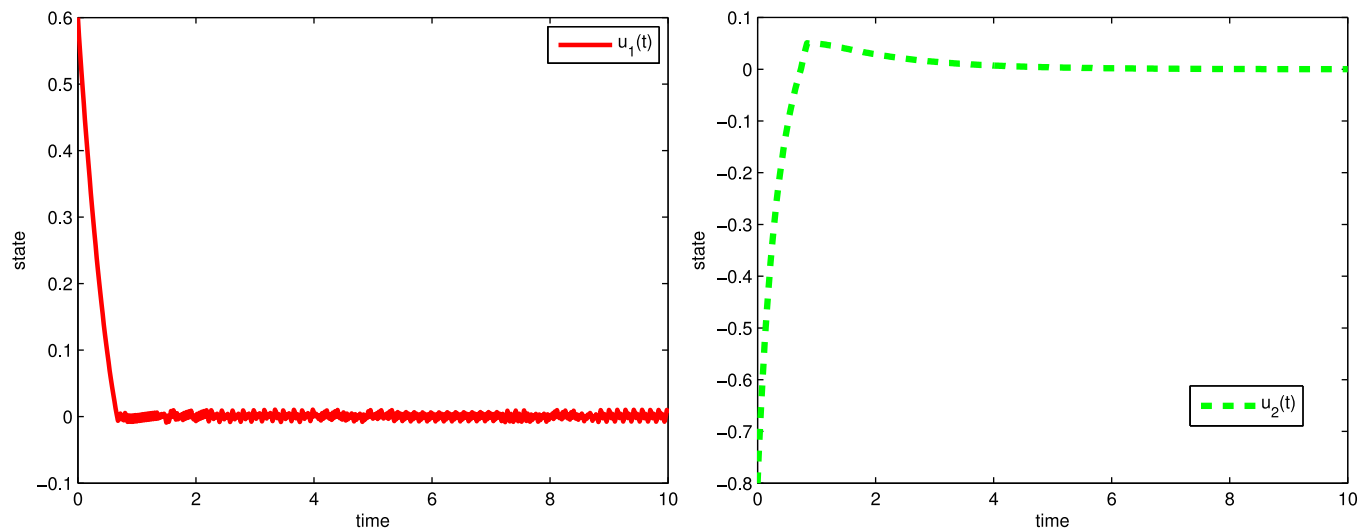


Fig. 7. The state responses $u_1(t)$ and $u_2(t)$ of (60) without distributed term.

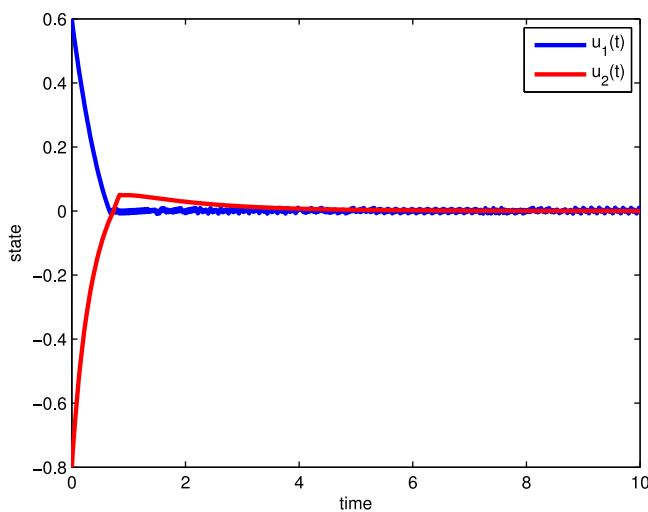


Fig. 8. The state trajectory of NNs (60) without distributed delay term.

Table 4
Maximum allowable upper bounds $\bar{\mu}$ for various μ , ς when $0 \leq \mu(t) \leq \frac{\mu}{\varsigma}$.

$\varsigma > 0$	$0 < \mu < 1$	$\bar{\mu} > 0$ [Corollary 3.3]
0.8	0.5	26.43
0.5	0.7	28.17
1	0.4	23.72
1.4	0.8	18.13
2.7	0.3	8.072
0.6	0.3	43.21

Table 5

Maximum allowable upper bounds $\bar{\mu}$ for various μ , ς when $\frac{\mu}{\varsigma} \leq \mu(t) \leq \frac{2}{3}\bar{\mu}$.

$\varsigma > 0$	$0 < \mu < 1$	$\bar{\mu} > 0$ [Corollary 3.3]
0.6	0.3	30.36
0.4	0.5	46.32
1.4	0.6	13.22
1.7	0.8	11.08
2.5	0.9	7.53

Table 6

Maximum allowable upper bounds $\bar{\mu}$ for various μ , ς when $\frac{2}{3}\bar{\mu} \leq \mu(t) \leq \bar{\mu}$.

$\varsigma > 0$	$0 < \mu < 1$	$\bar{\mu} > 0$ [Corollary 3.3]
0.6	0.3	45.6
0.3	0.4	59.75
0.5	0.5	60.43
0.4	0.7	64.91
3.8	0.9	5.14

Table 7

Average value of maximum allowable upper bounds $\bar{\mu}$ in Theorem 3.1.

Case (i)	Case (ii)	Case (iii)	Average
24.9731	26.425	26.742	26.0467

5. Conclusions

In this paper, via fractional delay intervals, the problem of exponential stability criteria for stochastic recurrent neural networks with impulsive effects and both time-delays have been investigated. By means of Lyapunov–Krasovskii functionals and M-matrix theory, several novel sufficient criteria in terms of LMIs have been derived to ensure the exponential stability of the considered neural networks. Further, depends on the system parameters, the exponential convergence rate is estimated. At last, the proposed methods can be easily validated by using the MATLAB LMI control toolbox in example section which are less conservative than some existing literature, also the numerical examples demonstrate the effectiveness and the merits of the proposed results.

Thus, the state responses of neural networks without stochastic noises (55) are depicted in Figs. 5 and 6 by the aid of control toolbox in MATLAB software. By solving the LMIs (56)–(59) in Corollary 3.3 the solutions with feasibility can be attained in three cases. In addition, the discrete time delayed upper bounds $\bar{\mu}$ for different μ and ς are classified in Tables 4–6. Therefore, under Corollary 3.3, one can conclude that the remodified neural networks (55) is exponentially stable in the mean square sense. This indicates the effectiveness and validity of our research work when compare with some existing literatures.

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Global Robust Synchronization of Fractional Order Complex Valued Neural Networks with Mixed Time Varying Delays and Impulses

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Abstract: In this article, we explore the theoretical issues on the drive-response synchronization of a class of fractional order uncertain complex valued neural networks (FOUCNNs) with mixed time varying delays and impulses. Based upon the contraction mapping principle, robust analysis techniques, as well as Riemann-Liouville (R-L) derivative, we derive a new set of novel sufficient conditions for the existence and uniqueness of equilibrium point of such neural network system, while by applying the Lyapunov functional approach, the global stability of the equilibrium solutions are obtained. Furthermore, the synchronization criterion of FOUCNNs is also attracted by means of the adaptive error feedback control strategy. Finally, two examples are provided along with the simulation results to demonstrate the effectiveness of our main proofs.

Keywords: Adaptive synchronization, asymptotic stability, complex valued neural networks, Riemann-Liouville derivative.

1. INTRODUCTION

The fresh concept of fractional order calculus and differential equations has three hundred years old of branch. For long period, the theory of fractional calculus is developed only on pure mathematics. Owing to lack of solution methods, the development of fractional order calculus has not much attracted more mathematicians in those periods. At present, fractional order dynamical system has attracted increasing interests of many researchers from various aspects such as porous media [2], viscoelasticity [19] and so on. As an extension of ordinary integer order calculus, fractional calculus has been acted more powerful tool because the results are more accurate than integer order in both theory part as well as application part. In continuous time integer order case, the common capacitance can be replaced by fractance, giving this issues is called the origin of non integer order neural network dynamical system [5]. Recently, fractional order calculus and their properties has been applied to neural networks, especially complex valued neural networks.

Complex valued neural network systems, the generalization of real valued recurrent neural network models is totally different properties and more complicated to real world neural network models because the connection

weight parameters, activation functions and state variables are mainly chosen in complex values. Many authors considered sigmoid activation functions in real world neural networks because these activation are continuously differentiable and bounded. Moreover, in complex valued neural networks, the continuously differentiable and bounded activation in complex domain is not convenient since they will reduce constants over entire $\mathbb{C}$ by means of Liouville's theorem [15]. In practice, detection of symmetry problem and XOR problem are not suitable for real world counterparts and it can be only solved by complex valued neural networks [32]. However, in practice, the stability may be affected by the parameter uncertainty [4], discrete delay [7, 9], synchronization errors [13], distributed delay [33], diffusion effects [14], impulsive effects [22, 29], stochastic effects [34] and in electronic network implementations, these widely exist in neural network models. For example, a lot of many excellent considerable results with fractional order and integer order complex valued neural networks have been reported in the existing literature, see [7, 8, 11, 12, 25].

Since the method of synchronization between two chaotic systems with different initial conditions was governed in Pecora and Carroll [10]. In past few days, the study of fractional order complex valued synchronization

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has a burgeoning research topic in the area of neural networks and complex networks. Up to now, there are numerous types of synchronization in fractional order has been proposed projective synchronization [3], global Mittag-Leffler synchronization [18], finite time synchronization [20], adaptive synchronization [23, 24], Finite time Mittag-Leffler synchronization [26], quasi uniform synchronization [27] and lag synchronization [28] by means of sliding mode control, state feedback control, adaptive controls and period intermittent control methods. There are few number of results are published in the synchronization analysis of complex valued neural networks with integer order and non integer order cases, see Ref [1, 31]. Through the review of literature, a few results have been available on robust stability and robust adaptive synchronization analysis of fractional order complex valued neural networks with mixed time delays and impulses. Motivated by the aforementioned arguments, the main objective of this article are listed as follows

- This is the first time to investigates the Riemann-Liouville sense for global robust synchronization of FOUCNNs with mixed time varying delay and impulses.
- By employing the contraction mapping principle, a suitable Lyapunov functional, Barbalat's Lemma and the properties of R-L derivative, some sufficient conditions which ensures the existence, uniqueness and stability of equilibrium point of the system are established.
- By means of adaptive feedback control, we have to show the occurrence of robust adaptive complex valued synchronization conditions between the drive-response systems and we have to introduces some special corollaries of obtained main results which is different from existing literatures.

Notations: Throughout this paper, $\mathbb{C}$ is the space of complex valued functions, $\mathbb{R}$ be the space of real valued functions, $\mathbb{C}^n$, $\mathbb{R}^n$ denotes n -dimensional unitary and n -dimensional Euclidean spaces. Let $x = p + iq$ be a complex number, where $i = \sqrt{-1}$ is the imaginary units, $p, q \in \mathbb{R}$. In this paper $\|\cdot\|$ denotes $\|\cdot\|_1$, $\mathbb{R}^{n \times n}$ and $\mathbb{C}^{n \times n}$ denotes the set of all $n \times n$ real and complex matrices. Let ${}^{RL}D^\beta$ is Riemann-Liouville operator, simply denoted by D^β and $C((-\infty, 0], \mathbb{R}^n)$ denotes the family of Banach space of all continuous functions mapping from $(-\infty, 0]$ to $\mathbb{R}^n$. Similarly, $C((-\infty, 0], \mathbb{C}^n)$ denotes the family of Banach space of all continuous functions mapping from $(-\infty, 0]$ to $\mathbb{C}^n$.

2. PROBLEM FORMULATION AND PRELIMINARIES

In this work, we consider the impulsive fractional order uncertain complex valued neural networks (FOUCNN's)

model as follows:

$$\begin{aligned} D^\beta x_h(t) &= -r_h x_h(t) + \sum_{l=1}^n (u_{hl} + \Delta u_{hl}) g_l(x_l(t)) \\ &\quad + \sum_{l=1}^n (v_{hl} + \Delta v_{hl}) g_l(x_l(t - \tau_l(t))) \\ &\quad + \sum_{l=1}^n (w_{hl} + \Delta w_{hl}) \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \\ &\quad \times g_l(x_l(s)) ds + I_h, \quad t \neq t_k, \quad t \geq 0, \\ \Delta x_h(t_k) &= x_h(t_k^+) - x_h(t_k^-) = \Phi_{hk}(x_h(t_k)), \end{aligned} \quad (1)$$

for $h = 1, 2, \dots, n$, $k = 1, 2, \dots$, where $x_h(t) = (x_1(t), x_2(t), \dots, x_n(t)) \in \mathbb{C}^n$ is the state vector of the of the h -th neuron at time t ; D^β denotes the Riemann Liouville operator with fractional order $0 < \beta < 1$; $\tau_l(t) = (\tau_1(t), \tau_2(t), \dots, \tau_n(t))$ denotes the corresponding discrete time varying delay of the l -th neuron to h -th neuron; $\mathbb{L}_{hl}(\cdot)$ denotes the delay kernel of infinite distributed delay defined on $[0, +\infty)$ and non negative bounded function; $g_l : \mathbb{C}^n \rightarrow \mathbb{C}^n$ describes the nonlinear complex valued activation function; $R = \text{diag}\{r_h\} > 0$, $h = 1, 2, \dots, n$ are constant matrices; $U = (u_{hl})_{n \times n} \in \mathbb{C}^{n \times n}$, $V = (v_{hl})_{n \times n} \in \mathbb{C}^{n \times n}$ and $W = (w_{hl})_{n \times n} \in \mathbb{C}^{n \times n}$ be the connection weight matrices; $I = (I_1, I_2, \dots, I_n) \in \mathbb{C}^n$ denotes the external inputs; t_k denotes the impulsive perturbations and it satisfies $0 < t_1 < t_2 < \dots$, $\lim_{k \rightarrow \infty} t_k = +\infty$; Φ_{hk} denotes the impulsive jumps, $x_h(t_k^-) = \lim_{t \rightarrow t_k^-} x(t)$ and $x_h(t_k^+) = \lim_{t \rightarrow t_k^+} x(t)$ stands for the left and right limits on impulsive moments at time $t = t_k$. Without loss of generality, the solution of network system (1) is left continuous at time t_k . i.e., $x_i(t_k^-) = x_i(t_k)$; Δu_{hl} , Δv_{hl} and Δw_{hl} be the complex uncertain parameter. The initial value associated with the system (1) is

$$D^{-(1-\beta)} x_h(t) = \rho_h(t) \in C((-\infty, 0], \mathbb{C}^n).$$

There are two kinds of approaches to solved complex valued concepts in neural network systems. The first one is connection weight parameters, activation functions and state variables are all defined in complex domain and the results can be obtained straight forward [17, 25]. Another one is separation of complex valued neural networks into real and imaginary parts of neural networks, which is the twice that dimensional of real valued neural networks [1]. In this paper, we moves with second type of approaches. Let $x_h(t) = p_h(t) + iq_h(t)$, $g_h(x_h(t)) = g_h^R(p_h(t), q_h(t)) + ig_h^I(p_h(t), q_h(t))$, $I_h = I_h^R + I_h^I$, where $p_h(t), q_h(t) \in \mathbb{R}$, $g_h^R(p_h(t), q_h(t)), g_h^I(p_h(t), q_h(t)) : \mathbb{R}^2 \rightarrow \mathbb{R}$. Therefore, the equation (1) can be splitted into real and imaginary parts. We have that

$$\begin{aligned} D^\beta p_h(t) &= -r_h p_h(t) + \sum_{l=1}^n (u_{hl}^R + \Delta u_{hl}^R) g_l^R(p_l(t), q_l(t)) \\ &\quad - \sum_{l=1}^n (u_{hl}^I + \Delta u_{hl}^I) g_l^I(p_l(t), q_l(t)) \end{aligned}$$

$$\begin{aligned}
& - \sum_{l=1}^n (u_{hl}^I + \Delta u_{hl}^I) g_l^I(p_l(t), q_l(t)) \\
& + \sum_{l=1}^n (v_{hl}^R + \Delta v_{hl}^R) g_l^R(p_l(t - \tau_l(t)), q_l(t - \tau_l(t))) \\
& - \sum_{l=1}^n (v_{hl}^I + \Delta v_{hl}^I) g_l^I(p_l(t - \tau_l(t)), q_l(t - \tau_l(t))) \\
& + \sum_{l=1}^n (w_{hl}^R + \Delta w_{hl}^R) \int_{-\infty}^t \mathbb{L}_{hl}(t-s) g_l^R(p_l(s), q_l(s)) ds \\
& - \sum_{l=1}^n (w_{hl}^I + \Delta w_{hl}^I) \int_{-\infty}^t \mathbb{L}_{hl}(t-s) g_l^I(p_l(s), q_l(s)) ds \\
& + I_h^R, \quad t \neq t_k, \quad t \geq 0, \\
\Delta p_h(t_k) &= p_h(t_k^+) - p_h(t_k^-) \\
&= \Phi_{hk}^R(p_h(t_k)), \quad k = 1, 2, \dots,
\end{aligned} \tag{2}$$

and

$$\begin{aligned}
D^\beta q_h(t) &= -r_h q_h(t) + \sum_{l=1}^n (u_{hl}^I + \Delta u_{hl}^I) g_l^I(p_l(t), q_l(t)) \\
&+ \sum_{l=1}^n (u_{hl}^R + \Delta u_{hl}^R) g_l^R(p_l(t), q_h(t)) \\
&+ \sum_{l=1}^n (v_{hl}^I + \Delta v_{hl}^I) g_l^I(p_l(t - \tau_l(t)), q_l(t - \tau_l(t))) \\
&+ \sum_{l=1}^n (v_{hl}^R + \Delta v_{hl}^R) g_l^R(p_l(t - \tau_l(t)), q_l(t - \tau_l(t))) \\
&+ \sum_{l=1}^n (w_{hl}^I + \Delta w_{hl}^I) \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \\
&\times g_l^R(p_l(s), q_l(s)) ds + \sum_{l=1}^n (w_{hl}^R + \Delta w_{hl}^R) \\
&\times \int_{-\infty}^t \mathbb{L}_{hl}(t-s) g_l^I(p_l(s), q_l(s)) ds \\
&+ I_h^I, \quad t \neq t_k, \quad t \geq 0, \\
\Delta q_h(t_k) &= q_h(t_k^+) - q_h(t_k^-) \\
&= \Phi_{hk}^I(q_h(t_k)), \quad k = 1, 2, \dots
\end{aligned} \tag{3}$$

The initial conditions of separating neural drive system (2)-(3) are defined as

$$\begin{aligned}
D^{-(1-\beta)} p_h(t) &= \varphi_h(t) \in C((-\infty, 0], \mathbb{R}^n), \\
D^{-(1-\beta)} q_h(t) &= \chi_h(t) \in C((-\infty, 0], \mathbb{R}^n).
\end{aligned}$$

Definition 1 [16]: The fractional order integral of order β for an integral function $y: [t_0, t) \rightarrow \mathbb{R}$ is defined as

$$D^{-\beta} y(t) = \frac{1}{\Gamma(\beta)} \int_{t_0}^t (t-m)^{\beta-1} y(m) dm,$$

where $\beta > 0$ and $\Gamma(\cdot)$ is the Gamma function.

Definition 2 [16]: The Riemann-Liouville (R-L) derivative of order β for a function $y(t)$ is defined as

$$D^\beta y(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_{t_0}^t (t-m)^{n-\beta-1} y(m) dm,$$

where $t \geq t_0$ and n is the positive integer such that $n-1 < \beta < n$.

Particularly, when $0 < \beta < 1$,

$$D^\beta y(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_{t_0}^t (t-m)^{-\beta} y(m) dm.$$

Definition 3: Let $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ is an equilibrium point of the system (1), if and only if

$$\begin{aligned}
0 &= -r_h x_h^* + \sum_{l=1}^n (u_{hl} + \Delta u_{hl} + v_{hl} + \Delta v_{hl}) g_l(x_l^*) \\
&+ \sum_{l=1}^n (w_{hl} + \Delta w_{hl}) \int_{-\infty}^t \mathbb{L}_{hl}(t-s) g_l(x_l^*) ds + I_h, \\
0 &= \Phi_{hk}(x_h^*), \quad h = 1, 2, \dots, n, \quad k = 1, 2, \dots
\end{aligned} \tag{4}$$

Lemma 1 [6]: If the time dependent differential function $\sigma(t)$ has finite limit as $t \rightarrow +\infty$, and if $\frac{d\sigma}{dt}$ is uniformly continuous, then $\frac{d\sigma}{dt} \rightarrow 0$ as $t \rightarrow +\infty$.

Assumption 1: Let $x_h(t) = p_h(t) + iq_h(t)$ and $\tilde{x}_h(t) = \tilde{p}_h(t) + i\tilde{q}_h(t)$. The nonlinear Lipschitz continuous activation function are given as follows: $g_h(x_h(t)) = g_h^R(p_h(t), q_h(t)) + ig_h^I(p_h(t), q_h(t))$, where $g_h^R(\cdot, \cdot): \mathbb{R}^2 \rightarrow \mathbb{R}$, $g_h^I(\cdot, \cdot): \mathbb{R}^2 \rightarrow \mathbb{R}$ with $p_h(t)$, $q_h(t)$, $\tilde{p}_h(t)$, $\tilde{q}_h(t) \in \mathbb{R}$, there exist some positive constants κ^{RR} , κ^{RI} , κ^{IR} , κ^{II} such that

$$\begin{aligned}
& |g_h^R(\tilde{p}_h(t), \tilde{q}_h(t)) - g_h^R(p_h(t), q_h(t))| \\
& \leq \kappa_h^{RR} |\tilde{p}_h(t) - p_h(t)| + \kappa_h^{RI} |\tilde{q}_h(t) - q_h(t)|, \\
& |g_h^I(\tilde{p}_h(t), \tilde{q}_h(t)) - g_h^I(p_h(t), q_h(t))| \\
& \leq \kappa_h^{IR} |\tilde{p}_h(t) - p_h(t)| + \kappa_h^{II} |\tilde{q}_h(t) - q_h(t)|.
\end{aligned}$$

Assumption 2: There exist some positive scalars $\check{\tau}$ and $\hat{\tau}$ such that $0 < \tau_l(t) < \check{\tau}$, $\dot{\tau}_l(t) \leq \hat{\tau} < 1$, $l = 1, 2, \dots, n$.

Assumption 3: For any $h, l = 1, 2, \dots, n$, there exists a positive constant ζ_{hl} such that

$$\int_0^{+\infty} \mathbb{L}_{hl}(m) dm = \zeta_{hl}.$$

Assumption 4: $\forall h, k = 1, 2, \dots, n$, there exist real scalars μ_{hl}^α , ϑ_{hl}^α , ω_{hl}^α ($\alpha = R, I$) such that

$$\Delta u_{hl}^\alpha = \mu_{hl}^\alpha \eta_{hl}^\alpha(t), \quad \Delta v_{hl}^\alpha = \vartheta_{hl}^\alpha \hat{\eta}_{hl}^\alpha(t), \quad \Delta w_{hl}^\alpha = \omega_{hl}^\alpha \check{\eta}_{hl}^\alpha(t).$$

where the time-varying uncertain real function η_{hl}^α , $\hat{\eta}_{hl}^\alpha$ and $\check{\eta}_{hl}^\alpha$ ($\alpha = R, I$) satisfies the conditions $[\eta_{hl}^\alpha(t)]^2 \leq 1$, $[\hat{\eta}_{hl}^\alpha(t)]^2 \leq 1$, $[\check{\eta}_{hl}^\alpha(t)]^2 \leq 1$, $h, l = 1, 2, \dots, n$.

Assumption 5: For any $h = 1, 2, \dots, n$ and $k = 1, 2, \dots$, there exist positive constants Υ_{hk}^R and Υ_{hk}^I such that functions Φ_{hk}^R and Φ_{hk}^I satisfying

$$\begin{aligned}
\Phi_{hk}^R &= -\Upsilon_{hk}^R(y_h(t)), \quad \Phi_{hk}^I = -\Upsilon_{hk}^I(z_h(t)), \\
0 &< \Upsilon_{hk}^R < 2, \quad 0 < \Upsilon_{hk}^I < 2,
\end{aligned}$$

where $y_h(t)$ and $z_h(t)$ are defined in later.

3. MINE RESULTS

3.1. Existence and uniqueness of the equilibrium point

Theorem 1: Suppose Assumptions 1, 2, 3, 4, and 5 hold, then there exists a unique equilibrium point of the system (1), $x^* = (p^*, q^*)$ is globally robust stable if $\Lambda_1 > 0$, $\Lambda_2 > 0$ and

$$\begin{aligned} \hat{\Upsilon} = \max_{1 \leq h \leq n} \left\{ \sum_{l=1}^n \left([\hat{u}_{lh}^R + \hat{u}_{lh}^I] \kappa_h^R + [\hat{v}_{lh}^R + \hat{v}_{lh}^I] \kappa_h^R \right. \right. \\ \left. \left. + \zeta_{lh} [\hat{w}_{lh}^R + \hat{w}_{lh}^I] \kappa_h^R \right), \sum_{l=1}^n \left([\hat{u}_{lh}^R + \hat{u}_{lh}^I] \kappa_h^I + [\hat{v}_{lh}^R \right. \right. \\ \left. \left. + \hat{v}_{lh}^I] \kappa_h^I + \zeta_{lh} [\hat{w}_{lh}^R + \hat{w}_{lh}^I] \kappa_h^I \right) \right\} < \min_{1 \leq h \leq n} \{r_h\}, \quad (5) \end{aligned}$$

where

$$\begin{aligned} \Lambda_1 = \min_{1 \leq h \leq n} \left\{ r_h - \sum_{l=1}^n \{ \hat{u}_{lh}^R \kappa_h^R + \hat{u}_{lh}^I \kappa_h^R \} \right. \\ \left. - \sum_{l=1}^n \frac{\{ \hat{v}_{lh}^R \kappa_h^R + \hat{v}_{lh}^I \kappa_h^R \}}{1 - \hat{\tau}} - \sum_{l=1}^n \{ \hat{w}_{lh}^R \kappa_h^R + \hat{w}_{lh}^I \kappa_h^R \} \zeta_{lh} \right\}, \\ \Lambda_2 = \min_{1 \leq h \leq n} \left\{ r_h - \sum_{l=1}^n \{ \hat{u}_{lh}^R \kappa_h^I + \hat{u}_{lh}^I \kappa_h^I \} \right. \\ \left. - \sum_{l=1}^n \frac{\{ \hat{v}_{lh}^R \kappa_h^I + \hat{v}_{lh}^I \kappa_h^I \}}{1 - \hat{\tau}} - \sum_{l=1}^n \{ \hat{w}_{lh}^R \kappa_h^I + \hat{w}_{lh}^I \kappa_h^I \} \zeta_{lh} \right\}, \end{aligned}$$

where $\hat{u}_{lh}^\alpha = |u_{lh}^\alpha| + |\mu_{lh}^\alpha|$, $\hat{v}_{lh}^\alpha = |v_{lh}^\alpha| + |\vartheta_{lh}^\alpha|$, $\hat{w}_{lh}^\alpha = |w_{lh}^\alpha| + |\omega_{lh}^\alpha|$ ($\alpha = R, I$), $\kappa_h^R = \kappa_h^{RR} + \kappa_h^{IR}$, $\kappa_h^I = \kappa_h^{IR} + \kappa_h^{II}$.

Proof: First, we can prove the existence and uniqueness of the equilibrium point.

Let $r_h p_h = \psi_h$ and $r_h q_h = \varepsilon_h$. Consider a mapping $\Pi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is $\Pi(\psi, \varepsilon) = (\Pi_1(\psi, \varepsilon), \dots, \Pi_n(\psi, \varepsilon), \Pi_{n+1}(\psi, \varepsilon), \dots, \Pi_{2n}(\psi, \varepsilon))^T$ defined by

$$\begin{aligned} \Pi_h(\psi_h, \varepsilon_h) = \sum_{l=1}^n (u_{hl}^R + \Delta u_{hl}^R) g_l^R \left(\frac{\psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \\ - \sum_{l=1}^n (u_{hl}^I + \Delta u_{hl}^I) g_l^I \left(\frac{\psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \\ + \sum_{l=1}^n (v_{hl}^R + \Delta v_{hl}^R) g_l^R \left(\frac{\psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \\ - \sum_{l=1}^n (v_{hl}^I + \Delta v_{hl}^I) g_l^I \left(\frac{\psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \\ + \sum_{l=1}^n (w_{hl}^R + \Delta w_{hl}^R) \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \\ \times g_l^R \left(\frac{\psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) ds - \sum_{l=1}^n (w_{hl}^I + \Delta w_{hl}^I) \\ \times \int_{-\infty}^t \mathbb{L}_{hl}(t-s) g_l^I \left(\frac{\psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) ds + I_h^R, \\ \Pi_{n+h}(\psi_h, \varepsilon_h) = \sum_{l=1}^n (u_{hl}^I + \Delta u_{hl}^I) g_l^R \left(\frac{\psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \end{aligned}$$

$$\begin{aligned} + \sum_{l=1}^n (u_{hl}^R + \Delta u_{hl}^R) g_l^I \left(\frac{\psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \\ + \sum_{l=1}^n (v_{hl}^I + \Delta v_{hl}^I) g_l^R \left(\frac{\psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \\ + \sum_{l=1}^n (v_{hl}^R + \Delta v_{hl}^R) g_l^I \left(\frac{\psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \\ + \sum_{l=1}^n (w_{hl}^I + \Delta w_{hl}^I) \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \\ \times g_l^R \left(\frac{\psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) ds + \sum_{l=1}^n (w_{hl}^R + \Delta w_{hl}^R) \\ \times \int_{-\infty}^t \mathbb{L}_{hl}(t-s) g_l^I \left(\frac{\psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) ds + I_h^I. \end{aligned}$$

Let us take any two complex vectors (ψ, ε) and $(\tilde{\psi}, \tilde{\varepsilon})$. Based Assumptions 1, 3, and 4, we gain

$$\begin{aligned} \left\| \Pi(\tilde{\psi}, \tilde{\varepsilon}) - \Pi(\psi, \varepsilon) \right\| \\ \leq \sum_{h=1}^n \left[\sum_{l=1}^n \frac{1}{r_l} \left((|u_{hl}^R| + |\mu_{hl}^R|) \kappa_l^R + (|v_{hl}^R| + |\vartheta_{hl}^R|) \kappa_l^R \right. \right. \\ \left. \left. + \zeta_{hl} (|w_{hl}^R| + |\omega_{hl}^R|) \kappa_l^R + (|u_{hl}^I| + |\mu_{hl}^I|) \kappa_l^R + (|v_{hl}^I| \right. \right. \\ \left. \left. + |\vartheta_{hl}^I|) \kappa_l^R + \zeta_{hl} (|w_{hl}^I| + |\omega_{hl}^I|) \kappa_l^R \right) |\tilde{\psi}_l - \psi_l| \right] \\ + \sum_{h=1}^n \left[\sum_{l=1}^n \frac{1}{r_l} \left((|u_{hl}^I| + |\mu_{hl}^I|) \kappa_l^I + (|v_{hl}^I| + |\vartheta_{hl}^I|) \kappa_l^I \right. \right. \\ \left. \left. + \zeta_{hl} (|w_{hl}^I| + |\omega_{hl}^I|) \kappa_l^I + (|u_{hl}^R| + |\mu_{hl}^R|) \kappa_l^I + (|v_{hl}^R| \right. \right. \\ \left. \left. + |\vartheta_{hl}^R|) \kappa_l^I + \zeta_{hl} (|w_{hl}^R| + |\omega_{hl}^R|) \kappa_l^I \right) |\tilde{\varepsilon}_l - \varepsilon_l| \right] \\ = \sum_{h=1}^n \frac{\sum_{l=1}^n \Lambda_{1l}}{r_h} |\tilde{\psi}_h - \psi_h| + \sum_{h=1}^n \frac{\sum_{l=1}^n \Lambda_{2l}}{r_h} |\tilde{\varepsilon}_h - \varepsilon_h|, \end{aligned}$$

where $\Lambda_{1l} = ([\hat{u}_{lh}^R + \hat{u}_{lh}^I] \kappa_h^R + [\hat{v}_{lh}^R + \hat{v}_{lh}^I] \kappa_h^R + \zeta_{lh} [\hat{w}_{lh}^R + \hat{w}_{lh}^I] \kappa_h^R)$, $\Lambda_{2l} = ([\hat{u}_{lh}^R + \hat{u}_{lh}^I] \kappa_h^I + [\hat{v}_{lh}^R + \hat{v}_{lh}^I] \kappa_h^I + \zeta_{lh} [\hat{w}_{lh}^R + \hat{w}_{lh}^I] \kappa_h^I)$.

By using (5), we know that

$$\frac{\sum_{l=1}^n \Lambda_{1l}}{r_h} < 1, \quad (6)$$

$$\frac{\sum_{l=1}^n \Lambda_{2l}}{r_h} < 1. \quad (7)$$

By using the inequality (6) and (7), we get

$$\begin{aligned} \left\| \Pi(\tilde{\psi}, \tilde{\varepsilon}) - \Pi(\psi, \varepsilon) \right\| &< \sum_{h=1}^n |\tilde{\psi}_h - \psi_h| + \sum_{h=1}^n |\tilde{\varepsilon}_h - \varepsilon_h| \\ &= \|(\tilde{\psi}, \tilde{\varepsilon}) - (\psi, \varepsilon)\|. \quad (8) \end{aligned}$$

It follows that $\Pi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a contraction mapping on $\mathbb{R}^{2n}$. Thus there exists a unique fixed point such that (ψ^*, ε^*) such that $\Pi(\psi^*, \varepsilon^*) = (\psi^*, \varepsilon^*)$, that is

$$\psi_h^* = \sum_{l=1}^n (u_{hl}^R + \Delta u_{hl}^R) g_l^R \left(\frac{\psi_l^*}{r_l}, \frac{\varepsilon_l^*}{r_l} \right)$$

$$\begin{aligned}
& - \sum_{l=1}^n (u_{hl}^I + \Delta u_{hl}^I) g_l^I \left(\frac{\Psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \\
& + \sum_{l=1}^n (v_{hl}^R + \Delta v_{hl}^R) g_l^R \left(\frac{\Psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \\
& - \sum_{l=1}^n (v_{hl}^I + \Delta v_{hl}^I) g_l^I \left(\frac{\Psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \\
& + \sum_{l=1}^n (w_{hl}^R + \Delta w_{hl}^R) \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \\
& \times g_l^R \left(\frac{\Psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) ds - \sum_{l=1}^n (w_{hl}^I + \Delta w_{hl}^I) \\
& \times \int_{-\infty}^t \mathbb{L}_{hl}(t-s) g_l^I \left(\frac{\Psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) ds + I_h^R, \\
\varepsilon_h^* &= \sum_{l=1}^n (u_{hl}^I + \Delta u_{hl}^I) g_l^I \left(\frac{\Psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \\
& + \sum_{l=1}^n (u_{hl}^R + \Delta u_{hl}^R) g_l^R \left(\frac{\Psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \\
& + \sum_{l=1}^n (v_{hl}^I + \Delta v_{hl}^I) g_l^I \left(\frac{\Psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \\
& + \sum_{l=1}^n (v_{hl}^R + \Delta v_{hl}^R) g_l^R \left(\frac{\Psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) \\
& + \sum_{l=1}^n (w_{hl}^I + \Delta w_{hl}^I) \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \\
& \times g_l^I \left(\frac{\Psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) ds + \sum_{l=1}^n (w_{hl}^R + \Delta w_{hl}^R) \\
& \times \int_{-\infty}^t \mathbb{L}_{hl}(t-s) g_l^R \left(\frac{\Psi_l}{r_l}, \frac{\varepsilon_l}{r_l} \right) ds + I_h^I.
\end{aligned}$$

Substituting $\Psi_h^* = r_h p_h^*$, $\varepsilon_h^* = r_h q_h^*$ into above equalities and by virtue of Definition 4, we conclude that, there exists a unique equilibrium point $x^* = (p^*, q^*)$ of a system (1).

Next, we shall prove to the unique equilibrium point of the system (1) is globally stable. Let $\hat{x}_h(t) = x_h(t) - x_h^* = y_h(t) + iz_h(t)$, where $y_h(t) = p_h(t) - p^*$, $z_h(t) = q_h(t) - q^*$. From (2)-(3), the error system can be obtained by

$$\begin{aligned}
D^\beta y_h(t) &= -r_h y_h(t) + \sum_{l=1}^n (u_{hl}^R + \Delta u_{hl}^R) \left[g_l^R(y_l(t) \right. \\
& \quad \left. + p_l^*, z_l(t) + q_l^*) - g_l^R(p_l^*, q_l^*) \right] \\
& - \sum_{l=1}^n (u_{hl}^I + \Delta u_{hl}^I) \left[g_l^I(y_l(t) + p_l^*, z_l(t) + q_l^*) \right. \\
& \quad \left. - g_l^I(p_l^*, q_l^*) \right] + \sum_{l=1}^n (v_{hl}^R + \Delta v_{hl}^R) \\
& \times \left[g_l^R(y_l(t - \tau_l(t))) + p_l^*, z_l(t - \tau_l(t)) \right. \\
& \quad \left. + q_l^*) - g_l^R(p_l^*, q_l^*) \right] - \sum_{l=1}^n (v_{hl}^I + \Delta v_{hl}^I) \\
& \times \left[g_l^I(y_l(t - \tau_l(t))) + p_l^*, z_l(t - \tau_l(t)) \right.
\end{aligned}$$

$$\begin{aligned}
& \left. + q_l^*) - g_l^I(p_l^*, q_l^*) \right] + \sum_{l=1}^n (w_{hl}^R + \Delta w_{hl}^R) \\
& \times \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \left[g_l^R(y_l(s) + p_l^*, z_l(s) + q_l^*) \right. \\
& \quad \left. - g_l^R(p_l^*, q_l^*) \right] ds - \sum_{l=1}^n (w_{hl}^I + \Delta w_{hl}^I) \\
& \times \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \left[g_l^I(y_l(s) + p_l^*, z_l(s) + q_l^*) \right. \\
& \quad \left. - g_l^I(p_l^*, q_l^*) \right] ds, \quad t \neq t_k, \quad t \geq 0,
\end{aligned}$$

$$\Delta y_h(t_k) = \Phi_{hk}^R(y_h(t_k)), \quad k = 1, 2, \dots, \quad (9)$$

$$\begin{aligned}
D^\beta z_h(t) &= -r_h z_h(t) + \sum_{l=1}^n (u_{hl}^I + \Delta u_{hl}^I) \left[g_l^I(y_l(t) \right. \\
& \quad \left. + p_l^*, z_l(t) + q_l^*) - g_l^I(p_l^*, q_l^*) \right] + \sum_{l=1}^n (u_{hl}^R \\
& \quad + \Delta u_{hl}^R) \left[g_l^R(y_l(t) + p_l^*, z_l(t) + q_l^*) \right. \\
& \quad \left. - g_l^R(p_l^*, q_l^*) \right] + \sum_{l=1}^n (v_{hl}^I + \Delta v_{hl}^I) \\
& \times \left[g_l^I(y_l(t - \tau_l(t))) + p_l^*, z_l(t - \tau_l(t)) \right. \\
& \quad \left. + q_l^*) - g_l^I(p_l^*, q_l^*) \right] + \sum_{l=1}^n (v_{hl}^R + \Delta v_{hl}^R) \\
& \times \left[g_l^R(y_l(t - \tau_l(t))) + p_l^*, z_l(t - \tau_l(t)) \right. \\
& \quad \left. + q_l^*) - g_l^R(p_l^*, q_l^*) \right] + \sum_{l=1}^n (w_{hl}^I + \Delta w_{hl}^I) \\
& \times \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \left[g_l^I(y_l(s) + p_l^*, z_l(s) + q_l^*) \right. \\
& \quad \left. - g_l^I(p_l^*, q_l^*) \right] ds + \sum_{l=1}^n (w_{hl}^R + \Delta w_{hl}^R) \\
& \times \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \left[g_l^R(y_l(s) + p_l^*, z_l(s) + q_l^*) \right. \\
& \quad \left. - g_l^R(p_l^*, q_l^*) \right] ds, \quad t \neq t_k, \quad t \geq 0,
\end{aligned}$$

$$\Delta z_h(t_k) = \Phi_{hk}^I(z_h(t_k)), \quad k = 1, 2, \dots, \quad (10)$$

Construct the following Lyapunov function:

$$\begin{aligned}
V(t) &= D^{-(1-\beta)} \left[\sum_{h=1}^n |y_h(t)| + \sum_{h=1}^n |z_h(t)| \right] \\
& + \frac{1}{(1-\hat{\tau})} \sum_{h=1}^n \Theta_1 \int_{t-\tau_h(t)}^t |y_h(m)| dm \\
& + \frac{1}{(1-\hat{\tau})} \sum_{h=1}^n \Theta_2 \int_{t-\tau_h(t)}^t |z_h(m)| dm \\
& + \sum_{h=1}^n \Theta_3 \int_{-\infty}^0 \int_{t+s}^t \mathbb{L}_{lh}(-s) |y_h(m)| dmds \\
& + \sum_{h=1}^n \Theta_4 \int_{-\infty}^0 \int_{t+s}^t \mathbb{L}_{lh}(-s) |z_h(m)| dmds, \quad (11)
\end{aligned}$$

where $\Theta_1 = \sum_{l=1}^n \{ \hat{v}_{lh}^R \kappa_h^R + \hat{v}_{lh}^I \kappa_h^R \}$, $\Theta_2 = \sum_{l=1}^n \{ \hat{v}_{lh}^R \kappa_h^I + \hat{v}_{lh}^I \kappa_h^I \}$, $\Theta_3 = \sum_{l=1}^n \{ \hat{w}_{lh}^R \kappa_h^R + \hat{w}_{lh}^I \kappa_h^R \}$, $\Theta_4 = \sum_{l=1}^n \{ \hat{w}_{lh}^R \kappa_h^I + \hat{w}_{lh}^I \kappa_h^I \}$.

$\hat{w}_{lh}^I \kappa_h^I$. On the other hand, from using Assumption 5 and we consider the case $t = t_k$, $k = 1, 2, 3, \dots$ and $t > 0$, one has

$$\begin{aligned}
V(t_k^+) &= D^{-(1-\beta)} \left[\sum_{h=1}^n |1 - \Upsilon_{hk}^R| |y_h(t_k^-)| \right. \\
&\quad \left. + \sum_{h=1}^n |1 - \Upsilon_{hk}^I| |z_h(t_k^-)| \right] \\
&\quad + \frac{1}{(1-\hat{\tau})} \sum_{h=1}^n \Theta_1 \int_{t_k^+ - \tau_h(t_k^+)}^{t_k^+} |y_h(m)| dm \\
&\quad + \frac{1}{(1-\hat{\tau})} \sum_{h=1}^n \Theta_2 \int_{t_k^+ - \tau_h(t_k^+)}^{t_k^+} |z_h(m)| dm \\
&\quad + \sum_{h=1}^n \Theta_3 \int_{-\infty}^0 \int_{t_k^+ + s}^{t_k^+} \mathbb{L}_{lh}(-s) |y_h(m)| dm ds \\
&\quad + \sum_{h=1}^n \Theta_4 \int_{-\infty}^0 \int_{t_k^+ + s}^{t_k^+} \mathbb{L}_{lh}(-s) |z_h(m)| dm ds \\
&< V(t_k^-), \\
\dot{V}(t) &\leq \sum_{h=1}^n \text{sgn}(y_h(t)) D^\beta y_h(t) + \sum_{h=1}^n \text{sgn}(z_h(t)) D^\beta z_h(t) \\
&\quad + \frac{1}{(1-\hat{\tau})} \sum_{h=1}^n \Theta_1 |y_h(t)| + \frac{1}{(1-\hat{\tau})} \sum_{h=1}^n \Theta_2 |z_h(t)| \\
&\quad - \sum_{h=1}^n \Theta_1 |y_h(t - \tau_h(t))| - \sum_{h=1}^n \Theta_2 |z_h(t - \tau_h(t))| \\
&\quad + \sum_{h=1}^n \Theta_3 \zeta_{lh} |y_h(t)| + \sum_{h=1}^n \Theta_4 \zeta_{lh} |z_h(t)| \\
&\quad - \sum_{h=1}^n \Theta_3 \int_{-\infty}^t \mathbb{L}_{lh}(t-s) |y_h(s)| ds \\
&\quad - \sum_{h=1}^n \Theta_4 \int_{-\infty}^t \mathbb{L}_{lh}(t-s) |z_h(s)| ds. \tag{12}
\end{aligned}$$

Noticing the error dynamical system (9)-(10), Assumptions 1 and 4, we can finally obtain the following inequality:

$$\begin{aligned}
&\sum_{h=1}^n \text{sgn}(y_h(t)) D^\beta y_h(t) + \sum_{h=1}^n \text{sgn}(z_h(t)) D^\beta z_h(t) \\
&\leq \sum_{h=1}^n \left\{ -r_h + \sum_{l=1}^n \{ \hat{u}_{lh}^R \kappa_h^R + \hat{u}_{lh}^I \kappa_h^I \} |y_h(t)| \right. \\
&\quad \left. + \sum_{l=1}^n \{ \hat{v}_{lh}^R \kappa_h^R + \hat{v}_{lh}^I \kappa_h^I \} |y_h(t - \tau_h(t))| \right. \\
&\quad \left. + \sum_{l=1}^n \{ \hat{w}_{lh}^R \kappa_h^R + \hat{w}_{lh}^I \kappa_h^I \} \int_{-\infty}^t \mathbb{L}_{lh}(t-s) |y_h(s)| ds \right\} \\
&\quad + \sum_{h=1}^n \left\{ -r_h + \sum_{l=1}^n \{ \hat{u}_{lh}^R \kappa_h^I + \hat{u}_{lh}^I \kappa_h^I \} |z_h(t)| \right. \\
&\quad \left. + \sum_{l=1}^n \{ \hat{v}_{lh}^R \kappa_h^I + \hat{v}_{lh}^I \kappa_h^I \} |z_h(t - \tau_h(t))| \right.
\end{aligned}$$

$$+ \sum_{l=1}^n \{ \hat{w}_{lh}^R \kappa_h^I + \hat{w}_{lh}^I \kappa_h^I \} \int_{-\infty}^t \mathbb{L}_{lh}(t-s) |z_h(s)| ds \Big\}. \tag{13}$$

From (13) substitute in (12), one has

$$\dot{V}(t) \leq -\Lambda \left[\sum_{l=1}^n |y_h(t)| + \sum_{l=1}^n |z_h(t)| \right] \quad \forall t \in [t_{k-1}, t_k), \tag{14}$$

where $\Lambda = \min\{\Lambda_1, \Lambda_2\}$. Taking integral on both sides of (14) from t to t_{k-1} , we get

$$\begin{aligned}
V(t) &\leq V(t_{k-1}) - \int_{t_{k-1}}^t \left[\sum_{l=1}^n |y_h(s)| + \sum_{l=1}^n |z_h(s)| \right] ds, \\
V(t) &\leq V(t_{k-1}^+) - \int_{t_{k-1}}^t \left[\sum_{l=1}^n |y_h(s)| + \sum_{l=1}^n |z_h(s)| \right] ds \\
&\leq V(t_{k-1}^-) - \int_{t_{k-1}}^t \left[\sum_{l=1}^n |y_h(s)| + \sum_{l=1}^n |z_h(s)| \right] ds \\
&\leq \dots \leq V(t_0) - \int_{t_0}^t \left[\sum_{l=1}^n |y_h(s)| + \sum_{l=1}^n |z_h(s)| \right] ds.
\end{aligned}$$

Thus, we have

$$V(t) + \int_{t_0}^t \left[\sum_{l=1}^n |y_h(s)| + \sum_{l=1}^n |z_h(s)| \right] ds \leq V(t_0). \tag{15}$$

Let $H(t) = \left[\sum_{l=1}^n |y_h(s)| + \sum_{l=1}^n |z_h(s)| \right]$. It easy to we can obtain, $\int_{t_0}^t H(s) ds$ has finite limit and $H(t)$ is bounded, it follows that $y_h(t)$ and $z_h(t)$ is also bounded. According to Eq. (9)-(10) and based on the previous description, there exists a scalars ς such that $|D^\beta H(t)| \leq \varsigma$. Next we will prove to $H(t)$ is uniformly continuous on the basis of Barbalat's lemma (2.7). For $t_0 \leq t_1 \leq t_2$, pointed has

$$|H(t_1) - H(t_2)| \leq 2\varsigma \frac{(t_2 - t_1)^\beta}{\Gamma(\beta + 1)}, \tag{16}$$

where $|t_2 - t_1| < \theta(\varepsilon) = [\varepsilon \frac{\Gamma(\beta+1)}{2\varsigma}]^{\frac{1}{\beta}}$. By virtue the definition of uniformly continuous, $H(t)$ is uniformly continuous.

According to Barbalat's lemma (2.7), we can get

$$\lim_{t \rightarrow \infty} [\|y(t)\| + \|z(t)\|] = 0.$$

Therefore, the equilibrium point $x^* = (p^*, q^*)$ of the system (1) is globally stable.

Remark 1: In [21], author addressed the asymptotic stability of delayed fractional-order neural networks with impulsive effects. In [22], asymptotic stability of delayed fractional-order BAM neural networks with impulsive effects were studied. In that two results, by applying Riemann-Liouville definitions and suitable Lyapunov approach, the equilibrium point of global asymptotic stability conditions was inspected. So, the main point in this work is to apply complex valued properties, parameter uncertainty and infinite time distributed delays.

3.2. Synchronization condition under adaptive feedback control

In this section, a novel sufficient conditions are established to ensure the global synchronization of fractional order UCNNs based on the adaptive feedback control. Next we shall consider the complex valued response system as follows:

$$\begin{aligned} D^\beta \tilde{x}_h(t) &= -r_h \tilde{x}_h(t) + \sum_{l=1}^n (u_{hl} + \Delta u_{hl}) g_l(\tilde{x}_l(t)) \\ &\quad + \sum_{l=1}^n (v_{hl} + \Delta v_{hl}) g_l(\tilde{x}_l(t - \tau_l(t))) \\ &\quad + \sum_{l=1}^n (w_{hl} + \Delta w_{hl}) \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \\ &\quad \times g_l(\tilde{x}_l(s)) ds + I_h + m_h(t), \quad t \neq t_k, \quad t \geq 0, \\ \Delta \tilde{x}_l(t_k) &= \tilde{x}_l(t_k^+) - \tilde{x}_l(t_k^-) = \Phi_{hk}(\tilde{x}_l(t_k)), \quad k = 1, \dots, \end{aligned} \quad (17)$$

where $\tilde{x}_h(t) = (\tilde{x}_1(t), \dots, \tilde{x}_n(t)) \in \mathbb{C}^n$ denotes the state variable of the drive response system and $m_h(t) = (m_1(t), \dots, m_n(t))$ denotes new designed controllers. All others are similar to defined in (1). The initial values associated with the system (17) is $D^{-(1-\beta)} \tilde{x}_h(t) = \tilde{\rho}_h(t) = (\tilde{\rho}_1(t), \dots, \tilde{\rho}_n(t))^T \in C((-\infty, 0], \mathbb{C}^n)$.

Denote $\tilde{x}_h(t) = \tilde{p}_h(t) + i\tilde{q}_h(t)$, $g_h(\tilde{x}_h(t)) = g_h^R(\tilde{p}_h(t), \tilde{q}_h(t)) + i g_h^I(\tilde{p}_h(t), \tilde{q}_h(t))$, where $\tilde{p}_h(t), \tilde{q}_h(t) \in \mathbb{R}$, $g_h^R(\tilde{p}_h(t), \tilde{q}_h(t)), g_h^I(\tilde{p}_h(t), \tilde{q}_h(t)) : \mathbb{R}^2 \rightarrow \mathbb{R}$. The adaptive linear feedback controller is defined as

$$\begin{cases} m_h^R(t) = -\lambda_h(t)[y_h(t)], \quad m_h^I(t) = -v_h(t)[z_h(t)], \\ \dot{\lambda}_h(t) = \frac{-\xi_h \varepsilon_h |y_h(t)| - \xi_h \varepsilon_h |y_h(t - \tau_h(t))|}{\lambda_h(t)} \\ \quad + \xi_h |y_h(t)|, \\ \dot{v}_h(t) = \frac{-\delta_h \phi_h |z_h(t)| - \delta_h \psi_h |z_h(t - \tau_h(t))|}{v_h(t)} \\ \quad + \delta_h |z_h(t)|, \end{cases} \quad (18)$$

where $\lambda_h(t), v_h(t)$ denotes the adaptive coupling strengths, $\varepsilon_h, \phi_h, \psi_h, \xi_h$ and δ_h are arbitrary positive scalars.

Let $\hat{x}_h(t) = \tilde{x}_h(t) - x_h(t) = y_h(t) + iz_h(t)$, where $y_h(t) = \tilde{p}_h(t) - p_h(t)$, $z_h(t) = \tilde{q}_h(t) - q_h(t)$. Based on drive-response systems (1) and (17), the error dynamical system can be expressed by the following form:

$$\begin{aligned} D^\beta y_h(t) &= -(r_h + \lambda_h(t))y_h(t) + \sum_{l=1}^n (u_{hl}^R + \Delta u_{hl}^R) \\ &\quad \times [g_l^R(y_l(t) + p_l(t), z_l(t) + q_l(t)) \\ &\quad - g_l^R(p_l(t), q_l(t))] - \sum_{l=1}^n (u_{hl}^I + \Delta u_{hl}^I) \\ &\quad \times [g_l^I(y_l(t) + p_l(t), z_l(t) + q_l(t)) \\ &\quad - g_l^I(p_l(t), q_l(t))] \end{aligned}$$

$$\begin{aligned} &- g_l^I(p_l(t), q_l(t))] + \sum_{l=1}^n (v_{hl}^R + \Delta v_{hl}^R) \\ &\quad \times [g_l^R(y_l(t - \tau_l(t)) + p_l(t - \tau_l(t)), \\ &\quad z_l(t - \tau_l(t)) + q_l(t - \tau_l(t))) \\ &\quad - g_l^R(p_l(t - \tau_l(t)), q_l(t - \tau_l(t)))] \\ &\quad - \sum_{l=1}^n (v_{hl}^I + \Delta v_{hl}^I) [g_l^I(y_l(t - \tau_l(t)) \\ &\quad + p_l(t - \tau_l(t)), z_l(t - \tau_l(t)) \\ &\quad + q_l(t - \tau_l(t))) - g_l^I(p_l(t - \tau_l(t)), \\ &\quad q_l(t - \tau_l(t)))] + \sum_{l=1}^n (w_{hl}^R + \Delta w_{hl}^R) \\ &\quad \times \int_{-\infty}^t \mathbb{L}_{hl}(t-s) [g_l^R(y_l(s) + p_l(s), z_l(s) \\ &\quad + q_l(s)) - g_l^R(p_l(s), q_l(s))] ds \\ &\quad - \sum_{l=1}^n (w_{hl}^I + \Delta w_{hl}^I) \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \\ &\quad \times [g_l^I(y_l(s) + p_l(s), z_l(s) + q_l(s)) \\ &\quad - g_l^I(p_l(s), q_l(s))] ds, \quad t \neq t_k, \quad t \geq 0, \\ \Delta y_h(t_k) &= \Phi_{hk}^R(y_h(t_k)), \quad k = 1, 2, \dots, \end{aligned} \quad (19)$$

and

$$\begin{aligned} D^\beta z_h(t) &= -(r_h + v_h(t))z_h(t) + \sum_{l=1}^n (u_{hl}^I + \Delta u_{hl}^I) \\ &\quad \times [g_l^R(y_l(t) + p_l(t), z_l(t) + q_l(t)) \\ &\quad - g_l^R(p_l(t), q_l(t))] + \sum_{l=1}^n (u_{hl}^R + \Delta u_{hl}^R) \\ &\quad \times [g_l^I(y_l(t) + p_l(t), z_l(t) + q_l(t)) \\ &\quad - g_l^I(p_l(t), q_l(t))] + \sum_{l=1}^n (v_{hl}^I + \Delta v_{hl}^I) \\ &\quad \times [g_l^R(y_l(t - \tau_l(t)) + p_l(t - \tau_l(t)), \\ &\quad z_l(t - \tau_l(t)) + q_l(t - \tau_l(t))) \\ &\quad - g_l^R(p_l(t - \tau_l(t)), q_l(t - \tau_l(t)))] \\ &\quad + \sum_{l=1}^n (v_{hl}^R + \Delta v_{hl}^R) [g_l^I(y_l(t - \tau_l(t)) \\ &\quad + p_l(t - \tau_l(t)), z_l(t - \tau_l(t)) \\ &\quad + q_l(t - \tau_l(t))) - g_l^I(p_l(t - \tau_l(t)), \\ &\quad q_l(t - \tau_l(t)))] + \sum_{l=1}^n (w_{hl}^I + \Delta w_{hl}^I) \\ &\quad \times \int_{-\infty}^t \mathbb{L}_{hl}(t-s) [g_l^R(y_l(s) + p_l(s), z_l(s) \\ &\quad + q_l(s)) - g_l^R(p_l(s), q_l(s))] ds \end{aligned}$$

$$\begin{aligned}
& + \sum_{l=1}^n (w_{hl}^R + \Delta w_{hl}^R) \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \\
& \times \left[g_l^I(y_l(s) + p_l(s), z_l(s) + q_l(s)) \right. \\
& \left. - g_l^I(p_l(s), q_l(s)) \right] ds, \quad t \neq t_k, \quad t \geq 0, \\
\Delta z_h(t_k) &= \Phi_{hk}^I(z_h(t_k)), \quad k = 1, 2, \dots
\end{aligned} \tag{20}$$

□

Theorem 2: Suppose Assumption 1-5 hold. Then the drive-response system (1) and (17) can be globally robust synchronized under the adaptive controller (18) if $\Lambda_1 > 0$ and $\Lambda_2 > 0$, where

$$\begin{aligned}
\Lambda_1 &= \min_{1 \leq h \leq n} \left\{ r_h + \varepsilon_h + \varepsilon_h - \sum_{l=1}^n \{ \hat{u}_{lh}^R + \hat{u}_{lh}^I \} \kappa_h^R \right. \\
&\quad \left. - \sum_{l=1}^n \frac{\{ \hat{v}_{lh}^R + \hat{v}_{lh}^I \} \kappa_h^R}{1 - \hat{\tau}} - \sum_{l=1}^n \{ \hat{w}_{lh}^R + \hat{w}_{lh}^I \} \kappa_h^R \zeta_{lh} \right\}, \\
\Lambda_2 &= \min_{1 \leq h \leq n} \left\{ r_h + \phi_h + \psi_h - \sum_{l=1}^n \{ \hat{u}_{lh}^R \kappa_h^I + \hat{u}_{lh}^I \kappa_h^I \} \right. \\
&\quad \left. - \sum_{l=1}^n \frac{\{ \hat{v}_{lh}^R + \hat{v}_{lh}^I \} \kappa_h^I}{1 - \hat{\tau}} - \sum_{l=1}^n \{ \hat{w}_{lh}^R + \hat{w}_{lh}^I \} \kappa_h^I \zeta_{lh} \right\}, \\
\hat{u}_{lh}^\alpha &= |\mu_{lh}^\alpha| + |\mu_{lh}^\alpha|, \quad \hat{v}_{lh}^\alpha = |\nu_{lh}^\alpha| + |\nu_{lh}^\alpha|, \quad \hat{w}_{lh}^\alpha = |\omega_{lh}^\alpha| \\
&\quad + |\omega_{lh}^\alpha|, \quad \kappa_h^R = \kappa_h^{RR} + \kappa_h^{IR}, \quad \kappa_h^I = \kappa_h^{RI} + \kappa_h^{II}, \\
&\quad (\alpha = R, I),
\end{aligned}$$

Proof: Construct the following Lyapunov functions:

$$\begin{aligned}
V(t) &= D^{-(1-\beta)} \left[\sum_{h=1}^n |y_h(t)| + \sum_{h=1}^n |z_h(t)| \right] \\
&\quad + \sum_{h=1}^n \frac{1}{2\xi_h} [\lambda_h(t)]^2 + \sum_{h=1}^n \frac{1}{2\delta_h} [v_h(t)]^2 \\
&\quad + \frac{1}{(1-\hat{\tau})} \sum_{h=1}^n \Theta_1 \int_t^{t-\tau_h(t)} |y_h(m)| dm \\
&\quad + \frac{1}{(1-\hat{\tau})} \sum_{h=1}^n \Theta_2 \int_t^{t-\tau_h(t)} |z_h(m)| dm \\
&\quad + \sum_{h=1}^n \Theta_3 \int_{-\infty}^0 \int_{t+s}^t \mathbb{L}_{lh}(-s) |y_h(m)| dmds \\
&\quad + \sum_{h=1}^n \Theta_4 \int_{-\infty}^0 \int_{t+s}^t \mathbb{L}_{lh}(-s) |z_h(m)| dmds, \quad (21)
\end{aligned}$$

where

$$\begin{aligned}
\Theta_1 &= \varepsilon_h - \sum_{l=1}^n \{ \hat{v}_{lh}^R + \hat{v}_{lh}^I \} \kappa_h^R, \quad \Theta_3 = \sum_{l=1}^n \{ \hat{w}_{lh}^R + \hat{w}_{lh}^I \} \kappa_h^R, \\
\Theta_2 &= \psi_h - \sum_{l=1}^n \{ \hat{v}_{lh}^R + \hat{v}_{lh}^I \} \kappa_h^I, \quad \Theta_4 = \sum_{l=1}^n \{ \hat{w}_{lh}^R + \hat{w}_{lh}^I \} \kappa_h^I.
\end{aligned}$$

On the other hand, according to Assumption 5 and we consider the case $t = t_k$, $k = 1, 2, 3, \dots$ and $t > 0$, one has

$$V(t_k^+) < D^{-(1-\beta)} \left[\sum_{h=1}^n |y_h(t_k^-)| + \sum_{h=1}^n |z_h(t_k^-)| \right]$$

$$\begin{aligned}
& + \sum_{h=1}^n \frac{1}{2\xi_h} [\lambda_h(t_k^+)]^2 + \sum_{h=1}^n \frac{1}{2\delta_h} [v_h(t_k^+)]^2 \\
& + \frac{1}{(1-\hat{\tau})} \sum_{h=1}^n \Theta_1 \int_{t_k^+}^{t_k^+ - \tau_h(t_k^+)} |y_h(m)| dm \\
& + \frac{1}{(1-\hat{\tau})} \sum_{h=1}^n \Theta_2 \int_{t_k^+}^{t_k^+ - \tau_h(t_k^+)} |z_h(m)| dm \\
& + \sum_{h=1}^n \Theta_3 \int_{-\infty}^0 \int_{t_k^+ + s}^{t_k^+} \mathbb{L}_{lh}(-s) |y_h(m)| dmds \\
& + \sum_{h=1}^n \Theta_4 \int_{-\infty}^0 \int_{t_k^+ + s}^{t_k^+} \mathbb{L}_{lh}(-s) |z_h(m)| dmds \\
& = V(t_k^-),
\end{aligned}$$

$$\begin{aligned}
\dot{V}(t) &\leq \sum_{h=1}^n \text{sgn}(y_h(t)) D^\beta y_h(t) \\
&\quad + \sum_{h=1}^n \text{sgn}(z_h(t)) D^\beta z_h(t) \\
&\quad - \sum_{h=1}^n [\varepsilon_h |y_h(t)| + \varepsilon_h |y_h(t - \tau_h(t))| + \lambda_h(t) \\
&\quad \times |y_h(t)|] - \sum_{h=1}^n [\phi_h |z_h(t)| + \psi_h |z_h(t - \tau_h(t))| \\
&\quad - v_h(t) |z_h(t)|] + \sum_{h=1}^n \Theta_1 |y_h(t - \tau_h(t))| \\
&\quad - \frac{1}{(1-\hat{\tau})} \sum_{h=1}^n \Theta_1 |y_h(t)| + \sum_{h=1}^n \Theta_2 |z_h(t - \tau_h(t))| \\
&\quad - \frac{1}{(1-\hat{\tau})} \sum_{h=1}^n \Theta_2 |z_h(t)| + \sum_{h=1}^n \Theta_3 \zeta_{lh} |y_h(t)| \\
&\quad + \sum_{h=1}^n \Theta_4 \zeta_{lh} |z_h(t)| - \sum_{h=1}^n \Theta_3 \int_{-\infty}^t \mathbb{L}_{lh}(t-s) \\
&\quad \times |y_h(s)| ds - \sum_{h=1}^n \Theta_4 \int_{-\infty}^t \mathbb{L}_{lh}(t-s) |z_h(s)| ds.
\end{aligned} \tag{22}$$

According to Assumptions 1, 4 and by proceeding the similar way of Theorem 1, we can get

$$\begin{aligned}
& \sum_{h=1}^n \text{sgn}(y_h(t)) D^\beta y_h(t) \\
& \leq \sum_{h=1}^n \left\{ -[r_h + \lambda_h(t)] |y_h(t)| + \sum_{l=1}^n \hat{u}_{hl}^R [\kappa_l^{RR} |y_l(t)| \right. \\
& \quad \left. + \kappa_l^{RI} |z_l(t)|] + \sum_{l=1}^n \hat{u}_{hl}^I \times [\kappa_l^{IR} |y_l(t)| + \kappa_l^{II} |z_l(t)|] \right\} \\
& \quad + \sum_{l=1}^n \hat{v}_{hl}^R [\kappa_l^{RR} |y_l(t - \tau_l(t))| + \kappa_l^{RI} |z_l(t - \tau_l(t))|] \\
& \quad + \sum_{l=1}^n \hat{v}_{hl}^I [\kappa_l^{IR} |y_l(t - \tau_l(t))| + \kappa_l^{II} |z_l(t - \tau_l(t))|] \\
& \quad + \sum_{l=1}^n \hat{w}_{hl}^R \int_{-\infty}^t \mathbb{L}_{hl}(t-s) [\kappa_l^{RR} |y_l(s)| + \kappa_l^{RI} |z_l(s)|] ds
\end{aligned}$$

$$+ \sum_{l=1}^n \hat{w}_{hl}^I \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \left[\kappa_l^{IR} |y_l(s)| + \kappa_l^{II} |z_l(s)| \right] ds \Big\}, \quad (23)$$

$$\begin{aligned} & \sum_{h=1}^n \text{sgn}(z_h(t)) D^\beta z_h(t) \\ & \leq \sum_{h=1}^n \left\{ -[r_h + v_h(t)] |z_h(t)| + \sum_{l=1}^n \hat{u}_{hl}^I \left[\kappa_l^{RR} |y_l(t)| \right. \right. \\ & \quad \left. \left. + \kappa_l^{RI} |z_l(t)| \right] + \sum_{l=1}^n \hat{u}_{hl}^R \left[\kappa_l^{IR} |y_l(t)| + \kappa_l^{II} |z_l(t)| \right] \right. \\ & \quad \left. + \sum_{l=1}^n \hat{v}_{hl}^I \left[\kappa_l^{RR} |y_l(t - \tau_l(t))| + \kappa_l^{RI} |z_l(t - \tau_l(t))| \right] \right. \\ & \quad \left. + \sum_{l=1}^n \hat{v}_{hl}^R \left[\kappa_l^{IR} |y_l(t - \tau_l(t))| + \kappa_l^{II} |z_l(t - \tau_l(t))| \right] \right. \\ & \quad \left. + \sum_{l=1}^n \hat{w}_{hl}^I \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \left[\kappa_l^{RR} |y_l(s)| + \kappa_l^{RI} |z_l(s)| \right] ds \right. \\ & \quad \left. + \sum_{l=1}^n \hat{w}_{hl}^R \int_{-\infty}^t \mathbb{L}_{hl}(t-s) \left[\kappa_l^{IR} |y_l(s)| + \kappa_l^{II} |z_l(s)| \right] ds \right\}. \end{aligned} \quad (24)$$

From (23) and (24) to (22), it yields that

$$\dot{V}(t) \leq -\Lambda \left[\sum_{l=1}^n |y_h(t)| + \sum_{l=1}^n |z_h(t)| \right] \quad \forall t \in [t_{k-1}, t_k).$$

Where $\Lambda = \min\{\Lambda_1, \Lambda_2\}$. By proceeding the similar way to the proof of Theorem 1 and we conclude that $\lim_{t \rightarrow \infty} [\|y(t)\| + \|z(t)\|] = 0$. Therefore, the drive-response systems are globally synchronized based on the controller (18). $\square$

Remark 2: In [1], authors proposed Caputo sense of synchronization conditions for fractional order delayed complex valued neural networks and derived some synchronization criteria. In [30], authors studied the complex projective synchronization of integer order complex-valued neural network with structure identification. In above mentioned papers, impulses, infinite time distributed delay, parameter uncertainty was ignored, while these parameters are considered in this paper by employing the adaptive feedback control and R-L properties. When the state variable, connection weights, activations are assumed to real values, the problem turn to robust adaptive synchronization of fractional order real valued neural networks with mixed time varying delays and impulses.

3.3. Numerical examples

In this section, two numerical simulations are presented to prove the effectiveness of the proposed main results.

Example 1: In system (1), choose $\beta = 0.97$, $x(t) = (x_1(t), x_2(t))^T$, $x_l(t) = p_l(t) + iq_l(t)$, $\tau(t) = \frac{\exp(t)}{1+\exp(t)}$, $\Upsilon_{1k}^R = \Upsilon_{2k}^R = 1.4$, $\Upsilon_{1k}^I = \Upsilon_{2k}^I = 0.9$, $\hat{\tau}_j(t) \leq \frac{1}{5} < \hat{\tau} =$

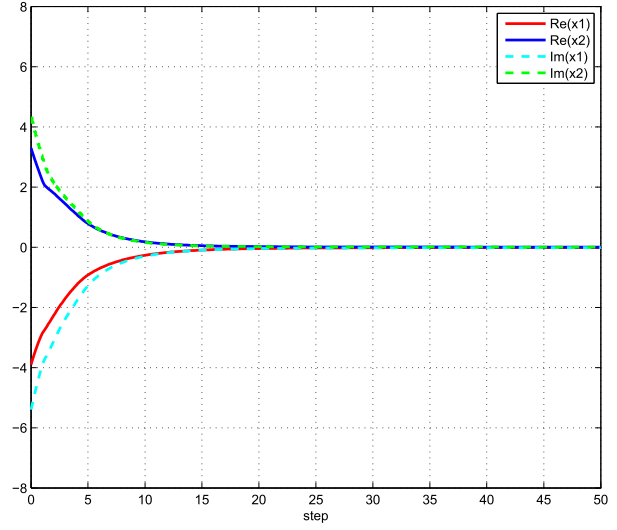


Fig. 1. Transient states of equilibrium point of the system in Example 1.

0.5 , $I_h = \frac{1}{7}(0.7 + 1.2i, 0.6i)^T$, $f_l(x_l(t)) = \frac{1 - \exp(-p_l(t))}{1 + \exp(-p_l(t))} + i \frac{1}{1 + \exp(-q_l(t))}$, ($l = 1, 2$), and

$$R = \begin{bmatrix} 7 & 0 \\ 0 & 9 \end{bmatrix}, \quad U = \frac{1}{7} \begin{bmatrix} 1 - 2i & -0.5 + 3i \\ -2 + 7i & 0.7 + 5i \end{bmatrix},$$

$$V = \frac{1}{7} \begin{bmatrix} 1 + 3i & -0.2 + 5i \\ 0.4 + 2i & -0.9 - 1.2i \end{bmatrix},$$

$$W = \frac{1}{9} \begin{bmatrix} -1.8 + 2i & 0.8 + 1.4i \\ 0.3 + 0.9i & -1 - 0.8i \end{bmatrix},$$

$$\Delta U = (0.1 \sin(t) - 0.3i \cos(t)) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\Delta V = (0.4 \sin(t) - 0.2i \cos(t)) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\Delta W = (0.3 \sin(t) - 0.6i \cos(t)) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

From Assumption 4, we can choose the uncertain parameter values given by $\mu_{hl}^R = 0.1$, $\mu_{hl}^I = -0.3$, $\vartheta_{hl}^R = 0.4$, $\vartheta_{hl}^I = -0.2$, $\omega_{hl}^R = 0.3$ and $\omega_{hl}^I = -0.6$, while we select $\kappa_h^{RR} = 0.75$, $\kappa_h^{II} = 1$, $\kappa_h^{RI} = \kappa_h^{IR} = 0$ and $\zeta_{lh} = 1$ for $h, l = 1, 2$, then Assumptions 1-5 are satisfied. After the simple manipulation, it is easy to derive that the conditions of Theorem 1 are $5.238 < 7$, $\Lambda_1 = 0.6765 > 0$ and $\Lambda_2 = 0.721 > 0$. Thus, the equilibrium point of the FOUC-NNs system (1) is globally robust stable, which is depicted in Fig. 1.

Example 2: In drive system (1), choose $\beta = 0.98$, $x(t) = (x_1(t), x_2(t))^T$, $x_l(t) = p_l(t) + q_l(t)$, $\tau(t) = \frac{\exp(t)}{1+\exp(t)}$, $\Upsilon_{1k}^R = 0.8$, $\Upsilon_{2k}^R = 1.6$, $\Upsilon_{1k}^I = 1.3$, $\Upsilon_{2k}^I = 0.9$, $\hat{\tau}_j(t) \leq \frac{1}{4} < \hat{\tau} = 0.4$, $I_h = (0.8, 1.7 + 0.6i)^T$, $f_l(x_l(t)) = \frac{1 - \exp(-p_l(t))}{1 + \exp(-p_l(t))} +$

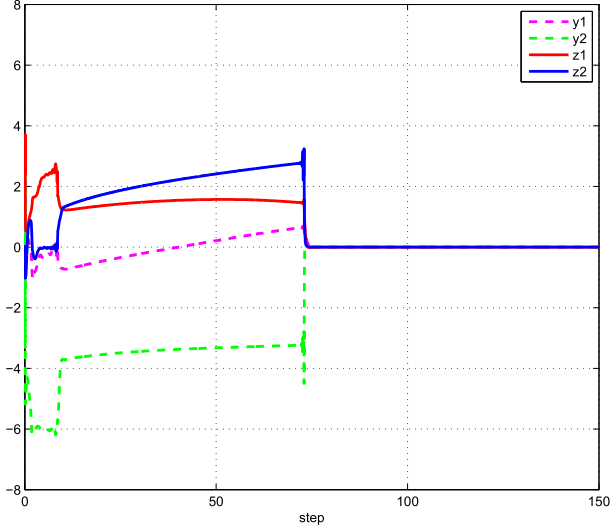


Fig. 2. Transient states of drive-response synchronization errors in Example 2.

$$i \frac{1}{1+\exp(-q_l(t))}, (l = 1, 2),$$

$$R = \begin{bmatrix} 3.5 & 0 \\ 0 & 4 \end{bmatrix}, \quad U = \begin{bmatrix} 1+2i & -1.5+2i \\ 0.5+i & 1.2+0.8i \end{bmatrix},$$

$$V = \begin{bmatrix} -1.3+i & 2.2+1.3i \\ 0.8+2i & 0.5+1.2i \end{bmatrix},$$

$$W = \begin{bmatrix} 3+1.3i & -0.9-2i \\ 0.7+1.5i & 1.7+2.1i \end{bmatrix},$$

$$\Delta U = (0.4 \sin(t) - 0.2i \cos(t)) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\Delta V = (0.9 \sin(t) - 2.1i \cos(t)) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\Delta W = (1.4 \sin(t) - 1.9i \cos(t)) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

From Assumption 4, we can choose the uncertain parameter values are $\mu_{hl}^R = 0.4$, $\mu_{hl}^I = -1.2$, $\vartheta_{hl}^R = 0.9$, $\vartheta_{hl}^I = -2.1$, $\omega_{hl}^R = 1.4$ and $\omega_{hl}^I = -1.9$, while we select $\kappa_h^{RR} = 0.25$, $\kappa_h^{II} = 0.5$, $\kappa_h^{RI} = \kappa_h^{IR} = 0$ and $\zeta_{lh} = 1.5$ for $h, l = 1, 2$. Assumptions 1-5 hold. In controller (18), we choose the control gains are $\lambda_h(0) = v_h(0) = 0.1$ ($h = 1, 2$), $\varepsilon_1 = 4.25$, $\varepsilon_2 = 3.5$, $\varepsilon_1 = 4.7$, $\varepsilon_2 = 3.8$, $\phi_1 = 9.1$, $\phi_2 = 7.6$, $\psi_1 = 8.2$, $\psi_2 = 8.8$. Thus the conditions of Theorem 2, it is easily to verified that $\Lambda_1 = 1.8 > 0$ and $\Lambda_2 = 1.375 > 0$. The simulations of the synchronization errors are showed in Fig. 3. Therefore the drive-response FOUCNNs systems are achieved to globally robust adaptive synchronization and the adaptive coupling strengths converges to some positive scalars, which confirm the effectiveness of Theorem 2.

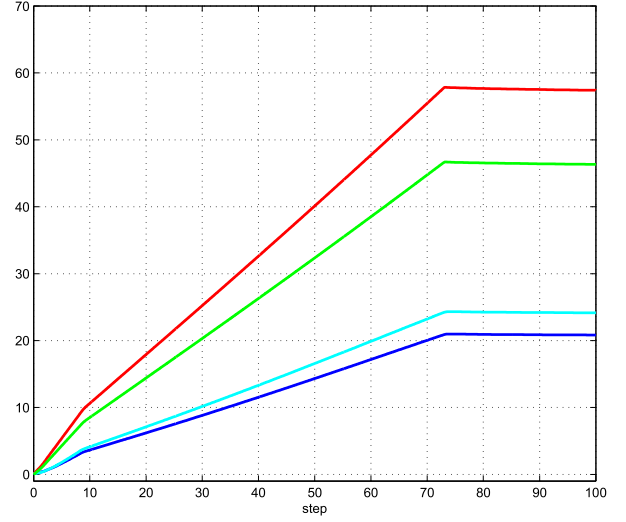


Fig. 3. Time response of $\lambda_h(t)$ and $v_h(t)$ in Example 2.

4. CONCLUSIONS

In this article, Robust adaptive synchronization of fractional order complex valued neural networks with mixed varying delays and impulses is investigated. By employing the Banach contraction mapping principle, the existence, uniqueness and global stability of equilibrium point for such UCNNs are established. A new sufficient criteria ensuring the robust synchronization of UCNNs have been investigated based on the valid adaptive control, Barbalat's lemma and the application of robust stability principle. At the end, we have presented two numerical simulations to provide the obtained theoretical main results. Furthermore, the adaptive state feedback control approach proposed in this paper can be applied for solving global projective synchronization of complex-valued competitive-type neural networks, and we will consider the interesting issue for future work.

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A New Global Robust Exponential Stability Criterion for H_∞ Control of Uncertain Stochastic Neutral-type Neural Networks with Both Time-varying Delays

Maharajan Chinnamuniyandi, Raja Ramachandran, Jinde Cao*, Grienggrai Rajchakit, and Xiaodi Li

Abstract: This paper mainly focuses on a novel H_∞ control design to handle the global robust exponential stability problem for uncertain stochastic neutral-type neural networks (USNNs) with mixed time-varying delays. Here the delays are assumed to be both discrete and distributed, which means that the lower and upper bounds can be derived. Firstly, we draw a control law for stabilized and stability of the neutral-type neural networks (NNs). Secondly, by employing the Lyapunov-Krasovskii functional(LKF) theory, Jensen's integral inequality, new required sufficient conditions for the global robust exponential stability of the given neural networks (NNs) are established in terms of delay-dependent linear matrix inequalities (LMIs), which can be easily checked in practice. The conditions obtained are expressed in terms of LMIs whose feasibility can be verified easily by MATLAB LMI control toolbox. Moreover, we have compared our work with previous one in the existing literature and showed that it reduces conservatism. Finally, one numerical example with their simulations is given to validate the effectiveness of our proposed theoretical results.

Keywords: Global robust exponential stability, H_∞ control, Linear matrix inequality, Lyapunov-Krasovskii functional, Mixed time-varying delays, Stochastic uncertain neutral-type neural networks.

1. INTRODUCTION

In recent years, researches on time delayed neural networks have found fruitful attention in several fields, such as biological systems, hydraulic systems, aircraft stabilization, electrical networks and nuclear reactor, [1, 2]. Obviously the time delay in the networks or dynamical systems frequently causes bad performance, oscillation and instability or chaos [3]. Thus many essential stability results on the dynamical behaviors have been reported for time delay systems, see [4–7] and references there in. Moreover, time delay can be categorized as two types: discrete and distributed time delays. Based on the huge length of the axon sizes we have taken both time delays into account when model our neural networks system. For instance many research authors in [8] dealt with both time delays for neural networks. Neutral time delay is another type of delay which has drawn a lot of attention nowa-

days. Because, a neutral-type delay phenomenon contains delays both in its derivatives and state variables. Practically, such phenomenon can be established in many areas including mechanics, automatic control, vibrating masses attached to an elastic bar, ecology, lossless transmission lines, robots in contact with rigid environments, heat exchangers, etc. Consequently, delayed neural networks with neutral terms has received much attention in recent years and a great number of conclusions on this issue have been reported [9–11]. In [12], the authors have been investigated about neutral type neural networks with time delays.

During the physical implementation among the parameter fluctuation, some external disturbances naturally affected a real system. In general, two kinds of external disturbances to be considered here, i.e., uncertain parameters and stochastic noises. In real nervous system, the connection weights of the neuron depend on certain resistance

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and capacitance values that include uncertain parameters (modeling errors). Moreover, the uncertainties are inevitable while modeling a neural networks. So, it is noteworthy that the parameter fluctuation and errors during their implementation, which also outcomes in poor performance and instability of the neural networks [13–15]. Hence the uncertain parameters of a system are only due to the deviations and perturbations of its parameters.

On the other hand, stochastic noises in neural networks built by introducing random variations into the network, either by giving them stochastic weights, or by giving the network's neurons stochastic transfer functions. This makes them useful tools for neural networks, since the random fluctuations help it to escape from instability. Stochastic neural networks have found applications in bio informatics, risk management, oncology, and other similar fields. Thus, the consideration of stochastic perturbations in dynamical systems is necessary for stability property, see [16–21]. In [17], the problem of stability criteria for neural networks with stochastic terms were analyzed. The neutral-type neural networks were designed in such a way that, for a given external input, they can reveal only one global asymptotic or exponential stable zero solution. Hence, some results have been proposed to guarantee the asymptotic stability or exponential stability for stochastic NNs with time delays. For example, the authors examined stability of hybrid impulsive stochastic differential systems [18]. The main difficulty stems from the question that how to represent stochastic and uncertain perturbations, to handle both mixed time-varying delays. Thus, the stability performance have been analyzed for time-delayed neural networks with stochastic noises or uncertain parameters.

Since Zames in [22] proposed the theory of H_∞ control initially. H_∞ designs are used in control theory to achieve stabilization with guaranteed performance, which has been successfully applied to an extensive variety of time-delay systems, the engineering fields and including uncertain systems. In system control, H_∞ control problem has attracted great research focus due to its theoretical and practical significance [14, 23–26]. The time-delay dependent problem of H_∞ control for stochastic neural networks with uncertainties have received substantial observation among control area for the past years [27]. However, there has been a small amount of results considering robust H_∞ control for time-delayed stochastic neural networks with uncertainties, which remains challenging but open. However, to the best of the knowledge, up to now, the global robust exponential stability analysis problem for H_∞ control of stochastic neutral-type neural networks with uncertainties and mixed time delays has not been properly addressed. This situation motivates our present research work.

The following summary having the major difficulties

and challenges of this paper:

- * To construct Lyapunov-Krasovskii functional for feasible, the neutral term needs to be considered. Accordingly, when the neutral term exists in the concerned neural networks, how can construct a feasible Lyapunov-Krasovskii functional.
- * Suppose the uncertain parameters, stochastic noises and both time-varying delays exists in USNNs, the corresponding stability performance becomes more complicated since a novel LKF is required to revealed these influences.
- * By introducing H_∞ control design approach, the globally robustly exponential stability of our proposed neural networks is obscure.

Inspired by the above discussions, our main aim of this paper is to investigate the global robust exponential stability problem for a class of uncertain stochastic neutral-type neural networks with discrete and distributed time-varying delays. The main contributions of this research work are highlighted as follows:

- Neutral-type, stochastic disturbances, uncertain parameters and discrete, distributed time-varying delays are taken into account in the stability analysis of addressed neural networks.
- By the implementation, Lyapunov-Krasovskii functionals, stability theory and some LMI techniques, a new-brand sufficient conditions for globally robustly exponential stability of H_∞ control USNNs are derived interms of LMIs.
- By handled the both time-varying delays and also neutral time delay terms in our uncertain stochastic neutral-type neural networks with H_∞ control design, the allowable upper bounds of discrete, distributed & neutral time-varying delays are very large. This explore the approach developed in this paper is effective and less conservative than some previous literature, which ensures the advantages of this research work.

2. MODEL DESCRIPTION AND PRELIMINARIES

Notations: Throughout this manuscript, $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n \times n}$ denotes the set of all real numbers, n -dimensional Euclidean space and the set of all $n \times n$ matrices with real entries, respectively. For a matrix $\mathcal{D}$ and two symmetric matrices $\mathcal{A}$ and $\mathcal{B}$, $\begin{bmatrix} \mathcal{A} & \mathcal{D} \\ * & \mathcal{B} \end{bmatrix}$ denote the symmetric matrix, where the symbol $*$ represents the entries implied by symmetry. The $\mathcal{A}^T$ and $\mathcal{A}^{-1}$ denotes the matrix transpose and inverse of $\mathcal{A}$, respectively. We say $\mathbb{T} > 0$ means that the matrix $\mathbb{T}$ is real symmetric positive definite with appropriate dimensions. $C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ is the family of all non negative functions $V(t, y(t))$ on $\mathbb{R}^+ \times \mathbb{R}^n$ which

are continuously twice differentiable in y and once differentiable in t . $(\mathbb{S}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ a complete probability space, where $\mathbb{S}$ is the sample space, $\mathcal{F}$ is the σ -algebra of subsets of the sample space, $\mathcal{P}$ is the probability measure on $\mathcal{F}$ and $\{\mathcal{F}_t\}_{t \geq 0}$ denotes the Filtration. $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator of y with respect to the given probability measure $\mathcal{P}$. $\text{diag}\{\dots\}$ denotes a block-diagonal matrix. I denotes the identity matrix with appropriate dimensions. Let $\|f\|_2 = \sqrt{\int_0^\infty \|f\|^2 dt}$, $f(t) \in L_2[0, \infty)$, $\|f\|$ refers to the Euclidean norm of the function $f(t)$ at time t . $L_2[0, \infty)$ is the space of square integrable vectors in $[0, \infty)$.

In this paper, we consider the stochastic neutral type neural networks with structured uncertainties and mixed time varying delays is described by the following integro-differential equation system :

$$\begin{aligned} & \dot{y}(t) - (U_1 + \Delta U_1(t))\dot{y}(t - d(t)) \\ &= -(A + \Delta A(t))y(t) + (V_0 + \Delta V_0(t))f(y(t)) \\ &+ (V_1 + \Delta V_1(t)) \times g(y(t - k(t))) \\ &+ (B + \Delta B(t))v(t) + Cu(t) \\ &+ (V_2 + \Delta V_2(t)) \times \int_{t-\sigma(t)}^t h(y(s))ds \\ &+ [\Delta E(t)y(t) + \Delta E_s(t)y(t - k(t))]d\omega(t), \\ &x(t) = Fy(t) + Dv(t), \quad t > 0, \\ &y(t) = \varphi(t), t \in [-K, 0], \end{aligned} \quad (1)$$

where $y(t) \in \mathbb{R}^n$ is the state; $v(t) \in \mathbb{R}^m$ is the control input; $u(t) \in \mathbb{R}^p$ is the disturbance input belonging to $L_2[0, \infty)$; $x(t) \in \mathbb{R}^q$ is the system output; $f(y(t)) = (f_1(y(t)), f_2(y(t)), \dots, f_n(y(t)))^T$, $g(y(t)) = (g_1(y(t)), g_2(y(t)), \dots, g_n(y(t)))^T$ and $h(y(t)) = (h_1(y(t)), h_2(y(t)), \dots, h_n(y(t)))^T$ are the neuron activation functions with $f(0) = g(0) = h(0) = 0$; the bound functions $k(t)$, $\sigma(t)$ and $d(t)$ respectively represents discrete, distributed and neutral delays of systems with $0 \leq k(t) \leq k_M$, $\dot{k}(t) \leq k_D < 1$, $0 \leq \sigma(t) \leq \sigma_M$, $\dot{\sigma}(t) \leq \sigma_D < 1$ and $0 \leq d(t) \leq d_M$, $\dot{d}(t) \leq d_D < 1$, $K = \max\{k_M, \sigma_M, d_M\}$; $\omega(t) = (\omega_1(t), \omega_2(t), \dots, \omega_n(t))^T$ is an n -dimensional Brownian motion defined on a complete probability space $(\mathbb{A}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with a Filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets) and $\mathbb{E}\{d\omega(t)\} = 0$, $\mathbb{E}\{d\omega^2(t)\} = dt$. It is assumed that $\varphi^T(t)\varphi(t) \leq \beta_0$, $t \in [-K, 0]$, where $\beta_0 > 0$ can be seen as an upper bound in the initial states. The diagonal matrix $A = \text{diag}\{a_1, a_2, \dots, a_n\}$ has positive entries $a_i > 0$. $V_0 = (V_{0ij})_{n \times n}$, $V_1 = (V_{1ij})_{n \times n}$, $V_2 = (V_{2ij})_{n \times n}$, $U_1 = (U_{1ij})_{n \times n}$, $B = (B_{ij})_{n \times n}$, $C = (C_{ij})_{n \times n}$, $F = (F_{ij})_{n \times n}$ and $D = (D_{ij})_{n \times n}$ are constant matrices with appropriate dimensions. $\Delta A(t)$, $\Delta V_0(t)$, $\Delta V_1(t)$, $\Delta V_2(t)$, $\Delta U_1(t)$, $\Delta B(t)$, $\Delta E(t)$ and $\Delta E_s(t)$ represents the structured uncertainties in stochastic neutral type neural networks (1).

Throughout this paper, to obtain sufficient criteria made the following assumptions.

Assumption 1: The neuron activation functions $f_i(\cdot)$, $g_i(\cdot)$ and $h_i(\cdot)$ ($i = 1, 2, \dots, n$) are bounded and there exists a positive diagonal matrix $G = \text{diag}(\zeta_1, \zeta_2, \dots, \zeta_n)$ such that

$$\begin{aligned} 0 &\leq \frac{f_i(\vartheta_1) - f_i(\vartheta_2)}{\vartheta_1 - \vartheta_2} \leq \zeta_i, \\ \forall \vartheta_1 &\neq \vartheta_2, \vartheta_1, \vartheta_2 \in \mathbb{R}, i = 1, 2, \dots, n; \end{aligned} \quad (2)$$

where $\zeta_i > 0$ ($i = 1, 2, \dots, n$) are some given constants. Similarly the functions $g_i(\cdot)$ and $h_i(\cdot)$ are satisfies the condition (2).

Assumption 2: The perturbed uncertain matrices $\Delta A(t)$, $\Delta V_0(t)$, $\Delta V_1(t)$, $\Delta V_2(t)$, $\Delta U_1(t)$, $\Delta B(t)$, $\Delta E(t)$ and $\Delta E_s(t)$ are time-varying functions satisfying: $\Delta V_0(t) = PF_0(t)N_{V_0}$, $\Delta V_1(t) = PF_1(t)N_{V_1}$, $\Delta V_2(t) = PF_2(t)N_{V_2}$, $[\Delta E(t), \Delta E_s(t)] = PF_3(t) [N_E, N_{E_s}]$ and $[\Delta A(t), \Delta B(t)] = PF_4(t)[N_A, N_B]$, $\Delta U_1(t) = P\bar{F}_1(t)N_{U_1}$, where P , N_A , N_B , N_{V_0} , N_{V_1} , N_{U_1} , N_{V_2} , N_E and N_{E_s} are given constant matrices. $F_l(t)$ ($l = 0, 1, 2, 3, 4$) and $\bar{F}_1(t)$ are unknown real time varying matrices which have the following structure: $F_l(t) = \text{blockdiag}\{\delta_{l_1}(t)I_{r_1}, \delta_{l_2}(t)I_{r_2}, \dots, \delta_{l_k}(t)I_{r_k}, F_{l_1}(t), \dots, F_{l_r}(t)\}$, $\delta_{l_i} \in \mathbb{R}$, $|\delta_{l_i}| \leq 1$, $1 \leq i \leq k$ and $F_{l_j}^T F_{l_j} \leq I$, $1 \leq j \leq r$. We define the set Δ_l as $\Delta_l = \{F_{l_j}^T F_{l_j} \leq I, F_l S_l = S_l F_l \text{ for all } S_l \in \Sigma_l\}$, where $\Sigma_l = \{S_l = \text{blockdiag}[S_{l_1}, S_{l_2}, \dots, S_{l_k}, s_{l_1}I_{f_{l_1}}, \dots, s_{l_r}I_{f_{l_r}}]\}$, S_{l_i} invertible $1 \leq i \leq k$ and $s_{l_j} \in \mathbb{R}$, $s_{l_j} \neq 0$ for $1 \leq j \leq r$. $\bar{F}_1(t)$ also have the same expression of $F_l(t)$ ($l = 0, 1, 2, 3, 4$).

In this paper, we implement the following control law to deal with the desired performance for the uncertain stochastic neutral-type neural networks (1):

$$v(t) = Ly(t), \quad (3)$$

where L denotes the gain matrices of the controller given by $L = XZ^{-1}$.

Remark 1: Note that Assumption 1 is used to linearize the nonlinear functions $f(y(t))$, $g(y(t))$, and $h(y(t))$. And also the activation functions in Assumption 1, are less restrictive than the descriptions on Lipschitz-type activation functions as well as sigmoid activation functions. These functions will playing in our proposed methods to get necessary conditions for the neural networks (1) which leads to globally robustly exponential stable.

Remark 2: The firing rates and the weight coefficients of the neurons depend on certain resistance and capacitance values, which are subject to uncertain parameters in practical implementation of neural networks. So, it is necessary to take uncertainties into account in the H_∞ control neutral type neural networks. However, in [23], the stability problem was only studied for H_∞ control neural networks with discrete delays without uncertain parameters was investigated, no distributed delays term was proposed. In addition to that neutral terms are also not considered in [10]. It means that the results, in [28] fail to deal with

this cases for globally robustly exponential stability conditions and thus Theorem 2 presented here is more general and practical than the existing results in [10, 23, 28].

Definition 1: The equilibrium point of time-delayed stochastic neural networks (1) is said to be asymptotically stable in the mean square sense if, when $u(t) = 0$, for all finite initial functions $\varphi \in \mathbb{R}^n$ defined on $[-K, 0]$, the following equation is satisfied: $\lim_{t \rightarrow \infty} \mathbb{E}\{|y(t, \varphi)|^2\} = 0$.

Definition 2: The time-delayed uncertain neural networks (1) is robustly mean square stabilized if the equilibrium point of the stochastic functional differential equation related to system (1) with $v(t) = Ly(t)$ is globally asymptotically stable for admissible uncertainties $\Delta A(t)$, $\Delta B(t)$, $\Delta E(t)$, $\Delta E_s(t)$, $\Delta V_0(t)$, $\Delta V_1(t)$, $\Delta V_2(t)$ and $\Delta U_1(t)$ in the mean square sense.

Definition 3: The zero solution of the system (1) when $u(t) = 0$ is said to be globally robustly exponential stable in the mean square sense if, there exist constants $\rho > 0$ and $\alpha > 0$ such that for $t \geq 0$

$$\mathbb{E}\{\|y_t\|^2\} \leq \rho e^{-2\alpha t} \mathbb{E}\left\{\sup_{-k_m, -d_m \leq s \leq 0} (\|\varphi(s)\|^2 + \|\dot{\varphi}(s)\|^2)\right\},$$

where $\varphi(t)$ is the initial function of y_t .

The following useful lemmas will play an important role in proving our main theorems as mentioned below:

Lemma 1 [13]: For any $z, y \in \mathbb{R}^n$ and for any symmetric positive-definite matrix $Q \in \mathbb{R}^{n \times n}$, we have

$$-2z^T y \leq z^T Q^{-1} z + y^T Q y.$$

Lemma 2 [13]: Let A, D, E, F and O be real matrices of appropriate dimensions with $O = O^T > 0$ and $F_l \in \Sigma_l$ satisfying $O^{-1} - D(N_l N_l^T) D^T > 0$, we will have

$$(A + DF_l E)^T O (A + DF_l E) \leq A^T (O^{-1} - D(N_l N_l^T) D^T)^{-1} A + E^T (N_l N_l^T)^{-1} E.$$

Lemma 3 [13]: Let D, E and F be real matrices of appropriate dimensions and $F_l \in \Sigma_l$, then for any block-structured matrix $N_l \in \Sigma_l$ we will have

$$DF_l E + (DF_l E)^T \leq D(N_l N_l^T) D^T + E^T (N_l N_l^T)^{-1} E.$$

Lemma 4 [29]: The LMI

$$\begin{pmatrix} S_{11} & S_{12} \\ * & S_{22} \end{pmatrix} < 0,$$

with $S_{11} = S_{11}^T$, and $S_{22} = S_{22}^T$, is equivalent to $S_{22} < 0$, and $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

Lemma 5 [30]: For any positive definite matrix $B > 0$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \rightarrow \mathbb{R}^n$ such that

the integrations concerned are well defined, the following inequality holds:

$$\begin{aligned} & \left(\int_0^\gamma \omega(s) ds \right)^T B \left(\int_0^\gamma \omega(s) ds \right) \\ & \leq \gamma \left(\int_0^\gamma \omega(s)^T B \omega(s) ds \right). \end{aligned}$$

Lemma 6 [14]: Let $R = R^T$, H and E be real matrices of appropriate dimensions, with F satisfying $F^T F \leq I$, then

$$R + HFE + E^T F^T H^T < 0$$

if and only if there exists a positive scalar $\varepsilon > 0$ such that

$$R + \frac{1}{\varepsilon} H H^T + \varepsilon E^T E < 0.$$

3. MAIN RESULTS

3.1. Robust stabilization in the mean square sense

Theorem 1: Under Assumptions 1 and 2, the time-delayed structured uncertain neutral-type neural networks (1) is robustly stabilized in the mean square sense with $u(t) = 0$ if $\exists$ symmetric positive definite matrices $Z, Q, M_1, J_0, J_1, J_2, J_3, J_4, M_2, M_3, M_4$ & M_5 and a matrix X satisfying the following LMIs:

$$\begin{pmatrix} -Z & P J_3 \\ * & -J_3 \end{pmatrix} < 0, \quad (4)$$

$$\Omega = \begin{pmatrix} \Pi & \Xi \\ * & \Theta \end{pmatrix} < 0 \quad (5)$$

holds, where

$$\begin{aligned} \Pi &= \begin{pmatrix} \hat{\Omega}_1^* & V_0 & V_1 & V_2 & 0 & 0 & 0 & \Phi_8 & \Phi_9 & \Phi_{10} \\ * & \hat{\Omega}_2' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \hat{\Omega}_3' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \hat{\Omega}_4' & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \hat{\Omega}_5' & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \hat{\Omega}_6' & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \hat{\Omega}_7' & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \hat{\Omega}_8' & 0 & 0 \\ * & * & * & * & * & * & * & * & \hat{\Omega}_9' & 0 \\ * & * & * & * & * & * & * & * & * & \hat{\Omega}_{10}' \end{pmatrix}, \end{aligned}$$

$$\Theta = \text{diag}\{\hat{\Omega}_{11}', \hat{\Omega}_{12}', \hat{\Omega}_{13}', \hat{\Omega}_{14}', \hat{\Omega}_{15}', \hat{\Omega}_{16}', \hat{\Omega}_{17}'\},$$

$$\hat{\Omega}_1^* = -ZA - AZ + BX + X^T B^T + P J_4 P^T - Z M_5 Z;$$

$$\hat{\Omega}_2' = -I + N_{V_0}^T J_0 N_{V_0}; \quad \hat{\Omega}_3' = -(1 - k_D) M_2 + N_{V_1}^T J_1 N_{V_1};$$

$$\hat{\Omega}_4' = -M_3 + N_{V_2}^T J_2 N_{V_2}; \quad \hat{\Omega}_5' = M_2; \quad \hat{\Omega}_6' = \sigma_M^2 M_3;$$

$$\hat{\Omega}_7' = M_4 + d_M^2 M_5; \quad \hat{\Omega}_8' = -M_5; \quad \hat{\Omega}_9' = -(1 - d_D) M_4;$$

$$\hat{\Omega}_{10}' = \frac{-1}{2} J_3; \quad \Phi_{17} = ZP; \quad \hat{\Omega}_{11}' = \frac{-(1 - k_D)}{2} J_3;$$

$$\hat{\Omega}_{12}' = -J_4; \quad \hat{\Omega}_{13}' = -I; \quad \hat{\Omega}_{14}' = -J_0; \quad \hat{\Omega}_{15}' = -J_1;$$

$$\hat{\Omega}'_{16} = -J_2; \quad \hat{\Omega}'_{17} = -T_1; \quad \Phi_8 = ZM_5Z; \quad \Phi_9 = ZU_1;$$

$$\Xi = \begin{pmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi_{14} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Phi_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Phi_{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Phi_{17} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Phi_{10} = ZN_E^T; \quad \Phi_{11} = ZN_{E_s}^T;$$

$$\Phi_{12} = (ZN_AZ - ZN_BX)^T; \quad \Phi_{13} = ZGZ;$$

$$\Phi_{14} = ZP; \quad \Phi_{15} = ZP; \quad \Phi_{16} = ZP;$$

and * denotes the symmetric terms of the matrix.

In the above LMIs, J_0, J_1, J_2, J_3, J_4 and T_1 respectively take the forms of $(S_0S_0^T)^{-1}, (S_1S_1^T)^{-1}, (S_2S_2^T)^{-1}, (S_3S_3^T), (S_4S_4^T)$ and $(\bar{S}_1\bar{S}_1^T)^{-1}$ for $S_l, \bar{S}_1 \in \Sigma_l$ ($l = 0, 1, 2, 3, 4$).

Proof: First, applying the controller $v(t) = Ly(t)$ to the system (1) with $u(t) = 0$, then we have the following system:

$$\begin{aligned} \dot{y}(t) = & \left[-\bar{A}y(t) + \bar{V}_0f(y(t)) + \bar{V}_1g(y(t-k(t))) \right. \\ & + \bar{V}_2 \int_{t-\sigma(t)}^t h(y(s))ds + [\Delta Ey(t) + \Delta E_s \\ & \times y(t-k(t))d\omega(t) + \bar{U}_1\dot{y}(t-d(t))] \Big], \end{aligned} \quad (6)$$

where $\bar{A} = (A + \Delta A(t)) - (B + \Delta B(t))L$, $\bar{V}_0 = V_0 + \Delta V_0(t)$, $\bar{V}_1 = V_1 + \Delta V_1(t)$, and $\bar{V}_2 = V_2 + \Delta V_2(t)$, $\bar{U}_1 = U_1 + \Delta U_1(t)$.

We introduce the Lyapunov-Krasovskii functional candidate $V(y_t, t)$ as follows:

$$\begin{aligned} V(y_t, t) = & y^T(t)Qy(t) + \int_{t-k(t)}^t y^T(s)M_1y(s)ds \\ & + \int_{t-k(t)}^t g^T(y(s))M_2g(y(s))ds \\ & + \sigma_M \int_{t-\sigma_M}^t (s - (t - \sigma_M))h^T(y(s))M_3 \\ & \times h(y(s))ds + \int_{t-d(t)}^t \dot{y}^T(s)M_4\dot{y}(s)ds \\ & + d_M \int_{-d_M}^0 \int_{t+\theta}^t \dot{y}^T(s)M_5\dot{y}(s)dsd\theta, \end{aligned} \quad (7)$$

where M_1 is given by $M_1 = \frac{2}{1-k_D}N_{E_s}^TJ_3^{-1}N_{E_s}$, then the averaged derivative will be given by the following expressions:

$$\begin{aligned} \mathcal{L}V(y_t) = & y^T(t)[-Q\bar{A} - \bar{A}^TQ]y(t) + 2y^T(t)Q\bar{V}_0f(y(t)) \\ & + 2y^T(t)Q\bar{V}_1g(y(t-k(t))) \\ & + 2y^T(t)Q\bar{V}_2 \int_{t-\sigma(t)}^t h(y(s))ds \end{aligned}$$

$$\begin{aligned} & + 2y^T(t)Q\bar{U}_1\dot{y}(t-d(t)) \\ & + [\Delta Ey(t) + \Delta E_sy(t-k(t))]^T \\ & \times Q[\Delta Ey(t) + \Delta E_sy(t-k(t))] + y^T(t)M_1y(t) \\ & - (1 - \dot{k}(t))y^T(t-k(t))M_1y(t-k(t)) \\ & + g^T(y(t))M_2g(y(t)) \\ & - (1 - \dot{k}(t))g^T(y(t-k(t)))M_2g(y(t-k(t))) \\ & + \sigma_M^2 h^T(y(t))M_3h(y(t)) \\ & - \sigma_M \int_{t-\sigma_M}^t h^T(y(s))M_3h(y(s))ds \\ & + f^T(y(t))f(y(t)) - f^T(y(t))f(y(t)) \\ & + \dot{y}^T(t)M_4\dot{y}(t) + d_M^2 \dot{y}^T(t)M_5\dot{y}(t) \\ & - (1 - d_D)\dot{y}^T(t-d(t))M_4\dot{y}(t-d(t)) \\ & - d_M \int_{t-d_M}^t \dot{y}^T(s)M_5\dot{y}(s)ds. \end{aligned} \quad (8)$$

Let $Z = Q^{-1}$, then by Lemma 4, the inequality (4) will be equivalent to

$$Q^{-1} - PJ_3P^T > 0. \quad (9)$$

Therefore, it follows from Lemmas 1 and 3 that,

$$\begin{aligned} & [\Delta Ey(t) + \Delta E_sy(t-k(t))]^T Q \\ & \times [\Delta Ey(t) + \Delta E_sy(t-k(t))] \\ & = \{PF_3(t)[N_Ey(t) + N_{E_s}y(t-k(t))]\}^T \\ & \times Q\{PF_3(t)[N_Ey(t) + N_{E_s}y(t-k(t))]\} \\ & \leq [N_Ey(t) + N_{E_s}y(t-k(t))]^T J_3^{-1} \\ & \times [N_Ey(t) + N_{E_s}y(t-k(t))] \\ & \leq [N_Ey(t)]^T J_3^{-1} [N_Ey(t)] \\ & + [N_{E_s}y(t-k(t))]^T J_3^{-1} [N_{E_s}y(t-k(t))] \\ & + [N_Ey(t)]^T J_3^{-1} [N_Ey(t)] \\ & + [N_{E_s}y(t-k(t))]^T J_3^{-1} J_3 J_3^{-1} [N_{E_s}y(t-k(t))] \\ & = 2[N_Ey(t)]^T J_3^{-1} [N_Ey(t)] + 2[N_{E_s} \\ & \times y(t-k(t))]^T J_3^{-1} [N_{E_s}y(t-k(t))]. \end{aligned} \quad (10)$$

From Lemma 5 and Assumption 1, we obtain

$$\begin{aligned} & -\sigma_M \int_{t-\sigma_M}^t h^T(y(s))M_3h(y(s))ds \\ & \leq -\left[\int_{t-\sigma(t)}^t h(y(s))ds \right]^T M_3 \left[\int_{t-\sigma(t)}^t h(y(s))ds \right], \end{aligned} \quad (11)$$

$$f^T(y(t))f(y(t)) \leq y^T(t)GGy(t), \quad (12)$$

$$\begin{aligned} & -d_M \int_{t-d_M}^t \dot{y}^T(s)M_5\dot{y}(s)ds \\ & \leq \begin{pmatrix} y(t) \\ y(t-d(t)) \end{pmatrix}^T \begin{pmatrix} -M_5 & M_5 \\ * & -M_5 \end{pmatrix} \begin{pmatrix} y(t) \\ y(t-d(t)) \end{pmatrix}. \end{aligned} \quad (13)$$

$$\Xi' = \begin{pmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Psi_{14} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Psi_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Psi_{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Psi_{17} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$\Theta' = \text{diag}\{\hat{\Omega}'_{11}, \hat{\Omega}'_{12}, \hat{\Omega}'_{13}, \hat{\Omega}'_{14}, \hat{\Omega}'_{15}, \hat{\Omega}'_{16}, \hat{\Omega}'_{17}\}$; $\hat{\Omega}'_1 = -Q(A - BL) - (A - BL)^T Q + (QP)J_4(QP)^T - M_5$; $\Psi_2 = QV_0$; $\Psi_3 = QV_1$; $\Psi_4 = QV_2$; $\Psi_9 = QU_1$; $\Psi_{10} = N_{E_s}^T$; $\Psi_{11} = N_{E_s}^T$; $\Psi_{13} = G$; $\Psi_{14} = QP$; $\Psi_{15} = QP$; $\Psi_{16} = QP$; $\Psi_{17} = QP$ and $\Psi_{12} = (N_A - N_B L)^T$.

Now pre and post-multiply the matrix Ω' by $\text{diag}(Z, I, I, I, I, I, I, I, I, I, I, I, I, I, I, I, I)$ and let $X = LZ$, we can see that $\Omega' < 0$ is equivalent to LMI (5).

Note that LMI (5) implies that there exists a scalar $\eta > 0$ such that

$$\Phi + \text{dia}(\eta I, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) < 0.$$

From this and the inequality (15), we obtain $\mathcal{LV}(y_t) \leq -\eta|y(t)|^2$

According to Itô's formula, the neural networks (1) is robustly stabilized in the mean square. The proof of this theorem is completed. $\square$

Remark 3: In Theorem 1, the time delay-dependent robust stabilized criterion for the system (1) with $u(t) = 0$ be established. For consequence, a different Lyapunov-Krasovskii functional handled in the following Theorem 2 to guarantee the stochastic neutral-type neural networks (1) with $u(t) = 0$ is globally robustly exponential stable in the mean square sense.

3.2. Globally robustly exponential stability in the mean square sense

Theorem 2: Under Assumptions 1 and 2, the equilibrium point of the time-delayed neural networks (1) is said to be globally robustly exponential stable with $u(t) = 0$ in the mean square sense, if there exist symmetric positive definite matrices $Z, J_3, Q, M_1, M_2, M_3, M_4, M_5$ and a matrix X satisfying the following LMIs:

$$\begin{pmatrix} -Z & PJ_3 \\ * & -J_3 \end{pmatrix} < 0, \quad (17)$$

$$\Omega^* = \begin{pmatrix} \Pi^* & \Xi^* \\ * & \Theta^* \end{pmatrix} < 0 \quad (18)$$

holds, where

$$\Pi^* = \begin{pmatrix} \hat{Y}_1 & \Lambda_2 & \Lambda_3 & \Lambda_4 & 0 & 0 & 0 & M_5 & \Lambda_9 & \Lambda_{10} \\ * & \hat{Y}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \hat{Y}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \hat{Y}_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \hat{Y}_5 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \hat{Y}_6 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \hat{Y}_7 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \hat{Y}_8 & 0 & 0 \\ * & * & * & * & * & * & * & * & \hat{Y}_9 & 0 \\ * & * & * & * & * & * & * & * & * & \hat{Y}_{10} \end{pmatrix},$$

$$\Xi^* = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Lambda_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda_{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{17} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$\Theta^* = \text{diag}\{\hat{Y}_{11}, \hat{Y}_{12}, \hat{Y}_{13}, \hat{Y}_{14}, \hat{Y}_{15}, \hat{Y}_{16}, \hat{Y}_{17}\}$; $\hat{Y}_1 = -Q(A - BL) - (A - BL)^T Q + 2N_{E_s}^T J_3^{-1} N_{E_s} + \frac{2}{1-k_D} N_{E_s}^T J_3^{-1} N_{E_s} + GG - M_5 - QP(N_A - N_B L) - QP(N_A - N_B L)^T$; $\hat{Y}_2 = -I$; $\hat{Y}_3 = -(1 - k_D)e^{-2\alpha_{k_M}} M_2$; $\hat{Y}_4 = -M_3$; $\hat{Y}_5 = M_2$; $\hat{Y}_6 = \sigma_M^2 M_3$; $\hat{Y}_7 = M_4 + d_M^2 M_5$; $\hat{Y}_8 = -M_5$; $\hat{Y}_9 = -(1 - d_D)e^{-2\alpha_{d_M}} M_4$; $\hat{Y}_{10} = -\varepsilon_1 I$; $\hat{Y}_{11} = -\varepsilon_2 I$; $\hat{Y}_{12} = -\varepsilon_3 I$; $\hat{Y}_{13} = -\varepsilon_4 I$; $\hat{Y}_{14} = -\varepsilon_1 I$; $\hat{Y}_{15} = -\varepsilon_2 I$; $\hat{Y}_{16} = -\varepsilon_3 I$; $\hat{Y}_{17} = -\varepsilon_4 I$; $\Lambda_2 = QV_0$; $\Lambda_3 = QV_1$; $\Lambda_4 = QV_2$; $\Lambda_9 = QU_1$; $\Lambda_{10} = \varepsilon_1 QP$; $\Lambda_{11} = \varepsilon_2 QP$; $\Lambda_{12} = \varepsilon_3 QP$; $\Lambda_{13} = \varepsilon_4 QP$; $\Lambda_{14} = N_{V_0}^T$; $\Lambda_{15} = N_{V_1}^T$; $\Lambda_{16} = N_{V_2}^T$; $\Lambda_{17} = N_{U_1}^T$ and * denotes the symmetric terms of the matrices.

Proof: Define the following the Lyapunov-Krasovskii functional :

$$\begin{aligned} V(y_t) = & e^{2\alpha t} y^T(t) Q y(t) + \int_{t-k(t)}^t e^{2\alpha s} y^T(s) M_1 y(s) ds \\ & + \int_{t-k(t)}^t e^{2\alpha s} g^T(y(s)) M_2 g(y(s)) ds \\ & + \sigma_M \int_{t-\sigma_M}^t e^{2\alpha \theta} (s - (t - \sigma_M)) h^T(y(s)) \\ & \times M_3 h(y(s)) ds + \int_{t-d(t)}^t e^{2\alpha s} \dot{y}^T(s) M_4 \dot{y}(s) ds \\ & + d_M \int_{-d_M}^0 \int_{t+\theta}^t e^{2\alpha \theta} \dot{y}^T(s) M_5 \dot{y}(s) ds d\theta, \quad (19) \end{aligned}$$

where M_1 is given by $M_1 = \frac{2}{1-k_D} N_{E_s}^T J_3^{-1} N_{E_s}$, then by the following expressions, the averaged derivative will be given

$$\mathcal{LV}(y_t)$$

$$\begin{aligned}
&= 2\alpha e^{2\alpha t} y^T(t) Q y(t) + e^{2\alpha t} y^T(t) [-Q\bar{A} \\
&\quad - \bar{A}^T Q] y(t) + 2e^{2\alpha t} y^T(t) Q \bar{V}_0 f(y(t)) \\
&\quad + 2e^{2\alpha t} y^T(t) Q \bar{V}_1 g(y(t-k(t))) \\
&\quad + 2e^{2\alpha t} y^T(t) Q \bar{V}_2 \int_{t-\sigma(t)}^t h(y(s)) ds + 2e^{2\alpha t} \\
&\quad \times y^T(t) Q \bar{U}_1 \dot{y}(t-d(t)) \\
&\quad + e^{2\alpha t} [\Delta E y(t) + \Delta E_s y(t-k(t))]^T Q \\
&\quad \times [\Delta E y(t) + \Delta E_s y(t-k(t))] \\
&\quad + e^{2\alpha t} y^T(t) M_1 y(t) - (1-\dot{k}(t)) e^{2\alpha t-k(t)} y^T \\
&\quad \times (t-k(t)) M_1 y(t-k(t)) \\
&\quad + e^{2\alpha t} g^T(y(t)) M_2 g(y(t)) \\
&\quad - (1-\dot{k}(t)) e^{2\alpha(t-k(t))} g^T(y(t-k(t))) \\
&\quad \times M_2 g(y(t-k(t))) \\
&\quad + \sigma_M^2 e^{2\alpha t} h^T(y(t)) M_3 h(y(t)) \\
&\quad - \sigma_M \int_{t-\sigma_M}^t e^{2\alpha s} h^T(y(t)) M_3 h(y(t)) ds \\
&\quad + e^{2\alpha t} f^T(y(t)) f(y(t)) - e^{2\alpha t} f^T(y(t)) f(y(t)) \\
&\quad + e^{2\alpha t} \dot{y}^T(t) M_4 \dot{y}(t) + d_M^2 e^{2\alpha t} \dot{y}^T(t) M_5 \dot{y}(t) \\
&\quad - (1-d_D) e^{2\alpha(t-d(t))} \dot{y}^T(t-d(t)) M_4 \dot{y}(t-d(t)) \\
&\quad - d_M \int_{t-d_M}^t e^{2\alpha s} \dot{y}^T(s) M_5 \dot{y}(s) ds. \quad (20)
\end{aligned}$$

Put $Z = Q^{-1}$, then by Lemma 4, inequality (17) will be equivalent to

$$Q^{-1} - P J_3 P^T > 0. \quad (21)$$

We proceed similar way of Theorem 1 and we get,

$$\begin{aligned}
\mathcal{L}V &\leq e^{2\alpha t} \{ y^T(t) [-Q(A-BL) \\
&\quad - (A-BL)^T Q + 2N_E^T J_3^{-1} N_E \\
&\quad + \frac{2}{1-k_D} N_{E_s}^T J_3^{-1} N_{E_s} + GG - M_5] y(t) \\
&\quad + f^T(y(t)) [-I] f(y(t)) + g^T(y(t-k(t))) \\
&\quad \times [-(1-\dot{k}(t)) M_2] g(y(t-k(t))) \\
&\quad + \left(\int_{t-\sigma(t)}^t h(y(s)) ds \right)^T [-M_3] \\
&\quad \times \left(\int_{t-\sigma(t)}^t h(y(s)) ds \right) \\
&\quad + g^T(y(t)) M_2 g(y(t)) \\
&\quad + h^T(y(t)) \sigma_M^2 M_3 h(y(t)) \\
&\quad + \dot{y}^T [M_4 + d_M^2 M_5] \dot{y}(t) \\
&\quad + \dot{y}^T(t-d(t)) [-(1-d_D) M_4] \\
&\quad \times \dot{y}(t-d(t)) \\
&\quad + y^T(t-d(t)) [-M_5] y(t-d(t)) \\
&\quad + 2y^T(t) Q \bar{V}_0 f(y(t)) \\
&\quad + 2y^T(t) Q \bar{V}_1 g(y(t-k(t)))
\end{aligned}$$

$$\begin{aligned}
&+ 2y^T(t) Q \bar{V}_2 \int_{t-\sigma}^t h(y(s)) ds + 2y^T(t) M_5 \\
&\quad \times y(t-d(t)) + 2y^T(t) Q \bar{U}_1 \dot{y}(t-d(t)) \}, \\
\mathcal{L}V(y_t) &\leq e^{2\alpha t} \Xi^T(t) \Omega^* \Xi(t), \quad (22)
\end{aligned}$$

where

$$\begin{aligned}
\Omega^* &= \Omega_1^* + Q_1 F_0(t) S_0 + S_0^T F_0^T(t) Q_1^T \\
&\quad + Q_1 F_1(t) S_1 + S_1^T F_1^T(t) Q_1^T + Q_1 F_2(t) S_2 \\
&\quad + S_2^T F_2^T(t) Q_1^T + Q_1 \bar{F}_1(t) \bar{S}_1 + \bar{S}_1^T \bar{F}_1^T(t) Q_1, \\
\Xi(t) &= \begin{bmatrix} y^T(t), f^T(y(t)), g^T(y(t-k(t))), \\ \left(\int_{t-\sigma(t)}^t h(y(s)) ds \right)^T, g^T(y(t)), h^T(y(t)), \\ \dot{y}^T(t), y^T(t-d(t)), \dot{y}^T(t-d(t)) \end{bmatrix}^T, \\
\Omega_1^* &= \begin{pmatrix} \Omega_{11}^* & QV_0 & QV_1 & QV_2 \\ * & -I & 0 & 0 \\ * & * & -(1-\dot{k}(t)) e^{-2\alpha k(t)} M_2 & 0 \\ * & * & * & -M_3 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & M_5 & QU_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ M_2 & 0 & 0 & 0 & 0 \\ * & \sigma_M^2 M_3 & 0 & 0 & 0 \\ * & * & M_4 + d_M^2 M_5 & 0 & 0 \\ * & * & * & -M_5 & 0 \\ * & * & * & * & \Omega_{99} \end{pmatrix}
\end{aligned}$$

with $\Omega_{11}^* = -Q(A-BL) - (A-BL)^T Q + 2N_E^T J_3^{-1} N_E + \frac{2}{1-k_D} N_{E_s}^T J_3^{-1} N_{E_s} + GG - M_5 - QP(N_A - N_B L) - QP(N_A - N_B L)^T$; $\Omega_{99} = -(1-d_D) e^{-2\alpha d(t)} M_4$;

$$\begin{aligned}
Q_1 &= (QP \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)^T, \\
S_0 &= (0 \ N_{V_0} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0), \\
S_1 &= (0 \ 0 \ N_{V_1} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0), \\
S_2 &= (0 \ 0 \ 0 \ N_{V_2} \ 0 \ 0 \ 0 \ 0 \ 0), \\
\bar{S}_1 &= (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ N_{U_1}).
\end{aligned}$$

Now by using Lemma 6, we get

$$\Omega \leq \Omega_1$$

$$\begin{aligned}
& + \varepsilon_1 \begin{pmatrix} QP \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} QP \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T + \varepsilon_1^{-1} \begin{pmatrix} 0 \\ N_{V_0}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ N_{V_0} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \\
& + \varepsilon_2 \begin{pmatrix} QP \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} QP \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T + \varepsilon_2^{-1} \begin{pmatrix} 0 \\ N_{V_1}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ N_{V_1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \\
& + \varepsilon_3 \begin{pmatrix} QP \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} QP \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T + \varepsilon_3^{-1} \begin{pmatrix} 0 \\ N_{V_2}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ N_{V_2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \\
& + \varepsilon_4 \begin{pmatrix} QP \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} QP \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T + \varepsilon_4^{-1} \begin{pmatrix} 0 \\ N_{U_1}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ N_{U_1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T.
\end{aligned}$$

Thus, we have

$$\mathcal{L}V(y_t) \leq -\lambda_1 \left\{ \|y_t\|^2 + \|\dot{y}_t\|^2 \right\}, \quad (23)$$

where $\lambda_1 = \lambda_{\min}\{\Omega\}$.

Taking expectation on both sides of (23) yields

$$\mathbb{E}\{\mathcal{L}V(y_t)\} \leq -\lambda_1 \mathbb{E}\left\{ \|\dot{y}_t\|^2 \right\}. \quad (24)$$

Now let us define $Y(t) = e^{2\alpha t}V(t)$. Its infinitesimal operator $\mathcal{L}$ is given by

$$\mathcal{L}Y(t) = 2\alpha e^{2\alpha t}V(t) + e^{2\alpha t}\mathcal{L}V(t). \quad (25)$$

Integrating on both sides of (25) and taking expectation gives

$$\mathbb{E}\{Y(t)\} - \mathbb{E}\{Y(0)\}$$

$$\begin{aligned}
& = \int_0^t \mathbb{E}\left\{ 2\alpha e^{2\alpha s}V(s) + e^{2\alpha s}\mathcal{L}V(s) \right\} ds \\
& \leq \int_0^t e^{2\alpha s} \mathbb{E}\left\{ (2\alpha\lambda_{\max}(Q) - \lambda_1) \sup\|y(s)\|^2 \right\} ds \quad (26)
\end{aligned}$$

Choose $2\alpha \leq \min\left\{ \frac{\lambda_1}{\lambda_{\max}(Q)} \right\}$ which implies $\mathbb{E}\{Y(t)\} - \mathbb{E}\{Y(0)\} \leq 0$, i. e.

$$\begin{aligned}
& e^{2\alpha t} \mathbb{E}\{V(t)\} \\
& \leq \mathbb{E}\{V(0)\} \\
& \leq \rho_1 \sup_{-k_M, -d_M \leq s \leq 0} \mathbb{E}\left\{ \|\varphi(s)\|^2 + \|\dot{\varphi}(s)\|^2 \right\}, \quad (27)
\end{aligned}$$

where

$$\begin{aligned}
\rho_1 = \max \big\{ & \lambda_{\max}(Q), k_0\lambda_{\max}(M_1) + k_0\lambda_{\max}(M_2) \\
& + \sigma_M^2\lambda_{\max}(M_3) + d_0\lambda_{\max}(M_4) + d_M^2\lambda_{\max}(M_5) \big\}.
\end{aligned}$$

Mean while, it follows from (19)

$$\mathbb{E}\{V(t)\} \geq \mathbb{E}\left\{ \lambda_{\min}(Q) \|y_t\|^2 \right\}. \quad (28)$$

Combining with (27) and (28) leads to

$$\begin{aligned}
\mathbb{E}\{\|y_t\|^2\} & \leq \rho e^{-2\alpha t} \mathbb{E}\left\{ \sup_{-k_M, -d_M \leq s \leq 0} (\|\varphi(s)\|^2 \right. \\
& \quad \left. + \|\dot{\varphi}(s)\|^2) \right\}, \quad (29)
\end{aligned}$$

where $\rho = \frac{\rho_1}{\lambda_{\min}(Q)}$.

This completes the proof. $\square$

Remark 4: As much as know, all the existing results concerning the dynamical behaviors of neural networks [17, 20, 28, 31, 32] have not considered the globally robustly exponential stability performance in the mean square and time-varying delayed situation, which are investigated via LMI approach in our paper. Therefore, our conclusions are brand-new and compare to previous results.

Remark 5: Table 1 shows that the maximum upper bounds of d_M which guaranteeing the globally robustly exponential stable of the addressed neural networks (1). These tables demonstrate the effectiveness of our proposed method.

Remark 6: Suppose, the uncertain parameters are not present in neural networks (1), then the modified nominal neutral type neural networks as follows:

$$\begin{aligned}
& \dot{y}(t) - U_1\dot{y}(t-d(t)) \\
& = -Ay(t) + V_0f(y(t)) + V_1g(y(t-k(t))) \\
& \quad + Bv(t) + Cu(t) + V_2 \int_{t-\sigma(t)}^t h(y(s))ds, \\
& x(t) = Fy(t) + Dv(t), \quad t > 0, \\
& y(t) = \varphi(t), \quad t \in [-K, 0]. \quad (30)
\end{aligned}$$

Then by Theorem 2, it is easy to have the following Corollary 1.

Corollary 1: Under the Assumptions 1 and 2, the equilibrium point of the time-delayed neutral NNs (1) is said to be globally robustly exponentially stable with $u(t) = 0$ in the mean square sense, if there exist symmetric positive definite matrices $Z, Q, M_1, M_2, M_3, M_4, M_5$ and a matrix X satisfying the following LMIs:

$$-Z < 0, \quad (31)$$

$$\Omega^* = \begin{pmatrix} \Pi_1^* & \Xi_1^* \\ * & \Theta_1^* \end{pmatrix} < 0, \quad (32)$$

hold, where

$$\Pi_1^* = \begin{pmatrix} \hat{\Upsilon}_1 & \Lambda_2 & \Lambda_3 & \Lambda_4 & 0 & 0 & 0 & M_5 & \Lambda_9 & \Lambda_{10} \\ * & \hat{\Upsilon}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \hat{\Upsilon}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \hat{\Upsilon}_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \hat{\Upsilon}_5 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \hat{\Upsilon}_6 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \hat{\Upsilon}_7 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \hat{\Upsilon}_8 & 0 & 0 \\ * & * & * & * & * & * & * & * & \hat{\Upsilon}_9 & 0 \\ * & * & * & * & * & * & * & * & * & \hat{\Upsilon}_{10} \end{pmatrix},$$

$$\Xi_1^* = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\Theta_1^* = \text{diag}\{\hat{\Upsilon}_{11}, \hat{\Upsilon}_{12}, \hat{\Upsilon}_{13}\}; \hat{\Upsilon}_1 = -Q(A - BL) - (A - BL)^T Q + GG - M_5; \hat{\Upsilon}_2 = -I; \hat{\Upsilon}_3 = -(1 - k_D)e^{-2\alpha_{k_M}} M_2; \hat{\Upsilon}_4 = -M_3; \hat{\Upsilon}_5 = M_2; \hat{\Upsilon}_6 = \sigma_M^2 M_3; \hat{\Upsilon}_7 = M_4 + d_M^2 M_5; \hat{\Upsilon}_8 = -M_5; \hat{\Upsilon}_9 = -(1 - d_D)e^{-2\alpha_{d_M}} M_4; \hat{\Upsilon}_{10} = -\varepsilon_1 I; \hat{\Upsilon}_{11} = -\varepsilon_2 I; \hat{\Upsilon}_{12} = -\varepsilon_3 I; \hat{\Upsilon}_{13} = -\varepsilon_4 I; \Lambda_2 = QV_0; \Lambda_3 = QV_1; \Lambda_4 = QV_2; \Lambda_9 = QU_1; \Lambda_{10} = \varepsilon_1 QP; \Lambda_{11} = \varepsilon_2 QP; \Lambda_{12} = \varepsilon_3 QP; \Lambda_{13} = \varepsilon_4 QP$ and $*$ denotes the symmetric terms of the matrices.

Remark 7: In case, the neutral terms and stochastic noises will not be considered in (1), then the mixed time delayed neural networks (1) reduces to

$$\begin{aligned} \dot{y}(t) = & -(A + \Delta A(t))y(t) + (V_0 + \Delta V_0(t))f(y(t)) \\ & + (V_1 + \Delta V_1(t))g(y(t - k(t))) \\ & + (B + \Delta B(t))v(t) + Cu(t) \\ & + (V_2 + \Delta V_2(t)) \int_{t-\sigma(t)}^t h(y(s))ds, \\ x(t) = & Fy(t) + Dv(t), t > 0, \\ y(t) = & \varphi(t), t \in [-K, 0], \end{aligned} \quad (33)$$

Corollary 2: Under the Assumptions 1 and 2, the equilibrium point of the time-delayed neural networks (1) is

said to be globally robustly exponentially stable with $u(t) = 0$ in the mean square sense, if there exist symmetric positive definite matrices $Z, J_3, Q, M_1, M_2, M_3, M_4, M_5$ and a matrix X satisfying the following LMIs:

$$\begin{pmatrix} -Z & PJ_3 \\ * & -J_3 \end{pmatrix} < 0, \quad (34)$$

$$\Omega^* = \begin{pmatrix} \Pi_2^* & \Xi_2^* \\ * & \Theta_2^* \end{pmatrix} < 0 \quad (35)$$

hold, where

$$\Pi_2^* = \begin{pmatrix} \hat{\Upsilon}_1^* & \Lambda_2 & \Lambda_3 & \Lambda_4 & 0 & 0 & 0 & M_5 & \Lambda_9 & \Lambda_{10} \\ * & \hat{\Upsilon}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \hat{\Upsilon}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \hat{\Upsilon}_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \hat{\Upsilon}_5 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \hat{\Upsilon}_6 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \hat{\Upsilon}_7 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \hat{\Upsilon}_8 & 0 & 0 \\ * & * & * & * & * & * & * & * & \hat{\Upsilon}_9 & 0 \\ * & * & * & * & * & * & * & * & * & \hat{\Upsilon}_{10} \end{pmatrix},$$

$$\Xi_2^* = \begin{pmatrix} \Lambda_{11} & 0 & 0 & 0 \\ 0 & \Lambda_{12} & 0 & 0 \\ 0 & 0 & \Lambda_{13} & 0 \\ 0 & 0 & 0 & \Lambda_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$\Theta_2^* = \text{diag}\{\hat{\Upsilon}_{11}, \hat{\Upsilon}_{12}, \hat{\Upsilon}_{13}, \hat{\Upsilon}_{14}\}; \hat{\Upsilon}_1^* = -Q(A - BL) - (A - BL)^T Q + GG - M_5 - QP(N_A - N_B L) - QP(N_A - N_B L)^T; \hat{\Upsilon}_2 = -I; \hat{\Upsilon}_3 = -(1 - k_D)e^{-2\alpha_{k_M}} M_2; \hat{\Upsilon}_4 = -M_3; \hat{\Upsilon}_5 = M_2; \hat{\Upsilon}_6 = \sigma_M^2 M_3; \hat{\Upsilon}_7 = M_4; \hat{\Upsilon}_8 = -M_5; \hat{\Upsilon}_9 = -\varepsilon_1 I; \hat{\Upsilon}_{10} = -\varepsilon_2 I; \hat{\Upsilon}_{11} = -\varepsilon_3 I; \hat{\Upsilon}_{12} = -\varepsilon_1 I; \hat{\Upsilon}_{13} = -\varepsilon_2 I; \hat{\Upsilon}_{14} = -\varepsilon_3 I; \Lambda_2 = QV_0; \Lambda_3 = QV_1; \Lambda_4 = QV_2; \Lambda_9 = \varepsilon_1 QP; \Lambda_{10} = \varepsilon_2 QP; \Lambda_{11} = \varepsilon_3 QP; \Lambda_{12} = N_{V_0}^T; \Lambda_{13} = N_{V_1}^T; \Lambda_{14} = N_{V_2}^T$ and $*$ denotes the symmetric terms of the matrices.

Remark 8: In case, the neutral terms and stochastic noises are not exists in neural networks (1), then we have the simplified neural networks as follows:

$$\begin{aligned} \dot{y}(t) = & -Ay(t) + V_0 f(y(t)) + V_1 g(y(t - k(t))) \\ & + Bv(t) + Cu(t) + V_2 \int_{t-\sigma(t)}^t h(y(s))ds, \\ x(t) = & Fy(t) + Dv(t), t > 0, \\ y(t) = & \varphi(t), t \in [-K, 0]. \end{aligned}$$

Then from Corollary 1, it is easy to have one Corollary for the above mentioned remark with $\Lambda_9 = 0$ and the other entries are same as Corollary 1.

Table 1. Maximum allowable upper bounds of neutral delays (d_M).

Methods	$d_M > 0$
In [33]	0.2951
In [34]	3
In [35]	4.6473
In [36]	5
Corollary 1	28

3.3. Numerical examples

In this section, providing one numerical example to illustrate the superiority and benefits of the proposed criteria.

Example 1: Consider the two-neuron neutral-type neural networks (30) with the following parameters

$$\begin{aligned}
 A &= \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, \quad V_0 = \begin{pmatrix} 0.01 & 0.07 \\ -0.04 & 0.02 \end{pmatrix}, \\
 V_1 &= \begin{pmatrix} 0.05 & 0.01 \\ -0.02 & 0.04 \end{pmatrix}, \quad B = \begin{pmatrix} 0.02 & -0.02 \\ 0.01 & -0.03 \end{pmatrix}, \\
 C &= \begin{pmatrix} 0.03 & -0.08 \\ 0.02 & 0.05 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0.03 & -0.04 \\ 0.02 & -0.02 \end{pmatrix}, \\
 U_1 &= \begin{pmatrix} 0.04 & 0.01 \\ -0.05 & 0.03 \end{pmatrix}, \quad F = \begin{pmatrix} 0.03 & 0.05 \\ 0.01 & 0.04 \end{pmatrix}, \\
 D &= \begin{pmatrix} 0.02 & 0.02 \\ 0.06 & 0.03 \end{pmatrix}, \quad X = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.3 \end{pmatrix}, \\
 Z &= \begin{pmatrix} 0.1 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad G = \begin{pmatrix} 0.03 & 0 \\ 0 & 0.05 \end{pmatrix}, \\
 L &= \begin{pmatrix} 0.02 & 0.04 \\ -0.03 & 0.05 \end{pmatrix}, \quad P = \begin{pmatrix} 0.04 & 0.07 \\ 0.01 & 0.02 \end{pmatrix}, \\
 I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
 \end{aligned}$$

Now, let $k_D = 0.5$, $\sigma_M = 25.5$, $d_D = 0.6$, $d_M = 28$, $k_M = 18.29$, $\alpha = 0.32$.

Here also the same activation functions in Theorem 1, are used for neural networks (30).

Taking the disturbance input $u(t) = 0.3 * \sin(0.2 * \exp(0.6 * t))$. Then solving the LMIs in Corollary 1 by aid of Matlab LMI control toolbox, one can obtain the following feasible solutions:

$$\begin{aligned}
 M_2 &= \begin{pmatrix} 3.0217 & -0.1811 \\ -0.1811 & 3.0243 \end{pmatrix}, \\
 M_3 &= \begin{pmatrix} 378.5930 & 17.0042 \\ 17.0042 & 391.0258 \end{pmatrix}, \\
 M_4 &= 10^5 \times \begin{pmatrix} 3.8890 & 0.7009 \\ 0.7009 & 4.5935 \end{pmatrix}, \\
 M_5 &= \begin{pmatrix} 408.2757 & 83.1658 \\ 83.1658 & 503.1650 \end{pmatrix},
 \end{aligned}$$

$$Q = \begin{pmatrix} 15.9011 & 7.8500 \\ 7.8500 & 6.3157 \end{pmatrix},$$

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 79.4453.$$

By utilizing Matlab LMI control toolbox, we have the comparison of upper bounds of neutral delay as listed in Table 1. Therefore, according to Corollary 1, we conclude that (30) is globally robustly exponentially stable in the mean square sense.

4. CONCLUSIONS

In this paper, the global robust exponential stability problem has been proposed for H_∞ control of stochastic neutral-type neural networks with parameter uncertainties and both time-varying delays. By a key role of the Lyapunov technique, M-matrix theory, stability analysis procedure and Inequality techniques, some brand-new LMIs conditions are derived to explore the uncertain neutral-type neural networks is globally robustly exponentially stable in the mean square sense. Additionally, our proposed method provide the allowable upper bounds of the discrete, distributed and neutral time-varying delays are very large when compared with the some previous literature, which may possess highly important significance in the design of time-delayed addressed neural networks with uncertain parameters. Finally, four numerical examples with simulations are given to demonstrate the superiority of the proposed theoretical outcomes.

To the best of our knowledge, there are no results on the stability criterion for H_∞ control analysis of descriptor-type neural networks with both time-varying delays, which might be our future research work.

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RESEARCH

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Global exponential stability of Markovian jumping stochastic impulsive uncertain BAM neural networks with leakage, mixed time delays, and α -inverse Hölder activation functions

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Abstract

This paper concerns the problem of enhanced results on robust finite time passivity for uncertain discrete time Markovian jumping BAM delayed neural networks with leakage delay. By implementing a proper Lyapunov–Krasovskii functional candidate, reciprocally convex combination method, and linear matrix inequality technique, we derive several sufficient conditions for varying the passivity of discrete time BAM neural networks. Further, some sufficient conditions for finite time boundedness and passivity for uncertainties are proposed by employing zero inequalities. Finally, the enhancement of the feasible region of the proposed criteria is shown via numerical examples with simulation to illustrate the applicability and usefulness of the proposed method.

Keywords: LMIs; Markovian jumping systems; Leakage delay; Bidirectional associative memory; Discrete time neural networks; Passivity and stability analysis

1 Introduction and problem statement with preliminaries

There has been a growing research interest in the field of recurrent neural networks (RNNs) largely studied by many researchers in recent years. The network architecture includes various types of neural networks such as bidirectional associative memory (BAM) neural networks, Hopfield neural networks, cellular neural networks, Cohen–Grossberg neural networks, neural and social networks which have received great attention due to their wide applications in the field of classification, signal and image processing, parallel computing, associate memories, optimization, cryptography, and so on. The bidirectional associative memory (BAM) neural network models were initially coined by Kosko, see [1, 2]. This network has an extraordinary class of RNNs which can have the ability to store bipolar vector pairs. It is composed of neurons and is arranged in two layers, one is the X-layer and the other is the Y-layer. The neurons in one layer are fully interconnected to the neurons in the other layer. The BAM neural networks are designed in such a way that, for a given external input, they can reveal only one global asymptotic or exponential stability equilibrium point. Hence, considerable efforts have been made in the study of stability

analysis of neural networks, and as a credit to this, a large number of sufficient conditions have been proposed to guarantee the global asymptotic or exponential stability for the addressed neural networks.

Furthermore, the existence of time delays in the network will result in bad performance, instability or chaos. Accordingly, the time delays can be classified into two types: discrete and distributed delays. Here, we have taken both the delays into account while modeling our network system, because the length of the axon sizes is too large. So, it is noteworthy to inspect the dynamical behaviors of neural systems with both time delays, see, for instance, [3–11].

In [12], Shu et al. considered the BAM neural networks with discrete and distributed time delays. Some sufficient conditions were obtained to ensure the global asymptotic stability [12]. Also, time delays in the leakage term have great impact on the dynamic behavior of neural networks. However, so far, there have been a very few existing works on neural networks with time delay in the leakage term, see, for instance, [13–17].

Further the stability performance of state variable with leakage time delays was discussed by Lakshmanan et al. in [18]. While modeling a real nervous system, stochastic noises and parameter uncertainties are inevitable and should be taken into account. In the real nervous system, the synaptic transmission has created a noisy process brought on by apparent variation from the release of neurotransmitters and the connection weights of the neuron completely depend on undisputed resistance and capacitance values. Therefore, it is of practical significance to investigate the stochastic disruption in the stability of time-delayed neural networks with parameter uncertainties, see references cited therein [19–22]. Moreover, the hasty consequence (impulsive effect) is probable to exist in a wide variety of evolutionary processes that in turn make changes in the states abruptly at certain moments of time [23–28].

The conversion of the parameters jump will lead to a finite-state Markov process. Recently, the researchers in [29, 30] investigated the existence of Markovian jumps in BAMNNs and exploited the stochastic LKF approach, the new sufficient conditions were derived for the global exponential stability in the mean square.

The BAM-type NNs with Markovian jumping parameters and leakage terms were described by Wang et al. in [31]. In [32], a robust stability problem was studied and some delay-dependent conditions were derived for the neutral-type NNs with time-varying delays. The authors in [33–35] developed some conditions for the stability analysis of neural networks with integral inequality approach. The criteria to obtain the stability result of neural networks with time-varying delays were checked in [36–38]. It should be noted that, with all the consequences reported in the literature above, they are concerned only with Markovian jumping SNNs with Lipschitz model neuron activation functions. Up to now, very little attention has been paid to the problem of the global exponential stability of Markovian jumping SBAMNNs with non-Lipschitz type activation functions, which frequently appear in realistic neural networks. This situation motivates our present problem, i.e., α -inverse holder activation functions.

Our main objective of this paper is to study the delay-dependent exponential stability problem for a class of Markovian jumping uncertain BAM neural networks with mixed time delays, leakage delays, and α -inverse Holder activation functions under stochastic noise perturbation.

To the best of authors knowledge, so far, no result on the global exponential stability of Markovian jumping stochastic impulsive uncertain BAM neural networks with leakage, mixed time delays, and α -inverse Hölder activation functions has been available in the existing literature, which motivates our research to derive the following BAM neural networks:

$$\begin{aligned} dx(t) &= \left[-Cx(t - \nu_1) + W_0 f(y(t)) + W_1 g(y(t - \tau_1(t))) + W_2 \int_{t-\sigma_1}^t h(y(s)) ds + I \right] dt \\ &\quad + \rho_1(x(t - \nu_1), y(t), y(t - \tau_1(t)), t) d\omega(t); \quad t > 0, t \neq t_k, \\ \Delta x(t_k) &= M_k(x(t_k^-), x_{t_k^-}); \quad t = t_k, k \in \mathbb{Z}_+, \\ dy(t) &= \left[-Dy(t - \nu_2) + V_0 \tilde{f}(x(t)) + V_1 \tilde{g}(x(t - \tau_2(t))) + V_2 \int_{t-\sigma_2}^t \tilde{h}(x(s)) ds + J \right] dt \\ &\quad + \rho_2(y(t - \nu_2), x(t), x(t - \tau_2(t)), t) d\tilde{\omega}(t); \quad t > 0, t \neq t_k, \\ \Delta y(t_k) &= N_k(y(t_k^-), y_{t_k^-}); \quad t = t_k, k \in \mathbb{Z}_+, \end{aligned} \quad (1)$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$ and $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T \in \mathbb{R}^n$ denote the states at time t ; $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ and $\tilde{f}(\cdot)$, $\tilde{g}(\cdot)$, $\tilde{h}(\cdot)$ denote the neuron activation functions, $C = \text{diag}\{c_i\}$, $D = \text{diag}\{d_j\}$ are positive diagonal matrices; $c_i > 0$, $d_j > 0$, $i, j = 1, 2, \dots, n$, are the neural self inhibitions; $W_0 = (W_{0ij})_{n \times n}$, $V_0 = (V_{0ij})_{n \times n}$ are the connection weight matrices; $W_1 = (W_{1ji})_{n \times n}$, $V_1 = (V_{1ij})_{n \times n}$ are the discretely delayed connection weight matrices; and $W_2 = (W_{2ji})_{n \times n}$, $V_2 = (V_{2ij})_{n \times n}$ are the distributively delayed connection weight matrices; $I = (I_1, I_2, \dots, I_n)^T$ and $J = (J_1, J_2, \dots, J_n)^T$ are the external inputs; $\tau_1(t)$ and $\tau_2(t)$ are the discrete time-varying delays which are bounded with $0 < \tau_1(t) < \bar{\tau}_1$, $\dot{\tau}_1(t) \leq \tau_1 < 1$, and $0 < \tau_2(t) < \bar{\tau}_2$, $\dot{\tau}_2(t) \leq \tau_2 < 1$, respectively; σ_1 and σ_2 are constant delays. The leakage delays $\nu_1 \geq 0$, $\nu_2 \geq 0$ are constants; $\rho_1 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ and $\rho_2 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ denote the stochastic disturbances $\omega(t) = (\omega_1(t), \omega_2(t), \dots, \omega_n(t))^T$ and $\tilde{\omega}(t) = (\tilde{\omega}_1(t), \tilde{\omega}_2(t), \dots, \tilde{\omega}_n(t))^T$ are n -dimensional Brownian motions defined on a complete probability space $(\mathbb{A}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets) and $\mathbb{E}\{d\omega(t)\} = \mathbb{E}\{d\tilde{\omega}(t)\} = 0$, $\mathbb{E}\{d\omega^2(t)\} = \mathbb{E}\{d\tilde{\omega}^2(t)\} = dt$; $M_k(\cdot) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $N_k(\cdot) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $k \in \mathbb{Z}_+$ are some continuous functions. The impulsive time t_k satisfies $0 = t_0 < t_1 < \dots < t_k \rightarrow \infty$, (i.e., $\lim_{k \rightarrow \infty} t_k = +\infty$) and $\inf_{k \in \mathbb{Z}_+} \{t_k - t_{k-1}\} > 0$.

The main contributions of this research work are highlighted as follows:

- * Uncertain parameters, Markovian jumping, stochastic noises, and leakage delays are taken into account in the stability analysis of designing BAM neural networks with mixed time delays.
- * By fabricating suitable LKF, the global exponential stability of addressed neural networks is checked via some less conserved stability conditions.
- * For novelty, some uncertain parameters are initially handled in Lyapunov–Krasovskii functional which ensures the sufficient conditions for global exponential stability of designed neural networks.
- * In our proposed BAM neural networks, by considering both the time delay terms, the allowable upper bounds of discrete time-varying delay is large when compared with some existing literature, see Table 1 of Example 4.1. This shows that the approach

developed in this paper is brand-new and less conservative than some available results.

Suppose that the initial condition of the stochastic BAM neural networks (1) has the form $x(t) = \phi(t)$ for $t \in [-\bar{\omega}, 0]$ and $y(t) = \psi(t)$ for $t \in [-\bar{\omega}, 0]$, where $\phi(t)$ and $\psi(t)$ are continuous functions, $\bar{\omega} = \max(\bar{\tau}_1, \nu_1, \sigma_1)$ and $\bar{\omega} = \max(\bar{\tau}_2, \nu_2, \sigma_2)$. Throughout this section, we assume that the activation functions $f_i, \tilde{f}_j, g_i, \tilde{g}_j, h_i, \tilde{h}_j; i, j = 1, 2, \dots, n$, satisfy the following assumptions:

Assumption 1

- (1) $f_i, \tilde{f}_j$ are monotonic increasing continuous functions.
- (2) For any $\rho_1, \rho_2, \theta_1, \theta_2 \in \mathbb{R}$, there exist the respective scalars $q_{\rho_1} > 0, r_{\rho_1} > 0$ and $q_{\rho_2} > 0, r_{\rho_2} > 0$ which are correlated with ρ_1, ρ_2 and $\alpha > 0, \beta > 0$ so that

$$\begin{aligned} |f_i(\theta_1) - f_i(\rho_1)| &\geq q_{i\rho_1} |\theta_1 - \rho_1|^\alpha, \quad \forall |\theta_1 - \rho_1| \leq r_{i\rho_1}, \quad \text{and} \\ |\tilde{f}_j(\theta_2) - \tilde{f}_j(\rho_2)| &\geq \tilde{q}_{j\rho_2} |\theta_2 - \rho_2|^\beta, \quad \forall |\theta_2 - \rho_2| \leq \tilde{r}_{j\rho_2}. \end{aligned}$$

Assumption 2 g_i, h_i and $\tilde{g}_j, \tilde{h}_j$ are continuous and satisfy

$$\begin{aligned} |g_i(s_1) - g_i(s_2)| &\leq e_i |f_i(s_1) - f_i(s_2)|; & |h_i(s_1) - h_i(s_2)| &\leq k_i |f_i(s_1) - f_i(s_2)|; \\ |\tilde{g}_j(\tilde{s}_1) - \tilde{g}_j(\tilde{s}_2)| &\leq \tilde{e}_j |\tilde{f}_j(\tilde{s}_1) - \tilde{f}_j(\tilde{s}_2)|; & |\tilde{h}_j(\tilde{s}_1) - \tilde{h}_j(\tilde{s}_2)| &\leq \tilde{k}_j |\tilde{f}_j(\tilde{s}_1) - \tilde{f}_j(\tilde{s}_2)|, \end{aligned}$$

$\forall s_1, s_2, \tilde{s}_1, \tilde{s}_2 \in \mathbb{R}, s_1 \neq s_2$ and $\tilde{s}_1 \neq \tilde{s}_2, i, j = 1, 2, 3, \dots, n$. Denote $E = \text{diag}\{e_i\}, K = \text{diag}\{k_i\}$ and $\tilde{E} = \text{diag}\{\tilde{e}_j\}, \tilde{K} = \text{diag}\{\tilde{k}_j\}$ respectively.

Remark 1.1 In [39], the function f_i used in Assumption 1 is said to be an α -inverse Holder activation function which is a non-Lipschitz function. This activation function plays an important role in the stability issues of neural networks, and there exists a great number of results in the engineering mathematics, for example, $f(\theta) = \arctan \theta$ and $f(\theta) = \theta^3 + \theta$ are 1-inverse Holder functions, $f(\theta) = \theta^3$ is 3-inverse Holder function.

Remark 1.2 From Assumption 2, we can get that $e_i, \tilde{e}_j$ and $k_i, \tilde{k}_j$ are positive scalars. So $E, \tilde{E}$ and $K, \tilde{K}$ are both positive definite diagonal matrices. The relations among the different activation functions $f_i, \tilde{f}_j$ (which are α -inverse Holder activation functions) $g_i, \tilde{g}_j$ and $h_i, \tilde{h}_j$ are implicitly established in Theorem 3.2. Such relations, however, have not been provided by any of the authors in the reported literature.

In order to guarantee the global exponential stability of system (1), we assume that the system tends to its equilibrium point and the stochastic noise contribution vanishes, i.e.,

Assumption 3 $\rho(x^*, y^*, y^*, t) = 0; i, j = 1, 2, \dots, n$.

For such deterministic BAM neural networks, we have the following system of equations:

$$\begin{aligned} dx(t) = & \left[-Cx(t - \nu_1) + W_0 f(y(t)) + W_1 (g(y(t - \tau_1(t)))) \right. \\ & \left. + W_2 \int_{t-\sigma_1}^t h(y(s)) ds + I \right] dt, \quad t > 0, t \neq t_k, \end{aligned}$$

$$\begin{aligned}\Delta x(t_k) &= M_k(x(t_{k-}), x_{t_{k-}}); \quad t = t_k, k \in \mathbb{Z}_+, \\ dy(t) &= \left[-Dy(t - v_2) + V_0 \tilde{f}(x(t)) + V_1 \tilde{g}(x(t - \tau_2(t))) \right. \\ &\quad \left. + V_2 \int_{t-\sigma_2}^t \tilde{h}(x(s)) ds + J \right] dt, \quad t > 0, t \neq t_k, \\ \Delta y(t_k) &= N_k(y(t_{k-}), y_{t_{k-}}); \quad t = t_k, k \in \mathbb{Z}_+.\end{aligned}\tag{2}$$

Thus system (1) admits one equilibrium point $(x^*, y^*) = (x_1^*, x_2^*, \dots, x_n^*, y_1^*, y_2^*, \dots, y_n^*)^T$ under Assumption 3. In this regard, let $u(t) = x(t) - x^*$ and $v(t) = y(t) - y^*$, then system (1) can be rewritten in the following form:

$$\begin{aligned}du(t) &= \left[-Cu(t - v_1) + W_0 \bar{f}(v(t)) + W_1 (\bar{g}(v(t - \tau_1(t)))) + W_2 \int_{t-\sigma_1}^t \bar{h}(v(s)) ds \right] dt \\ &\quad + \bar{\rho}_1(u(t - v_1), v(t), v(t - \tau_1(t)), t) d\bar{\omega}(t), \quad t > 0, t \neq t_k, \\ \Delta u(t_k) &= \bar{M}_k(u(t_k^-), u_{t_k^-}), \quad t = t_k, k \in \mathbb{Z}_+, \\ dv(t) &= \left[-Dv(t - v_2) + V_0 \tilde{\bar{f}}(u(t)) + V_1 \tilde{\bar{g}}(u(t - \tau_2(t))) + V_2 \int_{t-\sigma_2}^t \tilde{\bar{h}}(u(s)) ds \right] dt \\ &\quad + \tilde{\rho}_2(v(t - v_2), u(t), u(t - \tau_2(t)), t) d\tilde{\omega}(t), \quad t > 0, t \neq t_k, \\ \Delta v(t_k) &= \tilde{N}_k(v(t_k^-), v_{t_k^-}), \quad t = t_k, k \in \mathbb{Z}_+,\end{aligned}\tag{3}$$

where

$$\begin{aligned}u(t) &= (u_1(t), u_2(t), \dots, u_n(t))^T, \\ v(t) &= (v_1(t), v_2(t), \dots, v_n(t))^T, \\ u(t - v_1) &= (u_1(t - v_1), u_2(t - v_1), \dots, u_n(t - v_1))^T, \\ v(t - v_2) &= (v_1(t - v_2), v_2(t - v_2), \dots, v_n(t - v_2))^T, \\ \bar{f}(v(t)) &= (\bar{f}_1(v(t)), \bar{f}_2(v(t)), \dots, \bar{f}_n(v(t)))^T, \\ \tilde{\bar{f}}(u(t)) &= (\tilde{\bar{f}}_1(u(t)), \tilde{\bar{f}}_2(u(t)), \dots, \tilde{\bar{f}}_n(u(t)))^T, \\ \bar{g}(v(t - \tau_1(t))) &= (\bar{g}_1(v(t - \tau_1(t))), \bar{g}_2(v(t - \tau_1(t))), \dots, \bar{g}_n(v(t - \tau_1(t))))^T, \\ \tilde{\bar{g}}(u(t - \tau_2(t))) &= (\tilde{\bar{g}}_1(u(t - \tau_2(t))), \tilde{\bar{g}}_2(u(t - \tau_2(t))), \dots, \tilde{\bar{g}}_n(u(t - \tau_2(t))))^T, \\ \bar{h}(v(t)) &= (\bar{h}_1(v(t)), \bar{h}_2(v(t)), \dots, \bar{h}_n(v(t)))^T, \\ \tilde{\bar{h}}(u(t)) &= (\tilde{\bar{h}}_1(u(t)), \tilde{\bar{h}}_2(u(t)), \dots, \tilde{\bar{h}}_n(u(t)))^T, \\ \bar{f}_i(v(t)) &= f_i(v(t) + y^*) - f_i(y^*), \\ \tilde{\bar{f}}_i(u(t)) &= \tilde{f}_i(u(t) + x^*) - \tilde{f}_i(x^*), \\ \bar{g}_i(v(t - \tau_1(t))) &= g_i(v(t - \tau_1(t)) + y^*) - g_i(y^*), \\ \tilde{\bar{g}}_i(u(t - \tau_2(t))) &= \tilde{g}_i(u(t - \tau_2(t)) + x^*) - \tilde{g}_i(x^*), \\ \bar{h}_i(v(t)) &= h_i(v(t) + y^*) - h_i(y^*),\end{aligned}$$

$$\begin{aligned}
 \bar{\bar{h}}_i(u(t)) &= \bar{h}_i(u(t) + x^*) - \bar{h}_i(x^*), \\
 \bar{\rho}_1(u(t - v_1), v(t), v(t - \tau_1), t) \\
 &= \rho_1(u(t - v_1) + x^*, v(t) + y^*, v(t - \tau_1(t)) + y^*, t) - \rho_1(x^*, y^*, y^*, t), \\
 \bar{\rho}_2(v(t - v_2), u(t), u(t - \tau_2), t) \\
 &= \rho_2(v(t - v_2) + y^*, u(t) + x^*, u(t - \tau_2(t)) + x^*, t) - \rho_2(y^*, x^*, x^*, t), \\
 \bar{\rho}_1(u(t - v_1), v(t), v(t - \tau_1), t) \\
 &= (\bar{\rho}_{11}(u(t - v_1), v(t), v(t - \tau_1(t)), t), \dots, \bar{\rho}_{1n}(u(t - v_1), v(t), v(t - \tau_1(t)), t))^T, \\
 \bar{\rho}_2(v(t - v_2), u(t), u(t - \tau_2), t) \\
 &= (\bar{\rho}_{21}(v(t - v_2), u(t), u(t - \tau_2(t)), t), \dots, \bar{\rho}_{2n}(v(t - v_2), u(t), u(t - \tau_2(t)), t))^T.
 \end{aligned}$$

Apparently, $\bar{f}_i(s)$, $\bar{\bar{f}}_j(s)$ is also an α -inverse Holder function, and $\bar{f}_i(0) = \bar{g}_i(0) = \bar{h}_i(0) = \bar{\bar{f}}_j(0) = \bar{\bar{g}}_j(0) = \bar{\bar{h}}_j(0) = 0$, $i, j = 1, 2, \dots, n$.

Let $\{r(t), t \geq 0\}$ be a right-continuous Markov chain in a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and take values in a finite state space $\mathfrak{M} = \{1, 2, \dots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P\{r(t + \Delta t) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta t + O(\Delta t), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta t + O(\Delta t), & \text{if } i = j, \end{cases}$$

where $\Delta t > 0$ and $\lim_{\Delta t \rightarrow 0} \left(\frac{O(\Delta t)}{\Delta t}\right) = 0$. Here $\gamma_{ij} \geq 0$ is the transition probability rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{j=1}^N \gamma_{ij}$.

In this paper, we consider the following BAM neural networks with stochastic noise disturbance, leakage, mixed time delays, and Markovian jump parameters, which is actually a modification of system (3):

$$\begin{aligned}
 du(t) &= \left[-C(r(t))u(t - v_1) + W_0(r(t))\bar{f}(v(t)) \right. \\
 &\quad \left. + W_1(r(t))(\bar{g}(v(t - \tau_1(t)))) + W_2(r(t)) \int_{t-\sigma_1}^t \bar{h}(v(s)) ds \right] dt \\
 &\quad + \bar{\rho}_1(u(t - v_1), v(t), v(t - \tau_1(t)), t, r(t)) d\bar{\omega}(t), \quad t > 0, t \neq t_k, \\
 \Delta u(t_k) &= \bar{M}_k(r(t)) (u(t_k^-), u_{t_k^-}), \quad t = t_k, k \in \mathbb{Z}_+, \\
 dv(t) &= \left[-D(\tilde{r}(t))v(t - v_2) + V_0(\tilde{r}(t))\bar{\bar{f}}(u(t)) \right. \\
 &\quad \left. + V_1(\tilde{r}(t))\bar{\bar{g}}(u(t - \tau_2(t))) + V_2(\tilde{r}(t)) \int_{t-\sigma_2}^t \bar{\bar{h}}(u(s)) ds \right] dt \\
 &\quad + \bar{\rho}_2(v(t - v_2), u(t), u(t - \tau_2(t)), t, \tilde{r}(t)) d\bar{\omega}(t), \quad t > 0, t \neq t_k, \\
 \Delta v(t_k) &= \bar{N}_k(\tilde{r}(t)) (v(t_k^-), v_{t_k^-}), \quad t = t_k, k \in \mathbb{Z}_+,
 \end{aligned} \tag{4}$$

where $u(t - v_1)$, $\tau_1(t)$, $\tau_2(t)$, $v(t)$, $u(t)$, $v(t - v_2)$, $\bar{f}(v(t))$, $\bar{\bar{f}}(u(t))$, $\bar{g}(v(t - \tau_1(t)))$, $\bar{\bar{g}}(u(t - \tau_2(t)))$, $\bar{h}(v(t))$, $\bar{\bar{h}}(u(t))$ have the same meanings as those in (3), $\bar{\rho}_1(u(t - v_1), v(t), v(t - \tau_1(t)), t, r(t))$

and $\bar{\rho}_2(v(t - v_2), u(t), u(t - \tau_2(t)), t, \tilde{r}(t))$ are noise intensity function vectors, and for a fixed system mode, $C(r(t)), D(r(t)), W_0(r(t)), V_0(\tilde{r}(t)), W_1(r(t)), V_1(\tilde{r}(t)), W_2(r(t)), V_2(\tilde{r}(t)), \bar{M}_k(r(t)),$ and $\bar{N}_k(\tilde{r}(t))$ are known constant matrices with appropriate dimensions.

For our convenience, each possible value of $r(t)$ and $\tilde{r}(t)$ is denoted by i and j respectively; $i, j \in \mathfrak{M}$ in the sequel. Then we have $C_i = C(r(t)), D_j = D(\tilde{r}(t)), W_{0i} = W_0(r(t)), V_{0j} = V_0(\tilde{r}(t)), W_{1i} = W_1(r(t)), V_{1j} = V_1(\tilde{r}(t)), W_{2i} = W_2(r(t)), V_{2j} = V_2(\tilde{r}(t)), \bar{M}_{ki} = \bar{M}_k(r(t)), \bar{N}_{kj} = \bar{N}_k(\tilde{r}(t)),$ where $C_i, D_j, W_{0i}, V_{0j}, W_{1i}, V_{1j}, W_{2i}, V_{2j}, \bar{M}_{ki}, \bar{N}_{kj}$ for any $i, j \in \mathfrak{M}$.

Assume that $\bar{\rho}_1 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \times \mathfrak{M} \rightarrow \mathbb{R}^n$ and $\bar{\rho}_2 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \times \mathfrak{M} \rightarrow \mathbb{R}^n$ are locally Lipschitz continuous and satisfy the following assumption.

Assumption 4

$$\text{trace}[\bar{\rho}_1^T(u_1, v_1, v_2, t, i)\bar{\rho}_1(u_1, v_1, v_2, t, i)] \leq u_1^T R_{1i} u_1 + v_1^T R_{2i} v_1 + v_2^T R_{3i} v_2;$$

$$\text{trace}[\bar{\rho}_2^T(v_1, u_1, u_2, t, j)\bar{\rho}_2(v_1, u_1, u_2, t, j)] \leq v_1^T \tilde{R}_{1j} v_1 + u_1^T \tilde{R}_{2j} u_1 + u_2^T \tilde{R}_{3j} u_2;$$

for all $u_1, u_2, v_1, v_2 \in \mathbb{R}^n$ and $r(t) = i, \tilde{r}(t) = j, i, j \in \mathfrak{M}$, where $R_{1i}, \tilde{R}_{1j}, R_{2i}, \tilde{R}_{2j}, R_{3i},$ and $\tilde{R}_{3j}$ are known positive definite matrices with appropriate dimensions.

Consider a general stochastic system $dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))d\omega(t), t \geq 0$ with the initial value $x(0) = x_0 \in \mathbb{R}^n$, where $f : \mathbb{R}^n \times \mathbb{R}^+ \times \mathfrak{M} \rightarrow \mathbb{R}^n$ and $r(t)$ is the Markov chain. Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+ \times \mathfrak{M}; \mathbb{R}^+)$ denote a family of all nonnegative functions V on $\mathbb{R}^n \times \mathbb{R}^+ \times \mathfrak{M}$ which are twice continuously differentiable in x and once differentiable in t . For any $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+ \times \mathfrak{M}; \mathbb{R}^+)$, define $\mathcal{L}V : \mathbb{R}^n \times \mathbb{R}^+ \times \mathfrak{M} \rightarrow \mathbb{R}$ by

$$\begin{aligned} \mathcal{L}V(x(t), t, i) &= V_t(x(t), t, i) + V_x(x(t), t, i)f(x(t), t, i) \\ &\quad + \frac{1}{2} \text{trace}(g^T(x(t), t, i)V_{xx}(x(t), t, i)g(x(t), t, i)) \\ &\quad + \sum_{j=1}^N \gamma_{ij}V(x(t), t, j), \end{aligned}$$

where

$$\begin{aligned} V_t(x(t), t, i) &= \frac{\partial V(x(t), t, i)}{\partial t}, \\ V_x(x(t), t, i) &= \left(\frac{\partial V(x(t), t, i)}{\partial x_1}, \frac{\partial V(x(t), t, i)}{\partial x_2}, \dots, \frac{\partial V(x(t), t, i)}{\partial x_n} \right), \\ V_{xx}(x(t), t, i) &= \frac{\partial^2 V(x(t), t, i)}{\partial x_j \partial x_k}. \end{aligned}$$

By generalized Ito's formula, one can see that

$$\mathbb{E}V(x(t), y(t), t, r(t)) = \mathbb{E}V(x(0), y(0), 0, r(0)) + \mathbb{E} \int_0^t \mathcal{L}V(x(s), y(s), s, r(s)) ds.$$

Let $u(t; \xi)$ and $v(t; \tilde{\xi})$ denote the state trajectory from the initial data $u(\theta) = \xi(\theta)$ on $-\bar{\omega} \leq \theta \leq 0$ in $L^2_{\mathcal{F}_0}([-\bar{\omega}, 0]; \mathbb{R}^n)$ and $v(\theta) = \tilde{\xi}(\theta)$ on $-\bar{\omega} \leq \theta \leq 0$ in $L^2_{\mathcal{F}_0}([-\bar{\omega}, 0]; \mathbb{R}^n)$. Clearly, system

(4) admits a trivial solution $u(t, 0) \equiv 0$ and $v(t, 0) \equiv 0$ corresponding to the initial data $\xi = 0$ and $\tilde{\xi} = 0$, respectively. For simplicity, we write $u(t; \xi) = u(t)$ and $v(t; \tilde{\xi}) = v(t)$.

Definition 1.3 The equilibrium point of neural networks (4) is said to be globally exponentially stable in the mean square if, for any $\xi \in L^2_{\mathcal{F}_0}([-\bar{\omega}, 0]; \mathbb{R}^n)$, $\tilde{\xi} \in L^2_{\mathcal{F}_0}([-\bar{\omega}, 0]; \mathbb{R}^n)$, there exist positive constants η , $\mathcal{T}$, Π_ξ , and $\Theta_{\tilde{\xi}}$ correlated with ξ and $\tilde{\xi}$ such that, when $t > \mathcal{T}$, the following inequality holds:

$$\mathbb{E}\{\|u(t; \xi)\|^2\} + \mathbb{E}\{\|v(t; \tilde{\xi})\|^2\} \leq (\Pi_\xi + \Theta_{\tilde{\xi}})e^{-\eta t}.$$

Definition 1.4 We introduce the stochastic Lyapunov–Krasovskii functional $V \in C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathfrak{M}; \mathbb{R}^+)$ of system (4), the weak infinitesimal generator of random process $\mathcal{L}V$ from $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathfrak{M}$ to $\mathbb{R}^+$ defined by

$$\begin{aligned} \mathcal{L}V(t, u(t), v(t), i) = & \lim_{\Delta t \rightarrow 0^+} \frac{1}{\Delta t} [\mathbb{E}\{V((t + \Delta t), u(t + \Delta t), v(t + \Delta t), r(t + \Delta t)) | \\ & u(t), v(t), r(t) = i\} - V(t, u(t), v(t), r(t) = i)]. \end{aligned}$$

Lemma 1.5 ([39]) *If f_i is an α -inverse Holder function, then for any $\rho_0 \in \mathbb{R}$, one has*

$$\int_{\rho_0}^{+\infty} [f_i(\theta) - f_i(\rho_0)] d\theta = \int_{\rho_0}^{-\infty} [f_i(\theta) - f_i(\rho_0)] d\theta = +\infty.$$

Lemma 1.6 ([39]) *If f_i is an α -inverse Holder function and $f_i(0) = 0$, then there exist constants $q_{i_0} > 0$ and $r_{i_0} \geq 0$ such that $|f_i(\theta)| \geq q_{i_0}|\theta|^\alpha$, $\forall |\theta| \leq r_{i_0}$. Moreover, $|f_i(\theta)| \geq q_{i_0}r_{i_0}^\alpha$, $\forall |\theta| \geq r_{i_0}$.*

Lemma 1.7 ([21]) *For any real matrix $M > 0$, scalars a and b with $0 \leq a < b$, vector function $x(\alpha)$ such that the following integrals are well defined, we have*

$$\int_{-a}^{-b} \int_{t+\beta}^t x(\alpha)^T M x(\alpha) d\alpha d\beta \leq (b-a) \int_{t-b}^t x(\alpha)^T M x(\alpha) d\alpha.$$

Lemma 1.8 ([39]) *Let $x, y \in \mathbb{R}^n$, and G is a positive definite matrix, then*

$$2x^T y \leq x^T G x + y^T G^{-1} y.$$

Lemma 1.9 ([21]) *Given constant symmetric matrices Υ_1 , Υ_2 , and Υ_3 with appropriate dimensions, where $\Upsilon_1^T = \Upsilon_1$ and $\Upsilon_2^T = \Upsilon_2 > 0$, $\Upsilon_1 + \Upsilon_3^T \Upsilon_2^{-1} \Upsilon_3 < 0$ if and only if*

$$\begin{bmatrix} \Upsilon_1 & \Upsilon_3^T \\ \Upsilon_3 & -\Upsilon_2 \end{bmatrix} < 0 \quad [or] \quad \begin{bmatrix} \Upsilon_1 & \Upsilon_3 \\ \Upsilon_3^T & -\Upsilon_2 \end{bmatrix} < 0.$$

Lemma 1.10 ([21]) *For any constant matrix $\Omega \in \mathbb{R}^{n \times n}$, $\Omega = \Omega^T > 0$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \rightarrow \mathbb{R}^n$, such that the integrations concerned are well defined, then*

$$\frac{1}{\gamma} \left(\int_0^\gamma \omega(s) ds \right)^T \Omega \left(\int_0^\gamma \omega(s) ds \right) \leq \int_0^\gamma \omega^T(s) \Omega \omega(s) ds.$$

Lemma 1.11 ([33]) *For given matrices D , E , and F with $F^T F \leq I$ and scalar $\epsilon > 0$, the following inequality holds:*

$$DFE + E^T F^T D^T \leq \epsilon DD^T + \epsilon^{-1} EE^T.$$

Remark 1.12 Lakshmanan et al. in [18] analyzed the impact of time-delayed BAM neural networks for ensuring the stability performance when the leakage delay occurred. In [12], the authors discussed the stability behavior in the sense of asymptotic for BAM neural networks with mixed time delays and uncertain parameters. Moreover, the comparisons for maximum allowable upper bounds of discrete time-varying delays have been listed. Lou and Cui in [29] conversed the exponential stability conditions for time-delayed BAM NNs while Markovian jump parameters arose. Further, the stochastic effects on neural networks and stability criteria were conversed via exponential sense by Huang and Li in [40] by the aid of Lyapunov–Krasovskii functionals. In all the above mentioned references, the stability problem for BAM neural networks was considered only with leakage delays or mixed time delays, or stochastic effects, or Markovian jump parameters, or parameter uncertainties, but all the above factors have not been taken into one account and no one investigated exponential stability via delays at a time. Considering the above facts is very challenging and advanced in this research work.

2 Global exponential stability for deterministic systems

Theorem 2.1 *Under Assumptions 1 and 2, the neural network system (4) is globally exponentially stable in the mean square if, for given $\eta_i, \tilde{\eta}_j > 0$ ($i, j \in \mathfrak{M}$), there exist positive definite matrices $S, T, \tilde{S}, \tilde{T}, R_2, \tilde{R}_2, N_1, N_2, N_3, N_4, N_5, N_6$ and $H_i, \tilde{H}_j$ ($i, j \in \mathfrak{M}$), positive definite diagonal matrices P, Q , and positive scalars λ_i and μ_j ($i, j \in \mathfrak{M}$) such that the following LMIs are satisfied:*

$$H_i < \lambda_i I, \quad (5)$$

$$\tilde{H}_j < \mu_j I, \quad (6)$$

$$\bar{M}_k^T H_i \bar{M}_k - H_j \leq 0, \quad (7)$$

$$\bar{N}_k^T \tilde{H}_i \bar{N}_k - \tilde{H}_j \leq 0 \quad [\text{here } r(t_k) = i \text{ and } \tilde{r}(t_k) = j], \quad (8)$$

$$\Xi_i = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & 0 & 0 & 0 & 0 & 0 & \Xi_{19} & 0 & 0 & 0 & 0 & 0 \\ * & \Xi_{22} & 0 & 0 & \Xi_{25} & \Xi_{26} & \Xi_{27} & 0 & \Xi_{29} & 0 & 0 & 0 & 0 & 0 \\ * & * & \Xi_{33} & 0 & \Xi_{35} & \Xi_{36} & \Xi_{37} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55} & 0 & 0 & 0 & \Xi_{59} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Xi_{66} & 0 & 0 & \Xi_{69} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Xi_{77} & 0 & \Xi_{79} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Xi_{88} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Xi_{99} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Xi_{1010} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & \Xi_{1111} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Xi_{1212} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & \Xi_{1313} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & * & \Xi_{1414} \end{bmatrix} < 0, \quad (9)$$

$$\Omega_j = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & 0 & 0 & 0 & 0 & 0 & \Omega_{19} & 0 & 0 & 0 & 0 & 0 \\ * & \Omega_{22} & 0 & 0 & \Omega_{25} & \Omega_{26} & \Omega_{27} & 0 & \Omega_{29} & 0 & 0 & 0 & 0 & 0 \\ * & * & \Omega_{33} & 0 & \Omega_{35} & \Omega_{36} & \Omega_{37} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Omega_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Omega_{55} & 0 & 0 & 0 & \Omega_{59} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Omega_{66} & 0 & 0 & \Omega_{69} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Omega_{77} & 0 & \Omega_{79} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Omega_{88} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Omega_{99} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Omega_{1010} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & \Omega_{1111} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \Omega_{1212} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & \Omega_{1313} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & * & \Omega_{1414} \end{bmatrix}$$

$< 0,$ (10)

where

$$\begin{aligned} \Xi_{11} &= \lambda_i R_{1i}, & \Omega_{11} &= \mu_j \tilde{R}_{1j}, & \Xi_{22} &= -C_i H_i + \sum_{l=1}^N \gamma_{il} H_l + \frac{\lambda_i}{1-\tau_1} e^{-\eta_i \bar{\tau}_1} R_2 + \eta_i H_i, \\ \Omega_{22} &= -D_j \tilde{H}_j + \sum_{l=1}^N \tilde{\gamma}_{jl} \tilde{H}_l + \frac{\mu_j}{1-\tau_2} e^{-\tilde{\eta}_j \bar{\tau}_2} \tilde{R}_2 + \tilde{\eta}_j \tilde{H}_j, & \Xi_{33} &= \frac{1}{1-\tau_1} E^2 S + \sigma_1 K^2 T, \\ \Omega_{33} &= \frac{1}{1-\tau_2} \tilde{E}^2 \tilde{S} + \sigma_2 \tilde{K}^2 \tilde{T}, & \Xi_{44} &= -e^{\eta_i \bar{\tau}_1} S, & \Omega_{26} &= \tilde{H}_j V_{1j}, & \Omega_{1212} &= -\mu_j \tilde{R}_2, \\ \Omega_{44} &= -e^{-\tilde{\eta}_j \bar{\tau}_2} \tilde{S}, & \Omega_{55} &= N_2, & \Xi_{66} &= -(1-\tau_1) e^{-\eta_i \bar{\tau}_1} N_3, \\ \Omega_{66} &= -(1-\tau_2) e^{-\tilde{\eta}_j \bar{\tau}_2} N_4, & \Xi_{88} &= -\frac{1}{\sigma_1} T, & \Omega_{88} &= -\frac{1}{\sigma_2} \tilde{T}, \\ \Xi_{99} &= \sum_{l=1}^N \gamma_{il} C_i^T H_l C_i + \eta_i C_i^T H_i C_i, & \Xi_{13} &= \eta_i P - P C_i, & \Omega_{69} &= -D_j^T \tilde{H}_j V_{1j}, \\ \Omega_{99} &= \sum_{l=1}^N \tilde{\gamma}_{jl} D_j^T \tilde{H}_l D_j + \tilde{\eta}_j D_j^T \tilde{H}_j D_j, & \Omega_{1010} &= \mu_j \tilde{R}_{3j}, & \Omega_{37} &= Q V_{2j}, \\ \Omega_{13} &= \tilde{\eta}_j Q - Q D_j, & \Xi_{19} &= 0, & \Omega_{19} &= 0, & \Xi_{25} &= H_i W_{0i}, & \Omega_{25} &= \tilde{H}_j V_{0j}, \\ \Xi_{26} &= H_i W_{1i}, & \Xi_{27} &= H_i W_{2i}, & \Omega_{27} &= \tilde{H}_j V_{2j}, \\ \Xi_{29} &= C_i^T H_i C_i - \sum_{l=1}^N \gamma_{il} C_i H_l - \eta_i C_i^T H_i, & \Xi_{1212} &= -\lambda_i R_2, \\ \Xi_{35} &= P W_{0i}, & \Omega_{35} &= Q V_{0j}, & \Xi_{36} &= P W_{1i}, & \Omega_{36} &= Q V_{1j}, & \Xi_{37} &= P W_{2i}, \\ \Xi_{1010} &= \lambda_i R_{3i}, & \Xi_{59} &= -C_i^T H_i W_{0i}, & \Omega_{59} &= -D_j^T \tilde{H}_j V_{0j}, & \Xi_{69} &= -C_i^T H_i W_{1i}, \\ \Xi_{55} &= N_1, & \Omega_{1111} &= \mu_j \tilde{R}_{2j}, & \Xi_{79} &= -C_i^T H_i W_{2i}, & \Omega_{79} &= -D_j^T \tilde{H}_j V_{2j}, \\ \Omega_{29} &= D_j^T \tilde{H}_j D_j - \sum_{l=1}^N \tilde{\gamma}_{jl} D_j \tilde{H}_l - \tilde{\eta}_j D_j^T \tilde{H}_j, & \Omega_{12} &= 0, & \Xi_{1111} &= \lambda_i R_{2i}, \\ \Xi_{1313} &= N_3, & \Omega_{1313} &= N_4, & \Xi_{12} &= 0, & \Xi_{1414} &= \sigma_1 N_5, \\ \Xi_{77} &= -\frac{1}{\sigma_1} N_5, & \Omega_{1414} &= \sigma_2 N_6, & \Omega_{77} &= -\frac{1}{\sigma_2} N_6. \end{aligned}$$

Proof Let us construct the following Lyapunov–Krasovskii functional candidate:

$$V(t, u(t), v(t), i, j) = \sum_{l=1}^8 V_l(t, u(t), v(t), i, j),$$

where

$$\begin{aligned} V_1(t, u(t), v(t), i, j) &= e^{\eta_i t} \left\{ \left[u(t) - C_i \int_{t-\tau_1}^t u(s) ds \right]^T H_i \left[u(t) - C_i \int_{t-\tau_1}^t u(s) ds \right] \right. \\ &\quad \left. + e^{\tilde{\eta}_j t} \left\{ \left[v(t) - D_j \int_{t-\tau_2}^t v(s) ds \right]^T \tilde{H}_j \left[v(t) - D_j \int_{t-\tau_2}^t v(s) ds \right] \right\} \right\}, \\ V_2(t, u(t), v(t), i, j) &= \frac{\lambda_i}{1 - \tau_1} \int_{t-\tau_1(t)}^t e^{\eta_i(s+\tau_1(s))} u^T(s) R_2 u(s) ds \\ &\quad + \frac{\mu_j}{1 - \tau_2} \int_{t-\tau_2(t)}^t e^{\tilde{\eta}_j(s+\tau_2(s))} v^T(s) \tilde{R}_2 v(s) ds, \\ V_3(t, u(t), v(t), i, j) &= \frac{1}{1 - \tau_1} \int_{t-\tau_1(t)}^t e^{\eta_i(s)} \bar{g}^T(u(s)) S \bar{g}(u(s)) ds \\ &\quad + \frac{1}{1 - \tau_2} \int_{t-\tau_2(t)}^t e^{\tilde{\eta}_j(s)} \tilde{g}^T(v(s)) \tilde{S} \tilde{g}(v(s)) ds, \\ V_4(t, u(t), v(t), i, j) &= \int_{-\sigma_1}^0 \int_{t+s}^t e^{\eta_i \theta} \bar{h}^T(u(\theta)) T \bar{h}(u(\theta)) d\theta ds \\ &\quad + \int_{-\sigma_2}^0 \int_{t+s}^t e^{\tilde{\eta}_j \theta} \tilde{h}^T(v(\theta)) \tilde{T} \tilde{h}(v(\theta)) d\theta ds, \\ V_5(t, u(t), v(t), i, j) &= 2e^{\eta_i t} \sum_{l=1}^n p_l \int_0^{u_l(t)} \bar{f}_l(\theta) d\theta + 2e^{\tilde{\eta}_j t} \sum_{l=1}^n q_l \int_0^{v_l(t)} \tilde{f}_l(\theta) d\theta, \\ V_6(t, u(t), v(t), i, j) &= \int_0^t e^{\eta_i s} \bar{f}^T(v(s)) N_1 \bar{f}(v(s)) ds \\ &\quad + \int_0^t e^{\tilde{\eta}_j s} \tilde{f}^T(u(s)) N_2 \tilde{f}(u(s)) ds, \\ V_7(t, u(t), v(t), i, j) &= \int_{t-\tau_1(t)}^t e^{\eta_i s} \bar{g}^T(v(s)) N_3 \bar{g}(v(s)) ds \\ &\quad + \int_{t-\tau_2(t)}^t e^{\tilde{\eta}_j s} \tilde{g}^T(u(s)) N_4 \tilde{g}(u(s)) ds, \\ V_8(t, u(t), v(t), i, j) &= \int_{-\sigma_1}^0 \int_{t+s}^t e^{\eta_i \theta} \bar{h}^T(v(\theta)) N_5 \bar{h}(v(\theta)) d\theta ds \\ &\quad + \int_{-\sigma_2}^0 \int_{t+s}^t e^{\tilde{\eta}_j \theta} \tilde{h}^T(u(\theta)) N_6 \tilde{h}(u(\theta)) d\theta ds. \end{aligned}$$

By Assumption 4, (5) and (6), we obtain

$$\begin{aligned} \text{trace}[\bar{\rho}_1^T(u_1, v_1, v_2, t, i) H_i \bar{\rho}_1(u_1, v_1, v_2, t, i)] &\leq \lambda_i [u_1^T R_{1i} u_1 + v_1^T R_{2i} v_1 + v_2^T R_{3i} v_2], \\ \text{trace}[\tilde{\rho}_2^T(v_1, u_1, u_2, t, j) \tilde{H}_j \tilde{\rho}_2(v_1, u_1, u_2, t, j)] &\leq \mu_j [v_1^T \tilde{R}_{1j} v_1 + u_1^T \tilde{R}_{2j} u_1 + u_2^T \tilde{R}_{3j} u_2]. \end{aligned}$$

It is easy to prove that system (4) is equivalent to the following form:

$$\begin{aligned} d \left[u(t) - C_i \int_{t-v_1}^t u(s) ds \right] &= \left[-C_i u(t) + W_{0i} \bar{f}(v(t)) + W_{1i} \bar{g}(v(t - \tau_1(t))) \right. \\ &\quad \left. + W_{2i} \int_{t-\sigma_1}^t \bar{h}(v(s)) ds \right] dt \\ &\quad + \bar{\rho}_1(u(t - v_1), v(t), v(t - \tau_1(t)), t, i) d\bar{\omega}(t), \\ d \left[v(t) - D_j \int_{t-v_2}^t v(s) ds \right] &= \left[-D_j v(t) + V_{0j} \bar{\tilde{f}}(u(t)) + V_{1j} \bar{\tilde{g}}(u(t - \tau_2(t))) \right. \\ &\quad \left. + V_{2j} \int_{t-\sigma_2}^t \bar{\tilde{h}}(u(s)) ds \right] dt \\ &\quad + \bar{\rho}_2(v(t - v_2), u(t), u(t - \tau_2(t)), t, j) d\bar{\omega}(t). \end{aligned}$$

By utilizing Lemmas 1.6 and 1.10, from (4) and Definition 1.4, one has

$$\begin{aligned} \mathcal{L}V_1 &\leq e^{\eta_1 t} \left\{ u^T(t) (-2C_i H_i + \eta_i H_i) u(t) + 2u^T(t) H_i W_{0i} \bar{f}(v(t)) + 2u^T(t) H_i W_{1i} \right. \\ &\quad \times \bar{g}(v(t - \tau_1(t))) + 2u^T(t) H_i W_{2i} \left(\int_{t-\sigma_1}^t \bar{h}(v(s)) ds \right) + 2 \left(\int_{t-v_1}^t u(s) ds \right)^T C_i^T H_i \\ &\quad \times C_i u(t - v_1) - 2 \left(\int_{t-v_1}^t u(s) ds \right)^T C_i^T H_i W_{1i} \bar{g}(v(t - \tau_1(t))) - 2 \left(\int_{t-v_1}^t u(s) ds \right)^T \\ &\quad \times C_i^T H_i W_{2i} \left(\int_{t-\sigma_1}^t \bar{h}(v(s)) ds \right) + u^T(t) \sum_{l=1}^N \gamma_{il} H_i u(t) - 2u^T(t) \left(\sum_{l=1}^N \gamma_{il} C_i H_i \right. \\ &\quad \left. + \eta_i C_i^T H_i \right) \left(\int_{t-v_1}^t u(s) ds \right) + \left(\int_{t-v_1}^t u(s) ds \right)^T \left(\sum_{l=1}^N \gamma_{il} C_i^T H_i C_i + \eta_i C_i^T H_i C_i \right) \\ &\quad \times \left(\int_{t-v_1}^t u(s) ds \right) + u^T(t - v_1) \lambda_i R_{1i} u(t - v_1) + v^T(t) \lambda_i R_{2i} v(t) + v^T(t - \tau_1(t)) \\ &\quad \times \lambda_i R_{3i} v(t - \tau_1(t)) \left. \right\} + e^{\eta_j t} \left\{ v^T(t) (-2D_j \tilde{H}_j + \tilde{\eta}_j \tilde{H}_j) v(t) + 2v^T(t) \tilde{H}_j V_{0j} \bar{\tilde{f}}(u(t)) \right. \\ &\quad + 2v^T(t) \tilde{H}_j V_{1j} \bar{\tilde{g}}(u(t - \tau_2(t))) + 2v^T(t) \tilde{H}_j V_{2j} \left(\int_{t-\sigma_2}^t \bar{\tilde{h}}(u(s)) ds \right) \\ &\quad + 2 \left(\int_{t-v_2}^t v(s) ds \right)^T D_j^T \tilde{H}_j D_j v(t - v_2) - 2 \left(\int_{t-v_2}^t v(s) ds \right)^T \\ &\quad \times D_j^T \tilde{H}_j V_{1j} \bar{\tilde{g}}(u(t - \tau_2(t))) \\ &\quad - 2 \left(\int_{t-v_2}^t v(s) ds \right)^T D_j^T \tilde{H}_j V_{2j} \left(\int_{t-\sigma_2}^t \bar{\tilde{h}}(u(s)) ds \right) + v^T(t) \sum_{l=1}^N \tilde{\gamma}_{jl} \tilde{H}_l v(t) - 2v^T(t) \\ &\quad \times \left(\sum_{l=1}^N \tilde{\gamma}_{jl} D_j \tilde{H}_j + \tilde{\eta}_j D_j^T \tilde{H}_j \right) \left(\int_{t-v_2}^t v(s) ds \right) + \left(\int_{t-v_2}^t v(s) ds \right)^T \left(\sum_{l=1}^N \tilde{\gamma}_{jl} D_j^T \tilde{H}_j D_j \right. \end{aligned}$$

$$\begin{aligned} & + \tilde{\eta}_j D_j^T \tilde{H}_j D_j \Big) \left(\int_{t-\nu_2}^t v(s) ds \right) + v^T(t-\nu_2) \mu_j \tilde{R}_{1j} v(t-\nu_2) + u^T(t) \mu_j \tilde{R}_{2j} \\ & \times u(t) + u^T(t-\tau_2(t)) \mu_j \tilde{R}_{3j} u(t-\tau_2(t)) \Big\}, \end{aligned} \quad (11)$$

$$\begin{aligned} \mathcal{L}V_2 & \leq \frac{\lambda_i}{1-\tau_1} e^{\eta_i(t-\bar{\tau}_1)} u^T(t) R_2 u(t) - \lambda_i e^{\eta_i t} u^T(t-\tau_1(t)) R_2 u(t-\tau_1(t)) \\ & + \frac{\mu_j}{1-\tau_2} e^{\tilde{\eta}_j(t-\bar{\tau}_2)} v^T(t) \tilde{R}_2 v(t) - \mu_j e^{\tilde{\eta}_j t} v^T(t-\tau_2(t)) \tilde{R}_2 v(t-\tau_2(t)), \end{aligned} \quad (12)$$

$$\begin{aligned} \mathcal{L}V_3 & \leq \frac{1}{1-\tau_1} e^{\eta_i t} \tilde{f}^T(u(t)) E^2 \tilde{S} \tilde{f}(u(t) - e^{\eta_i(t-\bar{\tau}_1)} \tilde{g}^T(u(t-\tau_1(t)))) S \\ & \times \tilde{g}(u(t-\tau_1(t))) + \frac{1}{1-\tau_2} e^{\tilde{\eta}_j t} \tilde{f}^T(v(t)) \tilde{E}^2 \tilde{S} \tilde{f}(v(t) - e^{\tilde{\eta}_j(t-\bar{\tau}_2)} \\ & \times \tilde{g}^T(v(t-\tau_2(t)))) \tilde{S} \tilde{g}(v(t-\tau_2(t))), \end{aligned} \quad (13)$$

$$\begin{aligned} \mathcal{L}V_4 & \leq \sigma_1 e^{\eta_i t} \tilde{f}^T(u(t)) K^2 T \tilde{f}(u(t)) - \frac{1}{\sigma_1} e^{\eta_i t} \left(\int_{t-\sigma_1}^t \bar{h}(u(s)) ds \right)^T T \\ & \times \left(\int_{t-\sigma_1}^t \bar{h}(u(s)) ds \right) + \sigma_2 e^{\tilde{\eta}_j t} \tilde{f}^T(v(t)) \tilde{K}^2 \tilde{T} \tilde{f}(v(t)) - \frac{1}{\sigma_2} \\ & \times e^{\tilde{\eta}_j t} \left(\int_{t-\sigma_2}^t \bar{h}(v(s)) ds \right)^T \tilde{T} \left(\int_{t-\sigma_2}^t \bar{h}(v(s)) ds \right), \end{aligned} \quad (14)$$

$$\begin{aligned} \mathcal{L}V_5 & \leq 2e^{\eta_i t} \tilde{f}^T(u(t)) (\eta_i P - PC_i) u(t-\nu_1) + 2e^{\eta_i t} \tilde{f}^T(u(t)) PW_{0i} \tilde{f}(v(t)) \\ & + 2e^{\eta_i t} \tilde{f}^T(u(t)) PW_{1i} \tilde{g}(v(t-\tau_1(t))) + 2e^{\eta_i t} \tilde{f}^T(u(t)) PW_{2i} \int_{t-\sigma_1}^t \bar{h}(v(s)) ds \\ & + 2e^{\tilde{\eta}_j t} \tilde{f}^T(v(t)) (\tilde{\eta}_j Q - QD_j) v(t-\nu_2) + 2e^{\tilde{\eta}_j t} \tilde{f}^T(v(t)) QV_{0j} \tilde{f}(u(t)) \\ & + 2e^{\tilde{\eta}_j t} \tilde{f}^T(v(t)) QV_{1j} \tilde{g}(u(t-\tau_2(t))) + 2e^{\tilde{\eta}_j t} \tilde{f}^T(v(t)) QV_{2j} \int_{t-\sigma_2}^t \bar{h}(u(s)) ds, \end{aligned} \quad (15)$$

$$\mathcal{L}V_6 = e^{\eta_i t} \tilde{f}^T(v(t)) N_1 \tilde{f}(v(t)) + e^{\tilde{\eta}_j t} \tilde{f}^T(u(t)) N_2 \tilde{f}(u(t)), \quad (16)$$

$$\begin{aligned} \mathcal{L}V_7 & \leq e^{\eta_i t} \tilde{g}^T(v(t)) N_3 \tilde{g}(v(t)) \\ & - e^{\eta_i(t-\bar{\tau}_1)} \tilde{g}^T(v(t-\tau_1(t))) N_3 \tilde{g}(v(t-\tau_1(t))) (1-\tau_1) \\ & + e^{\tilde{\eta}_j t} \tilde{g}^T(u(t)) N_4 \tilde{g}(u(t)) \\ & - e^{\tilde{\eta}_j(t-\bar{\tau}_2)} \tilde{g}^T(u(t-\tau_2(t))) N_4 \tilde{g}(u(t-\tau_2(t))) (1-\tau_2), \end{aligned} \quad (17)$$

$$\begin{aligned} \mathcal{L}V_8 & \leq \sigma_1 e^{\eta_i t} \bar{h}^T(v(t)) N_5 \bar{h}(v(t)) - \frac{1}{\sigma_1} e^{\eta_i t} \left(\int_{t-\sigma_1}^t \bar{h}(v(s)) ds \right)^T N_5 \left(\int_{t-\sigma_1}^t \bar{h}(v(s)) ds \right) \\ & + \sigma_2 e^{\tilde{\eta}_j t} \bar{h}^T(u(t)) N_6 \bar{h}(u(t)) \\ & - \frac{1}{\sigma_2} e^{\tilde{\eta}_j t} \left(\int_{t-\sigma_2}^t \bar{h}(u(s)) ds \right)^T N_6 \left(\int_{t-\sigma_2}^t \bar{h}(u(s)) ds \right). \end{aligned} \quad (18)$$

By combining Eqs. (11)–(18), we can obtain that

$$\mathcal{L}V(t, u(t), v(t), i, j) \leq e^{\eta_i t} \Psi^T(t) \Xi_i \Psi(t) + e^{\tilde{\eta}_j t} \Phi^T(t) \Omega_j \Phi(t), \quad (19)$$

where

$$\begin{aligned} \Psi(t) = & \left[u^T(t - \nu_1), u^T(t), f^T(u(t)), \bar{g}^T(u(t - \tau_1(t))), \bar{f}^T(v(t)), \bar{g}^T(v(t - \tau_1(t))), \right. \\ & \left(\int_{t-\sigma_1}^t \bar{h}(v(s)) ds \right)^T, \left(\int_{t-\sigma_1}^t \bar{h}(u(s)) ds \right)^T, \left(\int_{t-\nu_1}^t u(s) ds \right)^T, v^T(t - \tau_1(t)), \\ & \left. v^T(t), u^T(t - \tau_1(t)), \bar{g}^T(v(t)), \bar{h}(v(t)) \right] \end{aligned}$$

and

$$\begin{aligned} \Phi(t) = & \left[v^T(t - \nu_2), v^T(t), \bar{f}^T(v(t)), \bar{g}^T(v(t - \tau_2(t))), \bar{f}^T(u(t)), \bar{g}^T(u(t - \tau_2(t))), \right. \\ & \left(\int_{t-\sigma_2}^t \bar{h}(u(s)) ds \right)^T, \left(\int_{t-\sigma_2}^t \bar{h}(v(s)) ds \right)^T, \left(\int_{t-\nu_2}^t v(s) ds \right)^T, u^T(t - \tau_2(t)), \\ & \left. u^T(t), v^T(t - \tau_2(t)), \bar{g}^T(u(t)), \bar{h}^T(u(t)) \right]. \end{aligned}$$

Let $\alpha = \min_{i \in S} \lambda_{\min}(-\Xi_i)$ and $\beta = \min_{j \in S} \mu_{\min}(-\Omega_j)$. From conditions (9) and (10), it is easy to see that $\alpha > 0$ and $\beta > 0$. This fact together with (19) gives

$$\mathcal{LV}(t, u(t), v(t), i, j) \leq -(\alpha e^{\eta_i t} (\|u(t)\|^2 + \|v(t)\|^2) + \beta e^{\tilde{\eta}_j t} (\|u(t)\|^2 + \|v(t)\|^2)). \quad (20)$$

Then, for $t = t_k$, by some simple calculations, one gets

$$V_1(t_k, u(t_k), v(t_k), i, j) - V_1(t_{k-}, u(t_{k-}), v(t_{k-}), i, j) < 0.$$

Therefore $V_1(t_k, u(t_k), v(t_k), i, j) \leq V_1(t_{k-}, u(t_{k-}), v(t_{k-}), i, j)$, $k \in \mathbb{Z}_+$, which implies that $V(t_k, u(t_k), v(t_k), i, j) \leq V(t_{k-}, u(t_{k-}), v(t_{k-}), i, j)$, $k \in \mathbb{Z}_+$. Using mathematical induction, we have that, for all $i, j \in \mathfrak{M}$ and $k \geq 1$,

$$\begin{aligned} \mathbb{E}V(t_k, u(t_k), v(t_k), i, j) & \leq \mathbb{E}V(t_{k-}, u(t_{k-}), v(t_{k-}), i, j) \\ & \leq \mathbb{E}V(t_{k-1}, u(t_{k-1}), v(t_{k-1}), r(t_{k-1}), \tilde{r}(t_{k-1})) \\ & \leq \mathbb{E}V(t_{k-1}^-, u(t_{k-1}^-), v(t_{k-1}^-), r(t_{k-1}^-), \tilde{r}(t_{k-1}^-)) \\ & \leq \mathbb{E}V(t_0, u(t_0), v(t_0), r(t_0), \tilde{r}(t_0)). \end{aligned}$$

Since $t > t'$, it follows from Dynkin's formula that we have

$$\begin{aligned} \mathbb{E}V(t, u(t), v(t), i, j) & \leq \mathbb{E}V(t', u(t'), v(t'), r(t'), \tilde{r}(t')), \\ \mathbb{E}V(t, u(t), v(t), i, j) & \leq \mathbb{E}V(0, u(0), v(0), r(0), \tilde{r}(0)). \end{aligned}$$

Hence it follows from the definition of $V(t, u(t), v(t), i, j)$, the generalized *Ito's* formula, and (20) that

$$\begin{aligned} & \lambda_{\min}(H_i) \mathbb{E}(\|u(t)\|^2 + \|v(t)\|^2) + \mu_{\min}(\tilde{H}_j) \mathbb{E}(\|u(t)\|^2 + \|v(t)\|^2) \\ & \leq e^{-\eta_i t} \mathbb{E}V(0, u(0), v(0), r(0), \tilde{r}(0)) + e^{-\eta_i t} \mathbb{E} \left[\int_0^t -\alpha e^{\eta_i s} (\|u(s)\|^2 + \|v(s)\|^2) ds \right] \\ & \quad + e^{-\tilde{\eta}_j t} \mathbb{E}V(0, u(0), v(0), r(0), \tilde{r}(0)) + e^{-\tilde{\eta}_j t} \mathbb{E} \left[\int_0^t -\beta e^{\tilde{\eta}_j s} (\|u(s)\|^2 + \|v(s)\|^2) ds \right] \\ & \leq e^{-\eta_i t} \mathbb{E}V(0, u(0), v(0), r(0), \tilde{r}(0)) + e^{-\tilde{\eta}_j t} \mathbb{E}V(0, u(0), v(0), r(0), \tilde{r}(0)). \end{aligned} \quad (21)$$

By (21), we can get that $\lim_{t \rightarrow +\infty} \mathbb{E}(\|u(t)\|^2 + \|v(t)\|^2) = 0$ and $\lim_{t \rightarrow +\infty} \mathbb{E}(\|u(t)\|^2 + \|v(t)\|^2) = 0$. Furthermore,

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \mathbb{E}(|u_l(t)|^2 + |v_l(t)|^2) = 0 \quad \text{and} \\ & \lim_{t \rightarrow +\infty} \mathbb{E}(|u_m(t)|^2 + |v_m(t)|^2) = 0, \quad l, m = 1, 2, \dots, n. \end{aligned} \quad (22)$$

For $\tilde{f}_l(\theta)$ and $\tilde{f}_m(\theta)$, by Lemma 1.6 there exist constants $q_{l0} > 0$, $\tilde{q}_{m0} > 0$, and $r_{l0} > 0$, $\tilde{r}_{m0} > 0$ such that $|\tilde{f}_l(\theta)| \geq q_{l0}|\theta|^\alpha$, $\forall |\theta| \leq r_{l0}$, $l = 1, 2, \dots, n$, and $|\tilde{f}_m(\theta)| \geq \tilde{q}_{m0}|\theta|^\alpha$, $\forall |\theta| \leq \tilde{r}_{m0}$, $m = 1, 2, \dots, n$.

By (22), there exists a scalar $\mathcal{T} > 0$, when $t \geq \mathcal{T}$, $\mathbb{E}\{u_l(t)\} \in [-\tilde{r}_0, \tilde{r}_0]$, $l = 1, 2, \dots, n$, where $\tilde{r}_0 = \min_{1 \leq l \leq n} r_{l0}$ and $\mathbb{E}\{v_m(t)\} \in [-\tilde{r}_0, \tilde{r}_0]$, $m = 1, 2, \dots, n$, where $\tilde{r}_0 = \min_{1 \leq m \leq n} \tilde{r}_{m0}$. Hence when $t \geq \mathcal{T}$, one gets

$$\begin{aligned} & e^{-\eta_i t} \mathbb{E}V(0, u(0), v(0), r(0)) + e^{-\tilde{\eta}_j t} \mathbb{E}V(0, u(0), v(0), \tilde{r}(0)) \\ & \geq \frac{2pq_0}{\alpha + 1} \left\{ \max_{1 \leq l \leq n} \mathbb{E}\{\|u_l(t)\|^2\} \right\}^{\frac{\alpha+1}{2}} + \frac{2\tilde{p}\tilde{q}_0}{\alpha + 1} \left\{ \max_{1 \leq m \leq n} \mathbb{E}\{\|v_m(t)\|^2\} \right\}^{\frac{\alpha+1}{2}}, \end{aligned} \quad (23)$$

where $p = \min_{1 \leq l \leq n} p_l$, $\tilde{p} = \min_{1 \leq m \leq n} \tilde{p}_m$ and $q_0 = \min_{1 \leq l \leq n} q_{l0}$, $\tilde{q}_0 = \min_{1 \leq m \leq n} \tilde{q}_{m0}$. By (23), we get

$$\begin{aligned} & \left\{ \max_{1 \leq l \leq n} \mathbb{E}\{\|u_l(t)\|^2\} \right\}^{\frac{\alpha+1}{2}} + \left\{ \max_{1 \leq m \leq n} \mathbb{E}\{\|v_m(t)\|^2\} \right\}^{\frac{\alpha+1}{2}} \\ & \leq \left(\frac{\alpha + 1}{2pq_0} \right)^* \left\{ (\mathbb{E}V(0, u(0), v(0), r(0), \tilde{r}(0))) e^{-\eta t} \right\} \\ & \quad \left(\text{Put } \left(\frac{\alpha + 1}{2pq_0} \right)^* = \max \left\{ \frac{\alpha + 1}{2pq_0}, \frac{\alpha + 1}{2\tilde{p}\tilde{q}_0} \right\} \text{ and } \eta = \min \left\{ \min_{i \in \mathfrak{M}} \eta_i, \min_{j \in \mathfrak{M}} \tilde{\eta}_j \right\} \right), \end{aligned}$$

where

$$\begin{aligned} & \mathbb{E}V(0, u(0), v(0), r(0), \tilde{r}(0)) \leq \lambda_{\max}(H_i) \mathbb{E}\|\xi\|^2 + \mu_{\max}(\tilde{H}_j) \mathbb{E}\|\tilde{\xi}\|^2 + \lambda_{\max}(R_2) \frac{\lambda_i}{1 - \tau_1} \\ & \quad \times \frac{e^{\eta_i \tilde{\tau}_1} - 1}{\eta_i(1 + \tau_1)} \mathbb{E}\|\xi\|^2 + \mu_{\max}(\tilde{R}_2) \frac{\mu_j}{1 - \tau_2} \frac{e^{\tilde{\eta}_j \tilde{\tau}_2} - 1}{\tilde{\eta}_j(1 + \tau_2)} \\ & \quad \times \mathbb{E}\|\tilde{\xi}\|^2 + \lambda_{\max}(S) \frac{1}{1 - \tau_1} \frac{1 - e^{\eta_i \tilde{\tau}_1}}{\eta_i} \mathbb{E}\|\tilde{g}(\xi)\|^2 + \mu_{\max}(\tilde{S}) \end{aligned}$$

$$\begin{aligned} & \times \frac{1}{1-\tau_2} \frac{1-e^{-\tilde{\eta}_j \tilde{\tau}_2}}{\tilde{\eta}_j} \mathbb{E} \|\tilde{g}(\tilde{\xi})\|^2 + \lambda_{\max}(T) \sigma_1 \frac{1-e^{-\eta_i t}}{\eta_i} \\ & \times \mathbb{E} \|\tilde{h}(\tilde{\xi})\|^2 + \mu_{\max}(\tilde{T}) \sigma_2 \frac{1-e^{-\tilde{\eta}_j t}}{\tilde{\eta}_j} \mathbb{E} \|\tilde{h}(\tilde{\xi})\|^2 \\ & + 2P \mathbb{E} |\tilde{\xi} \tilde{f}(\tilde{\xi})| + 2Q \mathbb{E} |\tilde{\xi} \tilde{f}(\tilde{\xi})|. \end{aligned} \quad (24)$$

Let

$$\pi_{\xi_i + \tilde{\xi}_j} = (\pi_{\xi_i} + \pi_{\tilde{\xi}_j})^{\frac{2}{\alpha+1}}, \quad (25)$$

where

$$\begin{aligned} \pi_{\xi_i} &= n \left\{ \left(\frac{\alpha+1}{2pq_0} \right)^* \left(\lambda_{\max}(H_i) \mathbb{E} \|\xi\|^2 + \lambda_{\max}(R_2) \frac{\lambda_i}{1-\tau_1} \frac{e^{\eta_i \tilde{\tau}_1} - 1}{\eta_i(1+\tau_1)} \mathbb{E} \|\xi\|^2 \right. \right. \\ & \quad + \lambda_{\max}(S) \frac{1}{1-\tau_1} \frac{1-e^{-\eta_i \tilde{\tau}_1}}{\eta_i} \mathbb{E} \|\tilde{g}(\xi)\|^2 \\ & \quad \left. \left. + \lambda_{\max}(T) \sigma_1 \frac{1-e^{-\eta_i \tilde{\tau}_1}}{\eta_i} \mathbb{E} \|\tilde{h}(\xi)\|^2 + 2P \mathbb{E} |\xi \tilde{f}(\xi)| \right) \right\}, \\ \pi_{\tilde{\xi}_j} &= n \left\{ \left(\frac{\alpha+1}{2pq_0} \right)^* \left(\mu_{\max}(\tilde{H}_j) \mathbb{E} \|\tilde{\xi}\|^2 + \mu_{\max}(\tilde{R}_2) \frac{\mu_j}{1-\tau_2} \frac{e^{-\tilde{\eta}_j \tilde{\tau}_2} - 1}{\tilde{\eta}_j(1+\tau_2)} \mathbb{E} \|\tilde{\xi}\|^2 \right. \right. \\ & \quad + \mu_{\max}(\tilde{S}) \frac{1}{1-\tau_2} \frac{1-e^{-\tilde{\eta}_j \tilde{\tau}_2}}{\tilde{\eta}_j} \mathbb{E} \|\tilde{g}(\tilde{\xi})\|^2 \\ & \quad \left. \left. + \mu_{\max}(\tilde{T}) \sigma_2 \frac{1-e^{-\tilde{\eta}_j \tilde{\tau}_2}}{\tilde{\eta}_j} \mathbb{E} \|\tilde{h}(\tilde{\xi})\|^2 + 2Q \mathbb{E} |\tilde{\xi} \tilde{f}(\tilde{\xi})| \right) \right\}. \end{aligned}$$

Let $\chi = \max_{i,j \in \mathbb{M}} \{\pi_{\xi_i + \tilde{\xi}_j}\}$. It follows from (23), (24), and (25) that

$$\mathbb{E} \{\|u(t)\|^2\} + \mathbb{E} \{\|v(t)\|^2\} \leq \chi e^{-\frac{2\eta}{\alpha+1} t}.$$

Therefore

$$\mathbb{E} \{\|u(t)\|^2\} + \mathbb{E} \{\|v(t)\|^2\} \leq \chi e^{-\frac{2\eta}{\alpha+1} t} \quad \text{for all } t > 0. \quad (26)$$

By Definition 1.3 and (26), we see that the equilibrium point of neural networks (4) is globally exponentially stable in the mean square sense. $\square$

Remark 2.2 To the best of our knowledge, the global exponential stability criteria for impulsive effects of SBAMNNs with Markovian jump parameters and mixed time, leakage term delays, α -inverse Holder activation functions have not been discussed in the existing literature. Hence this paper reports a new idea and some sufficient conditions for global exponential stability conditions of neural networks, which generalize and improve the outcomes in [9, 11, 21, 37, 38].

Remark 2.3 The criteria given in Theorem 2.1 are dependent on the time delay. It is well known that the delay-dependent criteria are less conservative than the delay-independent

criteria, particularly when the delay is small. Based on Theorem 2.1, the following result can be obtained easily.

Remark 2.4 If there are no stochastic disturbances in system (4), then the neural networks are simplified to

$$\begin{aligned} du(t) &= \left[-C(r(t))u(t - v_1) + W_0(r(t))\bar{f}(v(t)) \right. \\ &\quad + W_1(r(t))(\bar{g}(v(t - \tau_1(t)))) \\ &\quad \left. + W_2(r(t)) \int_{t-\sigma_1}^t \bar{h}(v(s)) ds \right] dt, \quad t > 0, t \neq t_k, \\ \Delta u(t_k) &= \bar{M}_k(u(t_k^-), u_{t_k^-}), \quad t = t_k, k \in \mathbb{Z}_+, \\ dv(t) &= \left[-D(\tilde{r}(t))v(t - v_2) + V_0(\tilde{r}(t))\bar{\bar{f}}(u(t)) \right. \\ &\quad + V_1(\tilde{r}(t))\bar{\bar{g}}(u(t - \tau_2(t))) \\ &\quad \left. + V_2(\tilde{r}(t)) \int_{t-\sigma_2}^t \bar{\bar{h}}(u(s)) ds \right] dt, \quad t > 0, t \neq t_k, \\ \Delta v(t_k) &= \bar{N}_k(v(t_k^-), v_{t_k^-}), \quad t = t_k, k \in \mathbb{Z}_+. \end{aligned} \quad (27)$$

3 Global exponential stability of uncertain system

Now consider the following BAM neural networks with stochastic noise disturbance, Markovian jump parameters, leakage and mixed time delays, which are in the uncertainty case system:

$$\begin{aligned} du(t) &= \left[-(C + \Delta C(t))(r(t))u(t - v_1) \right. \\ &\quad + (W_0 + \Delta W_0(t))(r(t))\bar{f}(v(t)) \\ &\quad + (W_1 + \Delta W_1(t))(r(t))(\bar{g}(v(t - \tau_1(t)))) \\ &\quad \left. + (W_2 + \Delta W_2(t))(r(t)) \int_{t-\sigma_1}^t \bar{h}(v(s)) ds \right] dt \\ &\quad + \bar{\rho}_1(u(t - v_1), v(t), v(t - \tau_1(t)), t, r(t)) d\bar{\omega}(t), \quad t > 0, t \neq t_k, \\ \Delta u(t_k) &= (\bar{M}_k + \Delta \bar{M}_k(t))(r(t))(u(t_k^-), u_{t_k^-}), \quad t = t_k, k \in \mathbb{Z}_+, \\ dv(t) &= \left[-(D + \Delta D(t))(\tilde{r}(t))v(t - v_2) \right. \\ &\quad + (V_0 + \Delta V_0(t))(\tilde{r}(t))\bar{\bar{f}}(u(t)) \\ &\quad + (V_1 + \Delta V_1(t))(\tilde{r}(t))\bar{\bar{g}}(u(t - \tau_2(t))) \\ &\quad \left. + (V_2 + \Delta V_2(t))(\tilde{r}(t)) \int_{t-\sigma_2}^t \bar{\bar{h}}(u(s)) ds \right] dt \\ &\quad + \bar{\rho}_2(v(t - v_2), u(t), u(t - \tau_2(t)), t, \tilde{r}(t)) d\bar{\bar{\omega}}(t), \quad t > 0, t \neq t_k, \\ \Delta v(t_k) &= (\bar{N}_k + \Delta \bar{N}_k(t))(\tilde{r}(t))(v(t_k^-), v_{t_k^-}), \quad t = t_k, k \in \mathbb{Z}_+. \end{aligned} \quad (28)$$

Assumption 5 The perturbed uncertain matrices $\Delta C(t)$, $\Delta D(t)$, $\Delta W_{0i}(t)$, $\Delta W_{1i}(t)$, $\Delta W_{2i}(t)$, $\Delta V_{0j}(t)$, $\Delta V_{1j}(t)$, and $\Delta V_{2j}(t)$ are time-varying functions satisfying: $\Delta W_{0i}(t) = MF_l(t)N_{W_{0i}}$, $\Delta W_{1i}(t) = MF_l(t)N_{W_{1i}}$, $\Delta W_{2i}(t) = MF_l(t)N_{W_{2i}}$, $\Delta V_{0j}(t) = MF_l(t)N_{V_{0j}}$, $\Delta V_{1j}(t) = MF_l(t)N_{V_{1j}}$, $\Delta V_{2j}(t) = MF_l(t)N_{V_{2j}}$, $\Delta C(t) = MF_l(t)N_{C_i}$ and $\Delta D(t) = MF_l(t)N_{D_j}$, where M , $N_{W_{0i}}$, $N_{W_{1i}}$, $N_{W_{2i}}$, $N_{V_{0j}}$, $N_{V_{1j}}$, $N_{V_{2j}}$, N_{C_i} , and N_{D_j} are given constant matrices, respectively. $F_{l_z}(t)$ ($l = 0, 1, 2, 3$) (where $z =$ either i or j) are unknown real time-varying matrices which have the following structure: $F_{l_z}(t) = \text{blockdiag}\{\delta_{l_1}(t)I_{z_{l_1}}, \dots, \delta_{l_k}(t)I_{z_{l_k}}, F_{l_1}(t), \dots, F_{l_s}(t)\}$, $\delta_{l_z} \in \mathbb{R}$, $|\delta_{l_z}| \leq 1$, $1 \leq z \leq \tilde{k}$ and $F_{l_p}^T F_{l_p} \leq I$, $1 \leq p \leq s$. We define the set Δ_l as $\Delta_l = \{F_{l_z}^T(t)F_{l_z}(t) \leq I, F_{l_z}N_{l_z} = N_{l_z}F_{l_z}, \forall N_{l_z} \in \Gamma_{l_z}\}$, where $\Gamma_{l_z} = \{N_{l_z} = \text{blockdiag}[N_{l_1}, \dots, N_{l_k}, n_{l_1}I_{f_{l_1}}, \dots, n_{l_s}I_{f_{l_s}}]\}$, N_{l_z} invertible for $1 \leq z \leq \tilde{k}$ and $n_{l_p} \in \mathbb{R}$, $n_{l_p} \neq 0$, for $1 \leq p \leq s$ and $p, \tilde{k}, s \in \mathfrak{M}$.

Also ΔH_i , $\Delta \tilde{H}_j$, ΔR_{1i} , ΔR_{2i} , ΔR_{3i} , $\Delta \tilde{R}_{1j}$, $\Delta \tilde{R}_{2j}$, $\Delta \tilde{R}_{3j}$, ΔR_2 , $\Delta \tilde{R}_2$, ΔS , ΔT , $\Delta \tilde{S}$, $\Delta \tilde{T}$, ΔN_1 , ΔN_2 , ΔN_3 , ΔN_4 , ΔN_5 , and ΔN_6 are positive definite diagonal matrices that are defined as follows: $\Delta H_i = \check{E}\Sigma F_{H_i}$, $\Delta \tilde{H}_j = \check{E}\Sigma F_{\tilde{H}_j}$ and $\Delta R_{1i} = \check{E}\Sigma F_{R_{1i}}$, where $\check{E}$, F_{H_i} , $F_{\tilde{H}_j}$, $F_{R_{1i}}$, $F_{R_{2i}}$, $F_{R_{3i}}$, $F_{\tilde{R}_{1j}}$, $F_{\tilde{R}_{2j}}$, $F_{\tilde{R}_{3j}}$, F_{R_2} , $F_{\tilde{R}_2}$, F_S , $F_{\tilde{S}}$, F_T , $F_{\tilde{T}}$, F_{N_1} , F_{N_2} , F_{N_3} , F_{N_4} , F_{N_5} , and F_{N_6} are positive diagonal matrices (i.e., $F_{H_i}F_{H_i}^T = \text{diag}(h_1^*, h_2^*, \dots, h_n^*)$, $F_{\tilde{H}_j}F_{\tilde{H}_j}^T = \text{diag}(\tilde{h}_1^*, \tilde{h}_2^*, \dots, \tilde{h}_n^*)$, where $h_i^*, \tilde{h}_j^* > 0$ ($i, j = 1, 2, \dots, n$)) and the remaining terms are defined in a similar way, which characterizes how the deterministic uncertain parameter in Σ enters the nominal matrices H_i , $\tilde{H}_j$, R_{b_i} ($b = 1, 2, 3$), $\tilde{R}_{c_j}$ ($c = 1, 2, 3$), S , $\tilde{S}$, T , $\tilde{T}$, N_1 , N_2 , N_3 , N_4 , N_5 , and N_6 . The matrix Σ with real entries, which may be time-varying, is unknown and satisfies $\Sigma^T \Sigma \leq I$.

Remark 3.1 Overall, the stability of time-delayed neural networks fully depends on the Lyapunov–Krasovskii functional and LMI concepts. In particular, based on the neural networks, different types of LKF are chosen or handled to lead to the system stability. Up to now, no one has considered uncertain parameters in Lyapunov–Krasovskii functional terms. Without loss of generality, the gap is initially filled in this work, and also this kind of approach gives more advanced and less conserved stability results.

Theorem 3.2 Under Assumptions 1, 2, and 5, the neural network system (28) is global robust exponentially stable in the mean square if, for given $\eta_i, \tilde{\eta}_j > 0$ ($i, j \in \mathfrak{M}$), there exist positive definite matrices S , T , $\tilde{S}$, $\tilde{T}$, R_2 , $\tilde{R}_2$, N_1 , N_2 , N_3 , N_4 , N_5 , N_6 and H_i , $\tilde{H}_j$ ($i, j \in \mathfrak{M}$), positive definite diagonal matrices ΔS , ΔT , ΔR_2 , $\Delta \tilde{S}$, $\Delta \tilde{T}$, $\Delta \tilde{R}_2$, ΔH_i , $\Delta \tilde{H}_j$, ΔN_1 , ΔN_2 , ΔN_3 , ΔN_4 , ΔN_5 , ΔN_6 , P , Q and positive scalars λ_i and μ_j ($i, j \in \mathfrak{M}$) such that the following LMIs are satisfied:

$$(H_i + \Delta H_i) < \lambda_i I, \quad (29)$$

$$(\tilde{H}_j + \Delta \tilde{H}_j) < \mu_j I, \quad (30)$$

$$(\bar{M}_k + \Delta \bar{M}_k)^T (H_i + \Delta H_i) (\bar{M}_k + \Delta \bar{M}_k) - (H_j + \Delta H_j) \leq 0, \quad (31)$$

$$(\bar{N}_k + \Delta \bar{N}_k)^T (\tilde{H}_i + \Delta \tilde{H}_i) (\bar{N}_k + \Delta \bar{N}_k) - (\tilde{H}_j + \Delta \tilde{H}_j) \leq 0 \quad [\text{here } r(t_k) = i \text{ and } \tilde{r}(t_k) = j], \quad (32)$$

$$\Omega^* = \begin{bmatrix} \Xi_1^* & \Xi_2^* & \Xi_3^* & \Xi_4^* & \Xi_5^* & \Xi_{10}^* \\ * & \Xi_6^* & 0 & 0 & 0 & 0 \\ * & * & \Xi_7^* & 0 & 0 & 0 \\ * & * & * & \Xi_8^* & 0 & 0 \\ * & * & * & * & \Xi_9^* & 0 \\ * & * & * & * & * & \Xi_{11}^* \end{bmatrix} < 0, \quad (33)$$

(34)

where

$$\begin{aligned} \Xi_6^* &= \text{diag}\{\Theta_i\}, & \Xi_7^* &= \text{diag}\{\Theta_{l_1}\}, & \Delta_6^* &= \text{diag}\{\Phi_j\}, & \Delta_7^* &= \text{diag}\{\Phi_{l_2}\}, \\ i, j &= 1, 2, 3, \dots, 12 \text{ and } l_1, l_2 = 13, 14, 15, \dots, 24; \\ \Xi_8^* &= \text{diag}\{\tilde{\Theta}_s\}, & \Xi_9^* &= \text{diag}\{\tilde{\Theta}_{l_3}\}, & \Delta_8^* &= \text{diag}\{\tilde{\Phi}_{s^*}\}, & \Delta_9^* &= \text{diag}\{\tilde{\Phi}_{l_4}\}, \\ s, s^* &= 1, 2, 3, \dots, 12 \text{ and } l_3, l_4 = 13, 14, 15, 16; \\ \Xi_{11}^* &= \text{diag}\{\Theta_{l_5}^*\}, & \Delta_{11}^* &= \text{diag}\{\Phi_{l_6}^*\}, & l_5, l_6 &= 1, 2, 3, \dots, 10; \end{aligned}$$

[illegible]

[illegible]

$$\Xi_4^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ FH_i^T & \epsilon_{13}\check{E} & 0 & \epsilon_{14}M & 0 & \Gamma_4^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Gamma_1^* & 0 & \Gamma_2^* & 0 & \Gamma_3^* & 0 & \vartheta_1^{*T} & \epsilon_{16}\check{E} & \Gamma_5^* & \Gamma_6^* & \vartheta_2^{*T} & \Gamma_7^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Delta_4^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ F\tilde{H}_j^T & \tilde{\epsilon}_{13}\check{E} & 0 & \tilde{\epsilon}_{14}M & 0 & \tilde{\Gamma}_4^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{\Gamma}_1^* & 0 & \tilde{\Gamma}_2^* & 0 & \tilde{\Gamma}_3^* & 0 & \tilde{\vartheta}_1^{*T} & \tilde{\epsilon}_{16}\check{E} & \tilde{\Gamma}_5^* & \tilde{\Gamma}_6^* & \tilde{\vartheta}_2^{*T} & \tilde{\Gamma}_7^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Xi_5^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ H_i^T N_{C_i}^T & \epsilon_{19}C_i^T M & \Gamma_8^* & C_i^T \check{E} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Delta_5^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \tilde{H}_j^T N_{D_j}^T & \tilde{\epsilon}_{19}D_j^T M & \tilde{\Gamma}_8^* & D_j^T \check{E} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Xi_{10}^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon_{21}\check{E} & F_{N_1}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_{22}\check{E} & \alpha^*F_{N_3}^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_{23}\check{E} & \Gamma_{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon_{24}\check{E} & F_{N_3}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon_{25}\check{E} & \sigma_1F_{N_5}^T \end{bmatrix},$$

$$\Delta_{10}^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{\epsilon}_{21}\check{E} & F_{N_2}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\epsilon}_{22}\check{E} & \beta^*F_{N_4}^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{\epsilon}_{23}\check{E} & \tilde{\Gamma}_{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\epsilon}_{24}\check{E} & F_{N_4}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\epsilon}_{25}\check{E} & \sigma_2F_{N_6}^T \end{bmatrix},$$

where

$$\begin{aligned} \Theta_1 &= \Theta_2 = -\epsilon_1 I, & \Theta_3 &= \Theta_4 = -\epsilon_2 I, & \Theta_7 &= \Theta_8 = -\epsilon_4 I, & \Theta_{11} &= \Theta_{12} = -\epsilon_6 I, \\ \Theta_{19} &= -\epsilon_{10} I, & \Theta_9 &= \Theta_{10} = -\epsilon_5 I, & \Theta_{13} &= \Theta_{14} = -\epsilon_7 I, & \Theta_{15} &= \Theta_{16} = -\epsilon_8 I, \\ \Theta_{17} &= \Theta_{18} = -\epsilon_9 I, & \Theta_{20} &= -\epsilon_{10} I, & \Theta_5 &= \Theta_6 = -\epsilon_3 I, & \Theta_{21} &= \Theta_{22} = -\epsilon_{11} I, \\ \Theta_{23} &= \Theta_{24} = -\epsilon_{12} I, & \Phi_1 &= \Phi_2 = \tilde{\epsilon}_1 I, & \Phi_3 &= \Phi_4 = \tilde{\epsilon}_2 I, & \Phi_5 &= \Phi_6 = \tilde{\epsilon}_3 I, \\ \Phi_7 &= \Phi_8 = \tilde{\epsilon}_4 I, & \Phi_{15} &= \Phi_{16} = \tilde{\epsilon}_8 I, & \Phi_{11} &= \Phi_{12} = \tilde{\epsilon}_6 I, & \Phi_{13} &= \Phi_{14} = \tilde{\epsilon}_7 I, \\ \Phi_9 &= \Phi_{10} = \tilde{\epsilon}_5 I, & \Phi_{17} &= \Phi_{18} = \tilde{\epsilon}_9 I, & \Phi_{19} &= \Phi_{20} = \tilde{\epsilon}_{10} I, & \Phi_{21} &= \Phi_{22} = \tilde{\epsilon}_{11} I, \\ \Phi_{23} &= \Phi_{24} = \tilde{\epsilon}_{12} I, & \Theta_1^* &= \Theta_2^* = -\epsilon_{21} I, & \Theta_3^* &= \Theta_4^* = -\epsilon_{22} I, & \Theta_5^* &= \Theta_6^* = -\epsilon_{23} I, \\ \Theta_7^* &= \Theta_8^* = -\epsilon_{24} I, & \Theta_9^* &= \Theta_{10}^* = -\epsilon_{25} I, & \Phi_1^* &= \Phi_2^* = -\tilde{\epsilon}_{21} I, & \Phi_3^* &= \Phi_4^* = -\tilde{\epsilon}_{22} I, \\ \Phi_5^* &= \Phi_6^* = -\tilde{\epsilon}_{23} I, & \Phi_7^* &= \Phi_8^* = -\tilde{\epsilon}_{24} I, & \Phi_9^* &= \Phi_{10}^* = -\tilde{\epsilon}_{25} I, & \Gamma_1 &= \epsilon_1 M H_i, \\ \Gamma_2 &= N_{C_i}^T F_{H_i}^T, & \Gamma_3 &= F_{H_i}^T W_{0i}^T, & \Gamma_4 &= F_{H_i}^T N_{W_{0i}}^T, & \Gamma_5 &= F_{H_i}^T W_{1i}^T, & \Gamma_6 &= F_{H_i}^T N_{W_{1i}}^T, \end{aligned}$$

$$\begin{aligned}
 \Gamma_7 &= -W_1^T F_{H_i}^T, & \Gamma_8 &= F_{H_i}^T W_{2i}^T, & \Gamma_{12} &= C F_{H_i}^T N_{W_{2i}}^T, & \Gamma_{10} &= -W_2^T F_{H_i}^T, \\
 \Gamma_{11} &= C F_{H_i}^T N_{W_{1i}}^T, & \Gamma_9 &= F_{H_i}^T N_{W_{2i}}^T, & \Gamma_{13} &= \frac{-1}{2\sigma_1} F_T^T, & \Gamma_{14} &= \frac{-\lambda_i}{2} F_{R_2}^T, \\
 \tilde{\Gamma}_1 &= \tilde{\epsilon}_1 M \tilde{H}_j, & \tilde{\Gamma}_2 &= N_{D_j}^T F_{H_j}^T, & \tilde{\Gamma}_3 &= F_{H_j}^T V_{0j}^T, & \tilde{\Gamma}_4 &= F_{H_j}^T N_{V_{0j}}^T, & \tilde{\Gamma}_5 &= F_{H_j}^T V_{1j}^T, \\
 \tilde{\Gamma}_6 &= F_{H_j}^T N_{V_{1j}}^T, & \tilde{\Gamma}_7 &= -V_1^T F_{H_j}^T, & \tilde{\Gamma}_8 &= F_{H_j}^T V_{2j}^T, & \tilde{\Gamma}_{10} &= -V_2^T F_{H_j}^T, \\
 \tilde{\Gamma}_9 &= F_{H_j}^T N_{V_{2j}}^T, & \tilde{\Gamma}_{11} &= D F_{H_j}^T N_{V_{1j}}^T, & \tilde{\Gamma}_{12} &= D F_{H_j}^T N_{V_{2j}}^T, & \tilde{\Gamma}_{13} &= \frac{-1}{2\sigma_2} F_T^T, \\
 \tilde{\Gamma}_{14} &= \frac{-\mu_j}{2} F_{R_2}^T, & \tilde{\Theta}_1 &= \tilde{\Theta}_2 = -\epsilon_{13} I, & \tilde{\Theta}_3 &= \tilde{\Theta}_4 = -\epsilon_{14} I, & \tilde{\Theta}_5 &= \tilde{\Theta}_6 = -\epsilon_{15} I, \\
 \tilde{\Theta}_7 &= \tilde{\Theta}_8 = -\epsilon_{16} I, & \tilde{\vartheta}_2^* &= D_j^T N_{D_j}^T F_{H_j}^T, & \tilde{\Theta}_9 &= \tilde{\Theta}_{10} = -\epsilon_{17} I, \\
 \tilde{\Theta}_{11} &= \tilde{\Theta}_{12} = -\epsilon_{18} I, & \tilde{\Theta}_{13} &= \tilde{\Theta}_{14} = -\epsilon_{19} I, & \tilde{\Theta}_{15} &= \tilde{\Theta}_{16} = -\epsilon_{20} I, \\
 \tilde{\Phi}_1 &= \tilde{\Phi}_2 = -\tilde{\epsilon}_{13} I, & \tilde{\Phi}_3 &= \tilde{\Phi}_4 = -\tilde{\epsilon}_{14} I, & \tilde{\Phi}_5 &= \tilde{\Phi}_6 = -\tilde{\epsilon}_{15} I, \\
 \tilde{\Phi}_7 &= \tilde{\Phi}_8 = -\tilde{\epsilon}_{16} I, & \tilde{\Gamma}_8^* &= \tilde{\epsilon}_{20} F_{H_j}^T M N_{D_j}^T, & \tilde{\Phi}_9 &= \tilde{\Phi}_{10} = -\tilde{\epsilon}_{17} I, \\
 \tilde{\Phi}_{11} &= \tilde{\Phi}_{12} = -\tilde{\epsilon}_{18} I, & \tilde{\Phi}_{13} &= \tilde{\Phi}_{14} = -\tilde{\epsilon}_{19} I, & \tilde{\Phi}_{15} &= \tilde{\Phi}_{16} = -\tilde{\epsilon}_{20} I, & \tilde{\Gamma}_2^* &= N_{D_j}^T \tilde{H}_j^T, \\
 \vartheta_1^* &= C_i^T F_{H_i} C_i + M^T N_{C_i}^T F_{H_i} M N_{C_i}, & \tilde{\vartheta}_1^* &= D_j^T F_{H_j} D_j + M^T N_{D_j}^T F_{H_j} M N_{D_j}, \\
 \vartheta_2^* &= C_i^T N_{C_i}^T F_{H_i}^T, & \Gamma_1^* &= -C_i^T F_{H_i}^T, & \Gamma_2^* &= N_{C_i}^T H_i^T, & \Gamma_3^* &= N_{C_i}^T F_{H_i}, \\
 \Gamma_4^* &= \epsilon_{15} M \check{E}, & \Gamma_5^* &= N_{C_i} H_i^T, & \tilde{\Gamma}_5^* &= N_{D_j} \tilde{H}_j^T, & \Gamma_6^* &= \epsilon_{17} C_i M^T, \\
 \Gamma_7^* &= \epsilon_{18} M^T \check{E}, & \Gamma_8^* &= \epsilon_{20} F_{H_i}^T M N_{C_i}^T, & \tilde{\Gamma}_6^* &= \tilde{\epsilon}_{17} D_j M^T, & \tilde{\Gamma}_7^* &= \tilde{\epsilon}_{18} M^T \check{E}, \\
 \tilde{\Gamma}_1^* &= -D_j^T F_{H_j}^T, \\
 \Omega_{99}^* &= \sum_{l=1}^N \tilde{\gamma}_{li} D_j^T \tilde{H}_l D_j + \sum_{l=1}^N \tilde{\gamma}_{li} M^T N_{D_j}^T \tilde{H}_j M N_{D_j} - \tilde{\eta}_j D_j^T \tilde{H}_j D_j + M^T N_{D_j}^T \tilde{H}_j M N_{D_j}, \\
 \Xi_{29}^* &= C_i^T H_i C_i - \sum_{l=1}^N \gamma_{il} C_i H_l + M^T N_{C_i}^T H_i M N_{C_i} - \eta_i C_i^T H_i, \\
 \Omega_{69}^* &= -D_j^T \tilde{H}_j V_{1j} - M^T N_{D_j}^T \tilde{H}_j M N_{V_{1j}}, & \Xi_{69}^* &= -C_i^T H_i W_{1i} - M^T N_{C_i}^T H_i M N_{W_{1i}}, \\
 \Omega_{29}^* &= D_j^T \tilde{H}_j D_j - \sum_{l=1}^N \tilde{\gamma}_{li} D_j \tilde{H}_l + M^T N_{D_j}^T \tilde{H}_j M N_{D_j} - \tilde{\eta}_j D_j^T \tilde{H}_j, \\
 \Xi_{99}^* &= \sum_{l=1}^N \gamma_{il} C_i^T H_l C_i + \sum_{l=1}^N \gamma_{il} M^T N_{C_i}^T H_i M N_{C_i} + \eta_i C_i^T H_i C_i + M^T N_{C_i}^T H_i M N_{C_i}, \\
 \tilde{\Gamma}_3^* &= N_{D_j}^T F_{H_j}^T, \\
 \Xi_{79}^* &= -C_i^T H_i W_{2i} - M^T N_{C_i}^T H_i M N_{W_{2i}}, & \Omega_{79}^* &= -D_j^T \tilde{H}_j V_{2j} - M^T N_{D_j}^T \tilde{H}_j M N_{V_{2j}}, \\
 \tilde{\Gamma}_4^* &= \tilde{\epsilon}_{15} M \check{E}, & \Gamma_{15} &= \frac{-1}{\sigma_1} F_{N_5}^T, & \tilde{\Gamma}_{15} &= \frac{-1}{\sigma_2} F_{N_6}^T, \\
 \alpha^* &= -(1 - \tau_1) e^{-\eta_i \bar{\tau}_1}, & \beta^* &= -(1 - \tau_2) e^{-\tilde{\eta}_j \bar{\tau}_2}.
 \end{aligned}$$

The remaining values of Ξ_1^* , Δ_1^* are the same as in Theorem 2.1, and $*$ means the symmetric terms.

Proof The matrices C_i , H_i , D_j , $\tilde{H}_j$, R_2 , $\tilde{R}_2$, S , $\tilde{S}$, T , $\tilde{T}$, N_1 , N_2 , N_3 , N_4 , N_5 , and N_6 in the Lyapunov–Krasovskii functional of Theorem 2.1 are replaced by $C_i + \Delta C_i(t)$, $H_i + \Delta H_i$, $D_j + \Delta D_j(t)$, $\tilde{H}_j + \Delta \tilde{H}_j$, $R_2 + \Delta R_2$, $\tilde{R}_2 + \Delta \tilde{R}_2$, $S + \Delta S$, $\tilde{S} + \Delta \tilde{S}$, $T + \Delta T$, $\tilde{T} + \Delta \tilde{T}$, $N_1 + \Delta N_1$, $N_2 + \Delta N_2$, $N_3 + \Delta N_3$, $N_4 + \Delta N_4$, $N_5 + \Delta N_5$, and $N_6 + \Delta N_6$, respectively.

Hence, by applying the same procedure of Theorem 2.1 and using Assumption 5, Lemmas 1.8, 1.9, 1.10 and 1.11 and putting $\eta = \max_{i,j \in \mathfrak{M}} \{\max_{i \in \mathfrak{M}} \eta_i, \max_{j \in \mathfrak{M}} \tilde{\eta}_j\}$, we have from (28) and Definition 2 (weak infinitesimal operator $\mathcal{L}V$) that

$$\mathcal{L}V \leq e^{\eta t} \{ \Psi^T(t) \Omega^* \Psi(t) + \Phi^T(t) \Pi \Phi(t) \},$$

where $\Psi(t)$ and $\Phi(t)$ are given in Theorem 2.1. The remaining proof of this theorem is similar to the procedure of Theorem 2.1, and we get that the uncertain neural network (28) is global robust exponentially stable in the mean square sense. $\square$

4 Numerical examples

In this section, we provide two numerical examples with their simulations to demonstrate the effectiveness of our results.

Example 4.1 Consider the second order stochastic impulsive BAM neural networks (4) with $u(t) = (u_1(t), u_2(t))^T$, $v(t) = (v_1(t), v_2(t))^T$; $\bar{\omega}(t)$, $\tilde{\omega}(t)$ are second order Brownian motions and $r(t)$, $\tilde{r}(t)$ denote right-continuous Markovian chains taking values in $\mathfrak{M} = \{1, 2\}$ with generator

$$\Gamma = \begin{bmatrix} 0.2 & 0.1 \\ 0.4 & 0.3 \end{bmatrix}, \quad \tilde{\Gamma} = \begin{bmatrix} 0.5 & 0.2 \\ 0.4 & 0.3 \end{bmatrix}.$$

The associated parameters of neural networks (4) take the values as follows:

$$\begin{aligned} C_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, & C_2 &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, & D_1 &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, & D_2 &= \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}, \\ W_{01} &= \begin{bmatrix} 0.02 & 0.01 \\ -0.02 & 0.01 \end{bmatrix}, & W_{02} &= \begin{bmatrix} 0.02 & 0.01 \\ 0.02 & 0.01 \end{bmatrix}, & W_{11} &= \begin{bmatrix} 0.03 & -0.02 \\ 0.03 & 0.02 \end{bmatrix}, \\ W_{12} &= \begin{bmatrix} 0.02 & 0.02 \\ -0.01 & 0.01 \end{bmatrix}, & W_{21} &= \begin{bmatrix} 0.03 & 0.04 \\ 0.03 & 0.02 \end{bmatrix}, & W_{22} &= \begin{bmatrix} 0.02 & -0.02 \\ 0.03 & -0.01 \end{bmatrix}, \\ V_{01} &= \begin{bmatrix} 0.02 & 0.02 \\ 0.01 & 0.03 \end{bmatrix}, & V_{02} &= \begin{bmatrix} 0.02 & 0.01 \\ 0.01 & 0.02 \end{bmatrix}, & V_{11} &= \begin{bmatrix} 0.01 & -0.02 \\ 0.02 & 0.01 \end{bmatrix}, \\ V_{12} &= \begin{bmatrix} 0.01 & 0.02 \\ 0.02 & -0.03 \end{bmatrix}, & V_{21} &= \begin{bmatrix} 0.02 & 0.01 \\ 0.02 & 0.01 \end{bmatrix}, & V_{22} &= \begin{bmatrix} 0.02 & 0.02 \\ 0.03 & 0.02 \end{bmatrix}, \\ R_2 &= \begin{bmatrix} 0.02 & 0 \\ 0 & 0.03 \end{bmatrix}, & \tilde{R}_2 &= \begin{bmatrix} 0.05 & 0 \\ 0 & 0.02 \end{bmatrix}, & R_{11} &= \begin{bmatrix} 0.06 & 0 \\ 0 & 0.04 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} R_{12} &= \begin{bmatrix} 0.03 & 0 \\ 0 & 0.02 \end{bmatrix}, & R_{21} &= \begin{bmatrix} 0.02 & 0 \\ 0 & 0.03 \end{bmatrix}, & R_{22} &= \begin{bmatrix} 0.05 & 0 \\ 0 & 0.02 \end{bmatrix}, \\ R_{31} &= \begin{bmatrix} 0.03 & 0 \\ 0 & 0.08 \end{bmatrix}, & R_{32} &= \begin{bmatrix} 0.07 & 0 \\ 0 & 0.04 \end{bmatrix}, & \tilde{R}_{11} &= \begin{bmatrix} 0.2341 & 0 \\ 0 & 0.3421 \end{bmatrix}, \\ \tilde{R}_{12} &= \begin{bmatrix} 0.2451 & 0 \\ 0 & 0.0251 \end{bmatrix}, & \tilde{R}_{21} &= \begin{bmatrix} 0.1802 & 0 \\ 0 & 0.0102 \end{bmatrix}, \\ \tilde{R}_{22} &= \begin{bmatrix} 0.1212 & 0 \\ 0 & 0.0140 \end{bmatrix}, & \tilde{R}_{31} &= \begin{bmatrix} 0.02 & 0 \\ 0 & 0.05 \end{bmatrix}, & \tilde{R}_{32} &= \begin{bmatrix} 0.03 & 0 \\ 0 & 0.04 \end{bmatrix}, \\ \bar{M}_k &= \begin{bmatrix} 0.04 & 0 \\ 0 & 0.05 \end{bmatrix}, & \bar{N}_k &= \begin{bmatrix} 0.05 & 0 \\ 0 & 0.03 \end{bmatrix}. \end{aligned}$$

Taking

$$\begin{aligned} &\bar{\rho}_1(u(t - \nu_1), v(t), v(t - \tau_1(t)), t, 1) \\ &= \begin{bmatrix} 0.04 * u_1(t - \nu_1) + 0.04 * v_1(t) + 0.03 * v_1(t - \bar{\tau}_1) & 0 \\ 0 & 0.05 * u_2(t - \nu_1) + 0.02 * v_2(t) + 0.03 * v_2(t - \bar{\tau}_1) \end{bmatrix}, \\ &\bar{\rho}_1(u(t - \nu_1), v(t), v(t - \tau_1(t)), t, 2) \\ &= \begin{bmatrix} 0.04 * u_1(t - \nu_1) + 0.03 * v_1(t) + 0.05 * v_1(t - \bar{\tau}_1) & 0 \\ 0 & 0.02 * u_2(t - \nu_1) + 0.04 * v_2(t) + 0.02 * v_2(t - \bar{\tau}_1) \end{bmatrix}, \\ &\bar{\rho}_2(v(t - \nu_2), u(t), u(t - \tau_2(t)), t, 1) \\ &= \begin{bmatrix} 0.02 * v_1(t - \nu_2) + 0.02 * u_1(t) + 0.02 * u_1(t - \bar{\tau}_2) & 0 \\ 0 & 0.03 * v_2(t - \nu_2) + 0.02 * u_2(t) + 0.05 * u_2(t - \bar{\tau}_2) \end{bmatrix}, \\ &\bar{\rho}_2(v(t - \nu_2), u(t), u(t - \tau_2(t)), t, 2) \\ &= \begin{bmatrix} 0.01 * v_1(t - \nu_2) + 0.03 * u_1(t) + 0.03 * u_1(t - \bar{\tau}_2) & 0 \\ 0 & 0.04 * v_2(t - \nu_2) + 0.03 * u_2(t) + 0.02 * u_2(t - \bar{\tau}_2) \end{bmatrix}, \\ &\nu_1 = \nu_2 = 1, \quad \bar{\tau}_1 = \bar{\tau}_2 = 7.46, \quad \sigma_1 = 0.6, \quad \sigma_2 = 0.8. \end{aligned}$$

The following activation functions play in neural network system (4):

$$\begin{aligned} \bar{f}(v) &= \sinh(v), & \bar{f}(u) &= \sinh(u), & \bar{g}(v) &= v, \\ \bar{g}(u) &= u, & \bar{h}(v) &= \sin(v), & \bar{h}(u) &= \sin(u). \end{aligned}$$

It is easy to obtain that, for any $a, b \in \mathbb{R}$ with $a < b$, there exists a scalar $c \in (a, b)$ such that

$$\frac{f_i(b) - f_i(a)}{b - a} = \frac{\sinh(b) - \sinh(a)}{b - a} = \cosh(c) \geq 1.$$

Therefore $f_i(\cdot)$ and $\bar{f}_i(\cdot)$, $i, j = 1, 2$, are 1-inverse Holder functions. In addition, for any $a, b \in \mathbb{R}$, it is easy to check that

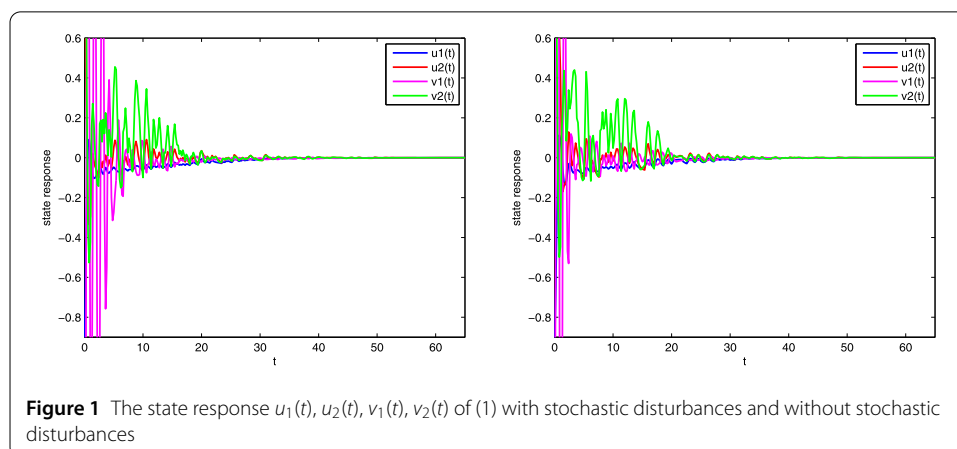
$$\begin{aligned} |h_i(b) - h_i(a)| &= |\sin(b) - \sin(a)| \leq |g_i(b) - g_i(a)| = |b - a| \\ &\leq |f_i(b) - f_i(a)| = |\sinh(b) - \sinh(a)|. \end{aligned}$$

By a similar way, we get the same inequalities for $\bar{g}_j(\cdot)$ and also $\bar{h}_j(\cdot)$ ($j = 1, 2$). That means the activation functions $\bar{f}_i(\cdot), \bar{f}_j(\cdot), \bar{g}_i(\cdot), \bar{g}_j(\cdot), \bar{h}_i(\cdot), \bar{h}_j(\cdot)$ ($i, j = 1, 2$) satisfy Assumptions 1 and 2.

Then, by Theorem 2.1, solving the LMIs using the Matlab LMI control toolbox, one can obtain the following feasible solutions:

$$\begin{aligned} S &= 10^{-4} \times \begin{bmatrix} 0.2968 & -0.0000 \\ -0.0000 & 0.2957 \end{bmatrix}, & T &= 10^{-4} \times \begin{bmatrix} 0.8435 & -0.0000 \\ -0.0000 & 0.8461 \end{bmatrix}, \\ N_1 &= \begin{bmatrix} 0.4948 & -0.0001 \\ -0.0001 & 0.4948 \end{bmatrix}, & N_3 &= \begin{bmatrix} 0.7465 & 0.0147 \\ 0.0147 & 0.7263 \end{bmatrix}, \\ N_4 &= \begin{bmatrix} 0.8333 & -0.0046 \\ -0.0046 & 0.8248 \end{bmatrix}, & N_5 &= \begin{bmatrix} 0.4424 & 0.0001 \\ 0.0001 & 0.4423 \end{bmatrix}, \\ \tilde{S} &= \begin{bmatrix} 0.0373 & 0.0000 \\ 0.0000 & 0.0373 \end{bmatrix}, & \tilde{T} &= 10^{-3} \times \begin{bmatrix} 0.2156 & -0.0001 \\ -0.0001 & 0.2155 \end{bmatrix}, \\ H_1 &= \begin{bmatrix} 0.3533 & -0.0069 \\ -0.0069 & 0.0936 \end{bmatrix}, & \tilde{H}_1 &= \begin{bmatrix} 0.0728 & 0 \\ 0 & 0.0076 \end{bmatrix}, \\ \tilde{H}_2 &= \begin{bmatrix} 0.0355 & -0.0030 \\ -0.0030 & 0.0350 \end{bmatrix}, & P &= 10^{-4} \times \begin{bmatrix} 0.4368 & 0 \\ 0 & 0.4487 \end{bmatrix}, \\ N_2 &= \begin{bmatrix} 0.4979 & 0.0215 \\ 0.0215 & 0.9157 \end{bmatrix}, & N_6 &= \begin{bmatrix} 0.4841 & 0.0000 \\ 0.0000 & 0.4840 \end{bmatrix}, \\ H_2 &= \begin{bmatrix} 0.0016 & -0.0000 \\ -0.0000 & 0.0017 \end{bmatrix}, & P_1 &= 10^{-4} \times \begin{bmatrix} 0.4947 & 0 \\ 0 & 0.4947 \end{bmatrix}, \\ Q_1 &= 10^{-3} \times \begin{bmatrix} 0.4947 & 0 \\ 0 & 0.4947 \end{bmatrix}, & Q &= 10^{-3} \times \begin{bmatrix} 0.5549 & 0 \\ 0 & 0.4465 \end{bmatrix}, \\ \lambda_1 &= 0.6805, & \lambda_2 &= 0.0023, & \mu_1 &= 2.0060, & \mu_2 &= 0.8234. \end{aligned}$$

Figure 1 narrates the time response of state variables $u_1(t), u_2(t), v_1(t), v_2(t)$ with and without stochastic noises, and Fig. 2 depicts the time response of Markovian jumps $r(t) = i$,



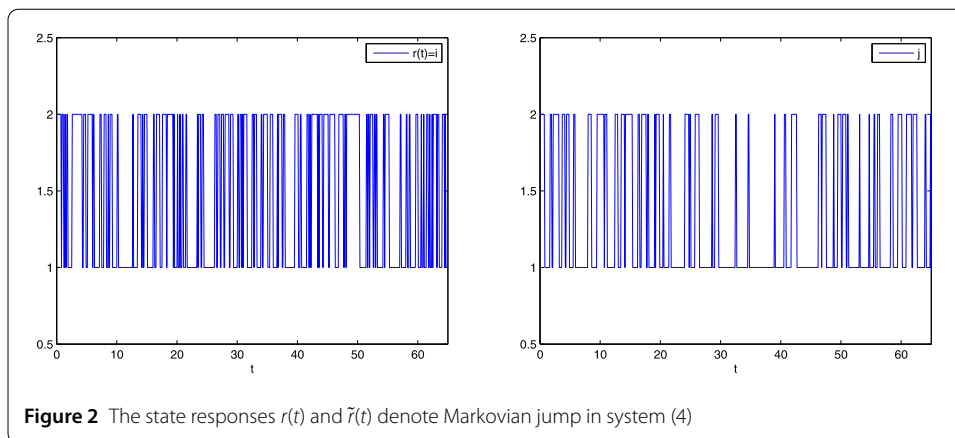


Table 1 Maximum allowable upper bounds of discrete time delays

Methods	$\bar{\tau}_1 = \bar{\tau}_2 > 0$	System status
In Ref. [41]	0.5784	feasible
In Ref. [42]	2.1	feasible
In Ref. [43]	4.822	feasible
In Ref. [44]	5	feasible
In Ref. [34]	5.912	feasible
In Ref. [45]	6.884	feasible
Theorem 2.1	7.46	feasible

$\tilde{r}(t) = j$. By solving LMIs (5)–(10), we get the feasible solutions. The obtained discrete time delay upper bounds of $\bar{\tau}_1$ and $\bar{\tau}_2$ for neural networks (4), which are given in Table 1, are very maximum. This shows that the contributions of this research work is more effective and less conservative than some existing results. Therefore, by Theorem 2.1, we can conclude that neural networks (4) are globally exponentially stable in the mean square for the maximum allowable upper bounds $\bar{\tau}_1 = \bar{\tau}_2 = 7.46$.

Example 4.2 Consider the second order uncertain stochastic impulsive BAM neural networks (28) with $u(t) = (u_1(t), u_2(t))^T$, $v(t) = (v_1(t), v_2(t))^T$; $\tilde{\omega}(t)$, $\tilde{\tilde{\omega}}(t)$ are second order Brownian motions and $r(t)$, $\tilde{r}(t)$ denote right-continuous Markovian chains taking values in $\mathfrak{M} = \{1, 2\}$ with generator

$$\Gamma = \tilde{\Gamma} = \begin{bmatrix} -3 & 3 \\ 4 & -2 \end{bmatrix}.$$

The associated parameters of neural networks (28) are as follows:

$$\begin{aligned} C_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, & C_2 &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, & D_1 &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, & D_2 &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \\ W_{01} &= \begin{bmatrix} 0.02 & 0.01 \\ -0.02 & 0.01 \end{bmatrix}, & W_{02} &= \begin{bmatrix} 0.02 & 0.01 \\ 0.02 & 0.01 \end{bmatrix}, & W_{11} &= \begin{bmatrix} 0.03 & -0.02 \\ 0.03 & 0.02 \end{bmatrix}, \\ W_{12} &= \begin{bmatrix} 0.02 & 0.02 \\ -0.01 & 0.01 \end{bmatrix}, & W_{21} &= \begin{bmatrix} 0.05 & 0.01 \\ 0.03 & 0.02 \end{bmatrix}, & W_{22} &= \begin{bmatrix} 0.02 & -0.01 \\ 0.03 & -0.01 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
 V_{01} &= \begin{bmatrix} 0.02 & 0.02 \\ 0.01 & 0.03 \end{bmatrix}, & V_{02} &= \begin{bmatrix} 0.02 & 0.01 \\ 0.01 & 0.02 \end{bmatrix}, & V_{11} &= \begin{bmatrix} 0.01 & -0.02 \\ 0.02 & 0.01 \end{bmatrix}, \\
 V_{12} &= \begin{bmatrix} 0.01 & 0.02 \\ 0.02 & -0.03 \end{bmatrix}, & V_{21} &= \begin{bmatrix} 0.02 & 0.01 \\ 0.02 & 0.01 \end{bmatrix}, & V_{22} &= \begin{bmatrix} 0.02 & 0.02 \\ 0.03 & 0.02 \end{bmatrix}, \\
 M &= \begin{bmatrix} 0.5 & 0.6 \\ 0.2 & 0.5 \end{bmatrix}, & N_{C_1} &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, & N_{C_2} &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \\
 N_{D_1} &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, & N_{D_2} &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, & N_{W_{01}} &= \begin{bmatrix} 0.05 & 0.06 \\ 0.02 & 0.02 \end{bmatrix}, \\
 N_{W_{02}} &= \begin{bmatrix} 0.02 & 0.06 \\ 0.02 & 0.02 \end{bmatrix}, & N_{W_{11}} &= \begin{bmatrix} 0.03 & 0.04 \\ 0.02 & 0.01 \end{bmatrix}, & N_{W_{12}} &= \begin{bmatrix} 0.01 & 0.03 \\ 0.03 & 0.01 \end{bmatrix}, \\
 N_{W_{21}} &= \begin{bmatrix} 0.04 & 0.03 \\ 0.03 & 0.02 \end{bmatrix}, & N_{W_{22}} &= \begin{bmatrix} 0.02 & 0.03 \\ 0.02 & 0.01 \end{bmatrix}, & N_{V_{01}} &= \begin{bmatrix} 0.03 & 0.06 \\ 0.02 & 0.02 \end{bmatrix}, \\
 N_{V_{02}} &= \begin{bmatrix} 0.02 & 0.04 \\ 0.01 & 0.03 \end{bmatrix}, & N_{V_{11}} &= \begin{bmatrix} 0.02 & 0.04 \\ 0.02 & 0.03 \end{bmatrix}, & N_{V_{12}} &= \begin{bmatrix} 0.06 & 0.03 \\ 0.01 & 0.04 \end{bmatrix}, \\
 N_{V_{21}} &= \begin{bmatrix} 0.05 & 0.05 \\ 0.03 & 0.01 \end{bmatrix}, & N_{V_{22}} &= \begin{bmatrix} 0.03 & 0.03 \\ 0.02 & 0.03 \end{bmatrix}.
 \end{aligned}$$

Taking

$$\begin{aligned}
 &\bar{\rho}_1(u(t-v_1), v(t), v(t-\tau_1(t)), t, 1) \\
 &= \begin{bmatrix} 0.3 * u_1(t-v_1) + 0.4 * v_1(t) + 0.4 * v_1(t-\bar{\tau}_1) & 0 \\ 0 & 0.3 * u_2(t-v_1) + 0.2 * v_2(t) + 0.3 * v_2(t-\bar{\tau}_1) \end{bmatrix}, \\
 &\bar{\rho}_1(u(t-v_1), v(t), v(t-\tau_1(t)), t, 2) \\
 &= \begin{bmatrix} 0.4 * u_1(t-v_1) + 0.3 * v_1(t) + 0.4 * v_1(t-\bar{\tau}_1) & 0 \\ 0 & 0.2 * u_2(t-v_1) + 0.5 * v_2(t) + 0.2 * v_2(t-\bar{\tau}_1) \end{bmatrix}, \\
 &\bar{\rho}_2(v(t-v_2), u(t), u(t-\tau_2(t)), t, 1) \\
 &= \begin{bmatrix} 0.3 * v_1(t-v_2) + 0.2 * u_1(t) + 0.2 * u_1(t-\bar{\tau}_2) & 0 \\ 0 & 0.4 * v_2(t-v_2) + 0.5 * u_2(t) + 0.3 * u_2(t-\bar{\tau}_2) \end{bmatrix}, \\
 &\bar{\rho}_2(v(t-v_2), u(t), u(t-\tau_2(t)), t, 2) \\
 &= \begin{bmatrix} 0.2 * v_1(t-v_2) + 0.3 * u_1(t) + 0.4 * u_1(t-\bar{\tau}_2) & 0 \\ 0 & 0.2 * v_2(t-v_2) + 0.3 * u_2(t) + 0.2 * u_2(t-\bar{\tau}_2) \end{bmatrix},
 \end{aligned}$$

$v_1 = v_2 = 1$, $\bar{\tau}_1 = \bar{\tau}_2 = 0.4$, $\sigma_1 = \sigma_2 = 0.3$. The following activation functions play in neural network system (28):

$$\begin{aligned}
 \bar{f}(v) &= \sinh(v), & \bar{f}(u) &= \sinh(u), & \bar{g}(v) &= v, \\
 \bar{g}(u) &= u, & \bar{h}(v) &= \sin(v), & \bar{h}(u) &= \sin(u).
 \end{aligned}$$

Therefore, by Theorem 3.2 in this paper, the uncertain delayed stochastic impulsive BAM neural networks (28) under consideration are global robust exponentially stable in the mean square.

5 Conclusions

In this paper, we have treated the problem of global exponential stability analysis for the leakage delay terms. By employing the Lyapunov stability theory and the LMI framework, we have attained a new sufficient condition to justify the global exponential stability of stochastic impulsive uncertain BAMNNs with two kinds of time-varying delays and leakage delays. The advantage of this paper is that different types of uncertain parameters were introduced into the Lyapunov–Krasovskii functionals, and the exponential stability behavior was studied. Additionally, two numerical examples have been provided to reveal the usefulness of our obtained deterministic and uncertain results. To the best of our knowledge, there are no results on the exponential stability analysis of inertial-type BAM neural networks with both time-varying delays by using Wirtinger based inequality, which might be our future research work.

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Notations

$\mathbb{R}$ is the set of real numbers; $\mathbb{R}^n$ is the n -dimensional Euclidean space; $\mathbb{R}^{n \times n}$ denotes the set of all $n \times n$ real matrices; $\mathbb{Z}_+$ is the set of all positive integers. For any matrix A , A^T is the transpose of A , A^{-1} is the inverse of A ; $*$ means the symmetric terms in a symmetric matrix. Positive definite matrix A is denoted by $A > 0$, negative definite A is denoted by $A < 0$, $\lambda_{\min}(\cdot)$ denotes minimum eigenvalues of a real symmetric matrix; $\lambda_{\max}(\cdot)$ is maximum eigenvalues of a real symmetric matrix; I_n denotes the $n \times n$ identity matrix; $x = (x_1, x_2, \dots, x_n)^T$, $y = (y_1, y_2, \dots, y_n)^T$ are column vectors; $x^T y = \sum_{i=1}^n x_i y_i$, $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$; $\dot{x}(t)$, $\dot{y}(t)$ denote the derivatives of $x(t)$ and $y(t)$, respectively; $*$ is the symmetric form of a matrix; $C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^n \times \mathfrak{M}; \mathbb{R}^+)$ is the family of all nonnegative functions $V(t, u(t), i)$ on $\mathbb{R}^+ \times \mathbb{R}^n \times \mathfrak{M}$ which are continuously twice differentiable in u and once differentiable in t ; $(\mathbb{A}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is a complete probability space, where $\mathbb{A}$ is the sample space, $\mathcal{F}$ is the σ -algebra of subsets of the sample space, $\mathbb{P}$ is the probability measure on $\mathcal{F}$ and $\{\mathcal{F}_t\}_{t \geq 0}$ denotes the filtration; $L^2_{\mathcal{F}_0}([-\bar{\omega}, 0]; \mathbb{R}^n)$ denotes the family of all $\mathcal{F}_0$ -measurable $C([-\bar{\omega}, 0]; \mathbb{R}^n)$ -valued random variables $\tilde{\xi} = \{\tilde{\xi}(\theta) : -\bar{\omega} \leq \theta \leq 0\}$ such that $\sup_{-\bar{\omega} \leq \theta \leq 0} \mathbb{E}|\tilde{\xi}(\theta)| < \infty$, where $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathbb{P}$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Robust extended dissipativity criteria for discrete-time uncertain neural networks with time-varying delays

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Abstract In this draft, we consider the problem of robust extended dissipativity for uncertain discrete-time neural networks (DNNs) with time-varying delays. By constructing appropriate Lyapunov–Krasovskii functional (LKF), sufficient conditions are established to ensure that the considered time-delayed uncertain DNN is extended dissipative. The derived conditions are presented in terms of linear matrix inequalities (LMIs). Numerical examples are provided to illustrate the superiority of this result.

Keywords Extended dissipativity analysis · Uncertain discrete-time neural networks · Lyapunov method · Linear matrix inequality

1 Introduction

Studies on neural networks (NNs) have phased great impact on research in the past two decades, which are generally impulsive that their application may happen in different fields, for example synthesis, automatic control, image reparation, and fault ascertainment [1]. Various results have been considered in the dynamic deportation of continuous-time NNs [2–6]. However, compared with continuous-time NNs, discrete-time neural networks (DNNs) equally have a robust engineering capitalization background for the benefit of computer-foundation simulation, and the dynamics of continuous-time NNs cannot be preserved by discretization (see [7]). Thus, it is essential to study the dynamic behavior of DNNs. The stability analysis of DNNs has been an significant and captivating research subject [8–11].

In additional field, dissipative theory has been greatly established in recent years, which was first introduced in [12]. The dissipativity theory is an essential subject for system synthesis and analysis. Dissipativity theorem extends many basic system theorems, such as passivity theorem, sector-bounded nonlinearity, and bounded real theorem. Dissipativity theory has presented a serious part in the control design and analysis of linear and nonlinear systems, specifically for high-order systems [13], and because of the applied capitalization point of view, many systems need to be dissipative for reach proficient noise extenuation [14, 15]. In addition, passivity is section of a general and a broader theory of dissipativeness. The principal concept of passivity theory is that the passive characteristic of a system can keep the system interior stable. In current years, passivity and dissipativity analysis for NNs was proposed in [16–23]. In particular, passivity and dissipativity criteria for complex-valued DNNs has been considered in [16]. In [18], dissipativity analysis of stochastic DNNs was presented.

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In [19], stochastic dissipativity analysis has been established for DNNs. In [21], Zhang et al. analyzed the passivity problem for DNNs with mixed time delays and randomly occurring quantization effects. Dissipativity problem for discrete-time fuzzy NNs with leakage delays was studied in [22].

Zhang et al. were the first to introduce the concept of extended dissipativity in [24], which is very useful approach for delayed NNs to analyze its dynamic behaviors. The stability, passivity, dissipativity, H_∞ performance, and L_2 - L_∞ performance are included in the extended dissipativity by modifying some weighting matrices via new performance index. Currently, extended dissipative approach has got great deal of research attention [25–28]. Nevertheless, robust extended dissipativity of uncertain DNNs with interval delays has not been completely studied, which is the inspiration and motivation of this work.

Remembering the past several years, although, there are few works that have been achieved on passivity and dissipativity analysis of DNNs with time-varying delays. Activated by these previous attempts, in this paper, we are connected with the issue of robust extended dissipativity analysis of DNNs with time-varying delays. A suitable Lyapunov–Krasovskii functional (LKF) is constructed, and then using new summation inequality, less conservative robust extended dissipativity criteria are derived in terms of linear matrix inequalities (LMIs). Finally, several numerical examples are provided to illustrate the superiority of the main results.

Notations $\mathbb{R}^n$ and $\mathbb{R}^{n \times n}$ represent the n -dimensional Euclidean space and the set of all $n \times n$ real matrices, respectively. The notation $*$ denotes the entries induced by symmetry. $\mathbb{A}^T$ denotes the matrix transpose of $\mathbb{A}$. $\mathcal{X} > 0$ ($\mathcal{X} \geq 0$) denotes that $\mathcal{X}$ is a real symmetric positive definite (positive semi-definite) matrix. $\text{diag}\{a, b, \dots, z\}$ denotes the block-diagonal matrix with elements $a, b, \dots, z$ in the diagonal entries. I is the identity matrix with appropriate dimensions.

2 Preliminaries

$$\begin{aligned} e(\varphi+1) &= (C + \nabla C)e(\varphi) + (D_0 + \nabla D_0)f(e(\varphi)) \\ &\quad + (D_1 + \nabla D_1)g(e(\varphi - \vartheta(\varphi))) + u(\varphi), \quad \varphi \geq 0, \quad (1) \\ z(\varphi) &= We(\varphi), \\ e(\varphi) &= \varphi(\varphi), \quad \varphi = -\vartheta_2, -\vartheta_2 + 1, \dots, 0, \end{aligned}$$

where $e(\varphi) = [e_1(\varphi), e_2(\varphi), \dots, e_n(\varphi)]^T \in \mathbb{R}^n$ is the state of the neural, $u(\varphi) \in \mathbb{R}^n$ is the input belonging to $L_2[0, \infty)$, $z(\varphi)$ is the output of the NN. φ is the initial value, n is the number of neurals, $f(e(\varphi)) = [f_1(e_1(\varphi)), f_2(e_2(\varphi)), \dots, f_n(e_n(\varphi))]^T$, $g(e(\varphi - \vartheta(\varphi))) = [g_1(e_1(\varphi - \vartheta(\varphi))), g_2(e_2(\varphi - \vartheta(\varphi))), \dots, g_n(e_n(\varphi - \vartheta(\varphi)))]^T$, is the neuron activation function; $C = \text{diag}\{c_1, c_2, \dots, c_n\}$, $c_i > 0$ represents the self-feedback term; D_0 and D_1 denotes the

connection weights and the discretely delayed connection weights, respectively. For every $\varphi \geq 0$, the variable delay $\vartheta(\varphi)$ is defined to be a positive integer and $\vartheta(\varphi) \in [\vartheta_1, \vartheta_2]$, $\forall \varphi \geq 0$ for some integers $\vartheta_2 \geq \vartheta_1 > 1$. ∇C , ∇D_0 , and ∇D_1 are real-valued unknown matrices, which represent the time-varying parameter uncertainties. These matrices are assumed to satisfy

$$[\nabla C \nabla D_0 \nabla D_1] = \mathcal{W}F(\varphi) [E_1 E_2 E_3], \quad (2)$$

where $\mathcal{W}$ and E_1, E_2, E_3 are known real constant matrices and $F(\varphi)$ is an unknown time-varying matrix functions satisfying

$$F^T(\varphi)F(\varphi) \leq I, \quad \forall \varphi \geq 0. \quad (3)$$

Assumption 2.1 ([29]) For any $s_1, s_2 \in \mathbb{R}$, $s_1 \neq s_2$, the continuous and bounded activation functions $f_i(\cdot)$ and $g_i(\cdot)$ satisfy

$$\begin{aligned} \mathcal{M}_i^- &\leq \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leq \mathcal{M}_i^+, \\ \mathcal{N}_i^- &\leq \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \leq \mathcal{N}_i^+, \quad i = 1, 2, \dots, n, \end{aligned}$$

where $\mathcal{M}_i^-, \mathcal{M}_i^+, \mathcal{N}_i^-,$ and $\mathcal{N}_i^+$ are known constants.

Now, the DNN (1) can be rewritten as:

$$\begin{aligned} e(\varphi+1) &= Ce(\varphi) + D_0f(e(\varphi)) + D_1g(e(\varphi - \vartheta(\varphi))) \\ &\quad + \mathcal{W}p(\varphi) + u(\varphi), \quad \varphi \geq 0, \quad (4) \\ p(\varphi) &= F(\varphi)q(\varphi), \\ q(\varphi) &= E_1e(\varphi) + E_2f(e(\varphi)) + E_3g(e(\varphi - \vartheta(\varphi))), \\ z(\varphi) &= We(\varphi), \\ e(\varphi) &= \varphi(\varphi), \quad \varphi = -\vartheta_2, -\vartheta_2 + 1, \dots, 0. \end{aligned}$$

Lemma 2.2 ([30]) Let $\mathcal{Y} \in \mathbb{R}^{n \times n}$ be a given positive definite matrix. Then for all $\omega_0, \omega_1, \omega_2, \dots, \omega_n \in \mathbb{R}^n$, the inequality

$$\begin{aligned} \sum_{\varphi=0}^n \omega_\varphi^T \mathcal{Y} \omega_\varphi &\geq \frac{1}{n+1} \left(\sum_{\varphi=0}^n \omega_\varphi^T \right) \mathcal{Y} \left(\sum_{\varphi=0}^n \omega_\varphi \right) \\ &\quad + \frac{3}{n(n+1)(n+2)} \widehat{\Omega}_0^T \mathcal{Y} \widehat{\Omega}_0, \end{aligned}$$

holds, where $\widehat{\Omega}_0 = \sum_{\varphi=0}^n (n-2\varphi) \omega_\varphi$.

Lemma 2.3 ([30]) Let $\mathcal{X} \in \mathbb{R}^{n \times n}$ be a given positive definite matrix. Then for all $\omega_0, \omega_1, \omega_2, \dots, \omega_n \in \mathbb{R}^n$, the inequality

$$\begin{aligned} \sum_{\varphi=0}^n \Delta \omega_\varphi^T \mathcal{X} \Delta \omega_\varphi &\geq \frac{1}{n+1} (\omega_{n+1} - \omega_0)^T \mathcal{X} (\omega_{n+1} - \omega_0) \\ &\quad + \frac{3}{n+1} \widehat{\Omega}_2^T \left(\frac{n+2}{n} \mathcal{X} \right) \widehat{\Omega}_2, \end{aligned}$$

holds, where $\Delta\omega_{\wp} = \omega_{\wp+1} - \omega_{\wp}$ and $\widehat{\Omega}_2 = \omega_{n+1} + \omega_0 - \frac{2}{n+2} \sum_{\wp=0}^{n+1} \omega_{\wp}$.

Lemma 2.4 ([31]) For $0 < M = M^T \in \mathbb{R}^{n \times n}$, two integers ϑ_1 and ϑ_2 satisfying $\vartheta_1 < \vartheta_2$, and a vector function $e(s) : [\wp - \vartheta_2, \wp - \vartheta_1 - 1] \rightarrow \mathbb{R}^n$, the following inequality holds

$$- \frac{\vartheta_{12}(\vartheta_{12} + 1)}{2} \sum_{s=-\vartheta_2}^{-\vartheta_1-1} \sum_{u=\wp+s}^{\wp-\vartheta_1-1} e^T(u) M e(u) \\ \leq - \left(\sum_{s=-\vartheta_2}^{-\vartheta_1-1} \sum_{u=\wp+s}^{\wp-\vartheta_1-1} e(u) \right)^T M \left(\sum_{s=-\vartheta_2}^{-\vartheta_1-1} \sum_{u=\wp+s}^{\wp-\vartheta_1-1} e(u) \right),$$

where $\vartheta_{12} = \vartheta_2 - \vartheta_1$.

Lemma 2.5 ([31]) Let $\mathbb{D}$ be an open subset of $\mathbb{R}^m$, $f_1, f_2, \dots, f_n : D \mapsto \mathbb{R}$ have positive values. Then, the reciprocally convex combination of f_i over $\mathbb{D}$ satisfies

$$\min_{\{\alpha_i | \alpha_i > 0, \sum_i \alpha_i = 1\}} \sum_i \frac{1}{\alpha_i} f_i(\wp) = \sum_i f_i(\wp) + \max_{g_{ij}(\wp)} \sum_{i \neq j} g_{ij}(\wp).$$

subject to

$$\left\{ g_{ij} : \mathbb{R}^m \mapsto \mathbb{R}, g_{ji}(\wp) = g_{ij}(\wp), \begin{bmatrix} f_i(\wp) & g_{ij}(\wp) \\ g_{ji}(\wp) & f_j(\wp) \end{bmatrix} \geq 0 \right\}.$$

$$\vartheta_{12} = \vartheta_2 - \vartheta_1, w_i = \begin{bmatrix} 0, 0, \dots, \overbrace{I}^i, \dots, 0 \end{bmatrix}_{14 \times n}^T, i = 1, 2, \dots, 14, \Delta e(\wp) = e(\wp + 1) - e(\wp),$$

$$\xi(\wp) = \left[e^T(\wp), e^T(\wp - \vartheta_1), e^T(\wp - \vartheta(\wp)), e^T(\wp - \vartheta_2), \Delta e^T(\wp), \Delta e^T(\wp - \vartheta_1), \Delta e^T(\wp - \vartheta_2), f^T(e(\wp)), \right.$$

$$\left. g^T(e(\wp - \vartheta(\wp))), \sum_{s=\wp-\vartheta_1}^{\wp-1} e^T(s), \sum_{s=\wp-\vartheta(\wp)}^{\wp-\vartheta_1-1} e^T(s), \sum_{s=\wp-\vartheta_2}^{\wp-\vartheta(\wp)-1} e^T(s), \sum_{i=\wp-\vartheta_1}^{\wp-1} \sum_{s=\wp-\vartheta_1}^{i-1} e^T(s), p^T(\wp) \right]^T,$$

$$\alpha(\wp) = \left[e^T(\wp), e^T(\wp - \vartheta_1), e^T(\wp - \vartheta_2), \sum_{s=\wp-\vartheta_1}^{\wp-1} e^T(s), \sum_{s=\wp-\vartheta_2}^{\wp-\vartheta_1-1} e^T(s) \right]^T,$$

$$\beta(\wp) = \left[e^T(\wp), \Delta e^T(\wp) \right]^T, \Omega_1 = [w_1 + w_5, w_2 + w_6, w_4 + w_7, w_1 - w_2 + w_{10}, w_2 - w_4 + w_{11} + w_{12}],$$

$$\Omega_2 = [w_1, w_2, w_4, w_{10}, w_{11} + w_{12}], \Omega_3 = [w_1, w_5], \Omega_4 = [w_2, w_6], \Omega_5 = [w_4, w_7], \Omega_6 = [w_{10}, w_1 - w_2],$$

$$\Omega_7 = [w_{11}, w_2 - w_3, w_{12}, w_3 - w_4], \Omega_8 = [w_{13}, w_{10} - \vartheta_1 w_2], \Theta_1 = \Omega_1 R \Omega_1^T - \Omega_2 R \Omega_2^T,$$

$$\Theta_2 = \Omega_3 U \Omega_3^T - \Omega_4 U \Omega_4^T + \Omega_4 V \Omega_4^T - \Omega_5 V \Omega_5^T, \Theta_3 = h_1^2 \Omega_3 T_1 \Omega_3^T + h_2^2 \Omega_4 T_2 \Omega_4^T,$$

$$\Theta_4 = \left(\frac{\vartheta_1(\vartheta_1 + 1)}{2} \right)^2 w_5 T_3 w_5^T + \left(\frac{\vartheta_{12}(\vartheta_{12} + 1)}{2} \right)^2 w_6 T_4 w_6^T - (\vartheta_1 w_1 - w_{10}) T_3 (\vartheta_1 w_1 - w_{10})^T$$

$$- (\vartheta_{12} w_2 - w_{11} - w_{12}) T_4 (\vartheta_{12} w_2 - w_{11} - w_{12})^T, \Theta_5 = -w_1 M_1 \Lambda_1 w_1^T + 2w_1 M_2 \Lambda_1 w_8^T - w_8 \Lambda_1 w_8^T,$$

$$\Theta_6 = -w_3 N_1 \Lambda_2 w_3^T + 2w_3 N_2 \Lambda_2 w_9^T - w_9 \Lambda_2 w_9^T, \Sigma = 2(w_5 Z_1 + w_1 Z_2) (C w_1^T + D_0 w_8^T + D_1 w_9^T + W w_{14}^T - w_5^T - w_1^T),$$

$$\Theta_7 = \epsilon \left[w_1 E_1^T E_1 w_1^T + w_1 E_1^T E_2 w_8^T + w_8 E_2^T E_1 w_1^T + w_1 E_1^T E_3 w_9^T + w_9 E_3^T E_1 w_1^T + w_8 E_2^T E_2 w_8^T \right. \\ \left. + w_8 E_2^T E_3 w_9^T + w_9 E_3^T E_2 w_8^T + w_9 E_3^T E_3 w_9^T - w_{14} w_{14}^T \right],$$

Assumption 2.6 For given symmetric matrices $\Upsilon_1, \Upsilon_2, \Upsilon_3$, and Υ_4 , such that

- (a) $\Upsilon_1 = \Upsilon_1^T, \Upsilon_3 = \Upsilon_3^T$, and $\Upsilon_4 = \Upsilon_4^T$
- (b) $\|W\| \cdot \|\Upsilon_4\| = 0$,
- (c) $(\|\Upsilon_1\| + \|\Upsilon_2\|) \cdot \|\Upsilon_4\| = 0$,
- (d) $W^T \Upsilon_1 W + W^T \Upsilon_2 + \Upsilon_2^T W + \Upsilon_3 > 0$ are satisfied.

Definition 2.7 ([24]) For given matrices $\Upsilon_1, \Upsilon_2, \Upsilon_3$, and Υ_4 , satisfying Assumption (2.6), DNN (1) attains the extended dissipativity if for any $\rho \geq 0$ and all $u(\wp) \in L_2[0, \infty)$, under the zero initial state, the inequality:

$$\sum_{i=0}^{\tau} J(i) \geq \sup_{0 \leq \wp \leq \rho} \mathfrak{z}^T(\wp) \Upsilon_4 \mathfrak{z}(\wp), \quad (5)$$

holds, where $J(i) = \mathfrak{z}^T(\wp) \Upsilon_1 \mathfrak{z}(\wp) + 2\mathfrak{z}^T(\wp) \Upsilon_2 u(\wp) + u^T(\wp) \Upsilon_3 u(\wp)$.

3 Main results

In this section, a new criterion on extended dissipativity analysis for DNNs is to be established.

Define the following notations:

$$T_{22} = T_2 + \begin{bmatrix} 0 & Q_2 \\ * & Q_2 \end{bmatrix}, \quad T_{23} = T_2 + \begin{bmatrix} 0 & Q_3 \\ * & Q_3 \end{bmatrix}, \quad \Psi_1 = -\Omega_6 \left(T_1 + \begin{bmatrix} 0 & Q_1 \\ * & Q_1 \end{bmatrix} \right) \Omega_6^T$$

$$-\Omega_7 \begin{bmatrix} T_{22} & S \\ * & T_{23} \end{bmatrix} \Omega_7^T - \frac{3}{h_1^2 - 1} (2\Omega_8 - (\vartheta_1 - 1)\Omega_6) \left(T_1 + \begin{bmatrix} 0 & Q_1 \\ * & Q_1 \end{bmatrix} \right) (2\Omega_8 - (\vartheta_1 - 1)\Omega_6)^T,$$

$$\Psi_2 = \vartheta_1 w_1 Q_1 w_1^T - \vartheta_1 w_2 Q_1 w_2^T + \vartheta_1 w_2 Q_2 w_2^T - \vartheta_1 w_3 Q_2 w_3^T + \vartheta_1 w_3 Q_3 w_3^T - \vartheta_1 w_4 Q_3 w_4^T, \quad \Phi = \sum_{i=1}^7 \Theta_i.$$

Theorem 3.1 For given integer ϑ_1, ϑ_2 satisfying $1 < \vartheta_1 \leq \vartheta_2$, matrices $\Upsilon_1, \Upsilon_2, \Upsilon_3$, and Υ_4 with Υ_1, Υ_3 , and Υ_4 being real symmetric, DNN (1) (with $\mathfrak{z}(\wp) = u(\wp) = 0$) is asymptotically stable for $\vartheta_1 < \vartheta(\wp) \leq \vartheta_2$, when there exists positive definite matrices $R \in \mathbb{R}^{5n \times 5n}, U \in \mathbb{R}^{2n \times 2n}, V \in \mathbb{R}^{2n \times 2n}, T_1 \in \mathbb{R}^{2n \times 2n}, T_2 \in \mathbb{R}^{2n \times 2n}, T_3 \in \mathbb{R}^{n \times n}, T_4 \in \mathbb{R}^{n \times n}$, any symmetric matrices $Q_i \in \mathbb{R}^{n \times n}$ ($i = 1, 2, 3$), diagonal matrices $\Lambda_1 > 0, \Lambda_2 > 0, Z_1, Z_2$ of appropriate dimensions, and any matrix $S \in \mathbb{R}^{2n \times 2n}$, such that

$$(\Phi + \Psi_j + \Sigma) < 0, \quad (j = 1, 2), \quad (6)$$

$$T_1 + \begin{bmatrix} 0 & Q_1 \\ * & Q_1 \end{bmatrix} > 0,$$

$$T = \begin{bmatrix} T_{22} & S \\ * & T_{23} \end{bmatrix} > 0.$$

Proof Define the following LKF candidate:

$$V(\wp) = \sum_{j=1}^4 V_j(\wp),$$

where

$$V_1(\wp) = \alpha^T(\wp) R \alpha(\wp),$$

$$V_2(\wp) = \sum_{s=\wp-\vartheta_1}^{\wp-1} \beta^T(s) U \beta(s) + \sum_{s=\wp-\vartheta_2}^{\wp-\vartheta_1-1} \beta^T(s) V \beta(s),$$

$$V_3(\wp) = \vartheta_1 \sum_{s=-\vartheta_1}^{-1} \sum_{u=\wp+s}^{\wp-1} \beta^T(s) T_1 \beta(s) \\ + \vartheta_{12} \sum_{s=-\vartheta_2}^{-\vartheta_1-1} \sum_{u=\wp+s}^{\wp-\vartheta_1-1} \beta^T(s) T_2 \beta(s),$$

$$V_4(\wp) = \frac{\vartheta_1(\vartheta_1 + 1)}{2} \sum_{s=-\vartheta_1}^{-1} \sum_{u=s}^{-1} \sum_{v=\wp+u}^{\wp-1} \Delta e^T(v) T_3 \Delta e(v) \\ + \frac{\vartheta_{12}(\vartheta_{12} + 1)}{2} \sum_{s=-\vartheta_2}^{-\vartheta_1-1} \sum_{u=s}^{-\vartheta_1-1} \sum_{v=\wp+u}^{\wp-\vartheta_1-1} \Delta e^T(v) T_4 \Delta e(v).$$

Calculating the derivative of $V(\wp)$.

$$\Delta V_1(\wp) = \xi^T(\wp) \Theta_1 \xi(\wp), \quad (7)$$

$$\Delta V_2(\wp) = \xi^T(\wp) \Theta_2 \xi(\wp). \quad (8)$$

And $\Delta V_3(\wp)$ calculated as

$$\Delta V_3(\wp) = (\vartheta_1)^2 \beta^T(\wp) T_1 \beta(\wp) - \vartheta_1 \sum_{s=\wp-\vartheta_1}^{\wp-1} \beta^T(s) T_1 \beta(s) \\ + (\vartheta_{12})^2 \beta^T(\wp - \vartheta_1) T_2 \beta(\wp - \vartheta_1) \\ - \vartheta_{12} \sum_{s=\wp-\vartheta_2}^{\wp-\vartheta_1-1} \beta^T(s) T_2 \beta(s). \quad (9)$$

For any symmetric matrices Q_1, Q_2 , and Q_3 , the following equations are satisfied:

$$0 = e^T(\wp) (\vartheta_1 p_1) e(\wp) - e^T(\wp - \vartheta_1) (\vartheta_1 Q_1) e(\wp - \vartheta_1) - \vartheta_1 \\ \sum_{s=\wp-\vartheta_1}^{\wp-1} \left(\Delta e^T(s) Q_1 \Delta e(s) + 2e^T(s) Q_1 \Delta e(s) \right), \quad (10)$$

$$0 = e^T(\wp - \vartheta_1) (\vartheta_{12} Q_2) e(\wp - \vartheta_1) \\ - e^T(\wp - \vartheta(\wp)) (\vartheta_{12} Q_2) e(\wp - \vartheta(\wp)) \\ - \vartheta_{12} \sum_{s=\wp-\vartheta(\wp)}^{\wp-\vartheta_1-1} \left(\Delta e^T(s) Q_2 \Delta e(s) + 2e^T(s) Q_2 \Delta e(s) \right), \quad (11)$$

$$0 = e^T(\wp - \vartheta(\wp)) (\vartheta_{12} Q_3) e(\wp - \vartheta(\wp)) \\ - e^T(\wp - \vartheta_2) (\vartheta_{12} Q_3) e(\wp - \vartheta_2) \\ - \vartheta_{12} \sum_{s=\wp-\vartheta_2}^{\wp-\vartheta(\wp)-1} \left(\Delta e^T(s) Q_3 \Delta e(s) + 2e^T(s) Q_3 \Delta e(s) \right). \quad (12)$$

Adding the equalities from (10) to (12) implies

$$\begin{aligned}
 \Delta V_3(\wp) = & e^T(\wp)(\vartheta_1 Q_1)e(\wp) - e^T(\wp - \vartheta_1)(\vartheta_1 Q_1)e(\wp - \vartheta_1) + e^T(\wp - \vartheta_1)(\vartheta_{12} Q_2)e(\wp - \vartheta_1) \\
 & - e^T(\wp - \vartheta(\wp))(\vartheta_{12} Q_2)e(\wp - \vartheta(\wp)) + e^T(\wp - \vartheta(\wp))(\vartheta_{12} Q_3)e(\wp - \vartheta(\wp)) \\
 & - e^T(\wp - \vartheta_2)(\vartheta_{12} Q_3)e(\wp - \vartheta_2) + (\vartheta_1)^2 \beta^T(\wp) T_1 \beta(\wp) \\
 & - (\vartheta_{12})^2 \beta^T(\wp - \vartheta_1) T_2 \beta(\wp - \vartheta_1) - \vartheta_1 \sum_{s=\wp-\vartheta_1}^{\wp-1} \beta^T(s) \left(T_1 + \begin{bmatrix} 0 & Q_1 \\ * & Q_1 \end{bmatrix} \right) \beta(s) \\
 & - \vartheta_{12} \sum_{s=\wp-\vartheta(\wp)}^{\wp-\vartheta_1-1} \beta^T(s) \left(T_2 + \begin{bmatrix} 0 & Q_2 \\ * & Q_2 \end{bmatrix} \right) \beta(s) \\
 & - \vartheta_{12} \sum_{s=\wp-\vartheta_2}^{\wp-\vartheta(\wp)-1} \beta^T(s) \left(T_2 + \begin{bmatrix} 0 & Q_3 \\ * & Q_3 \end{bmatrix} \right) \beta(s). \quad (13)
 \end{aligned}$$

Through Lemma 2.2, we get

$$\begin{aligned}
 & - \vartheta_1 \sum_{s=\wp-\vartheta_1}^{\wp-1} \beta^T(s) \left(T_1 + \begin{bmatrix} 0 & Q_1 \\ * & Q_1 \end{bmatrix} \right) \beta(s) \\
 & \leq -\xi^T(\wp) \Omega_6 \left(T_1 + \begin{bmatrix} 0 & Q_1 \\ * & Q_1 \end{bmatrix} \right) \Omega_6^T \xi(\wp) \\
 & - \frac{3}{h_1^2 - 1} \xi^T(\wp) (2\Omega_8 - (\vartheta_1 - 1)\Omega_6) \left(T_1 + \begin{bmatrix} 0 & Q_1 \\ * & Q_1 \end{bmatrix} \right) \\
 & \times (2\Omega_8 - (\vartheta_1 - 1)\Omega_6)^T \xi(\wp). \quad (14)
 \end{aligned}$$

Using Lemma 2.4 implies

$$\begin{aligned}
 & - \vartheta_{12} \sum_{s=\wp-\vartheta(\wp)}^{\wp-\vartheta_1-1} \beta^T(s) \left(T_2 + \begin{bmatrix} 0 & Q_2 \\ * & Q_2 \end{bmatrix} \right) \beta(s) \\
 & - \vartheta_{12} \sum_{s=\wp-\vartheta_2}^{\wp-\vartheta(\wp)-1} \beta^T(s) \left(T_2 + \begin{bmatrix} 0 & Q_3 \\ * & Q_3 \end{bmatrix} \right) \beta(s) \\
 & \leq -\xi^T(\wp) \Omega_7 \begin{bmatrix} T_{22} & S \\ * & T_{23} \end{bmatrix} \Omega_7^T \xi(\wp). \quad (15)
 \end{aligned}$$

Then,

$$\Delta V_3(\wp) \leq \xi^T(\wp) (\Theta_3 + \Psi_j) \xi(\wp), \quad (j = 1, 2). \quad (16)$$

Moreover,

$$\begin{aligned}
 \Delta V_4(\wp) = & \left(\frac{\vartheta_1(\vartheta_1 + 1)}{2} \right)^2 \Delta e^T(\wp) T_3 \Delta e(\wp) \\
 & - \frac{\vartheta_1(\vartheta_1 + 1)}{2} \sum_{s=-\vartheta_1}^{-1} \sum_{u=s}^{-1} \Delta e^T(\wp + u) T_3 \Delta e(\wp + u) \\
 & + \left(\frac{\vartheta_{12}(\vartheta_{12} + 1)}{2} \right)^2 \Delta e^T(\wp - \vartheta_1) T_4 \Delta e(\wp - \vartheta_1) \\
 & - \frac{\vartheta_{12}(\vartheta_{12} + 1)}{2} \sum_{s=-\vartheta_2}^{-\vartheta_1-1} \sum_{u=s}^{-\vartheta_1-1} \Delta e^T(\wp + u) T_4 \Delta e(\wp + u). \quad (17)
 \end{aligned}$$

Using Lemma 2.3, we get

$$\begin{aligned}
 & - \frac{\vartheta_1(\vartheta_1 + 1)}{2} \sum_{s=-\vartheta_1}^{-1} \sum_{u=s}^{-1} \Delta e^T(\wp + u) T_3 \Delta e(\wp + u) \\
 & - \xi^T(\wp) (\vartheta_1 w_1 - w_{10}) T_3 \\
 & \times (\vartheta_1 w_1 - w_{10})^T \xi(\wp). \quad (18)
 \end{aligned}$$

and

$$\begin{aligned}
 & - \frac{\vartheta_{12}(\vartheta_{12} + 1)}{2} \sum_{s=-\vartheta_2}^{-\vartheta_1-1} \sum_{u=s}^{-\vartheta_1-1} \Delta e^T(\wp + u) T_4 \Delta e(\wp + u) \\
 & \leq -\xi^T(\wp) (\vartheta_{12} w_2 - w_{11} - w_{12}) T_4 \\
 & \times (\vartheta_{12} w_2 - w_{11} - w_{12})^T \xi(\wp). \quad (19)
 \end{aligned}$$

Then,

$$\Delta V_4(\wp) \leq \xi^T(\wp) \Theta_4 \xi(\wp). \quad (20)$$

From Assumption 2.1, we have

$$\begin{bmatrix} e(\wp) \\ f(e(\wp)) \end{bmatrix}^T \begin{bmatrix} M_1 \Lambda_1 & -M_2 \Lambda_1 \\ -M_2^T \Lambda_1 & \Lambda_1 \end{bmatrix}^T \begin{bmatrix} e(\wp) \\ f(e(\wp)) \end{bmatrix} \leq 0, \quad (21)$$

$$\begin{bmatrix} e(\wp - \vartheta(\wp)) \\ g(e(\wp - \vartheta(\wp))) \end{bmatrix}^T \times \begin{bmatrix} N_1 \Lambda_2 & -N_2 \Lambda_2 \\ -N_2^T \Lambda_2 & \Lambda_2 \end{bmatrix}^T \times \begin{bmatrix} e(\wp - \vartheta(\wp)) \\ g(e(\wp - \vartheta(\wp))) \end{bmatrix} \leq 0, \quad (22)$$

where

$$\begin{aligned}\Lambda_1 &= \text{diag}\{\mu_{11}, \mu_{12}, \dots, \mu_{1n}\}, \Lambda_2 \\ &= \text{diag}\{\mu_{21}, \mu_{22}, \dots, \mu_{2n}\}, \\ M_1 &= \text{diag}\{\mathcal{M}_1^- \mathcal{M}_1^+, \mathcal{M}_2^- \mathcal{M}_2^+, \dots, \mathcal{M}_n^- \mathcal{M}_n^+\}, M_2 \\ &= \text{diag}\left\{\frac{\mathcal{M}_1^- + \mathcal{M}_1^+}{2}, \frac{\mathcal{M}_2^- + \mathcal{M}_2^+}{2}, \dots, \frac{\mathcal{M}_n^- + \mathcal{M}_n^+}{2}\right\}, \\ N_1 &= \text{diag}\{\mathcal{N}_1^- \mathcal{N}_1^+, \mathcal{N}_2^- \mathcal{N}_2^+, \dots, \mathcal{N}_n^- \mathcal{N}_n^+\}, N_2 \\ &= \text{diag}\left\{\frac{\mathcal{N}_1^- + \mathcal{N}_1^+}{2}, \frac{\mathcal{N}_2^- + \mathcal{N}_2^+}{2}, \dots, \frac{\mathcal{N}_n^- + \mathcal{N}_n^+}{2}\right\}.\end{aligned}$$

Since $\Delta e(\varphi) = e(\varphi + 1) - e(\varphi)$, and then, by choosing some appropriate dimensioned relaxation matrices Z_1 and Z_2 , we get

$$\begin{aligned}2(\Delta^T(\varphi)Z_1 + e^T(\varphi)Z_2)[Ce(\varphi) + D_0f(e(\varphi)) \\ + D_1g(e(\varphi - \vartheta(\varphi))) \\ + \mathcal{W}p(\varphi) - e(\varphi) - \Delta^T(\varphi)] = 0.\end{aligned}\quad (23)$$

From (2) and (3), the following inequality holds:

$$p^T(\varphi)p(\varphi) \leq q^T(\varphi)q(\varphi), \quad (24)$$

then, there exists a positive scalar ϵ satisfying the following inequality:

$$0 \leq \epsilon[q^T(\varphi)q(\varphi) - p^T(\varphi)p(\varphi)]. \quad (25)$$

Adding the equations from (7) to (25), then, we can get

$$\begin{aligned}\Delta V(\varphi) &\leq \xi^T(\varphi) \left(\sum_{i=1}^7 \Theta_i + \Psi_j + \Sigma \right) \xi(\varphi) \\ &= \xi^T(\varphi) (\Phi + \Psi_j + \Sigma) \xi(\varphi), \quad (j = 1, 2).\end{aligned}\quad (26)$$

From LMI (6), if $(\Phi + \Psi_j + \Sigma) < 0$, then DNN (1) is said to be asymptotically stable when $\mathfrak{z}(\varphi) = u(\varphi) = 0$. $\square$

4 Extended dissipative criteria

In this section, the extended dissipative criteria will be established for all nonzero $u(\varphi) \in L_2[0, \infty)$ by assuming zero initial conditions.

Theorem 4.1 For given integer ϑ_1, ϑ_2 satisfying $1 < \vartheta_1 \leq \vartheta_2$, matrices $\Upsilon_1, \Upsilon_2, \Upsilon_3$, and Υ_4 with Υ_1, Υ_3 , and Υ_4 being real symmetric, DNN (1) is asymptotically stable and extended dissipative for $\vartheta_1 < \vartheta(\varphi) \leq \vartheta_2$, when there exist positive definite matrices $R \in \mathbb{R}^{5n \times 5n}$, $U \in \mathbb{R}^{2n \times 2n}$, $V \in \mathbb{R}^{2n \times 2n}$, $T_1 \in \mathbb{R}^{2n \times 2n}$, $T_2 \in \mathbb{R}^{2n \times 2n}$, $T_3 \in \mathbb{R}^{n \times n}$, $T_4 \in \mathbb{R}^{n \times n}$, any symmetric matrices $Q_i \in \mathbb{R}^{n \times n}$ ($i = 1, 2, 3$), diagonal matrices $\Lambda_1 > 0$, $\Lambda_2 > 0$, Z_1, Z_2 of appropriate

dimensions, and any matrix $S \in \mathbb{R}^{2n \times 2n}$, such that

$$(\widehat{\Phi} + \widehat{\Psi}_j + \widehat{\Sigma}) < 0, \quad (j = 1, 2), \quad (27)$$

$$\begin{bmatrix} W^T \Upsilon_4 W & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} < R, \quad (28)$$

$$T_1 + \begin{bmatrix} 0 & Q_1 \\ * & Q_1 \end{bmatrix} > 0,$$

$$T = \begin{bmatrix} T_{22} & S \\ * & T_{23} \end{bmatrix} > 0,$$

where

$$\begin{aligned}\widehat{\Phi} &= \begin{bmatrix} \Phi - e_1^T W \Upsilon_1 W e_1^T & -\Phi_2 \\ * & -\Upsilon_3 \end{bmatrix}, \quad \Phi_2 \\ &= [\Upsilon_2^T W \quad 0_{1 \times 13}]^T, \quad \widehat{\Psi}_j = \begin{bmatrix} \Psi_j & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{\Sigma} = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & 0 \end{bmatrix}, \\ \widehat{\Sigma}_{11} &= 2(w_5 Z_1 + w_1 Z_2)(C w_1^T + D_0 w_8^T + D_1 w_9^T \\ &\quad + \mathcal{W} w_{14}^T + w_{15}^T - w_5^T - w_1^T).\end{aligned}$$

Proof From Theorem 3.1, $\Delta V(\varphi) \leq \xi^T(\varphi)(\Phi + \Psi_j + \Sigma)\xi(\varphi)$, then it is easy to see that

$$\begin{aligned}\widehat{\xi}^T(\varphi)[\widehat{\Phi} + \widehat{\Psi}_j + \widehat{\Sigma}]\widehat{\xi}(\varphi) &= \xi^T(\varphi)[\Phi + \Psi_j + \Sigma]\xi(\varphi) \\ &\quad - \sum_{i=0}^{\varphi-1} J(i),\end{aligned}$$

where $\widehat{\xi}^T(\varphi) = [\xi^T(\varphi) \quad u^T(\varphi)]^T$, and $J(i)$ is defined in definition 2.7. From LMI (27), we have

$$\Delta V(\varphi) - \sum_{i=0}^{\varphi-1} J(i) = \sum_{i=0}^{\varphi-1} (\Delta V(i) - J(i)) \leq 0, \quad \text{for any } \varphi > 0. \quad (29)$$

From Definition 2.7, we have

$$\begin{aligned}\mathfrak{z}^T(\varphi) \Upsilon_4 \mathfrak{z}(\varphi) &= e^T(\varphi) W^T \Upsilon_4 W e(\varphi) \\ &\leq \alpha^T(\varphi) \begin{bmatrix} W^T \Upsilon_4 W & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \alpha(\varphi) \\ &\leq \alpha^T(\varphi) R \alpha(\varphi) \leq V(\varphi).\end{aligned}\quad (30)$$

Then,

$$\mathfrak{z}^T(\varphi) \Upsilon_4 \mathfrak{z}(\varphi) \leq \sum_{i=0}^{\varphi-1} J(i). \quad (31)$$

For $\Upsilon_1, \Upsilon_2, \Upsilon_3$, and Υ_4 , we prove that the following inequality satisfies under Assumption 2.6:

$$\sup_{0 \leq \varphi \leq \rho} \mathfrak{z}^T(\varphi) \Upsilon_4 \mathfrak{z}(\varphi) \leq \sum_{i=0}^{\tau} J(i). \quad (32)$$

Consider $\Upsilon_4 = 0$ and $\Upsilon_4 > 0$ as two cases. For $\Upsilon_4 = 0$, $\sum_{i=0}^{\rho} J(i) \geq 0$ holds for all $\rho \geq 0$, then the extended dissipativity approach can be represented as passivity, strictly $(\Upsilon_1, \Upsilon_2, \Upsilon_3) - \gamma$ dissipativity, passivity, H_∞ performance. Thus, (32) holds. When $\Upsilon_4 > 0$ consider the Assumption 2.6, which implies that $\Upsilon_1 = 0$, $\Upsilon_2 = 0$, and $J(i) = u^T(i) \Upsilon_3 u(i) \geq 0$. From (31), we obtain the following inequality $\mathfrak{z}^T(\varphi) \Upsilon_4 \mathfrak{z}(\varphi) \leq \sum_{i=0}^{\varphi-1} J(i) \leq \sum_{i=0}^{\rho} J(i)$ for any $\rho \geq \varphi$. By taking the supremum over $0 \leq \varphi \leq \rho$ satisfies (32). From Definition 2.7, (31), and (32), DNN (1) is extended dissipative. $\square$

Remark 4.2 The extended dissipativity generalizes the stability, passivity, dissipativity, H_∞ performance, and L_2 - L_∞ performance, in a combined structure. In particular, the extended dissipativity reduces to passivity criteria by taking $\Upsilon_1 = 0$, $\Upsilon_2 = I_n$, $\Upsilon_3 = \gamma I_n$, and $\Upsilon_4 = 0$. When $\Upsilon_1 = I_n$, $\Upsilon_2 = I_n$, $\Upsilon_3 = \gamma I_n$, and $\Upsilon_4 = 0$ gives the dissipativity conditions by taking. When $\Upsilon_1 = I_n$, $\Upsilon_2 = I_n$, $\Upsilon_3 = \mathfrak{R} - \gamma I_3$, and $\Upsilon_4 = 0$, the performance reduces to $(\Upsilon_1, \Upsilon_2, \Upsilon_3) - \gamma$ -dissipative. When $\Upsilon_1 = -I_n$, $\Upsilon_2 = 0$, $\Upsilon_3 = \gamma^2 I_n$, and $\Upsilon_4 = 0$ gives the H_∞ performance analysis. When $\Upsilon_1 = 0$, $\Upsilon_2 = 0$, $\Upsilon_3 = \gamma^2 I_n$, and $\Upsilon_4 = I_n$, it gives L_2 - L_∞ performance analysis.

Remark 4.3 In Assumption (2.6), condition (a) ensures that the inequality (5) is well defined. The condition (d) is to investigate the dissipativity performance. On the other side, the output equation of the considered neural network should not include disturbance inputs (see [24]) when we consider the L_2 - L_∞ performance. Hence, it should be assumed that $W = 0$ when $\Upsilon_4 \neq 0$, which gives reasons for the necessity of the condition (b) and the condition (c) is technically required for the progress of extended dissipativity approach.

Remark 4.4 In [32], Karimi et. al investigated the exponential H_∞ synchronization problem for uncertain NNs with mixed time delays, where the mixed delays include different discrete, neutral, and distributed time delays. The polytopic and the norm-bounded uncertainties are separately taken into consideration. In [32], the problem of H_∞ synchronization was only considered. In this paper, we considered the robust extended dissipativity problem, which comprises the stability, passivity, dissipativity, and H_∞ and L_2 - L_∞ performances. Karimi et. al [33] studied the stability analysis and control synthesis problems of delayed neutral systems and nonlinear uncertainties. In [34], adaptive fuzzy tracking

control synthesis problem for nonstrict-feedback stochastic nonlinear time-delay systems was examined based on approximation method. In [33, 34], the time delays are either time-varying delay $\vartheta(t)$ or constant delay $\vartheta_2 > 0$, which belong to $0 < \vartheta(t) \leq \vartheta_2$ (an upper bound interval). The lower bound value of $\vartheta(t)$ is not needed to be zero always. Therefore, $\vartheta(t)$ has been chosen as $0 < \vartheta_1 \leq \vartheta(t) \leq \vartheta_2$ (interval with lower and upper bounds) in this paper. Hence, the proposed method in this paper is more general than the results published in literature, in particular [32–34].

5 Numerical examples

Example 5.1 Consider the time-delayed DNNs (4) with $p(\varphi) = 0$ and

$$C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$D_0 = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.004 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} -0.01 & 0.01 \\ -0.02 & -0.01 \end{bmatrix},$$

and the activation function $f(e) = \begin{bmatrix} \tanh(e_1) \\ \tanh(e_2) \end{bmatrix}$. For various values of ϑ_1 , maximum allowable upper bound (MAUB) of delay ϑ_2 is obtained through Theorem 3.1, which ensures the asymptotic stability of DNNs (4). From Table 1, it is easy to know that the results in this approach is less conservative than [35–37] (Fig. 1).

Example 5.2 Consider the DNNs (4) with $p(\varphi) = 0$ and

$$C = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, D_0 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}, M_1 = N_1 = 0_2, M_2 = N_2 = I_2.$$

For various values of ϑ_1 , MAUB ϑ_2 is given in Table 2. Hence, the results of Theorem 3.1 provide less conservative results than those of [36, 38–40]. Additionally, the allowable lower bounds ϑ_1 for different values of ϑ_2 have been listed in Table 3 (Fig. 2).

Table 1 MAUB ϑ_2 for given ϑ_1

Methods	2	4	6	8	10	20
[35]	13	16	17	19	22	31
[36]	15	17	18	20	23	32
[37]	30	32	34	36	38	48
Theorem 3.1	34	36	38	40	42	54

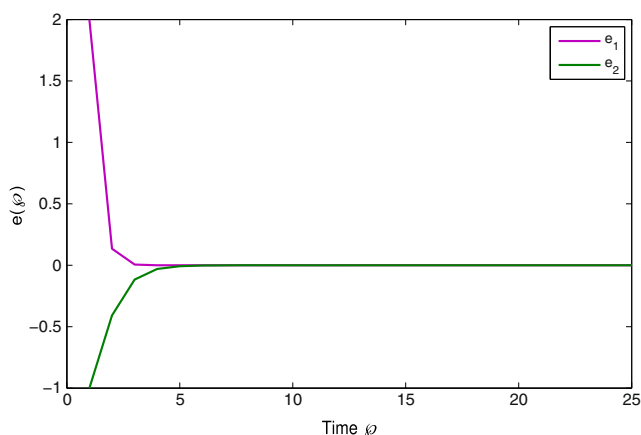


Fig. 1 State response $e(t)$ in Example 5.1

Example 5.3 Consider the DNNs (4) with $p(\varphi) = 0$ and

$$C = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.7 \end{bmatrix}, D_0 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}, M_1 = N_1 = 0_2, M_2 = N_2 = I_2.$$

For various values of ϑ_1 , MAUB ϑ_2 is given in Table 4. Hence, the results of Theorem 3. provide less conservative results than those of [41, 42] (Fig. 3).

Example 5.4 Consider the DNNs (4) with $p(\varphi) = 0$ and

$$C = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$

$$D_0 = \begin{bmatrix} -0.2 & -0.2 & 0.1 \\ 0 & -0.3 & 0.2 \\ -0.2 & -0.1 & -0.2 \end{bmatrix}, D_1 = \begin{bmatrix} -0.2 & 0.1 & 0 \\ -0.2 & 0.3 & 0.1 \\ 0.1 & -0.2 & 0.3 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.4 & 0 \\ 0 & 0 & -0.2 \end{bmatrix}, M_2 = \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Table 2 MAUB ϑ_2 with different ϑ_1 (5.2)

Methods	4	6	8	10	15	20
[38]	17	18	19	—	—	—
[36]	18	18	20	20	23	27
Thm 1 [39]	20	20	21	21	23	26
[40]	19	20	20	21	24	27
Thm 2 [39]	22	22	23	24	26	29
Theorem 3.1	20	25	28	29	32	35

Table 3 MAUB ϑ_1 with different ϑ_2 (Example 5.2)

Methods	16	18	19	20
[38]	3	5	7	9
Theorem 3.1	4	4	6	8

$$N_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.4 & 0 \\ 0 & 0 & -0.2 \end{bmatrix}, N_2 = \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For various values of ϑ_1 , MAUB ϑ_2 is given in Table 5. Hence, the results of Theorem 3.1 provide less conservative results than those of [10, 28, 39].

Example 5.5 Consider the DNNs (4) with $p(\varphi) = 0$ and

$$C = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, D_0 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}, W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$M_1 = N_1 = 0_2, M_2 = N_2 = I_2.$$

Passivity analysis: Choose $\Upsilon_1 = 0_2$, $\Upsilon_2 = I_2$, $\Upsilon_3 = \gamma I_2$, and $\Upsilon_4 = 0$, then through Theorem 4.1, the passivity performance is obtained. When $\vartheta_1 = 5$, minimum passivity performance γ listed in Table 6 for various values of ϑ_2 .

Dissipativity analysis: Choose $\Upsilon_1 = -0.5I_2$, $\Upsilon_2 = I_2$, $\Upsilon_3 = \Re - \gamma I_2$, and $\Upsilon_4 = 0$, then through Theorem 4.1, the $(\Upsilon_1, \Upsilon_2, \Re)$ - γ -dissipativity performance is obtained.

When $\Re = \begin{bmatrix} 4.5 & 0.15 \\ 0.15 & 5 \end{bmatrix}$, $\vartheta_1 = 4$, maximum dissipativity performance γ listed in Table 7 for various values of ϑ_2 .

H_∞ Performance: Choose $\Upsilon_1 = -I_2$, $\Upsilon_2 = 0$, $\Upsilon_3 = \gamma^2 I_2$, and $\Upsilon_4 = 0$, then through Theorem 4.1, the H_∞

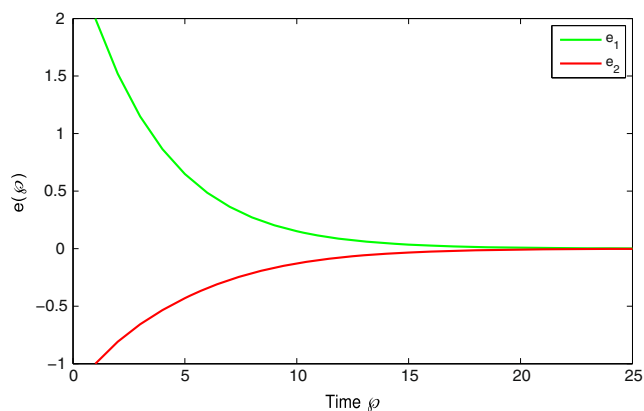


Fig. 2 State response $e(t)$ in Example 5.2

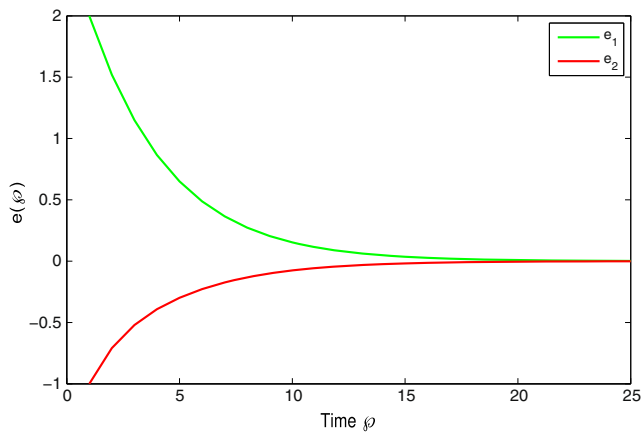


Fig. 3 State response $e(t)$ in Example 5.3

performance is obtained. When $\vartheta_1 = 3$, minimum H_∞ performance γ was listed in Table 8 for various values of ϑ_2 .

$L_2 - L_\infty$ Performance: Choose $\Upsilon_1 = 0_2$, $\Upsilon_2 = 0$, $\Upsilon_3 = \gamma^2 I_2$ and $\Upsilon_4 = I_2$, then through Theorem 4.1, the $L_2 - L_\infty$ performance is obtained. When $\vartheta_1 = 3$, minimum $L_2 - L_\infty$ performance γ listed in Table 9 for various values of ϑ_2 .

Example 5.6 Consider the DNNs (4) with the following matrix parameters:

$$C = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, D_0 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}, W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$M_1 = N_1 = 0_2, M_2 = N_2 = I_2,$$

$$W = E_1 = E_2 = E_3 = 0.1 I_2.$$

(A) Stability analysis: Considering the uncertain DNN (4) with $\vartheta_1 = 2$, and $\vartheta_1 = 10$ and applying Theorem 4.1, we can obtain the following feasible solutions to LMI (6):

Table 4 MAUB ϑ_2 with different ϑ_1

Methods	4	8	15	25
[41]	17	19	26	36
[42]	17	21	28	38
Theorem 3.1	16	22	29	39

Table 5 MAUB ϑ_2 with different ϑ_1

Methods	2	4	6	8	10
[10]	10	12	14	16	18
[39]	18	19	21	23	25
[28]	20	21	23	25	27
Theorem 3.1	21	22	24	26	28

Table 6 Minimum γ with different ϑ_2

Methods	7	9	11	13	15
[28]	1.9497	2.0501	2.1697	2.3031	2.4512
Theorem 4.1	1.4587	1.6384	1.8794	2.1564	2.2847

Table 7 Maximum γ with different ϑ_2

Methods	7	8	9	11	13
[28]	1.1435	1.0809	1.0106	0.8362	0.6032
Theorem 4.1	1.8547	1.6745	1.4932	1.1193	0.9138

Table 8 Minimum γ with different ϑ_2

Methods	9	11	13	15	17
[28]	2.3622	2.6071	2.8654	3.1382	3.4245
Theorem 4.1	1.9864	2.2674	2.5813	2.7916	3.1087

Table 9 Minimum γ with different ϑ_2

Methods	8	12	16	20	24
[28]	1.2672	1.4246	1.6030	1.8057	2.0362
Theorem 4.1	1.1698	1.3675	1.4715	1.6376	1.8653

Table 10 Minimum γ with different ϑ_2

Methods	6	7	8	9
Theorem 4.1	0.6536	0.7563	0.8367	0.9194

Table 11 Maximum γ with different ϑ_2

Methods	6	7	8	9
Theorem 4.1	1.9517	1.8436	1.6624	1.5351

Table 12 Minimum γ with different ϑ_2

Methods	6	7	8	9
Theorem 4.1	0.6125	0.7075	0.7896	0.8647

$$\begin{aligned}
R &= \begin{bmatrix} 550.7529 & -7.5350 & -0.6182 & 0.0930 & 0.7033 & -0.3592 & 5.4868 & 2.9667 & 0.8776 & -0.8059 \\ -7.5350 & 156.6214 & 1.4217 & 2.1460 & -0.1224 & 0.5072 & -0.0200 & 5.9011 & 0.0136 & 1.3198 \\ -0.6182 & 1.4217 & 3.7150 & 0.4247 & -0.0223 & 0.1031 & 1.0827 & 0.7771 & -0.2307 & -0.2632 \\ 0.0930 & 2.1460 & 0.4247 & 39.9100 & 0.1427 & -0.1039 & 1.8559 & 3.1263 & -1.6022 & -1.1467 \\ 0.7033 & -0.1224 & -0.0223 & 0.1427 & 23.3357 & -3.9623 & -0.0711 & -0.0617 & -0.1152 & -0.3877 \\ -0.3592 & 0.5072 & 0.1031 & -0.1039 & -3.9623 & 17.1464 & 0.0332 & -0.0287 & -0.2594 & 0.0704 \\ 5.4868 & -0.0200 & 1.0827 & 1.8559 & -0.0711 & 0.0332 & 12.3648 & 6.0029 & 0.2285 & -1.1127 \\ 2.9667 & 5.9011 & 0.7771 & 3.1263 & -0.0617 & -0.0287 & 6.0029 & 15.8560 & -1.5680 & -0.1941 \\ 0.8776 & 0.0136 & -0.2307 & -1.6022 & -0.1152 & -0.2594 & 0.2285 & -1.5680 & 6.3140 & 1.7217 \\ -0.8059 & 1.3198 & -0.2632 & -1.1467 & -0.3877 & 0.0704 & -1.1127 & -0.1941 & 1.7217 & 3.6234 \end{bmatrix} \\
U &= \begin{bmatrix} 76.8579 & -5.0214 & 44.8097 & -13.1207 \\ -5.0214 & 94.3738 & -9.0465 & 69.0846 \\ 44.8097 & -9.0465 & 177.0811 & -29.7092 \\ -13.1207 & 69.0846 & -29.7092 & 162.5331 \end{bmatrix}, \quad V = \begin{bmatrix} 26.6393 & -11.0598 & 16.4909 & -5.2473 \\ -11.0598 & 21.5599 & -4.2511 & 14.2829 \\ 16.4909 & -4.2511 & 49.0182 & -8.7099 \\ -5.2473 & 14.2829 & -8.7099 & 35.0100 \end{bmatrix}, \\
T_1 &= \begin{bmatrix} 11.8376 & -4.5358 & 8.0232 & -3.2886 \\ -4.5358 & 6.3784 & -3.0092 & 3.7276 \\ 8.0232 & -3.0092 & 14.4167 & -4.7619 \\ -3.2886 & 3.7276 & -4.7619 & 6.7413 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0.6719 & -0.2320 & 0.2781 & -0.0900 \\ -0.2320 & 0.3809 & -0.0771 & 0.1419 \\ 0.2781 & -0.0771 & 0.6069 & -0.1546 \\ -0.0900 & 0.1419 & -0.1546 & 0.3321 \end{bmatrix}, \\
T_3 &= \begin{bmatrix} 5.0062 & -1.1560 \\ -1.1560 & 2.5632 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 0.0484 & -0.0064 \\ -0.0064 & 0.0375 \end{bmatrix}, \\
T &= \begin{bmatrix} 11.7407 & 5.9815 & 1.4217 & 1.4217 & 5.9815 & 6.5488 & 5.3974 & 6.6940 \\ 5.9815 & 6.5488 & 1.4217 & 1.4217 & 3.4016 & 5.4808 & 3.6899 & 3.4023 \\ 1.4217 & 1.4217 & 11.7407 & 5.9815 & 1.7581 & 1.2609 & 0.4126 & 0.5448 \\ 1.4217 & 1.4217 & 5.9815 & 6.5488 & 1.1921 & 1.3805 & 0.6993 & 0.1929 \\ 5.9815 & 3.4016 & 1.7581 & 1.1921 & 6.6940 & 3.4016 & 3.7150 & 3.7150 \\ 6.5488 & 5.4808 & 1.2609 & 1.3805 & 3.4016 & 5.4808 & 3.7150 & 3.7150 \\ 5.3974 & 3.6899 & 0.4126 & 0.6993 & 3.7150 & 3.7150 & 6.6940 & 3.4016 \\ 6.6940 & 3.4023 & 0.5448 & 0.1929 & 3.7150 & 3.7150 & 3.4016 & 5.4808 \end{bmatrix}.
\end{aligned}$$

(B) Passivity analysis: Choose $\Upsilon_1 = 0_2$, $\Upsilon_2 = I_2$, $\Upsilon_3 = \gamma I_2$, and $\Upsilon_4 = 0$, then through Theorem 4.1, the passivity performance is obtained. When $\vartheta_1 = 5$, minimum passivity performance γ was listed in Table 10 for various values of ϑ_2 .

(C) Dissipativity analysis: Choose $\Upsilon_1 = -0.5I_2$, $\Upsilon_2 = I_2$, $\Upsilon_3 = \Re - \gamma I_2$, and $\Upsilon_4 = 0$, then through Theorem 4.1, the $(\Upsilon_1, \Upsilon_2, \Re)$ - γ -dissipativity performance is obtained. When $\Re = 2I_2$, $\vartheta_1 = 5$, maximum dissipativity performance γ was listed in Table 11 for various values of ϑ_2 .

(D) H_∞ Performance: Choose $\Upsilon_1 = -I_2$, $\Upsilon_2 = 0$, $\Upsilon_3 = \gamma^2 I_2$, and $\Upsilon_4 = 0$, then through Theorem 4.1, the H_∞ performance is obtained. When $\vartheta_1 = 5$, minimum H_∞ performance γ was listed in Table 12 for various values of ϑ_2 .

(E) $L_2 - L_\infty$ Performance: Choose $\Upsilon_1 = 0_2$, $\Upsilon_2 = 0$, $\Upsilon_3 = \gamma^2 I_2$, and $\Upsilon_4 = I_2$, then through Theorem 4.1,

the $L_2 - L_\infty$ performance is obtained. When $\vartheta_1 = 5$, minimum $L_2 - L_\infty$ performance γ was listed in Table 13 for various values of ϑ_2 .

Remark 5.1 The LKF constructed in Theorem 3.1 is in augmented form; thus, Theorem 3.1 contains more number of decision variables (NDVs) than the results in [35–38, 40, 41]. Hence, some computational complexity can occur in our paper. At the expense of a higher NDVs compared with previous results, a significant improvements of Theorem 3.1 and Theorem 3.1 are shown in the Tables 1 to 9 by comparing maximum delay upper bounds and convergence. Moreover, the approaches in [39, 42] contain larger NDVs than our results.

Remark 5.2 Lee et al. studied the extended dissipative criteria for delayed NNs (see [26]). Extended dissipative criteria for delayed NNs with general activation functions

Table 13 Minimum γ with different ϑ_2

Methods	6	7	8	9
Theorem 4.1	1.1763	1.2873	1.4562	1.5047

was proposed in [25]. Extended dissipative analysis of memristor-based NNs with two additive delays was established in [27]. In [28], Feng et al. considered extended dissipativity of discrete-time NNs for first time. The results in [25–27] studies only for continuous-time NNs and the result in [28] consider discrete-time NN without uncertainty. However, robust extended dissipative criteria for uncertain DNNs with interval time-varying delays has not been thoroughly considered. In addition, our results are less conservative than the results in [28]. Thus, our work is more general than the previous results.

Remark 5.3 In the recent years, there are some related important works that have been published [2–5]. In particular, cooperative containment control of multi-agent systems based on follower observers considered in [2]. Passivity of switched recurrent NNs was studied in [3]. Non-fragile H_∞ control for switched stochastic delay systems was established in [4]. In [5], passivity and passification problem of uncertain switched stochastic systems was studied.

6 Conclusion

In this paper, robust extended dissipativity analysis of time-delayed uncertain DNNs was studied. A novel delay-dependent criteria has been established to ensure the considered DNNs to be robust extended dissipative. By constructing the suitable Lyapunov functional and using LMI techniques, the results are obtained. Numerical examples are provided to show the feasibility and superiority of our approach. It is possible to extend this work to the robust extended dissipativity problem for Markovian jump stochastic DNNs.

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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Impulsive discrete-time BAM neural networks with random parameter uncertainties and time-varying leakage delays: an asymptotic stability analysis

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Abstract This proposal analyzes the problem of asymptotic approach on stability criteria for impulsive discrete-time BAM neural networks with random parameter uncertainties and time-varying leakage delays. Reciprocally convex combination technique is approached in this paper for the reduction of decision variables. This lemma is derived from the derivation of Jensen's inequality. Here, the uncertainties are considered as a randomly occurring parameter uncertainty and it obey certain mutually uncorrelated Bernoulli-distributed white noise sequences. A priori indicates the occurrence of uncertain parameters in the probability which is the valuable feature. Some novelty suf-

ficient conditions for ensuring the asymptotic stability of the addressed neural networks are attained in terms of linear matrix inequalities (LMIs) by the aid of Lyapunov–Krasovskii functionals approach, which can be easily checked by MATLAB LMI Toolbox. Finally, three illustrative examples are accomplished to manifest the effectiveness and fruitfulness of the proposed research work.

Keywords Discrete-time neural networks · Impulse · Stability · Random uncertainties · BAM neural networks · Linear matrix inequality · Leakage delay

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1 Introduction

Owing to expeditious advances in theoretical study and biological experiments regarding underlying networks mechanisms, synthetic biology has been more engaging consideration from the biology communities. The neural networks in the prior few years have been broadly considered and practiced in different areas such as pattern recognition, image processing, hetero and auto-associative memories, optimization problems, mechanics of structures & materials and other research fields (see [1–16]). The bidirectional associative memory (BAM) is an extension of auto-associator Hopfield neural networks, which was first initiated by Kosko [17]. It is composed of two layers such as X -layer and Y -layer, which are fully interconnected to each other. In recent times, due to their promising uti-

lization in associative memories, parallel computation and optimization problems [18, 19], the stability and periodicity of BAM neural networks have acquired much concentration among researchers.

Currently, instead of the bifurcation method, different mathematical techniques were engaged by Wang and Zou [20] to examine the stability and number of stable periodic solutions for discrete-time BAM neural networks. Mohamad [21] derived some conditions for continuous-time BAM neural networks to be exponentially stable by applying Lyapunov–Krasovskii functionals and Halanay-type inequalities. For continuous-time BAM neural networks, the analogues of discrete time were developed and their stability were studied in [21].

Note that, in practical, various types of neural networks have been assumed to act in a continuous-time manner. However, in appliance and applications of neural networks, discrete-time neural networks become more important than their continuous-time counterpart, because when it comes to implementation of continuous-time networks for the sake of computer-based simulation, experimentation or computation, it is usual to discretize the continuous-time networks. On the other hand, discrete-time network could be more suitable to model digitally transmitted signals in a dynamical way. Moreover, when the axon size is small, there exists a possibility for delay in the process of storing or sending information (i.e., discrete time delay). In [22, 23], authors deeply analyzed the delay in discrete term and obtained several new time-delayed conditions by using different LKFs. Generally speaking, if the size of the axon length is much larger, then it is not easier to compute the information size, and there occurs delay in the process (i.e., distributed time delay).

Among the stability of neural networks, systems with leakage delays become one of the important topics. While carrying information to the axon by the state vectors, a delay will emerge often called as leakage delay. Leakage delay also called as forgetting delay is the negative term in the state vector, which can be drawn back to 1992. The author in [24] noticed that the leakage delay had great impact on neural networks system. Since then, some interesting results have been proposed and much attention has paid to the NNs with leakage delay [25, 26]. For instance, Gopalsamy deliberated a population dynamics model with leakage delay in [27] and also found the destabilization of NNs. The

author in [28] initially explored the BAM neural networks with leakage delay as follows:

$$\begin{aligned}\frac{dx_i(t)}{dt} &= -a_i x_i(t - \tau_i) + \sum_{j=1}^n a_{ij} f_j(y_j(t - \sigma_j(t))) + I_i \\ \frac{dy_i(t)}{dt} &= -b_i y_i(t - \tau_i) + \sum_{j=1}^n b_{ij} g_j(x_j(t - \sigma_j(t))) + J_i\end{aligned}$$

An abrupt change at certain moments of time is called impulse, and this phenomenon is a wide variety of evolutionary process which exists universally [7, 8, 29]. We extend this phenomenon in neural networks. For example, suppose a stimulus from the external environment or the body is received by receptors during the implementation of electronic network, the electrical impulses will be conveyed to the neural net and impulsive phenomenon that is called impulsive perturbations arises naturally [30]. Moreover, impulsive perturbations as well as time delays can affect the dynamical behavior of neural networks system, see [31–34]. Furthermore, in practical phenomena, the uncertain parameters have been paid considerable attention in neural networks. In [35], the authors explored the state performance of the uncertain time-delayed neural networks. There are two main sources of uncertainty: (1) the deviations and (2) perturbations in parameters. Luo et al. [36] analyzed the concept of stability of the equilibrium point for the neural networks with uncertain parameters with the help of different LKFs.

Recently, in [37], the problem for stochastic neural networks with probability distribution to check the stability analysis has been explained. The problem of state estimation with Markovian jumping parameters has been proposed in [38, 39]. The authors in [40] delivered the stability analysis of stochastic NNs with leakage delay. Meanwhile, the problems on stability issues of NNs have received extensive attention; hence, some conclusions on the discrete-time neural networks with leakage delay in state vector have been reported. Hou and Zhu [41] investigate the stability conditions when the leakage delays affect the neural networks. In [40], the authors explored the concept of robust analysis with uncertain Markovian jumping discrete-time NNs with leakage term. The authors in [42] deliberated the stability in the sense of robust for time-delayed neural networks with leakage delays. In [23], Gao and Cui discussed the time-delayed BAM neural networks leading to robust stability or not. Liu [43] investigated the sta-

bility for discrete-time BAM neural networks, which is exponential, and in [31] authors proposed discrete-time neural networks with stochastic, mixed time delays and impulsive effects. However, to the best of authors' knowledge, up to now there are no results on the problem of impulsive discrete-time BAM neural networks with random parameter uncertainties and time-varying leakage delays—an asymptotic stability analysis.

Inspired by the above motivations, our proposed work is to find the robust stability conditions for a class of impulsive discrete-time BAM neural network systems with randomly occurring uncertainties and time-varying leakage delays. The augmented type LKF is chosen which exploit more information about the neural networks system. Some zero equations are added with Lyapunov–Krasovskii functionals for the stability of NNs, which can be exposed in terms of linear matrix inequalities. By LMI control toolbox in MATLAB software, we can easily verified the feasibility of proposed stability conditions. To propose the advantages of this research work, some illustrative examples are employed.

The main core of the proposed research work is as follows:

- Robust stability analysis of impulsive discrete-time BAM neural networks with randomly occurring parameter uncertainties and time-varying leakage delays using convex combination approach is proposed. An augmented type Lyapunov–Krasovskii functional terms are initiated, which leads less conservatism than previous work which are expressed in numerical examples with the help of MATLAB LMI Toolbox.
- More decision variables are considered in Lyapunov–Krasovskii functionals to employ the bounds of time-varying leakage delays as well as transmission delays. In our proposal, BAM NNs with randomly occurring uncertain parameters are engaged which is different from the existing literature.
- Reciprocally convex combination technique is used in our proposal. This lemma helps in reducing the number of decision variables without affecting the conservatism. The relevance of this lemma is from the derivation of Jensen's inequality for stability conditions.
- By handling the discrete-time-varying delay terms in our impulsive BAM neural networks with time-varying leakage delays and uncertain parame-

Table 1 Comparisons of upper bounds of time delays

Methods	Lower bounds	Upper bounds
In Ref. [23]	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M = 4$
In Ref. [49]	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M = 4$
In Ref. [26]	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M = 4$
In Ref. [32]	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M = 2$
In Ref. [48]	$\sigma_m = \tau_m = 3$	$\sigma_M = \tau_M = 5$
In Ref. [50]	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M > 0$
In Ref. [22]	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M = 6$
In this paper	$\alpha_m = \beta_m > 0$	$\alpha_M = \beta_M > 0$

ters, the allowable upper bounds of discrete-time-varying delays are very large, when compared with the previous results, see Table 1 in Examples 5.1, 5.2. This explores the approach developed in this paper, which is effective and less conservative than some existing literature, and also ensures the novelty of the research work.

- In general, leakage delay gives some prominent change in the dynamic behaviors of the neural networks. If the leakage delay is constant or time-varying [42], the system leads to be unstable. In our proposed work, time-varying leakage delays are considered along with parameters such as uncertain, impulse. If all the parameters affect the system simultaneously, then the system becomes unstable. This can be avoided with the help of augmented type Lyapunov–Krasovskii functional and some inequality techniques, which leads the system to be stable, and it also be very challenged.

The remaining of this manuscript is summarized as follows: In Sect. 2, problem formulation and preliminaries will be employed for the main results. The main results of discrete-time BAM NNs with time-varying leakage delays to check the asymptotic stability conditions are derived in Sect. 3. With the help of Theorem 1, Sect. 4 explains the concept of uncertain discrete-time BAM neural networks with time-varying leakage delays. Section 5 demonstrates our derived results with four illustrative examples. At last, conclusions are drawn in Sect. 6.

Notations Throughout this manuscript, the notations are quite standard. $\mathbb{R}^n$ be the n -dimensional Euclidean space and $\mathbb{R}^{n \times m}$ be the set of all $n \times m$ real matrices. I refers the identity matrix with appropriate dimensions. $\text{Diag}(\cdot)$ denotes the diagonal matrix. A^T means

the transpose of matrix A . k denotes the set of positive integers. For real symmetric matrices X and Y , the notation $X \geq Y$ (resp., $X > Y$) means that the matrix $X - Y$ is positive semidefinite (resp., positive definite). $\mathbb{N} = \{1, 2, \dots, n\}$ and $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^n$. (-1) be the matrix inverse. The mathematical expectation be $\mathbb{E}$. The symbol $*$ within a matrix represents the symmetric term of the matrix.

2 Problem description and preliminaries

In this section, with the help of some zero inequalities and convex combination approach, the asymptotic stability condition is derived for impulsive discrete-time BAM neural networks with time-varying leakage delays.

The impulsive discrete-time BAM neural network systems with time-varying leakage delays are as follows:

$$\begin{aligned} u_i(k+1) &= l_i u_i(k - \varrho(k)) + \sum_{j=1}^n m_{ji} \hat{f}(v_j(k)) \\ &+ \sum_{j=1}^n n_{ji} \hat{g}(v_j(k - \alpha(k))) + I \\ u_i(k_r) &= u_i(k_r^-) + \hat{e}_r(u_i(k_r^-)) \quad r = 1, 2, 3, \dots \\ v_j(k+1) &= a_j v_j(k - \rho(k)) \\ &+ \sum_{i=1}^n b_{ij} \tilde{f}(u_i(k)) + \sum_{i=1}^n c_{ij} \tilde{g}(u_i(k - \beta(k))) + J \\ v_j(k_r) &= v_j(k_r^-) + \tilde{e}_r(v_j(k_r^-)) \quad r = 1, 2, 3, \dots \quad (1) \end{aligned}$$

and the vector form is

$$\begin{aligned} u(k+1) &= Lu(k - \varrho(k)) + M\hat{f}(v(k)) \\ &+ N\hat{g}(v(k - \alpha(k))) + I \\ u(k_r) &= u(k_r^-) + \hat{e}_r(u(k_r^-)) \quad r = 1, 2, 3, \dots \\ v(k+1) &= Av(k - \rho(k)) + B\tilde{f}(u(k)) \\ &+ C\tilde{g}(u(k - \beta(k))) + J \\ v(k_r) &= v(k_r^-) + \tilde{e}_r(v(k_r^-)) \quad r = 1, 2, 3, \dots \quad (2) \end{aligned}$$

where $u(\cdot) = (u_1(\cdot), u_2(\cdot), \dots, u_n(\cdot))^T \in \mathbb{R}^n$, $v(\cdot) = (v_1(\cdot), v_2(\cdot), \dots, v_n(\cdot))^T \in \mathbb{R}^n$, are the state vectors.

The activation functions be $\hat{f}(\cdot) = [\hat{f}_1(\cdot), \hat{f}_2(\cdot), \dots, \hat{f}_n(\cdot)]^T \in \mathbb{R}^n$, $\hat{g}(\cdot) = [\hat{g}_1(\cdot), \hat{g}_2(\cdot), \dots, \hat{g}_n(\cdot)]^T \in \mathbb{R}^n$, $\tilde{f}(\cdot) =$

$[\tilde{f}_1(\cdot), \tilde{f}_2(\cdot), \dots, \tilde{f}_n(\cdot)]^T \in \mathbb{R}^n$, $\tilde{g}(\cdot) = [\tilde{g}_1(\cdot), \tilde{g}_2(\cdot), \dots, \tilde{g}_n(\cdot)]^T \in \mathbb{R}^n$. $I = [I_1, I_2, \dots, I_n]^T$ and $J = [J_1, J_2, \dots, J_n]^T$ are the input vectors from the external source. $\varrho(k)$ and $\rho(k)$ denote the leakage delays satisfying $0 < \varrho_m \leq \varrho(k) \leq \varrho_M$ and $0 < \rho_m \leq \rho(k) \leq \rho_M$, respectively, where $\varrho_m, \rho_m, \varrho_M, \rho_M$ denote the lower and upper bounds of $\varrho(k)$ and $\rho(k)$, respectively. $\alpha(k)$ and $\beta(k)$ represent the transmission delays satisfying $0 < \alpha_m \leq \alpha(k) \leq \alpha_M$ and $0 < \beta_m \leq \beta(k) \leq \beta_M$, respectively, $\alpha_m, \beta_m > 0$ and $\alpha_M, \beta_M > 0$ denote the lower and upper bounds of $\alpha(k)$ and $\beta(k)$, respectively.

The conditions $u(k_r) = u(k_r^-) + \hat{e}_r(u(k_r^-))$ and $v(k_r) = v(k_r^-) + \tilde{e}_r(v(k_r^-))$, $r = 1, 2, 3, \dots$ are the impulse dynamical activity caused by abrupt jumps at certain instants during the evolutionary process and $u(k_r^-) = \lim_{k \rightarrow k_r^-} u(k)$, $v(k_r^-) = \lim_{k \rightarrow k_r^-} v(k)$, $\hat{e}_r = \mathbb{R}^n \rightarrow \mathbb{R}$, $\tilde{e}_r = \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous over $\mathbb{R}^n$ and the impulse instant k_r is assumed to satisfy $0 = k_0 < k_1 < k_2 < \dots < k_{r-1} < k_r < \infty$ and $k_r - k_{r-1} > 1$ for $(r = 1, 2, \dots)$.

The following activation functions help to satisfy the following conditions throughout this paper.

Condition 1 $\hat{f}_j(\cdot), \hat{g}_j(\cdot), \tilde{f}_i(\cdot), \tilde{g}_i(\cdot)$, $j, i = 1, 2, 3, \dots, n$ are the neuron activation functions which are continuous and bounded.

Condition 2 For $j = 1, 2, 3, \dots, n$ and $i = 1, 2, 3, \dots, n$ and any $s_1, s_2, t_1, t_2 \in \mathbb{R}$, $s_1 \neq s_2$ and $t_1 \neq t_2$, the neuron activation function in condition 1 satisfies

$$\begin{aligned} F_j^- &\leq \frac{\hat{f}_j(s_1) - \hat{f}_j(s_2)}{s_1 - s_2} \leq F_j^+; \\ D_i^- &\leq \frac{\tilde{f}_i(t_1) - \tilde{f}_i(t_2)}{t_1 - t_2} \leq D_i^+; \\ G_j^- &\leq \frac{\hat{g}_j(s_1) - \hat{g}_j(s_2)}{s_1 - s_2} \leq G_j^+; \\ E_i^- &\leq \frac{\tilde{g}_i(t_1) - \tilde{g}_i(t_2)}{t_1 - t_2} \leq E_i^+. \end{aligned}$$

where $G_j^-, G_j^+, F_j^-, F_j^+, D_i^-, D_i^+, E_i^-, E_i^+$ are known constants.

Remark 2.1 In [44,45], authors proposed the aforementioned Condition 2, respectively. Here, the constants are allowed to be positive, negative or zero. We

use generalized activation functions which are more general than sigmoid and Lipschitz activation functions. It is very helpful for using LMI-based technique to reduce the possible conservatism.

In order to simplify our proof, we shift the equilibrium point $u^* = [u_1^*, u_2^*, \dots, u_n^*]^T$ and $v^* = [v_1^*, v_2^*, \dots, v_n^*]^T$ of system (2) to origin. Let $x_i(k) = u_i(k) - u_i^*$, $y_j(k) = v_j(k) - v_j^*$, $f_j(y_j(k)) = \hat{f}_j(v_j(k)) - \hat{f}_j(v_j^*)$, $g_j(y_j(k)) = \hat{g}_j(v_j(k)) - \hat{g}_j(v_j^*)$, $\hat{f}_i(x_i(k)) = \hat{f}_i(u_i(k)) - \hat{f}_i(u_i^*)$, $\hat{g}_i(x_i(k)) = \hat{g}_i(u_i(k)) - \hat{g}_i(u_i^*)$.

After transformation, the neural network of system (2) can be rewritten as:

$$\begin{aligned} x(k+1) &= Lx(k - \varrho(k)) + Mf(y(k)) \\ &\quad + Ng(y(k - \alpha(k))) \\ x(k_r) &= x(k_r^-) + e_r x(k_r^-) \\ y(k+1) &= Ay(k - \rho(k)) + B\hat{f}(x(k)) \\ &\quad + C\hat{g}(x(k - \beta(k))) \\ y(k_r) &= y(k_r^-) + \hat{e}_r y(k_r^-). \end{aligned} \quad (3)$$

where $x(k) = [x_1(k), x_2(k), \dots, x_n(k)]^T \in \mathbb{R}^n$, $y(k) = [y_1(k), y_2(k), \dots, y_n(k)]^T \in \mathbb{R}^n$ are the state vectors, $x(k - \varrho(k)) = [x_1(k - \varrho(k)), x_2(k - \varrho(k)), \dots, x_n(k - \varrho(k))]^T$, $y(k - \rho(k)) = [y_1(k - \rho(k)), y_2(k - \rho(k)), \dots, y_n(k - \rho(k))]^T$, $f(y(k)) = [f_1(y(k)), f_2(y(k)), \dots, f_n(y(k))]^T$, $g(y(k - \alpha(k))) = [g_1(y(k - \alpha(k))), g_2(y(k - \alpha(k))), \dots, g_n(y(k - \alpha(k)))]^T$, $\hat{f}(y(k)) = [\hat{f}_1(y(k)), \hat{f}_2(y(k)), \dots, \hat{f}_n(y(k))]^T$, $\hat{g}(x(k - \beta(k))) = [\hat{g}_1(y(k - \beta(k))), \hat{g}_2(y(k - \beta(k))), \dots, \hat{g}_n(y(k - \beta(k)))]^T$ be the activation functions.

From Condition 2, the activation functions $f_j(\cdot)$, $g_j(\cdot)$, $\hat{f}_i(\cdot)$, $\hat{g}_i(\cdot)$, $j=1, 2, 3, \dots, n$, $i=1, 2, 3, \dots, n$ gratify

$$F_j^- \leq \frac{f_j(s_1) - f_j(s_2)}{s_1 - s_2} \leq F_j^+;$$

$$D_i^- \leq \frac{\hat{f}_i(t_1) - \hat{f}_i(t_2)}{t_1 - t_2} \leq D_i^+.$$

$$G_j^- \leq \frac{g_j(s_1) - g_j(s_2)}{s_1 - s_2} \leq G_j^+;$$

$$E_i^- \leq \frac{\hat{g}_i(t_1) - \hat{g}_i(t_2)}{t_1 - t_2} \leq E_i^+.$$

For any $t_1 \neq t_2$, $s_1 \neq s_2$ and $f_j(0) = g_j(0) = 0$, $\hat{f}_i(0) = \hat{g}_i(0) = 0$

The initial conditions are as follows:

$$\begin{aligned} x(s) &= \varpi(s), \quad s = -\omega, -\omega + 1, \dots, 0 \\ y(t) &= \tau(t), \quad t = -\delta, -\delta + 1, \dots, 0. \end{aligned} \quad (4)$$

where $\omega = \max \{\varrho_M, \alpha_M\}$ and $\delta = \max \{\rho_M, \beta_M\}$.

To organize our theorems, the following lemmas will be needful.

Lemma 2.2 [46] *let the functions $h_1, h_2, \dots, h_N : \mathbb{R}^m \rightarrow \mathbb{R}$ in an open subset D of $\mathbb{R}^m$ have been positive values. Then the conditions satisfied by reciprocally convex combination of h_i over D as follows*

$$\begin{aligned} \min_{\{\gamma_i/\gamma_i > 0, \sum_i \gamma_i = 1\}} \sum_i \frac{1}{\gamma_i} h_i(k) &= \sum_i h_i(k) \\ &+ \max_{z_{ij}(k)} \sum_{i \neq j} z_{ij}(k), \end{aligned} \quad (5)$$

subject to

$$z_{ij} : \mathbb{R}^m \rightarrow \mathbb{R}, \quad z_{j,i}(k), \begin{pmatrix} h_i(k) & z_{i,j}(k) \\ z_{i,j}(k) & h_i(k) \end{pmatrix} \geq 0. \quad (6)$$

Proof The constraint in (6) implies that

$$\begin{pmatrix} \sqrt{\frac{\gamma_j}{\gamma_i}} \\ -\sqrt{\frac{\gamma_i}{\gamma_j}} \end{pmatrix}^T \begin{pmatrix} h_i(k) & z_{i,j}(k) \\ z_{i,j}(k) & h_j(k) \end{pmatrix} \begin{pmatrix} \sqrt{\frac{\gamma_j}{\gamma_i}} \\ -\sqrt{\frac{\gamma_i}{\gamma_j}} \end{pmatrix} \geq 0.$$

Then we have

$$\begin{aligned} \sum_i \frac{1}{\gamma_i} h_i(k) &= \sum_{i,j} \frac{\gamma_j}{\gamma_i} h_i(k) \\ &= \sum_i h_i(k) \\ &\quad + \frac{1}{2} \sum_{i \neq j} \left(\frac{\gamma_j}{\gamma_i} h_i(k) + \frac{\gamma_i}{\gamma_j} h_j(k) \right) \\ &\geq \sum_i h_i(k) + \sum_{i \neq j} z_{i,j}(k). \end{aligned}$$

Note that the inequality holds for

$$\gamma_i = \frac{\sqrt{h_i(k)}}{\sum_j \sqrt{h_j(k)}}, \quad z_{i,j}(k) = \sqrt{h_i(k)h_j(k)}$$

which completes the proof. $\square$

Lemma 2.3 [47] *Given constant matrices Π_1, Π_2 and Π_3 with appropriate dimensions where $\Pi_1^T = \Pi_1$ and $\Pi_2^T = \Pi_2 > 0$, then*

$$\Pi + \Pi_3^T \Pi_2^{-1} \Pi_3 < 0$$

if and only if

$$\begin{pmatrix} \Pi_1 & \Pi_3^T \\ * & -\Pi_2 \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -\Pi_2 & \Pi_3 \\ * & \Pi_1 \end{pmatrix} < 0.$$

3 Asymptotic stability results

To check the stability in the sense of asymptotic for the discrete-time BAM neural networks (3) with impulse and time-varying leakage delays, some sufficient conditions are obtained in this section.

Theorem 3.1 *Suppose that Condition 2 holds, NNs (3) are asymptotically stable, if $\exists$ symmetric matrices $Q_i > 0$ ($i = 1$ to 8), and $P_j > 0$ ($j = 1$ to 8), $O_1 > 0$, $O_2 > 0$, $O_3 > 0$, $O_4 > 0$, $Z_1 > 0$, $Z_2 > 0$, $Z_3 > 0$, $Z_4 > 0$, $H_j > 0$, $T_j > 0$, and $H_i > 0$, $T_i > 0$, diagonal matrices $\zeta_1 > 0$, $\zeta_2 > 0$, $\zeta_3 > 0$, $\zeta_4 > 0$, and matrices R_i ($i = 1$ to 4), R_j ($j = 5$ to 8) and Y_1, Y_2, Y_3, Y_4 of appropriate dimensions and positive constant ϵ ($\epsilon \leq 1$), then that the following LMIs hold.*

$$(i) \quad \sum_{i=1}^m \ln(1 + k_i)^2 (1 + \omega_i)^2 + k \ln(1 - \epsilon) \leq \vartheta(k),$$

$$\text{for any } k \in [k_r, k_{r+1}), \text{ then}$$

$$\lim_{k \rightarrow +\infty} \vartheta(k) = +\infty, \quad (7)$$

$$(ii) \quad \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} > 0, \quad (8)$$

$$(iii) \quad \begin{pmatrix} H_5 & H_6 \\ H_7 & H_8 \end{pmatrix} > 0, \quad (9)$$

$$(iv) \quad \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} > 0, \quad (10)$$

$$(v) \quad \begin{pmatrix} T_5 & T_6 \\ T_7 & T_8 \end{pmatrix} > 0, \quad (11)$$

$$(vi) \quad \Psi = \sum_{r=1}^{10} \varphi_r < 0, \quad (12)$$

$$(vii) \quad \chi = \sum_{r=1}^{10} \theta_r < 0, \quad (13)$$

where

$$\begin{aligned} \varphi_1 = & e_8^T Q_1 e_8 + 2e_8^T Q_1 e_1 \\ & + 2e_8^T Q_2 (e_2 - e_4) + 2e_8^T Q_2 (e_{11} + e_{12}) \\ & + 2e_8^T Q_3 (e_5 - e_7) + 2e_8^T Q_3 \\ & \times (e_{13} + e_{14}) + 2e_1^T Q_2 (e_2 - e_4) \\ & + 2e_1^T Q_3 (e_5 - e_7) \\ & + (e_2 - e_4)^T Q_4 (e_2 - e_4) + 2(e_2 - e_4) \\ & \times Q_4 (e_{11} + e_{12}) \\ & + 2(e_2 - e_4)^T Q_5 (e_5 - e_7) \\ & + 2(e_2 - e_4)^T Q_5 (e_{13} + e_{14}) \\ & + 2(e_{11} + e_{12})^T \\ & \times Q_5 (e_{13} + e_{14}) + (e_5 - e_7)^T Q_6 (e_5 - e_7) \\ & + 2(e_5 - e_7)^T Q_6 (e_{13} + e_{14}), \end{aligned}$$

$$\varphi_2 = (\varrho_1 + 1) e_1^T Q_7 e_1 - e_3^T Q_7 e_3,$$

$$\varphi_3 = (\alpha_1 + 1) e_1^T Q_8 e_1 - e_6^T Q_8 e_6,$$

$$\begin{aligned} \varphi_4 = & \varrho_1^2 e_1^T O_1 e_1 + \varrho_1 e_2^T R_3 e_2 \\ & + \varrho_1 e_3^T (R_1 - R_3) e_3 \\ & - \varrho_1 e_4^T R_1 e_4 + \varrho_1^2 e_8^T O_2 e_8, \end{aligned}$$

$$\begin{aligned} \varphi_5 = & - \left[e_{11}^T O_1 e_{11} + 2e_{11}^T R_1 (e_3 - e_4) \right. \\ & + 2e_{11}^T H_1 e_{12} + 2e_{11}^T H_2 (e_2 - e_3) \\ & + (e_3 - e_4)^T (O_2 + R_1) \\ & \times (e_3 - e_4) + 2(e_3 - e_4)^T H_3 e_{12} \\ & + 2(e_3 - e_4)^T H_4 (e_2 - e_3) \\ & + e_{12}^T O_1 e_{12} + 2e_{12}^T R_3 (e_2 - e_3) \\ & \left. + (e_2 - e_3)^T (O_2 + R_3) (e_2 - e_3) \right], \end{aligned}$$

$$\begin{aligned} \varphi_6 = & \alpha_1^2 e_1^T Z_1 e_1 + \alpha_1 e_5^T e_5 R_7 \\ & + \alpha_1 e_6^T (R_5 - R_7) e_6 - \alpha_1 e_7^T R_5 e_7 + \alpha_1^2 e_8^T Z_2 e_8, \end{aligned}$$

$$\begin{aligned} \varphi_7 = & - \left[e_{13}^T Z_1 e_{13} + 2e_{13}^T R_5 (e_6 - e_7) \right. \\ & + 2e_{13}^T T_1 e_{14} + 2e_{13}^T T_2 (e_5 - e_6) \\ & \left. + (e_6 - e_7)^T (Z_2 + R_5) \right] \end{aligned}$$

$$\begin{aligned}
 & \times (e_6 - e_7) + 2(e_6 - e_7)^T T_3 e_{14} \\
 & + 2(e_6 - e_7)^T T_4 (e_5 - e_6) + e_{14}^T Z_1 e_{14} \\
 & + 2e_{14}^T R_7 (e_5 - e_6) \\
 & + (e_5 - e_6)^T (Z_2 + R_7) (e_5 - e_6) \Big], \\
 \varphi_8 &= -e_1^T (F_1 \zeta_1) e_1 + 2e_1^T (F_2 \zeta_1) e_9 - e_9^T \zeta_1 e_9, \\
 \varphi_9 &= -e_6^T (G_1 \zeta_2) e_6 + 2e_6^T (G_2 \zeta_2) e_{10} - e_{10}^T \zeta_2 e_{10}, \\
 \varphi_{10} &= 2(e_8^T Y_1^T + e_1^T Y_2^T) (Le_3 + Me_9 + Ne_{10} - e_8 - e_1), \\
 \theta_1 &= e_8^{*T} P_1 e_8^* + 2e_8^{*T} P_1 e_1^* \\
 & + 2e_8^{*T} P_2 (e_2^* - e_4^*) + 2e_8^{*T} P_2 (e_{11}^* + e_{12}^*) \\
 & + 2e_8^{*T} P_3 (e_5^* - e_7^*) + 2e_8^{*T} \\
 & \times P_3 (e_{13}^* + e_{14}^*) + 2e_1^{*T} P_2 (e_2^* - e_4^*) \\
 & + 2e_1^{*T} P_3 (e_5^* - e_7^*) \\
 & + (e_2^* - e_4^*)^T P_4 (e_2^* - e_4^*) + 2(e_2^* - e_4^*) \\
 & \times P_4 (e_{11}^* + e_{12}^*) + 2(e_2^* - e_4^*)^T P_5 (e_5^* - e_7^*) \\
 & + 2(e_2^* - e_4^*)^T P_5 (e_{13}^* + e_{14}^*) + 2(e_{11}^* + e_{12}^*)^T P_5 \\
 & \times (e_{13}^* + e_{14}^*) + (e_5^* - e_7^*)^T P_6 (e_5^* - e_7^*) \\
 & + 2(e_5^* - e_7^*)^T P_6 (e_{13}^* + e_{14}^*), \\
 \theta_2 &= (\rho_1 + 1) e_1^{*T} P_7 e_1^* - e_3^{*T} P_7 e_3^*, \\
 \theta_3 &= (\beta_1 + 1) e_1^{*T} P_8 e_1^* - e_6^{*T} P_8 e_6^*, \\
 \theta_4 &= \rho_1^2 e_1^{*T} O_3 e_1^* \\
 & + \rho_1 e_2^{*T} R_4 e_2^* + \rho_1 e_3^{*T} (R_2 - R_4) e_3^* - \rho_1 e_4^{*T} R_2 e_4^* \\
 & + \rho_1^2 e_8^{*T} O_4 e_8^*, \\
 \theta_5 &= - \left[e_{11}^{*T} O_3 e_{11}^* \right. \\
 & + 2e_{11}^{*T} R_2 (e_3^* - e_4^*) + 2e_{11}^{*T} H_5 e_{12}^* \\
 & + 2e_{11}^{*T} H_6 (e_2^* - e_3^*) + (e_3^* - e_4^*)^T \\
 & \times (O_4 + R_2) (e_3^* - e_4^*) + 2(e_3^* - e_4^*)^T H_7 e_{12}^* \\
 & + 2(e_3^* - e_4^*)^T H_8 (e_2^* - e_3^*) \\
 & + e_{12}^{*T} O_3 e_{12}^* + 2e_{12}^{*T} \\
 & \times R_4 (e_2^* - e_3^*) + (e_2^* - e_3^*)^T (O_4 \\
 & \left. + R_4) (e_2^* - e_3^*) \right],
 \end{aligned}$$

$$\begin{aligned}
 \theta_6 &= \beta_1^2 e_1^{*T} Z_3 e_1^* \\
 & + \beta_1 e_5^{*T} e_5 R_8 + \beta_1 e_6^{*T} (R_6 - R_8) e_6^* \\
 & - \beta_1 e_7^{*T} R_6 e_7^* + \beta_1^2 e_8^{*T} Z_4 e_8^*, \\
 \theta_7 &= - \left[e_{13}^{*T} Z_3 e_{13}^* + 2e_{13}^{*T} R_6 (e_6^* - e_7^*) \right. \\
 & + 2e_{13}^{*T} T_5 e_{14}^* \\
 & + 2e_{13}^{*T} T_6 (e_5^* - e_6^*) + (e_6^* - e_7^*)^T (Z_4 + R_6) \\
 & \times (e_6^* - e_7^*) \\
 & + 2(e_6^* - e_7^*)^T T_7 e_{14}^* \\
 & + 2(e_6^* - e_7^*)^T T_8 (e_6^* - e_7^*) + e_{14}^{*T} Z_3 e_{14}^* \\
 & + 2e_{14}^{*T} R_8 (e_5^* - e_6^*) \\
 & \left. + (e_5^* - e_6^*)^T (Z_4 + R_8) (e_5^* - e_6^*) \right], \\
 \theta_8 &= -e_1^{*T} (D_1 \zeta_3) e_1^* + 2e_1^{*T} (D_2 \zeta_3) e_9^* - e_9^{*T} \zeta_3 e_9^*, \\
 \theta_9 &= -e_6^{*T} (E_1 \zeta_4) e_6^* + 2e_6^{*T} (E_2 \zeta_4) e_{10}^* - e_{10}^{*T} \zeta_4 e_{10}^*, \\
 \theta_{10} &= 2(e_8^{*T} Y_3^T + e_1^{*T} Y_4^T) (Ae_3^* + Be_9^* + Ce_{10}^* - e_8^* - e_1^*).
 \end{aligned}$$

Proof Consider the Lyapunov–Krasovskii functionals for neural networks (3) as follows:

$$\begin{aligned}
 V_1(k) &= \begin{pmatrix} x(k) \\ \sum_{i=k-Q_M}^{k-Q_m-1} x(i) \\ \sum_{i=k-\beta_M}^{k-\beta_m-1} x(i) \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_2 & Q_3 \\ * & Q_4 & Q_5 \\ * & * & Q_6 \end{pmatrix} \\
 &\times \begin{pmatrix} x(k) \\ \sum_{i=k-Q_M}^{k-Q_m-1} x(i) \\ \sum_{i=k-\beta_M}^{k-\beta_m-1} x(i) \end{pmatrix} + \begin{pmatrix} y(k) \\ \sum_{j=k-\rho_M}^{k-\rho_m-1} y(j) \\ \sum_{j=k-\alpha_M}^{k-\alpha_m-1} y(j) \end{pmatrix}^T \\
 &\times \begin{pmatrix} P_1 & P_2 & P_3 \\ * & P_4 & P_5 \\ * & * & P_6 \end{pmatrix} \begin{pmatrix} y(k) \\ \sum_{j=k-\rho_M}^{k-\rho_m-1} y(j) \\ \sum_{j=k-\alpha_M}^{k-\alpha_m-1} y(j) \end{pmatrix}, \quad (14)
 \end{aligned}$$

$$\begin{aligned}
 V_2(k) &= \sum_{i=k-Q(k)}^{k-1} x^T(i) Q_7 x(i) \\
 &+ \sum_{j=-Q_M+1}^{-Q_m} \sum_{i=k+j}^{k-1} x^T(i) Q_7 x(i) \\
 &+ \sum_{j=k-\rho(k)}^{k-1} y^T(j) P_7 y(j) \\
 &+ \sum_{i=-\rho_M+1}^{-\rho_m} \sum_{j=k+i}^{k-1} y^T(j) P_7 y(j), \quad (15)
 \end{aligned}$$

$$\begin{aligned}
V_3(k) &= \sum_{j=k-\alpha(k)}^{k-1} y^T(j) P_8 y(j) \\
&+ \sum_{i=-\alpha_M+1}^{-\alpha_m} \sum_{j=k+i}^{k-1} y^T(j) P_8 y(j) \\
&+ \sum_{i=k-\beta(k)}^{k-1} x^T(i) Q_8 x(i) \\
&+ \sum_{j=-\beta_M+1}^{-\beta_m} \sum_{i=k+j}^{k-1} x^T(i) Q_8 x(i), \quad (16)
\end{aligned}$$

$$\begin{aligned}
V_4(k) &= \varrho_1 \sum_{i=-\varrho_M}^{-\varrho_m-1} \sum_{j=k+i}^{k-1} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix}^T \begin{pmatrix} O_1 & 0 \\ 0 & O_2 \end{pmatrix} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix} \\
&+ \rho_1 \sum_{j=-\rho_M}^{-\rho_m-1} \sum_{i=k+j}^{k-1} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix}^T \begin{pmatrix} O_3 & 0 \\ 0 & O_4 \end{pmatrix} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix}, \quad (17)
\end{aligned}$$

$$\begin{aligned}
V_5(k) &= \alpha_1 \sum_{i=-\alpha_M}^{-\alpha_m-1} \sum_{j=k+i}^{k-1} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix}^T \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix} \\
&+ \beta_1 \sum_{j=-\beta_M}^{-\beta_m-1} \sum_{i=k+j}^{k-1} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix}^T \begin{pmatrix} Z_3 & 0 \\ 0 & Z_4 \end{pmatrix} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix}, \quad (18)
\end{aligned}$$

where $\mu(k) = x(k+1) - x(k)$, $v(k) = y(k+1) - y(k)$.

Calculating the difference of

$$\Delta V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + \Delta V_4(k) + \Delta V_5(k).$$

along the trajectories of neural networks system (3)

$$\begin{aligned}
\Delta V_1(k) &= \begin{pmatrix} \mu(k) + x(k) \\ \sum_{i=k-\varrho_M}^{k-\varrho_m-1} (\mu(i) + x(i)) \\ \sum_{i=k-\beta_M}^{k-\beta_m-1} (\mu(i) + x(i)) \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_2 & Q_3 \\ * & Q_4 & Q_5 \\ * & * & Q_6 \end{pmatrix} \\
&\times \begin{pmatrix} \eta(k) + x(k) \\ \sum_{i=k-\sigma_M}^{k-\sigma_m-1} (\mu(i) + x(i)) \\ \sum_{i=k-\beta_M}^{k-\beta_m-1} (\mu(i) + x(i)) \end{pmatrix} \\
&+ \begin{pmatrix} v(k) + y(k) \\ \sum_{j=k-\rho_M}^{k-\rho_m-1} (v(j) + y(j)) \\ \sum_{j=k-\alpha_M}^{k-\alpha_m-1} (v(j) + y(j)) \end{pmatrix}^T \begin{pmatrix} P_1 & P_2 & P_3 \\ * & P_4 & P_5 \\ * & * & P_6 \end{pmatrix} \\
&\times \begin{pmatrix} v(k) + y(k) \\ \sum_{j=k-\rho_M}^{k-\rho_m-1} (v(j) + y(j)) \\ \sum_{j=k-\alpha_M}^{k-\alpha_m-1} (v(j) + y(j)) \end{pmatrix} \\
&= \begin{pmatrix} \mu(k) \\ x(k) \\ \sum_{i=k-\varrho_M}^{k-\varrho_m-1} \mu(i) \\ \sum_{i=k-\varrho_M}^{k-\varrho_m-1} x(i) \\ \sum_{i=k-\beta_M}^{k-\beta_m-1} \mu(i) \\ \sum_{i=k-\beta_M}^{k-\beta_m-1} x(i) \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_1 & Q_2 & Q_2 & Q_3 & Q_3 \\ Q_1 & 0 & Q_2 & 0 & Q_3 & 0 \\ Q_2^T & Q_2^T & Q_4 & Q_4 & Q_5 & Q_5 \\ Q_3^T & Q_3^T & Q_5^T & Q_5^T & Q_6 & Q_6 \\ Q_3^T & 0 & Q_5^T & 0 & Q_6 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&\times \begin{pmatrix} \mu(k) \\ x(k) \\ \sum_{i=k-\varrho_M}^{k-\varrho_m-1} \mu(i) \\ \sum_{i=k-\varrho_M}^{k-\varrho_m-1} x(i) \\ \sum_{i=k-\beta_M}^{k-\beta_m-1} \mu(i) \\ \sum_{i=k-\beta_M}^{k-\beta_m-1} x(i) \end{pmatrix} \\
&+ \begin{pmatrix} v(k) \\ y(k) \\ \sum_{j=k-\rho_M}^{k-\rho_m-1} v(j) \\ \sum_{j=k-\rho_M}^{k-\rho_m-1} y(j) \\ \sum_{j=k-\alpha_M}^{k-\alpha_m-1} v(j) \\ \sum_{j=k-\alpha_M}^{k-\alpha_m-1} y(j) \end{pmatrix}^T \begin{pmatrix} P_1 & P_1 & P_2 & P_2 & P_3 & P_3 \\ P_1^T & 0 & P_2 & 0 & P_3 & 0 \\ P_2^T & P_2^T & P_4 & P_4 & P_5 & P_5 \\ P_3^T & P_3^T & P_5^T & P_5^T & P_6 & P_6 \\ P_3^T & 0 & P_5^T & 0 & P_6 & 0 \end{pmatrix} \\
&\times \begin{pmatrix} v(k) \\ y(k) \\ \sum_{j=k-\rho_M}^{k-\rho_m-1} v(j) \\ \sum_{j=k-\rho_M}^{k-\rho_m-1} y(j) \\ \sum_{j=k-\alpha_M}^{k-\alpha_m-1} v(j) \\ \sum_{j=k-\alpha_M}^{k-\alpha_m-1} y(j) \end{pmatrix} \\
&= \Upsilon^T(k) [\varphi_1] \Upsilon(k) + \Xi^T(k) [\theta_1] \Xi(k), \quad (19)
\end{aligned}$$

$$\begin{aligned}
\Delta V_2(k) &= (\varrho_1 + 1) x^T(k) Q_7 x(k) \\
&- x^T(k - \varrho(k)) Q_7 x(k - \varrho(k)) \\
&+ (\rho_1 + 1) y^T(k) P_7 y(k) \\
&- y^T(k - \rho(k)) P_7 y(k - \rho(k)), \\
&= \Upsilon^T(k) [\varphi_2] \Upsilon(k) \\
&+ \Xi^T(k) [\theta_2] \Xi(k), \quad (20)
\end{aligned}$$

$$\begin{aligned}
\Delta V_3(k) &= (\alpha_1 + 1) y^T(k) Q_8 y(k) \\
&- y^T(k - \alpha(k)) Q_8 y(k - \alpha(k)) \\
&+ (\beta_1 + 1) x^T(k) P_8 x(k) \\
&- x^T(k - \beta(k)) P_8 x(k - \beta(k)), \\
&= \Upsilon^T(k) [\varphi_3] \Upsilon(k) + \Xi^T(k) [\theta_3] \Xi(k), \quad (21)
\end{aligned}$$

$$\begin{aligned}
\Delta V_4(k) &= \varrho_1^2 \begin{pmatrix} x(k) \\ \mu(k) \end{pmatrix}^T \begin{pmatrix} O_1 & 0 \\ 0 & O_2 \end{pmatrix} \begin{pmatrix} x(k) \\ \mu(k) \end{pmatrix} \\
&- \varrho_1 \sum_{i=k-\varrho_M}^{k-\varrho_m-1} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix}^T \begin{pmatrix} O_1 & 0 \\ 0 & O_2 \end{pmatrix} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix} \\
&+ \rho_1^2 \begin{pmatrix} y(k) \\ v(k) \end{pmatrix}^T \begin{pmatrix} O_3 & 0 \\ 0 & O_4 \end{pmatrix} \begin{pmatrix} y(k) \\ v(k) \end{pmatrix} \\
&- \rho_1 \sum_{j=k-\rho_M}^{k-\rho_m-1} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix}^T \begin{pmatrix} O_3 & 0 \\ 0 & O_4 \end{pmatrix} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix} \\
&\Delta V_5(k) = \alpha_1^2 \begin{pmatrix} y(k) \\ v(k) \end{pmatrix}^T \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \begin{pmatrix} y(k) \\ v(k) \end{pmatrix} \\
&- \alpha_1 \sum_{i=k-\alpha_M}^{k-\alpha_m-1} \begin{pmatrix} y(i) \\ v(i) \end{pmatrix}^T \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \begin{pmatrix} y(i) \\ v(i) \end{pmatrix}
\end{aligned} \quad (22)$$

$$\begin{aligned}
 & + \beta_1^2 \begin{pmatrix} x(k) \\ \mu(k) \end{pmatrix}^T \begin{pmatrix} Z_3 & 0 \\ 0 & Z_4 \end{pmatrix} \begin{pmatrix} x(k) \\ \mu(k) \end{pmatrix} \\
 & - \beta_1 \sum_{j=k-\beta_M}^{k-\beta_m-1} \begin{pmatrix} x(j) \\ \mu(j) \end{pmatrix}^T \begin{pmatrix} Z_3 & 0 \\ 0 & Z_4 \end{pmatrix} \begin{pmatrix} x(j) \\ \mu(j) \end{pmatrix}, \quad (23)
 \end{aligned}$$

where $\varrho_1 = \varrho_M - \varrho_m$, $\rho_1 = \rho_M - \rho_m$ and $\alpha_1 = \alpha_M - \alpha_m$, $\beta_1 = \beta_M - \beta_m$. It is easy to see that

$$\begin{aligned}
 0 & = \varrho_1 \left[x^T(k - \varrho(k)) R_1 x(k - \varrho(k)) \right. \\
 & \quad - x^T(k - \varrho_M) R_1 x(k - \varrho_M) \\
 & \quad \left. - \sum_{i=k-\varrho_M}^{k-\varrho(k)-1} \mu^T(i) T_1 \mu(i) + 2x(i) \right], \quad (24)
 \end{aligned}$$

$$\begin{aligned}
 0 & = \rho_1 \left[y^T(k - \rho(k)) R_2 y(k - \rho(k)) \right. \\
 & \quad - y^T(k - \rho_M) R_2 y(k - \rho_M) \\
 & \quad \left. - \sum_{j=k-\rho_M}^{k-\rho(k)-1} v^T(j) T_2 v(j) + 2y(j) \right], \quad (25)
 \end{aligned}$$

$$\begin{aligned}
 0 & = \varrho_1 \left[x^T(k - \varrho_m) R_3 x(k - \varrho_m) \right. \\
 & \quad - x^T(k - \varrho(k)) R_3 x(k - \varrho(k)) \\
 & \quad \left. - \sum_{i=k-\varrho(k)}^{k-\varrho_m-1} \mu^T(i) T_3 \mu(i) + 2x(i) \right], \quad (26)
 \end{aligned}$$

$$\begin{aligned}
 0 & = \rho_1 \left[y^T(k - \rho_m) R_4 y(k - \rho_m) \right. \\
 & \quad - y^T(k - \rho(k)) R_4 y(k - \rho(k)) \\
 & \quad \left. - \sum_{j=k-\rho(k)}^{k-\rho_m-1} v^T(j) T_4 v(j) + 2y(j) \right], \quad (27)
 \end{aligned}$$

$$\begin{aligned}
 0 & = \alpha_1 \left[y^T(k - \tau(k)) R_5 y(k - \alpha(k)) \right. \\
 & \quad - y^T(k - \alpha_M) R_5 y(k - \alpha_M) \\
 & \quad \left. - \sum_{j=k-\alpha_M}^{k-\alpha(k)-1} v^T(j) R_5 v(j) + 2y(j) \right], \quad (28)
 \end{aligned}$$

$$\begin{aligned}
 0 & = \beta_1 \left[x^T(k - \beta(k)) R_6 x(k - \beta(k)) \right. \\
 & \quad - x^T(k - \beta_M) R_6 x(k - \beta_M) \\
 & \quad \left. - \sum_{i=k-\beta_M}^{k-\beta(k)-1} \mu^T(i) R_6 \mu(i) + 2x(i) \right], \quad (29)
 \end{aligned}$$

$$\begin{aligned}
 0 & = \alpha_1 \left[y^T(k - \alpha_m) R_7 y(k - \alpha_m) \right. \\
 & \quad - y^T(k - \alpha(k)) R_7 y(k - \alpha(k)) \\
 & \quad \left. - \sum_{j=k-\alpha(k)}^{k-\alpha_m-1} v^T(j) R_7 v(j) + 2y(j) \right], \quad (30)
 \end{aligned}$$

$$\begin{aligned}
 0 & = \beta_1 \left[x^T(k - \beta_m) R_8 x(k - \beta_m) \right. \\
 & \quad - x^T(k - \beta(k)) R_8 x(k - \beta(k)) \\
 & \quad \left. - \sum_{i=k-\beta(k)}^{k-\beta_m-1} \mu^T(i) R_8 \mu(i) + 2x(i) \right], \quad (31)
 \end{aligned}$$

Adding (25)–(28) to the second term of Eq. (23) and also adding (28)–(31) to the second term of Eq. (23), then we acquired.

$$\begin{aligned}
 \Delta V_4(k) & = \varrho_1^2 \begin{pmatrix} x(k) \\ \mu(k) \end{pmatrix}^T \begin{pmatrix} O_1 & 0 \\ 0 & O_2 \end{pmatrix} \begin{pmatrix} x(k) \\ \mu(k) \end{pmatrix} \\
 & - \varrho_1 \sum_{i=k-\varrho_M}^{k-\varrho(k)-1} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix}^T \begin{pmatrix} O_1 & 0 \\ 0 & O_2 \end{pmatrix} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix} \\
 & - \varrho_1 \sum_{i=k-\varrho(k)}^{k-\varrho_m-1} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix}^T \begin{pmatrix} O_1 & 0 \\ 0 & O_2 \end{pmatrix} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix} \\
 & + \varrho_1 x^T(k - \varrho(k)) R_1 x(k - \varrho(k)) \\
 & - x^T(k - \varrho_M) T_1 x(k - \varrho_M) \\
 & + x^T(k - \varrho_m) R_3 x(k - \varrho_m) - x^T(k - \varrho(k)) R_3 \\
 & \quad \times x(k - \varrho(k)) - \sum_{i=k-\varrho_M}^{k-\varrho(k)-1} \mu^T(i) R_1 \mu(i) \\
 & + 2x(i) - \sum_{i=k-\varrho(k)}^{k-\varrho_m-1} \mu^T(i) R_3 \mu(i) + 2x(i) \\
 & + \rho_1^2 \begin{pmatrix} y(k) \\ v(k) \end{pmatrix}^T \begin{pmatrix} O_3 & 0 \\ 0 & O_4 \end{pmatrix} \begin{pmatrix} y(k) \\ v(k) \end{pmatrix} \\
 & - \rho_1 \sum_{j=k-\rho_M}^{k-\rho(k)-1} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix}^T \begin{pmatrix} O_3 & 0 \\ 0 & O_4 \end{pmatrix} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix} \\
 & - \rho_1 \sum_{j=k-\rho(k)}^{k-\rho_m-1} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix}^T \begin{pmatrix} O_3 & 0 \\ 0 & O_4 \end{pmatrix} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix} \\
 & + \rho_1 y^T(k - \rho(k)) R_2 y(k - \rho(k)) \\
 & - y^T(k - \rho_M) R_2 y(k - \rho_M) \\
 & + y^T(k - \rho_m) R_4 y(k - \rho_m) - y^T(k - \rho(k)) R_4 \\
 & \quad \times y(k - \rho(k)) \\
 & - \sum_{j=k-\rho_M}^{k-\rho(k)-1} v^T(j) R_4 v(j) + 2y(j) \\
 & - \sum_{j=k-\rho(k)}^{k-\rho_m-1} v^T(j) R_4 v(j) + 2y(j) \\
 & < \Upsilon^T(k) [\varphi_4] \Upsilon(k) \\
 & - \frac{\varrho_1}{\varrho_M - \varrho(k)} \left(\sum_{i=k-\varrho_M}^{k-\varrho(k)-1} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix} \right)^T
 \end{aligned}$$

$$\begin{aligned}
& \times \begin{pmatrix} O_1 & R_1 \\ * & O_2 + R_1 \end{pmatrix} \left(\sum_{i=k-\varrho_M}^{k-\varrho(k)-1} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix} \right) \\
& - \frac{\varrho_1}{\varrho(k) - \varrho_m} \left(\sum_{i=k-\varrho_k}^{k-\varrho_m-1} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix} \right)^T \\
& \times \begin{pmatrix} O_1 & R_3 \\ * & O_2 + R_3 \end{pmatrix} \left(\sum_{i=k-\varrho_k}^{k-\varrho_m-1} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix} \right) \\
& + \Xi^T(k) \theta_4 \Xi(k) \\
& - \frac{\rho_1}{\rho_M - \rho(k)} \left(\sum_{j=k-\rho_M}^{k-\rho(k)-1} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix} \right)^T \\
& \times \begin{pmatrix} O_3 & R_2 \\ * & O_3 + R_2 \end{pmatrix} \left(\sum_{j=k-\rho_M}^{k-\rho(k)-1} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix} \right) \\
& - \frac{\rho_1}{\rho_k - \rho_m} \\
& \times \left(\sum_{j=k-\rho_k}^{k-\rho_m-1} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix} \right)^T \\
& \times \begin{pmatrix} O_3 & R_4 \\ * & O_3 + R_4 \end{pmatrix} \left(\sum_{j=k-\rho_k}^{k-\rho_m-1} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix} \right)
\end{aligned}$$

Let $\eta_1 = \frac{\varrho_M - \varrho(k)}{\varrho_1}$, $\eta_2 = \frac{\rho_M - \rho(k)}{\rho_1}$, $\eta_3 = \frac{\varrho(k) - \varrho_m}{\varrho_1}$, $\eta_4 = \frac{\rho(k) - \rho_m}{\rho_1}$. $\square$

Since $\eta_1 + \eta_2 = 1$, $\eta_3 + \eta_4 = 1$.

By Lemma 2.2, there $\exists$ a matrix

$$\begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} > 0, \text{ and } \begin{pmatrix} H_5 & H_6 \\ H_7 & H_8 \end{pmatrix} > 0.$$

such that

$$\begin{aligned}
\Delta V_4(k) &= \Upsilon^T(k) [\varphi_4] \Upsilon(k) \\
& - \left(\sum_{i=k-\varrho_M}^{k-\varrho(k)-1} \begin{pmatrix} x(i) \mu(i) \end{pmatrix} \right)^T \begin{pmatrix} O_1 & R_1 & H_1 & H_2 \\ * & O_2 + R_1 & H_3 & H_4 \\ * & * & O_1 & R_3 \\ * & * & * & O_2 + R_3 \end{pmatrix} \\
& \times \left(\sum_{i=k-\varrho_k}^{k-\varrho_m-1} \begin{pmatrix} x(i) \mu(i) \end{pmatrix} \right) \\
& + \Xi^T(k) \theta_4 \Xi(k) - \left(\sum_{j=k-\rho_M}^{k-\rho(k)-1} \begin{pmatrix} y(j) v(j) \end{pmatrix} \right)^T \\
& \times \begin{pmatrix} O_3 & R_2 & H_5 & H_6 \\ * & O_4 + R_2 & H_7 & H_8 \\ * & * & O_3 & R_4 \\ * & * & * & O_4 + R_4 \end{pmatrix} \left(\sum_{j=k-\rho_k}^{k-\rho_m-1} \begin{pmatrix} y(j) v(j) \end{pmatrix} \right) \\
& \leq \Upsilon^T(k) [\varphi_4] \Upsilon(k) - \begin{pmatrix} \Lambda_1(k) \\ x(k - \varrho(k)) - x(k - \varrho_M) \\ \Lambda_2(k) \\ x(k - \varrho_m) - x(k - \sigma(k)) \end{pmatrix}^T \\
& \times \begin{pmatrix} O_1 & R_1 & H_1 & H_2 \\ * & O_2 + R_1 & H_3 & H_4 \\ * & * & O_1 & R_3 \\ * & * & * & O_2 + R_3 \end{pmatrix} \begin{pmatrix} \Lambda_1(k) \\ x(k - \varrho(k)) - x(k - \varrho_M) \\ \Lambda_2(k) \\ x(k - \varrho_m) - x(k - \sigma(k)) \end{pmatrix} \\
& + \Xi^T(k) \theta_4 \Xi(k) \\
& - \begin{pmatrix} \Lambda_5(k) \\ y(k - \rho(k)) - y(k - \rho_M) \\ \Lambda_6(k) \\ y(k - \rho_m) - y(k - \rho(k)) \end{pmatrix}^T
\end{aligned}$$

$$\begin{aligned}
& \times \begin{pmatrix} O_3 & R_2 & H_5 & H_6 \\ * & O_4 + R_2 & H_7 & H_8 \\ * & * & O_3 & R_4 \\ * & * & * & O_4 + R_4 \end{pmatrix} \begin{pmatrix} \Lambda_5(k) \\ y(k - \rho(k)) - y(k - \rho_M) \\ \Lambda_6(k) \\ y(k - \rho_m) - y(k - \rho(k)) \end{pmatrix} \\
& \leq \Upsilon^T(k) [\varphi_4 + \varphi_5] \Upsilon(k) \\
& + \Xi^T(k) [\theta_4 + \theta_5] \Xi(k)
\end{aligned} \tag{32}$$

Similarly,

$$\begin{aligned}
\Delta V_5(k) &\leq \Upsilon^T(k) [\varphi_6] \Upsilon(k) \\
& - \begin{pmatrix} \Lambda_3(k) \\ y(k - \alpha(k)) - y(k - \alpha_M) \\ \Lambda_4(k) \\ y(k - \alpha_m) - y(k - \alpha(k)) \end{pmatrix}^T \\
& \times \begin{pmatrix} Z_1 & R_5 & T_1 & T_2 \\ * & Z_2 + R_5 & T_3 & T_4 \\ * & * & Z_1 & R_7 \\ * & * & * & Z_2 + R_7 \end{pmatrix} \begin{pmatrix} \Lambda_3(k) \\ y(k - \alpha(k)) - y(k - \alpha_M) \\ \Lambda_4(k) \\ y(k - \sigma_m) - y(k - \sigma(k)) \end{pmatrix} \\
& + \Xi^T(k) \theta_6 \Xi(k) - \begin{pmatrix} \Lambda_7(k) \\ x(k - \beta(k)) - x(k - \beta_M) \\ \Psi_8(k) \\ x(k - \beta_m) - x(k - \beta(k)) \end{pmatrix}^T \\
& \times \begin{pmatrix} Z_3 & R_6 & T_5 & T_6 \\ * & Z_4 + R_6 & T_7 & T_8 \\ * & * & Z_3 & R_8 \\ * & * & * & Z_4 + R_8 \end{pmatrix} \begin{pmatrix} \Lambda_7(k) \\ x(k - \beta(k)) - x(k - \beta_M) \\ \Lambda_8(k) \\ x(k - \beta_m) - x(k - \beta(k)) \end{pmatrix} \\
& \leq \Upsilon^T(k) [\varphi_6 + \varphi_7] \Upsilon(k) \\
& + \Xi^T(k) [\theta_6 + \theta_7] \Xi(k).
\end{aligned} \tag{33}$$

From Condition 2, we get

$$\begin{pmatrix} x(k) \\ \hat{f}(x(k)) \end{pmatrix}^T \begin{pmatrix} D_1 \zeta_1 & -D_2 \zeta_1 \\ -D_2 \zeta_1 & \zeta_1 \end{pmatrix} \begin{pmatrix} x(k) \\ \hat{f}(x(k)) \end{pmatrix} \leq 0, \tag{34}$$

$$\begin{pmatrix} x(k - \beta(k)) \\ \hat{g}(x(k - \beta(k))) \end{pmatrix}^T \begin{pmatrix} E_1 \zeta_2 & -E_2 \zeta_2 \\ -E_2 \zeta_2 & \zeta_2 \end{pmatrix} \begin{pmatrix} x(k - \gamma(k)) \\ \hat{g}(x(k - \beta(k))) \end{pmatrix} \leq 0, \tag{35}$$

$$\begin{pmatrix} y(k) \\ f(y(k)) \end{pmatrix}^T \begin{pmatrix} F_1 \zeta_3 & -F_2 \zeta_3 \\ -F_2 \zeta_3 & \zeta_3 \end{pmatrix} \begin{pmatrix} y(k) \\ f(y(k)) \end{pmatrix} \leq 0, \tag{36}$$

$$\begin{pmatrix} y(k - \alpha(k)) \\ g(y(k - \alpha(k))) \end{pmatrix}^T \begin{pmatrix} G_1 \zeta_4 & -G_2 \zeta_4 \\ -G_2 \zeta_4 & \zeta_4 \end{pmatrix} \begin{pmatrix} y(k - \alpha(k)) \\ g(y(k - \alpha(k))) \end{pmatrix} \leq 0, \tag{37}$$

where

$$\zeta_1 = \text{diag} \{ \zeta_{11}, \zeta_{12}, \dots, \zeta_{1n} \};$$

$$\zeta_2 = \text{diag} \{ \zeta_{21}, \zeta_{22}, \dots, \zeta_{2n} \}$$

$$\zeta_3 = \text{diag} \{ \zeta_{31}, \zeta_{32}, \dots, \zeta_{3n} \};$$

$$\zeta_4 = \text{diag} \{ \zeta_{41}, \zeta_{42}, \dots, \zeta_{4n} \}.$$

$$F_1 = \text{diag} \{ F_1^- F_1^+, F_2^- F_2^+, \dots, F_n^- F_n^+ \};$$

$$F_2 = \text{diag} \{ F_1^- F_1^+, F_2^- F_2^+, \dots, F_n^- F_n^+ \}$$

$$\begin{aligned} D_1 &= \text{diag}\{D_1^- D_1^+, D_2^- D_2^+, \dots, D_n^- D_n^+\}; \\ D_2 &= \text{diag}\{D_1^- D_1^+, D_2^- D_2^+, \dots, D_n^- D_n^+\}; \\ E_1 &= \text{diag}\{E_1^- E_1^+, E_2^- E_2^+, \dots, E_n^- E_n^+\}; \\ E_2 &= \text{diag}\{E_1^- E_1^+, E_2^- E_2^+, \dots, E_n^- E_n^+\}; \\ G_1 &= \text{diag}\{G_1^- G_1^+, G_2^- G_2^+, \dots, G_n^- G_n^+\}; \\ G_2 &= \text{diag}\{G_1^- G_1^+, G_2^- G_2^+, \dots, G_n^- G_n^+\} \end{aligned}$$

We know that, $\mu(k) = x(k+1) - x(k)$, and $v(k) = y(k+1) - y(k)$, and we attain the following zero equations by introducing relaxation matrices as Y_1, Y_2, Y_3, Y_4 .

$$\begin{aligned} 2(\mu^T(k)Y_1^T + x^T(k)Y_2^T) &\left[Lx(k - \varrho(k)) + Mf(y(k)) \right. \\ &\left. + Ng(y(k - \alpha(k))) - x(k) - \mu(k) \right] = 0, \end{aligned} \quad (38)$$

$$\begin{aligned} 2(v^T(k)Y_3^T + y^T(k)Y_4^T) &\left[Ay(k - \rho(k)) + B\hat{f}(y(k)) \right. \\ &\left. + Cg(y(k - \beta(k))) - y(k) - v(k) \right] = 0. \end{aligned} \quad (39)$$

By combining the above equations, we acquired

$$\begin{aligned} \Delta V(k) &\leq \Upsilon^T(k) \left[\sum_{r=1}^{10} \varphi_r \right] \Upsilon(k) \\ &+ \Xi^T(k) \left[\sum_{r=1}^{10} \theta_r \right] \Xi(k) < 0. \end{aligned} \quad (40)$$

Let $\xi^* = \xi_{\max}(\Psi) < 0$ and $\lambda^* = \lambda_{\max}(\chi) < 0$. Then we easily achieve

$$\begin{aligned} \Delta V(k) &\leq \xi^* [||x(k)||^2] + \lambda^* [||y(k)||^2], \\ x(k) &\neq 0, \quad y(k) \neq 0. \end{aligned} \quad (41)$$

Put $\kappa = \max(\lambda^*, \xi^*)$

$$\Delta V(k) \leq \kappa [||x(k)||^2 + ||y(k)||^2] \quad (42)$$

Therefore, the neural networks (3) are asymptotically stable. $\square$

Remark 3.2 Leakage delay or forgetting term sometimes leads worse dynamical behavior of neural networks system. very few results are proposed on neural networks with leakage delay in discrete time. The authors in [40, 46] considered the leakage term as constant delay. But in our proposed work the leakage delay is considered as time-varying.

Next, we are in position to prove the impulsive effect of NNs (3).

From the definition of the operator Δ , one can observe that

$$\begin{aligned} \Delta V(k, x(k), y(k)) &= V(k+1, x(k+1), y(k+1)) \\ &\quad - V(k, x(k), y(k)) < 0 \end{aligned}$$

Then there exists a positive constant $\epsilon \leq 1$ such that $V(k+1, x(k+1), y(k+1)) \leq (1-\epsilon)V(k, x(k), y(k))$, $k \neq k_r$

Then

$$\begin{aligned} V(k_0+1, x(k_0+1), y(k_0+1)) &\leq (1-\epsilon) \\ V(k_0, x(k_0), y(k_0)) &V(k_0+2, x(k_0+2), y(k_0+2)) \\ &\leq (1-\epsilon)^2 V(k_0, x(k_0), y(k_0)) \\ &\vdots \\ &\vdots \\ V(k_1, x(k_1), y(k_1)) &= V(k_0+k_1-k_0, x(k_0 \\ &\quad +k_1-k_0), y(k_0+k_1-k_0)) \end{aligned}$$

$$\leq (1-\epsilon)^{k_1-k_0} V(k_0, x(k_0), y(k_0))$$

Similarly,

$$\begin{aligned} V(k_2, x(k_2), y(k_2)) &= V(k_1+k_2-k_1, \\ &\quad x(k_1+k_2-k_1), y(k_1+k_2-k_1)) \\ &\leq (1-\epsilon)^{k_2-k_1} V(k_1, x(k_1), y(k_1)) \\ &\leq (1-\epsilon)^{k_2-k_1} (1-\epsilon)^{k_1-k_0} V(k_0, x(k_0), y(k_0)) \\ &= (1-\epsilon)^{k_2-k_0} V(k_0, x(k_0), y(k_0)) \end{aligned} \quad (43)$$

and hence

$$\begin{aligned} V(k_2, x(k), y(k)) &\leq (1-\epsilon)^{k-k_r} \\ V(k_r, x(k_r), y(k_r)) &, k \in [k_r, k_{r+1}). \end{aligned} \quad (44)$$

Here, we note that

$$\begin{aligned} &V(k, x(k_r), y(k_r)) \\ &= \left(\begin{array}{c} x(k_r^-) \\ \sum_{i=k_r^- - \varrho_M}^{k_r^- - \varrho_m - 1} x(i) \\ \sum_{i=k_r^- - \beta_M}^{k_r^- - \beta_m - 1} x(i) \end{array} \right)^T \left(\begin{array}{ccc} Q_1 & Q_2 & Q_3 \\ * & Q_4 & Q_5 \\ * & * & Q_6 \end{array} \right) \end{aligned}$$

$$\begin{aligned}
& \times \begin{pmatrix} x(k_r^-) \\ \sum_{i=k_r^- - \varrho_M}^{k_r^- - \varrho_M - 1} x(i) \\ \sum_{i=k_r^- - \beta_M}^{k_r^- - \beta_M - 1} x(i) \end{pmatrix} + \begin{pmatrix} y(k_r^-) \\ \sum_{j=k_r^- - \rho_M}^{k_r^- - \rho_M - 1} y(j) \\ \sum_{j=k_r^- - \alpha_M}^{k_r^- - \alpha_M - 1} y(j) \end{pmatrix}^T \\
& \times \begin{pmatrix} P_1 & P_2 & P_3 \\ * & P_4 & P_5 \\ * & * & P_6 \end{pmatrix} \times \begin{pmatrix} y(k_r^-) \\ \sum_{j=k_r^- - \rho_M}^{k_r^- - \rho_M - 1} y(j) \\ \sum_{j=k_r^- - \alpha_M}^{k_r^- - \alpha_M - 1} y(j) \end{pmatrix} \\
& + \sum_{i=k_r^- - \varrho(k_r^-)}^{k_r^- - 1} x^T(i) Q_7 x(i) \\
& + \sum_{j=-\varrho_M+1}^{-\varrho_M} \sum_{i=k_r^-+j}^{k_r^- - 1} x^T(i) Q_7 x(i) \\
& + \sum_{j=k_r^- - \rho(k_r^-)}^{k_r^- - 1} y^T(j) P_7 y(j) \\
& + \sum_{i=-\rho_M+1}^{-\rho_M} \sum_{j=k_r^-+i}^{k_r^- - 1} y^T(j) P_7 y(j) \\
& + \sum_{j=k_r^- - \alpha(k_r^-)}^{k_r^- - 1} y^T(j) P_8 y(j) \\
& + \sum_{i=-\alpha_M+1}^{-\alpha_M} \sum_{j=k_r^-+i}^{k_r^- - 1} y^T(j) P_8 y(j) \\
& + \sum_{i=k_r^- - \beta(k_r^-)}^{k_r^- - 1} x^T(i) Q_8 x(i) \\
& + \sum_{j=-\beta_M+1}^{-\beta_M} \sum_{i=k_r^-+j}^{k_r^- - 1} x^T(i) Q_8 x(i). \\
& + \varrho_1 \sum_{i=-\varrho_M}^{-\varrho_M - 1} \sum_{j=k_r^-+i}^{k-1} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix}^T \begin{pmatrix} O_1 & 0 \\ 0 & O_2 \end{pmatrix} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix} \\
& + \rho_1 \sum_{j=-\rho_M}^{-\rho_M - 1} \sum_{i=k_r^-+j}^{k_r^- - 1} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix}^T \begin{pmatrix} O_3 & 0 \\ 0 & O_4 \end{pmatrix} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix} \\
& + \alpha_1 \sum_{i=-\alpha_M}^{-\alpha_M - 1} \sum_{j=k_r^-+i}^{k_r^- - 1} \begin{pmatrix} y(j) \\ v(j) \end{pmatrix}^T \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \\
& \times \begin{pmatrix} y(j) \\ v(j) \end{pmatrix} + \beta_1 \sum_{j=-\beta_M}^{-\beta_M - 1} \sum_{i=k_r^-+j}^{k_r^- - 1}
\end{aligned}$$

$$\begin{aligned}
& \times \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix}^T \begin{pmatrix} Z_3 & 0 \\ 0 & Z_4 \end{pmatrix} \begin{pmatrix} x(i) \\ \mu(i) \end{pmatrix} \\
& = (1 + k_r)^2 (1 + \omega_r)^2 V(k_r^-, x(k_r^-), y(k_r^-)).
\end{aligned} \tag{45}$$

By applying Eqs. (44) and (45) successively in each interval of $[k_r, k_{r+1})$, the following equations yield:
For $k \in [k_0, k_1)$,

$$\begin{aligned}
& V(k, x(k), y(k)) \\
& \leq (1 - \epsilon)^k V(k_0, x(k_0), y(k_0)) \\
& \text{and } V(k_1, x(k_1), y(k_1)) \\
& = (1 + \mathcal{K}_1)^2 (1 + \omega_1)^2 V(k_1^-, x(k_1^-), y(k_1^-)) \\
& \leq (1 + \mathcal{K}_1)^2 (1 + \omega_1)^2 V(k_1^-, x(k_1^-), y(k_1^-))
\end{aligned}$$

For $k \in [k_1, k_2)$,

$$\begin{aligned}
& V(k, x(k), y(k)) \\
& \leq (1 - \epsilon)^{k-k_1} V(k_1, x(k_1), y(k_1)) \\
& \leq (1 + \mathcal{K}_1)^2 (1 + \omega_1)^2 (1 - \epsilon)^{k-k_0} V(k_0, x(k_0), y(k_0)) \\
& \text{and } V(k_2, x(k_2), y(k_2)) \\
& = (1 + \mathcal{K}_2)^2 (1 + \omega_2)^2 V(k_2^-, x(k_2^-), y(k_2^-)) \\
& \leq (1 + \mathcal{K}_1)^2 (1 + \mathcal{K}_2)^2 (1 + \omega_1)^2 (1 + \omega_2)^2 (1 - \epsilon)^{k_2-k_0} \\
& \quad \times V(k_0^-, x(k_0^-), y(k_0^-))
\end{aligned}$$

For $k \in [k_2, k_3)$,

$$\begin{aligned}
& V(k, x(k), y(k)) \\
& \leq (1 - \epsilon)^{k-k_2} V(k_2, x(k_2), y(k_2)) \\
& \leq (1 + \mathcal{K}_1)^2 (1 + \mathcal{K}_2)^2 (1 + \omega_1)^2 \\
& \quad \times (1 + \omega_2)^2 (1 - \epsilon)^{k-k_0} \\
& \quad \times V(k_0, x(k_0), y(k_0)) \\
& \text{and } V(k_3, x(k_3), y(k_3)) \\
& = (1 + \mathcal{K}_3)^2 (1 + \omega_3)^2 \\
& \quad \times V(k_3^-, x(k_3^-), y(k_3^-)) \\
& \leq (1 + \mathcal{K}_1)^2 (1 + \mathcal{K}_2)^2 \\
& \quad \times (1 + \mathcal{K}_3)^2 (1 + \omega_1)^2 \\
& \quad \times (1 + \omega_2)^2 (1 + \omega_3)^2 (1 - \epsilon)^{k_3-k_0}
\end{aligned}$$

$$V(k_0^-, x(k_0^-), y(k_0^-))$$

Therefore by induction, for any $[k_r, k_{r+1})$, ($r = 0, 1, 2, \dots$) we obtain

$$\begin{aligned} V(k, x(k), y(k)) &\leq (1 - \epsilon)^{k-k_r} V(k_r, x(k_r), y(k_r)) \\ &\leq \left[\prod_{i=1}^r (1 + \mathcal{K}_i)^2 (1 + \omega_i)^2 \right] (1 - \epsilon)^k V(k_0, x(k_0), y(k_0)) \\ &= V(k_0, x(k_0), y(k_0)) \\ &\quad \times \exp \left\{ 2 \sum_{i=1}^m \ln(k + \mathcal{K}_i)(1 + \omega_i) + k \ln(1 - \epsilon) \right\} \\ &\leq V(k_0, x(k_0), y(k_0)) \exp \{ \vartheta(k) \}. \end{aligned}$$

Remark 3.3 In our paper, some zero inequalities and convex combination technique are approached. Because convex combination lemma helps to reduce the decision variables in LMI, which is the relevance lemma of Jensen's inequality and zero inequalities help to reduce the conservatism of stability criterion than the existing literature.

Remark 3.4 In [22], Guo et al. delivered the exponential stability analysis of discrete-time BAM neural networks with uncertain parameters where the uncertainties are taken in the form of linear fractional form. Consequently, by the aid of some novel Lyapunov–Krasovskii functional, the stability in exponential sense is investigated for BAM neural networks with usual robustness in [48]. Different from the above literature, the uncertain parameters designed in this research are assumed to be randomly occurring type. By this consideration in our addressed BAM neural networks, one can obtain the allowable upper bounds of discrete-time delay are maximum when compared with the above-mentioned literature [22, 48].

4 Randomly occurring uncertain neural networks

The following neural networks (46) with randomly occurring uncertain parameters are the extension of NNs (3).

Now the remodified neural networks of (3) are as follows:

$$\begin{aligned} x(k+1) &= \left[L + \iota(k) \Delta L(k) \right] x(k - \varrho(k)) \\ &\quad + \left[M + \pi(k) \Delta M(k) \right] f(y(k)) \end{aligned}$$

$$\begin{aligned} &+ \left[N + \nu(k) \Delta N(k) \right] g(y(k - \alpha(k))) \\ x(k_r) &= x(k_r^-) + e_r x(k_r^-) \\ y(k+1) &= \left[A + \check{\iota}(k) \Delta A(k) \right] y(k - \rho(k)) \\ &\quad + \left[B + \check{\pi}(k) \Delta B(k) \right] \hat{f}(x(k)) \\ &\quad + \left[C + \check{\nu}(k) \Delta C(k) \right] \hat{g}(x(k - \beta(k))) \\ y(k_r) &= y(k_r^-) + \hat{e}_r y(k_r^-). \end{aligned} \quad (46)$$

where $\Delta L(k)$, $\Delta M(k)$, $\Delta N(k)$, $\Delta A(k)$, $\Delta B(k)$, $\Delta C(k)$ are the uncertain parameters which are defined as

$$\begin{aligned} &\begin{bmatrix} \Delta L(k) & \Delta M(k) & \Delta N(k) & \Delta A(k) & \Delta B(k) & \Delta C(k) \end{bmatrix} \\ &= XJ(k) \begin{bmatrix} W_l & W_m & W_n & W_a & W_b & W_c \end{bmatrix} \end{aligned} \quad (47)$$

From above, X , W_l , W_m , W_n , W_a , W_b , W_c are known constant matrices and $J(k)$ is the unknown time-varying matrix with

$$J^T(k)J(k) \leq I \quad (48)$$

where I be the identity matrix. $\iota(k)$, $\delta(k)$, $\nu(k)$, $\check{\iota}(k)$, $\check{\pi}(k)$, $\check{\nu}(k)$ are the stochastic variables for the random nature of uncertain parameters. Each of the parameters is independent mutually with Bernoulli-distributed white noise sequences which taking the values 0 and 1. The probabilities are as follows:

$$\begin{aligned} \text{prob}\{\iota(k) = 1\} &= \iota, \quad \text{prob}\{\iota(k) = 0\} = 1 - \iota \\ \text{prob}\{\pi(k) = 1\} &= \pi, \quad \text{prob}\{\delta(k) = 0\} = 1 - \pi \\ \text{prob}\{\nu(k) = 1\} &= \nu, \quad \text{prob}\{\nu(k) = 0\} = 1 - \nu \\ \text{prob}\{\check{\iota}(k) = 1\} &= \check{\iota}, \quad \text{prob}\{\check{\iota}(k) = 0\} = 1 - \check{\iota} \\ \text{prob}\{\check{\pi}(k) = 1\} &= \check{\pi}, \quad \text{prob}\{\check{\delta}(k) = 0\} = 1 - \check{\pi} \\ \text{prob}\{\check{\nu}(k) = 1\} &= \check{\nu}, \quad \text{prob}\{\check{\nu}(k) = 0\} = 1 - \check{\nu} \end{aligned}$$

With the help of (47) and (48), system (46) can be written as:

$$\begin{aligned} x(k+1) &= Lx(k - \varrho(k)) + Mf(y(k)) \\ &\quad + Ng(y(k - \alpha(k))) + X\zeta(k), \\ \zeta(k) &= J(k)\varphi(k), \\ \varphi(k) &= \iota(k)W_lx(k - \varrho(k)) + \pi(k)W_mf(y(k)) \\ &\quad + \nu(k)W_ng(y(k - \alpha(k))), \\ x(s) &= \varpi(s), \quad s = -\omega, -\omega + 1, \dots, 0 \end{aligned}$$

$$\begin{aligned}
x(k_r) &= x(k_r^-) + e_r x(k_r^-) \\
y(k+1) &= Ay(k - \rho(k)) + B\hat{f}(x(k)) \\
&\quad + C\hat{g}(x(k - \beta(k))) + X\zeta^*(k), \\
\zeta^*(k) &= J(k)\wp^*(k), \\
\wp^*(k) &= \check{\imath}(k)W_a y(k - \rho(k)) + \check{\pi}(k)W_b \hat{f}(x(k)) \\
&\quad + \check{\upsilon}(k)W_c g(x(k - \beta(k))), \\
y(t) &= \tau(t), t = -\delta, -\delta + 1, \dots, 0, \\
y(k_r) &= y(k_r^-) + \hat{e}_r y(k_r^-). \tag{49}
\end{aligned}$$

Theorem 4.1 Suppose that Condition 2 holds, NNs (49) are asymptotically stable, if $\exists$ symmetric matrices $Q_i > 0$ ($i = 1$ to 8), and $P_j > 0$ ($j = 1$ to 8), $O_1 > 0$, $O_2 > 0$, $O_3 > 0$, $O_4 > 0$, $Z_1 > 0$, $Z_2 > 0$, $Z_3 > 0$, $Z_4 > 0$, $H_j > 0$, $T_j > 0$, and $H_i > 0$, $T_i > 0$, diagonal matrices $\zeta_1 > 0$, $\zeta_2 > 0$, $\zeta_3 > 0$, $\zeta_4 > 0$, a scalars ε , ε^* and matrices R_i ($i = 1$ to 4), R_j ($j = 5$ to 8) and Y_1, Y_2, Y_3, Y_4 of appropriate dimensions and positive constant ϵ ($\epsilon \leq 1$), such that the following LMIs hold.

$$(i) \quad \sum_{i=1}^m \ln(1 + k_i)^2 (1 + \omega_i)^2 + k \ln(1 - \epsilon) \leq \vartheta(k)$$

for any $k \in [k_r, k_{r+1})$, then

$$\lim_{k \rightarrow +\infty} \vartheta(k) = +\infty. \tag{50}$$

$$(ii) \quad \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} > 0, \tag{51}$$

$$(iii) \quad \begin{pmatrix} H_5 & H_6 \\ H_7 & H_8 \end{pmatrix} > 0, \tag{52}$$

$$(iv) \quad \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} > 0, \tag{53}$$

$$(v) \quad \begin{pmatrix} T_5 & T_6 \\ T_7 & T_8 \end{pmatrix} > 0, \tag{54}$$

$$(vi) \quad \begin{pmatrix} \sum_{r=1}^{12} \varphi_r & \varepsilon \Gamma^T \\ * & -\varepsilon I \end{pmatrix} < 0, \tag{55}$$

$$(vii) \quad \begin{pmatrix} \sum_{r=1}^{12} \theta_r & \varepsilon^* \Gamma_1^T \\ * & -\varepsilon^* I \end{pmatrix} < 0 \tag{56}$$

where

$$\varphi_{11} = 2(e_8^T Y_1^T + e_1^T Y_2^T) X e_{15},$$

$$\varphi_{12} = -e_{15}^T \varepsilon I e_{15},$$

$$\theta_{11} = 2(e_8^{*T} Y_1^T + e_1^{*T} Y_2^T) X e_{15}^*,$$

$$\theta_{12} = -e_{15}^{*T} \varepsilon I e_{15}^*,$$

$$\Gamma = \begin{bmatrix} 0 & 0 & \iota W_l & 0 & 0 & 0 & 0 & \pi W_m & \upsilon W_n & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma_1 = \begin{bmatrix} 0 & 0 & \check{\imath} W_a & 0 & 0 & 0 & 0 & \check{\pi} W_b & \check{\upsilon} W_c & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof Replacing (3) with (49) and taking mathematical expectation in (49), we acquired that

$$\begin{aligned}
\mathbb{E}\{\Delta V(k)\} &\leq \mathbb{E}\left\{\tilde{\Upsilon}^T(k) \sum_{r=1}^{11} \varphi_r \tilde{\Upsilon}(k)\right\} \\
&\quad + \mathbb{E}\left\{\tilde{\Xi}^T(k) \sum_{r=1}^{11} \theta_r \tilde{\Xi}(k)\right\} < 0 \tag{57}
\end{aligned}$$

From Eq. (48), we get

$$\begin{aligned}
\zeta^T(k) \zeta(k) &\leq \wp^T(k) \wp(k) \\
\zeta^{*T}(k) \zeta^*(k) &\leq \wp^{*T}(k) \wp^*(k)
\end{aligned}$$

Then there are scalars $\varepsilon, \varepsilon^* > 0$ satisfies the inequalities as follows:

$$\varepsilon \left[\tilde{\Upsilon}^T(k) \Gamma^T \Gamma \tilde{\Upsilon}(k) - \zeta^T(k) \zeta(k) \right] \geq 0, \tag{58}$$

$$\varepsilon^* \left[\tilde{\Xi}^T(k) \Gamma_1^T \Gamma_1 \tilde{\Xi}(k) - \zeta^{*T}(k) \zeta^*(k) \right] \geq 0 \tag{59}$$

where $\tilde{\Upsilon}^T = \begin{bmatrix} \Upsilon^T(k) & \zeta^T(k) \end{bmatrix}$ and $\tilde{\Xi}^T = \begin{bmatrix} \Xi^T(k) & \zeta^{*T}(k) \end{bmatrix}$.

Equations (58) and (59) are added with (57), and applying S-procedure, we have

$$\begin{aligned}
\mathbb{E}\{\Delta V(k)\} &\leq \mathbb{E}\left\{\tilde{\Upsilon}^T(k) \left(\sum_{r=1}^{12} \varphi_r + \varepsilon \Gamma^T \Gamma \right) \tilde{\Upsilon}(k)\right\} \\
&\quad + \mathbb{E}\left\{\tilde{\Xi}^T(k) \left(\sum_{r=1}^{12} \theta_r + \varepsilon^* \Gamma_1^T \Gamma_1 \right) \tilde{\Xi}(k)\right\} < 0.
\end{aligned}$$

By the utilization of Lemma 2.2, we obtain the proof of Theorem 4.1. $\square$

Remark 4.2 If the leakage term in NNs (3) disappears (i.e., $\varrho(k), \rho(k) = 0$), then we get

$$\begin{aligned}
x(k+1) &= Lx(k) + Mf(y(k)) + Ng(y(k - \alpha(k))) \\
x(k_r) &= x(k_r^-) + e_r x(k_r^-) \\
y(k+1) &= Ay(k) + B\hat{f}(x(k)) + C\hat{g}(x(k - \beta(k))) \\
y(k_r) &= y(k_r^-) + \hat{e}_r y(k_r^-). \tag{60}
\end{aligned}$$

The following Corollary helps to check the asymptotic stability of the concerned NNs (60).

Corollary 4.3 Suppose that Condition 2 holds, NNs (60) are asymptotically stable, if $\exists$ symmetric matrices $Q_i > 0$ ($i = 1$ to 8), and $P_j > 0$ ($j = 1$ to 8), $Z_1 > 0$, $Z_2 > 0$, $Z_3 > 0$, $Z_4 > 0$, $T_j > 0$, and $T_i > 0$, diagonal matrices $\zeta_1 > 0$, $\zeta_2 > 0$, $\zeta_3 > 0$, $\zeta_4 > 0$, and matrices R_i ($i = 1$ to 4), R_j ($j = 5$ to 8) and Y_1, Y_2, Y_3, Y_4 of appropriate dimensions and positive constant ϵ ($\epsilon \leq 1$), such that the following LMIs hold.

$$(i) \sum_{i=1}^m \ln(1 + k_i)^2 (1 + \omega_i)^2 + k \ln(1 - \epsilon) \leq \vartheta(k),$$

for any $k \in [k_r, k_{r+1})$, then

$$\lim_{k \rightarrow +\infty} \vartheta(k) = +\infty, \quad (61)$$

$$(ii) \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} > 0, \quad (62)$$

$$(iii) \begin{pmatrix} T_5 & T_6 \\ T_7 & T_8 \end{pmatrix} > 0, \quad (63)$$

$$(iv) \Psi = \sum_{r=1}^5 \aleph_r < 0, \quad (64)$$

$$(v) \chi = \sum_{r=1}^5 \hbar_r < 0. \quad (65)$$

where

$$\begin{aligned} \aleph_1 &= e_5^T Q_1 e_5 + 2e_5^T Q_1 e_1 \\ &\quad + 2e_5^T Q_2 (e_3 - e_4) + 2e_5^T Q_2 (e_8 + e_9) \\ &\quad + 2e_1^T Q_2 (e_3 - e_4) (e_3 - e_4)^T \\ &\quad + Q_3 (e_3 - e_4) + 2(e_3 - e_4)^T Q_3 (e_8 + e_9), \\ \aleph_2 &= (\beta_1 + 1) e_1^T Q_4 e_1 - e_3^T Q_4 e_3, \\ \aleph_3 &= \beta_1^2 e_1^T Z_1 e_1 + \beta_1^2 e_5^T Z_2 e_5 \\ &\quad + \beta_1 e_3^T (R_5 - R_7) e_3 - \beta_1 e_4^T R_5 e_4 \\ &\quad + \beta_1 e_2^T R_7 e_2 - e_8^T Z_1 e_8 - 2e_8^T R_5 (e_3 - e_4) \\ &\quad - 2e_8^T T_1 e_9 - 2e_8^T T_2 (e_2 - e_3) \\ &\quad - (e_3 - e_4)^T (Z_2 + R_5) (e_3 - e_4) \\ &\quad - 2(e_3 - e_4)^T T_3 e_9 - 2(e_3 - e_4)^T \\ &\quad \times T_3 (e_2 - e_3) - e_9^T Z_1 e_9 \\ &\quad - 2e_9 R_7 (e_2 - e_3) - (e_2 - e_3)^T \end{aligned}$$

$$\begin{aligned} &\quad (Z_2 + R_7) (e_2 - e_3), \\ \aleph_4 &= 2(e_5^T Y_1^T + e_1^T Y_2^T) (Le_1 \\ &\quad + Me_6 + Ne_7 - e_1 - e_5), \\ \aleph_5 &= -e_1^T (F_1 \zeta_1) e_1 + 2e_1^T (F_2 \zeta_1) e_6 - e_6^T \zeta_1 e_6 \\ &\quad - e_3^T G_1 \zeta_2 e_3 + 2e_3^T G_2 \zeta_2 e_7 - e_7^T \zeta_2 e_7, \\ \hbar_1 &= e_5^{*T} P_1 e_5^* + 2e_5^{*T} P_1 e_1^* + 2e_5^{*T} P_2 (e_3^* - e_4^*) \\ &\quad + 2e_5^{*T} P_2 (e_8^* + e_9^*) + 2e_1^{*T} P_2 (e_3^* - e_4^*) \\ &\quad + (e_3^* - e_4^*)^T P_3 (e_3^* - e_4^*) \\ &\quad + 2(e_3^* - e_4^*)^T P_3 (e_8^* + e_9^*), \\ \hbar_2 &= (\alpha_1 + 1) e_1^{*T} P_4 e_1^* \\ &\quad - e_3^{*T} P_4 e_3^*, \\ \hbar_3 &= \alpha_1^2 e_1^{*T} Z_3 e_1^* + \alpha_1^2 e_5^{*T} Z_4 e_5^* \\ &\quad + \alpha_1 e_3^{*T} (R_6 - R_8) e_3^* - \alpha_1 e_4^{*T} R_6 e_4^* \\ &\quad + \alpha_1 e_2^{*T} R_8 e_2^* - e_8^{*T} Z_3 e_8^* - 2e_8^{*T} \\ &\quad \times R_6 (e_3^* - e_4^*) - 2e_8^{*T} T_5 e_9^* 2e_8^{*T} T_6 (e_2^* - e_3^*) \\ &\quad - (e_3^* - e_4^*)^T (Z_3 + R_6) (e_3^* - e_4^*) \\ &\quad - 2(e_3^* - e_4^*)^T T_7 e_9^* - 2(e_3^* - e_4^*)^T T_7 (e_2^* - e_3^*) \\ &\quad - e_9^{*T} Z_3 e_9^* \\ &\quad - 2e_9^* R_8 (e_2^* - e_3^*) - (e_2^* - e_3^*)^T \\ &\quad \times (Z_4 + R_8) (e_2^* - e_3^*), \\ \hbar_4 &= 2(e_5^{*T} Y_3^T + e_1^{*T} Y_4^T) (Ae_1^* \\ &\quad + Be_6^* + Ce_7^* - e_1^* - e_5^*), \\ \hbar_5 &= -e_1^{*T} (D_1 \zeta_3) e_1^* + 2e_1^{*T} (D_2 \zeta_3) e_6^* \\ &\quad - e_6^{*T} \zeta_3 e_6^* \\ &\quad - e_3^{*T} (E_1 \zeta_4) e_3^* \\ &\quad + 2e_3^{*T} (E_2 \zeta_4) e_7^* - e_7^{*T} \zeta_4 e_7^*. \end{aligned}$$

Proof The proof of this corollary follows from Theorem 3.1. $\square$

Remark 4.4 If the leakage term and impulse in NNs (3) disappears (i.e., $\varrho(k)$, $\rho(k) = 0$), then we get

$$\begin{aligned}x(k+1) &= Lx(k) + Mf(y(k)) + Ng(y(k - \alpha(k))) \\y(k+1) &= Ay(k) + B\hat{f}(x(k)) + C\hat{g}(x(k - \beta(k)))\end{aligned}\quad (66)$$

The following Corollary helps to check the stability of the neural networks system (66).

Corollary 4.5 Suppose that Condition 2 holds, NNs (66) are asymptotically stable, if $\exists$ symmetric matrices $Q_i > 0$ ($i = 1$ to 8), and $P_j > 0$ ($j = 1$ to 8), $Z_1 > 0$, $Z_2 > 0$, $Z_3 > 0$, $Z_4 > 0$, $T_j > 0$, and $T_i > 0$, diagonal matrices $\zeta_1 > 0$, $\zeta_2 > 0$, $\zeta_3 > 0$, $\zeta_4 > 0$, and matrices R_i ($i = 1$ to 4), R_j ($j = 5$ to 8) and Y_1, Y_2, Y_3, Y_4 of appropriate dimensions such that the following LMIs hold.

$$(i) \quad \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} > 0, \quad (67)$$

$$(ii) \quad \begin{pmatrix} T_5 & T_6 \\ T_7 & T_8 \end{pmatrix} > 0, \quad (68)$$

$$(iii) \quad \Psi = \sum_{r=1}^5 \aleph_r < 0, \quad (69)$$

$$(iv) \quad \chi = \sum_{r=1}^5 \hat{h}_r < 0. \quad (70)$$

where

$$\begin{aligned}\aleph_1 &= e_5^T Q_1 e_5 + 2e_5^T Q_1 e_1 \\&\quad + 2e_5^T Q_2 (e_3 - e_4) + 2e_5^T Q_2 (e_8 + e_9) \\&\quad + 2e_1^T Q_2 (e_3 - e_4) \\&\quad + (e_3 - e_4)^T Q_3 (e_3 - e_4) \\&\quad + 2(e_3 - e_4)^T Q_3 (e_8 + e_9), \\ \aleph_2 &= (\beta_1 + 1)e_1^T Q_4 e_1 - e_3^T Q_4 e_3, \\ \aleph_3 &= \beta_1^2 e_1^T Z_1 e_1 + \beta_1^2 e_5^T Z_2 e_5 \\&\quad + \beta_1 e_3^T (R_5 - R_7) e_3 - \beta_1 e_4^T R_5 e_4 \\&\quad + \beta_1 e_2^T R_7 e_2 - e_8^T Z_1 e_8 - 2e_8^T R_5 (e_3 - e_4) \\&\quad - 2e_8^T T_1 e_9 - 2e_8^T T_2 (e_2 - e_3) \\&\quad - (e_3 - e_4)^T (Z_2 + R_5) (e_3 - e_4) \\&\quad - 2(e_3 - e_4)^T T_3 e_9 - 2(e_3 - e_4)^T \\&\quad \times T_3 (e_2 - e_3) - e_9^T Z_1 e_9\end{aligned}$$

$$\begin{aligned}&\quad - 2e_9 R_7 (e_2 - e_3) - (e_2 - e_3)^T (Z_2 \\&\quad + R_7) (e_2 - e_3) Z_2 e_8, \\ \aleph_4 &= 2(e_5^T Y_1^T + e_1^T Y_2^T) (Le_1 + Me_6 + Ne_7 - e_1 - e_5), \\ \aleph_5 &= -e_1^T (F_1 \zeta_1) e_1 + 2e_1^T (F_2 \zeta_1) e_6 - e_6^T \zeta_1 e_6 - e_3^T G_1 \zeta_2 e_3 \\&\quad + 2e_3^T G_2 \zeta_2 e_7 - e_7^T \zeta_2 e_7, \\ \hat{h}_1 &= e_5^*{}^T P_1 e_5^* + 2e_5^*{}^T P_1 e_1^* + 2e_5^*{}^T P_2 (e_3^* - e_4^*) \\&\quad + 2e_5^*{}^T P_2 (e_8^* + e_9^*) + 2e_1^*{}^T P_2 (e_3^* - e_4^*) \\&\quad + (e_3^* - e_4^*)^T P_3 (e_3^* - e_4^*) \\&\quad + 2(e_3^* - e_4^*)^T P_3 (e_8^* + e_9^*), \\ \hat{h}_2 &= (\alpha_1 + 1)e_1^*{}^T P_4 e_1^* - e_3^*{}^T P_4 e_3^*, \\ \hat{h}_3 &= \alpha_1^2 e_1^*{}^T Z_3 e_1^* + \alpha_1^2 e_5^*{}^T Z_4 e_5^* \\&\quad + \alpha_1 e_3^*{}^T (R_6 - R_8) e_3^* - \alpha_1 e_4^*{}^T R_6 e_4^* \\&\quad + \alpha_1 e_2^*{}^T R_8 e_2^* - e_8^*{}^T Z_3 e_8^* - 2e_8^*{}^T \\&\quad \times R_6 (e_3^* - e_4^*) - 2e_8^*{}^T T_5 e_9^* 2e_8^*{}^T T_6 (e_2^* e_3^*) \\&\quad - (e_3^* - e_4^*)^T (Z_3 + R_6) (e_3^* - e_4^*) - 2(e_3^* - e_4^*)^T T_7 e_9^* \\&\quad - 2(e_3^* - e_4^*)^T T_7 (e_2^* - e_3^* - e_9^*{}^T Z_3 e_9^*) - 2e_9^*{}^T R_8 (e_2^* \\&\quad - e_3^*) - (e_2^* - e_3^*)^T (Z_4 + R_8) (e_2^* - e_3^*) Z_4 e_8^*, \\ \hat{h}_4 &= 2(e_5^*{}^T Y_3^T + e_1^*{}^T Y_4^T) (Ae_1^* + Be_6^* + Ce_7^* - e_1^* - e_5^*), \\ \hat{h}_5 &= -e_1^*{}^T (D_1 \zeta_3) e_1^* + 2e_1^*{}^T (D_2 \zeta_3) e_6^* \\&\quad - e_6^*{}^T \zeta_3 e_6^* - e_3^*{}^T (E_1 \zeta_4) e_3^* \\&\quad + 2e_3^*{}^T (E_2 \zeta_4) e_7^* - e_7^*{}^T \zeta_4 e_7^*.\end{aligned}$$

Proof The proof of Corollary 4.3 is omitted. Hence, it follows from Theorem 3.1. $\square$

5 Illustrative examples

In this section, we will present four numerical examples with their simulations to guarantee the superiority and validity of our theoretical results.

Example 5.1 Consider a two-dimensional impulsive discrete-time BAM neural networks with time-varying leakage delays. The parameters are as follows:

$$\begin{aligned}A &= \begin{pmatrix} 0.09 & 0 \\ 0 & 0.08 \end{pmatrix}, \quad B = \begin{pmatrix} 0.5 & 0.6 \\ 0.9 & 0.5 \end{pmatrix}, \\ C &= \begin{pmatrix} 0.01 & 0.6 \\ 0.5 & -0.55 \end{pmatrix},\end{aligned}$$

$$L = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.07 \end{pmatrix}, \quad N = \begin{pmatrix} 1.05 & 2.86 \\ 0.59 & 0.5 \end{pmatrix},$$

$$M = \begin{pmatrix} 0.4 & 0.8 \\ 0.9 & 1.05 \end{pmatrix},$$

$$I_m = -1, \quad J_m = -2,$$

$$\varrho(k) = \varrho = 2; \quad \rho(k) = \rho = 3;$$

The activation functions are taken as

$$f(y(k)) = g(y(k)) = \begin{pmatrix} \tanh(-0.2y) \\ \tanh(-0.5y) \end{pmatrix},$$

$$\hat{f}(x(k)) = \hat{g}(x(k)) = \begin{pmatrix} \tanh(-0.9x) \\ \tanh(-0.9x) \end{pmatrix}$$

Our main aim in this example is to estimate the maximum allowable upper bound delay α_M , β_M , for given lower bound α_m , β_m . By solving MATLAB LMI Toolbox, one can easily obtain the feasible solution for any time delay satisfying $0 < \alpha(k) < \alpha_M$ for any large finite value and $0 < \beta(k) < \alpha_M$ for any large finite value. For example, if we take $\alpha_m = \beta_m = 13$, $\alpha_M = \beta_M = 17$, $f_1^+ = 2$, $f_1^- = -4$, $f_2^+ = 1$, $f_2^- = -1$, $g_1^+ = 0.2$, $g_1^- = 0.2$, $g_2^+ = 0.2$, $g_2^- = 0.2$, $u_1^+ = 4$, $u_1^- = -0.2$, $u_2^+ = 4$, $u_2^- = 0.2$, $d_1^+ = 3$, $d_1^- = 0.2$, $d_2^+ = 3$, $d_2^- = -0.2$. with

$$F_1 = \begin{pmatrix} -8 & 0 \\ 0 & -1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.04 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 0.6 & 0 \\ 0 & -0.6 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1.6 & 0 \\ 0 & 1.4 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} -0.8 & 0 \\ 0 & 0.8 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1.9 & 0 \\ 0 & 2.1 \end{pmatrix},$$

Now we obtain the feasible solution as follows

$$P_1 = \begin{pmatrix} 292.8888 & -1.6270 \\ -1.6270 & 291.3657 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 271.1228 & -1.8430 \\ -1.8430 & 269.4223 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 272.0117 & -1.6260 \\ -1.6260 & 270.1301 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} 374.8426 & -3.5484 \\ -3.5484 & 367.0428 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 172.8637 & -1.4644 \\ -1.4644 & 170.3510 \end{pmatrix},$$

$$P_6 = \begin{pmatrix} 911.9019 & -6.1157 \\ -6.1157 & 903.5726 \end{pmatrix}$$

$$P_7 = \begin{pmatrix} 537.5761 & 5.3396 \\ 5.3396 & 368.9104 \end{pmatrix}, \quad P_8 = \begin{pmatrix} 681.5702 & -28.7999 \\ -28.7999 & 365.0425 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 288.9321 & 6.4160 \\ 6.4160 & 332.0618 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 268.5954 & 9.4023 \\ 9.4023 & 319.1253 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 265.0284 & -0.2672 \\ -0.2672 & 303.7453 \end{pmatrix},$$

$$Q_4 = \begin{pmatrix} 339.6602 & -38.3858 \\ -38.3858 & 359.9275 \end{pmatrix}$$

$$Q_5 = \begin{pmatrix} 159.4793 & 0.4181 \\ 0.4181 & 230.7959 \end{pmatrix}, \quad Q_6 = \begin{pmatrix} 962.0885 & -149.2273 \\ -149.2273 & 481.8919 \end{pmatrix},$$

$$Q_7 = \begin{pmatrix} 425.3142 & -35.9647 \\ -35.9647 & 474.0118 \end{pmatrix}$$

$$Q_8 = \begin{pmatrix} 110.9821 & -54.9147 \\ -54.9147 & 197.9181 \end{pmatrix}, \quad O_2 = \begin{pmatrix} 50.7973 & 0.5655 \\ 0.5655 & 54.6328 \end{pmatrix},$$

$$O_1 = \begin{pmatrix} 144.9491 & -4.2889 \\ -4.2889 & 139.9814 \end{pmatrix}$$

$$O_3 = \begin{pmatrix} 269.0453 & -4.0024 \\ -4.0024 & 289.3900 \end{pmatrix}, \quad O_4 = \begin{pmatrix} 51.5348 & 1.7405 \\ 1.7405 & 50.1377 \end{pmatrix},$$

$$H_1 = \begin{pmatrix} 145.1633 & 0.1545 \\ 0.1545 & 150.1215 \end{pmatrix}$$

$$H_2 = \begin{pmatrix} 702.6359 & 80.9577 \\ 80.9577 & 406.3829 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 380.3071 & 0.0000 \\ 0.0000 & 380.3071 \end{pmatrix},$$

$$H_4 = \begin{pmatrix} 374.7986 & -14.0906 \\ -14.0906 & 360.5113 \end{pmatrix}$$

$$H_5 = \begin{pmatrix} 150.5293 & -2.0946 \\ -2.0946 & 151.8571 \end{pmatrix}, \quad H_6 = \begin{pmatrix} 380.5555 & -14.4297 \\ -14.4297 & 394.1536 \end{pmatrix}$$

$$H_7 = \begin{pmatrix} 298.7321 & -24.2027 \\ -24.2027 & 331.7226 \end{pmatrix}$$

$$H_8 = \begin{pmatrix} 298.7321 & -24.2027 \\ -24.2027 & 331.7226 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} 112.3577 & 20.6775 \\ 20.6775 & 70.0531 \end{pmatrix},$$

$$Z_2 = \begin{pmatrix} 146.5759 & 38.9148 \\ 38.9148 & 74.3640 \end{pmatrix}$$

$$Z_3 = \begin{pmatrix} 1.4291 & -0.5006 \\ -0.5006 & 0.8734 \end{pmatrix}, \quad Z_4 = \begin{pmatrix} 350.1442 & -47.7362 \\ -47.7362 & 215.5953 \end{pmatrix},$$

$$T_1 = \begin{pmatrix} 269.6468 & -32.0684 \\ -32.0684 & 245.0229 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 264.3967 & -152.9522 \\ -152.9522 & 370.2717 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 380.5093 & 0.1144 \\ 0.1144 & 380.8204 \end{pmatrix},$$

$$T_4 = \begin{pmatrix} 220.5596 & -59.3386 \\ -59.3386 & 170.2364 \end{pmatrix}$$

$$T_5 = \begin{pmatrix} 320.0325 & -64.8638 \\ -64.8638 & 270.5913 \end{pmatrix}, \quad T_6 = \begin{pmatrix} 375.1138 & -6.9526 \\ -6.9526 & 374.0131 \end{pmatrix},$$

$$T_7 = \begin{pmatrix} 379.6538 & -0.2340 \\ -0.2340 & 379.6389 \end{pmatrix}$$

$$T_8 = \begin{pmatrix} 402.5238 & -9.3036 \\ -9.3036 & 306.2595 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 89.5992 & -0.0391 \\ -0.0391 & 93.3572 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 87.6563 & -1.9292 \\ -1.9292 & 90.0645 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 266.0215 & -152.5991 \\ -152.5991 & 380.5480 \end{pmatrix}, \quad R_4 = \begin{pmatrix} 290.5417 & 58.3600 \\ 58.3600 & 275.6008 \end{pmatrix},$$

$$R_5 = \begin{pmatrix} 784.4856 & 153.3889 \\ 153.3889 & 659.7415 \end{pmatrix}$$

$$R_6 = \begin{pmatrix} 715.2169 & 69.9024 \\ 69.9024 & 518.2362 \end{pmatrix}, \quad R_7 = \begin{pmatrix} 271.3480 & 107.7069 \\ 107.7069 & 121.4036 \end{pmatrix},$$

$$R_8 = \begin{pmatrix} 275.6604 & 144.2375 \\ 144.2375 & 169.5516 \end{pmatrix}$$

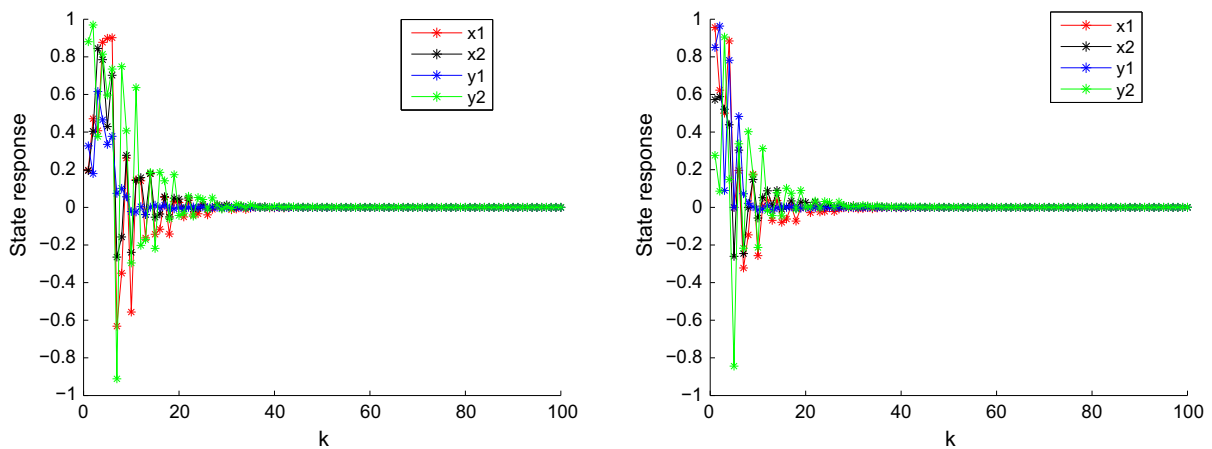


Fig. 1 State response of (3) with impulsive and non-impulsive effects

$$\begin{aligned}
 Y_1 &= \begin{pmatrix} 26.1885 & -40.4006 \\ -40.4006 & 62.3269 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 27.6678 & -42.2200 \\ -42.2200 & 64.4275 \end{pmatrix}, \\
 Y_3 &= \begin{pmatrix} 239.2988 & 122.3856 \\ 122.3856 & 174.2126 \end{pmatrix} \\
 Y_4 &= \begin{pmatrix} 239.2988 & 122.3856 \\ 122.3856 & 174.2126 \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} 79.9344 & 0 \\ 0 & 205.9890 \end{pmatrix}, \\
 \zeta_2 &= \begin{pmatrix} 63.4866 & 0 \\ 0 & 254.7480 \end{pmatrix} \\
 \zeta_3 &= \begin{pmatrix} 170.4501 & 0 \\ 0 & 216.2830 \end{pmatrix}, \quad \zeta_4 = \begin{pmatrix} 231.7904 & 0 \\ 0 & 165.9446 \end{pmatrix}
 \end{aligned}$$

Based on Theorem 3.1, we can attain that the neural networks system (3) with above given parameters are asymptotically stable with the help of Lyapunov–Krasovskii functionals, and state trajectories $x_1(k)$, $x_2(k)$, $y_1(k)$, $y_2(k)$ of above impulsive discrete-time BAM neural networks with time-varying leakage delays are depicted in Fig. 1.

Example 5.2 Consider a two-dimensional impulsive discrete-time BAM neural networks with randomly occurring uncertainties and time-varying leakage delays.

The following parameters help to check the feasibility of neural networks system (46)

$$\begin{aligned}
 A &= \begin{pmatrix} 0.09 & 0 \\ 0 & 0.08 \end{pmatrix}, \quad B = \begin{pmatrix} 0.5 & 0.6 \\ 0.9 & 0.5 \end{pmatrix}, \quad C = \begin{pmatrix} 0.01 & 0.6 \\ 0.5 & -0.55 \end{pmatrix}, \\
 L &= \begin{pmatrix} 0.05 & 0 \\ 0 & 0.07 \end{pmatrix}, \quad N = \begin{pmatrix} 1.05 & 2.86 \\ 0.59 & 0.5 \end{pmatrix}, \quad M = \begin{pmatrix} 0.4 & 0.8 \\ 0.9 & 1.05 \end{pmatrix} \\
 W_l &= \begin{pmatrix} 0.5 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}, \quad W_m = \begin{pmatrix} 0.6 & 0.7 \\ 0.8 & 0.9 \end{pmatrix}, \quad W_n = \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{pmatrix} \\
 W_a &= \begin{pmatrix} 0.9 & 0.4 \\ 0.6 & 0.3 \end{pmatrix}, \quad W_b = \begin{pmatrix} 0.8 & 0.7 \\ 0.6 & 0.5 \end{pmatrix}, \quad W_c = \begin{pmatrix} 0.8 & 0.9 \\ 0.6 & 0.7 \end{pmatrix} \\
 X &= \begin{pmatrix} 0.8 & 0 \\ 0 & 0.8 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 I_m &= -2, \quad J_m = -2, \\
 \varrho(k) &= \varrho = 5; \quad \rho(k) = \rho = 5; \\
 \delta &= 0.5; \quad \nu = 0.6; \quad \iota = 0.7; \\
 \check{\delta} &= 0.3; \quad \check{\nu} = 0.8; \quad \check{\iota} = 0.9;
 \end{aligned}$$

The activation functions are taken as

$$\begin{aligned}
 f(y(k)) &= g(y(k)) = \begin{pmatrix} \exp(-0.99y) \\ \exp(-0.99y) \end{pmatrix}, \\
 \hat{f}(x(k)) &= \hat{g}(x(k)) = \begin{pmatrix} \exp(-0.99x) \\ \exp(-0.99x) \end{pmatrix}
 \end{aligned}$$

Our main aim in this example is to estimate the maximum allowable upper bound delay α_M , β_M , for given lower bound α_m , β_m . By solving MATLAB LMI Toolbox, one can easily obtain the feasible solution for any time delay satisfying $0 < \alpha(k) < \alpha_M$ for any large finite value and $0 < \beta(k) < \beta_M$ for any large finite value. For example, if we take $\alpha_m = \beta_m = 7$, $\alpha_M = \beta_M = 13$, $f_1^+ = 2$, $f_1^- = -4$, $f_2^+ = 1$, $f_2^- = -1$, $g_1^+ = 0.2$, $g_1^- = 0.2$, $g_2^+ = 0.2$, $g_2^- = 0.2$, $u_1^+ = 4$, $u_1^- = -0.2$, $u_2^+ = 4$, $u_2^- = 0.2$, $d_1^+ = 3$, $d_1^- = 0.2$, $d_2^+ = 3$, $d_2^- = -0.2$. with

$$\begin{aligned}
 F_1 &= \begin{pmatrix} -8 & 0 \\ 0 & -1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \\
 G_1 &= \begin{pmatrix} 0.04 & 0 \\ 0 & 0.04 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \\
 D_1 &= \begin{pmatrix} 0.6 & 0 \\ 0 & -0.6 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1.6 & 0 \\ 0 & 1.4 \end{pmatrix}, \\
 E_1 &= \begin{pmatrix} -0.8 & 0 \\ 0 & 0.8 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1.9 & 0 \\ 0 & 2.1 \end{pmatrix},
 \end{aligned}$$

Now we obtain the feasible solution as follows:

$$\begin{aligned}
 P_1 &= \begin{pmatrix} 204.5727 & -1.1637 \\ -1.1637 & 203.4828 \end{pmatrix}, & P_2 &= \begin{pmatrix} 189.1772 & -1.3159 \\ -1.3159 & 187.9568 \end{pmatrix}, \\
 P_3 &= \begin{pmatrix} 189.8024 & -1.1654 \\ -1.1654 & 188.4585 \end{pmatrix} \\
 P_4 &= \begin{pmatrix} 262.5994 & -2.4950 \\ -2.4950 & 257.1018 \end{pmatrix}, & P_5 &= \begin{pmatrix} 119.8775 & -1.0533 \\ -1.0533 & 118.0974 \end{pmatrix}, \\
 P_6 &= \begin{pmatrix} 635.0379 & -4.3866 \\ -4.3866 & 629.1128 \end{pmatrix} \\
 P_7 &= \begin{pmatrix} 376.5259 & 3.7361 \\ 3.7361 & 258.4162 \end{pmatrix}, & P_8 &= \begin{pmatrix} 476.0092 & -21.5589 \\ -21.5589 & 252.0999 \end{pmatrix}, \\
 Q_1 &= \begin{pmatrix} 202.0052 & 4.6153 \\ 4.6153 & 232.2483 \end{pmatrix} \\
 Q_2 &= \begin{pmatrix} 187.6723 & 6.7504 \\ 6.7504 & 223.0553 \end{pmatrix}, & Q_3 &= \begin{pmatrix} 185.2238 & 0.0350 \\ 0.0350 & 212.2623 \end{pmatrix}, \\
 Q_4 &= \begin{pmatrix} 237.7200 & -27.1031 \\ -27.1031 & 252.0790 \end{pmatrix} \\
 Q_5 &= \begin{pmatrix} 110.9435 & 0.5117 \\ 0.5117 & 160.3469 \end{pmatrix}, & Q_6 &= \begin{pmatrix} 671.6334 & -102.7249 \\ -102.7249 & 336.8742 \end{pmatrix}, \\
 Q_7 &= \begin{pmatrix} 297.9458 & -25.1869 \\ -25.1869 & 332.1454 \end{pmatrix} \\
 Q_8 &= \begin{pmatrix} 76.5089 & -38.4080 \\ -38.4080 & 137.3497 \end{pmatrix}, & O_2 &= \begin{pmatrix} 36.1348 & 0.4562 \\ 0.4562 & 38.0656 \end{pmatrix}, \\
 O_1 &= \begin{pmatrix} 102.4601 & -2.5851 \\ -2.5851 & 97.4345 \end{pmatrix} \\
 O_3 &= \begin{pmatrix} 180.3395 & -2.5069 \\ -2.5069 & 193.8166 \end{pmatrix}, & O_4 &= \begin{pmatrix} 36.0224 & 1.2426 \\ 1.2426 & 35.1141 \end{pmatrix}, \\
 H_1 &= \begin{pmatrix} 102.8293 & 0.2484 \\ 0.2484 & 104.7745 \end{pmatrix} \\
 H_2 &= \begin{pmatrix} 489.9939 & 56.2302 \\ 56.2302 & 284.6589 \end{pmatrix}, & H_3 &= \begin{pmatrix} 266.4602 & -0.0000 \\ -0.0000 & 266.4602 \end{pmatrix}, \\
 H_4 &= \begin{pmatrix} 262.6814 & -9.8082 \\ -9.8082 & 252.5437 \end{pmatrix} \\
 H_5 &= \begin{pmatrix} 105.3490 & -1.4314 \\ -1.4314 & 106.3900 \end{pmatrix}, & H_6 &= \begin{pmatrix} 266.6557 & -10.0629 \\ -10.0629 & 276.2196 \end{pmatrix}, \\
 H_7 &= \begin{pmatrix} 209.1056 & -16.6238 \\ -16.6238 & 233.6975 \end{pmatrix} \\
 H_8 &= \begin{pmatrix} 209.1056 & -16.6238 \\ -16.6238 & 233.6975 \end{pmatrix} \\
 Z_1 &= \begin{pmatrix} 77.7712 & 15.0609 \\ 15.0609 & 49.0337 \end{pmatrix} \\
 Z_2 &= \begin{pmatrix} 101.7746 & 27.7696 \\ 27.7696 & 52.0144 \end{pmatrix} \\
 Z_3 &= \begin{pmatrix} 993.6567 & -341.6832 \\ -341.6832 & 598.1055 \end{pmatrix}, & Z_4 &= \begin{pmatrix} 242.1107 & -29.0098 \\ -29.0098 & 144.8830 \end{pmatrix}, \\
 T_1 &= \begin{pmatrix} 189.0556 & -22.3030 \\ -22.3030 & 171.2480 \end{pmatrix} \\
 T_2 &= \begin{pmatrix} 185.1430 & -107.2418 \\ -107.2418 & 259.5594 \end{pmatrix}, & T_3 &= \begin{pmatrix} 266.6014 & 0.0799 \\ 0.0799 & 266.8213 \end{pmatrix}, \\
 T_4 &= \begin{pmatrix} 154.5714 & -41.3974 \\ -41.3974 & 118.9310 \end{pmatrix} \\
 T_5 &= \begin{pmatrix} 224.3986 & -46.1427 \\ -46.1427 & 190.3638 \end{pmatrix} \\
 T_6 &= \begin{pmatrix} 262.8330 & -4.8681 \\ -4.8681 & 262.0440 \end{pmatrix}, \\
 T_7 &= \begin{pmatrix} 266.0039 & -0.1635 \\ -0.1635 & 265.9902 \end{pmatrix} \\
 T_8 &= \begin{pmatrix} 277.9033 & -3.6029 \\ -3.6029 & 210.9432 \end{pmatrix}, & R_1 &= \begin{pmatrix} 63.6434 & 0.0822 \\ 0.0822 & 65.0703 \end{pmatrix}, \\
 R_2 &= \begin{pmatrix} 61.3013 & -1.3008 \\ -1.3008 & 63.2007 \end{pmatrix} \\
 R_3 &= \begin{pmatrix} 186.1789 & -106.5294 \\ -106.5294 & 266.4642 \end{pmatrix} \\
 R_4 &= \begin{pmatrix} 203.3079 & 41.6707 \\ 41.6707 & 192.2140 \end{pmatrix} \\
 R_5 &= \begin{pmatrix} 548.9885 & 108.0646 \\ 108.0646 & 460.9957 \end{pmatrix} \\
 R_6 &= \begin{pmatrix} 492.9769 & 55.8313 \\ 55.8313 & 354.9638 \end{pmatrix} \\
 R_7 &= \begin{pmatrix} 187.7680 & 75.9434 \\ 75.9434 & 85.2374 \end{pmatrix} \\
 R_8 &= \begin{pmatrix} 193.2514 & 101.1284 \\ 101.1284 & 117.8250 \end{pmatrix} \\
 Y_1 &= \begin{pmatrix} 26.8546 & -41.4341 \\ -41.4341 & 63.9299 \end{pmatrix} \\
 Y_2 &= \begin{pmatrix} 27.9808 & -42.6842 \\ -42.6842 & 65.1151 \end{pmatrix} \\
 Y_3 &= \begin{pmatrix} 81.4225 & 24.2249 \\ 24.2249 & 67.4518 \end{pmatrix} \\
 Y_4 &= \begin{pmatrix} 81.4225 & 24.2249 \\ 24.2249 & 67.4518 \end{pmatrix} \\
 \zeta_1 &= \begin{pmatrix} 53.9465 & 0 \\ 0 & 142.5026 \end{pmatrix} \\
 \zeta_2 &= \begin{pmatrix} 42.6444 & 0 \\ 0 & 173.4502 \end{pmatrix} \\
 \zeta_3 &= \begin{pmatrix} 119.3567 & 0 \\ 0 & 151.3998 \end{pmatrix}, & \zeta_4 &= \begin{pmatrix} 162.0177 & 0 \\ 0 & 115.4791 \end{pmatrix} \\
 \varsigma &= 222.6251; & \varsigma_1 &= 215.3682
 \end{aligned}$$

According to Theorem 4.1, we can acquire that the neural networks system (46) with above given parameters is asymptotically stable with the help of Lyapunov–Krasovskii functionals, and state trajectories $x_1(k)$, $x_2(k)$, $y_1(k)$, $y_2(k)$ of above impulsive discrete-time BAM neural networks with time-varying leakage delays and randomly occurring parameter uncertainties are depicted in Fig. 2.

Remark 5.3 If the impulse disappears, then the neural networks system (3) is rewritten as follows:

$$\begin{aligned}
 x(k+1) &= Lx(k - \varrho(k)) + Mf(y(k)) + Ng(y(k - \alpha(k))) \\
 y(k+1) &= Ay(k - \rho(k)) + B\hat{f}(x(k)) + C\hat{g}(x(k - \beta(k)))
 \end{aligned} \tag{71}$$

From the above three diagrams in Fig. 3, the state trajectories of time-varying leakage delays are depicted. That is, the state trajectory of first diagram represents that NNs (71) have no leakage delay. The state trajectory of second diagram shows that NNs (71) with upper bound of leakage delay as $\varrho(k) = \varrho = 5$ and $\rho(k) = \rho = 5$ which leads the system with some

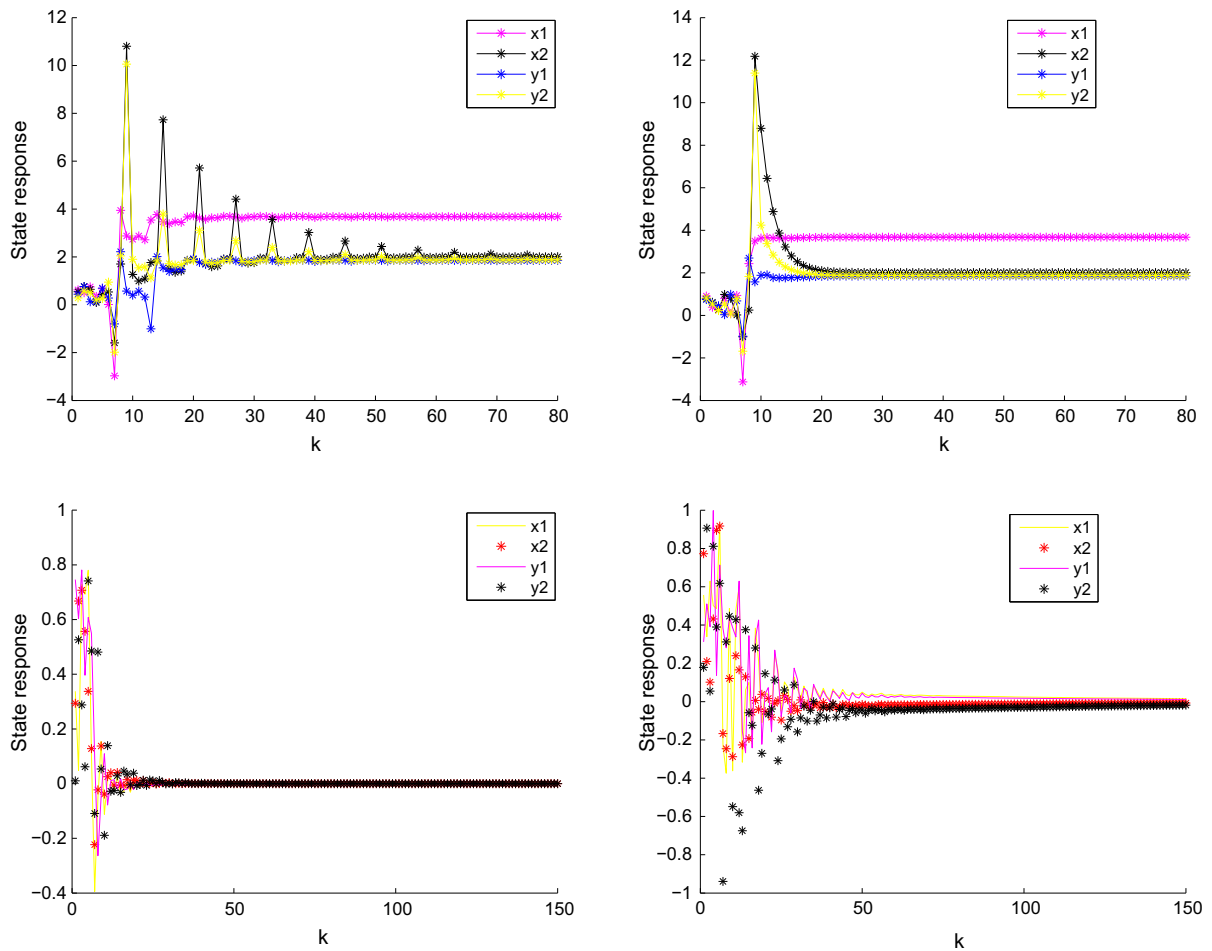


Fig. 2 State trajectories of (46) with impulsive and non-impulsive effects

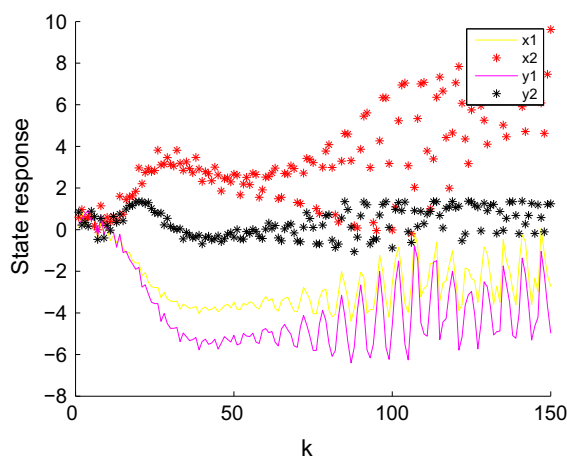


Fig. 3 State responses of (71)

disturbances and finally leads to stable. But the state trajectory of third diagram depicts that NNs (71) to be unstable with $\varrho(k) = \varrho = 8$ and $\rho(k) = \rho = 8$. From the three state trajectories, we conclude that if the leakage is high the system becomes worse and leads to unstable.

6 Conclusions

In this research work, we have explored the robust stability problem for impulsive discrete-time BAM neural networks with randomly occurring uncertainties and time-varying leakage delays. With the help of Lyapunov–Krasovskii functional technique, a novel conditions have been entrenched to guarantee the robust stability of the designed discrete-time neural

networks in terms of LMIs, which can be adequately checked by MATLAB LMI Toolbox. In addition, convex combination lemma is used which helps to reduce the decision variables without affecting the conservatism. At last, four numerical simulations have been delivered to illustrate the effectiveness of our developed theoretical work. It would be interesting to extend the results proposed in this paper to extend further to H_∞ control for neural networks with mode stochastic noise and Markovian jumping parameters and corresponding results will appear in near future.

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RESEARCH

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Approximation of state variables for discrete-time stochastic genetic regulatory networks with leakage, distributed, and probabilistic measurement delays: a robust stability problem

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Abstract

This work predominantly labels the problem of approximation of state variables for discrete-time stochastic genetic regulatory networks with leakage, distributed, and probabilistic measurement delays. Here we design a linear estimator in such a way that the absorption of mRNA and protein can be approximated via known measurement outputs. By utilizing a Lyapunov–Krasovskii functional and some stochastic analysis execution, we obtain the stability formula of the estimation error systems in the structure of linear matrix inequalities under which the estimation error dynamics is robustly exponentially stable. Further, the obtained conditions (in the form of LMIs) can be effortlessly solved by some available software packages. Moreover, the specific expression of the desired estimator is also shown in the main section. Finally, two mathematical illustrative examples are accorded to show the advantage of the proposed conceptual results.

Keywords: Genetic regulatory networks (GRNs); Time-varying delays; Distributed delays; Leakage delays; Probabilistic measurement delays

1 Introduction and system formulation

A gene is a physical structure made up of DNA, and most of the genes hold the data which is required to make molecules called as proteins. In the modern years, research in genetic regulatory networks (GRNs) has gained significance in both biological and bio-medical sciences, and a huge number of tremendous results have been issued. Distinct kinds of computational models have been applied to propagate the behaviors of GRNs; see, for instance, the Bayesian network models, the Petri net models, the Boolean models, and the differential equation models. Surrounded by the indicated models, the differential equation models describe the rate of change in the concentration of gene production, such as mRNAs and proteins, as constant values, whereas the other models do not have such a basis.

As one of the mostly investigated dynamical behaviors, the state estimation for GRNs has newly stirred increasing research interest (see [1, 2] and the references cited therein [1, 3–10]). In fact, this is an immense concern since GRNs are complex nonlinear systems. Due to the complication, it is frequently the case that only partial facts around the states of the nodes are accessible in the network outputs. In consideration of realizing the GRNs better, there has been a necessity to estimate the state of the nodes through securable measurements. In [1], the robust H_∞ problem was considered for a discrete-time stochastic GRNs with probabilistic measurement delays. In [2], the robust H_∞ state estimation problem was investigated for a general class of uncertain discrete-time stochastic neural networks with probabilistic measurement delays. By designing an adaptive controller, the authors investigated the problem of delayed GRNs stabilization in [7]. Xiao et al. discussed the stability, periodic oscillation, and bifurcation of two-gene regulatory networks with time delays [8]. The stability of continuous GRNs and discrete-time GRNs was discussed, respectively, in [11]. Huang et al. considered the bifurcation of delayed fractional GRNs by hybrid control [12].

Due to the limited signal communication speed, the measurement among the networks is always assumed to be a delayed one. So, the network measurement could not include instruction about the present gene states, while the delayed network measurement could. The most fashionable mechanism to relate the probabilistic measurement delay or some other kind of lacking measurement is to grab it as a Bernoulli distributed white classification [13–20]. The robust stochastic stability of stochastic genetic GRNs was considered, and some delay-dependent criteria were presented in the form of LMIs [18]. And the asymptotic stability of delayed stochastic GRNs with impulsive effect was discussed in [19]. The synchronization problem of dynamical system was also discussed in [21, 22]. The challenging task is how to draft the robust estimators when both uncertainties and probabilistic appeared in discrete-time GRN models.

More recently, in [23], Liu et al. developed a state estimation problem for a genetic regulatory network with Markovian jumping parameters and time delays:

$$\begin{aligned}\dot{m}(t) &= -A(r(t))m(t) + W(r(t))g(p(t - \sigma(t))), \\ \dot{p}(t) &= -C(r(t))p(t) + D(r(t))m(t - \tau(t)).\end{aligned}$$

Also in [24], Wan et al. proposed the state estimation of discrete-time GRN with random delays governed by the following equation:

$$\begin{aligned}M(k+1) &= AM(k) + Bf(P(k - d(k))) + V, \\ P(k+1) &= CP(k) + DM(k - \tau(k)).\end{aligned}$$

Considering the above referenced papers, the robustness of approximation of the stochastic GRNs with leakage delays, distributed delays, and probabilistic measurement delays has not been tackled. The main contributions of this paper are summarized as follows:

1. We examine the approximation concern for the discrete-time stochastic GRNs with the leakage delays, distributed delays, and probabilistic measurement delays into the problem and model the robust H_∞ state estimator for a class of discrete-time stochastic GRNs. Here, the probabilistic measurement delays, which narrate the

binary shifting sequence, are satisfied by the conditional probability distribution. So, the crisis of parameter uncertainties, including errors, stochastic disturbance, leakage delays, distributed delays, and the activation function of the addressed GRNs, is identified by sector-bounded nonlinearities.

2. By applying the Lyapunov stability theory and stochastic analysis techniques, sufficient conditions are first entrenched to assure the presence of the desired estimators in terms of a linear matrix inequality (LMI). These circumstances are reliant on both the lower and upper bounds of time-varying delays. Again, the absolute expression of the desired estimator is demonstrated to assure the estimation error dynamics to be robustly exponentially stable in the mean square for the consigned system.
3. Finally, twin mathematical examples beside with simulations are given to view the capability of the advanced criteria.

In this note, we consider the GRNs with leakage, discrete, and distributed delays described as follows:

$$\begin{aligned} x(k+1) &= -(\mathbb{A} + \Delta\mathbb{A}(k))x(k - \rho_1) + (\mathbb{B} + \Delta\mathbb{B}(k))\hat{g}(y(k - \delta(k))) \\ &\quad + (E + \Delta E(k)) \sum_{s=1}^{\infty} \mu_s h(y(k-s)) + \sigma(k, x(k - \rho_1))\omega(k) + L_x v_x(k), \\ y(k+1) &= -(\mathbb{C} + \Delta\mathbb{C}(k))y(k - \rho_2) + (\mathbb{D} + \Delta\mathbb{D}(k))x(k - \tau(k)) \\ &\quad + (F + \Delta F(k)) \sum_{n=1}^{\infty} \xi_n x(k-n) + L_y v_y(k), \end{aligned} \quad (1)$$

where $x(k - \rho_1) = [x_1(k - \rho_1), \dots, x_n(k - \rho_1)]^T \in \mathbb{R}^n$, $y(k - \rho_2) = [y_1(k - \rho_2), \dots, y_n(k - \rho_2)]^T \in \mathbb{R}^n$, $x_i(k - \rho_1)$, and $y_i(k - \rho_2)$ ($i = 1, 2, \dots, n$) denote the concentrations of mRNA and protein of the i th node at time t , respectively; $\mathbb{A} = \text{diag}\{a_1, a_2, \dots, a_n\}$, $\mathbb{C} = \text{diag}\{c_1, c_2, \dots, c_n\}$, and $\mathbb{D} = \text{diag}\{d_1, d_2, \dots, d_n\}$ are constant matrices; $a_i > 0$, $c_i > 0$, and $d_i > 0$ are the degradation rates of mRNAs, protein, and the translation rate of the i th gene, respectively; the coupling matrix of the genetic regulatory network is defined as $\mathbb{B} = (b_{ij}) \in \mathbb{R}^{n \times n}$; $E = \text{diag}\{e_1, e_2, \dots, e_n\}$, and $F = \text{diag}\{f_1, f_2, \dots, f_n\}$ are the weight matrices. $\Delta\mathbb{A}(k)$, $\Delta\mathbb{B}(k)$, $\Delta\mathbb{C}(k)$, $\Delta\mathbb{D}(k)$, $\Delta\mathbb{E}(k)$, and $\Delta\mathbb{F}(k)$ represent the parameter uncertainties; $h(y(k)) = [h_1(y(k)), \dots, h_n(y(k))]^T \in \mathbb{R}^n$ denotes the activation function; the exogenous disturbance signals $v_x(k), v_y(k) \in \mathbb{R}^n$ satisfy $v_i(\cdot) \in L_2[0, \infty)$. L_x and L_y are the known real constant matrices. $\delta(k)$ denotes the feedback regulation delay and $\tau(k)$ denotes the translation delay, which satisfy

$$0 \leq \delta_m \leq \delta(k) \leq \delta_M, \quad 0 \leq \tau_m \leq \tau(k) \leq \tau_M, \quad (2)$$

where the lower bound δ_m , τ_m and the upper bound δ_M , τ_M are known positive integers.

Furthermore, the nonlinear activation function $\hat{g}(y(k - \delta(k))) = [\hat{g}_1(y_1(k - \delta(k))), \dots, \hat{g}_n(y_n(k - \delta(k)))]^T \in \mathbb{R}^n$ represents the feedback regulation of the protein on the transcription. It is a monotonic function in the Hill form, that is, $\hat{g}_i(f) = \frac{f^{h_j}}{1+f^{h_j}}$ ($j = 1, 2, \dots, n$), where h_j is the Hill co-efficient and f is a positive constant. The noise intensity function vector

$\sigma(k, x(k)) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies

$$\sigma^T(k, x(k - \rho_1))\sigma(k, x(k - \rho_1)) \leq x^T(k - \rho_1)Hx(k - \rho_1), \quad (3)$$

where $H > 0$ is a known matrix. $\omega(k)$ is a Brownian motion with $\mathbb{E}\{\omega(k)\} = 0$, $\mathbb{E}\{\omega^2(k)\} = 1$ and $\mathbb{E}\{\omega(i)\omega(j)\} = 0$ ($i \neq j$).

For large-scale complex networks, information around the network nodes is not often fully attainable from the network outputs (see [25, 26]). We can assume that network measurements are described as follows:

$$\begin{aligned} Z_x(k) &= Mx(k), \\ Z_y(k) &= Ny(k), \end{aligned} \quad (4)$$

where M and N are known constant matrices. $Z_x(k), Z_y(k) \in \mathbb{R}^l$ are the complete outputs of the network. The network outputs are subjected to probabilistic delays that can be described by

$$\begin{aligned} \tilde{Z}_x(k) &= \alpha_k Z_x(k) + (1 - \alpha_k)Z_x(k - 1), \\ \tilde{Z}_y(k) &= \beta_k Z_y(k) + (1 - \beta_k)Z_y(k - 1), \end{aligned} \quad (5)$$

where the stochastic variables $\alpha_k, \beta_k \in \mathbb{R}$ are Bernoulli allocated with sequences directed by

$$\begin{aligned} \text{Prob}\{\alpha_k = 1\} &= \mathbb{E}\{\alpha_k\} = \alpha_0, & \text{Prob}\{\alpha_k = 0\} &= 1 - \mathbb{E}\{\alpha_k\} = 1 - \alpha_0, \\ \text{Prob}\{\beta_k = 1\} &= \mathbb{E}\{\beta_k\} = \beta_0, & \text{Prob}\{\beta_k = 0\} &= 1 - \mathbb{E}\{\beta_k\} = 1 - \beta_0. \end{aligned} \quad (6)$$

Here $\alpha_0, \beta_0 > 0$ are known constants. Obviously, for α_k, β_k , the variance $\sigma_\alpha = \alpha_0(1 - \alpha_0)$, $\sigma_\beta = \beta_0(1 - \beta_0)$.

The GRN state estimator to be designed is given as follows:

$$\begin{cases} \hat{x}(k + 1) = -\mathbb{A}_x \hat{x}(k) + \mathbb{B}_x \tilde{Z}_x(k), \\ \hat{y}(k + 1) = -\mathbb{A}_y \hat{y}(k) + \mathbb{B}_y \tilde{Z}_y(k), \end{cases} \quad (7)$$

where $\hat{x}(k), \hat{y}(k) \in \mathbb{R}^n$ are the estimations of $x(k)$ and $y(k)$, and $\mathbb{A}_x, \mathbb{A}_y, \mathbb{B}_x, \mathbb{B}_y$ are the estimator gain matrices to be determined.

Assume that the estimation error vectors are $\tilde{x}(k) = x(k) - \hat{x}(k)$ and $\tilde{y}(k) = y(k) - \hat{y}(k)$; the estimation error dynamics can be defined as follows from equations (1), (5), and (7):

$$\begin{aligned} \tilde{x}(k + 1) &= -(\mathbb{A} + \Delta\mathbb{A}(k))x(k - \rho_1) + (\mathbb{A}_x - \alpha_k \mathbb{B}_x M)x(k) + (\mathbb{B} + \Delta\mathbb{B}(k))\hat{g}(y(k - \delta(k))) \\ &\quad + (E + \Delta E(k)) \sum_{s=1}^{\infty} \mu_s h(y(k - s)) + \sigma(k, x(k - \rho_1))\omega(k) - \mathbb{A}_x \tilde{x}(k) \\ &\quad - (1 - \alpha_k)\mathbb{B}_x Mx(k - 1) + L_x v_x(k), \end{aligned}$$

$$\begin{aligned}\tilde{y}(k+1) = & -(\mathbb{C} + \Delta\mathbb{C}(k))y(k - \rho_2) + (\mathbb{A}_y - \beta_k\mathbb{B}_yN)y(k) + (\mathbb{D} + \Delta\mathbb{D}(k))x(k - \tau(k)) \\ & + (F + \Delta F(k)) \sum_{n=1}^{\infty} \xi_n x((k-n)) - \mathbb{A}_y \tilde{y}(k) - (1 - \beta_k)\mathbb{B}_y N y(k-1) \\ & + L_y v_y(k).\end{aligned}\quad (8)$$

For suitability, we denote

$$\begin{aligned}\tilde{x}(k) &= \begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix}, \quad \tilde{y}(k) = \begin{bmatrix} y(k) \\ \tilde{y}(k) \end{bmatrix}, \\ \tilde{x}(j) &= \psi(j), \quad j = -\tau_M, -\tau_{M+1}, \dots, -1, 0, \\ \tilde{y}(j) &= \varphi(j), \quad j = -\delta_M, -\delta_{M+1}, \dots, -1, 0,\end{aligned}$$

where $\psi(j)$, $j = -\tau_M, -\tau_{M+1}, \dots, -1, 0$ and $\varphi(j)$, $j = -\delta_M, -\delta_{M+1}, \dots, -1, 0$ are the initial conditions.

2 Preliminaries

Notations: Throughout the paper, $\mathbb{N}^+$ refers to the position for the set of nonnegative integers; $\mathbb{R}^n$ indicates the n -dimensional Euclidean space. The superscript “ T ” acts as the matrix transposition. The code $X \geq Y$ (each $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definitive (respectively positive definite). I means the identity matrix with consistent dimension. The symbol “ $*$ ” denotes the term symmetry. In addition, $E\{\cdot\}$ denotes the expectation operator. $L_2[0, \infty)$ is the amplitude of square-integrable vector functions over $[0, \infty)$. $\|\cdot\|$ denotes the Euclidean vector norm. Matrices, if not absolutely specified, are affected to have compatible dimensions.

Assumption 1 The parameter uncertainties $\Delta\mathbb{A}(k)$, $\Delta\mathbb{B}(k)$, $\Delta\mathbb{C}(k)$, $\Delta\mathbb{D}(k)$, $\Delta E(k)$, $\Delta F(k)$ are of the following form.

The admissible parameter uncertainties are assumed to be of the form:

$$\begin{aligned}& \begin{bmatrix} \Delta\mathbb{A}(k) & \Delta\mathbb{B}(k) & \Delta\mathbb{C}(k) & \Delta\mathbb{D}(k) & \Delta E(k) & \Delta F(k) \end{bmatrix} \\ &= RN(k) \begin{bmatrix} W_1 & W_2 & W_3 & W_4 & W_5 & W_6 \end{bmatrix},\end{aligned}$$

where R , W_i ($i = 1, 2, \dots, 6$) are the known constant matrices with appropriate dimensions. The uncertain matrix $N(k)$ satisfies $N^T(k)N(k) \leq I$, $\forall k \in \mathbb{N}^+$.

Assumption 2 The vector-valued function $\hat{g}_i(\cdot)$ is assumed to satisfy the following sector-bounded condition, namely for $\forall x, y \in \mathbb{R}^n$:

$$[\hat{g}(x) - \hat{g}(y) - N_1(x - y)]^T [\hat{g}(x) - \hat{g}(y) - N_2(x - y)] \leq 0,$$

where N_1 , N_2 are known real constant matrices, and $\tilde{N} = N_1 - N_2$ is a symmetric positive definite matrix.

Definition 2.1 If there exist constants $\alpha > 0$ and $0 < \mu < 1$, system (8) with $v_x(k) = 0$ and $v_y(k) = 0$ is global robust exponential state estimator of GRNs (1) with measurements (5)

in the mean square sense such that

$$\mathbb{E}\{|\bar{x}(k)|^2 + |\bar{y}(k)|^2\} \leq \alpha \mu^k \left(\max_{-\tau_M \leq k \leq 0} |\bar{x}(k)|^2 + \max_{-\delta_M \leq k \leq 0} |\bar{y}(k)|^2 \right).$$

Definition 2.2 If there exists a scalar $\gamma > 0$, system (8) is a robust H_∞ state estimator of GRNs (1) with measurements (5) in the mean square sense with zero initial conditions such that

$$\mathbb{E} \sum_{k=0}^{\infty} \{|\bar{x}(k)|^2 + |\bar{y}(k)|^2\} \leq \gamma^2 \mathbb{E} \sum_{k=0}^{\infty} (|v_x(k)|^2 + |v_y(k)|^2)$$

for all non-zero $v_x(k), v_y(k) \in L_2[0, \infty)$.

The following lemmas are crucial in implementing our main results.

Lemma 2.3 (see [2, 26]) *Let N and S be real constant matrices; matrix $F(k)$ satisfies $F^T(k)F(k) \leq 1$. Then we have:*

- (i) *For any $\epsilon > 0$, $NF(k)S + S^T F^T(k)N^T \leq \epsilon^{-1}NN^T + \epsilon S^T S$.*
- (ii) *For any $P > 0$, $\pm 2x^T y \leq x^T P^{-1}x + y^T P y$.*

Lemma 2.4 *Given the constant matrices $\hat{\Omega}_1, \hat{\Omega}_2$, and $\hat{\Omega}_3$, where $\hat{\Omega}_1^T = \hat{\Omega}_1$ and $\hat{\Omega}_2^T = \hat{\Omega}_2 > 0$, then $\hat{\Omega}_1 + \hat{\Omega}_3^T \hat{\Omega}_2^{-1} \hat{\Omega}_3 < 0$, if and only if*

$$\begin{bmatrix} \hat{\Omega}_1 & \hat{\Omega}_3^T \\ \hat{\Omega}_3 & -\hat{\Omega}_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\hat{\Omega}_2 & \hat{\Omega}_3 \\ \hat{\Omega}_3^T & \hat{\Omega}_1 \end{bmatrix} < 0.$$

Lemma 2.5 *Let $\mathbb{M} \in \mathbb{R}^{n \times n}$ be a positive semi-definite matrix, $x_i \in \mathbb{R}^n$, and $a_i \geq 0$ ($i = 1, 2, \dots$). If the series distressed are convergent, the following inequality holds:*

$$\left(\sum_{i=1}^{+\infty} a_i x_i \right)^T \mathbb{M} \left(\sum_{i=1}^{+\infty} a_i x_i \right) \leq \left(\sum_{i=1}^{+\infty} a_i \right) \sum_{i=1}^{+\infty} a_i x_i^T \mathbb{M} x_i.$$

Remark 2.1 In [1] Wang et al. investigated the robust state estimation for stochastic genetic regulatory networks with probabilistic delays in discrete sense, and Lv et al. [4] developed the robust distributed state estimation for genetic regulatory networks with Markovian jumping parameters. However, the inclusion of discrete-interval GRNs with leakage delays, probabilistic measurement delays, noise, and distributed delays has not been taken into account. So, the prime intention of this work is to elucidate that the state estimation problem for the improved system (8) with leakage delays is robustly exponentially stable.

3 Exponential stability criterion

In this part, we first introduce a sufficient condition under which the augmented system (8) is robustly mean-square exponentially stable with the exogenous disturbance signals $v_x(k) = 0$ and $v_y(k) = 0$.

Theorem 3.1 *Suppose that Assumptions 1 and 2 hold. Let the leakage delays ρ_1, ρ_2 and the estimation parameters $\mathbb{A}_x, \mathbb{B}_x, \mathbb{A}_y$, and $\mathbb{B}_y$ be given and also the acceptable conditions hold.*

Then the estimation error system (8) with $v_x(k) = 0$ and $v_y(k) = 0$ is robustly exponentially stable in the mean square if there exist positive definite matrices $R_{11}, R_{12}, R_{21}, R_{22}, R_{31}, R_{32}, R_{41}, R_{42}, R_{51}, R_{52}$ and three positive constant scalars λ, ε_1 , and ε_2 such that the following LMI holds:

$$\Lambda_1 = \begin{bmatrix} \Lambda'_{11} & * & * \\ S_1 & J_1 & * \\ 0 & \bar{T}_1^T & -\varepsilon_1 I \end{bmatrix} < 0, \quad \Lambda_2 = \begin{bmatrix} \Lambda'_{22} & * & * \\ S_2 & J_2 & * \\ 0 & \bar{T}_2^T & -\varepsilon_2 I \end{bmatrix} < 0, \quad (9)$$

where

$$\begin{aligned} \Lambda'_{11} &= \begin{bmatrix} \psi_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_{31} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_{41} + \varepsilon_1 W_4^T W_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & HR_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I(R_{12} + R_{22}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\xi} R_{52} \end{bmatrix}, \\ \Lambda'_{22} &= \begin{bmatrix} \psi_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_{42} - \lambda \tilde{N}_1 + \varepsilon_2 W_2^T W_2 & -\lambda \tilde{N}_2^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda \tilde{N}_2 & -\lambda I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I(R_{11} + R_{21}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\mu} R_{51} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\mu} R_{51} \end{bmatrix}, \\ S_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & \bar{\Xi}_{15} & 0 & 0 \\ \Xi_{21} & -\sqrt{2} R_{21} \mathbb{A}_x & \Xi_{23} & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\alpha} R_{21} \mathbb{B}_x M & 0 & \sqrt{\sigma_\alpha} R_{21} \mathbb{B}_x M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\Xi}_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\Xi}_{55} & 0 \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \bar{\Xi}_{15} &= -\sqrt{2}(R_{11} + R_{21})\mathbb{A}; & \bar{\Xi}_{44} &= \sqrt{2}(R_{12} + R_{22})\mathbb{D}; & \bar{\Xi}_{55} &= \sqrt{2}(R_{12} + R_{22})F; \\ \Xi_{21} &= \sqrt{2}R_{21}(\mathbb{A}_x - \alpha_0 \mathbb{B}_x M); & \Xi_{23} &= -\sqrt{2}R_{21}(1 - \alpha_0)\mathbb{B}_x M; \\ S_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \bar{\Theta}_{16} & 0 & 0 & 0 \\ \Theta_{21} & -\sqrt{2}R_{22}\mathbb{A}_y & \Theta_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\beta} R_{22} \mathbb{B}_y N & 0 & \sqrt{\sigma_\beta} R_{22} \mathbb{B}_y N & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{\Theta}_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Theta}_{57} & 0 & 0 \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \bar{\Theta}_{16} &= -\sqrt{2}(R_{12} + R_{22})\mathbb{C}; & \bar{\Theta}_{45} &= \sqrt{2}(R_{11} + R_{21})\mathbb{B}; & \bar{\Theta}_{57} &= \sqrt{2}(R_{11} + R_{21})E; \\ \Theta_{21} &= \sqrt{2}R_{22}(\mathbb{A}_y - \beta_0 \mathbb{B}_y N); & \Theta_{23} &= -\sqrt{2}R_{22}(1 - \beta_0)\mathbb{B}_y N, \\ J_1 &= \text{diag}\{-(R_{11} + R_{21}), -R_{21}, -R_{21}, -(R_{12} + R_{22}), -(R_{12} + R_{22})\}, \\ J_2 &= \text{diag}\{-(R_{12} + R_{22}), -R_{22}, -R_{22}, -(R_{11} + R_{21}), -(R_{11} + R_{21})\}, \end{aligned}$$

$$\bar{T}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -\sqrt{2}(R_{11} + R_{21})T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}(R_{12} + R_{22})T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}(R_{12} + R_{22})T & 0 \end{bmatrix},$$

$$\bar{T}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\sqrt{2}(R_{12} + R_{22})T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}(R_{11} + R_{21})T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}(R_{11} + R_{21})T & 0 & 0 \end{bmatrix},$$

$$\bar{\mu} = \sum_{s=1}^{\infty} \mu_s, \quad \bar{\xi} = \sum_{n=1}^{\infty} \xi_n,$$

$$\psi_{11} = -R_{11} + R_{31} + (\tau_M - \tau_m + 1)R_{41} + \bar{\xi}R_{52}; \quad \psi_{12} = -R_{12} + R_{32} + (\delta_M - \delta_m + 1)R_{42},$$

$$\tilde{N}_1 = \frac{(N_1^T N_2 + N_2^T N_1)}{2}; \quad \tilde{N}_2 = -\frac{(N_1^T + N_2^T)}{2}.$$

Proof Choose a Lyapunov–Krasovskii functional for the augmented system (8):

$$\mathbb{V}(k) = \mathbb{V}_1(k) + \mathbb{V}_2(k) + \mathbb{V}_3(k) + \mathbb{V}_4(k) + \mathbb{V}_5(k) + \mathbb{V}_6(k), \quad (10)$$

where

$$\begin{aligned} \mathbb{V}_1(k) &= x^T(k)R_{11}x(k) + y^T(k)R_{12}y(k), \\ \mathbb{V}_2(k) &= \tilde{x}^T(k)R_{21}\tilde{x}(k) + \tilde{y}^T(k)R_{22}\tilde{y}(k), \\ \mathbb{V}_3(k) &= x^T(k-1)R_{31}x(k-1) + y^T(k-1)R_{32}y(k-1), \\ \mathbb{V}_4(k) &= \sum_{i=k-\tau(k)}^{k-1} x^T(i)R_{41}x(i) + \sum_{i=k-\delta(k)}^{k-1} y^T(i)R_{42}y(i), \\ \mathbb{V}_5(k) &= \sum_{j=-\tau_M+1}^{-\tau_m} \sum_{i=k+j}^{k-1} x^T(i)R_{41}x(i) + \sum_{j=-\delta_M+1}^{-\delta_m} \sum_{i=k+j}^{k-1} y^T(i)R_{42}y(i), \\ \mathbb{V}_6(k) &= \sum_{i=1}^{\infty} \mu_i \sum_{j=k-i}^{k-1} h^T(y(j))R_{51}h(y(j)) + \sum_{i=1}^{\infty} \xi_i \sum_{j=k-i}^{k-1} x^T(i)R_{52}x(i). \end{aligned}$$

Calculate the difference of $\mathbb{V}_i(k)$ ($i = 1, 2, \dots, 6$) along the trajectories of model (8) with $v_x(k) = 0$, $v_y(k) = 0$ and

$$\mathbb{E}\{\Delta \mathbb{V}(k)\} = \sum_{i=1}^6 \mathbb{E}\{\mathbb{V}_i(k)\}. \quad (11)$$

Now, we have

$$\begin{aligned} \mathbb{E}\{\Delta \mathbb{V}_1(k)\} &= \mathbb{E}\{\mathbb{V}_1(k+1) - \mathbb{V}_1(k)\} \\ &= \mathbb{E}\left\{ \left[-(\mathbb{A} + \Delta \mathbb{A}(k))x(k - \rho_1) + (\mathbb{B} + \Delta \mathbb{B}(k))\hat{g}(y(k - \delta(k))) \right. \right. \end{aligned}$$

$$\begin{aligned}
& + (E + \Delta E(k)) \sum_{s=1}^{\infty} \mu_s h(y(k-s)) \Bigg]^T \\
& \times R_{11} \left[-(\mathbb{A} + \Delta \mathbb{A}(k))x(k - \rho_1) + (\mathbb{B} + \Delta \mathbb{B}(k))\hat{g}(y(k - \delta(k))) \right. \\
& \left. + (E + \Delta E(k)) \sum_{s=1}^{\infty} \mu_s h(y(k-s)) \right] \\
& + \sigma^T(k, x(k - \rho_1))R_{11}\sigma(k, x(k - \rho_1)) - x^T(k)R_{11}x(k) - y^T(k)R_{12}y(k) \\
& + \left[-(\mathbb{C} + \Delta \mathbb{C}(k))y(k - \rho_2) + (\mathbb{D} + \Delta \mathbb{D}(k))x(k - \tau(k)) \right. \\
& \left. + (F + \Delta F(k)) \sum_{n=1}^{\infty} \xi_n x(k - n) \right]^T \\
& \times R_{12} \left[-(\mathbb{C} + \Delta \mathbb{C}(k))y(k - \rho_2) + (\mathbb{D} + \Delta \mathbb{D}(k))x(k - \tau(k)) \right. \\
& \left. + (F + \Delta F(k)) \sum_{n=1}^{\infty} \xi_n x(k - n) \right] \Bigg\}, \tag{12}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\{\Delta V_2(k)\} &= \mathbb{E}\{V_2(k+1) - V_2(k)\} \\
&= \mathbb{E}\left\{ \left[-(\mathbb{A} + \Delta \mathbb{A}(k))x(k - \rho_1) + (\mathbb{A}_x - \alpha_k \mathbb{B}_x M)x(k) \right. \right. \\
&\quad + (\mathbb{B} + \Delta \mathbb{B}(k))\hat{g}(y(k - \delta(k))) + (E + \Delta E(k)) \sum_{s=1}^{\infty} \mu_s h(y(k-s)) \\
&\quad \left. \left. - \mathbb{A}_x \tilde{x}(k) - (1 - \alpha_k) \mathbb{B}_x Mx(k-1) \right]^T \right. \\
&\quad \times R_{21} \left[-(\mathbb{A} + \Delta \mathbb{A}(k))x(k - \rho_1) + (\mathbb{A}_x - \alpha_k \mathbb{B}_x M)x(k) \right. \\
&\quad + (\mathbb{B} + \Delta \mathbb{B}(k))\hat{g}(y(k - \delta(k))) + (E + \Delta E(k)) \sum_{s=1}^{\infty} \mu_s h(y(k-s)) \\
&\quad \left. \left. - \mathbb{A}_x \tilde{x}(k) - (1 - \alpha_k) \mathbb{B}_x Mx(k-1) \right] \right. \\
&\quad + \sigma_\alpha [\mathbb{B}_x Mx(k) + \mathbb{B}_x Mx(k-1)]^T R_{21} [\mathbb{B}_x Mx(k) + \mathbb{B}_x Mx(k-1)] \\
&\quad \left. - \tilde{x}^T(k)R_{21}\tilde{x}(k) + \left[-(\mathbb{C} + \Delta \mathbb{C}(k))y(k - \rho_2) + (\mathbb{A}_y - \beta_k \mathbb{B}_y N)y(k) \right. \right. \\
&\quad \left. + (\mathbb{D} + \Delta \mathbb{D}(k))x(k - \tau(k)) + (F + \Delta F(k)) \sum_{n=1}^{\infty} \xi_n x(k - n) \right. \\
&\quad \left. \left. - \mathbb{A}_y \tilde{y}(k) - (1 - \beta_k) \mathbb{B}_y Ny(k-1) \right]^T \right\}
\end{aligned}$$

$$\begin{aligned}
& \times R_{22} \left[-(\mathbb{C} + \Delta \mathbb{C}(k))y(k - \rho_2) + (\mathbb{A}_y - \beta_k \mathbb{B}_y N)y(k) \right. \\
& + (\mathbb{D} + \Delta \mathbb{D}(k))x(k - \tau(k)) + (F + \Delta F(k)) \sum_{n=1}^{\infty} \xi_n x((k - n)) \\
& \left. - \mathbb{A}_y \tilde{y}(k) - (1 - \beta_k) \mathbb{B}_y N y(k - 1) \right] \\
& + \sigma_{\beta} [\mathbb{B}_y N y(k) + \mathbb{B}_y N y(k - 1)]^T R_{22} [\mathbb{B}_y N y(k) + \mathbb{B}_y N y(k - 1)] \\
& - \tilde{y}^T(k) R_{22} \tilde{y}(k) \Big\}, \tag{13}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\{\Delta \mathbb{V}_3(k)\} &= \mathbb{E}\{\mathbb{V}_3(k + 1) - \mathbb{V}_3(k)\} \\
&= \mathbb{E}\{x^T(k) R_{31} x(k) - x^T(k - 1) R_{31} x(k - 1) \\
&\quad + y^T(k) R_{32} y(k) - y^T(k - 1) R_{32} y(k - 1)\}, \tag{14}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\{\Delta \mathbb{V}_4(k)\} &= \mathbb{E}\{\mathbb{V}_4(k + 1) - \mathbb{V}_4(k)\} \\
&\leq \mathbb{E}\left\{ x^T(k) R_{41} x(k) - x^T(k - \tau(k)) R_{41} x(k - \tau(k)) \right. \\
&\quad + \sum_{i=k-\tau_M+1}^{k-\tau_m} x^T(i) R_{41} x(i) \\
&\quad + y^T(k) R_{42} y(k) - y^T(k - \delta(k)) R_{42} y(k - \delta(k)) \\
&\quad \left. + \sum_{i=k-\delta_M+1}^{k-\delta_m} y^T(i) R_{42} y(i) \right\}, \tag{15}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\{\Delta \mathbb{V}_5(k)\} &= \mathbb{E}\{\mathbb{V}_5(k + 1) - \mathbb{V}_5(k)\} \\
&= \mathbb{E}\left\{ (\tau_M - \tau_m) x^T(k) R_{41} x(k) - \sum_{i=k-\tau_M+1}^{k-\tau_m} x^T(i) R_{41} x(i) \right. \\
&\quad \left. + (\delta_M - \delta_m) y^T(k) R_{42} y(k) - \sum_{i=k-\delta_M+1}^{k-\delta_m} y^T(i) R_{42} y(i) \right\}, \tag{16}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\{\Delta \mathbb{V}_6(k)\} &= \mathbb{E}\{\mathbb{V}_6(k + 1) - \mathbb{V}_6(k)\} \\
&= \sum_{i=1}^{\infty} \mu_i \sum_{j=k+1-i}^{k+1-1} h^T(y(j)) R_{51} h(y(j)) + \sum_{i=1}^{\infty} \xi_i \sum_{j=k+1-i}^{k+1-1} x^T(i) R_{52} x(i) \\
&\quad - \sum_{i=1}^{\infty} \mu_i \sum_{j=k-i}^{k-1} h^T(y(j)) R_{51} h(y(j)) - \sum_{i=1}^{\infty} \xi_i \sum_{j=k-i}^{k-1} x^T(i) R_{52} x(i) \\
&= \sum_{i=1}^{\infty} \mu_i [h^T(y(k)) R_{51} h(y(k)) - h^T(y(k - i)) R_{51} h(y(k - i))] \\
&\quad + \sum_{i=1}^{\infty} \xi_i [x^T(k) R_{52} x(k) - x^T(k - i) R_{52} x(k - i)].
\end{aligned}$$

Using Lemma 2.5, we get

$$\begin{aligned} \mathbb{E}\{\Delta \mathbb{V}_6(k)\} &\leq \bar{\mu} h^T(y(k)) R_{51} h(y(k)) - \bar{\mu} [\bar{\mu} h(y(k-s))]^T R_{51} [\bar{\mu} h(y(k-s))] \\ &\quad + \bar{\xi} x^T(k) R_{52} x(k) - \bar{\xi} [\bar{\xi} x(k-n)]^T R_{52} [\bar{\xi} x(k-n)]. \end{aligned} \quad (17)$$

Substituting equations (12)–(17) into equation (11) results in

$$\begin{aligned} \mathbb{E}\{\Delta \mathbb{V}(k)\} &\leq \mathbb{E}\{\varpi_0^T(k) [\Lambda_{11} + \sigma_\alpha \hat{W}_{01}^T R_{21} \hat{W}_{01} + 2\hat{G}_{01}^T(k)(R_{11} + R_{21})\hat{G}_{01}(k) \\ &\quad + 2\hat{F}_{01}^T(k)R_{21}\hat{F}_{01}(k) + 2\hat{G}_{11}^T(k)(R_{12} + R_{22})\hat{G}_{11}(k) \\ &\quad + 2\hat{S}_{01}^T(k)(R_{12} + R_{22})\hat{S}_{01}(k)] \varpi_0(k) \\ &\quad + \Gamma_0^T(k) [\Lambda_{12} + \sigma_\beta \hat{W}_{02}^T R_{22} \hat{W}_{02} + 2\hat{G}_{02}^T(k)(R_{12} + R_{22})\hat{G}_{02}(k) \\ &\quad + 2\hat{F}_{02}^T(k)R_{22}\hat{F}_{02}(k) + 2\hat{G}_{12}^T(k)(R_{11} + R_{21})\hat{G}_{12}(k) \\ &\quad + 2\hat{S}_{02}^T(k)(R_{11} + R_{21})\hat{S}_{02}(k)] \Gamma_0(k)\}, \end{aligned} \quad (18)$$

where

$$\begin{aligned} \varpi_0(k) &= [x^T(k), \tilde{x}^T(k), x^T(k-1), x^T(k-\tau(k)), x^T(k-\rho_1), x^T(k-n), [\bar{\xi} x(k-n)]^T], \\ \Gamma_0(k) &= [y^T(k), \tilde{y}^T(k), y^T(k-1), y^T(k-\delta(k)), g^T(y(k-\delta(k))), y^T(k-\rho_2), h^T(y(k-s)), \\ &\quad h^T(k), [\bar{\mu} h(y(k-s))]^T], \end{aligned}$$

where

$$\begin{aligned} \Lambda_{11} &= \begin{bmatrix} \psi_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_{31} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_{41} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & HR_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I(R_{12} + R_{22}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\xi} R_{52} \end{bmatrix}, \\ \Lambda_{12} &= \begin{bmatrix} \psi_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_{42} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I(R_{11} + R_{21}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\mu} R_{51} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\mu} R_{51} \end{bmatrix}, \end{aligned}$$

$$\psi_{11} = -R_{11} + R_{31} + (\tau_M - \tau_m + 1)R_{41} + \bar{\xi} R_{52}; \quad \psi_{12} = -R_{12} + R_{32} + (\delta_M - \delta_m + 1)R_{42},$$

$$\hat{W}_{01} = [\mathbb{B}_x M, 0, \mathbb{B}_x M, 0, 0, 0, 0]; \quad \hat{W}_{02} = [\mathbb{B}_y N, 0, \mathbb{B}_y N, 0, 0, 0, 0, 0, 0],$$

$$\hat{G}_{01}(k) = [0, 0, 0, 0, -(\mathbb{A} + \Delta \mathbb{A}(k)), 0, 0];$$

$$\begin{aligned}\hat{G}_{02}(k) &= [0, 0, 0, 0, 0, -(\mathbb{C} + \Delta C(k)), 0, 0, 0], \\ \hat{F}_{01}(k) &= [\mathbb{A}_x - \alpha_0 \mathbb{B}_x M, -\mathbb{A}_x, -(1 - \alpha_0) \mathbb{B}_x M, 0, 0, 0, 0]; \\ \hat{F}_{02}(k) &= [\mathbb{A}_y - \beta_0 \mathbb{B}_y N, -\mathbb{A}_y, -(1 - \beta_0) \mathbb{B}_y N, 0, 0, 0, 0, 0], \\ \hat{G}_{11}(k) &= [0, 0, 0, (\mathbb{D} + \Delta D(k)), 0, 0, 0]; \quad \hat{G}_{12}(k) = [0, 0, 0, 0, (\mathbb{B} + \Delta B(k)), 0, 0, 0, 0], \\ \hat{S}_{01}(k) &= [0, 0, 0, 0, 0, (F + \Delta F(k)), 0]; \quad \hat{S}_{02}(k) = [0, 0, 0, 0, 0, 0, (E + \Delta E(k)), 0, 0].\end{aligned}$$

From Assumption 2, we have

$$\begin{bmatrix} y(k - \delta(k)) \\ \hat{g}(y(k - \delta(k))) \end{bmatrix}^T \begin{bmatrix} \tilde{N}_1 & \tilde{N}_2 \\ \tilde{N}_2^T & I \end{bmatrix} \begin{bmatrix} y(k - \delta(k)) \\ \hat{g}(y(k - \delta(k))) \end{bmatrix} \leq 0, \quad (19)$$

where

$$\tilde{N}_1 = \frac{(N_1^T N_2 + N_2^T N_1)}{2}; \quad \tilde{N}_2 = -\frac{(N_1^T + N_2^T)}{2}.$$

Then, from equations (18) and (19), we have

$$\begin{aligned}\mathbb{E}\{\Delta V(k)\} &\leq \mathbb{E}\{\Delta V(k)\} - \mathbb{E}\left\{\lambda \begin{bmatrix} y(k - \delta(k)) \\ \hat{g}(y(k - \delta(k))) \end{bmatrix}^T \begin{bmatrix} \tilde{N}_1 & \tilde{N}_2 \\ \tilde{N}_2^T & I \end{bmatrix} \begin{bmatrix} y(k - \delta(k)) \\ \hat{g}(y(k - \delta(k))) \end{bmatrix}\right\} \\ &= \mathbb{E}\left\{\varpi_0^T(k) [\Lambda_{11} + \sigma_\alpha \hat{W}_{01}^T R_{21} \hat{W}_{01} + 2\hat{G}_{01}^T(k)(R_{11} + R_{21})\hat{G}_{01}(k) \right. \\ &\quad + 2\hat{F}_{01}^T(k)R_{21}\hat{F}_{01}(k) + 2\hat{G}_{11}^T(k)(R_{12} + R_{22})\hat{G}_{11}(k) \\ &\quad + 2\hat{S}_{01}^T(k)(R_{12} + R_{22})\hat{S}_{01}(k)]\varpi_0(k) \\ &\quad + \Gamma_0^T(k) [\Lambda_{22} + \sigma_\beta \hat{W}_{02}^T R_{22} \hat{W}_{02} + 2\hat{G}_{02}^T(k)(R_{12} + R_{22})\hat{G}_{02}(k) \\ &\quad + 2\hat{F}_{02}^T(k)R_{22}\hat{F}_{02}(k) + 2\hat{G}_{12}^T(k)(R_{11} + R_{21})\hat{G}_{12}(k) \\ &\quad \left. + 2\hat{S}_{02}^T(k)(R_{11} + P_{21})\hat{S}_{02}(k)]\Gamma_0(k)\right\}, \quad (20)\end{aligned}$$

where

$$\Lambda_{22} = \begin{bmatrix} \psi_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_{42} - \lambda \tilde{N}_1 & -\lambda \tilde{N}_2^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda \tilde{N}_2 & -\lambda I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I(R_{11} + R_{21}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\mu} R_{51} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\mu} R_{51} \end{bmatrix}.$$

Notice that, since $\Lambda_1 < 0$ and $\Lambda_2 < 0$, there are two scalars $\mu_1 > 0$ and $\mu_2 > 0$ such that

$$\begin{aligned}\hat{\Lambda}_1 &= \Lambda_1 + \mu_1 \begin{bmatrix} I_{2n \times 2n} & 0 \\ 0 & 0 \end{bmatrix} < 0, \\ \hat{\Lambda}_2 &= \Lambda_2 + \mu_2 \begin{bmatrix} I_{2n \times 2n} & 0 \\ 0 & 0 \end{bmatrix} < 0.\end{aligned} \quad (21)$$

Equation (21) implies

$$\begin{aligned} \Lambda_{11} + \mu_1 \begin{bmatrix} I_{2n \times 2n} & 0 \\ 0 & 0 \end{bmatrix} + \sigma_\alpha \hat{W}_{01}^T R_{21} \hat{W}_{01} + 2\hat{G}_{01}^T(k)(R_{11} + R_{21})\hat{G}_{01}(k) + 2\hat{F}_{01}^T(k)R_{21}\hat{F}_{01}(k) \\ + 2\hat{G}_{11}^T(k)(R_{12} + R_{22})\hat{G}_{11}(k) + 2\hat{S}_{01}^T(k)(R_{12} + R_{22})\hat{S}_{01}(k) < 0, \\ \Lambda_{22} + \mu_2 \begin{bmatrix} I_{2n \times 2n} & 0 \\ 0 & 0 \end{bmatrix} + \sigma_\beta \hat{W}_{02}^T R_{22} \hat{W}_{02} + 2\hat{G}_{02}^T(k)(R_{12} + R_{22})\hat{G}_{02}(k) + 2\hat{F}_{02}^T(k)R_{22}\hat{F}_{02}(k) \\ + 2\hat{G}_{12}^T(k)(R_{11} + R_{21})\hat{G}_{12}(k) + 2\hat{S}_{02}^T(k)(R_{11} + R_{21})\hat{S}_{02}(k) < 0. \end{aligned} \quad (22)$$

First we satisfy (21) before proving the exponential stability. Using Lemma 2.4, the above equalities are equivalent to

$$\Lambda_3(k) = \begin{bmatrix} \hat{\Lambda}_{11} & * \\ S_1(k) & J_1 \end{bmatrix} < 0, \quad \Lambda_4(k) = \begin{bmatrix} \hat{\Lambda}_{22} & * \\ S_2(k) & J_2 \end{bmatrix} < 0, \quad (23)$$

where

$$\begin{aligned} \hat{\Lambda}_{11} &= \Lambda_{11} + \mu_1 \begin{bmatrix} I_{2n \times 2n} & 0 \\ 0 & 0 \end{bmatrix}, \\ \hat{\Lambda}_{22} &= \Lambda_{22} + \mu_2 \begin{bmatrix} I_{2n \times 2n} & 0 \\ 0 & 0 \end{bmatrix}, \\ S_1(k) &= \begin{bmatrix} \sqrt{2}(R_{11} + R_{21})\hat{G}_{01}(k) \\ \sqrt{2}R_{21}\hat{F}_{01}(k) \\ \sqrt{\sigma_\alpha}R_{21}\hat{W}_{01} \\ \sqrt{2}(R_{12} + R_{22})\hat{G}_{11}(k) \\ \sqrt{2}(R_{12} + R_{22})\hat{S}_{01}(k) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & \Xi_{15} & 0 & 0 \\ \Xi_{21} & -\sqrt{2}R_{21}\mathbb{A}_x & \Xi_{23} & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\alpha}R_{21}\mathbb{B}_x M & 0 & \sqrt{\sigma_\alpha}R_{21}\mathbb{B}_x M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Xi_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Xi_{55} & 0 \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \Xi_{15} &= -\sqrt{2}(R_{11} + R_{21})(\mathbb{A} + \Delta\mathbb{A}(k)); \\ \Xi_{21} &= \sqrt{2}R_{21}(\mathbb{A}_x - \alpha_0\mathbb{B}_x M); \\ \Xi_{23} &= -\sqrt{2}R_{21}(1 - \alpha_0)\mathbb{B}_x M; \\ \Xi_{44} &= \sqrt{2}(R_{12} + R_{22})(\mathbb{D} + \Delta\mathbb{D}(k)); \\ \Xi_{55} &= \sqrt{2}(R_{12} + R_{22})(F + \Delta F(k)), \end{aligned}$$

$$S_2(k) = \begin{bmatrix} \sqrt{2}(R_{12} + R_{22})\hat{G}_{02}(k) \\ \sqrt{2}R_{22}\hat{F}_{02}(k) \\ \sqrt{\sigma_\beta}R_{22}\hat{W}_{02} \\ \sqrt{2}(R_{11} + R_{21})\hat{G}_{12}(k) \\ \sqrt{2}(R_{11} + R_{21})\hat{S}_{02}(k) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \Theta_{16} & 0 & 0 & 0 \\ \Theta_{21} & -\sqrt{2}R_{22}\mathbb{A}_y & \Theta_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\beta}R_{22}\mathbb{B}_yN & 0 & \sqrt{\sigma_\beta}R_{22}\mathbb{B}_yN & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Theta_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Theta_{57} & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} \Theta_{16} &= -\sqrt{2}(R_{12} + R_{22})(\mathbb{C} + \Delta\mathbb{C}(k)); & \Theta_{21} &= \sqrt{2}R_{22}(\mathbb{A}_y - \beta_0\mathbb{B}_yN); \\ \Theta_{23} &= -\sqrt{2}R_{22}(1 - \beta_0)\mathbb{B}_yN; & \Theta_{45} &= \sqrt{2}(R_{11} + R_{21})(\mathbb{B} + \Delta\mathbb{B}(k)); \\ \Theta_{57} &= \sqrt{2}(R_{11} + R_{21})(E + \Delta E(k)), \\ J_1 &= \text{diag}\{- (R_{11} + R_{21}), -R_{21}, -R_{21}, -(R_{12} + R_{22}), -(R_{12} + R_{22})\}, \\ J_2 &= \text{diag}\{- (R_{12} + R_{22}), -R_{22}, -R_{22}, -(R_{11} + R_{21}), -(R_{11} + R_{21})\}. \end{aligned}$$

Note that $S_1(k)$ and $S_2(k)$ can be decomposed as

$$\begin{aligned} S_1(k) &= S_1 + \Delta S_1(k), \\ S_2(k) &= S_2 + \Delta S_2(k), \end{aligned} \tag{24}$$

where

$$S_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \bar{\Xi}_{15} & 0 & 0 \\ \bar{\Xi}_{21} & -\sqrt{2}R_{21}\mathbb{A}_x & \bar{\Xi}_{23} & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\alpha}R_{21}\mathbb{B}_xM & 0 & \sqrt{\sigma_\alpha}R_{21}\mathbb{B}_xM & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\Xi}_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\Xi}_{55} & 0 \end{bmatrix},$$

$$\begin{aligned} \bar{\Xi}_{15} &= -\sqrt{2}(R_{11} + R_{21})\mathbb{A}; & \bar{\Xi}_{44} &= \sqrt{2}(R_{12} + R_{22})\mathbb{D}; \\ \bar{\Xi}_{55} &= \sqrt{2}(R_{12} + R_{22})F, \end{aligned}$$

$$\Delta S_1(k) = \begin{bmatrix} 0 & 0 & 0 & 0 & -\sqrt{2}(R_{11} + R_{21})\Delta\mathbb{A}(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}(R_{12} + R_{22})\Delta\mathbb{D}(k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}(R_{12} + R_{22})\Delta F(k) & 0 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \bar{\Theta}_{16} & 0 & 0 & 0 \\ \Theta_{21} & -\sqrt{2}R_{22}\mathbb{A}_y & \Theta_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\beta}R_{22}\mathbb{B}_yN & 0 & \sqrt{\sigma_\beta}R_{22}\mathbb{B}_yN & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{\Theta}_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Theta}_{57} & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned}\bar{\Theta}_{16} &= -\sqrt{2}(R_{12} + R_{22})\mathbb{C}; & \bar{\Theta}_{45} &= \sqrt{2}(R_{11} + R_{21})\mathbb{B}; & \bar{\Theta}_{57} &= \sqrt{2}(R_{11} + R_{21})E, \\ \Delta S_2(k) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \kappa_{16} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \kappa_{57} & 0 & 0 \end{bmatrix},\end{aligned}$$

where $\kappa_{16} = -\sqrt{2}(R_{12} + R_{22})\Delta C(k)$, $\kappa_{45} = \sqrt{2}(R_{11} + R_{21})\Delta B(k)$, $\kappa_{57} = \sqrt{2}(R_{11} + R_{21})\Delta E(k)$.

From Assumption 1, it follows readily that

$$\Delta S_1(k) = \bar{T}_1 N(k) \bar{W}_1, \quad \Delta S_2(k) = \bar{T}_2 N(k) \bar{W}_2, \quad (25)$$

where

$$\begin{aligned}\bar{T}_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & -\sqrt{2}(R_{11} + R_{21})T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}(R_{12} + R_{22})T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}(R_{12} + R_{22})T & 0 \end{bmatrix}, \\ \bar{T}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\sqrt{2}(R_{12} + R_{22})T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}(R_{11} + R_{21})T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}(R_{11} + R_{21})T & 0 & 0 \end{bmatrix}, \\ \bar{W}_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & W_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & W_6 & 0 \end{bmatrix}, \\ \bar{W}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & W_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & W_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & W_5 & 0 & 0 \end{bmatrix}.\end{aligned}$$

Note that $\Lambda_3(k)$ and $\Lambda_4(k)$ can be decomposed as follows:

$$\Lambda_3(k) = \Lambda_3 + \Delta \Lambda_3(k), \quad \Lambda_4(k) = \Lambda_4 + \Delta \Lambda_4(k), \quad (26)$$

where

$$\begin{aligned}\Lambda_3 &= \begin{bmatrix} \hat{\Lambda}_{11} & * \\ S_1 & J_1 \end{bmatrix} < 0, & \Delta \Lambda_3(k) &= \begin{bmatrix} 0 & * \\ \Delta S_1(k) & 0 \end{bmatrix}, \\ \Lambda_4 &= \begin{bmatrix} \hat{\Lambda}_{22} & * \\ S_2 & J_2 \end{bmatrix} < 0 \quad \text{and} \quad \Delta \Lambda_4(k) &= \begin{bmatrix} 0 & * \\ \Delta S_2(k) & 0 \end{bmatrix}.\end{aligned}$$

Let

$$\begin{aligned}\tilde{T}_1^T &= [0, \tilde{T}_1^T], & \tilde{W}_1 &= [\tilde{W}_1, 0], \\ \tilde{T}_2^T &= [0, \tilde{T}_2^T], & \tilde{W}_2 &= [\tilde{W}_2, 0].\end{aligned}$$

Using Lemma 2.3(i), $\Delta\Lambda_3(k)$ and $\Delta\Lambda_4(k)$ can be rewritten as

$$\begin{aligned}\Delta\Lambda_3(k) &= \tilde{T}_1 N(k) \tilde{W}_1 + \tilde{W}_1^T N^T(k) \tilde{T}_1^T \leq \varepsilon_1^{-1} \tilde{T}_1 \tilde{T}_1^T + \varepsilon_1 \tilde{W}_1^T \tilde{W}_1, \\ \Delta\Lambda_4(k) &= \tilde{T}_2 N(k) \tilde{W}_2 + \tilde{W}_2^T N^T(k) \tilde{T}_2^T \leq \varepsilon_2^{-1} \tilde{T}_2 \tilde{T}_2^T + \varepsilon_2 \tilde{W}_2^T \tilde{W}_2.\end{aligned}\quad (27)$$

It is clear from equations (26) and (27) that

$$\Lambda_3(k) \leq \Lambda'_3 + \varepsilon_1^{-1} \tilde{T}_1 \tilde{T}_1^T, \quad \Lambda_4(k) \leq \Lambda'_4 + \varepsilon_2^{-1} \tilde{T}_2 \tilde{T}_2^T, \quad (28)$$

where

$$\begin{aligned}\Lambda'_3 &= \begin{bmatrix} \Lambda'_{11} + \mu_1 \begin{bmatrix} I_{2n \times 2n} & 0 \\ 0 & 0 \end{bmatrix} & * \\ S_1 & J_1 \end{bmatrix}, \\ \Lambda'_4 &= \begin{bmatrix} \Lambda'_{22} + \mu_2 \begin{bmatrix} I_{2n \times 2n} & 0 \\ 0 & 0 \end{bmatrix} & * \\ S_2 & J_2 \end{bmatrix}.\end{aligned}$$

It follows from Lemma 2.4 that equation (22) is equivalent to the case that the right-hand side of equation (28) is negative definite. Hence, we come to the conclusion that $\Lambda_3(k) < 0$ and $\Lambda_4(k) < 0$, and therefore equation (22) holds. Moreover, the combination of equations (20) and (22) leads to

$$\mathbb{E}\{\Delta V(k)\} \leq \mu_1 \mathbb{E}\{|\bar{x}(k)|^2\} - \mu_2 \mathbb{E}\{|\bar{y}(k)|^2\}. \quad (29)$$

We are in a position to prove the stability of system (8). First, from equation (10), it is easily verified that

$$\begin{aligned}\mathbb{E}\{\Delta V(k)\} &\leq \varepsilon_{11} \mathbb{E}\{|\bar{x}(k)|^2\} + \varepsilon_{21} \sum_{i=k-\tau_M}^{k-1} \mathbb{E}\{|\bar{x}(i)|^2\} \\ &\quad + \varepsilon_{12} \mathbb{E}\{|\bar{y}(k)|^2\} + \varepsilon_{22} \sum_{i=k-\delta_M}^{k-1} \mathbb{E}\{|\bar{y}(i)|^2\},\end{aligned}\quad (30)$$

where

$$\begin{aligned}\varepsilon_{11} &= \max\{\lambda_{\max}(R_{11}), \lambda_{\max}(R_{21}), \lambda_{\max}(R_{52})\}, \\ \varepsilon_{21} &= (\tau_M - \tau_m + 1)(\lambda_{\max}(R_{31}) + \lambda_{\max}(R_{41})), \\ \varepsilon_{12} &= \max\{\lambda_{\max}(R_{12}), \lambda_{\max}(R_{22}), \lambda_{\max}(R_{51})\}, \\ \varepsilon_{22} &= (\delta_M - \delta_m + 1)(\lambda_{\max}(R_{32}) + \lambda_{\max}(R_{42})).\end{aligned}$$

For any scalar $\zeta > 1$, the above inequality, combined with equation (29), indicates that

$$\begin{aligned}
 \zeta^{k+1} \mathbb{E}\{\mathbb{V}(k+1)\} - \zeta^k \mathbb{E}\{\mathbb{V}(k)\} &= \zeta^{k+1} \mathbb{E}\{\Delta \mathbb{V}(k)\} + \zeta^k (\zeta - 1) \mathbb{E}\{\mathbb{V}(k)\} \\
 &\leq -\zeta^{k+1} (\mu_1 \mathbb{E}\{|\bar{x}(k)|^2\} - \mu_2 \mathbb{E}\{|\bar{y}(k)|^2\}) + \zeta^k (\zeta - 1), \\
 &\left(\varepsilon_{11} \mathbb{E}\{|\bar{x}(k)|^2\} + \varepsilon_{21} \sum_{i=k-\tau_M}^{k-1} \mathbb{E}\{|\bar{x}(i)|^2\} + \varepsilon_{12} \mathbb{E}\{|\bar{y}(k)|^2\} + \varepsilon_{22} \mathbb{E}\{|\bar{y}(i)|^2\} \right) \\
 &= \zeta^k \eta_{11}(\zeta) \mathbb{E}\{|\bar{x}(k)|^2\} + \zeta^k \eta_{21}(\zeta) \sum_{i=k-\tau_M}^{k-1} \mathbb{E}\{|\bar{x}(i)|^2\} \\
 &\quad + \zeta^k \eta_{12}(\zeta) \mathbb{E}\{|\bar{y}(k)|^2\} + \zeta^k \eta_{22}(\zeta) \sum_{i=k-\delta_M}^{k-1} \mathbb{E}\{|\bar{y}(i)|^2\}, \tag{31}
 \end{aligned}$$

where

$$\begin{aligned}
 \eta_{11}(\zeta) &= -\zeta \mu_1 + (\zeta - 1) \varepsilon_{11}, & \eta_{21}(\zeta) &= (\zeta - 1) \varepsilon_{21}, \\
 \eta_{12}(\zeta) &= -\zeta \mu_2 + (\zeta - 1) \varepsilon_{12} & \text{and} & \quad \eta_{22}(\zeta) = (\zeta - 1) \varepsilon_{22}.
 \end{aligned}$$

In addition, for any integer $N \geq \max\{\delta_M, \tau_M\} + 1$, summing both sides of equation (31) from 0 to $N - 1$ with respect to k , we have

$$\begin{aligned}
 \zeta^N \mathbb{E}\{\mathbb{V}(N)\} - \mathbb{E}\{\mathbb{V}(0)\} &\leq \eta_{11}(\zeta) \sum_{k=0}^{N-1} \zeta^k \mathbb{E}\{|\bar{x}(k)|^2\} + \eta_{21}(\zeta) \sum_{k=0}^{N-1} \sum_{i=k-\tau_M}^{k-1} \zeta^k \mathbb{E}\{|\bar{x}(i)|^2\} \\
 &\quad + \eta_{12}(\zeta) \sum_{k=0}^{N-1} \zeta^k \mathbb{E}\{|\bar{y}(k)|^2\} + \eta_{22}(\zeta) \sum_{k=0}^{N-1} \sum_{i=k-\delta_M}^{k-1} \zeta^k \mathbb{E}\{|\bar{y}(i)|^2\}. \tag{32}
 \end{aligned}$$

Note that, for $\tau_M, \delta_M \geq 1$,

$$\begin{aligned}
 \sum_{k=0}^{N-1} \sum_{i=k-\tau_M}^{k-1} \zeta^k \mathbb{E}\{|\bar{x}(i)|^2\} &\leq \tau_M \zeta^{\tau_M} \max_{-\tau_M \leq i \leq 0} \mathbb{E}\{|\Omega(i)|^2\} + \tau_M \zeta^{\tau_M} \sum_{i=0}^{N-1} \zeta^i \mathbb{E}\{|\bar{x}(k)|^2\}, \\
 \sum_{k=0}^{N-1} \sum_{i=k-\delta_M}^{k-1} \zeta^k \mathbb{E}\{|\bar{y}(i)|^2\} &\leq \delta_M \zeta^{\delta_M} \max_{-\delta_M \leq i \leq 0} \mathbb{E}\{|\Pi(i)|^2\} + \delta_M \zeta^{\delta_M} \sum_{i=0}^{N-1} \zeta^i \mathbb{E}\{|\bar{y}(k)|^2\}. \tag{33}
 \end{aligned}$$

Then, from equations (32) and (33), one has

$$\begin{aligned}
 \zeta^N \mathbb{E}\{\mathbb{V}(N)\} &\leq \mathbb{E}\{\mathbb{V}(0)\} + [\eta_{11}(\zeta) + \tau_M \zeta^{\tau_M} \eta_{21}(\zeta)] \sum_{k=0}^{N-1} \zeta^k \mathbb{E}\{|\bar{x}(k)|^2\} \\
 &\quad + \tau_M \zeta^{\tau_M} \eta_{21}(\zeta) \max_{-\tau_M \leq i \leq 0} \mathbb{E}\{|\Omega(i)|^2\} \\
 &\quad + [\eta_{12}(\zeta) + \delta_M \zeta^{\delta_M} \eta_{22}(\zeta)] \sum_{k=0}^{N-1} \zeta^k \mathbb{E}\{|\bar{y}(k)|^2\} \\
 &\quad + \delta_M \zeta^{\delta_M} \eta_{22}(\zeta) \max_{-\delta_M \leq i \leq 0} \mathbb{E}\{|\Pi(i)|^2\}. \tag{34}
 \end{aligned}$$

Let

$$\begin{aligned}\varepsilon_{01} &= \min\{\lambda_{\min}(R_{11}), \lambda_{\min}(R_{21}), \lambda_{\min}(R_{52})\}, & \tilde{\varepsilon}_1 &= \max\{\varepsilon_{11}, \varepsilon_{21}\}, \\ \varepsilon_{02} &= \min\{\lambda_{\min}(R_{12}), \lambda_{\min}(R_{22}), \lambda_{\min}(R_{51})\}, & \tilde{\varepsilon}_2 &= \max\{\varepsilon_{12}, \varepsilon_{22}\}.\end{aligned}$$

It is clear that

$$\mathbb{E}\{\mathbb{V}(N)\} \geq \varepsilon_{01} \mathbb{E}\{|\tilde{x}(N)|^2\} + \varepsilon_{02} \mathbb{E}\{|\tilde{y}(N)|^2\}. \quad (35)$$

It follows readily from equation (30) that

$$\mathbb{E}\{\mathbb{V}(0)\} \leq \tilde{\varepsilon}_1 \max_{-\tau_M \leq i \leq 0} \mathbb{E}\{|\Omega(i)|^2\} + \tilde{\varepsilon}_2 \max_{-\delta_M \leq i \leq 0} \mathbb{E}\{|\Pi(i)|^2\}. \quad (36)$$

Additionally, it can be verified that there exists a scalar $\zeta_0 > 1$ such that

$$\begin{aligned}\eta_{11}(\zeta_0) + \tau_M \zeta_0^{\tau_M} \eta_{21}(\zeta_0) &= 0, \\ \eta_{12}(\zeta_0) + \delta_M \zeta_0^{\delta_M} \eta_{22}(\zeta_0) &= 0.\end{aligned} \quad (37)$$

Substituting equations (35)–(37) into equation (34), we can get

$$\begin{aligned}\varepsilon_{01} \mathbb{E}\{|\tilde{x}(N)|^2\} + \varepsilon_{02} \mathbb{E}\{|\tilde{y}(N)|^2\} &\leq (\tilde{\varepsilon}_1 + \tau_M \zeta_0^{\tau_M} \eta_{21}(\zeta_0)) \max_{-\tau_M \leq i \leq 0} \mathbb{E}\{|\Omega(i)|^2\} \\ &\quad + (\tilde{\varepsilon}_2 + \delta_M \zeta_0^{\delta_M} \eta_{22}(\zeta_0)) \max_{-\delta_M \leq i \leq 0} \mathbb{E}\{|\Pi(i)|^2\}.\end{aligned} \quad (38)$$

The above equation (38) completes the proof of exponential stability with $\nu_x(k) = 0$ and $\nu_y(k) = 0$. $\square$

Remark 3.1 In this paper, we have considered the time-varying delays $\delta(k)$, $\tau(k)$ and the leakage delays ρ_1 , ρ_2 in the negative feedback term of the GRNs which lead to the instability of the systems with small amount of leakage delay. This paper is to establish techniques to accord with the robust H_∞ state estimation concern for uncertain discrete stochastic GRNs (equation (1)) with leakage delays, distributed delays, and probabilistic measurement delays.

Consider that the H_∞ attainment of the estimation error system (8) is robustly stochastically stable with non-zero exogenous disturbance signals $\nu_x(k), \nu_y(k) \in L_2[0, \infty)$.

Theorem 3.2 *Let Assumptions 1 and 2 hold. Let the leakage delays ρ_1 , ρ_2 and the estimation parameters $\mathbb{A}_x, \mathbb{B}_x, \mathbb{A}_y, \mathbb{B}_y$, and $\gamma > 0$ be given. Then the estimation error system (8) is robustly stochastically stable with disturbance attenuation γ , if there exist positive definite matrices $R_{11}, R_{12}, R_{21}, R_{22}, R_{31}, R_{32}, R_{41}, R_{42}, R_{51}, R_{52}$ and three positive constant scalars λ , ε_1 , and ε_2 such that the following LMI holds:*

$$\Lambda_1 = \begin{bmatrix} \Lambda'_{11} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ S_1 & 0 & J_1 & * \\ 0 & 0 & \tilde{T}_1^T & -\varepsilon_1 I \end{bmatrix} < 0, \quad \Lambda_2 = \begin{bmatrix} \Lambda'_{22} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ S_2 & 0 & J_2 & * \\ 0 & 0 & \tilde{T}_2^T & -\varepsilon_2 I \end{bmatrix} < 0, \quad (39)$$

and the other variables are described in Theorem 3.1.

Proof Choose the Lyapunov–Krasovskii function (equation (10)) as in Theorem 3.1. For given $\gamma > 0$, we define

$$T(n) = \mathbb{E} \sum_{k=0}^n [\bar{x}^T(k) \bar{x}(k) + \bar{y}^T(k) \bar{y}(k) - \gamma^2 v_x^T(k) v_x(k) - \gamma^2 v_y^T(k) v_y(k)]. \quad (40)$$

Here, n is a nonnegative integer. Our aim is to show $T(n) < 0$. Under the zero initial condition, we have

$$\begin{aligned} T(n) &= \mathbb{E} \sum_{k=0}^n [\bar{x}^T(k) \bar{x}(k) + \bar{y}^T(k) \bar{y}(k) - \gamma^2 v_x^T(k) v_x(k) - \gamma^2 v_y^T(k) v_y(k) + \Delta \mathbb{V}(k)] \\ &\quad - \mathbb{E} \mathbb{V}(n+1) \\ &\leq T(n) + \sum_{k=0}^n \mathbb{E}(\Delta \mathbb{V}(k)) \\ &= \sum_{k=0}^n \mathbb{E} \{ \varpi^T(k) [\tilde{\Lambda}_{11} + \sigma_\alpha \tilde{W}_{01}^T R_{21} \tilde{W}_{01} + 2\tilde{G}_{01}^T(k)(R_{11} + R_{21})\tilde{G}_{01}(k) \\ &\quad + 2\tilde{F}_{01}^T(k)R_{21}\tilde{F}_{01}(k) + 2\tilde{G}_{11}^T(k)(R_{12} + R_{22})\tilde{G}_{11}(k) \\ &\quad + 2\tilde{S}_{01}^T(k)(R_{12} + R_{22})\tilde{S}_{01}(k)] \varpi(k) \\ &\quad + \Gamma^T(k) [\tilde{\Lambda}_{22} + \sigma_\beta \tilde{W}_{02}^T R_{22} \tilde{W}_{02} + 2\tilde{G}_{02}^T(k)(R_{12} + R_{22})\tilde{G}_{02}(k) \\ &\quad + 2\tilde{F}_{02}^T(k)R_{22}\tilde{F}_{02}(k) + 2\tilde{G}_{12}^T(k)(R_{11} + R_{21})\tilde{G}_{12}(k) \\ &\quad + 2\tilde{S}_{02}^T(k)(R_{11} + R_{21})\tilde{S}_{02}(k)] \Gamma(k) \}, \end{aligned} \quad (41)$$

where

$$\begin{aligned} \varpi(k) &= [\varpi_0(k), v_x(k)]^T, \quad \Gamma(k) = [\Gamma_0(k), v_y(k)]^T, \quad \tilde{W}_{01} = [\hat{W}_{01}^T, 0], \\ \tilde{G}_{01}(k) &= [\hat{G}_{01}^T(k), 0], \quad \tilde{F}_{01}(k) = [\hat{F}_{01}^T(k), 0], \quad \tilde{G}_{11}(k) = [\hat{G}_{11}^T(k), 0], \\ \tilde{W}_{02} &= [\hat{W}_{02}^T, 0], \quad \tilde{G}_{02}(k) = [\hat{G}_{02}^T(k), 0], \quad \tilde{F}_{02}(k) = [\hat{F}_{02}^T(k), 0], \\ \tilde{G}_{12}(k) &= [\hat{G}_{12}^T(k), 0], \quad \tilde{S}_{01}(k) = [\hat{S}_{01}^T(k), 0], \quad \tilde{S}_{02}(k) = [\hat{S}_{02}^T(k), 0], \\ \tilde{\Lambda}_{11} &= \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \quad \text{and} \quad \tilde{\Lambda}_{22} = \begin{bmatrix} \Lambda_{22} & 0 \\ 0 & -\gamma^2 I \end{bmatrix}. \end{aligned}$$

By equation (41), in order to assure $T(n) < 0$, we just need to show

$$\begin{aligned} &\tilde{\Lambda}_{11} + \sigma_\alpha \tilde{W}_{01}^T R_{21} \tilde{W}_{01} + 2\tilde{G}_{01}^T(k)(R_{11} + R_{21})\tilde{G}_{01}(k) + 2\tilde{F}_{01}^T(k)R_{21}\tilde{F}_{01}(k) \\ &\quad + 2\tilde{G}_{11}^T(k)(R_{12} + R_{22})\tilde{G}_{11}(k) + 2\tilde{S}_{01}^T(k)(R_{12} + R_{22})\tilde{S}_{01}(k) < 0, \\ &\tilde{\Lambda}_{22} + \sigma_\beta \tilde{W}_{02}^T R_{22} \tilde{W}_{02} + 2\tilde{G}_{02}^T(k)(R_{12} + R_{22})\tilde{G}_{02}(k) + 2\tilde{F}_{02}^T(k)R_{22}\tilde{F}_{02}(k) \\ &\quad + 2\tilde{G}_{12}^T(k)(R_{11} + R_{21})\tilde{G}_{12}(k) + 2\tilde{S}_{02}^T(k)(R_{11} + R_{21})\tilde{S}_{02}(k) < 0, \end{aligned} \quad (42)$$

which, by Lemma 2.4, is equivalent to

$$\tilde{\Lambda}_3(k) = \begin{bmatrix} \tilde{\Lambda}_{11} & * \\ \tilde{S}_1(k) & J_1 \end{bmatrix} < 0 \quad \text{and} \quad \tilde{\Lambda}_4(k) = \begin{bmatrix} \Lambda_{22} & * \\ \tilde{S}_2(k) & J_2 \end{bmatrix} < 0, \quad (43)$$

where

$$\tilde{S}_1(k) = \bar{S}_1 + \Delta \bar{S}_1(k) = [S_1, 0] + [\Delta S_1(k), 0],$$

$$\tilde{S}_2(k) = \bar{S}_2 + \Delta \bar{S}_2(k) = [S_2, 0] + [\Delta S_2(k), 0]$$

and J_1 and J_2 are defined in Theorem 3.1. Note that $\tilde{\Lambda}_3(k)$ and $\tilde{\Lambda}_4(k)$ can be rearranged as follows:

$$\tilde{\Lambda}_3(k) = \tilde{\Lambda}_3 + \Delta \tilde{\Lambda}_3(k), \quad \tilde{\Lambda}_4(k) = \tilde{\Lambda}_4 + \Delta \tilde{\Lambda}_4(k), \quad (44)$$

where

$$\tilde{\Lambda}_3 = \begin{bmatrix} \tilde{\Lambda}_{11} & * \\ \tilde{S}_1 & J_1 \end{bmatrix} < 0 \quad \text{and} \quad \Delta \tilde{\Lambda}_3(k) = \begin{bmatrix} 0 & * \\ \Delta \tilde{S}_1(k) & 0 \end{bmatrix},$$

$$\tilde{\Lambda}_4 = \begin{bmatrix} \tilde{\Lambda}_{22} & * \\ \tilde{S}_2 & J_2 \end{bmatrix} < 0 \quad \text{and} \quad \Delta \tilde{\Lambda}_4(k) = \begin{bmatrix} 0 & * \\ \Delta \tilde{S}_2(k) & 0 \end{bmatrix}.$$

Let

$$\begin{aligned} \tilde{T}_1^T &= [0, \tilde{T}_1^T], & \tilde{W}_1 &= [\tilde{W}_1, 0], & \tilde{T}_2^T &= [0, \tilde{T}_2^T], & \tilde{W}_2 &= [\tilde{W}_2^T, 0], \\ \tilde{T}_1^T &= [0, 0, \tilde{T}_1^T], & \tilde{W}_1 &= [\tilde{W}_1, 0, 0], & \tilde{T}_2^T &= [0, 0, \tilde{T}_2^T] \quad \text{and} \quad \tilde{W}_2 &= [\tilde{W}_2, 0, 0]. \end{aligned}$$

Using Lemma 2.3(i), $\Delta \tilde{\Lambda}_3(k)$ and $\Delta \tilde{\Lambda}_4(k)$ can be rewritten as

$$\begin{aligned} \Delta \tilde{\Lambda}_3(k) &= \tilde{T}_1 N(k) \tilde{W}_1 + \tilde{W}_1^T N^T(k) \tilde{T}_1^T \leq \epsilon_1^{-1} \tilde{T}_1 \tilde{T}_1^T + \epsilon_1 \tilde{W}_1^T \tilde{W}_1, \\ \Delta \tilde{\Lambda}_4(k) &= \tilde{T}_2 N(k) \tilde{W}_2 + \tilde{W}_2^T N^T(k) \tilde{T}_2^T \leq \epsilon_2^{-1} \tilde{T}_2 \tilde{T}_2^T + \epsilon_1 \tilde{W}_2^T \tilde{W}_2. \end{aligned} \quad (45)$$

It is implied from equations (44) and (45) that

$$\begin{aligned} \tilde{\Lambda}_3(k) &\leq \begin{bmatrix} \Lambda'_{11} & * & * \\ 0 & -\gamma^2 I & * \\ S_1 & 0 & J_1 \end{bmatrix} + \epsilon_1^{-1} \tilde{T}_1 \tilde{T}_1^T, \\ \tilde{\Lambda}_4(k) &\leq \begin{bmatrix} \Lambda'_{22} & * & * \\ 0 & -\gamma^2 I & * \\ S_2 & 0 & J_2 \end{bmatrix} + \epsilon_2^{-1} \tilde{T}_2 \tilde{T}_2^T. \end{aligned} \quad (46)$$

Using Lemma 2.4, the above inequality (45) holds if and only if the right-hand side of (45) is negative definite, which implies $T(n) < 0$. Letting $n \rightarrow \infty$, we have

$$\mathbb{E} \sum_{k=0}^{\infty} \{ |\tilde{x}(k)|^2 + |\tilde{y}(k)|^2 \} \leq \gamma^2 \mathbb{E} \sum_{k=0}^{\infty} (|v_x(k)|^2 + |v_y(k)|^2).$$

Hence the proof of Theorem 3.2 is complete. $\square$

Theorem 3.3 *With the help of the assumptions, system (7) becomes a robust H_∞ state estimator of GRNs (1) with leakage delays, distributed delays, and probabilistic measurement delays (5) if there exist positive definite matrices $X_1, X_2, Y_1, Y_2, R_{11}, R_{12}, R_{21}, R_{22}, R_{31}, R_{32}, R_{41}, R_{42}, R_{51}$, and R_{52} and three positive constant scalars λ, ε_1 , and ε_2 such that the following LMIs hold:*

$$\Lambda_1 = \begin{bmatrix} \Lambda'_{11} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ S'_1 & 0 & J_1 & * \\ 0 & 0 & \bar{T}_1^T & -\varepsilon_1 I \end{bmatrix} < 0, \quad \Lambda_2 = \begin{bmatrix} \Lambda'_{22} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ S'_2 & 0 & J_2 & * \\ 0 & 0 & \bar{T}_2^T & -\varepsilon_2 I \end{bmatrix} < 0,$$

where

$$S'_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \Sigma_{15} & 0 & 0 \\ \sqrt{2}(X_1 - \alpha_0 X_2 M) & -\sqrt{2}X_1 & -\sqrt{2}(1 - \alpha_0)X_2 M & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\alpha}X_2 M & 0 & \sqrt{\sigma_\alpha}X_2 M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}(R_{12} + R_{22})\mathbb{D} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Sigma_{56} & 0 \end{bmatrix},$$

$$\Sigma_{15} = -\sqrt{2}(R_{11} + R_{21})A; \quad \Sigma_{56} = \sqrt{2}(R_{12} + R_{22})F,$$

$$S'_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \Upsilon_{16} & 0 & 0 & 0 \\ \sqrt{2}(Y_1 - \beta_0 Y_2 N) & -\sqrt{2}Y_1 & -\sqrt{2}(1 - \beta_0)Y_2 N & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\beta}Y_2 N & 0 & \sqrt{\sigma_\beta}Y_2 N & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}(R_{11} + R_{21})\mathbb{B} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_{57} & 0 & 0 \end{bmatrix},$$

$$\Upsilon_{16} = -\sqrt{2}(R_{12} + R_{22})C; \quad \Upsilon_{57} = \sqrt{2}(R_{11} + R_{21})E,$$

and the other variables are described in Theorem 3.1. Furthermore, the state estimator gain matrices can be described as follows:

$$\mathbb{A}_x = R_{21}^{-1}X_1, \quad \mathbb{B}_x = R_{21}^{-1}X_2, \quad \mathbb{A}_y = R_{22}^{-1}Y_1 \quad \text{and} \quad \mathbb{B}_y = R_{22}^{-1}Y_2.$$

Proof The rest of the proof of this theorem is the same as that of Theorem 3.2. Due to the limitation of the length of this paper, we omit it here. Then the proof of Theorem 3.3 is completed. $\square$

Consider the discrete-time genetic regulatory network system:

$$\begin{aligned} x(k+1) &= -\mathbb{A}x(k - \rho_1) + \mathbb{B}\hat{g}(y(k - \delta(k))) + E \sum_{s=1}^{\infty} \mu_s h(y(k - s)) \\ &\quad + \sigma(k, x(k - \rho_1))\omega(k) + L_x v_x(k), \\ y(k+1) &= -\mathbb{C}y(k - \rho_2) + \mathbb{D}x(k - \tau(k)) + F \sum_{n=1}^{\infty} \xi_n x(k - n) + L_y v_y(k). \end{aligned} \quad (47)$$

Corollary 3.1 *Let the leakage delays ρ_1, ρ_2 and the estimation parameters $\mathbb{A}_x, \mathbb{B}_x, \mathbb{A}_y$, and $\mathbb{B}_y$ be given and also the acceptable conditions hold. Then the estimation error system (8) with $v_x(k) = 0$ and $v_y(k) = 0$ is robustly exponentially stable in the mean square if there*

exist positive definite matrices $R_{11}, R_{12}, R_{21}, R_{22}, R_{31}, R_{32}, R_{41}, R_{42}, R_{51}, R_{52}$ and the positive constant scalar λ such that the following LMI holds:

$$\Lambda_1 = \begin{bmatrix} \Lambda'_{11} & * & * \\ S_1 & J_1 & * \\ 0 & 0 & I \end{bmatrix} < 0, \quad \Lambda_2 = \begin{bmatrix} \Lambda'_{22} & * & * \\ S_2 & J_2 & * \\ 0 & 0 & I \end{bmatrix} < 0, \quad (48)$$

where

$$\begin{aligned} \Lambda'_{11} &= \begin{bmatrix} \psi_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_{31} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_{41} + \varepsilon_1 W_4^T W_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & HR_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I(R_{12} + R_{22}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\xi} R_{52} \end{bmatrix}, \\ \Lambda'_{22} &= \begin{bmatrix} \psi_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_{42} - \lambda \tilde{N}_1 + \varepsilon_2 W_2^T W_2 & -\lambda \tilde{N}_2^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda \tilde{N}_2 & -\lambda I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I(R_{11} + R_{21}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\mu} R_{51} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\mu} R_{51} \end{bmatrix}, \\ S_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & \bar{\Xi}_{15} & 0 & 0 \\ \Xi_{21} & -\sqrt{2}R_{21}\mathbb{A}_x & \Xi_{23} & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\alpha}R_{21}\mathbb{B}_x M & 0 & \sqrt{\sigma_\alpha}R_{21}\mathbb{B}_x M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\Xi}_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\Xi}_{55} & 0 \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \bar{\Xi}_{15} &= -\sqrt{2}(R_{11} + R_{21})\mathbb{A}; & \bar{\Xi}_{44} &= \sqrt{2}(R_{12} + R_{22})\mathbb{D}; & \bar{\Xi}_{55} &= \sqrt{2}(R_{12} + R_{22})F; \\ \Xi_{21} &= \sqrt{2}R_{21}(\mathbb{A}_x - \alpha_0\mathbb{B}_x M); & \Xi_{23} &= -\sqrt{2}R_{21}(1 - \alpha_0)\mathbb{B}_x M, \\ S_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \bar{\Theta}_{16} & 0 & 0 & 0 \\ \Theta_{21} & -\sqrt{2}R_{22}\mathbb{A}_y & \Theta_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\beta}R_{22}\mathbb{B}_y N & 0 & \sqrt{\sigma_\beta}R_{22}\mathbb{B}_y N & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{\Theta}_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Theta}_{57} & 0 & 0 \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \bar{\Theta}_{16} &= -\sqrt{2}(R_{12} + R_{22})\mathbb{C}; & \bar{\Theta}_{45} &= \sqrt{2}(R_{11} + R_{21})\mathbb{B}; & \bar{\Theta}_{57} &= \sqrt{2}(R_{11} + R_{21})E; \\ \Theta_{21} &= \sqrt{2}R_{22}(\mathbb{A}_y - \beta_0\mathbb{B}_y N); & \Theta_{23} &= -\sqrt{2}R_{22}(1 - \beta_0)\mathbb{B}_y N, \\ J_1 &= \text{diag}\{- (R_{11} + R_{21}), -R_{21}, -R_{21}, -(R_{12} + R_{22}), -(R_{12} + R_{22})\}, \\ J_2 &= \text{diag}\{- (R_{12} + R_{22}), -R_{22}, -R_{22}, -(R_{11} + R_{21}), -(R_{11} + R_{21})\}, \\ \bar{\mu} &= \sum_{s=1}^{\infty} \mu_s, & \bar{\xi} &= \sum_{n=1}^{\infty} \xi_n, \end{aligned}$$

$$\begin{aligned}\psi_{11} &= -R_{11} + R_{31} + (\tau_M - \tau_m + 1)R_{41} + \bar{\xi}R_{52}; & \psi_{12} &= -R_{12} + R_{32} + (\delta_M - \delta_m + 1)R_{42}, \\ \tilde{N}_1 &= \frac{(N_1^T N_2 + N_2^T N_1)}{2}; & \tilde{N}_2 &= -\frac{(N_1^T + N_2^T)}{2}.\end{aligned}$$

4 Numerical examples

In this part, two mathematical examples with simulations are provided to show the effectiveness of the proposed robust state estimator.

Example 4.1 Consider the discrete-time GRN (1) with parameters given as follows:

$$\begin{aligned}A &= \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}, & B &= \begin{pmatrix} 0.08 & 0 \\ 0 & 0.2 \end{pmatrix}, \\ C = D &= \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, & E &= \begin{pmatrix} 0.36 & 0 \\ 0 & 0.1 \end{pmatrix}, \\ d_1 &= \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, & d_2 &= \begin{pmatrix} 0.28 & 0 \\ 0 & 0.135 \end{pmatrix}, \\ L_x &= \begin{pmatrix} 0.2 & 0 \\ 0 & 0.5 \end{pmatrix}, & L_y &= \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix}, \\ W_1 = W_2 = W_3 = W_4 = W_5 = W_6 &= \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix}, & \mu = \xi &= \exp(-2), \\ F = 0.4I, & R = 0.2I, & G &= \begin{pmatrix} \sin(k) & 0 \\ 0 & \cos(k) \end{pmatrix},\end{aligned}$$

and the leakage delays $\rho_1 = \rho_2 = 1$. The regulatory function is taken as $g(s) = \frac{s^2}{1+s^2}$. The time-varying delays are chosen as $\delta(k) = 3 + (2 * \sin(k * \pi/2))$ and $\tau(k) = 3 + (2 * \cos(k * \pi/2))$, and the exogenous disturbance inputs are selected as $v_x(k) = \sin(6k) \exp(-0.1k)$ and $v_y(k) = \cos(2k) \exp(-0.2k)$.

Now consider the estimation error system (8) with parameters given by

$$\begin{aligned}A &= 0.1I, & B &= \begin{pmatrix} -0.1 & 0 \\ 0 & 0.2 \end{pmatrix}, & E = F &= 0.3I, & C = D &= 0.2I, \\ M &= \begin{pmatrix} 0.6 & 0 \\ 0 & 0.1 \end{pmatrix}, & N &= \begin{pmatrix} 0.4 & 0 \\ 0.3 & 0.5 \end{pmatrix}, & R &= \begin{pmatrix} 0.1 & 0 \\ 0 & 0.3 \end{pmatrix}, \\ N(k) &= \begin{pmatrix} \sin(k * \pi/2) & 0 \\ 0 & \cos(k * \pi/2) \end{pmatrix}, & \alpha &= 0.001, & \beta &= 0.003, \\ d_1 &= \begin{pmatrix} 0.2 \times (\cos(\pi/2) - 2) & 0 \\ 0 & 0.1 \times (\sin(\pi/2) - 1) \end{pmatrix}, & d_2 &= \begin{pmatrix} 0.28 & 0 \\ 0 & 0.135 \end{pmatrix}, \\ L_x &= \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix}, & L_y &= \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \\ W_1 = W_2 = W_3 = W_4 = W_5 = W_6 &= 0.1I, & \mu = \xi &= \exp(-1),\end{aligned}$$

and the leakage delays $\rho_1 = \rho_2 = 1$. The exogenous disturbance inputs are selected as

$$v_x(k) = (\sin 6k) \exp(-0.1k), \quad v_y(k) = (\cos 2k) \exp(-0.1k).$$

The regulatory function is taken as $g(s) = \frac{s^2}{1+s^2}$. The time-varying delays are chosen as $\delta(k) = 3 + (2 * \sin(k * \pi/2))$ and $\tau(k) = 3 + (2 * \cos(k * \pi/2))$. By using the Matlab LMI toolbox, LMIs (40) and (41) are solved and a set of feasible solutions is obtained as follows:

$$\begin{aligned} X_1 &= \begin{pmatrix} 0.4338 & -0.0041 \\ -0.0041 & 0.2852 \end{pmatrix}, & X_2 &= \begin{pmatrix} 0.0210 & -0.0260 \\ -0.0260 & 0.0533 \end{pmatrix}, \\ R_{11} &= \begin{pmatrix} 9.3434 & 0.0025 \\ 0.0025 & 6.9265 \end{pmatrix}, & R_{21} &= \begin{pmatrix} 0.2169 & -0.4861 \\ -0.4861 & 0.9363 \end{pmatrix}, \\ Y_1 &= \begin{pmatrix} 1.1918 & -0.0128 \\ -0.0128 & 0.5832 \end{pmatrix}, & Y_2 &= \begin{pmatrix} 1.0836 & -0.2279 \\ -0.2279 & 0.1073 \end{pmatrix}. \end{aligned}$$

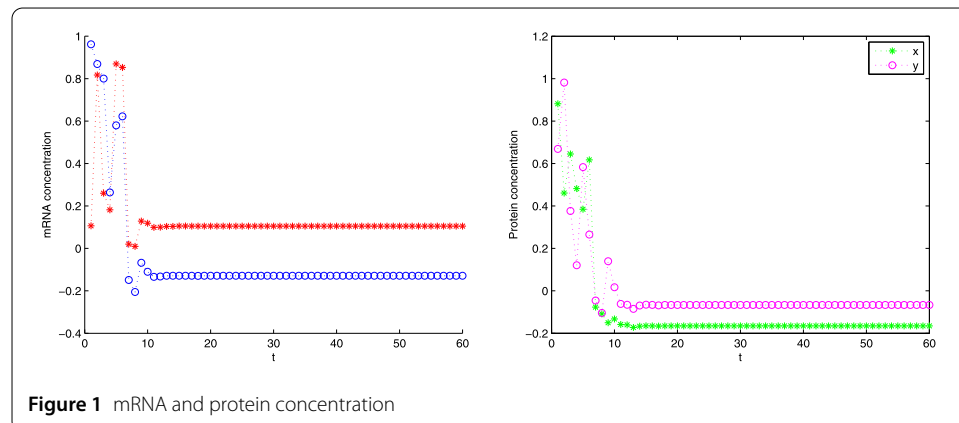
The state estimator gain matrices can be determined as follows:

$$\begin{aligned} A_x &= \begin{pmatrix} 1.2173 & 0.4060 \\ 0.6324 & 0.1804 \end{pmatrix}, & A_y &= \begin{pmatrix} 0.2203 & 0.0032 \\ 0.0063 & 0.2096 \end{pmatrix}, \\ B_x &= \begin{pmatrix} 2.1102 & 0.4831 \\ 1.3729 & 0.3185 \end{pmatrix}, & B_y &= \begin{pmatrix} 0.2005 & 0.4226 \\ 0.8342 & 0.3887 \end{pmatrix}. \end{aligned}$$

The concentration of mRNA and protein and their estimation error are illustrated in Figs. 1 and 2 with the initial conditions $\phi_1(k) = \{1, 0.1\}$, $\psi_1(k) = \{0.9, 0.7\}$, $\phi_2(k) = \{0.9, 0.8\}$, and $\psi_2(k) = \{0.15, 0.9\}$.

Example 4.2 Consider the discrete-time GRN (47) with parameters given by

$$A = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad B = \begin{pmatrix} -0.5 & 0 \\ 2.5 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad D = \begin{pmatrix} 0.08 & 0 \\ 0 & 0.2 \end{pmatrix},$$



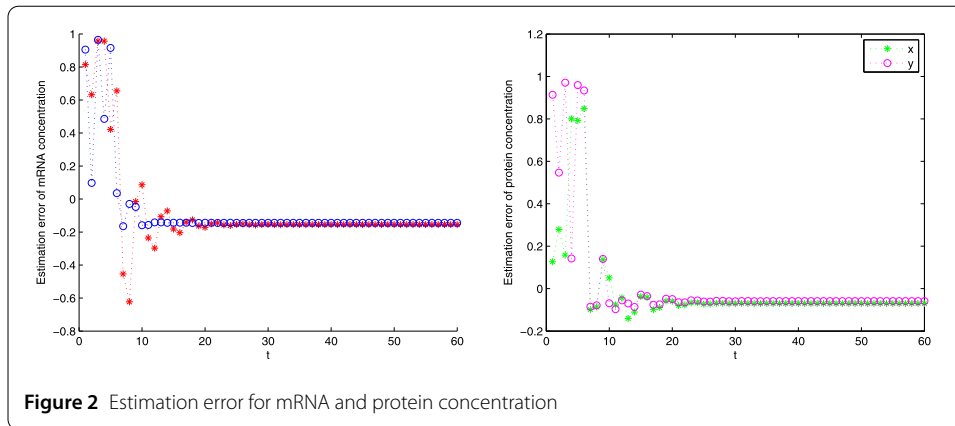


Figure 2 Estimation error for mRNA and protein concentration

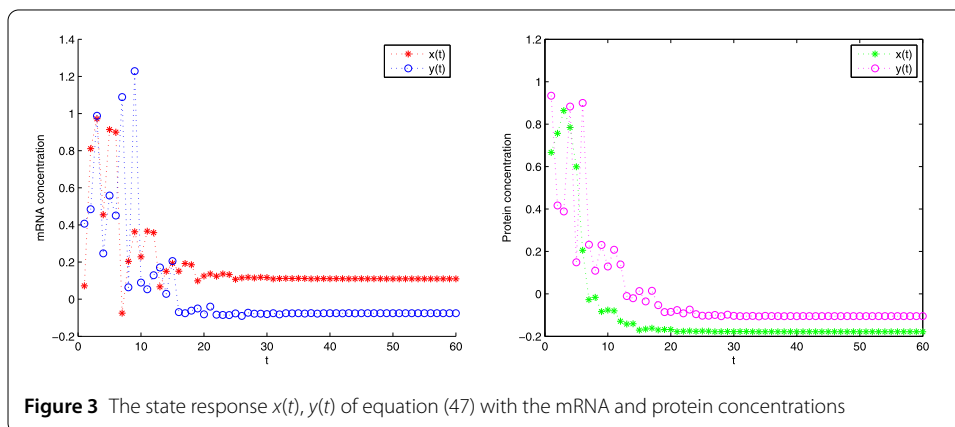


Figure 3 The state response $x(t)$, $y(t)$ of equation (47) with the mRNA and protein concentrations

$$E = \begin{pmatrix} 0.36 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad F = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.4 \end{pmatrix},$$

$$d_1 = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} 0.28 & 0 \\ 0 & 0.135 \end{pmatrix},$$

$$L_x = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad L_y = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix},$$

and the leakage delays $\rho_1 = \rho_2 = 1$. The regulatory function is taken as $g(s) = \frac{s^2}{1+s^2}$. The time-varying delays are chosen as $\delta(k) = 2$ and $\tau(k) = 1$, and the exogenous disturbance inputs are selected as $v_x(k) = \sin(6k) \exp(-0.1k)$ and $v_y(k) = \cos(2k) \exp(-0.2k)$. The state responses $x(t)$ and $y(t)$ are shown in Fig. 3.

5 Conclusions

In this paper, we have studied the approximation concern for the discrete-time stochastic GRNs with the leakage delays, distributed delays, and probabilistic measurement delays into the problem and modeled the robust H_∞ state estimator for a class of discrete-time stochastic GRNs. Here, the probabilistic measurement delays, which narrate the binary shifting sequence, are satisfied by the conditional probability distribution. So, the crisis of parameter uncertainties, including errors, stochastic disturbance, leakage delays, dis-

tributed delays, and the activation function of the addressed GRNs, is identified by sector-bounded nonlinearities. By applying the Lyapunov stability theory and stochastic analysis techniques, sufficient conditions are first entrenched to assure the presence of the desired estimators in terms of a linear matrix inequality (LMI). These circumstances are reliant on both the lower and upper bounds of time-varying delays. Again, the absolute expression of the desired estimator is demonstrated to assure the estimation error dynamics to be robustly exponentially stable in the mean square for the consigned system. Lastly, numerical simulations have been utilized to illustrate the suitability and usefulness of our advanced theoretical results.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Exponential stability of semi-Markovian jump generalized neural networks with interval time-varying delays

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Abstract This draft addresses the exponential stability problem for semi-Markovian jump generalized neural networks (S-MJGNNs) with interval time-varying delays. The exponential stability conditions are derived by establishing a suitable Lyapunov–Krasovskii functional and applying new analysis method. Improved results are obtained to guarantee the exponential stability of S-MJGNNs through improved reciprocally convex combination and new weighted integral inequality techniques. The method in this paper shows the advantages over some existing ones. To verify the advantages and benefits of employing proposed method is explained through numerical examples.

Keywords Exponential stability · Generalized neural networks · Interval time-varying delay · Semi-Markovian jump design

1 Introduction

Artificial neural networks (NNs) have recently been grandly studied in the past few decades since their broad range of applications in various fields, such as fault diagnosis, signal and image processing, pattern recognition, parallel computation, communication and industrial automation. It is familiar that the time delays are

inevitable in hardware implementations. The presence of time delays may lead the systems into oscillation, divergence and instability. Generally, time delays are found in electronic implementations of NNs due to the finite switching speed of the amplifiers and communication time [1–3]. In recent years, many useful subjects including passivity analysis, stability analysis, state estimation and control and filtering of NNs have been investigated [4–13]. Moreover, to characterize the dynamical evolution rule of NNs, according to the use of neural states or local field states of neurons, NNs can be classified into static NNs [14] and local field NNs [2]. These two networks can be transferred equivalently from one to the other through some assumptions, but these assumptions cannot always be satisfied in many applications [15]. That is, local field NNs and static NNs are not always equivalent. Thus, it is necessary and important to study them separately. Recently, some researchers constructed a new unified NN model by combining the above two NNs together, which can be called as generalized neural networks (GNNs) [16–18].

In the last few decades, the Markovian jump NNs have received noticeable attention in stability analysis, since Markovian jump NNs have a compatible structure for obtaining the behavior of numerous physical systems, such as maneuvered target tracking and fault-tolerant control systems. Moreover, these NNs can be designed by a Markov chain with a finite set. When the NN includes the sudden changes in its formation, Markovian jump NNs are very appropriate for describing its dynamics. These transitions may come from component repairs and failures, modification of the subsystem interconnection and unforeseen environmental interferences. In Markovian jump NNs, the jumping modes are governed by the Markov process [19]. Therefore, it is needful to analyze the stochastic effects on dynamic behaviors of the delayed

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NNs. But Markovian jump NNs have some limitations in its applications, since the jump time of a Markov chain is exponentially distributed, and the results obtained for Markovian jump NNs are intrinsically conservative due to constant transition rates [20]. Disparate from the Markovian jump NNs, semi-Markovian jump NNs are designed by fixed matrix of transition probabilities and a matrix of sojourn time probability density functions [21]. Due to their relaxed conditions on the probability distributions, semi-Markovian jump NNs have much broader applications than the conventional Markovian jump NNs. Thus, semi-Markovian jump NNs is significant not only in theoretical, but also in practical [22–24].

The stability problem for NNs became the important focus of numerous research activities. Many sufficient conditions are obtained for different types of stability, such as complete stability, asymptotic stability and exponential stability, see [25, 26]. The exponential stability is main interesting subject than asymptotic stability, because exponential stability provides faster convergence rate to the equilibrium point and gives the information about the decay rates of NNs. Correspondingly, it is important in particular, when the exponential stability property guarantees that whatever transformation occurs, the NN stability to store rapidly the activity pattern is left invariant by self-organization. Therefore, exponential stability of NNs with time-varying delays has worth to study, see [25, 27–30]. Moreover, to the authors' best knowledge, exponential stability analysis of S-MJGNNs with interval time-varying delay has not been considered yet.

For stability analysis, obtaining tight bounds of various functions, particularly integral terms of quadratic functions in the case of continuous-time systems, has had a key part in reducing the conservatism. There are many methods in the literature based on various mathematical tools such as Jensen's inequality [31], Delay-decomposition approach [32], Augmented Lyapunov approach [14] and free-weighting matrix method [18]. Recently, the stability studies have been presented based on the integral inequalities, such as Wirtinger-based inequality [33], Bessel–Legendre inequality [34], auxiliary function-based inequalities [35], because these approach requires only small number of decision variables than other existing approaches having a fine performance behavior. Most recently, a new weighted integral inequality has been proposed by Hien and Trinh [36].

Based on the above discussions, the exponential stability problem for S-MJGNNs with interval time-varying delays is considered in this paper. By using a new analysis method based on the lower and upper bound of the time delay and a appropriate LKF with improved RCC approach and a new

weighted integral inequality, a sufficient condition for the exponential stability is presented in terms of LMIs. Finally, numerical examples are given to illustrate the usefulness and effectiveness of the proposed method. The results are also compared with exiting methods to show less conservativeness.

Notations: Throughout this paper, $\mathcal{R}^n$ and $\mathcal{R}^{n \times n}$ denote, respectively, the n -dimensional Euclidean space and the set of all $n \times n$ real matrices. The notation $*$ represents the entries implied by symmetry. A^T and A^{-1} are the matrix transpose and inverse of A , respectively. We say $X > 0$ means that the matrix X is real symmetric positive definite with appropriate dimensions. For any matrix, $col[a, b, \dots, z]$ denotes $[a^T \ b^T \ \dots \ z^T]^T$. And $diag\{a, b, \dots, z\}$ denotes the block diagonal matrix with $a, b, \dots, z$ in the diagonal entries. I denotes the identity matrix with appropriate dimensions.

2 Problem statement and preliminaries

Let $\{u_t, t \geq 0\}$ be a right-continuous Markov chain on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ taking values in a finite-state space $\mathcal{M} = \{1, 2, \dots, M\}$ with generator $\Pi \triangleq \{\pi_{rs}(\Delta)\}$ given by

$$P\{u_{t+\Delta} = s | u_t = r\} = \begin{cases} \pi_{rs}(\Delta)\Delta + o(\Delta), & r \neq s, \\ 1 + \pi_{rr}(\Delta)\Delta + o(\Delta), & r = s, \end{cases}$$

where $\Delta > 0$, $\lim_{\Delta \rightarrow 0} \frac{o(\Delta)}{\Delta} = 0$, $\pi_{rs} \geq 0$ is the transition rate from r to s if $s \neq r$ while $\pi_{rr} = -\sum_{s=1, s \neq r}^m \pi_{rs}$.

Consider the following GNNs with interval time-varying delay:

$$\begin{aligned} \dot{x}(t) = & -A(u_t)x(t) + W_0(u_t)f(W_2(u_t)x(t)) \\ & + W_1(u_t)f(W_2(u_t)x(t - \tau(t))), \end{aligned} \quad (1)$$

where $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathcal{R}^n$ is the state. $A(u_t) = diag\{a_1(u_t), \dots, a_n(u_t)\}$ is a diagonal matrix with $a_i(u_t) > 0$, $i = 1, 2, \dots, n$. $f(W_2(u_t)x(t)) = [f_1(W_2(u_t)x_1(t)), \dots, f_n(W_2(u_t)x_n(t))]^T \in \mathbb{R}^n$ denotes the neuron activation function at time t . $W_0(u_t)$, $W_1(u_t)$ and $W_2(u_t)$ represents the connection weight matrix between neurons. The time delay, $\tau(t)$ is a time-varying differentiable function that satisfies

$$0 < \tau_1 \leq \tau(t) \leq \tau_2, \quad \dot{\tau}(t) \leq \mu, \quad (2)$$

where τ_1 , τ_2 and μ are positive constants.

In fact, for a fixed u_t , (1) becomes a delayed S-MJGNN. Therefore, the hybrid neural network model (1) can be regarded as a combination of M different delayed S-MJGNNs,

$$\dot{x}(t) = -A_r x(t) + W_{0r} f(W_{2r} x(t)) + W_{1r} f(W_{2r} x(t - \tau(t))), \quad (3)$$

with $r \in \mathcal{M}$. The switching between them follows the movement of the Markov chain u_t , $t \geq 0$.

Assumption (A1) ([37]) Each $f_i(t)$, $(i = 1, 2, \dots, n)$ are continuous and bounded and satisfy the following condition:

$$h_i^- \leq \frac{f_i(W_{2ri}n_1) - f_i(W_{2ri}n_2)}{W_{2ri}n_1 - W_{2ri}n_2} \leq h_i^+, \quad (4)$$

with $H_1 = \text{diag}(h_1^-, h_2^-, \dots, h_n^-)$ and $H_2 = \text{diag}(h_1^+, h_2^+, \dots, h_n^+)$ are some constants, and $n_1, n_2 \in \mathcal{R}$, $n_1 \neq n_2$.

Lemma 2.1 ([36]) For a given $n \times n$ matrix $R > 0$, scalar $\alpha > 0$, and a function $\phi \in \mathcal{C}([a, b], \mathcal{R}^n)$, the following inequality holds.

$$\int_a^b e^{\alpha(s-b)} \phi^T(s) R \phi(s) ds \geq \frac{\alpha}{\bar{\gamma}_0} \left(\int_a^b \phi(s) ds \right)^T R \times \left(\int_a^b \phi(s) ds \right) + \frac{\alpha}{\bar{\rho}_0} \zeta^T (K_1^T K_1 \otimes R) \zeta,$$

where $\rho_0 = \frac{\bar{\gamma}_0}{\bar{\gamma}_1} [\bar{\gamma}_0^2 - (\alpha l)^2 e^{\alpha l}]$, $\bar{\gamma}_0 = e^{\alpha l} - 1$, $\bar{\gamma}_1 = \bar{\gamma}_0 - \alpha l$, $l = b - a$, $\zeta = \text{col} \left\{ \int_a^b \phi(s) ds, \int_a^b \int_s^b \phi(u) du ds \right\}$, $K_1 = [1 \quad -\alpha \bar{\gamma}_0 / \bar{\gamma}_1]$.

Remark 2.2 The following facts can be found by using Taylor series of the exponential function

$$\lim_{\alpha \rightarrow 0^+} \frac{\alpha}{\bar{\rho}_0} = \frac{3}{l}, \quad \lim_{\alpha \rightarrow 0^+} K_1 = \bar{K}_1 = \begin{bmatrix} 1 & -\frac{2}{l} \end{bmatrix}.$$

Therefore, when α approaches zero, we obtain the following results which are the same as those derived by the following inequality

$$\int_a^b \phi^T(s) R \phi(s) ds \geq \frac{1}{l} \zeta^T (K_0^T K_0 + 3 \bar{K}_1^T \bar{K}_1) \otimes R \zeta, \quad (5)$$

where $K_0 = [1 \quad 0]$.

Lemma 2.3 (Improved reciprocally convex combination [38]). For given symmetric positive definite matrices $R_1 \in \mathcal{R}^{n \times n}$, $R_2 \in \mathcal{R}^{m \times m}$, if there exists a matrix $X \in \mathcal{R}^{n \times m}$ such that $\begin{bmatrix} R_1 & X \\ * & R_2 \end{bmatrix} \geq 0$, then the inequality

$$\begin{bmatrix} \frac{1}{\delta} R_1 & 0 \\ * & \frac{1}{1-\delta} R_2 \end{bmatrix} \geq \begin{bmatrix} R_1 & X \\ * & R_2 \end{bmatrix}$$

holds for all $\delta \in (0, 1)$.

Definition 2.4 Consider the S-MJGNN system (1). If there exist constants $\kappa \geq 1$ and $\alpha > 0$ such that

$$\|x(t)\| \leq \kappa \psi e^{-\alpha t}, \quad \forall t \geq 0, \quad \psi = \sup_{-\tau_2 \leq \theta \leq 0} \|x(\theta)\|,$$

then the origin of system (1) is exponentially stable, where α is called the exponential convergence rate.

Remark 2.5 It should be pointed that the S-MJGNN (1) contains LFNNs and SNNs. When $W_{0r} = 0$, $W_{1r} = I$, we obtain the following SNN from (3):

$$\dot{x}(t) = -A_r x(t) + f(W_{2r} x(t - \tau(t))). \quad (6)$$

When $W_{2r} = I$, we obtain the following LFNN:

$$\dot{x}(t) = -A_r x(t) + W_{0r} f(x(t)) + W_{1r} f(x(t - \tau(t))). \quad (7)$$

3 Main results

For simplicity, the vectors and matrices are represented as follows:

$$\xi(t) = \begin{bmatrix} x^T(t) & x^T(t - \tau_1) & x^T(t - \tau(t)) & x^T(t - \tau_2) \\ f^T(W_{2r} x(t)) & f^T(W_{2r} x(t - \tau(t))) & \frac{1}{\tau_1} \int_{t-\tau_1}^t x^T(s) ds \\ \frac{1}{\tau(t) - \tau_1} \int_{t-\tau(t)}^{t-\tau_1} x^T(s) ds & \frac{1}{\tau_2 - \tau(t)} \int_{t-\tau_2}^{t-\tau(t)} x^T(s) ds \end{bmatrix}^T,$$

$$\eta(t) = \begin{bmatrix} x^T(t) & \int_{t-\tau_1}^t x^T(s) ds & \int_{t-\tau_2}^{t-\tau_1} x^T(s) ds \end{bmatrix}^T,$$

$$e_\rho = [0_{n \times (\rho-1)n} \quad I_n \quad 0_{n \times (9-\rho)n}], \quad \rho = 1, 2, \dots, 9.$$

Based on the LKF and LMI techniques, the following result can be obtained.

Theorem 3.1 For given scalars $0 < \tau_1 < \tau_2$, $\mu > 0$, the S-MJGNN (1) is exponentially stable with the exponential convergence rate index α if there exist symmetric positive definite matrices $P_r \in \mathcal{R}^{3n \times 3n}$ ($r = 1, 2, \dots, M$), $Q_1, Q_2, Q_3, R, S \in \mathcal{R}^{n \times n}$, positive definite diagonal matrices $\Gamma_p \in \mathcal{R}^{n \times n}$, ($p = 1, 2, 3$), $\Lambda_q \in \mathcal{R}^{n \times n}$ ($q = 1, 2$) and a matrix $X \in \mathcal{R}^{2n \times 2n}$, such that the following LMI holds:

$$\Xi_r(\tau) < 0, \quad \Pi = \begin{bmatrix} \tilde{S} & X \\ 0 & \tilde{S} \end{bmatrix} \geq 0, \quad (8)$$

where

$$\begin{aligned}
\Xi_r(\tau) &= \Xi_{0r}(\tau) + \sum_{z=1}^5 \Xi_z - e^{-\alpha\tau_{21}} \Psi^T \Pi \Psi, \\
\Xi_{0r}(\tau) &= \Theta^T(\tau) P_r \Theta_{0r} + \Theta_{0r}^T P_r \Theta(\tau) + \alpha \Theta^T(\tau) P_r \Theta(\tau) + \Theta^T(\tau) \sum_{s \in \mathcal{M}} \pi_{rs}(\Delta) P_s \Theta_{0r}, \\
\Theta(\tau) &= \begin{bmatrix} e_1 \\ \tau_1 e_7 \\ (\tau(t) - \tau_1)e_8 + (\tau_2 - \tau(t))e_9 \end{bmatrix}, \quad \Theta_{0r} = \begin{bmatrix} \hat{A}_r \\ e_1 - e_2 \\ e_2 - e_4 \end{bmatrix}, \quad \Theta_1 = \begin{bmatrix} e_1 - e_2 \\ \tau_1(e_1 - e_7) \end{bmatrix}, \\
\Theta_2 &= \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_8 \end{bmatrix}, \quad \Theta_3 = \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_9 \end{bmatrix}, \quad \Xi_{1r} = [2[e_5 - H_1 W_{2r} e_1]^T \Lambda_1 \hat{A}_r \\
&\quad + 2[H_2 W_{2r} e_1 - e_5]^T \Lambda_2 \hat{A}_r], \quad \Xi_2 = e_1^T Q_1 e_1 - e^{-\alpha\tau_1} e_2^T Q_1 e_2 + e^{-\alpha\tau_1} e_2^T Q_2 e_2 \\
&\quad - e^{-\alpha\tau_2} e_4^T Q_2 e_4 + e_5^T Q_3 e_5 - (1 - \mu) e^{-\alpha\tau_2} e_6^T Q_3 e_6, \quad \Xi_{3r} = \hat{A}_r^T [\tau_1^2 R + \tau_{21}^2 e^{\alpha\tau_1} S] \hat{A}_r, \\
\Xi_4 &= - \left(\frac{\alpha\tau_1}{\gamma_0} (e_1 - e_2)^T R (e_1 - e_2) + \frac{\alpha\tau_1}{\rho_0} \Theta_1^T (K^T K \otimes R) \Theta_1 \right), \\
\Xi_{5r} &= - 2e_1^T W_{2r}^T H_1 \Gamma_1 H_2 W_{2r} e_1 + e_1^T W_{2r}^T (H_1 + H_2) \Gamma_1 e_5 + e_5^T \Gamma_2 (H_1 + H_2) W_{2r} e_1 - 2e_5^T \Gamma_1 e_5 \\
&\quad - 2e_2^T W_{2r}^T H_1 \Gamma_2 H_2 W_{2r} e_2 + e_2^T W_{2r}^T (H_1 + H_2) \Gamma_2 e_6 + e_6^T \Gamma_2 (H_1 + H_2) W_{2r} e_2 - 2e_6^T \Gamma_2 e_6 \\
&\quad - 2[e_5 - e_6 - H_1 W_{2r} (e_1 - e_2)]^T \Gamma_3 [e_5 - e_6 - H_2 W_{2r} (e_1 - e_2)], \\
\rho_0 &= \frac{\gamma_0}{\gamma_1^2} [\gamma_0^2 - (\alpha\tau_1)^2 e^{\alpha\tau_1}], \quad \gamma_0 = e^{\alpha\tau_1} - 1, \quad \gamma_1 = \gamma_0 - \alpha\tau_1, \quad K = \begin{bmatrix} 1 & -\alpha\gamma_0/\gamma_1 \end{bmatrix}, \\
\Psi &= [\Theta_2^T \quad \Theta_3^T]^T, \quad \tilde{S} = \text{diag}\{S, 3S\}, \quad \hat{A}_r = -A_r e_1 + W_{0r} e_5 + W_{1r} e_6, \quad \tau_{21} = \tau_2 - \tau_1.
\end{aligned}$$

Proof Define the LKF as follows,

$$\begin{aligned}
V(x_t) &= \eta^T(t) P(u_t) \eta(t) + 2 \sum_{i=1}^n \lambda_{1i} \int_0^{W_{2ri}x(t)} (f_i(s) - h_i^- s) ds \\
&\quad + 2 \sum_{i=1}^n \lambda_{2i} \int_0^{W_{2ri}x(t)} (h_i^+ s - f_i(s)) ds \\
&\quad + \int_{t-\tau_1}^t e^{\alpha(s-t)} x^T(s) Q_1 x(s) ds \\
&\quad + \int_{t-\tau_2}^{t-\tau_1} e^{\alpha(s-t)} x^T(s) Q_2 x(s) ds \\
&\quad + \int_{t-\tau(t)}^t e^{\alpha(s-t)} f^T(x(s)) Q_3 f(x(s)) ds \\
&\quad + \tau_1 \int_{-\tau_1}^0 \int_{t+s}^t e^{\alpha(\theta-t)} \dot{x}^T(\theta) R \dot{x}(\theta) d\theta ds \\
&\quad + \tau_{21} \int_{-\tau_2}^{-\tau_1} \int_{t+s}^t e^{\alpha(\theta+\tau_1-t)} \dot{x}^T(\theta) S \dot{x}(\theta) d\theta ds,
\end{aligned} \tag{9}$$

It follows from (9) that $V(x_t) \geq \lambda_{\min}(P_r) \|x(t)\|^2$. Then by calculating the time derivative of $V(x_t)$ along the trajectories of (3), we obtain

$$\begin{aligned}
\dot{V}(x_t) + \alpha V(x_t) &= \xi^T(t) \Xi_{0r}(\tau) \xi(t) + \xi^T(t) \\
&\quad [2[e_5 - H_1 W_{2r} e_1]^T \Lambda_1 \hat{A}_r \\
&\quad + 2[H_2 W_{2r} e_1 - e_5]^T \Lambda_2 \hat{A}_r] \xi(t) \\
&\quad + \leq \xi^T(t) [e_1^T Q_1 e_1 - e^{-\alpha\tau_1} \\
&\quad \times e_2^T Q_1 e_2 + e^{-\alpha\tau_1} e_2^T Q_2 e_2 \\
&\quad - e^{-\alpha\tau_1} e_4^T Q_2 e_4 + e_5^T \\
&\quad \times Q_3 e_5 - (1 - \mu) e^{-\alpha\tau_1} e_6^T Q_3 e_6] \xi(t) \\
&\quad + \xi^T(t) \hat{A}_r^T [\tau_1^2 R + \tau_{21}^2 e^{\alpha\tau_1} S] \hat{A}_r \xi(t) \\
&\quad - \tau_1 \int_{t-\tau_1}^t e^{\alpha(s-t)} \dot{x}^T(s) R \dot{x}(s) ds \\
&\quad - \tau_{21} \int_{t-\tau_2}^{t-\tau_1} e^{\alpha(s+\tau_1-t)} \dot{x}^T(s) S \dot{x}(s) ds \\
&\quad - \tau_{21} \int_{t-\tau(t)}^{t-\tau_1} e^{\alpha(s+\tau_1-t)} \dot{x}^T(s) S \dot{x}(s) ds.
\end{aligned} \tag{10}$$

By Lemma 2.1, we have

$$-\tau_1 \int_{t-\tau_1}^t e^{\alpha(s-t)} \dot{x}^T(s) R \dot{x}(s) ds \leq \xi^T(t) \Xi_4 \xi(t).$$

By using the inequality in (5) and Lemma 2.3, we have

$$\begin{aligned}
 & -\tau_{21} \int_{t-\tau_2}^{t-\tau(t)} e^{\alpha(s+\tau_1-t)} \dot{x}^T(s) S \dot{x}(s) ds \\
 & -\tau_{21} \int_{t-\tau(t)}^{t-\tau_1} e^{\alpha(s+\tau_1-t)} \dot{x}^T(s) S \dot{x}(s) ds \\
 & \leq -\frac{\tau_{21} e^{-\alpha\tau_{21}}}{\tau(t)-\tau_1} \zeta_1^T(t) \tilde{S} \zeta_1(t) - \frac{\tau_{21} e^{-\alpha\tau_{21}}}{\tau_2-\tau(t)} \zeta_2^T(t) \tilde{S} \zeta_2(t) \\
 & \leq -e^{-\alpha\tau_{21}} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}^T \begin{bmatrix} \frac{\tau_{21}}{\tau(t)-\tau_1} \tilde{S} & 0 \\ 0 & \frac{\tau_{21}}{\tau_2-\tau(t)} \tilde{S} \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} \\
 & = -e^{-\alpha\tau_{21}} \zeta^T(t) \Psi^T \Pi \Psi \zeta(t),
 \end{aligned}$$

where $\zeta_1(t) = \Theta_2 \zeta(t)$, $\zeta_2(t) = \Theta_3 \zeta(t)$. From Assumption (A1), we have

$$0 \leq 2[f(W_{2r}x(t)) - H_1 W_{2r}x(t)]^T \Gamma_1 [H_2 W_{2r}x(t) - f(W_{2r}x(t))], \quad (11)$$

$$0 \leq 2[f(W_{2r}x(t-\tau(t))) - H_1 W_{2r}x(t-\tau(t))]^T \Gamma_2 [H_2 W_{2r}x(t-\tau(t)) - f(W_{2r}x(t-\tau(t)))], \quad (12)$$

$$\begin{aligned}
 0 \leq & 2[f(W_{2r}x(t)) - f(W_{2r}x(t-\tau(t)))]^T \Gamma_3 \\
 & \times [f(W_{2r}x(t)) - f(W_{2r}x(t-\tau(t)))] \\
 & - H_2 (W_{2r}x(t) - W_{2r}x(t-\tau(t))). \quad (13)
 \end{aligned}$$

Combining (10) with (13), we obtain

$$\dot{V}(x_t) + \alpha V(x_t) \leq \zeta^T(t) \Xi_r(\tau) \zeta(t), \quad (14)$$

where $\Xi_r(\tau)$ is defined in (8). From (8), we obtain $\dot{V}(x_t) + \alpha V(x_t) \leq 0$. By integrating, we get $V(x_t) \leq V(\phi) e^{-\alpha t}$. Then by Definition 2.4, the equilibrium point is exponentially stable. $\square$

Remark 3.2 ([23]) It is noted that the condition in (8) is not LMI, which implies that it is difficult to solve the condition in (8) because of the existence of the time-varying terms $\sum_{s \in \mathcal{M}} \pi_{rs}(\Delta)$. In what follows, as pointed out in [20], the transition rate $\pi_{rs}(\Delta)$ is usually partly measurable in practice, that is, the $\pi_{rs}(\Delta)$ bounded and

satisfies $\pi_{rs}^- \leq \pi_{rr}(\Delta) \leq \pi_{rs}^+$. In this case, differently from [20], a natural assumption on the $\pi_{rr}(\Delta)$ can be raised as follows:

$$\pi_{rs}(\Delta) = \sum_{k=1}^{\mathcal{X}} \varsigma_k \pi_{rs,k}, \quad \sum_{k=1}^{\mathcal{X}} \varsigma_k = 1, \quad \varsigma_k \geq 1, \quad (15)$$

and

$$\pi_{rs,k} = \begin{cases} \pi_{rs}^- + (k+1) \frac{\pi_{rs}^- - \pi_{rs}^+}{\mathcal{X} - 1}, & r \neq s, s \in \mathcal{M} \\ \pi_{rs}^+ - (k+1) \frac{\pi_{rs}^- - \pi_{rs}^+}{\mathcal{X} - 1}, & r = s, s \in \mathcal{M}. \end{cases} \quad (16)$$

Remark 3.3 The generalized neural networks have now attracted much attention in recent years. GNN has been first introduced by Zhang and Han in [16]. In [17], the authors studied the delay-dependent stability analysis of GNN with two additive delay signals. Stability criteria for delayed GNN via a new integral inequality are investigated in [18]. Stability analysis of GNN with interval time-varying delays has been proposed in [39] by using new stability approach. In [40], Saravanakumar et al investigated the H_∞ state estimation problem for GNNs with interval time-varying delays. In the above methods, the improved asymptotic stability criteria were presented via different types of inequalities. However, those inequalities may not be useful to obtain improved exponential stability results. Thus, our main aim is to study the exponential stability of delayed GNNs using new WII technique.

Corollary 3.4 For given scalars $0 < \tau_1 < \tau_2$, $\mu > 0$, the S-MJGNN (1) is asymptotically stable if there exist symmetric positive definite matrices $P_r \in \mathcal{R}^{3n \times 3n}$ ($r = 1, 2, \dots, M$), $Q_1, Q_2, Q_3, R, S \in \mathcal{R}^{n \times n}$, positive definite diagonal matrices $\Gamma_p \in \mathcal{R}^{n \times n}$, ($p = 1, 2, 3$), $\Lambda_q \in \mathcal{R}^{n \times n}$ ($q = 1, 2$) and a matrix $X \in \mathcal{R}^{2n \times 2n}$, such that

$$\begin{aligned}
 & \Xi_r(\tau) < 0, \\
 & \Pi = \begin{bmatrix} \tilde{S} & X \\ 0 & \tilde{S} \end{bmatrix} \geq 0, \quad (17)
 \end{aligned}$$

where

$$\begin{aligned}
\Xi_r(\tau) &= \Xi_{0r}(\tau) + \sum_{z=1}^5 \Xi_z - \Psi^T \Pi \Psi, \\
\Xi_{0r}(\tau) &= \Theta^T(\tau) P_r \Theta_{0r} + \Theta_{0r}^T P_r \Theta(\tau) + \Theta^T(\tau) \sum_{s \in \mathcal{M}} \pi_{rs}(\Delta) P_s \Theta_{0r}, \\
\Theta(\tau) &= \begin{bmatrix} e_1 \\ \tau_1 e_7 \\ (\tau(t) - \tau_1)e_8 + (\tau_2 - \tau(t))e_9 \end{bmatrix}, \quad \Theta_{0r} = \begin{bmatrix} \hat{A}_r \\ e_1 - e_2 \\ e_2 - e_4 \end{bmatrix}, \quad \Theta_1 = \begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_7 \end{bmatrix}, \\
\Theta_2 &= \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_8 \end{bmatrix}, \quad \Theta_3 = \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_9 \end{bmatrix}, \quad \Xi_{1r} = [2[e_5 - H_1 W_{2r} e_1]^T \Lambda_1 \hat{A}_r \\
&\quad + 2[H_2 W_{2r} e_1 - e_5]^T \Lambda_2 \hat{A}_r], \quad \Xi_2 = e_1^T Q_1 e_1 - e_2^T Q_1 e_2 + e_2^T Q_2 e_2 - e_4^T Q_2 e_4 + e_5^T Q_3 e_5 \\
&\quad - (1 - \mu)e_6^T Q_3 e_6, \quad \Xi_{3r} = \hat{A}_r^T [\tau_1^2 R + \tau_{21}^2 S] \hat{A}_r, \quad \Xi_4 = -\Theta_1^T \bar{R} \Theta_1, \quad \bar{R} = \text{diag}\{R, 3R\}, \\
\Xi_{5r} &= -2e_1^T W_{2r}^T H_1 \Gamma_1 H_2 W_{2r} e_1 + e_1^T W_{2r}^T (H_1 + H_2) \Gamma_1 e_5 + e_5^T \Gamma_2 (H_1 + H_2) W_{2r} e_1 - 2e_5^T \Gamma_1 e_5 \\
&\quad - 2e_2^T W_{2r}^T H_1 \Gamma_2 H_2 W_{2r} e_2 + e_2^T W_{2r}^T (H_1 + H_2) \Gamma_2 e_6 + e_6^T \Gamma_2 (H_1 + H_2) W_{2r} e_2 - 2e_6^T \Gamma_2 e_6 \\
&\quad - 2[e_5 - e_6 - H_1 W_{2r} (e_1 - e_2)]^T \Gamma_3 [e_5 - e_6 - H_2 W_{2r} (e_1 - e_2)], \\
\Psi &= [\Theta_2^T \quad \Theta_3^T]^T, \quad \tilde{S} = \text{diag}\{S, 3S\}, \quad \hat{A}_r = -A_r e_1 + W_{0r} e_5 + W_{1r} e_6, \quad \tau_{21} = \tau_2 - \tau_1.
\end{aligned}$$

Proof Define the LKF as follows,

$$\begin{aligned}
V(x_t) &= \eta^T(t) P \eta(t) + 2 \sum_{i=1}^n \lambda_{1i} \int_0^{W_{2ri} x(t)} (f_i(s) - h_i^- s) ds \\
&\quad + 2 \sum_{i=1}^n \lambda_{2i} \int_0^{W_{2ri} x(t)} (h_i^+ s - f_i(s)) ds \\
&\quad + \int_{t-\tau_1}^t x^T(s) Q_1 x(s) ds + \int_{t-\tau_2}^{t-\tau_1} x^T(s) Q_2 x(s) ds \\
&\quad + \int_{t-\tau(t)}^t f^T(x(s)) Q_3 f(x(s)) ds \\
&\quad + \tau_1 \int_{-\tau_1}^0 \int_{t+s}^t \dot{x}^T(\theta) R \dot{x}(\theta) d\theta ds \\
&\quad + \tau_{21} \int_{-\tau_2}^{-\tau_1} \int_{t+s}^t \dot{x}^T(\theta) S \dot{x}(\theta) d\theta ds, \quad (18)
\end{aligned}$$

Calculating the time derivative of $V(x_t)$ along the trajectories of (3), we obtain

$$\begin{aligned}
\dot{V}(x_t) &= \xi^T(t) \Xi_{0r}(\tau) \xi(t) + \xi^T(t) [2[e_5 - H_1 W_{2r} e_1]^T \Lambda_1 \hat{A}_r \\
&\quad + 2[H_2 W_{2r} e_1 - e_5]^T \Lambda_2 \hat{A}_r] \xi(t) \\
&\quad + \leq \xi^T(t) [e_1^T Q_1 e_1 - e_2^T Q_1 e_2 + e_2^T Q_2 e_2 \\
&\quad - e_4^T Q_2 e_4 + e_5^T Q_3 e_5 - (1 - \mu)e_6^T Q_3 e_6] \xi(t) \\
&\quad + \xi^T(t) \hat{A}_r^T [\tau_1^2 R + \tau_{21}^2 S] \hat{A}_r \xi(t) - \tau_1 \int_{t-\tau_1}^t \dot{x}^T(s) R \dot{x}(s) ds \\
&\quad - \tau_{21} \int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s) S \dot{x}(s) ds - \tau_{21} \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s) S \dot{x}(s) ds. \quad (19)
\end{aligned}$$

By using the inequality in (5), we have

$$-\tau_1 \int_{t-\tau_1}^t \dot{x}^T(s) R \dot{x}(s) ds \leq \xi^T(t) \Xi_4 \xi(t).$$

Using Lemma 2.3 and the inequality in (5), we have

$$\begin{aligned}
& -\tau_{21} \int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s) S \dot{x}(s) ds - \tau_{21} \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s) S \dot{x}(s) ds \\
& \leq -\frac{\tau_{21}}{\tau(t) - \tau_1} \zeta_1^T(t) \tilde{S} \zeta_1(t) - \frac{\tau_{21}}{\tau_2 - \tau(t)} \zeta_2^T(t) \tilde{S} \zeta_2(t) \\
& \leq -\begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}^T \begin{bmatrix} \frac{\tau_{21}}{\tau(t) - \tau_1} \tilde{S} & 0 \\ 0 & \frac{\tau_{21}}{\tau_2 - \tau(t)} \tilde{S} \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} \\
& = -\xi^T(t) \Psi^T \Pi \Psi \xi(t).
\end{aligned}$$

where $\zeta_1(t) = \Theta_2 \xi(t)$, $\zeta_2(t) = \Theta_3 \xi(t)$. From Assumption (A1), we have

$$0 \leq 2[f(W_{2r} x(t)) - H_1 W_{2r} x(t)]^T \Gamma_1 [H_2 W_{2r} x(t) - f(W_{2r} x(t))], \quad (20)$$

$$0 \leq 2[f(W_{2r} x(t - \tau(t))) - H_1 W_{2r} x(t - \tau(t))]^T \Gamma_2 [H_2 W_{2r} x(t - \tau(t)) - f(W_{2r} x(t - \tau(t)))], \quad (21)$$

$$\begin{aligned}
0 &\leq 2[f(W_{2r} x(t)) - f(W_{2r} x(t - \tau(t))) - H_1 (W_{2r} x(t) \\
&\quad - W_{2r} x(t - \tau(t)))]^T \Gamma_3 \\
&\quad \times [f(W_{2r} x(t)) - f(W_{2r} x(t - \tau(t))) - H_2 (W_{2r} x(t) \\
&\quad - W_{2r} x(t - \tau(t)))]. \quad (22)
\end{aligned}$$

Combining (19) with (22), we obtain

$$\dot{V}(x_t) \leq \xi^T(t) \Xi_r(\tau) \xi(t), \quad (23)$$

where $\Xi_r(\tau)$ is defined in (17). This completes the proof.

4 Numerical examples

Example 4.1 Consider the S-MJGNN (1) with the following values:

Mode 1:

$$A_1 = \begin{bmatrix} 2.5 & 0 \\ 0 & 3.5 \end{bmatrix}, W_{01} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & -1 \end{bmatrix},$$

$$W_{11} = \begin{bmatrix} -0.5 & -0.1 \\ 0.6 & 0.8 \end{bmatrix}, W_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Mode 2:

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, W_{02} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$W_{12} = \begin{bmatrix} 0.1 & -0.2 \\ 0 & 0.1 \end{bmatrix}, W_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The transition rate matrix is chosen as $0.1 \leq \pi_{12}(\Delta) \leq 2.0$, $0.8 \leq \pi_{21}(\Delta) \leq 1.7$. From the equations (15) and (16), the transition rates $\pi_{12}(\Delta)$ and $\pi_{21}(\Delta)$ without loss of generality, when $\mathcal{X} = 2$, we have $\pi_{12,1} = 0.1$, $\pi_{12,2} = 2.0$, $\pi_{21,1} = 0.8$, $\pi_{21,2} = 1.7$. To obtain the exponential stability S-MJGNN (3), choose $\mu = 1.2$, $\tau_1 = 1$, $\tau_2 = 5$, $H_1 = 0_2$, $H_2 = I_2$ then apply Theorem 3.1, we can obtain the feasible solutions as follows:

$$P_1 = \begin{bmatrix} 1.6839 & -1.5227 & 1.0083 & 0.2759 & 0.0191 & -0.0127 \\ -1.5227 & 13.9732 & -1.1984 & 3.2852 & -0.0117 & -0.0020 \\ 1.0083 & -1.1984 & 4.0748 & -0.4916 & 0.0463 & -0.0125 \\ 0.2759 & 3.2852 & -0.4916 & 8.7830 & -0.0103 & 0.0590 \\ 0.0191 & -0.0117 & 0.0463 & -0.0103 & 0.0220 & -0.0058 \\ -0.0127 & -0.0020 & -0.0125 & 0.0590 & -0.0058 & 0.0266 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1.4908 & -1.3120 & -1.1318 & -1.1665 & -0.0104 & 0.0079 \\ -1.3120 & 1.8178 & 0.4134 & -3.3798 & 0.0041 & -0.0055 \\ -1.1318 & 0.4134 & -3.6932 & 0.1904 & -0.0387 & -0.0014 \\ -1.1665 & -3.3798 & 0.1904 & -7.6978 & -0.0078 & -0.0779 \\ -0.0104 & 0.0041 & -0.0387 & -0.0078 & -0.0724 & -0.0098 \\ 0.0079 & -0.0055 & -0.0014 & -0.0779 & -0.0098 & -0.1262 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 30.8975 & -0.7597 \\ -0.7597 & 45.0964 \end{bmatrix}, Q_2 = \begin{bmatrix} 10.0113 & 1.2028 \\ 1.2028 & 17.9069 \end{bmatrix}, Q_3 = \begin{bmatrix} 0.8718 & -0.7134 \\ -0.7134 & 3.2264 \end{bmatrix},$$

$$R = \begin{bmatrix} 1.2416 & -0.1050 \\ -0.1050 & 1.2125 \end{bmatrix}, S = \begin{bmatrix} 0.5187 & -0.0415 \\ -0.0415 & 0.5779 \end{bmatrix}.$$

Table 1 Example 4.1—the maximum exponential convergence rate $\alpha_{\max}$ for different μ

μ	0	0.6	0.8	1.5	Unknown μ
Ref. [29]	1.1605	0.9064	0.8381	0.8371	0.8359
Ref. [4]	1.1609	0.9428	0.8759	0.8739	0.9426
Theorem 3.1	1.5234	1.3654	1.2157	1.1034	0.9871

Table 2 Example 4.1—MAUB τ_2 for different τ_1 and α

Methods	α	1	2	4	6
Ref. [4]	0.05	1.8642	2.9492	4.8409	6.7416
Theorem 3.1		2.7698	3.8962	5.7865	7.5847
Ref. [4]	0.1	1.6314	2.7981	4.6218	6.4718
Theorem 3.1		2.5234	3.6154	5.5257	7.2632
Ref. [4]	0.2	1.4632	2.5712	4.3200	6.1321
Theorem 3.1		2.3984	3.4561	5.2157	6.9854

Therefore, the S-MJGNN (1) is exponentially stable with convergence rate $\alpha = 0.3625$. Additionally, when $r = 1$, consider the values in mode 1 with $\tau_1 = 0.8$, $\tau_2 = 1$, $H_1 = 0.998I_2$ and $H_2 = I_2$ the maximum exponential convergence rate $\alpha_{\max}$ obtained for different values of μ (see Table 1). When $\tau(t)$ is not differentiable, Table 2 lists the maximum admissible upper bound (MAUB) τ_2 for different values of the lower bound τ_1 and decay rate α . The time responses of the state variables in mode 1 is shown in Fig. 1 with $f(x(t)) = \tanh(x(t))$.

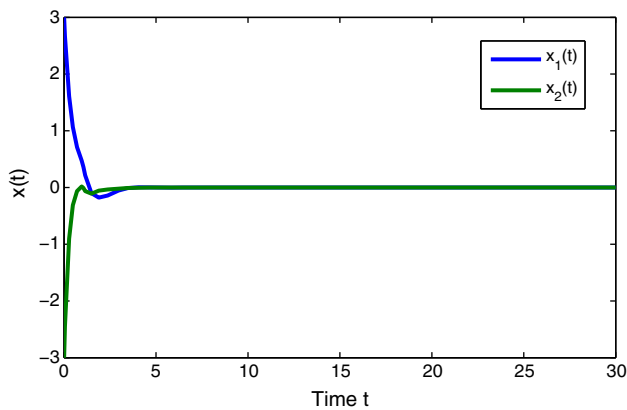


Fig. 1 Trajectories of the state $x(t)$ in mode 1

Table 3 Example 4.1—MAUB τ_2 for different τ_1 and α

Methods	α	$\mu = 0.8$	$\mu = 0.9$
Ref. [4]	0.03	36.5796	25.0271
Theorem 3.1		42.6321	31.3562
Ref. [4]	0.05	21.3416	14.4101
Theorem 3.1		28.3452	21.5461
Ref. [4]	0.642	16.2771	10.8788
Theorem 3.1		22.3849	17.4156

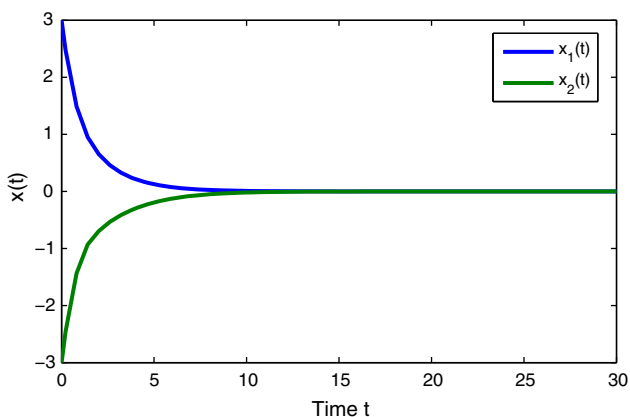


Fig. 2 Trajectories of the state $x(t)$ in mode 2.

Now, consider the values in mode 2 with $\tau_1 = 2$, $H_1 = 0_2$ and $H_2 = I_2$, then the Table 3 lists the MAUB τ_2 for different values of the μ and decay rate α . This shows that the established method is less conservative than the results

in [4, 29]. The time responses of the state variables in mode 2 are shown in Fig. 2 with $f(x(t)) = \tanh(x(t))$.

Example 4.2 Now, choose NN (7) with $r = 1$ and the matrix parameters in [41]: $A_1 = 2I$, $W_{01} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, $W_{11} = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}$, $H_1 = 0$, $H_2 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.8 \end{bmatrix}$.

By solving the LMI in Corollary 3.4, we obtain the MAUB τ_2 for different μ when $\tau_1 = 0$. From Table 4, it is easy to know that our results are significantly better than those in [41, 42].

Remark 4.1 In [3], receding horizon stabilization and disturbance attenuation for delayed NNs is established. A new finite horizon cost functional for receding horizon stabilization is presented for NNs with time-invariant delays using Wirtinger-type inequalities. In [10], the dissipative control and filtering problems for two-dimensional Roesser model have been considered. Stochastic stability problem for two-dimensional Roesser systems with multiplicative noise is studied in [5]. $L_2 - L_\infty$ suppression of limit cycles in interfered two-dimensional digital filters for a Fornasini–Marchesini model is considered in [9]. In the above articles, the authors studied some new interesting topics and they introduced new methods for solving dissipativity and $L_2 - L_\infty$ problems. We are very much interested to use the above interesting methods in our future works, for example, dissipativity and $L_2 - L_\infty$ problems.

Remark 4.2 In [4, 29], the assumptions that the time-varying delay is differentiable and the derivative is bounded or smaller than one are required. However, the assumptions are removed in this paper.

Remark 4.3 The LKFs (9) and (17) are established with few decision variables compared with the methods in [41, 42]. The number of decision variables in [41] and [42] are, respectively, $\frac{(11n(n+1))}{2} + 7n^2$ and $\frac{(24n(n+1))}{2} + 12n^2$. The number of decision variables in LKFs (9) and (17) is $\frac{(13n(n+1))}{2} + 3n^2$. Additionally, we have used new WII technique to obtain the maximum exponential convergence

Table 4 Example 4.2—MAUB τ_2 for different μ when $\tau_1 = 1$

μ	0.5	0.8	0.9	Unknown μ	# of variables
Ref. [41]	2.3068	2.4334	3.4846	1.3189	$\frac{(11n(n+1))}{2} + 7n^2$
Ref. [42]	3.4348	3.1034	2.1248	1.9897	$\frac{(24n(n+1))}{2} + 12n^2$
Corollary 3.4	4.4698	3.8965	3.4768	2.9456	$\frac{(13n(n+1))}{2} + 3n^2$

rate α_{max} and the maximum allowable delay upper bounds (MAUBs) τ_2 . From Tables 1, 2, 3 and 4, we conclude that the obtained results of this paper are less conservative than some existing methods [4, 29, 41, 42].

Remark 4.4 In [1] and [8], the authors investigated the passivity analysis problem for uncertain time-delayed NNs and dissipativity analysis problem for discrete-time stochastic time-delayed NNs, respectively. In the above papers, the time delay was assumed to be non-differentiable. In our result, the time-varying delay can be differentiable. However, by choosing the LKF $\int_{t-\tau}^t e^{\alpha(s-t)} f^T(x(s)) Q_3 f(x(s)) ds$ instead of $\int_{t-\tau(t)}^t e^{\alpha(s-t)} f^T(x(s)) Q_3 f(x(s)) ds$, one can extend this method to non-differentiable case with feasible solutions.

5 Conclusions

In this work, we have studied the exponential stability of S-MJGNNs with interval time-varying delays. An improved delay-dependent exponential stability criteria have been established in terms of LMI by constructing suitable LKFs. Numerical examples are given to show the effectiveness and superiority of the method proposed in this paper. The results are also compared with existing ones to exhibit the less conservativeness. In the future, it is possible that the above main results can be extended to stochastic S-MJGNNs with interval time-varying delays.

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Stabilization of Switched Stochastic Genetic Regulatory Networks with Leakage and Impulsive Effects

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Abstract In this paper, the global asymptotical stability analysis problem is considered for stabilization of switched stochastic genetic regulatory networks with leakage and impulsive effects. Using the method of Lyapunov function, sufficient conditions are derived based on the linear matrix inequality (LMI) technique, which can easily be checked by utilizing the numerically efficient Matlab LMI toolbox. Finally, two numerical examples are given to expose the capability and efficiency of our results.

Keywords Stochastic genetic regulatory networks (SGRNs) · Distributed delays · Leakage delays · Lyapunov–Krasovskii functional (LKF) · Impulses · Linear matrix inequalities (LMIs)

1 Introduction

In 1989, Macdonald was introduced the concept of genetic regulatory networks. GRN acts as a main role in our ordinary day life processes as well as cell discrimination, metabolism, the cell cycle and signal transduction; hence, significant efforts have been made to develop mathematical techniques for their analysis. Many types of model in GRNs have been studied such as Boolean network model, differential equation model, stochastic master equation model and Bayesian network model. Basically, Boolean network model and differential equation model are mainly used in the process of gene regulation. In [1, 4, 16, 26], the authors

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implemented the GRNs in differential equation model. Hence, we consider the differential equation model for switching SGRNs with the occurrence of time-delays. Because time delays can occur surely in transcription, translation and translocation processes of the slow reaction process. Considering that the time delays is decided in practice, we must take this case into account while modeling a network. Further, the existence of delays can be of discrete and distributed ones. In [7, 15, 21, 23], the authors discussed the distributed time-delays in modeling GRNs and in [2, 14, 20] the discrete time-delays studied extensively. On the other hand, due to the random chemical reactions, the internal noise arise in gene regulation process and the external noise arise in environment fluctuations [11, 25, 28]. In this way for relating the action of internal and external noise in GRNs, stochastic representations have been refined. Furthermore, when modeling the GRNs, the molecular noise stochastic process and parameter uncertainties play an important role in gene expression process [8, 10, 12, 13].

Recently, Gopalsamy [6] put forward a neural network system including time delays and leakage terms (i.e., negative feedback or decay terms which widely appeared in the models of neural networks, population dynamics, and GRNs). Along this line, a time delay will be taken into consideration in the decay terms of our GRNs model and we also call it as “leakage delay” [13].

Induced by the beyond deliberation, our main motivation of this paper is to investigate the switched stochastic genetic regulatory networks with leakage, multiple delays and impulses. By employing the stochastic analysis approach, stability theory and some inequality techniques, the asymptotic stability problem of the concerned genetic regulatory networks is transformed into feasibility problem of a set of linear matrix inequalities. Consequently, by constructing an appropriate novel Lyapunov–Krasovskii functional, some new sufficient criteria for asymptotic stability are formulated in terms of LMIs, which can be justified easily by LMI control toolbox in MATLAB software. Also, two numerical examples with their simulations are provided to demonstrate the feasibility with less conservatism of the proposed results. The contributions of this paper are given as follows:

- (1) The switched genetic regulatory networks covered by the deliberation are absolute that implicate the continuous-time case, stochastic disturbances, switching case, distributed delays, leakage delays and impulses;
- (3) Based on the contemporary Lyapunov–Krasovskii functional and integral inequality techniques, some sufficient conditions for asymptotic stability of switched genetic regulatory network are derived in the form of LMIs. In addition, compared to the existing results, the derived outcomes are different and advanced.
- (4) In this chapter, the feasibility of the obtained LMIs for asymptotical stability, can be easily solved by the aid of MATLAB LMI control toolbox.
- (3) Although, the existence of both time-varying delay and distributed time-varying delay terms in our concerned genetic regulatory networks, the allowable upper bounds of time-delays are maximum, when compared with some existing literatures, see Table 1 in Example 4.1. This can be expressed that the approach developed in this paper is more effective and less conserved.

This remaining part of the paper is classified as follows: In Sect. 2, the switched SGRNs with distributed delays, leakage delays and impulses are described and some assumptions, definitions and lemmas are introduced. In Sect. 3, we construct a Lyapunov function that consists of five parts, and derive the sufficient conditions expressed in terms of LMIs (Linear matrix inequalities). A mathematical example with simulation are shown in Sect. 4 to demonstrate the advantages of our results. Finally, Conclusions are drawn in Sect. 5.

Notations $\mathbb{R}^n$ denotes the n dimensional Euclidean space and the superscript T means that the matrix transposition. The symbol “*” is denoted by the symmetric term and I means that the unit matrix of appropriate dimensions. $\mathfrak{T}$ denotes continuous-time in GRNs and $\mathbb{E}\{x\}$ denotes the mathematical expectation of stochastic variable x . The representation $X \geq Y$ (consequently, $X > Y$), where X and Y are symmetric matrices, which are real. It manner that $X - Y$ is positive definite (or positive semi-definite).

2 Problem Formulation and Preparation

Deliberate the consequential SGRNs with leakage and impulses:

$$\begin{aligned} dx(\mathfrak{T}) &= \left(-Ax(\mathfrak{T} - \rho_1) + Bf(y(\mathfrak{T} - \delta_1)) + C\delta_2^{-1} \int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}} \hat{f}(y(\tau))d\tau \right) d\mathfrak{T} \\ &\quad + \eta(\mathfrak{T}, x(\mathfrak{T} - \rho_1), x(\mathfrak{T} - \zeta_1), y(\mathfrak{T} - \rho_2), y(\mathfrak{T} - \delta_1)) d\varpi(\mathfrak{T}) \\ x(\mathfrak{T}_k) &= \mathfrak{G}x(\mathfrak{T}_k^-), k \in z^+, \mathfrak{T} = \mathfrak{T}_k \\ dy(\mathfrak{T}) &= \left(-Dy(\mathfrak{T} - \rho_2) + Ex(\mathfrak{T} - \zeta_1) + F\zeta_2^{-1} \int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}} x(\tau)d\tau \right) d\mathfrak{T} \\ y(\mathfrak{T}_k) &= \mathfrak{H}y(\mathfrak{T}_k^-), k \in z^+, \mathfrak{T} = \mathfrak{T}_k \end{aligned} \quad (1)$$

where $x(\mathfrak{T}) \in \mathbb{R}^n$ is the concentrations of mRNA with n nodes and $y(\mathfrak{T}) \in \mathbb{R}^n$ is the concentrations of proteins with n nodes; The leakage delays are ρ_1 and ρ_2 ; $A = \text{diag}\{a_1, a_2, \dots, a_n\}$ is the decadence of mRNAs and $D = \text{diag}\{d_1, d_2, \dots, d_n\}$ is the decadence of proteins; Translation rates are $E = \text{diag}\{e_1, e_2, \dots, e_n\}$ and $F = \text{diag}\{f_1, f_2, \dots, f_n\}$. $\varpi(\mathfrak{T})$ is a Brownian motion on $(\Omega, \mathfrak{F}, \mathfrak{P})$. The time delays are $\delta_1, \delta_2, \zeta_1$ and ζ_2 ; $x(\mathfrak{T}_k)$ and $y(\mathfrak{T}_k)$ are the impulses of mRNAs and proteins respectively. The non-linear function f and $\hat{f}$ are constitutes the regulation of protein from their feedback in transcription period, which are in Hill form. If transcription factor j is an activator of gene i , then:

$$f_{ij}(y_j(\mathfrak{T} - \delta_1)) = \mathfrak{p}_{ij} \frac{(y_j(\mathfrak{T} - \delta_1))^{h_j}}{\alpha^{h_j} + (y_j(\mathfrak{T} - \delta_1))^{h_j}}, \quad \hat{f}_{ij}(y_j(\mathfrak{T} - \delta_1)) = \hat{\mathfrak{p}}_{ij} \frac{(y_j(\tau))^{h_j}}{\alpha^{h_j} + (y_j(\tau))^{h_j}}$$

If transcription factor j is an repressor of gene i , then:

$$\begin{aligned} f_{ij}(y_j(\mathfrak{T} - \delta_1)) &= \mathfrak{p}_{ij} \left(1 - \frac{(y_j(\mathfrak{T} - \delta_1))^{h_j}}{\alpha^{h_j} + (y_j(\mathfrak{T} - \delta_1))^{h_j}} \right) \\ \hat{f}_{ij}(y_j(\mathfrak{T} - \delta_1)) &= \hat{\mathfrak{p}}_{ij} \left(1 - \frac{(y_j(\tau))^{h_j}}{\alpha^{h_j} + (y_j(\tau))^{h_j}} \right) \end{aligned}$$

where $\mathfrak{p}_{ij}$ is the bounded constant which is the transcriptional rate of transcription factor j to i , α is positive constant and h_j is the Hill co-efficients.

The matrix $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ and $C = (c_{ij}) \in \mathbb{R}^{n \times n}$ are defined as follows:

$$b_{ij} = \begin{cases} \mathfrak{p}_{ij} & \text{if transcription factor } j \text{ is an activator of gene } i \\ 0 & \text{if there is no link from node } j \text{ to } i \\ -\mathfrak{p}_{ij} & \text{if transcription factor } j \text{ is a repressor of gene } i \end{cases} \quad (2)$$

$$c_{ij} = \begin{cases} \hat{p}_{ij} & \text{if transcription factor } j \text{ is an activator of gene } i \\ 0 & \text{if there is no link from node } j \text{ to } i \\ -\hat{p}_{ij} & \text{if transcription factor } j \text{ is a repressor of gene } i \end{cases} \quad (3)$$

where $x(\mathfrak{T} - \rho_1) = (x_1(\mathfrak{T} - \rho_1), x_2(\mathfrak{T} - \rho_1), \dots, x_n(\mathfrak{T} - \rho_1))^T$, $y(\mathfrak{T} - \rho_2) = (y_1(\mathfrak{T} - \rho_2), y_2(\mathfrak{T} - \rho_2), \dots, y_n(\mathfrak{T} - \rho_2))^T$, $f(y(\mathfrak{T} - \delta_1)) = (f_1(y_1(\mathfrak{T} - \delta_1)), f_2(y_2(\mathfrak{T} - \delta_1)), \dots, f_n(y_n(\mathfrak{T} - \delta_1)))^T$, $\hat{f}(y(\tau)) = (\hat{f}_1(y_1(\tau)), \hat{f}_2(y_2(\tau)), \dots, \hat{f}_n(y_n(\tau)))^T$ and $x(\mathfrak{T} - \zeta_1) = (x_1(\mathfrak{T} - \zeta_1), \dots, x_n(\mathfrak{T} - \zeta_1))^T$.

Now, we discuss the switched SGRNs with impulsive effects as follows:

$$\begin{aligned} dx(\mathfrak{T}) &= \sum_{\mathfrak{K}=1}^N \varsigma_{\mathfrak{K}}(\mathfrak{T})(-A_{\mathfrak{K}}x(\mathfrak{T} - \rho_1) \\ &\quad + B_{\mathfrak{K}}f(y(\mathfrak{T} - \delta_1)) + C_{\mathfrak{K}}\delta_2^{-1} \int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}} \hat{f}(y(\tau))d\tau)d\mathfrak{T} \\ &\quad + \eta(\mathfrak{T}, x(\mathfrak{T} - \rho_1), x(\mathfrak{T} - \zeta_1), y(\mathfrak{T} - \rho_2), y(\mathfrak{T} - \delta_1))d\varpi(\mathfrak{T}) \\ x(\mathfrak{T}_k) &= \mathfrak{G}x(\mathfrak{T}_k^-), k \in \mathbb{Z}^+, \mathfrak{T} = \mathfrak{T}_k \\ dy(\mathfrak{T}) &= \sum_{\mathfrak{K}=1}^N \varsigma_{\mathfrak{K}}(\mathfrak{T})(-D_{\mathfrak{K}}y(\mathfrak{T} - \rho_2) \\ &\quad + E_{\mathfrak{K}}x(\mathfrak{T} - \zeta_1) + F_{\mathfrak{K}}\zeta_2^{-1} \int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}} x(\tau)d\tau)d\mathfrak{T} \\ y(t_{\mathfrak{T}}) &= \mathfrak{H}y(\mathfrak{T}_k^-), k \in \mathbb{Z}^+, \mathfrak{T} = \mathfrak{T}_k \end{aligned} \quad (4)$$

where $\mathfrak{K}$ denotes the switching signal and its value in a finite set $S = \{1, 2, \dots, N\}$ and $\varsigma(\mathfrak{T}) = (\varsigma_1(\mathfrak{T}), \varsigma_2(\mathfrak{T}), \dots, \varsigma_N(\mathfrak{T}))^T$ is the indicator function, ie., $\forall \mathfrak{K} \in S$,

$$\varsigma_{\mathfrak{K}}(\mathfrak{T}) = \begin{cases} 1 & \text{when } A_{\mathfrak{K}}, B_{\mathfrak{K}}, C_{\mathfrak{K}}, D_{\mathfrak{K}}, E_{\mathfrak{K}}, \text{ and } F_{\mathfrak{K}} \text{ are activated} \\ 0 & \text{otherwise} \end{cases}$$

To prove our main results the following assumptions, definitions and lemmas are essential.

Assumption 1 $\eta : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ represents a noise intensity function vector, satisfying

$$\begin{aligned} \text{trace}(\eta^T \eta) &\leq x^T(\mathfrak{T} - \rho_1)J_1x(\mathfrak{T} - \rho_1) + x^T(\mathfrak{T} - \zeta_1)J_2x(\mathfrak{T} - \zeta_1) \\ &\quad + y^T(\mathfrak{T} - \rho_2)J_3y(\mathfrak{T} - \rho_2) \\ &\quad + y^T(\mathfrak{T} - \delta_1)J_4y(\mathfrak{T} - \delta_1) \end{aligned}$$

where J_1, J_2, J_3 and J_4 are the known matrices.

Assumption 2 The activation function $f_i(\cdot)$ is assumed to satisfy the following sector bounded condition by:

$$p_i \leq \frac{f_i(z_i)}{z_i} \leq q_i, \quad \forall z_i \in \mathbb{R}, z_i \neq 0, i = 1, 2, \dots, n.$$

Definition 2.1 If for any $\epsilon > 0$ there is $\delta(\epsilon) > 0$ then the system (4) is **stable** such that

$$\mathbf{E}||x^T(\mathfrak{T}), y^T(\mathfrak{T})||^2 < \epsilon \quad \text{when} \quad \sup_{-\tau \leq \delta \leq 0} \mathbf{E}||\varphi(s)||^2 < \delta$$

where $\varphi \in L^2([-\tau, 0]; \mathbb{R}^{2n})$.

The system (4) is **asymptotically stable** if $\lim_{t \rightarrow 0} \mathbf{E}||x^T(\mathfrak{T}), y^T(\mathfrak{T})||^2 = 0$.

Lemma 2.2 (Schur Complement [22]) *Given constant matrices Ξ_1, Ξ_2, Ξ_3 where $\Xi_1 = \Xi_1^T$ and $0 < \Xi_2 = \Xi_2^T$, then $\Xi_1 + \Xi_3^T \Xi_2^{-1} \Xi_3 < 0$ iff*

$$\begin{bmatrix} \hat{\Xi}_1 & \hat{\Xi}_3^T \\ \hat{\Xi}_3 & -\hat{\Xi}_2 \end{bmatrix} < 0 \text{ or } \begin{bmatrix} -\hat{\Xi}_2 & \hat{\Xi}_3 \\ \hat{\Xi}_3^T & \hat{\Xi}_1 \end{bmatrix} < 0$$

Lemma 2.3 (Jensen's Inequality [16]) *For any constant matrix $S > 0$, any scalar r, t with $r < t$, and a vector function $x(t) : [r, t] \rightarrow \mathbb{R}^n$ such that the integrals concerned are well defined, then the following condition holds.*

$$(t-r) \int_r^t x^T(p) S x(p) dp \geq \left(\int_r^t x(p) dp \right)^T S \left(\int_r^t x(p) dp \right). \quad (5)$$

Lemma 2.4 [10] *Let N, S and Q be the real matrices with $\epsilon > 0$. Then $N^T S + S^T N \leq \epsilon^{-1} N^T Q N + \epsilon S^T Q^{-1} S$ hold.*

Lemma 2.5 *The activation function $f_i(\cdot)$ satisfies assumption 2 and $f = (f_1, f_2, \dots, f_n)^T$. For any diagonal positive matrix Ω_1 such that,*

$$\begin{bmatrix} z \\ f(z) \end{bmatrix}^T \begin{bmatrix} -N\Omega_1 & N\Omega_2 \\ N\Omega_2 & -N \end{bmatrix} \begin{bmatrix} z \\ f(z) \end{bmatrix} \geq 0, \forall z \in \mathbb{R}^n. \quad (6)$$

where $\Omega_1 = \text{diag}\{p_1 q_1, p_2 q_2, \dots, p_n q_n\}$ and $\Omega_2 = \text{diag}\{\frac{p_1+q_1}{2}, \frac{p_2+q_2}{2}, \dots, \frac{p_n+q_n}{2}\}$.

3 Main Results

The following theorem will give the stability criterion for SGRNs.

Theorem 3.1 *With the help of Assumptions 1 and 2, the switched SGRNs (4) is globally asymptotically stable in mean square if there exist positive-definite matrices R_1 and S_1 , the matrices $R_2, R_3, R_4, S_2, S_3, S_4, T_1, T_2, X$ and positive definite diagonal matrix N such that the following LMIs hold:*

$$D_{ik}^T R_j D_{ik} - R_i < 0 \quad (7)$$

$$R_1 \leq \mu I \quad (8)$$

$$\Pi_{\mathcal{R}} = \begin{bmatrix} \varphi_{1,1} & 0 & -R_1 A_{\mathcal{R}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_1 B_{\mathcal{R}} & \varphi_{1,11} \\ * & \varphi_{2,2} & 0 & -S_1 D_{\mathcal{R}} & 0 & S_1 E_{\mathcal{R}} & 0 & \varphi_{2,8} & N \Omega_2 & 0 & 0 \\ * & * & \varphi_{3,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \varphi_{4,4} & -X D_{\mathcal{R}} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \varphi_{5,5} & X E_{\mathcal{R}} & 0 & \zeta_2^{-1} X F_{\mathcal{R}} & 0 & 0 & 0 \\ * & * & * & * & * & \varphi_{6,6} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \varphi_{7,7} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \varphi_{8,8} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \varphi_{9,9} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \varphi_{10,10} & 0 \\ * & * & * & * & * & * & * & * & * & * & \varphi_{11,11} \end{bmatrix}, \quad (9)$$

where $\varphi_{1,1} = \rho_1 R_2 + R_3 + \zeta_2 R_4$; $\varphi_{1,11} = \delta_2^{-1} R_1 C_{\mathcal{R}}$; $\varphi_{2,2} = \rho_2 S_2 + S_3 - \delta_1^{-1} S_4 - N \Omega_1$; $\varphi_{2,8} = \zeta_2^{-1} S_1 F_{\mathcal{R}}$; $\varphi_{3,3} = \mu J_1 - \rho_1 R_2$; $\varphi_{4,4} = \mu J_3 - \rho_2 S_2$; $\varphi_{5,5} = \delta_1 S_4 - 2X$; $\varphi_{5,8} = \zeta_2^{-1} X F_{\mathcal{R}}$;

$$\varphi_{6,6} = \mu J_2 - R_3; \varphi_{7,7} = \mu J_4 - S_3 - \delta_1^{-1} S_4; \varphi_{8,8} = -\zeta_2^{-1} R_4; \varphi_{9,9} = T_1 + \delta_2 T_2 - N; \\ \varphi_{10,10} = -T_1; \varphi_{11,11} = -\delta_2^{-1} T_2.$$

Proof We construct the LKF as follows:

$$\mathfrak{V}(\mathfrak{T}) = \mathfrak{V}_1(\mathfrak{T}) + \mathfrak{V}_2(\mathfrak{T}) + \mathfrak{V}_3(\mathfrak{T}) + \mathfrak{V}_4(\mathfrak{T}) + \mathfrak{V}_5(\mathfrak{T}) \quad (10)$$

where

$$\begin{aligned} \mathfrak{V}_1(\mathfrak{T}) &= x^T(\mathfrak{T}) R_1 x(\mathfrak{T}) + y^T(\mathfrak{T}) S_1 y(\mathfrak{T}) \\ \mathfrak{V}_2(\mathfrak{T}) &= \rho_1 \int_{\mathfrak{T}-\rho_1}^{\mathfrak{T}} x^T(s) R_2 x(s) ds + \rho_2 \int_{\mathfrak{T}-\rho_2}^{\mathfrak{T}} y^T(s) S_2 y(s) ds \\ \mathfrak{V}_3(\mathfrak{T}) &= \int_{\mathfrak{T}-\zeta_1}^{\mathfrak{T}} x^T(\tau) R_3 x(\tau) d\tau + \zeta_2^{-1} \int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}} \int_{\tau}^{\mathfrak{T}} x^T(s) R_4 x(s) ds d\tau \\ \mathfrak{V}_4(\mathfrak{T}) &= \int_{\mathfrak{T}-\delta_1}^{\mathfrak{T}} y^T(\tau) S_3 y(\tau) d\tau + \delta_1^{-1} \int_{\mathfrak{T}-\delta_1}^{\mathfrak{T}} \int_{\tau}^{\mathfrak{T}} \dot{y}^T(s) S_4 \dot{y}(s) ds d\tau \\ \mathfrak{V}_5(\mathfrak{T}) &= \int_{\mathfrak{T}-\delta_1}^{\mathfrak{T}} f^T(y(\tau)) T_1 f(y(\tau)) d\tau + \delta_2^{-1} \int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}} \int_{\tau}^{\mathfrak{T}} f^T(y(\tau)) T_2 f(y(\tau)) ds d\tau \end{aligned}$$

Here, we can find the consequential equation by using Ito's differential formula

$$d\mathfrak{V}(\mathfrak{T}) = \mathfrak{L}\mathfrak{V}d\mathfrak{T} + 2x^T(\mathfrak{T}) R_1 \eta(\mathfrak{T}, x(\mathfrak{T} - \rho_1), x(\mathfrak{T} - \zeta_1), y(\mathfrak{T} - \rho_2), y(\mathfrak{T} - \delta_1)) d\varpi(\mathfrak{T}) \quad (11)$$

□

Now, we can calculate $\mathfrak{L}\mathfrak{V}(\mathfrak{T})$ as:

$$\begin{aligned} \mathfrak{L}\mathfrak{V}(\mathfrak{T}) &= 2x^T(\mathfrak{T}) R_1 \sum_{\mathfrak{K}=1}^N \zeta_{\mathfrak{K}}(\mathfrak{T}) \left(-A_{\mathfrak{K}} x(\mathfrak{T} - \rho_1) + B_{\mathfrak{K}} f(y(\mathfrak{T} - \delta_1)) \right. \\ &\quad \left. + C_{\mathfrak{K}} \delta_2^{-1} \int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}} \hat{f}(y(\tau)) d\tau \right) \\ &\quad + \text{trac} \eta^T(\cdot) R_1 \eta(\cdot) + 2y^T(\mathfrak{T}) S_1 \sum_{\mathfrak{K}=1}^N \zeta_{\mathfrak{K}}(\mathfrak{T}) \left(-D_{\mathfrak{K}} y(\mathfrak{T} - \rho_2) \right. \\ &\quad \left. + E_{\mathfrak{K}} x(\mathfrak{T} - \zeta_1) \right. \\ &\quad \left. + F_{\mathfrak{K}} \zeta_2^{-1} \int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}} x(\tau) d\tau \right) d\mathfrak{T} + \rho_1 x^T(\mathfrak{T}) R_2 x(\mathfrak{T}) \\ &\quad - \rho_1 x^T(\mathfrak{T} - \rho_1) R_2 x(\mathfrak{T} - \rho_1) \\ &\quad + \rho_2 y^T(\mathfrak{T}) S_2 y(\mathfrak{T}) - \rho_2 y^T(\mathfrak{T} - \rho_2) S_2 y(\mathfrak{T} - \rho_2) \\ &\quad + x^T(\mathfrak{T}) R_3 x(\mathfrak{T}) - x^T(\mathfrak{T} - \zeta_1) R_3 \\ &\quad \times x(\mathfrak{T} - \zeta_1) + \zeta_2 x^T(\mathfrak{T}) R_4 x(\mathfrak{T}) \\ &\quad - \int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}} x^T(\tau) R_4 x(\tau) d\tau + y^T(\mathfrak{T}) S_3 y(\mathfrak{T}) \\ &\quad - y^T(\mathfrak{T} - \delta_1) S_3 y(\mathfrak{T} - \delta_1) + \delta_1 \dot{y}^T(\mathfrak{T}) S_4 \dot{y}(\mathfrak{T}) \\ &\quad - \int_{\mathfrak{T}-\delta_1}^{\mathfrak{T}} \dot{y}^T(\tau) S_4 \dot{y}(\tau) d\tau \end{aligned}$$

$$\begin{aligned}
 & + f^T(y(\mathfrak{T}))T_1 f(y(\mathfrak{T})) - f^T(y(\mathfrak{T} - \delta_1))T_1 f(y(\mathfrak{T} - \delta_1)) \\
 & + \delta_2 f^T(y(\mathfrak{T}))T_2 f(y(\mathfrak{T})) \\
 & - \int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}} f^T(y(\tau))T_2 f(y(\tau))d\tau
 \end{aligned} \quad (12)$$

We can find the following terms by using Lemma 2.3,

$$- \int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}} x^T(\tau)R_4 x(\tau)d\tau \leq -\frac{1}{\zeta_2} \left(\int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}} x(\tau)d\tau \right)^T R_4 \left(\int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}} x(\tau)d\tau \right) \quad (13)$$

$$\begin{aligned}
 - \int_{\mathfrak{T}-\delta_1}^{\mathfrak{T}} \dot{y}^T(\tau)S_4 \dot{y}(\tau)d\tau & \leq -\frac{1}{\delta_1} \left(\int_{\mathfrak{T}-\delta_1}^{\mathfrak{T}} \dot{y}(\tau)d\tau \right)^T S_4 \left(\int_{\mathfrak{T}-\delta_1}^{\mathfrak{T}} \dot{y}(\tau)d\tau \right) \\
 & \leq -\delta_1^{-1} (y(\mathfrak{T}) - y(\mathfrak{T} - \delta_1))^T S_4 (y(\mathfrak{T}) - y(\mathfrak{T} - \delta_1))
 \end{aligned} \quad (14)$$

$$- \int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}} f^T(y(\tau))T_2 f(y(\tau))d\tau \leq -\frac{1}{\delta_2} \left(\int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}} f(y(\tau))d\tau \right)^T T_2 \left(\int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}} f(y(\tau))d\tau \right) \quad (15)$$

According to Assumption 2, it is clear to view that,

$$\begin{aligned}
 \text{trace}((\eta(\cdot)R_1\eta(\cdot))) & \leq \mu(x^T(\mathfrak{T} - \rho_1)J_1 x(\mathfrak{T} - \rho_1) + x^T(\mathfrak{T} - \zeta_1)J_2 x(\mathfrak{T} - \zeta_1) \\
 & + y^T(\mathfrak{T} - \rho_2)J_3 y(\mathfrak{T} - \rho_2) + y^T(\mathfrak{T} - \delta_1)J_4 y(\mathfrak{T} - \delta_1))
 \end{aligned} \quad (16)$$

For any positive-definite diagonal matrix N, the following inequality hold.

$$\begin{bmatrix} y(\mathfrak{T}) \\ f(y(\mathfrak{T})) \end{bmatrix}^T \begin{bmatrix} -N\Omega_1 & N\Omega_2 \\ N\Omega_2 & -N \end{bmatrix} \begin{bmatrix} y(\mathfrak{T}) \\ f(y(\mathfrak{T})) \end{bmatrix} \geq 0 \quad (17)$$

In addition,

$$\begin{aligned}
 0 & = 2\dot{y}(\mathfrak{T})X \left(-\dot{y}(\mathfrak{T}) + \sum_{k=1}^N \varsigma_{\mathfrak{R}}(\mathfrak{T}) \left(-D_{\mathfrak{R}}y(\mathfrak{T} - \rho_2) \right. \right. \\
 & \quad \left. \left. + E_{\mathfrak{R}}x(\mathfrak{T} - \zeta_1) + F_{\mathfrak{R}}\zeta_2^{-1} \int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}} x(\tau)d\tau \right) \right)
 \end{aligned} \quad (18)$$

Combining Eqs. (12) to (18), we get

$$\begin{aligned}
 \mathfrak{L}\mathfrak{V}(\mathfrak{T}) & = \sum_{\mathfrak{R}=1}^N \varsigma_{\mathfrak{R}}(\mathfrak{T}) \left(-2x^T(\mathfrak{T})R_1 A_{\mathfrak{R}}x(\mathfrak{T} - \rho_1) \right. \\
 & \quad + 2x^T(\mathfrak{T})R_1 B_{\mathfrak{R}}f(y(\mathfrak{T} - \delta_1)) + 2x^T(\mathfrak{T})R_1 C_{\mathfrak{R}}\delta_2^{-1} \\
 & \quad \times \left. \int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}} \hat{f}(y(\tau))d\tau \right) + \mu \left(x^T(\mathfrak{T} - \rho_1)J_1 x(\mathfrak{T} - \rho_1) \right. \\
 & \quad + \mu x^T(\mathfrak{T} - \zeta_1)J_2 x(\mathfrak{T} - \zeta_1) \\
 & \quad + \mu y^T(\mathfrak{T} - \rho_2)J_3 y(\mathfrak{T} - \rho_2) + \mu y^T(\mathfrak{T} - \delta_1)J_4 y(\mathfrak{T} - \delta_1) \Big) \\
 & \quad + \sum_{\mathfrak{R}=1}^N \varsigma_{\mathfrak{R}}(\mathfrak{T}) \left(-2y^T(\mathfrak{T})S_1 D_{\mathfrak{R}}y(\mathfrak{T} - \rho_2) \right. \\
 & \quad + 2y^T(\mathfrak{T})S_1 E_{\mathfrak{R}}x(\mathfrak{T} - \zeta_1) + 2y^T(\mathfrak{T})S_1 F_{\mathfrak{R}}\zeta_2^{-1} \int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}} x(\tau)d\tau \Big) d\mathfrak{T} \\
 & \quad + \rho_1 x^T(\mathfrak{T})R_2 x(\mathfrak{T})
 \end{aligned}$$

$$\begin{aligned}
& -\rho_1 x^T(\mathfrak{T} - \rho_1) R_2 x(\mathfrak{T} - \rho_1) + \rho_2 y^T(\mathfrak{T}) S_2 y(\mathfrak{T}) \\
& -\rho_2 y^T(\mathfrak{T} - \rho_2) S_2 y(\mathfrak{T} - \rho_2) \\
& + x^T(\mathfrak{T}) R_3 x(\mathfrak{T}) - x^T(\mathfrak{T} - \zeta_1) R_3 x(\mathfrak{T} - \zeta_1) \\
& + \zeta_2 x^T(\mathfrak{T}) R_4 x(\mathfrak{T}) - \zeta_2^{-1} \left(\int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}} x(\tau) d\tau \right)^T \\
& \times R_4 \left(\int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}} x(\tau) d\tau \right) + y^T(\mathfrak{T}) S_3 y(\mathfrak{T}) - y^T(\mathfrak{T} - \delta_1) S_3 y(\mathfrak{T} - \delta_1) \\
& + \delta_1 \dot{y}^T(\mathfrak{T}) S_4 \dot{y}(\mathfrak{T}) - \delta_1^{-1} (y(\mathfrak{T}) - y(\mathfrak{T} - \delta_1))^T S_4 (y(\mathfrak{T}) - y(\mathfrak{T} - \delta_1)) \\
& + f^T(y(\mathfrak{T})) T_1 f(y(\mathfrak{T})) - f^T(y(\mathfrak{T} - \delta_1)) T_1 f(y(\mathfrak{T} - \delta_1)) + \delta_2 f^T(y(\mathfrak{T})) T_2 f(y(\mathfrak{T})) \\
& - \delta_2^{-1} \left(\int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}} f(y(\tau)) d\tau \right)^T T_2 \left(\int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}} f(y(\tau)) d\tau \right) + 2 \dot{y}^T(\mathfrak{T}) X (-\dot{y}(\mathfrak{T})) \\
& + \sum_{k=1}^N \varsigma_{\mathfrak{R}}(\mathfrak{T}) \left(-D_{\mathfrak{R}} y(\mathfrak{T} - \rho_2) + E_{\mathfrak{R}} x(\mathfrak{T} - \zeta_1) + F_{\mathfrak{R}} \zeta_2^{-1} \int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}} x(\tau) d\tau \right) \\
& + \begin{bmatrix} y(\mathfrak{T}) \\ f(y(\mathfrak{T})) \end{bmatrix}^T \begin{bmatrix} -N\Omega_1 & N\Omega_2 \\ N\Omega_2 & -N \end{bmatrix} \begin{bmatrix} y(\mathfrak{T}) \\ f(y(\mathfrak{T})) \end{bmatrix} \\
& \leq \sum_{k=1}^N \varsigma_{\mathfrak{R}}(\mathfrak{T}) \xi^T(\mathfrak{T}) \Pi_k \xi(\mathfrak{T}) < 0
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
\xi(\mathfrak{T}) = & (x(\mathfrak{T}), y(\mathfrak{T}), x(\mathfrak{T} - \rho_1), y(\mathfrak{T} - \rho_2), \dot{y}(\mathfrak{T}), x(\mathfrak{T} - \zeta_1), y(\mathfrak{T} - \delta_1), \\
& \int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}} x(\tau) d\tau, f(y(\mathfrak{T})), f(y(\mathfrak{T} - \delta_1)), \int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}} f(y(\tau)) d\tau)^T.
\end{aligned}$$

Using Itô differential equation, it is clear as in [25]:

$$\mathbf{E}\mathfrak{V}(\mathfrak{T}) - \mathbf{E}\mathfrak{V}(\mathfrak{T}_0) = \mathbf{E} \int_{\mathfrak{T}_0}^{\mathfrak{T}} \mathfrak{L}\mathfrak{V}(\mathfrak{T}) ds \tag{20}$$

By Definition 2.1, there must exist positive λ_1 such that

$$\begin{aligned}
\lambda_1 \mathbf{E} \| (x^T(\mathfrak{T}), y^T(\mathfrak{T}))^T \|^2 & \leq \mathbf{E}\mathfrak{V}(\mathfrak{T}) \leq \mathbf{E}\mathfrak{V}(\mathfrak{T}_0) + \mathbf{E} \int_{\mathfrak{T}_0}^{\mathfrak{T}} \mathfrak{L}\mathfrak{V}(\mathfrak{T}) ds \\
& \leq \mathbf{E}\mathfrak{V}(\mathfrak{T}_0) + \lambda_{\max} \mathbf{E} \int_{\mathfrak{T}_0}^{\mathfrak{T}} \| (x^T(\mathfrak{T}), y^T(\mathfrak{T}))^T \|^2 ds
\end{aligned} \tag{21}$$

where $\lambda_{\max}$ is negative and it is denoted by the maximal eigenvalue. Hence, the switched SGRNs (4) is globally asymptotically stable. On the other hand, from (10) and Theorem 3.1 conditions, we note that,

$$\begin{aligned}
V_1(\mathfrak{T}_k, x(\mathfrak{T}_k), j) - V_1(\mathfrak{T}_k^-, x(\mathfrak{T}_k^-), i) & = x_m^T(\mathfrak{T}_k) R_j x_m(\mathfrak{T}_k) - x_m^T(\mathfrak{T}_k^-) R_i x_m(\mathfrak{T}_k^-) \\
& = x_m^T(\mathfrak{T}_k^-) D_{ik}^T R_j D_{ik} x_m(\mathfrak{T}_k^-) - x_m^T(\mathfrak{T}_k^-) R_i x_m(\mathfrak{T}_k^-) \\
& = x_m^T(\mathfrak{T}_k^-) (D_{ik}^T R_j D_{ik} - R_i) x_m(\mathfrak{T}_k^-) \\
& \leq 0 \\
V_1(\mathfrak{T}_k, x_m(\mathfrak{T}_k), j) & \leq V_1(\mathfrak{T}_k^-, x_m(\mathfrak{T}_k^-), i), k \in \mathbb{Z}_+.
\end{aligned}$$

Which implies that,

$$V(\mathfrak{T}_k, x(\mathfrak{T}_k), j) \leq V(\mathfrak{T}_k^-, x(\mathfrak{T}_k^-), i), k \in \mathbb{Z}_+.$$
 (22)

The system (4) with impulsive effect is asymptotically stable in the sense of mean square. This concludes the proof.

Remark 3.1 For GRNs (2) without switching case, then it is treated as a particular expression for the switching system (4). And also it discusses the stability criteria when the leakage delay terms and impulses are involved.

Remark 3.2 The present model accommodates with leakage delays, impulses, stochastic disturbances and distributed delays and it can be used to express the continuous-time switched genetic regulatory networks in view of its generality. Yao et al [26] discuss the stability analysis for switched genetic regulatory networks. Moreover, in [9], Li et al. taken the switched systems via nonlinear measure. Recently, Wang et al. [23] considered the switched system for genetic regulatory networks in emigrate of leakage delays and impulses. However, this work is more purposeful and shows to be less cautious stability criterion than the ones in [9, 23, 26].

Corollary 3.4 *With the help of Assumptions 1 and 2, the switched SGRNs (4) is globally asymptotically stable in mean square if there exist a constant $\epsilon \in (0, 1)$, positive-definite matrices $R_1, R_2, R_3, R_4, S_1, S_2, S_3, S_4, T_1, T_2, X$ and positive definite diagonal matrix N such that the following LMIs hold:*

$$D_{ik}^T R_j D_{ik} - R_i < 0$$
 (23)

$$R_1 \leq \mu I$$
 (24)

$$\Psi_{\mathfrak{R}} = \begin{bmatrix} \Phi_{1,1} & 0 & -R_1 A_{\mathfrak{R}} & 0 & 0 & 0 & 0 & 0 & 0 & R_1 B_{\mathfrak{R}} \\ * & \Phi_{2,2} & 0 & -S_1 D_{\mathfrak{R}} & 0 & S_1 E_{\mathfrak{R}} & -\delta_1^{-1} S_4 & N \Omega_2 & 0 & \\ * & * & \Phi_{3,3} & 0 & 0 & 0 & 0 & 0 & 0 & \\ * & * & * & \Phi_{4,4} & -X D_{\mathfrak{R}} & X E_{\mathfrak{R}} & 0 & 0 & 0 & \\ * & * & * & * & \Phi_{5,5} & 0 & 0 & 0 & 0 & \\ * & * & * & * & * & \Phi_{6,6} & 0 & 0 & 0 & \\ * & * & * & * & * & * & \Phi_{7,7} & 0 & N \Omega_2 & \\ * & * & * & * & * & * & * & \Phi_{8,8} & 0 & \\ * & * & * & * & * & * & * & * & \Phi_{9,9} & \end{bmatrix},$$
 (25)

where

$$\begin{aligned} \Phi_{1,1} &= \rho_1 R_2 + R_3 + \zeta_2 R_4 + \delta_1^{-1} R_1 C_{\mathfrak{R}} T_2^{-1} C_{\mathfrak{R}}^T R_1^T; \\ \Phi_{2,2} &= \rho_2 S_2 + S_3 - \delta_1^{-1} S_4 - N \Omega_1 + \epsilon^{-1} \zeta_2^{-1} S_1 F_{\mathfrak{R}} R_4^{-1} F_{\mathfrak{R}}^T S_1^T; \\ \Phi_{3,3} &= \mu J_1 - \rho_1 R_2; \quad \Phi_{4,4} = \mu J_3 - \rho_2 S_2; \\ \Phi_{5,5} &= \delta_1 S_4 - 2X + (1 - \epsilon)^{-1} X F_{\mathfrak{R}} R_4^{-1} F_{\mathfrak{R}}^T X^T; \\ \Phi_{6,6} &= \mu J_2 - R_3; \quad \Phi_{7,7} = \mu J_4 - S_3 - \delta_1^{-1} S_4 - N \Omega_1; \\ \Phi_{8,8} &= T_1 + \delta_2 T_2 - N; \quad \Phi_{9,9} = -T_1 - N_1. \end{aligned}$$

Proof Choose the same LKF in the above theorem. We can find the following terms by using Lemma 2.4,

$$2x^T(\mathfrak{T})R_1C_{\mathfrak{R}}\delta_2^{-1}\int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}}\hat{f}(y(\tau))d\tau\leq\frac{1}{\delta_2}\left(\int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}}\hat{f}(y(\tau))d\tau\right)^T T_2\left(\int_{\mathfrak{T}-\delta_2}^{\mathfrak{T}}\hat{f}(y(\tau))d\tau\right) \\ +\delta_2^{-1}x^T(\mathfrak{T})R_1C_{\mathfrak{R}}T_2^{-1}C_{\mathfrak{R}}^TR_1^Tx(\mathfrak{T}); \quad (26)$$

$$2y^T(\mathfrak{T})S_1F_{\mathfrak{R}}\zeta_2^{-1}\int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}}x(\tau)d\tau\leq\frac{\epsilon}{\zeta_2}\left(\int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}}x(\tau)d\tau\right)^TR_4\left(\int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}}x(\tau)d\tau\right) \\ +\epsilon^{-1}\zeta_2^{-1}y^T(\mathfrak{T})S_1F_{\mathfrak{R}}R_4^{-1}F_k^TS_1^Ty(\mathfrak{T}); \quad (27)$$

$$2\dot{y}^T(\mathfrak{T})XF_{\mathfrak{R}}\zeta_2^{-1}\int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}}x(\tau)d\tau\leq\frac{1-\epsilon}{\zeta_2}\left(\int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}}x(\tau)d\tau\right)^TR_4\left(\int_{\mathfrak{T}-\zeta_2}^{\mathfrak{T}}x(\tau)d\tau\right) \\ +(1-\epsilon)^{-1}\zeta_2^{-1}\dot{y}^T(\mathfrak{T})XF_{\mathfrak{R}}R_4^{-1}F_{\mathfrak{R}}^TX^T\dot{y}(\mathfrak{T}) \quad (28)$$

We can obtain $\mathfrak{L}\mathfrak{V}(\mathfrak{T})$ as follows:

$$\begin{aligned} \mathfrak{L}\mathfrak{V}(\mathfrak{T}) &= \sum_{\mathfrak{R}=1}^N \varsigma_{\mathfrak{R}}(\mathfrak{T}) \left(-2x^T(\mathfrak{T})R_1A_{\mathfrak{R}}x(\mathfrak{T}-\rho_1) \right. \\ &\quad + 2x^T(\mathfrak{T})R_1B_{\mathfrak{R}}f(y(\mathfrak{T}-\delta_1)) + \delta_2^{-1}x^T(\mathfrak{T})R_1C_{\mathfrak{R}}T_2^{-1} \\ &\quad \times C_{\mathfrak{R}}^TR_1^Tx(\mathfrak{T}) \Big) + \mu \left(x^T(\mathfrak{T}-\rho_1)J_1x(\mathfrak{T}-\rho_1) \right. \\ &\quad + \mu x^T(\mathfrak{T}-\zeta_1)J_2x(\mathfrak{T}-\zeta_1) + \mu y^T(\mathfrak{T}-\rho_2) \\ &\quad \times J_3y(\mathfrak{T}-\rho_2) + \mu y^T(\mathfrak{T}-\delta_1)J_4y(\mathfrak{T}-\delta_1) \Big) \\ &\quad + \sum_{\mathfrak{R}=1}^N \varsigma_{\mathfrak{R}}(\mathfrak{T}) \left(-2y^T(\mathfrak{T})S_1D_{\mathfrak{R}}y(\mathfrak{T}-\rho_2) \right. \\ &\quad + 2y^T(\mathfrak{T})S_1E_{\mathfrak{R}}x(\mathfrak{T}-\zeta_1) + \epsilon^{-1}\zeta_2^{-1}y^T(\mathfrak{T})S_1F_{\mathfrak{R}}R_4^{-1}F_k^TS_1^Ty(\mathfrak{T}) \Big) \\ &\quad + \rho_2y^T(\mathfrak{T})S_2y(\mathfrak{T}) \\ &\quad - \rho_2y^T(\mathfrak{T}-\rho_2)S_2y(\mathfrak{T}-\rho_2) + x^T(\mathfrak{T})R_3x(\mathfrak{T}) \\ &\quad - x^T(\mathfrak{T}-\zeta_1)R_3x(\mathfrak{T}-\zeta_1) + \zeta_2x^T(\mathfrak{T})R_4x(\mathfrak{T}) \\ &\quad + y^T(\mathfrak{T})S_3y(\mathfrak{T}) - y^T(\mathfrak{T}-\delta_1)S_3y(\mathfrak{T}-\delta_1) + \delta_1\dot{y}^T(\mathfrak{T})S_4\dot{y}(\mathfrak{T}) \\ &\quad - \delta_1^{-1}(y(\mathfrak{T})-y(\mathfrak{T}-\delta_1))^T \\ &\quad \times S_4(y(\mathfrak{T})-y(\mathfrak{T}-\delta_1)) + f^T(y(\mathfrak{T}))T_1f(y(\mathfrak{T})) \\ &\quad - f^T(y(\mathfrak{T}-\delta_1))T_1f(y(\mathfrak{T}-\delta_1)) \\ &\quad + \delta_2f^T(y(\mathfrak{T}))T_2f(y(\mathfrak{T})) + \sum_{k=1}^N \varsigma_{\mathfrak{R}}(\mathfrak{T}) \left(-2\dot{y}^T(\mathfrak{T})X\dot{y}(\mathfrak{T}) - 2\dot{y}^T(\mathfrak{T})XD_{\mathfrak{R}}y(\mathfrak{T}-\rho_2) \right. \\ &\quad + 2\dot{y}^T(\mathfrak{T})XE_{\mathfrak{R}}x(\mathfrak{T}-\zeta_1) + (1-\epsilon)^{-1}\zeta_2^{-1}\dot{y}^T(\mathfrak{T})XF_{\mathfrak{R}}R_4^{-1}F_{\mathfrak{R}}^TX^T\dot{y}(\mathfrak{T}) \Big) \\ &\quad + \begin{bmatrix} y(\mathfrak{T}) \\ f(y(\mathfrak{T})) \end{bmatrix}^T \begin{bmatrix} -N\Omega_1 & N\Omega_2 \\ N\Omega_2 & -N \end{bmatrix} \begin{bmatrix} y(\mathfrak{T}) \\ f(y(\mathfrak{T})) \end{bmatrix} \\ &\quad + \begin{bmatrix} y(\mathfrak{T}-\delta_1) \\ f(y(\mathfrak{T}-\delta_1)) \end{bmatrix}^T \begin{bmatrix} -N\Omega_1 & N\Omega_2 \\ N\Omega_2 & -N \end{bmatrix} \begin{bmatrix} y(\mathfrak{T}-\delta_1) \\ f(y(\mathfrak{T}-\delta_1)) \end{bmatrix} \\ &\leq \sum_{k=1}^N \varsigma_{\mathfrak{R}}(\mathfrak{T}) \hat{\xi}^T(\mathfrak{T}) \Pi_k \hat{\xi}(\mathfrak{T}) < 0 \end{aligned} \quad (29)$$

where

$$\hat{\xi}(\mathfrak{T}) = (x(\mathfrak{T}), y(\mathfrak{T}), x(\mathfrak{T} - \rho_1), y(\mathfrak{T} - \rho_2), \dot{y}(\mathfrak{T}), x(\mathfrak{T} - \zeta_1), y(\mathfrak{T} - \delta_1), f(y(\mathfrak{T})), f(y(\mathfrak{T} - \delta_1)))^T.$$

Therefore we conclude that the switched SGRNs (4) is globally asymptotically stable in the sense of mean square. On the other hand, from (10) and Corollary 3.4 conditions, The proof is same to that in above theorem, we get

$$V(\mathfrak{T}_k, x(\mathfrak{T}_k), j) \leq V(\mathfrak{T}_k^-, x(\mathfrak{T}_k^-), i), k \in \mathbb{Z}_+.$$
 (30)

The system (4) with impulsive effect is asymptotically stable in the mean square. Hence the proof is concluded. $\square$

Corollary 3.5 *With the help of Assumptions 1 and 2, the switched SGRNs (4) with $N = 1$ is globally asymptotically stable in mean square if there exist a constant $\epsilon \in (0, 1)$, positive-definite matrices $R_1, R_2, R_3, R_4, S_1, S_2, S_3, S_4, T_1, T_2, X$ and positive definite diagonal matrix N such that the following LMIs hold:*

$$D_{ik}^T R_j D_{ik} - R_i < 0$$
 (31)

$$R_1 \leq \mu I$$
 (32)

$$\Psi = \begin{bmatrix} \Phi_{1,1} & 0 & -R_1 A & 0 & 0 & 0 & 0 & 0 & R_1 B \\ * & \Phi_{2,2} & 0 & -S_1 D & 0 & S_1 E & -\delta_1^{-1} S_4 & N \Omega_2 & 0 \\ * & * & \Phi_{3,3} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Phi_{4,4} & -X D & X E & 0 & 0 & 0 \\ * & * & * & * & \Phi_{5,5} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Phi_{6,6} & 0 & 0 & 0 \\ * & * & * & * & * & * & \Phi_{7,7} & 0 & N \Omega_2 \\ * & * & * & * & * & * & * & \Phi_{8,8} & 0 \\ * & * & * & * & * & * & * & * & \Phi_{9,9} \end{bmatrix},$$
 (33)

where

$$\begin{aligned} \Phi_{1,1} &= \rho_1 R_2 + R_3 + \zeta_2 R_4 + \delta_1^{-1} R_1 C T_2^{-1} C^T R_1^T; \\ \Phi_{2,2} &= \rho_2 S_2 + S_3 - \delta_1^{-1} S_4 - N \Omega_1 + \epsilon^{-1} \zeta_2^{-1} S_1 F R_4^{-1} F^T S_1^T; \\ \Phi_{3,3} &= \mu J_1 - \rho_1 R_2; \quad \Phi_{4,4} = \mu J_3 - \rho_2 S_2; \\ \Phi_{5,5} &= \delta_1 S_4 - 2X + (1 - \epsilon)^{-1} X F R_4^{-1} F^T X^T; \\ \Phi_{6,6} &= \mu J_2 - R_3; \quad \Phi_{7,7} = \mu J_4 - S_3 - \delta_1^{-1} S_4 - N \Omega_1; \\ \Phi_{8,8} &= T_1 + \delta_2 T_2 - N; \quad \Phi_{9,9} = -T_1 - N_1. \end{aligned}$$

Remark 3.3 The method enhanced in this paper is that the relationship between time-varying delays and their upper bounds are fully considered. In addition, some inequality techniques and Lyapunov–Krasovski functional, are used to obtain the maximum upper bounds for time-varying delays, see Example 4.1. Hence this approaches and methods used in our paper may reduces the conservative criterion with some existing works.

4 Numerical Examples

In this portion, an illustrative example to expose the capability and efficiency of our results.

Table 1 Comparisons of upper bounds of time-delays

Methods	Lower bounds	Upper bounds
In ref [26]	$\delta_m = \zeta_m = 0.3$	$\delta_M = \zeta_M = 9$
In ref [27]	$\delta_m = \zeta_m = 0.1$	$\delta_M = \zeta_M = 5$
In ref [18]	$\delta_m = \zeta_m = 1$	$\delta_M = \zeta_M = 10$
In Theorem 3.1	$\delta_m = \zeta_m = 1$	$\delta_M = \zeta_M > 0$

Example 4.1 Consider the switched SGRNs with impulsive effects as follows:

$$\begin{aligned}
 dx(t) &= \sum_{\mathfrak{K}=1}^N \varsigma_{\mathfrak{K}}(t) \left(-A_{\mathfrak{K}}x(t - \rho_1) + B_{\mathfrak{K}}f(y(t - \delta_1)) \right. \\
 &\quad \left. + C_{\mathfrak{K}}\delta_2^{-1} \int_{t-\delta_2}^t \hat{f}(y(\tau))d\tau \right) dt \\
 &\quad + \eta(t, x(t - \rho_1), x(t - \zeta_1), y(t - \rho_2), y(t - \delta_1))d\varpi(t) \\
 x(t_k) &= \mathfrak{G}x(t_k^-), k \in \mathbb{Z}^+, t = t_k \\
 dy(t) &= \sum_{\mathfrak{K}=1}^N \varsigma_{\mathfrak{K}}(t)(-D_{\mathfrak{K}}y(t - \rho_2) \\
 &\quad + E_{\mathfrak{K}}x(t - \zeta_1) + F_{\mathfrak{K}}\zeta_2^{-1} \int_{t-\zeta_2}^t x(\tau)d\tau)dt \\
 y(t_k) &= \mathfrak{H}y(t_k^-), k \in \mathbb{Z}^+, t = t_k,
 \end{aligned} \tag{34}$$

where

$$\begin{aligned}
 A_1 &= \begin{pmatrix} 0.3 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.5 & 0.2 \\ 0.2 & 0.8 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.9 \end{pmatrix} \\
 C_1 &= \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0.08 & 0.6 \\ 0 & 0.2 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0.36 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.489 \end{pmatrix} \\
 E_1 &= \begin{pmatrix} 0.6 & 0 \\ 0 & 0.6 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.6 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.9 \end{pmatrix} \\
 J_1 &= \begin{pmatrix} 0.36 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.8 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0.36 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.8 \end{pmatrix}
 \end{aligned}$$

and the leakage delays $\rho_1 = \rho_2 = 0.2$. The regulatory function is taken as $g(y) = \frac{y^2}{1+y^2}$. It can be easily checked that the derivative of $g(y)$ is less than 0.6. Assume that the feedback regulation delay $\delta_1 = 1, \delta_2 = 1$ and the translation delay $\zeta_1(t) = 2, \zeta_2(t) = 2$.

By Theorem 3.1, we can obtain the following feasible parameters. Due to the limitation of the length of this paper, we only provide a part of the feasible solutions here.

$$\begin{aligned}
 R_1 &= \begin{pmatrix} 0.0340 & 0.0030 \\ 0.0030 & 0.0019 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0.7328 & 0.0031 \\ 0.0031 & 0.8730 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0.1942 & 0.0000 \\ 0.0000 & 0.4591 \end{pmatrix} \\
 R_4 &= \begin{pmatrix} 0.5440 & 0.0003 \\ 0.0003 & 1.7873 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0.0050 & 0.0000 \\ 0.0000 & 0.0081 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0.2461 & 0.0186 \\ 0.0186 & 0.1769 \end{pmatrix} \\
 T_1 &= \begin{pmatrix} 0.1639 & 0.0001 \\ 0.0001 & 0.1249 \end{pmatrix}, \quad X = \begin{pmatrix} 0.1001 & 0.0020 \\ 0.0020 & 0.1615 \end{pmatrix}, \quad \mu = 0.7010.
 \end{aligned}$$

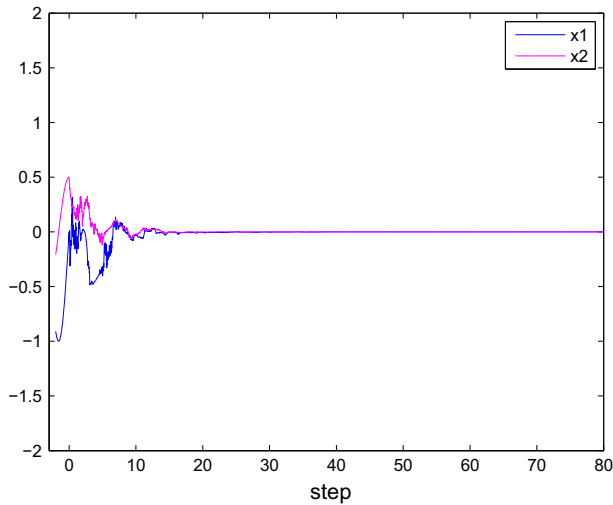


Fig. 1 mRNA concentrations with impulsive effects

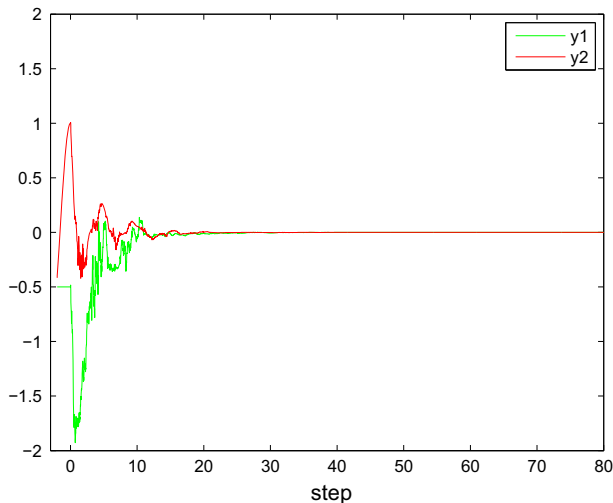


Fig. 2 mRNA concentrations without impulsive effects

Therefore, we know from Theorem 3.1, the GRNs (34) with distributed delays are asymptotically stable in the mean-square. The initial condition is $[-1.0 \ -0.1]^T$ for mRNAs with impulsive effects, at the same time, the initial condition is $[-1.0 \ -0.2]^T$ for mRNAs without impulsive effects. Similarly, the initial condition is $[-0.5 \ -0.4]^T$ for proteins with impulsive effects and the initial condition is $[-0.5 \ -0.3]^T$ for proteins without impulsive effects are given. The simulation results of the mRNA and protein concentrations are shown in Figs. 2 and 4, respectively. Furthermore, the impulsive effects of mRNA and protein concentrations are shown in Figs. 1 and 3, respectively.

Example 4.2 The leakage delay plays an important role in dynamic behaviors of GRNs such as in the process of transcription, translation and translocation. It is a fascinating attribute of

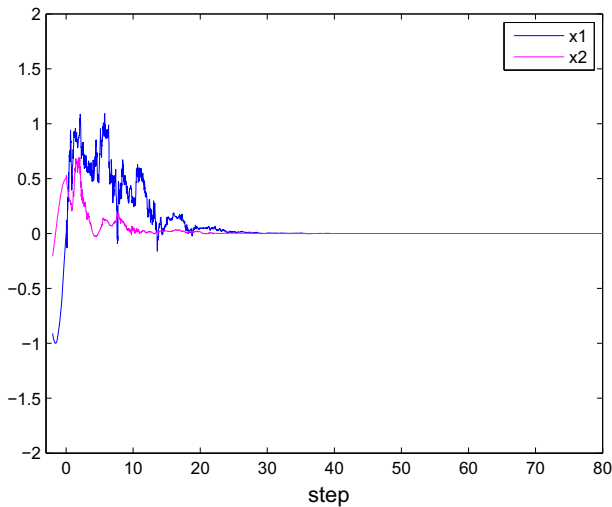


Fig. 3 Protein concentrations with impulsive effects

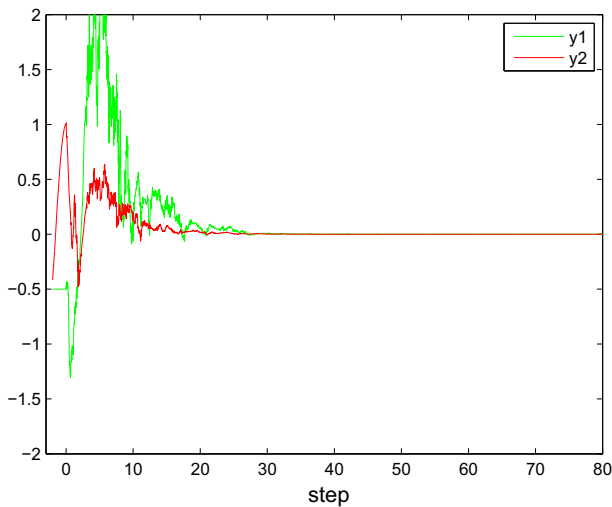


Fig. 4 Protein concentrations without impulsive effects

signal transmission and creating oscillation of the genetic network systems and poor performance for network stability. In [25], the authors considered the switched genetic regulatory networks with time-varying delays for stability. However, our main aim in this paper is to stabilize the switched genetic regulatory networks with the existence of leakage delays, multiple delays, noise and impulses.

Consider the switched genetic regulatory networks (4) with the following parameters:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0.2 \\ 0.2 & 0.1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.9 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0.5 & 0.6 \\ 0 & 0.2 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.9 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.6 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.9 \end{pmatrix}$$

$$J_1 = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0.1 & 0 \\ 0 & 2 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}.$$

The regulatory function is taken as $g(y) = \frac{y^2}{1+y^2}$. It can be easily checked that the derivative of $g(y)$ is less than 0.6. Assume that the feedback regulation delay $\delta_1 = 2$, $\delta_2 = 2$ and the translation delay $\zeta_1(t) = 1$, $\zeta_2(t) = 1$.

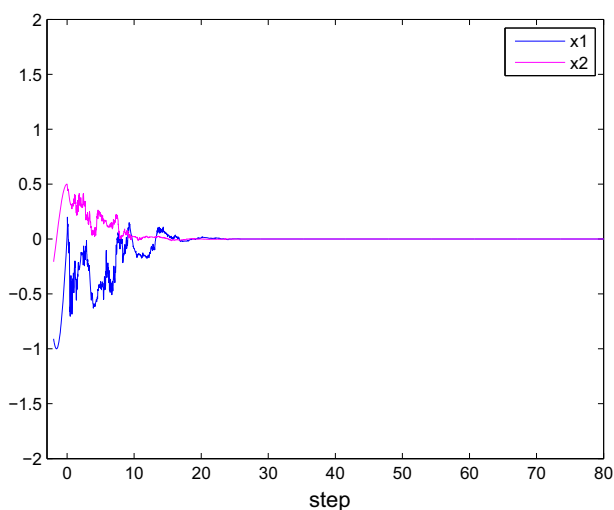


Fig. 5 mRNA concentrations without leakage effects ($\rho_1 = \rho_2 = 0$)

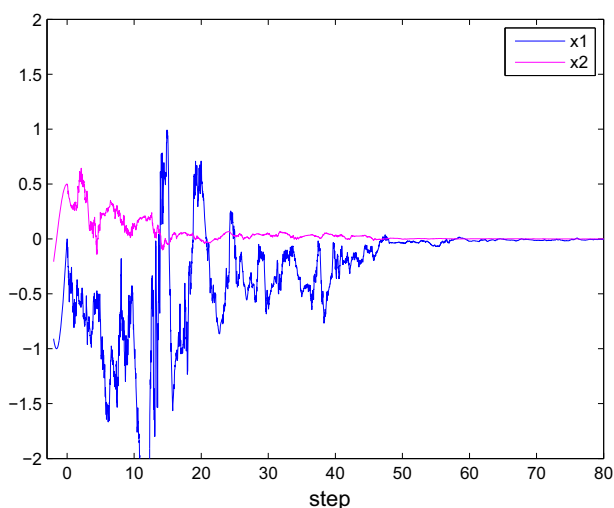


Fig. 6 mRNA concentrations with leakage effects ($\rho_1 = \rho_2 = 0.7$)

When $\rho_1 = \rho_2 = 0$, in other words, there is no leakage delay, the system (4) is stable (see, Fig. 5). However, when $\rho_1 = \rho_2 = 0.7$, the system (4) becomes oscillates then finally stable (see, Fig. 6). The following Figures shows how the leakage delays affects the stability of the system.

The initial conditions for mRNAs with and without leakage effects are $[-1.0 \ -0.19]^T$ $[-1.1 \ -0.2]^T$, respectively. Similarly, the initial condition for proteins with and without leakage effects are $[-0.5 \ -0.45]^T$ and $[-0.51 \ -0.4]^T$, respectively. Furthermore, the simulation results of the mRNA and protein concentrations with leakage effects are shown in Figs. 5, 6, 7 and 8 respectively.

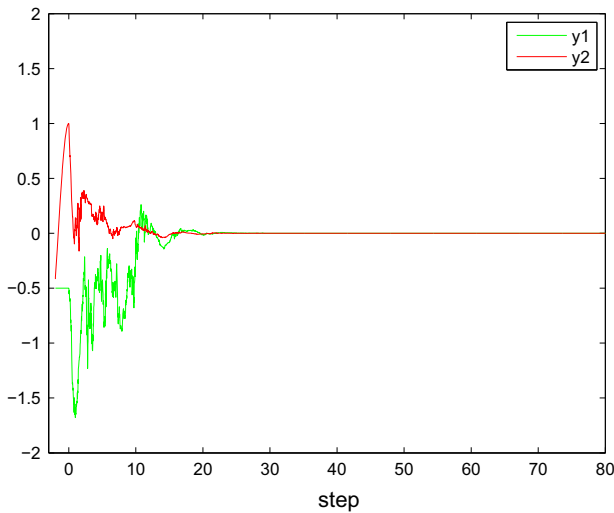


Fig. 7 Protein concentrations without leakage effects ($\rho_1 = \rho_2 = 0$)

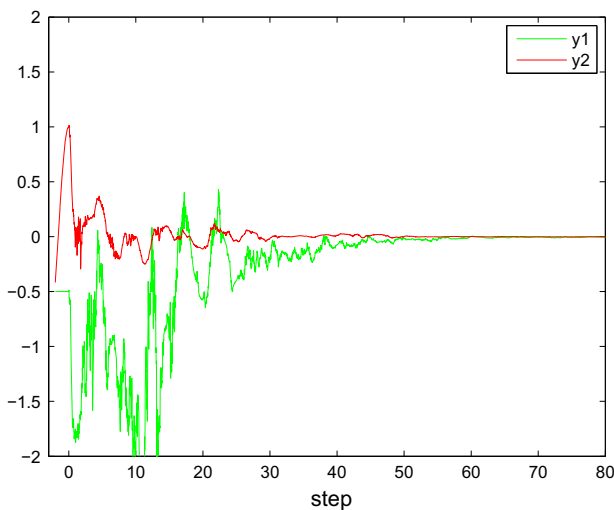


Fig. 8 Protein concentrations with leakage effects ($\rho_1 = \rho_2 = 0.7$)

5 Conclusions

In this paper, we have studied the stabilization of switched stochastic genetic regulatory networks with leakage and impulsive effects. Here the proposed work is proved by constructing a Lyapunov–Krasvoskii functional (LKF) as well as some analysis method, several sufficient conditions are derived in terms of linear matrix inequalities (LMIs) and it can be solved via LMI Control tool box. Finally, two numerical examples are given to expo the capability of our proposed results.

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Exponential dissipativity criteria for generalized BAM neural networks with variable delays

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Abstract This article evaluates the exponential dissipativity and passivity criteria for generalized bidirectional associative memory neural networks (BAMGNNs) including interval time-varying delayed signals. Exponential dissipativity and passivity criteria are proposed by making suitable Lyapunov–Krasovskii functional and proposing a novel approach. The improved reciprocally convex combination and weighted integral inequality techniques are utilized to obtain new exponential dissipativity and passivity conditions of such delayed BAMGNNs. The feasibility of the obtained results is clearly demonstrated by numerical examples.

Keywords Generalized BAM neural network · Exponential stability · Passivity and dissipativity analysis · Time-varying delay · Weighted integral inequality

1 Introduction

Artificial neural networks (NNs) have been successfully studied in the past 2 decades, since they have been applied and extended to numerous engineering systems, such as image processing, communication, fault diagnosis, parallel computations, fixed-point computations, and industrial automation [1]. During the implementation and application of NNs, time-delayed signals are unavoidable because of the inherent finite signal transfer time between the neurons. The existence of time-delayed signals in systems may cause divergence, instability, and the gradual degradation of the system performance. Thus, time-delayed NNs have received a great deal of research attention [2–10]. In previous years, static NNs and local field NNs have been studied separately. To avoid such doubly work, Zhang et al. [11] proposed a new combined NN model named as generalized neural network (GNN), which brings both static and local field NNs together. Recently, a number of stability problems for such GNNs have been investigated in [12–15].

Recently, the dissipativity problem has gained extensive research attention, since it is an important subject for physical systems, such as the design of group coordination, hybrid systems, nonlinear time-delay systems, and Internet-based control [16, 17]. Passivity, and its generalization dissipativity, characterizes the energy consumption of a system and is used in a variety of applications (e.g., electrical, mechanical, chemical, and communication systems). In most cases, passivity can be used to demonstrate that passive systems will be stable under specific criteria [18, 19]. The dissipativity criteria are a more general case of passivity and stability analyses. Thus, it is essential to consider dissipativity problems for NNs.

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Recently, asymptotic and exponential stability criteria have been studied for NNs [20]. As stated in [21], the exponential convergence rate can be utilized in ascertaining the speed of neural calculations while studying the exponential stability criteria. Hence, analyzing the exponential stability has practical significance, and it is useful to estimate and ensure the exponential convergence rate of NNs [22–25]. We know that the bidirectional associative memory NNs (BAMNNs) were introduced by Kosko [26], in which the two-layer pattern-matched hetero-associative circuit has been generalized from a single auto-associative Hebbian correlator. In current years, BAMNNs have got great research interest [27–32]. In particular, finite-time Mittag–Leffler synchronization problem for fractional-order delayed memristive BAMNNs has been investigated in [27]. The stability of BAMNNs with leakage delays and probabilistic delays was considered in [29]. Exponential passivity criteria for BAMNNs were studied in [30]. In [31], passivity analysis of Markovian jump fuzzy BAMNNs was investigated. Sakthivel et al. [32] studied the problem of robust passivity for delayed fuzzy Cohen–Grossberg BAMNNs. In [28], passivity analysis of memristor-based stochastic neutral-type BAMNNs was established. Nevertheless, the exponential dissipativity and passivity problem for delayed BAMGNNs has not yet been completely considered in the literature. Hence, our motivation is to study the exponential dissipativity and passivity of BAMGNNs with interval time-varying delays.

As discussed in previous paragraphs, the exponential dissipativity and passivity analysis of delayed BAMGNNs is studied in this work. A suitable Lyapunov–Krasovskii functional (LKF) is established, and then by using improved RCC and WII techniques desired linear matrix inequalities (LMIs) are constructed. A novel approach is utilized to obtain sufficient conditions of the exponential dissipativity and passivity of BAMGNNs. At last, some examples are provided to show the superiority of our results.

Notations: $\mathbb{R}^n$ denotes the n -dimensional Euclidean space, and $\mathbb{R}^{n \times n}$ denotes the set of all $n \times n$ real matrices. $*$ denotes the term that is induced by symmetry. X^T Denotes the matrix transpose of X . $Y > 0$ denotes that Y is a real symmetric positive definite matrix. I denotes the identity matrix with appropriate dimensions, and the Kronecker product is denoted by $\otimes$.

2 Preliminaries

Consider the delayed BAMGNNs:

$$\left. \begin{aligned} \dot{m}(t) &= -Cm(t) + U_0g(U_2h(t)) + U_1g(U_2h(t - \chi(t)) + u(t), \\ \dot{z}(t) &= g(U_2h(t)), \\ \dot{h}(t) &= -Dh(t) + V_0f(V_2m(t)) + V_1f(V_2m(t - \vartheta(t)) + v(t), \\ \dot{\eta}(t) &= f(V_2m(t)), \end{aligned} \right\} \quad (1)$$

where $m(t) = [m_1(t), m_2(t), \dots, m_n(t)]^T \in \mathbb{R}^n$ and $h(t) = [h_1(t), h_2(t), \dots, h_n(t)]^T \in \mathbb{R}^n$ are the neuron state vectors. $u(t), v(t) \in \mathbb{R}^n$ are the inputs belonging to $L_2[0, \infty)$, and $z(t), \eta(t)$ are the outputs of the NN. $g(U_2h(t)) = [g(U_2h_1(t)), \dots, g(U_2h_n(t))]^T \in \mathbb{R}^n$ and $f(V_2m(t)) = [f(V_2m_1(t)), \dots, f(V_2m_n(t))]^T \in \mathbb{R}^n$ represent the neuron activation function vectors. $C = \text{diag}\{c_1, \dots, c_n\}$ and $D = \text{diag}\{d_1, \dots, d_n\}$ are positive diagonal matrices with $c_j > 0$ and $d_j > 0, j = 1, 2, \dots, n$, respectively. Matrices U_0, U_1, U_2, V_0, V_1 , and V_2 are the connection weighting matrices. $\chi(t), \vartheta(t)$ are time-varying delayed signals and satisfy

$$0 < \chi_1 \leq \chi(t) \leq \chi_2, \quad \dot{\chi}(t) \leq \ell_1, \quad (2)$$

$$0 < \vartheta_1 \leq \vartheta(t) \leq \vartheta_2, \quad \dot{\vartheta}(t) \leq \ell_2, \quad (3)$$

where $\chi_1, \chi_2, \vartheta_1, \vartheta_2, \ell_1$, and ℓ_2 are positive constants.

Assumption (A1) [33] $g_j(t), f_j(t)$ ($j = 1, 2, \dots, n$) are continuous and bounded functions and satisfy:

$$\widehat{z}_j^- \leq \frac{g_j(U_{2j}a) - g_j(U_{2j}b)}{U_{2j}a - U_{2j}b} \leq \widehat{z}_j^+, \quad (4)$$

$$\overline{z}_j^- \leq \frac{f_j(V_{2j}a) - f_j(V_{2j}b)}{V_{2j}a - V_{2j}b} \leq \overline{z}_j^+, \quad (5)$$

where $g_j(0) = 0, f_j(0) = 0, Z_1 = \text{diag}(\widehat{z}_1^-, \widehat{z}_2^-, \dots, \widehat{z}_n^-), Z_2 = \text{diag}(\widehat{z}_1^+, \widehat{z}_2^+, \dots, \widehat{z}_n^+), Z_3 = \text{diag}(\overline{z}_1^-, \overline{z}_2^-, \dots, \overline{z}_n^-)$, and $Z_4 = \text{diag}(\overline{z}_1^+, \overline{z}_2^+, \dots, \overline{z}_n^+)$ are some constants and $a, b \in \mathbb{R}, a \neq b$.

Lemma 2.1 [34] Given scalar $\alpha > 0$, matrix $R > 0$ with $n \times n$ dimensions, and function $\phi \in C([a, b], \mathbb{R}^n)$, satisfies the following relation:

$$\int_a^b e^{\alpha(s-b)} \phi^T(s) R \phi(s) ds \geq \frac{\alpha}{\bar{\gamma}_0} \left(\int_a^b \phi(s) ds \right)^T R \left(\int_a^b \phi(s) ds \right) + \frac{\alpha}{\bar{\rho}_0} \zeta^T (K_1^T K_1 \otimes R) \zeta$$

where

$$\bar{\rho}_0 = \frac{\bar{\gamma}_0}{\bar{\gamma}_1^2} [\bar{\gamma}_0^2 - (\alpha l)^2 e^{\alpha l}], \bar{\gamma}_0 = e^{\alpha l} - 1, \bar{\gamma}_1 = \bar{\gamma}_0 - \alpha l, l = b - a,$$

$$\zeta = \left[\left(\int_a^b \phi(s) ds \right)^T \left(\int_a^b \int_s^b \phi(u) du ds \right)^T \right]^T, K_1 = [1 \quad -\alpha \bar{\gamma}_0 / \bar{\gamma}_1].$$

Based on the exponential function for Taylor series, the following facts can be found:

$$\lim_{\alpha \rightarrow 0^+} \frac{\alpha}{\bar{\rho}_0} = \frac{3}{l}, \lim_{\alpha \rightarrow 0^+} K_1 = \bar{K}_1 = \begin{bmatrix} 1 & -\frac{2}{l} \end{bmatrix}.$$

Hence, we can obtain the following results when α tends to zero

$$\int_a^b \phi^T(s) R \phi(s) ds \geq \frac{1}{l} \zeta^T (K_0^T K_0 + 3 \bar{K}_1^T \bar{K}_1) \otimes R \zeta, \quad (6)$$

where $K_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Lemma 2.2 (Improved RCC [35]) *Given matrices $R_1 \in \mathbb{R}_{>0}^{n \times n}$, $R_2 \in \mathbb{R}_{>0}^{r \times r}$, if there exists an appropriately dimensioned matrix $X \in \mathbb{R}^{n \times r}$ satisfying $\begin{bmatrix} R_1 & X \\ * & R_2 \end{bmatrix} \geq 0$ such that the following inequality*

$$\begin{bmatrix} \frac{1}{\delta} R_1 & 0 \\ * & \frac{1}{1-\delta} R_2 \end{bmatrix} \geq \begin{bmatrix} R_1 & X \\ * & R_2 \end{bmatrix}$$

holds for all $\delta \in (0, 1)$.

Definition 2.3 [30] The BAMGNN (1) is said to be exponentially passive, when there exists an exponential LKF $V(m_t, h_t)$, and a constant $\alpha > 0$ such that,

$$\dot{V}(m_t, h_t) + \alpha V(m_t, h_t) \leq 2(\beta^T(t)u(t) + \eta^T(t)v(t)), \quad t \geq t_0,$$

holds for all $u(t)$ and $v(t)$, and all initial conditions $m(t_0)$ and $h(t_0)$.

Definition 2.4 For given real known matrices $\mathbb{Q} = \mathbb{Q}^T$, $\mathbb{S}$, and $\mathbb{R} = \mathbb{R}^T$, the BAMGNN (1) is said to be exponential $(\mathbb{Q}, \mathbb{S}, \mathbb{R}) - \gamma$ -dissipative from input, if there exists a storage function (exponential Lyapunov function) $V(m_t, h_t)$, and a scalar $\gamma > 0$ such that for all $u(t)$ and $v(t)$, all the initial conditions $m(t_0)$ and $h(t_0)$, the following relation holds:

$$\dot{V}(m_t, h_t) + \alpha V(m_t, h_t) \leq J(t), \quad t \geq t_0, \quad (7)$$

where the quadratic supply rate $J(t)$ represents

$$J(t) = \begin{bmatrix} \beta(s) \\ \eta(s) \end{bmatrix}^T \mathbb{Q} \begin{bmatrix} \beta(s) \\ \eta(s) \end{bmatrix} + 2 \begin{bmatrix} \beta(s) \\ \eta(s) \end{bmatrix}^T \mathbb{S} \begin{bmatrix} u(s) \\ v(s) \end{bmatrix} + \begin{bmatrix} u(s) \\ v(s) \end{bmatrix}^T (\mathbb{R} - \gamma I) \begin{bmatrix} u(s) \\ v(s) \end{bmatrix},$$

and $\dot{V}(m_t, h_t)$ denotes the derivative of $V(m_t, h_t)$ along the trajectories $m(t)$ and $h(t)$ of BAMGNN (1).

3 Main results

We introduce the following definitions for simplicity:

$$\begin{aligned} \eta_1(t) &\triangleq \begin{bmatrix} m^T(t) & \int_{t-\chi_1}^t m^T(s) ds & \int_{t-\chi_2}^{t-\chi_1} m^T(s) ds \end{bmatrix}^T, \\ \eta_2(t) &\triangleq \begin{bmatrix} h^T(t) & \int_{t-\vartheta_1}^t h^T(s) ds & \int_{t-\vartheta_2}^{t-\vartheta_1} h^T(s) ds \end{bmatrix}^T, \\ \xi_1(t) &\triangleq \begin{bmatrix} m^T(t) & m^T(t-\chi_1) & m^T(t-\chi(t)) & m^T(t-\chi_2) \\ g^T(U_2 h(t)) & g^T(U_2 h(t-\chi(t))) & \frac{1}{\chi_1} \int_{t-\chi_1}^t m^T(s) ds & \frac{1}{\chi(t)-\chi_1} \int_{t-\chi(t)}^{t-\chi_1} m^T(s) ds \\ \frac{1}{\chi_2-\chi(t)} \int_{t-\chi_2}^{t-\chi(t)} m^T(s) ds & u^T(t) \end{bmatrix}^T, \\ \xi_2(t) &\triangleq \begin{bmatrix} h^T(t) & h^T(t-\vartheta_1) & h^T(t-\vartheta(t)) & h^T(t-\vartheta_2) \\ f^T(V_2 m(t)) & f^T(V_2 m(t-\vartheta(t))) & \frac{1}{\vartheta_1} \int_{t-\vartheta_1}^t h^T(s) ds & \frac{1}{\vartheta(t)-\vartheta_1} \int_{t-\vartheta(t)}^{t-\vartheta_1} h^T(s) ds \\ \frac{1}{\vartheta_2-\vartheta(t)} \int_{t-\vartheta_2}^{t-\vartheta(t)} h^T(s) ds & v^T(t) \end{bmatrix}^T, \\ e_r &\triangleq \begin{bmatrix} 0_{n \times (r-1)n} & I_n & 0_{n \times (10-r)n} \end{bmatrix}, \\ w_r &\triangleq \begin{bmatrix} 0_{n \times (r-1)n} & I_n & 0_{n \times (10-r)n} \end{bmatrix}, \quad r = 1, 2, \dots, 10. \end{aligned}$$

Based on the above definitions, we present the main result of this paper in the next theorem.

Theorem 3.1 *Given scalars $0 < \chi_1 < \chi_2$, $0 < \vartheta_1 < \vartheta_2$, $\ell_1 > 0$, $\ell_2 > 0$, BAMGNN (1) is exponentially dissipative, when there exist positive definite matrices $P_1, P_2 \in \mathbb{R}^{3n \times 3n}$, $Q_1, Q_2, Q_3, R, S, Q_4, Q_5, Q_6, R, S \in \mathbb{R}^{n \times n}$, $\Gamma_p = \text{diag}\{\gamma_{p1}, \gamma_{p2}, \dots, \gamma_{pn}\} \in \mathbb{R}^{n \times n}$, ($p = 1, 2, 3, 4, 5, 6$), $\Lambda_q = \text{diag}\{\Lambda_{q1}, \Lambda_{q2}, \dots, \Lambda_{qn}\} \in \mathbb{R}^{n \times n}$ ($q = 1, 2, 3, 4$) and any matrices $X, Y \in \mathbb{R}^{2n \times 2n}$ satisfying*

$$\Sigma(\chi) < 0, \quad (8)$$

$$\Upsilon(\vartheta) < 0,$$

$$\Pi_1 = \begin{bmatrix} \tilde{S} & X \\ 0 & \tilde{S} \end{bmatrix} \geq 0, \quad (9)$$

$$\Pi_2 = \begin{bmatrix} \tilde{S} & Y \\ 0 & \tilde{S} \end{bmatrix} \geq 0,$$

where

$$\begin{aligned}
 \Sigma(\chi) &= \Sigma_0(\chi) + \sum_{z=1}^6 \Sigma_z - e^{-\alpha\chi_{21}} \Psi^T \Pi_1 \Psi, \quad \Sigma_0(\chi) \\
 &= \Psi^T(\chi) P_1 \Psi_0 + \Psi_0^T P_1 \Psi(\chi) + \alpha \Psi^T(\chi) P_1 \Psi(\chi), \\
 \Psi(\chi) &= \begin{bmatrix} e_1 \\ \chi_1 e_7 \\ (\chi(t) - \chi_1) e_8 + (\chi_2 - \chi(t)) e_9 \end{bmatrix}, \quad \Psi_0 = \begin{bmatrix} \hat{C} \\ e_1 - e_2 \\ e_2 - e_4 \end{bmatrix}, \\
 \Psi_1 &= \begin{bmatrix} e_1 - e_2 \\ \chi_1 (e_1 - e_7) \end{bmatrix}, \quad \Psi_2 = \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_8 \end{bmatrix}, \quad \Psi_3 = \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_9 \end{bmatrix}, \\
 \Sigma_1 &= [2[e_5 - Z_1 U_2 e_1]^T \Lambda_1 \hat{C} + 2[Z_2 U_2 e_1 - e_5]^T \Lambda_2 \hat{C}], \\
 \Sigma_2 &= e_1^T Q_1 e_1 - e^{-\alpha\chi_1} e_2^T Q_1 e_2 + e^{-\alpha\chi_1} e_2^T Q_2 e_2 - e^{-\alpha\chi_1} e_4^T Q_2 e_4 \\
 &\quad + e_5^T Q_3 e_5 + \pi_1(\ell_1) e_6^T Q_3 e_6, \quad \Sigma_3 = \hat{C}^T [\chi_1^2 R + \chi_{21}^2 e^{\alpha\chi_{21}} S] \hat{C}, \\
 \Sigma_4 &= - \left(\frac{\alpha\chi_1}{\gamma_0} (e_1 - e_2)^T R (e_1 - e_2) + \frac{\alpha\chi_1}{\rho_0} \Psi_1^T (K_1^T K_1 \otimes R) \Psi_1 \right), \\
 \Sigma_5 &= - 2e_1^T U_2^T Z_1 \Gamma_1 Z_2 U_2 e_1 + e_1^T U_2^T (Z_1 + Z_2) \Gamma_1 e_5 + e_5^T \Gamma_1 (Z_1 + Z_2) U_2 e_1 \\
 &\quad - 2e_5^T \Gamma_1 e_5 \\
 &\quad - 2e_2^T U_2^T Z_1 \Gamma_2 Z_2 U_2 e_2 + e_2^T U_2^T (Z_1 + Z_2) \Gamma_2 e_6 + e_6^T \Gamma_2 (Z_1 + Z_2) U_2 e_2 \\
 &\quad - 2e_6^T \Gamma_2 e_6 \\
 &\quad - 2[e_5 - e_6 - Z_1 U_2 (e_1 - e_2)]^T \Gamma_3 [e_5 - e_6 - Z_2 U_2 (e_1 - e_2)], \\
 \Sigma_6 &= - e_5^T Q_4 e_5 - 2e_5^T S e_{10} - e_{10}^T (R - \gamma I) e_{10}, \\
 \rho_0 &= \frac{\gamma_0}{\gamma_1^2} [\gamma_0^2 - (\alpha\chi_1)^2 e^{\alpha\chi_1}], \quad \gamma_0 = e^{\alpha\chi_1} - 1, \quad \gamma_1 = \gamma_0 - \alpha\chi_1, \\
 \gamma_1^2 K_1 &= \begin{bmatrix} 1 & -\alpha\gamma_0/\gamma_1 \end{bmatrix}, \\
 \Psi &= [\Psi_2^T \quad \Psi_3^T]^T, \quad \tilde{S} = \text{diag}\{S, 3S\}, \quad \hat{C} = -Ce_1 + U_0 e_5 + U_1 e_6 + e_{10}, \\
 \chi_{21} &= \chi_2 - \chi_1, \\
 \Upsilon(\vartheta) &= \Upsilon_0(\vartheta) + \sum_{z=1}^6 \Upsilon_z - e^{-\alpha\vartheta_{21}} \Phi^T \Pi_2 \Phi, \quad \Upsilon_0(\vartheta) = \Phi^T(\vartheta) P_2 \Phi_0 + \Phi_0^T P_2 \Phi(\vartheta) \\
 &\quad + \alpha \Phi^T(\vartheta) P_2 \Phi(\vartheta), \\
 \Phi(\vartheta) &= \begin{bmatrix} w_1 \\ \vartheta_1 w_7 \\ (\vartheta(t) - \vartheta_1) w_8 + (\vartheta_2 - \vartheta(t)) w_9 \end{bmatrix}, \quad \Phi_0 = \begin{bmatrix} \tilde{D} \\ w_1 - w_2 \\ w_2 - w_4 \end{bmatrix}, \\
 \Phi_1 &= \begin{bmatrix} w_1 - w_2 \\ \vartheta_1 (w_1 - w_7) \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} w_2 - w_3 \\ w_2 + w_3 - 2w_8 \end{bmatrix}, \quad \Phi_3 = \begin{bmatrix} w_3 - w_4 \\ w_3 + w_4 - 2w_9 \end{bmatrix}, \\
 \Upsilon_1 &= [2[w_5 - Z_3 V_2 w_1]^T \Lambda_3 \tilde{D} + 2[Z_4 V_2 w_1 - w_5]^T \Lambda_3 \tilde{D}], \\
 \Upsilon_2 &= w_1^T Q_1 w_1 - e^{-\alpha\vartheta_1} w_2^T Q_1 w_2 + e^{-\alpha\vartheta_1} w_2^T Q_2 w_2 - e^{-\alpha\vartheta_1} w_4^T Q_2 w_4 \\
 &\quad + w_5^T Q_3 w_5 + \pi_2(\ell_2) w_6^T Q_3 w_6, \quad \Upsilon_3 = \tilde{D}^T [\vartheta_1^2 R + \vartheta_{21}^2 e^{\alpha\vartheta_{21}} S] \tilde{D}, \\
 \Upsilon_4 &= - \left(\frac{\alpha\vartheta_1}{\delta_0} (w_1 - w_2)^T R (w_1 - w_2) + \frac{\alpha\vartheta_1}{\varrho_0} \Phi_1^T (K_2^T K_2 \otimes R) \Phi_1 \right), \\
 \Upsilon_5 &= - 2w_1^T V_2^T Z_3 \Gamma_4 Z_4 V_2 w_1 + w_1^T V_2^T (Z_3 + Z_4) \Gamma_4 w_5 + w_5^T \Gamma_4 (Z_3 + Z_4) V_2 w_1 \\
 &\quad - 2w_5^T \Gamma_4 w_5 \\
 &\quad - 2w_2^T V_2^T Z_3 \Gamma_5 Z_4 V_2 w_2 + w_2^T V_2^T (Z_3 + Z_4) \Gamma_5 w_6 + w_6^T \Gamma_5 (Z_3 + Z_4) V_2 w_2 \\
 &\quad - 2w_6^T \Gamma_5 w_6 \\
 &\quad - 2[w_5 - w_6 - Z_3 V_2 (w_1 - w_2)]^T \Gamma_6 [w_5 - w_6 - Z_4 V_2 (w_1 - w_2)], \\
 \Upsilon_6 &= - w_5^T Q_4 w_5 - 2w_5^T S w_{10} - w_{10}^T (R - \gamma I) w_{10}, \\
 \varrho_0 &= \frac{\delta_0}{\delta_1^2} [\delta_0^2 - (\alpha\vartheta_1)^2 e^{\alpha\vartheta_1}], \quad \delta_0 = e^{\alpha\vartheta_1} - 1, \quad \delta_1 = \delta_0 - \alpha\vartheta_1, \\
 \delta_1^2 K_2 &= \begin{bmatrix} 1 & -\alpha\delta_0/\delta_1 \end{bmatrix}, \\
 \Phi &= [\Phi_2^T \quad \Phi_3^T]^T, \quad \tilde{S} = \text{diag}\{S, 3S\}, \quad \tilde{D} \\
 &= -Dw_1 + V_0 w_5 + V_1 w_6 + w_{10}, \quad \vartheta_{21} = \vartheta_2 - \vartheta_1.
 \end{aligned}$$

Proof Choose the following LKF candidate:

$$\begin{aligned}
 V(m_t, h_t) &= \eta_1^T(t) P_1 \eta_1(t) + 2 \sum_{j=1}^n \lambda_{1j} \int_0^{U_{2j} h(t)} (g_j(s) - \hat{z}_j^- s) ds \\
 &\quad + 2 \sum_{j=1}^n \lambda_{2j} \int_0^{U_{2j} h(t)} (\hat{z}_j^+ s - g_j(s)) ds, \\
 &\quad + \eta_2^T(t) P_2 \eta_2(t) + 2 \sum_{j=1}^n \lambda_{3j} \int_0^{V_{2j} m(t)} (f_j(s) - \hat{z}_j^- s) ds \\
 &\quad + 2 \sum_{j=1}^n \lambda_{4j} \int_0^{V_{2j} m(t)} (\hat{z}_j^+ s - f_j(s)) ds, \\
 &\quad + \int_{t-\chi_1}^t e^{\alpha(s-t)} m^T(s) Q_1 m(s) ds + \int_{t-\chi_2}^{t-\chi_1} e^{\alpha(s-t)} m^T(s) Q_2 m(s) ds \\
 &\quad + \int_{t-\chi(t)}^t e^{\alpha(s-t)} g^T(U_2 h(s)) Q_3 g(U_2 h(s)) ds \\
 &\quad + \chi_1 \int_{-\chi_1}^0 \int_{t+s}^t e^{\alpha(\theta-t)} \dot{m}^T(\theta) R \dot{m}(\theta) d\theta ds \\
 &\quad + \chi_{21} \int_{-\chi_2}^{-\chi_1} \int_{t+s}^t e^{\alpha(\theta+\chi_1-t)} \dot{m}^T(\theta) S \dot{m}(\theta) d\theta ds \\
 &\quad + \int_{t-\vartheta_1}^t e^{\alpha(s-t)} h^T(s) Q_1 h(s) ds + \int_{t-\vartheta_2}^{t-\vartheta_1} e^{\alpha(s-t)} h^T(s) Q_2 h(s) ds \\
 &\quad + \int_{t-\vartheta(t)}^t e^{\alpha(s-t)} f^T(V_2 m(s)) Q_3 f(V_2 m(s)) ds \\
 &\quad + \vartheta_1 \int_{-\vartheta_1}^0 \int_{t+s}^t e^{\alpha(\theta-t)} \dot{h}^T(\theta) R \dot{h}(\theta) d\theta ds \\
 &\quad + \vartheta_{21} \int_{-\vartheta_2}^{-\vartheta_1} \int_{t+s}^t e^{\alpha(\theta+\vartheta_1-t)} \dot{h}^T(\theta) S \dot{h}(\theta) d\theta ds.
 \end{aligned} \tag{10}$$

The time derivative of $V(m_t, h_t)$ is as follows:

$$\begin{aligned}
 \dot{V}(m_t, h_t) &+ \alpha V(m_t, h_t) \leq \xi_1^T(t) \Sigma_0(\chi) \xi_1(t) + \xi_1^T \\
 &\quad [2[e_5 - Z_1 U_2 e_1]^T \Lambda_1 \hat{C} + 2[Z_2 U_2 e_1 - e_5]^T \Lambda_2 \hat{C}] \xi_1(t) \\
 &\quad + \xi_1^T [e_1^T Q_1 e_1 - e^{-\alpha\chi_1} e_2^T Q_1 e_2 + e^{-\alpha\chi_1} e_2^T Q_2 e_2 \\
 &\quad - e^{-\alpha\chi_1} e_4^T Q_2 e_4 + e_5^T Q_3 e_5 + \pi_1(\ell_1) e_6^T Q_3 e_6] \xi_1(t) \\
 &\quad + \xi_1^T(t) \hat{C}^T [\chi_1^2 R + \chi_{21}^2 e^{\alpha\chi_{21}} S] \hat{C} \xi_1(t) \\
 &\quad - \chi_1 \int_{t-\chi_1}^t e^{\alpha(s-t)} \dot{m}^T(s) R \dot{m}(s) ds \\
 &\quad - \chi_{21} \int_{t-\chi_2}^{t-\chi_1} e^{\alpha(s+\chi_1-t)} \dot{m}^T(s) S \dot{m}(s) ds \\
 &\quad - \chi_{21} \int_{t-\chi(t)}^{t-\chi_1} e^{\alpha(s+\chi_1-t)} \dot{m}^T(s) S \dot{m}(s) ds \\
 &\quad + \xi_2^T(t) \Upsilon_0(\vartheta) \xi_2(t) + \xi_2^T [2[w_5 - Z_3 V_2 w_1]^T \Lambda_3 \tilde{D} \\
 &\quad + 2[Z_4 V_2 w_1 - w_5]^T \Lambda_4 \tilde{D}] \xi_2(t) + \xi_2^T [w_1^T Q_1 w_1 \\
 &\quad - e^{-\alpha\vartheta_1} w_2^T Q_1 w_2 + e^{-\alpha\vartheta_1} w_2^T Q_2 w_2 - e^{-\alpha\vartheta_1} w_4^T Q_2 w_4 \\
 &\quad + w_5^T Q_3 w_5 + \pi_2(\ell_2) w_6^T Q_3 w_6] \xi_2(t) + \xi_2^T(t) \tilde{D}^T \\
 &\quad [\vartheta_1^2 R + \vartheta_{21}^2 e^{\alpha\vartheta_{21}} S] \tilde{D} \xi_2(t) - \vartheta_1 \int_{t-\vartheta_1}^t e^{\alpha(s-t)} \dot{h}^T(s) R \dot{h}(s) ds \\
 &\quad - \vartheta_{21} \int_{t-\vartheta_2}^{t-\vartheta_1} e^{\alpha(s+\vartheta_1-t)} \dot{h}^T(s) S \dot{h}(s) ds - \vartheta_{21} \\
 &\quad \int_{t-\vartheta(t)}^{t-\vartheta_1} e^{\alpha(s+\vartheta_1-t)} \dot{h}^T(s) S \dot{h}(s) ds.
 \end{aligned} \tag{11}$$

where

$$\pi_1(\ell_1) = \begin{cases} -(1 - \ell_1)e^{-\alpha\chi_2}, & \ell_1 \leq 1, \\ -(1 - \ell_1)e^{-\sigma\chi_1}, & \ell_1 > 1, \end{cases}$$

$$\pi_2(\ell_2) = \begin{cases} -(1 - \ell_2)e^{-\alpha\vartheta_2}, & \ell_2 \leq 1, \\ -(1 - \ell_2)e^{-\sigma\vartheta_1}, & \ell_2 > 1. \end{cases}$$

Through Lemma 2.1, we obtain

$$-\chi_1 \int_{t-\chi_1}^t e^{\alpha(s-t)} \dot{m}^T(s) R \dot{m}(s) ds \leq \xi_1^T(t) \Sigma_4 \xi_1(t),$$

$$-\vartheta_1 \int_{t-\vartheta_1}^t e^{\alpha(s-t)} \dot{h}^T(s) \mathcal{R} \dot{h}(s) ds \leq \xi_2^T(t) \Upsilon_4 \xi_2(t).$$

Through Lemma 2.2 and relation (6), we obtain

$$\begin{aligned} & -\chi_{21} \int_{t-\chi_2}^{t-\chi(t)} e^{\alpha(s+\chi_1-t)} \dot{m}^T(s) S \dot{m}(s) ds \\ & -\chi_{21} \int_{t-\chi(t)}^{t-\chi_1} e^{\alpha(s+\chi_1-t)} \dot{m}^T(s) S \dot{m}(s) ds \\ & \leq -\frac{\chi_{21} e^{-\alpha\chi_{21}}}{\chi(t) - \chi_1} \xi_1^T(t) \tilde{S} \xi_1(t) - \frac{\chi_{21} e^{-\alpha\chi_{21}}}{\chi_2 - \chi(t)} \xi_2^T(t) \tilde{S} \xi_2(t) \\ & \leq -e^{-\alpha\chi_{21}} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}^T \begin{bmatrix} \frac{\chi_{21}}{\chi(t) - \chi_1} \tilde{S} & 0 \\ 0 & \frac{\chi_{21}}{\chi_2 - \chi(t)} \tilde{S} \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} \\ & = -e^{-\alpha\chi_{21}} \xi_1^T(t) \Psi^T \Pi_1 \Psi \xi_1(t), \\ & -\vartheta_{21} \int_{t-\vartheta_2}^{t-\vartheta(t)} e^{\alpha(s+\vartheta_1-t)} \dot{h}^T(s) S \dot{h}(s) ds - \vartheta_{21} \int_{t-\vartheta(t)}^{t-\vartheta_1} e^{\alpha(s+\vartheta_1-t)} \dot{h}^T(s) \\ & S \dot{h}(s) ds \\ & \leq -\frac{\vartheta_{21} e^{-\alpha\vartheta_{21}}}{\vartheta(t) - \vartheta_1} \xi_3^T(t) \tilde{S} \xi_3(t) - \frac{\vartheta_{21} e^{-\alpha\vartheta_{21}}}{\vartheta_2 - \vartheta(t)} \xi_4^T(t) \tilde{S} \xi_4(t) \\ & \leq -e^{-\alpha\vartheta_{21}} \begin{bmatrix} \xi_3(t) \\ \xi_4(t) \end{bmatrix}^T \begin{bmatrix} \frac{\vartheta_{21}}{\vartheta(t) - \vartheta_1} \tilde{S} & 0 \\ 0 & \frac{\vartheta_{21}}{\vartheta_2 - \vartheta(t)} \tilde{S} \end{bmatrix} \begin{bmatrix} \xi_3(t) \\ \xi_4(t) \end{bmatrix} \\ & = -e^{-\alpha\vartheta_{21}} \xi_2^T(t) \Phi^T \Pi_2 \Phi \xi_2(t), \end{aligned}$$

where $\xi_1(t) = \Psi_2 \xi_1(t)$, $\xi_2(t) = \Psi_3 \xi_1(t)$, $\xi_3(t) = \Phi_2 \xi_2(t)$, and $\xi_4(t) = \Phi_3 \xi_2(t)$. From (A1), we have

$$0 \leq 2[g(U_2 h(t)) - Z_1 U_2 m(t)]^T \Gamma_1 [Z_2 U_2 m(t) - g(U_2 h(t))], \quad (12)$$

$$0 \leq 2[g(U_2 h(t - \chi(t))) - Z_1 U_2 m(t - \chi(t))]^T \Gamma_2 [Z_2 U_2 m(t - \chi(t)) - g(U_2 h(t - \chi(t)))], \quad (13)$$

$$0 \leq 2[g(U_2 h(t)) - g(U_2 h(t - \chi(t))) - Z_1 (U_2 m(t) - U_2 m(t - \chi(t)))]^T \Gamma_3 \times [g(U_2 h(t)) - g(U_2 h(t - \chi(t))) - Z_2 (U_2 m(t) - U_2 m(t - \chi(t)))], \quad (14)$$

$$0 \leq 2[f(V_2 m(t)) - Z_3 V_2 h(t)]^T \Gamma_4 [Z_4 V_2 h(t) - f(V_2 m(t))], \quad (15)$$

$$0 \leq 2[f(V_2 m(t - \vartheta(t))) - Z_3 V_2 h(t - \vartheta(t))]^T \Gamma_5 [Z_4 V_2 h(t - \vartheta(t)) - f(V_2 m(t - \vartheta(t)))], \quad (16)$$

$$0 \leq 2[f(V_2 m(t)) - f(V_2 m(t - \vartheta(t))) - Z_3 (V_2 h(t) - V_2 h(t - \vartheta(t)))]^T \Gamma_6 \times [f(V_2 m(t)) - f(V_2 m(t - \vartheta(t))) - Z_4 (V_2 h(t) - V_2 h(t - \vartheta(t)))]. \quad (17)$$

Combining (11)–(17), we obtain

$$\dot{V}(m_t, h_t) + \alpha V(m_t, h_t) - J(t) \leq \xi_1^T(t) \Sigma(\chi) \xi_1(t) + \xi_2^T(t) \Upsilon(\vartheta) \xi_2(t), \quad (18)$$

where $\Sigma(\chi)$ and $\Upsilon(\vartheta)$ are defined in (8) and (9), respectively. From (8) and (9) we obtain $\dot{V}(m_t, h_t) + \alpha V(m_t, h_t) \leq 0$. By integrating both sides, we obtain $V(m_t, h_t) \leq V(\phi) e^{-\alpha t}$. Then, Definitions 2.3 and 2.4 show that the BAMGNN (1) is exponentially dissipative. $\square$

Remark 3.2 In Theorem 3.1, the exponential dissipativity and passivity analysis of delayed BAMGNN in (1) was considered. We now establish a corollary to show the reduced conservatism of the asymptotic stability condition. Let us consider the following system:

$$\dot{h}(t) = -Dh(t) + V_0 f(V_2 h(t)) + V_1 f(V_2 h(t - \vartheta(t))). \quad (19)$$

Corollary 3.3 Given scalars $0 < \vartheta_1 < \vartheta_2$, $\ell_2 > 0$, system (19) is asymptotically stable when there exist positive definite matrices $P_2 \in \mathbb{R}^{3n \times 3n}$, $Q_1, Q_2, Q_3, R, S \in \mathbb{R}^{n \times n}$, $\Gamma_p = \text{diag}\{\gamma_{p1}, \gamma_{p2}, \dots, \gamma_{pn}\} \in \mathbb{R}^{n \times n}$, ($p = 1, 2, 3$), $\Lambda_q = \text{diag}\{\Lambda_{q1}, \Lambda_{q2}, \dots, \Lambda_{qn}\} \in \mathbb{R}^{n \times n}$ ($q = 1, 2$), and a matrix $Y \in \mathbb{R}^{2n \times 2n}$ satisfying

$$\begin{aligned} & \bar{\mathcal{U}}(\vartheta) < 0, \\ & \Pi_2 = \begin{bmatrix} \tilde{S} & Y \\ 0 & \tilde{S} \end{bmatrix} \geq 0, \end{aligned} \quad (20)$$

where

$$\begin{aligned}
 \mathcal{U}(\vartheta) &= \mathcal{U}_0(\vartheta) + \sum_{z=1}^5 \mathcal{U}_z - \bar{\Theta}^T \Pi_2 \bar{\Theta}, \quad \mathcal{U}_0(\vartheta) = \Theta^T(\vartheta) \\
 &\quad P_2 \Theta_0 + \Theta_0^T P \Theta(\vartheta), \\
 \Theta(\vartheta) &= \begin{bmatrix} w_1 \\ \vartheta_1 w_7 \\ (\vartheta(t) - \vartheta_1) w_8 + (\vartheta_2 - \vartheta(t)) w_9 \end{bmatrix}, \quad \Theta_0 = \begin{bmatrix} \tilde{D} \\ w_1 - w_2 \\ w_2 - w_4 \end{bmatrix}, \\
 \Theta_1 &= \begin{bmatrix} w_1 - w_2 \\ w_1 + w_2 - 2w_7 \end{bmatrix}, \quad \Theta_2 = \begin{bmatrix} w_2 - w_3 \\ w_2 + w_3 - 2w_8 \end{bmatrix}, \\
 \Theta_3 &= \begin{bmatrix} w_3 - w_4 \\ w_3 + w_4 - 2w_9 \end{bmatrix}, \\
 \mathcal{U}_1 &= [2[w_5 - Z_3 V_2 w_1]^T \Lambda_1 \tilde{D} + 2[Z_4 V_2 w_1 - w_5]^T \Lambda_2 \tilde{D}], \\
 \mathcal{U}_2 &= w_1^T \mathcal{Q}_1 w_1 - w_2^T \mathcal{Q}_1 w_2 + w_2^T \mathcal{Q}_2 w_2 - w_4^T \mathcal{Q}_2 w_4 + w_5^T \mathcal{Q}_3 w_5 \\
 &\quad - (1 - \ell_2) w_6^T \mathcal{Q}_3 w_6, \\
 \mathcal{U}_3 &= \tilde{D}^T [\vartheta_1^2 \mathcal{R} + \vartheta_2^2 \mathcal{S}] \tilde{D}, \quad \mathcal{U}_4 = -\Theta_1^T \tilde{\mathcal{R}} \Theta_1, \quad \tilde{\mathcal{R}} = \text{diag}\{\mathcal{R}, 3\mathcal{R}\}, \\
 \mathcal{U}_5 &= -2w_1^T V_2^T Z_3 \Gamma_4 Z_4 V_2 w_1 + w_1^T V_2^T (Z_3 + Z_4) \Gamma_4 w_5 + w_5^T \Gamma_2 \\
 &\quad (Z_3 + Z_4) V_2 w_1 - 2w_5^T \Gamma_4 w_5 \\
 &\quad - 2w_2^T V_2^T Z_3 \Gamma_5 Z_4 V_2 w_2 + w_2^T V_2^T (Z_3 + Z_4) \Gamma_5 w_6 + w_6^T \\
 &\quad \Gamma_2 (Z_3 + Z_4) V_2 w_2 - 2w_6^T \Gamma_5 w_6 \\
 &\quad - 2[w_5 - w_6 - Z_3 V_2 (w_1 - w_2)]^T \Gamma_6 [w_5 - w_6 - Z_4 \\
 &\quad V_2 (w_1 - w_2)], \\
 \bar{\Theta} &= [\Theta_2^T \quad \Theta_3^T]^T, \quad \tilde{\mathcal{S}} = \text{diag}\{\mathcal{S}, 3\mathcal{S}\}, \quad \tilde{D} = -D w_1 + V_0 w_5 \\
 &\quad + V_1 w_6, \quad \vartheta_{21} = \vartheta_2 - \vartheta_1.
 \end{aligned}$$

Proof Choose the LKF candidate:

$$\begin{aligned}
 V(t) &= \eta_2^T(t) P_2 \eta_2(t) + 2 \sum_{j=1}^n \lambda_{1j} \int_0^{V_2 h(t)} (f_j(s) - \bar{z}_j s) ds \\
 &\quad + 2 \sum_{j=1}^n \lambda_{2j} \int_0^{V_2 h(t)} (\bar{z}_j^+ s - f_j(s)) ds \\
 &\quad + \int_{t-\vartheta_1}^t h^T(s) \mathcal{Q}_1 h(s) ds + \int_{t-\vartheta_2}^{t-\vartheta_1} h^T(s) \mathcal{Q}_2 h(s) ds \\
 &\quad + \int_{t-\vartheta(t)}^t f^T(V_2 h(s)) \mathcal{Q}_3 f(V_2 h(s)) ds \\
 &\quad + \vartheta_1 \int_{-\vartheta_1}^0 \int_{t+s}^t \dot{h}^T(\theta) \mathcal{R} \dot{h}(\theta) d\theta ds + \vartheta_{21} \\
 &\quad \int_{-\vartheta_2}^{-\vartheta_1} \int_{t+s}^t \dot{h}^T(\theta) \mathcal{S} \dot{h}(\theta) d\theta ds.
 \end{aligned} \tag{21}$$

Now, we obtain the time derivative of $V(t)$,

$$\begin{aligned}
 \dot{V}(t) &\leq \xi^T(t) \mathcal{U}_0(\vartheta) \xi(t) + \xi^T [2[w_5 - Z_3 V_2 w_1]^T \Lambda_1 \tilde{D} \\
 &\quad + 2[Z_4 V_2 w_1 - w_5]^T \Lambda_2 \tilde{D}] \xi(t) \\
 &\quad + \xi^T [w_1^T \mathcal{Q}_1 w_1 - w_2^T \mathcal{Q}_1 w_2 + w_2^T \mathcal{Q}_2 w_2 \\
 &\quad - w_4^T \mathcal{Q}_2 w_4 + w_5^T \mathcal{Q}_3 w_5 - (1 - \ell) w_6^T \mathcal{Q}_3 w_6] \xi(t) \\
 &\quad + \xi^T(t) \tilde{D}^T [\vartheta_1^2 \mathcal{R} + \vartheta_2^2 \mathcal{S}] \tilde{D} \xi(t) \\
 &\quad - \vartheta_1 \int_{t-\vartheta_1}^t \dot{h}^T(s) \mathcal{R} \dot{h}(s) ds - \vartheta_{21} \int_{t-\vartheta_2}^{t-\vartheta_1} \dot{h}^T(s) \mathcal{S} \dot{h}(s) ds \\
 &\quad - \vartheta_{21} \int_{t-\vartheta(t)}^{t-\vartheta_1} \dot{h}^T(s) \mathcal{S} \dot{h}(s) ds.
 \end{aligned} \tag{22}$$

Using the inequality in (6), we obtain

$$-\vartheta_1 \int_{t-\vartheta_1}^t e^{\alpha(s-t)} \dot{h}^T(s) \mathcal{R} \dot{h}(s) ds \leq \xi^T(t) \mathcal{U}_4 \xi(t).$$

Through inequality (6) and Lemma 2.2, we obtain

$$\begin{aligned}
 &-\vartheta_{21} \int_{t-\vartheta_2}^{t-\vartheta_1} \dot{h}^T(s) \mathcal{S} \dot{h}(s) ds - \vartheta_{21} \int_{t-\vartheta(t)}^{t-\vartheta_1} \dot{h}^T(s) \mathcal{S} \dot{h}(s) ds \\
 &\leq -\frac{\vartheta_{21}}{\vartheta(t) - \vartheta_1} \varpi_1^T(t) \tilde{\mathcal{S}} \varpi_1(t) - \frac{\vartheta_{21}}{\vartheta_2 - \vartheta(t)} \varpi_2^T(t) \tilde{\mathcal{S}} \varpi_2(t) \\
 &\leq -\begin{bmatrix} \varpi_1(t) \\ \varpi_2(t) \end{bmatrix}^T \begin{bmatrix} \frac{\vartheta_{21}}{\vartheta(t) - \vartheta_1} \tilde{\mathcal{S}} & 0 \\ 0 & \frac{\vartheta_{21}}{\vartheta_2 - \vartheta(t)} \tilde{\mathcal{S}} \end{bmatrix} \begin{bmatrix} \varpi_1(t) \\ \varpi_2(t) \end{bmatrix} \\
 &= -\xi^T(t) \bar{\Theta}^T \Pi_2 \bar{\Theta} \xi(t),
 \end{aligned}$$

where $\varpi_1(t) = \Theta_2 \xi(t)$, $\varpi_2(t) = \Theta_3 \xi(t)$. From (A1), we have

$$0 \leq 2[f(V_2 h(t)) - Z_3 V_2 h(t)]^T \Gamma_4 [Z_4 V_2 h(t) - f(V_2 h(t))], \tag{23}$$

$$0 \leq 2[f(V_2 h(t - \vartheta(t))) - Z_3 V_2 h(t - \vartheta(t))]^T \Gamma_5 [Z_4 V_2 h(t - \vartheta(t)) - f(V_2 h(t - \vartheta(t)))], \tag{24}$$

$$\begin{aligned}
 0 &\leq 2[f(V_2 h(t)) - f(V_2 h(t - \vartheta(t))) - Z_3 (V_2 h(t) \\
 &\quad - V_2 h(t - \vartheta(t)))]^T \Gamma_6 \times [f(V_2 h(t)) \\
 &\quad - f(V_2 h(t - \vartheta(t))) - Z_4 (V_2 h(t) - V_2 h(t - \vartheta(t)))].
 \end{aligned} \tag{25}$$

Combining (22)–(25), we obtain

$$\dot{V}(t) \leq \xi^T(t) \mathcal{U}(\vartheta) \xi(t), \tag{26}$$

where

$$\begin{aligned}
 \xi(t) &\triangleq [h^T(t) h^T(t - \vartheta_1) h^T(t - \vartheta(t)) h^T(t - \vartheta_2) f(V_2 h(t)) \\
 &\quad f^T(V_2 h(t - \vartheta(t))) \frac{1}{\vartheta_1} \int_{t-\vartheta_1}^t h^T(s) ds \frac{1}{\vartheta(t) - \vartheta_1} \\
 &\quad \int_{t-\vartheta(t)}^{t-\vartheta_1} h^T(s) ds \frac{1}{\vartheta_2 - \vartheta(t)} \int_{t-\vartheta_2}^{t-\vartheta(t)} h^T(s) ds]^T,
 \end{aligned}$$

and $\mathcal{U}(\vartheta)$ is defined in (20). This completes the proof of Corollary 3.3. $\square$

4 Examples

Example 4.1 Consider BAMGNN (1) with

$$C = \begin{bmatrix} 1.8 & 0 \\ 0 & 2.2 \end{bmatrix}, U_0 = \begin{bmatrix} 2.5 & 0 \\ 0 & 2.2 \end{bmatrix},$$

$$U_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, U_2 = \begin{bmatrix} 0.3 & 0.2 \\ 0 & 0.4 \end{bmatrix},$$

$$D = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.6 \end{bmatrix}, V_0 = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.5 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.4 \end{bmatrix}, V_2 = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.6 \end{bmatrix},$$

$$Z_1 = 0_2, Z_2 = \text{diag}\{0.5, 0.5\}, Z_3 = 0_2,$$

$$Z_4 = \text{diag}\{0.4, 0.8\}.$$

(A) *Exponential stability analysis* Let $\ell_1 = 1.1$, $\ell_2 = 1.2$, $\chi_1 = 1$, $\chi_2 = 5$, $\vartheta_1 = 2$, and $\vartheta_2 = 4$, then apply Theorem 3.1 to BAMGNN (1) (with $\mathfrak{z}(t) = \mathfrak{y}(t) = u(t) = v(t) = 0$), which gives feasible results for LMIs in (8) and (9) as follows:

states $m(t)$ and $h(t)$ for the delay $\chi(t) = 2.3 + 0.5 \cos(t)$ and $\vartheta(t) = 1.7 + 0.9 \cos(t)$.

(B) *Passivity analysis* When $\mathbb{Q} = 0$, $\mathbb{S} = I_2$, and $\mathbb{R} = 0$, the passivity conditions are analyzed for BAMGNN (1) by applying Theorem 3.1. By taking $\chi_1 = \vartheta_1 = 1$, the minimum passivity index γ is obtained for different values of χ_2, ϑ_2 (see Table 2).

(C) *Dissipativity analysis* When $\mathbb{Q} = 0.5I_2$, $\mathbb{S} = I_2$, and $\mathbb{R} = 2I_2$, the dissipativity conditions are analyzed for BAMGNN (1) by applying Theorem 3.1. By taking $\chi_1 = \vartheta_1 = 1$, the optimal dissipativity index γ is obtained for different values of χ_2, ϑ_2 (see Table 3).

Example 4.2 Consider the following NN

$$\dot{h}(t) = -Dh(t) + V_0 f(V_2 m(t)) + V_1 f(V_2 m(t - \vartheta(t))) \quad (27)$$

$$P_1 = \begin{bmatrix} 1.7389 & -0.0923 & 0.0866 & -0.0364 & 0.0082 & -0.0003 \\ -0.0923 & 1.6596 & 0.0027 & 0.0903 & 0.0023 & 0.0086 \\ 0.0866 & 0.0027 & 0.2532 & -0.0438 & 0.0377 & -0.0066 \\ -0.0364 & 0.0903 & -0.0438 & 0.2502 & -0.0049 & 0.0351 \\ 0.0082 & 0.0023 & 0.0377 & -0.0049 & 0.0400 & -0.0045 \\ -0.0003 & 0.0086 & -0.0066 & 0.0351 & -0.0045 & 0.0367 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1.4497 & -0.3334 & 0.1463 & -0.0423 & 0.0108 & -0.0156 \\ -0.3334 & 1.0061 & -0.0538 & 0.2356 & -0.0137 & 0.0270 \\ 0.1463 & -0.0538 & 0.1983 & -0.0334 & 0.0439 & -0.0169 \\ -0.0423 & 0.2356 & -0.0334 & 0.3154 & -0.0171 & 0.0758 \\ 0.0108 & -0.0137 & 0.0439 & -0.0171 & 0.0501 & -0.0094 \\ -0.0156 & 0.0270 & -0.0169 & 0.0758 & -0.0094 & 0.0702 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 1.0289 & -0.0764 \\ -0.0764 & 1.1097 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.6191 & -0.0787 \\ -0.0787 & 0.6215 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 0.4721 & 0.1297 \\ 0.1297 & 1.4934 \end{bmatrix}, R = \begin{bmatrix} 0.0718 & 0.0009 \\ 0.0009 & 0.0697 \end{bmatrix}, S = \begin{bmatrix} 0.0625 & -0.0061 \\ -0.0061 & 0.0450 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 1.1037 & 0.0530 \\ 0.0530 & 1.4197 \end{bmatrix}, Q_5 = \begin{bmatrix} 0.6418 & -0.0070 \\ -0.0070 & 0.7090 \end{bmatrix},$$

$$Q_6 = \begin{bmatrix} 0.5289 & -0.1634 \\ -0.1634 & 1.2454 \end{bmatrix}, R = \begin{bmatrix} 0.0747 & -0.0047 \\ -0.0047 & 0.0795 \end{bmatrix}, S = \begin{bmatrix} 0.1015 & -0.0087 \\ -0.0087 & 0.1122 \end{bmatrix}.$$

Therefore, the BAMGNN (1) is exponentially stable. In addition, comparison results for equal time delays χ_2, ϑ_2 are obtained for various values of χ_1, ϑ_1 and are listed in Table 1 when $\ell_1 = \ell_2 = 0$, $U_0 = V_0 = 0$, and $U_2 = V_2 = I_2$. Figure 1 shows the state trajectories of the

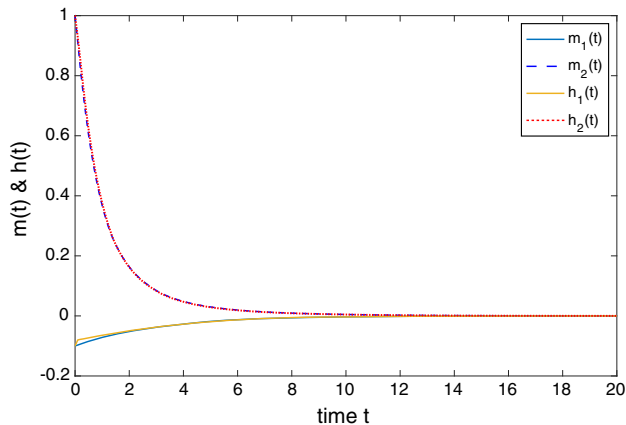
with

$$D = \begin{bmatrix} 2.5 & 0 \\ 0 & 3.5 \end{bmatrix}, V_0 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & -1 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} -0.5 & -0.1 \\ 0.6 & 0.8 \end{bmatrix}, V_2 = I_2, Z_3 = 0.998I_2, Z_4 = I_2.$$

Table 1 Maximum upper bounds of equal time delays χ_2, ϑ_2 for various values of χ_1, ϑ_1

Methods	1	2	3	4	5
[36]	9.2147	10.3357	12.7365	13.8763	15.2965
Theorem 3.1	11.1244	12.8547	14.2193	15.9476	17.2681

**Fig. 1** State response $m(t)$ and $h(t)$ in Example 4.1**Table 2** Minimum passivity index γ

$\chi_2 = \vartheta_2$	1.5	2.0	2.5	3.0
$\ell_1 = \ell_2 = 0.5$	0.3546	0.3845	0.4156	0.4536
$\ell_1 = \ell_2 = 0.8$	0.4236	0.4617	0.5073	0.5612
$\ell_1 = \ell_2 = 0.9$	0.4823	0.5137	0.5491	0.5814

Table 3 Optimal dissipativity index γ

$\chi_2 = \vartheta_2$	1.5	2.0	2.5	3.0
$\ell_1 = \ell_2 = 0.5$	1.7033	1.6921	1.6723	1.6648
$\ell_1 = \ell_2 = 0.8$	1.6736	1.6694	1.6573	1.6503
$\ell_1 = \ell_2 = 0.9$	1.6653	1.6594	1.6486	1.6415

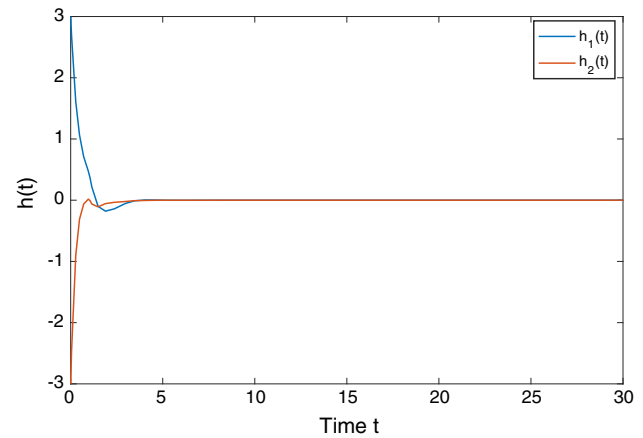
Table 4 α_{max} for various values of ℓ_2

Methods	0	0.6	0.8	1.5	Unknown ℓ_2
Theorem 3.1	1.5234	1.3654	1.2157	1.1034	0.9871
Ref. [2]	1.1609	0.9428	0.8759	0.8739	0.9426
Ref. [22]	1.1605	0.9064	0.8381	0.8371	0.8359

By using the LMI in (9), when $\vartheta_1 = 0.8$, $\vartheta_2 = 1$, and $v(t) = 0$, α_{max} is calculated for various values of ℓ_2 , which is listed in Table 4. If $\vartheta(t)$ is not differentiable, then Table 5 shows the MAUB ϑ_2 for various values of ϑ_1 and

Table 5 MAUB ϑ_2 for different values of ϑ_1 and α

Methods	α	1	2	4	6
Ref. [2] Theorem 3.1	0.05	1.8642	2.9492	4.8409	6.7416
		2.7698	3.8962	5.7865	7.5847
Ref. [2] Theorem 3.1	0.1	1.6314	2.7981	4.6218	6.4718
		2.5234	3.6154	5.5257	7.2632
Ref. [2] Theorem 3.1	0.2	1.4632	2.5712	4.3200	6.1321
		2.3984	3.4561	5.2157	6.9854

**Fig. 2** State response $h(t)$ in Example 4.2

α . The state trajectories of the state $h(t)$ for the delay are shown in Fig. 2.

Example 4.3 Consider the NN in (19) and the matrix values in [37]:

$$D = \text{diag}\{7.3458, 6.9987, 5.5949\},$$

$$V_2 = \begin{bmatrix} 13.6014 & -2.9616 & -0.6936 \\ 7.4736 & 21.6810 & 3.2100 \\ 0.7290 & -2.6334 & -20.1300 \end{bmatrix},$$

$$\bar{z}_1^- = \bar{z}_2^- = \bar{z}_3^- = 0, \bar{z}_1^+ = 0.3680,$$

$$\bar{z}_2^+ = 0.1795, \bar{z}_3^+ = 0.2876.$$

Solving Corollary 3.3, one can reach the MAUB ϑ_2 for various values of ℓ_2 when $\vartheta_1 = 0.5$ (see Table 6). Hence, our method is better than those in [15, 37–40].

Remark 4.4 LKF (21) is not conservative, since the LKF (21) is constructed using a smaller number of decision variables (NDVs) compared with the methods in [15, 37–40] to avoid the numerical computational complexity. In particular, the NDVs in [15, 37–40] are, respectively, $\frac{(26n(n+1))}{2} + 22n^2$, $\frac{(21n(n+1))}{2} + 4n^2$, $\frac{(23n(n+1))}{2} + 33n^2$, $\frac{(11n(n+1))}{2} + 23n^2$, and $\frac{(38n(n+1))}{2} + 22n^2$. The NDV in

Table 6 MAUB ϑ_2 for various values of ℓ_2 when $\vartheta_1 = 0.5$

ℓ_2	0.2	0.3	0.9	Unknown ℓ_2
[37]	0.6551	0.5886	0.5885	0.5885
[38]	0.7515	0.6021	0.6021	0.6021
[39]	0.7578	0.6053	0.6053	0.6053
[40]	0.8006	0.6344	–	0.6169
[15]	0.8413	0.7542	0.6547	0.6547
Corollary 3.3	0.9143	0.8429	0.7542	0.7542

Corollary 3.3 is $\frac{(13n(n+1))}{2} + 3n^2$. In addition, through the new WII technique, the maximum allowable upper bounds (MAUBs) are obtained. Thus, our presented results are significantly less conservative than those of the existing approaches in [15, 37–40].

Remark 4.5 In recent years, some researchers have used the integral inequality techniques, such as the RCC and Wirtinger's inequality techniques. These two inequalities are the best techniques to reduce conservatism, in which Jensen's inequality is one of the special cases. However, these techniques may not be accurate techniques for exponential stability problems, but WII inequality is a perfect technique exponential stability analysis. Therefore, we utilized the new WII for both single and double integral terms to study the exponential stability and dissipativity problem.

5 Conclusions

This article evaluates the exponential dissipativity and passivity problem for delayed BAMGNNs. Some improved results were proposed by constructing a proper LKF. All the proposed conditions are provided in terms of LMIs by using a new WII technique. To show the feasibility of the proposed results, some examples are added in the previous section. It is difficult to use the integral inequality techniques to solve the above problems for stochastic NNs. Therefore, we may extend this work to robust exponential dissipativity and passivity analysis of Markovian jump BAMGNNs with additive time delays.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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RESEARCH

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Enhanced robust finite-time passivity for Markovian jumping discrete-time BAM neural networks with leakage delay

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Abstract

This paper is concerned with the problem of enhanced results on robust finite-time passivity for uncertain discrete-time Markovian jumping BAM delayed neural networks with leakage delay. By implementing a proper Lyapunov-Krasovskii functional candidate, the reciprocally convex combination method together with linear matrix inequality technique, several sufficient conditions are derived for varying the passivity of discrete-time BAM neural networks. An important feature presented in our paper is that we utilize the reciprocally convex combination lemma in the main section and the relevance of that lemma arises from the derivation of stability by using Jensen's inequality. Further, the zero inequalities help to propose the sufficient conditions for finite-time boundedness and passivity for uncertainties. Finally, the enhancement of the feasible region of the proposed criteria is shown via numerical examples with simulation to illustrate the applicability and usefulness of the proposed method.

Keywords: LMIs; Markovian jumping systems; leakage delay; bidirectional associative memory; discrete-time neural networks; passivity and stability analysis

1 Introduction

Over the past decades, delayed neural networks have found successful applications in many areas such as signal processing, pattern recognition, associative memories and optimization solvers. In such applications quantitative behavior of dynamical systems is an important step for the practical design of neural networks [1]. Therefore, the dynamic characteristics of discrete-time neural networks have been extensively investigated, for example, see [2–12]. The study on neural networks is mostly in the continuous-time setting, but they are often discretized for experimental or computational purposes. Also, neural networks with leakage delay is one of the important types of neural networks. Hence, time delay in the leakage term has a great impact on the dynamics of neural networks. Although, time delay in the stabilizing negative feedback term has a tendency to destabilize a neural network system [13–16], the delay in the leakage term can destroy the stability neural networks. Gopalasamy [17] initially investigated the dynamics of bidirectional associative memory (BAM) network model with leakage delay. Based on this work, authors in [15] considered the global stability for a class of nonlinear systems with leakage delay. Li and Cao discussed the stability of memristive neural networks with both reaction-diffusion

term and leakage delay, and some easily checked criteria have been established by employing differential inclusion theory and Jensen's integral inequality [18].

The (BAM) neural network model, proposed by Kosko [19, 20] is a two-layered nonlinear feedback network model, where the neurons in one layer always interconnect with the neurons in the another layer, while there are no interconnections among neurons in the same layer. In the current scenario, due to its application in many fields, the study of bidirectional associative memory neural networks has attracted the attention of many researchers, and they have studied the stability properties of neural networks and presented various sufficient conditions for asymptotic or exponential stability of the BAM neural networks [4, 15, 21–23].

On the one hand, time delay is one of the main sources of instability, which is encountered in many engineering systems such as chemical processes, long transmission lines in pneumatic systems, networked control systems, *etc.* Over the past years, the study of time delay systems has received considerable attention, and a great number of research results on time delay systems exist in the literature. The stability of time delay systems is a fundamental problem because it is important in the synthesis and analysis of such neural network systems [3, 5, 24, 25]. The exponential stability of stochastic BAM networks with mixed delays was discussed by Lyapunov theory [25].

On the other hand, the theory of passivity was implemented first in circuit analysis and generates increasing interest among the researchers. It is a useful tool in obtaining the stability analysis of both linear and nonlinear systems, especially for high-order systems. It is evidently true that the passive properties can ideally keep the systems internally stable. Due to its importance and applicability, the problem of passivity analysis for delayed dynamic systems has been investigated, and lots of results have been reported in the literature [26–30]. For instance, in [28] the authors Wu *et al.* derived the passivity condition for discrete-time switched neural networks with various functions and mixed time delays. Moreover, the passivity and synchronization of switched neural networks were investigated in [30], and some delay-dependent as well as delay-independent criteria were provided. In [6, 7], the authors delivered the concept of passivity which is stable or not.

However, using the conventional Lyapunov asymptotic stability theory, it should be mentioned that all these existing studies about the passivity analysis are performed with definition over the infinite-time interval. The concept of finite-time (or short-time) analysis problem was first initiated by Dorato in 1961 [31]. Communication network system, missile system and robot control system are the examples of systems which work in a short-time interval. In the present years, many biologists have been focusing on the transient values of the actual network states. In [32–34], many interesting results for finite-time stability of various types of systems can be found. Recently, an extended finite-time H_∞ control problem for uncertain switched linear neutral systems with time-varying delays was investigated in [27], and also the concept of time-varying delays was proposed in [22, 35]. The results for finite-time stabilization of neural networks with discontinuous activations was proposed in [36].

Motivated by the aforementioned discussions, in this paper we focus on the finite-time boundedness and passivity of uncertain discrete-time Markovian jumping BAM neural networks with leakage time-delays. Here we use a new type of LKF to handle the given range of time delay interval together with free weighting matrix approach to derive the main results. Our main contributions are highlighted as follows:

- The finite-time passivity result for discrete-time Markovian jumping uncertain BAM neural networks with leakage delay is proposed for the first time.
- Reciprocally convex combination approach is used to handle the triple summation terms and a new type of zero inequalities is introduced.
- Delay-dependent results for finite-time boundedness and finite-time passivity are derived by using the finite-time stability method and the Lyapunov-Krasovskii functional approach.

The rest of this paper is well organized as follows. Problem formulation and mathematical preliminaries are presented in Section 2. Section 3 gives the main result of this paper, and it also contains the subsection on finite-time boundedness. Robust finite-time passivity is derived in Section 4. Numerical examples are demonstrated in Section 5 to illustrate the effectiveness of the proposed method. Finally, we give the conclusion of this paper in Section 6.

1.1 Notations

The notations in this paper are standard. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote, respectively, the n -dimensional Euclidean space and the set of all $n \times m$ real matrices. I denotes the identity matrix with appropriate dimensions and $\text{diag}(\cdot)$ denotes the diagonal matrix. A^T denotes the transpose of matrix A . k denotes the set of positive integers. For real symmetric matrices X and Y , the notation $X \geq Y$ (resp., $X > Y$) means that the matrix $X - Y$ is positive semi-definite (resp., positive definite). $\mathbb{N} = \{1, 2, \dots, n\}$ and $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^n$. $\lambda_{\max}(X)$ (resp., $\lambda_{\min}(X)$) stands for the maximum (resp., minimum) eigenvalue of the matrix X . I_n and 0_n represent the identity matrix and zero matrix, respectively. $l_2(0, \infty)$ denotes the space of square summable infinite vector sequences. The symbol $*$ within a matrix represents the symmetric term of the matrix.

2 Problem formulation and mathematical preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual condition (i.e., it is right continuous and $\mathcal{F}_0$ contains all P -null sets); $E\{\cdot\}$ stands for the mathematical expectation operator with respect to given probability measure P . Let $r(k)$, $k \geq 0$ be a Markovian chain taking values in a finite space $S = \{1, 2, 3, \dots, N\}$ with probability transition matrix $\pi = (\pi_{ij})_{N \times N}$ given by

$$\Pr[r(k+1) = j | r(k) = i] = \pi_{ij}, \quad j, i \in S,$$

where $\pi \geq 0$ ($i, j \in S$) is a transition rate from i to j and $\sum_{j=1}^N \pi_{ij} = 1$, $\forall i \in S$.

Consider the following BAM uncertain discrete-time Markovian jumping neural network with time-varying delays, and leakage delay is described by

$$x(k+1) = A(r(k))x(k - \gamma_1) + B(r(k))f(y(k)) + C(r(k))f(y(k - \tau(k))) + u(k), \quad (1)$$

$$g^*(k) = B_{g^*}(r(k))f(y(k)) + C_{g^*}(r(k))f(y(k - \tau(k))), \quad (2)$$

$$x(k) = \phi(k), \quad \text{for every } k \in [-\tau_M, 0],$$

$$y(k+1) = D(r(k))y(k - \gamma_2) + E(r(k))g(x(k)) + F(r(k))g(x(k - \sigma(k))) + v(k), \quad (3)$$

$$h(k) = E_h(r(k))g(x(k)) + F_h(r(k))g(x(k - \sigma(k))), \quad (4)$$

$$y(k) = \psi(k) \quad \text{for every } k \in [-\sigma_M, 0],$$

where $x(k), y(k) \in \mathbb{R}^n$ is the neural state vector, $v(k), u(k)$ is the exogenous disturbance input vector belonging to $\mathcal{L}_2[0, \infty)$ and $g^*(k), h(k)$ is the output vector of the neural network, $f(y(k)), g(x(k))$ is the neuron activation function, the positive integer $\tau(k), \sigma(k)$ denotes the time-varying delay satisfying $\tau_m \leq \tau(k) \leq \tau_M$ and $\sigma_m \leq \sigma(k) \leq \sigma_M$ for all $k \in N$, where τ_m and τ_M, σ_m and σ_M are constant positive scalars representing the minimum and maximum delays, respectively.

$$\begin{aligned} A_i(k) &= A_i + \Delta A_i(k), & B_i(k) &= B_i + \Delta B_i(k), \\ C_i(k) &= C_i + \Delta C_i(k), & D_i(k) &= D_i + \Delta D_i(k), \\ E_i(k) &= E_i + \Delta E_i(k), & F_i(k) &= F_i + \Delta F_i(k) \end{aligned}$$

in which $A = \text{diag}\{a_1, a_2, \dots, a_n\}$, $D = \text{diag}\{d_1, d_2, \dots, d_n\}$ represent the state feed back coefficient matrix with $|a_i| < 1$, $|d_i| < 1$, $B = [b_{ij}]_{n \times n}$, $E = [e_{ij}]_{n \times n}$, $C = [c_{ij}]_{n \times n}$, $F = [f_{ij}]_{n \times n}$, respectively, the connection weights and the delayed connection weights, the initial function $\phi(k), \psi(k)$ is continuous and defined on $[-\tau_M, 0], [-\sigma_M, 0]$. Further the uncertainty parameters are defined as follows:

$$[\Delta A(k), \Delta B(k), \Delta C(k), \Delta D(k), \Delta E(k), \Delta F(k)] = MN(k)[M_a, M_b, M_c, M_d, M_e, M_f], \quad (5)$$

where $M_a, M_b, M_c, M_d, M_e, M_f, M$ are known constant matrices of appropriate dimensions and $N(k)$ is an unknown time-varying matrix with Lebesgue measurable elements bounded by $N^T(k)N(k) \leq I$.

The following assumptions help to complete the main result.

Assumption 1 For any $i = 1, 2, 3, \dots, n$, there exist constraints $F_i^-, F_i^+, G_i^-, G_i^+$ such that

$$\begin{aligned} F_i^- &\leq \frac{f_i(x_1) - f_i(x_2)}{x_1 - x_2} \leq F_i^+ \quad \text{for all } x_1, x_2 \in \mathbb{R}, x_1 \neq x_2, \\ G_i^- &\leq \frac{g_i(y_1) - g_i(y_2)}{y_1 - y_2} \leq G_i^+ \quad \text{for all } y_1, y_2 \in \mathbb{R}, y_1 \neq y_2. \end{aligned}$$

For presentation convenience, in the following we denote

$$\begin{aligned} F_1 &= \text{diag}\{F_1^-, F_1^+, \dots, F_m^-, F_m^+\}, & F_2 &= \text{diag}\left\{\frac{F_1^- + F_1^+}{2}, \dots, \frac{F_n^- + F_n^+}{2}\right\}, \\ G_1 &= \text{diag}\{G_1^-, G_1^+, \dots, G_m^-, G_m^+\}, & G_2 &= \text{diag}\left\{\frac{G_1^- + G_1^+}{2}, \dots, \frac{G_n^- + G_n^+}{2}\right\}. \end{aligned}$$

Remark 2.1 In biologically inspired neural networks, the activation function is usually an abstraction representing the rate of action potential firing in the cell. Non-monotonic functions can be more suitable than other activation functions. In many electronic circuits, the input-output functions of amplifiers may be neither monotonically increasing nor continuously differentiable. The constants are positive, negative or zero in the above assumption. So, the activation function may be non-monotonic and more widespread than usual sigmoid functions and Lipschitz functions. Such conditions are discourteous in qualifying the lower and upper bounds of the activation functions. Therefore, by using

the LMI-based technique, the generalized activation function is considered to reduce the possible conservatism.

Assumption II The disturbance input vector $v(x)$ and $u(x)$ is time-varying and, for given $v > 0$, $u > 0$ satisfies $v^T(x)v(x) \leq v$, $u^T(x)u(x) \leq u$.

Before deriving our main results, the following definitions and lemmas will be stated.

Definition 2.2 ([37]) DNN (1), (3) is said to be robustly finite-time bounded with respect to $(\eta_1, \eta_2, \eta, \chi, L, Q, L_1, Q_1, u, v)$, where $0 < \eta_1 < \eta_2 < \eta < \chi$ and $L, L_1 > 0$, if

$$x^T(k_1)Lx(k_1) + y^T(k_2)L_1y(k_2) \leq \eta \Rightarrow x^T(k_3)Lx(k_3) + y^T(k_4)L_1y(k_4) \leq \chi,$$

$\forall k_1 \in \{-\tau_M, -\tau_M + 1, \dots, 0\}$, $k_2 \in \{-\sigma_M, -\sigma_M + 1, \dots, 0\}$, $k_3 = k_4 \in \{1, 2, \dots, N\}$ holds for any nonzero $v(x)$, $u(x)$ satisfying Assumption II.

Definition 2.3 ([37]) DNN (1), (3) with output (2), (4) is said to be robustly finite-time passive with respect to $(\eta_1, \eta_2, \eta, \chi, L, Q, L_1, Q_1, \gamma, \tilde{\gamma}_1, u, v)$, where $0 < \eta_1 < \eta_2 < \eta < \chi$, $\omega, \tilde{\omega}$ is a prescribed positive scalar and $L > 0$ and $L_1 > 0$, iff DNN (1), (3) with output (2), (4) is robustly finite-time bounded with respect to $(\eta_1, \eta_2, \eta, \chi, L, Q, L_1, Q_1, u, v)$ and under the zero initial condition the output $G(k)$, $H(k)$ satisfies

$$2 \left[\sum_{k=0}^N g^{*T}(k)u(k) + \sum_{k=0}^N h^T(k)v(k) \right] \geq \gamma \left[\sum_{k=0}^T u^T(k)u(k) + \sum_{k=0}^{T_1} v^T(k)v(k) \right]$$

for any nonzero $v(x)$, $u(x)$ satisfying Assumption II.

Remark 2.4 The concept of finite-time passivity is different from that of usual passivity. If the states in the system exceed their recommended bounds, it is usual passivity. Here, in this paper, Assumption II and Definition 2.2 should be in the given bounds, which helps to prove the finite-time passivity in the main result.

Lemma 2.5 ([12]) For any symmetric constant matrix $Z \in \mathbb{R}^{n \times n}$, $Z \geq 0$, two scalars τ_m and τ_M satisfying $\tau_m \leq \tau_M$, and a vector-valued function $\eta k = x(k+1) - x(k)$ ($k \in \mathbb{Z}^+$), we have

$$\begin{aligned} \text{(i)} \quad & - \sum_{i=k-\tau_M}^{k-\tau_m-1} \eta^T(i)Z\eta(i) \leq \frac{-1}{(\tau_M - \tau_m)} \sum_{i=k-\tau_M}^{k-\tau_m-1} \eta^T(i)Z \sum_{i=k-\tau_M}^{k-\tau_m-1} \eta(i); \\ \text{(ii)} \quad & - \sum_{j=-\tau_M}^{-\tau_m-1} \sum_{i=k+j}^{k-1} \eta^T(i)Z\eta(i) \\ & \leq \frac{-2}{(\tau_M - \tau_m)(\tau_M + \tau_m + 1)} \sum_{j=-\tau_M}^{-\tau_m-1} \sum_{i=k+j}^{k-1} \eta^T(i)Z \sum_{j=-\tau_M}^{-\tau_m-1} \sum_{i=k+j}^{k-1} \eta^T(i)\eta(i). \end{aligned}$$

Lemma 2.6 ([38]) Let $g_1, g_2, \dots, g_N : \mathbb{R}^m \rightarrow \mathbb{R}$ have positive values in an open subset $\mathbb{D}$ of $\mathbb{R}^m$. Then the reciprocally convex combination of g_i over $\mathbb{D}$ satisfies

$$\min_{\{\alpha_i | \alpha_i > 0, \sum_i \alpha_i = 1\}} \sum_i \frac{1}{\alpha_i} g_i(t) = \sum_i g_i(t) + \max_{h_{ij}(t)} \sum_{i \neq j} h_{ij}(t)$$

subject to

$$\left\{ h_{ij} : \mathbb{R}^m \rightarrow \mathbb{R}, h_{ij}(t) = h_{ji}(t), \begin{pmatrix} g_i(t) & h_{ij}(t) \\ h_{ij}(t) & g_j(t) \end{pmatrix} \right\}.$$

Remark 2.7 There are two main methods to find lower bounds. The first one is based on Moon's inequality. The second method is the so-called reciprocally convex combination lemma, and this approach also helps to reduce the number of decision variables. The conservatism induced by these two inequalities is independent. While, in some cases, such as stability conditions resulting from the application of Jensen's inequality, the reciprocally convex combination lemma is in general more conservative than Moon's inequality. Also, note that the reciprocally convex combination approach is successfully applied on double summation terms.

3 Robust finite-time boundedness

The main concern in this subsection is that the sufficient conditions for the finite boundedness of DNN (1), (3) and the LMI-based robust conditions will be derived using the Lyapunov technique.

Theorem 3.1 Under Assumptions I and II, for given scalars $\mu > 1$, $\rho > 1$, τ_m , τ_M , σ_m , σ_M , DNN model (1), (3) is robustly finite-time bounded with respect to $(\eta_1, \eta_2, \eta, \chi, L, Q, L_1, Q_1, u, v)$ if there exist symmetric positive definite matrices P_{1i} , P_{2i} , W , W_1 , R , S_1 , S_2 , S_3 , S_4 , Z_1 , Z_2 , Z_3 , Z_4 , matrices U_1 , U_2 , U_3 , U_4 , U_5 , U_6 , U_7 , U_8 , positive diagonal matrices H_1 , H_2 , H_3 , H_4 and positive scalars λ^{P_1} , λ^{P_2} , λ_{P_1} , λ_{P_2} , λ_{W_3} , λ_{W_4} , λ_{R_3} , λ_{R_4} , λ_{S_1} , λ_{S_2} , λ_{S_3} , λ_{S_4} , λ_{Z_1} , λ_{Z_2} , λ_{Z_3} , λ_{Z_4} , ϵ , ϵ_1 , $P_{1i} = P_1(r_k)$, $P_{2i} = P_2(r_k)$, $\bar{P}_{1i} = \sum_{j=1}^N \Pi_{ij} P_{1j}$, $\bar{P}_{2i} = \sum_{j=1}^N \Pi_{ij} P_{2j}$ such that the following LMIs hold for $r = 1, 2, 3, 4$:

$$\begin{pmatrix} \tilde{\Pi}_r & \tilde{\Pi}_1^T P_{1i} & \hat{\sigma} \tilde{\Pi}_2^T Z_1 & \frac{\check{\sigma}}{2} \tilde{\Pi}_2^T Z_2 & 0 & \epsilon \tilde{M}_{ABC}^T \\ * & -P_{1i} & 0 & 0 & M & 0 \\ * & * & -Z_1 & 0 & M & 0 \\ * & * & * & -Z_2 & M & 0 \\ * & * & * & * & -\epsilon I & 0 \\ * & * & * & * & * & -\epsilon I \end{pmatrix} < 0, \quad r = 1, 2, \quad (6)$$

$$\begin{pmatrix} \tilde{\Pi}_r & \tilde{\Pi}_1^T P_{2i} & \hat{\tau} \tilde{\Pi}_2^T Z_3 & \frac{\check{\tau}}{2} \tilde{\Pi}_2^T Z_4 & 0 & \epsilon^* \tilde{M}_{DEF}^T \\ * & -P_{2i} & 0 & 0 & M & 0 \\ * & * & -Z_1 & 0 & M & 0 \\ * & * & * & -Z_2 & M & 0 \\ * & * & * & * & -\epsilon^* I & 0 \\ * & * & * & * & * & -\epsilon^* I \end{pmatrix} < 0, \quad r = 3, 4, \quad (7)$$

$$\begin{pmatrix} Z_1 & U_1 \\ * & Z_1 \end{pmatrix} \geq 0, \quad \begin{pmatrix} Z_2 & U_2 \\ * & Z_2 \end{pmatrix} \geq 0, \quad \begin{pmatrix} Z_3 & U_3 \\ * & Z_3 \end{pmatrix} \geq 0, \quad \begin{pmatrix} Z_4 & U_4 \\ * & Z_4 \end{pmatrix} \geq 0, \quad (8)$$

$$\begin{pmatrix} S_1 & U_5 \\ * & S_1 \end{pmatrix} \geq 0, \quad \begin{pmatrix} S_2 & U_6 \\ * & S_2 \end{pmatrix} \geq 0, \quad \begin{pmatrix} S_3 & U_7 \\ * & S_3 \end{pmatrix} \geq 0, \quad \begin{pmatrix} S_4 & U_8 \\ * & S_4 \end{pmatrix} \geq 0, \quad (9)$$

$$\begin{aligned} \lambda^{P_1} I \leq P_1 \leq \lambda_{P_1} I, \quad \lambda^{P_2} I \leq P_2 \leq \lambda_{P_2} I, \quad 0 < W_3 \leq \lambda W_3 I, \\ 0 < W_4 \leq \lambda W_4 I, \quad 0 < R_3 \leq \lambda R_3 I, \quad 0 < R_4 \leq \lambda R_4 I, \end{aligned} \quad (10)$$

$$\begin{aligned} 0 < S_{r1} \leq \lambda_{S_r} I, \quad 0 < Z_r \leq \lambda_{Z_r} I, \\ \psi_1 \eta_1 + \psi_2 \kappa + \lambda_{Q_1} u + \psi_3 \eta_2 + \psi_4 \kappa_1 + \lambda_{Q_2} v < M^* \chi \mu^{-N} \rho^{-N}, \end{aligned} \quad (11)$$

where

$$\tilde{\Pi}_r = \begin{pmatrix} \tilde{\Pi}_{1,1,i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \tilde{\Pi}_{2,2,i} & 0 & 0 & 0 & G_2 H_1 & 0 & \tilde{\Pi}_{r2} & \tilde{\Pi}_{r3} & 0 \\ * & * & \tilde{\Pi}_{3,3,i} & \tilde{\Pi}_{3,4,i} & \mu^{\sigma_{m+1}} U_1^T & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \tilde{\Pi}_{4,4,i} & \tilde{\Pi}_{4,5,i} & 0 & G_2 H_2 & 0 & 0 & 0 \\ * & * & * & * & \tilde{\Pi}_{5,5,i} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \tilde{\Pi}_{6,6,i} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \tilde{\Pi}_{7,7,i} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \tilde{\Pi}_{8,8,i} & \tilde{\Pi}_{8,9,i} & 0 \\ * & * & * & * & * & * & * & * & \tilde{\Pi}_{9,9,i} & 0 \\ * & * & * & * & * & * & * & * & * & -Q \end{pmatrix},$$

$$r = 1, 2,$$

$$\check{\Pi}_r = \begin{pmatrix} \check{\Pi}_{1,1,i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \check{\Pi}_{2,2,i} & 0 & 0 & 0 & F_4 H_3 & 0 & \check{\Pi}_{r2} & \check{\Pi}_{r3} & 0 \\ * & * & \check{\Pi}_{3,3,i} & \check{\Pi}_{3,4,i} & \rho^{\tau_{m+1}} U_3^T & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \check{\Pi}_{4,4,i} & \check{\Pi}_{4,5,i} & 0 & G_4 H_4 & 0 & 0 & 0 \\ * & * & * & * & \check{\Pi}_{5,5,i} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \check{\Pi}_{6,6,i} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \check{\Pi}_{7,7,i} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \check{\Pi}_{8,8,i} & \check{\Pi}_{8,9,i} & 0 \\ * & * & * & * & * & * & * & * & \check{\Pi}_{9,9,i} & 0 \\ * & * & * & * & * & * & * & * & * & -Q_1 \end{pmatrix},$$

$$r = 3, 4,$$

$$\begin{aligned} \tilde{\Pi}_{2,2,i} &= (1 + \hat{\sigma}) W + \hat{\sigma} (S_1 + \mu^{\sigma_{m+1}} U_6) + \frac{\hat{\sigma}}{2} S_2 - \mu^{\sigma_{m+1}} \hat{\sigma}^2 Z_2 - \mu P_{1i} - G_1 H_1, \\ \tilde{\Pi}_{12} &= \mu^{\sigma_{m+1}} \hat{\sigma} U_2, \quad \tilde{\Pi}_{13} = \mu^{\sigma_{m+1}} \hat{\sigma} Z_2, \quad \tilde{\Pi}_{3,3,i} = \mu^{\sigma_{m+1}} (U_5 - Z_1), \\ \tilde{\Pi}_{3,4,i} &= \mu^{\sigma_{m+1}} (Z_1 - U_1^T), \quad \tilde{\Pi}_{4,4,i} = \mu^{\sigma_{m+1}} \text{sym}(U_1 - Z_1) - \mu^{\sigma_m} W - G_1 H_2, \\ \tilde{\Pi}_{4,5,i} &= \mu^{\sigma_{m+1}} (Z_1 - U_1^T), \quad \tilde{\Pi}_{5,5,i} = -\mu^{\sigma_{m+1}} (Z_1 + U_5), \quad \tilde{\Pi}_{6,6,i} = (1 + \hat{\sigma}) R - H_1, \\ \tilde{\Pi}_{7,7,i} &= -\mu^{\sigma_{m+1}} R - H_2, \quad \tilde{\Pi}_{8,8,i} = -\mu^{\sigma_{m+1}} \left(Z_2 + \frac{1}{\hat{\sigma}} U_6 \right), \\ \tilde{\Pi}_{8,9,i} &= -\mu^{\sigma_{m+1}} \left(U_2^T - \frac{1}{\hat{\sigma}} U_6 \right), \quad \tilde{\Pi}_{9,9,i} = -\mu^{\sigma_{m+1}} \left(Z_2 + \frac{1}{\hat{\sigma}} U_6 \right), \\ \tilde{\Pi}_{22} &= \mu^{\sigma_{m+1}} \hat{\sigma} Z_2, \quad \tilde{\Pi}_{23} = \mu^{\sigma_{m+1}} \hat{\sigma} U_2^T, \\ \check{\Pi}_{2,2,i} &= (1 + \hat{\tau}) W_1 + \hat{\tau} (S_3 + \rho^{\tau_{m+1}} U_8) + \frac{\hat{\tau}}{2} S_4 - \rho^{\tau_{m+1}} \hat{\tau}^2 Z_4 - \rho P_{2i} - F_3 H_3, \\ \check{\Pi}_{32} &= \rho^{\tau_{m+1}} \hat{\tau} U_4, \quad \check{\Pi}_{33} = \rho^{\tau_{m+1}} \hat{\tau} Z_4, \quad \check{\Pi}_{3,3,i} = \rho^{\tau_{m+1}} (U_7 - Z_3), \\ \check{\Pi}_{3,4,i} &= \rho^{\tau_{m+1}} (Z_3 - U_3^T), \quad \check{\Pi}_{4,4,i} = \rho^{\tau_{m+1}} \text{sym}(U_3 - Z_3) - \rho^{\tau_m} W_1 - G_3 H_4, \\ \check{\Pi}_{4,5,i} &= \rho^{\tau_{m+1}} (Z_3 - U_3^T), \quad \check{\Pi}_{5,5,i} = -\rho^{\tau_{m+1}} (Z_3 + U_7), \quad \check{\Pi}_{6,6,i} = (1 + \hat{\tau}) R_1 - H_3, \\ \check{\Pi}_{7,7,i} &= -\rho^{\tau_m} R_1 - H_4, \quad \check{\Pi}_{8,8,i} = -\rho^{\tau_{m+1}} \left(Z_4 + \frac{1}{\hat{\tau}} U_8 \right), \end{aligned}$$

$$\begin{aligned}
\check{\Pi}_{8,9,i} &= -\rho^{\tau_m+1} \left(U_4^T - \frac{1}{\hat{t}} U_8 \right), & \check{\Pi}_{9,9,i} &= -\rho^{\tau_m+1} \left(Z_4 + \frac{1}{\hat{t}} U_8 \right), \\
\check{\Pi}_{42} &= \rho^{\tau_m+1} \hat{t} Z_4, & \check{\Pi}_{43} &= \rho^{\tau_m+1} \hat{t} U_4^T, \\
\check{\Pi}_1 &= [A \quad 0 \quad 0 \quad 0 \quad 0 \quad B \quad C \quad 0 \quad 0 \quad I], \\
\check{\Pi}_2 &= [A - I \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad B \quad C \quad 0 \quad 0 \quad I] \\
\check{\Pi}_3 &= [D \quad 0 \quad 0 \quad 0 \quad 0 \quad E \quad F \quad 0 \quad 0 \quad I], \\
\check{\Pi}_4 &= [D - I \quad 0 \quad 0 \quad 0 \quad 0 \quad E \quad F \quad 0 \quad 0 \quad I], \\
\check{M}_{ABC} &= [M_A \quad 0 \quad 0 \quad 0 \quad 0 \quad M_B \quad M_C \quad 0 \quad 0 \quad 0], \\
\check{M}_{DEF} &= [M_D \quad 0 \quad 0 \quad 0 \quad 0 \quad M_E \quad M_F \quad 0 \quad 0 \quad 0], \\
\hat{t} &= \tau_M - \tau_m, & \check{t} &= (\tau_M - \tau_m)(\tau_M + \tau_m + 1), \\
\hat{\sigma} &= \sigma_M - \sigma_m, & \check{\sigma} &= (\sigma_M - \sigma_m)(\sigma_M + \sigma_m + 1), \\
\hat{f} &= \max_{1 \leq i \leq n} \{ |F_i^-|, |F_i^+| \}, & \hat{g} &= \max_{1 \leq i \leq n} \{ |G_i^-|, |G_i^+| \}, \\
\psi_1 &= \lambda_{P_1} + \left(\mu^{\sigma_M-1} \sigma_M + \mu^{\sigma_M} \frac{\hat{\sigma}(\sigma_M + \sigma_m - 1)}{2} \right) (\lambda W_3 + \hat{f}^2 \lambda_{R_3}) + \lambda_{S_{11}} \mu^{\sigma_m-1} \frac{\check{\sigma}}{2} \\
&\quad + \lambda_{S_{31}} \mu^{\sigma_M-1} \left[\left(\frac{1}{12} \sigma_m(\sigma_m + 1)(2\sigma_M + 1) - \sigma_m(\sigma_m + 1)(2\sigma_m + 1) \right) - \frac{\check{\sigma}}{4} \right], \\
\psi_2 &= \lambda_{Z_1} \mu^{\sigma_M-1} \frac{\hat{\sigma} \check{\sigma}}{2} + \lambda_{Z_2} \\
&\quad + \mu^{\sigma_m-1} \check{\sigma} \left[\left(\frac{1}{24} \sigma_m(\sigma_m + 1)(2\sigma_M + 1) - \sigma_m(\sigma_m + 1)(2\sigma_m + 1) \right) - \frac{\check{\sigma}}{8} \right], \\
\psi_3 &= \lambda_{P_2} + \left(\rho^{\tau_M-1} \tau_M + \rho^{\tau_M} \frac{\hat{t}(\tau_M + \tau_m - 1)}{2} \right) (\lambda W_4 + \hat{g}^2 \lambda_{R_4}) + \lambda_{S_{21}} \mu^{\tau_m-1} \frac{\check{t}}{2} \\
&\quad + \lambda_{S_{41}} \rho^{\tau_M-1} \left[\left(\frac{1}{12} \tau_m(\tau_m + 1)(2\tau_M + 1) - \tau_m(\tau_m + 1)(2\tau_m + 1) \right) - \frac{\check{t}}{4} \right], \\
\psi_4 &= \lambda_{Z_3} \rho^{\tau_M-1} \frac{\hat{t} \check{t}}{2} + \lambda_{Z_4} \\
&\quad + \rho^{\tau_m-1} \check{t} \left[\left(\frac{1}{24} \tau_m(\tau_m + 1)(2\tau_M + 1) - \tau_m(\tau_m + 1)(2\tau_m + 1) \right) - \frac{\check{t}}{8} \right].
\end{aligned}$$

Proof To prove the required results, the following LKF for finite-time passivity BAM DNN model (1)-(4) is considered:

$$V(k) = \sum_{i=1}^5 V_i(k), \quad (12)$$

where

$$\begin{aligned}
V_{1i}(k) &= V_{11}(k) + V_{12}(k) \\
&= x^T(k) P_{1i} x(k) + y^T(k) P_{2i} y(k),
\end{aligned}$$

$$\begin{aligned}
V_{2i}(k) &= V_{21}(k) + V_{22}(k) \\
&= \sum_{i=k-\sigma(k)}^{k-1} \mu^{k-i-1} x^T(i) W x(i) + \sum_{j=-\sigma_M+1}^{-\sigma_M} \sum_{i=k+j}^{k-1} \mu^{k-i-1} x^T(i) W x(i) \\
&\quad + \sum_{i=k-\tau(k)}^{k-1} \rho^{k-i-1} y^T(i) W_1 y(i) + \sum_{j=-\tau_M+1}^{-\tau_M} \sum_{i=k+j}^{k-1} \rho^{k-i-1} y^T(i) W_1 y(i), \\
V_{3i}(k) &= V_{31}(k) + V_{32}(k) \\
&= \sum_{i=k-\sigma(k)}^{k-1} \mu^{k-i-1} g^T(x(i)) R g(x(i)) + \sum_{j=-\sigma_M+1}^{-\sigma_M} \sum_{i=k+j}^{k-1} \mu^{k-i-1} g^T(x(i)) R g(x(i)) \\
&\quad + \sum_{i=k-\tau(k)}^{k-1} \rho^{k-i-1} f^T(y(i)) R_1 f(y(i)) + \sum_{j=-\tau_M+1}^{-\tau_M} \sum_{i=k+j}^{k-1} \rho^{k-i-1} f^T(y(i)) R_1 f(y(i)), \\
V_{4i}(k) &= V_{41}(k) + V_{42}(k) \\
&= \sum_{j=-\sigma_M}^{-\sigma_{m-1}} \sum_{i=k+j}^{k-1} \mu^{k-i-1} x^T(i) S_1 x(i) + \sum_{l=-\sigma_M}^{-\sigma_{m-1}} \sum_{j=l}^{-1} \sum_{i=k+j}^{k-1} \mu^{k-i-1} x^T(i) S_2 x(i) \\
&\quad + \sum_{j=-\tau(M)}^{-\tau_{m-1}} \sum_{i=k+j}^{k-1} \rho^{k-i-1} y^T(i) S_3 y(i) + \sum_{l=-\tau_M}^{-\tau_{m-1}} \sum_{j=l}^{-1} \sum_{i=k+j}^{k-1} \rho^{k-i-1} y^T(i) S_4 y(i), \\
V_{5i}(k) &= V_{51}(k) + V_{52}(k) \\
&= \hat{\sigma} \sum_{j=-\sigma_M}^{-\sigma_{m-1}} \sum_{i=k+j}^{k-1} \mu^{k-i-1} \eta^T(i) Z_1 \eta(i) + \frac{\check{\sigma}}{2} \sum_{l=-\sigma_M}^{-\sigma_{m-1}} \sum_{j=l}^{-1} \sum_{i=k+j}^{k-1} \mu^{k-i-1} \eta^T(i) Z_2 \eta(i) \\
&\quad + \hat{\tau} \sum_{j=-\tau(M)}^{-\tau_{m-1}} \sum_{i=k+j}^{k-1} \rho^{k-i-1} \zeta^T(i) Z_3 \zeta(i) + \frac{\check{\tau}}{2} \sum_{l=-\tau_M}^{-\tau_{m-1}} \sum_{j=l}^{-1} \sum_{i=k+j}^{k-1} \rho^{k-i-1} \zeta^T(i) Z_4 \zeta(i),
\end{aligned}$$

where $\eta(k) = x(k+1) - x(k)$ and $\zeta(k) = y(k+1) - y(k)$. Calculating the forward difference of $V(k)$ by defining $\Delta V(k) = V(k+1) - V(k)$ along the solution of (1) and (3), we obtain

$$\begin{aligned}
&\Delta V_{a1}(k) - (\mu - 1)V_{a1}(k) + \Delta V_{a2}(k) - (\rho - 1)V_{a2}(k) \\
&= \sum_{i=1}^5 \left[\Delta V_{a1_i}(k) - (\mu - 1)V_{a1_i}(k) + \Delta(V_{a2_i}(k)) - (\rho - 1)V_{a2_i}(k) \right], \\
&\quad \text{where } a = 1, 2, \dots, 5, \tag{13} \\
&\Delta V_{11_i}(k) - (\mu - 1)V_{11_i}(k) + \Delta(V_{12_i}(k)) - (\rho - 1)V_{12_i}(k) \\
&= x^T(k+1)P_{1i}x(k+1) - \mu x^T P_{1i}x(k) + y^T(k+1)P_{2i}y(k+1) \\
&\quad - \rho y^T P_{2i}y(k) \\
&= [A_i(k)x(k - \gamma_1) + B_i(k)f(y(k)) + C_i f(y(k - \tau(k))) + u(k)]^T \bar{P}_{1i} \\
&\quad \times [A_i(k)x(k - \gamma_1) + B_i(k)f(y(k)) + C_i f(y(k - \tau(k))) + u(k)] \\
&\quad - \mu x^T(k)P_{1i}x(k) [D_i(k)y(k - \gamma_2) + E_i(k)g(x(k)) + F_i(k)]
\end{aligned}$$

$$\begin{aligned} & \times g(x(k - \sigma(k))) + v(k)]^T \bar{P}_{2i} [D_i(k)y(k - \gamma_2) + E_i(k)g(x(k)) \\ & + F_i(k)g(x(k - \sigma(k))) + v(k)] - \rho y_T(k) P_{2i} y(k), \end{aligned} \quad (14)$$

$$\begin{aligned} & \Delta V_{21i}(k) - (\mu - 1)V_{21i}(k) + \Delta V_{22i}(k)(\rho - 1)V_{22i}(k) \\ & \leq x^T(k)Wx(k) - \mu^{\sigma_M} x^T(k - \sigma(k))Wx(k - \sigma(k)) \\ & + \sum_{i=k+1-\sigma_M}^{k-\sigma_m} \mu^{k-i} x^T(i)Wx(i) + \hat{\sigma} x^T(k)Wx(k) - \sum_{i=k+1-\sigma_M}^{k-\sigma_m} \mu^{k-i} x^T(i)Wx(i) \\ & + y^T(k)w_1 y(k) - \rho^{\tau_M} y^T(k - \tau(k))W_1 y(k - \tau(k)) \\ & + \sum_{i=k+1-\tau_M}^{k-\tau_m} \rho^{k-i} y^T(i)W_1 y(i) + \hat{\tau} y^T(k)W_1 y(k) - \sum_{i=k+1-\tau_M}^{k-\tau_m} \rho^{k-i} y^T(i)W_1 y(i), \end{aligned} \quad (15)$$

$$\begin{aligned} & \Delta V_{31i}(k) - (\mu - 1)V_{31i}(k) + \Delta V_{32i}(k) - (\rho - 1)V_{32i}(k) \\ & \leq (1 + \hat{\sigma})g^T(x(k))Rg(x(k)) - \mu^{\sigma_M} g^T(x(k - \sigma(k)))Rg(k - \sigma(k)) \\ & + \sum_{i=k+1-\sigma_M}^{k-\sigma_m} \mu^{k-i} g^T(x(i))Rg(x(i)) - \sum_{i=k+1-\sigma_M}^{k-\sigma_m} \mu^{k-i} g^T(x(i))Rg(x(i)) \\ & + (1 + \hat{\tau})f^T(y(k))R_1 f(y(k)) - \rho^{\tau_M} f^T(y(k - \tau(k)))R_1 f(y(k - \tau(k))) \\ & + \sum_{i=k+1-\tau_M}^{k-\tau_m} \rho^{k-i} f^T(y(i))R_1 f(y(i)) - \sum_{i=k+1-\tau_M}^{k-\tau_m} \rho^{k-i} f^T(y(i))R_1 f(y(i)), \end{aligned} \quad (16)$$

$$\begin{aligned} & \Delta V_{41i}(k) - (\mu - 1)V_{41i}(k) + \Delta V_{42i}(k) - (\rho - 1)V_{42i}(k) \\ & \leq (1 + \hat{\sigma})x^T(k)S_1 x(k) \\ & - \mu^{\sigma_m+1} g^T(x(k - \sigma(k)))Rg(k - \sigma(k)) \sum_{i=k-\sigma_M}^{k-\sigma_m-1} x^T(i)S_1 x(i) \\ & + \frac{\hat{\sigma}}{2} x^T(k)S_2 x(k) - \mu^{\sigma_m+1} \sum_{l=-\sigma_M}^{-\sigma_m-1} \sum_{j=k+l}^{k-1} x^T(j)S_2 x(j) + \hat{\tau} y^T(k)S_3 y(k) \\ & - \rho^{\tau_m+1} \sum_{i=k-\tau_M}^{k-\tau_m-1} y^T(i)S_3 y(i) \frac{\hat{\tau}}{2} y^T(k)S_4 y(k) - \rho^{\tau_m+1} \sum_{l=-\sigma_M}^{-\sigma_m-1} \sum_{j=k+l}^{k-1} y^T(j)S_4 y(j), \end{aligned} \quad (17)$$

$$\begin{aligned} & \Delta V_{51i}(k) - (\mu - 1)V_{51i}(k) + \Delta V_{52i}(k) - (\rho - 1)V_{52i}(k) \\ & \leq \hat{\sigma}^2 \eta^T(k)Z_1 \eta(k) - \hat{\sigma} \mu^{\sigma_m+1} \sum_{i=k-\sigma_M}^{k-\sigma_m-1} \eta^T(i)Z_1 \eta(i) \frac{\check{\sigma}^2}{4} \eta^T(k)Z_2 \eta(k) \\ & - \frac{\check{\sigma}}{2} \mu^{\sigma_m+1} \sum_{l=-\sigma_M}^{-\sigma_m-1} \sum_{j=k+l}^{k-1} \eta^T(j)Z_2 \eta(j) + \hat{\tau}^2 \zeta^T(k)Z_3 \zeta(k) \\ & - \hat{\tau} \rho^{\tau_m+1} \sum_{j=k-\tau_M}^{k-\sigma_m-1} \zeta^T(j)Z_3 \zeta(j) \frac{\check{\tau}^2}{4} \zeta^T(k)Z_3 \zeta(k) - \frac{\check{\tau}}{2} \rho^{\tau_m+1} \sum_{l=-\sigma_M}^{-\sigma_m-1} \sum_{j=k+l}^{k-1} \zeta^T(j)Z_4 \zeta(j) \end{aligned}$$

$$\begin{aligned}
&\leq \left[(A(k) - I)x(k - \gamma_1) + B(k)f(y(k)) + c(k)f(y(k - \tau(k))) + v(k) \right] \\
&\quad - \mu^{\sigma_m+1} \hat{\sigma} \sum_{i=k-\sigma_M}^{k-\sigma_m-1} \eta^T(i) Z_1 \eta(i) - \mu^{\sigma_m+1} \frac{\check{\sigma}}{2} \sum_{l=-\sigma_M}^{j=-\sigma_m-1} \sum_{j=k+l}^{k-1} \eta^T(j) Z_2 \eta(j) \\
&\quad + \left[E(k)g(x(k)) + F(k)g(y(k - \sigma(k))) + u(k) \right] \\
&\quad - \rho^{\tau_m} \hat{\tau} \sum_{i=k-\tau_M}^{k-\sigma_m-1} \zeta^T(i) Z_3 \zeta(i) \rho^{\tau_m+1} \frac{\check{\tau}}{2} \sum_{l=-\sigma_M}^{j=-\sigma_m-1} \sum_{j=k+l}^{k-1} \zeta^T(j) Z_4 \zeta(j). \tag{18}
\end{aligned}$$

Using Lemma 2.5(i), the first summation term in ΔV_{5i} can be written as

$$\begin{aligned}
-\hat{\sigma} \sum_{s=k-\sigma_M}^{k-\sigma_m-1} \eta^T(s) Z_1 \eta(s) &= -\hat{\sigma} \sum_{s=k-\sigma_M}^{k-\sigma(k)-1} \eta^T(s) Z_1 \eta(s) - \hat{\sigma} \sum_{s=k-\sigma(k)}^{k-\sigma_m-1} \eta^T(s) Z_1 \eta(s) \\
&\leq \frac{-\hat{\sigma}}{\sigma_1} \sum_{s=k-\sigma_M}^{k-\sigma(k)-1} \eta^T(s) Z_1 \sum_{s=k-\sigma_M}^{k-\sigma(k)-1} \eta(s) \\
&\quad - \frac{\hat{\sigma}}{\sigma_2} \sum_{s=k-d(k)}^{k-\sigma(m)-1} \eta^T(s) Z_1 \sum_{s=k-\sigma(k)}^{k-\sigma_m-1} \eta(s) \\
&\leq \frac{-\hat{\sigma}}{\sigma_1} [x(k - \sigma(k)) - x(k - \sigma_M)]^T Z_1 [x(k - \sigma(k)) - x(k - \sigma_M)] \\
&\quad - \frac{\hat{\sigma}}{\sigma_2} [x(k - \sigma_m) - x(k - \sigma(k))]^T Z_1 [x(k - \sigma_m) - x(k - \sigma(k))].
\end{aligned}$$

Similarly,

$$\begin{aligned}
-\hat{\tau} \sum_{s=k-\tau_M}^{k-\tau_m-1} \zeta^T(s) Z_3 \zeta(s) \\
\leq \frac{-\hat{\tau}}{\tau_1} [y(k - \tau(k)) - y(k - \tau_M)]^T Z_3 [y(k - \tau(k)) - y(k - \tau_M)] \\
\quad - \frac{\hat{\tau}}{\tau_2} [y(k - \tau_m) - y(k - \tau(k))]^T Z_3 [y(k - \tau_m) - y(k - \tau(k))]. \tag{19}
\end{aligned}$$

Further, using Lemma 2.5(ii), the second summation term in $\Delta V_{5i}(k)$ becomes

$$\begin{aligned}
\frac{\check{\sigma}}{2} \sum_{l=-\sigma_M}^{-\sigma_m-1} \sum_{j=k+l}^{k-1} \eta^T(j) Z_2 \eta(j) &= \frac{\check{\sigma}}{2} \sum_{l=-\sigma_M}^{-\sigma(k)-1} \sum_{j=k+l}^{k-1} \eta^T(j) Z_2 \eta(j) + \frac{\check{\sigma}}{2} \sum_{l=-\sigma(k)}^{-\sigma_m-1} \sum_{j=k+l}^{k-1} \eta^T(j) Z_2 \eta(j) \\
&\geq \frac{\check{\sigma}}{\sigma_3} \left[\sum_{l=-\sigma_M}^{-\sigma(k)-1} \sum_{j=k+l}^{k-1} \eta^T(j) Z_2 \sum_{l=-\sigma(k)}^{-\sigma_m-1} \sum_{j=k+l}^{k-1} \eta(j) \right] \\
&\quad + \frac{\check{\sigma}}{\sigma_4} \left[\sum_{l=-\sigma(k)}^{-\sigma_m-1} \sum_{j=k+l}^{k-1} \eta^T(j) Z_2 \sum_{l=-\sigma(k)}^{-\sigma_m-1} \sum_{j=k+l}^{k-1} \eta(j) \right]
\end{aligned}$$

$$\begin{aligned} &\geq \frac{\check{\sigma}}{\sigma_3} \left[\sigma_1 x(k) - \sum_{i=k-\sigma_M}^{k-\sigma(k)-1} x(i) \right]^T Z_2 \left[\sigma_1 x(k) - \sum_{i=k-\sigma_M}^{k-\sigma_M-1} x(i) \right] \\ &\quad + \frac{\check{\sigma}}{\sigma_4} \left[\sigma_2 x(k) - \sum_{i=k-\sigma(k)}^{k-\sigma_M-1} x(i) \right]^T Z_2 \left[\sigma_2 x(k) - \sum_{i=k-\sigma(k)}^{k-\sigma_M-1} x(i) \right]. \quad (20) \end{aligned}$$

By reciprocally convex combination Lemma 2.6, if LMIs in (8) hold, then the following inequalities hold:

$$\begin{aligned} &\left(\begin{array}{c} \sqrt{\frac{\alpha_1}{\alpha_2}} (x^T(k-\sigma(k)) - x^T(k-\sigma_M)) \\ -\sqrt{\frac{\alpha_2}{\alpha_1}} (x^T(k-\sigma_M) - x^T(k-\sigma(k))) \end{array} \right)^T \begin{pmatrix} Z_1 & U_1 \\ * & Z_1 \end{pmatrix} \\ &\quad \times \left(\begin{array}{c} \sqrt{\frac{\alpha_1}{\alpha_2}} (x^T(k-\sigma(k)) - x^T(k-\sigma_M)) \\ -\sqrt{\frac{\alpha_2}{\alpha_1}} (x^T(k-\sigma_M) - x^T(k-\sigma(k))) \end{array} \right) \geq 0, \\ &\left(\begin{array}{c} \sqrt{\frac{\beta_1}{\beta_2}} (\sigma_1 x^T(k) - \sum_{i=k-\sigma_M}^{k-\sigma(k)-1} x^T(i)) \\ -\sqrt{\frac{\beta_2}{\beta_1}} (\sigma_2 x^T(k) - \sum_{i=k-\sigma(k)}^{k-\sigma_M-1} x^T(i)) \end{array} \right)^T \begin{pmatrix} Z_2 & U_2 \\ * & Z_2 \end{pmatrix} \\ &\quad \times \left(\begin{array}{c} \sqrt{\frac{\beta_1}{\beta_2}} (\sigma_1 x^T(k) - \sum_{i=k-\sigma_M}^{k-\sigma(k)-1} x^T(i)) \\ -\sqrt{\frac{\beta_2}{\beta_1}} (\sigma_2 x^T(k) - \sum_{i=k-\sigma(k)}^{k-\sigma_M-1} x^T(i)) \end{array} \right) \geq 0, \end{aligned}$$

where $\alpha_1 = \frac{\sigma_1}{\sigma}$, $\alpha_2 = \frac{\sigma_2}{\sigma}$, $\beta_1 = \frac{\sigma_3}{\sigma}$, $\beta_2 = \frac{\sigma_4}{\sigma}$ with $\sigma_1 = (\sigma_M - \sigma(k))$, $\sigma_2 = (\sigma(k) - \sigma_m)$, $\sigma_3 = (\sigma_M - \sigma(k))(\sigma_M + \sigma(k) + 1)$, $\sigma_4 = (\sigma(k) - \sigma_m)(\sigma(k) + \sigma_m + 1)$.

Similarly,

$$\begin{aligned} \frac{\check{\tau}}{2} \sum_{l=-\tau_M}^{-\tau_m-1} \sum_{j=k+l}^{k-1} \zeta^T(j) Z_4 \zeta(j) &= \frac{\check{\tau}}{2} \sum_{l=-\tau_M}^{-\tau(k)-1} \sum_{j=k+l}^{k-1} \zeta^T(j) Z_4 \zeta(j) + \frac{\check{\tau}}{2} \sum_{l=-\tau(k)}^{-\tau_m-1} \sum_{j=k+l}^{k-1} \zeta^T(j) Z_4 \zeta(j) \\ &\geq \frac{\check{\tau}}{\tau_3} \left[\sum_{l=-\tau_M}^{-\tau(k)-1} \sum_{j=k+l}^{k-1} \zeta^T(j) Z_4 \sum_{l=-\tau(k)}^{-\tau_m-1} \sum_{j=k+l}^{k-1} \zeta(j) \right] \\ &\quad + \frac{\check{\tau}}{\tau_4} \left[\sum_{l=-\tau(k)}^{-\tau_m-1} \sum_{j=k+l}^{k-1} \zeta^T(j) Z_4 \sum_{l=-\tau(k)}^{-\tau_m-1} \sum_{j=k+l}^{k-1} \zeta(j) \right] \\ &\geq \frac{\check{\tau}}{\tau_3} \left[\tau_1 y(k) - \sum_{i=k-\tau_M}^{k-\tau(k)-1} y(i) \right]^T Z_4 \left[\tau_1 y(k) - \sum_{i=k-\tau_M}^{k-\tau_m-1} y(i) \right] \\ &\quad + \frac{\check{\tau}}{\tau_4} \left[\tau_2 y(k) - \sum_{i=k-\tau(k)}^{k-\tau_m-1} y(i) \right]^T Z_2 \left[\tau_2 y(k) - \sum_{i=k-\tau(k)}^{k-\tau_m-1} y(i) \right]. \end{aligned}$$

By reciprocally convex combination Lemma 2.6, if LMIs in (9) hold, then the following inequalities hold:

$$\begin{aligned} &\left(\begin{array}{c} \sqrt{\frac{\alpha_3}{\alpha_4}} (y^T(k-\tau(k)) - y^T(k-\tau_M)) \\ -\sqrt{\frac{\alpha_4}{\alpha_3}} (y^T(k-\tau_M) - y^T(k-\tau(k))) \end{array} \right)^T \begin{pmatrix} Z_3 & U_3 \\ * & Z_3 \end{pmatrix} \\ &\quad \times \left(\begin{array}{c} \sqrt{\frac{\alpha_3}{\alpha_4}} (y^T(k-\tau(k)) - y^T(k-\tau_M)) \\ -\sqrt{\frac{\alpha_4}{\alpha_3}} (y^T(k-\tau_M) - y^T(k-\tau(k))) \end{array} \right) \geq 0, \end{aligned}$$

$$\begin{pmatrix} \sqrt{\frac{\beta_3}{\beta_4}}(\tau_1 y^T(k) - \sum_{i=k-\sigma_M}^{k-\sigma(k)-1} y^T(i)) \\ -\sqrt{\frac{\beta_4}{\beta_3}}(\tau_2 y^T(k) - \sum_{i=k-\sigma(k)}^{k-\sigma_m-1} y^T(i)) \end{pmatrix}^T \begin{pmatrix} Z_4 & U_4 \\ * & Z_4 \end{pmatrix} \\ \times \begin{pmatrix} \sqrt{\frac{\beta_3}{\beta_4}}(\tau_1 y^T(k) - \sum_{i=k-\sigma_M}^{k-\sigma(k)-1} y^T(i)) \\ -\sqrt{\frac{\beta_4}{\beta_3}}(\tau_2 y^T(k) - \sum_{i=k-\sigma(k)}^{k-\sigma_m-1} y^T(i)) \end{pmatrix} \geq 0,$$

where $\alpha_3 = \frac{\tau_1}{\tau}$, $\alpha_4 = \frac{\tau_2}{\tau}$, $\beta_1 = \frac{\tau_3}{\tau}$, $\beta_1 = \frac{\tau_4}{\tau}$ with $\tau_1 = (\tau_M - \tau(k))$, $\tau_2 = (\tau(k) - \tau_m)$, $\tau_3 = (\tau_M - \tau(k))(\tau_M + \tau(k) + 1)$, $\tau_4 = (\tau(k) - \tau_m)(\tau(k) + \tau_m + 1)$.

Then inequalities (19) and (20) can be rewritten as

$$-\hat{\sigma} \sum_{s=k-\sigma_M}^{k-\sigma_m-1} \eta^T(s) Z_1 \eta(s) \leq - \begin{pmatrix} x^T(k - \sigma(k)) - x^T(k - \sigma_M) \\ x^T(k - \sigma_m) - x^T(k - \sigma(k)) \end{pmatrix}^T \begin{pmatrix} Z_1 & U_1 \\ * & Z_1 \end{pmatrix} \\ \times \begin{pmatrix} x(k - \sigma(k)) - x(k - \sigma_M) \\ x(k - \sigma_m) - x(k - \sigma(k)) \end{pmatrix}, \quad (21)$$

$$-\frac{\check{\sigma}}{2} \sum_{l=-\sigma_M}^{-\sigma_m-1} \sum_{j=k+l}^{k-1} \eta^T(j) Z_2 \eta(j) \leq - \begin{pmatrix} \sigma_1 x^T(k) - \sum_{i=k-\sigma_M}^{k-\sigma(k)-1} x^T(i) \\ \sigma_2 x^T(k) - \sum_{i=k-\sigma(k)}^{k-\sigma_m-1} x^T(i) \end{pmatrix}^T \begin{pmatrix} Z_2 & U_2 \\ * & Z_2 \end{pmatrix} \\ \times \begin{pmatrix} \sigma_1 x^T(k) - \sum_{i=k-\sigma_M}^{k-\sigma(k)-1} x^T(i) \\ \sigma_2 x^T(k) - \sum_{i=k-\sigma(k)}^{k-\sigma_m-1} x^T(i) \end{pmatrix}. \quad (22)$$

Similarly,

$$-\hat{\tau} \sum_{s=k-\sigma_M}^{k-\sigma_m-1} \zeta^T(s) Z_3 \zeta(s) \leq - \begin{pmatrix} y^T(k - \tau(k)) - y^T(k - \tau_M) \\ y^T(k - \tau_m) - y^T(k - \tau(k)) \end{pmatrix}^T \begin{pmatrix} Z_3 & U_3 \\ * & Z_3 \end{pmatrix} \\ \times \begin{pmatrix} y(k - \tau(k)) - y(k - \tau_M) \\ y(k - \tau_m) - y(k - \tau(k)) \end{pmatrix}, \quad (23)$$

$$-\frac{\check{\tau}}{2} \sum_{l=-\tau_M}^{-\tau_m-1} \zeta^T(j) Z_4 \zeta(j) \leq \begin{pmatrix} \tau_1 y^T(k) - \sum_{i=k-\tau_M}^{k-\tau(k)-1} y^T(i) \\ \tau_2 y^T(k) - \sum_{i=k-\tau(k)}^{k-\tau_m-1} y^T(i) \end{pmatrix}^T \begin{pmatrix} Z_4 & U_4 \\ * & Z_4 \end{pmatrix} \\ \times \begin{pmatrix} \tau_1 y^T(k) - \sum_{i=k-\tau_M}^{k-\tau(k)-1} y^T(i) \\ \tau_2 y^T(k) - \sum_{i=k-\tau(k)}^{k-\tau_m-1} y^T(i) \end{pmatrix}. \quad (24)$$

It is noted that when $\sigma(k) = \sigma_m$ or $\sigma(k) = \sigma_M$ and $\tau(k) = \tau_m$ or $\tau(k) = \tau_M$, we have $x(k - \sigma(k)) - x(k - \sigma_m) = 0$ or $x(k - \sigma_m) - x(k - \sigma(k)) = 0$ and $y(k - \tau(k)) - y(k - \tau_M) = 0$ or $y(k - \tau_m) - y(k - \tau(k)) = 0$, respectively. So, inequalities (20) and (21) still hold.

For any matrices u_5, u_6, u_7, u_8 , the following equalities hold:

$$\mu^{\sigma_m+1} \left[x^T(k - \sigma_m) U_5 x(k - \sigma_m) - x^T(k - \sigma_M) U_5 x(k - \sigma_M) \right. \\ \left. - \sum_{s=k-\sigma_M}^{k-\sigma_m-1} [\eta^T(s) U_5 \eta(s) + 2x^T(s) U_5 \eta(s)] \right] = 0, \quad (25)$$

$$\mu^{\sigma_m+1} \left[\hat{\sigma} x^T(k) U_6 x(k - \sigma_m) - x^T(k - \sigma_M) U_5 x(k - \sigma_M) - \sum_{s=k-\sigma_M}^{k-\sigma_m-1} [\eta^T(j) U_6 \eta(j) + 2x^T(j) U_6 \eta(j)] \right] = 0, \quad (26)$$

$$\rho^{\sigma_m+1} \left[y^T(k - \sigma_m) U_7 y(k - \tau_m) - y^T(k - \tau_M) U_7 y(k - \tau_M) - \sum_{s=k-\sigma_M}^{k-\sigma_m-1} [\zeta^T(s) U_7 \zeta(s) + 2y^T(s) U_7 \zeta(s)] \right] = 0, \quad (27)$$

$$\rho^{\sigma_m+1} \left[\hat{\tau} y^T(k) U_8 y(k) - \sum_{s=k-\sigma_M}^{k-\sigma_m-1} y^T(s) U_8 x(s) - \sum_{l=-\tau_M}^{-\sigma_m-1} \sum_{j=k+l}^{k-1} [\zeta^T(j) U_8 \zeta(j) + 2y^T(j) U_8 \zeta(j)] \right] = 0. \quad (28)$$

On the other hand, from Assumption I, we have

$$(f_i(x_i(k)) - F_i^-(x_i(k))) - (f_i(x_i(k)) - F_i^+(x_i(k))) \leq 0, \\ (g_i(y_i(k)) - G_i^-(y_i(k))) - (g_i(y_i(k)) - G_i^+(y_i(k))) \leq 0,$$

which is equivalent to

$$\begin{pmatrix} x(k) \\ g(x(k)) \end{pmatrix}^T \begin{pmatrix} G_i^- G_i^+ e_i e_i^T & -\frac{-G_i^- + G_i^+}{2} e_i e_i^T \\ \frac{G_i^- + G_i^+}{2} e_i e_i^T & e_i e_i^T \end{pmatrix} \begin{pmatrix} x(k) \\ g(x(k)) \end{pmatrix}_{i=1,2,\dots,k} \leq 0$$

and

$$\begin{pmatrix} y(k) \\ f(y(k)) \end{pmatrix}^T \begin{pmatrix} F_i^- F_i^+ e_i e_i^T & -\frac{F_i^- + F_i^+}{2} e_i e_i^T \\ \frac{F_i^- + F_i^+}{2} e_i e_i^T & e_i e_i^T \end{pmatrix} \begin{pmatrix} y(k) \\ f(y(k)) \end{pmatrix}_{i=1,2,\dots,n} \leq 0,$$

where e_i denotes the unit column vector having the element 1 on its i th row and zero elsewhere. Let $H_1 = \text{diag}\{h_{11}, h_{12}, \dots, h_{1n}\}$, $H_2 = \text{diag}\{h_{21}, h_{22}, \dots, h_{2n}\}$, $H_3 = \text{diag}\{h_{31}, h_{32}, \dots, h_{3n}\}$, $H_4 = \text{diag}\{h_{41}, h_{42}, \dots, h_{4n}\}$.

Then

$$\sum_{i=0}^n h_{1i} \begin{pmatrix} x(k) \\ g(x(k)) \end{pmatrix}^T \begin{pmatrix} G_i^- G_i^+ e_i e_i^T & -\frac{-G_i^- + G_i^+}{2} e_i e_i^T \\ \frac{G_i^- + G_i^+}{2} e_i e_i^T & e_i e_i^T \end{pmatrix} \begin{pmatrix} x(k) \\ g(x(k)) \end{pmatrix} \leq 0, \quad (29) \\ \begin{pmatrix} x(k) \\ g(x(k)) \end{pmatrix}^T \begin{pmatrix} G_1 H_1 & -G_2 H_1 \\ G_1 H_1 & H_1 \end{pmatrix} \begin{pmatrix} x(k) \\ g(x(k)) \end{pmatrix} \leq 0.$$

Similarly,

$$\begin{pmatrix} y(k) \\ f(y(k)) \end{pmatrix}^T \begin{pmatrix} F_3 H_3 & -F_4 H_3 \\ G_4 H_3 & H_3 \end{pmatrix} \begin{pmatrix} y(k) \\ f(y(k)) \end{pmatrix} \leq 0. \quad (30)$$

Similarly, one can get

$$\begin{pmatrix} x(k - \sigma(k)) \\ g(x(k - \sigma(k))) \end{pmatrix}^T \begin{pmatrix} G_1 H_2 & -G_2 H_2 \\ G_2 H_2 & H_2 \end{pmatrix} \begin{pmatrix} x(k - \sigma(k)) \\ g(x(k - \sigma(k))) \end{pmatrix} \leq 0$$

and

$$\begin{pmatrix} y(k - \tau(k)) \\ f(y(k - \tau(k))) \end{pmatrix}^T \begin{pmatrix} G_3 H_4 & -G_4 H_4 \\ -G_4 H_4 & H_4 \end{pmatrix} \begin{pmatrix} y(k - \tau(k)) \\ f(y(k - \tau(k))) \end{pmatrix} \leq 0.$$

Then from (13) and adding (25)-(30) gives

$$\begin{aligned} & \Delta V_{a1}(k) - (\mu - 1)V_{a1}(k) - V^T(k)QV(k) + \Delta V_{a2}(k) - (\rho - 1)V_{a2}(k) - u^T(k)Q_1 u(k) \\ & \leq \varpi^T(k) \left[\Psi + \Psi_1^T P_{1i} \Psi_1 + \hat{\sigma}^2 \Psi_2^T Z_1 \Psi_2 + \frac{\check{\sigma}^2}{4} \Psi_2^T Z_2 \Psi_2 \right] \varpi(k) \\ & \quad - \mu^{\sigma_m+1} \left[\sum_{s=k-\sigma_M}^{k-\sigma_m-1} \varpi_1^T(s) \Psi_3 \varpi_1^T(s) - \sum_{l=-\sigma_M}^{-\sigma_m-1} \sum_{j=k+l}^{k-1} \varpi_1^T(j) \Psi_4 \varpi_1^T(j) \right] \\ & \quad + \Xi^T(k) \left[\Phi + \Phi_1^T P_{2i} \Phi_1 + \hat{\tau}^2 \Phi_2^T Z_3 \Phi_2 + \frac{\check{\tau}^2}{4} \Phi_2^T Z_4 \Phi_2 \right] \Xi(k) \\ & \quad - \rho^{\tau_m+1} \left[\sum_{s=k-\tau_M}^{k-\tau_m-1} \Xi_1^T(s) \Phi_3 \Xi_1^T(s) - \sum_{l=-\sigma_M}^{-\sigma_m-1} \sum_{j=k+l}^{k-1} \Xi_1^T(j) \Phi_4 \Xi_1^T(j) \right], \end{aligned} \quad (31)$$

where

$$\Psi = \begin{pmatrix} \psi \Pi_{1,1,i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \Psi_{2,2,i} & 0 & 0 & 0 & G_2 H_1 & 0 & \Psi_{12} & \Psi_{13} & 0 \\ * & * & \Psi_{3,3,i} & \Psi_{3,4,i} & \mu^{\sigma_m+1} U_1^T & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Psi_{4,4,i} & \Psi_{4,5,i} & 0 & G_2 H_2 & 0 & 0 & 0 \\ * & * & * & * & \Psi_{5,5,i} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Psi_{6,6,i} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Psi_{7,7,i} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Psi_{8,8,i} & \Psi_{8,9,i} & 0 \\ * & * & * & * & * & * & * & * & \Psi_{9,9,i} & 0 \\ * & * & * & * & * & * & * & * & * & -Q \end{pmatrix},$$

$$\Phi = \begin{pmatrix} \phi \Pi_{1,1,i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \Phi_{2,2,i} & 0 & 0 & 0 & F_4 H_3 & 0 & \Phi_{32} & \Phi_{33} & 0 \\ * & * & \Phi_{3,3,i} & \Phi_{3,4,i} & \rho^{\tau_m+1} U_3^T & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Phi_{4,4,i} & \Phi_{4,5,i} & 0 & G_4 H_4 & 0 & 0 & 0 \\ * & * & * & * & \Phi_{5,5,i} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Phi_{6,6,i} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Phi_{7,7,i} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Phi_{8,8,i} & \Phi_{8,9,i} & 0 \\ * & * & * & * & * & * & * & * & \Phi_{9,9,i} & 0 \\ * & * & * & * & * & * & * & * & * & -Q_1 \end{pmatrix},$$

$$\Psi_{2,2,i} = (1 + \hat{\sigma})W + \hat{\sigma}(S_1 + \mu^{\sigma_m+1}U_6) + \frac{\hat{\sigma}}{2}S_2$$

$$- \mu^{\sigma_m+1}(\sigma_1^2 Z_2 - \sigma_2^2 Z_2 - \sigma_1 \sigma_2 (U_2 + U_2^2)) - \mu P_{1i} - G_1 H_1,$$

$$\Psi_{12} = \mu^{\sigma_m+1}(\sigma_1 Z_2 + \sigma_2 U_2), \quad \Psi_{13} = \mu^{\sigma_m+1}(\sigma_2 Z_2 + \sigma_1 U_2^T),$$

$$\begin{aligned}
\Psi_{3,3,i} &= \mu^{\sigma_m+1}(U_5 - Z_1), & \Psi_{3,4,i} &= \mu^{\sigma_m+1}(Z_1 - U_1^T), \\
\Psi_{4,4,i} &= \mu^{\sigma_m+1} \text{sym}(U_1 - Z_1) - \mu^{\sigma_m} W - G_1 H_2, \\
\Psi_{4,5,i} &= \mu^{\sigma_m+1}(Z_1 - U_1^T), & \Psi_{5,5,i} &= -\mu^{\sigma_m+1}(Z_1 + U_5), & \Psi_{6,6,i} &= (1 + \hat{\sigma})R - H_1, \\
\Psi_{7,7,i} &= -\mu^{\sigma_m} R - H_2, & \Psi_{8,8,i} &= -\mu^{\sigma_m+1}\left(Z_2 + \frac{1}{\hat{\sigma}}U_6\right), \\
\Psi_{8,9,i} &= -\mu^{\sigma_m+1}\left(U_2^T - \frac{1}{\hat{\sigma}}U_6\right), & \Psi_{9,9,i} &= -\mu^{\sigma_m+1}\left(Z_2 + \frac{1}{\hat{\sigma}}U_6\right), \\
\Phi_{2,2,i} &= (1 + \hat{\tau})W_1 + \hat{\tau}(S_3 + \rho^{\tau_m+1}U_8) + \frac{\hat{\tau}}{2}S_4 \\
&\quad - \rho^{\tau_m+1}(\tau_1^2 Z_4 - \tau_2^2 Z_4 - \tau_1 \tau_2) - \rho(U_4 + U_4^2)P_{2i} - F_3 H_3, \\
\Phi_{32} &= \rho^{\tau_m+1}(\tau_1 Z_4 + \tau_2 U_4), & \Phi_{33} &= \rho^{\tau_m+1}(\tau_2 Z_4 + \tau_1 U_4^T), \\
\Phi_{3,3,i} &= \rho^{\tau_m+1}(U_7 - Z_3), & \Phi_{3,4,i} &= \rho^{\tau_m+1}(Z_3 - U_3^T), \\
\Phi_{4,4,i} &= \rho^{\tau_m+1} \text{sym}(U_3 - Z_3) - \rho^{\tau_m} W_1 - G_3 H_4, \\
\Phi_{4,5,i} &= \rho^{\tau_m+1}(Z_3 - U_3^T), & \Phi_{5,5,i} &= -\rho^{\tau_m+1}(Z_3 + U_7), & \Phi_{6,6,i} &= (1 + \hat{\tau})R_1 - H_3, \\
\Phi_{7,7,i} &= -\rho^{\tau_m} R_1 - H_4, & \Phi_{8,8,i} &= -\rho^{\tau_m+1}\left(Z_4 + \frac{1}{\hat{\tau}}U_8\right), \\
\Phi_{8,9,i} &= -\rho^{\tau_m+1}\left(U_4^T - \frac{1}{\hat{\tau}}U_8\right), & \Phi_{9,9,i} &= -\rho^{\tau_m+1}\left(Z_4 + \frac{1}{\hat{\tau}}U_8\right), \\
\Psi_1 &= [A(k) \quad 0 \quad 0 \quad 0 \quad 0 \quad B(k) \quad C(k) \quad 0 \quad 0 \quad I], \\
\Psi_2 &= [A(k) - I \quad 0 \quad 0 \quad 0 \quad 0 \quad B(k) \quad C(k) \quad 0 \quad 0 \quad I], \\
\Phi_1 &= [D(k) \quad 0 \quad 0 \quad 0 \quad 0 \quad E(k) \quad F(k) \quad 0 \quad 0 \quad I], \\
\Phi_2 &= [D(k) - I \quad 0 \quad 0 \quad 0 \quad 0 \quad E(k) \quad F(k) \quad 0 \quad 0 \quad I], \\
\varpi &= \left[x^T(k - \gamma_1) x^T(k) x^T(k - \sigma_m) x^T(k - \sigma(k)) x^T(k - \sigma_M) g^T(x(k)) g^T(x(k - \sigma(k))) \right. \\
&\quad \times \left. \sum_{i=k-\sigma_M}^{k-\sigma(k)-1} x^T(i) \sum_{i=k-\sigma(k)}^{k-\sigma_m-1} x^T(i) v^T(k) \right]^T, \\
\Xi &= \left[y^T(k - \gamma_2) y^T(k) y^T(k - \tau_m) y^T(k - \tau(k)) y^T(k - \tau_M) f^T(y(k)) f^T(y(k - \tau(k))) \right. \\
&\quad \times \left. \sum_{i=k-\tau_M}^{k-\tau(k)-1} y^T(i) \sum_{i=k-\tau(k)}^{k-\tau_m-1} y^T(i) u^T(k) \right]^T, \\
\Psi_3 &= \begin{pmatrix} S_1 & U_5 \\ * & U_5 \end{pmatrix}, & \Psi_4 &= \begin{pmatrix} S_2 & U_6 \\ * & U_6 \end{pmatrix}, & \Phi_3 &= \begin{pmatrix} S_3 & U_7 \\ * & U_7 \end{pmatrix}, & \Phi_4 &= \begin{pmatrix} S_4 & U_8 \\ * & U_8 \end{pmatrix},
\end{aligned}$$

and

$$\varpi_1 = [x^T(k) \eta^T(k)]^T, \quad \Xi_1 = [y^T(k) \zeta^T(k)]^T.$$

Next, in view of Schur complement [39], the RHS of (31) can be written as

$$\Psi + \Psi_1^T P_{1i} \Psi_1 + \hat{\sigma}^2 \Psi_2^T Z_1 \Psi_2 + \frac{\check{\sigma}^2}{4} \Psi_2^T Z_2 \Psi_2^T Z_2 \Psi_2$$

$$\leq \begin{pmatrix} \Psi & \Psi_1^T P_{1i} & \hat{\sigma} \Psi_2^T Z_1 & \frac{\check{\sigma}}{2} \Psi_2^T Z_2 \\ * & -P_{1i} & 0 & 0 \\ * & * & -Z_1 & 0 \\ * & * & * & -Z_2 \end{pmatrix}$$

Similarly, for

$$\Phi + \Phi_1^T P_{2i} \Phi_1 + \hat{\tau}^2 \Phi_2^T Z_3 \Phi_2 + \frac{\check{\tau}^2}{4} \Phi_2^T Z_4 \Phi_2$$

$$\leq \begin{pmatrix} \Phi & \Phi_1^T P_{2i} & \hat{\tau} \Phi_2^T Z_3 & \frac{\check{\tau}}{2} \Phi_2^T Z_4 \\ * & -P_{2i} & 0 & 0 \\ * & * & -Z_3 & 0 \\ * & * & * & -Z_4 \end{pmatrix}$$

Then, by using uncertainty description (2), (4) and procedure as in Lemma 2.6, we have

$$\Psi + \Psi_1^T P \Psi_1 + \hat{\sigma}^2 \Psi_2^T Z_1 \Psi_2 + \frac{\check{\sigma}^2}{4} \Psi_2^T Z_2 \Psi_2^T Z_2 \Psi_2$$

$$\leq \begin{pmatrix} \Psi & \tilde{\Pi}_1^T P_{1i} & \hat{\sigma} \tilde{\Pi}_2^T Z_1 & \frac{\check{\sigma}}{2} \tilde{\Pi}_2^T Z_2 \\ * & -P_{1i} & 0 & 0 \\ * & * & -Z_1 & 0 \\ * & * & * & -Z_2 \end{pmatrix} + \text{sym}(\tilde{M}N(k)\tilde{M}_{abc})$$

$$\leq \begin{pmatrix} \Psi & \tilde{\Pi}_1^T P_{1i} & \hat{\sigma} \tilde{\Pi}_2^T Z_1^T & \frac{\check{\sigma}}{2} \tilde{\Pi}_2^T Z_2 \\ * & -P_{1i} & 0 & 0 \\ * & * & -Z_1 & 0 \\ * & * & * & -Z_2 \end{pmatrix} + \varepsilon^{-1} \tilde{M} \tilde{M}^T + \varepsilon \tilde{M}_{abc}^T \tilde{M}_{abc}, \quad (32)$$

where

$$\tilde{M} = [0_{1,10n} \quad M^T \quad M^T \quad M^T],$$

$$\tilde{M}_{abc} = [M_A \quad 0_{1,4n} \quad M_B \quad M_C \quad 0_{1,7n}],$$

$$\Phi + \Phi_1^T P \Phi_1 + \hat{\tau}^2 \Phi_2^T Z_3 \Phi_2 + \frac{\check{\tau}^2}{4} \Phi_2^T Z_4 \Phi_2^T Z_4 \Phi_2$$

$$\leq \begin{pmatrix} \Phi & \check{\Pi}_1^T P_{2i} & \hat{\tau} \check{\Pi}_2^T Z_3 & \frac{\check{\tau}}{2} \check{\Pi}_2^T Z_4 \\ * & -P_{2i} & 0 & 0 \\ * & * & -Z_3 & 0 \\ * & * & * & -Z_4 \end{pmatrix} + \text{sym}(\check{M}N(k)\check{M}_{def})$$

$$\leq \begin{pmatrix} \Phi & \check{\Pi}_1^T P_{2i} & \hat{\tau} \check{\Pi}_2^T Z_3^T & \frac{\check{\tau}}{2} \check{\Pi}_2^T Z_4 \\ * & -P_{2i} & 0 & 0 \\ * & * & -Z_3 & 0 \\ * & * & * & -Z_4 \end{pmatrix} + \varepsilon_1^{-1} \check{M} \check{M}^T + \varepsilon_1 \check{M}_{def}^T \check{M}_{def}, \quad (33)$$

where

$$\check{M} = \begin{bmatrix} 0_{1,10n} & M^T & M^T & M^T \end{bmatrix},$$

$$\check{M}_{def} = \begin{bmatrix} M_D & 0_{1,4n} & M_E & M_F & 0_{1,7n} \end{bmatrix}.$$

Hence if LMIs ((6), (7), (8), (9) hold, it is easy to get

$$\begin{aligned} \Delta V(k) &= \Delta V_{r1}(k) + \Delta V_{r2}(k) \\ \Delta V_{r1}(k) - (\mu - 1)\Delta V_{r1} - v^T(k)Qv(k) \\ &\quad + \Delta V_{r2}(k) - (\rho - 1)\Delta V_{r2} - u^T(k)Q_1u(k) \leq 0 \\ V(k+1) - V(k) &\leq [(\mu - 1) + (\rho - 1)]V(k) + u^T(k)Q_1u(k) \\ &\quad + v^T(k)Qv(k) \\ &\leq [(\mu - 1) + (\rho - 1)]v(k) + \lambda_{Q_1}u^T(k)u(k) \\ &\quad + \lambda_Qv^T(k)v(k), \end{aligned} \quad (34)$$

where $\lambda_{Q_1} = \lambda_{\max}(Q_1)$ and $\lambda_Q = \lambda_{\max}(Q)$. Simple computation gives

$$V(k+1) \leq \mu V(k) + \rho V(k) + \lambda_{Q_1}u^T(k)u(k) + \lambda_Qv^T(k)v(k).$$

Noticing $\mu \geq 1$ and $\rho \geq 1$, it follows that

$$\begin{aligned} V(k) &\leq \mu^k V(0) + \rho^k V(0) + \lambda_{Q_1} \sum_{n=0}^{k-1} \mu^{k-n-1} u^T(n)u(n) + \lambda_Q \sum_{n=0}^{k-1} \mu^{k-n-1} v^T(n)v(n) \\ &\leq \mu^k V(0) + \rho^k V(0) + \mu^k \lambda_{Q_1} u + \mu^k \lambda_Q v. \end{aligned} \quad (35)$$

Further, from (12), we can get

$$\begin{aligned} V(0) &= x^T(0)P_{1i}x(0) + y^T P_{2i}y(0) + \sum_{i=-\sigma(0)}^{-1} \mu^{-i-1} x^T(i)Wx(i) + \sum_{j=-\sigma_M+1}^{-\sigma_M} \sum_{i=j}^{-1} \mu^{-i-1} x^T(i)Wx(i) \\ &\quad + \sum_{i=k-\tau(0)}^{-1} \rho^{-i-1} y^T(i)W_1y(i) + \sum_{j=-\tau_M+1}^{-\tau_M} \sum_{i=j}^{-1} \rho^{-i-1} y^T(i)W_1y(i) \\ &\quad + \sum_{i=k-\sigma(0)}^{-1} \mu^{-i-1} g^T(x(i))Rg(x(i)) + \sum_{j=-\sigma_M+1}^{-\sigma_M} \sum_{i=j}^{-1} \mu^{-i-1} g^T(x(i))Rg(x(i)) \\ &\quad + \sum_{i=k-\tau(0)}^{-1} \rho^{-i-1} f^T(y(i))R_1f(y(i)) + \sum_{j=-\tau_M+1}^{-\tau_M} \sum_{i=j}^{-1} \rho^{-i-1} f^T(y(i))R_1f(y(i)) \\ &\quad + \sum_{j=-\sigma_M}^{-\sigma_M-1} \sum_{i=j}^{-1} \mu^{-i-1} x^T(i)S_1x(i) + \sum_{l=-\sigma_M}^{-\sigma_M-1} \sum_{j=l}^{-1} \sum_{i=j}^{-1} \mu^{-i-1} x^T(i)S_2x(i) \\ &\quad + \sum_{j=-\tau(M)}^{-\tau_M-1} \sum_{i=j}^{-1} \rho^{-i-1} y^T(i)S_3y(i) \end{aligned}$$

$$\begin{aligned}
& + \sum_{l=-\tau_M}^{-\tau_M-1} \sum_{j=l}^{-1} \sum_{i=j}^{-1} \rho^{-i-1} y^T(i) S_4 y(i) \hat{\sigma} \sum_{j=-\sigma_M}^{-\sigma_M-1} \sum_{i=j}^{-1} \mu^{-i-1} \eta^T(i) Z_1 \eta(i) \\
& + \frac{\check{\sigma}}{2} \sum_{l=-\sigma_M}^{-\sigma_M-1} \sum_{j=l}^{-1} \sum_{i=j}^{-1} \mu^{-i-1} \eta^T(i) Z_2 \eta(i) + \hat{\tau} \sum_{j=-\tau(M)}^{-\tau_M-1} \sum_{i=j}^{-1} \rho^{-i-1} \zeta^T(i) Z_3 \zeta(i) \\
& + \frac{\check{\tau}}{2} \sum_{l=-\tau_M}^{-\tau_M-1} \sum_{i=j}^{-1} \rho^{-i-1} \zeta^T(i) Z_4 \zeta(i).
\end{aligned}$$

Letting

$$\begin{aligned}
P_1 &= L^{\frac{-1}{2}} P_{1i} L^{\frac{-1}{2}}, & P_2 &= L_1^{\frac{1}{2}} P_{2i} L_1^{\frac{-1}{2}}; & W_3 &= L^{\frac{-1}{2}} W_1 L^{\frac{-1}{2}}, & W_4 &= L_1^{\frac{-1}{2}} W_1 L_1^{\frac{-1}{2}}, \\
R_3 &= L^{\frac{-1}{2}} R L^{\frac{-1}{2}}, & R_4 &= L_1^{\frac{-1}{2}} R_1 L^{\frac{-1}{2}}, & S_{11} &= L^{\frac{-1}{2}} S_1 L^{\frac{-1}{2}}, & S_{21} &= L_1^{\frac{-1}{2}} S_2 L^{\frac{-1}{2}}, \\
S_{31} &= L_1^{\frac{-1}{2}} S_3 L_1^{\frac{-1}{2}}, & S_{41} &= L_1^{\frac{-1}{2}} S_4 L^{\frac{-1}{2}},
\end{aligned}$$

we obtain

$$\begin{aligned}
V(0) &= x^T(0) L^{\frac{1}{2}} P_1 L^{\frac{1}{2}} x(0) + y^T(0) L_1^{\frac{1}{2}} P_2 L_1^{\frac{1}{2}} y(0) + \sum_{i=-\sigma(0)}^{-1} \mu^{-i-1} x^T L^{\frac{1}{2}}(i) W_3 L^{\frac{1}{2}} x(i) \\
& + \sum_{j=-\sigma_M+1}^{-\sigma_M} \sum_{i=j}^{-1} \mu^{-i-1} x^T(i) L^{\frac{1}{2}} W_3 L^{\frac{1}{2}} x(i) + \sum_{i=-\tau(0)}^{-1} \rho^{-i-1} y^T(i) L_1^{\frac{1}{2}} W_4 L_1^{\frac{1}{2}} L_1^{\frac{1}{2}} y(i) \\
& + \sum_{j=-\tau_M+1}^{-\tau_M} \sum_{i=j}^{-1} \rho^{-i-1} y^T(i) L_1^{\frac{1}{2}} W_4 L_1^{\frac{1}{2}} y(i) + \sum_{i=k-\sigma(0)}^{-1} \mu^{-i-1} g^T(x(i)) L^{\frac{1}{2}} R_3 L^{\frac{1}{2}} g(x(i)) \\
& + \sum_{j=-\sigma_M+1}^{-\sigma_M} \sum_{i=j}^{-1} \mu^{-i-1} g^T(x(i)) L^{\frac{1}{2}} R_3 L^{\frac{1}{2}} g(x(i)) + \sum_{i=k-\tau(0)}^{-1} \rho^{-i-1} f^T(y(i)) L_1^{\frac{1}{2}} R_4 L_1^{\frac{1}{2}} f(y(i)) \\
& + \sum_{j=-\tau_M+1}^{-\tau_M} \sum_{i=j}^{-1} \rho^{-i-1} f^T(y(i)) L_1^{\frac{1}{2}} R_4 L_1^{\frac{1}{2}} f(y(i)) + \sum_{j=-\sigma_M}^{-\sigma_M-1} \sum_{i=j}^{-1} \mu^{-i-1} x^T(i) L^{\frac{1}{2}} S_{11} L^{\frac{1}{2}} x(i) \\
& + \sum_{l=-\sigma_M}^{-\sigma_M-1} \sum_{j=l}^{-1} \sum_{i=j}^{-1} \mu^{-i-1} x^T(i) L^{\frac{1}{2}} S_{21} L^{\frac{1}{2}} x(i) + \sum_{j=-\tau(M)}^{-\tau_M-1} \sum_{i=j}^{-1} \rho^{-i-1} y^T(i) L_1^{\frac{1}{2}} S_{31} L_1^{\frac{1}{2}} y(i) \\
& + \sum_{l=-\tau_M}^{-\tau_M-1} \sum_{j=l}^{-1} \sum_{i=j}^{-1} \rho^{-i-1} y^T(i) L_1^{\frac{1}{2}} S_{41} L_1^{\frac{1}{2}} y(i) + \hat{\sigma} \sum_{j=-\sigma_M}^{-\sigma_M-1} \sum_{i=j}^{-1} \mu^{-i-1} \eta^T(i) Z_1 \eta(i) \\
& + \frac{\check{\sigma}}{2} \sum_{l=-\sigma_M}^{-\sigma_M-1} \sum_{j=l}^{-1} \sum_{i=j}^{-1} \mu^{-i-1} \eta^T(i) Z_2 \eta(i) + \hat{\tau} \sum_{j=-\tau(M)}^{-\tau_M-1} \sum_{i=j}^{-1} \rho^{-i-1} \zeta^T(i) Z_3 \zeta(i) \\
& + \frac{\check{\tau}}{2} \sum_{l=-\tau_M}^{-\tau_M-1} \sum_{i=j}^{-1} \rho^{-i-1} \zeta^T(i) Z_4 \zeta(i) \\
& \leq \lambda_{P_1} x^T(0) L x(0) + \lambda_{P_2} y^T(0) L y(0) + \lambda_{W_3} \sum_{i=-\sigma(0)}^{-1} \mu^{-i-1} x^T L x(i)
\end{aligned}$$

$$\begin{aligned}
& + \lambda_{W_3} \sum_{j=-\sigma_M+1}^{-\sigma_M} \sum_{i=j}^{-1} \mu^{-i-1} x^T(i) Lx(i) + \lambda_{W_4} \sum_{i=-\tau(0)}^{-1} \rho^{-i-1} y^T(i) L_1 y(i) \\
& + \lambda_{W_4} \sum_{j=-\tau_M+1}^{-\tau_M} \sum_{i=j}^{-1} \rho^{-i-1} y^T(i) L_1 y(i) + \hat{g}^2 \lambda_{R_3} \sum_{i=k-\sigma(0)}^{-1} s \mu^{-i-1} x^T(i) Lx(i) \\
& + \hat{g}^2 \lambda_{R_3} \sum_{j=-\sigma_M+1}^{-\sigma_M} \sum_{i=j}^{-1} \mu^{-i-1} x^T(i) Lx(i) + \hat{f}^2 \lambda_{R_4} \sum_{i=k-\tau(0)}^{-1} \rho^{-i-1} y^T(i) L_1 y(i) \\
& + \hat{f}^2 \lambda_{R_4} \sum_{j=-\tau_M+1}^{-\tau_M} \sum_{i=j}^{-1} \rho^{-i-1} y^T(i) L_1 y(i) + \lambda_{S_{11}} \hat{\sigma} \sum_{j=-\sigma_M}^{-\sigma_M-1} \sum_{i=j}^{-1} \mu^{-i-1} x^T(i) x(i) \\
& + \lambda_{S_{21}} \hat{\sigma} \sum_{l=-\sigma_M}^{-\sigma_M-1} \sum_{j=l}^{-1} \sum_{i=j}^{-1} \mu^{-i-1} x^T(i) x(i) + \lambda_{S_{31}} \hat{\tau} \sum_{j=-\tau(M)}^{-\tau_M-1} \sum_{i=j}^{-1} \rho^{-i-1} y^T(i) y(i) \\
& + \lambda_{S_{41}} \hat{\tau} \sum_{l=-\tau_M}^{-\tau_M-1} \sum_{j=l}^{-1} \sum_{i=j}^{-1} \rho^{-i-1} y^T(i) y(i) + \lambda_{Z_1} \hat{\sigma} \sum_{j=-\sigma_M}^{-\sigma_M-1} \sum_{i=j}^{-1} \mu^{-i-1} \eta^T(i) \eta(i) \\
& + \lambda_{Z_2} \frac{\check{\sigma}}{2} \sum_{l=-\sigma_M}^{-\sigma_M-1} \sum_{j=l}^{-1} \sum_{i=j}^{-1} \mu^{-i-1} \eta^T(i) \eta(i) + \lambda_{Z_3} \hat{\tau} \sum_{j=-\tau(M)}^{-\tau_M-1} \sum_{i=j}^{-1} \rho^{-i-1} \zeta^T(i) \zeta(i) \\
& + \lambda_{Z_4} \frac{\check{\tau}}{2} \sum_{l=-\tau_M}^{-\tau_M-1} \sum_{i=j}^{-1} \rho^{-i-1} \zeta^T(i) \zeta(i) \\
& \leq \left[\lambda_{P_1} + \left(\mu^{\sigma_M-1} \sigma_M + \mu^{\sigma_M} \frac{\hat{\sigma} + \sigma_M - 1}{2} \right) (\lambda_{W_3} + \hat{f}^2 \lambda_{R_3}) + \lambda_{S_{11}} \mu^{\sigma_M-1} \frac{\check{\sigma}}{2} \right. \\
& \quad \left. + \lambda_{S_{21}} \mu^{\sigma_M-1} \left[\left(\frac{1}{12} \sigma_M (\sigma_M + 1) (2\sigma_M + 1) - \sigma_m (\sigma_m + 1) (2\sigma_m + 1) \right) - \frac{\check{\sigma}}{4} \right] \right] \eta_1 \\
& \quad + \left[\lambda_{P_2} + \left(\rho^{\tau_M-1} \tau_M + \rho^{\tau_M} \frac{\hat{\tau} + \tau_M - 1}{2} \right) (\lambda_{W_4} + \hat{g}^2 \lambda_{R_4}) + \lambda_{S_{31}} \rho^{\tau_M-1} \frac{\check{\tau}}{2} \right. \\
& \quad \left. + \lambda_{S_{41}} \rho^{\tau_M-1} \left[\left(\frac{1}{12} \tau_M (\tau_M + 1) (2\tau_M + 1) - \tau_m (\tau_m + 1) (2\tau_m + 1) \right) - \frac{\check{\tau}}{4} \right] \right] \eta_2 \\
& \quad + \left[\lambda_{Z_1} \mu^{\sigma_M-1} \frac{\hat{\sigma} \check{\sigma}}{2} \right. \\
& \quad \left. + \lambda_{Z_2} \mu^{\sigma_M-1} \check{\sigma} \left[\left(\frac{1}{12} \sigma_M (\sigma_M + 1) (2\sigma_M + 1) - \sigma_m (\sigma_m + 1) (2\sigma_m + 1) \right) - \frac{\check{\sigma}}{8} \right] \right] \kappa_1 \\
& \quad + \left[\lambda_{Z_3} \rho^{\tau_M-1} \frac{\hat{\tau} \check{\tau}}{2} \right. \\
& \quad \left. + \lambda_{Z_4} \rho^{\tau_M-1} \check{\tau} \left[\left(\frac{1}{12} \tau_M (\tau_M + 1) (2\tau_M + 1) - \tau_m (\tau_m + 1) (2\tau_m + 1) \right) - \frac{\check{\tau}}{8} \right] \right] \kappa_2,
\end{aligned}$$

where $\lambda_{P_1} = \lambda_{\max}(P_1)$, $\lambda_{P_2} = \lambda_{\max}(P_2)$, $\lambda^{P_1} = \lambda_{\min}(P_1)$, $\lambda^{P_2} = \lambda_{\min}(P_2)$, $\lambda_{W_3} = \lambda_{\max}(W_3)$, $\lambda_{W_4} = \lambda_{\max}(W_4)$, $\lambda_{R_3} = \lambda_{\max}(R_3)$, $\lambda_{R_4} = \lambda_{\max}(R_4)$, $\lambda_{S_1} = \lambda_{\max}(S_{11})$, $\lambda_{S_2} = \lambda_{\max}(S_{21})$, $\lambda_{S_3} = \lambda_{\max}(S_{31})$, $\lambda_{S_4} = \lambda_{\max}(S_{41})$, $\lambda_{Z_1} = \lambda_{\max}(Z_1)$, $\lambda_{Z_2} = \lambda_{\max}(Z_2)$, $\lambda_{Z_3} = \lambda_{\max}(Z_3)$, $\lambda_{Z_4} = \lambda_{\max}(Z_4)$.

On the other hand, from (12), we can obtain that

$$\begin{aligned}
 V(k) &\geq x^T(k)P_{1i}x(k) + y^T(k)P_{2i}y(k) \\
 &\geq x^T(k)L^{\frac{1}{2}}P_1L^{\frac{1}{2}}x(k) + y^T(k)L_1^{\frac{1}{2}}P_2L_1^{\frac{1}{2}}y(k) \\
 &\geq \lambda^{P_1}x^T(k)Lx(k) + y^T(k)L_1^{\frac{1}{2}}P_2L_1^{\frac{1}{2}}y(k) \\
 &\geq M^*[x^T(k)Lx(k) + y^T(k)L_1y(k)].
 \end{aligned} \tag{36}$$

Put $M^* = \min\{\lambda^{P_1}, \lambda^{P_2}\}$.

From (35) to (36), we get

$$\begin{aligned}
 x^T(k)Lx(k) + y^T(k)L_1y(k) &< \frac{(\psi_1\eta_1 + \psi_2\kappa + \lambda_{Q_1}u)\mu^k + (\psi_2\eta_2 + \psi_3\kappa_1 + \lambda_{Q_2}v)\rho^k}{M^*} \\
 &< \frac{(\psi_1\eta + \psi_2\kappa + \lambda_{Q_1}u)\mu^k + (\psi_2\eta + \psi_3\kappa_1 + \lambda_{Q_2}v)\rho^k}{M^*}.
 \end{aligned}$$

Therefore, from (11), we get that

$$x^T(k)Lx(k) + y^T(k)L_1y(k) < \chi, \quad \forall k \in \{1, 2, \dots, N\}.$$

Then, using Definition 2.2, DNN (1), (3) is robustly finite-time bounded. $\square$

Remark 3.2 Leakage time delay in the stabilizing negative feedback term has a tendency to destabilize a system. The term $x(k - \gamma_1)$, $y(k - \gamma_2)$ in system (1) and (3) corresponds to a stabilizing negative feedback of the system which acts instantaneously with time delay. The term is variously known as leakage (or forgetting) term which is considered as a time delay.

4 Robust finite-time passivity

In this subsection, we focus on the robust finite-time passivity of DNN (1), (3) with output (2), (4). In order to deal this, we introduce $I = 2g^T(k)u(k) + u^T(k)Qu(k)$ and $J = 2h^T(k)v(k) + v^T(k)Q_1v(k)$.

Theorem 4.1 Under Assumptions I and II, for given scalars $\mu > 1$, $\rho > 1$, τ_m , τ_M , σ_m , σ_M , DNN model (1), (3) is robustly finite-time passive with respect to $(\eta_1, \eta_2, \eta, \chi, L, Q, L_1, Q_1, u, v)$, if there exist symmetric positive definite matrices P_{1i} , P_{2i} , W , W_1 , R_1 , R , S_1 , S_2 , S_3 , S_4 , Z_1 , Z_2 , Z_3 , Z_4 , matrices U_1 , U_2 , U_3 , U_4 , U_5 , U_6 , U_7 , U_8 , positive diagonal matrices H_1 , H_2 , H_3 , H_4 and positive scalars λ^{P_1} , λ^{P_2} , λ_{P_1} , λ_{P_2} , λ_{W_3} , λ_{W_4} , λ_{R_3} , λ_{R_4} , λ_{S_1} , λ_{S_2} , λ_{S_3} , λ_{S_4} , λ_{Z_1} , λ_{Z_2} , λ_{Z_3} , λ_{Z_4} , ϵ , ϵ^* , $P_{1i} = P_1(r_k)$, $P_{2i} = P_2(r_k)$, $\bar{P}_{1i} = \sum_{j=1}^N \Pi_{ij}P_{1j}$, $\bar{P}_{2i} = \sum_{j=1}^N \Pi_{ij}P_{2j}$, ω , ω_1 such that the following LMIs (8), (9) hold for $r = 1, 2, 3, 4$:

$$\begin{pmatrix}
 \tilde{\Lambda}_r & \tilde{\Pi}_1^T P_{1i} & \hat{\sigma} \tilde{\Pi}_2^T Z_1 & \frac{\hat{\sigma}}{2} \tilde{\Pi}_2^T Z_2 & 0 & \epsilon \tilde{M}_{ABC}^T \\
 * & -P_{1i} & 0 & 0 & M & 0 \\
 * & * & -Z_1 & 0 & M & 0 \\
 * & * & * & -Z_2 & M & 0 \\
 * & * & * & * & -\epsilon I & 0 \\
 * & * & * & * & * & -\epsilon I
 \end{pmatrix} < 0, \quad r = 1, 2, \tag{37}$$

$$\begin{pmatrix} \check{\Lambda}_r & \check{\Pi}_1^T P_{2i} & \hat{\tau} \check{\Pi}_2^T Z_3 & \frac{\check{\tau}}{2} \check{\Pi}_2^T Z_4 & 0 & \epsilon^* \check{M}_{DEF}^T \\ * & -P_{2i} & 0 & 0 & M & 0 \\ * & * & -Z_1 & 0 & M & 0 \\ * & * & * & -Z_2 & M & 0 \\ * & * & * & * & -\epsilon^* I & 0 \\ * & * & * & * & * & -\epsilon^* I \end{pmatrix} < 0, \quad r = 3, 4, \quad (38)$$

$$\psi_1 \eta_1 + \psi_2 \kappa + \mu^{-T} u + \psi_3 \eta_2 + \psi_4 \kappa_1 + \rho^{-T_1} v < M^* \chi \mu^{-T} \rho^{-T_1}, \quad (39)$$

where

$$\check{\Lambda}_r = \begin{pmatrix} \check{\Pi}_{1,1,i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \check{\Pi}_{2,2,i} & 0 & 0 & 0 & G_2 H_1 & 0 & \check{\Pi}_{r2} & \check{\Pi}_{r3} & 0 \\ * & * & \check{\Pi}_{3,3,i} & \check{\Pi}_{3,4,i} & \mu^{\sigma_m+1} U_1^T & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \check{\Pi}_{4,4,i} & \check{\Pi}_{4,5,i} & 0 & G_2 H_2 & 0 & 0 & 0 \\ * & * & * & * & \check{\Pi}_{5,5,i} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \check{\Pi}_{6,6,i} & 0 & 0 & 0 & -B_{g^*}^T \\ * & * & * & * & * & * & \check{\Pi}_{7,7,i} & 0 & 0 & -C_{g^*}^T \\ * & * & * & * & * & * & * & \check{\Pi}_{8,8,i} & \check{\Pi}_{8,9,i} & 0 \\ * & * & * & * & * & * & * & * & \check{\Pi}_{9,9,i} & 0 \\ * & * & * & * & * & * & * & * & * & -\omega I \end{pmatrix},$$

$$r = 1, 2,$$

$$\check{\Lambda}_r = \begin{pmatrix} \check{\Pi}_{1,1,i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \check{\Pi}_{2,2,i} & 0 & 0 & 0 & F_4 H_3 & 0 & \check{\Pi}_{r2} & \check{\Pi}_{r3} & 0 \\ * & * & \check{\Pi}_{3,3,i} & \check{\Pi}_{3,4,i} & \rho^{\tau_m+1} U_3^T & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \check{\Pi}_{4,4,i} & \check{\Pi}_{4,5,i} & 0 & G_4 H_4 & 0 & 0 & 0 \\ * & * & * & * & \check{\Pi}_{5,5,i} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \check{\Pi}_{6,6,i} & 0 & 0 & 0 & -E_h^T \\ * & * & * & * & * & * & \check{\Pi}_{7,7,i} & 0 & 0 & -F_h^T \\ * & * & * & * & * & * & * & \check{\Pi}_{8,8,i} & \check{\Pi}_{8,9,i} & 0 \\ * & * & * & * & * & * & * & * & \check{\Pi}_{9,9,i} & 0 \\ * & * & * & * & * & * & * & * & * & -\omega_1 I \end{pmatrix},$$

$$r = 3, 4$$

and the parameters are defined in the above theorem.

Proof The proof is followed from the theorem above by choosing $u = \omega \mu^{-T}$ in I and $v = \omega_1 \rho^{-T_1}$ in J . Using similar lines of (34), it follows that

$$\begin{aligned} & \Delta V_{r1}(k) - (\mu - 1)V_{r1}(k) - 2g^T(k)u(k) - \omega \mu^{-T} u^T(k)u(k) \\ & + \Delta V_{r2}(k) - (\rho - 1)V_{r2}(k) - 2h^T(k)v(k) - \omega_1 \rho^{-T_1} v^T(k)v(k) \leq 0, \\ & V(k+1) - V(k) \leq (\mu - 1)V_{r1}(k) + (\rho - 1)V_{r2}(k) + 2g^T(k)u(k) + 2h^T(k)v(k) \\ & + \omega \mu^{-T} u^T(k)u(k) + \omega_1 \mu^{-T_1} v^T(k)v(k). \end{aligned}$$

By simple computation,

$$\begin{aligned} v(k) & \leq \mu^k v_{r1}(0) + \rho^k v_{r2}(0) + 2 \sum_{i=0}^{k-1} \mu^{k-i-1} g^T(i)u(i) + 2 \sum_{i=0}^{k-1} \rho^{k-i-1} h^T(i)v(i) \\ & + w \mu^{-T} \sum_{i=0}^{k-1} \mu^{k-i-1} u^T(i)u(i) + w_1 \mu^{-T_1} \sum_{i=0}^{k-1} \rho^{k-i-1} v^T(i)v(i). \end{aligned}$$

Under the zero initial condition and noticing $v(k) \geq 0$ for all $k \in \{1, 2, 3, \dots, N\}$, we have

$$\begin{aligned} 2 \sum_{i=0}^{k-1} \mu^{k-i-1} g^T(i) u(i) + 2 \sum_{i=0}^{k-1} \rho^{k-i-1} h^T(i) v(i) &\geq -\omega \mu^{-T} \sum_{i=0}^{k-1} \mu^{k-i-1} u^T(i) u(i) \\ &\quad - \omega_1 \mu^{-T_1} \sum_{i=0}^{k-1} \rho^{k-i-1} v^T(i) v(i). \end{aligned}$$

Noticing that $\mu, \rho \geq 1$, we have

$$\begin{aligned} 2 \left[\sum_{k=0}^T \mu^{T-k} g^T(k) u(k) + \sum_{k=0}^{T_1} \rho^{T_1-k} h^T(k) v(k) \right] &\geq -\omega \mu^{-T} \sum_{k=0}^T \mu^{T-k} u^T(k) u(k) \\ &\quad - \omega_1 \rho^{-T_1} \sum_{k=0}^{T_1} \rho^{T_1-k} v^T(k) v(k). \quad (40) \end{aligned}$$

Let $\gamma^* = \max\{\omega, \omega_1\}$

$$\begin{aligned} &2 \left[\sum_{k=0}^T \mu^{T-k} g^{*T}(k) u(k) + \sum_{k=0}^{T_1} \rho^{T_1-k} h^T(k) v(k) \right] \\ &\geq \gamma^* \left[-\mu^{-T} \sum_{k=0}^T \mu^{T-k} g^{*T}(k) u(k) - \rho^{-T_1} \right]. \end{aligned}$$

Therefore, from (40), it is easy to get the inequality in Definition 2.2. Hence it can be concluded that DNN model (1), (3) is robustly finite-time passive. This completes the proof. $\square$

Remark 4.2 If leakage terms γ_1 and γ_2 become zero, then the neural networks system (1)-(4) is

$$\begin{aligned} x(k+1) &= A(r(k))x(k) + B(r(k))f(y(k)) + C(r(k))f(y(k-\tau(k))) + u(k), \\ g^*(k) &= B_{g^*}(r(k))f(y(k)) + C_{g^*}f(y(k-\tau(k))), \\ x(k) &= \phi(k) \quad \text{for every } k \in [-\tau_M, 0], \\ y(k+1) &= D(r(k))y(k) + E(r(k))g(x(k)) + F(r(k))g(x(k-\sigma(k))) + v(k), \\ h(k) &= E_h(r(k))g(x(k)) + F_h g(x(k-\sigma(k))), \\ y(k) &= \psi(k) \quad \text{for every } k \in [-\sigma_M, 0]. \end{aligned}$$

5 Numerical simulation

In this section, we present one numerical example with its simulations to guarantee the superiority and validity of our theoretical results.

Example 5.1 Consider two-dimensional robust finite-time passivity Markovian jumping discrete-time BAM neural networks with (1)-(4) with $x(k) = (x_1(k), x_2(k))^T$, $y(k) = (y_1(k), y_2(k))^T$; $r(k)$ denotes right-continuous Markovian chains taking values in $\mathfrak{M} = \{1, 2\}$

with generators

$$\begin{aligned}
 A &= \begin{pmatrix} 0.05 & 0 \\ 0 & 0.04 \end{pmatrix}, & B &= \begin{pmatrix} -0.02 & 0.08 \\ 0.09 & 0.03 \end{pmatrix}, & C &= \begin{pmatrix} 0.06 & 0.03 \\ -0.08 & 0.05 \end{pmatrix}, \\
 D &= \begin{pmatrix} 0.03 & 0 \\ 0 & 0.02 \end{pmatrix}, & E &= \begin{pmatrix} 0.01 & -0.1 \\ 0.02 & 1.1 \end{pmatrix}, & F &= \begin{pmatrix} 1 & 0.03 \\ 0.03 & -0.04 \end{pmatrix}, \\
 G_1 &= \begin{pmatrix} -8 & 2 \\ 2 & 1 \end{pmatrix}, & G_2 &= \begin{pmatrix} 1 & 5 \\ 4.23 & -4.3 \end{pmatrix}, \\
 F_1 &= \begin{pmatrix} 6 & 5 \\ 5 & 0.6 \end{pmatrix}, & F_2 &= \begin{pmatrix} 5 & 0.9 \\ 2 & 8 \end{pmatrix}, \\
 G_3 &= \begin{pmatrix} -4 & 0.9 \\ 2 & 7 \end{pmatrix}, & G_4 &= \begin{pmatrix} 2 & 4 \\ 6 & -2 \end{pmatrix}, \\
 F_3 &= \begin{pmatrix} 8 & 0.9 \\ 6.67 & 0.8 \end{pmatrix}, & F_4 &= \begin{pmatrix} 1.9 & 0.7 \\ 0.9 & 2.1 \end{pmatrix}, \\
 M_A &= \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix}, & M_B &= \begin{pmatrix} 0.05 & 0 \\ 0 & 0.05 \end{pmatrix}, & M_C &= \begin{pmatrix} 0.07 & 0 \\ 0 & 0.07 \end{pmatrix}, \\
 M_D &= \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, & M_E &= \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix}, & M_F &= \begin{pmatrix} 0.8 & 0 \\ 0 & 0.8 \end{pmatrix}.
 \end{aligned}$$

Leakage delay is defined as $\gamma_1 = \gamma_2 = 5$, and the scalars are as follows: $\mu = 5$; $\rho = 9$; $\eta_1 = 9$; $\eta_2 = 30$; $\eta = 8$; $\xi = 9$; $L = 5$; $Q = 4$; $L_1 = 3$; $Q_1 = 8$; $u = 10$; $v = 30$. Lower bounds and upper bounds of finite-time passivity BAM neural networks system (1)-(4) are $\tau_M = 50$, $\tau_m = 8$, $\sigma_m = 5$, and $\sigma_M = 30$.

Now, take the activation functions as follows:

$$f_1(y(k)) = g_1(y(k)) = \begin{pmatrix} \tanh(-0.8y) \\ \tanh(-0.8y) \end{pmatrix}, \quad f_2(x(k)) = g_2(x(k)) = \begin{pmatrix} \tanh(-0.8x) \\ \tanh(-0.8x) \end{pmatrix}.$$

Now, the feasible solutions are as follows:

$$\begin{aligned}
 P_{11} &= 10^3 \times \begin{pmatrix} 2.3666 & -0.1023 \\ -0.1023 & 2.0944 \end{pmatrix}, & P_{21} &= \begin{pmatrix} 5.8034 & -0.1288 \\ -0.1288 & 6.1511 \end{pmatrix}, \\
 P_{12} &= \begin{pmatrix} 3.3526 & 0 \\ 0 & 3.3526 \end{pmatrix}, & P_{22} &= \begin{pmatrix} 6.2695 & 0 \\ 0 & 6.4310 \end{pmatrix}, \\
 \bar{P}_{11} &= \begin{pmatrix} 3.3526 & 0 \\ 0 & 3.3526 \end{pmatrix}, & \bar{P}_{21} &= \begin{pmatrix} 3.3526 & 0 \\ 0 & 3.3526 \end{pmatrix}, \\
 \bar{P}_{12} &= \begin{pmatrix} 3.3526 & 0 \\ 0 & 3.3526 \end{pmatrix}, & \bar{P}_{22} &= \begin{pmatrix} 3.3526 & 0 \\ 0 & 3.3526 \end{pmatrix}, \\
 W &= \begin{pmatrix} 4.8570 & -0.3754 \\ -0.3754 & 3.7776 \end{pmatrix}, & W_1 &= 10^5 \times \begin{pmatrix} 0.4250 & 0.0005 \\ 0.0005 & 0.3395 \end{pmatrix},
 \end{aligned}$$

$$\begin{aligned}
R &= \begin{pmatrix} 1.7080 & -0.0980 \\ -0.0980 & 1.7150 \end{pmatrix}, & R_1 &= 10^6 \begin{pmatrix} 0.1283 & -0.0021 \\ -0.0021 & 0.1222 \end{pmatrix}, \\
S_1 &= \begin{pmatrix} 240.2276 & 29.0528 \\ 29.0528 & 317.4281 \end{pmatrix}, & S_2 &= \begin{pmatrix} 448.2930 & 49.5842 \\ 49.5842 & 579.8507 \end{pmatrix}, \\
S_3 &= \begin{pmatrix} 2.1983 & 0.0489 \\ 0.0489 & 2.0885 \end{pmatrix}, & S_4 &= \begin{pmatrix} 2.6581 & 0.0369 \\ 0.0369 & 2.5750 \end{pmatrix}, \\
Z_1 &= \begin{pmatrix} 0.7935 & -0.0290 \\ -0.0290 & 0.8393 \end{pmatrix}, & Z_2 &= 10^3 \times \begin{pmatrix} 0.0487 & -0.0023 \\ -0.0023 & 0.0434 \end{pmatrix}, \\
Z_3 &= \begin{pmatrix} 0.8915 & 0.0182 \\ 0.0182 & 0.9041 \end{pmatrix}, & Z_4 &= \begin{pmatrix} 0.4589 & -0.0047 \\ -0.0047 & 0.4605 \end{pmatrix}, \\
U_1 &= \begin{pmatrix} 0.5777 & -0.0141 \\ -0.0141 & 0.6023 \end{pmatrix}, & U_2 &= \begin{pmatrix} 0.0100 & -0.0013 \\ -0.0013 & 0.0110 \end{pmatrix}, \\
U_3 &= \begin{pmatrix} 0.8905 & -0.1991 \\ -0.1991 & 0.8906 \end{pmatrix}, & U_4 &= \begin{pmatrix} 0.1132 & -0.0011 \\ -0.0011 & 0.1111 \end{pmatrix}, \\
U_5 &= \begin{pmatrix} 2.2529 & 0.0058 \\ 0.0058 & 2.4217 \end{pmatrix}, & U_6 &= \begin{pmatrix} 0.1111 & 0.0105 \\ 0.0105 & 0.1469 \end{pmatrix}, \\
U_7 &= \begin{pmatrix} 0.1740 & 0.0144 \\ 0.0144 & 0.1768 \end{pmatrix}, & U_8 &= \begin{pmatrix} 0.1214 & 0.0037 \\ 0.0037 & 0.1112 \end{pmatrix}, \\
\epsilon = \epsilon_1 &= 10^3 \times (3.3526), & H_1 &= \begin{pmatrix} 383.2021 & 0 \\ 0 & 456.5170 \end{pmatrix}, \\
H_2 &= \begin{pmatrix} 559.7824 & 0 \\ 0 & 319.0622 \end{pmatrix}, & H_3 &= \begin{pmatrix} 922.4418 & 0 \\ 0 & 841.9595 \end{pmatrix}, \\
H_4 &= \begin{pmatrix} 352.9877 & 0 \\ 0 & 400.7669 \end{pmatrix}.
\end{aligned}$$

The trajectory of finite-time passivity BAM neural networks system (1)-(4) is shown in Figure 2.

According to Theorem 4.1, we can obtain that system (1)-(4) with the above given parameters is exponentially stable. With the help of Lyapunov functions and state trajectories $x_1(k)$, $x_2(k)$, $y_1(k)$, $y_2(k)$, the above finite-time passivity BAM neural networks are depicted in Figures 1, 2 and 3.

The performance of Markovian jumping for system (1)-(4) is given in Figure 3.

6 Conclusion

Passivity result for uncertain discrete-time Markovian jumping BAM neural networks with leakage delay has been investigated. By using the Lyapunov theory together with zero inequalities, convex combination and reciprocally convex combination approaches, the finite-time boundedness and passivity are derived in terms of LMI which can be easily verified via the LMI toolbox. Leakage delay has been considered as a time-varying delay. Utilizing the reciprocal convex technique, conservatism of the proposed criteria has been

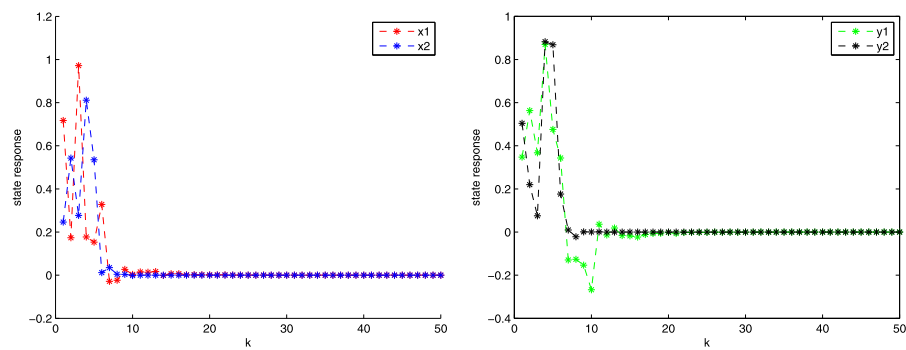


Figure 1 The state response $x(k), y(k)$ of (1)-(4) with leakage delay.

Figure 2 State trajectories of finite-time passivity BAM neural networks (1)-(4).

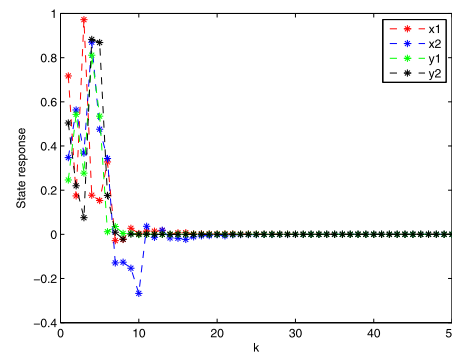


Figure 3 $r(k)$ denotes Markovian jump of (1)-(4).

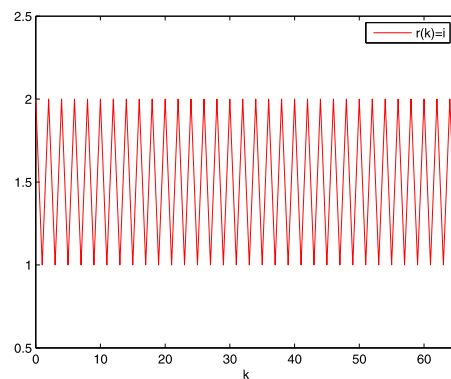


Table 1 Optimal values of χ for different τ_M and σ_M

τ_M :	12	14	16	18	20	22
σ_M :	10	12	14	16	18	20
χ	5.3	8	10.6	12	13.89	15.04

reduced significantly. A numerical example has been provided to illustrate the effectiveness of the results and their improvement over the existing results.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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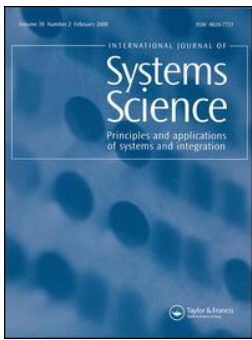
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Extended dissipativity of generalised neural networks including time delays

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ABSTRACT

This article explores the extended dissipativity conditions for generalised neural networks (GNNs) including interval time-varying delays. Extended dissipativity criteria are proposed by making proper Lyapunov–Krasovskii functional. The improved reciprocally convex combination and weighted integral inequality techniques are together applied in main results to establish the new extended dissipativity conditions of delayed GNNs. Finally, the feasibility and superiority of the proposed novel approach is clearly shown by numerical examples.

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weighted integral inequality

1. Introduction

Dynamic behaviours of neural networks (NNs) have recently been effectively studied since their useful applications in many areas, such as image processing, communication, fault diagnosis, fixed-point computations, parallel computations and industrial automation (Haykin, 1994). The applications of NNs usually depend on the behaviour and stability characteristic of the system (Qiu, Gao, & Ding, 2016b; Wang, Qiu, Yin, Gao, Fan, & Chai, 2016c; Wang, Gao, & Qiu, 2016a; Wang, Qiu, & Gao, 2016b). Thus, it is worth analysing the stability of NNs. When implementing artificial NNs, time-delayed signals often occur due to the inherent communication time between the finite switching speed of amplifiers and neurons, which can lead to divergence, oscillation and even instability. Thus, the stability problems of time-delayed NNs have become a major subject of research, and many interesting results have been proposed in recent years (Yang, Dong, Wang, Ren, & Alsaadi, 2016; Saravanakumar, Ali, Cao, & Huang, 2016b; Saravanakumar, Ali, & Karimi, 2017). In the earlier research, static NNs and local field NNs have been studied separately. To avoid such multiple works, Zhang and Han (2011) introduced a new model named as generalised neural network (GNN), which includes both static and local field NNs. A few number of stability problems for such GNNs has been studied in Zhang, He, Jiang, Wu, and Wu (2014b), Zeng, He, Wu, and Xiao (2015e), Liu, Lee, Kwon, and Park (2015) and Saravanakumar, Ali, Ahn, Karimi, and Shi (2016a).

An additional field, dissipative theory has been greatly established, which was first introduced in Fan, Taussky, and Todd (1955). In many engineering problems, stability issues are often associated with the theory of dissipative systems, which postulate that the energy dissipated inside a dynamic system is less than the one supplied from external source. Passivity, and its generalisation dissipativity, characterises the energy consumption of a system and is used in a variety of applications, e.g. electrical, mechanical, chemical and communication systems. In

most cases, passivity can be used to demonstrate that passive systems will be stable under specific criteria. Therefore, the dissipativity theory is an essential subject for system synthesis and analysis. The dissipativity theorem extends many basic system theorems, such as passivity theorem, sector-bounded nonlinearity and bounded real theorem. Dissipativity theory has presented a serious part in the control design and analysis of linear and nonlinear systems, specifically for high-order systems (Willems (1971)), because of the applied capitalisation point of view, many systems need to be dissipative to reach proficient noise extenuation (Shen, Park, Zhang, & Wu, 2014; Shen, Zhu, Zhang, & Park, 2017). In this decade, passivity and dissipativity analysis for NNs was proposed in Zhang, Xu, and Lam (2014a), Li, Zhong, Cheng, Shi, and Ren (2016b), Zeng, Park, and Shen (2015b), Thuan, Trinh, and Hien (2016), Wu, Park, Su, and Chu (2012b), Zeng, Park, and Xia (2015c), Zeng, He, Shi, Wu, and Xiao (2015a), Wu, Lam, Su, and Chu (2012a), Zeng, Park, Zhang, and Wang (2015d), Feng et al. (2011), Feng and Lam (2011). In particular, relaxed passivity analysis of delayed NNs was presented in Zhang et al. (2014a). In Zeng et al. (2015b), a new method has been proposed for robust passivity criteria for NNs with mixed delays. In Zeng et al. (2015c), improved results for dissipativity analysis of delayed NNs with randomly occurring uncertainties was established. In Zeng et al. (2015a), an improved result for dissipativity analysis of delayed NNs has been proposed. In Wu et al. (2012a), the stability and dissipativity problems of static NNs with time delays was investigated. In further, an improved stability and dissipativity conditions for static NNs including interval delays was studied in Zeng et al. (2015d).

The extended dissipativity approach was first introduced in Zhang, Zheng, and Xu (2013), which is very useful approach for delayed NNs to analyse its dynamic behaviours. The stability, passivity, dissipativity, H_∞ performance and L_2 – L_∞ performance are included in the extended dissipativity by modifying some weighting matrices via new performance index. Currently, extended dissipative approach has got great deal of

research attention (Wang, She, Zhong, & Cheng, 2016d; Lee, Park, Park, Kwon, & Lee, 2014; Wei, Li, Chen, & Tu, 2016a; Feng & Zheng, 2015). Particularly, the extended dissipativity for delayed NNs with general activation functions (Wang et al., 2016d), the extended dissipativity for time-delayed NNs (Lee et al., 2014), the extended dissipative analysis for delayed memristive NN (Wei et al., 2016a) and the extended dissipativity for discrete-time delayed NNs (Feng & Zheng, 2015) has been studied in existing literature. However, the extended dissipative criteria for GNNs with interval time-varying delays has not been thoroughly considered, which is the main inspiration of this work.

Motivated by the aforementioned analysis, the extended dissipativity analysis of delayed GNNs is considered in this work. To do so, a suitable Lyapunov–Krasovskii functional (LKF) is constructed, then by using improved RCC and weighted integral inequality (WII) techniques, desired linear matrix inequalities (LMIs) are constructed. A novel approach is utilised to obtain sufficient conditions of the extended dissipativity of considered GNNs. Finally, some examples are provided to show the feasibility and superiority of our results.

Notations: $\mathbb{R}^n$ denotes the n -dimensional Euclidean space, and $\mathbb{R}^{n \times n}$ denotes the set of all $n \times n$ real matrices. $*$ denotes the term that is induced by symmetry. X^T denotes the matrix transpose of X . $Y > 0$ denotes that Y is a real symmetric positive definite matrix. I denotes the identity matrix with appropriate dimensions and the Kronecker product is denoted by $\otimes$.

2. Preliminaries

Consider the delayed GNNs:

$$\begin{aligned} \dot{m}(t) &= -Dm(t) + V_0 f(V_2 m(t)) + V_1 f(V_2 m(t - \vartheta(t))) + z(t), \\ \eta(t) &= Bm(t), \end{aligned} \quad (1)$$

where $m(t) = [m_1(t), m_2(t), \dots, m_n(t)]^T \in \mathbb{R}^n$ is the neuron state vectors. $z(t) \in \mathbb{R}^n$ is the inputs belonging to $L_2[0, \infty)$, $\eta(t)$ is the output of the NN. $f(V_2 m(t)) = [f(V_2 m_1(t)), \dots, f(V_2 m_n(t))]^T \in \mathbb{R}^n$ represents the neuron activation function vectors. $D = \text{diag}\{d_1, \dots, d_n\}$ is a positive diagonal matrix $d_j > 0, j = 1, 2, \dots, n$. V_0, V_1 , and V_2 are the connection weighting matrices. $\vartheta(t)$ is time-varying delayed signal and satisfies

$$0 < \vartheta_1 \leq \vartheta(t) \leq \vartheta_2, \quad \dot{\vartheta}(t) \leq \ell, \quad (2)$$

where $\vartheta_1, \vartheta_2, \ell$ are positive constants.

Assumption (A1) (Liu, Wang, & Liu, 2006): $f_j(t)$ ($j = 1, 2, \dots, n$) is continuous and bounded function, and satisfies:

$$\bar{z}_j^- \leq \frac{f_j(V_{2j}a) - f_j(V_{2j}b)}{V_{2j}a - V_{2j}b} \leq \bar{z}_j^+, \quad (3)$$

where $f_j(0) = 0$, $Z_1 = \text{diag}(\bar{z}_1^-, \bar{z}_2^-, \dots, \bar{z}_n^-)$, and $Z_2 = \text{diag}(\bar{z}_1^+, \bar{z}_2^+, \dots, \bar{z}_n^+)$ are some constants and $a, b \in \mathbb{R}$, $a \neq b$.

Lemma 2.1 (Hien & Trinh, 2016): Given scalar $\alpha > 0$, matrix $R > 0$ with $n \times n$ dimensions, and function $\phi \in C([a, b], \mathbb{R}^n)$,

satisfies the following relation:

$$\begin{aligned} \int_a^b e^{\alpha(s-b)} \phi^T(s) R \phi(s) ds &\geq \frac{\alpha}{\bar{\gamma}_0} \left(\int_a^b \phi(s) ds \right)^T R \left(\int_a^b \phi(s) ds \right) \\ &\quad + \frac{\alpha}{\bar{\rho}_0} \zeta^T (K_1^T K_1 \otimes R) \zeta \end{aligned}$$

where

$$\begin{aligned} \bar{\rho}_0 &= \frac{\bar{\gamma}_0}{\bar{\gamma}_1^2} [\bar{\gamma}_0^2 - (\alpha l)^2 e^{\alpha l}], \quad \bar{\gamma}_0 \\ &= e^{\alpha l} - 1, \quad \bar{\gamma}_1 = \bar{\gamma}_0 - \alpha l, \quad l = b - a, \\ \zeta &= \left[\left(\int_a^b \phi(s) ds \right)^T \left(\int_a^b \int_s^b \phi(u) du ds \right)^T \right]^T, \\ K_1 &= [1 - \alpha \bar{\gamma}_0 / \bar{\gamma}_1]. \end{aligned}$$

Remark 2.1: Based on the exponential function for Taylor series, the following facts can be found:

$$\lim_{\alpha \rightarrow 0^+} \frac{\alpha}{\bar{\rho}_0} = \frac{3}{l}, \quad \lim_{\alpha \rightarrow 0^+} K_1 = \bar{K}_1 = [1 - \frac{2}{l}].$$

Hence, we can obtain the following results when α tends to zero

$$\int_a^b \phi^T(s) R \phi(s) ds \geq \frac{1}{l} \zeta^T (K_0^T K_0 + 3\bar{K}_1^T \bar{K}_1) \otimes R \zeta, \quad (4)$$

where $K_0 = [1 \quad 0]$.

Lemma 2.2 (Improved RCC [Seuret, Gouaisbaut, & Fridman, 2013]): Given matrices $R_1 \in \mathbb{R}_{>0}^{n \times n}$, $R_2 \in \mathbb{R}_{>0}^{r \times r}$, if there exists an appropriately dimensioned matrix $X \in \mathbb{R}^{n \times r}$ satisfying $\begin{bmatrix} R_1 & X \\ * & R_2 \end{bmatrix} \geq 0$ such that the following inequality

$$\begin{bmatrix} \frac{1}{\delta} R_1 & 0 \\ * & \frac{1}{1-\delta} R_2 \end{bmatrix} \geq \begin{bmatrix} R_1 & X \\ * & R_2 \end{bmatrix}$$

holds for all $\delta \in (0, 1)$.

Assumption (A2): For given symmetric matrices Y_1, Y_2, Y_3 , and Y_4 , such that

- (a) $Y_1 \leq 0, Y_3 > 0$, and $Y_4 \geq 0$,
- (b) $(\|Y_1\| + \|Y_2\|) \cdot \|Y_4\| = 0$, are holds.

Definition 2.1: Zhang et al. (2013) For given matrices Y_1, Y_2, Y_3 and Y_4 , satisfying assumption (A2), MJGNN (1) attains the extended dissipativity if for any $t_f \geq 0$ and all $z(t) \in L_2[0, \infty)$, under the zero initial state, the inequality:

$$\int_0^{t_f} J(t) dt \geq \sup_{0 \leq t \leq t_f} \eta^T(t) Y_4 \eta(t), \quad (5)$$

holds, where $J(t) = \eta^T(t) Y_1 \eta(t) + 2\eta^T(t) Y_2 z(t) + z^T(t) Y_3 z(t)$.

3. Main results

We introduce the following definitions for simplicity:

$$\begin{aligned}\eta(t) &\triangleq \left[m^T(t) \int_{t-\vartheta_1}^t m^T(s) ds \int_{t-\vartheta_2}^{t-\vartheta_1} m^T(s) ds \right]^T, \\ w_r &\triangleq [0_{n \times (r-1)n} \ I_n \ 0_{n \times (10-r)n}], \quad r = 1, 2, \dots, 10, \\ \xi(t) &\triangleq \left[m^T(t) m^T(t - \vartheta_1) m^T(t - \vartheta(t)) \right. \\ &\quad \times m^T(t - \vartheta_2) f(V_2 m(t)) f^T(V_2 m(t - \vartheta(t))) \frac{1}{\vartheta_1} \\ &\quad \times \int_{t-\vartheta_1}^t m^T(s) ds \frac{1}{\vartheta(t) - \vartheta_1} \int_{t-\vartheta(t)}^{t-\vartheta_1} m^T(s) ds \frac{1}{\vartheta_2 - \vartheta(t)} \\ &\quad \left. \times \int_{t-\vartheta_2}^{t-\vartheta(t)} m^T(s) ds \dot{m}^T(t) \right]^T.\end{aligned}$$

Theorem 3.1: Given scalars $0 < \vartheta_1 < \vartheta_2$, $\ell > 0$, the system (1) is asymptotically stable when there exist positive definite matrices $P \in \mathbb{R}^{3n \times 3n}$, $Q_1, Q_2, Q_3, \mathcal{R}, \mathcal{S} \in \mathbb{R}^{n \times n}$, $\Gamma_p = \text{diag}\{\gamma_{p1}, \gamma_{p2}, \dots, \gamma_{pn}\} \in \mathbb{R}^{n \times n}$, ($p = 1, 2, 3$), $\Lambda_q = \text{diag}\{\Lambda_{q1}, \Lambda_{q2}, \dots, \Lambda_{qn}\} \in \mathbb{R}^{n \times n}$ ($q = 1, 2$), and any matrices $W_q \in \mathbb{R}^{n \times n}$ ($q = 1, 2$) and $Y \in \mathbb{R}^{2n \times 2n}$ satisfying

$$\begin{aligned}\Sigma(\vartheta) + \Xi &< 0, \\ \bar{U} = \begin{bmatrix} \tilde{\mathcal{S}} & Y \\ 0 & \tilde{\mathcal{S}} \end{bmatrix} &\geq 0,\end{aligned}\quad (6)$$

where

$$\begin{aligned}\Sigma(\vartheta) &= \Sigma_0(\vartheta) + \sum_{z=1}^5 \Sigma_z - \Phi^T \bar{U} \Phi, \quad \Sigma_0(\vartheta) \\ &= \Omega^T(\vartheta) P \Omega_0 + \Omega_0^T P \Omega(\vartheta), \\ \Omega(\vartheta) &= \begin{bmatrix} w_1 \\ \vartheta_1 w_7 \\ (\vartheta(t) - \vartheta_1) w_8 + (\vartheta_2 - \vartheta(t)) w_9 \end{bmatrix}, \\ \Omega_0 &= \begin{bmatrix} w_{10} \\ w_1 - w_2 \\ w_2 - w_4 \end{bmatrix}, \quad \Omega_1 = \begin{bmatrix} w_1 - w_2 \\ w_1 + w_2 - 2w_7 \end{bmatrix}, \\ \Omega_2 &= \begin{bmatrix} w_2 - w_3 \\ w_2 + w_3 - 2w_8 \end{bmatrix}, \quad \Omega_3 = \begin{bmatrix} w_3 - w_4 \\ w_3 + w_4 - 2w_9 \end{bmatrix}, \\ \Sigma_1 &= [2[w_5 - Z_1 V_2 w_1]^T \Lambda_1 w_{10} + 2[Z_2 V_2 w_1 - w_5]^T \Lambda_2 w_{10}], \\ \Sigma_2 &= w_1^T Q_1 w_1 - w_2^T Q_1 w_2 \\ &\quad + w_2^T Q_2 w_2 - w_4^T Q_2 w_4 + w_5^T Q_3 w_5 - (1 - \ell) w_6^T Q_3 w_6, \\ \Sigma_3 &= w_{10}^T [\vartheta_1^2 \mathcal{R} + \vartheta_{21}^2 \mathcal{S}] w_{10}, \\ \Sigma_4 &= -\Omega_1^T \tilde{\mathcal{R}} \Omega_1, \quad \tilde{\mathcal{R}} = \text{diag}\{\mathcal{R}, 3\mathcal{R}\}, \\ \Sigma_5 &= -2w_1^T V_2^T Z_1 \Gamma_1 Z_2 V_2 w_1 + w_1^T V_2^T (Z_1 + Z_2) \Gamma_1 w_5 \\ &\quad + w_5^T \Gamma_1 (Z_1 + Z_2) V_2 w_1 - 2w_5^T \Gamma_1 w_5 \\ &\quad - 2w_2^T V_2^T Z_1 \Gamma_2 Z_2 V_2 w_2 + w_2^T V_2^T (Z_1 + Z_2) \Gamma_2 w_6 \\ &\quad + w_6^T \Gamma_2 (Z_1 + Z_2) V_2 w_2 - 2w_6^T \Gamma_2 w_6 \\ &\quad - 2[w_5 - w_6 - Z_1 V_2 (w_1 - w_2)]^T \Gamma_3 [w_5 \\ &\quad - w_6 - Z_2 V_2 (w_1 - w_2)], \quad \Phi = [\Omega_2^T \ \Omega_3^T]^T, \\ \tilde{\mathcal{S}} &= \text{diag}\{\mathcal{S}, 3\mathcal{S}\}, \quad \Xi = 2(w_1^T W_1 + w_{10}^T W_2) \\ &\quad \times (-Dw_1 + V_0 w_5 + V_1 w_6 - w_{10}), \quad \vartheta_{21} = \vartheta_2 - \vartheta_1.\end{aligned}$$

Proof: Choose the LKF candidate:

$$\begin{aligned}V(m_t) &= \eta^T(t) P \eta(t) + 2 \sum_{j=1}^n \lambda_{1j} \int_0^{V_{2j} m(t)} (f_j(s) - \bar{z}_j^- s) ds \\ &\quad + 2 \sum_{j=1}^n \lambda_{2j} \int_0^{V_{2j} m(t)} (\bar{z}_j^+ s - f_j(s)) ds \\ &\quad + \int_{t-\vartheta_1}^t m^T(s) Q_1 m(s) ds + \int_{t-\vartheta_2}^{t-\vartheta_1} m^T(s) Q_2 m(s) ds \\ &\quad + \int_{t-\vartheta(t)}^t f^T(V_2 m(s)) Q_3 f(V_2 m(s)) ds \\ &\quad + \vartheta_1 \int_{-\vartheta_1}^0 \int_{t+s}^t \dot{m}^T(\theta) \mathcal{R} \dot{m}(\theta) d\theta ds \\ &\quad + \vartheta_{21} \int_{-\vartheta_2}^{-\vartheta_1} \int_{t+s}^t \dot{m}^T(\theta) \mathcal{S} \dot{m}(\theta) d\theta ds.\end{aligned}\quad (7)$$

Now, we obtain the time derivative of $V(m_t)$,

$$\begin{aligned}\dot{V}(m_t) &\leq \xi^T(t) \Sigma_0(\vartheta) \xi(t) + \xi^T [2[w_5 - Z_1 V_2 w_1]^T \Lambda_1 w_{10} \\ &\quad + 2[Z_2 V_2 w_1 - w_5]^T \Lambda_2 w_{10}] \xi(t) \\ &\quad + \xi^T [w_1^T Q_1 w_1 - w_2^T Q_1 w_2 + w_2^T Q_2 w_2 \\ &\quad - w_4^T Q_2 w_4 + w_5^T Q_3 w_5 - (1 - \ell) w_6^T Q_3 w_6] \xi(t) \\ &\quad + \xi^T(t) w_{10}^T [\vartheta_1^2 \mathcal{R} + \vartheta_{21}^2 \mathcal{S}] w_{10} \xi(t) \\ &\quad - \vartheta_1 \int_{t-\vartheta_1}^t \dot{m}^T(s) \mathcal{R} \dot{m}(s) ds \\ &\quad - \vartheta_{21} \int_{t-\vartheta_2}^{t-\vartheta_1} \dot{m}^T(s) \mathcal{S} \dot{m}(s) ds \\ &\quad - \vartheta_{21} \int_{t-\vartheta(t)}^{t-\vartheta_1} \dot{m}^T(s) \mathcal{S} \dot{m}(s) ds.\end{aligned}\quad (8)$$

Using the inequality in (4), we obtain

$$-\vartheta_1 \int_{t-\vartheta_1}^t e^{\alpha(s-t)} \dot{m}^T(s) \mathcal{R} \dot{m}(s) ds \leq \xi^T(t) \Sigma_4 \xi(t).$$

Again using the inequality (4) and Lemma 2.2, we obtain

$$\begin{aligned}&-\vartheta_{21} \int_{t-\vartheta_2}^{t-\vartheta_1} \dot{m}^T(s) \mathcal{S} \dot{m}(s) ds - \vartheta_{21} \int_{t-\vartheta(t)}^{t-\vartheta_1} \dot{m}^T(s) \mathcal{S} \dot{m}(s) ds \\ &\leq -\frac{\vartheta_{21}}{\vartheta(t) - \vartheta_1} \zeta_1^T(t) \tilde{\mathcal{S}} \zeta_1(t) - \frac{\vartheta_{21}}{\vartheta_2 - \vartheta(t)} \zeta_2^T(t) \tilde{\mathcal{S}} \zeta_2(t) \\ &\leq -\begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}^T \begin{bmatrix} \frac{\vartheta_{21}}{\vartheta(t) - \vartheta_1} \tilde{\mathcal{S}} & 0 \\ 0 & \frac{\vartheta_{21}}{\vartheta_2 - \vartheta(t)} \tilde{\mathcal{S}} \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} \\ &\leq -\begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}^T \begin{bmatrix} \tilde{\mathcal{S}} & Y \\ 0 & \tilde{\mathcal{S}} \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}\end{aligned}\quad (9)$$

$$\leq -\xi^T(t) \Phi^T \bar{U} \Phi \xi(t), \quad (10)$$

where $\zeta_1(t) = \Omega_2 \xi(t)$, $\zeta_2(t) = \Omega_3 \xi(t)$. From (A1), we have

$$0 \leq 2[f(V_2 m(t)) - Z_1 V_2 m(t)]^T \Gamma_1 [Z_2 V_2 m(t) - f(V_2 m(t))], \quad (11)$$

$$0 \leq 2[f(V_2 m(t - \vartheta(t))) - Z_1 V_2 m(t - \vartheta(t))]^T \Gamma_2 \times [Z_2 V_2 m(t - \vartheta(t)) - f(V_2 m(t - \vartheta(t)))], \quad (12)$$

$$0 \leq 2[f(V_2 m(t)) - f(V_2 m(t - \vartheta(t)))]^T \Gamma_3 [f(V_2 m(t)) - f(V_2 m(t - \vartheta(t)))] - Z_2 (V_2 m(t) - V_2 m(t - \vartheta(t))). \quad (13)$$

Choose W_1, W_2 are some appropriate dimensioned relaxation matrices, then we have

$$2(m^T(t)W_1 + \dot{m}(t)^T W_2)[-Dm(t) + V_0 f(V_2 m(t)) + V_1 f(V_2 m(t - \vartheta(t)) - \dot{m}(t))] = 0. \quad (14)$$

Combining (8) to (14), we obtain

$$\dot{V}(m_t) \leq \xi^T(t)[\Sigma(\vartheta) + \Xi]\xi(t), \quad (15)$$

where $\Sigma(\vartheta)$ and Ξ are defined in (6). If $\Sigma(\vartheta) + \Xi < 0$ then $\dot{V}(m_t) < 0$. Hence, the considered GNN is asymptotically stable when $z(t) = 0$. ■

4. Extended dissipative criteria

In this section, the extended dissipative criteria will be established for all nonzero $z(t) \in L_2[0, \infty)$ by assuming the initial conditions for zero.

Theorem 4.1: Given scalars $0 < \vartheta_1 < \vartheta_2$, $\ell > 0$, GNN (1) is asymptotically stable and extended dissipative, when there exist positive definite matrices $P \in \mathbb{R}^{3n \times 3n}$, $Q_1, Q_2, Q_3, R, S \in \mathbb{R}^{n \times n}$, $\Gamma_p = \text{diag}\{\gamma_{p1}, \gamma_{p2}, \dots, \gamma_{pn}\} \in \mathbb{R}^{n \times n}$, ($p = 1, 2, 3$), $\Lambda_q = \text{diag}\{\Lambda_{q1}, \Lambda_{q2}, \dots, \Lambda_{qn}\} \in \mathbb{R}^{n \times n}$ ($q = 1, 2$), and any matrices $W_q \in \mathbb{R}^{n \times n}$ ($q = 1, 2$) and $Y \in \mathbb{R}^{2n \times 2n}$, such that

$$\tilde{\Sigma}(\vartheta) + \tilde{\Xi} < 0, \quad (16)$$

$$P - B^T \Upsilon_4 B \geq 0, \quad (17)$$

$$\tilde{\mathcal{U}} = \begin{bmatrix} \tilde{\mathcal{S}} & Y \\ 0 & \tilde{\mathcal{S}} \end{bmatrix} \geq 0,$$

where

$$\begin{aligned} \tilde{\Sigma}(\vartheta) &= \begin{bmatrix} \Sigma(\vartheta) - e_1^T B^T \Upsilon_1 B e_1 & \bar{\Sigma} \\ * & -\Upsilon_3 \end{bmatrix}, \\ \bar{\Sigma} &= [-\Upsilon_2^T B \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ \tilde{\Xi} &= 2(w_1^T W_1 + w_{10}^T W_2)(-Dw_1 + V_0 w_5 + V_1 w_6 - w_{10} + w_{11}). \end{aligned}$$

Proof: From Theorem 3.1, $\dot{V}(t) \leq \xi^T(t)[\Sigma(\vartheta) + \Xi]\xi(t)$, one can see that

$$\hat{\xi}^T(t)[\tilde{\Sigma}(\vartheta) + \tilde{\Xi}]\hat{\xi}(t) = \xi^T(t)[\Sigma(\vartheta) + \Xi]\xi(t) - J(t),$$

where $\hat{\xi}^T(t) = [\xi^T(t) \ z^T(t)]^T$, and $J(t)$ is defined in definition 2.1. From LMI (16), we get

$$\dot{V}(t) \leq \hat{\xi}^T(t)[\tilde{\Sigma}(\vartheta) + \tilde{\Xi}]\hat{\xi}(t) + J(t) \leq J(t), \quad (18)$$

Integrating both sides of (18) from 0 to $t \geq 0$, we get

$$\int_0^t J(s)ds \geq V(t) - V(0) \geq m^T(t)Pm(t). \quad (19)$$

Consider $\Upsilon_4 = 0$ and $\Upsilon_4 > 0$ as two separate cases. When $\Upsilon_4 = 0$ the extended dissipativity approach can be represented as passivity, strictly ($\Upsilon_1, \Upsilon_2, \Upsilon_3$) - γ dissipativity, passivity, H_∞ performance. When $\Upsilon_4 > 0$, L_2 - L_∞ , all criteria can be considered. First, consider that $\Upsilon_4 = 0$. From (19), we have

$$\int_0^{t_f} J(s)ds \geq 0. \quad (20)$$

Thus, Assumption (A2) satisfies with $\Upsilon_4 = 0$. Now, consider $\Upsilon_4 > 0$. From assumption (A2), we get $\Upsilon_1 = 0, \Upsilon_2 = 0, \Upsilon_3 > 0$. For any $0 \leq t \leq t_f$, (19) gives $\int_0^{t_f} J(s)ds \geq \int_0^t J(s)ds \geq m^T(t)Pm(t)$. Hence, from (17), we get

$$\begin{aligned} \eta^T(t)\Upsilon_4\eta(t) &= m^T(t)B^T\Upsilon_4Bm(t) \\ &\leq \eta^T(t) \begin{bmatrix} B^T\Upsilon_4B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \eta(t) \\ &\leq \eta^T(t)P\eta(t) \leq \int_0^{t_f} J(s)ds. \end{aligned} \quad (21)$$

From Definition 2.1, (20), and (21), the GNN (1) is extended dissipative. ■

Remark 4.1: To compare our results to some existing approaches, let us consider the following system:

$$\begin{aligned} \dot{m}(t) &= -Dm(t) + V_0 f(V_2 m(t)) \\ &\quad + V_1 f(V_2 m(t - \vartheta(t)) + z(t), \\ \eta(t) &= f(V_2 m(t)), \end{aligned} \quad (22)$$

Then using the similar method in Theorem 4.1 to GNNs (22) instead of GNNs (1), we can obtain the following Corollary for the case $\Upsilon_4 = 0$.

Corollary 4.1: Given scalars $0 < \vartheta_1 < \vartheta_2$, $\ell > 0$, GNN (22) is asymptotically stable and dissipative, when there exist positive definite matrices $P \in \mathbb{R}^{3n \times 3n}$, $Q_1, Q_2, Q_3, R, S \in \mathbb{R}^{n \times n}$, $\Gamma_p = \text{diag}\{\gamma_{p1}, \gamma_{p2}, \dots, \gamma_{pn}\} \in \mathbb{R}^{n \times n}$, ($p = 1, 2, 3$), $\Lambda_q = \text{diag}\{\Lambda_{q1}, \Lambda_{q2}, \dots, \Lambda_{qn}\} \in \mathbb{R}^{n \times n}$ ($q = 1, 2$) and any matrices $W_q \in \mathbb{R}^{n \times n}$ ($q = 1, 2$) and $Y \in \mathbb{R}^{2n \times 2n}$, such that

$$\hat{\Sigma}(\vartheta) + \tilde{\Xi} < 0, \quad (23)$$

$$\tilde{\mathcal{U}} = \begin{bmatrix} \tilde{\mathcal{S}} & Y \\ 0 & \tilde{\mathcal{S}} \end{bmatrix} \geq 0,$$

where

$$\widehat{\Sigma}(\vartheta) = \begin{bmatrix} \Sigma(\vartheta) - e_5^T \Upsilon_1 e_5 & \underline{\Sigma} \\ * & -\Upsilon_3 \end{bmatrix},$$

$$\underline{\Sigma} = [0 \ 0 \ 0 \ 0 \ -\Upsilon_2^T \ 0 \ 0 \ 0 \ 0]^T,$$

and $\widetilde{\Xi}$ is defined in [Theorem 4.1](#).

Remark 4.2: The extended dissipativity generalises the stability, passivity, dissipativity, H_∞ performance and L_2 - L_∞ performance, in a combined structure. In particular, the extended dissipativity reduces to passivity criteria by taking $\Upsilon_1 = 0$, $\Upsilon_2 = I_n$, $\Upsilon_3 = \gamma I_n$ and $\Upsilon_4 = 0$. When $\Upsilon_1 = I_n$, $\Upsilon_2 = I_n$, $\Upsilon_3 = \gamma I_n$ and $\Upsilon_4 = \Upsilon_{42} = 0$ gives the dissipativity conditions by taking. When $\Upsilon_3 = \mathbb{R} - \gamma I_3$, the performance reduces to $(\Upsilon_1, \Upsilon_2, \Upsilon_3) - \gamma$ -dissipative. When $\Upsilon_1 = -I_n$, $\Upsilon_2 = 0$, $\Upsilon_3 = \gamma^2 I_n$ and $\Upsilon_4 = 0$ gives the H_∞ performance analysis. When $\Upsilon_1 = 0$, $\Upsilon_2 = 0$, $\Upsilon_3 = \gamma^2 I_n$ and $\Upsilon_4 = I_n$ gives L_2 - L_∞ performance analysis.

Remark 4.3: As stated in the [remark 4.2](#), by applying $\Upsilon_3 = \mathbb{R} - \gamma I_3$ in (5), we can obtain $(\Upsilon_1, \Upsilon_2, \Upsilon_3) - \gamma$ -dissipative. γ is also an adjustable parameter to determine the strict dissipativity of the GNN (1) (see Feng, Lam, & Gao, 2011; Feng & Lam, 2011). The GNN (1) is strictly $(\Upsilon_1, \Upsilon_2, \Upsilon_3) - \gamma$ -dissipative only if there exists a scalar $\gamma > 0$ satisfying (5), even if it is small. Nevertheless, our aim is to check whether (5) is satisfied for a given γ and whether increasing the value of γ makes it harder to satisfy. Hence, to guarantee the strict $(\Upsilon_1, \Upsilon_2, \Upsilon_3) - \gamma$ -dissipativity of GNN (1), we can find the maximum allowable γ .

5. Examples

Example 5.1: Consider the GNN (22) with the matrix parameters in Li et al. (2016b)

$$D = \begin{bmatrix} 0.1137 & 0 \\ 0 & 0.1528 \end{bmatrix}, V_0 = \begin{bmatrix} 0.6277 & -0.7744 \\ 0.9266 & 1.4050 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} -0.5314 & 1.1948 \\ -0.6716 & -1.6170 \end{bmatrix},$$

$$V_2 = I_2, Z_1 = 0_2, Z_2 = \text{diag}\{0.1515, 0.1796\}.$$

Let $\Upsilon_1 = 0$, $\Upsilon_2 = I_2$, $\Upsilon_3 = \gamma I_2$ and $\Upsilon_4 = 0$, then by applying [Corollary 4.1](#) maximum admissible delay upper bound (MADUB) ϑ_2 is obtained for different values of ℓ when $\vartheta_1 = 0$, which is clearly presented in [Table 1](#).

Example 5.2: Considering GNN (22) with the matrix parameters in Thuan et al. (2016):

$$D = \begin{bmatrix} 2.2 & 0 \\ 0 & 1.8 \end{bmatrix}, V_0 = \begin{bmatrix} 1.2 & 1 \\ -0.2 & 0.3 \end{bmatrix}, V_1 = \begin{bmatrix} 0.8 & 0.4 \\ -0.2 & 0.1 \end{bmatrix},$$

$$V_2 = I_2, Z_1 = 0_2, Z_2 = I_2.$$

Table 1. MADUB ϑ_2 for different values of ℓ .

Methods	$\ell = 0$	$\ell = 0.3$	$\ell = 0.5$
Zhang et al. (2014a)	2.1009	1.5876	0.6306
Li et al. (2016b)	3.8126	2.2876	1.6898
Corollary 4.1	4.2134	2.7564	2.1384

Table 2. MADUB ϑ_2 for various values of ℓ .

Methods	$\ell = 0.5$	$\ell = 0.9$	$\ell = 1$
Zeng et al. (2015b)	3.0430	2.8428	2.8036
Thuan et al. (2016)	3.1127	2.9415	2.9059
Corollary 4.1	3.7251	3.4108	3.2317

Let $\Upsilon_1 = 0$, $\Upsilon_2 = I_2$, $\Upsilon_3 = \gamma I_2$ and $\Upsilon_4 = 0$, then by applying [Corollary 4.1](#) MADUB ϑ_2 is obtained for various values of ℓ when $\vartheta_1 = 0$, which is clearly presented in [Table 2](#). When $u(t) = 0$, one can obtain the state trajectories of the state $m(t)$ for the delay $\vartheta(t) = 1.7 + 0.9\cos(t)$, see [Figure 1](#).

Example 5.3: Considering GNN (22) with the matrix parameters in Zeng et al. (2015a):

$$D = \begin{bmatrix} 2.0 & 0 \\ 0 & 1.5 \end{bmatrix}, V_0 = \begin{bmatrix} -1 & 1 \\ 0.5 & -1 \end{bmatrix}, V_1 = \begin{bmatrix} -0.5 & 0.6 \\ 0.7 & 0.8 \end{bmatrix},$$

$$V_2 = I_2, Z_1 = -0.1I_2, Z_2 = 0.9I_2.$$

Letting $\Upsilon_1 = \begin{bmatrix} -0.9 & 0 \\ 0 & -0.9 \end{bmatrix}$, $\Upsilon_2 = \begin{bmatrix} 0.5 & 0 \\ 0.3 & 1 \end{bmatrix}$, $\Upsilon_3 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \gamma I_2$, then by applying [Corollary 4.1](#) the optimal dissipativity index γ is obtained for various values of ℓ when $\vartheta_1 = 0$, $\vartheta_2 = 0.4$. The results are stated in [Table 3](#). When $u(t) = 0$, one can obtain the state trajectories of the state $m(t)$ for the delay $\vartheta(t) = 1.7 + 0.9\cos(t)$, see [Figure 2](#).

Example 5.4: Considering GNN (22) with the matrix parameters in Wu et al. (2012a):

$$D = \begin{bmatrix} 7.0214 & 0 \\ 0 & 7.4367 \end{bmatrix}, V_2 = \begin{bmatrix} -6.4993 & -12.0275 \\ -0.6867 & 5.6614 \end{bmatrix},$$

$$V_0 = 0_2, V_1 = I_2, Z_1 = 0_2, Z_2 = 0.4I_2.$$

Letting $\Upsilon_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Upsilon_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\Upsilon_3 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \gamma I_2$, then by applying [Corollary 4.1](#) the optimal dissipativity index γ is obtained for various values of ℓ when $\vartheta_1 = 0.1$, $\vartheta_2 = 0.3$. The results are stated in [Table 4](#).

Example 5.5: Consider the GNNs (1) with the following matrix parameters:

$$D = \begin{bmatrix} 0.1137 & 0 \\ 0 & 0.1528 \end{bmatrix}, V_0 = \begin{bmatrix} 0.6277 & -0.7744 \\ 0.9266 & 1.4050 \end{bmatrix},$$

Table 3. Optimal dissipativity index γ for various values of ℓ .

Methods	$\ell = 0.2$	$\ell = 0.4$	$\ell = 0.6$	$\ell = 0.8$	$\ell = 1.0$
Wu et al. (2012b)	1.6470	1.5871	1.4500	1.2802	1.2461
Zeng et al. (2015c)	1.7186	1.7183	1.7180	1.7176	1.7174
Zeng et al. (2015a)	1.8570	1.8566	1.8561	1.8559	1.8558
Corollary 4.1	1.9893	1.9845	1.9824	1.9792	1.9761

Table 4. Optimal dissipativity index γ for various values of ℓ .

Methods	$\ell = 0.3$	$\ell = 0.5$	$\ell = 0.7$	$\ell = 0.9$
Wu et al. (2012a)	1.3371	1.2980	1.2148	1.0266
Zeng et al. (2015d)	1.5245	1.5104	1.5051	1.5037
Corollary 4.1	2.5184	2.4863	2.4531	2.4217

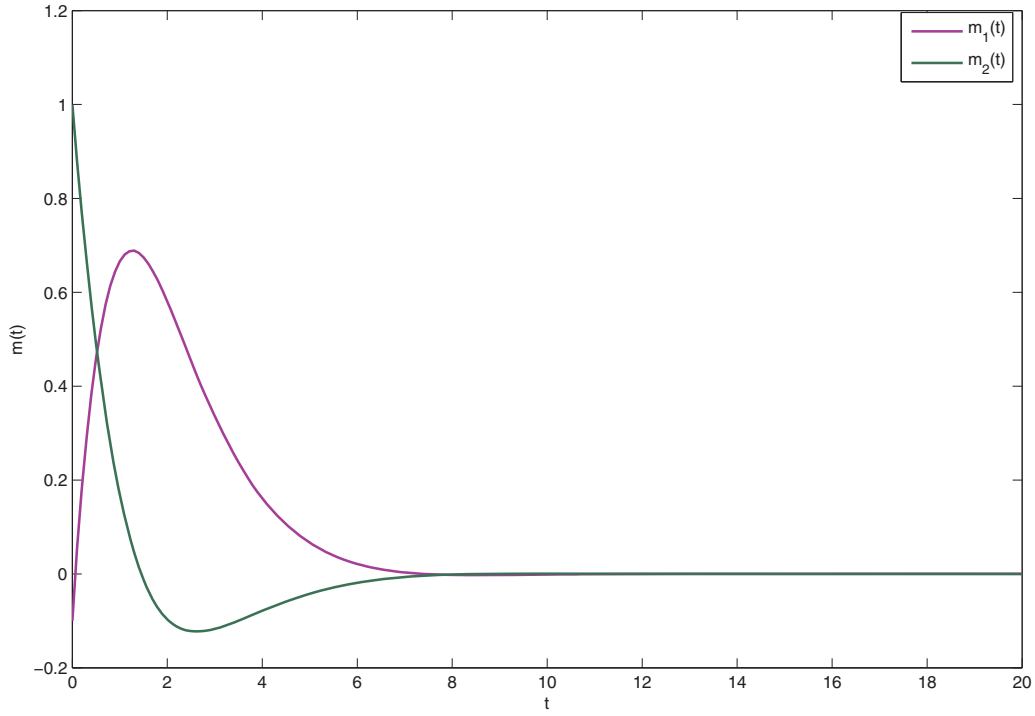


Figure 1. State response $m(t)$ in Example 5.2.

$$V_1 = \begin{bmatrix} -0.5314 & 1.1948 \\ -0.6716 & -1.6170 \end{bmatrix},$$

$$V_2 = \begin{bmatrix} -0.4993 & -0.0275 \\ -0.6867 & 1.6614 \end{bmatrix},$$

$$Z_1 = 0_2, \quad Z_2 = \text{diag}\{0.1515, 0.1796\}.$$

(A) *Passivity analysis*: Choose $\Upsilon_1 = 0_2$, $\Upsilon_2 = I_2$, $\Upsilon_3 = \gamma I_2$ and $\Upsilon_4 = 0$, then through Theorem 4.1, the passivity performance is obtained. When $\vartheta_1 = 1$, $\vartheta_2 = 2$, minimum passivity performance γ listed in Table 5 for various values of ℓ .

(B) *Dissipativity analysis*: Choose $\Upsilon_1 = -0.5I_2$, $\Upsilon_2 = I_2$, $\Upsilon_3 = \Re - \gamma I_2$ and $\Upsilon_4 = 0$, then through Theorem 4.1, the

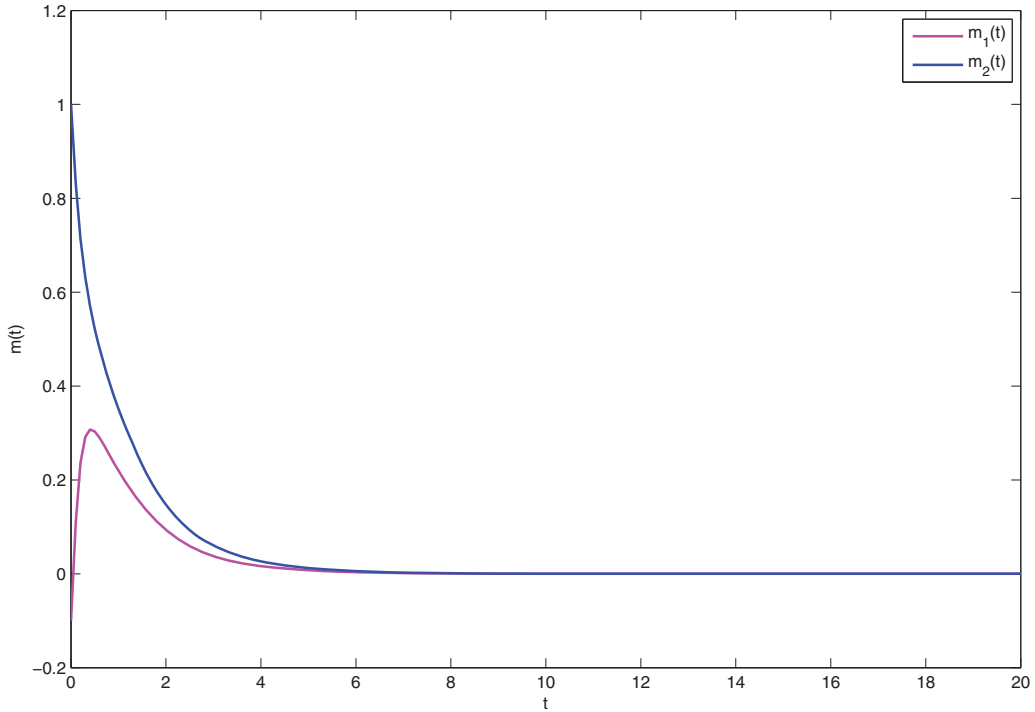


Figure 2. State response $m(t)$ in Example 5.3.

Table 5. Minimum γ for various ℓ .

Methods	0.3	0.5	0.7	0.9	≥ 1
Theorem 4.1	0.5193	0.5517	0.5803	0.6273	0.6547

Table 6. Optimal dissipativity index γ for various ℓ .

Methods	0.3	0.5	0.7	0.9	≥ 1
Theorem 4.1	1.8945	1.8439	1.7906	1.7463	1.7013

Table 7. Minimum γ for various ℓ .

Methods	0.3	0.5	0.7	0.9	≥ 1
Theorem 4.1	0.4482	0.4810	0.5173	0.5546	0.5874

Table 8. Minimum γ for various ℓ .

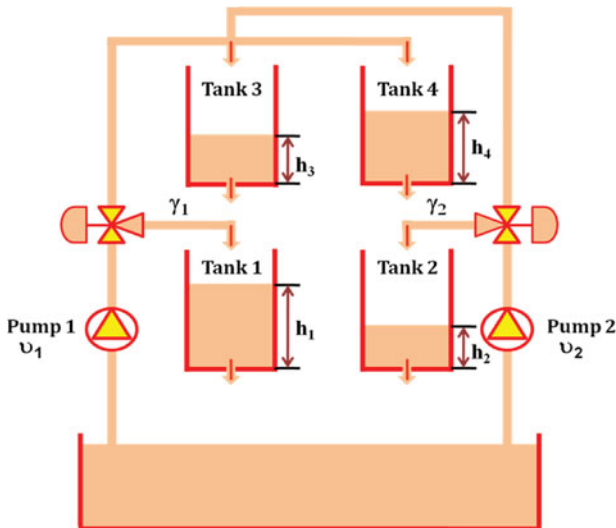
Methods	0.3	0.5	0.7	0.9	≥ 1
Theorem 4.1	0.3214	0.3473	0.3671	0.3816	0.4231

($\Upsilon_1, \Upsilon_2, \mathfrak{R}$)- γ -dissipativity performance is obtained. When $\mathfrak{R} = 2I_2$, $\vartheta_1 = 1$, $\vartheta_2 = 2$, maximum dissipativity performance γ listed in Table 6 for various values of ℓ .

(C) H_∞ Performance: Choose $\Upsilon_1 = -I_2$, $\Upsilon_2 = 0$, $\Upsilon_3 = \gamma^2 I_2$ and $\Upsilon_4 = 0$, then through Theorem 4.1, the H_∞ performance is obtained. When $\vartheta_1 = 1$, $\vartheta_2 = 2$, minimum H_∞ performance γ listed in Table 7 for various values of ℓ .

(D) $L_2 - L_\infty$ Performance: Choose $\Upsilon_1 = 0_2$, $\Upsilon_2 = 0$, $\Upsilon_3 = \gamma^2 I_2$ and $\Upsilon_4 = I_2$, then through Theorem 4.1, the $L_2 - L_\infty$ performance is obtained. When $\vartheta_1 = 1$, $\vartheta_2 = 2$, minimum $L_2 - L_\infty$ performance γ listed in Table 8 for various values of ℓ .

Example 5.6: Artificial NNs have the characteristics of the biological neurons in a nervous system that are functionally related. Moreover, NNs can be used in some practical systems. In this example, we choose the quadruple-tank process (QTP), as shown in Figure 3. The aim is to stabilise the liquid level in tanks 1 and 2 by using pumps 1 and 2, respectively. Therefore, v_1 and v_2

**Figure 3.** Schematic representation of QTP (see El Haoussi et al. (2011)).

are the input voltages to pumps 1 and 2, respectively, and the outputs are γ_1 and γ_2 (voltages from level measurement devices). The QTP can be represented by using the NN. The differential equation for the mass balances in the QTP can be expressed as follows (see El Haoussi, Tissir, Tadeo, & Hmamed, 2011):

$$\dot{\tilde{x}}(t) = \tilde{D}_0 \tilde{m}(t) + \tilde{D}_1 \tilde{m}(t - \vartheta_1) + \tilde{V}_0 \tilde{z}(t - \vartheta_2) + \tilde{V}_1 \tilde{z}(t - \vartheta_3), \quad (24)$$

where

$$\tilde{D}_0 = \begin{bmatrix} -0.0021 & 0 & 0 & 0 \\ 0 & -0.0021 & 0 & 0 \\ 0 & 0 & -0.0424 & 0 \\ 0 & 0 & 0 & -0.0424 \end{bmatrix},$$

$$\tilde{D}_1 = \begin{bmatrix} 0 & 0 & 0.0424 & 0 \\ 0 & 0 & 0 & 0.0424 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{V}_0 = \begin{bmatrix} 0.1113\gamma_1 & 0 & 0 & 0 \\ 0 & 0.1042\gamma_2 & 0 & 0 \end{bmatrix},$$

$$\tilde{V}_1 = \begin{bmatrix} 0 & 0 & 0 & 0.1113\gamma_1 \\ 0 & 0 & 0.1042\gamma_2 & 0 \end{bmatrix},$$

$$\tilde{K} = \begin{bmatrix} -0.1609 & -0.1765 & -0.0795 & -0.2073 \\ -0.1977 & -0.1579 & -0.2288 & -0.0772 \end{bmatrix},$$

$$\gamma_1 = 0.333, \gamma_2 = 0.307, \tilde{z}(t) = \tilde{K} \tilde{m}(t).$$

Generally speaking that the differential equations representing the mass balances in the delayed (transport delay) equations. The transport delays are usually included with the delay phenomenon in the inlet of water to the tanks. Furthermore, in this present study transport delays between valves and tanks being time-varying, it was assumed that $\vartheta_1 = 0$, $\vartheta_2 = 0$ and $\vartheta_3 = \vartheta(t)$. The control input variable $\tilde{z}(t)$ denotes the amount of water provided by the pumps. Hence, it is easy to see that $\tilde{z}(t)$ exhibits a threshold value because of the limited capacity area and pump hoses. Thus, we can consider the nonlinear function $\tilde{z}(t)$ as follows:

$$\begin{aligned} \tilde{z}(t) &= \tilde{K} \tilde{f}(\tilde{x}(t)), \\ \tilde{f}(\tilde{m}(t)) &= [\tilde{f}_1(\tilde{m}_1(t)), \dots, \tilde{f}_4(\tilde{m}_4(t))]^T, \\ \tilde{f}_i(\tilde{m}_i(t)) &= 0.01(|\tilde{m}_i(t) + 1| - |\tilde{m}_i(t) - 1|), \quad i = 1, \dots, 4. \end{aligned}$$

The QTP (24) can be transferred to SNN (1) with $W_2 = I$, as follows:

$$\begin{aligned} \dot{m}(t) &= -Dm(t) + V_0 f(m(t)) + V_1 f(m(t - \vartheta(t))) + z(t), \\ \eta(t) &= Bm(t), \end{aligned}$$

where $D = -\tilde{D}_0 - \tilde{D}_1$, $V_0 = \tilde{V}_0 \tilde{K}$, $V_1 = \tilde{V}_1 \tilde{K}$, and $f(\cdot) = \tilde{f}(\cdot)$. Additionally, taking the values $J = [0 \ 0 \ 0 \ 0]^T$, $Z_1 = 0_4$, $Z_2 = 0.01I_4$, $\mu = 0.5$, $\vartheta_1 = 0.5$, we obtained the delay bound $\vartheta_2 = 0.8942$, which ensures the feasibility of the LMI in Theorem 3.1.

Table 9. MADUB $\mathfrak{g}_2$ for various values of ℓ .

Methods	$\ell = 0$	$\ell = 0.5$	$\ell = 0.7$
Zhang et al. (2014a)	0.7340	0.6834	0.6355
Xiao, Zhong, and Li (2016)	1.9251	1.6214	1.5815
Corollary 4.1	2.5854	2.3758	2.2147

Example 5.7: Considering GNN (22) with the matrix parameters in Thuan et al. (2016):

$$D = \begin{bmatrix} 1.4 & 0 \\ 0 & 1.5 \end{bmatrix}, V_0 = \begin{bmatrix} 1.2 & 1 \\ -1.2 & 1.3 \end{bmatrix}, V_1 = \begin{bmatrix} -0.2 & 0.5 \\ 0.3 & -0.8 \end{bmatrix}, \\ V_2 = I_2, Z_1 = 0_2, Z_2 = I_2.$$

Let $Y_1 = 0, Y_2 = I_2, Y_3 = \gamma I_2$ and $Y_4 = 0$, then by applying Corollary 4.1 MADUB $\mathfrak{g}_2$ is obtained for various values of ℓ when $\mathfrak{g}_1 = 0$, which is clearly presented in Table 9.

Remark 5.1: The LKF (7) contains single and double integral terms, which means that the LKF (7) is not conservative. In this paper, the computation complexity has been significantly reduced than the methods published in literature. For example, the number of decision variables (NDVs) of Zhang et al. (2014a) and Li et al. (2016b) are $\frac{(14n(n+1))}{2} + 16n^2$ and $\frac{(18n(n+1))}{2} + 52n^2$, respectively. However, NDVs of this approach is $\frac{(13n(n+1))}{2} + 10n^2$. Hence, this paper provides less conservative results than the methods in Zhang et al. (2014a), Li et al. (2016b).

Remark 5.2: Mode-dependent nonrational output feedback control synthesis for semi-Markovian jump systems has been established in Wei, Qiu, and Fu (2015). H_∞ dynamic output feedback control problem was investigated for delayed Markovian jump systems with defective mode information by Wei, Qiu, Karimi, and Wang (2014b). Wei, Qiu, Shi, and Lam (2016b) studied the H_∞ piecewise filtering problem for discrete-time nonlinear time-varying delay systems via T-S fuzzy affine models. In the above papers, recently developed input-output approach and Wirtinger-based integral inequality has been utilised to solve the considered problems. Nevertheless, in this paper we have used the new WII technique, which helps us to obtain less conservative results than the existing ones in Zhang et al. (2014a), Li et al. (2016b), Zeng et al. (2015b), Thuan et al. (2016), Wu et al. (2012b), Zeng et al. (2015c), Zeng et al. (2015a), Wu et al. (2012a) and Zeng et al. (2015d). In the future, the current results can be extended into Markovian jump GNNs for the problems investigated in Wei, Wang, and Qiu (2013), Wei, Qiu, Karimi, and Wang (2014a), Li, Ding, Qiu, Yang, and Zhang (2016a), Li, Ding, Qiu, and Yang (2017) and Qiu, Ding, Gao, and Yin (2016a).

6. Conclusions

In this paper, the extended dissipativity conditions for GNNs with interval time delays was investigated. To do so, extended dissipativity conditions were established in the form of LMIs by constructing suitable LKFs. To show the feasibility and less conservativeness of the proposed results, some examples are added in the previous section. In addition, the MADUB of delays and maximum dissipativity performance index are obtained for

various cases. In the future, the above results can possibly be extended to stochastic GNNs.

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Exponential Stability, Passivity, and Dissipativity Analysis of Generalized Neural Networks With Mixed Time-Varying Delays

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Abstract—In this paper, we analyze the exponential stability, passivity, and $(\Omega, \mathcal{S}, \mathcal{R})$ - γ -dissipativity of generalized neural networks (GNNs) including mixed time-varying delays in state vectors. Novel exponential stability, passivity, and $(\Omega, \mathcal{S}, \mathcal{R})$ - γ -dissipativity criteria are developed in the form of linear matrix inequalities for continuous-time GNNs by constructing an appropriate Lyapunov–Krasovskii functional (LKF) and applying a new weighted integral inequality for handling integral terms in the time derivative of the established LKF for both single and double integrals. Some special cases are also discussed. The superiority of employing the method presented in this paper over some existing methods is verified by numerical examples.

Index Terms—Exponential passivity, generalized neural networks (GNNs), $(\Omega, \mathcal{S}, \mathcal{R})$ - γ -dissipativity, time-varying delay, weighted integral inequality (WII).

I. INTRODUCTION

ARTIFICIAL neural networks (NNs) have become a very popular research theme in the past two decades, and they are broadly studied, since they have many applications in several areas, such as image processing, communication, fault diagnosis, fixed-point computations, parallel computations, and industrial automation [1]–[9]. The applications of NNs usually depend on the behavior and stability characteristic of the system. Thus, it is worth analyzing the stability of NNs. When implementing artificial NNs, time-delayed signals often

occur due to the inherent communication time between the finite switching speed of amplifiers and neurons, which can lead to divergence, oscillation, and even instability. Thus, the stability problems of time-delayed NNs have become a major subject of research, and many interesting results have been proposed in recent years [10]–[25].

Xu *et al.* [26] stated that artificial NNs can be classified into two NN models: 1) scilicet and static NNs [27] and 2) local-field NNs [28]. Local-field NNs can be designed in various forms, such as cellular, Cohen–Grossberg, bidirectional associative memory (BAM), and Hopfield NNs. For instance, the stability of Hopfield NNs was investigated in [12]. Cheng *et al.* [13] studied the existence and stability problem of delayed discrete-time Cohen–Grossberg NNs. To obtain the uniqueness and global exponential stability, M-matrix theory and the contracting mapping principle were used. In [28], the passivity problem was studied for stochastic neutral-type memristor-based BAM NNs including mixed time-varying delayed signals, where the authors introduced the quadruple integral Lyapunov–Krasovskii functional (LKF) for stochastic stability. Compared with local field NNs, only a small number of stability problems have been studied for static NNs [27], [29]. For example, the improved stability of static NNs including interval time-delayed signals is presented in [27]. In [29], the global exponential stability problem was studied for delayed static NNs. Moreover, local-field and static NNs can be changed from one form to an equivalent form by introducing a few assumptions. However, these assumptions cannot be applied to numerous practical applications [9]. Therefore, the above NNs are not the same, and the two NN models are not always transferred equivalently. As a result, the stability problems for these two NN models are generally studied separately. Very recently, a unified model called a generalized neural network (GNN) was constructed by combining these two NN models (see [30]–[33]). Currently, many researchers are focusing their research on GNNs due to their variety of applications in biological networks, associative memories, signal processing, the tasks of pattern recognition, and other scientific areas.

Analyzing the stability of NNs has become the center of attention for various research proceedings. In recent years, asymptotic, complete, and global exponential stability problems have been investigated for NNs [29], [33]. As discussed in [34], the investigation of the exponential

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stability criteria is very significant, while the exponential convergence rate can be utilized to ensure the speed of neural calculations. Thus, studying the problem of exponential stability has some practical importance, and it is useful to estimate and ensure the exponential convergence rate of NNs. Therefore, analyzing exponential stability problems for delayed NNs has become important in both theory and practice [35]–[41].

Recently, the dissipativity problem has gained extensive research attention, since it is an important subject for physical systems, such as the design of group coordination, hybrid systems, nonlinear time-delay systems, and Internet-based control [42]–[48]. Passivity, and its generalization dissipativity, characterizes the energy consumption of a system and is used in a variety of applications (e.g., electrical, mechanical, chemical, and communication systems). In most cases, passivity can be used to demonstrate that passive systems will be stable under specific criteria [49]. Passivity is a section of a general and broader theory of dissipativeness. That is, the dissipativity criteria are a more general case of passivity and stability analyses. Thus, it is essential to consider dissipativity problems for NNs. In [43], the global dissipativity problem for delayed recurrent NNs was studied. In [44], the dissipativity problem was investigated for uncertain inertial NNs. In [45], the dissipativity criteria of delayed memristive NNs with randomly occurring uncertainties were established. In [46], the synchronization of uncertain chaotic NNs with actuator failures was studied based on extended dissipativity criteria. Strict dissipativity and stability problems were analyzed for direct-form digital filters in [50]. To the best of the authors' knowledge, the exponential stability, passivity, and dissipativity problems for delayed GNNs have not yet been completely considered in the literature, which partly motivates us to conduct study.

Many methods have been established using various mathematical methods (e.g., Jensen's inequality [51], the free-weighting matrix approach [28], and the delay-partitioning method [52]). Stability investigations have been conducted using some new inequalities (e.g., reciprocal convex combination (RCC) [53], Wirtinger-based inequality [54], and auxiliary function-based inequality [55]). It has been shown that these approaches provide less conservative results for the stability analysis of the problems under consideration and require few decision variables compared with other existing approaches in the literature. However, Hien and Trinh [56] proposed a new weighted integral inequality (WII) for exponential Lyapunov functionals.

Based on the above views, in this paper, the exponential stability, passivity, and $(\Omega, \mathfrak{S}, \mathfrak{R})$ - γ -dissipativity problems for delayed GNNs are considered. A novel improved method is established for the exponential stability of delayed GNNs by using a new WII-based mathematical analysis for both single and double integrals. To demonstrate the effectiveness of the established results, some numerical examples are given. The obtained results are compared with those of recent methods in the literature.

Some notations used in this paper are presented in the following table.

Notation	Description
$\mathcal{R}^n$	The Euclidean space with n -dimensions
$\mathcal{R}^{m \times n}$	The set of $m \times n$ dimension real matrices
$Y > 0$	Y is a positive-definite and symmetric matrix
$\text{diag}\{a_1, \dots, a_n\}$	A block diagonal matrix
$\mathcal{C}([-\tau, 0]; \mathcal{R}^n)$	Set of continuous functions ψ
$\mathcal{L}_{\mathcal{F}}^b([-\tau, 0]; \mathcal{R}^n)$	Mapping from $[-\tau, 0]$ to $\mathcal{R}^n$
$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$	Set of $\mathcal{F}$ -measurable
$\mathcal{E}(\cdot)$	$\mathcal{C}([-\tau, 0]; \mathcal{R}^n)$ -valued variables
$L_2[0, \infty)$	The complete probability space
	The mathematical expectation
	The space of square-integrable vector functions over $[0, \infty)$

II. SYSTEM FORMULATION

Let us consider GNNs with mixed time-varying delays

$$\begin{aligned} \dot{\varpi}(t) &= -C\varpi(t) + W_0 f(W_3 \varpi(t)) + W_1 f(W_3 \varpi(t - d(t))) \\ &\quad + W_2 \int_{t-\tau(t)}^t f(W_3 \varpi(\varepsilon)) d\varepsilon + z(t) \\ \mathfrak{z}(t) &= f(W_3 \varpi(t)) \end{aligned} \quad (1)$$

where $\varpi(t) = [\varpi_1(t), \varpi_2(t), \dots, \varpi_n(t)]^T \in \mathcal{R}^n$ is the neuron state vector signal. $C = \text{diag}\{c_1, \dots, c_n\} > 0$ is a known diagonal matrix. W_0 , W_1 , W_2 , and W_3 are the interconnection weight matrices of the neurons. $z(t) \in \mathcal{R}^n$ is the inputs belonging to $L_2[0, \infty)$. $\mathfrak{z}(t)$ is the output vector of the NNs. $f(W_3 \varpi(t)) = [f_1(W_3 \varpi_1(t)), \dots, f_n(W_3 \varpi_n(t))]^T \in \mathcal{R}^n$ denotes the activation function of neurons. $d(t)$ and $\tau(t)$ denote time-varying discrete delay and time-varying distributed delay, respectively. They are assumed to satisfy

$$0 < d(t) \leq d, \quad \dot{d}(t) \leq \mu, \quad 0 < \tau(t) \leq \tau \quad (2)$$

where d , τ , and μ are positive constants.

It should be noted that for $i = 1, 2, \dots, n$, $f_i(\cdot)$ satisfies the following assumption.

Assumption 1 [57]: For any $i = 1, 2, \dots, n$, $f_i(0) = 0$ and there exist constant matrices $H_1 = \text{diag}\{m_1^-, m_2^-, \dots, m_n^-\}$ and $H_2 = \text{diag}\{m_1^+, m_2^+, \dots, m_n^+\}$ such that

$$m_i^- \leq \frac{f_i(W_{3i}u) - f_i(W_{3i}v)}{W_{3i}u - W_{3i}v} \leq m_i^+ \quad (3)$$

holds for any $u, v \in \mathcal{R}$ with $u \neq v$.

Definition 1 [59], [60]: The GNN (1) is said to be exponentially dissipative from input $z(t)$ to output $\mathfrak{z}(t)$, and the quadratic supply rate $\gamma(z, \mathfrak{z}) = \mathfrak{z}^T \Omega \mathfrak{z} + 2\mathfrak{z}^T \mathfrak{S} z + z^T \mathfrak{R} z$, where $\Omega \leq 0$, $\mathfrak{R} = \mathfrak{R}^T > 0$, and $\mathfrak{S}$ are real known matrices, if there exists an exponential Lyapunov function (storage function) V , and a constant $\alpha > 0$ such that for any input $z(t)$ and any initial condition $\varpi(t_0)$, the following relation holds:

$$\dot{V}(\varpi(t)) + \alpha V(\varpi(t)) \leq \gamma(z, \mathfrak{z}), \quad t > t_0.$$

Remark 1: The global dissipativity and global exponential dissipativity criteria are established in [43] for Hopfield and cellular NNs with or without time delays. To achieve this, a radically unbounded and positive-definite Lyapunov function has been employed. Then, by solving and using the definition based on the attractive set, the dissipativeness was investigated. However, Definition 1 in this paper states the properties of the $(\Omega, \mathfrak{S}, \mathfrak{R})$ - γ -dissipativity criteria of the GNN (1). By using

the LKF and linear matrix inequality (LMI) techniques, the dissipativeness of the GNN has been analyzed.

Lemma 1 [56]: For a given matrix $R \in \mathcal{R}_{>0}^{n \times n}$, scalar $\alpha > 0$, and function $\chi \in \mathcal{C}([a, b], \mathcal{R}^n)$, satisfy

$$\int_a^b e^{\alpha(s-b)} \chi^T(\varepsilon) R \chi(\varepsilon) d\varepsilon \geq \frac{\alpha}{\bar{\eta}_0} \left(\int_a^b \chi(\varepsilon) d\varepsilon \right)^T R \left(\int_a^b \chi(\varepsilon) d\varepsilon \right) + \frac{\alpha}{\bar{\rho}_0} \zeta^T (K_1^T K_1 \otimes R) \zeta$$

where

$$\rho_0 = \frac{\bar{\eta}_0}{\bar{\eta}_1^2} [\bar{\eta}_0^2 - (\alpha l)^2 e^{\alpha l}], \bar{\eta}_0 = e^{\alpha l} - 1, \bar{\eta}_1 = \bar{\eta}_0 - \alpha l$$

$$l = b - a, \zeta = \left[\int_a^b \chi(\varepsilon) d\varepsilon \int_a^b \int_\varepsilon^b \chi(u) du d\varepsilon \right]^T$$

$$K_1 = \begin{bmatrix} 1 & -\alpha \bar{\eta}_0 / \bar{\eta}_1 \end{bmatrix}.$$

Lemma 2 [56]: For a given matrix $Z \in \mathcal{R}_{>0}^{n \times n}$, scalar $\alpha > 0$, and function $\chi \in \mathcal{C}([a, b], \mathcal{R}^n)$, satisfy

$$\int_a^b \int_\varepsilon^b e^{\alpha(\theta-b)} \chi^T(\theta) Z \chi(\theta) d\theta d\varepsilon \geq \frac{\alpha^2}{\bar{\eta}_1} \psi^T(t) (K_0^T K_0 \otimes Z) \psi(t) + \frac{4\alpha^2}{\bar{\rho}_1} \psi^T(t) (K_2^T K_2 \otimes Z) \psi(t)$$

where

$$\rho_1 = \frac{\bar{\eta}_1}{\bar{\eta}_2^2} [2\eta_1^2 - (\alpha d + (\alpha d - 1)\bar{\eta}_0)(\alpha d)^2]$$

$$\bar{\eta}_2 = \bar{\eta}_1 - \frac{(\alpha d)^2}{2}, l = b - a, K_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 1 & -\alpha \bar{\eta}_1 / \bar{\eta}_2 \end{bmatrix}$$

$$\psi = \left[\int_a^b \int_\varepsilon^b \chi(\theta) d\theta d\varepsilon \int_a^b \int_\varepsilon^b \int_\theta^b \chi(\theta) d\theta du d\varepsilon \right]^T.$$

III. MAIN RESULTS

Define

$$\vartheta(t) = \begin{bmatrix} \varpi^T(t) & \int_{t-d}^t \varpi^T(\varepsilon) d\varepsilon & \int_{-d}^0 \int_{t+\varepsilon}^t \varpi^T(u) du d\varepsilon \end{bmatrix}^T$$

$$e_\ell = [0_{n \times (\ell-1)n} \quad I_n \quad 0_{n \times (8-\ell)n}]^T, \ell = 1, 2, \dots, 8$$

$$\xi(t) = \begin{bmatrix} \varpi^T(t) & \varpi^T(t-d) & f(W_3 \varpi(t)) \\ \times f^T(W_3 \varpi(t-d)) & \int_{t-d}^t \varpi^T(\varepsilon) d\varepsilon \\ \times \int_{t-\tau(t)}^t f^T(W_3 \varpi(\varepsilon)) d\varepsilon & \int_{-d}^0 \int_{t+\varepsilon}^t \varpi^T(u) du d\varepsilon & z(t) \end{bmatrix}^T.$$

The following theorem has been proposed based on the LKF and LMI techniques.

Theorem 1: Given $\alpha > 0$ and some scalars $d > 0$, $\tau > 0$, $\mu > 0$, the GNN (1) is dissipative when there exist matrices, $P \in \mathcal{R}_{>0}^{3n \times 3n}$ and $Q, R, S, T, Z \in \mathcal{R}_{>0}^{n \times n}$, and diagonal matrices $\Gamma_p \in \mathcal{R}_{>0}^{n \times n}$, ($p = 1, 2, 3$), $\Lambda_q \in \mathcal{R}_{>0}^{n \times n}$ ($q = 1, 2$), such that

$$\sum_{m=1}^8 \Xi_m < 0 \quad (4)$$

where

$$\Phi_0 = \begin{bmatrix} e_1 \\ e_5 \\ e_7 \end{bmatrix}, \Phi_1 = \begin{bmatrix} \mathbb{A} \\ e_1 - e_2 \\ de_1 - e_5 \end{bmatrix}, \Phi_2 = \begin{bmatrix} e_1 - e_2 \\ de_1 - e_5 \end{bmatrix}$$

$$\Phi_3 = \begin{bmatrix} de_1 - e_5 \\ \frac{d^2}{2} e_1 - e_7 \end{bmatrix}, \Xi_1 = \Phi_0^T P \Phi_1 + \Phi_1^T P \Phi_0 + \alpha \Phi_0^T P \Phi_0$$

$$\Xi_2 = 2[e_3 - H_1 W_3 e_1]^T \Lambda_1 W_3 \mathbb{A} + 2[H_2 W_3 e_1 - e_3]^T \Lambda_2 W_3 \mathbb{A}$$

$$\Xi_3 = e_1^T Q e_1 - e^{-\alpha d} e_2^T Q e_2 + e_3^T [S + \tau^2 T] e_3$$

$$+ \pi(\mu) e_4^T S e_4 + \mathbb{A}^T \left[dR + \frac{d^2}{2} Z \right] \mathbb{A}$$

$$\Xi_4 = - \left(\frac{\alpha}{\eta_0} (e_1 - e_2)^T R (e_1 - e_2) + \frac{\alpha}{\rho_0} \Phi_2^T (K_1^T K_1 \otimes R) \Phi_2 \right)$$

$$\Xi_5 = -(1 - \mu) e^{-\alpha d} e_6^T T e_6$$

$$\Xi_6 = - \left(\frac{\alpha^2}{\eta_1} (de_1 - e_5)^T Z \right. \\ \left. \times (de_1 - e_5) + \frac{4\alpha^2}{\rho_1} \Phi_3^T (K_2^T K_2 \otimes Z) \Phi_3 \right)$$

$$\Xi_7 = -2e_1^T W_3^T H_1 \Gamma_1 H_2 W_3 e_1 + e_1^T W_3^T (H_1 + H_2) \Gamma_1 e_3$$

$$+ e_3^T \Gamma_1 (H_1 + H_2) W_3 e_1 - 2e_3^T \Gamma_1 e_3$$

$$- 2e_2^T W_3^T H_1 \Gamma_2 H_2 W_3 e_2 + e_2^T W_3^T (H_1 + H_2) \Gamma_2 e_4$$

$$+ e_4^T \Gamma_2 (H_1 + H_2) W_3 e_2 - 2e_4^T \Gamma_2 e_4 - 2[e_3 - e_4$$

$$- H_1 W_3 (e_1 - e_2)]^T \Gamma_3 [e_3 - e_4 - H_2 W_3 (e_1 - e_2)]$$

$$\Xi_8 = -e_3^T \mathfrak{Q} e_3 - 2e_3^T \mathfrak{S} e_8 - e_8^T \mathfrak{X} e_8, \mathbb{A} = -C e_1$$

$$+ W_0 e_3 + W_1 e_4 + W_2 e_6 + e_8, \eta_2 = \eta_1 - \frac{(\alpha d)^2}{2}$$

$$K_1 = \begin{bmatrix} 1 & -\alpha \eta_0 / \eta_1 \end{bmatrix}, K_2 = \begin{bmatrix} 1 & -\alpha \eta_1 / \eta_2 \end{bmatrix}$$

$$\eta_0 = e^{\alpha d} - 1, \eta_1 = \eta_0 - \alpha d, \rho_0 = \frac{\eta_0}{\eta_1^2} [\eta_0^2 - (\alpha d)^2 e^{\alpha d}]$$

$$\rho_1 = \frac{\eta_1}{\eta_2^2} [2\eta_1^2 - (\alpha d + (\alpha d - 1)\eta_0)(\alpha d)^2]$$

and

$$\pi(\mu) = \begin{cases} -(1 - \mu) e^{-\alpha d}, & \mu \leq 1 \\ -(1 - \mu), & \mu > 1. \end{cases}$$

Proof: Let the following LKF be:

$$V(\varpi_t) = \vartheta^T(t) P \vartheta(t) + 2 \sum_{i=1}^n \lambda_{1i} \int_0^{W_{3i} \varpi(t)} (f_i(\varepsilon) - m_i^- \varepsilon) d\varepsilon$$

$$+ 2 \sum_{i=1}^n \lambda_{2i} \int_0^{W_{3i} \varpi(t)} (m_i^+ \varepsilon - f_i(\varepsilon)) d\varepsilon$$

$$+ \int_{t-d}^t e^{\alpha(\varepsilon-t)} \varpi^T(\varepsilon) Q \varpi(\varepsilon) d\varepsilon$$

$$+ \int_{-d}^0 \int_{t+\varepsilon}^t e^{\alpha(\theta-t)} \dot{\varpi}^T(\theta) R \dot{\varpi}(\theta) d\theta d\varepsilon$$

$$+ \int_{t-d(t)}^t e^{\alpha(\varepsilon-t)} f^T(W_3 \varpi(\varepsilon)) S f(W_3 \varpi(\varepsilon)) d\varepsilon$$

$$+ \tau \int_{t-\tau}^0 \int_{t+\varepsilon}^t e^{\alpha(\theta-t)} f^T(W_3 \varpi(\theta)) T f(W_3 \varpi(\theta)) d\theta d\varepsilon$$

$$+ \int_{-d}^0 \int_\varepsilon^0 \int_{t+\theta}^t e^{\alpha(u-t)} \dot{\varpi}^T(u) Z \dot{\varpi}(u) du d\theta d\varepsilon. \quad (5)$$

It follows from (5) that $V(\varpi_t) \geq \lambda_{\min}(P_{11})\|\varpi(t)\|^2$. The derivative of $V(\varpi_t)$ along the trajectory of (1) can be obtained as follows:

$$\begin{aligned} \dot{V}(\varpi_t) + \alpha \bar{V}(\varpi_t) &\leq \xi^T(t) \left[\sum_{n=1}^3 \Xi_n \right] \xi(t) - \int_{t-d}^t e^{\alpha(\varepsilon-t)} \dot{\varpi}^T(\varepsilon) R \dot{\varpi}(\varepsilon) d\varepsilon \\ &\quad - \tau \int_{t-\tau}^t e^{\alpha(\varepsilon-t)} f^T(W_3 \varpi(\varepsilon)) T f(W_3 \varpi(\varepsilon)) d\varepsilon \\ &\quad - \int_{-d}^0 \int_{t+\varepsilon}^t e^{\alpha(\theta-t)} \dot{\varpi}^T(\theta) Z \dot{\varpi}(\theta) d\theta d\varepsilon \end{aligned} \quad (6)$$

where $\bar{V}(\varpi_t) = V(\varpi_t) - 2 \sum_{i=1}^n \lambda_{1i} \int_0^{W_{3i}\varpi(t)} (f_i(\varepsilon) - m_i^- \varepsilon) d\varepsilon - 2 \sum_{i=1}^n \lambda_{2i} \int_0^{W_{3i}\varpi(t)} (m_i^+ \varepsilon - f_i(\varepsilon)) d\varepsilon$. Through Lemmas 1 and 2 and Jensen's inequality, we can obtain

$$- \int_{t-d}^t e^{\alpha(\varepsilon-t)} \dot{\varpi}^T(\varepsilon) R \dot{\varpi}(\varepsilon) d\varepsilon \leq \xi^T(t) \Xi_4 \xi(t) \quad (7)$$

$$- \tau \int_{t-\tau}^t e^{\alpha(\varepsilon-t)} f^T(W_3 \varpi(\varepsilon)) T f(W_3 \varpi(\varepsilon)) d\varepsilon \leq \xi^T(t) \Xi_5 \xi(t) \quad (8)$$

$$\int_{-d}^0 \int_{t+\varepsilon}^t e^{\alpha(\theta-t)} \dot{\varpi}^T(\theta) Z \dot{\varpi}(\theta) d\theta d\varepsilon \leq \xi^T(t) \Xi_6 \xi(t). \quad (9)$$

From Assumption 1, we have

$$0 \leq 2[f(W_3 \varpi(t)) - H_1 W_3 \varpi(t)]^T \Gamma_1 [H_2 W_3 \varpi(t) - f(W_3 \varpi(t))] \quad (10)$$

$$0 \leq 2[f(W_3 \varpi(t-d(t))) - H_1 W_3 \varpi(t-d(t))]^T \Gamma_2 \times [H_2 W_3 \varpi(t-d(t)) - f(W_3 \varpi(t-d(t)))] \quad (11)$$

$$0 \leq 2[f(W_3 \varpi(t)) - f(W_3 \varpi(t-d(t)))]^T \Gamma_3 \times [f(W_3 \varpi(t)) - f(W_3 \varpi(t-d(t)))] \times [f(W_3 \varpi(t)) - f(W_3 \varpi(t-d(t)))] \times [H_2(W_3 \varpi(t) - W_3 \varpi(t-d(t)))] \quad (12)$$

By combining (6) with (12), we obtain

$$\dot{V}(\varpi_t) + \alpha \bar{V}(\varpi_t) - \gamma(z(t), \mathfrak{z}(t)) \leq \xi^T(t) \left[\sum_{m=1}^8 \Xi_m \right] \xi(t) \quad (13)$$

where Ξ_m , ($m = 1, 2, \dots, 8$) are defined in (4). If the LMI (4) holds, then we obtain $\dot{V}(\varpi_t) + \alpha \bar{V}(\varpi_t) \leq \gamma(z(t), \mathfrak{z}(t))$. Then, through Definition 1, the GNN (1) is dissipative. ■

Remark 2: In Theorem 1, the passivity and $(\mathfrak{Q}, \mathfrak{S}, \mathfrak{R})$ - γ -dissipativity problems for the GNN have been considered. To obtain the stability conditions, when $z(t) = \mathfrak{z}(t) = 0$, the GNN (1) becomes

$$\begin{aligned} \dot{\varpi}(t) &= -C\varpi(t) + W_0 f(W_3 \varpi(t)) + W_1 f(W_3 \varpi(t-d(t))) \\ &\quad + W_2 \int_{t-\tau(t)}^t f(W_3 \varpi(\varepsilon)) d\varepsilon. \end{aligned} \quad (14)$$

The following theorem states the stability conditions for the GNN (14).

Theorem 2: Given $\alpha > 0$ and some scalars $d > 0$, $\tau > 0$, $\mu > 0$, the GNN (14) is exponentially stable when there exist matrices, $P \in \mathcal{R}_{>0}^{3n \times 3n}$ and $Q, R, S, T, Z \in \mathcal{R}_{>0}^{n \times n}$, and

diagonal matrices, $\Gamma_p \in \mathcal{R}_{>0}^{n \times n}$, ($p = 1, 2, 3$), $\Lambda_q \in \mathcal{R}_{>0}^{n \times n}$ ($q = 1, 2$), such that

$$\sum_{m=1}^7 \Xi_m < 0 \quad (15)$$

where

$$\begin{aligned} \Phi_0 &= \begin{bmatrix} e_1 \\ e_5 \\ e_7 \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} \mathbb{A} \\ e_1 - e_2 \\ de_1 - e_5 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} e_1 - e_2 \\ de_1 - e_5 \end{bmatrix} \\ \Phi_3 &= \begin{bmatrix} de_1 - e_5 \\ \frac{d^2}{2} e_1 - e_7 \end{bmatrix}, \quad \Xi_1 = \Phi_0^T P \Phi_1 + \Phi_1^T P \Phi_0 + \alpha \Phi_0^T P \Phi_0 \\ \Xi_2 &= 2[e_3 - H_1 W_3 e_1]^T \Lambda_1 W_3 \mathbb{A} + 2[H_2 W_3 e_1 - e_3]^T \Lambda_2 W_3 \mathbb{A} \\ \Xi_3 &= e_1^T Q e_1 - e^{-\alpha d} e_2^T Q e_2 + e_3^T [S + \tau^2 T] e_3 \\ &\quad + \pi(\mu) e_4^T S e_4 + \mathbb{A}^T \left[dR + \frac{d^2}{2} Z \right] \mathbb{A} \\ \Xi_4 &= - \left(\frac{\alpha}{\eta_0} (e_1 - e_2)^T R (e_1 - e_2) + \frac{\alpha}{\rho_0} \Phi_2^T (K_1^T K_1 \otimes R) \Phi_2 \right) \\ \Xi_5 &= -(1 - \mu) e^{-\alpha d} e_6^T T e_6, \quad \Xi_6 = - \left(\frac{\alpha^2}{\eta_1} (de_1 - e_5)^T Z \right. \\ &\quad \times (de_1 - e_5) + \frac{4\alpha^2}{\rho_1} \Phi_3^T (K_2^T K_2 \otimes Z) \Phi_3 \left. \right) \\ \Xi_7 &= -2e_1^T W_3^T H_1 \Gamma_1 H_2 W_3 e_1 + e_1^T W_3^T (H_1 + H_2) \Gamma_1 e_3 \\ &\quad + e_3^T \Gamma_1 (H_1 + H_2) W_3 e_1 - 2e_3^T \Gamma_1 e_3 \\ &\quad - 2e_2^T W_3^T H_1 \Gamma_2 H_2 W_3 e_2 + e_2^T W_3^T (H_1 + H_2) \Gamma_2 e_4 \\ &\quad + e_4^T \Gamma_2 (H_1 + H_2) W_3 e_2 - 2e_4^T \Gamma_2 e_4 - 2[e_3 - e_4 \\ &\quad - H_1 W_3 (e_1 - e_2)]^T \Gamma_3 [e_3 - e_4 - H_2 W_3 (e_1 - e_2)] \\ \rho_0 &= \frac{\eta_0}{\eta_1^2} [\eta_0^2 - (\alpha d)^2 e^{\alpha d}], \quad \eta_0 = e^{\alpha d} - 1, \quad \eta_1 = \eta_0 - \alpha d \\ \rho_1 &= \frac{\eta_1}{\eta_2^2} [2\eta_1^2 - (\alpha d + (\alpha d - 1)\eta_0)(\alpha d)^2] \\ \eta_2 &= \eta_1 - \frac{(\alpha d)^2}{2}, \quad K_1 = \begin{bmatrix} 1 & -\alpha \eta_0 / \eta_1 \end{bmatrix} \\ K_2 &= \begin{bmatrix} 1 & -\alpha \eta_1 / \eta_2 \end{bmatrix} \\ \mathbb{A} &= -C e_1 + W_0 e_3 + W_1 e_4 + W_2 e_6. \end{aligned}$$

Proof: By using the same criteria in Theorem 1, we can obtain the above results. ■

Remark 3: When $W_0 = W_2 = 0$ and $W_1 = I$ in (14), we have the static NN

$$\dot{\varpi}(t) = -C\varpi(t) + f(W_3 \varpi(t-d(t))). \quad (16)$$

When $W_3 = I$ in (14), we have the local-field NN

$$\begin{aligned} \dot{\varpi}(t) &= -C\varpi(t) + W_0 f(\varpi(t)) + W_1 f(\varpi(t-d(t))) \\ &\quad + W_2 \int_{t-\tau(t)}^t f(\varpi(\varepsilon)) d\varepsilon. \end{aligned} \quad (17)$$

Corollary 1: Given $\alpha > 0$ and some scalars $d > 0$, $\mu > 0$, the static NN (16) is exponentially stable when there exist matrices, $P \in \mathcal{R}_{>0}^{3n \times 3n}$ and $Q, R, S, Z \in \mathcal{R}_{>0}^{n \times n}$, and diagonal

matrices, $\Gamma_p \in \mathcal{R}_{>0}^{n \times n}$, ($p = 1, 2, 3$), $\Lambda_q \in \mathcal{R}_{>0}^{n \times n}$ ($q = 1, 2$), Define such that

$$\sum_{m=1}^6 \Xi_m < 0 \quad (18)$$

where

$$\begin{aligned} \Phi_0 &= \begin{bmatrix} e_1 \\ e_5 \\ e_6 \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} \mathcal{A} \\ e_1 - e_2 \\ de_1 - e_5 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} e_1 - e_2 \\ de_1 - e_5 \end{bmatrix} \\ \Phi_3 &= \begin{bmatrix} de_1 - e_5 \\ \frac{d^2}{2}e_1 - e_6 \end{bmatrix}, \quad \Xi_1 = \Phi_0^T P \Theta_1 + \Phi_1^T P \Phi_0 + \alpha \Phi_0^T P \Phi_0 \\ \Xi_2 &= 2[e_3 - H_1 W_3 e_1]^T \Lambda_1 W_3 \mathcal{A} + 2[H_2 W_3 e_1 - e_3]^T \Lambda_2 W_3 \mathcal{A} \\ \Xi_3 &= e_1^T Q e_1 - e^{-\alpha d} e_2^T Q e_2 + e_3^T S e_3 + \pi(\mu) e_4^T S e_4 \\ &\quad + \mathcal{A}^T \left[dR + \frac{d^2}{2} Z \right] \mathcal{A} \\ \Xi_4 &= -\left(\frac{\alpha}{\eta_0} (e_1 - e_2)^T R (e_1 - e_2) + \frac{\alpha}{\rho_0} \Phi_2^T (K_1^T K_1 \otimes R) \Phi_2 \right) \\ \Xi_5 &= -\left(\frac{\alpha^2}{\eta_1} (de_1 - e_5)^T Z \right. \\ &\quad \left. \times (de_1 - e_5) + \frac{4\alpha^2}{\rho_1} \Phi_3^T (K_2^T K_2 \otimes Z) \Phi_3 \right) \\ \Xi_6 &= -2e_1^T W_3^T H_1 \Gamma_1 H_2 W_3 e_1 + e_1^T W_3^T (H_1 + H_2) \Gamma_1 e_3 \\ &\quad + e_3^T \Gamma_1 (H_1 + H_2) W_3 e_1 - 2e_3^T \Gamma_1 e_3 \\ &\quad - 2e_2^T W_3^T H_1 \Gamma_2 H_2 W_3 e_2 + e_2^T W_3^T (H_1 + H_2) \Gamma_2 e_4 \\ &\quad + e_4^T \Gamma_2 (H_1 + H_2) W_3 e_2 - 2e_4^T \Gamma_2 e_4 \\ &\quad - 2[e_3 - e_4 - H_1 W_3 (e_1 - e_2)]^T \Gamma_3 \\ &\quad \times [e_3 - e_4 - H_2 W_3 (e_1 - e_2)], \quad K_1 = \begin{bmatrix} 1 & -\alpha\eta_0/\eta_1 \end{bmatrix} \\ K_2 &= \begin{bmatrix} 1 & -\alpha\eta_1/\eta_2 \end{bmatrix}, \quad \mathcal{A} = -Ce_1, \quad \eta_0 = e^{\alpha d} - 1 \\ \eta_1 &= \eta_0 - \alpha d, \quad \eta_2 = \eta_1 - \frac{(\alpha d)^2}{2}, \quad \rho_0 = \frac{\eta_0}{\eta_1^2} \left[\eta_0^2 - (\alpha d)^2 e^{\alpha d} \right] \\ \rho_1 &= \frac{\eta_1}{\eta_2^2} \left[2\eta_1^2 - (\alpha d + (\alpha d - 1)\eta_0)(\alpha d)^2 \right]. \end{aligned}$$

Proof: Choose the following LKF:

$$\begin{aligned} V_1(\varpi_t) &= \vartheta^T(t) P \vartheta(t) + 2 \sum_{i=1}^n \lambda_{1i} \int_0^{W_{3i}\varpi(t)} (f_i(\varepsilon) - m_i^- \varepsilon) d\varepsilon \\ &\quad + 2 \sum_{i=1}^n \lambda_{2i} \int_0^{W_{3i}\varpi(t)} (m_i^+ \varepsilon - f_i(\varepsilon)) d\varepsilon \\ &\quad + \int_{t-d}^t e^{\alpha(s-t)} \varpi^T(\varepsilon) Q \varpi(\varepsilon) d\varepsilon \\ &\quad + \int_{-d}^0 \int_{t+\varepsilon}^t e^{\alpha(\theta-t)} \dot{\varpi}^T(\theta) R \dot{\varpi}(\theta) d\theta d\varepsilon \\ &\quad + \int_{t-d(t)}^t e^{\alpha(s-t)} f^T(W_3 \varpi(\varepsilon)) S f(W_3 \varpi(\varepsilon)) d\varepsilon \\ &\quad + \int_{-d}^0 \int_{\varepsilon}^0 \int_{t+\theta}^t e^{\alpha(u-t)} \dot{\varpi}^T(u) Z \dot{\varpi}(u) du d\theta d\varepsilon. \quad (19) \end{aligned}$$

$$\begin{aligned} e_j &= \begin{bmatrix} 0_{n \times (j-1)n} & I_n & 0_{n \times (6-j)n} \end{bmatrix}, \quad j = 1, 2, \dots, 6 \\ \xi_1(t) &= \begin{bmatrix} \varpi^T(t) & \varpi^T(t-d) & f(W_3 \varpi(t)) \\ \times f^T(W_3 \varpi(t-d(t))) & \int_{t-d}^t \varpi^T(\varepsilon) d\varepsilon \\ \times \int_{-d}^0 \int_{t+\varepsilon}^t \varpi^T(u) du d\varepsilon \end{bmatrix}^T. \end{aligned}$$

By using an approach similar to that used in Theorem 1, we can obtain the exponential stability conditions in Corollary 1. ■

When $W_2 = 0$, NN (17) becomes

$$\dot{\varpi}(t) = -C\varpi(t) + W_0 f(\varpi(t)) + W_1 f(\varpi(t-d(t))). \quad (20)$$

In what follows, a sufficient condition is obtained for the NN (20).

Corollary 2: Given $\alpha > 0$ and some scalars $d > 0$, $\mu > 0$, the GNN (20) is exponentially stable when there exist matrices $P \in \mathcal{R}_{>0}^{3n \times 3n}$ and $Q, R, S, T, Z \in \mathcal{R}_{>0}^{n \times n}$, and diagonal matrices, $\Gamma_p \in \mathcal{R}_{>0}^{n \times n}$, ($p = 1, 2, 3$), $\Lambda_q \in \mathcal{R}_{>0}^{n \times n}$ ($q = 1, 2$), such that

$$\sum_{m=1}^6 \Xi_m < 0 \quad (21)$$

where

$$\begin{aligned} \Phi_0 &= \begin{bmatrix} e_1 \\ e_5 \\ e_6 \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} \bar{\mathcal{A}} \\ e_1 - e_2 \\ de_1 - e_5 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} e_1 - e_2 \\ de_1 - e_5 \end{bmatrix} \\ \Phi_3 &= \begin{bmatrix} de_1 - e_5 \\ \frac{d^2}{2}e_1 - e_6 \end{bmatrix}, \quad \Xi_1 = \Phi_0^T P \Theta_1 + \Phi_1^T P \Phi_0 + \alpha \Phi_0^T P \Phi_0 \\ \Xi_2 &= 2[e_3 - H_1 e_1]^T \Lambda_1 \bar{\mathcal{A}} + 2[H_2 e_1 - e_3]^T \Lambda_2 \bar{\mathcal{A}} \\ \Xi_3 &= e_1^T Q e_1 - e^{-\alpha d} e_2^T Q e_2 + e_3^T S e_3 + \pi(\mu) e_4^T S e_4 \\ &\quad + \bar{\mathcal{A}}^T \left[dR + \frac{d^2}{2} Z \right] \bar{\mathcal{A}} \\ \Xi_4 &= -\left(\frac{\alpha}{\eta_0} (e_1 - e_2)^T R (e_1 - e_2) + \frac{\alpha}{\rho_0} \Phi_2^T (K_1^T K_1 \otimes R) \Phi_2 \right) \\ \Xi_5 &= -\left(\frac{\alpha^2}{\eta_1} (de_1 - e_5)^T Z \right. \\ &\quad \left. \times (de_1 - e_5) + \frac{4\alpha^2}{\rho_1} \Phi_3^T (K_2^T K_2 \otimes Z) \Phi_3 \right) \\ \Xi_6 &= -2e_1^T H_1 \Gamma_1 H_2 e_1 + e_1^T \Gamma_1 (H_1 + H_2) e_3 \\ &\quad + e_3^T (H_1 + H_2) \Gamma_2 e_1 - 2e_3^T \Gamma_1 e_3 \\ &\quad - 2e_2^T H_1 \Gamma_2 H_2 e_2 + e_2^T \Gamma_2 (H_1 + H_2) e_4 \\ &\quad + e_4^T (H_1 + H_2) \Gamma_2 e_2 - 2e_4^T \Gamma_2 e_4 \\ \rho_0 &= \frac{\eta_0}{\eta_1^2} \left[\eta_0^2 - (\alpha d)^2 e^{\alpha d} \right] \quad \eta_0 = e^{\alpha d} - 1, \quad \eta_1 = \eta_0 - \alpha d \\ \rho_1 &= \frac{\eta_1}{\eta_2^2} \left[2\eta_1^2 - (\alpha d + (\alpha d - 1)\eta_0)(\alpha d)^2 \right] \\ \eta_2 &= \eta_1 - \frac{(\alpha d)^2}{2}, \quad K_1 = \begin{bmatrix} 1 & -\alpha\eta_0/\eta_1 \end{bmatrix} \\ K_2 &= \begin{bmatrix} 1 & -\alpha\eta_1/\eta_2 \end{bmatrix}, \quad \bar{\mathcal{A}} = -Ce_1 + W_0 e_3 + W_2 e_4. \end{aligned}$$

Proof: Choose the LKF as follows:

$$\begin{aligned}
V_2(\varpi_t) = & \vartheta^T(t)P\vartheta(t) + 2 \sum_{i=1}^n \lambda_{1i} \int_0^{W_{3i}\varpi(t)} (f_i(\varepsilon) - m_i^- \varepsilon) d\varepsilon \\
& + 2 \sum_{i=1}^n \lambda_{2i} \int_0^{W_{3i}\varpi(t)} (m_i^+ \varepsilon - f_i(\varepsilon)) d\varepsilon \\
& + \int_{t-d}^t e^{\alpha(s-t)} \varpi^T(\varepsilon) Q \varpi(\varepsilon) d\varepsilon \\
& + \int_{-d}^0 \int_{t+\varepsilon}^t e^{\alpha(\theta-t)} \dot{\varpi}^T(\theta) R \dot{\varpi}(\theta) d\theta d\varepsilon \\
& + \int_{t-d(t)}^t e^{\alpha(s-t)} f^T(W_3 \varpi(\varepsilon)) S f(W_3 \varpi(\varepsilon)) d\varepsilon \\
& + \int_{-d}^0 \int_{\varepsilon}^t \int_{t+\theta}^t e^{\alpha(u-t)} \dot{\varpi}^T(u) Z \dot{\varpi}(u) du d\theta d\varepsilon. \quad (22)
\end{aligned}$$

Let $\Gamma_p = \text{diag}\{\eta_{p1}, \eta_{p2}, \dots, \eta_{pn}\} > 0$ and $(p = 1, 2)$ be positive-diagonal matrices. Then, from Assumption 1, we have

$$0 \leq -2f^T(\varpi(t))\Gamma_1 f(\varpi(t)) + 2\varpi^T(t)\Gamma_1(H_1 + H_2)f(\varpi(t)) - 2\varpi^T(t)H_1\Gamma_1H_2\varpi(t) \quad (23)$$

$$\begin{aligned}
0 \leq & -2f^T(\varpi(t - \tau(t)))\Gamma_2 f(\varpi(t - \tau(t))) + 2\varpi^T(t - \tau(t)) \\
& \times \Gamma_2(H_1 + H_2)f(\varpi(t - \tau(t))) \\
& - 2\varpi^T(t - \tau(t))H_1\Gamma_2H_2\varpi(t - \tau(t)). \quad (24)
\end{aligned}$$

Define

$$\begin{aligned}
e_j = & [0_{n \times (j-1)n} \ I_n \ 0_{n \times (6-j)n}], \ j = 1, 2, \dots, 6 \\
\xi_2(t) = & \begin{bmatrix} \varpi^T(t) & \varpi^T(t-d) & f^T(\varpi(t)) & f^T(\varpi(t-d(t))) \\ \times \int_{t-d}^t \varpi^T(\varepsilon) d\varepsilon & \int_{-d}^0 \int_{t+\varepsilon}^t \varpi^T(u) du d\varepsilon \end{bmatrix}^T.
\end{aligned}$$

By using an approach similar to that used in Theorem 1, we can obtain the above exponential stability conditions. ■

Remark 4: Recently, GNNs have attracted great research interest; for example, the stability analysis problem of GNNs including additive time-varying delayed signals was examined in [30]. In [31], the improved stability of GNNs including interval time-varying delayed states was studied. The H_∞ state estimation of GNNs including interval time-delayed signals was investigated in [32]. Very recently, in [33], the stability problem was revisited for Markovian jump delayed GNNs. In the above methods, the RCC and Wirtinger-based double integral inequality (WDI) were utilized to obtain less conservative results. Thus, the above two techniques are the best methods for reducing conservatism. However, the exponential stability analysis problem for delayed GNNs has not been completely considered in the above results. In addition, the RCC and WDI techniques may not be accurate for exponential stability problems. In this paper, we utilized the new WII for both single and double integral terms.

IV. NUMERICAL EXAMPLES

Example 1: Let us consider the following matrix values:

$$C = \begin{bmatrix} 6.5618 & 0 & 0 \\ 0 & 5.5784 & 0 \\ 0 & 0 & 7.3269 \end{bmatrix}$$

$$\begin{aligned}
W_0 = & \begin{bmatrix} 0.3256 & -0.1904 & 0.3322 \\ -0.1564 & 0.2446 & 0.3674 \\ -0.1753 & 0.2956 & -0.3115 \end{bmatrix} \\
W_1 = & \begin{bmatrix} 0.1981 & -0.1313 & 0.1185 \\ 0.1645 & 0.0901 & 0.1013 \\ 0.0274 & -0.1518 & 0.0742 \end{bmatrix} \\
W_2 = & \begin{bmatrix} 1.8647 & 0.7983 & -0.8192 \\ 0.6983 & 1.4083 & 1.2748 \\ -0.4132 & -0.6093 & 1.2183 \end{bmatrix} \\
W_3 = & \begin{bmatrix} 0.4631 & 1.7363 & -1.7335 \\ 1.8369 & 1.4830 & 1.7428 \\ -1.4521 & -0.9603 & 1.8736 \end{bmatrix} \\
H_1 = & \text{diag}\{0, 0, 0\}, \ H_2 = \text{diag}\{0.5, 0.5, 0.5\}.
\end{aligned}$$

A. Stability Analysis

By considering the GNN (14) with $\mu = 1.2$, $d = 1$, and $\tau = 0.5$ and applying Theorem 2, we can obtain the following feasible solutions to LMI (15):

$$\begin{aligned}
P = & \begin{bmatrix} 39.0691 & -18.5234 & 20.7314 & 1.9127 & -1.1049 \\ -18.5234 & 45.6462 & -3.5469 & -1.3266 & 1.2421 \\ 20.7314 & -3.5469 & 63.9502 & 2.8042 & -0.5195 \\ 1.9127 & -1.3266 & 2.8042 & 13.6448 & -3.7799 \\ -1.1049 & 1.2421 & -0.5195 & -3.7799 & 15.2429 \\ 2.6739 & -1.6862 & 5.3377 & 8.2933 & -3.6949 \\ 7.9196 & -4.9988 & 5.7114 & -2.0523 & -2.1909 \\ -5.5125 & 7.1939 & -1.7188 & -1.8112 & -5.0803 \\ 7.3552 & -3.5224 & 18.1741 & 1.1287 & -0.6741 \\ 2.6739 & 7.9196 & -5.5125 & 7.3552 & \\ -1.6862 & -4.9988 & 7.1939 & -3.5224 & \\ 5.3377 & 5.7114 & -1.7188 & 18.1741 & \\ 8.2933 & -2.0523 & -1.8112 & 1.1287 & \\ -3.6949 & -2.1909 & -5.0803 & -0.6741 & \\ 26.4836 & 0.5188 & -0.6916 & 1.9064 & \\ 0.5188 & 46.4903 & -13.2618 & 16.8349 & \\ -0.6916 & -13.2618 & 43.8566 & -6.1617 & \\ 1.9064 & 16.8349 & -6.1617 & 79.2727 & \end{bmatrix} \\
Q = & \begin{bmatrix} 114.3368 & -6.6495 & 2.7397 \\ -6.6495 & 90.7478 & -9.4362 \\ 2.7397 & -9.4362 & 161.1047 \end{bmatrix} \\
R = & \begin{bmatrix} 7.7549 & -1.1358 & 4.4495 \\ -1.1358 & 10.4849 & -1.8651 \\ 4.4495 & -1.8651 & 13.6709 \end{bmatrix} \\
S = & \begin{bmatrix} 20.7488 & -11.5626 & 5.8076 \\ -11.5626 & 19.0814 & 0.8206 \\ 5.8076 & 0.8206 & 26.6407 \end{bmatrix} \\
T = & \begin{bmatrix} 348.7732 & 136.4528 & 27.2418 \\ 136.4528 & 332.9313 & 66.3766 \\ 27.2418 & 66.3766 & 359.9980 \end{bmatrix} \\
Z = & \begin{bmatrix} 3.0029 & -0.4653 & 0.5172 \\ -0.4653 & 3.0143 & -0.1786 \\ 0.5172 & -0.1786 & 4.0146 \end{bmatrix}, \quad \alpha = 1.7768.
\end{aligned}$$

Therefore, the GNN (14) attains exponential stability. When $\tau = 0.3$, the maximum exponential convergence rates $\alpha_{\max}$ are acquired for various d and μ , which are given in Table I. Table II shows the $\alpha_{\max}$ for various d and τ when $d(t)$ is not differentiable.

TABLE I
MAXIMUM EXPONENTIAL CONVERGENCE RATES
 $\alpha_{\max}$ FOR DIFFERENT d AND μ

$d \backslash \mu$	0.1	0.3	0.6	0.9	1.2
0.5	5.3278	5.0984	4.9107	4.7387	4.5536
1.0	3.2214	3.1845	2.9629	2.6867	2.3849
1.5	2.1358	2.0735	1.9334	1.7635	1.5178
2.0	1.6874	1.5684	1.4732	1.3861	1.2814

TABLE II
MAXIMUM EXPONENTIAL CONVERGENCE RATES
 $\alpha_{\max}$ FOR DIFFERENT d AND τ

$d \backslash \tau$	0.1	0.3	0.5	0.7	0.9
0.5	6.7825	6.4684	4.7756	3.4654	2.4354
1.0	4.4342	3.1354	2.3565	1.7347	1.2267
1.5	3.1567	2.1731	1.5513	1.1613	0.8167
2.0	2.3534	1.6518	1.2021	0.8535	0.6196

TABLE III
MINIMUM PASSIVITY INDEX γ FOR DIFFERENT VALUES OF d

$d \backslash \gamma$	0.5	1	1.5	2	2.5
Theorem 3.1	0.3035	0.3463	0.3648	0.3847	0.4258

TABLE IV
OPTIMAL DISSIPATIVITY INDEX γ FOR DIFFERENT VALUES OF d

$d \backslash \gamma$	0.5	1	1.5	2	2.5
Theorem 3.1	1.7324	1.7215	1.7017	1.6814	1.6583

B. Passivity Analysis

When $\mathfrak{Q} = 0$, $\mathfrak{S} = I_3$, and $\mathfrak{R} = \gamma I_3$, the passivity conditions are analyzed for the GNN (1) by applying Theorem 1. By taking $\mu = 1.2$ and $\tau = 0.5$, the minimum passivity index γ is obtained for different values of d (see Table III).

C. $(\mathfrak{Q}, \mathfrak{S}, \mathfrak{R})$ - γ -Dissipativity Analysis

When $\mathfrak{Q} = 0.5I_3$, $\mathfrak{S} = I_3$, and $\mathfrak{R} = 2I_3 - \gamma I_3$, the dissipativity conditions are analyzed for the GNN (1) by applying Theorem 1. By taking $\mu = 1.2$ and $\tau = 0.5$, the optimal dissipativity index γ is obtained for different values of d (see Table IV).

Example 2: Let us consider a GNN (17) with the parameters given in [39]

$$C = \text{diag}\{2.3, 3.4, 2.5\}, \quad W_0 = \begin{bmatrix} 0.9 & -1.5 & 0.1 \\ -1.2 & 1 & 0.2 \\ 0.2 & 0.3 & 0.8 \end{bmatrix}$$

$$W_1 = \begin{bmatrix} 0.8 & 0.6 & 0.2 \\ 0.5 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.5 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.2 \end{bmatrix}$$

$$H_1 = 0, \quad H_2 = 0.2I_3.$$

When $d = 1$, by applying Theorem 2, the maximum allowable upper bound (MAUB) τ is obtained for different values of α (see Table V). The number of decision variables (NDV) in [39] is $((25n(n+1))/2) + 6n^2$, and the NDV in Theorem 2 is $((13n(n+1))/2) + 3n^2$. Therefore, the approach of this paper provides less conservative results than [39]. The state trajectories of the state $\varpi(t)$ is shown in Fig. 1.

TABLE V
MAUB τ FOR DIFFERENT VALUES OF α

$\tau \backslash \alpha$	0.1	0.3	0.5	0.8
Ref. [40]	9.127	4.895	3.421	2.313
Theorem 3.3	11.654	6.967	5.762	4.683

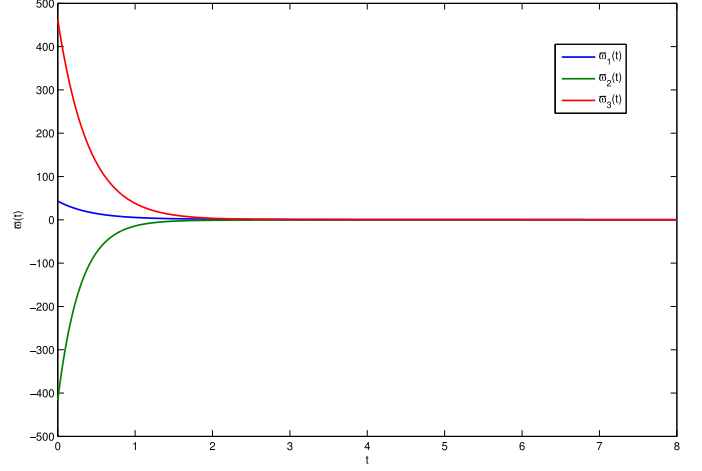


Fig. 1. State response $\varpi(t)$ in Example 2.

TABLE VI
MAXIMUM ADMISSIBLE CONVERGENCE RATES
 $\alpha_{\max}$ FOR DIFFERENT d , τ , AND μ

Methods	[37]	[38]	[39]	[40]	This paper
$d = 0.8, \tau = 0.2, \mu = 0.5$	0.10	0.11	0.30	0.21	0.43
$d = 0.6, \tau = 0.2, \mu = 0.5$	0.33	0.29	0.55	0.42	1.00
$d = 0.5, \tau = 0.2, \mu = 0.5$	0.35	0.45	0.73	0.59	1.39
$d = 0.5, \tau = 0.2, \mu = 0$	0.56	0.67	0.86	0.89	1.76

Example 3: Consider an NN (17) with the following parameters:

$$C = \text{diag}\{6, 5, 7\}, \quad W_0 = \begin{bmatrix} 1.2 & -0.8 & 0.6 \\ 0.5 & -1.5 & 0.7 \\ -0.8 & -1.2 & -1.4 \end{bmatrix}$$

$$W_1 = \begin{bmatrix} -1.4 & 0.9 & 0.5 \\ -0.6 & 1.2 & 0.8 \\ 0.5 & -0.7 & 1.1 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 1.8 & 0.7 & -0.8 \\ 0.6 & 1.4 & 1.0 \\ -0.4 & -0.6 & 1.2 \end{bmatrix}.$$

By applying Theorem 2, we obtain the maximum admissible convergence rate $\alpha_{\max}$. Table VI lists the maximum admissible convergence rates $\alpha_{\max}$ for different values of d , τ , and μ when $H_1 = \text{diag}\{-1.2, 0, -2.4\}$ and $H_2 = \{0, 1.4, 0\}$. Table VII lists the maximum admissible convergence rates $\alpha_{\max}$ for different values of μ when $d = \tau = 0.2$, $H_1 = \text{diag}\{0, 0, 0\}$, and $H_2 = \{-1.2, 1.4, -2.4\}$. Table VIII lists the maximum admissible convergence rates $\alpha_{\max}$ for different values of τ when $d = \mu = 0.2$, $H_1 = \text{diag}\{0, 0, 0\}$, and $H_2 = \{-1.2, 1.4, -2.4\}$. The NDVs in [36]–[41] are $((17n(n+1))/2) + 2n^2$, $((16n(n+1))/2) + 32n^2$, $((19n(n+1))/2) + 7n^2$, $((25n(n+1))/2) + 6n^2$, $((9n(n+1))/2)$, and

TABLE VII
MAXIMUM ADMISSIBLE CONVERGENCE RATES $\alpha_{\max}$
FOR DIFFERENT VALUES OF μ AND $\tau = 0.2$

$d = 0.2$	[41]	[42]	This paper
$\mu = 0.1$	0.76	1.48	3.24
$\mu = 0.3$	0.70	1.34	2.87
$\mu = 0.5$	0.67	1.18	2.54
unknown $\mu = 0.5$	0.64	0.78	1.96

TABLE VIII
MAXIMUM ADMISSIBLE CONVERGENCE RATES $\alpha_{\max}$
FOR DIFFERENT VALUES OF τ AND $d = 0.2, \mu = 0.2$

$d = 0.2$	[41]	[42]	This paper
$\tau = 0.1$	1.00	1.54	3.35
$\tau = 0.2$	0.73	1.41	2.98
$\tau = 0.3$	0.49	1.23	2.65

TABLE IX
MAXIMUM ADMISSIBLE CONVERGENCE RATES $\alpha_{\max}$
FOR VARIOUS VALUES OF μ WITH $d = 1$

$\alpha \backslash \mu$	0.0	0.1	0.2	unknown μ
Ref. [30]	0.2505	0.2378	0.2273	-
Corollary 3.5	2.3546	2.1954	2.0769	1.8769

$((13n(n+1))/2) + 6n^2$, respectively. The differences in the NDVs between the methods in [36]–[41] and Theorem 2 are $((4n(n+1))/2) - n^2$, $((3n(n+1))/2) + 29n^2$, $((6n(n+1))/2) + 4n^2$, $((12n(n+1))/2) + 3n^2$, $-((4n(n+1))/2) - 3n^2$, and $3n^2$, respectively. This demonstrates that the method established in this paper is less conservative than the methods given in [36]–[41].

Example 4: Choose the NN (16) with the following matrix values in [29]:

$$C = \text{diag}\{4.1989, 0.7160, 1.9985\}$$

$$W_3 = \begin{bmatrix} -0.1052 & -0.5069 & -0.1121 \\ -0.0257 & -0.2808 & 0.0212 \\ 0.1205 & -0.2153 & 0.1315 \end{bmatrix}$$

$$H_1 = \text{diag}\{0, 0, 0\}, H_2 = \{0.4219, 3.8993, 1.0160\}.$$

Table IX lists the $\alpha_{\max}$ for various values of μ when $d = 1$. The NDV in [29] is $((12n(n+1))/2) + 10n^2$, while that in Corollary 1 is $((12n(n+1))/2) + 3n^2$.

Example 5: Choose the NN (17) with the following parameters:

$$C = \begin{bmatrix} 3.99 & 0 \\ 0 & 2.99 \end{bmatrix}, W_1 = \begin{bmatrix} 0.009 & 0.14 \\ 0.05 & 0.09 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 0.45 & -0.2 \\ 0.3 & 0.42 \end{bmatrix}, W_3 = I_2$$

$$H_1 = \text{diag}\{0, 0\}, H_2 = \{1, 1\}.$$

Table X lists the MAUB d for various values of μ with $\alpha = 0.5, \tau = 0.1$ by using Theorem 2. It demonstrates that the method presented in this paper provides results with reduced conservatism compared with [36] and [39]. The state trajectories of the state $\varpi(t)$ is shown in Fig. 2.

TABLE X
MAUB d FOR VARIOUS μ WITH $\alpha = 0.5, \tau = 0.1$

$d \backslash \mu$	0.0	0.2	0.5	0.7
Ref. [37]	3.377	3.336	3.268	3.264
Ref. [40]	3.415	3.395	3.356	3.325
Theorem 3.3	5.346	4.876	4.592	4.264

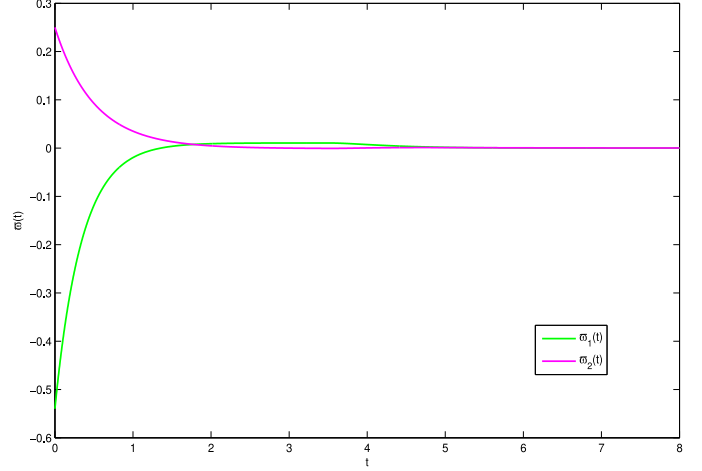


Fig. 2. State response $\varpi(t)$ in Example 5.

TABLE XI
 $\alpha_{\max}$ FOR VARIOUS μ

$\alpha \backslash \mu$	0.0	0.2	0.5	0.7
Ref. [62]	1.1540	0.8696	0.8354	0.8169
Ref. [59]	1.1544	0.8784	0.8484	0.8217
Corollary 3.6	1.6849	1.2893	1.2109	1.1938

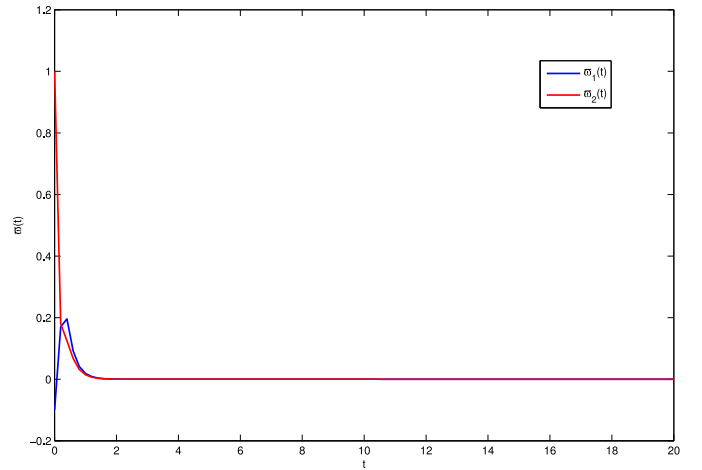


Fig. 3. State response $\varpi(t)$ in Example 6.

Example 6: Choose the NN (20) with the following parameters:

$$C = \begin{bmatrix} 2 & 0 \\ 0 & 3.5 \end{bmatrix}, W_0 = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1 \end{bmatrix}, W_1 = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$H_1 = \text{diag}\{0, 0\}, H_2 = \{1, 1\}.$$

By applying Corollary 2, we obtain the maximum allowable exponential convergence rates $\alpha_{\max}$ for various values of μ

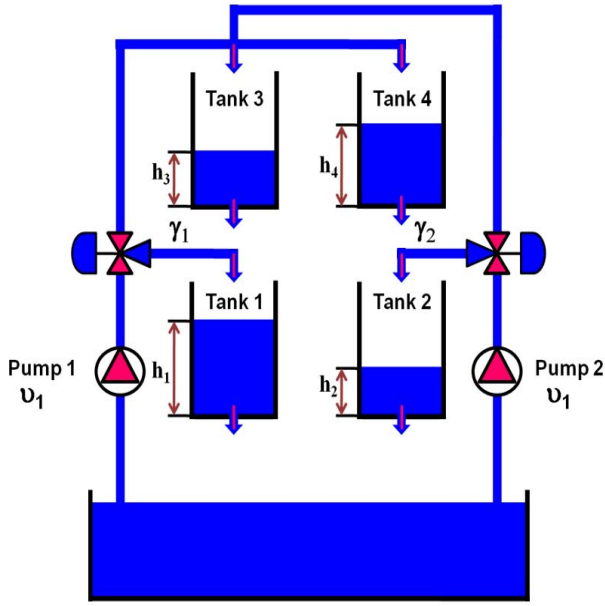


Fig. 4. Schematic representation of QTP (source [62]).

when $d = 1$, which are listed in Table XI. The state trajectories of the state $\varpi(t)$ is shown in Fig. 3.

Example 7: To show the application of artificial NNs, the quadruple-tank process (QTP) (see Fig. 4) has been chosen in this example. By using pumps 1 and 2, the target is to stabilize the liquid level in tanks 1 and 2, respectively. In the QTP, v_1 and v_2 are the input voltages to pumps 1 and 2, respectively. The outputs of pumps 1 and 2 are γ_1 and γ_2 (voltages from level measurement devices), respectively. The QTP has been expressed in the following system (see [62]):

$$\dot{\hat{x}}(t) = \hat{A}_0 \hat{x}(t) + \hat{A}_1 \hat{x}(t - d_1) + \hat{B}_0 \hat{u}(t - d_2) + \hat{B}_1 \hat{u}(t - d_3) \quad (25)$$

where

$$\begin{aligned} \hat{A}_0 &= \text{diag}\{-0.0021, -0.0021, -0.0424, -0.0424\} \\ \hat{A}_1 &= \begin{bmatrix} 0 & 0 & 0.0424 & 0 \\ 0 & 0 & 0 & 0.0424 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \hat{B}_0 &= \begin{bmatrix} 0.1113\gamma_1 & 0 & 0 & 0 \\ 0 & 0.1042\gamma_2 & 0 & 0 \end{bmatrix} \\ \hat{B}_1 &= \begin{bmatrix} 0 & 0 & 0 & 0.1113\gamma_1 \\ 0 & 0 & 0.1042\gamma_2 & 0 \end{bmatrix} \\ \hat{K} &= \begin{bmatrix} -0.1609 & -0.1765 & -0.0795 & -0.2073 \\ -0.1977 & -0.1579 & -0.2288 & -0.0772 \end{bmatrix} \\ \gamma_1 &= 0.333, \quad \gamma_2 = 0.307, \quad \hat{u}(t) = \hat{K}\hat{x}(t). \end{aligned}$$

Usually, by using differential equations the mass balances can be expressed. The transport delays are generally included with the delay phenomenon in the inlet of water to the tanks to obtain an interesting control problem. In addition, in this example transport delays between the valves and tanks are chosen as time-varying, and it is assumed that $d_1 = 0$, $d_2 = 0$, and $d_3 = d(t)$. The control input variable $\hat{u}(t)$ denotes the amount of water provided by the pumps. Therefore, it is easy

TABLE XII
UPPER BOUNDS OF d FOR VARIOUS VALUES OF γ_1 AND γ_2

$\gamma_2 \backslash \gamma_1$	0.3	0.4	0.5
0.3	0.9845	1.0058	1.0954
0.4	1.0084	1.0656	1.1072
0.5	1.0987	1.1193	1.1209

to see that $\hat{u}(t)$ exhibits a threshold value because of the limited capacity area and pump hoses. Thus, in this example the nonlinear function $\hat{u}(t)$ is considered as follows:

$$\begin{aligned} \hat{u}(t) &= \hat{K}\hat{f}(\hat{x}(t)) \\ \hat{f}(\hat{x}(t)) &= [\hat{f}_1(\hat{x}_1(t)), \dots, \hat{f}_4(\hat{x}_4(t))]^T \\ \hat{f}_i(\hat{x}_i(t)) &= 0.01(|\hat{x}_i(t) + 1| - |\hat{x}_i(t) - 1|), \quad i = 1, \dots, 4. \end{aligned}$$

The QTP (25) can be transferred to the GNN (1), with $W_3 = I$ and $W_2 = \tau = 0$, as follows:

$$\begin{aligned} \dot{\varpi}(t) &= -C\varpi(t) + W_0 f(\varpi(t)) + W_1 f(\varpi(t - d(t))) + J \\ \mathfrak{z}(t) &= f(\varpi(t)) \end{aligned} \quad (26)$$

where $C = -\hat{A}_0 - \hat{A}_1$, $W_0 = \hat{B}_0 \hat{K}$, $W_1 = \hat{B}_1 \hat{K}$, $f(\cdot) = \hat{f}(\cdot)$. Additionally, by taking the values

$$J = [0 \quad 0 \quad 0 \quad 0]^T, \quad H_1 = 0_4 \quad \text{and} \quad H_2 = 0.01I_4$$

the delay bound $d = 0.9867$ has been obtained; therefore, the LMI in Theorem 1 is feasible. In addition, the upper bounds of d for various values of γ_1 and γ_2 are given in Table XII.

Remark 5: The works in [38], [39], and [41] were restricted to the assumption that the derivative of the time-varying delay is either bounded or less than 1. However, this assumption is not required in this paper.

Remark 6: By establishing a suitable LKF and using the WII technique, the exponential stability, passivity, and $(\Omega, \mathfrak{S}, \mathfrak{R})$ - γ -dissipativity conditions are obtained without using any free-weighting matrices. To avoid numerical computational complexity, the LKF established in this paper has fewer NDVs than the methods in [36]–[39], [41], and [58]. In addition, through the new analysis method, we obtained the MAUBs. Therefore, the obtained results are less conservative than the approaches in [36]–[39], [41], and [58].

V. CONCLUSION

In this paper, we have examined the exponential stability, passivity, and $(\Omega, \mathfrak{S}, \mathfrak{R})$ - γ -dissipativity problems for delayed GNNs including discrete and distributed time-varying delayed states. Improved exponential stability, passivity, and $(\Omega, \mathfrak{S}, \mathfrak{R})$ - γ -dissipativity conditions were established by choosing a suitable LKF. The proposed conditions were represented in the form of LMIs by using a new WII technique for both single and double integrals and a new mathematical analysis method. Six numerical examples were used to explain the superiority of the presented results. The obtained results were compared with those of recent methods in the literature. We wish to extend this paper to the stochastic exponential stability, passivity, and $(\Omega, \mathfrak{S}, \mathfrak{R})$ - γ -dissipativity problems for GNNs including Markovian jumping parameters.

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Enhanced result on stability analysis of randomly occurring uncertain parameters, leakage, and impulsive BAM neural networks with time-varying delays: Discrete-time case

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Summary

In real-world problems, neural networks play an increasingly important role in terms of both theory and applications. In this paper, the asymptotic stability analysis issue is investigated for uncertain impulsive discrete-time bidirectional associative memory neural networks with leakage and time-varying delays. With the assistance of novel summation inequality, reciprocally convex combination technique, and triple Lyapunov-Krasovskii functionals terms, many cases of time-varying delays are examined to certify the stability of neural networks. Here, the uncertainties are considered as a randomly occurring parameter uncertainty, and it conforms certain mutually uncorrelated Bernoulli-distributed white noise sequences. An important feature of the results reported here is that the probability of occurrence of the parameter uncertainties specify a priori estimate. Finally, numerical examples are proposed to expose the capability and efficiency of our research work with the help of the LMI control toolbox in MATLAB.

KEYWORDS

BAM neural networks, leakage delay, linear matrix inequality (LMI), random uncertainties

1 | INTRODUCTION

As is known, the recurrent type neural network (NN) was elaborated in 1980. Since then, this type of NNs has been widely investigated and applied in many areas such as combinatorial optimization, signal processing, associative memory, pattern recognition, signal processing.^{1–6} The bidirectional associative memory (BAM) NN is another kind of recurrent NN, and it is a nonlinear feedback network model. It is of 2 different layers, X and Y . These layers are always interconnected with each other. Once the weights have been entrenched, input into layer X presents the pattern in layer Y and vice versa. Recently, because of its applications in many fields, BAM NNs have attracted the attention of many researchers. For solving optimization problems, it is anticipated that such type of NNs have obtained a unique zero solution, which is either globally exponentially or asymptotically stable.

In most of the physical dynamical systems and also in control schemes, there is an abrupt state change at certain moments of time. It is natural to assume that these changes occur promptly, namely in the form of impulses.^{7,8} In fact, in dynamic systems, impulsive phenomenon always exists in the evolution process such as in biological and financial models.

It is well known that impulses can show about complex impact on the system dynamics. Moreover, it is stated that the anxiety of impulsive NNs can damage the dynamical behaviors of the NNs and is the same as time delays affect, see other works.⁷⁻¹¹ Therefore, it is imperative to consider both the impulsive and delay effects on the dynamical behaviors of NNs. In addition, there have appeared many results on delayed NNs with impulsive perturbations (eg, see other works^{10,12-15}). One of the significant types of NNs is that with leakage delay. Forgetting delay or leakage delay is the negative feedback term in the state vector, which has the capacity to destabilize a NN system. For that reason, in dynamical NNs systems, leakage delay is handled. At first, BAM with leakage delay was inspected by Gopalasamy. There are some theoretical and technical complications while handling the leakage delay. Therefore, leakage delay has a great effect on the dynamics of NNs.¹⁶⁻¹⁹

From the past, continuous-time NNs have played an important role among researchers. However, owing to its applications in vast areas, discrete-time NNs become more crucial than continuous time²⁰⁻²² because discrete-time NNs help to discretize the continuous-time counterpart for the sake of computer-based simulations and computations. In a dynamical way, discrete time could be more appropriate to model digitally transmitted signals. Moreover, discrete- and continuous-time systems are often applied to computer realization and theoretical analysis, respectively. When modeling natural NNs, 2 main resources of the performance degradation are stochastic disturbances and parameter uncertainties. Therefore, the stability problem for discrete-time stochastic NNs with parameter uncertainties becomes more important. In practical systems, many uncertain factors exist. Uncertainty in a control system may be applied to modeling and measurement errors and parameter variations.²³ Thus, the study on the robust stability of uncertain discrete-time varying delayed NNs becomes more significant. The results acquired will be very helpful for further research, eg, see other works²⁴⁻³² and references therein.

By the assistance of the augmented Lyapunov-Krasovskii functional, Kwon et al discussed the discrete-time NNs with several time-varying delays.³³ With the help of some zero equalities and delay partitioning approach, the stability results have been proved in another work by Kwon et al.³⁴ Shu et al³⁵ proposed the concept of stability issues by the aid of novel summation inequality. Jin et al³⁶ delivered some stability conditions with the help of summation inequality and Wirtinger-based inequality. Moreover, there are many results on the stability analysis of Markovian jumping NNs with time-varying delays, eg, see other works.³⁷⁻⁴⁰ Zhang et al^{41,42} explored the concept of stability results in the sense of Lyapunov. Qui et al gave out some delay-dependent conditions for the delayed neutral system by the delay-dividing approach.⁴³ Moreover, there are many results on time-varying delays, eg, see other works.⁴⁴⁻⁴⁷ In other works,⁴⁸⁻⁵⁰ the authors studied the stability of uncertain parameters. However, leakage delay is not taken into account. However, there are many results on leakage delay. For example, in the work of Banu et al,⁵¹ the stability performance of NNs when affected by the leakage delay and parameter uncertainties are deliberated to analyze the robustness. Moreover, the leakage delay is fully analyzed in other works.⁵²⁻⁵⁵

From the above motivated deliberations, the stability analysis of uncertain impulsive discrete time BAM NNs with time-varying and leakage delay is addressed in this paper. With the aid of novel summation inequality and reciprocally convex technique, delay-dependent and uncertain conditions are derived, respectively, for stability criterion. In addition, the recommended method in this addressed work has reduced the computational burden because of not using delay partitioning approach. Finally, illustrative examples are derived to exhibit that the presented norms can afford a maximum upper bound when compared with previous literature.

The main tenors of this proposed work are compiled as follows:

- This is the first time, and we use Corollary 1, in which a new summation inequality is dealt in $\Delta V_4(k)$ and $\Delta V_5(k)$ for uncertain impulsive discrete-time BAM NNs with time-varying and leakage delays.
- The time-varying delays such as $\alpha(k)$ and $\beta(k)$ are classified into many cases for the first time in BAM NNs. Moreover, in our proposed research work, the trajectory of the system shows to be less conserved than the existing works in the literature.
- From Wirtinger-based integral inequality, a novel summation inequality is employed, which is the version of discrete time. A new inequality is disposed on the basis of the novel summation inequality and reciprocally convex approach. Furthermore, it helps us to organize the stability conditions as well.
- By handling the discrete-time varying delay term to our impulsive BAM NNs with leakage delay and uncertain parameters, we received the maximum allowable upper bounds of discrete-time varying delay that is very large, when compared with the previous results, see Tables 1 and 2 in Examples 1 and 2. This explores that the approach developed in this paper is effective and less conservative than some existing works, which ensures the novelty of our proposed research work.

TABLE 1 Comparisons of upper bounds of time delays

Methods	Lower Bounds	Upper Bounds
See the work of Liu et al ⁵⁶	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M = 4$
See the work of Gao and Cui ⁵⁷	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M = 4$
See the work of Raja et al ⁵⁸	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M = 2$
See the work of Li et al ⁵⁹	$\sigma_m = \tau_m = 3$	$\sigma_M = \tau_M = 5$
See the work of Raja et al ⁶⁰	$\sigma_m = \tau_m = 2$	$\sigma_M = \tau_M > 0$
In this paper	$\underline{\alpha} = \underline{\beta} = 1$	$\bar{\alpha} = \bar{\beta} > 0$

TABLE 2 Comparisons of upper bounds of time delays

Methods	Lower Bounds	Upper Bounds
See the work of Wu et al ⁶¹	$d_1 = 4$	$d_2 = 5$
See the work of Wang and Li ⁶²	$h_1 = 10$	$h_2 = 24$
In this paper	$\underline{\alpha} = \underline{\beta} = 1$	$\bar{\alpha} = \bar{\beta} > 0$

- Two special cases of the main results are derived in Corollaries 2 and 3, and by using MATLAB LMI control Toolbox, the LMIs can be checked properly for the respective corollaries are feasible as well as its simulations are provided in Examples 3 and 4.

The remainder of this manuscript is classified as follows. In Section 2, preliminaries are given, and the problem formulation will be employed. In addition, new sum inequality corollary and lemmas are provided. Section 3 explains the main results of discrete time BAM NNs with time-varying delays to check the asymptotic stability. Randomly occurring uncertain parameters with discrete-time BAM NNs are exposed in Section 4. Section 5 demonstrates our results with illustrative examples. At last, conclusions drawn in Section 6.

2 | PRELIMINARIES AND PROBLEM DESCRIPTION

Notation. Throughout this manuscript, the notations are quite standard. $\mathbb{R}^n$ is the n -dimensional Euclidean space, and $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices. I refers to the identity matrix with appropriate dimensions. $\text{Diag}(\cdot)$ denotes the diagonal matrix. A^T means the transpose of matrix A . k denotes the set of positive integers. For real symmetric matrices X and Y , the notation $X \geq Y$ (resp, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (resp, positive definite). $\mathbb{N} = \{1, 2, \dots, n\}$, and $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^n$. (-1) is the matrix inverse. The mathematical expectation is $\mathbb{E}$. The symbol $*$ within a matrix represents the symmetric term of the matrix.

In this paper, the impulsive discrete-time BAM NNs with time-varying and constant leakage delay will be considered as follows:

$$\begin{aligned}
 u_i(k+1) &= \left[l_i u_i(k-\rho) + \sum_{j=1}^n m_{ji} \hat{f}(v_j(k)) + \sum_{j=1}^n n_{ji} \hat{g}(v_j(k-\alpha(k))) + I \right] \\
 u_i(k_r) &= u_i(k_r^-) + \hat{e}_{ir}(u_i(k_r^-)), \quad r = 1, 2, 3, \dots \\
 v_j(k+1) &= \left[a_j v_j(k-\rho) + \sum_{i=1}^n b_{ij} \bar{f}(u_i(k)) + \sum_{i=1}^n c_{ij} \bar{g}(u_i(k-\beta(k))) + J \right] \\
 v_j(k_r) &= v_j(k_r^-) + \bar{e}_{jr}(v_j(k_r^-)), \quad r = 1, 2, 3, \dots \\
 u(k) &= \bar{\chi}(k), \quad k = -\kappa, -\kappa+1, \dots, 0 \\
 v(k) &= \bar{\varphi}, \quad k = -\iota, -\iota+1, \dots, 0,
 \end{aligned} \tag{1}$$

and the vector form is

$$\begin{aligned}
 u(k+1) &= [Lu(k-\rho) + M\hat{f}(v(k)) + N\hat{g}(v(k-\alpha(k))) + I] \\
 u(k_r) &= u(k_r^-) + \hat{e}_r(u(k_r^-)), \quad r = 1, 2, 3, \dots \\
 v(k+1) &= [Av(k-\rho) + B\hat{f}(u(k)) + C\hat{g}(u(k-\beta(k))) + J] \\
 v(k_r) &= v(k_r^-) + \tilde{e}_r(v(k_r^-)), \quad r = 1, 2, 3, \dots \\
 u(k) &= \bar{\chi}(k), \quad k = -\kappa, -\kappa+1, \dots, 0 \\
 v(k) &= \bar{\varphi}, \quad k = -\iota, -\iota+1, \dots, 0,
 \end{aligned} \tag{2}$$

where $u(\cdot) = [u_1(\cdot), u_2(\cdot), \dots, u_n(\cdot)]^T \in \mathbb{R}^n$ and $v(\cdot) = [v_1(\cdot), v_2(\cdot), \dots, v_n(\cdot)]^T \in \mathbb{R}^n$ are the state vectors. The activation functions are $\hat{f}(\cdot) = [\hat{f}_1(\cdot), \hat{f}_2(\cdot), \dots, \hat{f}_n(\cdot)]^T \in \mathbb{R}^n$, $\hat{g}(\cdot) = [\hat{g}_1(\cdot), \hat{g}_2(\cdot), \dots, \hat{g}_n(\cdot)]^T \in \mathbb{R}^n$, $\tilde{f}(\cdot) = [\tilde{f}_1(\cdot), \tilde{f}_2(\cdot), \dots, \tilde{f}_n(\cdot)]^T \in \mathbb{R}^n$, and $\tilde{g}(\cdot) = [\tilde{g}_1(\cdot), \tilde{g}_2(\cdot), \dots, \tilde{g}_n(\cdot)]^T \in \mathbb{R}^n$. $I = [I_1, I_2, \dots, I_n]^T$ and $J = [J_1, J_2, \dots, J_n]^T$ are the input vectors from the external source. ρ and ρ denote the leakage delays satisfying $\rho \geq 1$ and $\rho \geq 1$. $L = \text{diag}\{l_1, l_2, \dots, l_n\}$ and $A = \text{diag}\{a_1, a_2, \dots, a_n\} \in \mathbb{R}^{n \times n}$ with $|a_i| < 1, |l_i| < 1$ are the leakage delay connection weight matrices. M and B, N and $C \in \mathbb{R}^{n \times n}$ are the connection weight matrix and time-varying delay connection weight matrix, respectively. αk and βk represent the transmission delays satisfying $0 < \bar{\alpha} \leq \alpha(k) \leq \underline{\alpha}$ and $0 < \bar{\beta} \leq \beta(k) \leq \underline{\beta}$, respectively. $\underline{\alpha}, \underline{\beta} > 0$ and $\bar{\alpha}, \bar{\beta} > 0$ denote the lower and upper bounds of $\alpha(k)$ and $\beta(k)$, respectively. $\kappa = \max\{\rho, \bar{\alpha}\}$, $\iota = \max\{\rho, \bar{\beta}\}$.

The conditions $u(k_r) = u(k_r^-) + \hat{e}_r(u(k_r^-))$ and $v(k_r) = v(k_r^-) + \tilde{e}_r(v(k_r^-))$, $r = 1, 2, 3, \dots$, are the impulse dynamical activity caused by abrupt jumps at certain instants during the evolutionary process. $u(k_r^-) = \lim_{k \rightarrow k_r^-} u(k)$, $v(k_r^-) = \lim_{k \rightarrow k_r^-} v(k)$, $\hat{e}_r = \mathbb{R}^n \rightarrow \mathbb{R}$, $\tilde{e}_r = \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous over $\mathbb{R}^n$, and the impulse instant k_r is assumed to satisfy $0 = k_0 < k_1 < k_2 < \dots < k_{r-1} < k_r < \dots$ and $k_r - k_{r-1} > 1$ for $(r = 1, 2, \dots)$.

The following activation functions help satisfy the following conditions throughout this paper.

Assumption 1. $\hat{f}_j(\cdot), \hat{g}_j(\cdot), \tilde{f}_i(\cdot), \tilde{g}_i(\cdot), j, i = 1, 2, 3, \dots, n$ are the neuron activation functions, which are continuous and bounded.

Assumption 2. For $j = 1, 2, 3, \dots, n$ and $i = 1, 2, 3, \dots, n$ and any $s_1, s_2, t_1, t_2 \in \mathbb{R}$, $s_1 \neq s_2$, and $t_1 \neq t_2$, the neuron activation functions in Assumption 1 satisfying

$$\begin{aligned}
 F_j^- &\leq \frac{\hat{f}_j(s_1) - \hat{f}_j(s_2)}{s_1 - s_2} \leq F_j^+, \quad D_i^- \leq \frac{\tilde{f}_i(t_1) - \tilde{f}_i(t_2)}{t_1 - t_2} \leq D_i^+, \\
 G_j^- &\leq \frac{\hat{g}_j(s_1) - \hat{g}_j(s_2)}{s_1 - s_2} \leq G_j^+, \quad E_i^- \leq \frac{\tilde{g}_i(t_1) - \tilde{g}_i(t_2)}{t_1 - t_2} \leq E_i^+,
 \end{aligned}$$

where $G_j^-, G_j^+, F_j^-, F_j^+, D_i^-, D_i^+, E_i^-, E_i^+$ are known constants.

Remark 1. In 2006, Liu et al and Wang et al proposed the aforementioned Assumption 2, respectively. Here, the constants are allowed to be positive, negative, or zero. We use generalized activation functions, which is more general than Lipschitz and sigmoid activation functions. It is very helpful for using LMI-based technique to reduce the possible conservatism.

In order to simplify our proof, we shift the equilibrium point $u^* = [u_1^*, u_2^*, \dots, u_n^*]^T$ and $v^* = [v_1^*, v_2^*, \dots, v_n^*]^T$ of system (2) to the origin. Let $x_i(k) = u_i(k) - u_i^*$, $y_j(k) = v_j(k) - v_j^*$, $f_j(y_j(k)) = \hat{f}_j(v_j(k)) - \hat{f}_j(v_j^*)$, $g_j(y_j(k)) = \hat{g}_j(v_j(k)) - \hat{g}_j(v_j^*)$, $\tilde{f}_i(x_i(k)) = \tilde{f}_i(u_i(k)) - \tilde{f}_i(u_i^*)$, $\tilde{g}_i(x_i(k)) = \tilde{g}_i(u_i(k)) - \tilde{g}_i(u_i^*)$, $\chi(k) = \bar{\chi}(k) - u^*$, and $\varphi(k) = \bar{\varphi}(k) - v^*$.

After the transformation, the NN of system (2) can be rewritten as

$$\begin{aligned}
 x(k+1) &= Lx(k-\rho) + Mf(y(k)) + Ng(y(k-\alpha(k))) \\
 x(k_r) &= x(k_r^-) + e_r x(k_r^-) \\
 x(k) &= \chi(k), \quad k = -\kappa, -\kappa+1, \dots, 0 \\
 y(k+1) &= Ay(k-\rho(k)) + B\hat{f}(x(k)) + C\hat{g}(x(k-\beta(k))) \\
 y(k_r) &= y(k_r^-) + \tilde{e}_r y(k_r^-) \\
 y(k) &= \varphi(k), \quad k = -\iota, -\iota+1, \dots, 0,
 \end{aligned} \tag{3}$$

where $x(k) = [x_1(k), x_2(k), \dots, x_n(k)]^T \in \mathbb{R}^n$ and $y(k) = [y_1(k), y_2(k), \dots, y_n(k)]^T \in \mathbb{R}^n$ are the state vectors and $x(k - \rho) = [x_1(k - \rho), x_2(k - \rho), \dots, x_n(k - \rho)]^T$, $y(k - \rho) = [y_1(k - \rho), y_2(k - \rho), \dots, y_n(k - \rho)]^T$, $f(y(k)) = [f_1(y(k)), f_2(y(k)), \dots, f_n(y(k))]^T$, $g(y(k - \alpha(k))) = [g_1(y(k - \alpha(k))), g_2(y(k - \alpha(k))), \dots, g_n(y(k - \alpha(k)))]^T$, $\hat{f}(y(k)) = [\hat{f}_1(y(k)), \hat{f}_2(y(k)), \dots, \hat{f}_n(y(k))]^T$, $\hat{g}(x(k - \beta(k))) = [\hat{g}_1(y(k - \beta(k))), \hat{g}_2(y(k - \beta(k))), \dots, \hat{g}_n(y(k - \beta(k)))]^T$ are the activation functions.

From Assumption 2, the activation functions $f_j(\cdot), g_j(\cdot), \hat{f}_i(\cdot), \hat{g}_i(\cdot), j = 1, 2, 3, \dots, n, i = 1, 2, 3, \dots, n$ gratify

$$\begin{aligned} F_j^- &\leq \frac{f_j(s_1) - f_j(s_2)}{s_1 - s_2} \leq F_j^+, & D_i^- &\leq \frac{\hat{f}_i(t_1) - \hat{f}_i(t_2)}{t_1 - t_2} \leq D_i^+, \\ G_j^- &\leq \frac{g_j(s_1) - g_j(s_2)}{s_1 - s_2} \leq G_j^+, & E_i^- &\leq \frac{\hat{g}_i(t_1) - \hat{g}_i(t_2)}{t_1 - t_2} \leq E_i^+, \end{aligned}$$

for any $t_1 \neq t_2, s_1 \neq s_2$ and $f_j(0) = g_j(0) = 0, \hat{f}_i(0) = \hat{g}_i(0) = 0$.

To organize our theorems, the following lemmas and definition will be necessary.

Lemma 1. (see the work of Wang et al⁴⁷)

Let the functions $h_1, h_2, \dots, h_N : \mathbb{R}^m \rightarrow \mathbb{R}$ in an open subset D of $\mathbb{R}^m$ to be positive values. Then, the conditions satisfied by reciprocally convex combination of h_i over D are as follows:

$$\min_{\{\gamma_i/\gamma_i > 0, \sum_i \gamma_i = 1\}} \sum_i \frac{1}{\gamma_i} h_i(k) = \sum_i h_i(k) + \max_{z_{ij}(k)} \sum_{i \neq j} z_{ij}(k), \quad (4)$$

subject to

$$z_{ij} : \mathbb{R}^m \rightarrow \mathbb{R}, \quad z_{j,i}(k), \begin{pmatrix} h_i(k) & z_{i,j}(k) \\ z_{i,j}(k) & h_i(k) \end{pmatrix} \geq 0. \quad (5)$$

Lemma 2. (see the work of Banu et al⁵¹)

Two scalars $n_2 \geq n_1 > 0$ and $Z = Z^T$, and for any constant positive-definite matrix, $Z \in \mathbb{R}^{n \times n}$, such that the sums concerned are well defined. Then,

$$\begin{aligned} (i) \quad & \left[\sum_{i=k-n_2}^{k-n_1-1} \sum_{j=i}^{k-n_1-1} x(j) \right]^T Z \left[\sum_{i=k-n_2}^{k-n_1-1} \sum_{j=i}^{k-n_1-1} x(j) \right] \leq \frac{(n_2 - n_1)(n_2 - n_1 + 1)}{2} \sum_{i=k-n_2}^{k-n_1-1} \sum_{j=i}^{k-n_1-1} x^T(j) Z x(j), \\ (ii) \quad & \left[\sum_{i=-n_2}^{-n_1-1} \sum_{j=k+i}^{k-1} x(j) \right]^T Z \left[\sum_{i=-n_2}^{-n_1-1} \sum_{j=k+i}^{k-1} x(j) \right] \leq \frac{(n_2 - n_1)(n_2 + n_1 + 1)}{2} \sum_{i=-n_2}^{-n_1-1} \sum_{j=k+i}^{k-1} x^T(j) Z x(j). \end{aligned}$$

The following corollary helps to complete the proof of the asymptotic stability result.

Corollary 1. (see the work of Hou et al³⁷)

Let $G > 0$ be the $n \times n$ positive definite matrix. Then, for all $q_0, q_1, q_2, \dots, q_n \in \mathbb{R}^n$, the following inequality holds:

$$\sum_{k=0}^n \Delta q_k^T G \Delta q_k \geq \frac{1}{n+1} (q_{n+1} - q_0)^T G (q_{n+1} - q_0) + \frac{3}{n+1} \Omega_0^T \left(\frac{n+2}{n} G \right) \Omega_0,$$

where $\Delta q_k = q_{k+1} - q_k$ and $\Omega_0 = q_{n+1} + q_0 - \frac{2}{n+2} \sum_{k=0}^{n+1} q_k$.

Definition 1. The NNs (3) with discrete-time impulsive BAM is said to be globally asymptotically stable if every zero solution of the NNs satisfies the condition

$$\lim_{k \rightarrow \infty} \{ \|x(k)\|^2 + \|y(k)\|^2 \} = 0.$$

Remark 2. In our paper, some zero inequalities and reciprocally convex combination technique are approached because the convex combination lemma helps to reduce the decision variables in LMI, which is the relevance lemma of Jensen's inequality, and zero inequalities helps to reduce the conservatism of stability criterion than the existing literature.

3 | ASYMPTOTIC STABILITY RESULT

By the aid of Lyapunov-Krasovskii functions, NN system (3) with globally asymptotic stability will be derived. The following notations are proposed for our presentation convenience.

With the assistance of Corollary 1, we introduce $\hat{\alpha}$ and $\alpha(k)$ and $\hat{\beta}$ and $\beta(k)$ for solving the third and sixth terms of $\Delta V_4(k)$ and the latter four terms of $\Delta V_6(k)$.

Notations are as follows:

$$\begin{aligned} \varpi_1 &= \begin{cases} 1, & \text{when } \hat{\alpha} > 2, \quad \underline{\alpha} < \alpha(k) < \bar{\alpha} - 1 \\ 0, & \text{else} \end{cases}, \\ \varpi_2 &= \begin{cases} 1, & \text{when } \hat{\alpha} \geq 2, \quad \alpha(k) = \underline{\alpha}, \quad \bar{\alpha} - 1 \text{ or } \bar{\alpha} \\ 0, & \text{else} \end{cases}, \\ \varpi_3 &= \begin{cases} 1, & \text{when } \hat{\alpha} = 1, \quad \alpha(k) = \underline{\alpha} \text{ or } \bar{\alpha} \\ 0, & \text{else} \end{cases} \quad \tau_1 = \begin{cases} 1, & \text{when } \hat{\beta} > 2, \quad \underline{\beta} < \beta(k) < \bar{\beta} - 1 \\ 0, & \text{else} \end{cases}, \\ \tau_2 &= \begin{cases} 1, & \text{when } \hat{\beta} \geq 2, \quad \beta(k) = \underline{\beta}, \quad \bar{\beta} - 1 \text{ or } \bar{\beta} \\ 0, & \text{else} \end{cases}, \\ \tau_3 &= \begin{cases} 1, & \text{when } \hat{\beta} = 1, \quad \beta(k) = \underline{\beta} \text{ or } \bar{\beta} \\ 0, & \text{else} \end{cases}, \\ w(\underline{\alpha}) &= \begin{cases} 0, & \underline{\alpha} = 1 \\ \frac{1}{\underline{\alpha}-1}, & \underline{\alpha} > 1 \end{cases} \quad w^*(\underline{\beta}) = \begin{cases} 0, & \underline{\beta} = 1 \\ \frac{1}{\underline{\beta}-1}, & \underline{\beta} > 1 \end{cases} \quad h^* = \begin{cases} 0, & \bar{\alpha} = 1 \\ \frac{3(\bar{\alpha}+1)}{\bar{\alpha}-1}, & \bar{\alpha} \geq 2 \end{cases}, \\ h &= \begin{cases} 0, & \bar{\beta} = 1 \\ \frac{3(\bar{\beta}+1)}{\bar{\beta}-1}, & \bar{\beta} \geq 2 \end{cases}, \quad \zeta(\rho) = \begin{cases} 0, & \rho = 1 \\ \frac{1}{\rho-1}, & \rho > 1 \end{cases}, \quad \eta(\rho) = \begin{cases} 0, & \rho = 1 \\ \frac{1}{\rho-1}, & \rho > 1 \end{cases}, \\ \gamma_1 &= \begin{cases} 1, & \text{when } \hat{\alpha} \geq 2, \quad \underline{\alpha} < \alpha(k) < \bar{\alpha} - 1 \text{ or } \hat{\alpha} \geq 2, \quad \alpha(k) = \bar{\alpha} - 1 \\ 0, & \text{else} \end{cases}, \\ \gamma_2 &= \begin{cases} 1, & \text{when } \hat{\alpha} \geq 2, \quad \alpha(k) = \underline{\alpha}, \bar{\alpha} \text{ or } \hat{\alpha} = 1, \quad \alpha(k) = \underline{\alpha}, \bar{\alpha} \\ 0, & \text{else} \end{cases}, \\ \pi_1 &= \begin{cases} 1, & \text{when } \hat{\beta} \geq 2, \quad \underline{\beta} < \beta(k) < \bar{\beta} - 1 \text{ or } \hat{\beta} \geq 2, \beta(k) = \bar{\beta} - 1 \\ 0, & \text{else} \end{cases}, \\ \pi_2 &= \begin{cases} 1, & \text{when } \hat{\beta} \geq 2, \quad \beta(k) = \underline{\beta}, \bar{\beta} \text{ or } \hat{\beta} = 1, \beta(k) = \underline{\beta}, \bar{\beta} \\ 0, & \text{else} \end{cases}, \end{aligned}$$

$$\bar{\alpha} = \begin{cases} \bar{\alpha} - 2, & \varpi_1 = 1 \text{ or } \gamma_1 = 1 \\ \bar{\alpha}, & \varpi_2 = 1 \text{ or } \varphi_3 = 1 \text{ or } \gamma_2 = 1 \end{cases}, \quad \underline{\alpha} = \begin{cases} \underline{\alpha} + 1, & \varpi_1 = 1 \text{ or } \gamma_1 = 1 \\ \underline{\alpha}, & \varpi_2 = 1 \text{ or } \varphi_3 = 1 \text{ or } \gamma_2 = 1 \end{cases},$$

$$\bar{\beta} = \begin{cases} \bar{\beta} - 2, & \tau_1 = 1 \text{ or } \pi_1 = 1 \\ \bar{\beta}, & \tau_2 = 1 \text{ or } \tau_3 = 1 \text{ or } \pi_2 = 1 \end{cases}, \quad \underline{\beta} = \begin{cases} \underline{\beta} + 1, & \tau_1 = 1 \text{ or } \pi_1 = 1 \\ \underline{\beta}, & \tau_2 = 1 \text{ or } \tau_3 = 1 \text{ or } \pi_2 = 1 \end{cases}$$

$$\tilde{g}(k) = \begin{bmatrix} x^T(k), & \sum_{i=k-\underline{\beta}}^{k-1} x^T(i), & \sum_{i=k-\bar{\beta}}^{k-\underline{\beta}-1} x^T(i) \end{bmatrix}^T, \quad \vartheta^*(k) = \begin{bmatrix} y^T(k), & \sum_{j=k-\underline{\alpha}}^{k-1} y^T(j), & \sum_{j=k-\bar{\alpha}}^{k-\underline{\alpha}-1} y^T(j) \end{bmatrix}^T,$$

$$v(k) = \begin{bmatrix} x^T(k), & \sum_{i=k-\rho}^{k-1} x^T(i) \end{bmatrix}^T, \quad v^*(k) = \begin{bmatrix} y^T(k), & \sum_{j=k-\rho}^{k-1} y^T(j) \end{bmatrix}^T,$$

$$\mu(k) = x(k+1) - x(k), \quad v(k) = y(k+1) - y(k), \quad \beta^* = \frac{\hat{\beta}(\bar{\beta} + \underline{\beta} + 1)}{2},$$

$$\alpha^* = \frac{\hat{\alpha}(\bar{\alpha} + \underline{\alpha} + 1)}{2}, \quad \hat{\beta} = \bar{\beta} - \underline{\beta}, \quad \hat{\alpha} = \bar{\alpha} - \underline{\alpha},$$

$$\Gamma_i(\beta(k)) = \begin{cases} \Phi_{15}^T \begin{pmatrix} X_2 & 0 & W_{11} & W_{12} \\ * & X_2 l & W_{21} & W_{22} \\ * & * & X_2 & 0 \\ * & * & * & X_2 l \end{pmatrix} \Phi_{15}, & i = 1 \\ \Phi_{17}^T \begin{pmatrix} X_2 & 0 \\ * & X_2 l \end{pmatrix} \Phi_{17}, & i = 2, 3 \end{cases}$$

$$\tilde{\Gamma}_i(\alpha(k)) = \begin{cases} \Phi_{16}^T \begin{pmatrix} X_4 & 0 & Z_{11} & Z_{12} \\ * & X_4 l & Z_{21} & Z_{22} \\ * & * & X_4 & 0 \\ * & * & * & X_4 l \end{pmatrix} \Phi_{16}, & j = 1 \\ \Phi_{18}^T \begin{pmatrix} X_4 & 0 \\ * & X_4 l \end{pmatrix} \Phi_{18}, & j = 2, 3 \end{cases}$$

$$\aleph_i(\beta(k)) = \begin{cases} \Phi_{19}^T \begin{pmatrix} L_1 & T_1^T \\ * & L_1 \end{pmatrix} \Phi_{19} + \Phi_{21}^T \begin{pmatrix} L_2 & 0 \\ * & L_2 \end{pmatrix} \Phi_{21} & i = 1 \\ \Phi_{23}^T L_1 \Phi_{23} + \frac{1}{\hat{\beta}(\hat{\beta}+1)} \Phi_{24}^T L_2 \Phi_{24}, & i = 2 \end{cases}$$

$$\tilde{\aleph}_i(\alpha(k)) = \begin{cases} \Phi_{20}^T \begin{pmatrix} L_3 & T_2^T \\ * & L_3 \end{pmatrix} \Phi_{20} + \Phi_{22}^T \begin{pmatrix} L_4 & 0 \\ * & L_4 \end{pmatrix} \Phi_{22} & j = 1 \\ \Phi_{24}^T L_3 \Phi_{24} + \frac{1}{\hat{\alpha}(\hat{\alpha}+1)} \Phi_{25}^T L_4 \Phi_{25}, & j = 2 \end{cases}$$

$$\begin{aligned}
\Theta(k) &= \left[x^T(k), x^T(k - \underline{\beta}), x^T(k - \beta(k)), x^T(k - \bar{\beta}), \frac{1}{\underline{\beta} + 1} \sum_{i=k-\underline{\beta}}^k x^T(i), \frac{1}{\beta(k) - \underline{\beta} + 1}, \right. \\
&\quad \left. \sum_{i=k-\beta(k)}^{k-\underline{\beta}} x^T(i), \frac{1}{\bar{\beta} - \beta(k) + 1} \sum_{i=k-\bar{\beta}}^{k-\beta(k)} x^T(i), \hat{f}^T(x(k)), \hat{g}^T(x(k - \beta(k))), x^T(k - \rho), \frac{1}{\rho + 1} \sum_{i=k-\rho}^k x^T(i) \right]^T, \\
\Psi(k) &= \left[y^T(k), y^T(k - \underline{\alpha}), y^T(k - \alpha(k)), y^T(k - \bar{\alpha}), \frac{1}{\underline{\alpha} + 1} \sum_{j=k-\underline{\alpha}}^k y^T(j), \frac{1}{\alpha(k) - \underline{\alpha} + 1}, \right. \\
&\quad \left. \sum_{j=k-\alpha(k)}^{k-\underline{\alpha}} y^T(j), \frac{1}{\bar{\alpha} - \alpha(k) + 1} \sum_{j=k-\bar{\alpha}}^{k-\alpha(k)} y^T(j), f^T(y(k)), g^T(y(k - \alpha(k))), y^T(k - \rho), \frac{1}{\rho + 1} \sum_{j=k-\rho}^k y^T(j) \right]^T, \\
\Lambda &= \left(\underline{\beta}^2 X_1 + \hat{\beta}^2 X_2 \right) + \rho^2 Z_1 + (\beta^*)^2 L_1 + \frac{\hat{\beta}(\hat{\beta} + 1)}{4} L_2, \\
\Lambda^* &= \left(\underline{\alpha}^2 X_3 + \hat{\alpha}^2 X_4 \right) + \rho^2 Z_2 + (\alpha^*)^2 L_3 + \frac{\hat{\alpha}(\hat{\alpha} + 1)}{4} L_4.
\end{aligned}$$

Theorem 1. With the assistance of Assumption 2, NN system (3) is globally asymptotically stable for given integers $\rho \geq 1$, $\rho \geq 1$, $\underline{\beta}, \bar{\beta}, 0 < \underline{\beta} \leq \beta(k) \leq \bar{\beta}, \underline{\alpha}, \bar{\alpha}, 0 < \underline{\alpha} \leq \alpha(k) \leq \bar{\alpha}$ satisfying $0 < \underline{\beta} \leq \bar{\beta}$ and $0 < \underline{\alpha} \leq \bar{\alpha}$. Moreover, if there exist positive definite matrices $Q_i > 0, L_i > 0, i = 1, 2, 3, 4, P, P^*, X_i > 0, i = 1, 2, 3, 4, Z_i > 0, i = 1, 2$, positive definite diagonals matrices $\varsigma_1 > 0, \varsigma_2 > 0, \varsigma_3 > 0, \varsigma_4 > 0$ and any matrices $T_1, T_2, W_{11}, W_{12}, W_{21}, W_{22}, Z_{11}, Z_{12}, Z_{21}, Z_{22}$ positive constant $\epsilon (\epsilon \leq 1)$, then the following LMIs hold:

$$\begin{aligned}
(i) \quad & \sum_{i=1}^m \ln(1 + k_i)^2 (1 + \omega_i)^2 + k \ln(1 - \epsilon) \leq \vartheta(k), \\
& \text{for any } k \in [k_r, k_{r+1}), \text{ then } \lim_{k \rightarrow +\infty} \vartheta(k) = +\infty,
\end{aligned} \tag{6}$$

$$(ii) \quad \begin{pmatrix} X_2 & 0 & W_{11} & W_{12} \\ * & X_2 l & W_{21} & W_{22} \\ * & * & X_2 & 0 \\ * & * & * & X_2 l \end{pmatrix} \geq 0, \quad \begin{pmatrix} L_1 & T_1^T \\ * & L_1 \end{pmatrix} \geq 0, \tag{7}$$

$$(iii) \quad \begin{pmatrix} X_4 & 0 & Z_{11} & Z_{12} \\ * & X_4 l & Z_{21} & Z_{22} \\ * & * & X_4 & 0 \\ * & * & * & X_4 l \end{pmatrix} \geq 0, \quad \begin{pmatrix} L_4 & T_2^T \\ * & L_4 \end{pmatrix} \geq 0, \tag{8}$$

$$\Omega_1(\bar{\beta}^*) + \Omega_3 + \Omega_5 + \Omega_7(\bar{\beta}^*) + \Omega_9 + \Omega_{11}(\bar{\beta}^*) + \Omega_{13} + \Omega_{15} < 0,$$

$$\Omega_1(\underline{\beta}^*) + \Omega_3 + \Omega_5 + \Omega_7(\underline{\beta}^*) + \Omega_9 + \Omega_{11}(\underline{\beta}^*) + \Omega_{13} + \Omega_{15} < 0, \tag{9}$$

$$\Omega_2(\bar{\alpha}^*) + \Omega_4 + \Omega_6 + \Omega_8(\bar{\alpha}^*) + \Omega_{10} + \Omega_{12}(\bar{\alpha}^*) + \Omega_{14} + \Omega_{16} < 0,$$

$$\Omega_2(\underline{\alpha}^*) + \Omega_4 + \Omega_6 + \Omega_8(\underline{\alpha}^*) + \Omega_{10} + \Omega_{12}(\underline{\alpha}^*) + \Omega_{14} + \Omega_{16} < 0, \tag{10}$$

$$\Phi^* = [0, 0, 0, 0, 0, 0, 0, 0, M, N, L, 0], \quad \Phi^{**} = [0, 0, 0, 0, 0, 0, 0, 0, B, C, A, 0],$$

$$\Phi_1 = \left[\Phi^{*T}, (\underline{\beta} + 1)e_5^T - e_2^T, (\beta(k) - \underline{\beta} + 1)e_6^T + (\bar{\beta} - \beta(k) + 1)e_7^T - e_3^T - e_4^T \right]^T,$$

$$\Phi_3 = \left[\Phi^{**T}, (\underline{\alpha} + 1)r_5^T - r_2^T, (\alpha(k) - \underline{\alpha} + 1)r_6^T + (\bar{\alpha} - \alpha(k) + 1)r_7^T - r_3^T - r_4^T \right]^T,$$

$$\Phi_2 = \left[e_1^T, (\underline{\beta} + 1)e_5^T - e_1^T, (\beta(k) - \underline{\beta} + 1)e_6^T + (\bar{\beta} - \beta(k) + 1)e_7^T - e_3^T - e_2^T \right]^T$$

$$\Phi_4 = \left[r_1^T, (\underline{\alpha} + 1)r_5^T - r_1^T, (\alpha(k) - \underline{\alpha} + 1)r_6^T + (\bar{\alpha} - \alpha(k) + 1)r_7^T - r_3^T - r_2^T \right]^T$$

$$\Phi_5 = \left[\Phi^{*T}, (\rho + 1)e_{11}^T - e_{10}^T \right]^T, \quad \Phi_7 = \left[\Phi^{**T}, (\rho + 1)r_{11}^T - r_{10}^T \right]^T,$$

$$\phi_6 = \left[e_1^T, (\rho + 1)e_{11}^T - e_1^T \right]^T, \quad \phi_8 = \left[r_1^T, (\rho + 1)r_{11}^T - r_1^T \right]^T,$$

$$\Phi_9 = \left[\Phi^{*T} - e_1 \right], \quad \Phi_{10} = \left[\Phi^{**T} - r_1 \right]^T,$$

$$\Phi_{11} = \left[e_1^T - e_2^T, e_1^T + e_2^T - 2e_5^T \right]^T, \quad \Phi_{12} = \left[r_1^T - r_2^T, r_1^T + r_2^T - 2r_5^T \right]^T,$$

$$\Phi_{13} = \left[e_1^T - e_{10}^T, e_1^T + e_{10}^T - 2e_{11}^T \right]^T, \quad \Phi_{14} = \left[r_1^T - r_{10}^T, r_1^T + r_{10}^T - 2r_{11}^T \right]^T,$$

$$\Phi_{15} = \left[e_2^T - e_3^T, e_2^T + e_3^T - 2e_6^T, e_3^T - e_4^T, e_3^T + e_4^T - 2e_7^T \right]^T,$$

$$\Phi_{16} = \left[r_2^T - r_3^T, r_2^T + r_3^T - 2r_6^T, r_3^T - r_4^T, r_3^T + r_4^T - 2r_7^T \right]^T,$$

$$\Phi_{17} = \left[e_2^T - e_4^T, e_2^T + e_4^T - \frac{2}{\hat{\beta} + 1} \left((\bar{\beta} - \beta(k) + 1)e_7^T + (\beta(k) - \underline{\beta} + 1)e_6^T \right) - e_3^T \right]^T,$$

$$\Phi_{18} = \left[r_2^T - r_4^T, r_2^T + r_4^T - \frac{2}{\hat{\alpha} + 1} \left((\bar{\alpha} - \alpha(k) + 1)r_7^T + (\alpha(k) - \underline{\alpha} + 1)r_6^T \right) - r_3^T \right]^T,$$

$$\Phi_{19} = \left[(\beta(k) - \underline{\beta})e_1^T - (\bar{\beta} - \beta(k) + 1)e_7^T + e_3^T(\bar{\beta} - \beta(k))e_1^T - (\beta(k) - \underline{\beta} + 1)e_6^T + e_2^T \right]^T,$$

$$\Phi_{20} = \left[(\alpha(k) - \underline{\alpha})r_1^T - (\bar{\alpha} - \alpha(k) + 1)r_7^T + r_3^T(\bar{\alpha} - \alpha(k))r_1^T - (\alpha(k) - \underline{\alpha} + 1)r_6^T + r_2^T \right]^T,$$

$$\Phi_{21} = \left[e_3^T - e_7^T, e_2^T + e_6^T \right]^T, \quad \Phi_{22} = \left[r_3^T - r_7^T, r_2^T + r_6^T \right]^T,$$

$$\Phi_{23} = \left[e_1 - (\bar{\beta} - \beta(k) + 1)e_7 + e_3 \right]^T, \quad \Phi_{24} = \left[r_1 - (\bar{\alpha} - \alpha(k) + 1)r_7 + r_3 \right]^T,$$

$$\Phi_{25} = \left[\hat{\beta}e_2 - (\bar{\beta} - \beta(k) + 1)e_7 + e_3 \right]^T, \quad \Phi_{26} = \left[\hat{\alpha}r_2 - (\bar{\alpha} - \alpha(k) + 1)r_7 + r_3 \right]^T,$$

$$\Phi_{27} = \left[e_1^T, e_8^T \right]^T, \quad \Phi_{28} = \left[r_1^T, r_8^T \right]^T,$$

$$\Phi_{29} = \left[e_3^T, e_9^T \right]^T, \quad \Phi_{30} = \left[r_3^T, r_9^T \right]^T,$$

$$\Omega_1(\beta(k)) = \frac{1}{2} \text{sym} \left\{ (\Phi_1 + \Phi_2)^T Q_1 (\Phi_1 - \Phi_2) \right\}, \quad \Omega_2(\alpha(k)) = \frac{1}{2} \text{sym} \left\{ (\Phi_3 + \Phi_4)^T Q_2 (\Phi_3 - \Phi_4) \right\},$$

$$\Omega_3 = \frac{1}{2} \text{sym} \left\{ (\Phi_5 + \Phi_6)^T Q_3 (\Phi_5 - \Phi_6) \right\}, \quad \Omega_4 = \frac{1}{2} \text{sym} \left\{ (\Phi_7 + \Phi_8)^T Q_4 (\Phi_7 - \Phi_8) \right\},$$

$$\Omega_5 = (\hat{\beta} + 1)e_1^T P e_1 - e_3^T P e_3, \quad \Omega_6 = (\hat{\alpha} + 1)r_1^T P^* r_1 - r_3^T P^* r_3,$$

$$\begin{aligned}
\Omega_7(\beta(k)) &= \sum_{i=1}^3 \beta_i \Omega_{7i}(\beta(k)), \quad \Omega_8(\alpha(k)) = \sum_{i=1}^3 \alpha_i \Omega_{8i}(\alpha(k)), \\
e_i &= [0_{n \times (i-1)n}, I, 0_{n \times (11-i)n}]_{n \times 11n}, \quad i = 1, 2, 3, \dots, 11, \\
r_i &= [0_{n \times (i-1)n}, I, 0_{n \times (11-i)n}]_{n \times 11n}, \quad i = 1, 2, 3, \dots, 11, \\
\Omega_{7i}(\beta(k)) &= \Phi_9^T (\beta^2 X_1 + \hat{\beta}^2 X_2) \phi_9 - \Phi_{11}^T \begin{pmatrix} X_1 & 0 \\ * & 3(\underline{\beta} + 1)w^*(\underline{\beta} X_1) \end{pmatrix} \Phi_{11} - \Gamma_i(\beta(k)), \quad i = 1, 2, 3 \\
\Omega_{8j}(\alpha(k)) &= \Phi_{10}^T (\underline{\alpha}^2 X_3 + \hat{\alpha}^2 X_4) \phi_{10} - \Phi_{12}^T \begin{pmatrix} X_3 & 0 \\ * & 3(\underline{\alpha} + 1)w(\underline{\alpha} X_3) \end{pmatrix} \Phi_{12} - \tilde{\Gamma}_j(\alpha(k)), \quad j = 1, 2, 3 \\
\Omega_9 &= \Phi_9^T \rho^2 Z_1 \Phi_9 - \Phi_{13} \begin{pmatrix} Z_1 & 0 \\ * & 3(\rho + 1)\zeta(\rho)Z_1 \end{pmatrix} \Phi_{13} \\
\Omega_{10} &= \Phi_{10}^T \rho^2 Z_2 \Phi_{10} - \Phi_{14} \begin{pmatrix} Z_2 & 0 \\ * & 3(\rho + 1)\zeta(\rho)Z_2 \end{pmatrix} \Phi_{14} \\
\Omega_{11i}(\beta(k)) &= \Phi_9^T \left((\beta^*)^2 L_1 + \frac{\hat{\beta}(\hat{\beta} + 1)}{4} L_2 \right) \Phi_9 - \aleph_i(\beta(k)), \quad i = 1, 2, \\
\Omega_{12j}(\alpha(k)) &= \Phi_{10}^T \left((\alpha^*)^2 L_3 + \frac{\hat{\alpha}(\hat{\alpha} + 1)}{4} L_4 \right) \Phi_{10} - \tilde{\aleph}_j(\alpha(k)), \quad j = 1, 2, \\
\Omega_{13} &= \Phi_{27}^T \begin{pmatrix} D_1 \zeta_1 & -D_2 \zeta_1 \\ -D_2 \zeta_1 & \zeta_1 \end{pmatrix} \Phi_{27}, \quad \Omega_{14} = \Phi_{28}^T \begin{pmatrix} E_1 \zeta_2 & -E_2 \zeta_2 \\ -E_2 \zeta_2 & \zeta_2 \end{pmatrix} \Phi_{28} \\
\Omega_{15} &= \Phi_{29}^T \begin{pmatrix} F_1 \zeta_3 & -F_2 \zeta_3 \\ -F_2 \zeta_3 & \zeta_3 \end{pmatrix} \Phi_{29}, \quad \Omega_{16} = \Phi_{30}^T \begin{pmatrix} G_1 \zeta_4 & -G_2 \zeta_4 \\ -G_2 \zeta_4 & \zeta_4 \end{pmatrix} \Phi_{30}.
\end{aligned}$$

Proof. Lyapunov-Krasovskii functionals are constructed as follows:

$$V(k) = \sum_{i=1}^6 V_i(k),$$

where

$$\begin{aligned}
V_1(k) &= \vartheta^T(k) Q_1 \vartheta(k) + \vartheta^{*T}(k) Q_2 \vartheta^*(k), \\
V_2(k) &= v^T(k) Q_3 v(k) + v^{*T}(k) Q_4 v^*(k), \\
V_3(k) &= \sum_{i=k-\underline{\beta}(k)}^{k-1} x^T(i) P x(i) + \sum_{j=k-\underline{\beta}+1}^{\underline{\beta}} \sum_{i=k+j}^{k-1} x^T(i) P x(i) + \sum_{j=k-\underline{\alpha}(k)}^{k-1} y^T(j) P^* y(j) + \sum_{i=k-\underline{\alpha}+1}^{\underline{\alpha}} \sum_{j=k+i}^{k-1} y^T(j) P^* y(j) \\
V_4(k) &= \underline{\beta} \sum_{j=-\underline{\beta}+1}^0 \sum_{i=k+j-1}^{k-1} \mu^T(i) X_1 \mu(i) + \hat{\beta} \sum_{j=-\underline{\beta}+1}^{\underline{\beta}} \sum_{i=k+j-1}^{k-1} \mu^T(i) X_2 \mu(i) + \underline{\alpha} \sum_{i=-\underline{\alpha}+1}^0 \sum_{j=k+i-1}^{k-1} v^T(j) \\
&\quad \times X_3 v(j) + \hat{\alpha} \sum_{i=-\underline{\alpha}+1}^{\underline{\alpha}} \sum_{j=k+i-1}^{k-1} v^T(j) X_4 v(j) \\
V_5(k) &= \rho \sum_{j=-\rho+1}^0 \sum_{i=k+j-1}^{k-1} \mu^T(i) Z_1 \mu(i) + \rho \sum_{i=-\rho+1}^0 \sum_{j=k+i-1}^{k-1} v^T(j) Z_2 v(j) \\
V_6(k) &= \beta^* \sum_{i=-\beta^*}^{\underline{\beta}-1} \sum_{j=i}^{-1} \sum_{h=k+j}^{k-1} \mu^T(h) L_1 \mu(h) + \frac{1}{2} \sum_{i=k-\underline{\beta}}^{\underline{\beta}-1} \sum_{j=i}^{k-\underline{\beta}-1} \sum_{h=j}^{k-1} \mu^T(h) L_2 \mu(h) + \alpha^* \sum_{j=-\alpha^*}^{\underline{\alpha}-1} \sum_{i=j}^{-1} \sum_{h^*=k+i}^{k-1} v^T(h^*) L_3 v(h^*) \\
&\quad + \frac{1}{2} \sum_{j=k-\underline{\alpha}}^{\underline{\alpha}-1} \sum_{i=j}^{k-\underline{\alpha}-1} \sum_{h^*=i}^{k-1} v^T(h^*) L_4 v(h^*).
\end{aligned}$$

Along the trajectory of NN (3), $V_i(k)$ ($i = 1, 2, 3, 4, 5, 6$) is calculated as follows:

$$\Delta V(k) = \sum_{i=1}^6 \Delta V_i(k),$$

where

$$\begin{aligned} \Delta V_1(k) &= \Theta^T(k) [\Phi_1^T Q_1 \Phi_1 - \Phi_2 Q_1 \Phi_2] \Theta(k) + \Psi^T(k) [\Phi_3^T Q_2 \Phi_3 - \Phi_4 Q_2 \Phi_4] \Psi(k) \\ &= \Theta^T(k) \Omega_1(\beta(k)) \Theta(k) + \Psi^T(k) \Omega_2(\alpha(k)) \Psi(k), \end{aligned} \quad (11)$$

$$\begin{aligned} \Delta V_2(k) &= \Theta^T(k) [\Phi_5^T Q_3 \Phi_5 - \Phi_6^T Q_3 \Phi_6] \Theta(k) + \Psi^T(k) [\Phi_7^T Q_4 \Phi_7 - \Phi_8^T Q_4 \Phi_8] \Psi(k) \\ &= \Theta^T(k) \Omega_3 \Theta(k) + \Psi^T(k) \Omega_4 \Psi(k), \end{aligned} \quad (12)$$

$$\begin{aligned} \Delta V_3(k) &\leq (\hat{\beta} + 1) x^T(k) P x(k) - x^T(k - \beta(k)) P x(k - \beta(k)) + (\hat{\alpha} + 1) y^T(k) P^* y(k) - y^T(k - \alpha(k)) P^* y(k - \alpha(k)) \\ &= \Theta^T(k) \Omega_5 \Theta(k) + \Psi^T(k) \Omega_6 \Psi(k), \end{aligned} \quad (13)$$

$$\begin{aligned} \Delta V_4(k) &= \mu^T(k) (\underline{\beta}^2 X_1 + \hat{\beta}^2 X_2) \mu(k) - \underline{\beta} \sum_{i=k-\underline{\beta}}^{k-1} \mu^T(i) X_1 \mu(i) - \hat{\beta} \sum_{i=k-\hat{\beta}}^{k-\hat{\beta}-1} \mu^T(i) X_2 \mu(i) \\ &\quad + v^T(k) (\underline{\alpha}^2 X_3 + \hat{\alpha}^2 X_4) v(k) - \underline{\alpha} \sum_{j=k-\underline{\alpha}}^{k-1} v^T(j) X_3 v(j) - \hat{\alpha} \sum_{j=k-\hat{\alpha}}^{k-\hat{\alpha}-1} v^T(j) X_4 v(j) \end{aligned} \quad (14)$$

$$\Delta V_5(k) = \mu^T(k) \rho^2 Z_1 \mu(k) - \rho \sum_{i=k-\rho}^{k-1} \mu^T(i) Z_1 \mu(i) + v^T(k) \rho^2 Z_2 v(k) - \rho \sum_{j=k-\rho}^{k-1} v^T(j) Z_2 v(j) \quad (15)$$

$$\begin{aligned} \Delta V_6(k) &= (\beta^*)^2 \mu^T(k) L_1 \mu(k) + \frac{\hat{\beta}(\hat{\beta} + 1)}{4} \mu^T(k) L_2 \mu(k) - \beta^* \sum_{j=-\hat{\beta}}^{-\hat{\beta}-1} \sum_{i=k+j}^{k-1} \mu^T(i) L_1 \mu(i) \\ &\quad - \frac{1}{2} \sum_{j=k-\hat{\beta}}^{k-\hat{\beta}-1} \sum_{i=j}^{k-\hat{\beta}-1} \mu^T(i) L_2 \mu(i) + (\alpha^*)^2 v^T(k) L_3 v(k) \\ &\quad + \frac{\hat{\alpha}(\hat{\alpha} + 1)}{4} v^T(k) L_4 v(k) - \alpha^* \sum_{i=-\hat{\alpha}}^{-\hat{\alpha}-1} \sum_{j=k+i}^{k-1} v^T(j) L_3 v(j) - \frac{1}{2} \sum_{i=k-\hat{\alpha}}^{k-\hat{\alpha}-1} \sum_{j=i}^{k-\hat{\alpha}-1} v^T(j) L_4 v(j). \end{aligned} \quad (16)$$

From $\Delta V_4(k)$, ΔV_5 and ΔV_6 , we get

$$\begin{aligned} &\Theta^T(k) \Phi_9^T \Lambda \Phi_9 \Theta(k) + \Psi^T(k) \Phi_{10} \Lambda^* \Phi_{10} \Psi(k) \\ &= \mu^T(k) [\underline{\beta}^2 X_1 + \hat{\beta}^2 X_2] \mu(k) + \mu^T(k) \rho^2 Z_1 \mu(k) + (\beta^*)^2 \mu^T(k) L_1 \mu(k) + \frac{\hat{\beta}(\hat{\beta} + 1)}{4} \mu^T(k) L_2 \mu(k) \\ &\quad + v^T(k) [\underline{\alpha}^2 X_3 + \hat{\alpha}^2 X_4] v(k) + v^T(k) \rho^2 Z_2 v(k) + (\alpha^*)^2 v^T(k) L_3 v(k) + \frac{\hat{\alpha}(\hat{\alpha} + 1)}{4} v^T(k) L_4 v(k), \end{aligned}$$

With the help of Corollary 1, the second and fifth terms of $\Delta V_4(k)$ can be written as

$$\begin{aligned}
& -\underline{\beta} \sum_{i=k-\underline{\beta}}^{k-1} \mu^T(i) X_1 \mu(i) - \underline{\alpha} \sum_{j=k-\underline{\alpha}}^{k-1} v^T(j) X_2 v(j) \\
& \leq -\left[x(k) - x(k - \underline{\beta})\right]^T X_1 \left[x(k) - x(k - \underline{\beta})\right] - 3 \left[x(k) + x(k - \underline{\beta}) - \frac{2}{\underline{\beta} + 1} \sum_{i=k-\underline{\beta}}^k x(i)\right]^T (\underline{\beta} + 1) w^*(\underline{\beta}) \\
& \quad \times X_1 \left[x(k) + x(k - \underline{\beta}) - \frac{2}{\underline{\beta} + 1} \sum_{i=\underline{\beta}}^k x(i)\right] - \left[y(k) - y(k - \underline{\alpha})\right]^T X_3 \left[y(k) - y(k - \underline{\alpha})\right] \\
& \quad - 3 \left[y(k) + y(k - \underline{\alpha}) - \frac{2}{\underline{\alpha} + 1} \sum_{j=k-\underline{\alpha}}^k y(j)\right]^T (\underline{\alpha} + 1) w(\underline{\alpha}) X_3 \left[y(k) + y(k - \underline{\alpha}) - \frac{2}{\underline{\alpha} + 1} \sum_{j=\underline{\alpha}}^k y(j)\right] \\
& = -\Theta^T(k) \Phi_{11}^T \begin{pmatrix} X_1 & 0 \\ * & 3(\underline{\beta} + 1)w^*(\underline{\beta})X_1 \end{pmatrix} \Phi_{11} \Theta(k) - \Psi^T(k) \Phi_{12}^T \begin{pmatrix} X_3 & 0 \\ * & 3(\underline{\alpha} + 1)w(\underline{\alpha})X_3 \end{pmatrix} \Phi_{12} \Psi(k) \\
& \leq \Theta^T(k) \Omega_7 \Theta(k) + \Psi^T(k) \Omega_8 \Psi(k),
\end{aligned} \tag{17}$$

Next, in order to deal with second and fourth terms of $\Delta V_5(k)$, we acquire

$$\begin{aligned}
& -\rho \sum_{i=k-\rho}^{k-1} \mu^T(i) Z_1 \mu(i) - \rho \sum_{j=k-\rho}^{k-1} v^T(j) Z_2 v(j) \\
& \leq -\left[x(k) - x(k - \rho)\right]^T Z_1 \left[x(k) - x(k - \rho)\right] - \left[x(k) + x(k - \rho) - \frac{2}{\rho + 1} \sum_{i=k-\rho}^k x(i)\right]^T 3(\rho + 1) \zeta(\rho) \\
& \quad \times Z_1 \left[x(k) + x(k - \rho) - \frac{2}{\rho + 1} \sum_{i=k-\rho}^k x(i)\right] - \left[y(k) - y(k - \rho)\right]^T Z_2 \left[y(k) - y(k - \rho)\right] \\
& \quad - \left[y(k) + y(k - \rho) - \frac{2}{\rho + 1} \sum_{j=k-\rho}^k y(j)\right]^T 3(\rho + 1) \eta(\rho) Z_2 \left[y(k) + y(k - \rho) - \frac{2}{\rho + 1} \sum_{j=k-\rho}^k y(j)\right] \\
& = -\Theta^T(k) \Phi_{13}^T \begin{pmatrix} Z_1 & 0 \\ * & 3(\rho + 1)\zeta(\rho)Z_1 \end{pmatrix} \Phi_{13} \Theta(k) - \Psi^T(k) \Phi_{14}^T \begin{pmatrix} Z_2 & 0 \\ * & 3(\rho + 1)\eta(\rho)Z_2 \end{pmatrix} \Phi_{14} \Psi(k) \\
& \leq \Theta^T(k) \Omega_9 \Theta(k) + \Psi^T(k) \Omega_{10} \Psi(k).
\end{aligned} \tag{18}$$

Now, we are in position to deal with the third and sixth terms of $\Delta V_4(k)$ and the fourth and eighth terms of $\Delta V_6(k)$. First, we consider $\Delta V_4(k)$

$$\begin{aligned}
& -\hat{\beta} \sum_{i=k-\hat{\beta}}^{k-\hat{\beta}-1} \mu^T(i) X_2 \mu(i) - \hat{\alpha} \sum_{j=k-\hat{\alpha}}^{k-\hat{\alpha}-1} v^T(j) X_4 v(j) \\
& = -\hat{\beta} \sum_{i=k-\hat{\beta}}^{k-\hat{\beta}(k)-1} \mu^T(i) X_2 \mu(i) - \hat{\beta} \sum_{i=k-\hat{\beta}(k)}^{k-\hat{\beta}-1} \mu^T(i) X_2 \mu(i) - \hat{\alpha} \sum_{j=k-\hat{\alpha}}^{k-\hat{\alpha}(k)-1} v^T(j) X_4 v(j)
\end{aligned} \tag{19}$$

$$-\hat{\alpha} \sum_{j=k-\hat{\alpha}(k)}^{k-\hat{\alpha}-1} v^T(j) X_4 v(j). \tag{20}$$

Consider 3 cases as Case 1, Case 2, and Case 3 to solve 4 terms of Equation (20) on the right side.

Case 1:

If $\tau_1 = 1$, namely $\hat{\beta} > 2$, $\underline{\beta} < \beta(k) < \bar{\beta} - 1$, and $\varpi_1 = 1$, namely $\hat{\alpha} > 2$, $\underline{\alpha} < \alpha(k) < \bar{\alpha} - 1$ for first and third terms, respectively, in (20).

By Corollary 1, we attain

$$\begin{aligned}
& -\hat{\beta} \sum_{i=k-\bar{\beta}}^{k-\beta(k)-1} \mu^T(i) X_2 \mu(i) - \hat{\alpha} \sum_{j=k-\bar{\alpha}}^{k-\alpha(k)-1} \nu^T(j) X_4 \nu(j) \\
& \leq -\frac{\hat{\beta}}{\bar{\beta} - \beta(k)} [x(k - \beta(k)) - x(k - \bar{\beta})]^T X_2 [x(k - \beta(k)) - x(k - \bar{\beta})] - \frac{3\bar{\beta}}{\bar{\beta} - \beta(k)} \frac{\bar{\beta} - \beta(k) + 1}{\bar{\beta} - \beta(k) - 1} \\
& \quad \times \left[x(k - \beta(k)) + x(k - \bar{\beta}) - \frac{2}{\bar{\beta} - \beta(k) + 1} \sum_{i=k-\bar{\beta}}^{k-\beta(k)} x(i) \right]^T X_2 \left[x(k - \beta(k)) + x(k - \bar{\beta}) - \frac{2}{\bar{\beta} - \beta(k) + 1} \sum_{i=k-\bar{\beta}}^{k-\beta(k)} x(i) \right] \\
& \quad - \frac{\hat{\alpha}}{\bar{\alpha} - \alpha(k)} [y(k - \alpha(k)) - y(k - \bar{\alpha})]^T X_4 [y(k - \alpha(k)) - y(k - \bar{\alpha})] - \frac{3\bar{\alpha}}{\bar{\alpha} - \alpha(k)} \frac{\bar{\alpha} - \alpha(k) + 1}{\bar{\alpha} - \alpha(k) - 1} \\
& \quad \times \left[y(k - \alpha(k)) + y(k - \bar{\alpha}) - \frac{2}{\bar{\alpha} - \alpha(k) + 1} \sum_{j=k-\bar{\alpha}}^{k-\alpha(k)} y(j) \right]^T X_4 \left[y(k - \alpha(k)) + y(k - \bar{\alpha}) - \frac{2}{\bar{\alpha} - \alpha(k) + 1} \sum_{j=k-\bar{\alpha}}^{k-\alpha(k)} y(j) \right] \\
& \leq -\frac{\hat{\beta}}{\bar{\beta} - \beta(k)} [x(k - \beta(k)) - x(k - \bar{\beta})]^T X_2 [x(k - \beta(k)) - x(k - \bar{\beta})] \\
& \quad - \frac{\hat{\beta}}{\bar{\beta} - \beta(k)} \left[x(k - \beta(k)) + x(k - \bar{\beta}) - \frac{2}{\bar{\beta} - \beta(k) + 1} \sum_{i=k-\bar{\beta}}^{k-\beta(k)} x(i) \right]^T 3 \frac{\hat{\beta} + 1}{\hat{\beta} - 1} \\
& \quad X_2 \left[x(k - \beta(k)) + x(k - \bar{\beta}) - \frac{2}{\bar{\beta} - \beta(k) + 1} \sum_{i=k-\bar{\beta}}^{k-\beta(k)} x(i) \right] \\
& \quad - \frac{-\hat{\alpha}}{\bar{\alpha} - \alpha(k)} [y(k - \alpha(k)) - y(k - \bar{\alpha})]^T X_4 [y(k - \alpha(k)) - y(k - \bar{\alpha})] \\
& \quad - \frac{\hat{\alpha}}{\bar{\alpha} - \alpha(k)} \left[y(k - \alpha(k)) + y(k - \bar{\alpha}) - \frac{2}{\bar{\alpha} - \alpha(k) + 1} \sum_{j=k-\bar{\alpha}}^{k-\alpha(k)} y(j) \right]^T 3 \frac{\hat{\alpha} + 1}{\hat{\alpha} - 1} \\
& \quad X_4 \left[y(k - \alpha(k)) + y(k - \bar{\alpha}) - \frac{2}{\bar{\alpha} - \alpha(k) + 1} \sum_{j=k-\bar{\alpha}}^{k-\alpha(k)} y(j) \right]. \tag{21}
\end{aligned}$$

Similarly,

$$\begin{aligned}
& -\hat{\beta} \sum_{i=k-\beta(k)}^{k-\underline{\beta}-1} \mu^T(i) X_2 \mu(i) - \hat{\alpha} \sum_{j=k-\alpha(k)}^{k-\underline{\alpha}-1} \nu^T(j) X_4 \nu(j) \\
& \leq -\frac{\hat{\beta}}{\beta(k) - \underline{\beta}} [x(k - \underline{\beta}) - x(k - \beta(k))]^T X_2 [x(k - \beta(k)) - x(k - \bar{\beta})] \\
& \quad - \frac{\hat{\beta}}{\beta(k) - \underline{\beta}} \left[x(k - \underline{\beta}) + x(k - \beta(k)) - \frac{2}{\beta(k) - \underline{\beta} + 1} \sum_{i=k-\beta k}^{k-\underline{\beta}} x(i) \right]^T 3 \frac{\hat{\beta} + 1}{\hat{\beta} - 1} \\
& \quad X_2 \left[x(k - \underline{\beta}) + x(k - \beta(k)) - \frac{2}{\beta(k) - \underline{\beta} + 1} \sum_{i=k-\beta k}^{k-\underline{\beta}} x(i) \right] \\
& \quad - \frac{\hat{\alpha}}{\alpha(k) - \underline{\alpha}} [y(k - \underline{\alpha}) - y(k - \alpha(k))]^T X_4 [y(k - \alpha(k)) - y(k - \bar{\alpha})] \\
& \quad - \frac{\hat{\alpha}}{\alpha(k) - \underline{\alpha}} \left[y(k - \underline{\alpha}) + y(k - \alpha(k)) - \frac{2}{\alpha(k) - \underline{\alpha} + 1} \sum_{j=k-\alpha k}^{k-\underline{\alpha}} y(j) \right]^T 3 \frac{\hat{\alpha} + 1}{\hat{\alpha} - 1} \\
& \quad X_4 \left[y(k - \underline{\alpha}) + y(k - \alpha(k)) - \frac{2}{\alpha(k) - \underline{\alpha} + 1} \sum_{j=k-\alpha k}^{k-\underline{\alpha}} y(j) \right]. \tag{22}
\end{aligned}$$

By applying Lemma 1 in (21), (22) and with the help of (7), (8) we yields $-\hat{\beta} \sum_{i=k-\bar{\beta}}^{k-\beta(k)-1} \mu^T(i) X_2 \mu(i) - \hat{\beta} \sum_{i=k-\beta(k)}^{k-\bar{\beta}-1} \mu^T(i) X_2 \mu(i) - \hat{\alpha} \sum_{j=k-\bar{\alpha}}^{k-\alpha(k)-1} v^T(j) X_4 v(j) - \hat{\alpha} \sum_{j=k-\alpha(k)}^{k-\bar{\alpha}-1} v^T(j) X_4 v(j)$

$$\leq -\Theta^T(k) \Phi_{15}^T \begin{pmatrix} X_2 & 0 & W_{11} & W_{12} \\ * & \frac{3\hat{\beta}+1}{\hat{\beta}-1} & W_{21} & W_{22} \\ * & * & X_2 & 0 \\ * & * & * & \frac{3\hat{\beta}+1}{\hat{\beta}-1} X_2 \end{pmatrix} \Phi_{15} \Theta(k) - \Psi^T(k) \Phi_{16}^T \begin{pmatrix} X_4 & 0 & Z_{11} & Z_{12} \\ * & \frac{3\hat{\alpha}+1}{\hat{\alpha}-1} & Z_{21} & Z_{22} \\ * & * & X_4 & 0 \\ * & * & * & \frac{3\hat{\alpha}+1}{\hat{\alpha}-1} X_4 \end{pmatrix} \Phi_{16} \Psi(k) \quad (23)$$

$$\leq \Theta^T(k) \Omega_{71}(\beta(k)) \Theta(k) + \Psi^T(k) \Omega_{81}(\alpha(k)) \Psi(k). \quad (24)$$

Case 2:

If $\tau_2 = 1$, namely $\hat{\beta} \geq 2$, $\beta(k) = \underline{\beta}$, $\bar{\beta} - 1$ or $\bar{\beta}$, and $\varpi_2 = 1$, namely $\hat{\alpha} \geq 2$, $\alpha(k) = \underline{\alpha}$, $\bar{\alpha} - 1$ or $\bar{\alpha}$ for the second and fourth terms, respectively, in (20).

By the direct use of Corollary 1 in (20), we get

$$\begin{aligned} & -\hat{\beta} \sum_{i=k-\bar{\beta}}^{k-\underline{\beta}-1} \mu^T(i) X_2 \mu(i) - \hat{\alpha} \sum_{j=k-\bar{\alpha}}^{k-\underline{\alpha}-1} v^T(j) X_4 v(j) \\ & \leq -\left[x(k-\underline{\beta}) - x(k-\bar{\beta}) \right]^T X_2 \left[x(k-\underline{\beta}) - x(k-\bar{\beta}) \right] - \left[x(k-\underline{\beta}) + x(k-\beta(k)) - \frac{2}{\hat{\beta}+1} \sum_{i=k-\bar{\beta}}^{k-\underline{\beta}} x(i) \right] \\ & \quad - \left[x(k-\underline{\beta}) + x(k-\beta(k)) - \frac{2}{\hat{\beta}+1} \sum_{i=k-\bar{\beta}}^{k-\underline{\beta}} x(i) \right]^T 3 \frac{\hat{\beta}+1}{\hat{\beta}-1} X_2 \left[x(k-\underline{\beta}) + x(k-\bar{\beta}) - \frac{2}{\hat{\beta}+1} \sum_{i=k-\bar{\beta}}^{k-\underline{\beta}} x(i) \right] \\ & \quad - \left[y(k-\underline{\alpha}) - y(k-\bar{\alpha}) \right]^T X_4 \left[y(k-\underline{\alpha}) - y(k-\bar{\alpha}) \right] - \left[y(k-\underline{\alpha}) + y(k-\alpha(k)) - \frac{2}{\hat{\alpha}+1} \sum_{j=k-\bar{\alpha}}^{k-\underline{\alpha}} y(j) \right] \\ & \quad - \left[y(k-\underline{\alpha}) + y(k-\alpha(k)) - \frac{2}{\hat{\alpha}+1} \sum_{j=k-\bar{\alpha}}^{k-\underline{\alpha}} y(j) \right]^T 3 \frac{\hat{\alpha}+1}{\hat{\alpha}-1} X_4 \left[y(k-\underline{\alpha}) + y(k-\bar{\alpha}) - \frac{2}{\hat{\alpha}+1} \sum_{j=k-\bar{\alpha}}^{k-\underline{\alpha}} y(j) \right] \\ & = -\Theta^T(k) \Phi_{17}^T \begin{pmatrix} X_2 & 0 \\ * & \frac{3\hat{\beta}+1}{\hat{\beta}-1} X_2 \end{pmatrix} \Phi_{17} \Theta(k) - \Psi^T(k) \Phi_{18}^T \begin{pmatrix} X_4 & 0 \\ * & \frac{3\hat{\alpha}+1}{\hat{\alpha}-1} X_4 \end{pmatrix} \Phi_{18} \Psi(k) \\ & \leq \Theta^T(k) \Omega_{72}(\beta(k)) \Theta(k) + \Psi^T(k) \Omega_{82}(\alpha(k)) \Psi(k). \end{aligned} \quad (25)$$

Case 3:

If $\tau_3 = 1$, namely $\hat{\beta} = 1$, $\beta(k) = \underline{\beta}$ or $\bar{\beta}$, and $\varpi_3 = 1$, namely $\hat{\alpha} = 1$, $\alpha(k) = \underline{\alpha}$ or $\bar{\alpha}$ for the second and fourth terms, respectively, in (20)

$$\begin{aligned} & -\hat{\beta} \sum_{i=k-\bar{\beta}}^{k-\underline{\beta}-1} \mu^T(i) X_2 \mu(i) - \hat{\alpha} \sum_{j=k-\bar{\alpha}}^{k-\underline{\alpha}-1} v^T(j) X_4 v(j) - \mu^T(k-\bar{\beta}) X_2 \mu(k-\bar{\beta}) - v^T(k-\bar{\alpha}) X_4 v(k-\bar{\alpha}) \\ & = -\left[x(k-\underline{\beta}) - x(k-\bar{\beta}) \right]^T X_2 \left[x(k-\underline{\beta}) - x(k-\bar{\beta}) \right] \\ & \quad - \left[y(k-\underline{\alpha}) - y(k-\bar{\alpha}) \right]^T X_4 \left[y(k-\underline{\alpha}) - y(k-\bar{\alpha}) \right] \\ & = -\Theta^T(k) \Phi_{17}^T \begin{pmatrix} X_2 & 0 \\ 0 & 0 \end{pmatrix} \Phi_{17} \Theta(k) - \Psi^T(k) \Phi_{18}^T \begin{pmatrix} X_4 & 0 \\ 0 & 0 \end{pmatrix} \Phi_{18} \Psi(k) \end{aligned} \quad (26)$$

$$\leq \Theta^T(k) \Omega_{73}(\beta(k)) \Theta(k) + \Psi^T(k) \Omega_{83} \Psi(k). \quad (27)$$

Next, we are in position to prove the third, fourth, seventh, and eighth terms of $\Delta V_6(k)$ with the 2 cases as Case 1 and Case 2.

Case 1:

Consider $\pi_1 = 1$, namely $\hat{\beta} > 2$, $\underline{\beta} < \beta(k) < \bar{\beta} - 1$ or $\hat{\beta} \geq 2$, $\beta(k) = \bar{\beta} - 1$, and $\gamma_1 = 1$ namely $\hat{\alpha} > 2$, $\underline{\alpha} < \alpha(k) < \bar{\alpha} - 1$ or $\hat{\alpha} \geq 2$, $\alpha(k) = \bar{\alpha} - 1$ for the third and seventh terms, respectively, of $\Delta V_6(k)$.

With the help of Lemma 1, we acquire

$$\begin{aligned}
& -\beta^* \sum_{j=k-\bar{\beta}}^{\bar{\beta}-1} \sum_{i=k+j}^{k-1} \mu^T(i) L_1 \mu(i) - \alpha^* \sum_{i=k-\bar{\alpha}}^{\bar{\alpha}-1} \sum_{j=k+i}^{k-1} v^T(j) L_3 v(j) \\
& = \beta^* \frac{(\bar{\beta} - \beta(k))(\bar{\beta} + \beta(k) + 1)}{(\bar{\beta} - \beta(k))(\bar{\beta} + \beta(k) + 1)} \sum_{j=-\bar{\beta}}^{-\beta(k)-1} \sum_{i=k+j}^{k-1} \mu^T(i) L_1 \mu(i) - \beta^* \frac{(\beta(k) - \underline{\beta})(\beta(k) + \underline{\beta} + 1)}{(\beta(k) - \underline{\beta})(\beta(k) + \underline{\beta} + 1)} \sum_{j=-\beta(k)}^{-\bar{\beta}-1} \times \sum_{i=k+j}^{k-1} \mu^T(i) L_1 \mu(i), \\
& \leq -\frac{1}{\xi_1} \left(\sum_{j=-\bar{\beta}}^{-\beta(k)-1} \sum_{i=k+j}^{k-1} \mu(i) \right)^T L_1 \left(\sum_{j=-\bar{\beta}}^{-\beta(k)-1} \sum_{i=k+j}^{k-1} \mu(i) \right) - \frac{1}{\xi_2} \left(\sum_{j=-\beta(k)}^{-\bar{\beta}-1} \sum_{i=k+j}^{k-1} \mu(i) \right)^T L_1 \left(\sum_{j=-\beta(k)}^{-\bar{\beta}-1} \sum_{i=k+j}^{k-1} \mu(i) \right) \\
& \quad + \alpha^* \frac{(\bar{\alpha} - \alpha(k))(\bar{\alpha} + \alpha(k) + 1)}{(\bar{\alpha} - \alpha(k))(\bar{\alpha} + \alpha(k) + 1)} \sum_{i=-\bar{\alpha}}^{-\alpha(k)-1} \sum_{j=k+i}^{k-1} v^T(j) L_3 v(j) - \alpha^* \frac{(\alpha(k) - \underline{\alpha})(\alpha(k) + \underline{\alpha} + 1)}{(\alpha(k) - \underline{\alpha})(\alpha(k) + \underline{\alpha} + 1)} \sum_{i=-\alpha(k)}^{-\bar{\alpha}-1} \times \sum_{j=k+i}^{k-1} v^T(j) L_3 v(j), \\
& \leq -\frac{1}{\xi_3} \left(\sum_{i=-\bar{\alpha}}^{-\alpha(k)-1} \sum_{j=k+i}^{k-1} v(j) \right)^T L_3 \left(\sum_{i=-\bar{\alpha}}^{-\alpha(k)-1} \sum_{j=k+i}^{k-1} v(j) \right) - \frac{1}{\xi_4} \left(\sum_{i=-\alpha(k)}^{-\bar{\alpha}-1} \sum_{j=k+i}^{k-1} v(j) \right)^T L_3 \left(\sum_{i=-\alpha(k)}^{-\bar{\alpha}-1} \sum_{j=k+i}^{k-1} v(j) \right), \tag{28}
\end{aligned}$$

where

$$\begin{aligned}
\xi_1 &= \frac{(\bar{\beta} - \beta(k))(\bar{\beta} + \beta(k) + 1)}{(\bar{\beta} - \beta(k))(\bar{\beta} + \beta(k) + 1)} & \xi_2 &= \frac{(\beta(k) - \underline{\beta})(\beta(k) + \underline{\beta} + 1)}{(\bar{\beta} - \beta(k))(\bar{\beta} + \beta(k) + 1)}, \\
\xi_1 + \xi_2 &= 1, \\
\xi_3 &= \frac{(\bar{\alpha} - \alpha(k))(\bar{\alpha} + \alpha(k) + 1)}{(\bar{\alpha} - \alpha(k))(\bar{\alpha} + \alpha(k) + 1)} & \xi_4 &= \frac{(\alpha(k) - \underline{\alpha})(\alpha(k) + \underline{\alpha} + 1)}{(\bar{\alpha} - \alpha(k))(\bar{\alpha} + \alpha(k) + 1)}, \\
\xi_3 + \xi_4 &= 1.
\end{aligned}$$

By using Lemma 2, we attain

$$\begin{aligned}
& -\beta^* \sum_{j=k-\bar{\beta}}^{\bar{\beta}-1} \sum_{i=k+j}^{k-1} \mu^T(i) L_1 \mu(i) - \alpha^* \sum_{i=k-\bar{\alpha}}^{\bar{\alpha}-1} \sum_{j=k+i}^{k-1} v^T(j) L_3 v(j) \\
& \leq - \left(\sum_{j=-\bar{\beta}}^{-\beta(k)-1} \sum_{i=k+j}^{k-1} \mu(i) \right)^T \begin{pmatrix} L_1 & T_1^T \\ * & L_1 \end{pmatrix} \begin{pmatrix} \sum_{j=-\bar{\beta}}^{-\beta(k)-1} \sum_{i=k+j}^{k-1} \mu(i) \\ \sum_{j=-\beta(k)}^{-\bar{\beta}-1} \sum_{i=k+j}^{k-1} \mu(i) \end{pmatrix} \\
& \quad - \left(\sum_{i=-\bar{\alpha}}^{-\alpha(k)-1} \sum_{j=k+i}^{k-1} v(j) \right)^T \begin{pmatrix} L_3 & T_2^T \\ * & L_3 \end{pmatrix} \begin{pmatrix} \sum_{i=-\bar{\alpha}}^{-\alpha(k)-1} \sum_{j=k+i}^{k-1} v(j) \\ \sum_{i=-\alpha(k)}^{-\bar{\alpha}-1} \sum_{j=k+i}^{k-1} v(j) \end{pmatrix} \\
& = -\Theta^T(k) \Phi_{19}^T \begin{pmatrix} L_1 & T_1^T \\ * & L_1 \end{pmatrix} \Phi_{19} \Theta(k) - \Psi^T(k) \Phi_{20}^T \begin{pmatrix} L_3 & T_2^T \\ * & L_3 \end{pmatrix} \Phi_{20} \Psi(k). \tag{29}
\end{aligned}$$

Again, Consider case 1 as $\pi_1 = 1, \gamma_1 = 1$, and by utilizing Lemma 2 to the fourth and eighth terms of $\Delta V_6(k)$, respectively, we get

$$-\frac{1}{2} \sum_{j=k-\bar{\beta}}^{\bar{\beta}-1} \sum_{i=j}^{k-\bar{\beta}-1} \mu^T(i) L_2 \mu(i) - \frac{1}{2} \sum_{i=k-\bar{\alpha}}^{\bar{\alpha}-1} \sum_{j=i}^{k-\bar{\alpha}-1} v^T(j) L_4 v(j)$$

$$\begin{aligned}
&= -\frac{1}{2} \sum_{j=k-\bar{\beta}}^{k-\beta(k)-1} \sum_{i=j}^{k-\beta(k)-1} \mu^T(i) L_2 \mu(i) - \frac{1}{2} \sum_{j=k-\beta(k)}^{k-\bar{\beta}-1} \sum_{i=j}^{k-\bar{\beta}-1} \mu^T(i) L_2 \mu(i) - \frac{1}{2} \sum_{j=k-\bar{\beta}}^{k-\beta(k)-1} \sum_{i=\beta(k)}^{k-\bar{\beta}-1} \mu^T(i) L_2 \mu(i) \\
&\quad - \frac{1}{2} \sum_{i=k-\bar{\alpha}}^{k-\alpha(k)-1} \sum_{j=i}^{k-\alpha(k)-1} v^T(j) L_4 v(j) \\
&\quad - \frac{1}{2} \sum_{i=k-\alpha(k)}^{k-\alpha-1} \sum_{j=i}^{k-\alpha-1} v^T(j) L_4 v(j) - \frac{1}{2} \sum_{i=k-\bar{\alpha}}^{k-\alpha(k)-1} \sum_{j=\alpha(k)}^{k-\alpha-1} v^T(j) L_4 v(j) \\
&\leq - \left[x(k-\beta(k)) - \frac{1}{\bar{\beta}-\beta(k)+1} \sum_{i=k-\bar{\beta}}^{k-\beta(k)} x(i) \right]^T L_2 \left[x(k-\beta(k)) - \frac{1}{\bar{\beta}-\beta(k)+1} \sum_{i=k-\bar{\beta}}^{k-\beta(k)} x(i) \right] \\
&\quad - \left[x(k-\bar{\beta}) - \frac{1}{\beta(k)-\bar{\beta}+1} \sum_{i=k-\beta(k)}^{k-\bar{\beta}} x(i) \right]^T L_2 \left[x(k-\bar{\beta}) - \frac{1}{\beta(k)-\bar{\beta}+1} \sum_{i=k-\beta(k)}^{k-\bar{\beta}} x(i) \right] \\
&\quad - \left[y(k-\alpha(k)) - \frac{1}{\bar{\alpha}-\alpha(k)+1} \sum_{j=k-\bar{\alpha}}^{k-\alpha(k)} y(j) \right]^T L_4 \left[y(k-\alpha(k)) - \frac{1}{\bar{\alpha}-\alpha(k)+1} \sum_{j=k-\bar{\alpha}}^{k-\alpha(k)} y(j) \right] \\
&\quad - \left[y(k-\alpha) - \frac{1}{\alpha(k)-\alpha+1} \sum_{j=k-\alpha(k)}^{k-\alpha} y(j) \right]^T L_4 \left[y(k-\alpha) - \frac{1}{\alpha(k)-\alpha+1} \sum_{j=k-\alpha(k)}^{k-\alpha} y(j) \right] \\
&\leq -\Theta^T(k) \Phi_{21}^T \begin{pmatrix} L_2 & 0 \\ * & L_2 \end{pmatrix} \Phi_{21} \Theta(k) - \Psi^T(k) \Phi_{22}^T \begin{pmatrix} L_4 & 0 \\ * & L_4 \end{pmatrix} \Phi_{22} \Psi(k). \tag{30}
\end{aligned}$$

Therefore

$$\Delta V_6(k) \leq \Theta^T(k) \Omega_{111}(\beta(k)) \Theta(k) + \Psi^T(k) \Omega_{121}(\alpha(k)) \Psi(k). \tag{31}$$

Case 2:

Consider $\pi_2 = 1$, namely $\hat{\beta} \geq 2$, $\beta(k) = \beta, \bar{\beta}$ or $\hat{\beta} = 1$, $\beta(k) = \underline{\beta}, \bar{\beta}$ and $\gamma_2 = 1$, namely $\hat{\alpha} \geq 2$, $\alpha(k) = \underline{\alpha}, \bar{\alpha}$ or $\hat{\alpha} = 1$, $\alpha(k) = \underline{\alpha}, \bar{\alpha}$ for third and seventh terms respectively of $\Delta V_6(k)$.

Under the condition of Lemma 2, we acquire

$$\begin{aligned}
&-\beta^* \sum_{j=-\bar{\beta}}^{-\bar{\beta}-1} \sum_{i=k+j}^{k-1} \mu^T(i) L_1 \mu(i) - \alpha^* \sum_{i=-\bar{\alpha}}^{-\bar{\alpha}-1} \sum_{j=k+i}^{k-1} v^T(j) L_3 v(j) \\
&\leq - \left[\sum_{j=-\bar{\beta}}^{-\bar{\beta}-1} \sum_{i=k+j}^{k-1} \mu(i) \right]^T L_1 \left[\sum_{j=-\bar{\beta}}^{-\bar{\beta}-1} \sum_{i=k+j}^{k-1} \mu(i) \right] - \left[\sum_{i=-\bar{\alpha}}^{-\bar{\alpha}-1} \sum_{j=k+i}^{k-1} v(j) \right]^T L_3 \left[\sum_{i=-\bar{\alpha}}^{-\bar{\alpha}-1} \sum_{j=k+i}^{k-1} v(j) \right] \\
&= -\Theta^T(k) (e_1^T - (\bar{\beta} - \beta(k) + 1)e_7^T + e_3^T) L_1 (e_1 - (\bar{\beta} - \beta(k) + 1)e_7 + e_3) \Theta(k) \\
&\quad - \Psi^T(k) (r_1^T - (\bar{\alpha} - \alpha(k) + 1)r_7^T + r_3^T) L_3 (r_1 - (\bar{\alpha} - \alpha(k) + 1)r_7 + r_3) \Psi(k) \\
&= -\Theta^T(k) \Phi_{23}^T L_1 \Phi_{23} \Theta(k) - \Psi^T(k) \Phi_{24}^T L_3 \Phi_{24} \Psi(k). \tag{32}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&-\frac{1}{2} \sum_{j=k-\bar{\beta}}^{k-\bar{\beta}-1} \sum_{i=j}^{k-\bar{\beta}-1} \mu^T(i) L_2 \mu(i) - \frac{1}{2} \sum_{i=k-\bar{\alpha}}^{k-\alpha-1} \sum_{j=i}^{k-\alpha-1} v^T(j) L_4 v(j) \\
&\leq -\frac{1}{2} \frac{2}{(\bar{\beta}-\underline{\beta})(\bar{\beta}-\underline{\beta}+1)} \left[\sum_{j=k-\bar{\beta}}^{k-\bar{\beta}-1} \sum_{i=j}^{k-\bar{\beta}-1} \mu(i) \right]^T L_2 \left[\sum_{j=k-\bar{\beta}}^{k-\bar{\beta}-1} \sum_{i=j}^{k-\bar{\beta}-1} \mu(i) \right] - \frac{1}{2} \frac{2}{(\bar{\alpha}-\underline{\alpha})(\bar{\alpha}-\underline{\alpha}+1)} \left[\sum_{i=k-\bar{\alpha}}^{k-\alpha-1} \sum_{j=i}^{k-\alpha-1} v(j) \right]^T L_4 \left[\sum_{i=k-\bar{\alpha}}^{k-\alpha-1} \sum_{j=i}^{k-\alpha-1} v(j) \right] \\
&= -\Theta^T(k) \frac{1}{\hat{\beta}(\hat{\beta}+1)} [\hat{\beta}e_2^T - (\bar{\beta} - \beta(k) + 1)e_7^T + e_3^T] L_2 [\hat{\beta}e_2 - (\bar{\beta} - \beta(k) + 1)e_7 + e_3] \Theta(k) - \Psi^T(k) \\
&\quad \times \frac{1}{\hat{\alpha}(\hat{\alpha}+1)} [\hat{\alpha}r_2^T - (\bar{\alpha} - \alpha(k) + 1)r_7^T + r_3^T] L_4 [\hat{\alpha}r_2 - (\bar{\alpha} - \alpha(k) + 1)r_7 + r_3] \Psi(k) \\
&= -\Theta^T(k) \Phi_{24}^T L_2 \Phi_{24} \Theta(k) - \Psi^T(k) \Phi_{25}^T L_4 \Phi_{25} \Psi(k). \tag{33}
\end{aligned}$$

From above, we get

$$\Delta V_6(k) \leq \Theta^T(k) \Omega_{112}(\beta(k)) \Theta(k) + \Psi^T(k) \Omega_{122}(\alpha(k)) \Psi(k). \quad (34)$$

For any $i, j = 1, 2, 3, \dots, n$ and from Assumption 2, we have

$$-\left(\begin{array}{c} x(k) \\ \hat{f}(x(k)) \end{array} \right)^T \left(\begin{array}{cc} D_1 \varsigma_1 & -D_2 \varsigma_1 \\ -D_2 \varsigma_1 & \varsigma_1 \end{array} \right) \left(\begin{array}{c} x(k) \\ \hat{f}(x(k)) \end{array} \right) = \Theta^T(k) \Omega_{13} \Theta(k) \geq 0, \quad (35)$$

$$-\left(\begin{array}{c} x(k - \beta(k)) \\ \hat{g}(x(k - \beta(k))) \end{array} \right)^T \left(\begin{array}{cc} E_1 \varsigma_2 & -E_2 \varsigma_2 \\ -E_2 \varsigma_2 & \varsigma_2 \end{array} \right) \left(\begin{array}{c} x(k - \beta(k)) \\ \hat{g}(x(k - \beta(k))) \end{array} \right) = \Theta^T(k) \Omega_{15} \Theta(k) \geq 0, \quad (36)$$

$$-\left(\begin{array}{c} y(k) \\ f(y(k)) \end{array} \right)^T \left(\begin{array}{cc} F_1 \varsigma_3 & -F_2 \varsigma_3 \\ -F_2 \varsigma_3 & \varsigma_3 \end{array} \right) \left(\begin{array}{c} y(k) \\ f(y(k)) \end{array} \right) = \Psi^T(k) \Omega_{14} \Psi(k) \geq 0, \quad (37)$$

$$-\left(\begin{array}{c} y(k - \alpha(k)) \\ g(y(k - \alpha(k))) \end{array} \right)^T \left(\begin{array}{cc} G_1 \varsigma_4 & -G_2 \varsigma_4 \\ -G_2 \varsigma_4 & \varsigma_4 \end{array} \right) \left(\begin{array}{c} y(k - \alpha(k)) \\ g(y(k - \alpha(k))) \end{array} \right) = \Psi^T(k) \Omega_{16} \Psi(k) \geq 0, \quad (38)$$

where

$$\begin{aligned} \varsigma_1 &= \text{diag} \{ \varsigma_{11}, \varsigma_{12}, \dots, \varsigma_{1n} \}; & \varsigma_2 &= \text{diag} \{ \varsigma_{21}, \varsigma_{22}, \dots, \varsigma_{2n} \} \\ \varsigma_3 &= \text{diag} \{ \varsigma_{31}, \varsigma_{32}, \dots, \varsigma_{3n} \}; & \varsigma_4 &= \text{diag} \{ \varsigma_{41}, \varsigma_{42}, \dots, \varsigma_{4n} \}. \end{aligned}$$

$$\begin{aligned} F_1 &= \text{diag} \{ F_1^- F_1^+, F_2^- F_2^+, \dots, F_n^- F_n^+ \}; & F_2 &= \text{diag} \left\{ \frac{F_1^- + F_1^+}{2}, \frac{F_2^- + F_2^+}{2}, \dots, \frac{F_n^- + F_n^+}{2} \right\} \\ D_1 &= \text{diag} \{ D_1^- D_1^+, D_2^- D_2^+, \dots, D_n^- D_n^+ \}; & D_2 &= \text{diag} \left\{ \frac{D_1^- + D_1^+}{2}, \frac{D_2^- + D_2^+}{2}, \dots, \frac{D_n^- + D_n^+}{2} \right\} \\ E_1 &= \text{diag} \{ E_1^- E_1^+, E_2^- E_2^+, \dots, E_n^- E_n^+ \}; & E_2 &= \text{diag} \left\{ \frac{E_1^- + E_1^+}{2}, \frac{E_2^- + E_2^+}{2}, \dots, \frac{E_n^- + E_n^+}{2} \right\} \\ G_1 &= \text{diag} \{ G_1^- G_1^+, G_2^- G_2^+, \dots, G_n^- G_n^+ \}; & G_2 &= \text{diag} \left\{ \frac{G_1^- + G_1^+}{2}, \frac{G_2^- + G_2^+}{2}, \dots, \frac{G_n^- + G_n^+}{2} \right\}. \end{aligned}$$

□

Remark 3. In other works,^{2,25,33,54} the authors exposed the concept of time-varying delays in proving the stability analysis of NNs. Up to now, no results on different cases of time-varying delays in discrete-time BAM NNs have been tackled. Moreover, Wu et al⁶¹ and Wang and Li⁶² proposed the stability results with the assistance of free weighting matrix and delay partitioning approach. Therefore, in our work, time delays such as $\alpha(k)$ and $\beta(k)$ are considered with some cases, and by the aid of novel summation inequality, the maximum allowable upper bound is obtained in our research work when comparing with previous works in the literature, which is highlighted in Table 2.

From the previous equations, we obtain

$$\begin{aligned} \Delta V(k) &\leq \Theta^T(k) [\Omega_1(\beta(k)) + \Omega_3 + \Omega_5 + \Omega_7(\beta(k)) + \Omega_9 + \Omega_{11}(\beta(k)) + \Omega_{13} + \Omega_{15}] \Theta(k) \\ &\quad + \Psi^T(k) [\Omega_2(\alpha(k)) + \Omega_4 + \Omega_6 + \Omega_8(\alpha(k)) + \Omega_{10} + \Omega_{12}(\alpha(k)) + \Omega_{14} + \Omega_{16}] \Psi(k) < 0. \end{aligned} \quad (39)$$

Let $\lambda^* = \lambda_{\max} \{ \Omega_1(\beta(k)) + \Omega_3 + \Omega_5 + \Omega_7(\beta(k)) + \Omega_9 + \Omega_{11}(\beta(k)) + \Omega_{13} + \Omega_{15}, \Omega_2(\alpha(k)) + \Omega_4 + \Omega_6 + \Omega_8(\alpha(k)) + \Omega_{10} + \Omega_{12}(\alpha(k)) + \Omega_{14} + \Omega_{16} \}$.

Then, $Y^* = \lambda_{\max}(\lambda^*) < 0$. From above, we easily attain

$$\begin{aligned} \Delta V(k) &\leq Y^* \|\Theta(k) + \Psi(k)\|^2 \\ &\leq Y^* \|x(k) + y(k)\|^2 < 0, \quad \forall x(k), y(k) \neq 0. \end{aligned}$$

The NN system (3) proves to be globally asymptotically stable with the help of Lyapunov-Krasovskii functionals.

Remark 4. Leakage delay or forgetting term sometimes leads to a worse dynamical behavior of the NN system. Very few results are proposed on NNs with leakage delay in discrete time. However, in our paper, we assumed the leakage delay in the state vector as a constant one.

Remark 5. It is pointed out that, by the aid of convex combination approach and Corollary 1, we introduced 8 matrices $W_{11}, W_{12}, W_{21}, W_{22}, Z_{11}, Z_{12}, Z_{21}, Z_{22}$ to obtain the feasibility condition. Therefore, the proposed method largely reduces the computational burden. In addition, we have modified the Lyapunov function by including the leakage terms (ie, $-\rho \sum_{i=k-\rho}^{k-1} \mu^T(i) Z_1 \mu(i)$ and $-\rho \sum_{j=k-\rho}^{k-1} v^T(j) Z_2 v(j)$) and proved that it is less conserved than the stability conditions obtained in the work of Liu et al⁶³ (see Corollary 1).

Remark 6. Wu et al⁶¹ explored the time-delayed stability conditions for discrete-time NNs via delay partitioning approach and free-matrix technique. In addition, Wang and Li investigated the impact of Markovian jumping on discrete-time NNs with the help of delay partitioning approach.⁶² Moreover, the comparisons for the maximum allowable upper bounds has been listed. In this paper, by utilizing the novel summation inequality and convex combination technique, the stability criteria for addressed NNs have been derived, which is easily solved by the LMI control toolbox in the MATLAB software. Moreover, the obtained allowable upper bounds of discrete-time varying delay is very large when compared with some existing results, which are listed in Table 2.

Remark 7. Sakthivel et al⁶⁴ delivered the robust stability conditions by using uncertain parameters. Su et al proposed the stability results in the sense of asymptotic.⁶⁵ In the work of Mathiyalagan et al,⁶⁶ the problem of the stability issue of NNs with time-varying delays is explored. Song et al⁶⁷ proposed the NNs with uncertain parameters. From the motivation of the above references, in our work, discrete-time uncertain BAM NNs with leakage and time-varying delay is proposed.

Next, we are in position to prove the impulsive effect of NNs (3).

From the definition of the operator Δ , one can observe that

$$\Delta V(k, x(k), y(k)) = V(k+1, x(k+1), y(k+1)) - V(k, x(k), y(k)) < 0.$$

Then, there exists a positive constant $\epsilon \leq 1$ such that $V(k+1, x(k+1), y(k+1)) \leq (1-\epsilon)V(k, x(k), y(k))$, $k \neq k_r$.

Then,

$$\begin{aligned} V(k_0+1, x(k_0+1), y(k_0+1)) &\leq (1-\epsilon)V(k_0, x(k_0), y(k_0)) \\ V(k_0+2, x(k_0+2), y(k_0+2)) &\leq (1-\epsilon)^2 V(k_0, x(k_0), y(k_0)) \\ &\vdots \\ V(k_1, x(k_1), y(k_1)) &= V(k_0+k_1-k_0, x(k_0+k_1-k_0), y(k_0+k_1-k_0)) \\ &\leq (1-\epsilon)^{k_1-k_0} V(k_0, x(k_0), y(k_0)). \end{aligned}$$

Similarly,

$$\begin{aligned} V(k_2, x(k_2), y(k_2)) &= V(k_1+k_2-k_1, x(k_1+k_2-k_1), y(k_1+k_2-k_1)) \\ &\leq (1-\epsilon)^{k_2-k_1} V(k_1, x(k_1), y(k_1)) \\ &\leq (1-\epsilon)^{k_2-k_1} (1-\epsilon)^{k_1-k_0} V(k_0, x(k_0), y(k_0)) \\ &= (1-\epsilon)^{k_2-k_0} V(k_0, x(k_0), y(k_0)), \end{aligned} \tag{40}$$

and hence,

$$V(k_2, x(k), y(k)) \leq (1-\epsilon)^{k-k_r} V(k_r, x(k_r), y(k_r)), k \in [k_r, k_{r+1}). \tag{41}$$

Here, we note that

$$\begin{aligned}
& V(k, x(k_r), y(k_r)) \\
&= \vartheta^T(k_r^-) Q_1 \vartheta(k_r^-) + \vartheta^{*T}(k_r^-) Q_2 \vartheta^*(k_r^-) + v^T(k_r^-) Q_3 v(k_r^-) + v^{*T}(k_r^-) Q_4 v^*(k_r^-) + \sum_{i=k_r^- - \beta(k_r^-)}^{k_r^- - 1} \times x^T(i) P x(i) \\
&+ \sum_{j=k_r^- - \bar{\beta} + 1}^{-\bar{\beta}} \sum_{i=k_r^- + j}^{k_r^- - 1} x^T(i) P x(i) + \sum_{j=k - \alpha(k_r^-)}^{k_r^- - 1} y^T(j) P^* y(j) + \sum_{i=k_r^- - \bar{\alpha} + 1}^{-\bar{\alpha}} \sum_{j=k_r^- + i}^{k_r^- - 1} \times y^T(j) P^* y(j) \\
&+ \underline{\beta} \sum_{j=-\bar{\beta} + 1}^0 \sum_{i=k_r^- + j - 1}^{k - 1} \mu^T(i) X_1 \mu(i) + \hat{\beta} \sum_{j=-\bar{\beta} + 1}^{-\bar{\beta}} \sum_{i=k_r^- + j - 1}^{k - 1} \mu^T(i) X_2 \mu(i) \\
&+ \underline{\alpha} \sum_{i=-\bar{\alpha} + 1}^0 \sum_{j=k_r^- + i - 1}^{k - 1} v^T(j) X_3 v(j) + \hat{\alpha} \sum_{i=-\bar{\alpha} + 1}^{-\bar{\alpha}} \sum_{j=k_r^- + i - 1}^{k - 1} v^T(j) X_4 v(j) + \varrho \sum_{j=-\varrho + 1}^0 \sum_{i=k_r^- + j - 1}^{k_r^- - 1} \times \mu^T(i) Z_1 \mu(i) \\
&+ \rho \sum_{i=-\rho + 1}^0 \sum_{j=k_r^- + i - 1}^{k_r^- - 1} v^T(j) Z_2 v(j) + \beta^* \sum_{i=-\beta^*}^{-\beta - 1} \sum_{j=i}^{-1} \sum_{h=k_r^- + j}^{k_r^- - 1} \mu^T(h) L_1 \mu(h) + \frac{1}{2} \sum_{i=k_r^- - \bar{\beta}}^{k_r^- - \bar{\beta} - 1} \sum_{j=i}^{k_r^- - \bar{\beta} - 1} \sum_{h=j}^{k_r^- - 1} \mu^T(h) L_2 \mu(h) \\
&+ \alpha^* \sum_{j=-\alpha^*}^{-\alpha - 1} \sum_{i=j}^{-1} \sum_{h^*=k_r^- + i}^{k_r^- - 1} v^T(h^*) L_3 v(h^*) + \frac{1}{2} \sum_{j=k_r^- - \bar{\alpha}}^{k_r^- - \alpha - 1} \times \sum_{i=j}^{k_r^- - \alpha - 1} \sum_{h^*=i}^{k_r^- - 1} v^T(h^*) L_4 v(h^*) \\
&= (1 + k_r)^2 (1 + \omega_r)^2 V(k_r^-, x(k_r^-), y(k_r^-)) . \tag{42}
\end{aligned}$$

Applying Equations (41) and (42) successively in each interval of $[k_r, k_{r+1})$ yields the following equations:

For $k \in [k_0, k_1)$,

$$\begin{aligned}
V(k, x(k), y(k)) &\leq (1 - \epsilon)^k V(k_0, x(k_0), y(k_0)) \\
\text{and } V(k_1, x(k_1), y(k_1)) &= (1 + \mathcal{K}_1)^2 (1 + \omega_1)^2 V(k_1^-, x(k_1^-), y(k_1^-)) \\
&\leq (1 + \mathcal{K}_1)^2 (1 + \omega_1)^2 V(k_1^-, x(k_1^-), y(k_1^-)) .
\end{aligned}$$

For $k \in [k_1, k_2)$,

$$\begin{aligned}
V(k, x(k), y(k)) &\leq (1 - \epsilon)^{k - k_1} V(k_1, x(k_1), y(k_1)) \\
&\leq (1 + \mathcal{K}_1)^2 (1 + \omega_1)^2 (1 - \epsilon)^{k - k_0} V(k_0, x(k_0), y(k_0)) \\
V(k_2, x(k_2), y(k_2)) &= (1 + \mathcal{K}_2)^2 (1 + \omega_2)^2 V(k_2^-, x(k_2^-), y(k_2^-)) \\
&\leq (1 + \mathcal{K}_1)^2 (1 + \mathcal{K}_2)^2 (1 + \omega_1)^2 (1 + \omega_2)^2 (1 - \epsilon)^{k_2 - k_0} V(k_0^-, x(k_0^-), y(k_0^-)) .
\end{aligned}$$

For $k \in [k_2, k_3)$,

$$\begin{aligned}
V(k, x(k), y(k)) &\leq (1 - \epsilon)^{k - k_2} V(k_2, x(k_2), y(k_2)) \\
&\leq (1 + \mathcal{K}_1)^2 (1 + \mathcal{K}_2)^2 (1 + \omega_1)^2 (1 + \omega_2)^2 (1 - \epsilon)^{k - k_0} V(k_0, x(k_0), y(k_0)) \\
\text{and } V(k_3, x(k_3), y(k_3)) &= (1 + \mathcal{K}_3)^2 (1 + \omega_3)^2 V(k_3^-, x(k_3^-), y(k_3^-)) \\
&\leq (1 + \mathcal{K}_1)^2 (1 + \mathcal{K}_2)^2 (1 + \mathcal{K}_3)^2 (1 + \omega_1)^2 (1 + \omega_2)^2 (1 + \omega_3)^2 (1 - \epsilon)^{k_3 - k_0} \\
&\quad V(k_0^-, x(k_0^-), y(k_0^-)) .
\end{aligned}$$

Therefore, by induction, for any $[k_r, k_{r+1})$, ($r = 0, 1, 2, \dots$), we obtain

$$\begin{aligned} V(k, x(k), y(k)) &\leq (1 - \epsilon)^{k-k_r} V(k_r, x(k_r), y(k_r)) \\ &\leq \left[\prod_{i=1}^r (1 + \mathcal{K}_i)^2 (1 + \omega_i)^2 \right] (1 - \epsilon)^k V(k_0, x(k_0), y(k_0)) \\ &= V(k_0, x(k_0), y(k_0)) \exp \left\{ 2 \sum_{i=1}^m \ln(k + \mathcal{K}_i)(1 + \omega_i) + k \ln(1 - \epsilon) \right\} \\ &\leq V(k_0, x(k_0), y(k_0)) \exp \{ \vartheta(k) \}. \end{aligned}$$

Remark 8. In the recent literature for the stability criteria of NNs with BAM, different cases for time varying, leakage delays, and randomly occurring uncertainty have not been taken into account. However, in this paper, we examine the NNs with BAM and all other parameters simultaneously.

Remark 9. Li and Fu¹⁹ discussed about the new type of impulsive effects for nonlinear differential systems. This type of impulsive effects is more generalized to obtain the novel stability conditions. With the inspiration of the aforementioned work, we will ready to provide some new work on the basis of those discrete-time impulse for Cohen-Grossberg NNs with stochastic noises and mixed time delays in the near future.

4 | RANDOMLY OCCURRING UNCERTAIN DISCRETE TIME NEURAL NETWORKS

The following NNs (43) with randomly occurring uncertain parameters is the extension of NNs (3).

Now, the remodified NNs of (3) are as follows:

$$\begin{aligned} x(k+1) &= [L + \iota(k)\Delta L(k)] x(k - \rho) + [M + \wp(k)\Delta M(k)] f(y(k)) + [N + \theta(k)\Delta N(k)] g(y(k - \alpha(k))) \\ x(k_r) &= x(k_r^-) + e_r x(k_r^-) \\ y(k+1) &= [A + \check{\iota}(k)\Delta A(k)] y(k - \rho) + [B + \check{\wp}(k)\Delta B(k)] \hat{f}(x(k)) + [C + \check{\theta}(k)\Delta C(k)] \hat{g}(x(k - \beta(k))) \\ y(k_r) &= y(k_r^-) + \hat{e}_r y(k_r^-), \end{aligned} \tag{43}$$

where $\Delta L(k)$, $\Delta M(k)$, $\Delta N(k)$, $\Delta A(k)$, $\Delta B(k)$, and $\Delta C(k)$ are the uncertain parameters, which are defined as

$$[\Delta L(k), \Delta M(k), \Delta N(k), \Delta A(k), \Delta B(k), \Delta C(k)] = XJ(k) [W_l, W_m, W_n, W_a, W_b, W_c]. \tag{44}$$

From above, X , W_l , W_m , W_n , W_a , W_b , and W_c are known constant matrices, and $J(k)$ is the unknown time-varying matrix with

$$J^T(k)J(k) \leq I, \tag{45}$$

where I is the identity matrix. $\iota(k)$, $\wp(k)$, $\theta(k)$, $\check{\iota}(k)$, $\check{\wp}(k)$, $\check{\theta}(k)$ are the stochastic variables for the random nature of uncertain parameters. Each parameter is independent mutually with Bernoulli distributed white noise sequences, which take the values 0 and 1.

The probabilities are as follows:

$$\begin{aligned} \text{prob}\{\iota(k) = 1\}, \quad \text{prob}\{\iota(k) = 0\} &= 1 - \iota \\ \text{prob}\{\wp(k) = 1\}, \quad \text{prob}\{\wp(k) = 0\} &= 1 - \wp \\ \text{prob}\{\theta(k) = 1\}, \quad \text{prob}\{\theta(k) = 0\} &= 1 - \theta \\ \text{prob}\{\check{\iota}(k) = 1\}, \quad \text{prob}\{\check{\iota}(k) = 0\} &= 1 - \check{\iota} \\ \text{prob}\{\check{\wp}(k) = 1\}, \quad \text{prob}\{\check{\wp}(k) = 0\} &= 1 - \check{\wp} \\ \text{prob}\{\check{\theta}(k) = 1\}, \quad \text{prob}\{\check{\theta}(k) = 0\} &= 1 - \check{\theta} \end{aligned}$$

With the help of (44) and (45), system (43) can be written as

$$\begin{aligned}
 x(k+1) &= Lx(k - \rho(k)) + Mf(y(k)) + Ng(y(k - \alpha(k))) + X\sigma(k), \\
 \sigma(k) &= J(k)\delta(k), \\
 \delta(k) &= \iota(k)W_l x(k - \rho(k)) + \wp(k)W_m f(y(k)) + \theta(k)W_n g(y(k - \alpha(k))), \\
 x(k) &= \chi(k), \quad \kappa = -\kappa, -\kappa + 1, \dots, 0 \\
 x(k_r) &= x(k_r^-) + e_r x(k_r^-) \\
 y(k+1) &= Ay(k - \rho(k)) + B\hat{f}(x(k)) + C\hat{g}(x(k - \beta(k))) + X\sigma^*(k), \\
 \sigma^*(k) &= J(k)\delta^*(k), \\
 \delta^*(k) &= \check{\iota}(k)W_a y(k - \rho(k)) + \check{\wp}(k)W_b \hat{f}(x(k)) + \check{\theta}(k)W_c g(x(k - \beta(k))), \\
 y(k) &= \varphi(k), \quad k = -\iota, -\iota + 1, \dots, 0. \\
 y(k_r) &= y(k_r^-) + \hat{e}_r y(k_r^-).
 \end{aligned} \tag{46}$$

Theorem 2. With the assistance of assumption 2, NN (43) is globally asymptotically stable for given integers $\rho, \varrho \geq 1$, $\rho, \varrho \geq 1, 0 < \underline{\beta}, \bar{\beta}, 0 < \underline{\beta} \leq \beta(k) \leq \bar{\beta}, \underline{\alpha}, \bar{\alpha}, 0 < \underline{\alpha} \leq \alpha(k) \leq \bar{\alpha}$ satisfying $0 < \underline{\beta} \leq \bar{\beta}$ and $0 < \underline{\alpha} \leq \bar{\alpha}$. Moreover, if there exist positive definite matrices $Q_i > 0, L_i > 0, i = 1, 2, 3, 4, P, P^*, X_i > 0, i = \bar{1}, 2, 3, 4, Z_i > 0, i = 1, 2$; positive definite diagonals matrices $\varsigma_1 > 0, \varsigma_2 > 0, \varsigma_3 > 0, \varsigma_4 > 0$; a scalar $\tilde{\epsilon}, \epsilon^*$; and any matrices $T_1, T_2, W_{11}, W_{12}, W_{21}, W_{22}, Z_{11}, Z_{12}, Z_{21}, Z_{22}$ positive constant $\epsilon (\epsilon \leq 1)$, then the following LMIs hold:

$$\begin{aligned}
 (i) \quad & \sum_{i=1}^m \ln(1 + k_i)^2 (1 + \omega_i)^2 + k \ln(1 - \epsilon) \leq \vartheta(k), \\
 & \text{for any } k \in [k_r, k_{r+1}), \text{ then } \lim_{k \rightarrow +\infty} \vartheta(k) = +\infty,
 \end{aligned} \tag{47}$$

$$(ii) \quad \begin{pmatrix} X_2 & 0 & W_{11} & W_{12} \\ * & X_2 l & W_{21} & W_{22} \\ * & * & X_2 & 0 \\ * & * & * & X_2 l \end{pmatrix} \geq 0, \quad \begin{pmatrix} L_1 & T_1^T \\ * & L_1 \end{pmatrix} \geq 0, \tag{48}$$

$$(iii) \quad \begin{pmatrix} X_4 & 0 & Z_{11} & Z_{12} \\ * & X_4 l & Z_{21} & Z_{22} \\ * & * & X_4 & 0 \\ * & * & * & X_4 l \end{pmatrix} \geq 0, \quad \begin{pmatrix} L_4 & T_2^T \\ * & L_4 \end{pmatrix} \geq 0, \tag{49}$$

$$\begin{aligned}
 \Omega_1(\bar{\beta}^*) + \Omega_3 + \Omega_5 + \Omega_7(\bar{\beta}^*) + \Omega_9 + \Omega_{11}(\bar{\beta}^*) + \Omega_{13} + \Omega_{15} &< 0, \\
 \Omega_1(\underline{\beta}^*) + \Omega_3 + \Omega_5 + \Omega_7(\underline{\beta}^*) + \Omega_9 + \Omega_{11}(\underline{\beta}^*) + \Omega_{13} + \Omega_{15} &< 0,
 \end{aligned} \tag{50}$$

$$\begin{aligned}
 \Omega_2(\bar{\alpha}^*) + \Omega_4 + \Omega_6 + \Omega_8(\bar{\alpha}^*) + \Omega_{10} + \Omega_{12}(\bar{\alpha}^*) + \Omega_{14} + \Omega_{16} + \Omega_{17} &< 0, \\
 \Omega_2(\underline{\alpha}^*) + \Omega_4 + \Omega_6 + \Omega_8(\underline{\alpha}^*) + \Omega_{10} + \Omega_{12}(\underline{\alpha}^*) + \Omega_{14} + \Omega_{16} + \Omega_{18} &< 0,
 \end{aligned} \tag{51}$$

$$\begin{aligned}
 \tilde{h} &= [0 \ 0 \ \iota W_l \ 0 \ 0 \ 0 \ 0 \ 0 \ \wp W_m \ \theta W_n \ 0 \ 0 \ 0 \ 0 \ 0], \quad \tilde{h}_1 = [0 \ 0 \ \check{\iota} W_a \ 0 \ 0 \ 0 \ 0 \ 0 \ \check{\wp} W_b \ \check{\theta} W_c \ 0 \ 0 \ 0 \ 0 \ 0], \\
 \Omega_{17} &= \tilde{\epsilon} (\tilde{h}^T \tilde{h} - e_{12}^T e_{12}), \quad \Omega_{18} = \epsilon^* (\tilde{h}^T \tilde{h} - r_{12}^T r_{12}).
 \end{aligned}$$

Proof. From $J^T(k)J(k) \leq I$, we have $p^T p \leq \delta^T(k)\delta(k)$ and $q^T q \leq \sigma^T \sigma$. Then, $\exists$ a positive scalar $\tilde{\epsilon}, \epsilon^*$, which satisfies

$$\tilde{\epsilon} [\tilde{\Theta}^T(k) \tilde{h}^T \tilde{\Theta}(k) - P^T(k)P(k)] + \epsilon^* [\tilde{\Psi}^T(k) \tilde{h}^T \tilde{\Psi}(k) - q^T(k)q(k)] \geq 0, \tag{52}$$

where

$$\tilde{\Theta}^T(k) = [\Theta^T(k), \ p^T(k)], \quad \tilde{\Psi}^T(k) = [\psi^T(k), \ q^T(k)].$$

Therefore, we get

$$\begin{aligned}
 \Delta V(k) &\leq \tilde{\Theta}^T(k) [\Omega_1(\beta(k)) + \Omega_3 + \Omega_5 + \Omega_7(\beta(k)) + \Omega_9 + \Omega_{11}(\beta(k)) + \Omega_{13} + \Omega_{15} + \Omega_{17}] \tilde{\Theta}(k) \\
 &+ \tilde{\Psi}^T(k) [\Omega_2(\alpha(k)) + \Omega_4 + \Omega_6 + \Omega_8(\alpha(k)) + \Omega_{10} + \Omega_{12}(\alpha(k)) + \Omega_{14} + \Omega_{16} + \Omega_{18}] \tilde{\Psi}(k) < 0.
 \end{aligned} \tag{53}$$

□

Remark 10. There are no stability results on impulsive discrete-time BAM NNs with uncertain parameters, leakage, and different cases of time-varying delays (3) and (43) established via LMIs in available literature. Hence, in order to fulfill this gap, Theorems 1 and 2 is constructed with the assistance of a suitable novel Lyapunov-Krasovskii functional, novel summation inequality, and convex combination approach handled to guarantee the addressed BAM NNs (3) and (43) that are asymptotically stable in the mean square, which means that our results generalize the previous ones.

Remark 11. If the leakage term in NN (3) disappears (ie, $\rho, \rho = 0$), then we get

$$\begin{aligned} x(k+1) &= Lx(k) + Mf(y(k)) + Ng(y(k - \alpha(k))) \\ x(k_r) &= x(k_r^-) + e_r x(k_r^-) \\ y(k+1) &= Ay(k) + B\hat{f}(x(k)) + C\hat{g}(x(k - \beta(k))) \\ y(k_r) &= y(k_r^-) + \hat{e}_r y(k_r^-). \end{aligned} \quad (54)$$

The following corollary helps to check the asymptotic stability of the concerned NN (54).

Corollary 2. With the assistance of Assumption 2, NN system (54) is globally asymptotically stable for given integers $\rho, \rho \geq 1, \rho \geq 1, \beta, \bar{\beta}, 0 < \underline{\beta} \leq \beta(k) \leq \bar{\beta}, \alpha, \bar{\alpha}, 0 < \underline{\alpha} \leq \alpha(k) \leq \bar{\alpha}$ satisfying $0 < \underline{\beta} \leq \bar{\beta}$ and $0 < \underline{\alpha} \leq \bar{\alpha}$. Moreover, if there exist positive definite matrices $Q_i > 0, L_i > 0, i = 1, 2, 3, 4, P^*, X_i > 0, i = 1, 2, 3, 4, Z_i > 0, i = 1, 2$; positive definite diagonals matrices $\varsigma_1 > 0, \varsigma_2 > 0, \varsigma_3 > 0, \varsigma_4 > 0$; and any matrices $T_1, T_2, W_{11}, W_{12}, W_{21}, W_{22}, Z_{11}, Z_{12}, Z_{21}$, and Z_{22} positive constant $\epsilon (\epsilon \leq 1)$, then the following LMIs hold:

$$(i) \quad \sum_{i=1}^m \ln(1 + k_i)^2 (1 + \omega_i)^2 + k \ln(1 - \epsilon) \leq \vartheta(k), \quad (55)$$

for any $k \in [k_r, k_{r+1})$, then $\lim_{k \rightarrow +\infty} \vartheta(k) = +\infty$,

$$(ii) \quad \begin{pmatrix} X_2 & 0 & W_{11} & W_{12} \\ * & X_2 l & W_{21} & W_{22} \\ * & * & X_2 & 0 \\ * & * & * & X_2 l \end{pmatrix} \geq 0, \quad \begin{pmatrix} L_1 & T_1^T \\ * & L_1 \end{pmatrix} \geq 0, \quad (56)$$

$$(iii) \quad \begin{pmatrix} X_4 & 0 & Z_{11} & Z_{12} \\ * & X_4 l & Z_{21} & Z_{22} \\ * & * & X_4 & 0 \\ * & * & * & X_4 l \end{pmatrix} \geq 0, \quad \begin{pmatrix} L_4 & T_2^T \\ * & L_4 \end{pmatrix} \geq 0, \quad (57)$$

where

$$\Omega_1(\bar{\beta}^*) + \Omega_3 + \Omega_5 + \Omega_7(\bar{\beta}^*) + \Omega_9 + \Omega_{11}(\bar{\beta}^*) + \Omega_{13} + \Omega_{15} < 0, \quad (58)$$

$$\Omega_1(\underline{\beta}^*) + \Omega_3 + \Omega_5 + \Omega_7(\underline{\beta}^*) + \Omega_9 + \Omega_{11}(\underline{\beta}^*) + \Omega_{13} + \Omega_{15} < 0,$$

$$\Omega_2(\bar{\alpha}^*) + \Omega_4 + \Omega_6 + \Omega_8(\bar{\alpha}^*) + \Omega_{10} + \Omega_{12}(\bar{\alpha}^*) + \Omega_{14} + \Omega_{16} < 0, \quad (59)$$

$$\Omega_2(\underline{\alpha}^*) + \Omega_4 + \Omega_6 + \Omega_8(\underline{\alpha}^*) + \Omega_{10} + \Omega_{12}(\underline{\alpha}^*) + \Omega_{14} + \Omega_{16} < 0.$$

Proof. The parameters and the proof follows from Theorem 1. $\square$

Remark 12. If the leakage term and impulse in NNs(3) disappears (ie, $\rho, \rho = 0$), then we get

$$\begin{aligned} x(k+1) &= Lx(k) + Mf(y(k)) + Ng(y(k - \alpha(k))) \\ y(k+1) &= Ay(k) + B\hat{f}(x(k)) + C\hat{g}(x(k - \beta(k))). \end{aligned} \quad (60)$$

The following corollary helps to check the stability of the NN system (60).

Corollary 3. With the assistance of Assumption 2, NN system (60) is globally asymptotically stable for given integers $\rho, \rho \geq 1, \rho \geq 1, \beta, \bar{\beta}, 0 < \underline{\beta} \leq \beta(k) \leq \bar{\beta}, \alpha, \bar{\alpha}, 0 < \underline{\alpha} \leq \alpha(k) \leq \bar{\alpha}$ satisfying $0 < \underline{\beta} \leq \bar{\beta}$ and $0 < \underline{\alpha} \leq \bar{\alpha}$. Moreover, if there exist positive definite matrices $Q_i > 0, L_i > 0, i = 1, 2, 3, 4, P^*, X_i > 0, i = 1, 2, 3, 4, Z_i > 0, i = 1, 2$; positive definite

diagonals matrices $\varsigma_1 > 0, \varsigma_2 > 0, \varsigma_3 > 0, \varsigma_4 > 0$; and any matrices $T_1, T_2, W_{11}, W_{12}, W_{21}, W_{22}, Z_{11}, Z_{12}, Z_{21}$, and Z_{22} , then the following LMIs hold:

$$(i) \begin{pmatrix} X_2 & 0 & W_{11} & W_{12} \\ * & X_2 l & W_{21} & W_{22} \\ * & * & X_2 & 0 \\ * & * & * & X_2 l \end{pmatrix} \geq 0, \quad \begin{pmatrix} L_1 & T_1^T \\ * & L_1 \end{pmatrix} \geq 0, \quad (61)$$

$$(ii) \begin{pmatrix} X_4 & 0 & Z_{11} & Z_{12} \\ * & X_4 l & Z_{21} & Z_{22} \\ * & * & X_4 & 0 \\ * & * & * & X_4 l \end{pmatrix} \geq 0, \quad \begin{pmatrix} L_4 & T_2^T \\ * & L_4 \end{pmatrix} \geq 0, \quad (62)$$

$$\Omega_1(\bar{\beta}^*) + \Omega_3 + \Omega_5 + \Omega_7(\bar{\beta}^*) + \Omega_9 + \Omega_{11}(\bar{\beta}^*) + \Omega_{13} + \Omega_{15} < 0, \quad (63)$$

$$\Omega_1(\underline{\beta}^*) + \Omega_3 + \Omega_5 + \Omega_7(\underline{\beta}^*) + \Omega_9 + \Omega_{11}(\underline{\beta}^*) + \Omega_{13} + \Omega_{15} < 0,$$

$$\Omega_2(\bar{\alpha}^*) + \Omega_4 + \Omega_6 + \Omega_8(\bar{\alpha}^*) + \Omega_{10} + \Omega_{12}(\bar{\alpha}^*) + \Omega_{14} + \Omega_{16} < 0, \quad (64)$$

$$\Omega_2(\underline{\alpha}^*) + \Omega_4 + \Omega_6 + \Omega_8(\underline{\alpha}^*) + \Omega_{10} + \Omega_{12}(\underline{\alpha}^*) + \Omega_{14} + \Omega_{16} < 0.$$

Proof. The proof of the Corollary 3 is omitted. Hence, it follows from Theorem 1. $\square$

Remark 13. With the research motivation of function of the human brain, artificial NNs are able to learn from experience. These powerful problem solvers are highly effective where traditional formal analysis would be complex or impossible. Their strength lies in their ability to make sense out of difficult, noisy, or nonlinear data. Neural networks can provide robust solutions to problems in a wide range of disciplines, particularly areas involving filtering, pattern recognition, classification, prediction, function approximation, and optimization. Naturally, the passivity issue is internally stable for NNs. Time delay is unavoidable in real-world NNs. In practical implementations of electrical circuits, time delays occurred in the finite switching speed of amplifiers and signal transmissions among neurons. As is well known that the existence of time delays in discrete and leakage terms causes poor performance, it results in oscillation and instability of the concerned NNs. In order to overcome the situations like oscillations, chaos, and instability, a novel Lyapunov-Krasovskii functional have been constructed to obtain the stability in asymptotic sense.

Remark 14. Based on Examples 1 and 2, it is easy to see that the conclusion results are less conserved than the existing ones in other works.^{55,56,58-62} Therefore, the proposed method is an improvement over the available sources.

5 | ILLUSTRATIVE EXAMPLES

In this section, we will present 4 numerical examples with their simulations to guarantee the superiority and validity of our theoretical results.

Example 1. Consider a 2-dimensional impulsive discrete-time BAM NN with time-varying leakage delays.

The parameters are as follows:

$$A = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad B = \begin{pmatrix} 0.7 & -0.6 \\ 0.9 & 0.5 \end{pmatrix}, \quad C = \begin{pmatrix} 0.01 & 0.6 \\ 0.5 & -1.55 \end{pmatrix},$$

$$L = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.05 \end{pmatrix}, \quad N = \begin{pmatrix} 1.05 & 2.86 \\ 0.59 & 1.5 \end{pmatrix}, \quad M = \begin{pmatrix} 0.4 & 0.8 \\ -0.9 & 0.05 \end{pmatrix},$$

$$I_m = -2, \quad J_m = -1, \quad \rho = 2; \quad \rho = 3.$$

The activation functions are taken as

$$f(y(k)) = g(y(k)) = \begin{pmatrix} \tanh(-0.2y) \\ \tanh(-0.5y) \end{pmatrix}, \quad \hat{f}(x(k)) = \hat{g}(x(k)) = \begin{pmatrix} \tanh(-0.9x) \\ \tanh(-0.7x) \end{pmatrix}.$$

Our main aim in this example is to estimate the maximum allowable upper bound delay $\bar{\alpha}, \bar{\beta}$, for given lower bound $\underline{\alpha}, \underline{\beta}$. By solving MATLAB LMI toolbox, one can easily obtain the feasible solution for any time delay satisfying $0 < \alpha(k) < \bar{\alpha} =$ for any large finite value and $0 < \beta(k) < \bar{\beta} =$ for any large finite value. For example, if we take

$\underline{\alpha} = \underline{\beta} = 15$, $\alpha_M = \beta_M = 19$, $f_1^+ = 2$, $f_1^- = -4$, $f_2^+ = 1$, $f_2^- = -1$, $g_1^+ = 0.2$, $g_1^- = 0.2$, $g_2^+ = 0.2$, $g_2^- = 0.2$, $u_1^+ = 4$, $u_1^- = -0.2$, $u_2^+ = 4$, $u_2^- = 0.2$, $d_1^+ = 3$, $d_1^- = 0.2$, $d_2^+ = 3$, $d_2^- = -0.2$ with

$$F_1 = \begin{pmatrix} -8 & 0 \\ 0 & -1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.04 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 0.6 & 0 \\ 0 & -0.6 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1.6 & 0 \\ 0 & 1.4 \end{pmatrix}, \quad E_1 = \begin{pmatrix} -0.8 & 0 \\ 0 & 0.8 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1.9 & 0 \\ 0 & 2.1 \end{pmatrix},$$

now we obtain the feasible solution as follows:

$$P = \begin{pmatrix} 14.2307 & 0 \\ 0 & 14.2307 \end{pmatrix}, \quad P^* = \begin{pmatrix} 14.2307 & 9 \\ 0 & 14.2307 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 5.6476 & -0.2995 \\ -0.2995 & 5.2501 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 6.6733 & -0.2947 \\ -0.2947 & 7.3161 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 12.8182 & -1.1376 \\ -1.1376 & 11.7294 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 11.8916 & -1.0134 \\ -1.0134 & 13.7136 \end{pmatrix},$$

$$L_1 = \begin{pmatrix} 9.0090 & 0.2826 \\ 0.2826 & 8.4789 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 9.0090 & 0.2826 \\ 0.2826 & 8.4789 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 2.4052 & -1.7089 \\ -1.7089 & 5.7779 \end{pmatrix},$$

$$L_4 = \begin{pmatrix} 11.2146 & 0.4042 \\ 0.4042 & 9.7322 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0.3898 & -0.0543 \\ -0.0543 & 0.2728 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1.3131 & 0.5101 \\ 0.5101 & 1.4019 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 2.5162 & -0.9822 \\ -0.9822 & 3.4991 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 8.5432 & -7.5302 \\ -7.5302 & 23.5738 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} 4.6878 & 2.6326 \\ 2.6326 & 8.7477 \end{pmatrix},$$

$$Z_2 = \begin{pmatrix} 1.3037 & 0.0978 \\ 0.0978 & 1.0070 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 47.2020 & -0.0658 \\ -0.0658 & 47.3035 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 78.9319 & 0.9995 \\ 0.9995 & 80.8774 \end{pmatrix},$$

$$W_{11} = \begin{pmatrix} 25.9258 & 0.3999 \\ 0.3999 & 25.1516 \end{pmatrix}, \quad W_{12} = \begin{pmatrix} 57.7216 & -2.1420 \\ -2.1420 & 60.4385 \end{pmatrix}, \quad W_{21} = \begin{pmatrix} 25.9258 & 0.3999 \\ 0.3999 & 25.1516 \end{pmatrix},$$

$$W_{22} = \begin{pmatrix} 57.7216 & -2.1420 \\ -2.1420 & 60.4385 \end{pmatrix}, \quad Z_{11} = \begin{pmatrix} 50.6999 & 4.3520 \\ 4.3520 & 50.0765 \end{pmatrix}, \quad Z_{12} = \begin{pmatrix} 50.6999 & 4.3520 \\ 4.3520 & 50.0765 \end{pmatrix},$$

$$Z_{21} = \begin{pmatrix} 50.6999 & 4.3520 \\ 4.3520 & 50.0765 \end{pmatrix}, \quad Z_{22} = \begin{pmatrix} 28.0008 & -1.7275 \\ -1.7275 & 28.4720 \end{pmatrix}, \quad \varsigma_1 = \begin{pmatrix} 22.3801 & 0 \\ 0 & 25.7000 \end{pmatrix},$$

$$\varsigma_2 = \begin{pmatrix} 18.8123 & 0 \\ 0 & 17.0048 \end{pmatrix}, \quad \varsigma_3 = \begin{pmatrix} 3.9571 & 0 \\ 0 & 17.8650 \end{pmatrix}, \quad \varsigma_4 = \begin{pmatrix} 30.2156 & 0 \\ 0 & 30.2156 \end{pmatrix}.$$

On the basis of Theorem 1, we can attain that the NN system (3) with above given parameters are asymptotically stable with the help of Lyapunov-Krasovskii functionals, and the state trajectories $x_1(k)$, $x_2(k)$, $y_1(k)$, $y_2(k)$ of above impulsive discrete-time BAM NNs with time-varying leakage delays are depicted in Figure 1.

Example 2. Consider a 2-dimensional impulsive discrete-time BAM NN with randomly occurring uncertainties and time-varying leakage delays.

The following parameters help check the feasibility of NN system (43):

$$A = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad B = \begin{pmatrix} 0.7 & -0.6 \\ 0.9 & 0.5 \end{pmatrix}, \quad C = \begin{pmatrix} 0.01 & 0.6 \\ 0.5 & -1.55 \end{pmatrix},$$

$$L = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.05 \end{pmatrix}, \quad N = \begin{pmatrix} 1.05 & 2.86 \\ 0.59 & 1.5 \end{pmatrix}, \quad M = \begin{pmatrix} 0.4 & 0.8 \\ -0.9 & 0.05 \end{pmatrix},$$

$$W_l = \begin{pmatrix} 0.5 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}, \quad W_m = \begin{pmatrix} 0.6 & 0.7 \\ 0.8 & 0.9 \end{pmatrix}, \quad W_n = \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{pmatrix}$$

$$W_a = \begin{pmatrix} 0.9 & 0.4 \\ 0.6 & 0.3 \end{pmatrix}, \quad W_b = \begin{pmatrix} 0.8 & 0.7 \\ 0.6 & 0.5 \end{pmatrix}, \quad W_c = \begin{pmatrix} 0.8 & 0.9 \\ 0.6 & 0.7 \end{pmatrix}$$

$$X = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_m = -2, \quad J_m = -1, \quad \rho(k) = 2; \quad \rho(k) = 1;$$

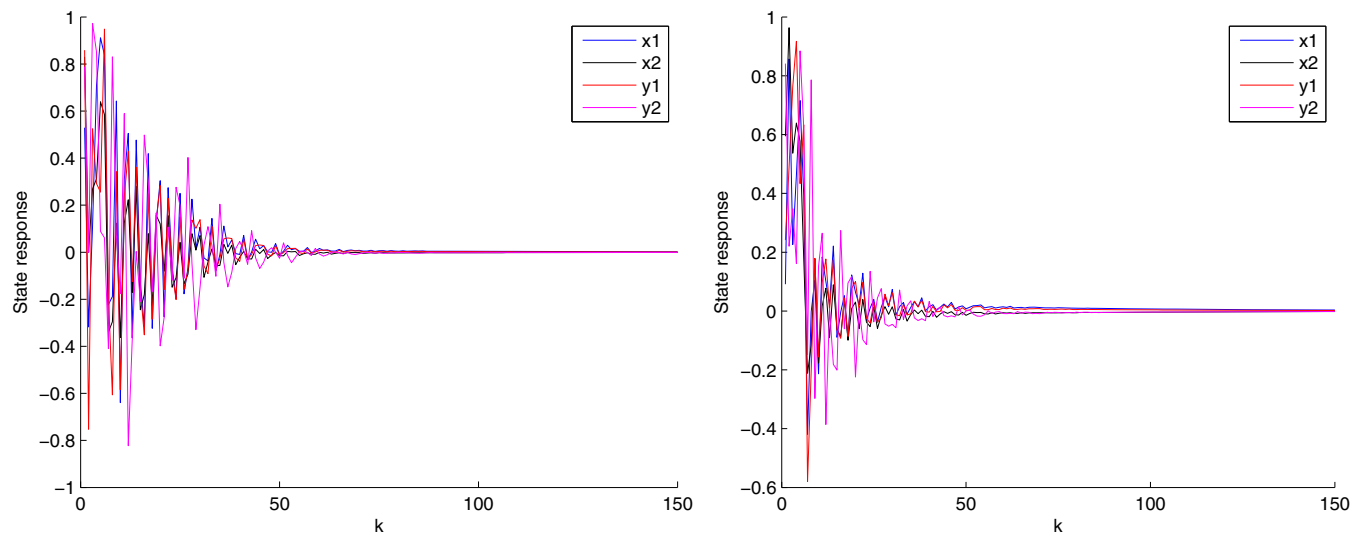


FIGURE 1 For $I_m = -2$, $J_m = -1$, the state response $x(k)$, $y(k)$ of (3) with impulsive and nonimpulsive effects [Colour figure can be viewed at wileyonlinelibrary.com]

$$\begin{aligned} \theta &= 0.3; & \varphi &= 0.9; & \iota &= 0.7; \\ \check{\theta} &= 0.5; & \check{\varphi} &= 0.8; & \check{\iota} &= 0.6. \end{aligned}$$

The activation functions are taken as

$$f(y(k)) = g(y(k)) = \begin{pmatrix} \exp(0.99y) \\ \exp(0.99y) \end{pmatrix}, \quad \hat{f}(x(k)) = \hat{g}(x(k)) = \begin{pmatrix} \exp(0.99x) \\ \exp(0.99x) \end{pmatrix}.$$

Our main aim in this example is to estimate the maximum allowable upper bound delay $\bar{\alpha}$, $\bar{\beta}$ for given lower bound $\underline{\alpha}$, $\underline{\beta}$. By solving MATLAB LMI toolbox, one can easily obtain the feasible solution for any time delay satisfying $0 < \alpha(k) < \bar{\alpha}$ for any large finite value and $0 < \beta(k) < \bar{\beta}$ for any large finite value. For example, if we take $\underline{\alpha} = \underline{\beta} = 17$, $\bar{\alpha} = \bar{\beta} = 19$, $f_1^+ = 2$, $f_1^- = -4$, $f_2^+ = 1$, $f_2^- = -1$, $g_1^+ = 0.2$, $g_1^- = 0.2$, $g_2^+ = 0.2$, $g_2^- = 0.2$, $u_1^+ = 4$, $u_1^- = -0.2$, $u_2^+ = 4$, $u_2^- = 0.2$, $d_1^+ = 3$, $d_1^- = 0.2$, $d_2^+ = 3$, $d_2^- = -0.2$ with

$$\begin{aligned} F_1 &= \begin{pmatrix} -8 & 0 \\ 0 & -1 \end{pmatrix}, & F_2 &= \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, & G_1 &= \begin{pmatrix} 0.04 & 0 \\ 0 & 0.04 \end{pmatrix}, & G_2 &= \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \\ D_1 &= \begin{pmatrix} 0.6 & 0 \\ 0 & -0.6 \end{pmatrix}, & D_2 &= \begin{pmatrix} 1.6 & 0 \\ 0 & 1.4 \end{pmatrix}, & E_1 &= \begin{pmatrix} -0.8 & 0 \\ 0 & 0.8 \end{pmatrix}, & E_2 &= \begin{pmatrix} 1.9 & 0 \\ 0 & 2.1 \end{pmatrix}, \end{aligned}$$

now we obtain the feasible solution as follows:

$$\begin{aligned} P &= \begin{pmatrix} 13.0984 & 0 \\ 0 & 13.0984 \end{pmatrix}, & P^* &= \begin{pmatrix} 13.0984 & 0 \\ 0 & 13.0984 \end{pmatrix}, & Q_1 &= \begin{pmatrix} 5.1647 & -0.5905 \\ -0.5905 & 5.2168 \end{pmatrix}, \\ Q_2 &= \begin{pmatrix} 6.6214 & -0.6480 \\ -0.6480 & 6.2852 \end{pmatrix}, & Q_3 &= \begin{pmatrix} 12.3656 & -2.6072 \\ -2.6072 & 12.2728 \end{pmatrix}, & Q_4 &= \begin{pmatrix} 12.6772 & -2.3017 \\ -2.3017 & 11.1416 \end{pmatrix}, \\ L_1 &= \begin{pmatrix} 8.7042 & 1.3365 \\ 1.3365 & 10.4892 \end{pmatrix}, & L_2 &= \begin{pmatrix} 8.7042 & 1.3365 \\ 1.3365 & 10.4892 \end{pmatrix}, & L_3 &= \begin{pmatrix} 5.0516 & -2.7125 \\ -2.7125 & 3.5762 \end{pmatrix}, \\ L_4 &= \begin{pmatrix} 10.8867 & 0.8061 \\ 0.8061 & 8.6134 \end{pmatrix}, & X_1 &= \begin{pmatrix} 1.1049 & -0.0913 \\ -0.0913 & 0.2742 \end{pmatrix}, & X_2 &= \begin{pmatrix} 10.1973 & 5.1026 \\ 5.1026 & 4.0741 \end{pmatrix}, \\ X_3 &= \begin{pmatrix} 4.1238 & -1.4093 \\ -1.4093 & 2.4421 \end{pmatrix}, & X_4 &= \begin{pmatrix} 17.0187 & -10.2750 \\ -10.2750 & 12.9432 \end{pmatrix}, & Z_1 &= \begin{pmatrix} 11.2846 & 7.9802 \\ 7.9802 & 8.5331 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
Z_2 &= \begin{pmatrix} 1.3791 & 0.3158 \\ 0.3158 & 0.8579 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 42.9432 & -0.6605 \\ -0.6605 & 42.1528 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 69.6692 & -0.4379 \\ -0.4379 & 76.5756 \end{pmatrix}, \\
W_{11} &= \begin{pmatrix} 29.0955 & 1.8672 \\ 1.8672 & 21.6173 \end{pmatrix}, \quad W_{12} = \begin{pmatrix} 40.0968 & -7.0883 \\ -7.0883 & 66.1002 \end{pmatrix}, \quad W_{21} = \begin{pmatrix} 29.0955 & 1.8672 \\ 1.8672 & 21.6173 \end{pmatrix}, \\
W_{22} &= \begin{pmatrix} 40.0968 & -7.0883 \\ -7.0883 & 66.1002 \end{pmatrix}, \quad Z_{11} = \begin{pmatrix} 41.4100 & 4.8665 \\ 4.8665 & 51.5403 \end{pmatrix}, \quad Z_{12} = \begin{pmatrix} 41.4100 & 4.8665 \\ 4.8665 & 51.5403 \end{pmatrix}, \\
Z_{21} &= \begin{pmatrix} 41.4100 & 4.8665 \\ 4.8665 & 51.5403 \end{pmatrix}, \quad Z_{22} = \begin{pmatrix} 28.2065 & -1.7872 \\ -1.7872 & 24.5841 \end{pmatrix}, \quad \varsigma_1 = \begin{pmatrix} 20.5704 & 0 \\ 0 & 23.6270 \end{pmatrix}, \\
\varsigma_2 &= \begin{pmatrix} 17.2641 & 0 \\ 0 & 15.6031 \end{pmatrix}, \quad \varsigma_3 = \begin{pmatrix} 3.6739 & 0 \\ 0 & 16.3911 \end{pmatrix}, \quad \varsigma_4 = \begin{pmatrix} 27.7475 & 0 \\ 0 & 27.7475 \end{pmatrix}, \\
\epsilon^* &= 15.2545, \quad \tilde{\epsilon} = 32.6248.
\end{aligned}$$

According to Theorem 2, we can acquire that the NN system (43) with above given parameters are asymptotically stable with the help of Lyapunov-Krasovskii functionals, and state trajectories $x_1(k)$, $x_2(k)$, $y_1(k)$, $y_2(k)$ of above impulsive discrete-time BAM NNs with time-varying leakage delays and randomly occurring parameter uncertainties are depicted in Figure 2.

Example 3. Consider a 2-dimensional impulsive discrete-time BAM NN without time-varying leakage delays with the following parameters:

$$\begin{aligned}
A &= \begin{pmatrix} 0.02 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad B = \begin{pmatrix} 0.7 & -0.6 \\ 0.9 & 0.5 \end{pmatrix}, \quad C = \begin{pmatrix} 0.01 & 0.6 \\ 0.5 & -1.55 \end{pmatrix}, \\
L &= \begin{pmatrix} 0.3 & 0 \\ 0 & 0.05 \end{pmatrix}, \quad N = \begin{pmatrix} 1.05 & 2.86 \\ 0.59 & 1.5 \end{pmatrix}, \quad M = \begin{pmatrix} 0.4 & 0.8 \\ -0.9 & 0.05 \end{pmatrix}.
\end{aligned}$$

The activation functions are taken as

$$f(y(k)) = g(y(k)) = \begin{pmatrix} \tanh(-0.9y) \\ \tanh(-0.9y) \end{pmatrix}, \quad \hat{f}(x(k)) = \hat{g}(x(k)) = \begin{pmatrix} \tanh(-0.9x) \\ \tanh(-0.9x) \end{pmatrix}.$$

According to Corollary 2, we can obtain that the NN system (54) with the above given parameters are asymptotically stable with the help of Lyapunov functionals, and state trajectories $x_1(k)$, $x_2(k)$, $y_1(k)$, $y_2(k)$ of above discrete-time impulsive BAM NNs are depicted in Figure 3.

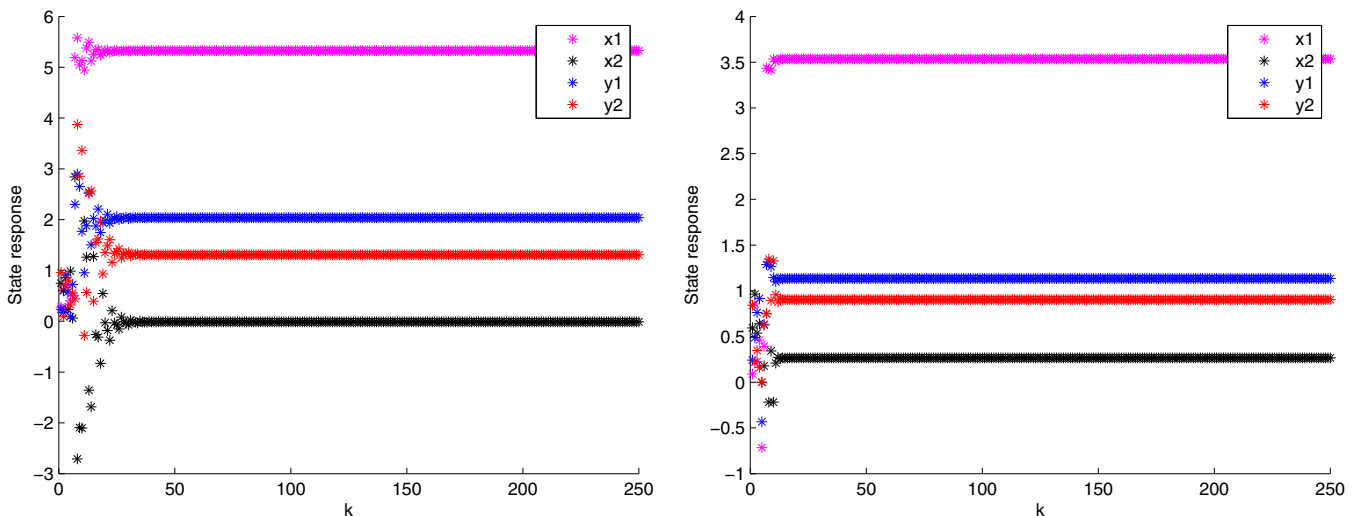


FIGURE 2 For $I_m = -2$ $J_m = -1$, the state response $x(k)$, $y(k)$ of (43) with impulsive and nonimpulsive effects [Colour figure can be viewed at wileyonlinelibrary.com]

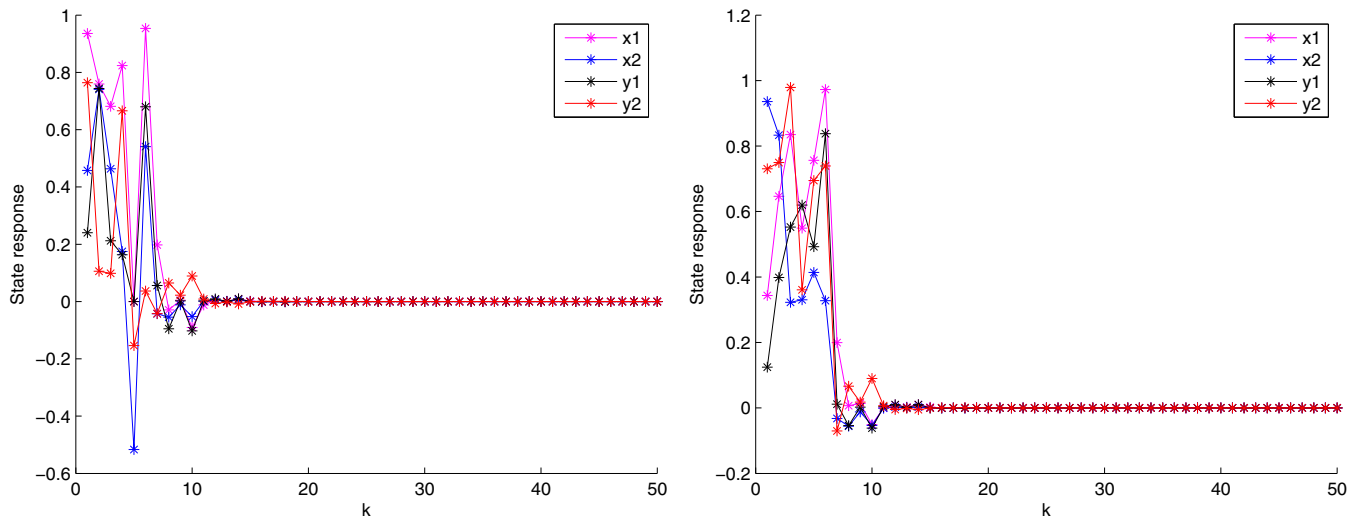


FIGURE 3 For $I_m = -1$ $J_m = -2$, the state response $x(k), y(k)$ of (54) with impulsive and nonimpulsive effects [Colour figure can be viewed at wileyonlinelibrary.com]

Example 4. Consider a 2-dimensional discrete-time BAM NN without time-varying leakage delays and impulse with the following parameters:

$$A = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad B = \begin{pmatrix} 0.7 & -0.6 \\ 0.9 & 0.5 \end{pmatrix}, \quad C = \begin{pmatrix} 0.01 & 0.6 \\ 0.5 & -1.55 \end{pmatrix},$$

$$L = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.05 \end{pmatrix}, \quad N = \begin{pmatrix} 1.05 & 2.86 \\ 0.59 & 1.5 \end{pmatrix}, \quad M = \begin{pmatrix} 0.4 & 0.8 \\ -0.9 & 0.05 \end{pmatrix}.$$

The activation functions are taken as

$$f(y(k)) = g(y(k)) = \begin{pmatrix} \tanh(-0.7y) \\ \tanh(-0.7y) \end{pmatrix}, \quad \hat{f}(x(k)) = \hat{g}(x(k)) = \begin{pmatrix} \tanh(-0.9x) \\ \tanh(-0.9x) \end{pmatrix}.$$

According to Corollary 4.5, we can obtain that the NN system (61) with the above given parameters are asymptotically stable with the help of Lyapunov functions, and state trajectories $x_1(k), x_2(k), y_1(k), y_2(k)$ of above discrete-time BAM NNs are depicted in Figure 4.

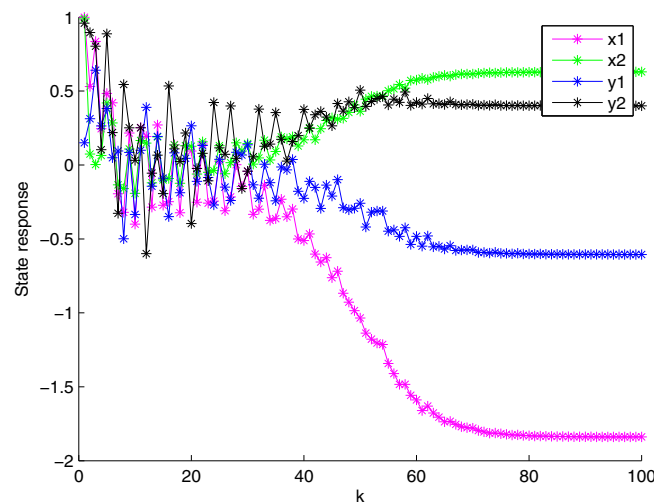


FIGURE 4 The state response $x(k), y(k)$ of (60) [Colour figure can be viewed at wileyonlinelibrary.com]

6 | CONCLUSIONS

This paper is concerned with the stability analysis of uncertain impulsive discrete-time BAM NNs with time-varying and leakage delays. By the aid of Lyapunov-Krasovskii functionals, novel conditions have been entrenched to guarantee the stability of the designed discrete-time BAM NNs in terms of LMIs, which can be adequately checked by MATLAB LMI toolbox. Novel summation inequality is approached, which is less conservative than Jensen's inequality. In addition, in this paper, leakage delay is considered as a constant delay. Illustrative examples are finally offered to exemplify the application and effectiveness of the obtained results. It would be interesting to extend the results proposed in this paper to H_∞ control for NNs with mode leakage time-varying and Markovian jumping parameters by Wirtinger-based inequality, and corresponding results will appear in the near future.

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